ROUGHNESS IN LA-SEMIGROUPS

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A Dissertation Submitted in the Partial Fulfillment of the Requirements for the Degree of

MASTERS OF PHILOSOPHY

IN

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CERTIFICATE

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF PHILOSOPHY

We accept this dissertation as conforming to the required standard

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2010

Dedicated

To

My Father & Mother

Who are the most precious gems of my life.
Who've always given me perpetual love, care, and cheers. Whose prayers have always been a source of great inspiration for me and whose sustained hope in me led me to where I stand today.

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Naveed Yaqoob

PREFACE

Zdzisław Pawlak [11] introduced the notion of rough sets, in the year 1982. The theory of rough sets has emerged as another major mathematical approach for managing uncertainty that arises from inexact, noisy or incomplete information. Later on, in connection with algebraic structures, R. Biswas and S. Nanda [1] gave the notion of rough subgroups, and N. Kuroki [6] introduced the rough ideals in semigroups and gave some properties of the lower and the upper approximations in the semigroups.

In this dissertation we have introduced the concept of rough subsets in LA-semigroups, which extends the notion of LA-semigroup by including the algebraic structures in rough sets and studied some of their properties. This dissertation consists of three chapters. The first chapter contains some basic definitions, results and examples of LA-semigroups which are relevant to our work. In second chapter, we have discussed preliminaries of rough sets and presented the equivalence relation on a set X and then presented the lower and the upper approximation of a subset and also presented the properties of approximations. In third chapter, we have defined rough subsets in LA-semigroup with respect to the congruence defined in LA-semigroup and discussed some of its properties. Rough ideals over the LA-semigroups are defined in natural way and rough sets with respect to idempotent congruences are also defined. We have also defined rough m-systems in LA-semigroups. Lastly, we have defined rough ideals in the quotient LA-semigroups.

Contents

Chapter 1

Fundamental Concepts

In this introductory chapter we present a brief summary of basic definitions and preliminary results of LA-semigroups which will be of our great help in further pursuits.

1.1 LA-semigroups, Basic Definitions and Examples

Definition 1 *[4] A groupoid $(S, \cdot)$ is called a left almost semigroup, abbreviated as an LA-semigroup, if it satisfies left invertive law:*

$$(a \cdot b) \cdot c = (c \cdot b) \cdot a \quad \text{for all } a, b, c \in S.$$

Similarly, a groupoid $(S, \cdot)$ is called a right almost semigroup, abbreviated as an RA-semigroup, if it satisfies the right invertive law:

$$a \cdot (b \cdot c) = c \cdot (b \cdot a) \quad \text{for all } a, b, c \in S.$$

LA-semigroups are also known as AG-groupoids.

Example 2 *[4] The set $S = \{x, y, z\}$, under "$\cdot$" defined below in the form of Cayley table is* *an LA-semigroup.*

$\cdot$	x	y	z
x	x	y	z
y	z	x	y
z	y	z	x

Example 3 *[7] Let $(\mathbb{Z},+)$ denote the commutative group of integers under addition. Define a binary operation "$*$" in $\mathbb{Z}$ as follows:*

$$a * b = b - a \text{ for all } a, b \in \mathbb{Z}.$$

where "$-$"denotes the ordinary subtraction of integers. Then $(\mathbb{Z}, *)$ is an LA-semigroup, because

$$\begin{aligned} (a * b) * c &= (b-a) * c = c - (b-a) = c - b + a \\ (c * b) * a &= (b-c) * a = a - (b-c) = a - b + c = c - b + a \\ So\ (a * b) * c &= (c * b) * a \end{aligned}$$

Example 4 *[7] Let $(\mathbb{R},+)$ denote the group of real numbers under ordinary addition. Define a binary operation "$*$" in $\mathbb{R}$ as follows:*

$$a * b = b - a \text{ for all } a, b \in \mathbb{R}.$$

where "$-$"denotes the ordinary subtraction of real numbers. Then $(\mathbb{R}, *)$ is an LA-semigroup.

Theorem 5 *[7] Let $(S, \cdot)$ be a commutative group with identity e. Let "$*$" be a binary operation defined in S as follows:*

$$a * b = ba^{-1} \text{ or } \left(a * b = a^{-1}b\right) \text{ for all } a,\ b \in S.$$

Then $(S, *)$ is an LA-semigroup with left identity e.

Proof. Let $a, b, c \in S$. Then

$$\begin{aligned}(a * b) * c &= (ba^{-1}) * c \\ &= c(ba^{-1})^{-1} \\ &= c(ab^{-1}) \\ &= ab^{-1}c, \text{ because } S \text{ is a commutative group.}\end{aligned}$$

$$\begin{aligned}\text{Also } (c * b) * a &= (bc^{-1}) * a \\ &= a(bc^{-1})^{-1} = a(cb^{-1}) \\ &= ab^{-1}c, \text{ because } S \text{ is a commutative group.}\end{aligned}$$

Thus $(a * b) * c = (c * b) * a$. Hence $(S, *)$ is an LA-semigroup. As

$$\begin{aligned}e * a &= ae^{-1} \\ &= ae = a \text{ for all } a \in S.\end{aligned}$$

It follows that e is the left identity in $(S, *)$. ■

Remark 6 *The LA-semigroup $(S, *)$ is referred to as an LA-semigroup defined by the commutative group $(S, \cdot)$.*

Definition 7 *[5] A nonempty subset B of an LA-semigroup S is called an LA-subsemigroup of S if $BB \subseteq B$.*

Definition 8 *[9] An element a of an LA-semigroup S is called an idempotent if $a = a^2$.*

Example 9 *[7] Let $S = \{e, f, a, b, c\}$ be an LA-semigroup S with multiplication defined by the*

Cayley table.

$\cdot$	e	f	a	b	c
e	e	f	a	b	c
f	f	f	f	b	c
a	a	f	e	b	c
b	c	c	c	f	b
c	b	b	b	c	f

Here e and f are idempotents in S.

Theorem 10 *[4] In an LA-semigroup S*

$$(ab)(cd) = (ac)(bd) \text{ for all } a, b, c, d \in S.$$

Proof. Let $a, b, c, d \in S$. Then

$$\begin{aligned} (ab)(cd) &= \{(cd)b\}a \\ &= \{(bd)c\}a \\ &= (ac)(bd). \end{aligned}$$

The law $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in S$, is known as medial law. ∎

Theorem 11 *[7] If an LA-semigroup S has left identity e, then it is unique.*

Proof. If there exists another left identity say f, then

$$ef = f \text{ and } fe = e$$

and

$$f = ef = (ee)f = (fe)e = ee = e.$$

This completes the proof. ∎

Following is an example of an LA-semigroup with left identity.

Example 12 *[4] Let $S = \{x, y, z\}$ be an LA-semigroup with the multiplication given in the following table.*

$\cdot$	x	y	z
x	x	y	z
y	z	x	y
z	y	z	x

S is an LA-semigroup with left identity x.

Lemma 13 *[8] If S is an LA-semigroup with left identity e, then*

$$a(bc) = b(ac) \text{ for all } a, b, c \in S.$$

Proof. If $a, b, c \in S$, then

$$\begin{aligned} a(bc) &= (ea)(bc) = (eb)(ac) \\ &= b(ac). \end{aligned}$$

This completes the proof. ∎

Definition 14 *[8] An LA-semigroup S is called a locally associative LA-semigroup if and only if $(aa)a = a(aa)$ for all $a \in S$.*

Theorem 15 *[7] If an LA-semigroup S has the right identity e, then e is also a left identity and hence is the identity in S.*

Theorem 16 *[7] An LA-semigroup S with right identity e, is a commutative semigroup with identity.*

Lemma 17 *[5] If S is an LA-semigroup with left identity e, then $SS = S$ and $S = eS = Se$.*

Proof. Suppose S is an LA-semigroup with left identity e and $x \in S$, then

$$\begin{aligned} x &= ex \in SS \text{ and so } S \subseteq SS. \text{ that is} \\ S &= SS \end{aligned}$$

Now using the facts $eS = S$, $SS = S$ and left invertive law.

$$Se = (SS)\,e = (eS)\,S = SS = S$$

Hence $S = eS = Se$. ■

Remark 18 *[5] If S is an LA-semigroup with left identity e, then $S = S^2$, but converse is not necessarily true.*

Definition 19 *[2] Let S and S' be two LA-semigroups. A mapping $f : S \to S'$ is said to be homomorphism if*

$$f\,(ab) = f\,(a)\,f\,(b) \quad \text{for all} \quad a, b \in S.$$

We denote by $e^{'}$ the identity of S' .

Definition 20 *[2] Let f and g be binary relations on an LA-semigroup S. Then the product fog of f and g is defined as follows:*

$$fog = \{(a, b) \in S \times S : (\exists\ c \in S)\ \ (a, c) \in f \ \ \text{and}\ \ (b, c) \in g\ \}$$

Assume f and g are congruence relations on an LA-semigroup S. Then fog is a congruence if and only if $fog = gof$.

Definition 21 *[2] If X and Y be any two sets and $f : X \longrightarrow Y$ is a map, then*

$$\begin{aligned} fof^{\,-1} &= \{(x, y) \in X \times X : (\exists\ z \in X)\ \ (x, z) \in f,\ \ (y, z) \in f\ \} \\ &= \{(x, y) \in X \times X :\ f\,(x) = f\,(y)\} \end{aligned}$$

is an equivalence relation.

Remark 22 *[2] The equivalence relation $fof^{\,-1}$ is called the kernel of f, and we write $fof^{\,-1} = \ker f$.*

Definition 23 *[9] Let S be an LA-semigroup. A relation ρ on the set S is called left compatible if*

$$(\text{ for all } s, t, a \in S) \quad (s, t) \in \rho \text{ implies } (as, at) \in \rho,$$

and is called right compatible if

$$(\text{ for all } s,t,a \in S) \quad (s,t) \in \rho \text{ implies } (sa,ta) \in \rho.$$

It is called compatible if

$$(\text{ for all } s,t,s',t' \in S) \quad (s,t) \in \rho \text{ and } (s',t') \in \rho \text{ implies } (ss',tt') \in \rho.$$

A left (right) compatible equivalence relation is called left (right) congruence. A compatible equivalence relation is called a congruence relation.

Definition 24 *[9] A congruence ρ on S is called complete congruence if*

$$[a]_\rho [b]_\rho = [ab]_\rho \quad \text{for all } a,b \in S$$

Definition 25 *[9] If ρ is a congruence relation on an LA-semigroup S, then we can define a binary operation in S/ρ in a natural way as*

$$(a\rho)(b\rho) = (ab)\rho.$$

The left invertive law holds in S/ρ.

Definition 26 *A congruence relation ρ on an LA-semigroup S is called an idempotent congruence if the quotient LA-semigroup S/ρ is an idempotent LA-semigroup.*

Theorem 27 *[2] Let S and T be two LA-semigroups, and $f : S \longrightarrow T$ be a homomorphism and*

$$\kappa = \ker f = \{(x,y) \in S \times S : f(x) = f(y)\}$$

is congruence relation on S. Also κ is called kernel of f.

1.2 Ideals in LA-semigroups

Definition 28 *[5] Let S be an LA-semigroup. A nonempty subset I of S is called a left (right) ideal of S if $SI \subseteq I$ $(IS \subseteq I)$, and is called an ideal if it is a left and a right ideal.*

Intersection of any family of left (right) ideals of an LA-semigroup S is either empty or a left (right) ideal of S. If A is a non empty subset of S, then intersection of all left (right) ideals of S which contains A is a left (right) ideal of S containing A. Of course this is the smallest left (right) ideal of S containing A and is called left (right) ideal of S generated by A. If $A = \{a\}$, a singleton subset of S, then the left (right) ideal of S generated by A is called principal left (right) ideal of S generated by A.

Union of left (right) ideals of an LA-semigroup S is a left (right) ideal of S.

Theorem 29 *[12] If S is an LA-semigroup with left identity e, then for $a \in S$. Sa is a principal left ideal of S generated by a.*

Proof. We have to prove that $S(Sa) \subseteq Sa$.

Let $x, y \in S$, then by using left invertive law and medial law, we obtain

$$\begin{aligned} x(ya) &= (ex)(ya) \\ &= ((ya)x)e \\ &= ((ya)(ex))e \\ &= ((ye)(ax))e \\ &= (e(ax))(ye) \\ &= (ax)(ye) \\ &= ((ye)x)a \in Sa \end{aligned}$$

This implies that

$$S(Sa) \subseteq Sa$$

Also $a = ea \in Sa$. If I is a left ideal of S containing a then $Sa \subseteq I$.

Hence Sa is the principal left ideal of S generated by a. ∎

Proposition 30 *[12] Let S be an LA-semigroup with left identity e. Then $aS \cup Sa$ is the smallest right ideal containing a.*

Proof. To show $aS \cup Sa$ is the smallest right ideal containing a.

$$\begin{aligned}
(aS \cup Sa)\,S &= (aS)\,S \cup (Sa)\,S \\
\text{Now } (aS)\,S &= (SS)\,a \subseteq Sa \\
\text{and } (Sa)\,S &= (Sa)\,(eS) \\
&= (Se)\,(aS) \text{ By medial law} \\
&\subseteq S\,(aS) = (eS)\,(aS) = a\,(SS) \subseteq aS \\
\text{So } (aS \cup Sa)\,S &\subseteq Sa \cup aS = aS \cup Sa.
\end{aligned}$$

Thus $aS \cup Sa$ is a right ideal of S. It is clear that a is contained in $aS \cup Sa$. Now let R be any right ideal of S containing a. Then

$$aS \subseteq RS \subseteq R \text{ implies } aS \subseteq R \tag{1.1}$$

$$\text{Also } Sa = (SS)\,a = (aS)\,S \subseteq (RS)\,S \subseteq RS \subseteq R$$

$$\text{This implies } Sa \subseteq R \tag{1.2}$$

By (1.1) and (1.2), $aS \cup Sa \subseteq R$. Hence $aS \cup Sa$ is the smallest right ideal containing a. ■

Definition 31 *[5] If B is an LA-subsemigroup of S, then B is said to be a bi-ideal of S if $(BS)B \subseteq B$.*

Definition 32 *[5] An ideal P of an LA-semigroup S is called prime if $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$, for all ideal A and B of S. Prime ideal can also be defined as, an ideal P of an LA-semigroup S is called prime ideal of S such that $ab \in P$ for some $a, b \in S$ implies $a \in P$ or $b \in P$.*

Definition 33 *[5] An ideal P of an LA-semigroup S is called semiprime if $I^2 \subseteq P$ implies that $I \subseteq P$, for any ideal I of S. Semiprime ideal can also be defined as, a subset P of an LA-semigroup S is called semiprime if $a^2 \in P$ $(a \in S)$ implies $a \in P$.*

Definition 34 *[5] Let P be a left ideal of an LA-semigroup S. P is called quasi-prime ideal if for left ideals A, B of S such that $AB \subseteq P$, we have $A \subseteq P$ or $B \subseteq P$.*

Definition 35 *[5] A nonempty subset A of an LA-semigroup S is called an interior ideal of S if $(SA)S \subseteq A$.*

Definition 36 *[5] An ideal I of an LA-semigroup S is called idempotent if $I^2 = I$.*

Proposition 37 *[5] If S is an LA-semigroup with left identity e, then every right ideal is a left ideal.*

Proof. Let I be a right ideal of S and $s \in S$, $i \in I$. Then by left invertive law, we have

$$si = (es)\,i = (is)\,e \in I$$

Hence I is a left ideal. ∎

Lemma 38 *[5] If I is a right ideal of an LA-semigroup S with left identity e, then I^2 is an ideal of S.*

Proof. By Proposition 37, I becomes a left ideal. So if $x \in I^2$ then $x = ij$ where $i, j \in I$. Now using left invertive law and definition of left ideals, we have

$$xs = (ij)\,s = (sj)\,i \in II = I^2$$

This implies that I^2 is a right ideal and by Proposition 37, I^2 becomes a left ideal. ∎

1.3 M-Systems in LA-semigroups

Definition 39 *[10] A subset M of an LA-semigroup S is called an m-system if for $a, b \in M$, there exists some x in S such that $a(xb) \in M$. A subset B of LA-semigroup S is called a p-system if for every b in B there exists some x in S such that $b(xb) \in B$. Every ideal is obviously an m-system and a p-system. Clearly every left ideal is an m-system.*

Example 40 *Let $S = \{1, 2, 3, 4, 5\}$ be an LA-semigroup defined by the following table*

$\cdot$	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	5	3	4
4	1	1	4	5	3
5	1	1	3	4	5

Here $(S, \bullet)$ is an LA-semigroup and $A = \{1\}$ and $B = \{1, 3, 4, 5\}$ are m-systems in S.

Proposition 41 *[10] Each m-system is a p-system.*

Proof. Let $a \in M$. Then there exists x in S such that $a(xa) \in M$ implying that M is a P-system. ■

Lemma 42 *[10] Every right ideal of an LA-semigroup S with left identity e is a P-system.*

Proof. Let I be a right ideal of S. Then by Proposition 37, I becomes an ideal of S. If $i \in I$ then $i(xi) \in I$ for all $x \in S$. Hence I is a P-system. ■

Chapter 2

Rough Sets

In this chapter we shall discuss the concept of rough sets which was given by Zdzisław Pawlak in 1982 in his paper [11].

2.1 Basic Concepts in Rough Sets

In this section, we will define some concepts related to rough set theory given by Zdzisław Pawlak.

Definition 43 *Suppose U be the set of objects, called the universe and an indiscernibility relation $R \subseteq U \times U$, for the sake of simplicity we assume that R such that xRy if and only if (x, y) is in R. R is an equivalence relation if it satisfies the following properties:*

i) Reflexive Property: (x, x) is in R for all x in U.

ii) Symmetric Property: if (x, y) is in R, then (y, x) is in R.

iii) Transitive Property: if (x, y) and (y, z) are in R, then (x, z) is in R.

Definition 44 *A partition P of U is a family of nonempty subsets of U such that each element of U is contained in exactly one element of P.*

i) $U = \bigcup_{i=1}^{n} U_i$

ii) $U_i \bigcap U_j = \phi$, for all $i \neq j$.

Definition 45 *[11] The indiscernibility relation:*

Rough set theory is based on the indiscernibility relation. Let us denote xRy, if we cannot discern x and y by their properties then R is called an indiscernibility relation on U. Usually indiscernibility relations are assumed to be equivalences. The equivalence class of $x \in U$, such that $[x]_R = \{y \in U : xRy\}$, consists of objects indiscernible from x. The indiscernibility relation $IND(P)$, is defined as follows

$IND(P) = \{(x, y) \in U \times U : \text{for all } a \in P, a(x) = a(y)\}$, here P is partition P of U.

In simple words, two objects are indiscernible if we can not discern between them, because they do not differ enough. The indiscernibility relation defines a partition in U.

Example 46 *Example of indiscernibility:*

Given an information system is

U	Headache	Temperature	Flu
1	yes	normal	no
2	yes	high	yes
3	yes	normal	no
4	yes	very high	no
5	no	high	no
6	no	very high	yes
7	no	high	yes
8	no	very high	yes

Here the possible indiscernibility relations are

Indiscernibility relation for HEADACHE:

$$R = \{(i, j) \in U \times U : HEAD(i) = HEAD(j)\}$$

The partition by HEADACHE:

$$IND(\{HEADACHE\}) = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}$$

Indiscernibility relation for TEMPERATURE:

$$R = \{(i, j) \in U \times U : TEMP(i) = TEMP(j)\}$$

The partition by TEMPERATURE:

$$IND(\{TEMPERATURE\}) = \{\{1,3\},\{2,5,7\},\{4,6,8\}\}$$

Indiscernibility relation for HEADACHE and TEMPERATURE:

$$R = \{(i,j) \in U \times U : HT(i) = HT(j)\}$$

The partition by HEADACHE and TEMPERATURE:

$$IND(\{HEAD\},\{TEMP\}) = \{\{1,3\},\{2\},\{4\},\{5,7\},\{6,8\}\}$$

2.2 Set Approximations

Definition 47 *[11] Lower approximation of a subset:*

The lower approximation of a set $X \subseteq U$ with respect to R is the set of all objects, which can be for certain classified as X with respect to R (are certainly X with respect to R). From the different representations of an equivalence relation, we obtain three constructive definitions of lower approximation

$i)\ R_-(X) = \{x \in U : [x]_R \subseteq X\}$

$ii)\ R_-(X) = \bigcup_{[x]_R \subseteq X} [x]_R$

$iii)\ R_-(X) = \bigcup\{A \in U/R : A \subseteq X\}$ where $[x]_R = \{y : xRy\}$

i) is element based definition, ii) is granule based definition, and iii) is subsystem based definition.

Definition 48 *[11] Upper approximation of a subset:*

The upper approximation of a set X with respect to R is the set of all objects which can be possibly classified as X with respect to R (are possibly X in view of R). From the different representations of an equivalence relation, we obtain three constructive definitions of upper approximation

$i)\ R^-(X) = \{x \in U : [x]_R \cap X \neq \varphi\}$

$ii)\ R^-(X) = \bigcup_{[x]_R \cap X \neq \phi} [x]_R$

$iii)\ R^-(X) = \bigcap\{A \in U/R : A \cap X \neq \phi\}$ where $[x]_R = \{y : xRy\}$

i) is element based definition, ii) is granule based definition, and iii) is subsystem based definition.

The lower and upper approximations, $R_{-}, R^{-} : 2^U \longrightarrow 2^U$, can be interpreted as a pair of unary set-theoretic operators. They are dual operators in the sense that $R_{-}(X) = (R^{-}(X^c))^c$ and $R^{-}(X) = (R_{-}(X^c))^c$, where X^c is set complement of X. The pair (U, R) is called approximation space.

Definition 49 *[14] Boundary Region.*

It is the collection of elementary sets defined by:

$\text{BND}(X) = R^{-}(X) - R_{-}(X)$

These sets are included in R-Upper but not in R-Lower approximations.

Based on the lower and upper approximations of a set $X \subseteq U$, the universe U can be divided into three disjoint regions, the positive region $\text{POS}(X)$, the negative region $\text{NEG}(X)$, and the boundary region $\text{BND}(X)$:

$\text{POS}(X) = R_{-}(X)$;

$\text{NEG}(X) = U - R^{-}(X) = (R^{-}(X))^c$;

$\text{BND}(X) = R^{-}(X) - R_{-}(X)$

As we can see from the granule based definition, approximations are expressed in terms of granules of knowledge. The lower approximation of a set is union of all granules which are entirely included in the set, the upper approximation is union of all granules which have non-empty intersection with the set, the boundary region of set is the difference between the upper and the lower approximation.

This definition is clearly depicted in Figure 1

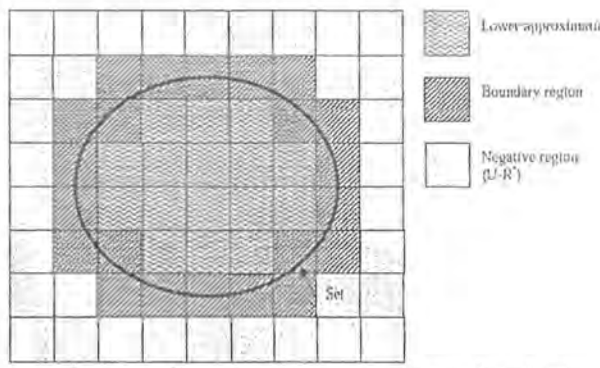


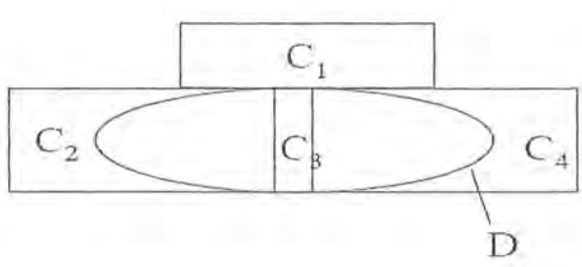
Fig. 1: *Illustration of the boundary region of rough set*

Figure 1 illustrates the approximation of a set X, and the positive, negative and boundary regions. Each small square represents an equivalence class. The upper approximation of a set X is the union of the positive and boundary regions, namely, $R^{-}(X) = \text{POS}(X) \cup \text{BND}(X)$.

Example 50 *Consider a set $U = \{1, 2, 3, 4, 5, 6\}$ as a universal set. Define R to be an equivalence relation such that, for an equivalence relation R on U :*

$$1R1,\quad 2R2,\quad 2R3,\quad 3R2,\quad 3R3,\quad 4R4,\quad 5R6,\quad 6R5,\quad 5R5,\quad 6R6$$

The equivalence relation induces four equivalence classes, which are the subsets $C_1 = \{1\}$, $C_2 = \{2, 3\}$, $C_3 = \{4\}$, $C_4 = \{5, 6\}$, here we want to characterize the set $D = \{3, 4, 5\}$ with respect to R. For this we have



Approximation of D by $\{C_1, C_2, C_3, C_4\}$

C_1 is definitely outside

$R_{-}(D) = \{4\} = C_3$, C_3 definitely inside.

$R^{-}(D) = \{2,3,4,5,6\} = C_2 \cup C_3 \cup C_4$

$C_2 \cup C_4 = \{2,3,5,6\}$ is the boundary region of D.

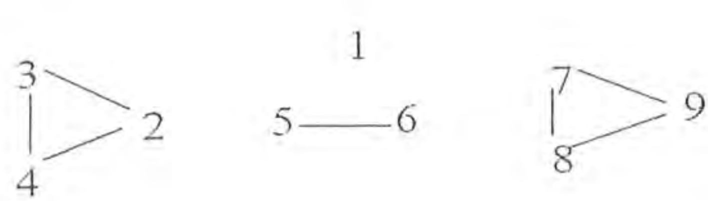
Example 51 *Consider the information system given in the table below*

Numbers	*Code*
1	$(0,0,1)$
2	$(0,1,1)$
3	$(0,1,1)$
4	$(0,1,1)$
5	$(1,0,0)$
6	$(1,0,0)$
7	$(1,1,0)$
8	$(1,1,0)$
9	$(1,1,0)$

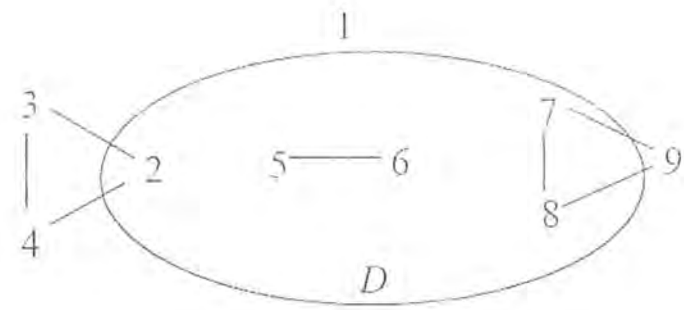
Now indiscernibility/equivalence relation on U is $R = \{(i,j) \in U \times U : Code(i) = Code(j)\}$, now here

$$\begin{aligned}
R(1) &= \{1\} &&= C_1\\
R(2) = R(3) = R(4) &= \{2,3,4\} &&= C_2\\
R(5) = R(6) &= \{5,6\} &&= C_3\\
R(7) = R(8) = R(9) &= \{7,8,9\} &&= C_4
\end{aligned}$$

here C_1, C_2, C_3 and C_4 are equivalence classes



Now take $D = \{2, 5, 6, 7, 8\}$, approximations of D are



$R_{-}(D) = \{5, 6\} = C_3$, C_3 definitely inside.

$R^{-}(D) = \{2, 3, 4, 5, 6, 7, 8, 9\} = C_2 \cup C_3 \cup C_4$

$C_2 \cup C_4 = \{2, 3, 4, 7, 8, 9\}$ is the boundary region of D.

And clearly C_1 is outside.

2.3 Properties of Approximations

Approximations have the following properties: [11]

1) $R_{-}(X) \subseteq X \subseteq R^{-}(X)$

2) $R_{-}(\phi) = \phi = R^{-}(\phi)$; $R_{-}(U) = U = R^{-}(U)$

3) $R^{-}(X \cup Y) = R^{-}(X) \cup R^{-}(Y)$

4) $R_{-}(X \cup Y) \supseteq R_{-}(X) \cup R_{-}(Y)$

5) $R^{-}(X \cap Y) \subseteq R^{-}(X) \cap R^{-}(Y)$

6) $R_{-}(X \cap Y) = R_{-}(X) \cap R_{-}(Y)$

7) $X \subseteq Y$ implies $R_{-}(X) \subseteq R_{-}(Y)$, $R^{-}(X) \subseteq R^{-}(Y)$

8) $R_{-}(-X) = -R^{-}(X)$

9) $R^{-}(-X) = -R_{-}(X)$

10) $R_{-}R_{-}(X) = R^{-}R_{-}(X) = R_{-}(X)$

11) $R^{-}R^{-}(X) = R_{-}R^{-}(X) = R^{-}(X)$

It is easily seen that approximations are in fact interior and closure operations in a topology generated by data.

Definition 52 *A subset X of U is called* ***Crisp*** *when its boundary region is empty i.e.* $R_{-}(X) = R^{-}(X)$.

Definition 53 *[11] Let S be a universal set and let R be an equivalence relation on S, then the set* $X \subseteq S$ *is called a rough with respect to R if* $R_{-}(X) \neq R^{-}(X)$.

Another definition is

Definition 54 *[14] A subset defined through its lower and upper approximations is called a Rough set. That is, when the boundary region is a nonempty set* $(R_{-}(X) \neq R^{-}(X))$.

Example 55 *Consider a set* $U = \{pen, book, bag\}$ *as a universal set, define R to be an equivalence relation such that* $R = \{(pen, pen), (book, bag), (bag, book), (book, book), (bag, bag)\}$, *now here we will characterize all the subsets of U with respect to R, the equivalence relation induces two equivalence classes* $\{pen\}$, $\{book, bag\}$.

$P(U) = \{\phi, \{pen\}, \{book\}, \{bag\}, \{pen, book\}, \{pen, bag\}, \{book, bag\}, U\}$ is the set of all subsets of U, now

$R_{-}(\phi) = \phi$

$R_{-}(\{pen\}) = \{pen\}$

$R_{-}(\{book\}) = \phi$

$R_{-}(\{bag\}) = \phi$

$R_{-}(\{pen, book\}) = \{pen\}$

$R_{-}(\{pen, bag\}) = \{pen\}$

$R_{-}(\{book, bag\}) = \{book, bag\}$

$R_{-}(U) = U$

$R^{-}(\phi) = \phi$

$R^{-}(\{pen\}) = \{pen\}$

$R^{-}(\{book\}) = \{book, bag\}$

$R^{-}(\{bag\}) = \{book, bag\}$

$R^{-}(\{pen, book\}) = U$

$R^{-}(\{pen, bag\}) = U$

$R^{-}(\{book, bag\}) = \{book, bag\}$

$R^-(U) = U$

Now here it is clear that

$$
\begin{aligned}
R_-(\{book\}) &\neq R^-(\{book\}) \\
R_-(\{bag\}) &\neq R^-(\{bag\}) \\
R_-(\{pen, book\}) &\neq R^-(\{pen, book\}) \\
R_-(\{pen, bag\}) &\neq R^-(\{pen, bag\})
\end{aligned}
$$

So $\{book\}$, $\{bag\}$, $\{pen, book\}$ and $\{pen, bag\}$ are rough sets with respect to R, and $\{pen\}$, $\{book, bag\}$ are crisp sets.

Example 56 *Let (U, R) is an approximation space, where $U = \{x_1, x_2, x_3, ..., x_8\}$ and an equivalence relation R with the following equivalence classes:*

$E_1 = \{x_1, x_4, x_8\}$

$E_2 = \{x_2, x_5, x_7\}$

$E_3 = \{x_3\}$

$E_4 = \{x_6\}$

Let $X = \{x_3, x_5\}$ and $Y = \{x_3, x_6\}$

$R_-(X) = \{x_3\}$ and $R^-(X) = \{x_2, x_3, x_5, x_7\}$

$R_-(Y) = \{x_3, x_6\}$ and $R^-(Y) = \{x_3, x_6\}$

So $R(X) = (\{x_3\}, \{x_2, x_3, x_5, x_7\})$ is a rough set and $R(Y)$ is a crisp set.

Example 57 *[14] This example illustrates the main ideas developed so far, consider a universe consisting of three elements $U = \{1, 2, 3\}$ and an equivalence relation R on U:*

$$1R1, \quad 2R2, \quad 1R3, \quad 3R1, \quad 3R3$$

The equivalence relation induces two equivalence classes $[1]_R = [3]_R = \{1, 3\}$, $[2]_R = \{2\}$, now $P(U) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, U\}$ is the set of all subsets of U. The following table summarizes the lower and upper approximations, the positive, negative and boundary regions for all subsets of U.

X	$R_-(X)$	$R^-(X)$	POS(X)	NEG(X)	BND(X)
ϕ	ϕ	ϕ	ϕ	U	ϕ
$\{1\}$	ϕ	$\{1,3\}$	ϕ	$\{2\}$	$\{1,3\}$
$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{1,3\}$	ϕ
$\{3\}$	ϕ	$\{1,3\}$	ϕ	$\{2\}$	$\{1,3\}$
$\{1,2\}$	$\{2\}$	U	$\{2\}$	ϕ	$\{1,3\}$
$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{2\}$	ϕ
$\{2,3\}$	$\{2\}$	U	$\{2\}$	ϕ	$\{1,3\}$
U	U	U	U	ϕ	ϕ

Now it is clear from the table that

$$\begin{aligned} R_-(\{1\}) &\neq R^-(\{1\}) \\ R_-(\{3\}) &\neq R^-(\{3\}) \\ R_-(\{1,2\}) &\neq R^-(\{1,2\}) \\ R_-(\{2,3\}) &\neq R^-(\{2,3\}) \end{aligned}$$

So $\{1\}$, $\{3\}$, $\{1,2\}$, $\{2,3\}$ are rough sets with respect to R, and $\{2\}$, $\{1,3\}$ are crisp sets with respect to R.

Two subsets X and Y of the universe U will be equal if $R_-(X) = R_-(Y)$ and $R^-(X) = R^-(Y)$.

Chapter 3

Roughness in LA-semigroups

In this chapter we have defined rough subsets in LA-semigroups and studied some of their properties. We have also defined rough ideals in an LA-semigroup and also defined rough M-Systems. Lastly, we have defined the rough ideals in the quotient LA-semigroup.

Throughout the chapter S will denote an LA-semigroup unless stated otherwise.

3.1 Rough Subsets in LA-semigroups

Definition 58 *Let ρ be a congruence relation on an LA-semigroup S. Let A be a nonempty subset of S. Then the sets*

$$\rho_{-}(A) = \left\{x \in S : [x]_{\rho} \subseteq A\right\}$$

and

$$\rho^{-}(A) = \left\{x \in S : [x]_{\rho} \cap A \neq \phi\right\}$$

are called ρ-lower and ρ-upper approximations of A respectively.

Definition 59 *Let us denote the set of all subsets of S by $P(S)$. For a nonempty subset A of S,*

$$\rho(A) = (\rho_{-}(A), \rho^{-}(A))$$

is called a rough set with respect to ρ *or simply a* ρ*-rough subset of* $P(S) \times P(S)$ *if*

$$\rho_-(A) \neq \rho^-(A).$$

Example 60 *Let* $S = \{1, 2, 3, 4, 5\}$ *be an LA-semigroup defined by the following table*

$\cdot$	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	5	3	4
4	1	1	4	5	3
5	1	1	3	4	5

Let us define a congruence ρ *on* S *by*

$$\rho = \{(1,1), (2,2), (3,3), (4,4), (5,5), (3,4), (4,3), (3,5), (5,3), (4,5), (5,4)\}$$

ρ*-congruence classes are* $[1]_\rho = \{1\}$, $[2]_\rho = \{2\}$, $[3]_\rho = [4]_\rho = [5]_\rho = \{3, 4, 5\}$, *let* $A = \{2, 4\}$ *be a subset of* S, *then*

$$\rho_-(A) = \{2\} \quad \text{and} \quad \rho^-(A) = \{2, 3, 4, 5\}$$

are respectively the ρ*-lower and* ρ*-upper approximations of* A *and here* $\rho_-(A) \neq \rho^-(A)$. *So*

$$\rho(A) = (\{2\}, \{2, 3, 4, 5\})$$

is a rough set.

Proposition 61 *Let* ρ *and* φ *be congruence relations on an LA-semigroup* S. *If* A *and* B *are nonempty subsets of* S, *then the following hold:*

(1) $\rho_-(A) \subseteq A \subseteq \rho^-(A)$;

(2) $\rho^-(A \cup B) = \rho^-(A) \cup \rho^-(B)$;

(3) $\rho_-(A \cap B) = \rho_-(A) \cap \rho_-(B)$;

(4) $A \subseteq B$ implies $\rho_-(A) \subseteq \rho_-(B)$;

(5) $A \subseteq B$ implies $\rho^-(A) \subseteq \rho^-(B)$;

(6) $\rho_-(A \cup B) \supseteq \rho_-(A) \cup \rho_-(B)$;

(7) $\rho^-(A \cap B) \subseteq \rho^-(A) \cap \rho^-(B)$;

(8) $\rho \subseteq \varphi$ implies $\rho_-(A) \supseteq \varphi_-(A)$;

(9) $\rho \subseteq \varphi$ implies $\rho^-(A) \subseteq \varphi^-(A)$;

Proof. (1) If $a \in \rho_-(A)$, then $a \in [a]_\rho \subseteq A$. Hence $\rho_-(A) \subseteq A$. Next, if $a \in A$, then, since $a \in [a]_\rho$, we have $[a]_\rho \cap A \neq \phi$, and so $a \in \rho^-(A)$. Thus $A \subseteq \rho^-(A)$.

(2) Note that

$$\begin{aligned} a \in \rho^-(A \cup B) &\iff [a]_\rho \cap (A \cup B) \neq \phi \\ &\iff ([a]_\rho \cap A) \cup ([a]_\rho \cap B) \neq \phi \\ &\iff [a]_\rho \cap A \neq \phi \quad \text{or} \quad [a]_\rho \cap B \neq \phi \\ &\iff a \in \rho^-(A) \quad \text{or} \quad a \in \rho^-(B) \\ &\iff a \in \rho^-(A) \cup \rho^-(B), \end{aligned}$$

Thus

$$\rho^-(A \cup B) = \rho^-(A) \cup \rho^-(B).$$

(3) Note that

$$\begin{aligned} a \in \rho_-(A \cap B) &\iff [a]_\rho \subseteq A \cap B \\ &\iff [a]_\rho \subseteq A \quad \text{and} \quad [a]_\rho \subseteq B \\ &\iff a \in \rho_-(A) \quad \text{and} \quad a \in \rho_-(B) \\ &\iff a \in \rho_-(A) \cap \rho_-(B) \end{aligned}$$

Thus

$$\rho_-(A \cap B) = \rho_-(A) \cap \rho_-(B).$$

(4) Since $A \subseteq B$ if and only if $A \cap B = A$, by (3) we have

$$\rho_-(A) = \rho_-(A \cap B) = \rho_-(A) \cap \rho_-(B)$$

This implies that

$$\rho_-(A) \subseteq \rho_-(B).$$

(5) Since $A \subseteq B$ if and only if $A \cup B = B$, by (2) we have

$$\rho^-(B) = \rho^-(A \cup B) = \rho^-(A) \cup \rho^-(B).$$

This implies that

$$\rho^-(A) \subseteq \rho^-(B).$$

(6) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by (4) we have

$$\rho_-(A) \subseteq \rho_-(A \cup B) \quad \text{and} \quad \rho_-(B) \subseteq \rho_-(A \cup B),$$

which yields

$$\rho_-(A) \cup \rho_-(B) \subseteq \rho_-(A \cup B).$$

(7) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by (5) we have

$$\rho^-(A \cap B) \subseteq \rho^-(A) \quad \text{and} \quad \rho^-(A \cap B) \subseteq \rho^-(B),$$

which yields

$$\rho^-(A \cap B) \subseteq \rho^-(A) \cap \rho^-(B).$$

(8) Since $\rho \subseteq \varphi$, then for each $a \in S$ we have

$$[a]_\rho = \{x \in S,\ (x, a) \in \rho\}$$

If $a \in \varphi_-(A)$ then $[a]_\varphi \subseteq A$. But $\rho \subseteq \varphi$, then

$$[a]_\rho \subseteq [a]_\varphi \subseteq A \quad \Rightarrow \quad [a]_\rho \subseteq A$$

Thus $a \in \rho_-(A)$. Hence

$$\varphi_-(A) \subseteq \rho_-(A).$$

(9) Let a be any element of $\rho^-(A)$. Then there exists $x \in [a]_\rho \cap A$. Then, since $\rho \subseteq \varphi$, we

have

$$x \in [a]_\rho \subseteq [a]_\varphi$$

and $x \in A$. Thus $x \in [a]_\varphi \cap A$ and so $a \in \varphi^-(A)$. Therefore

$$\rho^-(A) \subseteq \varphi^-(A).$$

This completes the proof. ■

Proposition 62 *Let ρ be a congruence relation on an LA-semigroup S, then*

$$\rho_-(S) = S = \rho^-(S)$$

Proof. The proof is straight forward. ■

Proposition 63 *Let ρ be a congruence relation on an LA-semigroup S. If A and B are nonempty subsets of S, then*

$$\rho^-(A)\rho^-(B) \subseteq \rho^-(AB).$$

Proof. Let c be any element of $\rho^-(A)\rho^-(B)$. Then $c = ab$ with $a \in \rho^-(A)$ and $b \in \rho^-(B)$. Thus there exist elements x, $y \in S$, such that

$$x \in [a]_\rho \cap A \quad \text{and} \quad y \in [b]_\rho \cap B.$$

Thus $x \in [a]_\rho$, $y \in [b]_\rho$, $x \in A$ and $y \in B$. Since ρ is a congruence on S

$$xy \in [a]_\rho[b]_\rho \subseteq [ab]_\rho$$

Since $xy \in AB$, we have

$$xy \in [ab]_\rho \cap AB$$

and so $ab \in \rho^-(AB)$. Thus we have

$$\rho^-(A)\rho^-(B) \subseteq \rho^-(AB).$$

This completes the proof. ■

Proposition 64 *Let ρ be a complete congruence relation on an LA-semigroup S. If A and B are nonempty subsets of S, then*

$$\rho_-(A)\rho_-(B) \subseteq \rho_-(AB)$$

Proof. Let c be any element of $\rho_-(A)\rho_-(B)$. Then $c = ab$ with $a \in \rho_-(A)$ and $b \in \rho_-(B)$. Thus we have

$$[a]_\rho \subseteq A \quad \text{and} \quad [b]_\rho \subseteq B.$$

Since ρ is complete congruence on S, we have

$$[ab]_\rho = [a]_\rho [b]_\rho \subseteq AB,$$

and so $ab \in \rho_-(AB)$. Thus

$$\rho_-(A)\rho_-(B) \subseteq \rho_-(AB).$$

This completes the proof. ■

Theorem 65 *Let ρ and φ be congruence relations on an LA-semigroup S. If A is a nonempty subset of S, then*

$$(\rho \cap \varphi)^-(A) \subseteq \rho^-(A) \cap \varphi^-(A).$$

Proof. Note that $\rho \cap \varphi$ is also a congruence relation on S. Let $c \in (\rho \cap \varphi)^-(A)$. Then

$$[c]_{\rho\cap\varphi} \cap A \neq \phi.$$

Then there exists an element $a \in [c]_{\rho\cap\varphi} \cap A$. Since $(a, c) \in \rho \cap \varphi$, we have

$$(a, c) \in \rho \quad \text{and} \quad (a, c) \in \varphi$$

Thus we have $a \in [c]_\rho$ and $a \in [c]_\varphi$. Since $a \in A$, we have

$$a \in [c]_\rho,\ a \in A \quad \text{and} \quad a \in [c]_\varphi,\ a \in A.$$

This implies that

$$a \in [c]_\rho \cap A \quad \text{and} \quad a \in [c]_\varphi \cap A$$

$$[c]_\rho \cap A \neq \phi \quad \text{and} \quad [c]_\varphi \cap A \neq \phi$$

$$c \in \rho^-(A) \quad \text{and} \quad c \in \varphi^-(A),$$

and so

$$c \in \rho^-(A) \cap \varphi^-(A).$$

Thus we obtain that

$$(\rho \cap \varphi)^-(A) \subseteq \rho^-(A) \cap \varphi^-(A).$$

This completes the proof. ■

Theorem 66 *Let ρ and φ be congruence relations on an LA-semigroup S. If A is a nonempty subset of S, then*

$$\rho_-(A) \cap \varphi_-(A) \subseteq (\rho \cap \varphi)_-(A)$$

Proof. Since $\rho \cap \varphi \subseteq \rho$ and $\rho \cap \varphi \subseteq \varphi$, which implies that

$$\rho_-(A) \subseteq (\rho \cap \varphi)_-(A) \quad \text{and} \quad \varphi_-(A) \subseteq (\rho \cap \varphi)_-(A)$$

$$\Longrightarrow \rho_-(A) \cap \varphi_-(A) \subseteq (\rho \cap \varphi)_-(A).$$

This completes the proof. ■

In paper [6] it is given that

$$\rho_-(A) \cap \varphi_-(A) = (\rho \cap \varphi)_-(A).$$

Next we show that the converse of the above theorem is not true. For this, we take an example

Example 67 *We consider an LA-semigroup $S = \{1,2,3,4,5\}$ with the following Cayley table*

$\cdot$	1	2	3	4	5
1	1	2	3	4	5
2	2	2	2	4	5
3	3	2	1	4	5
4	5	5	5	2	4
5	4	4	4	5	2

Let us define congruences ρ and φ on S by

$$\rho = \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$$

and the ρ-congruence classes are $[1]_\rho = [2]_\rho = [3]_\rho = \{1,2,3\}$, $[4]_\rho = \{4\}$, $[5]_\rho = \{5\}$,

$$\varphi = \{(1,1),(2,2),(3,3),(4,4),(5,5),(2,4),(4,2),(2,5),(5,2),(4,5),(5,4)\}$$

and the φ-congruence classes are $[2]_\varphi = [4]_\varphi = [5]_\varphi = \{2,4,5\}$, $[1]_\varphi = \{1\}$, $[3]_\varphi = \{3\}$

$$\rho \cap \varphi = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$$

Let $A = \{2,4,5\}$ be a nonempty subset of S, then

$$(\rho \cap \varphi)_-(A) = \{2,4,5\}$$

$$\rho_-(A) = \{4,5\} \quad \textit{and} \quad (\varphi)_-(A) = \{2,4,5\}$$

$$\Longrightarrow \rho_-(A) \cap (\varphi)_-(A) = \{4,5\}$$

This implies that

$$(\rho \cap \varphi)_-(A) \nsubseteq \rho_-(A) \cap (\varphi)_-(A)$$

Hence

$$(\rho \cap \varphi)_-(A) \neq \rho_-(A) \cap (\varphi)_-(A).$$

3.2 Rough Ideals in LA-semigroups

Definition 68 *Let ρ be a congruence relation on an LA-semigroup S. Then a nonempty subset A of S is called an upper rough LA-subsemigroup of S if $\rho^{-}(A)$ is an LA-subsemigroup of S, and A is called an upper rough left [right, two-sided] ideal of S if $\rho^{-}(A)$ is a left [right, two-sided] ideal of S.*

Theorem 69 *Let ρ and φ be congruence relations on an LA-semigroup S such that $\rho\circ\varphi = \varphi\circ\rho$. If A is an LA-subsemigroup of S, then*

$$\rho^{-}(A)\varphi^{-}(A) \subseteq (\rho \circ \varphi)^{-}(A)$$

Proof. Let c be any element of $\rho^{-}(A)\varphi^{-}(A)$. Then $c = ab$ where $a \in \rho^{-}(A)$ and $b \in \varphi^{-}(A)$. Then there exist elements $x, y \in S$, such that

$$x \in [a]_{\rho} \cap A \quad \text{and} \quad y \in [b]_{\varphi} \cap A$$

Thus $x \in [a]_{\rho}$, $y \in [b]_{\varphi}$, and $x, y \in A$. Since A is an LA-subsemigroup of S, we have $xy \in A$. Then $(x, a) \in \rho$ and $(y, b) \in \varphi$, and since ρ and φ be congruence relations, we have

$$(xy, ay) \in \rho \quad \text{and} \quad (ay, ab) \in \varphi$$

Thus we have

$$(xy, ab) \in \rho \circ \varphi$$

and so $xy \in [ab]_{\rho\circ\varphi}$. Therefore we have

$$xy \in [ab]_{\rho\circ\varphi} \cap A,$$

which yields

$$c = ab \in (\rho \circ \varphi)^{-}(A)$$

Thus we obtain

$$\rho^{-}(A)\varphi^{-}(A) \subseteq (\rho \circ \varphi)^{-}(A).$$

This completes the proof. ■

Theorem 70 *Let ρ be a congruence relation on an LA-semigroup S, then*

(1) If A is an LA-subsemigroup of S, then A is an upper rough LA-subsemigroup of S.

(2) If A is a left [right, two-sided] ideal of S, then A is an upper rough left [right, two-sided] ideal of S.

Proof. (1) Let A be an LA-subsemigroup of S. Then we have

$$\phi \neq A \subseteq \rho^{-}(A).$$

Then it follows from Proposition 63 that

$$\rho^{-}(A)\rho^{-}(A) \subseteq \rho^{-}(AA) \subseteq \rho^{-}(A).$$

This means that $\rho^{-}(A)$ is an LA-subsemigroup of S, that is, A is an upper rough LA-subsemigroup of S.

(2) Let A be a left ideal of S, that is, $SA \subseteq A$. Note that $\rho^{-}(S) = S$. Then by Proposition 63, we have

$$S\rho^{-}(A) = \rho^{-}(S)\rho^{-}(A) \subseteq \rho^{-}(SA) \subseteq \rho^{-}(A).$$

This means that $\rho^{-}(A)$ is a left ideal of S, that is, A is an upper rough left ideal of S. The other cases can be seen in a similar way. This completes the proof. ■

The above theorem shows that the notion of an upper rough LA-subsemigroup [left ideal, right ideal, two-sided ideal] is an extended notion of a usual LA-subsemigroup [left ideal, right ideal, two-sided ideal] of an LA-semigroup. The following example shows that the converse of above theorem does not hold in general.

Example 71 *Let $S = \{1, 2, 3, 4, 5\}$ be an LA-semigroup with the following multiplication table:*

$\cdot$	1	2	3	4	5
1	1	2	4	4	5
2	5	4	4	4	4
3	4	4	4	4	4
4	4	4	4	4	4
5	2	4	4	4	4

Let ρ be a congruence relation on S, such that the ρ-congruence classes are the subsets $\{1\}$, $\{3\}$, $\{2,4,5\}$. Then for $A = \{2,4\} \subseteq S$, $\rho^{-}(A) = \{2,4,5\}$, and here

$$\{2,4\}S = \{4,5\} \nsubseteq \{2,4\} \quad \textit{and} \quad S\{2,4\} = \{2,4\}$$

$$\textit{but} \;\; \{2,4,5\}S = S\{2,4,5\} = \{2,4,5\}$$

This means that the set $\{2,4,5\}$ is a two-sided ideal of S. It is clear that $A = \{2,4\}$ is not a two-sided ideal of S. Thus $A = \{2,4\}$ is an upper rough ideal but it is not an ideal.

Theorem 72 *Let ρ be a complete congruence relation on an LA-semigroup S, then*

(1) Let A be an LA-subsemigroup of S, then $\rho_{-}(A)$ is, if it is nonempty, an LA-subsemigroup of S.

(2) Let A be a left [right, two-sided] ideal of S, then $\rho_{-}(A)$ is, if it is nonempty, a left [right, two-sided] ideal of S.

Proof. (1) Since A is an LA-subsemigroup of S, by Proposition 64, we have

$$\rho_{-}(A)\rho_{-}(A) \subseteq \rho_{-}(AA) \subseteq \rho_{-}(A).$$

This means that $\rho_{-}(A)$ is, if it is nonempty, an LA-subsemigroup of S.

(2) Let A be a left ideal of S, that is, $SA \subseteq A$. Note that $\rho_{-}(S) = S$. Then by Proposition 64, we have

$$S\;\rho_{-}(A) = \rho_{-}(S)\rho_{-}(A) \subseteq \rho_{-}(SA) \subseteq \rho_{-}(A).$$

This means that $\rho_{-}(A)$ is, if it is nonempty, a left ideal of S. The other cases can be seen in a similar way. This completes the proof. ■

The following example shows that the converse of above theorem does not hold in general.

Example 73 *Let $S = \{1,2,3,4,5\}$ be an LA-subsemigroup with the following multiplication table:*

$\cdot$	1	2	3	4	5
1	5	2	1	2	3
2	2	2	2	2	2
3	3	2	5	2	1
4	2	2	2	2	2
5	1	2	3	2	5

Let ρ be a complete congruence relation on S, such that the ρ-congruence classes are the subsets $\{1,3,5\}$, $\{2\}$, $\{4\}$, then for $A = \{1,2\} \subseteq S$, $\rho_{-}(A) = \{2\}$, and here

$$\{2\}S = S\{2\} = \{2\}$$

$$\text{but } \{1,2\}S = S\{1,2\} = \{1,2,3,5\} \nsubseteq \{1,2\}$$

This means that the set $\{2\}$ is a two-sided ideal of S. It is clear that $A = \{1,2\}$ is not a two-sided ideal of S. Thus $A = \{1,2\}$ is a lower rough ideal but it is not an ideal.

Theorem 74 *Let ρ be a congruence relation on an LA-semigroup S. If A and B are a right ideal and a left ideal of S, respectively, then*

$$\rho^{-}(AB) \subseteq \rho^{-}(A) \cap \rho^{-}(B)$$

Proof. Since A is a right ideal of S, $AB \subseteq AS \subseteq A$, and since B is a left ideal of S, $AB \subseteq SB \subseteq B$, thus $AB \subseteq A \cap B$. Then it follows from Proposition 61(5) and Proposition 61(7) that

$$\rho^{-}(AB) \subseteq \rho^{-}(A \cap B) \subseteq \rho^{-}(A) \cap \rho^{-}(B).$$

Which completes the proof. ■

Theorem 75 *Let ρ be a congruence relation on an LA-semigroup S. If A and B are a right ideal and a left ideal of S, respectively, then*

$$\rho_{-}(AB) \subseteq \rho_{-}(A) \cap \rho_{-}(B)$$

Proof. Since A is a right ideal of S, $AB \subseteq AS \subseteq A$, and since B is a left ideal of S, $AB \subseteq SB \subseteq B$, thus $AB \subseteq A \cap B$. Then it follows from Proposition 61(3) and Proposition 61(4) that

$$\rho_{-}(AB) \subseteq \rho_{-}(A \cap B) = \rho_{-}(A) \cap \rho_{-}(B)$$

which implies that

$$\rho_{-}(AB) \subseteq \rho_{-}(A) \cap \rho_{-}(B).$$

This completes the proof. ■

Theorem 76 *Let ρ be a congruence relation on an LA-semigroup S with left identity, if A is a right ideal of S, then*

(1) $\rho^{-}(A)$ is a left ideal of S.

(2) If ρ is complete then, $\rho_{-}(A)$ is a left ideal of S.

Proof. (1) Let S be an LA-semigroup with left identity, and A is a right ideal of S, so by Theorem 70(2), $\rho^{-}(A)$ is a right ideal of S, such that

$$\rho^{-}(A)\ S \subseteq \rho^{-}(A)$$

Let $sa \in S\rho^{-}(A)$ for $s \in S$ and $a \in \rho^{-}(A)$. Now

$$\begin{aligned} sa &= (es)a \\ &= (as)e \quad \text{by left invertive law} \end{aligned}$$

As $\rho^{-}(A)$ is right ideal, so

$$as \in \rho^{-}(A)S \subseteq \rho^{-}(A)$$

Therefore we have

$$sa = (as)e \in \rho^{-}(A)S \subseteq \rho^{-}(A)$$

Hence

$$sa \in \rho^{-}(A)$$

This implies that

$$S\rho^{-}(A) \subseteq \rho^{-}(A)$$

This shows that $\rho^{-}(A)$ is left ideal. By using Theorem 72(2), the proof of (2) can be seen in a similar way. This completes the proof. ■

Lemma 77 *Let ρ be a congruence relation on a locally associative LA-semigroup S, then for a nonempty subset A of S,*

(1) $[\rho^{-}(A)]^n \subseteq \rho^{-}(A^n)$ for all $n \in \mathbb{N}$.

(2) If ρ is complete then, $[\rho_{-}(A)]^n \subseteq \rho_{-}(A^n)$ for all $n \in \mathbb{N}$.

Proof. (1) Let A be a nonempty subset of S, then for $n = 2$, and by Proposition 63, we get

$$[\rho^{-}(A)]^2 = \rho^{-}(A)\rho^{-}(A) \subseteq \rho^{-}(AA) = \rho^{-}(A^2)$$

Now for $n = 3$, we get

$$[\rho^{-}(A)]^3 = \rho^{-}(A)[\rho^{-}(A)]^2 \subseteq \rho^{-}(A)\rho^{-}(A^2) \subseteq \rho^{-}(AA^2) = \rho^{-}(A^3)$$

Suppose that the result is true for $n = k-1$, such that $[\rho^{-}(A)]^{k-1} \subseteq \rho^{-}(A^{k-1})$, then for $n = k$, we get

$$[\rho^{-}(A)]^k = \rho^{-}(A)[\rho^{-}(A)]^{k-1} \subseteq \rho^{-}(A)\rho^{-}(A^{k-1}) \subseteq \rho^{-}(AA^{k-1}) = \rho^{-}(A^k)$$

Hence this shows that $[\rho^{-}(A)]^k \subseteq \rho^{-}(A^k)$. This implies that $[\rho^{-}(A)]^n \subseteq \rho^{-}(A^n)$ is true for all $n \in \mathbb{N}$. By using Proposition 64, the proof of (2) can be seen in a similar way. This completes the proof. ■

Theorem 78 *Let ρ be a congruence relation on an LA-semigroup S, let A be a left ideal of S, then*

(1) A^2 is an upper rough right ideal of S.

(2) If ρ is complete then, A^2 is a lower rough right ideal of S.

Proof. (1) Let A be left ideal of an LA-semigroup S. Now

$$\begin{aligned}
\rho^-(A^2)S &= \rho^-(A^2)\rho^-(S) && (\ \rho^-(S) = S\ ) \\
&\subseteq \rho^-(A^2S) && \text{(by Proposition 63)} \\
&= \rho^-[(AA)S] && \\
&= \rho^-[(SA)A] && \text{(left invertive law)} \\
&\subseteq \rho^-(AA) && \text{(because } SA \subseteq A) \\
&= \rho^-(A^2)
\end{aligned}$$

Hence we get

$$\rho^-(A^2)S \subseteq \rho^-(A^2)$$

This shows that A^2 is an upper rough right ideal of S. By using Proposition 64, the proof of (2) can be seen in a similar way. This completes the proof. ∎

It can be prove that if A be a left ideal of a locally associative LA-semigroup then A^n is an upper rough right ideal and also a lower rough right ideal of S for all $n \in \mathbb{N}\backslash\{1\}$.

Theorem 79 *Let ρ be a congruence relation on an LA-semigroup S with left identity, let A be a left ideal of S, then*

(1) A^2 is an upper rough ideal of S.

(2) If ρ is complete then, A^2 is a lower rough ideal of S

Proof. (1) Let A be left ideal of an LA-semigroup S. By Theorem 78(1), we get that $\rho^-(A^2)$

is a right ideal of S. Now

$$\begin{aligned}
S\rho^{-}(A^2) &= \rho^{-}(S)\rho^{-}(A^2) && (\ \rho^{-}(S) = S\ )\\
&\subseteq \rho^{-}(SA^2) && \text{(by Proposition 63)}\\
&= \rho^{-}[S(AA)] \\
&= \rho^{-}[A(SA)] && \text{(by Lemma 13)}\\
&\subseteq \rho^{-}(AA) && (\text{because } SA \subseteq A)\\
&= \rho^{-}(A^2)
\end{aligned}$$

Hence we get

$$S\rho^{-}(A^2) \subseteq \rho^{-}(A^2)$$

This shows that A^2 is an upper rough ideal of S. By using Proposition 64 and Theorem 78(2), the proof of (2) can be seen in a similar way. This completes the proof. ■

Theorem 80 *Let ρ be a congruence relation on an LA-semigroup S, let A be a left ideal of S, then*

(1) $[\rho^{-}(A)]^2$ is a right ideal of S.

(2) If ρ is complete then, $[\rho_{-}(A)]^2$ is a right ideal of S.

Proof. (1) Let A be a left ideal of an LA-semigroup S. Now

$$\begin{aligned}
[\rho^{-}(A)]^2\ S &= [\rho^{-}(A)]^2\ S \\
&= [\rho^{-}(A)]^2\ \rho^{-}(S) && (\ \rho^{-}(S) = S\ )\\
&= [\rho^{-}(A)\ \rho^{-}(A)]\ \rho^{-}(S) \\
&= [\rho^{-}(S)\ \rho^{-}(A)]\ \rho^{-}(A) && \text{(left invertive law)}\\
&\subseteq \rho^{-}(SA)\ \rho^{-}(A) && \text{(by Proposition 63)}\\
&= \rho^{-}(A)\ \rho^{-}(A) && (\text{because } SA \subseteq A)\\
&= [\rho^{-}(A)]^2
\end{aligned}$$

Hence we get

$$[\rho^-(A)]^2\, S \subseteq [\rho^-(A)]^2$$

This shows that $[\rho^-(A)]^2$ is a right ideal of S. By using Proposition 64, the proof of (2) can be seen in a similar way. This completes the proof. ■

It can be proved that if A be a left ideal of a locally associative LA-semigroup then $[\rho^-(A)]^n$ and $[\rho_-(A)]^n$ are right ideals of S for all $n \in \mathbb{N}\backslash\{1\}$.

Theorem 81 *Let ρ be a congruence relation on an LA-semigroup S with left identity. If A is a left ideal of S, then*

(1) $[\rho^-(A)]^2$ is an ideal of S.

(2) If ρ is complete then, $[\rho_-(A)]^2$ is an ideal of S.

Proof. This follows from Theorem 80(1) and Proposition 63. ■

3.3 Rough Bi-ideals and Rough Interior-ideals in LA-semigroups

Definition 82 *A subset A of an LA-semigroup S is called a ρ-upper [ρ-lower] rough bi-ideal of S if $\rho^-(A)[\rho_-(A)]$ is a bi-ideal of S.*

Theorem 83 *Let ρ be a congruence relation on an LA-semigroup S. If A is a bi-ideal of S, then it is a ρ-upper rough bi-ideal of S.*

Proof. Let A be a bi-ideal of S. Then by Proposition 63, we have

$$\begin{aligned}(\rho^-(A)S)\rho^-(A) &= (\rho^-(A)\rho^-(S))\rho^-(A)\\ &\subseteq \rho^-((AS)A)\\ &\subseteq \rho^-(A).\end{aligned}$$

From this and Theorem 70(1), we obtain that $\rho^-(A)$ is a bi-ideal of S, that is, A is a ρ-upper rough bi-ideal of S. This completes the proof. ■

The following example shows that the converse of this theorem does not hold in general.

Example 84 *Let $S = \{1,2,3,4,5\}$ be an LA-semigroup with the following multiplication table:*

$\cdot$	1	2	3	4	5
1	1	2	4	4	5
2	5	4	4	4	4
3	4	4	4	4	4
4	4	4	4	4	4
5	2	4	4	4	4

Let ρ be a congruence relation on S such that ρ-congruence classes are the subsets $\{1\}, \{3\}, \{2,4,5\}$, then for $A = \{1,4\} \subseteq S$, $\rho^{-}(A) = \{1,2,4,5\}$. Here

$$(\{1,2,4,5\}S)\{1,2,4,5\} = \{1,2,4,5\}$$

$$\text{but} \quad (\{1,4\}S)\{1,4\} = \{1,2,4,5\} \nsubseteq \{1,4\}$$

It is clear that $\rho^{-}(A)$ is a bi-ideal of S. It is also clear that the LA-subsemigroup $\{1,4\}$ of S is not a bi-ideal of S.

Theorem 85 *Let ρ be a complete congruence relation on an LA-semigroup S. If A is a bi-ideal of S, then $\rho_{-}(A)$ is, if it is nonempty, a bi-ideal of S.*

Proof. *Let A be a bi-ideal of S, then by Proposition 64, we have*

$$\begin{aligned}(\rho_{-}(A)S)\rho_{-}(A) &= (\rho_{-}(A)\rho_{-}(S))\rho_{-}(A) \\ &\subseteq \rho_{-}((AS)A) \\ &\subseteq \rho_{-}(A).\end{aligned}$$

From this and Theorem 72(1), we obtain that $\rho_{-}(A)$ is, if it is nonempty, a bi-ideal of S. This completes the proof. ∎

The following example shows that the converse of this theorem does not hold in general.

Example 86 *Let $S = \{1,2,3,4,5\}$ be an LA-semigroup with the following multiplication table:*

$\cdot$	1	2	3	4	5
1	5	2	1	2	3
2	2	2	2	2	2
3	3	2	5	2	1
4	2	2	2	2	2
5	1	2	3	2	5

Let ρ be a complete congruence relation on S such that ρ-congruence classes are the subsets $\{1,3,5\},\{2\},\{4\}$, then for $A = \{2,5\} \subseteq S$, $\rho_-(A) = \{2\}$. Here

$$(\{2\}S)\{2\} = \{2\}$$

$$\text{but } (\{2,5\}S)\{2,5\} = \{1,2,3,5\} \nsubseteq \{2,5\}$$

It is clear that $\rho_-(A)$ is a bi-ideal of S. It is also clear that the LA-subsemigroup $\{2,5\}$ of S is not a bi-ideal of S.

Definition 87 *A subset A of an LA-semigroup S is called a ρ-upper [ρ-lower] rough interior-ideal of S if $\rho^-(A)[\rho_-(A)]$ is an interior-ideal of S.*

Theorem 88 *Let ρ be a congruence relation on an LA-semigroup S. If A is an interior-ideal of S, then it is a ρ-upper rough interior-ideal of S.*

Proof. Let A be an interior-ideal of S, then by Proposition 63, we have

$$\begin{aligned}(S\rho^-(A))S &= (\rho^-(S)\rho^-(A))\rho^-(S)\\ &\subseteq \rho^-((SA)S)\\ &\subseteq \rho^-(A).\end{aligned}$$

We obtain that $\rho^-(A)$ is an interior-ideal of S, that is, A is a ρ-upper rough interior-ideal of S. This completes the proof. ■

The following example shows that the converse of above theorem does not hold in general.

Example 89 *Let $S = \{1, 2, 3, 4, 5\}$ be an LA-semigroup with the following multiplication table:*

$\cdot$	1	2	3	4	5
1	1	4	4	4	4
2	4	3	5	4	2
3	4	2	3	4	5
4	4	4	4	4	4
5	4	5	2	4	3

Let ρ be a congruence relation on S such that ρ-congruence classes are the subsets $\{1\}, \{4\}, \{2,3,5\}$, then for $A = \{2,4\} \subseteq S$, $\rho^-(A) = \{2,3,4,5\}$. Here

$$(S\{2,3,4,5\})S = \{2,3,4,5\}$$

$$\text{but } \ (S\{2,4\})S = \{2,3,4,5\} \nsubseteq \{2,4\}$$

It is clear that $\rho^-(A)$ is an interior-ideal of S but the subset A of S is not an interior-ideal of S.

Theorem 90 *Let ρ be a complete congruence relation on an LA-semigroup S. If A is an interior-ideal of S, then $\rho_-(A)$ is, if it is nonempty, an interior-ideal of S.*

Proof. *Let A be an interior-ideal of S, then by Proposition 64, we have*

$$\begin{aligned}(S\rho_-(A))S &= (\rho_-(S)\rho_-(A))\rho_-(S) \\ &\subseteq \rho_-((SA)S) \\ &\subseteq \rho_-(A).\end{aligned}$$

We obtain that $\rho_-(A)$ is, if it is nonempty, an interior-ideal of S. This completes the proof. ■

The following example shows that the converse of above theorem does not hold in general.

Example 91 *Let $S = \{1, 2, 3, 4\}$ be an LA-semigroup with the following multiplication table:*

$\cdot$	1	2	3	4
1	1	2	3	4
2	3	1	2	4
3	2	3	1	4
4	4	4	4	4

Let ρ be a complete congruence relation on S such that ρ-congruence classes are the subsets $\{1,2,3\}, \{4\}$, then for $A = \{2,4\} \subseteq S$, $\rho_{-}(A) = \{4\}$. Here

$$(S\{4\})S = \{4\}$$

$$but \quad (S\{2,4\})S = \{1,2,3,4\} \nsubseteq \{2,4\}$$

It is clear that $\rho_{-}(A)$ is an interior-ideal of S but the subset A of S is not an interior-ideal of S.

3.4 Rough Prime and Rough Semiprime Ideals in LA-semigroups

Definition 92 *Let ρ be a congruence relation on an LA-semigroup S, then a subset A of S is called a lower [an upper] rough prime ideal of S if $\rho_{-}(A)$ $[\rho^{-}(A)]$ is a prime ideal of S.*

Theorem 93 *Let ρ be a complete congruence relation on an LA-semigroup S. If A is a prime ideal of S, then A is an upper rough prime ideal of S.*

Proof. Since A is a prime ideal of S, then by Theorem 70(2), we know that $\rho^{-}(A)$ is an ideal of S. Let

$$xy \in \rho^{-}(A) \quad \text{for some} \quad x, y \in S$$

then

$$[xy]_\rho \cap A = [x]_\rho [y]_\rho \cap A \neq \phi$$

Now there exist $x' \in [x]_\rho$ and $y' \in [y]_\rho$ such that $x'y' \in [xy]_\rho \cap A$, so $x'y' \in A$. Since A is a prime ideal, we have $x' \in A$ or $y' \in A$. Thus

$$x' \in [x]_\rho \cap A \quad \text{or} \quad y' \in [y]_\rho \cap A$$

So

$$[x]_\rho \cap A \neq \phi \quad \text{or} \quad [y]_\rho \cap A \neq \phi,$$

and so $x \in \rho^-(A)$ or $y \in \rho^-(A)$. Therefore $\rho^-(A)$ is a prime ideal of S. ∎

The following example shows that the converse of the above theorem does not hold in general.

Example 94 *Let $S = \{1, 2, 3, 4, 5\}$ be an LA-semigroup with the following multiplication table:*

$\cdot$	1	2	3	4	5
1	4	4	2	4	5
2	4	4	1	4	5
3	1	2	3	4	5
4	4	4	4	4	5
5	5	5	5	5	5

Let ρ be a complete congruence relation on S such that ρ-congruence classes are the subsets $\{1, 2, 3, 4\}, \{5\}$, then for $A = \{4, 5\} \subseteq S$, $\rho^-(A) = \{1, 2, 3, 4, 5\}$. It is clear that $\rho^-(A)$ is a prime ideal of S. The ideal A is not a prime ideal for $1 \cdot 2 = 4 \in A$ but $1 \notin A$ and $2 \notin A$.

Theorem 95 *Let ρ be a complete congruence relation on an LA-semigroup S and A a prime ideal of S, then $\rho_-(A)$ is, if it is nonempty, a prime ideal of S.*

Proof. Since A is a prime ideal of S, by Theorem 72(2), we know that $\rho_-(A)$ is an ideal of S. Let

$$xy \in \rho_-(A) \quad \text{for some} \quad x, y \in S,$$

then

$$[x]_\rho [y]_\rho = [xy]_\rho \subseteq A.$$

Suppose that $x \notin \rho_-(A)$ and $y \notin \rho_-(A)$. This implies that $[x]_\rho \nsubseteq A$ and $[y]_\rho \nsubseteq A$, there exist $a \in [x]_\rho$ and $b \in [y]_\rho$ such that $a, b \notin A$. Thus

$$ab \in [x]_\rho[y]_\rho = [xy]_\rho \subseteq A$$

Since A is a prime ideal, we have $a \in A$ or $b \in A$. It contradicts the supposition. This means that $\rho_-(A)$ is, if it is nonempty, a prime ideal of S. ∎

We call A a rough prime ideal of S if it is both a lower and an upper rough prime ideal of S. The following example shows that the converse of the above theorem does not hold in general.

Example 96 *Let $S = \{1, 2, 3, 4, 5\}$ be an LA-semigroup with the following multiplication table:*

$\cdot$	1	2	3	4	5
1	1	2	3	4	5
2	4	3	3	3	5
3	3	3	3	3	5
4	2	3	3	3	5
5	5	5	5	5	5

Let ρ be a complete congruence relation on S such that ρ-congruence classes are the subsets $\{1, 2, 3, 4\}$, $\{5\}$, then for $A = \{3, 5\} \subseteq S$, $\rho_-(A) = \{5\}$. It is clear that $\rho_-(A)$ is a prime ideal of S. The ideal A is not a prime ideal for $2 \cdot 4 = 3 \in A$ but $2 \notin A$ and $4 \notin A$.

Definition 97 *Let ρ be a congruence relation on an LA-semigroup S, then a subset A of S is called an upper [lower] rough semiprime ideal of S if $\rho^-(A)$ $[\rho_-(A)]$ is a semiprime ideal of S.*

Theorem 98 *Let ρ be a complete congruence relation on an LA-semigroup S. If A is a semiprime ideal of S then A is an upper rough semiprime ideal of S.*

Proof. Since A is a semiprime ideal of S, then by Theorem 70(2), we know that $\rho^-(A)$ is an ideal of S. Let $x^2 \in \rho^-(A)$ for some $x \in S$, then

$$[x]_\rho[x]_\rho \cap A = [xx]_\rho \cap A = [x^2]_\rho \cap A \neq \phi$$

Now there exist $y \in [x]_\rho$ such that $yy \in [x]_\rho[x]_\rho \cap A$, so $yy = y^2 \in A$. Since A is a semiprime ideal, we have $y \in A$. Thus

$$y \in [x]_\rho \cap A \implies [x]_\rho \cap A \neq \phi$$

so $x \in \rho^-(A)$. Therefore $\rho^-(A)$ is a semiprime ideal of S. ∎

It can be seen from Example 94, that, for $A = \{4, 5\} \subseteq S$, $\rho^-(A) = \{1, 2, 3, 4, 5\}$. It is clear that $\rho^-(A)$ is a semiprime ideal of S. The ideal A is not a semiprime ideal for $2^2 = 4 \in A$ but $2 \notin A$. This shows that the converse of above theorem does not hold in general.

Theorem 99 *Let ρ be a complete congruence relation on an LA-semigroup S and A a semiprime ideal of S, then $\rho_-(A)$ is, if it is nonempty, a semiprime ideal of S.*

Proof. Since A is a semiprime ideal of S, by Theorem 72(2), we know that $\rho_-(A)$ is an ideal of S. Let $x^2 \in \rho_-(A)$ for some $x \in S$, then

$$[x]_\rho[x]_\rho = [xx]_\rho = [x^2]_\rho \subseteq A.$$

We suppose that $\rho_-(A)$ is not a semiprime ideal, then there exist $x \in S$ such that $x^2 \in \rho_-(A)$ but $x \notin \rho_-(A)$. Thus $[x]_\rho \nsubseteq A$, then there exist

$$y \in [x]_\rho \text{ but } y \notin A$$

Thus

$$y^2 = yy \in [x]_\rho[x]_\rho \subseteq A.$$

Since A is a semiprime ideal, we have $y \in A$. It contradicts the supposition. This means that $\rho_-(A)$ is, if it is non-empty, a semiprime ideal of S. ∎

It can be seen from Example 96, that, for $A = \{3, 5\} \subseteq S$, $\rho_-(A) = \{5\}$. It is clear that $\rho_-(A)$ is a semiprime ideal of S. The ideal A is not a semiprime ideal for $4^2 = 3 \in A$ but $4 \notin A$. This shows that the converse of above theorem does not hold in general.

3.5 Rough Sets with Respect to Idempotent Congruences

Lemma 100 *Let ρ be an idempotent congruence relation on an LA-semigroup S. If A is a nonempty subset of S, then $\rho^-(A)$ is semiprime.*

Proof. Let $a^2 \in \rho^-(A)$. Then, since ρ is idempotent congruence,

$$[a]_\rho \cap A = [a]_\rho[a]_\rho \cap A = [a^2]_\rho \cap A \neq \phi.$$

This implies that $a \in \rho^-(A)$. Therefore $\rho^-(A)$ is semiprime. ∎

Theorem 101 *Let ρ be an idempotent congruence relation on an LA-semigroup S. If A and B are nonempty subsets of S, then*

$$\rho^-(A) \cap \rho^-(B) \subseteq \rho^-(AB).$$

Proof. Let c be any element of $\rho^-(A) \cap \rho^-(B)$. Then

$$c \in \rho^-(A) \quad \text{and} \quad c \in \rho^-(B)$$

$$[c]_\rho \cap A \neq \phi \quad \text{and} \quad [c]_\rho \cap B \neq \phi$$

Thus there exist elements $a, b \in S$ such that $a \in [c]_\rho$, $a \in A$, $b \in [c]_\rho$, and $b \in B$. Then, since ρ is an idempotent congruence on S, so

$$ab \in [c]_\rho[c]_\rho = [c]_\rho.$$

And since $ab \in AB$, we have

$$ab \in [c]_\rho \cap AB.$$

This implies that $c \in \rho^-(AB)$. Thus we obtain that

$$\rho^-(A) \cap \rho^-(B) \subseteq \rho^-(AB).$$

This completes the proof. ∎

Theorem 102 *Let ρ be an idempotent congruence relation on an LA-semigroup S. If A and B are nonempty subsets of S, then*

$$\rho_-(A) \cap \rho_-(B) \subseteq \rho_-(AB).$$

Proof. Let c be any element of $\rho_-(A) \cap \rho_-(B)$. Then $c \in \rho_-(A)$ and $c \in \rho_-(B)$. Thus $[c]_\rho \subseteq A$ and $[c]_\rho \subseteq B$. Since ρ is an idempotent congruence, we have

$$[c]_\rho = [c]_\rho[c]_\rho \subseteq AB,$$

which yields $c \in \rho_-(A)$. Therefore we obtain that

$$\rho_-(A) \cap \rho_-(B) \subseteq \rho_-(AB).$$

This completes the proof. ■

Theorem 103 *Let ρ be an idempotent congruence relation on an LA-semigroup S. If A and B are a right ideal and a left ideal of S, respectively, then*

$$\rho^-(A) \cap \rho^-(B) = \rho^-(AB).$$

Proof. This follows from Theorem 74, that

$$\rho^-(AB) \subseteq \rho^-(A) \cap \rho^-(B)$$

and also follows from Theorem 101, that

$$\rho^-(A) \cap \rho^-(B) \subseteq \rho^-(AB).$$

hence it follows

$$\rho^-(A) \cap \rho^-(B) = \rho^-(AB).$$

This completes the proof. ■

Theorem 104 *Let ρ be an idempotent congruence relation on an LA-semigroup S. If A and B are a right ideal and a left ideal of S, respectively, then*

$$\rho_{-}(A) \cap \rho_{-}(B) = \rho_{-}(AB).$$

Proof. This follows from Theorem 75, that

$$\rho_{-}(AB) \subseteq \rho_{-}(A) \cap \rho_{-}(B)$$

and also follows from Theorem 102

$$\rho_{-}(A) \cap \rho_{-}(B) \subseteq \rho_{-}(AB).$$

hence it follows

$$\rho_{-}(A) \cap \rho_{-}(B) = \rho_{-}(AB).$$

This completes the proof. ∎

3.6 Rough M-Systems in LA-semigroups

Definition 105 *A subset M of an LA-semigroup S is called a ρ-upper [ρ-lower] rough m-system in S if $\rho^{-}(M)[\rho_{-}(M)]$ is an m-system in S.*

Theorem 106 *Let ρ be a congruence relation on an LA-semigroup S. if M is an m-system in S, then M is an upper rough m-system in S.*

Proof. Let M is an m-system in S. Let $p, q \in \rho^{-}(M)$, then

$$[p]_{\rho} \cap M \neq \phi \quad \text{and} \quad [q]_{\rho} \cap M \neq \phi$$

Let $a \in [p]_{\rho} \cap M$ and $b \in [q]_{\rho} \cap M$, then $a \in [p]_{\rho}$, $a \in M$, $b \in [q]_{\rho}$ and $b \in M$. Since M is an m-system so there exist $r \in S$, such that $a(rb) \in M$. And also

$$r \in [r]_{\rho} \subseteq S$$

Now we have

$$\begin{aligned} a(rb) &\in [p]_\rho([r]_\rho[q]_\rho) \\ &\subseteq [p]_\rho[rq]_\rho \\ &\subseteq [p(rq)]_\rho \\ a(rb) &\in [p(rq)]_\rho \end{aligned}$$

Hence

$$a(rb) \in [p(rq)]_\rho \cap M$$

This implies that

$$[p(rq)]_\rho \cap M \neq \phi$$

Hence

$$p(rq) \in \rho^-(M)$$

This means that M is an upper rough m-system in S. This completes the proof. ∎

The following example shows that the converse of above theorem does not hold in general.

Example 107 *Let $S = \{1, 2, 3, 4, 5\}$ be an LA-semigroup with the following multiplication table:*

$\cdot$	1	2	3	4	5
1	4	4	4	4	4
2	4	4	4	4	4
3	4	4	4	4	4
4	4	4	4	4	4
5	4	4	3	4	4

Let ρ be a congruence relation on S such that ρ-congruence classes are the subsets $\{1\}$, $\{2\}$, $\{3,4\}, \{5\}$, then for $A = \{2,3\} \subseteq S$, $\rho^-(A) = \{2,3,4\}$, so now

$$\begin{aligned} S\{2,3,4\} &= \{3,4\} \subseteq \{2,3,4\} \quad \text{and} \quad \{2,3,4\}S = \{4\} \subseteq \{2,3,4\} \\ \text{but} \quad S\{2,3\} &= \{3,4\} \nsubseteq \{2,3\} \quad \text{and} \quad \{2,3\}S = \{4\} \nsubseteq \{2,3\} \end{aligned}$$

It is clear that $\rho^-(A)$ is an ideal and we know that every ideal is an m-system so $\rho^-(A)$ is an m-system in S but A is not an m-system in S because A is not an ideal and also there does not exist any $a \in S$ such that $x(ay) \in A$ for $x, y \in A$.

Now the lower approximation of an m-system is not an m-system in general. For this we take an example

Example 108 *Let $S = \{1, 2, 3, 4, 5\}$ be an LA-semigroup with the following multiplication table:*

$\cdot$	1	2	3	4	5
1	1	2	4	4	5
2	5	4	4	4	4
3	4	4	4	4	4
4	4	4	4	4	4
5	2	4	4	4	4

Let ρ be a congruence relation on S such that the ρ-congruence classes are the subsets $\{1\}$, $\{3\}$, $\{2, 4, 5\}$. Then for $A = \{3, 4\} \subseteq S$, $\rho_-(A) = \{3\}$, and here

$$\{3,4\}S = \{4\} \quad \textit{and} \quad S\{3,4\} = \{4\}$$

$$\textit{but} \quad \{3\}S = S\{3\} = \{4\} \nsubseteq \{3\}$$

This shows that the set $\{3, 4\}$ is an m-system of S but $\{3\}$ not an m-system of S. Thus $A = \{3, 4\}$ is an m-system but not lower rough m-system.

Here, if we take a complete congruence ρ on S, them the lower approximation of an m-system M is an m-system if M is also a left ideal, but this case is generally true by Theorem 72(2).

Theorem 109 *Let ρ be a congruence relation on an LA-semigroup S, if A is an LA-subsemigroup of S, then*

(1) $\rho^-(A)$ is an m-system in S.

(2) If ρ is complete then, $\rho_-(A)$ is an m-system in S.

Proof. (1) Let A is an LA-subsemigroup of S, then by Theorem 70(1), $\rho^{-}(A)$ is an LA-subsemigroup of S. Let $a, b \in \rho^{-}(A)$, then

$$a(ab) \in \rho^{-}(A)\rho^{-}(A) \subseteq \rho^{-}(AA) \subseteq \rho^{-}(A)$$

This means that $\rho^{-}(A)$ is an m-system in S. By using Theorem 72(1), we can prove (2) in a similar way. This completes the proof. ■

3.7 Rough Sets in the Quotient LA-semigroup

Definition 110 *Let ρ be a congruence relation on an LA-semigroup S and A be a subset of S. The lower and upper approximations can be presented in an equivalent form as shown below,*

$$\rho_{-}(A)/\rho = \{[x]_\rho \in S/\rho : [x]_\rho \subseteq A\}$$

$$\rho^{-}(A)/\rho = \{[x]_\rho \in S/\rho : [x]_\rho \cap A \neq \phi\}$$

Now we discuss these sets as subsets of a quotient LA-semigroup S/ρ of an LA-semigroup S.

Theorem 111 *Let ρ be a congruence relation on an LA-semigroup S. If A is an LA-subsemigroup of S, then $\rho^{-}(A)/\rho$ is an LA-subsemigroup of S/ρ.*

Proof. Let $[x]_\rho$ and $[y]_\rho$ be any elements of $\rho^{-}(A)/\rho$. Then

$$[x]_\rho \cap A \neq \phi \quad \text{and} \quad [y]_\rho \cap A \neq \phi.$$

Thus there exist elements $a, b \in S$ such that

$$a \in [x]_\rho \cap A \quad \text{and} \quad b \in [y]_\rho \cap A.$$

Then $a \in [x]_\rho$, $a \in A$, $b \in [y]_\rho$ and $b \in A$. Then $ab \in [x]_\rho[y]_\rho$. Since A is an LA-subsemigroup of S, we have $ab \in A$. Thus $ab \in [x]_\rho[y]_\rho \cap A$, so $[x]_\rho[y]_\rho \cap A \neq \phi$. This means that $[x]_\rho[y]_\rho \in \rho^{-}(A)/\rho$. Therefore $\rho^{-}(A)/\rho$ is an LA-subsemigroup of S/ρ. This completes the proof. ■

Theorem 112 *Let ρ be a congruence relation on an LA-semigroup S, and A be an LA-subsemigroup of S. Then $\rho_{-}(A)/\rho$ is, if it is nonempty, an LA-subsemigroup of S/ρ.*

Proof. Let $[x]_\rho$ and $[y]_\rho$ be any elements of $\rho_{-}(A)/\rho$. Then

$$[x]_\rho \subseteq A \quad \text{and} \quad [y]_\rho \subseteq A$$

Since A is an LA-subsemigroup of S, we have

$$[x]_\rho[y]_\rho \subseteq AA \subseteq A$$

and so $[x]_\rho[y]_\rho \in \rho_{-}(A)/\rho$. This means that $\rho_{-}(A)/\rho$ is, if it is nonempty, an LA-subsemigroup of S/ρ. This completes the proof. ■

Theorem 113 *Let ρ be a congruence relation on an LA-semigroup S. If A is a left [right, two-sided] ideal of S, then $\rho^{-}(A)/\rho$ is a left [right, two-sided] ideal of S/ρ.*

Proof. Assume that A is a left ideal of S. Let $[x]_\rho$ and $[s]_\rho$ be any elements of $\rho^{-}(A)/\rho$ and S/ρ, respectively. Then $[x]_\rho \cap A \neq \phi$, so there exists an element $a \in [x]_\rho \cap A$. Thus $a \in [x]_\rho$ and $a \in A$. Then, since A is a left ideal of S, for $t \in [s]_\rho$, we have

$$ta \in [s]_\rho A \subseteq SA \subseteq A$$

Since $ta \in [s]_\rho[x]_\rho$, we have

$$ta \in [s]_\rho[x]_\rho \cap A$$

Which implies that

$$[s]_\rho[x]_\rho \cap A \neq \phi$$

Thus $[s]_\rho[x]_\rho \in \rho^{-}(A)/\rho$. This means that $\rho^{-}(A)/\rho$ is a left ideal of S/ρ. The other cases can be seen in a similar way. This completes the proof. ■

Theorem 114 *Let ρ be a congruence relation on an LA-semigroup S. Let A be a left [right, two-sided] ideal of S. Then $\rho_{-}(A)/\rho$ is, if it is nonempty, a left [right, two-sided] ideal of S/ρ.*

Proof. Let A be a left ideal of S. Let $[x]_\rho$ and $[s]_\rho$ be any elements of $\rho_-(A)/\rho$ and S/ρ, respectively. Then $[x]_\rho \subseteq A$. Since A is a left ideal of S, we have

$$[s]_\rho[x]_\rho \subseteq SA \subseteq A.$$

Thus

$$[s]_\rho[x]_\rho \in \rho_-(A)/\rho.$$

This means that $\rho_-(A)/\rho$ is, if it is nonempty, a left ideal of S/ρ. The other cases can be seen in a similar way. This completes the proof. ■

Theorem 115 *Let ρ be a congruence relation on an LA-semigroup S. If A is a bi-ideal of S, then $\rho^-(A)/\rho$ is a bi-ideal of S/ρ.*

Proof. Let $[x]_\rho$ and $[y]_\rho$ be any elements of $\rho^-(A)/\rho$ and $[s]_\rho$ be any element of S/ρ. Then

$$[x]_\rho \cap A \neq \phi \quad \text{and} \quad [y]_\rho \cap A \neq \phi$$

and so there exist elements $a, b \in S$ such that

$$a \in [x]_\rho \cap A \quad \text{and} \quad b \in [y]_\rho \cap A.$$

Thus $a \in [x]_\rho$, $a \in A$, $b \in [y]_\rho$, and $b \in A$. Let t be element of $[s]_\rho$. Then, since A is a bi-ideal of S, so

$$(at)b \in (A[s]_\rho)A \subseteq (AS)A \subseteq A.$$

Since $(at)b \in ([x]_\rho[s]_\rho)[y]_\rho$, we have $(at)b \in ([x]_\rho[s]_\rho)[y]_\rho \cap A$. So $([x]_\rho[s]_\rho)[y]_\rho \cap A \neq \phi$. This implies $([x]_\rho[s]_\rho)[y]_\rho \in \rho^-(A)/\rho$. Then it follows from this and Theorem 111, that $\rho^-(A)/\rho$ is a bi-ideal of S/ρ. This completes the proof. ■

Theorem 116 *Let ρ be a congruence relation on an LA-semigroup S. Let A be a bi-ideal of S. Then $\rho_-(A)/\rho$ is, if it is nonempty, a bi-ideal of S/ρ.*

Proof. Let $[x]_\rho$ and $[y]_\rho$ be any elements of $\rho_-(A)/\rho$ and $[s]_\rho$ be any element of S/ρ. Then

$$[x]_\rho \subseteq A \quad \text{and} \quad [y]_\rho \subseteq A$$

Then for $[s]_\rho$, we have

$$([x]_\rho[s]_\rho)[y]_\rho \subseteq (AS)A \subseteq A.$$

Thus $([x]_\rho[s]_\rho)[y]_\rho \in \rho_-(A)/\rho$. Then it follows from this and Theorem 112 that $\rho_-(A)/\rho$ is, if it is nonempty, a bi-ideal of S/ρ. This completes the proof. ■

Theorem 117 *Let ρ be a congruence relation on an LA-semigroup S. If A is an interior ideal of S, then $\rho^-(A)/\rho$ is an interior ideal of S/ρ.*

Proof. Let $[x]_\rho$ be any element of $\rho^-(A)/\rho$, and let $[s]_\rho$ and $[t]_\rho$ be any elements of S/ρ. Then $[x]_\rho \cap A \neq \phi$, and so there exist element $a \in S$ such that $a \in [x]_\rho \cap A$. Thus $a \in [x]_\rho$, $a \in A$. Let s_1 and t_1 be any elements of $[s]_\rho$ and $[t]_\rho$. Then, since A is interior ideal of S, so

$$(s_1a)t_1 \in ([s]_\rho A)[t]_\rho \subseteq (SA)S \subseteq A.$$

Since $(s_1a)t_1 \in ([s]_\rho[x]_\rho)[t]_\rho$, we have $(s_1a)t_1 \in ([s]_\rho[x]_\rho)[t]_\rho \cap A$. So $([s]_\rho[x]_\rho)[t]_\rho \cap A \neq \phi$. This implies $([s]_\rho[x]_\rho)[t]_\rho \in \rho^-(A)/\rho$. Then it implies that $\rho^-(A)/\rho$ is an interior ideal of S/ρ. This completes the proof. ■

Theorem 118 *Let ρ be a congruence relation on an LA-semigroup S. Let A be an interior ideal of S. Then $\rho_-(A)/\rho$ is, if it is nonempty, an interior ideal of S/ρ.*

Proof. Let $[x]_\rho$ be any element of $\rho_-(A)/\rho$, and let $[s]_\rho$ and $[t]_\rho$ be any elements of S/ρ. Then $[x]_\rho \subseteq A$, now for $[s]_\rho$ and $[t]_\rho$, we have

$$([s]_\rho[x]_\rho)[t]_\rho \subseteq (SA)S \subseteq A.$$

Thus $([s]_\rho[x]_\rho)[t]_\rho \in \rho_-(A)/\rho$. Then it implies that $\rho_-(A)/\rho$ is, if it is nonempty, an interior ideal of S/ρ. This completes the proof. ■

Theorem 119 *Let ρ be a complete congruence relation on an LA-semigroup S, If A is an upper rough prime ideal of S, then $\rho^{-}(A)/\rho$ is a prime ideal of S/ρ.*

Proof. Since A is an upper rough ideal of S, by Theorem 113, we know that $\rho^{-}(A)/\rho$ is an ideal of S/ρ. Suppose

$$[x]_\rho[y]_\rho = [xy]_\rho \in \rho^{-}(A)/\rho \text{ for some } [x]_\rho, [y]_\rho \in S/\rho$$

then $[xy]_\rho \cap A \neq \phi$. Thus $xy \in \rho^{-}(A)$. Since A is an upper rough ideal of S, that is $\rho^{-}(A)$ is a prime ideal, we have

$$x \in \rho^{-}(A) \quad \text{or} \quad y \in \rho^{-}(A)$$

Hence

$$[x]_\rho \in \rho^{-}(A)/\rho \quad \text{or} \quad [y]_\rho \in \rho^{-}(A)/\rho.$$

Therefore $\rho^{-}(A)/\rho$ is a prime ideal of S/ρ. This completes the proof. ■

Theorem 120 *Let ρ be a complete congruence relation on an LA-semigroup S, If A is a lower rough prime ideal of S, then $\rho_{-}(A)/\rho$ is a prime ideal of S/ρ.*

Proof. Since A is a lower rough ideal of S, by Theorem 114, we know that $\rho_{-}(A)/\rho$ is an ideal of S/ρ. Suppose

$$[x]_\rho[y]_\rho = [xy]_\rho \in \rho_{-}(A)/\rho \text{ for some } [x]_\rho, [y]_\rho \in S/\rho$$

then $[xy]_\rho \subseteq A$. Thus $xy \in \rho_{-}(A)$. Since A is a lower rough ideal of S, that is $\rho_{-}(A)$ is a prime ideal, we have

$$x \in \rho_{-}(A) \quad \text{or} \quad y \in \rho_{-}(A)$$

Hence

$$[x]_\rho \in \rho_{-}(A)/\rho \quad \text{or} \quad [y]_\rho \in \rho_{-}(A)/\rho.$$

Therefore $\rho_{-}(A)/\rho$ is a prime ideal of S/ρ. This completes the proof. ■

Theorem 121 *Let ρ be a complete congruence relation on an LA-semigroup S, If A is an upper rough semiprime ideal of S, then $\rho^{-}(A)/\rho$ is a semiprime ideal of S/ρ.*

Proof. Since A is an upper rough ideal of S, by Theorem 113, we know that $\rho^{-}(A)/\rho$ is an ideal of S/ρ. Suppose

$$[x]_{\rho}[x]_{\rho} = [xx]_{\rho} = [x^2]_{\rho} \in \rho^{-}(A)/\rho \text{ for some } [x]_{\rho} \in S/\rho$$

then $[x^2]_{\rho} \cap A \neq \phi$. Thus $x^2 \in \rho^{-}(A)$. Since $\rho^{-}(A)$ is a semiprime ideal, we have $x \in \rho^{-}(A)$. Hence $[x]_{\rho} \in \rho^{-}(A)/\rho$. Therefore $\rho^{-}(A)/\rho$ is a semiprime ideal of S/ρ. This completes the proof. ■

Theorem 122 *Let ρ be a complete congruence relation on an LA-semigroup S, If A is a lower rough semiprime ideal of S, then $\rho_{-}(A)/\rho$ is a semiprime ideal of S/ρ.*

Proof. Since A is a lower rough ideal of S, by Theorem 114, we know that $\rho_{-}(A)/\rho$ is an ideal of S/ρ. Suppose

$$[x]_{\rho}[x]_{\rho} = [xx]_{\rho} = [x^2]_{\rho} \in \rho_{-}(A)/\rho \text{ for some } [x]_{\rho} \in S/\rho$$

then $[x^2]_{\rho} \subseteq A$. Thus $x^2 \in \rho_{-}(A)$. Since $\rho_{-}(A)$ is a semiprime ideal, we have $x \in \rho_{-}(A)$. Hence $[x]_{\rho} \in \rho_{-}(A)/\rho$. Therefore $\rho_{-}(A)/\rho$ is a semiprime ideal of S/ρ. This completes the proof. ■

Theorem 123 *Let ρ be a complete congruence relation on an LA-semigroup S. If M is an upper rough m-system in S, then $\rho^{-}(M)/\rho$ is an m-system of S/ρ.*

Proof. Suppose $[x]_{\rho}, [y]_{\rho} \in \rho^{-}(M)/\rho$, then

$$[x]_{\rho} \cap M \neq \phi \quad \text{and} \quad [y]_{\rho} \cap M \neq \phi.$$

Thus $x \in \rho^{-}(M)$ and $y \in \rho^{-}(M)$. Since $\rho^{-}(M)$ is an m-system, so for some $a \in S$, $[a]_{\rho} \in S/\rho$, we have $x(ay) \in \rho^{-}(M)$. Hence $[x(ay)]_{\rho} \cap M \neq \phi$, we have

$$[x(ay)]_{\rho} \in \rho^{-}(M)/\rho$$

$$[x]_{\rho}([a]_{\rho}[y]_{\rho}) \in \rho^{-}(M)/\rho.$$

Therefore $\rho^{-}(M)/\rho$ is an m-system in S/ρ. This completes the proof. ■

Theorem 124 *Let ρ be a complete congruence relation on an LA-semigroup S. If M is a lower rough m-system in S, then $\rho_{-}(M)/\rho$ is an m-system in S/ρ.*

Proof. Suppose $[x]_\rho, [y]_\rho \in \rho_{-}(M)/\rho$, then

$$[x]_\rho \subseteq M \quad \text{and} \quad [y]_\rho \subseteq M$$

we have

$$x \in \rho_{-}(M) \quad \text{and} \quad y \in \rho_{-}(M)$$

As $\rho_{-}(M)$ is an m-system of S, so for $a \in S$, $[a]_\rho \in S/\rho$, we have $x(ay) \in \rho_{-}(M)$. Hence $[x(ay)]_\rho \subseteq M$, This implies that

$$[x(ay)]_\rho \in \rho_{-}(M)/\rho$$

$$[x]_\rho([a]_\rho[y]_\rho) \in \rho_{-}(M)/\rho$$

Therefore $\rho_{-}(M)/\rho$ is an m-system in S/ρ. This completes the proof. ■

Bibliography

[1] R. Biswas and S. Nanda, Rough groups and rough subgroups, Bulletin of the Polish Academy of Sciences, Mathematics 42 (1994), no. 3, 251-254.

[2] J. M. Howie, Fundamentals of semigroup theory, Clarendon Press Oxford. 1995.

[3] S. Kamran; Structural properties of LA-semigroups, M. Phil. Desertation, Quaid-i-Azam University, Islamabad, 1987.

[4] M. A. Kazim and M. Naseerudin, On almost-semigroup, The Alig. Bull. Math, 2 (1972),1-7.

[5] M. Khan, Structural properties of LA-semigroups, M. Phil. Dissertation, Quaid-i-Azam University, Islamabad, 2003.

[6] N. Kuroki, Rough ideals in semigroups, Inform. Sci. 100 (1997) 139–163.

[7] Q. Mushtaq and S. M. Yusuf, On LA-semigroups, The Alig. Bull. Math, 8 (1978), 65-70.

[8] Q. Mushtaq and S. M. Yusuf, On locally associative LA-semigroups, J. Nat. Sci. Math. 19 (1979), 57-62.

[9] Q. Mushtaq and M. Iqbal, Partial ordering and congruences on LA-semigroups, Indian J. pure appl. Math.,22(4) (1991), 331-336.

[10] Q. Mushtaq and Madad Khan, M-Systems in LA-Semigroups, Southeast Asian Bulletin of Mathematics (2009) 33: 321–327.

[11] Z. Pawlak, Rough sets, http://grammars.grlmc.com/GRLMC/reports/rep29.doc

[12] P. V. Protic and N. Stevanovic, On AG-test and some general properties of Abel-Grassman's groupoids, P. U. M. A, 4, 6 (1995), 371-383.

[13] Q. M. Xiao and Z. L. Zhang, Rough prime ideals and rough fuzzy prime ideals in semi-groups, Information Sciences 176 (2006), no. 6 ,725-733.

[14] Yao, Y.Y., Wong, S.K.M., and Lin, T.Y., A review of rough set models in: Rough Sets and Data Mining: Analysis for Imprecise Data, Lin, T.Y. and Cercone, N. (Eds.), Kluwer Academic Publishers, Boston, pp. 47-75, 1997.