Homomorphic Images of Generalized Triangle Subgroups of PSL(2, $\mathbb{Z}$)

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Contents

Preface

The modular group generated by two linear fractional transformations, $u : z \longmapsto \frac{-1}{z}$ and $v : z \longmapsto \frac{z-1}{z}$, satisfying the relations $u^2 = v^3 = 1$ [46]. The linear transformation $t : z \mapsto \frac{1}{z}$ inverts u and v, i,e, $t^2 = (vt)^2 = (ut)^2 = 1$ and extends $PSL(2,\ \mathbb{Z})$ to $PGL\,(2, \mathbb{Z})$. In [72] a condition for the existence of t is explained.

G. Higman introduced coset diagrams for.$PSL(2,\ \mathbb{Z})$ and $PGL\,(2, \mathbb{Z})$. Since then, these have been used in several ways, particularly for finding the subgroups which arise as homomorphic images or quotients of $PGL\,(2, \mathbb{Z})$. The coset diagrams of the action of $PSL\,(2, \mathbb{Z})$ represent permutation representations of homomorphic images. In these coset diagrams the three cycles of the homomorphic image of v, say $\bar{v}$, are represented by small triangles Δ whose vertices are permuted counter-clockwise, any two vertices which are interchanged by homomorphic image of u, say $\bar{u}$, are joined by an edge ——, and $\bar{t}$ is denoted by symmetry along the vertical line. The fixed points of $\bar{u}$ and $\bar{v}$, if they exist are denoted by heavy dots. The fixed points of $\bar{t}$ lies on the vertical line of symmetry.

A real quadratic irrational field is denoted by $\mathbb{Q}\left(\sqrt{d}\right)$, where d is a square free positive integer. If $\alpha = \left(a_1 + b_1\sqrt{d}\right) \diagup c_1$ is an element of $\mathbb{Q}\left(\sqrt{d}\right)$, where $a_1, b_1, c_1, d,$ are integers, then α has a unique representation such that a_1, $\left(a_1^2 - d\right) \diagup c_1$ and c_1 are relatively prime. It is possible that α, and and its algebraic conjugate $\bar{\alpha} = \left(a_1 - \sqrt{d}\right) \diagup c_1$ have opposite signs. In this case α is called an ambiguous number by Q. Mushtaq in [69].

The coset diagrams of the action of $PSL\,(2, \mathbb{Z})$ on $\mathbb{Q}\left(\sqrt{d}\right)$ depict interesting results. It is shown in [69] that for a fixed value of d, there is only one circuit in the coset diagram of the orbit, corresponding to each α.

Any homomorphism $\rho_1 : PGL(2,\mathbb{Z}) \rightarrow PGL(2,q)$ give rise to an action on $PL(F_q)$. We denote the generators $(\mu)\rho_1$, $(\nu)\rho_1$ and $(t)\rho_1$ by $\bar{\mu}$, $\bar{\nu}$ and $\bar{t}$. If neither of the generators μ, ν and t lies in the kernel of ρ_1, so that $\bar{\mu}$, $\bar{\nu}$ and $\bar{t}$ are of order 2, 3 and 2 respectively, then ρ_1 is said to be a non-degenerate homomorphism. In addition to these relations, if another relation $(\bar{\mu}\bar{\nu})^k = 1$ is satisfied by it, then it has been proved in [74] that the conjugacy classes of non-degenerate homomorphisms of $PGL(2,\mathbb{Z})$ into $PGL(2,q)$ correspond into one to one way with the conjugacy classes of ρ_1 and an element θ of F_q. That is, the actions of $PGL(2,\mathbb{Z})$ on $PL(F_q)$ are parametrized by the elements of F_q. This further means that there is a unique coset diagram, for each conjugacy class corresponding to $\theta \in F_q$. Finally, by assigning a parameter $\theta \in F_q$ to the conjugacy class of ρ_1, there exists a polynomial $f(\theta)$ such that for each root θ_i of this polynomial, a triplet $\bar{\mu}$, $\bar{\nu}$, $\bar{t} \in PGL(2,q)$ satisfies the relations of the triangle group

$$\Delta(2,3,k) = \left\langle \bar{\mu}, \bar{\nu}, \bar{t} : \bar{\mu}^2 = \bar{\nu}^3 = (\bar{t})^2 = (\bar{\mu}\bar{\nu})^k = (\bar{\nu}\bar{t})^2 = (\bar{\mu}\bar{t})^2 = 1 \right\rangle.$$

Hence, we can obtain the triangle groups $\Delta(2,3,k)$ through the process of parametrization.

The generalized triangle group has the presentation $\langle u, v : u^r, v^s, W^k \rangle$,where $r,\ s,\ k$ are integers greater than 1, and $W = u^{\alpha_1}v^{\beta_1}...u^{\alpha_k}v^{\beta_k}$, where $k > 1, 0 < \alpha_i < r$ and $0 < \beta_i < s$ for all i. These groups are obtained by natural generalization of $\Delta(r,s,k)$ defined by the presentations $\left\langle u, v : u^r = v^s = (uv)^k = 1 \right\rangle$, where r, s and k are integers greater than one.

It was shown in [37] that G is infinite if $\frac{1}{r} + \frac{1}{s} + \frac{1}{k} \leq 1$ provided $r \geq 3$ or $k \geq 3$ and $s \geq 6$, or $(r,\ s, k) = (4,5,2)$. This was generalized in [4], where it was shown that G is infinite whenever $\frac{1}{r} + \frac{1}{s} + \frac{1}{k} \leq 1$. A proof of this last fact can be seen in [101].

A generalized triangle group may be infinite when $\frac{1}{r}+\frac{1}{s}+\frac{1}{k}>1$. The complete classification of finite generalized triangle groups is given in 1995 by J. Howie in [39] and later by L. Levai, G. Rosenberger, and B. Souvignier in [57] which are fourteen in number.

As there are fourteen, generalized triangle groups classified as finite [39], our area of interest is the set of groups which are homomorphic images or quotients of $PSL(2;\mathbb{Z})$. Out of these fourteen only eight groups are quotients of the modular group. In this study, we have extended parametrization of the action of $PSL(2;\mathbb{Z})$ on $PL(F_p)$, where p is a prime number, to obtain the finite generalized triangle groups $\left\langle \bar{\mu}^2=\bar{\nu}^3=\left(\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}^2\right)^3=1\right\rangle$ by this parametrization. By parametrization of action of $PGL(2;\mathbb{Z})$ on $PL(F_p)$ we have obtained the coset diagrams of

$$\left\langle \bar{\mu}^2=\bar{\nu}^3=\left(\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}^2\right)^3=1\right\rangle$$

for all $\theta\in F_p$.

This thesis is comprised of six chapters. The first chapter consists of some basic definitions and concepts along with examples. We have given brief introduction of linear groups, the modular and the extended modular group, real quadratic irrational fields, finite fields, coset diagrams, triangle groups, and generalized triangle groups.

In the second chapter, we show that entries of a matrix representing the element $g=\left((\mu v)^{m_1}\left(\mu v^2\right)^{m_2}\right)^l$ where $l\geq 1$ of $PSL\left(2,\mathbb{Z}\right)=\left\langle \mu,v:\mu^2=v^3=1\right\rangle$ are denominators of the convergents of the continued fractions related to the circuits of type $\left(m_1,m_2\right)$, for all $m_1,m_2\in\mathbb{N}$. We also investigate fixed points of a particular class of circuits of type $\left(m_1,m_2\right)$ and identify location of the Pisot numbers in a circuit of a coset diagram of the action of $PSL\left(2,\mathbb{Z}\right)$ on $\mathbb{Q}\left(\sqrt{d}\right)\cup\{\infty\}$, where d is a non-square positive integer.

In the third chapter we attempt to classify all those subgroups of the homomorphic image of $PSL\,(2,\mathbb{Z})$ which are depicted by coset diagrams containing circuits of the type $(m_1,\ m_2)$.

In the fourth chapter we devise a special parametrization of the action of modular group $PSL(2,\mathbb{Z})$ on $PL(F_p)$, where p is prime, to obtain the generalized triangle groups

$$\left\langle \bar{\mu}^2 = \bar{\nu}^3 = \left(\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}^2\right)^k = 1 \right\rangle$$

and by parametrization we obtain the coset diagrams of

$$\left\langle \bar{\mu}^2 = \bar{\nu}^3 = \left(\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}^2\right)^k = 1 \right\rangle$$

for all $\theta \in F_p$.

In the fifth chapter we investigate the action of $PSL(2,\mathbb{Z})$ on $PL(F_{7^n})$ for different values of n, where $n \in \mathbb{N}$, which yields $PSL(2,7)$. The coset diagrams for this action are obtained, by which the transitivity of the action is inspected in detail by finding all the orbits of the action. The orbits of the coset diagrams and the structure of prototypical $D168$ Schwarzite [48], are closely related to each other. So, we investigate in detail the relation of these coset diagram with the carbon allotrope structures with negative curvature $D168$ Schwarzite. Their relation reveals that the diagrammatic structure of these orbits is similar to the structure of hypothetical carbon allotrope $D56$ Protoschwarzite which has a C_{56} unit cell.

In the last chapter, we investigate the actions of the modular group $PSL(2,\mathbb{Z})$ on $PL(F_{11^m})$ for different values of m, where $m \in \mathbb{N}$ and draw coset diagrams for various orbits and prove some interesting results regarding the number of orbits that occur.

Chapter 1

Definitions and Concepts

This chapter comprises of some basic definitions along examples. We have included linear groups, modular and extended modular group, real quadratic irrational fields, finite fields, coset diagram, triangle groups, and generalized triangle groups.

1.1 Fields

1.1.1 Quadratic Fields

The solution of some quadratic equation with coefficients from rational numbers is called a quadratic irrational number. They are expressed as $\left(a_1 + b_1\sqrt{d}\right) \diagup c_1$, where a_1, b_1, c_1 are integers and d is a positive square-free integer. For a given d, they form a field of quadratic irrational numbers and it is defined as real quadratic irrational field.

A quadratic field is denoted by $\mathbb{Q}\left(\sqrt{d}\right)$, where d is a square-free positive integer. If $d > 0$, $\mathbb{Q}\left(\sqrt{d}\right)$ is said to be a real quadratic field, and if $d < 0$ it is called an imaginary quadratic field. The set of algebraic integers of $\mathbb{Q}\left(\sqrt{d}\right)$ is $\{a_1 + b_1\sqrt{d} : a_1, b_1 \in \mathbb{Z}\}$ if $d \equiv 2$

or 3(mod 4), and $\{a_1 + b_1\sqrt{d}/2 : a_1, b_1 \in \mathbb{Z},\ a_1 \equiv b_1 \pmod 2\}$ if $d \equiv 1 \pmod 4$ [65].

Each real quadratic irrational number is expressed uniquely as $\left(a_1 + \sqrt{d}\right) \diagup c_1$, where d is a square-free positive integer and $\gcd(a_1, \left(a_1^2 - d\right) \diagup c_1, c_1) = 1$. The algebraic conjugate of α, is defined as $\bar{\alpha} = \left(a_1 - \sqrt{d}\right) \diagup c_1$. If α and $\bar{\alpha}$ are both of negative (positive), then α is said to be a totally negative (positive) number. The element α is said to be an ambiguous number if both α and $\bar{\alpha}$ are of opposite signs (see [69]).

It is notable that the square-free integers are such type of integers which are not divisible by any perfect square, except 1.

1.1.2 Finite Fields

Fields having finite number of elements are of great importance in different branches of mathematics, like projective geometry, group theory, number theory, and many more. The most familiar examples of these type of fields are fields of integer modulo p, $\mathbb{Z}_p$, for some prime p. For an integer $s_1 > 0$ and a prime p, there is a field with p^{s_1} elements. The fields with other than p^{s_1} elements do not exist. The field with $q = p^{s_1}$ number of elements is denoted as F_q (or $GF(q)$).

The ring structure on the ring of integers modulo n, $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, is induced by the ring $\mathbb{Z}$. If n equals some prime p, then the structure $\mathbb{Z}_p$ is a field. Likewise $(\mathbb{Z}_p)^{s_1} = \{(c_0, c_1, c_2, ..., c_{s_1-1}) : c_i \in \mathbb{Z}_p\}$ is also a field. This field is obtained by associating the sequence $(c_0, c_1, c_2, ..., c_{s_1-1})$ with some polynomial $f(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{r-1} t^{s_1-1}$ in $\mathbb{Z}_p[t]$.

Let $g(t)$ be an irreducible polynomial in $\mathbb{Z}_p[t]$ of degree s_1. To construct a field with p^{s_1} elements we have to select $g(t)$ in such a manner that all non-zero elements of the field are the powers of t such that $t^{s_1-1} = 1$, where 1 is the multiplicative identity. This field is known as a Galois field and is denoted by $GF(p^{s_1})$ over $F_{p^{s_1}}$.

For instance, F_{3^3} is obtained by taking an irreducible polynomial $g(t) = t^3 + t^2 + 2t + 1$ over $\mathbb{Z}_3$. The elements of F_{3^3} are given in the table below.

$S.No$	$GF(3^3)$	$\mathbb{Z}_3[t]$	$S.No$	$GF(3^3)$	$\mathbb{Z}_3[t]$
1	0	0	15	t^{13}	2
2	1	1	16	t^{14}	$2t$
3	t	t	17	t^{15}	$2t^2$
4	t^2	t^2	18	t^{16}	t^2+2t+1
5	t^3	$2t^2+t+2$	19	t^{17}	t^2+2t+2
6	t^4	$2t^2+t+1$	20	t^{18}	t^2+2
7	t^5	$2t^2+1$	21	t^{19}	$2t^2+2$
8	t^6	t^2+1	22	t^{20}	t^2+t+1
9	t^7	$2t^2+2t+2$	23	t^{21}	$2t+2$
10	t^8	$t+1$	24	t^{22}	$2t^2+2t$
11	t^9	t^2+1	25	t^{23}	$2t+1$
12	t^{10}	$t+2$	26	t^{24}	$2t^2+t$
13	t^{11}	t^2+2t	27	t^{25}	$2t^2+2t+1$
14	t^{12}	t^2+t+2			

The relevant properties are as follows:

(i) The finite field F containing q elements is isomorphic to a Galois field $GF(q)$. Specifically, the field structure does not depend upon the selection of the irreducible polynomial $g(t)$.

$(ii)\, GF(p^{s_1})$ is a cyclic group having order $p^{s_1} - 1$ under multiplication. The generating element of $GF(p^{s_1})$ is known as primitive element.

1.1.3 Projective Lines

The space containing the set of all one dimensional subspaces of F^2 with natural action of $PGL\,(2, F_p)$ on it, is denoted by $PG\,(1, F_p)$, and is called projective line. If U is a subspace of F^2 of dimension one then either $U = \{(x_1, 0) : x_1 \in F_p\}$ or U is generated by $(x_1, 1)$ for some $x_1 \in F_p$. So $PG\,(1, F_p)$ is identified with $F_p \cup \{\infty\}$ using the map:

$$\{(x_1, 1) : x_1 \in F_p\} \to x_1$$

and

$$\{(0, 1) : x_1 \in F_p\} \to \infty$$

Every single coordinate is associated with every element of $PG\,(1, F_p)$. The coordinate of the subspace of F^2 generated by (y_1, z_1) is y_1/z_1 with the convention that $y_1/0 = \infty$, for $y_1 \neq 0$. Let $q = p^{s_1}$ for some positive integer s_1. Then $PL\,(F_q)$ consists of the elements of F_q with an extra element ∞. Particularly, for $s_1 = 1$, $F_q = F_p = \{0, 1, 2, 3, ..., p - 1\}$.

The group $PGL\,(2, F_p)$ naturally acts on $PG\,(1, F_p)$ in the following manner: let $\begin{pmatrix} c & e \\ d & f \end{pmatrix}$ be a non-singular matrix of $PGL\,(2, F_p)$, then

$$(y_1, z_1) \begin{pmatrix} c & e \\ d & f \end{pmatrix} = (cy_1 + ez_1, dy_1 + fz_1)$$

or

$$y_1/z_1 \to \frac{cy_1 + ez_1}{dy_1 + fz_1} = \frac{c\,(y_1/z_1) + e}{d\,(y_1/z_1) + f}.$$

defines the action of $PGL(2, F_p)$ on $PG(1, F_p)$. The group of transformations

$$T_{cedf} : z \rightarrow \frac{cx_1 + e}{dx_1 + f}.$$

is thus the linear fractional group acting on $F_p \cup \{\infty\}$, the projective line over F_p.

1.2 Linear Groups

The importance of linear groups is well known because of their influence and applications in different fields of science such as chemistry, physics and many others.

There is a relationship between linear groups, Galois theory, and the theory of Lie groups. Their connection with Galois theory leads to the classical groups over F_q and soluble groups. These groups are used widely in the group representations theory, in the study of polynomials and in spatial symmetries of vector spaces.

Consider an n dimensional vector space U over a field F. Then $Hom_F(U, U)$, the set of all linear transformations of U forms a vector space which also posses the ring structure. The multiplicative identity of $Hom_F(U, U)$ is identity mapping I on U. An element σ of $Hom_F(U, U)$ is known to be invertible if there exists a mapping δ in $Hom_F(U, U)$ such that $\sigma\delta = \delta\sigma = I$. This forms a group of all invertible elements of $Hom_F(U, U)$. This group is known to be the general linear group of degree n, and is denoted by $GL_n(U)$ or $GL(n, U)$.

The set of all matrices of order $n \times n$ with entries from F is $M_n(F)$. This set is directly linked with $Hom_F(U, U)$, that is, both $Hom_F(U, U)$ and $M_n(F)$ posses linear associative algebras so they are isomorphic. The n dimensional general linear group $GL(n, F)$ of all invertible matrices of order $n \times n$, is isomorphic to $GL_n(U)$.

The most important subgroups of $GL(n,\ F)$ is the special linear group, denoted by $SL(n, F)$ and presented as $SL(n, F) = \{[c_{ij}] : c_{ij} \in F,\ i, j = 1, 2,\ \det([c_{ij}]) = 1\}$. The importance of $SL(n, F)$ is associated with the fact that in a two dimensional lattice, bases $\{f_1,\ f_2\}$ and $\{e_1,\ e_2\}$ are correlated by the following equations:

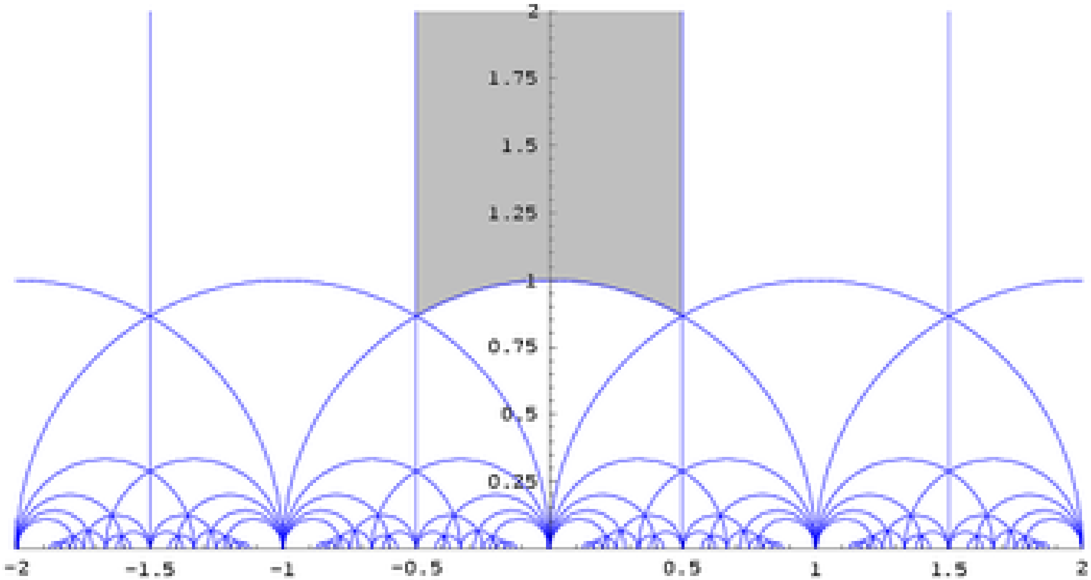
$$\begin{aligned} f_1 &= a_1e_1 + c_1e_2 \\ f_2 &= b_1e_1 + d_1e_2 \end{aligned}$$

where $a_1d_1 - b_1c_1 = \pm 1$ and $a_1,\ b_1,\ c_1,\ d_1 \in F$. It is necessary that the orientation from f_1 to f_2 is same as that from e_1 to e_2 to obtain $a_1d_1 - b_1c_1 = 1$.

1.2.1 Modular Group and Lobachevsky Plane

The modular group, denoted by $PSL(2, \mathbb{Z})$, is the quotient group of $SL(2, \mathbb{Z})$ by its center, thus $PSL(2, \mathbb{Z}) \cong SL(2, \mathbb{Z})/N$, where $N = \left\{ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

The fundamental domain can be observed as the shaded region of the modular region.



The upper half plane is known as the model of Lobachevsky plane $\{(\mu,\nu):\mu,\ \nu\in\mathbb{C},\nu>0\}$ and the orientation is persevered by the motion in it and is considered as transformation $z=\frac{\acute{a}z+\acute{b}}{\acute{c}z+\acute{d}}$, where $\acute{a}\acute{d}-\acute{b}\acute{c}=1$ and $\acute{a},\ \acute{d},\ \acute{b}$ and $\acute{c}$ are in $\mathbb{R}$. Consider the action of $PSL\,(2,\mathbb{Z})$ on upper half plane. Then in Lobachevsky plane, the modular group is considered as a discrete group of motions. Therefore, $PSL\,(2,\mathbb{Z})$ is a group generated by two linear transformations $\mu:z\longmapsto\dfrac{-1}{z}$ and $\nu:z\longmapsto\dfrac{z-1}{z}$ such that $\mu^2=\nu^3=1$ [29]. Thus $PSL\,(2,\mathbb{Z})$ can also be seen as a free product of the two cyclic groups $<\mu:\mu^2=1>$ and $<\nu:\nu^3=1>$. That is, $PSL\,(2,\mathbb{Z})\cong C_2*C_3$.

The linear transformation $t:z\mapsto\dfrac{1}{z}$ inverts μ and ν, that is, $t^2=(\nu t)^2=(\mu t)^2=1$ and extends $PSL(2,\ \mathbb{Z})$ to $PGL\,(2,\mathbb{Z})$. In [72] a condition for the existence of t in the action of modular group on $PL\,(F_q)$, is obtained.

Let p is a prime and $q=p^n$, then the group $PGL\,(2,q)$ is the group of all transformations $z = \frac{\acute{a}z+\acute{b}}{\acute{c}z+\acute{d}}$, where $\acute{a},\ \acute{d},\ \acute{b}$ and $\acute{c} \in F_q$, and $\acute{a}\acute{d} - \acute{b}\acute{c} \neq 0$. $PSL\left(2,q\right)$ is a subgroup of $PGL\left(2,q\right)$, consisting of all those linear fractional transformations $z = \frac{\acute{a}z+\acute{b}}{\acute{c}z+\acute{d}}$, where $\acute{a},\ \acute{d},\ \acute{b}$ and $\acute{c} \in F_q$, and $\acute{a}\acute{d} - \acute{b}\acute{c}$ is a non-zero square in F_q.

The order of the $PSL\left(n,q\right)$ is:

$$\left|PSL\left(n,q\right)\right| = \frac{1}{\left(n,\ q-1\right)} q^{n(n-1)/2} \left(q^2-1\right)\left(q^3-1\right)...\left(q^n-1\right).$$

Many mathematicians worked independently on linear fractional groups in several fields. J. A. Serret in 1866 [89], worked on the homomorphisms of general linear group of divisor 2 by following the pattern of E. Galios. A. Cayley in 1964 utilized it to find different properties of linear fractional transformations [105].

There is a well known classical relationship between the continued fractions and an action of $PSL\left(2,\mathbb{Z}\right)$ on real line. Many articles have been published upon the relationship between continued fraction and the geodesics on modular surface and have significance in the theory of approximation of real numbers by rationals [[7], [63]]. In [70], a connection between orbits of modular group and reduced indefinite binary quadratic forms has been established. In [77], using the action of modular group on real quadratic fields, the Lucas and Fibonacci numbers are determined. In [76], Pell numbers and Pell Lucas Numbers are found through the action of $PSL\left(2,\mathbb{Z}\right)$.

1.3 Coset Diagrams

The idea of using coset graphs, for depicting group actions, has a rich and long history. Coset graphs give us method to analyze a large range of topological and algebraic properties of different structures. For the analysis of groups that are finitely generated, graphical methods are explicitly used. Many important results are proved using graphical techniques as in [13, 15, 27]. For finite groups of small order, the coset graphs show the similar information as multiplication tables seen in [93] and [96]. They depict the same properties but in a more effective manner [95].

The concept of coset graphs for groups was first introduced in 1878 by A. Cayley [15]. After that in 1893, A. Hurwitz took coset graphs as a tool for representing groups. H. Maschke [66], in 1896 made use of Cayley's graphs in proving some useful results concerning representation of the finite groups, specially related to the groups of rotation of regular bodies in 3 and 4-dimensional space. In 1910, the Cayley's diagrams were reinvented by M. Dhen. Later, H. W. Kuhn [52] and O. Schreier [27] also used graphical methods to prove several results.

A. Cayley [15], by using coset graph of a given group with known generators illustrated the multiplication table of a group, and used different colours for different generators to draw the edges of the graph linked with those generators. The Cayley's diagram is a coset graph of a specific group where elements of the group are represented by vertices, which can also be seen as cosets of the identity subgroup $\{1\}$. O. Schreier [27] made generalization of this idea by taking into consideration the coset graph with vertices to be the cosets of a particular subgroup. H. S. M. Coxeter and W. J. Moser [27] in 1965, made use of both Schreier's and Cayley's diagrams for proving significant results about groups that are finitely generated.

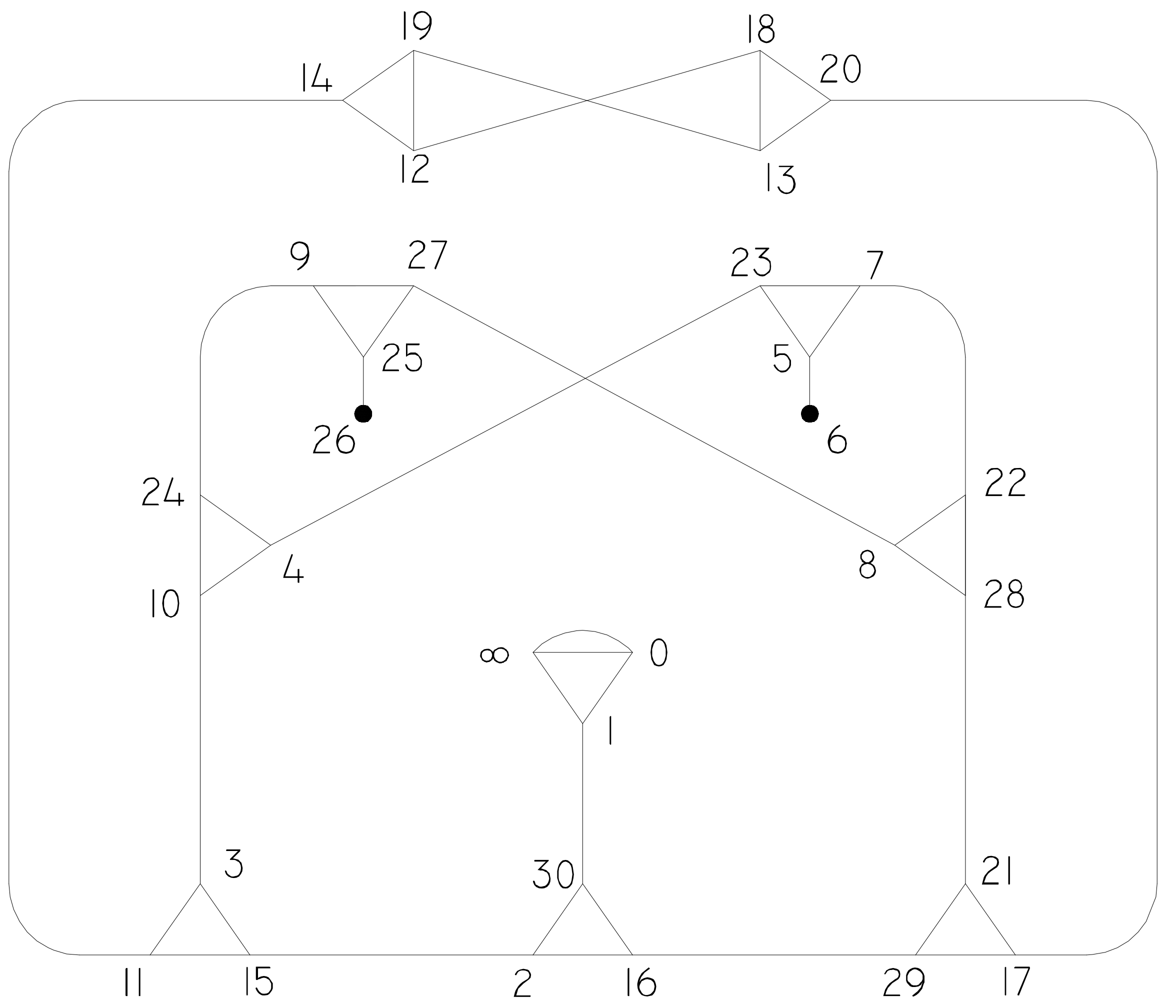
G. Higman, in 1978, defined coset diagrams for describing the actions of $PSL(2,\mathbb{Z})$ and $PGL(2,\mathbb{Z})$ (see [21, 22, 68, 72, 73, 74]). M. D. E. Conder in [21] and [25] showed that all but finitely many alternating groups are Hurwitz group by using special coset diagrams.

Coset diagrams of the action of $PSL(2,\mathbb{Z})$ are significant in several ways. Using colours can be avoided for $PSL(2,\mathbb{Z})$ because of the nature of these two generators that is $\mu^2 = \nu^3 = 1$. Since $\mu^2 = 1$, the generator μ is depicted by an edge which directs its both vertices towards each other. For the generator ν with $\nu^3 = 1$, it is required to differentiate ν and ν^2. Hence the cycles of ν of length 3 are portrayed by triangles, with the convention that ν permutes each vertex of a triangle anti-clockwise. This nature of the diagram makes the $\mu - edges$ and $\nu - edges$ different. The fixed points of μ and ν are denoted by heavy dots.

For example, take the action of $PGL(2,\mathbb{Z})$ acting on $PL(F_{31})$ illustrated by $\mu(\omega) = \frac{-1}{\omega}$, $\nu(\omega) = \frac{\omega-1}{\omega}$, where $\omega \in PL(F_{31})$. Then there is the following permutation representation of μ and ν

$$\bar{\mu} = (1,30)(2,15)(3,10)(4,23)(5,6)(7,22)(8,27)(9,24)(11,14)(12,18)$$
$$(13,19)(16,29)(17,20)(21,28)(25,26)(0,\infty),$$
$$\bar{\nu} = (0,\infty,1)(2,16,30)(3,11,15)(4,24,10)(5,7,23)(6)(8,28,22)(9,25,27)$$
$$(12,19,14)(13,20,18)(17,21,29)(26).$$

The action yield the following coset diagram:



W. W. Stothers [96], in 1977 studied the subgroups of $\Delta(2,3,7)$ using coset diagrams. To a subgroup of $\Delta(2,3,7)$ having finite index, he linked (a,b,e,f,h) a quintuple where $a,b,e,f,h \in \mathbb{Z}^+$ with $a \geq 1$ and $a = 84(b-1) + 21e + 28f + 36h$. It is also shown, except for three exceptions, that every quintuple fulfilling these conditions is associated to a subgroup. This was done by using coset diagrams and a technique of combining different or same diagrams through handles.

Q. Mushtaq [68] worked extensively on the modular group using coset diagrams as a basic technique and devised many important results ([**?**, 71, 46]). It is proved in [46] that coset diagram in the action of $PSL(2,\mathbb{Z})$ on rational projective line is connected and transitive. It is also shown that the linear fractional transformations μ and ν generate $PSL(2,\mathbb{Z})$ and that $\mu^2=\nu^3=1$ are defining relations for $PSL(2,\mathbb{Z})$ using coset diagrams. Coset diagrams also used to show that ambiguous numbers exist and they exist excessively in an orbit when $PSL(2,\mathbb{Z})$ acts on real quadratic fields.

In 1983, the actions of $PSL(2,\mathbb{Z})$ on different sets are studied and proved that for each value of $\theta \in F_q$, with q to be a power of a prime, a coset diagram of the action of $PGL(2,\mathbb{Z})$ over $PL(F_q)$ can be drawn [68]. In [71], Q. Mushtaq found a condition for the existence of a specific fragment in a coset diagram. That is he established a useful relation between a polynomial with coefficients from $\mathbb{Z}$ and a coset diagram containing the fragment.

Q. Mushtaq and F. Shaheen [79] proved the existence of some special circuits in the coset diagrams under action of a group with the presentation

$\left\langle \mu,\nu,t:\mu^2=\nu^n=t^2=(\mu t)^2=(\nu t)^2=1\right\rangle$ on projective lines over the Galois fields. Later on, Higman's question was answered for permutation representation of the symmetric and hyperbolic tessellation by using coset diagrams in [81]. M. D. E. Conder and Q. Mushtaq separately worked on the solutions of several identification problems with the help of these diagrams.

Coset diagrams are helpful in providing diagrammatic explanations of different concepts of combinatorial group theory including the Reidemeister-Schreier procedure, and to prove the theorem by Ree-Singerman regarding the cyclic structure for a transitive group of generating-permutations. To construct the infinite families of finite quotients of a particular group with finite presentation, similar methodology is helpful. For finding torsion-free subgroups of certain groups with finite presentation the coset diagrams have proved to be of great help for constructing the small volume hyperbolic 3-manifolds which show interesting behaviours. Coset diagrams can also be helpful in the formation of arc-transitive graphs and maximal automorphism groups of Riemann surface [25].

1.4 Triangle Groups

Triangle groups and their importance is explained in [27]. The relation between the triangle groups and $PSL(2,\mathbb{Z})$ is that, in certain cases they arise as quotients of $PSL(2,\mathbb{Z})$. Triangle groups can be presented as

$$\Delta(r,s,k) = \left\langle u,v : u^r = v^s = (uv)^k = 1 \right\rangle$$

where $r,s,k > 1$ and $r,s,k \in \mathbb{Z}$ (for details, see [27]).

The triangle groups $\Delta(r,s,k)$ are known to be finite precisely when $\gamma = \frac{1}{r} + \frac{1}{s} + \frac{1}{k} - 1 > 0$, and the groups obtained are A_4, S_4, D_{2n}, and C_n. If $\gamma = \frac{1}{r} + \frac{1}{s} + \frac{1}{k} - 1 = 0$, then $(r,s,k) = (2,4,4)$, $(3,3,3)$, or $(2,3,6)$. In this case $\Delta(r,s,k)$ is soluble but infinite. The triangle groups $\Delta(r,s,k)$ are infinite if and only if $\gamma = \frac{1}{r} + \frac{1}{s} + \frac{1}{k} - 1 \leq 0$.

A triangle group $\Delta(r,s,k)$ is a reflection group which means it is generated by the reflections of the three edges of the triangle through angles π/r, π/s and π/k. The group $\Delta(r,s,k)$ is *spherical* if $\gamma = \frac{1}{r} + \frac{1}{s} + \frac{1}{k} - 1 < 0$, if $\gamma = \frac{1}{r} + \frac{1}{s} + \frac{1}{k} - 1 = 0$, then the group $\Delta(r,s,k)$ obtained is *Euclidean*. If $\gamma = \frac{1}{r} + \frac{1}{s} + \frac{1}{k} - 1 > 0$ then the group $\Delta(r,s,k)$ is *hyperbolic*.

The triangle groups $\Delta(2,3,k)$ are significant as they arise as homomorphic images or quotients of $PSL(2,\mathbb{Z})$. For $k \geq 6$, the order of the triangle group is infinite. When $k \leq 5$, then hyperbolic triangle groups $\Delta(2,3,k)$ are A_4, S_4, A_5, $\{1\}$ and S_3 ([59, 74, 106]).

$\Delta(2,3,6)$ is an infinite soluble group for $k = 6$. It contains free abelian group as derived subgroup which is generated by two elements whose related factor-derived group is cyclic group having order 6 [74]. When $k = 7$, $\Delta(2,3,k)$ becomes the Hurwitz group which is widely studied by many mathematicians in [24, 60, 64, 73, 96].

1.4.1 Generalized Triangle Groups

The generalized triangle group has the presentation $\langle u, v : u^r, v^s, W^k \rangle$, where $r,\ s,\ k$ are integers greater than 1, and $W = u^{\alpha_1} v^{\beta_1} ... u^{\alpha_k} v^{\beta_k}$, where $k > 1, 0 < \alpha_i < r$ and $0 < \beta_i < s$ for all i These groups are obtained by natural generalization of the triangle groups $\Delta(r,s,k)$ defined by the presentations $\left\langle u, v : u^r = v^s = (uv)^k = 1 \right\rangle$, where r, s and k are integers greater than one.

It is proved in [4] and [35], that if $G = \langle u, v : u^r, v^s, W^k \rangle$, then there is a homomorphism Ψ: $G \rightarrow PSL(2,\mathbb{C})$ such that $\Psi(u)$, $\Psi(v)$ and $\Psi(W)$ are of orders r, s and k respectively. Almost at the same time Boyer in his paper [12] presents that there exist a homomorphism from $G \rightarrow SO(3)$ which posses the same property. He also proved that G is an infinite group if no two of $r,\ s,\ k$ are equal to 2 and $max\{r,s,k\} \geq 6$, with some restrictions on W.

It is shown in [37] that G is infinite if $\frac{1}{r} + \frac{1}{s} + \frac{1}{k} \leq 1$ provided $r \geq 3$ or $k \geq 3$ and $s \geq 6$, or $(r,\ s,k) = (4,5,2)$. This is generalized in [4], where it was shown in [101] that G is infinite whenever $\frac{1}{r} + \frac{1}{s} + \frac{1}{k} \leq 1$.

A generalized triangle group may be infinite when $\frac{1}{r} + \frac{1}{s} + \frac{1}{k} > 1$. The finite generalized groups with presentation $\left\langle u, v : u^r, v^s, \left(u^{\alpha_1} v^{\beta_1} ... u^{\alpha_k} v^{\beta_k} \right)^k \right\rangle$, are determined for $k \geq 3$ in [36]. In [17], it is shown that for $k = 1$ the group is a finite triangle group. The cases are also determined for $r = 2, k = 3$ *or* 4 in [58], $r = 2, k = 2$ in [83], and, if $(r, s) = (2, 3)$, for $kt \leq 12$ in [23] which are all finite generalized triangle groups.

The complete classification of finite generalized triangle groups is given in 1995 by J. Howie in [39] and later by L. Levai, G. Rosenberger, and B. Souvignier in [57]. The list of all finite generalized triangle groups is given below:

(1) $\left\langle u, v \mid u^2, v^3, \left(uvuvuv^2uv^2\right)^2 \right\rangle$, of order 576;

(2) $\left\langle u, v \mid u^2, v^3, \left(uvuvuv^2\right)^3 \right\rangle$, of order 1440;

(3) $\left\langle u, v \mid u^3, v^3, \left(uvuv^2\right)^2 \right\rangle \cong A_5 \times C_3$, of order 180;

(4) $\left\langle u, v \mid u^3, v^3, \left(uvu^2v^2\right)^2 \right\rangle$, of order 288;

(5) $\left\langle u, v \mid u^2, v^5, \left(uvuv^2\right)^2 \right\rangle$, of order 120;

(6) $\left\langle u, v \mid u^2, v^5, \left(uvuvuv^4\right)^2 \right\rangle$, of order 1200;

(7) $\left\langle u, v \mid u^2, v^5, \left(uvuv^2uv^4\right)^2 \right\rangle$, of order 1200;

(8) $\left\langle u, v \mid u^2, v^4, \left(uvuvuv^3\right)^2 \right\rangle$, of order 192;

(9) $\left\langle u, v \mid u^2, v^3, \left(uvuv^2\right)^2 \right\rangle$, of order 24;

(10) $\left\langle u, v \mid u^2, v^3, \left(uvuvuv^2\right)^2 \right\rangle$, of order 48;

(11) $\left\langle u, v \mid u^2, v^3, \left(uvuvuvuv^2\right)^2 \right\rangle$, of order 120;

(12) $\left\langle u, v \mid u^2, v^3, \left(uvuvuv^2uvuv^2\right)^2 \right\rangle$, of order 720;

(13) $\left\langle u, v \mid u^2, v^3, \left(uvuvuvuvuv^2uv^2\right)^2 \right\rangle$, of order 2880;

(14) $\left\langle u, v \mid u^2, v^3, \left(uvuvuvuv^2uv^2uvuv^2uv^2\right)^2 \right\rangle$. of order 17694720.

1.4.2 Hurwitz Groups

Any non-trivial quotient of an abstract triangle group $\Delta(2,3,7)$ with the finite presentation $\Delta(2,3,7) = \left\langle u, v \mid u^2 = v^3 = (uv)^7 = [u,v] = 1 \right\rangle$ is called a Hurwitz group. It means that a finite group with two generators u and v which satisfy relations $u^2 = v^3 = (uv)^7 = 1$ is a Hurwitz group. In 1893, Hurwitz's theorem originated the importance of such groups.

In 1990, M. D. E. Conder wrote a review on Hurwitz groups [24] and at that time the only finite simple groups identified as Hurwitz groups were the Ree groups ${}^2G^2({}^3p)$, where p is an odd prime, all but 64 of the alternating groups A_n, 11 of the 26 sporadic finite simple groups. It is also known that the remaining 15 sporadic finite simple groups were not Hurwitz groups and only for $q = 2$, $PSL(3,q)$ is a Hurwitz group.

In the recent times, the groups of Lie types are objects of attention. G. Malle in [64] shows that the Chevalley group $G_2(q)$, for every prime power $q \geq 5$, is a Hurwitz group and the Ree group ${}^2G_2(3^{2m+1})$, for every $m \geq 1$, is a Hurwitz group. Furthermore, it is also proved that the groups $G_2(2)$, $G_2(3)$, $G_2(4)$ and ${}^2G_2(3)$ are not Hurwitz groups, but can be viewed as factor groups of $PSL(2,\mathbb{Z})$.

It is shown for 'intermediate' ranks in [60] that for all prime powers q and 93 values of $n < 287$, $SL(n,q)$ is a Hurwitz group, and this result is extended by M. V. Semirnov by using $(2,3,7)-$generation of alternating groups to another 60 values of $n \leq 287$ in [54]. Particularly, $SL(49,q)$ for all q, are now known to be Hurwitz groups.

The method used in [60] is described very attractively in the review chapter in [100], later on, this approach is used by N. Semernov in his paper [88], to show that D_n

type Weyl groups contains subgroups which are Hurwitz groups for all suitably large values of n. A totally different method is presented by authors in [82] to obtain representations with zero characteristic of $(2,3,7)$ up to degree 7. This approach is taken to obtain Hurwitz groups which is embedded as subgroups of $GL(n,R)$ for some $n \leq 7$ and for an appropriate ring R.

A. Macbeath give a pleasant description of his work on Hurwitz groups and its action on the surfaces and curves in [61]. A very interesting work is done by M. Streit in [98], by investigating the Hurwitz groups and associated complex algebraic curves with it.

The action of automorphisms were considered by K. Magaard and H. Volklein in [62] on the set of Weierstrass points of a Hurwitz curves and proved that the action in not transitive when genus $g > 1$. Later, it is shown that it acts transitively for $g = 3$ in [91], and for $g = 7$ in [33]). In 2004, R. Vogeler in [103], develops a method for encoding and classifying the conjugacy classes of hyperbolic transformations in $\Delta(2,3,7)$. Further this work was extended to determine a large preliminary portion of the spectrum for $\Delta(2,3,7)$ and therefore for Hurwitz surfaces in [104].

1.5 Pisot Numbers

A real algebraic integer $\theta > 1$ is called a Pisot number if all its conjugates lie in the circle of radius 1. The set of all such numbers is denoted by S and is a closed set in the real line as mentioned in [84] and [90]. For investigation of S, a powerful method is introduced in [32] by J. Dufresony and C. Pisot to obtain all numbers in S in $[1, t^{'} + e]$, where $0 < e < 0.0004$ and $t^{'} = (1 + 51/2)/2$. It is shown that the smallest accumulation point $\check{r}$ of S is also an isolated point of S. Later, this method is practiced to explore the consecutive resultant sets of S as explained in [2], [8], [31] and [42].

The set of Salem numbers is symbolized by T which consists of all algebraic integers $\varkappa > 1$ for which all other conjugates lie on or in the circle of radius one, such that at least one of the conjugate lies on the circle. Thus R. Salem in [85] gives a reciprocal equation is satisfied by $\varkappa$ with the property that its roots $\varkappa$ and $\varkappa^{-1}$ lie on the circle of radius 1. A small number of Salem numbers are known as compared to Pisot numbers [9].

T. Vijayaraghava proves in [102] that the set S has infinite number of accumulation points. R. Salem shows in [84] that S contains derived sets of any finite order.

Elements of S are studied in a neighbourhood of a accumulation point of S by D. W. Boyer in [11]. He analyzes the infinite tree $\Im$ related to S in which paths to infinity, bounded by one, on the circle of radius one, correspond to certain rational functions.

Chapter 2

Pisot numbers and circuits of type (m_1, m_2)

In this chapter we show that coefficients of a matrix representing the element $g = \left((\mu v)^{m_1} \left(\mu v^2\right)^{m_2}\right)^l$ where $l \geq 1$ of $PSL\left(2, \mathbb{Z}\right) = \left\langle \mu, v : \mu^2 = v^3 = 1 \right\rangle$ are denominators of the convergents of the continued fractions related to the circuits of type (m_1, m_2), for all $m_1, m_2 \in \mathbb{N}$. We also investigate fixed points of a particular class of circuits of type (m_1, m_2) and identify location of the Pisot numbers in a circuit of a coset diagram for the action of the modular group $PSL\left(2, \mathbb{Z}\right)$ on $\mathbb{Q}\left(\sqrt{d}\right) \cup \{\infty\}$, where d is a non-square positive integer.

2.1 Introduction

By L. Euler, every real number has continued fractions $\delta = \pi_1 + \frac{1}{\pi_2 + \frac{1}{\pi_3 + \ldots}}$ which is infinite for irrational numbers and finite for rationals. The continued fractions can also be represented as $\delta = \gamma_1 + \frac{1}{\gamma_2 +} \frac{1}{\gamma_3 + \ldots}$ or $\delta = [\gamma_1; \gamma_2, \ldots]$. The irrationals whose continued fractions repeat after a certain stage such that $\delta = [\pi_1; \pi_2, ..., \pi_m; \overline{\pi_{m+1}, \pi_{m+2}, ..., \pi_{m+q}}]$ are the quadratic irrational numbers with $\begin{bmatrix} B_k & B_{k-1} \\ H_k & H_{k-1} \end{bmatrix}$, where $B_k = [\pi_1; \pi_2, ..., \pi_k]$ and $H_k = [\pi_2; \pi_3, .., \pi_k]$ are continuants of the convergent $\frac{B_k}{H_k}$. J. L. Lagrange proved the converse: if δ is a quadratic irrational, then the regular continued fraction expansion is periodic [28].

2.1.1 Circuits and Words

If $\rho = \{\sigma_o, n_1, \sigma_1, n_2, ..., n_k, \sigma_k\}$ is an alternating sequence of vertices and edges of a coset diagram then ρ is a path in the diagram joining σ_o and σ_k if n_i joins σ_{i-1} and σ_i for each i and $n_i \neq n_j\,(i \neq j)$. A path of triangle and edges is a word in which initial and terminal vertices are same, is called a circuit. For a sequence of positive integers $\eta_1, \eta_2, \eta_3, ..., \eta_{2k}$ the word $g = (\mu v)^{\eta_1} \left(\mu v^2\right)^{\eta_2} (\mu v)^{\eta_3} ... \left(\mu v^2\right)^{\eta_{2k}}$ where $\eta_i > 0$ fixes α vertex, is represented by $(\eta_1, \eta_2, \eta_3, ..., \eta_{2k})$. Such a circuit evolves an element of and fixes a specific vertex on the circuit.

It is important to mention some relevant results here which are proved by Q. Mushtaq in [?] for later use.

Theorem 1 *Every element of* $PSL(2,\mathbb{Z})$, *except the (group theoretic) conjugates of* μ *and* $v^{\pm 1}$ *and* $(\mu v)^n$, $n > 0$, *has a real quadratic irrational number as a fixed point.*

Theorem 2 *Ambiguous numbers in the coset diagram for the orbit of* α *form a single circuit and it is the only circuit contained in it.*

It is also note worthy from a result by Q. Mushtaq [69] that for every real quadratic irrational number under the action of $PSL(2,\mathbb{Z})$ on $\mathbb{Q}\left(\sqrt{d}\right) \cup \{\infty\}$, the value of d remains the same. Thus, if there is a real quadratic irrational number α we find a circuit in the orbit of α under the action of $PSL(2,\mathbb{Z})$.

2.2 Circuits of type (m_1, m_2) and relation with Pisot numbers

Consider circuits of the type $(m_1,\ m_2)$ where $m_1, m_2 \in \mathbb{N}$ and means that there are m_1 triangles with one vertex outside the circuit and m_2 triangles with one vertex inside the circuit. Let $h \in PSL(2,\mathbb{Z})$ be an element related to the circuit $(m_1,\ m_2)$ of the form

$$\begin{aligned} h &= \left(\mu v^2\right)^{m_2} (\mu v)^{m_1} \left(\mu v^2\right)^{m_2} (\mu v)^{m_1} \left(\mu v^2\right)^{m_2} (\mu v)^{m_1} \ldots (\mu v)^{m_1} \\ &= \left(\left(\mu v^2\right)^{m_2} (\mu v)^{m_1}\right)^l . \end{aligned}$$

Then:

Theorem 3 *If* α *be an ambiguous number and*

$h = \left(\left(\mu v^2\right)^{m_2} (\mu v)^{m_1}\right)^l \in PSL(2,\mathbb{Z})$ *fixes* α*, so that the orbit of* α *contains the circuit* $\underbrace{(m_1,\ m_2,\ m_1, ...,\ m_2)}_{2l}$*, then the matrix* $M(h)$ *has trace* $tr(M(h)) = m_1 J_{3l+1} + 2J_{3l-1}$*, and* $l \geq 1$.

Proof. As $\mu : \omega \to -1/\omega$ and $v : \omega \to (\omega - 1)/\omega$, so $\mu v : \omega \to \omega + 1$ and $\mu v^2 : \omega \to \omega/(\omega + 1)$, we have $M((\mu v)^{m_1}) = \begin{bmatrix} 1 & m_1 \\ 0 & 1 \end{bmatrix}$. Therefore

$$M\left(\left(\mu v^2\right)^{m_2}\right) = \begin{bmatrix} 1 & 0 \\ m_2 & 1 \end{bmatrix} \text{ and}$$

$$A = M\left(\mu v^2\right)^{m_2} M((\mu v)^{m_1}) = \begin{bmatrix} 1 & 0 \\ m_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & m_1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & m_1 \\ m_2 & 1 + m_2 m_1 \end{bmatrix} = \begin{bmatrix} J_2 & J_3 \\ J_4 & J_5 \end{bmatrix}.$$

This implies that $A^2 = \begin{bmatrix} 1 + m_2 m_1 & 2m_1 + m_2 m_1 \\ 2m_2 + m_2^2 m_1 & m_2^2 m_1^2 + 3m_2 m_1 + 1 \end{bmatrix} = \begin{bmatrix} J_5 & J_6 \\ J_7 & J_8 \end{bmatrix}$

and $A^3 = \begin{bmatrix} m_2^2 m_1^2 + 3m_2 m_1 + 1 & m_2^2 m_1^3 + 4m_2 m_1^2 + 3m_1 \\ m_2^3 m_1^2 + 4m_2^2 m_1 + 3m_2 & m_2^3 m_1^3 + 5m_2^2 m_1^2 + 6m_2 m_1 + 1 \end{bmatrix}$

$$= \begin{bmatrix} J_8 & J_9 \\ J_{10} & J_{11} \end{bmatrix}.$$

Hence $A^l = \begin{bmatrix} J_{3l-1} & J_{3l} \\ J_{3l+1} & J_{3l+2} \end{bmatrix}$,

where

$$J_{3l} = m_1 J_{3l-1} + J_{3l-3}$$

$J_{3l+1} = m_2 J_{3l-1} + J_{3l-2}$

$J_{3l+2} = m_1 J_{3l+1} + J_{3l-1}$, for $k \geq 3$ and $J_0 = J_1 = 0,\ J_2 = 1.$

It is then immediate that $Tr\left(A^l\right) = J_{3l-1} + J_{3l+2} = m_1 J_{3l+1} + 2J_{3l-1}$. The determinant of A^l, that is of $M\left(h\right),$ must be 1 being determinant of an element of $PSL(2,\mathbb{Z})$, which is given by $\begin{vmatrix} J_{3l-1} & J_{3l} \\ J_{3l+1} & J_{3l+2} \end{vmatrix} = 1$. The entries of A^l are the denominators of convergent for continued fractions corresponding to the circuits (m_1, m_2) and powers of $A\left(h\right)$ satisfying the recurrence relation

$$A^l = (1 + m_1 m_2)\left(A^{l-1} + A\right) + m_1 m_2\left(A^{l-2} + A^{l-3} + ... + A^2\right) - I \quad \blacksquare$$

By considering the circuits of coset diagrams, one may start with $(\mu v)^{m_1}$. So the element $g = \left((\mu v)^{m_1}\left(\mu v^2\right)^{m_2}\right)^l$ is considered instead of $h = \left(\left(\mu v^2\right)^{m_2}(\mu v)^{m_1}\right)^l$, where $l \geq 1$, of $PSL(2,\mathbb{Z})$. Therefore

$$B = M\left((\mu v)^{m_1}\left(\mu v^2\right)^{m_2}\right) = \begin{bmatrix} 1 & m_1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ m_2 & 1 \end{bmatrix} = \begin{bmatrix} 1 + m_2 m_1 & m_1 \\ m_2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} J_5 & J_3 \\ J_4 & J_2 \end{bmatrix}.$$

Thus, inductively matrix $B^l = \begin{bmatrix} J_{3l+2} & J_{3l} \\ J_{3l+1} & J_{3l-1} \end{bmatrix}$, where $l \geq 1$.

The matrix for $\left(\mu v^2\right)^{m_2}(\mu v)^{m_1}$ then turns out to be

$M\left(g\right) = B^l = \begin{bmatrix} J_{3l-1} & J_{3l} \\ J_{3l+1} & J_{3l+2} \end{bmatrix}$, having the same trace and determinant as of $M\left(h\right)$ and satisfying the recurrence relation

$B^l = (1 + m_1 m_2)\left(B^{l-1} + B\right) + m_1 m_2\left(B^{l-2} + B^{l-3} + ... + B^2\right) - I$. Also matrices

$\begin{bmatrix} 1+m_2m_1 & m_1 \\ m_2 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & m_1 \\ m & 1+m_2m_1 \end{bmatrix}$, having the same eigen values , given by the roots λ_1 and λ_2 of $\lambda^2-(m_2m_1+2)\,\lambda+1=0$ where $\lambda_1 = 1+\frac{m_2m_1+\sqrt{m_2^2m_1^2+4m_2m_1}}{2} = 1+\beta_1 = \frac{(\beta_1)^2}{4m_2m_1}$ and $\lambda_2 = 1+\frac{m_2m_1-\sqrt{m_2^2m_1^2+4m_2m_1}}{2} = 1+\bar{\beta}_1 = \frac{(\bar{\beta}_1)^2}{m_2m_1}$, $\beta_1 = \frac{m_2m_1+\sqrt{m_2^2m_1^2+4m_2m_1}}{2}$ and $\bar{\beta}_1$ is its algebraic conjugate.

Proposition 4 *If $\alpha \in Q\left(\sqrt{d}\right)$, and $h = \left(\left(\mu v^2\right)^{m_2}(\mu v)^{m_1}\right)^l$ or*

$g = \left((\mu v)^{m_1}\left(\mu v^2\right)^{m_2}\right)^l$ are elements of $PSL(2,\mathbb{Z})$ fixing α, then orbit of α contains the circuit of type $\underbrace{(m_1,\ m_2,\ m_1, ...,\ m_2)}_{2l}$ and $\alpha = m_1\beta_1^{-1},\, m_1\bar{\beta}_1^{-1}$

$\left(\text{or } \alpha = \frac{\beta_1}{m_1},\ \frac{\bar{\beta}_1}{m_1}\right)$, so $d = m_2^2m_1^2+4m_2m_1$. If h or g acts on $\mathbb{Q}\left(\sqrt{d}\right)$, then the circuit in the coset diagram contains only $2\,(m_2+m_1)$ ambiguous numbers.

Proof. Let $\alpha \in \mathbb{Q}\left(\sqrt{d}\right)$. If $h = \left(\left(\mu v^2\right)^{m_2}(\mu v)^{m_1}\right)^l$ fixes α, then $\frac{J_{3l-1}\alpha+J_{3l}}{J_{3l+1}\alpha+J_{3l+2}} = \alpha$, so that $J_{3l+1}\alpha^2+(J_{3l+2}-J_{3l-1})\,\alpha - J_{3l} = 0$. This implies that

$J_{3l-1}\left(m_2\alpha^2 - m_2m_1\alpha - m_1\right) = 0$, for $m_1, m_2, l \geq 1$.

Hence $\alpha = \frac{m_2m_1\pm\sqrt{d}}{2m_2}$, where $d = m_2^2m_1^2+4m_2m_1$, and the elements fixed by h are $\frac{\beta_1}{m_2} - m_1 = \frac{m_2m_1+\sqrt{d}}{2m_2} = m_1\beta_1^{-1}$ and $\frac{\beta_2}{m_2} - m_1 = \frac{m_2m_1-\sqrt{d_1}}{2m_2} = m_1\bar{\beta}_1^{-1}$, where

$\beta_1 = \frac{m_2m_1+\sqrt{d}}{2}$ and $\bar{\beta}_1$ is its algebraic conjugate. On the other hand, if $g = \left((xy)^{m_1}\left(xy^2\right)^{m_2}\right)^l$ fixes $\beta \in \mathbb{Q}\left(\sqrt{d}\right)$, then $\frac{J_{3l+2}\beta+J_{3l}}{J_{3l+1}\beta+J_{3l-1}} = \beta$. That is

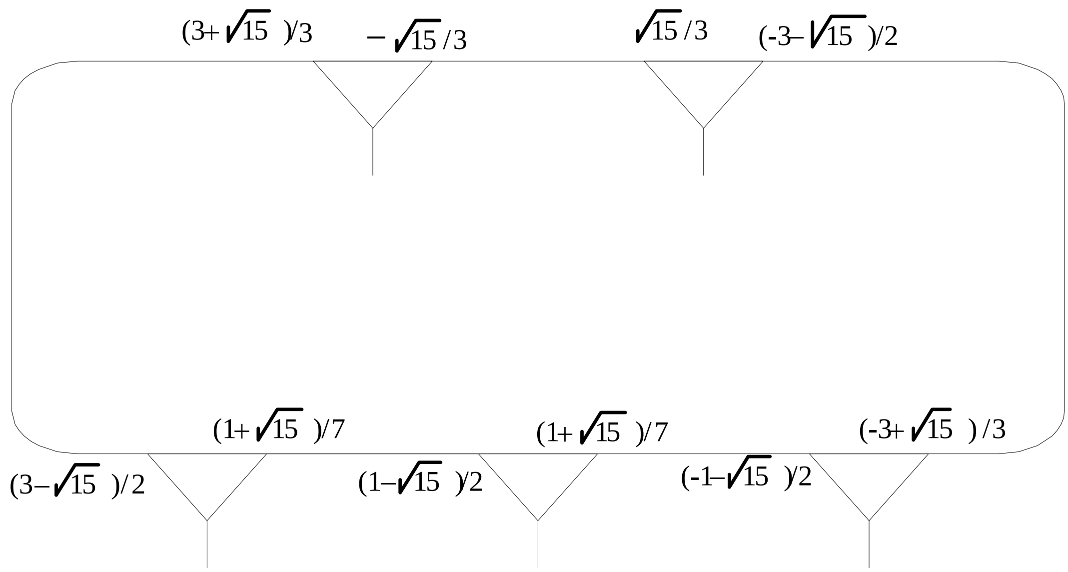
$J_{3l-1}\left(m_2\beta^2+m_2m_1\beta - m_1\right) = 0$. Hence $\beta = \frac{-m_2m_1\pm\sqrt{d}}{2m}$. This also implies that $d = m_2^2m_1^2+4m_2m_1$ and the elements fixed by g are $\frac{\beta_1}{m_2} = \frac{-m_2m_1+\sqrt{d}}{2m_2}$ and $\frac{\bar{\beta}_1}{m_2} = \frac{-m_2m_1-\sqrt{d}}{2m_2}$.

If the generators μ and v of $PSL(2,\mathbb{Z})$ act on $\mathbb{Q}\left(\sqrt{d}\right)$, where $d = m_2^2m_1^2+4m_2m_1$ then the circuit related to h or g is reduced to $\left(\left(\mu v^2\right)^{m_2}(\mu v)^{m_1}\right)$ or $\left((\mu v)^{m_1}\left(\mu v^2\right)^{m_2}\right)$ and hence contains only $2\,(m_1+m_2)$ ambiguous numbers. ∎

We illustrate this proposition with the example given below.

Example 5 *Let* $h_1 = \left((\mu v)^2 \left(\mu v^2\right)^3\right)^3$ *be an element of* $PSL(2, Z)$ *and* $\alpha \in \mathbb{Q}\left(\sqrt{d}\right)$ *be fixed by* h_1. *The polynomial corresponding to* h_1 *is* $3\alpha^2 + 6\alpha - 2 = 0$ *which is the same as that of the element* $(\mu v)^2 \left(\mu v^2\right)^3$*related to the circuit* $(2, 3)$.

Clearly the following circuit contains 10 *ambiguous numbers.*



Remark 6 *In the action of* $PSL(2, \mathbb{Z})$ *on* $\mathbb{Q}\left(\sqrt{d}\right) \cup \{\infty\}$, *then the circuit corresponding to* h *(or* g*) reduces to* $\left(\mu v^2\right)^{m_2} (\mu v)^{m_1}$ $\left(or\ (\mu v)^{m_1} \left(\mu v^2\right)^{m_2}\right)$.

Now considering the class of circuits of type $(\acute{m}_1, 1)$ representing the elements $g_1 = \left(\mu v^2\right)(\mu v)^{\acute{m}_1}$ of $PSL(2, \mathbb{Z})$, we have the following important result.

Theorem 7 *In the action of* $PSL(2, \mathbb{Z})$ *on* $\mathbb{Q}\left(\sqrt{d}\right)$, *the elements fixed by words of type* $\left(\left(\mu v^2\right)(\mu v)^{\acute{m}_1}\right)$ *are Pisot numbers.*

Proof. Let $g_1 = \left(\mu v^2\right)(\mu v)^{\acute{m}_1} \in PSL(2, \mathbb{Z})$ corresponding to $(\acute{m}_1, 1)$ fixes $\alpha \in$

$Q\left(\sqrt{d}\right)$. This means $\alpha^2 - \acute{m}_1\alpha - \acute{m}_1 = 0$ and thus we have $\alpha = \frac{\acute{m}_1 \pm \sqrt{\acute{m}_1^2 + 4\acute{m}_1}}{2}$. Consider $\alpha = \frac{\acute{m}_1 + \sqrt{\acute{m}_1^2 + 4\acute{m}_1}}{2}$ and its algebraic conjugate $\bar{\alpha} = \frac{\acute{m}_1 - \sqrt{\acute{m}_1^2 + 4\acute{m}_1}}{2}$.

As $\sqrt{\acute{m}_1^2 + 4\acute{m}_1} > \acute{m}_1$ this implies that $\acute{m}_1 + \sqrt{\acute{m}_1^2 + 4\acute{m}_1} > 2\acute{m}_1$, for all $\acute{m}_1 \geq 1$. So $\frac{\acute{m}_1 + \sqrt{\acute{m}_1^2 + 4\acute{m}_1}}{2} = \alpha > 1$, for all $m_1 \geq 1$.

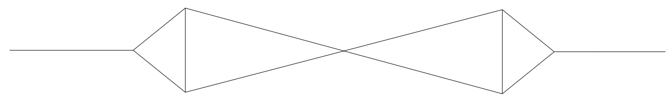
Now $\acute{m}_1^2 + 4\acute{m}_1 < (\acute{m}_1 + 2)^2$. This implies that $-\acute{m}_1 + \sqrt{\acute{m}_1^2 + 4\acute{m}_1} < 2$. Thus $\left|\frac{\acute{m}_1 - \sqrt{\acute{m}_1^2 + 4\acute{m}_1}}{2}\right| = |\bar{\alpha}| < 1$.

Hence α is a Pisot number. ■

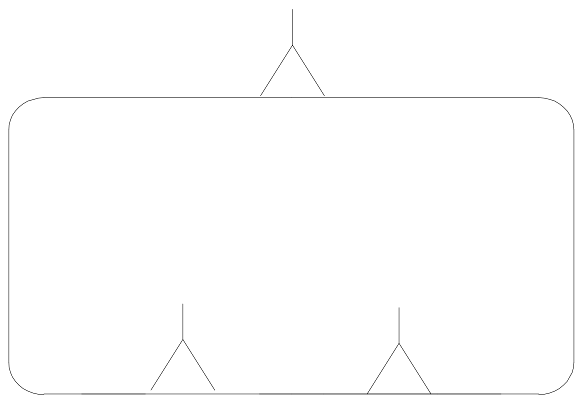
Remark 8 *If we consider element other than* $\left(\left(\mu v^2\right)\left(\mu v\right)^{\acute{m}_1}\right)$, *the resulting numbers will not be Pisot numbers.*

In the following, a list which describes the types of circuits, fixed vertices on the circuits, number of triangles on the circuits, discriminant and mod value of conjugate of fixed vertices of circuits is given.

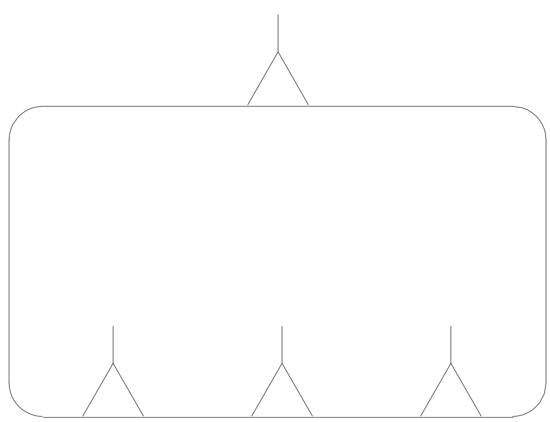
Type of circuit	No. of triangles on the circuit	Fixed vertex α	Disc	$\lvert\bar{\alpha}\rvert < 1$
$(1,\ 1)$	2	$\frac{1+\sqrt{5}}{2}$	5	0.61803398875
$(2,\ 1)$	3	$1+\sqrt{3}$	3	0.73205080757
$(3,\ 1)$	4	$\frac{3+\sqrt{21}}{2}$	21	0.79128784748
$(4,1)$	5	$2+2\sqrt{2}$	8	0.82842712475
$(5,\ 1)$	6	$\frac{5+\sqrt{45}}{2}$	45	0.85410196625
$(6,\ 1)$	7	$3+\sqrt{15}$	15	0.43649167310
$(7,\ 1)$	8	$\frac{7+\sqrt{77}}{2}$	77	0.88748219369
$(8,\ 1)$	9	$4+\sqrt{14}$	14	0.12917130661
$(9,1)$	10	$\frac{9+\sqrt{85}}{2}$	85	0.10977228646
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
$(1,m_1)$	$1+m_1$	$\frac{\acute{m}_1+\sqrt{\acute{m}_1^2+4\acute{m}_1}}{2}$	$\acute{m}_1^2+4\acute{m}_1$	$0.5\left(\acute{m}_1-\sqrt{\acute{m}_1^2+4\acute{m}_1}\right)$



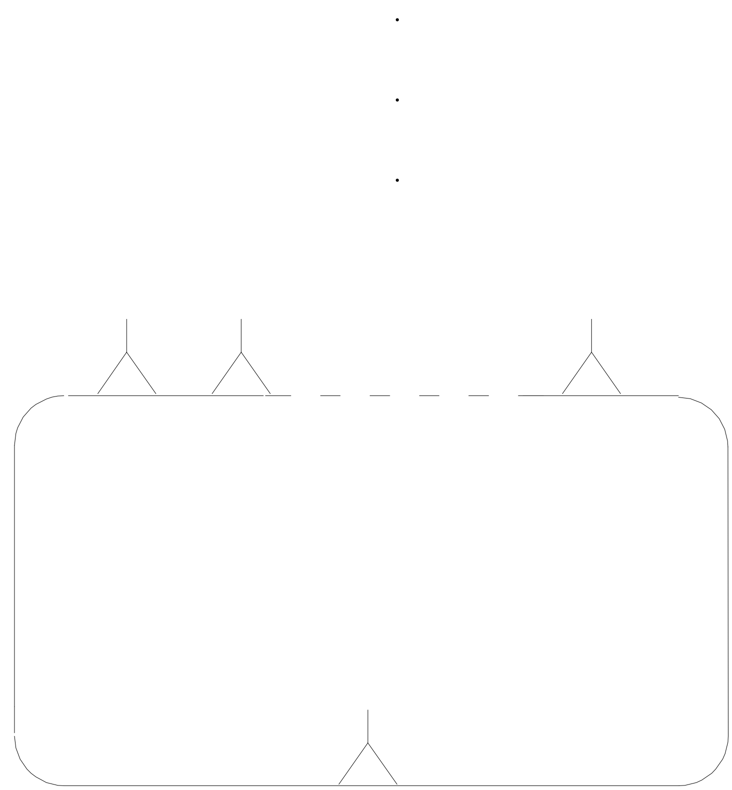
(1, 1)



(1, 2)



(1, 3)



$(1, m_1)$

2.3 Conclusion

By Theorem 3, matrices representing the element h or g of $PSL(2,\mathbb{Z})$ are matrices whose entries are denominators of the convergents of the continued fractions related to the circuits of type (m_1, m_2), for all $m_1, m_2 \in \mathbb{N}$. If an element h or g of $PSL(2,\mathbb{Z})$ acts on $\mathbb{Q}\left(\sqrt{d}\right)$, then $m_1\beta_1^{-1}$, $m_1\bar{\beta}_1^{-1}$ $\left(\text{or } \frac{\beta_1}{m_1},\ \frac{\bar{\beta}_1}{m_1}\right)$ where $\beta_1 = \frac{m_2 m_1 + \sqrt{m_2^2 m_1^2 + 4 m_2 m_1}}{2}$, are the fixed points and the corresponding reduced circuit is obtained in the coset diagram containing only $2(m_1 + m_2)$ ambiguous numbers. The only element of type $g_1 = \left(\left(\mu v^2\right)(\mu v)^{\acute{m}_1}\right)$ of

$PSL\left(2,\mathbb{Z}\right)$ corresponding to the circuits $(m_1, 1)$ where $m_1 \in \mathbb{N}$, gives Pisot numbers as fixed points. In case of elements other than $\left(\left(\mu v^2\right)\left(\mu v\right)^{\acute{m}_1}\right)$, fixed points are not Pisot numbers.

Chapter 3

A class of triangle subgroups of $PSL(2,p)$ related to circuits of type $(m_1,\ m_2)$

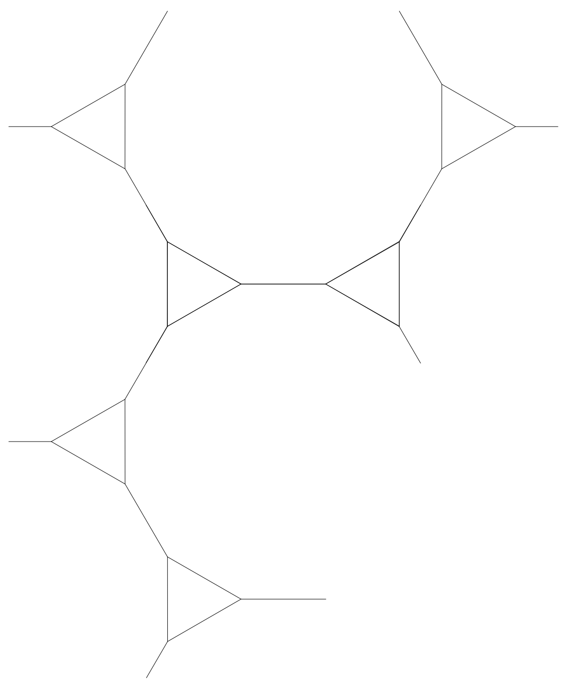
Coset diagrams for $PSL(2,\mathbb{Z})$ when acting on $PL(F_q)$ are composed of various types of well-defined fragments which are themselves composed of simple and non-periodic circuits connected together in a specified way. In this chapter, we attempt to classify all those subgroups of the homomorphic image of $PSL(2,\mathbb{Z})$ which are depicted by coset diagrams containing circuits of the type $(m_1,\ m_2)$.

3.1 Introduction

In [69], Q. Mushtaq proves that orbits of the action of $PSL(2,\mathbb{Z})$ on $\mathbb{Q}^*\left(\sqrt{d}\right) = \mathbb{Q}\left(\sqrt{d}\right) \cup \{\infty\}$ where d is a square free integer, contain ambiguous numbers. Coset diagrams for this action give a useful connection between orbits and the way these ambiguous numbers are located in orbits. In the following therefore, for the sake of completeness we explain coset diagrams as in [68].

3.1.1 Coset Diagrams and Fragments

The coset diagram for $PSL(2, Z)$ is represented by diagram Γ_C :

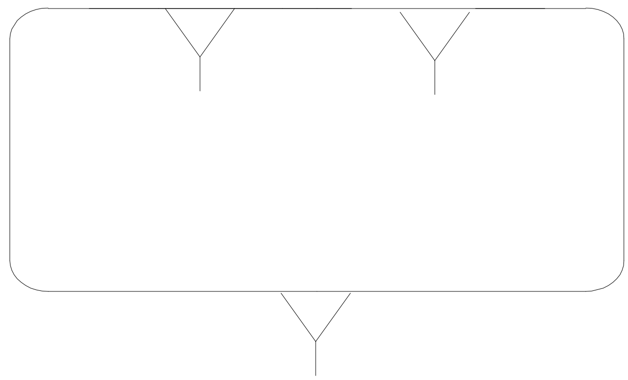


Γ_C

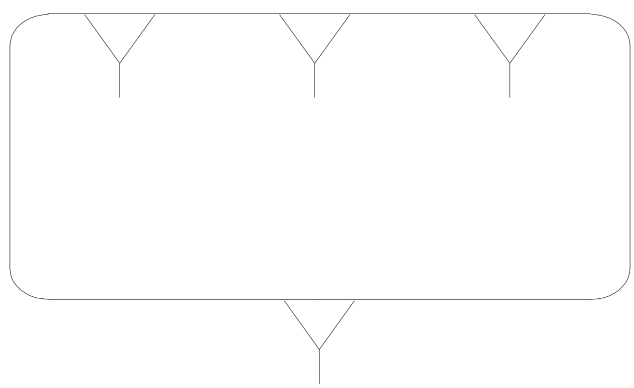
which is obtained by replacing every vertex in a tree of valence three by a triangle. The coset diagram for $PSL(2, \mathbb{Z})$ either in its regular representation, or in the representation for which a point stabilizer is $\langle t \rangle$, is exactly the same, except that one has to pick out an axis of symmetry. In the case Γ_C^1, where the representation is regular, the axis of symmetry contains no vertices, where as in the case Γ_C^2, where the representation is not regular, that is for which the a point stabilizer is $\langle t\rangle$, the line of symmetry contains two adjacent vertices.

Let $\hat{\alpha}$ is a vertex in Γ_C and $\hat{\alpha}'$ is a vertex of graph Γ'_C. It is always possible that for any vertex $\hat{\beta}$ in Γ_C is joined to $\hat{\alpha}$ by a unique path $\hat{\delta}$ if we do not allow consecutive $\mu - edges$ or consecutive $\upsilon - edges$. There is a related path in Γ'_C, starting with $\hat{\alpha}'$ and having $\mu - edges$ where $\hat{\delta}$ does and positive (or negative) $\upsilon - edges$ where $\hat{\delta}$ does. This path ends at a point $\hat{\beta}'$, exclusively determined by $\hat{\beta}$. Thus there is a mapping χ from Γ_C to Γ'_C, in which $\mu - edges$ correspond to $\mu - edges$, and positive $\upsilon - edges$ to positive $\upsilon - edges$. If Γ_C and Γ'_C are same then this map is one to one. If $\hat{\beta} \neq \hat{\alpha}$ maps on to $\hat{\alpha}'$, the path from $\hat{\alpha}$ to $\hat{\beta}$ in Γ_C maps on to a circuit in Γ'_C. The elements of $PSL\,(2,\mathbb{Z})$ are vertices of Γ_C. If $\hat{g} \in PSL\,(2,\mathbb{Z})$ is labelling $\hat{\beta}'$, then $\hat{\beta}'$ maps onto $\hat{\alpha}'$ if and only if $\hat{g}$ belongs to the stabilizer of $\hat{\alpha}'$ in the representation of $PSL\,(2,Z)$ on which Γ'_C is the diagram. Thus circuits in the diagram Γ'_C correspond to the elements of $PSL\,(2,\mathbb{Z})$ containing fixed points.

For example the circuit related to $\hat{g}_1 = (\mu\nu)\left(\mu\nu^2\right)^2$ and $\hat{g}_2 = \left(\mu\nu^2\right)^3(\mu\nu)$ attains the following form

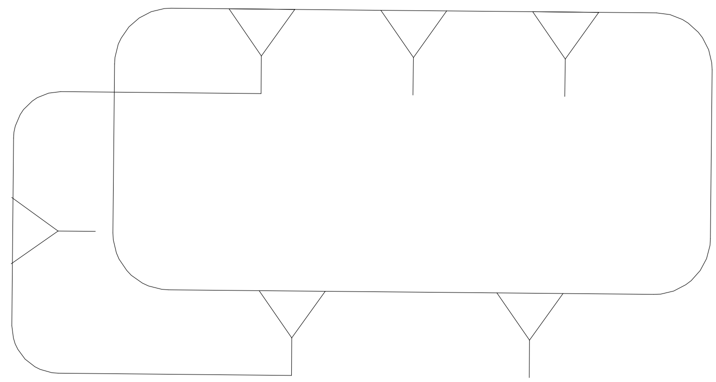


$\bar{\gamma}_1$



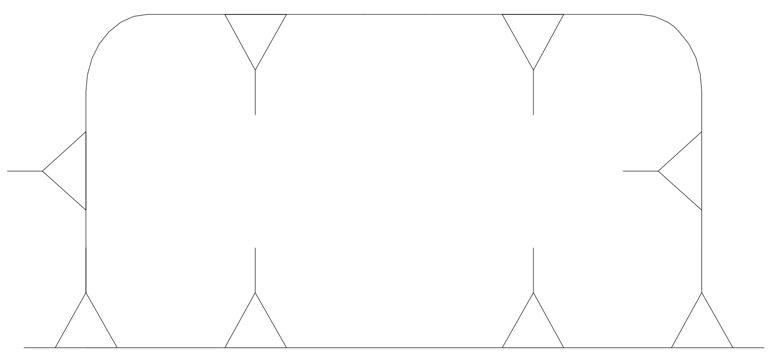
$\bar{\gamma}_2$

respectively. If $\hat{g}_1$ and $\hat{g}_2$ both have a fixed vertex $\hat{\alpha}'$ then by connecting these two circuits.the fragment $\bar{\gamma}_3$ is obtained.



$\bar{\gamma}_3$

Then $\hat{\alpha}'$ also fixed by $\hat{g}_1\hat{g}_2$. But Γ'_C contains a non simple circuit related to $\hat{g}_1\hat{g}_2$, which is given by $\bar{\gamma}_4$.



$\bar{\gamma}_4$

By γ we mean a non-simple fragment obtained by connecting, non trivial, non periodic, simple circuits. Q. Mushtaq proves in [69] that each orbit of the action always contains a single circuit and each vertex on the circuit is in fact an ambiguous number.

The coset diagrams for the action of $PSL\,(2,\mathbb{Z})$ on $Q^*\left(\sqrt{d}\right)$ are infinite. They become finite when the action of $PSL(2,\mathbb{Z})$ on $Q^*\left(\sqrt{d}\right)$ is transformed into the action of $PSL(2,\mathbb{Z})$ on $PL(F_q)$ naturally. The orbits get merged and circuits become fragments. The coset diagram $D(\theta,q)$ is obtained through the procedure described in [74] for every θ in F_q. The following result establish as a condition which proves the existence of γ or its homomorphic image $D(\theta,q)$.

Theorem 9 *Given a fragment γ, there is a polynomial f in $\mathbb{Z}[z]$ such that*

(i) if the fragment γ occurs in $D(\theta,q)$ then $f(\theta)=0$

(ii) If $f(\theta)=0$ then the fragment or a homomorphic image of it occurs in $D(\theta,q)$ or in $PL(F_q)$.

3.2 Results and discussions

Consider two circuits of type (n_1,m_1) and (n_2,m_2) where $m_1,m_2,n_1,n_2 \geq 1$. Each circuit corresponds to an element of $PSL(2,\mathbb{Z})$. The words corresponding to these circuits are $w_1=(\mu\upsilon)^{n_1}\left(\mu\upsilon^2\right)^{m_1}$ and $w_2=\left(\mu\upsilon^2\right)^{n_2}(\mu\upsilon)^{m_2}$ respectively. If X and Y are the matrices representing μ and υ of $PGL(2,q)$ and satisfying the relations

$$X^2=Y^3=\lambda I \tag{3.1}$$

where I is the identity matrix and λ is a scalar, then w_1,w_2 and w_1w_2 are represented as follows in terms of X and Y :

$$W_1=(XY)^{n_1}\left(XY^2\right)^{m_1}$$

$$W_2=\left(XY^2\right)^{n_2}(XY)^{m_2}$$

$$W_1W_2=(XY)^{n_1}\left(XY^2\right)^{n_2+m_1}(XY)^{m_2}.$$

Here we take $\bar{\mu}$ and $\bar{\upsilon}$ to be represented as $X = \begin{bmatrix} a_1 & k_1c_1 \\ c_1 & -a_1 \end{bmatrix}$, $Y = \begin{bmatrix} d_1 & k_1f_1 \\ f_1 & -d_1 - 1 \end{bmatrix}$ where $a_1, c_1, d_1, f_1, k_1 \in F_q$. We write

$$a_1^2 + k_1c_1^2 = -\Delta \neq 0$$

and require that

$$d_1^2 + d_1 + k_1f_1^2 + 1 = 0$$

These elements certainly satisfy relations (3.1). The trace of matrix representing $\bar{\mu}\bar{\upsilon}$ is $r = a\,(2d_1 + 1) + 2k_1c_1f_1$ and $\Delta = -\left(a_1^2 + k_1c_1^2\right)$ is the determinant. As $\det(X) = \Delta$ and $tr\,(X) = 0$, So

$$X^2 + \Delta I = 0 \tag{3.2}$$

and, $\det(Y) = 1$ and $tr\,(Y) = -1$, we have

$$Y^2 + Y + I = 0 \tag{3.3}$$

Further more, $\Delta = \det(XY)$ and $r = tr\,(XY)$, so, we have

$$(XY)^2 + r\,(XY) + \Delta I = 0 \tag{3.4}$$

and by equations $(3.2) - (3.3)$ give

$$XYX = rX + \Delta I + \Delta Y \tag{3.5}$$

$$YXY = X + rY \tag{3.6}$$

$$YX = rI - X - XY \tag{3.7}$$

By using equations (3.2) to (3.7),

$$W_1 = (-1)^{n_1+2}\left(\sum_{k=0}^{\infty}\sum_{l=0}^{k} C_l^{n_1-(l+2)} C_{k-l}^{m_1-(k+2-l)} r^{n_1+m_1-(2k+2)}\Delta^{k+1}\right) I +$$

$$(-1)^{m_1+1}\left[\begin{pmatrix} \Delta\sum_{k=0}^{\infty}\sum_{l=0}^{k} C_l^{n_1-(l+2)} C_{k-l}^{m_1-(k+1-l)} \\ -\sum_{k=0}^{\infty}\sum_{l=0}^{k} C_l^{n_1-(l+1)} C_{k-l}^{m_1-(k+1-l)} \end{pmatrix} r^{n_1+m_1-(2k+3)}\Delta^{k}\right] X$$

$$+(-1)^{m_1}\left(\sum_{k=0}^{\infty}\sum_{l=0}^{k} C_l^{n_1-(l+1)} C_{k-l}^{m_1-(k+2-l)} r^{n_1+m_1-(2k+2)}\Delta^{k+1}\right) Y$$

$$+(-1)^{m_1+1}\left[\begin{matrix} \left\{\begin{matrix} \Delta\begin{pmatrix} \sum_{k=0}^{\infty}\sum_{l=0}^{k} C_l^{n_1-(l+2)} C_{k-l}^{m_1-(k+1-l)} \\ +\sum_{k=0}^{\infty}\sum_{l=0}^{k} C_l^{n_1-(l+1)} C_{k-l}^{m_1-(k+2-l)} \end{pmatrix} \\ -\sum_{k=0}^{\infty}\sum_{l=0}^{k} C_l^{n_1-(l+1)} C_{k-l}^{m_1-(k+1-l)} \end{matrix}\right\} \\ r^{n_1+m_1-(2k+3)}\Delta^{k} \end{matrix}\right] XY$$

$$W_2 = (-1)^{n_2+1}\begin{pmatrix} \sum_{k'=0}^{\infty}\sum_{l'=0}^{k'} C_{l'}^{n_2-(l'+1)}\left(C_{k'-l'}^{m_2-(k'+1-l')} - C_{k'-l'}^{m_2-(k'+2-l')}\right) \\ r^{n_2+m_2-(2k'+2)}\Delta^{k'+1} \end{pmatrix} I$$

$$+(-1)^{n_2+1}\left(\sum_{k'=0}^{\infty}\sum_{l'=0}^{k'} C_{l'}^{n_2-(l'+1)} C_{k'-l'}^{m_2-(k'+2-l')} r^{n_2+m_2-(2k'+3)}\Delta^{k'+1}\right) X$$

$$+(-1)^{n_2+1}\left(\sum_{k'=0}^{\infty}\sum_{l'=0}^{k'} C_{l'}^{n_2-(l'+1)} C_{k'-l'}^{m_2-(k'+1-l')} r^{n_2+m_2-(2k'+2)}\Delta^{k'+1}\right) Y$$

$$+(-1)^{n_2+1}\left[\begin{matrix} \begin{pmatrix} \Delta\sum_{k'=0}^{\infty}\sum_{l'=0}^{k'} C_{l'}^{n_2-(l'+2)} C_{k'-l'}^{m_2-(k'+1-l')} \\ +\sum_{k'=0}^{\infty}\sum_{l'=0}^{k'} C_{l'}^{n_2-(l'+1)}\left(\Delta C_{k'-l'}^{m_2-(k'+2-l')} - C_{k'-l'}^{m_2-(k'+2-l')}\right) \end{pmatrix} \\ r^{n_2+m_2-(2k'+3)}\Delta^{k'+1} \end{matrix}\right] XY$$

$\hat{\lambda}_i$'s and $\hat{\mu}_i$'s are the coefficients of XY, Y, X and I in W_1 and W_2 where $i = 0, 1, 2, 3$.

$$\hat{\lambda}_0 = (-1)^{n_1+2}\left(\sum_{k=0}^{\infty}\sum_{l=0}^{k} C_l^{n_1-(l+2)} C_{k-l}^{m_1-(k+2-l)} r^{n_1+m_1-(2k+2)}\Delta^{k+1}\right)$$

$$\hat{\lambda}_1 = (-1)^{m_1+1}\left[\begin{pmatrix} \Delta\sum_{k=0}^{\infty}\sum_{l=0}^{k} C_l^{n_1-(l+2)}C_{k-l}^{m_1-(k+1-l)} \\ -\sum_{k=0}^{\infty}\sum_{l=0}^{k} C_l^{n_1-(l+1)}C_{k-l}^{m_1-(k+1-l)} \end{pmatrix} r^{n_1+m_1-(2k+3)}\Delta^k\right]$$

$$\hat{\lambda}_2 = (-1)^{m_1}\left(\sum_{k=0}^{\infty}\sum_{l=0}^{k} C_l^{n_1-(l+1)}C_{k-l}^{m_1-(k+2-l)} r^{n_1+m_1-(2k+2)}\Delta^{k+1}\right)$$

$$\hat{\lambda}_3 = (-1)^{m_1+1}\left[\begin{matrix}\left\{\begin{matrix} \Delta\begin{pmatrix} \sum_{k=0}^{\infty}\sum_{l=0}^{k} C_l^{n_1-(l+2)}C_{k-l}^{m_1-(k+1-l)} \\ +\sum_{k=0}^{\infty}\sum_{l=0}^{k} C_l^{n_1-(l+1)}C_{k-l}^{m_1-(k+2-l)} \end{pmatrix} \\ -\sum_{k=0}^{\infty}\sum_{l=0}^{k} C_l^{n_1-(l+1)}C_{k-l}^{m_1-(k+1-l)} \end{matrix}\right\} \\ r^{n_1+m_1-(2k+3)}\Delta^k \end{matrix}\right]$$

$$\hat{\mu}_0 = (-1)^{n_2+1}\begin{pmatrix} \sum_{k'=0}^{\infty}\sum_{l'=0}^{k'} C_{l'}^{n_2-(l'+1)}\left(C_{k'-l'}^{m_2-(k'+1-l')} - C_{k'-l'}^{m_2-(k'+2-l')}\right) \\ r^{n_2+m_2-(2k'+2)}\Delta^{k'+1} \end{pmatrix}$$

$$\hat{\mu}_1 = (-1)^{n_2+1}\left(\sum_{k'=0}^{\infty}\sum_{l'=0}^{k'} C_{l'}^{n_2-(l'+1)}C_{k'-l'}^{m_2-(k'+2-l')} r^{n_2+m_2-(2k'+3)}\Delta^{k'+1}\right)$$

$$\hat{\mu}_2 = (-1)^{n_2+1}\left(\sum_{k'=0}^{\infty}\sum_{l'=0}^{k'} C_{l'}^{n_2-(l'+1)}C_{k'-l'}^{m_2-(k'+1-l')} r^{n_2+m_2-(2k'+2)}\Delta^{k'+1}\right)$$

$$\hat{\mu}_3 = (-1)^{n_2+1}\left[\begin{matrix}\begin{pmatrix} \Delta\sum_{k'=0}^{\infty}\sum_{l'=0}^{k'} C_{l'}^{n_2-(l'+2)}C_{k'-l'}^{m_2-(k'+1-l')} \\ +\sum_{k'=0}^{\infty}\sum_{l'=0}^{k'} C_{l'}^{n_2-(l'+1)}\left(\Delta C_{k'-l'}^{m_2-(k'+2-l')} - C_{k'-l'}^{m_2-(k'+2-l')}\right) \end{pmatrix} \\ r^{n_2+m_2-(2k'+3)}\Delta^{k'+1} \end{matrix}\right]$$

Here $\hat{\mu}_i$ and $\hat{\lambda}_i$ for $i = 0, 1, 2, 3$ are terms involving Δ and r. Since $\tilde{v} = \tilde{v}w_1$ and $\tilde{v} = \tilde{v}w_2$, so matrices W_1 and W_2 have a common eigen vector .Thus the algebra generated by W_1 and W_2 has dimension 3. Whereas the algebra generated by $W_1W_2, W_2, W_1$and I is linearly dependent as given in [71].

Using equations (3.2) to (3.7), we have

$$W_1W_2 = \hat{\nu}_0 I + \hat{\nu}_1 X + \hat{\nu}_2 Y + \hat{\nu}_3 XY \tag{3.8}$$

where $\hat{\nu}_i$ for $i = 0, 1, 2, 3$ is calculated in terms of $\hat{\mu}_i$'s and $\hat{\lambda}_i$'s again by using equations (3.2) to (3.7) .The condition that $W_1W_2, W_2, W_1$and I are linearly dependent [71], is expressed as

$$\begin{vmatrix} \hat{\lambda}_1 & \hat{\lambda}_2 & \hat{\lambda}_3 \\ \hat{\mu}_1 & \hat{\mu}_2 & \hat{\mu}_3 \\ \hat{\nu}_1 & \hat{\nu}_2 & \hat{\nu}_3 \end{vmatrix} = 0 \tag{3.9}$$

If we calculate $\hat{\nu}_1, \hat{\nu}_2, \hat{\nu}_3$ in terms of $\hat{\mu}_i$'s and $\hat{\lambda}_i$'s and put in (3.9), we get

$$\begin{aligned} &\left(\hat{\lambda}_2\hat{\mu}_3 - \hat{\lambda}_3\hat{\mu}_2\right)^2 + \Delta\left(\hat{\lambda}_3\hat{\mu}_1 - \hat{\lambda}_1\hat{\mu}_3\right)^2 + \left(\hat{\lambda}_1\hat{\mu}_2 - \hat{\lambda}_2\hat{\mu}_1\right)^2 \\ &+r\left(\hat{\lambda}_2\hat{\mu}_3 - \hat{\lambda}_3\hat{\mu}_2\right)\left(\hat{\lambda}_3\hat{\mu}_1 - \hat{\lambda}_1\hat{\mu}_3\right) + \left(\hat{\lambda}_2\hat{\mu}_3 - \hat{\lambda}_3\hat{\mu}_2\right)\left(\hat{\lambda}_1\hat{\mu}_2 - \hat{\lambda}_2\hat{\mu}_1\right) = 0 \end{aligned} \tag{3.10}$$

By substituting these values in equation (3.10) one obtains a polynomial $f(\theta)$.

Remark 10 *Consider non-periodic simple circuits of type* (n_1, m_1) *and* (n_2, m_2) *corresponding to the elements* $g_1 = (\bar{\mu}\bar{\upsilon})^{n_1}\left(\bar{\mu}\bar{\upsilon}^{-1}\right)^{m_1}$ *and* $g_2 = \left(\bar{\mu}\bar{\upsilon}^{-1}\right)^{n_2}(\bar{\mu}\bar{\upsilon})^{m_2}$ *respectively, of* $PGL(2, \mathbb{Z})$. *One obtains a fragment by joining these circuits in such a way that this fragment will fix both* $\hat{g}_1$ *and* $\hat{g}_2$ *on a common fixed vertex. This fragment yields a polynomial* $f(\theta)$ *which is obtained by using the method developed in [71]. The roots of* $f(\theta)$ *in an appropriate finite field* F_p *where* p *is a prime number. Corresponding to each* θ*, where* θ *is a zero of* $f(\theta)$*, we use theorem in [74] to obtain a triplet* $\bar{\mu}, \bar{\upsilon}, \bar{t}$ *of linear fractional transformations such that* $(\bar{\mu})^2 = (\bar{\upsilon})^3 = (\bar{t})^2 = (\bar{\mu}\bar{t})^2 = (\bar{\upsilon}\bar{t})^2 = 1$ *and* $(\bar{\mu}\bar{\upsilon})^{p-1} = 1$

or $(\bar{\mu}\bar{\upsilon})^p = 1$. *That is, we obtain groups* $PSL(2,p)$, $PSL(2,p) \times C_2$, $\Delta(2,3,6)$, S_3, A_5 *or* A_4. *For* $\psi : PGL(2,\mathbb{Z}) \rightarrow PGL(2,p)$ *with parameters*

$$\theta = 0 \qquad \langle \bar{\mu},\ \bar{\upsilon},\ \bar{t} \rangle \tilde{=} S_3$$

$$\theta = 1 \qquad \langle \bar{\mu},\ \bar{\upsilon},\ \bar{t} \rangle \tilde{=} A_4$$

$$\theta = 2 \qquad \langle \bar{\mu},\ \bar{\upsilon},\ \bar{t} \rangle \tilde{=} S_4$$

and

$$\theta = 3 \qquad \langle \bar{\mu},\ \bar{\upsilon},\ \bar{t} \rangle \tilde{=} \Delta(2,3,6)$$

In all other cases

$$\langle \bar{\mu},\ \bar{\upsilon},\ \bar{t} \rangle \tilde{=} PSL(2,p) \ \text{ or } PSL(2,p) * C_2$$

3.3 Conclusion

Coset diagrams for $PSL(2,\mathbb{Z})$ when acting on F_p where p is a prime, are composed of various types of well-defined fragments which are themselves composed of simple and non-periodic circuits connected together in a specified way. We considered non-periodic simple circuits of type (n_1, m_1) and (n_2, m_2) corresponding to the elements $g_1 = (\mu\nu)^{n_1} (\mu\nu^2)^{m_1}$ and $g_2 = (\mu\nu^2)^{n_2} (\mu\nu)^{m_2}$ respectively, of $PGL(2,\mathbb{Z})$. We obtained a fragment by joining these circuits in such a way that both $\hat{g}_1$ and $\hat{g}_2$ have a common fixed vertex in this fragment. This fragment yield a polynomial $f(\theta)$ which is obtained by using the method introduced in [71]. We obtain the roots of $f(\theta)$ in an appropriate finite field F_p where p is a prime. Corresponding to each θ, where θ is a zero of $f(\theta)$, we use theorem in [74] to obtain a triplet

$\bar{\mu}, \bar{\upsilon}, \bar{t}$ such that $(\bar{\mu})^2 = (\bar{\upsilon})^3 = (\bar{t})^2 = (\bar{\upsilon}\bar{t})^2 = (\bar{\mu}\bar{t})^2 = 1$ and $(\bar{\mu}\bar{\upsilon})^{p-1} = 1$ or $(\bar{\mu}\bar{\upsilon})^p = 1$. Thus groups $PSL(2,p) * C_2$, $PSL(2,p)$, $\Delta(2,3,6)$, S_3, A_5 or A_4, are obtained.

Chapter 4

Generalized triangle groups as a homomorphic image of $PSL\,(2,\mathbb{Z})$

In this chapter we extend the parametrization of actions of the modular group $PSL(2,Z)$ on $PL(F_p)$, for various prime numbers p to obtain the generalized triangle groups, namely $\left\langle \bar{\mu},\bar{\nu} : \bar{\mu}^2 = \bar{\nu}^3 = \left(\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}^2\right)^k = 1\right\rangle$ and by the parametrization we obtain coset diagrams of $\left\langle \bar{\mu},\bar{\nu} : \bar{\mu}^2 = \bar{\nu}^3 = \left(\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}^2\right)^k = 1\right\rangle$ for all $\theta \in F_p$. We have also obtained the coset diagrams for three finite generalized triangle groups $\left\langle u,v : u^2 = v^3 = \left(uvuvuv^2\right)^2 = 1\right\rangle$, $\left\langle u,v : u^2 = v^3 = \left(uvuvuv^2uv^2\right)^2 = 1\right\rangle$ and $\left\langle u,v : u^2 = v^3 = \left(uvuv^2\right)^2 = 1\right\rangle$ by taking $\theta = 0$ as a parameter.

4.1 Introduction

In [83] G. Rosenberger conjectures that all generalized triangle groups satisfy the Tits Alternative. It is generally known that for a generalized triangle group $< u,v$:

$u^r, v^s, w^k >$ the Tits Alternative holds with the exception when (r, s, k) is of the form $(2, s, 2)$ where $s = 3, 4, 5, 6, 10, 12, 15, 20, 30, 60$ or one the forms $(r, s, k) = (3, 3, 2), (3, 4, 2)$, and $(3, 5, 2)$ [38]. In [40] it is shown that the Tits Alternative holds in the cases $(r, s, k) = (2, s, 2)$ for $q = 6, 10, 12, 15, 20, 30, 60$. In [80], Q. Mushtaq and F. Shaheen studied factor groups of the abstract group $\delta^{6,6,6}$ through coset diagrams by prameterizing its actions. The abstract group $\delta^{r,s,k}$ is defined for any positive integers r, s, k as

$$\left\langle \mu, \nu, t : \mu^2 = \nu^k = t^2 = (\mu\nu)^r = (\mu t)^2 = (\mu\nu t)^s = 1 \right\rangle$$

In [39] fourteen generalized triangle groups are classified as finite. Out of these fourteen only eight groups are quotients of the modular groups. Our aim is to obtain the coset diagrams of the action of the modular group on a projective line over a finite field, through parametrization. In this way we obtain one of these eight finite generalized triangle groups $\left\langle \mu, \nu : u^2 = v^3 = (w)^3 = 1 \right\rangle$ of order 1440, where $w = \mu\nu\mu\nu\mu\nu^2$.

4.1.1 Parametrization

Any homomorphism $\rho_1 : PGL(2, \mathbb{Z}) \to PGL(2, q)$ give rise to an action on $PL(F_q)$. We denote the generators $(\mu)\rho_1$, $(\nu)\rho_1$ and $(t)\rho_1$ by $\bar{\mu}$, $\bar{\nu}$ and $\bar{t}$. If neither of the generators μ and ν and t lies in the kernel of ρ_1, so that $\bar{\mu}$, $\bar{\nu}$ and $\bar{t}$ are of order 2, 3 and 2 respectively, then ρ_1 is called a non-degenerate homomorphism. It is proved in [74] that the conjugacy classes of non-degenerate homomorphism of $PGL(2, \mathbb{Z})$ into $PGL(2, q)$ are into one to one way with the conjugacy classes of non-trivial elements of $PGL(2, q)$. This of course parametrizes the conjugacy classes of homomorphism $\rho_1 : PGL(2, \mathbb{Z}) \to PGL(2, q)$ are parametrized except for $\theta = 0, 3 \in F_q$.

If ρ_1 is a such homomorphism, and $X,\ Y$ and T denote matrices of $PGL\,(2,q)$ yielding the elements $\bar{\mu}$, $\bar{\nu}$ and $\bar{t}$ in $PGL\,(2,q)$. The matrices $X,\ Y$ and T are

$$X = \begin{bmatrix} a_1 & k_1c_1 \\ & \\ c_1 & -a_1 \end{bmatrix},\ Y = \begin{bmatrix} d_1 & k_1f_1 \\ & \\ f_1 & -d_1-1 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0 & -k_1 \\ & \\ 1 & 0 \end{bmatrix}$$

where $a_1, c_1, d_1, f_1, k_1 \in F_q$.

This gives

$$\left(-a_1^2 - k_1c_1^2\right) = \Delta \neq 0$$

and require that

$$d_1^2 + d_1 + k_1f_1^2 = -1 \tag{4.2}$$

This yields elements satisfying the relations $X^2 = \lambda_1^{'}I,\ Y^3 = \lambda_2^{'}I$ and $T^2 = \lambda_3^{'}I$, where I is the identity matrix and $\lambda_1^{'}, \lambda_2^{'}$ and $\lambda_3^{'}$ are non zero scalers. The non-degenerate homomorphism ρ_1 is determined by $\bar{\mu}\bar{\nu}$. So, we must check the conjugacy class of $\bar{\mu}\bar{\nu}$. The trace of matrix XY is

$$r = a_1\,(2d_1 + 1) + 2k_1c_1f_1 \tag{4.3}$$

If $tr\,(TXY) = k_1s$, then

$$s = 2a_1f_1 - c_1\,(2d_1 + 1) \tag{4.4}$$

so that

$$3\Delta = r^2 + k_1s^2 \tag{4.5}$$

and set

$$\theta = \frac{r^2}{\Delta} \tag{4.6}$$

Thus, for known values of q and θ and by using the equations (4.1) to (4.6), the matrices $T,\ Y$ and X can be found.

Action of $PGL(2,\mathbb{Z})$ on $PL(F_q)$ through ρ_1 is shown by a coset diagram. The coset diagram $D(\theta,q)$ depicts the conjugacy class of actions of $PGL(2,\mathbb{Z})$ on $PL(F_q)$ corresponding to $\theta \in F_q$.

We explain the above discussion with the help of an example.

Example 11 *Let* $\theta = 4$ *and* $q = 17$.

By equation (4.6), $\theta = \frac{r^2}{\Delta}$ *and* $\theta = 4$ *implies that* $r^2 = 4\Delta$. *We can choose* $\Delta = 1$ *so that* $r = \pm 2$. *Let* $r = 2$ *and substituting the values of* Δ *and* r *in equation* (4.5) *to get* $s^2 = \frac{-1}{k_1}$. *Choosing* $k_1 = 1$, *we get* $s = \pm 4$. *choosing* $s = 4$. *Similarly by choosing* $d_1 = 0$, *we get* $f_1 = 4$. *Putting the values of* k_1, s, d_1 *and* f_1 *in equations* (4.3) *and* (4.4) *and solving these equations,* $a_1 = 0$ *and* $c_1 = 13$. *Thus*

$$X = \begin{bmatrix} 0 & 13 \\ 13 & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 & 4 \\ 4 & -1 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

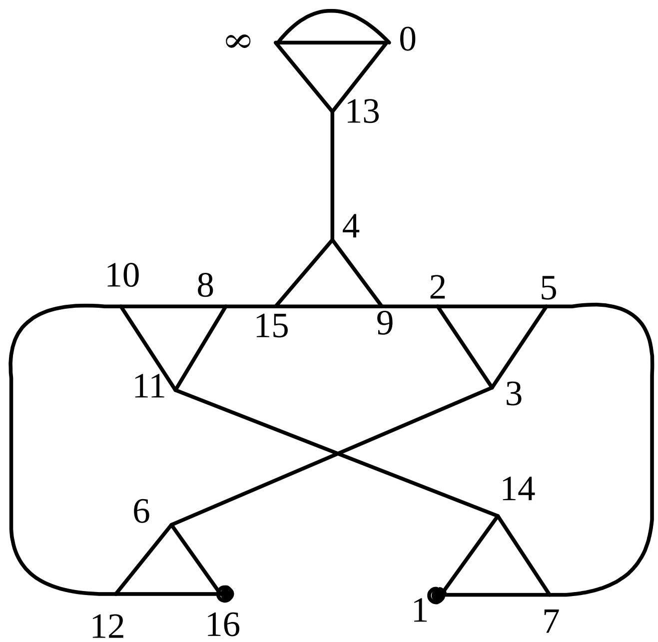
So, $\mu(z') = \frac{1}{z'}$ *and* $\nu(z') = \frac{4}{4z'-1}$. *The permutation representation of* $\bar{\mu}$ *and* $\bar{\nu}$ *is as follows.*

$\bar{\mu} = (0\ \ \infty)(2\ \ 9)(3\ \ 6)(4\ \ 13)(5\ \ 7)(8\ \ 15)(10\ \ 12)(11\ \ 14)(16)(1)$

$\bar{\nu} = (0\ \ 13\ \ \infty)(1\ \ 7\ \ 14)(2\ \ 3\ \ 5)(4\ \ 15\ \ 9)(6\ \ 12\ \ 16)(8\ \ 10\ \ 11)$

$\bar{t} = (0\ \ \infty)(1\ \ 16)(2\ \ 8)(3\ \ 11)(4)(5\ \ 10)(6\ \ 14)(7\ \ 12)(9\ \ 15)(13)$

The associated coset diagram $D(4,17)$ *is:*



4.2 Parametrization of generalized triangle groups

Lemma 12 *Homomorphism* ρ_1 *has two conjugacy classes and* $\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}^2$ *has order* 2, *and two others in which* $\bar{t}\bar{\mu}\bar{\nu}^2\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}$ *is of order* 2.

Proof. *Let* $w = \bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}^2$ *be of order* 2, *with* $\bar{\mu}^2 = \bar{\nu}^3 = (\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}^2)^2 = 1$ *such that* $\bar{\mu}$, $\bar{\nu}$ *produce a group having order* 48 *and* $\bar{t}$ *normalizes* $< \bar{\nu} >$, *and in this case it is characteristic in* $< \bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}^2 >$. *That is, actually maps* $PGL(2;\mathbb{Z})$ *into the normalizer in* $PGL(2;q)$ *of a cyclic group of order* 3. *The normalizer of this group is of order* 48. *We can take* $\bar{\nu}$ *to be fixed element of order* 3. *Any more conjugation can occur within* $N(< \bar{\nu} >)$. *In this group there are two classes of non-central involutions, and we select* $\bar{\mu}$. *Then* $\bar{\nu}\bar{t}$ *is of order* 2 *and it centralizes* $\bar{\mu}$ *and* $\bar{\nu}$. *It is the particular non-trivial element of the centre of* $N(< \bar{\nu} >)$. *Thus there are just two conjugacy classes of non-degenerate homomorphisms* *in which w is of order 2.*

As the dual ρ_1 and ρ_2 maps $\mu,\ \nu,\ t$ onto $\bar{\mu}\bar{t},\ \bar{\nu},\ \bar{t}$, therefore $\left(\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}^2\right)\rho_2 = \bar{\mu}\bar{t}\bar{\nu}\bar{\mu}\bar{t}\bar{\nu}\bar{\mu}\bar{t}\bar{\nu}^2 = \bar{t}(t\bar{\mu}\bar{\nu}^2\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu})\bar{t}$. If $t\bar{\mu}\bar{\nu}^2\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}$ is of order 2, so is $(\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}^2)\rho_2$. Hence there are only two conjugacy classes of non- degenerate homomorphisms: in which one $\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}^2$ is of order 2 and the other in which $t\bar{\mu}\bar{\nu}^2\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}$ is of order 2. ■

Theorem 13 *Any non trivial element g of $PGL(2,q)$, and its order is other than 2 or 3, which is the image of $w = \bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}^2$ under the homomorphism of $PGL(2,\mathbb{Z})$.*

Proof. We look for elements $\bar{\mu},\ \bar{\nu}$ and $\bar{t}$ of $PGL(2;q)$ satisfying the relations

$$\bar{\mu}^2 = \bar{\nu}^3 = (\ \bar{t})^2 = \left(\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}^2\right)^k = (\bar{\mu}\bar{t})^2 = (\bar{\nu}\bar{t})^2 = 1 \tag{4.7}$$

Take $\bar{\mu},\ \bar{\nu}$ and $\bar{t}$ to be represented by $X = \begin{bmatrix} a_1 & k_1c_1 \\ c & -a_1 \end{bmatrix}$,

$Y = \begin{bmatrix} d_1 & k_1f_1 \\ f_1 & -d_1-1 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & -k_1 \\ 1 & 0 \end{bmatrix}$ where $a_1, c_1, d_1, k_1, f_1 \in F_q$ with $k_1 \neq 0$. We shall take Δ as the determinant of matrix X

$$\det(X) = -a_1^2 - k_1c_1^2 = \Delta$$

Now we require the determinant of matrix Y to be 1, that is

$$d_1^2 + d_1 + k_1f_1^2 + 1 = 0$$

This clearly yields the elements which satisfy the relations $\bar{\mu}^2 = \bar{\nu}^3 = \left(\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}^2\right)^k = (\bar{\mu}\bar{t})^2 = (\bar{\nu}\bar{t})^2 = 1$. Therefore we just check the conjugacy class of $w = \bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}\bar{\mu}\bar{\nu}^2$. ■

Theorem 14 *The conjugacy classes of homomorphisms from* $PSL(2,\mathbb{Z})$ *to* $PL(F_q)$*, which gives* $\left\langle \mu^2 = \nu^3 = \left(\mu\nu\mu\nu\mu\nu^2\right)^k = 1\right\rangle$ *as a homomorphic image of* $PSL(2,q)$*, are parametrized by the elements of* F_q.

Proof. let $w = \mu\nu\mu\nu\mu\nu\nu = \mu\nu\mu\nu\mu\nu^2$

If X and Y are the matrices representing $\bar{\mu}$, $\bar{\nu}$ and $\bar{t}$ of $PGL(2,q)$ and satisfying the relations

$$(\bar{\mu})^2 = (\bar{\nu})^3 = \lambda I$$

Then w is represented as:

$M = Y^2XYXYX$

Here we take $\bar{\mu}$, $\bar{\nu}$ and $\bar{t}$ to be represented as $X = \begin{bmatrix} a_1 & k_1c_1 \\ c & -a_1 \end{bmatrix}$, $Y = \begin{bmatrix} d_1 & k_1f_1 \\ f_1 & -d_1-1 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & -k_1 \\ 1 & 0 \end{bmatrix}$ where $a_1, c_1, d_1, k_1, f_1 \in F_q$. We write

$$\det(X) = a_1^2 - k_1c_1^2 = \Delta \neq 0$$

and require that

$$d_1^2 + d_1 + k_1f_1^2 + 1 = 0$$

This gives the elements which satisfy the relations (4.1). We note that matrix XY representing $\bar{\mu}\bar{\nu}$ has the trace

$$r = a_1(2d_1+1) + 2k_1c_1f_1$$

because $\det(Y) = 1$. This means that $\det(X) = \Delta$ and $tr(X) = 0$, Then by using equations $(3.2) - (3.7)$ in Chapter 3, the matrix M is expressed linearly, as

$$M = (r\Delta - r^3)I + (0)\,X - r\Delta Y + (-\Delta + r^2)XY$$

$$trM = (r\Delta - r^3)trI - r\Delta trY + (-\Delta + r^2)trXY$$

$$trM = 2(r\Delta - 1r^3) - r\Delta\,(-1) + (-\Delta + 1r^2)\,(r)$$

$$trM = 2r\Delta - r^3 \tag{4.8}$$

This implies

$$trM = r\left(2\Delta - r^2\right)$$

this gives

$$\frac{trM}{r} = \left(2\Delta - r^2\right)$$

and

$$r^2 = 2\Delta - \frac{trM}{r} \tag{4.9}$$

$$r^2 - \Delta = \Delta - \frac{trM}{r}$$

implies that

$$r^2 - 3\Delta = -\Delta - \frac{trM}{r}. \tag{4.11}$$

Now

$$TM = (r\Delta - r^3)TI + (0)\,TX - r\Delta TY + (-\Delta + r^2)TXY$$

$$tr\,(TM) = (-\Delta + r^2)\,(trTXY)$$

$$tr\,(TM) = (-\Delta + r^2)\,(trTXY)$$

$$tr\,(TM) = (\Delta - \frac{trM}{r})\,(trTXY)\,. \tag{4.12}$$

As

$$(trTXY) = k_1 s$$

and

$$k_1 s^2 + r^2 = 3\Delta \Rightarrow k_1 s = \frac{3\Delta - r^2}{s} \tag{4.13}$$

using this value in equation (4.12)

$$\begin{aligned} tr\,(TM) &= (\Delta - \frac{trM}{r})\left(\frac{3\Delta - r^2}{s}\right) \\ &= \frac{1}{sr}(r\Delta - \frac{trM}{r})\left(3\Delta - r^2\right) \end{aligned}$$

using equation (4.11) , we get

$$\begin{aligned} &= \frac{1}{sr}(r\Delta - trM)\left(\Delta + \frac{trM}{r}\right) \\ &= \frac{1}{sr^2}(r\Delta - trM)(r\Delta + trM) \\ &= \frac{1}{sr^2}(r^2\Delta^2 - (trM)^2) \end{aligned}$$

$$tr\,(TM) = \frac{1}{sr^2}(r^2\Delta^2 - (trM)^2) \tag{4.14}$$

■

We illustrate this theorem by an example.

Example 15 *For the action of $PSL\,(2,\mathbb{Z})$ on $PL\,(F_7)$ for $\theta = 1$ of F_7.*

By equation $(4.6)\,, \theta = \frac{(trM)^2}{(\det M)}$ *and* $\det M = 1$

Let $trM = 1$*. As* $det(M) = \Delta^3$ *implies that* $\Delta^3 = 1$*.*

This gives $\Delta = 1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$*. Taking* $\Delta = 1$ *and by equation* $(4.8)\,, trM = 2r\Delta - r^3$ *implies* $1 = 2r - r^3 \Rightarrow r^3 - 2r + 1 = 0$*. Hence* $r = 1,\ \frac{-1+\sqrt{5}}{2},\ \frac{-1-\sqrt{5}}{2}$*.Take* $r = 1$ *and by equation* $(4.14)\,,\ tr\,(TM) = \frac{1}{s}(1-1) = 0$*.*

This shows $tr\,(TM) = 0$. *Let* $s = 1$, *then* $k_1 s^2 + r^2 = 3\Delta$ *implying* $k_1 + 1 = 3$ *or* $k_1 = 2$. *Take* $d_1 = 2$, *then* $d_1^2 + d_1 + k_1 f_1^2 + 1 = 0$ *or* $f_1 = 0$. *Now* $r = a_1\,(2d_1 + 1) + 2k_1 c_1 f_1$ *implies that* $a_1 = 3$ *and* $s = 2a_1 f_1 - c_1\,(2d_1 + 1)$, *that is* $c_1 = 4$.

So the matrices

$$X = \begin{bmatrix} 3 & 1 \\ 4 & 4 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$$

give

$\mu\,(z') = \frac{3z'+1}{4z'+4}$, $\nu\,(z') = \frac{z'}{-2}$ *and* $\bar{t} = \frac{-2}{z'}$.

Thus $\bar{\mu}$, $\bar{\nu}$ *and* $\bar{t}$ *are*

$$\bar{\mu} = (0 \quad 2)\,(1 \quad 4)\,(3 \quad 5)\,(6 \quad \infty)$$

$$\bar{\nu} = (1 \quad 4 \quad 2)\,(3 \quad 5 \quad 6)\,(0)\,(\infty)$$

$$\bar{t} = (0 \quad \infty)(1 \quad 5)(2 \quad 6)(3 \quad 4)$$

The associated coset diagram is:

4.3 Parametric equations for $\theta = 0$

We consider the parametrization of the homomorphisms of the actions for the group $\left\langle \mu^2 = \nu^3 = \left(\mu\nu\mu\nu\mu\nu^2\right)^3 = 1 \right\rangle$ in the section (4.2) for all the elements of the field F_q. In this section we consider case of $\theta = 0$ not only for the aforementioned group but also for two other finite generalized triangle groups $\left\langle \mu^2 = \nu^3 = \left(\mu\nu\mu\nu^2\right)^2 = 1 \right\rangle$ and $\left\langle \mu^2 = \nu^3 = \left(\mu\nu\mu\nu\mu\nu^2\mu\nu^2\right)^2 = 1 \right\rangle$ which are quotients of the modular group from the list of fourteen finite generalized triangle groups.

4.3.1 The group $\left\langle \mu^2 = \nu^3 = \left(\mu\nu\mu\nu\mu\nu^2\right)^2 = 1 \right\rangle$

Let $w = \mu\nu\mu\nu\mu\nu^2$, then by equations (4.1) – (4.14) are parametric equations for $\theta = 0$.

We consider the action of $PGL(2, \mathbb{Z})$ on $PL(F_7)$ and draw a coset diagram for $\theta = 0 \in F_{7.}$ Then

$\bar{\mu}(z') = \frac{2z'+4}{4z'-2}$, $\bar{\nu}(z') = \frac{-2z'}{3}$and $\bar{t} = \frac{-1}{z'}$.

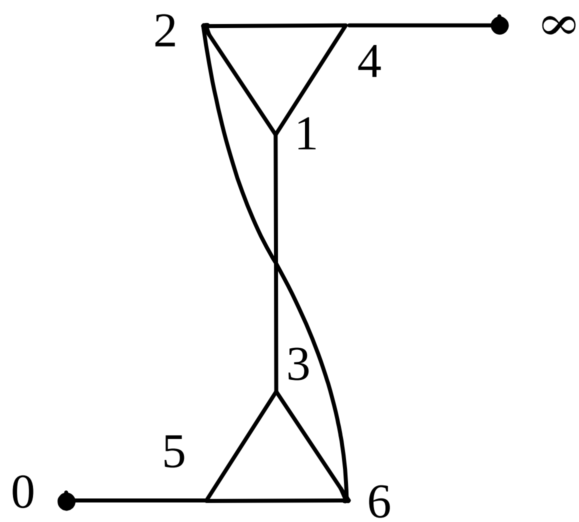
Thus $\bar{\mu}$, $\bar{\nu}$ and $\bar{t}$ are

$\bar{\mu} = (0 \quad 5)(1 \quad 3)(2 \quad 6)(4 \quad \infty)$

$\bar{\nu} = (1 \quad 4 \quad 2)(3 \quad 5 \quad 6)(0)(\infty)$

$\bar{t} = (0 \quad \infty)(1 \quad 6)(2 \quad 3)(5 \quad 4)$

where $z' \in PL(F_7)$. The associated coset diagram $D(0, 7)$ is:



4.3.2 The group $\left\langle \mu^2 = \nu^3 = (\mu\nu\mu\nu\mu\nu^2\mu\nu^2)^2 = 1 \right\rangle$

Let $w = \mu\nu\mu\nu\mu\nu^2\mu\nu^2$. Then the group $\left\langle \mu^2 = \nu^3 = (w)^2 = 1 \right\rangle$ represents a finite generalized triangle groups of order 576.

The word w is presented as $M = YYXYYXYXYX$ with the same matrices X, Y and T. By equations $(3.2) - (3.7)$ of Chapter 3, and matrix M is expressed as

$$M = (\Delta^2 - 2r^2\Delta + r^4)I + (r\Delta)\,X + r^2\Delta Y + (2r\Delta + r^3)XY,$$

where

$$trM = (\Delta^2 - 2r^2\Delta + r^4)tr\,(I) + (r\Delta)\,trX + r^2\Delta trY + (2r\Delta + r^3)trXY. \tag{4.15}$$

As $Tr\,(I) = 2$, $Tr\,(X) = 0$, $Tr\,(Y) = -1$ and $Tr\,(XY) = r$. Thus equation (4.15) implies that

$$TrM = 2\Delta^2 - 3r^2\Delta + r^4$$

that is

$$3\Delta - r^2 = \frac{1}{r^2}\left(2\Delta^2 - tr\,(M)\right) \tag{4.16}$$

Also

$$Tr\,(TM) = (2r\Delta + r^3)Tr\,(TXY)$$

By equations (4.1) – (4.5), we get

$$Tr\,(TM) = \frac{1}{sr}\left(2\Delta^2 - Tr\,(M)\right)(2\Delta - r^2) \tag{4.17}$$

Now, we consider the action of $PGL(2,\mathbb{Z})$ on $PL(F_{17})$ and draw its corresponding coset diagram.

Assuming $\theta = 0 \in F_{17}$, then by equations (4.1)−(4.5) and equations (4.15)−(4.17), $\bar{\mu}$, $\bar{\nu}$ and $\bar{t}$ are:

$\bar{\mu}\,(z') = \frac{12}{7z'}$, $\bar{\nu}\,(z') = \frac{3z'+11}{9z'-4}$ and $\bar{t} = \frac{-9}{z'}$. Thus $\bar{\mu}$, $\bar{\nu}$ and $\bar{t}$ are

$\bar{\mu} = (0 \quad \infty)\,(1 \quad 9)\,(2 \quad 13)\,(4 \quad 15)\,(5 \quad 12)$

$(6 \quad 10)\,(7 \quad 11)\,(8 \quad 16)\,(3)\,(14)$

$\bar{\nu} = (0 \quad 10 \quad 12)\,(1 \quad 14 \quad 16)\,(3 \quad 8 \quad 9)$

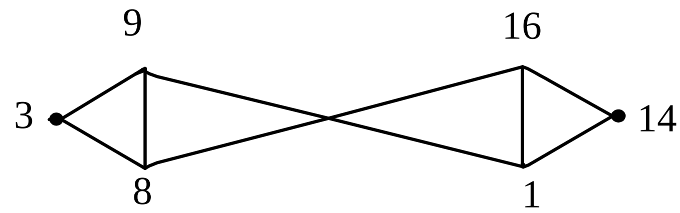
$(4 \quad 11 \quad \infty)\,(5 \quad 15 \quad 13)\,(6 \quad 7 \quad 12)$

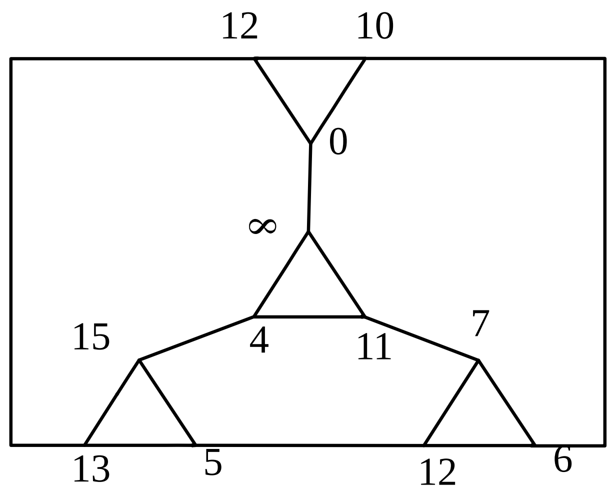
$\bar{t} = (0 \quad \infty)(1 \quad 8)(2 \quad 4)(3 \quad 14)\,(6 \quad 7)$

$(9 \quad 16)\,(10 \quad 11)\,(13 \quad 15)\,(5)\,(12)$

where $z' \in PL(F_{17})$.

The associated coset diagram $D\,(0,17)$ is:





4.3.3 The group $\left\langle \mu^2 = \nu^3 = (\mu\nu\mu\nu^2)^2 = 1 \right\rangle$

Let $w = \mu\nu\mu\nu^2$, then the group $\left\langle \mu^2 = \nu^3 = (w)^2 = 1 \right\rangle$ represents a finite generalized triangle groups of order 24. The word w can be presented as $M = Y^2XYX$ with the same matrices X, Y and T then by equations $(3.2) - (3.7)$ given in Chapter 3, the matrix M is expressed as

$$M = -r^2 I + (0)\, X - \Delta Y + rXY$$

$$TrM = \left(-r^2\right) TrI + (0)\, TrX - \Delta TrY + (r)\, TrXY \tag{4.18}$$

As $Tr\,(I) = 2$, $Tr\,(X) = 0$, $Tr\,(Y) = -1$ and $Tr\,(XY) = r$, thus equation (4.18) implies that

$$TrM = \Delta - r^2 \quad (4.19)$$

Also

$$Tr\,(TM) = (r)\,trXY$$

By equations (4.1) – (4.5) and equation (4.19), we get

$$Tr\,(TM) = \frac{r}{s}\,(2\Delta - Tr\,(M)) \quad (4.20)$$

Assuming $\theta = 0 \in F_7$, then by equations (4.18) – (4.20) and equations(4.15) – (4.17), we get $\bar{\mu}$, $\bar{\nu}$ and $\bar{t}$ respectively are:

$\mu\,(z') = \frac{3z'+1}{4z'-3}$, $\nu\,(z') = \frac{-2z'}{3}$ and $\bar{t} = \frac{-2}{z'}$.

Thus $\bar{\mu}$, $\bar{\nu}$ and $\bar{t}$ are

$\bar{\mu} = (0 \quad 2)\,(1 \quad 4)\,(3 \quad 5)\,(6 \quad \infty)$

$\bar{\nu} = (1 \quad 4 \quad 2)\,(3 \quad 5 \quad 6)\,(0)\,(\infty)$

$\bar{t} = (0 \quad \infty)(1 \quad 5)(2 \quad 6)(3 \quad 4)$, where $z' \in PL(F_7)$.

The associated coset diagram $D\,(0, 7)$ is:

4.4 Conclusion

In this work we extended the parametrization of action of $PSL\left(2,\mathbb{Z}\right)$ for that triangle group $\Delta\left(2,3,k\right)$ [74] to finite generalized triangle groups given by J. Howie in [39]. We considered only eight groups out of fourteen finite generalized triangle groups which are quotients of $PSL\left(2,\mathbb{Z}\right)$. We obtained coset diagrams of action of $PSL\left(2,\mathbb{Z}\right)$ on $PL\left(F_p\right)$ through parametrization which yield one of these eight finite generalized triangle groups, particularly $\left\langle \mu,\nu:\mu^2=v^3=\left(w\right)^k=1\right\rangle$ of order 1440, where $w=\mu\nu\mu\nu\mu\nu^2$. We also analyzed coset diagrams for parameter $\theta=0$ for three other finite generalized triangle groups $\left\langle \mu,\nu:\mu^2=v^3=\left(\mu\nu\mu\nu\mu\nu^2\right)^2=1\right\rangle$, $\left\langle \mu,\nu:\mu^2=v^3=\left(\mu\nu\mu\nu\mu\nu^2\mu\nu^2\right)^2=1\right\rangle$ and $\left\langle \mu,\nu:\mu^2=v^3=\left(\mu\nu\mu\nu^2\right)^2=1\right\rangle$.

Chapter 5

$\mathbf{PSL}(2,7)$ and carbon allotrope $D168$ Schwarzite

Coset diagrams for $PGL(2,\mathbb{Z})$, introduced by G. Higman in late sixties are used in understanding spatial symmetry of Fullerene molecules. We discuss their relation with the carbon allotrope structures with negative curvature $D168$ Schwarzite. We investigate actions of $PSL(2,\mathbb{Z})$ on $PL(\mathbf{F}_{7^n})$ for different values of n, where $n \in \mathbb{N}$, and draw coset diagrams for various orbits and prove some interesting results regarding the number of orbits that occur. We draw coset diagrams depicting meaningfully their relationship with the carbon allotrope structures with negative curvature $D168$ Schwarzite. Some related topological aspects of these diagrams are also highlighted.

5.1 Introduction

The use of point groups in chemistry is a well known application of group theory which portray the spatial symmetry of molecules [44, 45]. In this context the groups of the regular polyhedra are specifically noteworthy in view of their high symmetry. R. B. King discussed in [49] that these regular polyhedral groups are subgroups of larger permutation groups, which themselves are subgroups of the corresponding symmetric groups S_n. This methodology utilizes classical mathematics, which is by and large new to scientific experts. Of specific pertinence to chemists in [48] that these groups may be utilized to depict carbon allotrope structures with negative curvature built from hexagons and heptagons of sp2-hybridized carbon atoms [48, 19, 94].

$PSL(2,p)$ contains a special subset of groups for $p = 5, 7, 11$, in perspective of their specific structure of permutations. In three dimensional space the pollakispolyhedral groups can be viewed as multiples of regular polyhedral symmetry groups[19]. In this chapter we are interested in $PSL(2,7)$ having order 168 and is ${}^7O$ the *heptakisoctahedral* group. It has a subgroup of index 7 which is the octahedral group "O" and has many applications in physics and chemistry. The rotational symmetry of an idealization of the "plumber's nightmare" is $PSL(2,7)$, which is a representation for carbon allotropes "Schwarzite" [56].

Geometrical models, for the group $PSL(2,7)$ or heptakisoctahedral group of order 168 depict its transitive permutations on sets of 7 and 8 objects. A set of seven objects permuted transitively by the group $PSL(2,7)$ can be acquired when an equilateral triangle and a inscribed circle form the seven-point-seven-line geometry presented in D_3 symmetry [48]($Fig1$).

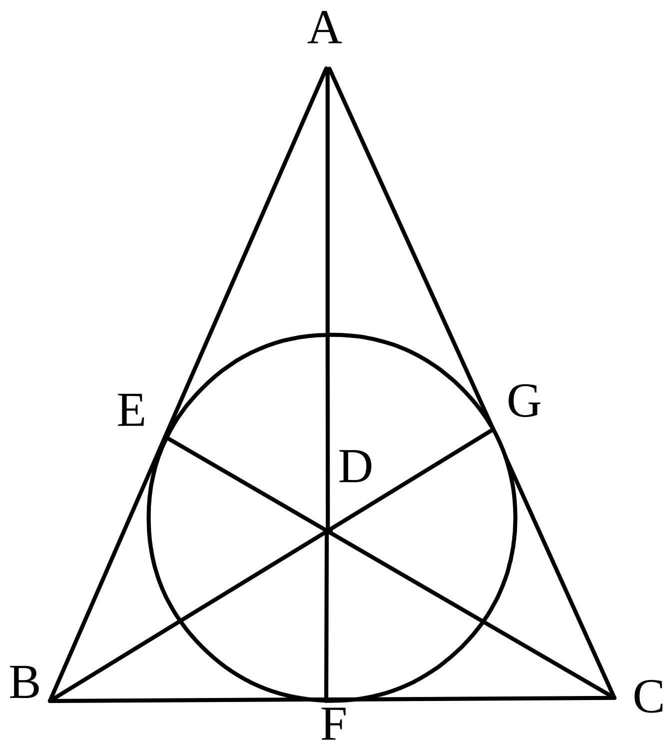


Fig1 : Seven point - Seven line geometry

The seven collineations $(AEB, AGC, BFC, BDG, ADF, EFG, CDE)$ preserved by the permutations of the seven vertex labels form the group $PSL(2,7)$. Note here that in this presentation the six straight lines making the three altitudes and the three edges of the triangle and the inscribed circle are treated on an equal basis. Eight objects permuted transitively by the heptakisoctahedral group are the vertex labels of a cuboid of D_2 point group symmetry which give a set of 168 nonsuperimposable cuboids, form the group $PSL(2,7)$. In analogy to the connection between the tetrahedral and icosahedral group, the octahedral rotation group O can be obtained from the heptakisoctahedral group or $PSL(2,7)$ by erasing all seven-fold symmetry elements [19].

The regular genus-3 Klein map group is an other representation of $PSL(2,7)$. Its high symmetry in association with the theory of multivalued functions is studied in [49]. This map shows the transitivity on a $7-set$, when sevenfold symmetry elements removed, seven octahedral structures are obtained which contain eight vertices. The relation of carbon structures with negative-curvature and this group is given in [1, 18].

$PSL(2,7)$ is important to analyze the permutational symmetry of $D168$ Schwarzite. Infact the prototypical role of $D168$ in Schwarzite series and C_{60} in fullerene series relate that the carbon atoms in their structure and the order of corresponding transitive permutational group are same. The structure of $D168$ is derived from a unit cell of 24 heptagons embedded in a surface of genus 3. These of 24 heptagons has 56 vertices. Every heptagon contains 7 vertices and three heptagons are connected with each other with one vertex. Infinite minimal surfaces with minimal Gaussian curvature and surfaces with genus 3 are discussed in [48].

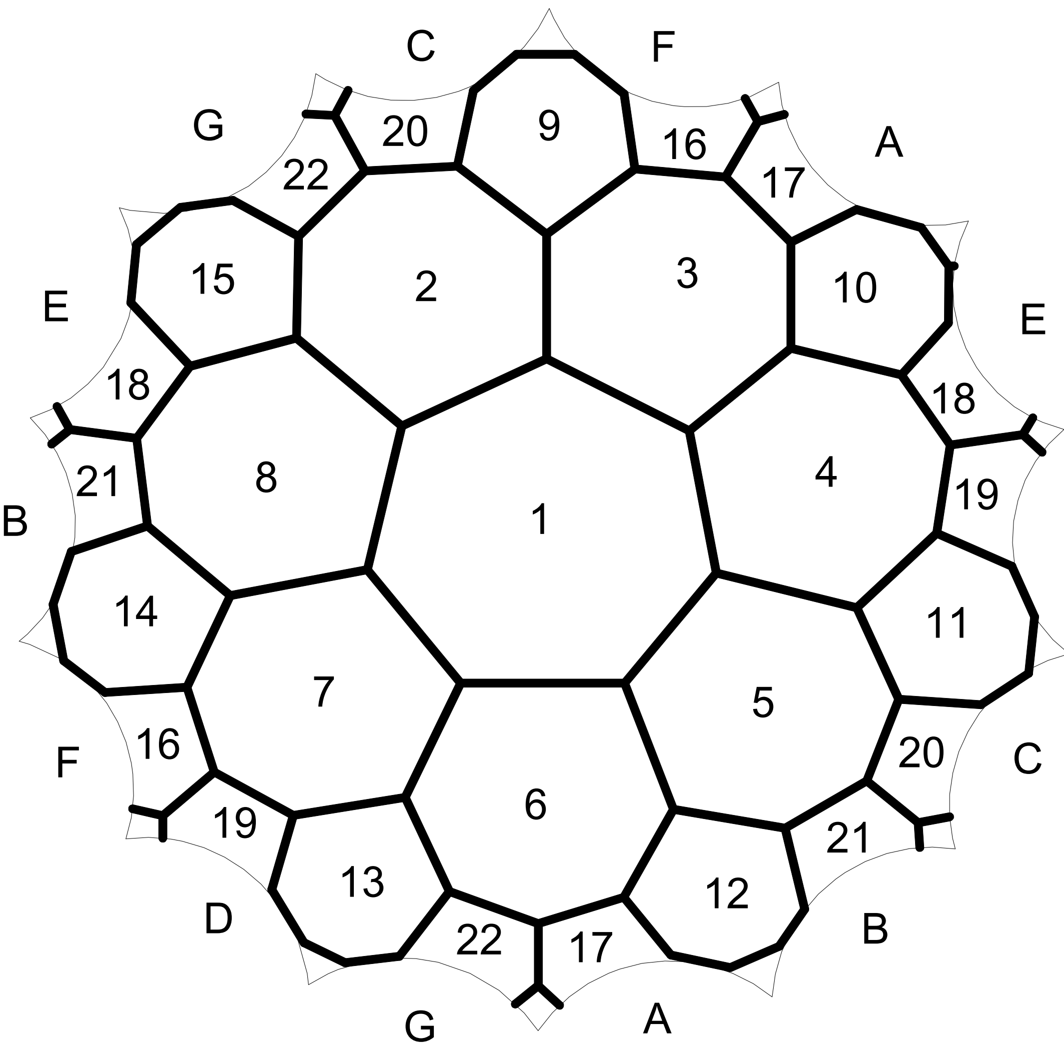


Fig2 : Klein graph

This *Fig* 2 (discussed by Klein in [50]) portrays an open network of full heptagons or their portions that can be modified into a negative curvature of genus 3. The unit cell with 24 heptagons, 84 edges and 56 vertices has an Euler's characteristic which corresponds to the genus 3. A carbon allotropes with such type of structure is known as $D56$ protoschwarzite.

This carbon allotrope structure leads to $D168$ structure, which is discussed in details in [48].

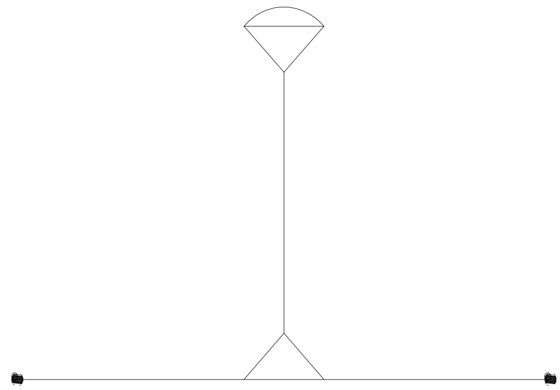
5.2 Action of G on $PL(F_{7^n})$

In this section we discuss the action of $\mathbf{G} = PSL(2, \mathbb{Z})$ on $PL(F_{7^n})$ where $n \in \mathbb{N}$. We use of coset diagrams to inspect the properties of this action and the orbits of the group thus obtained.

5.2.1 Action of G on $PL(F_7)$

Consider $\bar{\mathbf{G}}$ as a group generated by $\bar{\mu}$ and $\bar{\nu}$, where $\bar{\mu}$ and $\bar{\nu}$ are the permutation representations of μ and ν after the action of $\mathbf{G}$ on $PL(F_{7^n})$ for $n \in \mathbb{N}$. Taking $n = 1$, the action of $\mathbf{G}$ on $PL(F_7)$ gives

$\bar{\mu} = (0 \quad \infty)\,(1 \quad 6)\,(2 \quad 3)\,(4 \quad 5)$ and $\bar{\nu} = (1 \quad 0 \quad \infty)\,(2 \quad 4 \quad 6)\,(3)\,(5)$ .This yields the following coset diagram γ_1, which can be graphically represented as:



γ_1

This diagram is a representation of the well known simple group of order 168 [20].

5.2.2 Action of G on $PL(F_{7^2})$

We consider now the group **G** acting on $PL(F_{7^2})$. An irreducible polynomial of degree 2 in F_{7^2} is $t^2 + 2t + 3$. The elements of F_{49} are of the form $t_0 + t_1\zeta$, where $t_i \in \mathbb{Z}_7$, for $i = 0, 1$.

Let ζ be the 48^{th} root of unity of F_{49} satisfying $\zeta^2 = 5\zeta + 4$. When **G** acts on $PL(F_{7^2})$, μ and ν have the following permutation representation:

$\bar{\mu} = (0 \quad \infty)\left(1 \quad \zeta^{24}\right)\left(\zeta \quad \zeta^{23}\right)\left(\zeta^2 \quad \zeta^{22}\right)\left(\zeta^3 \quad \zeta^{21}\right)\left(\zeta^4 \quad \zeta^{20}\right)\left(\zeta^5 \quad \zeta^{19}\right)$

$\left(\zeta^6 \quad \zeta^{18}\right)\left(\zeta^7 \quad \zeta^{17}\right)\left(\zeta^8 \quad \zeta^{16}\right)\left(\zeta^9 \quad \zeta^{15}\right)\left(\zeta^{10} \quad \zeta^{14}\right)\left(\zeta^{11} \quad \zeta^{13}\right)\left(\zeta^{12} \quad \zeta^{12}\right)$

$\left(\zeta^{25} \quad \zeta^{47}\right)\left(\zeta^{26} \quad \zeta^{46}\right)\left(\zeta^{27} \quad \zeta^{45}\right)\left(\zeta^{28} \quad \zeta^{44}\right)\left(\zeta^{29} \quad \zeta^{43}\right)\left(\zeta^{30} \quad \zeta^{42}\right)$

$\left(\zeta^{31} \quad \zeta^{41}\right)\left(\zeta^{32} \quad \zeta^{40}\right)\left(\zeta^{33} \quad \zeta^{39}\right)\left(\zeta^{34} \quad \zeta^{38}\right)\left(\zeta^{35} \quad \zeta^{37}\right)\left(\zeta^{36} \quad \zeta^{36}\right)$

and

$\bar{\nu} = (1 \quad 0 \quad \infty)\left(\zeta \;\; \zeta^2 \;\; \zeta^{21}\right)\left(\zeta^3 \;\; \zeta^7 \;\; \zeta^{14}\right)\left(\zeta^4 \;\; \zeta^{31} \;\; \zeta^{37}\right)\left(\zeta^5 \;\; \zeta^{13} \;\; \zeta^6\right)\left(\zeta^8\right)$

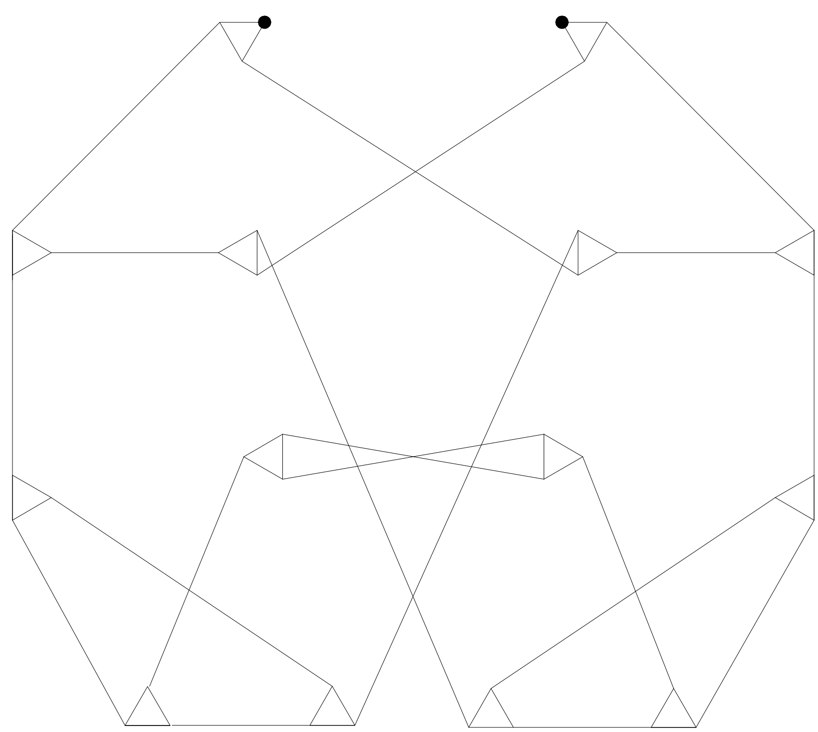
$(\zeta^{9}\ \ \zeta^{30}\ \ \zeta^{33})\,(\zeta^{10}\ \ \zeta^{36}\ \ \zeta^{26})\,(\zeta^{11}\ \ \zeta^{17}\ \ \zeta^{44})\,(\zeta^{12}\ \ \zeta^{38}\ \ \zeta^{22})\,(\zeta^{40})$

$(\zeta^{16}\ \ \zeta^{32}\ \ \zeta^{24})\,(\zeta^{15}\ \ \zeta^{18}\ \ \zeta^{39})\,(\zeta^{19}\ \ \zeta^{28}\ \ \zeta^{25})\,(\zeta^{20}\ \ \zeta^{29}\ \ \zeta^{23})\,(\zeta^{27}\ \ \zeta^{46}\ \ \zeta^{47})$

$(\zeta^{35}\ \ \zeta^{43}\ \ \zeta^{42})\,(\zeta^{34}\ \ \zeta^{41}\ \ \zeta^{45})$

which yield the following two orbits γ_1 and γ_2.

γ_2 can be graphically represented as:



γ_2

This coset diagram represents a group of order 168 [20] and consists of two orbits γ_1 and γ_2.

5.2.3 Action of G on $PL(F_{7^3})$

Let $\beta^3 + 6\beta + 2$ be the irreducible polynomial in F_{7^3}. The field has elements of type $\beta_0 + \beta_1\zeta + \beta_2\zeta^2$, where $\beta_i \in \mathbb{Z}_7$, for $i = 0, 1, 2$. Let ζ be the 342^{th} primitive root of unity satisfying $\zeta^3 = \zeta + 5$ of F_{343}. When **G** acts on $PL(F_{7^3})$, μ and ν have the following permutation representation:

$$\bar{\mu} = (0 \quad \infty)\,(1 \quad \zeta^{171})\,(\zeta \quad \zeta^{170})\,(\zeta^{2} \quad \zeta^{169})\,(\zeta^{3} \quad \zeta^{168})\,(\zeta^{4} \quad \zeta^{167})\,(\zeta^{5} \quad \zeta^{166})$$
$$(\zeta^{6} \quad \zeta^{165})\,(\zeta^{7} \quad \zeta^{164})\,(\zeta^{8} \quad \zeta^{163})\,(\zeta^{9} \quad \zeta^{162})\,(\zeta^{10} \quad \zeta^{161})\,(\zeta^{11} \quad \zeta^{160})\,(\zeta^{12} \quad \zeta^{159})$$
$$(\zeta^{13} \quad \zeta^{158})\,(\zeta^{14} \quad \zeta^{157})\,(\zeta^{15} \quad \zeta^{156})\,(\zeta^{16} \quad \zeta^{155})\,(\zeta^{17} \quad \zeta^{154})\,(\zeta^{18} \quad \zeta^{153})\,(\zeta^{19} \quad \zeta^{152})$$
$$(\zeta^{20} \quad \zeta^{40})\,(\zeta^{21} \quad \zeta^{151})\,(\zeta^{22} \quad \zeta^{150})\,(\zeta^{23} \quad \zeta^{149})\,(\zeta^{24} \quad \zeta^{148})\,(\zeta^{25} \quad \zeta^{147})\,(\zeta^{26} \quad \zeta^{146})$$
$$(\zeta^{27} \quad \zeta^{145})\,(\zeta^{28} \quad \zeta^{144})\,(\zeta^{29} \quad \zeta^{143})\,(\zeta^{30} \quad \zeta^{142})\,(\zeta^{31} \quad \zeta^{141})\,(\zeta^{32} \quad \zeta^{140})\,(\zeta^{33} \quad \zeta^{139})$$
$$(\zeta^{34} \quad \zeta^{138})\,(\zeta^{35} \quad \zeta^{137})\,(\zeta^{36} \quad \zeta^{136})\,(\zeta^{37} \quad \zeta^{135})\,(\zeta^{38} \quad \zeta^{134})\,(\zeta^{39} \quad \zeta^{133})\,(\zeta^{40} \quad \zeta^{132})$$
$$(\zeta^{41} \quad \zeta^{131})\,(\zeta^{42} \quad \zeta^{130})\,(\zeta^{43} \quad \zeta^{129})\,(\zeta^{44} \quad \zeta^{128})\,(\zeta^{45} \quad \zeta^{127})\,(\zeta^{46} \quad \zeta^{126})\,(\zeta^{47} \quad \zeta^{125})$$
$$(\zeta^{48} \quad \zeta^{124})\,(\zeta^{49} \quad \zeta^{123})\,(\zeta^{50} \quad \zeta^{122})\,(\zeta^{51} \quad \zeta^{121})\,(\zeta^{52} \quad \zeta^{120})\,(\zeta^{53} \quad \zeta^{119})\,(\zeta^{54} \quad \zeta^{118})$$
$$(\zeta^{55} \quad \zeta^{117})\,(\zeta^{56} \quad \zeta^{116})\,(\zeta^{57} \quad \zeta^{115})\,(\zeta^{58} \quad \zeta^{114})\,(\zeta^{59} \quad \zeta^{113})\,(\zeta^{60} \quad \zeta^{112})\,(\zeta^{61} \quad \zeta^{111})$$
$$(\zeta^{62} \quad \zeta^{110})\,(\zeta^{63} \quad \zeta^{109})\,(\zeta^{64} \quad \zeta^{108})\,(\zeta^{65} \quad \zeta^{107})\,(\zeta^{66} \quad \zeta^{106})\,(\zeta^{67} \quad \zeta^{105})\,(\zeta^{68} \quad \zeta^{104})$$
$$(\zeta^{69} \quad \zeta^{103})\,(\zeta^{70} \quad \zeta^{102})\,(\zeta^{71} \quad \zeta^{101})\,(\zeta^{72} \quad \zeta^{100})\,(\zeta^{73} \quad \zeta^{99})\,(\zeta^{74} \quad \zeta^{98})\,(\zeta^{75} \quad \zeta^{97})$$
$$(\zeta^{76} \quad \zeta^{96})\,(\zeta^{77} \quad \zeta^{95})\,(\zeta^{78} \quad \zeta^{94})\,(\zeta^{79} \quad \zeta^{93})\,(\zeta^{80} \quad \zeta^{92})\,(\zeta^{81} \quad \zeta^{91})\,(\zeta^{82} \quad \zeta^{90})$$
$$(\zeta^{82} \quad \zeta^{89})\,(\zeta^{83} \quad \zeta^{88})\,(\zeta^{84} \quad \zeta^{87})\,(\zeta^{85} \quad \zeta^{86})\,(\zeta^{172} \quad \zeta^{341})\,(\zeta^{173} \quad \zeta^{340})\,(\zeta^{174} \quad \zeta^{339})$$
$$(\zeta^{175} \quad \zeta^{338})\,(\zeta^{176} \quad \zeta^{337})\,(\zeta^{177} \quad \zeta^{336})\,(\zeta^{178} \quad \zeta^{335})\,(\zeta^{179} \quad \zeta^{334})\,(\zeta^{180} \quad \zeta^{333})$$
$$(\zeta^{181} \quad \zeta^{332})\,(\zeta^{182} \quad \zeta^{331})\,(\zeta^{183} \quad \zeta^{330})\,(\zeta^{184} \quad \zeta^{329})\,(\zeta^{185} \quad \zeta^{328})\,(\zeta^{186} \quad \zeta^{327})$$
$$(\zeta^{187} \quad \zeta^{326})\,(\zeta^{188} \quad \zeta^{325})\,(\zeta^{189} \quad \zeta^{324})\,(\zeta^{190} \quad \zeta^{323})\,(\zeta^{191} \quad \zeta^{322})\,(\zeta^{192} \quad \zeta^{321})$$
$$(\zeta^{193} \quad \zeta^{320})\,(\zeta^{194} \quad \zeta^{319})\,(\zeta^{195} \quad \zeta^{318})\,(\zeta^{196} \quad \zeta^{317})\,(\zeta^{197} \quad \zeta^{316})\,(\zeta^{198} \quad \zeta^{315})$$

$(\zeta^{199} \quad \zeta^{314})\,(\zeta^{200} \quad \zeta^{313})\,(\zeta^{201} \quad \zeta^{312})\,(\zeta^{202} \quad \zeta^{311})\,(\zeta^{203} \quad \zeta^{310})\,(\zeta^{204} \quad \zeta^{309})$

$(\zeta^{205} \quad \zeta^{308})\,(\zeta^{206} \quad \zeta^{307})\,(\zeta^{207} \quad \zeta^{306})\,(\zeta^{208} \quad \zeta^{305})\,(\zeta^{209} \quad \zeta^{304})\,(\zeta^{210} \quad \zeta^{303})$

$(\zeta^{211} \quad \zeta^{302})\,(\zeta^{212} \quad \zeta^{301})\,(\zeta^{213} \quad \zeta^{300})\,(\zeta^{214} \quad \zeta^{299})\,(\zeta^{215} \quad \zeta^{298})\,(\zeta^{216} \quad \zeta^{297})$

$(\zeta^{217} \quad \zeta^{296})\,(\zeta^{218} \quad \zeta^{295})\,(\zeta^{219} \quad \zeta^{294})\,(\zeta^{220} \quad \zeta^{293})\,(\zeta^{221} \quad \zeta^{292})\,(\zeta^{222} \quad \zeta^{291})$

$(\zeta^{223} \quad \zeta^{290})\,(\zeta^{224} \quad \zeta^{289})\,(\zeta^{225} \quad \zeta^{288})\,(\zeta^{226} \quad \zeta^{287})\,(\zeta^{227} \quad \zeta^{286})\,(\zeta^{228} \quad \zeta^{285})$

$(\zeta^{229} \quad \zeta^{284})\,(\zeta^{230} \quad \zeta^{283})\,(\zeta^{231} \quad \zeta^{282})\,(\zeta^{232} \quad \zeta^{281})\,(\zeta^{233} \quad \zeta^{180})\,(\zeta^{234} \quad \zeta^{279})$

$(\zeta^{235} \quad \zeta^{278})\,(\zeta^{236} \quad \zeta^{277})\,(\zeta^{237} \quad \zeta^{276})\,(\zeta^{238} \quad \zeta^{275})\,(\zeta^{239} \quad \zeta^{274})\,(\zeta^{240} \quad \zeta^{273})$

$(\zeta^{241} \quad \zeta^{272})\,(\zeta^{242} \quad \zeta^{271})\,(\zeta^{243} \quad \zeta^{270})\,(\zeta^{244} \quad \zeta^{269})\,(\zeta^{245} \quad \zeta^{268})\,(\zeta^{246} \quad \zeta^{267})$

$(\zeta^{247} \quad \zeta^{266})\,(\zeta^{248} \quad \zeta^{265})\,(\zeta^{249} \quad \zeta^{264})\,(\zeta^{250} \quad \zeta^{263})\,(\zeta^{251} \quad \zeta^{262})\,(\zeta^{252} \quad \zeta^{261})$

$(\zeta^{253} \quad \zeta^{260})\,(\zeta^{254} \quad \zeta^{259})\,(\zeta^{255} \quad \zeta^{258})\,(\zeta^{256} \quad \zeta^{257})$

and

$\bar{\nu} = (1 \quad 0 \quad \infty)\,(\zeta \quad \zeta^{12} \quad \zeta^{158})\,(\zeta^{341} \quad \zeta^{184} \quad \zeta^{330})\,(\zeta^{2} \quad \zeta^{54} \quad \zeta^{115})\,(\zeta^{340} \quad \zeta^{227} \quad \zeta^{288})$

$(\zeta^{3} \quad \zeta^{277} \quad \zeta^{233})\,(\zeta^{339} \quad \zeta^{109} \quad \zeta^{65})\,(\zeta^{4} \quad \zeta^{292} \quad \zeta^{217})\,(\zeta^{338} \quad \zeta^{125} \quad \zeta^{50})\,(\zeta^{5} \quad \zeta^{271} \quad \zeta^{237})$

$(\zeta^{337} \quad \zeta^{105} \quad \zeta^{71})\,(\zeta^{220} \quad \zeta^{287} \quad \zeta^{6})\,(\zeta^{336} \quad \zeta^{122} \quad \zeta^{55})\,(\zeta^{7} \quad \zeta^{84} \quad \zeta^{80})\,(\zeta^{335} \quad \zeta^{262} \quad \zeta^{258})$

$(\zeta^{9} \quad \zeta^{236} \quad \zeta^{268})\,(\zeta^{333} \quad \zeta^{74} \quad \zeta^{106})\,(\zeta^{8} \quad \zeta^{314} \quad \zeta^{191})\,(\zeta^{334} \quad \zeta^{151} \quad \zeta^{28})\,(\zeta^{10} \quad \zeta^{329} \quad \zeta^{174})$

$(\zeta^{332} \quad \zeta^{168} \quad \zeta^{13})\,(\zeta^{11} \quad \zeta^{128} \quad \zeta^{32})\,(\zeta^{331} \quad \zeta^{310} \quad \zeta^{214})\,(\zeta^{14} \quad \zeta^{36} \quad \zeta^{121})\,(\zeta^{328} \quad \zeta^{221} \quad \zeta^{306})$

$(\zeta^{59} \quad \zeta^{15} \quad \zeta^{97})\,(\zeta^{327} \quad \zeta^{245} \quad \zeta^{283})\,(\zeta^{16} \quad \zeta^{62} \quad \zeta^{93})\,(\zeta^{326} \quad \zeta^{249} \quad \zeta^{280})\,(\zeta^{17} \quad \zeta^{67} \quad \zeta^{87})$

$(\zeta^{325} \quad \zeta^{255} \quad \zeta^{275})\,(\zeta^{18} \quad \zeta^{273} \quad \zeta^{222})\,(\zeta^{324} \quad \zeta^{120} \quad \zeta^{69})\,(\zeta^{19} \quad \zeta^{253} \quad \zeta^{241})\,(\zeta^{323} \quad \zeta^{101} \quad \zeta^{89})$

$(\zeta^{20} \quad \zeta^{234} \quad \zeta^{259})\,(\zeta^{322} \quad \zeta^{83} \quad \zeta^{108})\,(\zeta^{21} \quad \zeta^{229} \quad \zeta^{263})\,(\zeta^{321} \quad \zeta^{79} \quad \zeta^{113})\,(\zeta^{22} \quad \zeta^{316} \quad \zeta^{175})$

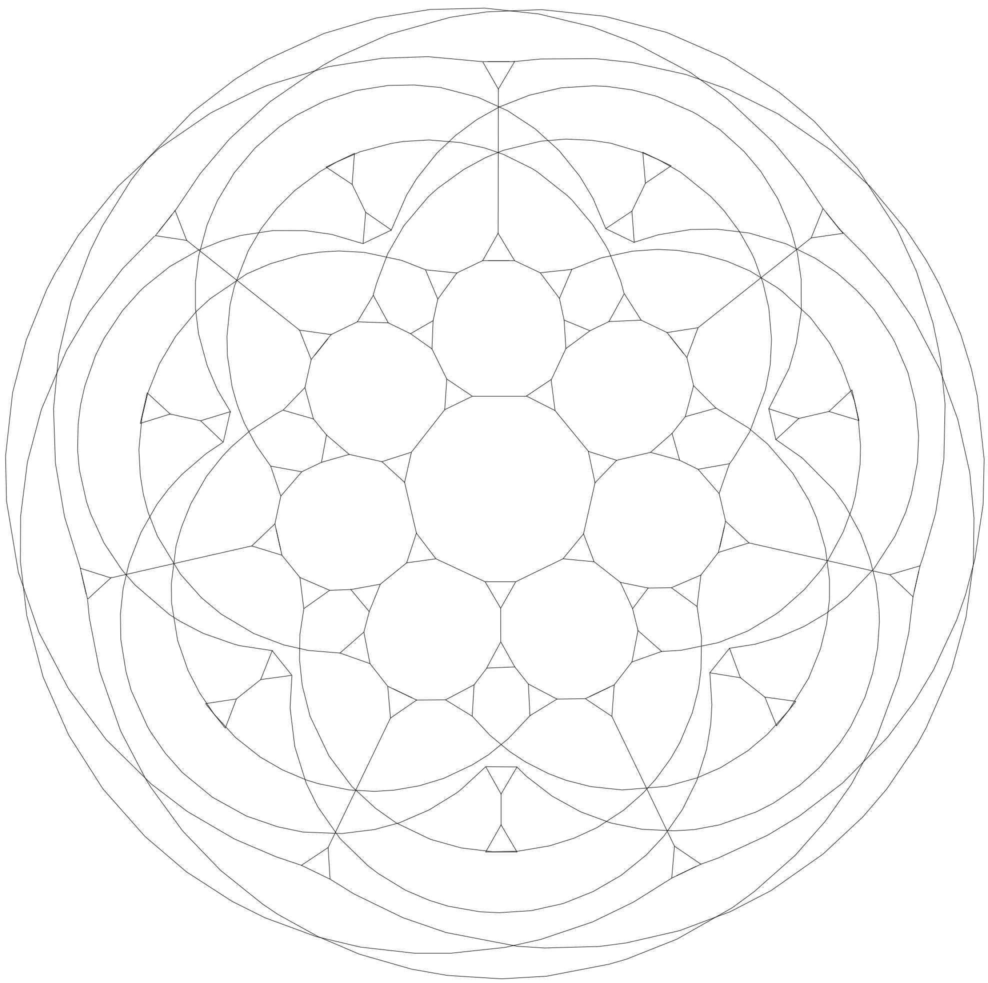
$(\zeta^{318} \quad \zeta^{148} \quad \zeta^{47})\,(\zeta^{320} \quad \zeta^{167} \quad \zeta^{26})\,(\zeta^{23} \quad \zeta^{281} \quad \zeta^{209})\,(\zeta^{57})\,(\zeta^{285})\,(\zeta^{27} \quad \zeta^{76} \quad \zeta^{68})$

$(\zeta^{315} \quad \zeta^{274} \quad \zeta^{266})\,(\zeta^{29} \quad \zeta^{82} \quad \zeta^{60})\,(\zeta^{313} \quad \zeta^{282} \quad \zeta^{260})\,(\zeta^{30} \quad \zeta^{185} \quad \zeta^{298})\,(\zeta^{312} \quad \zeta^{44} \quad \zeta^{157})$

$(\zeta^{31} \quad \zeta^{196} \quad \zeta^{286})\,(\zeta^{311} \quad \zeta^{56} \quad \zeta^{146})\,(\zeta^{33} \quad \zeta^{250} \quad \zeta^{230})\,(\zeta^{309} \quad \zeta^{42} \quad \zeta^{92})\,(\zeta^{34} \quad \zeta^{210} \quad \zeta^{269})$

$(\zeta^{307}\ \zeta^{51}\ \zeta^{155})(\zeta^{37}\ \zeta^{296}\ \zeta^{180})(\zeta^{305}\ \zeta^{162}\ \zeta^{46})(\zeta^{38}\ \zeta^{45}\ \zeta^{88})(\zeta^{304}\ \zeta^{254}\ \zeta^{297})$
$(\zeta^{39}\ \zeta^{276}\ \zeta^{198})(\zeta^{303}\ \zeta^{144}\ \zeta^{66})(\zeta^{40}\ \zeta^{242}\ \zeta^{231})(\zeta^{302}\ \zeta^{111}\ \zeta^{100})(\zeta^{41}\ \zeta^{178}\ \zeta^{249})$
$(\zeta^{301}\ \zeta^{48}\ \zeta^{164})(\zeta^{42}\ \zeta^{172}\ \zeta^{299})(\zeta^{300}\ \zeta^{170}\ \zeta^{43})(\zeta^{58}\ \zeta^{279}\ \zeta^{176})(\zeta^{284}\ \zeta^{166}\ \zeta^{63})$
$(\zeta^{70}\ \zeta^{251}\ \zeta^{192})(\zeta^{272}\ \zeta^{150}\ \zeta^{91})(\zeta^{49}\ \zeta^{246}\ \zeta^{218})(\zeta^{293}\ \zeta^{124}\ \zeta^{96})(\zeta^{72}\ \zeta^{202}\ \zeta^{239})$
$(\zeta^{270}\ \zeta^{103}\ \zeta^{140})(\zeta^{75}\ \zeta^{215}\ \zeta^{223})(\zeta^{267}\ \zeta^{119}\ \zeta^{127})(\zeta^{77}\ \zeta^{212}\ \zeta^{224})(\zeta^{265}\ \zeta^{118}\ \zeta^{130})$
$(\zeta^{78}\ \zeta^{203}\ \zeta^{232})(\zeta^{264}\ \zeta^{110}\ \zeta^{139})(\zeta^{85}\ \zeta^{181}\ \zeta^{247})(\zeta^{257}\ \zeta^{95}\ \zeta^{161})(\zeta^{86}\ \zeta^{289}\ \zeta^{138})$
$(\zeta^{256}\ \zeta^{204}\ \zeta^{53})(\zeta^{81}\ \zeta^{225}\ \zeta^{207})(\zeta^{261}\ \zeta^{135}\ \zeta^{117})(\zeta^{90}\ \zeta^{244}\ \zeta^{179})(\zeta^{252}\ \zeta^{163}\ \zeta^{98})$
$(\zeta^{99}\ \zeta^{278}\ \zeta^{136})(\zeta^{243}\ \zeta^{206}\ \zeta^{64})(\zeta^{104}\ \zeta^{169}\ \zeta^{240})(\zeta^{238}\ \zeta^{102}\ \zeta^{173})(\zeta^{107}\ \zeta^{195}\ \zeta^{211})$
$(\zeta^{235}\ \zeta^{181}\ \zeta^{147})(\zeta^{116}\ \zeta^{200}\ \zeta^{197})(\zeta^{226}\ \zeta^{145}\ \zeta^{142})(\zeta^{134}\ \zeta^{189}\ \zeta^{190})(\zeta^{208}\ \zeta^{152}\ \zeta^{153})$
$(\zeta^{129}\ \zeta^{165}\ \zeta^{219})(\zeta^{213}\ \zeta^{123}\ \zeta^{177})(\zeta^{26}\ \zeta^{201}\ \zeta^{186})(\zeta^{216}\ \zeta^{150}\ \zeta^{141})(\zeta^{137}\ \zeta^{193}\ \zeta^{183})$
$(\zeta^{205}\ \zeta^{159}\ \zeta^{149})(\zeta^{154}\ \zeta^{160}\ \zeta^{199})(\zeta^{188}\ \zeta^{143}\ \zeta^{182})(\zeta^{171}\ \zeta^{228}\ \zeta^{114})(\zeta^{308}\ \zeta^{73}\ \zeta^{132})$
$(\zeta^{35}\ \zeta^{187}\ \zeta^{291})$

We have following orbits γ_1 and γ_3. The graphical representation of γ_3 is:



γ_3

This coset diagram represents a group of order 168 [20] and consists of two orbits γ_1 and two copies of γ_3.

The orbit γ_3 with 24 heptagons has 56 triangles where each triangle is shared by three heptagons, $(24)(7)/2 + 56(3) = 252$ edges, 168 vertices and $24 + 56 = 80$ faces. Thus has Euler's characteristics $(168 - 252 + 80) = -4$ which corresponds to genus 3. The diagrammatic structure of this orbit is similar to the structure of $D168$ Schwazite as both have same genus. Also total number of carbon atoms in $D168$ schwarzite structure and the order of permutational group obtained are same.

5.2.4 Action of G on $PL(F_{7^n})$

Similarly we can draw coset diagrams for the action of **G** on $PL(F_{7^n})$ for any $n \in \mathbb{N}$, because the orbits of the action contain no new coset diagrams for the orbits other than γ_1, γ_2 and γ_3 in the coset diagram. In this section we show that the action of **G** on $PL(F_{7^n})$ evolves $PSL(2,7)$. We also prove some relevant results.

Theorem 16 *If $PSL(2,\mathbb{Z})$ acts on $PL(F_{7^n})$, then*

$$\bar{\mathbf{G}} = <\bar{\mu},\bar{\nu} : (\bar{\mu})^2 = (\bar{\nu})^3 = (\bar{\mu}\bar{\nu})^7 = [\bar{\mu} \quad \bar{\nu}]^4 = 1 > \cong PSL(2,7).$$

Proof. Indeed the actions considered are homomorphisms from $PSL(2,7)$ to $Sym(m)$, for $m = 8, 42, 168$, whose images are transitive subgroups. Obviously these images are isomorphic to $PSL(2,7)$, since this group is simple. ■

Existence of fixed points of $\bar{\mu}$ and $\bar{\nu}$ in these coset diagrams play an important role which is evident in the subsequent discussion.

Theorem 17 *If* **G** *acts on $PL(F_{7^n})$, then*

(1) *fixed points of $\bar{\mu}$ exist only for even n.*

(2) *fixed points of $\bar{\nu}$ exist for all n.*

Proof. (1) When n is even, 7^n+1 are the total number of elements in $PL(F_{7^n})$. As we have $7^n \equiv 1 \pmod 4$ and the permutation $\bar{\mu}$ is composed of two cycles leaving one element which becomes a fixed point of $\bar{\mu}$.

(2) $(\omega)\nu = (\omega-1)/\omega$ implies $(\omega-1)/\omega = \omega$, that is $\omega^2-\omega+1=0$. So $\omega \equiv 3,5 \pmod 7$ are the fixed points of $\bar{\nu}$ which exist for all n. ■

Remark 18 *The action of* $\mathbf{G}$ *on* $PL(F_{7^n})$ *gives three types of orbits* $\gamma_1,\ \gamma_2$ *and* γ_3*.*

The orbit γ_1 *consists of* 8 *elements.* $\zeta^{(q-1)/6}$ *and* $\zeta^{5(q-1)/6}$ *are fixed points of* $\bar{\nu}$ *in* γ_1 *where* $q=7^n$*. All coset diagrams for this action contain* γ_1 *for all* n *and* $\zeta^{(q-1)/4}$, $\zeta^{3(q-1)/4}$ *are fixed points of* $\bar{\mu}$ *which lie in the orbit* γ_2 *consisting of* 42 *elements. This orbit exists in the coset diagram only for even* n*. The third orbit* γ_3 *consists of* 168 *vertices but it does not contain any fixed points of* $\bar{\mu}$ *or* $\bar{\nu}$*. It exists in a coset diagram always in the form of symmetric pairs for all* $n \geqslant 3$*.*

Remark 19 *Let* $<\zeta : \zeta^{7^n-1}=1>$ *be a cyclic group of* F_{7^n}*. Then,*

(i) *the fixed points of* $\bar{\mu}$ *are* $\zeta^{(7^n-1)/4}$ *and* $\zeta^{3(7^n-1)/4,}$

(ii) *the fixed points of* $\bar{\nu}$ *are* $\zeta^{(7^n-1)/6}$ *and* $\zeta^{5(7^n-1)/6}$*, and*

(iii) $0,1,2,3,4,5,6$ *and* ∞ *are the vertices of* γ_1*, where* $2=\zeta^{(7^n-1)/3}, 4=\zeta^{2(7^n-1)/3}$ *and* $6=\zeta^{(7^n-1)/2}$.

Lemma 20 *The conjugacy class equation of* $\bar{\mathbf{G}}$ *is*

$\left|\bar{\mathbf{G}}\right| = \left|Z\left(\bar{\mathbf{G}}\right)\right| + \sum_{r=1}^{6} h_r = 1+21+56+42+24+24$, where $Z\left(\bar{\mathbf{G}}\right)$ is the centre of $\bar{\mathbf{G}}$ and $h_r = |x_r| = \left|\bar{\mathbf{G}} : \mathbf{N}_{\bar{\mathbf{G}}}(x_r)\right|$ for any element x_r in the $x_r th-$conjugacy class and $\mathbf{N}_{\bar{\mathbf{G}}}(x_r)$ is the centralizer of an element x_r in $\bar{\mathbf{G}}$.

Proof. The group obtained by the action of $PSL(2,\mathbb{Z})$ on $PL(F_{7^n})$ is isomorphic to $PSL(2,7)$ by theorem (19). So the elements of $PSL(2,7)$ are of orders $1,\ 2,\ 3,\ 4$ and 7. Since the orbit γ_1 lies in all the coset diagram for the action of $PSL(2,\mathbb{Z})$ on $PL(F_{7^n})$, we consider that orbit which, by remark (21) and remark (22), consists of eight elements which are $0, 1, 2, 3, 4, 5, 6,$ and ∞. There are six conjugacy classes of $\bar{\mathbf{G}}$ which partitions $\bar{\mathbf{G}}$. The only element which commutes with all other elements of $\bar{\mathbf{G}}$ is the identity element only. So 167 elements are left of order 2, 3, 4 and 7. The element $(1\ 2)(3\ 4)(5\ \infty)(6\ 7)$ of order 2 forms a conjugacy class containing the following 21 elements:

$(1\ 2)(3\ 4)(5\ \infty)(6\ 7)$, $(1\ 2)(3\ 6)(4\ 5)(7\ \infty)$, $(1\ 2)(3\ \infty)(4\ 7)(5\ 6)$, $(1\ 3)(2\ 4)(5\ 6)(7\ \infty)$, $(1\ 3)(2\ 5)(4\ 7)(6\ \infty)$, $(1\ 3)(2\ \infty)(4\ 6)(5\ 7)$, $(1\ 4)(2\ 3)(5\ 7)(6\ \infty)$, $(1\ 4)(2\ 5)(3\ \infty)(6\ 7)$, $(1\ 4)(2\ 6)(3\ 7)(5\ \infty)$, $(1\ 5)(2\ 4)(3\ 7)(6\ \infty)$, $(1\ 5)(2\ 6)(3\ 4)(7\ \infty)$, $(1\ 5)(2\ 7)(3\ 6)(4\ \infty)$, $(1\ 6)(2\ 5)(3\ 7)(4\ \infty)$, $(1\ 6)(2\ 7)(3\ \infty)(4\ 5)$, $(1\ 6)(2\ \infty)(3\ 5)(4\ 7)$, $(1\ 7)(2\ 3)(4\ 6)(5\ \infty)$, $(1\ 7)(2\ 6)(3\ 5)(4\ \infty)$, $(1\ 7)(2\ \infty)(3\ 4)(5\ 6)$, $(1\ \infty)(2\ 3)(4\ 5)(6\ 7)$, $(1\ \infty)(2\ 4)(3\ 6)(5\ 7)$, $(1\ \infty)(2\ 7)(3\ 5)(4\ 6)$.

The element $(3\ 5\ 7)(4\ 6\ \infty)$ of order three forms the conjugacy class containing the following 56 elements: $(3\ 5\ 7)(4\ 6\ \infty)$, $(3\ 7\ 5)(4\ \infty\ 6)$, $(2\ 3\ 4)(5\ \infty\ 7)$, $(2\ 3\ \infty)(4\ 6\ 7)$, $(2\ 4\ 3)(5\ 7\ \infty)$, $(2\ 4\ 5)(3\ \infty\ 6)$, $(2\ 5\ 4)(3\ 6\ \infty)$, $(2\ 5\ 6)(3\ 7\ 4)$, $(2\ 6\ 7)(4\ \infty\ 5)$, $(\ 6\ 5)(3\ 4\ 7)$, $(2\ 7\ 6)(4\ 5\ \infty)$, $(2\ 7\ \infty)(3\ 6\ 5)$, $(2\ \infty\ 3)(4\ 7\ 6)$, $(2\ \infty\ 7)(3\ 5\ 6)$, $(1\ 2\ 3)(5\ 6\ 7)$, $(1\ 2\ 4)(6\ 7\ \infty)$, $(1\ 2\ 5)(3\ 7\ \infty)$, $(1\ 2\ 6)(3\ 4\ \infty)$, $(1\ 2\ 7)(3\ 4\ 5)$, $(1\ 2\ \infty)(4\ 5\ 6)$, $(1\ 3\ 2)(5\ 7\ 6)$, $(1\ 3\ 4)(5\ \infty\ 6)$, $(1\ 3\ \infty)(4\ 5\ 7)$, $(1\ 3\ 6)(2\ 7\ 4)$, $(1\ 3\ 5)(2\ \infty\ 4)$, $(1\ 3\ 7)(2\ \infty\ 6)$, $(1\ 4\ 2)(6\ \infty\ 7)$, $(1\ 4\ 3)(5\ 6\ \infty)$, $(1\ 4\ 5)(3\ 7\ 6)$, $(1\ 4\ 6)(2\ 3\ 5)$, $(1\ 4\ \infty)(2\ 3\ 7)$, $(1\ 4\ 7)(2\ \infty\ 5)$, $(1\ 5\ 2)(3\ \infty\ 7)$, $(1\ 5\ 6)(4\ \infty\ 7)$, $(1\ 5\ 4)(3\ 6\ 7)$, $(1\ 5\ \infty)(2\ 3\ 6)$, $(1\ 5\ 3)(2\ 4\ \infty)$, $(1\ 5\ 7)(2\ 4\ 6)$, $(1\ 6\ 2)(3\ \infty\ 4)$, $(1\ 6\ 5)(4\ 7\ \infty)$, $(1\ 6\ 7)(3\ \infty\ 5)$, $(1\ 6\ 3)(2\ 4\ 7)$, $(1\ 6\ 4)(2\ 5\ 3)$, $(1\ 6\ \infty)(2\ 5\ 7)$, $(1\ 7\ 2)(3\ 5\ 4)$, $(1\ 7\ 6)(3\ 5\ \infty)$, $(1\ 7\ \infty)(3\ 6\ 4)$, $(1\ 7\ 4)(2\ 5\ \infty)$, $(1\ 7\ 3)(2\ 6\ \infty)$, $(1\ 7\ 5)(2\ 6\ 4)$, $(1\ \infty\ 2)(4\ 6\ 5)$, $(1\ \infty\ 3)(4\ 7\ 5)$, $(1\ \infty\ 7)(3\ 4\ 6)$,

$(1\ \infty\ 5)(2\ 6\ 3),\ (1\ \infty\ 4)(2\ 7\ 3), (1\ \infty\ 6)(2\ 7\ 5)$.

The class of element $(1\ 2\ 3\ 5)(4\ \infty\ 7\ 6)$, of order 4, consists of the following 42 elements: $(1\ 2\ 3\ 5)(4\ \infty\ 7\ 6),\ (1\ 2\ 4\ 6)(3\ \infty\ 7\ 5),\ (1\ 2\ 5\ 7)(3\ \infty\ 6\ 4), (1\ 2\ 6\ \infty)(3\ 7\ 5\ 4)$,

$(1\ 2\ 7\ 3)(4\ \infty\ 6\ 5),\ (1\ 2\ \infty\ 4)(3\ 7\ 6\ 5), (1\ 3\ 7\ 2)(4\ 5\ 6\ \infty),\ (1\ 3\ 6\ 7)(2\ 4\ 5\ \infty)$,

$(1\ 3\ \infty\ 5)(2\ 4\ 7\ 6), (1\ 3\ 2\ 6)(4\ 7\ 5\ \infty),\ (1\ 3\ 4\ \infty)(2\ 7\ 5\ 6),\ (1\ 3\ 5\ 4)(2\ 7\ 6\ \infty)$,

$(1\ 4\ \infty\ 2)(3\ 5\ 6\ 7), (1\ 4\ 7\ \infty)(2\ 5\ 6\ 3),\ (1\ 4\ 3\ 6)(2\ 5\ \infty\ 7), (1\ 4\ 2\ 7)(3\ 5\ \infty\ 6)$,

$(1\ 4\ 5\ 3)(2\ \infty\ 6\ 7),\ (1\ 4\ 6\ 5)(2\ \infty\ 7\ 3),\ (1\ 5\ 3\ 2)(4\ 6\ 7\ \infty),\ (1\ 5\ 6\ 4)(2\ 3\ 7\ \infty)$,

$(1\ 5\ 7\ 6)(2\ 3\ \infty\ 4), (1\ 5\ \infty\ 3)(2\ 6\ 7\ 4),\ (1\ 5\ 4\ 7)(2\ 6\ 3\ \infty),\ (1\ 5\ 2\ \infty)(3\ 7\ 4\ 6)$,

$(1\ 6\ 4\ 2)(3\ 5\ 7\ \infty),\ (1\ 6\ 2\ 3)(4\ \infty\ 5\ 7),\ (1\ 6\ 7\ 5)(2\ 4\ \infty\ 3),\ (1\ 6\ \infty\ 7)(2\ 4\ 3\ 5)$,

$(1\ 6\ 5\ \infty)(2\ 7\ 4\ 3),\ (1\ 6\ 3\ 4)(2\ 7\ \infty\ 5), (1\ 7\ 5\ 2)(3\ 4\ 6\ \infty),\ (1\ 7\ 2\ 4)(3\ 6\ \infty\ 5)$,

$(1\ 7\ \infty\ 6)(2\ 5\ 3\ 4),\ (1\ 7\ 3\ \infty)(2\ 5\ 4\ 6),\ (1\ 7\ 6\ 3)(2\ \infty\ 5\ 4),\ (1\ 7\ 4\ 5)(2\ \infty\ 3\ 6)$,

$(1\ \infty\ 6\ 2)(3\ 4\ 5\ 7),\ (1\ \infty\ 5\ 6)(2\ 3\ 4\ 7),\ (1\ \infty\ 7\ 4)(2\ 3\ 6\ 5), (1\ \infty\ 2\ 5)(3\ 6\ 4\ 7)$,

$(1\ \infty\ 4\ 3)(2\ 6\ 5\ 7),\ (1\ \infty\ 3\ 7)(2\ 6\ 4\ 5)$.

There are two conjugacy classes of order 7, each containing 24 elements. The class for element $(2\ 3\ 5\ 4\ 7\ \infty\ 6)$, contains the following 24 elements:

$(2\ 5\ 7\ 6\ 3\ 4\ \infty),\ (2\ 7\ 3\ \infty\ 5\ 6\ 4),\ (1\ 2\ 3\ 6\ \infty\ 4\ 7), (1\ 2\ 5\ \infty\ 4\ 6\ 3),\ (1\ 2\ 7\ 4\ 6\ \infty\ 5)$,

$(1\ 3\ 4\ 6\ 7\ 2\ 5),\ (1\ 3\ 5\ 2\ 6\ 7\ \infty), (1\ 3\ \infty\ 7\ 2\ 6\ 4), (1\ 4\ 7\ 5\ 3\ 6\ 2),\ (1\ 4\ \infty\ 3\ 6\ 5\ 7)$,

$(1\ 4\ 2\ 6\ 5\ 3\ \infty),\ (1\ 5\ 6\ \infty\ 3\ 2\ 7),\ (1\ 5\ 4\ 3\ 2\ \infty\ 6), (1\ 5\ 7\ 2\ \infty\ 3\ 4),\ (1\ 6\ 3\ 7\ 5\ \infty\ 2)$,

$(1\ 6\ 4\ 5\ \infty\ 7\ 3),\ (1\ 6\ 2\ \infty\ 7\ 5\ 4),\ (1\ 7\ \infty\ 4\ 5\ 2\ 3), (1\ 7\ 3\ 2\ 4\ 5\ 6), (1\ 7\ 6\ 5\ 2\ 4\ \infty)$,

$(1\ \infty\ 5\ 3\ 7\ 4\ 2),\ (1\ \infty\ 6\ 7\ 4\ 3\ 5),\ (1\ \infty\ 2\ 4\ 3\ 7\ 6),\ (2\ 3\ 5\ 4\ 7\ \infty\ 6)$.

The class of order 7 for the element $(2\ 4\ 6\ 5\ \infty\ 3\ 7)$, also consists of following 24 elements: $(2\ 4\ 6\ 5\ \infty\ 3\ 7), (2\ 6\ \infty\ 7\ 4\ 5\ 3),\ (2\ \infty\ 4\ 3\ 6\ 7\ 5),\ (1\ 2\ 4\ 7\ 3\ 5\ \infty)$,

$(1\ 2\ 6\ 3\ 5\ 7\ 4), (1\ 2\ \infty\ 5\ 7\ 3\ 6), (1\ 3\ 6\ 4\ \infty\ 5\ 2),\ (1\ 3\ 7\ \infty\ 5\ 4\ 6), (1\ 3\ 2\ 5\ 4\ \infty\ 7),$

$(1\ 4\ 5\ 7\ \infty\ 2\ 6), (1\ 4\ 6\ 2\ 7\ \infty\ 3), (1\ 4\ 3\ \infty\ 2\ 7\ 5), (1\ 5\ \infty\ 6\ 4\ 7\ 2),\ (1\ 5\ 3\ 4\ 7\ 6\ \infty),$

$(1\ 5\ 2\ 7\ 6\ 4\ 3), (1\ 6\ \infty\ 2\ 3\ 4\ 5), (1\ 6\ 5\ 4\ 2\ 3\ 7),\ (1\ 6\ 7\ 3\ 4\ 2\ \infty),\ (1\ 7\ 4\ \infty\ 6\ 3\ 2),$

$(1\ 7\ 5\ 6\ 3\ \infty\ 4), (1\ 7\ 2\ 3\ \infty\ 6\ 5), (1\ \infty\ 3\ 5\ 6\ 2\ 4),\ (1\ \infty\ 7\ 6\ 2\ 5\ 3),\ (1\ \infty\ 4\ 2\ 5\ 6\ 7).$

■

Theorem 21 *If* $\mathbf{G}$ *acts on* $PL(F_{7^n})$, *then*

(*i*) $\left|Orb_{PL(\mathbf{F}_{7^n})}\left(\bar{\mathbf{G}}\right)\right| = 1 + \frac{(7^n+1)-8}{168}$ *if* n *is odd,*

(*ii*) $\left|Orb_{PL(\mathbf{F}_{7^n})}\left(\bar{\mathbf{G}}\right)\right| = 2 + \frac{(7^n+1)-50}{168}$ *if* n *is even.*

Proof. By Remark 21, when n is odd, then the orbit γ_1 composed of 8 vertices exists for all n. So $(7^n+1)-8$ elements of $PL(F_{7^n})$ are left. By Theorem 19, $\bar{\mathbf{G}}$ is isomorphic to $PSL(2,7)$ containing elements of orders $2,3,4,7$ and the identity element. Theorem 20 shows that for odd n, there is no fixed point of $\bar{\mu}$. So there are 21 elements of order 2 which do not fix any element of $PSL(2,7)$. Also, when n is odd, $7^n \equiv 3 \pmod 4$ and so there are 42 elements of order 4 which do not fix any element. By Theorem 19, fixed points of $\bar{\nu}$ exist for all n. Therefore there are 56 elements of order 3 fixing 2 elements. Moreover, $7^n+1 \equiv 1 \pmod 7$ so there are $24+24=48$ elements of order 7 which fix one element because $PSL(2,7)$ contains two conjugacy classes of order 7. All (7^n+1) elements of $PL(F_{7^n})$ are fixed by the identity element. By Frobenius-Burnside lemma [41], the number of orbits including γ_1 are

$$\begin{aligned}\left|Orb_{PL(\mathbf{F}_{7^n})}\left(\bar{\mathbf{G}}\right)\right| &= \frac{1}{\left|\bar{\mathbf{G}}\right|}\left(\sum_{g\in\bar{\mathbf{G}}}\left|Fix_{PL(\mathbf{F}_{7^n})}(g)\right|\right)\\ &= \frac{1}{168}\left(21\times 0+42\times 0+56\times 2+48\times 1+1\times(7^n+1)\right)\\ &= 1+\frac{(7^n+1)-8}{168}.\end{aligned}$$

By Remark 21, when n is even, γ_1 containing 8 and γ_2 containing 42 vertices, are two orbits. Only when n is even, γ_2 exists in coset diagram. So $(7^n+1)-50$ elements of $PL(F_{7^n})$ are left. By Theorem 20 when n is even, fixed points of $\bar{\mu}$ exist so 21 elements of order 2 fix two elements. When n is even we have $7^n \equiv 1 \pmod 4$. Therefore 42 elements of order 4 fix 2 elements. Fixed points of $\bar{\nu}$ exist for all n so 56 elements of order 3 fix two elements. In addition $7^n+1 \equiv 1 \pmod 7$, so 48 elements of order 7 are fixing one element and all (7^n+1) elements are fixed by the identity element. By Frobenius-Burnside lemma [41], the number of orbits including γ_1 are

$$\left|Orb_{PL(\mathbf{F}_{7^n})}\left(\bar{\mathbf{G}}\right)\right| = \frac{1}{|\bar{\mathbf{G}}|}\left(\sum_{g\in\bar{\mathbf{G}}}\left|Fix_{PL(\mathbf{F}_{7^n})}(g)\right|\right)$$

$$= \frac{1}{168}\left(21\times 2+42\times 2+56\times 2+48\times 1+1\times(7^n+1)\right)$$

$$= 2+\frac{(7^n+1)-50}{168} \quad \blacksquare$$

Now, we have the following corollary.

Corollary 22 *The action of* $\mathbf{G}$ *on* $PL(F_{7^n})$ *is intransitive.*

Remark 23 (1) *If n is odd, we have* $1+\frac{(7^n+1)-8}{168}$ *number of orbits, including one orbit* γ_1 *containing* 8 *vertices. leftover elements are evenly divided into* $\frac{(7^n+1)-8}{168}$ *number of orbits. All of these orbits are copies of* γ_3 *consisting of* 168 *vertices.*

(2) *If n is even, we have* $2+\frac{(7^n+1)-50}{168}$ *number of orbits. One of these orbits is* γ_1 *containing* 8 *vertices and the other is* γ_2 *containing* 42 *vertices. Remaining elements are evenly divided into* $\frac{(7^n+1)-8}{168}$ *number of orbits. These* $\frac{(7^n+1)-8}{168}$ *orbits are copies of* γ_3 *containing* 168 *vertices.*

Theorem 24 $p^n = \{\ p+l.p\,(p-1)+s.\frac{p(p^2-1)}{2}\ \}$ *for any prime p.*

Proof. For $n = 1$, $p^n = p$ and for $n = 2$, we have $p^2 = p + p(p-1)$. Suppose for $n = k$, it is true, that is $p^k = \{p + l.p(p-1) + s.\frac{p(p^2-1)}{2}\}$, where $l = 0$ if $n\ is\ odd$, $l = 1\ if\ n\ is\ even$ and $s = 0$ for $n < 3$.

Next for $n = k+1$, consider $p^k.p = \{\ p + l.p(p-1) + s.\frac{p(p^2-1)}{2}\ \}.p$. Then $p^{k+1} = \{\ p^2 + l.p^2(p-1) + s.\frac{p^2(p^2-1)}{2}\ \} = p + p(p-1) + l.p(p-1) - l.p\left(p^2-1\right) + s.\frac{p^2(p^2-1)}{2} = p + (1+l)\{p(p-1)\} + (sp-2l)\frac{p(p^2-1)}{2} = p + l'.p(p-1) + s'\left(\frac{p(p^2-1)}{2}\right)$.

Therefore it is true for $n = k+1$. Hence it is true for all values of n. ■

5.3 Conclusion

The group $PSL(2,7)$ is an important group of order 168 and has many applications in carbon chemistry. It is useful to understand and analyze the structure of graphite and fullerenes having surface of negative curvature due to its link with polymeric carbon allotropes having unusually low density. We analyzed that the coset diagrams for the action of $PGL(2,\mathbb{Z})$ or $PSL(2,\mathbb{Z})$ on $PL(F_{7^n})$, are a diagrammatic view of $D168$ Schwarzite. The total number of orbits that exist in coset diagram are $1 + \frac{(7^n+1)-8}{168}$ if n is odd and $2 + \frac{(7^n+1)-50}{168}$ if n is even. The orbits of the coset diagram are closely related to the structure of $D168$ Schwarzite. The transitive action of $\mathbf{G}$ on a set of 7 elements for $n = 1$ gives us an orbit γ_1 having 8 vertices and also has octahedral "O" symmetry. It is ^7O *heptakisoctahedral* group [6]. For $n = 2$, $\mathbf{G}$ acts on $PL(F_{49})$ intransitively obtaining two orbits γ_1 and γ_2 containing 8 and 42 elements respectively and representing *heptakisoctahedral* group. When $\mathbf{G}$ acts on $PL(F_{7^n})$ for $n \geqslant 3$, we obtain orbits γ_1, γ_2 and copies of γ_3. The orbit γ_3 and $D168$ Shwarzite are topologically same as both have genus 3. The total number of carbon atoms in $D168$ schwarzite structure and the order of permutational group obtained are also same.

Chapter 6

PSL$(2, 11)$ and C_{60} graph

In this chapter we investigate the actions of the modular group $PSL(2, \mathbb{Z})$ on the projective line over finite fields $PL(F_{11^m})$ for different values of m, where $m \in \mathbb{N}$ and draw coset diagrams for various orbits and prove some interesting results regarding the number of orbits that occur.

6.1 Introduction

To determine the properties of molecules the use of point groups, is a well known technique and there is a immense literature on this subject. For a particular non-trivial molecule the group involved is the molecule symmetry group, which upto conjugacy can be regarded as a finite subgroup of $O(3)$.

The groups $PSL(2, 5)$, $PSL(2, 7)$ and $PSL(2, 11)$ form a special subset of $PSL(2, p)$ These groups have a particular permutational structure, so in three dimensional $(3D)$ space they are viewed as multiples of the symmetry groups of the regular polyhedra. These groups are also called the pollakispolyhedral groups [19]. $PSL(2,5)$ correspond to the pentakistetrahedral, ${}^5T$, and and $PSL(2,7)$ correspond to heptakisoctahedral group, ${}^7O$ and have applications in physics and chemistry. $PSL(2,5)$ is the rotation group of the icosahedron and fullerene C_{60}. $PSL(2,7)$ is the rotational symmetry group of $D168$ schwarzite an allotrope of carbon. The third group $PSL(2,11)$ forms the undecakisicosahedral group, ${}^{11}I$. $PSL(2,11)$ is more interesting than the first two groups, also defines trivalent frameworks and so has application to other hypothetical high-genus forms of carbon. M. Deza [56] studied realization of ${}^{11}I$ as the symmetry group of a 60-vertex regular map of genus 26. The connection between the skeleton of C_{60} and this map is very important to this realization. Applications of $PSL(2,11)$ in chemistry or physics up to now are limited. In view of the inherent relations between this group and the icosahedral lattice, this powerful symmetry in near future can be predictable to look directly in the description of physical occurrences.

6.2 Action of G on $PL(F_{11^m})$

In this section we discuss the action of $\mathbf{G} = PSL(2,\mathbb{Z})$ on $PL(F_{11^m})$ where $m \in \mathbb{N}$. We make use of coset diagrams to inspect the properties of this action and the orbits of the group thus obtained.

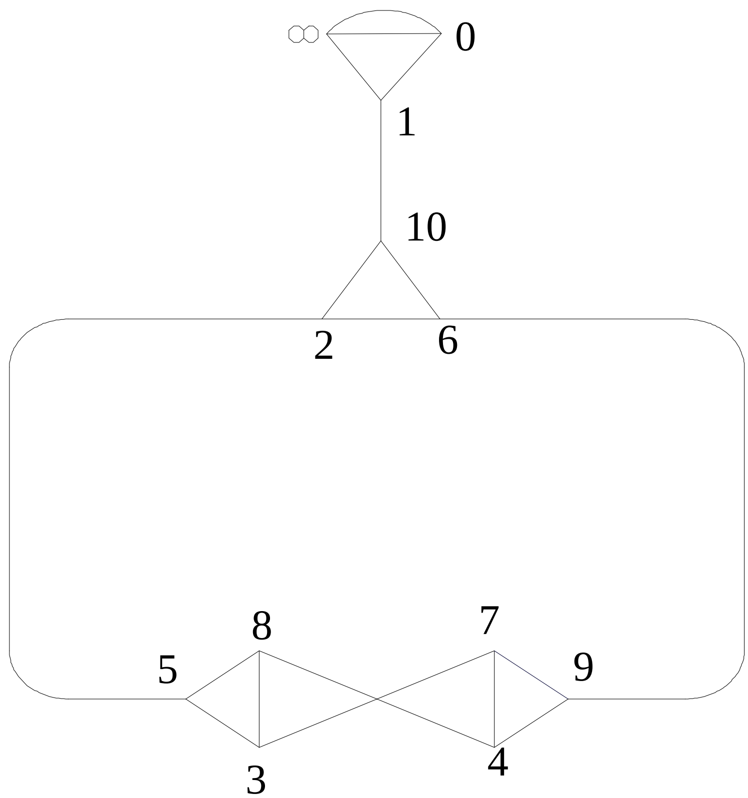
6.2.1 Action of G on $PL(F_{11})$

Consider $\bar{\mathbf{G}}_1$ is a group generated by $\bar{\mu}$ and $\bar{\nu}$, where $\bar{\mu}$ and $\bar{\nu}$ are the permutation representations of μ and ν after the action of $\mathbf{G}$ on $PL(F_{11^m})$ for $m \in \mathbb{N}$. Taking $m = 1$, we get

$\bar{\mu} = (0 \quad \infty)\,(1 \quad 10)\,(2 \quad 5)\,(3 \quad 7)\,(4 \quad 8)\,(6 \quad 9)$ and

$\bar{\nu} = (1 \quad 0 \quad \infty)\,(2 \quad 6 \quad 10)\,(3 \quad 8 \quad 5)\,(4 \quad 9 \quad 7)$. The associated coset diagram is $\pounds_1$, which is graphically represented as:

$\pounds_1$



$\pounds_1$

This diagram represents simple group of order 660 [20].

6.2.2 Action of G on $PL(F_{11^2})$

An irreducible polynomial of degree 2 in F_{11^2} is $\psi^2 + \psi + 7$. The action of **G** on $PL(F_{11^2})$ gives elements of F_{121} of the form $\psi_0 + \psi_1\wp$, where $\psi_j \in \mathbb{Z}_{11}$, for $j = 0, 1$.

Let $\wp$ be the primitive root of $GF\,(121)$ satisfying $\wp^2 = 5\wp + 4$. When **G** acts on $PL(F_{11^2})$, the permutation representation of μ and ν is:

$\bar{\mu} = (0\ \ \infty)\left(1\ \wp^{60}\right)\left(\wp^1\ \wp^{59}\right)\left(\wp^2\ \wp^{58}\right)\left(\wp^3\ \wp^{57}\right)\left(\wp^4\ \wp^{56}\right)\left(\wp^5\ \wp^{55}\right)$

$\left(\wp^6\ \wp^{54}\ \right)\left(\wp^7\ \wp^{53}\right)\left(\wp^8\ \wp^{52}\right)\left(\wp^9\ \wp^{51}\right)\left(\wp^{10}\ \wp^{50}\right)\left(\wp^{11}\ \wp^{49}\right)\left(\wp^{12}\ \wp^{48}\right)$

$\left(\wp^{13}\ \wp^{47}\right)\left(\wp^{14}\ \wp^{46}\right)\left(\wp^{15}\ \wp^{45}\right)\left(\wp^{16}\ \wp^{44}\right)\left(\wp^{17}\ \wp^{43}\right)\left(\wp^{18}\ \wp^{42}\right)\left(\wp^{19}\ \wp^{41}\right)$

$\left(\wp^{20}\ \wp^{40}\right)\left(\wp^{21}\ \wp^{39}\right)\left(\wp^{22}\ \wp^{38}\right)\left(\wp^{23}\ \wp^{37}\right)\left(\wp^{24}\ \wp^{36}\right)\left(\wp^{25}\ \wp^{35}\right)\left(\wp^{26}\ \wp^{34}\right)$

$\left(\wp^{27}\ \wp^{33}\right)\left(\wp^{28}\ \wp^{32}\right)\left(\wp^{29}\ \wp^{31}\right)\left(\wp^{30}\right)\left(\wp^{90}\right)\left(\wp^{61}\ \wp^{119}\right)\left(\wp^{62}\ \wp^{118}\right)\left(\wp^{63}\ \wp^{117}\right)$

$\left(\wp^{64}\ \wp^{116}\right)\left(\wp^{65}\ \wp^{115}\right)\left(\wp^{66}\ \wp^{114}\right)\left(\wp^{67}\ \wp^{113}\right)\left(\wp^{68}\ \wp^{112}\right)\left(\wp^{69}\ \wp^{111}\right)\left(\wp^{70}\ \wp^{110}\right)$

$\left(\wp^{71}\ \wp^{109}\right)\left(\wp^{72}\ \wp^{108}\right)\left(\wp^{73}\ \wp^{107}\right)\left(\wp^{74}\ \wp^{106}\right)\left(\wp^{75}\ \wp^{105}\right)\left(\wp^{76}\ \wp^{104}\right)\left(\wp^{77}\ \wp^{103}\right)$

$\left(\wp^{78}\ \wp^{102}\right)\left(\wp^{79}\ \wp^{101}\right)\left(\wp^{80}\ \wp^{100}\right)\left(\wp^{81}\ \wp^{99}\right)\left(\wp^{82}\ \wp^{98}\right)\left(\wp^{83}\ \wp^{97}\right)\left(\wp^{84}\ \wp^{96}\right)$

$\left(\wp^{85}\ \wp^{95}\right)\left(\wp^{86}\ \wp^{94}\right)\left(\wp^{87}\ \wp^{93}\right)\left(\wp^{88}\ \wp^{92}\right)\left(\wp^{89}\ \wp^{91}\right)$

and

$\bar{\nu} = (1\ 0\ \infty)\left(\wp^1\ \wp^{87}\ \wp^{92}\right)\left(\wp^{119}\ \wp^{28}\ \wp^{33}\right)\left(\wp^2\ \wp^{37}\ \wp^{21}\right)\left(\wp^{118}\ \wp^{99}\ \wp^{83}\right)$

$\left(\wp^3\ \wp^{109}\ \wp^{68}\right)\left(\wp^{117}\ \wp^{52}\ \wp^{11}\right)\left(\wp^4\ \wp^{101}\ \wp^{75}\right)\left(\wp^{116}\ \wp^{45}\ \wp^{19}\right)\left(\wp^5\ \wp^{78}\ \wp^{97}\right)$

$\left(\wp^{115}\ \wp^{23}\ \wp^{42}\right)\left(\wp^6\ \wp^{16}\ \wp^{38}\right)\left(\wp^7\ \wp^{39}\ \wp^{14}\right)\left(\wp^{113}\ \wp^{106}\ \wp^{81}\right)\left(\wp^8\ \wp^{35}\ \wp^{17}\right)$

$\left(\wp^{112}\ \wp^{103}\ \wp^{85}\right)\left(\wp^{36}\ \wp^{84}\ \wp^{60}\right)\left(\wp^9\ \wp^{73}\ \wp^{98}\right)\left(\wp^{111}\ \wp^{22}\ \wp^{47}\right)\left(\wp^{10}\ \wp^{91}\ \wp^{79}\right)$

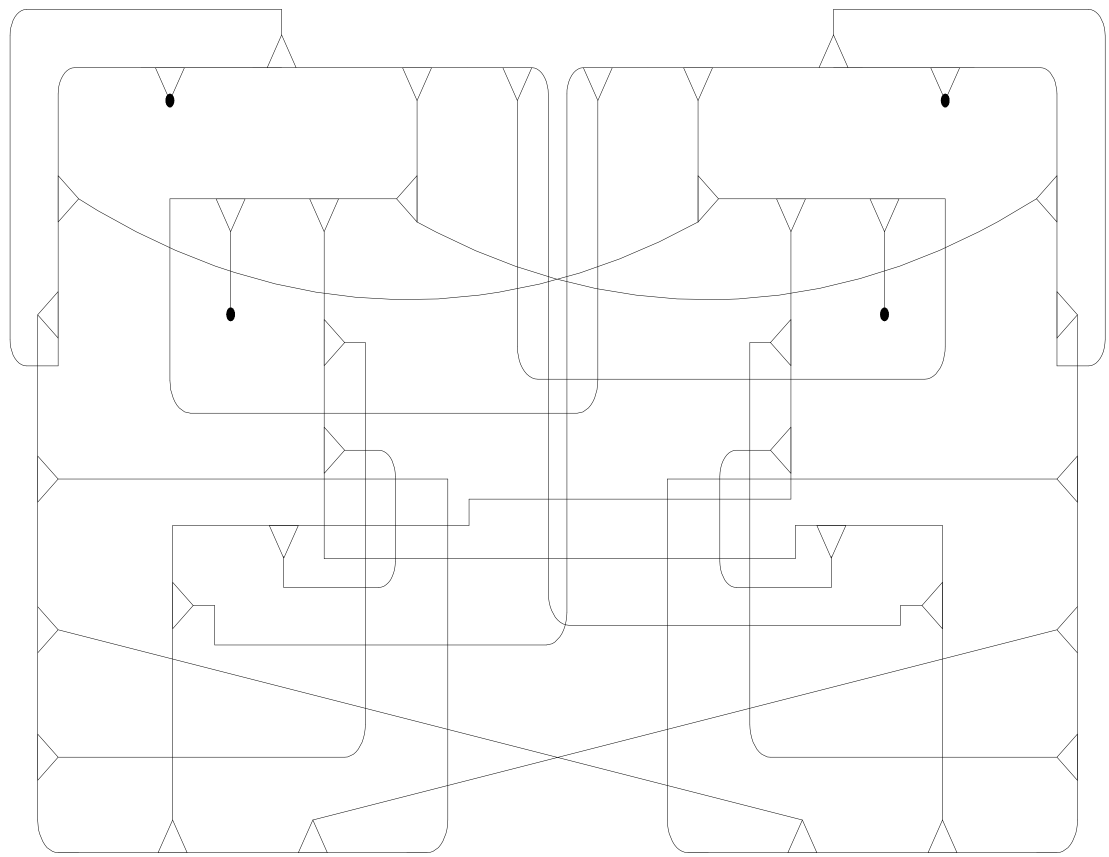
$\left(\wp^{110}\ \wp^{41}\ \wp^{29}\right)\left(\wp^{12}\ \wp^{72}\ \wp^{96}\right)\left(\wp^{108}\ \wp^{24}\ \wp^{48}\right)\left(\wp^{13}\ \wp^{102}\ \wp^{65}\right)\left(\wp^{107}\ \wp^{55}\ \wp^{18}\right)$

$\left(\wp^{15}\ \wp^{89}\ \wp^{76}\right)\left(\wp^{105}\ \wp^{44}\ \wp^{31}\right)\left(\wp^{26}\ \wp^{80}\ \wp^{74}\right)\left(\wp^{94}\ \wp^{46}\ \wp^{40}\right)\left(\wp^{27}\ \wp^{90}\ \wp^{63}\right)$

$\left(\wp^{93}\ \wp^{97}\ \wp^{30}\right)\left(\wp^{34}\ \wp^{77}\ \wp^{69}\right)\left(\wp^{86}\ \wp^{51}\ \wp^{43}\right)\left(\wp^{49}\ \wp^{61}\ \wp^{70}\right)\left(\wp^{71}\ \wp^{50}\ \wp^{59}\right)$

$\left(\wp^{53}\ \wp^{95}\ \wp^{32}\right)\left(\wp^{67}\ \wp^{88}\ \wp^{25}\right)\left(\wp^{56}\ \wp^{58}\ \wp^{66}\right)\left(\wp^{64}\ \wp^{54}\ \wp^{62}\right)\left(\wp^{20}\right)\left(\wp^{100}\right)$

which yield orbits $\pounds_1$ and $\pounds_2$. Graphically $\pounds_2$ can be represented as:



$\pounds_2$

This coset diagram is a representation of a group of order 660 [20] and contains two orbits $\pounds_1$ and $\pounds_2$.

6.2.3 Action of G on $PL(F_{11^3})$

let $\sigma^3 + \sigma^2 + 3$ be the irreducible polynomial in F_{11^3} for $m = 3$. The elements are of type $\sigma_0 + \sigma_1\wp + \sigma_2\wp^2$, where $\sigma_j \in \mathbb{Z}_{11}$, for $j = 0, 1, 2$. Let $\wp$ be the 1330^{th} primitive root of unity satisfying $\wp^3 = \wp + 5$ of F_{1331}. The permutation representation of μ and ν under the action of **G** on $PL(F_{11^2})$ is

$$\bar{\mu} = \bar{\mu} = (0 \ \ \infty)\left(1 \ \wp^{665}\right)\left(\wp \ \wp^{664}\right)\left(\wp^2 \ \wp^{663}\right)\left(\wp^3 \ \wp^{662}\right)\left(\wp^4 \ \wp^{661}\right)\left(\wp^5 \ \wp^{660}\right)$$

$$\left(\wp^6 \ \wp^{659}\right)\left(\wp^7 \ \wp^{658}\right)\left(\wp^8 \ \wp^{657}\right)\left(\wp^9 \ \wp^{656}\right)\left(\wp^{10} \ \wp^{655}\right)\left(\wp^{11} \ \wp^{654}\right)\left(\wp^{12} \ \wp^{653}\right)$$

$$\left(\wp^{13} \ \wp^{652}\right)\left(\wp^{14} \ \wp^{651}\right)\left(\wp^{15} \ \wp^{650}\right)\left(\wp^{16} \ \wp^{649}\right)\left(\wp^{17} \ \wp^{648}\right)\left(\wp^{18} \ \wp^{647}\right)\left(\wp^{19} \ \wp^{646}\right)$$

$$\left(\wp^{20} \ \wp^{645}\right)\left(\wp^{21} \ \wp^{644}\right)\left(\wp^{22} \ \wp^{643}\right)\left(\wp^{23} \ \wp^{642}\right)\left(\wp^{24} \ \wp^{641}\right)\left(\wp^{25} \ \wp^{640}\right)\left(\wp^{26} \ \wp^{639}\right)$$

$$\left(\wp^{27} \ \wp^{638}\right)\left(\wp^{28} \ \wp^{637}\right)\left(\wp^{29} \ \wp^{636}\right)\left(\wp^{30} \ \wp^{635}\right)\left(\wp^{30} \ \wp^{635}\right)\left(\wp^{31} \ \wp^{634}\right)\left(\wp^{32} \ \wp^{633}\right)$$

$$\left(\wp^{33} \ \wp^{632}\right)\left(\wp^{34} \ \wp^{631}\right)\left(\wp^{35} \ \wp^{630}\right)\left(\wp^{36} \ \wp^{629}\right)\left(\wp^{37} \ \wp^{628}\right)\left(\wp^{38} \ \wp^{627}\right)\left(\wp^{39} \ \wp^{626}\right)$$

$$\left(\wp^{40} \ \wp^{625}\right)\left(\wp^{41} \ \wp^{624}\right)\left(\wp^{42} \ \wp^{623}\right)\left(\wp^{43} \ \wp^{622}\right)\left(\wp^{44} \ \wp^{621}\right)\left(\wp^{45} \ \wp^{620}\right)\left(\wp^{46} \ \wp^{619}\right)$$

$$\left(\wp^{47} \ \wp^{618}\right)\left(\wp^{48} \ \wp^{617}\right)\left(\wp^{49} \ \wp^{616}\right)\left(\wp^{50} \ \wp^{615}\right)\left(\wp^{51} \ \wp^{614}\right)\left(\wp^{52} \ \wp^{613}\right)\left(\wp^{53} \ \wp^{612}\right)$$

$$\left(\wp^{53} \ \wp^{612}\right)\left(\wp^{54} \ \wp^{611}\right)\left(\wp^{55} \ \wp^{610}\right)\left(\wp^{56} \ \wp^{609}\right)\left(\wp^{57} \ \wp^{608}\right)\left(\wp^{58} \ \wp^{607}\right)\left(\wp^{59} \ \wp^{606}\right)$$

$$\left(\wp^{60} \ \wp^{605}\right)\left(\wp^{61} \ \wp^{604}\right)\left(\wp^{62} \ \wp^{603}\right)\left(\wp^{63} \ \wp^{602}\right)\left(\wp^{64} \ \wp^{601}\right)\left(\wp^{65} \ \wp^{600}\right)\left(\wp^{66} \ \wp^{599}\right)$$

$$\left(\wp^{67} \ \wp^{598}\right)\left(\wp^{68} \ \wp^{597}\right)\left(\wp^{69} \ \wp^{596}\right)\left(\wp^{70} \ \wp^{595}\right)\left(\wp^{71} \ \wp^{594}\right)\left(\wp^{72} \ \wp^{593}\right)\left(\wp^{73} \ \wp^{592}\right)$$

$$\left(\wp^{74} \ \wp^{591}\right)\left(\wp^{75} \ \wp^{590}\right)\left(\wp^{76} \ \wp^{589}\right)\left(\wp^{77} \ \wp^{588}\right)\left(\wp^{78} \ \wp^{587}\right)\left(\wp^{79} \ \wp^{586}\right)\left(\wp^{80} \ \wp^{585}\right)$$

$$\left(\wp^{81} \ \wp^{584}\right)\left(\wp^{82} \ \wp^{583}\right)\left(\wp^{83} \ \wp^{582}\right)\left(\wp^{84} \ \wp^{581}\right)\left(\wp^{85} \ \wp^{580}\right)\left(\wp^{86} \ \wp^{579}\right)\left(\wp^{87} \ \wp^{578}\right)$$

$$\left(\wp^{88} \ \wp^{577}\right)\left(\wp^{89} \ \wp^{576}\right)\left(\wp^{90} \ \wp^{575}\right)\left(\wp^{91} \ \wp^{574}\right)\left(\wp^{92} \ \wp^{573}\right)\left(\wp^{93} \ \wp^{572}\right)\left(\wp^{94} \ \wp^{571}\right)$$

$$\left(\wp^{95} \ \wp^{570}\right)\left(\wp^{96} \ \wp^{569}\right)\left(\wp^{97} \ \wp^{568}\right)\left(\wp^{98} \ \wp^{567}\right)\left(\wp^{99} \ \wp^{566}\right)\left(\wp^{100} \ \wp^{565}\right)\left(\wp^{101} \ \wp^{564}\right)$$

$$\left(\wp^{102} \ \wp^{563}\right)\left(\wp^{103} \ \wp^{562}\right)\left(\wp^{104} \ \wp^{561}\right)\left(\wp^{105} \ \wp^{560}\right)\left(\wp^{106} \ \wp^{559}\right)\left(\wp^{107} \ \wp^{558}\right)$$

$$\left(\wp^{108} \ \wp^{557}\right)\left(\wp^{109} \ \wp^{556}\right)\left(\wp^{110} \ \wp^{555}\right)\left(\wp^{111} \ \wp^{554}\right)\left(\wp^{112} \ \wp^{553}\right)\left(\wp^{113} \ \wp^{552}\right)$$

$$\left(\wp^{114} \ \wp^{551}\right)\left(\wp^{115} \ \wp^{550}\right)\left(\wp^{116} \ \wp^{549}\right)\left(\wp^{117} \ \wp^{548}\right)\left(\wp^{118} \ \wp^{547}\right)\left(\wp^{119} \ \wp^{546}\right)$$

$$\left(\wp^{120} \ \wp^{545}\right)\left(\wp^{121} \ \wp^{544}\right)\left(\wp^{122} \ \wp^{543}\right)\left(\wp^{123} \ \wp^{542}\right)\left(\wp^{124} \ \wp^{541}\right)\left(\wp^{125} \ \wp^{540}\right)$$

$$\left(\wp^{126} \ \wp^{539}\right)\left(\wp^{127} \ \wp^{538}\right)\left(\wp^{128} \ \wp^{537}\right)\left(\wp^{129} \ \wp^{536}\right)\left(\wp^{130} \ \wp^{535}\right)\left(\wp^{131} \ \wp^{534}\right)$$

$$\left(\wp^{132}\ \wp^{533}\right)\left(\wp^{133}\ \wp^{532}\right)\left(\wp^{134}\ \wp^{531}\right)\left(\wp^{135}\ \wp^{530}\right)\left(\wp^{136}\ \wp^{529}\right)\left(\wp^{137}\ \wp^{528}\right)$$

$$\left(\wp^{138}\ \wp^{527}\right)\left(\wp^{139}\ \wp^{526}\right)\left(\wp^{140}\ \wp^{525}\right)\left(\wp^{141}\ \wp^{524}\right)\left(\wp^{142}\ \wp^{523}\right)\left(\wp^{143}\ \wp^{522}\right)$$

$$\left(\wp^{144}\ \wp^{521}\right)\left(\wp^{145}\ \wp^{520}\right)\left(\wp^{146}\ \wp^{519}\right)\left(\wp^{147}\ \wp^{518}\right)\left(\wp^{148}\ \wp^{517}\right)\left(\wp^{149}\ \wp^{516}\right)$$

$$\left(\wp^{150}\ \wp^{515}\right)\left(\wp^{151}\ \wp^{514}\right)\left(\wp^{152}\ \wp^{513}\right)\left(\wp^{153}\ \wp^{512}\right)\left(\wp^{154}\ \wp^{511}\right)\left(\wp^{155}\ \wp^{510}\right)$$

$$\left(\wp^{156}\ \wp^{509}\right)\left(\wp^{157}\ \wp^{508}\right)\left(\wp^{158}\ \wp^{507}\right)\left(\wp^{159}\ \wp^{506}\right)\left(\wp^{160}\ \wp^{505}\right)\left(\wp^{161}\ \wp^{504}\right)$$

$$\left(\wp^{162}\ \wp^{503}\right)\left(\wp^{163}\ \wp^{502}\right)\left(\wp^{164}\ \wp^{501}\right)\left(\wp^{165}\ \wp^{500}\right)\left(\wp^{166}\ \wp^{499}\right)\left(\wp^{167}\ \wp^{498}\right)$$

$$\left(\wp^{168}\ \wp^{497}\right)\left(\wp^{169}\ \wp^{496}\right)\left(\wp^{170}\ \wp^{495}\right)\left(\wp^{171}\ \wp^{494}\right)\left(\wp^{172}\ \wp^{493}\right)\left(\wp^{173}\ \wp^{492}\right)$$

$$\left(\wp^{174}\ \wp^{491}\right)\left(\wp^{175}\ \wp^{490}\right)\left(\wp^{176}\ \wp^{489}\right)\left(\wp^{177}\ \wp^{488}\right)\left(\wp^{178}\ \wp^{487}\right)\left(\wp^{179}\ \wp^{486}\right)$$

$$\left(\wp^{180}\ \wp^{485}\right)\left(\wp^{181}\ \wp^{484}\right)\left(\wp^{182}\ \wp^{483}\right)\left(\wp^{183}\ \wp^{482}\right)\left(\wp^{184}\ \wp^{481}\right)\left(\wp^{185}\ \wp^{480}\right)$$

$$\left(\wp^{186}\ \wp^{479}\right)\left(\wp^{187}\ \wp^{478}\right)\left(\wp^{188}\ \wp^{477}\right)\left(\wp^{189}\ \wp^{476}\right)\left(\wp^{190}\ \wp^{475}\right)\left(\wp^{191}\ \wp^{474}\right)$$

$$\left(\wp^{192}\ \wp^{473}\right)\left(\wp^{193}\ \wp^{472}\right)\left(\wp^{194}\ \wp^{471}\right)\left(\wp^{195}\ \wp^{470}\right)\left(\wp^{196}\ \wp^{469}\right)\left(\wp^{197}\ \wp^{468}\right)$$

$$\left(\wp^{198}\ \wp^{467}\right)\left(\wp^{199}\ \wp^{466}\right)\left(\wp^{200}\ \wp^{465}\right)\left(\wp^{201}\ \wp^{464}\right)\left(\wp^{202}\ \wp^{463}\right)\left(\wp^{203}\ \wp^{462}\right)$$

$$\left(\wp^{204}\ \wp^{461}\right)\left(\wp^{205}\ \wp^{460}\right)\left(\wp^{206}\ \wp^{459}\right)\left(\wp^{207}\ \wp^{458}\right)\left(\wp^{208}\ \wp^{457}\right)\left(\wp^{209}\ \wp^{456}\right)$$

$$\left(\wp^{210}\ \wp^{455}\right)\left(\wp^{211}\ \wp^{454}\right)\left(\wp^{212}\ \wp^{453}\right)\left(\wp^{213}\ \wp^{452}\right)\left(\wp^{214}\ \wp^{451}\right)\left(\wp^{215}\ \wp^{450}\right)$$

$$\left(\wp^{216}\ \wp^{449}\right)\left(\wp^{217}\ \wp^{448}\right)\left(\wp^{218}\ \wp^{447}\right)\left(\wp^{219}\ \wp^{446}\right)\left(\wp^{220}\ \wp^{445}\right)\left(\wp^{221}\ \wp^{444}\right)$$

$$\left(\wp^{222}\ \wp^{443}\right)\left(\wp^{223}\ \wp^{442}\right)\left(\wp^{224}\ \wp^{441}\right)\left(\wp^{225}\ \wp^{440}\right)\left(\wp^{226}\ \wp^{439}\right)\left(\wp^{227}\ \wp^{438}\right)$$

$$\left(\wp^{228}\ \wp^{437}\right)\left(\wp^{229}\ \wp^{436}\right)\left(\wp^{230}\ \wp^{435}\right)\left(\wp^{231}\ \wp^{434}\right)\left(\wp^{232}\ \wp^{433}\right)\left(\wp^{233}\ \wp^{432}\right)$$

$$\left(\wp^{234}\ \wp^{431}\right)\left(\wp^{235}\ \wp^{430}\right)\left(\wp^{236}\ \wp^{429}\right)\left(\wp^{237}\ \wp^{428}\right)\left(\wp^{238}\ \wp^{427}\right)\left(\wp^{239}\ \wp^{426}\right)$$

$$\left(\wp^{240}\ \wp^{425}\right)\left(\wp^{241}\ \wp^{424}\right)\left(\wp^{242}\ \wp^{423}\right)\left(\wp^{243}\ \wp^{422}\right)\left(\wp^{244}\ \wp^{421}\right)\left(\wp^{245}\ \wp^{420}\right)$$

$$\left(\wp^{246}\ \wp^{419}\right)\left(\wp^{247}\ \wp^{418}\right)\left(\wp^{248}\ \wp^{417}\right)\left(\wp^{249}\ \wp^{416}\right)\left(\wp^{250}\ \wp^{415}\right)\left(\wp^{251}\ \wp^{414}\right)$$

$$\left(\wp^{252}\ \wp^{413}\right)\left(\wp^{253}\ \wp^{412}\right)\left(\wp^{254}\ \wp^{411}\right)\left(\wp^{255}\ \wp^{410}\right)\left(\wp^{256}\ \wp^{409}\right)\left(\wp^{257}\ \wp^{408}\right)$$

$$\left(\wp^{258}\ \wp^{407}\right)\left(\wp^{259}\ \wp^{406}\right)\left(\wp^{260}\ \wp^{405}\right)\left(\wp^{261}\ \wp^{404}\right)\left(\wp^{262}\ \wp^{403}\right)\left(\wp^{263}\ \wp^{402}\right)$$

$$\left(\wp^{264}\ \wp^{401}\right)\left(\wp^{265}\ \wp^{400}\right)\left(\wp^{266}\ \wp^{399}\right)\left(\wp^{267}\ \wp^{398}\right)\left(\wp^{268}\ \wp^{397}\right)\left(\wp^{269}\ \wp^{396}\right)$$

$$\left(\wp^{270}\ \wp^{395}\right)\left(\wp^{271}\ \wp^{394}\right)\left(\wp^{272}\ \wp^{393}\right)\left(\wp^{273}\ \wp^{392}\right)\left(\wp^{274}\ \wp^{391}\right)\left(\wp^{275}\ \wp^{390}\right)$$

$$\left(\wp^{276}\ \wp^{389}\right)\left(\wp^{277}\ \wp^{388}\right)\left(\wp^{278}\ \wp^{387}\right)\left(\wp^{279}\ \wp^{386}\right)\left(\wp^{280}\ \wp^{385}\right)\left(\wp^{281}\ \wp^{384}\right)$$

$$\left(\wp^{282}\ \wp^{383}\right)\left(\wp^{283}\ \wp^{382}\right)\left(\wp^{284}\ \wp^{381}\right)\left(\wp^{285}\ \wp^{380}\right)\left(\wp^{286}\ \wp^{379}\right)\left(\wp^{287}\ \wp^{378}\right)$$

$$\left(\wp^{288}\ \wp^{377}\right)\left(\wp^{289}\ \wp^{376}\right)\left(\wp^{290}\ \wp^{375}\right)\left(\wp^{291}\ \wp^{374}\right)\left(\wp^{292}\ \wp^{373}\right)\left(\wp^{293}\ \wp^{372}\right)$$

$$\left(\wp^{294}\ \wp^{371}\right)\left(\wp^{295}\ \wp^{370}\right)\left(\wp^{296}\ \wp^{369}\right)\left(\wp^{297}\ \wp^{368}\right)\left(\wp^{298}\ \wp^{367}\right)\left(\wp^{299}\ \wp^{366}\right)$$

$$\left(\wp^{300}\ \wp^{365}\right)\left(\wp^{301}\ \wp^{364}\right)\left(\wp^{302}\ \wp^{363}\right)\left(\wp^{303}\ \wp^{362}\right)\left(\wp^{304}\ \wp^{361}\right)\left(\wp^{305}\ \wp^{360}\right)$$

$$\left(\wp^{306}\ \wp^{359}\right)\left(\wp^{307}\ \wp^{358}\right)\left(\wp^{308}\ \wp^{357}\right)\left(\wp^{309}\ \wp^{356}\right)\left(\wp^{310}\ \wp^{355}\right)\left(\wp^{311}\ \wp^{354}\right)$$

$$\left(\wp^{312}\ \wp^{353}\right)\left(\wp^{313}\ \wp^{352}\right)\left(\wp^{314}\ \wp^{351}\right)\left(\wp^{315}\ \wp^{350}\right)\left(\wp^{316}\ \wp^{349}\right)\left(\wp^{317}\ \wp^{348}\right)$$

$$\left(\wp^{318}\ \wp^{347}\right)\left(\wp^{319}\ \wp^{346}\right)\left(\wp^{320}\ \wp^{345}\right)\left(\wp^{321}\ \wp^{344}\right)\left(\wp^{322}\ \wp^{343}\right)\left(\wp^{323}\ \wp^{342}\right)$$

$$\left(\wp^{324}\ \wp^{341}\right)\left(\wp^{325}\ \wp^{340}\right)\left(\wp^{326}\ \wp^{339}\right)\left(\wp^{327}\ \wp^{338}\right)\left(\wp^{328}\ \wp^{337}\right)\left(\wp^{329}\ \wp^{336}\right)$$

$$\left(\wp^{330}\ \wp^{335}\right)\left(\wp^{331}\ \wp^{334}\right)\left(\wp^{332}\ \wp^{333}\right)\left(\wp^{666}\ \wp^{1329}\right)\left(\wp^{667}\ \wp^{1328}\right)\left(\wp^{668}\ \wp^{1327}\right)$$

$$\left(\wp^{669}\ \wp^{1326}\right)\left(\wp^{670}\ \wp^{1325}\right)\left(\wp^{671}\ \wp^{1324}\right)\left(\wp^{672}\ \wp^{1323}\right)\left(\wp^{673}\ \wp^{1332}\right)\left(\wp^{674}\ \wp^{1331}\right)$$

$$\left(\wp^{675}\ \wp^{1320}\right)\left(\wp^{676}\ \wp^{1319}\right)\left(\wp^{677}\ \wp^{1318}\right)\left(\wp^{678}\ \wp^{1317}\right)\left(\wp^{679}\ \wp^{1316}\right)\left(\wp^{680}\ \wp^{1315}\right)$$

$$\left(\wp^{681}\ \wp^{1314}\right)\left(\wp^{682}\ \wp^{1313}\right)\left(\wp^{683}\ \wp^{1312}\right)\left(\wp^{684}\ \wp^{1311}\right)\left(\wp^{685}\ \wp^{1310}\right)\left(\wp^{686}\ \wp^{1309}\right)$$

$$\left(\wp^{687}\ \wp^{1308}\right)\left(\wp^{688}\ \wp^{1307}\right)\left(\wp^{689}\ \wp^{1306}\right)\left(\wp^{690}\ \wp^{1305}\right)\left(\wp^{691}\ \wp^{1304}\right)\left(\wp^{692}\ \wp^{1303}\right)$$

$$\left(\wp^{693}\ \wp^{1302}\right)\left(\wp^{694}\ \wp^{1301}\right)\left(\wp^{695}\ \wp^{1300}\right)\left(\wp^{696}\ \wp^{1299}\right)\left(\wp^{697}\ \wp^{1298}\right)\left(\wp^{698}\ \wp^{1297}\right)$$

$$\left(\wp^{699}\ \wp^{1296}\right)\left(\wp^{700}\ \wp^{1295}\right)\left(\wp^{701}\ \wp^{1294}\right)\left(\wp^{702}\ \wp^{1293}\right)\left(\wp^{703}\ \wp^{1292}\right)\left(\wp^{704}\ \wp^{1291}\right)$$

$$\left(\wp^{705}\ \wp^{1290}\right)\left(\wp^{706}\ \wp^{1289}\right)\left(\wp^{707}\ \wp^{1288}\right)\left(\wp^{708}\ \wp^{1287}\right)\left(\wp^{709}\ \wp^{1286}\right)\left(\wp^{710}\ \wp^{1285}\right)$$

$$\left(\wp^{711}\ \wp^{1284}\right)\left(\wp^{712}\ \wp^{1283}\right)\left(\wp^{713}\ \wp^{1282}\right)\left(\wp^{714}\ \wp^{1281}\right)\left(\wp^{715}\ \wp^{1280}\right)\left(\wp^{716}\ \wp^{1279}\right)$$

$$\left(\wp^{717}\ \wp^{1278}\right)\left(\wp^{718}\ \wp^{1277}\right)\left(\wp^{719}\ \wp^{1276}\right)\left(\wp^{720}\ \wp^{1275}\right)\left(\wp^{721}\ \wp^{1274}\right)\left(\wp^{722}\ \wp^{1273}\right)$$

$$\left(\wp^{723}\ \wp^{1272}\right)\left(\wp^{724}\ \wp^{1271}\right)\left(\wp^{725}\ \wp^{1270}\right)\left(\wp^{726}\ \wp^{1269}\right)\left(\wp^{727}\ \wp^{1268}\right)\left(\wp^{728}\ \wp^{1267}\right)$$

$(\wp^{729}\ \wp^{1266})\ (\wp^{730}\ \wp^{1265})\ (\wp^{731}\ \wp^{1264})\ (\wp^{732}\ \wp^{1263})\ (\wp^{733}\ \wp^{1262})\ (\wp^{734}\ \wp^{1261})$

$(\wp^{735}\ \wp^{1260})\ (\wp^{736}\ \wp^{1259})\ (\wp^{737}\ \wp^{1258})\ (\wp^{738}\ \wp^{1257})\ (\wp^{739}\ \wp^{1256})\ (\wp^{740}\ \wp^{1255})$

$(\wp^{741}\ \wp^{1254})\ (\wp^{742}\ \wp^{1253})\ (\wp^{743}\ \wp^{1252})\ (\wp^{744}\ \wp^{1251})\ (\wp^{745}\ \wp^{1250})\ (\wp^{746}\ \wp^{1249})$

$(\wp^{747}\ \wp^{1248})\ (\wp^{748}\ \wp^{1247})\ (\wp^{749}\ \wp^{1246})\ (\wp^{750}\ \wp^{1245})\ (\wp^{751}\ \wp^{1244})\ (\wp^{752}\ \wp^{1243})$

$(\wp^{753}\ \wp^{1242})\ (\wp^{754}\ \wp^{1241})\ (\wp^{755}\ \wp^{1240})\ (\wp^{756}\ \wp^{1239})\ (\wp^{757}\ \wp^{1238})\ (\wp^{758}\ \wp^{1237})$

$(\wp^{759}\ \wp^{1236})\ (\wp^{760}\ \wp^{1235})\ (\wp^{761}\ \wp^{1234})\ (\wp^{762}\ \wp^{1233})\ (\wp^{763}\ \wp^{1232})\ (\wp^{764}\ \wp^{1231})$

$(\wp^{765}\ \wp^{1230})\ (\wp^{766}\ \wp^{1229})\ (\wp^{767}\ \wp^{1228})\ (\wp^{768}\ \wp^{1227})\ (\wp^{769}\ \wp^{1226})\ (\wp^{770}\ \wp^{1225})$

$(\wp^{771}\ \wp^{1224})\ (\wp^{772}\ \wp^{1223})\ (\wp^{773}\ \wp^{1222})\ (\wp^{774}\ \wp^{1221})\ (\wp^{775}\ \wp^{1220})\ (\wp^{776}\ \wp^{1219})$

$(\wp^{777}\ \wp^{1218})\ (\wp^{778}\ \wp^{1217})\ (\wp^{779}\ \wp^{1216})\ (\wp^{780}\ \wp^{1215})\ (\wp^{781}\ \wp^{1214})\ (\wp^{782}\ \wp^{1213})$

$(\wp^{783}\ \wp^{1212})\ (\wp^{784}\ \wp^{1211})\ (\wp^{785}\ \wp^{1210})\ (\wp^{786}\ \wp^{1209})\ (\wp^{787}\ \wp^{1208})\ (\wp^{788}\ \wp^{1207})$

$(\wp^{789}\ \wp^{1206})\ (\wp^{790}\ \wp^{1205})\ (\wp^{791}\ \wp^{1204})\ (\wp^{792}\ \wp^{1203})\ (\wp^{793}\ \wp^{1202})\ (\wp^{794}\ \wp^{1201})$

$(\wp^{795}\ \wp^{1200})\ (\wp^{796}\ \wp^{1199})\ (\wp^{797}\ \wp^{1198})\ (\wp^{798}\ \wp^{1197})\ (\wp^{799}\ \wp^{1196})\ (\wp^{800}\ \wp^{1195})$

$(\wp^{801}\ \wp^{1194})\ (\wp^{802}\ \wp^{1193})\ (\wp^{803}\ \wp^{1192})\ (\wp^{804}\ \wp^{1191})\ (\wp^{805}\ \wp^{1190})\ (\wp^{806}\ \wp^{1189})$

$(\wp^{807}\ \wp^{1188})\ (\wp^{808}\ \wp^{1187})\ (\wp^{809}\ \wp^{1186})\ (\wp^{810}\ \wp^{1185})\ (\wp^{811}\ \wp^{1184})\ (\wp^{812}\ \wp^{1183})$

$(\wp^{813}\ \wp^{1182})\ (\wp^{814}\ \wp^{1181})\ (\wp^{815}\ \wp^{1180})\ (\wp^{816}\ \wp^{1179})\ (\wp^{817}\ \wp^{1178})\ (\wp^{818}\ \wp^{1177})$

$(\wp^{819}\ \wp^{1176})\ (\wp^{820}\ \wp^{1175})\ (\wp^{821}\ \wp^{1174})\ (\wp^{822}\ \wp^{1173})\ (\wp^{823}\ \wp^{1172})\ (\wp^{824}\ \wp^{1171})$

$(\wp^{825}\ \wp^{1170})\ (\wp^{826}\ \wp^{1169})\ (\wp^{827}\ \wp^{1168})\ (\wp^{828}\ \wp^{1167})\ (\wp^{829}\ \wp^{1166})\ (\wp^{830}\ \wp^{1165})$

$(\wp^{831}\ \wp^{1164})\ (\wp^{832}\ \wp^{1163})\ (\wp^{833}\ \wp^{1162})\ (\wp^{834}\ \wp^{1161})\ (\wp^{835}\ \wp^{1160})\ (\wp^{836}\ \wp^{1159})$

$(\wp^{837}\ \wp^{1158})\ (\wp^{838}\ \wp^{1157})\ (\wp^{839}\ \wp^{1156})\ (\wp^{840}\ \wp^{1155})\ (\wp^{841}\ \wp^{1154})\ (\wp^{842}\ \wp^{1153})$

$(\wp^{843}\ \wp^{1152})\ (\wp^{844}\ \wp^{1151})\ (\wp^{845}\ \wp^{1150})\ (\wp^{846}\ \wp^{1149})\ (\wp^{847}\ \wp^{1148})\ (\wp^{848}\ \wp^{1147})$

$(\wp^{849}\ \wp^{1146})\ (\wp^{850}\ \wp^{1145})\ (\wp^{851}\ \wp^{1144})\ (\wp^{852}\ \wp^{1143})\ (\wp^{853}\ \wp^{1142})\ (\wp^{854}\ \wp^{1141})$

$(\wp^{855}\ \wp^{1140})\ (\wp^{856}\ \wp^{1139})\ (\wp^{857}\ \wp^{1138})\ (\wp^{858}\ \wp^{1137})\ (\wp^{859}\ \wp^{1136})\ (\wp^{860}\ \wp^{1135})$

$(\wp^{861}\ \wp^{1134})\ (\wp^{862}\ \wp^{1133})\ (\wp^{863}\ \wp^{1132})\ (\wp^{864}\ \wp^{1131})\ (\wp^{865}\ \wp^{1130})\ (\wp^{866}\ \wp^{1129})$

$(\wp^{867}\ \wp^{1128})\ (\wp^{868}\ \wp^{1127})\ (\wp^{869}\ \wp^{1126})\ (\wp^{870}\ \wp^{1125})\ (\wp^{871}\ \wp^{1124})\ (\wp^{872}\ \wp^{1123})$

$(\wp^{873}\ \wp^{1122})\ (\wp^{874}\ \wp^{1121})\ (\wp^{875}\ \wp^{1120})\ (\wp^{876}\ \wp^{1119})\ (\wp^{877}\ \wp^{1118})\ (\wp^{878}\ \wp^{1117})$

$(\wp^{879}\ \wp^{1116})\ (\wp^{880}\ \wp^{1115})\ (\wp^{881}\ \wp^{1114})\ (\wp^{882}\ \wp^{1113})\ (\wp^{883}\ \wp^{1112})\ (\wp^{884}\ \wp^{1111})$

$(\wp^{885}\ \wp^{1110})\ (\wp^{886}\ \wp^{1109})\ (\wp^{887}\ \wp^{1108})\ (\wp^{888}\ \wp^{1107})\ (\wp^{889}\ \wp^{1106})\ (\wp^{890}\ \wp^{1105})$

$(\wp^{891}\ \wp^{1104})\ (\wp^{892}\ \wp^{1103})\ (\wp^{893}\ \wp^{1102})\ (\wp^{894}\ \wp^{1101})\ (\wp^{895}\ \wp^{1100})\ (\wp^{896}\ \wp^{1099})$

$(\wp^{897}\ \wp^{1098})\ (\wp^{898}\ \wp^{1097})\ (\wp^{899}\ \wp^{1096})\ (\wp^{900}\ \wp^{1095})\ (\wp^{901}\ \wp^{1094})\ (\wp^{902}\ \wp^{1093})$

$(\wp^{903}\ \wp^{1092})\ (\wp^{904}\ \wp^{1091})\ (\wp^{905}\ \wp^{1090})\ (\wp^{906}\ \wp^{1089})\ (\wp^{907}\ \wp^{1088})\ (\wp^{908}\ \wp^{1087})$

$(\wp^{909}\ \wp^{1086})\ (\wp^{910}\ \wp^{1085})\ (\wp^{911}\ \wp^{1084})\ (\wp^{912}\ \wp^{1083})\ (\wp^{913}\ \wp^{1082})\ (\wp^{914}\ \wp^{1081})$

$(\wp^{915}\ \wp^{1080})\ (\wp^{916}\ \wp^{1079})\ (\wp^{917}\ \wp^{1078})\ (\wp^{918}\ \wp^{1077})\ (\wp^{919}\ \wp^{1076})\ (\wp^{920}\ \wp^{1075})$

$(\wp^{921}\ \wp^{1074})\ (\wp^{922}\ \wp^{1073})\ (\wp^{923}\ \wp^{1072})\ (\wp^{924}\ \wp^{1071})\ (\wp^{925}\ \wp^{1070})\ (\wp^{926}\ \wp^{1069})$

$(\wp^{927}\ \wp^{1068})\ (\wp^{928}\ \wp^{1067})\ (\wp^{929}\ \wp^{1066})\ (\wp^{930}\ \wp^{1065})\ (\wp^{931}\ \wp^{1064})\ (\wp^{932}\ \wp^{1063})$

$(\wp^{933}\ \wp^{1062})\ (\wp^{934}\ \wp^{1061})\ (\wp^{935}\ \wp^{1060})\ (\wp^{936}\ \wp^{1059})\ (\wp^{937}\ \wp^{1058})\ (\wp^{938}\ \wp^{1057})$

$(\wp^{939}\ \wp^{1056})\ (\wp^{940}\ \wp^{1055})\ (\wp^{941}\ \wp^{1054})\ (\wp^{942}\ \wp^{1053})\ (\wp^{943}\ \wp^{1052})\ (\wp^{944}\ \wp^{1051})$

$(\wp^{945}\ \wp^{1050})\ (\wp^{946}\ \wp^{1049})\ (\wp^{947}\ \wp^{1048})\ (\wp^{948}\ \wp^{1047})\ (\wp^{949}\ \wp^{1046})\ (\wp^{950}\ \wp^{1045})$

$(\wp^{951}\ \wp^{1044})\ (\wp^{952}\ \wp^{1043})\ (\wp^{953}\ \wp^{1042})\ (\wp^{954}\ \wp^{1041})\ (\wp^{955}\ \wp^{1040})\ (\wp^{956}\ \wp^{1039})$

$(\wp^{957}\ \wp^{1038})\ (\wp^{958}\ \wp^{1037})\ (\wp^{959}\ \wp^{1036})\ (\wp^{960}\ \wp^{1035})\ (\wp^{961}\ \wp^{1034})\ (\wp^{962}\ \wp^{1033})$

$(\wp^{963}\ \wp^{1032})\ (\wp^{964}\ \wp^{1031})\ (\wp^{965}\ \wp^{1030})\ (\wp^{966}\ \wp^{1029})\ (\wp^{967}\ \wp^{1028})\ (\wp^{968}\ \wp^{1027})$

$(\wp^{969}\ \wp^{1026})\ (\wp^{970}\ \wp^{1025})\ (\wp^{971}\ \wp^{1024})\ (\wp^{972}\ \wp^{1023})\ (\wp^{973}\ \wp^{1022})\ (\wp^{974}\ \wp^{1021})$

$(\wp^{975}\ \wp^{1020})\ (\wp^{976}\ \wp^{1019})\ (\wp^{977}\ \wp^{1018})\ (\wp^{978}\ \wp^{1017})\ (\wp^{979}\ \wp^{1016})\ (\wp^{980}\ \wp^{1015})$

$(\wp^{981}\ \wp^{1014})\ (\wp^{982}\ \wp^{1013})\ (\wp^{983}\ \wp^{1012})\ (\wp^{984}\ \wp^{1011})\ (\wp^{985}\ \wp^{1010})\ (\wp^{986}\ \wp^{1009})$

$(\wp^{987}\ \wp^{1008})\ (\wp^{988}\ \wp^{1007})\ (\wp^{989}\ \wp^{1006})\ (\wp^{990}\ \wp^{1005})\ (\wp^{991}\ \wp^{1004})\ (\wp^{992}\ \wp^{1003})$

$\left(\wp^{993}\ \wp^{1002}\right)\left(\wp^{995}\ \wp^{1000}\right)\left(\wp^{996}\ \wp^{999}\right)\left(\wp^{997}\ \wp^{998}\right)$

and

$\bar{\nu} = (1\ 0\ \infty)\left(\wp^{1}\ \wp^{42}\ \wp^{622}\right)\left(\wp^{1329}\ \wp^{708}\ \wp^{1288}\right)\left(\wp^{2}\ \wp^{172}\ \wp^{491}\right)\left(\wp^{1328}\ \wp^{839}\ \wp^{1158}\right)$

$\left(\wp^{3}\ \wp^{47}\ \wp^{615}\right)\left(\wp^{1327}\ \wp^{715}\ \wp^{1283}\right)\left(\wp^{4}\ \wp^{1208}\ \wp^{783}\right)\left(\wp^{1326}\ \wp^{547}\ \wp^{122}\right)$

$\left(\wp^{5}\ \wp^{124}\ \wp^{536}\right)\left(\wp^{1325}\ \wp^{794}\ \wp^{1206}\right)\left(\wp^{7}\ \wp^{125}\ \wp^{533}\right)\left(\wp^{1323}\ \wp^{797}\ \wp^{1205}\right)$

$\left(\wp^{8}\ \wp^{1040}\ \wp^{947}\right)\left(\wp^{1322}\ \wp^{383}\ \wp^{290}\right)\left(\wp^{9}\ \wp^{456}\ \wp^{200}\right)\left(\wp^{1321}\ \wp^{1130}\ \wp^{874}\right)$

$\left(\wp^{10}\ \wp^{895}\ \wp^{1090}\right)\left(\wp^{1320}\ \wp^{240}\ \wp^{435}\right)\left(\wp^{11}\ \wp^{462}\ \wp^{192}\right)\left(\wp^{1319}\ \wp^{1138}\ \wp^{868}\right)$

$\left(\wp^{12}\ \wp^{1286}\ \wp^{697}\right)\left(\wp^{1318}\ \wp^{633}\ \wp^{44}\right)\left(\wp^{13}\ \wp^{949}\ \wp^{1033}\right)\left(\wp^{1317}\ \wp^{297}\ \wp^{381}\right)$

$\left(\wp^{14}\ \wp^{688}\ \wp^{1293}\right)\left(\wp^{1316}\ \wp^{37}\ \wp^{642}\right)\left(\wp^{15}\ \wp^{152}\ \wp^{498}\right)\left(\wp^{1315}\ \wp^{832}\ \wp^{1178}\right)$

$\left(\wp^{16}\ \wp^{871}\ \wp^{1108}\right)\left(\wp^{1314}\ \wp^{222}\ \wp^{459}\right)\left(\wp^{17}\ \wp^{928}\ \wp^{1050}\right)\left(\wp^{1313}\ \wp^{280}\ \wp^{402}\right)$

$\left(\wp^{18}\ \wp^{807}\ \wp^{1170}\right)\left(\wp^{1312}\ \wp^{160}\ \wp^{523}\right)\left(\wp^{19}\ \wp^{502}\ \wp^{144}\right)\left(\wp^{1311}\ \wp^{1186}\ \wp^{828}\right)$

$\left(\wp^{20}\ \wp^{110}\ \wp^{535}\right)\left(\wp^{1310}\ \wp^{795}\ \wp^{1220}\right)\left(\wp^{21}\ \wp^{911}\ \wp^{1063}\right)\left(\wp^{1308}\ \wp^{1249}\ \wp^{768}\right)$

$\left(\wp^{23}\ \wp^{421}\ \wp^{221}\right)\left(\wp^{1306}\ \wp^{150}\ \wp^{539}\right)\left(\wp^{25}\ \wp^{365}\ \wp^{275}\right)\left(\wp^{1305}\ \wp^{1055}\ \wp^{965}\right)$

$\left(\wp^{26}\ \wp^{710}\ \wp^{1259}\right)\left(\wp^{1304}\ \wp^{71}\ \wp^{620}\right)\left(\wp^{27}\ \wp^{881}\ \wp^{1087}\right)\left(\wp^{1303}\ \wp^{243}\ \wp^{449}\right)$

$\left(\wp^{28}\ \wp^{738}\ \wp^{1229}\right)\left(\wp^{1302}\ \wp^{101}\ \wp^{592}\right)\left(\wp^{29}\ \wp^{171}\ \wp^{465}\right)\left(\wp^{1301}\ \wp^{865}\ \wp^{1159}\right)$

$\left(\wp^{30}\ \wp^{764}\ \wp^{1201}\right)\left(\wp^{1300}\ \wp^{129}\ \wp^{566}\right)\left(\wp^{31}\ \wp^{1156}\ \wp^{808}\right)\left(\wp^{1299}\ \wp^{522}\ \wp^{174}\right)$

$\left(\wp^{32}\ \wp^{1268}\ \wp^{695}\right)\left(\wp^{1298}\ \wp^{635}\ \wp^{62}\right)\left(\wp^{33}\ \wp^{517}\ \wp^{115}\right)\left(\wp^{1297}\ \wp^{1215}\ \wp^{813}\right)$

$\left(\wp^{34}\ \wp^{576}\ \wp^{55}\right)\left(\wp^{1296}\ \wp^{1275}\ \wp^{754}\right)\left(\wp^{35}\ \wp^{888}\ \wp^{1072}\right)\left(\wp^{1295}\ \wp^{258}\ \wp^{442}\right)$

$\left(\wp^{36}\ \wp^{944}\ \wp^{1015}\right)\left(\wp^{1294}\ \wp^{315}\ \wp^{386}\right)\left(\wp^{38}\ \wp^{743}\ \wp^{1214}\right)\left(\wp^{1292}\ \wp^{116}\ \wp^{587}\right)$

$\left(\wp^{39}\ \wp^{300}\ \wp^{326}\right)\wp^{1291}\ \wp^{1004}\ \wp^{1030}\left(\wp^{40}\ \wp^{527}\ \wp^{98}\right)\left(\wp^{1290}\ \wp^{1232}\ \wp^{803}\right)$

$\left(\wp^{41}\ \wp^{148}\ \wp^{476}\right)\left(\wp^{1289}\ \wp^{854}\ \wp^{1182}\right)\left(\wp^{43}\ \wp^{1126}\ \wp^{826}\right)\left(\wp^{1287}\ \wp^{504}\ \wp^{204}\right)$

$\left(\wp^{45}\ \wp^{543}\ \wp^{77}\right)\left(\wp^{1255}\ \wp^{1253}\ \wp^{787}\right)\left(\wp^{46}\ \wp^{384}\ \wp^{235}\right)\left(\wp^{1284}\ \wp^{1095}\ \wp^{946}\right)$

$$\left(\wp^{48}\ \wp^{538}\ \wp^{79}\right)\left(\wp^{1282}\ \wp^{1251}\ \wp^{792}\right)\left(\wp^{49}\ \wp^{1086}\ \wp^{860}\right)\left(\wp^{1281}\ \wp^{470}\ \wp^{244}\right)$$

$$\left(\wp^{50}\ \wp^{1266}\ \wp^{679}\right)\left(\wp^{1280}\ \wp^{651}\ \wp^{64}\right)\left(\wp^{51}\ \wp^{1002}\ \wp^{942}\right)\left(\wp^{1279}\ \wp^{388}\ \wp^{328}\right)$$

$$\left(\wp^{52}\ \wp^{334}\ \wp^{279}\right)\left(\wp^{1278}\ \wp^{1051}\ \wp^{996}\right)\left(\wp^{53}\ \wp^{1058}\ \wp^{884}\right)\left(\wp^{1277}\ \wp^{446}\ \wp^{272}\right)$$

$$\left(\wp^{54}\ \wp^{418}\ \wp^{193}\right)\left(\wp^{1276}\ \wp^{1132}\ \wp^{912}\right)\left(\wp^{56}\ \wp^{393}\ \wp^{216}\right)\left(\wp^{1274}\ \wp^{1114}\ \wp^{937}\right)$$

$$\left(\wp^{57}\ \wp^{428}\ \wp^{180}\right)\left(\wp^{1273}\ \wp^{1150}\ \wp^{902}\right)\left(\wp^{58}\ \wp^{444}\ \wp^{163}\right)\left(\wp^{1272}\ \wp^{1167}\ \wp^{886}\right)$$

$$\left(\wp^{59}\ \wp^{763}\ \wp^{1173}\right)\left(\wp^{1271}\ \wp^{157}\ \wp^{567}\right)\left(\wp^{60}\ \wp^{796}\ \wp^{1139}\right)\left(\wp^{1270}\ \wp^{191}\ \wp^{534}\right)$$

$$\left(\wp^{61}\ \wp^{265}\ \wp^{339}\right)\left(\wp^{1269}\ \wp^{991}\ \wp^{1065}\right)\left(\wp^{63}\ \wp^{815}\ \wp^{1117}\right)\left(\wp^{1267}\ \wp^{213}\ \wp^{515}\right)$$

$$\left(\wp^{65}\ \wp^{963}\ \wp^{967}\right)\left(\wp^{1265}\ \wp^{363}\ \wp^{367}\right)\left(\wp^{66}\ \wp^{701}\ \wp^{1228}\right)\left(\wp^{1264}\ \wp^{102}\ \wp^{629}\right)$$

$$\left(\wp^{67}\ \wp^{167}\ \wp^{431}\right)\left(\wp^{1263}\ \wp^{899}\wp^{1163}\right)\left(\wp^{68}\ \wp^{391}\ \wp^{206}\right)\left(\wp^{1262}\ \wp^{1124}\ \wp^{939}\right)$$

$$\left(\wp^{69}\ \wp^{735}\ \wp^{1191}\right)\left(\wp^{1261}\ \wp^{139}\ \wp^{595}\right)\left(\wp^{70}\ \wp^{285}\ \wp^{310}\right)\left(\wp^{1260}\ \wp^{1020}\ \wp^{1045}\right)$$

$$\left(\wp^{72}\ \wp^{842}\ \wp^{1081}\right)\left(\wp^{1258}\ \wp^{249}\ \wp^{488}\right)\left(\wp^{73}\ \wp^{480}\ \wp^{112}\right)\left(\wp^{1257}\ \wp^{1218}\ \wp^{850}\right)$$

$$\left(\wp^{74}\ \wp^{887}\ \wp^{1034}\right)\left(\wp^{1256}\ \wp^{296}\ \wp^{443}\right)\left(\wp^{75}\ \wp^{1096}\ \wp^{824}\right)\left(\wp^{1255}\ \wp^{506}\ \wp^{234}\right)$$

$$\left(\wp^{76}\ \wp^{93}\ \wp^{496}\right)\left(\wp^{1254}\ \wp^{834}\ \wp^{1237}\right)\left(\wp^{78}\ \wp^{1080}\ \wp^{837}\right)\left(\wp^{1252}\ \wp^{493}\ \wp^{250}\right)$$

$$\left(\wp^{80}\ \wp^{818}\ \wp^{1097}\right)\left(\wp^{1250}\ \wp^{233}\ \wp^{512}\right)\left(\wp^{82}\ \wp^{1083}\ \wp^{830}\right)\left(\wp^{1248}\ \wp^{500}\ \wp^{247}\right)$$

$$\left(\wp^{83}\ \wp^{905}\ \wp^{1007}\right)\left(\wp^{1247}\ \wp^{323}\ \wp^{425}\right)\left(\wp^{84}\ \wp^{1032}\ \wp^{879}\right)\left(\wp^{1246}\ \wp^{451}\ \wp^{298}\right)$$

$$\left(\wp^{85}\ \wp^{274}\ \wp^{306}\right)\left(\wp^{1245}\ \wp^{1024}\ \wp^{1056}\right)\left(\wp^{86}\ \wp^{1084}\ \wp^{825}\right)\left(\wp^{1244}\ \wp^{505}\ \wp^{246}\right)$$

$$\left(\wp^{87}\ \wp^{392}\ \wp^{186}\right)\left(\wp^{1243}\ \wp^{1144}\ \wp^{938}\right)\left(\wp^{88}\ \wp^{800}\ \wp^{1107}\right)\left(\wp^{1242}\ \wp^{223}\ \wp^{530}\right)$$

$$\left(\wp^{89}\ \wp^{906}\ \wp^{1000}\right)\left(\wp^{1241}\ \wp^{330}\ \wp^{424}\right)\left(\wp^{90}\ \wp^{472}\ \wp^{103}\right)\left(\wp^{1240}\ \wp^{1227}\ \wp^{858}\right)$$

$$\left(\wp^{91}\ \wp^{1071}\ \wp^{833}\right)\left(\wp^{1239}\ \wp^{497}\ \wp^{259}\right)\left(\wp^{92}\ \wp^{107}\ \wp^{466}\right)\left(\wp^{1238}\ \wp^{864}\ \wp^{1223}\right)$$

$$\left(\wp^{94}\ \wp^{974}\ \wp^{927}\right)\left(\wp^{1236}\ \wp^{403}\ \wp^{356}\right)\left(\wp^{95}\ \wp^{840}\ \wp^{1060}\right)\left(\wp^{1235}\ \wp^{270}\ \wp^{490}\right)$$

$$\left(\wp^{96}\ \wp^{355}\ \wp^{214}\right)\left(\wp^{1234}\ \wp^{1116}\ \wp^{975}\right)\left(\wp^{97}\ \wp^{880}\ \wp^{1018}\right)\left(\wp^{1233}\ \wp^{312}\ \wp^{450}\right)$$

$$\left(\wp^{99}\ \wp^{1026}\ \wp^{870}\right)\left(\wp^{1231}\ \wp^{460}\ \wp^{304}\right)\left(\wp^{100}\ \wp^{1219}\ \wp^{676}\right)\left(\wp^{1230}\ \wp^{654}\ \wp^{111}\right)$$

$\left(\wp^{104}\ \wp^{448}\ \wp^{113}\right)\left(\wp^{1226}\ \wp^{1217}\ \wp^{882}\right)\left(\wp^{105}\ \wp^{1131}\ \wp^{759}\right)\left(\wp^{1225}\ \wp^{571}\ \wp^{199}\right)$

$\left(\wp^{106}\ \wp^{378}\ \wp^{181}\right)\left(\wp^{1224}\ \wp^{1149}\ \wp^{952}\right)\left(\wp^{108}\ \wp^{904}\ \wp^{983}\right)\left(\wp^{1222}\ \wp^{347}\ \wp^{426}\right)$

$\left(\wp^{109}\ \wp^{786}\ \wp^{1100}\right)\left(\wp^{1221}\ \wp^{230}\ \wp^{544}\right)\left(\wp^{117}\ \wp^{1134}\ \wp^{744}\right)\left(\wp^{1213}\ \wp^{586}\ \wp^{196}\right)$

$\left(\wp^{118}\ \wp^{1025}\ \wp^{852}\right)\left(\wp^{1212}\ \wp^{478}\ \wp^{305}\right)\left(\wp^{119}\ \wp^{387}\ \wp^{159}\right)\left(\wp^{1211}\ \wp^{1171}\ \wp^{943}\right)$

$\left(\wp^{120}\ \wp^{1110}\ \wp^{765}\right)\left(\wp^{1210}\ \wp^{565}\ \wp^{220}\right)\left(\wp^{121}\ \wp^{1092}\ \wp^{782}\right)\left(\wp^{1209}\ \wp^{548}\wp^{238}\right)$

$\left(\wp^{123}\ \wp^{401}\ \wp^{141}\right)\left(\wp^{1208}\ \wp^{1189}\ \wp^{929}\right)\left(\wp^{126}\ \wp^{1003}\ \wp^{866}\right)\left(\wp^{1204}\ \wp^{464}\ \wp^{327}\right)$

$\left(\wp^{127}\ \wp^{257}\ \wp^{281}\right)\left(\wp^{1203}\ \wp^{1049}\ \wp^{1073}\right)\left(\wp^{128}\ \wp^{340}\ \wp^{197}\right)\left(\wp^{1202}\ \wp^{1133}\ \wp^{990}\right)$

$\left(\wp^{131}\ \wp^{1200}\ \wp^{664}\right)\left(\wp^{1199}\ \wp^{666}\ \wp^{130}\right)\left(\wp^{132}\ \wp^{846}\ \wp^{1017}\right)\left(\wp^{1198}\ \wp^{313}\ \wp^{484}\right)$

$\left(\wp^{133}\ \wp^{1064}\ \wp^{798}\right)\left(\wp^{1197}\ \wp^{532}\ \wp^{266}\right)\left(\wp^{134}\ \wp^{969}\ \wp^{892}\right)\left(\wp^{1196}\ \wp^{438}\ \wp^{361}\right)$

$\left(\wp^{135}\ \wp^{713}\ \wp^{1147}\right)\left(\wp^{1195}\ \wp^{183}\ \wp^{617}\right)\left(\wp^{136}\ \wp^{836}\ \wp^{1023}\right)\left(\wp^{1194}\ \wp^{307}\ \wp^{494}\right)$

$\left(\wp^{137}\ \wp^{1175}\ \wp^{683}\right)\left(\wp^{1193}\ \wp^{647}\ \wp^{155}\right)\left(\wp^{138}\ \wp^{219}\ \wp^{308}\right)\left(\wp^{1192}\ \wp^{1022}\ \wp^{1111}\right)$

$\left(\wp^{140}\ \wp^{146}\ \wp^{379}\right)\left(\wp^{1190}\ \wp^{951}\ \wp^{1184}\right)\left(\wp^{142}\ \wp^{1157}\ \wp^{696}\right)\left(\wp^{1188}\ \wp^{634}\ \wp^{173}\right)$

$\left(\wp^{143}\ \wp^{1129}\ \wp^{723}\right)\left(\wp^{1187}\ \wp^{607}\ \wp^{201}\right)\left(\wp^{145}\ \wp^{1119}\ \wp^{731}\right)\left(\wp^{1185}\ \wp^{599}\ \wp^{211}\right)$

$\left(\wp^{147}\ \wp^{189}\wp^{329}\right)\left(\wp^{1183}\ \wp^{1001}\ \wp^{1141}\right)\left(\wp^{151}\ \wp^{699}\ \wp^{1145}\right)\left(\wp^{1179}\ \wp^{185}\ \wp^{631}\right)$

$\left(\wp^{149}\ \wp^{799}\ \wp^{1047}\right)\left(\wp^{1181}\ \wp^{283}\ \wp^{531}\right)\left(\wp^{153}\ \wp^{318}\ \wp^{194}\right)\left(\wp^{1177}\ \wp^{1136}\ \wp^{1012}\right)$

$\left(\wp^{154}\ \wp^{918}\ \wp^{923}\right)\left(\wp^{1176}\ \wp^{407}\ \wp^{412}\right)\left(\wp^{156}\ \wp^{964}\ \wp^{875}\right)\left(\wp^{1174}\ \wp^{455}\ \wp^{366}\right)$

$\left(\wp^{158}\ \wp^{165}\ \wp^{342}\right)\left(\wp^{1172}\ \wp^{988}\ \wp^{1165}\right)\left(\wp^{161}\ \wp^{673}\ \wp^{1161}\right)\left(\wp^{1169}\ \wp^{169}\ \wp^{657}\right)$

$\left(\wp^{162}\ \wp^{930}\ \wp^{903}\right)\left(\wp^{1168}\ \wp^{427}\ \wp^{400}\right)\left(\wp^{164}\ \wp^{669}\ \wp^{1162}\right)\left(\wp^{1166}\ \wp^{168}\ \wp^{661}\right)$

$\left(\wp^{166}\ \wp^{1216}\ \wp^{613}\right)\left(\wp^{1164}\ \wp^{717}\ \wp^{114}\right)\left(\wp^{170}\ \wp^{1044}\ \wp^{781}\right)\left(\wp^{1160}\ \wp^{549}\ \wp^{286}\right)$

$\left(\wp^{175}\ \wp^{961}\ \wp^{859}\right)\left(\wp^{1155}\ \wp^{471}\ \wp^{369}\right)\left(\wp^{176}\ \wp^{271}\ \wp^{218}\right)\left(\wp^{1154}\ \wp^{1112}\ \wp^{1059}\right)$

$\left(\wp^{177}\ \wp^{955}\ \wp^{863}\right)\left(\wp^{1153}\ \wp^{467}\ \wp^{375}\right)\left(\wp^{178}\ \wp^{872}\ \wp^{945}\right)\left(\wp^{1152}\ \wp^{385}\ \wp^{458}\right)$

$\left(\wp^{179}\ \wp^{210}\ \wp^{276}\right)\left(\wp^{1151}\ \wp^{1054}\ \wp^{1120}\right)\left(\wp^{182}\ \wp^{293}\ \wp^{190}\right)\left(\wp^{1148}\ \wp^{1140}\ \wp^{1037}\right)$

$(\wp^{184}\ \wp^{1113}\ \wp^{698})\ (\wp^{1146}\ \wp^{632}\ \wp^{217})\ (\wp^{187}\ \wp^{898}\ \wp^{910})\ (\wp^{1143}\ \wp^{420}\ \wp^{432})$

$(\wp^{188}\ \wp^{1079}\ \wp^{728})\ (\wp^{1142}\ \wp^{602}\ \wp^{251})\ (\wp^{198}\ \wp^{897}\ \wp^{900})\ (\wp^{1132}\ \wp^{430}\ \wp^{433})$

$(\wp^{195}\ \wp^{827}\ \wp^{973})\ (\wp^{1135}\ \wp^{357}\ \wp^{503})\ (\wp^{202}\ \wp^{254}\ \wp^{209})\ (\wp^{1128}\ \wp^{1121}\ \wp^{1076})$

$(\wp^{203}\ \wp^{982}\ \wp^{810})\ (\wp^{1127}\ \wp^{520}\ \wp^{348})\ (\wp^{205}\ \wp^{779}\ \wp^{1011})\ (\wp^{1125}\ \wp^{319}\ \wp^{551})$

$(\wp^{207}\ \wp^{968}\ \wp^{820})\ (\wp^{1123}\ \wp^{510}\ \wp^{362})\ (\wp^{212}\ \wp^{932}\ \wp^{851})\ (\wp^{1118}\ \wp^{479}\ \wp^{398})$

$(\wp^{215}\ \wp^{745}\ \wp^{1035})\ (\wp^{1115}\ \wp^{295}\ \wp^{585})\ (\wp^{224}\ \wp^{914}\ \wp^{857})\ (\wp^{1106}\ \wp^{473}\ \wp^{416})$

$(\wp^{225}\ \wp^{981}\ \wp^{789})\ (\wp^{1105}\ \wp^{541}\ \wp^{349})\ (\wp^{226}\ \wp^{678}\ \wp^{1091})\ (\wp^{1104}\ \wp^{239}\ \wp^{652})$

$(\wp^{227}\ \wp^{806}\ \wp^{962})\ (\wp^{1103}\ \wp^{368}\ \wp^{524})\ (\wp^{228}\ \wp^{845}\ \wp^{922})\ (\wp^{1102}\ \wp^{408}\ \wp^{485})$

$(\wp^{229}\ \wp^{689}\ \wp^{1077})\ (\wp^{1101}\ \wp^{253}\ \wp^{641})\ (\wp^{231}\ \wp^{711}\ \wp^{1053})\ (\wp^{1099}\ \wp^{277}\ \wp^{619})$

$(\wp^{232}\ \wp^{757}\ \wp^{1006})\ (\wp^{1098}\ \wp^{324}\ \wp^{573})\ (\wp^{236}\ \wp^{1029}\ \wp^{730})\ (\wp^{1094}\ \wp^{600}\ \wp^{301})$

$(\wp^{237}\ \wp^{766}\ \wp^{992})\ (\wp^{1093}\ \wp^{338}\ \wp^{564})\ (\wp^{241}\ \wp^{1070}\ \wp^{684})\ (\wp^{1089}\ \wp^{646}\ \wp^{260})$

$(\wp^{242}\ \wp^{862}\ \wp^{891})\ (\wp^{1088}\ \wp^{439}\ \wp^{468})\ (\wp^{245}\ \wp^{1048}\ \wp^{702})\ (\wp^{1085}\ \wp^{628}\ \wp^{282})$

$(\wp^{252}\ \wp^{853}\ \wp^{890})\ (\wp^{1078}\ \wp^{440}\ \wp^{477})\ (\wp^{248}\ \wp^{761}\ \wp^{986})\ (\wp^{1082}\ \wp^{344}\ \wp^{569})$

$(\wp^{255}\ \wp^{1069}\ \wp^{671})\ (\wp^{1075}\ \wp^{659}\ \wp^{261})\ (\wp^{256}\ \wp^{934}\ \wp^{805})\ (\wp^{1074}\ \wp^{525}\ \wp^{396})$

$(\wp^{262}\ \wp^{1009}\ \wp^{724})\ (\wp^{1068}\ \wp^{606}\ \wp^{321})\ (\wp^{263}\ \wp^{772}\ \wp^{960})\ (\wp^{1067}\ \wp^{370}\ \wp^{558})$

$(\wp^{264}\ \wp^{721}\ \wp^{1010})\ (\wp^{1066}\ \wp^{320}\ \wp^{609})\ (\wp^{268}\ \wp^{686}\ \wp^{1041})\ (\wp^{1062}\ \wp^{289}\ \wp^{644})$

$(\wp^{273}\ \wp^{1028}\ \wp^{694})\ (\wp^{1057}\ \wp^{636}\ \wp^{302})\ (\wp^{1052}\ \wp^{561}\ \wp^{382})\ (\wp^{278}\ \wp^{948}\ \wp^{769})$

$(\wp^{284}\ \wp^{997}\ \wp^{714})\ (\wp^{1046}\ \wp^{616}\ \wp^{333})\ (\wp^{287}\ \wp^{749}\ \wp^{959})\ (\wp^{1043}\ \wp^{371}\ \wp^{581})$

$(\wp^{288}\ \wp^{896}\ \wp^{811})\ (\wp^{1042}\ \wp^{519}\ \wp^{434})\ (\wp^{291}\ \wp^{999}\ \wp^{705})\ (\wp^{1039}\ \wp^{625}\ \wp^{331})$

$(\wp^{292}\ \wp^{667}\ \wp^{1036})\ (\wp^{1038}\ \wp^{294}\ \wp^{663})\ (\wp^{303}\ \wp^{831}\ \wp^{861})\ (\wp^{1027}\ \wp^{469}\ \wp^{499})$

$(\wp^{309}\ \wp^{809}\ \wp^{877})\ (\wp^{1021}\ \wp^{453}\ \wp^{521})\ (\wp^{311}\ \wp^{936}\ \wp^{748})\ (\wp^{1019}\ \wp^{582}\ \wp^{394})$

$(\wp^{314}\ \wp^{956}\ \wp^{725})\ (\wp^{1016}\ \wp^{605}\ \wp^{374})\ (\wp^{316}\ \wp^{758}\ \wp^{921})\ (\wp^{1014}\ \wp^{409}\ \wp^{572})$

$\left(\wp^{317}\ \wp^{693}\ \wp^{985}\right)\left(\wp^{1013}\ \wp^{345}\ \wp^{637}\right)\left(\wp^{322}\ \wp^{716}\ \wp^{957}\right)\left(\wp^{1008}\ \wp^{373}\ \wp^{614}\right)$

$\left(\wp^{325}\ \wp^{755}\ \wp^{915}\right)\left(\wp^{1005}\ \wp^{415}\ \wp^{575}\right)\left(\wp^{332}\ \wp^{747}\ \wp^{916}\right)\left(\wp^{998}\ \wp^{414}\ \wp^{583}\right)$

$\left(\wp^{335}\ \wp^{682}\ \wp^{978}\right)\left(\wp^{995}\ \wp^{352}\ \wp^{648}\right)\left(\wp^{336}\ \wp^{752}\ \wp^{907}\right)\left(\wp^{994}\ \wp^{423}\ \wp^{578}\right)$

$\left(\wp^{337}\ \wp^{739}\ \wp^{919}\right)\left(\wp^{993}\ \wp^{411}\ \wp^{591}\right)\left(\wp^{341}\ \wp^{746}\ \wp^{908}\right)\left(\wp^{989}\ \wp^{422}\ \wp^{584}\right)$

$\left(\wp^{343}\ \wp^{668}\ \wp^{984}\right)\left(\wp^{987}\ \wp^{346}\ \wp^{662}\right)\left(\wp^{353}\ \wp^{838}\ \wp^{804}\right)\left(\wp^{977}\ \wp^{526}\ \wp^{492}\right)$

$\left(\wp^{351}\ \wp^{970}\ \wp^{674}\right)\left(\wp^{979}\ \wp^{656}\ \wp^{360}\right)\left(\wp^{354}\ \wp^{706}\ \wp^{935}\right)\left(\wp^{976}\ \wp^{395}\ \wp^{624}\right)$

$\left(\wp^{358}\ \wp^{821}\ \wp^{816}\right)\left(\wp^{972}\ \wp^{514}\ \wp^{509}\right)\left(\wp^{359}\ \wp^{924}\ \wp^{712}\right)\left(\wp^{971}\ \wp^{618}\ \wp^{406}\right)$

$\left(\wp^{364}\ \wp^{788}\ \wp^{843}\right)\left(\wp^{966}\ \wp^{487}\ \wp^{542}\right)\left(\wp^{376}\ \wp^{639}\ \wp^{980}\right)\left(\wp^{954}\ \wp^{350}\ \wp^{691}\right)$

$\left(\wp^{377}\ \wp^{777}\ \wp^{841}\right)\left(\wp^{953}\ \wp^{489}\ \wp^{553}\right)\left(\wp^{380}\ \wp^{742}\ \wp^{873}\right)\left(\wp^{950}\ \wp^{457}\ \wp^{588}\right)$

$\left(\wp^{389}\ \wp^{784}\ \wp^{822}\right)\left(\wp^{941}\ \wp^{508}\ \wp^{546}\right)\left(\wp^{390}\ \wp^{876}\ \wp^{729}\right)\left(\wp^{940}\ \wp^{601}\ \wp^{454}\right)$

$\left(\wp^{397}\ \wp^{917}\ \wp^{681}\right)\left(\wp^{933}\ \wp^{649}\ \wp^{413}\right)\left(\wp^{404}\ \wp^{690}\ \wp^{901}\right)\left(\wp^{926}\ \wp^{429}\ \wp^{640}\right)$

$\left(\wp^{405}\ \wp^{849}\ \wp^{741}\right)\left(\wp^{925}\ \wp^{589}\ \wp^{481}\right)\left(\wp^{410}\ \wp^{878}\ \wp^{707}\right)\left(\wp^{920}\ \wp^{623}\ \wp^{452}\right)$

$\left(\wp^{417}\ \wp^{893}\ \wp^{685}\right)\left(\wp^{913}\ \wp^{645}\ \wp^{437}\right)\left(\wp^{6}\ \wp^{1031}\ \wp^{958}\right)\left(\wp^{1324}\ \wp^{372}\ \wp^{299}\right)$

$\left(\wp^{22}\ \wp^{562}\ \wp^{81}\right)\left(\wp^{1309}\ \wp^{267}\ \wp^{419}\right)\left(\wp^{24}\ \wp^{791}\ \wp^{1180}\right)\left(\wp^{1307}\ \wp^{1109}\ \wp^{909}\right)$

$\left(\wp^{208}\ \wp^{726}\ \wp^{1061}\right)\left(\wp^{1122}\ \wp^{269}\ \wp^{604}\right)\left(\wp^{441}\ \wp^{753}\ \wp^{801}\right)\left(\wp^{889}\ \wp^{529}\ \wp^{577}\right)$

$\left(\wp^{436}\ \wp^{692}\ \wp^{867}\right)\left(\wp^{894}\ \wp^{463}\ \wp^{638}\right)\left(\wp^{445}\ \wp^{817}\ \wp^{733}\right)\left(\wp^{885}\ \wp^{597}\ \wp^{513}\right)$

$\left(\wp^{447}\ \wp^{734}\ \wp^{814}\right)\left(\wp^{883}\ \wp^{516}\ \wp^{596}\right)\left(\wp^{461}\ \wp^{732}\ \wp^{802}\right)\left(\wp^{869}\ \wp^{528}\ \wp^{598}\right)$

$\left(\wp^{474}\ \wp^{709}\ \wp^{812}\right)\left(\wp^{856}\ \wp^{518}\ \wp^{621}\right)\left(\wp^{475}\ \wp^{750}\ \wp^{770}\right)\left(\wp^{855}\ \wp^{560}\ \wp^{580}\right)$

$\left(\wp^{482}\ \wp^{740}\ \wp^{773}\right)\left(\wp^{848}\ \wp^{557}\ \wp^{590}\right)\left(\wp^{483}\ \wp^{677}\ \wp^{835}\right)\left(\wp^{847}\ \wp^{495}\ \wp^{653}\right)$

$\left(\wp^{486}\ \wp^{790}\ \wp^{719}\right)\left(\wp^{844}\ \wp^{611}\ \wp^{540}\right)\left(\wp^{501}\ \wp^{774}\ \wp^{720}\right)\left(\wp^{829}\ \wp^{610}\ \wp^{556}\right)$

$\left(\wp^{507}\ \wp^{751}\ \wp^{737}\right)\left(\wp^{823}\ \wp^{593}\ \wp^{579}\right)\left(\wp^{511}\ \wp^{704}\ \wp^{780}\right)\left(\wp^{819}\ \wp^{550}\ \wp^{626}\right)$

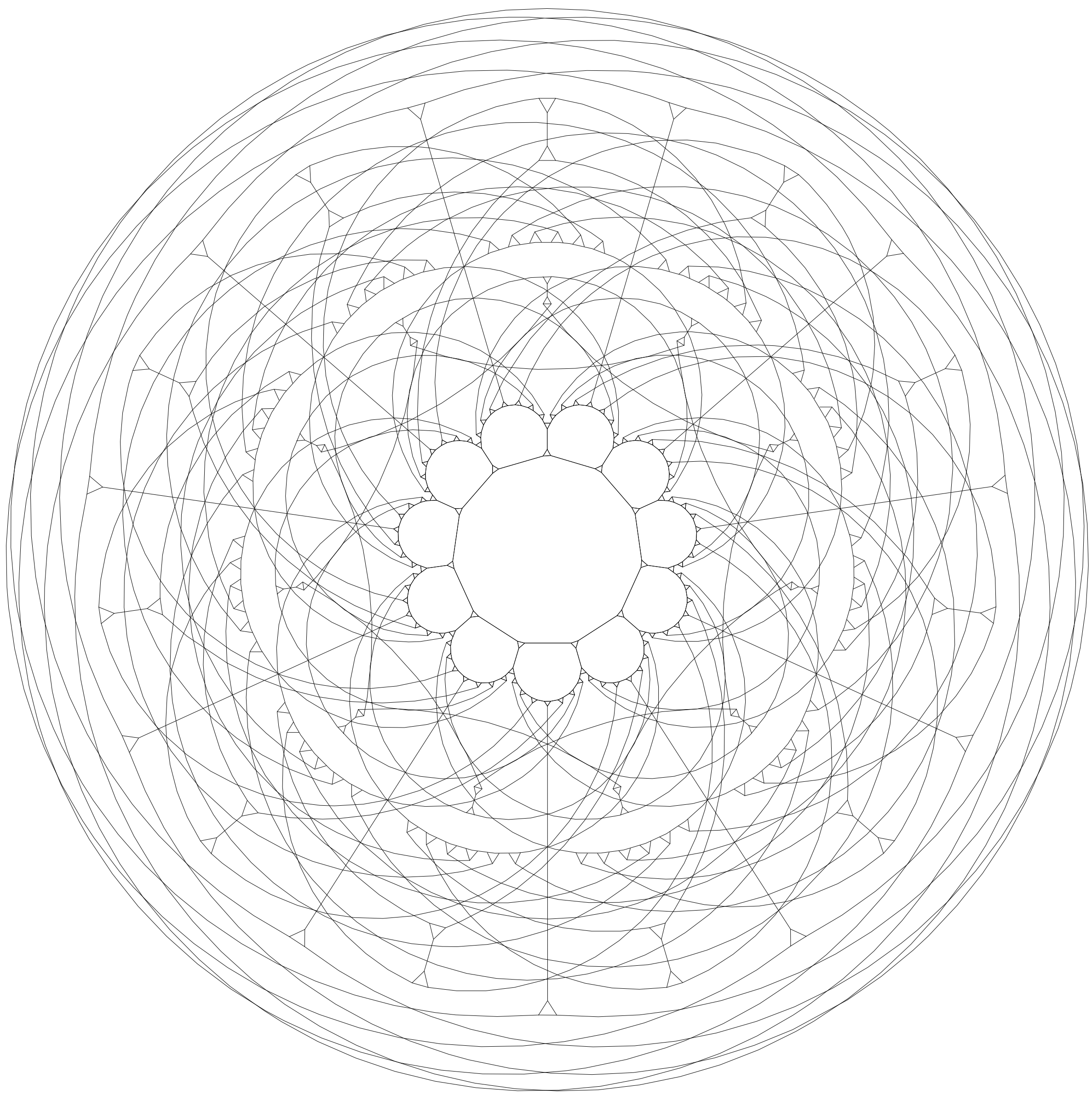
$\left(\wp^{537}\ \wp^{722}\ \wp^{736}\right)\left(\wp^{793}\ \wp^{594}\ \wp^{608}\right)\left(\wp^{555}\ \wp^{785}\ \wp^{655}\right)\left(\wp^{775}\ \wp^{675}\ \wp^{545}\right)$

$\left(\wp^{559}\ \wp^{660}\ \wp^{776}\right)\left(\wp^{771}\ \wp^{554}\ \wp^{670}\right)\left(\wp^{668}\ \wp^{700}\ \wp^{727}\right)\left(\wp^{762}\ \wp^{603}\ \wp^{630}\right)$

$\left(\wp^{563}\ \wp^{760}\ \wp^{672}\right)\left(\wp^{767}\ \wp^{658}\ \wp^{570}\right)\left(\wp^{643}\ \wp^{778}\ \wp^{574}\right)\left(\wp^{687}\ \wp^{756}\ \wp^{552}\right)$

$\left(\wp^{612}\ \wp^{703}\ \wp^{680}\right)\left(\wp^{718}\ \wp^{650}\ \wp^{627}\right)$

Under the action of $\mathbf{G}$ on $PL(F_{11^3})$, we get two orbits $\pounds_1$ and $\pounds_3$. Graphical representation of $\pounds_3$ is:



$\pounds_3$

This coset diagram also represents a group of order 660 [20] and contains two orbits $\pounds_1$ and $\pounds_3$.

6.2.4 Action of G on $PL(F_{11^m})$

Similarly we can draw coset diagrams for the action of $\mathbf{G}$ on $PL(F_{11^m})$ for any $m \in \mathbb{N}$, because the orbits of the action contain no new coset diagrams for the orbits other than $\pounds_{1,}$ $\pounds_2$ and $\pounds_3$ in the coset diagram. In this section we show that the actions of $\mathbf{G}$ on $PL(F_{11^m})$ evolves $PSL\,(2, 11)$. We also prove some relevant results.

Theorem 25 *If* $\mathbf{G}$ *acts on* $PL(F_{11^m})$, $m \in \mathbb{N}$

$$\bar{\mathbf{G}}_1 =< \bar{\mu}, \bar{\nu} : (\bar{\mu})^2 = (\bar{\nu})^3 = (\bar{\mu}\bar{\nu})^{11} = (\bar{\mu}\bar{\nu})^4 \left(\bar{\mu}\bar{\nu}^{-1}\right)^5 = 1 >\cong PSL\,(2, 11)\,.$$

Proof. Indeed the actions considered are homomorphisms from $PSL\,(2, 11)$ to $Sym(m)$, for $m = 12, 110, 660$, whose images are transitive subgroups. Obviously these images are isomorphic to $PSL\,(2, 11)$, since this group is simple. ■

Existence of fixed points of $\bar{\mu}$ and $\bar{\nu}$ in these coset diagrams play an important role which will be evident in the subsequent discussion.

Theorem 26 *If* $\mathbf{G}$ *acts on* $PL(F_{11^m})$*, then fixed points of* $\bar{\mu}$ *and* $\bar{\nu}$ *exist only for even* m*.*

Proof. When m is even, 11^m+1 are the total number of elements in $PL(F_{11^m})$. As we have $11^m \equiv 1 \pmod 4$ and the permutation $\bar{\mu}$ composed of two cycles leaving one element which becomes a fixed point of $\bar{\mu}$. Also we have $11^m \equiv 1 \pmod 3$ and the permutation $\bar{\nu}$ composed of three cycles leaving two elements which are fixed points of $\bar{\nu}$. ■

Remark 27 *The action of* $\mathbf{G}$ *on* $PL(F_{11^m})$ *gives three types of orbits namely* $\pounds_1$*,* $\pounds_2$ *and* $\pounds_3$.

The orbit $\pounds_1$ consist of 12 elements. All coset diagrams for this action contain $\pounds_1$ for all m. $\wp^{(11^m-1)/4}$ and $\wp^{3(11^m-1)/4}$ are fixed points of $\bar{\mu}$ and $\wp^{(11^m-1)/6}$ and $\zeta^{5(11^m-1)/6}$

are fixed points of $\bar{\nu}$ which lie in the orbit $\pounds_2$ consisting of 110 elements. This orbit exists in coset diagram only for even m. The third orbit $\pounds_3$ consist of 660 vertices but it does not contain any fixed points of $\bar{\mu}$ or $\bar{\nu}$. It exists in coset diagram always in the form of symmetric pairs for all $m \geqslant 3$.

Remark 28 *Let* $< \wp : \wp^{11^m-1} = 1 >$ *be the cyclic subgroup of* F_{11^m}. *Then,*

(*i*) *the fixed points of* $\bar{\mu}$ *are* $\wp^{(11^m-1)/4}$ *and* $\wp^{3(11^m-1)/4}$,

(*ii*) *the fixed points are* $\bar{\nu}$ *are* $\wp^{(11^m-1)/6}$ *and* $\zeta^{5(11^m-1)/6}$, *and*

(*iii*) $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ *and* ∞ *are the vertices of* $\pounds_1$.

Lemma 29 *The conjugacy class equation of* $\bar{\mathbf{G}}_1$ *is*

$\left|\bar{\mathbf{G}}_1\right| = \left|Z\left(\bar{\mathbf{G}}_1\right)\right| + \sum_{r_1=1}^{8} h'_{r_1} = 1 + 55 + 110 + 132 + 132 + 110 + 60 + 60$, where $Z\left(\bar{\mathbf{G}}_1\right)$ denotes the centre of $\bar{\mathbf{G}}_1$ and $h'_{r_1} = |x_{r_1}| = \left|\bar{\mathbf{G}}_1 \colon \mathbf{N}_{\bar{\mathbf{G}}_1}(x_{r_1})\right|$ for any element x_{r_1} in the $x_{r_1}th-$conjugacy class and $\mathbf{N}_{\bar{\mathbf{G}}_1}(x_{r_1})$ is the centralizer of an element x_{r_1} in $\bar{\mathbf{G}}_1$.

Proof. The group obtained by the action of $PSL(2, \mathbb{Z})$ on $PL(F_{11^m})$ is isomorphic to $PSL(2, 11)$ by Theorem 28. So the elements of $PSL(2, 11)$ are of order 1, 2, 3, 5, 6 and 11. Since the orbit $\pounds_1$ lies in all the coset diagram for the action of $PSL(2, \mathbb{Z})$ on $PL(F_{11^m})$. We consider that orbit which, by Remark 30 and Remark 31, consist of twelve elements which are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ and ∞. There are eight conjugacy classes of $\bar{\mathbf{G}}_1$ which partitions $\bar{\mathbf{G}}_1$. The only element which commutes with all other elements of $\bar{\mathbf{G}}_1$ is the identity element only. So 659 elements are left of order 2, 3, 5, 6 and 11. The element (1 2)(3 4)(5 12)(6 11)(7 10)(8 9) of order 2 forms a conjugacy class containing 55 elements. The element (1 2 3)(4 8 12)(5 10 9)(6 11 7) of order 3 forms the conjugacy class containing

110 elements. There are two classes of order 5 each containing 132 elements. So there are 264 elements on the whole in both classes of order 5. The class of order 6 contain 110 elements. There are two conjugacy classes of order 11, each containing 60 elements. So both classes contain 120 elements of order 11. ■

Theorem 30 *If* $\mathbf{G}$ *acts on* $PL(F_{7^n})$, *then*

(*i*) $\left|Orb_{PL(F_{11^m})}\left(\bar{\mathbf{G}}_1\right)\right| = 1 + \frac{(11^m+1)-12}{660}$ *if* m *is odd,*

(*ii*) $\left|Orb_{PL(F_{11^m})}\left(\bar{\mathbf{G}}_1\right)\right| = 2 + \frac{(11^m+1)-122}{168}$ *if* m *is even.*

Proof. By Remark 28, when m is odd, the orbit $\pounds_1$ composed of 12 vertices, exists for all m. So $(11^m+1)-12$ elements of $PL(F_{11^m})$ are left. By Theorem 26, $\bar{\mathbf{G}}_1$ is isomorphic to $PSL\,(2,11)$ containing elements of orders $2,3,5,6,11$ and the identity element. Theorem (27) shows that for odd m, there is no fixed point of $\bar{\mu}$ and $\bar{\nu}$. So, by Theorem 27 the total number of orbits for odd m is:

$$\left|Orb_{PL(F_{11^m})}\left(\bar{\mathbf{G}}_1\right)\right| = 1 + \frac{(11^m+1)-12}{660}.$$

By Remark 28, when m is even, $\pounds_1$ containing 12 and $\pounds_2$ containing 110 vertices, are two orbits. Only when m is even, $\pounds_2$ exists in coset diagram. So $(11^m+1)-122$ elements of $PL(\mathbf{F}_{11^m})$ are left. By Theorem 29 when m is even, fixed points of $\bar{\mu}$ and $\bar{\nu}$ exist. So, by Lemma 30 and Theorem 27, the total number of orbits for even m is:

$$\left|Orb_{PL(F_{11^m})}\left(\bar{\mathbf{G}}_1\right)\right| = 2 + \frac{(11^m+1)-122}{660} \quad ■$$

Thus we have the subsequent result.

Corollary 31 *The action of* $\mathbf{G}$ *on* $PL(F_{11^m})$ *is intransitive.*

Remark 32 (1) *If* m *is odd, we have* $1 + \frac{(11^m+1)-12}{660}$ *number of orbits, including one orbit*

$\pounds_1$ containing 12 *vertices. Remaining elements are evenly divided into* $\frac{(11^m+1)-12}{660}$ *number of orbits. All of these orbits are copies of $\pounds_3$ consisting of* 660 *vertices.*

(2) *If m is even, we have* $2+\frac{(11^m+1)-122}{660}$ *number of orbits. One of these orbits is $\pounds_1$ containing* 12 *vertices and the other is $\pounds_2$ containing* 110 *vertices. Remaining elements are evenly divided into* $\frac{(11^m+1)-122}{660}$ *number of orbits. All these orbits are copies of $\pounds_3$ containing* 660 *vertices.*

6.3 Conclusion

The group $PSL(2,11)$ is an important group of order 660 and has many applications in carbon chemistry. This group is useful to understand and analyze the structure of graphite and fullerenes. We analyzed the coset diagrams for the action of $PGL(2,\mathbb{Z})$ or $PSL(2,\mathbb{Z})$ on $PL\left(F_{11^m}\right)$. The total number of orbits that exist in coset diagram are $1+\frac{(11^m+1)-12}{660}$ if m is odd and $2+\frac{(11^m+1)-122}{660}$ if m is even. The transitive action of $\mathbf{G}$ on a set of 11 elements for $m=1$ gives us an orbit $\pounds_1$ having 12 vertices. It is ^{11}I *undecakisicosahedral* group [56]. For $m=2$, $\mathbf{G}$ acts on $PL\left(F_{121}\right)$ intransitively obtaining two orbits $\pounds_1$ and $\pounds_2$ containing 12 and 110 elements respectively and representing *undecakisicosahedral* group. When $\mathbf{G}$ acts on $PL\left(F_{11^m}\right)$ for $m \geqslant 3$, we obtain orbits $\pounds_1$, $\pounds_2$ and copies of $\pounds_3$.

Bibliography

[1] S. L. Altmann, Induced representations in crystals and molecules, Academic Press, London, 1977.

[2] M. Amara, Enclosed sets of algebraic numbers, Ann. ScL École Norm. Sup. (3), 83 (1966) , 215 − 270.

[3] M. Ashiq, Q. Mushtaq and T. Maqsood, Parametrization of actions of a subgroup of he modular group, Quasigroups and Related Systems, 20(2012), 21 − 28.

[4] G. Baumslag, J. W. Morgan and P. B. Shalen, Generalized triangle groups, Math. Proc. Cam. Philos. Soc., 102(1987), 25 − 31.

[5] A. F. Beardon, The geometry of discrete groups, Springer-Verlag, Berlin, 1980.

[6] L. Bellarosa, P. W. Fowler, E. Lijnen and M. Deza, Additional patterns in carbon allotropes: Independent numbers and d-codes in Klein and related graphs, Jour. Chem. Inf. Comput. Sci., 44 (2004) , 1314 − 1323.

[7] R. Bowen and C. Series, Markov maps associated with Fuschsian groups, Publ. Math. IHES, 50 (1979) , 153 − 170.

[8] D. W. Boyd, Pisot numbers and the width of meromorphic functions, Privately circulated manuscript, 1977.

[9] D. W. Boyd, Pisot sequences which satisfy no linear recurrence, Acta Arith., 32 (1977) , 89 – 98.

[10] D.W. Boyd, Small Salem numbers, Duke Math. Jour., 44(1977), 315 – 328.

[11] D. W. Boyd, Pisot Numbers in the neighbourhood of a limit point, I, Jour. No. theory, 21(1985), 17 – 43.

[12] S. Boyer, On proper powers in free products and Dehn surgery, Jour. Pure Appl. Algebra, 51(1988), 217 – 229.

[13] W. W. Burnside, Theory of groups of finite order, 2nd., Ed., Dover Pub., New York, 1995.

[14] J. W. S. Cassels, An Introduction to Diophantine Approximation, Camb., 1957.

[15] A. Cayley, The theory of groups: graphical representations, Amer. Jour. Math., 1 (1878) , 174 – 176.

[16] C. M. Campbell, M. D. E. Conder and E. F. Robertson, Defining relations for Hurwitz groups, Math. Jour., 36 (1990) , 363 – 370.

[17] C. M. Campbell and R. M. Thomas, On infinite groups of Fibonacci type, Proc. Edin. Math. Soc., 29(1986), 225 – 232.

[18] A. Ceulemans and P. W. Fowler, Symmetry extensions of Euler's theorem for

polyhedral, toroidal and benzenoid molecules , Jour. Chem. Soc. Faraday Trans., 91(1995), 3089 – 3093.

[19] A. Ceulemans, R. B. King, S. A. Bovin, K. M. Rogers, A. Troisi and P. W. Fowler, The heptakisoctahedral group and its relevance to carbon allotropes with negative curvature, Jour. Math. Chem., 23(1999), 101 – 123.

[20] The GAP Group, GAP–Groups, Algorithms and Programming, Version 4.2, Lehrstuhl D fur Mathematik, RWTH Aachen and School of Mathematical and Computational Sciences, University of St. Andrews, 1999. Available at http://www.gap.dcs.st-and.ac.uk∼gap.

[21] M. D. E. Conder, Generators for alternating and symmetric groups, Jour. London Math. Soc., (2) , 22 (1980) , 75 – 86.

[22] M. D. E. Conder, Some results on quotients of triangle groups, Bull. Aust. Math. Soc., 30 (1984) , 73 – 90.

[23] M. D. E. Conder, Three-relator quotients of the modular group, Quart. Jour. Math. Oxford, (2) , 38(1987), 427 – 447.

[24] M. D. E. Conder, Hurwitz groups: a brief survey, Bull. Amer. Math. Soc., 23(1990), 359 – 370.

[25] M. D. E. Conder, Schreier coset graphs and their applications, RIMS Kokyuroku, 794(1992), 169 – 175.

[26] H. S. M. Coxeter, The abstract groups $G^{m,n,p}$, Trans. Amer. Math. Soc., 45(1939), $73-150$.

[27] H. S. M. Coxeter and W. O. J. Moser, Generators and relations for discrete groups, Springer-Verlag, Berlin, 1965.

[28] H. Davenport, The higher arithmetic: An introduction to the theory of numbers, Camb. Uni. Press, (2008).

[29] L. E. Dickson, Linear Groups with an exposition of the Galios field theory, Dover Pub. Inc., New York, (1958).

[30] A. Dubickas, C. J. Smyth, On the lines passing through two conjugates of a Salem number, Math. Proc. Camb. Phil. Soc., 144(2008), $29-37$.

[31] J. Dufresony and C. Pisot, On the accumulation elements of a closed assembly of algebraic integers, Bull. Sci. Math. 2, $79\,(1955)$, $54-64$.

[32] J. Dufresony and C. Pisot, Study of some meromorphic functions bounded on the unit circle, application to a closed set of algebraic integers, Ann. Sci. École Norm. Sup., 3, $72\,(1955)$, $69-92$.

[33] N. D. Elkies, The Klein quartic in number theory, in: The eightfold way, Math. Sci. Res. Inst. Pub., 35, Camb. Uni. Press, 1999, $51-101$.

[34] B. Everitt, Alternating quotients of Fuchsian groups, Jour. Algebra, 223(2000), $457-476$.

[35] B. Fine, J. Howie and G. Rosenberger, One-relator quotients and free products of cyclics, Proc. Amer. Math. Soc., 2, 102(1988), $249-254$.

[36] B. Fine, F. Levin and G. Rosenberger, Free subgroups and decompositions of one-relator products of cyclics. Part 1: The Tits Alternative, Arch. Math., 50(1988), $97-109$.

[37] B. Fine and G. Rosenberger, A note on generalized triangle groups, Abh. Math. Sem. Uni. Hamburg, 56(1986), $233-244$.

[38] B. Fine, A. Hulpke, V. G. Rebel and G. Rosenberger, The Tits Alternative for spherical generalized tetrahedron groups, Algebra Colloq. 15, 541(2008), $541-554$.

[39] J. Howie, V. Metaftsis and R. M. Thomas, Finite generalized triangle groups, Trans. Amer. Math. Soc., 9, 347(1995), $3613-3623$.

[40] J. Howie and G. Williams, Free subgroups in certain generalized triangle groups of type $(2, m, 2)$, Geo. Dedi., 119(2006), $181-197$.

[41] C. F. Gardiner, A first course in group theory, Heidelberg: Springer-Verlag, 1980.

[42] M. Grandet-Hugot, Enclosed sets of algebraic integers, Ann. Sci. École Norm. Sup., 3, $82\,(1965)\,, 1-35$.

[43] L.S. Guimond, Z. Masáková, E. Pelantová, Arithmetics of beta-expansions, Acta Arith., 112(2004), $23-40$.

[44] I. Hargittai and M. Hargittai, Symmetry through the eyes of a chemist, Plenum Press, New York, 1995.

[45] E. Heilbronner and J.D. Dunitz, Reflections on symmetry in chemistry and elsewhere,Weinheim, 1993.

[46] G. Higman, Q. Mushtaq, Coset diagrams and relations for $PSL(2;\mathbb{Z})$, Arab Gulf Jour. Sci. Res., 1 (1983) , $159 - 164$.

[47] G. A. Jones and D. Singerman, Complex functions : an algebraic and geometric viewpoint, Camb. Uni. Press, 1987.

[48] R. B. King, Chemical applications of topology and group theory. 29. Low density polymeric carbon allotropes based on negative curvature structures, Jour. Phys. Chem., 100(1996), $15096 - 15104$.

[49] R. B. King, Some aspects of the symmetry and topology of possible carbon allotrope structures, Jour. Math. Chem., 23(1998), $197 - 227$.

[50] F. Klein, On the seventh-order transformations of elliptic functions, Math. Ann., 14 (1879) , $428 - 471$.

[51] B. Kostant, Structure of the truncated icosahedron (such as fullerene or viral coatings) and a 60-element conjugacy class in $PSL(2, 11)$, Proc. Nat. Acad. Sci. USA, 91(1994), $11714 - 11717$.

[52] H. W. Kuhn, Subgroup theorems for groups presented by generators and relations, Ann. Math., 56 (1952), $22 - 46$.

[53] P. Lakatos, Salem numbers, PV numbers and spectral radii of Coxeter transformations, C. R. Math. Rep. Acad. Sci. Canada, 23(2001), $71 - 77$.

[54] J. Leech and J. Mennicke, Note on a conjecture of Coxeter, Proc. Glasgow Math. Assoc., 5(1961), $25-29$.

[55] J. Leech, Note on the abstract group $(2,3,7;9)$, Proc. Camb. Philos. Soc., 62(1966), $7-10$.

[56] E. Lijnen, A. Ceulemans, P. Fowler and M. Deza, The undecakisicosahedral group and a 3-regular carbon network of genus 26, Jour. Math. Chem., 3, 42 (2007) , $617-644$.

[57] L. Levai, G. Rosenberger and B. Souvignier, All finite generalized triangle groups, Trans. Amer. Math. Soc., 1995, $3625-3627$.

[58] F. Levin and G. Rosenberger, On free subgroups of generalized triangle groups, Part II, Proceedings of the Ohio State-Denison conference on group theory (S. Seghal et al., eds.), World Sci., 1993, $206-222$.

[59] A. Lucchini, $(2,3,k)$-generated groups of large rank, Arch. Math., 73(1999), $241-248$.

[60] A. Lucchini, M. C. Tamburini and J. S. Wilson, Hurwitz groups of large rank, Jour. London Math. Soc., (2), 61(2000), $81-92$.

[61] A. M. Macbeath, Hurwitz groups and surfaces, in: The eightfold way, Math. Sci. Res. Inst. Pub., 35, Camb. Uni. Press, 1999, $103-113$.

[62] K. Magaard and H. Volklein, On Weierstrass points of Hurwitz curves. Jour. Algebra, 300(2006), $647-654$.

[63] R. Moeckel, Geodesics on modular surfaces and continued fractions, Ergodic Theory, Dover Pub. Inc. New York, 1976.

[64] G. Malle, Hurwitz groups and $G_2(q)$, Canad. Math. Bull., 33(1990), $349 - 357$.

[65] D. A. Marcus, Number fields, Springer-Verlag, 1977.

[66] H. Maschke, The representation of finite groups, especially for the rotation of the regular bodies in three- and four-dimentional space, Amer. Jour. Math., 18 (1896), $156 - 194$.

[67] M. Mignotte, On the conjugates of Pisot numbers, C. R. Acad. Sci. Paris Sér. I. Math., 298(1984), 21.

[68] Q. Mushtaq, Coset diagrams for modular groups, D. Phil. thesis, University of Oxford, 1983.

[69] Q. Mushtaq, Modular group acting on real quadratic field, Bull. Aust. Math. Soc., 2, 37 (1988), $303 - 309$.

[70] Q. Mushtaq, Reduced indefinite binary quadratic forms and orbits if the modular group, Radovi Mat, 4 (1988), $331 - 336$.

[71] Q. Mushtaq, A condition for the existence of a fragment of a coset diagram, Quart. Jour. Math. Oxford, 2, 39 (1988), $81 - 95$.

[72] Q. Mushtaq, Coset diagrams for the action of extended modular group on the projective line over a finite field, Jour. Pure Appl. Math., 8, 20 (1989), $747 - 754$.

[73] Q. Mushtaq, Coset diagrams for Hurwitz groups, Comm. Algebra, 18, 11(1990), $3857 - 3888$.

[74] Q. Mushtaq, Parametrization of all homomorphisms from $PGL(2;\mathbb{Z})$ into $PGL(2;q)$, Comm. Algebra, 20, 4(1992), $1023-1040$.

[75] Q. Mushtaq, On word structure of the modular group over finite and real quadratic fields, Disc. Math., 178 (1998), $155-164$.

[76] Q. Mushtaq and U. Hayat, Pell numbers, Pell-Lucas numbers and modular group, Algebra Colloq., 14, 1 (2007), $97-102$.

[77] Q. Mushtaq and U. Hayat, Horadam generalized Fibonacci numbers and the modular group, Indian Jour. Pure. Appl. Math., 5, 38 (2007), $345-352$.

[78] Q. Mushtaq and A. Rafiq, Adjacency matrices of $PSL(2,5)$ and resemblance of its coset diagrams with fullerene C_{60}, Algebra Colloq., 4, 20 (2013), $541-552$.

[79] Q. Mushtaq and F. Shaheen, On the subgroups of polyhedral groups, Math. Japonica, 35, 5(1990), $849-853$.

[80] Q. Mushtaq and F. Shaheen, Factor groups of $G^{6,6,6}$ through coset diagrams for an action on $PL(F_q)$, Disc. Math., 126 (1994), $1-3$.

[81] Q. Mushtaq and H. Servatius, Permutation representation of the symmetric groups of regular hyperbolic tesselations, Jour. London Math. Soc., 48 (1993), $77-86$.

[82] W. Plesken and D. Robertz, Representations, commutative algebra, and Hurwitz groups, Jour. Algebra, 300(2006), $223-247$.

[83] G. Rosenberger, On free subgroups of generalized triangle groups, Algebra i Logika, 28(1989), $227-240$.

[84] R. Salem, A remarkable class of algebraic integers. Proof of a conjecture of Vijayaraghavan, Duke Math. Jour., 11 (1944) , 103 – 107.

[85] R. Salem, Power series with integral coefficients, Duke Math. Jour., 12 (1945) , 153 – 171.

[86] R. Salem, Algebraic Numbers and Fourier Analysis, D.C. Heath, Boston, 1963.

[87] K. Schmidt, On periodic expansions of Pisot numbers and Salem numbers, Bull. London Math. Soc., 12(1980), 269 – 278.

[88] N.S. Semenov, On the Hurwitz property of Weyl groups, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat.Inst. Steklov. (POMI) 289 (2002), Vopr. Teor. Predst. Algebr. i Grupp. 9, 260 – 266, 304; translation in Jour. Math. Sci. (N.Y.), 124(2004), 4822 – 4825.

[89] J. A. Serret, Higher algebra course, 3rd ed., Gauthier-Villars, Paris, 1866.

[90] C. L. Siegel, Algebraic integers whose conjugates lie in the unit circle, Duke Math. Jour., 11 (1944) , 597 – 602.

[91] D. Singerman and P.D. Watson, Weierstrass points on regular maps, Geo. Dedicata, 66(1997), 69 – 88.

[92] A. Sinkov, On the group-defining relations $(2, 3, 7; p)$, Ann. of Math., 38(1937), 67–76.

[93] A. Sinkov, The number of abstract definitions of $LF(2, p)$ as a quotient groups of $(2, 3, n)$, Jour. Algebra, 12 (1969) , 525 – 532.

[94] S. Spadoni, L. Colombo, P. Milani and G. Benedek, Routes to carbon schwarzites from fullerene fragments., Europhys. Lett., 39(1997), $269 - 274$.

[95] R. Steinberg, Finite reflection groups, Trans. Amer. Math. Soc., 91 (1959) , $493 - 504$.

[96] W. W. Stothers, Subgroups of $(2, 3, 7)$ triangle groups, Manuscripta Math., 20 (1977) , $223 - 334$.

[97] K. Strambach and H. V¨olklein, On linearly rigid tuples, Jour. Reine Angew. Math., 510(1999), $57 - 62$.

[98] M. Streit, Field of definition and Galois orbits for the Macbeath-Hurwitz curves, Arch. Math. Basel, 74(2000), $342 - 349$.

[99] M. C. Tamburini and M. Vsemirnov, Irreducible $(2, 3, 7)$-subgroups of $PGL_n(F)$, $n \leq 7$, Jour. Algebra, 300(2006), $339 - 362$.

[100] M. C. Tamburini and M. Vsemirnov, Hurwitz groups and Hurwitz generation, invited chapter in: Handbook of Algebra, vol. 4, ed. M. Hazewinkel, Elsevier, 2006, $385 - 426$.

[101] R. M. Thomas, Cayley graphs and group presentations, Math. Proc. Camb. Philo. Soc., 103(1988), $385 - 387$.

[102] T. Vijayaraghava, On the fractional parts of the powers of a number, II, Proc. Camb. Phil. Soc., 37(1941), $349 - 357$.

[103] R. Vogeler, Combinatorics of curves on Hurwitz surfaces, Dissertation, University of Helsinki, Helsinki, 2004; Ann. Acad. Sci. Fenn. Math. Diss. No., 137(2004), 40.

[104] R. Vogeler, Explicit computation of Hurwitz spectra, in: Computational aspects of algebraic curves, Lecture Notes Ser. Compu., 13, World Sci. Pub. (Hackensack, NJ), 2005, $84 - 94$.

[105] H. Wielandt, Finite permutation groups, Academic press, New York, 1964.

[106] J. S. Wilson, Simple images of triangle groups, Quart. Jour. Math., Ser(2), 200, 50(1999), $523 - 531$.

[107] T. Zaimi, An arithmetical property of powers of Salem numbers, Jour. Number Theory, 120(2006), $179 - 191$.