Some Contributions to Rough Fuzzy Quantales and Quantale Modules

By

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Supervised

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Some Contributions to Rough Fuzzy Quantales and Quantale Modules

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A thesis Submitted in Partial Fulfillment of the requirement for the degree of doctor of philosophy in mathematics.

We accept this thesis as conforming to the required standard

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This is to certify that the research work presented in this thesis entitled "**Some Contributions to Rough Fuzzy Quantales and Quantale Modules**" was conducted by Mr. **Saqib Mazher Qurashi** under the supervision of **Prof. Dr. Muhammad Shabir**. No part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the Department of Mathematics, Quaid-i-Azam University, Islamabad in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Field of Mathematics from Department of Mathematics, Quaid-i-Azam University Islamabad, Pakistan.

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TO

MY PARENTS, SUPERVISOR,

MOTHER-IN-LAW,

WIFE

AND

CHILDREN.

Contents

0.1 Acknowledgment

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Saqib Mazher Qurashi

0.2 Research Profile

The thesis is based on the following research papers.

1. Roughness in Quantale Module, Journal of Intelligent and Fuzzy Systems. 35 (2018): 2359-2372.
2. Generalized Rough Fuzzy Ideals in Quantales, Discrete Dynamics in Nature and Society Volume 2018, Article ID 1085201, 11 pages.
3. Characterizations of Quantales by (α, β)-Fuzzy Ideals, Italian journal of pure and applied mathematics (In press).
4. Generalized Approximations of $(\in, \in \vee q)$-Fuzzy Ideals in Quantales Computational and Applied Mathematics. 37 (2018): 6821-6837.
5. $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Ideals in Quantales, Punjab University Journal of Mathematics. 51(8)(2019): 67-85.
6. On Generalized Fuzzy Filters in Quantales. (submitted).
7. Generalized approximation of $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Substructures in Quantales. (submitted).

0.3 Introduction

The theory of quantale was first introduced by Mulvey [57]. With algebraic structures and lattice-ordered structures, Quantale introduces a lattice setting of the study of non-commutative C*- algebra and an initiation of the study of quantum mechanics. A connection between quantale theory and linear logic was introduced by Yetter in 1990, in which he established a complete class of models for linear intuitionistic logic [102]. Quantales may be utilized in many interesting research topics like algebraic theory [44], rough set theory ([49, 67, 68, 70, 91, 96]), topological theory [30], theoretical computer science [77] and linear logic [28].

The idea of quantale module was introduced by Abramsky and Vickers [1]. The quantale module has attracted many scholars eyes. The idea of quantale module was motivated by the thought of module over a ring [5]. It replaces rings by quantales and abelian groups by complete lattices. The concept of quantale module showed up out of the blue for the first time as the key notion in the unified treatment of process semantics done by Abramsky and Vickers. A family of models of full linear logic is provided by modules over a commutative unital quantaleas as shown by Rosenthal [80].

Fuzzy set theory, at first proposed by Zadeh [105], has given an important scientific and mathematical tool to the description of those frameworks which are unreasonably perplexing or uncertain. Moreover, those conditions including vulnerabilities or ambiguities even more solidly, the unit interval $[0, 1]$ is replaced with a lattice and L-fuzzy sets were proposed by Goguen [29]. Gradually by applying fuzzy sets to the lattice-ordered environment, an important branch, has attracted consideration of researchers [114, 115], recently since fuzzy lattices have been extensively used as a part of designing, software engineering, topology, logic etc [64, 65]. Further, fuzzy algebra has furthermore transformed into a promising subject, since fuzzy algebraic structures have been viably associated with a wide range of fields [49, 67].

The idea of fuzziness is generally utilized in the theory of formal languages and automata. Numerous scientists utilized this idea to generalize notions of algebra. Rosenfeld defined fuzzy subgroups. Ahsan et al. proposed fuzzy semirings [2]. There are several authors who applied the theory of fuzzy sets to quantale, for instance, Luo and Wang [49] applied the fuzzy set theory to quantales. They defined fuzzy prime, fuzzy semi-prime and fuzzy primary ideals of quantales. They also introduced the notions of rough fuzzy (prime, semi-prime, primary) ideals of quantales.

The significance of fuzzy algebraic structures can be viewed by utilizing the notion of belongingness and quasi-coincidence with a fuzzy subset. Ming and Ming [66] presented the idea of quasi-coincidence of a fuzzy point with a fuzzy subset. The idea of a quasi-coincidence of a fuzzy point with a fuzzy set had a indispensable role to develop different types of fuzzy subgroups [6, 7]. Remembering this target, the concept of $(\in, \in \vee q)$-fuzzy sub-nearrings was introduced by Davvaz [17]. The idea of (α, β)-fuzzy ideals of hemirings was proposed by Dudek et al., [23]. In terms of $(\in, \in \vee q)$-fuzzy interior ideals, ordered semigroups was characterized by Khan et al., [40]. The generalization of fuzzy interior ideals of semigroup was initiated by Jun and Song [38]. The concept of (α, β)-fuzzy subalgebras (ideals) of a BCK/BCI algebra was suggested by Jun [35] and investigated the related results. An $(\in, \in \vee q_k)$-fuzzy ideals in ternary semigroups was studied by Shabir and Noor [86]. The general form of (α, β)-fuzzy ideals of hemirings were proposed by Jun et al., [36]. An $(\in, \in \vee q_k)$-type fuzzy ideals of semigroups were characterized by Shabir et al., [85]. An $(\overline{\in}, \overline{\in} \vee \overline{q})$-interval valued fuzzy H-ideals in BCK-algebras was described by Zulfiqar and Shabir [119]. Ma et al. studied $(\in, \in \vee q)$-fuzzy filters of RO-algebras [52]. For more details see [23, 37, 41, 42, 84].

In 2010, the more general forms of $(\in, \in \vee q)$-fuzzy filters and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy filters of BL-algebras were introduced by Yin and Zhan [103]. An $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy filters and $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy filters of BL-algebras were also defined by them. Some important results regarding these notions were incorporated also. An $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideals in ordered semigroups was proposed by Khan et al., [43]. The significance of these new types of notion is increased further by the work of Ma et al. They presented the idea of $(\in_\gamma, \in_\gamma \vee q_\delta)$-type fuzzy ideals of BCI-algebras and introduced a few essential results of BCI-algebras [53]. $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals in semigroups were investigated by Shabir and Ali jointly [83].

Rough set theory, introduced in 1982 by Pawlak [61], is a mathematical approach to imperfect knowledge. The methodology of rough set is concerned with the classification and analysis of imprecise, uncertain or incomplete information and knowledge. The subset generated by lower approximations is characterized by objects that will definitely form part of an interest subset, whereas the upper approximation is characterized by objects that will possibly form part of an interest subset. Every subset defined through upper and lower approximation is known as Rough set. After Pawlak's work, Yao [98, 99] and Zhu [116, 117, 118] provided some new views on rough set theory. Ali et al. [3] studied some properties of generalized rough sets. The applications of rough set theory used today is much wider than in the past, principally in the areas of cognitive sciences, medicine, knowledge acquisition, analysis of database attributes, automata theory, machine learning, pattern recognition and process control.

Although rough set theory and fuzzy set theory are two prominent notions to study about uncertainty, unpredictability and vagueness yet these theories are distinct in nature. It can be combined in a good manner to solve many problems. Theory of fuzzy sets proposes an exceptionally decent way to deal with vagueness. In 1990, Dubois and Prade [21], introduced the concepts of fuzzy rough and rough fuzzy sets.

There are several authors who introduced rough sets theory in algebraic structures and fuzzy algebraic structures. Investigation of algebraic properties of rough sets was started by Iwinski [32]. For instance, some results on rough subgroups were proposed by Biswas and Nanda [11]. Qurashi and Shabir introduced the roughness in Q_t-module [68]. Xiao and Li [91], considered a quantale as a ground set and presented the notions of generalized rough quantales and generalized rough subquantales. Rough prime, rough semi-prime and rough primary ideals of quantales were investigated by Yang and Xu [96]. Fuzzy ideals, fuzzy prime, fuzzy semi-prime and fuzzy primary ideals in quantales were introduced by Luo and Wang [49]. They also discussed rough fuzzy (prime, semi-prime, primary) ideals of quantales. Rough ideals in rings was proposed by Davvaz [16]. An algebraic T-rough sets were also proposed by Davvaz [19]. Yamak et al. [95], introduced the concept of set-valued mappings as the basis of the generalized upper and lower approximations of a ring with the help of an ideal. T-rough sets based on lattices were introduced by Hosseini et al [31]. They also investigated some results on T-rough (prime, primary) ideal and T-rough fuzzy (prime, primary) ideal on commutative rings. Roughness in Hemirings [4], was presented by Ali et al. Yaqoob et al. presented the rough prime bi-Γ-hyperideals of Γ-semihypergroups [100, 101]. Tahir et al. proposed the generalized roughness in fuzzy filters and fuzzy ideals with thresholds in ordered semigroups [54]. Generalized roughness in $(\in, \in \vee q)$-$f$uzzy ideals of hemirings was initiated by Rameez et al., [74]. Characterizations of Quantales by (α, β)-fuzzy ideals and its generalized approximations of $(\in, \in \vee q)$-fuzzy ideals in Quantales were proposed by Qurashi and Shabir [69, 70]. Kuroki [45] introduced the notion of rough ideal in a semigroup. Kuroki and Mordeson [46] studied the structure of rough sets and rough groups. Jun [34], applied the rough set theory to BCK-algebras.

0.4 Chapter-wise Study

This thesis comprises of eight chapters. Througout the thesis, Q_t and M denotes a quantale and quantale modules, unless and otherwise specified.

Chapter one having introductory nature, gives fundamental definitions and results, which are required for the consequent sections.

Chapter two represents the roughness in subsets of a Q_t-module with respect to Pawlak approximation space. Some basic properties of upper and lower approximations are discussed. We initiate the study of upper and lower rough approximations of Q_t-submodule of a Q_t-module and discuss the relations between the lower (upper) rough Q_t-submodules of Q_t-module and the lower (upper) approximations of their homomorphic images. The concept of set-valued homomorphism and strong set-valued homomorphism of Q_t-modules are presented in this chapter. At the end of this chapter, by using Q_t-module homomorphism, homomorphic images of generalized rough Q_t-submodules are introduced.

Chapter three is devoted to the study the generalized rough fuzzy ideals, generalized rough fuzzy prime ideals, generalized rough fuzzy semi-prime ideals and generalized rough fuzzy primary deals of quantales. Further, approximations of fuzzy ideals, fuzzy prime, fuzzy semi-prime and fuzzy primary ideals with the help of set-valued and strong set-valued homomorphisms are discussed. In addition, homomorphic images of generalized rough prime (semi-prime, primary) ideals which are established by quantale homomorphism, are examined.

Chapter four presents the study of (α, β)-type $f$uzzy subquantales (ideals) in quantale. Further, an $(\in, \in \vee q)$-type fuzzy ideals (subquantales) is discussed. It is investigated that homomorphic image of an $(\in, \in \vee q)$-fuzzy subquantales (ideal) under quantale homomorphism is an $(\in, \in \vee q)$-fuzzy subquantale (ideal). These fuzzy subquantales and fuzzy ideals are characterized by their level subquantales and ideals, respectively. Some important results about $(\in, \in \vee q)$-fuzzy prime and $(\in, \in \vee q)$-$f$uzzy semi prime ideals are discussed.

In the chapter five, we are starting the investigation of roughness in $(\in, \in \vee q)$-$f$uzzy subquantale and $(\in, \in \vee q)$-$f$uzzy ideal of quantales with respect to the generalized approximation space. Moreover, it is demonstrated that generalized upper and lower approximations of $(\in, \in \vee q)$-$f$uzzy ideal, $(\in, \in \vee q)$-$f$uzzy subquantale, $(\in, \in \vee q)$-$f$uzzy prime ideal and $(\in, \in \vee q)$-$f$uzzy semi-prime ideal are $(\in, \in \vee q)$-$f$uzzy ideal, $(\in, \in \vee q)$-$f$uzzy subquantale, $(\in, \in \vee q)$-$f$uzzy prime and $(\in, \in \vee q)$-$f$uzzy semi-prime ideal by using set-valued and strong set-valued homomorphism, respectively.

In the chapter six, we are presenting more general forms of $(\in, \in \vee q)$-$f$uzzy subquantale and $(\in, \in \vee q)$-$f$uzzy ideal of Quantales. We introduce the concepts of (α, β)-$f$uzzy subquantale, (α, β)-$f$uzzy ideal and some related properties are investigated. Special attention is given to $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy subquantale, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy prime, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy semi-prime ideals, and some interesting results about them are obtained. Furthermore, subquantale, prime, semi-prime and $f$uzzy subquantale, $f$uzzy prime, $f$uzzy semi-prime ideals of the types $(\in_\gamma, \in_\gamma \vee q_\delta)$ are linked by using level subsets.

In chapter seven, the concept of (α, β)-$f$uzzy filter is introduced and some related properties are discussed. Further, an $(\in, \in \vee q)$-type fuzzy filters are discussed. It is investigated that inverse image of an $(\in, \in \vee q)$-fuzzy filter under quantale homomorphism is an $(\in, \in \vee q)$-fuzzy filter. Moreover, these fuzzy filters are characterized by their level sets. Furthermore, in this chapter, we are presenting more general forms of $(\in, \in \vee q)$-$f$uzzy $f$ilters of Quantales. Special attention is given to $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy $f$ilters.

The goal of chapter eight is to study the the concept of generalized approximations of $(\in_\gamma, \in_\gamma \vee q_\delta)$-type fuzzy subquantales (ideals and filters) in quantales. With the help of set-valued and strong set-valued homomorphisms, respectively, it is observed that lower and upper approximations of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals (subquantale and filter) are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals (subquantale and filter), respectively. Some examples are added to convey these ideas.

Chapter 1

Preliminaries

In this chapter, we recall some definitions and results concerning with quantales, quantale modules, fuzzy sets and rough sets which are valuable for our consequent chapters. To start with, we portray complete lattice in light of the fact that nearly everything will be based on these, and then we address quantales and quantale modules.

In the first section, some fundamental definitions about the poset, lattice, sup-lattice, complete lattices and their homomorphisms are recalled. The definition of a quantale, ideal and filter of a quantale are presented in the second section. The Quantale homomorphism and its congruence with an example is given here. An example is added to demonstrate the definitions of subquantale, ideal and filters of a quantale. In the third section, the term quantale module and its examples are given. The Q_t-submodule and Q_t-submodule ideal of a quantale module are introduced. The congruence of quantale module and quantale module homomorphism with some related results are given. Some basic results about fuzzy set theory is introduced in the fourth section. Fuzzy ideals and fuzzy prime (semi-prime and primary) ideals are given in this section. In the last section, the notion of rough sets and generalized rough sets are presented.

1.1 Complete Lattices: Definitions and Examples

We start by recalling some basic definitions about partial orders and sup-lattices, as can be seen in [9].

Definition 1.1.1 *A partially ordered set (poset)* $(P_t, \preceq)$ *is a non-empty set* P_t *equipped with a binary relation* $\preceq$*, which fulfills the conditions below, for all* $w, u, z \in P_t$:

(1) $w \preceq w$. (Reflexivity)

(2) If $w \preceq z$ and $z \preceq w$, then $w = z$. (Antisymmetry)

(3) If $w \preceq z$ and $z \preceq u$, then $w \preceq u$. (Transitivity)

A poset $(P_t, \preceq)$ is bounded if P_t has a greatest element $\top \in P_t$ such that $w \preceq \top$ for all $w \in P_t$, and a least element $\bot \in P_t$ such that $\bot \preceq w$ for all $w \in P_t$. Sometimes we call greatest element as top element and least element as bottom element.

Example 1.1.2 *Some examples of poset are given below:*

(1) Consider the set of all non-negative integers $\mathbb{Z}^+$. Define "$\preceq$" by: $w \preceq u$ if and only if $w \mid u$. Then, $(\mathbb{Z}^+, \preceq)$ is a poset, but it is not bounded.

(2) Let $X \neq \emptyset$ and $P(X)$ be a power set of X. Then, it is easy to check that $(P(X), \subseteq)$ is a poset and it is bounded.

Definition 1.1.3 *Let* P_t *be a poset. Then* $z \in P_t$ *is an upper bound of a subset* X *of* P_t *if* $x \preceq z$ *for all* $x \in X$*. Similarly,* $w \in P_t$ *is a lower bound of a subset* X *of* P_t *if* $w \preceq x$ *for all* $x \in X$*.*

Let P_t *be a poset. Then* $z \in P_t$ *is the supremum, or join of a subset* X *of* P_t *if* z *is an upper bound of* X *and, for all upper bounds* z' *of* X*, we have* $z \preceq z'$*. Similarly,* $w \in P_t$ *is the infimum, or meet of a subset* X *of* P_t *if* w *is a lower bound of* X *and, for all lower bounds* w' *of* X*, we have* $w' \preceq w$*.*

The join (*resp.* meet) of X, if it exists, is unique and we denote it by $\bigvee X$ (*resp.* $\bigwedge X$), or, for sets of two elements $\{x, y\}$, $x \vee y$ (*resp.* $x \wedge y$).

Proposition 1.1.4 *If* $(P_t, \preceq)$ *is a poset, then:*

(1) $sup\{sup\{z, y\}, w\} = sup\{z, sup\{y, w\}\}$;

(2) $inf\{inf\{z, y\}, w\} = inf\{z, inf\{y, w\}\}$;

(3) $z \preceq y \Leftrightarrow sup\{z, y\} = y \Leftrightarrow inf\{z, y\} = z$.

Definition 1.1.5 *A poset* $(L_t, \preceq)$ *is called a lattice if* $sup\{z, w\}$ *and* $inf\{z, w\}$ *exist for any z,w in* L_t*. Clearly,* $(\mathbb{R},\preceq)$ *is a lattice, where* $\mathbb{R}$ *is the set of real numbers and* " $\preceq$ " *is the less than equal relation of real numbers.*

Definition 1.1.6 *A non-empty poset* L_t*, whose every subset has its supremum in* L_t*, will be called sup-lattice for simplicity in the following text.*

It is known that a set closed under joins contains arbitrary meets as well, and every sup-lattice is therefore a complete lattice. Considered as ordered sets, sup-lattices and complete lattices are thus identical, but a difference appears when we look at their homomorphisms.

Definition 1.1.7 *For sup-lattices* L_{t_1}*,* L_{t_2} *a map* $\sigma_S : L_{t_1} \longrightarrow L_{t_2}$ *is a sup-lattice homomorphism if it preserves arbitrary joins. Written more formally: for any* $\{z_i\} \subseteq L_{t_1}$*,* $(i \in I)$*, the following holds:*

$$\sigma_S(\vee_{i\in I} z_i) = \vee_{i\in I}\sigma_S(z_i).$$

Since any homomorphism σ_S *preserves suprema including a supremum of an empty subset, it holds that* $\sigma_S(\perp_{t_1}) = (\perp_{t_2})$*.*

Every homomorphism of complete lattices is certainly a sup-lattice homomorphism, too, but sup-lattice homomorphisms needn't preserve meets in general.

Definition 1.1.8 *A lattice* L_t *is complete when there is* $\bigvee X$ *and* $\bigwedge X$ *for every subset* X *of* L_t*.*

Example 1.1.9 $([0,1],\vee,\wedge)$ *is a complete lattice.*

1.2 Quantales: Definitions and Examples

In 1986, Mulvey initiated the notion of quantale, [57]. In 1990, Yetter connected quantale theory to linear logic and gave a sound and complete class of models for linear intuitionstic logic [104]. Quantales have played an important role in many research areas like algebraic theory [45], rough set theory [50, 68, 69, 71, 74, 92, 97], topological theory [31], theoretical computer science [78] and linear logic [29]. Here we present some definitions and examples relevant to the basics of the theory of quantale.

Definition 1.2.1 *[79] A quantale Q_t is a complete lattice equipped with an associative, binary operation $\otimes$ distributing over arbitrary joins. In other words, for any $w \in Q_t$ and $\{z_i\} \subseteq Q_t$, $(i \in I)$, it holds:*

$$\begin{aligned} w \otimes (\vee_{i\in I} z_i) &= \vee_{i\in I}(w \otimes z_i); \\ (\vee_{i\in I} w_i) \otimes z &= \vee_{i\in I}(w_i \otimes z). \end{aligned}$$

Let $X_i, X, Y \subseteq Q_t$, we define the followings;

$$\begin{aligned} X \vee Y &= \{x \vee y \mid x \in X,\ y \in Y\}; \\ X \otimes Y &= \{x \otimes y \mid x \in X,\ y \in Y\}; \\ \vee_{i\in I} X_i &= \{\vee_{i\in I} x_i \mid x_i \in X_i\}. \end{aligned}$$

Throughout the thesis, the symbole Q_t will be utilized for quantale, the symbol $\top$ will denote the top element and $\perp$ will stand for the bottom one for quantale, unless stated otherwise.

Definition 1.2.2 *[79, 80] Let Q_t be a quantale. An element $z \in Q_t$ is called:*

(1) idempotent if and only if $z \otimes z = z^2 = z$.

(2) right-sided (left-sided) if and only if $z \otimes \top \leq z$ ($\top \otimes z \leq z$).

(3) two-sided if it is both right-sided and left-sided.

(4) Let Q_t be a quantale. Then Q_t is commutative if $x \otimes z = z \otimes x$ for all $x, z \in Q_t$.

Example 1.2.3 *The following are the Examples of quantales:*

(1) Consider the ring $(R, +, \cdot)$. The set of left ideals of a ring R denoted by $LIdl(R)$ forms a quantale with joins as ideals generated by the union of ideals and multiplication realized as a product of two ideals given by: $A \cdot B = \{a_1 \cdot b_1 +, ..., + a_n \cdot b_n \mid a_i \in A, b_i \in B,\ 1 \leq i \leq n\}$. Of course, the sets $RIdl(R)$ and $Idl(R)$ of

right ideals and two sided ideals of R are quantales as well. Obviously, all these three notions merge when R is commutative. Thus, set of all ideals of a ring under inclusion ordering and standard multiplication of ideals form a quantale.

(2) Let $(Q_t, \star)$ be a semigroup and $P(Q_t)$ be the set of all its subsets. Then $P(Q_t)$ is a complete lattice under inclusion order. The multiplication $\otimes$ can be realized as: $U \otimes V = \{u \star v \mid u \in U,\, v \in V\}$. Thus, $(P(Q_t), \otimes)$ is a quantale.

(3) Binary relations on some set X under inclusion order form a complete lattice. With their composition defined as $R_1 \otimes R_2 = \{(z,w) | \exists\, u : (z,u) \in R_2 \;\&\; (u,w) \in R_1\}$ a quantale structure can be introduced as the composition distributes over suprema:

$$\begin{aligned} R \otimes (\vee_{i\in I} R_i) &= \{(z,w) \mid \exists u : (z,u) \in \vee_{i\in I} R_i \;\&\; (u,w) \in R\} \\ &= \{(z,w) \mid \exists u, \exists i : (z,u) \in R_i \;\&\; (u,w) \in R\} \\ &= \{(z,w) \mid \exists i : (z,w) \in R \otimes R_i\} \\ &= \vee_{i\in I}(R \otimes R_i) \end{aligned}$$

The case with a supremum in the left operand is analogous. Thus, binary relations on a non-empty set under inclusion ordering and composition of relations give a quantale.

(4) For a sup-lattice S_L, the set of all sup-lattices homomorphisms, $\pounds(S_L) = \{\sigma_S : S_L \to S_L \mid \sigma_S \text{ is a homomorphism}\}$ with pointwise ordering and map composition form a quantale. Sup-lattice S_L endomorphisms can be ordered pointwise: for σ_{S_1},$\sigma_{S_2} : S_L \to S_L$ we set $\sigma_{S_1} \preceq \sigma_{S_2} \Leftrightarrow \forall\, x \in S_L$: $\sigma_{S_1}(x) \preceq \sigma_{S_2}(x)$ what allows us to compute suprema: $(\vee_{i\in I}\sigma_{S_i})(x) = \vee_{i\in I}\sigma_{S_i}(x)$, but it has to be verified that $\vee_{i\in I}\sigma_{S_i}$ is a homomorphism:

$$\begin{aligned} (\vee_{i\in I}\sigma_{S_i})(\vee_{j\in I}x_j) &= (\vee_{i\in I})(\sigma_{S_i}(\vee_{j\in I}x_j)) \\ &= (\vee_{i\in I})(\vee_{j\in I}\sigma_{S_i}(x_j)) \\ &= (\vee_{j\in I})(\vee_{i\in I}\sigma_{S_i}(x_j)) \\ &= (\vee_{j\in I})((\vee_{i\in I}\sigma_{S_i})(x_j)). \end{aligned}$$

Multiplication is defined as a map composition $\sigma_{S_1} \otimes \sigma_{S_2} = \sigma_{S_1} \circ \sigma_{S_2}$ which is join-preserving since,

$$\begin{aligned}
\sigma_S \otimes (\vee_{i\in I}\sigma_{S_i})(x) &= \sigma_S \circ (\vee_{i\in I}\sigma_{S_i})(x) \\
&= \sigma_S(\vee_{i\in I}\sigma_{S_i}(x)) \\
&= \vee_{i\in I}(\sigma_S \circ \sigma_{S_i})(x) \\
&= \vee_{i\in I}(\sigma_S \otimes \sigma_{S_i})(x)
\end{aligned}$$

$$\begin{aligned}
((\vee_{i\in I}\sigma_{S_i}) \otimes \sigma_S)(x) &= ((\vee_{i\in I}\sigma_{S_i}) \circ \sigma_S)(x) \\
&= \vee_{i\in I}(\sigma_{S_i}(\sigma_S(x)) \\
&= \vee_{i\in I}(\sigma_{S_i} \circ \sigma_S)(x) \\
&= \vee_{i\in I}(\sigma_{S_i} \otimes \sigma_S)(x).
\end{aligned}$$

(5) Let Q_t be a complete lattice. Then Q_t become a quantale if $z \otimes x = x$ for all $x, z \in Q_t$. It also becomes a quantale if $z \otimes x = z$ for all $x, z \in Q_t$.

Throughout the thesis, the notations F_r, σ_t and η will be utilized for $f$ilter, quantale homomorphism and congruence in quantales, respectively.

Definition 1.2.4 *[79] Let $(Q_t, \otimes)$ and $(Q_t,\otimes')$ be two quantales. A map $\sigma_t : Q_t \longrightarrow Q_t'$ is called a quantale homomorphism if for every z,$w \in Q_t$,$\{z_i\} \subseteq Q_t$, $(i \in I)$, it holds that*

$$\begin{aligned}
\sigma_t(z \otimes w) &= \sigma_t(z) \otimes' \sigma_t(w); \\
\sigma_t(\vee_{i\in I} z_i) &= \vee_{i\in I}\sigma_t(z_i)).
\end{aligned}$$

A quantale homomorphism σ_t is an **epimorphism** if σ_t is onto Q_t' and σ_t is **monomorphism** if σ_t is one-one. If σ_t is bijective, then it is called an **isomorphism**. It is obvious that σ_t is order-preserving because if $w \preceq z$, then $\sigma_t(w) \preceq \sigma_t(z)$.

Definition 1.2.5 *[79] Let Q_t be a quantale. A binary relation η on Q_t is a congruence if η is an equivalence and for any a,b,c,$d \in Q_t$, $\{a_i\}$,$\{b_i\} \subseteq Q_t$, $(i \in I)$ it satisfies $a\eta b$ & $c\eta d \Longrightarrow (a \otimes c)\eta(b \otimes d)$ and also for all $i \in I$: $a_i\eta b_i \Longrightarrow (\vee_{i\in I}a_i)\eta(\vee_{i\in I}b_i)$. If η is a congruence on a quantale Q_t then Q_t/η is again a quantale where $Q_t/\eta = \{[a]_\eta : a \in Q_t\}$ while the operations $\vee$ and $\otimes$ on Q_t/η are defined as follows:*

(1) $[\vee_{i\in I}a_i]_\eta = \vee_{i\in I}[a_i]_\eta$.

(2) $[a]_\eta \otimes [b]_\eta = [a \otimes b]_\eta$ *for all* $a_i, a, b \in Q_t$ *and* $\{a_i\} \subseteq Q_t$.

Example 1.2.6 *[60] Let $\sigma_t : Q_t \longrightarrow Q'_t$ be a quantale homomorphism and $ker(\sigma_t) = \{(a,b) \mid a,b \in Q_t,\ \sigma_t(a) = \sigma_t(b)\}$. Then $ker(\sigma_t)$ is a congruence on Q_t.*

Proposition 1.2.7 *[60] If Q_t is a quantale and η a congruence on Q_t, the factor set Q_t/η is a quantale again and the mapping $\alpha : Q_t \longrightarrow Q_t/\eta$ defined by $\alpha(a) = [a]_\eta$ is a quantale homomorphism. The quantale Q_t/η is then called a quotient quantale of Q_t by the congruence η.*

1.2.1 Subquantales, Ideals and Filters

Now, we give definitions of subquantale, ideal and filter of quantale and some examples of them.

Definition 1.2.8 *[79] A subset Q of a quantale Q_t is called a subquantale of Q_t if it is closed under arbitrary sup and multiplication $\otimes$ induced by the quantale.*

Example 1.2.9 *[60] For any quantale Q_t the collection of right-sided, left-sided and two-sided elements of Q_t ($R(Q_t)$, $L(Q_t)$, $T(Q_t)$) are its subquantales.*

Definition 1.2.10 *[88, 89] Let Q_t be a quantale. A subset $\emptyset \neq C$ of Q_t is said to be an ideal of Q_t if the following conditions hold:*

(1) If z,$w \in C$, then $z \vee w \in C$;

(2) for all z, $w \in Q_t$ and $w \in C$ such that $z \leqslant w$ implies $z \in C$;

(3) for all $z \in Q_t$ and $w \in C$ implies $z \otimes w \in C$ and $w \otimes z \in C$.

Let C be an ideal of a quantale Q_t. Then, C is said to be a prime ideal if $z \otimes w \in C$ implies $z \in C$ or $w \in C$, $\forall\ z$,$w \in Q_t$. An ideal C is said to be a semi prime ideal if $z \otimes z \in C$ implies $z \in C$ for each $z \in Q_t$. Primary ideal is an ideal C of Q_t if $\forall$ x,$z \in Q_t$, $x \otimes z \in C$ and $x \notin C$ imply $z^n \in C$ for some positive integer n, where $z^n = \underbrace{z \otimes ... \otimes z}_{n}$.

Definition 1.2.11 *[79] Let Q_t be a quantale. A non-empty subset F_r of Q_t is said to be a filter of Q_t if F_r is an upper set and closed under $\otimes$. i.e., the following conditions hold:*

(1) for all $x \in Q_t$ and $z \in F_r$ such that $z \leq x$ implies $x \in F_r$;

(2) for all $z, x \in F_r$ implies $z \otimes x \in F_r$.

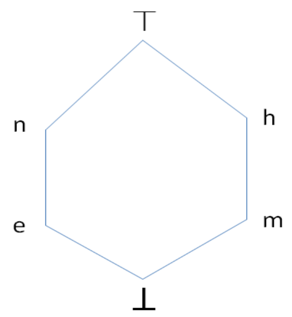

Fig.1

Table 1.

$\otimes$	$\perp$	e	m	n	h	$\top$
$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
e	$\perp$	e	$\perp$	e	$\perp$	e
m	$\perp$	$\perp$	m	$\perp$	m	m
n	$\perp$	e	$\perp$	n	$\perp$	n
h	$\perp$	$\perp$	m	$\perp$	h	h
$\top$	$\perp$	e	m	n	h	$\top$

Example 1.2.12 *Let Q_t be the complete lattice shown in Fig.1 and the operation $\otimes$ on Q_t is shown in Table.1. Then it is straightforward to verify that (Q_t,$\otimes$) is a quantale.*

(1) The subsets $Q_1 = \{\perp, m, h, \top\}$, $Q_2 = \{m, h\}$ and $Q_3 = \{\perp, e, n, \top\}$ of Q_t are examples of subquantales of Q_t.

(2) The subsets $C_1 = \{\perp, e, n\}$ and $C_2 = \{\perp, m, h\}$ of Q_t are examples of ideals of Q_t.

(3) The subsets $F_{r_1} = \{e, n, \top\}$, $F_{r_2} = \{m, h, \top\}$ and $F_{r_3} = \{n, \top\}$ of Q_t are examples of filters of Q_t.

1.3 Quantale Module: Definitions and Examples

The quantale modules were introduced by Abramsky and Vickers [1]. The idea of quantale module was motivated by the thought of module over a ring [5]. It replaces rings by quantales and abelian groups by complete lattices. The concept of quantale module showed up out of the blue for the first time as the key notion in the unified treatment of process semantics done by Abramsky and Vickers. A family of models of full linear logic is provided by modules over a commutative unital quantales as shown by Rosenthal [80]. The following is going to deal with quantale modules. Most of the theory is provided by [1, 16, 45, 60, 81].

Definition 1.3.1 *[1, 15, 44, 60, 81] Let Q_t be a quantale and M be a sup-lattice equipped with a left action $* : Q_t \times M \longrightarrow M$. Then M is called a left Q_t-module over the quantale Q_t if for any $a, b \in Q_t$, $\{a_i\} \subseteq Q_t, x \in M$,$\{x_j\} \subseteq M$ $(i \in I)$,$(j \in J)$, the following conditions hold:*

$$\begin{aligned} (\vee_{i\in I} a_i) * x &= \vee_{i\in I}(a_i * x); \\ a * (\vee_{j\in J} x_j) &= \vee_{j\in J}(a * x_j); \\ (a \otimes b) * x &= a * (b * x). \end{aligned}$$

Right modules can be defined in an analogous way. For the rest of the thesis, Q_t-module M will stand for a left module over the quantale Q_t. Let M be a Q_t-module, $A \subseteq Q_t$ and $m \in M$. We have,

$$\begin{aligned} A * m &= \{a * m \mid a \in A\}; \\ A * B &= \{a * b \mid a \in A, b \in B\} \text{ where } B \subseteq M. \end{aligned}$$

For $A, B, A_i \subseteq M$ and $i \in I$, we have,

$$\begin{aligned} A \vee B &= \{a \vee b \mid a \in A, b \in B\}; \\ \vee_{i\in I} A_i &= \{\vee_{i\in I} a_i \mid a_i \in A_i\}. \end{aligned}$$

The symbol $\top$ will denote the top element and $\bot$ will stand for the bottom element of the Q_t-modules as well, throughout the thesis, unless stated otherwise.

Example 1.3.2 *The following are the examples of Q_t-modules.*

(1) Let $Q_t = \{0, y, z, 1\}$ be a complete lattice where 0 is the bottom element and 1 is the top element of Q_t, as shown in $Fig.2$ and the operation $\otimes$ on Q_t is shown in Table 2. Then it is straightforward to verify that $(Q_t, \otimes)$ is a quantale. Let $M = \{\bot, x, \top\}$ be a sup-lattice. The order relation of M is given in $Fig.3$.

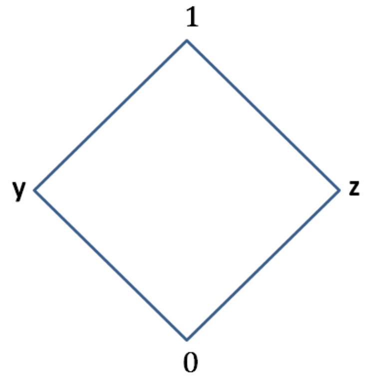

Fig. 2

Table.2

$\otimes$	0	y	z	1
0	0	0	0	0
y	0	y	0	y
z	0	0	z	z
1	0	y	z	1

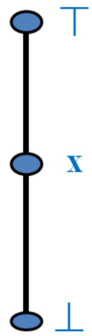

Fig. 3

Let $* : Q_t \times M \longrightarrow M$ be the left action on M as shown in the table 3.

Table. 3

$*$	$\bot$	x	$\top$
0	$\bot$	$\bot$	$\bot$
y	$\bot$	x	$\top$
z	$\bot$	x	x
1	$\bot$	x	$\top$

Then it is straightforward that M is a Q_t-module.

(2) Every quantale Q_t is certainly a Q_t-module over Q_t.

(3) We already know that if M is a sup-lattice, the set of all sup-lattice homomorphisms, $\pounds(M) = \{\rho_S : M \to M \mid \rho_S \text{ is a sup-lattice homomorphism}\}$ with pointwise ordering and composition of maps form a quantale. Let Q_t be another quantale and $\rho_m : Q_t \to \pounds(M)$ be a quantale homomorphism. Then we can define an action $a * z = \rho_m(a)(z)$ for $a \in Q_t$, $z \in M$ and M becomes a left Q_t-module. Now consider the following:

- $a * (b * z) = \rho_m(a)(b * z) = \rho_m(a)(\rho_m(b)(z)) = (\rho_m(a) \circ \rho_m(b))(z) = \rho_m(a * b)(z) = (a * b) * z$.
- $(\vee_{i\in I} a_i) * z = \rho_m(\vee_{i\in I}(a_i)(z) = (\rho_m(\vee_{i\in I}(a_i))(z) = \vee_{i\in I}(\rho_m(a_i)(z)) = \vee_{i\in I}(a_i * z)$.
- $a * (\vee_{i\in I} z_i) = \rho_m(a)(\vee_{i\in I} z_i) = \vee_{i\in I}(\rho_m(a)(z_i)) = \vee_{i\in I}(a * z_i)$.

(4) If Q_t is a quantale, $\pounds(Q_t)$ can be viewed as Q_t-module with multiplication $q*\rho_m(z) = (q * \rho_m)(z)$ where ρ_m is a Q_t-module homomorphism.

In the next, the notations M, ρ_m and η will be utilized for quantale module, quantale module homomorphism and congruence in quantale module, respectively.

Definition 1.3.3 *[1] Let M be a Q_t-module. A subset $M_1 \subseteq M$ is called a Q_t-submodule of M if for any $m \in M_1$, $\{m_i\} \subseteq M_1$, $q \in Q_t$, it holds that $\vee m_i \in M_1$ and $q * m \in M_1$.*

Example 1.3.4 *Let Q_t be a quantale and $a \in Q_t$. Then the set $Q_t*a = \{q*a \mid q \in Q_t\}$ is a left Q_t-submodule of Q_t.*

Definition 1.3.5 *[81] Let M be a Q_t-module and $\emptyset \neq C \subseteq M$. Then C is a Q_t-module-ideal of M provided*

(1) If $a_i \in C$ $(i \in I)$ then $\vee_{i\in I} a_i \in C$;

(2) $x \in C$ and $b \leq x$ imply $b \in C$;

(3) $x \in C$ implies $a * x \in C$, $\forall\ a \in Q_t$.

A Q_t-submodule-ideal is a Q_t-submodule as well.

Example 1.3.6 *Let Q_t be a complete lattice shown in $Fig.1$and $\otimes$ be an operation on Q_t defined as $x \otimes z = \perp$ for all $x, z \in Q_t$. Then it is straightforward to verify that $(Q_t, \otimes)$ is a quantale. Also Q_t is a Q_t-module over itself. Since $Q_1 = \{\perp, m, h, \top\}$ is a Q_t-submodule of Q_t but it is not a Q_t-submodule-ideal as $\top \in Q_1$ and $n \leq \top$ but $n \notin Q_1$.*

Definition 1.3.7 *[60, 78] Let M be a Q_t-module. A binary relation η on M is called a congruence on M if it is an equivalence relation on M and for any given $\{m_i\}$, $\{n_i\} \subseteq M$, m, $n \in M$ and $q \in Q_t$, it satisfies the following conditions $\forall\ i \in I$, $m_i \eta n_i$ implies $(\vee_{i\in I} m_i)\eta(\vee_{i\in I} n_i)$ and $m\eta n$ implies $(q * m)\eta(q * n)$.*

Definition 1.3.8 *[60, 78] Let M and M' be two Q_t-modules. A map $\rho_m : M \longrightarrow M'$ is a Q_t-module **homomorphism** if it is a sup-lattice **homomorphism** which also preserves scalar multiplication, i.e.*

$$\begin{aligned} \rho_m(\vee_{i\in I} m_i) &= \vee_{i\in I} \rho_m(m_i); \\ \rho_m(a * m) &= a * \rho_m(m) \end{aligned}$$

for any $a \in Q_t, m \in M$, $\{m_i\} \subseteq M$ $(i \in I)$.

A Q_t-module homomorphism $\rho_m : M \longrightarrow M'$ is called an **epimorphism** if ρ_m is onto M' and ρ_m is called a **monomorphism** if ρ_m is one-one. It is an **isomorphism**, if ρ_m is bijective.

Proposition 1.3.9 *[60, 78] Let M be a Q_t-module and η be a congruence on M. Then M/η is again a Q_t-module and a projection $\alpha : M \longrightarrow M/\eta$ is a module homomorphism. Let η be a congruence relation on a Q_t-module M. We define operations $\vee$ and $*$ on the quotient Q_t-module $M/\eta = \left\{[m]_\eta \mid m \in M\right\}$ as follows:*

(1) $\vee_{i\in I} [m_i]_\eta = [\vee_{i\in I} m_i]_\eta$ *and*

(2) $[q * m]_\eta = q * [m]_\eta$ *for all $m_i, m \in M$ and $q \in Q_t$.*

Theorem 1.3.10 *[60, 78] If ρ_m is a homomorphism of Q_t-modules from M to M', then*

$$ker(\rho_m) = \{(z, w) \in M \times M \mid \rho_m(z) = \rho_m(w)\}$$

is a congruence of Q_t-modules. The $ker(\rho_m)$ is called the kernel of ρ_m.

1.4 Fuzzy Sets and Fuzzy Ideals in Quantales

Numerous uses of fuzzy set theory have emerged over the years, for example, fuzzy logic, fuzzy cellular neural networks, fuzzy automata etc. A fuzzy subset g in a non-empty universe Z is defined with the help of a mapping $g : Z \longrightarrow [0,1]$ which associates a value $g(z)$ to each object z of the set Z. This value portrays the degree to which an object z is a member of the set Z, or the extent to which the object z satisfies the property of the set Z. The value $g(z)$ is known as the membership grade of the object z and the mapping g is known as the membership function of Z. As a generalization of the abstract set theory, Zadeh, [105] originated the theory of fuzzy sets. Numerous algebraic structures have been characterized by many authors to generalize these concepts. Let $\emptyset \neq Z$ be a universe of discourse. Then, the formal definitions of fuzzy subset and its operations, as established by Zadeh [105], are given below.

Definition 1.4.1 *A fuzzy subset g in Z is a function from Z to the unit closed interval $[0,1]$, that is $g : Z \longrightarrow [0,1]$. A fuzzy subset $g : Z \longrightarrow [0,1]$ is non-empty if g is not a zero map. Let $\mathcal{F}(Z)$ be the collection of all fuzzy subsets in Z.*

Definition 1.4.2 *Let g and f be two fuzzy subsets in Z. Then $g \subseteq f$ if and only if $g(z) \leq f(z)$ for all $z \in Z$. Clearly, $g = f$ if and only if $g \subseteq f$ and $f \subseteq g$.*

Definition 1.4.3 *The null fuzzy subset in Z is defined by the mapping $\emptyset_Z : Z \longrightarrow [0,1]$ such that $\emptyset_Z(z) = 0$ for all $z \in Z$. The whole fuzzy subset in Z is defined by the mapping $F_Z : Z \longrightarrow [0,1]$ such that $F_Z(z) = 1$ for all $z \in Z$.*

Definition 1.4.4 *Let g and f be any two fuzzy subsets in Z. Then, the union and intersection of g and f are defined as:*

$$\begin{aligned} (f \Cup g)(z) &= sup(g(z), f(z)) \text{ for all } z \in Z \text{ and} \\ (f \Cap g)(z) &= inf(g(z), f(z)) \text{ for all } z \in Z. \end{aligned}$$

Definition 1.4.5 *A fuzzy subset g in Z is said to be a constant fuzzy subset in Z if and only if $g : Z \longrightarrow [0,1]$ is a constant function.*

Definition 1.4.6 *For $\alpha \in [0,1]$, the sets*

$$g_\alpha = \{x \in Z \mid g(x) \geq \alpha\} \quad \text{and} \quad g_{\alpha^+} = \{x \in Z \mid g(x) > \alpha\}$$

are called, α-cut and strong α-cut of g, respectively.

Definition 1.4.7 *[36] Let $\sigma_t : Q_t \longrightarrow Q'_t$ be a mapping from a quantale Q_t to a quantale Q'_t, and let g and g' be fuzzy subsets in Q_t and Q'_t, respectively. Then the image of g under σ_t and the pre-image of g' under σ_t are the f-subsets $\sigma_t(B)$ and $\sigma_t^{-1}(B')$, respectively, defined as follows:*

$$(i)\ \sigma_t(g)(y) = \begin{cases} \underset{x \in \sigma_t^{-1}(y)}{Sup} \ g(x), & if\ \sigma_t^{-1}(y) \neq \emptyset\ \forall\ y \in Q'_t \\ 0, & otherwise \end{cases},$$

(ii) $\sigma_t^{-1}(g')(x) = g'(\sigma_t(x))\ \forall\ x \in Q_t$.

If σ_t is a quantale homomorphism, then $\sigma_t(g)$ is called the homomorphic image of g under σ_t and $\sigma_t^{-1}(g')$ is called the homomorphic pre-image of g'.

Next for fuzzy subset, $f$uzzy ideal, $f$uzzy prime ideal, $f$uzzy semiprime ideal and $f$uzzy primary ideals, the following shortened forms, f-subset, FI, FPI, $FSPI$ and $FPYI$, will be utilized, respectively.

Definition 1.4.8 *A non-empty f-subset g in Q_t is called a FI of Q_t, if the conditions bellow are satisfied:*

(1) $z \leq w \Longrightarrow g(w) \leq g(z)$;

(2) $inf\{g(z), g(w)\} \leq g(z \vee w)$;

(3) $sup\{g(z), g(w)\} \leq g(z \otimes w))$ for all z,$w \in Q_t$.

From (1) and (2) in Definition 1.4.8, it is observed that $g(z \vee w) = inf\{g(z), g(w)\}$, for all z,$w \in Q_t$. Thus, a f-subset g of Q_t is a FI of Q_t if and only if $g(z \vee w) = inf\{g(z),g(w)\}$ and $g(z \otimes w) \geq sup\{g(z), g(w)\}$, for all z,$w \in Q_t$.

The following definitions are taken from [49].

Definition 1.4.9 *Let g be a non-constant FI of a quantale Q_t. Then g, is called a FPI of Q_t if it satisfies;*

$$g(z \otimes w) = g(z)\ or\ g(z \otimes w) = g(w)\ for\ all\ z, w \in Q_t.$$

Definition 1.4.10 *Let g be a FI of a quantale Q_t. Then g is called a $FSPI$ of Q_t if the following assertion is satisfied:*

$$g(z \otimes z) = g(z) \text{ for all } z \in Q_t.$$

Definition 1.4.11 *A non-constant FI, g of a quantale Q_t is called a $FPYI$ of Q_t if,*

$g(z \otimes w) = g(z)$ *or* $g(z \otimes w) = g(w^n)$ *for all* $z, w \in Q_t$ *and for some positive integer* n.

Proposition 1.4.12 *Let g be a FI of a quantale Q_t. Then g is a FPI if and only if* $g(w \otimes z) = g(z) \vee g(w)$ *for all* $w, z \in Q_t$.

Proposition 1.4.13 *Let g be a f-subset of a quantale Q_t.*

(1) *Then g, is a FI of Q_t if and only if for each $\alpha \in [0, 1]$, g_α (res. g_{α^+}) is either empty or an ideal of Q_t.*

(2) *Then g, is a $FSPI$ of Q_t if and only if for each $\alpha \in [0, 1]$, g_α (res. g_{α^+}) is either empty or an SPI of Q_t.*

Proposition 1.4.14 *Let g be a FI of a quantale Q_t.*

(1) *Then g, is a FPI of Q_t if and only if for each $\alpha \in [0, 1]$, g_α (res. g_{α^+}) is either empty or an PI of Q_t.*

(2) *Then g, is a $FPYI$ of Q_t if and only if for each $\alpha \in [0, 1]$, g_α (res. g_{α^+}) is either empty or an PYI of Q_t.*

1.5 Rough Sets: Definitions and Examples

Pawlak at first proposed the theory of rough sets [62, 63]. It was utilized to deal with imprecision and deficiency in data frameworks. The initial methodology supported by Pawlak incorporates partitioning the universe set into granules (classes) of components, which are indistinguishable or indiscernible subject to the accessible data or information. With the help of these classes, the two definable subsets called the lower and upper approximations of an arbitrary subset of a universe can be approximated.

In this section, we will give a few ideas identified with rough set theory. An example is added to demonstrate these concepts.

Let Z be a non-empty set and η be an equivalence relation on Z. Let $[z]_\eta$ denotes the equivalence class of the relation η containing $z \in Z$. Any finite union of equivalence classes of Z is called a definable set in Z. Let X be any subset of Z, in general X is not a definable set in Z. However the set X can be approximated by two definable sets in Z. The first one is called η-lower approximation of X and the second is called η-upper approximation. They are defined as follows

$$\begin{aligned} \underline{\eta}(X) &= \{z \in Z \mid [z]_\eta \subseteq X\}; \\ \overline{\eta}(X) &= \{z \in Z \mid [z]_\eta \cap X \neq \emptyset\}. \end{aligned}$$

The η-upper approximation of X in Z is the least definable set in Z containing X. The η-lower approximation of X in Z is the greatest definable set in Z contained in X. For any non-empty subset X of Z, $\eta(X) = (\underline{\eta}(X), \overline{\eta}(X))$ is called a rough set with respect to η or simply an η-rough subset of $P(Z) \times P(Z)$ if $\underline{\eta}(X) \neq \overline{\eta}(X)$, where $P(Z)$ denotes the set of all subsets of Z.

The universe Z can be separated into three disjoint regions, by using the lower and upper approximations of a set $X \subseteq Z$.

(1) the positive region $(\mathcal{POS})_\eta(X) = \underline{\eta}(X)$;

(2) the negative region $(\mathcal{NEG})_\eta(X) = Z - \overline{\eta}(X) = (\overline{\eta}(X))^c$;

(3) the boundary region $(\mathcal{BND})_\eta(X) = \overline{\eta}(X) - \underline{\eta}(X)$.

The positive region contains all objects of Z that can be classified to the equivalence classes of Z with respect to the equivalence relation η. The boundary region, $(\mathcal{BND})_\eta(X)$, is the set of objects that can possibly, but not certainly, be classified in this way. The negative region, $(\mathcal{NEG})_\eta(X)$, is the set of objects that cannot be classified to classes of Z/η.

This is obviously delineated in Figure 4.

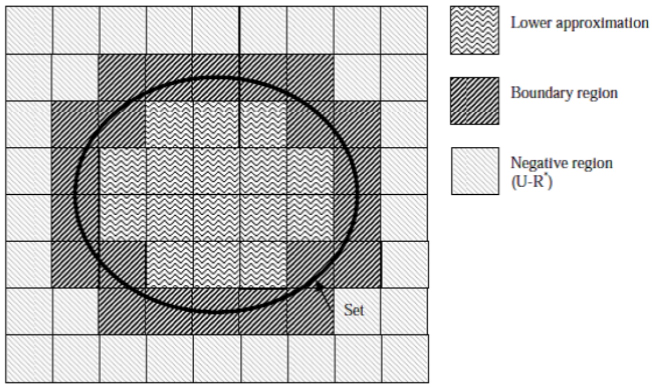

Fig.4 Illustration of the boundary region of Rough set

The approximation of a set X, and the negative, positive and boundary regions are expressed through Figure 1. Each small square regarded an equivalence class. The union of the positive and boundary regions constitute the upper approximation of a set N represented by $\overline{\eta}(X) = (\mathcal{POS})_\eta(X) \cup (\mathcal{BND})_\eta(X)$.

Proposition 1.5.1 *[62] Let (Z, η) be an approximations space. Then the lower and upper approximations for any $X, Y \subseteq Z$, are satisfied.*

1. $\underline{\eta}(X) \subseteq X \subseteq \overline{\eta}(X)$
2. $\underline{\eta}(\emptyset) = \emptyset = \overline{\eta}(\emptyset)\,;\ \underline{\eta}(Z) = Z = \overline{\eta}(Z)$
3. $\overline{\eta}(X \cup Y) = \overline{\eta}(X) \cup \overline{\eta}(Y)$
4. $\underline{\eta}\big(X) \cup \underline{\eta}(Y\big) \subseteq \underline{\eta}(X \cup Y)$
5. $\overline{\eta}(X) \cap \overline{\eta}(Y) \supseteq \overline{\eta}(X \cap Y)$
6. $\underline{\eta}(X \cap Y) = \underline{\eta}\big(X) \cap \underline{\eta}(Y\big)$
7. $X \subseteq Y$ implies $\underline{\eta}(X) \subseteq \underline{\eta}(Y)$, $\overline{\eta}(X) \subseteq \overline{\eta}(Y)$
8. $\underline{\eta}(-X) = -\overline{\eta}(X)$
9. $\overline{\eta}(-X) = -\underline{\eta}(X)$
10. $\underline{\eta\eta}(X) = \overline{\eta}\underline{\eta}(X) = \underline{\eta}(X)$

11. $\overline{\eta}\overline{\eta}\,(X) = \underline{\eta}\overline{\eta}\,(X) = \overline{\eta}\,(X)$.

Where $-X$ means the complement of X.

It is observed that approximations are in fact closure and interior operator in a topology generated by data.

Definition 1.5.2 *[99] A subset X of Z is called crisp when its boundary region is empty, i.e., $\underline{\eta}(X) = \overline{\eta}(X)$.*

Definition 1.5.3 *[62] Let Z be a universal set and let η be an equivalence relation on Z. Then the set $X \subseteq Z$ is called a rough set with respect to η if $\underline{\eta}(X) \neq \overline{\eta}(X)$.*

Another definition is

Definition 1.5.4 *[99] A subset defined through its lower and upper approximations is called a Rough set. That is, when the boundary region is a non-empty set ($\underline{\eta}(X) \neq \overline{\eta}(X)$).*

Example 1.5.5 *Let (Z, η) is an approximation space, and η an equivalence relation, where $Z = \{x_1,x_2,x_3,...,x_8\}$. Consider the following equivalence classes:*

$$\mathcal{E}_1 = \{x_1, x_4, x_8\},\ \mathcal{E}_2 = \{x_2, x_5, x_7\},\ \mathcal{E}_3 = \{x_3\},\ \mathcal{E}_4 = \{x_6\}.$$

Let $X = \{x_3, x_5\}$ and $Y = \{x_3, x_6\}$

$\underline{\eta}(X) = \{x_3\}$ and $\overline{\eta}(X) = \{x_2, x_3, x_5, x_7\}$

$\underline{\eta}(Y) = \{x_3, x_6\}$ and $\overline{\eta}(Y) = \{x_3, x_6\}$

So $\eta(X) = (\{x_3\},\{x_2,x_3,x_5,x_7\})$ is a rough set and $\eta(Y)$ is a crisp set.

1.5.1 Generalized Rough Sets:

Frequently, it is not possible to find a suitable equivalence relation among the elements of the universe set Z due to indefinite human knowledge. An equivalence relation is the essential prerequisite for lower and upper approximations while studying rough set theory. Therefore, there was need to generalize the rough set theory in a more general form to overcome this situation. The generalized rough set is the generalization of Pawlak's rough set. Yamak et al. proposed one of these generalizations. [95].

Definition 1.5.6 *Let Z and W be two non-empty universes and H be a set-valued mapping given by $H : Z \longrightarrow P^*(W)$ where $P^*(W) = P(W) \setminus \emptyset$. Then the triplet (Z, W, H) is called as generalized approximation space. Any set-valued function from Z to $P^*(W)$ defines a binary relation from Z to W by setting $\rho_H = \{(x, y) \in Z \times W \mid y \in H(x)\}$. Obviously, if ρ is an arbitrary relation from Z to W, then a set-valued mapping $H_\rho : Z \to P(W)$ can be defined by $H_\rho(x) = \{y \in W \mid (x, y) \in \rho\}$ for all $x \in Z$. For any set $A \subseteq W$, the lower and upper generalized approximations $\underline{H}(A)$ and $\overline{H}(A)$, are defined by*

$$\begin{aligned} \underline{H}(A) &= \{z \in Z \mid H(z) \subseteq A\}; \\ \overline{H}(A) &= \{z \in Z \mid H(z) \cap A \neq \emptyset\}. \end{aligned}$$

The pair $(\underline{H}(A), \overline{H}(A))$ is referred to as a generalized rough set where $\underline{H}$ and $\overline{H}$ are referred to as a lower and upper generalized approximation operators, respectively. If a subset $A \subseteq W$ satisfies that $\underline{H}(A) = \overline{H}(A)$, then A is called a definable set of (Z, W, H). From the definitions of lower and upper generalized approximation operators, the following theorem can be easily derived.

Theorem 1.5.7 *[95] Let (Z, W, H) be a generalized approximation space. Its lower and upper generalized approximation operators satisfy the following properties.*

For all $B, C \in P(W)$;

$$\begin{array}{ll} (L1)\ \underline{H}(C) = (\overline{H}(C^c))^c; & (U1)\ \overline{H}(C) = (\underline{H}(C^c))^c; \\ (L2)\ \underline{H}(W) = Z; & (U2)\ \overline{H}(\emptyset) = \emptyset; \\ (L3)\ \underline{H}(C \cap B) = \underline{H}(C) \cap \underline{H}(B); & (U3)\ \overline{H}(C \cup B) = \overline{H}(C) \cup \overline{H}(B); \\ (L4)\ C \subseteq B \Longrightarrow \underline{H}(C) \subseteq \underline{H}(B); & (U4)\ C \subseteq B \Longrightarrow \overline{H}(C) \subseteq \overline{H}(B); \\ (L5)\ \underline{H}(B) \cup \underline{H}(C) \subseteq \underline{H}(C \cup B); & (U5)\ \overline{H}(C \cap B) \subseteq \overline{H}(C) \cap \overline{H}(B). \end{array}$$

Where C^c is the complement of C.

Throughout the thesis, for generalized approximation space, generalized lower and upper approximations, lower and upper approximations, the following shortened forms GAS, GLA and GUA, LA and UA, respectively, will be used.

Chapter 2

Roughness in Quantale Modules

In this chapter, we study the roughness in subsets of a Q_t-module with respect to Pawlak approximation space. We present some basic properties of upper and lower approximations. We initiate the study of upper and lower rough approximations of Q_t-submodule of a Q_t-module and discuss the relations between the lower (upper) rough Q_t-submodules of Q_t-module and the lower (upper) approximations of their homomorphic images. Generalized roughness is also introduced in this chapter. The idea of set-valued homomorphism and strong set-valued homomorphism of Q_t-modules are presented.

In the first section, properties of lower and upper approximations of subsets of Q_t-modules are discussed. Next, complete congruence with respect to $\vee$-complete and $*$-complete is introduced. Further, upper and lower rough Q_t-submodules of Q_t-module are defined and their different properties are discussed. In the second section, the relations between the lower (upper) rough Q_t-submodules of Q_t-module and the lower (upper) approximations of their homomorphic images are discussed. Moreover, roughness in quotient of Q_t-module are proposed. In the third section, set-valued homomorphism and strong set-valued homomorphism of Q_t-modules are defined. Properties of lower and upper approximations of subsets of Q_t-modules are discussed. The last section shows the relation between homomorphic image of upper (lower) approximations of a subset of Q_t-module and the upper (lower) approximations of homomorphic image of of a subset of Q_t-module.

2.1 Pawlak Approximation of Q_t-module

In this section, we present the roughness in subsets of a Q_t-module regarding Pawlak approximation space. We contemplate some fundamental properties of lower approximation (LA) and upper approximation (UA). Additionally, we will present the idea of rough Q_t-submodules and discuss their properties. For quantale module homomorphism, quantale module isomorphism, set-valued map, set-valued homomorphism and strong set-valued homomorphism, the following shortened forms QMH, QMI, SVM, SVH and $SSVH$, respectively, will be utilized.

Definition 2.1.1 *Let η be a congruence relation on a Q_t-module M. Let A be a subset of M. Then the sets*

$$\begin{aligned} \underline{\eta}(A) &= \left\{m \in M \mid [m]_\eta \subseteq A\right\} \text{ and} \\ \overline{\eta}(A) &= \left\{m \in M \mid [m]_\eta \cap A \neq \emptyset\right\} \end{aligned}$$

are known as the LA and UA of A.

Example 2.1.2 *Take the Q_t-module M of Example 1.3.2. Let*

$$\alpha = \{(\bot, \bot), (x, x), (\top, \top), (x, \bot), (\bot, x)\}$$

be an equivalence relation on M. Then it is easy to check that α is a congruence on the Q_t-module M. The α-equivalence classes are $\{\bot, x\}$ and $\{\top\}$. Let $A = \{x, \top\}$. Then $\underline{\alpha}(A) = \{\top\}$ and $\overline{\alpha}(A) = M$. It is obvious that $\underline{\alpha}(A) \subseteq A \subseteq \overline{\alpha}(A)$.

Theorem 2.1.3 *Let η and λ be congruence relations on a Q_t-module M. If A and B are non-empty subsets of M, then the following hold*;

(1) $\underline{\eta}(A) \subseteq A \subseteq \overline{\eta}(A)$;

(2) $\overline{\eta}(A \cup B) = \overline{\eta}(A) \cup \overline{\eta}(B)$;

(3) $\underline{\eta}(A \cap B) = \underline{\eta}(A) \cap \underline{\eta}(B)$;

(4) $A \subseteq B$ *implies* $\underline{\eta}(A) \subseteq \underline{\eta}(B)$;

(5) $A \subseteq B$ *implies* $\overline{\eta}(A) \subseteq \overline{\eta}(B)$;

(6) $\underline{\eta}(A \cup B) \supseteq \underline{\eta}(A) \cup \underline{\eta}(B)$;

(7) $\overline{\eta}(A \cap B) \subseteq \overline{\eta}(A) \cap \overline{\eta}(B)$;

(8) $\eta \subseteq \lambda$ *implies* $\underline{\eta}(A) \supseteq \underline{\lambda}(B)$;

(9) $\eta \subseteq \lambda$ *implies* $\overline{\eta}(A) \subseteq \overline{\lambda}(B)$.

Proof. The proof is similar to Theorem 2.1 of [45]. ■

Theorem 2.1.4 *Let η be a congruence relation on a Q_t-module M. If A and B are non-empty subsets of M, then*

(1) $\overline{\eta}(A) \cup \overline{\eta}(B) \subseteq \overline{\eta}(A \vee B)$, *if* $\perp \in A \cap B$,

(2) $\underline{\eta}(A) \cap \underline{\eta}(B) \subseteq \underline{\eta}(A \vee B)$,

(3) $\underline{\eta}(A) \cup \underline{\eta}(B) \subseteq \underline{\eta}(A \vee B)$, *if* $\perp \in A \cap B$.

Proof. (1) Let $a \in A$, we have $a \vee \perp \in A \vee B$ because $\perp \in B$. Hence $A \subseteq A \vee B$. Similarly, $B \subseteq A \vee B$. Thus $A \cup B \subseteq A \vee B$. By Theorem 2.1.3, we get $\overline{\eta}(A) \cup \overline{\eta}(B) = \overline{\eta}(A \cup B) \subseteq \overline{\eta}(A \vee B)$.

(2) It is easy to prove that $A \cap B \subseteq A \vee B$. By Theorem 2.1.3, we have $\underline{\eta}(A) \cap \underline{\eta}(B) = \underline{\eta}(A \cap B) \subseteq \underline{\eta}(A \vee B)$.

(3) It is similar to part 1. ■

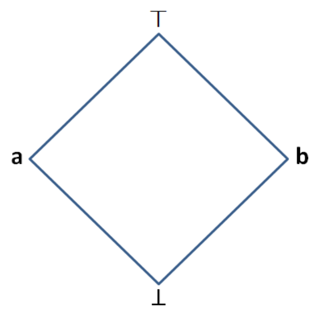

Fig. 5

Example 2.1.5 *Consider the complete lattice Q_1 as shown in Fig. 5 and the operation "$\otimes_1$" on Q_1 is defined as $z \otimes_1 w = \perp$ for all $z, w \in Q_1$. Then Q_1 is a quantale. Also, Q_1 is a Q_t-module over Q_1. Let η be an equivalence relation on a Q_t-module Q_1 with the η-equivalence classes being $\{\perp, b\}, \{a, \top\}$. It is easy to check that η is a congruence* *relation on* Q_1. *Let* $A = \{\bot, b\}$ *and* $B = \{\bot, a\}$. *Then* $\underline{\eta}(A) = \{\bot, b\}$, $\underline{\eta}(B) = \emptyset$, $\underline{\eta}(A \vee B) = Q_1$ *and* $\overline{\eta}(A) = \{\bot, b\}$, $\overline{\eta}(B) = Q_1$. *Thus converse of parts* 2 *and* 3 *of Theorem* 2.1.4, *are not true in general.*

Definition 2.1.6 *Let* η *be an equivalence relation on a* Q_t*-module* M. *Then* η *is called a weak congruence on* M *if for all* $a, b, c, d \in M$ *and* $q \in Q_t$, $a\eta b$ *and* $c\eta d$ *implies* $(a \vee c)\eta(b \vee d)$ *and* $a\eta b$ *imply* $(q * a)\eta(q * b)$.

Theorem 2.1.7 *Let* η *be a weak congruence relation on a* Q_t*-module* M. *If* A *and* B *are non-empty subsets of* M, *then*

(1) $\overline{\eta}(A) \vee \overline{\eta}(B) \subseteq \overline{\eta}(A \vee B)$;

(2) $\overline{\eta}(A) \cap \overline{\eta}(B) \subseteq \overline{\eta}(A \vee B)$.

Proof. (1) Suppose that $c \in \overline{\eta}(A) \vee \overline{\eta}(B)$. Then there exist $a \in \overline{\eta}(A)$, $b \in \overline{\eta}(B)$ such that $c = a \vee b$. So there exist $x \in [a]_\eta \cap A$ and $y \in [b]_\eta \cap B$ such that $x \vee y \in A \vee B$ and $x \vee y \in [a]_\eta \vee [b]_\eta \subseteq [a \vee b]_\eta$. We have $x \vee y \in [a \vee b]_\eta \cap A \vee B$. Thus, $c = a \vee b \in \overline{\eta}(A \vee B)$.

(2) Suppose that $w \in \overline{\eta}(A) \cap \overline{\eta}(B)$. Then there exist $a \in [w]_\eta \cap A$ and $b \in [w]_\eta \cap B$. Thus, we have $a \vee b \in [w]_\eta \vee [w]_\eta \subseteq [w]_\eta$. So $a \vee b \in (A \vee B) \cap [w]_\eta$. Hence $w \in \overline{\eta}(A \vee B)$.
■

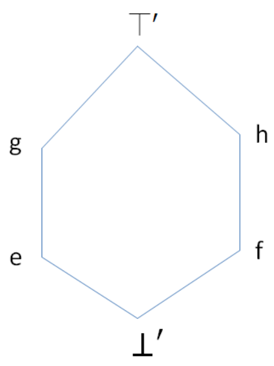

Fig. 6

Example 2.1.8 *Let* Q_2 *be a complete lattice as depicted in Fig* 6 *and operation "*$\otimes_2$*" on* Q_2 *is defined as* $x \otimes_2 y = \bot'$ *for all* $x, y \in Q_2$. *Then* Q_2 *is a quantale. Consider* Q_2 *as a* Q_t*-module over itself. Let*

$$\alpha = \{(\bot', \bot'), (e, e), (g, g), (h, h), (f, f), (\top', \top'), (f, \bot'), (\bot', f)\}.$$

Then α is an equivalence relation on a Q_t-module Q_2 with the α-equivalence classes being $\{\bot', f\}, \{e\}, \{g\}, \{h\}, \{\top'\}$. It is easy to verify that α is a congruence relation on Q_2. Let $A = \{\bot', e\}$ and $B = \{\bot', h\}$. Then $\overline{\alpha}(A) = \{\bot', f, e\}$, $\overline{\alpha}(B) = \{\bot', f, h\}$ and $\overline{\alpha}(A) \cup \overline{\alpha}(B) = \{\bot', e, f, h\}$. Also $A \vee B = \{\bot', h, e, \top'\}$ and $\overline{\alpha}(A \vee B) = \{\bot', f, h, e, \top'\}$. Hence converse of Theorem 2.1.4(1) is not valid in general. Suppose η is an equivalence relation on a Q_t-module Q_1 having η-equivalence classes $\{\bot'\}, \{a\}$ and $\{b, \top'\}$. Clearly η is a weak congruence relation on Q_1. Let $A = \{a\}$ and $B = \{\bot', b\}$. Then $\overline{\eta}(A) = \{a\}$, $\overline{\eta}(B) = \{\bot', b, \top'\}$, $\overline{\eta}(A \vee B) = \{a, b, \top'\}$ and $\underline{\eta}(A) = \{a\}$, $\underline{\eta}(B) = \{\bot'\}, \underline{\eta}(A \vee B) = \{a\}$. Hence $\underline{\eta}(A) \vee \underline{\eta}(B) = \{a\}$ and $\overline{\eta}(A) \cap \overline{\eta}(B) = \emptyset$. This concludes that converse of all parts of Theorem 2.1.7, are not true in general.

Theorem 2.1.9 *Let η be a congruence relation on a Q_t-module M and A, B be Q_t-submodule ideals of M. Then $\overline{\eta}(A \wedge B) = \overline{\eta}(A) \cap \overline{\eta}(B)$.*

Proof. It is easy to prove that $A \wedge B = A \cap B$. Hence $\underline{\eta}(A \wedge B) = \underline{\eta}(A \cap B) = \underline{\eta}(A) \cap \underline{\eta}(B) = \underline{\eta}(A) \wedge \underline{\eta}(B)$. ■

Definition 2.1.10 *A congruence relation η on a Q_t-module M is called $\vee$-complete if $\vee_{i\in I} [x_i]_\eta = [\vee_{i\in I} x_i]_\eta$ for $x_i \in M$, and is called "$*$" complete if it satisfies $q * [x]_\eta = [q * x]_\eta$ for $x \in M$ and $q \in Q_t$. η is a complete congruence if it is $\vee$-complete and $*$ complete.*

Definition 2.1.11 *Let M be a Q_t-module and η be an equivalence relation on M. A subset $M_1 \subseteq M$ is called an upper (lower) rough Q_t-submodule of M if $\overline{\eta}(M_1)$ ($\underline{\eta}(M_1)$) is a Q_t-submodule of M. If M_1 is both upper and a lower rough Q_t-submodule of M, then we say that M_1 is a rough Q_t-submodule of M.*

Theorem 2.1.12 *Let η be a congruence relation on a Q_t-module M and $\emptyset \neq M_1 \subseteq M$. If M_1 is a Q_t-submodule of M, then M_1 is also an upper rough Q_t-submodule of M.*

Proof. Clearly $\emptyset \neq M_1 \subseteq \overline{\eta}(M_1)$. Let $x_i \in \overline{\eta}(M_1)$ for $i \in I$. Then there exists $a_i \in M_1$ for $i \in I$ such that $x_i \eta a_i$. Since η is a congruence relation, we have $(\vee x_i)\eta(\vee a_i)$. But M_1 is a Q_t-submodule of M, we have $\vee a_i \in M_1$. This shows that $(\vee x_i) \in \overline{\eta}(M_1)$. Let $q \in Q_t$ and $x \in \overline{\eta}(M_1)$. Then there exists $y \in M_1$ with $y \eta x$. Since η is a congruence relation and M_1 is a Q_t-submodule of M, we have $q * y \in M_1$ and $(q * y)\eta(q * x)$. This implies $q * x \in \overline{\eta}(M_1)$. Therefore $\overline{\eta}(M_1)$ is a Q_t-submodule of M, that is M_1 is an upper rough Q_t-submodule of M. ■

Theorem 2.1.13 *Let η be a complete congruence on a Q_t-module M and $M_1 \subseteq M$. If M_1 is a Q_t-submodule of M and $\underline{\eta}(M_1) \neq \emptyset$, then M_1 is also a lower rough Q_t-submodule of M.*

Proof. Let $x_i \in \underline{\eta}(M_1)$ for $i \in I$. Then $[x_i]_\eta \subseteq M_1$ for all $i \in I$. Since η is a complete congruence on the Q_t-module M and M_1 is a Q_t-submodule of M, we have $[\vee_{i\in I} x_i]_\eta = \vee_{i\in I} [x_i]_\eta \subseteq M_1$. Hence $\vee_{i\in I} x_i \in \underline{\eta}(M_1)$. Assume $q \in Q_t$ and $x \in \underline{\eta}(M_1)$, then we have $[x]_\eta \subseteq M_1$. Since η is a complete congruence and M_1 is a Q_t-submodule of M, we have $[q * x]_\eta = q * [x]_\eta \subseteq M_1$. Thus, we have $q * x \in \underline{\eta}(M_1)$. Therefore $\underline{\eta}(M_1)$ is a Q_t-submodule of M, that is M_1 is a lower rough Q_t-submodule of M. ■

By the above two Theorems, we have the following Theorem.

Theorem 2.1.14 *Let η be a complete congruence on a Q_t-module M. If M_1 is a Q_t-submodule of M and $\underline{\eta}(M_1) \neq \emptyset$, then M_1 is also a rough Q_t-submodule of M.*

Proposition 2.1.15 *Let M be a Q_t-module and M_1 be a Q_t-submodule of M. Define a relation η_{M_1} on M by $a\eta_{M_1}b$ if and only if there exist $m_1, m_2 \in M_1$ such that $a \vee m_1 = b \vee m_2$. Then η_{M_1} is a congruence on the Q_t-module M. (η_{M_1} is also called congruence induced by M_1).*

Proof. We show that η_{M_1} is an equivalence relation on M. Since $\perp \in M_1$, we have that $a\eta_{M_1}a$ for each $a \in M$, i.e., η_{M_1} is reflexive. By the definition of η_{M_1}, it is clear that η_{M_1} is symmetric. Suppose that $a\eta_{M_1}b$ and $b\eta_{M_1}c$. Then there exist m_1, m_2, m_3, $m_4 \in M_1$ such that $a \vee m_1 = b \vee m_3$ and $b \vee m_2 = c \vee m_4$ and thus $a \vee (m_1 \vee m_2) = (a \vee m_1) \vee m_2 = (b \vee m_3) \vee m_2 = (b \vee m_2) \vee m_3 = (c \vee m_4) \vee m_3 = c \vee (m_3 \vee m_4)$. Furthermore, since $m_1 \vee m_2$, $m_3 \vee m_4 \in M_1$, we have $a\eta_{M_1}c$. This shows that η_{M_1} is transitive.

Next, we shall show that η_{M_1} is a congruence on M. Assume that $a\eta_{M_1}b$ and $q \in Q_t$. Then there exist m_1, $m_2 \in M_1$ such that $a \vee m_1 = b \vee m_2$ and thus $(q * a) \vee (q * m_1) = q * (a \vee m_1) = q * (b \vee m_2) = (q * b) \vee (q * m_2)$. Since $q * m_1$, $q * m_2 \in M_1$, we have $(q * a)\eta_{M_1}(q * b)$. Let $a_i\eta_{M_1}b_i$ for $i \in I$. Then there exist m_i, $m_i' \in M_1$ such that $a_i \vee m_i = b_i \vee m_i'$ but then $\vee_{i\in I}(a_i \vee m_i) = \vee_{i\in I}(b_i \vee m_i') \Rightarrow (\bigvee_{i\in I} a_i) \vee (\bigvee_{i\in I} m_i) = (\bigvee_{i\in I} b_i)$

$\vee\ (\underset{i\in I}{\vee} m_i')$. Since $\underset{i\in I}{\vee} m_i, \underset{i\in I}{\vee} m_i' \in M_1$, we have $(\underset{i\in I}{\vee} a_i)\eta_{M_1}(\underset{i\in I}{\vee} b_i)$. As a consequence, η_{M_1} is a congruence on M. ■

Proposition 2.1.16 *Let M be a Q_t-module and M_1 be a Q_t-submodule-ideal of M. Then*

(1) *for every $m \in M$, $[m]_{\eta_{M_1}} = M_1$ if and only if $m \in M_1$;*

(2) $\underline{\eta}_{M_1}(M_1) = M_1 = \overline{\eta}_{M_1}(M_1)$.

Proof. (1) Let $[m]_{\eta_{M_1}} = M_1$. Since η_{M_1} is reflexive, it can be concluded that $m \in M_1$. Conversely, assume $m \in M_1$. Let $m \in M_1$. Then $m \vee z = z \vee m$ for all $z \in M_1$, and thus $z \in [m]_{\eta_{M_1}}$, that is $M_1 \subseteq [m]_{\eta_{M_1}}$. On the other hand, if $z \in [m]_{\eta_{M_1}}$, then there exist $m_1, m_2 \in M_1$ such that $z \vee m_1 = m \vee m_2$. Since $m \vee m_2 \in M_1$, we have $z \vee m_1 \in M_1$ and $z \in M_1$. Therefore $[m]_{\eta_{M_1}} = M_1$.

(2) It is clear that $\underline{\eta}_{M_1}(M_1) \subseteq M_1 \subseteq \overline{\eta}_{M_1}(M_1)$. By part (1), we conclude that $\underline{\eta}_{M_1}(M_1) = M_1 = \overline{\eta}_{M_1}(M_1)$. ■

Proposition 2.1.17 *Let η be a congruence on a Q_t-module M. Then $[\bot]_\eta$ is a Q_t-submodule of M.*

Proof. Clearly $[\bot]_\eta \neq \emptyset$

(1) Let $a_i \in [\bot]_\eta$ for $i \in I$. Then $a_i\eta\bot$. Since η is a congruence, we have $\vee_{i\in I} a_i\eta\bot$, i.e., $\vee_{i\in I} a_i \in [\bot]_\eta$.

(2) Let $q \in Q_t$ and $w \in [\bot]_\eta$. Then $w\eta\bot$ and $(q * w)\eta\bot$. It follows that $q * w \in [\bot]_\eta$. Thus, $[\bot]_\eta$ is a Q_t-submodule of M. ■

Proposition 2.1.18 *Let η be a weak congruence relation on a Q_t-module M. Then $\eta_{[\bot]_\eta} \subseteq \eta$.*

Proof. Suppose $z\ \eta_{[\bot]_\eta} w$. Then there exist $v, t \in [\bot]_\eta$ such that $z \vee v = w \vee t$. Since $v\eta\bot$, $t\eta\bot$ and η is a weak congruence on M, we have $(z \vee v)\eta z$, $(w \vee t)\eta w$. Therefore, $z\eta w$ by transitivity. i.e., $\eta_{[\bot]_\eta} \subseteq \eta$. ■

Proposition 2.1.19 *Let M be a Q_t-module and M_1 be a Q_t-submodule-ideal of M. Then $[\bot]_{\eta_{M_1}} = M_1$.*

Proof. Let $z \in [\perp]_{\eta_{M_1}}$. Then $z\eta_{M_1}\perp$ and there exist $v_1, v_2 \in M_1$ such that $z \vee v_1 = \perp \vee v_2 = v_2$, and thus $z \in M_1$. Conversely, suppose $z \in M_1$. Then $z\eta_{M_1}\perp$, i.e., $z \in [\perp]_{\eta_{M_1}}$. Thus, $[\perp]_{\eta_{M_1}} = M_1$. ■

Lemma 2.1.20 *Let M_1 be a Q_t-submodule-ideal of a Q_t-module M and $\emptyset \neq B \subseteq M$. Then the statements below hold;*

(1) $\overline{\eta}_{M_1}(B) = M_1$ *if and only if* $B \subseteq M_1$;

(2) $M_1 \subseteq \underline{\eta}_{M_1}(B)$ *if and only if* $M_1 \subseteq B$;

(3) *If $M_1 \subseteq B$ and B is a Q_t-submodule-ideal of M then* $\underline{\eta}_{M_1}(B) = B = \overline{\eta}_{M_1}(B)$.

Proof. (1) Let $\overline{\eta}_{M_1}(B) = M_1$. Then, $B \subseteq \overline{\eta}_{M_1}(B) = M_1$. Conversely, let $B \subseteq M_1$, by Proposition 2.1.16(2), we get $\overline{\eta}_{M_1}(B) \subseteq \overline{\eta}_{M_1}(M_1) = M_1$. Let $m \in M_1$. Then from Proposition 2.1.16(1), it follows that $[m]_{\eta_{M_1}} \cap B = M_1 \cap B = B$, and $m \in \overline{\eta}_{M_1}(B)$. Thus, we have $M_1 \subseteq \overline{\eta}_{M_1}(B)$. Therefore $\overline{\eta}_{M_1}(B) = M_1$.

(2) It is obvious that $M_1 \subseteq B$ whenever $M_1 \subseteq \underline{\eta}_{M_1}(B)$. Suppose that $M_1 \subseteq B$. Then $M_1 = \underline{\eta}_{M_1}(M_1) \subseteq \underline{\eta}_{M_1}(B)$ by Proposition 2.1.16(2).

(3) It is clear that $\underline{\eta}_{M_1}(B) \subseteq B \subseteq \overline{\eta}_{M_1}(B)$, we need only to show that $B \subseteq \underline{\eta}_{M_1}(B)$ and $\overline{\eta}_{M_1}(B) \subseteq B$. Let $b \in B$. For $w \in [b]_{\eta_{M_1}}$, there exist m_1,$m_2 \in M_1$ such that $w \vee m_1 = b \vee m_2$. Since $b \vee m_2 \in B$, we have $w \in B$, which gives $[b]_{\eta_{M_1}} \subseteq B$, i.e., $b \in \underline{\eta}_{M_1}(B)$. Thus $B \subseteq \underline{\eta}_{M_1}(B)$. Similarly, we can show that $\overline{\eta}_{M_1}(B) \subseteq B$. As a consequence, $\underline{\eta}_{M_1}(B) = B = \overline{\eta}_{M_1}(B)$. ■

2.2 Problem of Homomorphism and Quotients of $\mathbf{Q}_t$-modules

In this section, relations between the upper (lower) rough Q_t-submodules of Q_t-module and the upper approximation (UA) of their homomorphic images will be discussed.

Theorem 2.2.1 *Let M and M' be Q_t-modules and $\rho_m : M \longrightarrow M'$ be a QMH. If B is a non-empty subset of M and $\eta = ker(\rho_m)$, then*

(1) $\rho_m(\overline{\eta}(B)) = \rho_m(B)$.

(2) *If ρ_m is one-one, then* $\rho_m(\underline{\eta}(B)) = \rho_m(B)$.

Proof. (1) Since $B \subseteq \overline{\eta}(B)$, then $\rho_m(B) \subseteq \rho_m(\overline{\eta}(B))$. To see that the reverse inclusion holds, let $y \in \rho_m(\overline{\eta}(B))$. Then there exists an element $w \in \overline{\eta}(B)$ such that $\rho_m(w) = y$. Thus there exists an element $b \in M$ such that $b \in [w]_\eta \cap B$, and so $b \in [w]_\eta$ and $b \in B$. Thus $(b,w) \in \eta$ such that $\rho_m(w) = \rho_m(b)$. Then $y = \rho_m(w) = \rho_m(b) \in \rho_m(B)$ and so $\rho_m(\overline{\eta}(B)) \subseteq \rho_m(B)$. Thus, we have $\rho_m(B) = \rho_m(\overline{\eta}(B))$.

(2) If ρ_m is one-one then $[x]_\eta = \{x\}$ because if $y \in [x]_\eta$ then $\rho_m(y) = \rho_m(x) \Longrightarrow y = x$ because ρ_m is one one. Thus in this case $\underline{\eta}(B) = B = \overline{\eta}(B)$. This implies that $\rho_m(\underline{\eta}(B)) = \rho_m(B) = \rho_m(\overline{\eta}(B))$. ■

Proposition 2.2.2 *Let M and N be Q_t-modules, $\rho_m : M \longrightarrow N$ a surjective QMH and η_2 be a congruence on N. Set $\eta_1 = \{(m_1, m_2) \in M \times M \mid (\rho_m(m_1), \rho_m(m_2)) \in \eta_2\}$, then*

(1) *η_1 is a congruence relation on M;*

(2) *$\overline{\eta}_2(\rho_m(B)) = \rho_m(\overline{\eta}_1(B))$ for each $B \subseteq M$;*

(3) *$\underline{\eta}_2(\rho_m(B)) \supseteq \rho_m(\underline{\eta}_1(B))$ for each $B \subseteq M$, if ρ_m is injective, then $\underline{\eta}_2(\rho_m(B)) = \rho_m(\underline{\eta}_1(B))$.*

Proof. (1) Clearly, η_1 is an equivalence relation. For congruence relation, let $w_i\eta_1 y_i$ for all $i \in I$. Then $\rho_m(w_i)\eta_2\rho_m(y_i)$ for all $i \in I$. Since ρ_m is a QMH, $\vee_{i\in I}\rho_m(w_i)\eta_2\vee_{i\in I}$ $\rho_m(y_i)$ implies that $\rho_m(\vee_{i\in I}w_i)\eta_2\rho_m(\vee_{i\in I}y_i)$, i.e., $(\rho_m(\vee_{i\in I}w_i),\ \rho_m(\vee_{i\in I}y_i)) \in \eta_2$. Thus we have, $((\vee_{i\in I}w_i),(\vee_{i\in I}y_i)) \in \eta_1$. Let $w\eta_1 y$. Then $\rho_m(w)\eta_2\rho_m(y)$. Let $a \in Q_t$, since η_2 is a congruence relation and ρ_m is a QMH, we have $(a * \rho_m(w))\eta_2(a * \rho_m(y))$ $\Longrightarrow \rho_m(a*w)\eta_2\rho_m(a*y)$. So, $(a*w)\eta_1(a*y)$, *i.e.*, $((a*w),(a*y)) \in \eta_1$. Consequently, η_1 is a congruence relation on M.

(2) Let $z \in \rho_m(\overline{\eta}_1(B))$. Then there exists $a \in \overline{\eta}_1(B)$ such that $\rho_m(a) = z$ and $[a]_{\eta_1} \cap B \neq \emptyset$. Thus there exists $x \in [a]_{\eta_1} \cap B$ such that $x \in B$ and $(x,a) \in \eta_1$. This shows that $(\rho_m(x), \rho_m(a)) \in \eta_2 \Rightarrow \rho_m(x) \in [\rho_m(a)]_{\eta_2}$. Also, $\rho_m(x) \in \rho_m(B)$. Thus $[\rho_m(a)]_{\eta_2} \cap \rho_m(B) \neq \emptyset \Rightarrow z = \rho_m(a) \in \overline{\eta}_2(\rho_m(B))$, that is $\rho_m(\overline{\eta}_1(B)) \subseteq \overline{\eta}_2(\rho_m(B))$. Conversely, let $w \in \overline{\eta}_2(\rho_m(B))$. Then there exists $a \in \rho_m(B)$ such that $(w,a) \in \eta_2$. Since ρ_m is surjective so there exist $x \in B$ and $s \in Q_t$ such that $a = \rho_m(x)$ and $w = \rho_m(s)$. Thus $(\rho_m(s), \rho_m(x)) = (w,a) \in \eta_2 \Rightarrow (s,x) \in \eta_1$. This implies $x \in [s]_{\eta_1} \cap B$, so we have $s \in \overline{\eta}_1(B)$, that is $w = \rho_m(s) \in \rho_m(\overline{\eta}_1(B))$. Thus $\overline{\eta}_2(\rho_m(B)) \subseteq \rho_m(\overline{\eta}_1(B))$. Hence $\rho_m(\overline{\eta}_1(B)) = \overline{\eta}_2(\rho_m(B))$.

(3) Let $b \in \rho_m\left(\underline{\eta_1}(B)\right)$. Then there exists $a \in \underline{\eta_1}(B)$ such that $\rho_m(a) = b$ and $[a]_{\eta_1} \subseteq B$. Let $y' \in [b]_{\eta_2}$. Then there exist $x' \in Q_t$ such that $\rho_m(x') = y'$ and $\rho_m(x') \in [\rho_m(a)]_{\eta_2}$, i.e., $(\rho_m(x'), \rho_m(a)) \in \eta_2$. Hence $(x', a) \in \eta_1$, i.e., $x' \in [a]_{\eta_1} \subseteq B$ and so $\rho_m(x') \in \rho_m(B)$. Thus, $[b]_{\eta_2} \subseteq \rho_m(B)$ which yields that $b \in \underline{\eta_2}(\rho_m(B))$. So we have $\rho_m\left(\underline{\eta_1}(B)\right) \subseteq \underline{\eta_2}(\rho_m(B))$.

Now, suppose that ρ_m is one one and let $b \in \underline{\eta_2}(\rho_m(B))$. Then there exists a unique $a \in Q_t$ such that $\rho_m(a) = b$ and $[\rho_m(a)]_{\eta_2} \subseteq \rho_m(B)$. Let $u' \in [a]_{\eta_1}$, i.e., $(a, u') \in \eta_1$. Then $(\rho_m(a), \rho_m(u') \in \eta_2$, i.e., $\rho_m(u') \in [\rho_m(a)]_{\eta_2} \subseteq \rho_m(B)$, and so $u' \in B$. Thus, $[a]_{\eta_1} \subseteq B$, which gives $a \in \underline{\eta_1}(B)$. Then $b = \rho_m(a) \in \rho_m(\underline{\eta_1}(B))$, and so $\underline{\eta_2}(\rho_m(B)) \subseteq \rho_m(\underline{\eta_1}(B))$. ∎

Lemma 2.2.3 *Let M and N be two Q_t-modules, $\rho_m : M \longrightarrow N$ be a surjective QMH and η_2 be a congruence relation on N and η_1 the congruence on M defined in Proposition* 2.2.2. *Then for each $w \in M$ and $A \subseteq M$, the following hold;*

(1) $w \in \overline{\eta}_1(A) \Longleftrightarrow \rho_m(w) \in \rho_m(\overline{\eta}_1(A))$.

(2) $w \in \underline{\eta}_1(A) \Longleftrightarrow \rho_m(w) \in \rho_m(\underline{\eta}_1(A))$.

Proof. (1) Let $w \in \overline{\eta}_1(A)$. Then $\rho_m(w) \in \rho_m(\overline{\eta}_1(A))$. Conversely, if $\rho_m(w) \in \rho_m(\overline{\eta}_1(A))$, then there exists $a \in \overline{\eta}_1(A)$ such that $\rho_m(w) = \rho_m(a)$, then $\rho_m(w)\eta_2\rho_m(a)$ and thus $w\eta_1 a$. Therefore, $w \in [a]_{\eta_1} \subseteq \overline{\eta}_1(A)$.

(2) Proof is similar to the part (1). ∎

Theorem 2.2.4 *Let ρ_m be a surjective QMH from a Q_t-module M to a Q_t-module M'. Let η_2 be a congruence relation on M' and A be a subset of M. If $\eta_1 = \{(m_1, m_2) \in M \times M \mid (\rho_m(m_1), \rho_m(m_2)) \in \eta_2\}$, then*

(1) *$\overline{\eta}_1(A)$ is a Q_t-submodule of M if and only if $\overline{\eta}_2(\rho_m(A))$ is a Q_t-submodule of M'.*

(2) *$\underline{\eta}_1(A)$ is a Q_t-submodule of M if and only if $\underline{\eta}_2(\rho_m(A))$ is a Q_t-submodule of M'.*

Proof. By Proposition 2.2.2(3), $\overline{\eta}_2(\rho_m(A)) = \rho_m(\overline{\eta}_1(A))$ for each $A \subseteq M$.

(1) Let $\rho_m(\overline{\eta}_1(A))$ is a Q_t-submodule of M'.

(i) Let $w_i \in \overline{\eta}_1(A)$ $(i \in I)$. Then $\rho_m(w_i) \in \rho_m(\overline{\eta}_1(A))$ $(i \in I)$. Since $\rho_m(\overline{\eta}_1(A))$ is a Q_t-submodule and ρ_m is a QMH, we have $\rho_m(\vee_{i\in I} w_i) = \vee_{i\in I}\rho_m(w_i) \in \rho_m(\overline{\eta}_1(A))$. By Lemma 2.2.3, we have $\vee_{i\in I} w_i \in \overline{\eta}_1(A)$.

(ii) Let $w \in \overline{\eta}_1(A)$ and $q \in Q_t$. Then $\rho_m(w) \in \rho_m(\overline{\eta}_1(A))$. Since $\rho_m(\overline{\eta}_1(A))$ is a Q_t-submodule of M', we have $\rho_m(q*w) = q*\rho_m(w) \in \rho_m(\overline{\eta}_1(A))$. Thus $q*w \in \overline{\eta}_1(A)$.

By *(i)*-*(ii)*, $\overline{\eta}_1(A)$ is a Q_t-submodule of M.

Conversely, suppose $\overline{\eta}_1(A)$ is a Q_t-submodule of M. We want to show that $\rho_m(\overline{\eta}_1(A))$ is a Q_t-submodule of M'.

(i) Let $y_i \in \rho_m(\overline{\eta}_1(A))$ $(i \in I)$. Then there exists $w_i \in \overline{\eta}_1(A)$ such that $y_i = \rho_m(w_i)$ $(i \in I)$. We have $\vee_{i\in I} y_i = \vee_{i\in I}\rho_m(w_i) = \rho_m(\vee_{i\in I} w_i)$. Since $\overline{\eta}_1(A)$ is a Q_t-submodule of M, $\vee_{i\in I} w_i \in \overline{\eta}_1(A)$ if and only if $\rho_m(\vee_{i\in I} w_i) = \vee_{i\in I} y_i \in \rho_m(\overline{\eta}_1(A))$. Thus, we have $\vee_{i\in I} y_i \in \rho_m(\overline{\eta}_1(A))$.

(ii) Let $y \in \rho_m(\overline{\eta}_1(A))$ and $q \in Q_t$. Then $w \in \overline{\eta}_1(A)$ such that $\rho_m(w) = y$. Since, $\overline{\eta}_1(A)$ is a Q_t-submodule of M and ρ_m is a QMH, we have $q*\rho_m(w) = \rho_m(q*w) = q*y$. Then $q*w \in \overline{\eta}_1(A)$ if and only if $q*y = \rho_m(q*w) \in \rho_m(\overline{\eta}_1(A))$.

By *(i)*-*(ii)*, $\rho_m(\overline{\eta}_1(A)) = \overline{\eta}_2(\rho_m(A))$ is a Q_t-submodule of M'.

(2) The proof is similar to that of (1). ■

Let η be a congruence relation on a Q_t-module M. We can define operations $\vee$ and $*$ on the quotient Q_t-module $M/\eta = \left\{[m]_\eta \mid m \in M\right\}$ as follows:

$$\vee_{i\in I}[m_i]_\eta = [\vee_{i\in I} m_i]_\eta \text{ and } [q*m]_\eta = q*[m]_\eta \text{ for all } m_i, m \in M \text{ and } q \in Q_t.$$

The LA and UA can be displayed in an alternative form as:

$$\underline{\eta}(A)/\eta = \left\{[w]_\eta \in M/\eta : [w]_\eta \subseteq A\right\}$$

$$\overline{\eta}(A)/\eta = \left\{[w]_\eta \in M/\eta : [w]_\eta \cap A \neq \emptyset\right\}.$$

Theorem 2.2.5 *Let η be a congruence relation on a Q_t-module M and $A \subseteq M$. Then*

(1) A is a lower rough Q_t-submodule of M if and only if $\underline{\eta}(A)/\eta$ is a Q_t-submodule of M/η.

(2) A is an upper rough Q_t-submodule of M if and only if $\overline{\eta}(A)/\eta$ is a Q_t-submodule of M/η.

Proof. (1) Assume that A is a lower rough Q_t-submodule of M. Let $[w_i]_\eta \in \underline{\eta}(A)/\eta$ for $i \in I$. Then $w_i \in \underline{\eta}(A)$. Since A is a lower rough Q_t-submodule of M, we have $\vee_{i\in I} w_i \in \underline{\eta}(A)$. Thus, $\vee_{i\in I}[w_i]_\eta = [\vee_{i\in I} w_i]_\eta \in \underline{\eta}(A)/\eta$. Let $[w]_\eta \in \underline{\eta}(A)/\eta$ and

$q \in Q_t$. Then $w \in \underline{\eta}(A)$ and $q * w \in \underline{\eta}(A)$ because A is a lower rough Q_t-submodule of M. So $[q * w]_\eta = q * [w]_\eta \in \underline{\eta}(A)/\eta$. Hence, $\underline{\eta}(A)/\eta$ is a Q_t-submodule of M/η.

Conversely, suppose that $\underline{\eta}(A)/\eta$ is a Q_t-submodule of M/η. Let $w_i \in \underline{\eta}(A)$ for $i \in I$. Then $[w_i]_\eta \in \underline{\eta}(A)/\eta$ for $i \in I$. Since $\underline{\eta}(A)/\eta$ is a Q_t-submodule, we have $[\vee_{i\in I} w_i]_\eta \in \underline{\eta}(A)/\eta$. So $\vee_{i\in I} w_i \in \underline{\eta}(A)$ for $i \in I$. Let $w \in \underline{\eta}(A)$ and $q \in Q_t$. Then $[w]_\eta \in \underline{\eta}(A)/\eta$ and $q * [w]_\eta = [q * w]_\eta \in \underline{\eta}(A)/\eta$ because $\underline{\eta}(A)/\eta$ is a Q_t-submodule. Hence $q * w \in \underline{\eta}(A)$. Thus $\underline{\eta}(A)$ is a Q_t-submodule of M. Hence A is a lower rough Q_t-submodule of M.

(2) The case of upper approximation can be seen in a similar way. ∎

Now we shall consider the relation between the approximation of a set and the approximation of its preimage. We may get the important results.

Theorem 2.2.6 *Let ρ_m be a surjective QMH from a Q_t-module M to a Q_t-module N and $\rho_m^{-1}(B) = \{w \in M \mid \rho_m(w) \in B\}$ for $B \subseteq N$. If η_1 is a congruence relation on M and set $\eta_2 = \{(\rho_m(w_1), \rho_m(w_2)) \in N \times N \mid (w_1, w_2) \in \eta_1\}$, then*

(1) *η_2 is a congruence relation on N;*

(2) $\overline{\eta}_1(\rho_m^{-1}(B)) = \rho_m^{-1}(\overline{\eta}_2(B))$;

(3) $\underline{\eta}_1(\rho_m^{-1}(B)) = \rho_m^{-1}(\underline{\eta}_2(B))$.

Proof. (1) is straightforward.

(2) Let $u \in \overline{\eta}_1(\rho_m^{-1}(B))$. Then $[u]_{\eta_1} \cap \rho_m^{-1}(B) \neq \emptyset$. Let $u' \in [u]_{\eta_1} \cap \rho_m^{-1}(B)$. Then $\rho_m(u') \in B$ and $(u', u) \in \eta_1$, so we have $(\rho_m(u'), \rho_m(u)) \in \eta_2$. Therefore $\rho_m(u') \in [\rho_m(u)]_{\eta_2} \cap B$. Thus $\rho_m(u) \in \overline{\eta}_2(B) \Rightarrow u \in \rho_m^{-1}(\overline{\eta}_2(B))$. This shows that $\overline{\eta}_1(\rho_m^{-1}(B)) \subseteq \rho_m^{-1}(\overline{\eta}_2(B))$. Let $v \in \rho_m^{-1}(\overline{\eta}_2(B))$. Then $\rho_m(v) \in \overline{\eta}_2(B)$. This shows that $[\rho_m(v)]_{\eta_2} \cap B \neq \emptyset$. Let $v' \in B$ be such that there exist $x \in M$ such that $\rho_m(x) = v'$. Thus $x \in \rho_m^{-1}(B)$ and $\rho_m(x) \in [\rho_m(v)]_{\eta_2}$. This implies that $x \in [v]_{\eta_1}$. So $[v]_{\eta_1} \cap \rho_m^{-1}(B) \neq \emptyset$. Thus $v \in \overline{\eta}_1(\rho_m^{-1}(B))$. This implies that $\rho_m^{-1}(\overline{\eta}_2(B)) \subseteq \overline{\eta}_1(\rho_m^{-1}(B))$. Thus, we have $\overline{\eta}_1(\rho_m^{-1}(B)) = \rho_m^{-1}(\overline{\eta}_2(B))$.

(3) Let $x \in \rho_m^{-1}(\underline{\eta}_2(B))$. Then there exist $y \in \underline{\eta}_2(B)$ such that $\rho_m(x) = y$. Since $\rho_m(x) = y \in \underline{\eta}_2(B) \Rightarrow [\rho_m(x)]_{\eta_2} \subseteq B$. Let $y' \in [\rho_m(x)]_{\eta_2}$. Then there exist $x' \in M$ such that $\rho_m(x') \in [\rho_m(x)]_{\eta_2}$, i.e., $(\rho_m(x'), \rho_m(x)) \in \eta_2$ then $(x', x) \in \eta_1$. This shows that $x' \in [x]_{\eta_1}$. But $y' = \rho_m(x') \in B \Rightarrow x' \in \rho_m^{-1}(B)$. Thus $[x]_{\eta_1} \subseteq \rho_m^{-1}(B)$. This concludes that $x \in \underline{\eta}_1(\rho_m^{-1}(B))$. Therefore $\rho_m^{-1}(\underline{\eta}_2(B)) \subseteq \underline{\eta}_1(\rho_m^{-1}(B))$. Let $u \in \underline{\eta}_1(\rho_m^{-1}(B))$.

Then $[u]_{\eta_1} \subseteq \rho_m^{-1}(B)$. Let $u' \in [u]_{\eta_1}$, i.e., $(u', u) \in \eta_1$. Then $(\rho_m(u'), \rho_m(u)) \in \eta_2$, i.e., $\rho_m(u') \in [\rho_m(u)]_{\eta_2}$. But $\rho_m(u') \in B$. Therefore $[\rho_m(u)]_{\eta_2} \subseteq B \Rightarrow \rho_m(u) \in \underline{\eta}_2(B)$. This shows that $u \in \rho_m^{-1}(\underline{\eta}_2(B))$. Thus $\underline{\eta}_1(\rho_m^{-1}(B)) \subseteq \rho_m^{-1}(\underline{\eta}_2(B))$. Finally, we have $\underline{\eta}_1(\rho_m^{-1}(B)) = \rho_m^{-1}(\underline{\eta}_2(B))$. ■

Theorem 2.2.7 *Let ρ_m be a surjective QMH from a Q_t-module M to a Q_t-module N and $\rho_m^{-1}(B) = \{w \in M \mid \rho_m(w) \in B\}$ for $B \subseteq N$. If η_1 is a congruence relation on M and $\eta_2 = \{(\rho_m(w_1),\ \rho_m(w_2)) \in N \times N \mid (w_1, w_2) \in \eta_1\}$, then*

(1) *$\overline{\eta}_2(B)$ is a Q_t-submodule of N if and only if $\overline{\eta}_1(\rho_m^{-1}(B))$ is a Q_t-submodule of M.*

(2) *$\underline{\eta}_2(B)$ is a Q_t-submodule of N if and only if $\underline{\eta}_1(\rho_m^{-1}(B))$ is a Q_t-submodule of M.*

Proof. (1) Let $\overline{\eta}_2(B)$ be a Q_t-submodule of N. We show that $\overline{\eta}_1(\rho_m^{-1}(B))$ is a Q_t-submodule of M. By Theorem 2.2.6(2), we have $\overline{\eta}_1(\rho_m^{-1}(B)) = \rho_m^{-1}(\overline{\eta}_2(B))$. Let $w_i \in \rho_m^{-1}(\overline{\eta}_2(B))$ for $i \in I$. Then $\rho_m(w_i) \in \overline{\eta}_2(B)$ for $i \in I$. Since $\overline{\eta}_2(B)$ is a Q_t-submodule of N, we have $\rho_m(\vee_{i \in I} w_i) = \vee_{i \in I}\ \rho_m(w_i) \in \overline{\eta}_2(B)$. Thus $\vee_{i \in I} w_i \in \rho_m^{-1}(\overline{\eta}_2(B))$. Let $w \in \rho_m^{-1}(\overline{\eta}_2(B))$ and $q \in Q_t$. Then $\rho_m(w) \in \overline{\eta}_2(B)$. Since $\overline{\eta}_2(B)$ is a Q_t-submodule of N, we have $\rho_m(q*w) = q*\rho_m(w) \in \overline{\eta}_2(B)$. Thus $q*w \in \rho_m^{-1}(\overline{\eta}_2(B))$. Hence $\rho_m^{-1}(\overline{\eta}_2(B))$ is a Q_t-submodule of M. But since $\overline{\eta}_1(\rho_m^{-1}(B)) = \rho_m^{-1}(\overline{\eta}_2(B))$. Thus $\overline{\eta}_1(\rho_m^{-1}(B))$ is a Q_t-submodule of M.

Conversely, suppose $\overline{\eta}_1(\rho_m^{-1}(B))$ is a Q_t-submodule of M. We show that $\overline{\eta}_2(B)$ is a Q_t-submodule of N. Let $y_i \in \overline{\eta}_2(B)$ such that $y_i = \rho_m(w_i)$. Then $w_i \in \rho_m^{-1}(\overline{\eta}_2(B))$. Since $\rho_m^{-1}(\overline{\eta}_2(B))$ is a Q_t-submodule, we get $\vee_{i \in I} w_i \in \rho_m^{-1}(\overline{\eta}_2(B))$ and then $\rho_m(\vee_{i \in I} w_i) \in \overline{\eta}_2(B)$. Now since ρ_m is QMH, we have $\vee_{i \in I} y_i = \vee_{i \in I}\ \rho_m(w_i) = \rho_m(\vee_{i \in I} w_i) \in \overline{\eta}_2(B)$. Let $y \in \overline{\eta}_2(B)$ and $q \in Q_t$. Then there exists $w \in M$ such that $y = \rho_m(w) \in \overline{\eta}_2(B)$ and $w \in \rho_m^{-1}(\overline{\eta}_2(B))$. Since $\rho_m^{-1}(\overline{\eta}_2(B))$ is a Q_t-submodule, we have $q * w \in \rho_m^{-1}(\overline{\eta}_2(B))$. Hence $q * y = q * \rho_m(w) = \rho_m(q * w) \in \overline{\eta}_2(B)$. Thus $\overline{\eta}_2(B)$ is a Q_t-submodule of N.

(2) Proof is similar to part 1. ■

2.3 Generalized Rough $\mathbf{Q}_t$-submodules

In this section, we define the concept of set-valued homomorphism (SVH) of Q_t-modules and give some examples of SVH. It is observed that QMH of a Q_t-module is a SVH. We also investigate some basic properties of GLA and GUA in Q_t-modules.

Definition 2.3.1 *Let M and N be Q_t-modules. A mapping $H : M \longrightarrow P(N)$ is called a SVH if*

(1) $\vee_{i\in I} H(m_i) \subseteq H(\vee_{i\in I} m_i)$;

(2) $q * H(m) \subseteq H(q * m)$ *for all* $m, m_i \in M$ *and* $q \in Q_t$.

A set-valued mapping $H : M \longrightarrow P(N)$ is called a $SSVH$ if

(1) $\vee_{i\in I} H(m_i) = H(\vee_{i\in I} m_i)$;

(2) $q * H(m) = H(q * m)$ *for all* $m, m_i \in M$ *and* $q \in Q_t$.

Example 2.3.2 *(i) Let η be a congruence on a Q_t-module M and $H : M \longrightarrow P(M)$ be a SVM defined by $H(m) = [m]_\eta$. Then H is a SVH.*

(ii) Let M and N be two Q_t-modules. Then the SVM, $H : M \longrightarrow P(N)$ defined by $H(m) = \{\perp\}$ is a $SSVH$.

(iii) Let $\rho_m : M \longrightarrow N$ be a QMH. Then the SVM, $H : M \longrightarrow P(N)$ defined by $H(m) = \{\rho_m(m)\}$ is a $SSVH$.

Note that, Example 2.3.2(i) point out that congruence relation may be consider as a SVH. So, SVH is important for pure algebraic systems.

Theorem 2.3.3 *Let M and N be two Q_t-modules and C be a subset of N. Then*

(1) Let $H : M \longrightarrow P^*(N)$ be a SVH. Let C be a Q_t-submodule of N and $\overline{H}(C)$ be a non-empty subset of M. Then $\overline{H}(C)$ is a Q_t-submodule of M.

(2) Let $H : M \longrightarrow P^*(N)$ be a $SSVH$. Let C be a Q_t-submodule of N and $\underline{H}(C)$ be a non-empty subset of M. Then $\underline{H}(C)$ is a Q_t-submodule of M.

Proof. (1) Let $w_i \in \overline{H}(C)$ for $i \in I$. Then $H(w_i) \cap C \neq \emptyset$ for $i \in I$. Hence, there exist $a_i \in H(w_i) \cap C$ $(i \in I)$ such that $\vee_{i\in I} a_i \in \vee_{i\in I} H(w_i) \subseteq H(\vee_{i\in I} w_i)$. Since, C is a Q_t-submodule, we have $\vee_{i\in I} a_i \in C$. So $H(\vee_{i\in I} w_i) \cap C \neq \emptyset$. Therefore, $\vee_{i\in I} w_i \in \overline{H}(C)$. Let $w \in \overline{H}(C)$ and $q \in Q_t$. Then, $H(w) \cap C \neq \emptyset$. Let $y \in H(w) \cap C$. Then we have $q * y \in C$ and $q * y \in q * H(w) \subseteq H(q * w)$. Thus, we have $H(q * w) \cap C \neq \emptyset$ and $q * w \in \overline{H}(C)$. This concludes that $\overline{H}(C)$ is a Q_t-submodule of M.

(2) Let $w_i \in \underline{H}(C)$ for $i \in I$. Then $H(w_i) \subseteq C$ for $i \in I$. Since $H(\vee_{i\in I} w_i) = \vee_{i\in I} H(w_i) \subseteq C$, we have $\vee_{i\in I} w_i \in \underline{H}(C)$. Let $z \in \underline{H}(C)$. Then $H(z) \subseteq C$. Now

$H(q * z) = q * H(z) \subseteq C$. Hence, $q * z \in \underline{H}(C)$. This shows that $\underline{H}(C)$ is a Q_t-submodule of M. ■

Definition 2.3.4 *Let M and N be two Q_t-modules and C be a subset of N. Let $H : M \longrightarrow P^*(N)$ be a SSVH. If $\underline{H}(C)$ and $\overline{H}(C)$ are Q_t-submodules of M, then we call $(\underline{H}(C),\ \overline{H}(C))$ a generalized rough Q_t-submodule.*

Proposition 2.3.5 *Let $H : M \longrightarrow P^*(N)$ be a SVM. If C, B are non-empty subsets of Q_t-module N, then*

(1) $\overline{H}(C) \cup \overline{H}(B) \subseteq \overline{H}(C \vee B)$; *if* $\bot \in C \cap B$

(2) $\underline{H}(C) \cap \underline{H}(B) \subseteq \underline{H}(C \vee B)$;

(3) $\underline{H}(C) \cup \underline{H}(B) \subseteq \underline{H}(C \vee B)$; *if* $\bot \in C \cap B$.

Proof. (1) Let $c \in C$. Then $c = c \vee \bot \in C \vee B$ for $\bot \in B$. So $C \subseteq C \vee B$. Similarly, $B \subseteq C \vee B$. So $C \cup B \subseteq C \vee B$. By Theorem 1.5.7, we have $\overline{H}(C) \cup \overline{H}(B) \subseteq \overline{H}(C \vee B)$.

(2) It is obvious that $C \cap B \subseteq C \vee B$. By Theorem 1.5.7, we have $\underline{H}(C) \cap \underline{H}(B) = \underline{H}(C \cap B) \subseteq \underline{H}(C \vee B)$.

(3) The proof is similar to the proof of (1). ■

Example 2.3.6 *Let Q_1 and Q_2 be two complete lattices as depicted in Fig 5 and 6. The operation "$\otimes$" on Q_1 and Q_2 is same and is defined as $x \otimes y = \bot$ for all $x, y \in Q_1$ and $x \otimes y = \bot'$ for all $x,y \in Q_2$. Then Q_1 and Q_2 are quantales and Q_t-modules over Q_1 and Q_2, respectively. Consider $H : Q_1 \longrightarrow P(Q_2)$ be a SVM defined as $H(\bot) = \{\bot'\}$, $H(\top) = \{\top'\}$, $H(a) = \{e\}$, $H(b) = \{f\}$. Let $A = \{\bot', e\} \subseteq Q_2$, $B = \{\bot', g, h\} \subseteq Q_2$. Then $A \vee B = \{\bot', g, e, h, \top'\}$, $\overline{H}(A) = \{\bot, a\}$, $\overline{H}(B) = \{\bot\}$, $\overline{H}(A \vee B) = \{\bot, a, \top\}$, $\underline{H}(A) = \{\bot, e\}$, $\underline{H}(B) = \{\bot\}$, $\underline{H}(A \vee B) = \{\bot, e, \top\}$. It is easily seen that converse of all parts of Proposition 2.3.5 are not true in general.*

Proposition 2.3.7 *Let $H : M \longrightarrow P^*(N)$ be a SSVH. If C, B are non-empty subsets of Q_t-module N, then*

(1) $\overline{H}(C) \vee \overline{H}(B) \subseteq \overline{H}(C \vee B)$;

(2) $\overline{H}(C) \cap \overline{H}(B) \subseteq \overline{H}(C \vee B)$;

(3) $\underline{H}(C) \vee \underline{H}(B) \subseteq \underline{H}(C \vee B)$.

Proof. (1) Let $x \in \overline{H}(C) \vee \overline{H}(B)$. Then $x = y \vee z$ with $y \in \overline{H}(C)$ and $z \in \overline{H}(B)$. Therefore $H(y) \cap C \neq \emptyset$ and $H(z) \cap B \neq \emptyset$. Then there exist elements a, b such that $a \in H(y) \cap C$ and $b \in H(z) \cap B$. Therefore $a \vee b \in C \vee B$, $a \vee b \in H(y) \vee H(z) = H(y \vee z) = H(x)$ which implies that $a \vee b \in H(x) \cap (C \vee B)$. Thus $x \in \overline{H}(C \vee B)$. Hence $\overline{H}(C) \vee \overline{H}(B) \subseteq \overline{H}(C \vee B)$.

(2) Let $y \in \overline{H}(C) \cap \overline{H}(B)$. Then $y \in \overline{H}(C)$ and $y \in \overline{H}(B)$. Let there exist $c \in C$ and $b \in B$ such that $c \vee b \in C \vee B$ and $c \vee b \in H(y) \vee H(y) = H(y \vee y) = H(y)$ which implies that $c \vee b \in H(y) \cap (C \vee B)$. Thus $y \in \overline{H}(C \vee B)$. Hence $\overline{H}(C) \cap \overline{H}(B) \subseteq \overline{H}(C \vee B)$.

(3) Let $x \in \underline{H}(C) \vee \underline{H}(B)$. Then $x = y \vee z$ with $y \in \underline{H}(C)$ and $z \in \underline{H}(B)$. Therefore $H(y) \subseteq C$ and $H(z) \subseteq B$. We get $H(y \vee z) = H(y) \vee H(z) \subseteq C \vee B$. Hence, $x \in \underline{H}(C \vee B)$. Therefore, we have $\underline{H}(C) \vee \underline{H}(B) \subseteq \underline{H}(C \vee B)$. ■

Example 2.3.8 *Let Q_1 be a complete lattice shown in Fig 5 and the operation "$\otimes$" on Q_1 is defined as $x \otimes y = \perp$ for all $x, y \in Q_1$. Then Q_1 is a quantale and Q_t-module over Q_1. Let $H : Q_1 \longrightarrow P^*(Q_1)$ be a SSVH as defined by $H(\perp) = \{\perp\}$, $H(\top) = H(a) = H(b) = \{\top\}$. Let $C = \{b\}$ and $B = \{\perp, a, \top\}$. Then $C \vee B = \{b, \top\}$, $\overline{H}(C) = \emptyset$, $\overline{H}(B) = Q_1$, $\overline{H}(C \vee B) = \{a, b, \top\}$, $\underline{H}(C) = \emptyset$, $\underline{H}(B) = Q_1$, $\underline{H}(C \vee B) = \{a, b, \top\}$. From above calculations, it is easily seen that converse of all parts of Proposition 2.3.7 are not true in general.*

Proposition 2.3.9 *Let $H : M \longrightarrow P^*(N)$ be a SSVH and $\rho_m : M' \longrightarrow M$ be a QMI. Then $H \circ \rho_m$ is a SSVH from M' to $P^*(N)$ such that $\overline{H \circ \rho_m}(B) = \rho_m^{-1}(\overline{H}(B))$ and $\underline{(H \circ \rho_m)}(B) = \rho_m^{-1}(\underline{H}(B))$ for all $B \in P^*(N)$.*

Proof. We show that $H \circ \rho_m$ is a $SSVH$ from M' to $P^*(N)$. Let $m_i \in M'$ for $i \in I$. Then

(1) $(H \circ \rho_m)(\vee_{i \in I} m_i) = H(\rho_m(\vee_{i \in I} m_i)) = H(\vee_{i \in I} \rho_m(m_i)) = \vee_{i \in I} H(\rho_m(m_i)) = \vee_{i \in I}(H \circ \rho_m)(m_i)$.

(2) $(H \circ \rho_m)(q * m) = H(\rho_m(q * m)) = H(q * \rho_m(m)) = q * H(\rho_m(m)) = q * (H \circ \rho_m)(m)$.

Hence, $H \circ \rho_m$ is a $SSVH$ from M' to $P^*(N)$.

Let $w \in \overline{H \circ \rho_m}(B) \iff (H \circ \rho_m)(w) \cap B \neq \emptyset \iff H(\rho_m(w)) \cap B \neq \emptyset \iff \rho_m(w) \in \overline{H}(B) \iff w \in \rho_m^{-1}(\overline{H}(B))$. Hence $\overline{H \circ \rho_m}(B) = \rho_m^{-1}(\overline{H}(B))$.

Let $w \in \underline{(H \circ \rho_m)}(B) \Longleftrightarrow (H \circ \rho_m)(w) \subseteq B \Longleftrightarrow H(\rho_m(w)) \subseteq B \Longleftrightarrow \rho_m(w) \in \underline{H}(B) \Longleftrightarrow w \in \rho_m^{-1}(\underline{H}(B))$. Hence $\underline{(H \circ \rho_m)}(B) = \rho_m^{-1}(\underline{H}(B))$ for all $B \in P^*(N)$. ■

Proposition 2.3.10 *Let $H : M \longrightarrow P^*(N)$ be a $SSVH$ and $\rho_m : N \longrightarrow M'$ be a QMI. Then H_{ρ_m} is a $SSVH$ from M to $P^*(M')$ defined by $H_{\rho_m}(m) = \rho_m(H(m))$ such that $\underline{H_{\rho_m}}(B) = \underline{H}(\rho_m^{-1}(B))$ and $\overline{H_{\rho_m}}(B) = \overline{H}(\rho_m^{-1}(B))$ for all $B \in P^*(M')$.*

Proof. We show that H_{ρ_m} is a $SSVH$ from M to $P^*(M')$. Let $m_i \in M$ for $i \in I$. Then

(1) $H_{\rho_m}(\vee_{i\in I} m_i) = \rho_m(H(\vee_{i\in I} m_i)) = \rho_m(\vee_{i\in I} H(m_i)) = \vee_{i\in I} \rho_m(H(m_i)) = \vee_{i\in I} H_{\rho_m}(m_i)$.

(2) $H_{\rho_m}(q * m) = \rho_m(H(q * m)) = \rho_m(q * H(m)) = q * \rho_m(H(m)) = q * H_{\rho_m}(m)$.

Hence, H_{ρ_m} is a $SSVH$ from M to $P^*(M')$.

Let $w \in \underline{H_{\rho_m}}(B) \Longleftrightarrow H_{\rho_m}(w) \subseteq B \Longleftrightarrow \rho_m(H(w)) \subseteq B \Longleftrightarrow H(w) \subseteq \rho_m^{-1}(B) \Longleftrightarrow w \in \underline{H}(\rho_m^{-1}(B))$. Hence $\underline{H_{\rho_m}}(B) = \underline{H}(\rho_m^{-1}(B))$.

Let $w \in \overline{H_{\rho_m}}(B) \Longleftrightarrow H_{\rho_m}(w) \cap B \neq \emptyset \Longleftrightarrow \rho_m(H(w)) \cap B \neq \emptyset \Longleftrightarrow H(w) \cap \rho_m^{-1}(B) \neq \emptyset \Longleftrightarrow w \in \overline{H}(\rho_m^{-1}(B))$. Hence $\overline{H_{\rho_m}}(B) = \overline{H}(\rho_m^{-1}(B))$. ■

Proposition 2.3.11 *Let $H : M \longrightarrow P^*(N)$ be a SVH and η be a congruence on a Q_t-module N. Define $H_\eta : M \longrightarrow P(N/\eta)$ by $H_\eta(m) = \Big\{[b]_\eta \mid b \in H(m)\Big\}$, where N/η is the quotient Q_t-module of N by η. Then H_η is a SVH.*

Proof. (1) We show that H_η is a SVH from M to $P^*(N/\eta)$. Let $m_i \in M$ for $i \in I$. Then

$$\begin{aligned} H_\eta(\vee_{i\in I} m_i) &= \Big\{[b]_\eta \mid b \in H(\vee_{i\in I} m_i)\Big\} \supseteq \Big\{[b]_\eta \mid b \in \vee_{i\in I} H(m_i)\Big\} \\ &= \Big\{[b]_\eta \mid b = v_1 \vee v_2, ..., \vee v_i,\ v_1 \in H(m_1), ...,\ v_i \in H(m_i)\ \Big\} \\ &= \Big\{[v_1]_\eta \mid v_1 \in H(m_1)\Big\} \vee, ..., \vee \Big\{[v_i]_\eta \mid v_i \in H(m_i)\Big\} \\ &= H_\eta(m_1) \vee H_\eta(m_2) \vee H_\eta(m_3) \vee, ..., \vee H_\eta(m_i) \\ &= \vee_{i\in I} H_\eta(m_i) \end{aligned}$$

Thus, $\vee_{i\in I} H_\eta(m_i) \subseteq H_\eta(\vee_{i\in I} m_i)$.

(2) $H_\eta(q * m) = \Big\{[b]_\eta \mid b \in H(q * m)\Big\} \supseteq \Big\{[b]_\eta \mid b \in q * H(m)\Big\} = q * H_\eta(m)$

Thus, we have $q * H_\eta(m) \subseteq H_\eta(q * m)$. It concludes that H_η is a SVH. Similarly, it can be shown that H_η is a $SSVH$ when H is a $SSVH$. ■

2.4 Homomorphic images of generalized rough Q_t-Submodules

In this section, we will discuss the images of lower and upper approximations under Q_t-module homomorphism (QMH) and SVH.

Theorem 2.4.1 *Let M and N be two Q_t-modules and $\rho_m : M \longrightarrow N$ be an epimorphism and $H_2 : N \longrightarrow P^*(N)$ be a SVH. If ρ_m is one to one and $H_1(x) = \{y \in M \mid \rho_m(y) \in H_2(\rho_m(x))\}$ for all $x \in M$, then H_1 is a SVH from M to $P^*(M)$.*

Proof. First, we show that H_1 is well defined mapping. Suppose $x_1 = x_2$ then we have, $y_1 \in H_1(x_1) \Longleftrightarrow \rho_m(y_1) \in H_2(\rho_m(x_1)) = H_2(\rho_m(x_2)) \Longleftrightarrow y_1 \in H_1(x_2)$. Thus we have $H_1(x_1) = H_1(x_2)$. Now we show that H_1 is a SVH. First, we show that $\vee_{i\in I} H_1(x_i) \subseteq H_1(\vee_{i\in I} x_i)$ for all $x_i \in M$ $(i \in I)$. Let $y \in \vee_{i\in I} H_1(x_i)$. Then there exist $a_i \in H_1(x_i)$ for all $i \in I$ such that $y = \vee_{i\in I} a_i$. Hence $\rho_m(y) = \rho_m(\vee_{i\in I} a_i) = \vee_{i\in I} \rho_m(a_i) \in \vee_{i\in I} H_2(\rho_m(x_i)) \subseteq H_2(\vee_{i\in I} \rho_m(x_i)) = H_2(\rho_m(\vee_{i\in I} x_i))$. Finally, we have, $y = \vee_{i\in I} a_i \in H_1(\vee_{i\in I} x_i)$. We have $\vee_{i\in I} H_1(x_i) \subseteq H_1(\vee_{i\in I}(x_i))$. Let $y \in q * H_1(x)$. Then there exists $a \in H_1(x)$ such that $y = q * a$. Since H_2 is a SVH and ρ_m is a QMH, we have $\rho_m(a) \in H_2(\rho_m(x))$ and $q * \rho_m(a) \in q * H_2(\rho_m(x)) \subseteq H_2(q * \rho_m(x)) = H_2(\rho_m(q * x))$. Therefore, $\rho_m(q * a) \in H_2(\rho_m(q * x))$. Hence $y = q * a \in H_1(q * x)$. Thus, we have $q * H_1(x) \subseteq H_1(q * x)$. So, H_1 is a SVH from M to $P^*(M)$. ■

Theorem 2.4.2 *Let M and N be Q_t-modules, $\rho_m : M \longrightarrow N$ be a surjective QMH from M to N and $H_2 : N \longrightarrow P^*(N)$ be a SVH. Set $H_1(w) = \{y \in M \mid \rho_m(y) \in H_2(\rho_m(w))\}$ for all $w \in M$ and for all $\emptyset \neq A \subseteq M$, then*

(1) $\rho_m(\overline{H}_1(A)) = \overline{H}_2(\rho_m(A))$;

(2) $\rho_m(\underline{H}_1(A)) = \underline{H}_2(\rho_m(A))$;

(3) *If ρ_m is one one then $\rho_m(w) \in \rho_m(\overline{H}_1(A)) \Longleftrightarrow w \in \overline{H}_1(A)$.*

Proof. Let $z \in \rho_m(\overline{H}_1(A))$. Then there exists $w \in \overline{H}_1(A)$ such that $\rho_m(w) = z$. So $H_1(w) \cap A \neq \emptyset$, then there exists $w' \in H_1(w) \cap A$ such that $\rho_m(w') \in \rho_m(A)$, and $\rho_m(w') \in H_2(\rho_m(w))$. So $H_2(\rho_m(w)) \cap \rho_m(A) \neq \emptyset$, which implies $z = \rho_m(w) \in \overline{H}_2(\rho_m(A))$.

Conversely, let $z \in \overline{H}_2(\rho_m(A))$. Then there exists $w \in M$ such that $\rho_m(w) = z$. Hence $H_2(\rho_m(w)) \cap \rho_m(A) \neq \emptyset$. So there exists $w' \in A$ such that $\rho_m(w') \in \rho_m(A)$

and $\rho_m(w') \in H_2(\rho_m(w))$. Then by H_1, we have $w' \in H_1(w)$. Thus $H_1(w) \cap A \neq \emptyset$, which implies $w \in \overline{H}_1(A)$. So $z = \rho_m(w) \in \rho_m(\overline{H}_1(A))$. It means that $\overline{H}_2(\rho_m(A)) \subseteq \rho_m(\overline{H}_1(A))$. From the above, we have $\rho_m(\overline{H}_1(A)) = \overline{H}_2(\rho_m(A))$.

(2) Let $z \in \rho_m(\underline{H}_1(A))$. Then there exists $w \in \underline{H}_1(A)$ such that $\rho_m(w) = z$, so we have $H_1(w) \subseteq A$. Let $z' \in H_2(\rho_m(w))$. Then there exists $w' \in M$ such that $\rho_m(w') = z'$ and $\rho_m(w') \in H_2(\rho_m(w))$. Hence $w' \in H_1(w) \subseteq A$ and so $z' = \rho_m(w') \in \rho_m(A)$. Thus $H_2(\rho_m(w)) \subseteq \rho_m(A)$ which gives that $\rho_m(w) \in \underline{H}_2(\rho_m(A))$, so we have $\rho_m(\underline{H}_1(A)) \subseteq \underline{H}_2(\rho_m(A))$.

Suppose $z \in \underline{H}_2(\rho_m(A))$. Then there exists $w \in M$ such that $\rho_m(w) = z$ and $H_2(\rho_m(w)) \subseteq \rho_m(A)$. Let $w' \in H_1(w)$. Then $\rho_m(w') \in H_2(\rho_m(w)) \subseteq \rho_m(A)$, and so $w' \in A$. Thus $H_1(w) \subseteq A$, which yields $w \in \underline{H}_1(A)$. Then $\rho_m(w) = z \in \rho_m(\underline{H}_1(A))$, and so $\underline{H}_2(\rho_m(A)) \subseteq \rho_m(\underline{H}_1(A))$. Hence we have $\rho_m(\underline{H}_1(A)) = \underline{H}_2(\rho_m(A))$.

(3) Let $w \in \overline{H}_1(A)$. Then $\rho_m(w) \in \rho_m(\overline{H}_1(A))$. Conversely suppose that $\rho_m(w) \in \rho_m(\overline{H}_1(A))$. Then there exists $w' \in \overline{H}_1(A)$ such that $\rho_m(w) = \rho_m(w')$. Since ρ_m is ono-one, we get $w = w' \in \overline{H}_1(A)$. ■

Remark 2.4.3 *From Theorem* 2.4.2(3), *it is easily found that* $\rho_m(x) \in \rho_m(\underline{H}_1(A)) \Longleftrightarrow x \in \underline{H}_1(A)$.

Theorem 2.4.4 *Let M and N be two Q_t-modules and $\rho_m : M \longrightarrow N$ be a surjective QMH and $H_2 : N \longrightarrow P^*(N)$ be a SVH. Set $H_1(x) = \{y \in M \mid \rho_m(y) \in H_2(\rho_m(x))\}$ for all $x \in M$ and for all $\emptyset \neq C \subseteq M$, then*

(1) *$\overline{H}_1(C)$ is a Q_t-submodule of M if and only if $\overline{H}_2(\rho_m(C))$ is a Q_t-submodule of N.*

(2) *$\underline{H}_1(C)$ is a Q_t-submodule of M if and only if $\underline{H}_2(\rho_m(C))$ is a Q_t-submodule of N.*

Proof. (1) Let $\overline{H}_1(C)$ be a Q_t-submodule of M. We show that $\overline{H_2}(\rho_m(C))$ is a Q_t-submodule of N. Let $y_i \in \rho_m(\overline{H}_1(C))$ $(i \in I)$. Then there exists $x_i \in \overline{H}_1(C)$ $(i \in I)$ such that $\rho_m(x_i) = y_i$. Since ρ_m is a QMH and $\overline{H}_1(C)$ is a Q_t-submodule of M, we have $\vee_{i\in I} y_i = \vee_{i\in I}\rho_m(x_i) = \rho_m(\vee_{i\in I} x_i)$. Therefore $\vee_{i\in I} x_i \in \overline{H}_1(C)$ if and only if $\vee_{i\in I} y_i = \rho_m(\vee_{i\in I} x_i) \in \rho_m(\overline{H}_1(C))$. Suppose $y \in \rho_m(\overline{H}_1(C))$ and q be an arbitrary element of Q_t. Then there exists $x \in \overline{H}_1(C)$ such that $\rho_m(x) = y$. Now $\rho_m(q * x) = q * \rho_m(x) = q * y$. Then $q * x \in \overline{H}_1(C)$ if and only if $q * y = \rho_m(q * x) \in \rho_m(\overline{H}_1(C))$. Since, $\rho_m(\overline{H}_1(C)) = \overline{H}_2(\rho_m(C))$ by Theorem 2.4.2(1). We have $\overline{H}_2(\rho_m(C))$ is a Q_t-submodule of N.

Conversely, suppose $\rho_m(\overline{H}_1(C)) = \overline{H}_2(\rho_m(C))$ is a Q_t-submodule of N. Let $x_i \in \overline{H}_1(C)$ for $i \in I$. Then $\rho_m(x_i) \in \rho_m(\overline{H}_1(C))$ $(i \in I)$. Since $\rho_m(\overline{H}_1(C))$ is a Q_t-submodule of N, we have $\vee_{i\in I}\rho_m(x_i) = \rho_m(\vee_{i\in I}x_i) \in \rho_m(\overline{H}_1(C))$. Then by Theorem 2.4.2(3), we have $\vee_{i\in I}x_i \in \overline{H}_1(C)$. Let $x \in \overline{H}_1(C)$. Then $\rho_m(x) \in \rho_m(\overline{H}_1(C))$. Since $\rho_m(\overline{H}_1(C))$ is a Q_t-submodule, we have $\rho_m(q * x) = q * \rho_m(x) \in \rho_m(\overline{H}_1(C))$ and thus $q * x \in \overline{H}_1(C)$ by theorem 2.4.2(3). So $\overline{H}_1(C)$ is a Q_t-submodule of M.

(2) The proof is similar to the part 1. ■

Chapter 3

Generalized Rough Fuzzy Ideals in Quantales

In this chapter, we define generalized rough fuzzy ideals, generalized rough fuzzy prime ideals, generalized rough fuzzy semi-prime ideals and generalized rough fuzzy primary deals of quantales. There are some intrinsic relations between fuzzy prime (fuzzy semi-prime, fuzzy primary) ideals and generalized rough fuzzy prime (generalized rough fuzzy semi-prime, generalized rough fuzzy primary) ideals of quantales. Further, approximations of fuzzy ideals, fuzzy prime, fuzzy semi-prime and fuzzy primary ideals with the help of SVH and $SSVH$ are discussed. In addition, homomorphic images of generalized rough prime (semi-prime, primary) ideals which are established by QH, are examined.

In the first section, by applying generalized rough set theory to fuzzy ideals of quantales, we introduce the notions of generalized rough fuzzy (prime, semi-prime, primary) ideals of quantales. By using SVH and $SSVH$, it is observed that generalized lower and upper approximations of fuzzy ideals (fuzzy prime, fuzzy semi-prime, fuzzy primary) are fuzzy ideals (fuzzy prime, fuzzy semi-prime, fuzzy primary). Some related results about fuzzy ideals are also discussed in this section. In the second section, a SVH is presented with the help of another SVH by using QH. It is also noted that homomorphic image of upper (lower) approximation of a subset of a quantale is equal to the upper (lower) approximation of homomorphic image of a subset of the quantale. Further, in this section, relations between the upper (lower) generalized rough (prime, semi-prime, primary) ideals of quantales and the upper (lower) approximations of their homomorphic images are studied. In the last section of this chapter, we will discuss relations between the upper (lower) generalized rough fuzzy (prime, semi-prime, primary) ideals of quantales and the upper (lower) approximations of their homomorphic images and give some theorems related to them.

3.1 Generalized Rough Fuzzy Prime (Primary) ideals in Quantale

This section presents the generalized rough fuzzy ideal in quantales and further properties of such ideals are displayed here. For fuzzy subset, generalized rough fuzzy set, generalized rough fuzzy ideal, generalized rough fuzzy prime ideal, generalized rough fuzzy semi-prime ideal and generalized rough fuzzy primary ideal, the following shortened forms, f-subset, $GRFS$, $GRFI$, $GRFPI$, $GRFSPI$ and $GRFPYI$ will be used.

Definition 3.1.1 *[21] Let (Z,η) be an approximation space and g be a f-subset of Z, that is g is a mapping from Z to $[0,1]$. Then for $z\in Z$, we define;*

$$\underline{\eta}(g)(z)=\bigwedge_{p\in[z]_\eta} g(p) \quad and \quad \overline{\eta}(g)(z)=\bigvee_{p\in[z]_\eta} g(p).$$

They are called, the lower approximation (LA) and the upper approximation (UA) of g, respectively. If $\underline{\eta}(g)\neq\overline{\eta}(g)$, then $\eta(g)=(\underline{\eta}(g),\overline{\eta}(g))$ is called a rough fuzzy set (RFS) with respect to η.

For $\alpha\in[0,1]$, the sets

$$g_\alpha=\{x\in Z\mid g(x)\geq\alpha\} \quad and \quad g_{\alpha^+}=\{x\in Z\mid g(x)>\alpha\}$$

are called, α-cut and strong α-cut of g, respectively.

Now we use the concept from definition 3.1.1 and generalize it in the following.

Definition 3.1.2 *Let $(Q_t,\otimes_1)$ and $(Q'_t,\otimes_2)$ be two quantales and $H:Q_t\longrightarrow P^*(Q'_t)$ be a SVH and g be any f-subset of Q'_t. Then for every $z\in Q_t$, we define,*

$$\underline{H}(g)(z)=\underset{a\in H(z)}{Inf}\ g(a) \quad and \quad \overline{H}(g)(z)=\underset{a\in H(z)}{Sup}\ g(a)$$

Here $\underline{H}(g)$ *is the* GLA *and* $\overline{H}(g)$ *is the* GUA *of the f-subset* g*. The pair* $(\underline{H}(g), \overline{H}(g))$ *is called generalized rough fuzzy set* $(GRFS)$ *of* Q_t *if* $\underline{H}(g) \neq \overline{H}(g)$*.*

Definition 3.1.3 *[91] Let* $(Q_t, \otimes_1)$ *and* $(Q'_t, \otimes_2)$ *be two quantales. A set-valued mapping* (SVM)*,* $H : Q_t \longrightarrow P^*(Q'_t)$*, where* $P^*(Q'_t)$ *means the collection of all non-empty subsets of* Q'_t*, is called a set-valued homomorphism if, for all* a_i*,* a*,* $b \in Q_t$*,*

(1) $H(a) \otimes_2 H(b) \subseteq H(a \otimes_1 b)$.

(2) $\vee_{i \in I} H(a_i) \subseteq H(\vee_{i \in I} a_i)$.

A set-valued mapping $H : Q_t \longrightarrow P^*(Q'_t)$ *is called a strong set-valued homomorphism if we replace inclusion by equality in* (1) *and* (2)*.*

Lemma 3.1.4 *Let* $H : Q_t \longrightarrow P^*(Q'_t)$ *be a* SVM*. Then for every collection* $\{g_i\}_{i \in I} \subseteq \mathcal{F}(Q'_t)$*;*

(1) $\underline{H}(\underset{i \in I}{Inf}\ g_i) = \underset{i \in I}{Inf}\underline{H}(g_i)$;

(2) $\overline{H}(\underset{i \in I}{Sup}\ g_i) = \underset{i \in I}{Sup}\ \overline{H}(g_i)$.

Proof. (1) For $x \in Q_t$, we have

$$\underline{H}(\underset{i \in I}{Inf}\ g_i)(x) = \underset{a \in H(x)}{Inf}\ \underset{i \in I}{Inf}\ g_i(a) = \underset{i \in I}{Inf}\ \underset{a \in H(x)}{Inf}\ g_i(a) = \underset{i \in I}{Inf}\ \underline{H}(g_i)(x).$$

The other part has the similar proof. ∎

Proposition 3.1.5 *Let* $(Q_t, \otimes_1)$ *and* $(Q'_t, \otimes_2)$ *be two quantales and* $H : Q_t \longrightarrow P^*(Q'_t)$ *be a* SVM*. Let* g *be a f-subset of* Q'_t*. Then for each* $\alpha \in [0, 1]$*, we have the following,*

(1) $\underline{H}(g_\alpha) = (\underline{H}(g))_\alpha$;

(2) $\overline{H}(g_\alpha) = (\overline{H}(g))_\alpha$;

(3) $\underline{H}(g_{\alpha^+}) = (\underline{H}(g))_{\alpha^+}$;

(4) $\overline{H}(g_{\alpha^+}) = (\overline{H}(g))_{\alpha^+}$.

Proof. (1) Let $z \in (\underline{H}(g))_\alpha \Longleftrightarrow \underline{H}(g)(z) \geq \alpha \Longleftrightarrow \underset{a \in H(z)}{\text{Inf}} g(a) \geq \alpha$

$$\Longleftrightarrow g(a) \geq \alpha \ \text{ for all } a \in H(z);$$

$$\Longleftrightarrow H(z) \subseteq g_\alpha \Longleftrightarrow z \in \underline{H}(g_\alpha).$$

Proofs of (2), (3) and (4) are similar to the proof of (1). ■

Definition 3.1.6 *Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a SVH. A f-subset g of the quantale Q'_t is said to be a lower [an upper] GRF ideal ($GRFI$) of Q'_t if $\underline{H}(g)$ [$\overline{H}(g)$] is a fuzzy ideal (FI) of Q_t. A f-subset g of Q'_t which is both an upper and a lower $GRFI$ of Q'_t, is called a $GRFI$ of Q'_t.*

Now, LA and UA of FI of quantales are being studied in the following.

Theorem 3.1.7 *Let g be a FI of Q'_t and $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$. Then $\underline{H}(g)$ is a FI of Q_t.*

Proof. As g is a FI of Q'_t, so by definition 1.4.8, we have, $g(a \otimes c) \geq sup\{g(a), g(c)\}$ and $g(a \vee c) = inf\{g(c), g(a)\}$ for all $a, c \in Q'_t$. Since H is a $SSVH$, so $H(z_1) \vee H(z_2) = H(z_1 \vee z_2)$ for all $z_1,\ z_2 \in Q_t$.

Therefore,

$$\begin{aligned}\underline{H}(g)(z_1 \vee z_2) &= \underset{e \in H(z_1 \vee z_2)}{Inf} g(e) \\ &= \underset{e \in H(z_1) \vee H(z_2)}{Inf} g(e)\end{aligned}$$

Since $e \in H(z_1) \vee H(z_2)$, there exist $c_1 \in H(z_1)$ and $c_2 \in H(z_2)$ such that $e = c_1 \vee c_2$.

Hence,

$$\begin{aligned}\underline{H}(g)(z_1 \vee z_2) &= \underset{c_1 \vee c_2 \in H(z_1) \vee H(z_2)}{Inf} g(c_1 \vee c_2) \\ &= \underset{c_1 \in H(z_1),\ c_2 \in H(z_2)}{Inf} (g(c_1) \wedge g(c_2)) \\ &= Inf \left[\underset{c_1 \in H(z_1)}{Inf} g(c_1),\ \underset{c_2 \in H(z_2)}{Inf} g(c_2) \right] \\ &= \underline{H}(g)(z_1) \wedge \underline{H}(g)(z_2).\end{aligned}$$

Hence $\underline{H}(g)(z_1 \vee z_2) = \underline{H}(g)(z_1) \wedge \underline{H}(g)(z_2)$ for all $z_1, z_2 \in Q_t$. (1)

Again since H is a $SSVH$, we have $H(z_1 \otimes_1 z_2) = H(z_1) \otimes_2 H(z_2)$ for all $z_1,\ z_2 \in Q_t$.

Thus we have,

$$\begin{aligned}\underline{H}(g)(z_1 \otimes_1 z_2) &= \underset{e \in H(z_1 \otimes_1 z_2)}{Inf} g(e) \\ &= \underset{e \in H(z_1) \otimes_2 H(z_2)}{Inf} g(e).\end{aligned}$$

Now since $e \in H(z_1) \otimes_2 H(z_2)$ so there exist $c_1 \in H(z_1)$, $c_2 \in H(z_2)$ such that $e = c_1 \otimes_2 c_2$.

Thus,

$$\begin{aligned}
\underline{H}(g)(z_1 \otimes_1 z_2) &= \underset{c_1 \otimes_2 c_2 \, \in \, H(z_1) \otimes_2 H(z_2)}{Inf} g(c_1 \otimes_2 c_2) \\
&\geq \underset{c_1 \otimes_2 c_2 \, \in \, H(z_1) \otimes_2 H(z_2)}{Inf} [g(c_1) \vee g(c_2)] \\
&= \underset{c_1 \, \in \, H(z_1), \, c_2 \in H(z_2)}{Inf} [g(c_1) \vee g(c_2)] \\
&= Sup \left[\underset{c_1 \in H(z_1)}{\text{Inf}} g(c_1), \underset{c_2 \in H(z_2)}{Inf} g(c_2) \right] \\
&= \underline{H}(g)(z_1) \vee \underline{H}(g)(z_2).
\end{aligned}$$

Hence, $\underline{H}(g)(z_1 \otimes_1 z_2) \geq \underline{H}(g)(z_1) \vee \underline{H}(g)(z_2)$ for all $z_1, z_2 \in Q_t$. (2)

Thus, by (1) and (2) $\underline{H}(g)$ is a FI of Q_t. ■

Theorem 3.1.8 *Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$ and g be a FI of Q'_t. Then $\overline{H}(g)$ is a FI of Q_t.*

Proof. Since H is a $SSVH$, therefore $H(z_1) \vee H(z_2) = H(z_1 \vee z_2)$ for all $z_1, z_2 \in Q_t$. Also g is FI of Q'_t, hence $inf\{g(a), g(b)\} = g(a \vee b)$ for all a, $b \in Q'_t$.

Consider,

$$\begin{aligned}
\overline{H}(g)(z_1 \vee z_2) &= \underset{c \in H(z_1 \vee z_2)}{Sup} g(c) \\
&= \underset{c \in H(z_1) \vee H(z_2)}{Sup} g(c)
\end{aligned}$$

For $c \in H(z_1) \vee H(z_2)$, we have $a \in H(z_1)$ and $b \in H(z_2)$ such that $c = a \vee b$.

Hence,

$$\begin{aligned}
\overline{H}(g)(z_1 \vee z_2) &= \underset{a \vee b \in H(z_1) \vee H(z_2)}{Sup} g(a \vee b) \\
&= \underset{a \in H(z_1), \, b \in H(z_2)}{Sup} [g(a) \wedge g(b)] \\
&= Inf \left[\underset{a \in H(z_1)}{Sup} g(a), \underset{b \in H(z_2)}{Sup} g(b) \right] \\
&= \overline{H}(g)(z_1) \wedge \overline{H}(g)(z_2).
\end{aligned}$$

Thus, $\overline{H}(g)(z_1 \vee z_2) = \overline{H}(g)(z_1) \wedge \overline{H}(g)(z_2)$ for all $z_1, z_2 \in Q_t$. (1)

Now for,

$$\begin{aligned}\overline{H}(g)(z_1 \otimes_1 z_2) &= \underset{c \in H(z_1 \otimes_2 z_2)}{Sup} g(c) \\ &= \underset{c \in H(z_1) \otimes_2 H(z_2)}{Sup} g(c).\end{aligned}$$

For $c \in H(z_1) \otimes_2 H(z_2)$, there exist $a \in H(z_1)$ and $b \in H(z_2)$ such that $c = a \otimes_2 b$.

Hence,

$$\begin{aligned}\overline{H}(g)(z_1 \otimes_1 z_2) &= \underset{a \otimes_2 b \in H(z_1) \otimes_2 H(z_2)}{Sup} g(a \otimes_2 b) \\ &\geq \underset{a \in H(z_1),\, b \in H(z_2)}{Sup} [g(a) \vee g(b)] \\ &= Sup \left(\underset{a \in H(z_1)}{Sup} g(a), \underset{b \in H(z_2)}{Sup} g(b) \right) \\ &= \overline{H}(g)(z_1) \vee \overline{H}(g)(z_2).\end{aligned}$$

Thus, $\overline{H}(g)(z_1 \otimes_1 z_2) \geq \overline{H}(g)(z_1) \vee \overline{H}(g)(z_2)$ for all $z_1, z_2 \in Q_t$. (2)

Hence by (1) and (2), we have $\overline{H}(g)$ is a FI of Q_t. ■

From the two theorems discussed above, we have the following corollary.

Corollary 3.1.9 *Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$ and let g be a FI of Q'_t. Then g is a $GRFI$ of Q'_t.*

Fig. 7

Table 4.

$\otimes_1$	$\bot$	a	$\top$
$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	a	a
$\top$	$\bot$	a	$\top$

Proposition 3.1.10 *Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$ and $\{g_i\}_{i\in I}$ be a family of FI of Q'_t. Then $\underline{H}(\underset{i\in I}{Inf}\ (g_i))$ is a FI of Q_t.*

Proof. By Lemma 3.1.4, we have $\underline{H}(\underset{i\in I}{Inf}\ g_i) = \underset{i\in I}{Inf}\underline{H}(g_i)$. Since every g_i is a FI for $i \in I$ and $\underline{H}(g_i)$ is a FI of Q_t by Theorem 3.1.7, hence intersection of FIs is a FI. Therefore $\underline{H}(\underset{i\in I}{Inf}\ (g_i))$ is a FI of Q_t. ■

Theorem 3.1.11 *Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$ and g be a f-subset of Q'_t. Then $\underline{H}(g)$ (respectively $\overline{H}(g)$) is a FI of Q_t if and only if for each $\alpha \in [0,1]$, $\underline{H}(g_\alpha)$ (respectively $\overline{H}(g_\alpha)$) where $g_\alpha \neq \emptyset$, is an ideal of Q_t.*

Proof. The Proof is similar to the proof of Proposition 1.4.13(1). ■

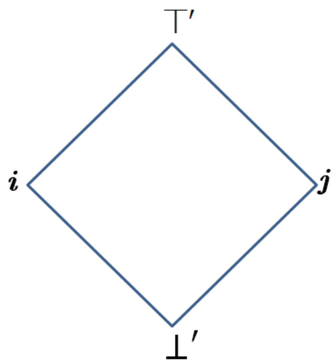

Fig. 8

Table. 5

$\otimes_2$	$\bot'$	i	j	$\top'$
$\bot'$	$\bot'$	$\bot'$	$\bot'$	$\bot'$
i	$\bot'$	i	$\bot'$	i
j	$\bot'$	$\bot'$	j	j
$\top'$	$\bot'$	i	j	$\top'$

Example 3.1.12 *Let $(Q_t, \otimes_1)$ and $(Q'_t, \otimes_2)$ be two quantales, where Q_t and Q'_t are depicted in Fig. 7 and 8 and the binary operations $\otimes_1$ and $\otimes_2$ on both the quantales are the same as the meet operation in the lattices Q_t and Q'_t as shown in the table 4 and 5.*

Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$ defined by $H(\bot) = \{\bot'\}$, $H(a) = \{i, j\}$, $H(\top) = \{\top'\}$. Let g be a FI of Q'_t defined by $g = \frac{0.9}{\bot'} + \frac{0.6}{i} + \frac{0.7}{j} + \frac{0.6}{\top'}$. Then GLA and GUA of g are as follows: $\underline{H}(g) = \frac{0.9}{\bot} + \frac{0.6}{a} + \frac{0.6}{\top}$ and $\overline{H}(g) = \frac{0.9}{\bot} + \frac{0.7}{a} + \frac{0.6}{\top}$. It is easily confirmed that $\underline{H}(g)$ and $\overline{H}(g)$ are FI of Q_t.

Consider $H : Q'_t \longrightarrow P^*(Q'_t)$ defined by $H(\bot') = H(i) = H(j) = \{\bot'\}$ and $H(\top') = Q'_t$. Then H is a SVH.

Let μ be a f-subset of Q'_t defined by $\mu(x) = \begin{cases} 1, & x = \bot' \\ 0.7, & x \neq \bot' \end{cases}$ for all $x \in Q'_t$. Then μ is a FI of Q'_t. Hence GLA and GUA of μ are $\underline{H}(\mu) = \frac{1}{\bot'} + \frac{1}{i} + \frac{1}{j} + \frac{0.7}{\top'}$ and $\overline{H}(\mu) = \frac{1}{\bot'} + \frac{1}{i} + \frac{1}{j} + \frac{1}{\top'}$. It is observed that $\underline{H}(\mu)$ is not a FI of Q'_t and $\overline{H}(\mu)$ is a constant FI. Hence it is important to take $SSVH$.

Definition 3.1.13 *Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a SVH and g be a f-subset of a quantale Q'_t. Then g is called an upper [a lower] generalized rough fuzzy prime ideal ($GRFPI$) of Q'_t if $\overline{H}(g)$ [$\underline{H}(g)$] is a fuzzy prime ideal (FPI) of Q_t. A f-subset g of Q'_t which is both an upper and a lower $GRFPI$, is called $GRFPI$ of Q'_t.*

Similarly, we can define upper [lower] generalized rough fuzzy semi-prime ideal ($GRFSPI$) and generalized rough fuzzy primary ideal ($GRFPYI$) of quantale.

Proposition 3.1.14 *Let g be a FPI of Q'_t and $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$. Then $\underline{H}(g)$ is a FPI of Q_t.*

Proof. As g is a FPI of Q'_t, therefore $g(c) = g(c \otimes_2 b)$ or $g(b) = g(c \otimes_2 b)$ for all $c, b \in Q'_t$. Since, g is a FPI of Q'_t, so g is a FI. By Theorem 3.1.7, $\underline{H}(g)$ is a FI of Q_t.

Consider,

$$\begin{aligned}\underline{H}(g)(x_1 \otimes_1 y_1) &= \underset{e \in H(x_1 \otimes_1 y_1)}{Inf} g(e) \\ &= \underset{e \in H(x_1) \otimes_2 H(y_1)}{Inf} g(e)\end{aligned}$$

Since H is a $SSVH$, therefore for $e \in H(x_1) \otimes_2 H(y_1)$ there exist $c \in H(x_1)$ and $b \in H(y_1)$ such that $e = c \otimes_2 b$.

Hence,

$$\begin{aligned}\underline{H}(g)(x_1 \otimes_1 y_1) &= \underset{c \otimes_2 b \in H(x_1) \otimes_2 H(y_1)}{Inf} g(c \otimes_2 b) \\ &= \underset{c \in H(x_1),\ b \in H(y_1)}{Inf} g(c \otimes_2 b) \\ &= \underset{c \in H(x_1),\ b \in H(y_1)}{Inf} [g(c) \text{ or } g(b)] \\ &= \underset{c \in H(x_1)}{Inf} g(c) \text{ or } \underset{b \in H(y_1)}{Inf} g(b) \\ &= \underline{H}(g)(x_1) \text{ or } \underline{H}(g)(y_1).\end{aligned}$$

Thus, $\underline{H}(g)(x_1 \otimes_1 y_1) = \underline{H}(g)(x_1)$ or $\underline{H}(g)(x_1 \otimes_1 y_1) = \underline{H}(g)(y_1)$ for all $x_1,\ y_1 \in Q_t$. Hence $\underline{H}(g)$ is a FPI of Q_t. ■

Theorem 3.1.15 *Let g be a FPI of Q'_t and $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$. Then $\overline{H}(g)$ is a FPI of Q_t.*

Proof. As g is a FPI of Q'_t, therefore $g(c) = g(c \otimes_2 b)$ or $g(b) = g(c \otimes_2 b)$ for all $c, b \in Q'_t$. Since, g is a FPI of Q'_t, so g is a FI. By Theorem 3.1.8, $\overline{H}(g)$ is a FI of Q_t.

Consider,

$$\begin{aligned}\overline{H}(g)(w \otimes_1 z) &= \underset{e \in H(w \otimes_1 z)}{Sup} g(e) \\ &= \underset{e \in H(w) \otimes_2 H(z)}{Sup} g(e)\end{aligned}$$

Since H is a $SSVH$, therefore for $e \in H(w) \otimes_2 H(z)$ there exist $c \in H(w)$ and $b \in H(z)$ such that $e = c \otimes_2 b$.

Hence,

$$\begin{aligned}\overline{H}(g)(w \otimes_1 z) &= \underset{c\otimes_2 b\in H(w)\otimes_2 H(z)}{Sup} g(c \otimes_2 b)\\ &= \underset{c\in H(w),\, b\in H(z)}{Sup} g(c \otimes_2 b)\\ &= \underset{c\in H(w),\, b\in H(z)}{Sup} [g(c) \text{ or } g(b)]\\ &= \underset{c\in H(w)}{Sup}\, g(c) \text{ or } \underset{b\in H(z)}{Sup}\, g(b)\\ &= \overline{H}(g)(w) \text{ or } \overline{H}(g)(z).\end{aligned}$$

Thus, $\overline{H}(g)(w \otimes_1 z) = \overline{H}(g)(w)$ or $\overline{H}(g)(w \otimes_1 z) = \overline{H}(g)(z)$ for all $w, z \in Q_t$. Hence $\overline{H}(g)$ is a FPI of Q_t. ■

Now, we have the following corollary.

Corollary 3.1.16 *Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$ and g be a FPI of Q'_t. Then g is a $GRFPI$ of Q'_t.*

Theorem 3.1.17 *Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$ and $\underline{H}(g)$ be a FI of Q_t. Then $\underline{H}(g)$ is a FPI of Q_t if and only if $\underline{H}(g)(w \otimes_1 z) = \underline{H}(g)(w) \vee \underline{H}(g)(z)$ for all z, $w \in Q_t$.*

Proof. Let $\underline{H}(g)$ be a FPI of Q_t. Then $\underline{H}(g)(w) = \underline{H}(g)(w \otimes_1 z)$ or $\underline{H}(g)(z) = \underline{H}(g)(w \otimes_1 z)$.

This implies that $\underline{H}(g)(w) \vee \underline{H}(g)(z) \geq \underline{H}(g)(w \otimes_1 z)$. (1)

As $\underline{H}(g)$ is a FI of Q_t, hence by definition of FI, we have $\underline{H}(g)(w \otimes_1 z) \geq \underline{H}(g)(w) \vee \underline{H}(g)(z)$. (2)

By (1) and (2), we obtain $\underline{H}(g)(w) \vee \underline{H}(g)(z) = \underline{H}(g)(w \otimes_1 z)$. Conversely, suppose that $\underline{H}(g)(w \otimes_1 z) = \underline{H}(g)(w) \vee \underline{H}(g)(z)$ for all w, $z \in Q_t$. We have to show that $\underline{H}(g)$ is a FPI. As [0,1] is a totally ordered so $\underline{H}(g)(w) \vee \underline{H}(g)(z) = \underline{H}(g)(w)$ or $\underline{H}(g)(w) \vee \underline{H}(g)(z) = \underline{H}(g)(z)$. Hence $\underline{H}(g)(w \otimes_1 z) = \underline{H}(g)(w)$ or $\underline{H}(g)(w \otimes_1 z) = \underline{H}(g)(z)$ for all w, $z \in Q_t$. This shows that $\underline{H}(g)$ is a FPI of Q_t. ■

Theorem 3.1.18 *Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$ and g be a FPI of Q'_t. Then $\underline{H}(g)$ (respectively $\overline{H}(g)$) is a FPI of Q_t if and only if for each $\alpha \in [0, 1]$, $\underline{H}(g_\alpha)$ (respectively $\overline{H}(g_\alpha)$) where $g_\alpha \neq \emptyset$, is a PI of Q_t.*

Proof. The proof is similar to the proof of Proposition 1.4.14(1). ■

Theorem 3.1.19 *Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$ and g be a $FSPI$ of Q'_t. Then $\underline{H}(g)$ is a $FSPI$ of Q_t.*

Proof. As g is a $FSPI$ of Q'_t, therefore $g(d^2) = g(d)$ for all $d \in Q'_t$ and g is a FI of Q'_t, so by Theorem 3.1.7, $\underline{H}(g)$ is a FI of Q_t.

Now consider,

$$\begin{aligned} \underline{H}(g)(w) = & \underset{d\in H(w)}{Inf} \; g(d) \\ = & \underset{d\in H(w)}{Inf} \; g(d^2) \\ = & \underset{d*_2 d\in H(w)*_2 H(w)}{Inf} \; g(d^2) \\ = & \underset{d*_2 d\in H(w*_1 w)}{Inf} \; g(d^2) \\ = & \underset{d^2\in H(w^2)}{Inf} \; g(d^2) \\ = & \underline{H}(g)(w^2). \end{aligned}$$

Thus $\underline{H}(g)(w) = \underline{H}(g)(w^2)$ for all $w \in Q_t$. Therefore $\underline{H}(g)$ is a $FSPI$ of Q_t. ■

Theorem 3.1.20 *Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$ and g be a $FSPI$ of Q'_t. Then $\overline{H}(g)$ is a $FSPI$ of Q_t.*

Proof. The Proof is similar as reported in Theorem 3.1.19. ■

Corollary 3.1.21 *Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$ and g be a $FSPI$ of Q'_t. Then g is a $GRFSPI$ of Q'_t.*

Theorem 3.1.22 *Let g be a $FSPI$ of Q'_t and $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$. Then $\underline{H}(g)$ (respectively $\overline{H}(g)$) is a $FSPI$ of Q_t if and only if for each $\alpha \in [0,1]$, $\underline{H}(g_\alpha)$ (respectively $\overline{H}(g_\alpha)$) where $g_\alpha \neq \emptyset$, is a SPI of Q_t.*

Proof. Proof is similar to the proof of Proposition 1.4.13(2). ■

Example 3.1.23 *Let $(Q_t,\otimes_1)$ and $(Q'_t, \otimes_2)$be two quantales, where Q_t and Q'_t are depicted in Fig. 7 and 8 and the binary operations $\otimes_1$ and $\otimes_2$ on both the quantales are the same as the meet operation in the lattices Q_t and Q'_t as shown in the table 4 and 5. Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$ as defined in Example 3.1.12. Let λ*

be a f-subset of Q'_t defined by $\lambda = \frac{0.9}{\bot'} + \frac{0.6}{i} + \frac{0.9}{j} + \frac{0.6}{\top'}$. *Then it is easy to confirm that λ is a FPI of Q'_t. Hence GUA and GLA of λ, are* $\overline{H}(\lambda) = \frac{0.9}{\bot} + \frac{0.9}{a} + \frac{0.6}{\top}$ *and* $\underline{H}(\lambda) = \frac{0.9}{\bot} + \frac{0.6}{a} + \frac{0.6}{\top}$. *It is observed that $\overline{H}(\lambda)$ and $\underline{H}(\lambda)$ are non-constant FPI of Q_t.*

Let g be a f-subset of Q'_t defined by $g(x) = \begin{cases} 1, & x = \bot' \\ 0.6, & x \neq \bot' \end{cases}$ *for all $x \in Q'_t$. Then g is a $FSPI$ of Q'_t. Hence GLA and GUA of g, are as follows* $\underline{H}(g) = \frac{1}{\bot} + \frac{0.6}{a} + \frac{0.6}{\top}$ *and* $\overline{H}(g) = \frac{1}{\bot} + \frac{0.6}{a} + \frac{0.6}{\top}$. *It is straightforward that $\overline{H}(g)$ and $\underline{H}(g)$ are $FSPI$ of Q_t.*

The next results are about the lower and upper approximations of fuzzy primary ideals ($FPYI$).

Theorem 3.1.24 *Let g be a $FPYI$ of Q'_t and H be a $SSVH$. Then $\underline{H}(g)$ is a $FPYI$ of Q_t.*

Proof. As g is a $FPYI$ of Q'_t, therefore $g(a) = g(a \otimes_2 b)$ or $g(b^n) = g(a \otimes_2 b)$ for all $a, b \in Q'_t$ and hence, g is a FI of Q'_t, so by Theorem 3.1.7, $\underline{H}(g)$ is a FI of Q_t. Since H is given as $SSVH$,

Consider,

$$\begin{aligned}
\underline{H}(g)(z \otimes_1 w) &= \underset{d \in H(z \otimes_1 w)}{Inf} g(d) \\
&= \underset{a \otimes_2 b \in H(z) \otimes_2 H(w)}{Inf} g(a \otimes_2 b) \\
&= \underset{a \in H(z),\ b \in H(w)}{Inf} g(a \otimes_2 b) \\
&= \underset{a \in H(z),\ b \in H(w)}{Inf} [g(a) \text{ or } g(b^n)] \\
&= \underset{a \in H(z)}{Inf} g(a) \text{ or } \underset{b \in H(w)}{Inf} g(b^n) \\
&= \underset{a \in H(z)}{Inf} g(a) \text{ or } \underset{b^n \in H(w^n)}{Inf} g(b^n) \\
&= \underline{H}(g)(z) \text{ or } \underline{H}(g)(w^n).
\end{aligned}$$

Here $b^n = b \otimes_2 b \otimes_2, ..., \otimes_2 b \in H(w) \otimes_2 H(w) \otimes_2, ..., \otimes_2 H(w) = H(w \otimes_1 w \otimes_1 w \otimes_1, ..., \otimes_1 w) = H(w^n)$ up to n times for some positive integer n. Thus $\underline{H}(g)(z \otimes_1 w) = \underline{H}(g)(z)$ or $\underline{H}(g)(z \otimes_1 w) = \underline{H}(g)(w^n)$ for all $z, w \in Q_t$. Therefore $\underline{H}(g)$ is a $FPYI$ of Q_t. ∎

Theorem 3.1.25 *Let g be a $FPYI$ of Q'_t and H be a $SSVH$. Then $\overline{H}(g)$ is a $FPYI$ of Q_t.*

Proof. The proof is similar to the proof of Theorem 3.1.24. ■

Theorem 3.1.26 *Let H be a $SSVH$ and g be a non-constant $FPYI$ of Q'_t. Then $\underline{H}(g)$ (respectively $\overline{H}(g)$) is a $FPYI$ of Q_t if and only if for each $\alpha \in [0,1]$, $\underline{H}(g_\alpha)$ (respectively $\overline{H}(g_\alpha)$) where $g_\alpha \neq \emptyset$, is a PYI of Q_t.*

Proof. The proof is similar to the proof of Proposition 1.4.14(2). ■

3.2 Homomorphic images of Generalized Rough Ideals based on Quantale Homomorphism

In this section, we will describe the images of GLA and GUA by using QH and SVH of quantales.

Proposition 3.2.1 *Let $(Q_t, \otimes_1)$ and $(Q'_t, \otimes_2)$ be two quantales, $\sigma_t : Q_t \longrightarrow Q'_t$ be an epimorphism and $H_2 : Q'_t \longrightarrow P^*(Q'_t)$ be a SVH. If σ_t is one-one and $H_1(x) = \{y \in Q_t \mid \sigma_t(y) \in H_2(\sigma_t(x))\}$ for all $x \in Q_t$, then H_1 is a SVH from Q_t to $P^*(Q_t)$.*

Proof. First of all, we show that H_1 is well defined mapping. Suppose $x_1 = x_2$, then we have, $y_1 \in H_1(x_1) \Longleftrightarrow \sigma_t(y_1) \in H_2(\sigma_t(x_1)) = H_2(\sigma_t(x_2)) \Longleftrightarrow y_1 \in H_1(x_2)$. Thus we have $H_1(x_1) = H_1(x_2)$. Now we show that H_1 is SVH. Suppose $y \in H_1(x_1) \otimes_1 H_1(x_2)$, then there exist $a \in H_1(x_1)$ and $b \in H_1(x_2)$ such that $y = a \otimes_1 b$. Since H_2 is a SVH and σ_t is a QH, we have $\sigma_t(a) \otimes_2 \sigma_t(b) \in H_2(\sigma_t(x_1)) \otimes_2 H_2(\sigma_t(x_2)) \subseteq H_2(\sigma_t(x_1) \otimes_2 (\sigma_t(x_2)) = H_2(\sigma_t(x_1 \otimes_1 x_2))$. Therefore, $\sigma_t(a \otimes_1 b) = \sigma_t(a) \otimes_2 \sigma_t(b) \in H_2(\sigma_t(x_1 \otimes_1 x_2))$. Hence $y = a \otimes_1 b \in H_1(x_1 \otimes_1 x_2)$. Thus, we have $H_1(x_1) \otimes_1 H_1(x_2) \subseteq H_1(x_1 \otimes_1 x_2)$. Now we show that $\vee_{i \in I} H_1(x_i) \subseteq H_1(\vee_{i \in I} x_i)$ for all $x_i \in Q_t$ $(i \in I)$. Let $y \in \vee_{i \in I} H_1(x_i)$, then there exist $a_i \in H_1(x_i)$ for all $i \in I$ such that $y = \vee_{i \in I} a_i$. Hence $\sigma_t(y) = \sigma_t(\vee_{i \in I} a_i) = \vee_{i \in I} \sigma_t(a_i) \in \vee_{i \in I} H_2(\sigma_t(x_i)) \subseteq H_2(\vee_{i \in I} \sigma_t(x_i)) = H_2(\sigma_t(\vee_{i \in I} x_i))$. Thus, $y = \vee_{i \in I} a_i \in H_1(\vee_{i \in I} x_i)$. Hence $\vee_{i \in I} H_1(x_i) \subseteq H_1(\vee_{i \in I} x_i))$. So, H_1 is a SVH from Q_t to $P^*(Q_t)$. ■

Theorem 3.2.2 *Let $\sigma_t : Q_t \longrightarrow Q'_t$ be a surjective QH and $H_2 : Q'_t \longrightarrow P^*(Q'_t)$ be a SVH. Set $H_1(m) = \{z \in Q_t \mid \sigma_t(z) \in H_2(\sigma_t(m))\}$ for all $m \in Q_t$ and for all $\emptyset \neq C \subseteq Q_t$, then*

(1) $\overline{H}_2(\sigma_t(C)) = \sigma_t(\overline{H}_1(C))$;

(2) $\underline{H}_2(\sigma_t(C)) = \sigma_t(\underline{H}_1(C))$;

(3) *If* $\sigma_t : Q_t \longrightarrow Q'_t$ *is also one-one, then* $\sigma_t(x) \in \sigma_t(\overline{H}_1(C)) \Longleftrightarrow x \in \overline{H}_1(C)$.

Proof. (1) Let $z \in \sigma_t(\overline{H}_1(C))$. Then there exist $x \in \overline{H}_1(C)$ such that $\sigma_t(x) = z$. Since $x \in \overline{H}_1(C)$, so $H_1(x) \cap C \neq \emptyset$. Suppose, $z' \in H_1(x) \cap C$, then $\sigma_t(z') \in \sigma_t(C)$, and by the definition of $H_1(x)$, we obtain $\sigma_t(z') \in H_2(\sigma_t(x))$. Thus, $H_2(\sigma_t(x)) \cap \sigma_t(C) \neq \emptyset$, and hence $z = \sigma_t(x) \in \overline{H}_2(\sigma_t(C))$. Thus, we obtain $\sigma_t(\overline{H}_1(C) \subseteq \overline{H}_2(\sigma_t(C))$. Now we take $y \in \overline{H}_2(\sigma_t(C))$, then there exist $m \in Q_t$ such that $\sigma_t(m) = y$. Hence $H_2(\sigma_t(m)) \cap \sigma_t(C) \neq \emptyset$. So there exists $z_1 \in C$ such that $\sigma_t(z_1) \in \sigma_t(C)$ and $\sigma_t(z_1) \in H_2(\sigma_t(m))$. By the definition of $H_1(m)$, we have $z_1 \in H_1(m)$. Thus $H_1(m) \cap C \neq \emptyset$. This gives $m \in \overline{H}_1(C)$. Hence, $y = \sigma_t(m) \in \sigma_t(\overline{H}_1(C))$. Thus $\overline{H}_2(\sigma_t(C) \subseteq \sigma_t(\overline{H}_1(C)$. Finally, we obtain $\sigma_t(\overline{H}_1(C) = \overline{H}_2(\sigma_t(C))$.

(2) Suppose $z \in \sigma_t(\underline{H}_1(C))$, then there exists $m \in \underline{H}_1(C)$ such that $\sigma_t(m) = z$ and $H_1(m) \subseteq C$. Suppose $z' \in H_2(\sigma_t(m))$, then there is $n' \in Q_t$ such that $\sigma_t(n') = z'$, hence $\sigma_t(n') \in H_2(\sigma_t(m))$. Thus $n' \in H_1(m) \subseteq C$ and so $z' = \sigma_t(n') \in \sigma_t(C)$. Hence, $H_2(\sigma_t(m)) \subseteq \sigma_t(C)$. Thus $z = \sigma_t(m) \in \underline{H}_2(\sigma_t(C))$, so we have $\sigma_t(\underline{H}_1(C)) \subseteq \underline{H}_2(\sigma_t(C))$. Now, let $y \in \underline{H}_2(\sigma_t(C))$. Then there exists $n \in Q_t$ such that $\sigma_t(n) = y$ and $H_2(\sigma_t(n)) \subseteq \sigma_t(C)$. Suppose $n' \in H_1(n)$, then $\sigma_t(n') \in H_2(\sigma_t(n)) \subseteq \sigma_t(C)$ and hence $n' \in C$. Thus $H_1(n) \subseteq C$ and we obtain $n \in \underline{H}_1(C)$. Hence $\sigma_t(n) = y \in \sigma_t(\underline{H}_1(C))$ and thus, $\underline{H}_2(\sigma_t(C)) \subseteq \sigma_t(\underline{H}_1(C))$. Hence, we have $\sigma_t(\underline{H}_1(C)) = \underline{H}_2(\sigma_t(C))$.

(3) Let $x \in \overline{H}_1(C)$. Then $\sigma_t(x) \in \sigma_t(\overline{H}_1(C))$. Conversely, suppose that $\sigma_t(x) \in \sigma_t(\overline{H}_1(C))$, then there exists $y \in \overline{H}_1(C)$ such that $\sigma_t(y) = \sigma_t(x)$. Since σ_t is ono-one, we have $x = y \in \overline{H}_1(C)$. ■

Lemma 3.2.3 *Let* $(Q_t, \otimes_1)$ *and* $(Q'_t, \otimes_2)$ *be two quantales,* $\sigma_t : Q_t \longrightarrow Q'_t$ *be an isomorphism,* $H_2 : Q'_t \longrightarrow P^*(Q'_t)$ *be a* SVH *and* $H_1 : Q_t \longrightarrow P^*(Q_t)$ *defined in Proposition 3.2.2. Then* $\sigma_t(x) \in \sigma_t(\underline{H}_1(C)) \Longleftrightarrow x \in \underline{H}_1(C)$.

Proof. The Proof is similar to the proof of Proposition 3.2.2(3). ■

Theorem 3.2.4 *Let* $\sigma_t : Q_t \longrightarrow Q'_t$ *be an isomorphism and* $H_2 : Q'_t \longrightarrow P^*(Q'_t)$ *be a* SVH. *Let* $H_1(x) = \{y \in Q_t \mid \sigma_t(y) \in H_2(\sigma_t(x))\}$ *for all* $x \in Q_t$. *Then for all* $\emptyset \neq C \subseteq Q_t$, *the following hold;*

(1) $\overline{H}_1(C)$ *is an ideal of* Q_t *if and only if* $\overline{H}_2(\sigma_t(C))$ *is an ideal of* Q'_t;

(2) $\overline{H}_1(C)$ *is a* PI *of* Q_t *if and only if* $\overline{H}_2(\sigma_t(C))$ *is a* PI *of* Q'_t;

(3) $\overline{H}_1(C)$ *is a* SPI *of* Q_t *if and only if* $\overline{H}_2(\sigma_t(C))$ *is a* SPI *of* Q'_t;

(4) $\overline{H}_1(C)$ *is a primary ideal* (PYI) *of* Q_t *if and only if* $\overline{H}_2(\sigma_t(C))$ *is a primary ideal* (PYI) *of* Q'_t.

Proof. By Theorem 3.2.2(1), $\sigma_t(\overline{H}_1(C)) = \overline{H}_2(\sigma_t(C))$ for each $C \subseteq Q_t$.

(1) Suppose $\overline{H}_1(C)$ is an ideal of Q_t.

(i) Let $x, z \in \sigma_t(\overline{H}_1(C))$. Then there exist $x_1, z_1 \in \overline{H}_1(C)$ such that $\sigma_t(x_1) = x$ and $\sigma_t(z_1) = z$. Since σ_t is a surjective QH and $\overline{H}_1(C)$ is an ideal of Q_t, we obtain $x \vee z = \sigma_t(x_1) \vee \sigma_t(z_1) = \sigma_t(x_1 \vee z_1) \in \sigma_t(\overline{H}_1(C))$. Therefore $x \vee z \in \sigma_t(\overline{H}_1(C))$ for all $x, z \in \sigma_t(\overline{H}_1(C))$.

(ii) Let $z \leq x \in \sigma_t(\overline{H}_1(C))$. Then there exist $x_1 \in \overline{H}_1(C)$ and $z_1 \in Q_t$ such that $\sigma_t(x_1) = x$ and $\sigma_t(z_1) = z$. Since $\sigma_t(z_1) \leq \sigma_t(x_1)$, we have $\sigma_t(x_1 \vee z_1) = \sigma_t(x_1) \vee \sigma_t(z_1) = \sigma_t(x_1) \in \sigma_t(\overline{H}_1(C))$. From part (3) in Theorem 3.2.2, it follows that $x_1 \vee z_1 \in \overline{H}_1(C)$. Since $\overline{H}_1(C)$ is an ideal and $z_1 \leq x_1 \vee z_1$, we have $z_1 \in \overline{H}_1(C)$. Thus $z = \sigma_t(z_1) \in \sigma_t(\overline{H}_1(C))$.

(iii) Let $x \in \sigma_t(\overline{H}_1(C))$ and $z \in Q'_t$. Then there exist $x_1 \in \overline{H}_1(C)$ and $z_1 \in Q_t$ such that $\sigma_t(x_1) = x$ and $\sigma_t(z_1) = z$. Since $\overline{H}_1(C)$ is an ideal and σ_t is a QH, we obtain $x_1 \otimes_1 z_1 \in \overline{H}_1(C)$. Hence $x \otimes_2 z = \sigma_t(x_1) \otimes_2 \sigma_t(z_1) = \sigma_t(x_1 \otimes_1 z_1) \in \sigma_t(\overline{H}_1(C))$. In a similar way, we have $z \otimes_2 x \in \sigma_t(\overline{H}_1(C))$. Hence, $\sigma_t(\overline{H}_1(C))$ is an ideal of Q'_t. But $\overline{H}_2(\sigma_t(C)) = \sigma_t(\overline{H}_1(C))$. So $\overline{H}_2(\sigma_t(C))$ is an ideal of Q'_t.

Conversely, suppose $\overline{H}_2(\sigma_t(C)) = \sigma_t(\overline{H}_1(C))$ is an ideal of Q'_t.

(i) Let $z_1, z_2 \in \overline{H}_1(C)$. Then $\sigma_t(z_1), \sigma_t(z_2) \in \sigma_t(\overline{H}_1(C))$. Since $\sigma_t(\overline{H}_1(C))$ is an ideal, $\sigma_t(z_1 \vee z_2) = \sigma_t(z_1) \vee \sigma_t(z_2) \in \sigma_t(\overline{H}_1(C))$. So by Theorem 3.2.2(3), we have $z_1 \vee z_2 \in \overline{H}_1(C)$.

(ii) Let $z_1 \leq z_2 \in \overline{H}_1(C)$. Then $\sigma_t(z_1) \leq \sigma_t(z_2) \in \sigma_t(\overline{H}_1(C))$. Since $\sigma_t(\overline{H}_1(C))$ is an ideal, we have $\sigma_t(z_1) \in \sigma_t(\overline{H}_1(C))$. By Theorem 3.2.2(3), we obtain $z_1 \in \overline{H}_1(C)$. So $\overline{H}_1(C)$ is a lower set.

(iii) Suppose $y' \in Q_t$ and $y \in \overline{H}_1(C)$, then $\sigma_t(y') \in Q'_t$ and $\sigma_t(y) \in \sigma_t(\overline{H}_1(C))$. But $\sigma_t(\overline{H}_1(C))$ is an ideal of Q'_t, we have $\sigma_t(y \otimes_1 y') = \sigma_t(y) \otimes_2 \sigma_t(y') \in \sigma_t(\overline{H}_1(C))$ and

$\sigma_t(y' \otimes_1 y) = \sigma_t(y') \otimes_2 \sigma_t(y) \in \sigma_t(\overline{H}_1(C))$. Thus, we have $y \otimes_1 y'$, $y' \otimes_1 y \in \overline{H}_1(C)$ by Theorem 3.2.2(3). Hence by (i)-(iii), $\overline{H}_1(C)$ is an ideal of Q_t.

(2) First we show that $\overline{H}_1(C) \neq Q_t \Leftrightarrow \sigma_t(\overline{H}_1(C)) \neq Q'_t$, that is $\overline{H}_1(C) = Q_t \Leftrightarrow \sigma_t(\overline{H}_1(C)) = Q'_t$. Assume that $\overline{H}_1(C) = Q_t$. Since σ_t is surjective, we have $\sigma_t(\overline{H}_1(C)) = \sigma_t(Q_t) = Q'_t$. Conversely, assume that $\sigma_t(\overline{H}_1(C)) = Q'_t$. For each $z \in Q_t$ we have $\sigma_t(z) \in \sigma_t(Q_t) = Q'_t = \sigma_t(\overline{H}_1(C))$. Then by Theorem 3.2.2(3), we have $z \in \overline{H}_1(C)$ and thus $\overline{H}_1(C) = Q_t$.

Let $\overline{H}_1(C)$ is a PI of Q_t. Then $\overline{H}_1(C)$ is obviously an ideal of Q_t and $\overline{H}_1(C) \neq Q_t$. By part (1), $\overline{H}_2(\sigma_t(C))$ is an ideal of Q'_t. We also have that $\overline{H}_2(\sigma_t(C)) = \sigma_t(\overline{H}_1(C)) \neq Q'_t$. Now suppose $y_1, y_2 \in Q'_t$ and $y_1 \otimes_2 y_2 \in \overline{H}_2(\sigma_t(C))$. Since σ_t is surjective, there are $z_1, z_2 \in Q_t$ such that $y_1 = \sigma_t(z_1)$, $y_2 = \sigma_t(z_2)$. Then $\sigma_t(z_1 \otimes_1 z_2) = \sigma_t(z_1) \otimes_2 \sigma_t(z_2) = y_1 \otimes_2 y_2 \in \sigma_t(\overline{H}_1(C))$. By Theorem 3.2.2(3), we have $z_1 \otimes_1 z_2 \in \overline{H}_1(C)$. Since $\overline{H}_1(C)$ is prime, we have $z_1 \in \overline{H}_1(C)$ or $z_2 \in \overline{H}_1(C)$. Thus $y_1 \in \sigma_t(\overline{H}_1(C)) = \overline{H}_2(\sigma_t(C))$ or $y_2 \in \sigma_t(\overline{H}_1(C)) = \overline{H}_2(\sigma_t(C))$. So $\overline{H}_2(\sigma_t(C))$ is a PI of Q'_t.

Conversely, let $\overline{H}_2(\sigma_t(C))$ is a PI of Q'_t. Then $\overline{H}_2(\sigma_t(C))$ is an ideal of Q'_t. Since $\sigma_t(\overline{H}_1(C) = \overline{H}_2(\sigma_t(C) \neq Q'_t$ and thus $\overline{H}_1(C) \neq Q_t$. By part (1), $\overline{H}_1(C)$ is an ideal of Q_t. Now suppose z_1, $z_2 \in Q_t$ and $z_1 \otimes_1 z_2 \in \overline{H}_1(C)$. So, $\sigma_t(z_1) \otimes_2 \sigma_t(z_2) = \sigma_t(z_1 \otimes_1 z_2) \in \sigma_t(\overline{H}_1(C))$. Since $\sigma_t(\overline{H}_1(C)) = \overline{H}_2(\sigma_t(C))$ is prime, we have $\sigma_t(z_1) \in \sigma_t(\overline{H}_1(C))$ or $\sigma_t(z_2) \in \sigma_t(\overline{H}_1(C))$. So by Theorem 3.2.2(3), we have $z_1 \in \overline{H}_1(C)$ or $z_2 \in \overline{H}_1(C)$. Thus $\overline{H}_1(C)$ is a PI of Q_t.

The proof of remaining parts (3) and (4) are similar to the proof (1) and (2). ∎

Theorem 3.2.5 *Let $\sigma_t : Q_t \longrightarrow Q'_t$ be an isomorphism and $H_2 : Q'_t \longrightarrow P^*(Q'_t)$ be a SVH. Set $H_1(x) = \{y \in Q_t \mid \sigma_t(y) \in H_2(\sigma_t(x))\}$ for all $x \in Q_t$. Then for all $\emptyset \neq B \subseteq Q_t$, the following hold,*

(1) *$\underline{H}_1(B)$ is an ideal of Q_t if and only if $\underline{H}_2(\sigma_t(B))$ is an ideal of Q'_t;*

(2) *$\underline{H}_1(B)$ is a PI of Q_t if and only if $\underline{H}_2(\sigma_t(B))$ is a PI of Q'_t;*

(3) *$\underline{H}_1(B)$ is a SPI of Q_t if and only if $\underline{H}_2(\sigma_t(B))$ is a SPI of Q'_t;*

(4) *$\underline{H}_1(B)$ is a PYI of Q_t if and only if $\underline{H}_2(\sigma_t(B))$ is a PYI of Q'_t.*

Proof. By Theorem 3.2.2(1), $\sigma_t(\underline{H}_1(B)) = \underline{H}_2(\sigma_t(B))$ for each $B \subseteq Q_t$.

(1) Suppose $\underline{H}_1(B)$ is an ideal of Q_t.

(i) Let $x, z \in \sigma_t(\underline{H}_1(B))$. Then there exist $x_1, z_1 \in \underline{H}_1(B)$ such that $\sigma_t(x_1) = x$ and $\sigma_t(z_1) = z$. Since σ_t is a surjective QH and $\underline{H}_1(B)$ is an ideal of Q_t, we obtain $x \vee z = \sigma_t(x_1) \vee \sigma_t(z_1) = \sigma_t(x_1 \vee z_1) \in \sigma_t(\underline{H}_1(B))$. Therefore $x \vee z \in \sigma_t(\underline{H}_1(B))$ for all $x, z \in \sigma_t(\underline{H}_1(B))$.

(ii) Let $z \leq x \in \sigma_t(\underline{H}_1(B))$. Then there exist $x_1 \in \underline{H}_1(B)$ and $z_1 \in Q_t$ such that $\sigma_t(x_1) = x$ and $\sigma_t(z_1) = z$. Since $\sigma_t(z_1) \leq \sigma_t(x_1)$, we have $\sigma_t(x_1 \vee z_1) = \sigma_t(x_1) \vee \sigma_t(z_1) = \sigma_t(x_1) \in \sigma_t(\underline{H}_1(B))$. From part (3) in Theorem 3.2.2, it follows that $x_1 \vee z_1 \in \underline{H}_1(B)$. Since $\underline{H}_1(B)$ is an ideal and $z_1 \leq x_1 \vee z_1$, we have $z_1 \in \underline{H}_1(B)$. Thus $z = \sigma_t(z_1) \in \sigma_t(\underline{H}_1(B))$.

(iii) Let $x \in \sigma_t(\underline{H}_1(B))$ and $z \in Q'_t$. Then there exist $x_1 \in \underline{H}_1(B)$ and $z_1 \in Q_t$ such that $\sigma_t(x_1) = x$ and $\sigma_t(z_1) = z$. Since $\underline{H}_1(B)$ is an ideal and σ_t is a QH, we obtain $x_1 \otimes_1 z_1 \in \underline{H}_1(B)$. Hence $x \otimes_2 z = \sigma_t(x_1) \otimes_2 \sigma_t(z_1) = \sigma_t(x_1 \otimes_1 z_1) \in \sigma_t(\underline{H}_1(B))$. In a similar way, we have $z \otimes_2 x \in \sigma_t(\underline{H}_1(B))$. Hence, $\sigma_t(\underline{H}_1(B))$ is an ideal of Q'_t. But $\underline{H}_2(\sigma_t(B)) = \sigma_t(\underline{H}_1(B))$. So $\underline{H}_2(\sigma_t(B))$ is an ideal of Q'_t.

Conversely, suppose $\underline{H}_2(\sigma_t(B)) = \sigma_t(\underline{H}_1(B))$ is an ideal of Q'_t.

(i) Let $z_1, z_2 \in \underline{H}_1(B)$. Then $\sigma_t(z_1), \sigma_t(z_2) \in \sigma_t(\underline{H}_1(B))$. Since $\sigma_t(\underline{H}_1(B))$ is an ideal, $\sigma_t(z_1 \vee z_2) = \sigma_t(z_1) \vee \sigma_t(z_2) \in \sigma_t(\underline{H}_1(B))$. So by Theorem 3.2.2(3), we have $z_1 \vee z_2 \in \underline{H}_1(B)$.

(ii) Let $z_1 \leq z_2 \in \underline{H}_1(B)$. Then $\sigma_t(z_1) \leq \sigma_t(z_2) \in \sigma_t(\underline{H}_1(B))$. Since $\sigma_t(\underline{H}_1(B))$ is an ideal, we have $\sigma_t(z_1) \in \sigma_t(\underline{H}_1(B))$. By Theorem 3.2.2(3), we obtain $z_1 \in \underline{H}_1(B)$. So $\underline{H}_1(B)$ is a lower set.

(iii) Suppose $y' \in Q_t$ and $y \in \underline{H}_1(B)$, then $\sigma_t(y') \in Q'_t$ and $\sigma_t(y) \in \sigma_t(\underline{H}_1(B))$. But $\sigma_t(\underline{H}_1(B))$ is an ideal of Q'_t, we have $\sigma_t(y \otimes_1 y') = \sigma_t(y) \otimes_2 \sigma_t(y') \in \sigma_t(\underline{H}_1(B))$ and $\sigma_t(y' \otimes_1 y) = \sigma_t(y') \otimes_2 \sigma_t(y) \in \sigma_t(\underline{H}_1(B))$. Thus, we have $y \otimes_1 y'$, $y' \otimes_1 y \in \underline{H}_1(B)$ by Theorem 3.2.2(3). Hence by (i)-(iii), $\underline{H}_1(B)$ is an ideal of Q_t.

(2) First we show that $\underline{H}_1(B) \neq Q_t \Leftrightarrow \sigma_t(\underline{H}_1(B)) \neq Q'_t$, that is $\underline{H}_1(B) = Q_t \Leftrightarrow \sigma_t(\underline{H}_1(B)) = Q'_t$. Assume that $\underline{H}_1(B) = Q_t$. Since σ_t is surjective, we have $\sigma_t(\underline{H}_1(B)) = \sigma_t(Q_t) = \sigma_t(Q'_t)$. Conversely, assume that $\sigma_t(\underline{H}_1(B)) = Q'_t$. For each $z \in Q_t$ we have $\sigma_t(z) \in \sigma_t(Q_t) = Q'_t = \sigma_t(\underline{H}_1(B))$. Then by Theorem 3.2.2(3), we have $z \in \underline{H}_1(B)$ and thus $\underline{H}_1(B) = Q_t$.

Let $\underline{H}_1(B)$ is a PI of Q_t. Then $\underline{H}_1(B)$ is obviously an ideal of Q_t and $\underline{H}_1(B) \neq Q_t$. By part (1), $\underline{H}_2(\sigma_t(B))$ is an ideal of Q'_t. We also have that $\underline{H}_2(\sigma_t(B)) = \sigma_t(\underline{H}_1(B)) \neq Q'_t$.

Now suppose $y_1, y_2 \in Q'_t$ and $y_1 \otimes_2 y_2 \in \underline{H}_2(\sigma_t(B))$. Since σ_t is surjective, there are $z_1, z_2 \in Q_t$ such that $y_1 = \sigma_t(z_1)$, $y_2 = \sigma_t(z_2)$. Then $\sigma_t(z_1 \otimes_1 z_2) = \sigma_t(z_1) \otimes_2 \sigma_t(z_2) = y_1 \otimes_2 y_2 \in \sigma_t(\underline{H}_1(B))$. By Theorem 3.2.2(3), we have $z_1 \otimes_1 z_2 \in \underline{H}_1(B)$. Since $\underline{H}_1(B)$ is prime, we have $z_1 \in \underline{H}_1(B)$ or $z_2 \in \underline{H}_1(B)$. Thus $y_1 \in \sigma_t(\underline{H}_1(B)) = \underline{H}_2(\sigma_t(B))$ or $y_2 \in \sigma_t(\underline{H}_1(B)) = \underline{H}_2(\sigma_t(B))$. So $\underline{H}_2(\sigma_t(B))$ is a PI of Q'_t.

Conversely, let $\underline{H}_2(\sigma_t(B))$ is a PI of Q'_t. Then $\underline{H}_2(\sigma_t(B))$ is an ideal of Q'_t. Since $\sigma_t(\underline{H}_1(B) = \underline{H}_2(\sigma_t(B) \neq Q'_t$ and thus $\underline{H}_1(B) \neq Q_t$. By part (1), $\underline{H}_1(B)$ is an ideal of Q_t. Now suppose z_1, $z_2 \in Q_t$ and $z_1 \otimes_1 z_2 \in \underline{H}_1(B)$. So, $\sigma_t(z_1) \otimes_2 \sigma_t(z_2) = \sigma_t(z_1 \otimes_1 z_2) \in \sigma_t(\underline{H}_1(B))$. Since $\sigma_t(\underline{H}_1(B)) = \underline{H}_2(\sigma_t(B))$ is prime, we have $\sigma_t(z_1) \in \sigma_t(\underline{H}_1(B))$ or $\sigma_t(z_2) \in \sigma_t(\underline{H}_1(B))$. So by Theorem 3.2.2(3), we have $z_1 \in \underline{H}_1(B)$ or $z_2 \in \underline{H}_1(B)$. Thus $\underline{H}_1(B)$ is a PI of Q_t.

The proof of remaining parts (3) and (4) are similar to the proof (2). ■

3.3 Generalized Rough Fuzzy Prime (Primary) Ideals Induced by Quantale Homomorphism

In this section, we will discuss relations between the upper (lower) generalized rough fuzzy (prime, semi-prime, primary) ideals of quantales and the upper (lower) approximations of their homomorphic images and give some Theorems related to them.

Theorem 3.3.1 *Let $\sigma_t : Q_t \longrightarrow Q'_t$ be a surjective QH, $H_2 : Q'_t \longrightarrow P^*(Q'_t)$ be a SVH and λ be a f-subset of Q_t. If $H_1(x) = \{y \in Q_t \mid \sigma_t(y) \in H_2(\sigma_t(x))\}$ for all $x \in Q_t$, then*

(1) *$\overline{H}_1(\lambda)$ is a FI of Q_t if and only if $\overline{H}_2(\sigma_t(\lambda))$ is a FI of Q'_t;*

(2) *$\overline{H}_1(\lambda)$ is a FPI of Q_t if and only if $\overline{H}_2(\sigma_t(\lambda))$ is a FPI of Q'_t;*

(3) *$\overline{H}_1(\lambda)$ is a $FSPI$ of Q_t if and only if $\overline{H}_2(\sigma_t(\lambda))$ is a $FSPI$ of Q'_t;*

(4) *$\overline{H}_1(\lambda)$ is a $FPYI$ of Q_t if and only if $\overline{H}_2(\sigma_t(\lambda))$ is a $FPYI$ of Q'_t.*

In the above,

$$\sigma_t(\lambda)(y) = \begin{cases} \underset{x \in \sigma_t^{-1}(y)}{Sup} \lambda(x), & \text{if } \sigma_t^{-1}(y) \neq \emptyset \ \forall \ y \in Q'_t \\ 0, & \text{otherwise} \end{cases}$$

i.e., $\sigma_t(\lambda)$ is the standard Zadeh image of the f-subset λ under the mapping σ_t. (see Definition 1.4.7).

Proof. (1) We first point out that for each $\alpha \in [0,1]$, $(\sigma_t(\lambda))_{\alpha^+} = \sigma_t(\lambda_{\alpha^+})$ and $(\overline{H}_1(\lambda))_{\alpha^+} \neq \emptyset$ if and only if $(\overline{H}_2(\sigma_t(\lambda)))_{\alpha^+} \neq \emptyset$.

Let $\overline{H}_1(\lambda)$ be a FI of Q_t. Then for all $\alpha \in (0,1]$, if $(\overline{H}_2(\sigma_t(\lambda))_{\alpha^+} \neq \emptyset$, then $(\overline{H}_1(\lambda))_{\alpha^+} \neq \emptyset$. By Theorem 3.1.11, we have $(\overline{H}_1(\lambda))_{\alpha^+}$ is an ideal of Q_t. Also by using Proposition 3.1.5, we obtain $\overline{H}_1(\lambda_{\alpha^+})$ is an ideal of Q_t. Now, by Theorem 3.2.4(1) and Proposition 3.1.5, we have $(\overline{H}_2(\sigma_t(\lambda)))_{\alpha^+} = \overline{H}_2((\sigma_t(\lambda))_{\alpha^+} = \overline{H}_2(\sigma_t(\lambda_{\alpha^+}))$ is an ideal of Q'_t. Thus, by Theorem 3.1.11, we have $\overline{H}_2(\sigma_t(\lambda))$ is a FI of Q'_t.

Conversely, suppose $\overline{H}_2(\sigma_t(\lambda))$ is a FI of Q'_t. We have $(\overline{H}_2(\sigma_t(\lambda)))_{\alpha^+} = \overline{H}_2(\sigma_t(\lambda))_{\alpha^+} = \overline{H}_2(\sigma_t(\lambda_{\alpha^+}))$ is an ideal of Q'_t by utilizing Theorem 3.1.11 and Proposition 3.1.5. Thus, $\overline{H}_1(\lambda_{\alpha^+})$ is an ideal of Q_t from Theorem 3.2.4(1). Hence $\overline{H}_1(\lambda)$ is a FI of Q_t by Theorem 3.1.11.

(2) Let $\overline{H}_1(\lambda)$ be a FPI of Q_t. Now for $\overline{H}_2(\sigma_t(\lambda))_{\alpha^+} \neq \emptyset$, then $(\overline{H}_1(\lambda))_{\alpha^+} \neq \emptyset$ for each $\alpha \in [0,1]$. Since $\overline{H}_1(\lambda)$ is a FPI of Q_t, then by Theorem 3.1.18 and Proposition 3.1.5, we have $(\overline{H}_1(\lambda))_{\alpha^+} = \overline{H}_1(\lambda))_{\alpha^+} = \overline{H}_1(\lambda_{\alpha^+})$ is a PI of Q_t. Hence $(\overline{H}_2(\sigma_t(\lambda)))_{\alpha^+} = \overline{H}_2((\sigma_t(\lambda))_{\alpha^+} = \overline{H}_2(\sigma_t(\lambda_{\alpha^+}))$ is a PI of Q'_t, by Theorem 3.2.4(2). Thus, by Theorem 3.1.18, we have $\overline{H}_2(\sigma_t(\lambda))$ is a FPI of Q'_t.

Conversely, suppose $\overline{H}_2(\sigma_t(\lambda))$ is a FPI of Q'_t. By Theorem 3.1.18, we have

$$(\overline{H}_2(\sigma_t(\lambda)))_{\alpha^+} = \overline{H}_2(\sigma_t(\lambda))_{\alpha^+} = \overline{H}_2(\sigma_t(\lambda_{\alpha^+}))$$

is a PI of Q'_t. Thus from Theorem 3.2.4(2), $\overline{H}_1(\lambda_{\alpha^+})$ is a PI of Q_t. Hence $\overline{H}_1(\lambda)$ is a FPI of Q_t by Theorem 3.1.18.

Proof of (3) and (4) is similar to the proof of (1) and (2). ∎

Theorem 3.3.2 *Let σ_t be a surjective QH from a quantale $(Q_t, \otimes_1)$ onto a quantale $(Q'_t, \otimes_2)$. Let $H_2 : Q'_t \longrightarrow P^*(Q'_t)$ be a SVH and λ be a f-subset of Q_t. If $H_1(x) = \{y \in Q_t \mid f(y) \in H_2(f(x))\}$ for all $x \in Q_t$, then*

(1) *$\underline{H}_1(\lambda)$ is a FI of Q_t if and only if $\underline{H}_2(\sigma_t(\lambda))$ is a FI of Q'_t;*

(2) *$\underline{H}_1(\lambda)$ is a FPI of Q_t if and only if $\underline{H}_2(\sigma_t(\lambda))$ is a FPI of Q'_t;*

(3) *$\underline{H}_1(\lambda)$ is a $FSPI$ of Q_t if and only if $\underline{H}_2(\sigma_t(\lambda))$ is a $FSPI$ of Q'_t;*

(4) *$\underline{H}_1(\lambda)$ is a $FPYI$ of Q_t if and only if $\underline{H}_2(\sigma_t(\lambda))$ is a $FPYI$ of Q'_t.*

Proof. The proof is similar to the proof of Theorem 3.3.1. ■

Chapter 4

Characterizations of Quantales by (α, β)-Fuzzy Ideals

In this chapter, we describe (α, β)-$f$uzzy subquantales and (α, β)-$f$uzzy ideals of quantale. Further, $(\in, \in \vee q)$-fuzzy ideal and $(\in, \in \vee q)$-fuzzy subquantale are discussed. It is investigated that homomorphic image of an $(\in, \in \vee q)$-fuzzy subquantale (ideal) under QH is an $(\in, \in \vee q)$-fuzzy subquantale (ideal). These fuzzy subquantales and fuzzy ideals are characterized by their level subquantales and ideals, respectively. Some important results about $(\in, \in \vee q)$-fuzzy prime and $(\in, \in \vee q)$-fuzzy semi prime ideals are discussed. Fuzzy quantale submodule is defined and its generalization that is an (α, β)-$f$uzzy Q_t-submodule of Q_t-module is also introduced in this chapter.

In the first section, (α, β)-$f$uzzy ideals and (α, β)-$f$uzzy subquantales are introduced. Moreover, $(\in, \in \vee q)$-fuzzy ideals and $(\in, \in \vee q)$-fuzzy subquantales are discussed in the second section. With the help of QH, it is proved that inverse image of $(\in, \in \vee q)$-fuzzy subquantale and $(\in, \in \vee q)$-fuzzy ideal are $(\in, \in \vee q)$-fuzzy subquantale and $(\in, \in \vee q)$-fuzzy ideal, respectively. In section three, we define the $(\in, \in \vee q)$-$f$uzzy prime and $(\in, \in \vee q)$-$f$uzzy semi prime ideals of Quantale. It is also investigated that if a f-subset g is an $(\in, \in \vee q)$-$f$uzzy prime (or $(\in, \in \vee q)$-$f$uzzy semi prime) ideal of Q'_t, then $\sigma^{-1}(g)$ is an $(\in, \in \vee q)$-$f$uzzy prime (or $(\in, \in \vee q)$-$f$uzzy semi prime) ideal of Q_t. In the last section, (α, β)-$f$uzzy Q_t-submodule of Q_t-module is introduced. Fuzzy Q_t-submodule is characterized by its level Q_t-subquantales.

4.1 (α,β)-Fuzzy Ideals of Quantale

In this section, let α and β be one of $\in$, q, $\in\vee q$ or $\in\wedge q$, unless otherwise specified. From here onward, we will write (α,β)-FI, (α,β)-FRI, (α,β)-FLI, (α,β)-FS, $(\in,\in\vee q)$-FI, $(\in,\in\vee q)$-FS, $(\in,\in\vee q)$-FRI and $(\in,\in\vee q)$-FLI for (α,β)-fuzzy ideal, (α,β)-fuzzy right ideal, (α,β)-fuzzy left ideal, (α,β)-fuzzy subquantale, $(\in,\in\vee q)$-fuzzy ideal, $(\in,\in\vee q)$-fuzzy subquantale, $(\in,\in\vee q)$-fuzzy right ideal and $(\in,\in\vee q)$-fuzzy left ideal, respectively.

Definition 4.1.1 *[66] A f-subset g of a quantale Q_t is called a fuzzy point if*

$$g(y)=\begin{cases} p, & \text{if } y=z \\ 0, & \text{otherwise} \end{cases} \quad \text{for all } z,y\in Q_t.$$

Then z is called the support of g and $p\in(0,1]$ is its value. A fuzzy point is represented by z_p. Pu and Liu [66], gave meaning to the symbol $z_p\alpha g$, where $\alpha\in\{\in,\ q,\in\vee q,\in\wedge q\}$ for a fuzzy point z_p and a f-subset g in a set Q_t.

(1) *When $g(z)\geq p$, then it means that z_p belongs to g and is represented as $z_p\in g$.*

(2) *When $g(z)+p>1$, then z_p is called quasi-coincident with g and is denoted as z_pqg.*

(3) *When $g(z)\geq p$ or $g(z)+p>1$, then z_p belongs to g or z_p is quasi-coincident with g and is denoted as $z_p\ (\in\vee q)g$. Similarly, $z_p\ (\in\wedge q)g$ denotes that $z_p\in g$ and z_pqg. When $z_p\overline{\alpha}g$ means that $z_p\alpha g$ does not hold.*

Each f-subset g defined on Q_t can be characterized by its level subsets, i.e., by the sets of the form $U(g;p)=\{z\in Q_t : g(z)\geq p\}$, where $p\in[0,1]$. An important part is played by the support of g, i.e., the set $g_\circ=\{z\in Q_t : g(z)>0\}$.

For a f-subset g of Q_t such that $g(z)\leq 0.5$ for any $z\in Q_t$, in this case $z_p(\in\wedge q)g$, we have $g(z)\geq p$ and $g(z)+p>1$. Thus, $1<g(z)+p\leq g(z)+g(z)=2g(z)$. This shows that $g(z)\geq 0.5$. Hence, $\{z_p : z_p(\in\wedge q)g\}=\emptyset$. Thus, the case $\alpha=\in\wedge q$ is omitted.

Definition 4.1.2 *[90] Let $\sigma_t : Q_t\longrightarrow Q'_t$ be a mapping from a quantale Q_t to a quantale Q'_t, and let g and g' be f-subsets in Q_t and Q'_t, respectively. Then the image of g under σ_t and the pre-image of g' under σ_t are the f-subsets $\sigma_t(g)$ and $\sigma_t^{-1}(g')$, respectively, defined as follows:*

$$(i)\ \sigma_t(g)(y) = \begin{cases} \underset{x \in \sigma_t^{-1}(y)}{Sup} g(x), & \text{if } \sigma_t^{-1}(y) \neq \emptyset \text{ for all } y \in Q'_t \\ 0, & \text{otherwise} \end{cases},$$

(ii) $\sigma_t^{-1}(g')(x) = g'(\sigma_t(x))$ *for all* $x \in Q_t$.

If σ_t *is a* QH, *then* $\sigma_t(g)$ *is called the homomorphic image of* g *under* σ_t *and* $\sigma_t^{-1}(g')$ *is called the homomorphic pre-image of* g'.

Definition 4.1.3 *Let* $(Q_t, \otimes)$ *be a quantale and* g *be a f-subset of* Q_t. *We say that* g *is a* FS *of* Q_t *if*

(i) $g(\vee_{i\in I} z_i) \geq \underset{i\in I}{inf}\ g(z_i)$,

(ii) $g(y \otimes z) \geq inf(g(y), g(z))$ *for all* z, $z_i, y \in Q_t$.

Proposition 4.1.4 *Let* g_1 *and* g_2 *be the* FSs *of* Q_t. *Then* $(g_1 \Cap g_2)$ *is a* FS *of* Q_t.

Proof. Let $z_i \in Q_t$ for some $i \in I$ and g_1 and g_2 be the FS's of Q_t, so by Definition 4.1.3, we have;

$$g_1(\vee_{i\in I} z_i) \geq \underset{i\in I}{inf}\ g_1(z_i) \text{ and } g_2(\vee_{i\in I} z_i) \geq \underset{i\in I}{inf}\ g_2(z_i)$$

$$\Longrightarrow inf\{g_1(\vee_{i\in I} z_i),\ g_2(\vee_{i\in I} z_i)\} \geq inf\{\underset{i\in I}{inf} g_1(z_i),\ \underset{i\in I}{inf} g_2(z_i)\}$$

$$\Longrightarrow inf\{g_1(\vee_{i\in I} z_i),\ g_2(\vee_{i\in I} z_i)\} \geq \underset{i\in I}{inf}\{inf(g_1(z_i),\ g_2(z_i))\}$$

$$\Longrightarrow (g_1 \Cap g_2)(\vee_{i\in I} z_i) \geq \underset{i\in I}{inf}(g_1 \Cap g_2)(z_i)$$

Next, as $g_1(z_1 \otimes z_2) \geq inf\{g_1(z_1), g_1(z_2)\}$ and $g_2(z_1 \otimes z_2) \geq inf\{g_2(z_1), g_2(z_2)\}$

$\Longrightarrow inf\{g_1(z_1 \otimes z_2), g_2(z_1 \otimes z_2)\} \geq inf(inf\{g_1(z_1), g_1(z_2)\}, inf\{g_2(z_1), g_2(z_2)\})$

$\Longrightarrow inf\{g_1(z_1 \otimes z_2), g_2(z_1 \otimes z_2)\} \geq inf(inf\{g_1(z_1), g_2(z_1)\}, inf\{g_1(z_2), g_2(z_2)\})$

$\Longrightarrow (g_1 \Cap g_2)(z_1 \otimes z_2) \geq inf\{(g_1 \Cap g_2)(z_1), (g_1 \Cap g_2)(z_2)\}$.

Therefore, $(g_1 \Cap g_2)$ is a FS of Q_t. ■

Definition 4.1.5 *A f-subset* g *of a quantale* Q_t *is called an* (α, β)-FS *of* Q_t, *if*

(i) $(z_i)_{p_i} \alpha g \longrightarrow (\vee_{i\ \in\ I} z_i)_{\underset{i\in I}{inf} p_i} \beta g$,

(ii) $z_p \alpha g$, *and* $w_v \alpha g \longrightarrow (z \otimes w)_{inf(p,v)} \beta g$, *for all* p_i, p, $v \in (0, 1]$ *and for all* z_i, z, $w \in Q_t$.

Lemma 4.1.6 *A f-subset g of a quantale Q_t is a FS of Q_t if and only if it satisfies $(z_i)_{p_i} \in g \longrightarrow (\vee_{i \in I} z_i)_{\inf_{i\in I} p_i} \in g$ and $z_p \in g$, $w_v \in g \longrightarrow (z \otimes w)_{inf(p,v)} \in g$ for all p_i, p, $v \in (0,1]$ and for all $z_i, z, w \in Q_t$.*

Proof. Let g be a FS of Q_t and $z_i \in Q_t$ and $p_i \in (0,1]$ be such that $(z_i)_{p_i} \in g$ for $i \in I$. Then $g(z_i) \geq p_i$, for all $i \in I$. Since g is a FS of Q_t, so $g(\vee_{i\in I} z_i) \geq \inf_{i\in I} g(z_i) \geq \inf_{i\in I} p_i$. Hence $(\vee_{i \in I} z_i)_{\inf_{i\in I} p_i} \in g$.

Let $p, v \in (0,1]$ and $z, w \in Q_t$ be such that $z_p \in g$ and $w_v \in g$. Then $g(z) \geq p$ and $g(w) \geq v$. But g is a FS of Q_t, hence $g(z \otimes w) \geq inf(g(z), g(w)) \geq inf(p, v)$. Thus $g(z \otimes w) \geq inf(p, v)$. This implies that $(z \otimes w)_{inf(p,v)} \in g$.

Conversely, suppose that g satisfies the given conditions. First we show that $g(\vee_{i\in I} z_i) \geq \inf_{i\in I} g(z_i)$ for $i \in I$. On contrary suppose that $g(\vee_{i\in I} z_i) < \inf_{i\in I} g(z_i)$ for $z_i \in Q_t$. Let $p \in (0,1]$ be such that $g(\vee_{i\in I} z_i) < p \leq \inf_{i\in I} g(z_i)$. Then $(z_i)_p \in g$ but $(\vee_{i \in I} z_i)_p \overline{\in} g$. This contradicts our hypothesis. Thus $g(\vee_{i\in I} z_i) \geq \inf_{i\in I} g(z_i)$ for $z_i \in Q_t$. Similarly, we show that $g(w \otimes z) \geq inf(g(z), g(w))$ for all $w, z \in Q_t$. Hence g is a FS of Q_t. ■

Remark 4.1.7 *The above Lemma shows that every FS of Q_t is an $(\in, \in)$-FS of Q_t and vice versa.*

Theorem 4.1.8 *Let g be a nonzero (α, β)- FS of Q_t. Then the set $g_\circ = \{y \in Q_t \mid g(y) > 0\}$ is a subquantale of Q_t.*

Proof. Let $y_i \in g_\circ$ for $i \in I$. Then $g(y_i) > 0$ for all $i \in I$. Let $g(\vee_{i\in I} y_i) = 0$. If $\alpha \in \{\in, \in \vee q\}$, then $(y_i)_{g(y_i)} \alpha g$ for all $i \in I$ but $g(\vee_{i\in I} y_i) = 0 < \inf_{i\in I} g(y_i)$ and $g(\vee_{i\in I} y_i) + \inf_{i\in I} g(y_i) \leq 0 + 1 = 1$. So $(\vee_{i\in I} y_i)_{\inf_{i\in I} g(y_i)} \overline{\beta} g$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, this gives a contradiction. Hence $g(\vee_{i\in I} y_i) > 0$, i.e., $\vee_{i\in I} y_i \in g_\circ$. Also $(y_i)_1 q g$ for all $i \in I$ but $(\vee_{i\in I} y_i)_1 \overline{\beta} g$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$. Hence $g(\vee_{i\in I} y_i) > 0$, i.e., $\vee_{i\in I} y_i \in g_\circ$. Thus $g_\circ$ is closed under arbitrary join. The proof is similar for $g_\circ$ to be closed under $\otimes$. ■

Definition 4.1.9 *A f-subset g of a quantale Q_t is said to be an (α, β)-FRI (FLI) of Q_t, if* *(i)* $z_p\alpha g,\ w_v\alpha g \longrightarrow (z\vee w)_{inf(p,v)}\beta g,$

(ii) $z_p\alpha g, w\in Q_t \longrightarrow (z\otimes w)_p\beta g,$ *[respectively,* $(w\otimes z)_p\beta g]$

(iii) $z_p\alpha g$ *and* $w\le z \longrightarrow w_p\beta g,$ *for all* $p,\ v\in(0,1]$ *and for all* $z,\ w\in Q_t$.

A f*-subset* g *of a quantale* Q_t *is called an* (α,β)*-*FI *of* Q_t *if it is both an* (α,β)*-*FRI *and* (α,β)*-*FLI *of* Q_t*.*

Example 4.1.10 *Let* $(Q_t,\otimes)$ *be a quantale, where* Q_t *is depicted in Fig.*9 *and the binary operation* $\otimes$*on* Q_t *is shown in the table* 7*. Ideals of* Q_t *are* $\{\bot\}$, $\{\bot,j\}$ *and* Q_t*.*

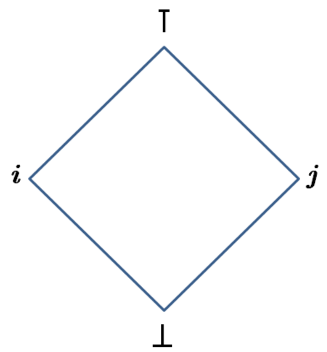

Fig. 9

Table. 6

$\otimes$	$\bot$	i	j	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
i	$\bot$	i	j	$\top$
j	$\bot$	j	j	j
$\top$	$\bot$	$\top$	j	$\top$

Define $g: Q_t \longrightarrow [0,1]$ by $g=\frac{0.8}{\bot}+\frac{0.7}{i}+\frac{0.6}{j}+\frac{0.5}{\top}$. Then clearly g is an $(\in,\in\vee q)$-FI of Q_t. But,

(i) g is not $(\in,\in)$-$FI\ of\ Q_t$, since

$$i_{0.68}\in g \text{ but } (i\otimes j)_{0.68}\overline{\in} g;$$

(ii) g is not $(q, \in)$-FI of Q_t, since

$$i_{0.61} q g \text{ but } (i \otimes j)_{0.61} \overline{\in} g;$$

(iii) g is not $(\in, q)$-FI of Q_t, since

$$\top_{0.3} \in g \text{ but } (\top \otimes j)_{0.3} \overline{q} g;$$

(iv) g is not $(q, \in \wedge q)$-FI of Q_t, since

$$\top_{0.6} q g \text{ but } (\top \otimes i)_{0.6} \overline{(\in \wedge q)} g;$$

(v) g is not $(\in \vee q, \in \wedge q)$-$FI$ of Q_t, since

$$i_{0.65} q g \text{ but } (\top \otimes i)_{0.65} \overline{(\in \wedge q)} g;$$

(vi) g is not $(\in \vee q, \in)$-FI of Q_t, since

$$i_{0.65} q g \text{ but } (\top \otimes i)_{0.65} \overline{\in} g;$$

(vii) g is not $(\in, \in \wedge q)$-FI of Q_t, since

$$i_{0.67} \in g \text{ but } (j \otimes i)_{0.67} \overline{(\in \wedge q)} g;$$

$(viii)$ g is not (q, q)-FI of Q_t, since

$$i_{0.5} q g \text{ but } (\top \otimes i)_{0.5}\ \overline{q} g;$$

Lemma 4.1.11 *A f-subset g in a quantale Q_t is a FRI (FLI) of Q_t if and only if the following hold:*

(1) $z_p, w_v \in g \longrightarrow (z \vee w)_{inf(p,v)} \in g$;

(2) $z_p \in g, w \in Q_t \longrightarrow (z \otimes w)_p \in g$ *[respectively,* $(w \otimes z)_p \in g$];

(3) $z_p \in g$ *and* $w \leq z \longrightarrow w_p \in g$, *for all* $p, v \in (0, 1]$ *and for all* $z, w \in Q_t$.

Proof. The proof is like the proof of Lemma 4.1.6. ■

Remark 4.1.12 *The above Lemma shows that every FRI (FLI) of Q_t is an $(\in, \in)$-FRI (FLI) of Q_t and vice versa.*

Theorem 4.1.13 *Let g be a nonzero (α, β)- FRI (FLI) of Q_t. Then $g_\circ = \{y \in Q_t \mid g(y) > 0\}$ is a right (left) ideal of Q_t.*

Proof. Let g be a nonzero (α, β)- FRI of Q_t. Let $w, z \in g_\circ$. Then $g(w) > 0$ and $g(z) > 0$. Let $g(w \vee z) = 0$. If $\alpha \in \{\in, \in \vee q\}$, then $(w)_{g(w)} \alpha g$ and $(z)_{g(z)} \alpha g$

but $g(w \vee z) = 0 < inf(g(w), g(z))$ and $g(w \vee z) + inf(g(w), g(z)) \leq 0 + 1 = 1$. So $(w \vee z)_{inf(g(w),g(z))}\overline{\beta}g$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Hence $g(w \vee z) > 0$, i.e., $w \vee z \in g_\circ$. Also $w_1 qB$ and $z_1 qB$ but $(w \vee z)_1\overline{\beta}g$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$. Hence $g(w \vee z) > 0$, that is $w \vee z \in g_\circ$. Thus $g_\circ$ is closed under join.

Let $w, z \in Q_t$ and $w \leq z$. If $z \in g_\circ$, then $g(z) > 0$. Assume that $g(w) = 0$. If $\alpha \in \{\in, \in \vee q\}$, then $(z)_{g(z)}\alpha g$ but $(w)_{g(w)}\overline{\beta}g$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Also $z_1 qg$ but $w_1\overline{\beta}g$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$. Hence $g(w) > 0$, i.e., $w \in g_\circ$.

Let $w \in g_\circ$ and for all $z \in Q_t$. Then $g(w) > 0$. We want to show that $g(w \otimes z) > 0$ for all $z \in Q_t$. Suppose that $g(w \otimes z) = 0$ and let $\alpha \in \{\in, \in \vee q\}$. Then $(w)_{g(w)}\alpha B$ but $(w \otimes z)_{g(w)}\overline{\beta}g$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, this is a contradiction. Also $w_1 qB$ but $(w \otimes z)_1\overline{\beta}g$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Therefore $g(w \otimes z) > 0$ and so $w \otimes z \in g_\circ$. Hence $g_\circ$ is a right ideal of a quantale Q_t. ■

Proposition 4.1.14 *Let A be a right (left) ideal of Q_t. Then a f-subset g of Q_t such that $g(z) \geq 0.5$ for $z \in A$ and $g(z) = 0$ otherwise is an $(\alpha, \in \vee q)$- FRI (FLI) of Q_t.*

Proof. Let A be a right ideal of Q_t.

(**a**) Suppose and $p, v \in (0, 1]$ and $y, z \in Q_t$ be such that $y_p \in g$ and $z_v \in g$. Then $g(y) \geq p$ and $g(z) \geq v$. Thus $y, z \in A$ and so $y \vee z \in A$, that is $g(y \vee z) \geq 0.5$. If $inf(p, v) \leq 0.5$, then $g(y \vee z) \geq 0.5 \geq inf(p, v)$. Hence $(y \vee z)_{inf(p,v)} \in g$. If $inf(p, v) > 0.5$, then $g(y \vee z) + inf(p, v) > 0.5 + 0.5 = 1$ and so $(y \vee z)_{inf(p,v)}qg$. Therefore $(y \vee z)_{inf(p,v)}(\in \vee q)g$.

Let $y, z \in Q_t$, $y \leq z$ and $v \in (0, 1]$ be such that $z_v \in g$. Then $g(z) \geq v$. Thus $z \in A$ and since A is a right ideal so $y \in A$, that is $g(y) \geq 0.5$. If $v \leq 0.5$, then $g(y) \geq 0.5 \geq v$. Hence $y_v \in g$. If $v > 0.5$, then $g(y) + v > 0.5 + 0.5 = 1$ and so $y_v qg$. It follows that $y_v(\in \vee q)g$.

Now let $y, z \in Q_t$ and $p \in (0, 1]$ be such that $y_p \in g$. Then $g(y) \geq p$, which implies $y \in A$, and so $y \otimes z \in A$, for all $z \in Q_t$. Consequently $g(y \otimes z) \geq 0.5$. If $p \leq 0.5$, then $g(y \otimes z) \geq 0.5 \geq p$. Hence $(y \otimes z)_p \in g$. If $p > 0.5$, then $g(y \otimes z) + p > 0.5 + 0.5 = 1$ and so $(y \otimes z)_p qg$. Thus $(y \otimes z)_p(\in \vee q)g$. Hence g is an $(\in, \in \vee q)$-FRI of Q_t.

(**b**) Suppose that $y, z \in Q_t$ and $p, v \in (0,1]$ be such that $y_p qg$ and $z_v qg$. Then $y, z \in A$, $g(y)+p > 1$ and $g(z)+v > 1$. Thus, $y, z \in A$ and since A is a right ideal so $y \vee z \in A$, we have $g(y \vee z) \geq 0.5$. If $inf(p, v) \leq 0.5$, then $g(y \vee z) \geq 0.5 \geq inf(p, v)$. Hence $(y \vee z)_{inf(p,v)} \in g$. If $inf(p, v) > 0.5$, then $g(y \vee z) + inf(p, v) > 0.5 + 0.5 = 1$ and so $(y \vee z)_{inf(p,v)} qg$. Therefore $(y \vee z)_{inf(p,v)} (\in \vee q) g$.

Let $y, z \in Q_t, y \leq z$ and $v \in (0,1]$ be such that $z_v qg$. Then $g(z) + v > 1$. Thus $z \in A$ and since A is a right ideal so $y \in A$, that is $g(y) \geq 0.5$. If $v \leq 0.5$, then $g(y) \geq 0.5 \geq v$. Hence $y_v \in g$. If $v > 0.5$, then $g(y) + v > 0.5 + 0.5 = 1$ and so $y_v qg$. It follows that $y_v (\in \vee q) g$.

Now, let $y, z \in Q_t$ and $p \in (0,1]$ be such that $y_p qg$, which implies that $g(y) + p > 1$. Thus $y \in A$ and so $y \otimes z$ is in A. This means that $g(y \otimes z) \geq 0.5$. If $p \leq 0.5$, then $g(y \otimes z) \geq 0.5 \geq p$. Hence $(y \otimes z)_p \in g$. If $p > 0.5$, then $g(y \otimes z) + p > 0.5 + 0.5 = 1$ and so $(y \otimes z)_p qg$. Thus $(y \otimes z)_p (\in \vee q) g$. Hence g is an $(q, \in \vee q)$-FRI of Q_t.

(**c**) Let $y, z \in Q_t$ and $p, v \in (0,1]$ be such that $y_p \in g$ and $z_v qg$. Then $g(y) \geq p$ and $g(z) + v > 1$. Thus, $y, z \in A$ and so $y \vee z \in A$, we have $g(y \vee z) \geq 0.5$. Thus, $(y \vee z)_{inf(p,v)} \in g$ for $inf(p, v) \leq 0.5$ and $(y \vee z)_{inf(p,v)} qg$ for $inf(p, v) > 0.5$. Thus $(y \vee z)_{inf(p,v)} (\in \vee q) g$. The rest is similar to the proof of parts (a) and (b). ■

Theorem 4.1.15 *Let C be a subquantale of Q_t. Then a f-subset g of Q_t such that $g(c) \geq 0.5$ for $c \in C$ and $g(c) = 0$ otherwise is an $(\alpha, \in \vee q)$-FS of Q_t.*

Proof. The proof is like the proof of Theorem 4.1.14. ■

Proposition 4.1.16 *Let g be a f-subset of a quantale Q'_t and $\sigma_t : Q_t \longrightarrow Q'_t$ be a QH. Then $(\sigma_t(w))_p \alpha g$ if and only if $w_p \alpha \sigma_t^{-1}(g)$ for all $w \in Q_t$ and $p \in (0,1]$.*

Proof. Let $\alpha = \in$. Then $(\sigma_t(w))_p \in g \Longleftrightarrow g(\sigma_t(w)) \geq p \Longleftrightarrow \sigma_t^{-1}(g)(w) \geq p \Longleftrightarrow w_p \in \sigma_t^{-1}(g)$. Let $\alpha = q$. Then $(\sigma_t(w))_p qg \Longleftrightarrow g(\sigma_t(w)) + p > 1 \Longleftrightarrow \sigma_t^{-1}(g)(w) + p > 1 \Longleftrightarrow w_p q \sigma^{-1}(g)$. Similarly, we can obtain the other cases. ■

Theorem 4.1.17 *Let $\sigma_t : Q_t \longrightarrow Q'_t$ be a QH and g be an (α, β)-FRI (FLI) of Q'_t. Then $\sigma_t^{-1}(g)$ is an (α, β)-FRI (FLI) of Q_t.*

Proof. Let $z, w \in Q_t$ and $p, v \in (0,1]$ be such that $z_p \alpha \sigma_t^{-1} g$ and $w_v \alpha \sigma_t^{-1} g$. Then $(\sigma_t(z))_p \alpha g$ and $(\sigma_t(w))_v \alpha g$ by Proposition 4.1.16. Since g is an (α, β)-FRI of Q'_t, we have $(\sigma_t(z) \vee \sigma_t(w))_{inf(p,v)}\beta g$ and $(\sigma_t(z \vee w))_{inf(p,v)}\beta g$ by using QH. Thus, $(z \vee w)_{inf(p,v)}\beta\sigma_t^{-1}g$ by Proposition 4.1.16. Let $z_p\alpha\sigma_t^{-1}g$ such that $w \leq z$. Then $(\sigma_t(z))_p\alpha g$ and $\sigma_t(w) \leq \sigma_t(z)$. Since g is an (α, β)-FRI of Q'_t, we have $\sigma_t(w)_p\beta g$. By Proposition 4.1.16, $w_p\beta\sigma_t^{-1}g$. Let $z_p\alpha\sigma_t^{-1}g$ and for all $y \in Q_t$. Then $(\sigma_t(z))_p\alpha g$ and $\sigma_t(y) \in Q'_t$. Hence, $(\sigma_t(z) \otimes' \sigma_t(y))_p\beta g \longrightarrow (\sigma_t(z \otimes y))_p\beta g$ as g is an (α, β)-FRI of Q'_t and σ_t is a QH. Again by Proposition 4.1.16, we have $(z \otimes y)_p\beta\sigma_t^{-1}g$. Hence $\sigma_t^{-1}(g)$ is an (α, β)-FRI of Q_t. ■

Proposition 4.1.18 *Let $(Q_t, \otimes)$ and $(Q'_t, \otimes')$ be two quantales and $\sigma_t : Q_t \longrightarrow Q'_t$ be a QH. Let g be (α, β)-FS of Q'_t. Then $\sigma_t^{-1}(g)$ be an (α, β)-FS of Q_t.*

Proof. The proof is similar to the proof of Theorem 4.1.17. ■

4.2 $(\in, \in \vee q)$- Fuzzy Ideals of Quantale

We introduce some results about $(\in, \in \vee q)$-FI and $(\in, \in \vee q)$-FS of quantale Q_t in this section. We will show that homomorphic image of $(\in, \in \vee q)$-FS is $(\in, \in \vee q)$-FS. Also with the help of QH, we will show that inverse image of $(\in, \in \vee q)$-FS ($(\in, \in \vee q)$-FI) is $(\in, \in \vee q)$-FS ($(\in, \in \vee q)$-FI).

Lemma 4.2.1 *For a f-subset g of a quantale Q_t, the conditions below are equivalent*:

$$(z_i)_{p_i} \in g \longrightarrow (\vee_{i \in I} z_i)_{\inf\limits_{i\in I} p_i}(\in \vee q)g, \tag{1}$$

$$g(\vee_{i\in I} z_i) \geq inf(\inf_{i\in I} g(z_i), 0.5). \tag{2}$$

Proof. (1) $\longrightarrow$ (2) Let $z_i \in Q_t$ for all $i \in I$. We consider the two cases:

(a°) $\inf\limits_{i\in I} g(z_i) < 0.5$,

(b°) $0.5 \leq \inf\limits_{i\in I} g(z_i)$.

First we consider the case when $\inf\limits_{i\in I} g(z_i) < 0.5$. Let $g(\vee_{i\in I} z_i) < inf(\inf\limits_{i\in I} g(z_i), 0.5)$, which implies that $g(\vee_{i\in I} z_i) < \inf\limits_{i\in I} g(z_i)$. Then we can select p such that $g(\vee_{i\in I} z_i) < p < \inf\limits_{i\in I} g(z_i)$, which means that $(z_i)_p \in g$ for all i but $(\vee_{i \in I} z_i)_p\overline{(\in \vee q)}g$. This contradicts (1). Hence, our supposition $g(\vee_{i\in I} z_i) < inf(\inf\limits_{i\in I} g(z_i), 0.5)$ is wrong.

Now consider the case $0.5 \leq \underset{i \in I}{inf}\ g(z_i)$. So, for $g(\vee_{i \in I} z_i) < 0.5$, we have $(z_i)_{0.5} \in g$ for all $i \in I$ and $(\vee_{i\ \in\ I} z_i)_{0.5} \overline{(\in \vee q)} g$, which is impossible. Hence, we have $g(\vee_{i \in I} z_i) \geq 0.5$. Thus $g(\vee_{i \in I} z_i) \geq 0.5 \geq inf(\underset{i \in I}{inf}\ g(z_i), 0.5)$.

(2) $\longrightarrow$ (1) Let $(z_i)_{p_i} \in g$ for all $i \in I$. Then $g(\vee_{i \in I} z_i) \geq inf(\underset{i \in I}{inf}\ g(z_i), 0.5) \geq inf(\underset{i \in I}{inf} p_i, 0.5)$. Hence we have $g(\vee_{i \in I} z_i) \geq \underset{i \in I}{inf}\ p_i$ when $\underset{i \in I}{inf}\ p_i \leq 0.5$ and $g(\vee_{i \in I} z_i) \geq 0.5$ for $\underset{i \in I}{inf}\ p_i > 0.5$. Thus $(\vee_{i\ \in\ I} z_i)_{\underset{i \in I}{inf} p_i} (\in \vee q) g$. ■

Lemma 4.2.2 *For any f-subset g of Q_t, the following conditions are equivalent:*

$$z_p \in g \text{ and } w_v \in g \longrightarrow (z \otimes w)_{inf(p,v)} (\in \vee q) g, \tag{3}$$

$$g(z \otimes w) \geq inf(g(z), g(w), 0.5). \tag{4}$$

Proof. The Proof is similar to the proof of Lemma 4.2.1. ■

Corollary 4.2.3 *A f-subset g of Q_t is an $(\in, \in \vee q)$-FS of Q_t if and only if the conditions* (2) *and* (4) *hold.*

Theorem 4.2.4 *Let $\sigma_t : Q_t \longrightarrow Q'_t$ be a QH. Let g_1 and g_2 be $(\in, \in \vee q)$-FS of Q_t and Q'_t, respectively. Then*

(1) *$\sigma_t(g_1)$ is an $(\in, \in \vee q)$-FS of Q'_t,*

(2) *$\sigma_t^{-1}(g_2)$ is an $(\in, \in \vee q)$-FS of Q_t.*

Proof. (1) For any $z_i \in Q'_t$, if $\sigma_t^{-1}(z_i) = \emptyset$ for $i \in I$, then $inf[\underset{i \in I}{inf}\ \sigma_t(g_1)(z_i), 0.5] = 0 \leq \sigma_t(g_1)(\vee_{i \in I} z_i)$ and if $\sigma_t^{-1}(z) = \emptyset$ or $\sigma_t^{-1}(w) = \emptyset$, then $inf(\sigma_t(g_1)(z), \sigma_t(g_1)(w), 0.5) = 0 \leq \sigma_t(g_1)(z \otimes w)$. Now suppose that $\sigma_t^{-1}(z_i) \neq \emptyset$ for each $i \in I$ and $\sigma_t^{-1}(\vee_{i \in I} z_i) \neq \emptyset$.

Thus,

$$\begin{aligned}
inf[\inf_{i\in I}(\sigma_t(g_1)(z_i)), 0.5] &= inf[inf[\sigma_t(g_1)(z_1), \sigma_t(g_1)(z_2), ..., \sigma_t(g_1)(z_i)], 0.5] \\
&= inf[inf[\sup_{a_1 \in \sigma_t^{-1}(z_1)} g_1(a_1), ..., \sup_{a_i \in \sigma_t^{-1}(z_i)} g_1(a_i)], 0.5] \\
&= \sup_{a_1 \in \sigma_t^{-1}(z_1), ..., a_i \in \sigma_t^{-1}(z_i)} inf[inf(g_1(a_1), ..., g_1(a_i)), 0.5] \\
&= \sup_{\sigma_t(a_1) = z_1, ..., \sigma_t(a_i) = z_i} inf[inf(g_1(a_1), ..., g_1(a_i)), 0.5] \\
&= \sup_{\vee_{i\in I}\sigma_t(a_i) = \vee_{i\in I} z_i} inf[\inf_{i\in I} g_1(a_i), 0.5] \\
&= \sup_{\sigma_t(\vee_{i\in I} a_i) = \vee_{i\in I} z_i} inf[\inf_{i\in I} g_1(a_i), 0.5], \ \sigma_t \text{ is a } QH \\
&\leq \sup_{\vee_{i\in I} a_i \in \sigma_t^{-1}(\vee_{i\in I} z_i)} g_1(\vee_{i\in I} a_i) \\
&= \sup_{y \in \sigma_t^{-1}(\vee_{i\in I} z_i)} g_1(y) \\
&= \sigma_t(g_1)(\vee_{i\in I} z_i)
\end{aligned}$$

Hence, $\sigma_t(g_1)(\vee_{i\in I} z_i) \geq inf[\inf_{i\in I} \sigma_t(g_1)(z_i), 0.5]$ for all $z_i \in Q'_t$ and

$$\begin{aligned}
inf[\sigma_t(g_1)(z), \sigma_t(g_1)(w), 0.5] &= inf[\sup_{a \in \sigma_t^{-1}(z)} g_1(a), \sup_{b \in \sigma_t^{-1}(w)} g_1(b), 0.5] \\
&= \sup_{a \in \sigma_t^{-1}(z),\, b \in \sigma_t^{-1}(w)} inf[g_1(a), g_1(b), 0.5] \\
&= \sup_{\sigma_t(a)=z,\, \sigma_t(b)=w} inf[g_1(a), g_1(b), 0.5] \\
&= \sup_{\sigma_t(a)\otimes'\sigma_t(b) = z\otimes' w} inf[g_1(a), g_1(b), 0.5] \\
&= \sup_{\sigma_t(a\otimes b) \in z\otimes' w} inf[g_1(a), g_1(b), 0.5], \ \sigma_t \text{ is a } QH \\
&\leq \sup_{a\otimes b \in \sigma_t^{-1}(z\otimes' w)} g_1(a \otimes b) \\
&= \sup_{y \in \sigma_t^{-1}(z \otimes' w)} g_1(y) \\
&= \sigma_t(g_1)(z \otimes' w)
\end{aligned}$$

So, $\sigma_t(g_1)(z \otimes' w) \geq inf[\sigma_t(g_1)(z), \sigma_t(g_1)(w), 0.5]$ for all $z, w \in Q'_t$. By Corollary 4.2.3, we have $\sigma_t(g_1)$ is an $(\in, \in \vee q)$-FS of Q'_t.

(2) Let $z_i \in Q_t$ for all $i \in I$. Then

$$\begin{aligned}
\sigma_t^{-1}(g_2)(\vee_{i\in I} z_i) &= g_2(\sigma_t(\vee_{i\in I} z_i)) \\
&= g_2(\vee_{i\in I}\sigma_t(z_i)), \ \sigma_t \text{ is a } QH \\
&\geq inf[\inf_{i\in I} g_2(\sigma_t(z_i)), 0.5] \\
&= inf[\inf_{i\in I} \sigma_t^{-1}(g_2)(z_i), 0.5].
\end{aligned}$$

Hence, $\sigma_t^{-1}(g_2)(\vee_{i\in I} z_i) \geq inf[\underset{i\in I}{inf}\ \sigma_t^{-1}(g_2)(z_i), 0.5]$ for all $z_i \in Q_t$.

Now

$$\begin{aligned} \sigma_t^{-1}(g_2)(z\otimes w) = &\ g_2(\sigma_t(z\otimes w)) \\ = &\ g_2(\sigma_t(z)\otimes' \sigma_t(w)),\ \sigma_t \text{ is a } QH \\ \geq &\ inf(g_2(\sigma_t(z)), g_2(\sigma_t(w)), 0.5) \\ = &\ inf(\sigma_t^{-1}(g_2)(z), \sigma_t^{-1}(g_2)(w), 0.5). \end{aligned}$$

Thus, $\sigma_t^{-1}(g_2)(z\otimes w) \geq inf(\sigma_t^{-1}(g_2)(z), \sigma_t^{-1}(g_2)(w), 0.5)$.

By Corollary 4.2.3, we have $\sigma_t^{-1}(g_2)$ is an $(\in, \in \vee q)$-FS of Q_t. ■

Lemma 4.2.5 *The following two conditions are equivalent, for any f-subset g of Q_t;*

$$z_p,\ w_v \in g \longrightarrow (z\vee w)_{inf(p,v)}(\in \vee q)g, \tag{5}$$

$$g(z\vee w) \geq inf(g(z), g(w), 0.5), \text{ for all } z,\ w \in Q_t \text{ and for all } p,\ v \in (0,1]. \tag{6}$$

Proof. (5) $\longrightarrow$ (6) On contrary assume that there exist $z, w \in Q_t$ such that $g(z\vee w) < inf(g(z), g(w), 0.5)$. Consider the following two cases.

Case:1 If $inf(g(z), g(w)) \leq 0.5$ then $g(z \vee w) < inf(g(z), g(w))$. We can find $p \in (0, 0.5)$ such that $g(z \vee w) < p \leq inf(g(z), g(w))$, which means that $z_p, w_p \in g$ but $(z \vee w)_p \overline{\in} g$. Also $g(z \vee w) + p < 0.5 + 0.5 = 1$ so $(z \vee w)_p \overline{q} g$. Thus, $(z \vee w)_p \overline{(\in \vee q)} g$, which is a contradiction.

Case:2 If $inf(g(z), g(w)) > 0.5$, then $g(z\vee w) < 0.5$. Now $z_{0.5}, w_{0.5} \in g$ but $(z\vee w)_{0.5} \overline{\in} g$ and $g(z\vee w)+0.5 < 1$, i.e., $(z\vee w)_p \overline{q} g$. Hence, $(z\vee w)_{inf(0.5,0.5)} \overline{(\in \vee q)} g$, a contradiction. Therefore $g(z \vee w) \geq inf(g(z), g(w), 0.5)$.

(6) $\longrightarrow$ (5) Let $z_p, w_v \in g$. Then $g(z \vee w) \geq inf(g(z), g(w), 0.5) \geq inf(p, v, 0.5)$. Consider the following two cases. Case:1 If $inf(p, v) \leq 0.5$, then $g(z \vee w) \geq inf(p, v)$. This shows that $(z \vee w)_{inf(p,v)} \in g$.

Case:2 If $inf(p, v) > 0.5$, then $g(z\vee w) \geq 0.5$. Hence, $g(z\vee w)+inf(p, v) > 0.5+0.5 = 1$, i.e., $(z \vee w)_{inf(p,v)} q g$. Thus $(z \vee w)_{inf(p,v)}(\in \vee q)g$. ■

Lemma 4.2.6 *The following conditions are equivalent, for any f-subset g of a quantale Q_t;*

$$z_p \in g\ ,\ w \in Q_t \longrightarrow (w\otimes z)_p(\in \vee q)g, \tag{7}$$

$$g(w\otimes z) \geq inf(g(z), 0.5) \text{ for all } z, w \in Q_t. \tag{8}$$

Proof. (7) $\longrightarrow$ (8) Let z, $w \in Q_t$ and $0.5 > g(z)$. Let $g(z) > g(z \otimes w)$. Then there is $p \in (0, 1]$ such that $g(z) > p > g(w \otimes z)$. This shows that $z_p \in g$ and $(w \otimes z)_p \overline{(\in \vee q)} g$. This is a contradiction against (7). So we have $g(w \otimes z) \geq g(z) = inf(g(z), 0.5)$. Now consider $g(z) \geq 0.5$. If $g(w \otimes z) < 0.5$, then $z_{0.5} \in g$ and $(w \otimes z)_{0.5} \overline{(\in \vee q)} g$ which is again a contradiction against (7). Hence $g(w \otimes z) \geq inf(g(z), 0.5)$.

(8) $\longrightarrow$ (7) Let $w \in Q_t$ and $z_p \in g$. Then $g(z) \geq p$. By supposition, $g(w \otimes z) \geq inf(g(z), 0.5) \geq inf(p, 0.5)$. Consider the following two cases.

Case:1 If $p \leq 0.5$, then $g(w \otimes z) \geq p$. Thus, $(w \otimes z)_p \in g$.

Case:2 If $p > 0.5$, then $g(w \otimes z) \geq 0.5$. Hence, $g(w \otimes z) + p > 0.5 + 0.5 = 1$, i.e., $(w \otimes z)_p q g$. Thus $(w \otimes z)_{inf(p,v)} (\in \vee q) g$. ■

Lemma 4.2.7 *The following two conditions are equivalent, for any f-subset g of a quantale Q_t;*

$$z_p \in g, w \in Q_t \longrightarrow (z \otimes w)_p (\in \vee q) g, \tag{9}$$

$$g(z \otimes w) \geq inf(g(z), 0.5) \text{ for all } z, w \in Q_t. \tag{10}$$

Proof. The Proof is similar to the proof of Lemma 4.2.6. ■

Lemma 4.2.8 *The following two conditions are equivalent for any f-subset g of a quantale Q_t;*

$$z_p \in g \text{ and } w \leq z \longrightarrow w_p (\in \vee q) g, \tag{11}$$

$$w \leq z, g(w) \geq inf(g(z), 0.5) \text{ for all } z, w \in Q_t. \tag{12}$$

Proof. (11) $\longrightarrow$ (12) Let w, $z \in Q_t$ and $w \leq z$. We consider two cases.

(a°) $0.5 > g(z)$,

(b°) $0.5 \leq g(z)$.

Consider the first case when $g(z) < 0.5$. Assume $g(w) < inf(g(z), 0.5)$. Then $g(w) < g(z)$. Take p such that $g(z) \geq p > g(w)$ and $g(w) + p < 1$. Then $z_p \in g$ but $w_p \overline{(\in \vee q)} g$ which is a contradiction. Hence $g(w) \geq inf(g(z), 0.5)$. For case (b°), let $w \leq z$ and $g(z) \geq 0.5$. If $g(w) < inf(g(z), 0.5) = 0.5$ and $g(w) + 0.5 < 1$, then $z_{0.5} \in g$ but $w_{0.5} \overline{(\in \vee q)} g$, we obtain a contradiction. Therefore $g(w) \geq inf(g(z), 0.5)$.

$(12) \longrightarrow (11)$ Let $w, z \in Q_t$ and $w \leq z$ be such that $z_p \in g$. Then $g(z) \geq p$ and by supposition, we have $g(w) \geq inf(g(z), 0.5) \geq inf(p, 0.5)$. This means that $g(w) \geq p$ or $g(w) \geq 0.5$, according to $p \leq 0.5$ or $p > 0.5$. Therefore $w_p(\in \vee q)g$. ■

Proposition 4.2.9 *A f-subset g of Q_t is an $(\in, \in \vee q)$-FRI (FLI) of Q_t if and only if the coditions below hold*

(1) $g(z \vee w) \geq inf(g(z), g(w), 0.5)$;

(2) $g(z \otimes w) \geq inf(g(z), 0.5)$, *[respectively* $g(w \otimes z) \geq inf(g(z), 0.5)$*]*;

(3) $w \leq z$, $g(w) \geq inf(g(z), 0.5)$, *for all* z, $w \in Q_t$.

Proof. Let g satisfy the conditions $(1), (2)$ and (3). Since, the conditions $(1), (2)$ and (3) are equivalent to the conditions $(6), (8)$ and (12), respectively $(4.2.5, 4.2.6, 4.2.7, 4.2.8)$. Thus, g is an $(\in, \in \vee q)$-FRI of Q_t.

Conversely, let g be an $(\in, \in \vee q)$-FRI of Q_t. Then g satisfies the the conditions $(6), (8)$ and (12), which are equivalent to the given conditions $(1), (2)$ and (3), respectively. ■

Theorem 4.2.10 *Let Q_t and Q'_t be two quantales and $\sigma_t : Q_t \longrightarrow Q'_t$ be a QH. Let g be an $(\in, \in \vee q)$-FRI (FLI) of Q'_t. Then $\sigma_t^{-1}(g)$ is an $(\in, \in \vee q)$-FRI (FLI) of Q_t.*

Proof. The proof is similar to the proof of Theorem 4.2.4(2). ■

Theorem 4.2.11 *Let $(Q_t, \otimes)$ be a quantale and $\{g_i\}_{i \in I}$ be a non-empty family of $(\in, \in \vee q)$-FRI (FLI) of Q_t. Then $\Cap_{i \in I} g_i$ is an $(\in, \in \vee q)$-FRI (FLI) of Q_t.*

Proof. Let $\{g_i\}_{i \in I}$ be a non-empty family of $(\in, \in \vee q)$-FRI of Q_t. Let $w, z \in Q_t$ be such that $w \leq z$. Then

$$\begin{aligned} (\Cap_{i \in I} g_i)(w) &= \inf_{i \in I} g_i(w) \\ &\geq \inf_{i \in I}[inf(g_i(z), 0.5)] \\ &= inf[\inf_{i \in I} g_i(z), 0.5] \\ &= inf[(\Cap_{i \in I} g_i)(z), 0.5] \end{aligned}$$

Thus, $\Cap_{i \in I} g_i(w) \geq inf[(\Cap_{i \in I} g_i)(z), 0.5]$.

Let $w, z \in Q_t$. Then

$$\begin{aligned}(\Cap_{i\in I} g_i)(w \vee z) &= \inf_{i\in I} g_i(w \vee z) \\ &\geq \inf_{i\in I}[\inf(g_i(w), g_i(z), 0.5)] \\ &= \inf[\inf_{i\in I} g_i(w), \inf_{i\in I} g_i(z)], 0.5] \\ &= \inf[\Cap_{i\in I} g_i(w), \Cap_{i\in I} g_i(z), 0.5]\end{aligned}$$

Hence $(\Cap_{i\in I} g_i)(w \vee z) \geq \inf[(\Cap_{i\in I} g_i)(w), (\Cap_{i\in I} g_i)(z), 0.5]$.

Also for $w, z \in Q_t$, we have,

$$\begin{aligned}(\Cap_{i\in I} g_i)(z \otimes w) &= \inf_{i\in I} g_i(z \otimes w) \\ &\geq \inf_{i\in I}[\inf(g_i(z), 0.5)] \\ &= \inf[\inf_{i\in I} g_i(z), 0.5] \\ &= \inf[\Cap_{i\in I} g_i(z), 0.5]\end{aligned}$$

Thus $(\Cap_{i\in I} g_i)(z \otimes w) \geq \inf[((\Cap_{i\in I} g_i)(z), 0.5]$.

Therefore $\Cap_{i\in I} g_i$ is an $(\in, \in \vee q)$-FRI of Q_t. ■

The following Proposition and Corollary are obvious.

Proposition 4.2.12 *Every $(\in \vee q, \in \vee q)$-FI of Q_t is an $(\in, \in \vee q)$-FI of Q_t.*

Corollary 4.2.13 *Every $(\in, \in)$-FI of Q_t is an $(\in, \in \vee q)$-FI of Q_t.*

The Example below shows that the converse of Proposition 4.2.12 and Corollary 4.2.13 are not true in general.

Example 4.2.14 *Consider the quantale Q_t as defined in Example 4.1.10 and taking $g = \frac{0.8}{\perp} + \frac{0.7}{i} + \frac{0.6}{j} + \frac{0.5}{\top}$. Then*

(1) *It is simple to confirm that g is an $(\in, \in \vee q)$-FI of Q_t.*

(2) *g is not an $(\in, \in)$-FI of Q_t, since $i_{0.68} \in g$ and $j_{0.59} \in g$ but $(i \vee j)_{inf(0.68,0.59)} = \top_{0.59} \overline{\in} g$.*

(3) *g is not an $(\in \vee q, \in \vee q)$-FI of Q_t, since $i_{0.68}(\in \vee q)g$ and $j_{0.59}(\in \vee q)g$ but $(i \vee j)_{inf(0.68,0.59)} = \top_{0.59}\overline{(\in \vee q)}g$.*

Definition 4.2.15 *Let C be a crisp subset of a quantale Q_t. We use K_C to denote the characteristic function of C, i.e., the mapping of a quantale Q_t into $[0, 1]$ defined* *by*

$$K_C(z) = \begin{cases} 1, & if \ z \ \in C, \\ 0, & if \ z \ \notin C. \end{cases}$$

The following results are about the characteristic function K_C of an ideal C of a quantale Q_t.

Lemma 4.2.16 *Let $\emptyset \neq C \subseteq Q_t$. Then K_C (the characteristic function) is an $(\in, \in)$-FI of Q_t if and only if C is an ideal of Q_t.*

Proof. Let C be an ideal of Q_t. Let $w, z \in Q_t$ and $p, v \in (0, 1]$ be such that $w_p \in K_C$ and $z_v \in K_C$. Then $K_C(w) \geq p > 0$ and $K_C(z) \geq v > 0$, which imply that $K_C(w) = K_C(z) = 1$. Thus $w, z \in C$ and C is an ideal so $w \vee z \in C$. It follows that $K_C(w \vee z) = 1 \geq inf(p, v)$ so that $(w \vee z)_{inf(p,v)} \in K_C$. Now let $b, z \in Q_t$ and $p \in (0,1]$ be such that $b_p \in K_C$. Then $K_C(b) \geq p > 0$, and so $K_C(b) = 1$, i.e., $b \in C$. Since C is an ideal of Q_t, we have $b \otimes z, z \otimes b \in C$ and hence $K_C(b \otimes z) = K_C(b \otimes z) = 1 \geq p$. Therefore $(b \otimes z)_p \in K_C$ and $(z \otimes b)_p \in K_C$. Let $w, z \in Q_t$, $z_p \in K_C$ and $w \leq z$. Then $K_C(z) \geq p > 0$, and so $K_C(z) = 1$, i.e., $z \in C$. Since C is an ideal, we have $w \in C$ and so $K_C(w) = 1 \geq p$. Therefore $w_p \in K_C$ and consequently K_C is an $(\in, \in)$-FI of Q_t.

Conversely, let K_C be an $(\in, \in)$-FI of Q_t and $w, z \in C$. Then $(w)_1 \in K_C$ and $(z)_1 \in K_C$ which show that $(w \vee z)_1 = (w \vee z)_{inf(1,1)} \in K_C$. Hence $K_C(w \vee z) > 0$, and so $w \vee z \in C$. Let $w, z \in Q_t$, $w \leq z$ and $z \in C$. Then $K_C(z) = 1$, and thus $(z)_1 \in K_C$. Since K_C is an $(\in, \in)$-FI, so we have $(w)_1 \in K_C$. Thus $K_C(w) = 1$. Hence $w \in C$. Now let $w \in Q_t$ and $z \in C$. Then $K_C(z) = 1$, and thus $(z)_1 \in K_C$. Since K_C is an $(\in, \in)$-FI, it follows that $(z \otimes w)_1 \in K_C$ so that $K_C(z \otimes w) = 1$. Hence $z \otimes w \in C$. Similarly, $w \otimes z \in C$ as C is an ideal of Q_t. ∎

Proposition 4.2.17 *Let $\emptyset \neq C \subseteq Q_t$. Then, C is an ideal of Q_t if and only if K_C is an $(\in, \in \vee q)$-FI of Q_t.*

Proof. Let C be an ideal of Q_t. Then K_C is an $(\in, \in)$-FI of Q_t by lemma 4.2.16, and therefore K_C is an $(\in, \in \vee q)$-FI of Q_t by Corollary 4.2.13.

Conversely, let K_C be an $(\in, \in \vee q)$-FI of Q_t. Let $w, z \in C$. Then $w_1 \in K_C$ and $z_1 \in K_C$ which show that $(w \vee z)_1 = (w \vee z)_{inf(1,1)} \ (\in \vee q) K_C$. Hence $K_C(w \vee z) > 0$, and so $w \vee z \in C$. Let $w, z \in Q_t$, $w \leq z$ and $z \in C$. Then $K_C(z) = 1$, and thus $z_1 \in K_C$. Since K_C is an $(\in, \in \vee q)$-FI, so we have $w_1 \in K_C$. Thus $K_C(w) = 1$. Hence $w \in C$. Now let $w \in Q_t$ and $z \in C$. Then $K_C(z) = 1$, and thus $z_1 \in K_C$. Since K_C is an $(\in, \in \vee q)$-FI, it follows that $(z \otimes w)_1 \in K_C$ so that $K_C(z \otimes w) = 1$. Hence $z \otimes w \in C$. Also, $w \otimes z \in C$ as C is an ideal of Q_t. ■

Proposition 4.2.18 *Let g be an $(\in, \in \vee q)$-FI of Q_t such that $g(w) < 0.5$ for all $w \in Q_t$. Then g is an $(\in, \in)$-FI of Q_t.*

Proof. Let g be an $(\in, \in \vee q)$-FI of Q_t such that $g(w) < 0.5$ for all $w \in Q_t$. Then by Proposition 4.2.9, we have

(1) $g(z \vee w) \geq inf(g(z), g(w), 0.5) = inf(g(z), g(w))$

(2) $g(z \otimes w) \geq inf(g(z), 0.5) = g(z)$ and $g(w \otimes z) \geq inf(g(z), 0.5) = g(z)$

(3) $w \leq z$, $g(w) \geq inf(g(z), 0.5) = g(z)$. Thus g is an $(\in, \in)$-FI of Q_t by Lemma 4.1.11. ■

Theorem 4.2.19 *Let Q_t be a quantale and g be a f-subset of Q_t. Then g is an $(\in, \in \vee q)$-FI of Q_t if and only if each non-empty $U(g; p)$ is an ideal of Q_t for all $p \in (0, 0.5]$.*

Proof. Consider g be an $(\in, \in \vee q)$-FI of Q_t and $p \in (0, 0.5]$. Let $w, z \in Q_t$ be such that $w \leq z$. If $z \in U(g; p)$ then $g(z) \geq p$. Since $g(w) \geq inf(g(z), 0.5) \geq inf(p, 0.5) = p$, we have $w \in U(g; p)$. Let $w, z \in Q_t$ be such that $w \in U(g; p)$. Then $g(w) \geq p$. Now since, $g(z \otimes w) \geq inf(g(w), 0.5) \geq inf(p, 0.5) = p$, so we have $z \otimes w \in U(g; p)$. Similarly, we can obtain $w \otimes z \in U(g; p)$. Let $w, y \in U(g; p)$. Then $g(w) \geq p$ and $g(y) \geq p$. Since g is an $(\in, \in \vee q)$-FI of Q_t, so we have $g(w \vee y) \geq inf(g(w), g(y), 0.5) \geq inf(p, 0.5) = p$. Thus $w \vee y \in U(g; p)$. Hence $U(g; p)$ is an ideal of Q_t.

Conversely, suppose $\emptyset \neq U(g; p)$ is an ideal of Q_t for all $p \in (0, 0.5]$. Let there exist $w, z \in Q_t$ such that $g(w \vee z) < inf(g(z), g(w), 0.5)$, then we can take p such that $g(w \vee z) < p < inf(g(z), g(w), 0.5)$. Thus $w, z \in U(g; p)$ and $p < 0.5$ and so $w \vee z \in U(g; p)$. This is a contradiction. Therefore $g(w \vee z) \geq inf(g(z), g(w), 0.5)$ for all $w, z \in Q_t$. Now if there exist $y, z \in Q_t$ such that $g(y \otimes z) < inf(g(z), 0.5)$, then we can choose $p \in (0, 0.5]$ such that $g(y \otimes z) < p < inf(g(z), 0.5)$. It concludes that

$z \in U(g;p)$ and $p < 0.5$ so that $y \otimes z \in U(g;p)$, similarly, we have $z \otimes y \in U(g;p)$, i.e., $g(y \otimes z) \geq p$ and $g(z \otimes y) \geq p$. This is a contradiction. Hence $g(y \otimes z) \geq inf(g(z), 0.5)$ and $g(z \otimes y) \geq inf(g(z), 0.5)$ for all $w, z \in Q_t$. Let $w, z \in Q_t$ and $w \leq z$. If $g(w) < inf(g(z), 0.5)$, we can find $p \in (0, 0.5]$ such that $g(w) < p < inf(g(z), 0.5)$. This implies that $z \in U(g;p)$ and $p < 0.5$. Since $U(g;p)$ is an ideal, so $w \in U(g;p)$. Hence $g(w) \geq p$. This gives a contradiction. So $g(w) \geq inf(g(z), 0.5)$ for all $w, z \in Q_t$. Using Proposition 4.2.9, g is an $(\in, \in \vee q)$-FI of Q_t. ■

4.3 $(\in, \in \vee q)$-Fuzzy Prime (semi prime) Ideals of Quantale

In this section, we define $(\in, \in \vee q)$-$f$uzzy prime and $(\in, \in \vee q)$-$f$uzzy semi prime ideals of a Quantale. It is also investigated that if a f-subset g is an $(\in, \in \vee q)$-$f$uzzy prime ($(\in, \in \vee q)$-$f$uzzy semi prime) ideal of Q'_t, then $\sigma_t^{-1}(g)$ is an $(\in, \in \vee q)$-$f$uzzy prime ($(\in, \in \vee q)$-$f$uzzy semi-prime) ideal of Q_t, where σ_t is a QH.

The following shortened forms $(\in, \in \vee q)$-FPI and $(\in, \in \vee q)$-$FSPI$ will be used for $(\in, \in \vee q)$-$f$uzzy prime ideals and $(\in, \in \vee q)$-$f$uzzy semi prime ideals, respectively.

Definition 4.3.1 *An (α, β)-FI, g of a quantale Q_t is called an (α, β)-FPI of Q_t if for all $p \in (0, 1]$ and $z, w \in Q_t$, $(z \otimes w)_p \alpha g \longrightarrow (z)_p \beta g$ or $(w)_p \beta g$. An (α, β)-FI, g of a quantale Q_t is called an (α, β)-$FSPI$ of Q_t if for all $z \in Q_t$ and $p \in (0, 1]$, $(z \otimes z)_p \alpha g \longrightarrow (z)_p \beta g$.*

Proposition 4.3.2 *A f-subset g of a quantale Q_t is a FPI if and only if g is an $(\in, \in)$-FPI.*

Proof. Let g be a FPI. Then $g(w \otimes z) = g(w)$ or $g(w \otimes z) = g(z)$ for all z, $w \in Q_t$. Let $(w \otimes z)_p \in g$ for some $p \in (0, 1]$. Then $g(w \otimes z) \geq p$. Thus $g(w) = g(w \otimes z) \geq p$ or $g(z) = g(w \otimes z) \geq p$. This implies that $w_p \in g$ or $z_p \in g$. Therefore g is an $(\in, \in)$-FPI.

Conversely, let g be an $(\in, \in)$-FPI. Let z, $w \in Q_t$ and $g(w \otimes z) = v$ for some $v \in (0, 1]$. Then $g(w \otimes z) \geq v$. This shows that $(z \otimes w)_v \in g$. This gives $w_v \in g$ or $z_v \in g$. So $g(w) \geq v$ or $g(z) \geq v$, i.e., $g(w) \geq g(w \otimes z)$ or $g(z) \geq g(w \otimes z)$ Thus we have, $sup(g(w), g(z)) \geq g(w \otimes z)$. But since g is an $(\in, \in)$-FPI, therefore g is a FPI by Proposition 1.4.12. ■

Theorem 4.3.3 *A f-subset g is a (q, q)-FPI of a quantale Q_t if and only if g is an $(\in, \in)$-FPI of Q_t.*

Proof. Let g be a (q, q)-FPI of the quantale Q_t. Let $p \in (0, 1]$ and $z, y \in Q_t$ be such that $(y \otimes z)_p \in g$. Then $g(y \otimes z) \geq p$. This implies that $g(y \otimes z) + \epsilon > p$, for some $\epsilon > 0 \longrightarrow g(y \otimes z) + \epsilon - p + 1 > 1 \longrightarrow (y \otimes z)_{(\epsilon - p+1)} qg$. Since g is a (q, q)-FPI, so $(y)_{(\epsilon-p+1)} qg$ or $(z)_{(\epsilon-p+1)} qg$. This implies that $g(y) + \epsilon - p + 1 > 1$ or $g(z)+\epsilon-p+1 > 1 \longrightarrow g(y)+\epsilon > p$ or $g(z)+\epsilon > p \longrightarrow g(y) \geq p$ or $g(z) \geq p \longrightarrow y_p \in g$ or $z_p \in g$. Hence $(y \otimes z)_p \in g \longrightarrow y_p \in g$ or $z_p \in g$. Thus g is an $(\in, \in)$-FPI of Q_t.

Conversely, assume that $(y \otimes z)_p qg \longrightarrow g(y \otimes z) + p > 1 \longrightarrow g(y \otimes z) > 1 - p \longrightarrow g(y \otimes z) \geq \epsilon - p + 1 > 1 - p$ for some $\epsilon > 0 \longrightarrow (y \otimes z)_{(\epsilon-p+1)} \in g$. Since g is an $(\in, \in)$-FPI of Q_t. Therefore, we have $y_{\epsilon-p+1} \in g$ or $z_{\epsilon-p+1} \in g$. Thus we have $g(y) \geq \epsilon - p + 1 > 1 - p$ or $g(z) \geq \epsilon - p + 1 > 1 - p \longrightarrow g(y) > 1 - p$ or $g(z) > 1 - p \longrightarrow g(y) + p > 1$ or $g(z) + p > 1 \longrightarrow y_p qg$ or $z_p qg$. Thus $(y \otimes z)_p qg \longrightarrow y_p qg$ or $z_p qg$. Hence g is a (q, q)-FPI of the quantale Q_t. ■

Proposition 4.3.4 *An $(\in, \in \vee q)$-FI, g of a quantale Q_t is an $(\in, \in \vee q)$-FPI if and only if $sup(g(z), g(w)) \geq inf(g(z \otimes w), 0.5)$ for all $w, z \in Q_t$.*

Proof. We want to show that $sup(g(z), g(w)) \geq inf(g(z \otimes w), 0.5)$ for all $w, z \in Q_t$. Let there exist $y, z \in Q_t$ such that $sup(g(z), g(y)) < inf(g(y \otimes z), 0.5)$. Then there exist v such that $sup(g(z), g(y)) < v < inf(g(y \otimes z), 0.5)$ for $v \in (0, 0.5]$. This means that $g(y \otimes z) > v \longrightarrow (y \otimes z)_v \in g$. But $g(y) < v$ and $g(z) < v$, i.e., $y_v \overline{\in} g$ and $z_v \overline{\in} g$. Also we have $g(y) + v < 2v < 2 \times 0.5 = 1 \longrightarrow y_v \overline{(\in \vee q)} g$, $z_v \overline{(\in \vee q)} g$. This gives a contradiction. Hence we have $sup(g(z), g(w)) \geq inf(g(z \otimes w), 0.5)$ for all $w, z \in Q_t$.

Conversely, suppose that the condition $sup(g(z), g(y)) \geq inf(g(z \otimes y), 0.5)$ holds for all $y, z \in Q_t$. Let $w, z \in Q_t$ be such that $(w \otimes z)_v \in g$, where $v \in (0, 1]$. Then $g(w \otimes z) \geq v$. Thus by supposition we have $sup(g(z), g(y)) \geq inf(g(z \otimes y), 0.5) \geq inf(v, 0.5)$. Now $sup(g(z), g(y)) \geq v$ if we suppose $v \leq 0.5$. Hence $g(z) \geq v$ or $g(y) \geq v$. This implies $y_v \in g$ or $z_v \in g$. If we suppose $v > 0.5$, then $sup(g(z), g(y)) \geq 0.5$. Thus $g(z) \geq 0.5$ or $g(y) \geq 0.5 \longrightarrow g(y) + v \geq 0.5 + v > 0.5 + 0.5 = 1$ or $g(z) + v \geq 0.5 + v > 0.5 + 0.5 = 1 \longrightarrow y_v qg$ or $z_v qg$. By combining the above two cases, we have $y_v(\in \vee q)g$ or $z_v(\in \vee q)g$. Hence $(w \otimes z)_v \in g \longrightarrow y_v(\in \vee q)g$ or $z_v(\in \vee q)g$. Therefore g is an $(\in, \in \vee q)$-FPI of Q_t. ■

The following Proposition gives a criteria for an $(\in, \in \vee q)$-FPI to be an $(\in, \in)$-FPI.

Proposition 4.3.5 *If a f-subset g of a quantale Q_t is an $(\in, \in \vee q)$-FPI of Q_t and $g(z) < 0.5$ for all $z \in Q_t$, then g is also an $(\in, \in)$-FPI of Q_t.*

Proof. Suppose g is an $(\in, \in \vee q)$-FPI of Q_t and $g(z) < 0.5$ for all $z \in Q_t$. Let $(x \otimes z)_v \in g$. Then $g(x \otimes z) \geq v$. Since $\otimes$ is a binary operation on Q_t so $x \otimes z \in Q_t$, hence we have $v \leq g(x \otimes z) < 0.5$, i.e., $v < 0.5$ and $g(x) < 0.5$, $g(z) < 0.5$. Also $g(z) + v < 0.5 + 0.5 = 1$ and $g(x) + v < 0.5 + 0.5 = 1$. This gives $x_v \overline{q} g$ and $z_v \overline{q} g$. So we have $x_v \in g$ or $z_v \in g$ as g is an $(\in, \in \vee q)$-FPI. Thus g is an $(\in, \in)$-FPI of Q_t. ■

Theorem 4.3.6 *An $(\in, \in \vee q)$-FI, g of a quantale Q_t is an $(\in, \in \vee q)$-FPI if and only if for all $0 < p \leq 0.5$, each non-empty $U(g; p)$ is a PI of Q_t.*

Proof. Let g be an $(\in, \in \vee q)$-FPI. Then g is an $(\in, \in \vee q)$-FI. Each $\emptyset \neq U(g; p)$ is an ideal of Q_t, by Theorem 4.2.19. Let $y \otimes z \in U(g; p)$. Then $g(y \otimes z) \geq p$. Now, by Proposition 4.3.4, we have $sup(g(y), g(z)) \geq inf(g(y \otimes z), 0.5) \geq inf(p, 0.5) = p$. So, $g(y) \geq p$ or $g(z) \geq p$. Thus $y \in U(g; p)$ or $z \in U(g; p)$. Hence $U(g; p)$ is a PI of Q_t.

Conversely, suppose that $U(g; p)$ is a PI of Q_t for all $p \in (0, 0.5]$ and assume that the condition $sup(g(z), g(w)) \geq inf(g(z \otimes w), 0.5)$ is not valid. Then there exist some $a, c \in Q_t$ such that $sup(g(a), g(c)) < inf(g(a \otimes c), 0.5)$ and we take $p \in (0, 0.5)$ such that $sup(g(a), g(c)) < p < inf(g(a \otimes c), 0.5)$. This implies that $a \otimes c \in U(g; p)$ but $a, c \notin U(g; p)$. This contradicts our supposition. Hence we must have $sup(g(a), g(c)) \geq inf(g(a \otimes c), 0.5)$. Consequently, g is an $(\in, \in \vee q)$-FPI of Q_t by Proposition 4.3.4.
■

Theorem 4.3.7 *Let $\emptyset \neq A \subseteq Q_t$ be a PI if and only if the f-subset g of Q_t defined by $g(z) = p \geq 0.5$ for $z \in A$ and $g(z) = 0$ otherwise is an $(\in, \in \vee q)$-FPI of Q_t.*

Proof. Proof is similar to the proof of Theorem 4.1.14. ■

The proof of following Proposition is similar to the proof of Proposition 4.2.17.

Theorem 4.3.8 *Let $\emptyset \neq A \subseteq Q_t$. Then K_A (the characteristic function) is an $(\in, \in \vee q)$-FPI of Q_t if and only if A is a PI of Q_t.*

Theorem 4.3.9 *Let $(Q_t, \otimes)$ and $(Q'_t, \otimes')$ be two quantales and $\sigma_t : Q_t \longrightarrow Q'_t$ be a QH. Let g be an $(\in, \in \vee q)$-FPI of Q'_t. Then $\sigma_t^{-1}(g)$ is an $(\in, \in \vee q)$-FPI of Q_t.*

Proof. Let g be an $(\in, \in \vee q)$-FPI of Q'_t. Then $\sigma_t^{-1}(g)$ is an $(\in, \in \vee q)$-FI of Q_t by Theorem 4.2.10. Let $x, z \in Q_t$ be such that $(x \otimes z)_p \in \sigma_t^{-1}(g)$. Then $\sigma_t^{-1}(g)(x \otimes z) \geq p \longrightarrow g(\sigma_t(x \otimes z)) \geq p \longrightarrow (\sigma_t(x \otimes z))_p \in g$. Since σ_t is a QH, we have $(\sigma_t(x) \otimes' \sigma_t(z))_p \in g$. As g is an $(\in, \in \vee q)$-FPI of Q'_t, so $(\sigma_t(x))_p(\in \vee q)g$ or $(\sigma_t(z))_p(\in \vee q)g \longrightarrow g(\sigma_t(z)) \geq p$ or $g(\sigma_t(z))+p > 1$ or $g(\sigma_t(x)) \geq p$ or $g(\sigma_t(x))+p > 1 \longrightarrow \sigma_t^{-1}(g)(x) \geq p$ or $\sigma_t^{-1}(g)(x) + p > 1$ or $\sigma_t^{-1}(g)(z) \geq p$ or $\sigma_t^{-1}(g)(z) + p > 1 \longrightarrow x_p \in \sigma_t^{-1}(g)$ or $x_p q \sigma^{-1}(g)$ or $z_p \in \sigma_t^{-1}(g)$ or $z_p q \sigma^{-1}(g) \longrightarrow x_p(\in \vee q)\sigma_t^{-1}(g)$ or $z_p(\in \vee q)\sigma_t^{-1}(g)$. Thus $(x \otimes z)_p \in \sigma_t^{-1}(g) \longrightarrow x_p(\in \vee q)\sigma_t^{-1}(g)$ or $z_p(\in \vee q)\sigma_t^{-1}(g)$. Thus, $\sigma_t^{-1}(g)$ is an $(\in, \in \vee q)$-FPI of Q_t. ■

The proof of following Propositions are similar to the proof of Proposition 4.3.2, Theorem 4.3.3, Proposition 4.3.4 and Theorem 4.3.6, respectively.

Proposition 4.3.10 *A f-subset g of Q_t is a $FSPI$ if and only if g is an $(\in, \in)$-$FSPI$.*

Proposition 4.3.11 *A f-subset g is a (q, q)-$FSPI$ of a quantale Q_t if and only if g is an $(\in, \in)$-$FSPI$ of Q_t.*

Proposition 4.3.12 *An $(\in, \in \vee q)$-FI, g of Q_t is an $(\in, \in \vee q)$-$FSPI$ if and only if $g(z) \geq inf(g(z \otimes z), 0.5)$ for all $z \in Q_t$.*

Proposition 4.3.13 *An $(\in, \in \vee q)$-FI, g of Q_t is an $(\in, \in \vee q)$-$FSPI$ if and only if for all $0 < p \leq 0.5$, each non-empty $U(g; p)$ is a SPI of Q_t.*

4.4 (α, β)-Fuzzy Q_t-submodule of Q_t-module

Now properties of (α, β)-fuzzy Q_t-submodule of Q_t-modules are introduced in this section.

Definition 4.4.1 *[60, 78] Let M and M' be two Q_t-modules. A map $\rho_m : M \longrightarrow M'$ is a Q_t-module* ***homomorphism*** *if it is a sup-lattice* ***homomorphism*** *which also preserves scalar multiplication, i.e.*

$$\begin{aligned} \rho_m(\vee_{i \in I} m_i) &= \vee_{i \in I} \rho_m(m_i); \\ \rho_m(a * m) &= a * \rho_m(m) \end{aligned}$$

for any $a \in Q_t, m \in M$, $\{m_i\} \subseteq M, (i \in I)$.

A Q_t*-module homomorphism* $\rho_m : M \longrightarrow M'$ *is called an* ***epimorphism*** *if* ρ_m *is onto* M' *and* ρ_m *is called a* ***monomorphism*** *if* ρ_m *is one-one. It is an* ***isomorphism****, if* ρ_m *is bijective.*

Definition 4.4.2 *Let* M *be a* Q_t*-module and* g *be a f-subset of* M*. We say that* g *is a fuzzy* Q_t*-submodule of* M *if*

(1) $g(\vee_{i\in I} m_i) \geq \underset{i\in I}{inf}\ g(m_i)$,

(2) $g(a * m) \geq g(m)$ *for all* m_i, $m \in M$ *and* $a \in Q_t$*(quantale).*

Definition 4.4.3 *A f-subset* g *of a* Q_t*-module* M *is called an* (α, β)*-fuzzy* Q_t*-submodule of* M*, if*

(1) $(m_i)_{p_i} \alpha g \longrightarrow (\vee_{i\,\in\, I} m_i)_{\underset{i\in I}{inf} p_i} \beta g$,

(2) $m_p \alpha g$*, and* $a \in Q_t \longrightarrow (a * m)_p \beta g$ *for all* p_i, $p \in (0, 1]$, m_i, $m \in M$ *and* $a \in Q_t$.

Lemma 4.4.4 *A f-subset* g *of a* Q_t*-module* M *is a fuzzy* Q_t*-submodule of* M *if and only if it satisfies*

(1) $(m_i)_{p_i} \in g \longrightarrow (\vee_{i\,\in\, I} m_i)_{\underset{i\in I}{inf} p_i} \in g$,

(2) $m_p \in g, a \in Q_t \longrightarrow (a * m)_p \in g$ *for all* $p_i, p \in (0, 1]$, $m_i, m \in M$ *and* $a \in Q_t$.

Proof. Let g be a fuzzy Q_t-submodule of a Q_t-module M. Let $m_i \in M$ and $p_i \in (0, 1]$ be such that $(m_i)_{p_i} \in g$ for $i \in I$. Then $g(m_i) \geq p_i$, for all $i \in I$. Since g is a fuzzy Q_t-submodule of M, so $g(\vee_{i\in I} m_i) \geq \underset{i\in I}{inf}\ g(m_i) \geq \underset{i\in I}{inf}\ p_i$. Hence $(\vee_{i\,\in\, I} m_i)_{\underset{i\in I}{inf} p_i} \in g$. Let $a \in Q_t$, $m \in M$ and $p \in (0, 1]$ be such that $m_p \in g$. Then $g(m) \geq p$. But g is a fuzzy Q_t-submodule of M, hence we have $g(a * m) \geq g(m) \geq p$. Thus $g(a * m) \geq p$. This implies that $(a * w)_p \in g$.

Conversely, suppose that g satisfies the conditions (1) and (2). First we show that $g(\vee_{i\in I} m_i) \geq \underset{i\in I}{inf}\ g(m_i)$ for $i \in I$. On contrary suppose that $g(\vee_{i\in I} m_i) < \underset{i\in I}{inf}\ g(m_i)$ for some $m_i \in M$. Let $p \in (0, 1]$ be such that $g(\vee_{i\in I} m_i) < p < \underset{i\in I}{inf}\ g(m_i)$. Then $(m_i)_p \in g$ but $(\vee_{i\,\in\, I} m_i)_p \overline{\in} g$. This contradicts our hypothesis. Thus $g(\vee_{i\in I} m_i) \geq \underset{i\in I}{inf}$ $g(m_i)$ for all $m_i \in M$. Now we show that $g(a * m) \geq g(m)$ for all $m \in M$ and $a \in Q_t$. Let $g(a * m) < g(m)$. Then there exist $v \in (0, 1]$ such that $g(a * m) < v < g(m)$. Thus

$m_v \in g$ and $(a * m)_v \overline{\in} g$, a contradiction. Hence $g(a * m) \geq g(m)$ for all $m \in M$ and $a \in Q_t$. This concludes that g is a fuzzy Q_t-submodule of M. ■

Remark 4.4.5 *It is concluded from the above Lemma that every fuzzy Q_t-submodule of M is an $(\in, \in)$-fuzzy Q_t-submodule of M.*

Theorem 4.4.6 *Let g be a nonzero (α, β)-fuzzy Q_t-submodule of M. Then the set $g_\circ = \{y \in Q_t \mid g(y) > 0\}$ is a Q_t-submodule of M.*

Proof. Let $m_i \in g_\circ$ for $i \in I$. Then $g(m_i) > 0$ for all $i \in I$. Let $g(\vee_{i \in I} m_i) = 0$. If $\alpha \in \{\in, \in \vee q\}$, then $(m_i)_{g(m_i)} \alpha g$ for all $i \in I$ but $g(\vee_{i \in I} m_i) = 0 < \underset{i \in I}{inf}\ g(m_i)$ and $g(\vee_{i \in I} m_i) + \underset{i \in I}{inf}\ g(m_i) \leq 0 + 1 = 1$. So $(\vee_{i \in I} m_i)_{\underset{i \in I}{inf} g(m_i)} \overline{\beta} g$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Hence $g(\vee_{i \in I} m_i) > 0$, that is $\vee_{i \in I} m_i \in g_\circ$. Also $(m_i)_1 q g$ for all $i \in I$ but $(\vee_{i \in I} m_i)_1 \overline{\beta} g$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$. Hence $g(\vee_{i \in I} m_i) > 0$, that is $\vee_{i \in I} m_i \in g_\circ$. Let $m \in g_\circ$ and for all $q \in Q_t$. Then $g(m) > 0$. We want to show that $g(q * m) > 0$ for all $q \in Q_t$. Suppose that $g(q * m) = 0$ and let $\alpha \in \{\in, \in \vee q\}$. Then $(m)_{g(m)} \alpha g$ but $(q * m)_{g(m)} \overline{\beta} g$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, this is a contradiction. Also $(m)_1 q g$ but $(q * m)_1 \overline{\beta} g$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Therefore $g(q * m) > 0$ and so $q * m \in g_\circ$. Hence $g_\circ$ is a Q_t-submodule of M. ■

Theorem 4.4.7 *Let A be a Q_t-submodule of M. Then a f-subset g of Q_t such that $g(c) \geq 0.5$ for $c \in A$ and $g(c) = 0$ otherwise, is an $(\alpha, \in \vee q)$-fuzzy Q_t-submodule of M.*

Proof. Let A be a Q_t-submodule of M.

(**a**) Let $m_i \in M$ and $v_i \in (0, 1]$ for $i \in I$ be such that $(m_i)_{v_i} \in g$. Then $g(m_i) \geq v_i$ for all $i \in I$. Thus $m_i \in A$ and so $\vee_{i \in I} m_i \in A$ because A is a Q_t-submodule of M, that is $g(\vee_{i \in I} m_i) \geq 0.5$. If $\underset{i \in I}{inf}(v_i) \leq 0.5$, then $g(\vee_{i \in I} m_i) \geq 0.5 \geq \underset{i \in I}{inf}(v_i)$. Hence $(\vee_{i \in I} m_i)_{\underset{i \in I}{inf}(v_i)} \in g$. If $\underset{i \in I}{inf}(v_i) > 0.5$, then $g(\vee_{i \in I} m_i) + \underset{i \in I}{inf}(v_i) > 0.5 + 0.5 = 1$ and so $(\vee_{i \in I} m_i)_{\underset{i \in I}{inf}(v_i)} q g$. Therefore $(\vee_{i \in I} m_i)_{\underset{i \in I}{inf}(v_i)} (\in \vee q) g$.

Now let $m \in M$ and $p \in (0, 1]$ be such that $m_p \in g$. Then $g(m) \geq p$, which implies $m \in A$, and so $q * m \in A$ for all $q \in Q_t$ because A is a Q_t-submodule of M. Consequently $g(q * m) \geq 0.5$. If $p \leq 0.5$, then $g(q * m) \geq 0.5 \geq p$. Hence $(q * m)_p \in g$. If $p > 0.5$, then $g(q*m)+p > 0.5+0.5 = 1$ and so $(q*m)_p qg$. Thus $(q*m)_p(\in \vee q)g$. Hence g is an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M.

(**b**) Let $m_i \in M$ and $p_i \in (0,1]$ be such that $(m_i)_{p_i} qg$. Then $g(m_i)+p_i > 1$ and $m_i \in A$. Since A is a Q_t-submodule of M so $\vee_{i\in I} m_i \in A$, we have $g(\vee_{i\in I} m_i) \geq 0.5$. If $\underset{i\in I}{inf}(p_i) \leq 0.5$, then $g(\vee_{i\in I} m_i) \geq 0.5 \geq \underset{i\in I}{inf}(p_i)$. Hence $(\vee_{i\in I} m_i)_{\underset{i\in I}{inf}(p_i)} \in g$. If $\underset{i\in I}{inf}(p_i) > 0.5$, then $g(\vee_{i\in I} m_i) + \underset{i\in I}{inf}(p_i) > 0.5+0.5 = 1$ and so $(\vee_{i\in I} m_i)_{\underset{i\in I}{inf}(p_i)} qg$. Therefore $(\vee_{i\in I} m_i)_{\underset{i\in I}{inf}(p_i)}(\in \vee q)g$.

Let $m \in M$ and $v \in (0,1]$ be such that $m_v qg$. Then, $g(m)+v > 1$. Thus $m \in A$ and so $q*m$ is in A for all $q \in Q_t$. This means that $g(q*m) \geq 0.5$. If $v \leq 0.5$, then $g(q*m) \geq 0.5 \geq v$. Hence $(q*m)_v \in g$. If $v > 0.5$, then $g(q*m)+v > 0.5+0.5 = 1$ and so $(q*m)_v qg$. Thus $(q*m)_v(\in \vee q)g$. Hence g is an $(q, \in \vee q)$-fuzzy Q_t-submodule of M.

(**c**) Let $m_i \in M$ and $p_i \in (0,1]$ be such that $(m_i)_{p_i} \in g$ or $(m_i)_{p_i} qg$. Then $g(m_i) \geq p_i$ and $g(m_i)+p_i > 1$. Since $m_i \in A$, we have that $\vee_{i\in I} m_i \in A$. Hence $g(\vee_{i\in I} m_i) \geq 0.5$. Thus, $(\vee_{i\in I} m_i)_{inf(p_i)} \in g$ for $inf(p_i) \leq 0.5$ and $(\vee_{i\in I} m_i)_{inf(p_i)} qg$ for $inf(p_i) > 0.5$. Thus $(\vee_{i\in I} m_i)_{inf(p_i)}(\in \vee q)g$. The rest is similar to the proof of parts (a) and (b). ■

Proposition 4.4.8 *Let g be a f-subset of a Q_t-module M and $\rho_m : M \longrightarrow M'$ be a Q_t-module homomorphism. Then $(\rho_m(m))_p \alpha g$ if and only if $m_p \alpha \rho_m^{-1}(g)$ for all $m \in M$ and $p \in (0,1]$.*

Proof. Let $\alpha = \in$. Then $(\rho_m(m))_p \in g \iff g(\rho_m(m)) \geq p \iff \rho_m^{-1}(g)(m) \geq p \iff m_p \in \rho_m^{-1}(g)$. Let $\alpha = q$. Then $(\rho_m(m))_p qg \iff g(\rho_m(m)) + p > 1 \iff \rho_m^{-1}(g)(m) + p > 1 \iff m_p q \rho_m^{-1}(g)$. Similarly, we can show the other cases. ■

Theorem 4.4.9 *Let $(M, *)$ and $(M', *')$ be Q_t-modules and $\rho_m : M \longrightarrow M'$ be a Q_t-module homomorphism. Let g be an (α, β)-fuzzy Q_t-submodule of M'. Then $\rho_m^{-1}(g)$ is an (α, β)-fuzzy Q_t-submodule of M.*

Proof. Let $m_i \in M$ and $p_i \in (0,1]$ for $i \in I$ be such that $(m_i)_{p_i} \alpha \rho_m^{-1}(g)$. Then $(\rho_m(m_i))_{p_i} \alpha g$ for all $i \in I$, by Proposition 4.4.8. Since g is an (α, β)-fuzzy Q_t-submodule of M', we have $(\vee_{i\in I} \rho_m(m_i))_{\underset{i\in I}{inf}(p_i)} \beta g$ and so $(\rho_m(\vee_{i\in I} m_i))_{\underset{i\in I}{inf}(p_i)} \beta g$ by using Q_t-module homomorphism. Thus, $(\vee_{i\in I} m_i)_{\underset{i\in I}{inf}(p_i)} \beta \rho_m^{-1} g$ by Proposition 4.4.8. Let $x_p \alpha \rho_m^{-1} g$ and for all $q \in Q_t$. Then $(\rho_m(x))_p \alpha g$. Hence, for all $q \in Q_t$, $(q *'$

$\rho_m(x))_p\beta g \longrightarrow (\rho_m(q*x))_p\beta g$ as g is an (α,β)-fuzzy Q_t-submodule of M' and ρ_m is a Q_t-module homomorphism. Again by Proposition 4.4.8, we have $(q*x)_p\beta\rho_m^{-1}(g)$. Hence $\rho_m^{-1}(g)$ is an (α,β)-fuzzy Q_t-submodule of M. ■

4.5 $(\in, \in \vee q)$-Fuzzy Q_t-submodule of Q_t-Module

In this section, we will present some results about $(\in, \in \vee q)$-fuzzy Q_t-submodules.

Lemma 4.5.1 *For a f-subset g of a Q_t-module M, the following two conditions are equivalent:*

$$(m_i)_{p_i} \in g \longrightarrow (\vee_{i\,\in\,I} m_i)_{\underset{i\in I}{inf} p_i}(\in \vee q)g, \tag{1}$$

$$g(\vee_{i\in I} m_i) \geq inf(\underset{i\in I}{inf}\, g(m_i), 0.5). \tag{2}$$

Proof. Proof is similar to the proof of Lemma 4.2.1. ■

Lemma 4.5.2 *The following conditions are equivalent, for any f-subset g of a Q_t-module M;*

$$m_p \in g\ ,\ q \in Q_t \longrightarrow (q{*}m)_p(\in \vee q)g, \tag{3}$$

$$g(q{*}m) \geq inf(g(m), 0.5) \text{ for all } m \in M, \text{ and } q \in Q_t. \tag{4}$$

Proof. The Proof is similar to the proof of Lemma 4.2.6. ■

Proposition 4.5.3 *A f-subset g of M is an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M if and only if it satisfies* (2) *and* (4).

Theorem 4.5.4 *Let M and M' be two Q_t-modules and $\rho_m : M \longrightarrow M'$ be a Q_t-module homomorphism. Let g_1 and g_2 be $(\in, \in \vee q)$-fuzzy Q_t-submodule sof M and M', respectively. Then*

(1) *$\rho_m(g_1)$ is an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M',*

(2) *$\rho_m^{-1}(g_2)$ is an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M.*

Proof. (1) For any $m_i, m \in M'$ and $q \in Q_t$, if $\rho_m^{-1}(m_i) = \emptyset$ for some $i \in I$, then $inf[\underset{i\in I}{inf}\, \rho_m(g_1)(m_i), 0.5] = 0 \leq \rho_m(g_1)(\vee_{i\in I} m_i)$ and if $\rho_m^{-1}(m) = \emptyset$, then $inf(\rho_m(g_1)(m), 0.5) = 0 \leq \rho_m(g_1)(q{*}m)$. Now suppose that $\rho_m^{-1}(m_i) \neq \emptyset$ for each $i \in I$ and $\rho_m^{-1}(\vee_{i\in I} m_i) \neq \emptyset$.

Thus,

$$\begin{aligned}
inf[\underset{i\in I}{inf}(\rho_m(g_1)(m_i)), 0.5] &= inf[inf[\rho_m(g_1)(m_1), \rho_m(g_1)(m_2), ..., \rho_m(g_1)(m_i)], 0.5] \\
&= inf[inf[\underset{a_1 \in \rho_m^{-1}(m_1)}{Sup} g_1(a_1), ..., \underset{a_i \in \rho_m^{-1}(m_i)}{Sup} g_1(a_i)], 0.5] \\
&= \underset{a_1 \in \rho_m^{-1}(m_1),..., a_i \in \rho_m^{-1}(m_i)}{Sup} inf[inf(g_1(a_1), ..., g_1(a_i)), 0.5] \\
&= \underset{\rho_m(a_1) = m_1,..., \rho_m(a_i) = m_i}{Sup} inf[inf(g_1(a_1), ..., g_1(a_i)), 0.5] \\
&= \underset{\vee_{i\in I}\rho_m(a_i) = \vee_{i\in I}m_i}{Sup} inf[\underset{i\in I}{inf} g_1(a_i), 0.5] \\
&= \underset{\rho_m(\vee_{i\in I}a_i) = \vee_{i\in I}m_i}{Sup} inf[\underset{i\in I}{inf} g_1(a_i), 0.5], \ \rho_m \text{ is a } Q_tMH \\
&\leq \underset{\vee_{i\in I}a_i \in \rho_m^{-1}(\vee_{i\in I}m_i)}{Sup} g_1(\vee_{i\in I}a_i) \\
&= \underset{y \in \rho_m^{-1}(\vee_{i\in I}m_i)}{Sup} g_1(y) \\
&= \rho_m(g_1)(\vee_{i\in I}m_i)
\end{aligned}$$

Hence, $\rho_m(g_1)(\vee_{i\in I}m_i) \geq inf[\underset{i\in I}{inf}\ \rho_m(g_1)(m_i), 0.5]$ for all $m_i \in M'$.

and

$$\begin{aligned}
inf[\rho_m(g_1)(z), 0.5] &= inf[\underset{a \in \rho_m^{-1}(z)}{Sup} g_1(a), 0.5] \\
&= \underset{a\in \rho_m^{-1}(z)}{Sup} inf[g_1(a), 0.5] \\
&= \underset{\rho_m(a)=z}{Sup} inf[g_1(a), 0.5] \\
&= \underset{q*'\rho_m(a)=q*'z}{Sup} inf[g_1(a), 0.5] \\
&= \underset{\rho_m(q*a)=q*'z}{Sup} inf[g_1(a), 0.5], \ \rho_m \text{ is a } Q_tMH \\
&\leq \underset{q*a \in \rho_m^{-1}(q*'z)}{Sup} g_1(q*a) \\
&= \underset{y \in \rho_m^{-1}(q*'z)}{Sup} g_1(y) \\
&= \rho_m(g_1)(q*' z)
\end{aligned}$$

So, $\rho_m(g_1)(q*' z) \geq inf[\rho_m(g_1)(z), 0.5]$ for all $z \in M'$ and $q \in Q_t$. Thus, we have $\rho_m(g_1)$ is an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M'.

(2) Proof is similar to the proof of Theorem 4.4.9. ■

Corollary 4.5.5 *Every $(\in \vee q, \in \vee q)$-fuzzy Q_t-submodule of M is an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M.*

Proof. Obvious. ■

Corollary 4.5.6 *Every* $(\in, \in)$*-fuzzy* Q_t*-submodule of* M *is an* $(\in, \in \vee q)$*-fuzzy* Q_t*-submodule of* M*.*

Proof. Straightforward. ■

Definition 4.5.7 *Let* C *be a crisp subset of a* Q_t*-module* M*. We use* K_C *to denote the characteristic function of* C*, i.e., the mapping from* M *into* $[0, 1]$ *defined by*

$$K_C(z) = \begin{cases} 1, & if\ z\ \in C, \\ 0, & if\ z\ \notin C. \end{cases}$$

The following results are about the characteristic function K_C of a Q_t-submodule C of a Q_t-module M.

Lemma 4.5.8 *Let* $\emptyset \neq C \subseteq Q_t$*. Then* K_C *(the characteristic function) is an* $(\in, \in)$*-fuzzy* Q_t*-submodule of* M *if and only if* C *is a* Q_t*-submodule of* M*.*

Proof. Let C be a Q_t-submodule of M. Let $w_i \in M$ and $p_i \in (0, 1]$ be such that $(w_i)_{p_i} \in K_C$. Then $K_C(w_i) \geq p_i > 0$, which imply that $K_C(w_i) = 1$. Thus $w_i \in C$ and C is a Q_t-submodule of M so $\vee_{i\in I} w_i \in C$. It follows that $K_C(\vee_{i\in I} w_i) = 1 \geq inf(p_i)$ so $(\vee_{i\in I} w_i)_{inf(p_i)} \in K_C$. Now let $b \in M$, $q \in Q_t$ and $p \in (0, 1]$ be such that $b_p \in K_C$. Then $K_C(b) \geq p > 0$, and so $K_C(b) = 1$, i.e., $b \in C$. Since C is a Q_t-submodule of M, we have $q * b \in C$ and hence $K_C(q * b) = 1 \geq p$. Therefore $(q * b)_p \in K_C$.

Conversely, let K_C be an $(\in, \in)$-fuzzy Q_t-submodule of M and $w_i \in C$. Then $(w_i)_1 \in K_C$. This shows that $(\vee_{i\in I} w_i)_1 = (\vee_{i\in I} w_i)_{inf(1,1)} \in K_C$. Hence $K_C(\vee_{i\in I} w_i) > 0$, and so $\vee_{i\in I} w_i \in C$. Now let $q \in Q_t$ and $z \in C$. Then $K_C(z) = 1$, and thus $z_1 \in K_C$. Since K_C is an $(\in, \in)$-fuzzy Q_t-submodule, it follows that $(q * z)_1 \in K_C$ so $K_C(q * z) = 1$. Hence $q * z \in C$. Thus, C is a Q_t-submodule of M. ■

Proposition 4.5.9 *Let* $\emptyset \neq C \subseteq Q_t$*. Then* K_C *is an* $(\in, \in \vee q)$*-fuzzy* Q_t*-submodule of* M *if and only if* C *is a* Q_t*-submodule of* M*.*

Proof. Let C be a Q_t-submodule of M. Then K_C is an $(\in, \in)$-fuzzy Q_t-submodule of M by lemma 4.5.8, and therefore K_C is an $(\in, \in \vee q)$-$f$uzzy Q_t-submodule of M by Corollary 4.5.6.

Conversely, let K_C be an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M. Let $z_i \in C$. Then $(z_i)_1 \in K_C$ which show that $(\vee_{i\in I} z_i)_1 = (\vee_{i\in I} z_i)_{inf(1,1)} (\in \vee q) K_C$. Hence $K_C(\vee_{i\in I} z_i) > 0$, and so $\vee_{i\in I} z_i \in C$. Now let $a \in Q_t$ and $z \in C$. Then $K_C(z) = 1$, and thus $z_1 \in K_C$. Since K_C is an $(\in, \in \vee q)$-fuzzy Q_t-submodule, it follows that $(a * z)_1 \in K_C$ so that $K_C(a * z) = 1$. Hence $a * z \in C$. Hence C is a Q_t-submodule of M. ■

Proposition 4.5.10 *Let g be an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M such that $g(w) < 0.5$ for all $w \in M$. Then g is an $(\in, \in)$-fuzzy Q_t-submodule of M.*

Proof. Suppose g is an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M such that $g(z) < 0.5$ for all $z \in M$. Let $(z_i)_{p_i} \in g$. Then $0.5 > g(z_i) \geq p_i$. Since $z_i \in M$ and M is closed under join, so $\vee_{i\in I} z_i \in M$ and $0.5 > g(\vee_{i\in I} z_i)$. Thus $g(\vee_{i\in I} z_i) + inf(p_i) < 0.5 + 0.5 = 1$, i.e., $(\vee_{i\in I} z_i)_{inf(p_i)} \overline{q} g$. But since g is an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M, this shows that $(\vee_{i\in I} z_i)_{inf(p_i)} \in g$. Similarly, we can show that $(a * z)_p \in g$ for $z_p \in g$ and for all $a \in Q_t$. ■

Theorem 4.5.11 *Let M be a Q_t-module and g be a f-subset of M. Then g is an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M if and only if each non-empty $U(g; p)$ is a Q_t-submodule of M for all $p \in (0, 0.5]$.*

Proof. Let g be an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M and $p \in (0, 0.5]$. Let $x \in M$ and $q \in Q_t$ be such that $x \in U(g; p)$. Then $g(x) \geq p$. Now since $g(q * x) \geq inf(g(x), 0.5) \geq inf(p, 0.5) = p$, so we have $q * x \in U(g; p)$. Let $x_i \in U(g; p)$. Then $g(x_i) \geq p$. Since g is an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M, so we have $g(\vee_{i\in I} x_i) \geq inf(\inf_{i\in I} g(x_i), 0.5) \geq inf(p, 0.5) = p$. Thus $\vee_{i\in I} x_i \in U(g; p)$. Hence $U(g; p)$ is Q_t-submodule of M.

Conversely, assume that each non-empty $U(g; p)$ is a Q_t-submodule of M for all $p \in (0, 0.5]$. Let there exist $m_i \in M$ such that $g(\vee_{i\in I} m_i) < inf(\inf_{i\in I} g(m_i), 0.5)$, then we can take p such that $g(\vee_{i\in I} m_i) < p \leq inf(\inf_{i\in I} g(m_i), 0.5)$. Thus $m_i \in U(g; p)$ and $p < 0.5$ but $\vee_{i\in I} m_i \notin U(g; p)$. This is a contradiction. Therefore $g(\vee_{i\in I} m_i) \geq inf(\inf_{i\in I} g(m_i), 0.5)$ for all $m_i \in M$. Now, if there exist $z \in M$ and $q \in Q_t$ such that $g(q * z) < inf(g(z), 0.5)$, then we can choose $p \in (0, 0.5]$ such that $g(q * z) < p \leq inf(g(z), 0.5)$. It follows that $z \in U(g; p)$ and $p < 0.5$ but $q * z \notin U(g; p)$. This is not possible. Hence $g(q * z) \geq inf(g(z), 0.5)$ for all $q \in Q_t$ and $z \in M$. Thus, g is an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M by Proposition 4.5.3. ■

Chapter 5

Generalized Approximations of $(\in, \in \vee q)$-Fuzzy Ideals and Subquantales in Quantale

In the present chapter, we are starting the investigation of roughness in $(\in, \in \vee q)$-FS and $(\in, \in \vee q)$-FI of quantales with respect to the generalized approximation space. Moreover, it is demonstrated that GLA and GUA of $(\in, \in \vee q)$-FI, $(\in, \in \vee q)$-FS, $(\in, \in \vee q)$-FPI and $(\in, \in \vee q)$-$FSPI$ are $(\in, \in \vee q)$-FI, $(\in, \in \vee q)$-FS, $(\in, \in \vee q)$-FPI and $(\in, \in \vee q)$-$FSPI$ by using SVH and $SSVH$, respectively.

In the first section, LA and UA of FS are introduced. It is also noted that GLA of a FS is not a FS while taking SVH. In the second section, initially the generalized approximations of $(\in, \in \vee q)$-FS are examined. Then, we study the generalized roughness of $(\in, \in \vee q)$-FI in terms of SVH and $SSVH$. It is observed that GLA of $(\in, \in \vee q)$-FI is not a $(\in, \in \vee q)$-FI while taking SVH and GUA of $(\in, \in \vee q)$-FI is $(\in, \in \vee q)$-FI while taking SVH. Further, generalized roughness being extended to $(\in, \in \vee q)$- FPI and $(\in, \in \vee q)$-$FSPI$. In the last sections approximations of fuzzy Q_t-submodules and approximations of $(\in, \in \vee q)$-fuzzy Q_t-submodules of Q_t-modules are introduced.

5.1 Lower and Upper Approximation of Fuzzy Subquantales [Ideals]

It is observed that SVM are very useful to study roughness in quantales [91]. In this section, initially the generalized approximations of FS are examined.

Theorem 5.1.1 *Let* $H : Q_t \longrightarrow P^*(Q'_t)$ *be a* $SSVH$ *and* g *be a* FS *of* Q'_t. *Then* $\underline{H}(g)$ *is a* FS *of* Q_t.

Proof. As g is given to be a FS of Q'_t, so by Definition 4.1.3, we have $g(\vee_{i\in I}t_i) \geq \wedge_{i\in I}g(t_i)$ and $g(y \otimes' t) \geq g(y) \wedge g(t)$ for all $y,\ t,\ t_i \in Q'_t$. As $H : Q_t \longrightarrow P^*(Q'_t)$ is a $SSVH$, so $\vee_{i\in I}H(t_i) = H(\vee_{i\in I}t_i)$.

Consider,

$$\begin{aligned}\underline{H}(g)(\vee_{i\in I}t_i) &= \underset{e\ \in\ H(\vee_{i\in I}t_i)}{Inf} g(e)\\ &= \underset{e\ \in\ \vee_{i\in I}H(t_i)}{Inf} g(e).\end{aligned}$$

Since $e \in \vee_{i\in I}H(t_i)$, there exist $a_1 \in H(t_1)$, $a_2 \in H(t_2)$,..., $a_i \in H(t_i)$ such that $e = \vee_{i\in I}a_i$.

Hence,

$$\begin{aligned}\underline{H}(g)(\vee_{i\in I}t_i) &= \underset{\vee_{i\in I}a_i\ \in\ \vee_{i\in I}H(t_i)}{Inf} g(\vee_{i\in I}a_i)\\ &\geq \underset{\vee_{i\in I}a_i\ \in\ \vee_{i\in I}H(t_i)}{Inf} [\wedge_{i\in I}g(a_i)]\\ &= \underset{a_1\ \in\ H(t_1),\ a_2\ \in\ H(t_2),...,\ a_i\ \in\ H(t_i)}{Inf} [g(a_1) \wedge g(a_2) \wedge, ..., \wedge g(a_i)]\\ &= \left(\underset{a_1\ \in\ H(t_1)}{Inf} g(a_1)\right) \wedge \left(\underset{a_2\ \in\ H(t_2)}{Inf} g(a_2)\right) \wedge, ..., \wedge \left(\underset{a_i\ \in\ H(t_i)}{Inf} g(a_i)\right)\\ &= Inf(\underline{H}(g)(t_1),\ \underline{H}(g)(t_2), ...,\ \underline{H}(g)(t_i))\\ &= \underset{i\in I}{Inf}\ \underline{H}(g)(t_i).\end{aligned}$$

Thus we have

$$\underline{H}(g)(\vee_{i\in I}t_i) \geq \underset{i\in I}{Inf}\ \underline{H}(g)(t_i) \text{ for all } t_i \in Q_t.$$

Now since $H : Q_t \longrightarrow P^*(Q'_t)$ is a $SSVH$, we have $H(t_1) \otimes' H(t_2) = H(t_1 \otimes t_2)$ for all $t_1,\ t_2 \in Q_t$.

Consider,

$$\begin{aligned}\underline{H}(g)(t_1 \otimes t_2) &= \underset{e \,\in\, H(t_1 \otimes t_2)}{Inf} \quad g(e) \\ &= \underset{e \,\in\, H(t_1) \,\otimes' H(t_2)}{Inf} \quad g(e)\end{aligned}$$

As $e \in H(t_1) \otimes' H(t_2)$, we obtain $a_1 \in H(t_1)$ and $a_2 \in H(t_2)$ such that $e = a_1 \otimes' a_2$.

Hence,

$$\begin{aligned}\underline{H}(g)(t_1 \otimes t_2) &= \underset{a_1 \otimes' a_2 \,\in\, H(t_1) \,\otimes' H(t_2)}{Inf} \quad g(a_1 \otimes' a_2) \\ &\geq \underset{a_1 \otimes' a_2 \,\in\, H(t_1) \,\otimes' H(t_2)}{Inf} \quad [g(a_1) \wedge g(a_2)] \\ &= \underset{a_1 \in\, H(t_1),\ a_2 \,\in\, H(t_2)}{Inf} \quad [g(a_1) \wedge g(a_2)] \\ &= [\underset{a_1 \in\, H(t_1)}{Inf} \quad g(a_1)] \wedge [\underset{a_2 \,\in\, H(t_2)}{Inf} \quad g(a_2)] \\ &= Inf(\underline{H}(g)(t_1), \underline{H}(g)(t_2)).\end{aligned}$$

Hence $\underline{H}(g)(t_1 \otimes t_2) \geq Inf(\underline{H}(g)(t_1), \underline{H}(g)(t_2))$ for all $t_1,\, t_2 \in Q_t$.

Thus, $\underline{H}(g)$ is a FS of Q_t. ■

Now we show that by using SVH, GLA of a FS is not a FS.

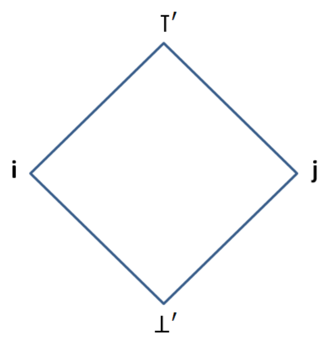

Fig. 10

Table. 7

$\otimes'$	$\bot'$	i	j	$\top'$
$\bot'$	$\bot'$	$\bot'$	$\bot'$	$\bot'$
i	$\bot'$	i	$\bot'$	i
j	$\bot'$	$\bot'$	j	j
$\top'$	$\bot'$	i	j	$\top'$

Example 5.1.2 *Let $(Q'_t, \otimes')$ be a quantale, where Q'_t is depicted in Fig. 10 and the binary operations $\otimes'$ on Q'_t is shown in the table 7.*

Let $H : Q'_t \longrightarrow P^*(Q'_t)$ be defined by $H(\bot') = H(i) = H(j) = \{\bot'\}$ and $H(\top') = Q'_t$. It is easily seen that H is a SVH. Consider a f-subset, g of Q'_t given by $g = \frac{1}{\bot'} + \frac{0.5}{i} + \frac{0.5}{j} + \frac{1}{\top'}$. It is easily verified that g is a FS of Q'_t. With the help of Definition 3.1.2, we have $\underline{H}(g) = \frac{1}{\bot'} + \frac{1}{i} + \frac{1}{j} + \frac{0.5}{\top'}$. As $\underline{H}(g)(t_1 \otimes' t_2) \geq \underline{H}(g)(t_1) \wedge \underline{H}(g)(t_2)$ is satisfied for all t_1, $t_2 \in Q'_t$. But $\underline{H}(g)(\vee_{i\in I} t_i) \geq \underset{i\in I}{Inf}\ \underline{H}(g)(t_i)$ for all $t_i \in Q'_t$ is not satisfied in this case, because $\underline{H}(g)(i \vee j) = \underline{H}(g)(\top') = 0.5$ and $\underline{H}(g)(i) \wedge \underline{H}(g)(j) = 1 \wedge 1 = 1$. Hence $\underline{H}(g)(i \vee j) \ngeq \underline{H}(g)(i) \wedge \underline{H}(g)(j)$. Hence GLA of a FS is not a FS while taking SVH.

Theorem 5.1.3 *Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a SVH and g be a FS of Q'_t. Then $\overline{H}(g)$ is a FS of Q_t.*

Proof. As g is a FS of Q'_t, so we have $g(\vee_{i\in I} t_i) \geq \wedge_{i\in I} g(t_i)$ and $g(y \otimes' t) \geq g(y) \wedge g(t)$ for all y, t, $t_i \in Q'_t$. Since $H : Q_t \longrightarrow P^*(Q'_t)$ is a SVH, we have $\vee_{i\in I} H(t_i) \subseteq H(\vee_{i\in I} t_i)$.

For this consider,

$$\begin{aligned}
\underset{i\in I}{Inf}\ \overline{H}(g)(t_i) &= \underset{i\in I}{Inf}\ (\overline{H}(g)(t_1), \overline{H}(g)(t_2), ...,\ \overline{H}(g)(t_i)) \\
&= \left(\underset{a_1 \in H(t_1)}{Sup}\ g(a_1) \right) \wedge \left(\underset{a_2 \in H(t_2)}{Sup}\ g(a_2) \right) \wedge, ..., \wedge \left(\underset{a_i \in H(t_i)}{Sup}\ g(a_i) \right) \\
&= \underset{a_1 \in H(t_1),\ a_2 \in H(t_2), ...,\ a_i \in H(t_i)}{Sup} [g(a_1) \wedge g(a_2) \wedge, ..., \wedge\ g(a_i)] \\
&= \underset{\vee_{i\in I} a_i \in \vee_{i\in I} H(t_i)}{Sup} [\wedge_{i\in I} g(a_i)] \\
&\leq \underset{\vee_{i\in I} a_i \in \vee_{i\in I} H(t_i)}{Sup} g(\vee_{i\in I} a_i) \\
&\leq \underset{\vee_{i\in I} a_i \in H(\vee_{i\in I} t_i)}{Sup} g(\vee_{i\in I} a_i) \\
&= \underset{e \in H(\vee_{i\in I} t_i)}{Sup}\ g(e) \\
&= \overline{H}(g)(\vee_{i\in I} t_i).
\end{aligned}$$

Hence $\overline{H}(g)(\vee_{i\in I} t_i) \geq \underset{i\in I}{Inf}\ \overline{H}(g)(t_i)$ for all $t_i \in Q_t$.

As $H : Q_t \longrightarrow P^*(Q'_t)$ is a SVH, so $H(t_1) \otimes' H(t_2) \subseteq H(t_1 \otimes t_2)$ for all t_1, $t_2 \in Q_t$.

Consider,

$$
\begin{aligned}
Inf(\overline{H}(g)(t_1), \overline{H}(g)(t_2)) &= [\underset{a_1 \in H(t_1)}{Sup} g(a_1)] \wedge [\underset{a_2 \in H(t_2)}{Sup} g(a_2)] \\
&= \underset{a_1 \in H(t_1),\, a_2 \in H(t_2)}{Sup} [g(a_1) \wedge g(a_2)] \\
&= \underset{a_1 \otimes' a_2 \in H(t_1) \otimes' H(t_2)}{Sup} g(a_1) \wedge g(a_2) \\
&\leq \underset{a_1 \otimes' a_2 \in H(t_1) \otimes' H(t_2)}{Sup} g(a_1 \otimes' a_2) \\
&\leq \underset{a_1 \otimes' a_2 \in H(t_1 \otimes t_2)}{Sup} g(a_1 \otimes' a_2) \\
&= \underset{e \in H(t_1 \otimes t_2)}{Sup} g(e) \\
&= \overline{H}(g)(t_1 \otimes t_2).
\end{aligned}
$$

Hence $\overline{H}(g)(t_1 \otimes t_2) \geq Inf(\overline{H}(g)(t_1), \overline{H}(g)(t_2))$ for all $t_1, t_2 \in Q_t$. Thus $\overline{H}(g)$ is a FS of Q_t. ∎

Theorem 5.1.4 *[67] Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$ and g be a FI of Q'_t. Then $\underline{H}(g)$ is a FI of Q_t.*

Similarly, we can show that by using SVH, the GLA of FI is not a FI.

Example 5.1.5 *Let $(Q'_t, \otimes')$be a quantale, where Q'_t is depicted in Fig.* 10 *and the binary operation $\otimes'$ on Q'_t is shown in the table* 7*. Let $H : Q'_t \longrightarrow P^*(Q'_t)$ be a SVH as defined in Example 5.1.2. Let λ be a f-subset of Q'_t defined by*

$$
\lambda(x) = \begin{cases} 1, & x = \perp' \\ 0.7, & x \neq \perp' \end{cases} \quad \textit{for all } x \in Q'_t.
$$

It is easy to verify that λ is a FI of Q'_t. Now GLA of λ is $\underline{H}(\lambda) = \frac{1}{\perp'} + \frac{1}{i} + \frac{1}{j} + \frac{0.7}{\top'}$. We observe that $\underline{H}(\lambda)(i \vee j) = \underline{H}(\lambda)(\top') = 0.7 \neq \underline{H}(\lambda)(i) \wedge \underline{H}(\lambda)(j) = 1$. Hence GLA of λ is not a FI while taking SVH.

Theorem 5.1.6 *[67] Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a SVH and g be a FI of Q'_t. Then $\overline{H}(g)$ is a FI of Q_t.*

5.2 Lower and Upper Approximations of $(\in, \in \vee q)$-Fuzzy Ideals

It is well-known that the notion of ideals is one of the powerful tools to characterize an algebraic structure. The idea of $(\in, \in \vee q)$-$f$uzzy structures was started by Bhakat and Das [6]. These, $(\in, \in \vee q)$-FI have significant role. Note that $(\in, \in \vee q)$-FI are the generalization of FI. In fuzzy algebraic structures, roughness has been considered broadly, however less investigation has been made for roughness in an $(\in, \in \vee q)$-FI and $(\in, \in \vee q)$-FS. In this section, at first the investigation of generalized roughness in $(\in, \in \vee q)$-FS is started.

Theorem 5.2.1 *Let g be an $(\in, \in \vee q)$-FS of Q'_t and $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$. Then $\underline{H}(g)$ is an $(\in, \in \vee q)$-FS of Q_t.*

Proof. As $H : Q_t \longrightarrow P^*(Q'_t)$ is a $SSVH$, so $\vee_{i\in I} H(z_i) = H(\vee_{i\in I} z_i)$. Let g be an $(\in, \in \vee q)$-FS of Q'_t.

Consider,

$$\begin{aligned}
\underline{H}(g)(\vee_{i\in I} z_i) &= \underset{e \,\in\, H(\vee_{i\in I} z_i)}{Inf} \quad g(e) \\
&= \underset{e \,\in\, \vee_{i\in I} H(z_i)}{Inf} \quad g(e).
\end{aligned}$$

As $e \in \vee_{i\in I} H(z_i)$, so there exist $a_1 \in H(z_1), a_2 \in H(z_2), ..., \ a_i \in H(z_i)$ such that $e = \vee_{i\in I} a_i$.

$$\begin{aligned}
\underline{H}(g)(\vee_{i\in I} z_i) &= \underset{\vee_{i\in I} a_i \,\in\, \vee_{i\in I} H(z_i)}{Inf} \quad g(\vee_{i\in I} a_i) \\
&\geq \underset{\vee_{i\in I} a_i \,\in\, \vee_{i\in I} H(z_i)}{Inf} \quad [\wedge_{i\in I} g(a_i) \wedge 0.5] \text{ by Lemma 4.2.1} \\
&= \underset{a_1 \,\in\, H(z_1),\ a_2 \,\in\, H(z_2), ...,\ a_i \,\in\, H(z_i)}{Inf} \quad [g(a_1) \wedge g(a_2) \wedge, ..., \wedge g(a_i] \wedge 0.5 \\
&= \left(\underset{a_1 \,\in\, H(z_1)}{Inf} \quad g(a_1) \right) \wedge \left(\underset{a_2 \,\in\, H(z_2)}{Inf} \quad g(a_2) \right) \wedge, ..., \wedge \left(\underset{a_i \,\in\, H(z_i)}{Inf} \quad g(a_i) \right) \wedge 0.5 \\
&= \underset{i\in I}{Inf} \ [\underline{H}(g)(z_1) \wedge \ \underline{H}(g)(z_2) \wedge, ..., \wedge \ \underline{H}(g)(z_i)] \wedge 0.5 \\
&= [\underset{i\in I}{Inf} \ \underline{H}(g)(z_i)] \wedge 0.5.
\end{aligned}$$

Hence $\underline{H}(g)(\vee_{i\in I} z_i) \geq Inf[(\underset{i\in I}{Inf} \ \underline{H}(g)(z_i)), 0.5]$ for all $z_i \in Q_t$.

As H is a $SSVH$, so $H(z \otimes w) = H(z) \otimes' H(w)$.

Consider,

$$\begin{aligned}
\underline{H}(g)(z \otimes w) &= \underset{e \,\in\, H(z \otimes w)}{Inf} \quad g(e) \\
&= \underset{e \,\in\, H(z) \otimes' H(w)}{Inf} \quad g(e).
\end{aligned}$$

As $e \in H(z) \otimes' H(w)$, so there exist $a_1 \in H(z)$, $a_2 \in H(w)$ such that $e = a_1 \otimes' a_2$.

Hence,

$$\begin{aligned}
\underline{H}(g)(z \otimes w) &= \underset{a_1 \otimes' a_2 \,\in\, H(z) \otimes' H(w)}{Inf} g(a_1 \otimes' a_2) \\
&\geq \underset{a_1 \otimes' a_2 \,\in\, H(z) \otimes' H(w)}{Inf} [g(a_1) \wedge g(a_2) \wedge 0.5] \text{ by Lemma 4.2.2} \\
&= \underset{a_1 \,\in\, H(z),\ a_2 \,\in\, H(w)}{Inf} [g(a_1) \wedge g(a_2)] \wedge 0.5 \\
&= \left[\left(\underset{a_1 \,\in\, H(z)}{\wedge} g(a_1)\right) \wedge \left(\underset{a_2 \,\in\, H(w)}{\wedge} g(a_2)\right)\right] \wedge 0.5 \\
&= \underline{H}(g)(z) \wedge \underline{H}(g)(w) \wedge 0.5.
\end{aligned}$$

Hence $\underline{H}(g)(z \otimes w) \geq Inf[\underline{H}(g)(z), \underline{H}(g)(w), 0.5]$ for all $z, w \in Q_t$. Thus, $\underline{H}(g)$ is an $(\in, \in \vee q)$-FS of Q_t. ■

Example 5.2.2 *Let $(Q'_t, \otimes')$ be the quantale depicted in Fig. 10 and the binary operations $\otimes'$ on Q'_t is shown in the table 7. Let $H : Q'_t \longrightarrow P^*(Q'_t)$ be the SVH as defined in Example 5.1.2. It is easily seen that H is a set-valued homomorphism. Let g be a f-subset of Q'_t given by $g = \frac{0.5}{\perp'} + \frac{0.3}{i} + \frac{0.3}{j} + \frac{0.5}{\top'}$. It is easily verified that g is a $(\in, \in \vee q)$-FS of Q'_t. GLA of g is as follows $\underline{H}(g) = \frac{0.5}{\perp'} + \frac{0.5}{i} + \frac{0.5}{j} + \frac{0.3}{\top'}$. As $i_{0.4} \in g$ and $j_{0.5} \in g$ but $(i \vee j)_{0.4} \overline{(\in \vee q)} g$. Thus, $\underline{H}(g)$ is not an $(\in, \in \vee q)$-FS of Q'_t, while using SVH.*

Theorem 5.2.3 *Let g be an $(\in, \in \vee q)$-FS of Q'_t and $H : Q_t \longrightarrow P^*(Q'_t)$ be a SVH. Then $\overline{H}(g)$ is an $(\in, \in \vee q)$-FS of Q_t.*

Proof. As $H : Q_t \longrightarrow P^*(Q'_t)$ is a SVH, so $\vee_{i \in I} H(z_i) \subseteq H(\vee_{i \in I} z_i)$.

Consider,

$$\begin{aligned}
Inf(\underset{i \in I}{Inf}\, \overline{H}(g)(z_i), 0.5) &= [\overline{H}(g)(z_1) \wedge \overline{H}(g)(z_2) \wedge, ..., \wedge\, \overline{H}(g)(z_i)] \wedge 0.5 \\
&= \left[\left(\underset{a_1 \,\in\, H(z_1)}{Sup} g(a_1)\right) \wedge, ..., \wedge \left(\underset{a_i \,\in\, H(z_i)}{Sup} g(a_i)\right)\right] \wedge 0.5 \\
&= \underset{a_1 \,\in\, H(z_1),\ a_2 \,\in\, H(z_2), ...,\ a_i \,\in\, H(z_i)}{Sup} [g(a_1) \wedge, ..., \wedge\, g(a_i] \wedge 0.5 \\
&= \underset{\vee_{i \in I} a_i \,\in\, \vee_{i \in I} H(z_i)}{Sup} [\wedge_{i \in I} g(a_i) \wedge 0.5] \\
&\leq \underset{\vee_{i \in I} a_i \,\in\, \vee_{i \in I} H(z_i)}{Sup} g(\vee_{i \in I} a_i) \\
&= \underset{e \,\in\, \vee_{i \in I} H(z_i)}{Sup} g(e) \\
&\leq \underset{e \,\in\, H(\vee_{i \in I} z_i)}{Sup} g(e) \\
&= \overline{H}(g)(\vee_{i \in I} z_i).
\end{aligned}$$

Hence $\overline{H}(g)(\vee_{i\in I} z_i) \geq Inf(\underset{i\in I}{Inf}\ \overline{H}(g)(z_i), 0.5)$ for all $z_i \in Q_t$.

Similarly, it can be shown that $\underline{H}(g)(z \otimes w) \geq Inf(\underline{H}(g)(z), \underline{H}(g)(w), 0.5)$ for all $z, w \in Q_t$. Thus, $\overline{H}(g)$ is an $(\in, \in \vee q)$-FS of Q_t. ■

Theorem 5.2.4 *Let g be an $(\in, \in \vee q)$-FRI (FLI) of Q'_t and $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$. Then $\underline{H}(g)$ is an $(\in, \in \vee q)$-FRI (FLI) of Q_t.*

Proof. Since $H : Q_t \longrightarrow P^*(Q'_t)$ is a $SSVH$, we have $H(z \vee w) = H(z) \vee H(w)$.

Consider,

$$\begin{aligned}\underline{H}(g)(z \vee w) &= \underset{e\,\in\, H(z\,\vee\, w)}{Inf} g(e)\\ &= \underset{e\,\in\, H(z)\vee H(w)}{Inf} g(e).\end{aligned}$$

As $e \in H(z) \vee H(w)$, therefore there are $t_1 \in H(z)$ and $t_2 \in H(w)$ such that $e = t_1 \vee t_2$.

Hence,

$$\begin{aligned}\underline{H}(g)(z \vee w) &= \underset{t_1\vee t_2\in\, H(z)\vee H(w)}{Inf} g(t_1 \vee t_2)\\ &\geq \underset{t_1\,\in\, H(z),\, t_2\,\in\, H(w)}{Inf} [g(t_1) \wedge g(t_2) \wedge 0.5] \text{ by Lemma 4.2.5}\\ &= \left(\underset{t_1\,\in\, H(z)}{Inf} g(t_1)\right) \wedge \left(\underset{t_2\,\in\, H(z)}{Inf} g(t_2)\right) \wedge 0.5\\ &= \underline{H}(g)(z) \wedge \underline{H}(g)(w) \wedge 0.5.\end{aligned}$$

Hence $\underline{H}(g)(z \vee w) \geq Inf(\underline{H}(g)(z), \underline{H}(g)(w), 0.5)$ for all $z, w \in Q_t$.

As H is a $SSVH$, so $H(z \otimes w) = H(z) \otimes' H(w)$.

Consider,

$$\begin{aligned}\underline{H}(g)(z \otimes w) &= \underset{e\,\in\, H(z\,\otimes\, w)}{Inf} g(e)\\ &= \underset{e\,\in\, H(z)\otimes' H(w)}{Inf} g(e).\end{aligned}$$

For $e \in H(z) \otimes' H(w)$, there exist $t_1 \in H(z)$ and $t_2 \in H(w)$ such that $e = t_1 \otimes' t_2$.

Hence we have,

$$\begin{aligned}\underline{H}(g)(z \otimes w) &= \underset{t_1 \otimes' t_2 \in\ H(z) \otimes' H(w)}{Inf} g(t_1 \otimes' t_2)\\ &\geq \underset{t_1\ \in\ H(z),\ t_2 \in\ H(w)}{Inf} [g(t_1) \wedge 0.5] \text{ by Lemma 4.2.6}\\ &= [\underset{t_1\ \in\ H(z)}{Inf}\ g(t_1)] \wedge 0.5\\ &= \underline{H}(g)(z) \wedge 0.5.\end{aligned}$$

Hence $\underline{H}(g)(z \otimes w) \geq Inf(\underline{H}(g)(z), 0.5)$ for all $z, w \in Q_t$. Similarly, we can show that $\underline{H}(g)(w \otimes z) \geq Inf(\underline{H}(g)(z), 0.5)$ for all $z, w \in Q_t$.

Let $w \leq z$. Then $w \vee z = z$. Since $H : Q_t \longrightarrow P^*(Q'_t)$ is a $SSVH$, so $H(z) = H(w \vee z) = H(w) \vee H(z)$.

Consider

$$\begin{aligned}Inf(\underline{H}(g)(z), 0.5) &= \underset{e \in\ H(z)}{Inf}\ g(e) \wedge 0.5\\ &= \left[\underset{e \in\ H(z) \vee H(w)}{Inf} g(e)\right] \wedge 0.5.\end{aligned}$$

Since $e \in H(z) \vee H(w)$ so there exist $t_1 \in H(z)$ and $t_2 \in H(w)$ such that $e = t_1 \vee t_2$. As $t_1 \vee t_2 \geq t_2$. So, by Lemma 4.2.8, we have

$$\begin{aligned}Inf(\underline{H}(g)(z), 0.5) &= \left[\underset{t_1 \vee t_2 \in\ H(z) \vee H(w)}{Inf} g(t_1 \vee t_2)\right] \wedge 0.5\\ &= \underset{t_1 \in\ H(z),\ t_2 \in\ H(w)}{Inf} [g(t_1 \vee t_2) \wedge 0.5]\\ &\leq \underset{t_2 \in\ H(w)}{Inf}\ g(t_2)\\ &= \underline{H}(g)(w).\end{aligned}$$

Thus, $\underline{H}(g)(w) \geq \underline{H}(g)(z) \wedge 0.5$. Therefore, $\underline{H}(g)$ is an $(\in, \in \vee q)$-FRI of Q_t. ∎

Example 5.2.5 *Let $(Q'_t, \otimes')$ be the quantale, where the binary operations $\otimes'$ on Q'_t is shown in the table 7 and Q'_t is depicted in Fig. 10. Let $H : Q'_t \longrightarrow P^*(Q'_t)$ be the SVH as defined in Example 5.1.2. It is easily seen that H is a set-valued homomorphism. Let g be a f-subset of Q'_t given by $g = \frac{0.8}{\perp'} + \frac{0.5}{i} + \frac{0.8}{j} + \frac{0.5}{\top'}$. It is easily verified that g is an $(\in, \in \vee q)$-FI of Q'_t. GLA of g is as follows $\underline{H}(g) = \frac{0.5}{\top'} + \frac{0.8}{j} + \frac{0.8}{i} + \frac{0.8}{\perp'}$. As $i_{0.55} \in \underline{H}(g)$ and $j_{0.75} \in \underline{H}(g)$ but $(i \vee j)_{0.55} \overline{(\in \vee q)} \underline{H}(g)$. Thus, $\underline{H}(g)$ is not an $(\in, \in \vee q)$-FI of Q'_t by using SVH.*

Theorem 5.2.6 *Let H be $SSVH$ and g be an $(\in, \in \vee q)$-FRI (FLI) ideal of Q'_t. Then $\overline{H}(g)$ is an $(\in, \in \vee q)$-FRI (FLI) of Q_t.*

Proof. Proof is similar as reported in Theorem 5.2.4. ■

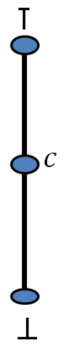

Fig 11.

Table 8.

$\otimes$	0	c	1
0	0	0	0
c	0	c	c
1	0	c	1

Example 5.2.7 *Let $(Q_t, \otimes)$ and $(Q'_t, \otimes')$ be two quantales, where Q_t and Q'_t are depicted in Fig. 10 and 11 and the binary operations $\otimes$ and $\otimes'$ on both the quantales are shown in the table 7 and 8. Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$ defined by $H(\perp) = \{\perp'\}$, $H(c) = \{i, j\}$ and $H(\top) = \{\top'\}$. Let g be an $(\in, \in \vee q)$-FI of Q'_t defined by $g = \frac{0.8}{\perp'} + \frac{0.7}{i} + \frac{0.8}{j} + \frac{0.7}{\top'}$. Then GLA and GUA of the $(\in, \in \vee q)$-FRI (FLI) g of Q'_t are as follows: $\underline{H}(g) = \frac{0.8}{\perp} + \frac{0.7}{c} + \frac{0.7}{\top}$ and $\overline{H}(g) = \frac{0.8}{\perp} + \frac{0.8}{c} + \frac{0.7}{\top}$. It can be verified that $\underline{H}(g)$ and $\overline{H}(g)$ are $(\in, \in \vee q)$-FI of Q_t.*

5.3 Approximations of $(\in, \in \vee q)$-Fuzzy Prime (Semi prime) Ideals

Now generalized roughness being extended to $(\in, \in \vee q)$-FPI and $(\in, \in \vee q)$-$FSPI$. First the LA and UA of $(\in, \in \vee q)$-FPI are being started.

Theorem 5.3.1 *Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$ and g be an $(\in, \in \vee q)$-FPI of Q'_t. Then $\underline{H}(g)$ is an $(\in, \in \vee q)$-FPI of Q_t.*

Proof. As g is an $(\in, \in \vee q)$-FPI of Q'_t, therefore g is an $(\in, \in \vee q)$-FI of Q'_t, hence by Theorem 5.2.4, $\underline{H}(g)$ is an $(\in, \in \vee q)$-FI of Q_t. Moreover by Proposition 4.3.4, we have $g(e) \vee g(c) \geq g(e \otimes' c) \wedge 0.5$ for all $e, c \in Q_t$.

Consider,

$$\begin{aligned}
Sup(\underline{H}(g)(z), \underline{H}(g)(w)) = &\ Sup[\underset{e \in H(z)}{Inf}\ g(e), \underset{c \in H(w)}{Inf}\ g(c)] \\
= &\ \underset{e \in H(z),\, c \in H(w)}{Inf} [g(e) \vee g(c)] \\
\geq &\ \underset{e \in H(z),\, c \in H(w)}{Inf} [g(e \otimes' c) \wedge 0.5] \\
= &\ [\underset{e \otimes' c \in H(z) \otimes' H(w)}{Inf} g(e \otimes' c)] \wedge 0.5] \\
= &\ [\underset{e \otimes' c \in H(z \otimes w)}{Inf} g(e \otimes' c)] \wedge 0.5] \\
= &\ \underline{H}(g)(z \otimes w) \wedge 0.5.
\end{aligned}$$

Thus $\underline{H}(g)(z) \vee \underline{H}(g)(w) \geq Inf(\underline{H}(g)(z \otimes w), 0.5)$ for all $z, w \in Q_t$. Therefore by Proposition 4.3.4, we obtain $\underline{H}(g)$ is an $(\in, \in \vee q)$-FPI of Q_t. ■

Theorem 5.3.2 *Let $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$ and g be an $(\in, \in \vee q)$-FPI of Q'_t. Then $\overline{H}(g)$ is an $(\in, \in \vee q)$-FPI of Q_t.*

Proof. The proof is similar to the proof of Theorem 5.3.1. ■

Theorem 5.3.3 *Let g be an $(\in, \in \vee q)$-$FSPI$ of Q'_t and $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$. Then $\underline{H}(g)$ is an $(\in, \in \vee q)$-$FSPI$ of Q_t.*

Proof. As g is an $(\in, \in \vee q)$-$FSPI$ of Q'_t, by Proposition 4.3.12, we have $g(e) \geq g(e \otimes' e) \wedge 0.5$, for all $e \in Q'_t$.

Consider the following,

$$
\begin{aligned}
\underline{H}(g)(z) = & \underset{e \in H(z)}{Inf} \; g(e) \\
\geq & \underset{e \in H(z)}{Inf} \; [g(e \otimes' e) \wedge 0.5] \\
= & [\underset{e \otimes' e \in H(z) \otimes' H(z)}{Inf} \; g(e \otimes' e)] \wedge 0.5 \\
= & [\underset{e^2 \in H(z \otimes z)}{Inf} \; g(e \otimes' e)] \wedge 0.5 \\
\geq & \underline{H}(g)(z \otimes z) \wedge 0.5.
\end{aligned}
$$

Thus, $\underline{H}(g)(z) \geq Inf(\underline{H}(g)(z \otimes z), 0.5)$ for all $z \in Q_t$. Hence by Proposition 4.3.12, $\underline{H}(g)$ is an $(\in, \in \vee q)$-$FSPI$ of Q_t. ■

Theorem 5.3.4 *Let g be an $(\in, \in \vee q)$-$FSPI$ of Q'_t and $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$. Then $\overline{H}(g)$ is an $(\in, \in \vee q)$-$FSPI$ of Q_t.*

Proof. The proof is similar to the proof of Theorem 5.3.3. ■

5.4 Approximation of Fuzzy Q_t-submodule of Q_t-Module

It is observed that SVM are very useful to study roughness in quantales [91]. In this section, initially the generalized approximations of fuzzy Q_t-submodule of a Q_t-module are examined.

Definition 5.4.1 *Let M and N be Q_t-modules. A mapping $H : M \longrightarrow P^*(N)$ is called a SVH of Q_t-modules if*

(1) $\vee_{i \in I} H(m_i) \subseteq H(\vee_{i \in I} m_i)$;

(2) $q * H(m) \subseteq H(q * m)$ *for all m, $m_i \in M$ and $q \in Q_t$.*

A set-valued mapping $H : M \longrightarrow P^*(N)$ is called a $SSVH$ of Q_t-modules if if we replace containment by equality in (1) *and* (2).

Theorem 5.4.2 *Let $H : M \longrightarrow P^*(N)$ be a $SSVH$ of Q_t-modules and g be a fuzzy Q_t-submodule of N. Then $\underline{H}(g)$ is a fuzzy Q_t-submodule of M.*

Proof. As g is given to be a fuzzy Q_t-submodule of N, so by Definition 4.4.2, we have $g(\vee_{i \in I} x_i) \geq \wedge_{i \in I} g(x_i)$ and $g(q *' x) \geq g(x)$ for all x, $x_i \in N$ and $q \in Q_t$. As $H : M \longrightarrow P^*(N)$ is a $SSVH$, so $\vee_{i \in I} H(m_i) = H(\vee_{i \in I} m_i)$ for all $m_i \in M$.

Consider,

$$\begin{aligned}\underline{H}(g)(\vee_{i\in I} m_i) &= \underset{e\,\in\, H(\vee_{i\in I} m_i)}{Inf} g(e)\\ &= \underset{e\,\in\, \vee_{i\in I} H(m_i)}{Inf} g(e).\end{aligned}$$

Since $e \in \vee_{i\in I} H(m_i)$, there exist $a_1 \in H(m_1)$, $a_2 \in H(m_2)$,..., $a_i \in H(m_i)$ such that $e = \vee_{i\in I} a_i$.

Hence,

$$\begin{aligned}\underline{H}(g)(\vee_{i\in I} m_i) &= \underset{\vee_{i\in I} a_i\,\in\, \vee_{i\in I} H(m_i)}{Inf} g(\vee_{i\in I} a_i)\\ &\geq \underset{\vee_{i\in I} a_i\,\in\, \vee_{i\in I} H(m_i)}{Inf} [\wedge_{i\in I} g(a_i)]\\ &= \underset{a_1\,\in\, H(m_1),\; a_2\,\in\, H(m_2),...,\; a_i\,\in\, H(m_i)}{Inf} [g(a_1) \wedge g(a_2) \wedge, ..., \wedge g(a_i]\\ &= \left(\underset{a_1\,\in\, H(m_1)}{Inf} g(a_1)\right) \wedge \left(\underset{a_2\,\in\, H(m_2)}{Inf} g(a_2)\right) \wedge, ..., \wedge \left(\underset{a_i\,\in\, H(m_i)}{Inf} g(a_i)\right)\\ &= Inf(\underline{H}(g)(m_1),\ \underline{H}(g)(m_2), ...,\ \underline{H}(g)(m_i))\\ &= \underset{i\in I}{Inf}\ \underline{H}(g)(m_i).\end{aligned}$$

Thus we have

$$\underline{H}(g)(\vee_{i\in I} m_i) \geq \underset{i\in I}{Inf}\ \underline{H}(g)(m_i) \text{ for all } m_i \in M.$$

Now, since $H : M \longrightarrow P^*(N)$ is a $SSVH$ of Q_t-modules, we have $q{*'}H(m) = H(q{*}m)$ for all $m \in M$ and $q \in Q_t$.

Consider,

$$\begin{aligned}\underline{H}(g)(q \otimes m) &= \underset{e\,\in\, H(q*m)}{Inf} g(e)\\ &= \underset{e\,\in\, q\,*' H(m)}{Inf} g(e)\end{aligned}$$

As $e \in q *' H(m)$, we obtain $n \in H(m)$ such that $e = q *' n$.

Hence,

$$\begin{aligned}\underline{H}(g)(q * m) &= \underset{q*'n\in q\,*' H(m)}{Inf} g(q *' n)\\ &\geq \underset{q*'n\in q\,*' H(m)}{Inf} g(n)\\ &= \underset{n\,\in\, H(m)}{Inf} g(n)\\ &= \underline{H}(g)(m)\end{aligned}$$

Hence $\underline{H}(g)(q * m) \geq \underline{H}(g)(m)$ for all $m \in M$ and $q \in Q_t$. Thus, $\underline{H}(g)$ is a fuzzy Q_t-submodule of M. ■

Theorem 5.4.3 *Let $H : M \longrightarrow P^*(N)$ be a SVH of Q_t-modules and g be a fuzzy Q_t-submodule of N. Then $\overline{H}(g)$ is a fuzzy Q_t-submodule of M.*

Proof. As g is a fuzzy Q_t-submodule of N, so we have $g(\vee_{i\in I} n_i) \geq \wedge_{i\in I} g(n_i)$ and $g(q *' n) \geq g(n)$ for all n, $n_i \in N$ and $q \in Q_t$. Since $H : M \longrightarrow P^*(N)$ is a SVH of Q_t-modules, so we have $\vee_{i\in I} H(m_i) \subseteq H(\vee_{i\in I} m_i)$ for all $m_i \in M$.

For this consider,

$$\begin{aligned}
\underset{i\in I}{Inf}\ (\overline{H}(g)(m_i)) &= \underset{i\in I}{Inf}\ (\overline{H}(g)(m_1), \overline{H}(g)(m_2), ...,\ \overline{H}(g)(m_i)) \\
&= \left(\underset{a_1\ \in\ H(m_1)}{Sup} g(a_1) \right) \wedge \left(\underset{a_2\ \in\ H(m_2)}{Sup} g(a_2) \right) \wedge, ..., \wedge \left(\underset{a_i\ \in\ H(m_i)}{Sup} g(a_i) \right) \\
&= \underset{a_1\ \in\ H(m_1),\ a_2\ \in\ H(m_2), ...,\ a_i\ \in\ H(m_i)}{Sup} [g(a_1) \wedge g(a_2) \wedge, ..., \wedge\ g(a_i] \\
&= \underset{\vee_{i\in I} a_i\ \in\ \vee_{i\in I} H(m_i)}{Sup} [\wedge_{i\in I} g(a_i)] \\
&\leq \underset{\vee_{i\in I} a_i\ \in\ \vee_{i\in I} H(m_i)}{Sup} g(\vee_{i\in I} a_i) \\
&\leq \underset{\vee_{i\in I} a_i\ \in\ H(\vee_{i\in I} m_i)}{Sup} g(\vee_{i\in I} a_i) \\
&= \underset{e\ \in\ H(\vee_{i\in I} m_i)}{Sup} g(e) \\
&= \overline{H}(g)(\vee_{i\in I} m_i).
\end{aligned}$$

Hence $\overline{H}(g)(\vee_{i\in I} m_i) \geq \underset{i\in I}{Inf}\ \overline{H}(g)(m_i)$ for all $m_i \in M$.

As $H : M \longrightarrow P^*(N)$ is a SVH, so $q *' H(m) \subseteq H(q * m)$ for all $m \in M$ and $q \in Q_t$.

Consider,

$$\begin{aligned}
\overline{H}(g)(m) &= \underset{a\in\ H(m)}{Sup}\ g(a) \\
&\leq \underset{a\in\ H(m)}{Sup}\ g(q *' a) \\
&= \underset{q\otimes' a\ \in q\otimes' H(m)}{Sup}\ g(q *' a) \\
&\leq \underset{q\otimes' a\ \in\ H(q*m)}{Sup}\ g(q *' a) \\
&= \underset{e\ \in\ H(q*m)}{Sup}\ g(e) \\
&= \overline{H}(g)(q * m).
\end{aligned}$$

Hence $\overline{H}(g)(q \otimes m) \geq \overline{H}(g)(m)$ for all $m \in M$ and $q \in Q_t$. Thus $\overline{H}(g)$ is a fuzzy Q_t-submodule of M. ■

5.5 Approximations of $(\in, \in \vee q)$-Fuzzy Q_t-submodule of Q_t-Module

In this section, the investigation of generalized roughness in $(\in, \in \vee q)$-fuzzy Q_t-submodule is started.

Theorem 5.5.1 *Let g be an $(\in, \in \vee q)$-fuzzy Q_t-submodule of N and $H : M \longrightarrow P^*(N)$ be a SSVH of Q_t-modules. Then $\underline{H}(g)$ is an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M.*

Proof. As $H : M \longrightarrow P^*(N)$ is a $SSVH$ of Q_t-modules, so we have $\vee_{i\in I}H(m_i) = H(\vee_{i\in I}m_i)$. Let g be an $(\in, \in \vee q)$-fuzzy Q_t-submodule of N.

Consider,

$$\begin{aligned}
\underline{H}(g)(\vee_{i\in I}m_i) &= \underset{e \,\in\, H(\vee_{i\in I}m_i)}{Inf} g(e) \\
&= \underset{e \,\in\, \vee_{i\in I}H(m_i)}{Inf} g(e).
\end{aligned}$$

As $e \in \vee_{i\in I}H(m_i)$, so there exist $a_1 \in H(m_1), a_2 \in H(m_2), ..., a_i \in H(m_i)$ such that $e = \vee_{i\in I}a_i$.

$$\begin{aligned}
\underline{H}(g)(\vee_{i\in I}m_i) &= \underset{\vee_{i\in I}a_i \,\in\, \vee_{i\in I}H(m_i)}{Inf} g(\vee_{i\in I}a_i) \\
&\geq \underset{\vee_{i\in I}a_i \,\in\, \vee_{i\in I}H(m_i)}{Inf} [\wedge_{i\in I}g(a_i) \wedge 0.5] \text{ by Lemma 4.5.1} \\
&= \underset{a_1 \,\in\, H(m_1),\, a_2 \,\in\, H(m_2),...,\, a_i \,\in\, H(m_i)}{Inf} [g(a_1)\wedge, ..., \wedge g(a_i)] \wedge 0.5 \\
&= \left(\underset{a_1 \,\in\, H(m_1)}{Inf} g(a_1)\right) \wedge, ..., \wedge \left(\underset{a_i \,\in\, H(m_i)}{Inf} g(a_i)\right) \wedge 0.5 \\
&= \underset{i\in I}{Inf} [\underline{H}(g)(m_1) \wedge \underline{H}(g)(m_2)\wedge, ..., \wedge \underline{H}(g)(m_i)] \wedge 0.5 \\
&= [\underset{i\in I}{Inf} \underline{H}(g)(m_i)] \wedge 0.5.
\end{aligned}$$

Hence $\underline{H}(g)(\vee_{i\in I}m_i) \geq Inf[(\underset{i\in I}{Inf} \underline{H}(g)(m_i)), 0.5]$ for all $m_i \in Q_t$.

As H is a $SSVH$ of Q_t-modules, so $H(q * m) = q *' H(m)$.

Consider,

$$\begin{aligned}\underline{H}(g)(q*m) &= \underset{e\,\in\,H(q*m)}{Inf}\; g(e)\\ &= \underset{e\in q*'H(m)}{Inf}\; g(e).\end{aligned}$$

As $e \in q *' H(m)$, so there exists $a \in H(m)$ such that $e = q *' a$.

Hence,

$$\begin{aligned}\underline{H}(g)(q*m) &= \underset{q*'a\in q*'H(m)}{Inf}\; g(q*'a)\\ &\geq \underset{q*'a\in q*'H(m)}{Inf}\; [g(a)\wedge 0.5] \text{ by Lemma 4.5.2}\\ &= \underset{a\in H(m)}{Inf}\; g(a)\wedge 0.5\\ &= \underline{H}(g)(m)\wedge 0.5.\end{aligned}$$

Hence $\underline{H}(g)(q*m) \geq Inf[\underline{H}(g)(z), 0.5]$ for all $q \in Q_t$ and $m \in M$. Thus, $\underline{H}(g)$ is an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M. ■

Theorem 5.5.2 *Let g be an $(\in, \in \vee q)$-fuzzy Q_t-submodule of N and $H : M \longrightarrow P^*(N)$ be a SVH of Q_t-modules. Then $\overline{H}(g)$ is an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M.*

Proof. As $H : M \longrightarrow P^*(N)$ is a SVH of Q_t-modules, so $\vee_{i\in I}H(m_i) \subseteq H(\vee_{i\in I}m_i)$.

Consider,

$$\begin{aligned}Inf(\underset{i\in I}{Inf}\;\overline{H}(g)(m_i), 0.5) &= [\overline{H}(g)(m_1)\wedge \overline{H}(g)(m_2)\wedge, ..., \wedge\, \overline{H}(g)(m_i)]\wedge 0.5\\ &= \left[\left(\underset{a_1\,\in\,H(m_1)}{Sup}\; g(a_1)\right)\wedge, ..., \wedge\left(\underset{a_i\,\in\,H(m_i)}{Sup}\; g(a_i)\right)\right]\wedge 0.5\\ &= \underset{a_1\,\in\,H(m_1),\;a_2\,\in\,H(m_2),...,\;a_i\,\in\,H(m_i)}{Sup}\; [g(a_1)\wedge, ..., \wedge\, g(a_i)]\wedge 0.5\\ &= \underset{\vee_{i\in I}a_i\,\in\,\vee_{i\in I}H(m_i)}{Sup}\; [\wedge_{i\in I}g(a_i)\wedge 0.5]\\ &\leq \underset{\vee_{i\in I}a_i\,\in\,\vee_{i\in I}H(m_i)}{Sup}\; g(\vee_{i\in I}a_i)\\ &= \underset{e\,\in\,\vee_{i\in I}H(m_i)}{Sup}\; g(e)\\ &\leq \underset{e\,\in\,H(\vee_{i\in I}m_i)}{Sup}\; g(e)\\ &= \overline{H}(g)(\vee_{i\in I}m_i).\end{aligned}$$

Hence $\overline{H}(g)(\vee_{i\in I}m_i) \geq Inf(\underset{i\in I}{Inf}\;\overline{H}(g)(m_i), 0.5)$ for all $m_i \in Q_t$.

Similarly, it can be shown that $\overline{H}(g)(q * m) \geq Inf(\overline{H}(g)(m), 0.5)$ for all $q \in Q_t$ and $m \in M$. Thus, $\overline{H}(g)$ is an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M. ■

Chapter 6

$(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Ideals in Quantales

In the present chapter, we are presenting more general forms of $(\in, \in \vee q)$-$f$uzzy subquantale and $(\in, \in \vee q)$-$f$uzzy ideal of Quantales. We introduce the concepts of (α, β)-$f$uzzy subquantale, (α, β)-$f$uzzy ideal and some related properties are investigated. Special attention is given to $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy subquantale, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy prime, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy semi-prime ideals, and some interesting results about them are obtained. Furthermore, subquantale, prime, semi-prime and $f$uzzy subquantale, $f$uzzy prime, $f$uzzy semi-prime ideals of the types $(\in_\gamma, \in_\gamma \vee q_\delta)$ are linked by using level subsets.

In the first section, (α, β)-$f$uzzy subquantale and (α, β)-$f$uzzy ideal of Quantales are introduced and some related results are discussed. An $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy subquantale and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy ideal are presented in the second section. Relation between $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy subquantale, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy ideal and subquantale, ideal are also discussed in this section. In the third section, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy prime and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy semi-prime ideals are given. We also discuss the relationship between prime, semi-prime ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy prime, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy semi-prime ideal of Quantale. In the fourth and fifth sections, (α, β)-$f$uzzy Q_t-submodules and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy Q_t-submodules of Q_t-modules are introduced.

6.1 (α, β)-Fuzzy Subquantales (Ideals) of Quantale

In this section, we introduce some new relationships between $f$uzzy points and f-subsets, and investigate (α, β)-$f$uzzy subquantale and (α, β)-$f$uzzy ideal of Quantales.

Throughout the remaining paper $\gamma, \delta \in [0, 1]$, where $\gamma < \delta$ and $\alpha, \beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$. For a $f$uzzy point z_p and a f-subset g of Q_t, we say that

1. $z_p \in_\gamma g$ if $g(z) \geq p > \gamma$.
2. $z_p q_\delta g$ if $g(z) + p > 2\delta$.
3. $z_p(\in_\gamma \vee q_\delta)g$ if $z_p \in_\gamma g$ or $z_p q_\delta g$.
4. $z_p(\in_\gamma \wedge q_\delta)g$ if $z_p \in_\gamma g$ and $z_p q_\delta g$.
5. $z_p \overline{\alpha} g$ if $z_p \alpha g$ does not hold for $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$.

Note that the case when $\alpha = \in_\gamma \wedge q_\delta$ is omitted. Suppose that g is a f-subset of a quantale Q_t such that $g(z) \leq \delta$ for all $z \in Q_t$. Suppose $z \in Q_t$ and $p \in [0, 1]$ be such that $z_p(\in_\gamma \wedge q_\delta)g$. Then it follows that $g(z) \geq p > \gamma$ and $g(z) + p > 2\delta$. Hence, $2\delta < g(z) + p \leq g(z) + g(z) = 2g(z)$, that is $g(z) > \delta$. This means that $\{z_p : z_p(\in_\gamma \wedge q_\delta)g\} = \emptyset$. Therefore, we are not taking the case when $\alpha = \in_\gamma \wedge q_\delta$.

If we take $\gamma = 0$ and $\delta = 0.5$ then $\in_\gamma$ and q_δ becomes $\in$ and q as defined in Chapter 4.

From here onward, we will write (α, β)-FI, (α, β)-FS, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$ for (α, β)-$f$uzzy ideals, (α, β)-$f$uzzy subquantales, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy subquantale, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy prime ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy semi-prime ideal, respectively.

Definition 6.1.1 *A f-subset g of a quantale Q_t is called an (α, β)-FS of Q_t, if*

(F_1) $(z_i)_{p_i} \alpha g \longrightarrow (\vee_{i \in I} z_i)_{\underset{i \in I}{inf} p_i} \beta g$;

(F_2) $z_p \alpha g$, $w_v \alpha g \longrightarrow (z \otimes w)_{inf(p,v)} \beta g$ for all $z, w \in Q_t, \{z_i\} \subseteq Q_t$ $(i \in I)$, and $p_i \in (0, 1]$.

Theorem 6.1.2 *Let g be a non-zero (α, β)-FS of Q_t and $2\delta = 1 + \gamma$. Then $g_\gamma = \{y \in Q_t \mid g(y) > \gamma\}$ is a subquantale of Q_t.*

Proof. Let $y_i \in g_\gamma$ for $i \in I$. Then $g(y_i) > \gamma$ for all $i \in I$. Let $g(\vee_{i\in I} y_i) \leq \gamma$. If $\alpha \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$, then $(y_i)_{g(y_i)}\alpha g$ for all $i \in I$ but $g(\vee_{i\in I} y_i) \leq \gamma < \underset{i\in I}{inf}\ g(y_i)$ and $g(\vee_{i\in I} y_i) + \underset{i\in I}{inf}\ g(y_i) \leq \gamma + \underset{i\in I}{inf} g(y_i) \leq \gamma + 1 = 2\delta$. So $(\vee_{i\in I} y_i)_{\underset{i\in I}{inf} g(y_i)}\overline{\beta} g$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, a contradiction. Hence $g(\vee_{i\in I} y_i) > \gamma$, i.e., $\vee_{i\in I} y_i \in g_\gamma$. If $\alpha = q_\delta$ then $(y_i)_1 q_\delta g$ for all $i \in I$ because $g(y_i) + 1 > 1 + \gamma = 2\delta$, but $(\vee_{i\in I} y_i)_1 \overline{\beta} g$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, because $g(\vee_{i\in I} y_i) \leq \gamma$, so $(\vee_{i\in I} y_i)_1 \overline{\in}_\gamma g$ and $g(\vee_{i\in I} y_i) + 1 \leq \gamma + 1 = 2\delta$, so $(\vee_{i\in I} y_i)_1 \overline{q}_\delta g$. Hence $g(\vee_{i\in I} y_i) > \gamma$, that is $\vee_{i\in I} y_i \in g_\gamma$. Thus g_γ is closed under arbitrary join. The proof is similar for g_γ to be closed under $\otimes$. This shows that g_γ is a subquantale of Q_t. ■

Definition 6.1.3 *A f-subset g of a quantale Q_t is said to be an (α, β)-FLI (FRI) of Q_t, if*

(1) $z_p \alpha g, w_v \alpha g \longrightarrow (z \vee w)_{inf(p,v)} \beta g$;

(2) $z_v \alpha g$ *and* $w \leq z \longrightarrow w_v \beta g$;

(3) $z_v \alpha g, w \in Q_t \longrightarrow (w \otimes z)_v \beta g, ((z \otimes w)_v \beta g)$ *for all* $z, w \in Q_t$ *and* $p, v \in (0, 1]$.

A f-subset g of a quantale Q_t is called an (α, β)-FI of Q_t if it is both an (α, β)-FRI and (α, β)-FLI of Q_t.

Theorem 6.1.4 *Let $2\delta = 1 + \gamma$ and g be a non-zero (α, β)-FLI (FRI) of Q_t. Then $g_\gamma = \{y \in Q_t \mid g(y) > \gamma\}$ is a left (right) ideal of Q_t.*

Proof. Let g be a nonzero (α, β)-FLI of Q_t. Let $y, z \in g_\gamma$. Then $g(y) > \gamma$ and $g(z) > \gamma$. Let $\gamma \geq g(y \vee z)$. If $\alpha \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$, then $(y)_{g(y)}\alpha g$ and $(z)_{g(z)}\alpha g$ but $(y \vee z)_{inf(g(y),g(z))}\overline{\beta} g$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, (because $g(y \vee z) \leq \gamma < inf(g(y), g(z))$ so $(y \vee z)_{inf(g(y),g(z))}\overline{\in}_\gamma g$ and $g(y \vee z) + inf(g(y), g(z)) \leq \gamma + inf(g(y), g(z)) \leq \gamma + 1 = 2\delta$, so $(y \vee z)_{inf(g(y),g(z))}\overline{q}_\delta g)$, a contradiction. Hence $g(y \vee z) > \gamma$, that is $y \vee z \in g_\gamma$. If $\alpha = q_\delta$ then $y_1 q_\delta g$ and $z_1 q_\delta g$ (because $g(y) + 1 > 1 + \gamma = 2\delta$ and $g(z) + 1 > 1 + \gamma = 2\delta$) but $(y \vee z)_1 \overline{\beta} g$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, (because $g(y \vee z) \leq \gamma$, so $(y \vee z)_1 \overline{\in}_\gamma g$ and $g(y \vee z) + 1 \leq 1 + \gamma = 2\delta$), a contradiction. Hence $g(y \vee z) > \gamma$, that is $y \vee z \in g_\gamma$. Thus g_γ is closed under join.

Let $y, z \in Q_t$ and $y \leq z$. If $z \in g_\gamma$, then $g(z) > \gamma$. Assume that $g(y) \leq \gamma$. If $\alpha \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$, then $(z)_{g(z)}\alpha g$ but $(y)_{g(y)}\overline{\beta} g$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, a contradiction. Also $z_1 q_\delta g$ but $y_1 \overline{\beta} g$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, (because $g(y) \leq \gamma$ so $y_1 \overline{\in} g$ and $g(y) + 1 \leq \gamma + 1 = 2\delta$, so $y_1 \overline{q}_\delta g)$. Hence $g(y) > \gamma$, i.e., $y \in g_\gamma$.

Let $y \in g_\gamma$ and $z \in Q_t$. Then $g(y) > \gamma$. We want to show that $g(z \otimes y) > \gamma$. Suppose that $g(z \otimes y) \leq \gamma$ and let $\alpha \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$. Then $(y)_{g(y)} \alpha g$ but $(z \otimes y)_{g(y)} \overline{\beta} g$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, this is a contradiction again. Also $y_1 q_\delta g$ but $(z \otimes y)_1 \overline{\beta} g$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, a contradiction. Therefore $g(z \otimes y) > \gamma$ and so $z \otimes y \in g_\gamma$. Hence g_γ is a LI of the quantale Q_t. ■

Theorem 6.1.5 *Let $2\delta = 1 + \gamma$ and $\emptyset \neq C \subseteq Q_t$. Then C is a LI (RI) of Q_t if and only if the f-subset g of Q_t defined by*

$$g(w) = \begin{cases} \geq \delta \ if \ w \in C \\ \gamma \ otherwise \end{cases} \quad for \ all \ w \in Q_t.$$

is an $(\alpha, \in_\gamma \vee q_\delta)$-$FLI$ (FRI) of Q_t.

Proof. Let C be a LI of Q_t.

(**a**) Let $w, z \in Q_t$ and $p, v \in (\gamma, 1]$ be such that $w_p \in_\gamma g$ and $z_v \in_\gamma g$. Then $g(w) \geq p > \gamma$ and $g(z) \geq v > \gamma$. Hence $g(w) \geq \delta$ and $g(z) \geq \delta$. Thus $w, z \in C$ and so $w \vee z \in C$, that is $g(w \vee z) \geq \delta$. If $inf\{p, v\} \leq \delta$, then $g(w \vee z) \geq \delta \geq inf\{p, v\} > \gamma$. Hence $(w \vee z)_{inf(p,v)} \in_\gamma g$. If $inf\{p, v\} > \delta$, then $g(w \vee z) + inf\{p, v\} > \delta + \delta = 2\delta$ and so $(w \vee z)_{inf(p,v)} q_\delta g$. Therefore $(w \vee z)_{inf(p,v)} (\in_\gamma \vee q_\delta) g$.

Let $w, z \in Q_t$, $w \leq z$ and $v \in (\gamma, 1]$ be such that $z_v \in_\gamma g$. Then $g(z) \geq v > \gamma$. Thus $z \in C$ and since C is a LI so $w \in C$, that is $g(w) \geq \delta$. If $v \leq \delta$, then $g(w) \geq \delta \geq v > \gamma$. Hence $w_v \in_\gamma g$. If $v > \delta$, then $g(w) + v > \delta + \delta = 2\delta$ and so $w_v q_\delta g$. It follows that $w_v (\in_\gamma \vee q_\delta) g$.

Now let $w, z \in Q_t$ and $p \in (\gamma, 1]$ be such that $w_p \in_\gamma g$. Then $g(w) \geq p > \gamma$, which implies $w \in C$, and so $z \otimes w \in C$, for all $z \in Q_t$. Consequently $g(z \otimes w) \geq \delta$. If $p \leq \delta$, then $g(z \otimes w) \geq \delta \geq p > \gamma$. Hence $(z \otimes w)_p \in_\gamma g$. If $p > \delta$, then $g(z \otimes w) + p > \delta + \delta = 2\delta$ and so $(z \otimes w)_p q_\delta g$. Thus $(z \otimes w)_p (\in_\gamma \vee q_\delta) g$. Hence g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FLI$ of Q_t.

(**b**) Let $w, z \in Q_t$ and $p, v \in (\gamma, 1]$ be such that $w_p q_\delta g$ and $z_v q_\delta g$. Since, $g(w) + p > 2\delta$ and $g(z) + v > 2\delta$, and so $g(w) > 2\delta - p \geq 2\delta - 1 = \gamma$ and $g(z) > 2\delta - v \geq 2\delta - 1 = \gamma$, it follows that $g(w) > \gamma$ and $g(z) > \gamma$, i.e., $w, z \in C$. Since C is a LI so $w \vee z \in C$, hence we have $g(w \vee z) \geq \delta$. If $inf\{p, v\} \leq \delta$, then $g(w \vee z) \geq \delta \geq inf\{p, v\} > \gamma$. Hence $(w \vee z)_{inf(p,v)} \in_\gamma g$. If $inf\{p, v\} > \delta$, then $g(w \vee z) + inf\{p, v\} > \delta + \delta = 2\delta$ and so $(w \vee z)_{inf(p,v)} q_\delta g$. Therefore $(w \vee z)_{inf(p,v)} (\in_\gamma \vee q_\delta) g$.

Let $w, z \in Q_t$, $w \leq z$ and $v \in (\gamma, 1]$ be such that $z_v q_\delta g$. Then $g(z) + v > 2\delta$ so $g(z) > 2\delta - v \geq 2\delta - 1 = \gamma$. Thus $z \in C$ and since C is a LI so $w \in C$, that is $g(w) \geq \delta$. If $v \leq \delta$, then $g(w) \geq \delta \geq v > \gamma$. Hence $w_v \in_\gamma g$. If $v > \delta$, then $g(w) + v > \delta + \delta = 2\delta$ and so $w_v q_\delta g$. It follows that $w_v(\in_\gamma \vee q_\delta)g$.

Now, let $w, z \in Q_t$ and $p \in (\gamma, 1]$ be such that $w_p q_\delta g$, which implies that $g(w)+p > 2\delta$. Thus $w \in C$ and so $z \otimes w$ is in C. This means that $g(z \otimes w) \geq \delta$. If $p \leq \delta$, then $g(z \otimes w) \geq \delta \geq p > \gamma$. Hence $(z \otimes w)_p \in_\gamma g$. If $p > \delta$, then $g(z \otimes w) + p > \delta + \delta = 2\delta$ and so $(z \otimes w)_p q_\delta g$. Thus $(z \otimes w)_p(\in_\gamma \vee q_\delta)g$. Hence g is $(q_\delta, \in_\gamma \vee q_\delta)$-$FLI$ of Q_t.

(**c**) Let $w, z \in Q_t$ and $p, v \in (\gamma, 1]$ be such that $w_p \in_\gamma g$ and $z_v q_\delta g$. Then $g(w) \geq p > \gamma$ and $g(z) + v > 2\delta$. Thus, $w,\ z \in C$, implies that $w \vee z \in C$. Hence $g(w \vee z) \geq \delta$. In a similar way we obtain $(w \vee z)_{inf(p,v)} \in_\gamma g$ for $inf\{p, v\} \leq \delta$ and $(w \vee z)_{inf(p,v)} q_\delta g$ for $inf\{p, v\} > \delta$. Thus $(w \vee z)_{inf(p,v)}(\in_\gamma \vee q_\delta)g$. The rest is similar to the proof of parts (a) and (b).

Conversely, suppose that g is an $(\alpha, \in_\gamma \vee q_\delta)$-$FLI$ of Q_t. It is easy to prove that $C = g_\gamma$. Hence, from Theorem 6.1.4, C is a LI of Q_t. ■

The proof of the following Theorem can be obtained in a similar way.

Theorem 6.1.6 *Let $2\delta = 1 + \gamma$ and $\emptyset \neq C \subseteq Q_t$. Then C is a subquantale of Q_t if and only if the f-subset g of Q_t defined by*

$$g(w) = \begin{cases} \geq \delta \ if \ w \in C \\ \gamma \ otherwise \end{cases} \quad for\ all\ w \in Q_t.$$

is an $(\alpha, \in_\gamma \vee q_\delta)$-$FS$ of Q_t.

6.2 $(\in_\gamma, \in_\gamma \vee q_\delta)$- Fuzzy Suquantales (Ideals) of Quantale

In this section, we present an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of quantale Q_t and discuss some of their properties.

Definition 6.2.1 *A f-subset g of Q_t is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t, if*

(F_1) $(z_i)_{p_i} \in_\gamma g \longrightarrow (\vee_{i \in I} z_i)_{\underset{i \in I}{inf} p_i}(\in_\gamma \vee q_\delta)g$;

(F_2) *$z_p \in_\gamma g$ and $w_v \in_\gamma g \longrightarrow (z \otimes w)_{inf(p,v)}(\in_\gamma \vee q_\delta)g$ for all $\{z_i\} \subseteq Q_t$ $(i \in I)$, $z, w \in Q_t$ and $p_i, p, v \in (\gamma, 1]$.*

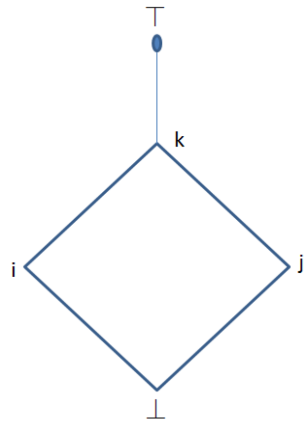

Fig. 12

Table 9.

$\otimes$	$\bot$	i	j	k	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
i	$\bot$	i	$\bot$	i	i
j	$\bot$	$\bot$	j	j	j
k	$\bot$	i	j	k	k
$\top$	$\top$	i	j	k	$\top$

Example 6.2.2 *Let $(Q_t, \otimes)$ be a quantale, where Q_t is delineated in Fig.12 and the binary operation $\otimes$ on Q_t is shown in the Table 9. Taking $g = \frac{0.9}{\bot} + \frac{0.5}{i} + \frac{0.5}{j} + \frac{0.5}{k} + \frac{0.6}{\top}$. Then by routine calculations g is an $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-$FS$ of Q_t.*

Theorem 6.2.3 *Let g be a f-subset of Q_t. If g is an $(q_\delta, \in_\gamma \vee q_\delta)$-$FS$ of Q_t, then conditions below hold:*

(1) $sup\ \{g(\vee_{i\in I} z_i), \gamma\} \geq inf\ \{\underset{i\in I}{inf} g(z_i), \delta\})$

(2) $sup\ \{g(z \otimes y), \gamma\} \geq inf\ \{g(z), g(y), \delta\}$ *for all* $\{z_i\} \subseteq Q_t$ $(i \in I)$ *and* $z, y \in Q_t$.

Proof. Let g be a $(q_\delta, \in_\gamma \vee q_\delta)$-$FS$ of Q_t. Assume that there exist $z_i \in Q_t$ such that $sup\{g(\vee_{i\in I} z_i), \gamma\} < inf\ \{\underset{i\in I}{inf} g(z_i), \delta\}$. Then for all $\gamma < v \leq 1$

such that

$$2\delta - sup\{g(\vee_{i\in I} z_i), \gamma\} > v \geq 2\delta - inf\{\underset{i\in I}{inf} g(z_i), \delta\}$$

and so

$$2\delta - g(\vee_{i\in I} z_i) \geq 2\delta - sup\{g(\vee_{i\in I} z_i), \gamma\} > v \geq sup\{2\delta - \underset{i\in I}{inf} g(z_i), \delta\}$$

That is, $2\delta - g(\vee_{i\in I} z_i) > v$, $2\delta - \underset{i\in I}{inf} g(z_i) < v$.

Thus,

$$\underset{i\in I}{inf} g(z_i) + v > 2\delta, \ g(\vee_{i\in I} z_i) + v < 2\delta$$

and $g(\vee_{i\in I} z_i) < \delta < v$. Hence $(z_i)_v q_\delta g$ for all $i \in I$, but $(\vee_{i\in I} z_i)_v \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Therefore $sup\{g(\vee_{i\in I} z_i), \gamma\} \geq inf \ \{\underset{i\in I}{inf} g(z_i), \delta\}$.

Let there exist $z, y \in Q_t$ be such that $sup\{g(z \otimes y), \gamma\} < inf \ \{g(z), g(y), \delta\}$. Then for all $\gamma < t \leq 1$ such that

$$2\delta - sup\{g(z \otimes y), \gamma\} > t \geq 2\delta - inf\{g(z), g(y), \delta\}$$

we have

$$2\delta - g(z \otimes y) \geq 2\delta - sup\{g(z \otimes y), \gamma\} > t \geq sup\{2\delta - g(z), 2\delta - g(y), \delta\}$$

That is, $2\delta - g(z) < t$, $2\delta - g(w) < t$, $2\delta - g(z \otimes y) > t$.

and so

$$g(z) + t > 2\delta, \ g(y) + t > 2\delta, \ g(z \otimes y) + t < 2\delta$$

and $g(z \otimes y) < \delta < t$. Hence $z_t q_\delta g$, $y_t q_\delta g$ but $(z \otimes y)_t \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Therefore, $sup\{g(z \otimes y), \gamma\} \geq inf\{g(z), g(y), \delta\}$ for all $z, y \in Q_t$. ■

Theorem 6.2.4 *A f-subset g of Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t if and only if the conditions below hold:*

(1) *$sup \ \{g(\vee_{i\in I} z_i), \gamma\} \geq inf\{\underset{i\in I}{inf} g(z_i), \delta\}$;*

(2) *$sup \ \{g(z \otimes y), \gamma\} \geq inf\{g(z), g(y), \delta\}$ for all $\{z_i\} \subseteq Q_t$ $(i \in I)$ and $z, y \in Q_t$.*

Proof. Let g be a $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t. Let there exist $z_i \in Q_t$ and $v \in (\gamma, \delta]$ such that $sup\{g(\vee_{i\in I} z_i), \gamma\} < v \leq inf\{\underset{i\in I}{inf} g(z_i), \delta\}$. Then $g(z_i) \geq v > \gamma$ for all $i \in I$, $g(\vee_{i\in I} z_i) < v$ and $g(\vee_{i\in I} z_i) + v < 2v \leq 2\delta$, *i.e.*, $(z_i)_v \in_\gamma g$ for all $i \in I$ but $(\vee_{i\in I} z_i)_v \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Thus, $sup\{g(\vee_{i\in I} z_i), \gamma\} \geq inf\ \{\underset{i\in I}{inf} g(z_i), \delta\}$ for all $z_i \in Q_t$. Let $z, y \in Q_t$ and $v \in (\gamma, \delta]$ be such that $sup\{g(z \otimes y), \gamma\} < v \leq inf\ \{g(z), g(y), \delta\}$. Then $g(z) \geq v > \gamma$, $g(y) \geq v > \gamma$, $g(z \otimes y) < v$ and $g(z \otimes y) + v < 2v \leq 2\delta$, *i.e.*, $z_v \in_\gamma g$, $y_v \in_\gamma g$ but $(z \otimes y)_v \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Thus, $sup\{g(z \otimes y), \gamma\} \geq inf\ \{g(z), g(y), \delta\}$ for all $z, y \in Q_t$.

Conversely, suppose that the above two conditions are true. We show that g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t. Let $z_i \in Q_t$ and $v_i \in (\gamma, \delta]$ be such that $(z_i)_{v_i} \in_\gamma g$ but $(\vee_{i\ \in\ I} z_i)_{\underset{i\in I}{inf} v_i} \overline{(\in_\gamma \vee q_\delta)} g$. Then $g(z_i) \geq v_i$ for all $i \in I$, $g(\vee_{i\in I} z_i) < \underset{i\in I}{inf} v_i$ and $g(\vee_{i\in I} z_i) + \underset{i\in I}{inf} v_i \leq 2\delta$. It follows that $g(\vee_{i\in I} z_i) < \delta$ and so $sup\ \{g(\vee_{i\in I} z_i), \gamma\} < inf\ \{\underset{i\in I}{inf} g(z_i), \delta\}$, a contradiction. Hence $(\vee_{i\ \in\ I} z_i)_{\underset{i\in I}{inf} p_i} (\in_\gamma \vee q_\delta) g$. Similarly, it can be shown that if $z_p \in_\gamma g$, and $w_v \in_\gamma g$ then $g(z \otimes w)_{inf(p,v)} (\in_\gamma \vee q_\delta) g$. ■

Proposition 6.2.5 *Let g_1 and g_2 be $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$'s of Q_t. Then, $(g_1 \Cap g_2)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t.*

Proof. Let $z_i \in Q_t$ for some $i \in I$ and $\gamma, \delta \in (0, 1]$ with $\gamma < \delta$. Since g_1 and g_2 are $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t, so, $sup\{g_1(\vee_{i\in I} z_i), \gamma\} \geq inf\{\underset{i\in I}{inf} g_1(z_i), \delta\}$ and $sup\{g_2(\vee_{i\in I} z_i), \gamma\} \geq inf\{\underset{i\in I}{inf} g_2(z_i), \delta\}$

Now, consider

$$\begin{aligned} sup\{(g_1 \Cap g_2)(\vee_{i\in I} z_i), \gamma\} &= sup\{g_1(\vee_{i\in I} z_i) \wedge g_2(\vee_{i\in I} z_i), \gamma\} \\ &= sup\{g_1(\vee_{i\in I} z_i), \gamma\} \wedge sup\{g_2(\vee_{i\in I} z_i), \gamma\} \\ &\geq inf\{\underset{i\in I}{inf} g_1(z_i), \delta\} \wedge inf\{\underset{i\in I}{inf} g_2(z_i), \delta\} \\ &= inf\{\underset{i\in I}{inf}(g_1(z_i) \wedge g_2(z_i)), \delta\} \end{aligned}$$

That is, $sup\{(g_1 \Cap g_2)(\vee_{i\in I} z_i), \gamma\} \geq inf\{\underset{i\in I}{inf}(g_1 \Cap g_2)(z_i), \delta\}$

Next, as $sup\{g_1(z_1 \otimes z_2), \gamma\} \geq inf\{g_1(z_1), g_1(z_2), \delta\}$ and

$$sup\{g_2(z_1 \otimes z_2), \gamma\} \geq inf\{g_2(z_1), g_2(z_2), \delta\}$$

Now, consider

$$\begin{aligned} sup\{(g_1 \Cap g_2)(z_1 \otimes z_2), \gamma\} = &\ sup\{g_1(z_1 \otimes z_2) \wedge g_2(z_1 \otimes z_2), \gamma\} \\ = &\ sup\{g_1(z_1 \otimes z_2), \gamma\} \wedge sup\{g_2(z_1 \otimes z_2), \gamma\} \\ \geq &\ inf\{g_1(z_1), g_1(z_2), \delta\} \wedge inf\{g_2(z_1), g_2(z_2), \delta\} \\ = &\ inf\{g_1(z_1) \wedge g_2(z_1), g_1(z_2) \wedge g_2(z_2), \delta\} \end{aligned}$$

Hence, $sup\{(g_1 \Cap g_2)(z_1 \otimes z_2), \gamma\} \geq inf\{(g_1 \Cap g_2)(z_1),\ (g_1 \Cap g_2)(z_2), \delta\}$

Therefore, $g_1 \Cap g_2$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t by Theorem 6.2.4. ■

The following Propositions are obvious.

Proposition 6.2.6 *Every* $((\in_\gamma \vee q_\delta), \in_\gamma \vee q_\delta))$-$FS$ *of* Q_t *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ *of* Q_t.

Proposition 6.2.7 *Every* $(\in_\gamma, \in_\gamma)$-FS *of* Q_t *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ *of* Q_t.

The Example below demonstrates that the converses of Propositions 6.2.6 and 6.2.7 may not be true in general.

Example 6.2.8 *Consider the quantale* Q_t *as defined in Example* 6.2.2 *and taking* $g = \frac{0.9}{\perp} + \frac{0.7}{i} + \frac{0.65}{j} + \frac{0.54}{k} + \frac{0.31}{\top}$. *Then*

(1) *It is easy to verify that* g *is an* $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$-$FS$ *of* Q_t.

(2) g *is not an* $(\in_{0.3}, \in_{0.3})$-FS *of* Q_t, *since* $i_{0.68} \in_{0.3} g$ *and* $j_{0.61} \in_{0.3} g$ *but* $(i \vee j)_{inf(0.68,0.61)} = k_{0.61} \overline{\in}_{0.3} g$.

(3) g *is not an* $(\in_{0.3} \vee q_{0.6}, \in_{0.3} \vee q_{0.6})$-$FS$ *of* Q_t, *since* $i_{0.68}(\in_{0.3} \vee q_{0.6})g$ *and* $j_{0.59}(\in_{0.3} \vee q_{0.6})g$ *but* $(i \vee j)_{inf(0.68,0.59)} = k_{0.59}\overline{(\in_{0.3} \vee q_{0.6})}g$.

Definition 6.2.9 *A* f*-subset* g *of a quantale* Q_t *is said to be an* $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FLI$ (FRI) *of* Q_t, *if*

(F_3) $z_p \in_\gamma g, w_v \in_\gamma g \longrightarrow (z \vee w)_{inf(p,v)}(\in_\gamma \vee q_\delta)g$;

(F_4) $z_v \in_\gamma g$ *and* $w \leq z \longrightarrow w_v(\in_\gamma \vee q_\delta)g$;

(F_5) $z_v \in_\gamma g, w \in Q_t \longrightarrow (w \otimes z)_v(\in_\gamma \vee q_\delta)g)$, $((z \otimes w)_p(\in_\gamma \vee q_\delta)g)$ *for all* $z, w \in Q_t$ *and* $p, v \in (\gamma, 1]$.

A f*-subset* g *of a quantale* Q_t *is called an* $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ *of* Q_t *if it is both an* $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FRI$ *and* $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FLI$ *of* Q_t.

Theorem 6.2.10 *Let g be a f-subset of Q_t and g be an $(q_\delta, \in_\gamma \vee q_\delta)$-$FLI$ (FRI) of Q_t. Then the conditions below are satisfied:*

(1) $sup\{g(z \vee w), \gamma\} \geq inf\ \{g(z), g(w), \delta\}$;

(2) $sup\{g(w), \gamma\} \geq inf\ \{g(z), \delta\}$ *with* $w \leq z$;

(3) $sup\{g(w \otimes z), \gamma\} \geq inf\ \{g(z), \delta\})$, $(sup\ \{g(z \otimes w), \gamma\} \geq inf\ \{g(z), \delta\})$ *for all* $z, w \in Q_t$.

Proof. If there exist $z, w \in Q_t$ such that $sup\{g(z \vee w), \gamma\} < inf\ \{g(z), g(w), \delta\}$. Then for all $\gamma < v \leq 1$ such that

$$2\delta - sup\{g(z \vee w), \gamma\} > v \geq 2\delta - inf\{g(z), g(w), \delta\}$$

Thus, we have

$$2\delta - g(z \vee w) \geq 2\delta - sup\{g(z \vee w), \gamma\} > v \geq sup\{2\delta - g(z), 2\delta - g(w), \delta\}$$

That is, $2\delta - g(z) < v$, $2\delta - g(w) < v$, $2\delta - sup\{g(z \vee w) > v$.

and so,

$$g(z) + v > 2\delta,\ g(w) + v > 2\delta,\ g(z \vee w) + v < 2\delta$$

and $g(z \vee w) < \delta < v$. Hence $w_v q_\delta g$, $z_v q_\delta g$ but $(z \vee w)_v \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Therefore

$$sup\{g(z \vee w), \gamma\} \geq inf\{g(z), g(w), \delta\} \text{ for all } z, y \in Q_t.$$

Let $z, y \in Q_t$ be such that $sup\{g(w \otimes z), \gamma\} < inf\{g(z), \delta\}$. Then for all $\gamma < p \leq 1$ such that

$$2\delta - sup\{g(w \otimes z), \gamma\} > p \geq 2\delta - inf\{g(z), \delta\}$$

we have

$$2\delta - g(w \otimes z) \geq 2\delta - sup\{g(w \otimes z), \gamma\} > p \geq sup\{2\delta - g(z), \delta\}$$

That is, $2\delta - g(z) < p$, $2\delta - g(w \otimes z) > p$.

and so

$$g(z) + p > 2\delta,\ g(w \otimes z) + p < 2\delta$$

and $g(w \otimes z) < \delta < p$. Hence $z_p q_\delta g$ but $(w \otimes z)_p \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Therefore $sup\{g(w \otimes z), \gamma\} \geq inf\{g(z), \delta\}$ for all $z, y \in Q_t$. Similarly, we can prove that sup $\{g(w), \gamma\} \geq inf$ $\{g(z), \delta\}$ with $w \leq z$ for all $z, y \in Q_t$. ∎

Theorem 6.2.11 *A f-subset g of Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-FRI (FLI) of Q_t if and only if the conditions below are satisfied:*

(1) sup $\{g(z \vee w), \gamma\} \geq inf$ $\{g(z), g(w), \delta\}$;

(2) sup $\{g(w), \gamma\} \geq inf$ $\{g(z), \delta\}$ *with* $w \leq z$;

(3) sup $\{g(w \otimes z), \gamma\} \geq inf$ $\{g(z), \delta\})$, $(sup$ $\{g(z \otimes w), \gamma\} \geq inf$ $\{g(z), \delta\})$, *for all* $z, w \in Q_t$.

Proof. $(\mathbf{F}_3) \Longrightarrow (\mathbf{1})$. If there exist $z, w \in Q_t$ such that sup $\{g(z \vee w), \gamma\} < v \leq inf$ $\{g(z), g(w), \delta\}$ for some $v \in (\gamma, \delta]$. Then $g(z) \geq v > \gamma$, $g(w) \geq v > \gamma$, $g(z \vee w) < v$ and $g(z \vee w) + v < 2v \leq 2\delta$, i.e., $z_v \in_\gamma g$, $w_v \in_\gamma g$ but $(z \vee w)_v \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Hence $sup\{g(z \vee w), \gamma\} \geq inf\{g(z), g(w), \delta\}$ for all $z, w \in Q_t$.

$(\mathbf{1}) \Longrightarrow (\mathbf{F}_3)$. Let there be $z, w \in Q_t$ and $s, t \in (\gamma, \delta]$ be such that $z_s \in_\gamma g$ and $w_t \in_\gamma g$ but $(z \vee w)_{inf(s,t)} \overline{(\in_\gamma \vee q_\delta)} g$, then $g(z) \geq s > \gamma$, $g(w) \geq t > \gamma$, $g(z \vee w) < inf\{s, t\}$ and $g(z \vee w) + inf\{s, t\} \leq 2\delta$. Thus, we have $g(z \vee w) < \delta$ and so $sup\{g(z \vee w), \gamma\} < inf\{g(z), g(w), \delta\}$, a contradiction. Hence (F_3) is valid.

$(\mathbf{F}_4) \Longrightarrow (\mathbf{2})$. If there exist $z, w \in Q_t$ with $w \leq z$ such that $sup\{g(w), \gamma\} < p \leq inf$ $\{g(z), \delta\}$ for some $p \in (\gamma, \delta]$. Then $g(z) \geq p > \gamma$, $g(w) < p$ and $g(w) + p < 2p \leq 2\delta$, i.e., $z_p \in_\gamma g$ but $w_p \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Hence (2) is valid.

$(\mathbf{2}) \Longrightarrow (\mathbf{F}_4)$. Assume that there exist $z, w \in Q_t$ with $w \leq z$ and $v \in (\gamma, \delta]$ such that $z_p \in_\gamma g$ but $w_p \overline{(\in_\gamma \vee q_\delta)} g$, then $g(z) \geq p > \gamma$, $g(w) < p$ and $g(w) + p \leq 2\delta$. It follows that $g(w) < \delta$ and hence, $sup\{g(w), \gamma\} < inf\{g(z), \delta\}$, a contradiction.

$(\mathbf{F}_5) \Longrightarrow (\mathbf{3})$. If there exist $z, y \in Q_t$ such that sup $\{g(w \otimes z), \gamma\} < v \leq inf$ $\{g(z), \delta\}$. Then $g(z) \geq v > \gamma$, $g(w \otimes z) < v$ and $g(w \otimes z) + v < 2v \leq 2\delta$, i.e., $z_v \in_\gamma g$ but $(w \otimes z)_v \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Hence sup $\{g(w \otimes z), \gamma\} \geq inf$ $\{g(z), \delta\}$ for all $z, y \in Q_t$.

$(\mathbf{3}) \Longrightarrow (\mathbf{F}_5)$. Let there be $z, y \in Q_t$ and $s \in (\gamma, \delta]$ be such that $z_s \in_\gamma g$ but $(w \otimes z)_s \overline{(\in_\gamma \vee q_\delta)} g$. Then $g(z) \geq s > \gamma$, $g(w \otimes z) < s$ and $g(w \otimes z) + s \leq 2\delta$. This shows $g(w \otimes z) < \delta$ and so $sup\{g(w \otimes z), \gamma\} < inf$ $\{g(z), \delta\}$, a contradiction. Hence (F_5) is valid. ∎

Proposition 6.2.12 *If g_1 and g_2 are $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FRI$ (FLI) of Q_t, then, $g_1 \Cap g_2$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FRI$ (FLI) of Q_t.*

Proof. Let $z, y \in Q_t$ and $\gamma, \delta \in (0, 1]$ with $\gamma < \delta$. Since g_1 and g_2 are $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FRI$ of Q_t, so by Theorem 6.2.11, we have $sup\{g_1(z), \gamma\} \geq inf\{g_1(y), \delta\}$ and $sup\{g_2(z), \gamma\} \geq inf\{g_2(y), \delta\}$ with $z \leq y$.

Now, consider

$$\begin{aligned} sup\{(g_1 \Cap g_2)(z), \gamma\} &= sup\{g_1(z) \wedge g_2(z), \gamma\} \\ &= sup\{g_1(z), \gamma\} \wedge sup\{g_2(z), \gamma\} \\ &\geq inf\{g_1(y), \delta\} \wedge inf\{g_2(y), \delta\} \\ &= inf\{g_1(y) \wedge g_2(y), \delta\}. \end{aligned}$$

That is, $sup\{(g_1 \Cap g_2)(z), \gamma\} \geq inf\{(g_1 \Cap g_2)(y), \delta\}$.

Next, as g_1 and g_2 are $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FRI$ of Q_t, so we have

$$sup\{g_1(z \vee w), \gamma\} \geq inf\{g_1(z), g_1(w), \delta\}$$

and

$$sup\{g_2(z \vee w), \gamma\} \geq inf\{g_2(z), g_2(w), \delta\}.$$

Now, consider

$$\begin{aligned} sup\{(g_1 \Cap g_2)(z \vee w), \gamma\} &= sup\{g_1(z \vee w) \wedge g_2(z \vee w), \gamma\} \\ &= sup\{g_1(z \vee w), \gamma\} \wedge sup\{g_2(z \vee w), \gamma\} \\ &\geq inf\{g_1(z), g_1(w), \delta\} \wedge inf\{g_2(z), g_2(w), \delta\} \\ &= inf\{g_1(z) \wedge g_2(z), g_1(w) \wedge g_2(w), \delta\}. \end{aligned}$$

Hence, $sup\{(g_1 \Cap g_2)(z \vee w), \gamma\} \geq inf\{(g_1 \Cap g_2)(z), (g_1 \Cap g_2)(w), \delta\}$. Similarly, we can show that $sup\{(g_1 \Cap g_2)(z \otimes w), \gamma\} \geq inf\{(g_1 \Cap g_2)(z), \delta\}$ for all $z, w \in Q_t$. Therefore, $g_1 \Cap g_2$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FRI$ of Q_t by Theorem 6.2.11. ■

For any $g \in \mathcal{F}(Q_t)$, where $\mathcal{F}(Q_t)$ denotes the set of all f-subsets of Q_t, we define

$$g_v = \{y \in Q_t \mid y_v \in_\gamma g\} \text{ for all } v \in (\gamma, 1];$$

$$g_v^\delta = \{y \in Q_t \mid y_v q_\delta g\} \text{ for all } v \in (\gamma, 1];$$

and

$$[g]_v^\delta = \{y \in Q_t \mid y_v(\in_\gamma \vee q_\delta)g\} \ for \text{ all } v \in (\gamma, 1].$$

It follows that $[g]_v^\delta = g_v \cup g_v^\delta$.

The following Theorem gives the relation between $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ and crisp subquantale of Q_t.

Theorem 6.2.13 *For any f-subset g of quantale Q_t, the following are equivalent:*

(F_6) *g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t;*

(F_7) *$g_v(\neq \emptyset)$ is a subquantale of Q_t for all $v \in (\gamma, \delta]$.*

Proof. $(\mathbf{F}_6) \Longrightarrow (\mathbf{F}_7)$. Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t. Let $z_i \in Q_t$ and $v \in (\gamma, \delta]$ be such that $z_i \in g_v$ for all $i \in I$. Then $(z_i)_v \in_\gamma g$ for all $i \in I$ and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t, therefore $(\vee_{i \in I} z_i)_v(\in_\gamma \vee q_\delta)g$. If $(\vee_{i \in I} z_i)_v \in_\gamma g$, then $\vee_{i \in I} z_i \in g_v$ and if $(\vee_{i \in I} z_i)_v q_\delta g$, then $g(\vee_{i \in I} z_i) > 2\delta - v \geq v > \gamma$; that is, $\vee_{i \in I} z_i \in g_v$. Let $x, z \in Q_t$ be such that $x, z \in g_v$ for some $v \in (\gamma, \delta]$. Then $z_v \in_\gamma g$ and $x_v \in_\gamma g$, and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t, therefore $(z \otimes x)_v(\in_\gamma \vee q_\delta)g$. If $(z \otimes x)_v \in_\gamma g$, then $z \otimes x \in g_v$ and if $(z \otimes x)_v q_\delta g$, then $g(z \otimes x) > 2\delta - v \geq v > \gamma$; that is, $z \otimes x \in g_v$. Therefore g_v is a subquantale of Q_t.

$(\mathbf{F}_7) \Longrightarrow (\mathbf{F}_6)$. Assume that $\emptyset \neq g_v$ is a subquantale of Q_t for all $v \in (\gamma, \delta]$. Suppose that there exist $z_i \in Q_t$ for $i \in I$ such that $sup\{g(\vee_{i\in I} z_i), \gamma\} < inf\{\underset{i\in I}{inf} g(z_i), \delta\}$; then there exist $v \in (\gamma, \delta]$ such that $sup\ \{g(\vee_{i\in I} z_i), \gamma\} < v \leq inf\ \{\underset{i\in I}{inf} g(z_i), \delta\}$. This shows that $(z_i)_v \in_\gamma g$ for all $i \in I$; that is, $z_i \in g_v$ for all $i \in I$ but $(\vee_{i\in I} z_i) \notin g_v$, a contradiction. Therefore, $sup\{g(\vee_{i\in I} z_i), \gamma\} \geq inf\ \{\underset{i\in I}{inf} g(z_i), \delta\}$ for all $z_i \in Q_t, (i \in I)$. Let $z, y \in Q_t$ and $sup\{g(z \otimes y), \gamma\} < inf\{g(z), g(y), \delta\}$. Then $sup\{g(z \otimes y), \gamma\} < v \leq inf\ \{g(z), g(y), \delta\}$ for some $v \in (\gamma, \delta]$. This implies that $z \in g_v$ and $y \in g_v$ but $(z \otimes y) \notin g_v$, a contradiction. Therefore, $\{g(z \otimes y), \gamma\} \geq inf\ \{g(z), g(y), \delta\}$. By Theorem 6.2.4, g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t. ■

Theorem 6.2.14 *Let $2\delta = 1 + \gamma$. Then a f-subset g of Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ if and only if $\emptyset \neq g_v^\delta$ is a subquantale of Q_t for all $v \in (\delta, 1]$.*

Proof. Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t. Let $z_i \in Q_t$ for all $i \in I$ and $v \in (\delta, 1]$ be such that $z_i \in g_v^\delta$ for all $i \in I$. Then $(z_i)_v q_\delta g$ for all $i \in I$; that is $g(z_i) > 2\delta - v \geq$

$2\delta - 1 = \gamma$. Thus, $g(z_i) > \gamma$. Since $v \in (\delta, 1]$, we have $2\delta - v < \delta < v$. By hypothesis, we have,

$$\begin{aligned} sup\{g(\vee_{i\in I} z_i), \gamma\} &\geq inf\{\underset{i\in I}{inf} g(z_i),\ \delta\}; \\ g(\vee_{i\in I} z_i) &> inf\{2\delta - v,\ \delta\}; \\ &= 2\delta - v. \end{aligned}$$

that is, $g(\vee_{i\in I} z_i) > 2\delta - v$. Hence $\vee_{i\in I} z_i \in g_v^\delta$.

Let $w, z \in Q_t$ be such that $w, z \in g_v^\delta$ for some $v \in (\delta, 1]$. Then $z_v q_\delta g$ and $w_v q_\delta g$, that is $g(z) > 2\delta - v \geq 2\delta - 1 = \gamma$, $g(w) > 2\delta - v \geq 2\delta - 1 = \gamma$ and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t, therefore,

$$\begin{aligned} sup\{g(z \otimes w), \gamma\} \geq\ & inf\{g(z), g(w), \delta\} \\ >\ & inf\{2\delta - v, 2\delta - v, \delta\} \\ =\ & 2\delta - v; \end{aligned}$$

that is, $g(z \otimes w) > 2\delta - v$. Hence $z \otimes w \in g_v^\delta$. So, g_v^δ is a subquantale of Q_t.

Conversely, assume that $\emptyset \neq g_v^\delta$ is a subquantale of Q_t for all $v \in (\delta, 1]$. Suppose that there exist $z_i \in Q_t$ for $i \in I$ such that $sup\{g(\vee_{i\in I} z_i), \gamma\} < inf\{\underset{i\in I}{inf} g(z_i), \delta\} \Rightarrow 2\delta - inf\{\underset{i\in I}{inf} g(z_i), \delta\} < 2\delta - sup\{g(\vee_{i\in I} z_i), \gamma\} \Rightarrow sup\ \{2\delta - \underset{i\in I}{inf} g(z_i), \delta\} < inf\{2\delta - g(\vee_{i\in I} z_i), 2\delta - \gamma\}$ Take $v \in (\delta, 1]$ such that $sup\{2\delta - \underset{i\in I}{inf} g(z_i), \delta\} < v \leq inf\{2\delta - g(\vee_{i\in I} z_i), 2\delta - \gamma\}$. Then $2\delta - \underset{i\in I}{inf} g(z_i) < v$ and $2\delta - g(\vee_{i\in I} z_i) \geq v \Rightarrow \underset{i\in I}{inf} g(z_i) + v > 2\delta$ but $g(\vee_{i\in I} z_i) + v \leq 2\delta$. This shows that $(z_i)_v q_\delta g$ for $i \in I$, that is $z_i \in g_v^\delta$ for all $i \in I$ but $(\vee_{i\in I} z_i)_v \overline{q} g$, i.e., $(\vee_{i\in I} z_i) \notin g_v^\delta$, a contradiction. Therefore, $sup\{g(\vee_{i\in I} z_i), \gamma\} \geq inf\ \{\underset{i\in I}{inf} g(z_i), \delta\}$ for all $z_i \in Q_t, (i \in I)$. By the same arguments, we have $z \in g_v^\delta$ and $y \in g_v^\delta$ but $(z \otimes y) \notin g_v^\delta$, a contradiction. Therefore, $\{g(z \otimes y), \gamma\} \geq inf\ \{g(z), g(y), \delta\}$. Hence g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t by Theorem 6.2.4. ■

If we take $\gamma = 0$ and $\delta = 0.5$ in Theorem 6.2.13, we have the following Theorem.

Theorem 6.2.15 *[69] Let g be a f-subset of a quantale Q_t. Then g is an $(\in, \in \vee q)$-FS of Q_t if and only if each $\emptyset \neq U(g; p)$ is a subquantale of Q_t for all $p \in (0, 0.5]$.*

Theorem 6.2.16 *Let $2\delta = 1 + \gamma$. Then a f-subset g of a quantale Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ if and only if $\emptyset \neq [g]_v^\delta$ is a subquantale of Q_t for all $v \in (\gamma, 1]$.*

Proof. The proof is similar to the proof of Theorem 6.2.13 and 6.2.14. ■

Corollary 6.2.17 *Let $\gamma, \gamma', \delta, \delta' \in [0,1]$ be such that $\gamma < \delta, \gamma' < \delta'$, $\gamma' < \gamma$ and $\delta' < \delta$. Then every $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t is an $(\in_{\gamma'}, \in_{\gamma'} \vee q_{\delta'})$-$FS$ of Q_t.*

The Example below demonstrates that the converse of Corollary 6.2.17 is not true in general.

Example 6.2.18 *Let Q_t be a quantale and g be a f-subset as discussed in Example 6.2.8. Then g is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$-$FS$ of Q_t but not an $(\in_{0.3}, \in_{0.3} \vee q_{0.9})$-$FS$ of Q_t.*

Theorem 6.2.19 *Let $g \in \mathcal{F}(Q_t)$. Then*

(1) g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t if and only if $\emptyset \neq g_v$ is an ideal of Q_t for all $v \in (\gamma, \delta]$.

(2) If $2\delta = 1 + \gamma$, then g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ if and only if $\emptyset \neq g_v^\delta$ is an ideal of Q_t for all $v \in (\delta, 1]$.

(3) If $2\delta = 1 + \gamma$, then g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ if and only if $\emptyset \neq [g]_v^\delta$ is an ideal of Q_t for all $v \in (\gamma, 1]$.

Proof. (1). Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t. Let $z, w \in Q_t$ with $w \leq z$ and $v \in (\gamma, \delta]$ be such that $z \in g_v$. Then $z_v \in_\gamma g$ and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t, so $w_v(\in_\gamma \vee q_\delta)g$. If $w_v \in_\gamma g$, then $w \in g_v$ and if $w_v q_\delta g$, then $g(w) > 2\delta - v > v > \gamma$, that is, $w \in g_v$. Now we have to show that $z \vee w \in g_v$, for all $z, w \in g_v$. Let $z, w \in Q_t$ be such that $z, w \in g_v$ for some $v \in (\gamma, \delta]$. Then $w_v \in_\gamma g$ and $z_v \in_\gamma g$, and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t, therefore $(w \vee z)_v(\in_\gamma \vee q_\delta)g$. If $(w \vee z)_v \in_\gamma g$, then $(w \vee z) \in g_v$ and if $(w \vee z)_v q_\delta g$, then $g(w \vee z) > 2\delta - v > v > \gamma$, that is, $w \vee z \in g_v$. Let $z \in Q_t$ and $z' \in g_v$ for some $v \in (\gamma, \delta]$. Then $z'_v \in_\gamma g$ and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t, therefore $(z' \otimes z)_v(\in_\gamma \vee q_\delta)g$ and $(z \otimes z')_v(\in_\gamma \vee q_\delta)g$. If $(z' \otimes z)_v \in_\gamma g$, then $(z' \otimes z) \in g_v$ and if $(z' \otimes z)_v q_\delta g$, then $g(z' \otimes z) > 2\delta - v > v > \gamma$, that is, $z' \otimes z \in g_v$. Similarly, $z \otimes z' \in g_v$. Thus, g_v is an ideal of Q_t.

Conversely, suppose that $\emptyset \neq g_v$ is an ideal of Q_t for all $v \in (\gamma, \delta]$. Let $z, w \in Q_t$ with $w \leq z$ and $sup\{g(w), \gamma\} < inf\{g(z), \delta\}$; then there exists $v \in (\gamma, \delta]$ such that $sup\{g(w), \gamma\} < v \leq inf\{g(z), \delta\}$. This shows that $z_v \in_\gamma g$; that is $z \in g_v$ but $w \notin g_v$, a contradiction. Hence, $sup\{g(w), \gamma\} \geq inf\ \{g(z), \delta\}$ for all $z, w \in Q_t$ with $w \leq z$. Let

$z, w \in Q_t$ and $sup\ \{g(z \vee w), \gamma\} < inf\ \{g(z), g(w), \delta\}$, then $sup\ \{g(z \vee w), \gamma\} < v \leq inf$ $\{g(z), g(w), \delta\}$ for some $v \in (\gamma, \delta]$. This implies that $z \in g_v$ and $w \in g_v$ but $(z \vee w) \notin g_v$, a contradiction. Therefore, $sup\{g(z \vee w), \gamma\} \geq inf\{g(z), g(w), \delta\}$.

Similarly, we can show that $sup\ \{g(y \otimes z), \gamma\} \geq inf\{g(z), \delta\}$, [respectively, $(sup\ \{g(z \otimes y), \gamma\} \geq inf\ \{g(z), \delta\})$] for all $z, y \in Q_t$. Consequently, g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t.

(2). Proof of (2) is similar to the proof of Theorem 6.2.14.

(3). Suppose g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t. Let $z, w \in Q_t$ with $w \leq z$ and $v \in (\gamma, 1]$ be such that $z \in [g]_v^\delta$. Then $z_v(\in_\gamma \vee q_\delta)g$, that is $g(z) \geq v > \gamma$ or $g(z) + v > 2\delta$. Thus, $g(z) \geq v$ or $g(z) > 2\delta - v \geq 2\delta - 1 = \gamma$. If $v \in (\gamma, \delta]$, then $\gamma < v \leq \delta$. This implies $2\delta - v \geq \delta \geq v$. Then it follows from above that $g(z) \geq v$. By hypothesis;

$$\begin{aligned} sup\{g(w), \gamma\} \geq &\ inf\{g(z), \delta\} \\ \Rightarrow g(w) \geq &\ inf\{g(z), \delta\} \geq inf\{v, v\} = v \end{aligned}$$

and so $w_v \in_\gamma g$. Thus, $w \in [g]_v^\delta$. If $v \in (\delta, 1]$, then $\delta < v \leq 1$. This implies $2\delta - v < \delta < v$. It follows that $g(z) > 2\delta - v \geq 2\delta - 1 = \gamma$. Now by hypothesis;

$$\begin{aligned} sup\{g(w), \gamma\} \geq &\ inf\{g(z), \delta\} \\ \Rightarrow g(w) \geq &\ inf\{g(z), \delta\} > inf\{2\delta - v, 2\delta - v\} \\ \Rightarrow g(w) > &\ 2\delta - v \end{aligned}$$

Thus, $g(w) + v > 2\delta \Rightarrow w_v q_\delta g$. This implies $w \in [g]_v^\delta$. Now we show that $z \vee w \in [g]_v^\delta$ for all $z, w \in [g]_v^\delta$. Let $z, w \in Q_t$ be such that $z, w \in [g]_v^\delta$ for some $p \in (\gamma, 1]$. Then $z_p(\in_\gamma \vee q_\delta)g$, $w_p(\in_\gamma \vee q_\delta)g$, i.e., $g(z) \geq p > \gamma$ or $g(z) + p > 2\delta$ and $g(w) \geq p > \gamma$ or $g(w) + p > 2\delta$. Thus, $g(z) \geq p$ or $g(z) > 2\delta - p \geq 2\delta - 1 = \gamma$ and $g(w) \geq p$ or $g(w) > 2\delta - p \geq 2\delta - 1 = \gamma$. If $p \in (\gamma, \delta]$, then $\gamma < p \leq \delta$. Thus we have, $2\delta - p \geq \delta \geq p$. Then it follows from above that $g(z) \geq p$ and $g(w) \geq p$. By hypothesis;

$$\begin{aligned} sup\{g(z \vee w), \gamma\} \geq &\ inf\{g(z), g(w), \delta\} \\ \Rightarrow g(z \vee w) \geq &\ inf\{g(z), g(w), \delta\} \geq inf\{p, p, p\} \\ \Rightarrow g(z \vee w) \geq &\ p \end{aligned}$$

and so $(z \vee w)_p \in_\gamma g$. Thus, $z \vee w \in [g]_v^\delta$. If $p \in (\delta, 1]$, then $\delta < p \leq 1$. This implies $2\delta - p < \delta < p$. It follows that $g(z) > 2\delta - p$, $g(w) > 2\delta - p$. Now by hypothesis;

$$\begin{aligned} sup\{g(z \vee w), \gamma\} \geq &\ inf\{g(z), g(w), \delta\} \\ \Rightarrow g(z \vee w) \geq &\ inf\{2\delta - p, 2\delta - p, 2\delta - p\} = 2\delta - p \end{aligned}$$

Thus, $g(z \vee w) + v > 2\delta \Rightarrow (z \vee w)_p q_\delta g$. This implies $(z \vee w) \in [g]_v^\delta$. Similarly, we can show that for $z \in Q_t$ and $z' \in [g]_v^\delta$, we have $z' \otimes z$ and $z \otimes z' \in [g]_v^\delta$.

Conversely, suppose that $\emptyset \neq [g]_v^\delta$ is an ideal of Q_t for all $v \in (\gamma, 1]$. Let $z, w \in Q_t$ with $w \leq z$ and $sup\{g(w), \gamma\} < inf\{g(z), \delta\}$; then there exists $v \in (\gamma, 1]$ such that $sup\{g(w), \gamma\} < v \leq inf\{g(z), \delta\}$. This shows that $z_v \in_\gamma g$; that is $z \in [g]_v^\delta$ but $w_v \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Hence, $sup\{g(w), \gamma\} \geq inf\ \{g(z), \delta\}$ for all $z, w \in Q_t$ with $w \leq z$. Let $z, w \in Q_t$ and $sup\{g(z \vee w), \gamma\} < inf\ \{g(z), g(w), \delta\}$. Then select $p \in (\gamma, 1]$ such that $sup\{g(z \vee w), \gamma\} < p \leq inf\ \{g(z), g(w), \delta\}$. This implies that $z_p \in_\gamma g$ and $w_p \in_\gamma g$ but $(z \vee w)_p \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Therefore, $sup\{g(z \vee w), \gamma\} \geq inf\{g(z), g(w), \delta\}$. Similarly, we can show that $sup\{g(y \otimes z), \gamma\} \geq inf\{g(z), \delta\}$, $(sup\ \{g(z \otimes y), \gamma\} \geq inf\ \{g(z), \delta\})$ for all $z, y \in Q_t$. Consequently, g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t. ■

If we take $\gamma = 0$ and $\delta = 0.5$ in Theorem 6.2.19, we have,

Theorem 6.2.20 *[69] Let g be a f-subset of a quantale Q_t. Then g is an $(\in, \in \vee q)$-FI of Q_t if and only if each $\emptyset \neq U(g; p)$ is an ideal of Q_t for all $p \in (0, 0.5]$.*

Corollary 6.2.21 *Let $\gamma, \gamma', \delta, \delta' \in [0, 1]$ be such that $\gamma < \delta, \gamma' < \delta'$, $\gamma' < \gamma$ and $\delta' < \delta$. Then every $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t is an $(\in_{\gamma'}, \in_{\gamma'} \vee q_{\delta'})$-$FI$ of Q_t.*

The Example below demonstrates that above Corollary is not valid in general

Example 6.2.22 *Consider the quantale given in Example 6.2.2 and define a f-subset g of Q_t as follows:*

$$g = \frac{1}{\bot} + \frac{0.75}{i} + \frac{0.67}{j} + \frac{0.54}{k} + \frac{0.32}{\top}.$$

Then g is an $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-$FI$ of Q_t but not an $(\in_{0.3}, \in_{0.3} \vee q_{0.95})$-$FI$ of Q_t.

The following Propositions are straightforward.

Proposition 6.2.23 *Every $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-$FI$ of Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t.*

Proposition 6.2.24 *Every $(\in_\gamma, \in_\gamma)$-FI of Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t.*

Converses of Propositions 6.2.23 and 6.2.24 do not hold in general as given in the Example below.

Example 6.2.25 *Consider the quantale Q_t as discussed in Example 6.2.2 and take $g = \frac{0.9}{\perp} + \frac{0.7}{i} + \frac{0.65}{j} + \frac{0.54}{k} + \frac{0.31}{\top}$. Then*

(1) *It is simple to confirm that g is an $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-$FI$ of Q_t.*

(2) *g is not an $(\in_{0.3}, \in_{0.3})$-FI of Q_t, since $i_{0.68} \in_{0.3} g$ and $j_{0.61} \in_{0.3} g$ but $(i \vee j)_{inf(0.68,0.61)} = k_{0.61} \overline{\in}_{0.3} g$.*

(3) *g is not an $(\in_{0.3} \vee q_{0.6}, \in_{0.3} \vee q_{0.6})$-$FI$ of Q_t, since $i_{0.68}(\in_{0.3} \vee q_{0.6})g$ and $j_{0.59}(\in_{0.3} \vee q_{0.6})g$ but $(i \vee j)_{inf(0.68,0.59)} = k_{0.59}\overline{(\in_{0.3} \vee q_{0.6})}g$.*

The following Lemma and Proposition describe the relation between characteristic function K_C and $(\in_\gamma, \in_\gamma)$-FI, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t.

Lemma 6.2.26 *If C is an ideal of Q_t, then the characteristic function K_C of C is an $(\in_\gamma, \in_\gamma)$-FI of Q_t.*

Proof. Let $w, z \in Q_t$ and $p, v \in (\gamma, 1]$ be such that $w_p \in_\gamma K_C$ and $z_v \in_\gamma K_C$. Then $K_C(w) \geq p > \gamma$ and $K_C(z) \geq v > \gamma$, which imply that $K_C(w) = K_C(z) = 1$. As C is an ideal and $w, z \in C$, so $w \vee z \in C$. It follows that $K_C(w \vee z) = 1 \geq inf\{p, v\} > \gamma$ so that $(w \vee z)_{inf(p,v)} \in_\gamma K_C$. Now let $b, z \in Q_t$ and $p \in (\gamma, 1]$ be such that $b_p \in_\gamma K_C$. Then $K_C(b) \geq p > \gamma$, and so $K_C(b) = 1$, i.e., $b \in C$. Since C is an ideal of Q_t, we have $b \otimes z$, $z \otimes b \in C$ and hence $K_C(b \otimes z) = K_C(z \otimes b) = 1 \geq p > \gamma$. Therefore $(b \otimes z)_p \in_\gamma K_C$ and $(z \otimes b)_p \in_\gamma K_C$. Let $w, z \in Q_t$, $z_p \in_\gamma K_C$ with $w \leq z$. Then $K_C(z) \geq p > \gamma$, and so $K_C(z) = 1$, i.e., $z \in C$. Since C is a lower set, we have $w \in C$ and so $K_C(w) = 1 \geq p > \gamma$. Therefore $w_p \in_\gamma K_C$ and consequently K_C is an $(\in_\gamma, \in_\gamma)$-FI of Q_t. ■

Proposition 6.2.27 *Let $\emptyset \neq C \subseteq Q_t$. Then K_C (the characteristic function) is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t if and only if C is an ideal of Q_t.*

Proof. Let K_C be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t, $p, v \in (\gamma, 1]$ and $w, z \in C$. Then $w_1 \in_\gamma K_C$ and $z_1 \in_\gamma K_C$ which show that $(w \vee z)_1 = (w \vee z)_{inf(1,1)} (\in_\gamma \vee q_\delta) K_C$. Hence $K_C(w \vee z) > \gamma$, and so $w \vee z \in C$. Let $w, z \in Q_t$ with $w \leq z$ and $z \in C$. Then $K_C(z) = 1$, and thus $z_1 \in_\gamma K_C$. Since K_C is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$, so we have $w_1 \in_\gamma K_C$. Thus $K_C(w) = 1$. Hence $w \in C$. Now let $w \in Q_t$ and $z \in C$. Then $K_C(z) = 1$, and thus $z_1 \in_\gamma K_C$. Since K_C is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$, it follows that $(z \otimes w)_1 \in_\gamma K_C$ so that $K_C(z \otimes w) = 1$. Hence $z \otimes w \in C$ and similarly, $w \otimes z \in C$. Thus, C is an ideal of Q_t.

Conversely, if C is an ideal of Q_t, then K_C is an $(\in_\gamma, \in_\gamma)$-FI of Q_t by lemma 6.2.26, and therefore K_C is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t by Proposition 6.2.24. ■

6.3 $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Prime (Semi Prime) Ideals of Quantale

$(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$ of a quantale Q_t are introduced in this section. We also discuss the relationship between prime (semi-prime) and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ ($FSPI$) of Quantale.

Definition 6.3.1 *An (α, β)-FI, g of a quantale Q_t is called an (α, β)-FPI of Q_t if for all $p \in (\gamma, 1]$ and $z, w \in Q_t$, $(z \otimes w)_p \alpha g \longrightarrow z_p \beta g$ or $w_p \beta g$. An (α, β)-FI, g of a quantale Q_t is called an (α, β)-$FSPI$ of Q_t if for all $z \in Q_t$ and $p \in (\gamma, 1]$, $(z \otimes z)_p \alpha g \longrightarrow z_p \beta g$.*

Proposition 6.3.2 *An $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$, g of a quantale Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ if and only if $sup\{g(z), g(w), \gamma\} \geq inf\{g(z \otimes w), \delta\}$ for all $w, z \in Q_t$ and $v \in (\gamma, \delta]$.*

Proof. Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ of a quantale Q_t. We want to show that $sup\{g(z), g(w), \gamma\} \geq inf\{g(z \otimes w), \delta\}$ for all $w, z \in Q_t$. Let there exist $y, z \in Q_t$ and $v \in (\gamma, \delta]$ be such that $sup\{g(z), g(y), \gamma\} < v \leq inf\{g(z \otimes y), \delta\}$. Then $g(z \otimes y) \geq v > \gamma$, $g(z) < v$, $g(y) < v$ and $g(z) + v < 2v \leq 2\delta$, $g(y) + v < 2v \leq 2\delta$. This means that $(z \otimes y)_v \in_\gamma g$, but $y_v \overline{(\in_\gamma \vee q_\delta)} g$ and $z_v \overline{(\in_\gamma \vee q_\delta)} g$. This gives a contradiction. Hence we have, $sup\{g(z), g(w), \gamma\} \geq inf\{g(z \otimes w), \delta\}$ for all $w, z \in Q_t$.

Conversely, suppose that the condition $sup\{g(z), g(w), \gamma\} \geq inf\{g(z \otimes w), \delta\}$ for all $w, z \in Q_t$ hold. Let $w, z \in Q_t$ and $v \in (\gamma, \delta]$ be such that $(w \otimes z)_v \in_\gamma g$ but $w_v \overline{(\in_\gamma \vee q_\delta)} g$ and $z_v \overline{(\in_\gamma \vee q_\delta)} g$, then $g(w \otimes z) \geq v > \gamma$, $g(w) < v$ and $g(w) + v < 2\delta$, similarly, $g(z) < v$ and $g(z) + v < 2\delta$. It follows that $g(w) < \delta$, $g(z) < \delta$ and so $sup\{g(z), g(w), \gamma\} < inf\{g(z \otimes w), \delta\}$, a contradiction. Therefore, g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ of Q_t. ■

Theorem 6.3.3 *Let g be a f-subset of a quantale Q_t. Then g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ if and only if g_v is a PI of Q_t for all $v \in (\gamma, \delta]$.*

Proof. Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ of Q_t. Let $y, z \in Q_t$ and $v \in (\gamma, \delta]$ be such that $y \otimes z \in g_v$. Then $(y \otimes z)_v \in_\gamma g$ and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ of Q_t, therefore $y_v(\in_\gamma \vee q_\delta)g$ or $z_v(\in_\gamma \vee q_\delta)g$. If $y_v \in_\gamma g$ then $y \in g_v$ and if $y_v q_\delta g$, then $g(y) > 2\delta - v \geq v > \gamma$; that is, $y \in g_v$. Similarly $z \in g_v$. Hence g_v is a PI of Q_t.

Conversely, suppose that g_v is a PI of Q_t for all $v \in (\gamma, \delta]$ and assume that the condition $sup\{g(z), g(w), \gamma\} \geq inf\{g(z \otimes w), \delta\}$ is not valid, then there exist some $a, c \in Q_t$ such that $sup\{g(a), g(c), \gamma\} < inf\{g(a \otimes c), \delta\}$, then there exists $v \in (\gamma, \delta]$ such that $sup\{g(a), g(c), \gamma\} < v \leq inf\{g(a \otimes c), \delta\}$. This implies that $(a \otimes c)_v \in_\gamma g$, that is $a \otimes c \in g_v$. Since g_v is a PI of Q_t, we have $a \in g_v$ or $c \in g_v$, i.e., $g(a) \geq v$ or $g(c) \geq v$, which contradicts the condition. Hence we must have $sup\{g(z), g(w), \gamma\} \geq inf\{g(z \otimes w), \delta\}$. Consequently g is an $(\in, \in \vee q)$-FPI of Q_t by Proposition 6.3.2. ■

Proposition 6.3.4 *An $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$, g of a quantale Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$ if and only if $sup\{g(z), \gamma\} \geq inf\{g(z \otimes z), \delta\}$ for all $z \in Q_t$.*

Proof. Proof is obtained in a similar way from Proposition 6.3.2. ■

Proposition 6.3.5 *Let g be a f-subset of a quantale Q_t. Then g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$ if and only if g_v is a SPI of Q_t for all $v \in (\gamma, \delta]$.*

Proof. Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$. Let $(y \otimes y) \in g_v$. Then $g(y \otimes y) \geq v$. Thus by Proposition 6.3.4, we have $sup\{g(z), \gamma\} \geq inf\{g(z \otimes z), \delta\} \geq inf\{v, \delta\} = v$. So, $g(z) \geq v$. Thus $z \in g_v$. Hence g_v is a SPI of Q_t.

Conversely, suppose that g_v is a SPI of Q_t for all $v \in (\gamma, \delta]$ and assume that condition $sup\{g(z), \gamma\} \geq inf\{g(z \otimes z), \delta\}$ is not valid, then there exist some $c \in Q_t$ such that $sup\{g(c), \gamma\} < inf\{g(c \otimes c), \delta\}$ and we take $v \in (\gamma, \delta]$ such that $sup\{g(c), \gamma\} < v \leq inf\{g(c \otimes c), \delta\}$. This implies that $(c \otimes c) \in g_v$. Since g_v is a SPI of Q_t, we have $c \in g_v$, i.e., $g(c) \geq v$, which contradicts the condition. Hence we must have $sup\{g(z), \gamma\} \geq inf\{g(z \otimes z), \delta\}$ for all $z \in Q_t$. Consequently, g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$ of Q_t by Proposition 6.3.4. ■

6.4 (α, β)-Fuzzy Q_t-Submodule of Q_t-Module

Some new relationships between $f$uzzy points and f-subsets regarding (α, β)-fuzzy Q_t-submodule of Q_t-module are introduced in this section.

If we take $\gamma = 0$ and $\delta = 0.5$ then $\in_\gamma$ and q_δ becomes $\in$ and q as defined in section 4.4 of Chapter 4.

Definition 6.4.1 *A f-subset g of a Q_t-module M is called an (α, β)-fuzzy Q_t-submodule of M, if*

(F_1) $(m_i)_{p_i} \alpha g \longrightarrow (\vee_{i \in I} m_i)_{\underset{i\in I}{inf}\{p_i\}} \beta g$;

(F_2) $m_p \alpha g \longrightarrow (a * m)_p \beta g$ *for all* $m_i, m \in M$, $p_i, p \in (0, 1]$ *and* $a \in Q_t$.

Theorem 6.4.2 *Let g be a non-zero (α, β)-fuzzy Q_t-submodule of a Q_t-module M and $2\delta = 1 + \gamma$. Then $g_\gamma = \{m \in M \mid g(m) > \gamma\}$ is a Q_t-submodule of M.*

Proof. Let $m_i \in g_\gamma$ for $i \in I$. Then $g(m_i) > \gamma$ for all $i \in I$. Let $g(\vee_{i\in I} m_i) \leq \gamma$. If $\alpha \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$, then $(m_i)_{g(m_i)} \alpha g$ for all $i \in I$ but $g(\vee_{i\in I} m_i) \leq \gamma < \underset{i\in I}{inf}\ g(m_i)$ and $g(\vee_{i\in I} m_i) + \underset{i\in I}{inf}\ g(m_i) \leq \gamma + \underset{i\in I}{inf} g(m_i) \leq \gamma + 1 = 2\delta$. So $(\vee_{i\in I} m_i)_{\underset{i\in I}{inf} g(m_i)} \overline{\beta} g$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, a contradiction. Hence $g(\vee_{i\in I} m_i) > \gamma$, i.e., $\vee_{i\in I} m_i \in g_\gamma$. If $\alpha = q_\delta$ then $(m_i)_1 q_\delta g$ for all $i \in I$ because $g(m_i) + 1 > 1 + \gamma = 2\delta$, but $(\vee_{i\in I} m_i)_1 \overline{\beta} g$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, because $g(\vee_{i\in I} m_i) \leq \gamma$, so $(\vee_{i\in I} m_i)_1 \overline{\in}_\gamma g$ and $g(\vee_{i\in I} m_i) + 1 \leq \gamma + 1 = 2\delta$, so $(\vee_{i\in I} m_i)_1 \overline{q}_\delta g$. Hence $g(\vee_{i\in I} m_i) > \gamma$, that is $\vee_{i\in I} m_i \in g_\gamma$. Thus g_γ is closed under arbitrary join. Let $m \in g_\gamma$. Then $g(m) > \gamma$. Suppose $g(q * m) \leq \gamma$ for all $q \in Q_t$. If $\alpha \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$, then $(m)_{g(m)} \alpha g$ but $g(q * m) \leq \gamma < g(m)$ and $g(q * m) + g(m) \leq \gamma + g(m) \leq \gamma + 1 = 2\delta$. So $(q * m)_{g(m)} \overline{\beta} g$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, a contradiction. Hence $g(q * m) > \gamma$, i.e., $q * m \in g_\gamma$. If $\alpha = q_\delta$ then $(m)_1 q_\delta g$ because $g(m) + 1 > 1 + \gamma = 2\delta$, but $(q * m)_1 \overline{\beta} g$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, since $g(q*m) \leq \gamma$, so $(q*m)_1 \overline{\in}_\gamma g$ and $g(q*m) + 1 \leq \gamma + 1 = 2\delta$, so $(q * m)_1 \overline{q}_\delta g$. Hence $g(q * m) > \gamma$, that is $q * m \in g_\gamma$. Thus, g_γ is a Q_t-submodule of M. ■

Theorem 6.4.3 *Let $2\delta = 1 + \gamma$ and $\emptyset \neq C \subseteq M$. Then C is a Q_t-submodule of Q_t-module M if and only if the f-subset g of M defined by*

$$g(m) = \begin{cases} \geq \delta \ if \ m \in C \\ \gamma \ otherwise \end{cases} \quad for\ all\ m \in M.$$

is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M.

Proof. Let C be a Q_t-submodule of M.

(**a**) Let $m_i \in M$ and $v_i \in (\gamma, 1]$ be such that $(m_i)_{v_i} \in_\gamma g$. Then $g(m_i) \geq v_i > \gamma$. Hence $g(m_i) \geq \delta$. Thus $m_i \in C$ and so $\vee_{i\in I} m_i \in C$, that is $g(\vee_{i\in I} m_i) \geq \delta$. If $inf\{v_i\} \leq \delta$, then $g(\vee_{i\in I} m_i) \geq \delta \geq inf\{v_i\} > \gamma$. Hence $(\vee_{i\in I} m_i)_{iinf\{v_i\}} \in_\gamma g$. If $inf\{v_i\} > \delta$, then $g(\vee_{i\in I} m_i) + inf\{v_i\} > \delta + \delta = 2\delta$ and so $(\vee_{i\in I} m_i)_{inf\{v_i\}} q_\delta g$. Therefore $(\vee_{i\in I} m_i)_{inf\{v_i\}} (\in_\gamma \vee q_\delta) g$.

Now let $m \in M$ and $p \in (\gamma, 1]$ be such that $m_p \in_\gamma g$. Then $g(m) \geq p > \gamma$. This shows $m \in C$, and so $a * m \in C$ for all $a \in Q_t$. Consequently $g(a * m) \geq \delta$. If $p \leq \delta$, then $g(a*m) \geq \delta \geq p > \gamma$. Hence $(a*m)_p \in_\gamma g$. If $p > \delta$, then $g(a*m)+p > \delta+\delta = 2\delta$ and so $(a*m)_p q_\delta g$. Thus $(a*m)_p (\in_\gamma \vee q_\delta) g$. Hence g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M.

(**b**) Let $m_i \in M$ and $p_i \in (\gamma, 1]$ be such that $(m_i)_{p_i} q_\delta g$. Then $g(m_i) + p_i > 2\delta$ and so $g(m_i) > 2\delta - p_i \geq 2\delta - 1 = \gamma$. It follows that $g(m_i) > \gamma$, i.e.,$m_i \in C$. Since C is a Q_t-submodule of M, so $\vee_{i\in I} m_i \in C$, hence we have $g(\vee_{i\in I} m_i) \geq \delta$. If $inf\{p_i\} \leq \delta$, then $g(\vee_{i\in I} m_i) \geq \delta \geq inf\{p_i\} > \gamma$. Hence $(\vee_{i\in I} m_i)_{inf\{p_i\}} \in_\gamma g$. If $inf\{p_i\} > \delta$, then $g(\vee_{i\in I} m_i) + inf\{p_i\} > \delta + \delta = 2\delta$ and so $(\vee_{i\in I} m_i)_{inf\{p_i\}} q_\delta g$. Therefore $(\vee_{i\in I} m_{iinf\{p_i\}} (\in_\gamma \vee q_\delta) g$. Let $m \in M$ and $p \in (\gamma, 1]$ be such that $m_p q_\delta g$. Then $g(m) + p > 2\delta$ and so $g(m) > 2\delta - p \geq 2\delta - 1 = \gamma$. Thus $m \in C$ and so $a * m$ is in C for all $a \in Q_t$. This means that $g(a*m) \geq \delta$. If $p \leq \delta$, then $g(a*m) \geq \delta \geq p > \gamma$. Hence $(a*m)_p \in_\gamma g$. If $p > \delta$, then $g(a*m)+p > \delta+\delta = 2\delta$ and so $(a*m)_p q_\delta g$. Thus $(a*m)_p (\in_\gamma \vee q_\delta) g$. Hence g is $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M.

(**c**) Let $m_i \in M$ and $p_i \in (\gamma, 1]$ be such that $(m_i)_{v_i} \in_\gamma g$ or $(m_i)_{v_i} q_\delta g$. Then $g(m_i) \geq v_i > \gamma$ and $g(m_i) + v_i > 2\delta$. This shows that $m_i \in C$ and $\vee_{i\in I} m_i \in C$. Hence $g(\vee_{i\in I} m_i) \geq \delta$. Thus, in a similar way, we have $(\vee_{i\in I} m_i)_{inf\{p_i\}} \in_\gamma g$ for $inf\{p_i\} \leq \delta$ and $(\vee_{i\in I} m_i)_{inf\{p_i\}} q_\delta g$ for $inf\{p_i\} > \delta$. Thus $(\vee_{i\in I} m_i)_{inf\{p_i\}} (\in_\gamma \vee q_\delta) g$. The rest is similar to the proof of parts (a) and (b).

Conversely, suppose that g is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M. It is easy to prove that $C = g_\gamma$. Hence, by Theorem 6.4.2, C is a Q_t-submodule of M. ■

6.5 $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Q_t-Submodule of Q_t-Module

In this section, we present an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of Q_t-module M and discuss some of their properties.

Definition 6.5.1 *A f-subset g of Q_t-module M is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M, if*

(F_1) $(m_i)_{p_i} \in_\gamma g \longrightarrow (\vee_{i\,\in\, I} m_i)_{\underset{i\in I}{inf}\{p_i\}}(\in_\gamma \vee q_\delta)g$;

(F_2) $m_p \in_\gamma g \longrightarrow (q{*}m)_p(\in_\gamma \vee q_\delta)g$ *for all* $\{m_i\} \subseteq M$ $(i \in I)$, $m \in M$ *and* $p_i, p \in (\gamma, 1]$.

Example 6.5.2 *Let $(Q_t, \otimes)$ be a quantale, where Q_t is delineated in Fig.*12 *and the binary operation $\otimes$ on Q_t is shown in the Table* 9. *Then Q_t is a Q_t-module over Q_t. Taking $g = \frac{0.9}{\perp} + \frac{0.63}{i} + \frac{0.63}{j} + \frac{0.63}{k} + \frac{0.65}{\top}$. Then by routine calculations g is an $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-fuzzy Q_t-submodule of M.*

Theorem 6.5.3 *Let g be a f-subset of a Q_t-module M. If g is an $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule M, then conditions below hold:*

(1) $sup\ \{g(\vee_{i\in I} m_i), \gamma\} \geq inf\ \{\underset{i\in I}{inf} g(m_i), \delta\}$;

(2) $sup\{g(q * m), \gamma\} \geq inf\{g(m), \delta\}$ *for all* $\{m_i\} \subseteq Q_t$ $(i \in I)$, $m \in M$ *and* $q \in Q_t$.

Proof. Let g be a $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M. Assume that there exist $m_i \in M$ such that $sup\{g(\vee_{i\in I} m_i), \gamma\} < inf\ \{\underset{i\in I}{inf} g(m_i), \delta\}$. Then for all $\gamma < v \leq 1$ such that

$$2\delta - sup\{g(\vee_{i\in I} m_i), \gamma\} > v \geq 2\delta - inf\{\underset{i\in I}{inf} g(m_i), \delta\}$$

and so

$$2\delta - g(\vee_{i\in I} m_i) \geq 2\delta - sup\{g(\vee_{i\in I} m_i), \gamma\} > v \geq sup\{2\delta - \underset{i\in I}{inf} g(m_i), \delta\}$$

That is, $2\delta - g(\vee_{i\in I} m_i) > v$, $2\delta - \underset{i\in I}{inf} g(m_i) < v$.

Thus,

$$\underset{i\in I}{inf} g(m_i) + v > 2\delta,\ \ g(\vee_{i\in I} m_i) + v < 2\delta$$

and $g(\vee_{i\in I} m_i) < \delta < v$. Hence $(m_i)_v q_\delta g$ for all $i \in I$, but $(\vee_{i\in I} m_i)_v \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Therefore $sup\{g(\vee_{i\in I} m_i), \gamma\} \geq inf\ \{\underset{i\in I}{inf} g(m_i), \delta\}$.

Let there exist $m \in M$ and for all $q \in Q_t$ be such that $sup\{g(q * m), \gamma\} < inf$ $\{g(m), \delta\}$. Then for all $\gamma < t \leq 1$ such that

$$2\delta - sup\{g(q * m), \gamma\} > t \geq 2\delta - inf\{g(m), \delta\}$$

we have

$$2\delta - g(q * m) \geq 2\delta - sup\{g(q * m), \gamma\} > t \geq sup\{2\delta - g(m), \delta\}$$

That is, $2\delta - g(m) < t$, $2\delta - g(q * m) > t$.

and so

$$g(m) + t > 2\delta, \; g(q * m) + t < 2\delta$$

and $g(q * m) < \delta < t$. Hence $m_t q_\delta g$ but $(q * m)_t \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Therefore, $sup\{g(q * m), \gamma\} \geq inf\{g(z), g(y), \delta\}$ for all $m \in M$ and $q \in Q_t$. ■

Theorem 6.5.4 *A f-subset g of Q_t-module M is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M if and only if the conditions below hold:*

(1) $sup\ \{g(\vee_{i \in I} m_i), \gamma\} \geq inf\{\underset{i \in I}{inf} g(m_i), \delta\}$;

(2) $sup\{g(q * m), \gamma\} \geq inf\{g(m), \delta\}$ *for all* $\{m_i\} \subseteq Q_t$ $(i \in I)$, $m \in M$ *and* $q \in Q_t$.

Proof. Let g be a $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M. Let there exist $m_i \in M$ and $v \in (\gamma, \delta]$ be such that $sup\{g(\vee_{i \in I} m_i), \gamma\} < v \leq inf\{\underset{i \in I}{inf} g(m_i), \delta\}$. Then $g(m_i) \geq v > \gamma$ for all $i \in I$, $g(\vee_{i \in I} m_i) < v$ and $g(\vee_{i \in I} m_i) + v < 2v \leq 2\delta$, *i.e.*, $(m_i)_v \in_\gamma g$ for all $i \in I$ but $(\vee_{i \in I} m_i)_v \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Thus, $sup\{g(\vee_{i \in I} m_i), \gamma\} \geq inf$ $\{\underset{i \in I}{inf} g(m_i), \delta\}$ for all $m_i \in Q_t$. Let $z, y \in Q_t$ and $v \in (\gamma, \delta]$ be such that $sup\{g(q * m), \gamma\} < v \leq inf\{g(m), \delta\}$. Then $g(m) \geq v > \gamma$, $g(q * m) < v$ and $g(q * m) + v < 2v \leq 2\delta$, *i.e.*, $m_v \in_\gamma g$ but $(q * m)_v \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Thus, $sup\{g(q * m), \gamma\} \geq inf$ $\{g(m), \delta\}$ for all $m \in M$ and $q \in Q_t$.

Conversely, suppose above conditions are true. We show that g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M. Let $m_i \in M$ and $v_i \in (\gamma, \delta]$ be such that $(m_i)_{v_i} \in_\gamma g$ but $(\vee_{i \in I} m_i)_{\underset{i \in I}{inf}\{v_i\}} \overline{(\in_\gamma \vee q_\delta)} g$. Then $g(m_i) \geq v_i$ for all $i \in I$, $g(\vee_{i \in I} m_i) < \underset{i \in I}{inf}\{v_i\}$ and $g(\vee_{i \in I} m_i) + \underset{i \in I}{inf}\{v_i\} \leq 2\delta$. It follows that $g(\vee_{i \in I} m_i) < \delta$ and so $sup\{g(\vee_{i \in I} m_i), \gamma\} < inf\{\underset{i \in I}{inf} g(m_i), \delta\}$, a contradiction. Hence $(\vee_{i \in I} m_i)_{\underset{i \in I}{inf}\{p_i\}} (\in_\gamma \vee q_\delta) g$. Similarly, it can be shown that if $z_p \in_\gamma g$, and $q \in Q_t$ then $g(q * m)_p (\in_\gamma \vee q_\delta) g$. ■

Proposition 6.5.5 *Let g_1 and g_2 be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodules of M. Then, $(g_1 \Cap g_2)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M.*

Proof. Let $m_i \in M$ for some $i \in I$ and $\gamma, \delta \in (0,1]$ with $\gamma < \delta$. Since g_1 and g_2 are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodules of M., so, $sup\{g_1(\vee_{i\in I} m_i), \gamma\} \geq inf\{\underset{i\in I}{inf} g_1(m_i), \delta\}$ and $sup\{g_2(\vee_{i\in I} m_i), \gamma\} \geq inf\{\underset{i\in I}{inf} g_2(m_i), \delta\}$

Now, consider

$$\begin{aligned} sup\{(g_1 \Cap g_2)(\vee_{i\in I} m_i), \gamma\} &= sup\{g_1(\vee_{i\in I} m_i) \wedge g_2(\vee_{i\in I} m_i), \gamma\} \\ &= sup\{g_1(\vee_{i\in I} m_i), \gamma\} \wedge sup\{g_2(\vee_{i\in I} m_i), \gamma\} \\ &\geq inf\{\underset{i\in I}{inf} g_1(m_i), \delta\} \wedge inf\{\underset{i\in I}{inf} g_2(m_i), \delta\} \\ &= inf\{\underset{i\in I}{inf}(g_1(m_i) \wedge g_2(m_i)), \delta\} \end{aligned}$$

That is, $sup\{(g_1 \Cap g_2)(\vee_{i\in I} m_i), \gamma\} \geq inf\{\underset{i\in I}{inf}(g_1 \Cap g_2)(m_i), \delta\}$

Next, as

$$\begin{aligned} sup\{g_1(a * m), \gamma\} &\geq inf\{g_1(m), \delta\} \text{ and} \\ sup\{g_2(a * m), \gamma\} &\geq inf\{g_2(m), \delta\}. \end{aligned}$$

Now, consider

$$\begin{aligned} sup\{(g_1 \Cap g_2)(a \otimes m), \gamma\} &= sup\{g_1(a \otimes m) \wedge g_2(a \otimes m), \gamma\} \\ &= sup\{g_1(a \otimes m), \gamma\} \wedge sup\{g_2(a \otimes m), \gamma\} \\ &\geq inf\{g_1(m), \delta\} \wedge inf\{g_2(m), \delta\} \\ &= inf\{g_1(m) \wedge g_2(m), \delta\} \end{aligned}$$

Hence, $sup\{(g_1 \Cap g_2)(a * m), \gamma\} \geq inf\{(g_1 \Cap g_2)(m), \delta\}$

Therefore, $g_1 \Cap g_2$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodules of M by Theorem 6.5.4. ■

The following Propositions are obvious.

Proposition 6.5.6 *Every $((\in_\gamma \vee q_\delta), \in_\gamma \vee q_\delta))$-fuzzy Q_t-submodule of M is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M.*

Proposition 6.5.7 *Every $(\in_\gamma, \in_\gamma)$-fuzzy Q_t-submodule of M is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M.*

The Example below describes that the converses of Propositions 6.5.6 and 6.5.7 may not be true in general.

Example 6.5.8 *Let Q_t be a quantale defined in Example 6.2.2. Then Q_t is a Q_t-module over itself and taking $g = \frac{0.9}{\perp} + \frac{0.7}{i} + \frac{0.65}{j} + \frac{0.54}{k} + \frac{0.41}{\top}$. Then*

(1) *It is easy to verify that* g *is an* $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$*-fuzzy* Q_t*-submodule of* Q_t.

(2) g *is not an* $(\in_{0.3}, \in_{0.3})$*-fuzzy* Q_t*-submodule of* Q_t, *since* $i_{0.68} \in_{0.3} g$ *and* $j_{0.61} \in_{0.3} g$ *but* $(i \vee j)_{inf(0.68,0.61)} = k_{0.61} \overline{\in}_{0.3} g$.

(3) g *is not an* $(\in_{0.3} \vee q_{0.6}, \in_{0.3} \vee q_{0.6})$*-fuzzy* Q_t*-submodule of* Q_t, *since* $i_{0.68}(\in_{0.3} \vee q_{0.6})g$ *and* $j_{0.59}(\in_{0.3} \vee q_{0.6})g$ *but* $(i \vee j)_{inf(0.68,0.59)} = k_{0.59}\overline{(\in_{0.3} \vee q_{0.6})}g$.

The following Theorem gives the relation between $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M and crisp Q_t-submodule of M.

Theorem 6.5.9 *The following are equivalent for any* f*-subset* g *of* Q_t*-module* M:

(1) g *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy* Q_t*-submodule of* M;

(2) $g_v(\neq \emptyset)$ *is a* Q_t*-submodule of* M *for all* $v \in (\gamma, \delta]$.

Proof. $(\mathbf{1}) \Longrightarrow (\mathbf{2})$. Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M. Let $m_i \in M$ and $v \in (\gamma, \delta]$ be such that $m_i \in g_v$ for all $i \in I$. Then $(m_i)_v \in_\gamma g$ for all $i \in I$ and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M, therefore $(\vee_{i \in I} m_i)_v (\in_\gamma \vee q_\delta) g$. If $(\vee_{i \in I} m_i)_v \in_\gamma g$, then $\vee_{i \in I} m_i \in g_v$ and if $(\vee_{i \in I} m_i)_v q_\delta g$, then $g(\vee_{i \in I} m_i) > 2\delta - v \geq v > \gamma$; that is, $\vee_{i \in I} m_i \in g_v$. Let $m \in M$ and $a \in Q_t$ be such that $m \in g_v$ for some $v \in (\gamma, \delta]$. Then $m_v \in_\gamma g$ and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M, therefore $(a * m)_v (\in_\gamma \vee q_\delta) g$. If $(a * m)_v \in_\gamma g$, then $a * m \in g_v$ and if $(a * m)_v q_\delta g$, then $g(a * m) > 2\delta - v \geq v > \gamma$; that is, $a * m \in g_v$. Therefore g_v is a Q_t-submodule of M.

$(\mathbf{2}) \Longrightarrow (\mathbf{1})$. Assume that $\emptyset \neq g_v$ is a Q_t-submodule of M for all $v \in (\gamma, \delta]$. Suppose that there exist $m_i \in M$ for $i \in I$ such that $sup\{g(\vee_{i \in I} m_i), \gamma\} < inf\{\underset{i \in I}{inf} g(m_i), \delta\}$; then there exist $v \in (\gamma, \delta]$ such that $sup\{g(\vee_{i \in I} m_i), \gamma\} < v \leq inf\{\underset{i \in I}{inf} g(m_i), \delta\}$. This shows that $(m_i)_v \in_\gamma g$ for all $i \in I$; that is, $m_i \in g_v$ for all $i \in I$ but $(\vee_{i \in I} m_i) \notin g_v$, a contradiction. Therefore, $sup\{g(\vee_{i \in I} m_i), \gamma\} \geq inf\ \{\underset{i \in I}{inf} g(m_i), \delta\}$ for all $m_i \in M, (i \in I)$. Let $m \in M$ and $q \in Q_t$ be such that $sup\{g(a * m), \gamma\} < inf\{g(m), \delta\}$. Then $sup\{g(a * m), \gamma\} < v \leq inf\ \{g(m), \delta\}$ for some $v \in (\gamma, \delta]$. This implies that $m \in g_v$ and but $(a * m) \notin g_v$, a contradiction. Therefore, $\{g(a * m), \gamma\} \geq inf\{g(m), \delta\}$. By Theorem 6.5.4, g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M. ■

Theorem 6.5.10 *Let* $2\delta = 1 + \gamma$. *Then a* f*-subset* g *of* Q_t*-module* M *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy* Q_t*-submodule of* M *if and only if* $\emptyset \neq g_v^\delta$ *is a* Q_t*-submodule of* M *for all* $v \in (\delta, 1]$.

Proof. Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M. Let $m_i \in M$ for all $i \in I$ and $v \in (\delta, 1]$ be such that $m_i \in g_v^\delta$ for all $i \in I$. Then $(m_i)_v q_\delta g$ for all $i \in I$; that is $g(m_i) > 2\delta - v \geq 2\delta - 1 = \gamma$. Thus, $g(m_i) > \gamma$. Since $v \in (\delta, 1]$, we have $2\delta - v < \delta < v$. By hypothesis, we have,

$$\begin{aligned} sup\{g(\vee_{i\in I} m_i), \gamma\} &\geq inf\{\underset{i\in I}{inf} g(m_i),\ \delta\}; \\ g(\vee_{i\in I} m_i) &> inf\{2\delta - v,\ \delta\}; \\ &= 2\delta - v. \end{aligned}$$

that is, $g(\vee_{i\in I} m_i) > 2\delta - v$. Hence $\vee_{i\in I} m_i \in g_v^\delta$.

Let $x \in M$ be such that $x \in g_v^\delta$ for some $v \in (\delta, 1]$. Then $x_v q_\delta g$ that is $g(x) > 2\delta - v \geq 2\delta - 1 = \gamma$ and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M, therefore,

$$\begin{aligned} sup\{g(a * x), \gamma\} &\geq inf\{g(x), \delta\} \\ &> inf\{2\delta - v, \delta\} \\ &= 2\delta - v \end{aligned}$$

that is, $g(a * x) > 2\delta - v$. Hence $a * x \in g_v^\delta$. So, g_v^δ is a Q_t-submodule of M.

Conversely, assume that $\emptyset \neq g_v^\delta$ is a Q_t-submodule of M for all $v \in (\delta, 1]$. Suppose that there exist $m_i \in M$ for $i \in I$ such that $sup\{g(\vee_{i\in I} m_i), \gamma\} < inf\{\underset{i\in I}{inf} g(m_i), \delta\} \Rightarrow 2\delta - inf\{\underset{i\in I}{inf} g(m_i), \delta\} < 2\delta - sup\{g(\vee_{i\in I} m_i), \gamma\} \Rightarrow sup\ \{2\delta - \underset{i\in I}{inf} g(m_i), \delta\} < inf\{2\delta - g(\vee_{i\in I} m_i), 2\delta - \gamma\}$ Take $v \in (\delta, 1]$ such that $sup\{2\delta - \underset{i\in I}{inf} g(m_i), \delta\} < v \leq inf\{2\delta - g(\vee_{i\in I} m_i), 2\delta - \gamma\}$. Then $2\delta - \underset{i\in I}{inf} g(m_i) < v$ and $2\delta - g(\vee_{i\in I} m_i) \geq v \Rightarrow \underset{i\in I}{inf} g(m_i) + v > 2\delta$ but $g(\vee_{i\in I} m_i) + v \leq 2\delta$. This shows that $(m_i)_v q_\delta g$ for i $\in I$, that is $m_i \in g_v^\delta$ for all $i \in I$ but $(\vee_{i\in I} m_i)_v \overline{q} g$, i.e., $(\vee_{i\in I} m_i) \notin g_v^\delta$, a contradiction. Therefore, $sup\{g(\vee_{i\in I} m_i), \gamma\} \geq inf\ \{\underset{i\in I}{inf} g(m_i), \delta\}$ for all $m_i \in M, (i \in I)$. By the same arguments, we have $m \in g_v^\delta$ but $(a * m) \notin g_v^\delta$, a contradiction. Therefore, $\{g(a * m), \gamma\} \geq inf\ \{g(m), \delta\}$. Hence g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M by Theorem 6.5.4. ■

If we take $\gamma = 0$ and $\delta = 0.5$ in Theorem 6.5.9, we have Theorem 4.5.11.

Theorem 6.5.11 *[69] Let M be a Q_t-module and g be a f-subset of M. Then g is an $(\in, \in \vee q)$-fuzzy Q_t-submodule of M if and only if each $\emptyset \neq U(g; p)$ is a Q_t-submodule of M for all $p \in (0, 0.5]$.*

Theorem 6.5.12 *Let $2\delta = 1+\gamma$. Then a f-subset g of a Q_t-module M is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M if and only if $\emptyset \neq [g]_v^\delta$ is a Q_t-submodule of M for all $v \in (\gamma, 1]$.*

Proof. The proof is similar to the proof of Theorem 6.5.9 and 6.5.10. ■

Lemma 6.5.13 *Let S be a Q_t-submodule of M. Then the characteristic function K_S of S is an $(\in_\gamma, \in_\gamma)$-fuzzy Q_t-submodule of M.*

Proof. Let $m_i \in M$ and $v_i \in (\gamma, 1]$ be such that $s_{v_i} \in_\gamma K_S$. Then $K_S(m_i) \geq v_i > \gamma$. This implies that $K_S(m_i) = 1$. As S is a Q_t-submodule of M and $m_i \in S$, so $\vee_{i\in I} m_i \in S$. It follows that $K_S(\vee_{i\in I} m_i) = 1 \geq inf\{v_i\} > \gamma$ so that $(\vee_{i\in I} m_i)_{inf\{v_i\}} \in_\gamma K_S$. Now let $m \in M$ and $p \in (\gamma, 1]$ be such that $m_p \in_\gamma K_S$. Then $K_S(m) \geq p > \gamma$ and so $K_S(m) = 1$, i.e., $m \in S$. Since S is Q_t-submodule of M, we have $q * m \in S$ for all $q \in Q_t$ and hence $K_S(q * m) = 1 \geq p > \gamma$. Therefore $(q * m)_p \in_\gamma K_S$. Therefore $w_p \in_\gamma K_S$. Thus, K_S is an $(\in_\gamma, \in_\gamma)$-fuzzy Q_t-submodule of M. ■

Proposition 6.5.14 *Let $\emptyset \neq S \subseteq Q_t$. Then the characteristic function K_S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M if and only if S is a Q_t-submodule of M.*

Proof. Let K_S be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M such that $p_i \in (\gamma, 1]$ and $s_i \in S$. Then $(s_i)_1 \in_\gamma K_S$ which shows that $(\vee_{i\in I} s_i)_1 = (\vee_{i\in I} s_i)_{inf(1,1)}\ (\in_\gamma \vee q_\delta) K_S$. Hence $K_S(\vee_{i\in I} s_i) > \gamma$, and so $\vee_{i\in I} s_i \in S$. Now let $q \in Q_t$ and $s \in S$. Then $K_S(s) = 1$, and thus $s_1 \in_\gamma K_S$. Since K_S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M, it follows that $(q * s)_1 \in_\gamma K_S$ so that $K_S(q * s) = 1$. Hence $q * s \in S$. Thus, S is a Q_t-submodule of M.

Conversely, Let S be a Q_t-submodule of M. Then K_S is an $(\in_\gamma, \in_\gamma)$-fuzzy Q_t-submodule of M by lemma 6.5.13, and therefore K_S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M by Proposition 6.5.7. ■

Chapter 7

On Generalized Fuzzy Filters in Quantales

In this chapter, the concept of (α, β)-$f$uzzy filter is introduced and some related properties are discussed. Further, $(\in, \in \vee q)$-fuzzy filters are discussed. It is investigated that inverse image of an $(\in, \in \vee q)$-fuzzy filter under QH is an $(\in, \in \vee q)$-fuzzy filter. Moreover, these fuzzy filters are characterized by their level sets. Furthermore, in this chapter, we are presenting more general forms of $(\in, \in \vee q)$-$f$uzzy $f$ilters of Quantales. Special attention is given to $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy $f$ilters.

In the first section, (α, β)-$f$uzzy $f$ilters are introduced. It is shown that inverse image of an (α, β)-$f$uzzy $f$ilter under QH is an (α, β)-$f$uzzy filter. Moreover, $(\in, \in \vee q)$-$f$uzzy $f$ilters are discussed in the second section. It is also investigated that if a f-subset g is an $(\in, \in \vee q)$-$f$uzzy $f$ilter of Q'_t, then $\sigma_t^{-1}(g)$ is an $(\in, \in \vee q)$-$f$uzzy $f$ilter of Q_t. In the last section, we define the $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy $f$ilters of a Quantale Q_t. Relation among $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy $f$ilter, $(\in, \in \vee q)$-$f$uzzy $f$ilter and ordinary $f$uzzy $f$ilters are also discussed.

7.1 (α, β)-Fuzzy Filters in Quantales

In this section, α and β will mean any one of $\in, q, \in \vee q$ and $\in \wedge q$, unless otherwise specified. From here onward, we will write (α, β)-FF, $(\in, \in \vee q)$-FF and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ for (α, β)-$f$uzzy $f$ilter, $(\in, \in \vee q)$-$f$uzzy $f$ilter and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy $f$ilter, respectively.

Definition 7.1.1 *[79] A non-empty subset F_r of a quantale Q_t is said to be a filter of Q_t if F_r is closed under $\otimes$ and an upper set i.e., the following conditions hold;*

(1) *for all $z_1 \in Q_t$ and for all $z_2 \in F_r$, $z_2 \leq z_1$ implies $z_1 \in F_r$;*

(2) *for all $z_1, z_2 \in F_r$ implies $z_1 \otimes z_2 \in F_r$.*

Definition 7.1.2 *A f-subset g of a quantale Q_t is called a FF of Q_t if the following assertions hold:*

(1) *for all $z_1, z_2 \in Q_t$, $z_1 \leq z_2$ implies $g(z_1) \leq g(z_2)$;*

(2) *for all $z_1, z_2 \in Q_t$, $g(z_1 \otimes z_2) \geq inf(g(z_1), g(z_2))$.*

Proposition 7.1.3 *Let g_1 and g_2 be FF of Q_t. Then $(g_1 \Cap g_2)$ is a FF of Q_t.*

Proof. Let $z_1, z_2 \in Q_t$ with $z_1 \leq z_2$. As g_1 and g_2 are the FF of Q_t, so

$$g_1(z_1) \leq g_1(z_2) \text{ and } g_2(z_1) \leq g_2(z_2)$$

$$\Longrightarrow inf\{g_1(z_1), g_2(z_1)\} \leq inf\{g_1(z_2), g_2(z_2)\}$$

$$\Longrightarrow (g_1 \Cap g_2)(z_1) \leq (g_1 \Cap g_2)(z_2).$$

Next, as $g_1(z_1 \otimes z_2) \geq inf\{g_1(z_1), g_1(z_2)\}$ and $g_2(z_1 \otimes z_2) \geq inf\{g_2(z_1), g_2(z_2)\}$.

$\Longrightarrow inf\{g_1(z_1 \otimes z_2), g_2(z_1 \otimes z_2)\} \geq inf(inf\{g_1(z_1), g_1(z_2)\}, inf\{g_2(z_1), g_2(z_2)\})$

$\Longrightarrow inf\{g_1(z_1 \otimes z_2), g_2(z_1 \otimes z_2)\} \geq inf(inf\{g_1(z_1), g_2(z_1)\}, inf\{g_1(z_2), g_2(z_2)\})$

$\Longrightarrow (g_1 \Cap g_2)(z_1 \otimes z_2) \geq inf\{(g_1 \Cap g_2)(z_1), (g_1 \Cap g_2)(z_2)\}$.

Therefore, $(g_1 \Cap g_2)$ is a FF of Q_t. ■

Definition 7.1.4 *Let Q_t be a quantale and $\emptyset \neq F_r \subseteq Q_t$. Then the characteristic $function$ K_{Fr} of F_r is defined by*

$$K_{F_r} : Q_t \longrightarrow (0,1], \qquad z \mapsto \begin{cases} 1 & if\ z \in F_r \\ 0 & if\ z \notin F_r. \end{cases}$$

Clearly, a non-empty subset F_r of Q_t is a $f$ilter if and only if the characteristic function K_{F_r} of F_r is a FF of Q_t.

The proof of the theorem below is easy and so excluded.

Theorem 7.1.5 *A f-subset g of Q_t is a FF of Q_t if and only if $\emptyset \neq U(g;p)$ for all $p \in (0,1]$ is a filter of Q_t.*

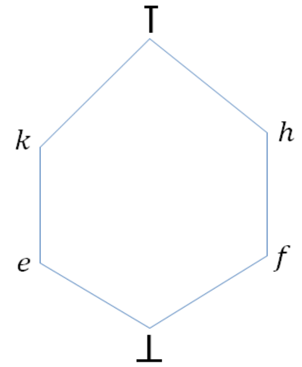

Fig.13

Table. 10

$\otimes$	$\bot$	e	f	k	h	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
e	$\bot$	e	$\bot$	e	$\bot$	e
f	$\bot$	$\bot$	f	$\bot$	f	f
k	$\bot$	e	$\bot$	k	$\bot$	k
h	$\bot$	$\bot$	f	$\bot$	h	h
$\top$	$\bot$	e	f	k	h	$\top$

Example 7.1.6 *Let $(Q_t, \otimes)$ be a Quantale, where Q_t is illustrated in Figure 13 and the binary operations $\otimes$ on the quantale is the same as the meet operation in the lattice Q_t as shown in Table 10. Filters of Q_t are $\{f, h, \top\}, \{h, \top\}, \{\top\}$ and Q_t. Define a f-subset $g : Q_t \longrightarrow (0,1]$ by $g = \frac{0.5}{\bot} + \frac{0.5}{e} + \frac{0.6}{f} + \frac{0.5}{k} + \frac{0.7}{h} + \frac{0.9}{\top}$. Then*

$$U(g,p) = \begin{cases} Q_t & if \;\; 0 < p \leq 0.5 \\ \{f,h,\top\} & if \;\; 0.5 < p \leq 0.6 \\ \{h,\top\} & if \;\; 0.6 < p \leq 0.7 \\ \{\top\} & if \;\; 0.7 < p \leq 0.9 \\ \emptyset & if \;\; 0.9 < p \leq 1. \end{cases}$$

Thus, by Theorem 7.1.5, g *is a* FF *of* Q_t.

Theorem 7.1.7 *Let* g *be a* f*-subset of* Q_t. *Then* $\emptyset \neq U(g;p)$ *is a filter of* Q_t *for all* $p \in (0.5, 1]$ *if and only if* g *satisfies the conditions below:*

(1) $sup(g(y), 0.5) \geq g(z)$ *with* $z \leq y$;

(2) $sup(g(z \otimes y), 0.5) \geq inf(g(z), g(y)$ *for all* $z, y \in Q_t$.

Proof. Assume that $U(g;p)$ is a filter of Q_t for all $p \in (0.5, 1]$. If there exist $z, w \in Q_t$ with $z \leq w$ such that the condition (1) is not valid, then $sup(g(w), 0.5) < g(z) = r$. Then $r \in (0.5, 1]$, $z \in U(g;r)$. But $r > g(w)$ implies that $w \notin U(g;r)$, we get a contradiction. Hence condition (1) is valid.

If there are $z, w \in Q_t$ such that $inf(g(z), g(w)) = s > sup(g(z \otimes w), 0.5)$, then $z, w \in U(g;s)$ and $s \in (0.5, 1]$. But $g(z \otimes w) < s$. Thus, $z \otimes w \notin U(g;s)$, a contradiction. Hence condition (2) is valid.

Conversely, suppose that g satisfies the conditions (1) and (2). Let $w, z \in Q_t$ with $w \leq z$ be such that $w \in U(g;p)$ for some $p \in (0.5, 1]$. Then $g(w) \geq p$. Since $w \leq z$ so it follows by condition (1)

$$sup(g(z), 0.5) \geq g(w) \geq p > 0.5$$

so that $g(z) \geq p$, i.e., $z \in U(g;p)$. Now, for $w, z \in U(g;p)$, we have,

$$sup(g(w \otimes z), 0.5) \geq inf(g(w), g(z)) \geq p > 0.5$$

and so $g(w \otimes z) \geq p$. It follows that $w \otimes z \in U(g;p)$. Thus, $U(g;p)$ is a filter of Q_t for all $p \in (0.5, 1]$. ■

Definition 7.1.8 *A* f*-subset* g *of a quantale* Q_t *is called an* (α, β)*-*FF *of* Q_t, *if it satisfies the conditions below;*

(1) $z_p \alpha g \rightarrow w_p \beta g$ *with* $z \leq w$;

(2) $z_p \alpha g, w_v \alpha g \rightarrow (z \otimes w)_{inf(p,v)} \beta g$, *for all* $z, w \in Q_t$ *and* $p, v \in (0, 1]$.

Theorem 7.1.9 *Let* g *be a non-zero* (α, β)*-*FF *of* Q_t. *Then* $g_\circ = \{z \in Q_t \mid g(z) > 0\}$ *is a filter of* Q_t.

Proof. Let $z, w \in Q_t$ and $z \leq w$ be such that $z \in g_\circ$. Then $g(z) > 0$. Assume that $g(w) = 0$. If $\alpha \in \{\in, \in \vee q\}$, then $z_{g(z)}\alpha g$ but $w_{g(w)}\overline{\beta} g$ for every $\beta \in \{\in,\ q,\ \in \wedge q, \in \vee q\}$, a contradiction. Further, $z_1 qg$, but $w_1\overline{\beta} g$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Hence $g(w) > 0$, that is $w \in g_\circ$. Now let z, $w \in g_\circ$. Then $g(w) > 0$ and $g(z) > 0$. Assume that $g(z \otimes w) = 0$ and let $\alpha \in \{\in, \in \vee q\}$, then $z_{g(z)}\alpha g$, $w_{g(w)}\alpha g$ but $(z \otimes w)_{inf(g(z),g(w))}\overline{\beta} g$ for every $\beta \in \{\in \wedge q, \in, \in \vee q, q\}$, a contradiction. Also $z_1 qg$ and $w_1 qg$ but $(z \otimes w)_1\overline{\beta} g$ for every $\beta \in \{\in \wedge q, \in, \in \vee q, q\}$, a contradiction. Thus, $g(z \otimes w) > 0$, it follows that, $z \otimes w \in g_\circ$. Therefore $g_\circ$ is a $f$ilter of Q_t. ■

Proposition 7.1.10 *Let F_r be a filter of Q_t. Then a f-subset g of Q_t such that*

$$g(w) = \begin{cases} \geq 0.5 & if\ w \in F_r \\ 0 & if\ w \in Q_t \backslash F_r. \end{cases} \quad \textit{for all } w \in Q_t.$$

is an $(\alpha, \in \vee q)$-FF of Q_t.

Proof. Suppose F_r is a filter of Q_t.

(**i**) Let $w, z \in Q_t$, $w \leq z$ and $v \in (0, 1]$ be such that $w_v \in g$. Then $w \in F_r$ and we have $z \in F_r$. If $v \leq 0.5$ then $g(z) \geq 0.5 \geq v$ implies $g(z) \geq v$, and so $z_v \in g$. If $v > 0.5$ then $g(z) + v > 0.5 + 0.5 = 1$ and $z_v qg$. Hence $z_v(\in \vee q)g$. Let $w, z \in Q_t$ and $v, r \in (0, 1]$ be such that $w_v \in g$ and $z_r \in g$. Then $w, z \in F_r$ and we have $w \otimes z \in F_r$. If $inf(v, r) \leq 0.5$ then $g(w \otimes z) \geq 0.5 \geq inf(v, r)$ and so $g(w \otimes z) \geq inf(v, r)$ implies $(w \otimes z)_{inf(v,r)} \in g$. If $inf(v, r) > 0.5$ then $g(w \otimes z) + inf(v, r) > 0.5 + 0.5 = 1$ and so $(w \otimes z)_{inf(v,r)} qg$. Hence $(w \otimes z)_{inf(v,r)}(\in \vee q)g$.

(**ii**) Let $w, z \in Q_t$ and $v \in (0, 1]$ with $w \leq z$ be such that $w_v qg$. Then $w \in F_r$ and $z \geq w \in F_r$ implies that $z \in F_r$. If $0.5 \geq v$ then $g(z) \geq 0.5 \geq v$ implies that $g(z) \geq v$ and so $z_v \in g$. If $0.5 < v$ then $g(z) + v > 0.5 + 0.5 = 1$ and $z_v qg$. Hence $z_v(\in \vee q)g$. Let $u, v \in (0, 1]$ and $w, z \in Q_t$ be such that $w_u qg$ and $z_v qg$. Then $w, z \in F_r$ and so $w \otimes z \in F_r$. If $0.5 \geq inf(u, v)$ then $g(w \otimes z) \geq 0.5 \geq inf(u, v)$ and so $g(w \otimes z) \geq inf(u, v)$ implies $(w \otimes z)_{inf(u,v)} \in g$. If $inf(u, v) > 0.5$ then $g(w \otimes z) + inf(u, v) > 0.5 + 0.5 = 1$ and so $(w \otimes z)_{inf(u,v)} qg$. Thus, $(w \otimes z)_{inf(u,v)}(\in \vee q)g$.

(**iii**) Let $y, z \in Q_t$ and $p, v \in (0, 1]$ be such that $y_p \in g$ and $z_v qg$. Then $g(y) \geq p$ and $g(z) + v > 1$. Thus, $y, z \in F_r$ and so $y \otimes z \in F_r$, we have $g(y \otimes z) \geq 0.5$. Thus, $(y \otimes z)_{inf(p,v)} \in g$ for $inf(p, v) \leq 0.5$ and $(y \otimes z)_{inf(p,v)} qg$ for $inf(p, v) > 0.5$. Thus $(y \otimes z)_{inf(p,v)}(\in \vee q)g$. ■

Lemma 7.1.11 *A f-subset g in a quantale Q_t is a FF of Q_t if and only if it satisfies*;

(1) $w_v \in g$ *and* $w \leq z \longrightarrow z_v \in g$;

(2) $z_p, w_v \in g \longrightarrow (z \otimes w)_{inf(p,v)} \in g$ *for all* $z, w \in Q_t$ *and* $p, v \in (0, 1]$.

Proof. Let g be a FF of Q_t. Let $w_v \in g$ for some $v \in (0, 1]$. Then $g(w) \geq v$. Since g is a FF of Q_t so, for $w \leq z$, we have $v \leq g(w) \leq g(z)$. This shows that $g(z) \geq v$. Hence $z_v \in g$. Consider $z, w \in Q_t$, $p, v \in (0, 1]$ be such that $z_p \in g$ and $w_v \in g$. Then $g(z) \geq p$ and $g(w) \geq v$. But g is a FF of Q_t so, we have $g(z \otimes w) \geq inf(g(z), g(w))$ $\geq inf(p, v)$. Thus $g(z \otimes w) \geq inf(p, v)$. This implies that $(z \otimes w)_{inf(p,v)} \in g$.

Conversely, suppose that g satisfies the conditions (1) and (2). First we show that for all $z, w \in Q_t$, $z \leq w$ implies $g(z) \leq g(w)$. Suppose that $g(z) > g(w)$ for some $z, w \in Q_t$, then there exists $v \in (0, 1]$ such that $g(z) \geq v > g(w)$. Then $z_v \in g$ but $w_v \overline{\in} g$, a contradiction to the hypothesis (1). Now we show that $inf(g(z), g(w)) \leq g(z \otimes w)$ for all $w, z \in Q_t$. On contrary suppose that $g(a \otimes c) < inf(g(a), g(c))$ for some $a, c \in Q_t$. Let $p \in (0, 1]$ be such that $g(a \otimes c) < p \leq inf(g(a), g(c))$. Then $g(a) > p$ and $g(c) > p$ but $(a \otimes c)_p \overline{\in} g$. This contradicts our hypothesis (2). Thus, $inf(g(z), g(w)) \leq g(z \otimes w)$ for all $z, w \in Q_t$. Hence g is a FF of a quantale Q_t. ■

Remark 7.1.12 *A f-subset g of a quantale Q_t is a FF of Q_t if and only if g is an $(\in, \in)$-FF of Q_t.*

Proposition 7.1.13 *Let $\sigma_t : Q_t \longrightarrow Q'_t$ be a QH and g be an (α, β)-FF of Q'_t. Then $\sigma_t^{-1}(g)$ is an (α, β)-FF of Q_t.*

Proof. Let $z, w \in Q_t$ and $p, v \in (0, 1]$ be such that $z_p \alpha \sigma_t^{-1} g$ and $w_v \alpha \sigma_t^{-1} g$. Then $(\sigma_t(z))_p \alpha g$ and $(\sigma_t(w))_v \alpha g$ by Proposition 4.1.16. Since g is an (α, β)-FF of Q'_t, we have $(\sigma_t(z) \otimes' \sigma_t(w))_{inf(p,v)} \beta g$ and $(\sigma_t(z \otimes w))_{inf(p,v)} \beta g$ by using QH. Thus, $(z \otimes w)_{inf(p,v)} \beta \sigma_t^{-1} g$ by Proposition 4.1.16. Let $z_p \alpha \sigma_t^{-1} g$ such that $z \leq w$. Then $(\sigma_t(z))_p \alpha g$. As σ_t is an order preserving hence $\sigma_t(z) \leq \sigma_t(w)$. Since g is an (α, β)-FF of Q'_t, we have $\sigma_t(w)_p \beta g$. By Proposition 4.1.16, $w_p \beta \sigma_t^{-1} g$. Hence $\sigma_t^{-1}(g)$ is an (α, β)-FF of Q_t. ■

7.2 $(\in, \in \vee q)$-Fuzzy Filters of Quantale

Now, the concept of $(\in, \in \vee q)$-FF in quantale is introduced in this section and we characterize the $f$ilters of Quantale in terms of $(\in, \in \vee q)$- FF. Also with the help of QH, we will show that inverse image of $(\in, \in \vee q)$-FF is $(\in, \in \vee q)$-FF.

Definition 7.2.1 *A f-subset g of a quantale Q_t is called an $(\in, \in \vee q)$-FF of Q_t if it satisfies:*

(1) $z \leq y, z_p \in g \rightarrow y_p(\in \vee q)g$;

(2) $z_p \in g, y_v \in g \rightarrow (z \otimes y)_{inf(p,v)}(\in \vee q)g$ *for all* $z, y \in Q_t$ *and* $p, v \in (0, 1]$.

Example 7.2.2 *Let $(Q_t, \otimes)$ be a quantale, where Q_t is depicted in Figure* 13 *and the binary operation $\otimes$ on the quantale is the same as the meet operation in the lattice Q_t as shown in Table* 11. *Define a f-subset g of Q_t as* $g = \frac{0.5}{\perp} + \frac{0.6}{e} + \frac{0.65}{f} + \frac{0.6}{k} + \frac{0.7}{h} + \frac{0.9}{\top}$. *Then g is an $(\in, \in \vee q)$-FF of Q_t. But*

(1) *g is not an $(\in, \in)$-FF of Q_t, since $e_{0.58} \in g$ and $f_{0.63} \in g$ but* $(e \otimes f)_{inf(0.63,0.58)} = \perp_{0.58}\overline{\in}g$.

(2) *g is not an $(q, \in)$-FF of Q_t, since $f_{0.52}qg$ and $k_{0.51}qg$ but* $(f \otimes k)_{inf(0.52,0.51)} = \perp_{0.51}\overline{\in}g$.

(3) *g is not an $(\in, q)$-FF of Q_t, since $k_{0.57} \in g$ and $h_{0.4} \in g$ but* $(k \otimes h)_{inf(0.57,0.4)} \in g = \perp_{0.4}\overline{q}g$.

Theorem 7.2.3 *A f-subset g of Q_t is an $(\in, \in \vee q)$-FF of Q_t if and only if it satisfies the conditions below:*

(1) $z \leq y$, $g(y) \geq inf(g(z), 0.5)$;

(2) $g(z \otimes y) \geq inf(g(z), g(y), 0.5)$ *for all* $z, y \in Q_t$.

Proof. Let g be an $(\in, \in \vee q)$-FF and $z, y \in Q_t$ be such that $z \leq y$. If $g(z) = 0$, then $g(y) \geq inf(g(z), 0.5)$. Let $g(z) \neq 0$ and assume, on the contrary that $g(y) < inf(g(z), 0.5)$. Take $v \in (0, 1]$ such that $g(y) < v \leq inf(g(z), 0.5)$. Case-1 If $g(z) < 0.5$, then $g(y) < v \leq g(z)$ and so $z_v \in g$ but $y_v\overline{\in}g$. Also $g(y)+v < 0.5+0.5 = 1$ so $y_v\overline{q}g$. Thus, $z_v \in g$ but $y_v(\overline{\in \vee q})g$, a contradiction. Case-2 If $g(z) \geq 0.5$ then $g(y) < 0.5$ and so $z_{0.5} \in g$ but $y_{0.5} \overline{\in} g$ and $g(y) + 0.5 < 1$, i.e., $y_{0.5}\overline{q}g$, again a contradiction. Hence

$g(y) \geq inf(g(z), 0.5)$ for all $z, y \in Q_v$ with $z \leq y$. Let $w, y \in Q_t$ be such that $g(w \otimes y) < inf(g(w), g(y), 0.5)$. Take $p \in (0, 1]$ such that $g(w \otimes y) < p \leq inf(g(w), g(y), 0.5)$. Case-1 If $inf(g(w), g(y)) < 0.5$ then $g(w \otimes y) < p \leq inf(g(w), g(y))$ and $w_p, y_p \in g$ but $(w \otimes y)_p \overline{\in} g$. Also we have, $g(w \otimes y) + p < 0.5 + 0.5 = 1$, so $(w \otimes y)_p \overline{q} g$, a contradiction. Let $0.5 \leq inf(g(w), g(y))$. Then $w_{0.5}, y_{0.5} \in g$ but $(w \otimes y)_{0.5} \overline{\in} g$ and $g(w \otimes y) + 0.5 < 1$, i.e., $(w \otimes y)_{0.5} \overline{q} g$, again a contradiction. Thus, $g(w \otimes y) \geq inf(g(w), g(y), 0.5)$ for all $w, y \in Q_t$.

Conversely suppose that the conditions (1) and (2) are satisfied. Let $w, z \in Q_t$ and $w_v \in g$ with $w \leq z$ for some $v \in (0, 1]$. Then $g(w) \geq v$. By hypothesis, $g(z) \geq inf(g(w), 0.5) \geq inf(v, 0.5)$. Case-1. If $v \leq 0.5$, then $g(z) \geq v$ and $z_v \in g$. If $v > 0.5$ then $g(z) + v > 0.5 + 0.5 = 1$ and so $z_v q g$, i.e., z_v $(\in \vee q) g$. Let $v_1, v_2 \in (0, 1]$ and $w, z \in Q_t$ be such that w_{v_1}, $z_{v_2} \in g$. Then $g(w) \geq v_1$ and $g(z) \geq v_2$ and so by hypothesis we have, $inf(v_1, v_2, 0.5) \leq inf(g(w), g(z), 0.5) \leq g(w \otimes z)$. Case-1. If $inf(v_1, v_2) \leq 0.5$ then $g(w \otimes z) \geq inf(v_1, v_2)$ and $(w \otimes z)_{inf(v_1, v_2)} \in g$. Case-2. If $inf(v_1, v_2) > 0.5$ then $g(w \otimes z) + inf(v_1, v_2) > 0.5 + 0.5 = 1$ and so $(w \otimes z)_{inf(v_1, v_2)} q g$. Hence $(w \otimes z)_{inf(v_1, v_2)}$ $(\in \vee q) g$. Consequently, g is an $(\in, \in \vee q)$-FF of Q_t. ■

Remark 7.2.4 *A f-subset g of a quantale Q_t is an $(\in, \in \vee q)$-FF of Q_t if and only if it satisfies the conditions* (1) *and* (2) *of Theorem* 7.2.3.

Lemma 7.2.5 *Every $(\in, \in)$-FF of Q_t is an $(\in, \in \vee q)$-FF of Q_t.*

Proof. Obvious. ■

For $(\in, \in \vee q)$-FF to be an $(\in, \in)$-FF of Q_t, some condition is imposed in the next Proposition.

Proposition 7.2.6 *Let g be an $(\in, \in \vee q)$-FF of Q_t such that $g(z) < 0.5$ for all $z \in Q_t$. Then g is an $(\in, \in)$-FF of Q_t.*

Proof. Let g be an $(\in, \in \vee q)$-FF of Q_t such that $g(z) < 0.5$ for all $z \in Q_t$. Then by Theorem 7.2.3, if $z \leq y$ then $g(y) \geq inf(g(z), 0.5) = g(z)$. Now if $z, w \in Q_t$ then $g(z \otimes y) \geq inf(g(z), g(y), 0.5) = inf(g(z), g(y)$. Hence g is an $(\in, \in)$-FF of Q_t by Lemma 7.1.11. ■

Lemma 7.2.7 *Let $(Q_t, \otimes)$ be a quantale and $\emptyset \neq F_r \subseteq Q_t$. Then the characteristic function K_{F_r} is an $(\in, \in)$-FF of Q_t if and only if F_r is a filter of Q_t.*

Proof. Let $w, z \in Q_t$ be such that $z \leq w$ and $z_p \in K_{F_r}$ where $p \in (0,1]$. Then $K_{F_r}(z) \geq p > 0$, and so $K_{F_r}(z) = 1$, i.e., $z \in F_r$. Since F_r is a filter, we have $w \in F_r$ and so $K_{F_r}(w) = 1 \geq p$. Therefore $w_p \in K_{F_r}$. Suppose $p, v \in (0,1]$ and $w, z \in Q_t$ be such that $w_p \in K_{F_r}$ and $z_v \in K_{F_r}$. Then $K_{F_r}(w) \geq p > 0$ and $K_{F_r}(z) \geq v > 0$, which show that $K_{F_r}(w) = K_{F_r}(z) = 1$. Thus $w, z \in F_r$ and F_r is a $f$ilter so $w \otimes z \in F_r$. It shows that $K_{F_r}(w \otimes z) = 1 \geq inf(p, v)$ so that $(w \otimes z)_{inf(p,v)} \in K_{F_r}$ and consequently K_{F_r} is an $(\in, \in)$-FF of Q_t.

Conversely, let K_{F_r} be an $(\in, \in)$-FF of Q_t and $w, z \in F_r$. Then $w_1 \in K_{F_r}$ and $z_1 \in K_{F_r}$ which show that $(w \otimes z)_1 = (w \otimes z)_{inf(1,1)} \in K_{F_r}$. Hence $K_{F_r}(w \otimes z) = 1$, and so $w \otimes z \in F_r$. Let $w, z \in Q_t$ and $w \leq z$ be such that $w \in F_r$. Then $K_{F_r}(w) = 1$, and thus $w_1 \in K_{F_r}$. Since K_{F_r} is an $(\in, \in)$-FF, so we have $z_1 \in K_{F_r}$. Thus $K_{F_r}(z) = 1$ and $z \in F_r$. Hence F_r is a filter of Q_t. ■

Theorem 7.2.8 *The characteristic function K_{F_r} is an $(\in, \in \vee q)$-FF of a quantale Q_t if and only if F_r is a filter of Q_t, for any $\emptyset \neq F_r \subseteq Q_t$.*

Proof. Suppose K_{F_r} is an $(\in, \in \vee q)$-FF of Q_t and $w, z \in F_r$. Then $w_1 \in K_{F_r}$ and $z_1 \in K_{F_r}$ which show that $(w \otimes z)_1 = (w \otimes z)_{inf(1,1)}\ (\in \vee q) K_{F_r}$. Hence $K_{F_r}(w \otimes z) > 0$, and so $w \otimes z \in F_r$. Let $w, z \in Q_t$ and $z \in F_r$ be such that $z \leq w$. Then $K_{F_r}(z) = 1$, and thus $z_1 \in K_{F_r}$. Since K_{F_r} is an $(\in, \in \vee q)$-FF, so we have $w_1 \in K_{F_r}$. Thus $K_{F_r}(w) = 1$. Hence $w \in F_r$.

Conversely, if F_r is a filter of Q_t, then K_{F_r} is an $(\in, \in)$-FF of Q_t by lemma 7.2.7, and therefore K_{F_r} is an $(\in, \in \vee q)$-FF of Q_t by Corollary 7.2.5. ■

Theorem 7.2.9 *A f-subset g of Q_t is an $(\in, \in \vee q)$-FF of Q_t if and only if $U(g; p) = \{w \in Q_t : g(w) \geq p\}$ is a filter of Q_t for all $p \in (0, 0.5]$.*

Proof. Suppose g is an $(\in, \in \vee q)$-FF of Q_t. Let $w, b \in Q_t$ be such that $w \leq b$, and let $p \in (0, 0.5]$ be such that $w \in U(g; p)$. Then $g(w) \geq p$ and it is clear from Theorem 7.2.3(1) that

$$g(b) \geq inf(g(w), 0.5) \geq inf(p, 0.5) = p$$

and so $b \in U(g; p)$. Let $w, a \in U(g; p)$ for some $p \in (0, 0.5]$. Thus from Theorem 7.2.3(2), we have $g(w \otimes a) \geq inf(g(w), g(a), 0.5) \geq inf(p, 0.5) = p$, and so $w \otimes a \in U(g; p)$.

Conversely, let $U(g;p)$ be a $f$ilter of Q_t for all $p \in (0, 0.5]$. If there exist $a, y \in Q_t$ with $a \leq y$ such that $g(y) < inf(g(a), 0.5)$, then select $v \in (0, 0.5]$ such that $g(y) < v \leq inf(g(a), 0.5)$, then $a \in g_v$ but $y \notin U(g;p)$, a contradiction. Hence $g(y) \geq inf(g(a), 0.5)$ for all $a, y \in Q_t$ with $a \leq y$. If there exist $z, y \in Q_t$ such that $g(z \otimes y) < inf(g(z), g(y), 0.5)$. We can choose $s \in (0, 0.5]$ such that $inf(g(z), g(y), 0.5) \geq s > g(z \otimes y)$. Then $z, y \in U(g;s)$ but $z \otimes y \notin U(g;s)$, a contradiction. Hence $inf(g(z), g(y), 0.5) \leq g(z \otimes y)$ for all $z, y \in Q_t$. By Theorem 7.2.3, g is an $(\in, \in \vee q)$-FF of Q_t. ■

Theorem 7.2.10 *Let Q_t and Q'_t be two quantales and $\sigma_t : Q_t \longrightarrow Q'_t$ be a QH. Let g be an $(\in, \in \vee q)$-FF of Q'_t. Then $\sigma_t^{-1}(g)$ is an $(\in, \in \vee q)$-FF of Q_t.*

Proof. Suppose $z, y \in Q_t$ with $y \leq z$. Then $\sigma_t(y) \leq \sigma_t(z)$.

$$\begin{aligned} \sigma_t^{-1}(g)(z) &= g(\sigma_t(z)) \\ &\geq inf(g(\sigma_t(y)), 0.5) \\ &= inf(\sigma_t^{-1}(g)(y), 0.5). \end{aligned}$$

Hence, $\sigma_t^{-1}(g)(z) \geq inf(\sigma_t^{-1}(g)(y), 0.5)$.

Now,

$$\begin{aligned} \sigma_t^{-1}(g)(z \otimes w) &= g(\sigma_t(z \otimes w)) \\ &= g(\sigma_t(z) \otimes' \sigma_t(w)) \\ &\geq inf(g(\sigma_t(z)), g(\sigma_t(w)), 0.5) \\ &= inf(\sigma_t^{-1}(g)(z), \sigma_t^{-1}(g)(w), 0.5). \end{aligned}$$

Thus, $\sigma_t^{-1}(g)(z \otimes w) \geq inf(\sigma_t^{-1}(g)(z), \sigma_t^{-1}(g)(w), 0.5)$ for all $z, w \in Q_t$.

By Theorem 7.2.3, we have $\sigma_t^{-1}(g)$ is an $(\in, \in \vee q)$-FF of Q_t. ■

7.3 $(\in_\gamma, \in_\gamma \vee q_\delta)$- Fuzzy Filters of Quantale

In this section, some more general forms of $(\in, \in \vee q)$-FF are introduced and we introduce the notion of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ in quantale. Furthermore, filter and fuzzy filter (FF) of the types $(\in_\gamma, \in_\gamma \vee q_\delta)$ are linked by using level subsets.

Definition 7.3.1 *A f-subset g of a quantale Q_t is said to be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t, if*

$(F_r)_1$ $w_v \in_\gamma g \longrightarrow z_v(\in_\gamma \vee q_\delta)g$ *with* $w \leq z$;

$(F_r)_2$ $z_p \in_\gamma g, w_v \in_\gamma g \longrightarrow (z \otimes w)_{inf(p,v)}(\in_\gamma \vee q_\delta)g$ *for all* $z, w \in Q_t$ *and* $p, v \in (\gamma, 1]$.

Example 7.3.2 *Consider the quantale as given in Example* 7.1.6. *Taking* $g = \frac{0.5}{\bot} + \frac{0.6}{e} + \frac{0.65}{f} + \frac{0.6}{k} + \frac{0.72}{h} + \frac{0.91}{\top}$. *Then g is an* $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$*-FF of* Q_t.

Theorem 7.3.3 *Let g be a f-subset of a quantale* Q_t *and g be a* $(q_\delta, \in_\gamma \vee q_\delta)$*-FF of* Q_t. *Then the following conditions hold:*

(1) $sup(g(w), \gamma) \geq inf(g(z), \delta)$ *with* $z \leq w$;

(2) $sup(g(z \otimes w), \gamma) \geq inf(g(z), g(w), \delta)$ *for all* $z, y, w \in Q_t$.

Proof. Let $z, w \in Q_t$ be such that $sup(g(w), \gamma) < inf\ (g(z), \delta)$ with $z \leq w$. Then for all $\gamma < p \leq 1$ such that

$$2\delta - sup(g(w), \gamma) > p \geq 2\delta - inf(g(z), \delta)$$

we have,

$$2\delta - g(w) \geq 2\delta - sup(g(w), \gamma) > p \geq sup(2\delta - g(z), \delta)$$

That is, $2\delta - g(w) > p$, $2\delta - g(z) < p$

and so,

$$g(z) + p > 2\delta,\ g(w) + p < 2\delta$$

and $g(w) < \delta < p$. Hence $z_p q_\delta g$ but $w_p\overline{(\in_\gamma \vee q_\delta)}g$, a contradiction. Hence, sup $(g(w), \gamma) \geq inf\ (g(z), \delta)$ with $z \leq w$.

If there exist $z, w \in Q_t$ such that $sup(g(z \otimes w), \gamma) < inf\ (g(z), g(w), \delta)$. Then for all $\gamma < v \leq 1$ such that

$$2\delta - sup(g(z \otimes w), \gamma) > v \geq 2\delta - inf(g(z), g(w), \delta)$$

we have,

$$2\delta - g(z \otimes w) \geq 2\delta - sup(g(z \otimes w), \gamma) > v \geq sup(2\delta - g(z), 2\delta - g(w), \delta)$$

We have, $2\delta - g(z \otimes w) > v$, $2\delta - g(z) < v$, $2\delta - g(w) < v$

and so,

$$g(z) + v > 2\delta,\ g(w) + v > 2\delta,\ g(z \otimes w) + v < 2\delta$$

and $g(z \otimes w) < \delta < v$. Hence $w_v q_\delta g, z_v q_\delta g$ but $(z \otimes w)_v \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Therefore $sup(g(z \otimes w), \gamma) \geq inf(g(z), g(w), \delta)$ for all $z, w \in Q_t$. ■

Theorem 7.3.4 *A f-subset g of a quantale Q_t is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t if and only if the conditions below hold:*

(1) *$sup\ (g(w), \gamma) \geq inf\ (g(z), \delta)$ with $z \leq w$;*

(2) *$sup\ (g(z \otimes w), \gamma) \geq inf\ (g(z), g(w), \delta)$ for all $z, w \in Q_t$.*

Proof. $(F_r)_1 \implies (1)$. If there exist $z, w \in Q_t$ with $z \leq w$ such that $sup(g(w), \gamma) < p \leq inf\ (g(z), \delta)$. Then, $g(z) \geq p > \gamma$, $g(w) < p$ and $g(w) + p < 2p \leq 2\delta$. This implies that $z_p \in_\gamma g$ but $w_p \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Hence (1) is valid.

$(1) \implies (F_r)_1$. Assume that there exist $z, w \in Q_t$ with $z \leq w$ and $v \in (\gamma, \delta]$ such that $z_p \in_\gamma g$ but $w_p \overline{(\in_\gamma \vee q_\delta)} g$, then $g(z) \geq p > \gamma$ and $g(w) < p$ and $g(w) + p \leq 2\delta$. It follows that $g(w) < \delta$ and hence, $sup(g(w), \gamma) < inf(g(z), \delta)$, a contradiction.

$(F_r)_2 \implies (2)$. If there exist $z, w \in Q_t$ such that $sup\ (g(z \otimes w), \gamma) < v \leq inf\ (g(z), g(w), \delta)$. Then $g(z) \geq v > \gamma$, $g(w) \geq v > \gamma$, but $g(z \otimes w) < v$ and $g(z \otimes w) + v < 2v \leq 2\delta$, i.e., $z_v \in_\gamma g$, $w_v \in_\gamma g$ but $(z \otimes w)_v \overline{(\in_\gamma \vee q_\delta)} g$, a contradiction. Hence $sup(g(z \otimes w), \gamma) \geq inf(g(z), g(w), \delta)$ for all $z, w \in Q_t$.

$(2) \implies (F_r)_2$. Suppose there exist $z, w \in Q_t$ and $u, v \in (\gamma, \delta]$ such that $z_u \in_\gamma g$ and $w_v \in_\gamma g$ but $(z \otimes w)_{inf(u,v)} \overline{(\in_\gamma \vee q_\delta)} g$, then $g(z) \geq u > \gamma$, $g(w) \geq v > \gamma$, $g(z \otimes w) < inf(u, v)$ and $g(z \otimes w) + inf(u, v) \leq 2\delta$. It concludes that $g(z \otimes w) < \delta$ and so $sup(g(z \otimes w), \gamma) < inf(g(z), g(w), \delta)$, a contradiction. Hence $(F_r)_2$ is valid. ■

Proposition 7.3.5 *If g_1 and g_2 are $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t, then $(g_1 \Cap g_2)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t.*

Proof. Let $z_1, z_2 \in Q_t$ and $\gamma, \delta \in (0, 1]$ with $\gamma < \delta$. Since g_1 and g_2 are $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t, so by Theorem 7.3.4, we have $sup\{g_1(z_2), \gamma\} \geq inf\{g_1(z_1), \delta\}$ with $z_1 \leq z_2$ and $sup\{g_2(z_2), \gamma\} \geq inf\{g_2(z_1), \delta\}$.

Now, consider

$$\begin{aligned} sup\{(g_1 \Cap g_2)(z_2), \gamma\} &= sup\{g_1(z_2) \wedge g_2(z_2), \gamma\} \\ &= sup\{g_1(z_2), \gamma\} \wedge sup\{g_2(z_2), \gamma\} \\ &\geq inf\{g_1(z_1), \delta\} \wedge inf\{g_2(z_1), \delta\} \\ &= inf\{g_1(z_1) \wedge g_2(z_1), \delta\}. \end{aligned}$$

That is, $sup\{(g_1 \Cap g_2)(z_2), \gamma\} \geq inf\{(g_1 \Cap g_2)(z_1), \delta\}$.

Next, as $sup\{g_1(z_1 \otimes z_2), \gamma\} \geq inf\{g_1(z_1), g_1(z_2), \delta\}$ and

$sup\{g_2(z_1 \otimes z_2), \gamma\} \geq inf\{g_2(z_1), g_2(z_2), \delta\}$.

Now, consider

$$\begin{aligned} sup\{(g_1 \Cap g_2)(z_1 \otimes z_2), \gamma\} &= sup\{g_1(z_1 \otimes z_2) \wedge g_2(z_1 \otimes z_2), \gamma\} \\ &= sup\{g_1(z_1 \otimes z_2), \gamma\} \wedge sup\{g_2(z_1 \otimes z_2), \gamma\} \\ &\geq inf\{g_1(z_1), g_1(z_2), \delta\} \wedge inf\{g_2(z_1), g_2(z_2), \delta\} \\ &= inf\{g_1(z_1) \wedge g_2(z_1), g_1(z_2) \wedge g_2(z_2), \delta\}. \end{aligned}$$

Hence, $sup\{(g_1 \Cap g_2)(z_1 \otimes z_2), \gamma\} \geq inf\{(g_1 \Cap g_2)(z_1), (g_1 \Cap g_2)(z_2), \delta\}$.

Therefore, $g_1 \Cap g_2$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t by Theorem 7.3.4. ■

For any $g \in \mathcal{F}(Q_t)$, where $\mathcal{F}(Q_t)$ denotes the set of all f-subsets of Q_t, we define

$$g_v = \{y \in Q_t \mid y_v \in_\gamma g\} \text{ for all } v \in (\gamma, 1];$$

$$g_v^\delta = \{y \in Q_t \mid y_v q_\delta g\} \text{ for all } v \in (\gamma, 1];$$

and

$$[g]_v^\delta = \{y \in Q_t \mid y_v(\in_\gamma \vee q_\delta) g\} \text{ for all } v \in (\gamma, 1].$$

It follows that $[g]_v^\delta = g_v \cup g_v^\delta$.

Corollary 7.3.6 *Let $\gamma, \gamma', \delta, \delta' \in [0,1]$ be such that $\gamma < \delta, \gamma' < \delta'$, $\gamma' < \gamma$ and $\delta' < \delta$. Then every $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t is an $(\in_{\gamma'}, \in_{\gamma'} \vee q_{\delta'})$-$FF$ of Q_t.*

Example 7.3.7 *Consider the quantale Q_t as given in Example 7.1.6 and define a f-subset g of Q_t as follows:*

$$g = \frac{0.5}{\perp} + \frac{0.65}{e} + \frac{0.7}{f} + \frac{0.65}{k} + \frac{0.75}{h} + \frac{0.95}{\top}.$$

Then g is an $(\in_{0.3}, \in_{0.3} \vee q_{0.4})$-$FF$ of Q_t but it is not an $(\in_{0.3}, \in_{0.3} \vee q_{0.9})$-$FF$ of Q_t.

Now, we characterize $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t by their level sets.

Theorem 7.3.8 *Let $g \in \mathcal{F}(Q_t)$. Then*

(1) g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t if and only if $\emptyset \neq g_v$ is a filter of Q_t for all $v \in (\gamma, \delta]$.

(2) If $2\delta = 1 + \gamma$, then g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ if and only if $g_v^\delta (\neq \emptyset)$ is a filter of Q_t for all $v \in (\delta, 1]$.

(3) If $2\delta = 1 + \gamma$, then g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ if and only if $[g]_v^\delta$ $(\neq \emptyset)$ is a filter of Q_t for all $v \in (\gamma, 1]$.

Proof. (**1**). Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t. Suppose $z, w \in Q_t$ with $w \leq z$ and $v \in (\gamma, \delta]$ be such that $w \in g_v$. Then $w_v \in_\gamma g$ and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t, so $z_v(\in_\gamma \vee q_\delta)g$. If $z_v \in_\gamma g$, then $z \in g_v$ and if $z_v q_\delta g$, then $g(z) > 2\delta - v \geq v > \gamma$, that is, $z \in g_v$. Now we have to show that $z \otimes w \in g_v$ for all $z, w \in g_v$. Let $z, w \in Q_t$ be such that $z, w \in g_v$ for some $v \in (\gamma, \delta]$. Then $w_v \in_\gamma g$ and $z_v \in_\gamma g$, and since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t, therefore $(w \otimes z)_v(\in_\gamma \vee q_\delta)g$. If $(w \vee z)_v \in_\gamma g$, then $(w \otimes z) \in g_v$ and if $(w \otimes z)_v q_\delta g$, then $g(w \otimes z) > 2\delta - v \geq v > \gamma$, that is, $w \otimes z \in g_v$. Thus g_v is $f$ilter of Q_t.

Conversely, suppose that $\emptyset \neq g_v$ is a $f$ilter of Q_t for all $v \in (\gamma, \delta]$. Let $z, w \in Q_t$ with $z \leq w$ and $sup(g(w), \gamma) < inf(g(z), \delta)$. Then there exist $v \in (\gamma, \delta]$ such that $sup(g(w), \gamma) < v \leq inf(g(z), \delta)$. This shows that $z_v \in_\gamma g$; that is $z \in g_v$ but $w \notin g_v$, a contradiction. Thus, $sup(g(w), \gamma) \geq inf\ (g(z), \delta)$ with $z \leq w$. Let $z, w \in Q_t$ and sup $(g(z \otimes w), \gamma) < inf\ (g(z), g(w), \delta)$. Then $sup(g(z \otimes w), \gamma) < v \leq inf\ (g(z), g(w), \delta)$ for some $v \in (\gamma, \delta]$. This implies that $z \in g_v$ and $w \in g_v$ but $(z \otimes w) \notin g_v$, a contradiction. Therefore, $sup(g(z \otimes w), \gamma) \geq inf(g(z), g(w), \delta)$. Consequently, g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t by Theorem 7.3.4.

(**2**). Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t. Let $z, w \in Q_t$ with $w \leq z$ be such that $w \in g_v^\delta$. Then $w_v q_\delta g$, that is $g(w) + v > 2\delta \Rightarrow g(w) > 2\delta - v \geq 2\delta - 1 = \gamma$. Thus, $g(w) > \gamma$. By hypothesis, we have

$$\begin{aligned} sup(g(z), \gamma) \geq & \quad inf(g(w), \delta) \\ \Rightarrow g(z) > & \quad inf(2\delta - v, \delta) \end{aligned}$$

Since $v \in (\delta, 1]$, $\delta < v \leq 1 \Rightarrow 2\delta - v < \delta < v$. Thus, $g(z) > 2\delta - v \Rightarrow g(z) + v > 2\delta$. Hence, $z \in g_v^\delta$.

Now we have to show that $z \otimes w \in g_v^\delta$ for all $z, w \in g_v^\delta$. Let $z, w \in Q_t$ be such that $z, w \in g_v^\delta$. Then $w_v q_\delta g$ and $z_v q_\delta g$, that is $g(w) + v > 2\delta \Rightarrow g(w) > 2\delta - v \geq 2\delta - 1 = \gamma$ and similarly $g(z) > \gamma$. By assumption, we have

$$\begin{aligned} sup(g(z \otimes w), \gamma) \geq & \quad inf(g(w), g(z), \delta) \\ \Rightarrow g(z \otimes w) > & \quad inf(2\delta - v, 2\delta - v, \delta) \end{aligned}$$

Since $v \in (\delta, 1]$, $\delta < v \leq 1 \Rightarrow 2\delta - v < \delta < v$. So, $g(z \otimes w) > 2\delta - v \Rightarrow g(z \otimes w) + v > 2\delta$. Hence, $z \otimes w \in g_v^\delta$.

Conversely, suppose that $\emptyset \neq g_v^\delta$ is a *f*ilter of Q_t for all $v \in (\delta, 1]$. We show that g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$. Let $z, w \in Q_t$ with $z \leq w$ be such that $z_p q_\delta g$. Let $sup(g(w), \gamma) < inf(g(z), \delta)$. Then

$$\begin{aligned} 2\delta - inf(g(z), \delta) < & \quad 2\delta - sup(g(w), \gamma) \\ \Rightarrow sup(2\delta - g(z), \delta) < & \quad inf(2\delta - g(w), 2\delta - \gamma). \end{aligned}$$

Take $p \in (\delta, 1]$ such that $sup(2\delta - g(z), \delta) < p \leq inf(2\delta - g(w), 2\delta - \gamma)$. Then $2\delta - g(z) < p$ and $2\delta - g(w) \geq p \Rightarrow g(z) + p > 2\delta$ and $g(w) + p \leq 2\delta$. This shows that $z_p q_\delta g$; that is $z \in g_v^\delta$ but $w \notin g_v^\delta$, a contradiction. Hence, $sup(g(w), \gamma) \geq inf\ (g(z), \delta)$ with $z \leq w$. Let $z, w \in Q_t$ and $sup\ (g(z \otimes w), \gamma) < inf\ (g(z), g(w), \delta)$. Then $2\delta - inf(g(z), g(w), \delta) < 2\delta - sup(g(z \otimes w), \gamma) \Rightarrow sup(2\delta - g(z), 2\delta - g(w), \delta) < inf(2\delta - g(z \otimes w), 2\delta - \gamma)$. There exist $u \in (\delta, 1]$ such that $sup(2\delta - g(z), 2\delta - g(w), \delta) < u \leq inf(2\delta - g(z \otimes w), 2\delta - \gamma)$. Then $2\delta - g(z) < u$, $2\delta - g(w) < u$ and $2\delta - g(z \otimes w) \geq u \Rightarrow g(z) + u > 2\delta$, $g(w) + u > 2\delta$ but $g(z \otimes w) + u \leq 2\delta$. Thus, $z \in g_v^\delta$ and $w \in g_v^\delta$ but $(z \otimes w) \notin g_v^\delta$, a contradiction. Therefore, $sup(g(z \otimes w), \gamma) \geq inf(g(z), g(w), \delta)$. Consequently, g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t by Theorem 7.3.4.

(**3**). The proof of part 3 is similar to the proof of parts 1 and 2. ■

Chapter 8

Generalized Approximations of $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Substructers in Quantales

The concept of generalized approximations (GA) of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ in quantales are presented in this chapter. With the help of SVH and $SSVH$, it is observed that GLA and GUA of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$, respectively.

In the first section, GLA and GUA of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ are introduced. It is observed that GLA of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ are not $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$, respectively, while taking SVH. Furthermore, GUA of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ are presented by using SVH. In the second section, at first, GLA (and GUA) of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FIs$ is introduced. In the third section, GLA and GUA of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$ are discussed. GLA and GUA of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodules of a Q_t-module are being presented at the end of this chapter.

8.1 Approximations of $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Filters and $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Subquantales

The idea of generalized roughness (GR) of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of a quantale Q_t is being presented, in the following. The investigation of GLA and GUA in $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of a quantale Q_t is being first started in the following. However, we begin with the result.

Theorem 8.1.1 *Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q'_t and $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$. Then $\underline{H}(g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t.*

Proof. Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q'_t. As $H : Q_t \longrightarrow P^*(Q'_t)$ is a $SSVH$, so $\vee_{i\in I} H(z_i) = H(\vee_{i\in I} z_i)$. Consider the following:

$$\begin{aligned} sup\{\underline{H}(g)(\vee_{i\in I} z_i), \gamma\} &= sup\left\{\underset{a\in H(\vee_{i\in I} z_i)}{\wedge} g(a), \gamma\right\} \\ &= \underset{a\in H(\vee_{i\in I} z_i)}{\wedge} sup\{g(a), \gamma\} \\ &= \underset{a\in \vee_{i\in I} H(z_i)}{\wedge} sup\{g(a), \gamma\} \end{aligned}$$

Since $a \in \vee_{i\in I} H(z_i)$, there exist $a_1 \in H(z_1), a_2 \in H(z_2), ..., a_i \in H(z_i)$ such that $a = \vee_{i\in I} a_i$.

$$\begin{aligned} sup\{\underline{H}(g)(\vee_{i\in I} z_i), \gamma\} &= \underset{\vee_{i\in I} a_i \in \vee_{i\in I} H(z_i)}{\wedge} sup\{g(\vee_{i\in I} a_i), \gamma\} \\ &\geq \underset{\vee_{i\in I} a_i \in \vee_{i\in I} H(z_i)}{\wedge} inf\{\underset{i\in I}{inf} g(a_i), \delta\} \\ &= \underset{a_1 \in H(z_1), ..., a_i \in H(z_i)}{\wedge} inf\{\underset{i\in I}{inf}[g(a_1), ..., g(a_i)], \delta\} \\ &= inf\{\underset{i\in I}{inf}[\underset{a_1 \in H(z_1)}{\wedge} g(a_1), ..., \underset{a_i \in H(z_i)}{\wedge} g(a_i)], \delta\} \\ &= inf\{\underset{i\in I}{inf}[\underline{H}(g)(z_1), ..., \underline{H}(g)(z_i)], \delta\} \\ &= inf\{\underset{i\in I}{inf}[\underline{H}(g)(z_i)], \delta\}. \end{aligned}$$

Thus, we have $sup\{\underline{H}(g)(\vee_{i\in I} z_i), \gamma\} \geq inf\{\underset{i\in I}{inf}[\underline{H}(g)(z_i)], \delta\}$.

As $H : Q_t \longrightarrow P^*(Q_t)$ is a $SSVH$, so $H(z \otimes w) = H(z) \otimes' H(w)$.

Now, consider

$$\begin{aligned} sup\,\{\underline{H}(g)(z\otimes w),\gamma\} &= sup\left\{\bigwedge_{a\in H(z\otimes w)} g(e),\gamma\right\} \\ &= \bigwedge_{e\in H(z\otimes w)} sup\,\{g(e),\gamma\} \\ &= \bigwedge_{e\in H(z)\otimes' H(w)} sup\,\{g(e),\gamma\}\,. \end{aligned}$$

As $e \in H(z)\otimes' H(w)$, we obtain $a \in H(z)$ and $b \in H(w)$ such that $e = a\otimes' b$.

$$\begin{aligned} sup\,\{\underline{H}(g)(z\otimes w),\gamma\} &= \bigwedge_{a\otimes' b\in H(z)\otimes' H(w)} sup\,\{g(a\otimes' b),\gamma\} \\ &= \bigwedge_{a\,\in\, H(z),\; b\,\in\, H(w)} sup\,\{g(a\otimes' b),\gamma\} \\ &\geq \bigwedge_{a\,\in\, H(z),\; b\,\in\, H(w)} inf\{g(a),g(b),\delta\} \\ &= inf\{\bigwedge_{a\,\in\, H(z)} g(a), \bigwedge_{b\,\in\, H(w)} g(b),\delta\} \\ &= inf\{\underline{H}(g)(z),\underline{H}(g)(w),\delta\}. \end{aligned}$$

Thus, $sup\,\{\underline{H}(g)(z\otimes w),\gamma\} \geq inf\{\underline{H}(g)(z),\underline{H}(g)(w),\delta\}$. Therefore, $\underline{H}(g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t. ■

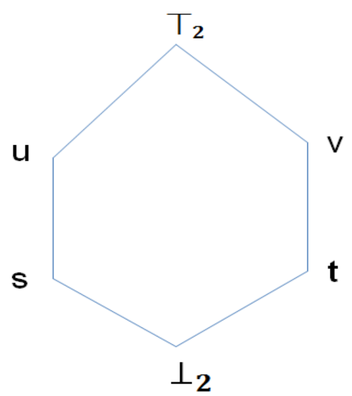

Fig. 14

Table. 11

$\otimes_2$	$\perp_2$	s	t	u	v	$\top_2$
$\perp_2$	$\perp_2$	s	t	u	v	$\top_2$
s	$\perp_2$	s	t	u	v	$\top_2$
t	$\perp_2$	s	t	u	v	$\top_2$
u	$\perp_2$	s	t	u	v	$\top_2$
v	$\perp_2$	s	t	u	v	$\top_2$
$\top_2$	$\perp_2$	s	t	u	v	$\top_2$

The example below shows that, if H is a SVH and g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$, then its lower approximations $\underline{H}(g)$, may not be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$.

Example 8.1.2 *Let $\gamma, \delta \in (0,1]$ with $\gamma < \delta$. Let $Q'_t = \{\perp_2, s, t, u, v, \top_2\}$ be a sup-lattice depicted in Fig.*14 *and the binary operation $\otimes_2$ on Q'_t is shown in Table* 11*. Then $(Q'_t, \otimes_2)$ is a quantale. Define a f-subset $g : Q'_t \to [0,1]$ by $g = \frac{1}{\perp_2} + \frac{0.5}{s} + \frac{0.6}{t} + \frac{0.7}{u} + \frac{0.8}{v} + \frac{1}{\top_2}$. Then g is an $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-$FS$ of Q'_t. Now, consider $H : Q'_t \to P^*(Q'_t)$ defined as $H(\perp_2) = H(s) = H(t) = H(u) = H(v) = \{\perp_2\}$ and $H(\top_2) = Q'_t$. It is easily seen that $H : Q'_t \to P^*(Q'_t)$ is a SVH. With the help of Definition* 3.1.1*, we have $\underline{H}(g) = \frac{1}{\perp_2} + \frac{1}{s} + \frac{1}{t} + \frac{1}{u} + \frac{1}{v} + \frac{0.5}{\top_2}$. Now, for $u \le \top_2$ and $v \le \top_2$ with $\gamma = 0.3$ and $\delta = 0.6$, but $sup\{\underline{H}(g)(\vee_{i\in I} z_i), \gamma\} \ge inf\{\underset{i\in I}{inf}[\underline{H}(g)(z_i)], \delta\}$ for all $z_i \in Q'_t$ is not satisfied, because $sup\{\underline{H}(g)(u \vee v), \gamma\} = sup\{\underline{H}(g)(\top_2), \gamma\} \ngeq inf\{inf[\underline{H}(g)(u), \underline{H}(g)(v)], \delta\}$. Also, $sup\{\underline{H}(g)(s \vee t), \gamma\} = sup\{\underline{H}(g)(\top_2), \gamma\} \ngeq inf\{inf[\underline{H}(g)(s), \underline{H}(g)(t)], \delta\}$.*

Theorem 8.1.3 *Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q'_t and $H : Q_t \longrightarrow P^*(Q'_t)$ be a SVH. Then $\overline{H}(g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t.*

Proof. Let $z_i \in Q_t$ for $i \in I$. Since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q'_t and $H : Q_t \longrightarrow P^*(Q'_t)$ is a SVH, so we have $\vee_{i\in I} H(z_i) \subseteq H(\vee_{i\in I} z_i)$. Consider the following:

$$\begin{aligned}
inf\{\underset{i\in I}{inf}[\overline{H}(g)(z_i)],\delta\} &= inf\{\underset{i\in I}{inf}[\overline{H}(g)(z_1),\overline{H}(g)(z_2),...,\overline{H}(g)(z_i)],\delta\}\\
&= inf\{\underset{i\in I}{inf}[\bigvee_{a_1\,\in\,H(z_1)} g(a_1),...,\bigvee_{a_i\,\in\,H(z_i)} g(a_i)],\delta\}\\
&= \bigvee_{a_1\,\in\,H(z_1),...,\,a_i\,\in\,H(z_i)} inf\{\underset{i\in I}{inf}[g(a_1),...,g(a_i)],\delta\}\\
&= \bigvee_{\vee_{i\in I}a_i\in\vee_{i\in I}H(z_i)} inf\{\underset{i\in I}{inf} g(a_i),\delta\}\\
&\leq \bigvee_{\vee_{i\in I}a_i\in\vee_{i\in I}H(z_i)} sup\{g(\vee_{i\in I}a_i),\gamma\}\\
&= \bigvee_{e\in\vee_{i\in I}H(z_i)} sup\{g(e),\gamma\}\\
&\leq \bigvee_{e\in H(\vee_{i\in I}z_i)} sup\{g(e),\gamma\}\\
&= sup\left\{\bigvee_{e\in H(\vee_{i\in I}z_i)} g(e),\gamma\right\}\\
&= sup\left\{\overline{H}(g)(\vee_{i\in I}z_i),\gamma\right\}.
\end{aligned}$$

Thus, we have $sup\left\{\overline{H}(g)(\vee_{i\in I}z_i),\gamma\right\} \geq inf\{\underset{i\in I}{inf}[\overline{H}(g)(z_i)],\delta\}$.

As $H : Q_t \longrightarrow P^*(Q_t)$ is a SVH, so $H(z)\otimes' H(w) \subseteq H(z\otimes w)$.

Furthermore, consider

$$\begin{aligned}
inf\{\overline{H}(g)(z),\overline{H}(g)(w),\delta\} &= inf\{\bigvee_{a\,\in\,H(z)} g(a),\bigvee_{b\,\in\,H(w)} g(b),\delta\}\\
&= \bigvee_{a\,\in\,H(z),\,b\,\in\,H(w)} inf\{g(a),g(b),\delta\}\\
&\leq \bigvee_{a\,\in\,H(z),\,b\,\in\,H(w)} sup\{g(a\otimes' b),\gamma\}\\
&= \bigvee_{a\otimes' b\in H(z)\otimes' H(w)} sup\{g(a\otimes' b),\gamma\}\\
&\leq \bigvee_{a\otimes' b\in H(z\otimes w)} sup\{g(a\otimes' b),\gamma\}\\
&= \bigvee_{c\in H(z\otimes w)} sup\{g(c),\gamma\}\\
&= sup\left\{\bigvee_{c\in H(z\otimes w)} g(c),\gamma\right\}\\
&= sup\left\{\overline{H}(g)(z\otimes w),\gamma\right\}.
\end{aligned}$$

Thus, $sup\left\{\overline{H}(g)(z\otimes w),\gamma\right\} \geq inf[\overline{H}(g)(z),\overline{H}(g)(w),\delta\}$. Therefore, $\overline{H}(g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t. ■

Proposition 8.1.4 *Let g_1 and g_2 be $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q'_t and $H : Q_t \longrightarrow P^*(Q'_t)$ be a SVH. Then $\overline{H}(g_1) \Cap \overline{H}(g_2)$ and $\underline{H}(g_1) \Cap \underline{H}(g_2)$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FS$ of Q_t.*

Proof. Proof follows from Proposition 6.2.5 and Theorems 8.1.1, 8.1.3. ■

Now, we discuss GLA and GUA of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FFs$. First the GLA is being presented.

Theorem 8.1.5 *Let* g *be an* $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ *of* Q'_t *and* $H : Q_t \longrightarrow P^*(Q'_t)$ *be a* $SSVH$. *Then* $\underline{H}(g)$ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ *of* Q_t.

Proof. Consider $z, w \in Q_t$ and $\gamma, \delta \in (0, 1]$ such that $\gamma < \delta$. Since $H : Q_t \longrightarrow P^*(Q_t)$ is a $SSVH$, so, $H(z) \otimes' H(w) = H(z \otimes w)$. Consider the following:

$$\begin{aligned} sup\{\underline{H}(g)(z \otimes w), \gamma\} &= sup\left\{\bigwedge_{e \in H(z \otimes w)} g(e), \gamma\right\} \\ &= \bigwedge_{e \in H(z \otimes w)} sup\{g(e), \gamma\} \\ &= \bigwedge_{e \in H(z) \otimes' H(w)} sup\{g(e), \gamma\}. \end{aligned}$$

Since $e \in H(z) \otimes' H(w)$, there exist $a_1 \in H(z)$ and $a_2 \in H(w)$ such that $e = a_1 \otimes' a_2$. So,

$$\begin{aligned} sup\{\underline{H}(g)(z \otimes w), \gamma\} &= \bigwedge_{a_1 \otimes' a_2 \in H(z) \otimes' H(w)} sup\{g(a_1 \otimes' a_2), \gamma\} \\ &\geq \bigwedge_{a_1 \otimes' a_2 \in H(z) \otimes' H(w)} inf\{g(a_1), g(a_2), \delta\} \\ &= \bigwedge_{a_1 \in H(z),\, a_2 \in H(w)} inf\{g(a_1), g(a_2), \delta\} \\ &= inf\{\bigwedge_{a_1 \in H(z)} g(a_1), \bigwedge_{a_2 \in H(w)} g(a_2), \delta\} \\ &= inf\{\underline{H}(g)(z), \underline{H}(g)(w), \delta\}. \end{aligned}$$

Thus, we have $sup\{\underline{H}(g)(z \otimes w), \gamma\} \geq inf\{\underline{H}(g)(z), \underline{H}(g)(w), \delta\}$.

Furthermore, let $w \leq z$. Then $w \vee z = z$. Since $H : Q_t \longrightarrow P^*(Q_t)$ is a $SSVH$, so $H(z) = H(w \vee z) = H(w) \vee H(z)$.

Consider,

$$\begin{aligned} sup\{\underline{H}(g)(z), \gamma\} &= sup\{\bigwedge_{e \in H(z)} g(e), \gamma\} \\ &= \bigwedge_{e \in H(z) \vee H(w)} sup\{g(e), \gamma\}. \end{aligned}$$

Since $e \in H(z) \vee H(w)$ so there exist $c \in H(z)$ and $d \in H(w)$ such that $e = c \vee d$. As $c \vee d \geq d$. We have,

$$\begin{aligned}
sup\{\underline{H}(g)(z),\gamma\} &= \bigwedge_{c\vee d\in\ H(z)\vee H(w)} sup\{g(c\vee d),\gamma\}\\
&= \bigwedge_{c\in\ H(z),\ d\in\ H(w)} sup\{g(c\vee d),\gamma\}\\
&\geq \bigwedge_{c\in\ H(z),\ d\in\ H(w)} inf\{g(d),\delta\}\\
&= inf\{\bigwedge_{d\in\ H(w)} g(d),\delta\}\\
&= inf\{\underline{H}(g)(w),\delta\}.
\end{aligned}$$

Thus, we have $sup\{\underline{H}(g)(z),\gamma\} \geq inf\{\underline{H}(g)(w),\delta\}$. Therefore, $\underline{H}(g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t. ■

The GLA of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ is not necessarily a FF by using SVH, as illustrated by the example below.

Example 8.1.6 *Let $(Q'_t, \otimes_2)$ be a quantale, where Q'_t is depicted in $Fig.14$ and the binary operation $\otimes_2$ on Q'_t is shown in the Table 11. Let $\gamma, \delta \in (0,1]$ with $\gamma < \delta$. Now, consider $H : Q'_t \longrightarrow P^*(Q'_t)$ a SVH defined as $H(\perp_2) = \{\perp_2, u\}, H(s) = \{u, v, \top_2\}, H(t) = \{u, v, \top_2\}, H(u) = \{\perp_2, u, v, \top_2\}, H(v) = \{u, v, \top_2\}$ and $H(\top_2) = \{v, u, \top_2\}$. Let g be a f-subset of Q'_t given by $g = \frac{0.5}{\perp_2} + \frac{0.5}{s} + \frac{0.7}{t} + \frac{0.8}{u} + \frac{0.8}{v} + \frac{1}{\top_2}$. Then it is easy to verify that g is an $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-$FF$ of Q'_t. Now, GLA of the $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-$FF$ is $\underline{H}(g) = \frac{0.5}{\perp_2} + \frac{0.8}{s} + \frac{0.8}{t} + \frac{0.5}{u} + \frac{0.8}{v} + \frac{0.8}{\top_2}$. We observe that for $s \leq u$ with $\gamma = 0.3$, $\delta = 0.6$, we have, $sup\{\underline{H}(g)(u), \gamma\} \ngeq inf\{\underline{H}(g)(s), \delta\}$.*

Theorem 8.1.7 *If g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q'_t and $H : Q_t \longrightarrow P^*(Q'_t)$ be a SVH. Then $\overline{H}(g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t.*

Proof. Let $z_1, z_2 \in Q_t$ and $\gamma, \delta \in (0,1]$ be such that $\gamma < \delta$. Let $z_1 \leq z_2$. Then $z_1 \vee z_2 = z_2$.

Consider,

$$\begin{aligned}
inf\{\overline{H}(g)(z_1),\delta\} &= inf\{\bigvee_{x\in\ H(z_1)} g(x),\delta\}\\
&= \bigvee_{x\in\ H(z_1)} inf\{g(x),\delta\}.
\end{aligned}$$

Since H is a SVH, so $H(z_1) \vee H(z_2) \subseteq H(z_1 \vee z_2) = H(z_2)$. As $x \vee y \geq x$, we have,

$$\begin{aligned}
inf\{\overline{H}(g)(z_1), \delta\} &= \bigvee_{x\in\, H(z_1)} inf\{g(x), \delta\} \\
&\leq \bigvee_{x\in\, H(z_1),\, y\in\, H(z_2)} sup\{g(x \vee y), \gamma\} \\
&= \bigvee_{x\vee y\in\, H(z_1)\vee H(z_2)} sup\{g(x \vee y), \gamma\} \\
&\leq \bigvee_{x\vee y\in\, H(z_1\vee z_2)} sup\{g(x \vee y), \gamma\} \\
&= \bigvee_{e\in\, H(z_1\vee z_2)} sup\{g(e), \gamma\} \\
&= \bigvee_{e\in\, H(z_2)} sup\{g(e), \gamma\} \\
&= sup\{\bigvee_{e\in\, H(z_2)} g(e), \gamma\} \\
&= sup\{\overline{H}(g)(z_2), \gamma\}.
\end{aligned}$$

Thus, we have $sup\{\overline{H}(g)(z_2), \gamma\} \geq inf\{\overline{H}(g)(z_1), \delta\}$.

Next, Consider the following:

$$\begin{aligned}
inf\{\overline{H}(g)(z_1), \overline{H}(g)(z_2)), \delta\} &= inf\{\bigvee_{a\in\, H(z_1)} g(a), \bigvee_{b\in\, H(z_2)} g(b), \delta\} \\
&= \bigvee_{a\in\, H(z_1), b\in\, H(z_2)} inf\{g(a), g(b), \delta\}
\end{aligned}$$

Since H is a SVH, so $H(z_1) \otimes' H(z_2) \subseteq H(z_1 \otimes z_2)$. We have,

$$\begin{aligned}
inf\{\overline{H}(g)(z_1), \overline{H}(g)(z_2)), \delta\} &= \bigvee_{a\in\, H(z_1),\, b\in\, H(z_2)} inf\{g(a), g(b), \delta\} \\
&\leq \bigvee_{a\in\, H(z_1),\, b\in\, H(z_2)} sup\{g(a \otimes' b), \gamma\} \\
&= \bigvee_{a\otimes' b\in\, H(z_1)\otimes' H(z_2)} sup\{g(a \otimes' b), \gamma\} \\
&\leq \bigvee_{a\otimes' b\in\, H(z_1\otimes' z_2)} sup\{g(a \otimes' b), \gamma\} \\
&= sup\{\bigvee_{e\in\, H(z_1\otimes' z_2)} g(e), \gamma\} \\
&= sup\{\overline{H}(g)(z_1 \otimes z_2), \gamma\}
\end{aligned}$$

Thus, we have $sup\left\{\overline{H}(g)(z_1 \otimes z_2), \gamma\right\} \geq inf\{\overline{H}(g)(z_1), \overline{H}(g)(z_2), \delta\}$. Therefore, $\overline{H}(g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t. ■

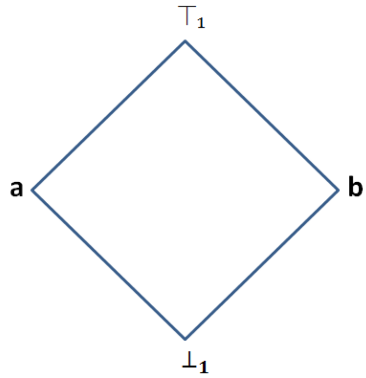

Fig. 15

Table 12.

$\otimes_1$	$\bot_1$	a	b	$\top_1$
$\bot_1$	$\bot_1$	a	b	$\top_1$
a	$\bot_1$	a	b	$\top_1$
b	$\bot_1$	a	b	$\top_1$
$\top_1$	$\bot_1$	a	b	$\top_1$

Example 8.1.8 *Let $(Q_t, \otimes_1)$ and $(Q'_t, \otimes_2)$ be two quantales, where Q_t and Q'_t are depicted in Figures 14 and 15 and the binary operations $\otimes_1$ and $\otimes_2$ on both the quantales are shown in Tables 11 and 12. Let $\gamma, \delta \in (0, 1]$ with $\gamma < \delta$. Now, consider $H : Q_t \longrightarrow P^*(Q'_t)$ defined as $H(\bot_1) = \{\bot_2\}, H(a) = \{u, s\}, H(b) = \{u, v\}$ and $H(\top_1) = \{u, \top_2\}$. Then, H is a $SSVH$. Let g be a f-subset of Q'_t given by $g = \frac{0.5}{\bot_2} + \frac{0.5}{s} + \frac{0.8}{t} + \frac{0.5}{u} + \frac{0.8}{v} + \frac{1}{\top_2}$. Then it is easy to verify that g is an $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-$FF$ of Q'_t. Now, GLA and GUA of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$, g of Q_t are as follows: $\underline{H}(g) = \frac{0.5}{\bot_1} + \frac{0.5}{a} + \frac{0.5}{b} + \frac{0.5}{\top_1}$ and $\overline{H}(g) = \frac{0.5}{\bot_1} + \frac{0.5}{a} + \frac{0.8}{b} + \frac{1}{\top_1}$. It can be verified that $\underline{H}(g)$ and $\overline{H}(g)$ are $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-$FF$ of Q_t.*

Proposition 8.1.9 *Let g_1 and g_2 be $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q'_t and $H : Q_t \longrightarrow P^*(Q'_t)$ be a SVH. Then $\overline{H}(g_1) \Cap \overline{H}(g_2)$ and $\underline{H}(g_1) \Cap \underline{H}(g_2)$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FF$ of Q_t.*

Proof. Follows from Proposition 7.3.5 and Theorems 8.1.5, 8.1.7. ■

8.2 Approximations of $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Ideals in Quantales

Now in the following discussion, the concept of GLA and GUA of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$, in quantales are being introduced.

Theorem 8.2.1 *Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q'_t and $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$. Then, $\underline{H}(g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t.*

Proof. Let $z, w \in Q_t$ and $\gamma, \delta \in (0, 1]$ be such that $\gamma < \delta$. Since $H : Q_t \longrightarrow P^*(Q_t)$ is a $SSVH$, so we have $H(z \vee w) = H(z) \vee H(w)$. Consider the following:

$$\begin{aligned} sup\{\underline{H}(g)(z \vee w), \gamma\} &= sup\left\{\bigwedge_{e \in H(z \vee w)} g(e), \gamma\right\} \\ &= \bigwedge_{e \in H(z \vee w)} sup\{g(e), \gamma\} \\ &= \bigwedge_{e \in H(z) \vee H(w)} sup\{g(e), \gamma\} \end{aligned}$$

Since $e \in H(z) \vee H(w)$, there exist $a_1 \in H(z)$ and $a_2 \in H(w)$ such that $e = a_1 \vee a_2$. So,

$$\begin{aligned} sup\{\underline{H}(g)(z \vee w), \gamma\} &= \bigwedge_{a_1 \vee a_2 \in H(z) \vee H(w)} sup\{g(a_1 \vee a_2), \gamma\} \\ &\geq \bigwedge_{a_1 \vee a_2 \in H(z) \vee H(w)} inf\{g(a_1), g(a_2), \delta\} \\ &= \bigwedge_{a_1 \in H(z),\ a_2 \in H(w)} inf\{g(a_1), g(a_2), \delta\} \\ &= inf\{\bigwedge_{a_1 \in H(z)} g(a_1), \bigwedge_{a_2 \in H(w)} g(a_2), \delta\} \\ &= inf\{\underline{H}(g)(z), \underline{H}(g)(w), \delta\} \end{aligned}$$

Thus, we have $sup\{\underline{H}(g)(z \vee w), \gamma\} \geq inf\{\underline{H}(g)(z), \underline{H}(g)(w), \delta\}$.

Furthermore, let $w \leq z$. Then $w \vee z = z$. Since $H : Q_t \longrightarrow P^*(Q_t)$ is a $SSVH$, so $H(z) = H(w \vee z) = H(w) \vee H(z)$.

Consider,

$$\begin{aligned} inf\{\underline{H}(g)(z), \delta\} &= inf\{\bigwedge_{e \in H(z)} g(e), \delta\} \\ &= \bigwedge_{e \in H(z)} inf\{g(e), \delta\} \\ &= \bigwedge_{e \in H(z) \vee H(w)} inf\{g(e), \delta\}. \end{aligned}$$

Since $e \in H(z) \vee H(w)$ so there be $c \in H(z)$ and $d \in H(w)$ such that $e = c \vee d$. As $c \vee d \geq d$. We have,

$$\begin{aligned}
inf\{\underline{H}(g)(z), \gamma\} &= \bigwedge_{c\vee d\in\ H(z)\vee H(w)} inf\{g(c\vee d), \delta\} \\
&= \bigwedge_{c\in\ H(z),\ d\in\ H(w)} inf\{g(c\vee d), \delta\} \\
&\leq \bigwedge_{c\in\ H(z),\ d\in\ H(w)} sup\{g(d), \gamma\} \\
&= sup\{\bigwedge_{d\in\ H(w)} g(d), \gamma\} \\
&= sup\{\underline{H}(g)(w), \gamma\}.
\end{aligned}$$

Thus, we have $sup\{\underline{H}(g)(w), \gamma\} \geq inf\{\underline{H}(g)(z), \delta\}$.

As $H : Q_t \longrightarrow P^*(Q_t)$ is a $SSVH$, so $H(z \otimes w) = H(z) \otimes' H(w)$.

Furthermore, consider

$$\begin{aligned}
sup\{\underline{H}(g)(w\otimes z), \gamma\} &= sup\left\{\bigwedge_{e\in H(z\otimes w)} g(e), \gamma\right\} \\
&= \bigwedge_{e\in H(z\otimes w)} sup\{g(e), \gamma\} \\
&= \bigwedge_{e\in H(z)\otimes' H(w)} sup\{g(e), \gamma\}
\end{aligned}$$

As $e \in H(z) \otimes' H(w)$, we obtain $a \in H(z)$ and $b \in H(w)$ such that $e = b \otimes' a$.

$$\begin{aligned}
sup\{\underline{H}(g)(w\otimes z), \gamma\} &= \bigwedge_{a\otimes' b\in H(z)\otimes' H(w)} sup\{g(b\otimes' a), \gamma\} \\
&= \bigwedge_{a\ \in\ H(z),\ b\ \in\ H(w)} sup\{g(b\otimes' a), \gamma\} \\
&\geq \bigwedge_{a\ \in\ H(z),\ b\ \in\ H(w)} inf\{g(a), \delta\} \\
&= inf\{\bigwedge_{a\ \in\ H(z)} g(a), \delta\} \\
&= inf[\underline{H}(g)(z), \delta\}
\end{aligned}$$

Thus, $sup\{\underline{H}(g)(w\otimes z), \gamma\} \geq inf[\underline{H}(g)(z), \delta\}$. Also, $sup\{\underline{H}(g)(z\otimes w), \gamma\} \geq inf[\underline{H}(g)(z), \delta\}$. Therefore, $\underline{H}(g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t by Theorem 6.2.11. ■

The next example shows that, if H is a SVH, and g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t, then $\underline{H}(g)$ may not be a an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t.

Example 8.2.2 *Let $\gamma, \delta \in (0, 1]$ with $\gamma < \delta$. Let $Q'_t = \{\bot_2, s, t, u, v, \top_2\}$ be a sup-lattice with the multiplication Table* 11 *and order relation as shown in the Fig.* 14. *Then $(Q'_t, \otimes_2)$ is a quantale. Define $g : Q'_t \to [0, 1]$ by*

$$g(z) = \begin{cases} 1, & x = \bot_2 \\ 0.5, & x \neq \bot_2 \end{cases} \quad \text{for all } z \in Q'_t$$

Then g is an $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-$FI$ of Q'_t. Let $H : Q'_t \to P^*(Q'_t)$ be a SVH defined as in Example 8.1.2. *Now LA of g is $\underline{H}(g) = \frac{1}{\bot_2} + \frac{1}{s} + \frac{1}{t} + \frac{1}{u} + \frac{1}{v} + \frac{0.5}{\top_2}$. Now, for $\gamma = 0.5$*

and $\delta = 0.7$, *the following are not satisfied:* $sup\{\underline{H}(g)(u \vee v), \gamma\} = sup\{\underline{H}(g)(\top_2), \gamma\} \ngeq inf\{\underline{H}(g)(u), \underline{H}(g)(v), \delta\}$. *Also,*

$$sup\{\underline{H}(g)(s \vee t), \gamma\} = sup\{\underline{H}(g)(\top_2), \gamma\} \ngeq inf\{\underline{H}(g)(s), \underline{H}(g)(t), \delta\}.$$

Theorem 8.2.3 *Let* g *be an* $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ *of* Q'_t *and* $H : Q_t \longrightarrow P^*(Q'_t)$ *be a* SVH. *Then,* $\overline{H}(g)$ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ *of* Q_t.

Proof. The proof is like the proof of Theorem 8.2.1. ■

Example 8.2.4 *Let* $(Q_t, \otimes_1)$ *and* $(Q'_t, \otimes_2)$ *be two quantales, where* Q_{t_1} *and* Q'_t *are depicted in Figures* 14 *and* 15 *and the binary operations* $\otimes_1$ *and* $\otimes_2$ *on both the quantales are shown in Tables* 11 *and* 12. *Let* $\gamma, \delta \in (0,1]$ *with* $\gamma < \delta$. *Now, consider* $H : Q_t \longrightarrow P^*(Q'_t)$ *defined as* $H(\perp_1) = \{\perp_2\}, H(a) = \{u, s\}, H(b) = \{u, v\}$ *and* $H(\top_1) = \{u, \top_2\}$. *Then,* H *is a* $SSVH$. *Let* g *be a f-subset of* Q'_t *given by* $g(z) = \begin{cases} 1, & z = \perp_2 \\ 0.5, & z \neq \perp_2 \end{cases}$ *for all* $z \in Q'_t$. *Then it is easy to verify that* g *is an* $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-$FI$ *of* Q'_t. *Now,* LA *and* UA *of* $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-$FI$ *of* Q'_t *are as follows:* $\underline{H}(g) = \frac{1}{\perp_1} + \frac{0.5}{a} + \frac{0.5}{b} + \frac{0.5}{\top_1}$ *and* $\overline{H}(g) = \frac{1}{\perp_1} + \frac{0.5}{a} + \frac{0.5}{b} + \frac{1}{\top_1}$. *It can be verified that* $\underline{H}(g)$ *and* $\overline{H}(g)$ *are* $(\in_{0.3}, \in_{0.3} \vee q_{0.6})$-$FI$ *of* Q_t.

Proposition 8.2.5 *Let* g_1 *and* g_2 *be* $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ *of* Q_t *and* $H : Q_t \longrightarrow P^*(Q'_t)$ *be a* SVH. *Then* $\overline{H}(g_1) \Cap \overline{H}(g_2)$ *and* $\underline{H}(g_1) \Cap \underline{H}(g_2)$ *are* $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ *of* Q_t.

Proof. The proof follows from Proposition 6.2.12 and Theorems 8.2.1, 8.2.3. ■

8.3 Approximations of $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Prime (Semi-Prime) Ideals in Quantales

Now, GLA and GUA being extended to $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$. First the GLA and GUA of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ are being started.

Theorem 8.3.1 *Let* g *be an* $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ *of* Q'_t *and* $H : Q_t \longrightarrow P^*(Q'_t)$ *be a* $SSVH$. *Then* $\underline{H}(g)$ *is a* $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ *of* Q_t.

Proof. Let $w, z \in Q_t$ and $\gamma, \delta \in (0,1]$ be such that $\gamma < \delta$. As g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ of Q'_t, therefore g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q'_t, hence by Theorem 8.2.1, $\underline{H}(g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FI$ of Q_t. Moreover by Proposition 6.3.2, we have $sup\{g(e), g(c), \gamma\} \geq inf\{g(e \otimes c), \delta\}$ for all $e, c \in Q_t$.

Consider,

$$\begin{aligned}
sup\{\underline{H}(g)(z), \underline{H}(g)(w), \gamma\} &= sup\{\bigwedge_{e \in H(z)} g(e), \bigwedge_{d \in H(w)} g(d), \gamma\} \\
&= \bigwedge_{e \in H(z), d \in H(w)} sup\{g(e), g(d), \gamma\} \\
&\geq \bigwedge_{e \in H(z),\, d \in H(w)} inf\{g(e \otimes' d), \delta\} \\
&= \bigwedge_{e \otimes' d \in H(z) \otimes' H(w)} inf\{g(e \otimes' d), \delta\} \\
&= inf\{\bigwedge_{e \otimes' d \in H(z \otimes w)} g(e \otimes' d), \delta\} \\
&= inf\{\underline{H}(g)(z \otimes w), \delta\}.
\end{aligned}$$

Thus $sup\{\underline{H}(g)(z), \underline{H}(g)(w), \gamma\} \geq sup\{\underline{H}(g)(z \otimes w), \delta\}$ for all $z, w \in Q_t$. ■

Proposition 8.3.2 *Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ of Q_t and $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$. Then, $\overline{H}(g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FPI$ of Q_t.*

Proof. The proof is simple and is similar to the Theorem 8.3.1. ■

Theorem 8.3.3 *Lat g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$ of Q_t and $H : Q_t \longrightarrow P^*(Q'_t)$ be a $SSVH$. Then, $\underline{H}(g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$ of Q_t.*

Proof. Let $z \in Q_t$ and $\gamma, \delta \in (0,1]$ be such that $\gamma < \delta$. Since g is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$ of Q'_t, by Proposition 6.3.4, we have $sup\{g(e), \gamma\} \geq inf\{g(e \otimes' e), \delta\}$ for all $e \in Q_t$.

Consider the following:

$$\begin{aligned} sup\{\underline{H}(g)(z),\gamma\} &= sup\{\bigwedge_{e\,\in\,H(z\,)} g(e),\gamma\} \\ &= \bigwedge_{e\,\in\,H(z\,)} sup\{g(e),\gamma\} \\ &\geq \bigwedge_{e\,\in\,H(z\,)} inf\{g(e\otimes' e),\delta\} \\ &= \bigwedge_{e\otimes' e\,\in\,H(z\,)\otimes' H(z)} inf\{g(e\otimes' e),\delta\} \\ &= \bigwedge_{e\otimes' e\,\in\,H(z\,\otimes z)} inf\{g(e\otimes' e),\delta\} \\ &= inf\{\bigwedge_{e^2\,\in\,H(z\,\otimes z)} g(e\otimes' e)],\delta\} \\ &= inf\{\underline{H}(g)(z\otimes z),\delta\}. \end{aligned}$$

Thus, $sup\{g(z),\gamma\} \geq inf\{\underline{H}(g)(z\otimes z),\delta\}$ for all $z \in Q_t$. Hence, $\underline{H}(g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$ of Q_t. ■

Proposition 8.3.4 *Let g be a $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$ of Q_t and $H : Q_t \longrightarrow P^*(Q_t)$ be a $SSVH$. Then, $\overline{H}(g)$ is a $(\in_\gamma, \in_\gamma \vee q_\delta)$-$FSPI$ of Q_t.*

Proof. The proof is similar to the proof of Theorem 8.3.3. ■

8.4 Approximations of $(\in_\gamma, \in_\gamma \vee q_\delta)$-Fuzzy Q_t-submodules of Q_t-Module

GLA and GUA of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodules of a Q_t-module is being presented in this section.

Theorem 8.4.1 *Let $H : M \longrightarrow P^*(N)$ be a $SSVH$ of Q_t-modules and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of N. Then $\underline{H}(g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule M.*

Proof. Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of N. As $H : M \longrightarrow P^*(N)$ is a $SSVH$ of Q_t-modules, so $\vee_{i\in I} H(m_i) = H(\vee_{i\in I} m_i)$. Consider the following:

$$\begin{aligned} sup\left\{\underline{H}(g)(\vee_{i\in I} m_i),\gamma\right\} &= sup\left\{\bigwedge_{c\in H(\vee_{i\in I} m_i)} g(c),\gamma\right\} \\ &= \bigwedge_{c\in H(\vee_{i\in I} m_i)} sup\left\{g(c),\gamma\right\} \\ &= \bigwedge_{c\in \vee_{i\in I} H(m_i)} sup\left\{g(c),\gamma\right\} \end{aligned}$$

Since $c \in \vee_{i\in I} H(m_i)$, there exist $c_1 \in H(m_1), c_2 \in H(m_2), ..., c_i \in H(m_i)$ such that $c = \vee_{i\in I} c_i$.

$$\begin{aligned}
sup\{\underline{H}(g)(\vee_{i\in I} m_i), \gamma\} &= \bigwedge_{\vee_{i\in I} c_i \in \vee_{i\in I} H(m_i)} sup\{g(\vee_{i\in I} c_i), \gamma\} \\
&\geq \bigwedge_{\vee_{i\in I} c_i \in \vee_{i\in I} H(m_i)} inf\{\inf_{i\in I} g(c_i), \delta\} \\
&= \bigwedge_{c_1 \in H(m_1),..., c_i \in H(m_i)} inf\{\inf_{i\in I}[g(c_1), ..., g(c_i)], \delta\} \\
&= inf\{\inf_{i\in I}[\bigwedge_{c_1 \in H(m_1)} g(c_1), ..., \bigwedge_{c_i \in H(m_i)} g(c_i)], \delta\} \\
&= inf\{\inf_{i\in I}[\underline{H}(g)(m_1), ..., \underline{H}(g)(m_i)], \delta\} \\
&= inf\{\inf_{i\in I}[\underline{H}(g)(m_i)], \delta\}.
\end{aligned}$$

Thus, we have $sup\{\underline{H}(g)(\vee_{i\in I} m_i), \gamma\} \geq inf\{\inf_{i\in I}[\underline{H}(g)(m_i)], \delta\}$.

As $H : M \longrightarrow P^*(N)$ is a $SSVH$ of Q_t-modules, so $H(q * m) = q *' H(m)$.

Now, consider

$$\begin{aligned}
sup\{\underline{H}(g)(q * m), \gamma\} &= sup\left\{\bigwedge_{a\in H(q*m)} g(e), \gamma\right\} \\
&= \bigwedge_{e\in H(q*m)} sup\{g(e), \gamma\} \\
&= \bigwedge_{e\in q*' H(m)} sup\{g(e), \gamma\}.
\end{aligned}$$

As $e \in q *' H(m)$, there is $a \in H(m)$ such that $e = q *' a$.

$$\begin{aligned}
sup\{\underline{H}(g)(q * m), \gamma\} &= \bigwedge_{q*' a\in q*' H(m)} sup\{g(q *' a), \gamma\} \\
&\geq \bigwedge_{q*' a\in q*' H(m)} inf\{g(a), \delta\} \\
&= inf\{\bigwedge_{a \in H(m)} g(a), \delta\} \\
&= inf\{\underline{H}(g)(m), \delta\}.
\end{aligned}$$

Thus, $sup\{\underline{H}(g)(q * m), \gamma\} \geq inf\{\underline{H}(g)(m), \delta\}$. Therefore, $\underline{H}(g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M. ■

Theorem 8.4.2 *Let g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of N and $H : M \longrightarrow P^*(N)$ be a SVH of Q_t-modules. Then $\overline{H}(g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M.*

Proof. Let $H : M \longrightarrow P^*(N)$ be a SVH of Q_t-modules. Then, we have $\vee_{i\in I} H(m_i) \subseteq H(\vee_{i\in I} m_i)$. Let $m_i \in M$ for $i \in I$ and g be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of N. Consider the following:

$$
\begin{aligned}
inf\{\underset{i\in I}{inf}[\overline{H}(g)(m_i)], \delta\} &= inf\{\underset{i\in I}{inf}[\overline{H}(g)(m_1), \overline{H}(g)(m_2), ..., \overline{H}(g)(m_i)], \delta\} \\
&= inf\{\underset{i\in I}{inf}[\underset{a_1 \in H(m_1)}{\vee} g(a_1), ..., \underset{a_i \in H(m_i)}{\vee} g(a_i)], \delta\} \\
&= \underset{a_1 \in H(m_1), ..., a_i \in H(m_i)}{\vee} inf\{\underset{i\in I}{inf}[g(a_1), ..., g(a_i)], \delta\} \\
&= \underset{\vee_{i\in I}a_i \in \vee_{i\in I}H(m_i)}{\vee} inf\{\underset{i\in I}{inf} g(a_i), \delta\} \\
&\le \underset{\vee_{i\in I}a_i \in \vee_{i\in I}H(m_i)}{\vee} sup\{g(\vee_{i\in I}a_i), \gamma\} \\
&= \underset{e \in \vee_{i\in I}H(m_i)}{\vee} sup\{g(e), \gamma\} \\
&\le \underset{e \in H(\vee_{i\in I}m_i)}{\vee} sup\{g(e), \gamma\} \\
&= sup\left\{\underset{e \in H(\vee_{i\in I}m_i)}{\vee} g(e), \gamma\right\} \\
&= sup\left\{\overline{H}(g)(\vee_{i\in I}m_i), \gamma\right\}.
\end{aligned}
$$

Thus, we have $sup\left\{\overline{H}(g)(\vee_{i\in I}m_i), \gamma\right\} \ge inf\{\underset{i\in I}{inf}[\overline{H}(g)(m_i)], \delta\}$.

As $H : Q_t \longrightarrow P^*(Q_t)$ is a SVH of Q_t-modules, so $q *' H(m) \subseteq H(q * m)$.

Furthermore, consider

$$
\begin{aligned}
inf\{\overline{H}(g)(m), \delta\} &= inf\{\underset{a \in H(m)}{\vee} g(a), \delta\} \\
&= \underset{a \in H(m)}{\vee} inf\{g(a), \delta\} \\
&\le \underset{a \in H(z)}{\vee} sup\{g(q *' a), \gamma\} \\
&= \underset{q*'a \in q*'H(z))}{\vee} sup\{g(q *' a), \gamma\} \\
&\le \underset{q*'a \in H(q*m)}{\vee} sup\{g(q *' a), \gamma\} \\
&= \underset{c \in H(q*m)}{\vee} sup\{g(c), \gamma\} \\
&= sup\left\{\underset{c \in H(q*m)}{\vee} g(c), \gamma\right\} \\
&= sup\left\{\overline{H}(g)(q * m), \gamma\right\}.
\end{aligned}
$$

Thus, $sup\left\{\overline{H}(g)(q * m), \gamma\right\} \ge inf[\overline{H}(g)(m), \delta\}$. Therefore, $\overline{H}(g)$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M. ■

Proposition 8.4.3 *Let g_1 and g_2 be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of N and $H : M \longrightarrow P^*(N)$ be a SVH of Q_t-modules. Then $\overline{H}(g_1) \Cap \overline{H}(g_2)$ and $\underline{H}(g_1) \Cap \underline{H}(g_2)$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy Q_t-submodule of M.*

Proof. The proof follows from Proposition 6.2.5 and Theorems 8.4.1, 8.4.2. ■

Conclusion

In this thesis at first, we contributed to the roughness in the subsets of a Q_t-module with respect to Pawlak approximation space. Further complete congruence with respect to $\vee$-complete and $*$- complete is introduced. Upper and lower rough Q_t-submodules of Q_t-module are defined and their different properties are discussed. Moreover, roughness in quotient of Q_t-module are proposed. Then we generalized this concept and provided the concept of generalized roughness in the subsets of Q_t-module. The idea of set-valued homomorphism and strong set-valued homomorphism of Q_t-module are also proposed.

As a generalization of rough fuzzy ideals in quantale [49], the concept of generalized rough fuzzy ideals, generalized rough fuzzy prime ideals, generalized rough fuzzy semi-prime ideals and generalized rough fuzzy primary deals of quantales were proposed in the third chapter. Further, approximations of fuzzy ideals, fuzzy prime, fuzzy semi-prime and fuzzy primary ideals with the help of set-valued homomorphism and strong set-valued homomorphism are discussed. In addition, homomorphic images of generalized rough prime (semi-prime, primary) ideals which are established by quantale homomorphism, are examined.

Next, in chapter four, we define (α, β)-$f$uzzy subquantales and (α, β)-$f$uzzy ideals of quantale which are the generalization of fuzzy subquantales and fuzzy ideals in quantale [49]. Further, an $(\in, \in \vee q)$-fuzzy ideals and $(\in, \in \vee q)$-fuzzy subquantales are discussed. These fuzzy subquantales and fuzzy ideals are characterized by their level subquantales and ideals, respectively. Some important results about $(\in, \in \vee q)$-$f$uzzy prime and $(\in, \in \vee q)$-$f$uzzy semi-prime ideals are discussed. Fuzzy quantale submodule is defined and its generalization that is an (α, β)-$f$uzzy Q_t-submodule of Q_t-module is also introduced in this chapter. Fuzzy Q_t-submodule is characterized by its level Q_t-subquantales. Further, approximations of fuzzy Q_t-submodule and approximations of $(\in, \in \vee q)$-fuzzy Q_t-submodule of Q_t-module are introduced.

The concept of (α, β)-$f$uzzy $f$ilter and some related properties are discussed in chapter five. Further, an $(\in, \in \vee q)$-$f$uzzy $f$ilters are discussed. It is investigated that under quantale homomorphism, inverse image of an $(\in, \in \vee q)$-$f$uzzy filter is an $(\in, \in \vee q)$-$f$uzzy filter. Moreover, these $f$uzzy $f$ilters are characterized by their level sets. Furthermore, in this chapter, we are presenting more general forms of $(\in, \in \vee q)$-$f$uzzy $f$ilters of Quantales. Particular attention is given to $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy $f$ilters.

In the chapter six, we started the investigation of roughness in $(\in, \in \vee q)$-$f$uzzy ideal and $(\in, \in \vee q)$-$f$uzzy $f$ilter of quantales with respect to the generalized approximation space. Moreover, it is demonstrated that generalized lower and upper approximations of $(\in, \in \vee q)$-$f$uzzy ideal, $(\in, \in \vee q)$-$f$uzzy $f$ilter, $(\in, \in \vee q)$-$f$uzzy prime ideal and $(\in, \in \vee q)$-$f$uzzy semi-prime ideal are $(\in, \in \vee q)$-$f$uzzy ideal, $(\in, \in \vee q)$-$f$uzzy $f$ilter, $(\in, \in \vee q)$-$f$uzzy prime ideal and $(\in, \in \vee q)$-$f$uzzy semi-prime ideal by using set-valued and strong set-valued homomorphism, respectively.

In chapter seven, we are presenting more general forms of $(\in, \in \vee q)$-$f$uzzy subquantale and $(\in, \in \vee q)$-$f$uzzy ideal of quantales. We introduce the concepts of (α, β)-$f$uzzy subquantale, (α, β)-$f$uzzy ideal and some related properties are investigated. Special attention is given to $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy subquantale, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy prime, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy semi-prime ideals, and some interesting results about them are obtained. Furthermore, subquantale, prime, semi-prime and $f$uzzy subquantale, $f$uzzy prime, $f$uzzy semi-prime ideals of the types $(\in_\gamma, \in_\gamma \vee q_\delta)$ are linked by using level subsets.

The concept of generalized approximations (GA) of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy subquantale and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy filter in quantales were presented in chapter eight. With the help of set-valued and strong set-valued homomorphism, it is observed that GLA and GUA of $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy subquantale and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy filter are $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy subquantale and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy filter, respectively. To extend this work, one may consider the following topics:

(1) Generalized Rough Fuzzy Ideals in near-ring.

(2) Generalized Rough Fuzzy Q_t-submodules of Q_t-module.

(3) Generalized roughness in $(\in, \in \vee q)$-fuzzy ideals of BCK algebra.

(4) Generalized roughness in $(\in, \in \vee q)$-fuzzy ideals of of near-ring.

(5) $(\in_\gamma, \in_\gamma \vee q_\delta)$-$f$uzzy ideals in near-ring.

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