Black Holes:
Quartic Quasi-Topological Gravity and Greybody Factor

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by

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I Jamil Ahmed hereby state that my PhD thesis titled **Black Holes: Quartic Quasi-Totoplogical Gravity and Greybody Factor** is my own work and has not been submitted previously by me for taking any degree from the Quaid-i-Azam University Islamabad, Pakistan or anywhere else in the country/world.

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To my family and my supervisor . . .

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Preface

Alternative theories of gravity are those which provide description of gravitational interaction different from the description of the usual general theory of relativity. In 1920, soon after the birth of general relativity, efforts were made to propose these modified theories of gravity. In those days, work in this direction was mainly due to the curiosity to challenge the newly presented general theory of relativity, but with time, interest and motivation to work on modified gravities waxed and waned. Overall, however, a continuous activity in this direction over the past ninety years may still be found. A part of this thesis is another such effort in the aforementioned approach.

In this thesis, higher curvature gravity theories are considered which modify Einstein's theory of gravity due to the addition of higher order curvature terms to the Einstein-Hilbert term. The primary motivation for this modification comes from the appearance of quadratic curvature terms in the low-energy effective action of string theory and is also due to black hole solutions in higher curvature gravities. Furthermore, these black hole solutions exhibit certain properties which are unavailable in the usual Einstein's theory of gravity.

Black hole emission and absorption phenomena is related to an important quantity, known as greybody factor. Due to greybody factor Hawking radiation's spectrum deviates from the spectrum of black body radiations. Some studies of greybody factor are presented in this thesis which elaborate the characteristics of the radiations emitted from black holes.

The outline of the thesis is as follows: In Chapter 1, historical background of general relativity is introduced followed by current challenges and then a brief description of higher curvature gravities. This chapter also introduces the notion of greybody factor and its importance. In Chapter 2, the quartic version of generalized cubic quasi-topological gravity is constructed. This class of theories includes Lovelock gravity and a known form of quartic quasi-topological gravity as special cases and possesses a number of remarkable properties:

- In vacuum, or in the presence of suitable matter, there is a single independent field equation which is a total derivative.
- At the linearized level, the equations of motion on a constant curvature background are second order, coinciding with the linearized Einstein equations up to a redefinition of Newton's constant. Therefore, these theories propagate only the massless, transverse graviton on a constant curvature background.
- While the Lovelock and quasi-topological terms are trivial in four dimensions, there exist four new generalized quasi-topological terms (the quartet) that are nontrivial, leading to interesting higher curvature theories in $d \geq 4$ dimensions that appear well suited for holographic study.

In Chapter 3, four dimensional black hole solutions to the theory are constructed and their properties are studied. Further, black brane solutions in general dimensions of the theory are studied. Results of this study may lead to interesting consequences for dual conformal field theories. In Chapter 4, generalized Reissner-Nordström anti-de Sitter black hole solution for generalized cubic quasi-topological gravity is constructed. Asymptotic and near-horizon solutions for this theory are found in d spacetime dimensions. A form of extended first law of black hole thermodynamics for these black holes is also presented. Critical values of volume, pressure and temperature are presented.

In Chapter 5, a general expression for the greybody factor of non-minimally coupled scalar fields in Reissner-Nordström-de Sitter spacetime in low frequency approximation is derived. Greybody factor as a characteristic of effective potential barrier is also presented. The role of cosmological constant, both in the absence as well as in the presence of non-minimal coupling, is presented. Considering the non-minimal coupling as a mass term, its effect on the greybody factor is discussed. The significance of the results are elaborated by giving formulae for differential energy rates and general absorption cross sections. The greybody factor gives insight into the spectrum of Hawking radiations.

In Chapter 6, greybody factor of massless, uncharged scalar fields is worked out in the background of cylindrically symmetric spacetime, in the low-energy approximation. Two cases are discussed. In the first case, analytical expression for absorption probability is derived with the spacetime kinetically coupled with the Einstein tensor. In the second case, an analysis is performed in the absence of the coupling constant by using the wave equation, which is derived from Klein-Gordon equation. The radial part of the wave equation is solved in the form of hypergeometric function in the near-horizon region whereas in the far region, the solution is of the form of Bessel's function. Finally considering the continuity of wave function, the two solutions in the low energy approximation are smoothly matched to get a formula for absorption probability. The last chapter concludes the thesis by summarizing the outcomes of this work.

List of Publications from the Thesis

As publication is one of the requirements of the Higher Education Commission of Pakistan, we give here a list of publications from the thesis:

1. J. Ahmed, R. A. Hennigar, R. B. Mann and M. Mir, Quintessential quartic quasi-topological quartet, *JHEP*, **2017** (2017) 134.

2. J. Ahmed and K. Saifullah, Greybody factor of scalar fields from black strings, *Eur. Phys. J. C*, **77** (2017) 885.

3. J. Ahmed and K. Saifullah, Greybody factor of scalar field from Reissner-Nordström-de Sitter black hole, *Eur. Phys. J. C*, **78** (2018) 316.

4. M. Mir, R. A. Hennigar,J. Ahmed and R. B. Mann, Black hole chemistry and holography in generalized quasi-topological gravity, arXiv:1902.02005 (2019).

5. J. Ahmed and K. Saifullah, Absorption cross section of scalar fields for three-dimensional black holes, *In preparation*, (2019).

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Chapter 1

Introduction

1.1 Historical background of general relativity

Gravity is a fundamental force which has puzzled the man for ages despite the fact that it is related to phenomena which can be experienced in everyday life. It is also an established fact that the gravitational interaction was the first of its kind which underwent experimental investigations. This was its simplicity that allowed the construction of an experimental apparatus. At the end of the16^{th} century, it was Galileo Galilei who first introduced pendulums and inclined planes to study gravity. In fact, it appears that gravity played an important role in building in Galileo's mind ideas related to the requirement of experiments in science; his experiments had a great impact on modern scientific thinking. However, gravity was not well understood until 1665, when Isaac Newton introduced his famous "inverse square gravitational law". Newton's view of gravity was based on two concepts. First, space is absolute, i.e., a fixed, unaffected structure with physical phenomena taking place in a rigid arena. Second, inertial and gravitational mass coincide. The first twenty years of Newton's gravity managed to explain all the aspects of gravity known at that time. However, soon it was realized how large a portion of physical world was insufficiently described by this theory.

In 19^{th} century, there were certain experimental results which could not be predicted by Newton's theory of gravity. These include Verrier's observation of a 35 arc-seconds excess precession of Mercury's orbit in 1855 and later on, Newcomb's more precise observation of 43 arc-seconds. Also, Mach's idea in 1893 about Newton's absolute space is considered as the first constructive attack on Newtonian physics. In the form of initial formulation of Mach, the idea was rather vague but Einstein put it in mainstream physics in a more elegant way. According to Einstein, "inertia originates in a kind of interaction between bodies". This idea totally contradicts Newtonian picture of inertia, according to which inertia was a property of absolute space. Furthermore, Dicke suggested that the gravitational constant should be a function of mass distribution, contrary to Newton, who thought that it should be a universal constant. These were the developments which compelled physicists of that time to reconsider the basic axioms of Newtonian picture of gravity.

Einstein in 1905 postulated his theory of special relativity, which successfully explained many non-gravitational phenomena, but it was in contradiction with Newtonian physics. Since relative motion and related concepts of Newton and Galileo were ruled out by the theory of special relativity, it was soon realized that one has to generalize the special theory of relativity to accommodate non-inertial frames of reference as well. In 1907, Einstein predicted gravitational redshift by his idea of equivalence of gravity and inertia. In 1915, he succeeded in formulating his general theory of relativity which included gravity. Remarkably, this theory matched perfectly with the experimental results about the precision of Mercury's orbit and the deflection of light from the Sun, which later measured in 1919 by A. Eddington.

Despite general theory of relativity being well-accepted and a practised theory of gravity, Newton's theory could not be abandoned completely. This was because it was realized that Newton's ideas were valid in limited cases and were sufficient for most practical applications. Also it is important to note that general relativity reduces to Newtonian gravity in the limit of certain gravitational field strengths and velocities. In other words, it can be said that Einstein reformulated some of Newton's ideas in a more suitable way, for example, his equivalence principle.

Nowadays, general relativity is facing questions similar to Newton's gravity, some of which are briefly presented above. These include issues of being inefficient at providing explanation to particular observations, being incompatible with established theories and lack of uniqueness.

1.2 Current challenges of general relativity

Modern physics is based on two great theories: general relativity and quantum field theory. Both these theories are successful in their own arenas of physical phenomena. As general relativity is based on a classical viewpoint, it describes gravitational systems and non-inertial frames. On the other hand, quantum field theory is quite successful in revealing the mysteries of high energy or small scales, where classical description is not valid. In quantum theory, spacetime is considered as flat whereas in quantum field theory, this is generalized to a curved spacetime and that quantum fields play in a rigid arena. On the other hand, general relativity does not take into account the quantum nature of matter. In fact, the behaviour of gravitational fields at quantum scales itself is not clear. An interesting question which follows is the extent to which these two great theories are compatible.

The real problem is that there is no idea whether gravity retains its nature as a force at small scales or not. This is because it is the weakest interaction compared with other interactions, so that the characteristic scale under which one could experience relevant non-classical effects of gravity is too small and is of the order of 10^{-33} cm, the Planck scale, which is of course a big problem.

There are many reasons why physicists are trying to fit together general relativity and quantum mechanics [1]. Curiosity is the main motivation for cutting edge scientific triumphs. In the present context, curiosity lies in finding how the theory of quantum gravity would look like and what modifications in general relativity and in quantum mechanics are required, in order to make these theories compatible.

Some of the theories which modify general relativity are considered in this thesis. In particular, those theories which are obtained by adding higher curvature terms in the usual Einstein-Hilbert action are investigated.

1.3 Higher curvature theories

The Einstein-Hilbert action in general relativity is the action that gives Einstein field equations. The gravitational part of this action is given by

$$\mathcal{I}_{EH} = \frac{c^4}{16\pi G}\int R\sqrt{-g}d^4x, \tag{1.1}$$

where $g = det(g_{ab})$ is the determinant of the metric tensor, R is the Ricci scalar, G is the gravitational constant and c is the speed of light in vacuum. Although it is in the simplest possible form, but it has all the important required properties, including the one which requires equations of motion to be of second-order. Despite the fact that general relativity is one of the most successful theories of all times, it has some limitations. For example, it is not renormalizable. To overcome this limitation, loop quantum corrections were suggested [2], which is done by adding counter terms in the action, which are higher order in curvature scalars. Furthermore, in string theory, one can obtain the Einstein-Hilbert action from α' expansion of string theory, where $\sqrt{\alpha'}$ is the string length. If one considers higher order in α' expansion of string theory, then higher curvature corrections appear. String theory makes other strong predictions about the nature of addition of higher curvature corrections to the Einstein's gravity [3]. The most important predictions include existence of super-symmetry and extra dimensions. According to one of the requirements of super-symmetry, spacetime has 10 dimensions. This convinces physicists to look for the modifications of general relativity in higher dimensions to solve the problems of quantum gravity.

Such studies have enough motivation to extend gravity beyond Einstein's theory and study some interesting black hole solutions in the resulting theories. A part of this thesis is devoted to those theories in which terms are added to the Einstein-Hilbert part, which are of higher order in curvature.

1.4 $f(R)$ theories

When square of the Ricci scalar is added to the Einstein-Hilbert part of the action, we get the simplest possible extension of Einstein's theory of gravity. The action in this case is given by

$$\mathcal{I}_{f(R)} = \int (R + \alpha R^2) d^d x. \tag{1.2}$$

This action is usually known as the Starobinsky mode [4] and is a particular case of more general class of Lagrangians, which are simply polynomial functions of the Ricci scalar. Theories of these general class of Lagrangians are called $f(R)$ theories. These theories provide very good toy models to test the laws of black hole thermodynamics. If one couples these theories minimally to scalar fields, then these are equivalent to Einstein's gravity. Due to this property, these theories are usually known as scalar-tensor theories. The natural restriction on scalar fields is that they are proportional to the first derivative of the function, which in turn ensure that these theories are ghost-free. If we consider general $f(R)$ theories, then equations of motion will be of fourth order and hence propagate ghosts, but if we consider a scalar-tensor subset of $f(R)$ theories, then they are ghost-free.

1.5 Lovelock gravity

Lovelock theory of gravity is regarded as the generalization of Einstein's theory of general relativity. It was introduced by D. Lovelock [5, 6]. In arbitrary dimensions d it is the most general theory of gravity which yield second order equations of motion. Therefore, Lovelock's theory is the natural generalization of Einstein's theory in higher dimensions. Both the theories coincide in three and four dimensions but are different in higher dimensions. Since Einstein-Hilbert action is one of the terms that constitute the Lovelock action, therefore Einstein's theory is a special case of Lovelock's theory in dimensions greater then four. Lagrangian of the theory is given by

$$\mathcal{L} = \sqrt{-g}\sum_{n=0}^{t} a_n R^n, R^n = \frac{1}{2^n}\delta^{a_1 b_1 \ldots a_n b_n}_{c_1 d_1 \ldots c_n d_n}\prod_{r=1}^{n} R^{e_r f_r}_{a_r b_r}, \tag{1.3}$$

where R^{ab}_{cd} represents Riemann tensor and $\delta^{a_1 b_1 \ldots a_n b_n}_{c_1 d_1 \ldots c_n d_n}$ is the generalized Kronecker delta which is defined as

$$\delta^{a_1 b_1 \ldots a_n b_n}_{c_1 d_1 \ldots c_n d_n} = n!\delta^{a_1}_{[c_1}\delta^{b_1}_{d_1}\ldots\delta^{a_n}_{c_n}\delta^{b_n}_{d_n]}. \tag{1.4}$$

The coupling constants a_n appearing in the above Lagrangian have dimensions of $[length]^{2n-d}$. Expansion of the product in equation (1.3) gives the following

$$\mathcal{L} = \sqrt{-g}(a_0 + a_1 R + a_2(R^2 + R_{abcd}R^{abcd} - 4R_{ab}R^{ab}) + a_3\mathcal{O}(R^3)). \tag{1.5}$$

It is clear from above equation that a_0 corresponds to the cosmological constant, while a_n with $n \geqslant 2$ are coupling constants for additional terms that correspond to the ultraviolet corrections to Einstein's theory, involving higher order contractions of the Riemann tensor R^{ab}_{cd}. In particular, second order term is $R^2 + R_{abcd}R^{abcd} - 4R_{ab}R^{ab}$, which is precisely the quadratic Gauss-Bonnet term, which is the dimensionally extended version of the four-dimensional Euler density.

Lovelock and $f(R)$ gravities have received a lot of attention in the quest to modify general relativity. There is a class of theories in which both these classes appear as special cases. This class is known as $f(Lovelock)$ theories. The general form of the action of $f(Lovelock)$ gravities can be written as

$$\mathcal{S}_{f(Lovelock)} = \frac{1}{16\pi G}\int (d^d x\sqrt{|g|}f(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, ..., \mathcal{L}_{\frac{d}{2}})), \tag{1.6}$$

where f is a differentiable function of dimensionally extended Euler densities given by

$$\mathcal{L}_n = \delta^{b_1 b_2 ... b_{2n}}_{a_1 a_2 ... a_{2n}} R^{a_1 a_2}_{b_1 b_2} ... R^{a_{2n-1} a_{2n}}_{b_{2n-1} b_{2n}}. \tag{1.7}$$

We can see that, in particular, $\mathcal{L}_1$ is the usual Einstein-Hilbert term and $\mathcal{L}_2$ is the Gauss-Bonnet term. We will see the form of $\mathcal{L}_3$ in later sections and develop the form of $\mathcal{L}_4$ in later chapters.

It is evident from the above that we can reduce $f(Lovelock)$ theories to the usual $f(R)$ and Lovelock theories if we choose f to be a linear combination of the sum of arbitrary functions of Ricci scalar and Euler density, respectively. As we have seen above, Lovelock gravities are the most general theories having second order equations of motion but in the $f(Lovelock)$ case, the equations of motion are of fourth order. In a particular case, if one restricts the number of dimensions to four, then the action will contain the Ricci scalar and Gauss-Bonnet terms alone. Such theories are usually known as $f(R, G)$ gravities.

1.6 Einsteinian cubic gravity

After the great success of Lovelock and other theories in modifying Einstein's gravity, physicists devoted efforts to construct new theories of gravity, which are free of ghosts. These efforts also include the work in the direction to construct theories of cubic curvature corrections to Einstein's gravity. Quite recently, a new model of cubic curvature gravity has been presented [7]. This gives a unique model when it is added to quadratic and cubic Lovelock terms. It also provides Einstenian spectrum at linearized level, meaning that the theory is ghost-free. Furthermore, it has dimension-independent coupling constants. This new model is coined as "Einsteinian cubic gravity". The Lagrangian density for this new theory is given by

$$\mathcal{L}_{ECG} = \frac{1}{2\kappa}[-2\Lambda + R] + \beta_1 X_4 + \kappa[\beta_2 X_6 + \lambda P], \tag{1.8}$$

where

$$P = 12 R_a{}^c{}_b{}^d R_c{}^e{}_d{}^f R_e{}^a{}_f{}^b + R_{ab}{}^{cd} R_{cd}{}^{ef} R_{ef}{}^{ab} - 12 R_{abcd} R^{ac} R^{bd} + 8 R_a^b R_b^c R_c^a. \tag{1.9}$$

Also, X_4 and X_6 are four- and six-dimensional Euler densities, respectively, representing Lovelock terms and have the explicit forms given by

$$X_4 = -\frac{1}{4}\delta^{a_1 a_2 a_3 a_4}_{c_1 c_2 c_3 c_4} R^{c_1 c_2}_{a_1 a_2} R^{c_3 c_4}_{a_3 a_4}, \tag{1.10}$$

and

$$X_6 = -\frac{1}{4}\delta^{a_1 a_2 a_3 a_4 a_5 a_6}_{c_1 c_2 c_3 c_4 c_5 c_6} R^{c_1 c_2}_{a_1 a_2} R_{a_3 a_4}{}^{c_3 c_4} R^{c_5 c_6}_{a_5 a_6}. \tag{1.11}$$

An interesting thing about the contribution of this new P term is that it is active in four dimensions. Remarkably, P contributes in all dimensions except $d = 3$ and $d = 6$. Therefore it allows one to study the affects of cubic order curvature in four dimensions. It was claimed [7] that P is the unique theory for cubic curvature in which the linearized equations of motion match the Einstein's gravity. Therefore, it provides a useful holographic toy model and also one can seek interesting black hole solutions of this new theory.

1.7 Generalized cubic quasi-topological gravity

Quasi-topological theories are very interesting gravity theories because when evaluated on a spherically symmetric background, the field equations of these theories reduce to second order differential equations and also produce exact solutions, similar to those of the Lovelock gravity. This is peculiar, because when the condition of spherical symmetry is relaxed, the order of field equation rises to four. Another important thing which makes quasi-topological theories more relevant than the Lovelock theories is that for a given order of curvature, these theories are active in dimensions less than the corresponding dimensions in which Lovelock theories are active for a given order of curvature. For example, if we consider cubic order in curvature, Lovelock gravity will be active in dimensions seven and higher, whereas quasi-topological gravity will be active in dimensions five and higher. Furthermore, on the background of a constant curvature spacetime, the linearized equations of motion of the theory coincide with that of the linearized equations of motion of Einstein's gravity up to an overall pre-factor. This match of linearized equations of motion guarantees that the theory is physical, free of ghosts and does not posses extra degrees of freedom other then Einsteinian gravity. This is very important, because if a theory propagates extra degrees of freedom, then some of them may carry negative kinetic energy, which in turn is equivalent to breakdown of unitarity in quantum theory. Furthermore, this match of equations of motion with Einstein's theory of gravity implies that the holographic studies of the theory are significantly simplified.

The gravitational action containing cubic order in curvature possesses generalized Schwarzschild solution (usual Schwarzschild solution can be recovered by turning off the coupling constants with cubic interaction terms), that is, possesses a vacuum static spherically symmetric solution, characterized by a single metric function and having the most general form of Lagrangian, which can be written as [8]

$$\mathcal{L}_{CQTG} = -2\Lambda + R + \alpha X_4 + \beta X_6 + \mu Z_d - \lambda S_d. \tag{1.12}$$

Here $\alpha, \beta, \mu, \lambda$ are coupling constants and Λ is cosmological constant. The forms of Z_d and S_d are given by [8]

$$\begin{aligned}\mathcal{Z}_d \;=\;& R_a{}^b{}_c{}^d R_b{}^e{}_d{}^f R_e{}^a{}_f{}^c + \frac{1}{(2d-3)(d-4)}\Big(\frac{3(3d-8)}{8} R_{abcd}R^{abcd}R - \frac{3(3d-4)}{2} R_a{}^c R_c{}^a R \\ & -3(d-2)R_{acbd}R^{acb}{}_e R^{de} + 3dR_{acbd}R^{ab}R^{cd} + 6(d-2)R_a{}^c R_c{}^b R_b{}^a + \frac{3d}{8}R^3\Big). \end{aligned} \tag{1.13}$$

and

$$\begin{aligned}\mathcal{S}_d \;=\;& 14R_a{}^e{}_c{}^f R^{abcd}R_{bedf} + 2R^{ab}R_a{}^{cde}R_{bcde} - \frac{4(66-35d+2d^2)}{3(d-2)(2d-1)} R_a{}^c R^{ab} R_{bc} \\ & -\frac{2(-30+9d+4d^2)}{(d-2)(2d-1)} R^{ab}R^{cd}R_{acbd} - \frac{(38-29d+4d^2)}{4(d-2)(2d-1)} R R_{abcd}R^{abcd} \\ & +\frac{(34-21d+4d^2)}{(d-2)(2d-1)} R_{ab}R^{ab}R - \frac{(30-13d+4d^2)}{12(d-2)(2d-1)} R^3. \end{aligned} \tag{1.14}$$

This new theory has many remarkable properties:

1. If one wishes to recover Einstein's gravity from it, it can be done by simply setting α, β, μ and λ equal to zero.

2. In four dimensions S_d reduces to "Einsteinian cubic gravity" and in all dimensions, this theory provides the generalized Schwarzschild solution.

3. In the background of a constant curvature spacetime, linearized equations of motion match with linearized equations of motion of Einstein's gravity.

1.8 Black hole solutions

Higher curvature gravities contain whole classes of new black hole solutions which are unavailable in the classical Einstein's gravity. It was observed that addition of quadratic curvature terms in the Einstein-Hilbert action did not produce any modification to the vacuum Einstein equations. Therefore, the effect of adding the terms containing three Riemann curvatures was investigated. Results of this analysis [9] showed that, there is a modification in the relation of the mass of the black hole and its thermodynamic parameters. Particulary, the entropy of the black hole is not proportional to the area of the horizon which is in contrast to the usual Einstein's gravity.

Furthermore, perturbative analyses were developed to consider black holes in the context of string theory [10–12]. In these references, authors have examined affects of curvature squared terms on spherically symmetric black holes in general dimensions and also considered four-dimensional black hole solutions with angular momentum or with charge. An important observation was that, apart from other thermodynamic modifications made by the higher curvature terms, these terms also generate various new long-range scalar fields "hair" on the black holes. However, these hair are not primary as they can be completely determined by, for example, mass and charge. Therefore, these solutions are not violating “no hair theorems”. In fact, new hair arise because the scalar fields have non-minimal coupling with higher curvature terms.

Due to strong implications in string theory, physicists have focussed on Lovelock gravity. Remarkably, exact spherically symmetric solutions were found for Gauss-Bonnet theory [13]. These results are extended to arbitrary Lovelock theories [14–16] and also to include charged black hole solutions [17]. These solutions exhibit certain peculiar properties such as multiple horizons and unusual thermodynamical aspects. For example, some of these solutions exhibit vanishing Hawking temperature [18]. The next interesting aspect which was considered and explored was whether scalar hair found in the earlier work survived beyond perturbative theory or not. In this direction, four-dimensional exact dilatonic solution in Gauss-Bonnet theory was investigated [19]. The full equations of motion of this analysis are difficult and thus analytic solution is not possible. Therefore, relying on the numerical evidences, it was found that modified black holes do carry these secondary scalar hair as do the charged black holes [20].

Some black hole solutions in $f(R)$ theories are also known [21, 22]. Particularly, for the case $f = R + a_2 R^2$, it was observed that, for $a_2 > 0$, no new hair can arise. Thus, for this condition, black hole solutions are same as that of Einstein's gravity. This work was extended [22] to include spherically symmetric solutions in arbitrary dimensions for general polynomial action. Results of this study revealed that, regardless of the other terms in the action, the only constraints on quadratic terms, as above i.e., $a_2 > 0$, ensured the existence of asymptotically flat black holes which are like the Schwarzschild solution.

1.9 Black hole thermodynamics

Black hole thermodynamics is certainly one of the most remarkable features of black hole physics, because it provides a common point to thermodynamics, quantum field theory and general relativity and thus provides hope that it might give insights into quantum gravity. Originally, it was developed in the context of Einstein's gravity. Afterwards, it was easy to extend similar framework for the case of higher curvature theories. It was the work of Hawking [23] who first showed that, black holes could actually radiate with temperature T, proportional to its surface gravity κ:

$$k_\beta T = \frac{\hbar\kappa}{2\pi c}. \tag{1.15}$$

Here k_β is, $\hbar$ Planck's constant and c is speed of light. This was the prediction of quantum field theory for spacetimes possessing horizon, which is independent of the dynamics of the gravity theory. Using the Euclidean path integral approach [24] in higher curvature gravity, the first law of black hole thermodynamics can be written as

$$\frac{\kappa}{2\pi c}\delta S = c^2\delta M - \Omega\delta J, \tag{1.16}$$

where M, J and Ω are mass of black hole, angular momentum and angular velocity at the horizon, respectively. Once black hole temperature and surface gravity are known, black hole entropy can be calculated. In the context of Einstein's gravity, the famous Bekenstein-Hawking entropy relation is given by [23–25]

$$S_{BH} = \frac{\kappa^3 c}{\hbar G}\frac{A_H}{4}, \tag{1.17}$$

where A_H represents the area of the event horizon. As will be seen in the next section, this formula is not valid for the case of higher curvature gravities.

1.9.1 Wald entropy

In higher curvature gravities, it was found that temperature is still proportional to the surface gravity, but entropy is no longer proportional to the area of the event horizon. Wald [26] generalized the usual Bekenstein-Hawking entropy formula for black holes in higher curvature gravities. To see this generalization, consider a Lagrangian of the form $\mathcal{L}(g_{ab}, R_{abce})$. Then, Wald entropy is given by [27]

$$S = -2\pi \int P^{abce}\epsilon_{ab}\epsilon_{ce}d^d x, \tag{1.18}$$

where $\epsilon_{ab} = k_a l_b - l_a k_b$ is the binormal to bifurcate the Killing horizon. For the case of Einstein's gravity

$$P^{abce} = \frac{\partial L}{\partial R_{abce}} = \frac{1}{32\pi G}(g^{ac}g^{be} - g^{ae}g^{bc}). \tag{1.19}$$

From this relation and the fact $\epsilon_{ab}\epsilon^{ab} = -2$, we can get the usual area-entropy relation $S = A/4$.

1.10 Greybody factor

If one considers black holes as a thermal system, then black holes will have temperature and entropy. This implies that black holes can radiate. As thermal systems, black holes have an associated temperature and entropy and therefore they radiate, and the radiations are called Hawking radiations. The emission rate in a mode of frequency ω, at the event horizon, is given by [23]

$$\Gamma(\omega) = \left(\frac{1d^3k}{e^{\beta\omega} \pm 1(2\pi)^3}\right). \tag{1.20}$$

In this relation β is used to denote the inverse of the Hawking temperature and minus (plus) sign is for bosons (fermions, respectively). This formula for emission rate can be generalized for any dimension and it is valid for massive and massless particles. Spectrum of the radiations from black holes at the event horizon is perfectly same as that of the black body spectrum. Due to this, it gives rise to the information loss paradox. The important fact is that the geometry of the spacetime around a black is non-trivial. This non-trivial geometry modifies the spectrum of Hawking radiations. In fact, the non-trivial geometry acts as a potential barrier which allows some of the radiations to transmit and reflect the rest to the black hole. The mathematical expression that summarizes all the above discussion is

$$\Gamma(\omega) = \left(\frac{\gamma(\omega)d^3k}{e^{\beta\omega} \pm 1(2\pi)^3}\right), \tag{1.21}$$

where $\gamma(\omega)$ is known as the greybody factor, which is frequency-dependent.

Physically, greybody factor originates from an effective potential barrier by a black hole spacetime. For example, the potential barrier for massless scalars from Schwarzschild spacetime is

$$V_{eff}(r) = \left(1 - \frac{r_H}{r}\right)\left(\frac{r_H}{r^3} + \frac{l(l+1)}{r^2}\right), \tag{1.22}$$

where r_H is the horizon's radius and l is angular momentum of the scalar. It is this potential which transmits or reflects radiations from black holes. Therefore, it gives rise to the frequency dependent greybody factor.

Greybody factor not only accounts for the deviation of Hawking radiations from black body spectrum, but is also important in working out energy emission rates and is also relevant for computing the partial absorption cross sections of black holes. In this thesis we investigate some spherically symmetric and axially symmetric spacetimes for these effects. The main procedure to get expression for greybody factor is to derive the solution of wave equation in the near horizon and asymptotic regions separately and then match them to an appropriate intermediate point [28–38]. In this thesis we investigate some spherically symmetric and axially symmetric spacetimes for these effects.

Chapter 2

Quartic Quasi-Topological Gravity

2.1 Introduction

Physicists expect that modification of Einstein-Hilbert action by the addition of higher curvature terms could lead to the formulation of the theory of quantum gravity. In this context, Gauss-Bonnet term, Lovelock class and various other theories are developed by various authors [3, 5, 10, 39, 40]. Higher curvature gravity is interesting area of research for many reasons. For example, it has been known for more than forty years that these theories allow for renormalizable quantum gravity [41]. Also, in the holographic context, the study of these theories has led to the discovery of numerous interesting properties [42] of conformal field theories [43–47]. Details of these studies revealed that, the inclusion of quadratic terms has been shown to lead to violations of the Kovtun–Son–Starinets (KSS) viscosity/entropy ratio bound [47, 48] and studies of cubic curvature theories have led to holographic c-theorems [49], valid in arbitrary dimensions [50].

Thermodynamic studies reveal that black holes in higher curvature theories have peculiar behaviour. Recently, there has been a renewed interest in the thermodynamics and phase structure of black holes. Motivated by the basic thermodynamic scaling arguments, it has been realized that when describing AdS black holes, cosmological constant should be treated as a thermodynamic black hole parameter i.e., the pressure and its conjugate quantity is known as the volume. It was found that, in this framework, the critical behaviour of the charged AdS black hole becomes a physical analogy with the van der Waals fluids such that the liquid/gas becomes analogous to small/large black hole phase transition [51–53].

Quasi-topological theories are those for which, in vacuum or in the presence of suitable matter, in the spherically symmetric case, there is a single independent field equation for one metric function that is a total derivative. Also, for spherical symmetry, the field equation is in the form of a polynomial of the single metric function. Whereas, for generalized quasi-topological theories, in spherical symmetry, field equation contain first two derivatives of the single metric function. As we will present in this chapter, there are four new generalized quasi-topological theories which are non-trivial in four dimensions. This is intriguing finding because quartic Lovelock and several quartic quasi-topological theories are trivial in four dimensions.

The motivations stated above led to the construction of a new cubic theory of gravity, *quasi-topological gravity* [54–62], which possesses a number of remarkable properties. As we know that the cubic Lovelock term – the six dimensional Euler density – is gravitationally non-trivial only in $d > 6$, this new cubic quasi-topological term contributes to the field equations in five dimensions and higher. The equations of motion, which are fourth order on general backgrounds, reduce to second order under the restriction to spherical symmetry. The theory admits exact spherically symmetric black hole solutions with the metric function determined by a polynomial equation very similar to the Wheeler polynomial of Lovelock gravity. Remarkably, despite the field equations being fourth order on general backgrounds, the linearized equations of motion describing graviton propagation in a constant curvature background are second order and match the linearized Einstein equations, up to a redefinition of Newton's constant [56, 63]. In other words, the additional massive scalar mode and massive, ghost-like graviton are absent.

In a recent work Ref. [8] it was shown that cubic quasi-topological gravity and cubic Lovelock gravity can be understood as members of a class of gravitational theories—*generalized quasi-topological gravity*—which, under the restriction of spherical symmetry, have a single independent field equation. This is a sufficient condition to allow vacuum static spherically symmetric (VSSS) solutions described by a single metric function; that is, solutions of the form

$$ds^2 = -N^2 f dt^2 + \frac{dr^2}{f} + r^2 d\Sigma^2_{(d-2),k} \,, \tag{2.1}$$

with $N = const.$, i.e., the solution is characterized in terms of a single metric function f [64]. Here $d\Sigma^2_{(d-2),k}$ is the line element on a surface of constant scalar curvature $k = +1, 0, -1$ corresponding to spherical, flat, and hyperbolic metrices, respectively. In Ref. [8] it was demonstrated that the most general theory to cubic order in curvature having this property is given by the action

$$\mathcal{I} = \frac{1}{16\pi G} \int d^d x \sqrt{-g} \big[-2\Lambda + R + \alpha \mathcal{X}_4 + \beta \mathcal{X}_6 + \mu \mathcal{Z}_d - \lambda \mathcal{S}_d \big] \,. \tag{2.2}$$

Here, Λ is the cosmological constant and $\alpha, \beta, \mu, \lambda$ are arbitrary coupling constants. R stands for the Ricci scalar and $\mathcal{X}_4$ and $\mathcal{X}_6$ are the four- and six-dimensional Euler densities,

$$\begin{aligned}
\mathcal{X}_4 &= -\frac{1}{4} \delta^{a_1 b_1 a_2 b_2}_{c_1 d_1 c_2 d_2} R_{a_1 b_1}{}^{c_1 d_1} R_{a_2 b_2}{}^{c_2 d_2} \,, \\
\mathcal{X}_6 &= -\frac{1}{8} \delta^{a_1 b_1 a_2 b_2 a_3 b_3}_{c_1 d_1 c_2 d_2 c_3 d_3} R_{a_1 b_1}{}^{c_1 d_1} R_{a_2 b_2}{}^{c_2 d_2} R_{a_3 b_3}{}^{c_3 d_3} \,,
\end{aligned} \tag{2.3}$$

corresponding to the standard Gauss-Bonnet and cubic Lovelock terms, respectively. $\mathcal{Z}_d$ is the cubic quasi-topological term given by (2.10) below, and $\mathcal{S}_d$ is a new term whose explicit form

$$\begin{aligned}\mathcal{S}_d \;=\;& 14 R_a{}^e{}_c{}^f R^{abcd} R_{bedf} + 2 R^{ab} R_a{}^{cde} R_{bcde} - \frac{4(66-35d+2d^2)}{3(d-2)(2d-1)} R_a{}^c R^{ab} R_{bc} \\ & - \frac{2(-30+9d+4d^2)}{(d-2)(2d-1)} R^{ab} R^{cd} R_{acbd} - \frac{(38-29d+4d^2)}{4(d-2)(2d-1)} R R_{abcd} R^{abcd} \\ & + \frac{(34-21d+4d^2)}{(d-2)(2d-1)} R_{ab} R^{ab} R - \frac{(30-13d+4d^2)}{12(d-2)(2d-1)} R^3, \end{aligned} \tag{2.4}$$

was elucidated for the first time in Ref. [8]. Interestingly, while both the cubic Lovelock and quasi-topological terms vanish in four dimensions, the new term $\mathcal{S}_4$ makes a non-trivial contribution to the field equations, reducing to the contribution of *Einsteinian cubic gravity* [65]. However, while Einsteinian cubic gravity does not permit solutions of the form (2.1) in $d > 4$, $\mathcal{S}_d$ does. In this sense, $\mathcal{S}_d$ can be viewed as the d-dimensional generalization of the four-dimensional Einsteinian cubic term.

In Ref. [8] it was observed that the linearized equations of motion derived from the action (2.2) coincide with the linearized Einstein equations, up to a redefinition of Newton's constant. Thus, to cubic order in curvature, the entire class of theories which have a single independent field equation for a VSSS ansatz enjoy the property of propagating only the massless, transverse graviton familiar from Einstein's gravity. It was also conjectured In Ref. [8] that this would be a general feature for this class of theories to all orders in the curvature. Shortly after this, it was demonstrated [66] that this is indeed the case for any theory for which the metric (2.1) describes the gravitational field outside a spherically symmetric mass distribution. This caveat explains why some theories, such as $f(R)$ gravity, admit solutions of the form (2.1) with $N = 1$ but also propagate additional modes on the vacuum; in these theories, the metric (2.1) does not describe the gravitational field of a spherical mass [66].

The aim of the present chapter is to provide the quartic version of generalized quasi-topological gravity describing all quartic Lagrangian densities which, under the restriction of a VSSS ansatz, have only a single independent field equation.

This chapter is organized as follows. In section 2.2 we first review the procedure by which the generalized quasi-topological gravities can be constructed, and present the results of this construction for the quartic case. In section 2.3 we discuss the linearized theory and in section 2.4 we derive the field equations from the actions we construct.

2.2 Construction of the quartic theories

2.2.1 Review of the construction

We begin by briefly reviewing the construction used to obtain the generalized quasi-topological theories in [8]. The central idea is to construct theories that supplement Einstein gravity with higher curvature terms in a manner such that these terms can be “turned off" by a suitable adjustment of parameters in the action. Our conditions are the same as those mentioned in [7]. Explicitly, the conditions are:

1. The solution is not an ‘embedding’ of an Einstein gravity black hole into a higher order gravity [67–69]. That is, the solution must be modified by the addition of the higher curvature terms.

2. The solution is not of a pure higher order gravity, but includes the Einstein-Hilbert term. For example, pure Weyl-squared gravity allows for four dimensional solutions with $N = 1$ [67, 70–72].

3. Further, the theory must admit an Einstein-Gravity limit, i.e. reduce to the Einstein-Hilbert action upon setting some of the parameters in the action to zero. This excludes

certain theories that tune the couplings between the various orders of curvature terms [73, 74].

These are, effectively, designed so that the most general solution of the theory takes the form of (2.1) with $N = 1$. That is, we demand that,

$$\left(\mathcal{E}_t^t - \mathcal{E}_r^r\right)\big|_{N=1} = 0\,, \tag{2.5}$$

where

$$\mathcal{E}_{ab} = \frac{1}{\sqrt{-g}}\frac{\delta \mathcal{I}}{\delta g^{ab}} \tag{2.6}$$

is the generalized Einstein tensor. We emphasize once again that in enforcing equation (2.5) we do not place any constraints on the metric function f. In a general quartic theory, evaluating the field equations in full generality is an arduous task. It is more convenient to enforce (2.5) by taking advantage of the Weyl method [75, 76]. Here, one inserts the metric ansatz (2.1) into the action, integrates by parts to remove boundary terms, considering the action as functional of N and f, $\mathcal{I}[N, f]$ and varies the action with respect to N and f to obtain the two field equations. A simple application of the chain rule reveals that

$$\frac{\delta \mathcal{I}}{\delta N} = \omega^{(k)}_{(d-2)} r^{d-2}\frac{2\mathcal{E}_{tt}}{f N^2}\,, \quad \frac{\delta \mathcal{I}}{\delta f} = -\frac{\omega^{(k)}_{(d-2)} r^{d-2}}{f}\left[\mathcal{E}_t^t - N\mathcal{E}_r^r\right], \tag{2.7}$$

and so condition (2.5) becomes

$$\frac{\delta \mathcal{I}}{\delta f}\bigg|_{N=1} = 0 \tag{2.8}$$

as was pointed out in [66].

Carrying out this procedure for a cubic theory of gravity, one is led to the following action

$$\mathcal{I} = \frac{1}{16\pi G}\int d^dx\sqrt{-g}\big[-2\Lambda + R + \alpha\mathcal{X}_4 + \beta\mathcal{X}_6 + \mu\mathcal{Z}_d - \lambda\mathcal{S}_d\big]\,, \tag{2.9}$$

where Λ is the cosmological constant and $\alpha, \beta, \mu, \lambda$ are arbitrary coupling constants. R stands for the Ricci scalar and $\mathcal{X}_4$ and $\mathcal{X}_6$ are the four- and six-dimensional Euler densities, corresponding to the standard Gauss-Bonnet and cubic Lovelock terms, respectively. $\mathcal{Z}_d$ is the cubic quasi-topological term,

$$\begin{aligned}\mathcal{Z}_d \;=\;& R_a{}^b{}_c{}^d R_b{}^e{}_d{}^f R_e{}^a{}_f{}^c + \frac{1}{(2d-3)(d-4)}\Big(\frac{3(3d-8)}{8}R_{abcd}R^{abcd}R - \frac{3(3d-4)}{2}R_a{}^cR_c{}^aR \\ & -3(d-2)R_{acbd}R^{acb}{}_eR^{de} + 3dR_{acbd}R^{ab}R^{cd} + 6(d-2)R_a{}^cR_c{}^bR_b{}^a + \frac{3d}{8}R^3\Big)\,. \end{aligned} \tag{2.10}$$

and $\mathcal{S}_d$ is a new term, written explicitly in (2.4).

Here we are interested in constructing the quartic generalization of this action. The possible quartic curvature interactions are given by [63, 77]:

$$\begin{aligned}
&\mathcal{L}_1 = R_a{}^e{}_c{}^f R^{abcd} R_e{}^j{}_b{}^h R_{fjdh}\,, \quad \mathcal{L}_2 = R_a{}^e{}_c{}^f R^{abcd} R_{bjdh} R_e{}^j{}_f{}^h\,, \quad \mathcal{L}_3 = R_{ab}{}^{ef} R^{abcd} R_c{}^j{}_e{}^h R_{djfh}\,,\\
&\mathcal{L}_4 = R_{ab}{}^{ef} R^{abcd} R_{ce}{}^{jh} R_{dfjh}\,, \quad \mathcal{L}_5 = R_{ab}{}^{ef} R^{abcd} R_{cdjh} R_{ef}{}^{jh}\,, \quad \mathcal{L}_6 = R_{abc}{}^e R^{abcd} R_{fhjd} R^{fhj}{}_e\,,\\
&\mathcal{L}_7 = (R_{abcd}R^{abcd})^2\,, \quad \mathcal{L}_8 = R^{ab} R_c{}^h{}_{ea} R^{cdef} R_{dhfb}\,, \quad \mathcal{L}_9 = R^{ab} R_{cd}{}^h{}_a R^{cdef} R_{efhb}\,,\\
&\mathcal{L}_{10} = R^{ab} R_a{}^c{}_b{}^d R_{efhc} R^{efh}{}_d\,, \quad \mathcal{L}_{11} = R R_a{}^c{}_b{}^d R_c{}^e{}_d{}^f R_e{}^a{}_f{}^b\,, \quad \mathcal{L}_{12} = R R_{ab}{}^{cd} R_{cd}{}^{ef} R_{ef}{}^{ab},\\
&\mathcal{L}_{13} = R^{ab} R^{cd} R_{ebfd} R^e{}_a{}^f{}_c\,, \quad \mathcal{L}_{14} = R^{ab} R^{cd} R_{ecfd} R^e{}_a{}^f{}_b\,, \quad \mathcal{L}_{15} = R^{ab} R^{cd} R_{efbd} R^{ef}{}_{ac}\,,\\
&\mathcal{L}_{16} = R^{ab} R_b{}^c R_{defc} R^{def}{}_a\,, \quad \mathcal{L}_{17} = R_{ef} R^{ef} R_{abcd} R^{abcd}\,, \quad \mathcal{L}_{18} = R^{de} R R_{abcd} R^{abc}{}_e\,,\\
&\mathcal{L}_{19} = R^2 R_{abcd} R^{abcd}\,, \quad \mathcal{L}_{20} = R^{ab} R_e{}^d R^{ec} R_{acbd}\,, \quad \mathcal{L}_{21} = R^{ac} R^{bd} R R_{abcd}\,,\\
&\mathcal{L}_{22} = R_a{}^b R_b{}^c R_c{}^d R_d{}^a\,, \quad \mathcal{L}_{23} = (R_{ab}R^{ab})^2\,, \quad \mathcal{L}_{24} = R_a{}^b R_b{}^c R_c{}^a R\,,\\
&\mathcal{L}_{25} = R_{ab} R^{ab} R^2\,, \quad \mathcal{L}_{26} = R^4\,.
\end{aligned} \tag{2.11}$$

It is worth noting that in dimensions less than eight, the above 26 curvature invariants are not independent. The reason is because a certain linear combination of these yields the eight dimensional Euler density,

$$\mathcal{X}_8 = \frac{1}{2^4}\delta^{a_1b_1a_2b_2a_3b_3a_4b_4}_{c_1d_1c_2d_2c_3d_3c_4d_4} R_{a_1b_1}{}^{c_1d_1} R_{a_2b_2}{}^{c_2d_2} R_{a_3b_3}{}^{c_3d_3} R_{a_4b_4}{}^{c_4d_4} \tag{2.12}$$

which vanishes identically in dimensions less than eight. Furthermore, under the restriction to spherical symmetry, there are additional, subtle degeneracies. There exist certain combinations of the above curvature invariants that identically vanish for spherically symmetric metrics [78]. Thus, we can expect certain degeneracies of theories in the spherically symmetric case: The field equations will not change upon the addition of one of these terms to the action. However, we should note that the resulting theories *will be different* when one moves away from spherical symmetry.

In what follows we focus on the quartic contributions to the action and write the following action

$$\mathcal{I} = \frac{1}{16\pi G}\int d^dx\sqrt{-g}\left[-2\Lambda + R + \sum_{i=1}^{26} c_i\mathcal{L}_i\right] \tag{2.13}$$

turning off the quadratic and cubic terms for the time being. In the following subsections we will enforce condition (2.8) on this theory by fixing the constants c_i such that the condition is satisfied for any metric function f. From a practical perspective, we first compute the action (2.13) in complete generality by explicitly determining each of the 26 terms in arbitrary dimensions for a VSSS ansatz. This procedure is made significantly more manageable via a simple Maple script that we have used to determine the dimensional dependence. Our results have been cross-checked up to (in some cases) 19 dimensions. All subsequent calculations were then performed working directly with this completely general action.

We now split our discussion into two parts. First, we will consider the case of dimensions greater than four and then we will consider the four dimensional case. This is because, as was observed in the cubic theory [8], the four dimensional case is somewhat special, while all other dimensions can be treated on equal footing.

2.2.2 The case for dimensions larger than four

Enforcing the condition (2.8) on action (2.13) revealed that, in five and higher dimensions, there are nine constraints that determine the class of theories with this property i.e., those which possess VSSS. Somewhat arbitrarily, we solve the constraints for c_{12}, c_{17}, c_{19}, c_{20}, c_{21}, c_{22}, c_{23}, c_{24} and c_{25}. The expressions for these constraints are lengthy and the results are given in Appendix A. After these, we are left with seventeen free parameters that we have to adjust to get interesting quartic curvature terms. Learning from the existing literature, to classify such theories, one must divide these theories into two categories:

1. Theories for which the field equation is in the form of total derivative of the polynomial of the metric function.

2. Theories which contain more than one derivative of the metric function.

For each of the above categories, we will present field equations in § 2.4. For now, we will present suitable forms which fulfill the conditions of either of above categories, for the 17 quartic theories.

2.2.2.1 Lovelock and quasi-topological theories

In this section, we determine the number of constraints required, in addition to the nine constraints mentioned above, to allow all terms in the action to vanish (for those which lead to more than one derivative in the field equations for the VSSS metric function f, see § 2.4). The resulting field equation is then a total derivative of a polynomial in f. We conclude that two additional constraints are the *minimum number required* in order to get rid of these higher derivative terms in the action. Furthermore, we find that this conclusion does not depend on the particular choice of the constraints and therefore choosing arbitrarily c_9 and c_{15}, we get

$$\begin{aligned} c_9 = & -\frac{2\left(-8+2d^4-23d+39d^2-16d^3\right)}{(3d-4)(d-4)(d^2-6d+11)}c_1 - \frac{2\left(122+130d^2-207d-37d^3+4d^4\right)}{(3d-4)(d-4)(d^2-6d+11)}c_2 \\ & - \frac{(5d-7)(d-4)}{(3d-4)(d^2-6d+11)}c_3 - \frac{8\left(82+82d^2-21d^3+2d^4-139d\right)}{(3d-4)(d-4)(d^2-6d+11)}c_4 \\ & - \frac{16\left(82+82d^2-21d^3+2d^4-139d\right)}{(3d-4)(d-4)(d^2-6d+11)}c_5 - \frac{4(d-2)(d-3)\left(4d^2-17d+16\right)}{(3d-4)(d-4)(d^2-6d+11)}c_6 \\ & - \frac{32(d-1)(d-3)(d-2)^2}{(3d-4)(d-4)(d^2-6d+11)}c_7 - \frac{(d-4)}{2(d^2-6d+11)}c_8 - \frac{(d-3)d}{2(d^2-6d+11)}c_{10}\,, \end{aligned} \tag{2.14}$$

$$\begin{aligned} c_{15} = & \frac{8d^7-111d^6+570d^5-1190d^4+210d^3+2725d^2-3308d+1024}{(3d-4)(d-4)(d^2-6d+11)(d^3-7d^2+14d-4)}c_1 \\ & + \frac{2(d-1)\left(8d^6-116d^5+689d^4-2141d^3+3661d^2-3197d+988\right)}{(3d-4)(d-4)(d^2-6d+11)(d^3-7d^2+14d-4)}c_2 \\ & + \frac{13d^5-167d^4+781d^3-1615d^2+1396d-384}{(3d-4)(d^2-6d+11)(d^3-7d^2+14d-4)}c_3 \\ & + \frac{8\left(4d^7-70d^6+513d^5-2022d^4+4566d^3-5760d^2+3557d-716\right)}{(3d-4)(d-4)(d^2-6d+11)(d^3-7d^2+14d-4)}(c_4+2c_5) \\ & + \frac{4(d-1)\left(8d^6-116d^5+673d^4-1966d^3+2983d^2-2148d+512\right)}{(3d-4)(d-4)(d^2-6d+11)(d^3-7d^2+14d-4)}c_6 \\ & + \frac{32(d-1)\left(2d^4-15d^3+32d^2-9d-4\right)(d-2)(d-3)}{(3d-4)(d-4)(d^2-6d+11)(d^3-7d^2+14d-4)}c_7 \\ & + \frac{(d-3)^2\left(d^2-6d+2\right)}{(d^2-6d+11)(d^3-7d^2+14d-4)}c_8 + \frac{(d-4)\left(3d^3-21d^2+37d-11\right)}{(d^2-6d+11)(d^3-7d^2+14d-4)}c_{10} \\ & - \frac{d^3-8d^2+19d-8}{2(d^3-7d^2+14d-4)}c_{13} - \frac{d(d-3)}{d^3-7d^2+14d-4}c_{14} + \frac{(d-1)(d-4)}{d^3-7d^2+14d-4}c_{16} \end{aligned} \tag{2.15}$$

Two additional constraints can be imposed to eliminate more than two derivatives of $N(r)$ in the action. These will make variational principle more manageable. These terms vanish anyway because the theory is constructed in such a way that $N = 1$ solves one of the field equations. The remarkable property of these theories that we can kill higher derivatives of $N(r)$ and also that the field equations are algebraic, is consistent with Conjecture 2 of Ref. [66]. To ensure this we choose c_{18} and c_{26}, given by

$$
\begin{aligned}
c_{18} = {} & \frac{1}{(3d-4)\,(d-4)\,(d^2-6d+11)\,(d^3-7d^2+14d-4)\,(d^3-9d^2+26d-22)} \\
& \times \big[-2(2d^{10}-112640d^2+6558d^6+71315d^3-2329d^7-16827d^4+447d^8 \\
& -46d^9-6654d^5+87822d-28032)c_1-4(2d^{10}+156501d^2+16490d^6 \\
& -158736d^3-3749d^7+107067d^4+562d^8-50d^9-50145d^5-92828d+25270)c_2 \\
& -2(d-4)(4d^8-92d^7+900d^6-4901d^5+16264d^4-33711d^3+42690d^2 \\
& -30290d+9280)c_3-16(d^{10}+81391d^2+10461d^6-93331d^3-2292d^7 \\
& +67198d^4+325d^8-27d^9-32176d^5-39268d+7550)c_4-32(d^{10}+81391d^2 \\
& +10461d^6-93331d^3-2292d^7+67198d^4+325d^8-27d^9-32176d^5 \\
& -39268d+7550)c_5-8(2d^{10}+88410d^2+15059d^6-106219d^3-3597d^7 \\
& +81940d^4+555d^8-50d^9-42522d^5-42618d+9088)c_6 \\
& -32\,(d-2)\,(d-3)\,(d^8-18d^7+137d^6-573d^5+1436d^4-2191d^3+1884d^2 \\
& -584d-164)c_7+\tfrac{1}{2}(3d-4)(d-4)\,(d-3)\,(5d^5-60d^4+281d^3-626d^2 \\
& +632d-184)c_8\big]-\frac{(d-5)\left(d^3-6d^2+10d-6\right)}{(d^2-6d+11)\,(d^3-7d^2+14d-4)}c_{10} \\
& +\frac{3\,(d-1)\,(d-3)}{d^3-9d^2+26d-22}c_{11}-\frac{(d-3)^2}{d^3-7d^2+14d-4}c_{13}+2\frac{(d-3)}{d^3-7d^2+14d-4}c_{14} \\
& -\frac{2\,(d-3)^2}{d^3-7d^2+14d-4}c_{16}
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
c_{26} = & \frac{1}{(3d-4)(d-2)^3(d^2-6d+11)(d^3-9d^2+26d-22)(d^2-7d+14)(d-4)} \\
& \times \Big[\frac{c_1}{24}(147d^{10} + 340880d^2 + 2457d^6 - 20d^{11} - 475564d^3 + 6300d^7 + 317426d^4 \\
& - 927d^8 - 366d^9 + d^{12} - 97214d^5 - 87616d - 2048) + \frac{c_2}{12}(200d^{10} - 467516d^2 \\
& - 29192d^6 - 22d^{11} + 359578d^3 + 3084d^7 - 208682d^4 + 1757d^8 - 919d^9 + d^{12} \\
& + 94783d^5 + 393168d - 155456) + \frac{c_3}{12}(d-4)(d^{10} - 20d^9 + 150d^8 - 440d^7 \\
& - 401d^6 + 6292d^5 - 16150d^4 + 12280d^3 + 13312d^2 - 26080d + 10240) \\
& + \frac{c_4}{6}(d-2)(d^{11} - 22d^{10} + 194d^9 - 806d^8 + 995d^7 + 4130d^6 - 13426d^5 - 20342d^4 \\
& + 181192d^3 - 412060d^2 + 442000d - 194240) + \frac{c_5}{3}(d-2)(d^{11} - 22d^{10} + 194d^9 \\
& - 806d^8 + 995d^7 + 4130d^6 - 13426d^5 - 20342d^4 + 181192d^3 - 412060d^2 \\
& + 442000d - 194240) + \frac{c_6}{6}(195d^{10} + 499536d^2 - 13335d^6 - 22d^{11} - 403948d^3 \\
& + 3582d^7 + 163666d^4 + 1153d^8 - 826d^9 + d^{12} - 14578d^5 - 316736d + 79872) \\
& + \frac{c_7}{3}(d-2)(d^{11} - 18d^{10} + 122d^9 - 324d^8 - 169d^7 + 2302d^6 + 810d^5 - 28868d^4 \\
& + 88832d^3 - 152480d^2 + 168128d - 87552) \Big] \\
& - \frac{\left(3d^4 - 28d^3 + 105d^2 - 176d + 120\right)(d-4)^2}{4(d^2-7d+14)(d^3-9d^2+26d-22)(d^2-6d+11)(d-2)^3} c_8 \\
& - \frac{\left(d^4 - 6d^3 + 8d^2 + 18d - 24\right)}{2(d^2-7d+14)(d^2-6d+11)(d-2)^3} c_{10} \\
& + \frac{\left(d^5 - 10d^4 + 29d^3 + 16d^2 - 172d + 152\right)}{2(d^2-7d+14)(d^3-9d^2+26d-22)(d-2)^3} c_{11} + \frac{(d-4)}{2(d^2-7d+14)(d-2)^3} c_{13} \\
& - \frac{\left(d^2 - 6d + 12\right)}{2(d^2-7d+14)(d-2)^3} c_{14} + \frac{(d-4)}{(d^2-7d+14)(d-2)^3} c_{16} \\
& - \frac{\left(d^3 - 8d^2 + 20d - 8\right)}{2(d^2-7d+14)(d-2)^3} c_{18}
\end{aligned} \tag{2.17}
$$

Resulting theories will have a field equation with a form similar to that of Lovelock gravity, as presented in Ref. [61] – the quartic quasi-topological gravity. The field equation, which will be presented in § 2.4, and it will be shown there that it is a total derivative of a polynomial in $f(r)$.

We are left with thirteen free parameters after the two additional constraints (2.16) and (2.17) are imposed, given the constraints in Appendix A and equations (2.14) and (2.15). We remark that only seven of these terms make non-trivial contributions to the field equations; these are characterized by the constants c_1, c_2, c_3, c_4, c_5, c_6 and c_7. Of these seven non-trivial theories, one will be quartic Lovelock gravity, i.e., the choice of these constraints that yield eight dimensional Euler density is

$$\begin{aligned}\mathcal{X}_8: \quad & c_1 = 96, \quad c_2 = -48, \quad c_3 = 96, \quad c_4 = -48, \quad c_5 = -6, \quad c_6 = 48, \quad c_7 = -3, \\ & c_8 = -384, \quad c_{10} = -192, \quad c_{11} = 32, \quad c_{13} = -192, \quad c_{14} = 192, \quad c_{16} = -192. \end{aligned} \tag{2.18}$$

Another selection which possesses non-trivial field equation is

$$\begin{aligned}\mathcal{Z}_d^{(1)}: \quad & c_1 = 0, \quad c_2 = 8(d-2)(860 - 2113d + 1959d^2 - 810d^3 + 102d^4 + 30d^5 - 11d^6 + d^7) \\ & c_3 = 0, \quad c_4 = 0, \quad c_6 = 0, \\ & c_5 = -(d-2)(1108 - 2723d + 2639d^2 - 1224d^3 + 235d^4 + 10d^5 - 10d^6 + d^7), \\ & c_7 = -1292 + 2929d - 2741d^2 + 1527d^3 - 684d^4 + 276d^5 - 82d^6 + 14d^7 - d^8, \\ & c_8 = 0, \quad c_{10} = 0, \quad c_{11} = 0, \quad c_{13} = 0, \\ & c_{14} = 16(d-2)^3(274 - 389d + 183d^2 - 34d^3 + 2d^4), \quad c_{16} = 0\,. \end{aligned} \tag{2.19}$$

This choice gives quartic quasi-topological gravity [61].

All of the above presented theories are available in literature and we made choices of c's to recover their forms. Now we will make choices of c's to get new theories of desired properties which are mentioned above for quasi-topological and generalized quasi-topological theories. Therefore, we find five new quartic quasi-topological theories which are not available in literature previously. We make the following simple choice of c's for these new terms given by:

$$
\begin{aligned}
\mathcal{Z}_d^{(2)} &: \quad c_1 = 1, \quad \text{other } c_i = 0 \text{ except those constrained in Appendix A and equations (2.14)-(2.17);} \\
\mathcal{Z}_d^{(3)} &: \quad c_2 = 1\,, \quad \text{other } c_i = 0 \text{ except those constrained in Appendix A and equations (2.14)-(2.17);} \\
\mathcal{Z}_d^{(4)} &: \quad c_3 = 1\,, \quad \text{other } c_i = 0 \text{ except those constrained in Appendix A and equations (2.14)-(2.17);} \\
\mathcal{Z}_d^{(5)} &: \quad c_4 = 1\,, \quad \text{other } c_i = 0 \text{ except those constrained in Appendix A and equations (2.14)-(2.17);} \\
\mathcal{Z}_d^{(6)} &: \quad c_5 = 1\,, \quad \text{other } c_i = 0 \text{ except those constrained in Appendix A and equations (2.14)-(2.17).}
\end{aligned} \tag{2.20}
$$

Lagrangian densities for these theories have very complicated dimension, dependent on the spacetime dimension. Their explicit forms for any dimension $d > 4$ are included in Appendix B. Each of the quasi-topological theories contribute to the field equations in dimensions $d \geq 5$, but are 'quasi-topological' in $d = 8$, i.e., they do not contribute to the equations of motion for eight dimensional spherically symmetric metrics.

As mentioned above, we have found seven quasi-topological Lagrangians for the quartic order in curvature. One of these is the quartic Lovelock term, one is previously known term [61] and five are new. All of these six theories fall into the class of theories known as quasi-topological gravity. It is not possible to "move between" these quasi-topological terms by adding a term proportional to the eight-dimensional Euler density: there is no linear combination of the $\mathcal{Z}_d^{(i)}$ terms we have defined which can yield $\mathcal{X}_8$. This is in notable contrast to the cubic case where, in five dimensions, there are two contributions to the field equations, namely, in the notation of Ref. [56], $\mathcal{Z}_5$ and $\mathcal{Z}'_5$. However these densities obey the relationship [56]

$$
\mathcal{X}_6 = 4\mathcal{Z}'_5 - 8\mathcal{Z}_5, \tag{2.21}
$$

and since the six dimensional Euler density identically vanishes in five dimensions for any metric, it follows that there are not really two independent theories. *Cubic quasi-topological gravity is unique.* The fact that in the quartic case

$$\mathcal{X}_8 \neq \sum_{i=1}^{6} c_i \mathcal{Z}_d^{(i)}, \tag{2.22}$$

for any choice of the coefficients c_i means that each of these theories are distinct for general metrics. However, as mentioned at the beginning of this section, there is a sense in which these theories are degenerate. Under the constraint of spherical symmetry, there exist invariants that vanish for any spherically symmetric metric [78]. In fact, the combination

$$I^{(ij)} = \frac{\hat{\mu}_{(i)}}{\mu_{(i)}} \mathcal{Z}^{(i)} - \frac{\hat{\mu}_{(j)}}{\mu_{(j)}} \mathcal{Z}^{(j)}, \tag{2.23}$$

will always be such a term [the quantities with the hats are defined below in equation (2.31)]. Thus, in spherical symmetry, there is a "unique" quasi-topological theory in the sense that each of the $\mathcal{Z}_d^{(i)}$ terms makes the same contribution to the field equations and are related to one another by the addition of a term that vanishes on *spherically symmetric* metrics. We emphasize, however, that these theories are ultimately distinct because they will each yield different dynamics when spherical symmetry is not imposed.

The quartic quasi-topological term (2.19) was also claimed to be unique; however, this does not appear to be the case, at least in the sense originally described [61]. That theory is unique only in the sense described above: terms vanishing under the constraint of spherical symmetry can be added to the action without altering the field equations. However, apart from spherical symmetry, these are distinct theories, even in dimensions less than eight.

2.2.2.2 Generalized quasi-topological terms

We now move on to consider generalized quasi-topological terms. Of the 13 free parameters remaining under the restrictions imposed by the constraints in Appendix A and equations (2.14)-(2.17), the Lovelock and six quasi-topological terms comprise of the only seven non-trivial theories. The remaining six terms do not contribute to the field equations of a VSSS ansatz. Here, we do not explicitly present the Lagrangians for these terms, but rather simply indicate choices of constants by which they are produced. We make the following choices:

$$\begin{aligned}
&\mathcal{C}_d^{(1)}: \quad c_8 = 1\,, \quad \text{other } c_i = 0 \text{ except those constrained in Appendix A and equations (2.14)-(2.17);}\\
&\mathcal{C}_d^{(2)}: \quad c_{10} = 1\,, \quad \text{other } c_i = 0 \text{ except those constrained in Appendix A and equations (2.14)-(2.17);}\\
&\mathcal{C}_d^{(3)}: \quad c_{11} = 1\,, \quad \text{other } c_i = 0 \text{ except those constrained in Appendix A and equations (2.14)-(2.17);}\\
&\mathcal{C}_d^{(4)}: \quad c_{13} = 1\,, \quad \text{other } c_i = 0 \text{ except those constrained in Appendix A and equations (2.14)-(2.17);}\\
&\mathcal{C}_d^{(5)}: \quad c_{14} = 1\,, \quad \text{other } c_i = 0 \text{ except those constrained in Appendix A and equations (2.14)-(2.17);}\\
&\mathcal{C}_d^{(6)}: \quad c_{16} = 1\,, \quad \text{other } c_i = 0 \text{ except those constrained in Appendix A and equations (2.14)-(2.17).}
\end{aligned} \tag{2.24}$$

In (2.24) and in § 2.2.2.1, we presented choices of constants that yield 13 of the 17 theories. We find that seven of these theories are non-trivial on a spherically symmetric metric and belong to the Lovelock or quasi-topological classes.

These remaining six terms yield vanishing contributions to the field equations when $N = 1$ is permitted (e.g., in vacuum or for electromagnetic matter), but would make non-vanishing contributions in the presence of more general matter distributions.

We shall now relax the additional constraints imposed in equations (2.14)-(2.17) in order to obtain a full family of theories satisfying the constraint (2.5). These four distinct new theories

– the quartet – have a field equation that is a total derivative of a quantity that is a polynomial in both $f(r)$ and its first two derivatives and it will be elaborated in subsequent section.

We make the following selections:

$$
\begin{aligned}
\mathcal{S}_d^{(1)}: \quad & c_1 = 1\,, \\
& c_9 = -\frac{2d^6 - 23d^5 + 106d^4 - 292d^3 + 588d^2 - 709d + 320}{d\,(d-3)\,(3d^2 - 18d + 19)\,(d^2 - 6d + 11)}\,, \\
& c_{15} = \frac{1}{(d-3)^2\,(d^3 - 9d^2 + 26d - 22)\,d\,(3d^2 - 18d + 19)\,(d^2 - 6d + 11)} \times \\
& \quad \times \big[d^{10} - 20d^9 + 188d^8 - 1211d^7 + 6287d^6 - 25778d^5 + 75674d^4 \\
& \quad - 146251d^3 + 172418d^2 - 110076d + 28160\big]\,, \\
& \text{all other } c_i = 0 \text{ except those constrained in Appendix A}\,;
\end{aligned} \tag{2.25}
$$

$$
\begin{aligned}
\mathcal{S}_d^{(2)}: \quad & c_2 = 1\,, \\
& c_9 = -\frac{2\left(2d^6 - 24d^5 + 103d^4 - 161d^3 - 67d^2 + 409d - 274\right)}{d\,(d-3)\,(3d^2 - 18d + 19)\,(d^2 - 6d + 11)}\,, \\
& c_{15} = \frac{2}{(d-3)^2\,(d^3 - 9d^2 + 26d - 22)\,d\,(3d^2 - 18d + 19)\,(d^2 - 6d + 11)} \times \\
& \quad \times \big[d^{10} - 16d^9 + 55d^8 + 601d^7 - 7258d^6 + 35933d^5 - 102275d^4 \\
& \quad + 177665d^3 - 184591d^2 + 104237d - 24112\big]\,, \\
& \text{all other } c_i = 0 \text{ except those constrained in Appendix A}\,;
\end{aligned} \tag{2.26}
$$

$$\mathcal{S}_d^{(3)}: \quad c_4 = 1\,,$$
$$c_9 = -\frac{4(2d^6 - 25d^5 + 112d^4 - 185d^3 - 70d^2 + 494d - 340)}{d\,(d-3)\,(3d^2 - 18d + 19)\,(d^2 - 6d + 11)}\,,$$
$$c_{15} = \frac{4}{(d-3)^2\,(d^3 - 9d^2 + 26d - 22)\,d\,(3d^2 - 18d + 19)\,(d^2 - 6d + 11)} \times$$
$$\times \left[d^{10} - 18d^9 + 99d^8 + 193d^7 - 5212d^6 + 30115d^5 - 93864d^4\right.$$
$$\left. + 175930d^3 - 196892d^2 + 120000d - 29920\right]\,,$$
$$\text{all other } c_i = 0 \text{ except those constrained in Appendix A}\,; \qquad (2.27)$$

$$\mathcal{S}_d^{(4)}: \quad c_5 = 1\,,$$
$$c_9 = -\frac{8(2d^6 - 25d^5 + 112d^4 - 185d^3 - 70d^2 + 494d - 340)}{d\,(d-3)\,(3d^2 - 18d + 19)\,(d^2 - 6d + 11)}\,,$$
$$c_{15} = \frac{8}{(d-3)^2\,(d^3 - 9d^2 + 26d - 22)\,d\,(3d^2 - 18d + 19)\,(d^2 - 6d + 11)} \times$$
$$\times \left[d^{10} - 18d^9 + 99d^8 + 193d^7 - 5212d^6 + 30115d^5 - 93864d^4\right.$$
$$\left. + 175930d^3 - 196892d^2 + 120000d - 29920\right],$$
$$\text{all other } c_i = 0 \text{ except those constrained in Appendix A}\,. \qquad (2.28)$$

We have chosen these constants to render the field equations in general dimensions as simple as possible. Our choices have been further motivated by the four dimensional case, which will be presented in § 2.2.3. The explicit Lagrangian densities that result for these terms are presented in Appendix C.

Although, we have made many different attempts, it does not seem possible to select additional constraints such that the reduced Lagrangian of these generalized quasi-topological theories takes the form,

$$L_{N,f} = NF_0 + N'F_1 + N''F_2 \qquad (2.29)$$

where F_i are functions of f and its derivatives with respect to r. In other words, it does not seem possible to eliminate terms that are higher order in the derivatives of N (e.g., N'^2/N, etc.) without also eliminating the theory. This adds further support to Conjecture 2 made in Ref. [66] since we also find that the field equations for these theories are not algebraic.

We have now listed choices of constants for all 17 theories which satisfy condition (2.5) at the quartic level. We are now able to write down the explicit action for the full theory in five and higher dimensions. This takes the form

$$\mathcal{I} = \frac{1}{16\pi G}\int d^dx\sqrt{-g}\Big[-2\Lambda + R + \alpha_2\mathcal{X}_4 + \alpha_3\mathcal{X}_6 + \mu\mathcal{Z}_d - \lambda\mathcal{S}_d + \alpha_4\mathcal{X}_8 + \sum_{i=1}^{6}\hat{\mu}_{(i)}\mathcal{Z}_d^{(i)} - \sum_{i=1}^{4}\hat{\lambda}_{(i)}\mathcal{S}_d^{(i)} + \sum_{i=1}^{6}\gamma_{(i)}\mathcal{C}_d^{(i)}\Big]. \tag{2.30}$$

For any situation in which the stress-energy tensor satisfies $T_t^t = T_r^r$ (including the vacuum), the $\mathcal{C}_d^{(i)}$ terms will make no contribution to the field equations: their contributions to the generalized Einstein tensor all contain derivatives of N. For this reason, we shall not include these terms in our action in any of the discussion to follow. In the above, we have made the following rescalings of the coupling constants to simplify the resulting field equations:

$$
\begin{aligned}
\hat{\mu}_{(1)} &= \frac{1}{(d-1)(d-2)(d-3)^2(d-4)(d-8)P}\mu_{(1)}\,,\\
\hat{\mu}_{(2)} &= \frac{24(3d-4)}{(d-3)(d-8)(28d^3-173d+160+d^5+18d^2-10d^4)}\mu_{(2)}\,,\\
\hat{\mu}_{(3)} &= \frac{12(3d-4)}{(d-3)(d-8)(-12d^4+61d^3+242d-167d^2+d^5-137)}\mu_{(3)}\,,\\
\hat{\mu}_{(4)} &= \frac{12(3d-4)}{(d-3)(d-8)(d-4)(d^3-10d^2+31d-26)}\mu_{(4)}\,,\\
\hat{\mu}_{(5)} &= \frac{6(3d-4)}{(d-3)(d-8)(d^5+79d^3-14d^4+316d-170-224d^2)}\mu_{(5)}\,,\\
\hat{\mu}_{(6)} &= \frac{3(3d-4)}{(d-3)(d-8)(d^5+79d^3-14d^4+316d-170-224d^2)}\mu_{(6)}\,,\\
\hat{\lambda}_{(1)} &= \frac{d\left(3d^3-27d^2+73d-57\right)}{(2d^5-20d^4+56d^3+36d^2-346d+320)}\lambda_{(1)}\,,\\
\hat{\lambda}_{(2)} &= \frac{\left(3d^3-27d^2+73\,d-57\right)d}{4\,(d^5-12d^4+61d^3-167d^2+242\,d-137)}\lambda_{(2)}\,,\\
\hat{\lambda}_{(3)} &= \frac{\left(3d^3-27d^2+73\,d-57\right)d}{8\,(d^5-14d^4+79d^3-224d^2+316\,d-170)}\lambda_{(3)}\,,\\
\hat{\lambda}_{(4)} &= \frac{\left(3d^3-27d^2+73\,d-57\right)d}{16\,(d^5-14d^4+79d^3-224d^2+316\,d-170)}\lambda_{(4)}\,.
\end{aligned}
\tag{2.31}
$$

In the first term above we have defined

$$
P = \left(d^5 - 20\,d^4 + 142\,d^3 - 472d^2 + 743\,d - 436\right). \tag{2.32}
$$

This concludes our discussion of the theories in dimensions larger than four. We now turn to a discussion of the four dimensional case.

2.2.3 The case for four dimensions

As was observed in the cubic version of generalized quasi-topological gravity, the four dimensional case is somewhat special, with only seven constraints as opposed to nine. We find that the most general four dimensional theory satisfying (2.5) is given by placing the following seven constraints on the quartic terms in the action:

$$
\begin{aligned}
c_{12} &= -\frac{19}{60}c_1 - \frac{1}{2}c_2 - \frac{1}{12}c_3 - \frac{4}{5}c_4 - \frac{8}{5}c_5 - \frac{14}{15}c_6 - \frac{56}{15}c_7 - \frac{1}{8}c_8 - \frac{1}{4}c_9 - \frac{1}{2}c_{11}, \\
c_{17} &= -\frac{23}{30}c_1 - \frac{4}{3}c_2 - \frac{1}{12}c_3 - \frac{11}{5}c_4 - \frac{22}{5}c_5 - \frac{41}{15}c_6 - \frac{28}{5}c_7 - \frac{1}{24}c_8 - c_9 - \frac{11}{12}c_{10} \\
&\quad - \frac{1}{6}c_{13} - \frac{1}{3}c_{14} - \frac{1}{12}c_{15} - \frac{1}{4}c_{16}\,, \\
c_{19} &= \frac{11}{30}c_1 + \frac{7}{12}c_2 + \frac{1}{12}c_3 + \frac{9}{10}c_4 + \frac{9}{5}c_5 + \frac{17}{15}c_6 + \frac{16}{5}c_7 + \frac{5}{48}c_8 + \frac{1}{4}c_9 + \frac{1}{6}c_{10} \\
&\quad + \frac{3}{8}c_{11} - \frac{1}{48}c_{13} + \frac{1}{12}c_{14} - \frac{1}{24}c_{15} - \frac{1}{4}c_{18}\,, \\
c_{20} &= \frac{36}{5}c_1 + \frac{32}{3}c_2 + 2c_3 + \frac{72}{5}c4 + \frac{144}{5}c_5 + \frac{104}{5}c_6 + \frac{1088}{15}c_7 + \frac{7}{3}c_8 + \frac{4}{3}c_{10} + 8c_{11} \\
&\quad - \frac{8}{3}c_{13} + \frac{2}{3}c_{14} - \frac{10}{3}c_{15} - 2c_{16} - 4c_{18} + 2c_{24} + 8c_{25} + 32c_{26}\,, \\
c_{21} &= -\frac{5}{3}c_1 - \frac{8}{3}c_2 - \frac{1}{3}c_3 - 4c_4 - 8c_5 - \frac{16}{3}c_6 - \frac{32}{3}c_7 - \frac{1}{3}c_8 - c_9 - \frac{4}{3}c_{10} - c_{11} \\
&\quad + \frac{1}{6}c_{13} - \frac{2}{3}c_{14} + \frac{1}{3}c_{15} - c_{24} - 4c_{25} - 16c_{26}\,, \\
c_{22} &= -\frac{7}{3}c_1 - \frac{10}{3}c_2 - \frac{2}{3}c_3 - 4c_4 - 8c_5 - \frac{20}{3}c_6 - \frac{64}{3}c_7 - \frac{2}{3}c_8 - \frac{2}{3}c_{10} - 2c_{11} \\
&\quad + \frac{1}{3}c_{13} - \frac{1}{3}c_{14} + \frac{2}{3}c_{15} - 2c_{24} + 16c_{26}\,, \\
c_{23} &= \frac{1}{15}c_1 + \frac{1}{3}c_2 - \frac{1}{6}c_3 + \frac{4}{5}c_4 + \frac{8}{5}c_5 + \frac{14}{15}c_6 - \frac{28}{5}c_7 - \frac{1}{3}c_8 + c_9 + \frac{7}{6}c_{10} - \frac{3}{2}c_{11} \\
&\quad + \frac{5}{12}c_{13} + \frac{1}{3}c_{14} + \frac{1}{3}c_{15} + \frac{1}{2}c_{16} + c_{18} - 2c_{25} - 12c_{26}\,.
\end{aligned}
\tag{2.33}
$$

Thus, one is left with a 19 parameter family of quartic densities whose solutions are of the form equation (2.1) with $N =$ constant. We shall now discuss a useful choices for these theories.

In general, only the six terms corresponding to c_1, c_2, c_4, c_5, c_6 and c_7 make nonzero contribution to field equations in the context of VSSS metrics. Furthermore, each of these six terms make the same contributions to the field equations, up to overall constants. These six terms provide the quartic generalizations of the cubic $\mathcal{S}_4$ term in four dimensions. The remaining 13 terms do not contribute to the field equations of a VSSS ansatz, or in any case where the stress-energy tensor satisfies $T_t^t = T_r^r$. Our focus here will be to present the six non-vanishing contributions.

In the previous subsection, we presented four Lagrangian densities, $\mathcal{S}_d^{(i)}$ with $i = 1, 2, 3, 4$. These terms account for four of the six contributions in four dimensions, upon setting $d = 4$ in the expressions presented in Appendix C. The two additional non-trivial contributions can be obtained by the following selection of free parameters.

$$\begin{aligned}
\mathcal{S}_4^{(5)}: \quad & c_6 = 1\,, \quad c_9 = -\frac{56}{15}, \\
& \text{all other } c_i = 0 \text{ except those constrained in equation (2.33)}\,; \\
\mathcal{S}_4^{(6)}: \quad & c_7 = 1\,, \quad c_9 = -\frac{224}{15}\,, \\
& \text{all other } c_i = 0 \text{ except those constrained in equation (2.33)}\,.
\end{aligned} \tag{2.34}$$

We have presented explicit forms for these expressions in Appendix C. In addition to the six non-trivial terms, there are 13 terms that are the four dimensional analogs of the $\mathcal{C}_d^{(i)}$ terms. We do not present full expressions for these terms here since they have no effect on the field equations in the situations we are interested in. A simple choice for these terms is obtained simply by taking $c_i = 1$ and all other $c_j = 0$ (except those which are constrained) for each of the constants that have not been fixed by the above considerations.

We note again that the imposition of spherical symmetry yields a degeneracy amongst these theories: they differ by terms that vanish for a spherically symmetric metric. However, this degeneracy is lifted if spherical symmetry is relaxed and so the theories are ultimately distinct.

The action for the non-trivial contributions to the field equations in four dimensions reads

$$\mathcal{I} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[-2\Lambda + R - \lambda\mathcal{S}_4 - \sum_{i=1}^{6}\hat{\lambda}_{(i)}\mathcal{S}_4^{(i)}\right], \tag{2.35}$$

where

$$\hat{\lambda}_{(5)} = -\frac{5}{24}\lambda_{(5)}\,, \quad \hat{\lambda}_{(6)} = -\frac{5}{96}\lambda_{(6)}, \tag{2.36}$$

with all other $\hat{\lambda}_{(i)}$ as defined in equation (2.31) with $d = 4$. These choices of normalization have been made to simplify the form of the field equations.

2.3 Linear theory and vacua

In this section, we provide a brief discussion of the linearized equations of motion for the theories presented in § 2.2. As conjectured in Ref. [8] and then demonstrated in Ref. [66], a theory satisfying equation (2.5) must necessarily have linearized equations of motion that agree with the linearized Einstein equations on a constant curvature background, up to an overall constant. The only caveat is that, in this theory, the metric (2.1) describes the gravitational field outside of a spherically symmetric mass distribution. Thus, this section provides a useful check of the correctness of the theories, and the results may be useful in future studies of these theories.

In what follows, we will closely follow Ref. [63], adopting the conventions therein.

We consider a perturbation h_{ab} away from a constant curvature spacetime $\bar{g}_{ab}$ such that,

$$g_{ab} = \bar{g}_{ab} + h_{ab}\,. \tag{2.37}$$

The curvature of the constant curvature background is given by,

$$\bar{R}_{abcd} = 2K\bar{g}_{a[c}\bar{g}_{d]b}\,, \tag{2.38}$$

for some constant K. Adopting the procedure explained in [63], the linearized equations of motion for h_{ab} are then given by,

$$\frac{1}{2}\mathcal{E}^L_{ab} = \big[e - 2K\left(a(d-1)+c\right) + (2a+c)\bar{\Box}\big]G^L_{ab} + [a+2b+c]\big[\bar{g}_{ab}\bar{\Box} - \bar{\nabla}_a\bar{\nabla}_b\big]R^L$$
$$- K\big[a(d-3) - 2b(d-1) - c\big]\bar{g}_{ab}R^L = \frac{1}{4}T^L_{ab}, \tag{2.39}$$

where a, b, c and e are convenient choices of parameters based on the linearization procedure which can be computed from the following two relationships

$$(\frac{\partial\mathcal{L}}{\partial\alpha})|_{\alpha=0} = 2e\chi(\chi-1), \tag{2.40}$$

$$(\frac{\partial^2\mathcal{L}}{\partial\alpha^2})|_{\alpha=0} = 4\chi(\chi-1)(a + b\chi(\chi-1) + c(\chi-1)), \tag{2.41}$$

here, α is a parameter and χ is an arbitrary integer number, which will remain undetermined because values of a, b, c and e do not depend on χ, therefore we will not compute its value. In the above, all quantities with a bar correspond to those defined for the background metric, $\bar{g}_{ab}$, while

$$G^L_{ab} = R^L_{ab} - \frac{1}{2}\bar{g}_{ab}R^L - (d-1)Kh_{ab}\,,$$
$$R^L_{ab} = \frac{1}{2}\big[\bar{\nabla}_a\bar{\nabla}_c h_b{}^c + \bar{\nabla}_b\bar{\nabla}_c h_a{}^c - \bar{\Box}h_{ab} - \bar{\nabla}_a\bar{\nabla}_b h\big] + dKh_{ab} - Kh\bar{g}_{ab}$$
$$R^L = \bar{\nabla}^a\bar{\nabla}^b h_{ab} - \bar{\Box}h - (d-1)Kh, \tag{2.42}$$

where $h = \bar{g}^{ab}h_{ab}$. The additional scalar and massive graviton modes will be absent provided $2a+c=0$ and $4b+c=0$ [63]. In other words, these terms will be absent provided the linearized equations are proportional to the linearized Einstein tensor (plus cosmological term) on the same background. Let us now explicitly present the linearized equations for the theories we have constructed. Specifically, we consider the theory

$$\mathcal{I} = \frac{1}{16\pi G}\int d^dx\sqrt{-g}\Big[-2\Lambda + R - \sum_{i=1}^{4}\hat{\lambda}_{(i)}\mathcal{S}_d^{(i)} + \sum_{i=1}^{6}\hat{\mu}_{(i)}\mathcal{Z}_d^{(i)}\Big]\,, \tag{2.43}$$

which includes all of the non-trivial contributions at the quartic level, except for the Lovelock term, which has been thoroughly studied previously. The results can be easily extended to cases with additional terms appearing in the action by simply adding the contributions arising from these terms to the relevant equations below.

We will define, for convenience, the following constants:

$$\mu\sum_{i=1}^{6}\mu_{(i)}\,, \quad \lambda\sum_{i=1}^{4+2\delta_{d,4}}\lambda_{(i)}\,. \tag{2.44}$$

Then, it is a matter of calculation to show that the linearized equations are given by,

$$\mathcal{E}_{ab}^{L} = \frac{1}{2}\left[1 + 4\left(\mu + \frac{(d-8)}{3}\lambda\right)K^3\right]G_{ab}^{L}. \tag{2.45}$$

Note that in the above, it is the couplings without hats that appear; the definitions made in equation (2.31) significantly simplify the form of the linearized equations. As expected, we see the result is proportional to the Einstein tensor linearized on the same background.

In four dimensions, the additional terms $\mathcal{S}_4^{(5)}$ and $\mathcal{S}_4^{(6)}$ also contribute, while the quasi-topological terms make no contribution. The linearized field equations then become

$$\mathcal{E}_{ab}^{L} = \frac{1}{2}\left[1 - \frac{16}{3}\lambda K^3\right]G_{ab}^{L}\,, \qquad \text{in } d = 4 \tag{2.46}$$

where the sum defining λ now runs over all six couplings, $\lambda_{(i)}$.

The full field equations will relate the curvature of the background, K, to the length scale introduced by the cosmological constant, Λ. This dependence can be obtained by evaluating the field equations (see next section) on the constant curvature background. One finds that the following relationship must hold,

$$-\frac{2\Lambda}{(d-1)(d-2)}+K+\left(\mu+\frac{(d-8)}{3}\lambda\right)K^4=0\,, \tag{2.47}$$

with μ and λ defined by the sums above. Note that when the higher curvature terms are switched off, the cosmological constant uniquely determines the curvature of the constant curvature solutions of the theory. However, when the higher curvature terms are present there will generically be multiple solutions for K: four in this quartic theory. In general, only a single one of these solutions will have a smooth limit to the vacuum of Einstein's gravity upon sending $\mu, \lambda \to 0$.

In order to ensure the proper coupling to matter, the prefactor appearing in front of G^L_{ab} in the linearized equations must have the same sign as in Einstein's gravity. If this were not the case, then the graviton would be a ghost. For the theory discussed here, this requirement demands that

$$1+4\left(\mu+\frac{(d-8)}{3}\lambda\right)K^3>0\,. \tag{2.48}$$

This condition must be satisfied by any physically reasonable solution to the equations of motion.

We close this section by noting that, in a $d>4$ theory that contains both the quasi-topological and generalized quasi-topological terms, the value

$$\mu=-\frac{(d-8)}{3}\lambda \tag{2.49}$$

seems to be special. When the couplings are constrained in this way, the theory has a unique vacuum coinciding with the Einstein's gravity vacuum. Furthermore, the above inequality for the absence of ghosts is trivially satisfied.

2.4 Nonlinear field equations in spherical symmetry

Here we present the field equations that are derived from the actions presented in Section 2.2. We consider first the theory defined in dimensions larger than four, and then close with the four dimensional case.

2.4.1 The field equations in dimensions larger than four

2.4.1.1 Quasi-topological theories

We consider first the field equations for the quasi-topological gravities constructed in Section 2.2. The field equations of Lovelock gravity are well known and we do not discuss them here. We consider the following action,

$$\mathcal{I} = \frac{1}{16\pi G}\int d^d x\sqrt{-g}\bigg[\frac{(d-1)(d-2)}{\ell^2} + R + \sum_{i=1}^{6}\hat{\mu}_{(i)}\mathcal{Z}_d^{(i)}\bigg], \tag{2.50}$$

where the $\hat{\mu}_{(i)}$ terms are as in equation (2.31). A spherically symmetric metric (2.1) satisfies the field equations $F' = 0$ with $N = 1$ and

$$F = (d-2)r^{d-1}\left[\frac{1}{\ell^2} - \psi + \sum_{i=1}^{6}\mu_{(i)}\psi^4\right], \tag{2.51}$$

where we have defined

$$\psi = \frac{f-k}{r^2}\,. \tag{2.52}$$

Equation (2.51) is a total derivative and can be easily integrated revealing that f is determined by the following algebraic relationship,

$$(d-2)r^{d-1}\left[\frac{1}{\ell^2} - \psi + \sum_{i=1}^{6}\mu_{(i)}\psi^4\right] = m, \tag{2.53}$$

where m is an integration constant which is related to the mass of a black hole. Note that in passing from the action to the field equations, the hats have been removed from the μ's. It was for this simplification that the $\hat{\mu}_{(i)}$ terms were defined as in equation (2.31). These equations only hold for $d > 4$ but $d \neq 8$: in eight dimensions, the quasi-topological terms are trivial.

2.4.1.2 Generalized quasi-topological theories

We next present the field equations for the four non-trivial generalized quasi-topological terms that were presented in § 2.2. The field equations for these theories are not algebraic, but rather, in vacuum, integrate to a second order differential equation that the metric function f must satisfy.

We consider now the following action,

$$\mathcal{I} = \frac{1}{16\pi G}\int d^dx\sqrt{-g}\left[\frac{(d-1)(d-2)}{\ell^2} + R - \sum_{i=1}^{4}\hat{\lambda}_{(i)}\mathcal{S}_d^{(i)}\right], \tag{2.54}$$

where the $\hat{\lambda}_{(i)}$ terms are as defined in equation (2.31). The field equations of this theory can be written in the following simple total derivative form,

$$F' = 0, \tag{2.55}$$

where

$$F = (d-2)r^{d-3}\left[k - f + \frac{r^2}{\ell^2}\right] + (d-2)\left(\sum_{i=1}^{4}\lambda_{(i)}\right)F_{\mathcal{S}_d}, \tag{2.56}$$

and the $F_{\mathcal{S}_d}$ represents the contribution from each $\mathcal{S}_d^{(i)}$ to the field equations, which is the same for each term $\mathcal{S}_d^{(i)}$ due to the choices made in equations (2.25)- (2.28). Note that, once again, in passing from the action to the field equations, the hats have been removed from the λ's. It was for this simplification that they were normalized in equation (2.31). Explicitly, the contribution made to the field equations from each $\mathcal{S}_d^{(i)}$ is given by

$$
\begin{aligned}
F_{\mathcal{S}_d} = & (k-f)\left[(d-4)\, f\,(k-f)\, f'' + f'^2\left(\left(d^2-\frac{23}{2}d+32\right)f-\frac{1}{2}\,k\,(d-4)\right)\right] r^{d-7} \\
& + 2\, f f' f''\left((k-f)\,(d-5)\, r^{d-6}+\frac{f'}{8}\,(3d-16)\, r^{d-5}\right) \\
& + f f'\,(k-f)^2\,(d-4)\,(d-7)\, r^{d-8}+\frac{f'^3}{12}\Big[((3d-16)\, f-8k)\,(d-5)\, r^{d-6} \\
& - 3\frac{f'}{4}\,(3d-16)\, r^{d-5}\Big] ,
\end{aligned}
\tag{2.57}
$$

where $f = f(r)$ and its prime denotes a derivative with respect to r.

2.4.2 The field equations in four dimensions

In four dimensions, the only non-trivial contributions to the field equations come from the generalized quasi-topological terms. Considering the action

$$
\mathcal{I} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[-2\Lambda + R - \sum_{i=1}^{6}\hat{\lambda}_{(i)}\mathcal{S}_4^{(i)}\right], \tag{2.58}
$$

with the $\hat{\lambda}_{(i)}$ terms defined in equations (2.31) and (2.36), the field equations read

$$
F' = 0, \tag{2.59}
$$

where

$$
F = 2r\left[k - f + \frac{r^2}{\ell^2}\right] + 2\left(\sum_{i=1}^{6}\lambda_{(i)}\right) F_{\mathcal{S}_4}, \tag{2.60}
$$

and $F_{\mathcal{S}_4}$ is given by the same expression as in equation (2.57) evaluated in $d = 4$. Explicitly, this takes the relatively simple form,

$$
F_{\mathcal{S}_4} = 2\frac{f f' f''}{r^2}\left(f - \frac{1}{2} r f' - k\right) + \frac{f'^4}{4r} + \frac{f'^3}{3r^2}\,(f+2k) + 2\frac{f f'^2}{r^3}\,(k-f)\,. \tag{2.61}
$$

This completes the presentation of field equations for the new quartic theories. In the next chapter we move on to a discussion of their black hole and black brane solutions.

Chapter 3

Black Hole and Black Brane Solutions in Quartic Quasi-Topological Gravity

3.1 Black hole solutions in four dimensions

To explore features of the newly constructed quartic quasi-topological gravities in previous chapter, here we will present black hole solutions in four dimensions. We consider the simplest case i.e., the generalization of Schwarzschild solution for these theories. Therefore we aim to study the vacuum field equations. An exact solution for the field equations is very difficult to obtain and our efforts in this regard failed. Therefore we rely on perturbative solutions. We work out asymptotically flat black hole solutions in four dimensions. It is a very important characteristic of the generalized quasi-topological theories that higher curvature modifications occur in four dimensions while maintaining relatively simple field equations. Here we only work on the spherically symmetric solutions which can be easily generalized to other cases (i.e. those with $k = 0$

or -1). We consider the action

$$\mathcal{I} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\big[-2\Lambda + R + K\mathcal{S}_d^{(quartic)}\big] \ , \tag{3.1}$$

where K is the quartic coupling. The field equations naively contain up to third derivatives of the metric function, but after imposing spherical symmetry and setting $N = 1$ they reduce to a single equation $F' = 0$, where

$$\begin{aligned} F = \frac{r}{\kappa}(k-f) + \frac{24\kappa^2}{5}K\Bigg[\frac{ff'f''}{r^2}\left(f - \frac{1}{2}rf' - k\right) + \frac{f'^4}{8r} \\ + \frac{f'^3}{6r^2}\left(f + 2k\right) + \frac{ff'^2}{r^3}(k-f)\Bigg], \end{aligned} \tag{3.2}$$

with $\kappa = 8\pi G$, $f = f(r)$ and we have set $\Lambda = 0$. We have made a choice for K given by,

$$K = \frac{5}{6\kappa^3}\sum_{i=1}^{6}\lambda_{(i)}. \tag{3.3}$$

A notable thing here is that the Einstein's gravity can be recovered from here by simply setting $K = 0$. In calculations we will keep k for the general case but at the end we will put $k = 1$ to focus only on the asymptotically flat spherical black hole.

Integration of field equation gives

$$F = \frac{C}{\kappa}, \tag{3.4}$$

where C is the integration constant and the factor $1/\kappa$ gives us the valid contribution to the mass coming from the large r solution, as we will see shortly.

For the asymptotic flat solution we consider as $r \to \infty$, we also assume that the K terms have small corrections, so that use the following expansion of the metric function

$$f(r) = k - \frac{C}{r} + \epsilon h(r), \tag{3.5}$$

where ϵ accounts for the order of contribution of $h(r)$. We use this expansion into (3.2) and only retain the linear terms in $h(r)$. Therefore, we get a second order inhomogeneous differential equation after setting $\epsilon = 1$. A particular solution upto the first order of K is given by

$$h_p(r) = k - \frac{C}{r} + \frac{108\kappa^3 kKC^3}{5r^9} - \frac{97\kappa^3 KC^4}{5r^{10}} + \mathcal{O}\left(\frac{K^2C^5}{r^{17}}\right). \tag{3.6}$$

Now, we consider the homogeneous equation which has the form,

$$h_h'' - \frac{5}{r}h_h' - \omega^2 r^6 h_h = 0, \tag{3.7}$$

where

$$\omega^2 = \frac{5}{36\kappa^3 kC^2|K|}. \tag{3.8}$$

Here the assumption is that ω^2 is positive, which, in turn, implies that K is negative. The homogeneous equation can be solved exactly in terms of Bessel functions, but in this case it is the approximate solution which is more relevant to capture the behaviour. The first derivative term is small and hence negligible for large r, so the approximate solution can be written as,

$$h_h(r) \approx A \exp\left(\frac{\omega r^4}{4}\right) + B \exp\left(-\frac{\omega r^4}{4}\right). \tag{3.9}$$

For asymptotic flat spacetime we get $A = 0$, so at leading order we obtain

$$h(r) \approx h_p(r) + B \exp\left(-\frac{\omega r^4}{4}\right). \tag{3.10}$$

The solution obtained for the homogeneous equation is similar to Yukawa-type terms and it is also exponentially decaying. Therefore we can neglect it and hence we are only left with the correction that appears to the particular solution. Another justification for this is that the theory which we have developed does not propagate massive ghosts.

The ADM (Arnowitt, Deser and Misner) mass corresponding to the asymptotic solution is given by [79]

$$M = \frac{d-2}{2\kappa}\omega_{(k)d-2} \lim_{r\to\infty} r^{d-3}(k - g_{tt}) = \frac{\omega_{(k)2}C}{8\pi G}, \tag{3.11}$$

where $\omega_{(k)d-2}$ represents the volume of the space with the line element $d\Sigma_{(k)d-2}$; for a two-sphere this is just $\omega_2 = 4\pi$.

Next we will explore the behaviour of the solution near the event horizon. To do this we consider the following expansion for the metric function

$$f(r) = 4\pi T(r - r_+) + \sum_{n=2} a_n (r - r_+)^n, \tag{3.12}$$

where $T = f'(r_+)/(4\pi)$ represents the Hawking temperature and we opt to use temperature instead of $f'(r_+)$. We use this expansion in the field equation (3.2) and perform series expansion in $(r - r_+)$. Retaining only zero and first order expansion we get the two relations given by

$$\begin{aligned}
\frac{C}{\kappa} &= \frac{1}{5\kappa r_+^2}\left(5kr_+^3 + 512\pi^3 kK\kappa^3 T^3 + 768\pi^4 K\kappa^3 r_+ T^4\right), \\
0 &= \frac{1}{5\kappa r_+^3}(5kr_+^3 - 20\pi r_+^4 T + 512\pi^3 kK\kappa^3 T^3 + 256\pi^4 K\kappa^3 r_+ T^4).
\end{aligned}$$

These two equations allow us to determine physical quantities. The first equation gives the value of C which is related to mass (3.11), and from the second equation we could determine T in terms of the horizon radius, r_+. We can observe that the second equation is quartic but only one root is real and approaches to nonnegative value for $K \to 0$. Therefore this branch is interesting and appropriate one due to the fact that it possesses smooth Einsteinian limit, so we will consider it in our discussions.

We now use numerical solution of the field equation to match the two solutions i.e., asymptotic and near-horizon solutions. The initial data for numerical scheme will come from the expansion very near to the horizon. To get this data we have to use higher orders in the expansion (3.12). Although the higher order terms are more complicated, but good thing is that at each order one can solve for the new parameter a_n in terms of parameters in the previous orders that themselves are eventually related to the single free parameter a_2 at second order. We consider the value of this free parameter as [80]

$$a_2 = \frac{f''(r_+)}{2} = -\frac{1}{r_+^2}[1+\delta], \tag{3.13}$$

where δ accounts for the corrections, which appear due to higher curvature terms, to the usual Schwarzschild solution which can be recovered by setting $\delta = 0$. It is important to choose the values of δ very carefully as it should be consistent with the boundary conditions (i.e. $f(r) \to 1$ as $r \to \infty$). We use the shooting method to determine it. In (3.12), our expansion runs for twelve orders. Good results can be obtained with fewer terms, but the construction of these terms is easily automated and therefore working to a high order comes with no extra difficulty. We have found that determining δ to ten significant digits is sufficient to integrate the solution to the point where the large r expansions become accurate (see Figure 3.1).

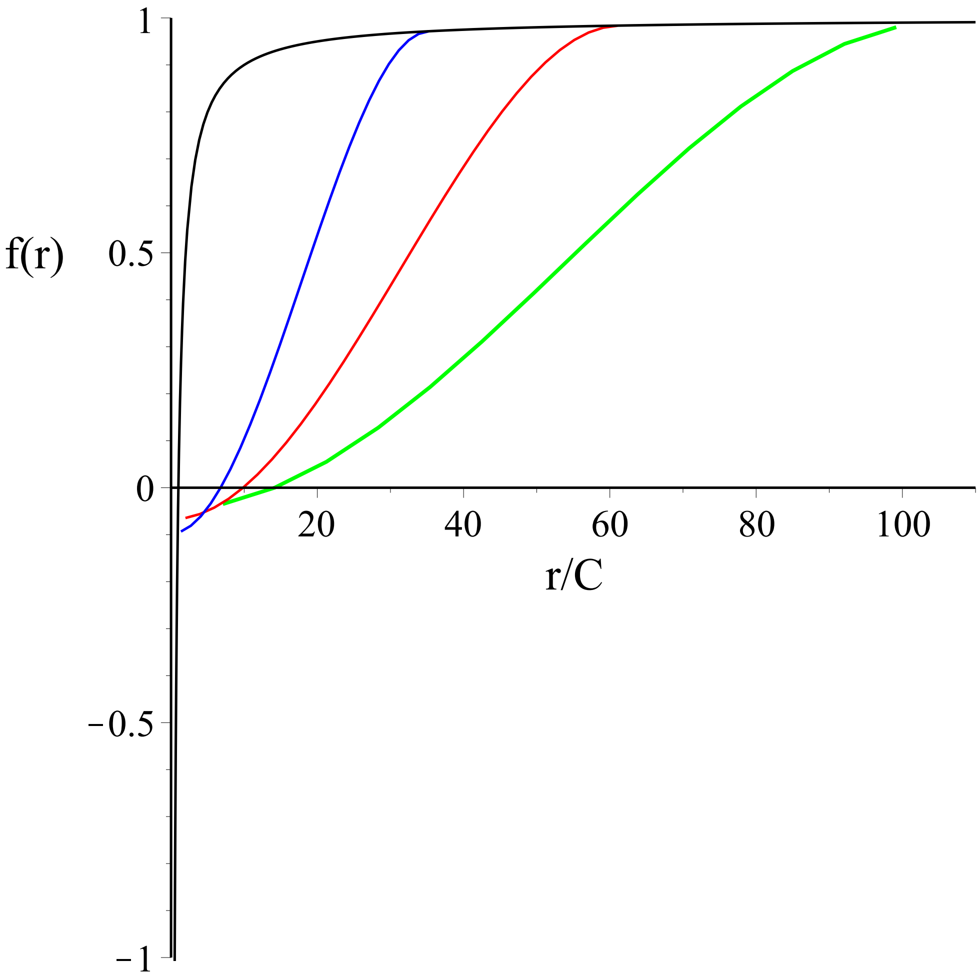

FIGURE 3.1: Asymptotically flat solution in quartic theory, black curve is for usual Schwarzschild solution and blue, red and green curves are for $K = -0.32, -1, -3$ respectively.

The results of numerical solution are presented in Figure 3.1. Here behaviour of $f(r)$ against the dimensionless distance r/C is presented for various values of K. It can be observed from the graph that the solution approaches the expected value for asymptotically flat black holes.

When quartic coupling is increased we get more outward displacement of the horizon compared to the Schwarzschild black hole. This behaviour appears due to the addition of higher curvature terms in the action which in fact removes metric singularity as $r \to 0$. However, the curvature singularity remains, because it is observed that the Kretschmann scalar diverging as $1/r^4$ and Ricci scalar as $1/r^2$. This can be confirmed through an expansion of $f(r)$ as $r \to 0$. These results are same as those observed for the case of cubic theory [7, 8].

Another interesting aspect that we want to elaborate from the black hole solutions is that, despite the equation of motion for $f(r)$ in these theories being a third order differential equation (though a total derivative), asymptotically the black hole solutions we have obtained are characterized only by their mass. In the near-horizon solution, the free parameter a_2 is equivalent to a choice of boundary conditions, and it appears that there is a unique value yielding asymptotically flat conditions. One might naively expect that since the equations of motion are third order and the black holes may possess "higher derivative hair". However, the above discussion shows that this is clearly not the case for this class of theories. The black holes are characterized by a single free parameter (after fixing boundary conditions) and that is the black hole mass. In Ref. [66] it was suggested that this may be a general feature of this class of theories.

3.1.1 Black hole entropy

As described in Chapter 1, the usual Bekenstein-Hawking entropy gets modified due to the addition of higher curvature terms. These corrections were incorporated by Wald and the entropy relation is given by (1.18).

We start by considering the quartic quasi-topological theories. Our analysis shows that each

$\mathcal{Z}_d^{(i)}$ makes the same contribution to the Wald entropy. For the theory written down in equation (2.50), we find the expression for entropy given by,

$$s_4^{(i)} = \frac{(d-2)}{(d-8)}\frac{k^3}{r_+^6}\mu_{(i)} r_+^{d-2}, \qquad \text{for } \mathcal{Z}_d^{(i)}, \tag{3.14}$$

giving

$$s = \frac{r_+^{d-2}}{4}\left[1 + \frac{(d-2)}{(d-8)}\frac{k^3}{r_+^6}\left(\sum_{i=1}^{6}\mu_{(i)}\right)\right], \tag{3.15}$$

for the theory (2.50). For a theory containing additional terms, the above entropy density is simply modified by the addition of the entropy densities corresponding to the extra terms.

In the case of the generalized quasi-topological terms, $\mathcal{S}_d^{(i)}$, due to our choices of constraints and the normalization of $\hat{\lambda}_{(i)}$ in equation (2.31), we find the same contribution to the entropy from each $\mathcal{S}_d^{(i)}$. For the particular case of the theory presented in equation (2.54) this relation becomes,

$$s_4^{(i)} = -\frac{\pi(d-2)T}{r_+^3} r_+^{d-2}\left[\frac{(d-4)k^2}{r_+^2} + \frac{4\pi(d-5)kT}{r_+} + \frac{(4\pi)^2}{12}(3d-16)T^2\right]\lambda_{(i)}, \tag{3.16}$$

giving for the theory in equation (2.54),

$$s = \frac{r_+^{d-2}}{4}\left\{1 - \sum_{i=1}^{4}\frac{4\pi(d-2)T}{r_+^3}\left[\frac{(d-4)k^2}{r_+^2} + \frac{4\pi(d-5)kT}{r_+} + \frac{(4\pi)^2}{12}(3d-16)T^2\right]\lambda_{(i)}\right\}, \tag{3.17}$$

where $T = f'(r_+)/4\pi$ is the temperature. In all the above cases we have set $G = 1$. Again, if the Lagrangian contains additional terms, then the corresponding entropy densities of these terms will be simply added to the above.

The results presented above are valid for the case of four dimensions, where the only modification is the addition of the two additional terms corresponding to the contributions from $\mathcal{S}_4^{(5)}$ and $\mathcal{S}_4^{(6)}$

yielding

$$s_{d=4} = \frac{r_+^2}{4}\left[1 + \frac{32\pi^2 T^2}{r_+^4}\sum_{i=1}^{6}\lambda_{(i)}\left(k + \frac{4\pi}{3}r_+ T\right)\right]. \tag{3.18}$$

As a consistency check of the calculations, one can verify that the first law of black hole thermodynamics holds. Using near horizon solution presented above, and replacing C with M, from equation (3.11), we find that

$$\delta M = T\delta S, \tag{3.19}$$

where

$$S = \frac{\omega_2 r_+^2}{2\kappa}\left[1 + \frac{192\pi^2 T^2}{5r_+^4}\kappa^3 K\left(k + \frac{4\pi}{3}r_+ T\right)\right], \tag{3.20}$$

from above. Since each of the terms going into the first law was computed independently, the fact that this relationship holds provides an important check of our calculations.

3.2 Black brane solutions in quartic quasi-topological gravity

In this section, we aim to construct the black brane solutions of the quartic generalized quasi-topological gravities presented in Chapter 2. In order to get a more direct implication, we will only consider quartic generalized quasi-topological terms supplemented to the Einstein-Hilbert term. Therefore, we consider the action (3.1). Also, the metric ansatz is given by

$$ds^2 = \frac{r^2}{\ell^2}\left[-N(r)^2 f(r)dt^2 + \sum_{i=1}^{d-2} dx_i^2\right] + \frac{\ell^2 dr^2}{r^2 f(r)}, \tag{3.21}$$

where l is the AdS length. The resulting field equation reads $F' = 0$ with

$$F = (d-2)(r\ell)^{d-3}\left[r^2(f-1)\right] + \frac{(d-2)\lambda}{4}\ell^{d-3}r^{d-1}\Bigg[\left((3d-16)rf' + 4(d-6)f\right)r^3 ff'$$
$$-\frac{1}{4}(3d-16)r^4 f'^4 + \frac{(3d-16)(d+1)}{3}r^3 ff'^3 + 2d(d-6)f^2r^2f'^2 + \frac{4f^4}{3}(d-8)\Bigg], \quad (3.22)$$

and we have set $k = 0$ and rescaled λ with powers of ℓ so that it is dimensionless. Integration of this gives $F = C$, where C is an integration constant, depending on the mass. Here, we will use $N = 1/\sqrt{f_\infty}$ which means that the speed of light in dual CFT (conformal field theories) is equal to one.

3.2.1 Perturbative solution

Similar to the case presented in previous the section, here again we are not able to solve the field equation exactly and, therefore, choose to go for an approximate solution. To obtain asymptotic solution, we use the following expansion of the metric function $f(r)$

$$f(r) = f_\infty - \frac{\ell^2 C}{(r\ell)^{d-1}} + \epsilon h(r)\,. \quad (3.23)$$

In the above the quantity, f_∞ represents the asymptotic value of the metric function which can be obtained by solving

$$1 - f_\infty + \frac{(d-8)}{3}\lambda f_\infty^4 = 0\,. \quad (3.24)$$

Defined this way, the black branes asymptote to an AdS space with curvature radius $\tilde{\ell} = \ell/\sqrt{f_\infty}$.

We substitute this choice for $f(r)$ into the field equations. An inhomogeneous equation in $h(r)$ is obtained by setting all terms ϵ^n equal to zero for $n > 1$ and then, at the end, by setting $\epsilon = 1$. The general form of the equation is complicated and not presented here for the sake of simplicity.

The form of particular solution of the resulting differential have the form,

$$
\begin{aligned}
h_p(r) = &\frac{4(d-8)f_\infty^3\lambda}{4(d-8)f_\infty^3\lambda-3}\frac{\ell^2 C}{(r\ell)^{d-1}} \\
&-\frac{3(d^4-8d^3+13d^2-10d+32)\left(4(d-8)\lambda f_\infty^3+3\right)\lambda f_\infty^2}{2\left(4(d-8)\lambda f_\infty^3-3\right)^2}\frac{\ell^4 C^2}{(r\ell)^{2d-2}}+\mathcal{O}\left(\frac{\lambda C^3}{r^{3d-3}}\right). \quad (3.25)
\end{aligned}
$$

We note that, provided $f_\infty \neq 0$, there are corrections of the same order as the mass of the black brane from the higher curvature terms.

Next, we consider the homogenous part of the differential equation for large r. We are led to two separate cases. First, we consider the $d \neq 6$ case. In this case, the differential equation has the form,

$$
h'' - \frac{(3\,d-16)\,(d-1)^2\,C}{4\,(d-6)\,f_\infty r^d \ell^{d-3}}h' - \omega_d^2 r^{d-3} h = 0\,, \quad (\text{for } d \neq 6) \tag{3.26}
$$

with

$$
\omega_d^2 = \frac{3-4\,(d-8)\,\lambda f_\infty^3}{3\lambda\,(d-1)\,(d-6)\,C f_\infty^2}\ell^{d-3}\,. \tag{3.27}
$$

To get a valid solution and make sure that the AdS boundary conditions are satisfied, we must have $\omega^2 > 0$ which constrains the coupling to satisfy

$$
\frac{3-4(d-8)\lambda f_\infty^3}{(d-6)\lambda} > 0\,. \tag{3.28}
$$

Physical solution has to satisfy this inequality. An approximate solution to the homogenous equation meeting the above condition is given by

$$
h(r) \approx A\exp\left(\frac{2\omega_d r^{(d-1)/2}}{d-1}\right) + B\exp\left(-\frac{2\omega_d r^{(d-1)/2}}{d-1}\right)\,. \tag{3.29}
$$

To ensure the boundary conditions are met, we have to set $A = 0$. Also, it can be observed that the second term is suppressed and can be ignored.

For the case of six dimensions, calculations give the following form of the homogeneous equation,

$$h'' - \frac{5}{r}h' - \omega_6^2 r^8 h = 0, \tag{3.30}$$

where

$$\omega_6^2 = \frac{2(3 + 8\lambda f_\infty^3)}{75\lambda C^2 f_\infty}\ell^6, \tag{3.31}$$

and we must have

$$\frac{3 + 8\lambda f_\infty^3}{\lambda} > 0 \quad \text{for } d = 6. \tag{3.32}$$

In this case the approximate solution can be written as,

$$h(r) \approx A \exp\left(\frac{\omega_6 r^5}{5}\right) + B \exp\left(-\frac{\omega_6 r^5}{5}\right). \tag{3.33}$$

Again, we set $A = 0$ to ensure consistency with the boundary conditions, and also discard the contribution of second term due to its enormous suppression.

The importance of the consideration of homogenous equation is evident from the above results. These calculations give the restriction on couplings which are not available with particular solution alone. Furthermore, we observe that we can drop the exponential terms from our considerations.

3.2.2 Near-horizon solution

Here, we will present the near-horizon solution for the black brane case. We use the following expansion for the metric function,

$$f(r) = \frac{4\pi T\sqrt{f_\infty}\ell^2}{r_+^2}(r - r_+) + \sum_{i=2} a_n (r - r_+)^n, \tag{3.34}$$

assuming that the metric function vanishes linearly as $r \to r_+$ for non-extremal black holes. Using this ansatz into the field equations allows one to determine the coefficients from the resulting recurrence relation. The first two of these are

$$C = \ell^{d-3} r_+^{d-1} - 16(3d-16)\pi^4 \lambda r_+^{d-5} \ell^{d+5} f_\infty^2 T^4 \,,$$
$$0 = (d-1) r_+^4 - 4\pi \sqrt{f_\infty} T r_+^3 \ell^2 + \frac{16\lambda}{3}(d-5)(3d-16) f_\infty^2 \ell^8 \pi^4 T^4. \tag{3.35}$$

These two relations allow us to determine C, the integration constant, and the temperature in terms of the horizon radius. The rest of the terms become more and more complicated and, in fact, are not required for thermodynamic considerations. It is well established now from some extensive studies that the thermodynamics of black objects in the generalized quasi-topological theories can be studied exactly, despite the lack of an exact solution to the full field equations [7, 8, 66, 81].

To close this section, we remark that the above result shows that, in five dimensions, temperature is same as that of the Einstein's gravity, but in all other dimensions, it gets modified by these higher curvature terms.

3.2.3 Thermodynamical considerations

The expression for the entropy for these theories has the following form

$$s = \frac{1}{4}\left(\frac{r_+}{\ell}\right)^{d-2}\left[1 - \frac{16\lambda}{3}\frac{(d-2)(3d-16)\pi^3 \ell^6 f_\infty^{3/2} T^3}{r_+^3}\right], \tag{3.36}$$

which is not simply given by the Bekenstein-Hawking area law, but rather contains corrections due to the generalized quasi-topological contributions. This is remarkably different from what is observed in both Lovelock and quasi-topological gravity, where the area law remains unaffected for black branes, and may have interesting holographic consequences. For $d \leq 5$ the entropy is larger than that in the Einstein's gravity ($\lambda = 0$), whereas for $d \geq 6$, it is smaller.

As a check, we can demonstrate that (3.36) satisfies the first law,

$$d\varepsilon = T ds, \tag{3.37}$$

where the energy density has the form,

$$\begin{aligned} \varepsilon &= \frac{(d-2)}{16\pi\sqrt{f_\infty}\ell^{2d-3}} C\,, \\ &= \frac{(d-2)}{16\pi\sqrt{f_\infty}\ell^{2d-3}} \big[\ell^{d-3} r_+^{d-1} - 16(3d-16)\pi^4 \lambda r_+^{d-5} \ell^{d+5} f_\infty^2 T^4\big]. \end{aligned} \tag{3.38}$$

The factors of r_+ appearing in the entropy and energy densities can be eliminated by solving the second equation of (3.35). This is made easier by writing,

$$r_+ = \gamma_\lambda \frac{4\pi\ell^2\sqrt{f_\infty}}{d-1} T \tag{3.39}$$

where γ_λ solves the equation,

$$\gamma_\lambda^4 - \gamma_\lambda^3 + \frac{\lambda}{48}(d-1)^3(d-5)(3d-16) = 0\,, \tag{3.40}$$

which is obtained by substituting r_+ for the above definition in the second equation of (3.35). Here, we have included the subscript λ to illustrate that this quantity depends directly on the coupling λ.

The entropy and energy densities now have the forms,

$$s = \frac{12\gamma_\lambda^3 - \lambda(d-1)^3(d-2)(3d-16)}{48\gamma_\lambda^3}\left(\gamma_\lambda \frac{4\pi\ell\sqrt{f_\infty}T}{d-1}\right)^{d-2},$$
$$\varepsilon = \frac{(d-2)\ell^{d-4}}{256\pi\gamma_\lambda^4}\left[16\gamma_\lambda^4 - (d-1)^4(3d-16)\lambda\right]\left(\gamma_\lambda \frac{4\pi\ell^2\sqrt{f_\infty}}{d-1}T\right)^{d-1}. \tag{3.41}$$

The analysis of the polynomial (3.40) reveals that there will be real, positive solutions for γ_λ, provided the coupling satisfies

$$\lambda \leq \frac{81}{16(d-1)^3(d-5)(3d-16)}, \tag{3.42}$$

with equality corresponding to a positive, real double root. This does not apply in $d = 5$, but in this case, λ does not contribute to the polynomial, and the only valid solution is $\gamma_\lambda = 1$, which is valid for any value of the coupling.

Free energy density can be constructed from energy densities and have the form

$$\mathcal{F} = \varepsilon - Ts = -\frac{12\gamma_\lambda^3 - \lambda(d-1)^3(d-2)(3d-16)}{192\pi\ell\sqrt{f_\infty}\gamma_\lambda^4}\left(\gamma_\lambda \frac{4\pi\ell\sqrt{f_\infty}T}{d-1}\right)^{d-1}. \tag{3.43}$$

The entropy and energy densities can be shown [using equation (3.40)] to satisfy the relation

$$\varepsilon = \frac{d-2}{d-1}Ts, \tag{3.44}$$

which is expected for a CFT living in $d-1$ dimensions.

An interesting aspect of the above results is that the entropy and energy densities are modified from the Einstein's gravity result. Similar results were noted for five dimensional black branes in cubic generalized quasi-topological gravity [66]. In Lovelock and quasi-topological gravity, this is not the case: the expressions are identical, apart from the appearance of the term f_∞, characterizing the curvature of the AdS space [56]. In a sense, the properties of black branes in these latter theories are 'universal'.

Chapter 4

Charged Anti-de Sitter Black Hole Solutions in Cubic Generalized Quasi-Topological Gravity

4.1 Charged black hole solutions in cubic GQG

In this chapter we shall study charged static, spherically symmetric AdS black holes in generalized quasi-topological gravity. This includes a more thorough study of the results presented for asymptotically flat solutions and AdS black branes in recent works [66, 82].

The most general cubic theory satisfying the condition $g_{tt}g_{rr} = -1$, ensuring dependence on a single metric function, includes the cubic Lovelock and quasi-topological terms, in addition to the GQG term. Since both Lovelock and quasi-topological terms have been previously studied, here we take Einstein's gravity accompanied only by the cubic generalized quasi-topological term and a Maxwell field. In d spacetime dimensions, the action is given by [8]

$$\mathcal{I} = \frac{1}{16\pi G}\int d^dx\sqrt{-g}\left[\frac{(d-1)(d-2)}{\ell^2} + R + \frac{12(2d-1)(d-2)}{d-3}\mu\mathcal{S}_{3,d} - \frac{1}{4}F_{ab}F^{ab}\right], \quad (4.1)$$

where the cosmological constant $\Lambda = -\frac{(d-1)(d-2)}{2\ell^2}$ and

$$\begin{aligned}\mathcal{S}_{3,d} = {} & 14R_a{}^e{}_c{}^f R^{abcd}R_{bedf} + 2R^{ab}R_a{}^{cde}R_{bcde} - \frac{4(66-35d+2d^2)}{3(d-2)(2d-1)}R_a{}^cR^{ab}R_{bc} \\ & - \frac{2(-30+9d+4d^2)}{(d-2)(2d-1)}R^{ab}R^{cd}R_{acbd} - \frac{(38-29d+4d^2)}{4(d-2)(2d-1)}RR_{abcd}R^{abcd} \\ & + \frac{(34-21d+4d^2)}{(d-2)(2d-1)}R_{ab}R^{ab}R - \frac{(30-13d+4d^2)}{12(d-2)(2d-1)}R^3. \end{aligned} \quad (4.2)$$

The ansatz for the metric is in the following form

$$ds^2 = -N(r)^2f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Sigma^2_{(d-2),k}, \quad (4.3)$$

and the field equations permit $N(r)$ =constant [8]; we set $N(r) = 1$ for simplicity. In the above, $d\Sigma^2_{(d-2),k}$ denotes the line element of the $(d-2)$-dimensional transverse space, which we take to be a surface of constant scalar curvature $k = +1, 0, -1$, associated with spherical, flat, and hyperbolic topologies, respectively.[1]

A particular case of the metric (4.3) is a constant curvature space, for which the metric function is given by,

$$f_{\mathrm{AdS}}(r) = k + f_\infty\frac{r^2}{\ell^2}\,. \quad (4.4)$$

[1] The case $k = 0$ has been previously investigated [83] and so we only concentrate on non-planar black holes.

Here, ℓ is the length scale associated with the cosmological constant, while f_∞ is a constant that solves the following polynomial equation:

$$1 - f_\infty + \frac{\mu}{\ell^4}(d-6)(4d^4 - 49d^3 + 291d^2 - 514d + 184)f_\infty^3 = 0\,, \tag{4.5}$$

which is insensitive to the value of k. With $\mu \neq 0$, f_∞ will differ from unity, indicating that the higher curvature terms contribute to the radius of curvature of the space. In general, the real solutions to this polynomial may be positive or negative — we discard any negative solutions, since these would correspond to dS vacua. Restricting to only $f_\infty > 0$, the effective radius of the AdS space is then given by $\ell_{\text{eff}} = \ell/\sqrt{f_\infty}$.

The negative of the derivative of equation (4.5) with respect to f_∞ coincides with the prefactor appearing in the linearized equations of motion [8], and therefore must be positive

$$P(f_\infty) = 1 - 3\frac{\mu}{\ell^4}(d-6)(4d^4 - 49d^3 + 291d^2 - 514d + 184)f_\infty^2 > 0, \tag{4.6}$$

to ensure that the graviton is not a ghost in these backgrounds.

As our aim is to study charged black holes, we introduce a Maxwell field $F_{ab} = \partial_a A_b - \partial_b A_a$, with electromagnetic one-form defined as,

$$A \;=\; qE(r)dt. \tag{4.7}$$

By substitution of the above expression in the Maxwell equation, the unknown function is determined as

$$E(r) \;=\; \sqrt{\frac{2(d-2)}{(d-3)}}\frac{1}{r^{d-3}}, \tag{4.8}$$

where the specific choice of the prefactor is made to simplify the thermodynamic expressions.

The only independent field equation from (4.1) becomes

$$\frac{d}{dr}F[f,f',f'']=0, \tag{4.9}$$

with

$$F \;=\; r^{d-3}\left(k-f(r)+\frac{r^2}{\ell^2}\right)+\mu F_{\mathcal{S}_{3,d}}+r^{3-d}q^2\,. \tag{4.10}$$

The term $F_{\mathcal{S}_{3,d}}$ is the contribution from the cubic generalized quasi-topological term to the field equation and is given by

$$\begin{aligned}F_{\mathcal{S}_{3,d}} \;=\;& \frac{32232(2d-1)k}{13(d-3)}\delta_{d,7}+12\Big[(d^2+5d-15)\Big(\frac{4}{3}r^{d-4}f'^3-8r^{d-5}ff''\big(\frac{rf'}{2}+k-f\big)\\&-2r^{d-5}((d-4)f-2k)f'^2+8(d-5)r^{d-6}ff'(f-k)\Big)-\frac{1}{3}(d-4)r^{d-7}(k-f)^2\\&\times\Big(\big(-d^4+\frac{57}{4}d^3-\frac{261}{4}d^2+312d-489\big)f+k\big(129-192d+\frac{357}{4}d^2-\frac{57}{4}d^3+d^4\big)\Big)\Big]\,.\end{aligned} \tag{4.11}$$

Since the above equation is a total derivative, it can be directly integrated to yield

$$F=m, \tag{4.12}$$

where m is an integration constant with dimensions of $[\text{length}]^{d-3}$ and we shall see shortly that it is related to the mass of black hole. Although exact solutions to these field equations are not possible, we can study the asymptotic behaviour and near-horizon behaviour of the metric perturbatively. From the near-horizon expansion it will be possible to completely characterize the thermodynamics of the black holes.

4.1.1 Asymptotic solution

The asymptotic form of the metric function is

$$f(r) = k + f_\infty \frac{r^2}{\ell^2} - \frac{m}{r^{d-3}} + \frac{q^2}{r^{2d-6}} + \epsilon h(r), \tag{4.13}$$

where $h(r)$ describes a correction to the Einstein's gravity solution due to the cubic terms and ϵ is used to track the order of these contributions. Inserting the above expression into the field equation, at first order in ϵ we obtain an inhomogenous second order differential equation for the modified function $h(r)$. Assuming that $\mu \neq 0$,[2] the homogenous part of the equation at large r is

$$h_h'' - \frac{4}{r} h_h' - \gamma^2 r^{d-3} h_h = 0, \tag{4.14}$$

where

$$\gamma^2 \quad = \quad -\frac{3l^2 P(f_\infty)}{144(d-1)\left(d^2+5d-15\right) f_\infty \mu \; m} \,. \tag{4.15}$$

First, consider the case of $\gamma^2 > 0$. In this case the solution to (4.14) takes the form,[3]

$$h_{h+} = A r^{5/2} I_{\frac{5}{d-1}} \left(\frac{2\gamma r^{\frac{d-1}{2}}}{d-1} \right) + B r^{5/2} K_{\frac{5}{d-1}} \left(\frac{2\gamma r^{\frac{d-1}{2}}}{d-1} \right), \tag{4.16}$$

[2] Of course, the coefficients of h'' and h' appearing in the differential equation are proportional to μ, and therefore, in the $\mu \to 0$ limit, these terms are simply absent, and the full solution limits to the d-dimensional RN-AdS black hole solution of Einstein's gravity.

[3] Note that the first and the second terms in this equation contribute to the same order at asymptotic infinity and so pure exponential solutions (without some power of r) are not valid.

where I and K denote the modified Bessel functions of the first and second kinds, respectively, and A and B are constants. Schematically, in the limit of large r, the behaviour is

$$h_h \sim Ar^{5/2} \exp\left(\frac{2\gamma r^{\frac{d-1}{2}}}{d-1}\right) + Br^{5/2} \exp\left(-\frac{2\gamma r^{\frac{d-1}{2}}}{d-1}\right), \tag{4.17}$$

which shows that by imposing $A = 0$, the homogenous solution falls off super-exponentially in the asymptotic region — this can be viewed as a consequence of the fact that the theory does not propagate ghosts on AdS. The super-exponential fall off of the second term also justifies our dropping of the homogenous solution below.

Note that k is not present in the approximate large r solution.

Consider next $\gamma^2 < 0$; the homogenous solution at large r becomes

$$h_{h-} = C_1 r^{5/2} J_{\frac{5}{d-1}}\left(\frac{2|\gamma| r^{\frac{d}{2}-\frac{1}{2}}}{d-1}\right) + C_2 r^{5/2} Y_{\frac{5}{d-1}}\left(\frac{2|\gamma| r^{\frac{d}{2}-\frac{1}{2}}}{d-1}\right), \tag{4.18}$$

where J and Y are the Bessel functions of the first and second kinds, respectively. Note that the radial dependence is such that, in any dimension, we get solutions that oscillate rapidly and grow faster than r^2/ℓ^2, and thus do not approach AdS at infinity. The only consistent possibility would be to impose $C_1 = C_2 = 0$, eliminating the homogenous part of the solution. Since in all but this finely tuned case the asymptotic structure of the black holes is not AdS, in what follows, we disregard solutions with $\gamma^2 < 0$, imposing the constraint $\gamma^2 > 0$, thereby restricting our attention to solutions that do not have oscillating behaviour near infinity.

The differential equation for the metric perturbation $h(r)$ has a complicated particular solution given by

$$\begin{aligned}
h_p(r) \;=\; & \frac{P(f_\infty)-1}{P(f_\infty)}\frac{m}{r^{d-3}} - \frac{P(f_\infty)-1}{P(f_\infty)}\frac{q^2}{r^{2d-6}} + \frac{8058k\mu}{P(f_\infty)r^4}\delta_{d,7} \\
& -\frac{\mu}{2}(72d^5-294d^4+2358d^3-11880d^2+18888d-6624)\frac{P(f_\infty)-2}{P(f_\infty)^2}\frac{f_\infty m^2}{\ell^2 r^{2d-4}} \\
& +\mu(216d^5-342d^4-2442d^3+5064d^2-1992d+2016)\frac{P(f_\infty)-2}{P(f_\infty)^2}\frac{f_\infty m q^2}{\ell^2 r^{3d-7}} \\
& -24\mu(d-2)(d-1)^2\left(d^2+5d-15\right)\frac{P(f_\infty)-2}{P(f_\infty)^2}\frac{km^2}{r^{2d-2}} \\
& +\mathcal{O}\left(\frac{g_1(\mu,d,l)m^3}{P(f_\infty)^3 r^{3d-5}}, \frac{g_2(\mu,d,l)kq^2 m}{P(f_\infty)^2 r^{3d-5}}, \frac{g_3(\mu,d,l)q^4}{P(f_\infty)^2 r^{4d-10}}\right),
\end{aligned} \tag{4.19}$$

where $P(f_\infty)$ is defined in (4.6). We have written the five leading terms, indicating the fall off structure of the next corrections to $h_p(r)$. It is easy to see that as $\mu \to 0$, $h_p \to 0$ because, in this limit, $P(f_\infty) \to 1$ eliminates the first and second terms, while $\mu \to 0$ removes the remaining terms.

From the particular solution we note the leading order corrections are of the same order as the mass and charge, similar to Gauss-Bonnet gravity. Furthermore, as long as the graviton is not a ghost, the denominators are positive (and non-vanishing). Finally, since the particular solution falls off polynomially in r, it is the dominant contribution to the general solution for $h(r) = h_{h+} + h_p$ and we can safely neglect h_{h+} in equation (4.16).

4.1.2 Near-horizon solution

Next, we look at the solution near the horizon, which is achieved by performing the following expansion for the metric function:

$$f(r) = 4\pi T(r-r_+) + \sum_{i=2} a_n(r-r_+)^n, \tag{4.20}$$

where T is Hawking temperature of the black hole, which can be written as

$$T = \frac{f'}{4\pi}\,. \tag{4.21}$$

Inserting the near-horizon expansion of the metric function into the field equation and demanding it satisfy the field equations at each order of $(r - r_+)$ we find from the first two orders

$$\begin{aligned}
\omega^{d-3} &= kr_+^{d-3} + \frac{r_+^{d-1}}{\ell^2} + \frac{q^2}{r_+^{d-3}} - \frac{12(2d-1)}{(d-3)}\mu\Big[-\frac{2686}{13}k\delta_{d,7} - \frac{(d-3)}{2d-1} \\
&\quad \times\Big(-\frac{k(d-4)(129 - 192d + \frac{357}{4}\,d^2 - \frac{57}{4}\,d^3 + d^4)r_+^{d-7}}{3} \\
&\quad +(d^2+5d-15)\big(64k\pi^2 r_+^{d-5} + \frac{256}{3}\pi^3 T r_+^{d-4}\big)T^2\Big)\Big]\,, \\
0 &= (d-3)kr_+^{d-4} + (d-1)\frac{r_+^{d-2}}{\ell^2} - (d-3)\frac{q^2}{r_+^{d-2}} - 4\pi T r_+^{d-3} + \frac{12\mu}{(d-3)} \\
&\quad \times\Big[-\frac{k}{12}\,(d-3)\,(d-4)\,(d-7)\left(516 - 768d + 357d^2 - 57d^3 + 4d^4\right) r_+^{d-8} \\
&\quad -\frac{128}{3}\pi^3\,(d-4)\,(d-3)\left(d^2 + 5\,d - 15\right) r_+^{d-5}T^3 \\
&\quad -64\pi^2(d-3)(d-5)\left(d^2 + 5\,d - 15\right) kr_+^{d-6}T^2 \\
&\quad +(d-3)\,(d-4)\,(d-6)\,\pi\left(4d^3 - 33d^2 + 127d - 166\right) k^2 r_+^{d-7}T\Big]\,,
\end{aligned} \tag{4.22}$$

which determine the mass parameter and temperature in terms of the horizon radius and coupling. These formulae are enough to determine the thermodynamic properties of the black hole. Going to higher orders we eventually get all the series coefficients in terms of a_2, which is a free parameter whose value is fixed from the boundary condition at infinity.

Consider next the behaviour of the metric function for $r \to 0$. Expanding the field equations near the origin we find

$$f(r) = a_0 + ra_1 + r^2 a_2 + \cdots\,, \tag{4.23}$$

where these coefficients depend on the mass and charge parameters. Therefore the metric is regular at the origin, implying that the Kretschmann scalar $R_{abcd}R^{abcd} \sim r^{-4}$. This is a milder singularity than that which appears in Einstein's gravity, for which the metric in these coordinates is singular at $r = 0$ and for which $R_{abcd}R^{abcd} \sim r^{-6}$. It would be interesting to study if it is possible to remove this singularity completely by adding matter fields.

4.2 Thermodynamic considerations

In this section we investigate the thermodynamic properties of charged black holes in cubic generalized quasi-topological gravity. Applying the black hole chemistry formalism [53], we start by investigating the first law and Smarr relation, taking both Λ and μ to be thermodynamic variables. We then look at the physical constraints between the cubic coupling and the charge and present the domain for parameters to get physical critical points. We also illustrate the critical behaviour for the black holes here.

4.2.1 First law and Smarr relation

The near-horizon expansion of the metric function discussed in Section 4.1.2 above allows for the mass and temperature of the black holes to be determined algebraically by (4.22), despite the lack of an exact solution. However, except for $d = 4$ an explicit solution for the temperature is complicated, so we shall use the second equation implicitly instead to show that the first law is satisfied.

To calculate the entropy, we use the Iyer-Wald formalism explained in Chapter 1. Calculation yields the form of the entropy for the action (4.1),

$$\begin{aligned} S &= \frac{\Sigma_{(d-2),k}}{4} r_+^{d-2} \Big[1 + \frac{48\mu}{r_+^4}(d-2)\Big(8\pi\left(d^2+5d-15\right) r_+ T(k+\pi r_+ T) \\ &\quad -\frac{1}{16}(d-4)\left(4d^3-33d^2+127d-166\right)k^2\Big)\Big] , \end{aligned} \tag{4.24}$$

where $\Sigma_{(d-2),k}$ is the volume of the submanifold with line element $d\Sigma_{(k)d-2}$. When $k=1$, this is just the volume of the $(d-2)$-dimensional sphere, while for $k=0$ and $k=-1$ the numeric answer depends on what type of identifications are performed. The pressure is defined in the standard way,

$$P = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi\ell^2}, \tag{4.25}$$

with other thermodynamic quantities given by

$$\begin{aligned} V &= \frac{\Sigma_{(d-2),k} r_+^{d-1}}{(d-1)}, \quad Q = \Sigma_{(d-2),k} \frac{\sqrt{2(d-2)(d-3)}}{16\pi} q, \quad \Phi = \sqrt{\frac{2(d-2)}{d-3}} \frac{q}{r_+^{d-3}}, \\ \Psi_\mu &= -32(d-2)(d^2+5d-15)\Sigma_{(d-2),k}\left(\pi^2 r_+^{d-4} T^3 + \frac{3}{2}\pi k T^2 r_+^{d-5}\right) \\ &\quad + \frac{(d-2)(d-4)\Sigma_{(d-2),k}}{4}\Big[3\left(4d^3-33d^2+127d-166\right)k^2 T r_+^{d-6} \\ &\quad - \left(129 - 192d + \frac{357}{4}d^2 - \frac{57}{4}d^3 + d^4\right)\frac{k^3 r_+^{d-7}}{\pi}\Big], \end{aligned} \tag{4.26}$$

and [79] the mass

$$M = \frac{d-2}{16\pi G}\Sigma_{(d-2),k}\left[m - 8058 k\mu\delta_{d,7}\right], \tag{4.27}$$

where we have absorbed the $\delta_{d,7}$ part into the mass since it occurs at the same order of m.

These quantities satisfy the first law of black hole thermodynamics, as it is then straightforward to confirm

$$dM = TdS + VdP + \Phi dQ + \Psi_\mu d\mu, \tag{4.28}$$

which is the extended first law, with V the thermodynamic volume conjugate to the pressure and Ψ_μ the potential conjugate to the coupling μ. The quantities also satisfy the Smarr formula that is expected from scaling,

$$(d-3)M = (d-2)TS - 2PV + (d-3)\Phi Q + 4\mu\Psi_\mu\,. \tag{4.29}$$

Our aim is to study the critical behaviour of these black holes, and so we must obtain the equation of state. This is constructed by replacing ℓ^2 in the second equation in equation (4.22) in terms of pressure, yielding

$$\begin{aligned} P &= \frac{T}{v} - \frac{(d-3)}{\pi(d-2)}\frac{k}{v^2} + \frac{e^2}{v^{2d-4}} + (d-7)(d-4)\frac{\beta_0}{v^6} - (d-6)(d-4)\beta_1\frac{T}{v^5} \\ &\quad +(d-5)\frac{\beta_2}{v^4}T^2 + (d-4)\frac{\beta_3}{v^3}T^3, \end{aligned} \tag{4.30}$$

where, to simplify the resulting expressions we have introduced

$$\begin{aligned} v &= \frac{4r_+}{(d-2)}, \qquad e^2 = \frac{16^{d-3}}{\pi}(d-3)(d-2)^{5-2d}q^2 \\ \beta_0 &= \frac{2^8\left(4d^4 - 57d^3 + 357d^2 - 768d + 516\right)k}{\pi(d-2)^5}\mu, \quad \beta_2 = \frac{3\times 2^{12}\pi\left(d^2+5d-15\right)k}{(d-2)^3}\mu, \\ \beta_1 &= \frac{3\times 2^8\left(4d^3 - 33d^2 + 127d - 166\right)k^2}{(d-2)^4}\mu, \quad \beta_3 = \frac{2^{11}\pi^2\left(d^2+5d-15\right)}{(d-2)^2}\mu, \end{aligned} \tag{4.31}$$

where we refer to v as the specific volume and the others are rescaled physical parameters. In the sequel we choose β_3 and e as the free parameters.

The non-linear dependence of the equation of state on the temperature in (4.30) has been observed for the generalized quasi-topological theories in Ref. [81]. We shall study how including cubic generalized quasi-topological terms modify the results for Einstein's gravity in four and higher dimensions up to seven. To facilitate the study of the thermodynamics, we present the explicit form of the Gibbs free energy $\mathcal{G} = M - TS$

$$\begin{aligned}\mathcal{G} &= \left[\frac{4}{d-2}\right]^{d-1} \frac{G}{\Sigma_{(d-2),k}} = \frac{v^{d-1}P}{d-1} + \frac{v^{d-3}k}{\pi(d-2)} + \frac{e^2}{(d-3)v^{d-3}} - \beta_0(d-4)v^{d-7} - \Big(\frac{v^{d-2}}{d-2} \\ &\quad -\beta_1(d-4)v^{d-6}\Big)T - \beta_0\frac{48\pi^2(d-2)^2\left(d^2+5d-15\right)v^{d-5}}{4d^4-57d^3+357d^2-768d+516}T^2 - \beta_3 v^{d-4}T^3 \end{aligned} \tag{4.32}$$

where the overall positive factor is suppressed in the new definition to simplify the expression and other parameters are defined in equation (4.31). In stable equilibrium, the preferred state of the system is that which minimizes the Gibbs free energy at constant temperature and pressure.

The equation of state in general d using the rescaled parameters given in equation (4.31) is

$$\begin{aligned} p &= \frac{T}{v} - \frac{(d-3)k}{\pi(d-2)v^2} + \frac{\beta_3(d-4)T^3}{v^3} + \frac{6\beta_3(d-5)kT^2}{\pi(d-2)v^4} + \frac{e^2}{v^{2d-4}} - \frac{3\beta_3(d-6)(d-4)k^2T}{8\pi^2(d-2)^2} \\ &\quad \times\frac{\left(4d^3-33d^2+127d-166\right)}{\left(d^2+5d-15\right)v^5} + \frac{\beta_3(d-7)(d-4)\left(4d^4-57d^3+357d^2-768d+516\right)k}{8\pi^3(d-2)^3\left(d^2+5d-15\right)v^6} \\ &, \end{aligned} \tag{4.33}$$

where we note that all quantities depend only on e^2, and so the same results hold for both positive and negative charge.

The general idea for observing phase transitions is to see whether the coefficients of different powers of v in the equation of state have signs that allow for various maxima and minima of p. A necessary condition for a critical point to occur is that

$$\frac{\partial p}{\partial v} = \frac{\partial^2 p}{\partial^2 v} = 0 \ , \tag{4.34}$$

which will generally have non-degenerate solutions. An exception to this occurs when

$$\frac{\partial p}{\partial T} = 0, \tag{4.35}$$

which corresponds to a singular critical point. We find for general d that $\partial p/\partial T$ is proportional to a power of $1/v$ and so we do not obtain any singular critical points.

Unfortunately, the equation of state is quite complicated and does not admit an analytic solution for the critical values in arbitrary dimensions. However, some insight can still be gleaned from its form. We begin by isolating $\partial p/\partial v = 0$ for e^2, and then plug this result into $\partial^2 p/\partial v^2$. (Due to the non-linear temperature dependence, it's not possible to isolate either $\partial p/\partial v$ or $\partial^2 p/\partial v^2$ for the temperature or volume.) This produces the following result

$$\begin{aligned}
0 = & -\pi T (2d-5)(d-2)^5 v_c^5 + 4k(d-3)^2(d-2)^4 v_c^4 \\
& - 6144\,\mu T^3 \pi^3 (d-4)(2d-7)\left(d^2+5d-15\right)(d-2)^3 v_c^3 \\
& - 98304\,\pi^2 T^2 k \mu\,(d-4)(d-5)\left(d^2+5d-15\right)(d-2)^2 v_c^2 \\
& + 3840\,\mu T k^2 \pi\,(d-2)(d-4)(d-6)(2d-9)\left(4d^3-33d^2+127d-166\right) v_c \\
& - 3072\,k\mu\,(d-5)(d-7)(d-4)\left(4d^4-57d^3+357d^2-768d+516\right),
\end{aligned} \tag{4.36}$$

which fixes the critical volume, v_c, when the other quantities are known. In the following, we concentrate only on the case of four dimensions and discuss thermodynamic behaviour in some detail.

4.2.2 Critical behaviour in four dimensions

The existence of critical points for four dimensional charged black holes has been previously pointed out in both Einstein's gravity [84] and in ECG [81]. The equation of state (4.30) takes the following relatively simple form:

$$p = \frac{T}{v} - \frac{k}{2\pi v^2} + \frac{e^2}{v^4} - \frac{3\beta_3 k T^2}{\pi v^4}, \tag{4.37}$$

where for small v (i.e. for small black holes) the contributions from the electromagnetic and higher curvature terms equally dominate.

Solving equation (4.34) we find for general values of β_3 and e^2 that the critical temperature, pressure and volume are

$$T^2_{c\,\pm} = \frac{3\pi^3 e^2 \pm \pi\sqrt{9\pi^4 e^4 - 4k^4\beta_3}}{18\pi^2 k\beta_3}, \quad p_{c\,\pm} = \frac{3\pi^2 e^2 \pm \sqrt{9\pi^4 e^4 - 4k^4\beta_3}}{32k^2\beta_3}, \quad v_{c\,\pm} = \frac{2k}{3\pi T_{c\,\pm}}, \tag{4.38}$$

which result in two choices because the equation of state is quadratic in T. Note that if $\beta_3 < 0$ then if $k > 0$ only T_{c-}, p_{c-} and v_{c-} exist; if $k < 0$ then only T_{c+} exists, but both p_{c-} and v_{c-} are negative. Note that no solution exists for $k = 0$, recovering the result that no critical points exist for black branes.

From the above relations, it is obvious that the constraint

$$\beta_3 < \frac{9\pi^4 e^4}{4k^4}, \tag{4.39}$$

must hold to get a well defined critical solution. We also find that the ratio of critical quantities in (4.38) is independent of the black hole

$$\frac{p_c v_c}{T_c} = \frac{3}{8}, \tag{4.40}$$

and in this sense is universal. Note that this ratio is independent of choice of spherical or hyperbolic geometry, though in the latter case we do not have critical points since p_{c-} and v_{c-} are negative.

Remarkably, the ratio (4.40) is precisely the same as that first observed for charged black holes in four dimensional Einstein's gravity [84]; higher curvature corrections have not affected this universal value for spherical black holes. However, for four dimensional black branes the van der Waals ratio can differ from this value [83].

Chapter 5

Greybody Factor of Scalar Field from Reissner-Nordström-de Sitter Black Hole

5.1 Introduction

The study of asymptotically non-flat spacetime geometries received a lot of attention after it was discovered that our Universe has entered into a new phase of accelerated expansion [85]. Among these non-flat geometries de Sitter spacetime is of great interest due to its rich symmetries and also because it could incorporate the accelerated expansion of the Universe due to the presence of non-zero cosmological constant in the Einstein field equations. As predicted, the Universe is in continuous expansion, so in far future it will pass through a de Sitter phase. Further, de Sitter geometry could also approximate the inflationary phase of our Universe [86]. The fact that all black hole spacetimes of Kerr-Newman family can be generalized to include a cosmological constant makes black hole de Sitter spacetimes an interesting field of investigation.

De Sitter spacetime is the maximally symmetric Lorentzian space having positive curvature. In four dimensions the symmetry group is $SO(1,4)$ and topology is $R \times S$. Due to the structure of de Sitter spacetime, inertial observers are surrounded by cosmological horizons, which are a characteristic of spacetimes having positive cosmological constant. Also dS/CFT correspondence enhances the interest in the study of de Sitter spacetimes as they provide connection with conformal field theories.

Thermodynamically, in particular, and otherwise, in general, black holes are the most interesting and relevant objects in any gravitational theory. Their thermodynamics and entropy have been investigated by taking into account quantum mechanical effects [87, 88]. Also, Hawking temperature of radiations emitted from variety of different black holes was calculated [89–93]. Greybody factor defined as the probability for a given wave coming from infinity to be absorbed by the black hole (technically, rate of absorption probability) is directly connected to absorption cross section [23, 28–33, 94–97]. Black hole emission and absorption phenomena is related to this important quantity known as greybody factor. It is this quantity that makes it different from emission and absorption of a black body. The question is, how this quantity originated? It is generated by an effective potential barrier by black hole spacetimes. This potential quantum mechanically allows some of the radiation to transmit and remaining to reflect back. This leads to the frequency dependent greybody factor. Due to this factor black hole thermal radiation formula is different from the black body radiation formula. Greybody factor not only alters the thermal radiation formula but is also important to compute the partial absorption cross section of black holes [35, 98, 99]. In the literature there are investigations for greybody factor of scalar fields for Schwarzschild-de Sitter black hole. These include the cases of lowest partial modes in low energy regimes [36, 38, 95, 100, 101]. The absorption and emission spectra of a Schwarzschild black hole was studied in Ref. [38]. There has been a considerable interest in the study of greybody factor of scalar and fermionic radiations from asymptotically flat spacetimes and black strings [28, 30–33, 94, 102–107].

In this chapter, we use the simple matching technique to solve the radial equation resulting from the Klein-Gordon equation in the background of the Riessner-Nordström-de Sitter black hole. In this method we divide the space into two regimes, namely near the black hole horizon and near the cosmological horizon and find solutions for radial equations in both the regimes separately. Then we stretch these solutions to an intermediate point r_m. The choice of r_m is such that

$$r_h < r_m, \quad r_c > r_m \quad \text{and } \omega r_m \ll 1, \tag{5.1}$$

where r_h and r_c correspond to black hole and cosmological horizons respectively and ω is the frequency.

The rest of the chapter is organized as follows. In Section 5.2 we will discuss the Klein-Gordon equation and the profile of effective potential in the background of Reissner-Nordström-de Sitter black hole. In Section 5.3 we compute the greybody factor, starting from near black hole horizon solution, near cosmological horizon solution, and then matching them at an intermediate point. This yields expressions for greybody factor and spectrum of Hawking radiations.

5.2 Klein-Gordon equation and profile of effective potential

The spacetime metric for Reissner-Nordström-de Sitter black hole is given by,

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{5.2}$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}, \tag{5.3}$$

and related electromagnetic field is given by the four-potential

$$A_\mu = \frac{Q}{r}\delta^t_\mu. \tag{5.4}$$

Here M is the mass and Q is the charge of the black hole. Introducing a dimensionless cosmological parameter $\lambda = 1/3\Lambda M^2$, a dimensionless charge $e = \frac{Q}{M}$ and dimensionless coordinates $t \longrightarrow t/M$, $r \longrightarrow r/M$. It is equivalent to putting $M = 1$ [108]. The horizons are determined by the condition

$$1 - \frac{2}{r} + \frac{e^2}{r^2} - \lambda r^2 = 0. \tag{5.5}$$

It is clear from the above equation that in the special case of $e = 0$, the black hole spacetimes exist for $0 < \lambda \leqslant \frac{1}{27}$.

Before going into detailed calculations of analytic result of greybody factor we comment about its validity. It is interesting to note that it is valid for arbitrary quantum number l and coupling ξ. On the other hand the accuracy of the result is guaranteed only if the two asymptotic regions overlap, which implies that it is only valid for small frequencies. Also the approximation which we have used is justified for "small" black holes (compared with characteristic dS scale) that is $\lambda << 1$. Therefore the result of greybody factor is valid only in complementary regions of the parameter space.

We consider a scalar field theory in which the field is either minimally or non-minimally coupled to gravity and described by the following action

$$\mathcal{S} = \int d^4x\sqrt{-g}[R - \xi R\Phi^2 - \partial_\mu\Phi\partial^\mu\Phi]. \tag{5.6}$$

The equation of motion for the above theory can be written as

$$\frac{1}{\sqrt{-g}}\partial_{\mu}\left[\sqrt{-g}g^{\mu\nu}\partial_{\nu}\Phi(t,r,\theta,\varphi)\right]=-4\xi\Lambda\Phi. \tag{5.7}$$

In the above, we have used $R=-4\Lambda$ and ξ is a coupling constant determining the magnitude of coupling between the scalar and gravitational field, with $\xi=0$ corresponding to the minimal coupling. In matrix form the above line element can be written as

$$g_{\mu\nu}=\begin{pmatrix} f(r) & 0 & 0 & 0 \\ 0 & -\frac{1}{f(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}. \tag{5.8}$$

Also,

$$\sqrt{-g}=r^2\sin\theta. \tag{5.9}$$

Using these values in equation (5.7), it takes the form

$$\frac{1}{r^2\sin\theta}\partial_t\left(\frac{r^2\sin\theta}{f(r)}\partial_t\Phi\right)+\frac{1}{r^2\sin\theta}\partial_r\left(r^2\sin\theta\left(-f(r)\right)\partial_r\Phi\right)+\frac{1}{r^2\sin\theta}\partial_\theta\left(-\sin\theta\partial_\theta\Phi\right)$$

$$+\frac{1}{r^2\sin\theta}\partial_\varphi\left(\partial_\varphi\Phi\right)=-4\xi\Lambda\Phi. \tag{5.10}$$

Let

$$\Phi(t,r,\theta,\varphi)=e^{-\iota\omega t}R(r)Y(\theta,\varphi), \tag{5.11}$$

therefore, the radial part of equation (5.10) is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2f(r)\right)\frac{dR(r)}{dr}+\left[\frac{\omega^2}{f(r)}-\frac{l(l+1)}{r^2}+4\xi\Lambda\right]R(r)=0, \tag{5.12}$$

where $l(l+1)$ are the eigenvalues coming from the (θ, φ) part.

Before solving equation (5.12) we will discuss the profile of effective potential due to which greybody factor originates. We employ the following transformation on equation (5.12)

$$R\left(r\right) = \frac{U\left(r\right)}{r} \tag{5.13}$$

and the tortoise coordinate

$$x \equiv \int \frac{dr}{f}, \tag{5.14}$$

such that

$$\frac{d}{dx} = f\frac{d}{dr}, \frac{d^2}{dx^2} = f^2\frac{d^2}{dr^2} + ff'\frac{d}{dr}.$$

Thus equation (5.12) takes the form

$$\left(\frac{d^2}{dx^2} + \omega^2 - V_{eff}\left(r\right)\right) U\left(r\right) = 0, \tag{5.15}$$

with

$$V_{eff}\left(r\right) = f\left(r\right)\left(\frac{l\left(l+1\right)}{r^2} - 4\xi\Lambda + \frac{f'}{r}\right). \tag{5.16}$$

In Fig. 5.1, we draw the profile of effective potential for different values of the cosmological constant for $\xi = 0.01$, $q = 1$ and $l = 0$. It is observed that an increase in the value of cosmological constant, decreases the height of gravitational barrier and thus enhances the greybody factor. In Fig. 5.2, the effective potential is depicted for different values of the coupling constant and $\Lambda = 0.01$, $q = 1$ and $l = 0$. It is observed that increase in the value of the coupling constant leads to the increase in gravitational barrier, which subsequently suppresses the emission of scalar fields.

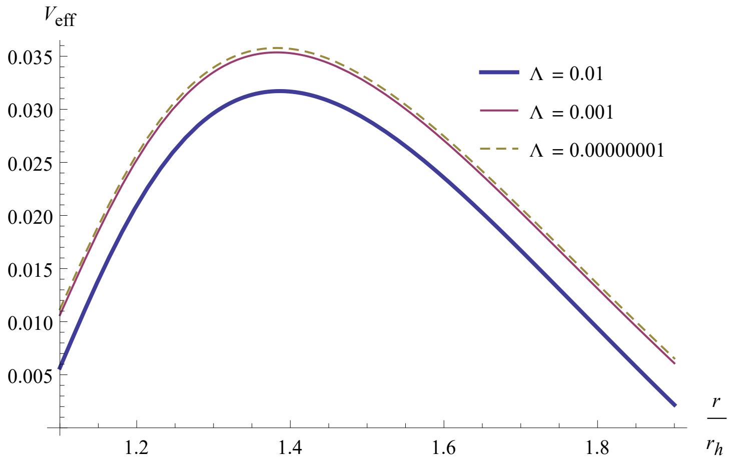

FIGURE 5.1: Profile of effective potential for different values of the cosmological constant for $\xi = 0.01$, $q = 1$ and $l = 0$.

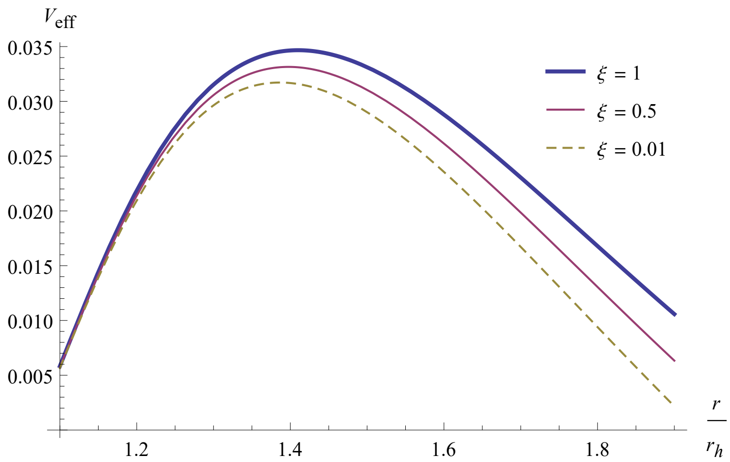

FIGURE 5.2: Profile of effective potential for different values of the coupling constant for $\Lambda = 0.01$, $q = 1$ and $l = 0$.

5.3 Greybody factor computation

5.3.1 Near black hole horizon solution

Equation (5.12) is the master equation of our interest. We will solve this equation in two regions separately, namely, near the black hole horizon and the cosmological horizon by using a semi-classical approach known as simple matching technique. Then we will match both the solutions to an intermediate region to get the analytical expression for the greybody factor.

For the near-horizon region $r \sim r_h$, we will perform the following transformation to simplify the radial equation [101]

$$r \to g = \frac{f(r)}{1 - \Xi r^2}, \tag{5.17}$$

where

$$\Xi = \frac{\Lambda}{3}.$$

Thus we get

$$\frac{dg}{dr} = (1-g)\frac{B(r_{h)}}{r_h\left(1 - \Xi r_h^2\right)}, \tag{5.18}$$

where, in the above

$$B(r_{h)} = \left(\frac{-6\Lambda M r_h^3 + 4\Lambda Q^2 r_h^2 + 6Mr_h - 6Q^2}{6Mr_h - 3Q^2}\right). \tag{5.19}$$

Using equations (5.17) and (5.18) in (5.12), we obtain

$$g(1-g)\frac{d^2R(g)}{df^2} + (1 - C_* g)\frac{dR(g)}{dg} + \left[\frac{F_*^2}{B^2(r_{h)}(1-g)g} - \frac{\lambda_h\left(1 - \Xi r_h^2\right)}{B^2(r_{h)}(1-g)}\right]R(g) = 0. \tag{5.20}$$

Here

$$F_* = \omega r_h, \tag{5.21}$$

$$C_* = \frac{r_h\left(1 - \Xi r_h^2\right)}{(1-g)^2 B(r_h)} - \frac{2r_h^2\Xi\left(1 - \Xi r_h^2\right)}{(1-g)B(r_h)}, \tag{5.22}$$

and

$$\lambda_h = [l(l+1) - 4\xi\Lambda]\, r_h^2. \tag{5.23}$$

In order to further simplify the above equation we use field redefinition

$$R(g) = g^{\mu_1}(1-g)^{\nu_1} F(g). \tag{5.24}$$

In equation (5.20) we use this definition of $R(f)$ to get

$$g\,(1-g)\,\frac{d^2F(g)}{dg^2}+[1+2\mu_1-(2\mu_1+2\nu_1+C_*)\,g]\,\frac{dF}{dg}+(\frac{\mu_1^2}{g}-\mu_1^2+\mu_1-2\mu_1\nu_1+\frac{\nu_1^2}{1-g}-\nu_1^2-\frac{2\nu_1}{1-g}$$

$$+\nu_1-\mu_1C_*+\frac{\nu_1C_*}{1-g}-\nu_1C_*+\frac{F_*^2}{B^2(r_h)g}+\frac{F_*^2}{B^2(r_h)\,(1-g)}-\frac{\lambda_h\left(1-\Xi r_h^2\right)}{B^2(r_{h)}\,(1-g)})F(g)=0. \quad (5.25)$$

Now, define

$$a_1=\mu_1+\nu_1+C_*-1, \quad (5.26)$$

$$b_1=\mu_1+\nu_1, \quad (5.27)$$

$$c_1=1+2\mu_1. \quad (5.28)$$

Also, constraints coming from the coefficients of $F(f)$ give

$$\mu_1^2+\frac{F_*^2}{B^2(r_h)}=0, \quad (5.29)$$

and

$$\nu_1^2+\nu_1(C_*-2)+\frac{F_*^2}{B^2(r_h)}-\frac{\lambda_h\left(1-\Xi r_h^2\right)}{B^2(r_{h)}}=0. \quad (5.30)$$

From here we find the values of μ_1 and ν_1 as

$$\mu_1=\pm\iota\frac{F_*}{B(r_h)}, \quad (5.31)$$

and

$$\nu_1=\frac{1}{2}\left[(2-C_*)\pm\sqrt{(2-C_*)^2-4\left(\frac{F_*^2}{B^2(r_h)}-\frac{4\lambda_h\left(1-\Xi r_h^2\right)\lambda_h}{B^2(r_{h)}}\right)}\right]. \quad (5.32)$$

Thus equation (5.25) by virtue of equations (5.26) − (5.28) and constraints (5.29) , (5.30) becomes

$$g\,(1-g)\,\frac{d^2F(g)}{dg^2}+\left[c_1-(1+a_1+b_1)\,g\right]\frac{dF(g)}{dg}-a_1b_1F(g)=0. \tag{5.33}$$

In the near-horizon region the solution can be written in the form of general hypergeometric function, which has the form

$$R(g)_{NH}=A_1g^{\mu_1}(1-g)^{\nu_1}F\,(a,b,c;g)+A_2g^{-\mu_1}(1-g)^{\nu_1}F\,(a-c+1,b-c+1,2-c;g)\,, \tag{5.34}$$

where A_1 and A_2 are arbitrary constants. For the near-horizon case there exists no outgoing mode, we choose $A_2=0$, thus we get

$$R(g)_{NH}=A_1g^{\mu_1}(1-g)^{\nu_1}F\,(a,b,c;g)\,. \tag{5.35}$$

5.3.2 Near cosmological horizon solution

We now solve the radial equation (5.12) close to the cosmological horizon r_c. In this case we choose the radial function $h\,(r)$, in place of $g\,(r)$, which is defined as

$$h\,(r)=1-\Xi r^2. \tag{5.36}$$

We employ the following transformation on equation (5.12)

$$r\rightarrow h\,(r)\,, \tag{5.37}$$

so that

$$\frac{dh}{dr}=\frac{(1-h)}{r}\,(-2)\,. \tag{5.38}$$

Using this, equation (5.12) becomes

$$h(1-h)\frac{d^2R(h)}{dh^2}+(1-2h)\frac{dR(h)}{dh}+\left[\frac{F_c^2}{B_c^2\,(1-h)\,h}-\frac{\lambda_c}{(1-h)\,B_c^2}\right]R(h)=0, \tag{5.39}$$

where, $F_c=\omega r_c^2$, $B_c=-2$ and $\lambda_c=\left[l\,(l+1)-4\xi\Lambda\right]r_c^2$. We redefine the radial function as

$$R(h)=h^{\mu_2}\,(1-h)^{\nu_2}\,X\,(h)\,. \tag{5.40}$$

Using equation (5.40) in (5.39) gives

$$h\,(1-h)\,\frac{d^2X(h)}{dh^2}+\left[c_2-(1+a_2+b_2)\,f\right]\frac{dX(h)}{dh}-a_2b_2X(h)=0. \tag{5.41}$$

In the above we defined

$$a_2=\mu_2+\nu_2+1, \tag{5.42}$$

$$b_2=\mu_2+\nu_2, \tag{5.43}$$

$$c_2=1+2\mu_2. \tag{5.44}$$

Also the constraints coming from the coefficients of $X\,(h)$ are

$$\mu_2^2+\frac{F_c^2}{B_c^2}=0, \tag{5.45}$$

and

$$\nu_2^2+\frac{F_c^2}{B_c^2}-\frac{\lambda_c}{B_c^2}=0. \tag{5.46}$$

From the above two equations we obtain the values of power coefficients as

$$\mu_{2\pm}=\pm\iota\frac{F_c}{B_c},\nu_{2\pm}=\pm\sqrt{\frac{F_c^2}{B_c^2}-\frac{\lambda_c}{B_c^2}}. \tag{5.47}$$

Equation (5.41) is again a hypergeometric equation and in order to ensure the convergence of the hypergeometric function, we choose negative sign of ν_2. Near the cosmological constant both the modes, incoming and outgoing exist, so the general solution can be written as

$$R(h)_{CH} = B_1 h^{\mu_2}(1-h)^{\nu_2} F\left(a_2, b_2, c_2; h\right) + B_2 h^{-\mu_2}(1-h)^{\nu_2} F\left(a_2 - c_2 + 1, b_2 - c_2 + 1, 2 - c_2; h\right). \tag{5.48}$$

5.3.3 Matching to an intermediate region

In order to match the above two solutions of the radial equation i.e., the near-horizon and near cosmological horizon solutions, to an intermediate region we use matching technique. We first shift the near-horizon solution given in equation (5.35) to an intermediate region, for which we change the argument of the hypergeometric function from g to $1-g$. This gives the following [105, 109]

$$R(g)_{NH} = A_1 g^{\mu_1} (1-g)^{\nu_1} \{ \frac{\Gamma(c_1)\Gamma(c_1 - a_1 - b_1)}{\Gamma(c-a)\Gamma(c-b)} F\left(a_1, b_1, a_1 + b_1 - c_1 + 1; 1-g\right)$$
$$+ (1-g)^{c_1 - a_1 - b_1} \frac{\Gamma(c_1)\Gamma(a_1 + b_1 - c_1)}{\Gamma(a_1)\Gamma(b_1)} F\left(c_1 - a_1, c_1 - b_1, c_1 - a_1 - b_1 + 1; 1-g\right) \}. \tag{5.49}$$

In the limit $r \gg r_h$, $g \to 1$ and we can use

$$(1-g)^{\nu_1} \simeq \left(\frac{r_h}{r}\right)^{\nu_1} \simeq \left(\frac{r}{r_h}\right)^{l}, \tag{5.50}$$

and

$$(1-g)^{\nu_1 + c_1 - a_1 - b_1} \simeq \left(\frac{r_h}{r}\right)^{2 - B_h - \nu_1} \simeq \left(\frac{r}{r_h}\right)^{-(l+1)}. \tag{5.51}$$

So, in an intermediate region the solution will take the following form

$$R(r)_{BH} = A_1 \frac{\Gamma(c_1)\Gamma(c_1 - a_1 - b_1)}{\Gamma(c_1 - a_1)\Gamma(c_1 - b_1)} \left(\frac{r}{r_h}\right)^{l} + A_1 \frac{\Gamma(c_1)\Gamma(a_1 + b_1 - c_1)}{\Gamma(a_1)\Gamma(b_1)} \left(\frac{r}{r_h}\right)^{-(l+1)}, \tag{5.52}$$

or

$$R(r)_{BH} = \mathcal{F}_1 r^{-(l+1)} + \mathcal{F}_2 r^{l}. \tag{5.53}$$

In the above we have used

$$\mathcal{F}_1 = A_1 \frac{\Gamma(c_1)\Gamma(a_1 + b_1 - c_1)}{\Gamma(a_1)\Gamma(b_1)}, \tag{5.54}$$

$$\mathcal{F}_2 = A_1 \frac{\Gamma(c_1)\Gamma(c_1 - a_1 - b_1)}{\Gamma(c_1 - a_1)\Gamma(c_1 - b_1)}. \tag{5.55}$$

Now we turn to equation (5.48) and shift its argument of hypergeometric function from h to $1-h$. Thus near cosmological horizon, we have $h(r_c) \to 0$, therefore

$$(1-h)^{\nu_2} \simeq \left(\frac{r}{r_c}\right)^{\nu_2} \simeq \left(\frac{r}{r_c}\right)^{-(l+1)}, \tag{5.56}$$

and

$$(1-h)^{\nu_2 + c_2 - a_2 - b_2} \simeq \left(\frac{r}{r_c}\right)^{-(1+2\nu_2)} \simeq \left(\frac{r}{r_c}\right)^{l}. \tag{5.57}$$

By using the above approximations and properties of hypergeometric functions [109] we can write equation (5.48) as

$$R(r)_C = \left[B_1 \frac{\Gamma(c_2)\Gamma(c_2 - a_2 - b_2)}{\Gamma(c_2 - a_2)\Gamma(c_2 - b_2)} + B_2 \frac{\Gamma(2 - c_2)\Gamma(c_2 - a_2 - b_2)}{\Gamma(1 - a_2)\Gamma(1 - b_2)}\right] \left(\frac{r}{r_c}\right)^{-(l+1)}$$

$$\left[B_1 \frac{\Gamma(c_2)\Gamma(a_2 + b_2 - c_2)}{\Gamma(a_2)\Gamma(b_2)} + B_2 \frac{\Gamma(2 - c_2)\Gamma(a_2 - b_2 - c_2)}{\Gamma(a_2 + 1 - c_2)\Gamma(b_2 + 1 - c_2)}\right] \left(\frac{r}{r_c}\right)^{l}, \tag{5.58}$$

or

$$R(r)_C = (\mathcal{F}_3 B_1 + \mathcal{F}_4 B_2) \left(\frac{r}{r_c}\right)^{-(l+1)} + (\mathcal{F}_5 B_1 + \mathcal{F}_6 B_2) \left(\frac{r}{r_c}\right)^{l}. \tag{5.59}$$

Here we have used

$$F_3 = \frac{\Gamma(c_2)\Gamma(c_2 - a_2 - b_2)}{\Gamma(c_2 - a_2)\Gamma(c_2 - b_2)}, F_4 = \frac{\Gamma(2 - c_2)\Gamma(c_2 - a_2 - b_2)}{\Gamma(1 - a_2)\Gamma(1 - b_2)}, \tag{5.60}$$

$$F_5 = \frac{\Gamma(c_2)\Gamma(a_2 + b_2 - c_2)}{\Gamma(a_2)\Gamma(b_2)}, F_6 = \frac{\Gamma(2 - c_2)\Gamma(a_2 - b_2 - c_2)}{\Gamma(a_2 + 1 - c_2)\Gamma(b_2 + 1 - c_2)}. \tag{5.61}$$

5.3.4 Greybody factor

Greybody factor $|A_l|^2$ for the emission of scalar fields can be defined by the amplitudes of the waves at the stretched cosmological horizon solution [101] that is

$$|A_l|^2 = 1 - \left|\frac{B_2}{B_1}\right|^2. \tag{5.62}$$

As the power law in (5.53) and (5.59) is same, we match these two solutions to get the values of B_1 and B_2 as

$$B_1 = \frac{F_1 F_6 - F_2 F_4}{F_3 F_6 - F_4 F_5}, \tag{5.63}$$

$$B_2 = \frac{F_2 F_3 - F_1 F_5}{F_3 F_6 - F_4 F_5}. \tag{5.64}$$

Using values from equations (5.63) and (5.64) in (5.62), we get the analytic formula for greybody factor for arbitrary mode l as

$$|A_l|^2 = 1 - \left|\frac{F_2 F_3 - F_1 F_5}{F_1 F_6 - F_2 F_4}\right|^2. \tag{5.65}$$

In the following plots we depict the greybody factor $|A_l|^2$ as a function of ωr_h. In Fig. 5.3, we take $\Lambda = 0.001$, $q = 3.5$, $l = 0$, for different values of ξ; in Fig. 5.4 $\Lambda = 0.05$, $q = 3.5$, $l = 0$, for different values of ξ; in Fig. 5.5 $\Lambda = 0.01$, $q = 3.5$ and $l = 0$, for different values of ξ; in Fig. 5.6 $\xi = 0.2$, $\Lambda = 0.001$ and $l = 0$, for different values of q; in Fig. 5.7 $\xi = 1$, $\Lambda = 0.001$, $l = 0$, for different values of q. An increase in value of the coupling parameter, decreases the greybody factor. This is due to the fact that non-minimal coupling plays the role of an effective mass and hence suppresses the greybody factor.

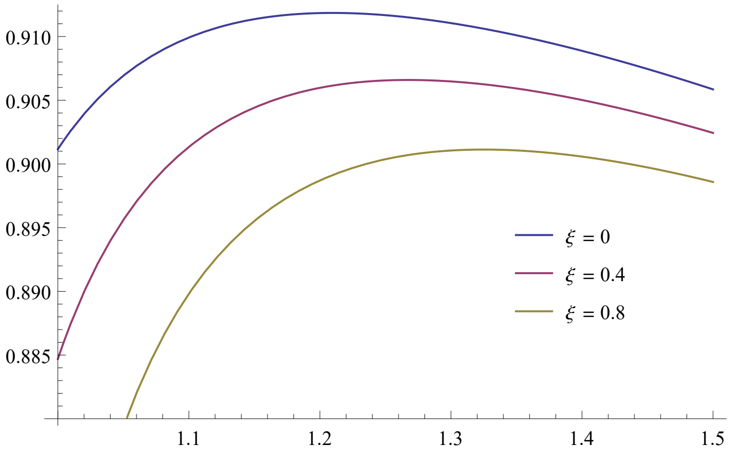

FIGURE 5.3: Greybody factor as a function of ωr_h for different values of ξ, when $\Lambda = 0.001$, $q = 3.5$ and $l = 0$.

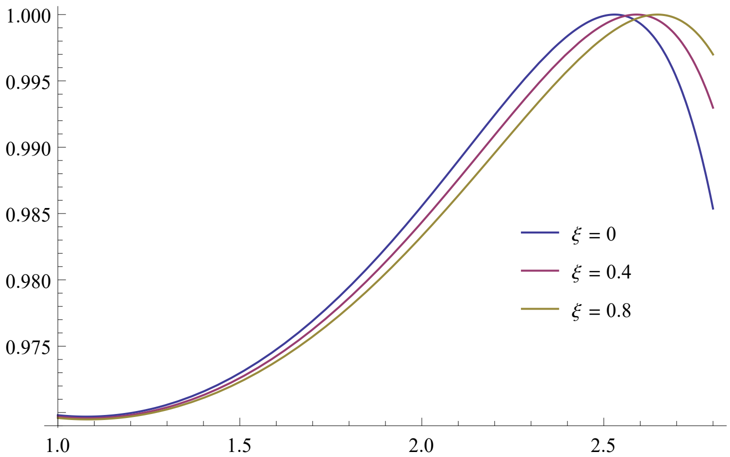

FIGURE 5.4: Greybody factor as a function of ωr_h for different values of ξ , when $\Lambda = 0.05$, $q = 3.5$ and $l = 0$.

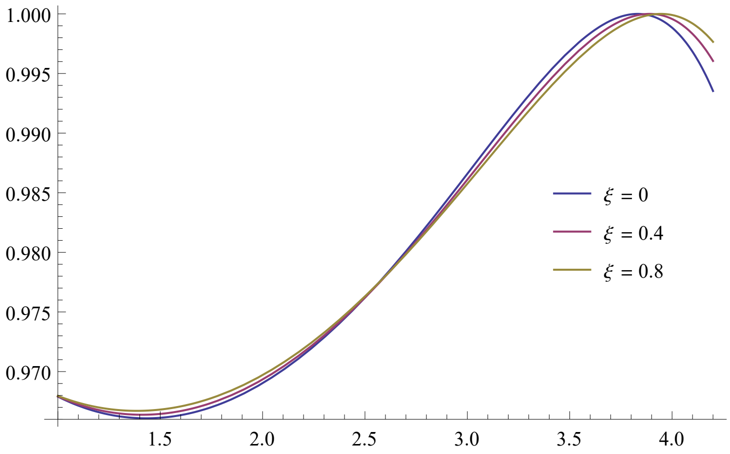

FIGURE 5.5: Greybody factor as a function of ωr_h for different values of ξ , when $\Lambda = 0.01$, $q = 3.5$ and $l = 0$.

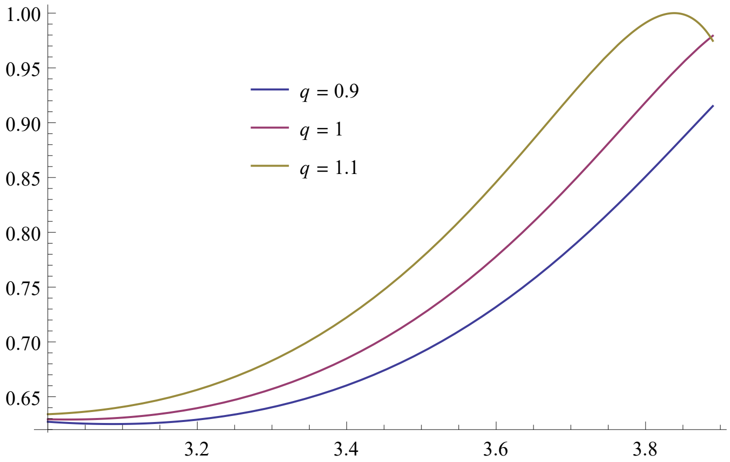

FIGURE 5.6: Greybody factor as a function of ωr_h for different values of q, when $\xi = 0.2$, $\Lambda = 0.001$ and $l = 0$.

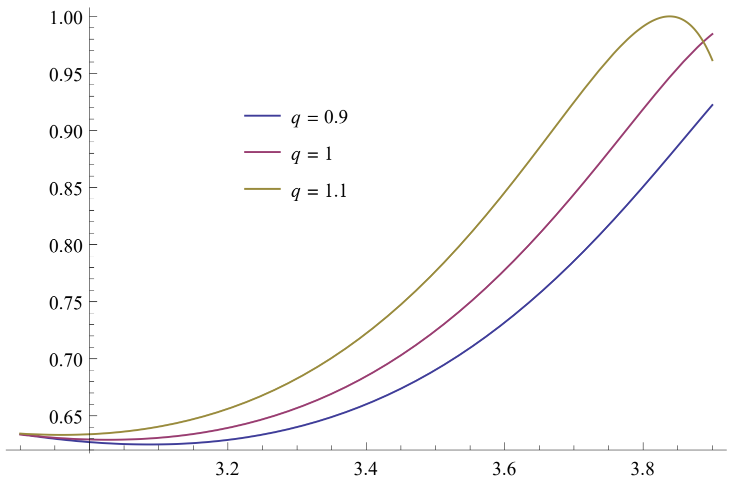

FIGURE 5.7: Greybody factor as a function of ωr_h for different values of q, when $\xi = 1$, $\Lambda = 0.001$ and $l = 0$.

5.3.5 Energy emission

The flux spectrum, that is, the number of massless scalar particles emitted by the black hole per unit time is given by [100]

$$\frac{dN(\omega)}{dt} = \frac{d\omega}{2\pi} \frac{1}{e^{\frac{\omega}{T_H}} - 1} \sum_{l=0}^{\infty} (2l+1) A_l(\omega) . \tag{5.66}$$

By using the value from equation (5.65) into (5.66) we get

$$\frac{dN(\omega)}{dt} = \frac{d\omega}{2\pi} \frac{1}{e^{\frac{\omega}{T_H}} - 1} \sum_{l=0}^{\infty} (2l+1) \sqrt{\left(1 - \left|\frac{F_2 F_3 - F_1 F_5}{F_1 F_6 - F_2 F_4}\right|^2\right)}. \tag{5.67}$$

Also, the differential energy rate is given by [100]

$$\frac{d^2 E(\omega)}{dt d\omega} = \frac{1}{2\pi} \frac{\omega}{e^{\frac{\omega}{T_H}} - 1} \sum_{l=0}^{\infty} (2l+1) A_l(\omega) . \tag{5.68}$$

On using the value of the greybody factor we get

$$\frac{d^2 E(\omega)}{dt d\omega} = \frac{1}{2\pi} \frac{\omega}{e^{\frac{\omega}{T_H}} - 1} \sum_{l=0}^{\infty} (2l+1) \sqrt{\left(1 - \left|\frac{F_2 F_3 - F_1 F_5}{F_1 F_6 - F_2 F_4}\right|^2\right)}. \tag{5.69}$$

The relevance of the low frequency limit is evident from the above results, that the coupling is only significant in this regime. As the coupling to scalar field is irrelevant in high frequency limits, the enhancement in emission rate occurs only at low frequencies.

5.3.6 Generalized absorption cross section

The definition of absorption cross section for asymptotically flat spacetimes is not valid for asymptotically non-flat spacetimes. For these cases the general formula for absorption cross section is given by [100, 110]

$$\sigma = \sum_{l=0}^{\infty} \sigma_l = \frac{\pi}{\omega^2} \sum_{l=0}^{\infty} (2l+1)\, A_l\,(\omega)\,. \tag{5.70}$$

Using the value from equation (5.65) we obtain

$$\sigma = \sum_{l=0}^{\infty} \sigma_l = \frac{\pi}{\omega^2} \sum_{l=0}^{\infty} (2l+1) \sqrt{\left(1 - \left|\frac{F_2 F_3 - F_1 F_5}{F_1 F_6 - F_2 F_4}\right|^2\right)}.$$

Chapter 6

Greybody Factor of Scalar Fields from Black Strings

6.1 Introduction

Scalar fields, non-minimally coupled with gravity, have shown significant features, both for inflation and dark energy. Also, the non-minimal couplings between derivatives of the scalar fields and the curvature reveal interesting cosmological behaviours. In general, the scalar-tensor theories give both the Einstein equation and the equation of motion for the scalar in the form of fourth-order differential equations. But if the kinetic term is only coupled to the Einstein tensor, the equation of motion for scalars is reduced to a second-order differential equation. Therefore, from the point of view of physics, considering such a coupling can be interpreted as a good theory because it is very simple. In the light of the earlier results [111–113] there is a need for more efforts to be focussed on the study of scalar fields coupled with tensors for more general cases. In order to fill the gap in the literature, for the case of cylindrically symmetric black holes, we have studied the properties of the scalar field when

- they are kinetically coupled to the Einstein tensor and
- they are without any coupling.

Our efforts are organized as follows: in Section 6.2, Klein-Gordon equation in a charged, black string background is calculated with a coupling to the Einstein tensor. In Section 6.3, solutions of the radial equation resulting from the Klein-Gordon equation in the near-horizon region and the far horizon regime will be presented. We will also match the solutions to an intermediate region to get the value of absorption probability (greybody factor). In Section 6.4, all the above analysis is then performed in the absence of the coupling parameter.

6.2 Klein-Gordon equation in the background of a charged black string

The Klein-Gordon equation when the Einstein tensor is coupled to a massless, uncharged scalar field is

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\left(g^{\mu\nu}+\eta\epsilon^{\mu\nu}\right)\partial_\nu\Psi\right]=0, \tag{6.1}$$

where η is a coupling constant and $\epsilon^{\mu\nu}$ is Einstein tensor. The charged black string having non-zero components of Einstein tensor is [37, 114]

$$ds^2=-f(r)dt^2+\frac{1}{f(r)}dr^2+r^2d\theta^2+\alpha^2r^2dz^2, \tag{6.2}$$

where

$$f(r)=\alpha^2r^2-\frac{4M}{\alpha r}+\frac{4Q^2}{\alpha^2r^2}. \tag{6.3}$$

Here M is the mass, Q is the charge and $\alpha = -\Lambda/3$, with Λ being the cosmological constant. For the above metric the Einstein tensor $\epsilon^{\mu\nu}$ in matrix form can be written as

$$\epsilon^{\mu\nu} = \frac{4Q^2}{\alpha^4 r^4}\begin{pmatrix} -\frac{1}{f} & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{\alpha^2 r^2} \end{pmatrix}. \tag{6.4}$$

Also

$$\sqrt{-g} = \alpha r^2. \tag{6.5}$$

Substituting the components of the Einstein tensor and spacetime metric in equation (6.1), it takes the form

$$\frac{1}{\alpha r^2}\partial_t\left[\alpha r^2\left(-\frac{1}{f} - \frac{4\eta Q^2}{\alpha^4 r^4 f}\right)\partial_t\Psi\right] + \frac{1}{\alpha r^2}\partial_r\left[\alpha r^2\left(f + \frac{4\eta Q^2 f}{\alpha^4 r^4}\right)\partial_r\Psi\right] +$$
$$\frac{1}{\alpha r^2}\partial_\theta\left[\alpha r^2\left(\frac{1}{r^2} - \frac{4\eta Q^2}{\alpha^4 r^6}\right)\partial_\theta\Psi\right] + \frac{1}{\alpha r^2}\partial_z\left[\alpha r^2\left(\frac{1}{\alpha r^2} - \frac{4\eta Q^2}{\alpha^6 r^6}\right)\partial_\theta\Psi\right] = 0. \tag{6.6}$$

Using the form of cylindrical harmonics

$$\Psi(t, r, \theta, z) = e^{-\iota\omega t}R(r)Y(\theta, z), \tag{6.7}$$

we get from the radial part of equation (6.6) as

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\left(1 + \frac{4\eta Q^2}{\alpha^4 r^4}\right)f\right]\frac{dR(r)}{dr} + \left[\left(1 + \frac{4\eta Q^2}{\alpha^4 r^4}\right)\frac{\omega^2}{f} - \left(1 - \frac{4\eta Q^2}{\alpha^4 r^4}\right)\frac{F_{lm}}{\alpha^2 r^2}\right]R(r) = 0, \tag{6.8}$$

where $F_{lm} = l(l+1)$ are the eigenvalues coming from the (θ, z) part.

6.3 Greybody factor computation

6.3.1 Near-horizon solution

Equation (6.8) is the master equation of our interest. We will solve this equation in two regions separately, namely, the near horizon region and the far region by using a semi-classical approach known as the simple matching technique. We will match both solutions to an intermediate region to get the analytical expression for the absorption probability.

For the near-horizon region $r \sim r_+$, we will perform the following transformation to simplify the radial equation [115–119]:

$$r \to f, \tag{6.9}$$

which implies

$$\frac{df}{dr} = (1-f)\,\frac{B(r_+)}{r_+}, \tag{6.10}$$

where r_+ is the horizon and

$$B(r_+) = 1 - \frac{4Q^2 - 2\alpha^4 r_+^4}{4M\alpha r_+ - 4Q^2}. \tag{6.11}$$

Using the above, equation (6.8) takes the form

$$\begin{aligned} f(1-f)\frac{d^2R(f)}{df^2} + (1 - C_* f)\,\frac{dR(f)}{df} \\ + \left[\frac{F_*^2}{B^2(r_+)\,(1-f)\,f} - \left(\frac{\alpha^4 r_+^4 - 4\eta Q^2}{\alpha^4 r_+^4 + 4\eta Q^2}\right)\frac{F_{lm}}{B^2(r_+)\alpha^2\,(1-f)}\right] R(f) = 0. \end{aligned} \tag{6.12}$$

Here

$$F_* = \omega r_*, \tag{6.13}$$

and

$$C_* = 2 - \frac{2}{B(r_+)}\left(\frac{\alpha^4 r_+^4 - 4\eta Q^2}{\alpha^4 r_+^4 + 4\eta Q^2}\right). \tag{6.14}$$

In order to further simplify the above equation, we use the field redefinition

$$R(f) = f^{\mu}\,(1-f)^{\nu}\,F(f). \tag{6.15}$$

Using this in equation (6.12), we obtain

$$\begin{aligned} f\,(1-f)\,\frac{d^2F(f)}{df^2} + [1+2\mu-(2\mu+2\nu+C_*)\,f]\,\frac{dF}{df} \\ +[\frac{\mu^2}{f} - \mu^2 + \mu - 2\mu\nu + \frac{\nu^2}{1-f} - \nu^2 - \frac{2\nu}{1-f} + \nu - \mu C_* + \frac{\nu C_*}{1-f} - \nu C_* \\ + \frac{F_*^2}{B^2(r_+)f} + \frac{F_*^2}{B^2(r_+)\,(1-f)} - \left(\frac{\alpha^4 r_+^4 - 4\eta Q^2}{\alpha^4 r_+^4 + 4\eta Q^2}\right)\frac{F_{lm}}{B^2(r_+)\alpha^2\,(1-f)}]F(f) = 0. \end{aligned} \tag{6.16}$$

We define

$$a = \mu+\nu+C_*-1, \quad b = \mu+\nu, \quad c = 1+2\mu. \tag{6.17}$$

Also the constraints coming from the coefficients of $F(f)$ give

$$\mu^2 + \frac{F_*^2}{B^2(r_+)} = 0, \tag{6.18}$$

and

$$\nu^2 + \nu(C_*-2) + \frac{F_*^2}{B^2(r_+)} - \left(\frac{\alpha^4 r_+^4 - 4\eta Q^2}{\alpha^4 r_+^4 + 4\eta Q^2}\right)\frac{F_{lm}}{B^2(r_+)\alpha^2} = 0. \tag{6.19}$$

From this we get the values of μ and ν:

$$\mu_{\pm} = \pm\iota\frac{F_*}{B(r_+)}, \tag{6.20}$$

and

$$\nu_{\pm} = \frac{1}{2}\left[(2 - C_*) \pm \sqrt{(2 - C_*)^2 - 4\left(\frac{F_*^2}{B^2(r_+)} - \left(\frac{\alpha^4 r_+^4 - 4\eta Q^2}{\alpha^4 r_+^4 + 4\eta Q^2}\right)\frac{F_{lm}}{B^2(r_+)\alpha^2}\right)}\right]. \tag{6.21}$$

Equation (6.16) by virtue of (6.17) and the constraints (6.18)-(6.19) becomes

$$f\,(1 - f)\,\frac{d^2F(f)}{df^2} + \left[c - (1 + a + b)\,f\right]\frac{dF(f)}{df} - abF(f) = 0. \tag{6.22}$$

For the near-horizon case there exists no outgoing mode, which means $\mu_+ = \mu_-$ and $\nu_+ = \nu_-$. So in the near-horizon region the solution can be written in the form of the general hypergeometric function, which has the form

$$R(f)_{NH} = C_- f^{\mu}(1 - f)^{\nu} F\,(a, b, c; f)\,, \tag{6.23}$$

where C_- is an arbitrary constant.

6.3.2 Far horizon solution

Now we find the solution of the radial equation for the far region. In this case the radial part will have the form

$$\frac{d^2R(r)_{FR}}{dr^2} + \frac{4}{r}\frac{dR(r)_{FR}}{dr} + \left(\omega^2 - \frac{F_{lm}}{\alpha^2 r^2}\right)R(r)_{FR} = 0. \tag{6.24}$$

This is the well-known Bessel equation, and in a far field its solution can be written as

$$R_{FR}\,(r) = \frac{1}{\sqrt{r\alpha\omega}}\left[B_1 J_{\gamma}(\omega\alpha r) + B_2 Y_{\gamma}(\omega\alpha r)\right]. \tag{6.25}$$

In the above solution J_γ and Y_γ are Bessel's functions. For $\gamma = l + 1/2$, and in the limit $r \to 0$, the above solution can be written as

$$R_{FR}\left(r\right) \simeq \frac{B_1\left(\frac{\omega\alpha r}{4}\right)^{\gamma}}{\sqrt{\omega\alpha r}\Gamma\left(\nu+1\right)} - \frac{B_2\Gamma\left(\gamma\right)}{\pi\sqrt{\omega\alpha r}\left(\frac{\omega\alpha r}{4}\right)^{\nu}}. \tag{6.26}$$

6.3.3 Matching the two solutions

We now stretch the near-horizon solution to an intermediate region [105, 109] which gives

$$\begin{aligned} R(f)_{NH} = C_- f^{\mu}\left(1-f\right)^{\nu} \Bigg[& \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F\left(a,b,a+b-c+1;1-f\right) \\ & + \left(1-f\right)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F\left(c-a,c-b,c-a-b+1;1-f\right)\Bigg]. \end{aligned} \tag{6.27}$$

We can approximate $1-f$ for the case $r \gg r_+$ as

$$1 - f \simeq \frac{4M}{\alpha r}. \tag{6.28}$$

So, the form of the final solution for the near-horizon case becomes

$$R(r)_{NH} \simeq A_1 r^{\nu} + A_2 r^{-(\nu + C_* - 2)}. \tag{6.29}$$

Here we have chosen

$$A_1 = C_- \left(\frac{4M}{\alpha}\right)^{\nu} \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{6.30}$$

and

$$A_2 = C_- \left(\frac{4M}{\alpha}\right)^{-(\nu + C_* - 2)} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}. \tag{6.31}$$

In the low-energy limit we can use the approximation

$$-\nu \simeq l + O(\omega^2), \tag{6.32}$$

$$\nu + C_* - 2 \simeq -(l+1) + O(\omega^2). \tag{6.33}$$

From equations (6.26) and (6.29) matching the coefficients and eliminating C_- give

$$B = \frac{B_1}{B_2} = -\frac{1}{\pi}\frac{1}{(\alpha\omega M)^{2l+1}}\frac{\Gamma(c-a-b)\Gamma(a)\Gamma(b)}{\Gamma(c-a)\Gamma(c-b)\Gamma(a)\Gamma(b)}\Gamma^2(l+1/2). \tag{6.34}$$

The greybody factor can now be given by [100]

$$\gamma_l(\omega) = |P_l|^2 = \frac{2\iota\,(B^* - B)}{|B|^2}. \tag{6.35}$$

By using the value of B we can find the expression of absorption probability of the radiations emitted from the charged black string. This relation gives a measure of how much the radiations are different (or modified) from the spectrum of the black body radiation.

6.4 Absorption probability for scalar field without coupling to the Einstein tensor

In this section we find an analytical expression of the absorption probability for scalar field from the charged black string without coupling to the Einstein tensor. The Klein-Gordon equation for a massless, uncharged scalar field is

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\,(g^{\mu\nu})\,\partial_\nu\Psi\right] = 0. \tag{6.36}$$

Using the values of each component of the spacetime considered in the previous section, we get the following equation:

$$\frac{1}{\alpha r^2}\partial_t\left[\alpha r^2\left(-\frac{1}{f}\right)\partial_t\Psi\right]+\frac{1}{\alpha r^2}\partial_r\left[\alpha r^2\left(f\right)\partial_r\Psi\right]+$$
$$\frac{1}{\alpha r^2}\partial_\theta\left[\alpha r^2\left(\frac{1}{r^2}\right)\partial_\theta\Psi\right]+\frac{1}{\alpha r^2}\partial_z\left[\alpha r^2\left(\frac{1}{\alpha r^2}\right)\partial_\theta\Psi\right]=0. \tag{6.37}$$

Considering cylindrical harmonics, we can separate the radial part of equation (6.37), which has the form

$$\frac{1}{\alpha r^2}\frac{d}{dr}\left(\alpha r^2 f\right)\frac{dR(r)}{dr}+\left[\frac{\omega^2}{f}-\frac{F_{lm}}{\alpha^2 r^2}\right]R(r)=0. \tag{6.38}$$

As in the previous case we will find two solutions of the radial equation (6.38), one for the near-horizon and the other for the far horizon regime. In the case of the near-horizon region, we use the transformation $r \to f$, which implies

$$\frac{df}{dr}=(1-f)\frac{B(r_+)}{r_+}, \tag{6.39}$$

where

$$B(r_+)=1-\frac{4Q^2-2\alpha^4 r_+^4}{4M\alpha r_+-4Q^2}. \tag{6.40}$$

Using equation (6.39), equation (6.38) takes the form

$$f(1-f)\frac{d^2R(f)}{df^2}+(1-C_* f)\frac{dR(f)}{df}+\left[\frac{F_*^2}{B^2(r_+)\,(1-f)\,f}-\frac{F_{lm}}{B^2(r_+)\alpha^2\,(1-f)}\right]R(f)=0. \tag{6.41}$$

Here

$$F_*=\omega r_*, \quad C_*=2-\frac{2}{B(r_+)}. \tag{6.42}$$

In order to further simplify the above equation, we use field redefinition

$$R(f) = f^{\mu}\,(1-f)^{\nu}\,F(f). \tag{6.43}$$

In equation (6.41) we use this definition of $R(f)$ to obtain

$$f\,(1-f)\,\frac{d^2F(f)}{df^2} + \Big[1+2\mu-(2\mu+2\nu+C_*)\,f\Big]\frac{dF}{df} + \Big[\frac{\mu^2}{f} - \mu^2 + \mu - 2\mu\nu + \frac{\nu^2}{1-f} - \nu^2 - \frac{2\nu}{1-f} + \nu - \mu C_* + \frac{\nu C_*}{1-f} - \nu C_* + \frac{F_*^2}{B^2(r_+)f} + \frac{F_*^2}{B^2(r_+)\,(1-f)} - \left(\frac{\alpha^4 r_+^4 - 4\eta Q^2}{\alpha^4 r_+^4 + 4\eta Q^2}\right)\frac{F_{lm}}{B^2(r_+)\alpha^2\,(1-f)}\Big]F(f) = 0. \tag{6.44}$$

We again use the definitions given in (6.17). The constraints coming from the coefficients of $F(f)$ yield

$$\mu^2 + \frac{F_*^2}{B^2(r_+)} = 0, \tag{6.45}$$

and

$$\nu^2 + \nu(C_* - 2) + \frac{F_*^2}{B^2(r_+)} - \frac{F_{lm}}{B^2(r_+)\alpha^2} = 0. \tag{6.46}$$

These give the values of μ and ν as

$$\mu_{\pm} = \pm\iota\frac{F_*}{B(r_+)}, \tag{6.47}$$

$$\nu_{\pm} = \frac{1}{2}\left[(2-C_*) \pm \sqrt{(2-C_*)^2 - 4\left(\frac{F_*^2}{B^2(r_+)} - \frac{F_{lm}}{B^2(r_+)\alpha^2}\right)}\right]. \tag{6.48}$$

Equation (6.44) by virtue of the above constraints becomes

$$f\left(1-f\right)\frac{d^2F(f)}{df^2}+\left[c-\left(1+a+b\right)f\right]\frac{dF(f)}{df}-abF(f)=0. \tag{6.49}$$

For the near-horizon case there exists no outgoing mode, which means $\mu_+ = \mu_-$ and $\nu_+ = \nu_-$. So, in the near-horizon region the solution can be written in the form of the general hypergeometric function, being of the form

$$R(f)_{NH} = C_{1-}f^{\mu}(1-f)^{\nu}F\left(a,b,c;f\right), \tag{6.50}$$

where C_{1-} is an arbitrary constant. We now stretch the near horizon solution to an intermediate region [105, 109] so that

$$\begin{aligned} R(f)_{NH} = C_- f^{\mu}\left(1-f\right)^{\nu}\Bigg[&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F\left(a,b,a+b-c+1;1-f\right)+ \\ &(1-f)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F\left(c-a,c-b,c-a-b+1;1-f\right)\Bigg]. \end{aligned} \tag{6.51}$$

We again approximate $1-f$ for the case $r \gg r_+$, as before, and obtain the final form of the solution given in (6.29).

Now, as in the previous section, the radial equation for the far region reduces to the form of Bessel's equation, and the form of the final solution in this region is

$$R_{FR}\left(r\right) \simeq \frac{B_1\left(\frac{\omega\alpha r}{4}\right)^{\gamma}}{\sqrt{\omega\alpha r}\Gamma\left(\nu+1\right)} - \frac{B_2\Gamma\left(\gamma\right)}{\pi\sqrt{\omega\alpha r}\left(\frac{\omega\alpha r}{4}\right)^{\nu}}. \tag{6.52}$$

Using the same procedure as in the previous case, we find

$$B = \frac{B_1}{B_2} = -\frac{1}{\pi}\frac{1}{\left(\alpha\omega M\right)^{2l+1}}\frac{\Gamma(c-a-b)\Gamma(a)\Gamma(b)}{\Gamma(c-a)\Gamma(c-b)\Gamma(a)\Gamma(b)}\Gamma^2(l+1/2). \tag{6.53}$$

The absorption probability and hence greybody factor can be found by using the value of B in equation (6.35).

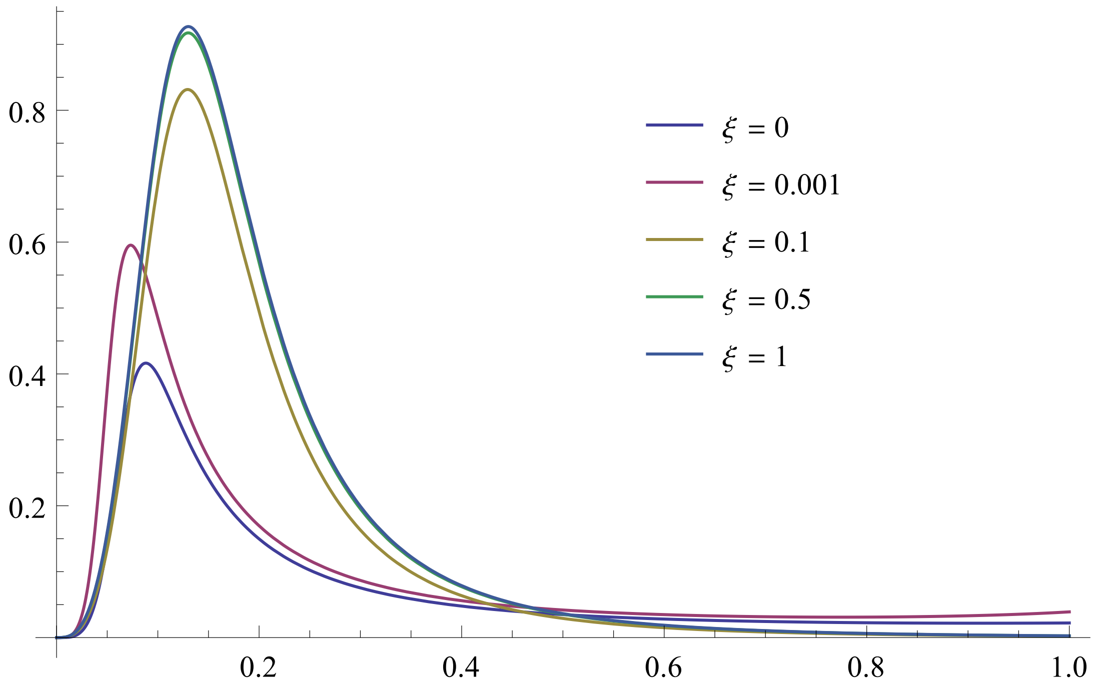

FIGURE 6.1: Greybody factor as a function of the frequency for $\xi = 0, 0.1, 0.001, 0.5, 1$ and for $l = 1$.

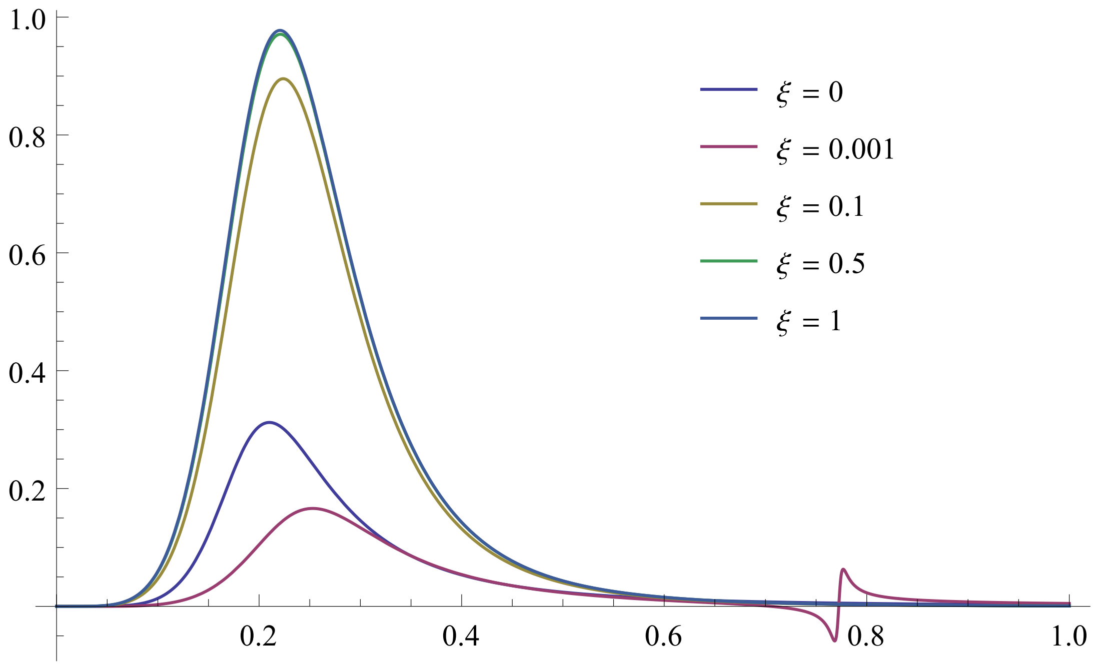

FIGURE 6.2: Greybody factor as a function of the frequency for $\xi = 0, 0.1, 0.001, 0.5, 1$ and for $l = 2$.

The effect of the coupling constant on the greybody factor is also analyzed graphically for different partial modes. In Figure 6.1, we have drawn the graph of greybody factor as a function frequency for different values of coupling constant and for $l = 1$. In Figure 6.2, the same is depicted for $l = 2$. It is observed that for different modes, stronger coupling enhances the absorption probability in low frequency approximation.

Chapter 7

Summary and Conclusion

We have constructed a complete set of quartic curvature theories of gravity. Under the restriction of spherical symmetry, the field equations of each of these theories reduce to the total derivative of a single metric function. In the case of four dimensions, we found that there are six generalized quasi-topological theories which have non-trivial contribution and these are given in equations (2.25)-(2.28) and (2.34). The equations of motion of these theories are in the form of total derivative of a polynomial of single metric function $f(r)$ and its first two derivatives.

In the case of dimensions five and higher, theories constructed here break up into the following categories:

1. Quartic Lovelock gravity: the explicit form of the Lagrangian for this class is given by the eight dimensional Euler density $\mathcal{X}_8$. An interesting aspect of theories of this class is that the equations of motion are always of second order. Furthermore, if we impose the restriction of spherical symmetry, the equation of motion will be unique and in the form of a total derivative of a single metric function.

2. Six quasi-topological theories, with Lagrangians given by equations (2.19) and (2.20). One of these, given in equation (2.19), is already known [61]; the remaining five given in equation (2.20) are new. For all these theories, in a general background, the equations of motion will be different and are of fourth order. On the other hand, if we impose the condition of spherical symmetry, the equations of motion are of second order and contributions of each of six Lagrangians coincide. This is due to the fact that the Lagrangians are equivalent up to the terms which vanish for spherically symmetric metrics.

3. Four generalized quasi-topological theories are found whose Lagrangians are given by equations (2.25)-(2.28). For this quartet, if we impose the condition of spherical symmetry, field equation will be same and in the form of a total derivative of a polynomial of single metric function f and its first and second derivatives.

4. The Lagrangians for six theories, whose field equations vanish when one sets N as constant, are given by equation (2.24). For situations where the stress-energy tensor has $T_t^t \neq T_r^r$, there will be two non-trivial field equations that determine N and f.

We have presented a generalized charged anti-de Sitter black hole solution for cubic quasi-topological gravity and also elaborated its thermodynamic aspect.

Furthermore, we have derived the analytic expression of greybody factor for non-minimally coupled scalar fields from Reissner-Nordström-de Sitter black hole in low energy approximation. This expression is valid for general, partial modes. For coupling to scalar curvature, which can be regarded as mass or charge terms, greybody factor tend to zero in low frequency regime, irrespective of the values of the coupling parameter. Non-zero greybody factor in low frequency regime means that there is non-zero Hawking emission rate of Hawking radiations. The matching technique is used in deriving formula for greybody factor. The significance of the results is elaborated by giving formulae of differential rate of energy and generalized absorption cross section from the greybody factor.

The results of the present study reduce to those of Ref. [101] in appropriate limiting case, i.e., if we put charge $Q = 0$, we recover the previously reported results. The effective potential and greybody factor are also analyzed graphically. We observe that the height of gravitational barrier increases with the increase of ξ, the coupling parameter, whereas in the absence of the coupling parameter, it is decreased by increasing the values of the cosmological constant. Also, from the plots of greybody factor, it is observed that an increase in the value of the coupling parameter decreases the greybody factor. This is due to the fact that non-minimal coupling plays the role of effective mass and hence suppresses the greybody factor.

In the previous chapter, we have presented a study of the greybody factor for a scalar field which are coupled to the Einstein tensor in the background of a charged black string, considering low energy approximation. We demonstrated that the greybody factor depends on the coupling between Einstein tensor and scalar field. It is observed that the presence of coupling enhances the greybody factor of the scalar field in the black string spacetime. Furthermore, for weaker coupling, greybody factor decreases with increase in charge of black string. In the second case, we discussed this work without considering coupling of scalar field and the Einstein tensor. It is trivial from results that the later case reduces to the result of former in absence of coupling constant.

In the case of three-dimensional topological black holes [90, 120] like the charged BTZ (Bañados , Teitelboim, Zanelli) black holes, we find the propagation of scalar fields with non-minimal coupling to gravity obeys the Universality theorem. This means that under the restrictions of zero angular-momentum, low energy regime, massless/chargeless scalar field and minimal coupling, the greybody factor approaches to a constant value. However, Universality theorem does not hold for zero angular-momentum if any of the above restrictions is relaxed.

We have thus explored theories in several aspects. Consideration of linearized spectrum of these theories revealed that on a constant curvature background, it is only the massless graviton that is propagated by these theories. We also have found the explicit forms of field equations of these theories in general spacetime dimension d which are valid for spherical symmetric background. Also, explicit form of black hole entropy in general spacetime dimension is presented. The consequence of this particular result is very interesting; for the case of black brane solutions, it modifies the usual Bekenstein-Hawking area law. It was observed previously that this aspect was not seen before for theories like Lovelock and quasi-topological gravities. Therefore, holographic consideration of these generalized quasi-topological theories may have interesting implications. Furthermore, we have found four dimensional asymptotic, flat, black solutions for these theories. This solution revealed that it is characterized only by mass, implying that it does not give rise to higher derivative "hair". We have presented perturbative and numerical solutions, but interestingly, thermodynamics can be studied analytically. In this regard, we found that first law of black hole thermodynamics holds. We presented black brane solutions of these theories in general d dimensions. Expected thermodynamic relations for a CFT (without chemical potential) are satisfied by these solutions, living in one dimension less. We also found the peculiar thermodynamical behaviour of these black brane solutions which is in contrast to the corresponding black brane solutions in Lovelock and quasi-topological gravities. For this reason, this result may have interesting consequences in holographic studies.

This class of theories (which has now been constructed to cubic [8, 65] and quartic order) provides interesting generalizations of Einstein's gravity that are non-trivial in four (and higher) dimensions. This contrasts with previous constructions of Lovelock and quasi-topological gravities, which vanish on four dimensional (spherically symmetric) metrics. The generalized quasi-topological terms can be thought of as the theories which have many of the interesting properties observed for Einsteinian cubic gravity [65] in four dimensions [7, 81], but in higher dimensions and/or to higher orders in the curvature. These theories necessarily [66] propagate only a massless, transverse graviton on a constant curvature spacetime. Furthermore, they admit black hole solutions which are characterized only by their mass. The thermodynamics of the black holes can be studied exactly, despite the lack of an exact, analytic solution to the field equations.

Construction of these theories has opened many problems which deserve further study. These problems include further investigations of the properties of four and higher dimensional black hole solutions in these theories. Also, as we know that the Birkhoff theorem holds for Lovelock and quasi-topological gravities [71, 72, 121]; it would be interesting to see whether this is the case for these theories. More interestingly, these theories seem well-suited for holographic study and therefore can serve as a good toy model in such investigations. A study in the context of holography could better shed further light on the stability of the solutions of these theories and the allowed values of coupling constants, and may reveal novel features in the case of black brane solutions of the theory. An ambitious undertaking would be to elucidate the general structure of the Lagrangians in this class of theories. This has been long known in the case of Lovelock gravity [5] but remains an open problem in the (generalized) quasi-topological cases.

Appendices

Appendix A

The Constraints in General Dimensions

We get the following values of constraints on c_{12}, c_{17}, c_{19}, c_{20}, c_{21}, c_{22}, c_{23}, c_{24} and c_{25} in order to ensure that condition (2.5) is met for the quartic action given by equation (2.13) in dimensions d larger than four.

$$
\begin{aligned}
c_{12} = &-\frac{(19-40d+38d^2-15d^3+2d^4)}{3(3d-2)(-22+26d-9d^2+d^3)}c_1 - \frac{(2-69d+83d^2-32d^3+4d^4)}{3(3d-2)(-22+26d-9d^2+d^3)}c_2 \\
&- \frac{(13-2d-4d^2+d^3)}{3(3d-2)(-22+26d-9d^2+d^3)}c_3 - \frac{4(d-2)(-2+22d-13d^2+2d^3)}{3(3d-2)(-22+26d-9d^2+d^3)}c_4 \\
&- \frac{8(d-2)(-2+22d-13d^2+2d^3)}{3(3d-2)(-22+26d-9d^2+d^3)}c_5 - \frac{8(d-3)(-1+6d-5d^2+d^3)}{3(3d-1)(-22+26d-9d^2+d^3)}c_6 \\
&- \frac{(d-4)(d-3)(d-1)^2}{2(3d-2)(-22+26d-9d^2+d^3)}c_{10} - \frac{8(d-3)(d-2)^2(2d-1)}{3(3d-2)(-22+26d-9d^2+d^3)}c_7 \\
&- \frac{(d-2)(1-7d+2d^2)}{4(3d-2)(-22+26d-9d^2+d^3)}c_8 - \frac{(5-28d+27d^2-9d^3+d^4)}{(3d-2)(-22+26d-9d^2+d^3)}c_9 \\
&- \frac{(16-15d+3d^2)}{4(-22+26d-9d^2+d^3)}c_{11}
\end{aligned} \tag{A.1}
$$

$$
\begin{aligned}
c_{17} = &-\frac{(-200+430d+566d^2-2677d^3+3194d^4-1807d^5+524d^6-74d^7+4d^8)}{2(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_1 \\
&-\frac{(272-1572d+4104d^2-5617d^3+4420d^4-2042d^5+536d^6-73d^7+4d^8)}{(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_2 \\
&+\frac{(336-1418d+2520d^2-2107d^3+885d^4-168d^5+7d^6+d^7)}{2(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_3 \\
&-\frac{4(148-790d+1986d^2-2683d^3+2126d^4-991d^5+262d^6-36d^7+2d^8)}{(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_4 \\
&-\frac{8(148-790d+1986d^2-2683d^3+2126d^4-991d^5+262d^6-36d^7+2d^8)}{(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_5 \\
&-\frac{2(136-988d+3086d^2-4784d^3+4079d^4-1975d^5+531d^6-73d^7+4d^8)}{(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_6 \\
&-\frac{8(20+58d-134d^2+74d^3-15d^4+d^5)}{(d-2)(3d-2)(-22+26d-9d^2+d^3)}c_7 \\
&+\frac{(24+36d-296d^2+683d^3-698d^4+368d^5-94d^6+9d^7)}{4(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_8 \\
&-\frac{3(40-546d+2232d^2-3935d^3+3633d^4-1870d^5+538d^6-81d^7+5d^8)}{2(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_9 \\
&-\frac{(-240+184d+2538d^2-7062d^3+7893d^4-4550d^5+1418d^6-228d^7+15d^8)}{4(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_{10} \\
&+\frac{3(d-4)(d-3)(d-1)^2(-1+3d)}{2(d-2)^2(2d-1)(-22+26d-9d^2+d^3)}c_{11}-\frac{(-4-13d+39d^2-24d^3+4d^4)}{4(d-2)^2(d-1)(2d-1)}c_{13} \\
&-\frac{(d-3)d^2}{4(d-2)^2(d-1)}c_{14}-\frac{(-2-8d+23d^2-13d^3+2d^4)}{2(d-2)^2(d-1)(2d-1)}c_{15} \\
&-\frac{(2+7d-9d^2+2d^3)}{2(d-2)^2(2d-1)}c_{16}-\frac{(d-4)(d-1)(-1+3d)}{2(d-2)^2(2d-1)}c_{18}
\end{aligned} \tag{A.2}
$$

$$
\begin{aligned}
c_{19} = {} & \frac{(-344 + 2086d - 4878d^2 + 5109d^3 - 2618d^4 + 590d^5 - 10d^6 - 17d^7 + 2d^8)}{4(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_1 \\
& + \frac{(-1000 + 4684d - 7926d^2 + 6691d^3 - 2963d^4 + 595d^5 - d^6 - 18d^7 + 2d^8)}{2(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_2 \\
& + \frac{(552 - 2214d + 2702d^2 - 1489d^3 + 358d^4 - 14d^5 - 8d^6 + d^7)}{4(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_3 \\
& + \frac{(-1096 + 5100d - 8504d^2 + 7072d^3 - 3046d^4 + 579d^5 + 8d^6 - 19d^7 + 2d^8)}{(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_4 \\
& + \frac{2(-1096 + 5100d - 8504d^2 + 7072d^3 - 3046d^4 + 579d^5 + 8d^6 - 19d^7 + 2d^8)}{(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_5 \\
& + \frac{2(-448 + 2150d - 3790d^2 + 3299d^3 - 1488d^4 + 302d^5 - d^6 - 9d^7 + d^8)}{(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_6 \\
& + \frac{2(368 - 580d + 262d^2 - 30d^3 - 5d^4 + d^5)}{(d-2)(3d-2)(-22+26d-9d^2+d^3)} c_7 \\
& + \frac{(360 - 1388d + 1568d^2 - 593d^3 - 124d^4 + 146d^5 - 36d^6 + 3d^7)}{8(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_8 \\
& + \frac{3(-776 + 3662d - 6388d^2 + 5545d^3 - 2543d^4 + 576d^5 - 38d^6 - 7d^7 + d^8)}{4(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_9 \\
& + \frac{(-1152 + 6104d - 12474d^2 + 12330d^3 - 6321d^4 + 1592d^5 - 128d^6 - 18d^7 + 3d^8)}{8(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_{10} \\
& + \frac{3(d-3)(16 - 49d + 41d^2 - 11d^3 + d^4)}{4(d-2)^2(2d-1)(-22+26d-9d^2+d^3)} c_{11} + \frac{(-11 + 25d - 14d^2 + 2d^3)}{4(d-2)^2(d-1)(2d-1)} c_{13} \\
& + \frac{(d-3)d}{4(d-2)^2(d-1)} c_{14} + \frac{(-7 + 16d - 8d^2 + d^3)}{2(d-2)^2(d-1)(2d-1)} c_{15} + \frac{(d-4)(d-1)}{2(d-2)^2(2d-1)} c_{16} \\
& - \frac{(-8 + 17d - 6d^2 + d^3)}{4(d-2)^2(2d-1)} c_{18}
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
c_{20} = &-\frac{8(66-106d-27d^2+99d^3-52d^4+8d^5)}{(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_1 \\
&-\frac{8(-220+638d-716d^2+427d^3-133d^4+16d^5)}{(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_2 \\
&-\frac{4(264-634d+494d^2-175d^3+23d^4)}{(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_3 \\
&-\frac{32(-132+362d-384d^2+227d^3-69d^4+8d^5)}{(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_4 \\
&-\frac{64(-132+362d-384d^2+227d^3-69d^4+8d^5)}{(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_5 \\
&-\frac{16(-132+422d-548d^2+373d^3-127d^4+16d^5)}{(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_6 \\
&-\frac{256(d-3)(d-1)d}{(3d-2)(-22+26d-9d^2+d^3)}c_7-\frac{4(44-80d+26d^2+13d^3-14d^4+3d^5)}{(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_8 \\
&-\frac{12(-44+116d-148d^2+116d^3-41d^4+5d^5)}{(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_9 \\
&-\frac{2(176-416d+242d^2+22d^3-53d^4+9d^5)}{(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_{10} \\
&-\frac{24(d-3)(d-1)^2d}{(d-2)(2d-1)(-22+26d-9d^2+d^3)}c_{11}+\frac{2(-5+2d)}{(d-2)(2d-1)}c_{13} \\
&+\frac{4}{d-2}c_{14}-\frac{8}{(d-2)(2d-1)}c_{15}+\frac{4(-3+2d)}{(d-2)(2d-1)}c_{16}+\frac{8(d-1)d}{(d-2)(2d-1)}c_{18}
\end{aligned}
\tag{A.4}
$$

$$\begin{aligned}
c_{21} = & \frac{4(-96+509d-1068d^2+1031d^3-516d^4+141d^5-23d^6+2d^7)}{(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_1 \\
& + \frac{4(-332+1560d-2593d^2+2194d^3-1041d^4+288d^5-48d^6+4d^7)}{(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_2 \\
& + \frac{4(64-253d+246d^2-101d^3+20d^4-5d^5+d^6)}{(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_3 \\
& + \frac{16(-180+850d-1408d^2+1191d^3-559d^4+153d^5-25d^6+2d^7)}{(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_4 \\
& + \frac{32(-180+850d-1408d^2+1191d^3-559d^4+153d^5-25d^6+2d^7)}{(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_5 \\
& + \frac{32(-80+381d-657d^2+566d^3-268d^4+73d^5-12d^6+d^7)}{(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_6 \\
& + \frac{32(d-3)(-24+26d-5d^2+d^3)}{(3d-2)(-22+26d-9d^2+d^3)}c_7 \\
& + \frac{(116-464d+557d^2-326d^3+115d^4-36d^5+6d^6)}{(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_8 \\
& + \frac{12(-148+689d-1170d^2+989d^3-452d^4+115d^5-16d^6+d^7)}{(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_9 \\
& + \frac{2(-256+1284d-2475d^2+2301d^3-1132d^4+304d^5-45d^6+3d^7)}{(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_{10} \\
& + \frac{3(-28+78d-51d^2+3d^3+2d^4)}{(d-2)(2d-1)(-22+26d-9d^2+d^3)}c_{11} - \frac{2(5-10d+4d^2)}{(d-2)(d-1)(2d-1)}c_{13} \\
& - \frac{2d}{(d-2)(d-1)}c_{14} - \frac{4(3-6d+2d^2)}{(d-2)(d-1)(2d-1)}c_{15} - \frac{8(d-1)}{(d-2)(2d-1)}c_{16} \\
& - \frac{4(d-1+d^2)}{(d-2)(2d-1)}c_{18}
\end{aligned} \tag{A.5}$$

$$
\begin{aligned}
c_{22} = &\frac{(-9504 + 28040d - 26710d^2 + 7806d^3 + 2763d^4 - 2722d^5 + 1012d^6 - 222d^7 + 15d^8 + 2d^9)}{12(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_1 \\
&+ \frac{(11616 - 38016d + 46080d^2 - 27850d^3 + 10107d^4 - 3153d^5 + 1051d^6 - 235d^7 + 14d^8 + 2d^9)}{6(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_2 \\
&+ \frac{(-15840 + 49168d - 53582d^2 + 25858d^3 - 5241d^4 + 356d^5 - 62d^6 + 14d^7 + d^8)}{12(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_3 \\
&+ \frac{(14784 - 47304d + 55236d^2 - 31612d^3 + 10792d^4 - 3404d^5 + 1165d^6 - 248d^7 + 13d^8 + 2d^9)}{3(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_4 \\
&+ \frac{2(14784 - 47304d + 55236d^2 - 31612d^3 + 10792d^4 - 3404d^5 + 1165d^6 - 248d^7 + 13d^8 + 2d^9)}{3(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_5 \\
&+ \frac{2(3168 - 10564d + 13436d^2 - 9026d^3 + 4002d^4 - 1539d^5 + 537d^6 - 118d^7 + 7d^8 + d^9)}{3(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_6 \\
&+ \frac{2(d-2)d(-264 + 102d - 48d^2 + 13d^3 + d^4)}{3(3d-2)(-22+26d-9d^2+d^3)} c_7 \\
&+ \frac{(-1760 + 4280d - 2348d^2 - 960d^3 + 1127d^4 - 150d^5 - 76d^6 + 14d^7 + d^8)}{8(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_8 \\
&+ \frac{(1408 - 1224d - 3726d^2 + 4544d^3 - 245d^4 - 1603d^5 + 784d^6 - 134d^7 + 3d^8 + d^9)}{4(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_9 \\
&+ \frac{(-10560 + 34480d - 39088d^2 + 17370d^3 + 582d^4 - 3803d^5 + 1726d^6 - 338d^7 + 12d^8 + 3d^9)}{24(d-2)(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)} c_{10} \\
&+ \frac{d(-220 + 309d - 68d^2 - 38d^3 + 8d^4 + d^5)}{4(d-2)(2d-1)(-22+26d-9d^2+d^3)} c_{11} \\
&- \frac{(72 - 77d - d^2 + 11d^3 + d^4)}{6(d-2)(d-1)(2d-1)} c_{13} - \frac{1}{d-1} c_{14} - \frac{(60 - 47d - 13d^2 + 11d^3 + d^4)}{6(d-2)(d-1)(2d-1)} c_{15} \\
&- \frac{(d+7)(-12+5d+d^2)}{6(d-2)(2d-1)} c_{16} - \frac{d(d+7)(-12+5d+d^2)}{12(d-2)(2d-1)} c_{18} - 2(d-2)d c_{26} \qquad \text{(A.6)}
\end{aligned}
$$

$$
\begin{aligned}
c_{23} = &-\frac{(7104-35088d+67484d^2-52018d^3+8628d^4+11187d^5-7810d^6+2236d^7-288d^8+3d^9+2d^{10})}{24(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_1\\
&-\frac{(13824-65760d+107232d^2-80268d^3+21812d^4+7485d^5-7491d^6+2305d^7-295d^8+2d^9+2d^{10})}{12(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_2\\
&-\frac{(-6144+28032d-37964d^2+27334d^3-11888d^4+2799d^5-64d^6-98d^7+8d^8+d^9)}{24(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_3\\
&-\frac{(15168-73200d+120720d^2-92556d^3+26636d^4+7372d^5-8018d^6+2449d^7-302d^8+d^9+2d^{10})}{6(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_4\\
&-\frac{(15168-73200d+120720d^2-92556d^3+26636d^4+7372d^5-8018d^6+2449d^7-302d^8+d^9+2d^{10})}{3(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_5\\
&-\frac{(6624-31560d+52724d^2-39676d^3+10258d^4+4194d^5-3873d^6+1167d^7-148d^8+d^9+d^{10})}{3(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_6\\
&-\frac{(-5856+7872d-1872d^2-1008d^3+630d^4-128d^5+5d^6+d^7)}{3(d-2)(3d-2)(-22+26d-9d^2+d^3)}c_7\\
&-\frac{(-1856+8016d-10784d^2+6260d^3-618d^4-1205d^5+722d^6-160d^7+8d^8+d^9)}{16(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_8\\
&-\frac{(12096-55344d+88108d^2-60318d^3+9758d^4+10317d^5-6659d^6+1620d^7-152d^8-3d^9+d^{10})}{8(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_9\\
&-\frac{(20352-98400d+177616d^2-139276d^3+34518d^4+16968d^5-14063d^6+3850d^7-410d^8-6d^9+3d^{10})}{48(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_{10}\\
&-\frac{(672-2232d+2438d^2-1299d^3+448d^4-86d^5+2d^6+d^7)}{8(d-2)^2(2d-1)(-22+26d-9d^2+d^3)}c_{11}\\
&+\frac{(-96+168d-29d^2-31d^3+5d^4+d^5)}{12(d-2)^2(d-1)(2d-1)}c_{13}\\
&+\frac{(d-4)d}{2(d-2)^2(d-1)}c_{14}+\frac{(-120+222d-41d^2-31d^3+5d^4+d^5)}{12(d-2)^2(d-1)(2d-1)}c_{15}\\
&+\frac{(72-42d-25d^2+6d^3+d^4)}{12(d-2)^2(2d-1)}c_{16}+\frac{(96-168d+66d^2-49d^3+6d^4+d^5)}{24(d-2)^2(2d-1)}c_{18}\\
&+(12-6d+d^2)c_{26} \qquad \text{(A.7)}
\end{aligned}
$$

$$
\begin{aligned}
c_{24} = &-\frac{(1768-5050d+5912d^2-3707d^3+1391d^4-285d^5+17d^6+2d^7)}{3(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_1 \\
&-\frac{2(-560-376d+2678d^2-2965d^3+1408d^4-299d^5+16d^6+2d^7)}{3(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_2 \\
&-\frac{(1696-3274d+2116d^2-371d^3-63d^4+15d^5+d^6)}{3(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_3 \\
&-\frac{4(-920+196d+2624d^2-3272d^3+1548d^4-313d^5+15d^6+2d^7)}{3(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_4 \\
&-\frac{8(-920+196d+2624d^2-3272d^3+1548d^4-313d^5+15d^6+2d^7)}{3(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_5 \\
&-\frac{8(36-888d+1898d^2-1668d^3+727d^4-150d^5+8d^6+d^7)}{3(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_6 \\
&-\frac{8(d-2)(-168+190d-88d^2+13d^3+d^4)}{3(3d-2)(-22+26d-9d^2+d^3)}c_7 \\
&-\frac{(120+44d-416d^2+417d^3-149d^4+15d^5+d^6)}{2(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_8 \\
&-\frac{(632-3322d+5142d^2-3529d^3+1162d^4-170d^5+4d^6+d^7)}{(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_9 \\
&-\frac{(2544-8320d+10510d^2-6760d^3+2395d^4-419d^5+15d^6+3d^7)}{6(d-2)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_{10} \\
&-\frac{(d-1)(-4+55d-43d^2+5d^3+d^4)}{(d-2)(2d-1)(-22+26d-9d^2+d^3)}c_{11} \\
&+\frac{2(-15+8d+d^2)}{3(d-2)(2d-1)}c_{13}+\frac{2(-15+8d+d^2)}{3(d-2)(2d-1)}c_{15} \\
&+\frac{2(-15+8d+d^2)}{3(d-2)(2d-1)}c_{16}+\frac{d(-15+8d+d^2)}{3(d-2)(2d-1)}c_{18}+8(d-2)c_{26} \qquad \text{(A.8)}
\end{aligned}
$$

$$
\begin{aligned}
c_{25} = &+\frac{(3600-16408d+31034d^2-30162d^3+16863d^4-5794d^5+1228d^6-114d^7-9d^8+2d^9)}{12(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_1 \\
&+\frac{(912-7080d+15960d^2-18454d^3+12579d^4-5229d^5+1243d^6-115d^7-10d^8+2d^9)}{6(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_2 \\
&+\frac{(1824-6056d+9634d^2-7274d^3+2439d^4-184d^5-50d^6+2d^7+d^8)}{12(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_3 \\
&+\frac{(480-5880d+14820d^2-18556d^3+13312d^4-5672d^5+1333d^6-116d^7-11d^8+2d^9)}{3(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_4 \\
&+\frac{2(480-5880d+14820d^2-18556d^3+13312d^4-5672d^5+1333d^6-116d^7-11d^8+2d^9)}{3(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_5 \\
&+\frac{2(888-5176d+10700d^2-11444d^3+7212d^4-2805d^5+639d^6-58d^7-5d^8+d^9)}{3(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_6 \\
&+\frac{2(-1248+1968d-1116d^2+366d^3-62d^4-d^5+d^6)}{3(d-2)(3d-2)(-22+26d-9d^2+d^3)}c_7 \\
&+\frac{(-32+600d-1476d^2+1936d^3-1469d^4+614d^5-112d^6+2d^7+d^8)}{8(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_8 \\
&+\frac{(2448-12968d+25506d^2-25744d^3+14727d^4-4859d^5+840d^6-38d^7-9d^8+d^9)}{4(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_9 \\
&+\frac{(6432-30800d+58832d^2-57438d^3+31686d^4-10223d^5+1810d^6-86d^7-24d^8+3d^9)}{24(d-2)^2(d-1)(2d-1)(3d-2)(-22+26d-9d^2+d^3)}c_{10} \\
&+\frac{(d-3)(-16+12d+37d^2-29d^3-d^4+d^5)}{4(d-2)^2(2d-1)(-22+26d-9d^2+d^3)}c_{11} \\
&-\frac{(-33+61d-25d^2-d^3+d^4)}{6(d-2)^2(d-1)(2d-1)}c_{13} \\
&+\frac{d}{2(d-2)^2(d-1)}c_{14}-\frac{(-36+67d-25d^2-d^3+d^4)}{6(d-2)^2(d-1)(2d-1)}c_{15}-\frac{(30-25d+d^3)}{6(d-2)^2(2d-1)}c_{16} \\
&-\frac{(d-1)d(-36+d+d^2)}{12(d-2)^2(2d-1)}c_{18}-2dc_{26}
\end{aligned}
\tag{A.9}
$$

Appendix B

Quasi-Topological Lagrangian Densities

In this appendix we provide a list of the explicit forms of the quasi-topological Lagrangian densities, which are obtained by the choices made in equations (2.19) and (2.20).

$$
\begin{aligned}
\mathcal{Z}_d^{(1)} \quad &= 16(d-2)(244-451d+306d^2-91d^3+10d^4)R_a{}^cR^{ab}R_b{}^dR_{cd} \\
&-64(d-2)(7-5d+d^2)(14-14d+3d^2)R_a{}^cR^{ab}R_{bc}R \\
&+8(-388+931d-856d^2+379d^3-82d^4+7d^5)R_{ab}R^{ab}R^2 \\
&+(-980+1683d-1060d^2+302d^3-36d^4+d^5)R^4 \\
&-32(d-4)^2(d-2)^2(14-14d+3d^2)R^{ab}R^{cd}RR_{acbd} \\
&+2(2764-6289d+5788d^2-2776d^3+736d^4-103d^5+6d^6)R^2R_{abcd}R^{abcd} \\
&+64(d-3)(d-2)^2(-58+75d-30d^2+4d^3)R_a{}^cR^{ab}R^{de}R_{bdce} \\
&-48(d-3)(d-2)(4-31d+37d^2-15d^3+2d^4)R^{ab}R^{cd}R_{ac}{}^{ef}R_{bdef} \\
&+16(d-2)^3(274-389d+183d^2-34d^3+2d^4)R^{ab}R^{cd}R_a{}^e{}_b{}^fR_{cedf} \\
&-4(d-4)(118-596d+876d^2-581d^3+195d^4-32d^5+2d^6)R_{ab}R^{ab}R_{cdef}R^{cdef} \\
&+16(d-4)(d-3)(d-2)(d-1)(14-14d+3d^2)R^{ab}R_a{}^{cde}R_{bc}{}^{fh}R_{defh} \\
&-(d-2)(1108-2723d+2639d^2-1224d^3 \\
&+235d^4+10d^5-10d^6+d^7)R_{ab}{}^{ef}R^{abcd}R_{cd}{}^{hi}R_{efhi}+8(d-2)(860-2113d \\
&+1959d^2-810d^3+102d^4+30d^5-11d^6+d^7)R_a{}^e{}_c{}^fR^{abcd}R_b{}^h{}_d{}^iR_{ehfi} \\
&+(-1292+2929d-2741d^2+1527d^3-684d^4+276d^5-82d^6 \\
&+14d^7-d^8)R_{abcd}R^{abcd}R_{efhi}R^{efhi}
\end{aligned} \tag{B.1}
$$

$$
\begin{aligned}
\mathcal{Z}_d^{(2)} \;=\; & \frac{1}{(d-4)(d-2)^3(3d-4)(11-6d+d^2)(-4+14d-7d^2+d^3)(-22+26d-9d^2+d^3)} \\
& \times\big[(d-4)(d^3-9d^2+26d-22)(d-1)(2d^8-36d^7+264d^6-969d^5+1486d^4+1289 \\
& \times d^3-8530d^2+11948d-5632)R^{ab}R_{ab}R^{cd}R_{cd}-(d-2)(-22+26d-9d^2+d^3)(3840 \\
& -9872d+13772d^2-12446d^3+6133d^4-795d^5-639d^6+327d^7-60d^8+4d^9)R_a{}^cR^{ab} \\
& \times R_b{}^dR_{cd}+(d-1)(d-4)(-22+26d-9d^2+d^3)(-5632+11948d-8530d^2+1289d^3 \\
& +1486d^4-969d^5+264d^6-36d^7+2d^8)R_a{}^cR^{ab}R_b{}^dR_{cd}+\frac{4}{3}(d-2)(92672-459640d \\
& +851460d^2-741570d^3+245584d^4+91339d^5-122856d^6+51524d^7-10130d^8+451 \\
& \times d^9+192d^{10}-36d^{11}+2d^{12})R_a{}^cR^{ab}R_{bc}R-(-19968+129856d-351080d^2+486664 \\
& \times d^3-350864d^4+91452d^5+48784d^6-50566d^7+18113d^8-2536d^9-243d^{10}+143d^{11} \\
& -20d^{12}+d^{13})R_{ab}R^{ab}R^2+\frac{1}{24}(385024-1950016d+3753760d^2-3555864d^3+1582172 \\
& \times d^4-4394d^5-370858d^6+206017d^7-59436d^8+10909d^9-1522d^{10}+195d^{11}-20d^{12} \\
& +d^{13})R^4+4(d-2)^2(-36480+97652d-90614d^2+16524d^3+30278d^4-24508d^5 \\
& +6916d^6+152d^7-625d^8+173d^9-21d^{10}+d^{11})RR^{ab}R^{cd}R_{acbd}+\frac{1}{4}(d-2)(-328704 \\
& +1158096d-1701488d^2+1286084d^3-430702d^4-82374d^5+157229d^6-79874d^7 \\
& +23397d^8-4346d^9+507d^{10}-34d^{11}+d^{12})R^2R_{abcd}R^{abcd}-2(d-2)^3(-28032+87822d \\
& -112640d^2+71315d^3-16827d^4-6654d^5+6558d^6-2329d^7+447d^8-46d^9+2d^{10})R^{ab} \\
& \times RR_a{}^{cde}R_{bcde}-8(d-2)^2(-22+26d-9d^2+d^3)(-64+1592d-2909d^2+1743d^3-58d^4 \\
& -371d^5+167d^6-30d^7+2d^8)R_a{}^cR^{ab}R^{de}R_{bdce}+(d-2)^3(-22+26d-9d^2+d^3)(1024 \\
& -3308d+2725d^2+210d^3-1190d^4+570d^5-111d^6+8d^7)R^{ab}R^{cd}R_{ac}{}^{ef}R_{bdef}+\frac{1}{3}(d-2)^3 \\
& \times(1792-3743d+2678d^2-531d^3-247d^4+150d^5-29d^6+2d^7)(-4+14d-7d^2+d^3) \\
& \times RR_{ab}{}^{ef}R^{abcd}R_{cdef}+\frac{1}{2}(d-2)(-22+26d-9d^2+d^3)(9216-31760d+41152d^2-22702 \\
& \times d^3+914d^4+5611d^5-3201d^6+839d^7-111d^8+6d^9)R_{ab}R^{ab}R_{cdef}R^{cdef}\big] \\
& -\frac{2(-8-23d+39d^2-16d^3+2d^4)}{(d-4)(3d-4)(11-6d+d^2)}R^{ab}R_a{}^{cde}R_{bc}{}^{fh}R_{defh}+R_a{}^e{}_c{}^fR^{abcd}R_b{}^h{}_e{}^jR_{dhfj} \qquad \text{(B.2)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Z}_d^{(3)} = &\frac{1}{12(-4+d)(-2+d)^3(-4+3d)(11-6d+d^2)(-22+26d-9d^2+d^3)(-4+14d-7d^2+d^3)} \\
&\times \big[-24(-2+d)(-22+26d-9d^2+d^3)(-3408+9452d-13070d^2+12869d^3 \\
&-9751d^4+5409d^5-2053d^6+496d^7-68d^8+4d^9)R_a{}^cR^{ab}R_b{}^dR_{cd} \\
&+24(-4+d)(-3+d)(-22+26d-9d^2+d^3)(1716-5894d+8839d^2-7538d^3 \\
&+4008d^4-1364d^5+291d^6-36d^7+2d^8)R_{ab}R^{ab}R_{cd}R^{cd}+32(-2+d)(-71704 \\
&+400996d-956122d^2+1301340d^3-1128581d^4+652069d^5-251257d^6 \\
&+60923d^7-7184d^8-444d^9+290d^{10}-40d^{11}+2d^{12})R_a{}^cR^{ab}R_{bc}R-24(15680 \\
&-106664d+323592d^2-568168d^3+638164d^4-479674d^5+243364d^6-80096d^7 \\
&+14246d^8+229d^9-800d^{10}+196d^{11}-22d^{12}+d^{13})R_{ab}R^{ab}R^2+(-302144 \\
&+1720608d-4189176d^2+5863660d^3-5304058d^4+3284002d^5-1431861d^6 \\
&+445160d^7-99552d^8+16457d^9-2171d^{10}+248d^{11}-22d^{12}+d^{13})R^4 \\
&+48(-2+d)^2(70472-240892d+359520d^2-299804d^3+143976d^4-30793d^5 \\
&-6594d^6+7094d^7-2428d^8+453d^9-46d^{10}+2d^{11})R^{ab}R^{cd}RR_{acbd} \\
&+6(-2+d)(316048-1340112d+2613908d^2-3095774d^3+2474698d^4 \\
&-1403521d^5+577724d^6-173518d^7+37673d^8-5759d^9+588d^{10} \\
&-36d^{11}+d^{12})R^2R_{abcd}R^{abcd}-48(-2+d)^3(25270-92828d+156501d^2 \\
&-158736d^3+107067d^4-50145d^5+16490d^6-3749d^7+562d^8-50d^9 \\
&+2d^{10})R^{ab}RR_a{}^{cde}R_{bcde}-192(-2+d)^2(-22+26d-9d^2+d^3)(-40-1069d \\
&+3085d^2-3689d^3+2463d^4-1001d^5+247d^6-34d^7+2d^8)R_a{}^cR^{ab}R^{de}R_{bdce} \\
&+24(-2+d)^3(-1+d)(-22+26d-9d^2+d^3)(988-3197d+3661d^2-2141d^3 \\
&+689d^4-116d^5+8d^6)R^{ab}R^{cd}R_{ac}{}^{ef}R_{bdef}+4(-2+d)^3(-4+14d \\
&-7d^2+d^3)(-1654+4528d-5595d^2+4003d^3-1751d^4+459d^5-66d^6 \\
&+4d^7)RR_{ab}{}^{ef}R^{abcd}R_{cdef}+12(-2+d)(-22+26d-9d^2+d^3)(-8144+33248d \\
&-60534d^2+64054d^3-43371d^4+19504d^5-5824d^6+1112d^7 \\
&-123d^8+6d^9)R_{ab}R^{ab}R_{cdef}R^{cdef}-24(-2+d)^3(-22+26d-9d^2+d^3)(-4 \\
&+14d-7d^2+d^3)(122-207d+130d^2-37d^3+4d^4)R^{ab}R_a{}^{cde}R_{bc}{}^{fh}R_{defh}\big] \\
&+R_a{}^e{}_c{}^fR^{abcd}R_b{}^h{}_d{}^jR_{ehfj} \qquad \text{(B.3)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Z}_d^{(4)} = {} & \frac{1}{12(-2+d)^2(-4+3d)(11-6d+d^2)(-22+26d-9d^2+d^3)(-4+14d-7d^2+d^3)} \\
& \times \big[-48(-2+d)(-22+26d-9d^2+d^3)(136-230d+271d^2-248d^3 \\
& +119d^4-26d^5+2d^6)R_a{}^cR^{ab}R_b{}^dR_{cd}+48(-1+d)(-22+26d-9d^2+d^3)(968 \\
& -2030d+1645d^2-689d^3+161d^4-20d^5+d^6)R_{ab}R^{ab}R_{cd}R^{cd} \\
& +16(-2+d)(16144-75888d+132572d^2-115700d^3+54596d^4-13179d^5 \\
& +902d^6+277d^7-64d^8+4d^9)R_a{}^cR^{ab}R_{bc}R-24(-1520+9492d-24910d^2 \\
& +33458d^3-24719d^4+9944d^5-1768d^6-118d^7+110d^8-18d^9+d^{10})R_{ab}R^{ab}R^2 \\
& +(33280-158960d+285656d^2-263172d^3+139206d^4-44518d^5+8963d^6 \\
& -1272d^7+162d^8-18d^9+d^{10})R^4+96(-2+d)^2(-3112+7497d-6676d^2 \\
& +2251d^3+265d^4-441d^5+138d^6-19d^7+d^8)R^{ab}R^{cd}RR_{acbd} \\
& +6(-2+d)(-28000+90828d-127196d^2+100724d^3-49778d^4+15961d^5 \\
& -3326d^6+434d^7-32d^8+d^9)R^2R_{abcd}R^{abcd}-24(-2+d)^2(9280-30290d \\
& +42690d^2-33711d^3+16264d^4-4901d^5+900d^6-92d^7+4d^8)R^{ab}RR_a{}^{cde}R_{bcde} \\
& -48(-2+d)^2(-22+26d-9d^2+d^3)(-24+538d-817d^2+444d^3-101d^4 \\
& +8d^5)R_a{}^cR^{ab}R^{de}R_{bdce}+12(-2+d)^2(-22+26d-9d^2+d^3)(-384+1396d \\
& -1615d^2+781d^3-167d^4+13d^5)R^{ab}R^{cd}R_{ac}{}^{ef}R_{bdef}+4(-2+d)^2(-4+14d \\
& -7d^2+d^3)(-496+1049d-844d^2+321d^3-58d^4+4d^5)RR_{ab}{}^{ef}R^{abcd}R_{cdef} \\
& +12(-4+d)(-2+d)(-1+d)(-22+26d-9d^2+d^3)(194-370d+237d^2 \\
& -63d^3+6d^4)R_{ab}R^{ab}R_{cdef}R^{cdef}-12(-4+d)(-2+d)^2(-7+5d)(-22+26d \\
& -9d^2+d^3)(-4+14d-7d^2+d^3)R^{ab}R_a{}^{cde}R_{bc}{}^{fh}R_{defh}\big]+R_{ab}{}^{ef}R^{abcd}R_c{}^h{}_e{}^jR_{dhfj} \qquad \text{(B.4)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Z}_d^{(5)} = {} & \frac{1}{6(-4+d)(-2+d)^3(-4+3d)(11-6d+d^2)(-22+26d-9d^2+d^3)(-4+14d-7d^2+d^3)} \\
& \times \big[-48(-2+d)(-22+26d-9d^2+d^3)(-144+188d-1374d^2+4021d^3 \\
& -5045d^4+3387d^5-1325d^6+304d^7-38d^8+2d^9) R_a{}^c R^{ab} R_b{}^d R_{cd} \\
& +48(-4+d)(-22+26d-9d^2+d^3)(-1804+9398d-18611d^2+19639d^3 \\
& -12568d^4+5162d^5-1380d^6+234d^7-23d^8+d^9) R_{ab} R^{ab} R_{cd} R^{cd} \\
& +64(-2+d)(-39608+263092d-689234d^2+982248d^3-860735d^4 \\
& +487978d^5-179717d^6+40454d^7-4102d^8-390d^9+179d^{10} \\
& -22d^{11}+d^{12}) R_a{}^c R^{ab} R_{bc} R - 24(4736-58896d+273136d^2-618416d^3 \\
& +804696d^4-652724d^5+339308d^6-109530d^7+18000d^8+735d^9-1064d^{10} \\
& +234d^{11}-24d^{12}+d^{13}) R_{ab} R^{ab} R^2 + (-318080+2132288d-5677552d^2 \\
& +8307160d^3-7598852d^4+4634276d^5-1950058d^6+577374d^7-122648d^8 \\
& +19523d^9-2542d^{10}+286d^{11}-24d^{12}+d^{13}) R^4 + 96(-2+d)^2(17960-85612d \\
& +156176d^2-144780d^3+69622d^4-10116d^5-7511d^6+5207d^7-1574d^8 \\
& +268d^9-25d^{10}+d^{11}) R^{ab} R^{cd} R R_{acbd} + 6(-2+d)(178336-1018016d+2405768d^2 \\
& -3225004d^3+2781796d^4-1644450d^5+687962d^6-206196d^7+44081d^8 \\
& -6568d^9+648d^{10}-38d^{11}+d^{12}) R^2 R_{abcd} R^{abcd} - 96(-2+d)^3(7550-39268d \\
& +81391d^2-93331d^3+67198d^4-32176d^5+10461d^6-2292d^7+325d^8 \\
& -27d^9+d^{10}) R^{ab} R R_a{}^{cde} R_{bcde} - 384(-2+d)^2(-1+d)(-22+26d-9d^2 \\
& +d^3)(-152+929d-1562d^2+1239d^3-542d^4+135d^5-18d^6+d^7) R_a{}^c R^{ab} R^{de} R_{bdce} \\
& +48(-2+d)^3(-22+26d-9d^2+d^3)(-716+3557d-5760d^2+4566d^3-2022d^4 \\
& +513d^5-70d^6+4d^7) R^{ab} R^{cd} R_{ac}{}^{ef} R_{bdef} + 8(-2+d)^3(-4+14d-7d^2+d^3)(-878 \\
& +3064d-4182d^2+2976d^3-1215d^4+288d^5-37d^6+2d^7) R R_{ab}{}^{ef} R^{abcd} R_{cdef} \\
& +24(-2+d)(-22+26d-9d^2+d^3)(-2704+14944d-32382d^2+37746d^3 \\
& -26667d^4+12018d^5-3491d^6+635d^7-66d^8+3d^9) R_{ab} R^{ab} R_{cdef} R^{cdef} \\
& -48(-2+d)^3(-22+26d-9d^2+d^3)(-4+14d-7d^2+d^3)(82-139d+82d^2 \\
& -21d^3+2d^4) R^{ab} R_a{}^{cde} R_{bc}{}^{fh} R_{defh} \big] + R_{ab}{}^{ef} R^{abcd} R_{ce}{}^{hj} R_{dfhj} \qquad \text{(B.5)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Z}_d^{(6)} = & \frac{1}{3(-4+d)(-2+d)^3(-4+3d)(11-6d+d^2)(-22+26d-9d^2+d^3)(-4+14d-7d^2+d^3)} \\
& \times \big[-48(-2+d)(-22+26d-9d^2+d^3)(-144+188d-1374d^2+4021d^3-5045d^4 \\
& +3387d^5-1325d^6+304d^7-38d^8+2d^9)R_a{}^c R^{ab} R_b{}^d R_{cd} + 48(-4+d)(-22 \\
& +26d-9d^2+d^3)(-1804+9398d-18611d^2+19639d^3-12568d^4+5162d^5 \\
& -1380d^6+234d^7-23d^8+d^9)R_{ab}R^{ab}R_{cd}R^{cd} + 64(-2+d)(-39608+263092d \\
& -689234d^2+982248d^3-860735d^4+487978d^5-179717d^6+40454d^7-4102d^8 \\
& -390d^9+179d^{10}-22d^{11}+d^{12})R_a{}^c R^{ab} R_{bc} R - 24(4736-58896d+273136d^2 \\
& -618416d^3+804696d^4-652724d^5+339308d^6-109530d^7+18000d^8+735d^9 \\
& -1064d^{10}+234d^{11}-24d^{12}+d^{13})R_{ab}R^{ab}R^2 + (-318080+2132288d-5677552d^2 \\
& +8307160d^3-7598852d^4+4634276d^5-1950058d^6+577374d^7-122648d^8+19523d^9 \\
& -2542d^{10}+286d^{11}-24d^{12}+d^{13})R^4 + 96(-2+d)^2(17960-85612d+156176d^2 \\
& -144780d^3+69622d^4-10116d^5-7511d^6+5207d^7-1574d^8+268d^9 \\
& -25d^{10}+d^{11})R^{ab}R^{cd}RR_{acbd} + 6(-2+d)(178336-1018016d+2405768d^2 \\
& -3225004d^3+2781796d^4-1644450d^5+687962d^6-206196d^7+44081d^8 \\
& -6568d^9+648d^{10}-38d^{11}+d^{12})R^2R_{abcd}R^{abcd} - 96(-2+d)^3(7550-39268d \\
& +81391d^2-93331d^3+67198d^4-32176d^5+10461d^6-2292d^7+325d^8 \\
& -27d^9+d^{10})R^{ab}RR_a{}^{cde}R_{bcde} - 384(-2+d)^2(-1+d)(-22+26d-9d^2 \\
& +d^3)(-152+929d-1562d^2+1239d^3-542d^4+135d^5-18d^6+d^7)R_a{}^c R^{ab} R^{de} R_{bdce} \\
& +48(-2+d)^3(-22+26d-9d^2+d^3)(-716+3557d-5760d^2+4566d^3-2022d^4 \\
& +513d^5-70d^6+4d^7)R^{ab}R^{cd}R_{ac}{}^{ef}R_{bdef} + 8(-2+d)^3(-4+14d-7d^2 \\
& +d^3)(-878+3064d-4182d^2+2976d^3-1215d^4+288d^5-37d^6+2d^7)RR_{ab}{}^{ef}R^{abcd}R_{cdef} \\
& +24(-2+d)(-22+26d-9d^2+d^3)(-2704+14944d-32382d^2+37746d^3-26667d^4 \\
& +12018d^5-3491d^6+635d^7-66d^8+3d^9)R_{ab}R^{ab}R_{cdef}R^{cdef} \\
& -48(-2+d)^3(-22+26d-9d^2+d^3)(-4+14d-7d^2+d^3)(82-139d+82d^2 \\
& -21d^3+2d^4)R^{ab}R_a{}^{cde}R_{bc}{}^{fh}R_{defh}\big] + R_{ab}{}^{ef}R^{abcd}R_{cd}{}^{hj}R_{efhj} \qquad \text{(B.6)}
\end{aligned}
$$

Appendix C

Generalized Quasi-Topological Lagrangian Densities

Here we present the explicit forms of the quartet of generalized quasi-topological theories, which are obtained by the choices made in equations (2.25)-(2.28) and (2.34).

$$\mathcal{S}_d^{(1)} = \frac{1}{6(d-3)^2(d-2)^2(d-1)d(11-6d+d^2)(19-18d+3d^2)(-22+26d-9d^2+d^3)}$$
$$\times \big[-2(d-2)^2(675840-1895902d+2220384d^2-1342691d^3+370480d^4$$
$$+36380d^5-68962d^6+27252d^7-6100d^8+862d^9-74d^{10}+3d^{11})R_a{}^c R^{ab} R_b{}^d R_{cd}$$
$$-2(1332480-3880512d+4484792d^2-2299414d^3+114412d^4+452234d^5$$
$$-195096d^6-509d^7+26111d^8-9952d^9+1830d^{10}-175d^{11}+7d^{12})R_{ab}R^{ab}R_{cd}R^{cd}$$
$$+8(d-2)^2(d-1)(8160-19934d+18411d^2-6271d^3-1872d^4+2790d^5$$
$$-1261d^6+301d^7-38d^8+2d^9)R_a{}^c R^{ab} R_{bc} R + 2(374400-1072928d$$
$$+1257694d^2-724744d^3+156052d^4+53793d^5-49657d^6+17344d^7$$
$$-3698d^8+525d^9-47d^{10}+2d^{11})R_{ab}R^{ab}R^2+24(d-2)(-128640+368958d$$
$$-429005d^2+239408d^3-43691d^4-22101d^5+15982d^6-4406d^7+625d^8$$
$$-43d^9+d^{10})R^{ab}R^{cd}RR_{acbd}-3(361600-1116656d+1410902d^2$$
$$-875630d^3+208502d^4+51581d^5-38382d^6-577d^7+6668d^8-2637d^9$$
$$+500d^{10}-49d^{11}+2d^{12})R^2R_{abcd}R^{abcd}-24(d-2)(d-1)(-119680$$
$$+338440d-401078d^2+240034d^3-58237d^4-13906d^5+14831d^6$$
$$-4890d^7+849d^8-78d^9+3d^{10})R_a{}^c R^{ab} R^{de} R_{bdce}$$
$$+6(d-2)^2(d-1)(28160-110076d+172418d^2-146251d^3$$
$$+75674d^4-25778d^5+6287d^6-1211d^7+188d^8-20d^9+d^{10})R^{ab}R^{cd}R_{ac}{}^{ef}R_{bdef}$$
$$+2(d-3)(d-2)^2(d-1)(-2400+9201d-9929d^2+2690d^3+1954d^4$$
$$-1667d^5+507d^6-72d^7+4d^8)RR_{ab}{}^{ef}R^{abcd}R_{cdef}+3(-113920$$
$$+801792d-1837992d^2+2067094d^3-1242116d^4+346968d^5+4985d^6$$
$$-18628d^7-8905d^8+9138d^9-3089d^{10}+546d^{11}-51d^{12}+2d^{13})R_{ab}R^{ab}R_{cdef}R^{cdef}$$
$$-6(d-3)(d-2)^2(d-1)(-22+26d-9d^2+d^3)(320-709d+588d^2$$
$$-292d^3+106d^4-23d^5+2d^6)R^{ab}R_a{}^{cde}R_{bc}{}^{fh}R_{defh}\big]$$
$$+R_a{}^e{}_c{}^f R^{abcd} R_b{}^h{}_e{}^j R_{dhfj} \tag{C.1}$$

$$
\begin{aligned}
\mathcal{S}_d^{(2)} = & \frac{1}{3(d-3)^2(d-2)^2(d-1)d(11-6d+d^2)(19-18d+3d^2)(-22+26d-9d^2+d^3)} \times \\
& \times \big[-2(d-2)^2(-578688+2025158d-3185710d^2+2977426d^3-1839784d^4 \\
& +791721d^5-244086d^6+54763d^7-8972d^8+1049d^9-80d^{10}+3d^{11})R_a{}^c R^{ab} R_b{}^d R_{cd} \\
& +(2281872-8031408d+12067376d^2-9693872d^3+3903996d^4+22113d^5 \\
& -946024d^6+572163d^7-189362d^8+39599d^9-5244d^{10}+405d^{11}-14d^{12})R_{ab}R^{ab}R_{cd}R^{cd} \\
& +8(d-2)^2(d-1)(-6987+21346d-29861d^2+26093d^3-15931d^4+6882d^5 \\
& -2031d^6+385d^7-42d^8+2d^9)R_a{}^c R^{ab} R_{bc} R + 2(-320580+1132666d \\
& -1781245d^2+1646682d^3-998922d^4+421855d^5-128958d^6+29348d^7 \\
& -5024d^8+627d^9-51d^{10}+2d^{11})R_{ab}R^{ab}R^2 - 12(d-2)(-220296 \\
& +774954d-1199885d^2+1070366d^3-604828d^4+223750d^5-53844d^6 \\
& +7998d^7-628d^8+12d^9+d^{10})R^{ab}R^{cd}RR_{acbd} - 3(-309620+1149158d \\
& -1856955d^2+1675917d^3-875073d^4+209908d^5+40520d^6-53295d^7 \\
& +21313d^8-4912d^9+693d^{10}-56d^{11}+2d^{12})R^2 R_{abcd}R^{abcd} \\
& -24(d-2)(d-1)(102476-371148d+606224d^2-585295d^3+368632d^4 \\
& -157824d^5+46423d^6-9251d^7+1194d^8-90d^9+3d^{10})R_a{}^c R^{ab} R^{de} R_{bdce} \\
& +6(d-2)^2(d-1)(-24112+104237d-184591d^2+177665d^3-102275d^4 \\
& +35933d^5-7258d^6+601d^7+55d^8-16d^9+d^{10})R^{ab}R^{cd}R_{ac}{}^{ef}R_{bdef} \\
& +(d-3)(d-2)^2(d-1)(4110-23613d+44912d^2-42687d^3+23334d^4 \\
& -7715d^5+1532d^6-169d^7+8d^8)RR_{ab}{}^{ef}R^{abcd}R_{cdef} + 3(97544-765604d \\
& +2080704d^2-2942717d^3+2459345d^4-1222083d^5+288468d^6+44796d^7 \\
& -62477d^8+24383d^9-5446d^{10}+743d^{11}-58d^{12}+2d^{13})R_{ab}R^{ab}R_{cdef}R^{cdef} \\
& -6(d-3)(d-2)^2(d-1)(-22+26d-9d^2+d^3)(-274+409d-67d^2 \\
& -161d^3+103d^4-24d^5+2d^6)R^{ab}R_a{}^{cde}R_{bc}{}^{fh}R_{defh}\big] \\
& +R_a{}^e{}_c{}^f R^{abcd} R_b{}^h{}_d{}^j R_{ehfj}
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
\mathcal{S}_d^{(3)} = &\frac{1}{3(d-3)^2(d-2)(d-1)d(11-6d+d^2)(19-18d+3d^2)(-22+26d-9d^2+d^3)}\times \\
&\times \big[-4(d-2)(-718080+2405582d-3666144d^2+3359133d^3-2057938d^4 \\
&+887142d^5-276120d^6+62662d^7-10296d^8+1182d^9-86d^{10}+3d^{11})R_a{}^cR^{ab}R_b{}^dR_{cd} \\
&-4(707880-2115012d+2700668d^2-1809780d^3+561468d^4+61133d^5-134394d^6 \\
&+60426d^7-15005d^8+2238d^9-189d^{10}+7d^{11})R_{ab}R^{ab}R_{cd}R^{cd} \\
&+16(d-2)(d-1)(-8670+30262d-47247d^2+43299d^3-25747d^4+10271d^5 \\
&-2734d^6+466d^7-46d^8+2d^9)R_a{}^cR^{ab}R_{bc}R+4(198900-592178d+790224d^2 \\
&-617415d^3+313537d^4-109500d^5+27237d^6-4900d^7+624d^8-51d^9+2d^{10})R_{ab}R^{ab}R^2 \\
&+48(d-2)(-68340+203532d-268574d^2+203038d^3-95967d^4+29190d^5 \\
&-5665d^6+667d^7-42d^8+d^9)R^{ab}R^{cd}RR_{acbd}-6(192100-603774d+820554d^2 \\
&-605255d^3+237492d^4-22951d^5-24843d^6+14329d^7-3890d^8+609d^9-53d^{10} \\
&+2d^{11})R^2R_{abcd}R^{abcd}-48(d-2)(d-1)(-63580+183572d-244118d^2 \\
&+192444d^3-97734d^4+32893d^5-7308d^6+1032d^7-84d^8+3d^9)R_a{}^cR^{ab}R^{de}R_{bdce} \\
&+12(d-2)(d-1)(-29920+120000d-196892d^2+175930d^3-93864d^4 \\
&+30115d^5-5212d^6+193d^7+99d^8-18d^9+d^{10})R^{ab}R^{cd}R_{ac}{}^{ef}R_{bdef} \\
&+4(d-3)(d-2)(d-1)(2550-15414d+28633d^2-26167d^3+13715d^4 \\
&-4351d^5+830d^6-88d^7+4d^8)RR_{ab}{}^{ef}R^{abcd}R_{cdef}+6(-60520+414664d \\
&-945458d^2+1097752d^3-719367d^4+242784d^5-3125d^6-36155d^7+17569d^8 \\
&-4430d^9+659d^{10}-55d^{11}+2d^{12})R_{ab}R^{ab}R_{cdef}R^{cdef} \\
&-12(d-3)(d-2)(d-1)(-22+26d-9d^2+d^3)(-340+494d-70d^2-185d^3 \\
&+112d^4-25d^5+2d^6)R^{ab}R_a{}^{cde}R_{bc}{}^{fh}R_{defh}\big] \\
&+R_{ab}{}^{ef}R^{abcd}R_{ce}{}^{hj}R_{dfhj}
\end{aligned}
\tag{C.3}
$$

$$\mathcal{S}_d^{(4)} = \frac{1}{3(d-3)^2(d-2)(d-1)d(11-6d+d^2)(19-18d+3d^2)(-22+26d-9d^2+d^3)} \times$$
$$\times \big[-8(d-2)(-718080+2405582d-3666144d^2+3359133d^3-2057938d^4$$
$$+887142d^5-276120d^6+62662d^7-10296d^8+1182d^9-86d^{10}+3d^{11})R_a{}^cR^{ab}R_b{}^dR_{cd}$$
$$-8(707880-2115012d+2700668d^2-1809780d^3+561468d^4+61133d^5$$
$$-134394d^6+60426d^7-15005d^8+2238d^9-189d^{10}+7d^{11})R_{ab}R^{ab}R_{cd}R^{cd}$$
$$+32(d-2)(d-1)(-8670+30262d-47247d^2+43299d^3-25747d^4+10271d^5$$
$$-2734d^6+466d^7-46d^8+2d^9)R_a{}^cR^{ab}R_{bc}R+8(198900-592178d+790224d^2$$
$$-617415d^3+313537d^4-109500d^5+27237d^6-4900d^7+624d^8-51d^9+2d^{10})R_{ab}R^{ab}R^2$$
$$+96(d-2)(-68340+203532d-268574d^2+203038d^3-95967d^4+29190d^5$$
$$-5665d^6+667d^7-42d^8+d^9)R^{ab}R^{cd}RR_{acbd}-12(192100-603774d+820554d^2$$
$$-605255d^3+237492d^4-22951d^5-24843d^6+14329d^7-3890d^8+609d^9-53d^{10}$$
$$+2d^{11})R^2R_{abcd}R^{abcd}-96(d-2)(d-1)(-63580+183572d-244118d^2$$
$$+192444d^3-97734d^4+32893d^5-7308d^6+1032d^7-84d^8+3d^9)R_a{}^cR^{ab}R^{de}R_{bdce}$$
$$+24(d-2)(d-1)(-29920+120000d-196892d^2+175930d^3-93864d^4$$
$$+30115d^5-5212d^6+193d^7+99d^8-18d^9+d^{10})R^{ab}R^{cd}R_{ac}{}^{ef}R_{bdef}$$
$$+8(d-3)(d-2)(d-1)(2550-15414d+28633d^2-26167d^3+13715d^4$$
$$-4351d^5+830d^6-88d^7+4d^8)RR_{ab}{}^{ef}R^{abcd}R_{cdef}+12(-60520+414664d$$
$$-945458d^2+1097752d^3-719367d^4+242784d^5-3125d^6-36155d^7+17569d^8$$
$$-4430d^9+659d^{10}-55d^{11}+2d^{12})R_{ab}R^{ab}R_{cdef}R^{cdef}$$
$$-24(d-3)(d-2)(d-1)(-22+26d-9d^2+d^3)(-340+494d-70d^2$$
$$-185d^3+112d^4-25d^5+2d^6)R^{ab}R_a{}^{cde}R_{bc}{}^{fh}R_{defh}\big]$$
$$+R_{ab}{}^{ef}R^{abcd}R_{cd}{}^{hj}R_{efhj} \tag{C.4}$$

The following two Lagrangian densities are relevant only for the four dimensional theory.

$$\begin{aligned}\mathcal{S}_4^{(5)} &= -\frac{14}{5}R_{ab}R^{ab}R_{cd}R^{cd} - \frac{20}{3}R_a{}^bR_b{}^cR_c{}^dR_d{}^a - \frac{8}{5}R^{ac}R^{bd}RR_{abcd} \\ &+ \frac{104}{5}R^{ab}R_e{}^dR^{ec}R_{acbd} + R_{ef}R^{ef}R_{abcd}R^{abcd} + \frac{1}{5}R^2R_{abcd}R^{abcd} \\ &- \frac{56}{15}R^{ab}R_{cd}{}^h{}_aR^{cdef}R_{efhb} + R_{abc}{}^eR^{abcd}R_{fhjd}R^{fhj}{}_e \end{aligned} \tag{C.5}$$

$$\begin{aligned}\mathcal{S}_4^{(6)} &= -\frac{308}{15}R_{ab}R^{ab}R_{cd}R^{cd} - \frac{64}{3}R_a{}^bR_b{}^cR_c{}^dR_d{}^a + \frac{64}{15}R^{ac}R^{bd}RR_{abcd} + \frac{1088}{15}R^{ab}R_e{}^dR^{ec}R_{acbd} \\ &+ \frac{28}{3}R_{ef}R^{ef}R_{abcd}R^{abcd} - \frac{8}{15}R^2R_{abcd}R^{abcd} - \frac{224}{15}R^{ab}R_{cd}{}^h{}_aR^{cdef}R_{efhb} \\ &+ R_{abcd}R^{abcd}R_{fhje}R^{fhje} \end{aligned} \tag{C.6}$$

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