Conservation Laws For Dynamical System Via Differential Variational principles.

By

Muhammad Asim

Supervised By

Dr. Amjad Hussain.

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
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Dedication

TO MY FAMILY AND TEACHERS WHO HELPED, SUPPORTED AND BOOSTED MY CAPABILITIES

Acknowledgment

In the name of Allah, the Most Gracious and the Most Merciful, all praises to Allah for His blessings in completing this thesis. I find no words to express my deepest appreciation to my beloved Holy Prophet Muhammad (Peace Be Upon Him), the foundation of knowledge and guidance for all.

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Muhammad Asim

Contents

Abstract

The purpose of this thesis is to study the conservation law of a dynamical system which is an important research area in the investigation of analytical dynamics.The great importance of the conservation law is to reduce the order of the differential equation of motion. To find the conservation we can used two methods, the first one is based on the invariance of Hamilton's integral and the other is differential variational principle.

In this thesis we used the differential variational principle to find the conservation law of a dynamical system. To find the conservation law of a dynamical system we will face two problems, either the motion of the dynamical system is under the ideal constraints or non-ideal constraints. For this purpose, we divides this thesis into four chapters.

The first three section of chapter one are devoted to the fundamental concepts of analytical dynamics including the notion of constraints and the generalized coordinates. Then the idea of variable mass is presented with some examples. After that the idea of fundamental and synchronous virtual variation (actual and virtual) is introduced and the exchange rule between actual and the virtual variation is established.

In the chapter second, we calculated the conservation law of a non-conservative dynamical system for Jourdian and Gauss principle under the ideal constraint, in which the virtual work is zero.

Chapter third deals with the calculation of conservative law of a dynamical system for the D'Alembert principle under the non-ideal constraints. Besides the conservation law we also find the magnitude of reaction force on the dynamical system. In this chapter we used ideas of supplementary virtual displacement and supplementary generalized coordinates.

In the last chapter we developed the theory of conservation law for non conservative dynamical system of the Jourdian and Guass differential variational principle under the non ideal constraints.

Chapter 1

Literature Revisited

1.1 General Consideration

Prior to the investigation of conservation laws of mechanical systems with variable mass, we need some fundamental and essential concepts which are important for non-technical readers in the studies of analytical dynamics. So the aim of this chapter is to collect some basic notions to familiarize the readers with some information which are necessary for the investigation in the subject of analytical dynamics. First we give a brief introduction of the constraints and their classification for the sake of completeness. The concept of generalized coordinates is given and the idea of variable mass is presented with some important examples. The notions of actual and virtual variation are obtained and the corresponding exchange rule $d\delta = \delta d$ is formulated. Finally, the general equation of dynamics under ideal constraints for constant mass, expressing Langrange-d'Alembert's principle is established in term of the generalized coordinates.

1.2 Fundamental Concepts

1.2.1 Dynamical System

Any physical phenomenon that moves under the action of certain laws of forces is called a dynamical system. For instance, the motion of a particle relative to a fixed point in a straight line under the action of an opposing force proportional to the distance is a dynamical system. Generally, a dynamical system may consist of N-particles in which each particle is assumed to maintain a distinguishable identity throughout the motion.

1.2.2 Constraint

Any condition or restriction that restrain the action of any physical phenomenon in certain region of space is called a constraint. In general the constraints are expressed by means of the equation or inequalities which provide a functional relation between the quantities which define the position of the particles of the system or the kinematics of the dynamical systems.

In order to give precise expression of the constraints analytically, let us consider a dynamical system consisting of N-particles. Let at any time t the position of the i-particle p_i $(i = 1, 2, ..., N)$ of the system to be defined by the cartesian coordinates

$$x_i = x_{3i-2}, \quad y_i = x_{3i-1}, \quad z_i = x_{3i}, \quad (i = 1, 2, ..., N)$$

relative to an inertial frame of reference $OXYZ$. A constraint can then be expressed in the form

$$f(x_1, ..., x_{3N}; \dot{x}_1, \dot{x}_2, ..., \dot{x}_{3N}) \lesseqqgtr 0. \tag{1.1}$$

Where f is assumed to be smooth function of at least class C^2 with respect to all of its arguments.

In our subsequent discussion, dot $(\cdot)$ over a quantity will imply the total derivative with respect to time.

It is obvious from the (1.1) of the constraint equation that this is an ordinary differential equation which may or may not be integrable. If (1.1) is integrable, then it implies that there exist a function $\phi(x_1, ..., x_{3N})$ of at least class C^1, such that

$$\frac{d\phi}{dt} \lesseqqgtr 0,$$

$$\phi(x_1, x_2, ..., x_{3N}) \lesseqqgtr C. \tag{1.2}$$

Where C is an arbitrary constant of integration and may be determined by given initial condition.

Apart from this there may be a functional relation among the coordinates $x_1, x_2, ..., x_{3N}$ and the possibly the time t which is not derivable from any function f as in (1.1) but intrinsically associated with the system and may be given in the form

$$\Psi(x_1, ..., x_{3N}, t) \lesseqqgtr 0, \tag{1.3}$$

where ψ is a function of at least class C^1.

We observe that the expression (1.2) and (1.3) furnish a relation between the position variable $x_1, ..., x_{3N}$ and the possibly the time t. Keeping in view, the discussion of preceding section, we now give the classification of the constraints.

1.2.3 Classification Of Constraints

There are two major classes of the constraints; namely, the holonomic and the non-holonomic constraints. We now furnish their definitions

Definition

A condition expressing a functional relationship between the quantities describing the position of the particles of system and possibly the time t and is given in the form of equation or inequalities (1.2) or (1.3), respectively is called a holonomic constraint.

Definition

A condition imposed on the kinematics of a dynamical system of particles and is described by mean of the non-integrable equations of the form (1.1) are called the non-holonomic constraints.

We now give further classification of the constraints. The constraint (1.2) and (1.3) are called Bilateral according as they are equations or inequalities. Moreover, if the function f and Φ depend upon the time explicitly, then the constraints are called reheonomic or time dependent, otherwise constraints are called sceleronomic or time independent [20]. Non-holonomic constraints may be non linear or linear according as function $f(x_1, ..., x_{3N}, \dot{x}_1, ..., \dot{x}_{3N}; t)$ is non linear or linear with respect to $\dot{x}$, the velocity component of the particles of the system.

We remarks that a dynamical system which moves subject to non-holonomic (or only holonomic) constraints is called a non-holonomic (or holonomic) dynamical system. If there are no constraints, the system is called a free system. Moreover in nature there is no dynamical system which is free in its motion, it always has to move under some restrictions which do not allow the system to move in space in any way it likes. Therefor, the natural motion are the constrained motions.

1.3 Generalized Coordinates

In order to specify the position of a dynamical system at a certain instant of time we require the value of a number of parameters. For instance, to specify the position moving freely in plane, we need two variable x, y; the cartesian coordinates. If a particle is moving in space, three parameters x, y, z are needed to define its position. Similarly, if we consider a mechanical system which is composed of N free particles, the $3N$ rectangular cartesian coordinates x_i, y_i, z_i are expressed in term of some other $3N$ quantities

$$q_1, q_2, ..., q_{3N}$$

and then these quantities are determined as functions of the time and certain arbitrary constants to furnish the solution of a dynamical problem.

As mentioned by Lancoz [21] that the set of $3N$ variable $q_1, q_2, ..., q_{3N}$ must have some geometrical relationship with $3N$ cartesian coordinates. Generally, the relation between these two set of variables are expressed by means of the invertible transformations of the form

$$\begin{aligned} x_1 &= f_1(q_1, q_2, ..., q_{3N}) \\ y_1 &= f_1(q_1, q_2, ..., q_{3N}) \\ &= .... \\ &= .... \\ z_N &= f_{3N}(q_1, q_2, ..., q_{3N}). \end{aligned} \tag{1.4}$$

This mean that the set of equation (1.4) express a one-to-one correspondence between these two sets of variable and that the matrix of transformation, that is

$$\|\frac{\partial(x_1, y_1, z_1, ..., x_N, y_N, z_N)}{\partial(q_1, q_2, ..., q_{3N-1}, q_{3N})}\| = \begin{bmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & \cdots & \frac{\partial x_1}{\partial q_{3N}} \\ \frac{\partial y_1}{\partial q_1} & \frac{\partial y_1}{\partial q_2} & \cdots & \frac{\partial y_1}{\partial q_{3N}} \\ \vdots & \ddots & \vdots & \\ \frac{\partial z_N}{\partial q_1} & \frac{\partial z_N}{\partial q_2} & \cdots & \frac{\partial z_N}{\partial q_{3N}} \end{bmatrix}, \tag{1.5}$$

is non singular. It implies that the Jacobian of the this matrix is non-zero, namely

$$\frac{\partial(x_1, y_1, z_1, ..., x_N, y_N, z_N)}{\partial(q_1, q_2, ..., q_{3N-1}, q_{3N})} \neq 0 \tag{1.6}$$

so that the quantities q_λ are expressible in the form

$$q_\lambda = q_\lambda(x_1, y_1, z_1, ..., x_N, y_N, z_N), \ (\lambda = 1, 2, ..., 3N) \tag{1.7}$$

thus, we give the following:

Definition If there exist geometrical relation of the form (1.4) between the cartesian coordinates and the new independent quantities q_λ and if, under the condition (1.6), they are expressible in the form (1.7), then $q_\lambda(\lambda = 1, 2, ..., 3N)$ are called the generalized coordinates.

So far we have not taken into taken into account the constraints. We now turn to the consideration of constrained dynamical system. Suppose that the system moves subject to the holonomic constraints that are expressed by mean of m independent equations

$$\Psi_\lambda(x_1, x_2, ..., x_{3N}) = 0, \ (\lambda = 1, 2, ..., m < 3N). \tag{1.8}$$

If we introduce the following notation $x_1 = x_1, \ x_2 = y_1, \ x_3 = z_1, ..., x_{3N-2} = x_N, \ x_{3N-1} = y_N, \ x_{3N} = z_N$, and assume that the function Ψ_λ satisfying (1.8) are of class C^1 at least and that the rank of the functional matrix

$$\|\frac{\partial(\Psi_1, \Psi_2, \Psi_3, ..., \Psi_m)}{\partial(x_1, x_2, ..., x_{3N-1}, x_{3N})}\| = \begin{bmatrix} \frac{\partial \Psi_1}{\partial x_1} & \frac{\partial \Psi_1}{\partial x_2} & \cdots & \frac{\partial \Psi_1}{\partial x_{3N}} \\ \frac{\partial \Psi_2}{\partial x_1} & \frac{\partial \Psi_2}{\partial x_2} & \cdots & \frac{\partial \Psi_1}{\partial x_{3N}} \\ \vdots & \ddots & \vdots & \\ \frac{\partial \Psi_m}{\partial x_1} & \frac{\partial \Psi_m}{\partial x_2} & \cdots & \frac{\partial \Psi_m}{\partial x_{3N}} \end{bmatrix}$$

is equal to m. Then by implicit function theorem, we can solve the system of equation (1.8) for the m x' s in terms of the remaining $(3N - m)$ x' s in the form

$$x_\lambda = x_\lambda(x_{m+1}, x_{m+2}, ..., x_{3N}; t), \ (\lambda = 1, 2, ..., m < 3N), \tag{1.9}$$

where the dependent variable $x_\lambda(\lambda = 1, 2, ..., m)$ are expressed in term of the $(3N-m)$ independent variables $x_k(k = m + 1, ..., 3N)$.

This provided that the Jacobian of the functional matrix is different from zero, that is

$$\frac{\partial(\Psi_1, \Psi_2, \Psi_3, ..., \Psi_m)}{\partial(x_1, x_2, ..., x_{3N-1}, x_{3N})} \neq 0. \tag{1.10}$$

Thus, the position of the system can be described by the $n = 3N - m$ number of independent parameters. The number n is defined to be the degrees of freedom of the holonomic system.

Further, in addition to the m holonomic constraints, if the system moves subject to a non-holonomic constraints of the type

$$F_\alpha(x_1, x_2, ..., x_{3N}; \dot{x}_1, \dot{x}_2, ..., \dot{x}_{3N}; t = 0). \tag{1.11}$$

$$(\alpha = 1, 2, ..., s < n)$$

Then the degree of freedom of the non-holonomic system is defined by the the number

$$n' = n - s.$$

It is also to be noted that it is not always advisable to use the independent cartesian coordinates $x_{m+1}, x_{m+2}, ..., x_{3N}$ for determining the position of the system. In place of these coordinates we can introduce some other $n = 3N - m$ independent parameters $q_1, q_2, ..., q_n$ which must be connected with $3N - m$ cartesian coordinates by mean of the relation of the type (1.4) that is

$$x_k = x_k(q_1, q_2, ..., q_n), \ (K = 1, 2, ..., n) \tag{1.12}$$

and satisfy the relation of the form (1.7) under the condition similar to (1.6). If the constraint equation (1.8) involve the time t explicitly, namely

$$F_\gamma(x_1, x_2, ..., x_{3N}; t) = 0. \ (\gamma = 1, 2, ..., m) \tag{1.13}$$

then (2.9) will take the following more general form

$$x_k = x_k(q_1, q_2, ..., q_n; t). \ (k = 1, 2, ..., n) \tag{1.14}$$

Taking into account the relation (1.10) and (2.39), all the cartesian coordinates $x_\gamma(\gamma = 1, 2, ..., 3N)$ can be expressed as function of the $q's$ and the time t.

More precisely,

$$x_\lambda = x_\lambda(q_1, q_2, ..., q_n; t) = x_\lambda(q_k, t) \tag{1.15}$$

$$(k = 1, 2, ..., n; \lambda = 1, 2, ..., 3N)$$

In consequence of these equations of transformation, the position vectors of all the particles of the system can be obtained in the form

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, ..., q_n; t) \equiv \mathbf{r}_i. \tag{1.16}$$

Keeping in view the preceding analysis, we introduce the following:

Definition A set of the minimum number of independent parameter $q_1, q_2, ..., q_n$ which are obtained after taking into account the holonomic constraint (2.38) and satisfy the relation of the form (1.9) and (1.14) are called the Lagrangian coordinates [13].

The constraint equation (1.11), in view of the relation (1.15), can be expressed in the form

$$F_\alpha(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n; t), \ (\alpha = 1, 2, ..., s),$$

for the non-holonomic system. Here the quantities $\dot{q}_1, \dot{q}_2, ..., \dot{q}_n$ are called the generalized velocities.

1.4 Variable Mass Dynamical System

There are various reason (e.g. temporal, spatial or kinematical) which cause the change in the mass of a particle or rigid body. In order to deal with the dynamics of such bodies, in the sequel, we give a brief discussion of the associated concepts.

1.4.1 General Variable Mass System

A body in which the number of particles change with time, generalized coordinates or generalized velocities is called a variable mass body. Precisely, the mass of a typical i-th particle of a variable mass system of N-particles is expressed analytically as

$$m_i = m_i(q_i, \dot{q}_i; t). \ (i = 1, 2, ..., N)$$

For example the mass of a rocket change with time [3], the mass of a homogenous sphere which burns uniformly and roll on a rough horizontal plane changes with respect to the generalized coordinates [23], similarly the mass of a raindrop falling from a stationary cloud varies with the position [24]. In the studies of special relativity, the relativistic mass of the particle varies with to the speed and is usually given by the expression

$$m = \frac{m_o}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where m_o is the rest mass and c is the speed of light, a positive constant. In most cases $\frac{v}{c}$ is very small ($i.e. \frac{v}{c} << 1$), and then the variation of m from rest mass m_0 may be negligible. But the variation becomes significant when $\frac{v}{c}$ is nearly equal to unity. In the case, where the mass change with the speed, we call such dynamical system as a relativistic variable mass system.

1.5 Fundamental Variations

Roughly speaking, the change in the configuration of a dynamical system is called a displacement. This change, in fact, depend on the changes which occur in the variable that define the configuration of the dynamical system at any time t. In what follows, we shall give a precise definition of actual and virtual change in the generalized coordinates.

Let us consider a holonomic dynamical system of N-particles whose configuration at any time t is determined by mean of the generalized coordinates $q_1, q_2, ..., q_n$. The solution of a dynamical problem implies determination of these generalized coordinates as function of the time. The set of functions

$$q_1(t), q_2(t), ..., q_n(t), \tag{1.17}$$

furnished the actual motion of the system. The differential dq_k $(k = 1, 2, ..., n)$ of the generalized coordinates present their infinitesimal change the actual path during an infinitesimal interval of time dt, which we write analytically as

$$\begin{aligned} dq_k &= q_k(t + dt) - q_k(t) \\ dq_k &= \dot{q}_k dt, \ (k = 1, 2, ..., n) \end{aligned} \tag{1.18}$$

where we have used the taylor's theorem and neglected the higher order terms of the infinitesimal quantity dt.

The result (2.45) yield the actual variation dq_k in the generalized coordinate along the actual path and $\dot{q}_k$ is called the actual generalized velocity.

In general formulation of the laws of analytical dynamics, it is useful to consider the infinitesimal quantities of another kind. The set of quantities (2.46) determine the actual configuration of the system at a given moment of the time t. There are an infinite number of possible configuration, but we examine only those configuration which are infinitely close to the actual configuration of the system. If we denote by $\delta q_1, \delta q_2, ..., \delta q_n$ the infinitesimal increments in the Langragian coordinates, then the number of configurations at a given time t can be determined by the set of quantities

$$\delta q_1 = \delta q^*(t) - q_1(t), ..., \delta q_n = \delta q_n^*(t) - q_n(t), \tag{1.19}$$

where the difference $\delta q_k (k = 1, 2, ..., n)$ are called the virtual variations of the Lagrangain coordinates and are determined by keeping t as fixed, that is $\delta t = 0$. We also assume them to be infinitely many times differential function of time.

Since the geometrical picture is of great help to our thinking, we may regard the variables $q_1, q_2, ..., q_n$ as the rectangular coordinates in the n-dimensional Euclidean space of a point (say) P and introduce the following:

1.5.1 Lagrangian Configuration Space

A set of n-independent parameters $q_k (k = 1, 2, ..., n)$ is said to form the Lagrangian configuration space, if the conditions

$$\delta t = 0, \ \delta q_k \neq 0, \ \delta \dot{q}_k \neq 0, \ (k = 1, 2, ..., n) \tag{1.20}$$

hold throughout the motion of the system. Let us consider a dynamical system consisting of N-particles. Let $\mathbf{r_i}$ denotes the vector that determines the position of a typical particle P_i of the system relative to an origin O of an inertial frame $OXYZ$. As the system moves the vector $\mathbf{r_i}$ in general, is a function of the generalized coordinates $q_k (k = 1, 2, ..., n)$ and possibly the time t, that is

$$\mathbf{r_i} = \mathbf{r_i}(q_k, t). \ (k = 1, 2, ..., n; i = 1, 2, ..., N)$$

Throughout our work we assume $\mathbf{r_i}$ to be a function of class C^2 with respect to all of its arguments.

1.5.2 Actual Displacement

Let $\mathbf{r_i}$ and $\mathbf{r_i} + d\mathbf{r_i}$ denotes the position vectors of the neighboring position $P_i(q_{k,t})$ and $Q_i(q_{k+dq_k,t+dt})$ of the particles corresponding to the time t and $t + dt$, where dt

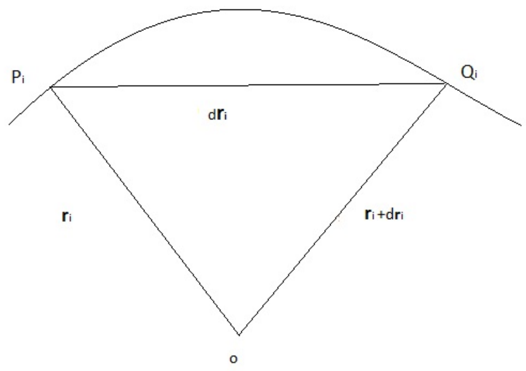

Figure 1.1:

is a small differential quantity with respect to time along the actual motion of the particle, as shown in the fig.1.

Then the differential $d\mathbf{r_i}$ of the position vector $\mathbf{r_i}$ with respect the generalized coordinates is given by

$$d\mathbf{r_i} = \mathbf{r_i}(q_k + dq_k, t + dt) - \mathbf{r_i}(q_k, t).\ (k = 1, 2, ..., n)$$

Applying taylor's theorem, we get

$$d\mathbf{r_i} = \frac{\partial \mathbf{r_i}}{\partial q_k} dq_k + \frac{\partial \mathbf{r_i}}{\partial t} dt + H.O$$

where $H.O$ denotes the term of order higher than one and we use the summation convention over the repeated index k and, in the sequel, we shall employ it throughout our work. Retaining only the first order terms, we obtain

$$d\mathbf{r_i} = \frac{\partial \mathbf{r_i}}{\partial q_k} dq_k + \frac{\partial \mathbf{r_i}}{\partial t} dt, \tag{1.21}$$

which defines the infinitesimal displacement of the particle P_i in the actual motion.

Definition The change $d\mathbf{r_i}$ of arbitrary position vector $\mathbf{r_i}(q_k, t)$ during the actual variation dq_k in time dt of the system is determined by the formula (1.21). The velocity vector of P_i is given by

$$d\dot{\mathbf{r}}_\mathbf{i} = \frac{\partial \mathbf{r_i}}{\partial q_k}\dot{q}_k + \frac{\partial \mathbf{r_i}}{\partial t}.$$

By differentiating with respect to $\dot{q}_k$, we get

$$\frac{\partial \dot{\mathbf{r}}_\mathbf{i}}{\partial \dot{q}_k} = \frac{\partial \mathbf{r_i}}{\partial q_k}.$$

This is known as the law of cancelation of dots.

The path along which the displacement $d\mathbf{r_i}$ takes place is known as the trajectory or the actual path.

1.5.3 Virtual Displacement

As discussed earlier, we introduce the virtual displacement $\delta\mathbf{r_i}$ on the basis of the fundamental variation δq_s of the generalized coordinates q_s. Let C denotes the actual path and C and C^* an infinitely close path to C obtained by the simultaneous projection of the position on C to the corresponding position on C^* as shown in the adjoining fig.2.

We denote the difference δr_i, in view of (1.20) by the relations

$$\delta\mathbf{r_i} = \mathbf{r_i}(q_k + \delta q_k, t) - \mathbf{r_i}(q_k, t),\ (k = 1, 2, ..., n)$$

which is called the synchronous displacement.

Again applying Tylor's theorem, we find that

$$\delta\mathbf{r_i} = \frac{\partial \mathbf{r_i}}{\partial q_k}\delta q_k + H.O$$

where $H.O$ denotes the terms containing the square and the higher power of δq_k. Neglecting the term of higher order, we get

$$\delta\mathbf{r_i} = \frac{\partial \mathbf{r_i}}{\partial q_k}\delta q_k \tag{1.22}$$

which describes the virtual displacement of the particle P_i.

definition In a simultaneous virtual variation displacement $\delta q_1, \delta q_2, ..., \delta q_n$ of the system, the variation $\delta\mathbf{r_i}$ in an arbitrary position vector $\mathbf{r_i}(q_k, t)$ is determined by formula (1.22).

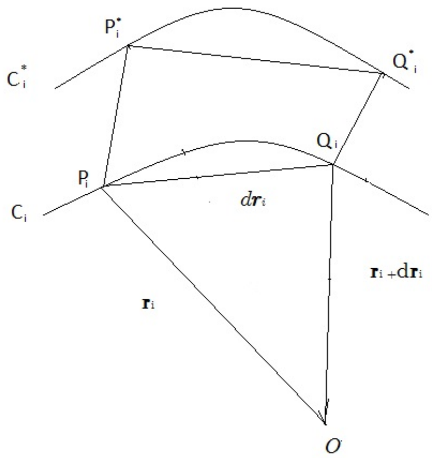

Figure 1.2:

These type of variation yield a path infinitely close to the actual path and is called the varied path.

The preceding analysis about the vector $\mathbf{r_i}$ can be applied to any function $G(q_1, q_2, ..., q_n; t)$ of the generalized coordinates and the time t. The differential of the this function, that is, its increment in the actual motion of the system during the interval dt of time t is given by

$$dG = \frac{\partial G}{\partial q_k} dq_k + \frac{\partial G}{\partial t} dt$$

on the other hand, the variation

$$\delta G = \frac{\partial G}{\partial q_k} \delta q_k, \ (k = 1, 2, ..., n)$$

is the infinitesimal change at a fixed moment of time t which takes G from one configuration to an infinitely close other configuration of the system.

1.5.4 The Exchange Rule $d\delta = \delta d$

Now we proceed to discuss the important rule $d\delta = \delta d$. In order to establish this rule, we assume that virtual displacement $\delta\mathbf{r_i}$ are differentiable function of time t. Operating variational equation (1.21) and (1.22) by d and δ, respectively, we get

$$\delta(d\mathbf{r_i}) = \frac{\partial^2\mathbf{r_i}}{\partial q_k \partial q_s} dq_k \delta q_s + \frac{\partial^2\mathbf{r_i}}{\partial q_k \partial t}\delta q_k dt + \frac{\partial \mathbf{r_i}}{\partial q_k}\delta(dq_k). \tag{1.23}$$

$$d(\delta\mathbf{r_i}) = \frac{\partial^2\mathbf{r_i}}{\partial q_k \partial q_s} dq_s \delta q_k + \frac{\partial^2\mathbf{r_i}}{\partial q_k \partial t}\delta q_k dt + \frac{\partial \mathbf{r_i}}{\partial q_k} d(\delta q_k). \tag{1.24}$$

$$(k, s = 1, 2, ..., n)$$

Interchanging the indices k and s in the first term on the right hand side of (1.23), we have

$$\delta(d\mathbf{r_i}) = \frac{\partial^2\mathbf{r_i}}{\partial q_s \partial q_k} dq_s \delta q_k + \frac{\partial^2\mathbf{r_i}}{\partial q_k \partial t}\delta q_k dt + \frac{\partial \mathbf{r_i}}{\partial q_k}\delta(dq_k). \tag{1.25}$$

Since $\mathbf{r_i}$ is a function of class C^2 in the domain of the variation $q_k (k = 1, 2, ..., n)$, equation (1.25) take the following form

$$\delta(d\mathbf{r_i}) = \frac{\partial^2\mathbf{r_i}}{\partial q_k \partial q_s} dq_s \delta q_k + \frac{\partial^2\mathbf{r_i}}{\partial q_k \partial t}\delta q_k dt + \frac{\partial \mathbf{r_i}}{\partial q_k}\delta(dq_k). \tag{1.26}$$

From (1.24) and (1.26), it follow that

$$d(\delta\mathbf{r_i}) - \delta(d\mathbf{r_i}) = \frac{\partial \mathbf{r_i}}{\partial q_k}[d(\delta q_k) - \delta(dq_k)]. \tag{1.27}$$

Let C_i denotes the trajectory of the i-th particle; P_i and Q_i denote its position on C_i at time t and $t + dt$, respectively. Let C_i^* denotes the varied path obtained from the actual path by means of the δ-variation. Let P_i^* and Q_i^* denote the position on C_i^* corresponding to P_i and Q_i at time t and $t + dt$ on the varied path, respectively. Let

$$\overline{OP_i} = \mathbf{r_i},$$

$$\overline{OP_i^*} = \mathbf{r_i} + \delta\mathbf{r_i},$$

and

$$\overline{P_i P_i^*} = \delta\mathbf{r_i}.$$

Again

$$\overline{OQ_i} = \mathbf{r_i} + d\mathbf{r_i},$$

and

$$\overline{P_iQ_i} = \overline{OQ_i} - OP_i = d\mathbf{r_i}.$$

In going from P_i to Q_i or P_i^* to Q_i^* we do not consider the change in time, therefore

$$\overline{P_i^*Q_i^*} = d(\mathbf{r_i} + \delta\mathbf{r_i}) = d\mathbf{r_i} + d\delta\mathbf{r_i},$$

$$\overline{Q_i^*Q_i} = \delta(\mathbf{r_i} + d\mathbf{r_i}) = \delta\mathbf{r_i} + \delta d\mathbf{r_i}.$$

From fig.3, it can easily be seen that

$$\overline{P_iQ_i^*} = \overline{P_iQ_i} + \overline{Q_iQ_i^*} = d\mathbf{r_i} + \delta\mathbf{r_i} + \delta d\mathbf{r_i} \tag{1.28}$$

and also

$$\overline{P_iQ_i^*} = \overline{P_iP_i^*} + \overline{P_i^*Q_i^*} = \delta\mathbf{r_i} + d\mathbf{r_i} + d\delta\mathbf{r_i}. \tag{1.29}$$

But from (1.28) and (1.29), it follows that

$$d\delta\mathbf{r_i} - \delta d\mathbf{r_i} = 0,$$

which may be written as

$$(d\delta - \delta d)\mathbf{r_i} = 0.$$

Since $\mathbf{r_i}$ is an arbitrary vector, therefore we have

$$d\delta = \delta d. \tag{1.30}$$

Consequently from (1.27) in conjunction with (3.42), it follows that

$$\frac{\partial\mathbf{r_i}}{\partial q_k}[d(\delta q_k) - \delta(dq_k)] = 0, \tag{1.31}$$

which in terms of the cartesian coordinates may be expressed as

$$\frac{\partial\mathbf{r}_\lambda}{\partial q_k}[d(\delta q_k) - \delta(dq_k)] = 0.$$

$$(\lambda = 1, 2, ..., 3N; k = 1, 2, ..., n)$$

Here the Cartesian coordinates are geometrically connected with $q's$ by means of the equation of transformation

$$x_\lambda = x_\lambda(q_k; t).$$

Since the rank of the matrix

$$\begin{bmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & \cdots & \frac{\partial x_1}{\partial q_n} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} & \cdots & \frac{\partial x_2}{\partial q_n} \\ \vdots & \ddots & \vdots & \\ \frac{\partial z_{3N}}{\partial q_1} & \frac{\partial z_{3N}}{\partial q_2} & \cdots & \frac{\partial z_{3N}}{\partial q_n} \end{bmatrix}$$

is n, it follows that the only solution of the system of equation (3.43) is the trivial solution.

Hence

$$d(\delta q_k) - \delta(dq_k) = 0$$

or

$$d(\delta q_k) = \delta(dq_k).$$

Thus we have proved that the rule $d\delta = \delta d$ holds for the generalized coordinates when the system is holonomic.

1.5.5 Actual And Virtual Work

Let $d\mathbf{r}(\delta\mathbf{r_i})$ denotes a small displacement along the actual (virtual) path. Then the small work done by the reaction force R along the actual(virtual) small displacement denoted by $dW(\delta W)$ is given by

$$dW = \mathbf{R} \cdot d\mathbf{r} \quad (\delta W = \mathbf{R} \cdot \delta\mathbf{r}). \tag{1.32}$$

1.5.6 Ideal Constraint

The presence of constraint implies that exist certain forces called the actual reaction or the forces of constraints, which are responsible for keeping a dynamical system in the state of equilibrium and these forces are denoted by $\mathbf{R}$. So on the basis of this we give the following:

Definition

If the work done (1.32) by the reaction (or the forces of constraints) $\mathbf{R}$ vanishes then the constraint is said to be ideal (smooth or perfect). Example of ideal constraint include, the mutual reaction of two particles which are rigidly connected together, the reactions of fixed smooth surface and the reactions of fixed perfectly rough surfaces etc ([25],[24]).

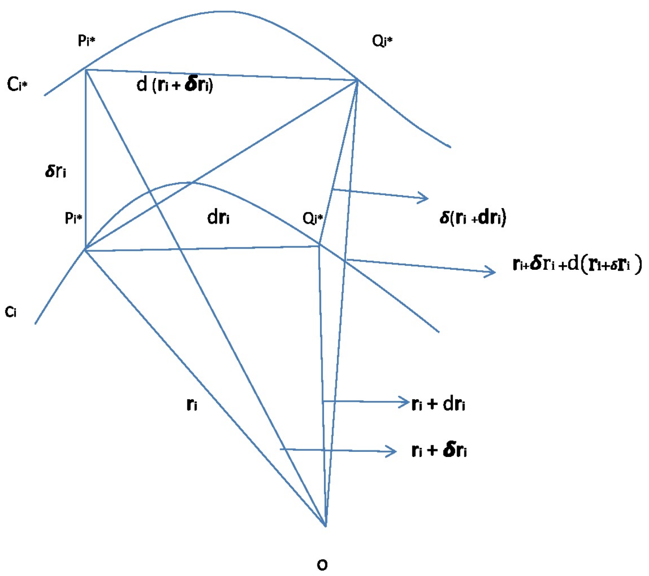
Pi*
Qj*
Ci*
d (ri + δri)
δri
Pi*
dri
Qj*
δ(ri +dri)
ri+δri +d(ri+δri)
ci
ri
ri + dri
ri + δri
o

Figure 1.3:

Chapter 2

Conservation Laws For Variable Mass Dynamical System Via Differential Variational Principles

2.1 Introduction

The study of conservation law for the conservative and the non-conservative dynamical system with the finite degree of freedom has great importance in physics and engineering. Two well known approaches are commonly used to obtain conservation laws for such dynamical system. The first approach involves integral variational principle while the second one involves differential variational principles.

The first is also known as Neotherian approach which is based on the invariant properties of the Hamilton's action integral with respect to it's infinitesimal transformation of the generalized coordinates and the time (see Vujanovic and Jones [17]). This approach can be used only for those cases in which the system is completely specified by the Lagrangian function, which is discussed in [19]. Here the invariant is the gauge invariant. By using the variational principle the conservation law for the purely non-conservative dynamical system explained in [18].

The other one is differential principle, this approach is used by B. Vujanovic, in [4] to find the conservation law of the dynamical system which is so called D'Alembert's principle. The importance of D'Alembert's principle is that, it is equally valid for the both conservative and non-conservative system. An attempt have been made in [20] to find the conservation law for the Jourdain and Gauss variation principle. In [7] Aftab Ahmed used differential variation principle to find the conservation law for the partially conserve variable mass system. An attempt have been made in [20] to find the conservative law for the Jourdain and Gauss variation principle. Now here question arise, can a differential variation principle of Jourdain and Gauss' work for the conservation law of the dynamical with variable mass? In this thesis, we will answer this question, we will use these principles to find the conserved quantities of the various kinds of a conservative and the non conservative dynamical system with variable mass. To achieve this goal, we will use the approach of differential variational principle by extending the idea of Vujanovic, [20].

2.2 Generalization Of Jourdian's Principle For The Variable Mass System

In this section we will find the Lagrangian-Jourdian principle on the basis of asynchronous virtual variation. Here we will find the generalization of Jourdian and Gauss principle to cover the holonomic dynamical system of the variable mass in conjunction with a generalized variation which includes the variation not only for the position but also the time variation. So, therefore, we introduce the asynchronous variation and the concept of infinitesimal transformation along with the gauge-variance to obtain conservation law for variable mass dynamical system.

2.2.1 Asynchronous Variation

Consider the motion of a dynamical system consisting of N-particles with the generalized coordinate q_1,q_2,...,q_n which are the continuous function on time t. Let C and C^1 denote the actual and virtual paths, respectively. Consider a point Q(P_k, t) on the actual path and a point Q^*(q_k+δq_k,t) on the virtual path, if these points are correlated at same time then the variation is the synchronous variation and is denoted by δ. This implies that the δ-variation allows change in the generalized coordinates but does not allow change in the time, that is, δt=0.

Now suppose that $\bar{q}_k$ $k = 1, 2, ..., n$(t) represent the coordinates of a point on the virtual path and q_k(t) represent coordinates of the point on the actual path, then by [16]

$$\bar{q}_k(t) = q_k(t) + \delta q_k, \qquad (2.1)$$
$$\bar{t} = t.$$

At the same time, in order to express the internal symmetry of the dynamical system properly, we must take into account the asynchronous variation. If the generalized coordinate q_k(k=1,2,......,n) of the point on the actual path, so we will determine the corresponding infinitesimally close motion by $\bar{q}_k$(t+Δt). So we have

$$\bar{q}_k(t + \Delta t) = q_k(t + \Delta t) + \delta q_k,$$
$$= q_k(t) + \dot{q}_k(t)\Delta t + \delta q_k,$$

using the equation (2.1), we obtain from last equation

$$\bar{q}_k(t + \Delta t) = \bar{q}_k(t) - \delta q_k(t) + \dot{q}_k\Delta t + \delta q_k,$$
$$\bar{q}_k(t + \Delta t) = \bar{q}_k(t) + \dot{q}_k\Delta t. \qquad (2.2)$$

Let us define the asynchronous variation by the equation

$$\Delta q_k = \bar{q}_k(t + \Delta t) - q_k(t), \qquad (2.3)$$

again using (2.1)

$$\Delta q_k = \delta q_k + \dot{q}_k\Delta t. \qquad (2.4)$$

For any function the asynchronous variation of equation (2.4) can written as,

$$\Delta F = \delta F + \dot{F}\Delta t.$$

Therefore for generalized velocity we can infer from (2.4)

$$\Delta \dot{q}_k = \delta \dot{q}_k + \ddot{q}_k \Delta t. \tag{2.5}$$

Now, differentiating (2.4) with respect to t we have

$$(\Delta q_k)^{\dot{}} = \delta \dot{q}_k + \ddot{q}_k \Delta t + \dot{q}_k (\Delta t)^{\dot{}}. \tag{2.6}$$

From (2.5) and (2.6) eliminate $\delta \dot{q}_k$, we get

$$(\Delta q_k)^{\dot{}} = \Delta \dot{q}_k - \ddot{q}_k \Delta t + \ddot{q}_k \Delta t + \dot{q}_k (\Delta t)^{\dot{}}$$

$$(\Delta q_k)^{\dot{}} = \Delta \dot{q}_k + \dot{q}_k (\Delta t)^{\dot{}}. \tag{2.7}$$

For the generalized variation of the acceleration vector

$$\Delta \ddot{q}_k = \delta \ddot{q}_k + \dddot{q}_k \Delta t. \tag{2.8}$$

Whose time derivative can be written as

$$(\Delta q_k)^{\ddot{}} = \delta \ddot{q}_k + \dddot{q}_k \Delta t + \ddot{q}_k (\Delta t)^{\dot{}} + \ddot{q}_k (\Delta t)^{\dot{}} + \dot{q}_k (\Delta t)^{\ddot{}},$$

this time eliminating $\delta \ddot{q}_k$ by using (2.8) in the last equation

$$(\Delta q_k)^{\ddot{}} = \Delta \ddot{q}_k - \dddot{q}_k \Delta t + \dddot{q}_k \Delta t + 2\ddot{q}_k (\Delta t)^{\dot{}} + \dot{q}_k (\Delta t)^{\ddot{}},$$

$$(\Delta q_k)^{\ddot{}} = \Delta \ddot{q}_k + 2\ddot{q}_k (\Delta t)^{\dot{}} + \dot{q}_k (\Delta t)^{\ddot{}}. \tag{2.9}$$

As discuss by Vujanovic, B.[12], a closed observation of the concept of asynchronous variation reveal that the quantities Δq_k and Δt, given by the following equation

$$\bar{q}_k(\bar{t}) = q_k(t) + \Delta q_k. \tag{2.10}$$

$$\bar{t} = t + \Delta t. \tag{2.11}$$

Since, equations (2.10) and (2.11) has great importance in the study of conservation laws. So in order to discussed the internal symmetry of the dynamical system and constraints, the transformation in the equations (2.10) and (2.11) may be extended to include the position variable q_k and the dynamical variables $\dot{q}_k$ together with the time t. We will suppose that this structure in the form

$$\bar{q}_k(\bar{t}) = q_k + \epsilon F_k(\dot{q}_k, q_k; t), \tag{2.12}$$

$$\bar{t} = t + \epsilon f(\dot{q}_k, q_k; t). \tag{2.13}$$

Where $F_k(\dot{q}_k, q_k; t) = F_k(\dot{q}_1, ..., \dot{q}_n, q_1, ..., q_n; t)$ and $f(\dot{q}_k, q_k; t)$ the space and the time generators of the infinitesimal transformations respectfully, and the ϵ is a small constant positive number.

By comparing the equations (2.10) and (2.11) with the equations (2.12) and (2.13) respectively, so we get that

$$\Delta q_k = \epsilon F_k(\dot{q}_k, q_k; t), \tag{2.14}$$

$$\Delta t = \epsilon f(\dot{q}_k, q_k; t). \tag{2.15}$$

Since we consider the Jourdain's phase space, so $\Delta \dot{q}_k$, $(\Delta q_k)^{\cdot}$ and $(\Delta t)^{\cdot}$ together with

$$\Delta q_k = 0, \Delta t = 0. \tag{2.16}$$

We obtain from equations (2.5) and (2.6),

$$\Delta \dot{q}_k = \delta \dot{q}_k \tag{2.17}$$

$$\Delta \dot{q}_k = (\Delta q_k)^{\cdot} - \dot{q}_k (\Delta t)^{\cdot}. \tag{2.18}$$

Here we use an infinitesimal transformation of the generalized coordinates, velocity and time for the better understanding of the nature of the Jourdian variations:

$$\begin{aligned} \bar{q}_k &= q_k, \ \bar{t} = t, \\ \frac{d\bar{q}_k}{dt} - \frac{dq_k}{dt} &= \delta \dot{q}_k = \Delta \dot{q}_k, \\ \frac{d\bar{q}_k}{dt} - \frac{dq_k}{dt} &= (\Delta q_k)^{\cdot}. \end{aligned} \tag{2.19}$$

Consider the infinitesimal quantities $(\Delta q_k)^{\cdot}$ and $(\Delta t)^{\cdot}$ as the primitively quantities for the Jourdain's infinitesimal transformation, so that

$$(\Delta q_k)^{\cdot} = \epsilon F_k(q_k, \dot{q}_k, t), \tag{2.20}$$

$$(\Delta t)^{\cdot} = \epsilon f(q_k, \dot{q}_k, t). \tag{2.21}$$

By virtue of equations (2.20) and (2.21) in the equation (2.18) becomes

$$\delta \dot{q}_k = \epsilon [F_k(q_k, \dot{q}_k, t) - \dot{q}_k f(q_k, \dot{q}_k, t)], \tag{2.22}$$

where $q_k = q_1, ..., q_n$ and $\dot{q}_k = \dot{q}_1, ..., \dot{q}_n$,
comparing equation (2.17) and equation (2.22), we obtain

$$\Delta \dot{q}_k = \epsilon [F_k(q_k, \dot{q}_k, t) - \dot{q}_k f(q_k, \dot{q}_k, t)]. \tag{2.23}$$

2.3 Equation Of Motion For The Variable Mass Dynamical System

Let us consider a variable mass dynamical system consisting of N-particles . Suppose $q_k, \dot{q}_k$ and $\ddot{q}_k$ be the position, velocity and acceleration of the $i-th$ particle of the system. If $\mathbf{Y}_i, \mathbf{R}_i$ denote the applied force and the constraint force. Let $\mathbf{F}_{ij}, \mathbf{F}_{ji}$ denote the mutually attractive and repulsive forces within the system of N-particle.

Then by the Newton's second law can be written as

$$\mathbf{Y}_i + \mathbf{R}_i + \mathbf{F}_{ij} + \mathbf{F}_{ji} = \frac{d\mathbf{p}_i}{dt}, \tag{2.24}$$

where $\mathbf{p}_i$ is the linear momentum of the particle is given by

$$\mathbf{p}_i = m_i \mathbf{v}_i,$$

and $\mathbf{v}_i$ is the velocity of the particle which can be written as

$$\mathbf{v}_i = \dot{\mathbf{r}}_i.$$

Hence, from (2.24) we have

$$\mathbf{Y}_i + \mathbf{R}_i + \mathbf{F}_{ij} + \mathbf{F}_{ji} = \frac{d(m\dot{\mathbf{r}}_i)}{dt}. \tag{2.25}$$

The classification is performed in such away that $\mathbf{R}_i(constriant)$ are ideal, so the virtual work is zero

$$\sum_{i}^{N} \mathbf{R}_i \cdot \delta \mathbf{r}_i = 0.$$

Taking into the account the virtual velocity $\delta \dot{q}_i$ of the i-th particle and the dot multiplying with the equation (2.24), we get

$$\left[\mathbf{Y}_i + (\mathbf{F}_{ij} + \mathbf{F}_{ji}) - \frac{d(m_i \dot{\mathbf{r}}_i)}{dt}\right] \cdot \delta \dot{\mathbf{r}}_i = 0. \tag{2.26}$$

For the system of N-particle, we sum over i and j from 1 to N, $i.e$

$$\sum_{i=1}^{N} \left[\mathbf{Y}_i + \sum_{j=1}^{N} (\mathbf{F}_{ij} + \mathbf{F}_{ji}) - \frac{d(m_i \dot{\mathbf{r}}_i)}{dt}\right] \cdot \delta \dot{\mathbf{r}}_i = 0. \tag{2.27}$$

If attracting forces $\mathbf{F}_{ij}$ and $\mathbf{F}_{ji}$ satisfy Newton's third law, then we must have

$$\sum_{i=1}^{N}\sum_{j=1}^{N}(\mathbf{F}_{ij}+\mathbf{F}_{ji})\cdot\delta\dot{\mathbf{r}}_i=0.$$

So

$$\sum_{i=1}^{N}\left(\mathbf{Y}_i-\frac{d(m_i\dot{\mathbf{r}}_i)}{dt}\right)\cdot\delta\dot{\mathbf{r}}_i=0, \tag{2.28}$$

which is the required Jourdain principle.

In order to transform equation (2.28) in term of generalized coordinates $q_k(t)$ $(k=1,2.....n)$, whose value at time t determine the configuration of the system. We let

$\mathbf{r}_i=\mathbf{r}_i(q_k,t)$ with $i=1,2...N$ and $k=1,2....n$,

$$\delta\mathbf{r}_i=\frac{\partial\mathbf{r}_i}{\partial q_k}\delta q_k,$$

whose derivative with respect to time is given by

$$\delta\dot{\mathbf{r}}_i=(\frac{\partial^2\mathbf{r}_i}{\partial q_k\partial q_l}\dot{q}_k+\frac{\partial^2\mathbf{r}_i}{\partial q_k\partial t})\delta q_k+\frac{\partial\mathbf{r}_i}{\partial q_k}\delta\dot{q}_k, \tag{2.29}$$

but in jourdian formulism $\delta q_k=0$, so

$\delta\dot{\mathbf{r}}_i=\frac{\partial\mathbf{r}_i}{\partial q_k}\delta\dot{q}_k$

putting the value of $\delta\dot{r}_i$ in equation (2.28) we obtain

$$\sum_{i=1}^{N}\left(\frac{d(m_i\dot{\mathbf{r}}_i)}{dt}-\mathbf{Y}_i\right)\cdot\frac{\partial\mathbf{r}_i}{\partial q_k}\delta\dot{q}_k=0. \tag{2.30}$$

2.4 Transformation Of Jourdain Principle

Let us consider a dynamical system with N number of particles and we assume that the forces $Y_1,.......Y_n$ act at some points of the system. The virtual velocity of these points will be denoted by $\delta\dot{\mathbf{r}}_1,.....,\delta\dot{\mathbf{r}}_N$.

So we make the following transformation of the Jourdain's principle (2.29) into an expression involving the Jourdain's asynchronous variation of the generalized velocity and the time elements. By putting the (2.22) into the equation (2.30), the standard procedure followed in L. A Parse Introduction to dynamics [13] we get,

$$\epsilon\sum_{i=1}^{N}(\mathbf{Y}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_k}-\frac{d(m_i\dot{\mathbf{r}}_i)}{dt}\cdot\frac{\partial\mathbf{r}_i}{\partial q_k})(F_k-\dot{q}_kf)=0. \tag{2.31}$$

using $\frac{\partial \mathbf{r}_i}{\partial q_k} = \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}$, we consider

$$\begin{aligned}
&\frac{d}{dt}(m_i\dot{\mathbf{r}}_i)\cdot\frac{\partial \mathbf{r}_i}{\partial q_k}(F_k-\dot{q}_k f)\\
&=\frac{d}{dt}\left[m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_k}(F_k-\dot{q}_k)\right]-m_i\dot{\mathbf{r}}_i\cdot\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)^{\cdot}(F_k-\dot{q}_k f)-m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_k}(F_k-\dot{q}_k f)^{\cdot}\\
&=\frac{d}{dt}\left[m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}(F_k-\dot{q}_k f)\right]-m_i\dot{\mathbf{r}}_i\cdot\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)^{\cdot}(F_k-\dot{q}_k f)-m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}(F_k-\dot{q}_k f)^{\cdot},\\
&=\frac{d}{dt}\left[\left(\frac{\partial}{\partial \dot{q}_k}(\frac{1}{2}m_i\dot{\mathbf{r}}_i^{\,2})-\frac{1}{2}\frac{\partial m_i}{\partial \dot{q}_k}\dot{\mathbf{r}}_i^{\,2}\right)(F_k-\dot{q}_k f)\right]-m_i\dot{\mathbf{r}}_i\cdot\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)^{\cdot}(F_k-\dot{q}_k f)\\
&-\ m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}(F_k-\dot{q}_k f)^{\cdot}.
\end{aligned}$$

As we know $T_i = \frac{1}{2}m_i\dot{\mathbf{r}}_i^{\,2}$,

$$\begin{aligned}
&\frac{d}{dt}\left[m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}(F_k-\dot{q}_k f)\right]\\
&=\frac{d}{dt}\left[(\frac{\partial T_i}{\partial \dot{q}_k}-\frac{1}{2}\frac{\partial m_i}{\partial \dot{q}_k}\dot{\mathbf{r}}_i^{\,2})(F_k-\dot{q}_k f)\right]-m_i\dot{\mathbf{r}}_i\cdot\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)^{\cdot}(F_k-\dot{q}_k f)\\
&-m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}(F_k-\dot{q}_k f)^{\cdot},\\
&=\frac{d}{dt}\left[\frac{\partial T_i}{\partial \dot{q}_k}(F_k-\dot{q}_k f)\right]-\frac{1}{2}\frac{d}{dt}\left[\frac{\partial m_i}{\partial \dot{q}_k}\dot{\mathbf{r}}_i^{\,2}(F_k-\dot{q}_k f)\right]\\
&-m_i\dot{\mathbf{r}}_i\cdot\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)^{\cdot}(F_k-\dot{q}_k f)-m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}(F_k-\dot{q}_k f)^{\cdot},\\
&=\frac{d}{dt}\left[\frac{\partial T_i}{\partial \dot{q}_k}(F_k-\dot{q}_k f)\right]-\frac{d}{dt}\left[\frac{1}{2m_i}\frac{\partial m_i}{\partial \dot{q}_k}m_i\dot{\mathbf{r}}_i^{\,2}(F_k-\dot{q}_k f)\right]\\
&-m_i\dot{\mathbf{r}}_i\cdot\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)^{\cdot}(F_k-\dot{q}_k f)-m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}(F_k-\dot{q}_k f)^{\cdot},
\end{aligned}$$

by using $T_i = \frac{1}{2}m_i\dot{\mathbf{r}}_i^2$,

$$\begin{aligned}\frac{d}{dt}(m_i\dot{\mathbf{r}}_i)\cdot\frac{\partial \mathbf{r}_i}{\partial q_k}(F_k-\dot{q}_kf) &= \frac{d}{dt}\left[\frac{\partial T_i}{\partial \dot{q}_k}(F_k-\dot{q}_kf)\right]-\frac{d}{dt}\left[\frac{1}{m_i}\frac{\partial m_i}{\partial \dot{q}_k}T_i(F_k-\dot{q}_kf)\right]\\ &-m_i\dot{\mathbf{r}}_i\cdot\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)^{\cdot}(F_k-\dot{q}_kf)-m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}(F_k-\dot{q}_kf)^{\cdot},\\ &=\frac{d}{dt}\left[\frac{\partial T_i}{\partial \dot{q}_k}(F_k-\dot{q}_kf)\right]-\frac{d}{dt}\left[\frac{\partial ln(m_i)}{\partial \dot{q}_k}T_i(F_k-\dot{q}_kf)\right]\\ &-m_i\dot{\mathbf{r}}_i\cdot\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)^{\cdot}(F_k-\dot{q}_kf)-m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}(F_k-\dot{q}_kf)^{\cdot},\\ &=\frac{d}{dt}\left[\frac{\partial T_i}{\partial \dot{q}_k}(F_k-\dot{q}_kf)\right]-\frac{d}{dt}\left[\frac{\partial ln(m_i)}{\partial \dot{q}_k}T_i(F_k-\dot{q}_kf)\right]\\ &-m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \dot{\mathbf{r}}_i}{\partial q_k}(F_k-\dot{q}_kf)-m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}(F_k-\dot{q}_kf)^{\cdot},\\ &=\frac{d}{dt}\left[\frac{\partial T_i}{\partial \dot{q}_k}(F_k-\dot{q}_kf)\right]-\frac{d}{dt}\left[\frac{\partial ln(m_i)}{\partial \dot{q}_k}T_i(F_k-\dot{q}_kf)\right]\\ &-\left[\frac{1}{2}\frac{\partial m_i\dot{\mathbf{r}}_i^2}{\partial q_k}-\frac{1}{2}\frac{\partial m_i}{\partial q_k}\dot{\mathbf{r}}_i^2\right](F_k-\dot{q}_kf)\\ &-m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}(F_k-\dot{q}_kf)^{\cdot},\end{aligned}$$

by using $T_i = \frac{1}{2}m_i\dot{\mathbf{r}}_i^2$ in the above equation's

$$\begin{aligned}&\frac{d}{dt}(m_i\dot{\mathbf{r}}_i)\cdot\frac{\partial \mathbf{r}_i}{\partial q_k}(F_k-\dot{q}_kf)\\ &=\frac{d}{dt}\left[\frac{\partial T_i}{\partial \dot{q}_k}(F_k-\dot{q}_kf)\right]\\ &-\frac{d}{dt}\left[\frac{\partial ln(m_i)}{\partial \dot{q}_k}T_i(F_k-\dot{q}_kf)\right]-\left[\frac{\partial T_i}{\partial q_k}-\frac{\partial ln(m_i)}{\partial q_k}T_i\right](F_k-\dot{q}_kf)\\ &-m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}(F_k-\dot{q}_kf)^{\cdot},\\ &=\frac{d}{dt}\left[\frac{\partial T_i}{\partial \dot{q}_k}(F_k-\dot{q}_kf)\right]-\frac{d}{dt}\left[\frac{\partial ln(m_i)}{\partial \dot{q}_k}T_i(F_k-\dot{q}_kf)\right]\\ &-\left[\frac{\partial T_i}{\partial q_k}-\frac{\partial ln(m_i)}{\partial q_k}T_i\right](F_k-\dot{q}_kf)-\left(\frac{1}{2}\frac{\partial m_i\dot{\mathbf{r}}_i^2}{\partial \dot{q}_k}-\frac{1}{2}\frac{\partial m_i}{\partial \dot{q}_k}\dot{\mathbf{r}}_i^2\right)(F_k-\dot{q}_kf)^{\cdot},\\ &=\frac{d}{dt}\left[\frac{\partial T_i}{\partial \dot{q}_k}(F_k-\dot{q}_kf)\right]-\frac{d}{dt}\left[\frac{\partial ln(m_i)}{\partial \dot{q}_k}T_i(F_k-\dot{q}_kf)\right]\\ &-\left[\frac{\partial T_i}{\partial q_k}-\frac{\partial ln(m_i)}{\partial q_k}T_i\right](F_k-\dot{q}_kf)-\left(\frac{\partial T_i}{\partial \dot{q}_k}-\frac{\partial ln(m_i)}{\partial \dot{q}_k}T_i\right)(F_k-\dot{q}_kf)^{\cdot},\end{aligned}$$

put the value of $\frac{d(m_i\dot{\mathbf{r}}_i)}{dt}\cdot\frac{\partial\ \mathbf{r}_i}{\partial q_k}(F_k-\dot{q}_kf)$ in the equation (2.31)

$$\epsilon\sum_{i=1}^{N}[\mathbf{Y}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_k}(F_k-\dot q_k f)-\frac{d}{dt}\left[\frac{\partial T_i}{\partial\dot q_k}(F_k-\dot q_k f)\right]+\frac{d}{dt}\left[\frac{\partial ln(m_i)}{\partial\dot q_k}T_i(F_k-\dot q_k f)\right]$$
$$+\left[\frac{\partial T_i}{\partial q_k}-\frac{\partial ln(m_i)}{\partial q_k}T_i\right](F_k-\dot q_k f)+\left(\frac{\partial T_i}{\partial\dot q_k}-\frac{\partial ln(m_i)}{\partial\dot q_k}T_i\right)(F_k-\dot q_k \dot{f)}]=0. \tag{2.32}$$

Let us suppose that each particle of the dynamical system is partially conserved, so it has conserved force $\frac{-\partial V}{\partial q_k}$ and the none conservative force $Q_k(q,\dot q,t)$. Thus we have

$$\sum_{i=1}^{N}\mathbf{Y}_i\cdot\frac{\partial\mathbf{r}_i}{\partial q_k}=\frac{-\partial V}{\partial q_k}+Q_k(q,\dot q,t). \tag{2.33}$$

Where $V=V(x,t)$ is the potential function,
so put equation (2.32) in the equation (2.33)

$$\epsilon\sum_{i=1}^{N}[(Q_k-\frac{\partial V}{\partial q_k})(F_k-\dot q_k f)-\frac{d}{dt}\left[\frac{\partial T_i}{\partial\dot q_k}(F_k-\dot q_k f)\right]+\frac{d}{dt}\left[\frac{\partial ln(m_i)}{\partial\dot q_k}T_i(F_k-\dot q_k f)\right]$$
$$+\left[\frac{\partial T_i}{\partial q_k}-\frac{\partial ln(m_i)}{\partial q_k}T_i\right](F_k-\dot q_k f)+\left(\frac{\partial T_i}{\partial\dot q_k}-\frac{\partial ln(m_i)}{\partial\dot q_k}T_i\right)(F_k-\dot q_k \dot{f)}]=0, \tag{2.34}$$

now introduce the Lagrangian function $L=T-V$

$$\frac{\partial L}{\partial\dot q_k}=\frac{\partial T_i}{\partial\dot q_k},$$

$\epsilon\sum_{i=1}^{N}[(Q_k-\frac{\partial V}{\partial q_k})(F_k-\dot q_k f)-\frac{d}{dt}\left[\frac{\partial L}{\partial\dot q_k}(F_k-\dot q_k f)\right]+\frac{d}{dt}\left[\frac{\partial ln(m_i)}{\partial\dot q_k}T_i(F_k-\dot q_k f)\right]+\left[\frac{\partial(T_i)}{\partial q_k}-\frac{\partial ln(m_i)}{\partial q_k}T_i\right](F_k-\dot q_k f)+\left(\frac{\partial L}{\partial\dot q_k}-\frac{\partial ln(m_i)}{\partial\dot q_k}T_i\right)(F_k-\dot q_k \dot{f)}]=0,$

$$\epsilon\sum_{i=1}^{N}[Q_k(F_k-\dot q_k f)-\frac{d}{dt}\left[\frac{\partial L}{\partial\dot q_k}(F_k-\dot q_k f)\right]+\frac{d}{dt}\left[\frac{\partial ln(m_i)}{\partial\dot q_k}T_i(F_k-\dot q_k f)\right]$$
$$+\left[\frac{\partial L}{\partial q_k}-\frac{\partial ln(m_i)}{\partial q_k}T_i\right](F_k-\dot q_k f)+\left(\frac{\partial L}{\partial\dot q_k}-\frac{\partial ln(m_i)}{\partial\dot q_k}T_i\right)(\dot F_k-\ddot q_k f-\dot q_k\dot f)]=0,$$

$$\epsilon\sum_{i=1}^{N}[Q_k(F_k-\dot q_k f)-\frac{d}{dt}\left[\frac{\partial L}{\partial\dot q_k}(F_k-\dot q_k f)\right]+\frac{d}{dt}\left[\frac{\partial ln(m_i)}{\partial\dot q_k}T_i(F_k-\dot q_k f)\right]$$
$$+\left[\frac{\partial L}{\partial q_k}-\frac{\partial ln(m_i)}{\partial q_k}T_i\right](F_k-\dot q_k f)+\left(\frac{\partial L}{\partial\dot q_k}-\frac{\partial ln(m_i)}{\partial\dot q_k}T_i\right)(\dot F_k-\ddot q_k f-\dot q_k\dot f)]=0,$$

$$\epsilon\sum_{i=1}^{N}[Q_k(F_k-\dot{q}_kf)-\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{q}_k}(F_k-\dot{q}_kf)\right]+\frac{d}{dt}\left[\frac{\partial ln(m_i)}{\partial\dot{q}_k}T_i(F_k-\dot{q}_kf)\right]$$
$$-\frac{\partial ln(m_i)}{\partial q_k}T_i(F_k-\dot{q}_kf)-\frac{\partial ln(m_i)}{\partial\dot{q}_k}T_i)(\dot{F}_k-\ddot{q}_kf-\dot{q}_k\dot{f})$$
$$-f(\frac{\partial L}{\partial q_k}\dot{q}_k+\frac{\partial L}{\partial\dot{q}_k}\ddot{q}_k+\frac{\partial L}{\partial t})+\frac{\partial L}{\partial t}f+\frac{\partial L}{\partial\dot{q}_k}(\dot{F}_k-\dot{q}_k\dot{f})+\frac{\partial L}{\partial q_k}F_k]=0.$$

But $\dot{L}(q_k,\dot{q}_k,t)=\frac{\partial L}{\partial q_k}\dot{q}_k+\frac{\partial L}{\partial\dot{q}_k}\ddot{q}_k+\frac{\partial L}{\partial t}$, and dot represent $\frac{d}{dt}$
then we will get that
$\epsilon\sum_{i=1}^{N}[Q_k(F_k-\dot{q}_kf)-\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{q}_k}(F_k-\dot{q}_kf)\right]+\frac{d}{dt}\left[\frac{\partial ln(m_i)}{\partial\dot{q}_k}T_i(F_k-\dot{q}_kf)\right]-f\dot{L}+\frac{\partial L}{\partial t}f+\frac{\partial L}{\partial\dot{q}_k}(\dot{F}_k-\dot{q}_k\dot{f})+\frac{\partial L}{\partial q_k}F_k+T_if(\frac{\partial ln(m_i)}{\partial\dot{q}_k}\ddot{q}_k+\frac{\partial ln(m_i)}{\partial q_k}\dot{q}_k+\frac{\partial ln(m_i)}{\partial t})-T_if\frac{\partial ln(m_i)}{\partial t}-\frac{\partial ln(m_i)}{\partial\dot{q}_k}T_i(\dot{F}_k-\dot{q}_k\dot{f})-\frac{\partial ln(m_i)}{\partial q_k}T_iF_k]=0,$

$$\begin{aligned}&\epsilon\sum_{i=1}^{N}[Q_k(F_k-\dot{q}_kf)-\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{q}_k}(F_k-\dot{q}_kf)\right]\\&+\frac{d}{dt}\left[\frac{\partial ln(m_i)}{\partial\dot{q}_k}T_i(F_k-\dot{q}_kf)\right]-f\dot{L}+\frac{\partial L}{\partial t}f+\frac{\partial L}{\partial\dot{q}_k}(\dot{F}_k-\dot{q}_k\dot{f})+\frac{\partial L}{\partial q_k}F_k\\&+T_if\frac{dln(m_i)}{dt}-T_if\frac{\partial ln(m_i)}{\partial t}-\frac{\partial ln(m_i)}{\partial\dot{q}_k}T_i(\dot{F}_k-\dot{q}_k\dot{f})\\&-\frac{\partial ln(m_i)}{\partial q_k}T_iF_k]=0,\end{aligned}\tag{2.35}$$

because $\frac{dln(m_i)}{dt}=\frac{\partial ln(m_i)}{\partial\dot{q}_k}\ddot{q}_k+\frac{\partial ln(m_i)}{\partial q_k}\dot{q}_k+\frac{\partial ln(m_i)}{\partial t}$ and $m_i=m_i(q_k,\dot{q}_k,t)$,

An arbitrary function $\epsilon\dot{p}(q_k,\dot{q}_k,t)$ called the gauge variant function in the classical field theory can be add and subtract in the equation (2.35), the function depends on the generalized coordinates, velocity and the time. But ϵ is a small number so it must be $\epsilon\neq 0$. Then equation's (2.35) becomes
$\sum_{i=1}^{N}[Q_k(F_k-\dot{q}_kf)+\frac{\partial L}{\partial t}f+\frac{\partial L}{\partial\dot{q}_k}(\dot{F}_k-\dot{q}_k\dot{f})+\frac{\partial L}{\partial q_k}F_k-T_if\frac{\partial ln(m_i)}{\partial t}-\frac{\partial ln(m_i)}{\partial\dot{q}_k}T_i(\dot{F}_k-\dot{q}_kf)-\frac{\partial ln(m_i)}{\partial q_k}T_iF_k-\dot{p}(q_k,\dot{q}_k,t)+\dot{f}L-ln(m_i)\frac{d(T_if)}{dt}]-\frac{d}{dt}[\frac{\partial L}{\partial\dot{q}_k}(F_k-\dot{q}_kf)-\frac{\partial ln(m_i)}{\partial\dot{q}_k}T_i(F_k-\dot{q}_kf)+fL-T_ifln(m_i)-p]=0.$

Which is the required transformation of the Jourdian's principle for the variable mass

2.4.1 Condition For The Existence Of Conserved Quantity

From the transform form of the Jourdian's principle, so it is obvious that if

$$\sum_{i=1}^{N}[Q_k(F_k - \dot{q}_k f) + \frac{\partial L}{\partial t} f + \frac{\partial L}{\partial \dot{q}_k}(\dot{F}_k - \dot{q}_k \dot{f}) + \frac{\partial L}{\partial q_k} F_k - T_i f \frac{\partial ln(m_i)}{\partial t}$$
$$- \frac{\partial ln(m_i)}{\partial \dot{q}_k} T_i(\dot{F}_k - \dot{q}_k f) - \frac{\partial ln(m_i)}{\partial q_k} T_i F_k - \dot{p}(q_k, \dot{q}_k, t) + \dot{f} l - ln(m_i) \frac{d(T_i f)}{dt}] = 0, \tag{2.36}$$

then the conservation of the form is
$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}(F_k - \dot{q}_k f) - \frac{\partial ln(m_i)}{\partial \dot{q}_k} T_i(F_k - \dot{q}_k f) + fL - T_i f ln(m_i) - p\right] = 0$

$$\frac{\partial L}{\partial \dot{q}_k}(F_k - \dot{q}_k f) - \frac{\partial ln(m_i)}{\partial \dot{q}_k} T_i(F_k - \dot{q}_k f) + fL - T_i f ln(m_i) - p = c. \tag{2.37}$$

The equation (2.36) must be satisfied by every infinitesimal transformation in the form (2.20) and (2.21) and the gauge variant function $p = (q_k, \dot{q}_k, t)$, so there exist a conserved quantity in the form (2.37). Since for the case of a conservative dynamical system, i.e q_k, the equation (2.36) and (2.37) constitute the classical form of the Noether's theorem.

we call the partial differential equations in the form of equation (2.36) is the generalized Killing's equation for the dynamical system of the variable mass, which help to determine the gauge-variant function. If this system of linear partial differential equations admits a solution say $F_k(q, \dot{q}, t), f(q, \dot{q}, t)$ and $p(q, \dot{q}, t)$, then a constant quantity of the form (2.37) will must be exists which furnishes the conservation law of the variable mass non-conservative dynamical system under consideration.

2.5 Guass Principle

Let us introduce the Gauss asynchronous variations, $\Delta\ddot{q}_k, (\Delta q_k)^{\cdot\cdot}, (\Delta \dot{q}_k)^{\cdot}$ and $(\Delta t)^{\cdot\cdot}$ together with the following requirements

$\delta q_k = 0, \Delta t = 0, \Delta \dot{q}_k = 0, (\Delta q_k)^{\cdot} = 0, (\Delta t)^{\cdot} = 0$, putting in the equations (2.8) and (2.9) becomes as

$$\Delta \ddot{q}_k = \delta \ddot{q}_k, \tag{2.38}$$

and

$$(\Delta q_k)^{\cdot\cdot} = \Delta \ddot{q}_k + \dot{q}_k (\Delta t)^{\cdot\cdot}, \tag{2.39}$$

using (2.38) in (2.39)

$$(\Delta q_k)^{\cdot\cdot} = \delta\ddot{q}_k + \dot{q}_k(\Delta t)^{\cdot\cdot},$$

$$\delta\ddot{q}_k = (\Delta q_k)^{\cdot\cdot} - \dot{q}_k(\Delta t)^{\cdot\cdot}. \tag{2.40}$$

We use the infinitesimally transformation of the acceleration vector and time variation for better understanding of Gauss'

$$\bar{q}_k = q_k$$

,

$$\bar{t} = t,$$

$$\frac{d\bar{q}_k}{dt} - \frac{dq_k}{dt} = \frac{d\bar{q}_k}{d\bar{t}} - \frac{dq_k}{dt},$$

$$\frac{d^2\bar{q}_k}{dt^2} - \frac{d^2q_k}{dt^2} = \delta\ddot{q}_k = (\delta q_k)^{\cdot\cdot} = (\Delta\dot{q}_k)^{\cdot} = \Delta\ddot{q}_k,$$

$$\frac{d^2\bar{q}_k}{d\bar{t}^2} - \frac{d^2q_k}{dt^2} = (\Delta q_k)^{\cdot\cdot}.$$

The primitively infinitesimal coordinate $(\Delta q_k)^{\cdot\cdot}$ and $(\Delta t)^{\cdot\cdot}$ as for the Gauss' infinitesimal transformation, so we introduce the Gauss' generator of the transformation

$$(\Delta q_k)^{\cdot\cdot} = \epsilon F_k(q_k, \dot{q}_k, t), \tag{2.41}$$

$$(\Delta t)^{\cdot\cdot} = \epsilon f(q_k, \dot{q}_k, t), \tag{2.42}$$

put equations (2.41) and (2.42) in the equation (2.39)

$$\Delta\ddot{q}_k = \epsilon[F_k(q_k, \dot{q}_k, t) - \dot{q}_k f(q_k, \dot{q}_k, t)], \tag{2.43}$$

by using the gauss' principle we have

$$\sum_{i=1}^{N}(\mathbf{Y}_i - \frac{d(m_i\dot{\mathbf{r}}_i)}{dt}) \cdot \delta\ddot{\mathbf{r}}_i = 0. \tag{2.44}$$

Where $\delta\ddot{\mathbf{r}}_i$ is represented the gauss' infinitesimal variation of the acceleration vector. As we know the total derivative of the velocity vector is

$$\mathbf{v}^i = \frac{\partial\mathbf{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial\mathbf{r}_i}{\partial t}, \tag{2.45}$$

equation (2.45) can be written for the acceleration vector as

$$\ddot{\mathbf{r}}_i = \frac{\partial \mathbf{r}_i}{\partial q_k}\ddot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_l}\dot{q}_k\dot{q}_l + 2\frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial t}\dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial t^2}, \tag{2.46}$$

the jourdian variation of the equation (2.46) can found as

$$\delta\ddot{\mathbf{r}}_i = \frac{\partial \mathbf{r}_i}{\partial q_k}\delta\ddot{q}_k + 2(\frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_l}\dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial t})\delta\dot{q}_k. \tag{2.47}$$

By putting $\delta\dot{q}_k = 0$ we will get

$$\delta\ddot{\mathbf{r}}_i = \frac{\partial \mathbf{r}_i}{\partial q_k}\delta\ddot{q}_k, \tag{2.48}$$

$\delta\ddot{\mathbf{r}}_i$ represented the gauss' infinitesimal variation of the acceleration vector by supposing that

$$\delta\ddot{\mathbf{r}}_i = (\mathbf{r}_i)^{\cdot\cdot} \neq 0, \delta\dddot{\mathbf{r}}_i = (\delta\mathbf{r}_i)^{\cdots} \neq 0,$$

$$\delta\mathbf{r}_i = 0, \delta\dot{\mathbf{r}}_i = 0, \delta t = 0,$$

put equation (2.48) in the equation (2.44)

$$\sum_{i=1}^{N}(\frac{d(m_i\dot{\mathbf{r}}_i)}{dt} - \mathbf{Y}_i)\cdot\frac{\partial \mathbf{r}_i}{\partial q_k}\delta\ddot{q}_k = 0.$$

$$\epsilon\sum_{i=1}^{N}(\mathbf{Y}_i\frac{\partial \mathbf{r}_i}{\partial q_k} - \frac{d(m_i\dot{\mathbf{r}}_i)}{dt}\cdot\frac{\partial \mathbf{r}_i}{\partial q_k})(F_k - \dot{q}_k f) = 0.$$

This equation is identical to the equation (2.31) so the repeat the same process as above to get the equation (2.36) and (2.37) for the gauss' infinitesimal transformation

Chapter 3

Conservation Laws Of Dynamical Systems By Generalized D'Alembert Principle

3.1 Introduction

When we deal with the motion of a dynamical system, we usual encounter two kinds forces under the observation, first one is the applied force due to which the rest body can be set into motion and the second one is the reaction force which resist the motion of the dynamical system. It mean that the motion of the dynamical system is moving under the reaction forces due to the constraints. In such type of problems we usually determine the motion of the system as well as magnitude of the reaction of the constraint. In 1743, Jean and D'Alembert developed a principle so called the lagrange D'Alembert principle. By using this discuss in [2]-[3] we can analysis different kind of problems in different areas of physics. According to this principle the motion under the ideal constraint is such that virtual work done by these constraints is zero. In this way the reaction force will be disappeared. Now for the generalizations of the lagrange d'Alembert principle we will used the non-ideal constraint as discussed in [1]. In this situation we will consider the following basic ideas,

1):Virtual displacement and supplementary virtual displacements,

2):The principle of libration of constraints,

3):Ideal constraints,

4):Generalized coordinates and supplementary generalized coordinates.

The supplementary virtual displacement are called the normal virtual displacement because normal virtual displacement are in the direction consistent with the ignored constraints. Corresponding to these supplementary virtual displacement, we introduced supplementary generalized coordinates equal to the numbers of constraints. Whose measured is made along the normal virtual displacement. Now, the classical virtual displacement in the D'Alembert principle of virtual work are replaced by the sum of the classical and normal virtual displacement. The equation of Lagrange and the dynamical equilibrium in the normal direction are derived from the generalized D'Alembert principle. The force of constraint are computed from the equation dynamical equilibrium. The equation of material particles can be taken into considerent by using generalized D'Alembert principle. This principle gives connection between the applied forces at equilibrium state and the forces of constraints. Some special problems are solved to check the validity of this principle.

In this chapter we will use the generalized D'Alembert principle which is used in [1] in order to find conserve quantities of various kinds for conservative and non-conservative systems. The conservative law can actually find by two methods. First one is the differential equation of motion, which used by B, Vujanovic in [4] and seconde one is the transformation properties of Hamilton's variation principle, which is used by E. L. Hill in [5]. But all these principle are used for the ideal constraints.

Now in this paper we make an extension for the generalized D'Alembert principle to find the conservative quantity of a non-conservative holonomic dynamical system with finite degree of freedom by using differential variation principle. This principle is equally valid for both the conservative and the non-conservative system.

3.2 Generalized Virtual Displacement

Consider N-particle holonomic dynamical system with mass m_a and (x_a, y_a, z_a) be the cartesian coordinate of the particle. The system is under the action of forces $\mathbf{Y}_a$ and constraint force $\mathbf{R}_a$. Let us consider $\mathbf{R}_a$ is the non-ideal reaction force, so their virtual work is non-zero,

$$\sum_{i}^{N}\mathbf{R}_a \cdot \delta\mathbf{\Psi}_a \neq 0.$$

Here virtual displacement is $\delta\mathbf{\Psi}_a$. Since the system is subject to the holonomic constraints

$$g_i(t, x_a, y_a, z_a) = 0, i = 1, ..., k. \tag{3.1}$$

Where t is the time and (x_a, y_a, z_a) is the cartesian coordinate of the particles. The system has $n = 3N - k$ degree of freedom and $p_j (j = 1, ..., n)$ be the corresponding generalized coordinate. The position of each particle in three dimension is $\mathbf{r}_a = x_a\mathbf{i} + y_a\mathbf{j} + z_a\mathbf{k}$. Where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is the unite vector of the cartesian coordinate system x,y,z. The simultaneous variation of constraint equation (3.1) is

$$\delta g_i = \sum_a \mathbf{f}_{ai} \cdot \delta\mathbf{r}_a = 0. \tag{3.2}$$

Where

$$\mathbf{f}_{ai} = \frac{\partial g_i}{\partial x_a}\mathbf{i} + \frac{\partial g_i}{\partial y_a}\mathbf{j} + \frac{\partial g_i}{\partial z_a}\mathbf{k}.$$

If the system is second order then we introduce a relation

$$\sum_a \mathbf{f}_{ai} \cdot \mathbf{f}_{a\Pi} = d_{i\Pi}, \tag{3.3}$$

and $\delta\mathbf{r}_a = \delta x_a\mathbf{i} + \delta y_a\mathbf{j} + \delta z_a\mathbf{k}$ is the simultaneous variation of the position vector, but $\mathbf{f}_{ai}$ is the normal component of the virtual variation which is perpendicular to the constraints. Let us make each particle in the system free of constraint and make every particle displacement $\mathbf{f}_{ai}p_{n+i}$ in the perpendicular direction $\mathbf{f}_{ai}$, here p_{n+i} is the supplementary generalized coordinate. If the motion is in the direction of constraint then $p_{n+i} = \dot{p}_{n+i} = \ddot{p}_{n+i} = 0$. The quantity which is calculated free of constraint is denoted by lower index c. The position of the particle which is free of constraint is,

$$\mathbf{\Psi}_a(t, p_u) = \mathbf{r}_a(t, p_j) + \mathbf{f}_{ai}(t, p_j)p_{n+i},$$

with

$$u = 1, ..., n + k.$$

Whose the simultaneous variation is

$$\delta\mathbf{\Psi}_a = \delta\mathbf{r}_a + \delta\mathbf{f}_{ai}p_{n+i} + \mathbf{f}_{ai}\delta p_{n+i}.$$

If the motion of the particle in the direction of constraint, then we have

$$(\delta\mathbf{\Psi}_a)_c = \delta\mathbf{r}_a + \mathbf{f}_{ai}\delta p_{n+i}. \tag{3.4}$$

Which is the generalized virtual displacement of any particle of the dynamical system. Decomposed equation (3.4) into tangent and normal component, so we have

$$(\delta\mathbf{\Psi}_a)_c = (\delta\mathbf{\Psi}_a)_{oT} + (\delta\mathbf{\Psi}_a)_{oN}, \tag{3.5}$$

but

$$(\delta\mathbf{\Psi}_a)_{oT} = \delta\mathbf{r}_a, (\delta\mathbf{\Psi}_a)_{oN} = \mathbf{f}_{ai}\delta p_{n+i}.$$

So it is obviously that

$$\sum_a (\delta\mathbf{\Psi}_a)_{oT}.(\delta\mathbf{\Psi}_a)_{oN} = \delta p_{n+i}\sum_a \mathbf{f}_{ai} \cdot \delta\mathbf{r}_a = 0.$$

Here we note that $\delta\mathbf{r}_a$ and $\mathbf{f}_{ai}p_{n+i}$ are perpendicular to each other.

3.3 Simultaneous And Non Simultaneous Variation Infinitesimal Transformation

Let $p_1, p_2, ..., p_n$ be the generalized coordinates of the position of a dynamical system with n-degree of freedom. These coordinate is continuous function of time t. Consider a point K on the actual path which is correlated to infinitesimally a point L on the virtual path at the same time by the relation

$$\overline{p}_j(t) = p_j(t) + \delta p_j, \tag{3.6}$$

where $\overline{p}_j(t)$ and $p_j(t)$ will represent the coordinate of the point k and the point L respectively, and δ represent synchronous variation. If the variation is non-simultaneously then we introduce $\overline{p}_j(t+\Delta t)$ which infinitesimally close to actual path but with small change of time Δt. Developing and ignore the higher order term, so we will get that

$$\overline{p}_j(t + \Delta t) = \overline{p}_j(t) + \dot{p}_j \Delta t,$$

but

$$\overline{p}_j(t + \Delta t) = p_j(t + \Delta t) + \delta p_j, \tag{3.7}$$

put equation (3.6) in the equation (3.7)

$$\overline{p}_j(t + \Delta t) = p_j(t) + \dot{p}_j(t)\Delta t + \overline{p}_j(t) - p_j(t).$$

$$\overline{p}_j(t + \Delta t) = \dot{p}_j(t)\Delta t + \overline{p}_j(t). \tag{3.8}$$

Let us Δp_j be the asynchronous variation define by

$$\Delta p_j = \overline{p}_j(t + \Delta t) - p_j(t), \tag{3.9}$$

$$\Delta p_j = p_j(t + \Delta t) - p_j(t) + \delta p_j,$$

$$\Delta p_j = p_j(t) + \dot{p}_j(t)\Delta t + \delta p_j - p_j(t),$$

$$\Delta p_j = \dot{p}_j(t)\Delta t + \delta p_j. \tag{3.10}$$

For any function the non-simultaneous Δp_j, equation (3.10) can be write as

$$\Delta F_j = \dot{F}_j(t)\Delta t + \delta F_j. \tag{3.11}$$

For the generalized velocity equation (3.10) can written as

$$\Delta \dot{p}_j = \ddot{p}_j(t)\Delta t + \delta \dot{p}_j. \tag{3.12}$$

Here we consider a variation so called the generalized variation, in which the infinitesimal transformation of the generalized coordinate $p_1, ..., p_j$ on the actual path which are correlated to the point $p_j+\Delta p_j, ..., p_j+\Delta p_j$ on the virtual path at the time $t+\Delta t$, which give the equations

$$\overline{p}_j(\overline{t}) = p_j(t) + \Delta p_j \tag{3.13}$$

.

$$\overline{t} = t + \Delta t \tag{3.14}$$

.

These two equation, the equation (3.14) and (3.13) has great important in the study of conservation law and similarly, we can take the infinitesimal transformation are of the form

$$\overline{p}_j(\overline{t}) = p_j(t) + \epsilon F_j(p, \dot{p}, t), \tag{3.15}$$

$$\overline{t} = t + \epsilon f(p, \dot{p}, t). \tag{3.16}$$

By using (3.14) and (3.13) we find

$$\Delta p_j = \epsilon F_j(p_j, \dot{p}_j, t). \tag{3.17}$$

$$\Delta t = \epsilon f(p_j, \dot{p}_j, t). \tag{3.18}$$

And similarly we can write these transformation for the supplementary generalized coordinate as

$$\Delta p_{n+i} = \epsilon F_{n+i}(p_{n+i}, \dot{p}_{n+i}, t). \tag{3.19}$$

$$\Delta t = \epsilon f(p_{n+i}, \dot{p}_{n+i}, t). \tag{3.20}$$

Now we take derivative of equation (3.10) and subtract from equation (3.12) we get that

$$\Delta \dot{p}_j = \epsilon[\dot{F}_j(p, \dot{p}_j, t) - \dot{p}_j \dot{f}(p, \dot{p}_j, t)]. \tag{3.21}$$

And for the supplementary generalized coordinate

$$\Delta \dot{p}_{n+i} = \epsilon[\dot{F}_{n+i}(p, \dot{p}_{n+i}, t) - \dot{p}_{n+i} \dot{f}(p, \dot{p}_{n+i}, t)]. \tag{3.22}$$

3.4 Kinetic Energy And The Generalized Force Of The Dynamical System

The position of a particle of the dynamical system, which is free of constraint, is denoted by $\mathbf{\Psi}_a$. Then the kinetic energy for such system is

$$T_k = \frac{1}{2}\sum_a m_a \dot{\mathbf{\Psi}}_a \cdot \dot{\mathbf{\Psi}}_a. \tag{3.23}$$

Since the position of the particle free of constraint is $\mathbf{\Psi}_a = \mathbf{r}_a + \mathbf{f}_{ai}p_{n+i}$. The corresponding velocity is

$$\dot{\mathbf{\Psi}}_a = \dot{\mathbf{r}}_a + \dot{\mathbf{f}}_{ai}p_{n+i} + \mathbf{f}_{ai}\dot{p}_{n+i}. \tag{3.24}$$

Where $\dot{\mathbf{r}}_a$ is the velocity which is not librated of constraints.

By substituting equation (3.24) in the equation (3.23) we will get

$$T_k = \frac{1}{2}\sum_a m_a(\dot{\mathbf{r}}_a + \dot{\mathbf{f}}_{ai}p_{n+i} + \mathbf{f}_{ai}\dot{p}_{n+i}) \cdot (\dot{\mathbf{r}}_a + \dot{\mathbf{f}}_{ai}p_{n+i} + \mathbf{f}_{ai}\dot{p}_{n+i}),$$

$$T_k = \frac{1}{2}\sum_a m_a(\dot{\mathbf{r}}_a) \cdot (\dot{\mathbf{r}}_a) + \sum_a m_a\dot{\mathbf{r}}_a(\dot{\mathbf{f}}_{ai}p_{n+i} + \mathbf{f}_{ai}\dot{p}_{n+i}) + nonlinearterm,$$

$$T_k = (T_k)_c + (\overline{T}_k),$$

by ignoring the nonlinear term which is nonlinear with respect to p_{n+i} and $\dot{p}_{n+i}$, $(T_k)_c = \frac{1}{2}\sum_a m_a(\dot{\mathbf{r}}_a) \cdot (\dot{\mathbf{r}}_a)$ is the kinetic energy with constraint and $\overline{T}_k = \sum_a m_a\dot{\mathbf{r}}_a \cdot (\dot{\mathbf{f}}_{ai}p_{n+i} + \mathbf{f}_{ai}\dot{p}_{n+i})$ which is linear with respect to p_{n+i} and $\dot{p}_{n+i}$.

If $\mathbf{Y}_a$ be the applied force, the virtual work with respect to the generalized virtual displacement is,

$$W^1 = \sum_a (\mathbf{Y}_a)_c \cdot (\delta\mathbf{\Psi}_a)_c, \tag{3.25}$$

by using equation (3.5) in equation (3.25)

$$W^1 = \sum_a (\mathbf{Y}_a)_c \cdot (\delta\mathbf{r}_a + \mathbf{f}_{ai}\delta p_{n+i}). \tag{3.26}$$

Since the position of the particle is the function of time and the generalized coordinate i.e, $\mathbf{r}_a = \mathbf{r}_a(t, p_j)$, then their virtual displacement is

$$\delta \mathbf{r}_a = \frac{\partial \mathbf{r}_a}{\partial p_j} \delta p_j, \tag{3.27}$$

substitute equation (3.27) in (3.28)

$$W^1 = \sum_a (\mathbf{Y}_a)_c \cdot (\frac{\partial \mathbf{r}_a}{\partial p_j} \delta p_j + \mathbf{f}_{ai} \delta p_{n+i}), \tag{3.28}$$

$$W^1 = Q_j \delta p_j + N_i \delta p_{n+i}. \tag{3.29}$$

With $Q_j = \sum_a (\mathbf{Y}_a \cdot \frac{\partial \mathbf{r}_a}{\partial p_j})_c$ is the classical generalized forces and the generalized force along the constraint is $N_i = \sum_a (\mathbf{Y}_a)_c \cdot \mathbf{f}_{ai}$.

3.5 Transformation Of Generalized D'Alembert Principle

Let $\mathbf{Y}_a$ be the applied force act on the particle of the system and $\mathbf{R}_a$ be the reaction force of the constraint, them for such system the generalized D'Alembert principle is

$$\sum_a \left(\mathbf{Y}_a + \mathbf{R}_a - m_a \ddot{\mathbf{\Psi}}_a \right)_c \cdot (\delta \mathbf{\Psi}_a)_c = 0. \tag{3.30}$$

Where m_a is the mass of the system and $\ddot{\mathbf{\Psi}}$ is the acceleration which is free of constraint. Now by introducing the generalized coordinate and applying the standard process we will get the lagrange-D'Alembert principle of the virtual work as discussed by Dorde Duckic. [1], so equation (3.30) becomes,

$$\left[(\frac{d}{dt} \frac{\partial T_k}{\partial \dot{p}_j} - \frac{\partial T_k}{\partial p_j})_c - Q_j \right] \delta p_j + \left[(\frac{d}{dt} \frac{\partial T_k}{\partial \dot{p}_{n+i}} - \frac{\partial T_k}{\partial p_{n+i}})_c - N_i \right] \delta p_{n+i} + R_i \delta p_{n+i} = 0. \tag{3.31}$$

Since the generalized coordinate and the supplementary generalized coordinate does not depend each other, so their virtual change also be independent on each other and cannot be zero mean that, $\delta p_j \neq 0, \delta p_{n+i} \neq 0$. It possible only if

$$(\frac{d}{dt} \frac{\partial T_k}{\partial \dot{p}_a} - \frac{\partial T_k}{\partial p_a})_c - Q_k = 0.$$

$$(\frac{d}{dt} \frac{\partial T_k}{\partial \dot{p}_{n+i}} - \frac{\partial T_k}{\partial p_{n+i}})_c - N_i - R_i = 0.$$

Where $\mathbf{R}_i$ is the reaction force and Q_i and N_i is the generalized force.

Let us take each particle of the system is partially conserved then the system have the conserved part is $\frac{-\partial V}{\partial P_j}, \frac{-\partial V}{\partial p_{n+i}}$ and the non-conservative part is Q_j, N_i, then we have

$$\sum_{a=1}^{N} \mathbf{Y}_a \cdot \frac{\partial \mathbf{v}_a}{\partial p_j} = Q_j - \frac{\partial V}{\partial p_j}, \sum_a (\mathbf{Y}_a)_c \cdot \mathbf{f}_{ai} = N_i - \frac{\partial V}{\partial p_{n+i}}. \tag{3.32}$$

Put (3.32) in (3.31)

$$\left[(\frac{d}{dt}\frac{\partial T_k}{\partial \dot{p}_j} - \frac{\partial T_k}{\partial p_j})_c - Q_j + \frac{\partial V}{\partial p_j} \right] \delta p_j + \left[(\frac{d}{dt}\frac{\partial T_k}{\partial \dot{p}_{n+i}} - \frac{\partial T_k}{\partial p_{n+i}})_c - N_i + \frac{\partial V}{\partial p_{n+i}} \right] \delta p_{n+i} + R_i \delta p_{n+i} = 0,$$

$$\left[(\frac{d}{dt}\frac{\partial (T_k - V)}{\partial \dot{p}_j} - \frac{\partial (T_k - V)}{\partial p_j})_c - Q_j \right] \delta p_j + \left[(\frac{d}{dt}\frac{\partial (T_k - V)}{\partial \dot{p}_{n+i}} - \frac{\partial (T_k - V)}{\partial p_{n+i}})_c - N_i \right] \delta p_{n+i} + R_i \delta p_{n+i} = 0. \tag{3.33}$$

But

$$L = T_k - V$$

$$\left[(\frac{d}{dt}\frac{\partial L}{\partial \dot{p}_j} - \frac{\partial L}{\partial p_j})_c - Q_j \right] \delta p_j + \left[(\frac{d}{dt}\frac{\partial L}{\partial \dot{p}_{n+i}} - \frac{\partial L}{\partial p_{n+i}})_c - N_i \right] \delta p_{n+i} + R_i \delta p_{n+i} = 0. \tag{3.34}$$

Where $L = L(p_j, \dot{p}_j, p_{n+i}, \dot{p}_{n+i}, t)$ is the lagrangian function of the given system which explain the conservative part of the system. Write (3.34) in the form

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_j}\delta p_j\right) - \frac{\partial L}{\partial \dot{p}_j}\delta \dot{p}_j - \frac{\partial L}{\partial p_j}\delta p_j - Q_j \delta p_j + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_{n+i}}\delta p_{n+i}\right) \\ &- \frac{\partial L}{\partial \dot{p}_{n+i}}\delta \dot{p}_{n+i} - \frac{\partial L}{\partial p_{n+i}}\delta p_{n+i} - N_j \delta p_{n+i} - R_i \delta p_{n+i} = 0. \end{aligned} \tag{3.35}$$

Using equation (3.10) and (3.12) in the equation (3.35)

$$\begin{aligned} &\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{p}_j}(\Delta p_j - \dot{p}_j \Delta t)\right] - \frac{\partial L}{\partial \dot{p}_j}(\Delta \dot{p}_j - \ddot{p}_j \Delta t) - \frac{\partial L}{\partial p_j}(\Delta p_j - \dot{p}_j \Delta t) \\ &- Q_j(\Delta p_j - \dot{p}_j \Delta t) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_{n+i}}(\Delta p_{n+i} - \dot{p}_{n+i}\Delta t)\right) - \frac{\partial L}{\partial \dot{p}_{n+i}}(\Delta \dot{p}_{n+i} - \ddot{p}_{n+i}\Delta t) \\ &- \frac{\partial L}{\partial p_{n+i}}(\Delta p_{n+i} - \dot{p}_{n+i}\Delta t) - N_j(\Delta p_{n+i} - \dot{p}_{n+i}\Delta t) - R_i(\Delta p_{n+i} - \dot{p}_{n+i}\Delta t) = 0. \end{aligned} \tag{3.36}$$

$$\begin{aligned}&\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{p}_j}(\Delta p_j + \dot{p}_j \Delta t)\right] - (\frac{\partial L}{\partial \dot{p}_j}\Delta \dot{p}_j + \frac{\partial L}{\partial p_j}\Delta p_j + \frac{\partial L}{\partial t}\Delta t) + (\frac{\partial L}{\partial \dot{p}_j}\ddot{p}_j + \frac{\partial L}{\partial p_j}\dot{p}_j + \frac{\partial L}{\partial t})\\&\Delta t - Q_j(\Delta p_j - \dot{p}_j \Delta t) + \frac{d}{dt}[\frac{\partial L}{\partial \dot{p}_{n+i}}(\Delta p_{n+i} + \dot{p}_{n+i}\Delta t)]\\&-(\frac{\partial L}{\partial \dot{p}_{n+i}}\Delta \dot{p}_{n+i} + \frac{\partial L}{\partial p_{n+i}}\Delta p_{n+i} + \frac{\partial L}{\partial t}\Delta t) + (\frac{\partial L}{\partial \dot{p}_{n+i}}\ddot{p}_{n+i} + \frac{\partial L}{\partial p_{n+i}}\dot{p}_{n+i} + \frac{\partial L}{\partial t})\Delta t\\&- N_i(\Delta p_{n+i} - \dot{p}_{n+i}\Delta t) - R_i(\Delta p_{n+i} - \dot{p}_{n+i}\Delta t) = 0.\end{aligned} \tag{3.37}$$

Let us consider

$$\Delta L = \frac{\partial L}{\partial \dot{p}_j}\Delta \dot{p}_j + \frac{\partial L}{\partial p_j}\Delta p_j + \frac{\partial L}{\partial t}\Delta t$$

$$\dot{L} = \frac{\partial L}{\partial \dot{p}_j}\ddot{p}_j + \frac{\partial L}{\partial p_j}\dot{p}_j + \frac{\partial L}{\partial t}$$

and

$$\Delta L_1 = \frac{\partial L}{\partial \dot{p}_{n+i}}\Delta \dot{p}_{n+i} + \frac{\partial L}{\partial p_{n+i}}\Delta p_{n+i} + \frac{\partial L}{\partial t}\Delta t$$

$$\dot{L}_1 = \frac{\partial L}{\partial \dot{p}_{n+i}}\ddot{p}_{n+i} + \frac{\partial L}{\partial p_{n+i}}\dot{p}_{n+i} + \frac{\partial L}{\partial t}$$

So we can (3.37) in the form

$$\begin{aligned}&\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{p}_j}(\Delta p_j - \dot{p}_j \Delta t) + \frac{\partial L}{\partial \dot{p}_{n+i}}(\Delta p_{n+i} - \dot{p}_{n+i}\Delta t) + L\Delta t\right] - \Delta L - L(\Delta t)^{\cdot} - \Delta L_1\\&+ \dot{L}_1(\Delta t) - Q_j(\Delta p_j - \dot{p}_j \Delta t) - N_i(\Delta p_{n+i} - \dot{p}_{n+i}\Delta t) - R_i(\Delta p_{n+i} - \dot{p}_{n+i}\Delta t) = 0.\end{aligned} \tag{3.38}$$

We will define a function is called gauge variant function, denoted by M which is function of the generalized coordinate, generalized velocity, supplementary generalized coordinate, supplementary generalized velocity and time written as, $M = M(p_j, \dot{p}_j, p_{n+i}, \dot{p}_{n+i}, t)$. We will add and subtract $\epsilon \dot{M}$ in equation (3.38). Where ϵ is the small positive number. We will get that

$$\begin{aligned}&\frac{d}{dt}[\frac{\partial L}{\partial \dot{p}_j}(\Delta p_j - \dot{p}_j \Delta t) + \frac{\partial L}{\partial \dot{p}_{n+i}}(\Delta p_{n+i} - \dot{p}_{n+i}\Delta t) + L\Delta t - \epsilon M] - [\Delta L + L(\Delta t)^{\cdot}\\&+ \Delta L_1 - \dot{L}_1(\Delta t) + Q_j(\Delta p_j - \dot{p}_j \Delta t) + N_i(\Delta p_{n+i} - \dot{p}_{n+i}\Delta t) - \epsilon \dot{M}]\\&- R_i(\Delta p_{n+i} - \dot{p}_{n+i}\Delta t) = 0.\end{aligned} \tag{3.39}$$

Which is the required transformation of the generalized D'Alembert principle. Equation (3.39) is the transformed form of the generalized D'Alembert principle it is clear that if

$$\Delta L + L(\Delta t)^{\cdot} + \Delta L_1 - \dot{L}_1(\Delta t) + Q_j(\Delta p_j - \dot{p}_j \Delta t) + N_i(\Delta p_{n+i} - \dot{p}_{n+i}\Delta t) - \epsilon \dot{M} + R_i(\Delta p_{n+i} - \dot{p}_{n+i}\Delta t) = 0, \tag{3.40}$$

is satisfied, the dynamical system admits a conservation law of the form

$$\frac{\partial L}{\partial \dot{p}_j}(\Delta p_j - \dot{p}_j \Delta t) + \frac{\partial L}{\partial \dot{p}_{n+i}}(\Delta p_{n+i} - \dot{p}_{n+i}\Delta t) + L\Delta t - \epsilon M = costant = c. \tag{3.41}$$

By substituting (3.17), (3.18), (3.19) and (3.20) in (3.40) , (3.41) take the form

$$\frac{\partial L}{\partial p_j}F_j + \frac{\partial L}{\partial \dot{p}_j}(\dot{F}_j - \dot{p}_j \dot{f}) + \frac{\partial L}{\partial t} f + L\dot{f} + \frac{\partial L}{\partial p_{n+i}}F_{n+i} + \frac{\partial L}{\partial \dot{p}_{n+i}}(\dot{F}_{n+i} - \dot{p}_{n+i}\dot{f}) + \frac{\partial L}{\partial t} f - \dot{L}_1 f + Q_j(F_j - \dot{p}_j f) + N_i(F_{n+i} - \dot{p}_{n+i} f) + R_i(F_{n+i} - \dot{p}_{n+i} f) - \dot{M} = 0 \tag{3.42}$$

$$\frac{\partial L}{\partial \dot{p}_j}(F_j - \dot{p}_j f) + \frac{\partial L}{\partial \dot{p}_{n+i}}(F_{n+i} - \dot{p}_{n+i} f) + Lf - M = c. \tag{3.43}$$

3.6 Conservation Law For Mathematical Pendulum With Length a

Here we will consider a mathematical pendulum with length a and mass m, the constraint equation for such pendulum is $g_1 = x^2 + y^2 - a^2 = 0$, x and y is the horizontal and vertical axis respectfully, where vertical axis is the oriented down. From constraint equation we have $\mathbf{f}_{ai} = 2a\sin\alpha\mathbf{i} + 2a\cos\alpha\mathbf{j}$, the position of the particle is,

$$\mathbf{r} = a\sin\alpha\mathbf{i} + a\cos\alpha\mathbf{j}.$$

And the velocity of the corresponding particle is

$$\dot{\mathbf{r}} = (a\cos\alpha\mathbf{i} - a\sin\alpha\mathbf{j})\dot{\alpha}.$$

Here only the gravitational force apply on the particle, which is $\mathbf{F} = mg\mathbf{j}$, where g is the gravitational acceleration. The corresponding potential energy is $V = mga(1 -$

$\cos\alpha$). Therefore, by using equation (4.24) the generalized forces are $Q_i = -mga\sin\alpha$, $N_i = 2mga\cos\alpha$.

$$(\frac{d}{dt}\frac{\partial T_k}{\partial \dot{p}_{n+i}} - \frac{\partial T_k}{\partial p_{n+i}})_c - N_i - R_i = 0,$$
$$R_i = -2m(a\dot{\alpha})^2 - 2mga\cos\alpha,$$

is the reaction force. Fore the kinetic energy we will use equation (3.23), so we will get that

$$T_k = \frac{m}{2}(a\dot{\alpha})^2 + 2m(a\dot{\alpha})^2 p_2.$$

Hence the Lagrangian function is

$$L = \frac{m}{2}(a\dot{\alpha})^2 + 2m(a\dot{\alpha})^2 p_2 - mga(1 - \cos\alpha)$$

Let suppose that the function F_j, F_{n+i}, f and M depend only the time t and position, so $F_j = F_j(t, p_j)$, $F_{n+i} = F_{n+i}(t, p_{n+i})$, $M = M(t, p_j, p_{n+i})$, $f = f(t, p_j)$ and $f = f(t, p_{n+i})$, then we can write equation (3.42) as

$$\begin{aligned}
&\frac{\partial L}{\partial p_j}F_j + \frac{\partial L}{\partial \dot{p}_j}(\frac{\partial F_j}{\partial p_j}\dot{p}_j + \frac{\partial F_j}{\partial t}) - \dot{p}_j\frac{\partial L}{\partial \dot{p}_j}(\frac{\partial f}{\partial p_j}\dot{p}_j + \frac{\partial f}{\partial t}) + \frac{\partial L}{\partial t}f + L(\frac{\partial f}{\partial p_j}\dot{p}_j + \frac{\partial f}{\partial t})\\
&+ \frac{\partial L}{\partial p_{n+i}}F_{n+i} + \frac{\partial L}{\partial \dot{p}_{n+i}}(\frac{\partial F_{n+i}}{\partial p_{n+i}}\dot{p}_{n+i} + \frac{\partial F_{n+i}}{\partial t}) - \dot{p}_{n+i}\frac{\partial L}{\partial \dot{p}_{n+i}}(\frac{\partial f}{\partial p_{n+i}}\dot{p}_{n+i} + \frac{\partial f}{\partial t})\\
&- f(\frac{\partial L}{\partial p_{n+i}}\dot{p}_{n+i} + \frac{\partial L}{\partial t}) + Q_i(F_j - \dot{p}_j f) + N_i(F_{n+i} - \dot{p}_{n+i}f) + R_i(F_{n+i} - \dot{p}_{n+i}f)\\
&- (\frac{\partial M}{\partial p_i}\dot{p}_i + \frac{\partial M}{\partial p_{n+i}}\dot{p}_{n+i} + \frac{\partial M}{\partial t}) = 0.
\end{aligned}$$

$$\begin{aligned}
&F_j(-ma\sin\alpha) + (\frac{\partial F_j}{\partial \alpha}\dot{\alpha} + \frac{\partial F_j}{\partial t})(ma\dot{\alpha} + 4ma\alpha p_2) - (\dot{\alpha})^2\frac{\partial f}{\partial \alpha}(ma\dot{\alpha} + 4ma\dot{\alpha}p_2)\\
&- \dot{\alpha}\frac{\partial f}{\partial t}(ma\dot{\alpha} + 4ma\dot{\alpha}p_2) + (\frac{m}{2}(a\dot{\alpha})^2 + 2m(a\dot{\alpha})^2 p_2 - mga(1 - \cos\alpha))(\frac{\partial f}{\partial \alpha}\dot{\alpha} + \frac{\partial f}{\partial t})\\
&+ F_2(2m(a\alpha)^2) - f(2m(a\dot{\alpha})^2 p_2\dot{p}_2) - mga\sin\alpha(F_j - \dot{\alpha}f) + 2mga\cos\alpha(F_2 - \dot{p}_2 f)\\
&- (2m(a\alpha)^2 + 2mga\cos\alpha)(F_2 - \dot{p}_2 f) - (\frac{\partial M}{\partial \alpha}\dot{\alpha} + \frac{\partial M}{\partial p_2}\dot{p}_2 + \frac{\partial M}{\partial t}) = 0.
\end{aligned}$$

Now by comparing various power of $\dot{\alpha}$ and $\dot{p}_2$, we will get that

$$\frac{\partial F}{\partial t}(ma + 4map_2) - mga\sin\alpha f - \frac{\partial M}{\partial \alpha} = 0. \tag{3.44}$$

$$\frac{\partial F}{\partial \alpha}(ma+4map_2)-\frac{\partial f}{\partial t}(ma+4map_2)+\frac{\partial f}{\partial t}(\frac{1}{2}ma^2+2ma^2p_2)+2ma^2(F_2-\dot{p}_2f)$$
$$-F_2 2ma^2-f(2ma^2p_2\dot{p}_2)=0. \tag{3.45}$$

$$-2m(a\alpha)^2fp_2-f(2p_2mga\cos\alpha)+f(2m(a\alpha)^2+2mga\cos\alpha)-\frac{\partial M}{\partial p_2}=0. \tag{3.46}$$

$$\frac{\partial f}{\partial t}(4ma\dot{\alpha})+f(2m(a\dot{\alpha})^2\dot{p}_2)=0. \tag{3.47}$$
$$\frac{\partial f}{\partial t}+f(\frac{1}{2}a\dot{\alpha}\dot{p}_2)=0,$$
$$\frac{d}{dt}(f\exp^{\frac{1}{2}a\dot{\alpha}\dot{P}_2t})=0,$$
$$f=C_1\exp^{-\frac{1}{2}a\dot{\alpha}\dot{P}_2t}.$$

Where C_1 is constant of integration and using value of f in equation (3.45), we find

$$F_j=\frac{C_1\exp^{-\frac{1}{2}a\dot{\alpha}\dot{p}_2t}}{1+4p_2}(2ap_2+2ap_2\dot{p}_2)\alpha-\frac{C_1\exp^{-\frac{1}{2}a\dot{\alpha}\dot{p}_2t}}{(1+4p_2)\dot{\alpha}\dot{p}_2a}(1+4p_2-\frac{a}{2}-2ap_2)\alpha$$
$$\frac{\partial F_j}{\partial t}=\frac{-a\dot{\alpha}\dot{p}_2C_1\exp^{-\frac{1}{2}a\dot{\alpha}\dot{p}_2t}}{2(1+4p_2)}(2ap_2+2ap_2\dot{p}_2)\alpha+\frac{C_1p_2\exp^{-\frac{1}{2}a\dot{\alpha}\dot{p}_2t}}{(1+4p_2)\dot{p}_2}(1+4p_2-\frac{a}{2}-2ap_2)\alpha.$$

Substitute the value of $\frac{\partial F_j}{\partial t}$ and f in equation (3.44) to find M.

$$M=(ma+4map_2)(\frac{a\dot{\alpha}\dot{p}_2C_1\exp^{-\frac{1}{2}a\dot{\alpha}\dot{p}_2t}}{2(1+4p_2)}(2ap_2+2ap_2\dot{p}_2)\frac{\alpha^2}{2}$$
$$-\frac{C_1p_2\exp^{-\frac{1}{2}a\dot{\alpha}\dot{p}_2t}}{(1+4p_2)\dot{p}_2}(1+4p_2-\frac{a}{2}-2ap_2)\frac{\alpha^2}{2})-mga\cos\alpha C_1\exp^{-\frac{1}{2}a\dot{\alpha}\dot{p}_2t},$$

and $F_2=0$. Now we can find conservation law by substituting the value F_j, F_2 and f in equation (3.43).

$$\left(ma^2\dot{\alpha}+4ma^2\dot{\alpha}p_2\right)(\frac{C_1\exp^{-\frac{1}{2}a\dot{\alpha}\dot{p}_2t}}{1+4p_2}(2ap_2+2ap_2\dot{p}_2)\alpha$$
$$-\frac{C_1\exp^{-\frac{1}{2}a\dot{\alpha}\dot{p}_2t}}{(1+4p_2)\dot{\alpha}\dot{p}_2a}(1+4p_2-\frac{a}{2}-2ap_2)\alpha)-\dot{\alpha}(ma^2\dot{\alpha}+4ma^2\dot{\alpha}p_2)C_1\exp^{-\frac{1}{2}a\dot{\alpha}\dot{P}_2t}$$
$$+\left(\frac{m}{2}(a\dot{\alpha})^2+2m(a\dot{\alpha})^2p_2-mga(1-\cos\alpha)\right)C_1\exp^{-\frac{1}{2}a\dot{\alpha}\dot{P}_2t}$$
$$-(ma+4map_2)(\frac{a\dot{\alpha}\dot{p}_2C_1\exp^{-\frac{1}{2}a\dot{\alpha}\dot{p}_2t}}{2(1+4p_2)}(2ap_2+2ap_2\dot{p}_2)\frac{\alpha^2}{2}$$
$$-\frac{C_1p_2\exp^{-\frac{1}{2}a\dot{\alpha}\dot{p}_2t}}{(1+4p_2)\dot{p}_2}(1+4p_2-\frac{a}{2}-2ap_2)\frac{\alpha^2}{2})$$
$$-mga\cos\alpha C_1\exp^{-\frac{1}{2}a\dot{\alpha}\dot{p}_2t}=consttant.$$

Chapter 4

Conservation Law Of Dynamical Systems By Mean Of The Differential Variational Principle Of Generalized Jourdian and Gauss

4.1 Introduction

In the analysis of motion of a mechanical system under the action of constraints there arise two problems. The first one is related to the determination of the motion of the system and the second one is related to the calculation of reaction forces due to the presence of constraints during the motion. In the subject of analytical dynamics these two problems are solved by the D'Alembert Lagrange principle and the notion of ideal constraints. The notion of ideal constraints implies that the virtual work done by the reaction forces vanish. In this way we eliminate reaction forces from the analysis of motion, which is based on D'Alembert Lagrange principal. For the reaction forces whose virtual work do not vanish, then such forces are summed up with the given forces and the corresponding constraints are called non-ideal.

So in this chapter we find conservation law by differential variational principle of the Guass and Jourdian under constraints. Conservation law or the first integral of both conservative and non-conservative dynamical system with finite degree of freedom has a very important role in physics and engineering since for both theoretical and practical science. Simply conservation law is the first integral of a differential equation of motion. When particles of the dynamical system are in the state of motion, then there is two forces must be under discussion, the applied force and the reaction force. The reaction force is actually the force of constraint. First time reaction force discussed by Dorde [1] to find the differential variational principle called generalized Lagrange-D'Alembert principle in which he find the magnitude of the reaction force. Vojanovic in [2] used a differential principle in which he find the conservation law but without discussing any constraint force on the dynamical system. So the virtual work was eliminated by Vojanovic in his work. Vujanovic in [20] also used differential variational principle of Jourdian and Gauss to find the conservation law but in this work he also ignore the reaction force. An attempt has been made by Aftab [7] to find the conservation law of the dynamical system whose mass is varies with respect to generalized velocities, generalized coordinates and time by using the virtual work of the reaction force is zero. If the virtual work is non-zero for some reaction force, then this reaction will add to the given applied force, the corresponding constraints are called non ideal constraints.

In this chapter we makes an attempt to find the conservation law of a dynamical system by using differential variational principle of Gauss' and Jourdian by introducing the non-ideal constraints, before finding the conservation law we will define some important following ideas: 1):Virtual displacement and supplementary virtual displacements, 2):The principle of libration of constraints,

3):Ideal constraints,

4):Generalized coordinates and supplementary generalized coordinates.

The supplementary virtual displacement are called the normal virtual displacement because normal virtual displacement are in the direction consistent with the ignored constraints. Corresponding to these supplementary virtual displacement, we introduced supplementary generalized coordinates equal to the numbers of constraints. Whose measured is made along the normal virtual displacement. Now, the classical virtual displacement in the D'Alembert principle of virtual work are replaced by the sum of the classical and normal virtual displacement. The equation of Lagrange and the dynamical equilibrium in the normal direction are derived from the generalized D'Alembert principle. The force of constraint are computed from the equation dynamical equilibrium. The equation of material particles can be taken into considerent by using generalized D'Alembert principle. This principle gives connection between the applied forces at equilibrium state and the forces of constraints. Some special problems are solved to check the validity of this principle.

4.2 Synchronous And Asynchronous Variation Infinitesimal Transformation

In this section we will consider the position of the particle with n-degree of freedom $(n = 3N - k)$ of the dynamical system is denoted by set of coordinates $y_1, y_2, ..., y_n$ and the set of supplementary generalized coordinates $y_{n+1}, y_{n+2}, ..., y_{n+k}$, which are continuous function of time t. Here we will denote the synchronous variation by δ, defined as a point on the actual path correlate at the same a point on the varied path by the relation

$$\overline{y}_l(t) = y_l(t) + \delta y_l, l = 1, ..., n. \tag{4.1}$$

Where $\overline{y}_l(t)$ and $y_l(t)$ is the coordinate of the point on the varied and the actual path respectfully. Now if we discuss the internal symmetry of the given dynamical system then we must take the time t change during the process of variation by the relation

$$\overline{y}_l(t + \Delta t) = y_l(t + \Delta t) + \delta y_l, \tag{4.2}$$

$$\overline{y}_l(t + \Delta t) = y_l(t) + \dot{y}_l \Delta t + \delta y_l, \tag{4.3}$$

eliminate δy_l from equation (4.3) using equation (4.1) in equation (4.3)

$$\overline{y}_l(t+\Delta t) = \overline{y}_l(t) + \dot{y}_l(t)\Delta t.$$

Where $\overline{y}_l(t+\Delta t)$ is the point on the varied path which is correlated with the point on the actual path asynchronously. Let us take an equation $\Delta y_l = \overline{y}_l(t+\Delta t) - y_l(t)$ which define the asynchronous variation and over dot represent derivative with respect to time. So

$$\Delta y_l = \overline{y}_l(t+\Delta t) - y_l(t), \tag{4.4}$$

using equation (4.2) in equation (4.4)

$$\Delta y_l = y_l(t+\Delta t) + \delta y_l - y_l(t)$$

$$\Delta y_l = y_l(t) + \dot{y}_l(t)\Delta t + \delta y_l - y_l(t)$$

$$\Delta y_l = \dot{y}_l(t)\Delta t + \delta y_l. \tag{4.5}$$

For any function equation (4.5) can be written as

$$\Delta F = \dot{F}(t)\Delta t + \delta F.$$

For generalized velocity equation (4.5) can be written as

$$\Delta \dot{y}_l = \ddot{y}_l(t)\Delta t + \delta \dot{y}_l. \tag{4.6}$$

Let suppose if δ and differentiation d commute each other then (4.6) will be written as

$$\Delta \dot{y}_l = \ddot{y}_l(t)\Delta t + \frac{d}{dt}(\delta y_l).$$

Now take derivative of equation (4.5) with respect to time t

$$(\Delta y_l)^{\dot{}} = \delta \dot{y}_l + \ddot{y}_l \Delta t + \dot{y}_l(\Delta t)^{\dot{}}. \tag{4.7}$$

From equation (4.6) put $\delta \dot{y}_l$ in equation (4.7)

$$(\Delta y_l)^{\dot{}} = \Delta \dot{y}_l + \dot{y}_l(\Delta t)^{\dot{}},$$

$$\Delta \dot{y}_l = (\Delta y_l)^{\dot{}} - \dot{y}_l(\Delta t)^{\dot{}}. \tag{4.8}$$

For the generalized acceleration equation (4.5) can be write in the form

$$\Delta \ddot{y}_l = \dddot{y}_l(t)\Delta t + \delta \ddot{y}_l. \tag{4.9}$$

Take derivative with respect to time of equation (4.7)

$$(\Delta y_l)^{\cdot\cdot} = \delta \ddot{y}_l + \dddot{y}_l \Delta t + 2\ddot{y}_l (\Delta t)^{\cdot} + \dot{y}_l (\Delta t)^{\cdot\cdot}, \tag{4.10}$$

from equation (4.9) put $\delta \ddot{y}_l$ into the equation (4.16), we will get

$$\Delta \ddot{y}_l = (\Delta y_l)^{\cdot\cdot} - 2\ddot{y}_l (\Delta t)^{\cdot} - \dot{y}_l (\Delta t)^{\cdot\cdot} \tag{4.11}$$

At this point it should be noted that the infinitesimal transformation which is shown by equation (4.1) and (4.4) has great important in the study of conservation law. We let these transformation in the form

$$(\Delta y_l) = \epsilon F_j(t, \dot{y}, y), \tag{4.12}$$

$$(\Delta t) = \epsilon f(t, \dot{y}, y). \tag{4.13}$$

For the supplementary generalized coordinate these transformation take the form

$$(\Delta y_{n+i}) = \epsilon F_{n+i}(t, \dot{y}, y), \tag{4.14}$$

$$(\Delta t) = \epsilon f(t, \dot{y}, y). \tag{4.15}$$

f and F is infinitesimal transformation of space and time.

1) : To define the jourdian generalized variation take $\Delta \dot{y}_l$, $(\Delta y_l)^{\cdot}$ and $(\Delta t)^{\cdot}$ with the $\Delta y_l = 0$, Δt=0 replace these value in equation (4.6) and (4.7) then we have

$$\Delta \dot{y}_l = \delta \dot{y}_l,$$

$$(\Delta y_l)^{\cdot} = \delta \dot{y}_l + \dot{y}_l (\Delta t)^{\cdot},$$

$$\Delta \dot{y}_l = (\Delta y_l)^{\cdot} - \dot{y}_l (\Delta t)^{\cdot}. \tag{4.16}$$

For better understanding the nature of Jourdian variation we will use the following transformation

$$\overline{y}_l = y_l, \; \overline{t} = t,$$

$$\overline{y}_l - y_l = \Delta y_l.$$

For the velocity vector we have

$$\overline{\dot{y}_l} - \dot{y}_l = \Delta \dot{y}_l,$$

$$\frac{d}{dt}(\overline{y}_l) - \frac{d}{dt}(y_l) = \Delta \dot{y}_l = \delta \dot{y}_l.$$

We will introduce the following space and time Jourdian generator of transformation which has great important in the study of conservation law, we will take structure in the form as discussed in the [4]

$$(\Delta y_l)^{\cdot} = \epsilon F_l(t, \dot{y}, y), \tag{4.17}$$

$$(\Delta t)^{\cdot} = \epsilon f(t, \dot{y}, y), \tag{4.18}$$

using equation (4.17) and (4.18) in equation (4.16)

$$\Delta \dot{y}_l = \epsilon[F_l(t, \dot{y}, y) - \dot{y}_l f(t, \dot{y}, y)], \tag{4.19}$$

now same as for the supplementary generalized coordinates

$$\Delta \dot{y}_{n+i} = \epsilon[F_{n+i}(t, \dot{y}, y) - \dot{y}_{n+i} f(t, \dot{y}, y)]. \tag{4.20}$$

2) : For the Gauss' we will take $\Delta \ddot{y}_l$, $(\Delta y_l)^{\cdot\cdot}$, $(\Delta t)^{\cdot\cdot}$ with $\Delta y_l = 0$, $\Delta t = 0$, $\Delta y_l = 0$, $(\Delta y_l)^{\cdot} = 0$ and $(\Delta t)^{\cdot} = 0$, substitute these value in equation (4.11) and (4.9)

$$\Delta \ddot{y}_l = \delta \ddot{y}_l,$$

$$\delta \ddot{y}_l = \Delta \ddot{y}_l = (\Delta y_l)^{\cdot\cdot} - \dot{y}_l (\Delta t)^{\cdot\cdot}. \tag{4.21}$$

In the sense of gauss' we will introduce the following infinitesimal transformation

$$\overline{y}_l = y_l, \overline{t} = t,$$

$$\overline{y}_l = y_l = \Delta y_l,$$

for the acceleration vector this written as

$$\overline{\ddot{y}}_l - \ddot{y}_l = \Delta \ddot{y}_l,$$

$$\frac{d^2}{dt^2}(\overline{\ddot{y}}_l) - \frac{d^2}{dt^2}(\ddot{y}_l) = \Delta \ddot{y}_l = \delta \ddot{y}_l,$$
$$\frac{d^2}{dt^2}(\overline{\ddot{y}}_l) - \frac{d^2}{dt^2}(\ddot{y}_l) = \Delta y_l^{\cdot\cdot}.$$

Let us define the space and time Gauss' generator F_l and f of the infinitesimal transformation of the generalized acceleration vector

$$(\Delta y_l)^{\cdot\cdot} = \epsilon F_l(t, \dot{y}_l, y_l),$$

$$(\Delta t)^{\cdot\cdot} = \epsilon f(t, \dot{y}_l, y_l).$$

So the simultaneous variation in term of above transformation is

$$\delta \ddot{y}_l = \epsilon[F_l(t, \dot{y}_l, y_l) - \dot{y}_l f(t, \dot{y}_l, y_l)]. \tag{4.22}$$

Similarly the infinitesimal transformation of the supplementary generalized coordinates can be written as

$$(\Delta y_{n+i})^{\cdot\cdot} = \epsilon F_{n+i}(t, \dot{y}_{n+i}, y_{n+i}),$$

$$(\Delta t)^{\cdot\cdot} = \epsilon f(t, \dot{y}_{n+i}, y_{n+i}).$$

$$\delta \ddot{y}_{n+i} = \epsilon[F_{n+i}(t, \dot{y}_{n+i}, y_{n+i}) - \dot{y}_{n+i} f(t, \dot{y}_{n+i}, y_{n+i})].$$

4.3 Generalized Virtual Displacement

Let us take a dynamical system with N-particles and take $y_l, \dot{y}_l, \ddot{y}_l$ be the position, velocity and acceleration of the generalized coordinate and similarly $y_{n+i}, \dot{y}_{n+i}, \ddot{y}_{n+i}$ be the position, velocity and the acceleration of the supplementary generalized coordinate of the particle, where $n = 3N - k$ is the degree of freedom. Here we will consider a system with mass m_j which is under action of applied force $\mathbf{F_j}$ and the reaction force $\mathbf{A_j}$ in such away that the reaction is non-ideal, then it must have the non-zero virtual work

$$\sum_j^N A_j \delta \mathbf{x}_j \neq 0,$$

$\delta \mathbf{x}_j$ represent the virtual displacement. It mean that the system is under the action of holonomic constraints

$$B_i(t, a_j, b_j, c_j) = 0, i = 1, ..., k. \tag{4.23}$$

Where (a_j, b_j, c_j) is the cartesian coordinate and t is the time. Let us suppose that $\mathbf{x}_j = a_j\mathbf{i} + b_j\mathbf{j} + c_j\mathbf{k}$ be the position of each particle in three dimension and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unite vectors. The simultaneous variation of the constraints equation are

$$\delta B_i = \sum_j \mathbf{R}_{ji} \cdot \delta \mathbf{x}_j = 0, \tag{4.24}$$

but

$$\mathbf{R}_{ji} = \frac{\partial B_i}{\partial a_j}\mathbf{i} + \frac{\partial B_i}{\partial b_j}\mathbf{j} + \frac{\partial B_i}{\partial c_j}\mathbf{k}.$$

If we will introduce the the second order system then we must take

$$\sum_j \mathbf{R}_{ji} \cdot \mathbf{R}_{j\pi} = d_{i\pi}.$$

Since $\delta \mathbf{x}_j$ is the simultaneous variation of the position vector and $\mathbf{R}_{ji}$ is the normal component of the virtual variation of the constraint. In this note any quantity which is calculated along the constraint is denoted by the lower index c. The position of the particles which is librated of constraint is

$$\mathbf{\Gamma}_j(t, y_u) = \mathbf{x}_j(t, y_l) + \mathbf{R}_{ji}(t, y_l) y_{n+i},$$

with the virtual change

$$\delta\mathbf{\Gamma}_j(t, y_u) = \delta\mathbf{x}_j(t, y_l) + \mathbf{R}_{ji}(t, y_l)\delta y_{n+i} + \delta\mathbf{R}_{ji}(t, y_l) y_{n+i}.$$

For the motion along the constraint, we have

$$(\delta\mathbf{\Gamma}_j)_c = \delta\mathbf{x}_j + \mathbf{R}_{ji}\delta y_{n+i}, \tag{4.25}$$

which is the required generalized virtual displacement. Now we separate equation (4.25) as a normal and the tangential direction, then

$$(\delta\mathbf{\Gamma}_j)_c = (\delta\mathbf{\Gamma}_j)_{OT} + (\delta\mathbf{\Gamma}_j)_{ON},$$

$$(\delta\mathbf{\Gamma}_j)_{OT} = \delta\mathbf{x}_j, (\delta\mathbf{\Gamma}_j)_{ON} = \mathbf{R}_{ji}\delta y_{n+i}.$$

By using equation (4.24) here it will show that

$$\sum_j (\delta\mathbf{\Gamma}_j)_{OT} \cdot (\delta\mathbf{\Gamma}_j)_{ON} = \delta y_{n+i} \sum_j \delta\mathbf{x}_j \cdot \mathbf{R}_{ji} = 0.$$

Which show $(\delta\mathbf{\Gamma}_j)_{OT}$ and $(\delta\mathbf{\Gamma}_j)_{ON}$ are perpendicular to each other, so it justify equation (4.25) of decomposition.

4.4 Kinetic Energy And The Generalized Force Of The Dynamical System

As we know that $\mathbf{\Gamma}_j$ be the position of the particles then the velocity for such position is

$$\dot{\mathbf{\Gamma}}_j(t, y_u) = \dot{\mathbf{x}}_j(t, y_l) + \dot{\mathbf{R}}_{ji}(t, y_l) y_{n+i} + \mathbf{R}_{ji}(t, y_l)\dot{y}_{n+i},$$

$$u = 1, ..., n + k.$$

Now here we will define the kinetics energy as

$$T_k = \frac{1}{2}\sum_j m_j \dot{\mathbf{\Gamma}}_j \cdot \dot{\mathbf{\Gamma}}_j$$

$$T_k = \frac{1}{2}\sum_j m_j \Big(\dot{\mathbf{x}}_j(t, y_l) + \dot{\mathbf{R}}_{ji}(t, y_l)y_{n+i} + \mathbf{R}_{ji}(t, y_l)\dot{y}_{n+i}\Big) \cdot (\dot{\mathbf{x}}_j(t, y_l)$$

$$+ \dot{\mathbf{R}}_{ji}(t, y_l)y_{n+i} + \mathbf{R}_{ji}(t, y_l)\dot{y}_{n+i}),$$

$$T_k = (T_k)_c + (\bar{T}_k) + nonlinearterm. \tag{4.26}$$

Where $(T_k)_c = \frac{1}{2}\sum_j m_j\dot{\mathbf{x}}_j \cdot \dot{\mathbf{x}}_j$ is the kinetic energy which is under the constraint and $\bar{T}_k = \sum_j m_j\dot{\mathbf{x}}_j \cdot (\dot{\mathbf{R}}_{ji}y_{n+i} + \mathbf{R}_{ji}\dot{y}_{n+i})$ is also part of kinetic energy which is linear with respect to $\dot{y}_{n+i}$ and y_{n+i} and here the non linear term mean that, that the term which is non linear with respect to $\dot{y}_{n+i}$ and y_{n+i}.

Since $\mathbf{F}_j$ be the applied force then the work done will be

$$W = \sum_j (\mathbf{F}_j)_c \cdot (\delta\mathbf{\Gamma}_j)_c. \tag{4.27}$$

By using equation (4.25) in equation (4.27)

$$W = \sum_j (\mathbf{F}_j)_c \cdot (\delta\mathbf{x}_j + \mathbf{R}_{ji}\delta y_{n+i})_c. \tag{4.28}$$

As we know $\mathbf{x}_j = \mathbf{x}_j(y_l, t)$ then the simultaneous variation is

$$\delta\mathbf{x}_j = \frac{\partial\mathbf{x}_j}{\partial y_l}\delta y_l. \tag{4.29}$$

By putting equation (4.28) into (4.29) we will gets

$$W = \sum_j (\mathbf{F}_j)_c \cdot (\frac{\partial\mathbf{x}_j}{\partial y_l}\delta y_l + \mathbf{R}_{ji}\delta y_{n+i})_c. \tag{4.30}$$

$$W = Q_l'\delta y_l + N_i'\delta y_{n+i},$$

with $Q_l' = \sum_j (\mathbf{F}_j)_c \cdot \frac{\partial\mathbf{x}_j}{\partial y_l}$ is the classical generalized force and $N_i' = \sum_j (\mathbf{F}_j)_c \cdot \mathbf{R}_{ji}$ is the force which is perpendicular in direction to the classical generalized force.

4.5 Differential Variation Principle

1): Let us consider m_j is the mass of the particle in the motion and $\mathbf{F}_j$ be the applied force which is resist by reaction force $\mathbf{A}_j$, so after applying the Newton's second law of motion under the constraint is $(\mathbf{F}_j + \mathbf{R}_j)_c = (m_j\ddot{\mathbf{\Gamma}}_j)_c$, sum this equation for all the particles over the index j and take dot product with the equation (4.25)

$$\sum_j (\mathbf{F}_j + \mathbf{R}_j - m_j\ddot{\mathbf{\Gamma}}_j)_c \cdot (\delta\mathbf{\Gamma}_j)_c = 0,$$

but from equation (4.25) we have

$$(\delta\mathbf{\Gamma}_j)_c = \delta\mathbf{x}_j + \mathbf{R}_{ji}\delta y_{n+i}, \tag{4.31}$$

but $\mathbf{x}_j = \mathbf{x}_j(t, y_l)$ then

$$\delta\mathbf{x}_j = \frac{\partial\mathbf{x}_j}{\partial y_l}\delta y_l$$

use in equation (4.31)

$$(\delta\mathbf{\Gamma}_j)_c = \frac{\partial\mathbf{x}_j}{\partial y_l}\delta y_l + \mathbf{R}_{ji}\delta y_{n+i}, \tag{4.32}$$

take derivative of equation (4.32) with respect to time

$$(\delta\dot{\mathbf{\Gamma}}_j)_c = (\frac{\partial^2\mathbf{x}_j}{\partial y_l\partial y_m}\dot{y}_m + \frac{\partial^2\mathbf{x}_j}{\partial y_l\partial t})\delta y_l + \frac{\partial\mathbf{x}_j}{\partial y_l}\delta\dot{y}_l + \dot{\mathbf{R}}_{ji}\delta y_{n+i} + \mathbf{R}_{ji}\delta\dot{y}_{n+i}, \tag{4.33}$$

$(\delta\dot{\mathbf{\Gamma}}_j)$ represent the generalized Jourdian variation which depend only the infinitesimal arbitrary change of the velocity and does not depend on the time and the space deformations:

$\delta\dot{y}_l \neq 0$, $\delta\dot{y}_{n+i} \neq 0$, $\delta y_l = 0$, $\delta t = 0$ and $\delta y_{n+i} = 0$ after replacing in equation (4.33) we will get

$$(\delta\dot{\mathbf{\Gamma}}_j)_c = \frac{\partial\mathbf{x}_j}{\partial y_l}\delta\dot{y}_l + \mathbf{R}_{ji}\delta\dot{y}_{n+i}. \tag{4.34}$$

Where $(\delta\dot{\mathbf{\Gamma}}_j)_c$ is the generalized Jourdian infinitesimal variation and take dot product with $(\mathbf{F}_j + \mathbf{R}_j)_c = (m_j\ddot{\mathbf{\Gamma}}_j)_c$ we will get

$$\sum_j (\mathbf{F}_j + \mathbf{R}_j - m_j\ddot{\mathbf{\Gamma}}_j)_c \cdot (\delta\dot{\mathbf{\Gamma}}_j)_c = 0, \tag{4.35}$$

which is the required generalized Jourdian variational principle.

2): Now let us introduce the generalized Gauss' differential variation principle in the form

$$\sum_j (\mathbf{F}_j + \mathbf{R}_j - m_j\ddot{\mathbf{\Gamma}}_j)_c \cdot (\delta\ddot{\mathbf{\Gamma}}_j)_c = 0, \tag{4.36}$$

but $(\delta\ddot{\mathbf{\Gamma}}_j)_c$ represent the generalized Gauss' infinitesimal variation of the acceleration vector which depend only on the infinitesimal arbitrary changes of the acceleration vector then we must have $(\delta y_l)^{\ddot{}} = \delta\ddot{y}_l \neq 0$, $(\delta y_{n+i})^{\ddot{}} = \delta\ddot{y}_{n+i} \neq 0$, $\delta\dot{y}_l = 0$, $\delta y_l = 0$, $\delta\dot{y}_{n+i} = \delta y_{n+i} = 0$ and $\delta t = 0$. First we will take derivative with respect to time of equation (4.34) and then the replace these value to get the generalized Gauss' infinitesimal variation.

$$(\delta\ddot{\mathbf{\Gamma}}_j)_c = (\frac{\partial^2 \mathbf{x}_j}{\partial y_l \partial y_m}\dot{y}_m + \frac{\partial \mathbf{x}_j}{\partial t})\delta\dot{y}_l + \frac{\partial \mathbf{x}_j}{\partial y_l}\delta\ddot{y}_l + \dot{\mathbf{R}}_{ij}\delta\dot{y}_{n+i} + \mathbf{R}_{ij}\delta\ddot{y}_{n+i},$$

put here $\delta\dot{y}_l = 0$ and $\delta\dot{y}_{n+i} = 0$ we will get generalized Gauss' infinitesimal variation.

$$(\delta\ddot{\mathbf{\Gamma}}_j)_c = \frac{\partial \mathbf{x}_j}{\partial y_l}\delta\ddot{y}_l + \mathbf{R}_{ij}\delta\ddot{y}_{n+i}. \tag{4.37}$$

Which is required generalized Gauss' infinitesimal variation.

4.6 Transformation Of Generalized Gauss' Variational Principle

In this section we consider a N particles dynamical system which is subject to holonomical constraint and $\mathbf{F}_j$ be the applied force which act on some point of dynamical system $\mathbf{A}_j$ be the force of constraint, then virtual acceleration for such a point is $(\delta\ddot{\mathbf{\Gamma}}_j)_c$. Now by using the generalized Gauss' principle is

$$\sum_j (\mathbf{F}_j + \mathbf{R}_j - m_j\ddot{\mathbf{\Gamma}}_j)_c \cdot (\delta\ddot{\mathbf{\Gamma}}_j)_c = 0, \tag{4.38}$$

after applying the standard process on equation (4.38) same as discussed by Dorde Duckic [1] we will get the Gauss' variational principle

$$\begin{aligned}(Z_l + Q_l')\delta\ddot{y}_l + (Z_{n+i} + N_i' + A_i)\delta\ddot{y}_{n+i} = 0,\\ \left[\left(\frac{d}{dt}(\frac{\partial T_k}{\partial\dot{y}_l}) - \frac{\partial T_k}{\partial y_l}\right)_c - Q_l'\right]\delta\ddot{y}_l + \left[\left(\frac{d}{dt}(\frac{\partial T_k}{\partial\dot{y}_{n+i}}) - \frac{\partial T_k}{\partial y_{n+i}}\right)_c - N_i'\right]\delta\ddot{y}_{n+i}\\ + A_i\delta\ddot{y}_{n+i} = 0.\end{aligned} \tag{4.39}$$

Since the generalized coordinate and the supplementary generalized coordinate are independent to each other as well as their virtual change also independent to each other and cannot be zero $\delta y_l \neq 0$, $\delta y_{n+i} \neq 0$, the virtual work is equal to zero for the virtual changes δy_l, δy_{n+i}

$$\frac{d}{dt}(\frac{\partial T_k}{\partial\dot{y}_l} - \frac{\partial T_k}{\partial y_l})_c - Q_l' = 0.$$

$$\frac{d}{dt}(\frac{\partial T_k}{\partial \dot{y}_{n+i}} - \frac{\partial T_k}{\partial y_{n+i}})_c - N_i' - A_i = 0.$$

Since A_i is the reaction force, Q_l' and N_i' is the generalized force. If the system is partially conserved then it must have the conserved part $-\frac{\partial V}{\partial y_l}$, $-\frac{\partial V}{\partial y_{n+i}}$ with the non-conserved part Q_l, N_i, so the generalized forces will take the form

$$\sum_j (\mathbf{F}_j)_c \cdot \frac{\partial \mathbf{x}_j}{\partial y_l} = Q_l - \frac{\partial V}{\partial y_l},$$

$$\sum_j (\mathbf{F}_j)_c \cdot \mathbf{R}_{ji} = N_i - \frac{\partial V}{\partial y_{n+i}},$$

put these generalized force in equation (4.39)

$$\left[\frac{d}{dt}(\frac{\partial T_k}{\partial \dot{y}_l}) - \frac{\partial T_k}{\partial y_l} - Q_l + \frac{\partial V}{\partial y_l}\right]\delta \ddot{y}_l + \left[\frac{d}{dt}(\frac{\partial T_k}{\partial \dot{y}_{n+i}}) - \frac{\partial T_k}{\partial y_{n+i}} - N_i + \frac{\partial V}{\partial y_{n+i}}\right]\delta \ddot{y}_{n+i}$$
$$- A_i \delta \ddot{y}_{n+i} = 0.$$

$$\left[\frac{d}{dt}(\frac{\partial T_k}{\partial \dot{y}_l}) - \frac{\partial (T_k - V)}{\partial y_l} - Q_l\right]\delta \ddot{y}_l + \left[\frac{d}{dt}(\frac{\partial T_k}{\partial \dot{y}_{n+i}}) - \frac{\partial (T_k - V)}{\partial y_{n+i}} - N_i\right]\delta \ddot{y}_{n+i}$$
$$- A_i \delta \ddot{y}_{n+i} = 0. \tag{4.40}$$

Here we will introduce the lagrangian function $L = T - V$ and use $\frac{\partial L}{\partial y_l} = \frac{\partial T}{\partial \dot{y}_l}$ and $\frac{\partial L}{\partial y_{n+i}} = \frac{\partial T}{\partial \dot{y}_{n+i}}$, so equation (4.40) will become

$$\left[\frac{d}{dt}(\frac{\partial L_k}{\partial \dot{y}_l}) - \frac{\partial L}{\partial y_l} - Q_l\right]\delta \ddot{y}_l + \left[\frac{d}{dt}(\frac{\partial L}{\partial \dot{y}_{n+i}}) - \frac{\partial L}{\partial y_{n+i}} - N_i\right]\delta \ddot{y}_{n+i} - A_i \delta \ddot{y}_{n+i} = 0. \tag{4.41}$$

Where $L = L(y_l, \dot{y}_l, y_{n+i}, \dot{y}_{n+i}, t)$ is the lagrangian function which is used to explain the conservation law.

We can write equation (4.41) in the form

$$[\frac{d}{dt}(\frac{\partial L}{\partial \dot{y}_l}\delta \ddot{y}_l) - \frac{\partial L}{\partial \dot{y}_l}\delta \dddot{y}_l - \frac{\partial L}{\partial y_l}\delta \ddot{y}_l - Q_l \delta \ddot{y}_l] + [\frac{d}{dt}(\frac{\partial L}{\partial \dot{y}_{n+i}}\delta \ddot{y}_{n+i}) - \frac{\partial L}{\partial \dot{y}_{n+i}}\delta \dddot{y}_{n+i}$$
$$- \frac{\partial L}{\partial y_{n+i}}\delta \ddot{y}_{n+i} - N_i \delta \ddot{y}_{n+i}] - A_i \delta \ddot{y}_{n+i} = 0.$$

Put equation (4.21) here

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_l}\left((\Delta y_l)^{\cdot\cdot} - \dot{y}_l(\Delta t)^{\cdot\cdot}\right) - \frac{\partial L}{\partial \dot{y}_l}((\Delta y_l)^{\cdot\cdot} - \dot{y}_l(\Delta t)^{\cdot\cdot})^{\cdot} - \frac{\partial L}{\partial y_l}((\Delta y_l)^{\cdot\cdot} - \dot{y}_l(\Delta t)^{\cdot\cdot}) \\
&- Q_l((\Delta y_l)^{\cdot\cdot} - \dot{y}_l(\Delta t)^{\cdot\cdot}) + \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_{n+i}}\left((\Delta y_{n+i})^{\cdot\cdot} - \dot{y}_{n+i}(\Delta t)^{\cdot\cdot}\right) \\
&- \frac{\partial L}{\partial \dot{y}_{n+i}}\left((\Delta y_{n+i})^{\cdot\cdot} - \dot{y}_{n+i}(\Delta t)^{\cdot\cdot}\right)^{\cdot} - \frac{\partial L}{\partial y_{n+i}}\left((\Delta y_{n+i})^{\cdot\cdot} - \dot{y}_{n+i}(\Delta t)^{\cdot\cdot}\right) \\
&- N_i((\Delta y_{n+i})^{\cdot\cdot} - \dot{y}_{n+i}(\Delta t)^{\cdot\cdot}) - A_i\left((\Delta y_{n+i})^{\cdot\cdot} - \dot{y}_{n+i}(\Delta t)^{\cdot\cdot}\right) = 0
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_l}\left((\Delta y_l)^{\cdot\cdot} - \dot{y}_l(\Delta t)^{\cdot\cdot}\right) - \frac{\partial L}{\partial \dot{y}_l}((\Delta y_l)^{\cdot\cdot\cdot} - \ddot{y}_l(\Delta t)^{\cdot\cdot} - \dot{y}_l(\Delta t)^{\cdot\cdot\cdot}) \\
&- \frac{\partial L}{\partial y_l}((\Delta y_l)^{\cdot\cdot} - \dot{y}_l(\Delta t)^{\cdot\cdot}) - Q_l((\Delta y_l)^{\cdot\cdot} - \dot{y}_l(\Delta t)^{\cdot\cdot}) + \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_{n+i}}\left((\Delta y_{n+i})^{\cdot\cdot} - \dot{y}_{n+i}(\Delta t)^{\cdot\cdot}\right) \\
&- \frac{\partial L}{\partial \dot{y}_{n+i}}((\Delta y_{n+i})^{\cdot\cdot\cdot} - \ddot{y}_{n+i}(\Delta t)^{\cdot\cdot} - \dot{y}_{n+i}(\Delta t)^{\cdot\cdot\cdot}) - \frac{\partial L}{\partial y_{n+i}}\left((\Delta y_{n+i})^{\cdot\cdot} - \dot{y}_{n+i}(\Delta t)^{\cdot\cdot}\right) \\
&- N_i((\Delta y_{n+i})^{\cdot\cdot} - \dot{y}_{n+i}(\Delta t)^{\cdot\cdot}) - A_i\left((\Delta y_{n+i})^{\cdot\cdot} - \dot{y}_{n+i}(\Delta t)^{\cdot\cdot}\right) = 0,
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_l}\left((\Delta y_l)^{\cdot\cdot} - \dot{y}_l(\Delta t)^{\cdot\cdot}\right) - \left[\frac{\partial L}{\partial \dot{y}_l}(\Delta y_l)^{\cdot\cdot\cdot} + \frac{\partial L}{\partial y_l}(\Delta y_l)^{\cdot\cdot} + \frac{\partial L}{\partial t}(\Delta t)^{\cdot\cdot}\right] + \\
&(\Delta t)^{\cdot\cdot}\left[\frac{\partial L}{\partial \dot{y}_l}\ddot{y}_l + \frac{\partial L}{\partial y_l}\dot{y}_l + \frac{\partial L}{\partial t}\right] + \frac{\partial L}{\partial \dot{y}_l}\dot{y}_l(\Delta t)^{\cdot\cdot\cdot} + \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_{n+i}}\left((\Delta y_{n+i})^{\cdot\cdot} - \dot{y}_{n+i}(\Delta t)^{\cdot\cdot}\right) \\
&- \left[\frac{\partial L}{\partial \dot{y}_{n+i}}(\Delta y_{n+i})^{\cdot\cdot\cdot} + \frac{\partial L}{\partial y_{n+i}}(\Delta y_{n+i})^{\cdot\cdot} + \frac{\partial L}{\partial t}(\Delta t)^{\cdot\cdot}\right] \\
&+ (\Delta t)^{\cdot\cdot}\left[\frac{\partial L}{\partial \dot{y}_{n+i}}\ddot{y}_{n+i} + \frac{\partial L}{\partial y_{n+i}}\dot{y}_{n+i} + \frac{\partial L}{\partial t}\right] + \frac{\partial L}{\partial \dot{y}_{n+i}}\dot{y}_{n+i}(\Delta t)^{\cdot\cdot\cdot} - Q_l((\Delta y_l)^{\cdot\cdot} - \dot{y}_l(\Delta t)^{\cdot\cdot}) \\
&- N_i((\Delta y_{n+i})^{\cdot\cdot} - \dot{y}_{n+i}(\Delta t)^{\cdot\cdot}) - A_i\left((\Delta y_{n+i})^{\cdot\cdot} - \dot{y}_{n+i}(\Delta t)^{\cdot\cdot}\right) = 0,
\end{aligned}$$

but $\dot{L} = \frac{\partial L}{\partial \dot{y}_l}\ddot{y}_l + \frac{\partial L}{\partial y_l}\dot{y}_l + \frac{\partial L}{\partial t}$ and $\dot{L}_1 = \frac{\partial L}{\partial \dot{y}_{n+i}}\ddot{y}_{n+i} + \frac{\partial L}{\partial y_{n+I}}\dot{y}_{n+i} + \frac{\partial L}{\partial t}$

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_l}\left((\Delta y_l)^{\cdot\cdot} - \dot{y}_l(\Delta t)^{\cdot\cdot}\right) - \left[\frac{\partial L}{\partial \dot{y}_l}(\Delta y_l)^{\cdot\cdot\cdot} + \frac{\partial L}{\partial y_l}(\Delta y_l)^{\cdot\cdot} + \frac{\partial L}{\partial t}(\Delta t)^{\cdot\cdot}\right] + (\Delta t)^{\cdot\cdot}\dot{L} \\
&+ \frac{\partial L}{\partial \dot{y}_l}\dot{y}_l(\Delta t)^{\cdot\cdot\cdot} + \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_{n+i}}\left((\Delta y_{n+i})^{\cdot\cdot} - \dot{y}_{n+i}(\Delta t)^{\cdot\cdot}\right) \\
&- \left[\frac{\partial L}{\partial \dot{y}_{n+i}}(\Delta y_{n+i})^{\cdot\cdot\cdot} + \frac{\partial L}{\partial y_{n+i}}(\Delta y_{n+i})^{\cdot\cdot} + \frac{\partial L}{\partial t}(\Delta t)^{\cdot\cdot}\right] \\
&+ (\Delta t)^{\cdot\cdot}L_1 + \frac{\partial L}{\partial \dot{y}_{n+i}}\dot{y}_{n+i}(\Delta t)^{\cdot\cdot\cdot} - Q_l((\Delta y_l)^{\cdot\cdot} - \dot{y}_l(\Delta t)^{\cdot\cdot}) - N_i((\Delta y_{n+i})^{\cdot\cdot} - \dot{y}_{n+i}(\Delta t)^{\cdot\cdot}) \\
&- A_i\left((\Delta y_{n+i})^{\cdot\cdot} - \dot{y}_{n+i}(\Delta t)^{\cdot\cdot}\right) = 0,
\end{aligned}$$

put $(\Delta y_l)^{\ddot{}} = \epsilon F_l$, $(\Delta y_{n+i})^{\ddot{}} = \epsilon F_{n+i}$ and $(\Delta t)^{\ddot{}} = \epsilon f$ in the above equation

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_l}(F_l - \dot{y}_l f) - \left[\frac{\partial L}{\partial \dot{y}_l}\dot{F}_l + \frac{\partial L}{\partial y_l}F + \frac{\partial L}{\partial t}f\right] + f\dot{L} \\
&+ \frac{\partial L}{\partial \dot{y}_l}\dot{y}_l\dot{f} + \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_{n+i}}(F_{n+i} - \dot{y}_{n+i}f) - \left[\frac{\partial L}{\partial \dot{y}_{n+i}}\dot{F}_{n+i} + \frac{\partial L}{\partial y_{n+i}}F_{n+i} + \frac{\partial L}{\partial t}f\right] \\
&+ \dot{f}L_1 + \frac{\partial L}{\partial \dot{y}_{n+i}}\dot{y}_{n+i}\dot{f} - Q_l(F_l - \dot{y}_l f) - N_i(F_{n+i} - \dot{y}_{n+i}f) \\
&- A_i\left(F_{n+i} - \dot{y}_{n+i}f\right) = 0,
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{y}_l}(F_l - \dot{y}_l f) + fL + \frac{\partial L}{\partial \dot{y}_{n+i}}(F_{n+i} - \dot{y}_{n+i}f)\right] - \left[\frac{\partial L}{\partial \dot{y}_l}\dot{F}_l + \frac{\partial L}{\partial y_l}F + \frac{\partial L}{\partial t}f\right] - \dot{f}L \\
&+ \frac{\partial L}{\partial \dot{y}_l}\dot{y}_l\dot{f} - \left[\frac{\partial L}{\partial \dot{y}_{n+i}}\dot{F}_{n+i} + \frac{\partial L}{\partial y_{n+i}}F_{n+i} + \frac{\partial L}{\partial t}f\right] \\
&+ \dot{f}L_1 + \frac{\partial L}{\partial \dot{y}_{n+i}}\dot{y}_{n+i}\dot{f} - Q_l(F_l - \dot{y}_l f) - N_i(F_{n+i} - \dot{y}_{n+i}f) \\
&- A_i\left(F_{n+i} - \dot{y}_{n+i}f\right) = 0,
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{y}_l}(F_l - \dot{y}_l f) + fL + \frac{\partial L}{\partial \dot{y}_{n+i}}(F_{n+i} - \dot{y}_{n+i}f)\right] - [\frac{\partial L}{\partial y_l}F_l + \frac{\partial L}{\partial t}f + \frac{\partial L}{\partial \dot{y}_l}(\dot{F}_l - \dot{y}_l\dot{f}) \\
&+ \frac{\partial L}{\partial y_{n+i}}F_{n+i} + \frac{\partial L}{\partial t}f + \frac{\partial L}{\partial \dot{y}_{n+i}}(\dot{F}_{n+i} - \dot{y}_{n+i}\dot{f}) + \dot{f}L - f\dot{L}_1 + Q_l(F_l - \dot{y}_l f) \\
&+ N_i(F_{n+i} - \dot{y}_{n+i}f) + A_i(F_{n+i} - \dot{y}_{n+i}f)] = 0.
\end{aligned}$$

In the above equation we will introduce Gauge variant function, which is represented by $P = P(y_l, \dot{y}_l, y_{n+i}, \dot{y}_{n+i}, t)$, so by adding and subtracting $\epsilon\dot{P}$ in the above equation, we will get

$$\begin{aligned}
&\epsilon\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{y}_l}(F_l - \dot{y}_l f) + fL + \frac{\partial L}{\partial \dot{y}_{n+i}}(F_{n+i} - \dot{y}_{n+i}f) - P\right] \\
&- \epsilon[\frac{\partial L}{\partial y_l}F_l + \frac{\partial L}{\partial t}f + \frac{\partial L}{\partial \dot{y}_l}(\dot{F}_l - \dot{y}_l\dot{f}) + \frac{\partial L}{\partial y_{n+i}}F_{n+i} + \frac{\partial L}{\partial t}f + \frac{\partial L}{\partial \dot{y}_{n+i}}(\dot{F}_{n+i} - \dot{y}_{n+i}\dot{f}) \\
&+ \dot{f}L - f\dot{L}_1 + Q_l(F_l - \dot{y}_l f) \\
&+ N_i(F_{n+i} - \dot{y}_{n+i}f) + A_i(F_{n+i} - \dot{y}_{n+i}f) + \dot{P}] = 0.
\end{aligned}$$

$$\begin{aligned}
&\epsilon[\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{y}_l}(F_l - \dot{y}_l f) + fL + \frac{\partial L}{\partial \dot{y}_{n+i}}(F_{n+i} - \dot{y}_{n+i}f) - P\right] - (\frac{\partial L}{\partial y_l}F_l + \frac{\partial L}{\partial t}f \\
&+ \frac{\partial L}{\partial \dot{y}_l}(\dot{F}_l - \dot{y}_l\dot{f}) + \frac{\partial L}{\partial y_{n+i}}F_{n+i} + \frac{\partial L}{\partial t}f + \frac{\partial L}{\partial \dot{y}_{n+i}}(\dot{F}_{n+i} - \dot{y}_{n+i}\dot{f}) + \dot{f}L - f\dot{L}_1 \\
&+ Q_l(F_l - \dot{y}_l f) + N_i(F_{n+i} - \dot{y}_{n+i}f) + A_i(F_{n+i} - \dot{y}_{n+i}f) - \dot{P})] = 0.
\end{aligned}$$

Where ϵ is small positive number, which can not be equal to zero $\epsilon \neq 0$. Then

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{y}_l}(F_l - \dot{y}_l f) + fL + \frac{\partial L}{\partial \dot{y}_{n+i}}(F_{n+i} - \dot{y}_{n+i} f) - P\right] - (\frac{\partial L}{\partial y_l}F_l + \frac{\partial L}{\partial t}f$$
$$+ \frac{\partial L}{\partial \dot{y}_l}(\dot{F}_l - \dot{y}_l \dot{f}) + \frac{\partial L}{\partial y_{n+i}}F_{n+i} + \frac{\partial L}{\partial t}f + \frac{\partial L}{\partial \dot{y}_{n+i}}(\dot{F}_{n+i} - \dot{y}_{n+i}\dot{f}) + \dot{f}L - f\dot{L}_1 \tag{4.42}$$
$$+ Q_l(F_l - \dot{y}_l f) + N_i(F_{n+i} - \dot{y}_{n+i} f) + A_i(F_{n+i} - \dot{y}_{n+i} f) - \dot{P}) = 0.$$

Which is the required transformation of the differential principle of Gauss' and juordain. Now the condition for the existence of a conserved quantity, it is obvious that if

$$\frac{\partial L}{\partial y_l}F_l + \frac{\partial L}{\partial t}f + \frac{\partial L}{\partial \dot{y}_l}(\dot{F}_l - \dot{y}_l \dot{f}) + \frac{\partial L}{\partial y_{n+i}}F_{n+i} + \frac{\partial L}{\partial t}f + \frac{\partial L}{\partial \dot{y}_{n+i}}(\dot{F}_{n+i} - \dot{y}_{n+i}\dot{f})$$
$$+ \dot{f}L - f\dot{L}_1 + Q_l(F_l - \dot{y}_l f) + N_i(F_{n+i} - \dot{y}_{n+i} f) + A_i(F_{n+i} - \dot{y}_{n+i} f) - \dot{P} = 0, \tag{4.43}$$

is satisfied , then the dynamical system admits the conservation law in the form

$$\frac{\partial L}{\partial \dot{y}_l}(F_l - \dot{y}_l f) + fL + \frac{\partial L}{\partial \dot{y}_{n+i}}(F_{n+i} - \dot{y}_{n+i} f) - P = constant. \tag{4.44}$$

Equation (4.43) can be decomposed into the a system of partial differential equation of the first order with respect to F_l, F_{n+i}, f and P, we will call this system is generalized killing's equation, for more explanation see [4], [13], [10], [11], [9] and [8].

4.7 Transformation Of Jourdain's Principle

In this section we will explained that the generalized principle (4.35) can be transformed into the relation similar as (4.42) in order to obtained the conservation law, so the Jourdian's principle is

$$\sum_j (F_j + R_j - m_j \ddot{\mathbf{\Gamma}}_j)_c \cdot (\delta \dot{\mathbf{\Gamma}}_j)_c = 0,$$

so repeat the same process as in the previous section, we can easily find the relation same as in equation (4.42).

4.8 Linear First Integrals

In this section we will find that type of integral which is linear with respect to the generalized velocities. In order to find linear first integral, we must take the infinitesimal transformation and the Guage- variant function in the form

$f = 0$, $F_l = F_l(y_l, t)$, $F_{n+i} = F_{n+i}(y_{n+i}, t)$, $P = P(y_l, y_{n+i}, t)$.

For such special form of the function f, F_l, F_{n+i} and P, the existence of linear first integral in equation (4.43) is

$$\frac{\partial L}{\partial y_l}F_l + \frac{\partial L}{\partial \dot{y}_l}\dot{F}_l + \frac{\partial L}{\partial y_{n+i}}F_{n+i} + \frac{\partial L}{\partial \dot{y}_{n+i}}\dot{F}_{n+i} + Q_lF_l + N_iF_{n+i} + A_iF_{n+i} - \dot{P} = 0, \quad (4.45)$$

and in equation (4.44) linear first integral is

$$\frac{\partial L}{\partial \dot{y}_l} + \frac{\partial L}{\partial \dot{y}_{n+i}}F_{n+i} - P = constant \quad (4.46)$$

Let us consider a mathematical pendulum with length r and mass m, the constraint equation is $B_1 = a^2 + b^2 - r^2 = 0$, a and b are the horizontal and vertical axis respectfully, where vertical axis is the oriented down. From the variation of constraint equation we have $\mathbf{R}_{ji} = 2r\sin\alpha\mathbf{i} + 2r\cos\alpha\mathbf{j}$, the position of the particle is

$$\mathbf{r}_1 = r\sin\alpha\mathbf{i} + r\cos\alpha\mathbf{j}.$$

And the velocity of the corresponding particle is,

$$\dot{\mathbf{r}}_\mathbf{1} = (r\cos\alpha\mathbf{i} - r\sin\alpha\mathbf{j})\dot{\alpha}.$$

Here only the gravitational force apply on the particle, which is $\mathbf{F} = mg\mathbf{j}$, where g is the gravitational acceleration. The corresponding potential energy is $V = mgr(1 - \cos\alpha)$. The generalized forces are $Q_l = -mgr\sin\alpha$, $N_i = 2mgr\cos\alpha$.

$$(\frac{d}{dt}\frac{\partial T_k}{\partial \dot{p}_{n+i}} - \frac{\partial T_k}{\partial p_{n+i}})_c - N_i - A_i = 0,$$
$$A_i = -2m(r\dot{\alpha})^2 - 2mgr\cos\alpha$$

is the reaction force. Fore the kinetic energy we will use equation (4.26), so we will get that

$$T_k = \frac{m}{2}(r\dot{\alpha})^2 + 2m(r\dot{\alpha})^2y_2.$$

Hence the Lagrangian function is

$$L = \frac{m}{2}(r\dot{\alpha})^2 + 2m(r\dot{\alpha})^2y_2 - mgr(1 - \cos\alpha).$$

Since the function F_l, F_{n+i}, and P depend only the time t and position, so $F_l = F_l(t, y_l)$, $F_{n+i} = F_{n+i}(t, y_{n+i})$, $P = P(t, y_l, y_{n+i})$, then we can write equation (4.45) as

$$\frac{\partial L}{\partial y_l}F_l + \frac{\partial L}{\partial \dot{y}_l}(\frac{\partial F_l}{\partial \alpha}\dot{\alpha} + \frac{\partial F_l}{\partial t}) + \frac{\partial L}{\partial y_{n+i}}F_{n+i} + \frac{\partial L}{\partial \dot{y}_{n+i}}(\frac{\partial F}{\partial y_{n+i}} + \frac{\partial F}{\partial t})$$
$$+ Q_lF_l + N_iF_{n+i} + A_iF_{n+i} - \dot{P} = 0.$$

$$F_l(-mgr\sin\alpha)+(\frac{\partial F_l}{\partial\alpha}\dot{\alpha}+\frac{\partial F_l}{\partial t})(mr^2\dot{\alpha}+4mr^2\dot{\alpha}y_2)+2m(r\dot{\alpha})^2F_2-(2m(r\dot{\alpha})^2$$
$$+2mgr\cos\alpha)F_2-F_lmgr\sin\alpha+2mgr\cos\alpha F_2-(\frac{\partial P}{\partial\alpha}\dot{\alpha}+\frac{\partial P}{\partial y_2}\dot{y}_2+\frac{\partial P}{\partial t})=0.$$

By comparing various power of $\dot{\alpha}$, we have

$$-2mgr\sin\alpha F_l-\frac{\partial P}{\partial t}=0. \tag{4.47}$$

$$\frac{\partial F_l}{\partial t}(mr^2+4mr^2y_2)-\frac{\partial P}{\partial\alpha}=0. \tag{4.48}$$

$$\frac{F_l}{\partial t}(mr^2+4mr^2y_2)+2mr^2F_2-2mr^2F_2=0, \tag{4.49}$$
$$\frac{F_l}{\partial t}(mr^2+4mr^2y_2)=0$$
$$F_l=C_1.$$

Where C_1 is constant of integration. Put value of F_l in equation (4.47)

$$-2mgr\sin\alpha C_1=\frac{\partial P}{\partial t}$$
$$P=-2mgr\sin\alpha C_1t.$$

Substitute the value of F_l and P in equation (4.46)

$$mr^2\dot{\alpha}+4mr^2\dot{\alpha}y_2+2mgr\sin\alpha C_1t=constant.$$

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