Some Results on Hyperspace Topologies

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A thesis submitted in the partial fulfillment of the requirement for the degree of

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Certificate

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Dedicated to

My Beloved

parents

بسم الله الرحمن الرحيم

In the name Of Allah, the most beneficent, the eternally merciful

Preface

There have been world of good achievements in hyperspace topologies. Since the start of last century, some hyperspace topologies have been introduced and developed. Specifically, Hausdorff metric and Veitoris topologies [1, 22]. The mentioned topologies are impeccable, in the sense of their usage at the least. It is a monumental belief that the most imperative hyperspace topologies have risen as topologies determined by families of geometric set functionals refer to [4].

In fact, hyperspace topologies and related set convergence notation have been considered at the outset of last century, the way we consider to the subject reflect ultimate modren contribution by mathematicians whose mandatory research interests exist outside the general topology. The revival of the subject comes from work of R. Wijsman [25] in the mid of 1960's, and its advancements over the next fifteen years was to a gigantic breakthrough in the hands of U. Mosco, R. Wets, H. Attouch, and their associates. This new approach was advanced for the most part in North America, Europe, Italy and France, in particular. This monumental interest is due to fruitfulness of these various areas of application (such as probability, statistics or variational problem, for example). It also describes the effort in comprehending their structure, common feature and general pattern in order to find a common description for them. About this latter view, we refer the papers [4, 23, 24], devoted to a description and classification of the hyperspace topologies as the outset topology, namely as the weakest topologies which makes continuous families of real-valued functionals defined on nonempty closed subsets of Y. Not only this is helpful in order to have a common description of the hyperspace topologies, but also enables us to tackle some application in an orthodox way (see [5] and [21]).

Three types of hyperspace topologies which comprises familiar topologies are as follows: the hit-and-miss, the proximal hit-and-miss [1, 6, 7] and the weak topologies generated by gap and excess functionals on nonempty closed subsets of Y [1, 4, 15], respectively. As a prototype of weak hyperspace topologies, we must recognize the Wijsman topology, which is the weak topology determined by the distance functionals seen as functionals of set argument. It is a basic tool in the construction of the lattice of hyperspace topologies, above mentioned and many other familiar hyperspace topologies has risen as supreme and infima, respectively of appropriate Wijsman topologies [3].

Let (Y, τ) be an arbitrary Hausdorff topological space. We denote the collection of nonempty, closed subset of Y by $C(Y)$. We will investigate topologies on $C(Y)$ such topologies are called hyperspace topologies. The focus of the thesis is to explain the hyperspace topologies. This thesis is divided into three chapters. In chapter 1, we will discuss the Wijsman and Hausdorff metric topologies. Chapter 1 also helps to explain the normality of the Wijsman topology. The chapter 2 deals with the hit-and-miss and the proximal hit-and-miss topologies. The most of the known Fell, Vietoris, proximal and ball proximal topologies are discussed. In chapter 2, the normality of the Fell and Vietoris topologies is also discussed. The last chapter describes the relationship among hyperspace topologies.

Abstract

The aim of this thesis is to study the hyperspace topologies. In this thesis we will discuss the Wijsman, Hausdorff metric, Fell, Vietoris, Proximal and Ball proximal topologies. Some results concerning the normality of the Wijsman, Fell and Vietoris topologies are also discussed. Furthermore, we describe the relationship among above mentioned hyperspace topologies.

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Chapter 1

The Hausdorff metric and Wijsman Topologies

In this chapter, hyperspace topologies on the metric space are studied. There are many such hyperspace topologies that have been studied extensively. Our aim of this chapter is to study two of these hyperspace topologies, the Hausdorff metric topology and the Wijsman topology determined by family of distance functionals. The Wijsman topology depends on the base space Y and the metric P. The Wijsman topology, defined first for some application in statistics and together used in many applications linked to variational problems. Moreover, the Wijsman topology is not only useful in applications but also as the building block of many other hyperspace topologies. The Hausdorff metric topology is one of the oldest and best-studied hyperspace topologies because of its applicability to different areas of mathematics [1]. Many completeness type properties of the Hausdorff metric topology are stock theorems in topology. For example, the Hausdorff metric topology is compact (resp. totally bounded) iff Y is. Moreover, the local compactness [11] and cofinal completeness [2] are described in Hausdorff metric topology. The Hausdorff metric topology amounts to the uniform convergence of distance functionals while the Wijsman topology is the topology of pointwise convergence of distance functionals corresponding to fixed metric P. Notice that two uniformly equivalent metrics fail to determine the same Wijsman topologies, but this fact is true for the Hausdorff metric topology.

We first present the appropriate terms. Let (Y, P) be a metric space and let U and V be two nonempty subset of Y. The gap $D_P(U, V)$ between U and V is given by

$$D_P(U, V) = \inf\{P(a, b) : a \in U, b \in V\},$$

the excess of U over V is defined by

$$e_P(U, V) = \sup\{P(a, V) : a \in U\},$$

where $P(a, V)$ is the distance functional defined as

$$P(a, V) = \inf\{P(a, v) : v \in U\}.$$

Our basic references for general topology and set theoretic are [13] and [19]. We will denote the open ball and closed ball centered at y_0 in the P metric by $B_P(y_0, \epsilon) = \{y \in Y : P(y, y_0) < \epsilon\}$ and $\overline{B}_P(y_0, \epsilon) = \{y \in Y : P(y, y_0) \leq \epsilon\}$ for $\epsilon > 0$, respectively.

A topological space Y is said to be first-countable if it has a countable local base at each point of Y. If a space Y has a countable base for its topology, then Y is said to be second-countable. Obviously, second-countable implies first-countable space. A topological space Y is said to be a Lindelöf space if every open cover of Y is reducible to a countable subcover.

We say that a topological space (Y, τ) is metrizable if there exist a metric P on Y such that $\tau = \tau_P$, where τ_P is the topology induced by metric P. A topological space Y is said to be separable if there exist a countable subset A of Y such that $\overline{A} = Y$.

A topological space Y is said to be Hausdorff if for each pair u, v of distinct points of Y, there exist disjoint open sets G and H such that $u \in G$ and $v \in H$. A topological space Y is said to be normal if for each pair of disjoint closed sets A and B of Y, there exist disjoint open sets containing A and B respectively. A space Y is said to be hereditary normal if every subspace of Y is normal.

Theorem 1.0.1. *Every metrizable space is normal.*

Theorem 1.0.2. *Every compact Hausdorff space is normal.*

[10] A uniformity on a set Y is a filter U on $Y \times Y$ such that

$(Q1)$ $\triangle(y) \subset V$, for all $V \in U$, where $\triangle(y) = \{(y, y) : y \in Y\}$.

$(Q2)$ for all $V \in U$, $\exists\ W \in U$, such that $W \circ W \in V$, where $W \circ W = \{(x, y) \in Y \times Y : \exists z \in Y$ with $(x, z) \in W$ and $(z, y) \in W\}$.

$(Q3)$ for all $V \in U$ implies $V^{-1} \in U$, where $V^{-1} = \{(x, y) \in Y \times Y : (y, x) \in V\}$. The sets in U are called entourages. The couple (Y, U) is called a uniform space.

1.1 Weak topology

Definition 1.1.1. *[1] Suppose $\{(Y_j, \tau_j) : j \in I\}$ is a collection of a Hausdorff spaces and Y is a nonempty set. Assume that $\Re = \{g_j : j \in I\}$ is a collection of functions, where for every j, $g_j : Y \to Y_j$. Then the weak topology $\tau_\Re$ on Y determined by $\Re$ is the smallest topology τ on Y such that each g_j is continuous.*

It would seem all sets of the type $g_j^{-1}(U_j)$ where U_j is open in τ_j is a subbase for $\tau_\Re$.

Example 1.1.2. *Suppose $\{(Y_i, \tau_i) : i \in I\}$ is a collection of Hausdorff spaces. The product topology on $\prod_{i \in I} Y_i$ is the weak topology determined by the family of projection maps $\{p_j : j \in I\}$ where $p_j : \prod_{i \in I} Y_i \to Y_j$ is defined by $p_j(y) = y(j)$.*

Example 1.1.3. *Consider a topological space (Y, τ). Let $B \subset Y$, then the relative topology on B is a weak topology determined by single inclusion map $i_B : B \to Y$ be defined by $i_B(b) = b$.*

Now, we describe the convergence of a sequence or net in weak topologies in next theorem.

Theorem 1.1.4. *Let $\Re = \{g_j : j \in I\}$ induced a topology $\tau_\Re$ on Y. Then a net $< y_\lambda >$ in Y is $\tau_\Re$-convergent to $y \in Y$ iff $\lim_\lambda g_j(y_\lambda) = g_j(y)$ for every $j \in I$.*

Proof. Assume that $y = \tau_\Re - \lim y_\lambda$, then for every $j \in I$, implies $\lim_\lambda g_j(y_\lambda) = g_j(y)$ by the continuity of every g_j. Conversely, assume that $\lim_\lambda g_j(y_\lambda) = g_j(y)$ for every $j \in I$. Suppose W is a $\tau_\Re$ open subset of Y. So there exist $j_1, j_2, j_3, ..., j_n \in I$ and open sets $W_{j_1}, W_{j_2}, W_{j_3}, ..., W_{j_n}$ in a mark spaces of $g_{j_1}, g_{j_2}, g_{j_3}, ..., g_{j_n}$ such that

$$y \in \cap_{m=1}^{n} g^{-1}(W_{j_m}) \subset W.$$

Since by continuity of g_{j_m} for $m = 1, 2, ..., n$ there exist $\lambda_0 \in \Lambda$ such that for each m and $\lambda \geq \lambda_0$, we have $g_{j_m}(y_\lambda) \in W_{j_m}$. Consequently, $y_\lambda \in W$ for $\lambda \geq \lambda_0$. □

1.2 Wijsman topology

Definition 1.2.1. *[1] Suppose (Y, P) is a metric space. The lower Wijsman topology (resp. upper Wijsman topology) on $C(Y)$ defined by the metric P, denoted by $\tau^-_{W_P}$(resp. $\tau^+_{W_P}$) is the smallest topology on $C(Y)$ such that $y \in Y$ the distance functional $P(y, .) : C(Y) \to [0, +\infty)$ is upper semicontinuous (resp. lower semicontinuous). The Wijsman topology of (Y, P), denoted by τ_{W_P} is the smallest topology on $C(Y)$ such that functional $P(y, .) : C(Y) \to [0, +\infty)$ is continuos for $y \in Y$.*

A Wijsman topology τ_{W_P} on $C(Y)$ has $\{B \in C(Y) : P(y, B) > k\}$ and $\{B \in C(Y) : P(y, B) < k\}$, $k > 0$ and $y \in Y$ as a subbase. Moreover, the local base for τ_{W_P} at $B \in C(Y)$ consist of all sets of the type

$$U_{(B,F,\epsilon)} = \{A \in C(Y) : |P(y, B) - P(y, A)| < \epsilon \text{ for all } y \in F\},$$

where $\epsilon > 0$ and F is finite subset of Y.

A net $(G_\lambda)_{\lambda\in\Lambda}$ in $C(Y)$ is $\tau_{W_P}-$convergent to G in $C(Y)$, if for all $y \in Y$ implies $\lim_\lambda P(y, G_\lambda) = P(y, G)$. Let us first define $f_G : Y \to R$ by $f_G(y) = P(y, G)$, where $G \in C(Y)$.

Theorem 1.2.2. *A net (G_λ) of closed subset of Y converges to G in $(C(Y), \tau_{W_P})$ iff $(f_{G_\lambda}) \to f_G$ pointwise.*

Proof. Suppose $(G_\lambda) \to G$ in $(C(Y), \tau_{W_P})$. Consider $y \in Y$ and $\epsilon > 0$. There exist $\lambda_0 \in \Lambda$ such that $G_\lambda \in U_{(G,y,\epsilon)}$ for $\lambda \geq \lambda_0$. Hence $|P(y, G_n) - P(y, G)| < \epsilon$ for $\lambda \geq \lambda_0 \Rightarrow |f_{G_n}(y) - f_G(y)| < \epsilon$. Conversely, assume $(f_{G_\lambda}) \to f_G$ pointwise. So for each $y \in Y$ and $\epsilon > 0$, there exist $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $|f_{G_\lambda}(y) - f_G(y)| < \epsilon$. This means that if $\lambda \geq \lambda_0$ then $|P(y_i, G_\lambda) - P(y_i, G)| < \epsilon$, for $i = 1, 2, ..., n$. Hence $G_\lambda \in U_{(G,y_i,\epsilon)}$ for $\lambda \geq \lambda_0$. □

Therefore, Wijsman convergence of a net of closed sets amounts to the pointwise convergence of the affiliate net of distance functionals. On the other hand, the mapping $B \to P(., B)$ is an embedding of $(C(Y), \tau_{W_P})$ into the space of continuous functions $C(Y, R)$, equipped with the topology of poinwise convergence.

Now, we characterize the Wijsman convergence with the help of following lemma.

Lemma 1.2.3. *[1] For a metric space (Y, P) a net $(G_\lambda)_{\lambda \in \Lambda}$ in $C(Y)$ is $\tau_{W_P}-$convergent to G in $C(Y)$ iff the following axioms are satisfied.*

(C_1) If $G \cap U \neq \phi$ then $G_\lambda \cap U \neq \phi$ eventually for every nonempty open set U.

(C_2) If $0 < \delta < \beta$ and then $B_P(y, \beta) \cap G = \phi$ implies $B_P(y, \delta) \cap G_\lambda = \phi$ eventually.

Proof. Claim the lemma is proved by showing that (C_1) and (C_2) are equivalent to the conditions that for all $y \in Y$, $P(y, G) \geq \limsup_\lambda P(y, G_\lambda)$ and $P(y, G) \leq \liminf_\lambda P(y, G_\lambda)$, respectively. Assume that $P(y, G) \geq \limsup_\lambda P(y, G_\lambda)$ holds, for each $y \in Y$ and suppose $G \cap U \neq \phi$ for open subset U of Y. Take $x \in G$ and $\delta > 0$ such that $B_P(x, \delta) \subseteq U$. As $P(x, G) = 0$, eventually $P(x, G_\lambda) < \delta$ will hold, and for each such λ $B_P(y, \delta) \cap G_\lambda \neq \phi$. Hence (C_1) is satisfied. Suppose (C_1) holds, fix $y \in Y$ and $\beta > 0$ and take $x \in G$ with $P(y, x) < P(y, G) + \frac{\beta}{2}$. Since by (C_1), $B_P(x, \delta) \cap G_\lambda \neq \phi$ for all large λ, and for all such λ $P(y, G_\lambda) \leq P(y, x) + P(x, G_\lambda) < P(y, x) + \frac{\beta}{2}$, therefore $P(y, G_\lambda) < P(y, G) + \beta$. Similarly, we can prove the other equivalence. □

Obviously, axiom (C_2) implies the closure condition given below. If all neighborhoods U of $y \in Y$ intersect G_λ for cofinal set of indices λ, then $y \in G$. Notice that this condition with (C_1) does not implies the Wijsman convergence in general.

Example 1.2.4. *Suppose $Y = \{y_n, n \in N\}$, let a metric P on Y is defined as*

$$P(y_1, y_n) = 2 \text{ for } n > 1, \text{ and } P(y_n, y_m) = 1 \text{ for } 1 < n < m.$$

Let $G_n = \{y_1, y_{n+1}, y_{n+2}, ...\}$ for each $n \in N$, and $G = \{y_1\}$. Thus $G, G_1, ...$ satisfies axiom (C_1) of Lemma 1.2.3 and also above closure condition, but G_n does not converges to G with respect to τ_{W_P}, because for all $n \geq 2$ we have $P(y_2, G_n) = 1$, and $P(y_2, G) = 2$.

We say that Wijsman topology τ_{W_P} on $C(Y)$ is admissible. If the relative topology that Y inherits from τ_{W_P} under the identification $y \to \{y\}$, coincides τ_P.

Lemma 1.2.5. *[1] Suppose (Y, P) is a metric space. Then τ_{W_P} on $C(Y)$ is Hausdorff, completely regular and admissible.*

We say a subset W of a metric space (Y, P) is ϵ-discrete if for any $w_1, w_2 \in W$ and $w_1 \neq w_2$, we have $P(w_1, w_2) > \epsilon$. By Zorn's lemma, for any $\epsilon > 0$, Y has maximal ϵ-discrete subset W_ϵ with respect to set inclusion, so by maximality of this set, $\overline{B}_P(W_\epsilon, \epsilon) = Y$.

Theorem 1.2.6. *Suppose (Y, P) is a metric space. Then the following are equivalent:*

(1) *Y is separable.*

(2) *τ_{W_P} on $C(Y)$ is second countable.*

(3) *τ_{W_P} on $C(Y)$ is metrizable.*

(4) *τ_{W_P} on $C(Y)$ is first countable.*

Proof. Suppose that (1) holds. Assume that D is a countable dense subset of Y. Since by the inequality $|P(y, B) - P(x, B)| \leq P(x, y)$ the sets of the form $\{B \in C(Y) : P(y, B) < \delta\}$ and $\{B \in C(Y) : P(y, B) > \delta\}$ for ($y \in D$ and δ is positive rational), form a countable subbase for τ_{W_P}.

(2) ⇒ (3). Since τ_{W_P} is completely regular, it follows from Urysohn metrization theorem, τ_{W_P} is metrizable.

(3) ⇒ (4). Obviously.

(4) ⇒ (1). If for all ϵ-discrete subset of Y are countable, then for any $n \in N$, there would exist $\frac{1}{n}$-discrete set D_n such that $\overline{B}_P(D_n, \frac{1}{n}) = Y$. Thus Y would be separable. If Y is not separable. Then there exist uncountable ϵ-discrete set D for some $\epsilon > 0$. Since $P(y, Y) = 0$ for $y \in Y$. So for $Y \in C(Y)$ and a finite subset E of Y and $\delta > 0$, the collection

$$B(E, \delta) = \{B \in C(Y) : P(x, B) < \delta,\ \forall\ x \in E\}$$

form a local base for τ_{W_P} at Y. Take a countable family $\{B(E_i, \delta_i) : i \in N\}$ of such sets, since D is uncountable so there exist $x_0 \in D$ such that $P(y, x_0) > \frac{\epsilon}{2}$ for $y \in \bigcup E_i$. As a result, for any i, $\{B \in C(Y) : P(x_0, B) < \frac{\epsilon}{2}\}$ is a neighborhood of Y in τ_{W_P} fails to contain E_i, and so fails to contain $B(E_i, \delta_i)$. Therefore, this particular local base is not countable, and hence τ_{W_P} fails to be first countable. □

Notice that, if $D = \{y_n : n \in N\}$ is any dense subset of Y, the metric δ_P on $C(Y)$ defined by

$$\delta_P(U, V) = \textstyle\sum_{j=1}^{\infty} 2^{-j} min\{1, |P(y_j, U) - P(y_j, V)|\}$$

is compatible with τ_{W_P}. The following example shows that completeness of the metric P does not implies the completeness of δ_P. Consider (Y, P) and G_n defined above in Example 1.2.4, and take $D = Y$ be countable dense subset. If $n < m$ then $\delta_P(U_n, V_m) = \sum_{i=j+1}^{m} 2^{-i}$, hence $< G_n >$ is δ_P-cauchy. If $< G_n > -\tau_{W_P}$ converges to some $B \in C(Y)$. So we must have for each $y \in Y$ $P(y, G_n) \to P(y, B)$. But $P(., G_n)$ converges pointwise to the function $g : Y \to R$ defined by

$$g(y) = \begin{cases} 0, & \text{If } y = y_1 \\ 1, & \text{If } y \neq y_1 \end{cases}$$

It seems, for any nonempty closed subset of Y, g is not distance functional. The natural question arises. When is a pointwise limit of a net a distance functional? An answer is given in the next proposition.

Proposition 1.2.7. *Let (Y, P) be a metric space and $g \in C(Y, R)$ be in the closure of $\{P(., B) : B \in C(Y)\}$ with respect to the topology of pointwise convergence. Let $G = \{y \in Y : g(y) = 0\}$ is nonempty, and for all $y \in Y$, we have $P(y, G) \leq g(y)$. Then g is distance functional for the set G.*

Proof. We need to prove that $g(y) \leq P(y, G)$ for $y \in Y$. By supposition $P(y, G) \leq g(y)$. Fix $y \in Y$. Assume on contrary $P(y, G) < g(y)$ holds. Let $\eta = g(y) - P(y, G)$, and take $a \in G$ with $P(y, a) < P(y, G) + \frac{\eta}{3}$. As g is pointwise limit of distance functional, there exist $F \in C(Y)$ for which $|g(a) - P(a, F)| < \frac{\eta}{3}$ and $|g(y) - P(y, F)| < \frac{\eta}{3}$. Since $g(a) = 0$, this implies,

$$g(y) < P(y, F) + \tfrac{\eta}{3} \leq P(y, a) + P(a, F) + \tfrac{\eta}{3} < P(y, a) + g(a) + \tfrac{2\eta}{3} < P(y, G) + \eta.$$

Which is contradiction to the definition of η. □

The metrics, P and P'', on a set Y is said to be equivalent if the corresponding metric topologies are the same, and is said to be uniformly equivalent if they determine the same uniformity.

Remark 1.2.8. *If two metrics are equivalent even metrics are uniformly equivalent need not determine the same Wijsman topologies. Suppose $Y = Z^+$ and a metric P on Y defined by $P(u, v) = |\frac{1}{u} - \frac{1}{v}|$, $\forall\ u, v \in Y$. Then P is equivalent to discrete metric δ. Take $G_n = \{n, n+1, ...\}$ for $n \geq 1$ and $G = \{1\}$. Therefore G_n $\tau_{W_\delta}-$convergent to G. On the other hand, G_n does not $\tau_{W_P}-$convergent to G.*

More interestingly we have two metrics they are not uniformly equivalent but give the same Wijsman topologies. Let us define a metrics P and δ on $Y = Z^+$ as follows, $P(u, v) = |\frac{1}{u} - \frac{1}{v}|$ and $\delta(u, v) = 1 + |\frac{1}{u} - \frac{1}{v}|$, for $u \neq v$. i.e $\tau_{W_\delta} = \tau_{W_P}$ Since P and δ are equivalent metrics, *so we only prove upper Wijsman convergence. Obviously,* $\tau_{W_P} \subseteq \tau_{W_\delta}$*. Suppose* $G_n \in C(Y)$ *converges to* $G \in C(Y)$ *with respect to* τ_{W_δ} *and* G_n *does not converges to* G *with respect to* τ_{W_P}*. Then there exist* $x_0 \in Y, 0 < \epsilon < \mu$ *such that for each* n *there is* $m_n \geq n$ *with* $G \cap B_P(x_0, \mu) = \phi$ *and* $G_{m_n} \cap B_P(x_0, \epsilon) \neq \phi$*. For each* n *let us take* $x_{m_n} \in G_{m_n} \cap B_P(x_0, \epsilon)$*. We take* α, β *such that* $0 < \alpha = \epsilon + 1 < \beta = \mu + 1$*, it follows that* $G_{m_n} \cap B_\delta(x_0, \alpha) \neq \phi$ *but* $G \cap B_\delta(x_0, \beta) = \phi$ *because* $B_\delta(x_0, \beta) = B_P(x_0, \mu)$*, a contradiction. Hence* G_n *converges to* G *with respect to* τ_{W_P}*. Thus* $\tau_{W_\delta} = \tau_{W_P}$*.*

We discuss the necessary and sufficient conditions on metrics P and δ which ensure that P and δ will come up with same Wijsman topologies.

Let (Y, P) be a metric space and U and V be two nonempty subset of Y such that $U \subseteq V$. We call U is strictly P-included in V if there exist a finite subset $y_1, y_2, ..., y_n$ of V and $0 < \beta_j < \eta_j$, $j = 1, 2, ...n$ such that

$$U \subset \cup_{j=1}^{n} B_P(y_j, \beta_j) \subset \cup_{j=1}^{n} B_P(y_j, \eta_j) \subset V.$$

We call U is P-included in V if there exist a finite subset $y_1, y_2, ..., y_n$ of V and $0 < \beta_j$, $j = 1, 2, ...n$ such that

$$U \subset \cup_{j=1}^{n} B_P(y_j, \beta_j) \subset V.$$

Theorem 1.2.9. *Suppose* (Y, P) *and* (Y, δ) *are equivalent metric spaces. Then* $\tau_{W_P} = \tau_{W_\delta}$ *on* $C(Y)$ *iff each proper open* P*-ball strictly* δ*-includes each concentric open* P *ball of smaller radius, and each proper open* δ*-ball strictly* P*-includes each concentric open* δ *ball of smaller radius.*

Proof. We prove the following.

(a) $\forall\ y \in Y$, if $B_P(y, \epsilon) \neq Y$ and $0 < \beta < \epsilon$, the ball $B_P(y, \beta)$ is strictly δ-included in $B_P(y, \epsilon) \Rightarrow \tau_{W_P} \subset \tau_{W_\delta}$.

(*b*) $\forall\ y \in Y$, if $B_\delta(y, \epsilon) \neq Y$ and $0 < \beta < \epsilon$, the ball $B_\delta(y, \beta)$ is strictly P-included in $B_\delta(y, \epsilon) \Rightarrow \tau_{W_\delta} \subset \tau_{W_P}$.

Assume that (a) holds, and let a net $(G_\lambda)_{\lambda\in\Lambda}$ in $C(Y)$ is $\tau_{W_\delta}-$convergent to G in $C(Y)$. We now use Lemma 1.2.3 for Wijsman convergence. Let $G \cap U \neq \phi$, where U is τ_P open set. Take $y \in G$ and $0 < \beta < \epsilon$ such that $B_P(y, \beta) \subset B_P(y, \epsilon) \subset U$. So by (a), there exist $v \in B_P(y, \epsilon)$ and $\alpha > 0$ such that $y \in B_\delta(v, \alpha) \subset B_P(y, \epsilon)$. As $G \cap B_\delta(v, \alpha) \neq \phi$ and $(G_\lambda)_{\lambda\in\Lambda}$ τ_{W_δ}-convergent to G. Thus $(G_\lambda) \cap B_\delta(v, \alpha) \neq \phi$ eventually. Since $B_\delta(v, \alpha) \subset U$. Hence $(G_\lambda) \cap U \neq \phi$ eventually. Fix $y \in Y$ and $0 < \beta < \epsilon$ such that $G \cap B_P(y, \epsilon) = \phi$. As $B_P(y, \epsilon)$ is proper ball, therefore by (a) we can find $y_1, y_2, ..., y_n$ in $B_P(y, \epsilon)$ and $0 < \beta_j < \epsilon_j$, $j = 1, 2, ...n$, such that

$$B_P(y, \beta) \subset \cup_{j=1}^n B_\delta(y_j, \beta_j) \subset \cup_{j=1}^n B_\delta(y_j, \epsilon_j) \subset B_P(y, \epsilon).$$

Since for all j, $G \cap B_\delta(y_j, \epsilon_j) = \phi$. Hence by τ_{W_δ}-convergence implies that for all j, $G_\lambda \cap B_\delta(y_j, \beta_j) = \phi$ eventually. Thus $G_\lambda \cap B_P(y, \beta) = \phi$ eventually. Analogously, we can prove the statement (b).

Conversely, we claim that if the statements (a) (resp. (b)) fails, then $\tau_{W_P} \not\subset \tau_{W_\delta}$ $(resp.\ \tau_{W_\delta} \not\subset \tau_{W_P})$ respectively. Assume that (a) is not holds, and in particular take $y_0 = y, \beta_0 = \beta$, and $\epsilon_0 = \epsilon$ for which $B_P(y, \beta)$ is not strictly δ-included in $B_P(y, \epsilon) \neq Y$. Let $v \in B_P(y_0, \epsilon_0)$, and $\delta(v, B) = \psi(v)$, where B is the complement of $B_P(y_0, \epsilon_0)$. So for each $y_1, y_2, ..., y_n$ in $B_P(y_0, \epsilon_0)$ and $0 < \beta_1, \beta_2, ..., \beta_n$ we have

$$B_P(y_0, \beta_0) \not\subset \cup_{j=1}^n B_\delta(y_j, \psi(y_j) - \beta_j).$$

Thus for any $y_1, y_2, ..., y_n$ in $B_P(y_0, \epsilon_0)$ and $m \in Z^+$, there exist an element $y = \sigma(y_1, y_2, ..., y_n, m)$ in $B_P(y_0, \beta_0)$ such that for $j = 1, 2, ..., n$

$$y \notin \{v \in Y : \delta(v, y_j) < (1 - \tfrac{1}{m})\psi(y_j)\}.$$

Suppose Σ be the collection of finite subset of $B_P(y_0, \epsilon_0)$. Thus $(\Sigma, \subseteq)$ is a poset. As $(N, \leq)$ is a poset. Equipping $\Sigma \times N$ with the product partial order, and

$$(S, m) \to B \cup \sigma(S, m)$$

where $\sigma(S,m)$ is defined above is a net from $\Sigma \times N$ to Y. Then a net $\sigma(S,m)$ τ_{W_δ}-converge to B. Obviously, first axiom of Lemma 1.2.3 for τ_{W_δ}-convergence is satisfied. Now we prove the second axiom, let $\overline{y} \in Y$ and $0 < \alpha < \eta$ with $B_\delta(\overline{y}, \eta) \cap B = \phi$. Since $\eta \leq \psi(\overline{y})$, and we can take $m \in N$ such that

$$(1 - \tfrac{1}{m})\psi(\overline{y}) \geq \alpha.$$

Thus if $(S,k) \geq (\{\overline{y}\}, m)$, we have $\sigma(S,k) \notin \{v \in Y : \delta(v,y) < (1 - \frac{1}{m})\psi(y)\}$, $\forall\ y \in S$, and in particular, $\sigma(S,k) \notin \{v \in Y : \delta(v,\overline{y}) < (1 - \frac{1}{m})\psi(\overline{y})\}$. Consequently, $\sigma(S,k)$ can not lie in $B_\delta(\overline{y}, \alpha)$. But on the other hand, $\sigma(S,m)$ does not converges to B with respect to τ_{W_P}. Because $\limsup P(y_0, \sigma(S,m)) \leq \beta_0 < \epsilon_0 \leq P(y_0, B)$. □

1.3 Normality of the Wijsman topology

In this section we will discuss the normality and metrizability of the Wijsman topology. About the normality of the Wijsman topology, we mention the papers [16] and [8]. By a well known result [20], if (Y,P) is a separable metric space iff $(C(Y), \tau_{W_P})$ is metrizable and hence normal. Di Maio In [12], raised the following problem. Is $(C(Y), \tau_{W_P})$ normal iff $(C(Y), \tau_{W_P})$ metrizable? We will present the solution of this problem in many classes of metric spaces given below.

Lemma 1.3.1. *Suppose $\delta > 0$ and (Y,P) is a $0-\delta$ metric space. If $(C(Y), \tau_{W_P})$ is normal then Y is countable.*

Proof. Since (Y,P) has nice closed balls, then by Theorem 3.0.13, we have $\tau_{W_P} = \tau_F$. By Theorem 2.3.6, Y is Lindelöf. Hence Y is countable. □

Remark 1.3.2. *In the above result we observe that the normality of τ_{W_P} is equivalent to metrizablity.*

Lemma 1.3.3. *Let (Y,P) be a metric space and take a closed discrete subset $S \subset Y$. Consider for each $y \in Y \setminus S$, the following property is satisfied. There is ϵ_y and at most one $s_y \in S$*

with $P(y, s_y) < \epsilon_y$, for all other $s \in S$ satisfied $P(y, s) = \epsilon_y$. Then $(C(S), \tau_{W_{P|S}})$ is a closed subspace of $(C(X), \tau_{W_P})$.

Proof. Since S is a closed, then $C(S)$ is a closed set in $(C(Y), U^-)$, so in $(C(Y), \tau_{W_P})$. Let $G_\lambda \in C(S)$ is τ_{W_P}-converge to $G \in C(S)$. Then $P(y, G_\lambda) \to P(y, G)$, for all $y \in Y$. Since $S \subset Y$ implies G_λ converges to G with respect to $\tau_{W_{P|S}}$. Now let $G_\lambda \in C(S)$ converges to $G \in C(S)$ with respect to $\tau_{W_{P|S}}$ and take $y \in Y \backslash S$. We have the following cases.

(1) $P(y, s) = \epsilon_y$ for all $s \in S$

(2) There is $s_y \in S$ with $P(y, s_y) = \lambda < \epsilon_y$ and $P(y, G) < \epsilon_y$.

(3) There is $s_y \in S$ with $P(y, s_y) = \lambda < \epsilon_y$ and $P(y, G) = \epsilon_y$.

In (1) $P(y, G_\lambda) = \epsilon_y \to \epsilon_y = P(y, G)$.

In (2) $s_y \in G$. So $s_y \in G_\lambda$ eventually and hence $P(y, G_\lambda) \to \lambda = P(y, G)$. In (3) $s_y \notin G$. Eventually, $s_y \notin G_\lambda$, and hence $P(y, G_\lambda) \to \epsilon_y = P(y, G)$. □

Consider a metric space (Y, P) and $\delta > 0$, a uniformly equivalent metric space (Y, P_δ) defined as $P_\delta(u, v) = min\{P(u, v), \delta\}$ for $u, v \in Y$.

Theorem 1.3.4. *Suppose (Y, P) is a metric space. If for each $\epsilon > 0$, there is $\delta \in (0, \epsilon)$ such that $(C(Y), \tau_{W_{P_\delta}})$ is normal, then Y is separable.*

Proof. Let Y be a non separable, so there exist an $\epsilon > 0$ and uncountable ϵ-discrete subset D of Y. Let $\delta < \frac{\epsilon}{2}$ such that $(C(Y), \tau_{W_{P_\delta}})$ is normal. Then by Lemma 1.3.3 $(C(D), \tau_{W_{P_{\delta|D}}})$ is a closed subset of $(C(X), \tau_{W_{P_\delta}})$ and thus normal. By Lemma 1.3.1 $|D| = \aleph_0$, which is a contradiction. Hence Y is separable. □

A subset A of Y is said to be totally bounded if for every $\epsilon > 0$, there exist a finite subset G of A such that $A \subset B_P(G, \epsilon)$.

Lemma 1.3.5. *Consider every ball in a metric space (Y, P) is totally bounded. If N is uniformly equivalent metric space on Y, then $\tau^+_{W_P} \subseteq \tau^+_{W_N}$.*

Proof. Suppose $G_n \in C(Y)$ converges to $G \in C(Y)$ with respect to τ_{W_N} and G_n does not converges to G with respect to τ_{W_P}, so there exist $x_0 \in Y, 0 < \epsilon < \mu$ such that for each n there is $m_n \geq n$ with $G \cap B_P(x_0, \mu) = \phi$ and $G_{m_n} \cap B_P(x_0, \epsilon) \neq \phi$. For each n let us take $x_{m_n} \in G_{m_n} \cap B_P(x_0, \epsilon)$. Since P and N are uniformly equivalent, there exist λ_1 and $\lambda_2 > 0$ such that for each $x, y \in Y$, $N(x, y) < \lambda_1$ implies $P(x, y) < \mu - \epsilon$ and $P(x, y) < \lambda_2$ implies $N(x, y) < \frac{\lambda_1}{2}$. Since $B_P(x_0, \epsilon)$ is totally bounded in (Y, P), we can find $y_1, y_2, ..., y_k \in B_P(x_0, \epsilon)$ with $B_P(x_0, \epsilon) \subseteq \cup_{i=1}^{k} B_P(y_i, \lambda_2)$. Consequently, there is i, $1 \leq i \leq k$, such that $x_{m_n} \in B_P(y_i, \lambda_2)$ frequently. Since $B_P(y_i, \lambda_2) \subseteq B_N(y_i, \frac{\lambda_1}{2})$, it follows that $G_{m_n} \cap B_N(y_i, \frac{\lambda_1}{2}) \neq \phi$ frequently. On the other hand, $z \in B_N(y_i, \lambda_1)$ implies $N(y_i, z) < \lambda_1$ then $P(y_i, z) < \mu - \epsilon$. $P(x_0, z) \leq P(x_0, y_i) + P(y_i, z) < \mu - \epsilon + \epsilon = \mu$, so $z \in B_P(x_0, \mu)$, i.e $B_N(y_i, \lambda_1) \subseteq B_P(x_0, \mu)$ and hence $G \cap B_N(y_i, \lambda_1) \neq \phi$, which is a contradiction. □

For a metric space (Y, P) and $k > 0$, a uniformly equivalent metric space (Y, N_k) defined as $N_k(x, y) = min\{P(x, y), k\}$. We have $\tau^+_{W_{N_k}} \subseteq \tau^+_{W_P}$, suppose that $G_i \in C(Y)$ converges to $G \in C(Y)$ with respect to $\tau^+_{W_P}$ and take any $x \in Y$ and $0 < \epsilon < \mu$. If $G \cap B_{N_k}(x, \mu) = \phi$, then $B_{N_k}(x, \mu) \neq Y$; thus $B_{N_k}(x, \mu) = B_P(x, \mu)$ and $B_{N_k}(x, \epsilon) = B_P(x, \epsilon)$.

Theorem 1.3.6. *Consider a metric space (Y, P). The following are equivalent.*

(1) *For every $k > 0$, $\tau^+_{W_P} = \tau^+_{W_{N_k}}$.*

(2) *For every $k > 0$, $\tau^+_{W_P} \subseteq \tau^+_{W_{N_k}}$*

(3) *Every proper closed ball in (Y, P) is totally bounded.*

Proof. By Lemma 1.3.5 we only need to show that $1 \Rightarrow 2$. Assume that there is proper closed $\overline{B}_P(x, t)$ which is not totally bounded. Then there exist $k > 0$ such that $\overline{B}_P(x, t) \notin B_P(G, 2k)$ for every finite subset G of $\overline{B}_P(x, t)$. We can take a sequence y_n of elements of $\overline{B}_P(x, t)$ such that $P(y_i, y_j) > 2k$ for $i \neq j$. Take $y_0 \in Y - \overline{B}_P(x, t)$ and number λ, δ such that $t < \lambda < \delta$ and $P(x, y_0) > \delta > \lambda > t$. Let $G_n = \{x_n\}$ and $G = \{y_0\}$ for $n \in N$. Then G_n converges to G with respect to $\tau^+_{N_k}$; take $x_0 \in Y$, and $0 < \epsilon < \mu$. If $G \cap B_{N_k}(x_0, \mu) = \phi$ then $B_{N_k}(x_0, \mu) = B_P(x_0, \mu)$ and $\mu \leq k$. Suppose for each $n \in N$, there exist $k_n \in N$

such that $y_{n_k} \in B_{N_k}(x_0, \epsilon) = B_P(x_0, \epsilon)$. Then $P(x_0, y_{n_k}) < \epsilon < \mu \leq k$ for $n \in N$ and hence $P(y_{n_k}, y_{n_m}) \leq P(y_{n_k}, x_0) + P(x_0, y_{n_m}) < 2k$, a contradiction. On the other hand, $G \cap B_P(x, \delta) = \phi$ but $G_n \cap B_P(x, \lambda) \neq \phi$, for all $n \in N$ and hence G_n does not converges to G with respect to $\tau^+_{W_P}$. □

Corollary 1.3.7. *Suppose (Y, P) is a metric space. Then every proper closed ball in Y is totally bounded iff for each $k > 0$ $\tau_{W_P} = \tau_{W_{N_k}}$.*

Corollary 1.3.8. *Consider every proper closed ball in a metric space (Y, P) is totally bounded. If $(C(X), \tau_{W_P})$ is normal. Then Y is separable.*

Furthermore, we will discuss normality of Wijsman topology which can be dealt with as partial answer to above Problem. For this, we will use the following results.

Lemma 1.3.9. *Consider a metric space (Y, P). If $\Re$ is a directed family in $C(Y)$ such that $G = \bigcup \Re$ is closed. Then G is a accumulation point of the net $(\Re, \subseteq)$ in $(C(Y), \tau_{W_P})$.*

Consider the ordinals ω_1 and $\omega_1 + 1$ as a topological spaces equipped with the order topology.

Proposition 1.3.10. *Consider a non-separable metric space (Y, P). Then the subspace $C(Y) \backslash Y$ of $(C(Y), \tau_{W_P})$ contains a closed copy of the space $\omega_1 \times (\omega_1 + 1)$.*

Proof. Since (Y, P) is not separable, there exist $\epsilon > 0$ and uncountable ϵ- discrete subset

$$U = \{x_{\mu,\nu} : \mu < \omega_1 \text{ and } \nu \leq \omega_1\}$$

of Y, with $x_{\mu,\nu} \neq x_{\mu',\nu'}$ for $(\mu, \nu) \neq (\mu', \nu')$. For every $\mu < \omega_1$, let $U_\mu = \{x_{\mu,\nu} : \nu \leq \omega_1\}$ and $D_\mu = B_P(U_\mu, \epsilon/2)$. Set $A_0 = \{x_{0,0}\}$. $\forall$ $\mu < \omega_1$ and $0 < \nu \leq \omega_1$, assume

$$G_\mu = Y \setminus (\bigcup_{\kappa \geq \mu} D_\kappa) \text{ and } A_\nu = \{x_{\kappa,\lambda} : \kappa < \omega_1 \text{ and } \lambda < \nu\}.$$

The families $\mathcal{F} = \{G_\mu : \mu < \omega_1\}$ and $S' = \{A_\nu : \nu \leq \omega_1\}$ are continuously increasing, in the sense $G_\mu = \bigcup\{G_{\kappa+1} : \kappa < \mu\}$ and $A_\nu = \bigcup\{A_{\lambda+1} : \lambda < \nu\}$, for each $0 < \mu < \omega_1$ and $0 < \nu \leq \omega_1$. We prove that the subspace

$$H' = \{G_\mu \cup A_\nu : \mu < \omega_1 \text{ and } \nu \leq \omega_1\}$$

of $(C(Y), \tau_{W_P})$ is homeomorphic to the product space $\omega_1 \times (\omega_1+1)$. Let $\psi : \omega_1 \times (\omega_1+1) \to H'$ be defined by $\psi(\mu, \nu) = G_\mu \cup A_\nu$. Obviously, ψ is one-to-one and onto. To prove that ψ is continuous, let $E \subseteq \omega_1 \times (\omega_1+1)$, and $(\mu, \nu) \in \overline{E}$. Suppose $E' = \{(\kappa, \lambda) \in E : \kappa \leq \mu \text{ and } \lambda \leq \nu\}$, and notice that $(\mu, \nu) \in \overline{E'}$. Since for all $(\kappa, \lambda), (\kappa', \lambda') \in E'$, there exists $(\alpha, \beta) \in E'$ such that $\alpha \geq \max(\kappa, \kappa')$ and $\beta \geq \max(\lambda, \lambda')$. Consequently, the collection $\{G_\kappa \cup A_\lambda : (\kappa, \lambda) \in E'\}$ is directed. As $(\mu, \nu) \in \overline{E'}$, so for each $\mu' < \mu$ and $\nu' < \nu$ that there exists $(\kappa, \lambda) \in E'$ such that $\kappa \geq \mu'$ and $\lambda \geq \nu'$. Since $\mathcal{F}$ and S' are continuously increasing, we have

$$\bigcup\{G_\kappa \cup A_\lambda : (\kappa, \lambda) \in E'\} = G_\mu \cup A_\nu.$$

From Lemma 1.3.9 the net $(\{G_\kappa \cup A_\lambda : (\kappa, \lambda) \in E'\}, \subseteq)$ converges to $F_\mu \cup A_\nu$ in τ_{W_P}. Consequently $\psi(\mu, \nu) \in \overline{\psi(E')} \subseteq \overline{\psi(E)}$. Hence ψ is continuous. Now we prove that ψ is open, notice that

($\star$) Since for all $x \in U, \mu < \omega_1$ and $\nu \leq \omega_1$, either $x \in G_\mu \cup A_\nu$ or $P(x, G_\mu \cup A_\nu) \geq \frac{\epsilon}{2}$. Let V be an open subset of $\omega_1 \times (\omega_1 + 1)$. Let $(\mu, \nu) \in V$. The element $\psi(\mu, \nu) = G_\mu \cup A_\nu$ of the set $\psi(V)$ is denoted by J. There exist $\kappa < \mu$ and $\lambda < \nu$ such that $(\kappa, \mu] \times (\lambda, \nu] \subseteq V$.

Let $S = \{x_{\mu,\nu}, x_{\mu,\lambda}, x_{\kappa,\nu}\}$ and

$$N_{J,S,\frac{\epsilon}{4}} = \{W \in H' : |P(y, W) - P(y, J)| < \frac{\epsilon}{4} \text{ for every } y \in S\}.$$

Since $N_{J,S,\frac{\epsilon}{4}}$ is a neighborhood of J in H'. We want to prove $N_{J,S,\epsilon} \subseteq \psi(V)$. Suppose $W \in N_{J,S,\frac{\epsilon}{4}}$, and let $\alpha < \omega_1$ and $\beta \leq \omega_1$ be such that $W = G_\alpha \cup A_\beta$. To show that $W \in \psi(V)$, we need to show that $\kappa < \alpha \leq \mu$ and $\lambda < \beta \leq \nu$. Since for $x_{\mu,\nu} \in S$, so $x_{\mu,\nu} \in D_\mu \subseteq Y \setminus G_\mu$

and $x_{\mu,\nu} \notin A_\nu$. Thus $x_{\mu,\nu} \notin J$, therefore, by ($\star$), we have $P(x_{\mu,\nu}, J) \geq \frac{\epsilon}{2}$. Indeed, we have $P(x_{\mu,\nu}, W) > 0$. Thus $x_{\mu,\nu} \notin W$, it follows that $x_{\mu,\nu} \notin G_\alpha$ and $x_{\mu,\nu} \notin A_\beta$. We deduce that $\alpha \leq \mu$ and $\beta \leq \nu$. Now for $x_{\mu,\lambda} \in S$, thus $x_{\mu,\lambda} \in A_\nu \subseteq J$, and therefore, $P(x_{\mu,\lambda}, J) = 0$. So we have $P(x_{\mu,\lambda}, W) < \frac{\epsilon}{4}$, therefore, by ($\star$), we have $x_{\mu,\lambda} \in W$. Since $W = G_\alpha \cup A_\beta$, this implies that either $x_{\mu,\lambda} \in G_\alpha$ or $x_{\mu,\lambda} \in A_\beta$. If $x_{\mu,\lambda} \in G_\alpha$, but $x_{\mu,\lambda} \in U_\mu \subseteq D_\mu$, we must have $\mu < \alpha$; however, we showed above that $\alpha \leq \mu$. Therefore, $x_{\mu,\lambda} \in A_\beta$. Thus $\lambda < \beta$. Consequently, $\lambda < \beta \leq \nu$. Similarly, for $x_{\kappa,\nu} \in S$, so $x_{\kappa,\nu} \in G_\mu \subseteq J$, therefore $P(x_{\kappa,\nu}, J) = 0$. Then we have $P(x_{\kappa,\nu}, W) < \frac{\epsilon}{4}$. Therefore, $x_{\kappa,\nu} \in W$, this implies that either $x_{\kappa,\nu} \in G_\alpha$ or $x_{\kappa,\nu} \in A_\beta$. If $x_{\kappa,\nu} \in A_\beta$, in this case $\nu < \beta$, but we showed above that $\beta \leq \nu$. So $x_{\kappa,\nu} \in G_\alpha$, thus $\kappa < \alpha$. Hence, $\kappa < \alpha \leq \mu$. Therefore, ψ is open.

Hence we proved that the subspace H' of $(C(Y), \tau_{W_P})$ is homeomorphic to the product space $\omega_1 \times (\omega_1 + 1)$. Since for all μ and ν $x_{\mu,\omega_1} \notin G_\mu \cup A_\nu$. Resultantly, $Y \notin H'$. It remain to show that H' is closed in the subspace $C(Y) \setminus \{Y\}$ of $(C(Y), \tau_{W_P})$. Let $K \in \overline{H'} \setminus H'$. We need to prove $K = Y$, suppose on contrary that $K \neq Y$. Assume $y \in Y \setminus K$. There exists $\mu_0 < \omega_1$ such that $y \notin \bigcup_{\kappa > \mu_0} D_\kappa$. Notice that $y \in G_\mu$ for every $\mu > \mu_0$. The subset $W_0 = \{G_\mu \cup S_\nu : \mu \leq \mu_0 \text{ and } \nu \leq \omega_1\} \subseteq H'$ is compact. Thus $K \in \overline{H' \setminus W_0}$. As $y \notin K$, suppose $P(y, K) = \epsilon$, for $\epsilon > 0$. Let

$$N_{K,\{y\},\epsilon} = \{W \in C(Y) : |P(y, W) - P(y, K)| < \epsilon\}$$

then it is neighborhood of K in $(C(Y), \tau_{W_P})$. Then there exist $\mu > \mu_0$ and $\nu \leq \omega_1$ such that $G_\mu \cup A_\nu \in N_{K,\{y\},\epsilon}$. Since $y \in G_\mu$ and thus $P(y, G_\mu \cup A_\nu) = 0$. As $G_\mu \cup A_\nu \in N_{K,\{y\},\epsilon}$, thus $P(y, G_\mu \cup A_\nu) < \epsilon$, which is a contradiction. We have shown that H' is closed in $C(Y) \setminus \{Y\}$. □

Theorem 1.3.11. *Consider a metric space (Y, P). The following are equivalent.*

(1) $(C(Y), \tau_{W_P})$ *is metrizable.*

(2) $(C(Y), \tau_{W_P})$ *is hereditarily normal.*

(3) $(C(Y) \setminus \{Y\}, \tau_{W_P})$ *is normal.*

Proof. It suffices to prove that (3) $\Rightarrow$ (1). Suppose (3) holds, suppose that $(C(Y), \tau_{W_P})$ is not metrizable. Then (Y, P) is non-separable. Thus by proposition1.3.10 $(C(Y) \setminus \{Y\}, \tau_{W_P})$ contains a closed copy of $\omega_1 \times (\omega_1 + 1)$. Since $\omega_1 \times (\omega_1 + 1)$ is not normal, which is a contradictions to the fact that $(C(Y) \setminus \{Y\}, \tau_{W_P})$ is normal. Hence this completes the proof. □

Proposition 1.3.12. *For non-separable metric space* (Y, P)*. Then for any* $n \geq 1$*,* $(C(Y), \tau_{W_P})$ *contains a copy of* $(\omega_1 + 1)^n$*.*

Proof. Suppose $H = (\omega_1 + 1)^n$ for $n \geq 1$, since (Y, P) is not separable, there exist an $\epsilon > 0$ and uncountable ϵ-discrete subset G of Y. We write $G = \bigcup_{r=0}^{n} G_r$ as a disjoint union such that $G_0 = \{d\}$ and $|G_r| = \aleph_1 \ \forall\ 1 \leq r \leq n$. For each $1 \leq r \leq n$, let $G_r = \{x^r_\mu : \mu < \omega_1\}$, and for each $\mu \leq \omega_1$, we write $M^r_\mu = \{x^r_\lambda \in G_r : \lambda < \mu\}$. Obviously each M^r_μ is closed in Y. Let $\psi : H \to (C(Y), \tau_{W_P})$ be defined as $\psi((\mu_r)) = G_0 \cup \bigcup_{r=1}^{n} M^r_{\mu_r}$. Obviously ψ is one-to-one. Now we show that ψ is continuous, let U^- is open in $(C(Y), \tau_{W_P})$. If $G_0 \cap U \neq \phi$, then $\psi^{-1}(U^-) = H$. Suppose $G_0 \cap U = \phi$. Let $(\mu_r) \in \psi^{-1}(U^-)$. Then $(G_0 \cup \bigcup_{r=1}^{n} M^r_{\mu_r}) \cap U \neq \phi$, so there exist $1 \leq j \leq n$ such that $M^j_{\mu_j} \cap U \neq \phi$. Suppose $\lambda < \mu_j$ be such that $x^j_\lambda \in U$. Thus $x^j_\lambda \in M^j_\kappa \cap U$ for each $\kappa > \lambda$. Consequently, the neighborhood

$$\prod_{r<j}[0, \mu_r] \times (\lambda, \mu_j] \times \prod_{r>j}[0, \mu_r]$$

of (μ_r) is contained in $\psi^{-1}(U^-)$. Hence $\psi^{-1}(U^-)$ is open. Furthermore, $P(x, \psi(\mu_r)) > \lambda$, for $x \in Y$ and $\lambda > 0$, then we have $P(x, \psi(\nu_r)) > \lambda$, whenever $\nu_r \leq \mu_r$ for every r. Hence, we have verified continuity of ψ. As H is compact, continuous one-to-one function ψ is an embedding. □

Corollary 1.3.13. *Consider a metric space* (Y, P)*. The following are equivalent.*

(*a*) $(C(Y), \tau_{W_P})$ *is metrizable.*

(*b*) $(C(Y) \setminus \{Y\}, \tau_{W_P})$ *is metacompact.*

(*c*) $(C(Y) \setminus \{Y\}, \tau_{W_P})$ *is meta-Lindelöf.*

(*d*) $(C(Y) \setminus \{Y\}, \tau_{W_P})$ *is orthocompact.*

Proof. We prove only $(d) \to (a)$, as $(c) \to (a)$ can be shown in a similar fashion and other statements are straightforward. Suppose that (d) holds, let $(C(Y), \tau_{W_P})$ is not metrizable. Then (Y, P) is non-separable. So by Proposition 1.3.10 $(C(Y) \setminus \{Y\}, \tau_{W_P})$ contains a closed copy of $\omega_1 \times (\omega_1+1)$. As $(C(Y)\setminus\{Y\}, \tau_{W_P})$ is orthocompact, thus $\omega_1 \times (\omega_1+1)$ is orthocompact, which is contradiction. □

Corollary 1.3.14. *For a metric space* (Y, P)*. Each of the conditions* (1) *Sequentiality,* (2) *countable tightness,* (3) *frechetnessis, is equivalent to metrizability of the space* $(C(Y), \tau_{W_P})$.

Proof. Suppose contrary $(C(Y), \tau_{W_P})$ is not metrizable. Then (Y, P) is non-separable. By Proposition 1.3.12, $(C(Y), \tau_{W_P})$ contains a copy of $\omega_1 + 1$. As $\omega_1 + 1$ does not have countable tightness. This concludes the proof. □

1.4 Hausdorff metric topology

The aim of this section is to study the Hausdorff metric topology (see [1]). We begin with the definition of the Hausdorff metric on $C(Y)$.

Definition 1.4.1. *Suppose* (Y, P) *is a bounded metric space. The Hausdorff distance* H_P *on* $C(Y)$ *of metric* (Y, P) *is defined by*

$$H_P(U, V) = max\{e_P(U, V), e_P(V, U)\}.$$

Then $(C(Y), H_P)$ *form a metric space, called Hausdorff metric space determined by* (Y, P).

Alternatively, we define the Hausdorff distance H_P on $C(Y)$ as,

$$H_P(U, V) = \sup_{y \in Y} |P(y, U) - P(y, V)|.$$

Our next lemma, characterize the convergence of a sequence in the Hausdorff metric space.

Lemma 1.4.2. *A sequence $(G_n) \in C(Y)$ converges in H_P to G iff $(f_{G_n}) \to f_G$ uniformly on Y.*

Proof. Suppose $(G_n) \in C(Y)$ which converges in H_P to G. Let $\epsilon > 0$, there exist $n_0 \in N$ such that $n \geq n_0$ then $H_P(G_n, G) < \epsilon$. Thus

$$H_P(G_n, G) = \sup_{y \in Y} |P(y, G_n) - P(y, G)| < \epsilon \text{ implies } \sup_{y \in Y} |f_{G_n}(y) - f_G(y)| < \epsilon.$$

Since this is true for supremum, so it holds for every $y \in Y$. Hence

$$|f_{G_n}(y) - f_G(y)| < \epsilon \text{ for } n \geq n_0.$$

As this holds for all $\epsilon > 0$, hence $(f_{G_n}) \to f_G$ uniformly.

Conversely, assume that $(f_{G_n}) \to f_G$ uniformly on Y. Then for each $\epsilon > 0$, there exist $n_0 \in N$ such that $n \geq n_0$ implies $|f_{G_n}(y) - f_G(y)| < \epsilon$ for each $y \in Y$. So that $|P(y, G_n) - P(y, G)| < \epsilon$ for all $y \in Y$. Therefore

$$\sup_{y \in Y} |P(y, G_n) - P(y, G)| \leq \epsilon$$

and $H_P(G_n, G) < \epsilon$. Hence $(G_n) \to G$ in Hausdorff metric. □

Definition 1.4.3. *Suppose (Y, P) is a metric space. The Hausdorff metric topology on $C(Y)$ is denoted by τ_{H_P} that $C(Y)$ inherits from τ_{uc} (Where τ_{uc} is a topology of uniform convergence on $C(X, R)$), under the identification $B \leftrightarrow P(., B)$.*

Proposition 1.4.4. *Consider a metric space (Y, P). Let $A \in C(Y)$ then the functionals $e_P(A, .) : (C(Y), H_P) \to [0, +\infty]$, $e_P(., A) : (C(Y), H_P) \to [0, +\infty]$, and $D_P(A, .) : (C(Y), H_P) \to [0, +\infty)$, are each Lipschitz continuous with constant one.*

Theorem 1.4.5. *Consider a metric space (Y, P). The Hausdorff metric topology on $C(Y)$ is the weakest topology τ on $C(Y)$ such that for each $A \in C(Y)$*

$e_P(A, .) : (C(Y), \tau) \to [0, +\infty]$,

$e_P(., A) : (C(Y), \tau) \to [0, +\infty]$,

$D_P(A, .) : (C(Y), \tau) \to [0, +\infty)$,

are all τ continuous.

Proof. We denotes τ_w as a weak topology determined by the family of all such functionals. Since by Proposition 1.4.4 these functionals are continuous on Hausdorff metric topology and hence $\tau_w \subset \tau_{H_P}$. Now we prove that continuity of all functionals of the form $e_P(., A)$ and $e_P(A, .)$ alone is enough to prove the reverse inclusion. Fix G_0, and let $\tau_w - \lim G_\lambda = G_0$. Since by convergence of a net in weak topology with $A = G_0$, we have $\lim_\lambda e_P(G_0, G_\lambda) = e_P(G_0, G_0) = 0$ and $\lim_\lambda e_P(G_\lambda, G_0) = e_P(G_0, G_0) = 0$. Consequently,

$$\lim_\lambda H_P(G_\lambda, G_0) = max\{e_P(G_0, G_\lambda), e_P(G_\lambda, G_0)\} = 0,$$

and we get Hausdorff metric convergence of a net. Hence $\tau_{H_P} \subset \tau_w$. □

Theorem 1.4.6. *Consider a metric space (Y, P) and if d is another compatible metric. Then $\tau_{H_P} = \tau_{H_d}$ on $C(Y)$ iff P and d are uniformly equivalent.*

Proof. Assume that P and d are uniformly equivalent. Then for every $\epsilon > 0$, there exist $\delta_1 = \delta_1(\epsilon)$ and $\delta_2 = \delta_2(\epsilon)$ such that for each subset B of Y, we have both

$$B_P(\delta_1, B) \subset B_d(\epsilon, B) \text{ and } B_d(\delta_2, B) \subset B_P(\delta_1, B).$$

In consequence, uniform equivalence of the metrics not only produce equality of the hyperspace, but also uniform equivalence of the induced Hausdorff distance. Conversely, assume that P and d are not uniformly equivalent. So the identity mapping $i : (Y, P) \to (Y, d)$ fails to be bi-uniformly continuous. Thus we can find $\epsilon > 0$ and sequences $< x_n >$ and $< y_n >$ such that for every n, $P(x_n, y_n) < \frac{1}{n}$ but $d(x_n, y_n) > \epsilon$. By the Efremovic Lemma and by passing to a subsequence, we may let $D_d(\{(x_n : n \in N\}, \{y_n : n \in N\}) \geq \frac{\epsilon}{4}$. For $k = 1, 2, ...,$

take $G_k = \{x_n : n \in N\} \cup \{y_n : n \geq k\}$ and let $G = \{x_n : n \in N\}$. By bicontinuity of the identity, neither $< x_n >$ nor $< y_n >$ can have limit points, whence G and every G_k is closed. Obviously $H_P(G, G_k) \to 0$, whereas for every k, $H_d(G, G_k) \geq \frac{\epsilon}{4}$, thus $\tau_{H_P} \neq \tau_{H_d}$. $\square$

Chapter 2

Hit-and-Miss and Proximal Hit-and-Miss Topologies

In this chapter, we will discuss hit-and-miss and proximal hit-and-miss hyperspace topologies. In particular, the Vietoris topology, Fell topology ,proximal topology and ball proximal topology are discussed. First we define some notations. Let Y be a Hausdorff space, denotes $C(Y)$ and $K(Y)$ be the set of nonempty closed and compact subsets of Y, respectively. Let $S \subseteq Y$, we write $S^+ = \{B \in C(Y) : B \subset S\}$, $S^- = \{B \in C(Y) : B \cap S \neq \phi\}$. If (Y, U) be a uniform space write $S^{++} = \{B \in C(Y) : \exists V \in U \text{ with } V[B] \subset S\}$, where $V[B] = \{y \in Y : \exists b \in B \text{ with } (y, b) \in V\}$. Further, if (Y, P) be a metric space, put $S^{++} = \{B \in C(Y) : \exists \delta > 0 B_P(B, \delta) \subset S\}$. The subbase for hit-and-miss topology on $C(Y)$, consist of all sets of the type V^- and $(B^c)^+$, where V is open in Y and B ranges over subfamily $\Delta \subset C(Y)$.

Let (Y, U) be a uniform space. The subbase for proximal hit-and-miss topology on $C(Y)$, consist of all sets of the type V^- and $(B^c)^{++}$, where V is open in Y and $B \in \Delta \subset C(Y)$.

2.1 Vietoris and Fell topologies

Definition 2.1.1. *The Vietoris topology on $C(Y)$ is denoted by τ_V. All sets of the type U^- and W^+, where U and W are open in Y is a subbase for τ_V.*

Definition 2.1.2. *The Fell topology on $C(Y)$, denoted by τ_F has a subbase all sets of the type W^- and $(K^c)^+$, where W and K are nonempty open and compact subset of Y, respectively.*

In the next result, we prove that the Vietoris topology is a weak topology for metrizable space.

Theorem 2.1.3. *Let Y be a metrizable space and denotes D be the set of metrics compatible for Y. Then τ_V on $C(Y)$ is a weak topology determined by the family $\{P(y,.) : y \in Y, P \in D\}$.*

Proof. We denotes τ_w as a weak topology determined by the above family of metrics. Suppose $P \in D$ and $y \in Y$ be fixed. Since $(B_P(y,\epsilon))^- = \{B \in C(X) : P(y,B) < \epsilon\} \in \tau_V$, for $\epsilon > 0$. Also, if $F \in \{B \in C(X) : P(y,B) > \epsilon\}$, so for some $\delta > \epsilon$ implies $F \cap \overline{B}_P(y,\delta) = \phi$. Thus,

$$F \in (\overline{B}_P(y,\delta)^c)^+ \subset \{B \in C(X) : P(y,B) > \epsilon\}.$$

This shows that each open set of τ_{W_P} in τ_V, as P was arbitrary, hence $\tau_w \subset \tau_V$. Now we prove that $\tau_V \subset \tau_w$. Since by Lemma 1.2.3 each set U^-, where U is open in Y belongs to each τ_{W_P}, and hence to τ_w. If $V = Y$, then $V^+ = C(Y) \in \tau_w$, and if $V = \phi$, then $V^+ = \phi \in \tau_w$. Suppose V be an open proper subset of Y, and $y_0 \in V^c$. Take fix $A \in V^+$, we introduce a compatible metric d for which

$$A \in \{B \in C(Y) : d(y_0, A) - \tfrac{1}{4} < d(y_0, B)\} \subset V^+.$$

This means that V^+ contains a τ_w neighborhood of each of its points. Since $A \cap V^c = \phi$, so by Urysohn's Lemma $\psi \in C(Y, [0,1])$ such that $\psi(A) = 0$ and $\psi(V^c) = 1$. The metric $d : Y \times Y \to [0, \frac{3}{2}]$ defined as

$$d(u,v) = min\{\tfrac{1}{2}, P(u,v)\} + |\psi(u) - \psi(v)|.$$

Obviously, d is a metric on Y equivalent to P. Assume that $\{B \in C(Y) : d(y_0, A) - \frac{1}{4} < d(y_0, B)\} \not\subset V^+$. So there exist $B \in C(Y)$ such that $d(y_0, A) - \frac{1}{4} < d(y_0, B)$, and $B \cap V^c \neq \phi$. Take $c \in B \cap V^c$. As c and y_0 are both in V^c, we have $d(c, y_0) \leq \frac{1}{2}$, thus

$$1 \leq d(y_0, A) < d(y_0, B) + \tfrac{1}{4} \leq d(y_0, c) + \tfrac{1}{4} \leq \tfrac{3}{4}.$$

Which is contradiction, thus V^+ is τ_w open. Hence $\tau_V \subset \tau_w$. Consequently, τ_V is a weak topology. $\square$

Remark 2.1.4. *Since by Theorem 2.1.3, τ_V is a weak topology. Thus we can express the convergence of a net in terms of distance functionals. i.e, a net (G_λ) in $C(Y)$ is τ_V convergent to $G \in C(Y)$ iff for all $y \in Y$, for all $P \in D$, implies $P(y, G) = \lim_\lambda P(y, G_\lambda)$.*

We intend to present the convergence of a net in the Fell topology for Hausdorff uniform space.

Theorem 2.1.5. *Let (Y, U) be a Hausdorff uniform space. A net $(G_\lambda)_{\lambda \in \Lambda}$ in $C(Y)$ is τ_F-convergent to G in $C(Y)$ iff for each $V \in U$ and $B \in K(Y)$, there exist $\lambda_0 \in \Lambda$ such that for each $\lambda \geq \lambda_0$, we have both $G_\lambda \cap B \subset V(G)$ and $G \cap B \subset V(G_\lambda)$.*

Proof. For $V \in U$, $B \in K(Y)$ and $G \in C(Y)$, write

$$[B, V](G) = \{A \in C(Y) : A \cap B \subset V(G) \text{ and } G \cap B \subset V(A)\}.$$

Thus the convergence of the net G_λ to G means that every set $[B, V](G)$ contains G_λ eventually. Suppose that the convergence in this sense holds. We prove G_λ τ_F-convergent to G, It suffices to work with subbasic τ_F open sets. Suppose $G \in W^-$, where W is open in Y. Take symmetric $V \in U$ and $x \in G$ with $x \in V(x) \subset W$. Then $G \in [\{x\}, V](G)$, if $G_\lambda \in [\{x\}, V](G)$, we have $G_\lambda \in W^-$. Let $B \in K(Y)$, and $G \in (B^c)^+$. Since B is compact, so there exist a symmetric entourage V such that $V[G] \cap B = \phi$. Hence, if $S \in [B, V](G)$, we have

$$S \cap B = (S \cap B) \cap B \subset V[G] \cap B = \phi.$$

Therefore, $G_\lambda \in [B,V](G) \Rightarrow G_\lambda \in (B^c)^+$, thus G_λ τ_F-convergent to G. Conversely, suppose G_λ τ_F-convergent to G holds. Take fix $B \in K(Y)$ and open symmetric $V \in U$. If $G \cap B = \phi$, so for each $A \in C(Y)$ implies that $G \cap B \subset V(A)$. In the case $G \cap B \neq \phi$, take an open symmetric entourage V_0 with $V_0 \circ V_0 \subset V$. As $G \cap B$ is nonempty compact set, thus we can find a finite subset $\{y_1, y_2, ..., y_n\}$ of $G \cap B$ such that $G \cap B \subset V_0(\{y_1, y_2, ..., y_n\})$. Therefore

$$A \in \cap_{j=1}^n V_0(y_j)^- \Rightarrow G \cap B \subset V(A).$$

Let $F = B \cap V(G)^c$, a compact set. If $A \in (F^c)^+$, then

$$A \cap B = (A \cap F) \cup (A \cap B \cap V(G)) = \phi \cup (A \cap B \cap V(G)) \subset V(G).$$

Thus, take only $(F^c)^+$ or $(\cap_{j=1}^n V_0(y_j)^-) \cap (F^c)^+$, we get τ_F-neighborhood of G contained in $[B,V](G)$. Hence $G_\lambda \in [B,V](G)$ eventually. □

In the previous theorem, if we take metric space instead of Hausdorff uniform space then the convergence of a net is characterized below.

Corollary 2.1.6. *Suppose (Y,P) is a metric space. A net $(G_\lambda)_{\lambda\in\Lambda}$ in $C(Y)$ is τ_F-convergent to G in $C(Y)$ iff for each $B \in K(Y)$, we have both $\lim_\lambda e_P(G \cap B, G_\lambda) = 0$ and $\lim_\lambda e_P(G_\lambda \cap B, G) = 0$.*

2.2 Ball proximal and Proximal topologies

Definition 2.2.1. *Consider a metric space (Y,P). The ball proximal topology on $C(Y)$ is denoted by τ_{B_P}. All sets of the type W^- and $(B^c)^{++}$, where W is open in Y and B is closed ball is a subbase for τ_{B_P}.*

Definition 2.2.2. *Consider a metric space (Y,P). The proximal topology on $C(Y)$ is denoted by τ_{δ_P}. All sets of the type U^- and W^{++}, where U and W are open in Y is a subbase for τ_{δ_P}.*

Remark 2.2.3. *Notice that for equivalent metric d the sets S^{++} are not necessarily conserve. For instance if P is $0-1$ metric on N and the equivalent metric d is defined by $d(a,b) = |\frac{1}{a} - \frac{1}{b}|$.*

The set $S = \{2n : n \in N\} \in S_P^{++}$, *but* $S \notin S_d^{++}$. *If* d *is uniformly equivalent to* P *then the sets* S^{++} *are unchanged. Because it is well known if the metrics* P *and* d *are uniformly equivalent then for each* $\delta > 0$, *we can fined* $\epsilon_1 = \epsilon_1(\delta)$ *and* $\epsilon_2 = \epsilon_2(\delta)$ *such that for each* $S \subset Y$, *we have both*

$$B_P(S, \epsilon_1) \subset B_d(S, \delta) \text{ and } B_d(S, \epsilon_2) \subset B_P(S, \delta).$$

Thus the fact that for uniformly equivalent metric d *to* P, *we have* $\tau_{\delta_P} = \tau_{\delta_d}$.

Theorem 2.2.4. *Consider a metric space* (Y, P), *and denotes* D *be the set of metrics which uniformly equivalent to* P. *Then the weak topology determined by the family* $\{d(y, .) : y \in Y, d \in D\}$ *equal to* τ_{δ_P} *generated by* P.

Proof. We denotes τ_w as a weak topology determined by the above family of metrics. Suppose for each $d \in D$, we have $\tau_{\delta_P} = \tau_{\delta_d} \supset \tau_{W_d}$, thus $\tau_{\delta_P} \supset \sup\{\tau_{W_d} : d \in D\} = \tau_w$. Now we prove that $\tau_{\delta_P} \subset \tau_w$. Since for each open set U in Y and $d \in D$ implies U^- belongs to τ_{W_d}, and hence to τ_w. If $V = Y$, then $V^{++} = C(Y) \in \tau_w$, and if $V = \phi$, then $V^{++} = \phi \in \tau_w$. Take fix $A \in V^{++}$, $y_0 \in V^c$, and $\epsilon < \frac{1}{2}$ such that $B_d(A, 2\epsilon) \subset V$. We introduce $d \in D$ such that

$$A \in \{B \in C(Y) : d(y_0, A) - \epsilon < d(y_0, B)\} \subset (B_d(A, \epsilon))^+ \subset (B_d(A, 2\epsilon))^{++} \subset V^{++}. \quad (2.2.1)$$

Since $\{B \in C(Y) : d(y_0, A) - \epsilon < d(y_0, B)\} \in \tau_{W_d} \subset \tau_w$, this would prove that $V^{++} \in \tau_w$. Let the function $\psi : Y \to R$ be defined as

$$\psi(y) = \begin{cases} P(y, A) & if y \in B_d(A, \epsilon) \\ \epsilon & if y \notin B_d(A, \epsilon) \end{cases}$$

As ψ is Lipschitz continuous with constant one, it is uniformly continuous, also the metric d on Y be defined as

$$d(a, b) = min\{\tfrac{1}{2}, P(a, b)\} + \tfrac{1}{\epsilon}|\psi(a) - \psi(b)|.$$

Obviously, d is a metric on Y, which is uniformly equivalent to P. Note that $d(x,y) \leq \frac{1}{2}$ provided $x, y \in (B_d(A,\epsilon))^c$, whereas if $x \in A$ and $y \in (B_d(A,\epsilon))^c$, then $d(x,y) \geq 1$. To establish 2.2.1, we only prove $\{B \in C(Y) : d(y_0, A) - \epsilon < d(y_0, B)\} \subset (B_d(A,\epsilon))^+$. Assume that $\{B \in C(Y) : d(y_0, A) - \epsilon < d(y_0, B)\} \not\subset (B_d(A,\epsilon))^+$. Then there exist $B \in C(Y)$ such that $d(y_0, A) - \epsilon < d(y_0, B)$, and $B \cap (B_d(A,\epsilon))^c \neq \phi$. Take $c \in B \cap (B_d(A,\epsilon))^c$. Now $y_0 \in V^c \subset (B_d(A,\epsilon))^c$. As c and y_0 are both in $B_d(A,\epsilon)^c$, and $\epsilon < \frac{1}{2}$, thus

$$1 \leq d(y_0, A) < d(y_0, B) + \epsilon \leq d(y_0, c) + \epsilon \leq \tfrac{1}{2} + \epsilon < 1.$$

Which is contradiction, thus 2.2.1 is valid and hence $\tau_{\delta_P} \subset \tau_w$. □

Therefore, in previous Theorem, a net (G_λ) in $C(Y)$ is τ_{δ_P} convergent to $G \in C(Y)$ iff for all $y \in Y$, for all $P \in D$, implies $P(y, G) = \lim_\lambda P(y, G_\lambda)$.

2.3 Normality of Fell and Vietoris topologies

Our attention of this section is to study the normality of Fell and Vietoris topologies refer to [18, 14]. We prove that $(C(Y), \tau_F)$ is normal iff Y is Lindelöf and local compact. For this, we will use the following results.

Proposition 2.3.1. *[6] Let Y be a Hausdorff space. The following are equivalent.*

(1) $(C(Y), \tau_F)$ *is Hausdorff.*

(2) $(C(Y), \tau_F)$ *is regular.*

(3) $(C(Y), \tau_F)$ *is completely regular.*

(4) Y *is locally compact.*

Lemma 2.3.2. *Let Y be a Hausdorff σ-compact space. Then $(C(Y), \tau_F)$ is σ-compact.*

Proof. Consider $\{G_n : n \in \omega\}$ is a sequence of compact sets in Y such that $Y = \bigcup\{G_n : n \in \omega\}$. Therefore $C(Y) = \bigcup\{\overline{G}_n : n \in \omega\}$. Since by [1] $\overline{G}_n$ is compact in $(C(Y), \tau_F)$ for each $n \in \omega$. Hence $(C(Y), \tau_F)$ is σ-compact. □

Consider κ be an ordinal. An open cover $B = \{V_\alpha : \alpha \in \kappa\}$ of Y is called well-monotone cover if $V_\alpha \not\subset V_\beta$ whenever $\alpha < \beta$, $\alpha, \beta \in \kappa$. For any well-monotone cover B we will select the sub collection $U(B)$ of $C(Y)$ as follows. Suppose κ is an ordinal such that $B = \{V_\alpha : \alpha \in \kappa\}$. for each $\alpha \in \kappa$ take $U_\alpha = Y \backslash \bigcup\{V_\beta : \beta < \alpha\}$ and set $U(B) = \{U_\alpha : \alpha \in \kappa\}$. Note that $U_0 = Y$ and for each limit ordinal $\alpha \in \kappa$ we have $U_\alpha = \bigcap\{U_\beta : \beta < \alpha\}$.

Lemma 2.3.3. *Consider a well-monotone cover B of a Hausdorff space Y. Then $U(B)$ is a closed set in $(C(Y), \tau_F)$.*

Proof. Let $B = \{V_\alpha : \alpha \in \kappa\}$ be a well-monotone cover. Let $A \in C(Y) \backslash U(B)$. Take $\mu = min\{\beta \in \kappa : A \not\subset U_\beta\}$. Recall $\bigcap\{U_\beta : \beta \in \kappa\} = \phi$ and $A \neq \phi$. Obviously, μ cannot be limit, thus $\mu = \lambda + 1$. Hence $A \subset U_\lambda$, thus $U_\lambda \backslash A \neq \phi$. Let $y \in U_\lambda \backslash A$ and define $S = (Y \backslash U_\mu)^- \cap (Y \backslash \{y\})^+$. Therefore $A \in S$ and $S \cap U(B) = \phi$. □

Lemma 2.3.4. *Consider a well-monotone cover $B = \{V_\alpha : \alpha \in \omega\}$ of a Hausdorff space Y. If $(C(Y), \tau_F)$ is normal then Y is σ-compact.*

Proof. Take $H = \{\{y\} : y \in Y\}$. So $U(B)$ and H are τ_F closed sets. Since $(C(Y), \tau_F)$ is normal, then there exist disjoint τ_F-open sets W_1 and W_2 such that

$$U(B) \subset W_1 \text{ and } H \subset W_2.$$

For every $U_n \in U(B)$ there are open sets $V_1^n, ..., V_{j_n}^n$ and a compact set K_n such that

$$U_n \in \bigcap\{(V_i^n)^- : i = 1, ..., j_n\} \cap ((K_n)^c)^+ \subseteq W_1.$$

We claim $Y = \bigcup\{K_n : n \in \omega\}$. Let there is $x \in Y \backslash \bigcup\{K_n : n \in \omega\}$. $\{x\} \in W_2$, then there is an open neighbourhood S of x and a compact set M with

$$\{x\} \in S^- \cap (M^c)^+ \subseteq W_2.$$

Since M is compact and $\bigcap\{U_n : n \in \omega\} = \phi$ so there is $m \in \omega$ with $U_m \cap M = \phi$. Hence $U_m \cup \{x\} \in W_1 \cap W_2$, a contradiction. □

Lemma 2.3.5. *Suppose Y is a Hausdorff space and $(C(Y), \tau_F)$ is a normal space. There is no well-monotone cover $B = \{V_\alpha : \alpha \in \kappa\}$ of Y with cofinality of κ greater than ω.*

Proof. Assume on contrary. Take $H = \{\{y\} : y \in Y\}$. So $U(B)$ and H are τ_F closed sets. Since $(C(Y), \tau_F)$ is normal, thus there is continuous mapping $g : C(Y) \to [0, 1]$ such that

$$g(U(B)) = 0 \text{ and } g(H) = 1.$$

For every $n \in \omega$ and let $y_n \in Y$, $E_n \in K(Y)$, an open neighbourhood V_n of y_n and $\eta_n \in E_n$ such that

(*a*) $g[V_n^- \cap ((E_n)^c)^+] \subset (1 - \frac{1}{n+2}, 1], y_n \in (E_n)^c$,

(*b*) $U_{\eta_n} \cap (\bigcup\{E_j \cup \{y_j\} : j \leq n\}) = \phi$,

(*c*) $y_{n+1} \in U_{\eta_n}, \eta_{n+1} \geq \eta_n$.

Suppose $n = 0$ and for any $y_0 \in Y$. Since by continuity of g and $g(\{y_0\}) = 1$ so there are an open neighbourhood V_0 of y_0 and $E_0 \in K(Y)$ such that $y_0 \notin E_0$ and $g[V_0^- \cap ((E_0)^c)^+] \subset (\frac{1}{2}, 1]$. Suppose $\eta_0 \in \kappa$ such that $U_{\eta_0} \cap (E_0 \cup \{y_0\}) = \phi$. As $\bigcap\{U_\beta : \beta \in \kappa\} = \phi$ and $E_0 \cup \{y_0\}$ is compact so such η_0 always exist. Consider we defined $y_0, y_1, ..., y_{n-1}, V_0, V_1, ..., V_{n-1}, E_0, E_1, ..., E_{n-1}$ and $\eta_0, \eta_1, ..., \eta_{n-1}$. Let y_n be any point of $U_{\eta_{n-1}}$. Since by continuity of g and $g(\{y_n\}) = 1$ thus there exist V_n and E_n which verify (a). There is $\eta_n \in \kappa$ with $\eta_n > \eta_{n-1}$ and $U_{\eta_n} \cap (\bigcup\{E_j \cup \{y_j\} : j \leq n\}) = \phi$.

Take $\eta = \sup\{\eta_n : n \in \omega\}$. Thus $\eta \in \kappa$ and $U_\eta = \bigcap\{U_{\eta_n} : n \in \omega\}$. For each $n \in \omega$ take $M_n = U_{\eta_n} \cup \{y_n\}$. Then $g(M_n) \in (1 - \frac{1}{n+2}, 1]$ by (*a*), as $M_n \in (V_n)^- \cap ((E_n)^c)^+$. Suppose $B \in K(Y)$ such that $B \cap U_\eta = \phi$. So there is $n \in \omega$ with $U_{\eta_n} \cup B = \phi$. Hence $\{M_n : n \in \omega\}$ τ_F-converges to U_η, which contradict the fact that $g(U_\eta) = 0$.

□

Theorem 2.3.6. *Let Y be a Hausdorff space and $(C(Y), \tau_F)$ is a normal space. Then Y is Lindelöf space.*

Proof. Assume that Y is not Lindelöf. In the collection of all open cover of Y without any countable subcover there is an open cover G of Y with the minimal cardinality $|G|$. Suppose κ

be the first ordinal having cardinality $|G|$. Then $G = \{V_\mu : \mu < \kappa\}$. For each $\mu < \kappa$ we define $L_\mu = \bigcup\{V_\eta : \eta \leq \mu\}$. Then $L_\mu \neq Y$ for every $\mu < \kappa$. There is a subfamily $\{L_{\mu_\lambda} : \lambda < \kappa\}$ of $\{L_\mu : \mu < \kappa\}$ which is well-monotone cover of Y. By means of transfinite induction we define a sequence $\{\mu_\lambda : \lambda < \kappa\} \subset [0, \kappa)$ in the following aspect. Suppose $\mu_0 = 0$. Having defined μ_λ let $\mu_{\lambda+1}$ be the first $\mu > \mu_\lambda$ such that $L_{\mu_\lambda} \subset L_\mu$. For λ a limit ordinal, let $\mu_\lambda = \sup\{\mu_\alpha : \alpha < \lambda\}$. If $\sup\{\mu_\alpha : \alpha < \lambda\} = \kappa$, then $\{L_{\mu_\alpha} : \alpha < \lambda\}$ is an open cover of Y with the cardinality less than $|G|$, thus there must exist a countable subcover of $\{L_{\mu_\alpha} : \alpha < \lambda\}$ which leads to a contradiction by Lemma 2.3.4. So $\alpha_\lambda < \kappa$. For each $\lambda < \kappa$ take $M_\lambda = L_{\mu_\lambda}$. Therefore $\{M_\lambda : \lambda < \kappa\}$ is a well-monotone cover of Y. Hence by Lemma 2.3.5, κ is cofinal with ω. Let $\kappa_n \nearrow \kappa$, $n \in \omega$. Define

$$T_n = \bigcup\{M_c : c \leq \kappa_n\}.$$

Thus $\{T_n : n \in \omega\}$ is a well-monotone cover of Y. Hence Y is σ-compact by Lemma 2.3.4, a contradiction. □

Theorem 2.3.7. *Consider a Hausdorff topological space Y. The following statements are equivalent.*

(1) *Y is locally compact and Lindelöf.*

(2) *$(C(Y), \tau_F)$ is σ-compact and regular.*

(3) *$(C(Y), \tau_F)$ is Lindelöf.*

(4) *$(C(Y), \tau_F)$ is paracompact.*

(5) *$(C(Y), \tau_F)$ is normal.*

Proof. (1) $\Rightarrow$ (2) if Y is locally compact. Then by Proposition 2.3.1 $(C(Y), \tau_F)$ is regular. Since Y is locally compact and Lindelöf, implies Y is σ-compact. Thus by Lemma 2.3.2 $(C(Y), \tau_F)$ is σ-compact.

(2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5) are obvious. (5) $\Rightarrow$ (1) if $(C(Y), \tau_F)$ is normal, then by Theorem 2.3.6 Y is Lindelöf. Since Y is Hausdorff, thus $(C(Y), \tau_F)$ is T_1. Hence $(C(Y), \tau_F)$ is regular. Finally by Proposition 2.3.1 Y is local compact. □

Now we will discuss the normality of the Vietoris topology. It is known that if Y is compact then $(C(Y), \tau_V)$ is compact Hausdorff and thus normal. Ivanova in [17], showed that if Y is a well order space with order topology then $(C(Y), \tau_V)$ is normal implies Y is compact. Now we show that $(C(Y), \tau_V)$ is normal iff Y is compact with by assuming continuum hypothesis. First we need some results.

All the upcoming results given below have the condition of CH(continuum hypothesis).

Proposition 2.3.8. *Let Y be a separable and countably compact but not compact. Then $[0, \omega_1)$ can be imbedded in $(C(Y), \tau_V)$ as closed subset.*

Proposition 2.3.9. *Let Y be a separable, not first countable and countably compact. Then $[0, \omega_1]$ can be imbedded in $(C(Y), \tau_V)$.*

Theorem 2.3.10. *$(C(Y), \tau_V)$ is normal iff Y is compact.*

Proof. Suppose that $(C(Y), \tau_V)$ is normal. Suppose $\overline{W} = G$, where G be any countable subset of Y. So $(C(W), \tau_V)$ is normal. If we can show for some separable space K, $(C(K), \tau_V)$ is normal implies K is compact, thus W will be compact. Then Y would be strongly compact and compact by [[18], corollary 2.6(d)]. We claim that if K is separable and $(C(K), \tau_V)$ is normal, then K is compact. Suppose K is separable and $(C(K), \tau_V)$ is normal, but K is not compact. By [[18], corollary 2.6(a)], K is not first countable. Assume that K is not countable at y. Suppose V is an open set and $y \in V$ such that $K - V$ is not compact. Such V exists becuse Y is not compact. Assume U be an open set and $y \in U$ such that $\overline{U} \subset V$. Suppose $M = K - \overline{(K - \overline{U})}$. Then M has the property that $K - M$ is separable and not compact. Let O be an open set and $y \in O$ with $\overline{O} \subset M$. Suppose $S_1 = \overline{O}$ and $S_2 = Y - M$. Let $W = S_1 \cup S_2$. Now W is closed subset of K also $(C(W), \tau_V)$ is homeomorphic to $(C(S_1), \tau_V) \times (C(S_2), \tau_V)$. Since by Propositon 2.3.8 $[0, \omega_1)$ can be imbedded in $(C(S_2), \tau_V)$ as closed subset and by Propositon 2.3.9 $[0, \omega_1]$ can be imbedded in $(C(S_1), \tau_V)$ as closed subset. So $[0, \omega_1] \times [0, \omega_1)$ is closed subset of $(C(W), \tau_V)$ and hence of $(C(X), \tau_V)$. Since $[0, \omega_1] \times [0, \omega_1)$ is not normal. Thus $(C(X), \tau_V)$ is not normal, which is contradiction. Hence this completes the proof. $\square$

Chapter 3

Relationship among Hyperspace Topologies

In this chapter, we will study connections among hyperspace topologies. We have introduced six topologies on $C(Y)$ namely, Wijsman topology, Hausdorff metric topology, Vietoris topology, Fell topology, ball proximal topology, and proximal topology. The main focus of this chapter is to completely characterize the relationship among above mentioned hyperspace topologies. Clearly, $\tau_F \subseteq \tau_V$. Since τ_{H_P}- convergence of a net (G_λ) to G is uniform convergence of $(P(y, G_\lambda))$ to $P(y, G)$. Consequently, $\tau_{W_P} \subseteq \tau_{H_P}$. By a well known that each of the statements (1) $\tau_{W_P} = \tau_V$, (2) $\tau_F = \tau_V$, (3) $\tau_{H_P} = \tau_V$, is equivalent to compactness of Y.

Theorem 3.0.1. *Suppose (Y, P) is a metric space. Then*

(1) $\tau_{W_P} \subseteq \tau_{B_P}$.

(2) $\tau_{W_P} = \tau_{B_P}$ on $C(Y)$ iff every closed ball B in Y is strictly P-included in each of its open ϵ-enlargements $B_P(B, \epsilon)$.

Proof. First we prove $\tau_{W_P} \subseteq \tau_{B_P}$. For this we need to show that each subbasic τ_{W_P}-open set lies in τ_{B_P}. Assume that $B_0 \in \{B \in C(Y) : P(y, B) < k\}$. So for some $b \in B_0$ implies $P(y, b) < k$. Take $\alpha = k - P(y, b)$, every point in $B_P(b, \alpha)$ has distance less than k from y, then $B_0 \in (B_P(b, \alpha))^- \subset \{B \in C(Y) : P(y, B) < k\}$. This shows $\{B \in C(Y) : P(y, B) < k\}$ is open in τ_{B_P}. Next we prove that $\{B \in C(Y) : P(y, B) > k\}$ is open in τ_{B_P}. Let $B_0 \in \{B \in C(Y) : P(y, B) > k\}$. Suppose $\epsilon = \frac{1}{2}(k + P(y, B_0))$, we have

$$B_0 \in \{x \in Y : P(y, x) > \epsilon\}^{++} \subset \{x \in Y : P(y, x) > \epsilon\}^{+} \subset \{B \in C(Y) : P(y, B) > k\}.$$

Since $\{x \in Y : P(y, x) > \epsilon\}$ is the complement of a closed ball, therefore $\{B \in C(Y) : P(y, B) > k\}$ is open in τ_{B_P}. Hence $\tau_{W_P} \subseteq \tau_{B_P}$.

Now we prove (2), since by Lemma 1.2.3 $U^- \in \tau_{W_P}$ where U is open in τ_P, requires no condition on the metric. Therefore for (2) we only prove $(B^c)^{++} \in \tau_{W_P}$ iff every closed ball B in Y is strictly P-included in $B_P(B, \epsilon)$.

Assume $(B^c)^{++} \in \tau_{W_P}$ for each closed ball B in Y. Suppose $B = \overline{B}_P(y, \delta)$ is a fixed closed ball and let $\epsilon > 0$. If $B_P(B, \epsilon) = Y$, then

$$B \subset \overline{B}_P(y, \delta + 1) \subset \overline{B}_P(y, \delta + 2) \subset B_P(B, \epsilon),$$

and thus B is strictly P-included in $B_P(B, \epsilon)$. Otherwise take $S = B_P(B, \epsilon)^c \in C(Y)$. Since $S \in (B^c)^{++}$ and since by supposition $(B^c)^{++} \in \tau_{W_P}$, then we can find $y_1, y_2, ..., y_n \in Y$ and $\mu > 0$ such that

$$S \in \cap_{j=1}^{n} \{F \in C(Y) : P(y_j, F) > P(y_j, S) - \mu\} \subset (B^c)^{++}.$$

Now let $M = \{j \in \{1, 2, ..., n\} : P(y_j, S) > 0\} \neq \phi$, because $(B^c)^{++} \neq C(Y)$. Suppose $0 < \alpha < \mu$ with $\alpha < min\{P(y_j, S) : j \in M\}$, and take $\epsilon_j = P(y_j, S) - \alpha$. We claim that $B \subset \cup_{j \in M} B_P(y_j, \epsilon_j)$. If not, there exist $b_0 \in B$ such that for every $j \in M$, $P(y_j, b_0) \geq \epsilon_j > P(y_j, S) - \mu$. This means that

$$\{b_0\} \in \cap_{j=1}^{n} \{F \in C(Y) : P(y_j, F) > P(y_j, S) - \mu\} \subset (B^c)^{++},$$

which is contradiction. With $\alpha_j = P(y_j, S) > \epsilon_j$ for $j \in M$, we have

$$B \subset \cup_{j \in M} B_P(y_j, \epsilon_j) \subset \cup_{j \in M} B_P(y_j, \alpha_j) \subset S^c = B_P(B, \epsilon).$$

Hence B is strictly P-included in $B_P(B, \epsilon)$.

Conversely, suppose each closed ball B in Y is strictly P-included in their enlargements. We need to prove $(B^c)^{++} \in \tau_{W_P}$. Let $B_0 \in (B^c)^{++}$, so for some $\epsilon > 0$ implies $B_P(B, \epsilon) \cap B_0 = \phi$, and by supposition, there exist a finite set $\{y_1, y_2, ..., y_n\}$ and $0 < \epsilon_j < \alpha_j$, $j = 1, 2, ..., n$ such that

$$B \subset \cup_{j=1}^{n} B_P(y_j, \epsilon_j) \subset \cup_{j=1}^{n} B_P(y_j, \alpha_j) \subset B_0^c.$$

Hence

$$B_0 \in \cap_{j=1}^{n} \{F \in C(Y) : P(y_j, F) > \tfrac{1}{2}(\epsilon_j + \alpha_{j)}\} \subset (B^c)^{++}.$$

Which is required. □

Remark 3.0.2. *By definition proximal topology τ_{δ_P} contains the ball proximal topology τ_{B_P} determined by metric P. Also by Theorem 3.0.1 τ_{B_P} contains τ_{W_P}. Thus it follows that τ_{δ_P} contains τ_{W_P}.*

Remark 3.0.3. *As a weak topologies illustrate by Theorem 2.1.3 and 2.2.4, the Vietoris topology τ_V contains the metric proximal topology τ_{δ_P} because τ_V is induce by bigger class of functionals. Consequently, the Vietoris topology is the largest topology among the hit-and-miss and proximal hit-and-miss topologies on $C(Y)$.*

The equality of τ_V and τ_{δ_P} on $C(Y)$ is discussed in the next proposition.

Proposition 3.0.4. *Suppose (Y, P) is a metric space. Then $\tau_V = \tau_{\delta_P}$ on $C(Y)$ iff whenever $U, V \in C(Y)$ are disjoint, then U and V are far.*

Proof. Assume that nonempty disjoint closed sets are far. Then for every open set W in Y, we have $W^+ = W^{++}$, and thus $\tau_V = \tau_{\delta_P}$ on $C(Y)$. Conversely, let $U, V \in C(Y)$ are disjoint and $D_P(U, V) = 0$. Take $U_n = \overline{B}_P(U, \frac{1}{n})$, implies $U = \tau_{\delta_P} - \lim U_n$ but $U \neq \tau_V - \lim U_n$, because $U_n \notin (V^c)^+$. Hence for such a metric space, τ_V properly contains τ_{δ_P}. □

Remark 3.0.5. *Thus from Theorem 2.1.3 and 2.2.4 we have* $\tau_V = \sup\{\tau_{W_P} : P \in D\}$ *and* $\tau_{\delta_P} = \sup\{\tau_{W_P} : P \in D\}$ *respectively.*

Corollary 3.0.6. *Consider a metrizable space* Y *and denotes* D *is the set of metrics compatible for* Y*. Then* $\tau_V = \sup\{\tau_{\delta_P} : P \in D\}$.

Therefore, at the same time the Vietoris topology is the supremum of the Wijsman topologies and proximal topologies corresponding to compatible metrics for a metrizable space Y i.e.,

$$\tau_V = \sup\{\tau_{W_P} : P \in D\} = \sup\{\tau_{\delta_P} : P \in D\}.$$

Lemma 3.0.7. *Consider a metric space* (Y, P)*. The following statements are equivalent.* (1) (Y, P) *is totally bounded.* (2) $\tau_{\delta_P} = \tau_{H_P}$. (3) $\tau_{H_P} \subset \tau_V$.

Theorem 3.0.8. *Consider* (Y, P) *is a metric space. The following are equivalent.*

(1) (Y, P) *is totally bounded.*

(2) $\tau_{H_P} = \tau_{W_P}$.

(3) $(C(Y), \tau_{H_P})$ *is second countable.*

Lemma 3.0.9. *If a metric space* (Y, P) *is not second countable. Then* $\tau_{\delta_P} \neq \tau_{W_P}$.

Theorem 3.0.10. *Consider a metric space* (Y, P)*. The following are equivalent.*

(1) (Y, P) *is totally bounded.*

(2) $\tau_{\delta_P} = \tau_{W_P}$.

Proof. Assume that (1) holds, then by Theorem 3.0.8 $\tau_{H_P} = \tau_{W_P}$. Conversely, Assume that (2) holds, as τ_{W_P} is metrizable (and second countable) iff Y is second countable. So by Lemma 3.0.9 τ_{δ_P} is second countable. Hence by [[3], Theorem 4.3], (Y, P) is totally bounded. □

Lemma 3.0.11. *Suppose* (Y, P) *is a metric space. Then* $\tau_F \subseteq \tau_{W_P}$.

Proof. By Lemma 1.2.3 $U^- \in \tau_{W_P}$ where U is open in τ_P. Take a compact subset K of Y and $B \in (K^c)^+$. Suppose $D_P(B, K) = \lambda > 0$, we can find a finite subset G of K such that $K \subset B_P(G, \frac{\lambda}{2})$. Then $B \in \bigcap_{x \in G}\{A \in C(Y) : P(x, A) > \frac{\lambda}{2}\} \subseteq (K^c)^+$. This shows that $(K^c)^+$ contains a τ_{W_P} neighborhood of each of its points and hence $(K^c)^+ \in \tau_{W_P}$. □

Definition 3.0.12. *A metric space (Y, P) is said to have nice closed balls. If B is proper closed ball in Y. Then B is compact.*

Theorem 3.0.13. *Suppose (Y, P) is a metric space. Then $\tau_{W_P} = \tau_F$ on $C(Y)$ iff (Y, P) has nice closed balls.*

Proof. Suppose (Y, P) has nice closed balls. Since for $x \in Y$ and $\epsilon > 0$ implies $\{U \in C(Y) : P(x, U) < \epsilon\} = (B_P(x, \epsilon))^- \in \tau_F$. If $\overline{B}_P(x, \epsilon)$ is proper closed ball in Y. Then there exist $\delta > \epsilon$ such that $\overline{B}_P(x, \delta)$ is compact. If $U \in C(Y)$ and $P(x, U) = \epsilon$, then $\overline{B}_P(x, \epsilon) \cap U \neq \phi$. It follows that $\{U \in C(Y) : P(x, U) > \epsilon\} = (\overline{B}_P(x, \epsilon)^c)^+ \in \tau_F$. Finally, if $\overline{B}_P(x, \epsilon) = Y$, then $\{U \in C(Y) : P(x, U) > \epsilon\} = \phi \in \tau_F$. Conversely, suppose (Y, P) fails to have nice closed balls. So there exist x and y in Y and $\epsilon > 0$ such that $\overline{B}_P(x, \epsilon)$ is noncompact and $P(x, y) > \epsilon$. Let $\{y_n\}$ be sequence in $\overline{B}_P(x, \epsilon)$ with no limit point. Then $\{y_n, y\}$ is τ_F-convergent to $\{y\}$, whereas $\lim_{n\to\infty} P(x, \{y_n, y\}) \neq P(x, \{y\})$. This shows that $\tau_{W_P} \neq \tau_F$ □

Corollary 3.0.14. *Suppose (Y, P) is a metric space. Then $\tau_{H_P} = \tau_F$ on $C(Y)$ iff Y is compact.*

Proof. It follows easily from Theorems 3.0.8 and 3.0.13, because a totally bounded metric space with nice closed balls can be expressed as a finite union of compact balls and is thus compact. □

Corollary 3.0.15. *Suppose (Y, P) is a metric space. Then $\tau_{\delta_P} = \tau_F$ on $C(Y)$ iff Y is compact.*

Proof. Assume that Y is compact. Then $\tau_F = \tau_V$, and τ_{δ_P} lies between them. Conversely, assume that $\tau_{\delta_P} = \tau_F$, we have both $\tau_{W_P} = \tau_F$ and $\tau_{\delta_P} = \tau_{W_P}$, thus (Y, P) has nice closed balls, and (Y, P) is totally bounded. Hence Y must be compact. □

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