Electromagnetic Model of Atmospheric Lightning

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CERTIFICATE

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Dedicated to my family

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Abstract

The TE, TM and TEM modes are explored for dielectric atmosphere with electric primitivity ε_o, bounded by two PEC shells that are Earth and ionosphere. The TE and TM modes of this Earth ionosphere spherical cavity are investigated using expanded form of Maxwell's equations and scalar electromagnetic potentials. The excitation of the TM, TE and TEM modes by a point current source $\vec{J}_o\delta(\vec{r}-\vec{r}^{\,\prime})$ are also explored using homogenous Green Functions in spherical coordinates system. The approximate equation is obtained for resonant model frequencies valid in both cases of the cavity by solving a transcendental equation. The electric and magnetic energy density are graphically presented for some of the resonant modes.

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Chapter 1

Introduction

The electromagnetic wave propagation in the atmosphere of Earth is important for different scientific and technological purposes. The Earth along with its ionosphere is considered as resonant cavity, filled with atmospheric gases as dielectric medium. The cavity model may be used in different meteorological calculations. The model may be useful to calculate the distribution of energy due to excitation by lightning disturbance or radio frequency sources. The ionosphere plays a vital role in radio waves propagation that changes with frequency bands, which is critically important for long range communication and navigation system. The electromagnetic propagation is important in a wide spectral range of frequency bands that can exist from VLF(3kHz-30kHz) to HF.

The planet Earth with radius of 6400km, is a good conducting sphere (PEC) for almost all frequency bands. The atmosphere covers the Earth in different layers, one of the layers is ionosphere layer with radius 6500km, which is also PEC for almost all frequency bands less than 10MHz. With the increase of atmospheric conductivity, this cavity can be treated as the dielectric layer, with electric permittivity ε_o, bounded between two PEC shells. The electromagnetic modes are excited in this cavity due to lightning thunderstorm or man made RF sources.

Cavities are usually excited by small loops, short monopoles or apertures and a complete set of modes is required in the cavity. This has been explained by Kurokawa [1] and Collin [2]. Liu [3] explained different theories of modal expansion method for the transient electromagnetic field in a closed volume and analyzed that the transient responses of the electromagnetic field can be derived in terms of resonance in this specified volume. He explained that modal expansion is an important tool in theoretical and numerical electromagnetics for computation of electromagnetic field in cavities. Omer et al [4] presented the idea of two types of field expansion for the electromagnetic field due to the radiation of electric and magnetic currents in a cavity resonator. The first type of expansion utilizes the guided resonant modes excited by the source current in the wave guide, while the second type is expressed as same modes as well as irrotational modes [5]. Schumann [6] introduced some theoretical and analytical solutions for the propagation of resonant modes in Earth ionosphere cavity. Later on, he derived the resonant frequencies and excitation of modes in ULF and ELF range numerically, which shows that lightning is main source of EM modes oscillation [7]. Wait [8] has solved some electromagnetic propagation problems analytically for ELF and VLF bands and analyzed numerically many realistic propagation problems for the low frequencies which is less than 1.5kHz [9]. Galejs [10] explained terrestrial propagation of electromagnetic wave in earth ionosphere cavity. Budden [11] gave the theory of radio propagation in which he derived conditions for existence of modes, excitation factors, and polarization of the waves by evaluating a contour integral. Barrick [12] gave the idea of spherical harmonics solution for electromagnetic field due to the dipole in Earth ionosphere cavity using speed-up numerical convergence algorithms. Cummer [13] presented a comparison among modal theory, FDTD and approximately analytical simulation of modal propagation in Earth ionosphere cavity for different frequency bands. Chapman and Jones [14] interpreted experimental results in terms of the waveguide modal theory by assuming that electrons and ions are sources of excitation in Earth ionosphere resonant cavity. Bouwkamp and Casimir [15] developed numerical methods for expansion of the EM field in multipole components due to radiating currents. They represented electromagnetic field in terms of Debye potentials, which related the radial components of the electric and magnetic vectors outside a sphere containing all sources. Pappert [16] derived formulas for the excitation of electromagnetic field due to magnetic and electric dipoles in Earth ionosphere cavity at satellite height and furthermore, he compared the results of his simulation with modal excitation theory of Budden [17]. They had examined inhomogenous Earth ionosphere spherical cavity modes due to radial excitation of dipole current at satellite height. Navarro et al [18] used the method of finite differences (FDTD) in the time domain with Fourier transform to determine TM and TE modes of the dielectric resonator. Price [19] presented both theoretical and experimental results of the global electromagnetic resonance phenomenon in Earth ionosphere cavity.

In present work the TM and TE modes are investigated for Earth ionosphere cavity in chapter 2. The approximate resonant frequency equation is derived using transcendental equations of TM and TE modes. In chapter 3, a point current source is placed in this cavity and solved through homogenous Green Function for TM, TE and TEM modes. The TM, TE and TEM mode expressions have been derived, due to excitation of this cavity by source current in three different directions. In chapter 4, the electric and magnetic energy density plots of TM and TE modes for excited and unexcited cavity have been plotted. It is shown graphically that electric and magnetic energy densities oscillate between TM and TE modes. In chapter 5, summary and concluding results about the Earth ionosphere spherical cavity have been discussed.

Chapter 2

Resonant Modes and Eigen Frequencies in Earth Ionosphere Cavity

In Earth ionosphere cavity electromagnetic modes are excited by atmospheric lightning. In this chapter electromagnetic modes of homogenous Earth ionosphere cavity are discussed and approximate resonant frequencies, which lie in frequency band of atmospheric lightning (3-30kHz) are calculated, which impart energy to this cavity. The transverse electric and transverse magnetic modes are investigated for high frequency bands.

2.1 Cavity Description

The cavity is modelled with two concentric spheres, having two different radii. The inner sphere has radius labelled with $r = a$, representing the radius of planet Earth and the outer sphere has radius labelled with $r = b$, representing the radius of ionosphere layer. The radius of Earth is about 6400km and that of the ionosphere is 6500km. The Earth ionosphere cavity is unique from other concentric spherical cavities for the reason, that it is assumed small cavity as compared with cavity assumed for modes propagation in the range of ELF radio frequency band and has very lossy boundaries. The Earth surface is generally assumed as PEC because its conductivity is of order 0.01 and 1Sm^{-1} on land and sea respectively. The ionosphere layer is also assumed PEC for all frequency bands less than 10-20 MHz. The electromagnetic field boundary conditions are satisfied at the interface of Earth and also at the ionosphere layer.

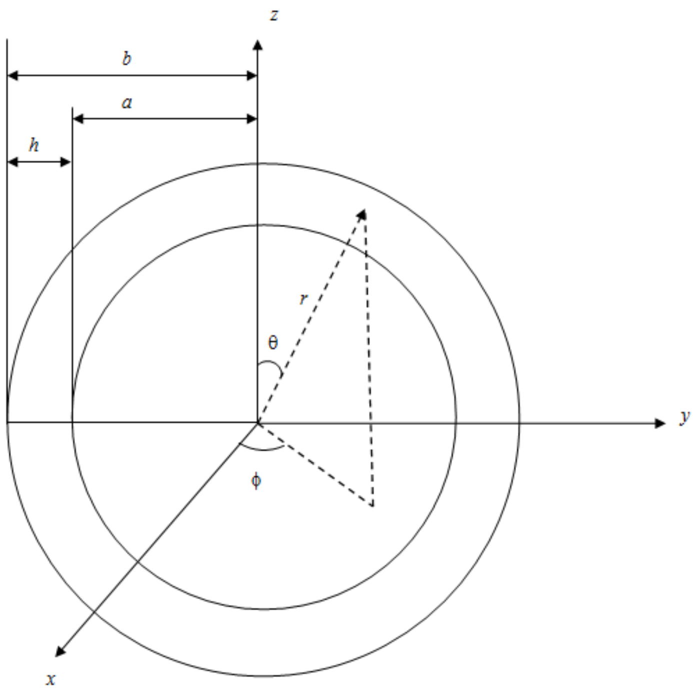

Figure 2.1 Geometry of concentric spherical cavity

2.2 General Formulation

The simplified formulation of a general spherical concentric cavity is used that leads to Eigenfunctions and Eigen-frequencies of the cavity. The method of separation of variables is applied to the electric and magnetic scalar potentials to analyse the electromagnetic field expressions of the concentric PEC spherical cavity analytically. The scalar potentials are derived from Maxwell's equations and boundary conditions are expressed in the spherical coordinate system. To determine the resonant modes of this spherical cavity that is the channel between two PEC spheres, the differential forms of Maxwell's equations are used. The Maxwell's equation used by Tia [20] or Harrington [21] are as;

$$\nabla \times \vec{E} = -\mu_o \frac{\partial H}{\partial t} \tag{2.1}$$

$$\nabla \times \vec{H} = \frac{\partial D}{\partial t} \tag{2.2}$$

$$\nabla . \vec{D} = 0 \tag{2.3}$$

$$\nabla.\vec{B} = 0 \tag{2.4}$$

E and H are the electric and magnetic field where D is electric flux density and B is magnetic flux density. The spherical polar coordinates (r, θ, ϕ) is used to describe the geometry. The Maxwell's equations can be expanded in the spherical coordinate system, the tangential components E_θ ,E_ϕ H_ϕ, H_θ can be expressed in terms of radial components E_r and H_r. We can write that Maxwell's equations assume the form for the source free region are,

$$\frac{1}{r\sin\theta}(\frac{\partial}{\partial r}r\sin\theta E_\phi - \frac{\partial}{\partial\phi}E_r) = i\omega\mu_o H_\theta \tag{2.5}$$

$$\frac{1}{r}(\frac{\partial}{\partial r}rE_\theta - \frac{\partial}{\partial\theta}E_r) = -i\omega\mu_o H_\phi \tag{2.6}$$

$$\frac{1}{r\sin\theta}(\frac{\partial}{\partial r}r\sin\theta H_\phi - \frac{\partial}{\partial\phi}H_r) = i\omega\varepsilon_o E_\theta \tag{2.7}$$

$$\frac{1}{r}(\frac{\partial}{\partial r}rH_\theta - \frac{\partial}{\partial\theta}H_r) = -i\omega\varepsilon_o H_\phi \tag{2.8}$$

$$\frac{1}{r\sin\theta}(\frac{\partial}{\partial\theta}\sin\theta E_\phi - \frac{\partial}{\partial\phi}E_\theta) = -i\omega\mu_o H_r \tag{2.9}$$

$$\frac{1}{r\sin\theta}(\frac{\partial}{\partial\theta}\sin\theta H_\phi - \frac{\partial}{\partial\phi}H_\theta) = -i\omega\varepsilon o E_r \tag{2.10}$$

Now using scalar Electric and Magnetic potential U and V as,

$$E_r = (\frac{\partial^2}{\partial r^2} + \frac{\omega^2}{c^2})rU \tag{2.11}$$

$$H_r = (\frac{\partial^2}{\partial r^2} + \frac{\omega^2}{c^2})rV \tag{2.12}$$

$$E_\theta = \frac{1}{r}\frac{\partial^2}{\partial\theta\partial r}rU + \frac{i\omega\mu_o}{\sin\theta}\frac{\partial}{\partial\phi}V \tag{2.13}$$

$$H_\theta = \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}rV + \frac{i\omega\varepsilon o}{\sin\theta}\frac{\partial}{\partial\phi}U \tag{2.14}$$

$$E_\phi = i\omega\mu_o \frac{\partial}{\partial\theta} V + \frac{1}{r\sin\theta}\frac{\partial^2}{\partial r\partial\phi}(rU) \tag{2.15}$$

$$H_\phi = i\omega\varepsilon_o \frac{\partial}{\partial\theta} U + \frac{1}{r\sin\theta}\frac{\partial^2}{\partial r\partial\phi}(rV) \tag{2.16}$$

Substituting from equations (2.11) to (2.16) in equation (2.9) and (2.10). The general equation for potentials is obtained like as,

$$\Big[\frac{1}{r^2}\Big(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Big) + \frac{\partial^2}{\partial r^2} + \frac{\omega^2}{c^2}\Big] rF = 0 \tag{2.17}$$

where "F" may be U or V. Using the method of separation of variables, the scalar potential F as,

$$F(r,\theta,\phi) = R(r)H(\theta)\Phi(\phi) \tag{2.18}$$

By substituting (2.18) into (2.17), dividing by F and multiplying by $r^2\sin^2\theta$, we obtained an ordinary differential equation given below,

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{\sin\theta}{H}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial H}) + \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} + k^2r^2\sin^2\theta = 0 \tag{2.19}$$

The ϕ dependence in (2.19) is separated by use of integer m as,

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m^2 \tag{2.20}$$

Also the equation for $H(\theta)$ is simplified as,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{dH}{d\theta}) + \Big[n(n+1) - \frac{m^2}{\sin^2\theta}\Big]H = 0 \tag{2.21}$$

The solution of equation (2.21) is,

$$H(\theta) = P_n^m(\cos\theta) \tag{2.22}$$

For $n \geq m$ the period for the above solution is $0 \leq \theta \leq \pi$. It is more convenient to consider the normalized harmonics.

$$Y_{mn}(\theta,\phi) = \Big[\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}\Big]^{1/2} P_n^m(\cos\theta)e^{im\phi} \tag{2.23}$$

Now the equation for radial function *R(r)* is,

$$\left[\frac{\partial^2}{\partial r^2}+\frac{\omega^2}{c^2}-\frac{n(n+1)}{r^2}\right]rR(r)=0 \tag{2.24}$$

Generally the solution of equation (2.24) will be a linear combination of two independent spherical Hankel functions $h_n^{(1)}(kr)$ and $h_n^{(2)}(kr)$. Substituting the solution of *R(r)*, $H(\theta)$ and $\Phi(\phi)$ in equation (2.18). The general and compact wave solution are obtained as given below,

$$F=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}[A_{mn}h_n^{(1)}(k_nr)+B_{mn}h_n^{(2)}(k_nr)]Y_{mn}(\theta,\phi) \tag{2.25}$$

where as "k_n" is unknown wave number will be given by the characteristics equation and F may be U or V.

The complete solution of the problem needs boundary conditions, which will be satisfied. The boundary conditions for surface of Earth $(r=a)$ and ionosphere layer $(r=b)$ must satisfy both electric potential (U) and magnetic potential (V). The boundary conditions given by Bliokh et al [22], are used like as,

$$V(a,\theta,\phi)\mid_{r=a}=0 \tag{2.26}$$

$$V(b,\theta,\phi)\mid_{r=b}=0 \tag{2.27}$$

$$\frac{\partial}{\partial r}rU\mid_{r=a}=0 \tag{2.28}$$

$$\frac{\partial}{\partial r}rU\mid_{r=b}=0 \tag{2.29}$$

2.3 Electric or TM Modes

In transverse magnetic modes, there is no radial component of magnetic field in the direction of propagation. To investigate the TM propagating modes in this cavity, H_r will be assumed to be zero. The equations (2.11-2.17) and (2.25) have been split into two independent subsets describing electric or TM oscillating modes. The equation for electric or TM oscillating modes is described as,

$$U=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}[A_{mn}h_n^{(1)}(k_nr)+B_{mn}h_n^{(2)}(k_nr)]Y_{mn}(\theta,\phi) \tag{2.30}$$

The boundary conditions impose on equation (2.32) are,

$$\frac{\partial}{\partial r} rU \mid_{r=a} = 0 \tag{2.31}$$

$$\frac{\partial}{\partial r} rU \mid_{r=b} = 0 \tag{2.32}$$

The field components in term of scalar electric potential can be written as,

$$E_r^{mn} = (\frac{\partial^2}{\partial r^2} + \frac{\omega^2}{c^2}) rU \tag{2.33}$$

$$E_\theta^{mn} = (\frac{1}{r} \frac{\partial^2}{\partial r \partial \theta}) rU \tag{2.34}$$

$$E_\phi^{mn} = (\frac{1}{r \sin\theta} \frac{\partial^2}{\partial r \partial \phi}) rU \tag{2.35}$$

$$H_r^{mn} = 0 \tag{2.36}$$

$$H_\theta^{mn} = \frac{i\omega\varepsilon_0}{\sin\theta} \frac{\partial}{\partial \phi} U \tag{2.37}$$

$$H_\phi^{mn} = -i\omega\varepsilon_0 \frac{\partial}{\partial \theta} U \tag{2.38}$$

Now for TM modes applying the boundary conditions, the following equations transcendental are obtained,

$$A_{mn}[ah_n^{(1)}(k_n a)]' + B_{mn}[ah_n^{(2)}(k_n a)]' = 0 \tag{2.39}$$

$$A_{mn}[bh_n^{(1)}(k_n b)]' + B_{mn}[bh_n^{(2)}(k_n b)]' = 0 \tag{2.40}$$

For non-zero solution the determinant of equations (2.39) and (2.40) must be equal to zero which yields a transcendental equation for k_n.

$$\left[[ah_n^{(1)}(k_n a)]'\right]\left[[bh_n^{(2)}(k_n b)]'\right] - \left[[bh_n^{(1)}(k_n b)]'\right]\left[[ah_n^{(2)}(k_n a)]'\right] = 0 \tag{2.41}$$

The modal fields can be obtained by obtaining value of A_{mn} from equation (2.39) or (2.40) and using in equations (2.34-2.38). The following expressions for fields corresponding to n$^{\text{th}}$

modes are given as,

$$E_r^{mn} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[A_{mn} \frac{n(n+1)}{k_n r^2} W_n(k_n r) Y_{mn}(\theta, \phi) \right] \tag{2.42}$$

$$E_\theta^{mn} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[A_{mn} \frac{W_n^{'}(k_n r)}{r} \frac{\partial}{\partial \theta} Y_{mn}(\theta, \phi) \right] \tag{2.43}$$

$$E_\phi^{mn} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[\frac{im}{r \sin\theta} A_{mn} W_n^{'}(k_n r) Y_{mn}(\theta, \phi) \right] \tag{2.44}$$

$$H_r^{mn} = 0 \tag{2.45}$$

$$H_\theta^{mn} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[\frac{-imc\varepsilon_o}{r \sin\theta} A_{mn} W_n(k_n r) Y_{mn}(\theta, \phi) \right] \tag{2.46}$$

$$H_\phi^{mn} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[\frac{-ic\varepsilon_o}{r} A_{mn} W_n(k_n r) \frac{\partial}{\partial \theta} Y_{mn}(\theta, \phi) \right] \tag{2.47}$$

The field as function of radial distance can be represented by,

$$W_n(k_n r) = \frac{\left[h_n^{(1)}(k_n r) \right] \left[[a h_n^{(2)}(k_n a)]' \right] - \left[h_n^{(2)}(k_n r) \right] \left[[a h_n^{(1)}(k_n a)]' \right]}{\left[[a h_n^{(2)}(k_n a)]' \right]} \tag{2.48}$$

which is the combination of spherical hankel function of 1^{st} and 2^{nd} kinds with its derivatives w-r-t to arguments. The $W_n^{'}(k_n r)$ is the derivative of $W_n(k_n r)$ w-r-t $k_n r$ and A_{mn} is the modal amplitude which is free parameter.

2.4 Magnetic or TE Modes

In transverse electric modes, there is no radial component of electric field in the direction of propagation. To investigate the TE propagating modes in this cavity, E_r will be assumed to be zero. The equations (2.11-2.17) and (2.26) have been split into two independent subsets describing magnetic or TE oscillating modes. The magnetic or TE modes expressions are obtained using the scalar magnetic potential like as,

$$V = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[C_{mn} h_n^{(1)}(k_n r) + D_{mn} h_n^{(2)}(k_n r) \right] Y_{mn}(\theta, \phi) \tag{2.49}$$

In case of TE modes, the boundary conditions are like as,

$$V(a,\theta,\phi)\mid_{r=a}=0 \tag{2.50}$$

$$V(b,\theta,\phi)\mid_{r=b}=0 \tag{2.51}$$

The field components in term of scalar magnetic potentials can be written as,

$$H_r^{mn}=(\frac{\partial^2}{\partial r^2}+\frac{\omega^2}{c^2})rV \tag{2.52}$$

$$H_\theta^{mn}=(\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta})rV \tag{2.53}$$

$$H_\phi^{mn}=(\frac{1}{r\sin\theta}\frac{\partial^2}{\partial r\partial\phi})rV \tag{2.54}$$

$$E_r^{mn}=0 \tag{2.55}$$

$$E_\theta^{mn}=\frac{i\omega\varepsilon_0}{\sin\theta}\frac{\partial}{\partial\phi}V \tag{2.56}$$

$$E_\phi^{mn}=-i\omega\varepsilon_o\frac{\partial}{\partial\theta}V \tag{2.57}$$

By applying boundary conditions, the eigen equations for TE modes are obtained as,

$$C_{mn}h_n^{(1)}(k_na)+D_{mn}h_n^{(2)}(k_na)=0 \tag{2.58}$$

$$C_{mn}h_n^{(1)}(k_nb)+D_{mn}h_n^{(2)}(k_nb)=0 \tag{2.59}$$

These equations will be solved simultaneously for the eigen values. For non-zero solution, the determinant of equations (2.58) and (2.59) must be equal to zero which yields a transcendental equation for k_n like as,

$$h_n^{(1)}(k_na)h_n^{(2)}(k_nb)-h_n^{(1)}(k_nb)h_n^{(2)}(k_nb)=0 \tag{2.60}$$

The modal fields can now be obtained using value of C_{mn} from (2.58) or (2.59) and using in equations (2.52-2.57). The following expressions for electromagnetic field corresponding to n$^{\text{th}}$ modes are given as,

$$H_r^{mn}=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[C_{mn}\frac{n(n+1)}{k_nr^2}X_n(k_nr)Y_{mn}(\theta,\phi)\right] \tag{2.61}$$

$$H_{\theta}^{mn} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[C_{mn}\frac{X_n^{'}(k_n r)}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right] \tag{2.62}$$

$$H_{\phi}^{mn} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[\frac{im}{r\sin\theta}C_{mn}X_n^{'}(k_n r)Y_{mn}(\theta,\phi)\right] \tag{2.63}$$

$$E_r^{mn} = 0 \tag{2.64}$$

$$E_{\theta}^{mn} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[\frac{-imc\varepsilon_o}{r\sin\theta}C_{mn}X_n(k_n r)Y_{mn}(\theta,\phi)\right] \tag{2.65}$$

$$E_{\phi}^{mn} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[\frac{-ic\varepsilon_o}{r}C_{mn}X_n(k_n r)\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right] \tag{2.66}$$

The constant C_{mn} is the modal amplitude of fields in all TE modes expressions and is a free parameter. The field as function of radial distance can be represented by,

$$X_n(k_n r) = \frac{h_n^{(1)}(k_n r)h_n^{(2)}(k_n a) - h_n^{(2)}(k_n r)h_n^{(1)}(k_n a)}{h_n^{(2)}(k_n a)} \tag{2.67}$$

$X_n(k_n r)$ is the combination of spherical hankel function of 1^{st} and 2^{nd} kinds which contain product of propagation constant and radial distance in its arguments. The $X_n^{'}(k_n r)$ is the derivative of $X_n(k_n r)$ w-r-t $k_n r$.

2.5 Resonant Frequencies

The resonant frequencies of Earth ionosphere cavity depend upon the size of this cavity. There are two approximate techniques to find the model frequencies in this cavity.

In first approximate technique, cavity is considered extremely large and has very lossy boundaries. Such that $r = a + h$, h is height of ionosphere above earth surface and a is radius of Earth. This case has been studied by several authors like Shumman [6], Wait [9] and Jakson [23] for ELF communication and called it the Shumman Resonance(SR). They considered that lightning discharge is the primary natural source of SR. The vertical lightning channels behave like huge antennas that radiate electromagnetic energy at frequencies below 100 kHz. In SR, lightning signals below 100 Hz are considered which is very weak and the attenuation is only 0.5 dB/Mm, and hence the electromagnetic waves from an individual discharge can be propagated a number of times around the globe before decaying into the background noise. In SR, the Earth-ionosphere waveguide behaves like a resonator at ELF frequencies, and amplifies the spectral signals from lightning at the resonance frequencies due to constructive interference of EM waves propagating around the globe in opposite directions. The resonance peaks occur when the wavelength of the ELF waves is comparable with the Earth's circumference (40,000 km), with the direct and antipodal waves resulting in constructive interference at the SR frequencies. Then the resonant frequencies f_n are determined by the Earth's radius and the speed of light c as shown,

$$f_n = (\frac{c}{2\pi a})\sqrt{n(n-1)} \tag{2.68}$$

where as $n = 1,2,3,4....$ for different modes frequencies. The resonance frequency in equation (2.68) depend upon the longitudinal dimension of earth, which is the circumference($2\pi a$) of the Earth. The Schumann made these assumptions and arrived at the expected SR first mode of 10 Hz. However, the Earth-ionosphere cavity is not a perfect electromagnetic cavity. The ELF radio waves are partially reflected over a large interval of altitudes. Heavy ions and ion complexes play a key role in determining the losses due to the finite ionosphere conductivity, resulting in the system resonating at lower frequencies than would be expected in an ideal case (7.8 Hz), with observed peaks wider than expected. In addition, there are a number of horizontal asymmetries of day–night transition, latitudinal changes in the Earth magnetic field, ground conductivity, etc that complicate the SR frequencies of modal fields.

In second approximate, we assumed the cavity as a thin dielectric atmosphere that is the only height of ionosphere which is in the transverse dimension of Earth. Such that $h = b - a$ which is about 100km then the resonant frequencies of this cavity will be in VLF and HF radio frequency range. Because the maximum radiated energy occurs around 10 kHz, the attenuation at these frequencies is about 10 dB/Mm. Hence these frequencies can only be detected at a range of thousands of km from the lightning discharge. This approximate technique is our area of research. The solution of transcendental equations for real values of k_n give us resonant frequencies of earth ionosphere cavity. By rewriting transcendental equation (2.41) of TM modes,

$$\left[[ah_n^{(1)}(k_n a)]'\right]\left[[bh_n^{(2)}(k_n b)]'\right] - \left[[bh_n^{(1)}(k_n b)]'\right]\left[[ah_n^{(2)}(k_n a)]'\right] = 0 \tag{2.69}$$

The derivatives of hankel function w-r-t to its specific arguments, yield the equation like as,

$$\frac{n^2}{k^2ab}\left[h_n^{(1)}(k_na)h_n^{(2)}(k_nb)-h_n^{(1)}(k_na)h_n^{(2)}(k_nb)\right]-\frac{n}{ka}\left[h_n^{(1)}(k_na)h_{n-1}^{(2)}(k_nb)-h_{n-1}^{(1)}(k_nb)h_n^{(2)}(k_na)\right]-\frac{n}{kb}\left[h_{n-1}^{(1)}(k_na)h_n^{(2)}(k_nb)-h_n^{(1)}(k_nb)h_{n-1}^{(2)}(k_na)\right]+\left[h_{n-1}^{(1)}(k_na)h_{n-1}^{(2)}(k_nb)-h_{n-1}^{(1)}(k_nb)h_{n-1}^{(2)}(k_na)\right]=0 \tag{2.70}$$

In order to find resonant frequencies, for large arguments of the spherical Hankel function, the following approximate identities [24] are used in above equation,

$$h_n^{(1)}(x)\approx -\frac{i}{x}e^{i[x-(\frac{n\pi}{2})]} \tag{2.71}$$

$$h_n^{(2)}(x)\approx \frac{i}{x}e^{-i[x-(\frac{n\pi}{2})]} \tag{2.72}$$

where as $x=k_na$ and k_nb in equations yield the following equations,

$$h_n^{(1)}(k_na)\approx -\frac{i}{k_na}e^{i[k_na-(\frac{n\pi}{2})]} \tag{2.73}$$

$$h_n^{(2)}(k_na)\approx \frac{i}{k_na}e^{-i[k_na-(\frac{n\pi}{2})]} \tag{2.74}$$

$$h_n^{(1)}(k_nb)\approx -\frac{i}{k_nb}e^{i[k_nb-(\frac{n\pi}{2})]} \tag{2.75}$$

$$h_n^{(2)}(k_nb)\approx \frac{i}{k_nb}e^{-i[k_nb-(\frac{n\pi}{2})]} \tag{2.76}$$

By substituting equations (2.73-2.76) in transcendental equation (2.70) of TM mode , yields the equation like as,

$$\left[e^{i[k_n(a-b)]}-e^{-i[k_n(a-b)]}\right]\left[\frac{n^2}{k_n^2ab}+\frac{n(i+1)}{k_na}+\frac{n(i+1)}{k_nab}+1\right]=0 \tag{2.77}$$

The solution of equation (2.77) gave equation for the propagation constant of TM modes as,

$$k_n=\frac{n\pi}{b-a} \tag{2.78}$$

Similarly, the solution of transcendental equation (2.60) for TE modes, yields the same propagation constant equation as for TM modes. Now using $k_n=\frac{\omega_n}{c}$ and $\omega_n=2\pi f_n$ in above equation will yields resonant mode frequencies equation. Where as b is the radius of ionosphere and a is radius of earth $(b-a=h=100\text{km})$.

$$f_n = \frac{n \times c}{2h} \tag{2.79}$$

for $c = 3 \times 10^5 \text{km/s}$ and $n = 1,2,3,4$, the resonant frequencies are 1.5kHz , 3kHz , 4.5kHz and 6kHz.

$$\triangle f = f_{n+1} - f_n = \frac{c}{2h} = 1.5\text{kHz} \tag{2.80}$$

Similarly, the wave length for each single propagating mode is obtained from the f_n as,

$$\lambda_n = \frac{2h}{n \times c} \tag{2.81}$$

$$\triangle \lambda = \lambda_{n+1} - \lambda_n = \frac{2h}{n(n+1)c} \tag{2.82}$$

From the above equations (2.79) and (2.81) it is clear that TM and TE modes are propagating in Earth ionosphere spherical cavity, having specific frequencies and wavelength.

2.6 Summary and Discussions

In this chapter, TM and TE modes of Earth ionosphere spherical cavity are discussed. For this purpose, electric and magnetic scalar potentials, and method of separation of variables are used. The resonant frequencies are also investigated for both TM and TE modes using the approximate large arguments of spherical Hankel function. The atmosphere is assumed as a dielectric waveguide having the 100km height from the Earth surface to the ionosphere. In this cavity, the TM and TE modes for VLF and HF radio frequency bands are observed. Each n^{th} TM and TE has its own model amplitude, wavelength, and the resonant frequency with which it is propagating in this cavity.

Chapter 3

Resonant Modes Excitation in Earth Ionosphere Cavity

The steady state electromagnetic excitation of Earth ionosphere cavity by the point source carrying current is studied. The resulting electromagnetic modes are analytically calculated using Green' functions. The homogenous wave equation is not valid after introducing a point source in this cavity. Therefore, the cavity will split into two source free regions, in order to utilize the model solution derivatives. The approximate resonant modal frequencies of respective electromagnetic modes for source free cavity discussed are also assumed in this cavity as resonant modal frequencies.

3.1 Cavity Description and Fields Formulation

The cavity is an approximate atmospheric waveguide above the Earth surface and below ionosphere layer having height h=100km. A point source, carrying current $\vec{J}_0$ is placed at $r = r^{'}$, $\theta^{'}$,$\phi^{'}$ inside the cavity. To facilitate the calculation of excited fields, the cavity space is divided into two regions. Region I is space below the location of point source and above the Earth surface for $a < r < r^{'}$ and the resulting electric field is denoted by $\vec{E}^{mn[I]}_{(r,\theta,\phi)}$ while the corresponding magnetic field is denoted by $\vec{H}^{mn[I]}_{(r,\theta,\phi)}$. Region II is space above the point source and beneath ionosphere layer that is for $r^{'} < r < b$ and in this region electric field is denoted by $\vec{E}^{mn[II]}_{(r,\theta,\phi)}$ while the associated magnetic field is denoted by $\vec{H}^{mn[II]}_{(r,\theta,\phi)}$. The modal fields derived for homogenous Earth ionosphere cavity discussed in the previous chapter are assumed for both regions I and region II.

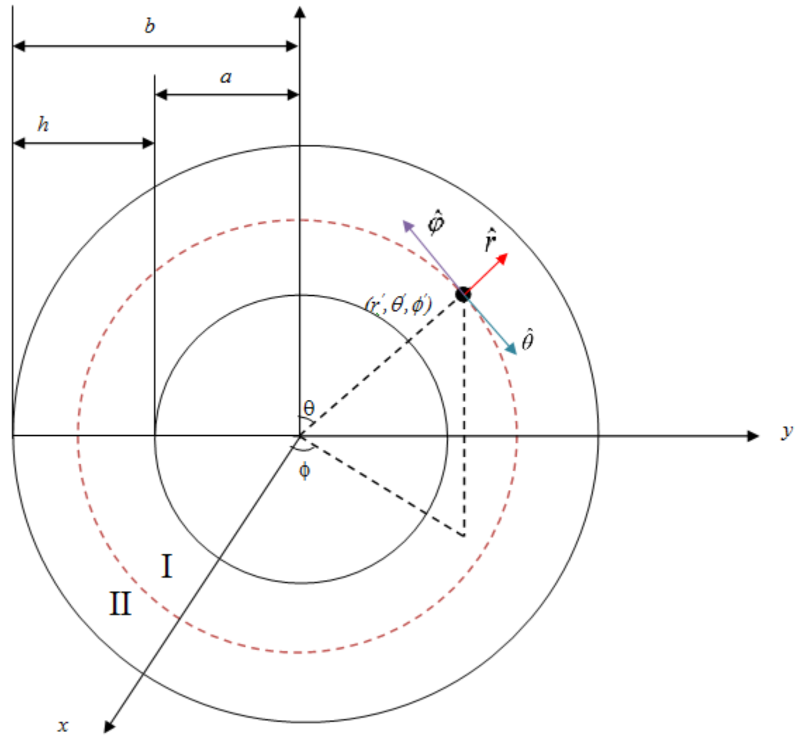

Figure 3.1 Geometry of concentric spherical cavity containing current source

3.2 Electric or TM modes

The field expressions for TM modes, derived in Chapter 2 are used to calculate the TM modes excitation by source in this cavity. The following expressions for the fields corresponding to n[th] propagating TM modes are assumed for the homogeneous region I of the cavity. The resulting fields assumed as Green Functions for region I can be written as,

$$E_r^{mn[I]} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[A_{mn}^{[I]} \frac{n(n+1)}{k_n r^2} [W_n(k_n r)]^{[I]} Y_{mn}(\theta, \phi) \right] \tag{3.1}$$

$$E_\theta^{mn[I]} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[A_{mn}^{[I]} \frac{[W_n'(k_n r)]^{[I]}}{r} \frac{\partial}{\partial \theta} Y_{mn}(\theta, \phi) \right] \tag{3.2}$$

$$E_\phi^{mn[I]} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[\frac{im}{r \sin\theta} [A_{mn}^{[I]} [W_n'(k_n r)]^{[I]} Y_{mn}(\theta, \phi) \right] \tag{3.3}$$

$$H_r^{mn[I]} = 0 \tag{3.4}$$

$$H_{\theta}^{mn[I]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[\frac{-imc\varepsilon_o}{r\sin\theta}A_{mn}^{[I]}[W_n(k_nr)]^{[I]}Y_{mn}(\theta,\phi)\right] \tag{3.5}$$

$$H_{\phi}^{mn[I]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[\frac{-ic\varepsilon_o}{r}A_{mn}^{[I]}[W_n(k_nr)]^{[I]}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right] \tag{3.6}$$

where as $A_{mn}^{[I]}$ is the modal amplitude of respective fields in region I. The term $[W_n(k_nr)]^{[I]}$ represent field as function of radial distances for region I which is the combination of spherical hankel function of 1st and 2nd kinds and $[W_n^{'}(k_nr)]^{[I]}$ is the derivative of $[W_n(k_nr)]^{[I]}$ w-r-t k_nr. The field as function of radial distances is like as,

$$[W_n(k_nr)]^{[I]} = \frac{\left[h_n^{(1)}(k_nr)\right]\left[[ah_n^{(2)}(k_na)]^{'}\right] - \left[h_n^{(2)}(k_nr)\right]\left[[ah_n^{(1)}(k_na)]^{'}\right]}{\left[[ah_n^{(2)}(k_na)]^{'}\right]} \tag{3.7}$$

The following expressions for the field corresponding to nth TM modes are used for homogeneous region II of the excited cavity.

$$E_r^{mn[II]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[A_{mn}^{[II]}\frac{n(n+1)}{k_nr^2}[W_n(k_nr)]^{[II]}Y_{mn}(\theta,\phi)\right] \tag{3.8}$$

$$E_{\theta}^{mn[II]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[A_{mn}^{[II]}\frac{[W_n^{'}(k_nr)]^{[II]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right] \tag{3.9}$$

$$E_{\phi}^{mn[II]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[\frac{im}{r\sin\theta}A_{mn}^{[II]}[W_n^{'}(k_nr)]^{[II]}Y_{mn}(\theta,\phi)\right] \tag{3.10}$$

$$H_r^{mn[II]} = 0 \tag{3.11}$$

$$H_{\theta}^{mn[II]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[\frac{-imc\varepsilon_o}{r\sin\theta}A_{mn}^{[II]}[W_n(k_nr)]^{[II]}Y_{mn}(\theta,\phi)] \tag{3.12}$$

$$H_{\phi}^{mn[II]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[\frac{-ic\varepsilon_o}{r}A_{mn}^{[II]}[W_n(k_nr)]^{[II]}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right] \tag{3.13}$$

where as $A_{mn}^{[II]}$ is the modal amplitude representation for corresponding fields in region II. The term $[W_n(k_nr)]^{[II]}$ has contribution due to radial field as function of radial distance for region II which is the combination of spherical hankel function of 1st and 2nd kinds and $[W_n^{'}(k_nr)]^{[II]}$ is the derivative of $[W_n(k_nr)]^{[II]}$ w-r-t k_nr. The field as function of radial distances for region

II can be written as,

$$[W_n(k_nr)]^{[II]} = \frac{\left[h_n^{(1)}(k_nr)\right]\left[[ah_n^{(2)}(k_nb)]'\right] - \left[h_n^{(2)}(k_nr)\right]\left[[ah_n^{(1)}(k_na)]'\right]}{\left[[ah_n^{(2)}(k_nb)]'\right]} \tag{3.14}$$

The utilized modal fields already satisfy the boundary conditions at interface of Earth $(r = a)$ and at the interface of ionosphere layer $(r = b)$. At interface between region I and II at $(r = r')$, the tangential electric fields are continuous while tangential magnetic fields are discontinuous by the amount of current placed on that interface. These can be expressed as,

$$\vec{E}^{mn[II]}_{(r,\theta,\phi)} - \vec{E}^{mn[I]}_{(r,\theta,\phi)} = 0 \tag{3.15}$$

$$\vec{H}^{mn[II]}_{(r,\theta,\phi)} - \vec{H}^{mn[I]}_{(r,\theta,\phi)} = \vec{J}_o\delta(\vec{r} - \vec{r}') \tag{3.16}$$

In spherical coordinates system we used the delta function, as expressed by [25]. The most general representation of delta function will be used as given,

$$\delta(\vec{r} - \vec{r}') = \begin{cases} \frac{1}{r^2}\delta(r - r')\delta(\cos\theta - \cos\theta')\delta(\phi - \phi') & \\ \frac{1}{2r^2}\delta(r - r')\delta(\phi - \phi') & \text{for no } \theta \text{ dependence} \\ \frac{1}{2\pi r^2}\delta(r - r')\delta(\cos\theta - \cos\theta') & \text{for no } \phi \text{ dependence} \\ \frac{1}{4\pi r^2}\delta(r - r') & \text{for neither } \theta \text{ nor } \phi \text{ dependence} \end{cases} \tag{3.17}$$

The delta function which represents the current distribution in all directions from the above equations are like as,

$$\delta(\vec{r} - \vec{r}') = \delta(r - r')\delta(\cos\theta - \cos\theta')\delta(\phi - \phi') \tag{3.18}$$

The delta function can be converted in to its series of discrete scalar deltas and summation of the scalar delta can be represented using [26] as shown below,

$$\delta(r - r')\delta(\cos\theta - \cos\theta')\delta(\phi - \phi') = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[\frac{e^{in(r-r')}}{2\pi}Y_{mn}(\theta,\phi)Y^*_{mn}(\theta',\phi')\right] \tag{3.19}$$

The current $\vec{J}_o$ is vector and it has radial, θ and ϕ components in spherical coordinate system. This current source excites the cavity due to its current in three different dimensions. The placement of current in all direction simultaneously and calculating the excited fields collectively is quite complex analytical process. That is why, the placement of current will be divide into three different direction distinctly. The current $\vec{J_o}$ is written in its components form as,

$$\vec{J_o} = J_{or}\hat{e_r} + J_{o\theta}\hat{e_\theta} + J_{o\phi}\hat{e_\phi} \tag{3.20}$$

The current placing position in each direction have electromagnetic field components in all directions. The current $\vec{J_o}$ can be assumed in the cavity in three different directions that is r, θ and ϕ which can excite the TM modes. The fields formulation for TM modes have three different cases.

3.2.1 Case I ($J_{or} \neq 0$, $J_{o\theta} = 0$, $J_{o\phi} = 0$)

The current source is placed only in radial direction then the resulting electric and magnetic fields have components in radial, θ and ϕ directions. Electric field can be resolved into its components form in region I are given as,

$$\vec{E}^{mn[I]r}_{(r,\theta,\phi)} = E_r^{mn[I]r}\hat{e_r} + E_\theta^{mn[I]r}\hat{e_\theta} + E_\phi^{mn[I]r}\hat{e_\phi} \tag{3.21}$$

By substituting equations (3.1-3.3) in (3.21), the components of electric field excited by radial current source in region I may be written as,

$$\vec{E}^{mn[I]r}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[A_{mn}^{[I]r}\left\{\frac{n(n+1)}{k_n r^2}[W_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_r} + A_{mn}^{[I]\theta}\left\{\frac{[W_n'(k_n r)]^{[I]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta} + A_{mn}^{[I]\phi}\left\{\frac{im}{r\sin\theta}[W_n'(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \tag{3.22}$$

The magnetic field can be decomposed in components form for region I is expressed as,

$$\vec{H}^{mn[I]r}_{(r,\theta,\phi)} = H_r^{mn[I]r}\hat{e_r} + H_\theta^{mn[I]r}\hat{e_\theta} + H_\phi^{mn[I]r}\hat{e_\phi} \tag{3.23}$$

The magnetic field and its components excited by current source can be obtained by using equations (3.4-3.6) in equation (3.23),

$$\vec{H}^{mn[I]r}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[A_{mn}^{[I]\theta}\left\{\frac{-imc\varepsilon_o}{r\sin\theta}[W_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta} + A_{mn}^{[I]\phi}\left\{\frac{-ic\varepsilon_o}{r}[W_n(k_n r)]^{[I]}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \tag{3.24}$$

where as $A_{mn}^{[I]r}$ is model amplitude of radial component of electric and magnetic field, $A_{mn}^{[I]\theta}$ and $A_{mn}^{[I]\phi}$ is the possible amplitude of tangential components of both the fields. The superscript r at the left side with electric and magnetic field equations (3.21-3.24) is due to the radial directed current while the unit vectors represent the direction of associated field components.

Electric field can be resolved into its components due source current for region II can be written as,

$$\vec{E}_{(r,\theta,\phi)}^{mn[II]r} = E_r^{mn[II]r}\hat{e_r} + E_\theta^{mn[II]r}\hat{e_\theta} + E_\phi^{mn[II]r}\hat{e_\phi} \tag{3.25}$$

The components of electric field excited due radial current in region II is obtained by substituting equations (3.8-3.10) in (3.25) like as,

$$\vec{E}_{(r,\theta,\phi)}^{mn[II]r} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Bigg[A_{mn}^{[II]r}\left\{\frac{n(n+1)}{k_n r^2}[W_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_r} + A_{mn}^{[II]\theta}\left\{\frac{[W_n'(k_n r)]^{[II]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta} + A_{mn}^{[II]\phi}\left\{\frac{im}{r\sin\theta}[W_n'(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\Bigg] \tag{3.26}$$

The components magnetic field excited by the radial current are like as,

$$\vec{H}_{(r,\theta,\phi)}^{mn[II]r} = H_r^{mn[II]r}\hat{e_r} + H_\theta^{mn[II]r}\hat{e_\theta} + H_\phi^{mn[II]r}\hat{e_\phi} \tag{3.27}$$

The radial placement of current source yields the magnetic field in components form for region II, by using equations (3.11-3.12) in equation (3.27),

$$\vec{H}_{(r,\theta,\phi)}^{mn[II]r} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Bigg[A_{mn}^{[II]\theta}\left\{\frac{-imc\varepsilon_o}{r\sin\theta}[W_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta} + A_{mn}^{[II]\phi}\left\{\frac{-ic\varepsilon_o}{r}[W_n(k_n r)]^{[II]}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\Bigg] \tag{3.28}$$

where as $A_{mn}^{[II]r}$ is model amplitude of radial component of electric and magnetic field, $A_{mn}^{[II]\theta}$ and $A_{mn}^{[II]\phi}$ is the possible amplitude of θ and ϕ components of electric and magnetic fields. The superscript r at the left side with electric and magnetic field equations (3.25-3.28) is due to the radial directed current while the unit vectors represent the direction of associated field components due to excitation of radial current.

In order to calculate the total electric and magnetic field expressions for TM mode excitation in this cavity, all unknown amplitude associated with the electric and magnetic field components are necessary. These unknown coefficients can be obtained by imposing boundary conditions. The boundary conditions are,

$$\hat{n} \times \left(\vec{E}^{mn[II]}_{(r,\theta,\phi)} - \vec{E}^{mn[I]}_{(r,\theta,\phi)}\right) = 0 \tag{3.29}$$

$$\hat{n} \times \left(\vec{H}^{mn[II]}_{(r,\theta,\phi)} - \vec{H}^{mn[I]}_{(r,\theta,\phi)}\right) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} [\frac{e^{in(r-r')}}{2\pi} Y_{mn}(\theta,\phi) Y^{*}_{mn}(\theta',\phi')] J_{or} \hat{e_r} \tag{3.30}$$

By solving the equations (3.29) and (3.30) simultaneously and using method of comparing coefficients, unknown modal amplitudes are obtained like as,

$$A^{[I]r}_{mn} = A^{[II]r}_{mn} = A^{[I]\theta}_{mn} = A^{[II]\theta}_{mn} = A^{[I]\phi}_{mn} = A^{[II]\phi}_{mn} = 0 \tag{3.31}$$

Now putting these values in equation (3.22) and (3.24), expressions for electric and magnetic fields for TM modes are,

$$\vec{E}^{mn[I+II]r}_{(r,\theta,\phi)} = 0 \tag{3.32}$$

$$\vec{H}^{mn[I+II]r}_{(r,\theta,\phi)} = 0 \tag{3.33}$$

It is observed that excited fields for TM modes vanishes in this cavity which conclude that, due to radial current the field components modal amplitudes become zero. The total electric and magnetic field evanescent for TM mode due excitation of radial current source.

3.2.2 Case II ($J_{or} = 0$, $J_{o\theta} \neq 0$, $J_{o\phi} = 0$)

In this case, the TM modes are obtained from the excitation of θ directed current $J_{o\theta}$. The excitation current is placed in θ-direction which produce the electric and magnetic field components in radial, θ and ϕ directions. Electric field in its components form for region I are,

$$\vec{E}^{mn[I]\theta}_{(r,\theta,\phi)} = E^{mn[I]\theta}_{r} \hat{e_r} + E^{mn[I]\theta}_{\theta} \hat{e_\theta} + E^{mn[I]\theta}_{\phi} \hat{e_\phi} \tag{3.34}$$

By using equations (3.1-3.3) in (3.34) electric field components due θ-directed excitation of current in region I is obtained like as,

$$\vec{E}^{mn[I]\theta}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \Bigg[A^{[I]r}_{mn} \left\{ \frac{n(n+1)}{k_n r^2} [W_n(k_n r)]^{[I]} Y_{mn}(\theta,\phi) \right\} \hat{e_r} +$$
$$A^{[I]\theta}_{mn} \left\{ \frac{[W_n'(k_n r)]^{[I]}}{r} \frac{\partial}{\partial\theta} Y_{mn}(\theta,\phi) \right\} \hat{e_\theta} + A^{[I]\phi}_{mn} \left\{ \frac{im}{r\sin\theta} [W_n'(k_n r)]^{[I]} Y_{mn}(\theta,\phi) \right\} \hat{e_\phi} \Bigg] \tag{3.35}$$

Similarly magnetic field can be split into its components form due to source current for region I can be written as,

$$\vec{H}^{mn[I]\theta}_{(r,\theta,\phi)} = H^{mn[I]\theta}_{r}\hat{e_r} + H^{mn[I]\theta}_{\theta}\hat{e_\theta} + H^{mn[I]\theta}_{\phi}\hat{e_\phi} \tag{3.36}$$

Using equations (3.4-3.6) in equation (3.36) the magnetic field and its excited components due to source are,

$$\vec{H}^{mn[I]\theta}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[A^{[I]\theta}_{mn}\left\{\frac{-imc\varepsilon_o}{r\sin\theta}[W_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta}\right.$$
$$\left. + A^{[I]\phi}_{mn}\left\{\frac{-ic\varepsilon_o}{r}[W_n(k_n r)]^{[I]}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \tag{3.37}$$

where as $A^{[I]r}_{mn}$ is unknown coefficient which can be the model amplitude of radial component of electric and magnetic field, $A^{[I]\theta}_{mn}$ and $A^{[I]\phi}_{mn}$ coefficients are the possible amplitude of tangential components of both the fields. The superscript θ at the left side in the equations of electric and magnetic field (3.34-3.37) is due to the θ directed current while the unit vectors represent the direction of corresponding excited field components.

Similarly, in region II, the electric field can be split into its components form is written as,

$$\vec{E}^{mn[II]\theta}_{(r,\theta,\phi)} = E^{mn[II]\theta}_{r}\hat{e_r} + E^{mn[II]\theta}_{\theta}\hat{e_\theta} + E^{mn[II]\theta}_{\phi}\hat{e_\phi} \tag{3.38}$$

By putting equations (3.8-3.10) in (3.38), fields components are obtained due to excited θ-directed current in region II. The field components are,

$$\vec{E}^{mn[II]\theta}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[A^{[II]r}_{mn}\left\{\frac{n(n+1)}{k_n r^2}[W_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_r} + \right.$$
$$\left. A^{[II]\theta}_{mn}\left\{\frac{[W_n'(k_n r)]^{[II]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta} + A^{[II]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[W_n'(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \tag{3.39}$$

The magnetic field can be split into its components form due excitation of θ-directed source current in region II, can be represented as,

$$\vec{H}^{mn[II]\theta}_{(r,\theta,\phi)} = H^{mn[II]\theta}_{r}\hat{e_r} + H^{mn[II]\theta}_{\theta}\hat{e_\theta} + H^{mn[II]\theta}_{\phi}\hat{e_\phi} \tag{3.40}$$

Using equations (3.11-3.12) in equation (3.40) the magnetic field in its components form due to source current for region II are,

$$\vec{H}^{mn[II]\theta}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[A^{[II]\theta}_{mn}\left\{\frac{-imc\varepsilon_o}{r\sin\theta}[W_n(k_nr)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta}\right.$$
$$\left. + A^{[II]\phi}_{mn}\left\{\frac{-ic\varepsilon_o}{r}[W_n(k_nr)]^{[II]}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \quad (3.41)$$

where as $A^{[II]r}_{mn}$, $A^{[II]\phi}_{mn}$, $A^{[II]\theta}_{mn}$ are unknown coefficient which represent the model amplitude of radial, θ, ϕ components of electric and magnetic field. The superscript θ at the left side in the equations of electric and magnetic field (3.38-3.41) is due to the θ directed current while the unit vectors represent the direction of corresponding excited field components.

The unknown coefficients associated with the components of electric and magnetic field for both regions can be find out using the boundary conditions at the interface between region I and II. The boundary conditions are,

$$\hat{n}\times\left(\vec{E}^{mn[II]\theta}_{(r,\theta,\phi)} - \vec{E}^{mn[I]\theta}_{(r,\theta,\phi)}\right) = 0 \quad (3.42)$$

$$\hat{n}\times\left(\vec{H}^{mn[II]\theta}_{(r,\theta,\phi)} - \vec{H}^{mn[I]\theta}_{(r,\theta,\phi)}\right) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[\frac{e^{in(r-r')}}{2\pi}Y_{mn}(\theta,\phi)Y^*_{mn}(\theta',\phi')]J_{o\theta}\hat{e_\theta} \quad (3.43)$$

By putting the equations (3.35), (3.39), (3.37) and (3.41) in equation (3.42) and (3.43), two equations are derived. By solving the obtained two equations linearly and comparing the coefficients on both sides, the unknown modal amplitudes are obtained. The modal amplitude can be written as,

$$A^{[I]r}_{mn} = A^{[II]r}_{mn} = 0 \quad (3.44)$$

$$A^{[I]\theta}_{mn} = \frac{-r\sin\theta e^{in(r-r')}Y^*_{mn}(\theta',\phi')[W'_n(k_nr)]^{[II]}}{2\pi imc\varepsilon_o T_n(k_nr)}J_{o\theta} \quad (3.45)$$

$$A^{[II]\theta}_{mn} = \frac{-r\sin\theta e^{in(r-r')}Y^*_{mn}(\theta',\phi')[W'_n(k_nr)]^{[I]}}{2\pi imc\varepsilon_o T_n(k_nr)}J_{o\theta} \quad (3.46)$$

$$A^{[I]\phi}_{mn} = A^{[II]\phi}_{mn} = 0 \quad (3.47)$$

where as $T_n(k_n r)$ is field as a function of radial distances for region I and region II. This can be written as,

$$T_n(k_n r) = [W_n^{'}(k_n r)]^{[I]}[W_n(k_n r)]^{[II]} - [W_n^{'}(k_n r)]^{[II]}[W_n(k_n r)]^{[I]} \tag{3.48}$$

By putting the coefficient values in equations (3.37), (3.39), (3.41) and (3.42), equations for electric and magnetic field are obtained for region I and region II. In order to get the total electric and magnetic fields, the electric fields of both regions are added and similarly the magnetic fields of both regions are added. The total electric and magnetic field expression for TM_{mn}^{θ} become as,

$$\vec{E}_{(r,\theta,\phi)}^{mn[I+II]\theta} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Big[\frac{\sin\theta e^{in(r-r^{'})}Y_{mn}^{*}(\theta^{'},\phi^{'})}{i\pi mc\varepsilon_o T_n(k_n r)}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)[W_n^{'}(k_n r)]^{[II]}[W_n^{'}(k_n r)]^{[I]}\Big]J_{o\theta}\hat{e}_{\theta} \tag{3.49}$$

$$\vec{H}_{(r,\theta,\phi)}^{mn[I+II]\theta} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Big[-\frac{e^{in(r-r')}}{2\pi}Y_{mn}(\theta,\phi)Y_{mn}^{*}(\theta^{'},\phi^{'})\Big]J_{o\theta}\hat{e}_{\theta} \tag{3.50}$$

where as $\vec{E}_{(r,\theta,\phi)}^{mn[I+II]\theta}$ represent the total electric field while $\vec{H}_{(r,\theta,\phi)}^{mn[I+II]\theta}$ show the total magnetic field for TM_{mn}^{θ}. The $J_{o\theta}\hat{e}_{\theta}$ in equation of electric and magnetic fields show that the tangential $\theta-$ directed placement of current source excite modes in this cavity.

3.2.3 Case III ($J_{or} = 0$, $J_{o\theta} = 0$, $J_{o\phi} \neq 0$)

In this case, the expressions for TM modes are obtained from the excitation of only ϕ-directed current source. When the current source is placed only in ϕ direction then electric and magnetic field have components in radial, θ and ϕ directions. Electric field in its components form in region I can be written as,

$$\vec{E}_{(r,\theta,\phi)}^{mn[I]\phi} = E_r^{mn[I]\phi}\hat{e_r} + E_{\theta}^{mn[I]\phi}\hat{e_{\theta}} + E_{\phi}^{mn[I]\phi}\hat{e_{\phi}} \tag{3.51}$$

The substitution of equations (3.1-3.3) in (3.51) resulted the field excitation by source current in region I as,

$$\vec{E}_{(r,\theta,\phi)}^{mn[I]\phi} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Bigg[A_{mn}^{[I]r}\left\{\frac{n(n+1)}{k_n r^2}[W_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_r} + A_{mn}^{[I]\theta}\left\{\frac{[W_n^{'}(k_n r)]^{[I]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_{\theta}} + A_{mn}^{[I]\phi}\left\{\frac{im}{r\sin\theta}[W_n^{'}(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_{\phi}}\Bigg] \tag{3.52}$$

Similarly, magnetic field can be split into components from by the excitation of source current in region I. The magnetic field in its components from can be written as,

$$\vec{H}^{mn[I]\phi}_{(r,\theta,\phi)} = H_r^{mn[I]\phi}\hat{e_r} + H_\theta^{mn[I]\phi}\hat{e_\theta} + H_\phi^{mn[I]\phi}\hat{e_\phi} \tag{3.53}$$

The resulting magnetic field expression in its components form due to source can be written by using equations (3.4-3.6) in equation (3.53) like as,

$$\vec{H}^{mn[I]\phi}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Bigg[A_{mn}^{[I]\theta}\left\{\frac{-imc\varepsilon_o}{r\sin\theta}[W_n(k_nr)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta} + A_{mn}^{[I]\phi}\left\{\frac{-ic\varepsilon_o}{r}[W_n(k_nr)]^{[I]}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\Bigg] \tag{3.54}$$

where as $A_{mn}^{[I]r}, A_{mn}^{[I]\phi}, A_{mn}^{[I]\theta}$ are unknown coefficient which represent the model amplitude of radial, θ, ϕ components of electric and magnetic field. The superscript ϕ at the left side in the equations of electric and magnetic field (3.51-3.54) is due to the ϕ directed current while the unit vectors represent the direction of corresponding excited field components.

In region II, electric field can be expanded into its components form due to ϕ-directed source current, can be written as,

$$\vec{E}^{mn[II]\phi}_{(r,\theta,\phi)} = E_r^{mn[II]\phi}\hat{e_r} + E_\theta^{mn[II]\phi}\hat{e_\theta} + E_\phi^{mn[II]\phi}\hat{e_\phi} \tag{3.55}$$

The resulted fields obtained by excitation of ϕ-directed current in region II is obtained by substituting equations (3.8-3.10) in (3.55). The electric field expression can be written as,

$$\vec{E}^{mn[II]\phi}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Bigg[A_{mn}^{[II]r}\left\{\frac{n(n+1)}{k_nr^2}[W_n(k_nr)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_r} + A_{mn}^{[II]\theta}\left\{\frac{[W_n'(k_nr)]^{[II]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta} + A_{mn}^{[II]\phi}\left\{\frac{im}{r\sin\theta}[W_n'(k_nr)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\Bigg] \tag{3.56}$$

Similarly resulted magnetic field due to source current in components form for region II is shown in (3.57) like as,

$$\vec{H}^{mn[II]\phi}_{(r,\theta,\phi)} = H_r^{mn[II]\phi}\hat{e_r} + H_\theta^{mn[II]\phi}\hat{e_\theta} + H_\phi^{mn[II]\phi}\hat{e_\phi} \tag{3.57}$$

Using equations (3.11-3.12) in equation (3.57) the magnetic field in its components form due to source are obtained as,

$$\vec{H}^{mn[II]\phi}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[A^{[II]\theta}_{mn}\left\{\frac{-imc\varepsilon_o}{r\sin\theta}[W_n(k_nr)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta}\right.$$
$$\left.+A^{[II]\phi}_{mn}\left\{\frac{-ic\varepsilon_o}{r}[W_n(k_nr)]^{[II]}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \quad (3.58)$$

where as $A^{[II]r}_{mn}$, $A^{[II]\phi}_{mn}$, $A^{[II]\theta}_{mn}$ are unknown model amplitude of radial, θ, ϕ components of electric and magnetic field. The boundary conditions used to find the unknown amplitude are,

$$\hat{n}\times\left(\vec{E}^{mn[II]\phi}_{(r,\theta,\phi)} - \vec{E}^{mn[I]\phi}_{(r,\theta,\phi)}\right) = 0 \quad (3.59)$$

$$\hat{n}\times\left(\vec{H}^{mn[II]\phi}_{(r,\theta,\phi)} - \vec{H}^{mn[I]\phi}_{(r,\theta,\phi)}\right) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[\frac{e^{in(r-r')}}{2\pi}Y_{mn}(\theta,\phi)Y^*_{mn}(\theta',\phi')]J_{o\phi}\hat{e_\phi} \quad (3.60)$$

The unknown modal amplitudes obtained are like as,

$$A^{[I]r}_{mn} = A^{[II]r}_{mn} = 0 \quad (3.61)$$

$$A^{[I]\theta}_{mn} = A^{[II]\theta}_{mn} = 0 \quad (3.62)$$

$$A^{[I]\phi}_{mn} = \frac{-re^{in(r-r')}Y_{mn}(\theta,\phi)Y^*_{mn}(\theta',\phi')[W'_n(k_nr)]^{[II]}}{2\pi ic\varepsilon_oT_n(k_nr)\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)}J_{o\phi} \quad (3.63)$$

$$A^{[II]\phi}_{mn} = \frac{-re^{in(r-r')}Y_{mn}(\theta',\phi')Y^*_{mn}(\theta',\phi')[W'_n(k_nr)]^{[I]}}{2\pi ic\varepsilon_oT_n(k_nr)\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)}J_{o\phi} \quad (3.64)$$

where as,

$$T_n(k_nr) = [W'_n(k_nr)]^{[I]}[W_n(k_nr)]^{[II]} - [W'_n(k_nr)]^{[II]}[W_n(k_nr)]^{[I]} \quad (3.65)$$

The substituting of modal amplitudes in equations (3.52), (3.54), (3.56) and (3.58) yield electric and magnetic field expression in region I and region II. The summation of electric field expressions for both regions gave the total electric field expression for TE^{ϕ}_{mn} mode. Similarly, expression of total magnetic field is obtained by the addition of magnetic field expression of region I and region II. The total electric and magnetic field expression for

TM_{mn}^{ϕ} written as,

$$\vec{E}_{(r,\theta,\phi)}^{mn[I+II]\phi} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Big[\frac{-me^{in(r-r^{'})}[Y_{mn}(\theta,\phi)]^2 Y_{mn}^{*}(\theta^{'},\phi^{'})[W_n^{'}(k_n r)]^{[II]}[W_n^{'}(k_n r)]^{[I]}}{\pi c\varepsilon_o \frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)T_n(k_n r)}\Big]J_{o\phi}\hat{e}_{\phi} \tag{3.66}$$

$$\vec{H}_{(r,\theta,\phi)}^{mn[I+II]\phi} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Big[\frac{e^{in(r-r^{'})}[Y_{mn}(\theta,\phi)]Y_{mn}^{*}(\theta^{'},\phi^{'})}{2\pi}\Big]J_{o\phi}\hat{e}_{\phi} \tag{3.67}$$

where as $\vec{E}_{(r,\theta,\phi)}^{mn[I+II]\phi}$ is total electric field and $\vec{H}_{(r,\theta,\phi)}^{mn[I+II]\phi}$ is total magnetic field. The total electric and magnetic field expression gave observation about TM modal fields expressions, that TM modes will be excite due to the ϕ-directed current of source.

3.3 Magnetic or TE modes

The electric and magnetic field expressions for TE modes in unexcited cavity are assumed to calculate the field expressions for TE modes in this excited cavity. The region of excited cavity is divided into two homogenous regions that is why the expressions of unexcited cavity are assumed for each region in excited cavity. The calculated electric field expressions for n^{th} TE modes in region I can written as,

$$H_r^{mn[I]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[C_{mn}^{[I]}\frac{n(n+1)}{k_n r^2}[X_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)] \tag{3.68}$$

$$H_\theta^{mn[I]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[C_{mn}^{[I]}\frac{[X^{'}{}_n(k_n r)]^{[I]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)] \tag{3.69}$$

$$H_\phi^{mn[I]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[\frac{im}{r\sin\theta}C_{mn}^{[I]}[X^{'}{}_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)] \tag{3.70}$$

$$E_r^{mn[I]} = 0 \tag{3.71}$$

$$E_\theta^{mn[I]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[\frac{-imc\varepsilon_o}{r\sin\theta}C_{mn}^{[I]}[X_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)] \tag{3.72}$$

$$E_\phi^{mn[I]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[\frac{-ic\varepsilon_o}{r}C_{mn}^{[I]}[X_n(k_n r)]^{[I]}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)] \tag{3.73}$$

where as $C_{mn}^{[I]}$ is the modal amplitude of respective fields in region I. The term $[X_n(k_n r)]^{[I]}$ represent field as function of radial distances for region I which is the combination of spherical hankel function of 1st and 2nd kinds and $[X_n^{'}(k_n r)]^{[I]}$ is the derivative of $[X_n(k_n r)]^{[I]}$ w-r-t $k_n r$. The field as function of radial distance for region I can be represented by equation,

$$[X_n(k_n r)]^{[I]} = \frac{h_n^{(1)}(k_n r)h_n^{(2)}(k_n a) - h_n^{(2)}(k_n a)h_n^{(1)}(k_n r)}{h_n^{(2)}(k_n a)} \tag{3.74}$$

Similarly, electric and magnetic field expressions corresponding to nth TE modes in homogenous region II of excited can be assumed as,

$$H_r^{mn[II]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[C_{mn}^{[II]}\frac{n(n+1)}{k_n r^2}[X_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)] \tag{3.75}$$

$$H_\theta^{mn[II]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[C_{mn}^{[II]}\frac{[X'_n(k_n r)]^{[II]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)] \tag{3.76}$$

$$H_\phi^{mn[II]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[\frac{im}{r\sin\theta}C_{mn}^{[II]}[X'_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)] \tag{3.77}$$

$$E_r^{mn[II]} = 0 \tag{3.78}$$

$$E_\theta^{mn[II]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[\frac{-imc\varepsilon_o}{r\sin\theta}C_{mn}^{[II]}[X_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)] \tag{3.79}$$

$$E_\phi^{mn[II]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[\frac{-ic\varepsilon_o}{r}C_{mn}^{[II]}[X_n(k_n r)]^{[II]}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)] \tag{3.80}$$

where as $C_{mn}^{[II]}$ is the modal amplitude of respective fields in region I. The term $[X_n(k_n r)]^{[II]}$ represent field as function of radial distances for region I which is the combination of spherical hankel function of 1st and 2nd kinds and $[X_n^{'}(k_n r)]^{[II]}$ is the derivative of $[X_n(k_n r)]^{[II]}$ w-r-t $k_n r$. The field as function of radial distance for region II can be represented as,

$$[X_n(k_n r)]^{[II]} = \frac{h_n^{(1)}(k_n r)h_n^{(2)}(k_n b) - h_n^{(2)}(k_n r)h_n^{(1)}(k_n b)}{h_n^{(2)}(k_n b)} \tag{3.81}$$

The boundary conditions are already satisfied by the modal fields at the interface of Earth($r = a$) and at interface of ionosphere($r = b$). The interface($r = r'$) between region I and region II, tangential electric field are continuous while the magnetic field are discontinuous by the amount of current placed on that interface. These can be expressed as,

$$\vec{E}^{mn[II]}_{(r,\theta,\phi)} - \vec{E}^{mn[I]}_{(r,\theta,\phi)} = 0 \tag{3.82}$$

$$\vec{H}^{mn[II]}_{(r,\theta,\phi)} - \vec{H}^{mn[I]}_{(r,\theta,\phi)} = \vec{J}_o \delta(\vec{r} - \vec{r}^{'}) \tag{3.83}$$

where as $\vec{J_o}$ is a source current and is a vector which create the field components in all directions. The vectorial representation of delta function are like as,

$$\delta(\vec{r} - \vec{r}^{'}) = \delta(r - r^{'})\delta(\cos\theta - \cos\theta^{'})\delta(\phi - \phi^{'}) \tag{3.84}$$

while its summation form can be written as,

$$\delta(r - r^{'})\delta(\cos\theta - \cos\theta^{'})\delta(\phi - \phi^{'}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[\frac{e^{in(r-r')}}{2\pi} Y_{mn}(\theta,\phi) Y^{*}_{mn}(\theta^{'},\phi^{'})] \tag{3.85}$$

3.3.1 Case I ($J_{or} \neq 0$, $J_{o\theta} = 0$, $J_{o\phi} = 0$)

In case I, to obtained the modes expressions for TE modes, the current source is assumed only in radial direction then the magnetic and electric fields have components in radial, θ and ϕ direction. The magnetic field in components form for region I are,

$$\vec{H}^{mn[I]r}_{(r,\theta,\phi)} = H^{mn[I]r}_{r} \hat{e_r} + H^{mn[I]r}_{\theta} \hat{e_\theta} + H^{mn[I]r}_{\phi} \hat{e_\phi} \tag{3.86}$$

By substituting equations (3.68-3.70) in (3.86), magnetic fields components due radial current in region I is obtained as,

$$\vec{H}^{mn[I]r}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Bigg[C^{[I]r}_{mn}\left\{\frac{n(n+1)}{k_n r^2}[X_n(k_n r)]^{[I]} Y_{mn}(\theta,\phi)\right\}\hat{e_r} + C^{[I]\theta}_{mn}\left\{\frac{[X^{'}_n(k_n r)]^{[I]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta} + C^{[I]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[X^{'}_n(k_n r)]^{[I]} Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\Bigg] \tag{3.87}$$

The electric field can be split into its components form due to radial directed source current in region I is expressed as,

$$\vec{E}^{mn[I]r}_{(r,\theta,\phi)} = E^{mn[I]r}_{r} \hat{e_r} + E^{mn[I]r}_{\theta} \hat{e_\theta} + E^{mn[I]r}_{\phi} \hat{e_\phi} \tag{3.88}$$

Using equations (3.71-3.73) in equation (3.88) the electric field has components due to source are like as,

$$\vec{E}^{mn[I]r}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[C^{[I]\theta}_{mn}\left\{\frac{-imc\varepsilon_o}{r\sin\theta}[X_n(k_nr)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta}\right.$$
$$\left. + C^{[I]\phi}_{mn}\left\{\frac{-ic\varepsilon_o}{r}[X_n(k_nr)]^{[I]}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \quad (3.89)$$

where as $C^{[I]r}_{mn}$, $C^{[I]\phi}_{mn}$, $C^{[I]\theta}_{mn}$ are unknown coefficient which represent the model amplitude of radial, θ, ϕ components of electric and magnetic field. The superscript r at the left side in the equations of electric and magnetic field (3.87-3.89) is due to the excitation of radial directed current while the unit vectors represent the direction of corresponding excited field components. Similarly, the magnetic field can be resolved into its components due to source current for region II can be written as,

$$\vec{H}^{mn[II]r}_{(r,\theta,\phi)} = H^{mn[II]r}_{r}\hat{e_r} + H^{mn[II]r}_{\theta}\hat{e_\theta} + H^{mn[II]r}_{\phi}\hat{e_\phi} \quad (3.90)$$

By substituting equations (3.75-3.77) in (3.90) magnetic field due to radial directed current in region II is obtained as,

$$\vec{H}^{mn[II]r}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[C^{[II]r}_{mn}\left\{\frac{n(n+1)}{k_nr^2}[X_n(k_nr)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_r}+\right.$$
$$\left. C^{[II]\theta}_{mn}\left\{\frac{[X_n'(k_nr)]^{[II]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta} + C^{[II]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[X_n'(k_nr)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \quad (3.91)$$

Electric field in component for region II can be written as,

$$\vec{E}^{mn[II]r}_{(r,\theta,\phi)} = E^{mn[II]r}_{r}\hat{e_r} + E^{mn[II]r}_{\theta}\hat{e_\theta} + E^{mn[II]r}_{\phi}\hat{e_\phi} \quad (3.92)$$

By substituting equations (3.78-3.80) in equation (3.92) the electric field in components form due to source current can be written as,

$$\vec{E}^{mn[II]r}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[C^{[II]\theta}_{mn}\left\{\frac{-imc\varepsilon_o}{r\sin\theta}[X_n(k_nr)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta}\right.$$
$$\left. + C^{[II]\phi}_{mn}\left\{\frac{-ic\varepsilon_o}{r}[X_n(k_nr)]^{[II]}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \quad (3.93)$$

where as $C_{mn}^{[II]r}$, $C_{mn}^{[II]\phi}$, $C_{mn}^{[II]\theta}$ are the model amplitude of radial, θ, ϕ components of electric and magnetic field in region II. The superscript r at the left side in the equations of electric and magnetic field (3.91-3.93) is due to the radial source current.

The expressions for total electric and magnetic field for TE modes in this excited cavity need the description of all known modal amplitudes of field components. These unknown can be obtained by imposing the boundary conditions. The boundary conditions are,

$$\vec{n} \times \left(\vec{E}^{mn[II]}_{(r,\theta,\phi)} - \vec{E}^{mn[I]}_{(r,\theta,\phi)}\right) = 0 \tag{3.94}$$

$$\vec{n} \times \left(\hat{H}^{mn[II]}_{(r,\theta,\phi)} - \vec{H}^{mn[I]}_{(r,\theta,\phi)}\right) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} [\frac{e^{in(r-r')}}{2\pi} Y_{mn}(\theta,\phi) Y^{*}_{mn}(\theta^{'},\phi^{'})] J_{or} \hat{e_r} \tag{3.95}$$

Using this boundary condition unknown amplitude vanishes as,

$$C_{mn}^{[I]r} = C_{mn}^{[II]r} = C_{mn}^{[I]\theta} = C_{mn}^{[II]\theta} = C_{mn}^{[I]\phi} = C_{mn}^{[II]\phi} = 0 \tag{3.96}$$

By substituting these modal amplitude values in equation (3.92) and (3.93). The expressions of total electric and magnetic field for TE modes are obtained as,

$$\vec{E}^{mn[I+II]r}_{(r,\theta,\phi)} = 0 \tag{3.97}$$

$$\vec{H}^{mn[I+II]r}_{(r,\theta,\phi)} = 0 \tag{3.98}$$

where as $\vec{E}^{mn[I+II]r}_{(r,\theta,\phi)}$ is the total electric field expression while $\vec{H}^{mn[I+II]r}_{(r,\theta,\phi)}$ is total magnetic field expression. It is observed from equations (3.97) and (3.98) that TE modes in this cavity are not excited due to radial current. The total electric and magnetic field vanishes due to radial directed current.

3.3.2 Case II $\left(J_{or} = 0 \, , J_{o\theta} \neq 0, J_{o\phi} = 0 \right)$

In this case, the cavity is excited by current source is only in θ-direction and resulted electric and magnetic field have components in radial, θ and ϕ direction. Magnetic field in its components form in region I due to excitation of source current can be written as,

$$\vec{H}^{mn[I]\theta}_{(r,\theta,\phi)} = H_r^{mn[I]\theta} \hat{e_r} + H_\theta^{mn[I]\theta} \hat{e_\theta} + H_\phi^{mn[I]\theta} \hat{e_\phi} \tag{3.99}$$

By putting equations (3.68-3.70) in (3.99) magnetic field expression for region I. This magnetic field in its components form due to current source in region I can be written as,

$$\vec{H}^{mn[I]\theta}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[C^{[I]r}_{mn}\left\{\frac{n(n+1)}{k_n r^2}[X_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_r}+\right.$$
$$\left. C^{[I]\theta}_{mn}\left\{\frac{[X_n'(k_n r)]^{[I]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta}+C^{[I]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[X_n'(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \quad (3.100)$$

Similarly, for electric field can be decomposed in components form for region I due source current in θ-direction are,

$$\vec{E}^{mn[I]\theta}_{(r,\theta,\phi)} = E^{mn[I]\theta}_{r}\hat{e_r} + E^{mn[I]\theta}_{\theta}\hat{e_\theta} + E^{mn[I]\theta}_{\phi}\hat{e_\phi} \quad (3.101)$$

The substitution of equations (3.71-3.73) in equation (3.101) yield the electric field in the form of its component due to source current in region I like as,

$$\vec{E}^{mn[I]\theta}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[C^{[I]\theta}_{mn}\left\{\frac{-imc\varepsilon_o}{r\sin\theta}[X_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta}\right.$$
$$\left.+C^{[I]\phi}_{mn}\left\{\frac{-ic\varepsilon_o}{r}[X_n(k_n r)]^{[I]}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \quad (3.102)$$

where as $C^{[I]r}_{mn}$, $C^{[I]\phi}_{mn}$, $C^{[I]\theta}_{mn}$ are unknown coefficients which represent the model amplitudes of radial, θ, ϕ components of electric and magnetic field. Magnetic Field for region II can be decomposed into its components form like as,

$$\vec{H}^{mn[II]\theta}_{(r,\theta,\phi)} = H^{mn[II]\theta}_{r}\hat{e_r} + H^{mn[II]\theta}_{\theta}\hat{e_\theta} + H^{mn[II]\theta}_{\phi}\hat{e_\phi} \quad (3.103)$$

The substitution of equations (3.75-3.77) in (3.103) yields magnetic field in components form due to θ-directed source current in region II. The magnetic field in components form can be written as,

$$\vec{H}^{mn[II]\theta}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[C^{[II]r}_{mn}\left\{\frac{n(n+1)}{k_n r^2}[X_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_r}+\right.$$
$$\left. C^{[II]\theta}_{mn}\left\{\frac{[X_n'(k_n r)]^{[II]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta}+C^{[II]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[X_n'(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \quad (3.104)$$

Similarly, the electric field in its components form for region II are,

$$\vec{E}^{mn[II]\theta}_{(r,\theta,\phi)} = E^{mn[II]\theta}_{r}\hat{e_r} + E^{mn[II]\theta}_{\theta}\hat{e_\theta} + E^{mn[II]\theta}_{\phi}\hat{e_\phi} \tag{3.105}$$

Using equations (3.78-3.80) in equation (3.105) the electric field and its excited components in region II due to source are,

$$\vec{E}^{mn[II]\theta}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Bigg[C^{[II]\theta}_{mn}\left\{\frac{-imc\varepsilon_o}{r\sin\theta}[X_n(k_nr)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta} + C^{[II]\phi}_{mn}\left\{\frac{-ic\varepsilon_o}{r}[X_n(k_nr)]^{[II]}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\Bigg] \tag{3.106}$$

where as $C^{[II]r}_{mn}$, $C^{[II]\phi}_{mn}$, $C^{[II]\theta}_{mn}$ are unknown coefficient which represent the model amplitude of radial, θ, ϕ components of electric and magnetic field.

The boundary conditions at the interface between both regions, determine the all unknown modal amplitude like as,

$$\vec{n}\times\left(\hat{E}^{mn[II]\theta}_{(r,\theta,\phi)} - \vec{E}^{mn[I]\theta}_{(r,\theta,\phi)}\right) = 0 \tag{3.107}$$

$$\vec{n}\times\left(\vec{H}^{mn[II]\theta}_{(r,\theta,\phi)} - \vec{H}^{mn[I]\theta}_{(r,\theta,\phi)}\right) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[\frac{e^{in(r-r')}}{2\pi}Y_{mn}(\theta,\phi)Y^{*}_{mn}(\theta',\phi')]J_{o\theta}\hat{e_\theta} \tag{3.108}$$

The substitution of equation (3.100),(3.102),(3.104) and (3.106) in equation of given boundary conditions yield the other two equations. The unknown modal amplitudes are obtained from derived equations by method of comparing coefficient. The obtained modal amplitudes are,

$$C^{[I]r}_{mn} = C^{[II]r}_{mn} = 0 \tag{3.109}$$

$$C^{[I]\theta}_{mn} = \frac{re^{in(r-r')}Y_{mn}(\theta,\phi)Y^{*}_{mn}(\theta',\phi')[X'_n(k_nr)]^{[II]}}{2\pi\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)T_n(k_nr)}J_{o\theta} \tag{3.110}$$

$$C^{[II]\theta}_{mn} = \frac{re^{in(r-r')}Y_{mn}(\theta,\phi)Y^{*}_{mn}(\theta',\phi')[X'_n(k_nr)]^{[I]}}{2\pi\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)T_n(k_nr)}J_{o\theta} \tag{3.111}$$

$$C^{[I]\phi}_{mn} = C^{[II]\phi}_{mn} = 0 \tag{3.112}$$

where as $T_n(k_n r)$ is the field as function of radial distance for both regions and can be written as,

$$T_n(k_n r) = [X_n'(k_n r)]^{[I]}[X_n(k_n r)]^{[II]} - [X_n'(k_n r)]^{[II]}[X_n(k_n r)]^{[I]} \tag{3.113}$$

By substituting the modal amplitudes in equations equation (3.100), (3.102), (3.104) and (3.106), expressions of electric and magnetic field are obtained for both the regions. The electric fields of both region I and region II similarly magnetic fields of both regions are added to get total electric and magnetic fields in this excited cavity. The field expressions for TE modes due to excitation of cavity can be written as,

$$\vec{E}^{mn[I+II]\theta}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Big[\frac{-imc\varepsilon_o e^{in(r-r')}[Y_{mn}(\theta,\phi)]^2 Y^*_{mn}(\theta',\phi')[X_n(k_n r)]^{[I]}[X_n(k_n r)]^{[II]}}{\pi\sin\theta T_n(k_n r)\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)}\Big]J_{o\theta}\hat{e}_\theta \tag{3.114}$$

$$\vec{H}^{mn[I+II]\theta}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Big[\frac{e^{in(r-r')}Y_{mn}(\theta,\phi)Y^*_{mn}(\theta',\phi')}{2\pi}\Big]J_{o\theta}\hat{e}_\theta \tag{3.115}$$

where as $\vec{E}^{mn[I+II]\theta}_{(r,\theta,\phi)}$ is total electric field and $\vec{H}^{mn[I+II]\theta}_{(r,\theta,\phi)}$ is total magnetic field due to excitation of θ-directed source current $J_{o\theta}\hat{e}_\theta$. These are expressions for TE modes in this excited cavity.

3.3.3 Case III ($J_{or} = 0$, $J_{o\theta} = 0$, $J_{o\phi} \neq 0$)

In case I, the current source is assumed only in ϕ direction then the resulting magnetic and electric fields have components in radial, θ and ϕ direction. The magnetic field in components form for region I is,

$$\vec{H}^{mn[I]\phi}_{(r,\theta,\phi)} = H_r^{mn[I]\phi}\hat{e_r} + H_\theta^{mn[I]\phi}\hat{e_\theta} + H_\phi^{mn[I]\phi}\hat{e_\phi} \tag{3.116}$$

By substituting equations (3.68-3.70) in (3.116), the magnetic field components form due ϕ-directed current in region I is,

$$\vec{H}^{mn[I]\phi}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Bigg[C^{[I]r}_{mn}\left\{\frac{n(n+1)}{k_n r^2}[X_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_r}+ C^{[I]\theta}_{mn}\left\{\frac{[X_n'(k_n r)]^{[I]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta} + C^{[I]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[X_n'(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\Bigg] \tag{3.117}$$

The electric field can be split into components form in region I is,

$$\vec{E}^{mn[I]\phi}_{(r,\theta,\phi)} = E_r^{mn[I]\phi}\hat{e_r} + E_\theta^{mn[I]\phi}\hat{e_\theta} + E_\phi^{mn[I]\phi}\hat{e_\phi} \tag{3.118}$$

Using equations (3.71-3.73) in equation (3.118) the electric field and its components due to source are like as,

$$\vec{E}^{mn[I]\phi}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[C^{[I]\theta}_{mn}\left\{\frac{-imc\varepsilon_o}{r\sin\theta}[X_n(k_nr)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta}\right.$$
$$\left. + C^{[I]\phi}_{mn}\left\{\frac{-ic\varepsilon_o}{r}[X_n(k_nr)]^{[I]}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \quad (3.119)$$

where as $C^{[I]r}_{mn}$, $C^{[I]\phi}_{mn}$, $C^{[I]\theta}_{mn}$ are unknown coefficients which represent the model amplitudes of radial, θ, ϕ components of electric and magnetic field. Similarly, magnetic field in components form for region II can be written as,

$$\vec{H}^{mn[II]\phi}_{(r,\theta,\phi)} = H^{mn[II]\phi}_{r}\hat{e_r} + H^{mn[II]\phi}_{\theta}\hat{e_\theta} + H^{mn[II]\phi}_{\phi}\hat{e_\phi} \quad (3.120)$$

By substituting equations (3.75-3.77) in (3.120), the resulting components of magnetic field excited by ϕ-directed current in region II are obtained as,

$$\vec{H}^{mn[II]\phi}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[C^{[II]r}_{mn}\left\{\frac{n(n+1)}{k_nr^2}[X_n(k_nr)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_r}+\right.$$
$$\left. C^{[II]\theta}_{mn}\left\{\frac{[X'_n(k_nr)]^{[II]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta} + C^{[II]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[X'_n(k_nr)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \quad (3.121)$$

The electric field in its components in region II are,

$$\vec{E}^{mn[II]\phi}_{(r,\theta,\phi)} = E^{mn[II]\phi}_{r}\hat{e_r} + E^{mn[II]\phi}_{\theta}\hat{e_\theta} + E^{mn[II]\phi}_{\phi}\hat{e_\phi} \quad (3.122)$$

By using equations (3.78-3.80) in equation (3.122), the electric field in its components form is obtained due to source current are like as,

$$\vec{E}^{mn[II]\phi}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[C^{[II]\theta}_{mn}\left\{\frac{-imc\varepsilon_o}{r\sin\theta}[X_n(k_nr)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta}\right.$$
$$\left. + C^{[II]\phi}_{mn}\left\{\frac{-ic\varepsilon_o}{r}[X_n(k_nr)]^{[II]}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \quad (3.123)$$

where as $C^{[I]r}_{mn}$, $C^{[I]\phi}_{mn}$, $C^{[I]\theta}_{mn}$ are unknown the model amplitudes of radial, θ, ϕ components of electric and magnetic field. The boundary conditions are used to find the unknown amplitude can be obtained. These are written as,

$$\hat{n} \times \left(\vec{E}^{mn[II]\phi}_{(r,\theta,\phi)} - \vec{E}^{mn[I]\phi}_{(r,\theta,\phi)}\right) = 0 \tag{3.124}$$

$$\hat{n} \times \left(\vec{H}^{mn[II]\phi}_{(r,\theta,\phi)} - \vec{H}^{mn[I]\phi}_{(r,\theta,\phi)}\right) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[\frac{e^{in(r-r')}}{2\pi}Y_{mn}(\theta,\phi)Y^{*}_{mn}(\theta^{'},\phi^{'})]J_{o\phi}\hat{e_{\phi}} \tag{3.125}$$

The substitution of equations (3.117), (3.119), (3.121) and (3.123) in above equations yield the unknown modal amplitudes. These modal amplitudes can be written as,

$$C^{[I]r}_{mn} = C^{[II]r}_{mn} = 0 \tag{3.126}$$

$$C^{[I]\theta}_{mn} = C^{[II]\theta}_{mn} = 0 \tag{3.127}$$

$$C^{[I]\phi}_{mn} = \frac{r\sin\theta e^{in(r-r^{'})}Y^{*}_{mn}(\theta^{'},\phi^{'})[X_n(k_nr)]^{[II]}}{2\pi imT_n(k_nr)}J_{o\phi} \tag{3.128}$$

$$C^{[II]\phi}_{mn} = \frac{r\sin\theta e^{in(r-r^{'})}Y^{*}_{mn}(\theta^{'},\phi^{'})[X_n(k_nr)]^{[I]}}{2\pi imT_n(k_nr)}J_{o\phi} \tag{3.129}$$

Where as

$$T_n(k_nr) = [X^{'}_n(k_nr)]^{[I]}[X_n(k_nr)]^{[II]} - [X^{'}_n(k_nr)]^{[II]}[X_n(k_nr)]^{[I]} \tag{3.130}$$

By substituting the modal amplitudes in equations (3.117), (3.119), (3.121) and (3.123), expressions of electric and magnetic field are obtained for both the regions. The electric fields of region I and region II similarly magnetic fields of both regions are added respectively, to get total electric and magnetic fields in this excited cavity. The field expressions for TE modes due to excitation of cavity can be written as,

$$\vec{E}^{mn[I+II]\phi}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Big[\frac{-c\varepsilon_o e^{in(r-r^{'})}Y^{*}_{mn}(\theta^{'},\phi^{'})[X_n(k_nr)]^{[II]}[X_n(k_nr)]^{[I]}}{\pi T_n(k_nr)}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\Big]J_{o\phi}\hat{e}_{\phi} \tag{3.131}$$

$$\vec{H}^{mn[I+II]\phi}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Big[\frac{e^{in(r-r^{'})}[Y_{mn}(\theta,\phi)]Y^{*}_{mn}(\theta^{'},\phi^{'})}{2\pi}\Big]J_{o\phi}\hat{e}_{\phi} \tag{3.132}$$

where as $\vec{E}^{mn[I+II]\phi}_{(r,\theta,\phi)}$ represent total electric field for both regions and $\vec{H}^{mn[I+II]\phi}_{(r,\theta,\phi)}$ is total magnetic field for the expression of TE modes. The current $J_{o\phi}\hat{e}_{\phi}$ is source of excitation in cavity which generate TE modes.

3.4 Transverse Electromagnetic or TEM modes

In the Transverse Electric and Magnetic (TEM) mode, both the electric field and the magnetic field (which are always perpendicular to one another in free space) are transverse to the direction of propagation. To calculate the field expressions for TEM modes due to source current in Earth ionosphere cavity, the magnetic field expressions for TE modes and electric field expressions for TM modes in unexcited cavity are assumed to calculate the field expressions for TEM modes. The following expressions for electric field corresponding to n^{th} TEM modes are assumed for homogeneous region I of cavity.

$$E_r^{mn[I]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[D_{mn}^{[I]}\frac{n(n+1)}{k_n r^2}[W_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right] \tag{3.133}$$

$$E_\theta^{mn[I]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[D_{mn}^{[I]}\frac{[W_n^{'}(k_n r)]^{[I]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right] \tag{3.134}$$

$$E_\phi^{mn[I]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[\frac{im}{r\sin\theta}[D_{mn}^{[I]}[W_n^{'}(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right] \tag{3.135}$$

The constant $D_{mn}^{[I]}$ is the modal amplitude of resonant mode in region I and is free parameter. The $[W_n(k_n r)]^{[I]}$ represent the field as a function of radial distance for region I, can be written as,

$$[W_n(k_n r)]^{[I]} = \frac{\left[h_n^{(1)}(k_n r)\right]\left[[ah_n^{(2)}(k_n a)]^{'}\right] - \left[h_n^{(2)}(k_n r)\right]\left[[ah_n^{(1)}(k_n a)]^{'}\right]}{\left[[ah_n^{(2)}(k_n a)]^{'}\right]} \tag{3.136}$$

Similarly magnetic field expression representation for homogenous region I are,

$$H_r^{mn[I]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[D_{mn}^{[I]}\frac{n(n+1)}{k_n r^2}[X_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)] \tag{3.137}$$

$$H_\theta^{mn[I]} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[D_{mn}^{[I]}\frac{[X^{'}{}_n(k_n r)]^{[I]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)] \tag{3.138}$$

$$H_{\phi}^{mn[I]} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} [\frac{im}{r\sin\theta} D_{mn}^{[I]} [X'_n(k_n r)]^{[I]} Y_{mn}(\theta, \phi)] \tag{3.139}$$

where as $[X_n(k_n r)]^{[I]}$ is the field as a function radial distance for region I and can be written as,

$$[X_n(k_n r)]^{[I]} = \frac{h_n^{(1)}(k_n r) h_n^{(2)}(k_n a) - h_n^{(2)}(k_n r) h_n^{(1)}(k_n a)}{h_n^{(2)}(k_n a)} \tag{3.140}$$

The electric Field expression for Region II are,

$$E_r^{mn[II]} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[D_{mn}^{[II]} \frac{n(n+1)}{k_n r^2} [W_n(k_n r)]^{[II]} Y_{mn}(\theta, \phi) \right] \tag{3.141}$$

$$E_{\theta}^{mn[II]} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[D_{mn}^{[II]} \frac{[W_n'(k_n r)]^{[II]}}{r} \frac{\partial}{\partial\theta} Y_{mn}(\theta, \phi) \right] \tag{3.142}$$

$$E_{\phi}^{mn[II]} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[\frac{im}{r\sin\theta} [D_{mn}^{[II]} [W_n'(k_n r)]^{[II]} Y_{mn}(\theta, \phi) \right] \tag{3.143}$$

The $[W_n(k_n r)]^{[II]}$ show field as a function of radial distance for region II and may written as,

$$[W_n(k_n r)]^{[II]} = \frac{\left[h_n^{(1)}(k_n r) \right] \left[[b h_n^{(2)}(k_n b)]' \right] - \left[h_n^{(2)}(k_n r) \right] \left[[b h_n^{(1)}(k_n b)]' \right]}{\left[[a h_n^{(2)}(k_n a)]' \right]} \tag{3.144}$$

where as $[W_n'(k_n r)]^{[II]}$ is derivative of $[W_n(k_n r)]^{[II]}$ w-r-t argument $k_n r$. Magnetic fields expression for region II can be written as,

$$H_r^{mn[II]} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} [D_{mn}^{[II]} \frac{n(n+1)}{k_n r^2} [X_n(k_n r)]^{[II]} Y_{mn}(\theta, \phi)] \tag{3.145}$$

$$H_{\theta}^{mn[II]} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} [D_{mn}^{[II]} \frac{[X'_n(k_n r)]^{[II]}}{r} \frac{\partial}{\partial\theta} Y_{mn}(\theta, \phi)] \tag{3.146}$$

$$H_{\phi}^{mn[II]} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} [\frac{im}{r\sin\theta} D_{mn}^{[II]} [X'_n(k_n r)]^{[II]} Y_{mn}(\theta, \phi)] \tag{3.147}$$

where as $[X_n(k_n r)]^{[II]}$ represent field as a function of radial distance in region I. This can be written as,

$$[X_n(k_n r)]^{[I]} = \frac{h_n^{(1)}(k_n r) h_n^{(2)}(k_n b) - h_n^{(2)}(k_n r) h_n^{(1)}(k_n b)}{h_n^{(2)}(k_n b)} \tag{3.148}$$

where as $[X'_n(k_n r)]^{[II]}$ derivative of $[X_n(k_n r)]^{[II]}$ w-r-t $k_n r$. The unknown coefficients $D_{mn}^{[I]}$ and $D_{mn}^{[II]}$ are the modal amplitudes of corresponding electric and magnetic field in region I and region II respectively. These unknown coefficients can be obtained by using the boundary condition at the interfaces.

The utilized modal fields already satisfy the boundary conditions at interface of Earth $(r = a)$ and at the interface of ionosphere layer $(r = b)$. At interface between region I and II at $(r = r')$, the tangential electric fields are continuous while tangential magnetic fields are discontinuous by the amount of current placed on that interface. These can be expressed as,

$$\vec{E}^{mn[II]}_{(r,\theta,\phi)} - \vec{E}^{mn[I]}_{(r,\theta,\phi)} = 0 \tag{3.149}$$

$$\vec{H}^{mn[II]}_{(r,\theta,\phi)} - \vec{H}^{mn[I]}_{(r,\theta,\phi)} = \vec{J}_o \delta(\vec{r} - \vec{r}') \tag{3.150}$$

The delta function which represent the current distribution in all direction can be written as,

$$\delta(\vec{r} - \vec{r}') = \delta(r - r')\delta(\cos\theta - \cos\theta')\delta(\phi - \phi') \tag{3.151}$$

The scalar delta function in sum form is represented as,

$$\delta(r - r')\delta(\cos\theta - \cos\theta')\delta(\phi - \phi') = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}[\frac{e^{in(r-r')}}{2\pi}Y_{mn}(\theta,\phi)Y^*_{mn}(\theta',\phi')] \tag{3.152}$$

The current $\vec{J}_o$ can be written as,

$$\vec{J}_o = J_{or}\hat{e_r} + J_{o\theta}\hat{e_\theta} + J_{o\phi}\hat{e_\phi} \tag{3.153}$$

As $\vec{J}_o$ excite the field in three different directions that is why, there are three cases for observation to investigate modes in this cavity.

3.4.1 Case I $(J_{or} \neq 0, J_{o\theta} = 0, J_{o\phi} = 0)$

The current source is placed only in radial direction then the resulting electric and magnetic fields have components in radial, θ and ϕ directions. Electric field can be resolved into its components form in region I are given as,

$$\vec{E}^{mn[I]r}_{(r,\theta,\phi)} = E_r^{mn[I]r}\hat{e_r} + E_\theta^{mn[I]r}\hat{e_\theta} + E_\phi^{mn[I]r}\hat{e_\phi} \tag{3.154}$$

By substituting equations (3.133-3.135) in (3.154), the components of electric field excited by radial current source in region I may be written as,

$$\vec{E}^{mn[I]r}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[D^{[I]r}_{mn}\left\{\frac{n(n+1)}{k_n r^2}[W_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_r}+\right.$$
$$\left. D^{[I]\theta}_{mn}\left\{\frac{[W_n^{'}(k_n r)]^{[I]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta}+D^{[I]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[W_n^{'}(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \quad (3.155)$$

The magnetic field can be decomposed in components form for region I is expressed as,

$$\vec{H}^{mn[I]r}_{(r,\theta,\phi)} = H^{mn[I]r}_{r}\hat{e_r}+H^{mn[I]r}_{\theta}\hat{e_\theta}+H^{mn[I]r}_{\phi}\hat{e_\phi} \quad (3.156)$$

The magnetic field and its components excited by current source can be obtained by using equations(3.137-3.139) in equation (3.156). The magnetic field in components form can be written as,

$$\vec{H}^{mn[I]r}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[D^{[I]r}_{mn}\left\{\frac{n(n+1)}{k_n r^2}[X_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_r}\right.$$
$$\left.+D^{[I]\theta}_{mn}\left\{\frac{[X^{'}{}_n(k_n r)]^{[I]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta}+D^{[I]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[X^{'}{}_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \quad (3.157)$$

where as $D^{[I]r}_{mn}$ is model amplitude of radial component of electric and magnetic field, $D^{[I]\theta}_{mn}$ and $D^{[I]\phi}_{mn}$ is the possible amplitude of tangential components of both the fields. Electric field can be resolved into its components due source current for region II can be written as,

$$\vec{E}^{mn[II]r}_{(r,\theta,\phi)} = E^{mn[II]r}_{r}\hat{e_r}+E^{mn[II]r}_{\theta}\hat{e_\theta}+E^{mn[II]r}_{\phi}\hat{e_\phi} \quad (3.158)$$

The components of electric field excited due radial current in region II is obtained by substituting equations (3.141-3.143) in (3.158) are as,

$$\vec{E}^{mn[II]r}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[D^{[II]r}_{mn}\left\{\frac{n(n+1)}{k_n r^2}[W_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_r}+\right.$$
$$\left. D^{[II]\theta}_{mn}\left\{\frac{[W_n^{'}(k_n r)]^{[II]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta}+D^{[II]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[W_n^{'}(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \quad (3.159)$$

The components magnetic field excited by the radial current can be written as,

$$\vec{H}^{mn[II]r}_{(r,\theta,\phi)} = H^{mn[II]r}_{r}\hat{e_r} + H^{mn[II]r}_{\theta}\hat{e_\theta} + H^{mn[II]r}_{\phi}\hat{e_\phi} \tag{3.160}$$

Using equations (3.145-3.147) in equation (3.159) the magnetic field in components form due to source current can be written as,

$$\vec{H}^{mn[II]r}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[D^{[II]r}_{mn}\left\{\frac{n(n+1)}{k_n r^2}[X_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_r}\right.$$
$$\left. + D^{[II]\theta}_{mn}\left\{\frac{[X'_n(k_n r)]^{[II]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta} + D^{[II]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[X'_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \tag{3.161}$$

where as $D^{[II]r}_{mn}$ is model amplitude of radial component of electric and magnetic field, $D^{[II]\theta}_{mn}$ and $D^{[II]\phi}_{mn}$ is the possible amplitude of θ and ϕ components of electric and magnetic fields.

In order to calculate the total electric and magnetic field expressions for TM mode excitation in this cavity, all unknown amplitude associated with the electric and magnetic field components are necessary. These unknown coefficients can be obtained by imposing boundary conditions from equations (3.149) and (3.150). The obtained modal amplitudes are,

$$D^{[I]r}_{mn} = \left[\frac{k_n r^2 e^{in(r-r')}Y^*_{mn}(\theta',\phi')[W_n(k_n r)]^{[II]}}{2\pi n(n+1)[Z_n(k_n r)]}\right]J_{or} \tag{3.162}$$

$$D^{[II]r}_{mn} = \left[\frac{k_n r^2 e^{in(r-r')}Y^*_{mn}(\theta',\phi')[W_n(k_n r)]^{[I]}}{2\pi n(n+1)[Z_n(k_n r)]}\right]J_{or} \tag{3.163}$$

$$D^{[I]\theta}_{mn} = D^{[II]\theta}_{mn} = 0 \tag{3.164}$$

$$D^{[I]\phi}_{mn} = D^{[II]\phi}_{mn} = 0 \tag{3.165}$$

where as $[Z_n(k_n r)]$ is fields as function of radial distance in TEM modes excitation and can be written as,

$$[Z_n(k_n r)] = [W_n(k_n r)]^{[I]}[X_n(k_n r)]^{[II]} - [W_n(k_n r)]^{[II]}[X_n(k_n r)]^{[I]} \tag{3.166}$$

As the source of excitation is only in radial direction that why modal amplitudes associated with the tangential components vanish out in the boundary conditions. Now putting these values in equation (3.159) and (3.161), the expression for electric and magnetic fields for TEM modes in this cavity are obtained. These can be written as,

$$\vec{E}^{mn[I+II]r}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[\frac{e^{in(r-r')}Y^{*}_{mn}(\theta',\phi')[W_n(k_nr)]^{[II]}[W_n(k_nr)]^{[I]}}{\pi[Z_n(k_nr)]}\right]J_{or}\hat{e}_r \tag{3.167}$$

$$\vec{H}^{mn[I+II]r}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[\frac{e^{in(r-r')}Y^{*}_{mn}(\theta',\phi')Y_{mn}(\theta,\phi)}{-2\pi}\right]J_{or}\hat{e}_r \tag{3.168}$$

where as $\vec{E}^{mn[I+II]r}_{(r,\theta,\phi)}$ represent the total electric field while $\vec{H}^{mn[I+II]r}_{(r,\theta,\phi)}$ show the total magnetic field for TEM^{r}_{mn}. The $J_{or}\hat{e}_r$ in equation of electric and magnetic fields show that the radial directed placement of current source excite modes in this cavity.

3.4.2 Case II $\left(J_{or} = 0\,, J_{o\theta} \neq 0, J_{o\phi} = 0 \right)$

In this case, the cavity is excited by current source is only in θ-direction and resulted electric and magnetic field have components in radial, θ and ϕ direction. Electric field in components form for region I can be written as,

$$\vec{E}^{mn[I]\theta}_{(r,\theta,\phi)} = E^{mn[I]\theta}_{r}\hat{e}_r + E^{mn[I]\theta}_{\theta}\hat{e_\theta} + E^{mn[I]\theta}_{\phi}\hat{e_\phi} \tag{3.169}$$

Substituting equations (3.133-3.135) in (3.169), electric field components due to current in region I is obtained like as,

$$\vec{E}^{mn[I]\theta}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Bigg[D^{[I]r}_{mn}\left\{\frac{n(n+1)}{k_nr^2}[W_n(k_nr)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e}_r + \\ D^{[I]\theta}_{mn}\left\{\frac{[W'_n(k_nr)]^{[I]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta} + D^{[I]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[W'_n(k_nr)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\Bigg] \tag{3.170}$$

Similarly, the corresponding magnetic field can be decomposed in components form for region I due source current in θ-direction are,

$$\vec{H}^{mn[I]\theta}_{(r,\theta,\phi)} = H^{mn[I]\theta}_{r}\hat{e}_r + H^{mn[I]\theta}_{\theta}\hat{e_\theta} + H^{mn[I]\theta}_{\phi}\hat{e_\phi} \tag{3.171}$$

Using equations (3.145-3.147) in equation (3.171) the magnetic field in components form due to source in region can be written as,

$$\vec{H}^{mn[I]\theta}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[D^{[I]r}_{mn}\left\{\frac{n(n+1)}{k_n r^2}[X_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\right.$$
$$\left. + D^{[I]\theta}_{mn}\left\{\frac{[X'_n(k_n r)]^{[I]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\} + D^{[I]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[X'_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\right] \quad (3.172)$$

Electric Field for region II can be decomposed into its components form due to source current which is in θ-direction. Fields for region II can be written as,

$$\vec{E}^{mn][II]\theta}_{(r,\theta,\phi)} = E^{mn[II]\theta}_{r}\hat{e_r} + E^{mn[II]\theta}_{\theta}\hat{e_\theta} + E^{mn[II]\theta}_{\phi}\hat{e_\phi} \quad (3.173)$$

Substituting equations (3.141-3.143) in (3.173) fields excited by θ-directed current in region II is obtained like as,

$$\vec{E}^{mn[II]\theta}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[D^{[II]r}_{mn}\left\{\frac{n(n+1)}{k_n r^2}[W_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_r}+\right.$$
$$\left. D^{[II]\theta}_{mn}\left\{\frac{[W'_n(k_n r)]^{[II]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta} + D^{[II]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[W'_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \quad (3.174)$$

$$\vec{H}^{mn[II]\theta}_{(r,\theta,\phi)} = H^{mn[II]\theta}_{r}\hat{e_r} + H^{mn[II]\theta}_{\theta}\hat{e_\theta} + H^{mn[II]\theta}_{\phi}\hat{e_\phi} \quad (3.175)$$

Using equations (3.145-3.147) in equation (3.175) the magnetic field and its component due to source are like as,

$$\vec{H}^{mn[I]\theta}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[D^{[I]r}_{mn}\left\{\frac{n(n+1)}{k_n r^2}[X_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\right.$$
$$\left. + D^{[I]\theta}_{mn}\left\{\frac{[X'_n(k_n r)]^{[I]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\} + D^{[I]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[X'_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\right] \quad (3.176)$$

where as $D^{[II]r}_{mn}$, $D^{[II]\phi}_{mn}$, $D^{[II]\theta}_{mn}$ are unknown coefficient which represent the model amplitude of radial, θ, ϕ components of electric and magnetic field. Now using the given boundary conditions given in equations (3.149) and (3.150), the unknown amplitudes can be obtained as,

$$D^{[I]r}_{mn} = D^{[II]r}_{mn} = 0 \quad (3.177)$$

$$D_{mn}^{[I]\theta} = \left[\frac{re^{in(r-r^{'})}Y_{mn}(\theta,\phi)Y_{mn}^{*}(\theta^{'},\phi^{'})[W_{n}^{'}(k_{n}r)]^{[II]}}{2\pi\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)[Z_{n}^{'}(k_{n}r)]}\right]J_{o\theta} \tag{3.178}$$

$$D_{mn}^{[II]\theta} = \left[\frac{re^{in(r-r^{'})}Y_{mn}(\theta,\phi)Y_{mn}^{*}(\theta^{'},\phi^{'})[W_{n}^{'}(k_{n}r)]^{[I]}}{2\pi\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)[Z_{n}^{'}(k_{n}r)]}\right]J_{o\theta} \tag{3.179}$$

$$D_{mn}^{[I]\phi} = D_{mn}^{[II]\phi} = 0 \tag{3.180}$$

Where as $Z_{n}^{'}(k_{n}r)$ is field as function of radial distance for both the regions and can be written as,

$$Z_{n}^{'}(k_{n}r) = [W_{n}^{'}(k_{n}r)]^{[I]}[X_{n}^{'}(k_{n}r)]^{[II]} - [W_{n}^{'}(k_{n}r)]^{[II]}[X_{n}^{'}(k_{n}r)]^{[I]} \tag{3.181}$$

By substituting the modal amplitudes in equations (3.170), (3.172), (3.174) and (3.176), expressions of electric and magnetic field are obtained for both the regions. The electric fields of both region I and region II similarly magnetic fields of both regions are added to get total electric and magnetic fields in this excited cavity. The field expressions for TEM modes due to excitation of cavity can be written as,

$$\vec{E}_{(r,\theta,\phi)}^{mn[I+II]\theta} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[\frac{e^{in(r-r^{'})}Y_{mn}(\theta,\phi)Y_{mn}^{*}(\theta^{'},\phi^{'})[W_{n}^{'}(k_{n}r)]^{[II]}[W_{n}^{'}(k_{n}r)]^{[I]}}{\pi Z_{n}^{'}(k_{n}r)}\right]J_{o\theta}\hat{e}_{\theta} \tag{3.182}$$

$$\vec{H}_{(r,\theta,\phi)}^{mn[I+II]\theta} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[-\frac{e^{in(r-r')}}{2\pi}Y_{mn}(\theta,\phi)Y_{mn}^{*}(\theta^{'},\phi^{'})\right]J_{o\theta}\hat{e}_{\theta} \tag{3.183}$$

where as $\vec{E}_{(r,\theta,\phi)}^{mn[I+II]\theta}$ is total electric field and $\vec{H}_{(r,\theta,\phi)}^{mn[I+II]\theta}$ is total magnetic field due to excitation of θ-directed source current $J_{o\theta}\hat{e}_{\theta}$. These are expressions for TEM modes in this excited cavity.

3.4.3 Case III ($J_{or} = 0$, $J_{o\theta} = 0$, $J_{o\phi} \neq 0$)

In this case, the current source is assumed only in ϕ direction then the resulting magnetic and electric fields have components in radial, θ and ϕ direction. The electric field in components form for region I is,

$$\vec{E}_{(r,\theta,\phi)}^{mn[I]\phi} = E_{r}^{mn[I]\phi}\hat{e_{r}} + E_{\theta}^{mn[I]\phi}\hat{e_{\theta}} + E_{\phi}^{mn[I]\phi}\hat{e_{\phi}} \tag{3.184}$$

Substituting equations (3.133-3.135) in (3.184) components of magnetic field due to excitation of current in region I can be written as,

$$\vec{E}^{mn[I]\phi}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[D^{[I]r}_{mn}\left\{\frac{n(n+1)}{k_n r^2}[W_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_r}+ D^{[I]\theta}_{mn}\left\{\frac{[W_n'(k_n r)]^{[I]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta}+D^{[I]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[W_n'(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \quad (3.185)$$

The magnetic field can be split into components form in region I is,

$$\vec{H}^{mn[I]\phi}_{(r,\theta,\phi)} = H^{mn[I]\phi}_r\hat{e_r}+H^{mn[I]\phi}_\theta\hat{e_\theta}+H^{mn[I]\phi}_\phi\hat{e_\phi} \quad (3.186)$$

Using equations (3.137-3.139) in equation (3.188) the magnetic field in component form due to source current can be written as,

$$\vec{H}^{mn[I]\phi}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[D^{[I]r}_{mn}\left\{\frac{n(n+1)}{k_n r^2}[X_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\} +D^{[I]\theta}_{mn}\left\{\frac{[X'_n(k_n r)]^{[I]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}+D^{[I]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[X'_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\right] \quad (3.187)$$

The electric field in components from for region II can be written as,

$$\vec{E}^{mn[II]\phi}_{(r,\theta,\phi)} = E^{mn[II]\phi}_r\hat{e_r}+E^{mn[II]\phi}_\theta\hat{e_\theta}+E^{mn[II]\phi}_\phi\hat{e_\phi} \quad (3.188)$$

By substituting equations (3.141-3.143) in (3.188) fields excited by ϕ-directed current in region II can be written as,

$$\vec{E}^{mn[II]\phi}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[D^{[II]r}_{mn}\left\{\frac{n(n+1)}{k_n r^2}[W_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_r}+ D^{[II]\theta}_{mn}\left\{\frac{[W_n'(k_n r)]^{[II]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}\hat{e_\theta}+D^{[II]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[W_n'(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}\hat{e_\phi}\right] \quad (3.189)$$

Similarly, magnetic field in region II can be decomposed into components form as,

$$\vec{H}^{mn[II]\phi}_{(r,\theta,\phi)} = H^{mn[II]\phi}_r\hat{e_r}+H^{mn[II]\phi}_\theta\hat{e_\theta}+H^{mn[II]\phi}_\phi\hat{e_\phi} \quad (3.190)$$

Using equations (3.145-3.147) in equation (3.190) the magnetic field in component form due to source current are like as,

$$\vec{H}^{mn[II]\phi}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Bigg[D^{[II]r}_{mn}\left\{\frac{n(n+1)}{k_n r^2}[X_n(k_n r)]^{[II]}Y_{mn}(\theta,\phi)\right\}$$

$$+D^{[II]\theta}_{mn}\left\{\frac{[X'_n(k_n r)]^{[I]}}{r}\frac{\partial}{\partial\theta}Y_{mn}(\theta,\phi)\right\}+D^{[II]\phi}_{mn}\left\{\frac{im}{r\sin\theta}[X'_n(k_n r)]^{[I]}Y_{mn}(\theta,\phi)\right\}\Bigg] \quad (3.191)$$

where as $D^{[II]r}_{mn}$, $D^{[II]\phi}_{mn}$ and $D^{[II]\theta}_{mn}$ are unknown coefficients which represent the model amplitudes of radial, θ, ϕ components of electric and magnetic field. Now using the given boundary conditions given in equations (3.149) and (3.150), the unknown amplitudes can be obtained as,

$$D^{[I]r}_{mn} = D^{[II]r}_{mn} = D^{[I]\theta}_{mn} = D^{[II]\theta}_{mn} = 0 \quad (3.192)$$

$$D^{[I]\phi}_{mn} = \left[\frac{r\sin\theta e^{in(r-r')}Y^{*}_{mn}(\theta',\phi')[X'_n(k_n r)]^{[II]}}{2\pi im[Z'_n(k_n r)]}\right]J_{o\phi} \quad (3.193)$$

$$D^{[II]\phi}_{mn} = \left[\frac{r\sin\theta e^{in(r-r')}Y^{*}_{mn}(\theta',\phi')[X'_n(k_n r)]^{[I]}}{2\pi im[Z'_n(k_n r)]}\right]J_{o\phi} \quad (3.194)$$

Where as,

$$Z'_n(k_n r) = [W'_n(k_n r)]^{[I]}[X'_n(k_n r)]^{[II]} - [W'_n(k_n r)]^{[II]}[X'_n(k_n r)]^{[I]} \quad (3.195)$$

By substituting the modal amplitudes in equations (3.185), (3.189), (3.187) and (3.191), expressions of electric and magnetic field are obtained for both the regions. The electric fields of region I and region II similarly magnetic fields of both regions are added respectively, to get total electric and magnetic fields in this excited cavity. The field expressions for TE modes due to excitation of cavity can be written as,

$$\vec{E}^{mn[I+II]\phi}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Big[-\frac{e^{in(r-r')}}{2\pi}Y_{mn}(\theta,\phi)Y^{*}_{mn}(\theta',\phi')\Big]J_{o\phi}\hat{e}_{\phi} \quad (3.196)$$

$$\vec{H}^{mn[I+II]\phi}_{(r,\theta,\phi)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[\frac{e^{in(r-r')}Y_{mn}(\theta,\phi)Y^{*}_{mn}(\theta',\phi')[X'_n(k_n r)]^{[II]}[X'_n(k_n r)]^{[I]}}{\pi Z'_n(k_n r)}\right]J_{o\phi}\hat{e}_{\phi} \quad (3.197)$$

where as $\vec{E}^{mn[I+II]\phi}_{(r,\theta,\phi)}$ represent total electric field for both regions and $\vec{H}^{mn[I+II]\phi}_{(r,\theta,\phi)}$ is total magnetic field for the expression of TEM modes. The current $J_{o\phi}\hat{e}_{\phi}$ is source of excitation in cavity which generate TE modes.

3.5 Summary and Discussion

In this chapter, excitation of modes due to delta current source are explained in Earth ionosphere cavity. The cavity is divided into two regions I and II for the simplification of the solution. The Green functions for both the regions are assumed with some changes, from the general formulation used in the earlier chapter. Due to delta current source placement, the electric and magnetic fields have components in r, θ and ϕ direction. When the current source is placed in the radial direction only TEM modes are excited, TE and TM modes vanish out. TM, TE and TEM modes are excited when the current source is tangential, placed in θ or ϕ direction.

Chapter 4

Results

In this chapter the electric and magnetic energy density of unexcited and excited Earth ionosphere cavity is discussed. In case of unexcited cavity, the expression for TM and TE modes have been derived in Chapter 2. The description of electromagnetic field in this cavity is quite complex that is why the electric and magnetic energy density of TM and TE modes have been graphically presented. In case of excited cavity, a point source carrying current $\vec{J_o}$ is placed in this cavity. The current $\vec{J_o}$ is a vector due to which the electromagnetic field components are in radial, θ and ϕ directions. In Chapter 3, TM, TE and TEM modes expressions for excited cavity are derived. The excitation of cavity is due to current $\vec{J_o}$, therefore the energy density for TM, TE and TEM modes have been plotted for three different cases. The electric and magnetic energy density for each case is calculated using [27]. The electric energy density u_E, magnetic energy density u_M and total energy density u may be calculated as,

$$u_E = \frac{1}{\varepsilon_o}\frac{|E_{mn}|^2}{2} \tag{4.1}$$

$$u_M = \frac{1}{\mu_o}\frac{|H_{mn}|^2}{2} \tag{4.2}$$

$$u = u_E + u_H \tag{4.3}$$

In both excited and unexcited cavity, the all expressions of electromagnetic modes have radial, θ and ϕ parameters dependency. In simulation set up, the ϕ dependency is assumed fixed because the sphere is rotationally symmetric. The θ-dependency of modes varies on both m and n while radial dependency is only on n. The electromagnetic modes for high frequency are obtained using $n \geq 1$ and $m \geq 1$. The number of resonant frequencies increase with the size of cavity. The electric and magnetic energy density of TM and TE modes in component form are separately graphically displayed for each case in this cavity, this means that the electric energy density of E_r^{mn}, E_θ^{mn} and E_ϕ^{mn} while magnetic energy density of H_r^{mn}, H_θ^{mn} and H_ϕ^{mn} have been shown. The total magnetic and electric energy density is sum of all these energy densities according to the given equation,

$$\vec{E}.\vec{E}^* = |E_r|^2 + |E_\theta|^2 + |E_\phi|^2 \tag{4.4}$$

$$\vec{H}.\vec{H}^* = |H_r|^2 + |H_\theta|^2 + |H_\phi|^2 \tag{4.5}$$

4.1 Energy Density of Unexcited Cavity

In unexcited cavity, the magnetic energy density of H_θ and H_ϕ field components for TM_{11} mode is shown in figure (4.1) and (4.2) as a function of radius r and polar angle θ respectively. The total magnetic energy density of TM_{11} mode is graphically presented in figure (4.1) and (4.2) using equation (4.5).

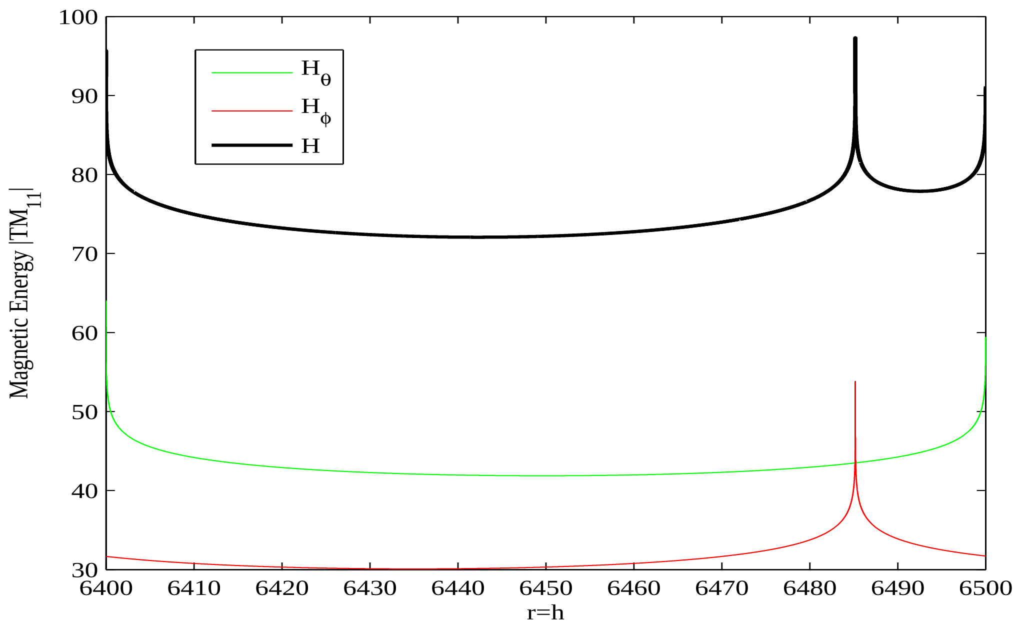

Figure 4.1 Magnetic Energy Density of TM_{11}, for $n = 1$, $m = 1$ and $\phi = 360^o$ with $r = h$ at $x-$axis

In figure (4.1) , the energy density due H_ϕ field has a peak in its energy density level while due to H_θ field show little variation in energy density level. Similarly, it is observed in figure

(4.2) that magnetic energy density due to H_θ field has a little variation while due to H_ϕ field component, it has maximum variation in energy density verses polar angle θ at $x-$axis. A single peak in energy density due to H_ϕ is also observed clearly in the plot of total magnetic energy density. The peak value of energy density is important while rest of the variation in energy density level is assumed as a noise. The peak value of magnetic energy density is depend upon the number of resonant frequency.

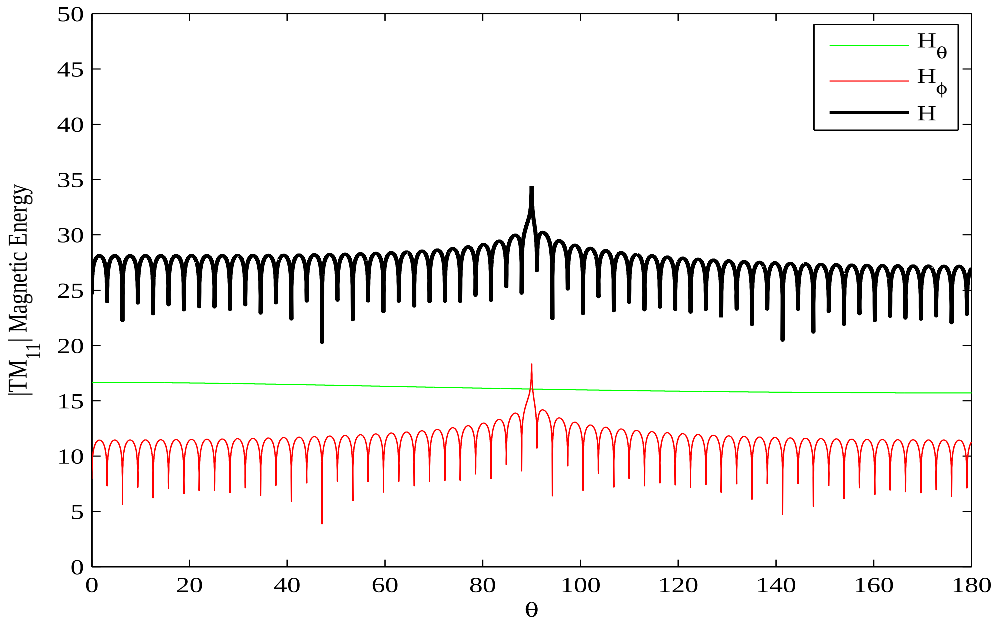

Figure 4.2 Magnetic Energy Density of TM_{11}, for $n=1$, $m=1$ and $\phi=360^o$ with θ at $x-$axis

The graphical representation of electric energy density for TM_{11} mode is shown in figures (4.3) and (4.4) verses radial distance r and polar angle θ at $x-$axis respectively. The azimuthal angle ϕ is assumed fixed at 360°. In both these plots the electric energy density due E_r, E_θ and E_ϕ fields for TM_{11} are shown distinctly. The plot of electric energy density due to E_θ field has slightly little variation verses radial distance and polar angle θ. The energy density due to E_ϕ has maximum variation verses θ and also has energy density peak value while verses radial distance r contain only a single peak of electric energy density. The electric energy density due to E_r and E_ϕ field verses radial distance r, has also energy density peak while E_θ field has a little downward change in its level of energy density. The total electric energy density is also shown in these figures, which is the sum of E_r, E_θ and E_ϕ according to the equation (4.4).

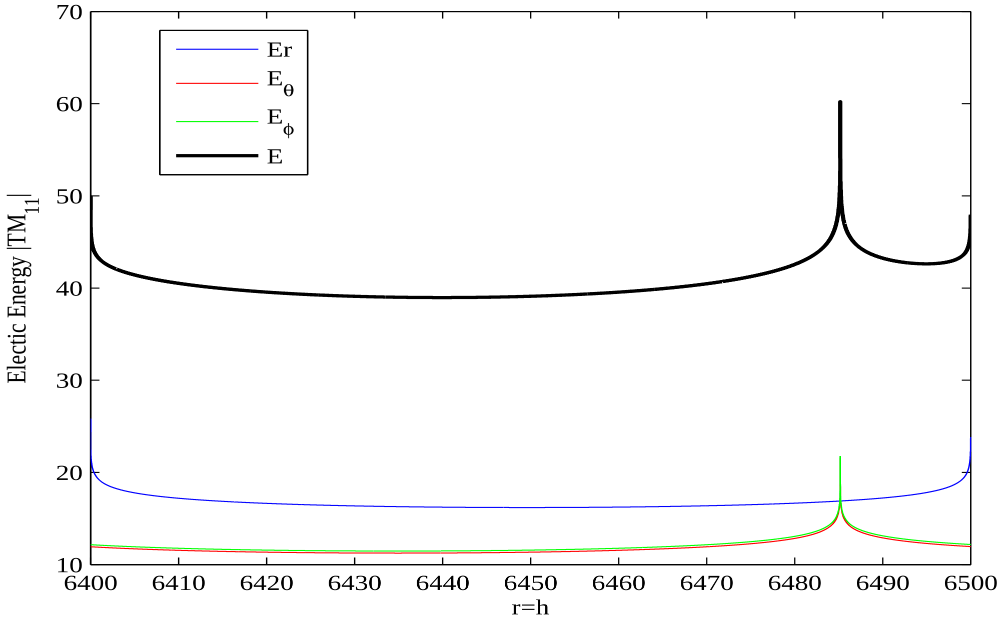

Figure 4.3 Electric Energy Density of TM_{11}, for $n = 1$, $m = 1$ and $\phi = 360^o$ with $r = h$ at $x -$ axis

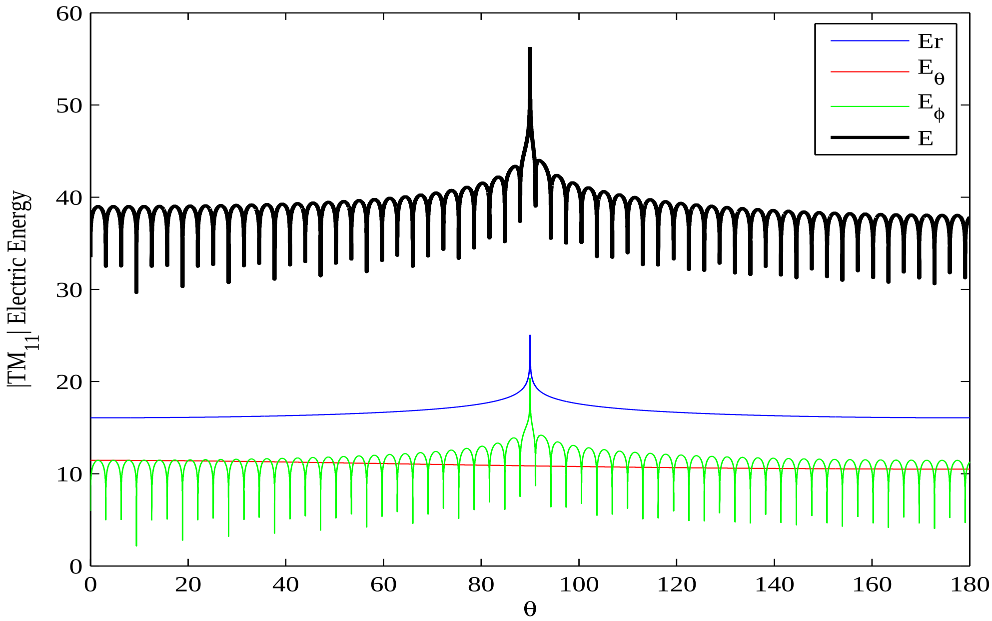

Figure 4.4 Electric Energy Density of TM_{11}, for $n = 1$, $m = 1$ and $\phi = 360^o$ with θ at $x -$ axis

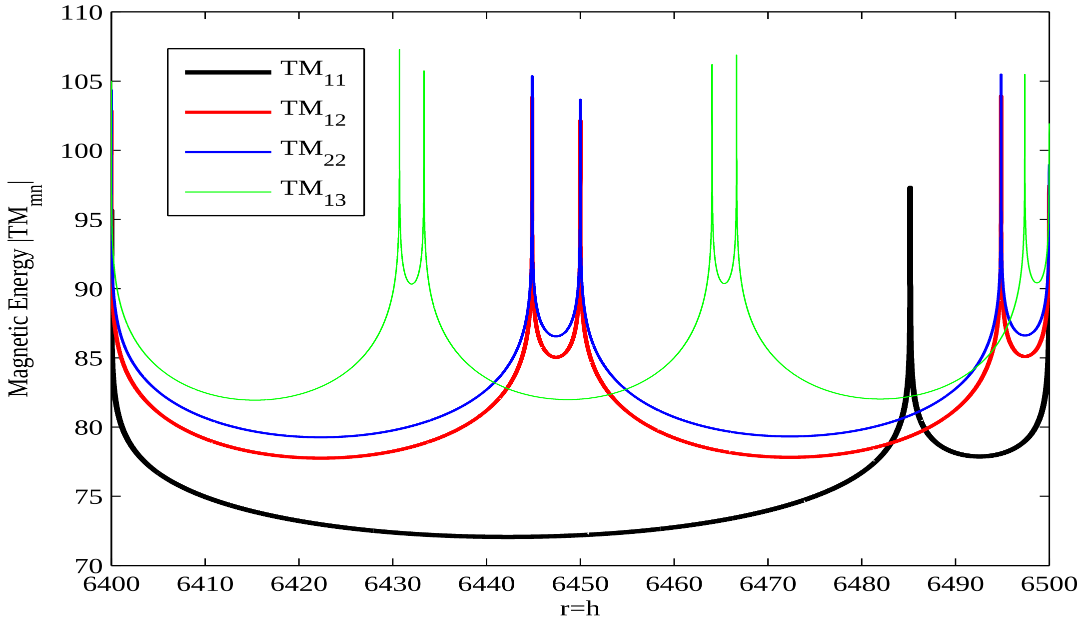

Figure 4.5 Comparison of Magnetic Energy Density of TM_{mn} for different m, n at $\phi = 360^{o}$ with $r = h$ at $x-$axis

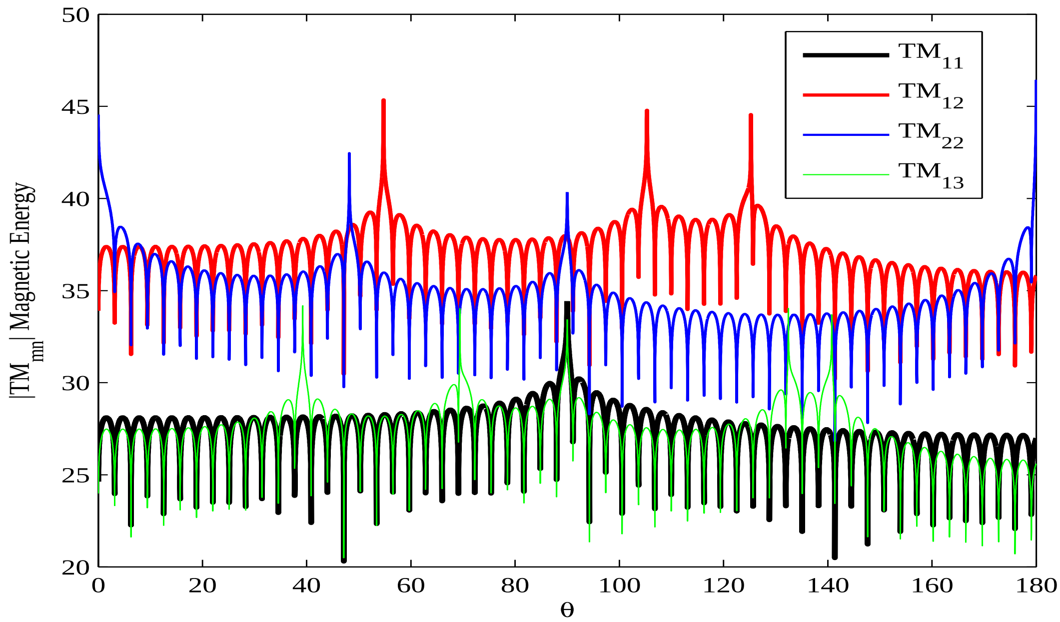

Figure 4.6 Comparison of Magnetic Energy Density of TM_{mn} for different m, n at $\phi = 360^{o}$ with θ at $x-$axis

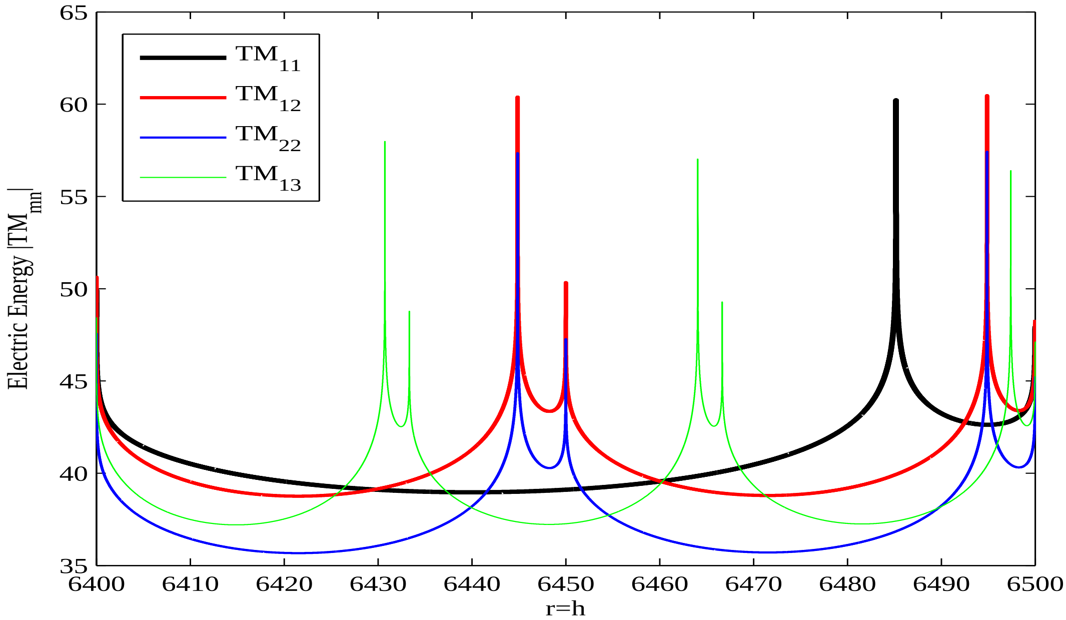

Figure 4.7 Comparison of Electric Energy Density of TM_{mn} for different m, n at $\phi = 360^o$ with $r = h$ at $x-$axis

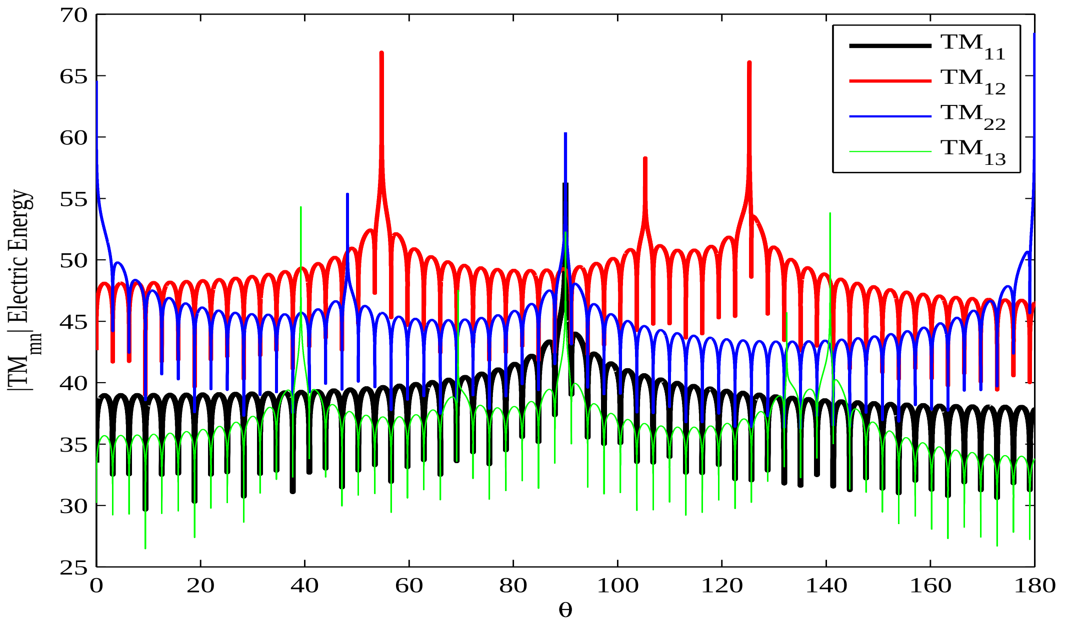

Figure 4.8 Comparison of Electric Energy Density of TM_{mn} for different m, n at $\phi = 360^o$ with θ at $x-$axis

In unexcited cavity, the total electric and magnetic energy densities are distributed between energy densities for TM_{mn} and TE_{mn} modes for different values of m and n. Therefore, electric and magnetic energy densities for TE_{mn} modes are graphically presented using the same simulation set up, like TM_{mn}. In figure (4.9) and (4.10), the magnetic energy density for TE_{11} mode is plotted verses radial distance r and polar angle θ. Similarly, the electric energy density for TE_{11} mode is shown verses radial distance r and polar angle θ in figure (4.11) and (4.12). The electric energy density for TE_{11} mode has resemblance with magnetic energy density for TM_{11} while electric energy density for TM_{11} mode has correspondence with the magnetic energy density for TE_{11} mode. The peak magnetic energy density for TE_{11} is comparable with the peak electric energy density for TM_{11} mode and peak magnetic energy density for TM_{11} shows resemblance with the peak electric energy density for TE_{11} mode. The results for electric and magnetic energy density for different values of m and n are also compared in figures (4.13), (4.14), (4.15) and (4.16). The number of peak values in energy density increase as resonant frequency increases with the increase in value of n. It is clear that the electric and magnetic energy densities reciprocate in each other for TE_{mn} and TM_{mn} modes respectively.

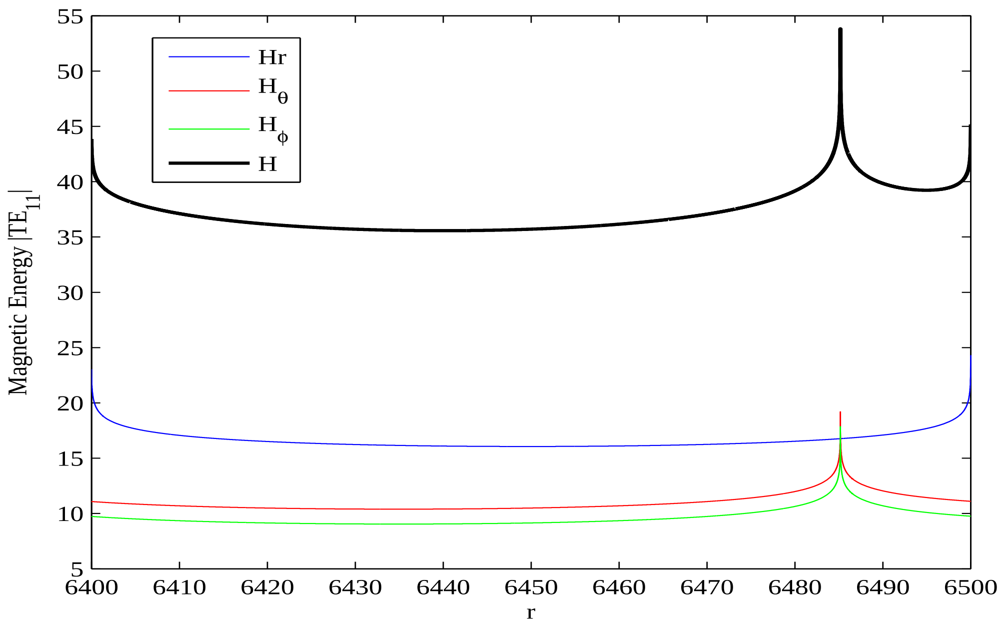

Figure 4.9 Magnetic Energy Density of TE_{11}, for $n = 1$, $m = 1$ and $\phi = 360^o$ with $r = h$ at $x-$axis

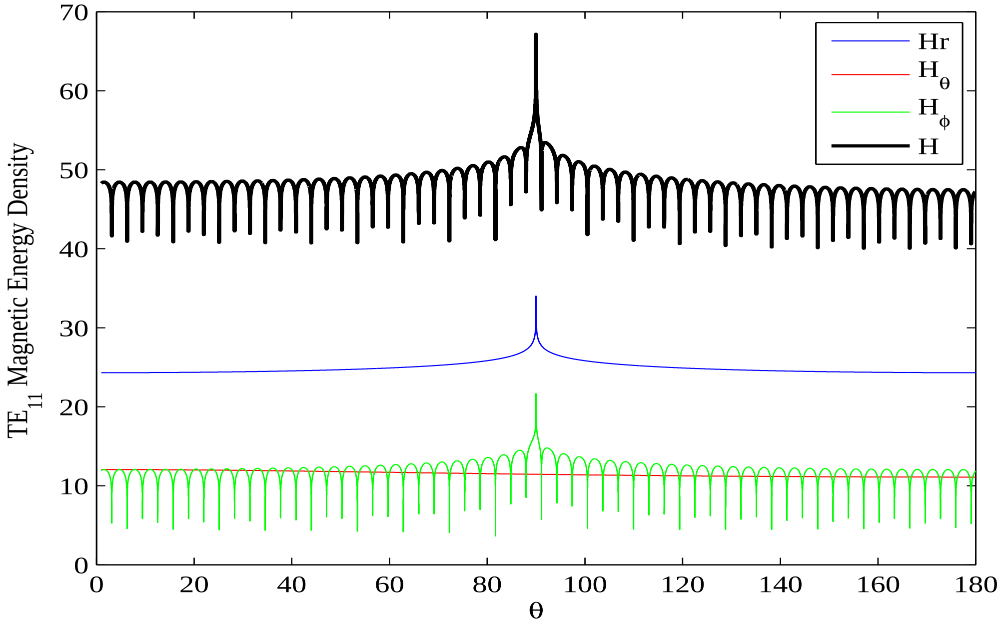

Figure 4.10 Magnetic Energy Density of TE_{11}, for $n = 1$, $m = 1$ and $\phi = 360^{o}$ with θ at $x-$axis

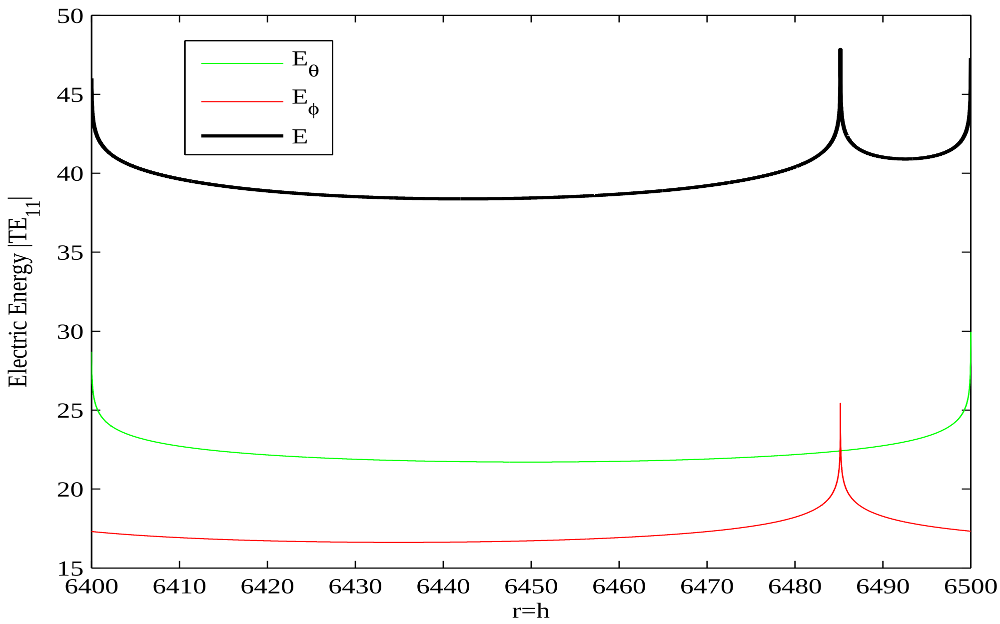

Figure 4.11 Electric Energy Density of TE_{11}, for $n = 1$, $m = 1$ and $\phi = 360^{o}$ with $r = h$ at $x-$axis

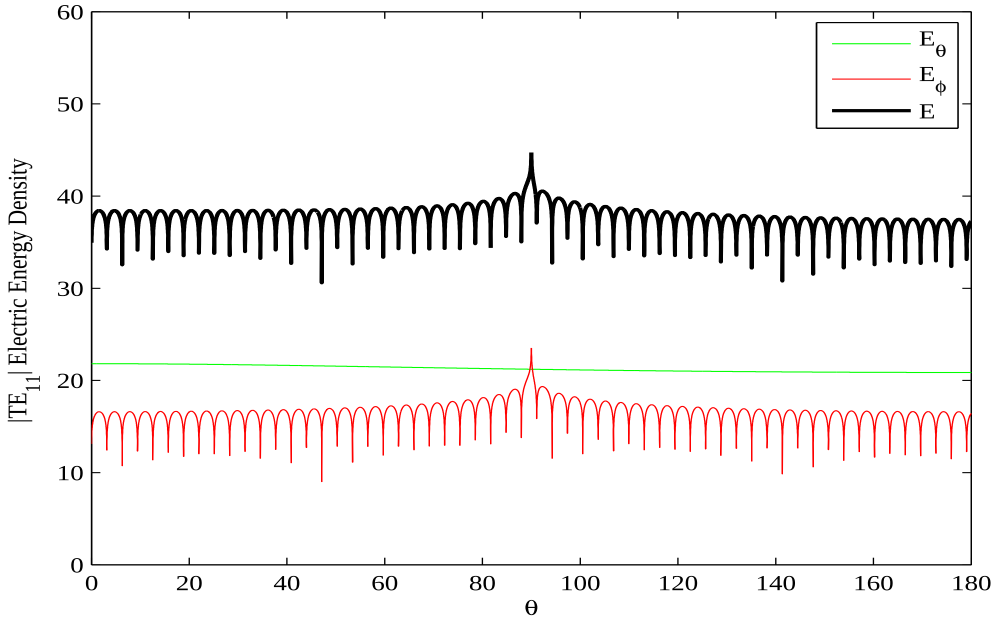

Figure 4.12 Electric Energy Density of TE_{11}, for $n = 1$, $m = 1$ and $\phi = 360^{o}$ with θ at $x-$axis

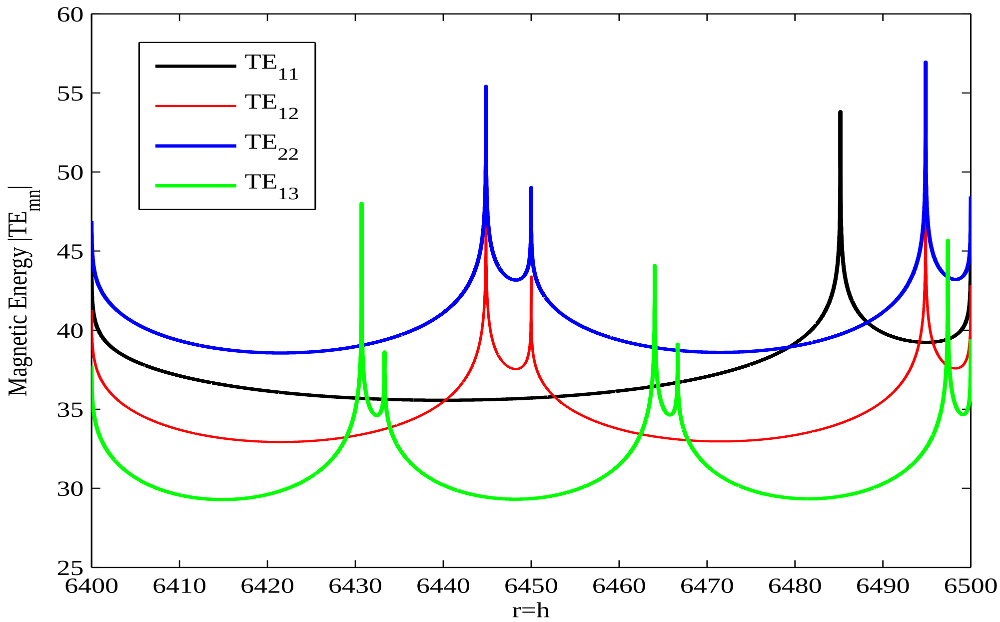

Figure 4.13 Comparison of Magnetic Energy Density of TE_{mn} for different m, n at $\phi = 360^{o}$ with $r = h$ at $x-$axis

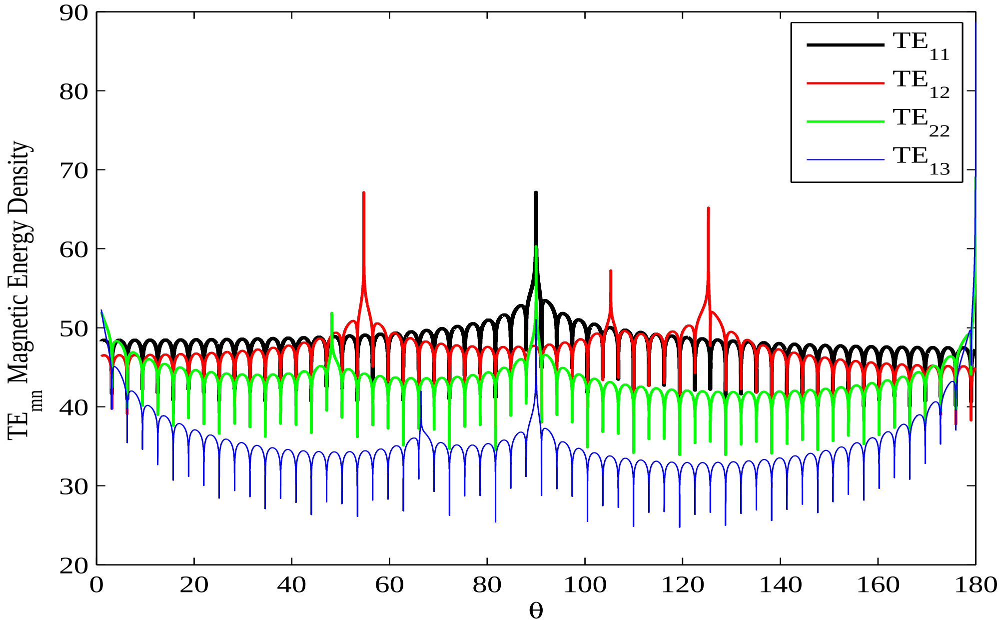

Figure 4.14 Comparison of Magnetic Energy Density of TE_{mn} for different m, n at $\phi = 360^{o}$ with θ at $x-$axis

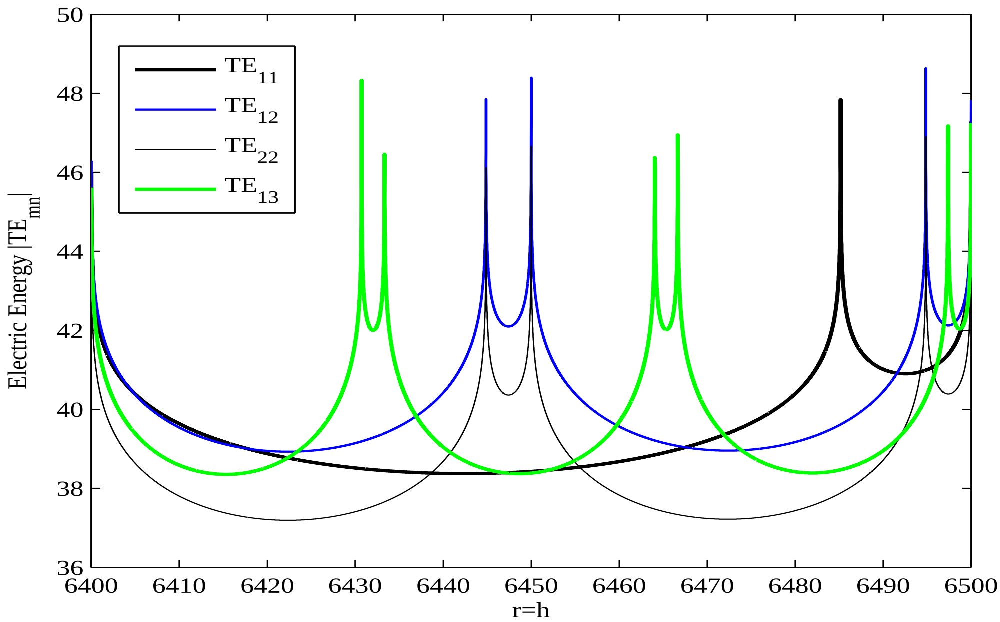

Figure 4.15 Comparison of Electric Energy Density of TE_{mn} for different m, n at $\phi = 360^{o}$ with $r = h$ at $x-$axis

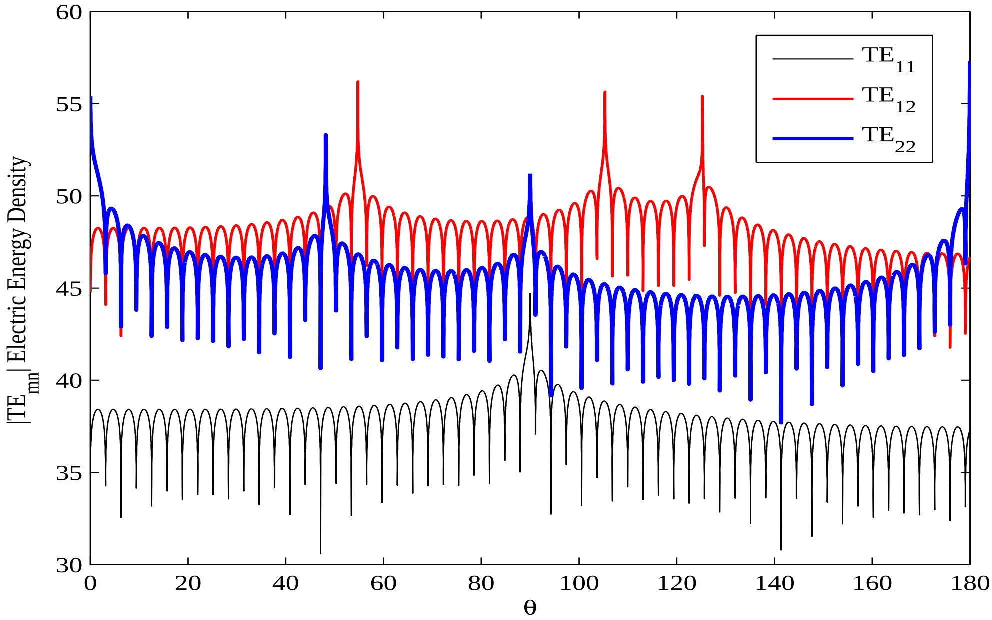

Figure 4.16 Comparison of Electric Energy Density of TE_{mn} for different m, n at $\phi = 360^{o}$ with θ at $x-$axis

4.2 Energy density of Excited Cavity

In case of excited cavity, results for the sum of electric and magnetic energy density for the TM_{mn}, TE_{mn} and TEM_{mn} modes have been presented. In Chapter 3, it is proved analytically that cavity is excited by tangential current in case of TM_{mn} and TE_{mn} modes. The comparison of total energy density due excitation of tangential current for TM^{θ}_{mn}, TM^{ϕ}_{mn}, TE^{θ}_{mn} and TE^{ϕ}_{mn} modes is shown in figures (4.17), (4.20), (4.22) and (4.25). In figure (4.17), comparison of total energy density for TM^{θ}_{11} and TM^{θ}_{22} is shown. The level and peak value of total energy density for TM^{θ}_{11} mode is greater than TM^{θ}_{22} mode but number of peaks in total energy density of TM^{θ}_{22} mode is more than TM^{θ}_{11} mode. It is due to reason that resonant frequency of TM^{θ}_{22} mode is higher than TM^{θ}_{11} mode. Similarly, same results are obtained for total energy density of TM^{ϕ}_{11} mode and TM^{ϕ}_{22}, shown in figure (4.20). In figure (4.18) and (4.19), pattern of energy density is shown for TM^{θ}_{11} and TM^{θ}_{22} modes, respectively. The figure (4.18) shows that energy density repeats its pattern after every 2km verses radial height r. Similarly, energy density pattern for TM^{θ}_{22} mode repeat after every 1km verses radial height r. The figure (4.21) showed that energy density pattern for TM^{ϕ}_{11} and TM^{ϕ}_{22} modes is also repeated like TM^{θ}_{11} and TM^{θ}_{22} verses radial height r. These electromagnetic modes depend on the radial, θ and ϕ components of field expressions.

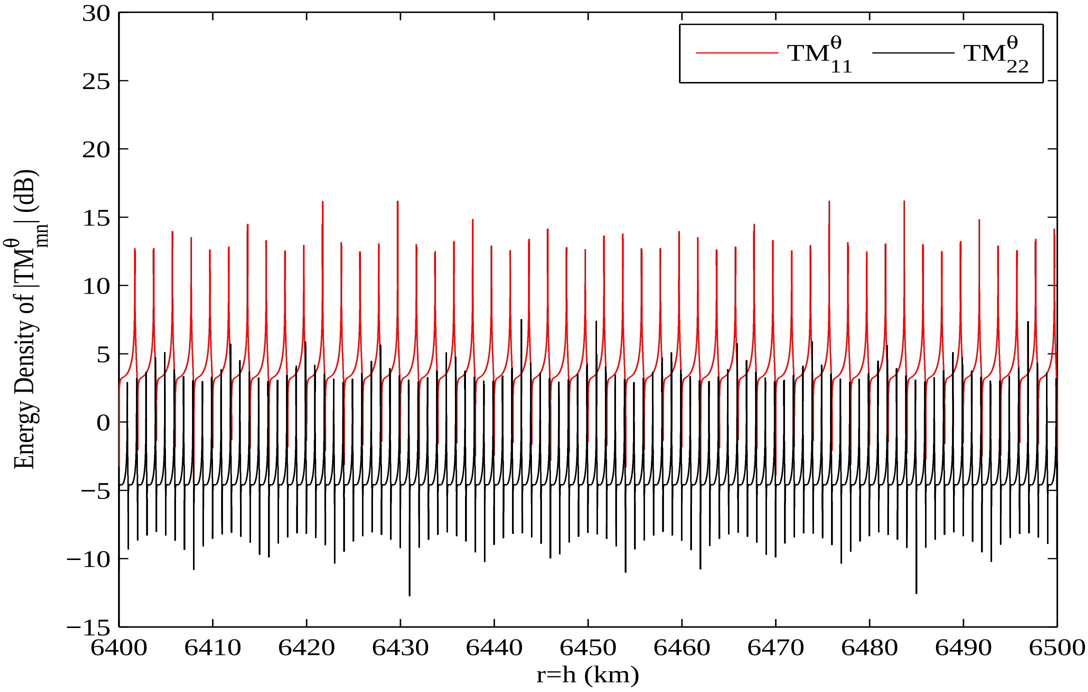

Figure 4.17 Comparison of Total Energy Density due to point source of TM^{θ}_{mn} mode, excitation in θ with $r = h$ at $x-$axis

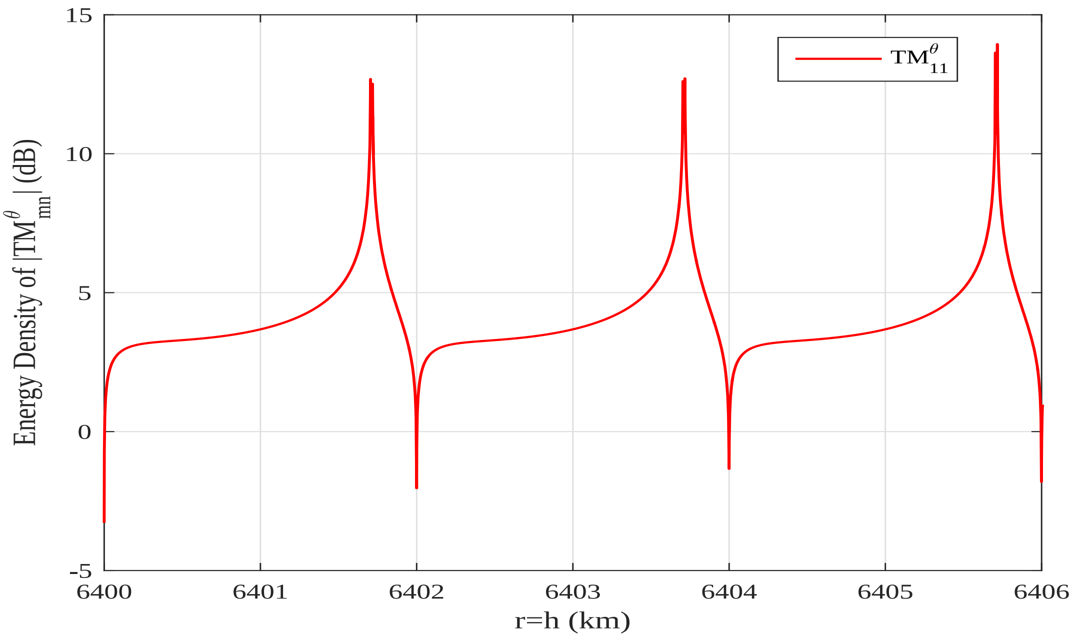

Figure 4.18 Total Energy Density Pattern for TM^{θ}_{11} mode, excitation in θ with $r = h$ at $x-$axis

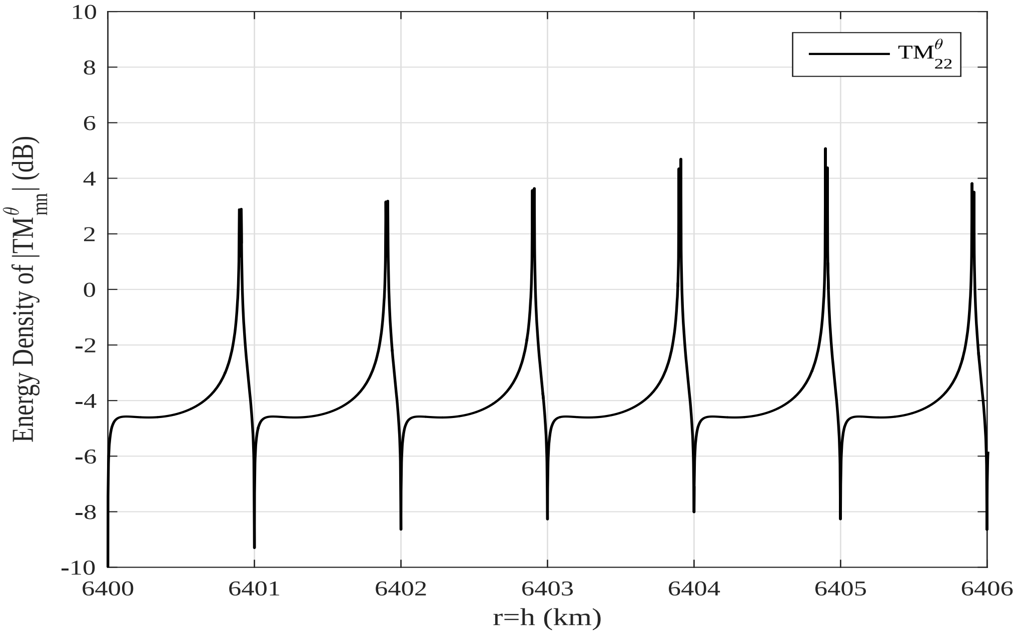

Figure 4.19 Total Energy Density pattern for TM^{θ}_{22} mode, excitation in θ with $r = h$ at $x-$axis

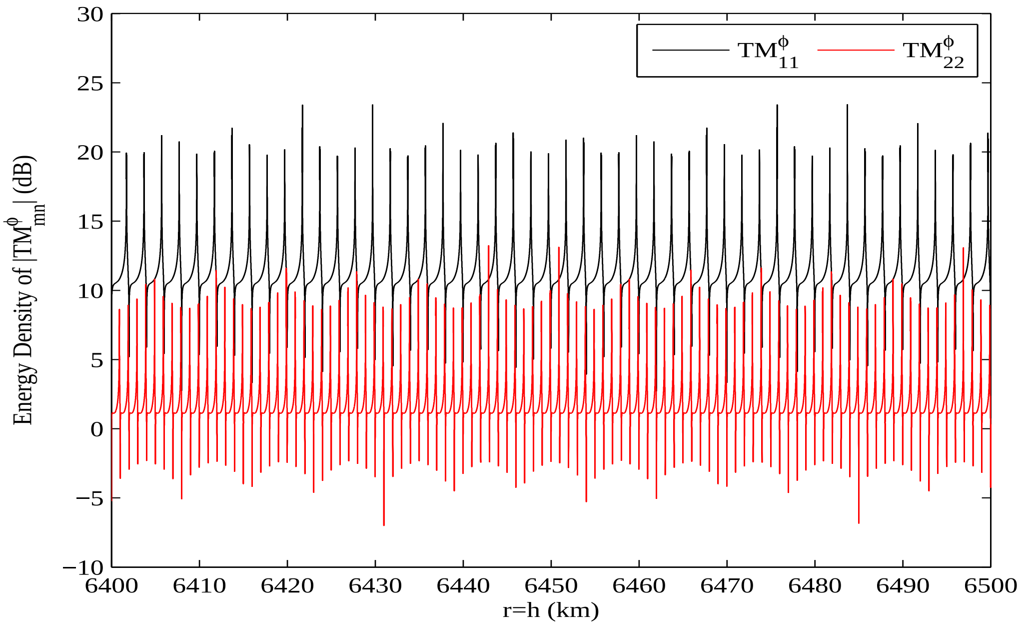

Figure 4.20 Comparison of Total Energy Density due to point source of TM^{ϕ}_{mn} mode, excitation in ϕ with $r = h$ at $x-$axis

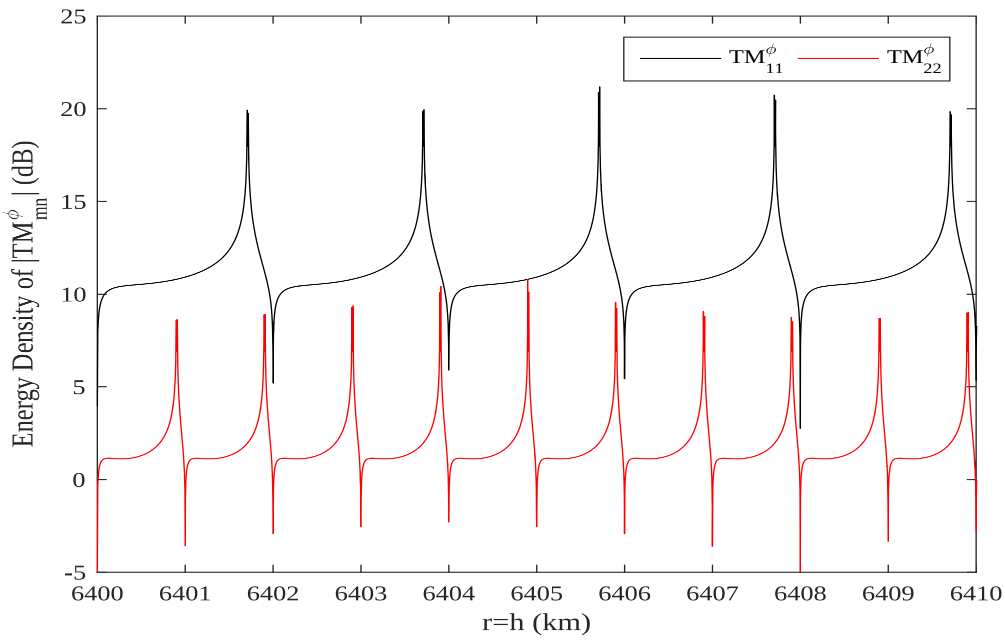

Figure 4.21 Comparison of Total Energy Density pattern between TM^{ϕ}_{11} and TM^{ϕ}_{22} mode, excitation in ϕ with $r = h$ at $x-$axis

In figure (4.22), the total energy density for TE^{θ}_{11} and TE^{θ}_{22} mode is presented. The level of total energy density decreases as the frequency increases due to increase in value of n. The reason for decrease of total energy density is that the energy density distribution occurs in most numbers of peaks for TE^{θ}_{22}. The figures (4.23) and (4.24)shows the pattern of electric energy density for TE^{θ}_{11} and TE^{θ}_{22} mode respectively, which show repetition like TM^{θ}_{mn} modes. Similarly in figure (4.25), the total energy density for TE^{ϕ}_{11} and TE^{ϕ}_{22} mode show the same variation like TE^{θ}_{mn}, with the increase in resonant frequency. The energy density pattern for TE^{ϕ}_{11} and TE^{ϕ}_{22} mode is presented in figure (4.26). The radial current didn't excite the electromagnetic field in cavity that is why the TM^{r}_{mn} and TE^{r}_{mn} mode have not any contribution in total electric and magnetic energy density.

In TEM case, the electromagnetic modes are excited by radial current and tangential current. The figure (4.27) shows the total energy density for TEM^{r}_{11} and TEM^{r}_{12}. The change in energy density for TEM^{r}_{12} is same like TE_{mn} and TM_{mn} modes. The main difference is that, in TEM^{r}_{mn} mode the total energy density level increases as the frequency increases while TE_{mn} and TM_{mn} modes it decreases. The sum of electric and magnetic energy density of TEM^{θ}_{mn} and TEM^{ϕ}_{mn} for different value of m and n can be graphically presented using same simulation set up used for TEM^{r}_{mn}.

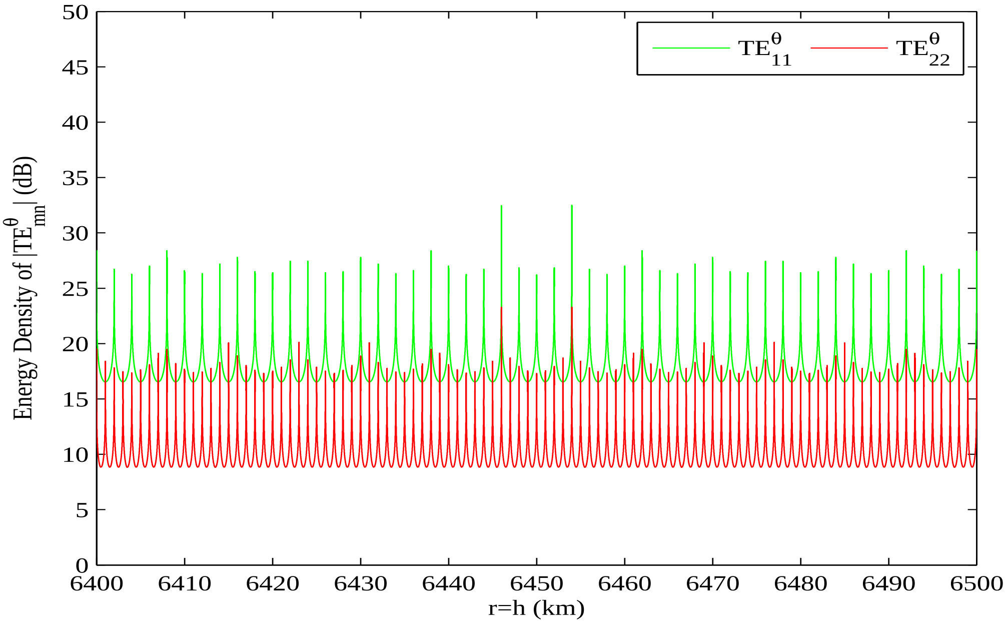

Figure 4.22 Comparison of Total Energy Density due to point source of TE^{θ}_{mn} mode, excitation in θ with $r = h$ at $x-$axis

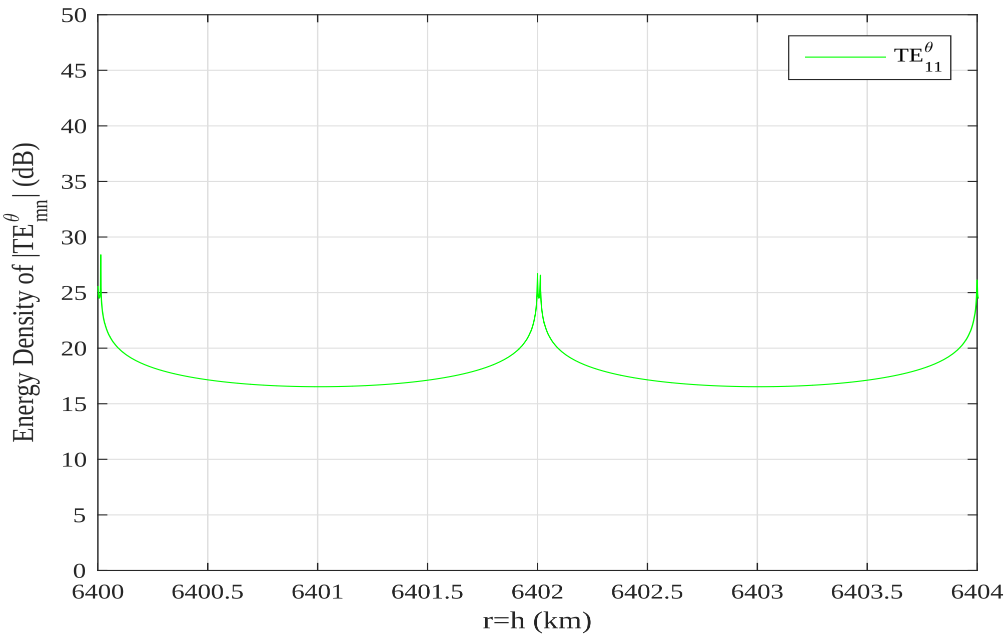

Figure 4.23 Total Energy Density pattern for TE^{θ}_{mn} mode, excitation in θ with $r = h$ at $x-$axis

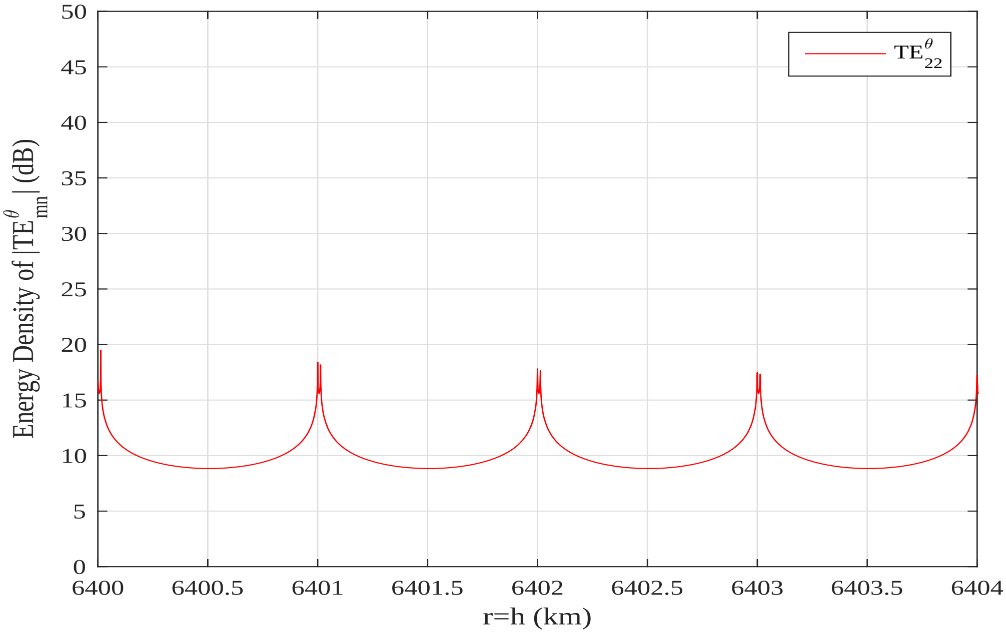

Figure 4.24 Total Energy Density pattern for TM_{mn}^{θ} mode, excitation in θ with $r = h$ at $x-$axis

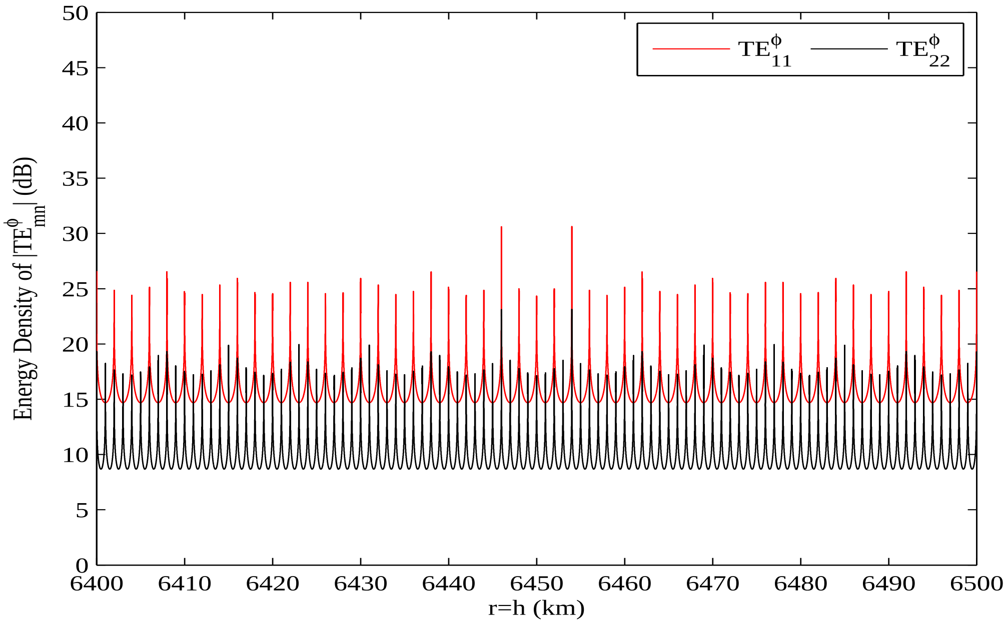

Figure 4.25 Comparison of Total Energy Density due to point source of TE_{mn}^{ϕ} mode, excitation in ϕ with $r = h$ at $x-$axis

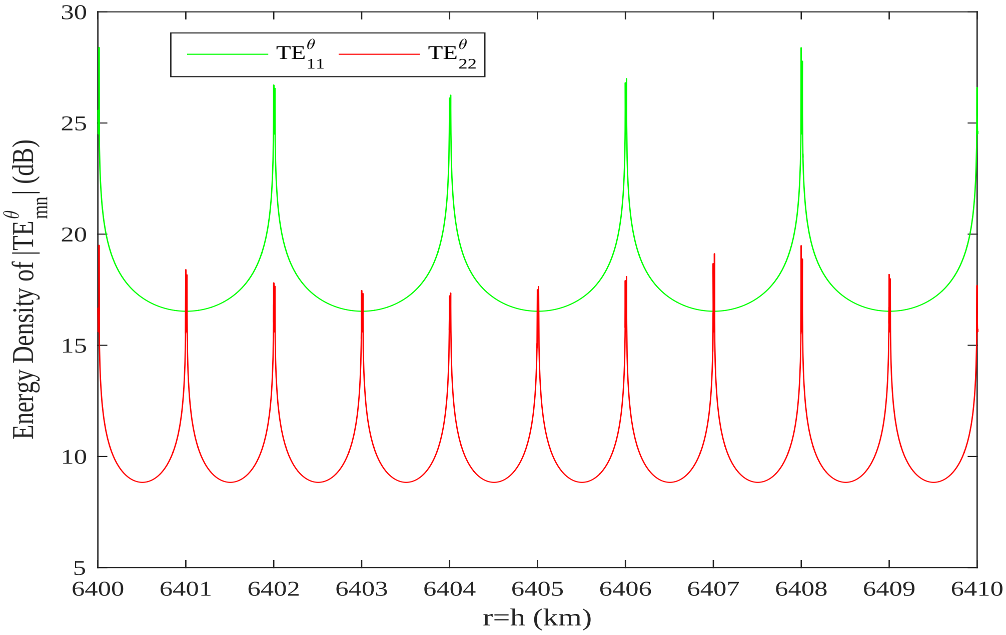

Figure 4.26 Total Energy Density pattern for TM^{ϕ}_{mn} mode, excitation in ϕ with $r = h$ at $x-$axis

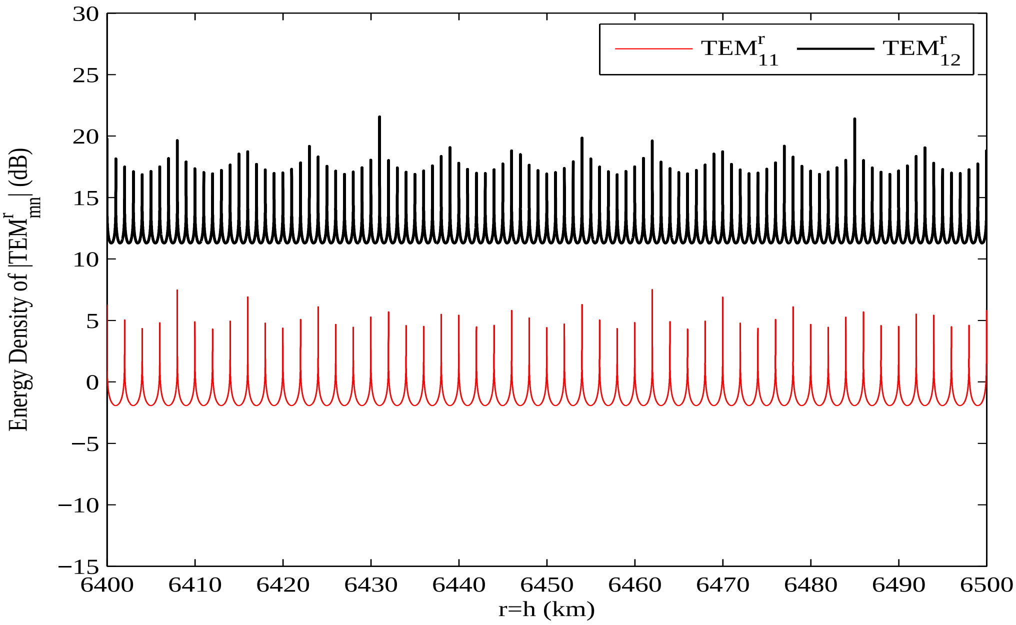

Figure 4.27 Comparison of Total Energy Density due to point source of TEM^{r}_{mn} mode, excitation in r with $r = h$ at $x-$axis

Chapter 5

Conclusion

In the first portion of this thesis, TM and TE modes in Earth ionosphere spherical cavity have been investigated. The resonant frequencies of the modes have been obtained through an approximate technique. These resonant frequencies are found to be very high as compared with Shummann frequencies in Earth ionosphere cavity. These frequencies span a range from VLF to HF, i.e., from 1.5kHz to 10kHz. The graphical representation of electric and magnetic energy densities for TM and TE modes have shown a decrease in energy densities with an increase in resonant frequency.

In the second portion, the TM, TE and TEM modes have been explored due to excitation of Earth ionosphere cavity by a infinitesimal current source at some specific location. The electric and magnetic field have components in r, θ, ϕ directions due to current source. Sine the current source may be placed in three mutually orthogonal directions, response of the cavity in all three cases is vectorial in nature. Ir may be noted that response of cavity is a superposition of TM, TE and TEM modes. It is observed analytically that due to radial placement of current, $E_{(r,\theta,\phi)}$ and $H_{(r,\theta,\phi)}$ vanish out and modal excitation doesn't take place as mentioned by Kurakawa [1]. The TM and TE modes are obtained only due to excitation of Earth ionosphere cavity by tangential current, $J_{o\theta}$ in θ-direction and $J_{o\phi}$ in ϕ-direction. The TEM modes are obtained due to excitation of cavity by current source $\vec{J_o}(r,\theta,\phi)$ in all directions. It is shown graphically that the electric and magnetic energy densities for TE, TM and TEM modes increase as the frequency increases.

In future work, the cavity losses will be calculated in case of unexcited and excited cavity. The energy loss and energy storage in this cavity will be determined. Since the total energy passes between electric and magnetic fields, we may calculate it from electric and magnetic fields. The lightning phenomena can be assumed as source of excitation in this cavity instead of infinitesimal current source. As the lightning follows the random path and is a complex process that is why probability density function of its path will be calculated. This probability density function of lightning will be used as source of excitation in this cavity.

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