Bayesian Inference using Record Values from Inverted Probability Distributions

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In The Name of Allah The Most Merciful and The Most Beneficent

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Certificate

I hereby solemnly declare that the thesis entitled "Bayesian Inference using Record Values from Inverted Probability Distributions" submitted by me for the partial fulfillment of Master of Philosophy in statistics, is original work and has not been submitted concurrently or latterly to this or any other university for any other degree.

Muhammad Anwar

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Muhammad Anwar

Abstract

In real life the record values arise in a natural way, for example, records in sports, weather, temperatures, economic, etc. It is thus important to model those records with the help of probability distributions for prediction purposes. In this thesis, the parameters of the inverted Topp-Leone and inverted Gompertz distributions are estimated using upper and lower record values, respectively, with the help of maximum likelihood, Bayes and empirical Bayes methods of estimation.The Bayes and empirical Bayes estimators are developed under both symmetric and asymmetric loss functions using informative and non-informative priors. For the empirical Bayes estimators, the hyperparameters are estimated using the methods of moments and maximum likelihood. Moreover, the prediction intervals for the future upper and lower record values are also derived.

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Chapter 1

Introduction

In our daily life, record values appear in a natural way. For instance, the hottest day of the month, the longest wining strip in Olympics games, the strongest level of earthquake, and lowest stock market figure are dealing with upper or lower record values. Every day, we wait for such record breaking performances in the newspaper, Internet, TV announcements, etc. The record value or record statistic is the smallest or largest value obtained from a sequence of random variables. An enormous amount of literature on the record values is available, because in some real life situations we have to deal with the datasets that comprise only record values. For example, in sports, the record-breaking performances are only noticed.

The topic of records has become very popular among the specialists in probability and statistics. Chandler (1952) was the first one to introduce a model for successive extremes (upper record values) arising from a sequence of independent and identically distributed random variables. In a very short time, after Chandler (1952), many researchers contributed to the theory of record values, which now has got broad applications in different fields, like medicines, traffic, stock prices, science, engineering, climatology, sports, industry, etc (cf., Balakrishnan et al., 1992; Ahsanullah, 1995; Arnold et al., 1998; Balakrishnan and Chan, 1998). Wilks (1959) posed a question that how much observations are required to obtain an extreme value which exceeds the maximum of n observations, where n is a positive integer. Dziubdziela and Kopocinski (1976) introduced the concept of k-record times, where k is a positive integer. Carlin and Gelfand (1993) analyzed the record breaking data sets, where only the exceeding observations or falling below observations from the current extreme values are recorded. The likelihood function based on the record values was derived by Arnold et al.

(1998).

In the existing literature, many researchers have shown keen interest in developing the statistical inference when the records follow a certain probability distribution. In this regards, one may refer to Kamps (1992) who considered a generalized class of distributions for modeling the record values, including the power function, Pareto, Weibull, Burr-XII and logistic distributions. For some important related works and extensive discussion on the record values we refer to Nagaraja (1988); Balakrishnan et al. (1992); Balakrishnan and Ahsanullah (1994); Franco and Ruiz (1997); Balakrishnan and Chan (1998); Sultan and Balakrishnan (1999); Raqab (2002); Jaheen (2003); Nadarajah and Ahsanullah (2004); Soliman et al. (2006); Raqab et al. (2007); Ahsanullah and Shakil (2012), and the references cited therein.

In many real life situations, the prediction of future record values is also of great interest. For example, to observe the record value until the present time, we naturally get interested in predicting the future expected record for which the present record can be broken for the first time. In the existence literature, several parametric and non-parametric procedures are available for the prediction of future record values. In many practical situations, one can construct a prediction interval for the future record value using the current available records. The prediction of future records when the observed records follow a certain probability distribution are discussed by many researchers. For example, Dunsmore (1983) studied the problem of predicting future record for two exponential models under the Bayesian framework. Awad and Raqab (2000) used first $m(m < n)$ observed record values from exponential distribution and made prediction for future nth record. For more research works, we refer to Chan (1998); Berred (1998); Mousa et al. (2002); Madi and Raqab (2004, 2007); Raqab and Balakrishnan (2008); Sultan et al. (2008); Raqab (2009); Soliman et al. (2010); Salehi et al. (2015), and the references cited therein.

In this thesis, our objective is to consider inverted probability distributions, and then observe record values from them. Using these record values, we make statistical inferences about the unknown parameters of these probability distributions. The rest of the thesis is organized as follows:

In Chapter 2, by applying an inverse variable transformation to the Topp-Leone distributed random variable, we propose an inverted Topp-Leone distribution (ITLD). Using upper record values from ITLD, the unknown parameter is estimated using maximum likelihood, Bayes and empirical Bayes estimation methods. Moreover, prediction intervals for the future upper record values are also derived.

In Chapter 3, with a set of lower record values from inverted Gompertz distribution, the unknown parameter is estimated using the maximum likelihood, Bayes and empirical Bayes estimation methods. Moreover, prediction intervals for the future lower record values are also derived.

Finally, in Chapter 4, we provide the conclusion and future works.

Chapter 2

Bayes and empirical Bayes estimation and prediction using upper record values from inverted Topp-Leone distribution

In this chapter, with an inverse transformation on the Topp-Leone distributed random variable, an inverted Topp-Leone distribution (ITLD) is derived. Using a set of upper record values from an ITLD, the maximum likelihood, Bayes and empirical Bayes methods are used to obtain the estimators of the unknown parameter. The Bayes and empirical Bayes estimators are developed under symmetric, linear-exponential and general entropy loss functions. The Bayes estimators are derived using informative and non-informative priors. For the empirical Bayes estimators, the hyperparameters are estimated using the methods of moments and maximum likelihood. A simulation study is conducted to compare the performances of these estimators. It turns out that the Bayes estimators under the symmetric and asymmetric loss functions are superior to the maximum likelihood estimator. Moreover, Bayes and empirical Bayes prediction intervals of the future record values are obtained and discussed. Some practical examples are also given to explain the applications of the results.

2.1 Introduction

Record value or record statistics is the consecutive maxima or minima obtained from a sequence of random variables. Record values arise naturally in many real life situations. Examples include meteorology, seismology, hydrology, reliability studies and stock market analysis etc. Moreover, record values are also important to all kinds of extreme phenomenon and talents. In these situations, it should be noted that the amount of information provided by records is considerable (cf., Ahmadi and Arghami, 2001). For some of the families of distributions, they have shown that the upper record values may contain more information than the same number of independent and identically distributed (IID) random variables. The statistical study of record values started with Chandler (1952) and now has spread in different directions. Record values can be described as follows: suppose that $X_1, X_2, ...$ be a sequence of IID random variables, with a probability density function (PDF) and cumulative distribution function (CDF). By definition, X_j is an upper record value if it exceeds all the previous observations, i.e., X_j is an upper record value if $X_j > X_i$ for every $j > i$. For more details, one may refer to Arnold et al. (1998).

In the existing literature, many researchers have shown keen interest in developing the statistical inference when the records follow a certain probability distribution. In this regards, one may refer to Sultan and Balakrishnan (1999) for higher order moments of record values arising from Rayleigh and Weibull distributions. Raqab (2002) derived exact expression for single and product moments of record statistics for a three-parameter generalized exponential distribution, and also obtained recurrence relations for single and product moments of record statistics. The Bayesian and empirical Bayesian estimation of generalized exponential distribution based on lower record values have been discussed by Jaheen (2004). He also derived the Bayes and empirical Bayes prediction intervals for a future lower record values. Ahmadi and Doostparast (2006) described Bayesian estimation and prediction for some lifetime distributions, including the exponential, Pareto, Weibull and Burr type-XII. Soliman et al. (2006) estimated the unknown parameters and lifetime parameters (reliability and hazard rate functions) of the Weibull distribution using upper record values under Bayesian and non-Bayesian methods. Moreover, they also constructed predictive intervals for a future upper record value. On similar lines, Soliman and Al-Aboud (2008) used Bayesian and non-Bayesian approaches to estimate the unknown parameters of the Rayleigh distribution using upper record values. Nadar et al. (2013) provided statistical analysis of Kumaraswamy distribution when upper record values are available. The Bayesian inference of the Topp-Leone distribution (TLD) with lower record values has been provided by MirMostafaee et al. (2016). They have obtained the Bayes point and interval estimators of the unknown shape parameter, survival and hazard rate functions of the TLD.

The TLD is a unimodal continuous distribution with j-shaped density function and bathtub shaped hazard rate function. The TLD with support in the standard unit interval $(0, 1)$ has received much attention of the statisticians as an alternative to the beta distribution. In our daily life, this distribution is used to model rates, percentages and some chemical process yield data. The two parameters TLD was first introduced by Topp and Leone (1955). After a long time, the TLD is reestablished by Kotz and Van D (2004), who derived an expression for the characteristic function and introduced a bivariate generalization of TLD. Ghitany et al. (2005) and Ghitany (2007) discussed asymptotic distribution of order statistics and some reliability measures of their stochastic ordering of the TLD. Vicari et al. (2008) introduced a two-sided generalized version of TLD and also discussed the moments and maximum likelihood methods for parameter estimation. Furthermore, they provided a comparison between two-sided generalized TLD and mixture of Gaussian distributions. Al-Zahrani (2012) discussed a class of goodness-of-fit tests for the TLD with estimated parameters.

In this chapter, with an inverse transformation on the Topp-Leone distributed random variable, an inverted TLD (ITLD) with its variable support $(0, \infty)$ is derived for modeling real life datasets. The unknown parameter of the ITLD is estimated using a set of upper record values, which include the maximum likelihood estimator (MLE), Bayes and empirical Bayes estimators. These Bayes estimators are obtained under both symmetric and asymmetric loss functions, including the squared-error loss function (SELF), linear-exponential loss function (LELF), and general entropy loss function (GELF). Both informative and non-informative priors are also used to derive the Bayes and empirical Bayes highest posterior density (HPD) intervals. Moreover, explicit mathematical expressions for the Bayes and empirical Bayes prediction intervals for a future upper record value are also derived. A numerical study with simulated and real datasets is given to illustrate the applications of the results.

The rest of the chapter is organized as follows: in Section 2.2 an ITLD distribution is derived. In Section 2.3, both the classical and Bayes methods are used to derive estimators of the unknown parameter of ITLD. The prediction of future record values is considered in Section 2.4. In Sections 2.5 and 2.6, simulated and real datasets are used for numerical comparisons of the derived estimators. Finally, Section 2.7 summarizes the main findings and concludes the chapter.

2.2 Inverted Topp-Leone distribution

Let Y be a Topp-Leone distributed random variable. The PDF and the CDF of Y are

$$\begin{aligned} f^*(y|\theta) &= 2\theta(1-y)(2y-y^2)^{\theta-1}, \ 0<y<1, \theta>0, \ \text{and} \\ F^*(y|\theta) &= (2y-y^2)^{\theta}, \end{aligned}$$

respectively.

Most of the probability distributions with random variables support $(0,\infty)$ are more flexible and practical when modeling positive data than those whose random variables support is $(0,1)$. For instance, exponential, Weibull, and gamma distributions are mostly used in queuing theory, reliability theory and reliability engineering for modeling times to failure of electronic components. On the contrary, the TLD is not very flexible as $Y \in (0,1)$. In particular, it may not be used to model the life of an electronic component. Thus, we devise an ITLD with its random variable support $(0,\infty)$ that could model lifetimes of different components. If Y follows TLD, then $X=(1-Y)/Y$ has an ITLD. The PDF, CDF, hazard rate and survival function of the ITLD are, respectively, given by

$$\begin{aligned} f(x|\theta) &= \frac{2x\theta(1+2x)^{\theta-1}}{(1+x)^{2\theta+1}}, \ x>0, \ \theta>0, && (2.1) \\ F(x|\theta) &= 1-\left(\frac{1+2x}{(1+x)^2}\right)^{\theta}, && (2.2) \\ h(x|\theta) &= \frac{2x\theta}{(1+x)(1+2x)}, \ \text{and} \\ S(x|\theta) &= \left(\frac{1+2x}{(1+x)^2}\right)^{\theta}. \end{aligned}$$

The above functions are plotted in Figure 2.1. The PDF $f(x|\theta)$ is unimodal and right tailed for different values of θ. The failure/hazard rate function of the ITLD is shown to be monotonic, where it first increases and then decreases, approaching a constant value as the lifetime approaches infinite. This property is often found in case of lifetime dominated by early occurrence of an event.

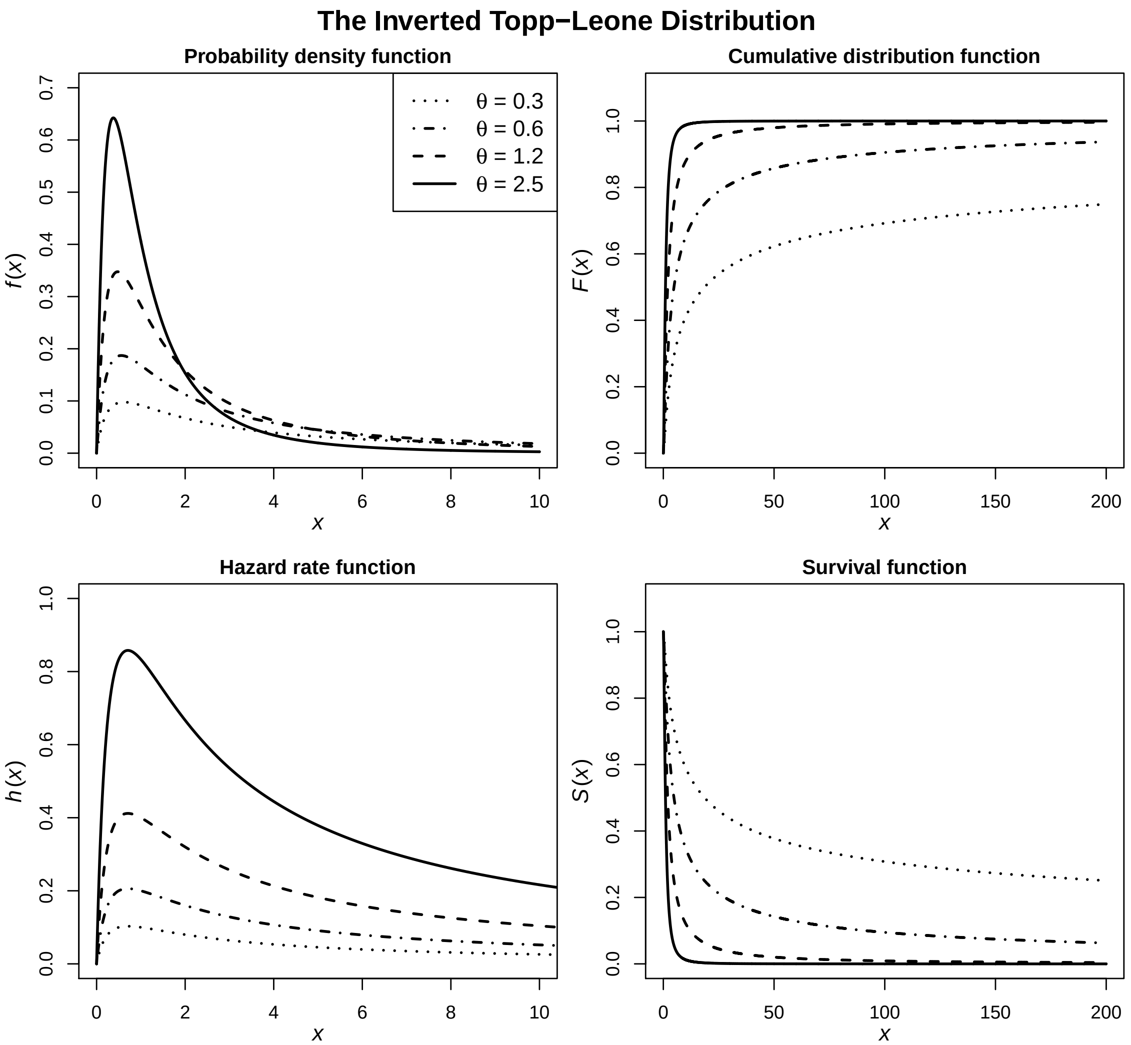

Figure 2.1: The PDF, CDF, hazard rate and survival functions of the ITLD

2.3 Estimation of the parameter

In this section, we estimate the unknown parameter θ of the ITLD using maximum likelihood, Bayes and empirical Bayes methods using k upper record values. Let $X_1 = x_1$, $X_2 = x_2$,..., $X_k = x_k$ be the k upper record values arising from a sequences of an IID random variables following an ITLD. According to Arnold et al. (1998), the likelihood function of θ given the k upper record values $\mathbf{x} = (x_1, x_2, ..., x_k)'$ is

$$L(\theta|\mathbf{x}) = f(x_k|\theta) \prod_{i=1}^{k-1} \frac{f(x_i|\theta)}{1 - F(x_i|\theta)}. \tag{2.3}$$

By substituting (2.1) and (2.2) in (2.3), we obtain

$$L(\theta|\mathbf{x}) = \theta^k \exp\left(-\theta\eta_1(x_k)\right)\eta_2(\mathbf{x}), \tag{2.4}$$

where

$$\eta_1(x_k) = -\ln\left(\frac{1+2x_k}{(1+x_k)^2}\right) \text{ and } \eta_2(\mathbf{x}) = \prod_{i=1}^{k} \frac{2x_i}{(1+x_i)(1+2x_i)}.$$

2.3.1 Maximum likelihood estimation

The natural logarithm of the likelihood function given in (2.4) is

$$\ln L(\theta|\mathbf{x}) = k\ln(\theta) - \theta\eta_1(x_k) + \ln\left(\eta_2(\mathbf{x})\right). \tag{2.5}$$

The MLE of θ can be obtained by minimizing (2.5), given by

$$\hat{\theta}_{\mathrm{ML}} = \frac{k}{\eta_1(x_k)}, \tag{2.6}$$

where $\hat{\theta}_{\mathrm{ML}}$ is the MLE of θ. Now we find the PDF of $\eta_1(x_k)$ to study the properties of $\hat{\theta}_{\mathrm{ML}}$. According to Arnold et al. (1998), the PDF of X_k is

$$\begin{aligned} f_{X_k}(x|\theta) &= \frac{1}{\Gamma(k+1)}\{-\ln(1-F(x|\theta))\}^k f(x|\theta),\ x>0 \\ &= \frac{\theta^{k+1}}{\Gamma(k+1)}(\eta_1(x_k))^k \exp\left(-\theta\eta_1(x_k)\right)\frac{2x_k}{(1+x_k)(1+2x_k)}. \end{aligned} \tag{2.7}$$

Using (2.7), the PDF of $Z = k/\eta_1(X_k)$ is

$$f_Z(z|\theta) = \frac{(k\theta)^{k+1}}{\Gamma(k+1)}\frac{1}{z^{k+2}}\exp\left(-\frac{k\theta}{z}\right),\ z>0, \tag{2.8}$$

which is an inverted gamma distribution with the shape parameter $k+1$ and the scale parameter $k\theta$. The mean and variance of $\hat{\theta}_{\mathrm{ML}}$ are

$$\begin{aligned} E(\hat{\theta}_{\mathrm{ML}}) &= \theta, \text{ and} \\ Var(\hat{\theta}_{\mathrm{ML}}) &= \frac{\theta^2}{k-1},\ k>1, \end{aligned}$$

respectively. It is clear that $\hat{\theta}_{\mathrm{ML}}$ is an unbiased estimator of θ. Moreover, as the value of k increases, $Var(\hat{\theta}_{\mathrm{ML}})$ decreases. This shows $\hat{\theta}_{\mathrm{ML}}$ is a weakly-consistent estimator of θ.

2.3.2 Bayesian estimation

In this subsection, we derive the Bayes estimators of θ using k upper record values under symmetric and asymmetric loss functions. In the Bayesian approach, θ is considered as a random variable, and it follows a prior distribution. It is a well known fact that the performance of the Bayes estimator depends on a suitable choice of prior distribution and an appropriate loss function considered for the study. Here, the posterior distributions of θ are derived using informative and non-informative priors. For the informative prior, a conjugate prior for θ, is considered and for the non-informative prior, Jeffreys and uniform priors are considered. A natural conjugate prior assumed for θ is a gamma distribution, with the PDF:

$$g_1(\theta; a, b) \quad = \quad \frac{b^a}{\Gamma(a)}\theta^{a-1}\exp(-\theta b), \;\; \theta > 0, \tag{2.9}$$

where $a > 0$ and $b > 0$ are the hyperparameters. The posterior distribution of θ is obtained by

$$g(\theta|\mathbf{x}) \quad = \quad \frac{L(\theta|\mathbf{x})g(\theta; a, b)}{\int_\Theta L(\theta|\mathbf{x})g(\theta; a, b)\,\mathrm{d}\theta}.$$

Using (2.4) and (2.9), we obtain

$$g_1(\theta|\mathbf{x}) \quad = \quad \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)}\theta^{\alpha_1-1}\exp(-\theta\beta_1), \;\; \theta > 0, \tag{2.10}$$

which is a gamma distribution with the shape parameters $\alpha_1 = a + k$ and the scale parameter $\beta_1 = b + \eta_1(x_k)$.

One of the most commonly used loss function is the SELF, given by

$$L_1(\hat{\theta}, \theta) \quad = \quad (\hat{\theta} - \theta)^2,$$

where $\hat{\theta}$ is an estimator of θ. The Bayes estimator of θ using SELF, denoted by $\hat{\theta}_{\text{BS}_1}$, is the mean of the posterior distribution, given by

$$\begin{aligned}\hat{\theta}_{\text{BS}_1} &= E_{\theta|\mathbf{x}}(\theta) \\ &= \frac{\alpha_1}{\beta_1} = \frac{a+k}{b+\eta_1(x_k)},\end{aligned}$$

where $E_{\theta|\mathbf{x}}(\cdot)$ is the mathematical expectation with respect to the posterior distribution of θ. The SELF is symmetric in nature and it gives equal weights whether there is overestimation or underestimation. Overestimation of a parameter may lead to more severe or less severe consequences than the underestimation. For example, Basu and Ebrahimi (1991) have mentioned that, when estimating the reliability function or average failure time, overestimation is more serious than underestimation. Also an underestimation of the failure rate results in more serious consequences than that with an overestimation. In such situations the use of a symmetric loss function may be inappropriate. Therefore, in order to cope up with such situations, the use of an asymmetric loss function has been suggested by many researchers, which include Varian (1975) and Zellner (1986) to name a few. A number of asymmetric loss functions are available in the literature, the most popular asymmetric loss functions among them are LELF and GELF. The LELF was first time introduced by Varian (1975) and extensively discussed by Calabria and Pulcini (1996). The LELF for estimating θ is

$$L_2(\Delta) = \exp(c\Delta) - c\Delta - 1,\ c \neq 0,$$

where $\Delta = \hat{\theta} - \theta$. The sign and magnitude of the shape parameter c shows the direction and degree of symmetry, respectively.

It is also notified that

- if $c = 1$, then $L_2(\Delta)$ becomes asymmetric about zero which leads to the overestimation,
- if $c = 0$, then $L_2(\Delta)$ becomes symmetric and not far from SELF.
- if $c < 0$, $L_2(\Delta)$ increases almost exponentially when $\Delta < 0$ and almost linearly when $\Delta > 0$.

The Bayes estimator of θ under LELF, denoted by $\hat{\theta}_{\text{BL}_1}$, is

$$\hat{\theta}_{\text{BL}_1} = -\frac{1}{c}\ln\Big(E_{\theta|\mathbf{x}}\left(\exp(-c\theta)\right)\Big) \tag{2.11}$$

provided that the posterior expectation $E_{\theta|\mathbf{x}}(\exp(-c\theta))$ exists and is finite, see Calabria and Pulcini (1996). The Bayes estimator $\hat{\theta}_{\mathrm{BL}_1}$ is obtained using (2.10) and (2.11):

$$\hat{\theta}_{\mathrm{BL}_1} = \frac{\alpha_1}{c}\ln\left(1+\frac{c}{\beta_1}\right) = \frac{a+k}{c}\ln\left(1+\frac{c}{b+\eta_1(x_k)}\right), \; c \neq 0.$$

Despite the flexibility and popularity of the LELF for the location parameter estimation, it appears to be unsuitable for scale parameter and other quantities (cf., Basu and Ebrahimi, 1991; Parsian and Sanjari F, 1993). Therefore, a suitable alternative to LELF is GELF. The GELF was proposed by Calabria and Pulcini (1996) and is given by

$$L_3(\hat{\theta},\theta) = \left(\frac{\hat{\theta}}{\theta}\right)^d - d\ln\left(\frac{\hat{\theta}}{\theta}\right) - 1, \; d \neq 0. \tag{2.12}$$

The general form in (2.12) allows different shapes of the LELF. The loss is minimum when $\theta = \hat{\theta}$. The constant d is used for overestimation or underestimation of the parameter. If $d > 0$, overestimation is more serious than underestimation, and if $d < 0$, underestimation is more serious than overestimation. The Bayes estimator of θ under GELF, denoted by $\hat{\theta}_{\mathrm{BG}_1}$, is given by

$$\begin{aligned}\hat{\theta}_{\mathrm{BG}_1} &= \left(E_{\theta|\mathbf{x}}\left(\theta^{-d}\right)\right)^{-\frac{1}{d}} \\ &= \frac{1}{\beta_1}\left(\frac{\Gamma(\alpha_1-d)}{\Gamma(\alpha_1)}\right)^{-\frac{1}{d}} = \frac{1}{b+\eta_1(x_k)}\left(\frac{\Gamma(a+k-d)}{\Gamma(a+k)}\right)^{-\frac{1}{d}}.\end{aligned} \tag{2.13}$$

It can be shown that, when $d = 1$, $\hat{\theta}_{\mathrm{BG}_1}$ coincides with the Bayes estimator under the weighted squared error loss function. Similarly, when $d = -1$, then $\hat{\theta}_{\mathrm{BG}_1}$ is equal to $\hat{\theta}_{\mathrm{BS}_1}$.

Sometimes when the life tester has no prior information about the parameter θ, then it is customary to use non-informative priors to estimate θ. The quasi prior is often used as a non-informative prior when estimating θ, given by

$$g(\theta) \propto \frac{1}{\theta^e}, \; \theta, \; e \geq 0. \tag{2.14}$$

In (2.14) with $e = 0$, the quasi prior becomes a uniform prior, say $g_2(\theta)$, given by

$$g_2(\theta) \propto 1, \; \theta > 0. \tag{2.15}$$

The posterior distribution using (2.4) and (2.15) is

$$g_2(\theta|\mathbf{x}) = \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)}\theta^{\alpha_2-1}\exp(-\theta\beta_2),\ \theta > 0, \tag{2.16}$$

which is a gamma distribution with the shape parameter $\alpha_2 = k+1$ and scale the parameter $\beta_2 = \eta_1(x_k)$.

The Bayes estimators under SELF, LELF, and GELF with the uniform prior are given by

$$\begin{aligned}
\hat{\theta}_{\mathrm{BS}_2} &= \frac{\alpha_2}{\beta_2} = \frac{k+1}{\eta_1(x_k)}, \\
\hat{\theta}_{\mathrm{BL}_2} &= \frac{\alpha_2}{c}\ln\left(1+\frac{c}{\beta_2}\right) = \frac{k+1}{c}\ln\left(1+\frac{c}{\eta_1(x_k)}\right), \text{ and} \\
\hat{\theta}_{\mathrm{BG}_2} &= \frac{1}{\beta_2}\left(\frac{\Gamma(\alpha_2-d)}{\Gamma(\alpha_2)}\right)^{-\frac{1}{d}} = \frac{1}{\eta_1(x_k)}\left(\frac{\Gamma(k+1-d)}{\Gamma(k+1)}\right)^{-\frac{1}{d}},
\end{aligned}$$

respectively.

Similarly, in (2.14) with $e = 1$, the quasi prior becomes the Jeffreys prior, say $g_3(\theta)$, given by

$$g_3(\theta) \propto \frac{1}{\theta},\ \theta > 0. \tag{2.17}$$

The posterior distribution using (2.4) and (2.17) is

$$g_3(\theta|\mathbf{x}) = \frac{\beta_3^{\alpha_3}}{\Gamma(\alpha_3)}\theta^{\alpha_3-1}\exp(-\theta\beta_3),\ \theta > 0, \tag{2.18}$$

which is a gamma distribution with the shape parameter $\alpha_3 = k$ and the scale parameter $\beta_3 = \eta_1(x_k)$.

Bayes estimators under SELF, LELF, and GELF with the Jeffreys prior are given by

$$\begin{aligned}
\hat{\theta}_{\mathrm{BS}_3} &= \frac{\alpha_3}{\beta_3} = \frac{k}{\eta_1(x_k)}, \\
\hat{\theta}_{\mathrm{BL}_3} &= \frac{\alpha_3}{c}\ln\left(1+\frac{c}{\beta_3}\right) = \frac{k}{c}\ln\left(1+\frac{c}{\eta_1(x_k)}\right), \text{ and} \\
\hat{\theta}_{\mathrm{BG}_3} &= \frac{1}{\beta_3}\left(\frac{\Gamma(\alpha_3-d)}{\Gamma(\alpha_3)}\right)^{-\frac{1}{d}} = \frac{1}{\eta_1(x_k)}\left(\frac{\Gamma(k-d)}{\Gamma(k)}\right)^{-\frac{1}{d}},
\end{aligned}$$

respectively.

2.3.3 Empirical Bayes estimation

The choice of hyperparameters for the prior distribution is quite sensitive in the Bayesian inference. In usual practice, it may be possible that the prior information is not available. Therefore, an alternative approach used to avoid this issue is the empirical Bayes estimation. When the hyperparameters a and b are unknown, we may use an empirical Bayes method to estimate them. The hyperparameters are usually estimated by the method of maximum likelihood or method of moments, see Carlin and Louis (1997). In this chapter, both these estimation methods are used to estimate the hyperparameters a and b.

1. Maximum likelihood estimation

Suppose that m past upper records samples, each of size k, are available, denoted by $\mathbf{X}_j$ with the past realization θ_j where $j = 1, 2, 3, ..., m$. Then the MLE of θ_j $(1 \leq j \leq m)$ is

$$\hat{\theta}_{\mathrm{ML,j}} = \frac{k}{\eta_1(X_{kj})} = z_j \text{ (say)}. \tag{2.19}$$

Using (2.8), the conditional distribution of Z_j given θ_j is

$$f(z_j|\theta_j) = \frac{(k\theta_j)^{k+1}}{\Gamma(k+1)} \frac{1}{z_j^{k+2}} \exp\left(-\frac{k\theta_j}{z_j}\right), \; z_j > 0. \tag{2.20}$$

According to Schafer and Feduccia (1972), the marginal PDF of Z_j is obtained by solving

$$\begin{aligned} f_{Z_j}(z_j) &= \int_{\Theta} f_{Z_j|\theta_j}(z_j|\theta_j) g_1(\theta_j; a, b)\, \mathrm{d}\theta_j \\ &= \frac{k^{k+1} b^a}{\mathrm{Beta}(k+1, a)} \frac{z_j^{a-1}}{(k + b z_j)^{a+k+1}}, \end{aligned} \tag{2.21}$$

which is a PDF of 3-parameter inverted beta distribution, denoted by inverted-Beta $(k+1, a, k/b)$. Here, $\mathrm{Beta}(\cdot, \cdot)$ is the usual beta function. The likelihood function of using (2.21) is

$$\begin{aligned} L(a, b|\mathbf{z}) &= \prod_{j=1}^{m} f_{Z_j}(z_j) \\ &= \left(\frac{k^{k+1} b^a}{\mathrm{Beta}(k+1, a)}\right)^m \prod_{j=1}^{m} \frac{z_j^{a-1}}{(k + b z_j)^{a+k+1}}, \end{aligned} \tag{2.22}$$

The log likelihood function, say $l = \log L(a, b|\mathbf{z})$, using (2.22) is

$$\begin{aligned} l &= m\left[(k+1)\ln(k) + a\ln(b) - \ln\left(\mathrm{Beta}(k+1,a)\right)\right] \\ &\quad +(a-1)\sum_{j=1}^{m}\ln(z_j) - (a+k+1)\sum_{j=1}^{m}\ln(k+bz_j). \end{aligned}$$

The MLEs of a and b, say $\hat{a}$ and $\hat{b}$, can be obtained by simultaneously solving the following equations:

$$m\left(\psi(a+k+1) - \psi(k+1) + \ln(\hat{b})\right) - \sum_{j=1}^{m}\ln(k+\hat{b}z_j) + \sum_{j=1}^{m}\ln(z_j) = 0, \tag{2.23}$$

$$\frac{am}{\hat{b}} - (\hat{a}+k+1)\sum_{j=1}^{m}\frac{z_j}{k+\hat{b}z_j} = 0, \tag{2.24}$$

where $\psi(\cdot)$ is the digamma function. The empirical Bayes estimators of the parameter θ under SELF, LELF, and GELF are

$$\begin{aligned} \hat{\hat{\theta}}_{\mathrm{BS}_1} &= \frac{\hat{\alpha}_1}{\hat{\beta}_1}, \\ \hat{\hat{\theta}}_{\mathrm{BL}_1} &= \frac{\hat{\alpha}_1}{c}\ln\left(1+\frac{c}{\hat{\beta}_1}\right), \text{ and} \\ \hat{\hat{\theta}}_{\mathrm{BG}_1} &= \frac{1}{\hat{\beta}_1}\left(\frac{\Gamma(\hat{\alpha}_1-d)}{\Gamma(\hat{\alpha}_1)}\right)^{-\frac{1}{d}}, \end{aligned}$$

respectively, where $\hat{\alpha}_1 = \hat{a}+k$ and $\hat{\beta}_1 = \hat{b}+\eta_1(x_k)$.

2. Method of moments

It is also possible to estimate a and b by using the method of moments. In order to obtain the moment estimators of a and b, the mean and variance of (2.21) are equated with the sample mean and variance, given by

$$\tilde{a} = \frac{k\,(k+1)^2\,t_1^2}{(k+1)\,t_2 - k\,(k+1)^2\,t_1^2} \quad \text{and} \quad \tilde{b} = \frac{(k+1)^2\,t_1}{(k-1)\,t_2 - k\,(k+1)^2\,t_1^2},$$

where

$$t_1 = \sum_{j=1}^{m}\frac{z_j}{m} \quad \text{and} \quad t_2 = \frac{(k-1)}{k}\sum_{j=1}^{m}\frac{z_j^2}{m}.$$

The empirical Bayes estimators of the parameter θ under SELF, LELF, and GELF are

$$\begin{aligned}
\tilde{\theta}_{\mathrm{BS}_1} &= \frac{\tilde{\alpha}_1}{\tilde{\beta}_1}, \\
\tilde{\theta}_{\mathrm{BL}_1} &= \frac{\tilde{\alpha}_1}{c} \ln\left(1+\frac{c}{\tilde{\beta}_1}\right), \text{ and} \\
\tilde{\theta}_{\mathrm{BG}_1} &= \frac{1}{\tilde{\beta}_1}\left(\frac{\Gamma(\tilde{\alpha}_1-d)}{\Gamma(\tilde{\alpha})}\right)^{-\frac{1}{d}},
\end{aligned}$$

respectively, where $\tilde{\alpha}_1 = \tilde{a}+k$ and $\tilde{\beta}_1 = \tilde{b}+\eta_1(x_k)$.

2.3.4 The HPD intervals

In this section, the HPD intervals for the unknown parameter θ of the ITLD are derived. For the Bayesian estimators it is important to account the posterior uncertainty. The most powerful tool that helps to measure such uncertainty is the HPD interval. This interval has the capability to account more probable values of the parameter than less probable ones. An HPD interval is a shortest interval which offers a pertinent summary of the posterior knowledge about θ. The $(1-\tau)100\%$ HPD interval, say $[H_1, H_2]$, for θ is obtained by solving:

$$\int_{H_1}^{H_2} g_i(\theta|\mathbf{x})\,\mathrm{d}\theta = 1-\tau \text{ and} \tag{2.25}$$

$$g_i(H_1|\mathbf{x}) = g_i(H_2|\mathbf{x}) \tag{2.26}$$

simultaneously, for $i = 1, 2, 3$.

After some simplification, of the (2.25) and (2.26) are simplified to

$$\frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)}\sum_{r=0}^{\infty}(-1)^r\frac{\beta_i^r}{r!(\alpha_i+r)}(H_2^{\alpha_i+r}-H_1^{\alpha_i+r}) = 1-\tau \text{ and} \tag{2.27}$$

$$\exp\left(\beta_i(H_1-H_2)\right) = \left(\frac{H_1}{H_2}\right)^{\alpha_i-1}, \tag{2.28}$$

respectively. Since a closed form of the HPD is difficult to derive, (2.27) and (2.28) are solved numerically to get the values of H_1 and H_2. The empirical Bayes HPD intervals can be obtained by replacing a and b by their respective maximum likelihood or moment estimators.

2.4 Prediction of the future records

In many real life situations, the prediction of future records values has a great interest. For example, to observe the record value until the present time, we naturally interested in predicting the future expected record for which the present record can be broken for the first time. This problem arises in many environmental process including rainfall, snowfall, drought, disaster, and many more. The future records will be based on the previous records that are observed from the same distribution. However, practically we need some threshold limits to indicate the range of future predicted records at a certain level of confidence. Several authors have discussed the Bayesian prediction bounds for some record statistics that are based on certain distributions. For example, Dunsmore (1983) studied the problem of predicting future records for two exponential models under the Bayesian framework.

2.4.1 Bayes and empirical Bayes prediction

Suppose that $\mathbf{x} = (x_1, x_2, ..., x_k)'$ be the first k upper record values observed from the ITLD. Then, we may be interested in predicting the value of next record $X_{k+1} = x_{k+1}$, or more generally, the value of sth record X_s for some $s > k$. Let $X_s = y$, following Ahsanullah (1995), the conditional PDF of Y given x_k is given by

$$f(y|x_k, \theta) = \frac{[\omega(y|\theta) - \omega(x_k|\theta)]^{\mathrm{s}-k-1}}{\Gamma(\mathrm{s}-k)} \frac{f(y|\theta)}{1 - F(x_k|\theta)}, \; x_k < y, \tag{2.29}$$

where $\omega(\cdot|\theta) = -\ln F(\cdot|\theta)$.

Using (2.1) and (2.2), (2.29) can be written as

$$f(y|x_k, \theta) = \frac{\theta^{\mathrm{s}-k}}{\Gamma(\mathrm{s}-k)} \left(\zeta(y, x_k)\right)^{\mathrm{s}-k-1} \exp\left(-\theta\zeta(y, x_k)\right) \times \xi(y), \tag{2.30}$$

where $\zeta(y, x_k) = (\eta_1(y) - \eta_1(x_k))$ and $\xi(y) = 2y/\left\{(1+y)(1+2y)\right\}$.

According to Arnold et al. (1998), the Bayes predictive density function of Y given $\mathbf{x}$ is

$$\begin{aligned} f(y|\mathbf{x}) &= \int_\theta f(y|x_k, \theta) g_i(\theta|\mathbf{x})\, \mathrm{d}\theta \\ &= \frac{1}{\mathrm{Beta}(\alpha_i, \mathrm{s}-k)} \frac{\beta_i^{\alpha_i}}{(\beta_i + \zeta(y, x_k))^{\mathrm{s}+a}} \left(\zeta(y, x_k)\right)^{\mathrm{s}-k-1} \times \xi(y), \end{aligned} \tag{2.31}$$

which is a PDF of 3-parameter inverted beta distribution, denoted by inverted-Beta $(\alpha_i, s-k, \beta_i)$, where $i = 1, 2, 3$. The $100\tau\%$ lower and upper prediction bounds for Y can be derived using the predictive survival function, obtained from (2.31), given by

$$P(Y \geq \lambda|\mathbf{x}) \quad = \quad \int_{\lambda}^{\infty} f(y|\mathbf{x})\,\mathrm{d}y. \tag{2.32}$$

The predictive bounds for a two-sided interval that cover the probability τ for the future record Y may be thus obtained by solving:

$$P(Y \geq L_i|\mathbf{x}) = \frac{1+\tau}{2} \text{ and } P(Y \geq U_i|\mathbf{x}) = \frac{1-\tau}{2},$$

where L_i and U_i are lower and upper bounds, respectively. In most cases, we predict the first unobserved record value X_{k+1}, the predictive survival function for $Y = X_{k+1}$ is obtained from (2.31) and (2.32) by replacing $s = k+1$, given by

$$P(Y \geq \lambda|\mathbf{x}) \quad = \quad \left(\frac{\beta_i}{\eta_1(\lambda) - \eta_1(x_k) + \beta_i}\right)^{\alpha_i}. \tag{2.33}$$

Thus an $100\tau\%$ Bayesian prediction interval for $Y = X_{k+1}$ satisfies

$$L_i \quad = \quad \left(1 - (L_i^*(\tau))^{\frac{1}{2}}\right)^{-1} - 1 \quad \text{and} \quad U_i = \left(1 - (U_i^*(\tau))^{\frac{1}{2}}\right)^{-1} - 1,$$

where

$$\begin{aligned} L_i^*(\tau) \quad &= \quad 1 - \exp\left(\beta_i - \eta_1(x_k) - \frac{\beta_i}{\left(\frac{1+\tau}{2}\right)^{\frac{1}{\alpha_i}}}\right) \text{ and} \\ U_i^*(\tau) \quad &= \quad 1 - \exp\left(\beta_i - \eta_1(x_k) - \frac{\beta_i}{\left(\frac{1-\tau}{2}\right)^{\frac{1}{\alpha_i}}}\right). \end{aligned}$$

On similar lines, empirical Bayes prediction bounds for the future upper record value X_{k+1} are obtained by replacing hyperparameters with their corresponding estimators, given by

$$\begin{aligned} \hat{L}_1 \quad &= \quad \left(1 - \sqrt{\hat{L}_i^*(\tau)}\right)^{-1} - 1 \text{ and } \hat{U}_1 = \left(1 - \sqrt{\hat{U}_i^*(\tau)}\right)^{-1} - 1, \\ \tilde{L}_2 \quad &= \quad \left(1 - \sqrt{\tilde{L}_i^*(\tau)}\right)^{-1} - 1 \text{ and } \tilde{U}_2 = \left(1 - \sqrt{\tilde{U}_i^*(\tau)}\right)^{-1} - 1, \end{aligned}$$

where

$$
\begin{aligned}
\hat{L}_i^*(\tau) &= 1-\exp\left(\hat{\beta}_1-\eta_1(x_k)-\frac{\hat{\beta}_1}{\left(\frac{1+\tau}{2}\right)^{\frac{1}{\hat{\alpha}_1}}}\right) \text{ and} \\
\hat{U}_i^*(\tau) &= 1-\exp\left(\hat{\beta}_1-\eta_1(x_k)-\frac{\hat{\beta}_1}{\left(\frac{1-\tau}{2}\right)^{\frac{1}{\hat{\alpha}_1}}}\right). \\
\tilde{L}_i^*(\tau) &= 1-\exp\left(\tilde{\beta}_1-\eta_1(x_k)-\frac{\tilde{\beta}_1}{\left(\frac{1+\tau}{2}\right)^{\frac{1}{\tilde{\alpha}_1}}}\right) \text{ and} \\
\tilde{U}_i^*(\tau) &= 1-\exp\left(\tilde{\beta}_1-\eta_1(x_k)-\frac{\tilde{\beta}_1}{\left(\frac{1-\tau}{2}\right)^{\frac{1}{\tilde{\alpha}_1}}}\right).
\end{aligned}
$$

Here, $(\hat{L}_i, \hat{U}_i)$ and $(\tilde{L}_i, \tilde{U}_i)$ are the empirical Bayes predictions bounds when the hyperparameters are estimated using the maximum likelihood method and method of moments, respectively.

2.5 Simulation study

In this section, a simulation study is conducted to compare the performances of the maximum likelihood, Bayes (with informative and non-informative priors) and empirical Bayes estimators of θ, in terms of their estimated risks (ERs) under different loss functions. The upper record values of sizes $k = 5, 7, 10$ are generated from an ITLD with $\theta = 0.50$. In order to compute the ERs of the estimators, 100,000 samples of size k are generated from an ITLD. The estimator values are then replaced into the loss function, and then the mean of the loss function is considered as an estimate of the risk function of that estimator. Following Pakhteev and Stepanov (2016), we provide a formula to generate k upper record values from an ITLD, given by

$$
(X_1, X_2, ..., X_k) = \left(\frac{1+\sqrt{1-\left(\prod_{i=1}^k U_i\right)^{1/\theta}}-\left(\prod_{i=1}^k U_i\right)^{1/\theta}}{\left(\prod_{i=1}^k U_i\right)^{1/\theta}}\right), \ k = 1, 2, ...,
$$

where $(U_1, U_2, ..., U_k)$ are IID uniform $(0, 1)$ random variables.

Table 2.1: Estimated risks of the MLE and Bayes estimators under different loss functions

		Loss	SELF	LELF		GELF	
				$c=-0.30$	$c=0.30$	$d=-0.90$	$d=0.90$
Estimator	Prior	Hyperparameters			$k=5$		
MLE	No use	No use	0.14097	0.01465	0.02381	0.10259	0.18031
Bayes	Informative	$(a=0.50, b=1.00)$	0.08185	0.01092	0.00959	0.08217	0.16021
Bayes	Informative	$(a=1.00, b=2.00)$	0.05752	0.00771	0.00670	0.06797	0.14241
Bayes	Informative	$(a=2.00, b=4.00)$	0.03379	0.00454	0.00393	0.04916	0.12749
Bayes	Empirical	No use	0.14097	0.01560	0.02235	0.10259	0.44614
Bayes	Jeffreys	No use	0.14097	0.01848	0.01716	0.10259	0.19914
Bayes	Uniform	No use	0.24242	0.03000	0.03168	0.11793	0.37802
					$k=7$		
MLE	No use	No use	0.07604	0.00842	0.01167	0.07309	0.11587
Bayes	Informative	$(a=0.50, b=1.00)$	0.05513	0.00713	0.00667	0.06261	0.18066
Bayes	Informative	$(a=1.00, b=2.00)$	0.04273	0.00557	0.00512	0.05444	0.17177
Bayes	Informative	$(a=2.00, b=4.00)$	0.02882	0.00377	0.00344	0.04222	0.16452
Bayes	Empirical	No use	0.07604	0.00896	0.01095	0.07309	0.32485
Bayes	Jeffreys	No use	0.07604	0.00967	0.00950	0.07309	0.19663
Bayes	Uniform	No use	0.11802	0.01458	0.01530	0.08160	0.33671
					$k=10$		
MLE	No use	No use	0.04173	0.00486	0.00565	0.05083	0.07408
Bayes	Informative	$(a=0.50, b=1.00)$	0.03495	0.00446	0.00428	0.04572	0.21530
Bayes	Informative	$(a=1.00, b=2.00)$	0.02962	0.00379	0.00361	0.04108	0.21307
Bayes	Informative	$(a=2.00, b=4.00)$	0.02294	0.00295	0.00279	0.03449	0.21450
Bayes	Empirical	No use	0.04173	0.00514	0.00533	0.05083	0.28741
Bayes	Jeffreys	No use	0.04173	0.00529	0.00515	0.05083	0.21936
Bayes	Uniform	No use	0.05913	0.00739	0.00741	0.05527	0.32970

Table 2.1 provides the ERs of the MLE, Bayes and empirical Bayes estimators under the SELF, LELF, and GELF. For the Bayes estimators under the informative gamma prior, different set of values of (a, b) are taken. Under each pair, the prior mean is equivalent to 0.50, which is the considered value of θ. For the empirical Bayes estimators, following Soliman and Al-Aboud (2008), the hyperparameters are estimated with the approximation: $E(\theta) \approx \hat{\theta}_{\mathrm{ML}}$ with $Var(\theta) = 0.2$. From Table 2.1 it is observed that as the value of k increases, generally, the values of ERs decrease and vice versa. Moreover, when the suitable prior information is available, the Bayes estimates with the informative priors are more precise than the MLE. It is to be noted that when the value of (a, b) increases, the prior variance decreases, and as a result the ERs of the Bayes estimates tend to decrease. In all cases, the empirical Bayes and Bayes (using Jeffreys prior) estimates are better than those with the uniform prior as the former possess less ERs than that of the latter.

2.6 An application to real data

In this section, a real dataset is taken for the application of the obtained results. The dataset is taken from MirMostafaee et al. (2016), where it has been shown that this dataset follows a TLD.

Table 2.2: Temporal Capacities for the month of February for Shasta Reservoir

Year	Capacities	Proportion to total capacity: (Y)	Inverse proportion to total capacity: (X)
1995	3448519	0.757583	0.319987
1996	3694201	0.811556	0.232201
1999	3594861	0.785339	0.273335
1998	3594861	0.783660	0.276064
1999	3569220	0.815627	0.226051
2000	3912933	0.847413	0.180062
2001	3859423	0.768007	0.302071
2002	3495969	0.843485	0.185558
2003	3584830	0.787408	0.269990
2004	3868600	0.849868	0.176653
2005	3168056	0.695970	0.436844
2006	3834224	0.842316	0.189203
2009	3992193	0.828689	0.206725
2008	2641041	0.580219	0.723561
2009	1960458	0.430681	1.321904
2010	3380149	0.742563	0.346687

Inverse proportion to total capacity : $(X = 1/Y - 1)$

The dataset comprises the monthly capacities for the months of February in years 1995 to 2010, while the maximum capacity of the reservoir is 4552000 acre-foot. According to the domain of TLD, the dataset is first transformed to lie within the unit interval (0,1), i.e., by dividing the capacities over the maximum capacity of the reservoir. In order to use this dataset for ITLD, an other transformation is considered, i.e., subtract one from the inverse proportion to total capacity. The Kolomogorov-Simirnov (K-S) test is used to check whether the transformed data follow an ITLD or not. The K-S test gives a p-value of 0.2106, which shows that the transformed dataset follows an ITLD. Using these data, a set of four upper record values is observed, given by

$$0.319987,\ 0.436844,\ 0.723561,\ 1.321904.$$

This dataset is used to obtain the estimators of the unknown parameter θ.

Table 2.3: MLE, Bayes estimates, HPD and prediction intervals using real data

Estimator	Loss / Prior	SELF	LELF $c=-0.50$	LELF $c=0.50$	GELF $d=-1.00$	GELF $d=1.10$
MLE	No use	10.21070				
Bayes	Empirical	10.21070	10.24059	10.18105	3.10560	3.10129
Bayes	Jeffreys	10.21070	19.35455	7.58141	7.76992	6.30339
Bayes	Uniform	12.76338	24.19319	9.47676	9.32694	7.92872
		HPD with $\tau = 0.90$			Prediction with $\tau = 0.90$	
		Interval	Length		Interval	Length
Bayes	Empirical	$(9.47648, 10.94112)$	1.46464		$(1.33801, 2.38304)$	1.04503
Bayes	Jeffreys	$(2.39261, 17.73118)$	15.33857		$(1.33811, 3.00823)$	1.67012
Bayes	Uniform	$(3.85113, 21.32862)$	17.47749		$(1.33485, 2.49761)$	1.16276

In order to find the Bayes estimators of θ under different loss functions with an informative priors, the values of hyperparameters must be known in advance. Here, as a matter of fact, the values of hyperparameters are unknown. In what follows, following Soliman and Al-Aboud (2008), the hyperparameters are estimated with the help of an approximation: $E(\theta) \approx \hat{\theta}_{\mathrm{ML}} = 10.2109$ with $Var(\theta) = 0.2$, which gives $\hat{a} = 521.292$ and $\hat{b} = 51.0535$. Table 2.3 reports the MLE, Bayes (using non-informative priors) and empirical Bayes estimators of θ under the SELF, LELF, and GELF. As expected, under SELF the empirical Bayes estimate of θ is equivalent to those of the MLE and Bayes with the Jeffreys prior. With the LELF and GELF, the Bayes estimates are over (under) estimated when $c < 0(> 0)$ and $d < 0(> 0)$. In addition, it is observed that the 90% HPD intervals with the informative prior (though its parameters are estimated from data) are shorter than those under the Jeffreys and uniform priors. Similarly, the 90% empirical Bayes prediction interval has small length than those with the non-informative priors.

2.7 Conclusion

In this chapter, with an inverse transformation on the Topp-Leone distributed random variable, an inverted Topp-Leone distribution (ITLD) has been derived. The maximum likelihood, Bayes and empirical Bayes methods have been used to obtain the estimators of the unknown parameter using upper record values from an ITLD. The Bayes and empirical Bayes estimators have been developed under SELF, LELF, and GELF. The Bayes estimators have been derived using both informative and non-informative priors. For obtaining the empirical Bayes estimators, the hyperparameters were estimated using the methods of moments and maximum likelihood. A simulation study was conducted to compare the performances of these estimators. It has turned out that, when suitable prior information is available, the Bayes estimators under the symmetric and asymmetric loss functions were superior to the maximum likelihood estimator. Moreover, explicit mathematical expression for the Bayes and empirical Bayes prediction intervals were also derived.

The current work could be extended to develop statistical inference when samples are available from an ITLD under different censoring schemes or generalized order statistics. Moreover, there is a possibility to estimate the unknown parameters when data are coming from a two or more component mixture of ITLDs.

Chapter 3

Bayes and empirical Bayes estimation and prediction using lower record values from inverted Gompertz distribution

In this chapter, we consider lower record values from an inverted Gompertz distribution (IGD). The maximum likelihood, Bayes and empirical Bayes methods are used to obtain the unknown parameter of IGD using a set of lower record values. The Bayes and empirical Bayes estimators are developed under the squared-error, linear-exponential and general entropy loss functions. The informative and non-informative priors are considered to derive the Bayes estimators. The methods of maximum likelihood and moments are used to estimate the hyperparameters. In addition, we provide the Bayes and empirical Bayes highest posterior density and prediction intervals. Furthermore, we have shown that the Bayes estimators under the symmetric and asymmetric loss functions are superior to the maximum likelihood estimator. A real life dataset is used to explain the applications of the mathematical results.

3.1 Introduction

In many disciplines the collected data are more easily accessed in terms of record values. For instance, weather, industry, sports, athletic events, economics, and life-tests among others. Record value or record statistics are the smallest or largest value obtained from a sequence of independent and identically distributed (IID) random variables. The theory of records is much discussed and well developed, due to its wide real life applications. Moreover, records become more important when the complete list of the observations is difficult to obtain or when they are being destroyed if the observations are subjected to an experimental test. Record values and their basic properties were first examined by Chandler (1952). In the existing literature, many researchers have attracted attention towards the estimation and inferential problem, when record values follow a given probability distribution. Record values can be defined as:

Consider $X_1, X_2, ...$ be a sequence of IID random variables with a probability density function (PDF) and cumulative distribution function (CDF). If an observation x_j is greater (smaller) than all the previous observations say $x_1, x_2, ..., x_{j-1}$, then X_j will be upper (lower) record value, i.e., $X_i < (>) X_j$, $\forall\ i < (>) j$. Let consider X_j is an upper as well as lower record value that is we can transform lower record value to an upper record by replacing the original sequence by $-X_j$. For further details on record values and their applications, see Ahsanullah (1995), Arnold et al. (1998), and the references cited therein.

Inference problems based on record values with many probability distributions were considered by several researchers. In this regards, one may refer to Raqab (2002), who derived an exact expression for single and product moments of record statistics for three parameter generalized exponential distribution. They have also discussed the recurrence relations for single and product moments of record statistics. The Bayesian and non-Bayesian estimators of the unknown parameters of the Weibull distribution are compared and discussed by Soliman et al. (2006) using upper record values. They have provided the estimation of some survival time parameters, e.g., reliability and hazard rate function, by considering a conjugate prior for the scale parameter and a discrete prior for the shape parameter, using symmetric and asymmetric loss functions. Moreover, predictive interval for the future upper record value was also constructed. The estimation and prediction of some lifetime distributions, including Exponential, Weibull, Pareto, and Burr type XII under the Bayesian framework were studied by Ahmadi and Doostparast (2006). Furthermore, they derived the Bayesian prediction interval for future upper record values for all these probability distributions. The Bayesian inference for the unknown parameter of Rayleigh distribution using upper record values was developed by Soliman and Al-Aboud (2008). They used both Bayesian and non-Bayesian approaches for the estimating the unknown scale and some lifetime parameters, e.g., reliability and hazard rate functions. Sultan (2008) discussed different methods of estimation for unknown parameters of an inverted Weibull distribution using upper record values. Nadar et al. (2013) used a set of m upper record values for statistical analysis of the Kumaraswamy distribution. Singh et al. (2016) discussed the Bayesian estimation and predictions of log-normal distribution based on lower record values and lower record values with inter-record times. They also computed the maximum likelihood estimators and asymptotic confidence intervals for the unknown parameters. In addition, the Bayes estimators and HPD intervals are derived using informative and non-informative priors. Furthermore, they provided the Bayes prediction interval under one-sample and two-sample framework. Seo and Kim (2016) derive unbiased estimators and confidence intervals for the parameters of the extreme value distribution with the help of different pivotal quantities using upper record values. Besides this, they also discussed Bayes inference using the Jeffreys and reference priors.

The Gompertz distribution plays an important role to model survival times, human motility and actuarial tables. In literature the mathematical model of Gompertz distribution was first introduced by Gompertz (1825), to fit mortality tables. The Gompertz distribution is used as a growth model, usually used to fit tumor growth, see Winsor (1932) and Casey (1934). Al-Hussaini et al. (2000) described the applications of the Gompertz distribution. Osman (1987) derived a compound Gompertz distribution from a two parameter Gompertz distribution, taking one of the parameters as a random variable following gamma distribution. Gompertz distribution can be seen as an extension of the exponential distributions because exponential distribution is limit of sequence of Gompertz distribution. Ghitany et al. (2014) considered a progressive type-II censored sample from a two parameter Gompertz distribution, and established a necessary and sufficient conditions for the existence and uniqueness of the likelihood estimates of the shape and scale parameters. A simulation based Bayesian approach is used by Soliman et al. (2011) for the estimation of the coefficient of variation under progressive first failure censored data from Gompertz distribution. They proposed a Markov chain Monte Carlo method for finding the point and interval estimation of the coefficient of variation using progressive first failure censored data. The Bayesian analysis of records statistics that follows a two parameter Gompertz distribution was provided by Jaheen (2003). Minimol and Thomas (2014) developed characterization of the Gompertz and inverted Gompertz distributions (IGDs) using record values.

In this chapter, a set of lower record values is used to find the estimators of the unknown parameter of IGD, which include the maximum likelihood estimator (MLE), Bayes and empirical Bayes estimators. Informative and non-informative priors are used to obtain the Bayes estimators under both symmetric and asymmetric loss functions, including the squared-error loss function (SELF), linear-exponential loss function (LELF), and general entropy loss function (GELF). Moreover, the Bayes and empirical Bayes highest posterior density (HPD) interval for the unknown parameter θ and prediction intervals of a future lower record value are derived. A real dataset is used to illustrate the applications of the results.

This rest of the work is organized as follows: In Section 3.2, we consider an IGD. The MLE, Bayes and empirical Bayes estimators of the unknown parameter of IGD are derived in Section 3.3. Section 3.4 deals with the computation of prediction interval for the future lower record value. An application to real data set is given in Section 3.5, and finally we conclude the chapter in Section 3.6.

3.2 Inverted Gompertz distribution

Let Y be a Gompertz distributed random variable with the PDF and CDF given by

$$\begin{aligned} f^*(y|\theta) &= \theta\left[\exp\left(y-\theta\left(\exp\left(y\right)-1\right)\right)\right],\ y\geq 0,\ \theta>0, \quad \text{and} \\ F^*(y|\theta) &= 1-\left[\exp\left(-\theta\left(\exp\left(y\right)-1\right)\right)\right], \end{aligned}$$

respectively. Gompertz distribution is unimodal with increasing hazard rate, and it is a positively skewed distribution. According to Ananda et al. (1996), the two parameter Gompertz distribution is mostly used as a survival time distribution in actuarial science, reliability, and life testing.

If Y is a Gompertz distributed random variable, then $X = 1/Y$ has an IGD (cf., Minimol and Thomas, 2014). The PDF, CDF, hazard rate and survival function of the IGD are, respectively, given by

$$
\begin{aligned}
f(x|\theta) &= \frac{\theta}{x^2}\exp\left(\frac{1}{x}-\theta\zeta(x)\right),\ x>0,\ \theta>0, && (3.1)\\
F(x|\theta) &= \exp\left(-\theta\zeta(x)\right), && (3.2)\\
h(x|\theta) &= \frac{\theta}{x^2}\frac{\exp\left(\frac{1}{x}-\theta\zeta(x)\right)}{1-\exp\left(-\theta\zeta(x)\right)} \quad \text{and}\\
S(x|\theta) &= 1-\exp\left(-\theta\zeta(x)\right),
\end{aligned}
$$

where $\zeta(x) = \exp(1/x) - 1$. In Figure 3.1, we plot the PDF, CDF, hazard rate and survival function of the IGD. It is observed that the IGD is also a unimodal and positively skewed distribution. As expected, there are different shapes of IGD with different values of θ.

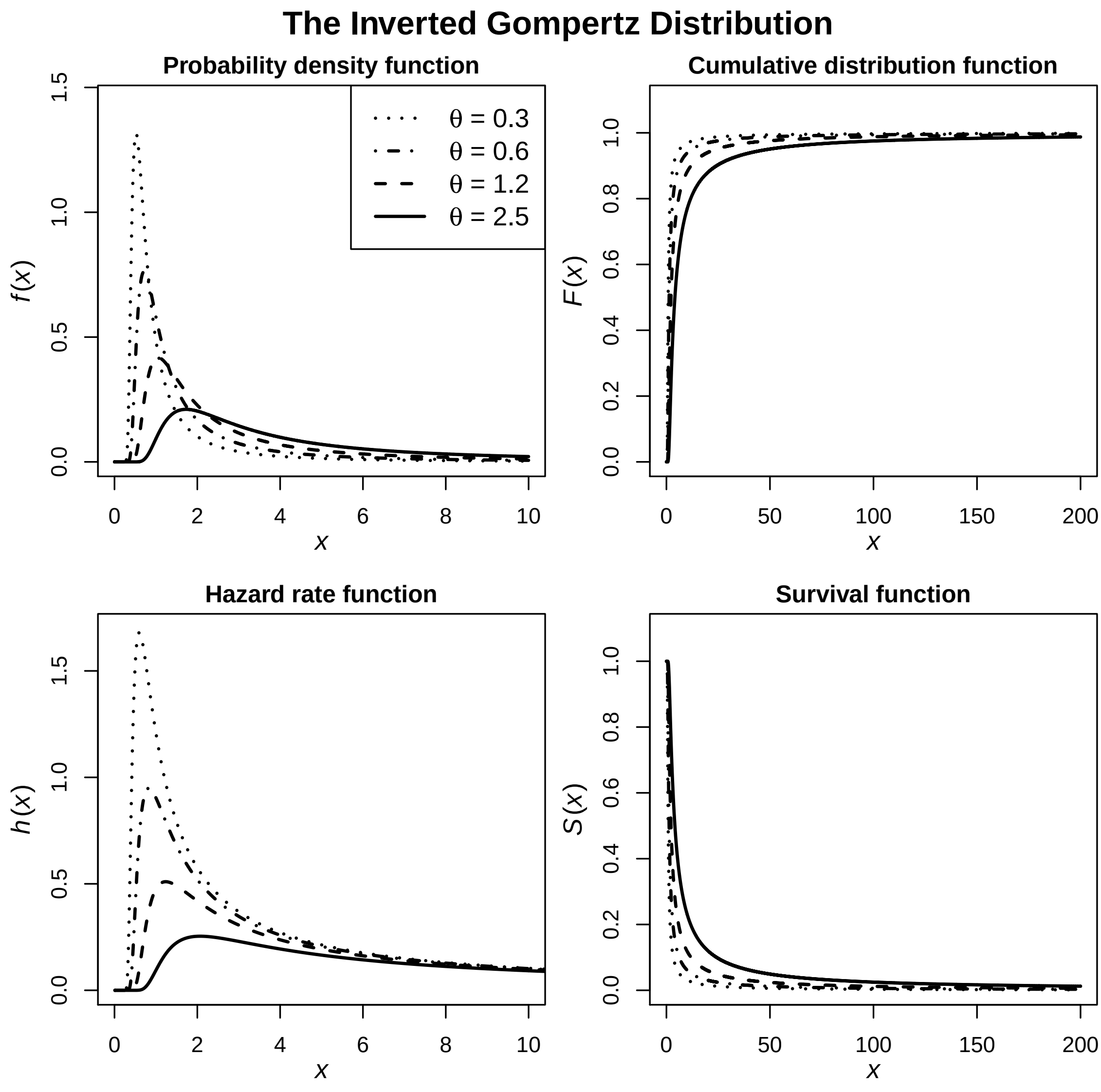

Figure 3.1: The PDF, CDF, hazard rate and survival functions of the IGD

3.3 Estimation of the parameter

In this section, we discuss the maximum likelihood, Bayes and empirical Bayes estimation of the unknown parameter θ of IGD, when data on k lower record values are available. Consider a set of k lower record values: $X_1^* = x_1$, $X_2^* = x_2$,..., $X_k^* = x_k$, that are observed from a sequence of an IID random variables following an IGD. The likelihood function of θ using k lower record values, say $\mathbf{x} = (x_1, x_2, ..., x_k)'$, is given by (cf., Arnold et al., 1998)

$$\begin{aligned} L(\theta|\mathbf{x}) &= f(x_k|\theta) \prod_{i=1}^{k-1} \frac{f(x_i|\theta)}{F(x_i|\theta)} && (3.3) \\ &= \theta^k \exp\left(-\theta\zeta(x_k)\right) \exp\left(\sum_{i=1}^{k} \frac{1}{x_i}\right) \prod_{i=1}^{k} \frac{1}{x_i^2}, && (3.4) \end{aligned}$$

where

$$\zeta(x_k) = \exp\left(\frac{1}{x_k}\right) - 1.$$

3.3.1 Maximum likelihood estimation

The log-likelihood function using (3.4), say $l = \ln L(\theta|\mathbf{x})$, is

$$l = k\ln(\theta) - \theta\zeta(x_k) + \sum_{i=1}^{k} \frac{1}{x_i} + \sum_{i=1}^{k} \ln\left(\frac{1}{x_i^2}\right). \quad (3.5)$$

The MLE of θ is obtained by minimizing (3.5), given by

$$\hat{\theta}_{\mathrm{ML}} = \frac{k}{\zeta(x_k)}, \quad (3.6)$$

where $\hat{\theta}_{\mathrm{ML}}$ is the MLE of θ. To study the properties of $\hat{\theta}_{\mathrm{ML}}$, we need to find the distribution of $\zeta(X_k)$. The PDF of X_k is given by (cf., Ahsanullah, 1995)

$$\begin{aligned} f_{X_k}(x|\theta) &= \frac{1}{\Gamma(k)} \{-\ln\left(F(x|\theta)\right)\}^{k-1} f(x|\theta), \;\; x > 0 \\ &= \frac{\theta^k}{\Gamma(k)} \left(\zeta(x)\right)^{k-1} \exp\left(-\theta\zeta(x)\right) \times \xi(x), && (3.7) \end{aligned}$$

where

$$\xi(x) = \exp\left(1/x\right)\left(1/x^2\right).$$

From (3.7), the PDF of $Z = k/\zeta(X_k)$ is

$$f_Z(z|\theta) = \frac{(k\theta)^k}{\Gamma(k)} \frac{1}{z^{k+1}} \exp\left(-\frac{k\theta}{z}\right), \ z > 0, \tag{3.8}$$

which is the PDF of an inverted gamma distribution with parameters k and $k\theta$. The mean and variance of $\hat{\theta}_{\mathrm{ML}}$ are, respectively, given by

$$\begin{aligned} E(\hat{\theta}_{\mathrm{ML}}) &= \frac{k\theta}{k-1}, \ k > 1, \\ Var(\hat{\theta}_{\mathrm{ML}}) &= \frac{(k\theta)^2}{(k-1)^2(k-2)}, \ k > 2. \end{aligned}$$

The bias and MSE of $\hat{\theta}_{\mathrm{ML}}$ are given by

$$\begin{aligned} \mathrm{Bias}(\hat{\theta}_{\mathrm{ML}}) &= \frac{\theta}{k-1} \quad \text{and} \\ \mathrm{MSE}(\hat{\theta}_{\mathrm{ML}}) &= \frac{\theta^2(k+2)}{(k-1)(k-2)}, \end{aligned}$$

respectively. It is clear that $\hat{\theta}_{\mathrm{ML}}$ is a weakly-consistent estimator of θ.

3.3.2 Bayesian estimation

In most of the practical situations, the size of observed record values is not so large, in such circumstances, the use of maximum likelihood estimation may provide biased estimates. In order to make a more reliable estimates from small samples, one can use the Bayesian method to estimate the unknown parameter(s).

In this subsection, the Bayesian approach is used to obtain the estimators of θ using a set of lower record values from IGD. In the Bayesian framework, θ is treated as a random variable, and it follows a prior distribution. In our study, we first derive the posterior distributions of θ using informative and non-informative priors. The gamma prior is taken as a conjugate informative prior, and the Jeffreys and uniform priors are taken as non-informative priors. Assume that θ follows a gamma distribution with the hyperparameters $a > 0$ and $b > 0$, given by

$$g_1(\theta; a, b) = \frac{b^a}{\Gamma(a)} \theta^{a-1} \exp(-\theta b), \ \theta > 0, \tag{3.9}$$

The posterior distribution of θ given $\mathbf{x}$ is obtained by solving

$$\begin{aligned} g_1(\theta|\mathbf{x}) &= \frac{L(\theta|\mathbf{x})g_1(\theta;a,b)}{\int_\Theta L(\theta|\mathbf{x})g_1(\theta;a,b)\,\mathrm{d}\theta} \\ &= \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)}\theta^{\alpha_1-1}\exp(-\theta\beta_1),\ \ \theta>0, \end{aligned} \tag{3.10}$$

which is the PDF of gamma distribution with parameters $\alpha_1 = a+k$ and $\beta_1 = b+\zeta(x_k)$. One of the most commonly used symmetric loss function is SELF, given by

$$L_1(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2,$$

where $\hat{\theta}$ is an estimator of θ. The SELF is often used in the Bayes inference because of its simplicity and ease in computation. The Bayes estimator of θ using SELF is

$$\begin{aligned} \hat{\theta}_{\mathrm{BS}_1} &= E_{\theta|\mathbf{x}}(\theta) \\ &= \frac{\alpha_1}{\beta_1} = \frac{a+k}{b+\zeta(x_k)}, \end{aligned}$$

where $E_{\theta|\mathbf{x}}(\cdot)$ denotes the mathematical expectation with respect to the posterior distribution of θ.

The SELF is symmetric in nature, and it gives equal magnitude to overestimation and underestimation. However, this restriction may be impractical, because in some situations, overestimation is considered to be more or less serious than underestimation. Under such situations, Varian (1975) and Zellner (1986) have suggested to consider an asymmetric loss function, called LELF. This loss function may associate more weight to overestimation or underestimation as it rises almost exponentially on one side of zero and approximately linear on the other side. The LELF for estimating θ is given by

$$L_2(\Delta) = \exp(c\Delta) - c\Delta - 1,\ \ \mathrm{c}\neq 0,$$

where $\Delta = \hat{\theta}-\theta$ is the scalar estimation error. The sign and magnitude of the shape parameter c show the direction and degree of symmetry, respectively. For $c = 1$, LELF is symmetric about zero with overestimation being more serious than underestimation. The LELF is approximately equal to SELF for $c = 0$. If $c < 0$, then $L_2(\Delta)$ increases almost exponentially when $\Delta < 0$ and almost linearly when $\Delta > 0$. The Bayes estimator of θ under LELF is

$$\hat{\theta}_{\mathrm{BL}_1} = -\frac{1}{c}\ln\left(E_{\theta|\mathbf{x}}\left(\exp(-c\theta)\right)\right) \tag{3.11}$$

provided that $E_{\theta|\mathbf{x}}(\exp(-c\theta))$ exists and is finite. The Bayes estimator $\hat{\theta}_{\mathrm{BL}_1}$ is obtained using (3.10) and (3.11), given by

$$\hat{\theta}_{\mathrm{BL}_1} = \frac{\alpha_1}{c}\ln\left(1+\frac{c}{\beta_1}\right) = \frac{a+k}{c}\ln\left(1+\frac{c}{b+\zeta(x_k)}\right),\ c \neq 0.$$

The LELF is more flexible among the asymmetric loss functions when estimating the location parameter. However, it appears to be unsuitable for scale parameter and some other quantities, (cf., Basu and Ebrahimi, 1991; Parsian and Sanjari F, 1993). Therefore, a suitable alternative to LELF was proposed by Calabria and Pulcini (1996), called GELF. The GELF for estimating θ is given by

$$L_3(\hat{\theta},\theta) = \left(\frac{\hat{\theta}}{\theta}\right)^d - d\ln\left(\frac{\hat{\theta}}{\theta}\right) - 1,\ d \neq 0. \tag{3.12}$$

As expected, the GELF is zero at $\hat{\theta} = \theta$. The overestimation or underestimation of θ depends on the sign and magnitude of d. For $d > 0$ ($d < 0$), overestimation is more (less) serious than underestimation. The Bayes estimator of θ under GELF is

$$\begin{aligned}\hat{\theta}_{\mathrm{BG}_1} &= \left(E_{\theta|\mathbf{x}}\left(\theta^{-d}\right)\right)^{-\frac{1}{d}} \\ &= \frac{1}{\beta_1}\left(\frac{\Gamma(\alpha_1-d)}{\Gamma(\alpha_1)}\right)^{-\frac{1}{d}} = \frac{1}{b+\zeta(x_k)}\left(\frac{\Gamma(a+k-d)}{\Gamma(a+k)}\right)^{-\frac{1}{d}}.\end{aligned}$$

It can be shown that, for $d = 1$, $\hat{\theta}_{\mathrm{BG}_1}$ is equivalent to the Bayes estimator under the weighted squared error loss function. Similarly, for $d = -1$, $\hat{\theta}_{\mathrm{BG}_1}$ is equivalent to $\hat{\theta}_{\mathrm{BS}_1}$.

There are situations where the suitable/reliable prior information is not available or it is difficult to quantify the prior information with the help of an informative prior. Under such situations, it is possible to derive Bayes estimators of θ with the help of non-informative priors. The most commonly used non-informative priors are Jeffreys, uniform, Quasi priors, etc. In what follows, we use these priors to derive Bayes estimators of θ under the aforementioned loss functions. The quasi prior for θ is

$$g(\theta) \quad \propto \quad \frac{1}{\theta^e}, \; \theta, \; e > 0. \tag{3.13}$$

In (3.13) by setting $e = 0$, a uniform prior for θ is obtained, given by

$$g_2(\theta) \quad \propto \quad 1, \; \theta > 0. \tag{3.14}$$

Using (3.4) and (3.13), the posterior distribution of θ given $\mathbf{x}$ is

$$g_2(\theta|\mathbf{x}) = \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} \, \theta^{\alpha_2 - 1} \exp\left(-\theta\beta_2\right), \; \theta > 0, \tag{3.15}$$

which is the PDF of gamma distribution with parameters $\alpha_2 = k+1$ and $\beta_2 = \zeta(x_k)$. The Bayes estimators under the symmetric (SELF) and asymmetric loss functions (LELF and GELF) are

$$\begin{aligned}
\hat{\theta}_{\mathrm{BS}_2} &= \frac{\alpha_2}{\beta_2} = \frac{k+1}{\zeta(x_k)}, \\
\hat{\theta}_{\mathrm{BL}_2} &= \frac{\alpha_2}{c} \ln\left(1 + \frac{c}{\beta_2}\right) = \frac{k+1}{c} \ln\left(1 + \frac{c}{\zeta(x_k)}\right) \quad \text{and} \\
\hat{\theta}_{\mathrm{BG}_2} &= \frac{1}{\beta_2}\left(\frac{\Gamma(\alpha_2 - d)}{\Gamma(\alpha_2)}\right)^{-\frac{1}{d}} = \frac{1}{\zeta(x_k)}\left(\frac{\Gamma(k-d+1)}{\Gamma(k+1)}\right)^{-\frac{1}{d}},
\end{aligned}$$

respectively.

Similarly, in (3.13) with $e = 1$, a non-informative Jeffreys prior for θ is obtained, given by

$$g_3(\theta) \quad \propto \quad \frac{1}{\theta}, \; \theta > 0. \tag{3.16}$$

Using (3.4) and (3.15), the posterior distribution of θ given $\mathbf{x}$ is

$$g_3(\theta|\mathbf{x}) = \frac{\beta_3^{\alpha_3}}{\Gamma(\alpha_3)} \, \theta^{\alpha_3 - 1} \exp\left(-\theta\beta_3\right), \; \theta > 0, \tag{3.17}$$

which is the PDF of gamma distribution with parameters $\alpha_3 = k$ and $\beta_3 = \zeta(x_k)$. The Bayes estimators under the symmetric (SELF) and asymmetric loss functions (LELF and GELF) are

$$\begin{aligned}
\hat{\theta}_{\mathrm{BS}_3} &= \frac{\alpha_3}{\beta_3} = \frac{k}{\zeta(x_k)}, \\
\hat{\theta}_{\mathrm{BL}_3} &= \frac{\alpha_3}{c}\ln\left(1+\frac{c}{\beta_3}\right) = \frac{k}{c}\ln\left(1+\frac{c}{\zeta(x_k)}\right) \quad \text{and} \\
\hat{\theta}_{\mathrm{BG}_3} &= \frac{1}{\beta_3}\left(\frac{\Gamma(\alpha_3-d)}{\Gamma(\alpha_3)}\right)^{-\frac{1}{d}} = \frac{1}{\zeta(x_k)}\left(\frac{\Gamma(k-d)}{\Gamma(k)}\right)^{-\frac{1}{d}},
\end{aligned}$$

respectively.

3.3.3 Empirical Bayes estimation

In Bayesian inference the selection of hyperparameters of the prior distribution is very important and sensitive. Sometime a life tester has no prior information, then empirical Bayes estimation is used to avoid this issue. When the hyperparameters a and b are unknown, we may use an empirical Bayes method to estimate them. The method of maximum likelihood or method of moments are usually used to estimate the hyperparameters, (cf., Carlin and Louis, 1997). Here, these methods are used to estimate the hyperparameters a and b.

1. Maximum likelihood estimation

Suppose that past m samples x_js with the past realization θ_js are available, where $j = 1, 2, ..., m$. Then, using (3.5), the MLE of θ_j $(1 \le j \le m)$ is

$$\hat{\theta}_{\mathrm{ML},j} = \frac{k}{\zeta(x_j)} = z_j \text{ (say)}. \tag{3.18}$$

The conditional distribution of Z_j given θ_j is obtained by replacing Z and θ with Z_j and θ_j in (3.8), respectively, given by

$$f_{Z|\theta_j}(z|\theta_j) = \frac{(k\theta_j)^k}{\Gamma(k)}\frac{1}{z_j^{k+1}}\exp\left(-\frac{k\theta_j}{z_j}\right), \; z_j > 0. \tag{3.19}$$

According to Schafer and Feduccia (1972), the marginal PDF of Z_j is obtained by solving

$$\begin{aligned}
f_{Z_j}(z_j) &= \int_0^\infty f_{z_j|\theta_j}(z_j|\theta_j)g_1(\theta_j)\,\mathrm{d}\theta_j \\
&= \frac{k^k b^a}{\mathrm{Beta}(k,a)}\frac{z_j^{a-1}}{(k+bz_j)^{a+k}},
\end{aligned} \tag{3.20}$$

which is the PDF of 3-parameter inverted beta distribution, say $Z_j \sim$ inverted-Beta$(k, a, k/b)$ for $j = 1, 2, ..., m$, where Beta$(\cdot, \cdot)$ is the usual beta function. The likelihood function using $\mathbf{z} = (z_1, z_2, ..., z_m)'$, Z_j is given by

$$\begin{aligned} L(a, b|\mathbf{z}) &= \prod_{j=1}^{m} f_{Z_j}(z_j) \\ &= \left(\frac{k^k b^a}{\text{Beta}(k, a)}\right)^m \prod_{j=1}^{m} \frac{z_j^{a-1}}{(k + b z_j)^{a+k}}, \end{aligned} \tag{3.21}$$

The log likelihood function, say $l = \ln \mathrm{L}(a, b|\mathbf{z})$, using (3.21) is

$$\begin{aligned} l &= m\left[(k)\ln(k) + a\left(\ln(b)\right) - \ln\left(\text{Beta}(k, a)\right)\right] \\ &\quad +(a-1)\sum_{j=1}^{m} \ln(z_j) - (a+k)\sum_{j=1}^{m} \ln(k + b z_j). \end{aligned}$$

The MLEs of a and b say $\hat{a}$ and $\hat{b}$, are obtained by solving the following equations simultaneously:

$$m\left(\psi(a+k) - \psi(k) + \ln(\hat{b})\right) - \sum_{j=1}^{m} \ln(k + \hat{b} z_j) + \sum_{j=1}^{m} \ln(z_j) = 0, \tag{3.22}$$

$$\frac{am}{\hat{b}} - (\hat{a} + k)\sum_{j=1}^{m} \frac{z_j}{k + \hat{b} z_j} = 0, \tag{3.23}$$

where $\psi(\cdot)$ is the digamma function. The empirical Bayes estimators of θ under the symmetric (SELF) and asymmetric loss functions (LELF and GELF) are

$$\begin{aligned} \hat{\hat{\theta}}_{\mathrm{BS}_1} &= \frac{\hat{\alpha}_1}{\hat{\beta}_1}, \\ \hat{\hat{\theta}}_{\mathrm{BL}_1} &= \frac{\hat{\alpha}_1}{c} \ln\left(1 + \frac{c}{\hat{\beta}_1}\right) \quad \text{and} \\ \hat{\hat{\theta}}_{\mathrm{BG}_1} &= \frac{1}{\hat{\beta}_1}\left(\frac{\Gamma(\hat{\alpha}_1 - d)}{\Gamma(\hat{\alpha}_1)}\right)^{-\frac{1}{d}}, \end{aligned}$$

respectively, where $\hat{\alpha}_1 = \hat{a} + k$ and $\hat{\beta}_1 = \hat{b} + \zeta(x_k)$.

2. Method of moments

The hyperparameters can also be estimated by the method of moments. The moment estimators of a and b are obtained by equating the mean and variance of (3.20) with the sample mean and variance of MLEs Z_js, given by

$$\tilde{a} = \frac{s_1^2}{s_2 - s_1^2} \quad \text{and} \quad \tilde{b} = \frac{ks_1}{(k-1)(s_2 - s_1^2)},$$
$$\text{where} \quad s_1 = \sum_{j=1}^{m} \frac{z_j}{m} \quad \text{and} \quad s_2 = \frac{(k-2)}{(k-1)} \sum_{j=1}^{m} \frac{z_j^2}{m}.$$

The moment estimators of θ under the symmetric (SELF) and asymmetric loss functions (LELF and GELF) are

$$\tilde{\tilde{\theta}}_{\mathrm{BS}_1} = \frac{\tilde{\alpha}_1}{\tilde{\beta}_1},$$
$$\tilde{\tilde{\theta}}_{\mathrm{BL}_1} = \frac{\tilde{\alpha}_1}{c} \ln\left(1 + \frac{c}{\tilde{\beta}_1}\right) \quad \text{and}$$
$$\tilde{\tilde{\theta}}_{\mathrm{BG}_1} = \frac{1}{\tilde{\beta}_1} \left(\frac{\Gamma(\tilde{\alpha}_1 - d)}{\Gamma(\tilde{\alpha}_1)}\right)^{-\frac{1}{d}},$$

respectively, where $\tilde{\alpha}_1 = \tilde{a} + k$ and $\tilde{\beta}_1 = \tilde{b} + \zeta(x_k)$.

3.3.4 The HPD intervals

In this subsection, we derive the HPD and empirical HPD intervals for the unknown parameter of IGD. An interval which is based on the posterior distribution is called credible interval, while an HPD interval is the shortest possible credible interval. The HPD interval is the most powerful tool that helps to measure the posterior uncertainty. Moreover, the HPD interval has the capability to account the more probable values of the parameter than less probable ones. An $100\tau\%$ HPD interval, say (H_1, H_2), for θ is obtained by solving:

$$\int_{H_1}^{H_2} g_i(\theta|\mathbf{x})\, \mathrm{d}\theta = \tau \quad \text{and} \tag{3.24}$$
$$g_i(H_1|\mathbf{x}) = g_i(H_2|\mathbf{x}) \tag{3.25}$$

simultaneously, for $i = 1, 2, 3$. The above equations (3.24) and (3.25) are simplified as follows:

$$\frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} \sum_{r=0}^{\infty} \frac{(-1)^r \beta_i^r}{r!(\alpha_i + r)} (H_2^{\alpha_i + r} - H_1^{\alpha_i + r}) = \tau \quad \text{and} \tag{3.26}$$
$$\exp\left((H_1 - H_2)\, \beta_i\right) = \left(\frac{H_1}{H_2}\right)^{\alpha_i - 1}, \tag{3.27}$$

respectively. Since an explicit expressions for the HPD interval is difficult to obtain, therefore (3.26) and (3.27) are solved numerically to get the values of H_1 and H_2. For the empirical HPD intervals, the values of hyperparameters a and b are replaced by their maximum likelihood or moment estimators, and the rest of its computation remains the same.

3.4 Prediction of the future records

This section is devoted for deriving the Bayes and empirical Bayes prediction interval of future lower records. The prediction of future lower records is a problem of great interest. For example, a meteorologist is usually interested to predict the future minimum low temperature or in sports a fastest event (shortest time duration), etc. For such practical situations, one can construct a prediction interval for the future lower record value using the current available lower records. In most of the situations the available record observations and the predictable future record possess same nature. In what follows, we consider the prediction of future lower record value using Bayes and empirical Bayes methods.

One of the important features of the Bayes inference is that the predictive distribution depends only on the observed data. Suppose that $\boldsymbol{X} = (X_1^* = x_1,\ X_2^* = x_2, ..., X_k^* = x_k)$ with $X_1^* \geq X_2^* \geq \cdots \geq X_k^*$ be the k observed lower record values from IGD. Then, the interest lies in predicting the value of next lower record, say X_{k+1}^* or X_s^* for some $s < k$. Let $X_s^* = y$ be the future lower record value, then the conditional PDF of Y given x_k is (cf., Ahsanullah, 1995)

$$f(y|x_k, \theta) = \frac{(\Phi(y) - \Phi_1(x_k))^{s-k-1}}{\Gamma(s-k)} \frac{f(y|\theta)}{F(x_k|\theta)},\ 0 < y < x_k < \infty, \tag{3.28}$$

where $\Phi(\cdot) = -\ln F(\cdot|\theta)$. Using (3.1) and (3.2), (3.28) can be written as

$$f(y|x_k, \theta) = \frac{\theta^{s-k}}{\Gamma(s-k)} (\eta(y, x_k))^{s-k-1} \exp\left(-\theta\eta(y, x_k)\right) \times \xi(y), \tag{3.29}$$

where $\eta(y, x_k) = \zeta(y) - \zeta(x_k)$ and $\xi(y) = \exp\left(1/y\right)\left(1/y^2\right)$.
The Bayes predictive distribution of Y given x_k (cf., Arnold et al., 1998) is

$$\begin{aligned} f(y|x_k) &= \int_0^\infty f(y|x_k, \theta) g_i(\theta|\mathbf{x})\, \mathrm{d}\theta \\ &= \frac{1}{\mathrm{Beta}(\alpha_i, s-k)} \frac{\beta_i^{\alpha_i}}{(\beta_i + \eta(y, x_k))^{s-k+\alpha_i}} (\eta(y, x_k))^{s-k-1} \times \xi(y), \end{aligned} \tag{3.30}$$

where i=1,2,3.

In order to obtained lower and upper $100\tau\%$ prediction bounds for $Y = X_s$ or X_{k+1} given x_k can be derived using the following predictive survival function:

$$P(Y \geq \lambda | x_k) = \int_{\lambda}^{x_k} f(y|x_k)\, \mathrm{d}y. \tag{3.31}$$

The predictive bound of a two-sided interval that covers τ for the future record Y may thus be obtained by solving the following two equations for lower (L) and upper (U) limits:

$$P(Y \geq \ L|x_k) = \frac{1+\tau}{2} \quad \text{and} \quad P(Y \geq U|x_k) = \frac{1-\tau}{2}.$$

In most cases, the interest lines in predicting the first unobserved record value X_{k+1}, the predictive survival function for $Y = X^*_{k+1}$ is obtained from (3.28) with $s = k+1$, which simplifies to

$$P(Y \geq \lambda | x_k) = 1 - \left(\frac{\beta_i}{\zeta(\lambda) - \zeta(x_k) + \beta_i} \right)^{\alpha_i}. \tag{3.32}$$

Thus an $100\tau\%$ Bayes prediction interval for $Y = X_{k+1}$ satisfies the following limits:

$$\begin{aligned} L &= \left\{ \ln \left(1 + \zeta(x_k) - \beta_i + \frac{\beta_i}{(\frac{1-\tau}{2})^{\frac{1}{\alpha_i}}} \right) \right\}^{-1} \quad \text{and} \\ U &= \left\{ \ln \left(1 + \zeta(x_k) - \beta_i + \frac{\beta_i}{(\frac{1+\tau}{2})^{\frac{1}{\alpha_i}}} \right) \right\}^{-1}. \end{aligned}$$

Similarly, the empirical Bayes prediction intervals are obtained by replace the hyperparameters values a and b by their respective maximum likelihood or moment estimators. The empirical Bayes prediction intervals for X_{k+1} with probability τ are

$$\begin{aligned} \hat{L} &= \left\{ \ln \left(1 + \zeta(x_k) - \hat{\beta}_1 + \frac{\hat{\beta}_1}{(\frac{1-\tau}{2})^{\frac{1}{\hat{\alpha}_1}}} \right) \right\}^{-1}, \\ \hat{U} &= \left\{ \ln \left(1 + \zeta(x_k) - \hat{\beta}_1 + \frac{\hat{\beta}_1}{(\frac{1+\tau}{2})^{\frac{1}{\hat{\alpha}_1}}} \right) \right\}^{-1}, \end{aligned}$$

and

$$\begin{aligned}
\tilde{L} &= \left\{ \ln \left(1 + \zeta(x_k) - \tilde{\beta}_1 + \frac{\tilde{\beta}_1}{(\frac{1-\tau}{2})^{\frac{1}{\tilde{\alpha}_1}}} \right) \right\}^{-1}, \\
\tilde{U} &= \left\{ \ln \left(1 + \zeta(x_k) - \tilde{\beta}_1 + \frac{\tilde{\beta}_1}{(\frac{1+\tau}{2})^{\frac{1}{\tilde{\alpha}_1}}} \right) \right\}^{-1}.
\end{aligned}$$

3.5 An application to real data

In this section, real life dataset are taken to explain the applications of the obtained results. The dataset comprises the survival time in (days) of a group of lungs cancer patients reported by Lawless (1982), given by

6.96, 9.30, 6.96, 7.24, 9.30, 4.90, 8.42, 6.05, 10.18, 6.82, 8.58, 7.77, 11.94, 11.25, 12.94, 12.94.

This dataset is used by Soliman and Al-Aboud (2008) for the Bayesian inference using upper record values from the Rayleigh distribution. In order to use this dataset for IGD, we consider the following transformation:
If W be a Rayleigh distributed random variable, then $1/\ln(1+W^2)$ follows an IGD. Using this transformation, the transformed dataset is given by:

0.25636, 0.22364, 0.25636, 0.25137, 0.22364, 0.31063, 0.23390, 0.27570,
0.21503, 0.25900, 0.23189, 0.24290, 0.20134, 0.20625, 0.19506, 0.19506.

The adequacy of the fitness of IGD is checked using the Kolomogorov-Simirnov (K-S) test. The K-S test gives a p-value of 0.4152, which indicates that the transform dataset follows IGD reasonably. Using these data the following set of lower record values is observed:

0.25636, 0.22364, 0.21503, 0.20134, 0.19506.

Using these record values, we obtain the maximum likelihood and Bayes estimators of θ.

Table 3.1: MLE, Bayes estimates, HPD and prediction intervals using real data

	Loss	SELF	LELF		GELF	
Estimator	Prior		$c = -0.50$	$c = 0.50$	$d = -1.00$	$d = 1.10$
MLE	No use	0.02986				
Bayes	Empirical	0.02986	0.02989	0.02983	0.02181	0.01854
Bayes	Jeffreys	0.02986	0.02983	0.03587	0.01854	0.02532
Bayes	Uniform	0.03583	0.02989	0.03580	0.02182	0.02218
		HPD with $\tau = 0.90$			Prediction with $\tau = 0.90$	
		Interval	Length		Interval	Length
Bayes	Empirical	$(0.00902, 0.04989)$	0.04087		$(0.00902, 0.0499)$	0.0408743
Bayes	Jeffreys	$(0.00901, 0.04990)$	0.04090		$(0.00901, 0.04990)$	0.0408898
Bayes	Uniform	$(0.01272, 0.05807)$	0.045353		$(0.01272, 0.0581)$	0.045350

In order to obtain the Bayes estimators of θ under the informative prior, the values of hyperparameters must be known in advance. Since the hyperparameters values are unknown then following Soliman and Al-Aboud (2008), we consider an approximation: Set $E(\theta) \approx \hat{\theta}_{\mathrm{ML}} = 0.029861$ with $Var(\theta) = 0.2$, which gives the hyperparameter's values as $a = 0.004458$ and $b = 0.1493$. The MLE, Bayes (using non-informative priors) and the empirical Bayes estimators of θ under SELF, LELF and GELF are computed and reported in Table 3.1. It is to be noted that the empirical Bayes estimator under SELF is equal to the MLE and the Bayes estimator with the Jeffreys prior. However, under LELF and GELF, the Bayes estimators are over (under) estimating when $c < 0$ (> 0) and $d < 0$ (> 0). Moreover, the length of 90% empirical HPD interval less than those with the Jeffreys and uniform priors. In addition, the length of the empirical Bayes prediction interval is also less than those with the non-informative priors. Here, the prediction interval is constructed for the next 6th lower record value.

3.5.1 Records generation

Suppose X be an inverted Gompertz distributed random variable. One can generate a random sample of k lower records from the IGD. In the existence literature, many researchers have provided records generations algorithms. In this regards, following Pakhteev and Stepanov (2016), we derive a formula for generation of k lower record values following an IGD, given by

$$(X_1, X_2, ..., X_k) = \frac{1}{\ln\left(1 - \ln\left(\prod_{i=1}^{k} U_i\right)^{1/\theta}\right)}, \quad k = 1, 2, ...,$$

where $(U_1, U_2, ..., U_k)$ are IID uniform $(0, 1)$ random variables.

3.6 Conclusion

In this chapter, using a set of k lower record values, we have estimated the unknown parameter of the IGD using maximum likelihood, Bayes and empirical Bayes estimation methods. The methods of the maximum likelihood and moments were used to estimate the hyperparameters. The Bayes estimators were derived using informative and non-informative priors under symmetric and asymmetric loss functions. Moreover, HPD intervals for the unknown parameter θ and prediction intervals for the future lower record values were also derived. Finally, a real life dataset was used to compute the numerical results using the derived mathematical formulas.

Chapter 4

Conclusion and Future Works

4.1 Conclusion

Every day we heard about the new records in the real life phenomena. For example, a new record value of the summer temperature in Pakistan, the strongest level of earthquake in India, and lowest stock market figure of china, etc., are dealing with upper or lower record values. The record values or record statistics are the successive extremes that occur in a sequence of independent and identically distributed random variables. In this thesis, the unknown parameters of two inverted probability distributions have been estimated using upper and lower record values.

In Chapter 2, an inverted Topp-Leone distribution (ITLD) has been derived from an inverse variable transformation on the Topp-Leone distributed random variable. The maximum likelihood, Bayes and empirical Bayes methods have been used to estimate the unknown parameter of ITLD using upper record values. The Bayes and empirical Bayes estimators have been derived under symmetric (SELF) and asymmetric (LELF and GELF) loss functions. Both informative and non-informative priors were used to compute the Bayes estimators. The unknown hyperparameters have been estimated by the methods of maximum likelihood and moments. The performances of these estimators have been compared using both real and simulated datasets. Through simulation analysis, it was verified that, when suitable prior information is available, the Bayes and empirical Bayes estimators under symmetric and asymmetric loss functions were more precise than the MLE. In addition, we have provided the Bayes and empirical Bayes HPD and prediction intervals for the unknown parameter of ITLD and future upper record value, respectively.

On similar lines, in Chapter 3, the unknown parameter of inverted Gompertz distribution (IGD) has been estimated by maximum likelihood, Bayes and empirical Bayes estimation methods when a set of lower record values is available. The symmetric and asymmetric loss functions were used to obtain the Bayes and empirical Bayes estimators. The Bayes estimators have been obtained under both informative and non-informative priors. Moreover, explicit mathematical expressions for the Bayes and empirical Bayes prediction intervals were also derived. Finally, the mathematical results were applied on a real dataset that followed IGD.

4.2 Future works

The current work could be extended to develop statistical inference when samples are available from ITLD and IGD under different censoring schemes or generalized order statistics. Moreover, there is a possibility to estimate the unknown parameters when data follow a two or more component mixture of ITLDs or IGDs.

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