On solutions of non-linear equations arising in Rivlin-Ericksen fluids

By

Muhammad Raheel Mohyuddin

Supervised by

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Department of Mathematics
Quaid-i-Azam University, Islamabad
PAKISTAN

2005

بسم الله الرحمن الرحيم

Sayings of Prophet Muhammad
(Peace Be Upon Him)

To acquire knowledge is binding upon all Muslims, whether male or female.

The ink of the scholar is more holy than the blood of the martyr.

He who travels in the search of knowledge, to him God shows the way of Paradise.

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A Thesis
Submitted in the Partial Fulfillment of the
Requirements for the Degree of
DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

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Certificate

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A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE DOCTOR OF PHILOSOPHY

We accept this thesis as conforming to the required standard

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Dedicated to

My parents, brothers, sisters, bhabies Sherjeel, Andleeb, Haseeb and Areba, for their many sacrifices:
whose prayers have always been a source of great inspiration for me
and whose sustained hope in me has led me to where I stand today;

and

My Wife
Who inculcated in me the spirit, struggle and continuous effort in order to achieve
This target
I Love Her

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Preface

Navier-Stokes equations are the most fundamental equations in Newtonian fluid mechanics. But for the last few decades it has been generally accepted that the Newtonian fluids, which have a relationship between the stress and the rate of strain, do not explain several phenomena observed for the fluids in industry and other technological applications. Rheological properties of non-Newtonian fluids are described by their so- called constitutive equations. Due to complexity of fluids, several models mainly based on the empirical observations have been proposed. Amongst the several non-Newtonian fluid models, the second and third grade fluid models have attracted the attention of many researchers. The attraction of such fluid models stems largely due to the fact that their constitutive equations have been derived on the basis of first principle and unlike many other 'phenomenological' models, there are no curve fittings or parameters to adjust. Though in both of these grades, there are material parameters that need to be measured.

The equations of motion for second and third grade fluids are highly non-linear and much complicated than the Navier-Stokes equations. There are very few cases in which exact analytical solutions of the Navier-Stokes equation can be obtained. These are even rare when the governing equations for non-Newtonian fluids are considered. Moreover, the equations for second and third grade are of higher order than the Navier-Stokes equations. However, there is no corresponding increase in the number of boundary conditions. In these methods, solutions can be found by assuming certain physical or geometrical properties of the flow field.

It is necessary here to mention that in chapters through 2-7 (which are all published/accepted papers), there are number of contributing authors but the major contribution is of the author of this dissertation.

Keeping all the above motivations in mind, the layout of this thesis is as follows:

1) Inverse solutions for modeled non-linear equations that govern the steady flows of a second grade fluid are discussed in chapter 2. The solution for stream function, velocity components and pressure are obtained from the non-linear equation by considering two illustrative forms of the stream function. The presented graphical results indicate that increasing magnitude of viscoelasticity decreases the velocity.
2) In chapter 3, the non-linear compatibility equation for the swirling flows of a second grade fluid is modeled. The studies of swirling viscoelastic flows have been

motivated by applications in rheology and tribology. The analytical solutions for the steady and unsteady axisymmetric flows of Newtonian and second grade fluids are obtained. The analytical solutions are built for the streamlines, velocity and vorticity components. Finally, the results are also compared with the corresponding solutions for the Newtonian fluid.

3) Chapter 4 deals with the modeling of equations for the unsteady flow of a second grade fluid in plane polar, axisymmetric cylindrical and spherical polar coordinates. The expressions for the streamlines and velocity components are given through the solution of the involved highly non-linear equations. Inverse methods have been employed for the solutions. Several previous results have been deduced from the presented analysis.

4) The work of chapter 5 is concerned with the unsteady flow of a third grade fluid over an infinite plate. The velocity field is obtained by solving a non-linear equation. Two flows induced by the plate are considered. These flows are generated due to the shear stress. Both analytical and numerical solutions of non-linear equation with non-linear boundary conditions are developed. It is observed that there is a very good agreement between the numerical and perturbation solution for small values of t $(t<1)$. For t greater than 3, there is a sufficient discrepancy in the results that the perturbation solution can no longer be accepted and the results from the numerical solution should only be used. However, when shear stress has an oscillatory character, then perturbation results are acceptable.

5) The objective of chapter 6 is to discuss the unsteady flow of a third grade fluid over an infinite plate with variable suction. The non-linear equation resulting from the momentum equation has been solved using similarity transformation and perturbation technique. It is noted that for short time $(\tau=4)$, a strong non-Newtonian effect is present in the velocity field and velocity behaves as a Newtonain case for large time $(\tau=100)$.

6) Chapter 7 is devoted to the flow of a third grade fluid over a porous plate, which executes oscillations in its own plane with superimposed blowing or suction. The modeled non-linear equation has been solved for the velocity field. Moreover, increasing or decreasing velocity amplitude of the oscillating porous plate is examined. Finally, it is seen that several interesting results of the previous studies can be taken as the special cases of the presented analysis.

Publish Work from the Thesis of the Author

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Chapter 0

Introduction

Multicomponent flows whether occurring in nature such us debris flows, avalanches, and mud slides, or in industrial applications, such as fluidized beds, solids transport and many other chemical and agricultural processes, present a formidable challenge to engineers and scientists. To model and study the flow and behavior of such complex fluids, one can use either statistical theories or continuum theories, in addition to the phenomenological/experimental approaches.

Due to various properties of real fluids there are many models. The simplest model is Navier-Stokes model which is used for fluids of low molecular weight. However, it is well known that materials with complex structures such as solutions and melts of polymers, plastic and synthetic fibers, certain oils and greases, soap and detergents, certain pharmaceutical and biological fluids fall into the category of non-Newtonian fluids. During the last several years, generalization of Navier-Stokes model to highly non-linear constitutive laws have been proposed because of their interest in applications to industry and technology. In order to explain several non-standard features, such as normal stress effects, rod climbing, shear thinning and shear thickening, Rivlin-Ericksen fluids [1] of differential type are introduced. These fluids are rather complex from the point of view of partial differential equation theory. Nevertheless, several authors in fluid mechanics are now engaged with the equations of motion on non-Newtonian fluids of second and third grade. In particular, some authors are interested in studying n-grade fluids as self-consistent models and not as approximating models. Therefore, in studying dynamics they ask that all the flows meet the Clausius-Duhem inequality and that the specific Helmholtz free energy of the fluid is a minimum at equilibrium [2]. On the other hand, it is under the same hypothesis that the Navier-Stokes model is studied. That is, it is always assumed that some real fluids exist such that Navier-Stokes or $n-$grade fluids are exact models, and not truncations of viscoelastic fluids. Moreover (as noted in refs. [3, 5]), different assumptions could heavily affect the rest state stability. Under these thermodynamically hypothesis, several results concerning existence and stability have been obtained [3, 4, 5].

The formulation of shear stress for non-Newtonian fluids is a difficult problem, which has not progressed very far from a theoretical standpoint. However, there is no single model which clearly exhibits all the properties of non-Newtonian fluids. For a more fundamental understanding, several empirical descriptions have established rheological models. For example, in most of these models, a significant drag past solid walls has been observed. A discussion of the various differential, rate-type, and integral models can be found in Schowalter [6], Huilgol [7], and Rajagopal [8].

The flow of non-Newtonian fluids has gained considerable importance because of its applications in various branches of science, engineering, and technology: particularly in material processing, chemical industries, geophysics, and bio-engineering. The study of non-Newtonian fluid flow is also of significant interest in oil reservoir engineering. For a variety of reasons, non-Newtonian fluids are classified on the basis of their behavior in shear. A fluid with a linear relationship between the shear stress and the shear rate, giving rise to a constant viscosity, is always characterized to be a Newtonian fluid. As a constant viscosity relation is not always a Newtonian fluid relation because there are fluids like a second grade fluid, a convected Maxwell fluid, and an Oldroyd fluid A and B that are certainly non-Newtonian, but also show a constant viscosity. Second grade fluid model is a subclass of differential type fluids for which one can reasonably hope to obtain analytical solutions. The fluids of differential type have usually higher order partial differential equations than the Navier-Stokes equations. So the issue of whether the 'no-slip' boundary condition is sufficient to have a well-posed problem is very important. This question cannot be answered in any generalization for fluids of grade 2 or grade 3, one can provide some definite answers, while some partial answers are possible for fluids of grade n [9].

In general, for fluids of the differential type of grade n, the equations of motion are of order $(n+1)$. Thus, if $n > 1$, then the adherence boundary condition is insufficient for determinacy. The standard method used to overcome this difficulty is to resort the perturbation that lowers the order of the equation [10 – 16], which is not mathematically rigorous. In fact, the workers are aware of this, but in the absence of any rational method for generating additional boundary conditions, they have no other way out of the impasse. It is possible that in flows in unbounded domains, we can obtain additional conditions based on the asymptotic structure of the flow at infinity. Mansutti *et al.* [17] showed that results by perturbation method and of augmenting the boundary conditions agree remarkably well. Rajagopal and Gupta have also discussed this issue in the reference [18] and studied the steady flow of a second-grade fluid past a porous plate. In another paper, Rajagopal [19] studied some unidirectional flows of a second grade fluid. In [20], Foote *et al.* studied the problem for the flow of an elastico-viscous fluid on an oscillating porous plate. Hayat *et al.* [21 – 25], Asghar *et al.* [26] and Siddiqui *et al.* [27 – 29] discussed the flows of differential type fluids in various geometrical configurations.

In many fields, such as food industry, drilling operations and bio-engineering, the fluids, either synthetic or natural, are mixtures of different constituents such as water, particle, oils, red cells and other long chain molecules; this combination imparts strong non-Newtonian characteristics to the resulting liquids; the viscosity function varies non-linearly with the shear rate; elasticity is felt through elongational effects and time-dependent effects. In these cases, the fluids have been treated as viscoelastic fluids. Because of the difficulty to suggest a single model which exhibits all properties of viscoelastic fluids, they cannot be described as simply as Newtonian fluids. For this reason, many models or constitutive equations have been proposed and most of them are empirical or semi-empirical. For more general three-dimensional representation, the method of continuum mechanics is needed. One of the most popular models for non-Newtonian fluids is the model that is called the second-grade fluid. Several authors [30 – 36] in fluid mechanics are now engaged with the equations of motion of second grade fluids.

Exact solutions are very important not only because they are solutions of some fundamental flows but also because they serve as accuracy checks for experimental, numerical and asymptotic methods. Navier-Stokes equations are non-linear partial differential equations for viscous fluids. For this reason, there exist only a limited number of exact solutions in which the non-linear inertial terms do not disappear automatically. These analytic solutions become even rare if non-Newtonian constitutive equations are considered in the equations of motion. This is because the resulting equations are highly non-linear partial differential equations. While studying second grade fluid the equations, in general, are one order higher than the Navier-Stokes equations. The third order equations of the second grade fluid flows, in general, require an additional boundary and/or initial condition in addition to those required for solving the Navier-Stokes equations. The necessity of this extra condition can be avoided by the application of the inverse method. This provides the motivation that, in some specific situations, the inverse method becomes attractive in studying the non-Newtonian fluids.

Usually, in the inverse method, the boundary conditions are not prescribed and solution of the differential equations are sought by assuming specific geometrical or physical properties of the field. Nemenyi [37] has given an excellent survey along with the applications in various fields of the mechanics of continua. Kaloni and Huschilt [38] used the inverse methods to study plane steady flow problems of a second grade fluid. Siddiqui and Kaloni [28] employed this approach to find the exact solutions for steady flows of a second grade fluid in plane polar, axisymmetric cylindrical and axisymmetric spherical coordinates.

There is a large class of processes which can be considered from the mathematical point of view as the motion of the fluid (liquid) between two parallel plates, moving towards each other or in opposite directions with a constant velocity. These include such processes as the motion of a fluid through a hydraulic pump and the motion of the underground fluid. We can observe that when the plates are approaching each other in a second grade fluid, the effort required is smaller than that when the plates are moving apart. When the plates are approaching each other it is of potential type and when they are moving away then it is of rotational nature. For such considerations the horizontal components of the velocity u, v, do not depend on the vertical components z, whereas the vertical velocity w depends linearly on the distance between the plates. This brings the motivation to model this situation in a second grade fluid and then to discuss few specific solutions of our interest in chapter 2. The contents of this work has been published in **Archives of Mechanics**, 55, 373 – 387 (2003).

The swirling flows have great importance in a number of industrial and practical applications. Spiral galaxies, atmospheric or oceanic circulation, bathtub vortices, or even stirring tea in a cup, are examples that illustrate the ubiquity of swirling flows at all scales in nature. In such flows the flow is usually axisymmetric (independent of the meridional angle θ) and second component of the velocity V_θ, is expressed in terms of the swirl Ω (angular momentum per unit mass). The resulting equations arising from the balance of linear momentum, are highly non-linear partial differential equations whose general solution in closed form is not possible to obtain. Few specific situations are considered in order to find the analytical solution of these equations both for Newtonian and non-Newtonian cases. Eleven steady and non-steady flows are discussed. This work has been published in **Nonlinear Dynamics**, **35**, 229 – 248 (2004).

In chapter 4, the time dependent flow equations are modeled in plane polar, axisymmetric cylindrical, and axisymmetric spherical coordinates. The obtained equations are coupled by introducing the stream functions into a single equation. The governing equations thus obtained are highly non-linear partial differential equations, whose general solution is not possible even for the Newtonian fluid. The solutions of these equations help to understand the properties and behavior of the non-linear fluids. Applying the inverse method on the most general equation we have proposed solutions to that equations, and in return the conditions are obtained on the fluid, which have the given solutions. Several limiting situations along with their amplifications are deduced and are compared with the known results already given in the literature (both for Newtonian and second grade fluids). This attempt is accepted for publication in **Mathematical Problems in Engineering.**

Although the second-grade fluid model is able to predict the normal stress differences, which are characteristics of non-Newtonian fluids, it does not take into account the shear thinning and thickening phenomena that many show. The third-grade fluid model represents a further, although inconclusive, attempt toward a comprehensive description of the properties of viscoelastic fluids. With this in view the flow of an incompressible unidirectional thermodynamically compatible third grade fluid over an infinite plate is analyzed in chapter 5. The infinite plate is placed along $x-$axis and $y-$axis is perpendicular to it. The plate is under a variable shear stress depending upon time. Incidentally, the time-dependent shear stress makes the boundary conditions non-linear. Two different situations are discussed when shear stress is proportional to $e^{\lambda t}$ and $e^{i\omega t}$ respectively. In the former case for positive λ, numerical and perturbation solutions are obtained whereas in the latter case, only perturbation solution is given. In the former case, it is observed that there is a very good agreement between the numerical and perturbation solution for small values of t ($t < 1$). For t greater than 3, there is a sufficient discrepancy in the results that the perturbation solution can no longer be accepted and the results from the numerical solution only should be used. However, when shear stress has an oscillatory character, the perturbation results are acceptable. It is found graphically that with an increase in second and third grade parameter the velocity decreases and the boundary layer thickness decreases. This analysis has been accepted in **Canadian Journal of Physics**.

Chapter 6 is devoted to study the unsteady problem of an incompressible third grade fluid past a porous plate. The infinite porous plate is aligned along the $x-$axis and flow is planar. The flow is induced due to sudden motion of a plate. The modeled flow equation is a highly non-linear partial differential equation with all non-zero third grade material parameters. Also the equation is of fourth order and there are only two boundary conditions. Here, the partial differential equation is converted into an ordinary differential equation using similarity transformation, which has been solved using perturbation in the inverse powers of time. It is observed that with an increase in suction, the boundary layer thickness decreases and with an increase in blowing, the boundary layer thickness increases. It is also noted that for short time ($\tau = 4$), a strong non-Newtonian effect is present in the velocity field and velocity behaves as a Newtonian case for large time ($\tau = 100$). These observations are published in **Mathematical and Computer Modelling 38**, 201 – 208 (2003).

The flow of a third grade fluid induced due to the oscillations of a porous plate is presented in chapter 7. We have considered the thermodynamical third grade model and flow is unidirectional with constant suction/blowing. The modeled equation is a third order partial differential equation which is solved by perturbation method. The porous plate is executing oscillations in its own plane with superimposed blowing or suction. An increasing or decreasing velocity amplitude of the oscillating porous plate is also examined. It is found that with the increase in material parameters of the third-grade fluid the velocity boundary layer thickness decreases in the case of suction and increases in the case of blowing and the amplitude of oscillation decreases for acceleration and increases for deceleration. Results for second grade and viscous fluids are obtained from the present analysis as the special cases. The contents of this chapter have been published in **Mathematical Problems in Engineering 2**, 133 – 143 (2004).

Chapter 1

Preliminaries and basic equations

1.1 Introduction

This chapter deals with basic definitions and derivations of the governing equations which will provide background for the succeeding chapters. The general expression for the nth-Rivlin-Ericksen tensor is also derived.

1.2 Non-Newtonian fluids

An abundance of literature deals with the solution of various types of fluids. Amongst these fluids, the Newtonian fluid is the simplest to be solved, not only numerically but also analytically. The governing equation that describes the flow of a Newtonian fluid is the Navier-Stokes equation. A literature survey indicated that applications of Newtonian fluid is very limited. This is due to the fact that many fluids used in the chemical, mechanical and other industries deviate from the Newtonian fluids. They are non-Newtonian fluids and there has been relatively scarce information about these. Now, non-Newtonian fluids are increasingly being recognized as more appropriate in modern technological applications in comparison with Newtonian fluids. Because of the non-linear nature of the dependence of stresses on the rate of strain for non-Newtonian fluids, the solution of flow problems for these fluids in general are more difficult to obtain. This is not only true of exact analytical solutions but even of numerical solutions. The non-Newtonian nature of the fluids also increases the order of the differential equation in general. Due to complexity of fluids, there are several proposed models. In the present thesis we will consider subclasses of the differential type fluids namely second and third-grade fluids.

The constitutive equation for second-grade fluid is

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1.1}$$

where $\mathbf{T}$ is the Cauchy stress, $-p\mathbf{I}$ is the spherical part of the stress due to constraint of incompressibility, p is the scalar pressure, $\mathbf{I}$ is the identity tensor, μ, α_1 and α_2 are measurable material constants. They denote, respectively, the viscosity, elasticity and cross-viscosity. $\mathbf{A}_1$ and $\mathbf{A}_2$ are Rivlin-Ericksen kinematical tensors [1] and they denote, respectively, the rate of strain and acceleration. The Rivlin-Ericksen kinematical tensors $\mathbf{A}_n$, are described as [1]

$$\begin{aligned} \mathbf{A}_0 &= \mathbf{I}, \\ \mathbf{A}_1 &= (\text{grad}\mathbf{V}) + (\text{grad}\mathbf{V})^{\top}, \\ \mathbf{A}_{n+1} &= \frac{d\mathbf{A}_n}{dt} + \mathbf{A}_n(\text{grad}\mathbf{V}) + (\text{grad}\mathbf{V})^{\top}\mathbf{A}_n, \quad n \geq 1, \end{aligned} \tag{1.2}$$

in which $\mathbf{V}$ denotes the velocity field, grad is the gradient operator, $\top$ is the transpose and $d/dt(\cdot) = \frac{\partial}{\partial t}(\cdot) + (\mathbf{V}\cdot\text{grad})(\cdot)$ is the material time derivative, where first term describes the local part and the second term is the convective part. Using Eqs. (1.1) and (1.2) into the balance of linear momentum

$$\rho\frac{d\mathbf{V}}{dt} = \rho\chi + \text{div}\mathbf{T}, \tag{1.3}$$

and making use of some vector identities we get the following equation

$$\begin{aligned} &\text{grad}\left[\frac{1}{2}\rho|\mathbf{V}|^2 + p - \alpha_1\left(\mathbf{V}\cdot\nabla^2\mathbf{V} + \frac{1}{4}|\mathbf{A}_1|^2\right)\right] + \rho\left[\mathbf{V}_t - \mathbf{V}\times(\nabla\times\mathbf{V})\right] \\ &= \mu\nabla^2\mathbf{V} + \alpha_1\left[\nabla^2\mathbf{V}_t + \nabla^2(\nabla\times\mathbf{V})\times\mathbf{V}\right] + (\alpha_1+\alpha_2)\,\text{div}\mathbf{A}_1^2 + \rho\chi. \end{aligned} \tag{1.4}$$

In above equations ∇^2 is the Laplacian operator, ρ is the constant density, χ is the body force, $\mathbf{V}_t = \partial\mathbf{V}/\partial t$, and $|\mathbf{A}_1|$ is the usual norm of matrix $\mathbf{A}_1$. We name above equation a *Master equation* as it will help us to model the governing equations in Cartesian, plane polar, axisymmetric cylindrical and axisymmetric spherical coordinates, which will be used in the next chapters.

Second order fluids are dilute polymeric solutions (e.g. polyisobutylene, methyl-methacrylate in n butyl acetate, polyethylene oxide in water, etc.). The equation is frame invariant and applicable for low shear rates. A detailed account on the characteristics of second grade fluids is well documented by Dunn and Rajagopal [3]. Theoretical investigations by Dunn and Fosdick [39] and Fosdick and Rajagopal [2] have indicated that for an exact model, satisfying the Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy be a minimum in equilibrium, the following conditions must hold:

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_1 + \alpha_2 = 0. \tag{1.5}$$

A detailed discussion regarding the signs of the material parameter has been given in Dunn and Rajagopal [3]. For third grade fluid , the expression for $\mathbf{T}$ is

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2\left(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1\right) + \beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1, \tag{1.6}$$

where β_1, β_2, and β_3 are additional material constants. Fosdick and Rajagopal [2] has discussed the thermodynamics of fluids modeled exactly through Eq. (1.6). The Clausius-Duhem inequality and the assumption that the specific Helmholtz free energy is minimum at equilibrium provide the following restrictions

$$\mu \geq 0, \ \alpha_1 \geq 0, \ \mid \alpha_1 + \alpha_2 \mid \leq \sqrt{24\mu\beta_3}, \ \beta_1 = \beta_2 = 0, \ \beta_3 \geq 0. \tag{1.7}$$

1.3 Equation of continuity

The conservation of mass for compressible fluid is

$$\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho\mathbf{V}) = 0. \tag{1.8}$$

For incompressible fluid above equation simplifies to

$$\nabla \cdot \mathbf{V} = 0. \tag{1.9}$$

1.4 Strain rate and vorticity tensors

The velocity gradient tensor $\nabla\mathbf{V}$ can be decomposed into a symmetric part $\mathbf{D}$ and antisymmetric part $\mathbf{W}$

$$\mathbf{D} = \frac{1}{2}(\nabla\mathbf{V} + \nabla\mathbf{V}^{\mathsf{T}}) = \frac{1}{2}\dot{\gamma}, \quad \mathbf{W} = \frac{1}{2}(\nabla\mathbf{V} - \nabla\mathbf{V}^{\mathsf{T}}) = \frac{1}{2}\omega, \tag{1.10}$$

where $\dot{\gamma}$ is called the rate of strain tensor and ω is called the vorticity tensor. Also it is noted that $\dot{\gamma} = (\nabla\mathbf{V} + \nabla\mathbf{V}^{\mathsf{T}})$ is equal to the first Rivlin-Ericksen kinematic tensor. The vorticity is defined by

$$\omega = \nabla\mathbf{V} - \nabla\mathbf{V}^{\mathsf{T}} = \nabla \times \mathbf{V}. \tag{1.11}$$

The reason for this is two fold. First, the rules governing the evaluation of vorticity are somewhat simpler than those governing the velocity field. For example, pressure gradient appear as a source of linear momentum in Eq. (1.3), yet the pressure itself depends on the instantaneous distribution of $\mathbf{V}$. By focusing on vorticity, on the other hand, we may dispense with the pressure field entirely. The second reason for studying vorticity is that many flows are characterized by localized regions of intense rotation (i.e. vorticity). Smoke rings, dust whirls in the street, trailing vortices on aircraft wings, whirlpools, tidal vortices, tornadoes, hurricanes and the great red spot of Jupiter represent just a few examples.

1.5 Gradient operator

The gradient operator is defined as

$$\nabla = \mathbf{e}_j \frac{\partial}{\partial x_j} = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3}, \tag{1.12}$$

in which $\mathbf{e}_j$ $(j = 1, 2, 3)$ are unit vectors.

1.6 Gradient of a scalar

The gradient of a scalar point function can be calculated as

$$\nabla\overline{\phi} = e_j\frac{\partial\overline{\phi}}{\partial x_j} = e_1\frac{\partial\overline{\phi}}{\partial x_1} + e_2\frac{\partial\overline{\phi}}{\partial x_2} + e_3\frac{\partial\overline{\phi}}{\partial x_3} \tag{1.13}$$

1.7 Gradient of velocity

The gradient of velocity **V** is defined as

$$\nabla\mathbf{V} = \left(e_i\frac{\partial}{\partial x_i}\right)(V_j e_j) = e_i e_j\frac{\partial V_j}{\partial x_i}, \tag{1.14}$$

where a matrix representation is given by

$$\begin{bmatrix} \frac{\partial V_1}{\partial x_1} & \frac{\partial V_2}{\partial x_1} & \frac{\partial V_3}{\partial x_1} \\ \frac{\partial V_1}{\partial x_2} & \frac{\partial V_2}{\partial x_2} & \frac{\partial V_3}{\partial x_2} \\ \frac{\partial V_1}{\partial x_3} & \frac{\partial V_2}{\partial x_3} & \frac{\partial V_3}{\partial x_3} \end{bmatrix}$$

and V_i $(i = 1, 2, 3)$ are the velocity components.

1.8 Divergence of a vector

The divergence of a vector is defined by

$$\begin{aligned} \nabla\cdot\mathbf{V} &= \left(e_i\frac{\partial}{\partial x_i}\right)\cdot(V_j e_j) = e_i\cdot e_j\frac{\partial V_j}{\partial x_i} = \delta_{ij}\frac{\partial V_j}{\partial x_i} \\ &= \frac{\partial V_i}{\partial x_i} = \frac{\partial V_1}{\partial x_1} + \frac{\partial V_2}{\partial x_2} + \frac{\partial V_3}{\partial x_3}. \end{aligned} \tag{1.15}$$

1.9 Curl of a vector

The curl of a vector is

$$\begin{aligned}\nabla \times \mathrm{V} &= \left(\mathrm{e}_i \frac{\partial}{\partial x_i}\right) \times (V_j \mathrm{e}_j) = \mathrm{e}_i \times \mathrm{e}_j \frac{\partial V_j}{\partial x_i} \\ &= \mathrm{e}_1 \left(\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3}\right) + \mathrm{e}_2 \left(\frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}\right) + \mathrm{e}_3 \left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right).\end{aligned} \tag{1.16}$$

1.10 Divergence of a tensor

The divergence of a tensor is defined by

$$\nabla \cdot \mathrm{S} = \left(\mathrm{e}_k \frac{\partial}{\partial x_k}\right) \cdot (S_{ij} \mathrm{e}_i \mathrm{e}_j) = \mathrm{e}_j \frac{\partial S_{ij}}{\partial x_i}. \tag{1.17}$$

1.11 Non-Cartesian frames

All the definitions for gradient and divergence of a tensor remain valid in a non-Cartesian frame, provided that the derivative operation is also applied to the basis vectors as well. We illustrate this process in two important frames, cylindrical and spherical coordinate systems.

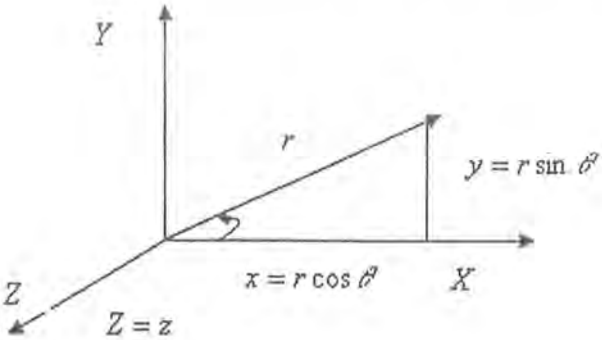

Fig. 1.1. Cylindrical coordinate system

1.11.1 Cylindrical coordinates

In cylindrical coordinate system, points are located by giving them values to $\{r, \theta, z\}$, which are related to $\{x = x_1,\ y = x_2,\ z = x_3\}$ by (see Fig. 1.1)

$$
\begin{aligned}
x &= r\cos\theta, \quad y = r\sin\theta, \quad z = z, \\
r &= \left(x^2 + y^2\right)^{\frac{1}{2}}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right), \quad z = z.
\end{aligned}
$$

The basis vectors in this frame are related to the Cartesian ones by

$$
\begin{aligned}
\mathbf{e}_r &= \cos\theta\ \mathbf{e}_x + \sin\theta\ \mathbf{e}_y, \quad \mathbf{e}_x = \cos\theta\ \mathbf{e}_r - \sin\theta\ \mathbf{e}_\theta, \\
\mathbf{e}_\theta &= -\sin\theta\ \mathbf{e}_x + \cos\theta\ \mathbf{e}_y, \quad \mathbf{e}_y = \sin\theta\ \mathbf{e}_r + \cos\theta\ \mathbf{e}_\theta.
\end{aligned}
$$

The velocity $\mathbf{V}$, a tensor $\mathbf{S}$, gradient operator, grad$\mathbf{V}$ and div$\mathbf{V}$ in terms of these coordinates are respectively given by

$$
\mathbf{V} = V_r\mathbf{e}_r + V_\theta\mathbf{e}_\theta + V_z\mathbf{e}_z = (V_r, V_\theta, V_z), \tag{1.18}
$$

$$
\begin{aligned}
\mathbf{S} &= S_{rr}\mathbf{e}_r\mathbf{e}_r + S_{r\theta}\mathbf{e}_r\mathbf{e}_\theta + S_{rz}\mathbf{e}_r\mathbf{e}_z + S_{\theta r}\mathbf{e}_\theta\mathbf{e}_r + S_{\theta\theta}\mathbf{e}_\theta\mathbf{e}_\theta + S_{\theta z}\mathbf{e}_\theta\mathbf{e}_z + S_{zr}\mathbf{e}_z\mathbf{e}_r \\
&\quad + S_{z\theta}\mathbf{e}_z\mathbf{e}_\theta + S_{zz}\mathbf{e}_z\mathbf{e}_z,
\end{aligned}
$$

where a matrix representation is given by

$$
\begin{bmatrix}
S_{rr} & S_{r\theta} & S_{rz} \\
S_{\theta r} & S_{\theta\theta} & S_{\theta z} \\
S_{zr} & S_{z\theta} & S_{zz}
\end{bmatrix},
$$

$$
\begin{aligned}
\nabla &= (\cos\theta\ \mathbf{e}_r - \sin\theta\ \mathbf{e}_\theta)\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) \\
&\quad + (\sin\theta\ \mathbf{e}_r + \cos\theta\ \mathbf{e}_\theta)\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right) + \mathbf{e}_z\frac{\partial}{\partial z} \\
&= \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{\partial}{\partial z}\right),
\end{aligned}
$$

$$
\begin{aligned}
\nabla\mathbf{V} &= \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right)(V_r\mathbf{e}_r + V_\theta\mathbf{e}_\theta + V_z\mathbf{e}_z) \\
&= \mathbf{e}_r\mathbf{e}_r\frac{\partial V_r}{\partial r} + \mathbf{e}_r\mathbf{e}_\theta\frac{\partial V_\theta}{\partial r} + \mathbf{e}_r\mathbf{e}_z\frac{\partial V_z}{\partial r} + \mathbf{e}_\theta\mathbf{e}_r\left(\frac{1}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta}{r}\right) \\
&\quad + \mathbf{e}_\theta\mathbf{e}_\theta\left(\frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r}{r}\right) + \mathbf{e}_\theta\mathbf{e}_z\frac{1}{r}\frac{\partial V_z}{\partial\theta} + \mathbf{e}_z\mathbf{e}_r\frac{\partial V_r}{\partial z} + \mathbf{e}_z\mathbf{e}_\theta\frac{\partial V_\theta}{\partial z} + \mathbf{e}_z\mathbf{e}_z\frac{\partial V_z}{\partial z}, \quad (1.20)
\end{aligned}
$$

where a matrix representation is given by

$$
\begin{bmatrix}
\frac{\partial V_r}{\partial r} & \frac{\partial V_\theta}{\partial r} & \frac{\partial V_z}{\partial r} \\
\frac{1}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta}{r} & \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r}{r} & \frac{1}{r}\frac{\partial V_z}{\partial\theta} \\
\frac{\partial V_r}{\partial z} & \frac{\partial V_\theta}{\partial z} & \frac{\partial V_z}{\partial z}
\end{bmatrix},
$$

$$
\nabla\cdot\mathbf{V} = \frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r}{r} + \frac{\partial V_z}{\partial z}, \quad (1.21)
$$

where we have used the following relations

$$
\begin{aligned}
\frac{\partial}{\partial r}\mathbf{e}_r &= 0, \quad \frac{\partial}{\partial r}\mathbf{e}_\theta = 0, \quad \frac{\partial}{\partial r}\mathbf{e}_r = 0, \quad \frac{\partial}{\partial\theta}\mathbf{e}_r = \mathbf{e}_\theta, \quad \frac{\partial}{\partial\theta}\mathbf{e}_\theta = -\mathbf{e}_r, \\
\frac{\partial}{\partial\theta}\mathbf{e}_z &= 0, \quad \frac{\partial}{\partial z}\mathbf{e}_r = 0, \quad \frac{\partial}{\partial z}\mathbf{e}_\theta = 0, \quad \frac{\partial}{\partial z}\mathbf{e}_z = 0.
\end{aligned}
$$

1.11.2 Spherical coordinates

In a spherical coordinate system, points are located by giving them values to $\{r, \theta, \phi\}$, which are related to $\{x = x_1,\, y = x_2,\, z = x_3\}$ by

$$\begin{aligned} x &= r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \\ r &= \sqrt{x^2+y^2+z^2}, \quad \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right), \quad \phi = \tan^{-1}\left(\frac{y}{x}\right). \end{aligned}$$

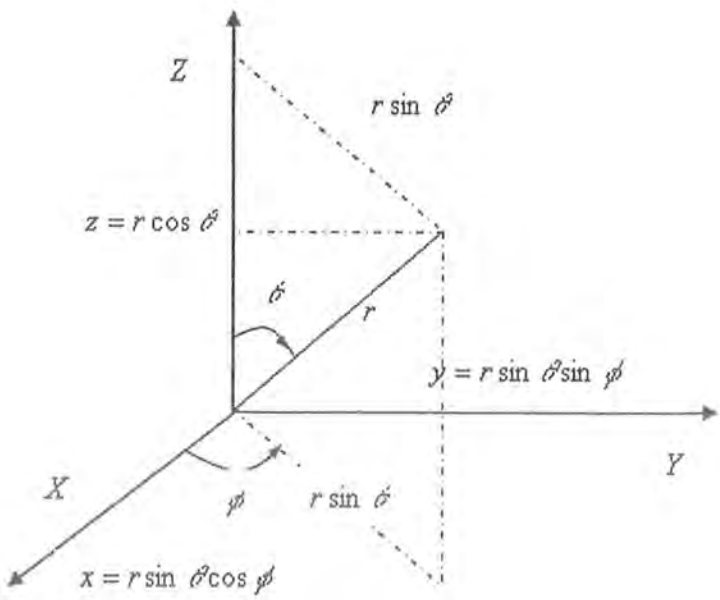

Fig. 1.2. Spherical frame of reference

The basis vectors are related by

$$\begin{aligned} \mathbf{e}_r &= \mathbf{e}_1\sin\theta\cos\phi + \mathbf{e}_2\sin\theta\sin\phi + \mathbf{e}_3\cos\theta, \\ \mathbf{e}_\theta &= \mathbf{e}_1\cos\theta\cos\phi + \mathbf{e}_2\cos\theta\sin\phi - \mathbf{e}_3\sin\theta, \\ \mathbf{e}_\phi &= -\mathbf{e}_1\sin\phi + \mathbf{e}_2\cos\phi, \end{aligned}$$

and

$$
\begin{aligned}
\mathbf{e}_1 &= \mathbf{e}_r \sin\theta\cos\phi + \mathbf{e}_\theta\cos\theta\cos\phi - \mathbf{e}_\phi\sin\phi, \\
\mathbf{e}_2 &= \mathbf{e}_r \sin\theta\sin\phi + \mathbf{e}_\theta\cos\theta\sin\phi + \mathbf{e}_\phi\cos\phi, \\
\mathbf{e}_3 &= \mathbf{e}_r\cos\theta - \mathbf{e}_\theta\sin\phi.
\end{aligned}
$$

In spherical coordinates we have the following:

$$
\nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}, \tag{1.22}
$$

$$
\begin{aligned}
\nabla\mathbf{V} &= \left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)(V_r\mathbf{e}_r + V_\theta\mathbf{e}_\theta + V_\phi\mathbf{e}_\phi) \\
&= \mathbf{e}_r\frac{\partial}{\partial r}(V_r\mathbf{e}_r + V_\theta\mathbf{e}_\theta + V_\phi\mathbf{e}_\phi) + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}(V_r\mathbf{e}_r + V_\theta\mathbf{e}_\theta + V_\phi\mathbf{e}_\phi) \\
&\quad +\mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(V_r\mathbf{e}_r + V_\theta\mathbf{e}_\theta + V_\phi\mathbf{e}_\phi) \qquad (1.23) \\
&\quad \mathbf{e}_r\mathbf{e}_r\frac{\partial V_r}{\partial r} + \mathbf{e}_r\mathbf{e}_\theta\frac{\partial V_\theta}{\partial r} + \mathbf{e}_r\mathbf{e}_\phi\frac{\partial V_\phi}{\partial r} + \mathbf{e}_\theta\mathbf{e}_r\left(\frac{1}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta}{r}\right) + \mathbf{e}_\theta\mathbf{e}_\theta\left(\frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r}{r}\right) \\
&\quad +\mathbf{e}_\phi\mathbf{e}_r\left(\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{V_\phi}{r}\right) + \mathbf{e}_\theta\mathbf{e}_\phi\frac{1}{r}\frac{\partial V_\phi}{\partial\theta} + \mathbf{e}_\phi\mathbf{e}_\theta\left(\frac{1}{r\sin\theta}\frac{\partial V_\theta}{\partial\phi} - \frac{V_\phi}{r}\cot\theta\right) \\
&\quad +\mathbf{e}_\phi\mathbf{e}_\phi\left(\frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi} + \frac{V_r}{r} + \frac{V_\theta}{r}\cot\theta\right) \\
&= \begin{bmatrix}
\frac{\partial V_r}{\partial r} & \frac{\partial V_\theta}{\partial r} & \frac{\partial V_\phi}{\partial r} \\
\frac{1}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta}{r} & \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r}{r} & \frac{1}{r}\frac{\partial V_\phi}{\partial\theta} \\
\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{V_\phi}{r} & \frac{1}{r\sin\theta}\frac{\partial V_\theta}{\partial\phi} - \frac{V_\phi}{r}\cot\theta & \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi} + \frac{V_r}{r} + \frac{V_\theta}{r}\cot\theta
\end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\nabla \cdot \mathbf{V} &= \left(\mathbf{e}_r\frac{\partial}{\partial r}+\mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}+\mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(V_r\mathbf{e}_r+V_\theta\mathbf{e}_\theta+V_\phi\mathbf{e}_\phi\right) \qquad (1.24)\\
&= \left(\mathbf{e}_r\frac{\partial}{\partial r}\right)\cdot\left(V_r\mathbf{e}_r+V_\theta\mathbf{e}_\theta+V_\phi\mathbf{e}_\phi\right)+\left(\mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)\cdot\left(V_r\mathbf{e}_r+V_\theta\mathbf{e}_\theta+V_\phi\mathbf{e}_\phi\right)\\
&\quad+\left(\mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(V_r\mathbf{e}_r+V_\theta\mathbf{e}_\theta+V_\phi\mathbf{e}_\phi\right)\\
&= \frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}+\frac{u_\theta\cos\theta}{r\sin\theta}+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}\\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2u_r\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(u_\theta\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}.
\end{aligned}
$$

In deriving above expressions we have used the following relations

$$
\begin{aligned}
\frac{\partial}{\partial r}\mathbf{e}_r &= 0,\ \frac{\partial}{\partial r}\mathbf{e}_\theta=0,\ \frac{\partial}{\partial r}\mathbf{e}_\phi=0,\ \frac{\partial}{\partial\theta}\mathbf{e}_r=\mathbf{e}_\theta,\ \frac{\partial}{\partial\theta}\mathbf{e}_\theta=-\mathbf{e}_r,\ \frac{\partial}{\partial\theta}\mathbf{e}_\phi=0,\\
\frac{\partial}{\partial\phi}\mathbf{e}_r &= \mathbf{e}_\phi\sin\theta,\ \frac{\partial}{\partial\phi}\mathbf{e}_\theta=\mathbf{e}_\phi\cos\theta,\ \frac{\partial}{\partial\phi}\mathbf{e}_\phi=-\mathbf{e}_r\sin\theta-\mathbf{e}_\theta\cos\theta.
\end{aligned}
$$

1.12 Symmetric and antisymmetric part of the velocity gradient

Eulerian description of acceleration is given by

$$
\mathbf{a}=\frac{\partial\mathbf{V}}{\partial t}+\left(\mathbf{V}\cdot\nabla\right)\mathbf{V}=\frac{d\mathbf{V}}{dt}. \qquad (1.25)
$$

Since $\mathbf{L}=\nabla\mathbf{V}=L_{ij}$ is a second rank tensor and as every tensor of rank 2 can be written as a sum of symmetric and antisymmetric tensors, therefore

$$
\mathbf{L}=\frac{1}{2}\left(\mathbf{L}+\mathbf{L}^\top\right)+\frac{1}{2}\left(\mathbf{L}-\mathbf{L}^\top\right)=\mathbf{D}+\mathbf{W}, \qquad (1.26)
$$

where the symmetric part $\mathbf{D}$ is called the rate of strain tensor and antisymmetric part $\mathbf{W}$ is called the vorticity tensor. We know that the strain tensor is defined by

$$
\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)=\frac{1}{2}\left(\mathbf{L}+\mathbf{L}^\top\right).
$$

Hence

$$\mathbf{D} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \varepsilon_{ij}, \tag{1.27}$$

$$\mathbf{W} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) = w_{ij}. \tag{1.28}$$

1.13 Rivlin-Ericksen tensor

The $n-$th Rivlin Ericksen tensor is defined as

$$\mathbf{A}_n(t) = \frac{d^n}{d\tau^n}\mathbf{C}_t(\tau)\mid_{\tau=t}, \quad n = 1, 2, ... \tag{1.29}$$

At start we assume that there is no deformations at $\tau = t$

$$\therefore \quad \mathbf{A}_0 = \mathbf{C}_t(\tau)\mid_{\tau=t} = \mathbf{I} \tag{1.30}$$

and

$$\mathbf{A}_1(t) = \frac{d}{d\tau}\left[\mathbf{C}_t(\tau)\right] = \frac{d}{d\tau}\left[\{\mathbf{F}_t(\tau)\}^\top \mathbf{F}_t(\tau)\right]. \tag{1.31}$$

In above equation we have used $\mathbf{C} = \mathbf{F}^\top\mathbf{F}$ as the right Cauchy-Green tensor.

Consider

$$\frac{d}{d\tau}\left(\mathbf{F}_t(\tau)\right) = \frac{d}{d\tau}\frac{\partial \xi_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{d\xi_i}{d\tau}\right) = \frac{\partial u_i}{\partial x_j} = \mathbf{L}, \tag{1.32}$$

where the position of the particle is ξ_i and

$$\frac{d}{d\tau}\left[\mathbf{F}_t(\tau)\right]^\top = \left[\frac{d}{d\tau}\mathbf{F}_t(\tau)^\top\right] = \mathbf{L}^\top. \tag{1.33}$$

Therefore

$$\begin{aligned} \frac{d}{d\tau}\left[\mathbf{C}_t(\tau)\right] &= \frac{d}{d\tau}\left[\mathbf{F}_t(\tau)^\top \mathbf{F}_t(\tau)\right] = \mathbf{F}_t(\tau)^\top \frac{d}{d\tau}\mathbf{F}_t(\tau) + \mathbf{F}_t(\tau)\frac{d}{d\tau}\mathbf{F}_t(\tau)^\top \\ &= \mathbf{F}_t(\tau)^\top\mathbf{L} + \mathbf{L}^\top\mathbf{F}_t(\tau). \end{aligned} \tag{1.34}$$

At $\tau = t$, $\mathbf{F}_t(\tau) = \mathbf{I}$, and thus

$$\mathbf{A} = \mathbf{L} + \mathbf{L}^\top. \tag{1.35}$$

Similarly from the definition (1.29) for $n = 2$ we have

$$\begin{aligned} \mathbf{A}_2(t) &= \frac{d^2}{d\tau^2}\left[\mathbf{C}_t(\tau)\right]|_{\tau=t} = \frac{d}{d\tau}\{\frac{d}{d\tau}\left[\mathbf{C}_t(\tau)\right]|_{\tau=t}\} \\ &= \frac{d}{d\tau}\left[\mathbf{F}_t(\tau)^\top \mathbf{L} + \mathbf{L}\mathbf{F}_t(\tau)\right] \\ &= \left[\frac{d}{d\tau}\left(\mathbf{F}_t(\tau)\right)^\top \mathbf{L} + \mathbf{F}_t(\tau)^\top \frac{d\mathbf{L}}{d\tau} + \frac{d\mathbf{L}^\top}{d\tau}\mathbf{F}_t(\tau) + \mathbf{L}^\top \frac{d}{d\tau}\mathbf{F}_t(\tau)\right] \\ &= \mathbf{L}^\top \mathbf{L} + \mathbf{F}_t(\tau)^\top \frac{d\mathbf{L}}{d\tau} + \frac{d\mathbf{L}^\top}{d\tau}\mathbf{F}_t(\tau) + \mathbf{L}^\top \mathbf{L}. \end{aligned} \tag{1.36}$$

Consider

$$\begin{aligned} \frac{d\mathbf{L}}{d\tau} &= \frac{d}{d\tau}\left(\frac{\partial V_i}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{dV_i}{d\tau}\right) = \frac{\partial}{\partial x_j}\left[\frac{\partial \mathbf{V}}{\partial \tau} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right] \\ &= \frac{\partial}{\partial \tau}\frac{\partial V_i}{\partial x_j} + \frac{\partial}{\partial x_j}V_i \cdot \nabla V_i + V_i \cdot \frac{\partial}{\partial x_j}\nabla V_i = \frac{\partial \mathbf{L}}{\partial \tau} + \nabla\mathbf{V}\cdot\nabla\mathbf{V} + \mathbf{V}\cdot\nabla\frac{\partial V_i}{\partial x_j} \\ &= \frac{\partial \mathbf{L}}{\partial \tau} + \mathbf{L}\cdot\mathbf{L} + \mathbf{V}\cdot\nabla\mathbf{L}. \end{aligned} \tag{1.37}$$

Similarly

$$\frac{d\mathbf{L}^\top}{d\tau} = \frac{\partial \mathbf{L}^\top}{\partial \tau} + \mathbf{L}^\top \cdot \mathbf{L}^\top + \mathbf{V}\cdot\nabla\mathbf{L}^\top, \tag{1.38}$$

and thus, Eq. (1.36) becomes

$$\mathbf{A}_2(t) = \mathbf{L}^\top \mathbf{L} + \mathbf{F}_t(\tau)^\top \left[\frac{\partial \mathbf{L}}{\partial \tau} + \mathbf{L}\mathbf{L} + \mathbf{V}\cdot\nabla\mathbf{L} + \left(\frac{\partial \mathbf{L}^\top}{\partial \tau} + \mathbf{L}^\top \cdot \mathbf{L}^\top + \mathbf{V}\cdot\nabla\mathbf{L}^\top\right)\mathbf{F}_t(\tau) + \mathbf{L}^\top \mathbf{L}\right].$$

Again at $\tau = t$, $\mathbf{F}_t(\tau) = \mathbf{I}$ and, therefore

$$\begin{aligned} \mathbf{A}_2(t) &= \mathbf{L}^\top \mathbf{L} + \frac{\partial \mathbf{L}}{\partial \tau} + \mathbf{L}^\top \mathbf{L} + \mathbf{V}\cdot\nabla\mathbf{L} + \frac{\partial \mathbf{L}^\top}{\partial \tau} + \mathbf{L}^\top \mathbf{L}^\top + \mathbf{V}\cdot\nabla\mathbf{L}^\top + \mathbf{L}^\top \mathbf{L} \\ &= \frac{\partial}{\partial \tau}\left(\mathbf{L} + \mathbf{L}^\top\right) + \mathbf{L}^\top\left(\mathbf{L} + \mathbf{L}^\top\right) + \left(\mathbf{L} + \mathbf{L}^\top\right)\mathbf{L} + \mathbf{V}\cdot\nabla\mathbf{L} + \mathbf{V}\cdot\nabla\mathbf{L}^\top \\ &= \frac{\partial}{\partial \tau}\mathbf{A}_1 + \mathbf{L}^\top \mathbf{A}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{V}\cdot\nabla\left(\mathbf{L} + \mathbf{L}^\top\right) = \frac{\partial \mathbf{A}_1}{\partial \tau} + \mathbf{L}^\top \mathbf{A}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{V}\cdot\nabla\mathbf{A}_1 \\ &= \left(\frac{\partial}{\partial \tau} + \mathbf{V}\cdot\nabla\right)\mathbf{A}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^\top \mathbf{A}_1 = \frac{d}{dt}\mathbf{A}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^\top \mathbf{A}_1. \end{aligned} \tag{1.39}$$

Similarly

$$\mathbf{A}_3(t) = \frac{d}{dt}\mathbf{A}_2 + \mathbf{A}_2\mathbf{L} + \mathbf{L}^\top \mathbf{A}_2. \tag{1.40}$$

We use a different procedure to find a recurrence relation. We know that the strain in the fluid is measured by looking at the length of a fluid element $t = 0$ to t. Let dX be the fluid element at X and dx at x then the length at time t is

$$dx^2(t) = \mathbf{C}(t) : d\mathbf{X}d\mathbf{X}. \tag{1.41}$$

Also

$$d\xi^2(\tau) = \mathbf{C}(\tau) : d\mathbf{X}d\mathbf{X}. \tag{1.42}$$

We know that

$$\begin{aligned}
\mathbf{C}(\tau) &= \mathbf{F}_t^\top(\tau)\,\mathbf{F}_t(\tau) = \left[\mathbf{F}_t(\tau)\,\mathbf{F}_t(\tau)^{-1}\right]^\top \mathbf{F}_t(\tau)\,\mathbf{F}_t(\tau)^{-1} \\
&= \left(\mathbf{F}^{-1}(\tau)\right)^\top \mathbf{F}_t^\top(\tau)\cdot\mathbf{F}_t(\tau)\,\mathbf{F}^{-1}(\tau) = \left(\mathbf{F}^{-1}(\tau)\right)^\top \mathbf{C}(\tau)\,\mathbf{F}_t^{-1}(\tau)
\end{aligned}$$

and

$$\frac{d^n}{d\tau^n}(\mathbf{C}(\tau)) = \left(\mathbf{F}^{-1}(\tau)\right)^\top \frac{d^n}{d\tau^n}\mathbf{C}(\tau)\,\mathbf{F}^{-1}(\tau)$$

$$\mathbf{F}^\top(\tau)\left[\frac{d^n}{d\tau^n}(\mathbf{C}_t(\tau))\right]\mathbf{F}(\tau) = \frac{d^n}{d\tau^n}(\mathbf{C}(\tau))$$

at $\tau = t$

$$\mathbf{F}^\top(\tau)\,\mathbf{A}_n\mathbf{F}(\tau) = \frac{d^n}{d\tau^n}(\mathbf{C}(\tau))$$

$$d\mathbf{X}^\top\mathbf{F}^\top(\tau)\,\mathbf{A}_n\mathbf{F}(\tau)\,d\mathbf{X} = \frac{d^n}{d\tau^n}\mathbf{C}(\tau) : d\mathbf{X}d\mathbf{X}$$

$$\mathbf{A}_n : d\mathbf{x}d\mathbf{x} = \frac{d^n}{d\tau^n}d\xi^2(\tau)\,|_{\tau=t}\,.$$

For $n = n + 1$

$$\begin{aligned}
\mathbf{A}_{n+1} &: d\mathbf{x}d\mathbf{x} = \frac{d}{d\tau}\left(\frac{d^n}{d\tau^n}d\xi^2(\tau)\right) \\
&= \frac{d}{d\tau}(\mathbf{A}_n : d\mathbf{x}d\mathbf{x}) = \frac{d\mathbf{A}_n}{d\tau} : d\mathbf{x}d\mathbf{x} + \mathbf{A}_n : \frac{d}{d\tau}(d\mathbf{x})\,d\mathbf{x} + \mathbf{A}_n : d\mathbf{x}\frac{d}{d\tau}(d\mathbf{x}).
\end{aligned}$$

As

$$\frac{d}{d\tau}(d\mathbf{x}) = \frac{d}{d\tau}(\mathbf{F}d\mathbf{x}) = \mathbf{L}\mathbf{F}d\mathbf{x} = \mathbf{L}d\mathbf{x}$$

so

$$\begin{aligned} \mathbf{A}_{n+1} \ &: \ d\mathbf{x}d\mathbf{x} = \frac{d\mathbf{A}_n}{d\tau} : d\mathbf{x}d\mathbf{x} + \mathbf{A}_n : \mathbf{L}d\mathbf{x}d\mathbf{x} + \mathbf{A}_n : d\mathbf{x}\mathbf{L}d\mathbf{x} \\ &= \ \left(\frac{d\mathbf{A}_n}{d\tau} + \mathbf{A}_n\mathbf{L} + \mathbf{L}^{\top}\mathbf{A}_n\right) : d\mathbf{x}d\mathbf{x} \end{aligned}$$

and thus

$$\mathbf{A}_{n+1} = \frac{d\mathbf{A}_n}{d\tau} + \mathbf{A}_n\mathbf{L} + \mathbf{L}^{\top}\mathbf{A}_n, \tag{1.43}$$

where (:) indicates the product of two tensors.

Chapter 2

Few inverse solutions involving second grade fluid

2.1 Introduction

In general the second grade flow equations are more complicated because of the addition of non-linearities in the stress function. As a result the solutions are smaller in number. As the non-linearities grow the complexities in solving these equations and their interpretation also grow. As a result the solutions are further restricted in comparison to viscous fluids in terms of the methods available. One such attempt has been made in this chapter, where we have considered the two dimensional flow equations and then introduced the stream function to obtain the compatibility equation. Solutions then are found by assuming specific forms of the stream function giving way to a large class of exact solutions. In each case, the expressions are constructed for the streamlines, velocity components and pressure distributions. Finally, the obtained expressions are compared with the known results in the literature.

Thus the problem at hand is the two-dimensional flow of a second-grade fluid near a stagnation point which has been discussed over the last few years. Actually, in 1911, one of the Prandtl's students, Hiemenz, found the stagnation point flow which are analyzed exactly by the Navier-Stokes equations. With this motivation we extend the work for the second grade fluid. In stagnation point flow, a rigid wall occupies the entire $x-$axis, the fluid domain is $y > 0$ and the flow impinges on the wall orthogonally. The $y-$axis behaves as an imaginary wall and fluid flows on both sides of this wall. Thus, the flow near $y-$axis needs to be analyzed. The dividing streamline is given by $\psi(x,y) = xF(y) + G(y)$.

2.2 Governing non-linear equation

Let us consider two parallel plates (see Figs. 2a,b,c) in some incompressible fluid (liquid) whose size is much larger than the distance between them $h << l$ (where h is the distance between the plates and l is the length of the plates) and suppose that they are moving towards each other or in opposite directions. We note that when the plates are moving towards each other (see Fig. 2a.) the force required is lesser as compared to that when they are moving against each other (see Fig. 2b.). Of course it varies with the different character or grade of the fluid (liquid). For Newtonian fluids (liquids) like water these experiments are much easier to perform than the non-Newtonian fluids (liquids). For general analysis since we are dealing with viscoelastic fluid in this chapter, so that the fluid considered between the impermeable or permeable plates is having the viscous as well as elastic properties, and one will have to put extra stress while approaching the plates towards each other or in opposite directions.

We also assume that the horizontal velocity does not depend on the vertical coordinate $(u \neq u(z), v \neq v(z))$ whereas the vertical velocity depends linearly on the distance between the plates $(w \propto z)$. Thus, the velocity field become [40]

$$u = u(x,y,t), \quad v = v(x,y,t), \quad w = -2\phi z, \tag{2.1}$$

where ϕ is the relative velocity of the plates (assumed constant).

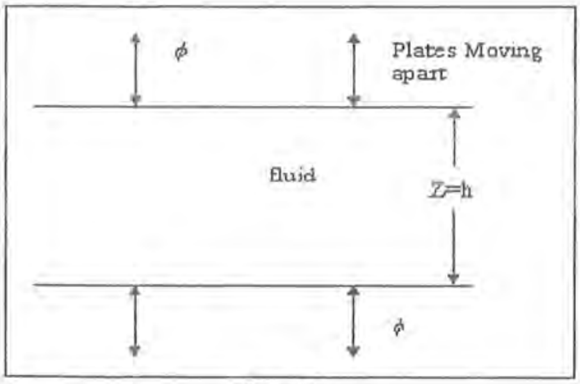

Fig. 2a.

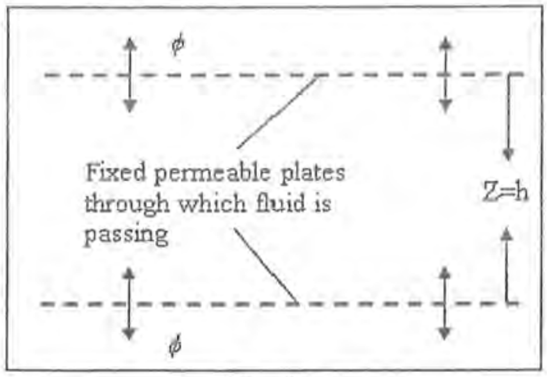

Fig. 2b.

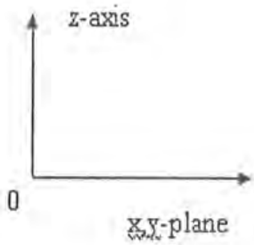

Fig. 2c.

Fig. 2. Geometry of the problem: (a) *moving impermeable plates* (b) *fixed permeable plates* and (c) *horizontal and vertical coordinates*

Using the velocity components defined in Eq. (2.1), the continuity equation (1.9) and Eq. (1.4) in component form give

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=2\phi. \tag{2.2}$$

$$\frac{\partial p_1}{\partial x}+\rho\left[\frac{\partial u}{\partial t}-v\omega\right]=\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\nabla^2 u-\alpha_1 v\nabla^2\omega+\rho\chi_1, \tag{2.3}$$

$$\frac{\partial p_1}{\partial y}+\rho\left[\frac{\partial v}{\partial t}+u\omega\right]=\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\nabla^2 v+\alpha_1 u\nabla^2\omega++\rho\chi_2, \tag{2.4}$$

$$\frac{\partial p_1}{\partial z}=\rho\chi_3, \tag{2.5}$$

where $\chi=(\chi_1,\chi_2,\chi_3)$ is the body force, and the modified pressure and the strength of the vorticity are defined as

$$p_1=p+\frac{1}{2}\rho\left(u^2+v^2+4\phi^2z^2\right)-\alpha_1\left[u\nabla^2u+v\nabla^2v+\frac{1}{4}\left|\mathrm{A}_1^2\right|\right], \tag{2.6}$$

$$\omega = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \tag{2.7}$$

in which

$$|\mathbf{V}|^2 = \mathbf{V}\cdot\mathbf{V} = u^2 + v^2 + 4\phi^2 z^2,$$

$$|\ \mathbf{A}_1^2\ | = \mathrm{tr}\left(\mathbf{A}_1\cdot\mathbf{A}_1^T\right) = \mathrm{tr}\left(\mathbf{A}_1^2\right) = 4\left(\frac{\partial u}{\partial x}\right)^2 + 4\left(\frac{\partial v}{\partial y}\right)^2 + 16\phi^2 + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2.$$

Differentiating Eq. (2.3) with respect to y and Eq. (2.4) with respect to x and using integrability condition $p_{1xy} = p_{1yx}$ we obtain the following compatibility equation

$$\begin{aligned}\rho\left[\frac{\partial\omega}{\partial t} + 2\phi\omega + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\omega\right] &= \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\nabla^2\omega + \rho\left(\frac{\partial}{\partial x}\chi_2 - \frac{\partial}{\partial y}\chi_1\right) \\ &\quad + \alpha_1\left[\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\nabla^2\omega + 2\phi\nabla^2\omega\right].\end{aligned} \tag{2.8}$$

Defining the velocity component in terms of the Stokes' stream function ψ through the following relations

$$u = \phi x + \frac{\partial\psi}{\partial y}, \quad v = \phi y - \frac{\partial\psi}{\partial x} \tag{2.9}$$

we see that the Eq. (2.2) is satisfied identically and Eq. (2.8) become

$$\begin{aligned}&\rho\left[\left(2\phi + \frac{\partial}{\partial t}\right)\nabla^2\psi + \phi\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\nabla^2\psi - \{\psi, \nabla^2\psi\}\right] \\ = &\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\nabla^4\psi + \alpha_1\left[2\phi\nabla^4\psi + \phi\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\nabla^4\psi - \{\psi, \nabla^4\psi\}\right] \\ &+ \rho\left(\frac{\partial}{\partial x}\chi_2 - \frac{\partial}{\partial y}\chi_1\right),\end{aligned} \tag{2.10}$$

in which

$$\nabla^4 = \nabla^2\cdot\nabla^2$$

and

$$\{\psi, \nabla^2\psi\} = \frac{\partial\psi}{\partial x}\frac{\partial\left(\nabla^2\psi\right)}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\left(\nabla^2\psi\right)}{\partial x}$$

Remark 1 *The solution $\psi = 0$ of Eq. (2.10), corresponds to liquid potential motion, known as the motion near the stagnation point.*

We now consider the following special cases:

- For steady case $\partial/\partial t = 0$ and Eq. (2.10) becomes

$$\begin{aligned} & \rho\left[2\phi\nabla^2\psi + \phi\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\nabla^2\psi - \{\psi, \nabla^2\psi\}\right] \\ = & \ \mu\nabla^4\psi + \alpha_1\left[2\phi\nabla^4\psi + \phi\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\nabla^4\psi - \{\psi, \nabla^4\psi\}\right] - \rho\left(\frac{\partial}{\partial x}\chi_2 - \frac{\partial}{\partial y}\chi_1\right). \end{aligned} \tag{2.11}$$

Note that for steady cases, the continuity equation, the modified pressure fields, the velocity components in terms of stream function and the vorticity vector remains the same whereas the velocity field becomes independent of time.

- For $\phi = 0$ Eq. (2.10) gives

$$\rho\left[\frac{\partial}{\partial t}\nabla^2\psi - \{\psi, \nabla^2\psi\}\right] = \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\nabla^4\psi - \alpha_1\{\psi, \nabla^4\psi\} - \rho\left(\frac{\partial}{\partial x}\chi_2 - \frac{\partial}{\partial y}\chi_1\right). \tag{2.12}$$

Here it is stated that the Eq. (2.12) is obtained when the velocity field, modified pressure, velocity components in terms of stream function and the continuity equation are

$$\begin{aligned} \mathbf{V}(x,y,t) &= [u(x,y,t),\ v(x,y,t),\ 0], \\ p_1 &= p + \frac{1}{2}\rho\left(u^2+v^2\right) - \alpha_1\left[u\nabla^2 u + v\nabla^2 v + \frac{1}{4}\left|\mathbf{A}_1^2\right|\right], \\ \left|\mathbf{A}_1^2\right| &= 4\left(\frac{\partial u}{\partial x}\right)^2 + 4\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2, \\ u &= \frac{\partial\psi}{\partial y},\ v = -\frac{\partial\psi}{\partial x},\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \end{aligned} \tag{2.13}$$

- For steady case Eqs. (2.12) reduces

$$-\rho\{\psi, \nabla^2\psi\} = \mu\nabla^4\psi - \alpha_1\{\psi, \nabla^4\psi\} - \rho\left(\frac{\partial}{\partial x}\chi_2 - \frac{\partial}{\partial y}\chi_1\right). \tag{2.14}$$

- Eq. (2.10) for unsteady viscous case is (see ref. [40])

$$\rho\left[\left(2\phi + \frac{\partial}{\partial t}\right)\nabla^2\psi + \phi\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\nabla^2\psi - \{\psi, \nabla^2\psi\}\right] = \mu\nabla^4\psi - \rho\left(\frac{\partial}{\partial x}\chi_2 - \frac{\partial}{\partial y}\chi_1\right). \tag{2.15}$$

- When $\phi = 0$ Eq. (2.15) reads as

$$\rho\left[\frac{\partial}{\partial t}\nabla^2\psi - \{\psi, \nabla^2\psi\}\right] = \mu\nabla^4\psi - \rho\left(\frac{\partial}{\partial x}\chi_2 - \frac{\partial}{\partial y}\chi_1\right). \tag{2.16}$$

- When $\phi = 0$ and the flow is steady then Eq. (2.15) gives (see ref. [41])

$$-\rho\{\psi, \nabla^2\psi\} = \mu\nabla^4\psi - \rho\left(\frac{\partial}{\partial x}\chi_2 - \frac{\partial}{\partial y}\chi_1\right). \tag{2.17}$$

- For creeping unsteady flow of second grade fluid when $u = \phi x + \frac{\partial\psi}{\partial y}$, $v = \phi y - \frac{\partial\psi}{\partial x}$ we have

$$\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\nabla^4\psi + \alpha_1\left[2\phi\nabla^4\psi + \phi\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\nabla^4\psi - \{\psi, \nabla^4\psi\}\right] = \rho\left(\frac{\partial}{\partial x}\chi_2 - \frac{\partial}{\partial y}\chi_1\right). \tag{2.18}$$

- For $\phi = 0$ Eq. (2.18) is

$$\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\nabla^4\psi - \alpha_1\{\psi, \nabla^4\psi\} = \rho\left(\frac{\partial}{\partial x}\chi_2 - \frac{\partial}{\partial y}\chi_1\right). \tag{2.19}$$

- For steady flow above expression is

$$\mu\nabla^4\psi - \alpha_1\{\psi, \nabla^4\psi\} = \rho\left(\frac{\partial}{\partial x}\chi_2 - \frac{\partial}{\partial y}\chi_1\right). \tag{2.20}$$

- For viscous fluid Eq. (2.18) is (see ref. [41])

$$\mu\nabla^4\psi = \rho\left(\frac{\partial}{\partial x}\chi_2 - \frac{\partial}{\partial y}\chi_1\right). \tag{2.21}$$

Note that the creeping flow for unsteady and steady viscous cases is the same. Also the velocity components, continuity equation, vorticity function and velocity components remain the same as in non-creeping flows but the modified pressure is slightly changed, that is, for second-grade fluid

$$p_1 = p - \alpha_1\left[u\nabla^2 u + v\nabla^2 v + \frac{1}{4}|\mathbf{A}_1^2|\right] \tag{2.22}$$

and for viscous fluid the modified pressure is

$$p_1 = p. \tag{2.23}$$

2.3 Solutions of some special types

Here we note that the compatibility equation (2.10) is highly nonlinear differential equation and it is not possible to find its analytic solution in closed form. Even, Eq. (2.10) has no closed form analytic solution for the Newtonian fluid. In order to obtain the solution various workers [24, 26, 38, 41, 42] assumed particular form of the stream function. Our interest in this chapter lies in finding the analytic solutions for the following two forms of the stream function: that is, flow where the stream function is linear with respect to x or y

$$\psi(x, y) = y\xi(x), \tag{2.24}$$

$$\psi(x, y) = y\xi(x) + \eta(x). \tag{2.25}$$

These type of flows are called the plane stagnation flows. Equation (2.24) represents the flow of a fluid in the neighbourhood of a stagnation point; the motion can be joined at a distance with a potential flow about a stagnation point. Here, the stream function is linear in y and once it strikes the boundary it becomes stagnant and then moves towards horizontally. Then it does not remain linear in y rather purely becomes a function of x.

2.3.1 Solution when $\psi(x, y) = y\xi(x)$

Substituting the value of ψ given in Eq. (2.24) into Eq. (2.10) we obtain

$$\rho\left[\phi\left(3\xi'' + x\xi'''\right) - \left(\xi'\xi'' - \xi\xi'''\right)\right] = \mu\xi^{IV} + \alpha_1\left[\phi\left(3\xi^{IV} + x\xi^{V}\right) - \left(\xi'\xi^{IV} - \xi\xi^{V}\right)\right], \tag{2.26}$$

where $\xi(x)$ is an arbitrary function of x and primes denote the derivative with respect to x. The above equation can also be written as

$$\frac{d}{dx}\left[\mu\xi''' + \rho\left\{\left(\xi'^2 - \xi\xi''\right) - \phi\left(2\xi' + x\xi''\right)\right\}\right] = \alpha_1\frac{d}{dx}\left\{\left(-\xi\xi^{IV} + 2\xi'\xi''' - \xi''^2\right) - \phi\left(2\xi''' + x\xi^{IV}\right)\right\}.$$

Integration of above equation yields

$$\mu\xi''' + \rho\left[\left(\xi'^2 - \xi\xi''\right) - \phi\left(2\xi' + x\xi''\right)\right] = \alpha_1\left[\left(-\xi\xi^{IV} + 2\xi'\xi''' - \xi''^2\right) - \phi\left(2\xi''' + x\xi^{IV}\right)\right]. \quad (2.27)$$

Let us assume a particular choice for the function ξ as:

$$\xi(x) = \delta\left(1 + \lambda e^{\sigma_5 x}\right) - \phi x \quad (2.28)$$

in which δ, σ_5 and λ are arbitrary real constants. Making use of Eq. (2.28) into Eq. (2.27) we easily find

$$\delta = \frac{\mu\sigma_5}{\rho - \alpha_1\sigma_5^2} - \frac{4\phi}{\sigma_5}. \quad (2.29)$$

Putting the value of δ in Eq. (2.28) we get

$$\xi(x) = \left(\frac{\mu\sigma_5}{\rho - \alpha_1\sigma_5^2} - \frac{4\phi}{\sigma_5}\right)\left(1 + \lambda e^{\sigma_5 x}\right) - \phi x \quad (2.30)$$

and the stream function ψ given by Eq. (2.24) become

$$\psi(x, y) = \left[\frac{\mu\sigma_5}{\rho - \alpha_1\sigma_5^2} - \frac{4\phi}{\sigma_5}\right] y\left(1 + \lambda e^{\sigma_5 x}\right) - \phi x y. \quad (2.31)$$

It is remarked here that the stream function (2.31) for $\alpha_1 = \phi = 0$ gives the results as discussed by Berker [41], and for $\alpha_1 = \phi = 0$, $\lambda = -1$, $\frac{\mu\sigma_5}{\rho} = -U$ $(U > 0)$ we recover the solution of Riabouchinsky [42].

Using Eq. (2.9) the velocity components are

$$u = \left[\frac{\mu\sigma_5}{\rho - \alpha_1\sigma_5^2} - \frac{4\phi}{\sigma_5}\right]\left(1 + \lambda e^{\sigma_5 x}\right), \quad (2.32)$$

$$v = 2\phi y - \left[\frac{\mu\sigma_5}{\rho - \alpha_1\sigma_5^2} - \frac{4\phi}{\sigma_5}\right] y\lambda\sigma_5 e^{\sigma_5 x}. \quad (2.33)$$

In order to find the pressure field we substitute Eqs. (2.32) and (2.33) into Eqs. (2.3) and (2.4), and then integrating the resulting equations we obtain

$$p_1 = p_0 + \mu\overline{a}\lambda\sigma_5\left(1 - \frac{\sigma_5^2 y^2}{2}\right)e^{\sigma_5 x} - \frac{1}{2}\rho\left[\overline{a}^2 + 4\phi^2\left(y^2+z^2\right) - \overline{a}^2\lambda^2 e^{2\sigma_5 x}\right] \quad (2.34)$$
$$+\alpha_1\left[\overline{a}\lambda\sigma_5\left(\overline{a}\sigma_5 - 2\phi\sigma_5^2 y^2 - 4\phi\right)e^{\sigma_5 x} + \overline{a}^2\lambda^2\sigma_5^2\left(3 + \frac{\sigma_5^2 y^2}{2}\right)e^{2\sigma_5 x} + 8\phi^2\right],$$

where p_0 is an arbitrary constant, known as the reference pressure.

In order to understand the streamline flow pattern we keep the stream function fixed i.e., $\psi(x,y) = \Omega_{11}$ (say) and solve the resulting expression for y in terms of the variable x. This particular procedure in two-dimensional flow in which one variable is expressed in terms of the other variable is called the functional form. In this way one can see the streamline flow pattern through graphs.

Eq. (2.1.6) for $\psi(x,y) = \Omega_{11}$ gives the following expression

$$y = \frac{\Omega_{11}}{\left(1 + \lambda e^{\sigma_5 x}\right)\varepsilon - x\phi}, \quad (2.35)$$

where

$$\varepsilon = \frac{\nu\sigma_5}{1 - \Lambda\sigma_5^2} - \frac{4\phi}{\sigma_5} \quad (2.36)$$

in which $\nu = \mu/\rho$ is the kinematic coefficient of viscosity and $\Lambda = \alpha_1/\rho$ is the second-grade parameter.

Fig. 2.1. is plotted for $\phi = \sigma_5 = \lambda = 1$, $\mu/\rho = 0.5$, $\alpha_1/\rho = 0.1$, $\psi = 15, 20, 25, 30, 40$. Fig. 2.1. describes the continuous streamline flow pattern. It should be mentioned here that these graphs simply defines the pattern of the flow for a particular choice of the stream function.

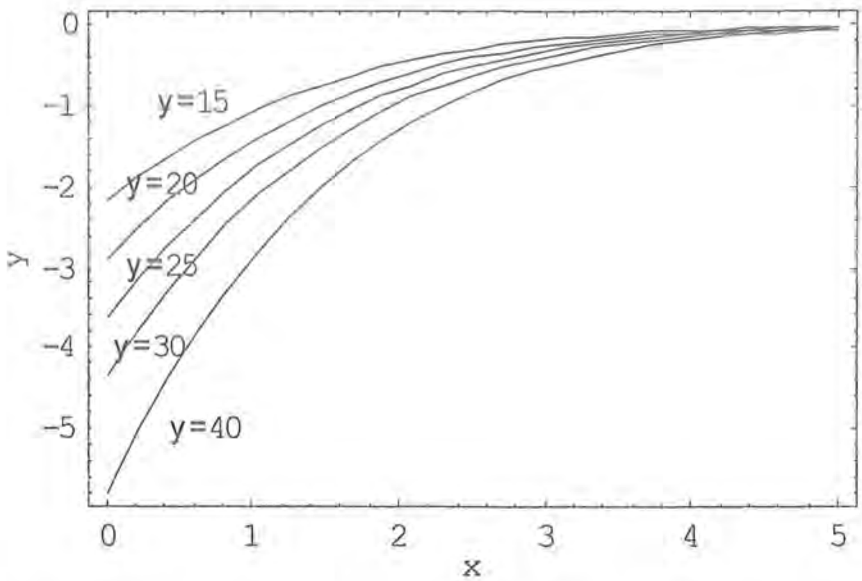

Fig. 2.1. Streamline flow pattern for $\psi(x,y) = \left[\frac{\mu\sigma_5}{\rho - \alpha_1\sigma_5^2} - 4\frac{\phi}{\sigma_5}\right] y\left(1 + \lambda e^{\sigma_5 x}\right) - \phi x y$

2.3.2 Solutions when $\psi(x,y) = y\xi(x) + \eta(x)$

To find another class of solution of Eq. (2.10) we use Eq. (2.25) into Eq. (2.10) to get the following nonlinear differential equation

$$\rho\left[\begin{array}{c} 2\phi\left(y\xi'' + \eta''\right) + \phi\{y\left(\xi'' + x\xi'''\right) + x\eta'''\} \\ -\{y\left(\xi'\xi'' - \xi\xi'''\right) + \left(\eta'\xi'' - \xi\eta'''\right)\} \end{array}\right] \tag{2.37}$$

$$= \mu\left(y\xi^{IV} + \eta^{IV}\right) + \alpha_1\left[\begin{array}{c} 2\phi\left(y\xi^{IV} + \eta^{IV}\right) + \phi\{y\left(\xi^{IV} + x\xi^{V}\right) + x\eta^{V}\} \\ -\{y\left(\xi'\xi^{IV} - \xi\xi^{V}\right) + \left(\eta'\xi^{IV} - \xi\eta^{V}\right)\} \end{array}\right].$$

From Eq. (2.37) we have the following equations:

$$\rho\left[\left(\xi'\xi'' - \xi\xi'''\right) - \phi\left(3\xi'' + x\xi'''\right)\right] + \mu\xi^{IV} - \alpha_1\left[\begin{array}{c} \left(\xi'\xi^{IV} - \xi\xi^{V}\right) \\ -\phi\left(3\xi^{IV} + x\xi^{V}\right) \end{array}\right] = 0, \tag{2.38}$$

and

$$\rho\left[\left(\eta'\xi''-\xi\eta'''\right)-\phi\left(2\eta''+x\eta'''\right)\right]+\mu\eta^{IV}-\alpha_1\left[\begin{array}{c}\left(\eta'\xi^{IV}-\xi\eta^{V}\right)\\-\phi\left(3\eta^{IV}+x\eta^{V}\right)\end{array}\right]=0, \tag{2.39}$$

where $\xi(x)$ and $\eta(x)$ are arbitrary functions of the variable x. Integrating these equations with respect to x and then taking the constants of integration equal to zero we have

$$\mu\xi'''+\rho\left[\left(\xi'^2-\xi\xi''\right)-\phi\left(2\xi'+x\xi''\right)\right]-\alpha_1\left[\begin{array}{c}\left(-\xi\xi^{IV}+2\xi'\xi'''-\xi''^2\right)\\-\phi\left(2\xi'''+x\xi^{IV}\right)\end{array}\right]=0, \tag{2.40}$$

$$\mu\eta'''+\rho\left[\left(\eta'\xi'-\xi\eta''\right)-\phi\left(2\eta'+x\eta''\right)\right]-\alpha_1\left[\begin{array}{c}\xi'\eta'''-\xi\eta^{IV}+\eta'\xi'''\\-\eta''\xi''-\phi\left(2\eta'''+x\eta^{IV}\right)\end{array}\right]=0. \tag{2.41}$$

Here it can be seen that Eq. (2.40) is exactly the same as of Eq. (2.26). The solution of Eq. (2.26) is given in Eq. (2.30). In order to obtain the solution of Eq. (2.41) we substitute the solution given in Eq. (2.30) into Eq. (2.41) and get

$$\alpha_1\delta\left(1+\lambda e^{\sigma_5 x}\right)\eta^{V}+\left(\mu+3\alpha_1\phi\right)\eta^{IV}-\rho\delta\left(1+\lambda e^{\sigma_5 x}\right)\eta'''-2\rho\phi\eta''+\delta\lambda\sigma_5^2\left(\rho-\alpha_1\sigma_5^2\right)e^{\sigma_5 x}\eta'=0. \tag{2.42}$$

Clearly to obtain the general solution of Eq. (2.42) is not easy. For analytic solution of above equation we consider the following cases:

Case 1 When $\alpha_1\neq 0$, $\phi\neq 0$, $\sigma_5=1$, $\lambda=0$

We have from Eq. (2.42) as

$$\alpha_1\overline{a}\eta^{V}+\left(\mu+3\alpha_1\phi\right)\eta^{IV}-\rho\overline{a}\eta'''-2\rho\phi\eta''=0. \tag{2.43}$$

The above equation is of fifth order and for solution we substitute $\eta''=\overline{A}(x)$ and get

$$\alpha_1\overline{a}\overline{A}'''+\left(\mu+3\alpha_1\phi\right)\overline{A}''-\rho\overline{a}\overline{A}'-2\rho\phi\overline{A}=0. \tag{2.44}$$

Taking $\overline{A}(x) = \widehat{P}(x) e^{x}$ the above equation becomes

$$\alpha_1 \delta \widehat{P}''' + \{3\alpha_1 (\delta + \phi) + \mu\} \widehat{P}'' + \{3\alpha_1 (\delta + 2\phi) + 2\mu - \rho\delta\} \widehat{P}' = 0. \tag{2.45}$$

Writing $\widehat{P}'(x) = R(x)$, Eq. (2.45) reduces to

$$\alpha_1 \delta R'' + \{\mu + 3\alpha_1 (\phi + \delta)\} R' - \{(3\alpha_1 - \rho) \delta + 2\mu + 6\alpha_1 \phi\} R = 0. \tag{2.46}$$

The above equation is second order and its solution can be written as

$$R(x) = A_3 \exp\left(\frac{-c - \sqrt{c^2 - 4d}}{2}\right) x + A_4 \exp\left(\frac{-c + \sqrt{c^2 - 4d}}{2}\right) x, \tag{2.47}$$

where A_3 and A_4 are arbitrary constants and

$$c = \frac{3\alpha_1 (\delta + \phi) + \mu}{\alpha_1 \delta}, \; d = \frac{3\alpha_1 (\delta + 2\phi) + 2\mu - \rho\delta}{\alpha_1 \delta}, \; \delta = \frac{\mu}{\rho - \alpha_1} - 4\phi.$$

Equation (2.47) can also be written as

$$R(x) = A_3 e^{m_1 x} + A_4 e^{m_2 x},$$

where

$$m_1 = \left(\frac{-c - \sqrt{c^2 - 4d}}{2}\right), \; m_2 = \left(\frac{-c + \sqrt{c^2 - 4d}}{2}\right).$$

In order to find $\eta(x)$ we make back substitutions to proceed as

$$\widehat{P}(x) = \int R(x)\, dx = \int \left(A_3 e^{m_1 x} + A_4 e^{m_2 x}\right) dx = \frac{A_3}{m_1} e^{m_1 x} + \frac{A_4}{m_2} e^{m_2 x} + A_5$$

and $\overline{A}(x) = \widehat{P}(x) e^{x}$ implies that

$$\overline{A}(x) = \frac{A_3}{m_1} e^{(1+m_1)x} + \frac{A_4}{m_2} e^{(1+m_2)x} + A_5 e^{x}$$

which after taking $\eta'' = \overline{A}(x)$ gives

$$\eta(x) = \frac{A_3}{m_1(1+m_1)^2} e^{(1+m_1)x} + \frac{A_4}{m_2(1+m_2)^2} e^{(1+m_2)x} + A_5 e^x + A_6 x + A_7, \tag{2.48}$$

where A_i $(i = 5, 6, 7)$ are constants of integrations. Substituting Eqs. (2.30) and (2.48) into Eq. (2.25) one obtains

$$\begin{aligned} \psi(x,y) &= y\left[\frac{\mu}{\rho-\alpha_1} - \phi(4+x)\right] + A_5 e^x + A_6 x + A_7 \\ &+ \frac{A_3}{m_1(1+m_1)^2} e^{(1+m_1)x} + \frac{A_4}{m_2(1+m_2)^2} e^{(1+m_2)x}. \end{aligned} \tag{2.49}$$

The velocity components and the pressure field are respectively given by

$$u = \frac{\mu}{\rho-\alpha_1} - 4\phi, \tag{2.50}$$

$$v = 2\phi y - \left[\frac{A_3}{m_1(1+m_1)} e^{(1+m_1)x} + \frac{A_4}{m_2(1+m_2)} e^{(1+m_2)x} + A_5 e^x + A_6\right]. \tag{2.51}$$

$$\begin{aligned} p &= p_0 - \frac{1}{2}\rho\left[a_1^2 + A_6^2 + 4\phi^2\left(y^2+z^2\right) - 4y\phi A_6\right] \\ &+\alpha_1 \left[\begin{array}{c} \frac{A_3^2}{m_1^2} e^{2(1+m_1)x} + \frac{2A_3A_4}{m_1 m_2} e^{(2+m_1+m_2)x} + \frac{A_4^2}{m_2^2} e^{2(1+m_2)x} + A_5^2 e^{2x} \\ 4\phi^2 + \frac{2A_3A_5(2+3m_1+m_1^2)}{(2+m_1)m_1(1+m_1)} e^{(2+m_1)x} + \frac{2A_4A_5(2+3m_2+m_2^2)}{(2+m_2)m_2(1+m_2)} e^{(2+m_2)x} \end{array}\right] \end{aligned} \tag{2.52}$$

The streamline flow pattern for $\psi = \Omega_{22}$ (say) is given as

$$y = -\frac{1}{\varepsilon_1 - x\phi}\left[\begin{array}{c} -\Omega_{22} + \frac{A_3}{m_1(1+m_1)^2} e^{(1+m_1)x} \\ + \frac{A_4}{m_2(1+m_2)^2} e^{(1+m_2)x} + A_5 e^x + A_6 x + A_7 \end{array}\right], \tag{2.53}$$

where

$$\varepsilon_1 = \frac{\nu}{1-\Lambda} - 4\phi, \; a_1 = \frac{\mu}{\rho-\alpha_1} - 4\phi.$$

The streamline flow pattern is plotted in Fig. 2.2. for $\phi = a = \lambda = 1$, $\mu/\rho = 0.5$, $\alpha_1/\rho = 0.1$, $A_i = 1$ $(i = 3 - 7)$, $\psi = 15, 20, 25, 30, 40$.

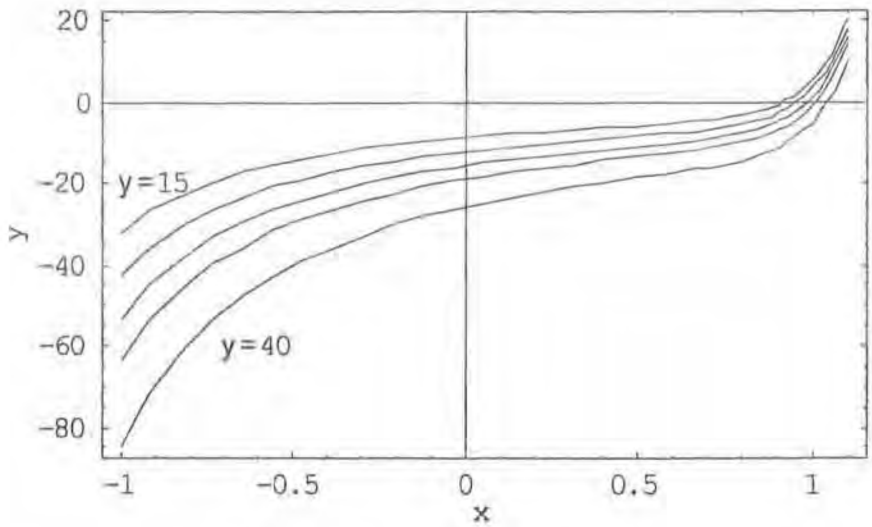

Fig. 2.2. Streamline flow pattern for
$\psi(x,y) = y\left[\frac{\mu}{\rho-\alpha_1} - \phi(4+x)\right] + \frac{A_3}{m_1(1+m_1)^2}e^{(1+m_1)x} + \frac{A_4}{m_2(1+m_2)^2}e^{(1+m_2)x} + A_5e^x + A_6x + A_7$

Case 2 When $\alpha_1 \neq 0$, $\phi = 0$, $\sigma_5 = 1$, $\lambda \neq 0$

then Eq. (2.42) after using $\delta = \frac{\mu}{\rho-\alpha_1}$ becomes

$$\alpha_1(1+\lambda e^x)\eta^V + (\rho-\alpha_1)\eta^{IV} - \rho(1+\lambda e^x)\eta''' + (\rho-\alpha_1)\lambda e^x\eta' = 0. \tag{2.54}$$

To find the solution of Eq. (2.54) we make few substitutions to reduce its order. For this purpose we put $\eta' = \widehat{A}(x)$ which leaves it into a form which is one order less, that is

$$\alpha_1(1+\lambda e^x)\widehat{A}^{IV} + (\rho-\alpha_1)\widehat{A}''' - \rho(1+\lambda e^x)\widehat{A}'' + (\rho-\alpha_1)\lambda e^x\widehat{A} = 0. \tag{2.55}$$

Now substituting $\widehat{A}(x) = \overline{P}(x)e^x$ in Eq. (2.55) and then $\overline{P}'(x) = R(x)$ into the resulting expression we get

$$\alpha_1\left[\begin{array}{c}(1+\lambda e^x)R''' + (3+4\lambda e^x)R'' \\ +(3+6\lambda e^x)R' + (1+4\lambda e^x)R\end{array}\right] = \rho\left[R'' + (2-\lambda e^x)R' + (1-2\lambda e^x)R\right]. \tag{2.56}$$

The Eq. (2.56) is third order. Its order can be reduced further by multiplying by e^x and then integrating. After this process we have

$$\alpha_1 (1 + \lambda e^x) R'' + [(2\alpha_1 + \rho) + 2\alpha_1 \lambda e^x] R' + [\alpha_1 + \rho - (2\alpha_1 - \rho) \lambda e^x] R = 0, \tag{2.57}$$

where for simplicity the constant of integration is taken equal to zero.

The solution of Eq. (2.57) for $\lambda = 0$ is given by

$$R(x) = C_5 e^{-x} + C_6 e^{-[(\alpha_1+\rho)/\alpha_1]x}. \tag{2.58}$$

Employing the same procedure as in case 1 we can write

$$\eta(x) = -C_5 x + \frac{\alpha_1^2}{\rho(\alpha_1 + \rho)} C_6 e^{-(\rho/\alpha_1)x} + C_7 e^x + C_8, \tag{2.59}$$

where C_r $(r = 5, 6, 7, 8)$ are arbitrary constants. The stream function, the velocity components and the pressure field in this case are respectively given as

$$\psi(x, y) = \frac{\mu}{\rho - \alpha_1} y + \left[-C_5 x + \frac{\alpha_1^2}{\rho(\alpha_1 + \rho)} C_6 e^{-(\rho/\alpha_1)x} + C_7 e^x + C_8 \right], \tag{2.60}$$

$$u = \frac{\mu}{\rho - \alpha_1}, \tag{2.61}$$

$$v = C_5 + \frac{\alpha_1}{(\alpha_1 + \rho)} C_6 e^{-(\rho/\alpha_1)x} - C_7 e^x, \tag{2.62}$$

$$\begin{aligned} p &= p_0 - \frac{1}{2}\rho \left[a_2^2 + C_5^2 + 2C_7 \overline{\alpha} e^{(1-\rho/\alpha_1)x} + 2\frac{(1-\alpha_1)}{(\alpha_1 - \rho)} C_7 \overline{\alpha} e^{(1-\rho/\alpha_1)x} \right] \\ &\quad + \alpha_1 \left[\begin{array}{c} C_7^2 e^{2x} + \frac{\rho^2 \overline{\alpha}^2}{\alpha_1^2} e^{-2(\rho/\alpha_1)x} - C_7 \frac{\rho^2 \overline{\alpha}}{\alpha_1^2} e^{(1-\rho/\alpha_1)x} \\ + \left(\frac{\rho}{\alpha_1} - 1 \right) C_7 \overline{\alpha} e^{(1-\rho/\alpha_1)x} \\ + C_7 \frac{\overline{\alpha}\alpha_1}{\alpha_1 - \rho} e^{(1-\rho/\alpha_1)x} - C_7 \frac{\overline{\alpha}\rho^3}{\alpha_1^2(\alpha_1 - \rho)} e^{(1-\rho/\alpha_1)x} \end{array} \right], \end{aligned} \tag{2.63}$$

where

$$\overline{\alpha} = \frac{\alpha_1}{\alpha_1 + \rho}, \; a_2 = \frac{\mu}{\rho - \alpha_1},$$

and the stream function for $\psi = \Omega_{33}$ (constant) is given by the following functional form

$$y = -\frac{1}{\varepsilon_2}\left[-\Omega_{33} - C_5 x + \frac{\Lambda^2}{1+\Lambda} C_6 e^{-(1/\Lambda)x} + C_7 e^x + C_8\right], \tag{2.64}$$

where

$$\varepsilon_2 = \frac{\nu}{1-\Lambda}.$$

The streamline flow pattern is sketched in Fig. 2.3. for $\phi = \lambda = 0$, $\sigma_5 = 1$, $\mu/\rho = 0.5$, $\alpha_1/\rho = 0.1$, $C_r = 1$ $(r = 5-8)$, $\psi = 15, 20, 25, 30, 40$.

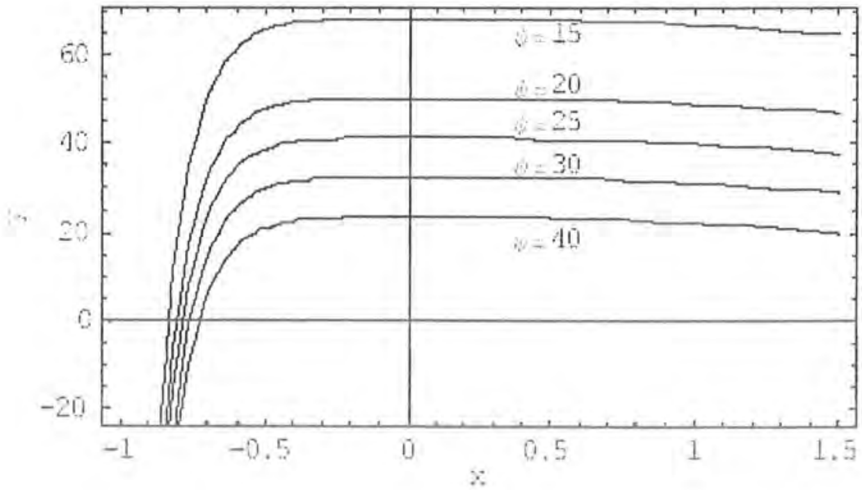

Fig. 2.3. Streamline flow pattern for
$\psi(x,y) = \frac{\mu}{\rho-\alpha_1} y + \left[-C_5 x + \frac{\alpha_1^2}{\rho(\alpha_1+\rho)} C_6 e^{-(\rho/\alpha_1)x} + C_7 e^x + C_8\right]$

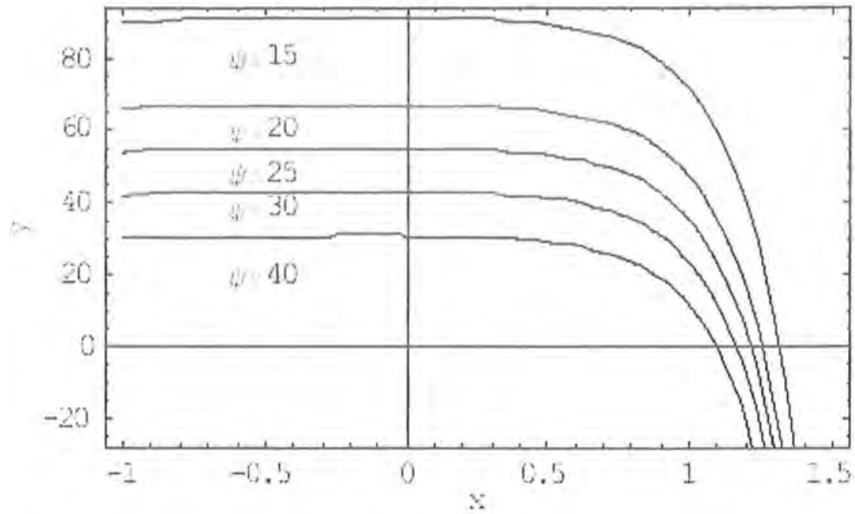

Fig. 2.4. Streamline flow pattern for negative second-grade parameter for $\psi(x,y) = \frac{\mu}{\rho-\alpha_1}y + \left[-C_5x + \frac{\alpha_1^2}{\rho(\alpha_1+\rho)}C_6e^{-(\rho/\alpha_1)x} + C_7e^x + C_8\right]$

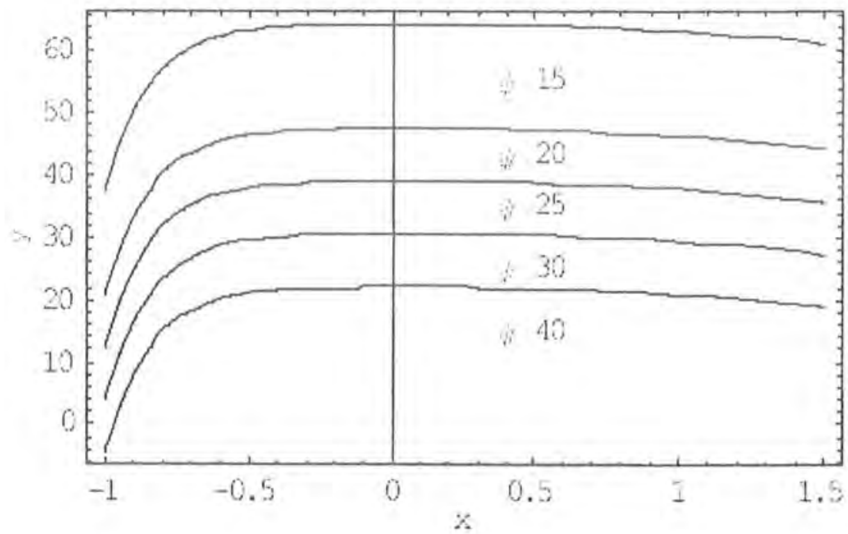

Fig. 2.5. Streamline flow pattern for positive second-grade parameter for $\psi(x,y) = \frac{\mu}{\rho-\alpha_1}y + \left[-C_5x + \frac{\alpha_1^2}{\rho(\alpha_1+\rho)}C_6e^{-(\rho/\alpha_1)x} + C_7e^x + C_8\right]$

2.4 Concluding remarks

In this chapter, the analytical solutions of non-linear equations governing the flow for a second-grade fluid are obtained. Two different forms of the stream function are taken. In each problem of stream function, the various possibilities of getting the analytical solutions are discussed. The expressions for velocity profile, streamline and pressure distribution are constructed in each case. Our results indicate that velocity, stream function and pressure are strongly dependent upon the material parameter α_1 of the second grade fluid. It is shown through graphs that increase in second-grade parameter ($\alpha_1 = 0.15$) leads to decrease in velocity. Also decrease in second grade parameter ($\alpha_1 = -0.5$) increases the velocity (see Figs. 2.4 and 2.5). The present analysis are more general and several results of various authors (Aristov and Gitman [40], Berker [41] and Riabouchinsky [42]) can be recovered in the limiting cases.

Chapter 3

On solutions of some non-linear differential equations arising in Newtonian and non-Newtonian fluids

3.1 Introduction

This work is motivated by the analysis of Lakshmana [43] . Lakshmana's [43] work is extended by considering the unattended parameters and then extended to second grade fluid. The mathematical modelling and the solution given are important from the understanding of second grade fluids. The work has great importance in a number of industrial or practical applications. Spiral galaxies, atmospheric or ocean circulation, bathtub vortices, or even stirring tea in a cup, are examples that illustrate the ubiquity of swirling flows at all scales in nature.

In this chapter, we develop the governing equation for an axisymmetric swirling flow of a second grade fluid, which is highly non-linear. The primary purpose of this chapter is to establish some analytical steady and unsteady solutions of the non-linear equation arising in the swirling flows both in Newtonian and non-Newtonian fluids. The solutions are obtained using various analytical methods including the Lie group method. The expressions for streamlines and velocity components are given in each case explicitly. The obtained solutions are also compared in context of second grade without swirl and with the results of viscous fluid.

3.2 Governing equation for swirling flow

Let us consider the swirling flows in a second grade fluid in which the second component ($\theta-$component) of the velocity is not zero. The compatibility equation obtained from the Stokes' stream function is to be solved by assuming a specific form of the vorticity. This gives us two unknown velocity components i.e., V_r and V_z. In order to obtain the V_θ component of the velocity we substitute the stream function and its derivatives in the compatibility equation which comes from the Stokes' stream function. The angular momentum per unit mass about the axis of symmetry of the flow is

$$\Omega = rV_\theta. \tag{3.1}$$

Note that we take the $z-$axis along the line of symmetry of the flow or we can say that flow is symmetric about $z-$axis. The velocity components (V_r, V_θ, V_z) are independent of the meridional angle θ. If the meridional component of velocity V_θ vanishes at every point of the flow, whereas V_r and V_z are non-zero, then we obtain an axisymmetric flow and if the meridional component of velocity V_θ does not vanish in the flow field region then the flow field is usually expressed in terms of the swirl Ω.

For the axisymmetric flow with the swirling motion, the velocity field is

$$\mathbf{V} = \left[V_r\left(r,z,t\right), \ \frac{\Omega\left(r,z,t\right)}{r}, \ V_z\left(r,z,t\right)\right] \tag{3.2}$$

On using above equation into Eqs. (1.4) and (1.9) we can write

$$\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{\partial V_z}{\partial z} = 0, \tag{3.3}$$

$$\frac{\partial \widehat{p}}{\partial r} + \rho\left[\frac{\partial V_r}{\partial t} - \omega V_z - \frac{\Omega}{r^2}\frac{\partial \Omega}{\partial r}\right] = -\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\left(\frac{\partial \omega}{\partial z}\right) - \alpha_1\left[V_z\left(\nabla^2\omega - \frac{\omega}{r^2}\right) + \frac{\Omega}{r^2}\frac{\partial}{\partial r}E^2\Omega\right], \tag{3.4}$$

$$\frac{1}{r}\frac{\partial \widehat{p}}{\partial \theta} + \frac{\rho}{r}\left[\frac{\partial \Omega}{\partial t} + \frac{V_r}{r}\frac{\partial \Omega}{\partial r} + \frac{V_z}{r}\frac{\partial \Omega}{\partial z}\right] = \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\frac{1}{r}E^2\Omega + \alpha_1\left[\frac{V_z}{r}\frac{\partial}{\partial z}E^2\Omega + \frac{V_r}{r}\frac{\partial}{\partial r}E^2\Omega\right], \tag{3.5}$$

$$\frac{\partial \widehat{p}}{\partial z}+\rho\left[\frac{\partial V_z}{\partial t}+\omega V_r-\frac{\Omega}{r^2}\frac{\partial \Omega}{\partial z}\right]=\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\nabla^2 V_z+\alpha_1\left[V_r\left(\nabla^2\omega-\frac{\omega}{r^2}\right)-\frac{\Omega}{r^2}\frac{\partial}{\partial z}E^2\Omega\right], \quad (3.6)$$

where

$$\begin{aligned}
\widehat{p} &= p+\frac{1}{2}\rho\left(V_r^2+V_z^2+\frac{\Omega^2}{r^2}\right)-\alpha_1\left[V_r\left(\nabla^2 V_r-\frac{V_r}{r^2}\right)+\frac{\Omega}{r^2}E^2\Omega+V_z\nabla^2 V_z+\frac{1}{4}\left|\mathbf{A}_1\right|^2\right],\\
\left|\mathbf{A}_1^2\right| &= 4\left(\frac{\partial V_r}{\partial r}\right)^2+4\left(\frac{\partial V_z}{\partial z}\right)^2+2\left(\frac{\partial V_r}{\partial z}+\frac{\partial V_z}{\partial r}\right)^2+4\left(\frac{V_r}{r}\right)^2+2\left(\frac{1}{r}\frac{\partial \Omega}{\partial r}-\frac{2\Omega}{r^2}\right)^2+\frac{2}{r^2}\left(\frac{\partial \Omega}{\partial z}\right)^2,\\
\nabla^2 &= \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2},\quad E^2=\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2},\quad \omega=\frac{\partial V_z}{\partial r}-\frac{\partial V_r}{\partial z}. \qquad (3.7)
\end{aligned}$$

Differentiating Eq. (3.4) with respect to z and Eq. (3.6) with respect to r and then subtracting the resulting equations we obtain

$$\begin{aligned}
&\rho\left[\frac{\partial \omega}{\partial t}+\frac{\partial}{\partial r}\left(\omega V_r\right)+\frac{\partial}{\partial z}\left(\omega V_z\right)-\frac{\partial}{\partial r}\left(\frac{\Omega}{r^2}\frac{\partial \Omega}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\Omega}{r^2}\frac{\partial \Omega}{\partial r}\right)\right] \qquad (3.8)\\
&= \left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\left(\nabla^2\omega-\frac{\omega}{r^2}\right)+\alpha_1\left[\begin{array}{c}\frac{\partial}{\partial r}\left\{\left(\nabla^2\omega-\frac{\omega}{r^2}\right)V_r\right\}+\frac{\partial}{\partial z}\left\{\left(\nabla^2\omega-\frac{\omega}{r^2}\right)V_z\right\}\\ -\frac{\partial}{\partial r}\left(\frac{\Omega}{r^2}\frac{\partial E^2\Omega}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\Omega}{r^2}\frac{\partial E^2\Omega}{\partial r}\right)\end{array}\right].
\end{aligned}$$

Introducing the Stokes' stream function $\psi(r,z,t)$ through

$$V_r=-\frac{1}{r}\frac{\partial \psi}{\partial z},\quad V_z=\frac{1}{r}\frac{\partial \psi}{\partial r} \qquad (3.9)$$

the continuity equation (1.9) is identically satisfied and Eq. (3.8) gives

$$\begin{aligned}
&\rho\left[E^2\psi_t+\frac{1}{r}\frac{\partial\left(\psi,E^2\psi\right)}{\partial(r,z)}+\frac{2}{r^2}\frac{\partial \psi}{\partial z}E^2\psi+\frac{2}{r^2}\Omega\frac{\partial \Omega}{\partial z}\right] \qquad (3.10)\\
&= \left(\mu+\alpha_1\frac{\partial}{\partial t}\right)E^4\psi+\alpha_1\left[\frac{2}{r^2}\Omega\frac{\partial E^2\Omega}{\partial z}-\frac{1}{r}\frac{\partial\left(\Omega,E^2\Omega\right)}{\partial(r,z)}+\frac{1}{r}\frac{\partial\left(\psi,E^2\psi\right)}{\partial(r,z)}+\frac{2}{r^2}\frac{\partial \psi}{\partial z}E^2\psi\right].
\end{aligned}$$

Assuming $\widehat{p}\neq\widehat{p}(\theta)$ will reduce Eq. (3.5) in the following form

$$\begin{aligned}
\frac{\rho}{r}\left[\frac{\partial \Omega}{\partial t}+\frac{1}{r}\left\{-\frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial \Omega}{\partial r}+\frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial \Omega}{\partial z}\right\}\right] &= \left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\frac{1}{r}E^2\Omega\\
&\quad+\frac{\alpha_1}{r}\left[-\frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial}{\partial z}E^2\Omega+\frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial}{\partial r}E^2\Omega\right],
\end{aligned}$$

or

$$\rho\left[\frac{\partial\Omega}{\partial t}+\frac{1}{r}\frac{\partial\left(\psi,\Omega\right)}{\partial\left(r,z\right)}\right]=\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)E^2\Omega+\frac{\alpha_1}{r}\frac{\partial\left(\psi,E^2\Omega\right)}{\partial\left(r,z\right)}, \tag{3.11}$$

where

$$\frac{\partial\left(\psi,\Omega\right)}{\partial\left(r,z\right)}=\frac{\partial\psi}{\partial r}\frac{\partial\Omega}{\partial z}-\frac{\partial\psi}{\partial z}\frac{\partial\Omega}{\partial r}. \tag{3.12}$$

It should be pointed out that the Eqs. (3.10) and (3.11) are the compatibility equations for the present axisymmetric swirling flows. These equations for $\alpha_1=0$ reduces to the results of Goldstein [51]. In the next section we will find the solutions of these equations for both $\alpha_1=0$ and $\alpha_1\neq 0$.

3.3 Analytic solutions

It is clear that the general solution of Eqs. (3.10) and (3.11) is not possible because of the high nonlinearity. Thus, we discuss the special cases of these highly nonlinear partial differential equations by imposing specific conditions on the stream function ψ and Ω. Let us first begin to find the particular solutions of Newtonian fluid both for steady and non-steady cases. Then we employ similar procedure in order to obtain the steady and unsteady solutions for the second grade fluid.

3.4 Steady cases $\partial/\partial t(\cdot)=0$

3.4.1 For viscous case $\alpha_1=0$, $\psi=\psi(r,z)$ and $\Omega=\Omega(r)$

Here Eqs. (3.10) and (3.11) reduce to

$$\frac{1}{r}\frac{\partial(\psi,E^2\psi)}{\partial(r,z)}+\frac{2}{r^2}\frac{\partial\psi}{\partial z}E^2\psi = \nu E^4\psi, \tag{3.13}$$

$$\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{d\Omega}{dr}+\nu\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)\Omega = 0. \tag{3.14}$$

Differentiating Eq. (3.14) with respect to variable z we get

$$\frac{\partial^2\psi}{\partial z^2}=0$$

which upon integration gives

$$\Psi(r,z) = f(r)z + g(r), \tag{3.15}$$

where $f(r)$ and $g(r)$ are arbitrary functions to be determined. If we substitute Eq. (3.15) into Eq. (3.13), we get two non linear differential equations for $f(r)$ and $g(r)$ which is not possible to have the solutions. In order to get rid of this difficulty we let

$$E^2\Psi = ar^2, \tag{3.16}$$

which leaves Eq. (3.13) as an identity and Eq. (3.15) along with Eq. (3.16) gives

$$E^2\left(f(r)z + g(r)\right) = \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)(fz+g) = ar^2,$$

or

$$z\left(\frac{d^2f}{dr^2} - \frac{1}{r}\frac{df}{dr}\right) + \left(\frac{d^2g}{dr^2} - \frac{1}{r}\frac{dg}{dr}\right) = ar^2,$$

which finally helps in writing

$$\frac{d^2f}{dr^2} - \frac{1}{r}\frac{df}{dr} = 0, \tag{3.17}$$

$$\frac{d^2g}{dr^2} - \frac{1}{r}\frac{dg}{dr} = ar^2. \tag{3.18}$$

The solutions of above equations are

$$f(r) = Ar^2 + B, \tag{3.19}$$

$$g(r) = \frac{ar^4}{\delta} + Cr^2 + D, \tag{3.20}$$

in which A, B, C and D are constants of integration. On using the values of $f(r)$ and $g(r)$ from Eqs. (3.19) and (3.20) into Eq. (3.15) we obtain the following expression for the stream function

$$\psi(r,z) = (Ar^2+B)z + \left(Cr^2 + D + \frac{ar^4}{\delta_1}\right). \tag{3.21}$$

Substituting Eq. (3.21) into Eq. (3.14) we obtain

$$\frac{1}{r}(Ar^2+B)\frac{d\Omega}{dr}+\nu\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)\Omega=0. \qquad (3.22)$$

Clearly V_r and V_z can be obtained through Eq. (3.21). For the determination of $V_\theta=\Omega/r$ we have to solve Eq. (3.22) for Ω. In order to find Ω we substitute $\eta=d\Omega/dr$ to get

$$\frac{d\eta}{dr}=\eta\left(\frac{1}{r^2}-\frac{A}{\nu}-\frac{B}{r\nu}\right),$$

which gives

$$\eta=\overline{c}r^{(1-\frac{B}{\nu})}e^{-\frac{Ar^2}{2\nu}}.$$

From $d\Omega/dr=\eta$ we now write

$$\Omega(r)=\overline{c}\int r^{(1-\frac{B}{\nu})}e^{-\frac{Ar^2}{2\nu}}dr+\delta_1, \qquad (3.23)$$

in which $\overline{c}$ and δ_1 are constants.

On setting $B=0$ and $\overline{c}=A\delta_1/\nu$, Eq. (3.23) gives the solution of Lakshmana [43] i.e.

$$\Omega(r)=\delta_1\left(1-e^{-\frac{Ar^2}{2\nu}}\right). \qquad (3.24)$$

From Eq. (3.18) the velocity components are

$$V_r = -\frac{1}{r}\frac{\partial\Psi}{\partial z}=-\frac{1}{r}Ar^2=-Ar, \qquad (3.25)$$

$$V_z = \frac{1}{r}\frac{\partial\Psi}{\partial r}=\frac{1}{r}(2Arz+2Cr+\frac{1}{2}ar^3)=2(Az+c)+\frac{1}{2}ar^2, \qquad (3.26)$$

whereas the velocity component $V_\theta=\Omega(r)/r$ can be obtained through Eq. (3.24) as

$$V_\theta=\frac{\Omega}{r}=\frac{\delta_1}{r}\left(1-e^{-\frac{Ar^2}{2\nu}}\right). \qquad (3.27)$$

The vorticity components are defined through

$$\xi_r=-\frac{1}{r}\frac{\partial\Omega}{\partial z},\ \xi_\theta=\omega=\frac{\partial V_r}{\partial z}-\frac{\partial V_z}{\partial r},\ \xi_z=\frac{1}{r}\frac{\partial\Omega}{\partial r}, \qquad (3.28)$$

which in the present case take the following form

$$\zeta_r = 0,\ \zeta_\theta = \omega = -ar,\ \zeta_z = \frac{A\delta_1}{\nu} e^{-\frac{Ar^2}{2\nu}}. \tag{3.29}$$

When $B \neq 0$, then solution of Eq. (3.23) by using *Mathematica* is given by

$$\Omega(r) = \delta - \overline{c}\left[2^{-\frac{B}{2\nu}}\left(\frac{A}{\nu}\right)^{-1+\frac{B}{2\nu}} \Gamma\left\{\left(1-\frac{B}{2\nu}\right), \frac{Ar^2}{2\nu}\right\}\right], \tag{3.30}$$

where $\Gamma(\alpha, x)$ is the *incomplete gamma function* which is the generalization of the *gamma function* $\Gamma(\alpha)$ (see Appendix 1). The velocity components and the vorticity components for $B \neq 0$ are

$$\begin{aligned} V_r &= -(Ar + \frac{B}{r}),\ V_z = 2(Az + C) + \frac{1}{2}ar^2, \\ V_\theta &= \frac{\delta}{r} - \frac{\overline{c}}{r}\left[2^{-\frac{B}{2\nu}}\left(\frac{A}{\nu}\right)^{-1+\frac{B}{2\nu}} \Gamma\left\{\left(1-\frac{B}{2\nu}\right), \frac{Ar^2}{2\nu}\right\}\right], \end{aligned} \tag{3.31}$$

$$\zeta_r = 0,\ \zeta_\theta = \omega = -ar,\ \zeta_z = \frac{A\delta_1}{\nu} e^{-\frac{Ar^2}{2\nu}}. \tag{3.32}$$

Here we remark that on setting $B = 0$ in Eqs. (3.30) to (3.32) we readily recover the solution given by Lakshmana [43]. Moreover, by letting $a = 0$ in equations (3.31) and (3.32) we recover the result of Roy [44].

On setting $A = 0$ in Eq. (3.22), we obtain the following solution for $\Omega(r)$

$$\Omega(r) = \overline{c}_1 + \overline{c}_2\left(\frac{\nu}{2\nu - B}\right) r^{\frac{2\nu - B}{\nu}} \tag{3.33}$$

and the corresponding velocity and vorticity components are respectively given by

$$V_r = -\frac{B}{r},\ V_z = 2C + \frac{1}{2}ar^2,\ V_\theta = \frac{\overline{c}_1}{r} + \frac{\overline{c}_2}{r}\left(\frac{\nu}{2\nu - B}\right) r^{\frac{2\nu - B}{\nu}}, \tag{3.34}$$

$$\zeta_r = 0,\ \zeta_\theta = -ar,\ \zeta_z = \overline{c}_2 r^{-\frac{B}{\nu}}, \tag{3.35}$$

where $\overline{c}_1$ and $\overline{c}_2$ are constants.

3.4.2 For $\alpha_1 \neq 0$, $\Omega = \Omega(r)$, $\psi = \psi(r,z)$

For this case Eq. (3.11) is

$$\beta\left[\frac{\partial\psi}{\partial z}\frac{d}{dr}\left(\frac{d^2\Omega}{dr^2}-\frac{1}{r}\frac{d\Omega}{dr}\right)\right]-\frac{\partial\psi}{\partial z}\frac{d\Omega}{dr}=\nu r\left(\frac{d^2\Omega}{dr^2}-\frac{1}{r}\frac{d\Omega}{dr}\right), \tag{3.36}$$

where $\beta = \alpha_1/\rho$ is the second grade parameter. Differentiating Eq. (3.36) with respect to z and then integrating twice we get

$$\psi(r,z) = f(r)z + g(r). \tag{3.37}$$

Again assuming $E^2\psi = ar^2$ leaves Eq. (3.19) as an identity and Eq. (3.37) becomes

$$\psi(r,z) = (Ar^2+B)z + \left(Cr^2 + D + \frac{a}{\delta_1}r^4\right). \tag{3.38}$$

On differentiating Eq. (3.38) with respect to z and substituting in Eq. (3.36) we have

$$\beta(Ar^2+B)\frac{d^3\Omega}{dr^3}-\left(\frac{\beta}{r}(Ar^2+B)+\nu r\right)\frac{d^2\Omega}{dr^2}=\left((1-\frac{\beta}{r^2})(Ar^2+B)-\nu\right)\frac{d\Omega}{dr}. \tag{3.39}$$

The general solution of Eq. (3.39) is not easy to obtain, therefore, we give some specific cases:

Equation (3.39) for $B = 0$ can be written as

$$\frac{Ar}{\nu}\left[\beta\frac{d}{dr}\left(\frac{d^2\Omega}{dr^2}-\frac{1}{r}\frac{d\Omega}{dr}\right)-\frac{d\Omega}{dr}\right]=\frac{d^2\Omega}{dr^2}-\frac{1}{r}\frac{d\Omega}{dr}. \tag{3.40}$$

In order to find the solution of Eq. (3.40) we put $d\Omega/dr = \eta_2$ to have the following equation

$$r^2\frac{d^2\eta_2}{dr^2}+\lambda_1 r\frac{d\eta_2}{dr}-(\lambda_1+r^2)\eta_2 = 0, \tag{3.41}$$

where $\lambda_1 = -\left(1+\frac{\nu}{\beta A}\right)$.

The solution of Eq. (3.41) is given as

$$\eta_2 = c_3 r^{\frac{1-\lambda}{2}} J\left[\frac{1+\lambda_1}{2}, -ir\right] + c_4 r^{\frac{1-\lambda}{2}} Y\left[\frac{1+\lambda_1}{2}, -ir\right] \tag{3.42}$$

and by putting $\eta = d\Omega/dr$ we have the following

$$\Omega(r) = \int \eta_2(r)\, dr, \tag{3.43}$$

where c_3 and c_4 are arbitrary constants, and J and Y are Bessel functions of first and second kind, respectively.

The velocity and vorticity components are

$$V_r = -Ar, \;\; V_z = 2(Az + C) + \frac{1}{2}ar^2, \;\; V_\theta = \frac{\Omega}{r} = \frac{1}{r}\int \eta_2(r)\, dr, \tag{3.44}$$

$$\zeta_r = 0, \;\; \zeta_\theta = -ar, \;\; \zeta_z = \frac{1}{r}\frac{\partial}{\partial r}\int \eta_2(r)\, dr. \tag{3.45}$$

Equation (3.39) for $A = 0$ can be written as

$$\frac{B}{r\nu}\left[\beta\frac{d}{dr}\left(\frac{d^2\Omega}{dr^2} - \frac{1}{r}\frac{d\Omega}{dr}\right) - \frac{d\Omega}{dr}\right] = \frac{d^2\Omega}{dr^2} - \frac{1}{r}\frac{d\Omega}{dr}. \tag{3.46}$$

The solution of Eq. (3.46) is obtained through *Mathematica* and is directly given as

$$\begin{aligned} \Omega(r) &= \bar{c}_5 + \bar{c}_3\frac{r^2}{2}\, pFq\left[\frac{B}{2\nu}, 2, \frac{r^2\nu}{2B\beta}\right] \\ &+ \bar{c}_4 r\, G\left[\left\{\left\{\frac{1}{2}\right\}, \left\{\frac{3}{2} - \frac{B}{2\nu}\right\}\right\}, \left\{\left\{\frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}\right\}\right\}, \frac{r^2\nu}{4B\beta}\right], \end{aligned} \tag{3.47}$$

where $\bar{c}_3$, $\bar{c}_4$, and $\bar{c}_5$ are constants. The velocity and vorticity components are

$$\begin{aligned} V_r &= 0, \;\; V_z = 2C + \frac{1}{2}ar^2, \;\; V_\theta = \bar{c}_5 + \bar{c}_3\frac{r}{2}\, pFq\left[\frac{B}{2\nu}, 2, \frac{r^2\nu}{2B\beta}\right] \\ &+ \bar{c}_4\, G\left[\left\{\left\{\frac{1}{2}\right\}, \left\{\frac{3}{2} - \frac{B}{2\nu}\right\}\right\}, \left\{\left\{\frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}\right\}\right\}, \frac{r^2\nu}{4B\beta}\right], \end{aligned} \tag{3.48}$$

$$
\begin{aligned}
\zeta_r &= 0, \ \zeta_\theta = -ar, \\
\zeta_z &= \frac{r^2 \overline{c}_3}{8\beta}\, pFq\left[1+\frac{B}{2\nu}, 3, \frac{r^2\nu}{2B\beta}\right] + \overline{c}_3\, pFq\left[\frac{B}{2\nu}, 2, \frac{r^2\nu}{2B\beta}\right] \\
&+ \frac{\overline{c}_4}{r}\, G\left[\left\{\{\,\}, \left\{\frac{3}{2}-\frac{B}{2\nu}\right\}\right\}, \left\{\left\{\frac{1}{2}, \frac{1}{2}\right\}, \{\,\}\right\}, \frac{r^2\nu}{2B\beta}\right] \\
&+ \frac{1}{2r}\left(\overline{c}_4 - 1\right) G\left[\left\{\left\{\frac{1}{2}\right\}, \left\{\frac{3}{2}-\frac{B}{2\nu}\right\}\right\}, \left\{\left\{\frac{1}{2}, \frac{1}{2}\right\}, \left\{-\frac{1}{2}\right\}\right\}, \frac{r^2\nu}{2B\beta}\right],
\end{aligned} \tag{3.49}
$$

where pFq (generalized hypergeometric function) and G (the Meijer function) are defined in Appendix 2.

3.4.3 For $\alpha_1 \neq 0$, $\Omega = 0$, $\psi = \psi(r, z)$

Here Eq. (3.11) is automatically satisfied and Eq. (3.10) becomes

$$
\rho\left[\frac{1}{r}\frac{\partial(\psi, E^2\psi)}{\partial(r,z)} + \frac{2}{r^2}\frac{\partial\psi}{\partial z}E^2\psi\right] - \mu E^4\psi = \alpha_1\left[\frac{1}{r}\frac{\partial(\psi, E^4\psi)}{\partial(r,z)} + \frac{2}{r^2}\frac{\partial\psi}{\partial z}E^4\psi\right]. \tag{3.50}
$$

The above equation can also be written as

$$
\rho\left[\frac{1}{r}\frac{\partial(\psi, E^2\psi/r^2)}{\partial(r,z)}\right] - \frac{\mu}{r}E^4\psi = \alpha_1\left[\frac{\partial(\psi, E^4\psi/r^2)}{\partial(r,z)}\right]. \tag{3.51}
$$

For solution of above equation let us take

$$
E^2\psi = \varphi(r) \tag{3.52}
$$

and obtain

$$
\frac{\partial\psi}{\partial z}\left[r\rho\left(2\varphi - r\varphi'\right) + \alpha_1\left(r^2\varphi''' - 3r\varphi'' + 3\varphi'\right)\right] - \mu r^2\left(r\varphi'' - \varphi'\right) = 0, \tag{3.53}
$$

where primes indicate differentiation with respect to r. Differentiating Eq. (3.53) with respect to z and then solving the resulting equation we have

$$
\psi(r,z) = \lambda(r)z + \widetilde{\alpha}(r), \tag{3.54}
$$

where $\lambda(r)$ and $\widetilde{\alpha}(r)$ are functions of integration.

Since $E^2\psi = \varphi(r)$, so Eq. (3.54) becomes

$$\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)(\lambda(r)z + \widetilde{\alpha}(r)) = \varphi(r),$$

or

$$\left(\lambda'' - \frac{1}{r}\lambda'\right)z + \left(\widetilde{\alpha}'' - \frac{1}{r}\widetilde{\alpha}'\right) = \varphi(r)$$

or

$$\lambda'' - \frac{1}{r}\lambda' = 0, \; \widetilde{\alpha}'' - \frac{1}{r}\widetilde{\alpha}' = \varphi(r). \tag{3.55}$$

The solution of first equation is

$$\lambda = c_1\frac{r^2}{2} + D_1 \tag{3.56}$$

and so Eq. (3.54) becomes

$$\psi(r, z) = \left(C_1 r^2 + D_1\right)z + \widetilde{\alpha}(r), \tag{3.57}$$

in which $C_1 = c_1/2$ and D_1 are arbitrary constants.

Using Eq. (3.57) into Eq. (3.53) one obtains

$$r\rho\left(C_1 r^2 + D_1\right)\left(2\varphi - r\varphi'\right) - \mu r^2\left(r\varphi'' - \varphi'\right) + \alpha_1\left(C_1 r^2 + D_1\right)\left(r^2\varphi''' - 3r\varphi'' + 3\varphi'\right) = 0. \tag{3.58}$$

For viscous (Newtonian) case we get the following equation from Eq. (3.58)

$$\nu r^2\varphi'' + (C_1 r^2 + D_1 - \nu)r\varphi' - 2(C_1 r^2 + D_1)\varphi = 0. \tag{3.59}$$

The particular solutions of Eq. (3.59) are obtained by Berker [41]. However, we give the general solution of Eq. (3.59) with the help of *Mathematica* as follows.

$$\varphi = -\frac{C_1 r^2 \overline{C}_1}{2\nu} - \frac{1}{4\nu^2}\left[C_1 r^2 (D + 2\nu)\overline{C}_2\left\{\Gamma\left(-1 - \frac{D_1}{2\nu}\right) - \Gamma\left(-1 - \frac{D_1}{2\nu}, \frac{C_1 r^2}{2\nu}\right)\right\}\right], \tag{3.60}$$

where Γ is the gamma function and $\overline{C}_1$, $\overline{C}_2$ are arbitrary constants.

For $\alpha_1 \neq 0$ and $D_1 = 0$ the solution of Eq. (3.58) is

$$\begin{aligned}\varphi(r) &= c_2 r^2 + c_1 r^{4-\nu/C_1\beta} \times pFq\left[\left\{1-\frac{\nu}{2C_1\beta}\right\},\left\{2-\frac{\nu}{2C_1\beta},3-\frac{\nu}{2C_1\beta}\right\},\frac{r^2}{4\beta}\right] \\ &+c_3\, G\left[\{\{\ \},\{2\}\},\left\{\{0,1\},\left\{2-\frac{\nu}{2C_1\beta}\right\}\right\},\frac{r^2}{4\beta}\right].\end{aligned} \quad (3.61)$$

Equation (3.61) can be written as

$$\varphi(r) = \varepsilon_2 r^2 + \varepsilon_1\varphi_1(r) + \varepsilon_3\varphi_2(r), \quad (3.62)$$

where

$$\begin{aligned}\varphi_1(r) &= r^{\frac{4-\nu}{C_1\beta}} pF_q\left[\left\{1-\frac{\nu}{2C_1\beta}\right\},\left\{2-\frac{\nu}{2C_1\beta},3-\frac{\nu}{2C_1\beta}\right\},\frac{r^2}{4\beta}\right], \\ \varphi_2(r) &= G\left[\left\{\{\},\{2\}\},\{\{0,1\},\{2-\frac{\nu}{2C_1\beta}\}\right\},\frac{r^2}{4\beta}\right].\end{aligned}$$

In order to find $\alpha(r)$ we write

$$\widetilde{\alpha}'' - \frac{\widetilde{\alpha}'}{r} = r\frac{d}{dr}\left(\frac{1}{r}\widetilde{\alpha}'\right) = \varphi(r) = \varepsilon_2 r^2 + \varepsilon_1\varphi_1(r) + \varepsilon_3\varphi_2(r),$$

and integration yields

$$\widetilde{\alpha}(r) = \varepsilon_2\frac{r^4}{8} + \varepsilon_1\int r\int\frac{1}{r}\varphi_1(r)dr + \varepsilon_3\int r\int\frac{1}{r}\varphi_2(r)dr + \varepsilon_4\frac{r^2}{2} + \varepsilon_5. \quad (3.63)$$

Using Eq. (3.63) in equation Eq. (3.57) we obtain

$$\psi(r,z) = Ar^2z + \varepsilon_2\frac{r^4}{8} + \varepsilon_1\int r\int\frac{1}{r}\varphi_1(r)dr + \varepsilon_3\int r\int\frac{1}{r}\varphi_2(r)dr + \varepsilon_4\frac{r^2}{2} + \varepsilon_5. \quad (3.64)$$

Thus, the velocity and vorticity components in this case are

$$V_r = -Ar,\ V_\theta = 0,\ V_z = 2Az + \varepsilon_2\frac{r^2}{2} + \varepsilon_4 + \varepsilon_1\int\frac{1}{r}\varphi_1(r)dr + \varepsilon_3\int\frac{1}{r}\varphi_2(r)dr, \quad (3.65)$$

$$\xi_r = 0,\ \xi_\theta = -\varepsilon_2 r - \varepsilon_1\frac{1}{r}\varphi_1(r) - \varepsilon_3\frac{1}{r}\varphi_2(r),\ \xi_z = 0. \quad (3.66)$$

The Eq. (3.58) for $C_1 = 0$ can be written as

$$\alpha_1 D_1 r^2 \varphi''' - r(\mu r^2 + 3\alpha_1 D_1) + (\mu r^2 - r^2 \rho D_1 + 3\alpha_1 D_1)\varphi' + 2r\rho D_1 \varphi = 0. \tag{3.67}$$

The solution of Eq. (3.67) is identical to that given by Siddiqui *et al.* [28]. The stream functions is thus given

$$\psi = zD_1 + \varepsilon_8 r^4 + \varepsilon_9 r^2 + \varepsilon_{10} + \varepsilon_6 \int r \left(\int r \left(\int \phi_3(r)\, dr \right) dr \right) dr + \varepsilon_7 \int r \left(\int r \left(\int \phi_4(r)\, dr \right) dr \right) dr. \tag{3.68}$$

In Eq. (3.68)

$$\phi_3(r) = r\ 1F1\left[\frac{4\mu - \rho D_1}{2\mu}, 3, \frac{\mu r^2}{2\alpha_1 D_1}\right],\ \phi_4(r) = -\frac{2\alpha_1 D_1}{\mu r}\ 1F1\left[\frac{\rho D_1}{2\mu}, -1, \frac{\mu r^2}{2\alpha_1 D_1}\right],$$

and ε_6, ε_7, ε_8, ε_9 and ε_{10} are constants and $1F1$ is the confluent hypergeometric function of the first kind and is the special case of pFq for $p = 1$ and $q = 1$ (see Appendix 2). The confluent hypergeometric function can be obtained from the series expansion

$$1F1(\theta, \mathrm{b}; z_1) = 1 + \frac{\theta z_1}{b} + \frac{\theta(\theta+1)}{2!} z_1^2 + \ldots = \sum_{k=0}^{\infty} \frac{(\theta)_k}{(b)_k} \frac{z_1^k}{k!}. \tag{3.69}$$

Remark 2 *Some special results are obtained when θ and b are both integers.*

1. If $\theta < 0$, and either $b > 0$ or $b < \theta$, the series yields a polynomial with a finite number of terms.

2. If $b = 0$ or negative integer, then $1F1(\theta, \mathrm{b}; z_1)$ itself is infinite.

3.4.4 $\Omega(r) = \Omega_0 r^2 + \Omega_1$

We now specify our problem by considering the particular choice of $\Omega(r)$,

$$\Omega(r) = \Omega_0 r^2 + \Omega_1, \tag{3.70}$$

where Ω_0 and Ω_1 are constants. Using Eq. (3.70) in Eq. (3.11) we get

$$-2\rho\Omega_0\frac{\partial\Psi}{\partial z}=0.$$

As $\rho\neq 0$, $\Omega_0\neq 0$ it implies that $\frac{\partial\Psi}{\partial z}=0$ and thus $\Psi=\Psi(r)$.

With this Eq. (3.10) becomes

$$E^4\Psi=0, \tag{3.71}$$

which can also be written as

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\right)\Psi=0. \tag{3.72}$$

Integrating four times we get

$$\Psi(r)=\overline{A}+\overline{B}r^2+\overline{C}r^4+\overline{D}r^2\ln r, \tag{3.73}$$

and the corresponding velocity and vorticity components are

$$V_r=0,\ \ V_\theta=\Omega_0 r+\frac{\Omega_1}{r},\ \ V_z=2\overline{B}+4\overline{C}r^2+\overline{D}\left(1+2\ln r\right), \tag{3.74}$$

$$\zeta_r=0,\ \ \zeta_\theta=-8\overline{C}r-2\frac{\overline{D}}{r},\ \ \zeta_z=2\Omega_0, \tag{3.75}$$

where $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$, Ω_0 and Ω_1 are constants.

3.5 Unsteady cases

In this section our interest lies in obtaining the unsteady solutions for viscous and second grade fluids.

3.5.1 When $\alpha_1=0$, $\Omega=\Omega(r,t)$ and $\psi=\psi(r,z)$

Eq. (3.11) become

$$\frac{\partial\Omega}{\partial t}-\frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial\Omega}{\partial r}-\nu\left(\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}\right)\Omega=0 \tag{3.76}$$

which gives

$$\psi(r,z) = f(r)z + g(r), \tag{3.77}$$

and on putting $E^2\psi = ar^2$, we get

$$\psi(r,z) = (Ar^2 + B)z + (Cr^2 + D + \frac{a}{8}r^4). \tag{3.78}$$

As before, using Eq. (3.78) into Eq. (3.76) we obtain

$$\frac{\partial \Omega}{\partial t} - \frac{1}{r}(Ar^2 + B)\frac{\partial \Omega}{\partial r} - \nu(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r})\Omega = 0. \tag{3.79}$$

In order to find the solution of Eq. (3.79) we use three different methods:

Method 1

On introducing

$$\sigma_1 = h(t)r,\ \Omega = \Omega(\sigma_1), \tag{3.80}$$

into Eq. (3.79) we readily obtain

$$\sigma_1 \left[\frac{1}{h^3}\left\{\frac{dh}{dt} - Ah\left(t\right)\right\} - \frac{B}{\sigma_1^2}\right]\frac{d\Omega}{d\sigma_1} - \nu\left[\frac{d^2\Omega}{d\sigma_1^2} - \frac{1}{\sigma_1}\frac{d\Omega}{d\sigma_1}\right] = 0. \tag{3.81}$$

Choosing $h(t)$ such that

$$\frac{dh}{dt} - Ah(t) = -A\overline{\lambda}h^3(t), \tag{3.82}$$

we get from Eq. (3.81) the following ordinary differential equation

$$\nu\frac{d^2\Omega}{d\sigma_1^2} + \left(A\overline{\lambda}\sigma_1 + \left(\frac{B-\nu}{\sigma_1}\right)\right)\frac{d\Omega}{d\sigma_1} = 0, \tag{3.83}$$

where $\overline{\lambda}$ is a constant. The Eq. (3.83) is similar to that discussed in section 3.2. The solution of Eq. (3.83) is found through *Mathematica* and is given as

$$\Omega\left(\sigma_1\right) = c_7 - c_6 2^{-\frac{B}{2\nu}}\left(\frac{A\overline{\lambda}}{\nu}\right)^{-1+\frac{B}{2\nu}}\Gamma\left[\left(1 - \frac{B}{2\nu}\right), \frac{A\overline{\lambda}\sigma_1^2}{2\nu}\right], \tag{3.84}$$

where c_6 and c_7 are constants.

To obtain the solution of Eq. (3.82) we put $h^{-2} = \Phi$ to have the following equation

$$\frac{d\Phi}{dt} + 2A\Phi = 2A\overline{\lambda}. \tag{3.85}$$

The solution of Eq. (3.85) is

$$\Phi = (\overline{\lambda} + c_8 e^{-2At}). \tag{3.86}$$

Since $\Phi = h^{-2}$ so

$$h(t) = (\overline{\lambda} + c_8 e^{-2At})^{-\frac{1}{2}}. \tag{3.87}$$

Also $\sigma_1 = h(t)r$ gives

$$\sigma_1 = r(\overline{\lambda} + c_8 e^{-2At})^{-\frac{1}{2}}. \tag{3.88}$$

Using the value of σ_1^2 from Eq. (3.88) in Eq. (3.84) we get the unsteady solution of Eq. (3.79) as

$$\Omega(r,t) = c_9 + c_{10} 2^{-\frac{B}{2\nu}} \left(\frac{A\overline{\lambda}}{\nu}\right)^{-1+\frac{B}{2\nu}} \Gamma\left[\left(1 - \frac{B}{2\nu}\right), \frac{A\overline{\lambda} r^2}{2\nu\left(\overline{\lambda} + c_8 e^{-2At}\right)}\right], \tag{3.89}$$

where c_8, c_9 and c_{10} are constants.

The velocity and vorticity components are respectively given by

$$\begin{aligned} V_r &= -Ar, \quad V_z = 2(Az + C) + \frac{1}{2} a r^2, \\ V_\theta &= \frac{c_9}{r} + \frac{c_{10}}{r} 2^{-\frac{B}{2\nu}} \left(\frac{A\overline{\lambda}}{\nu}\right)^{-1+\frac{B}{2\nu}} \Gamma\left[\left(1 - \frac{B}{2\nu}\right), \frac{A\overline{\lambda} r^2}{2\nu\left(\overline{\lambda} + c_8 e^{-2At}\right)}\right], \end{aligned} \tag{3.90}$$

$$\zeta_r = 0, \quad \zeta_\theta = -ar, \quad \zeta_z = c_{10} r^{-\frac{B}{\nu}} \left(\overline{\lambda} + c_8 e^{-2At}\right)^{\frac{1}{2}\left(\frac{B}{\nu} - 1\right)} e^{-\frac{Ar^2}{2\nu} \frac{\lambda}{\left(\overline{\lambda} + c_8 e^{-2At}\right)}}. \tag{3.91}$$

For $B = 0$, Eq. (3.83) become

$$\frac{d^2\Omega}{d\sigma_1^2} - \frac{1}{\sigma_1}\frac{d\Omega}{d\sigma_1} + \frac{A\overline{\lambda}}{\nu}\sigma_1 \frac{d\Omega}{d\sigma_1} = 0, \tag{3.92}$$

which is exactly the same as discussed by Lakshmana [43]. The solution is given as

$$\Omega(\sigma_1) = \delta_1 \left[1 - \exp\left(-\frac{A\overline{\lambda}}{2\nu}\sigma_1^2\right)\right]. \tag{3.93}$$

Using Eq. (3.88) in Eq. (3.93) we have

$$\Omega(r,t) = \delta_1 \left[1 - \exp\left(-\frac{A\overline{\lambda}}{2\nu}\frac{r^2}{(\overline{\lambda} + c_8 e^{-2At})}\right)\right] \tag{3.94}$$

and the corresponding velocity and vorticity components are

$$V_r = -Ar, \;\; V_\theta = \frac{\delta_1}{r}\left[1 - \exp\left(-\frac{Ar^2}{2\nu\,(1 + c_{11}e^{-2At})}\right)\right], \; V_z = 2\,(Az + C) + \frac{1}{2}ar^2, \tag{3.95}$$

$$\zeta_r = 0, \;\; \zeta_\theta = -ar, \;\; \zeta_z = \frac{A\delta_1}{\nu\,(1 + c_{11}e^{-2At})} e^{-\frac{Ar^2}{2\nu(1+c_{11}e^{-2At})}}, \tag{3.96}$$

where δ_1 is the constant of integration and $c_{11} = c_8/\overline{\lambda}$.

When $A = B = 0$, then Eq. (3.83) become

$$\frac{\partial \Omega}{\partial t} - \nu\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)\Omega = 0 \tag{3.97}$$

which upon using the similarity transform

$$\eta_1 = \frac{r}{2\sqrt{\nu t}} \tag{3.98}$$

reduces to

$$\frac{d^2\Omega}{d\eta_1^2} + (2\eta_1 - \frac{1}{\eta_1})\frac{d\Omega}{d\eta_1} = 0. \tag{3.99}$$

The solution of Eq. (3.99) is

$$\Omega(\eta_1) = -\frac{\Omega_2}{2}e^{-\eta_1^2} + \Omega_3, \tag{3.100}$$

in which Ω_2 and Ω_3 are constants of integration and which on using $\eta_1 = \frac{r}{2\sqrt{\nu t}}$ gives

$$\Omega(\eta_1) = -\frac{\Omega_2}{2}e^{-\frac{r^2}{4\nu t}} + \Omega_3. \tag{3.101}$$

Consequently, the velocity and vorticity components are

$$V_r = 0,\ V_\theta = \frac{1}{r}\left[\Omega_3 - \frac{\Omega_2}{2}e^{-\frac{r^2}{4\nu t}}\right],\ V_z = 2C + \frac{1}{2}ar^2, \tag{3.102}$$

$$\zeta_r = 0,\ \zeta_\theta = -ar,\ \zeta_z = \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}. \tag{3.103}$$

Method 2

Here we apply separation of variable method to obtain the solution of Eq. (3.79). For that let us assume

$$\Omega(r,t) = \widetilde{\xi}(r)\overline{\eta}(t), \tag{3.104}$$

into Eq. (3.79) to have the following equation

$$\widetilde{\xi}\frac{\overline{\eta}'(t)}{\overline{\eta}(t)} - \frac{1}{r}(Ar^2 + B)\frac{d\widetilde{\xi}}{dr} - \nu\left(\frac{d^2\widetilde{\xi}}{dr^2} - \frac{1}{r}\frac{d\widetilde{\xi}}{dr}\right) = 0, \tag{3.105}$$

where prime denotes the differentiation with respect to time. We now discuss two cases in order to study Eq. (3.105).

Case 1 $\overline{\eta}'(t) = 0$ implies that $\overline{\eta}(t)$ =constant= $\overline{\eta}_0$ (say) and Eq. (3.105) becomes a steady case which is already discussed in section 3.4.1.

Case 2 If $\overline{\eta}'(t) \neq 0$ then we choose $\overline{\eta}$ such that

$$\frac{\overline{\eta}'(t)}{\overline{\eta}(t)} = -\lambda_1(\text{constant}),$$

which is solved to give the solution

$$\overline{\eta}(t) = \lambda_0 e^{-\lambda_1 t}, \tag{3.106}$$

where λ_0 is an arbitrary integration constant and Eq. (3.105) become

$$\nu\frac{d^2\widetilde{\xi}}{dr^2} + \frac{1}{r}\left((Ar^2 + B) - \nu\right)\frac{d\widetilde{\xi}}{dr} + \lambda_1\widetilde{\xi} = 0. \tag{3.107}$$

When $\lambda_1 = 0$, we again have the steady case discussed in section 3.4.1. For $\lambda_1 \neq 0$ we discuss few possible cases which are described as:

Subcase 1 For $A \neq 0$, $B \neq 0$, we have the following solution

$$\begin{aligned} \widetilde{\xi}_1(r) &= \delta_3 \, 1F1\left[\frac{\lambda_1}{2A}, \frac{B}{2\nu}, -\frac{Ar^2}{2\nu}\right] \\ &\quad + \delta_4 \, 1F1\left[1 + \frac{\lambda_1}{2A} - \frac{B}{2\nu}, 2 - \frac{B}{2\nu}, -\frac{Ar^2}{2\nu}\right]. \end{aligned} \tag{3.108}$$

Subcase 2 For $A \neq 0$, $B = 0$, we have the following solution

$$\begin{aligned} \widetilde{\xi}_2(r) &= r^2 \delta_5 \, 1F1\left[1 + \frac{\lambda_1}{2A}, 2, -\frac{Ar^2}{2\nu}\right] \\ &\quad + \delta_6 \, G\left[\left\{\{\,\}, \left\{1 - \frac{\lambda_1}{2A}\right\}\right\}, \{\{0,1\}, \{\,\}\}, \frac{Ar^2}{2\nu}\right]. \end{aligned} \tag{3.109}$$

Subcase 3 For $A = 0$, $B \neq 0$, we have the following solution

$$\begin{aligned} \widetilde{\xi}_3(r) &= \frac{1}{\sqrt{\pi}} 2^{-(B-2\nu)/2\nu} r^{1-B/2\nu} \nu^{(\nu-B)/2\nu} \left(-\frac{\lambda_1}{2A}\right)^{-(B-2\nu)/4\nu} \\ &\quad \times \left[\delta_7 \, K\left[\frac{B-2\nu}{2\nu}, r\sqrt{\frac{-\lambda_1}{\nu}}\right] + \delta_8 \sqrt{\pi} 2^{(B-2\nu)/\nu} \, I\left[\frac{B-2\nu}{2\nu}, r\sqrt{\frac{-\lambda_1}{\nu}}\right] \Gamma\left(\frac{B}{2\nu}\right)\right], \end{aligned} \tag{3.110}$$

where δ_3, δ_4, δ_5, δ_6, δ_7, and δ_8 are constants, K is the modified Bessel function of the second kind, I is the modified Bessel function of the first kind, $1F1$ is the confluent hypergeometric of first kind, and G is the Meijer function. The complete solution in all the subcases is given by

$$\Omega_1(r,t) = \widetilde{\xi}_1(r)\,\lambda_0 e^{-\lambda_1 t}, \; \Omega_2(r,t) = \widetilde{\xi}_2(r)\,\lambda_0 e^{-\lambda_1 t}, \; \Omega_3(r,t) = \widetilde{\xi}_3(r)\,\lambda_0 e^{-\lambda_1 t} \tag{3.111}$$

and the velocity and vorticity components for subcases 1 and 2 are

$$V_r = -Ar, \;\; V_z = 2(Az + C) + \frac{1}{2}ar^2, \; V_{1\theta} = \frac{1}{r}\widetilde{\xi}_1(r)\,\lambda_0 e^{-\lambda_1 t}, \; V_{2\theta} = \frac{1}{r}\widetilde{\xi}_2(r)\,\lambda_0 e^{-\lambda_1 t}, \tag{3.112}$$

$$\zeta_r = 0, \;\; \zeta_\theta = -ar, \;\; \zeta_{1z} = \frac{1}{r}\frac{d\widetilde{\xi}_1}{dr}\lambda_0 e^{-\lambda_1 t}, \; \zeta_{2z} = \frac{1}{r}\frac{d\widetilde{\xi}_2}{dr}\lambda_0 e^{-\lambda_1 t}. \tag{3.113}$$

The velocity and vorticity components for subcase 3 are

$$V_r = 0, \;\; V_z = 2C + \frac{1}{2}ar^2, \; V_{3\theta} = \frac{1}{r}\widetilde{\xi}_3(r)\,\lambda_0 e^{-\lambda_1 t}, \tag{3.114}$$

$$\zeta_r = 0, \;\; \zeta_\theta = -ar, \;\; \zeta_{3z} = \frac{1}{r}\frac{d\widetilde{\xi}_3}{dr}\lambda_0 e^{-\lambda_1 t} \tag{3.115}$$

3.5.2 For $\alpha_1 \neq 0$, $\Omega = \Omega(r,t)$ and $\psi = \psi(r,z)$

Eq. (3.11) become

$$\begin{aligned} &\beta\left[\frac{\partial}{\partial t}\left(\frac{\partial^2\Omega}{\partial r^2} - \frac{1}{r}\frac{\partial\Omega}{\partial r}\right) - \frac{1}{r}\frac{\partial\psi}{\partial z}\left(\frac{\partial^3\Omega}{\partial r^3} - \frac{1}{r}\frac{\partial^2\Omega}{\partial r^2} + \frac{1}{r^2}\frac{\partial\Omega}{\partial r}\right)\right] \\ = &\left[\frac{\partial\Omega}{\partial t} - \frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial\Omega}{\partial r}\right] - \nu\left(\frac{\partial^2\Omega}{\partial r^2} - \frac{1}{r}\frac{\partial\Omega}{\partial r}\right) \end{aligned} \tag{3.116}$$

Differentiating Eq. (3.116) with respect to z and then integrating twice with respect to z we obtain as before

$$\psi(r,z) = f(r)\,z + g(r) \tag{3.117}$$

and $E^2\psi = ar^2$ gives

$$\psi(r,z) = \left(Ar^2 + B\right)z + \left(Cr^2 + D + \frac{a}{8}r^4\right). \tag{3.118}$$

Using the value of ψ from Eq. (3.118) in Eq. (3.116) we have the linear differential equation for determination of Ω

$$\begin{aligned} &\beta\left[\frac{\partial}{\partial t}\left(\frac{\partial^2\Omega}{\partial r^2} - \frac{1}{r}\frac{\partial\Omega}{\partial r}\right) - \frac{1}{r}\left(Ar^2 + B\right)\left(\frac{\partial^3\Omega}{\partial r^3} - \frac{1}{r}\frac{\partial^2\Omega}{\partial r^2} + \frac{1}{r^2}\frac{\partial\Omega}{\partial r}\right)\right] \\ = &\left[\frac{\partial\Omega}{\partial t} - \frac{1}{r}\left(Ar^2 + B\right)\frac{\partial\Omega}{\partial r}\right] - \nu\left(\frac{\partial^2\Omega}{\partial r^2} - \frac{1}{r}\frac{\partial\Omega}{\partial r}\right). \end{aligned} \tag{3.119}$$

In order to get the solution of Eq. (3.119), we introduce product of two functions as in viscous case

$$\Omega(r,t) = \widehat{\xi}(r)\widehat{\eta}(t), \tag{3.120}$$

which on inserting in Eq. (3.119) gives

$$\beta\left[\frac{\widehat{\eta}'(t)}{\widehat{\eta}(t)}\left(\frac{d^2\widehat{\xi}}{dr^2}-\frac{1}{r}\frac{d\widehat{\xi}}{dr}\right)-\frac{1}{r}(Ar^2+B)\left(\frac{d^3\widehat{\xi}}{dr^3}-\frac{1}{r}\frac{d^2\widehat{\xi}}{dr^2}+\frac{1}{r^2}\frac{d\widehat{\xi}}{dr}\right)\right]$$
$$=\left[\widehat{\xi}\frac{\widehat{\eta}'(t)}{\widehat{\eta}(t)}-\frac{1}{r}(Ar^2+B)\frac{d\widehat{\xi}}{dr}\right]-\nu\left(\frac{d^2\widehat{\xi}}{dr^2}-\frac{1}{r}\frac{d\widehat{\xi}}{dr}\right). \quad (3.121)$$

For $\beta=\frac{\alpha_1}{\rho}=0$, $B=0$,we obtain the case already discussed in previous section. For $\beta\neq 0$, $B=0$, we discuss two cases $\widehat{\eta}'(t)=0$ and $\widehat{\eta}'(t)\neq 0$.

Case 1 If $\widehat{\eta}'(t)=0$ then $\widehat{\eta}(t)$ =constant= $\widehat{\eta}_0$ and we obtain the case already discussed in section previous section.

Case 2 If $\widehat{\eta}'(t)\neq 0$ then we choose $\widehat{\eta}$ such that

$$\frac{\widehat{\eta}'}{\widehat{\eta}}=\text{constant}=-\lambda_2\,(\text{say}) \quad (3.122)$$

which leaves Eq. (3.121) in the following form

$$\beta Ar^2\frac{d^3\widehat{\xi}}{dr^3}-[\beta(\lambda_2-A)-\nu]\,r\frac{d^2\widehat{\xi}}{dr^2}-\left[\nu-\beta(\lambda_2-A)-Ar^2\right]\frac{d\widehat{\xi}}{dr}-\lambda_2 r\widehat{\xi}=0. \quad (3.123)$$

The solution of Eq. (3.123) is given by

$$\begin{aligned}\widehat{\xi}(r) &= c_{24}r^2\, pFq\left[\begin{array}{c}\left\{1+\frac{\lambda_2}{2A}\right\},\\ \left\{2,1+\frac{\lambda_2}{2A}-\frac{\nu}{2A\beta}\right\},\frac{r^2}{4\beta}\end{array}\right] \quad (3.124)\\ &+c_{25}r^{(2A\beta-\beta\lambda_2+\nu)/A\beta}\times pFq\left[\begin{array}{c}\left\{1+\frac{\nu}{2A\beta}\right\},\\ \left\{1-\frac{\lambda_2}{2A}+\frac{\nu}{2A\beta},2-\frac{\lambda_2}{2A}+\frac{\nu}{2A\beta}\right\},\frac{r^2}{4\beta}\end{array}\right]\\ &+c_{26}\,G\left[\left\{\left\{1-\frac{\lambda_2}{2A}\right\},\{\ \}\right\},\left\{\{0,1\},\left\{\frac{2A\beta-\beta\lambda_2+\nu}{2A\beta}\right\}\right\},\frac{r^2}{4\beta}\right],\end{aligned}$$

where c_{24}, c_{25} and c_{26} are constants and the velocity and vorticity components are

$$V_r=-Ar,\ \ V_z=2(Az+C)+\frac{1}{2}ar^2,\ \ V_\theta=\overline{\lambda}_0 e^{-\lambda_2 t}\frac{1}{r}\widehat{\xi}(r), \quad (3.125)$$

$$\zeta_r = 0, \ \ \zeta_\theta = -ar, \ \ \zeta_z = \overline{\lambda}_0 e^{-\lambda_2 t} \frac{1}{r} \frac{d\widehat{\xi}}{dr}. \tag{3.126}$$

where $\overline{\lambda}_0$ is an arbitrary constant.

3.5.3 For $V_r = -Ar$, $V_z = 2(Az + C) + \frac{1}{2}ar^2$, $\alpha_1 = 0$

we observe that $V_\theta = V(r,t)$ is governed by (3.11) with $\Omega = rV$. Since

$$\zeta = (\zeta_r, \zeta_\theta, \zeta_z) = \left[-\frac{1}{r}\frac{\partial \Omega}{\partial z}, -\omega, \frac{1}{r}\frac{\partial \Omega}{\partial r}\right]. \tag{3.127}$$

Writing

$$\zeta_z = \frac{1}{r}\frac{\partial}{\partial r}(rV) = \zeta \tag{3.128}$$

we obtain from (3.11) the following equation

$$\frac{\partial \zeta}{\partial t} - A\left(r\frac{\partial \zeta}{\partial r} + 2\zeta\right) = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \zeta}{\partial r}\right). \tag{3.129}$$

In order to find a class of exact solutions of Eq. (3.129) we apply symmetry group methods to find its symmetries and its reduction. The basis of our discussion is a theory conceived by S. Lie. Lie developed a general theory dealing with symmetries and group properties of differential equations. The theory of Lie is a valuable tool for solving ordinary differential equations and partial differential equations. The word symmetry is used in our everyday language in different meanings. In the one sense symmetric means something like well proportioned and well-balanced. The symmetry generators of (3.129) are

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \zeta\frac{\partial}{\partial \zeta}, \\ X_3 &= e^{2At}\frac{\partial}{\partial t} + Are^{2At}\frac{\partial}{\partial r} - \frac{A^2r^2}{\nu}e^{2At}\zeta\frac{\partial}{\partial \zeta}, \\ X_4 &= e^{-2At}\frac{\partial}{\partial t} - Are^{-2At}\frac{\partial}{\partial r} + 2A\zeta e^{-2At}\frac{\partial}{\partial \zeta}, \\ X_\alpha &= \alpha(t,r)\frac{\partial}{\partial \zeta}, \end{aligned} \tag{3.130}$$

as well as the infinite superposition symmetries $X_\alpha = \alpha(t,r)\,\partial/\partial\zeta$, where α satisfies Eq. (3.129) (see for e.g. [16] and refs. therein for symmetries of evolution equations). The linear parabolic equation

$$\frac{\partial\bar{\zeta}}{\partial\bar{t}} = \frac{\partial^2\bar{\zeta}}{\partial\bar{r}^2} + \frac{1}{4\bar{r}^2}\bar{\zeta} \tag{3.131}$$

also admits a similar Lie algebra of symmetry operators given by

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial\bar{t}}, \\ Y_2 &= \bar{\zeta}\frac{\partial}{\partial\bar{\zeta}}, \\ Y_3 &= 2\bar{t}\frac{\partial}{\partial\bar{t}} + \bar{r}\frac{\partial}{\partial\bar{r}}, \\ Y_4 &= 4\bar{t}^2\frac{\partial}{\partial\bar{t}} + 4\bar{r}\bar{t}\frac{\partial}{\partial\bar{r}} - \left(\bar{r}^2 + 2\bar{t}\right)\bar{\zeta}\frac{\partial}{\partial\bar{\zeta}}, \\ Y_\alpha &= \alpha\left(\bar{t},\bar{r}\right)\frac{\partial}{\partial\bar{\zeta}}, \end{aligned} \tag{3.132}$$

where α satisfies Eq. (3.131). The Lie algebra of symmetry operators can be represented by the following table:

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_α
X_1	0	0	$2X_1$	$2X_4 - 2X_3$	$X_{\alpha\bar{t}}$
X_2	0	0	0	0	$-X_\alpha$
X_3	$-2X_1$	0	0		X_α
X_4	$2X_3 - 4X_4$	0	$-2X_6$	0	X_α
X_α	$-X_{\alpha t}$	X_α	$-X_\alpha$	$-X_\alpha$	0

Table 3.1.

Here

$$\alpha' = \bar{r}\alpha_{\bar{r}} + 2\bar{t}\alpha_{\bar{t}}, \tag{3.132a}$$

$$\alpha''' = 4\bar{t}\bar{r}\alpha_{\bar{r}} + 2\bar{t}^2\alpha_{\bar{t}} + \left(\bar{r}^2 + 2\bar{t}\right)\alpha, \tag{3.132b}$$

where α satisfies Eq. (3.131). Thus if we know a solution to Eq. (3.131), Eqs. (3.132a, b) enables us to generate new solutions. For example, if $\bar{\zeta} = \bar{r}^{1/2}$ is a solution, then so is $4\bar{t}\bar{r}^{1/2} + \bar{r}^{5/2}$, which is solution from Eq. (3.132b). This means that we can generate an infinite number of polynomial solutions by repeatedly using Eqs. (3.132a, b).

As a consequence of the similarity of the Lie algebras of operators for both the parabolic equations, one can transform Eq. (3.129) to the simpler form Eq. (3.131). The invertible point transformation that reduces Eqs. (3.129) to (3.131) is

$$\begin{aligned} \bar{t} &= \frac{1}{2A}\left(1 - e^{2At}\right)^{-1}, \\ \bar{r} &= \frac{e^{At}}{\sqrt{\nu}\left(1 - e^{2At}\right)}r, \\ \bar{\zeta} &= \sqrt{r}\nu^{-1/4}e^{-\frac{3}{2}At}\left(1 - e^{2At}\right)^{1/2}\zeta\exp\left[-\frac{Ar^2}{2\nu}\left(\frac{e^{2At}}{1 - e^{2At}}\right)\right]. \end{aligned} \tag{3.133}$$

To obtain the solution of Eq. (3.131) we write

$$\bar{\zeta} = \overline{X}\left(\bar{r}\right)\overline{T}\left(\bar{t}\right) \tag{3.134}$$

and get

$$\bar{r}^2\overline{X}'' + \left(\frac{1}{4} + \lambda\bar{r}^2\right)\overline{X} = 0, \tag{3.135a}$$

$$\frac{1}{\overline{T}}\frac{d\overline{T}}{d\bar{t}} = -\widehat{\lambda}, \tag{3.135b}$$

where $\widehat{\lambda}$ is the arbitrary separation constant and prime denotes the differentiation with respect to r. The transformation $\overline{\overline{X}} = \bar{r}^{-1/2}\overline{X}$ converts Eq. (3.135a) into Bessel function of order zero i.e.,

$$\bar{r}^2\overline{\overline{X}}'' + \bar{r}\overline{\overline{X}}' + \widehat{\lambda}\overline{\overline{X}} = 0. \tag{3.136}$$

The solution of (3.136) is given by

$$\overline{\overline{X}}(\overline{r}) = \left[\overline{\delta}_1 \overline{J}_0\left(\sqrt{\widehat{\lambda}}\overline{r}\right) + \overline{\delta}_2 \overline{Y}_0\left(\sqrt{\widehat{\lambda}}\overline{r}\right)\right], \tag{3.137}$$

or

$$\overline{X}(\overline{r}) = \sqrt{\overline{r}}\left[\overline{\delta}_1 \overline{J}_0\left(\sqrt{\widehat{\lambda}}\overline{r}\right) + \overline{\delta}_2 \overline{Y}_0\left(\sqrt{\widehat{\lambda}}\overline{r}\right)\right],$$

and finally, from Eqs. (3.134), (3.135*b*) and (3.137) we have

$$\overline{\zeta}(\overline{r}, \overline{t}) = e^{-\lambda \overline{t}}\sqrt{\overline{r}}\left[\delta_1 \overline{J}_0\left(\sqrt{\widehat{\lambda}}\overline{r}\right) + \delta_2 \overline{Y}_0\left(\sqrt{\widehat{\lambda}}\overline{r}\right)\right], \tag{3.138}$$

where $\overline{J}_0\left(\sqrt{\widehat{\lambda}}\overline{r}\right)$ is the Bessel function of the first kind of order zero and $\overline{Y}_0\left(\sqrt{\widehat{\lambda}}\overline{r}\right)$ is the Bessel function of the second kind of order zero. The solution of Eq. (3.131) viz. Eq. (3.132) is in the transformed coordinate system $(\overline{t}, \overline{r}, \overline{\zeta})$. In the usual coordinate system (t, r, ζ) we have the following form via Eq. (3.133)

$$\begin{aligned}\zeta(r,t) &= e^{2At - \frac{\lambda}{2A(1-e^{2At})}} \exp\left[\frac{Ar^2 e^{2At}}{2\nu(1-e^{2At})}\right] \\ &\quad \times \left[\delta_1 J_0\left(\sqrt{\frac{\widehat{\lambda}}{\nu}}\frac{e^{At}}{1-e^{2At}}r\right) + \delta_2 Y_0\left(\sqrt{\frac{\widehat{\lambda}}{\nu}}\frac{e^{At}}{1-e^{2At}}r\right)\right],\end{aligned} \tag{3.139}$$

where δ_1, δ_2 and $\widehat{\lambda} > 0$ are constants and the velocity components are

$$V_r = -Ar,\ V_\theta = \frac{\Omega}{r} = \frac{1}{r}\int r\zeta dr,\ V_z = 2(Az + C) + \frac{1}{2}ar^2. \tag{3.140}$$

Remark 3 *One can use Eq.* (3.132*b*) *together with* (3.139) *to generate infinitely many solutions of* (3.8.3).

3.6 Conclusions

We have developed the governing equations of motion for the axially symmetric swirling flow of a second-grade fluid. Some exact, analytical, steady, and non-steady solutions for the non-linear equations of Newtonian and second-grade fluids are obtained. Various methods are used for obtaining the solutions of non-linear equations. The model and the analytical methods employed in this chapter have been shown to be useful for the theory analysis of viscoelastic fluid. Our analysis shows that the results obtained here are more general and several results obtained by different authors such Lakshmana [43], Roy [44], Berker [41], Siddiqui *et al.* [28], and Goldstein [51] can be recovered as special cases.

Chapter 4

Inverse solutions for unsteady flows of a second grade fluid

This work contains two parts In first part we develop the equations of motion in unsteady plane polar, axisymmetric cylindrical and axisymmetric spherical coordinates is given In the second part we solve these equations by choosing specific forms of the stream function in these coordinate system. The fluid equations and their solutions are important in the sense that the entire geometry of the system changes as one moves Cartesian coordinate system to these coordinate systems. For example, the flow in a pipe, flow around a cylinder, flow into a thin slit, flow around a sphere and flow between coaxial cylinders and spheres, can not be demonstrated in Cartesian coordinates.

This chapter is concerned with the modelling for the unsteady flow of a second grade fluid in unsteady plane polar, axisymmetric cylindrical and axisymmetric spherical polar coordinates. The analytical solution in each case are obtained by taking appropriate forms of the stream functions. The governing non-linear equations are solved in order to obtain the velocity components for flows in plane polar, axisymmetric cylindrical and spherical coordinates. The solutions obtained by the present analysis are also compared with the existing results in the literature.

4.1 Modelling for second-grade fluid in plane polar coordinates

The unsteady velocity field is defined by

$$\mathbf{V} = [u(r,\theta,t), v(r,\theta,t), 0]. \tag{4.1}$$

On substituting

$$\nabla\widehat{p} = \frac{\partial\widehat{p}}{\partial r} + \frac{1}{r}\frac{\partial\widehat{p}}{\partial\theta} + \frac{\partial\widehat{p}}{\partial z}, \tag{4.2}$$

$$\widehat{p} = p + \frac{1}{2}\rho \mid \mathbf{V} \mid^2 - \alpha_1 (\mathbf{V}\cdot\nabla^2\mathbf{V}) - \frac{1}{4} \mid \mathbf{A}_1^2 \mid, \tag{4.3}$$

$$\nabla^2\mathbf{V} = \left[\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial\theta}\right), \left(\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial\theta}\right), 0\right], \tag{4.4}$$

$$\frac{\partial}{\partial t}[\nabla^2\mathbf{V}] = \left[\frac{\partial}{\partial t}\left(\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial\theta}\right), \frac{\partial}{\partial t}\left(\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial\theta}\right), 0\right], \tag{4.5}$$

$$\nabla\times\mathbf{V} = \left[0, 0, \omega = \frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial\theta}\right], \tag{4.6}$$

$$\mathbf{V}\times(\nabla\times\mathbf{V}) = [v\omega, -u\omega, 0], \tag{4.7}$$

$$\nabla^2(\nabla\times\mathbf{V})\times\mathbf{V} = \left[-v\nabla^2\omega, u\nabla^2\omega, 0\right], \tag{4.8}$$

into Eqs. (1.4) and (1.9), we obtain the continuity equation and component form of momentum equation, in the absence of body forces, as follows

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta} = 0, \tag{4.9}$$

$$\frac{\partial \widehat{p}}{\partial r}+\rho\left[\frac{\partial u}{\partial t}-v\omega\right]=\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\left(\nabla^2 u-\frac{u}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial \theta}\right)-\alpha_1 v\nabla^2\omega+(\alpha_1+\alpha_2)\left[\mathrm{div}\mathbf{A}_1^2\right]_r, \tag{4.10}$$

$$\frac{1}{r}\frac{\partial \widehat{p}}{\partial \theta}+\rho\left[\frac{\partial v}{\partial t}+u\omega\right]=\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\left(\nabla^2 v-\frac{v}{r^2}+\frac{2}{r^2}\frac{\partial u}{\partial \theta}\right)+\alpha_1 u\nabla^2\omega+(\alpha_1+\alpha_2)\left[\mathrm{div}\mathbf{A}_1^2\right]_\theta, \tag{4.11}$$

$$\frac{\partial \widehat{p}}{\partial z}=0. \tag{4.12}$$

On using the following results

$$\begin{aligned}\nabla^2 u-\frac{u}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial \theta} &= -\frac{1}{r}\frac{\partial \omega}{\partial \theta},\ \nabla^2 v-\frac{v}{r^2}+\frac{2}{r^2}\frac{\partial u}{\partial \theta}=\frac{\partial \omega}{\partial r},\\ \left[\mathrm{div}\mathbf{A}_1^2\right]_r &= \frac{\partial}{\partial r}\left(\frac{|\mathbf{A}_1|^2}{2}\right),\ \left[\mathrm{div}\mathbf{A}_1^2\right]_\theta=\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{|\mathbf{A}_1|^2}{2}\right)\end{aligned} \tag{4.13}$$

the Eqs. (4.10) and (4.11) become

$$\frac{\partial \widehat{p}}{\partial r}+\rho\left[\frac{\partial u}{\partial t}-v\omega\right]=\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\left(-\frac{1}{r}\frac{\partial \omega}{\partial \theta}\right)-\alpha_1 v\nabla^2\omega+(\alpha_1+\alpha_2)\frac{\partial}{\partial r}\left(\frac{|\mathbf{A}_1^2|}{2}\right), \tag{4.14}$$

$$\frac{1}{r}\frac{\partial \widehat{p}}{\partial \theta}+\rho\left[\frac{\partial v}{\partial t}+u\omega\right]=\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\frac{\partial \omega}{\partial r}+\alpha_1 u\nabla^2\omega+(\alpha_1+\alpha_2)\frac{\partial}{\partial \theta}\left(\frac{|\mathbf{A}_1^2|}{2}\right). \tag{4.15}$$

Equations (4.14) and (4.15) can also be written as

$$\frac{\partial}{\partial r}S_1+\rho\left[\frac{\partial u}{\partial t}-v\omega\right]=\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\left(-\frac{1}{r}\frac{\partial \omega}{\partial \theta}\right)-\alpha_1 v\nabla^2\omega, \tag{4.16}$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}S_1+\rho\left[\frac{\partial v}{\partial t}+u\omega\right]=\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\frac{\partial \omega}{\partial r}+\alpha_1 u\nabla^2\omega, \tag{4.17}$$

where

$$
\begin{aligned}
S_1 &= p+\frac{1}{2}\rho\left(u^2+v^2\right)+\alpha_1\left(\frac{u}{r}\frac{\partial w}{\partial \theta}-v\frac{\partial w}{\partial r}\right) \\
&-\frac{(3\alpha_1+2\alpha_2)}{4}\left[4\left(\frac{\partial u}{\partial r}\right)^2+4\left(\frac{1}{r}\frac{\partial v}{\partial \theta}-\frac{u}{r}\right)^2+2\left(\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right)^2\right].
\end{aligned} \tag{4.18}
$$

In order to obtain the compatibility equation we define the stream function $\psi=\psi(r,\theta,t)$ through

$$
u=\frac{1}{r}\frac{\partial \psi}{\partial \theta},\quad v=-\frac{\partial \psi}{\partial r} \tag{4.19}
$$

we see that the continuity equation is satisfied identically and vorticity equation becomes

$$
\omega=\frac{\partial v}{\partial r}+\frac{v}{r}-\frac{1}{r}\frac{\partial u}{\partial \theta}=-\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\psi=-\nabla^2\psi. \tag{4.20}
$$

Using Eqs. (4.19) and (4.20) into Eqs. (4.16) and (4.17) we obtain

$$
\frac{\partial S_1}{\partial r}+\rho\left[\frac{1}{r}\frac{\partial^2\psi}{\partial t\partial \theta}-\frac{\partial \psi}{\partial r}\nabla^2\psi\right]=\frac{1}{r}\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\frac{\partial}{\partial \theta}\nabla^2\psi-\alpha_1\frac{\partial \psi}{\partial r}\nabla^4\psi, \tag{4.21}
$$

$$
\frac{\partial S_1}{\partial \theta}-\rho\left[r\frac{\partial^2\psi}{\partial t\partial \theta}+\frac{\partial \psi}{\partial r}\nabla^2\psi\right]=-r\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\frac{\partial}{\partial r}\nabla^2\psi-\alpha_1\frac{\partial \psi}{\partial \theta}\nabla^4\psi. \tag{4.22}
$$

To obtain the single equation in terms of stream function we use the integrability condition $S_{1r\theta}=S_{1\theta r}$ to eliminate the pressure gradient. This can be obtained by differentiating Eq. (4.21) with respect to θ and Eq. (4.22) with respect to r and then subtracting the resulting expressions i.e.

$$
\rho\left[r\frac{\partial}{\partial t}\nabla^2\psi-\left\{\psi,\nabla^2\psi\right\}\right]=r\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\nabla^4\psi-\alpha_1\left\{\psi,\nabla^4\psi\right\}, \tag{4.23}
$$

where

$$
\begin{aligned}
\nabla^2\psi &= \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\psi,\ \nabla^4\psi=\nabla^2\nabla^2\psi, \\
\frac{\partial\left(\psi,\nabla^2\psi\right)}{\partial(r,\theta)} &= \frac{\partial \psi}{\partial r}\frac{\partial \nabla^2\psi}{\partial \theta}-\frac{\partial \psi}{\partial \theta}\frac{\partial \nabla^2\psi}{\partial r}.
\end{aligned} \tag{4.24}
$$

It is to be noted that the Eq. (4.23) for steady case reduces to the equation as discussed by Siddiqui et al. [28].

4.2 Modelling of second grade fluid in axisymmetric cylindrical coordinates

Here the velocity field is

$$\mathbf{V} = [u(r,z,t), 0, w(r,z,t)]. \tag{4.25}$$

Using above equation, we have

$$\nabla \widetilde{p} = \left[\frac{\partial \widetilde{p}}{\partial r}, \frac{1}{r}\frac{\partial \widetilde{p}}{\partial \theta}, \frac{\partial \widetilde{p}}{\partial z}\right], \tag{4.26}$$

$$\nabla^2 \mathbf{V} = \left[\nabla^2 u - \frac{u}{r^2}, 0, \nabla^2 w\right], \tag{4.27}$$

$$\frac{\partial}{\partial t}\left[\nabla^2 \mathbf{V}\right] = \nabla^2 \mathbf{V}_t = \left[\frac{\partial}{\partial t}\left(\nabla^2 u - \frac{u}{r^2}\right), 0, \frac{\partial}{\partial t}\nabla^2 w\right], \tag{4.28}$$

$$\nabla \times \mathbf{V} = \left[0, \widetilde{\Omega} = -\left(\frac{\partial w}{\partial r} - \frac{\partial u}{\partial z}\right), 0\right], \tag{4.29}$$

$$\nabla^2(\nabla \times \mathbf{V}) = \left[0, -\left(\nabla^2 \widetilde{\Omega} - \frac{\widetilde{\Omega}}{r^2}\right), 0\right], \tag{4.30}$$

$$\nabla^2(\nabla \times \mathbf{V}) \times \mathbf{V} = \left[-w\left(\nabla^2 \widetilde{\Omega} - \frac{\widetilde{\Omega}}{r^2}\right),\ 0,\ u\left(\nabla^2 \widetilde{\Omega} - \frac{\widetilde{\Omega}}{r^2}\right)\right], \tag{4.31}$$

$$\mathbf{V} \times (\nabla \times \mathbf{V}) = \left(w\widetilde{\Omega},\ 0,\ -u\widetilde{\Omega}\right). \tag{4.32}$$

From Eqs. (1.4), (1.9) and (4.25) to (4.232) we get

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{4.33}$$

$$\frac{\partial \widetilde{p}}{\partial r}+\rho\left[\frac{\partial u}{\partial t}-w\widetilde{\Omega}\right]=\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\left(\nabla^2 u-\frac{u}{r^2}\right)-\alpha_1 w\left(\nabla^2\widetilde{\Omega}-\frac{\widetilde{\Omega}}{r^2}\right)+(\alpha_1+\alpha_2)\,\mathrm{div}\left(\mathbf{A}_1^2\right)_r, \tag{4.34}$$

$$\frac{1}{r}\frac{\partial \widetilde{p}}{\partial \theta}=0, \tag{4.35}$$

$$\frac{\partial \widetilde{p}}{\partial z}+\rho\left[\frac{\partial w}{\partial t}+u\widetilde{\Omega}\right]=\left(\mu\nabla^2 w+\alpha_1\frac{\partial}{\partial t}\right)+\alpha_1 u\left(\nabla^2\widetilde{\Omega}-\frac{\widetilde{\Omega}}{r^2}\right)+(\alpha_1+\alpha_2)\,\mathrm{div}\left(\mathbf{A}_1^2\right)_z, \tag{4.36}$$

where

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2},\quad \widetilde{\Omega}=\frac{\partial w}{\partial r}-\frac{\partial u}{\partial z}. \tag{4.37}$$

Using the results

$$\begin{aligned}
\nabla^2 u-\frac{u}{r^2} &= -\frac{\partial\widetilde{\Omega}}{\partial z},\ \nabla^2 w=\frac{\partial\widetilde{\Omega}}{\partial r}+\frac{\widetilde{\Omega}}{r},\ \mathrm{div}\left(A_1^2\right)_r=\frac{1}{2}\frac{\partial}{\partial r}|\mathbf{A}_1|^2+\frac{2}{r}\frac{\partial}{\partial z}\left(u\widetilde{\Omega}\right)+\frac{\widetilde{\Omega}^2}{r},\\
\mathrm{div}\left(\mathbf{A}_1^2\right)_\theta &= 0,\ \mathrm{div}\left(\mathbf{A}_1^2\right)_z=\frac{1}{2}\frac{\partial}{\partial z}|\mathbf{A}_1|^2-\frac{2}{r}\frac{\partial}{\partial r}\left(u\widetilde{\Omega}\right),\\
\mathrm{tr}\left(\mathbf{A}_1^2\right) &= \mathrm{tr}\left(\mathbf{A}_1\mathbf{A}_1^{\top}\right)=|\mathbf{A}_1^2|=4\left(\frac{\partial u}{\partial r}\right)^2+4\left(\frac{\partial w}{\partial z}\right)^2+4\left(\frac{u}{r}\right)^2+2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2,
\end{aligned} \tag{4.38}$$

the Eqs. (4.34) and (4.36) can be rewritten as

$$\begin{aligned}
\frac{\partial \widetilde{p}}{\partial r}+\rho\left[\frac{\partial u}{\partial t}+w\widetilde{\Omega}\right] &= -\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\frac{\partial\widetilde{\Omega}}{\partial z}-\alpha_1 w\left(\nabla^2\widetilde{\Omega}-\frac{\widetilde{\Omega}}{r^2}\right)\\
&\quad+(\alpha_1+\alpha_2)\left[\frac{1}{2}\frac{\partial}{\partial r}|\mathbf{A}_1^2|+\frac{2}{r}\frac{\partial}{\partial z}\left(u\widetilde{\Omega}\right)+\frac{\widetilde{\Omega}^2}{r}\right],
\end{aligned} \tag{4.39}$$

$$\begin{aligned}
\frac{\partial \widetilde{p}}{\partial z}+\rho\left[\frac{\partial w}{\partial t}+u\widetilde{\Omega}\right] &= -\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\left(\frac{\partial\widetilde{\Omega}}{\partial r}+\frac{\widetilde{\Omega}}{r}\right)+\alpha_1 u\left(\nabla^2\widetilde{\Omega}-\frac{\widetilde{\Omega}}{r^2}\right)\\
&\quad+(\alpha_1+\alpha_2)\left[\frac{1}{2}\frac{\partial}{\partial z}|\mathbf{A}_1^2|-\frac{2}{r}\frac{\partial}{\partial r}\left(u\widetilde{\Omega}\right)\right].
\end{aligned} \tag{4.40}$$

Defining the generalized pressure

$$S_2 = \frac{1}{2}\rho\left(u^2 + w^2\right) + p - \alpha_1\left[u\left(\nabla^2 u - \frac{u}{r^2}\right) + w\nabla^2 w\right] - \frac{(3\alpha_1 + 2\alpha_2)}{u}\left|A_1^2\right| \tag{4.41}$$

where

$$\left|A_1^2\right| = 4\left(\frac{\partial u}{\partial r}\right)^2 + 4\left(\frac{\partial w}{\partial z}\right)^2 + 4\left(\frac{u}{r}\right)^2 + 2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2.$$

we rewrite Eqs. (4.39) and (4.40) in the form

$$\frac{\partial S_2}{\partial r} + \rho\left[\frac{\partial u}{\partial t} - w\widetilde{\Omega}\right] = -\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\frac{\partial\widetilde{\Omega}}{\partial z} - \alpha_1 w\left(\nabla^2\widetilde{\Omega} - \frac{\widetilde{\Omega}}{r^2}\right) + (\alpha_1 + \alpha_2)\left[\frac{2}{r}\frac{\partial}{\partial z}\left(u\widetilde{\Omega}\right) + \frac{\widetilde{\Omega}^2}{r}\right], \tag{4.42}$$

$$\frac{\partial S_2}{\partial z} + \rho\left[\frac{\partial w}{\partial t} + u\widetilde{\Omega}\right] = \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\left(\frac{\partial\widetilde{\Omega}}{\partial r} + \frac{\widetilde{\Omega}}{r}\right) + \alpha_1 u\left(\nabla^2\widetilde{\Omega} - \frac{\widetilde{\Omega}}{r^2}\right) - \frac{2(\alpha_1 + \alpha_2)}{r}\frac{\partial}{\partial r}\left(u\widetilde{\Omega}\right), \tag{4.43}$$

To find the compatibility equation we define

$$u = \frac{1}{r}\frac{\partial\widetilde{\psi}}{\partial z}, \quad w = -\frac{1}{r}\frac{\partial\widetilde{\psi}}{\partial r} \tag{4.44}$$

and note that the continuity equation is satisfied identically and vorticity function is

$$\widetilde{\Omega} = \frac{\partial w}{\partial r} - \frac{\partial u}{\partial z} = -\frac{1}{r}\left(\frac{\partial^2\widetilde{\psi}}{\partial r^2} - \frac{1}{r}\frac{\partial\widetilde{\psi}}{\partial r} + \frac{\partial^2\widetilde{\psi}}{\partial z^2}\right) = -\frac{1}{r}E^2\widetilde{\psi}, \tag{4.45}$$

where

$$E^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{4.46}$$

Using Eqs. (4.44) and (4.45) in Eqs. (4.42) and (4.43) we obtain

$$\frac{\partial S_2}{\partial r}+\rho\left[\frac{1}{r}\frac{\partial^2\widetilde{\psi}}{\partial t\partial z}-\frac{\partial\widetilde{\psi}}{\partial r}\frac{E^2\widetilde{\psi}}{r^2}\right] \tag{4.47}$$
$$=\frac{1}{r}\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\frac{\partial}{\partial z}E^2\widetilde{\psi}-\alpha_1\frac{\partial\widetilde{\psi}}{\partial r}E^4\widetilde{\psi}-\frac{1}{r}(\alpha_1+\alpha_2)\left[2\frac{\partial}{\partial z}\left(\frac{\partial\widetilde{\psi}}{\partial z}\frac{E^2\widetilde{\psi}}{r^2}\right)-\left(\frac{E^2\widetilde{\psi}}{r^2}\right)^2\right]$$

$$\frac{\partial S_2}{\partial z}-\rho\left[r\frac{\partial^2\widetilde{\psi}}{\partial t\partial r}+\frac{\partial\widetilde{\psi}}{\partial z}\frac{E^2\widetilde{\psi}}{r^2}\right]=-\frac{1}{r}\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\frac{\partial}{\partial r}E^2\widetilde{\psi}-\alpha_1\frac{\partial\widetilde{\psi}}{\partial z}E^4\widetilde{\psi}+\frac{2}{r}(\alpha_1+\alpha_2)\frac{\partial}{\partial r}\left(\frac{\partial\widetilde{\psi}}{\partial z}\frac{E^2\widetilde{\psi}}{r^2}\right). \tag{4.48}$$

Differentiating Eq. (4.47) with respect to z and Eq. (4.48) with respect to r and then subtracting Eq. (4.47) from Eq. (4.48), we obtain

$$\rho\left[\frac{1}{r}\frac{\partial}{\partial t}E^2\Psi-\left\{\widetilde{\psi},E^2\widetilde{\psi}/r^2\right\}\right] = \frac{1}{r}\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)E^4\widetilde{\psi}-\alpha_1\left\{\widetilde{\psi},E^4\widetilde{\psi}/r^2\right\} \tag{4.49}$$
$$-(\alpha_1+\alpha_2)\left[\frac{2}{r}E^2\left(\frac{\partial\widetilde{\psi}}{\partial z}\frac{E^2\widetilde{\psi}}{r^2}\right)-\frac{1}{r}\frac{\partial}{\partial z}\left(\frac{E^2\widetilde{\psi}}{r}\right)^2\right],$$

where

$$\left\{\widetilde{\psi},E^2\widetilde{\psi}/r^2\right\}=\frac{\partial\widetilde{\psi}}{\partial r}\frac{\partial}{\partial z}\left(\frac{E^2\widetilde{\psi}}{r^2}\right)-\frac{\partial\widetilde{\psi}}{\partial z}\frac{\partial}{\partial r}\left(\frac{E^2\widetilde{\psi}}{r^2}\right). \tag{4.50}$$

Equation (4.23) for steady case reduces to Siddiqui *et al.* [28].

4.3 Modelling of second-grade fluid in axisymmetric spherical coordinates

The velocity field for this case is

$$\mathbf{V}=[u(R,\theta,t),\,v(R,\theta,t),\,0]. \tag{4.51}$$

Using above we have

$$\nabla\cdot\mathbf{V}=\frac{\partial u}{\partial R}+\frac{2u}{R}+\frac{1}{R}\frac{\partial v}{\partial\theta}+\frac{v}{R}\cot\theta, \tag{4.52}$$

$$\nabla\overline{p} = \left[\frac{\partial\overline{p}}{\partial R}, \frac{1}{R}\frac{\partial\overline{p}}{\partial\theta}, \frac{1}{R\sin\theta}\frac{\partial\overline{p}}{\partial\phi}\right], \tag{4.53}$$

$$\nabla^2\mathbf{V} = \left[\left(\nabla^2 u - \frac{2u}{R^2} - \frac{2v}{R^2}\cot\theta - \frac{2}{R^2}\frac{\partial v}{\partial\theta}\right), \left(\nabla^2 v - \frac{v}{R^2\sin^2\theta} + \frac{2}{R^2}\frac{\partial u}{\partial\theta}\right), 0\right], \tag{4.54}$$

$$\nabla^2\mathbf{V}_t = \left[\left(\nabla^2 u_t - \frac{2u_t}{R^2} - \frac{2v_t}{R^2}\cot\theta - \frac{2}{R^2}\frac{\partial v_t}{\partial\theta}\right), \left(\nabla^2 v_t - \frac{v_t}{R^2\sin^2\theta} + \frac{2}{R^2}\frac{\partial u_t}{\partial\theta}\right), 0\right], \tag{4.55}$$

$$\nabla\times\mathbf{V} = \left[0, 0, \overline{\Omega} = \frac{\partial v}{\partial R} + \frac{v}{R} - \frac{1}{R}\frac{\partial u}{\partial\theta}\right], \tag{4.56}$$

$$\mathbf{V}\times(\nabla\times\mathbf{V}) = \left[v\overline{\Omega}, -v\overline{\Omega}, 0\right], \tag{4.57}$$

$$\nabla^2(\nabla\times\mathbf{V}) = \left[0, 0, \nabla^2\overline{\Omega} - \frac{\overline{\Omega}}{R^2\sin^2\theta}\right], \tag{4.58}$$

$$\nabla^2(\nabla\times\mathbf{V})\times\mathbf{V} = \left[-v\left(\nabla^2\overline{\Omega} - \frac{\overline{\Omega}}{R^2\sin^2\theta}\right), u\left(\nabla^2\overline{\Omega} - \frac{\overline{\Omega}}{R^2\sin^2\theta}\right), 0\right]. \tag{4.59}$$

On using the results (4.52) to (4.59) into Eqs. (1.4) and (1.9) we obtain, in the absence of body forces, the following equations

$$\frac{\partial u}{\partial R} + \frac{2u}{R} + \frac{1}{R}\frac{\partial v}{\partial\theta} + \frac{v}{R}\cot\theta = 0, \tag{4.60}$$

$$\begin{aligned}\frac{\partial\overline{p}}{\partial R} + \rho\left[\frac{\partial u}{\partial t} - v\overline{\Omega}\right] &= \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\left(\nabla^2 u - \frac{2u}{R^2} - \frac{2v}{R^2}\cot\theta - \frac{2}{R^2}\frac{\partial v}{\partial\theta}\right) \\ &\quad -\alpha_1 v\left(\nabla^2\overline{\Omega} - \frac{\overline{\Omega}}{R^2\sin^2\theta}\right) + (\alpha_1+\alpha_2)\,\mathrm{div}\left[\mathbf{A}_1^2\right]_R,\end{aligned} \tag{4.61}$$

$$\begin{aligned}\frac{1}{R}\frac{\partial\overline{p}}{\partial\theta} + \rho\left[\frac{\partial v}{\partial t} + u\overline{\Omega}\right] &= \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\left(\nabla^2 v - \frac{v}{R^2\sin^2\theta} + \frac{2}{R^2}\frac{\partial u}{\partial\theta}\right) \\ &\quad +\alpha_1 u\left(\nabla^2\overline{\Omega} - \frac{\overline{\Omega}}{R^2\sin^2\theta}\right) + (\alpha_1+\alpha_2)\,\mathrm{div}\left[\mathbf{A}_1^2\right]_\theta,\end{aligned} \tag{4.62}$$

$$\frac{1}{R\sin\theta}\frac{\partial\overline{p}}{\partial\phi} = (\alpha_1+\alpha_2)\,\mathrm{div}\left[\mathbf{A}_1^2\right]_\phi, \tag{4.63}$$

where

$$\overline{p} = p+\frac{1}{2}\rho\mid \mathbf{V}\mid^2-\alpha_1\left(\mathbf{V}\cdot\nabla^2\mathbf{V}+\frac{1}{4}\mid \mathbf{A}_1^2\mid\right), \tag{4.64}$$

$$|\mathbf{A}_1|^2 = \mathrm{tr}\mathbf{A}_1^2=4\left(\frac{\partial u}{\partial R}\right)^2+4\left(\frac{u}{R}+\frac{1}{R}\frac{\partial v}{\partial \theta}\right)^2+4\left(\frac{u}{R}+\frac{v}{R}\cot\theta\right)^2+2\left(\frac{\partial v}{\partial R}-\frac{v}{R}+\frac{1}{R}\frac{\partial u}{\partial \theta}\right)^2,$$

$$\nabla^2 = \frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right)+\frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right), \tag{4.65}$$

and $\overline{\Omega}$ is the vorticity function.

On using the following values

$$\begin{aligned}
\nabla^2 u-\frac{2u}{R^2}-\frac{2v}{R^2}\cot\theta-\frac{2}{R^2}\frac{\partial v}{\partial\theta} &= -\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(\overline{\Omega}\sin\theta\right), \qquad (4.66)\\
\nabla^2 v-\frac{v}{R^2\sin^2\theta}+\frac{2}{R^2}\frac{\partial u}{\partial\theta} &= \frac{\partial\overline{\Omega}}{\partial R}+\frac{\overline{\Omega}}{R},\\
\left[\mathrm{div}\mathbf{A}_1^2\right]_R &= \frac{1}{2}\frac{\partial}{\partial R}|\mathbf{A}_1|^2+\frac{2}{R\sin\theta}\frac{\partial}{\partial\theta}\left[\left(\frac{u}{R}+\frac{v}{R}\cot\theta\right)\overline{\Omega}\sin\theta\right]+\frac{\overline{\Omega}^2}{R},\\
\left[\mathrm{div}\mathbf{A}_1^2\right]_\theta &= \frac{1}{2}\frac{\partial}{\partial\theta}|\mathbf{A}_1|^2-\frac{2}{R}\frac{\partial}{\partial R}\left[(u+v\cot\theta)\overline{\Omega}+\frac{\overline{\Omega}^2}{R}\cot\theta\right],\\
\left[\mathrm{div}\mathbf{A}_1^2\right]_\phi &= 0,\ \ \mathrm{tr}\mathbf{A}_1=\nabla\cdot\mathbf{V}=0,
\end{aligned}$$

equations (4.61) to (4.63) are

$$\begin{aligned}
\frac{\partial\overline{p}}{\partial R}+\rho\left[\frac{\partial u}{\partial t}-v\overline{\Omega}\right] &= -\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\left[\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(\overline{\Omega}\sin\theta\right)\right] \qquad (4.67)\\
&\quad -\alpha_1 v\left(\nabla^2\overline{\Omega}-\frac{\overline{\Omega}}{R^2\sin^2\theta}\right)+\frac{(\alpha_1+\alpha_2)}{2}\frac{\partial}{\partial R}|\mathbf{A}_1|^2\\
&\quad +(\alpha_1+\alpha_2)\left[\frac{2}{R\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{u}{R}+\frac{v}{R}\cot\theta\right)\overline{\Omega}\sin\theta+\frac{\overline{\Omega}^2}{R}\right],
\end{aligned}$$

$$\begin{aligned}
\frac{1}{R}\frac{\partial\overline{p}}{\partial\theta}+\rho\left[\frac{\partial v}{\partial t}+u\overline{\Omega}\right] &= \mu\left(\frac{\partial\overline{\Omega}}{\partial R}+\frac{\overline{\Omega}}{R}\right)+\alpha_1 u\left(\nabla^2\overline{\Omega}-\frac{\overline{\Omega}}{R^2\sin^2\theta}\right)+\frac{(\alpha_1+\alpha_2)}{2R}\frac{\partial}{\partial\theta}|\mathbf{A}_1|^2\\
&\quad +\frac{(\alpha_1+\alpha_2)}{R}\left[-2\frac{\partial}{\partial R}(u+v\cot\theta)\overline{\Omega}+\overline{\Omega}^2\cot\theta\right], \qquad (4.68)
\end{aligned}$$

$$\frac{1}{R\sin\theta}\frac{\partial\overline{p}}{\partial\phi}=0. \tag{4.69}$$

On defining

$$S_3 = p + \frac{1}{2}\rho\,|\mathbf{V}|^2 - \alpha_1\left(\mathbf{V}\cdot\nabla^2\mathbf{V}\right) - \frac{(3\alpha_1 + 2\alpha_2)}{4}\,|\mathbf{A}_1|^2 \tag{4.70}$$

the Eqs. (4.67) and (4.68) can be rewritten as

$$\begin{aligned}\frac{\partial S_3}{\partial R} + \rho\left[\frac{\partial u}{\partial t} - v\overline{\Omega}\right] &= -\frac{1}{R\sin\theta}\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\frac{\partial}{\partial\theta}\left(\overline{\Omega}\sin\theta\right) - \alpha_1 v\left(\nabla^2\overline{\Omega} - \frac{\overline{\Omega}}{R^2\sin^2\theta}\right)\\ &\quad + (\alpha_1 + \alpha_2)\left[\frac{2}{R\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{u}{R} + \frac{v}{R}\cot\theta\right)\overline{\Omega}\sin\theta + \frac{\overline{\Omega}^2}{R}\right], \end{aligned}\tag{4.71}$$

$$\begin{aligned}\frac{1}{R}\frac{\partial S_3}{\partial\theta} + \rho\left[\frac{\partial v}{\partial t} + u\overline{\Omega}\right] &= \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\left(\frac{\partial\overline{\Omega}}{\partial R} + \frac{\overline{\Omega}}{R}\right) + \alpha_1 u\left(\nabla^2\overline{\Omega} - \frac{\overline{\Omega}}{R^2\sin^2\theta}\right)\\ &\quad + \frac{(\alpha_1 + \alpha_2)}{R}\left[-2\frac{\partial}{\partial R}\left(u + v\cot\theta\right)\overline{\Omega} + \overline{\Omega}^2\cot\theta\right]. \end{aligned}\tag{4.72}$$

Introducing the Stokes' stream function

$$u = \frac{1}{R^2}\frac{\partial\overline{\psi}}{\partial\sigma},\; v = \frac{1}{R\sqrt{1-\sigma^2}}\frac{\partial\overline{\psi}}{\partial R},\;\; \sigma = \cos\theta \tag{4.73}$$

the continuity equation is identically satisfied and vorticity function become

$$\overline{\Omega} = \frac{\partial v}{\partial R} + \frac{v}{R} - \frac{1}{R}\frac{\partial u}{\partial\theta} = \frac{1}{R\sqrt{1-\sigma^2}}D^2\overline{\psi}, \tag{4.74}$$

where

$$D^2 = \frac{\partial^2}{\partial R^2} + \frac{1-\sigma^2}{R^2}\frac{\partial^2}{\partial\sigma^2}. \tag{4.75}$$

Substituting Eqs. (4.73) and (4.74) in Eqs. (4.71) and (4.72) we obtain

$$\begin{aligned}&\frac{\partial S_3}{\partial R} + \rho\left[\frac{1}{R^2}\frac{\partial}{\partial t}\frac{\partial\overline{\psi}}{\partial\sigma} - \frac{1}{R^2(1-\sigma^2)}\frac{\partial\overline{\psi}}{\partial R}D^2\overline{\psi}\right]\\ &= \frac{1}{R^2}\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\frac{\partial}{\partial\sigma}D^2\overline{\psi} - \frac{\alpha_1 D^4\overline{\psi}}{R^2(1-\sigma^2)}\frac{\partial\overline{\psi}}{\partial R}\\ &\quad + (\alpha_1 + \alpha_2)\left[-\frac{2}{R}\frac{\partial}{\partial\sigma}\left\{\left(\frac{1}{R^4}\frac{\partial\overline{\psi}}{\partial\sigma} + \frac{\sigma}{R^3(1-\sigma^2)}\frac{\partial\overline{\psi}}{\partial R}\right)D^2\overline{\psi}\right\} + \frac{\left(D^2\overline{\psi}\right)^2}{R^3(1-\sigma^2)}\right],\end{aligned}\tag{4.76}$$

$$
\begin{aligned}
& -\frac{\partial S_3}{\partial \sigma}+\rho\left[\frac{\partial}{\partial t}\frac{1}{1-\sigma^2}\frac{\partial \overline{\psi}}{\partial R}+\frac{1}{R^2\left(1-\sigma^2\right)}\frac{\partial \overline{\psi}}{\partial \sigma}D^2\overline{\psi}\right] \\
= & \left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\frac{1}{\left(1-\sigma^2\right)}\frac{\partial}{\partial R}D^2\overline{\psi}+\frac{\alpha_1 D^4\overline{\psi}}{R^2\left(1-\sigma^2\right)}\frac{\partial \overline{\psi}}{\partial \sigma} \\
& -\frac{\left(\alpha_1+\alpha_2\right)}{\sqrt{1-\sigma^2}}\left[2\frac{\partial}{\partial R}\left(\frac{1}{R^3}\frac{\partial \overline{\psi}}{\partial \sigma}+\frac{\sigma}{R^2\left(1-\sigma^2\right)}\frac{\partial \overline{\psi}}{\partial R}\right)\frac{D^2\overline{\psi}}{\sqrt{1-\sigma^2}}-\frac{\sigma}{R^2\left(1-\sigma^2\right)^{\frac{3}{2}}}\left(D^2\overline{\psi}\right)^2\right].
\end{aligned}
\tag{4.77}
$$

Differentiating Eq. (4.76) with respect to σ and Eq. (4.77) with respect to R and then adding the resulting equation we get

$$
\begin{aligned}
& \rho\left[\frac{\partial}{\partial t}\frac{D^2\overline{\psi}}{1-\sigma^2}-\left\{\overline{\psi},\frac{D^2\overline{\psi}}{R^2\left(1-\sigma^2\right)}\right\}\right] \\
= & \left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\frac{D^4\overline{\psi}}{1-\sigma^2}-\alpha_1\left\{\overline{\psi},\frac{D^4\overline{\psi}}{R^2\left(1-\sigma^2\right)}\right\}\pm\frac{2\left(\alpha_1+\alpha_2\right)}{1-\sigma^2}\left[\begin{array}{c} D^2\left(\frac{\partial\overline{\psi}}{\partial\sigma}\frac{D^2\overline{\psi}}{R^3}\right)+\frac{\sigma}{R^3(1-\sigma^2)}\frac{\partial\overline{\psi}}{\partial R}D^2\overline{\psi} \\ -\left(\frac{D^2\overline{\psi}}{R^3}\frac{\partial}{\partial\sigma}+\frac{\sigma D^2\overline{\psi}}{R^2(1-\sigma^2)}\frac{\partial}{\partial R}\right)D^2\overline{\psi}\end{array}\right],
\end{aligned}
\tag{4.78}
$$

where

$$
\left\{\overline{\psi},\frac{D^2\overline{\psi}}{R^2\left(1-\sigma^2\right)}\right\}=\frac{\partial\overline{\psi}}{\partial R}\frac{\partial}{\partial\sigma}\left(\frac{D^2\overline{\psi}}{R^2\left(1-\sigma^2\right)}\right)-\frac{\partial\overline{\psi}}{\partial\sigma}\frac{\partial}{\partial R}\left(\frac{D^2\overline{\psi}}{R^2\left(1-\sigma^2\right)}\right). \tag{4.79}
$$

Note that the Eq.(4.78) for steady case reduces to Siddiqui *et al.* [28].

4.4 Solutions

In this section we apply inverse method to obtain the solution of non-linear partial differential equations in sections 4.1, 4.2, and 4.3, by considering the specific forms of the stream function.

4.4.1 Flow where $\psi(r,\theta,t)=r^nF(\theta,t)$

We choose

$$
\psi\left(r,\theta,t\right)=r^nF\left(\theta,t\right) \tag{4.80}
$$

in which the arbitrary function F depends upon θ and t and n is an integer. Using Eq. (4.80) into Eq. (4.23) we obtain

$$
\begin{aligned}
\rho\left[\frac{\partial G}{\partial t}-\left\{nF\frac{\partial G}{\partial\theta}-(n-2)\frac{\partial F}{\partial\theta}G\right\}r^{n-2}\right] = & \left(\mu+\alpha_1\frac{\partial}{\partial t}\right)Hr^{-2} \\
& -\alpha_1\left[nF\frac{\partial H}{\partial\theta}-(n-4)\frac{\partial F}{\partial\theta}H\right]r^{n-4}.
\end{aligned}
\tag{4.81}
$$

In above equation

$$G(\theta,t) = n^2 F(\theta,t) + \frac{\partial^2 F(\theta,t)}{\partial\theta^2} = \left(n^2 + \frac{\partial}{\partial\theta^2}\right) F(\theta,t), \tag{4.82}$$
$$H(\theta,t) = \left((n-2)^2 + \frac{\partial^2}{\partial\theta^2}\right) G(\theta,t).$$

Taking $n=0$, Eqs. (4.81) *and* (4.82) *yield*

$$\rho\left[\frac{\partial G}{\partial t} - 2\frac{\partial F}{\partial\theta} G r^{-2}\right] = \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right) H r^{-2} - 4\alpha_1 \frac{\partial F}{\partial\theta} H r^{-4}, \tag{4.83}$$
$$G(\theta,t) = \frac{\partial^2 F(\theta,t)}{\partial\theta^2} = H(\theta,t) = \left(4 + \frac{\partial^2}{\partial\theta^2}\right) G(\theta,t)$$

and which gives the following equations

$$\rho\frac{\partial G}{\partial t} = 0, \tag{4.83a}$$
$$2\rho\frac{\partial F}{\partial\theta} G + \mu H + \alpha_1 \frac{\partial H}{\partial t} = 0, \tag{4.83b}$$
$$4\alpha_1 H \frac{\partial F}{\partial\theta} = 0. \tag{4.83c}$$

where

$$G = \frac{\partial^2 F}{\partial\theta^2},\ H = 4G + \frac{\partial^2 G}{\partial\theta^2}. \tag{4.84}$$

It is worth mentioning that for $\alpha_1 = 0$ and $\partial_t(\cdot) = 0$ we get Jeffery-Hamel flows [46] and for $\partial_t(\cdot) = 0$ we recover the analysis of reference [41].

Equation (4.83*a*) implies that $G \neq G(t)$ which shows that G is steady and hence from Eq. (4.84) H is steady. From Eq. (4.83*c*) we assume $\frac{\partial F}{\partial\theta} \neq 0$ (since $\frac{\partial F}{\partial\theta} = 0 \Rightarrow F \neq F(\theta)$ and which contradicts the assumption (4.80)) which implies $H = 0$. Using these informations in Eq. (4.83*b*) we get

$$2\rho\frac{\partial F}{\partial\theta}\frac{\partial^2 F}{\partial\theta^2} = 0. \tag{4.85}$$

The solution of above equation is

$$F(\theta,t) = A_7(t)\,\theta + B_7(t), \tag{4.86}$$

where $A_7(t)$ and $B_7(t)$ are arbitrary functions.

Now the expressions for stream function and velocity components are given through Eqs. (4.80) and (4.19) as

$$\psi(r,\theta,t) = A_7(t)\,\theta + B_7(t), \tag{4.87}$$

$$u = r^{-1}A_7(t), \tag{4.88}$$

$$v = 0. \tag{4.89}$$

For $n = 1$ Eq. (4.81) *becomes*

$$\rho\left[\frac{\partial G}{\partial t} - \left\{F\frac{\partial G}{\partial \theta} + \frac{\partial F}{\partial \theta}G\right\}r^{-1}\right] = \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)Hr^{-2} - \alpha_1\left[F\frac{\partial H}{\partial \theta} + 3\frac{\partial F}{\partial \theta}H\right]r^{-4}, \tag{4.90}$$

which give rise to the following equations

$$\frac{\partial G}{\partial t} = 0, \tag{4.90a}$$

$$\frac{\partial}{\partial \theta}(FG) = 0, \tag{4.90b}$$

$$\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)H = 0, \tag{4.90c}$$

$$F\frac{\partial H}{\partial \theta} + 3\frac{\partial F}{\partial \theta}H = 0, \tag{4.90d}$$

where

$$G = F + \frac{\partial^2 F}{\partial \theta^2},\ H = G + \frac{\partial^2 G}{\partial \theta^2}. \tag{4.91}$$

Now Eq. (4.90*a*) indicates that G is steady and hence through Eq. (4.91) H is steady and from Eq.(4.90*c*) we get

$$\frac{\partial^2 G}{\partial \theta^2} + G = 0 \tag{4.92}$$

whose general solution is

$$G(\theta,t) = A_8(t)\cos\theta + B_8(t)\sin\theta, \tag{4.93}$$

where $A_8(t)$ and $B_8(t)$ are arbitrary functions of t. Substitution of Eq. (4.93) into Eq. (4.90*b*) yield

$$F(\theta,t) = C(t)\left[A_8(t)\cos\theta + B_8(t)\sin\theta\right]^{-1}, \tag{4.94}$$

where $C(t)$ is function of integration. The stream function (4.80) and velocity components (4.19) are respectively given by

$$\psi(r,\theta,t) = rC(t)\left[A_8(t)\cos\theta + B_8(t)\sin\theta\right]^{-1}, \tag{4.95}$$

$$u = C(t)\left[A_8(t)\sin\theta - B_8(t)\cos\theta\right]\left[A_8(t)\cos\theta + B_8(t)\sin\theta\right]^{-2}, \tag{4.96}$$

$$v = -C(t)\left[A_8(t)\cos\theta + B_8(t)\sin\theta\right]^{-1}. \tag{4.97}$$

For $n = 2$ we have following from Eq. (4.81)

$$\rho\left[\frac{\partial G}{\partial t} - 2F\frac{\partial G}{\partial \theta}\right] = \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)Hr^{-2} - \alpha_1\left[2F\frac{\partial H}{\partial \theta} + 2\frac{\partial F}{\partial \theta}H\right]r^{-2}, \tag{4.98}$$

which yields

$$\frac{\partial G}{\partial t} - 2F\frac{\partial G}{\partial \theta} = 0, \tag{4.98a}$$

$$\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)H - 2\alpha_1\frac{\partial}{\partial \theta}(FH) = 0, \tag{4.98b}$$

where

$$G = 4F + \frac{\partial^2 F}{\partial \theta^2},\ H = \frac{\partial^2 G}{\partial \theta^2}. \tag{4.99}$$

For the special case $\alpha_1 = 0$ we get

$$\left(\frac{\partial}{\partial t} - 2F\frac{\partial}{\partial \theta}\right)\left(4F + \frac{\partial^2 F}{\partial \theta^2}\right) = 0,\ \frac{\partial^2}{\partial \theta^2}\left(4F + \frac{\partial^2 F}{\partial \theta^2}\right) = 0. \tag{4.100}$$

In order to solve Eq. (4.100) we let

$$F(\theta,t) = \frac{a_0}{2} + Q(s),\ s = \theta + a_0 t \tag{4.101}$$

to get (for $Q \neq 0$)

$$\frac{d^3Q}{ds^3} + 4\frac{dQ}{ds} = 0. \tag{4.102}$$

Solving Eq. (4.102) and then inserting in Eq. (4.101) we obtain

$$F(\theta,t) = a_4 + a_2\cos 2(\theta + at) + a_3\sin 2(\theta + at), \tag{4.103}$$

where a_0, a_2, a_3 and a_4 are the arbitrary constants. The stream function and velocity components are

$$\begin{aligned}
\psi(r,\theta,t) &= r^2\left[a_4 + a_2\cos 2(\theta + a_0 t) + a_3 \sin 2(\theta + a_0 t)\right], && (4.104)\\
u &= 2r\left[-a_2 \sin 2(\theta + a_0 t) + a_3 \cos 2(\theta + a_0 t)\right], && (4.105)\\
v &= -2r\left[a_4 + a_2\cos 2(\theta + a_0 t) + a_3 \sin 2(\theta + a_0 t)\right]. && (4.106)
\end{aligned}$$

For $\alpha_1 \neq 0$, Eq. $(98a, b)$ gives

$$\begin{aligned}
4\frac{\partial F}{\partial t} + \frac{\partial^3 F}{\partial t \partial \theta^2} - 8F\frac{\partial F}{\partial \theta} - 2F\frac{\partial^3 F}{\partial \theta^3} &= 0, && (4.107)\\
\left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)\left(4\frac{\partial F}{\partial \theta} + \frac{\partial^3 F}{\partial \theta^3}\right) - 2\alpha_1\left(4F\frac{\partial^2 F}{\partial \theta^2} + \frac{\partial^4 F}{\partial \theta^4}\right) &= a_5(t), && (4.108)
\end{aligned}$$

where $a_5(t)$ is arbitrary function. The possible solution of Eq. (4.108) for $a_5(t) = 0$ is given by

$$\begin{aligned}
\psi(r,\theta,t) &= r^2\left[\theta_0 \cos 2\theta + \theta_1 \sin 2\theta\right] e^{\lambda_2 t},\ u = 2r\left[-\theta_0 \sin 2\theta + \theta_1 \cos 2\theta\right] e^{\lambda_2 t},\\
v &= -2r\left[\theta_0 \cos 2\theta + \theta_1 \sin 2\theta\right] e^{\lambda_2 t}, && (4.109)
\end{aligned}$$

where λ_2, θ_0 and θ_1 are arbitrary constants.

For other value of n, Eq. (4.81) *requires to satisfy*

$$\begin{aligned}
\frac{\partial G}{\partial t} &= 0,\ \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right) H = 0, && (4.109a)\\
nF\frac{\partial G}{\partial \theta} - (n-2)\frac{\partial F}{\partial \theta} G &= 0, && (4.109b)\\
nF\frac{\partial H}{\partial \theta} - (n-4)\frac{\partial F}{\partial \theta} H &= 0. && (4.109c)
\end{aligned}$$

where G and H are described in Eq. (4.82). From Eq. $(4.109a)$ we get $G \neq G(t)$ and hence from Eq. $(4.82)_2$ $H \neq H(t)$ and we get $H = 0$. Eq. $(4.109c)$ is solved to get

$$G(\theta, t) = C_1(t) F^{\frac{n-2}{2}},\ n \neq 0. \qquad (4.110)$$

in which $C_1(t)$ is function of integration.

Eq. (4.110) together with Eq. (4.82) forms a non-linear partial differential equation for the determination of F (except when $n = 2$), which is given as

$$\frac{\partial^2 F}{\partial \theta^2} + n^2 F = C_1(t) F^{\frac{n-2}{2}}. \tag{4.111}$$

The solution (stream function and velocity components) of Eq. (4.111) for $n = 1$ and $C_1(t) = 0$ is given by

$$\psi = r\left(A_9(t)\cos\theta + B_9(t)\sin\theta\right), \; u = -A_9(t)\sin\theta + B_9(t)\cos\theta, \; v = -\left(A_9(t)\cos\theta + B_9(t)\sin\theta\right). \tag{4.112}$$

The solution (stream function and velocity components) of Eq. (4.111) for $n = 2$ and $C_1(t) \neq 0$ is as follows

$$\begin{aligned} \psi &= r^2\left[\frac{C_1(t)}{4} + A_{10}(t)\cos 2\theta + B_{10}(t)\sin 2\theta\right], \; u = 2r\left[-A_{10}(t)\sin 2\theta + B_{10}(t)\cos 2\theta\right], \\ v &= -2r\left[\frac{C_1(t)}{4} + A_{10}(t)\cos 2\theta + B_{10}(t)\sin 2\theta\right], \end{aligned} \tag{4.113}$$

in which $A_i(t)$ and $B_i(t)$ $(i = 9, 10)$ are arbitrary functions.

For $\psi = \psi(r,t)$, Eq. (4.111) becomes

$$\left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)\nabla^4\psi - \rho\nabla^2\psi_t = 0. \tag{4.114}$$

On letting

$$\psi(r,t) = \Phi_1(r)\, e^{\lambda_5 t} \tag{4.115}$$

equation (4.114) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi_1}{\partial r}\right)\right\}\right] - \xi^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi_1}{\partial r}\right) = 0, \tag{4.116}$$

which on simplification gives

$$r^2\frac{d^2\Phi_1}{dr^2} + r\frac{d\Phi_1}{dr} - r^2\xi^2\Phi_1 = \left(A_4 \ln r + B_4\right) r^2, \tag{4.117}$$

where

$$\xi_5^2 = \frac{\rho\lambda_5}{\mu + \alpha_1\lambda_5}.$$

For steady case the solution of Eq. (4.114) is

$$\psi(r) = A_5 r^2 \ln r + B_5 r^2 + C_3 \ln r, \tag{4.118}$$

where A_5, B_5 and C_3 are arbitrary constants and the velocity components are

$$u = 0, \quad v = -\left(C_3 r^{-1} + (A_5 + 2B_5) r + 2A_5 r \ln r\right). \tag{4.119}$$

Here we remark that the solution given in Eq. (4.118) is in agreement to that given in Siddiqui and Kaloni [28].

The solution of Eq. (4.117), after substituting in Eq. (4.115), for $A_4 = B_4 = 0$ is given as

$$\psi(r,t) = \left[\overline{A}_4 I_0(r\xi_5) + \overline{B}_4 K_0(r\xi_5)\right] e^{\lambda_5 t} \tag{4.120}$$

and the velocity components are

$$u = 0, \quad v = \xi_5\left[-\overline{A}_4 I_1(r\xi_5) + \overline{B}_4 K_1(r\xi_5)\right] e^{\lambda_5 t}, \tag{4.121}$$

where $I_n(x)$ and $K_n(x)$ are the modified Bessel functions of first and second kind, respectively.

4.4.2 Flow where $\widehat{\psi}(r,z,t) = r^n F(z,t)$

Inserting

$$\widehat{\psi}(r,z,t) = r^n F(z,t) \tag{4.122}$$

into Eq. (4.49) we get

$$\begin{aligned}
&\rho\left[\begin{array}{c} n(n-2)r^{n-3}\frac{\partial F}{\partial t}+r^{n-1}\frac{\partial^3 F}{\partial z^2\partial t}-4n(n-2)r^{2n-5}F\frac{\partial F}{\partial z} \\ -\left\{nF\frac{\partial^3 F}{\partial z^3}-(n-2)\frac{\partial F}{\partial z}\frac{\partial^2 F}{\partial z^2}\right\}r^{2n-3}\end{array}\right] \\
&= \left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\left[\begin{array}{c} n(n-2)^2(n-4)r^{n-5}F+2n(n-2)r^{n-3}\frac{\partial^2 F}{\partial z^2} \\ +r^{n-1}\frac{\partial^4 F}{\partial z^4}\end{array}\right] \\
&-\alpha_1\left[\begin{array}{c} 6n(n-2)^2(n-4)r^{2n-7}F\frac{\partial F}{\partial z} \\ +2n(n-2)\left\{nF\frac{\partial^3 F}{\partial z^3}-(n-4)\frac{\partial F}{\partial z}\frac{\partial^2 F}{\partial z^2}\right\}r^{2n-5} \\ +\left\{nF\frac{\partial^5 F}{\partial z^5}-(n-2)\frac{\partial F}{\partial z}\frac{\partial^4 F}{\partial z^4}\right\}r^{2n-3}\end{array}\right] \\
&-2(\alpha_1+\alpha_2)\left[\begin{array}{c} 3n(n-2)^2(n-4)r^{2n-7}F\frac{\partial F}{\partial z}+2(n-2)(3n-2)r^{2n-5} \\ \frac{\partial F}{\partial z}\frac{\partial^2 F}{\partial z^2}+\left(\frac{\partial F}{\partial z}\frac{\partial^4 F}{\partial z^4}+2\frac{\partial^2 F}{\partial z^2}\frac{\partial^3 F}{\partial z^3}\right)r^{2n-3}\end{array}\right].
\end{aligned} \tag{4.123}$$

For $n = 2$, *Eq.* (4.123) *reduces to*

$$\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\frac{\partial^4 F}{\partial z^4}+\rho\left(2F\frac{\partial^3 F}{\partial z^3}+\frac{\partial^4 F_t}{\partial z^4}\right)=2\alpha_1\left(F\frac{\partial^5 F}{\partial z^5}\right)+2(\alpha_1+\alpha_2)\left(\frac{\partial F}{\partial z}\frac{\partial^4 F}{\partial z^4}+2\frac{\partial^2 F}{\partial z^2}\frac{\partial^3 F}{\partial z^3}\right). \tag{4.124}$$

The first integral of Eq. (4.5.3) is

$$\begin{aligned}
\mu\frac{\partial^3 F}{\partial z^3}+\rho\left(2F\frac{\partial^2 F}{\partial z^2}-\left(\frac{\partial F}{\partial z}\right)^2-\frac{\partial F_t}{\partial z}\right) &= 2\alpha_1\left(F\frac{\partial^4 F}{\partial z^4}+\left(\frac{\partial^2 F}{\partial z^2}\right)^2-\frac{1}{2}\frac{\partial^3 F_t}{\partial z^3}\right) \\
&+\alpha_2\left(2F\frac{\partial^2 F}{\partial z^2}+\left(\frac{\partial^2 F}{\partial z^2}\right)^2\right),
\end{aligned} \tag{4.125}$$

where we have taken the function of integration equal to zero. In order to solve Eq. (4.124) we define

$$F(z,t)=N+Q(z+2Nt)=N+Q(s),\ s=z+2Nt \tag{4.126}$$

to obtain the following equation

$$\mu\frac{d^3Q}{ds^3}+\rho\left(2Q\frac{d^2Q}{ds^2}-\left(\frac{dQ}{ds}\right)^2\right)=2\alpha_1\left(Q\frac{d^4Q}{ds^4}+\left(\frac{d^2Q}{ds^2}\right)^2\right)+\alpha_2\left(2\frac{dQ}{ds}\frac{d^3Q}{ds^3}+\left(\frac{d^2Q}{ds^2}\right)^2\right). \tag{4.127}$$

Letting $\alpha_1 = \alpha_2 = 0$ in Eq. (4.127) and assuming

$$Q = A_{12} s^{\lambda_6} \tag{4.128}$$

we get the following relation

$$\mu\lambda_6 (\lambda_6 - 1)(\lambda_6 - 2) s^{\lambda_6 - 3} + A_{12}\rho\lambda_6 (2(\lambda_6 - 1) + 1) s^{2(\lambda_6 - 1)} = 0. \tag{4.129}$$

On choosing $\lambda_6 = -1$ we readily obtain $A_{12} = 2\nu$ (ν is the kinematic viscosity). The expressions for stream function (4.122) and velocity components (4.44) are

$$\widehat{\psi}(r,z,t) = r^2 \left[N + 2\nu (z + 2Nt)^{-1}\right], \tag{4.130}$$

$$u = -2\nu r (z + 2Nt)^{-2}, \tag{4.131}$$

$$v = -2 \left[N + 2\nu (z + 2Nt)^{-1}\right]. \tag{4.132}$$

It is noted that the solutions (4.130) to (4.132) reduces to that of Berker solution [41] when $N = 0$.

For $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ we assume [46, 47]

$$Q = A_0 (1 + C_0 e^{a_9 s}), \quad s = z + 2Nt \tag{4.133}$$

into Eq. (4.127) and get, after a straight forward calculations, the following solution

$$Q(s) = \frac{-\mu a_9}{2(\rho - \alpha_1 a_9^2)} (1 + C_0 e^{a_9 s}), \tag{4.134}$$

where

$$a_9 = \sqrt{\rho (4\alpha_1 + 3\alpha_2)^{-1}}$$

and C_0 is a constant. The stream function (4.122) and velocity components (4.44) in this case become

$$\widehat{\psi}(r,z,t) = \left[N - \frac{\mu a_9}{2\left(\rho - \alpha_1 a_9^2\right)}\left(1 + C_0 e^{a_9(z+2Nt)}\right)\right] r^2, \tag{4.135}$$

$$u = -\frac{\mu r}{2\left(\rho - \alpha_1 a_9^2\right)} C_0 e^{a_9(z+2Nt)}, \tag{4.136}$$

$$v = \left[-N + \frac{\mu a_9}{2\left(\rho - \alpha_1 a_9^2\right)}\left(1 + C_0 e^{a_9(z+2Nt)}\right)\right] 2r. \tag{4.137}$$

For $n = 0$, *Eq.* (4.123) *gives*

$$\rho \frac{\partial^2 F_t}{\partial z^3} = \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right) \frac{\partial^4 F}{\partial z^4}, \tag{4.138}$$

$$\rho \frac{\partial F}{\partial z}\frac{\partial^2 F}{\partial z^2} = \alpha_1 \frac{\partial F}{\partial z}\frac{\partial^4 F}{\partial z^4} + (\alpha_1 + \alpha_2)\left(\frac{\partial F}{\partial z}\frac{\partial^4 F}{\partial z^4} + 2\frac{\partial^2 F}{\partial z^2}\frac{\partial^3 F}{\partial z^3}\right), \tag{4.139}$$

$$(5\alpha_1 + 4\alpha_2)\frac{\partial F}{\partial z}\frac{\partial^2 F}{\partial z^2} = 0. \tag{4.140}$$

Since $\frac{\partial F}{\partial z} \neq 0$ thus Eq. (4.140) implies that

$$F(z,t) = a_{11}(t)\, z + a_{12}(t), \tag{4.141}$$

where $a_{11}(t)$ and $a_{12}(t)$ are arbitrary functions of time and above expression leads to the following values of the stream function (4.122) and velocity components (4.44)

$$\widehat{\psi} = F,\ u = r^{-1} a_{11}(t),\ v = 0. \tag{4.142}$$

Writing

$$F(z,t) = \Phi_2(z)\, e^{\lambda_7 t} \tag{4.143}$$

in Eq. (4.138) and then solving the resulting equation we obtain

$$F(z,t) = \left(a_{15} e^{\eta z} + a_{16} e^{-\eta z} - \frac{(a_{13} z + a_{14})}{\eta_5^2}\right) e^{\lambda_7 t}, \tag{4.144}$$

in which a_i $(i = 13, 14, 15, 16)$ are arbitrary integration constants and

$$\eta_5^2 = \frac{\rho\lambda_7}{\mu + \alpha_1\lambda_7}.$$

The corresponding stream function (4.122) and velocity components (4.44) are

$$\widehat{\psi} = F,\ u = r^{-1}\frac{\partial F}{\partial z},\ v = 0. \tag{4.145}$$

4.4.3 Flow where $\overline{\psi}(R, \sigma, t) = R^n F(\sigma, t)$

On specializing the solution of Eq. (4.78) of the form

$$\overline{\psi}(R, \sigma, t) = R^n F(\sigma, t) \tag{4.146}$$

we obtain

$$\begin{aligned}
&\rho\left[R^{n-2}\frac{\partial G_1}{\partial t} - \left\{nF\frac{\partial G_1}{\partial \sigma} - (n-4)\frac{\partial F}{\partial \sigma}G_1\right\}R^{2n-5}\right] \qquad (4.147)\\
=\ & \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)H_1 R^{n-4} - \alpha_1\left[nF\frac{\partial H_1}{\partial \sigma} + H_1\frac{\partial F}{\partial \sigma}(n-6)\right]R^{2n-7}\\
&+\frac{2(\alpha_1+\alpha_2)}{1-\sigma^2}\left[\begin{array}{c}\left\{(1-\sigma^2)\frac{\partial^2}{\partial\sigma^2} + (2n-5)(2n-6)\left(G\frac{\partial F}{\partial \sigma} + n\sigma F G_1\right)\right\}\\ -\left\{G\frac{\partial G}{\partial \sigma} + \frac{(n-2)\sigma}{1-\sigma^2}G^2\right\}\end{array}\right]R^{2n-7},
\end{aligned}$$

where

$$\begin{aligned}
G_1 &= \frac{G}{1-\sigma^2},\ H_1 = \frac{H}{1-\sigma^2},\ \sigma = \cos\theta, \qquad (4.148)\\
G(\sigma,t) &= n(n-1)F + (1-\sigma^2)\frac{\partial^2 F}{\partial \sigma^2},\ H(\sigma,t) = (n-2)(n-3)G + (1-\sigma^2)\frac{\partial^2 G}{\partial \sigma^2},\\
D^2\overline{\psi} &= \left(\frac{\partial^2}{\partial R^2} + \frac{1-\sigma^2}{R^2}\frac{\partial^2}{\partial \sigma^2}\right)\overline{\psi},\ u = \frac{1}{R^2}\frac{\partial\overline{\psi}}{\partial \sigma},\ v = \frac{1}{R\sqrt{1-\sigma^2}}\frac{\partial \overline{\psi}}{\partial R}.
\end{aligned}$$

For $n=0$, Eqs. (4.147) *and* (4.148) *become*

$$\rho\left[R^{-2}\frac{\partial G_1}{\partial t}-4\frac{\partial F}{\partial \sigma}G_1R^{-5}\right] = \left(\mu+\alpha_1\frac{\partial}{\partial t}\right)H_1R^{-4}-\alpha_1\left(6H_1\frac{\partial F}{\partial \sigma}\right)R^{-7} + \frac{2(\alpha_1+\alpha_2)}{1-\sigma^2}\left[\begin{array}{c}(1-\sigma^2)\frac{\partial^2}{\partial\sigma^2}\left(G\frac{\partial F}{\partial \sigma}\right)+30G\frac{\partial F}{\partial \sigma}\\ -\left(G\frac{\partial F}{\partial \sigma}-\frac{2\sigma}{1-\sigma^2}G^2\right)\end{array}\right]R^{-7}, \tag{4.149}$$

$$G=(1-\sigma^2)\frac{\partial^2 F}{\partial \sigma^2},\ G_1=\frac{G}{1-\sigma^2},\ H=6G+(1-\sigma^2)\frac{\partial^2 G}{\partial \sigma^2},\ H_1=\frac{H}{1-\sigma^2}. \tag{4.150}$$

Equation (4.149) give rise to the following partial differential equations

$$\begin{aligned}\frac{\partial G_1}{\partial t} &= 0,\ \frac{\partial F}{\partial \sigma}G_1=0,\ \left(\mu+\alpha_1\frac{\partial}{\partial t}\right)H_1=0, \\ 3\alpha_1H_1\frac{\partial F}{\partial \sigma} &= \frac{(\alpha_1+\alpha_2)}{1-\sigma^2}\left[\begin{array}{c}(1-\sigma^2)\frac{\partial^2}{\partial\sigma^2}\left(G\frac{\partial F}{\partial \sigma}\right)\\ +30G\frac{\partial F}{\partial \sigma}-\left(G\frac{\partial G}{\partial \sigma}-\frac{2\sigma}{1-\sigma^2}G^2\right)\end{array}\right].\end{aligned} \tag{4.151}$$

Equations $(4.151)_{1,3}$ and Eqs. $(4.150)_{2,4}$ imply that G and H are not functions of t and hence Eq. $(4.151)_2$ with the help of Eqs. $(4.150)_{1,2}$ become

$$\frac{\partial F}{\partial \sigma}G_1=\frac{\partial F}{\partial \sigma}\left(\frac{G}{1-\sigma^2}\right)=\frac{1}{1-\sigma^2}\frac{\partial F}{\partial \sigma}(1-\sigma^2)\frac{\partial^2 F}{\partial \sigma^2}=\frac{\partial F}{\partial \sigma}\frac{\partial^2 F}{\partial \sigma^2}=0.$$

Since $\frac{\partial F}{\partial \sigma}\neq 0$, we get $\frac{\partial^2 F}{\partial \sigma^2}=0$ and whose general solution is

$$F(\sigma,t)=\overline{C}_0(t)\,\sigma+\overline{C}_1(t), \tag{4.152}$$

where $\overline{C}_0(t)$ and $\overline{C}_1(t)$ are arbitrary constants and the stream function (4.146) and velocity components (4.73) are found as

$$\overline{\psi}=\overline{C}_0(t)\,\sigma+\overline{C}_1(t),\ u=\frac{\overline{C}_0(t)}{R^2},\ v=0. \tag{4.153}$$

For $n=1$, Eqs. (4.147) and (4.148) yield

$$\begin{aligned}
&\rho\left[R^{-1}\frac{\partial G_1}{\partial t}-\left\{F\frac{\partial G_1}{\partial\sigma}+3\frac{\partial F}{\partial\sigma}G_1\right\}R^{-3}\right] \\
=&\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)H_1R^{-3}+\alpha_1\left[F\frac{\partial H_1}{\partial\sigma}+5H_1\frac{\partial F}{\partial\sigma}\right]R^{-5} \\
&+\frac{2(\alpha_1+\alpha_2)}{1-\sigma^2}\left[\begin{array}{c}\left(1-\sigma^2\right)\frac{\partial^2}{\partial\sigma^2}\left(G\frac{\partial F}{\partial\sigma}+\sigma FG_1\right) \\ +12\left(G\frac{\partial F}{\partial\sigma}+\sigma FG_1\right)-\left(G\frac{\partial F}{\partial\sigma}-\frac{\sigma}{1-\sigma^2}G^2\right)\end{array}R^{-5}\right],
\end{aligned}\tag{4.154}$$

$$G=\left(1-\sigma^2\right)\frac{\partial^2F}{\partial\sigma^2},\ G_1=\frac{G}{1-\sigma^2},\ H=2G+\left(1-\sigma^2\right)\frac{\partial^2G}{\partial\sigma^2},\ H_1=\frac{H}{1-\sigma^2}.\tag{4.155}$$

Equation (4.156) gives the following partial differential equations

$$\begin{aligned}
\frac{\partial G_1}{\partial t}&=0,\ -\rho\left(F\frac{\partial G_1}{\partial\sigma}+3\frac{\partial F}{\partial\sigma}G_1\right)=\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)H_1, \\
\alpha_1&\left(F\frac{\partial H_1}{\partial\sigma}+5H_1\frac{\partial F}{\partial\sigma}\right)=\frac{2(\alpha_1+\alpha_2)}{1-\sigma^2}\left[\begin{array}{c}\left(1-\sigma^2\right)\frac{\partial^2}{\partial\sigma^2}\left(G\frac{\partial F}{\partial\sigma}+\sigma FG_1\right) \\ +12\left(G\frac{\partial F}{\partial\sigma}+\sigma FG_1\right)-\left(G\frac{\partial F}{\partial\sigma}-\frac{\sigma}{1-\sigma^2}G^2\right)\end{array}\right].
\end{aligned}\tag{4.156}$$

Again Eqs. $(4.156)_{1,2}$ give that G and hence H is steady, so that the Eq. $(4.156)_2$ becomes

$$\mu H_1+\rho\left(F\frac{\partial G_1}{\partial\sigma}+3\frac{\partial F}{\partial\sigma}G_1\right)=0$$

or

$$\mu\left[2\frac{\partial^2F}{\partial\sigma^2}+\frac{\partial^2}{\partial\sigma^2}\left\{\left(1-\sigma^2\right)\frac{\partial^2F}{\partial\sigma^2}\right\}\right]+\rho\left[F\frac{\partial^3F}{\partial\sigma^3}+3\frac{\partial F}{\partial\sigma}\frac{\partial^2F}{\partial\sigma^2}\right]=0.\tag{4.157}$$

In order to find the solution of Eq. (4.157), we write this, after some lengthy calculations, the following expression

$$\frac{\partial^3}{\partial\sigma^3}\left[2\mu\left(1-\sigma^2\right)\frac{\partial F}{\partial\sigma}+4\sigma\mu F+\rho F^2\right]=0.\tag{4.158}$$

Integrating Eq. (4.158) we obtain

$$2\mu\left(1-\sigma^2\right)\frac{\partial F}{\partial\sigma}+4\sigma\mu F+\rho F^2=C_{11}(t)\,\sigma^2+C_{12}(t)\,\sigma+C_{13}(t),\tag{4.159}$$

where $C_{11}(t)$, $C_{12}(t)$ and $C_{13}(t)$ are arbitrary functions of the variable t.

Following Landau and Liftshitz [47] (by setting functions of integration equal to zero) we assume the solution as

$$F(\sigma,t) = -\lambda_8\left(1-\sigma^2\right)(\sigma-a_{17})^{-1}, \tag{4.160}$$

where λ_8 and a_{17} are arbitrary real constants. Inserting Eq. (4.160) into Eq. (4.159) it follows

$$\lambda_8 = -2\mu/\rho. \tag{4.161}$$

Using the value of λ_8 in Eq. (4.160) one obtains

$$F(\sigma,t) = \frac{2\nu\left(1-\sigma^2\right)}{\sigma-a_{17}}. \tag{4.162}$$

Substituting Eq. (4.162) into Eq. $(4.156)_3$ we obtain the following

$$96\nu^2\left[\frac{1}{(\sigma-a_{17})^2}-\frac{2\sigma}{(\sigma-a_{17})^3}-\frac{1-\sigma^2}{(\sigma-a_{17})^4}\right]\left[\begin{array}{c}\alpha_1\left\{\frac{11\sigma^2-1}{\sigma-a_{17}}+\frac{21\sigma\left(1-\sigma^2\right)}{(\sigma-a_{17})^2}+\frac{10\left(1-\sigma^2\right)^2}{(\sigma-a_{17})^3}\right\}\\ (\alpha_1+\alpha_2)\left\{2\sigma+\frac{1+\sigma^2}{\sigma-a_{17}}+\frac{7\sigma\left(1-\sigma^2\right)}{(\sigma-a_{17})^2}+\frac{4\sigma\left(1-\sigma^2\right)^2}{(\sigma-a_{17})^3}\right\}\end{array}\right]=0. \tag{4.163}$$

It can be noted from Eq. (4.163) that solution cannot be obtained for all values of the parameter a_{17}. Siddiqui *et al.* [28] found the solution of Eq. (4.163) for steady cases when $a_{17} = -1, 1, 0$. We are recasting the solution for the completeness. On setting $a_{17} = \pm 1$, Eq. (4.163) is satisfied identically and Eq. (4.162) gives

$$F_{1,2} = \mp 2\nu(1\pm\sigma), \text{ for } a_{17} = \pm 1 \tag{4.164}$$

and for $a_{17} = 0$ and $7\alpha_1 + 2\alpha_2 = 0$, Eq. (4.162) become

$$F_3 = \frac{2\nu}{\sigma}\left(1-\sigma^2\right). \tag{4.165}$$

The stream function (4.146) and the velocity components (4.73) for F_1, F_2 and F_3 are respectively given as

$$\overline{\psi} = RF_1(\sigma,t),\ u = -\frac{2\nu}{R},\ v = -\frac{2\nu}{R}\frac{1+\sigma}{\sqrt{1-\sigma^2}}, \tag{4.166}$$

$$\overline{\psi} = RF_2(\sigma,t),\ u = -\frac{2\nu}{R},\ v = \frac{2\nu}{R}\frac{1-\sigma}{\sqrt{1-\sigma^2}}, \tag{4.167}$$

$$\overline{\psi} = RF_3(\sigma,t),\ u = -\frac{2\nu}{R}\frac{1+\sigma^2}{\sigma^2},\ v = \frac{2\nu}{R}\frac{\sqrt{1-\sigma^2}}{\sigma}. \tag{4.168}$$

For $n = 2$, Eqs. (4.147) *and* (4.148) *reduce to*

$$\begin{aligned} &\rho\left[\frac{\partial G_1}{\partial t} - \left\{2F\frac{\partial G_1}{\partial\sigma} + 2\frac{\partial F}{\partial\sigma}G_1\right\}R^{-1}\right] \\ = &\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)H_1R^{-2} - \alpha_1\left[2F\frac{\partial H_1}{\partial\sigma} + 4H_1\frac{\partial F}{\partial\sigma}\right]R^{-3} \\ &+\frac{2(\alpha_1+\alpha_2)}{1-\sigma^2}\left[\begin{array}{c}(1-\sigma^2)\frac{\partial^2}{\partial\sigma^2}\left(G\frac{\partial F}{\partial\sigma} + 2\sigma FG_1\right) \\ +2\left(G\frac{\partial F}{\partial\sigma} + 2\sigma FG_1\right) - G\frac{\partial G}{\partial\sigma}\end{array}\right]R^{-3}, \end{aligned} \tag{4.169}$$

$$G = 2F + (1-\sigma^2)\frac{\partial^2 F}{\partial\sigma^2},\ G_1 = \frac{1}{1-\sigma^2}G,\ H = (1-\sigma^2)\frac{\partial^2 G}{\partial\sigma^2},\ H_1 = \frac{1}{1-\sigma^2}H. \tag{4.170}$$

On comparing the coefficients of R, Eq. (4.169) gives the following equations

$$\begin{aligned} \frac{\partial G_1}{\partial t} &= 0,\ \frac{\partial}{\partial\sigma}(G_1F) = 0,\ \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)H_1 = 0, \\ \alpha_1\left(F\frac{\partial H_1}{\partial\sigma} + 2H_1\frac{\partial F}{\partial\sigma}\right) &= \frac{2(\alpha_1+\alpha_2)}{1-\sigma^2}\left[\begin{array}{c}(1-\sigma^2)\frac{\partial^2}{\partial\sigma^2}\left(G\frac{\partial F}{\partial\sigma} + 2\sigma FG_1\right) \\ +2\left(G\frac{\partial F}{\partial\sigma} + 2\sigma FG_1\right) - G\frac{\partial G}{\partial\sigma}\end{array}\right]. \end{aligned} \tag{4.171}$$

Equations $(4.171)_{1,3}$ together with Eq. (4.170) imply that G_1 and hence H_1 is steady. From Eq. $(4.171)_2$ we get

$$(1-\sigma^2)^{-1}F\left[2F + (1-\sigma^2)\frac{\partial^2 F}{\partial\sigma^2}\right] = \overline{C}(t). \tag{4.172}$$

The solution of Eq. (4.172) is given as

$$F(\sigma,t) = (\sigma^2-1)\widetilde{C}_1(t) + \frac{1}{4}\widetilde{C}_2(t)\left[-2\sigma + (1-\sigma^2)\{\ln(\sigma-1) - \ln(\sigma+1)\}\right] \tag{4.173}$$

and the stream function and velocity components respectively are

$$\begin{aligned} \overline{\psi} &= R^2F(\sigma,t),\ v = 2(1-\sigma^2)^{-1/2}F(\sigma,t), \\ u &= \frac{\sigma}{2}\left[4\widetilde{C}_1(t) + \widetilde{C}_2(t)\left\{\ln(\sigma+1) - \ln(\sigma-1) - \frac{2}{\sigma}\right\}\right], \end{aligned} \tag{4.174}$$

where $\widetilde{C}_1(t)$and $\widetilde{C}_2(t)$ are arbitrary functions.

For $n=3$, Eqs. (4.147) and (4.148) lead to the following

$$\begin{aligned} &\rho\left[R\frac{\partial G_1}{\partial t}-\left\{3F\frac{\partial G_1}{\partial \sigma}+G_1\frac{\partial F}{\partial \sigma}\right\}R\right] \\ = &\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)H_1R^{-1}-\alpha_1\left[3F\frac{\partial H_1}{\partial \sigma}+3H_1\frac{\partial F}{\partial \sigma}\right]R^{-1} \\ &+\frac{2(\alpha_1+\alpha_2)}{1-\sigma^2}\left[\left(1-\sigma^2\right)\frac{\partial^2}{\partial \sigma^2}\left(G\frac{\partial F}{\partial \sigma}+3\sigma FG_1\right)-\left(G\frac{\partial G}{\partial \sigma}+\frac{\sigma}{1-\sigma^2}G^2\right)\right]R^{-1}, \end{aligned} \tag{4.175}$$

$$G=6F+\left(1-\sigma^2\right)\frac{\partial^2 F}{\partial \sigma^2},\ G_1=\left(1-\sigma^2\right)^{-1}G,\ H=\left(1-\sigma^2\right)\frac{\partial^2 G}{\partial \sigma^2},\ H_1=\left(1-\sigma^2\right)^{-1}H. \tag{4.176}$$

Equation (4.175) gives rise to the following

$$\frac{\partial G_1}{\partial t}-\left(3F\frac{\partial G_1}{\partial \sigma}+G_1\frac{\partial F}{\partial \sigma}\right)=0, \tag{4.177}$$

$$\begin{aligned} &\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)H_1-\alpha_1\left[3F\frac{\partial H_1}{\partial \sigma}+3H_1\frac{\partial F}{\partial \sigma}\right] \\ = &\frac{2(\alpha_1+\alpha_2)}{1-\sigma^2}\left[\left(1-\sigma^2\right)\frac{\partial^2}{\partial \sigma^2}\left(G\frac{\partial F}{\partial \sigma}+3\sigma FG_1\right)-\left(G\frac{\partial G}{\partial \sigma}+\frac{\sigma}{1-\sigma^2}G^2\right)\right]. \end{aligned} \tag{4.178}$$

For steady $\left(\frac{\partial}{\partial t}(\cdot)=0\right)$ and viscous case $(\alpha_1=0)$, Eq. (4.178) gives $\mu H_1=0$ which on using Eq. (4.176) becomes

$$\mu\frac{\partial^2}{\partial \sigma^2}\left[6F+\left(1-\sigma^2\right)\frac{\partial^2 F}{\partial \sigma^2}\right]=0$$

which on integration gives

$$6F+\left(1-\sigma^2\right)\frac{\partial^2 F}{\partial \sigma^2}=k_1\sigma+k_2, \tag{4.179}$$

where $k_1$and k_2 are constants of integration. It is observed that the solution F obtained in Eq. (4.179) is only satisfied through Eq. (4.177) when the constants $k_1$and k_2 are fixed to zero. So the solution of Eq. (4.177) is

$$F(\sigma)=\sigma\left(\sigma^2-1\right)\widetilde{C}_3-\frac{1}{4}\widetilde{C}_4\left[-4+6\sigma^2+3\sigma\left(\sigma^2-1\right)\{1-\ln(1+\sigma)\}+\ln(\sigma-1)\right]. \tag{4.180}$$

The stream function and velocity components are

$$\begin{aligned}
\overline{\psi} &= R^3 F(\sigma,t),\ v = 3R\left(1-\sigma^2\right)^{-1/2} F(\sigma),\\
u &= R\left[\left(3\sigma^2-1\right)\widetilde{C}_3 - \frac{9}{2}\widetilde{C}_4\sigma + \frac{3}{4}\left(3\sigma^2-1\right)\widetilde{C}_4\left\{\ln(1+\sigma)\right\} - \ln(\sigma-1)\right]. \quad (4.181)
\end{aligned}$$

When $k_1 \neq 0$, $k_2 \neq 0$, we have the following solution of Eq. (4.179)

$$F(\sigma) = \frac{1}{24}\left[\begin{array}{c} 6k_2\sigma^2 + 4\left\{k_1\sigma^3 + 6\sigma\left(\sigma^2-1\right)\widetilde{C}_5 + 3\widetilde{C}_6\left(2-3\sigma^2\right)\right\} \\ +3\sigma\left(\sigma^2-1\right)\left(k_2 - 6\widetilde{C}_6\right)\left\{\ln(\sigma-1) - \ln(\sigma+1)\right\} \end{array}\right], \quad (4.182)$$

where $\widetilde{C}_i\,(i = 3-6)$ are constants. The stream function and the velocity components in this case are

$$\overline{\psi} = R^3 F(\sigma,t),\ u = R\frac{dF}{d\sigma},\ v = 3R\left(1-\sigma^2\right)^{-1/2} F(\sigma). \quad (4.183)$$

For $n = 4$, *Eqs.* (4.147) *and* (4.148) *become*

$$\begin{aligned}
\rho\left[R^2\frac{\partial G_1}{\partial t} - 4F\frac{\partial G_1}{\partial \sigma}R^3\right] &= \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)H_1 - \alpha_1\left[4F\frac{\partial H_1}{\partial \sigma} + 2H_1\frac{\partial F}{\partial \sigma}\right]R \quad (4.184)\\
&+\frac{2(\alpha_1+\alpha_2)}{1-\sigma^2}\left[\begin{array}{c} \left(1-\sigma^2\right)\frac{\partial^2}{\partial\sigma^2}\left(G\frac{\partial F}{\partial \sigma} + 4\sigma F G_1\right) \\ +6\left(G\frac{\partial F}{\partial \sigma} + 4\sigma F G_1\right) \\ -\left\{G\frac{\partial G}{\partial \sigma} + 2\sigma\left(1-\sigma^2\right)^{-1}G^2\right\} \end{array}\right]R,
\end{aligned}$$

$$\begin{aligned}
G &= 12F + \left(1-\sigma^2\right)\frac{\partial^2 F}{\partial \sigma^2},\ G_1 = \left(1-\sigma^2\right)^{-1}G,\\
H &= \left(1-\sigma^2\right)\frac{\partial^2 G}{\partial \sigma^2} + 2G,\ H_1 = \left(1-\sigma^2\right)^{-1}H. \quad (4.185)
\end{aligned}$$

Following equations are obtained from Eq. (4.184)

$$\begin{aligned}
\frac{\partial G_1}{\partial t} &= 0,\ F\frac{\partial G_1}{\partial \sigma} = 0,\ \left(\mu + \alpha_1\frac{\partial}{\partial t}\right)H_1 = 0, \quad (4.186)\\
\alpha_1\left[4F\frac{\partial H_1}{\partial \sigma} + 2H_1\frac{\partial F}{\partial \sigma}\right] &= \frac{2(\alpha_1+\alpha_2)}{1-\sigma^2}\left[\begin{array}{c} \left\{\left(1-\sigma^2\right)\frac{\partial^2}{\partial\sigma^2} + 6\right\}\left(G\frac{\partial F}{\partial \sigma} + 4\sigma F G_1\right) \\ -\left\{G\frac{\partial G}{\partial \sigma} + 2\sigma\left(1-\sigma^2\right)^{-1}G^2\right\} \end{array}\right].
\end{aligned}$$

The first and third equations in Eq. (4.186) imply that G_1 and hence H_1 is steady (not a function of t). Since $F \neq 0$, Eq. $(4.186)_2$ gives $\frac{\partial G_1}{\partial t} = 0$, which on using Eq. $(4.185)_2$ gives the following solution

$$G = A_{16}\left(1 - \sigma^2\right). \tag{4.187}$$

Using Eq. $(4.185)_1$ in Eq. (4.187) we get

$$\left(1 - \sigma^2\right)\frac{\partial^2 G}{\partial \sigma^2} + 2G = A_{16}\left(1 - \sigma^2\right). \tag{4.188}$$

The solution of Eq. (4.188) for steady case is given by Berker [41] and in order to avoid repetition we directly give the solution with stream function and velocity components as

$$\begin{aligned} F &= k_3\sigma^2\left(1 - \sigma^2\right), \quad (4.189) \\ \overline{\psi} &= k_3\sigma\left(1 - 2\sigma^2\right)R^2,\ u = 2k_3\sigma\left(1 - 2\sigma^2\right)R^2,\ v = 4k_3\sigma^2\sqrt{1 - \sigma^2}R^2, \end{aligned}$$

where k_3 is a constant.

4.5 Conclusions

In this chapter, the governing time dependent equations for plane polar, axisymmetric cylindrical and spherical coordinates are derived. By assuming certain forms of the stream function in different coordinate system, we obtained closed to eleven solutions of the resulting differential equations. The solutions obtained are found to be in well agreement to that of the previous solutions for viscous and second grade fluids. The modeled compatibility equations in all the three coordinate systems for steady cases reduce to Siddiqui *et al.* [28], whereas the solutions successfully verifies the results of Jaffery-Hamel [46], Berker [41], Squire [46], and Landau and Liftshitz [47].

Chapter 5

Flow of a third grade fluid induced by a variable shear stress

5.1 Introduction

The solution of third grade fluid is far more complicated than the Navier-Stokes equations and for the second grade fluids. The second grade fluids though complicated are sometimes amenable to certain solution methods, whereas third grade fluids do not yield solutions for these problems. Physically, if the second grade fluids are important by shear thickening properties, third grade fluids have the significance because of the shear thinning properties. The nonlinearity enters further through the boundary conditions as well.

This chapter comprises the flow of an incompressible third grade fluid over an infinite wall. The flow is induced due to a variable shear stress. The variable shear stress of the third grade fluid make the boundary condition non-linear. This chapter is arranged as follows:

In section 5.2, the modelling of the governing equation for flow of a third grade fluid is given. Section 5.3 deals with the formulation of the problem. Section 5.4 is decomposed into four subsections. In subsection 5.4.1, the solution is given when the shear stress is proportional to $e^{\lambda t}$ (λ is real and positive constant). Subsection 5.4.3 gives the analytical solution of the problem when shear stress is proportional to $e^{i\omega t}$ (ω is imposed frequency). Both the series and numerical solutions are given in subsection 5.4.1, whereas only series solution is obtained in subsection 5.4.3. Moreover, the results and discussion are presented in subsections 5.4.2.

and 5.4.4. Section 5.5 synthesis the concluding remarks. It is found that with an increase in second-grade parameter and third-grade parameter, the velocity decreases and thus boundary layer thickness increases.

5.2 Modeling for variable suction in third grade fluid

Consider the flow of a third grade fluid over a plate. The wall is infinite in extent and thus the velocity field depends only y and t, i.e.

$$\mathbf{V} = [u(y,t),\ 0,\ 0], \tag{5.1}$$

which satisfies the equation of continuity. Making use of Eq. (5.1), one can write

$$\mathbf{A}_1 = \begin{bmatrix} 0 & \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial y} & 0 \end{bmatrix}, \tag{5.2}$$

$$\mathbf{A}_1^2 = \begin{bmatrix} \left(\frac{\partial u}{\partial y}\right)^2 & 0 \\ 0 & \left(\frac{\partial u}{\partial y}\right)^2 \end{bmatrix}, \tag{5.3}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & \frac{\partial^2 u}{\partial y \partial t} \\ \frac{\partial^2 u}{\partial y \partial t} & 2\left(\frac{\partial u}{\partial y}\right)^2 \end{bmatrix}, \tag{5.4}$$

$$\mathbf{A}_3 = \begin{bmatrix} 0 & \frac{\partial^3 u}{\partial y \partial t^2} \\ \frac{\partial^3 u}{\partial y \partial t^2} & 6\frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial y \partial t}\right) \end{bmatrix}, \tag{5.5}$$

and thus through Eqs. (1.6) and (1.7) we can write

$$\mathbf{T} = \begin{bmatrix} -p + \alpha_2\left(\frac{\partial u}{\partial y}\right)^2 & \mu\frac{\partial u}{\partial y} + \alpha_1\left(\frac{\partial^2 u}{\partial y \partial t}\right) + 2\beta_3\left(\frac{\partial u}{\partial y}\right)^3 \\ \mu\frac{\partial u}{\partial y} + \alpha_1\frac{\partial^2 u}{\partial y \partial t} + 2\beta_3\left(\frac{\partial u}{\partial y}\right)^3 & -p + 2\alpha_1\left(\frac{\partial u}{\partial y}\right)^2 + \alpha_2\left(\frac{\partial u}{\partial y}\right)^2 \end{bmatrix}. \tag{5.6}$$

From above equation

$$(\mathrm{div}\mathbf{T})_x = \frac{\partial}{\partial x}\left[-p + \alpha_2\left(\frac{\partial u}{\partial y}\right)^2\right] + \frac{\partial}{\partial y}\left[\mu\frac{\partial u}{\partial y} + \alpha_1\left(\frac{\partial^2 u}{\partial y \partial t} + V\frac{\partial^2 u}{\partial y^2}\right) + 2\beta_3\left(\frac{\partial u}{\partial y}\right)^3\right], \tag{5.7}$$

$$(\mathrm{div}\mathbf{T})_y = -\frac{\partial p}{\partial y} + (2\alpha_1 + \alpha_2)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^2 \tag{5.8}$$

In absence of body forces, the momentum equation satisfies the following equations

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right]_x = (\mathrm{div}\mathbf{T})_x, \tag{5.9}$$

$$\rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right]_y = (\mathrm{div}\mathbf{T})_y, \tag{5.10}$$

where subscripts indicates the x and y components of the momentum equation.

From Eqs. (5.1) and Eqs. (5.7) to (5.10) one can write

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial \widehat{p}_2}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} + \alpha_1\frac{\partial^3 u}{\partial y^2 \partial t} + 6\beta_3\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}, \tag{5.11}$$

$$0 = -\frac{\partial \widehat{p}_2}{\partial y}, \tag{5.12}$$

where

$$\widehat{p}_2 = p - (2\alpha_1 + \alpha_2)\left(\frac{\partial u}{\partial y}\right)^2. \tag{5.13}$$

Eliminating the pressure gradient between Eqs. (5.11) and (5.12) finally yields

$$\rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2} + \alpha_1\frac{\partial^3 u}{\partial y^2 \partial t} + 6\beta_3\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} \tag{5.14}$$

It should be pointed out Eq. (5.14) holds for second grade fluid when $\beta_3 = 0$. The equation which governs the viscous flow can be taken for $\alpha_1 = 0$ and $\beta_3 = 0$.

5.3 Problem formulation

Let us consider the flow of a thermodynamic third grade fluid over an infinite plate at $y = 0$. Choose the $y-$axis perpendicular to the plate. The plate is assumed under a variable shear stress with magnitude $c_1\tau(t)$ where c_1 is a constant having the dimension ρU_0 (ρ is the density and U_0 is some reference velocity). The governing non-linear equation is taken from Eq. (5.14) as

$$\rho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial y^2} + \alpha_1\frac{\partial^3 u}{\partial y^2 \partial t} + 6\beta_3\left[\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial u}{\partial y}\right)^2\right]. \tag{5.15}$$

The non-linear boundary conditions for the flow under consideration are

$$\left[\mu\frac{\partial u}{\partial y}+\alpha_1\frac{\partial^2 u}{\partial y\partial t}+2\beta_3\left(\frac{\partial u}{\partial y}\right)^3\right]_{y=0}=c_1\tau(t),\quad t>0, \tag{5.16}$$

$$u(y,t)\to 0,\qquad \text{as}\quad y\to\infty. \tag{5.17}$$

We shall now write the field equation and the boundary conditions. For that we use

$$\overline{\alpha}_1=\frac{\alpha_1 U_0^2}{\rho\nu^2},\ \varepsilon=\frac{6\beta_3 U_0^4}{\rho\nu^3},\overline{u}=\frac{u}{U_0},\ \overline{t}=\frac{U_0^2 t}{\nu},\ \eta=\frac{U_0}{\nu}y, \tag{5.18}$$

in Eq. (5.15) and the boundary conditions (5.16) and (5.17), and then omitting the bars for simplicity we get

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial \eta^2}+\alpha_1\frac{\partial^3 u}{\partial\eta^2\partial t}+\varepsilon\left[\frac{\partial u}{\partial\eta}\left(\frac{\partial u}{\partial\eta}\right)^2\right], \tag{5.19}$$

$$\left[\frac{\partial u}{\partial\eta}+\alpha_1\frac{\partial^2 u}{\partial\eta\partial t}+\frac{1}{3}\varepsilon\left(\frac{\partial u}{\partial\eta}\right)^3\right]_{\eta=0}=\tau(t),\quad t>0, \tag{5.20}$$

$$u(\eta,t)\to 0\quad \text{as}\quad \eta\to\infty. \tag{5.21}$$

In this chapter we discuss two cases (i) $\tau(t)=e^{\lambda t}$ (λ is real and positive constant) and (ii) $\tau(t)=e^{\imath\omega t}$ (ω is imposed frequency). In the former case since λ is positive, it is prudent to obtain a numerical solution besides an analytical solution in the form of perturbation series in terms of ε. In the latter case, on the other hand since the solution is essentially bounded therefore, a perturbation solution should give acceptable results.

5.4 Solution of the problem

5.4.1 Solution for case 1: $\tau(t)=e^{\lambda t}$, λ is purely real (acceleration)

Numerical solution

In problems of this type, usually no initial condition is given at $t=0$. For example, for a second grade fluid ($\varepsilon=0$) Hayat *et al.* [21] and Rajagopal [19] derived the analytical solutions for a number of unsteady unidirectional flow problems, without using any initial condition. The initial condition, if derived, can be obtained from the solution.

Because of the nonlinearity introduced on account of the third grade fluid parameter, a closed form analytical solution, in general, is not feasible to obtain, and a numerical solution should be sought. For the latter, it appears that an initial condition must be prescribed at $t = 0$. However, as Ariel [48] has recently demonstrated in an analogous situation, the initial condition can be deduced if appropriate transformations are used.

We choose

$$u(\eta, t) = e^{\lambda t} f(\eta, t), \tag{5.22}$$

so that the differential equation for f takes the form

$$\frac{\partial f}{\partial t} + \lambda f = \frac{\partial^2 f}{\partial \eta^2} + \alpha_1 \left(\frac{\partial^3 f}{\partial t \partial \eta^2} + \lambda \frac{\partial^2 f}{\partial \eta^2} \right) + \varepsilon e^{2\lambda t} \left(\frac{\partial f}{\partial \eta} \right)^2 \frac{\partial^2 f}{\partial \eta^2} \tag{5.23}$$

and the boundary conditions become

$$(1 + \alpha_1 \lambda) \frac{\partial f(0, t)}{\partial \eta} + \frac{1}{3} \varepsilon e^{2\lambda t} \left[\frac{\partial f(0, t)}{\partial \eta} \right]^3 = 1, \quad f(\infty, t) = 0. \tag{5.24}$$

Next we introduce the transformation

$$\xi = e^{2\lambda t} \tag{5.25}$$

which leads us the boundary value problem

$$2\lambda \xi \frac{\partial f}{\partial \xi} + \lambda f = (1 + \alpha_1 \lambda) \frac{\partial^2 f}{\partial \eta^2} + 2\alpha_1 \lambda \xi \frac{\partial^3 f}{\partial \xi \partial \eta^2} + \varepsilon \xi \left(\frac{\partial f}{\partial \eta} \right)^2 \frac{\partial^2 f}{\partial \eta^2}, \tag{5.26}$$

$$(1 + \alpha_1 \lambda) \frac{\partial f(0, t)}{\partial \eta} + \frac{1}{3} \varepsilon \xi \left[\frac{\partial f(0, t)}{\partial \eta} \right]^3 = 1, \quad f(\infty, \xi) = 0. \tag{5.27}$$

Equation (5.26) has only the boundary conditions at $\eta = 0$ and $\eta = \infty$, but not the initial condition at $\xi = 0$. But if we make the reasonable assumption that f is regular at $\xi = 0$, we do not need the initial condition to get the integration started at $\xi = 0$. Equation (5.26) can thus be integrated in the entire domain $0 \leq \xi < \infty \cap 0 \leq \eta \leq \infty$. When one reaches $\xi = 1$, the initial condition is recovered for the problem. Now either the original equation (5.19) can be integrated in the usual manner, or the integration can be further carried out of equation (5.26) beyond $\xi = 1$. We have chosen the latter approach in the present work.

The details of the integration scheme have been furnished in Ariel [48] and are omitted here, except that in the present work the situation is slightly complicated on account of the boundary condition at $\eta = 0$. Now we have

$$(1+\alpha_1\lambda)\frac{\partial f(0,\xi)}{\partial \eta} + \frac{1}{3}\varepsilon\xi\left[\frac{\partial f(0,\xi)}{\partial \eta}\right]^3 = 1 \qquad (5.28)$$

which is a cubic in $\partial f(0,\xi)/\partial\eta$, that must be solved for each value of ξ. Also at $\xi = 0$, the solution for f is

$$f(\eta,0) = -\frac{1}{\sqrt{\lambda}\sqrt{1+\lambda\alpha_1}}\exp\left(-\sqrt{\frac{\lambda}{1+\lambda\alpha_1}}\eta\right). \qquad (5.30)$$

u(0)											
λ=0.5											
		t=0		t=1		t=2		t=3		t=5	
α_1	ε	Exact	Pertur-bation	Exact	Pertur-bation	Exact	Pertur-bation	Exact	Pertur-bation	Exact	Pertur-bation
0.2	0.1	-1.339476	-1.342384	-2.186549	-2.197751	-3.529042	-3.570781	-5.596185	-5.853466	-13.222440	-60.142478
	0.2	-1.331485	-1.336777	-2.157930	-2.177357	-3.443206	-3.537030	-5.384342	-6.404501	-12.355852	-213.784337
	0.5	-1.311407	-1.322416	-2.095671	-2.146119	-3.282987	-3.800568	-5.033027	-12.501669	-11.154726	-1334.267421
	1	-1.285830	-1.306674	-2.027334	-2.193869	-3.126898	-5.455769	-4.727634	-37.477161	-10.240992	-5400.264261
0.5	0.1	-1.257607	-1.261516	-2.055104	-2.070731	-3.322842	-3.377852	-5.285384	-5.466806	-12.620225	-25.166529
	0.2	-1.250940	-1.258241	-2.030566	-2.057429	-3.247671	-3.335021	-5.097938	-5.478324	-11.854309	-66.928162
	0.5	-1.233809	-1.249130	-1.976187	-2.026241	-3.107124	-3.312736	-4.790754	-6.809757	-10.776368	-384.242304
	1	-1.211545	-1.236331	-1.916194	-2.003322	-2.972230	-3.629622	-4.522355	-13.341755	-9.942527	-1553.196693
1	0.1	-1.149875	-1.153137	-1.863191	-1.896866	-3.056709	-3.106968	-4.885331	-5.055345	-11.798322	-14.513506
	0.2	-1.145326	-1.151598	-1.865411	-1.890251	-2.997434	-3.082781	-4.727786	-4.960096	-11.147837	-21.551781
	0.5	-1.133065	-1.147127	-1.823182	-1.872203	-2.880334	-3.026095	-4.466168	-5.020847	-10.224618	-82.218010
	1	-1.116092	-1.140168	-1.773710	-1.848109	-2.765489	-3.004536	-4.238184	-5.977082	-9.498322	-315.166407
2	0.1	-0.997836	-0.999508	-1.639238	-1.646527	-2.678170	-2.708566	-4.324706	-4.439598	-10.638175	-11.632051
	0.2	-0.995728	-0.999019	-1.630394	-1.644363	-2.644389	-2.699227	-4.216987	-4.402104	-10.107024	-11.763908
	0.5	-0.989722	-0.997566	-1.606962	-1.638058	-2.566931	-2.673475	-4.015461	-4.317206	-9.359660	-16.253280
	1	-0.980624	-0.995196	-1.575504	-1.628170	-2.480478	-2.638103	-3.830112	-4.267656	-8.773695	-37.381574

Table 5.1 Illustrating the variation of $u(0)$, the velocity at the plate with α_1, the viscoelastic fluid parameter and ε, the second-grade fluid parameter for $\lambda = 0.5$ using (i) exact numerical solution and (ii) perturbation solution

From Table 5.1, we observe that there is a very good agreement between the numerical solution and the perturbation solution for $t = 0$ and small values of t ($t < 1$). For the values of t greater than 3, there is sufficient discrepancy in the results that the perturbation solution can no longer be accepted and the results from the numerical solution only should be used.

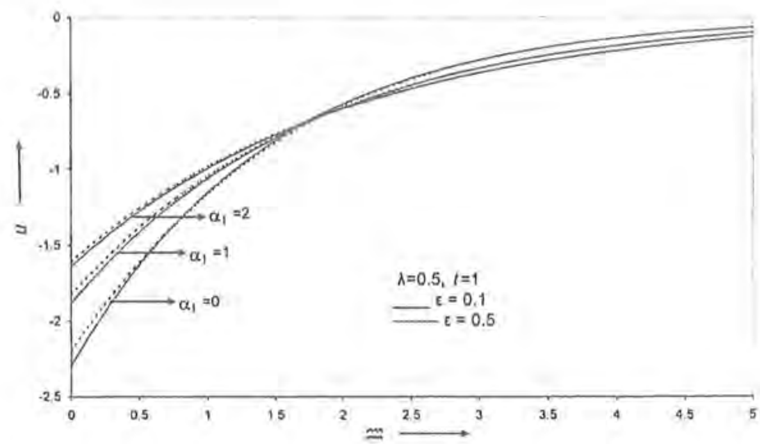

Fig. 5.1. Variation of velocity profile u with η for $t = 1$.

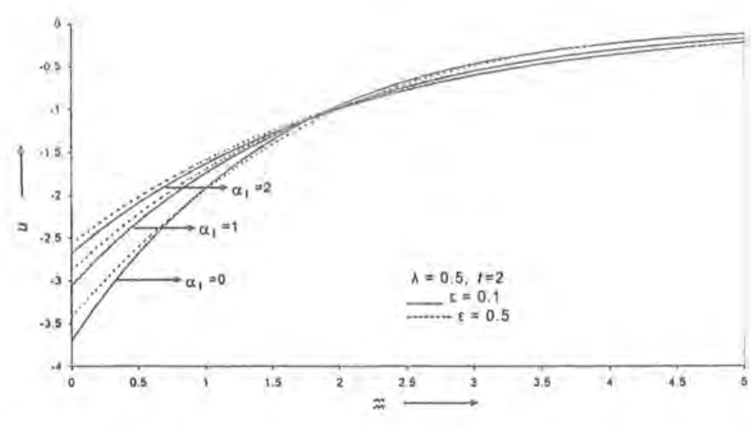

Fig. 5.2. Variation of velocity profile u with η for $t = 2$.

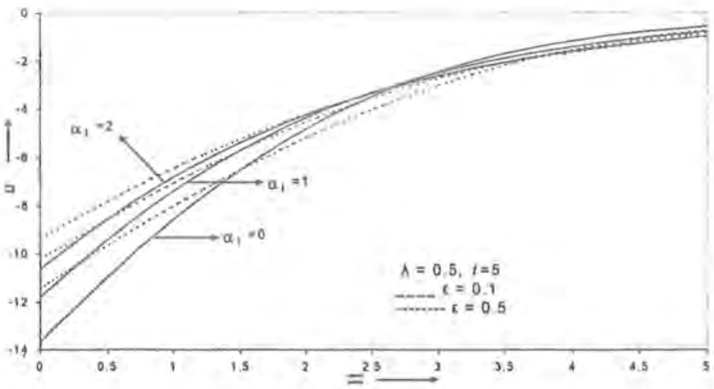

Fig. 5.3. Variation of velocity profile u with η for $t = 5$.

Perturbation solution

We perturb the velocity field u in third grade parameter ε as follows [49].

$$u(\eta, t; \varepsilon) = u_0(\eta, t) + \varepsilon u_1(\eta, t) + \varepsilon^2 u_2(\eta, t) + \cdots. \tag{5.30}$$

For $\varepsilon = 0$, Eq. (5.32) gives an exact solution for the reduced problem corresponding to a second-grade fluid. Using Eq. (5.30) into Eqs. (5.19) and the boundary conditions (5.20) and (5.21) and then comparing the coefficients of like powers of ε one obtains the following systems up to $O(\varepsilon^2)$ as:

Zeroth order system

$$\frac{\partial u_0}{\partial t} = \frac{\partial^2 u_0}{\partial \eta^2} + \alpha_1 \frac{\partial^3 u_0}{\partial \eta^2 \partial t}, \tag{5.31}$$

$$\left. \frac{\partial u_0}{\partial \eta} + \alpha_1 \frac{\partial^2 u_0}{\partial \eta \partial t} \right|_{\eta=0} = e^{\lambda t}, \quad t > 0, \tag{5.32}$$

$$u_0(\eta, t) \to 0 \quad \text{as} \quad \eta \to \infty. \tag{5.33}$$

First order system

$$\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial \eta^2} + \alpha_1 \frac{\partial^3 u_1}{\partial \eta^2 \partial t} + \frac{\partial^2 u_0}{\partial \eta^2} \left(\frac{\partial^2 u_0}{\partial \eta^2} \right)^2, \tag{5.34}$$

$$\left. \frac{\partial u_1}{\partial \eta} + \alpha_1 \frac{\partial^2 u_1}{\partial \eta \partial t} + \frac{1}{3} \left(\frac{\partial u_0}{\partial \eta} \right)^3 \right|_{\eta=0} = 0, \tag{5.35}$$

$$u_1(\eta, t) \to 0 \quad \text{as} \quad \eta \to \infty. \tag{5.36}$$

Second order system

$$\frac{\partial u_2}{\partial t} = \frac{\partial^2 u_2}{\partial \eta^2} + \alpha_1 \frac{\partial^3 u_2}{\partial \eta^2 \partial t} + \frac{\partial^2 u_1}{\partial \eta^2} \left(\frac{\partial^2 u_0}{\partial \eta^2} \right)^2 + 2 \frac{\partial u_0}{\partial \eta} \frac{\partial u_1}{\partial \eta} \frac{\partial^2 u_0}{\partial \eta^2}, \tag{5.37}$$

$$\left. \frac{\partial u_2}{\partial \eta} + \alpha_1 \frac{\partial^2 u_2}{\partial \eta \partial t} + \left(\frac{\partial u_0}{\partial \eta} \right)^2 \frac{\partial u_1}{\partial \eta} \right|_{\eta=0} = 0, \tag{5.38}$$

$$u_2(\eta, t) \to 0 \quad \text{as} \quad \eta \to \infty. \tag{5.39}$$

The above systems after using the transformations

$$u_0(\eta, t) = f_0(\eta) e^{\lambda t}, \quad u_1(\eta, t) = f_1(\eta) e^{3\lambda t}, \quad u_2(\eta, t) = f_2(\eta) e^{5\lambda t}. \tag{5.40}$$

reduce to the following:

$$(1 + \lambda \alpha_1) f_0''(\eta) - \lambda f_0 = 0, \tag{5.41}$$

$$(1 + \lambda \alpha_1) f_0'(\eta) = 1, \tag{5.42}$$

$$f_0(\eta) \to 0 \quad \text{as} \quad \eta \to \infty, \tag{5.43}$$

$$(1 + 3\lambda \alpha_1) f_1''(\eta) - 3\lambda f_1 = -\left(f_0'\right)^2 f_0'', \tag{5.44}$$

$$(1+3\lambda\alpha_1) f_1'(0) + \frac{1}{3}(f_0'(0))^3 = 0, \tag{5.45}$$

$$f_1(\eta) \to 0 \quad \text{as} \quad \eta \to \infty, \tag{5.46}$$

$$(1+5\lambda\alpha_1) f_2''(\eta) - 5\lambda f_1 = -(f_0')^2 f_1'' - 2f_0' f_1' f_0'', \tag{5.47}$$

$$(1+5\lambda\alpha_1) f_2'(0) + (f_0'(0))^3 f_0'(0) = 0, \tag{5.48}$$

$$f_2(\eta) \to 0 \quad \text{as} \quad \eta \to \infty, \tag{5.49}$$

where prime denotes the differentiation with respect to η.

After lengthy but straightforward calculations, the solutions of the above systems are

$$f_0(\eta) = -A_0 c_0 e^{-c_0\eta}, \tag{5.50}$$

$$f_1(\eta) = A_1\left(c_1 e^{-c_1\eta} - 3c_0 e^{-3c_0\eta}\right), \tag{5.51}$$

$$f_2(\eta) = A_2\left(c_2 e^{-c_2\eta} - 5c_0 e^{-5c_0\eta}\right) + B_2\left[c_2 e^{-c_2\eta} - (2c_0+c_1) e^{-(2c_0+c_1)\eta}\right], \tag{5.52}$$

where

$$\begin{aligned} A_0 &= \frac{1}{\lambda}, \; c_0 = \sqrt{\frac{\lambda}{1+\lambda\alpha_1}}, \; A_1 = -\frac{A_0^3 c_0^4}{18(1+4\lambda\alpha_1)}, \; c_1 = \sqrt{\frac{3\lambda}{1+3\lambda\alpha_1}}, \\ A_2 &= \frac{-9A_0^2 A_1 c_0^4}{20(1+6\lambda\alpha_1)}, \; B_2 = \frac{A_0^2 A_1 c_0^4 c_1^2}{(2c_0+c_1)^2(1+5\lambda\alpha_1) - 5\lambda}, \; c_2 = \sqrt{\frac{5\lambda}{1+5\lambda\alpha_1}}. \end{aligned}$$

The expression for *skin friction* is given as

$$\tau_1 = \overline{\tau}_1/\rho U_0^2 = \left[e^{\lambda t} f_0'(0) + \varepsilon e^{3\lambda t} f_1'(0) + \varepsilon^2 e^{5\lambda t} f_2'(0) + \cdots\right]. \tag{5.53}$$

From Eqs. (5.50) to (5.52) we can obtain

$$f_0'(\eta) = A_0 c_0^2 e^{-c_0\eta}, \tag{5.54}$$

$$f_1'(\eta) = A_1\left(-c_1^2 e^{-c_1\eta} + 9c_0^2 e^{-3c_0\eta}\right), \tag{5.55}$$

$$f_2'(\eta) = A_2\left(-c_2^2 e^{-c_2\eta} + 25c_0^2 e^{-5c_0\eta}\right) + B_2\left[-c_2^2 e^{-c_2\eta} + (2c_0+c_1)^2 e^{-(2c_0+c_1)\eta}\right]. \tag{5.56}$$

5.4.2 Results and discussion

Fig. 5.1 is plotted for the velocity field u against η for ($\alpha_1 = 0, 1, 2; t = 1; \lambda = 0.5$ and $\varepsilon = 0.1; 0.5$). It is observed that with an increase in the viscoelastic parameter α_1 the velocity increases near the boundary but then decreases away from the boundary thus causing the boundary layer thickness to increase. Also it is found that when α_1 is fixed i.e. ($\alpha_1 = 0$) and the third-grade parameter is increased from $\varepsilon = 0.1$ to $\varepsilon = 0.5$ the velocity is again increased near the plate and then decreased away from the boundary, though the effect of third-grade fluid parameter is not as pronounced as that of the viscoelastic fluid parameter. Same behavior is observed when $\alpha_1 = 1$ and $\alpha_1 = 2$. In Figs. 5.2 and 5.3 the velocity field u is plotted against η for ($\alpha_1 = 0, 1, 2; t = 2; \lambda = 0.5$ and $\varepsilon = 0.1; 0.5$) and ($\alpha_1 = 0, 1, 2; t = 5; \lambda = 0.5$ and $\varepsilon = 0.1; 0.5$), respectively. The similar observations for the velocity field and the boundary layer thickness are seen in these figures as in Fig. 5.1 except that the difference between the velocity profiles for $\varepsilon = 0.1$ and $\varepsilon = 0.5$ become prominent as we increase t from 2 to 5.

5.4.3 Solution for case 2: $\tau(t) = e^{\lambda t}$, λ is purely imaginary (oscillations)

Perturbation solution

We now discuss the case when the shear stress at the plate has an oscillating nature. For that we put $\lambda = i\omega$ in $\tau(t)$ and employing the same procedure as in section **5.3.1** one obtains

$$\begin{aligned} u(\eta, t; \varepsilon) &= \left[f_{0R}(\eta)\cos\omega t - f_{0I}(\eta)\sin\omega t\right] \\ &\quad + \varepsilon\left[f_{1R}(\eta)\cos 3\omega t - f_{1I}(\eta)\sin 3\omega t\right] \\ &\quad + \varepsilon^2\left[f_{2R}(\eta)\cos 5\omega t - f_{2I}(\eta)\sin 5\omega t\right] + \cdots, \end{aligned} \tag{5.57}$$

where the expressions for the functions $f_{0R}, f_{0I}, f_{1R}, f_{1I}, f_{2R}, f_{2I}$ are straightforward to obtain, which can be easily obtained from Eqs. (5.50) to (5.52) by letting $\lambda = i\omega$. Separating the real and imaginary parts we obtain

$$
\begin{aligned}
f_{0R} &= \frac{-e^{-R_1\eta}}{R_2^2+I_2^2}\left(R_2\cos I_1\eta - I_2\sin I_1\eta\right),\ f_{0I} = \frac{e^{-R_1\eta}}{R_2^2+I_2^2}\left(R_2\sin I_1\eta + I_2\cos I_1\eta\right),\\
f_{1R} &= \frac{e^{-3R_1\eta}}{R_4^2+I_4^2}\left[R_4\left(R_1\cos 3I_1\eta + I_1\sin 3I_1\eta\right) - I_4\left(I_1\cos 3I_1\eta - R_1\sin 3I_1\eta\right)\right]\\
&\quad -\frac{e^{-R_3\eta}}{3\left(R_4^2+I_4^2\right)}\left[R_4\left(R_3\cos I_3\eta + I_3\sin 3I_3\eta\right) - I_4\left(I_3\cos I_3\eta - R_3\sin I_3\eta\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
f_{1I} &= \frac{e^{-3R_1\eta}}{R_4^2+I_4^2}\left[R_4\left(I_1\cos 3I_1\eta - R_1\sin 3I_1\eta\right) + I_4\left(R_1\cos 3I_1\eta + I_1\sin 3I_1\eta\right)\right]\\
&\quad -\frac{e^{-R_3\eta}}{3\left(R_4^2+I_4^2\right)}\left[R_4\left(I_3\cos I_3\eta - R_3\sin I_3\eta\right) + I_4\left(R_3\cos I_3\eta + I_3\sin 3I_3\eta\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
f_{2R} &= R_9 + R_{13} + R_{14},\quad f_{2I} = I_9 + I_{13} + I_{14},\\
R_1 &= \frac{1}{\sqrt{2\left(1+\omega^2\alpha_1^2\right)}}\sqrt{\sqrt{\left(\omega^2\alpha_1\right)^2+\omega^2}+\omega^2\alpha_1},\\
I_1 &= \frac{1}{\sqrt{2\left(1+\omega^2\alpha_1^2\right)}}\sqrt{\sqrt{\left(\omega^2\alpha_1\right)^2+\omega^2}-\omega^2\alpha_1},\\
R_2 &= \frac{1}{\sqrt{2}}\sqrt{\sqrt{\left(\omega^2\alpha_1\right)^2+\omega^2}+\omega^2},\ I_2 = \pm\frac{1}{\sqrt{2}}\sqrt{\sqrt{\left(\omega^2\alpha_1\right)^2+\omega^2}-\omega^2},
\end{aligned}
$$

$$
\begin{aligned}
R_3 &= \frac{1}{\sqrt{2\left(1+9\omega^2\alpha_1^2\right)}}\sqrt{\sqrt{\left(9\omega^2\alpha_1\right)^2+9\omega^2}+9\omega^2\alpha_1},\\
I_3 &= \frac{1}{\sqrt{2\left(1+9\omega^2\alpha_1^2\right)}}\sqrt{\sqrt{\left(9\omega^2\alpha_1\right)^2+9\omega^2}-9\omega^2\alpha_1},\\
R_4 &= 2\omega^2\alpha_1\left(2\omega^2\alpha_1^2-3\right),\ I_4 = \omega\left(1-9\omega^2\alpha_1^2\right),
\end{aligned}
$$

$$R_{14} = \left(\overline{a}_7 \cos I_5\eta - \overline{b}_7 \sin I_5\eta\right) e^{-R_5\eta}, \; I_{14} = \left(\overline{b}_7 \cos I_5\eta + \overline{a}_7 \sin I_5\eta\right) e^{-R_5\eta},$$
$$R_{13} = R_{10} - \left(\overline{a}_{18}R_{12} - \overline{b}_{18}I_{12}\right), \; I_{13} = I_{10} - \left(\overline{b}_{18}R_{12} + \overline{a}_{18}I_{12}\right),$$

$$R_{12} = R_{11} \cos I_5\eta + I_{11} \sin I_5\eta, \; I_{12} = I_{11} \cos I_5\eta - R_{11} \sin I_5\eta,$$
$$R_{11} = \frac{e^{-R_5\eta}}{R_5^2 + I_5^2} \left[(R_3 + 2R_1) R_5 + I_5 (I_3 + 2I_1)\right],$$
$$I_{11} = \frac{e^{-R_5\eta}}{R_5^2 + I_5^2} \left[(I_3 + 2I_1) R_5 - I_5 (R_3 + 2R_1)\right],$$
$$R_{10} = e^{-(2R_1+R_3)\eta} \left[\overline{a}_{18} \cos(2I_1 + I_3)\eta + \overline{b}_{18} \sin(2I_1 + I_3)\eta\right],$$
$$I_{10} = e^{-(2R_1+R_3)\eta} \left[\overline{b}_{18} \cos(2I_1 + I_3)\eta - \overline{a}_{18} \sin(2I_1 + I_3)\eta\right],$$

$$\overline{a}_{18} = \frac{\overline{a}_{15}\overline{a}_{17} + \overline{b}_{15}\overline{b}_{17}}{\overline{a}_{17}^2 + \overline{b}_{17}^2}, \; \overline{b}_{18} = \frac{\overline{b}_{15}\overline{b}_{17} - \overline{a}_{15}\overline{a}_{17}}{\overline{a}_{17}^2 + \overline{b}_{17}^2}, \; \overline{a}_{17} = 3\left(\overline{a}_{16} - 5\omega\alpha_1\overline{b}_{16}\right),$$
$$\overline{a}_{16} = (2R_1 + R_3)^2 - (2I_1 + I_3)^2 - \left(R_5^2 + I_5^2\right)^2,$$
$$\overline{b}_{16} = (2R_1 + R_3)(2I_1 + I_3) - 2R_5I_5, \; \overline{b}_{17} = 3\left(\overline{b}_{16} + 5\omega\alpha_1\overline{a}_{16}\right),$$
$$\overline{a}_{15} = (2R_1 + R_3)\left(\overline{a}_{13}\overline{a}_{14} - \overline{b}_{13}\overline{b}_{14}\right) - (2I_1 + I_3)\left(\overline{b}_{13}\overline{a}_{14} + \overline{a}_{13}\overline{a}_{14}\right),$$

$$\overline{b}_{15} = (2I_1 + I_3)\left(\overline{a}_{13}\overline{a}_{14} - \overline{b}_{13}\overline{b}_{14}\right) + (2R_1 + R_3)\left(\overline{b}_{13}\overline{a}_{14} + \overline{a}_{13}\overline{a}_{14}\right),$$
$$\overline{a}_{14} = \left(R_6^2 - I_6^2\right)\left(R_1^2 - I_1^2\right) - 4R_1I_1R_6I_6,$$
$$\overline{b}_{14} = 2R_1I_1\left(R_6^2 - I_6^2\right) + 2R_6I_6\left(R_1^2 - I_1^2\right),$$
$$\overline{a}_{13} = \overline{a}_2\left(R_3^2 - I_3^2\right) - 2\overline{b}_2R_3I_3, \; \overline{b}_{13} = \overline{b}_2\left(R_3^2 - I_3^2\right) + 2\overline{a}_2R_3I_3,$$

$$
\begin{aligned}
R_9 &= R_7 - (R_8 \cos I_5\eta + I_8 \sin I_5\eta),\ I_9 = I_7 - (I_8 \cos I_5\eta - R_8 \sin I_5\eta),\\
R_8 &= \frac{5e^{-R_5\eta}}{R_5^2 + I_5^2}\left[(R_1R_5 + I_1I_5)\,\overline{a}_{12} - \overline{b}_{12}\,(I_1R_5 - R_1I_5)\right],\\
I_8 &= \frac{5e^{-R_5\eta}}{R_5^2 + I_5^2}\left[(I_1R_5 - R_1I_5)\,\overline{a}_{12} + \overline{b}_{12}\,(R_1R_5 + I_1I_5)\right],\\
R_7 &= e^{-5R_1\eta}\left(\overline{a}_{12}\cos 5I_1\eta + \overline{b}_{12}\sin 5I_1\eta\right),\ \overline{a}_{12} = \frac{15\left(\overline{a}_9\overline{a}_{11} + \overline{b}_9\overline{b}_{11}\right)}{\overline{a}_{11}^2 + \overline{b}_{11}^2},\\
I_7 &= e^{-5R_1\eta}\left(\overline{b}_{12}\cos 5I_1\eta - \overline{a}_{12}\sin 5I_1\eta\right),\ \overline{b}_{12} = \frac{15\left(\overline{b}_9\overline{a}_{11} - \overline{a}_9\overline{b}_{11}\right)}{\overline{a}_{11}^2 + \overline{b}_{11}^2},
\end{aligned}
$$

$$
\begin{aligned}
\overline{a}_{11} &= \overline{a}_{10} - 5\omega\alpha_1\overline{b}_{10},\ \overline{b}_{11} = \overline{b}_{10} + 5\omega\alpha_1\overline{a}_{10},\\
\overline{a}_{10} &= 25\left(R_1^2 - I_1^2\right) - \left(R_5^2 - I_5^2\right),\ \overline{b}_{10} = 50R_1I_1 - 2R_5I_5,\\
\overline{a}_9 &= \left(R_6^2 - I_6^2\right)\left(\overline{a}_2\overline{a}_8 - \overline{b}_2\overline{b}_8\right) - 2R_6I_6\left(\overline{b}_2\overline{a}_8 + \overline{a}_2\overline{b}_8\right),\\
\overline{b}_9 &= 2R_6I_6\left(\overline{a}_2\overline{a}_8 - \overline{b}_2\overline{b}_8\right) + \left(R_6^2 - I_6^2\right)\left(\overline{b}_2\overline{a}_8 + \overline{a}_2\overline{b}_8\right),
\end{aligned}
$$

$$
\begin{aligned}
\overline{a}_8 &= R_1\left\{\left(R_1^2 - I_1^2\right)^2 - 4R_1^2I_1^2\right\} - 4\left(R_1^2 - I_1^2\right)R_1I_1^2,\ d = \overline{a}_2 + i\overline{b}_2,\\
\overline{b}_8 &= I_1\left\{\left(R_1^2 - I_1^2\right)^2 - 4R_1^2I_1^2\right\} + 4\left(R_1^2 - I_1^2\right)R_1^2I_1,\ c = R_6 + iI_6,\\
\overline{a}_3 &= \left(R_3^2 - I_3^2\right) - 9\left(R_1^2 - I_1^2\right),\ \overline{b}_3 = 2R_3I_3 + 2R_1I_1,\ \sqrt{\theta} = R_5 + iI_5,\\
\overline{a}_4 &= \left(R_6^2 - I_6^2\right)\left(R_1^2 - I_1^2\right) - 4R_1I_1R_6I_6,\ \overline{b} = R_3 + iI_3,\\
\overline{b}_4 &= 2R_1I_1\left(R_6^2 - I_6^2\right) + 2R_6I_6\left(R_1^2 - I_1^2\right),\ \overline{a} = R_1 + iI_1,
\end{aligned}
$$

$$
\begin{aligned}
\overline{a}_5 &= R_5 - 5\omega\alpha_1I_5,\ \overline{b}_5 = I_5 + 5\omega\alpha_1R_5,\ \overline{a}_7 = \frac{\overline{a}_5\overline{a}_6 + \overline{b}_5\overline{b}_6}{3\left(\overline{a}_5^2 + \overline{b}_5^2\right)},\ \overline{b}_7 = \frac{\overline{a}_5\overline{b}_6 - \overline{a}_6\overline{b}_5}{3\left(\overline{a}_5^2 + \overline{b}_5^2\right)},\\
\overline{a}_6 &= \overline{a}_4\left(\overline{a}_2\overline{a}_3 - \overline{b}_2\overline{b}_3\right) - \overline{b}_4\left(\overline{b}_3\overline{a}_2 + \overline{a}_3\overline{b}_2\right),\ \overline{a}_2 = \frac{-30\omega^2\alpha_1}{\left(30\omega^2\alpha_1\right)^2 + \left(6\omega - 24\omega^3\alpha_1^2\right)^2},\\
\overline{b}_6 &= \overline{b}_4\left(\overline{a}_2\overline{a}_3 - \overline{b}_2\overline{b}_3\right) + \overline{a}_4\left(\overline{b}_3\overline{a}_2 + \overline{a}_3\overline{b}_2\right),\ \overline{b}_2 = \frac{24\omega^3\alpha_1^2 - 6\omega}{\left(30\omega^2\alpha_1\right)^2 + \left(6\omega - 24\omega^3\alpha_1^2\right)^2},
\end{aligned}
$$

$$R_6 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{\overline{a}_1^2+\overline{b}_1^2}+\overline{a}_1}, \; I_6 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{\overline{a}_1^2+\overline{b}_1^2}-\overline{a}_1},$$

$$R_5 = \frac{1}{\sqrt{2\left(1+25\omega^2\alpha_1^2\right)}}\sqrt{\sqrt{\left(25\omega^2\alpha_1\right)^2+25\omega^2}+25\omega^2\alpha_1},$$

$$I_5 = \frac{1}{\sqrt{2\left(1+25\omega^2\alpha_1^2\right)}}\sqrt{\sqrt{\left(25\omega^2\alpha_1\right)^2+25\omega^2}-25\omega^2\alpha_1},$$

in which f_{0R}, f_{0I}, f_{1R}, f_{1I}, and f_{2R}, f_{2I} indicate the real and imaginary parts of f_0, f_1 and f_2, respectively.

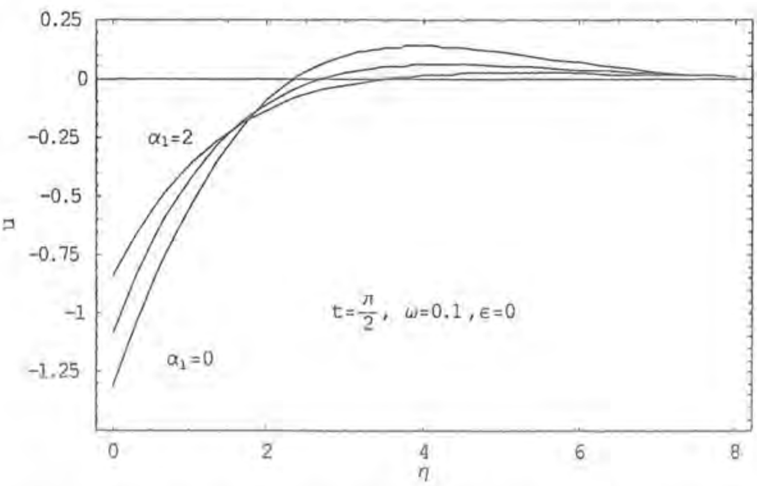

Fig. 5.4. Variation of velocity profile u with η for $t=\pi/2$, $\varepsilon=0$, $\omega=0.1$ and $\alpha_1=0;1;2$.

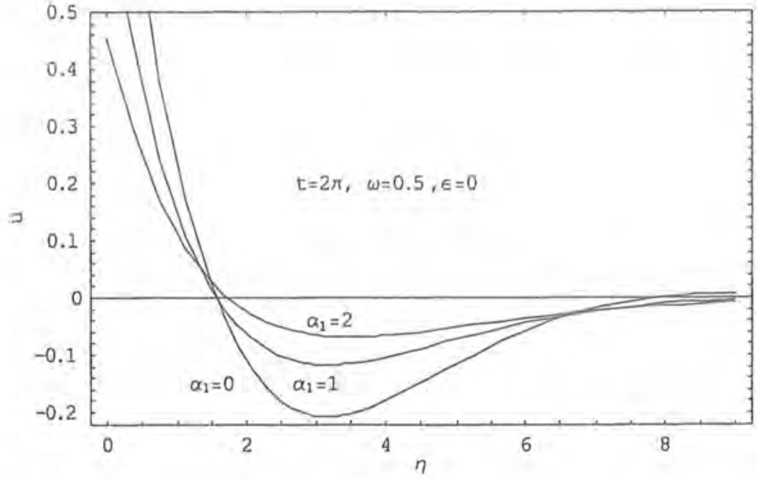

Fig. 5.5. Variation of velocity profile u with η for $t = 2\pi$, $\varepsilon = 0$, $\omega = 0.5$ and $\alpha_1 = 0; 1; 2$.

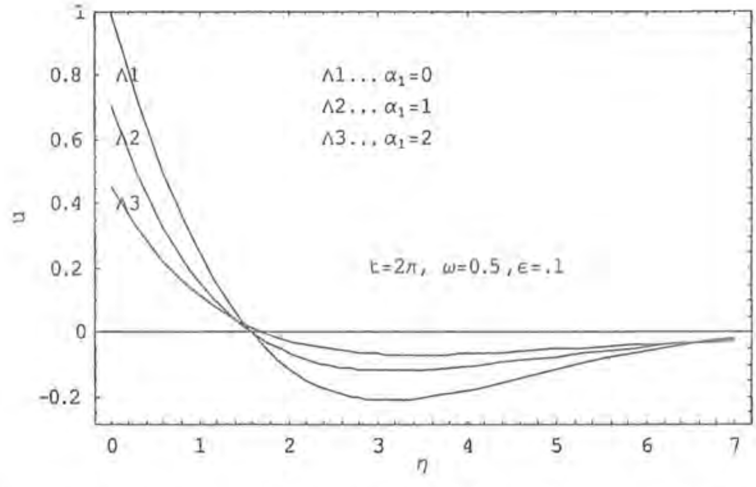

Fig. 5.6. Variation of velocity profile u with η for $t = 2\pi$, $\varepsilon = 0.1$, $\omega = 0.5$ and $\alpha_1 = 0; 1; 2$.

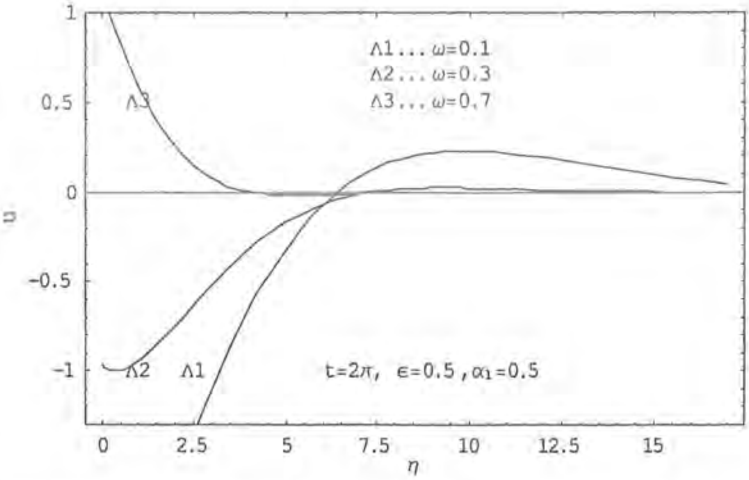

Fig. 5.7. Variation of velocity profile u with η for $t = 2\pi$, $\varepsilon = 0.5$, $\alpha_1 = 0.5$ and $\omega = 0.1; 0.3; 0.7$.

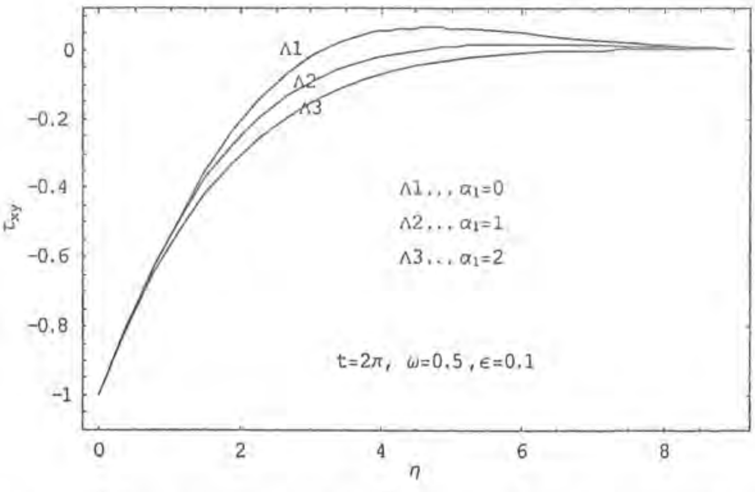

Fig. 5.8. Variation of shear stress τ_{xy} with η for $t = 2\pi$, $\varepsilon = 0.1$, $\omega = 0.5$ and $\alpha_1 = 0; 1; 2$.

5.4.4 Results and discussion

In Fig. 5.4 u the velocity is plotted against η for a second-grade fluid ($\alpha_1 = 0, 1, 2; t = \pi/2; \omega = 0.5; \varepsilon = 0$). It is observed that with an increase in the viscoelastic fluid parameter α_1 the velocity decreases and thus boundary layer thickness increases. Similar effects are seen in Fig. 5.5 in which $t = 2\pi$ and $\omega = 0.5$ are taken instead of $t = \pi/2$ and $\omega = 0.1$. In Fig. 5.6 the velocity u is plotted against η for a third-grade fluid ($\alpha_1 = 0, 1, 2; t = 2\pi; \omega = 0.5; \varepsilon = 0.1$). Figure 6 shows that with the increase in third-grade parameter the velocity decreases and the boundary layer thickness further increases. In Fig. 5.7 the velocity u is plotted against η for $\alpha_1 = 0.5$, $t = 2\pi$, $\varepsilon = 0.5$, and for various values of oscillating frequency ($\omega = 0.1, 0.3, 0.7$). It is clear from Fig. 5.7 that the amplitude of the velocity decreases with an increase of the oscillating frequency. Fig. 5.8 is plotted for the stress τ_{xy} at any point in the fluid against η for various values of α_1.

The *skin friction* at the plate $\eta = 0$ can be obtained by finding the real part in the following equation

$$\tau_2 = \overline{\tau}_2/\rho U_0^2 = \begin{bmatrix} e^{i\omega t}\{f'_{0R}(0) + if'_{0I}(0)\} + \varepsilon e^{3i\omega t}\{f'_{1R}(0) + if'_{1I}(0)\} \\ +\varepsilon e^{5i\omega t}\{f'_{2R}(0) + if'_{2I}(0)\} + \cdots \end{bmatrix}, \tag{5.58}$$

where $f'_{0R}(0)$, $f'_{1R}(0)$, $f'_{2R}(0)$, $f'_{0I}(0)$, $f'_{1I}(0)$, $f'_{2I}(0)$ are given as:

$$\begin{aligned} f'_{0R}(0) &= \frac{R_1R_2 + I_1I_1}{R_2^2 + I_2^2}, \quad f'_{0I}(0) = \frac{I_1R_2 - R_1I_2}{R_2^2 + I_2^2}, \\ f'_{1R}(0) &= \frac{1}{R_4^2 + I_4^2}\left[6R_1I_1I_4 + 3R_4\left(I_1^2 - R_1^2\right) + \frac{1}{3}R_4\left(R_3^2 - I_3^2\right) - \frac{2}{3}R_3I_1I_4\right], \\ f'_{1I}(0) &= \frac{1}{R_4^2 + I_4^2}\left[-6R_1I_1I_4 + 3I_4\left(I_1^2 - R_1^2\right) + \frac{1}{3}I_4\left(R_3^2 - I_3^2\right) + \frac{2}{3}R_3I_3R_4\right], \\ f'_{2R}(0) &= R'_9(0) + R'_{13}(0) + R'_{14}(0), \quad f'_{2I}(0) = I'_9(0) + I'_{13}(0) + I'_{14}(0). \end{aligned}$$

5.5 Concluding remarks

Here we have constructed the results for the flow of a third-grade fluid on a plate. The flow is generated due to a variable shear stress of the plate and the solution of non-linear partial differential equation is presented. The problem considered is more general and several limiting cases are obtained as the particular problem of the presented analysis. Specifically, the results for viscous and second-grade fluid flows due to a variable shear stress (which are not yet in the literature to the best of our knowledge) can be recovered by taking $\alpha_1 = \varepsilon = 0$ and $\varepsilon = 0$, respectively. Our investigation shows that the perturbation technique is adequate for the case when the variable shear stress has as oscillatory character, however, if the shear stress grows exponentially with time then the perturbation solution can be accepted only for small values of time. For moderate to large values of time, the numerical solution must be used.

Chapter 6

Time dependent flow of a third grade fluid in the case of suction

6.1 Introduction

We emphasize that the process of suction/blowing has its importance in many engineering applications such as in the design of thrust bearing and radial diffusers and thermal oil recovery. Suction is applied to chemical processes to remove reactants. Blowing is used to add reactants, cool the surfaces, prevent corrosion or scaling and reduce the drag. Hopefully, the subsequent analysis will help understand the phenomena in some more details.

This chapter examines the flow of an incompressible third grade fluid over an infinite porous plate. The flow analysis has been carried out for sudden motion of a plate. The governing non-linear partial differential equation resulting from the momentum equation is solved analytically. For the analytic solution, the perturbation method has been employed. Special emphasis has been given to the influence of suction and the material parameter of the third grade fluid on the flow. Several known results of interest are found to follow as particular cases of the solution of the problem considered. It is observed from the solution that non-Newtonian effects on the velocity are present for small time. For large time the velocity and shear stress for the Newtonian and non-Newtonian fluids are the same.

6.2 Governing problem

We consider an infinite permeable plate aligned along the $x-$axis. We mean by permeable plates that the plates with very fine holes distributed uniformly throughout the plate through which fluid can flow freely and continuously. Suddenly, the plate is set into motion with velocity U_0 along the $x-$axis. The fluid at $y > 0$ is at rest far away from the plate. The velocity field for the present flow analysis is

$$\mathbf{V} = (u(y,t),\ v(y,t),\ 0), \tag{6.1}$$

which together with continuity equation (1.9) gives

$$v = V(t)$$

in which $V(t) < 0$ corresponds to the variable suction velocity and thus Eq. (6.1) now becomes

$$\mathbf{V} = (u(y,t),\ V(t),\ 0). \tag{6.2}$$

Using above definition of velocity, Eq. (1.4) yields

$$\begin{aligned} \rho\left[\frac{\partial u}{\partial t} + V(t)\frac{\partial u}{\partial y}\right] &= -\frac{\partial \widehat{p}_3}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} + \alpha_1\left(\frac{\partial^3 u}{\partial y^2 \partial t} + V\frac{\partial^3 u}{\partial y^3}\right) \\ &+\beta_1\left(\frac{\partial^4 u}{\partial y^2 \partial t^2} + V'\frac{\partial^3 u}{\partial y^3} + 2V\frac{\partial^4 u}{\partial y^3 \partial t} + V^2\frac{\partial^4 u}{\partial y^4}\right) + 6(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}, \end{aligned} \tag{6.3}$$

$$\rho\frac{\partial V(t)}{\partial t} = -\frac{\partial \widehat{p}_3}{\partial y}, \tag{6.4}$$

$$\widehat{p}_3 = p - (2\alpha_1+\alpha_2)\left(\frac{\partial u}{\partial y}\right)^2 + 2(3\beta_1+\beta_2)\frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial y \partial t} + V\frac{\partial^2 u}{\partial y^2}\right). \tag{6.5}$$

Eliminating the pressure $\widehat{p}_3$ from Eqs. (6.3) and (6.4) one can write

$$\frac{\partial u}{\partial t}+V(t)\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}+\beta\left(\frac{\partial^3 u}{\partial y^2\partial t}+V(t)\frac{\partial^3 u}{\partial y^3}\right)+\gamma\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2} \tag{6.6}$$
$$+\varepsilon_1\left(\frac{\partial^2 u}{\partial y^2\partial t^2}+V'(t)\frac{\partial^3 u}{\partial y^3}+2V(t)\frac{\partial^4 u}{\partial y^3\partial t}+V^2\frac{\partial^4 u}{\partial y^4}\right)$$

where $\beta=\alpha_1/\rho$, $\gamma=6(\beta_2+\beta_3)/\rho$ and $\varepsilon_1=\beta_1/\rho$. Note that $\widehat{p}_3$ is a linear function in y and in writing Eq. (6.6) we have used the expression (1.6) for the Cauchy stress tensor.

The appropriate boundary conditions are

$$u(0,t) = U_0 \quad \text{for} \quad t>0, \tag{6.7}$$
$$u(y,t) \rightarrow 0 \quad \text{as} \quad y\rightarrow\infty.$$

Introducing the non-dimensional variables

$$u=U_0 f, \qquad y=\frac{\nu}{U_0}\xi, \qquad t=\frac{\nu}{U_0^2}\tau, \tag{6.8}$$

the governing problem becomes

$$\frac{\partial f}{\partial \tau}+\overline{V}(\tau)\frac{\partial f}{\partial \xi} = \frac{\partial^2 f}{\partial \xi^2}+\overline{\beta}\left[\frac{\partial^3 f}{\partial \xi^2\partial \tau}+\overline{V}(\tau)\frac{\partial^3 f}{\partial \xi^3}\right]+\overline{\gamma}\left(\frac{\partial f}{\partial \xi}\right)^2\frac{\partial^2 f}{\partial \xi^2} \tag{6.9}$$
$$+\overline{\varepsilon}\left[\frac{\partial^4 f}{\partial \xi^2\partial \tau^2}+\overline{V}'(\tau)\frac{\partial^3 f}{\partial \xi^3}+2\overline{V}\frac{\partial^4 f}{\partial \xi^3\partial \tau}+\overline{V}^2\frac{\partial^4 f}{\partial \xi^4}\right],$$

$$f(0,\tau) = 1, \quad \text{for} \quad \tau>0, \tag{6.10}$$
$$f \rightarrow 0 \quad \text{as} \quad \xi\rightarrow\infty,$$

in which

$$V(t)=U_0\overline{V}(\tau), \qquad \beta=\frac{\nu^2}{U_0^2}\overline{\beta}, \qquad \gamma=\frac{\nu^3}{U_0^4}\overline{\gamma}, \qquad \varepsilon=\frac{\nu^3}{U_0^4}\overline{\varepsilon}. \tag{6.11}$$

6.3 Perturbation solution

Writing

$$\eta = \frac{y - \int V(t)dt}{2\sqrt{\nu t}} = \frac{\xi - \int \overline{V}(\tau)d\tau}{2\sqrt{\tau}}, \tag{6.12}$$

$$f(\eta, \tau) = f_0(\eta) + \frac{1}{\tau} f_1(\eta) + \frac{1}{\tau^2} f_2(\eta) + \cdots, \tag{6.13}$$

into Eq. (6.9) and conditions (6.10) and then equating the coefficients of like powers of $1/\tau$ we get the following systems:

System of order zero

$$f_0^{''} + 2\eta f_0^{'} = 0, \tag{6.14}$$

$$\begin{aligned} f_0(0, \tau) &= 1, \qquad \text{for} \qquad \tau > 0, \\ f_0(\infty, \tau) &= 0. \end{aligned} \tag{6.15}$$

System of order one

$$f_1^{''} + 2\eta f_1^{'} + 4f_1 = \overline{\beta}\left(f_0^{''} + \frac{1}{2}\eta f_0^{'''}\right) - \frac{\overline{\gamma}}{4} f_0^{''} \left(f_0^{'}\right)^2, \tag{6.16}$$

$$\begin{aligned} f_1(0, \tau) &= 1, \qquad \text{for} \qquad \tau > 0, \\ f_1(\infty, \tau) &= 0. \end{aligned} \tag{6.17}$$

The solution of the zeroth order system is

$$\begin{aligned} f_0 &= 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\xi_1^2} d\xi_1 \\ &= 1 - \text{erf}(\eta) = \text{erf}\, c(\eta) \end{aligned} \tag{6.18}$$

where $\text{erf}(\eta)$ is the error function and $\text{erfc}(\eta)$ is the complementary error function.

Substitution of Eq. (6.18) into Eq. (6.16) we have

$$f_1'' + 2\eta f_1' + 4f_1 = \frac{2\overline{\beta}}{\sqrt{\pi}}\eta e^{-\eta^2}(3 - 2\eta^2) - \frac{4\overline{\gamma}}{\pi^{3/2}}\eta e^{-3\eta^2}. \tag{6.19}$$

Note that the solution of Eq. (6.19) is the sum of complementary function and particular integral. The complementary function satisfies

$$f_1'' + 2\eta f_1' + 4f_1 = 0. \tag{6.20}$$

Writing

$$f_1 = \sum_{j=0}^{\infty} a_j \eta^j \tag{6.21}$$

and using into Eq. (6.20) we obtain

$$a_{j+2} = \frac{-2a_j}{j+1}, \qquad j = 0, 1, 2, 3, 4, .,. \tag{6.22}$$

On substituting Eq. (6.22) into Eq. (6.21) the complementary function and particular integral are of the following type

$$f_{1c} = a_0\left(1 - 2\eta^2 + \frac{4}{3}\eta^4 - \frac{8}{15}\eta^6 + \cdots\right) + a_1\eta e^{-\eta^2}, \tag{6.23}$$

$$f_{1p} = \frac{\overline{\beta}}{\sqrt{\pi}}\eta^3 e^{-\eta^2} - \frac{\overline{\gamma}}{\pi^{3/2}}\left(\frac{2}{3}\eta^3 + \frac{2.8}{3.5}\eta^5 + \frac{2.4.74}{3.5.7.6}\eta^7 + \frac{4.2.184}{5.7.9.8}\eta^9\right)e^{-3\eta^2}. \tag{6.24}$$

The general solution is

$$f_1 = a_0\left(1 - 2\eta^2 + \frac{4}{3}\eta^4 - \frac{8}{15}\eta^6 + \cdots\right) + \left(a_1\eta + \frac{\overline{\beta}}{\sqrt{\pi}}\eta^3\right)e^{-\eta^2} + F(\eta)e^{-3\eta^2}, \tag{6.25}$$

where

$$F(\eta) = -\frac{\overline{\gamma}}{\pi^{3/2}}\left(\frac{2}{3}\eta^3 + \frac{2.8}{3.5}\eta^5 + \frac{2.4.74}{3.5.7.6}\eta^7 + \frac{4.2.184}{5.7.9.8}\eta^9\right). \tag{6.26}$$

Using $f_1(0) = 0$, we have $a_0 = 0$ and thus the expression for f_1 is

$$f_1 = \left(a_1\eta + \frac{\overline{\beta}}{\sqrt{\pi}}\eta^3\right)e^{-\eta^2} + F(\eta)e^{-3\eta^2}. \tag{6.27}$$

Let us write

$$F(\eta) = \sum_{n=1}^{\infty} a_{2n+1}\eta^{2n+1} = a_3\eta^3 + a_5\eta^5 + a_7\eta^7 + a_9\eta^9 + \cdots. \tag{6.28}$$

On comparing Eqs. (6.26) and (6.28) we can write

$$\frac{a_3}{\overline{\gamma}} = -\frac{2}{3\pi^{3/2}} = -0.1172475, \qquad a_5 = -\frac{\overline{\gamma}}{\pi^{3/2}}\frac{2.8}{3.5} = a_3.\frac{8}{5} \Rightarrow \frac{a_5}{a_3} = \frac{8}{5}. \tag{6.29}$$

The calculations of coefficients a_{2n+1} are described in terms of a_3 in Table 6.1. as follows:

$\frac{a_5}{a_3}$	1.60000000	$\frac{a_{15}}{a_3}$	0.05858760	$\frac{a_{25}}{a_3}$	0.00005584
$\frac{a_7}{a_3}$	1.40952381	$\frac{a_{17}}{a_3}$	0.01769331	$\frac{a_{27}}{a_3}$	0.00001094
$\frac{a_9}{a_3}$	0.87619048	$\frac{a_{19}}{a_3}$	0.00478697	$\frac{a_{29}}{a_3}$	0.00000201
$\frac{a_{11}}{a_3}$	0.42528139	$\frac{a_{21}}{a_3}$	0.00117728	$\frac{a_{31}}{a_3}$	0.00000035
$\frac{a_{13}}{a_3}$	0.17053169	$\frac{a_{23}}{a_3}$	0.00026620	$\frac{a_{33}}{a_3}$	0.00000006

Table 6.1

There is one constant a_1 in Eq. (6.27). The value of a_1 is calculated by imposing the fact that the displacement thickness has to vanish at $t = 0$ [50] and get

$$a_1 = -\frac{\overline{\beta}}{\sqrt{\pi}} - \sum_{n=1} \frac{n!}{3^{n+1}}a_{2n+1}. \tag{6.30}$$

The graphs are shown in Figs. 6.1, 6.2 and 6.3 in which velocity varies with respect to the non-dimensional distance to the plate, for various values of time and suction parameter i.e., $\tau = 4$, $\overline{\beta} = -4$, $\overline{\gamma} = 4$,....$\tau = 4$, $\overline{\beta} = -2$, $\overline{\gamma} = 2$, ...$\tau = 100$, $\overline{\beta} = -2$, $\overline{\gamma} = 2$ and $\overline{V} = -0.05$, 0, 0.05. For $\overline{V} = 0$ we get the result of Erdogan [50].

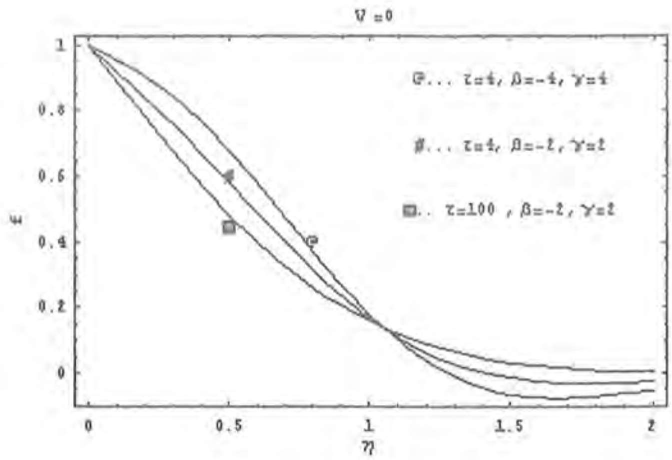

Fig. 6.1. Variation of f with η for $V(t) = 0$.

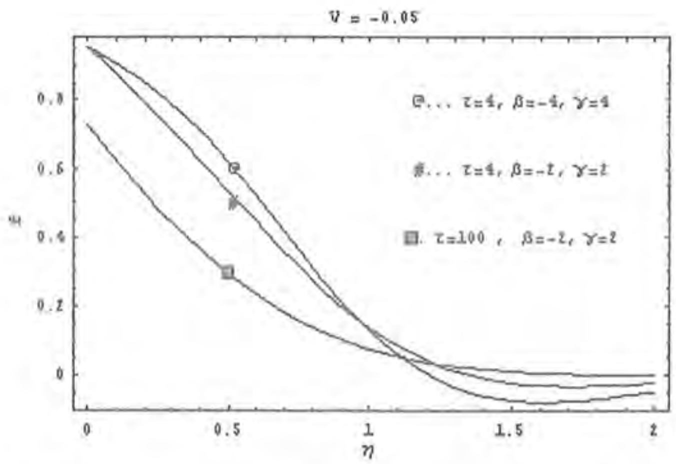

Fig. 6.2. Variation of f with η for $V(t) = -0.05$.

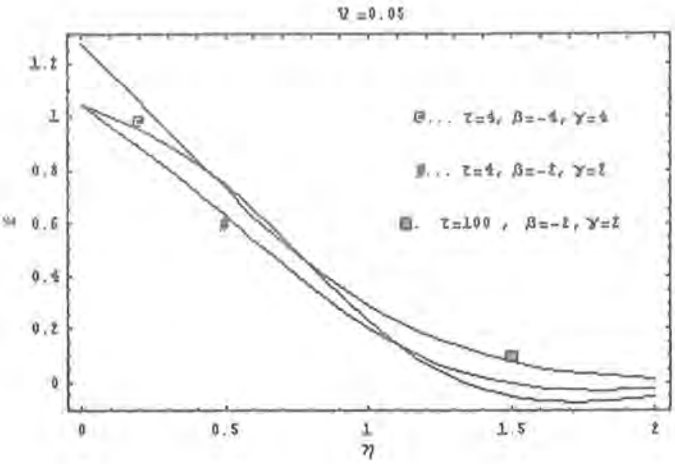

Fig. 6.3. Variation of f with η for $V(t) = 0.05$.

6.4 Shear stress at the plate

The shear stress at the plate is

$$\begin{aligned}\tau_{xy} &= \mu\frac{\partial u}{\partial y} + \alpha_1\left[\frac{\partial^2 u}{\partial y\partial t} + V(t)\frac{\partial^2 u}{\partial y^2}\right] + 2(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^3 \\ &\quad +\beta_1\left[\frac{\partial^3 u}{\partial y\partial t^2} + \frac{dV}{dt}\frac{\partial^2 u}{\partial y^2} + 2V(t)\frac{\partial^3 u}{\partial y^2\partial t} + (V(t))^2\frac{\partial^3 u}{\partial y^3}\right]\end{aligned} \tag{6.31}$$

which in terms of non-dimensional variables is

$$\begin{aligned}\frac{\tau_{xy}}{\rho U^2} &= \frac{\partial f}{\partial \xi} + \bar{\beta}\left[\frac{\partial^2 f}{\partial \xi\partial \tau} + \overline{V}(\tau)\frac{\partial^2 f}{\partial \xi^2}\right] + \frac{1}{3}\bar{\gamma}\left(\frac{\partial f}{\partial \xi}\right)^3 \\ &\quad +\bar{\varepsilon}\left[\frac{\partial^3 f}{\partial \xi\partial \tau^2} + \frac{d\overline{V}}{d\tau}\frac{\partial^2 f}{\partial \xi^2} + 2\overline{V}(\tau)\frac{\partial^3 f}{\partial \xi^2\partial \tau} + \left(\overline{V}(\tau)\right)^2\frac{\partial^3 f}{\partial \xi^3}\right].\end{aligned} \tag{6.32}$$

Using Eqs. (6.12) and (6.13) into above we obtain at $y=0$ the following

$$\frac{[\Upsilon_{xy}]_{y=0}}{\rho U^2} = -\frac{1}{\sqrt{\pi}}\left(\frac{\nu}{U^2 t}\right)^{\frac{1}{2}} e^{-\lambda^2} + \frac{1}{\sqrt{\pi}}\left(\frac{\nu}{U^2 t}\right)^{\frac{3}{2}} \left[\begin{array}{c} \left(0.5\overline{\beta} - \frac{1}{3\pi}\overline{\gamma} + 1.5\frac{\overline{\beta}}{\sqrt{\pi}}\lambda\right) \\ -\lambda\left\{\frac{\overline{\beta}}{\sqrt{\pi}}\left(\lambda^3 - 1\right) + 0.1197\overline{\gamma}\right\} \end{array}\right] e^{-\lambda^2} + \cdots, \tag{6.33}$$

where

$$\lambda(t) = -\frac{\int V(t)dt}{2\sqrt{\nu t}}. \tag{6.34}$$

Figs. 6.4 and 6.5 shows the variation of the shear stresses at the plate for various values of time and the suction parameter. In these graphs we have taken $\overline{\beta} = -1$, $\overline{\gamma} = 1$ and $\overline{\beta} = 0$, $\overline{\gamma} = 0$ and $V(t) = 0,\ -1$. It is clear from the graphs that for small times $\left(\frac{U^2 t}{\nu} < 5\right)$ non-Newtonian effects occur and for large times $\left(\frac{U^2 t}{\nu} \geq 5\right)$ it become weak and behaves like a Newtonian fluid. Moreover, it is shown in Fig. 6.5 that with the introduction of the suction parameter V, the boundary layer thickness decreases.

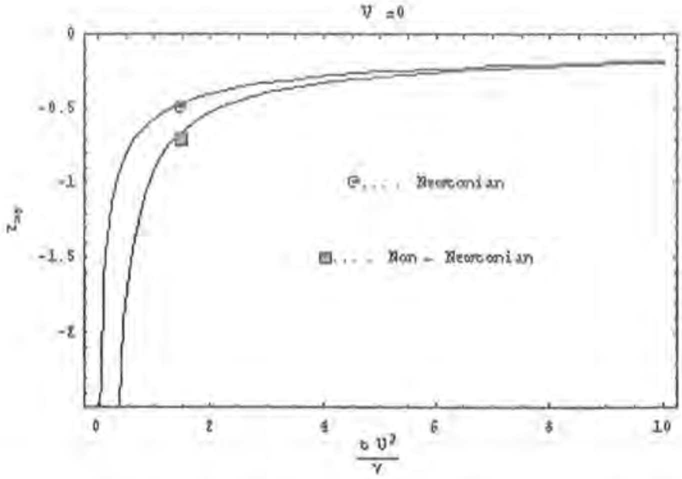

Fig. 6.4. Variation of shear stress at the plate for various values of time and for $V(t) = 0$.

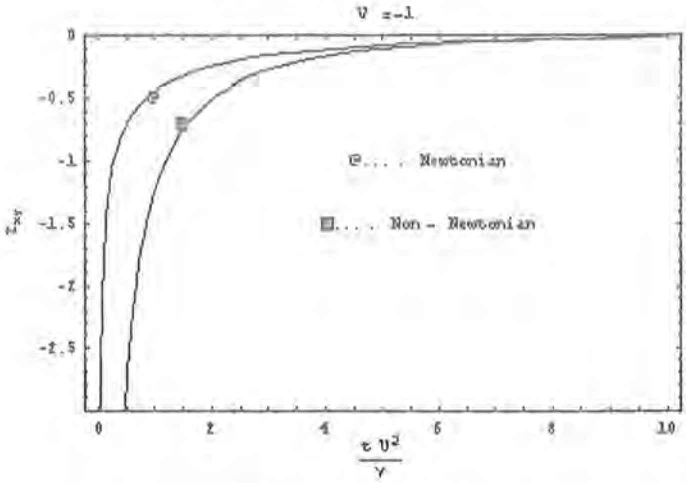

Fig. 6.5. Variation of shear stress at the plate for various values of time and for $V(t) = -1$.

6.5 Special cases

6.5.1 Case 1

At $\overline{V}(\tau) = 0$, $\overline{\beta} = 0$, $\overline{\gamma} = 0$ and $\overline{\varepsilon} = 0$, we obtain the familiar first Stokes' problem [51, 52] of a plate suddenly set into motion. The solution is given by

$$f(\eta) = U\left[1 - \operatorname{erf}(\eta^*)\right], \tag{6.35}$$

where

$$\eta^* = \frac{\xi}{2\sqrt{\tau}}. \tag{6.36}$$

6.5.2 Case 2

At $\overline{V}(\tau) = 0$, $\overline{\beta} \neq 0$, $\overline{\gamma} = 0$ and $\overline{\varepsilon} = 0$, we readily recover the result of Teipel [30], the impulsive motion of a flat plate in a viscoelastic fluid and the solution is given by

$$f = U\left[f_0(\eta) + \left(\frac{\beta}{vt}\right) f_1(\eta) + \left(\frac{\beta}{vt}\right)^2 f_2(\eta) + \cdots\right], \tag{6.37}$$

$$f_0(\eta)=\left(1-\frac{2}{\sqrt{\pi}}\int_0^{\eta}e^{-\xi^2}d\xi\right),\quad f_1(\eta)=\left(a_{01}\eta+\frac{1}{\sqrt{\pi}}\eta^3\right)e^{-\eta^2},$$

$$f_2(\eta)=\left[a_{02}\eta+\frac{5}{2}\left(\frac{1}{\sqrt{\pi}}-a_{01}-\frac{4}{15}a_{02}\right)\eta^3-\frac{1}{4}\left(\frac{11}{\sqrt{\pi}}-4a_{01}\right)\eta^5+\frac{1}{2\sqrt{\pi}}\eta^7\right]e^{-\eta^2},$$

where

$$a_{01}=-\frac{1}{2\sqrt{\pi}},\quad a_{02}=-\frac{3}{2}\frac{1}{\sqrt{\pi}}.$$

6.5.3 Case 3

For $\overline{V}(\tau)=\frac{KU}{2\sqrt{\tau}}$, $\overline{\beta}=0$, $\overline{\gamma}=0$ and $\overline{\varepsilon}=0$, we obtain the solution of the form [53]

$$f(\eta)=U\left[1-\frac{\operatorname{erf}\left(\frac{\xi}{2\sqrt{\tau}}-\frac{K}{2}\right)}{1+\operatorname{erf}\left(\frac{K}{2}\right)}\right]. \tag{6.38}$$

The graph is shown for $K=-2,0,1,2,4,6$ in Fig. 6. It is observed that for $K>23$, $f(\eta)$ is exactly one and for $K<-11$ it is no more real.

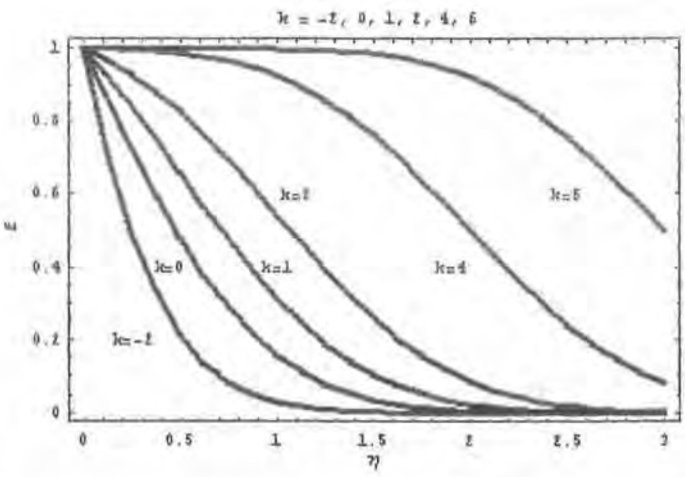

Fig. 6.6. Variation of the function f with η for different values of parameter K.

6.6 Concluding remarks

In this chapter, an analysis is made for the flow of a third grade fluid on the plate with variable suction. The following conclusions can be drawn from the present study.

1. It is found that with an increase in suction, the boundary layer thickness decreases and with an increase in blowing the boundary layer thickness increases.

2. From Eq. (6.12), it is again noted that for short time ($\tau = 4$) a strong non-Newtonian effect is present in the velocity field and velocity behaves as a Newtonian case for large time ($\tau = 100$).

3. Introduction of the similarity parameter η leads to an exact solution of the governing non-linear partial differential equation.

Chapter 7

Flow of a third grade fluid induced due to the oscillations of a porous plate

7.1 Introduction

This chapter describes the flow of a third-grade fluid on a porous plate which executes oscillations in its own plane with superimposed injection (blowing) or suction. The analysis also examines the behavior of an increasing or decreasing velocity amplitude of the oscillating porous plate. The non-linear problem has been solved using perturbation method. The obtained results are compared with those known from the literature. The result indicates that a combination of suction/injection and decreasing/increasing velocity amplitude is possible for a third-grade fluid.

7.2 Problem formulation

Here, we consider a thermodynamic compatible third grade fluid flow on a porous plate. We choose $x-$axis along and $y-$axis perpendicular to the plate. For $t > 0$, the plate starts oscillating. The governing equation for constant suction $V_0\,(< 0)$ can be obtained from Eq. (6.6) as

$$\frac{\partial u}{\partial t}+V_0\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}+\beta\left[V_0\frac{\partial^3 u}{\partial y^3}+\frac{\partial^3 u}{\partial y^2\partial t}\right]+\gamma\left[\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial u}{\partial y}\right)^2\right], \tag{7.1}$$

where $V_0>0$ is the blowing velocity. The above equation holds for a thermodynamic third grade fluid.

The expression for the shear stress is

$$\overline{\tau}_{xy}=\mu\frac{\partial u}{\partial y}+\alpha_1\left\{V_0\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial y\partial t}\right\}+2\beta_3\left(\frac{\partial u}{\partial y}\right)^3. \tag{7.2}$$

The boundary conditions are [54]

$$u(0,t)=U(t)=U_0e^{(\beta_0-i\omega)t},\quad \omega>0,\quad t>0,\quad \beta_0=\text{constant}\neq 0, \tag{7.3}$$

$$u(y,t)\rightarrow 0\quad \text{as}\quad y\rightarrow\infty, \tag{7.4}$$

where β_0 is the accelerating/decelerating parameter.

Defining the following non-dimensional parameters

$$\begin{aligned}&\eta=y\sqrt{\tfrac{\omega}{2\nu}},\quad d=\tfrac{V_0}{2\sqrt{\nu\omega}},\quad \tau=\omega t,\quad c=\tfrac{\beta_0}{\omega},\\ &f=\tfrac{u}{U_0},\quad \phi_1=\tfrac{\omega}{2\nu}\beta,\quad \phi_2=\tfrac{\omega U_0^2}{4\nu^2}\gamma\end{aligned} \tag{7.5}$$

equations (7.1) to (7.4) give

$$\frac{\partial f}{\partial \tau}+\sqrt{2}d\frac{\partial f}{\partial \eta}=\frac{1}{2}\frac{\partial^2 f}{\partial \eta^2}+\phi_1\left[\frac{\partial^3 f}{\partial \eta^2\partial \tau}+\sqrt{2}d\frac{\partial^3 f}{\partial \eta^3}\right]+\phi_2\frac{\partial^2 f}{\partial \eta^2}\left(\frac{\partial f}{\partial \eta}\right)^3, \tag{7.6}$$

$$f(0,\tau)=e^{(c-i)\tau}, \tag{7.7}$$

$$f\rightarrow 0\quad \text{as}\quad \eta\rightarrow\infty, \tag{7.8}$$

$$\tau_{xy}=\frac{1}{\sqrt{2\omega\nu\rho}U_0}\overline{\tau}_{xy}=\frac{1}{2}\frac{\partial f}{\partial \eta}+\phi_1\left[\frac{\partial^2 f}{\partial \eta\partial \tau}+\sqrt{2}d\frac{\partial^2 f}{\partial \eta^2}\right]+\frac{1}{2}\phi_2\left(\frac{\partial f}{\partial \eta}\right)^3. \tag{7.9}$$

7.3 Solution of the problem

We suppose that the non-dimensional velocity f can be expanded in power series in ϕ_2 as:

$$f(\eta,\tau;\phi_2) = f_0(\eta,\tau) + \phi_2 f_1(\eta,\tau) + \cdots. \tag{7.10}$$

On substituting Eq. (7.10) into Eqs. (7.6) to (7.9) and then equating the like powers of ϕ_2 we obtain the following systems:

Zeroth order system

$$\frac{\partial f_0}{\partial \tau} + \sqrt{2}d\frac{\partial f_0}{\partial \eta} = \frac{1}{2}\frac{\partial^2 f}{\partial \eta^2} + \phi_1\left[\frac{\partial^3 f_0}{\partial \eta^2 \partial \tau} + \sqrt{2}d\frac{\partial^3 f_0}{\partial \eta^3}\right], \tag{7.11}$$

$$f_0(0,\tau) = e^{(c-i)\tau}, \tag{7.12}$$

$$f_0 \to 0 \quad \text{as} \quad \eta \to \infty, \tag{7.13}$$

$$\tau_0 = \frac{1}{\sqrt{2\omega\nu\rho}U_0}\overline{\tau}_0 = \frac{1}{2}\frac{\partial f_0}{\partial \eta} + \phi_1\left[\frac{\partial^2 f_0}{\partial \eta \partial \tau} + \sqrt{2}d\frac{\partial^2 f_0}{\partial \eta^2}\right]. \tag{7.14}$$

First order system

$$\frac{\partial f_1}{\partial \tau} + \sqrt{2}d\frac{\partial f_1}{\partial \eta} = \frac{1}{2}\frac{\partial^2 f_1}{\partial \eta^2} + \phi_1\left[\frac{\partial^3 f_1}{\partial \eta^2 \partial \tau} + \sqrt{2}d\frac{\partial^3 f_1}{\partial \eta^3}\right] + \phi_2\frac{\partial^2 f_0}{\partial \eta^2}\left(\frac{\partial f_0}{\partial \eta}\right)^3, \tag{7.15}$$

$$f_1(0,\tau) = 0, \tag{7.16}$$

$$f_1 \to 0 \quad \text{as} \quad \eta \to \infty, \tag{7.17}$$

$$\tau_1 = \frac{1}{\sqrt{2\omega\nu\rho}U_0}\overline{\tau}_1 = \frac{1}{2}\frac{\partial f_1}{\partial \eta} + \phi_1\left[\frac{\partial^2 f_1}{\partial \eta \partial \tau} + \sqrt{2}d\frac{\partial^2 f_1}{\partial \eta^2}\right] + \frac{1}{2}\phi_2\left(\frac{\partial f_0}{\partial \eta}\right)^3. \tag{7.18}$$

Zeroth order solution

We write

$$f_0(\eta,\tau) = g_0(\eta)\, e^{(c-i)\tau}. \tag{7.19}$$

Making use of above equation, Eqs. (7.11) to (7.13) give

$$\phi_1 \frac{d^3 g_0}{d\eta^3} + \frac{1+2\phi_1(c-i)}{2\sqrt{2}d}\frac{d^2 g_0}{d\eta^2} - \frac{dg_0}{d\eta} - \frac{(c-i)}{\sqrt{2}d} g_0 = 0, \tag{7.20}$$

$$g_0(0) = 1, \tag{7.21}$$

$$g_0(\infty) = 0. \tag{7.22}$$

The solution of Eq. (7.20) along with conditions (7.21) and (7.22) is obtained by employing a procedure used by Foote *et al.* [20]. The solution of Eq. (7.20) subject to conditions (7.21) and (7.22) reads as

$$g_0(\eta) = e^{A\eta}. \tag{7.23}$$

The real part of the zeroth order solution is

$$f_0(\eta) = \exp\left[c\tau + \left(C_{0R} + C_{1R}\phi_1 + C_{2R}\phi_1^2\right)\eta\right]\cos\left[\tau - \left(C_{0I} + C_{1I}\phi_1 + C_{2I}\phi_1^2\right)\eta\right], \tag{7.24}$$

in which

$$\begin{aligned}
A &= \left(C_0 + C_1\phi_1 + C_2\phi_1^2\right), \; C_0 = \sqrt{2}\left(d - \sqrt{d^2 + c - i}\right), \; C_1 = \frac{\sqrt{2}dC_0^3 + (c-i)C_0^2}{\sqrt{2}d - C_0}, \\
C_2 &= \frac{C_1^2 + 6\sqrt{2}dC_0^2C_1 + 4(c-i)C_1C_0}{2\left(\sqrt{2}d - C_0\right)}, \; c_{0I} = \pm\sqrt{\sqrt{(d^2+c)^2+1} - d^2 - c}, \\
c_{0R} &= \sqrt{2}d \pm \sqrt{\sqrt{(d^2+c)^2+1} + (d^2+c)},
\end{aligned}$$

$$\begin{aligned}
C_{1R} &= \frac{P}{\sqrt{2}d - C_{0R}} - \frac{C_{0I}}{\sqrt{2}d - C_{0R}}\left[\frac{PC_{0I} + Q\left(\sqrt{2}d - C_{0R}\right)}{C_{0I}^2 + \left(\sqrt{2}d - C_{0R}\right)^2}\right], \\
C_{1I} &= \left[\frac{PC_{0I} + Q\left(\sqrt{2}d - C_{0R}\right)}{C_{0I}^2 + \left(\sqrt{2}d - C_{0R}\right)^2}\right], \; C_{2I} = \left[\frac{P_1C_{0I} + Q_1\left(\sqrt{2}d - C_{0R}\right)}{C_{0I}^2 + \left(\sqrt{2}d - C_{0R}\right)^2}\right],
\end{aligned}$$

$$
\begin{aligned}
C_{2R} &= \frac{P_1}{\sqrt{2}d - C_{0R}} - \frac{C_{0I}}{\sqrt{2}d - C_{0R}}\left[\frac{P_1 C_{0I} + Q_1\left(\sqrt{2}d - C_{0R}\right)}{C_{0I}^2 + \left(\sqrt{2}d - C_{0R}\right)^2}\right],\\
P &= \sqrt{2}d\left(C_{0R}^3 - 3C_{0R}C_{0I}^2\right) + c\left(C_{0R}^2 - C_{0I}^2\right) + 2C_{0R}C_{0I},\\
Q &= \sqrt{2}d\left(3C_{0R}^2 C_{0I} - C_{0I}^3\right) + 2cC_{0R}C_{0I} - \left(C_{0R}^2 - C_{0I}^2\right),\\
P_1 &= 3\sqrt{2}d\left(C_{0R}^2 C_{1R} - C_{0I}^2 C_{1R} - 2C_{0R}C_{0I}C_{1I}\right) + \frac{1}{2}\left(C_{1R}^2 - C_{1I}^2\right)\\
&\quad + 2c\left(C_{0R}C_{1R} - C_{0I}C_{1I}\right) + 2\left(C_{0R}C_{1I} + C_{0I}C_{1R}\right),\\
Q_1 &= 3\sqrt{2}d\left(2C_{0R}C_{0I}C_{1R} + C_{0R}^2 C_{1I} - C_{0I}^2 C_{1I}\right) + C_{1R}C_{1I}\\
&\quad + 2c\left(C_{0R}C_{1I} + C_{0I}C_{1R}\right) - 2\left(C_{0R}C_{1R} - C_{0I}C_{1I}\right).
\end{aligned}
$$

The expression for shear stress at the plate now is

$$
\tau_0 = \frac{1}{\sqrt{2\nu\omega}\rho U_0}\bar{\tau}_0 = Ae^{(c-i)\tau}\left[\frac{1}{2} + \phi_1\left\{\sqrt{2}dA + (c-i)\right\}\right]. \tag{7.25}
$$

First order solution

Using Eq. (7.24) into Eq. (7.15) we can write

$$
\frac{\partial f_1}{\partial \tau} + \sqrt{2}d\frac{\partial f_1}{\partial \eta} = \frac{1}{2}\frac{\partial^2 f_1}{\partial \eta^2} + \phi_1\left[\frac{\partial^3 f_1}{\partial \eta^2 \partial \tau} + \sqrt{2}d\frac{\partial^3 f_1}{\partial \eta^3}\right] + A^4 e^{3A\eta} e^{3(c-i)\tau}. \tag{7.26}
$$

We take the solution of above equation in the form

$$
f_1(\eta, \tau) = g_1(\eta)\, e^{3(c-i)\tau} \tag{7.27}
$$

and obtain

$$
\sqrt{2}d\phi_1 g_1''' + \frac{1 + 6\phi_1(c-i)}{2} g_1'' - \sqrt{2}d g_1' - 3(c-i) g_1 = -A^4 e^{3A\eta}. \tag{7.28}
$$

The corresponding boundary conditions are

$$g_1(0) = 0, \tag{7.29}$$

$$g_1(\infty) = 0. \tag{7.30}$$

The solution of Eq. (7.28) consists of complementary function and particular integral. The solution of Eq. (7.28) subject to conditions (7.29) and (7.30) is

$$g_1(\eta) = \frac{A^4}{A^*}\left(e^{B\eta} - e^{3A\eta}\right). \tag{7.31}$$

The real part of f_1 through Eq. (7.27) is

$$\begin{aligned} f_1(\eta,\tau) &= e^{(3c\tau+k_3\eta)}\left[q_1\cos\left(3\tau-k_4\eta\right)+q_2\sin\left(3\tau-k_4\eta\right)\right] \\ &\quad -e^{3(c\tau+k_1\eta)}\left[q_1\cos 3\left(\tau-k_2\eta\right)+q_2\sin 3\left(\tau-k_2\eta\right)\right], \end{aligned} \tag{7.32}$$

where

$$\begin{aligned} q_1 &= \frac{al+bn}{a^2+b^2},\ q_2=\frac{an-bl}{a^2+b^2},\ A^*=27A^3\overline{A}+9A^2\overline{B}+3A\overline{C}+\overline{D}, \\ \overline{A} &= \sqrt{2}d\phi_1,\ \overline{B}=\frac{1}{2}\left[1+6\phi_1\left(c-i\right)\right],\ \overline{C}=-\sqrt{2}d,\ \overline{D}=-3\left(c-i\right), \\ a &= \left[27\sqrt{2}d\phi_1\left(k_1^3-3k_1k_2^2\right)\right]+\frac{9}{2}\left[\left(k_1^2-k_2^2\right)\left(1+6\phi_1c\right)+12k_1k_2\phi_1\right]-3\sqrt{2}dk_1-3c, \\ b &= \left[27\sqrt{2}d\phi_1\left(3k_1^2k_2-k_2^3\right)\right]+\frac{9}{2}\left[2k_1k_2\left(1+6\phi_1c\right)-6\phi_1\left(k_1^2-k_2^2\right)\right]-3\sqrt{2}dk_2+3, \end{aligned}$$

$$\begin{aligned} l &= k_1^4+k_2^4-6k_1^2k_2^2,\ n=4k_1^3k_2-4k_1k_2^3,\ k_1=\left(C_{0R}+C_{1R}\phi_1+C_{2R}\phi_1^2\right), \\ k_2 &= \left(C_{0I}+C_{1I}\phi_1+C_{2I}\phi_1^2\right),\ k_3=\left(E_{0R}+E_{1R}\phi_1+E_{2R}\phi_1^2\right),\ k_4=\left(E_{0I}+E_{1I}\phi_1+E_{2I}\phi_1^2\right), \\ E_{0R} &= \sqrt{2}d-\sqrt{\sqrt{\left(d^2+3C\right)^2+9}+d^2+3C},\ E_{0I}=-\sqrt{\sqrt{\left(d^2+3C\right)^2+9}-d^2-3C}, \end{aligned}$$

$$
\begin{aligned}
E_{1R} &= \frac{P_2}{\sqrt{2}d - E_{0R}} - \frac{E_{0I}}{\sqrt{2}d - E_{0R}} \left[\frac{P_2 E_{0I} + Q_2 \left(\sqrt{2}d - E_{0R}\right)}{E_{0I}^2 + \left(\sqrt{2}d - E_{0R}\right)^2} \right], \\
E_{1I} &= \left[\frac{P_2 E_{0I} + Q_2 \left(\sqrt{2}d - E_{0R}\right)}{E_{0I}^2 + \left(\sqrt{2}d - E_{0R}\right)^2} \right], \; E_{2I} = \left[\frac{P_3 E_{0I} + Q_3 \left(\sqrt{2}d - E_{0R}\right)}{E_{0I}^2 + \left(\sqrt{2}d - E_{0R}\right)^2} \right], \\
E_{2R} &= \frac{P_3}{\sqrt{2}d - E_{0R}} - \frac{E_{0I}}{\sqrt{2}d - E_{0R}} \left[\frac{P_3 E_{0I} + Q_3 \left(\sqrt{2}d - E_{0R}\right)}{E_{0I}^2 + \left(\sqrt{2}d - E_{0R}\right)^2} \right],
\end{aligned}
$$

$$
\begin{aligned}
P_2 &= \sqrt{2}d \left(E_{0R}^3 - 3E_{0R}E_{0I}^2\right) + 3c\left(E_{0R}^2 - E_{0I}^2\right) + 6E_{0R}E_{0I}, \\
Q_2 &= \sqrt{2}d \left(3E_{0R}^2 E_{0I} - E_{0I}^3\right) + 6cE_{0R}E_{0I} - 3\left(E_{0R}^2 - E_{0I}^2\right), \\
P_3 &= 3\sqrt{2}d \left(E_{0R}^2 E_{1R} - E_{0I}^2 E_{1R} - 2E_{0R}E_{0I}E_{1I}\right) + \frac{1}{2}\left(E_{1R}^2 - E_{1I}^2\right) \\
&\quad + 6c\left(E_{0R}E_{1R} - E_{0I}E_{1I}\right) + 6\left(E_{0R}E_{1I} + E_{0I}E_{1R}\right), \\
Q_3 &= 3\sqrt{2}d \left(2E_{0R}E_{0I}E_{1R} + E_{0R}^2 E_{1I} - E_{0I}^2 E_{1I}\right) + E_{1R}E_{1I} \\
&\quad + 6c\left(E_{0R}E_{1I} + E_{0I}E_{1R}\right) - 6\left(E_{0R}E_{1R} - E_{0I}E_{1I}\right).
\end{aligned}
$$

Hence the velocity profile up to the first order is obtained by combining the zeroth order and first order solution in Eq. (7.10) as

$$
f(\eta, \tau) = \mathrm{Re}\left[e^{A\eta} e^{(c-i)\tau} + \phi_2 \left\{ \frac{A^4}{A^*} \left(e^{B\eta} - e^{3A\eta}\right) \right\} e^{3(c-i)\tau} \right]. \tag{7.33}
$$

We observe from Eq. (7.33) that this solution satisfies the boundary conditions given in Eqs. (7.3) and (7.4).

The non-dimensional stress at the plate ($\eta = 0$) is given by

$$
\begin{aligned}
\tau_1 &= \frac{1}{\sqrt{2\nu\omega}\rho U_0} \bar{\tau}_1 \qquad (7.34) \\
&= \frac{A^4 e^{3(c-i)\tau}}{A^*} \left[B\left\{ \frac{1}{2} + \phi_1 \left(\sqrt{2}dB + 3(c-i)\right) \right\} - A\left\{ \frac{3}{2} + 9\phi_1 (A + c - i) \right\} \right] + \frac{1}{3} A^3 e^{3(c-i)\tau}.
\end{aligned}
$$

7.4 Special cases

To understand the different physical aspects of the solution (7.33), we discuss some special cases:

7.4.1 Oscillating plate (Newtonian fluid with $c = d = 0$)

Stokes' second problem [55, 56] can be obtained by taking $c = d = \phi_1 = \phi_2 = 0$, i.e.,

$$f_{NS}(\eta,\tau) = \exp(-\eta)\cos(\tau - \eta), \tag{7.35}$$

where NS in the subscript stands for Navier-Stokes.

7.4.2 Oscillating plate (Newtonian fluid with $c = d \neq 0$)

New solutions of Stokes second problem [54] are recovered by taking $\phi_1 = \phi_2 = 0$ and $c = d \neq 0$ from solution (7.33) i.e.,

$$\begin{aligned} f_{NT}(\eta,\tau) &= \exp\left[c\tau + \left(\sqrt{2}d - \sqrt{\sqrt{(d^2+c)^2+1} + d^2 + c}\right)\eta\right] \\ &\quad \times \cos\left[\tau - \left(\sqrt{\sqrt{(d^2+c)^2+1} + d^2 + c}\right)\eta\right], \end{aligned} \tag{7.36}$$

where NT in the subscript indicates new solutions of Turbatu et al. [54].

7.4.3 Oscillating plate (Viscoelastic fluid with $c = d \neq 0$, $\phi_1 \neq 0$)

The results of viscoelastic second-grade fluid [25] are readily obtained by taking $\phi_2 = 0$ in the solution (7.33), that is

$$f_{VE}(\eta,\tau) = \exp\left[c\tau + \left(C_{0R} + C_{1R}\phi_1 + C_{2R}\phi_1^2\right)\eta\right] \times \cos\left[\tau - \left(C_{0I} + C_{1I}\phi_1 + C_{2I}\phi_1^2\right)\eta\right]. \tag{7.37}$$

Here VE stands for viscoelastic.

7.4.4 Oscillating porous plate (Third grade fluid with $c = 0$, $d \neq 0$, $\phi_1 \neq 0$, $\phi_2 \neq 0$)

For $c = 0$, $d \neq 0$, $\phi_1 \neq 0$ and $\phi_2 \neq 0$, solution (7.33) gives

$$\begin{aligned} f(\eta,\tau) &= \exp\left[\left(\widehat{C}_{0R}+\widehat{C}_{1R}\phi_1+\widehat{C}_{2R}\phi_1^2\right)\eta\right]\times\cos\left[\tau-\left(\widehat{C}_{0I}+\widehat{C}_{1I}\phi_1+\widehat{C}_{2I}\phi_1^2\right)\eta\right] \\ &\quad +\phi_2\{e^{\left(3c\tau+\widehat{k}_3\eta\right)}\left[\widehat{q}_1\cos\left(3\tau-\widehat{k}_4\eta\right)+\widehat{q}_2\sin\left(3\tau-\widehat{k}_4\eta\right)\right] \\ &\quad -e^{3\left(c\tau+\widehat{k}_1\eta\right)}\left[\widehat{q}_1\cos 3\left(\tau-\widehat{k}_2\eta\right)+\widehat{q}_2\sin 3\left(\tau-\widehat{k}_2\eta\right)\right]\}, \end{aligned} \tag{7.38}$$

where

$$\begin{aligned} \widehat{q}_1 &= \frac{\widehat{a}\widehat{l}+\widehat{b}\widehat{n}}{\widehat{a}^2+\widehat{b}^2},\ \widehat{q}_2=\frac{\widetilde{a}\widetilde{n}-\widetilde{b}\widetilde{l}}{\widehat{a}^2+\widehat{b}^2}, \\ \widehat{a} &= \left[27\sqrt{2}d\phi_1\left(\widehat{k}_1^3-3\widehat{k}_1\widehat{k}_2^2\right)\right]+\frac{9}{2}[\left(\widehat{k}_1^2-\widehat{k}_2^2\right)+12\widehat{k}_1\widehat{k}_2\phi_1]-3\sqrt{2}d\widehat{k}_1, \\ \widehat{b} &= \left[27\sqrt{2}d\phi_1\left(3\widehat{k}_1^2\widehat{k}_2-\widehat{k}_2^3\right)\right]+\frac{9}{2}[2\widehat{k}_1\widehat{k}_2-6\phi_1\left(\widehat{k}_1^2-\widehat{k}_2^2\right)]-3\sqrt{2}d\widehat{k}_2+3, \\ \widehat{l} &= \widehat{k}_1^4+\widehat{k}_2^4-6\widehat{k}_1^2\widehat{k}_2^2,\ \widehat{n}=4\widehat{k}_1^3\widehat{k}_2-4\widehat{k}_1\widehat{k}_2^3, \end{aligned}$$

$$\begin{aligned} \widehat{k}_1 &= \left(\widehat{C}_{0R}+\widehat{C}_{1R}\phi_1+\widehat{C}_{2R}\phi_1^2\right),\ \widehat{k}_2=\left(\widehat{C}_{0I}+\widehat{C}_{1I}\phi_1+\widehat{C}_{2I}\phi_1^2\right), \\ \widehat{k}_3 &= \left(\widehat{E}_{0R}+\widehat{E}_{1R}\phi_1+\widehat{E}_{2R}\phi_1^2\right),\ \widehat{k}_4=\left(\widehat{E}_{0I}+\widehat{E}_{1I}\phi_1+\widehat{E}_{2I}\phi_1^2\right), \\ \widehat{C}_{0R} &= \sqrt{2}d-\sqrt{\sqrt{d^4+1}+d^2},\ \widehat{C}_{0I}=-\sqrt{\sqrt{d^4+1}-d^2}, \end{aligned}$$

$$\begin{aligned}
\widehat{C}_{1R} &= \frac{\widehat{P}}{\sqrt{2}d-\widehat{C}_{0R}} - \frac{\widehat{C}_{0I}}{\sqrt{2}d-\widehat{C}_{0R}}\left[\frac{\widehat{P}\widehat{C}_{0I}+\widehat{Q}\left(\sqrt{2}d-\widehat{C}_{0R}\right)}{\widehat{C}_{0I}^2+\left(\sqrt{2}d-\widehat{C}_{0R}\right)^2}\right],\\
\widehat{C}_{1I} &= \left[\frac{\widehat{P}\widehat{C}_{0I}+\widehat{Q}\left(\sqrt{2}d-\widehat{C}_{0R}\right)}{\widehat{C}_{0I}^2+\left(\sqrt{2}d-\widehat{C}_{0R}\right)^2}\right],\ \widehat{C}_{2I} = \left[\frac{\widehat{P}_1\widehat{C}_{0I}+\widehat{Q}_1\left(\sqrt{2}d-\widehat{C}_{0R}\right)}{\widehat{C}_{0I}^2+\left(\sqrt{2}d-\widehat{C}_{0R}\right)^2}\right],\\
\widehat{C}_{2R} &= \frac{\widehat{P}_1}{\sqrt{2}d-\widehat{C}_{0R}} - \frac{\widehat{C}_{0I}}{\sqrt{2}d-\widehat{C}_{0R}}\left[\frac{\widehat{P}_1\widehat{C}_{0I}+\widehat{Q}_1\left(\sqrt{2}d-\widehat{C}_{0R}\right)}{\widehat{C}_{0I}^2+\left(\sqrt{2}d-\widehat{C}_{0R}\right)^2}\right],
\end{aligned}$$

$$\begin{aligned}
\widehat{P} &= \sqrt{2}d\left(\widehat{C}_{0R}^3-3\widehat{C}_{0R}^2\widehat{C}_{0I}^2\right)+2\widehat{C}_{0R}\widehat{C}_{0I},\ \widehat{Q}=\sqrt{2}d\left(3\widehat{C}_{0R}^2\widehat{C}_{0I}-\widehat{C}_{0I}^3\right)-\left(\widehat{C}_{0R}^2-\widehat{C}_{0I}^2\right),\\
\widehat{P}_1 &= 3\sqrt{2}d\left(\widehat{C}_{0R}^2\,\widehat{C}_{1R}-\widehat{C}_{0I}^2\,\widehat{C}_{1R}-2\widehat{C}_{0R}\,\widehat{C}_{0I}\,\widehat{C}_{1I}\right)+\frac{1}{2}\left(\widehat{C}_{1R}^2-\widehat{C}_{1I}^2\right)+2\left(\widehat{C}_{0R}\,\widehat{C}_{1I}+\widehat{C}_{0I}\,\widehat{C}_{1R}\right),\\
\widehat{Q}_1 &= 3\sqrt{2}d\left(2\widehat{C}_{0R}\,\widehat{C}_{0I}\,\widehat{C}_{1R}+\widehat{C}_{0R}^2\,\widehat{C}_{1I}-\widehat{C}_{0I}^2\,\widehat{C}_{1I}\right)+\widehat{C}_{1R}\,\widehat{C}_{1I}-2\left(\widehat{C}_{0R}\,\widehat{C}_{1R}-\widehat{C}_{0I}\,\widehat{C}_{1I}\right),\\
\widehat{E}_{0R} &= \sqrt{2}d-\sqrt{\sqrt{d^4+9}+d^2},\ \widehat{E}_{0I}=-\sqrt{\sqrt{d^4+9}-d^2},
\end{aligned}$$

$$\begin{aligned}
\widehat{E}_{1R} &= \frac{\widehat{P}_2}{\sqrt{2}d-\widehat{E}_{0R}} - \frac{\widehat{E}_{0I}}{\sqrt{2}d-\widehat{E}_{0R}}\left[\frac{\widehat{P}_2\widehat{E}_{0I}+\widehat{Q}_2\left(\sqrt{2}d-\widehat{E}_{0R}\right)}{\widehat{E}_{0I}^2+\left(\sqrt{2}d-\widehat{E}_{0R}\right)^2}\right],\\
\widehat{E}_{1I} &= \left[\frac{\widehat{P}_2\widehat{E}_{0I}+\widehat{Q}_2\left(\sqrt{2}d-\widehat{E}_{0R}\right)}{\widehat{E}_{0I}^2+\left(\sqrt{2}d-\widehat{E}_{0R}\right)^2}\right],\ \widehat{E}_{2I} = \left[\frac{\widehat{P}_3\widehat{E}_{0I}+\widehat{Q}_3\left(\sqrt{2}d-\widehat{E}_{0R}\right)}{\widehat{E}_{0I}^2+\left(\sqrt{2}d-\widehat{E}_{0R}\right)^2}\right],\\
\widehat{E}_{2R} &= \frac{\widehat{P}_3}{\sqrt{2}d-\widehat{E}_{0R}} - \frac{\widehat{E}_{0I}}{\sqrt{2}d-\widehat{E}_{0R}}\left[\frac{\widehat{P}_3\widehat{E}_{0I}+\widehat{Q}_3\left(\sqrt{2}d-\widehat{E}_{0R}\right)}{\widehat{E}_{0I}^2+\left(\sqrt{2}d-\widehat{E}_{0R}\right)^2}\right],
\end{aligned}$$

$$\begin{aligned}
\widehat{P}_2 &= \sqrt{2}d\left(\widehat{E}_{0R}^3-3\widehat{E}_{0R}\widehat{E}_{0I}^2\right)+6\widehat{E}_{0R}\widehat{E}_{0I},\ \widehat{Q}_2=\sqrt{2}d\left(3\widehat{E}_{0R}^2\widehat{E}_{0I}-\widehat{E}_{0I}^3\right)-3\left(\widehat{E}_{0R}^2-\widehat{E}_{0I}^2\right),\\
\widehat{P}_3 &= 3\sqrt{2}d\left(\widehat{E}_{0R}^2\widehat{E}_{1R}-\widehat{E}_{0I}^2\widehat{E}_{1R}-2\widehat{E}_{0R}\widehat{E}_{0I}\widehat{E}_{1I}\right)+\frac{1}{2}\left(\widehat{E}_{1R}^2-\widehat{E}_{1I}^2\right)+6\left(\widehat{E}_{0R}\widehat{E}_{1I}+\widehat{E}_{0I}\widehat{E}_{1R}\right),\\
\widehat{Q}_3 &= 3\sqrt{2}d\left(2\widehat{E}_{0R}\widehat{E}_{0I}\widehat{E}_{1R}+\widehat{E}_{0R}^2\widehat{E}_{1I}-\widehat{E}_{0I}^2\widehat{E}_{1I}\right)+\widehat{E}_{1R}\widehat{E}_{1I}-6\left(\widehat{E}_{0R}\widehat{E}_{1R}-\widehat{E}_{0I}\widehat{E}_{1I}\right).
\end{aligned}$$

The solution for the velocity component f is plotted in Figs. 7.1 and 7.2 for different values of ϕ_1 and ϕ_2 and for a fixed time $\tau = 2\pi$ as a function of the suction/blowing velocity V_0, given by $d = \frac{V_0}{2\sqrt{\nu\omega}}$. The values $d = 0$, $\phi_1 = 0$ and $\phi_2 = 0$ refer to the classical Stokes problem. It is noted that the boundary layer thickness is controlled by the suction velocity ($V_0 < 0$) i.e., it decreases with an increase in the suction velocity.

In case of blowing ($V_0 > 0$), the boundary layer thickness becomes large as is expected physically.

Figure 7.2 gives the effect of material parameter of third grade fluid. It is observed that with an increase in third grade parameter ϕ_2, the boundary layer thickness rapidly decreases in the case of suction ($V_0 < 0$) and increases in the case of blowing ($V_0 > 0$), when compared with the viscoelastic case [25] and viscous case [54].

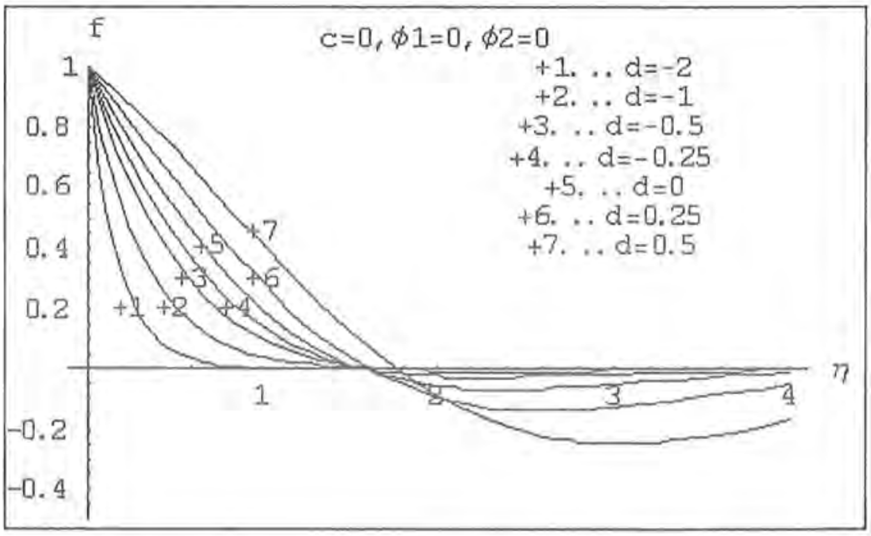

Fig. 7.1. Influence of suction/blowing on the velocity distribution at $\tau = 2\pi$

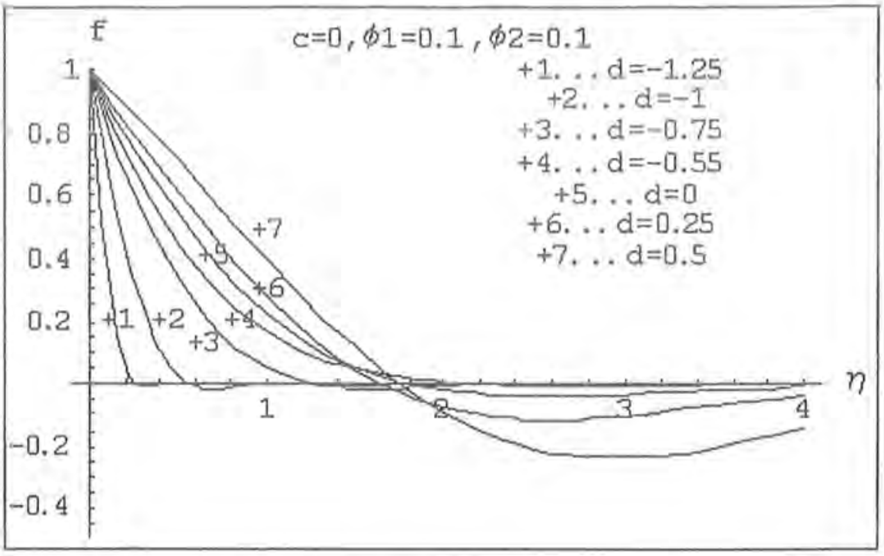

Fig. 7.2. Influence of suction/blowing on the velocity distribution at $\tau = 2\pi$

7.4.5 Oscillating plate with acceleration/deceleration (Third grade fluid with $d = 0$, $c \neq 0$, $\phi_1 \neq 0$, $\phi_2 \neq 0$)

In this section, the superposition of two time dependent functions is taken into account. One of which is due to the oscillation of the plate with imposed frequency ω and the second is an exponential increase or decrease of the velocity amplitude of the plate with the parameter β_0.

For $d = 0$, $c \neq 0$, $\phi_1 \neq 0$ and $\phi_2 \neq 0$, solution (7.33) takes the following form

$$\begin{aligned} f(\eta,\tau) &= \exp\left[c\tau + \left(\widetilde{C}_{0R} + \widetilde{C}_{1R}\phi_1 + \widetilde{C}_{2R}\phi_1^2\right)\eta\right] \times \cos\left[\tau - \left(\widetilde{C}_{0I} + \widetilde{C}_{1I}\phi_1 + \widetilde{C}_{2I}\phi_1^2\right)\eta\right] \\ &\quad +\phi_2\{e^{(3c\tau+\widetilde{k}_3\eta)}\left[\widetilde{q}_1\cos\left(3\tau - \widetilde{k}_4\eta\right) + \widetilde{q}_2\sin\left(3\tau - \widetilde{k}_4\eta\right)\right] \\ &\quad -e^{3(c\tau+\widetilde{k}_1\eta)}\left[\widetilde{q}_1\cos 3\left(\tau - \widetilde{k}_2\eta\right) + \widetilde{q}_2\sin 3\left(\tau - \widetilde{k}_2\eta\right)\right]\}, \end{aligned}$$

or

$$\begin{aligned}
g(\eta,\tau) &= \frac{f(\eta,\tau)}{\exp(c\tau)} = \exp\left[\left(\widetilde{C}_{0R}+\widetilde{C}_{1R}\phi_1+\widetilde{C}_{2R}\phi_1^2\right)\eta\right]\times\cos\left[\tau-\left(\widetilde{C}_{0I}+\widetilde{C}_{1I}\phi_1+\widetilde{C}_{2I}\phi_1^2\right)\eta\right] \\
&+\phi_2\{e^{\left(2c\tau+\widetilde{k}_3\eta\right)}\left[\widetilde{q}_1\cos\left(3\tau-\widetilde{k}_4\eta\right)+\widetilde{q}_2\sin\left(3\tau-\widetilde{k}_4\eta\right)\right] \qquad (7.39)\\
&-e^{\left(2c\tau+3\widetilde{k}_1\eta\right)}\left[\widetilde{q}_1\cos 3\left(\tau-\widetilde{k}_2\eta\right)+\widetilde{q}_2\sin 3\left(\tau-\widetilde{k}_2\eta\right)\right]\},
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{q}_1 &= \frac{\widetilde{a}\widetilde{l}+\widetilde{b}\widetilde{n}}{\widetilde{a}^2+\widetilde{b}^2},\ \widetilde{q}_2=\frac{\widetilde{a}\widetilde{n}-\widetilde{b}\widetilde{l}}{\widetilde{a}^2+\widetilde{b}^2},\ \widetilde{a}=\frac{9}{2}[\left(\widetilde{k}_1^2-\widetilde{k}_2^2\right)(1+6\phi_1 c)+12\widetilde{k}_1\widetilde{k}_2\phi_1]-3c,\\
\widetilde{b} &= \frac{9}{2}[2\widetilde{k}_1\widetilde{k}_2(1+6\phi_1 c)-6\phi_1\left(\widetilde{k}_1^2-\widetilde{k}_2^2\right)]+3,\ \widetilde{l}=\widetilde{k}_1^4+\widetilde{k}_2^4-6\widetilde{k}_1^2\widetilde{k}_2^2,\ \widetilde{n}=4\widetilde{k}_1^3\widetilde{k}_2-4\widetilde{k}_1\widetilde{k}_2^3,\\
\widetilde{k}_1 &= \left(\widetilde{C}_{0R}+\widetilde{C}_{1R}\phi_1+\widetilde{C}_{2R}\phi_1^2\right),\ \widetilde{k}_2=\left(\widetilde{C}_{0I}+\widetilde{C}_{1I}\phi_1+\widetilde{C}_{2I}\phi_1^2\right),\ \widetilde{C}_{0R}=-\sqrt{\sqrt{c^2+1}+c},\\
\widetilde{k}_3 &= \left(\widetilde{E}_{0R}+\widetilde{E}_{1R}\phi_1+\widetilde{E}_{2R}\phi_1^2\right),\ \widetilde{k}_4=\left(\widetilde{E}_{0I}+\widetilde{E}_{1I}\phi_1+\widetilde{E}_{2I}\phi_1^2\right),\ \widetilde{C}_{0I}=-\sqrt{\sqrt{c^2+1}-c},
\end{aligned}$$

$$\begin{aligned}
\widetilde{C}_{1R} &= \frac{\widetilde{P}}{-\widetilde{C}_{0R}}-\frac{\widetilde{C}_{0I}}{-\widetilde{C}_{0R}}\left[\frac{\widetilde{P}\widetilde{C}_{0I}-\widetilde{Q}\widetilde{C}_{0R}}{\widetilde{C}_{0I}^2-\widetilde{C}_{0R}^2}\right],\ \widetilde{C}_{1I}=\left[\frac{\widetilde{P}\widetilde{C}_{0I}-\widetilde{Q}\widetilde{C}_{0R}}{\widetilde{C}_{0I}^2-\widetilde{C}_{0R}^2}\right],\ \widetilde{E}_{0R}=-\sqrt{\sqrt{9C^2+9}+3C},\\
\widetilde{C}_{2R} &= \frac{\widetilde{P}_1}{-\widetilde{C}_{0R}}-\frac{\widetilde{C}_{0I}}{-\widetilde{C}_{0R}}\left[\frac{\widetilde{P}_1\widetilde{C}_{0I}-\widetilde{Q}_1\widetilde{C}_{0R}}{\widetilde{C}_{0I}^2-\widetilde{C}_{0R}^2}\right],\ \widetilde{C}_{2I}=\left[\frac{\widetilde{P}_1\widetilde{C}_{0I}-\widetilde{Q}_1\widetilde{C}_{0R}}{\widetilde{C}_{0I}^2-\widetilde{C}_{0R}^2}\right],\ \widetilde{E}_{0I}=-\sqrt{\sqrt{9C^2+9}-3C},
\end{aligned}$$

$$\begin{aligned}
\widetilde{E}_{1R} &= \frac{\widetilde{P}_2}{-\widetilde{E}_{0R}}-\frac{\widetilde{E}_{0I}}{-\widetilde{E}_{0R}}\left[\frac{\widetilde{P}_2\widetilde{E}_{0I}-\widetilde{Q}_2\widetilde{E}_{0R}}{\widetilde{E}_{0I}^2-\widetilde{E}_{0R}^2}\right],\ \widetilde{E}_{2R}=\frac{\widetilde{P}_3}{-\widetilde{E}_{0R}}-\frac{\widetilde{E}_{0I}}{-\widetilde{E}_{0R}}\left[\frac{\widetilde{P}_3\widetilde{E}_{0I}-\widetilde{Q}_3\widetilde{E}_{0R}}{\widetilde{E}_{0I}^2-\widetilde{E}_{0R}^2}\right],\\
\widetilde{P} &= c\left(\widetilde{C}_{0R}^2-\widetilde{C}_{0I}^2\right)+2\widetilde{C}_{0R}\widetilde{C}_{0I},\ \widetilde{Q}=2c\widetilde{C}_{0R}\widetilde{C}_{0I}-\left(\widetilde{C}_{0R}^2-\widetilde{C}_{0I}^2\right),\\
\widetilde{P}_1 &= \frac{1}{2}\left(\widetilde{C}_{1R}^2-\widetilde{C}_{1I}^2\right)+2c\left(\widetilde{C}_{0R}\,\widetilde{C}_{1R}-\widetilde{C}_{0I}\,\widetilde{C}_{1I}\right)+2\left(\widetilde{C}_{0R}\,\widetilde{C}_{1I}+\widetilde{C}_{0I}\,\widetilde{C}_{1R}\right),\\
\widetilde{Q}_1 &= \widetilde{C}_{1R}\,\widetilde{C}_{1I}+2c\left(\widetilde{C}_{0R}\,\widetilde{C}_{1I}+\widetilde{C}_{0I}\,\widetilde{C}_{1R}\right)-2\left(\widetilde{C}_{0R}\,\widetilde{C}_{1R}-\widetilde{C}_{0I}\,\widetilde{C}_{1I}\right),
\end{aligned}$$

$$\widetilde{P}_2 = 3c\left(E_{0R}^2 - E_{0I}^2\right) + 6E_{0R}E_{0I},\ \widetilde{Q}_2 = 6cE_{0R}E_{0I} - 3\left(E_{0R}^2 - E_{0I}^2\right),\ \widetilde{E}_{1I} = \left[\frac{\widetilde{P}_2\widetilde{E}_{0I} - \widetilde{Q}_2\widetilde{E}_{0R}}{\widetilde{E}_{0I}^2 - \widetilde{E}_{0R}^2}\right],$$

$$\widetilde{P}_3 = \frac{1}{2}\left(E_{1R}^2 - E_{1I}^2\right) + 6c\left(E_{0R}E_{1R} - E_{0I}E_{1I}\right) + 6\left(E_{0R}E_{1I} + E_{0I}E_{1R}\right),$$

$$\widetilde{Q}_3 = E_{1R}E_{1I} + 6c\left(E_{0R}E_{1I} + E_{0I}E_{1R}\right) - 6\left(E_{0R}E_{1R} - E_{0I}E_{1I}\right),\ \widetilde{E}_{2I} = \left[\frac{\widetilde{P}_3\widetilde{E}_{0I} - \widetilde{Q}_3\widetilde{E}_{0R}}{\widetilde{E}_{0I}^2 - \widetilde{E}_{0R}^2}\right].$$

The parameter $c = \beta_0/\omega$ gives the variation of the amplitude of the plate velocity and $c = 0$, $\phi_1 = 0$ and $\phi_2 = 0$ implies the classical viscous case. The solution (7.39) is plotted in Figs. 7.3 and 7.4 for $(\tau = 2\pi,\ \phi_1 = 0,\ \phi_2 = 0)$and $(\tau = 2\pi,\ \phi_1 = 0.1,\ \phi_2 = 0.1)$, respectively. Figs. 7.3 and 7.4 show the variation of β_0, ϕ_1 and ϕ_2. It is noted that with an increase in third grade parameter ϕ_2 the amplitude of the oscillations rapidly increases/decreases according to $\beta_0 > 0/\beta_0 < 0$.

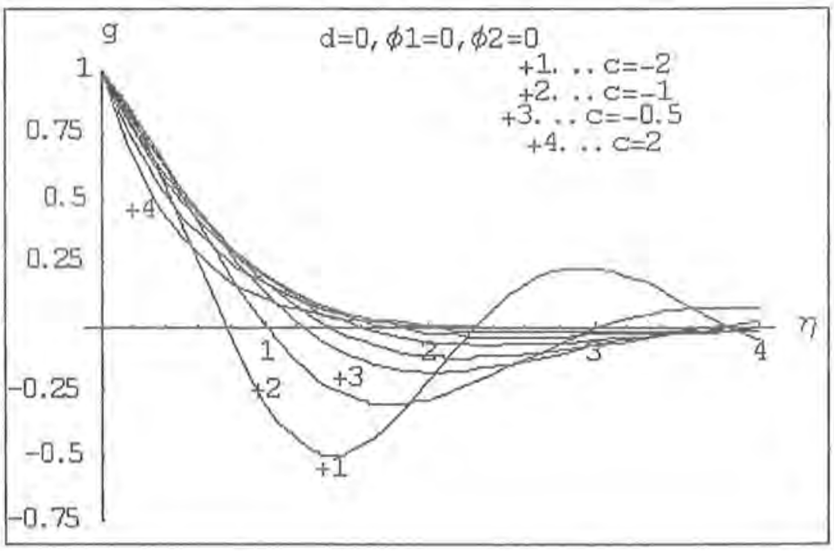

Fig. 7.3. Influence of increasing or decreasing amplitude of the plate on the normalized velocity distribution at $\tau = 2\pi$.

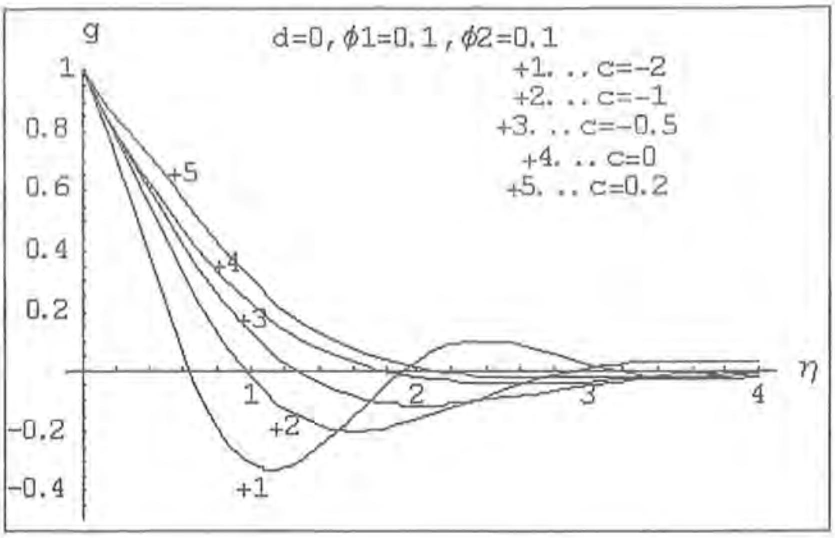

Fig. 7.4. Influence of increasing or decreasing amplitude of the plate on the normalized velocity distribution at $\tau = 2\pi$.

7.5 Conclusions

We have presented here results for the flow field of a fluid, which is called the third order fluid or the fluid of grade three, on an oscillating plate with superimposed blowing or suction. The analysis presented is further concerned with an increasing or decreasing velocity amplitude of the oscillating plate. It is noted that suction causes reduction in the boundary layer thickness as expected. Also, the amplitude of the oscillation decreases for acceleration and increases for deceleration when there is an increase in the material parameters of the second and third grade fluids. In addition it is also found that the results in references [54] and [25] can be recovered as the special cases of the problem considered by taking the parameters ϕ_1 and ϕ_2 equal to zero and ϕ_2 to be zero, respectively. This provides a useful mathematical check.

Chapter 8

Concluding remarks of the thesis

In this dissertation, the analytical solutions of non-linear equations governing the flow for a second-grade and third-grade fluids are obtained.

1. For second grade fluid two dimensional unsteady equations are derived in Cartesian, Plane-Polar, Axisymmetric Cylindrical in terms of swirl, and Axisymmetric Spherical Coordinates. Equations then are coupled in terms of the stream function so-called the compatibility equations. Several different forms of the stream function are taken. In each problem of stream function, the various possibilities of getting the analytical solutions are discussed. The expressions for velocity profile, streamline and pressure distribution are constructed in each case.

2. The present second grade models as well as solutions are more general and several results of various authors Aristov and Gitman [40], Berker [41], Riabouchinsky [42], Lakshmana [43], Roy [44], Siddiqui *et al.* [28], Goldstein [51], Jaffery-Hamel (by Squire) [46], Jahnke *et al.* [53] and Landau and Liftshitz [47] .can be recovered in the limiting cases.

3. For a third grade fluid since the equations are much more complicated, so only the uni-directional flows are considered in three different situations:

4. In the first case flow is generated due to a variable shear stress and our investigation shows that the perturbation technique is adequate for the case when the variable shear stress has as oscillatory character, however, if the shear stress grows exponentially with

time then the perturbation solution can be accepted only for small values of time. For moderate to large values of time, the numerical solution must be used.

5. In the second case the concept of variable suction is used when all the three third grade material parameters are non-zero and the introduction of the similarity parameter leads to the solution. It is found that with an increase in suction, the boundary layer thickness decreases and with an increase in blowing the boundary layer thickness increases. Furhter, it is noted that for short time ($\tau = 4$) a strong non-Newtonian effect is present in the velocity field and velocity behaves as a Newtonian case for large time ($\tau = 100$).

6. Finally, in the third case the third grade thermodynamic model is considered with superimposed blowing or suction and with an increasing or decreasing velocity amplitude of the oscillating plate. It is noted that suction causes reduction in the boundary layer thickness as expected. Also, the amplitude of the oscillation decreases for acceleration and increases for deceleration when there is an increase in the material parameters of the second and third grade fluids. In addition it is also found that the results of viscous and second grade fluid are recovered as a special cases of the present analysis.

7. The results of Stokes I & II problem [55], Teipel [30], Erdogan [50], Turbatu *et al.* [54], Hayat *et al.* [25], can be recovered as special cases of the present analysis of third grade study.

Bibliography

[1] R.S. Rivlin, J.L. Ericksen, Stress deformation relations for isotropic materials, J. Rat. Mech.impass Anal. **3**, 323 (1955).

[2] R.L. Fosdick, K.R. Rajagopal, Thermodynamic and stability of fluids of third grade, Proc. Roy. Soc. Lond. **A339**, 351 (1980).

[3] J.E. Dunn, K.R. Rajagopal, Fluids of differential type, critical review and thermodynamic analysis, Int. J. Eng. Sci. **33**, 689 (1995).

[4] C. Truesdell, W. Noll, *The non-linear field theories of mechanics*, in: W. Flugge (Ed.), Handbuch der Physik, Vol. III/3, Springer, Berlin, 1965.

[5] K.R. Rajagopal, On the stability of third grade fluids, Arch. Rat. Mech. Anal. **32**, 867 (1980).

[6] W.R. Schowalter, *Mechanics of non-Newtonian fluids*, Oxford, Pergamon, 1978.

[7] R.R. Huilgol, *Continuum mechanics of viscoelastic liquids*, Delhi, Hindustan publishing corporation, 1975.

[8] K.R. Rajagopal, Mechanics of non-Newtonian fluids. In: *Recent developments in theoretical fluid mechanics* (G.P. Gladi, J. Necas, eds.) Pitman Res. Notes math. Ser. **291**, 129 (1993).

[9] K.R. Rajagopal, On boundary conditions for fluids of differential type, in: Sequira (Eds.), *Navier-Stokes Equations and Related Nonlinear Problems*, Plenum Press, New York, 1995.

[10] A.C. Srivastava, The flow of a non-Newtonian liquid near a stagnation point. Z. Angew. Math. Phys. **9**, 80 (1958).

[11] D.W. Beard, K. Walters, Elastic-viscous boundary layer flow. Proc. Camb. Phil. Soc. **60**, 667 (1964).

[12] J. Astill, R.S. Jones, P. Lockeyer, Bounday layers in non-Newtonian fluids. J. Mecanique. **12**, 527 (1973).

[13] K.R. Frater, A boundary layer in an elastico-viscous fluid. Z. Angew. Math. Phys. **20**, 712 (1969).

[14] G.K. Rajeswari, S.L. Rathna, Flow of a particular class of non-Newtonian viscoelastic and visco-inelastic fluids near a stagnation point, Z. Angew. Math. Phys. **13**, 43 (1962).

[15] M.E. Erdogan, On the flow of a non-Newtonian fluid past a porous flat plate. *ZAMM*, **55**, 128 (1975).

[16] K.R. Rajagopal, A.S. Gupta, T.Y. Na, A note on the Falkner-Skan flows of a non-Newtonian fluid, Int. J. Non-Linear Mech. **17**, 113 (1983).

[17] D. Mansutti, G. Pontrelli, K.R. Rajagopal, Steady flows of non-Newtonian fluids past a porous plate with suction or injection, Int. J. Numerical Methods in fluids. **17**, 927 (1993).

[18] K.R. Rajagopal, A.S. Gupta, An exact solution for the flow of a non-Newtonian fluid past an infinite porous plate, Meccanica. **19**, 158 (1984).

[19] K.R. Rajagopal, A note on Unsteady unidirectional flows of a non-Newtonian fluid, Int. J. Non-Linear Mech. **17**, 373 (1982).

[20] J. R. Foote, P. Puri, P.K. Kythe, Some exact solutions of the Stokes problem for a elastico-viscous fluid, Acta Mech. **68**, 223 (1987).

[21] T. Hayat, S. Asghar, A. M. Siddiqui, Periodic unsteady flows of a non-Newtonian fluid, Acta Mech. **131**, 169 (1998).

[22] T. Hayat, S. Asghar, A. M. Siddiqui, On the moment of a plane disk in a non-Newtonian fluid, Acta Mech. **136**, 125 (1999).

[23] T. Hayat, Muhammad R. Mohyuddin, S. Asghar, Note on non-Newtonian flow due to a variable shear stress, Far East Journal of Applied Mathematics. **18**, 313 (2005).

[24] T. Hayat, Muhammad R. Mohyuddin, S. Asghar, Some inverse solutions unsteady flows of a non-Newtonian fluid, Tamsui Oxford Journal of Mathematics **21**, 1 (2005).

[25] T. Hayat, M.R. Mohyuddin, S. Asghar, A.M. Siddiqui, The flow of a viscoelastic fluid on an oscillating plate, ZAMM, Z. Angew Math. Mech. **84**, 1 (2004).

[26] S. Asghar, Muhammad R. Mohyuddin, P.D. Ariel, T. Hayat, On Stokes' problem for flow of a third grade fluid induced by a variable shear stress. Canadian J of Physics (In press).

[27] A.M. Siddiqui, M.R. Mohyuddin, T. Hayat, S. Asghar, Some more inverse solutions for steady flows of a second-grade fluid, Arch. Mech. **55**, 371 (2003).

[28] A.M. Siddiqui, P.N. Kaloni, Certain inverse solutions of a non-Newtonian fluid. Int. J. Non-linear Mech. **21**, 459 (1986).

[29] A.M. Siddiqui, Some more inverse solutions of a non-Newtonian fluid, Mech. Res. Commun. **17**, 157 (1990).

[30] I. Teipel, The impulsive motion of a flat plate in a viscoelastic fluid, Acta Mech. **39**, 277 (1981).

[31] R. Bandelli, K.R. Rajagopal, G.P. Galdi, On some unsteady motions of fluids of second-grade, Arch. Mech. **47**, 661 (1995).

[32] K.R. Rajagopal, On the decay of vortices in a second-grade fluid, Meccanica **9**, 185 (1980).

[33] K.R. Rajagopal, A.S. Gupta, On a class of exact solutions to the equations of motion of a second grade fluid, Int. J. Eng. Sci. **19**, 1009 (1981).

[34] F. Labropulu, A few more exact solutions of a second-grade fluid past a wedge, Mech. Res. Commun. **27**, 713 (2000).

[35] V.K. Garg, K.R. Rajagopal, Flow of a non-Newtonian fluid past a wedge, Acta Mech. **88**, 113 (1991).

[36] K.R. Rajagopal, On the creeping flow of second order fluid, J. Non-Newtonian Fluid Mech. **15**, 239 (1984).

[37] P.N. Nemenyi, *Recent developments in inverse and semi-inverse methods in the mechanics of continua.* Advances in applied mechanics, 2, New York 1951.

[38] P.N. Kaloni, K. Huschilt, Semi inverse solutions of a non-Newtonian fluid, Int. J. Non-linear Mech. **19**, 373 (1984).

[39] J.E. Dunn, R.L. Fosdick, Thermodynamics, stability and boundedness of fluids of complexity 2 and fluids of second grade, Arch. Rat. Mech. Anal. **56**, 191 (1974).

[40] S.N. Aristov, I.M. Gitman, Viscous flow between two moving parallel disks: exact solutions and stability analysis, J. Fluid Mech. **464**, 209, (2002).

[41] R. Berker, *Integration des equations du mouvemont d'un fluide visqueux incompressible*, Handbuk der Physik *VII*. Springer, Berlin 1963.

[42] D. Riabouchinsky, Some considerations regarding plane irrotational motion of a liquid C.R.S. Acad., Paris **179**, 1133 (1936).

[43] S. K. Lakshamana Rao, Some special solutions in viscous fluid motion, Proceedings of the Royal Irish Academy, 11 (1961).

[44] Durga Roy, 'A special solution in viscous fluid motion', Minutes of proceedings, R. Irish Acad. **16** (1960).

[45] E. Momoniat, F.M. Mahomed, The existence of contact transformations for evolution-type equations, J. Phys. A: Math & Gen. **32**, 8721 (1999).

[46] H.B. Squire, The round laminar jet, Q. J Appl. Math. Mech. **69**, 335 (1979).

[47] L.D. Landau, E.M. Lifshitz, *Fluid Mechanics*, Pergamon Press, London 1959.

[48] P.D. Ariel, Transient flow of a third grade fluid over a moving plate, Computer methods in Engng. (forthcoming).

[49] K.R.. Rajagopal, T.Y. Na, On Stokes problem for a non-Newtonian fluid, Acta Mech. **48**, 233 (1983).

[50] M.E. Erdogan, Plane surface suddenly set into motion in a non-Newtonian fluid, Acta Mech. **108**, 179 (1995).

[51] Goldstein (ed.), *Modern Developments in Fluid Dynamics*, (Oxford), **1**, 1938.

[52] H. Schlichting, *Boundary layer theory*, sixth ed., McGraw-Hill, New York, 1968.

[53] E. Jahnke, F. Emde, Losch.: *Tables of higher functions*, McGraw-Hill. 1960 (Rus. transl. from German, 1977).

[54] S. Tarbatu, K. Buhler, J. Zierep, New solutions of the *II*. Stokes problem for an oscillating flat plate, Acta Mechanica **129**, 25 (1998).

[55] G.G. Stokes, On the effect of the internal friction of fluids on the motion of pendulumns, Cambr. Phil. Trans. IX, 8 (1851); Math. and Phys. Papers, Cambridge, III, 1 (1901).

[56] R.B., Bird, R.C. Armstrong, O. Hassager, *Dynamics of polymer liquids*, Vol. 1. Wiley, New York 1977.

TABLE A

Summary of the differential operators involving the ∇ – operator in rectangular Cartesian coordinate system (x, y, z)

$$(\nabla \cdot \mathbf{V}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}, \tag{A1}$$

$$(\nabla^2 s) = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2}, \tag{A2}$$

$$\begin{aligned}\tau : \nabla\mathbf{V} = {} & \tau_{xx}\left(\frac{\partial v_x}{\partial x}\right) + \tau_{xy}\left(\frac{\partial v_x}{\partial y}\right) + \tau_{xz}\left(\frac{\partial v_x}{\partial z}\right) + \tau_{yx}\left(\frac{\partial v_y}{\partial x}\right) + \tau_{yy}\left(\frac{\partial v_y}{\partial y}\right) \\ & + \tau_{yz}\left(\frac{\partial v_y}{\partial z}\right) + \tau_{zx}\left(\frac{\partial v_z}{\partial x}\right) + \tau_{zy}\left(\frac{\partial v_z}{\partial y}\right) + \tau_{zz}\left(\frac{\partial v_z}{\partial z}\right),\end{aligned} \tag{A3}$$

$$[\nabla s]_x = \frac{\partial s}{\partial x}, \quad [\nabla s]_y = \frac{\partial s}{\partial y}, \quad [\nabla s]_z = \frac{\partial s}{\partial z}, \tag{A4}$$

$$[\nabla \times \mathbf{V}]_x = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \quad [\nabla \times \mathbf{V}]_y = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \quad [\nabla \times \mathbf{V}]_z = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}, \tag{A5}$$

$$\begin{aligned}[\nabla \cdot \tau]_x &= \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}, \quad [\nabla \cdot \tau]_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}, \\ [\nabla \cdot \tau]_z &= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z},\end{aligned} \tag{A6}$$

$$\begin{aligned}[\nabla^2 \mathbf{V}]_x &= \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}, \quad [\nabla^2 \mathbf{V}]_y = \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}, \\ [\nabla^2 \mathbf{V}]_z &= \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2},\end{aligned} \tag{A7}$$

$$
\begin{aligned}
[\mathbf{V}\cdot\nabla\mathbf{W}]_x &= v_x\left(\frac{\partial w_x}{\partial x}\right)+v_y\left(\frac{\partial w_x}{\partial y}\right)+v_z\left(\frac{\partial w_x}{\partial z}\right),\\
[\mathbf{V}\cdot\nabla\mathbf{W}]_y &= v_x\left(\frac{\partial w_y}{\partial x}\right)+v_y\left(\frac{\partial w_y}{\partial y}\right)+v_z\left(\frac{\partial w_y}{\partial z}\right),\\
[\mathbf{V}\cdot\nabla\mathbf{W}]_z &= v_x\left(\frac{\partial w_z}{\partial x}\right)+v_y\left(\frac{\partial w_z}{\partial y}\right)+v_z\left(\frac{\partial w_z}{\partial z}\right),
\end{aligned}
\tag{A8}
$$

$$
\begin{aligned}
\{\nabla\mathbf{V}\}_{xx} &= \frac{\partial v_x}{\partial x},\ \{\nabla\mathbf{V}\}_{xy}=\frac{\partial v_y}{\partial x},\ \{\nabla\mathbf{V}\}_{xz}=\frac{\partial v_z}{\partial x},\ \{\nabla\mathbf{V}\}_{yx}=\frac{\partial v_x}{\partial y},\ \{\nabla\mathbf{V}\}_{yy}=\frac{\partial v_y}{\partial y},\\
\{\nabla\mathbf{V}\}_{yz} &= \frac{\partial v_y}{\partial z}\ \{\nabla\mathbf{V}\}_{zx}=\frac{\partial v_z}{\partial x},\ \{\nabla\mathbf{V}\}_{zy}=\frac{\partial v_z}{\partial y},\ \{\nabla\mathbf{V}\}_{zz}=\frac{\partial v_z}{\partial z},
\end{aligned}
\tag{A9}
$$

$$
\begin{aligned}
\{\mathbf{V}\cdot\nabla\tau\}_{xx} &= (\mathbf{V}\cdot\nabla)\,\tau_{xx},\ \{\mathbf{V}\cdot\nabla\tau\}_{xy}=(\mathbf{V}\cdot\nabla)\,\tau_{xy},\ \{\mathbf{V}\cdot\nabla\tau\}_{xz}=(\mathbf{V}\cdot\nabla)\,\tau_{xz},\\
\{\mathbf{V}\cdot\nabla\tau\}_{yx} &= (\mathbf{V}\cdot\nabla)\,\tau_{yx},\ \{\mathbf{V}\cdot\nabla\tau\}_{yy}=(\mathbf{V}\cdot\nabla)\,\tau_{yy},\ \{\mathbf{V}\cdot\nabla\tau\}_{yz}=(\mathbf{V}\cdot\nabla)\,\tau_{yz},\\
\{\mathbf{V}\cdot\nabla\tau\}_{zx} &= (\mathbf{V}\cdot\nabla)\,\tau_{zx},\ \{\mathbf{V}\cdot\nabla\tau\}_{yy}=(\mathbf{V}\cdot\nabla)\,\tau_{zy},\ \{\mathbf{V}\cdot\nabla\tau\}_{zz}=(\mathbf{V}\cdot\nabla)\,\tau_{zz},
\end{aligned}
\tag{A10}
$$

where the operator $(\mathbf{V}\cdot\nabla)=v_x\dfrac{\partial}{\partial x}+v_y\dfrac{\partial}{\partial y}+v_z\dfrac{\partial}{\partial z}$.

TABLE B

Summary of the differential operators involving the ∇ – operator in cylindrical coordinate system (r, θ, z)

$$(\nabla \cdot \mathbf{V}) = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}, \tag{B1}$$

$$(\nabla^2 s) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2} + \frac{\partial^2 s}{\partial z^2}, \tag{B2}$$

$$\begin{aligned}(\tau : \nabla \mathbf{V}) = {} & \tau_{rr}\left(\frac{\partial v_r}{\partial r}\right) + \tau_{r\theta}\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}\right) + \tau_{rz}\left(\frac{\partial v_r}{\partial z}\right) + \tau_{\theta r}\left(\frac{\partial v_\theta}{\partial r}\right) + \tau_{\theta\theta}\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} - \frac{v_r}{r}\right) \\ & + \tau_{\theta z}\left(\frac{\partial v_\theta}{\partial z}\right) + \tau_{zr}\left(\frac{\partial v_z}{\partial r}\right) + \tau_{z\theta}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) + \tau_{zz}\left(\frac{\partial v_z}{\partial z}\right),\end{aligned} \tag{B3}$$

$$[\nabla s]_r = \frac{\partial s}{\partial r}, \quad [\nabla s]_\theta = \frac{1}{r}\frac{\partial s}{\partial \theta}, \quad [\nabla s]_z = \frac{\partial s}{\partial z}, \tag{B4}$$

$$[\nabla \times \mathbf{V}]_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}, \quad [\nabla \times \mathbf{V}]_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \quad [\nabla \times \mathbf{V}]_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}, \tag{B5}$$

$$\begin{aligned}[\nabla \cdot \tau]_r &= \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta r} - \frac{\tau_{\theta\theta}}{r}, \\ [\nabla \cdot \tau]_\theta &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta} + \frac{\partial}{\partial z}\tau_{z\theta} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r}, \\ [\nabla \cdot \tau]_z &= \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta z} + \frac{\partial}{\partial z}\tau_{zz},\end{aligned} \tag{B6}$$

$$\begin{aligned}[\nabla^2 \mathbf{V}]_r &= \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}, \\ [\nabla^2 \mathbf{V}]_\theta &= \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}, \\ [\nabla^2 \mathbf{V}]_z &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2},\end{aligned} \tag{B7}$$

$$
\begin{aligned}
[\mathbf{V}\cdot\nabla\mathbf{W}]_r &= v_r\left(\frac{\partial w_r}{\partial r}\right)+v_\theta\left(\frac{1}{r}\frac{\partial w_r}{\partial\theta}-\frac{w_\theta}{r}\right)+v_z\left(\frac{\partial w_r}{\partial z}\right),\\
[\mathbf{V}\cdot\nabla\mathbf{W}]_\theta &= v_r\left(\frac{\partial w_\theta}{\partial r}\right)+v_\theta\left(\frac{1}{r}\frac{\partial w_\theta}{\partial\theta}+\frac{w_r}{r}\right)+v_z\left(\frac{\partial w_\theta}{\partial z}\right),\\
[\mathbf{V}\cdot\nabla\mathbf{W}]_z &= v_r\left(\frac{\partial w_z}{\partial r}\right)+v_\theta\left(\frac{1}{r}\frac{\partial w_z}{\partial\theta}\right)+v_z\left(\frac{\partial w_z}{\partial z}\right),
\end{aligned}
\tag{B8}
$$

$$
\begin{aligned}
\{\nabla\mathbf{V}\}_{rr} &= \frac{\partial v_r}{\partial r},\quad \{\nabla\mathbf{V}\}_{r\theta}=\frac{\partial v_\theta}{\partial r},\quad \{\nabla\mathbf{V}\}_{rz}=\frac{\partial v_z}{\partial r},\\
\{\nabla\mathbf{V}\}_{\theta r} &= \frac{1}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r},\quad \{\nabla\mathbf{V}\}_{\theta\theta}=\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r},\quad \{\nabla\mathbf{V}\}_{\theta z}=\frac{1}{r}\frac{\partial v_z}{\partial\theta},\\
\{\nabla\mathbf{V}\}_{zr} &= \frac{\partial v_r}{\partial z},\quad \{\nabla\mathbf{V}\}_{z\theta}=\frac{\partial v_\theta}{\partial z},\quad \{\nabla\mathbf{V}\}_{zz}=\frac{\partial v_z}{\partial z},
\end{aligned}
\tag{B9}
$$

$$
\begin{aligned}
\{\mathbf{V}\cdot\nabla\tau\}_{rr} &= (\mathbf{V}\cdot\nabla)\,\tau_{rr}-\frac{\mathbf{V}_\theta}{r}\left(\tau_{r\theta}+\tau_{\theta r}\right),\quad \{\mathbf{V}\cdot\nabla\tau\}_{r\theta}=(\mathbf{V}\cdot\nabla)\,\tau_{r\theta}+\frac{\mathbf{V}_\theta}{r}\left(\tau_{rr}-\tau_{\theta\theta}\right),\\
\{\mathbf{V}\cdot\nabla\tau\}_{rz} &= (\mathbf{V}\cdot\nabla)\,\tau_{rz}-\frac{\mathbf{V}_\theta}{r}\tau_{\theta z},\quad \{\mathbf{V}\cdot\nabla\tau\}_{\theta r}=(\mathbf{V}\cdot\nabla)\,\tau_{\theta r}+\frac{\mathbf{V}_\theta}{r}\left(\tau_{rr}-\tau_{\theta\theta}\right),\\
\{\mathbf{V}\cdot\nabla\tau\}_{\theta\theta} &= (\mathbf{V}\cdot\nabla)\,\tau_{\theta\theta}+\frac{\mathbf{V}_\theta}{r}\left(\tau_{r\theta}+\tau_{\theta r}\right),\quad \{\mathbf{V}\cdot\nabla\tau\}_{\theta z}=(\mathbf{V}\cdot\nabla)\,\tau_{\theta z}+\frac{\mathbf{V}_\theta}{r}\tau_{rz},\\
\{\mathbf{V}\cdot\nabla\tau\}_{zr} &= (\mathbf{V}\cdot\nabla)\,\tau_{zr}-\frac{\mathbf{V}_\theta}{r}\tau_{z\theta},\quad \{\mathbf{V}\cdot\nabla\tau\}_{z\theta}=(\mathbf{V}\cdot\nabla)\,\tau_{z\theta}+\frac{\mathbf{V}_\theta}{r}\tau_{rz},\\
\{\mathbf{V}\cdot\nabla\tau\}_{zz} &= (\mathbf{V}\cdot\nabla)\,\tau_{zz},
\end{aligned}
\tag{B10}
$$

where the operator

$$
(\mathbf{V}\cdot\nabla)=v_r\frac{\partial}{\partial r}+\frac{v_\theta}{r}\frac{\partial}{\partial\theta}+v_z\frac{\partial}{\partial z}.
$$

TABLE C

Summary of the differential operators involving the ∇ – operator in spherical coordinate system (r, θ, ϕ)

$$(\nabla \cdot \mathrm{V}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial\left(v_\theta \sin\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}, \tag{C1}$$

$$(\nabla^2 s) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial s}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial s}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 s}{\partial\theta^2}, \tag{C2}$$

$$\begin{aligned}(\tau : \nabla v) = {} & \tau_{rr}\left(\frac{\partial v_r}{\partial r}\right) + \tau_{r\theta}\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r}\right) + \tau_{r\phi}\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{v\phi}{r}\right) \\ & +\tau_{\theta r}\left(\frac{\partial v_\theta}{\partial r}\right) + \tau_{\theta\theta}\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right) + \tau_{\theta\phi}\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi} - \frac{v\phi}{r}\cot\theta\right) \\ & +\tau_{\phi r}\left(\frac{\partial v_\phi}{\partial r}\right) + \tau_{\phi\theta}\left(\frac{1}{r}\frac{\partial v_\phi}{\partial\theta}\right) + \tau_{\phi\phi}\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{v_r}{r} + \frac{v_\theta}{r}\cot\theta\right),\end{aligned} \tag{C3}$$

$$[\nabla s]_r = \frac{\partial s}{\partial r}, \quad [\nabla s]_\theta = \frac{1}{r}\frac{\partial s}{\partial\theta}, \quad [\nabla s]_\phi = \frac{1}{r\sin\theta}\frac{\partial s}{\partial\phi}, \tag{C4}$$

$$\begin{aligned}[\nabla \times \mathrm{V}]_r &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v_\phi\sin\theta\right) - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}, \quad [\nabla \times \mathrm{V}]_\theta = \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{1}{r}\frac{\partial\left(r v_\phi\right)}{\partial r}, \\ [\nabla \times \mathrm{V}]_\phi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r v_\theta\right) - \frac{1}{r}\frac{\partial v_\phi}{\partial\theta},\end{aligned} \tag{C5}$$

$$\begin{aligned}[\nabla \cdot \tau]_r &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\tau_{\theta r}\sin\theta\right) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\tau_{\phi r} - \frac{\tau_{\theta\theta} - \tau_{\phi\phi}}{r}, \\ [\nabla \cdot \tau]_\theta &= \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\tau_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\tau_{\theta\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\tau_{\theta\phi} + \frac{\left(\tau_{\theta r} - \tau_{r\theta}\right) - \tau_{\phi\phi}\cot\theta}{r}, \\ [\nabla \cdot \tau]_\phi &= \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\tau_{r\phi}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\tau_{\theta\phi}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\tau_{\phi\phi} + \frac{\left(\tau_{\phi r} - \tau_{r\phi}\right) + \tau_{\phi\theta}\cot\theta}{r},\end{aligned} \tag{C6}$$

$$\begin{aligned}
\left[\nabla^2\mathbf{V}\right]_r &= \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2} \\
&\quad -\frac{2}{r^2\sin\theta}\frac{\partial\left(v_\theta\sin\theta\right)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial v_\theta}{\partial\phi}, \\
\left[\nabla^2\mathbf{V}\right]_\theta &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}v_\theta\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta\sin\theta\right)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2} \\
&\quad +\frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}, \\
\left[\nabla^2\mathbf{V}\right]_\phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}v_\phi\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(v_\phi\sin\theta\right)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2} \\
&\quad +\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial\phi},
\end{aligned} \tag{C7}$$

$$\begin{aligned}
\left[\mathbf{V}\cdot\nabla\mathbf{W}\right]_r &= v_r\left(\frac{\partial w_r}{\partial r}\right) + v_\theta\left(\frac{1}{r}\frac{\partial w_r}{\partial\theta} - \frac{w_\theta}{r}\right) + v_\phi\left(\frac{1}{r\sin\theta}\frac{\partial w_r}{\partial\phi} - \frac{w_\phi}{r}\right), \\
\left[\mathbf{V}\cdot\nabla\mathbf{W}\right]_\theta &= v_r\left(\frac{\partial w_\theta}{\partial r}\right) + v_\theta\left(\frac{1}{r}\frac{\partial w_\theta}{\partial\theta} + \frac{w_r}{r}\right) + v_\phi\left(\frac{1}{r\sin\theta}\frac{\partial w_\theta}{\partial\phi} - \frac{w_\phi}{r}\cot\theta\right), \\
\left[\mathbf{V}\cdot\nabla\mathbf{W}\right]_\phi &= v_r\left(\frac{\partial w_\phi}{\partial r}\right) + v_\theta\left(\frac{1}{r}\frac{\partial w_\phi}{\partial\theta}\right) + v_\phi\left(\frac{1}{r\sin\theta}\frac{\partial w_\phi}{\partial\phi} + \frac{w_r}{r} + \frac{w_\theta}{r}\cot\theta\right),
\end{aligned} \tag{C8}$$

$$\begin{aligned}
\{\nabla\mathbf{V}\}_{rr} &= \frac{\partial v_r}{\partial r}, \quad \{\nabla\mathbf{V}\}_{r\theta} = \frac{\partial v_\theta}{\partial r}, \quad \{\nabla\mathbf{V}\}_{r\phi} = \frac{\partial v_\phi}{\partial r}, \\
\{\nabla\mathbf{V}\}_{\theta r} &= \frac{1}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r}, \quad \{\nabla\mathbf{V}\}_{\theta\theta} = \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}, \quad \{\nabla\mathbf{V}\}_{\theta\phi} = \frac{1}{r}\frac{\partial v_\phi}{\partial\theta}, \\
\{\nabla\mathbf{V}\}_{\phi r} &= \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{v_\phi}{r}, \quad \{\nabla\mathbf{V}\}_{\phi\theta} = \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi} - \frac{v_\phi}{r}\cot\theta, \\
\{\nabla\mathbf{V}\}_{\phi\phi} &= \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} + \frac{v_r}{r} + \frac{v_\theta}{r}\cot\theta,
\end{aligned} \tag{C9}$$

$$\begin{aligned}
\{\mathbf{V}\cdot\nabla\tau\}_{rr} &= (\mathbf{V}\cdot\nabla)\,\tau_{rr} - \left(\frac{v_\theta}{r}\right)(\tau_{r\theta}+\tau_{\theta r}) - \left(\frac{v_\phi}{r}\right)(\tau_{r\phi}+\tau_{\phi r}), \\
\{\mathbf{V}\cdot\nabla\tau\}_{r\theta} &= (\mathbf{V}\cdot\nabla)\,\tau_{r\theta} + \left(\frac{v_\theta}{r}\right)(\tau_{rr}-\tau_{\theta\theta}) - \left(\frac{v_\phi}{r}\right)(\tau_{\phi\theta}+\tau_{r\phi}\cot\theta), \\
\{\mathbf{V}\cdot\nabla\tau\}_{r\phi} &= (\mathbf{V}\cdot\nabla)\,\tau_{r\phi} - \left(\frac{v_\theta}{r}\right)\tau_{\theta\phi} + \left(\frac{v_\phi}{r}\right)\left[(\tau_{rr}-\tau_{\theta\theta})+\tau_{r\theta}\cot\theta\right], \\
\{\mathbf{V}\cdot\nabla\tau\}_{\theta r} &= (\mathbf{V}\cdot\nabla)\,\tau_{\theta r} + \left(\frac{v_\theta}{r}\right)(\tau_{rr}-\tau_{\theta\theta}) - \left(\frac{v_\phi}{r}\right)(\tau_{\theta\phi}+\tau_{\phi r}\cot\theta), \\
\{\mathbf{V}\cdot\nabla\tau\}_{\theta\theta} &= (\mathbf{V}\cdot\nabla)\,\tau_{\theta\theta} + \left(\frac{v_\theta}{r}\right)(\tau_{r\theta}+\tau_{\theta r}) - \left(\frac{v_\phi}{r}\right)(\tau_{\theta\phi}+\tau_{\phi\theta})\cot\theta,
\end{aligned} \tag{C10}$$

$$
\begin{aligned}
\{\mathbf{V}\cdot\nabla\tau\}_{\theta\phi} &= (\mathbf{V}\cdot\nabla)\,\tau_{\theta\phi} + \left(\frac{v_\theta}{r}\right)\tau_{r\phi} + \left(\frac{v_\phi}{r}\right)\left[\tau_{\theta r} + (\tau_{\theta\theta} - \tau_{\phi\phi})\cot\theta\right], \\
\{\mathbf{V}\cdot\nabla\tau\}_{\phi r} &= (\mathbf{V}\cdot\nabla)\,\tau_{\phi r} - \left(\frac{v_\theta}{r}\right)\tau_{\phi\theta} + \left(\frac{v_\phi}{r}\right)\left[(\tau_{rr} - \tau_{\phi\phi}) + \tau_{\theta r}\cot\theta\right], \\
\{\mathbf{V}\cdot\nabla\tau\}_{\phi\theta} &= (\mathbf{V}\cdot\nabla)\,\tau_{\phi\theta} + \left(\frac{v_\theta}{r}\right)\tau_{\phi r} + \left(\frac{v_\phi}{r}\right)\left[\tau_{r\theta} + (\tau_{\theta\theta} - \tau_{\phi\phi})\cot\theta\right], \\
\{\mathbf{V}\cdot\nabla\tau\}_{\phi\phi} &= (\mathbf{V}\cdot\nabla)\,\tau_{\phi\phi} + \left(\frac{v_\phi}{r}\right)\left[(\tau_{r\phi} + \tau_{\phi r}) + (\tau_{\theta\phi} + \tau_{\phi\theta})\cot\theta\right],
\end{aligned}
\tag{C11}
$$

where the operator

$$
(\mathbf{V}\cdot\nabla) = v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}.
$$

TABLE D

Summary of the vector identities

$$
\begin{aligned}
\nabla rs &= r\nabla s + s\nabla r \\
(\nabla\cdot s\mathbf{V}) &= (\nabla s\cdot\mathbf{V}) + s\,(\nabla\cdot\mathbf{V}) \\
\nabla\cdot[\mathbf{V}\times\mathbf{W}] &= \mathbf{W}\cdot[\nabla\times\mathbf{V}] - \mathbf{V}\cdot[\nabla\times\mathbf{W}] \\
[\nabla\times s\mathbf{V}] &= [\nabla s\times\mathbf{V}] + s\,[\nabla\times\mathbf{V}] \\
(\nabla\cdot\nabla)\,\mathbf{V} &= \\
[\mathbf{V}\cdot\nabla\mathbf{V}] &= \frac{1}{2}\nabla\,(\mathbf{V}\cdot\mathbf{V}) - [\mathbf{V}\times[\nabla\times\mathbf{V}]] \\
[\nabla\cdot\mathbf{V}\mathbf{W}] &= [\mathbf{V}\cdot\nabla\mathbf{W}] + \mathbf{W}\,(\nabla\cdot\mathbf{V}) \\
(s\delta:\nabla\mathbf{V}) &= s\,(\nabla\cdot\mathbf{V}) \\
[\nabla\cdot s\delta] &= \nabla s \\
[\nabla\cdot s\tau] &= [\nabla s\cdot\tau] + s\,[\nabla\cdot\tau] \\
\nabla\,(\mathbf{V}\cdot\mathbf{W}) &= [(\nabla\mathbf{V})\cdot\mathbf{W}] + [(\nabla\mathbf{W})\cdot\mathbf{V}] \\
(\tau:\nabla\mathbf{V}) &= (\nabla\cdot[\tau\cdot\mathbf{V}]) - (\mathbf{V}\cdot[\nabla\cdot\tau]) \\
[\delta\cdot\mathbf{V}] &= [\mathbf{V}\cdot\delta] = \mathbf{V} \\
[\mathbf{U}\mathbf{V}\cdot\mathbf{W}] &= \mathbf{U}\,(\mathbf{V}\cdot\mathbf{W}) \\
[\mathbf{W}\cdot\mathbf{U}\mathbf{V}] &= (\mathbf{W}\cdot\mathbf{U})\,\mathbf{V} \\
(\mathbf{U}\mathbf{V}:\mathbf{W}\mathbf{Z}) &= (\mathbf{U}\mathbf{W}:\mathbf{V}\mathbf{Z}) = (\mathbf{U}\cdot\mathbf{Z})\,(\mathbf{V}\cdot\mathbf{W}) \\
(\tau:\mathbf{U}\mathbf{V}) &= ([\tau\cdot\mathbf{U}]\cdot\mathbf{V}) \\
(\mathbf{U}\mathbf{V}:\tau) &= (\mathbf{U}\cdot[\mathbf{V}\cdot\tau]) \\
\tau\cdot\tau &= \tau^2, \quad \tau\cdot\tau^2 = \tau^3, \ldots \\
(\delta:\tau) &= \sum_i\sum_j \delta_{ij}\tau_{ij}
\end{aligned}
$$

Appendix

1. The incomplete gamma function and the gamma function are related through

$$\Gamma_U(\widehat{\alpha}, x) + \Gamma_L(\widehat{\alpha}, x) = \Gamma(\widehat{\alpha}),$$

where $\Gamma_U(\widehat{\alpha}, x)$ is the upper incomplete gamma function and $\Gamma_L(\widehat{\alpha}, x)$ is the lower incomplete gamma function and are defined by

$$\Gamma_U(\widehat{\alpha}, x) = \int_x^{\infty} t^{\widehat{\alpha}-1} e^{-t} dt,$$

$$\Gamma_L(\widehat{\alpha}, x) = \int_0^x t^{\widehat{\alpha}-1} e^{-t} dt = \widehat{\alpha}^{-1} x^{\widehat{\alpha}} 1F1(\widehat{\alpha}; 1 + \widehat{\alpha}; -x),$$

where $1F1$ is the confluent hypergeometric function of the first kind. For "$\widehat{\alpha}$" an integer n

$$\Gamma_U(n, x) = (n-1)! e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!} = (n-1)! e^{-x} e_{n-1}(x),$$

$$\Gamma_L(n, x) = (n-1)! \left[1 - e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!}\right] = (n-1)! \left[1 - e^{-x} e_{n-1}(x)\right],$$

where $e_n(x)$ is the exponential sum function and is defined by

$$e_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} = e^x \frac{\Gamma(n+1, x)}{\Gamma(n+1)}.$$

2. Hypergeometric$PFQ\left[\{a_1, ..., a_p\}, \{b_1, ..., b_q\}, \widehat{z}\right]$ is the generalized hypergeometric function $pFq(\mathrm{a}, \mathrm{b}; \widehat{z})$. For example

$$\text{Hypergeometric}PFQ\left[\{1, 2, 1\}, \{2, 1\}, x\right] = \frac{1}{1-x}.$$

Hypergeometric$PFQ\left[\{a_1, ..., a_p\}, \{b_1, ..., b_q\}, \widehat{z}\right]$ has series expansion

$$pFq = \sum_{k=0}^{\infty} \frac{(a_1)_k, ..., (a_p)_k}{(b_1)_k, ..., (b_q)_k} \frac{\widehat{z}^k}{k!}.$$

We can differentiate and integrate HypergeometricPFQ as follows:

$$
\begin{aligned}
& \text{Differentiate}\left[\text{Hypergeometric}PFQ\left[\{a_1,a_2,a_3\},\{b_1,b_2\},x\right],x\right] \\
= \; & \frac{1}{b_1 b_2}\left(a_1 a_2 a_3 \text{Hypergeometric}PFQ\left[\begin{array}{c}\{1+a_1,1+a_2,1+a_3\}, \\ \{1+b_1,1+b_2\},x\end{array}\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
& \text{Integrate}\left[\text{Hypergeometric}PFQ\left[\{a_1,a_2,a_3\},\{b_1,b_2\},x\right],x\right] \\
= \; & x\Gamma(b_1)\Gamma(b_2)\{\left[\frac{1}{x\Gamma(b_1-1)\Gamma(b_2-1)(-1+a_1)(-1+a_2)(-1+a_3)}\right] \\
& + \frac{\text{Hypergeometric}PFQ\text{Regularized}\left[\begin{array}{c}\{a_1-1,a_2-1,a_3-1\}, \\ \{b_1-1,b_2-1\},x\end{array}\right]}{x(-1+a_1)(-1+a_2)(-1+a_3)}\},
\end{aligned}
$$

where HypergeometricPFQRegularized$[\{a_1,......,a_p\},\{b_1,......,b_q\},\widehat{z}]$ is the regularized generalized hypergeometric function $pFq(\mathbf{a},\mathbf{b};\widehat{z})\,/\,\{\Gamma(b_1),.....,\Gamma(b_q)\}$ and MeijerG

$$
\text{MeijerG}\left[\{\{a_1,......,a_n\},\ \{a_{n+1},......,a_p\}\},\{\{b_1,......,b_m\},\ \{b_{m+1},......,b_q\}\},\ \widehat{z}\right]
$$

is the MeijerG function

$$
G_{p\ \ q}^{m\ \ n}\left(\widehat{z}\Big/{}^{a_1,......,a_p}_{b_1,......,b_q}\right).
$$

For example

$$
\begin{aligned}
\text{Meijer}G\left[\{\{1,1\},\{\ \}\},\{\{1\},\{0\}\},x\right] &= \log(1+x), \\
\text{Meijer}G\left[\{\{\ \},\{\ \}\},\left\{\{0\},\left\{\frac{1}{2}\right\}\right\},\frac{x}{2}\right] &= \frac{\cos\sqrt{2x}}{\sqrt{x}}.
\end{aligned}
$$

For $m=1$, $n=2$, $p=2$, $q=2$, we have the following properties:

$$
\begin{aligned}
& \text{Differentiate}\left[\text{Meijer}G\left[\{\{a_1,a_2\},\{a_3,a_4\}\},\{\{b_1\},\{b_3,b_4\}\},x\right],x\right] \\
= \; & \text{Meijer}G\left[\begin{array}{c}\{\{-1,a_1-1,a_2-1\},\{a_3-1,a_4-1\}\}, \\ \{\{b_1-1\},\{0,b_2-1,b_3-1\}\},x\end{array}\right],
\end{aligned}
$$

$$
\begin{aligned}
& \text{Integrate}\left[\text{MeijerG}\left[\{\{a_1, a_2\}, \{a_3, a_4\}\}, \{\{b_1\}, \{b_3, b_4\}\}, x\right], x\right] \\
= \; & \text{MeijerG}\left[\begin{array}{c} \{\{1, 1+a_1, 1+a_2\}, \{1+a_3, 1+a_4\}\}, \\ \{\{1+b_1\}, \{0, 1+b_2, 1+b_3\}\}, x \end{array}\right].
\end{aligned}
$$