Some Steady and Unsteady Flow Problems Involving Newtonian and Non-Newtonian Fluids

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A Thesis
Submitted in the Partial Fulfillment of the
Requirements for the Degree of
DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

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We accept this thesis as conforming to the required standard.

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Earnestly Dedicated To---------------

All those who love and care me

espacilly

The one WHO never die

Acknowledgments

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Contents

Preface

The equations which govern the flows of Newtonian fluids are Navier-Stokes equations. This nonlinear set of partial differential equations has no general solution and a few number of exact solutions are found. Exact solutions are very important for many reasons. They provide a standard for checking the accuracies of many approximate methods such as numerical and empirical. Although computer techniques make the complete numerical integration of the Navier-Stokes equations feasible, the accuracy of the results can be established by a comparison with exact solution.

Recently it has generally been acknowledged that non-Newtonian fluids exhibiting a nonlinear relationship between the stresses and the rate of strain are more appropriate in technological applications than Newtonian fluids. Many industrial fluids are non-Newtonian in their flow characteristics and referred to as rheological fluids. Most particulate slurries (china clay and coal in water, sewage sludge, etc.), multiphase mixtures (oil-water emulsions, gas-liquid dispersions, such as froths and foams, butter), are non-Newtonian fluids. Further examples displaying a variety of non-Newtonian characteristics include pharmaceutical formulations, cosmetics and toiletries, paints, synthetics lubricants, biological fluids (blood, synovial fluid, salvia), and food stuffs (jams, jellies, soups, marmalades). Indeed, behavior of non-Newtonian flow is so widespread that it would be no exaggeration to say that the Newtonian fluid behavior is an exception rather than the rule. Such rheological fluids cannot be adequately described by the Navier-Stokes theory. Because of this reason, several models, mainly based on empirical observations have been developed for these fluids. One of the important classes of non-Newtonian fluids which has acquired a special status is the viscoelastic fluids. The major attractiveness of these fluids is the fact that the constitutive relations, whether we take the second-or the third grade fluids since they have been studied the most, are that they are derived based on first principles and unlike many other phenomenological models, there are no curve fittings of parameters to adjust. But the flows of second-and the third grade fluids present some interesting challenges to researchers in engineering, applied mathematics and computer science. The equations of these fluids are very complex involving a number of parameters, highly nonlinear and of higher order than the Navier-Stokes equations. Due to complexity of these equations, finding analytical solution is not easy.

An understanding of the dynamics of fluids in the presence of a magnetic field has principal interest because these flows are quite prevalent in nature. Such flows are important in many engineering applications. Specifically they are employed, for example to drive flows, induce stirring, levitation or to suppress turbulence, in the casting of metals and the growth of semiconductor crystal.

Motivated by these facts, this thesis comprises seven chapters. The objective of chapter one is to provide background for constitutive equations of second- and third grade fluids and the homotopy analysis method (HAM).

The aim of chapter two is to provide the exact analytical solutions for magneto-hydrodynamic (MHD) periodic flows induced by non-coaxial rotations of porous disk and Newtonian fluid at infinity. Both unsteady and steady solutions are obtained in suction and blowing for all values of the frequencies including the resonant frequency. It is found that the existing diffusive hydromagnetic waves decay within the ultimate steady state boundary layers and the external magnetic field expedites the decay process of these waves.

Chapter three has been prepared to see the influence of Hall current on the flow due to non-coaxial rotations of porous, non-conducting oscillating disk and a Newtonian fluid at infinity. Exact analytical solution describing the flow at large and small times after the start is obtained. The three solutions when the angular velocity is greater than, smaller than or equal to the frequency of oscillations are developed for sine and cosine oscillations of the disk. It is observed that the magnitude of the primary velocity increases while secondary velocity decreases by increasing the Hall parameter. It is further found that boundary layer thickness for sine oscillations are smaller when compared with cosine oscillations.

Chapter four looks the analytical solution for the hydrodynamic flow caused by non-coaxial rotations of a porous disk executing non-torsional oscillations and a fluid at infinity. The considered fluid is incompressible and second grade. Unsteady and steady series solutions are obtained for suction and blowing when angular velocity is greater than or smaller than the frequency of sine and cosine oscillations. Moreover, the meaningful steady blowing solution does not exist for both sine and cosine oscillations of the disk.

In chapter five, analysis of chapter four is extended to the case when fluid is electrically conducting in the presence of a uniform applied magnetic field. Here it is found that for uniform suction and blowing at the disk, shear oscillations are confined to Ekman-Hartman layer near the disk for all values of the frequencies. In fact, applied magnetic field provides a mechanism for the existence of meaningful steady blowing solution in the resonant case.

In chapter six, an analysis is performed to study the flow and heat transfer analysis of a second grade fluid past a porous plate. A generalized second grade fluid model which arises from a modification of the second grade constitutive equation incorporating a shear dependent viscosity is taken into consideration. The governing nonlinear flow equations are solved using HAM. Expressions for temperature field are developed for the two cases when the plate has constant temperature or it is insulated. The results indicate that the velocity boundary layer thickness increases for large values of normal stress coefficient. The thermal boundary layer thickness increases by increasing Eckert number whereas it decreases by increasing normal stress coefficient and Prandtl number.

Chapter seven is a contribution in order to obtain HAM solution for the flow of a third grade fluid in a porous channel. Expression for velocity is first constructed and then compared with the exact numerical solution for the various values of the physical parameters. It is observed that a proper choice of the auxiliary parameter in HAM solution gives very close results.

List of Figures

Nomenclature

$\mathbf{T}$	Cauchy stress tensor
p_1	Pressure (scalar function)
$\mathbf{I}$	Unit matrix
$\mathbf{D}$	Deformation tensor
$\mathbf{V}$	Velocity
$\mathbf{A}_1, ..., \mathbf{A}_n$	Rivlin-Ericksen tensors
d/dt	Material time derivative
$\mathbf{F}$	An isotropic function of degree n
μ	Dynamic viscosity
α_1, α_2	Material moduli
β_1,β_2, β_3	Material constants for third grade fluid
σ	Electric conductivity
Ω	Constant angular velocity
u, v, w	Velocity components in the x, y and $z-$ directions
ρ	The fluid density
ν	The kinematic viscosity
$\mathbf{J}$	The current density
$\mathbf{B}$	Total magnetic field
B_0	Magnitude of applied magnetic field
μ_m	Magnetic permeability
$\mathbf{E}$	Electric field

$h(t)$	General periodic oscillation of a disk
$n = \frac{2\pi}{T_0}$	Non zero-oscillating frequency
$\{a_k\}$	Fourier series coefficients
$W_0 > 0$	Suction velocity
$W_0 < 0$	Blowing velocity
$erfc(\widetilde{x})$	Complementary error function
s	Laplace parameter
e	Electron charge
p_e	Electron pressure
n_e	Electron number density
w_e	Cyclotron frequency
τ_e	Electron collision time
τ_i	Collision time for ions
$m = w_e\tau_e$	Hall parameter
$\widehat{p}$	Modified pressure
U	Reference velocity
N_1	Hartman number
N_2	Magnetic parameter
r	Radiant heating
$\mathbf{q}$	Heat flux vector
$\widetilde{m}$	Material fluid parameter of generalized second grade
c	Specific heat
θ	Temperature

k	Constant thermal conductivity
$\mathcal{L}, \mathcal{L}_1$	Auxiliary linear operators
p	Embedding parameter
$\hbar$	Non-zero auxiliary parameter
P_r	Prandtl number
E_c	Eckert numbers
θ_b	Bulk temperature
R	Reynold's number
K	Viscoelastic fluid parameter
$M_1 - M_{45}$	Constants in calculations

Chapter 1

Introduction

The ability of electromagnetic fields to influence fluid flow has long been known and used with varying degrees of success. The equations governing the flow consist of the Navier-Stokes equations of fluid motion coupled with Maxwell's equations of electro-magnetics and material constitutive relations. The study of these flows is Electro-Magneto-Fluid Dynamics (EMFD). The field of EMFD has traditionally been divided into flows influenced only by electric fields and electric charges, and flows influenced only by magnetic fields and without electric charges. The former are called Electrohydrodynamic (EHD) flows and the latter Magnetohydrodynamic (MHD) flows. However, the full system of equations in each case has, up until recently, been far too complex to solve generally. The Navier-Stokes equations alone become complex when analysis of flows in more dimensions is taken into account. Moreover, the governing equations become much complicated when hydrodynamic flows of realistic interest are desired (that is, turbulent, chemically reacting, multi-constituent, and/or non-Newtonian fluids). Coupled with Maxwell's equations, the complexity of the system is raised by orders of magnitude. The field is too vast to exhaustively cover the subject, so the intent is to provide some analytical solutions of the field. To accomplish this, chapters two and five provide the analytic solutions of velocity fields for MHD flows. Effort is made to provide a physical understanding of the field material interactions causing magnetization. Chapter two contains a general developed formula for any periodic disk oscillations. Asymptotic analysis has also been carried out to determine the viscous solutions for the larg time. Besides the engineering applications, the presented analysis in this chapter possesses geophysical applications. The contents of chapter two have been **published in Mathematical and Computer Modelling 40 (2004) 173-179.**

In an ionized gas where the density is low and/or the magnetic field is very strong, the conductivity normal to the magnetic field is reduced due to the free spiraling of electrons and ions about the magnetic lines of force before suffering collision; also, a current is induced in a direction normal to both the electric and the magnetic fields. This phenomenon, well known in the literature, is called the Hall effect. The study of MHD flows with Hall current has important engineering applications in problems of magnetohydrodynamic generators and of Hall accelerators as well as in flight magnetohydrodynamics. With this fact in mind, the effect of Hall current on the time-dependent flow induced by non-coaxial rotations of a porous oscillating disk and a viscous fluid is analyzed in chapter three. Exact analytical solution describing the flow at large and small times after the start is given. The combined effects of Hall current, rotation, suction or blowing are shown on the velocity. It is noted that layer thickness in case of sine oscillations is smaller than for the cosine oscillations. These results have been submitted for publication in **Canadian Journal of Physics.**

In many fields, such as food industry, drilling operations and bio-engineering, the fluids either synthetic or natural, are mixtures of different stuffs such as water, particle, oils, red cells and other long chain molecules. This combination imparts strong non-Newtonian characteristics to the resulting liquids; the viscosity function varies non-linearly with the shear rates; elasticity is felt through an elongational effects and time-dependent effects. In these cases, the fluids have been treated as viscoelastic fluids [1-16].

Understanding of the problems concerning the flow of viscoelastic fluids, such as crystal growth, polymer injection and molding, and food or chemical processing and transmission, etc. has become important in recent years. However, most of the applications treat the viscoelastic fluids as Newtonian fluids and use the Newtonian model to analyze, predict and simulate the behavior of viscoelastic fluids. Due to a widely expanded domain of applications, the flow characteristics of viscoelastic fluids have been found to be quite different from that of Newtonian fluids. This situation suggests that the flow behaviors of viscoelastic fluids cannot be replaced by the results from the Newtonian flows. Therefore, the fundamental flow behaviors of viscoelastic fluids should be studied in order to gain better understanding and to enhance applications in various industries. But due to complexity of fluids, it is very difficult to suggest a single model which exhibits all properties of viscoelastic fluids. They cannot be described as simple as Newtonian fluids. Rheological properties of material are specified in general by their so-called constitutive equations. The simplest constitutive equation for a fluid is a Newtonian one and the classical Navier-

Stokes theory is based on this equation. The mechanical behavior of many real fluids, especially those of low molecular weight, is well enough described by this theory. There are, however, many rheological complex fluids such as polymer solutions, special soap solutions, blood, paint, certain oils and greases which are not well described by a Newtonian constitutive equation. For this reason, many models or constitutive equations for complex fluids have been proposed and most of them are empirical or semiempirical.

The non-linear response of complex fluids constitute an important area of mathematical modeling in non-Newtonian fluid mechanics. The most widely used model by the researchers is the power-law constitutive relation. Although the power-law model adequately fits the shear stress and the shear rate measurements for many non-Newtonian fluids, it cannot be used to accurately describe phenomena such as "die-swelling" and "rod-climbing" which are manifestations of the stresses that develop orthogonal to planes of shear in the flow of these complex fluids. The power law model does not predict these normal stress effect; though it can predict some of the usual characteristics of non-Newtonian fluids such as shear thinning and shear thickening. The simplest model which can capture the normal effects is the second grade fluid or the Rivlin-Ericksen fluid of grade two. One of the recent advances in the theoretical studies in rheology is the development of generalized differential grade models. The simplicity of the form and the fact that these modified constitutive relations can be used to study shear-thining/thickening, the decrease/increase in viscosity with increasing/decreasing shear rate, as well as predicting normal stress differences, have opened the way for the solution to the engineering problems [17-19]. Specifically the generalized second grade and third grade fluid models exhibit the property of shear thinning/thickenning for both steady and unsteady flows. With these facts in mind, the steady flow of a generalized second grade fluid past a porous plate is presented in chapter six. Expression for velocity has been developed using homotopy analysis method (HAM). It is found that boundary layer thickness increases by increasing the normal stress coefficient. Chapter seven looks the channel flow of a third grade fluid. The fluid is induced by imposing constant pressure gradient. HAM solution is obtained for velocity and compared with the existing numerical solution. It is noted that HAM solution holds even for those values of the Deborah number (large values) for which numerical solution [20] has a convergence problem. The analysis of this chapter has been **published in " Non-Linear Dynamics" 45 (2005), 55-64.**

Heat transfer plays an important role during handling and processing of non-Newtonian fluids. The understanding of heat transfer in boundary layer flows of non-Newtonian fluids is of important in many engineering applications such as the design of thrust bearings and radial diffusers, transpiration cooling, drag reduction, thermal recovery of oil, etc. Because of this motivation, a systematic study of heat transfer of generalized second grade fluid has been provided in chapter six. Solutions for temperature distributions have been obtained for isothermal and insulated walls. It is observed that the behavior of temperature for insulated wall is similar to that of an isothermal wall. Further, thermal boundary layer thickness decreases for large values of the normal stress coefficient and Prandtl number. These results have been **accepted for publication in" Int. J. Applied Mechanics and Engineering".**

The viscous flow due to non-coaxial rotations of disk and a fluid at infinity or due to eccentric rotating disks has been considered by a number of workers [21]. Exact solution to the Navier-Stokes equations for such flow has been implied by Berker [22]. For flow between two coaxially or non-coaxially rotating infinite parallel disks with same angular velocity, he showed that there are infinite number of non-trivial solutions of the Navier-Stokes equations. He further pointed out that a single unique solution requires an extra condition. He assumed that flow is symmetric with respect to the origin. This condition helps for a unique solution if the boundary conditions on the disk are symmetric. However, if pressure gradient is prescribed in the Navier-Stokes equations then all the constants appearing in the solution can be determined and one has a unique solution. But, single disk flow problems do not need extra condition for a unique solution. The flow induced by a disk and fluid at infinity which are rotating non-coaxially at slightly different angular velocities has been studied by Coirier [23]. In continuation, exact solutions for the flow of three dimensional Navier-Stokes equations have been provided by Erdogan [24,25] when the porous disk and a viscous fluid at infinity rotate with the constant angular velocity and are in a state of non-coaxial rotation. Murthy and Ram [26] extended the analysis of reference [24] for the magnetohydrodynamic fluid. Additionally, they also discussed the effects of heat transfer due to eccentric rotations of a porous disk and a viscous fluid at infinity.

The studies of non-Newtonian fluids for this type of flow have also been available in the literature under various conditions and fluid models. Mention may be made to the works of Dai et al. [27], Blyler and Kurtz [28], Bird and Harris [29], Bower et al. [30], Abbott and Walters [31], Rajagopal [32], Rajagopal and Gupta [33], Kaloni and Siddiqui [34], Zhang and Goddard [35], Rajagopal and Wineman [36], Ersoy [37,38] and Asghar et al. [39].

The unsteady flow due to non-coaxially rotating disks or due to non-coaxial rotations of a disk and a fluid at infinity has also received considerable attention. Smith [40] has extended the work of Berker [41] to the case of unsteady motions. Later, the unsteady viscous flows between two eccentric disks or due to single disk and a fluid at infinity have been investigated by Rao and Kasiviswanathan [42], Kasiviswanathan and Rao [43], Pop [44], Erdogan [45-47], Ersoy [48] and Hayat et al. [49,50]. Very little attention has been given for such flows involving non-Newtonain fluids with unsteady motions [51,52]. Thus, the purpose of chapters four and five is to analyze the unsteady second grade flow due to non-coaxial rotations of porous disk and a fluid at infinity. Chapter four deals with the analysis for a hydrodynamic fluid. Several limiting cases are deduced from the present analysis. It is noted that steady blowing solution for resonant frequency does not exist. This is because of the thickening of a boundary layer at sufficiently large distance from the leading edge. The contents of this chapter have been **published in " Applied Mathematical Modelling" 28 (2004) 591-605**. In chapter five, the effects of an applied magnetic field on the flow of a second grade fluid is studied. It is found that for uniform suction and blowing meaningful unsteady and steady solutions valid for all values of the frequency are possible. The observations of chapter five have been **accepted in " Chemical Engineering Communications"**.

1.1 Theoretical background for constitutive equations of second and third grade fluids

For the homogenous incompressible fluids, the constitutive law satisfies the following equation

$$\mathbf{T} = -\,p_1\mathbf{I} + \mathbf{F}(\mathbf{D}), \tag{1.1}$$

in which $\mathbf{T}$ is the Cauchy stress tensor, p_1 is the pressure (scalar function) and $\mathbf{I}$ is the identity tensor. The function $\mathbf{F}$ depends upon the deformation tensor $\mathbf{D}$ given by

$$\mathbf{D} = \mathbf{L} + \mathbf{L}^T, \quad \mathbf{L} = \mathbf{gradV}, \tag{1.2}$$

where $\mathbf{V}$ is the velocity field and T in the superscript indicates the matrix transpose. Also, the function $\mathbf{F}$ verifies the Stokes hypothesis $\mathbf{F}(\mathbf{0}) = \mathbf{0}$. Note that, if $\mathbf{F}$ is non-linear then Eq. (1.1) defines a non-Newtonian fluid. Let us recall some definitions concerning a particular class of non-Newtonian fluids, namely the differential fluids of complexity n. For further details we refer the reader to Noll and Truesdell [53]. The constitutive law of fluids of complexity n is

$$\mathbf{T} = -\,p_1\mathbf{I} + \mathbf{F}(\mathbf{A}_1, ..., \mathbf{A}_n), \tag{1.3}$$

where $\mathbf{A}_1, ..., \mathbf{A}_n$ are the first Rivlin-Ericksen tensors defined by

$$\mathbf{A}_1 = 2\mathbf{D}, \tag{1.4}$$

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{L}^T\mathbf{A}_{n-1} + \mathbf{A}_{n-1}\mathbf{L}, \quad n > 1, \tag{1.5}$$

where d/dt is the material time derivative.

The principle of material frame indifference requires $\mathbf{F}$ to be an isotropic function. If in Eq. (1.3), the function $\mathbf{F}$ is a polynomial of degree n then we have the subclass of fluids of grade n. Taking into account the isotropy of $\mathbf{F}$, the most general constitutive laws for the fluids of grades 1 to 3 are, respectively,

$$\mathbf{T} = -\,p_1\mathbf{I} + \mu\mathbf{A}_1 \qquad \text{(grade 1)}, \tag{1.6}$$

$$\mathbf{T} = -\,p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 \quad \text{(grade 2)}, \tag{1.7}$$

$$\mathbf{T} = -p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2\left(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1\right) + \beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1 \quad \text{(grade 3)}, \tag{1.8}$$

where μ is the dynamic viscosity, α_1, α_2, β_1,β_2 and β_3 are the specific material moduli. Note that the fluids of grade 1 are in fact the classical Newtonian ones. In constitutive law for second grade fluid in Eq. (1.7), the α_1 and α_2 can be related to the first and second normal stress differences, $\tilde{N}_1$ and $\tilde{N}_2$, respectively. Experimental data available for the large number of viscoelastic fluids all suggest that $\tilde{N}_1$ is positive [54]. On the other hand, $\tilde{N}_2$ is often found to be either negative or zero. Also when $\tilde{N}_2$ is measured to be non-zero, it is usually found to be much smaller than $\tilde{N}_1$. This means that for a second grade fluid to comply with the experimental observations, one should have $\alpha_1 > 0$ and $\alpha_2 \leq 0$. Having said this, it should be mentioned that there are some controversies around this rheological model, particularly about the sign of α_1 and size of α_2. Fosdick and Rajagopal [55] argue that for a second grade rheological model to be thermodynamically compatible, the Clausius-Duhem inequality should hold together with the Helmholtz free energy being at its minimum whenever, the fluid is locally at rest. These thermodynamical constraints put some severe restrictions on the sign and magnitude of the material moduli [55]:

$$\mu \geq 0; \quad \alpha_1 \geq 0; \; \alpha_1 + \alpha_2 = 0. \tag{1.9}$$

The sign proposed above for α_1 is tantamount to saying that $\bar{N}_1$ is negative. If this sign is accepted for α_1, then based on Eq. (1.9) the sign for $\bar{N}_2$ should be positive. Both sign are in direct contradiction with experimental data available for viscoelastic fluids. The last relationship in Eq. (1.9) also suggests that the absolute values of $\bar{N}_1$ and $\bar{N}_2$ are equal to each other which simply cannot be confirmed experimentally. Obviously, there are certain important issues still unresolved about this controversial rheological model. For a critical review of the second grade fluid the reader is referred to Dunn and Rajagopal [56]. In the present thesis, we have decided to take the restrictions given in Eq. (1.9) for theoretical analysis.

Similar considerations for fluids of third grade led Rajagopal [57] and Fosdick and Rajagopal [58] to give the following restrictions on μ, α_1, $\alpha_2, \beta_1, \beta_2$, and β_3 appearing in definition (1.8):

$$\mu \geq 0; \; \beta_1 = \beta_2 = 0, \quad \beta_3 \geq 0, \; \alpha_1 \geq 0; \; |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}. \tag{1.10}$$

Some useful informations regarding the signs of α_1 have also been presented by Fosdick and Straughan [59], Straughan [60-63] and Franchi and Straughan [64,65]. With above restrictions, the constitutive law (1.8) of fluids of grade 3 takes the form

$$\mathbf{T} = -\,p_1\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3(tr\mathbf{A}_1^2)\mathbf{A}_1. \tag{1.11}$$

It is noted that this constitutive relation not only predicts the normal stress differences, but can also predict the "shear thickening" phenomenon (since $\beta_3 \geq 0$) which is the increase in viscosity with increasing shear rate. That is, we can rewrite Eq. (1.11) as

$$\mathbf{T} = -p_1\mathbf{I} + \left[\mu + \beta_3(tr\mathbf{A}_1^2)\right]\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 \tag{1.12}$$

and the quantity in the bracket can be thought of as an effective shear-dependent viscosity, that is,

$$\mu_{eff} = \mu + \beta_3 tr\mathbf{A}_1^2. \tag{1.13}$$

1.2 Homotopy analysis method

After the appearance of supercomputers, it is easy to obtain the analytical solutions of linear problems. Such solutions for the non-linear problems present special challenges to engineers, mathematicians and physicists alike. Researchers usually used the perturbation methods in the construction of analytical solutions of non-linear problems. It is out of question that perturbation methods play important role in the development of science and engineering. But, like other non-linear analytical techniques, perturbation methods have their own limitations. These methods are based upon the existence of small/large parameter which is often known as the perturbation quantity. Unfortunately, many non-linear problems do not contain such perturbation quantities at all. Some nonperturbative methods, such as the artificial small parameter method [66], the $\delta-$expansion method [67] and the Adomian's decomposition method [68], have been developed. Unlike from the perturbation methods, these nonperturbative methods do not depend upon small parameters. However, all these methods (perturbative and nonperturbative) cannot provide us greater freedom and larger flexibility to adjust and control convergence regions of series of approximations. Besides, the efficiency of approximating a non-linear problem has not been taken into enough account. So, it is necessary to develop some more efficient analytic methods.

A kind of analytic method namely the homotopy analysis method (HAM) [69,70] has been proposed by introducing an auxiliary parameter to construct a new kind of homotopy in a more general form. HAM has already been successfully applied to several non-linear problems [71-83].The HAM has the following advantages:

- It is valid if a given non-linear problem does not contain small/large parameter at all.
- It provides us with a convenient way to adjust and control convergence regions of series of analytical approximations.
- It can be employed to efficiently approximate a non-linear problem by choosing different sets of base functions.

After the introduction of HAM now we simply introduce the basic concept of homotopy. Two mathematical objects are said to be homotopic if one can be continuous deformation into another. For example, the real line is homotopic to a single point, as in any tree. However, the circle is not contractible, but is homotopic to a cylinder. The basic version of homotopy is between mapping of functions.

In topology, two continuous functions form one topological space to another called homotopic if one can be "continuous deformation" into another, such a deformation being called a homotopy between the two functions.

Definition: Homotopy is a continuous transformation from one function to another. A homotopy $\mathcal{H}$ between two continuous functions f and g from a topological space X to a topological space Y is defined to be continuous function

$$\mathcal{H}: \quad X \times [0,1] \to Y$$

such that for all points $x \in X$

$$\mathcal{H}(x,0) = f(x) \quad \text{and} \quad \mathcal{H}(x,1) = g(x).$$

If f is homotopic to g, then there exist one parameter family

$$\{\mathcal{H}_p;\ p \in [0,1]\}$$

of continuous functions such that

$$\mathcal{H}: \quad \mathbb{R} \times [0,1] \to \mathbb{R}$$

defined by

$$\mathcal{H}_p(x) = (1-p)\, f(x) + pg(x), \qquad \forall\, x \in \mathbb{R}\ \&\ p \in [0,1].$$

This function is well defined since both $(1-p)\, f(x)$ and $pg(x)$ belong to $\mathbb{R}$ and so $(1-p)\, f(x) + pg(x)$ belongs to $\mathbb{R}$. Further, we observe that

$$\mathcal{H}_0(x) = f(x) \quad \text{and} \quad \mathcal{H}_1(x) = g(x).$$

Usually $\{\mathcal{H}_p;\ p \in [0,1]\}$ is called the linear homotopy between f and g.

Chapter 2

Unsteady MHD periodic flows due to non-coaxial rotations of porous disk and a viscous fluid at infinity

In this chapter an analysis is carried out to study the magnetohydrodynamic (MHD) viscous fluid flows characteristics. The flows are subjected due to non-coaxial rotations of porous disk and a fluid at infinity. The general periodic oscillation of a porous disk has been taken into account. Exact solution for the velocity field has been developed using Laplace transform method. In addition, the velocity fields corresponding to five special cases of the considered general periodic oscillation are also included. It is found that the obtained steady and transient solutions in suction and blowing cases are meaningful.

2.1 Mathematical formulation

Consider the flow of an incompressible viscous fluid of finite electric conductivity σ. The fluid is electrically conducting in the presence of a uniform magnetic field $\mathbf{B}_0$ which is applied perpendicular to the disk. The disk ($z = 0$) is porous. The fluid fills the space $z > 0$ and is in contact with the disk. The axes of rotation, of both the disk and fluid, are assumed to be in the plane $x = 0$, with the distance between the axes being l. The disk and the fluid are initially rotating about the $z' - axis$ with constant angular velocity $\mathbf{\Omega}$ and at time $t = 0$, the disk and the fluid start to rotate at z and $z'-$axes respectively with $\mathbf{\Omega}$. For $t > 0$, the disk also oscillates in its own plane with frequency n.

In Cartesian coordinate system, the unsteady motion of the conducting viscous incompressible fluid is governed by the conservation laws of momentum and of mass which are

$$\frac{d\mathbf{V}}{dt} = -\frac{1}{\rho}\nabla p_1 + \nu\nabla^2\mathbf{V} + \frac{1}{\rho}\mathbf{J}\times\mathbf{B}, \tag{2.1}$$

$$div\ \mathbf{V} = 0, \tag{2.2}$$

in which $\mathbf{V} = (u,\ v,\ w)$ is the fluid velocity with u, v, and w as the velocity components in the x, y and $z-$ directions respectively, ρ is the fluid density, p_1 is the scalar pressure, d/dt is the material derivative, ν is the kinematic viscosity, $\mathbf{J}$ is the current density and $\mathbf{B}$ is the total magnetic field which is the sum of applied $\mathbf{B}_0$ and induced $\mathbf{b}$ magnetic fields.

In addition to Eqs. (2.1) and (2.2), the governing flow consists of the Maxwell equations and a generalized Ohm's law which after neglecting the displacement cur-

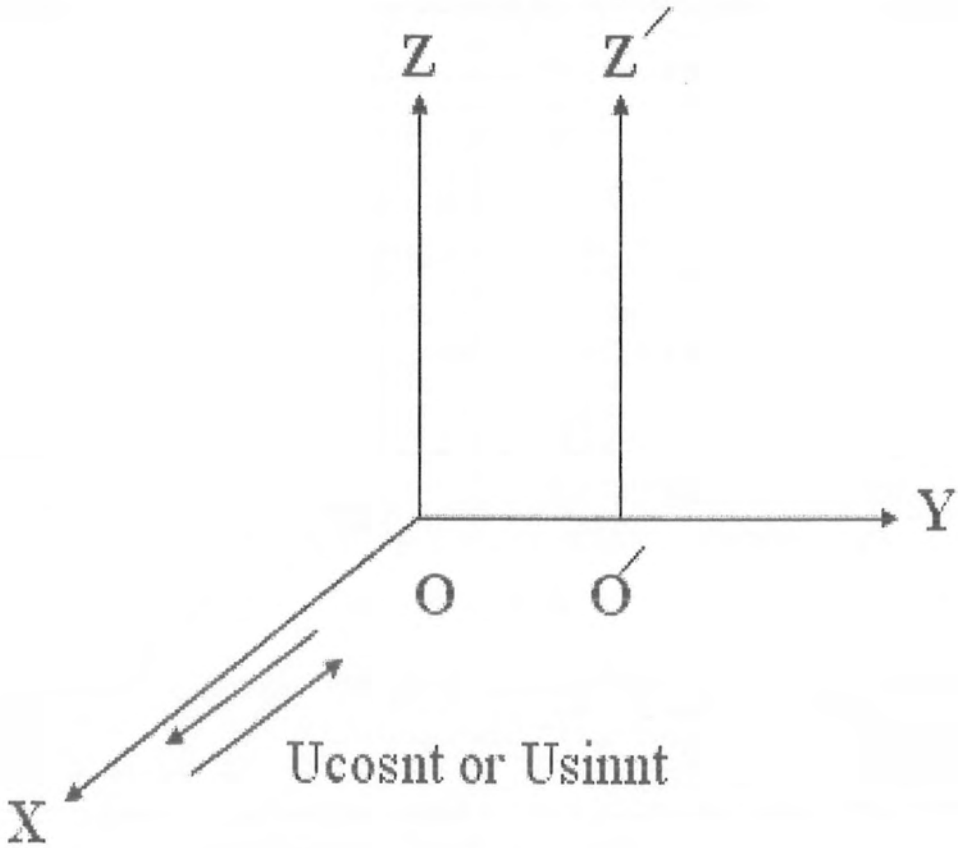

Fig. 2.1 Flow geometry

rents are given by the following equations

$$divB = 0, \tag{2.3}$$

$$curl\ B = \mu_m J, \tag{2.4}$$

$$curl\ E = -\frac{\partial B}{\partial t}, \tag{2.6}$$

$$J = \sigma (E + V \times B). \tag{2.6}$$

In above equations μ_m is the magnetic permeability and **E** is the electric field. For the derivation of Lorentz force in Eq. (2.1), we assume that the magnetic field is perpendicular to the velocity field, the electric field is negligible and the induced magnetic field is small compared with the applied magnetic field. The last assumption is valid when the magnetic Reynolds number is very small and there is no displacement current [84]. With these assumptions, one can write

$$\frac{1}{\rho}(J \times B) = -\frac{\sigma B_0^2}{\rho} V. \tag{2.7}$$

The relevant boundary conditions are

$$\left.\begin{array}{lll} u = -\Omega y + U h(t), & v = \Omega x & at\ z = 0\ \ for\ \ t > 0, \\ u = -\Omega(y - l), & v = \Omega x & as\ z \to \infty\ \ for\ all\ t, \\ u = -\Omega(y - l), & v = \Omega x & at\ t = 0\ \ for\ z > 0, \end{array}\right\} \tag{2.8}$$

in which U is the velocity and $h(t)$ is the general periodic oscillation of a disk. The Fourier series representation of $h(t)$ is given by

$$h(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{iknt}, \tag{2.9}$$

where

$$a_k = \frac{1}{T_0}\int_{T_0} h(t)e^{-iknt}\, dt, \tag{2.10}$$

with non zero oscillating frequency $n = 2\pi/T_0$. Eq. (2.9) is referred to as the synthesis equation and Eq. (2.10) as the analysis equation. The coefficients $\{a_k\}$ are the Fourier series coefficients or the spectral coefficients of $h(t)$. In practice the fluid motion would be set up from the rest, and, for some time after the initiation of the motion, the flow field contains "transients" determined by these initial conditions. It may be shown that the fluid velocity gradually becomes a harmonic function of t, with the same frequency as the velocity of the boundary, and only this periodic state will be considered here.

We seek a velocity field of the form

$$u = -\Omega y + f(z,t), \quad v = \Omega x + g(z,t). \tag{2.11}$$

Upon making use of above equation into Eq. (2.2), we have for uniform porous disk [85] that

$$w = -W_0 \tag{2.12}$$

where ($W_0 > 0$ is the suction velocity and $W_0 < 0$ corresponds to the blowing velocity).

In view of Eqs. (2.1), (2.7), (2.11) and (2.12) one can write

$$\nu\frac{\partial^3 F}{\partial z^3} + W_0\frac{\partial^2 F}{\partial z^2} - \frac{\partial^2 F}{\partial t\partial z} - \left(i\Omega + \frac{\sigma}{\rho}B_0^2\right)\frac{\partial F}{\partial z} = 0, \tag{2.13}$$

in which

$$F = f + ig. \tag{2.14}$$

With the help of above equation, the boundary and initial conditions are specified as

$$\left.\begin{aligned} F(0,t) &= Uh(t), \\ F(\infty,t) &= \Omega l, \end{aligned}\right\} \tag{2.15}$$

$$F(z,0) = \Omega l. \tag{2.16}$$

2.2 Exact analytical solution

To facilitate the solution of Eq. (2.13) subject to Eqs. (2.15) and (2.16), we define the Laplace transform pair as

$$\overline{H}(z,s) = \int_0^\infty F(z,t)e^{-st}dt, \tag{2.17}$$

$$F(z,t) = \frac{1}{2\pi i}\int_{\bar{\lambda}-i\infty}^{\bar{\lambda}+i\infty} \overline{H}(z,s)\, e^{st}ds. \tag{2.18}$$

In term of Laplace parameter s, the Eqs. (2.13) and (2.15) after taking into account Eq. (2.16) are transformed into

$$\left[\frac{d^3}{dz^3} + \frac{W_0}{\nu}\frac{d^2}{dz^2} - \left(\frac{M+s}{\nu}\right)\frac{d}{dz}\right]\overline{H}(z,s) = 0, \tag{2.19}$$

$$\overline{H}(0,s) = U\sum_{k=-\infty}^{\infty}\frac{a_k}{s-ikn}, \tag{2.20}$$

$$\overline{H}(z,s) = \frac{\Omega l}{s} \text{ as } z \to \infty, \tag{2.21}$$

where

$$M = i\Omega + \frac{\sigma B_0^2}{\rho}. \tag{2.22}$$

The general solution of the ordinary differential equation (2.19) is

$$\overline{H}(z,s) = C_1 + C_2 e^{-\left[\frac{W_0}{2\nu}+\sqrt{\left(\frac{W_0}{2\nu}\right)^2+\left(\frac{M+s}{\nu}\right)}\right]z} + C_3 e^{\left[-\frac{W_0}{2\nu}+\sqrt{\left(\frac{W_0}{2\nu}\right)^2+\left(\frac{M+s}{\nu}\right)}\right]z}, \tag{2.23}$$

where C_1, C_2, and C_3 are the arbitrary constants which after using Eqs. (2.20) and (2.21) can be determined as

$$C_1=\frac{\Omega l}{s},\quad C_2=U\sum_{k=-\infty}^{\infty}\frac{a_k}{s-ikn}-\frac{\Omega l}{s},\quad C_3=0. \tag{2.24}$$

In view of the above values of the constants, our solution (2.23) reduces to

$$\begin{aligned}\overline{H}(z,s) &= \frac{\Omega l}{s}\left[1-\exp\left\{-\left(\frac{W_0}{2\nu}+\sqrt{\left(\frac{W_0}{2\nu}\right)^2+\left(\frac{M+s}{\nu}\right)}\right)z\right\}\right] \\ &\quad +U\sum_{k=-\infty}^{\infty}\frac{a_k}{s-ikn}\left[1-\exp\left\{-\left(\frac{W_0}{2\nu}+\sqrt{\left(\frac{W_0}{2\nu}\right)^2+\left(\frac{M+s}{\nu}\right)}\right)z\right\}\right].\end{aligned} \tag{2.25}$$

Inverting the above result by means of Laplace transform we get the velocity field

$$F(z,t)=\Omega l\left[\begin{array}{c} 1-\frac{e^{-\frac{W_0}{2\nu}z}}{2}\left\{\begin{array}{c} e^{-z\sqrt{\frac{W_0^2}{4\nu^2}+\frac{\sigma B_0^2}{\rho\nu}+\frac{i\Omega}{\nu}}}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}+i\Omega\right)t}\right) \\ +e^{z\sqrt{\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho\nu}+\frac{i\Omega}{\nu}}}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}+i\Omega\right)t}\right)\end{array}\right\} \\ +U\sum\limits_{k=-\infty}^{\infty}a_k e^{-\frac{W_0}{2\nu}z}L^{-1}\left[\frac{e^{-z\sqrt{\left(\frac{W_0}{2\nu}\right)^2+\frac{M+s}{\nu}}}}{s-ikn}\right]\end{array}\right], \tag{2.26}$$

where L^{-1} indicates the inverse Laplace transform. We know that

$$L^{-1}\left[e^{-z\sqrt{\left(\frac{W_0}{2\nu}\right)^2+\frac{s}{\nu}}}\right]=\frac{z}{2\sqrt{\pi\nu t^3}}e^{-\left\{\left(\frac{W_0}{2\nu}\right)^2+\frac{s}{\nu}\right\}\nu t-\frac{z^2}{4\nu t}},\quad L^{-1}\left[\frac{1}{s-ikn}\right]=e^{iknt} \tag{2.27}$$

and thus by the convolution theorem of Laplace transform we can write

$$L^{-1}\left[\frac{e^{-z\sqrt{\left(\frac{W_0}{2\nu}\right)^2+\frac{s}{\nu}}}}{s-ikn}\right]=e^{iknt}*\frac{z}{2\sqrt{\pi\nu t^3}}e^{-\left\{\left(\frac{W_0}{2\nu}\right)^2+\frac{s}{\nu}\right\}\nu t-\frac{z^2}{4\nu t}}, \tag{2.28}$$

where asterik in above equation indicates the convolution.

The simplification of above equation yields

$$\left.\begin{array}{c} L^{-1}\left[\frac{e^{-z\sqrt{\left(\frac{W_0}{2\nu}\right)^2+\frac{s}{\nu}}}}{s-ikn}\right] \\ =\frac{e^{iknt}}{2}\left(\begin{array}{c} e^{z\sqrt{\left(\frac{W_0}{2\nu}\right)^2+\frac{s}{\nu}+\frac{ikn}{\nu}}}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\left(\frac{W_0}{2\nu}\right)^2+\frac{s}{\nu}+\frac{ikn}{\nu}\right)\nu t}\right) \\ +e^{-z\sqrt{\left(\frac{W_0}{2\nu}\right)^2+\frac{s}{\nu}+\frac{ikn}{\nu}}}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\left(\frac{W_0}{2\nu}\right)^2+\frac{s}{\nu}+\frac{ikn}{\nu}\right)\nu t}\right)\end{array}\right)\end{array}\right\}. \tag{2.29}$$

From Eqs. (2.14), (2.26) and (2.29) one obtains

$$\left.\begin{array}{c} \frac{f}{\Omega l}+i\frac{g}{\Omega l}=1-\frac{e^{-\frac{W_0}{2\nu}z}}{2}\left(\begin{array}{c} e^{-z\sqrt{\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho\nu}+\frac{i\Omega}{\nu}}}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho\nu}+\frac{i\Omega}{\nu}\right)t}\right) \\ +e^{z\sqrt{\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho\nu}+\frac{i\Omega}{\nu}}}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho\nu}+\frac{i\Omega}{\nu}\right)t}\right) \end{array}\right) \\ +\frac{Ue^{-\frac{W_0}{2\nu}z}}{2\Omega l}\sum_{k=-\infty}^{\infty}a_k e^{ik\omega_0 t} \\ \times\left(\begin{array}{c} e^{-z\sqrt{\frac{W_0^2}{4\nu^2}+\frac{\sigma B_0^2}{\rho\nu}+\frac{i(\Omega+kn)}{\nu}}}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-\sqrt{\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}+i\left(\frac{\Omega+kn}{\nu}\right)\right)t}\right) \\ +e^{z\sqrt{\frac{W_0^2}{4\nu^2}+\frac{\sigma B_0^2}{\rho\nu}+\frac{i(\Omega+kn)}{\nu}}}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+\sqrt{\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}+i\left(\frac{\Omega+kn}{\nu}\right)\right)t}\right) \end{array}\right) \end{array}\right\}, \tag{2.30}$$

where $erfc(\widetilde{x})$ is the complementary error function defined by

$$\operatorname{erf}c(\widetilde{x})=1-\operatorname{erf}(\widetilde{x})=\int_{\widetilde{x}}^{\infty}e^{-\tau_1^2}d\tau_1 \tag{2.31}$$

and the real and imaginary parts of Eq. (2.30) give $\frac{f}{\Omega l}$ and $\frac{g}{\Omega l}$, respectively.

Substituting

$$\sqrt{\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho\nu}+\frac{i\Omega}{\nu}}=x_1+iy_1 \tag{2.32}$$

$$\sqrt{\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}+i\left(\frac{\Omega+kn}{\nu}\right)}=r_k+i\delta_k \tag{2.33}$$

Eq. (2.30) becomes

$$\left.\begin{array}{c} \frac{f}{\Omega l}+i\frac{g}{\Omega l}=H^*+\frac{Ue^{-\frac{W_0}{2\nu}z}}{2\Omega l}\sum_{k=-\infty}^{\infty}a_k e^{iknt} \\ \times\left(\begin{array}{c} e^{-\frac{z}{\sqrt{\nu}}(r_k+i\delta_k)}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-(r_k+i\delta_k)\sqrt{t}\right) \\ +e^{\frac{z}{\sqrt{\nu}}(r_k+i\delta_k)}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+(r_k+i\delta_k)\sqrt{t}\right) \end{array}\right) \end{array}\right\} \tag{2.34}$$

in which

$$H^*=1-\frac{e^{-\frac{W_0}{2\nu}z}}{2}\left(\begin{array}{c} e^{-\frac{z}{\sqrt{\nu}}(x_1+iy_1)}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-(x_1+iy_1)\sqrt{t}\right) \\ +e^{\frac{z}{\sqrt{\nu}}(x_1+iy_1)}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+(x_1+iy_1)\sqrt{t}\right) \end{array}\right), \tag{2.35}$$

$$x_1 = \left[\frac{1}{2}\left\{\sqrt{\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}\right)^2+\Omega^2}+\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}\right)\right\}\right]^{\frac{1}{2}}, \tag{2.36}$$

$$y_1 = \left[\frac{1}{2}\left\{\sqrt{\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}\right)^2+\Omega^2}-\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}\right)\right\}\right]^{\frac{1}{2}}, \tag{2.37}$$

$$r_k = \left[\frac{1}{2}\left\{\sqrt{\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}\right)^2+(\Omega+nk)^2}+\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}\right)\right\}\right]^{\frac{1}{2}}, \tag{2.38}$$

$$\delta_k = \left[\frac{1}{2}\left\{\sqrt{\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}\right)^2+(\Omega+nk)^2}-\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}\right)\right\}\right]^{\frac{1}{2}}. \tag{2.39}$$

Equation (2.34) gives the complete analytical solution for the velocity field due to the porous disk oscillating periodically in its own plane. As a special case of this oscillation, the flow fields for different disk oscillations are obtained by an appropriate choice of the Fourier coefficients which give rise to different disk oscillations.

The periodic oscillations and their corresponding Fourier coefficients are given below.

Oscillation	Fourier coefficients
$f(t)$	a_k
i. e^{int}	$a_1 = 1, \quad a_k = 0, \ (k \neq 1)$
ii. $\cos nt$	$a_1 = a_{-1} = \frac{1}{2}, \quad a_k = 0,$ otherwise
iii. $\sin nt$	$a_1 = -a_{-1} = \frac{1}{2i}, \quad a_k = 0,$ otherwise
iv. $\begin{cases} 1, & \lvert t\rvert < \frac{T_1}{2} \\ 0, & \lvert T_1\rvert < \lvert t\rvert < \frac{T_0}{2} \end{cases}$	$a_0 = \frac{2T_1}{T_0}, \quad a_k = \frac{\sin(knT_1)}{\pi k}, \quad k \neq 0$
v. $\sum_{k=-\infty}^{\infty} \delta(t - kT_0)$	$a_k = \frac{1}{T_0} \quad \forall \ k$

The results for these oscillations are respectively given in the following expressions

$$\frac{f_1}{\Omega l}+i\frac{g_1}{\Omega l}=H^*+\frac{Ue^{-\frac{W_0}{2\nu}z+int}}{2\Omega l}\left(\begin{array}{c} e^{-\frac{z}{\sqrt{\nu}}(r_1+i\delta_1)}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-(r_1+i\delta_1)\sqrt{t}\right) \\ +e^{\frac{z}{\sqrt{\nu}}(r_1+i\delta_1)}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+(r_1+i\delta_1)\sqrt{t}\right) \end{array}\right), \quad (2.40)$$

$$\frac{f_2}{\Omega l}+i\frac{g_2}{\Omega l}=H^*+\frac{Ue^{-\frac{W_0}{2\nu}z}}{4\Omega l}$$
$$\times\left\{\begin{array}{c} e^{int}\left(\begin{array}{c} e^{-\frac{z}{\sqrt{\nu}}(r_1+i\delta_1)}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-(r_1+i\delta_1)\sqrt{t}\right) \\ +e^{\frac{z}{\sqrt{\nu}}(r_1+i\delta_1)}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+(r_1+i\delta_1)\sqrt{t}\right) \end{array}\right) \\ +e^{-int}\left(\begin{array}{c} e^{-\frac{z}{\sqrt{\nu}}(r_{-1}+i\delta_{-1})}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-(r_{-1}+i\delta_{-1})\sqrt{t}\right) \\ +e^{\frac{z}{\sqrt{\nu}}(r_{-1}+i\delta_{-1})}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+(r_{-1}+i\delta_{-1})\sqrt{t}\right) \end{array}\right) \end{array}\right\}, \quad (2.41)$$

$$\frac{f_3}{\Omega l}+i\frac{g_3}{\Omega l}=H^*-\frac{iUe^{-\frac{W_0}{2\nu}z}}{4\Omega l}$$
$$\times\left\{\begin{array}{c} e^{int}\left(\begin{array}{c} e^{-\frac{z}{\sqrt{\nu}}(r_1+i\delta_1)}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-(r_1+i\delta_1)\sqrt{t}\right) \\ +e^{\frac{z}{\sqrt{\nu}}(r_1+i\delta_1)}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+(r_1+i\delta_1)\sqrt{t}\right) \end{array}\right) \\ -e^{-int}\left(\begin{array}{c} e^{-\frac{z}{\sqrt{\nu}}(r_{-1}+i\delta_{-1})}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-(r_{-1}+i\delta_{-1})\sqrt{t}\right) \\ +e^{\frac{z}{\sqrt{\nu}}(r_{-1}+i\delta_{-1})}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+(r_{-1}+i\delta_{-1})\sqrt{t}\right) \end{array}\right) \end{array}\right\}, \quad (2.42)$$

$$\left.\begin{array}{c} \frac{f_4}{\Omega l}+i\frac{g_4}{\Omega l}=H^*+\frac{Ue^{-\frac{W_0}{2\nu}z}}{2\Omega l}\sum\limits_{k=-\infty}^{\infty}\left(\frac{\sin knT_1}{k\pi}\right)e^{iknt} \\ \times\left(\begin{array}{c} e^{-\frac{z}{\sqrt{\nu}}(r_k+i\delta_k)}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-(r_k+i\delta_k)\sqrt{t}\right) \\ +e^{\frac{z}{\sqrt{\nu}}(r_k+i\delta_k)}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+(r_k+i\delta_k)\sqrt{t}\right) \end{array}\right), k\neq 0 \end{array}\right\}, \quad (2.43)$$

$$\left.\begin{array}{c} \frac{f_5}{\Omega l}+i\frac{g_5}{\Omega l}=H^*+\frac{Ue^{-\frac{W_0}{2\nu}z}}{2\Omega lT_0}\sum\limits_{k=-\infty}^{\infty}e^{iknt} \\ \times\left(\begin{array}{c} e^{-\frac{z}{\sqrt{\nu}}(r_k+i\delta_k)}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}-(r_k+i\delta_k)\sqrt{t}\right) \\ +e^{\frac{z}{\sqrt{\nu}}(r_k+i\delta_k)}\operatorname{erf}c\left(\frac{z}{2\sqrt{\nu t}}+(r_k+i\delta_k)\sqrt{t}\right) \end{array}\right) \end{array}\right\}. \quad (2.44)$$

For the case of blowing we use $W_0 < 0$ in Eqs. (2.34) to (2.44). Thus solutions (2.34) and (2.40) to (2.44) clearly describe the general features of the unsteady hydromagnetic flow including the case of uniform suction or blowing according as $W_0 > 0$ or $W_0 < 0$ respectively.

2.3 The ultimate steady state flows and the boundary layers

Taking $t \to \infty$ in Eqs. (2.34) and (2.40) to (2.44) and using asymptotic formula for the complementary error function i. e.

$$\begin{bmatrix} \operatorname{erf} c\left(\frac{z}{2\sqrt{\nu t}} \pm (x_1 + iy_1)\right) ---> & (0,2), \\ \operatorname{erf} c\left(\frac{z}{2\sqrt{\nu t}} \pm (r_k + i\delta_k)\right) ---> & (0,2) \end{bmatrix} \tag{2.45}$$

we have

$$\frac{f}{\Omega l} + i\frac{g}{\Omega l} = \widetilde{H} + \frac{Ue^{-\frac{W_0}{2\nu}z}}{\Omega l} \sum_{k=-\infty}^{\infty} a_k e^{iknt - \frac{z}{\sqrt{\nu}}(r_k + i\delta_k)}, \tag{2.46}$$

$$\frac{f_1}{\Omega l} + i\frac{g_1}{\Omega l} = \widetilde{H} + \frac{Ue^{-\frac{W_0}{2\nu}z + int - \frac{z}{\sqrt{\nu}}(r_1 + i\delta_1)}}{\Omega l}, \tag{2.47}$$

$$\frac{f_2}{\Omega l} + i\frac{g_2}{\Omega l} = \widetilde{H} + \frac{Ue^{-\frac{W_0}{2\nu}z}}{2\Omega l}\left[e^{int - \frac{z}{\sqrt{\nu}}(r_1 + i\delta_1)} + e^{-int - \frac{z}{\sqrt{\nu}}(r_{-1} + i\delta_{-1})}\right], \tag{2.48}$$

$$\frac{f_3}{\Omega l} + i\frac{g_3}{\Omega l} = \widetilde{H} - \frac{iUe^{-\frac{W_0}{2\nu}z}}{2\Omega l}\left[e^{int - \frac{z}{\sqrt{\nu}}(r_1 + i\delta_1)} - e^{-int - \frac{z}{\sqrt{\nu}}(r_{-1} + i\delta_{-1})}\right], \tag{2.49}$$

$$\frac{f_4}{\Omega l} + i\frac{g_4}{\Omega l} = \widetilde{H} + \frac{Ue^{-\frac{W_0}{2\nu}z}}{\Omega l} \sum_{k=-\infty}^{\infty} \left(\frac{\sin knT_1}{k\pi}\right) e^{iknt - \frac{z}{\sqrt{\nu}}(r_k + i\delta_k)}, \; k \neq 0, \tag{2.50}$$

$$\frac{f_5}{\Omega l} + i\frac{g_5}{\Omega l} = \widetilde{H} + \frac{Ue^{-\frac{W_0}{2\nu}z}}{\Omega l T_0} \sum_{k=-\infty}^{\infty} e^{iknt - \frac{z}{\sqrt{\nu}}(r_k + i\delta_k)}, \tag{2.51}$$

where

$$\widetilde{H} = 1 - \frac{e^{-\frac{W_0}{2\nu}z - \frac{z}{\sqrt{\nu}}(x_1 + iy_1)}}{2}. \tag{2.52}$$

It is interesting to note that Eq. (2.46) is of the oscillatory nature. This solution represents transverse waves. In other words, hydromagnetic diffusive waves occur in a rotating system. For this flow, the thicknesses of the boundary layers are of the order $\left(\frac{W_0}{2\nu}+\frac{x_1}{\sqrt{\nu}}\right)^{-1}$ and $\left(\frac{W_0}{2\nu}+\frac{r_k}{\sqrt{\nu}}\right)^{-1}$. It is important to appreciate that the thicknesses of the boundary layers decrease with an increase of the magnetic field or suction parameter and remain bounded for all values of the frequency. When $n=0$ then $x_1=r_k$ and the distinct boundary layers combine into a single layer of thickness of the order $\left(\frac{W_0}{2\nu}+\frac{x_1}{\sqrt{\nu}}\right)^{-1}$.

In the case of blowing, the steady state solutions are obtained from Eqs. (2.34) and (2.40) to (2.44) by replacing W_0 by $-W_1$, $(W_1>0)$. Taking the limit $t\to\infty$ and then using the asymptotic formula for the complementary error function we have

$$\frac{f}{\Omega l}+i\frac{g}{\Omega l}=\widetilde{X}+\frac{Ue^{\frac{W_1}{2\nu}z}}{\Omega l}\sum_{k=-\infty}^{\infty}a_k e^{iknt-\frac{z}{\sqrt{\nu}}\left(\widetilde{r}_k+i\widetilde{\delta}_k\right)}, \tag{2.53}$$

$$\frac{f_1}{\Omega l}+i\frac{g_1}{\Omega l}=\widetilde{X}+\frac{Ue^{\frac{W_1}{2\nu}z+int-\frac{z}{\sqrt{\nu}}\left(\widetilde{r}_1+i\widetilde{\delta}_1\right)}}{\Omega l}, \tag{2.54}$$

$$\frac{f_2}{\Omega l}+i\frac{g_2}{\Omega l}=\widetilde{X}+\frac{Ue^{\frac{W_1}{2\nu}z}}{2\Omega l}\left[e^{int-\frac{z}{\sqrt{\nu}}\left(\widetilde{r}_1+i\widetilde{\delta}_1\right)}+e^{-int-\frac{z}{\sqrt{\nu}}\left(\widetilde{r}_1+i\widetilde{\delta}_1\right)}\right], \tag{2.55}$$

$$\frac{f_3}{\Omega l}+i\frac{g_3}{\Omega l}=\widetilde{X}-\frac{iUe^{\frac{W_1}{2\nu}z}}{2\Omega l}\left[e^{int-\frac{z}{\sqrt{\nu}}\left(\widetilde{r}_1+i\widetilde{\delta}_1\right)}-e^{-int-\frac{z}{\sqrt{\nu}}\left(\widetilde{r}_1+i\widetilde{\delta}_1\right)}\right], \tag{2.56}$$

$$\frac{f_4}{\Omega l}+i\frac{g_4}{\Omega l}=\widetilde{X}+\frac{Ue^{\frac{W_1}{2\nu}z}}{\Omega l}\sum_{k=-\infty}^{\infty}\left(\frac{\sin knT_1}{k\pi}\right)e^{iknt-\frac{z}{\sqrt{\nu}}\left(\widetilde{r}_k+i\widetilde{\delta}_k\right)},\ k\neq 0, \tag{2.57}$$

$$\frac{f_5}{\Omega l}+i\frac{g_5}{\Omega l}=\widetilde{X}+\frac{Ue^{\frac{W_1}{2\nu}z}}{\Omega l T_0}\sum_{k=-\infty}^{\infty}e^{iknt-\frac{z}{\sqrt{\nu}}\left(\widetilde{r}_k+i\widetilde{\delta}_k\right)}, \tag{2.58}$$

where $\widetilde{x}_1$, $\widetilde{y}_1$, $\widetilde{r}_k$, $\widetilde{\delta}_k$ and $\widetilde{X}$ are obtained by replacing W_0 by $-W_1$ in the expressions (2.36) to (2.39) and (2.52) respectively.

For hydrodynamic fluid $B_0=0$ and the solutions (2.53) to (2.58) do not satisfy the boundary condition at infinity in the resonant case $(n=\Omega)$. However, these solution satisfy all the boundary conditions in the non-resonant case *i.e.* for $n\neq\Omega$.

Actually one of the boundary layers becomes infinitely thick when $n = \Omega$. Consequently, the oscillations generated by the disk are no longer confined to the ultimate boundary layers. In hydromagnetic situation, the solutions (2.53) to (2.58) satisfy the boundary conditions for all values of frequency including the resonant frequency and unbounded spreading of shear oscillations away from the disk in blowing and resonance is controlled by the external magnetic field (all the boundary layers remain finite for all values of the frequencies). Physically, the diffusive hydromagnetic waves exist in the magnetohydrodynamic system. These waves are found to decay within the ultimate steady state boundary layers. The external magnetic field expedites the decay process of these waves.

2.4 Concluding remarks

The presented analysis in this chapter consists of an exact analytical transient and steady solutions for periodic flows induced by the non-coaxial rotations of porous disk and viscous fluid at infinity. The solutions for suction and blowing cases are derived for all values of frequencies including the resonant frequency. The effects of the magnetic parameter and suction/blowing parameter on the velocity are seen, from where we observe that an increase in the magnetic parameter leads to a decrease in the boundary layer thickness. The effect of suction parameter on the velocity is similar to that of the magnetic parameter. Moreover, it is further noted that diffusive waves occur in the hydromagnetic system.

Chapter 3

The influence of Hall current on the unsteady flow due to non-coaxially rotating porous disk and a viscous fluid at infinity

This chapter contains the analysis for flow due to the oscillating porous disk rotating non-coaxially with a viscous fluid at infinity. The fluid is electrically conducting in the presence of a uniform strong transverse magnetic field and the Hall effects are taken into account. Governing equations are implied with reasonable approximations and solved analytically to get the expressions for the velocity fields in closed form. Graphical results are presented for the velocity components for various values of parameters namely, the Hall, suction and blowing and a discussion is provided. It is important to note that the presented results are valid for all values of the frequencies.

3.1 Problem description and mathematical formulation

Let us consider an incompressible viscous fluid which fills the space $z > 0$ and is in contact with an infinite porous disk making oscillations in its own plane. We introduce a Cartesian coordinate system with the $z-$axis normal to the disk, which lies in the plane $z = 0$. The axis of rotation of both the disk and the fluid, are assumed to be in the plane $x = 0$, with the distance between the axes being l. The geometry of the problem is shown in Fig.3.1. Initially, the disk and the fluid at infinity are rotating with the same angular velocity Ω about the $z'-$axis and at time $t = 0$, the disk start to oscillate suddenly along the $x-$axis and to rotate impulsively about the $z-$axis with Ω and the fluid continues to rotate with Ω about the $z'-$axis. A uniform magnetic field $\mathbf{B}_0$ is applied in the positive $z-$direction.

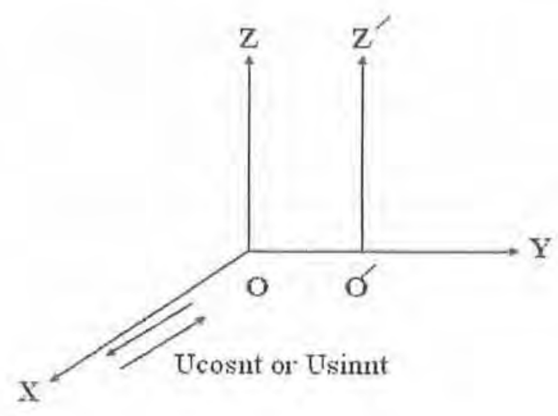

Fig. 3.1 Flow geometry

The corresponding boundary and initial conditions are taken as

$$u=-\Omega y+U\cos nt \;\text{ or }\; u=-\Omega y+U\sin nt;\;\; v=\Omega x \; at\; z=0 \;\text{ for }\; t>0,$$
$$u=-\Omega\left(y-l\right),\;\; v=\Omega x \;\; as\;\; z\longrightarrow\infty \;\text{ for all } t,$$
$$u=-\Omega\left(y-l\right),\;\; v=\Omega x \;\;\text{ at }\; t=0,\;\text{ for }\; z>0, \tag{3.1}$$

in which n being the frequency of the non-torsional oscillations.

The equations governing the flow are (2.1) to (2.5), (2.11), (2.12) and the following generalized Ohm's law which includes the Hall current [86]

$$\mathbf{J}+\frac{w_e\tau_e}{B_0}\left(\mathbf{J}\times\mathbf{B}\right)=\sigma[\mathbf{E}+\mathbf{V}\times\mathbf{B}+\frac{1}{en_e}\nabla p_e], \tag{3.2}$$

where e is the electron charge, p_e is the electron pressure, n_e is the electron number density, w_e is the cyclotron frequency and τ_e is the electron collision time. Note that the ion-slip and thermoelectric effects are not included in Eq.(3.2). In the absence of an external applied electric field and with negligible effects of polarization of the ionized gas we take $\mathbf{E}=0$. The induced magnetic field is negligible which is a valid consideration on the laboratory scale. Further, it is assumed that $w_e\tau_e\approx O(1)$ and $w_i\tau_i << 1$, where w_i and τ_i are cyclotron frequency and collision time for ions respectively.

Proceeding with the Eqs. (2.1), (2.11), (2.12) and (3.2) and then using the above assumptions, the flow with Hall effects is governed by the following scalar equations:

$$\frac{\partial f}{\partial t}-\Omega g-W_0\frac{\partial f}{\partial z}=-\frac{1}{\rho}\frac{\partial\widehat{p}}{\partial x}+v\frac{\partial^2 f}{\partial z^2}-\frac{\sigma B_0^2}{\rho\left(1-im\right)}\left(f-\Omega y\right), \tag{3.3}$$

$$\frac{\partial g}{\partial t}+\Omega f-W_0\frac{\partial g}{\partial z}=-\frac{1}{\rho}\frac{\partial\widehat{p}}{\partial y}+v\frac{\partial^2 g}{\partial z^2}-\frac{\sigma B_0^2}{\rho\left(1-im\right)}\left(g-\Omega x\right), \tag{3.4}$$

$$\frac{1}{\rho}\frac{\partial \widehat{p}}{\partial z} = \frac{\sigma B_0^2}{\rho\,(1-im)} W_0, \tag{3.5}$$

where $m = w_e \tau_e$ is the Hall parameter and the modified pressure $\widehat{p}$ is

$$\widehat{p} = p_1 - \frac{\rho r^2 \Omega^2}{2}, \quad r^2 = x^2 + y^2. \tag{3.6}$$

From Eqs. (2.11) and (3.1), we may write

$$\left.\begin{aligned} f\,(0,t) = U\cos nt \text{ or } f\,(0,t) = U\sin nt;\; g\,(0,t) = 0 \text{ for } t > 0, \\ f\,(z,t) = \Omega l;\; g\,(z,t) = 0 \text{ as } z \to \infty \text{ for all } t, \\ f\,(z,0) = \Omega l;\; g\,(z,0) = 0 \text{ for } z > 0. \end{aligned}\right\} \tag{3.7}$$

Eliminating $\widehat{p}$ from Eqs. (3.3) to (3.5), using boundary conditions $(3.7)_2$ and then combining the resulting equations one can write the following problem:

$$\upsilon\frac{\partial^2 G}{\partial z^2} - \frac{\partial G}{\partial t} + W_0\frac{\partial G}{\partial z} - (N + i\Omega)\, G = 0, \tag{3.8}$$

$$\left.\begin{aligned} G\,(0,t) = \tfrac{U}{\Omega l}\cos nt - 1 \text{ or } G\,(0,t) = \tfrac{U}{\Omega l}\sin nt - 1;\; t > 0, \\ G\,(z,t) = 0, \text{ as } z \to \infty \text{ for all } t, \\ G\,(z,0) = 0 \text{ for } z > 0 \end{aligned}\right\} \tag{3.9}$$

in which

$$G = \frac{f}{\Omega l} + i\frac{g}{\Omega l} - 1 \tag{3.10}$$

and

$$N = \frac{\sigma B_0^2\,(1 + im)}{\rho\,(1 + m^2)}.$$

Introducing

$$H = Ge^{i\Omega t} \tag{3.11}$$

the governing problem becomes

$$\upsilon\frac{\partial^2 H}{\partial z^2} - \frac{\partial H}{\partial t} + W_0\frac{\partial H}{\partial z} - NH = 0, \tag{3.12}$$

$$\begin{aligned} H(0,t) &= -1+\frac{U}{\Omega l}\cos nt \ \text{ or } \ H(0,t)=-1+\frac{U}{\Omega l}\sin nt;\ t>0, \qquad (3.13)\\ H(z,t) &= 0 \text{ as } z\to\infty \text{ for all } t;\ H(z,0)=0. \end{aligned}$$

3.2 Exact solution to the problem for non-resonant frequencies $(k = n/\Omega \neq 1)$

By means of the Laplace transform, we obtain the following solutions for the resulting transformed problems for $U\cos nt$

$$\overline{H}(z,s)=\left[-\frac{1}{s-i\Omega}+\frac{U}{2\Omega l}\left\{\frac{1}{s+i(n-\Omega)}+\frac{1}{s-i(n+\Omega)}\right\}\right]e_1;\ n>\Omega, \qquad (3.14)$$

$$\overline{H}(z,s)=\left[-\frac{1}{s-i\Omega}+\frac{U}{2\Omega l}\left\{\frac{1}{s-i(\Omega-n)}+\frac{1}{s-i(n+\Omega)}\right\}\right]e_1;\ n<\Omega \qquad (3.15)$$

and for $U\sin nt$

$$\overline{H}(z,s)=\left[-\frac{1}{s-i\Omega}+\frac{iU}{2\Omega l}\left\{\frac{1}{s+i(n-\Omega)}-\frac{1}{s-i(n+\Omega)}\right\}\right]e_1;\ n>\Omega, \qquad (3.16)$$

$$\overline{H}(z,s)=\left[-\frac{1}{s-i\Omega}+\frac{iU}{2\Omega l}\left\{\frac{1}{s-i(\Omega-n)}-\frac{1}{s-i(n+\Omega)}\right\}\right]e_1;\ n<\Omega, \qquad (3.17)$$

in which

$$e_1=e^{-\left[\frac{W_0}{2v}+\sqrt{\left(\frac{W_0}{2v}\right)^2+\frac{N}{v}}\right]z}. \qquad (3.18)$$

After the inversion for the Laplace transform, the Eqs. (3.14) to (3.18) yield the following suction solutions for $U\cos nt$, $n > \Omega$

$$\frac{f}{\Omega l}+i\frac{g}{\Omega l}=1+e^{-\sqrt{2}W\xi}\left\{\begin{array}{c} -\frac{1}{2}\left(\begin{array}{c} e^{(x_2+iy_2)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_2+iy_2)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_2+iy_2)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_2+iy_2)\sqrt{\frac{\tau}{2}}\right)\end{array}\right) \\ +\frac{U}{2\Omega l}e^{-ik\tau}\left(\begin{array}{c} e^{(x_3+iy_3)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_3+iy_3)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_3+iy_3)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_3+iy_3)\sqrt{\frac{\tau}{2}}\right)\end{array}\right) \\ +\frac{U}{2\Omega l}e^{ik\tau}\left(\begin{array}{c} e^{(x_4+iy_4)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_4+iy_4)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_4+iy_4)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_4+iy_4)\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\} \tag{3.19}$$

and for $n < \Omega$

$$\frac{f}{\Omega l}+i\frac{g}{\Omega l}=1+e^{-\sqrt{2}W\xi}\left\{\begin{array}{c} -\frac{1}{2}\left(\begin{array}{c} e^{(x_2+iy_2)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_2+iy_2)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_2+iy_2)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_2+iy_2)\sqrt{\frac{\tau}{2}}\right)\end{array}\right) \\ +\frac{U}{2\Omega l}e^{-ik\tau}\left(\begin{array}{c} e^{(x_5+iy_5)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_5+iy_5)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_5+iy_5)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_5+iy_5)\sqrt{\frac{\tau}{2}}\right)\end{array}\right) \\ +\frac{U}{2\Omega l}e^{ik\tau}\left(\begin{array}{c} e^{(x_4+iy_4)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_4+iy_4)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_4+iy_4)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_4+iy_4)\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}. \tag{3.20}$$

For $U \sin nt$, $n > \Omega$

$$\frac{f}{\Omega l}+i\frac{g}{\Omega l} = 1+e^{-\sqrt{2}W\xi}\left\{\begin{array}{c} -\frac{1}{2}\left(\begin{array}{c} e^{(x_2+iy_2)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_2+iy_2)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_2+iy_2)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_2+iy_2)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ +i\frac{U}{2\Omega l}e^{-ik\tau}\left(\begin{array}{c} e^{(x_3+iy_3)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_3+iy_3)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_3+iy_3)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_3+iy_3)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ -i\frac{U}{2\Omega l}e^{ik\tau}\left(\begin{array}{c} e^{(x_4+iy_4)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_4+iy_4)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_4+iy_4)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_4+iy_4)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \end{array}\right\}, \tag{3.21}$$

and for $n < \Omega$

$$\frac{f}{\Omega l}+i\frac{g}{\Omega l} = 1+e^{-\sqrt{2}W\xi}\left\{\begin{array}{c} -\frac{1}{2}\left(\begin{array}{c} e^{(x_2+iy_2)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_2+iy_2)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_2+iy_2)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_2+iy_2)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ +i\frac{U}{2\Omega l}e^{-ik\tau}\left(\begin{array}{c} e^{(x_5+iy_5)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_5+iy_5)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_5+iy_5)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_5+iy_5)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ -i\frac{U}{2\Omega l}e^{ik\tau}\left(\begin{array}{c} e^{(x_4+iy_4)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_4+iy_4)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_4+iy_4)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_4+iy_4)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \end{array}\right\}, \tag{3.22}$$

where

$$\begin{aligned}
x_2 &= \left[\sqrt{\left(W^2+\frac{N_1}{1+m^2}\right)^2+\left(1+\frac{N_1 m}{1+m^2}\right)^2}+\left(W^2+\frac{N_1}{1+m^2}\right)\right]^{\frac{1}{2}},\\
x_3 &= \left[\sqrt{\left(W^2+\frac{N_1}{1+m^2}\right)^2+\left(k-1-\frac{N_1 m}{1+m^2}\right)^2}+\left(W^2+\frac{N_1}{1+m^2}\right)\right]^{\frac{1}{2}},\\
x_4 &= \left[\sqrt{\left(W^2+\frac{N_1}{1+m^2}\right)^2+\left(k+1+\frac{N_1 m}{1+m^2}\right)^2}+\left(W^2+\frac{N_1}{1+m^2}\right)\right]^{\frac{1}{2}},\\
x_5 &= \left[\sqrt{\left(W^2+\frac{N_1}{1+m^2}\right)^2+\left(1+\frac{N_1 m}{1+m^2}-k\right)^2}+\left(W^2+\frac{N_1}{1+m^2}\right)\right]^{\frac{1}{2}},\\
y_2 &= \left[\sqrt{\left(W^2+\frac{N_1}{1+m^2}\right)^2+\left(1+\frac{N_1 m}{1+m^2}\right)^2}-\left(W^2+\frac{N_1}{1+m^2}\right)\right]^{\frac{1}{2}},\\
y_3 &= \left[\sqrt{\left(W^2+\frac{N_1}{1+m^2}\right)^2+\left(k-1-\frac{N_1 m}{1+m^2}\right)^2}-\left(W^2+\frac{N_1}{1+m^2}\right)\right]^{\frac{1}{2}},\\
y_4 &= \left[\sqrt{\left(W^2+\frac{N_1}{1+m^2}\right)^2+\left(k+1+\frac{N_1 m}{1+m^2}\right)^2}-\left(W^2+\frac{N_1}{1+m^2}\right)\right]^{\frac{1}{2}},\\
y_5 &= \left[\sqrt{\left(W^2+\frac{N_1}{1+m^2}\right)^2+\left(1+\frac{N_1 m}{1+m^2}-k\right)^2}-\left(W^2+\frac{N_1}{1+m^2}\right)\right]^{\frac{1}{2}},
\end{aligned}$$

and

$$\xi=\sqrt{\frac{\Omega}{2\upsilon}}z,\;\; k=\frac{n}{\Omega},\;\; \tau=\Omega t,\; N_1=\frac{\sigma B_0^2}{\rho\Omega},\; W=\frac{W_0}{2\sqrt{\upsilon\Omega}}. \tag{3.23}$$

Note that in obtaining Eqs. (3.19) to (3.22), the Eqs. (3.10) and (3.11) have also been used. The solutions (3.19) to (3.22) are unsteady and valid for the suction case. For blowing, the unsteady solutions can be directly taken from the suction case *i.e.* from Eqs. (3.19) to (3.22) by replacing W by $-W_1$ ($W_1 > 0$). Further, the steady state solutions in the respective case can be obtained by using the following asymptotic values of the complementary error function

$$\operatorname{erf} c\left(\frac{\xi}{\sqrt{2\tau}} \pm (x_j + iy_j)\sqrt{\frac{\tau}{2}}\right) --- > \quad (0,2)\,, \qquad j = 1 \text{ to } 4. \tag{3.24}$$

3.2.1 Exact solution for the resonant case ($n/\Omega = 1$)

Employing the same methodology of solution as in the previous section, the unsteady suction solutions for $U\cos nt$ and $U\sin nt$ can be respectively written as

$$\frac{f}{\Omega l}+i\frac{g}{\Omega l} = 1+e^{-\sqrt{2}W\xi}\left\{\begin{array}{l} -\frac{1}{2}\left(\begin{array}{c} e^{(x_2+iy_2)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_2+iy_2)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_2+iy_2)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_2+iy_2)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ +\frac{U}{2\Omega l}e^{-ik\tau}\left(\begin{array}{c} e^{(x_6+iy_6)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_6+iy_6)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_6+iy_6)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_6+iy_6)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ +\frac{U}{2\Omega l}e^{ik\tau}\left(\begin{array}{c} e^{(x_7+iy_7)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_7+iy_7)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_7+iy_7)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_7+iy_7)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \end{array}\right\}, \tag{3.25}$$

$$\frac{f}{\Omega l}+i\frac{g}{\Omega l} = 1+e^{-\sqrt{2}W\xi}\left\{\begin{array}{l} -\frac{1}{2}\left(\begin{array}{c} e^{(x_2+iy_2)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_2+iy_2)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_2+iy_2)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_2+iy_2)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ +i\frac{U}{2\Omega l}e^{-ik\tau}\left(\begin{array}{c} e^{(x_6+iy_6)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_6+iy_6)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_6+iy_6)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_6+iy_6)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ -i\frac{U}{2\Omega l}e^{ik\tau}\left(\begin{array}{c} e^{(x_7+iy_7)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(x_7+iy_7)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_7+iy_7)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(x_7+iy_7)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \end{array}\right\}, \tag{3.26}$$

in which

$$x_6 = \left[\sqrt{\left(W^2+\frac{N_1}{1+m^2}\right)^2+\left(\frac{N_1 m}{1+m^2}\right)^2}+\left(W^2+\frac{N_1}{1+m^2}\right)\right]^{\frac{1}{2}},$$

$$x_7 = \left[\sqrt{\left(W^2+\frac{N_1}{1+m^2}\right)^2+\left(2+\frac{N_1 m}{1+m^2}\right)^2}+\left(W^2+\frac{N_1}{1+m^2}\right)\right]^{\frac{1}{2}},$$

$$y_6 = \left[\sqrt{\left(W^2+\frac{N_1}{1+m^2}\right)^2+\left(\frac{N_1 m}{1+m^2}\right)^2}-\left(W^2+\frac{N_1}{1+m^2}\right)\right]^{\frac{1}{2}},$$

$$y_7 = \left[\sqrt{\left(W^2+\frac{N_1}{1+m^2}\right)^2+\left(2+\frac{N_1 m}{1+m^2}\right)^2}-\left(W^2+\frac{N_1}{1+m^2}\right)\right]^{\frac{1}{2}}.$$

For blowing $W = -W_1$ ($W_1 > 0$) and the respective unsteady solutions for $U\cos nt$ and $U\sin nt$ are

$$\frac{f}{\Omega l}+i\frac{g}{\Omega l} = 1+e^{\sqrt{2}W_1\xi}\left\{\begin{array}{c} -\frac{1}{2}\left(\begin{array}{c} e^{(\widetilde{x}_2+i\widetilde{y}_2)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(\widetilde{x}_2+i\widetilde{y}_2)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\widetilde{x}_2+i\widetilde{y}_2)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(\widetilde{x}_2+i\widetilde{y}_2)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ +\frac{U}{2\Omega l}e^{-ik\tau}\left(\begin{array}{c} e^{(\widetilde{x}_3+i\widetilde{y}_3)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(\widetilde{x}_3+i\widetilde{y}_3)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\widetilde{x}_3+i\widetilde{y}_3)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(\widetilde{x}_3+i\widetilde{y}_3)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ +\frac{U}{2\Omega l}e^{ik\tau}\left(\begin{array}{c} e^{(\widetilde{x}_4+i\widetilde{y}_4)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(\widetilde{x}_4+i\widetilde{y}_4)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\widetilde{x}_4+i\widetilde{y}_4)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(\widetilde{x}_4+i\widetilde{y}_4)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \end{array}\right\}, \tag{3.27}$$

$$\frac{f}{\Omega l}+i\frac{g}{\Omega l} = 1+e^{\sqrt{2}W_1\xi}\left\{\begin{array}{c} -\frac{1}{2}\left(\begin{array}{c} e^{(\widetilde{x}_2+i\widetilde{y}_2)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(\widetilde{x}_2+i\widetilde{y}_2)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\widetilde{x}_2+i\widetilde{y}_2)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(\widetilde{x}_2+i\widetilde{y}_2)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ +i\frac{U}{2\Omega l}e^{-ik\tau}\left(\begin{array}{c} e^{(\widetilde{x}_3+i\widetilde{y}_3)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(\widetilde{x}_3+i\widetilde{y}_3)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\widetilde{x}_3+i\widetilde{y}_3)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(\widetilde{x}_3+i\widetilde{y}_3)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ -i\frac{U}{2\Omega l}e^{ik\tau}\left(\begin{array}{c} e^{(\widetilde{x}_4+i\widetilde{y}_4)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}+(\widetilde{x}_4+i\widetilde{y}_4)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\widetilde{x}_4+i\widetilde{y}_4)\xi}\operatorname{erfc}\left(\frac{\xi}{\sqrt{2\tau}}-(\widetilde{x}_4+i\widetilde{y}_4)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \end{array}\right\}, \tag{3.28}$$

where

$$\tilde{x}_3 = \left[\sqrt{\left(W_1^2+\frac{N_1}{1+m^2}\right)^2+\left(\frac{N_1 m}{1+m^2}\right)^2}+\left(W_1^2+\frac{N_1}{1+m^2}\right)\right]^{\frac{1}{2}},$$

$$\tilde{x}_4 = \left[\sqrt{\left(W_1^2+\frac{N_1}{1+m^2}\right)^2+\left(2+\frac{N_1 m}{1+m^2}\right)^2}+\left(W_1^2+\frac{N_1}{1+m^2}\right)\right]^{\frac{1}{2}}$$

$$\tilde{y}_3 = \left[\sqrt{\left(W_1^2+\frac{N_1}{1+m^2}\right)^2+\left(\frac{N_1 m}{1+m^2}\right)^2}-\left(W_1^2+\frac{N_1}{1+m^2}\right)\right]^{\frac{1}{2}},$$

$$\tilde{y}_4 = \left[\sqrt{\left(W_1^2+\frac{N_1}{1+m^2}\right)^2+\left(2+\frac{N_1 m}{1+m^2}\right)^2}-\left(W_1^2+\frac{N_1}{1+m^2}\right)\right]^{\frac{1}{2}}.$$

In order to determine the steady state solutions, we use Eq. (3.24) and get for $U\cos nt$,

$$\frac{f}{\Omega l}+i\frac{g}{\Omega l}=1+e^{\sqrt{2}W_1\xi}\left\{-e^{(\tilde{x}_2+i\tilde{y}_2)\xi}+\frac{U}{\Omega l}e^{-i\tau}e^{(\tilde{x}_3+i\tilde{y}_3)\xi}+\frac{U}{\Omega l}e^{i\tau}e^{(\tilde{x}_4+i\tilde{y}_4)\xi}\right\} \quad (3.29)$$

and for $U\sin nt$

$$\frac{f}{\Omega l}+i\frac{g}{\Omega l}=1+e^{\sqrt{2}W_1\xi}\left\{-e^{(\tilde{x}_2+i\tilde{y}_2)\xi}+i\frac{U}{\Omega l}e^{-i\tau}e^{(\tilde{x}_3+i\tilde{y}_3)\xi}-i\frac{U}{\Omega l}e^{i\tau}e^{(\tilde{x}_4+i\tilde{y}_4)\xi}\right\}. \quad (3.30)$$

3.3 Results and discussion

In this chapter, we study the effects of Hall current on the flow due to non-coaxial rotation of an oscillating disk and a fluid at infinity in the presence of suction and blowing. For small times, the analytic solutions have been obtained using Laplace transform method. The analytic solutions for large times have been computed through asymptotic behavior of the complementary error function. We also compared the present velocity profiles mathematically and graphically with those given in the reference [50]. The results are found in well agreement. The mathematical problem contains altogether six dimensionless parameters (ξ, k, τ, W, N_1 and ϵ). A detailed investigation of the Hall parameter for cosine and sine oscillations when the angular velocity is greater than, smaller than or equal to the frequency of oscillations for both suction and blowing is analyzed.

To illustrate how the Hall effect modifies the structure of flow, the profiles of velocity are plotted for both cosine and sine oscillations when $\tau = 0.3$, $\frac{U}{4\Omega l} = 1$, $N_1 = 5$. The effect of Hall parameter $m = 0.5$, 1, 2 on velocity profiles for cosine oscillations when $W = 0$ and $k = -5$, 1, 5 are shown in Figs.3.2 (i, ii) , respectively. It is seen from these Figs. that the magnitudes of $f/\Omega l$ increases and $g/\Omega l$ decreases with the increase of m. Moreover, it is observed that boundary layer thickness for $k = 1$ is smallest when compared with $k < 1$ and $k > 1$. Also, the boundary layer thickness in case of $k < 1$ is smaller than that of $k > 1$.

In order to see the variation of Hall parameter $m = 0.5$, 1, 2 on the velocity profiles in presence of suction $W = 0.5$ and cosine oscillation, we display Figs.3.2 (i, ii)for $k = -5$, 1, 5. It appears that when the applied magnetic field is strong, both $f/\Omega l$ and $g/\Omega l$ depend strongly on the Hall parameter m. The magnitude of $f/\Omega l$ increases while $g/\Omega l$ decreases for large m. Here, the boundary layer thickness is minimum and the velocity profiles are maximum when $k = 1$. Also, the velocity profiles for $k > 1$ are greater than for $k < 1$. Further, the comparison of Figs. 3.2 (i, ii) and 3.3 (i, ii) indicate that velocity profiles and boundary layer thicknesses are smaller in case of suction. This is not surprising; it is known that suction causes reduction in the boundary layer thickness.

To demonstrate the effect of Hall parameter $m = 0.5$, 1, 2 on velocity profiles in blowing $W = -0.5$ and cosine oscillations, the Figs.3.4 (i, ii) are prepared for $k = -5$, 1, 5. These Figs. elucidate that the magnitude of velocity profiles is largest for $k = 1$ while the boundary layer thickness is smallest. It is also evident that velocity profiles $f/\Omega l$ and $g/\Omega l$ are larger for $k > 1$ when compared with that of $k < 1$.

Figs 3.5 (i, ii) to 3.7 (i, ii) illustrate the variation of Hall parameter $m = 0.5, 1, 2$ on $f/\Omega l$ and $g/\Omega l$ for $W = 0$, $W > 0$ and $W < 0$ for sine oscillations. It is obvious from Figs. 3.2 (i, ii) to 3.7 (i, ii) that boundary layer thickness in case of sine oscillations are smaller than that of cosine oscillations.

3.4 Conclusions

The investigation of the effects of Hall current on the flow characteristics due to non-coaxial rotations of disk and a fluid has been investigated in this chapter. The following conclusions have been emerged:

- One important point which should be taken into account is that in the presence of Hall parameter, the asymptotic steady solution for blowing and resonance exists.
- When the external magnetic field is strong, the role of Hall parameter becomes more significant.
- As the Hall parameter increases the magnitude of primary velocity $f/\Omega l$ increases while both secondary velocity $g/\Omega l$ and boundary layer thickness decreases.

- The layer thickness in case of sine oscillations are smaller than for cosine oscillations.
- The present analysis of small magnetic Reynolds number is valid for flow of liquid metals or slightly ionized gas. Note that the term involving electron pressure gradient in Eq. (3.2) is negligible for ionized gas and $p_e = p/2$ for fully ionized gas (ignoring ion-slip term).
- The results in the absence of Hall effects can be recovered by taking $m = 0$. This shows that the obtained results are in good agreement with the existing results [50].

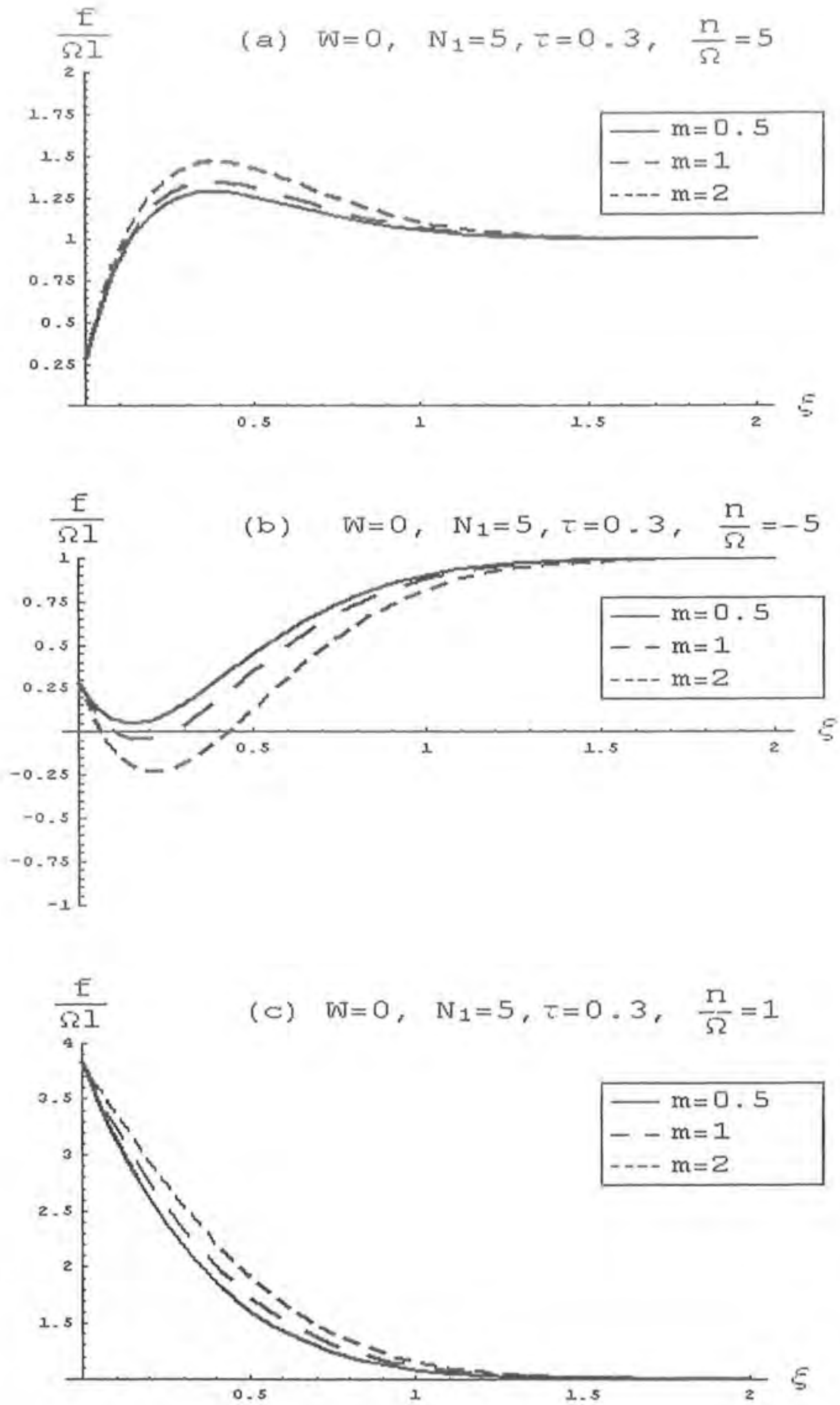

Fig.3.2(i) The effect of Hall parameter on $\frac{f}{\Omega l}$ for cosine oscillation in the absence of suction and blowing at $\left(\frac{U}{4\Omega l}=1\right)$.

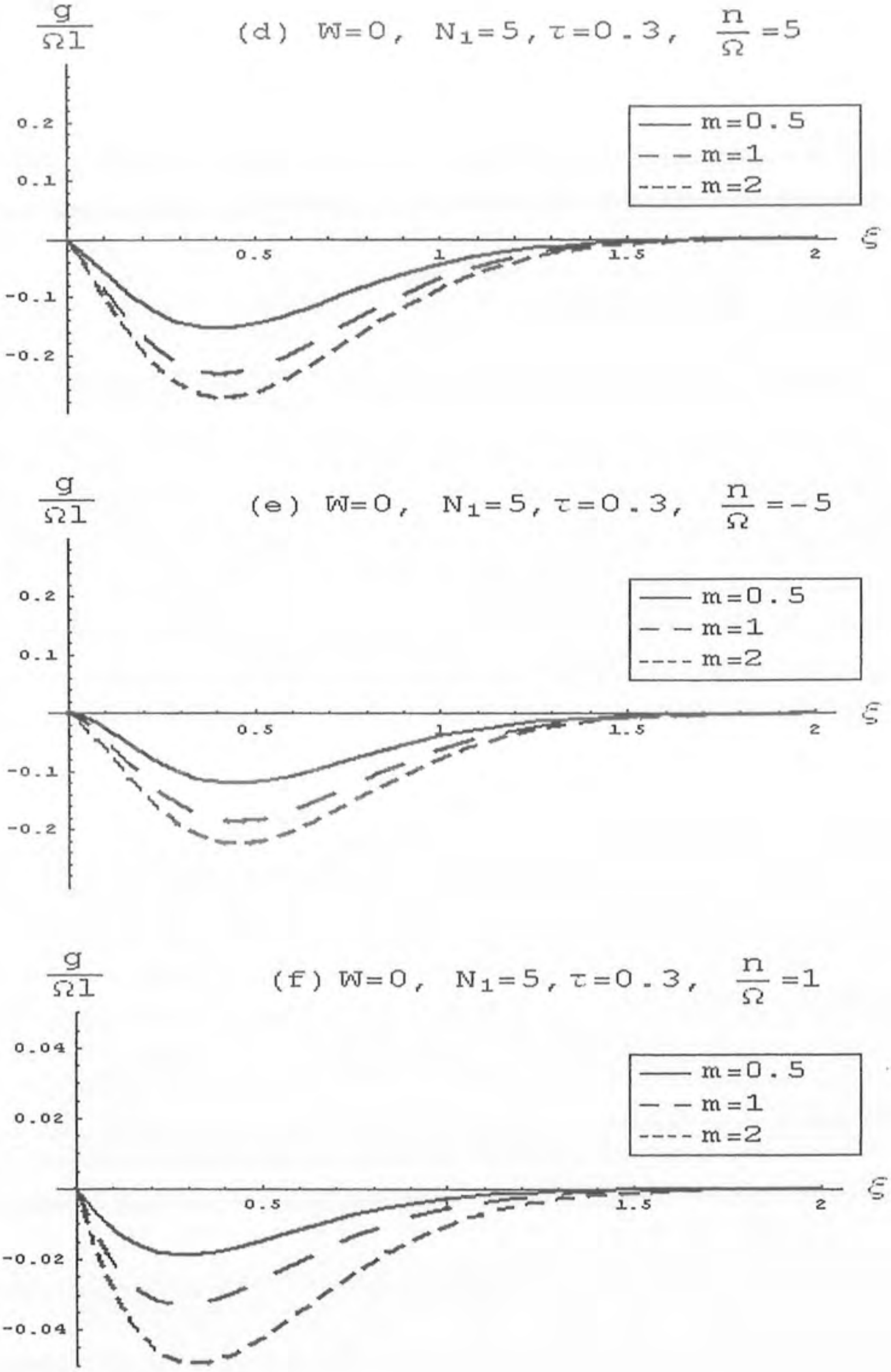

Fig.3.2(ii) The effect of Hall parameter on $\frac{g}{\Omega l}$ for cosine oscillation in the absence of suction and blowing at $\left(\frac{U}{4\Omega l}=1\right)$.

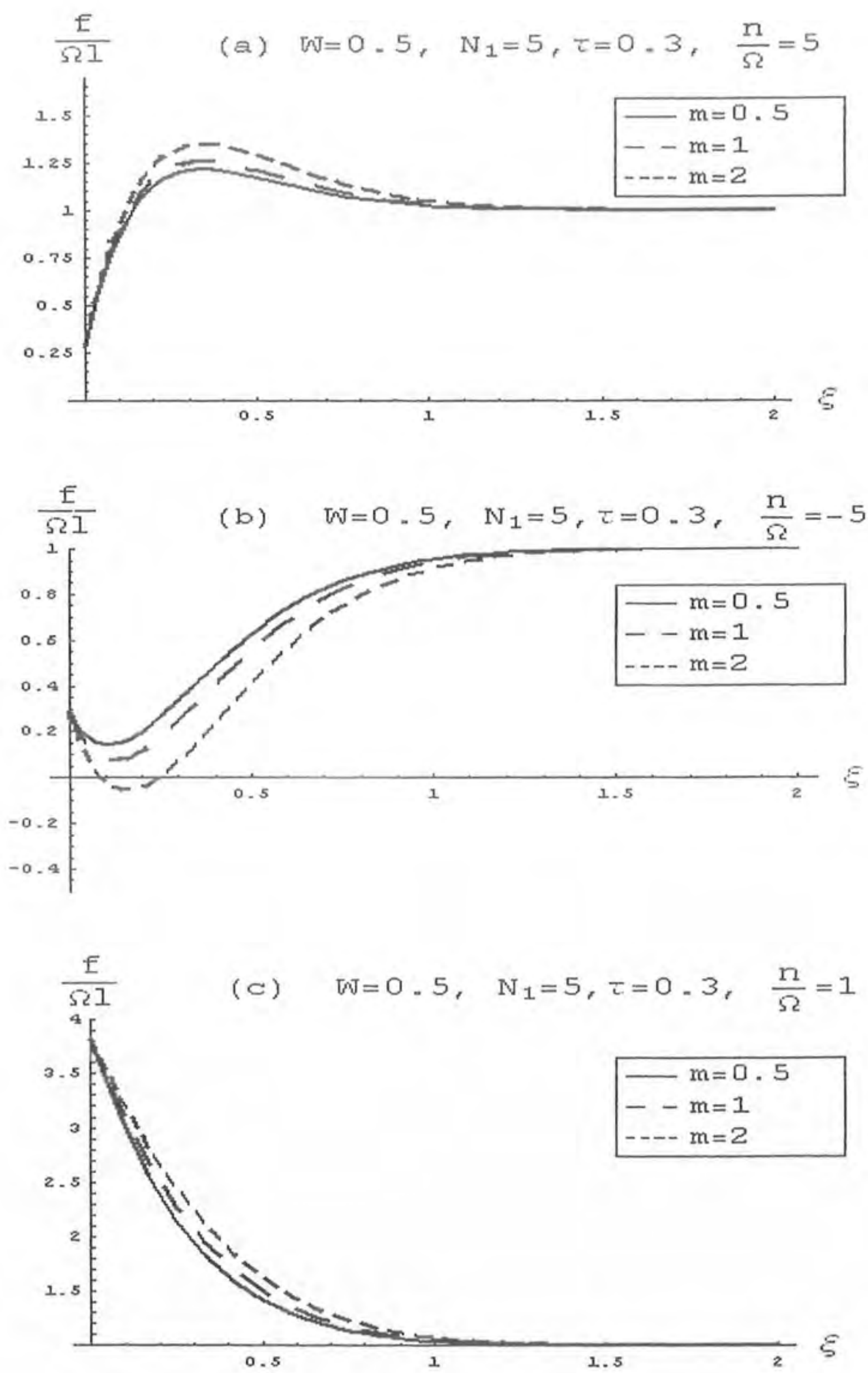

Fig.3.3(i) The effect of Hall parameter on $\frac{f}{\Omega l}$ for cosine oscillation in the presence of suction at $\left(\frac{U}{4\Omega l}=1\right)$.

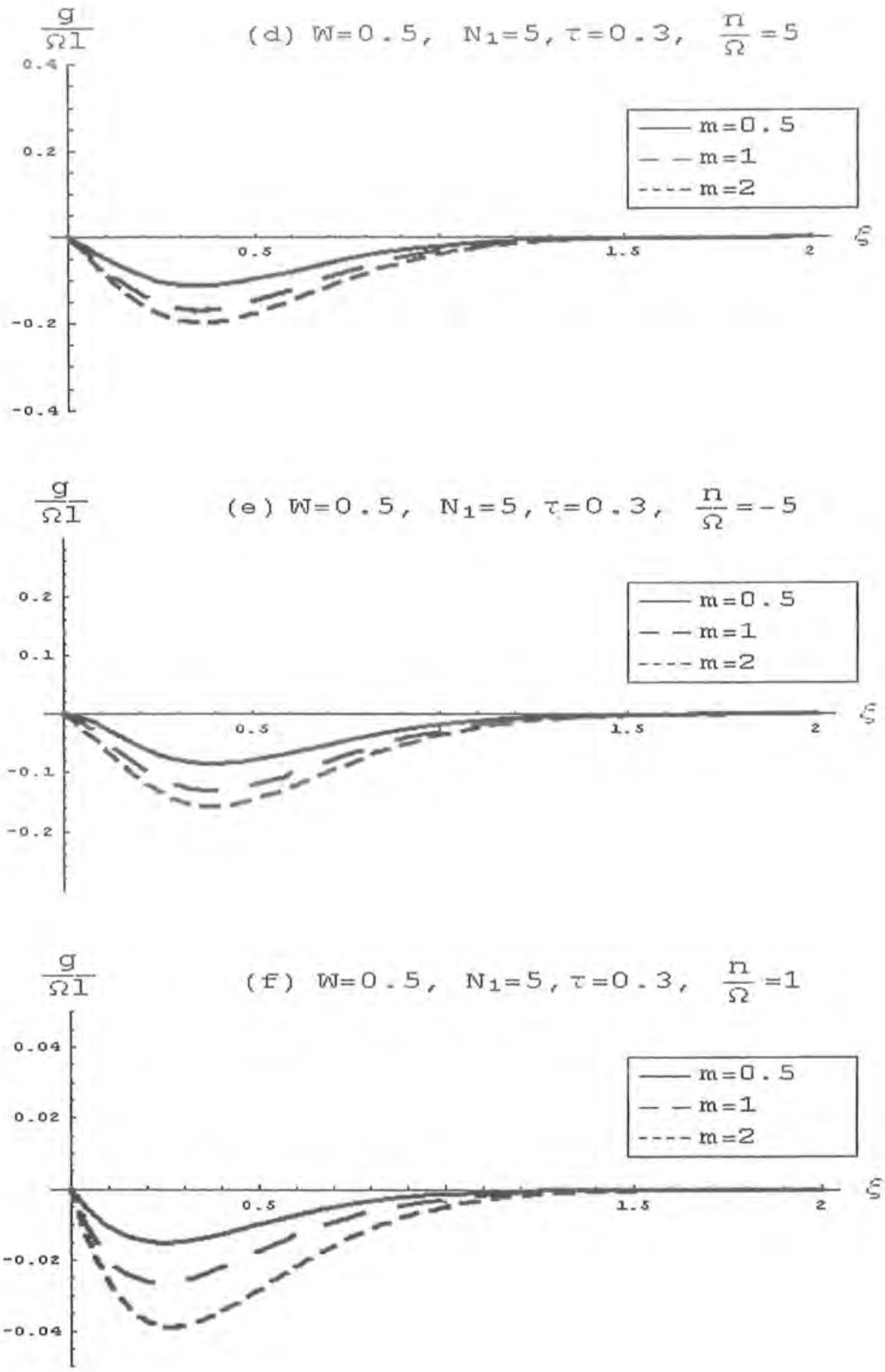

Fig.3.3(ii) The effect of Hall parameter on $\frac{g}{\Omega l}$ for cosine oscillation in the presence of suction at $\left(\frac{U}{4\Omega l}=1\right)$.

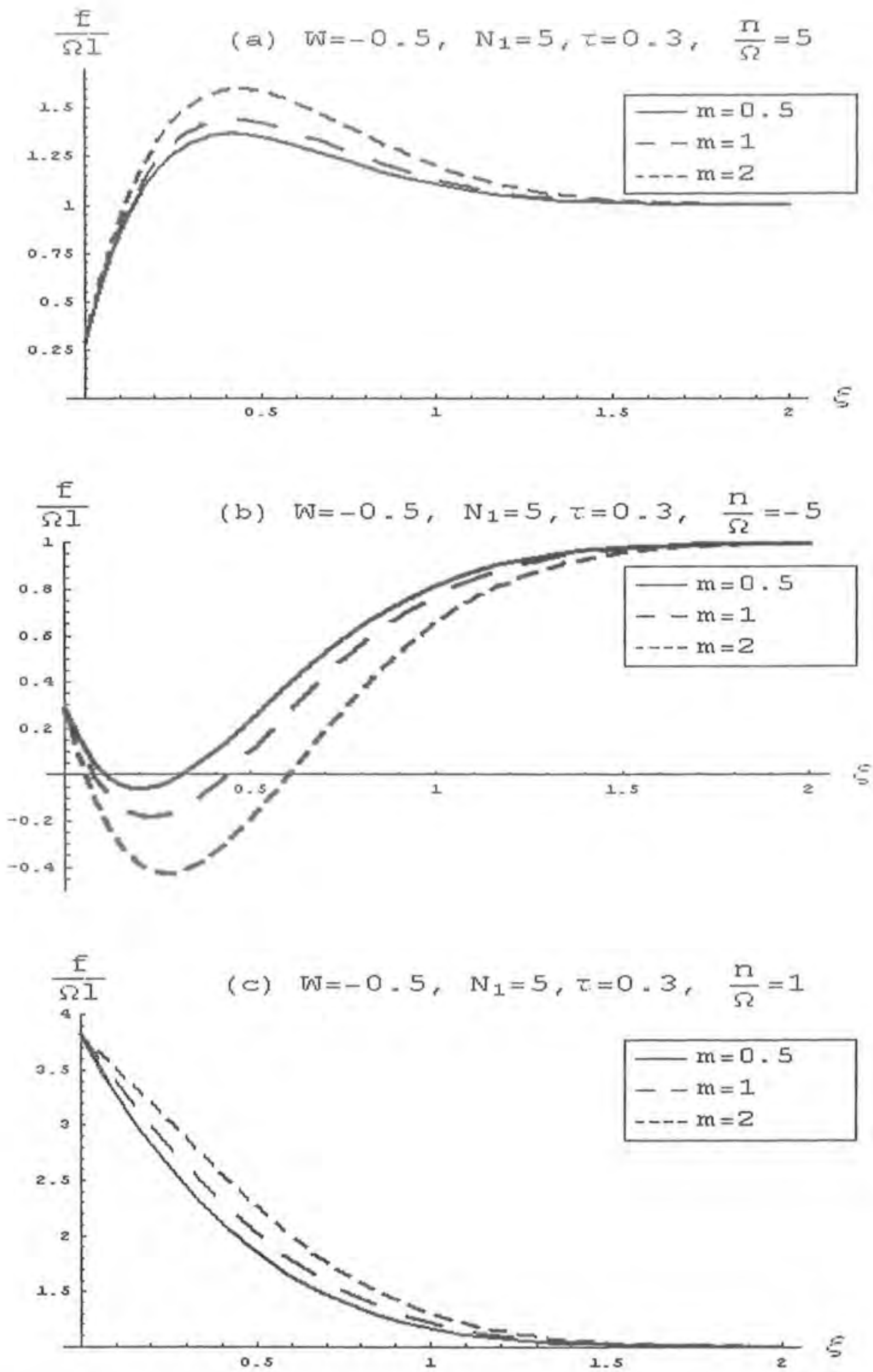

Fig.3.4(i) The effect of Hall parameter on $\frac{f}{\Omega l}$ for cosine oscillation in the presence of blowing at $\left(\frac{U}{4\Omega l}=1\right)$.

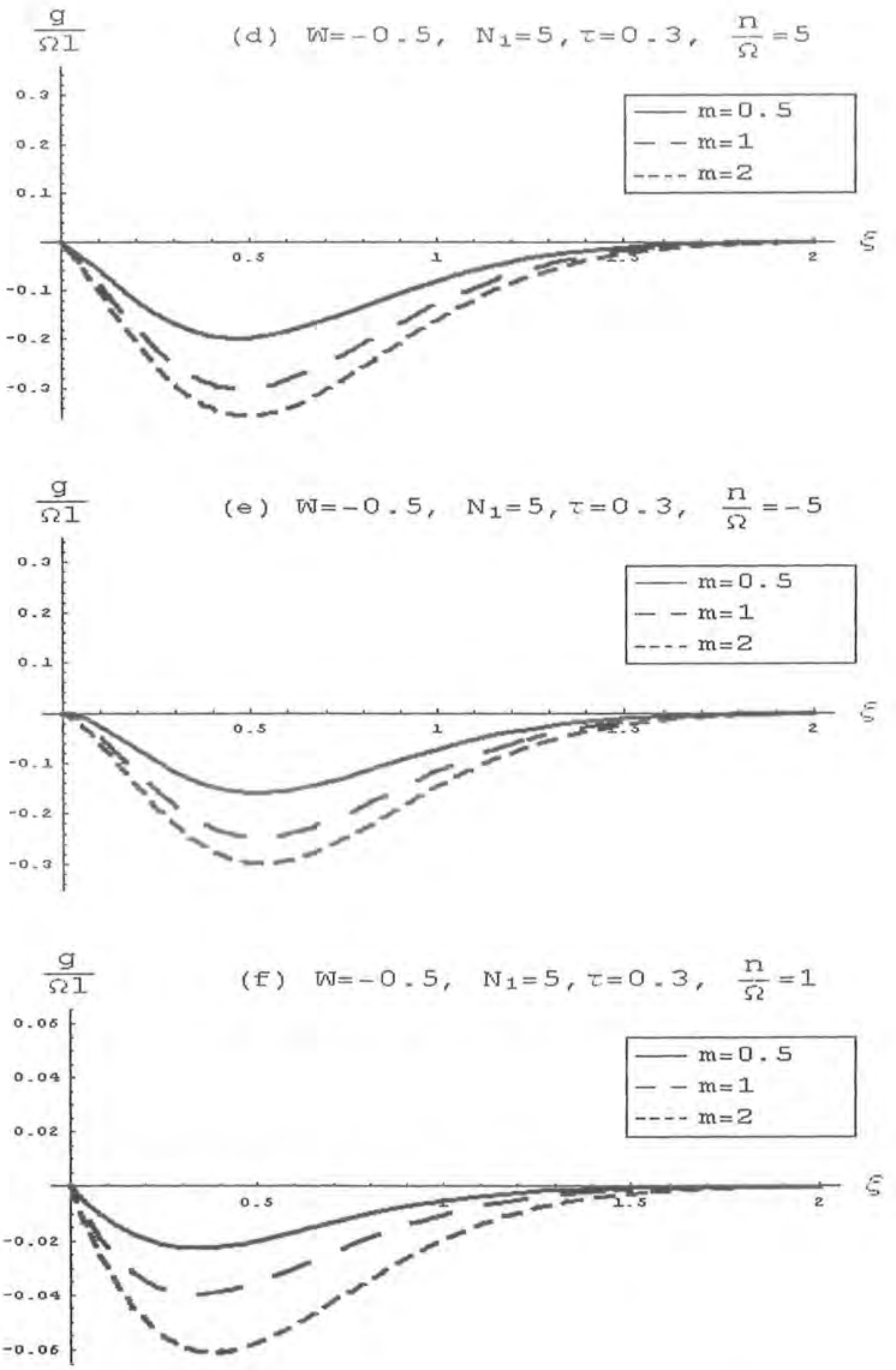

Fig.3.4(ii) The effect of Hall parameter on $\frac{g}{\Omega l}$ for cosine oscillation in the presence of blowing at $\left(\frac{U}{4\Omega l}=1\right)$.

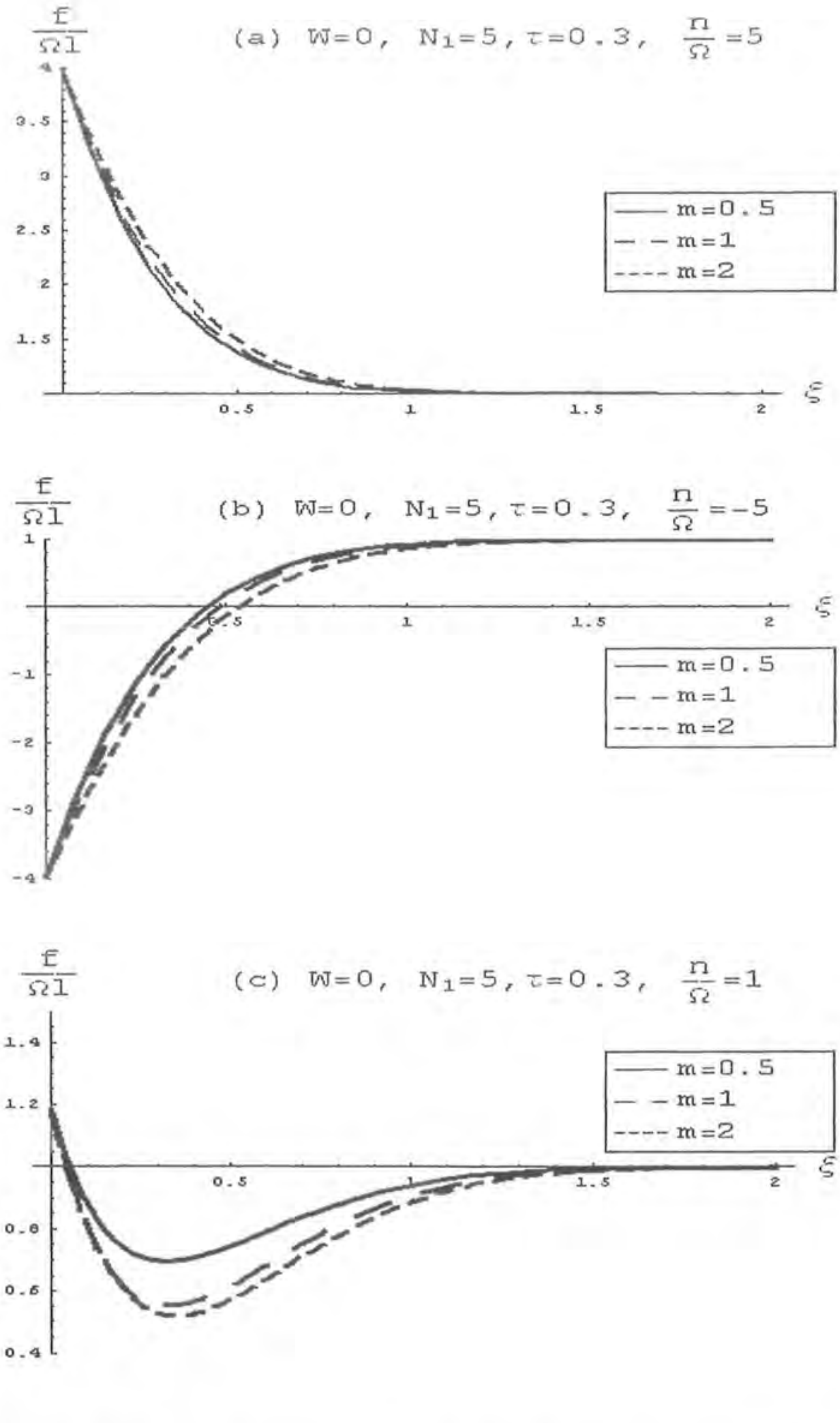

Fig.3.5(i) The effect of Hall parameter on $\frac{f}{\Omega l}$ for the sine oscillation in the absence of suction and blowing at $\left(\frac{U}{4\Omega l}=1\right)$.

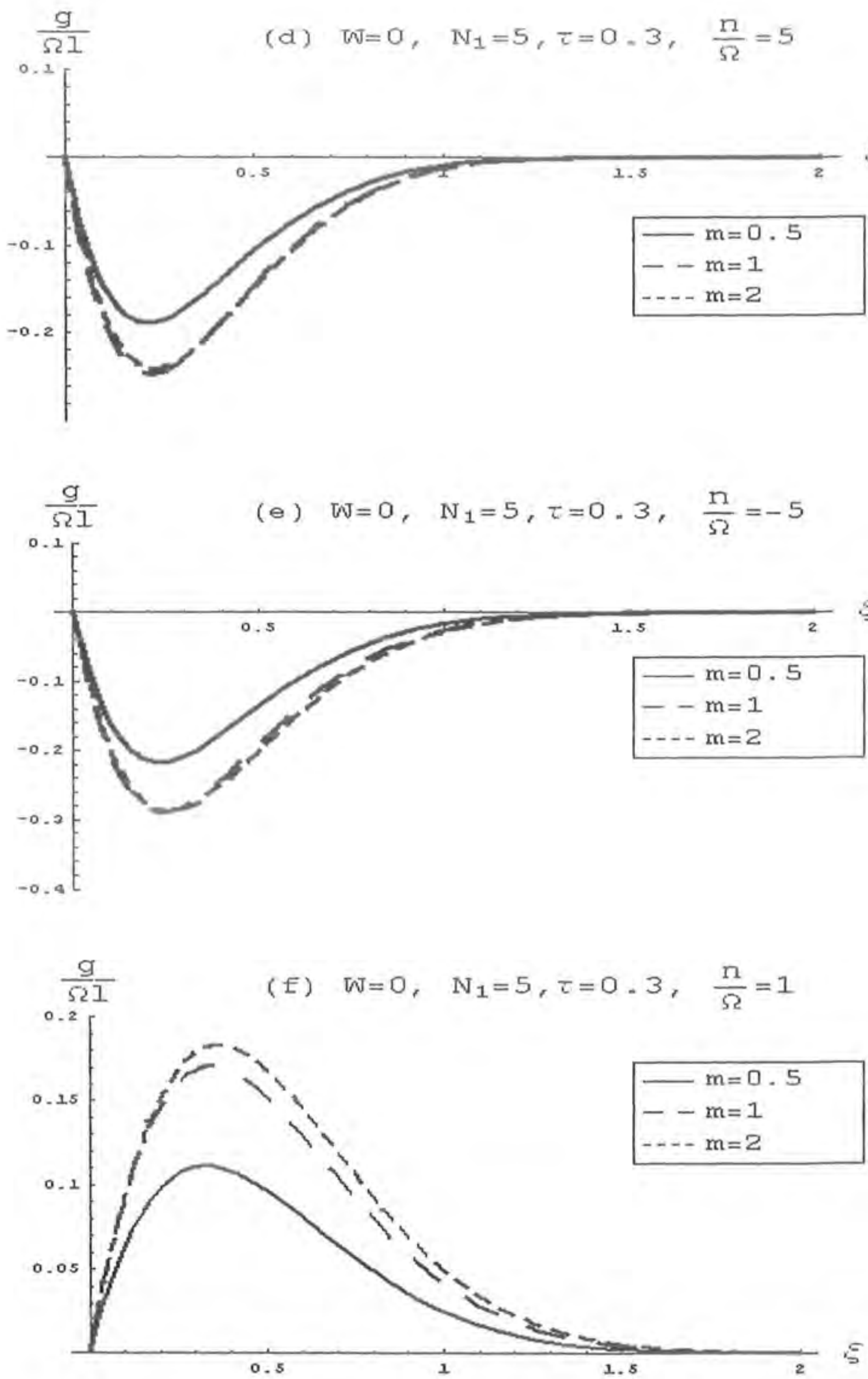

Fig.3.5(ii) The effect of Hall parameter on $\frac{g}{\Omega l}$ for the sine oscillation in the absence of suction and blowing at $\left(\frac{U}{4\Omega l}=1\right)$.

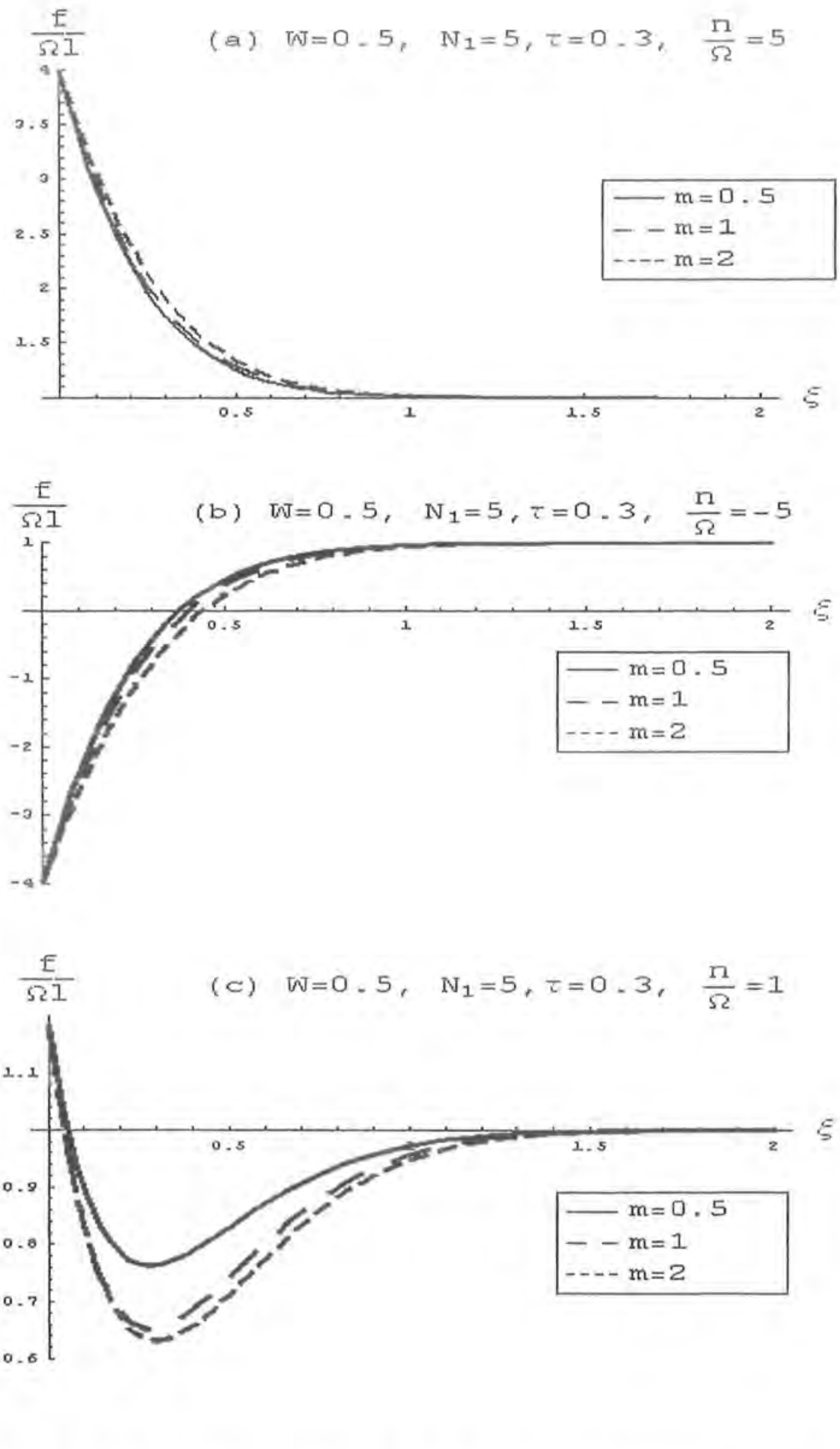

Fig.3.6(i) The effect of Hall parameter on $\frac{f}{\Omega l}$ for sine oscillation in the presence of suction at $\left(\frac{U}{4\Omega l}=1\right)$.

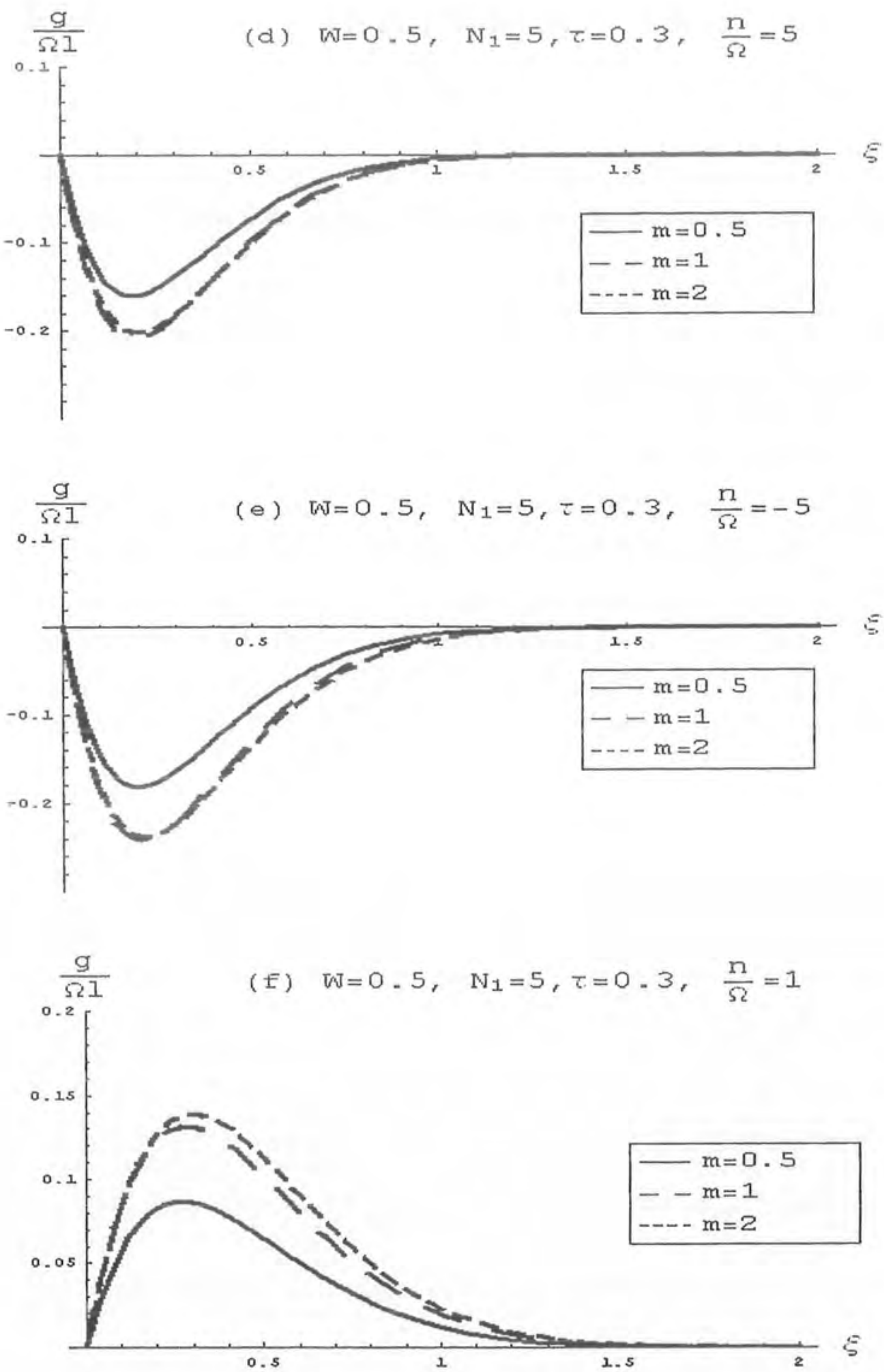

Fig.3.6(ii) The effect of Hall parameter on $\frac{g}{\Omega l}$ for sine oscillation in the presence of suction at $\left(\frac{U}{4\Omega l} = 1\right)$.

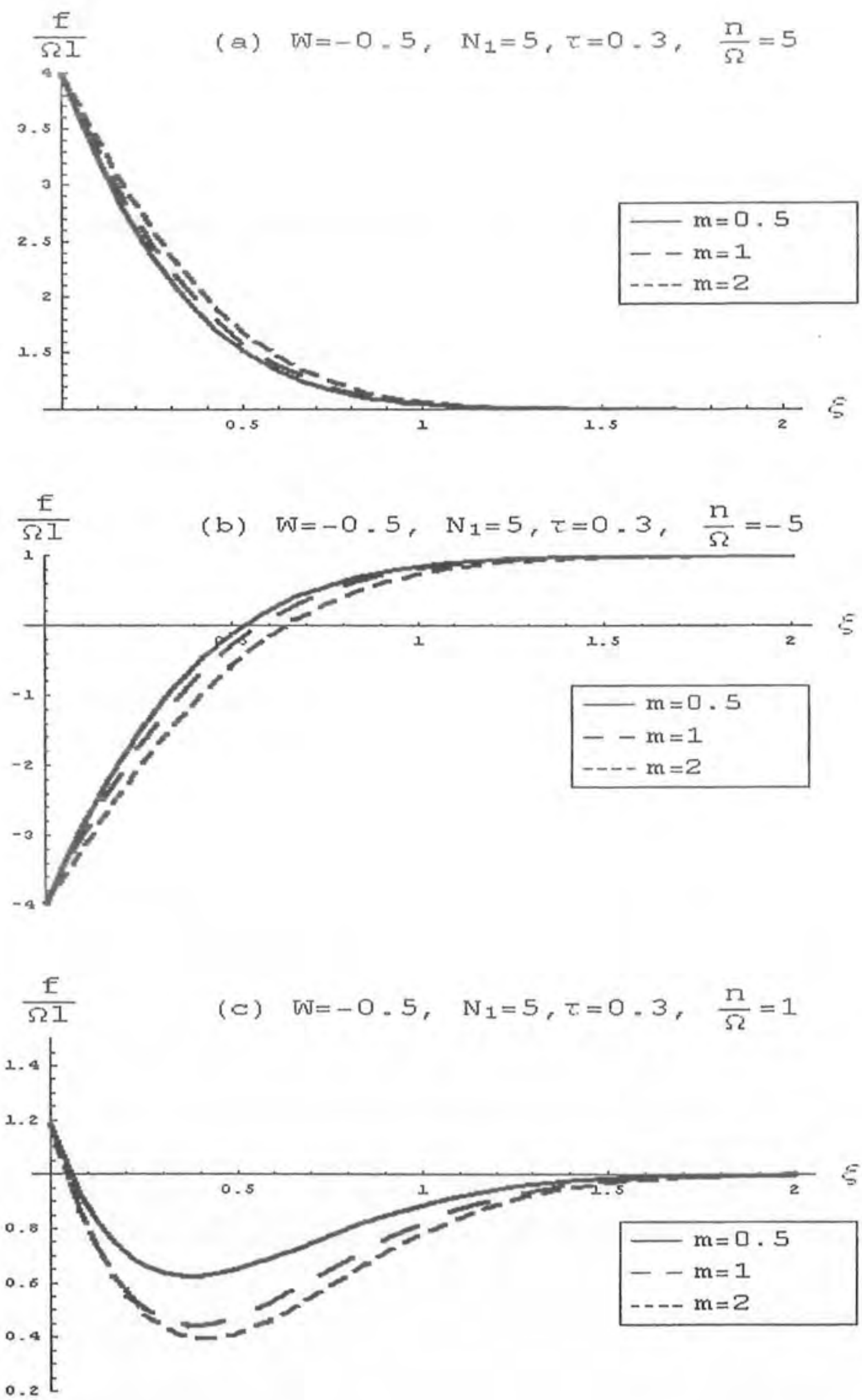

Fig.3.7(i) The effect of Hall parameter on $\frac{f}{\Omega l}$ for sine oscillation in the presence of blowing at $\left(\frac{U}{4\Omega l} = 1\right)$.

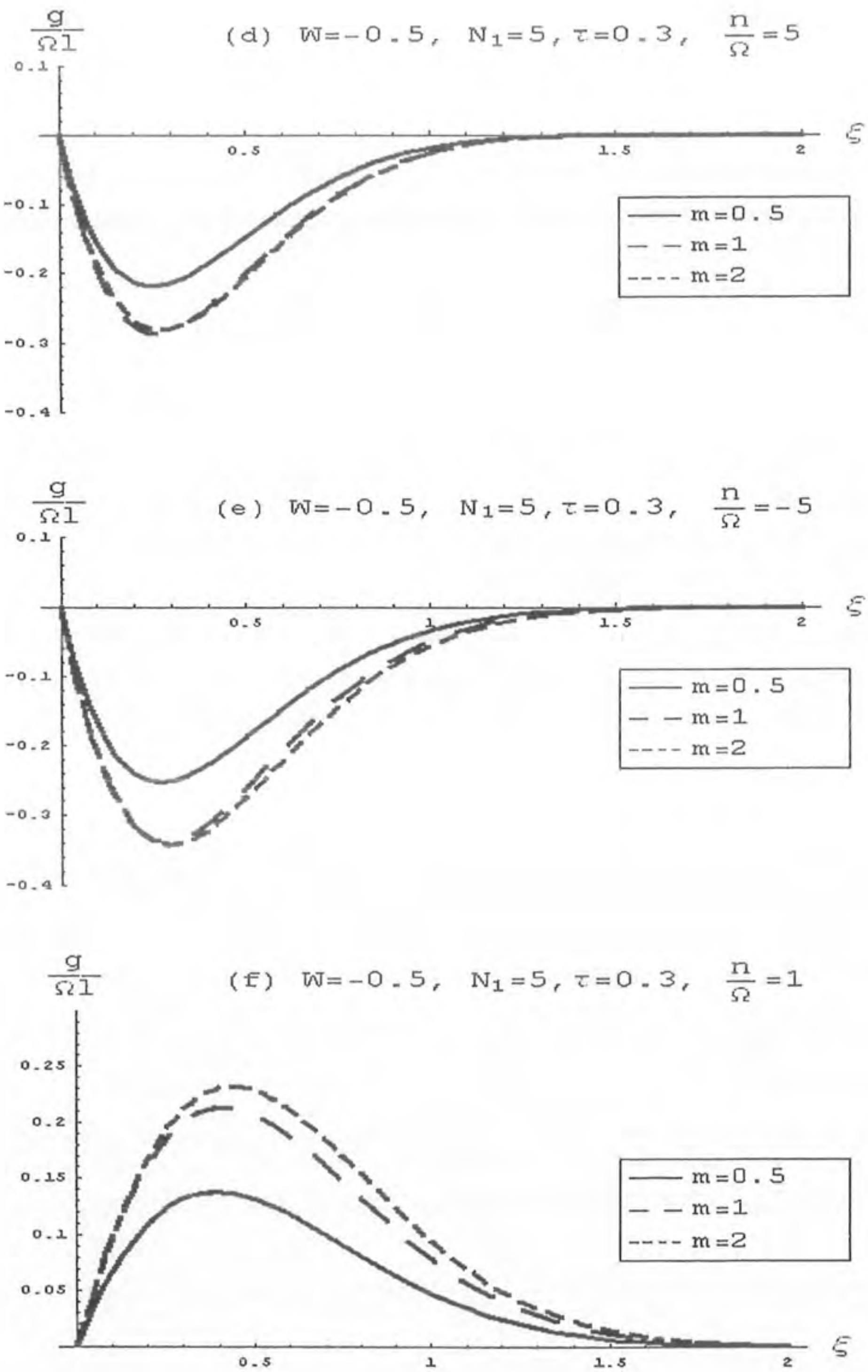

Fig.3.7(ii) The effect of Hall parameter on $\frac{g}{\Omega l}$ for sine oscillation in the presence of blowing at $\left(\frac{U}{4\Omega l}=1\right)$.

Chapter 4

Second grade fluid flow induced by non-coaxial rotations of porous oscillating disk and a fluid at infinity

In this chapter, the developed flow of hydrodynamic non-Newtonian fluid induced by the non-coaxial rotations of porous oscillating disk and a fluid at infinity has been studied with the emphasis of rheological effect. The developed velocity field is predicted by a second grade fluid model. Adopting the Laplace transform and perturbation series methods, the unsteady and steady solutions for velocity have been established analytically for all values of the frequencies in suction and blowing cases. It is also found in hydrodynamic second grade fluid that the steady blowing solution does not satisfy the boundary condition at infinity.

4.1 Description and formulation of the flow problem

Here, the flow field of the problem is bounded by an infinite porous disk located at $z = 0$. The fluid considered is non-Newtonian (second grade), incompressible, hydrodynamic and occupies the semi-infinite space $z > 0$. The disk and the fluid at infinity rotate initially about the z'−axis with the same $\mathbf{\Omega}$. But at $t = 0$, the fluid rotates about z'−axis with $\mathbf{\Omega}$ and the disk suddenly starts to rotate about the z−axis with the same $\mathbf{\Omega}$ and start to execute non-torsional oscillations along the x−axis. Thus, the flow equations here is described through Eqs.(1.7), (2.2), (2.11) and

$$\frac{d\mathbf{V}}{dt} = \frac{1}{\rho} div\mathbf{T}. \tag{4.1}$$

The resulting scalar equations in terms of f and g are

$$\frac{\partial f}{\partial t} - W_0\frac{\partial f}{\partial z} - \Omega g = -\frac{1}{\rho}\frac{\partial \widetilde{p}}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 f}{\partial z^2} + \frac{\alpha_1}{\rho}\left[\frac{\partial^3 f}{\partial t \partial z^2} - W_0\frac{\partial^3 f}{\partial z^3} - \Omega\frac{\partial^2 g}{\partial z^2}\right], \tag{4.2}$$

$$\frac{\partial g}{\partial t} - W_0\frac{\partial g}{\partial z} + \Omega f = -\frac{1}{\rho}\frac{\partial \widetilde{p}}{\partial y} + \frac{\mu}{\rho}\frac{\partial^2 f}{\partial z^2} + \frac{\alpha_1}{\rho}\left[\frac{\partial^3 g}{\partial t \partial z^2} - W_0\frac{\partial^3 g}{\partial z^3} - \Omega\frac{\partial^2 f}{\partial z^2}\right], \tag{4.3}$$

$$0 = -\frac{1}{\rho}\frac{\partial \widetilde{p}}{\partial z}, \tag{4.4}$$

in which the modified pressure including the rheological effects is

$$\widetilde{p} = p_1 - \frac{\rho\Omega^2 r^2}{2} - (2\alpha_1 + \alpha_2)\left[\left(\frac{\partial f}{\partial z}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2\right].$$

The boundary and initial conditions corresponding to the problem are given in Eq. (3.7). It can also be noted that $\widetilde{p}$ is independent upon z.

Using the properties of the partial derivatives of the modified pressure and Eq. $(3.7)_2$, one obtains

$$\frac{\alpha_1}{\rho}\frac{\partial^3 F}{\partial t \partial z^2} - \frac{\alpha_1}{\rho} W_0 \frac{\partial^3 F}{\partial z^3} + \left(\nu - \frac{i\alpha_1 \Omega}{\rho}\right)\frac{\partial^2 F}{\partial z^2} + W_0 \frac{\partial F}{\partial z} - \frac{\partial F}{\partial t} - i\Omega F = -i\Omega^2 l, \quad (4.5)$$

where F is defined in Eq. (2.14) and $\nu = \frac{\mu}{\rho}$.

By virtue of Eqs. (2.14) and (3.7), we have

$$\left.\begin{aligned} F(0,t) = \Omega l + U\cos nt \quad \text{or} \quad F(0,t) = \Omega l + U \sin nt;\ t > 0, \\ F(z,t) = 0 \text{ as } z \to \infty \text{ for all } t, \\ F(z,0) = 0,\ z > 0. \end{aligned}\right\} \quad (4.6)$$

Let us introduce the following non-dimensional quantities

$$\beta = \frac{\Omega \alpha_1}{\nu \rho}, \quad H = \left(\frac{F}{\Omega l} - 1\right) e^{i\tau}. \quad (4.7)$$

Invoking the non-dimensional quantities in Eq. (3.23) and above, Eqs. (4.5) and (4.6) are reduced to the non-dimensional form as

$$\beta \frac{\partial^3 H}{\partial \tau \partial \xi^2} - \beta W \frac{\partial^3 H}{\partial \xi^3} + (1 - 2i\beta)\frac{\partial^2 H}{\partial \xi^2} + 2W\frac{\partial H}{\partial \xi} - 2\frac{\partial H}{\partial \tau} = 0, \quad (4.8)$$

$$\begin{aligned} H(0,\tau) &= e^{i\tau}\left(-1 + \frac{U}{\Omega l}\cos k\tau\right) \quad \text{or} \quad H(0,\tau) = e^{i\tau}\left(-1 + \frac{U}{\Omega l}\sin k\tau\right); (4.9) \\ H(\infty,\tau) &= 0, H(\xi,0) = 0. \end{aligned}$$

4.1.1 Solution of the problem for non-resonant case ($k \neq 1$)

Taking the Laplace transform of both sides of Eqs. (4.8) and (4.9), we obtain

$$\beta W \overline{H}''' - (1 - 2i\beta + \beta s)\overline{H}'' - 2W\overline{H}' + 2s\overline{H} = 0, \quad (4.10)$$

$$\overline{H}(0,s) = -\frac{1}{s-i} + \frac{U}{2\Omega l}\left[\frac{1}{s-i(k+1)} + \frac{1}{s+i(k-1)}\right], \tag{4.11}$$

or

$$\overline{H}(0,s) = -\frac{1}{s-i} + \frac{U}{2i\Omega l}\left[\frac{1}{s-i(k+1)} - \frac{1}{s+i(k-1)}\right], \tag{4.12}$$

$$\overline{H}(\infty,s) = 0. \tag{4.13}$$

In Eq. (4.10), prime denotes differentiation with respect to ξ. Equation (4.10) is the third-order differential equation for $\beta \neq 0$ and $W \neq 0$ and for $\beta = 0$ or $W = 0$ this reduces to equation governing the Newtonian fluid or no suction respectively. The analysis of the flow of the second grade fluids, in particular, and the viscoelastic fluids, in general, is more challenging mathematically and computationally. The equations of motion are higher order than the Navier-Stokes equations. Also, the order of the differential equation characterizing the flow is more than the number of the available boundary conditions. Thus, the adherence boundary condition is insufficient for determinacy. The method used to overcome this difficulty is to expand the solution as a power series in the non-Newtonian parameter. This is indeed how most of the flow problems were solved since the initial effort of Beard and Walters [87] who considered the two-dimensional flow of a viscoelastic fluid near a stagnation point. One may also refer, for example, to the works of Shrestha [88], Mishra and Mohapatra [89], Rajagopal et al.[90], Verma et al. [91] and Erdogan [92] for other problems in various geometries. Thus for the solution of Eq. (4.10), we expand $\overline{H}$ as follows:

$$\overline{H} = \overline{H}_1 + \beta\overline{H}_2 + O\left(\beta^2\right) \tag{4.14}$$

which is valid for small values of β only. On substituting Eq. (4.14) into Eqs. (4.10) to (4.13) and then comparing coefficients of like powers of β one obtains the following systems of differential equations along with the appropriate boundary conditions for $U\cos nt$ and $U\sin nt$.

4.1.2 Zeroth and first-order systems

$$\overline{H}_1'' + 2W\overline{H}_1' - 2s\overline{H}_1 = 0, \tag{4.15}$$

$$\overline{H}_1(0,s) = -\frac{1}{s-i} + \frac{U}{2\Omega l}\left(\frac{1}{s-i(k+1)} + \frac{1}{s+i(k-1)}\right), \tag{4.16}$$

or

$$\overline{H}_1(0,s) = -\frac{1}{s-i} + \frac{U}{2i\Omega l}\left(\frac{1}{s-i(k+1)} - \frac{1}{s+i(k-1)}\right), \tag{4.17}$$

$$\overline{H}_1(\infty,s) = 0. \tag{4.18}$$

$$W\overline{H}_1''' - \overline{H}_2'' - (s-2i)\overline{H}_1'' - 2W\overline{H}_2' + 2s\overline{H}_2 = 0, \tag{4.19}$$

$$\overline{H}_2(0,s) = 0, \quad \overline{H}_2(\infty,s) = 0. \tag{4.20}$$

4.1.3 Zeroth and first order solutions

Without going into detail, the solutions of zeroth and first order systems can be shown as:

For $U\cos nt$ and $k > 1$

$$\overline{H}_1(\xi,s) = e^{-m_1\xi}\left\{-\frac{1}{s-i} + \frac{U}{2\Omega l}\left(\frac{1}{s+i(k-1)} + \frac{1}{s-i(k+1)}\right)\right\}, \tag{4.21}$$

$$\overline{H}_2 = \left\{ \begin{array}{c} \left(\begin{array}{c} -2W^2\sqrt{W^2+2s} - 2W^3 - 2Ws - \frac{s^2}{\sqrt{W^2+2s}} \\ +2i\sqrt{W^2+2s} + 2iW - \frac{2is}{\sqrt{W^2+2s}} \end{array} \right) \xi \frac{e^{-m_1\xi}}{s-i} \\ - \left(\begin{array}{c} -2W^2\sqrt{W^2+2s} - 2W^3 - 2Ws - \frac{s^2}{\sqrt{W^2+2s}} \\ +2i\sqrt{W^2+2s} + 2iW - \frac{2is}{\sqrt{W^2+2s}} \end{array} \right) \\ \times \left(\frac{1}{s+i(k-1)} + \frac{1}{s-i(k+1)} \right) U\xi \frac{e^{-m_1\xi}}{2\Omega l} \end{array} \right\}, \tag{4.22}$$

where

$$m_1 = W + \sqrt{W^2+2s}. \tag{4.23}$$

For $U \sin nt$ and $k > 1$

$$\overline{H}_1 = e^{-m_1\xi} \left\{ -\frac{1}{s-i} + \frac{iU}{2\Omega l} \left(\frac{1}{s+i(k-1)} - \frac{1}{s-i(k+1)} \right) \right\}, \tag{4.24}$$

$$\overline{H}_2 = \left\{ \begin{array}{c} \left(\begin{array}{c} -2W^2\sqrt{W^2+2s} - 2W^3 - 2Ws - \frac{s^2}{\sqrt{W^2+2s}} \\ +2i\sqrt{W^2+2s} + 2iW - \frac{2is}{\sqrt{W^2+2s}} \end{array} \right) \xi \frac{e^{-m_1\xi}}{s-i} \\ - \left(\begin{array}{c} -2W^2\sqrt{W^2+2s} - 2W^3 - 2Ws - \frac{s^2}{\sqrt{W^2+2s}} \\ +2i\sqrt{W^2+2s} + 2iW - \frac{2is}{\sqrt{W^2+2s}} \end{array} \right) \\ \times iU\xi \frac{e^{-m_1\xi}}{2\Omega l} \left(\frac{1}{s+i(k-1)} - \frac{1}{s-i(k+1)} \right) \end{array} \right\}. \tag{4.25}$$

For $U \cos nt$ and $k < 1$

$$\overline{H}_1(\xi, s) = e^{-m_1\xi} \left\{ -\frac{1}{s-i} + \frac{U}{2\Omega l} \left(\frac{1}{s-i(1-k)} + \frac{1}{s-i(k+1)} \right) \right\}, \tag{4.26}$$

$$\overline{H}_2 = \left\{ \begin{array}{c} \left(\begin{array}{c} -2W^2\sqrt{W^2+2s} - 2W^3 - 2Ws - \frac{s^2}{\sqrt{W^2+2s}} \\ +2i\sqrt{W^2+2s} + 2iW - \frac{2is}{\sqrt{W^2+2s}} \end{array} \right) \xi \frac{e^{-m_1\xi}}{s-i} \\ - \left(\begin{array}{c} -2W^2\sqrt{W^2+2s} - 2W^3 - 2Ws - \frac{s^2}{\sqrt{W^2+2s}} \\ +2i\sqrt{W^2+2s} + 2iW - \frac{2is}{\sqrt{W^2+2s}} \end{array} \right) \\ \times \left(\frac{1}{s-i(1-k)} + \frac{1}{s-i(k+1)} \right) U\xi \frac{e^{-m_1\xi}}{2\Omega l} \end{array} \right\}. \tag{4.27}$$

For $U \sin nt$ and $k < 1$

$$\overline{H}_1 = e^{-m_1\xi}\left\{-\frac{1}{s-i} + \frac{iU}{2\Omega l}\left(\frac{1}{s-i(1-k)} - \frac{1}{s-i(k+1)}\right)\right\}, \tag{4.28}$$

$$\overline{H}_2 = \left\{\begin{array}{c} \left(\begin{array}{c} -2W^2\sqrt{W^2+2s} - 2W^3 - 2Ws - \frac{s^2}{\sqrt{W^2+2s}} \\ +2i\sqrt{W^2+2s} + 2iW - \frac{2is}{\sqrt{W^2+2s}} \end{array}\right)\xi\frac{e^{-m_1\xi}}{s-i} \\ -\left(\begin{array}{c} -2W^2\sqrt{W^2+2s} - 2W^3 - 2Ws - \frac{s^2}{\sqrt{W^2+2s}} \\ +2i\sqrt{W^2+2s} + 2iW - \frac{2is}{\sqrt{W^2+2s}} \end{array}\right) \\ iU\xi\frac{e^{-m_1\xi}}{2\Omega l}\left(\frac{1}{s-i(1-k)} - \frac{1}{s-i(k+1)}\right) \end{array}\right\}. \tag{4.29}$$

In view of the above solutions, and Eq. (4.14) we have

For $U \cos nt$ and $k > 1$

$$\overline{H} = \left\{\begin{array}{c} e^{-m_1\xi}\left(-\frac{1}{s-i} + \frac{U}{2\Omega l}\left(\frac{1}{s+i(k-1)} + \frac{1}{s-i(k+1)}\right)\right) \\ +\beta\frac{\xi e^{-m_1\xi}}{s-i}\left(\begin{array}{c} -2w^2\sqrt{W^2+2s} - 2W^3 - 2Ws - \frac{s^2}{\sqrt{W^2+2s}} \\ +2i\sqrt{W^2+2s} + 2iW - \frac{2is}{\sqrt{W^2+2s}} \end{array}\right) \\ -\frac{\beta U\xi e^{-m_1\xi}}{2\Omega l}\left(\begin{array}{c} -2W^2\sqrt{W^2+2s} - 2W^3 - 2Ws - \frac{s^2}{\sqrt{W^2+2s}} \\ +2i\sqrt{W^2+2s} + 2iW - \frac{2is}{\sqrt{W^2+2s}} \end{array}\right) \\ \times\left(\frac{1}{s+i(k-1)} + \frac{1}{s-i(k+1)}\right) \end{array}\right\}. \tag{4.30}$$

For $U \sin nt$ and $k > 1$

$$\overline{H} = \left\{\begin{array}{c} e^{-m_1\xi}\left(-\frac{1}{s-i} + \frac{iU}{2\Omega l}\left(\frac{1}{s+i(k-1)} - \frac{1}{s-i(k+1)}\right)\right) \\ +\beta\frac{\xi e^{-m_1\xi}}{s-i}\left(\begin{array}{c} -2W^2\sqrt{W^2+2s} - 2W^3 - 2Ws - \frac{s^2}{\sqrt{W^2+2s}} \\ +2i\sqrt{W^2+2s} + 2iW - \frac{2is}{\sqrt{W^2+2s}} \end{array}\right) \\ -\frac{\beta iU\xi e^{-m_1\xi}}{2\Omega l}\left(\begin{array}{c} -2W^2\sqrt{W^2+2s} - 2W^3 - 2Ws - \frac{s^2}{\sqrt{w^2+2s}} \\ +2i\sqrt{W^2+2s} + 2iW - \frac{2is}{\sqrt{W^2+2s}} \end{array}\right) \\ \times\left(\frac{1}{s+i(k-1)} - \frac{1}{s-i(k+1)}\right) \end{array}\right\}. \tag{4.31}$$

For $U\cos nt$ and $k<1$

$$\overline{H}=\left\{\begin{array}{c} e^{-m_1\xi}\left(-\frac{1}{s-i}+\frac{U}{2\Omega l}\left(\frac{1}{s-i(1-k)}+\frac{1}{s-i(k+1)}\right)\right) \\ +\beta\frac{\xi e^{-m_1\xi}}{s-i}\left(\begin{array}{c} -2W^2\sqrt{W^2+2s}-2W^3-2Ws-\frac{s^2}{\sqrt{W^2+2s}} \\ +2i\sqrt{W^2+2s}+2iW-\frac{2is}{\sqrt{W^2+2s}} \end{array}\right) \\ -\frac{\beta U\xi e^{-m_1\xi}}{2\Omega l}\left(\begin{array}{c} -2W^2\sqrt{W^2+2s}-2W^3-2Ws-\frac{s^2}{\sqrt{W^2+2s}} \\ +2i\sqrt{W^2+2s}+2iW-\frac{2is}{\sqrt{W^2+2s}} \end{array}\right) \\ \times\left(\frac{1}{s-i(1-k)}+\frac{1}{s-i(k+1)}\right) \end{array}\right\}. \tag{4.32}$$

For $U\sin nt$ and $k<1$

$$\overline{H}=\left\{\begin{array}{c} e^{-m_1\xi}\left(-\frac{1}{s-i}+\frac{iU}{2\Omega l}\left(\frac{1}{s-i(1-k)}-\frac{1}{s-i(k+1)}\right)\right) \\ +\beta\frac{\xi e^{-m_1\xi}}{s-i}\left(\begin{array}{c} -2W^2\sqrt{W^2+2s}-2W^3-2Ws-\frac{s^2}{\sqrt{W^2+2s}} \\ +2i\sqrt{W^2+2s}+2iW-\frac{2is}{\sqrt{W^2+2s}} \end{array}\right) \\ -\frac{i\beta U\xi e^{-m_1\xi}}{2\Omega l}\left(\begin{array}{c} -2W^2\sqrt{W^2+2s}-2W^3-2Ws-\frac{s^2}{\sqrt{W^2+2s}} \\ +2i\sqrt{W^2+2s}+2iW-\frac{2is}{\sqrt{W^2+2s}} \end{array}\right) \\ \times\left(\frac{1}{s-i(1-k)}-\frac{1}{s-i(k+1)}\right) \end{array}\right\}. \tag{4.33}$$

Making use of Eqs. (4.30) to (4.33) in Eq. (2.18) and then evaluating the arising integrals by the substitution method we obtain after carrying the lengthy calculations the following expressions

$$\left.\begin{array}{c} -\frac{1}{2\pi i}\int\frac{e^{-m_1\xi+s\tau}}{s-i}ds= \\ -\frac{e^{-W\xi+i\tau}}{2}\left(\begin{array}{c} e^{\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right) \\ +e^{-\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \end{array}\right\}. \tag{4.34}$$

$$\left.\begin{array}{c} \frac{1}{2\pi i}\int\frac{e^{-m_1\xi+s\tau}}{s-i(k+1)}ds=-\frac{e^{-W\xi+i(k+1)\tau}}{2} \\ \times\left(\begin{array}{c} e^{\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right) \\ +e^{-\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \end{array}\right\}. \tag{4.35}$$

$$\left.\begin{array}{c}\frac{1}{2\pi i}\int\frac{e^{-m_1\xi+s\tau}}{s+i(k-1)}ds=-\frac{e^{-W\xi-i(k-1)\tau}}{2}\\ \times\left(\begin{array}{c}e^{\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2-2\left(k-1\right)i}\sqrt{\frac{\tau}{2}}\right)\\ +e^{-\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2-2\left(k-1\right)i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right).\end{array}\right\}. \tag{4.36}$$

$$\left.\begin{array}{c}\frac{1}{2\pi i}(-2)W^2\int\frac{\sqrt{W^2+2s}\xi e^{-m_1\xi+s\tau}}{s-i}ds=\\ -2\sqrt{\frac{2}{\pi}}\xi W^2\frac{e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}}{\sqrt{\tau}}-\xi W^2\sqrt{W^2+2i}e^{-W\xi+i\tau}\\ \times\left(\begin{array}{c}e^{-\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\\ -e^{\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}. \tag{4.37}$$

$$\left.\begin{array}{c}-\frac{1}{\pi i}W^3\int\frac{\xi e^{-m_1\xi+s\tau}}{s-i}ds=\\ -W^3\xi e^{-W\xi-+i\tau}\left(\begin{array}{c}e^{\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\\ +^{-\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}. \tag{4.38}$$

$$\left.\begin{array}{c}-\frac{1}{\pi i}W\int\frac{se^{-m_1\xi+s\tau}}{s-i}ds=\\ -\sqrt{\frac{2}{\pi}}\xi^2W\frac{e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}}{\tau^{\frac{3}{2}}}-iW\xi e^{-W\xi+i\tau}\\ \times\left(\begin{array}{c}e^{-\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\\ +e^{\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}. \tag{4.39}$$

$$\left.\begin{array}{l}-\frac{1}{2\pi i}\xi\int\frac{s^2e^{-m_1\xi+s\tau}}{(s-i)\sqrt{W^2+2s}}ds=\frac{\xi W^2}{2\sqrt{2\pi\tau}}e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}-i\xi\frac{e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}\\ -i\frac{\sqrt{W^2+2i}}{4}e^{-W\xi+i\tau}\left(\begin{array}{c}e^{-\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\\ -e^{\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\\ +i\frac{\xi W^2}{4\sqrt{W^2+2i}}e^{-W\xi+i\tau}\left(\begin{array}{c}e^{-\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\\ -e^{\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}. \tag{4.40}$$

$$\left.\begin{array}{c}\frac{1}{2\pi i}\int\frac{2i\xi\sqrt{W^2+2s}e^{m_1\xi+s\tau}}{s-i}ds=2i\xi\sqrt{\frac{2}{\pi\tau}}e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}\\+i\xi e^{-W\xi+i\tau}\sqrt{W^2+2i}\left(\begin{array}{c}e^{-\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\\-e^{\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}. \tag{4.41}$$

$$\left.\begin{array}{c}\frac{1}{2\pi i}\int\frac{2i\xi We^{-m_1\xi+s\tau}}{s-i}ds=\\\frac{2iW\xi e^{-W\xi+i\tau}}{2}\times\left(\begin{array}{c}e^{\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\\+e^{-\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}. \tag{4.42}$$

$$\left.\begin{array}{c}\frac{-1}{2\pi i}\int\frac{2i\xi se^{-m_1\xi+s\tau}}{(s-i)\sqrt{W^2+2s}}ds=-i\xi\sqrt{\frac{2}{\pi\tau}}e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}-i\xi e^{-W\xi+i\tau}\sqrt{W^2+2i}\\\times\left(\begin{array}{c}e^{-\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\\-e^{\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\\+2i\frac{\xi W^2}{4\sqrt{W^2+2i}}e^{-W\xi+i\tau}\left(\begin{array}{c}e^{-\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\\-e^{\sqrt{W^2+2i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}. \tag{4.43}$$

$$\left.\begin{array}{c}\frac{-W^2}{\pi i}\int\frac{\sqrt{W^2+2s}\xi e^{-m_1\xi+s\tau}}{s-i(k+1)}ds=\\-2\sqrt{\frac{2}{\pi}}\xi W^2\frac{e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}}{\sqrt{\tau}}-\xi W^2\sqrt{W^2+2\left(k+1\right)i}e^{-W\xi+i(k+1)\tau}\\\times\left(\begin{array}{c}e^{-\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right)\\-e^{\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}. \tag{4.44}$$

$$\left.\begin{array}{c}\frac{-W^3}{\pi i}\int\frac{\xi e^{-m_1\xi+s\tau}}{s-i(k+1)}ds=-\xi W^3e^{-W\xi+i(k+1)\tau}\\\times\left(\begin{array}{c}e^{\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right)\\+e^{-\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}. \tag{4.45}$$

$$\left.\begin{array}{c}\frac{-W\xi}{\pi i}\int\frac{se^{-m_1\xi+s\tau}}{s-i(k+1)}ds=\\-\sqrt{\frac{2}{\pi}}\xi^2W\frac{e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}}{\tau^{\frac{3}{2}}}-iW\left(k+1\right)\xi e^{-W\xi+i(k+1)\tau}\\\times\left(\begin{array}{c}e^{-\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right)\\+e^{\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}\tag{4.46}$$

$$\left.\begin{array}{c}\frac{-\xi}{2\pi i}\int\frac{s^2e^{-m_1\xi+s\tau}}{\sqrt{W^2+2s}(s-i(k+1))}ds=\frac{\xi W^2}{2\sqrt{2\pi\tau}}e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}\\-\frac{i(k+1)\xi e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}-\frac{i(k+1)\xi\sqrt{W^2+2i(k+1)}e^{-W\xi+i(k+1)\tau}}{4}\\\times\left(\begin{array}{c}e^{-\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right)\\-e^{\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\\+\frac{i(k+1)\xi W^2e^{-W\xi+i(k+1)\tau}}{4\sqrt{W^2+2i(k+1)}}\\\times\left(\begin{array}{c}e^{-\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right)\\-e^{\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}.\tag{4.47}$$

$$\left.\begin{array}{c}\frac{1}{2\pi i}\int\frac{2i\xi\sqrt{W^2+2s}e^{-m_1\xi+s\tau}}{s-i(k+1)}ds=\\2i\xi e^{-W\xi}\left(\sqrt{\frac{2}{\pi\tau}}e^{--\frac{\xi^2}{2\tau}-\frac{W^2\tau}{2}}\frac{\sqrt{W^2+2i(k+1)}}{2}e^{i(k+1)\tau}\right)\\+2i\xi e^{-W\xi}\left(\begin{array}{c}e^{-\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right)\\-e^{\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}.\tag{4.48}$$

$$\left.\begin{array}{c}\frac{1}{2\pi i}\int\frac{2iW\xi e^{-m_1\xi+s\tau}}{s-i(k+1)}=iW\xi e^{-W\xi+i(k+1)\tau}\\\times\left(\begin{array}{c}e^{\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right)\\+e^{-\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}.\tag{4.49}$$

$$\left.\begin{array}{c} \frac{-1}{\pi i}\int \frac{is\xi e^{-m_1\xi+s\tau}}{\sqrt{W^2+2s}(s-i(k+1))}ds = \\ -i\xi\sqrt{\frac{2}{\pi\tau}}e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}} - \frac{i\xi}{2}\sqrt{W^2+2i\left(k+1\right)}e^{-W\xi+i(k+1)\tau} \\ \times\left(\begin{array}{c} e^{-\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right) \\ -e^{\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ +\frac{i\xi W^2 e^{-W\xi+i(k+1)\tau}}{2\sqrt{W^2+2i(k+1)}} \\ \times\left(\begin{array}{c} e^{-\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right) \\ -e^{\sqrt{W^2+2(k+1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2+2\left(k+1\right)i}\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \end{array}\right\}. \tag{4.50}$$

$$\left.\begin{array}{c} \frac{-W^2}{\pi i}\int \frac{\sqrt{W^2+2s}\xi e^{-m_1\xi+s\tau}}{s+i(k-1)}ds = \\ -2\sqrt{\frac{2}{\pi}}\xi W^2\frac{e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}}{\sqrt{\tau}} - \xi W^2\sqrt{W^2-2\left(k-1\right)i}e^{-W\xi-i(k-1)\tau} \\ \times\left(\begin{array}{c} e^{-\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2-2\left(k-1\right)i}\sqrt{\frac{\tau}{2}}\right) \\ -e^{\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2-2\left(k-1\right)i}\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \end{array}\right\}. \tag{4.51}$$

$$\left.\begin{array}{c} \frac{-W^2}{\pi i}\int \frac{\xi e^{-m_1\xi+s\tau}}{s+i(k-1)}ds = -\xi W^3 e^{-W\xi-i(k-1)\tau} \\ \times\left(\begin{array}{c} e^{\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2-2\left(k-1\right)i}\sqrt{\frac{\tau}{2}}\right) \\ +e^{-\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2-2\left(k-1\right)i}\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \end{array}\right\}. \tag{4.52}$$

$$\left.\begin{array}{c} \frac{-W\xi}{\pi i}\int \frac{se^{-m_1\xi+s\tau}}{s+i(k-1)}ds = \\ -\sqrt{\frac{2}{\pi}}\xi^2 W\frac{e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}}{\tau^{\frac{3}{2}}} + iW\left(k-1\right)\xi e^{-W\xi-i(k-1)\tau} \\ \times\left(\begin{array}{c} e^{-\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2-2\left(k-1\right)i}\sqrt{\frac{\tau}{2}}\right) \\ +e^{\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2-2\left(k-1\right)i}\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \end{array}\right\}. \tag{4.53}$$

$$\left.\begin{array}{c}
\frac{-1}{2\pi i}\int\frac{s^2\xi e^{-m_1\xi+s\tau}}{\sqrt{W^2+2s}(s+i(k-1))}ds= \\
\frac{\xi W^2}{2\sqrt{2\pi\tau}}e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}+\frac{i(k-1)\xi e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} \\
+\frac{i(k-1)\xi\sqrt{W^2-2i(k-1)}e^{-W\xi-i(k-1)\tau}}{4} \\
\times\left(\begin{array}{c} e^{-\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2-2\left(k-1\right)i}\sqrt{\frac{\tau}{2}}\right) \\ -e^{\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2-2\left(k-1\right)i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right) \\
-\frac{i(k-1)\xi W^2e^{-W\xi-i(k-1)\tau}}{4\sqrt{W^2-2i(k-1)}} \\
\times\left(\begin{array}{c} e^{-\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2-2\left(k-1\right)i}\sqrt{\frac{\tau}{2}}\right) \\ -e^{\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2-2\left(k-1\right)i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)
\end{array}\right\}. \tag{4.54}$$

$$\left.\begin{array}{c}
\frac{2i}{2\pi i}\int\frac{\sqrt{W^2+2s}\xi e^{-m_1\xi+s\tau}}{s+i(k-1)}ds= \\
2i\xi e^{-W\xi}\left(\sqrt{\frac{2}{\pi\tau}}e^{-\frac{\xi^2}{2\tau}-\frac{W^2\tau}{2}}+\frac{\sqrt{W^2-2i(k-1)}}{2}e^{-i(k-1)\tau}\right) \\
+2i\xi e^{-W\xi}\left(\begin{array}{c} e^{-\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2-2\left(k-1\right)i}\sqrt{\frac{\tau}{2}}\right) \\ -e^{\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2-2\left(k-1\right)i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)
\end{array}\right\}. \tag{4.55}$$

$$\left.\begin{array}{c}
\frac{iW\xi}{\pi i}\int\frac{e^{-m_1\xi+s\tau}}{s+i(k-1)}ds=-i\xi W^3e^{-W\xi-i(k-1)\tau} \\
\times\left(\begin{array}{c} e^{\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2-2\left(k-1\right)i}\sqrt{\frac{\tau}{2}}\right) \\ +e^{-\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2-2\left(k-1\right)i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)
\end{array}\right\}. \tag{4.56}$$

$$\left.\begin{array}{c}\frac{-i}{\pi i}\int\frac{s\xi e^{-m_1\xi+s\tau}}{\sqrt{W^2+2s}(s+i(k-1))}ds=\\ -i\xi\sqrt{\frac{2}{\pi\tau}}e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}-\frac{i\xi}{2}\frac{\sqrt{W^2-2i(k-1)}e^{-W\xi-i(k-1)\tau}}{4}\\ \times\left(\begin{array}{c}e^{-\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2-2(k-1)i}\sqrt{\frac{\tau}{2}}\right)\\ -e^{\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2-2(k-1)i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\\ +\frac{i\xi W^2e^{-W\xi-i(k-1)\tau}}{4\sqrt{W^2-2i(k-1)}}\\ \times\left(\begin{array}{c}e^{-\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}-\sqrt{W^2-2(k-1)i}\sqrt{\frac{\tau}{2}}\right)\\ -e^{\sqrt{W^2-2(k-1)i}\xi}\operatorname{erf}\left(\frac{\xi}{\sqrt{2\tau}}+\sqrt{W^2-2(k-1)i}\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}. \tag{4.57}$$

Upon making use of Eqs. (4.34) to (4.57), the following expressions of $H(\xi,\tau)$ may be obtained.

For $U\cos nt$ and $k>1$

$$H(\xi,\tau)=\widetilde{H}_1(\xi,\tau)e^{i\tau}+\beta\widetilde{H}_2(\xi,\tau)+\frac{\beta U}{2\Omega l}\left(\widetilde{H}_3(\xi,\tau)+\widetilde{H}_4(\xi,\tau)\right). \tag{4.58}$$

For $U\sin nt$ and $k>1$

$$H(\xi,\tau)=\widetilde{H}_5(\xi,\tau)e^{i\tau}+\beta\widetilde{H}_2(\xi,\tau)+\frac{i\beta U}{2\Omega l}\left(\widetilde{H}_3(\xi,\tau)-\widetilde{H}_4(\xi,\tau)\right). \tag{4.59}$$

For $U\cos nt$ and $k<1$

$$H(\xi,\tau)=\widetilde{H}_6(\xi,\tau)e^{i\tau}+\beta\widetilde{H}_2(\xi,\tau)+\frac{\beta U}{2\Omega l}\left(\widetilde{H}_3(\xi,\tau)+\widetilde{H}_7(\xi,\tau)\right). \tag{4.60}$$

For $U\sin nt$ and $k<1$

$$H(\xi,\tau)=\widetilde{H}_8(\xi,\tau)e^{i\tau}+\beta\widetilde{H}_2(\xi,\tau)+\frac{i\beta U}{2\Omega l}\left(\widetilde{H}_3(\xi,\tau)-\widetilde{H}_7(\xi,\tau)\right). \tag{4.61}$$

In above equations

$$\widetilde{H}_1(\xi,\tau)=\left\{\begin{array}{c}\frac{-e^{-W\xi}}{2}\left(\begin{array}{c}e^{(x_8+iy_8)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_8+iy_8)\sqrt{\frac{\tau}{2}}\right)\\+e^{-(x_8+iy_8)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_8+iy_8)\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\\+\frac{Ue^{-W\xi+ik\tau}}{4\Omega l}\left(\begin{array}{c}e^{(x_9+iy_9)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_9+iy_9)\sqrt{\frac{\tau}{2}}\right)\\+e^{-(x_9+iy_9)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_9+iy_9)\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\\+\frac{Ue^{-W\xi-ik\tau}}{4\Omega l}\left(\begin{array}{c}e^{(x_{10}+iy_{10})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_{10}+iy_{10})\sqrt{\frac{\tau}{2}}\right)\\+e^{-(x_{10}+iy_{10})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_{10}+iy_{10})\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\},\tag{4.62}$$

$$\widetilde{H}_2(\xi,\tau)=\left\{\begin{array}{c}-\sqrt{\frac{2}{\pi\tau}}\xi\left(\frac{i}{2}-\frac{7W^2}{4}-\frac{W\xi}{\tau}\right)e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}+\xi e^{-W\xi+i\tau}\\\times\left(\begin{array}{c}(\eta_2-\eta_1)\,e^{(x_8+iy_8)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_8+iy_8)\sqrt{\frac{\tau}{2}}\right)\\+(\eta_2+\eta_1)\,e^{-(x_8+iy_8)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_8+iy_8)\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\},\tag{4.63}$$

$$\widetilde{H}_3(\xi,\tau)=\left\{\begin{array}{c}-\xi\sqrt{\frac{2}{\pi\tau}}\left(q_1-\frac{W\xi}{\tau}\right)e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}+\xi e^{-W\xi+i(1+k)\tau}\\\times\left(\begin{array}{c}(\eta_4-\eta_3)\,e^{(x_9+iy_9)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_9+iy_9)\sqrt{\frac{\tau}{2}}\right)\\+(\eta_3+\eta_4)\,e^{-(x_9+iy_9)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_9+iy_9)\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\},\tag{4.64}$$

$$\widetilde{H}_4(\xi,\tau)=\left\{\begin{array}{c}-\xi\sqrt{\frac{2}{\pi\tau}}\left(q_2-\frac{W\xi}{\tau}\right)e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}+\xi e^{-W\xi-i(k-1)\tau}\\\times\left(\begin{array}{c}(\eta_6-\eta_5)\,e^{(x_{10}+iy_{10})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_{10}+iy_{10})\sqrt{\frac{\tau}{2}}\right)\\+(\eta_6+\eta_5)\,e^{-(x_{10}+iy_{10})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_{10}+iy_{10})\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\},\tag{4.65}$$

$$\widetilde{H}_5(\xi,\tau)=\left\{\begin{array}{c} -\frac{e^{-W\xi}}{2}\left(\begin{array}{c} e^{(x_8+iy_8)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_8+iy_8)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_8+iy_8)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_8+iy_8)\sqrt{\frac{\tau}{2}}\right)\end{array}\right) \\ -\frac{iU}{4\Omega l}e^{-W\xi+ik\tau}\left(\begin{array}{c} e^{(x_9+iy_9)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_9+iy_9)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_9+iy_9)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_9+iy_9)\sqrt{\frac{\tau}{2}}\right)\end{array}\right) \\ +\frac{iU}{4\Omega l}e^{-W\xi-ik\tau}\left(\begin{array}{c} e^{(x_{10}+iy_{10})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_{10}+iy_{10})\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_{10}+iy_{10})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_{10}+iy_{10})\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}, \tag{4.66}$$

$$\widetilde{H}_6(\xi,\tau)=\left\{\begin{array}{c} -\frac{e^{-W\xi}}{2}\left(\begin{array}{c} e^{(x_8+iy_8)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_8+iy_8)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_8+iy_8)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_8+iy_8)\sqrt{\frac{\tau}{2}}\right)\end{array}\right) \\ +\frac{U}{4\Omega l}e^{-W\xi+ik\tau}\left(\begin{array}{c} e^{(x_9+iy_9)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_9+iy_9)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_9+iy_9)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_9+iy_9)\sqrt{\frac{\tau}{2}}\right)\end{array}\right) \\ +\frac{U}{4\Omega l}e^{-W\xi-ik\tau}\left(\begin{array}{c} e^{(x_{11}+iy_{11})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_{11}+iy_{11})\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_{11}+iy_{11})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_{11}+iy_{11})\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}, \tag{4.67}$$

$$\widetilde{H}_7(\xi,\tau)=\left\{\begin{array}{c} -\xi\sqrt{\frac{2}{\pi\tau}}\left(q_2-\frac{W\xi}{\tau}\right)e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}+\xi e^{-W\xi+i(1-k)\tau} \\ \times\left(\begin{array}{c} (\eta_6-\eta_7)\,e^{(x_{11}+iy_{11})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_{11}+iy_{11})\sqrt{\frac{\tau}{2}}\right) \\ +(\eta_6+\eta_7)\,e^{-(x_{11}+iy_{11})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_{11}+iy_{11})\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}, \tag{4.68}$$

$$\widetilde{H}_8(\xi,\tau) = \left\{ \begin{array}{c} -\frac{e^{-W\xi}}{2} \left(\begin{array}{c} e^{(x_8+iy_8)\xi} \operatorname{erf} c \left(\frac{\xi}{\sqrt{2\tau}} + (x_8+iy_8)\sqrt{\frac{\tau}{2}} \right) \\ +e^{-(x_8+iy_{81})\xi} \operatorname{erf} c \left(\frac{\xi}{\sqrt{2\tau}} - (x_8+iy_8)\sqrt{\frac{\tau}{2}} \right) \end{array} \right) \\ -\frac{iU}{4\Omega l} e^{-W\xi+ik\tau} \left(\begin{array}{c} e^{(x_9+iy_9)\xi} \operatorname{erf} c \left(\frac{\xi}{\sqrt{2\tau}} + (x_9+iy_9)\sqrt{\frac{\tau}{2}} \right) \\ +e^{-(x_9+iy_9)\xi} \operatorname{erf} c \left(\frac{\xi}{\sqrt{2\tau}} - (x_9+iy_9)\sqrt{\frac{\tau}{2}} \right) \end{array} \right) \\ +\frac{iU}{4\Omega l} e^{-W\xi-ik\tau} \left(\begin{array}{c} e^{(x_{11}+iy_{11})\xi} \operatorname{erf} c \left(\frac{\xi}{\sqrt{2\tau}} + (x_{11}+iy_{11})\sqrt{\frac{\tau}{2}} \right) \\ +e^{-(x_{11}+iy_{11})\xi} \operatorname{erf} c \left(\frac{\xi}{\sqrt{2\tau}} - (x_{11}+iy_{11})\sqrt{\frac{\tau}{2}} \right) \end{array} \right) \end{array} \right\}, \tag{4.69}$$

$$q_1 = \frac{i(1-k)}{2} - \frac{7W^2}{4}, \tag{4.70}$$

$$q_2 = \frac{i(1+k)}{2} - \frac{7W^2}{4}, \tag{4.71}$$

$$x_8 = \left[\frac{\sqrt{W^4+4}+W^2}{2} \right]^{\frac{1}{2}}, \tag{4.72}$$

$$x_9 = \left[\frac{\sqrt{W^4+4(k+1)^2}+W^2}{2} \right]^{\frac{1}{2}}, \tag{4.73}$$

$$x_{10} = \left[\frac{\sqrt{W^4+4(k-1)^2}+W^2}{2} \right]^{\frac{1}{2}}, \tag{4.74}$$

$$x_{11} = \left[\frac{\sqrt{W^4+4(1-k)^2}+W^2}{2} \right]^{\frac{1}{2}}, \tag{4.75}$$

$$y_8 = \left[\frac{\sqrt{W^4+4}-W^2}{2} \right]^{\frac{1}{2}}, \tag{4.76}$$

$$y_9 = \left[\frac{\sqrt{W^4+4(k+1)^2}-W^2}{2} \right]^{\frac{1}{2}}, \tag{4.77}$$

$$y_{10} = \left[\frac{\sqrt{W^4+4(k-1)^2}-W^2}{2} \right]^{\frac{1}{2}}, \tag{4.78}$$

$$y_{11} = \left[\frac{\sqrt{W^4 + 4\,(1-k)^2} - W^2}{2} \right]^{\frac{1}{2}}, \tag{4.79}$$

$$\eta_1 = (x_8 + iy_8)\left(\frac{i}{4} - W^2\right) + \frac{3iW^2}{4\,(x_8 + iy_8)}, \tag{4.80}$$

$$\eta_2 = -W^3, \tag{4.81}$$

$$\eta_3 = (y_9 + iy_9)\left[W^2 + \frac{(k+1)\,i}{4} - \frac{i}{2}\right] - \frac{iW^2\,(k+1+2)}{4\,(x_9 + iy_9)}, \tag{4.82}$$

$$\eta_4 = W^3 + ikW, \tag{4.83}$$

$$\eta_5 = (x_{10} + iy_{10})\left[W^2 - \frac{(k-1)\,i}{4} - \frac{i}{2}\right] + \frac{iW^2\,(k-1-2)}{4\,(x_{10} + iy_{10})}, \tag{4.84}$$

$$\eta_6 = W^3 - iWk, \tag{4.85}$$

$$\eta_7 = (x_{11} + iy_{11})\left[W^2 + \frac{(1-k)\,i}{4} - \frac{i}{2}\right] - \frac{iW^2\,(1-k+2)}{4\,(x_{11} + iy_{11})}. \tag{4.86}$$

Upon making use of Eq. (4.7), the solutions (4.58) to (4.61) can be written as

$$\frac{F}{\Omega l} - 1 = \widetilde{H}_1(\xi,\tau) + \beta e^{-i\tau}\widetilde{H}_2(\xi,\tau) + \frac{\beta U e^{-i\tau}}{2\Omega l}\left(\widetilde{H}_3(\xi,\tau) + \widetilde{H}_4(\xi,\tau)\right);\ U\cos nt,\quad k>1, \tag{4.87}$$

$$\frac{F}{\Omega l} - 1 = \widetilde{H}_5(\xi,\tau) + \beta e^{-i\tau}\widetilde{H}_2(\xi,\tau) + \frac{i\beta U e^{-i\tau}}{2\Omega l}\left(\widetilde{H}_3(\xi,\tau) - \widetilde{H}_4(\xi,\tau)\right);\ U\sin nt,\quad k>1, \tag{4.88}$$

$$\frac{F}{\Omega l} - 1 = \widetilde{H}_6(\xi,\tau) + \beta e^{-i\tau}\widetilde{H}_2(\xi,\tau) + \frac{\beta U e^{-i\tau}}{2\Omega l}\left(\widetilde{H}_3(\xi,\tau) + \widetilde{H}_7(\xi,\tau)\right);\ U\cos nt,\quad k<1, \tag{4.89}$$

$$\frac{F}{\Omega l} - 1 = \widetilde{H}_8(\xi,\tau) + \beta e^{-i\tau}\widetilde{H}_2(\xi,\tau) + \frac{i\beta U e^{-i\tau}}{2\Omega l}\left(\widetilde{H}_3(\xi,\tau) - \widetilde{H}_7(\xi,\tau)\right);\ U\sin nt,\quad k<1. \tag{4.90}$$

In above equations, we give exact solutions of zeroth and first order, but in view of the lack of convergence it is not sure that the obtained solution is close to the full solution. The solutions (4.87) to (4.90) after using Eq. (2.14) lead to the following expressions

$$\frac{f+ig}{\Omega l}=\left\{\begin{array}{l} 1+\widetilde{H}_1(\xi,\tau)+\beta e^{-i\tau}\widetilde{H}_2(\xi,\tau)+\frac{\beta U e^{-i\tau}}{2\Omega l}\left(\widetilde{H}_3(\xi,\tau)+\widetilde{H}_4(\xi,\tau)\right), U\cos nt, k>1 \\ 1+\widetilde{H}_5(\xi,\tau)+\beta e^{-i\tau}\widetilde{H}_2(\xi,\tau)+\frac{i\beta U e^{-i\tau}}{2\Omega l}\left(\widetilde{H}_3(\xi,\tau)-\widetilde{H}_4(\xi,\tau)\right), U\sin nt, k>1 \\ 1+\widetilde{H}_6(\xi,\tau)+\beta e^{-i\tau}\widetilde{H}_2(\xi,\tau)+\frac{\beta U e^{-i\tau}}{2\Omega l}\left(\widetilde{H}_3(\xi,\tau)+\widetilde{H}_7(\xi,\tau)\right), U\cos nt, k<1 \\ 1+\widetilde{H}_8(\xi,\tau)+\beta e^{-i\tau}\widetilde{H}_2(\xi,\tau)+\frac{i\beta U e^{-i\tau}}{2\Omega l}\left(\widetilde{H}_3(\xi,\tau)-\widetilde{H}_7(\xi,\tau)\right), U\sin nt, k<1 \end{array}\right\} \tag{4.91}$$

The real part gives $\frac{f}{\Omega l}$ and imaginary part gives $\frac{g}{\Omega l}$. The results for blowing can be obtained directly by replacing W with $-W_1$.

4.1.4 Solution for the resonant case ($k = 1$)

Following the same procedure as for $k \neq 1$, the solution expressions here are given by

$$\left.\begin{array}{r} \frac{f+ig}{\Omega l}=1+\widehat{H}_1(\xi,\tau)+\beta e^{-i\tau}\widetilde{H}_2(\xi,\tau) \\ +\frac{\beta U e^{-i\tau}}{2\Omega l}\left(\widehat{H}_3(\xi,\tau)+\widehat{H}_4(\xi,\tau)\right), U\cos nt, \\ \frac{f+ig}{\Omega l}=1+\widehat{H}_5(\xi,\tau)+\beta e^{-i\tau}\widetilde{H}_2(\xi,\tau) \\ +\frac{i\beta U e^{-i\tau}}{2\Omega l}\left(\widehat{H}_3(\xi,\tau)-\widehat{H}_4(\xi,\tau)\right), U\sin nt, \end{array}\right\} \tag{4.92}$$

where

$$\widehat{H}_1(\xi,\tau)=\left\{\begin{array}{c} \frac{-e^{-W\xi}}{2}\left(\begin{array}{c} e^{(x_8+iy_8)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_8+iy_8)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_8+iy_8)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_8+iy_8)\sqrt{\frac{\tau}{2}}\right)\end{array}\right) \\ +\frac{Ue^{-W\xi+i\tau}}{4\Omega l}\left(\begin{array}{c} e^{(\widehat{x}_5+i\widehat{y}_5)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(\widehat{x}_5+i\widehat{y}_5)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\widehat{x}_5+i\widehat{y}_5)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(\widehat{x}_5+i\widehat{y}_5)\sqrt{\frac{\tau}{2}}\right)\end{array}\right) \\ +\frac{Ue^{-W\xi-i\tau}}{4\Omega l}\left(\begin{array}{c} e^{W\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+iW\sqrt{\frac{\tau}{2}}\right) \\ +e^{-W\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-iW\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}, \tag{4.93}$$

$$\widehat{H}_3(\xi,\tau)=\left\{\begin{array}{c}-\xi\sqrt{\frac{2}{\pi\tau}}\left(\widehat{q}_1-\frac{W\xi}{\tau}\right)e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}+\xi e^{-W\xi+2i\tau}\\ \times\left(\begin{array}{c}(\widehat{\eta}_2-\widehat{\eta}_1)\,e^{(\widehat{x}_5+i\widehat{y}_5)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(\widehat{x}_5+i\widehat{y}_5)\sqrt{\frac{\tau}{2}}\right)\\ +(\widehat{\eta}_2+\widehat{\eta}_1)\,e^{-(\widehat{x}_5+i\widehat{y}_5)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(\widehat{x}_5+i\widehat{y}_5)\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}, \tag{4.94}$$

$$\widehat{H}_4(\xi,\tau)=\left\{\begin{array}{c}-\xi\sqrt{\frac{2}{\pi\tau}}\left(\widehat{q}_2-\frac{W\xi}{\tau}\right)e^{-W\xi-W^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}\\ +2\widehat{\eta}_3\xi e^{-2W\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-iW\sqrt{\frac{\tau}{2}}\right)\end{array}\right\}, \tag{4.95}$$

$$\widehat{H}_5(\xi,\tau)=\left\{\begin{array}{c}-\frac{e^{-W\xi}}{2}\left(\begin{array}{c}e^{(x_8+iy_8)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_8+iy_8)\sqrt{\frac{\tau}{2}}\right)\\ +e^{-(x_8+iy_8)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_8+iy_8)\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\\ -\frac{iU}{4\Omega l}e^{-W\xi+i\tau}\left(\begin{array}{c}e^{(\widehat{x}_9+i\widehat{y}_9)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(\widehat{x}_9+i\widehat{y}_9)\sqrt{\frac{\tau}{2}}\right)\\ +e^{-(\widehat{x}_9+i\widehat{y}_9)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(\widehat{x}_9+i\widehat{y}_9)\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\\ +\frac{iU}{4\Omega l}e^{-W\xi-i\tau}\left(\begin{array}{c}e^{W\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+iW\sqrt{\frac{\tau}{2}}\right)\\ +e^{-W\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-iW\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\} \tag{4.96}$$

and $\widetilde{H}_2$ is given in Eq. (3.72). In above expressions

$$\widehat{q}_1=-\frac{7W^2}{4}, \tag{4.97}$$

$$\widehat{q}_2=i-\frac{7W^2}{4}, \tag{4.98}$$

$$\widehat{\eta}_1=(\widehat{x}_5+i\widehat{y}_5)\,W^2-\frac{iW^2}{(\widehat{x}_5+i\widehat{y}_5)}, \tag{4.99}$$

$$\widehat{\eta}_2=W^3+iW, \tag{4.100}$$

$$\widehat{\eta}_3=W^3-iW, \tag{4.101}$$

$$\widehat{x}_5=\left[\frac{\sqrt{W^4+16}+W^2}{2}\right]^{\frac{1}{2}}, \tag{4.102}$$

$$\widehat{y}_5=\left[\frac{\sqrt{W^4+16}-W^2}{2}\right]^{\frac{1}{2}}. \tag{4.103}$$

4.2 Results and discussion

The starting solution for the case of suction is

$$H = H_1 + \beta H_2. \tag{4.104}$$

The above solution describes the general features of the unsteady boundary layer flow in a fluid bounded by a porous disk for small values of β. This solution clearly brings out the contribution due to the material parameter of the second grade fluid. It should be noted that Newtonian solution [93] can be recovered as a special case for $\beta = 0$. We also note that for $\beta = 0 = U$, the velocity field is identical to that of Erdogan [45]. This provides a useful mathematical check.

The solution given by Eq. (4.91) is general and independent of the assumption of the form of the steady state solution. In order to determine the steady structure of the solution (4.101) we use Eq. (3.24) and thus solution (4.91) takes the following form:

For $U \cos nt$, $k > 1$

$$\frac{f_s + ig_s}{\Omega l} = \left\{ \begin{array}{c} 1 + \left[-1 + 2\beta\xi\left(\eta_1 + \eta_2\right)\right] e^{-W\xi-(x_8+iy_8)\xi} \\ +\frac{U}{2\Omega l}\left[1 + 2\beta\xi\left(\eta_3 + \eta_4\right)\right] e^{-W\xi-(x_9+iy_9)\xi+ik\tau} \\ +\frac{U}{2\Omega l}\left[1 + 2\beta\xi\left(\eta_5 + \eta_6\right)\right] e^{-W\xi-(x_{10}+iy_{10})\xi-ik\tau} \end{array} \right\}. \tag{4.105}$$

For $U \sin nt$, $k > 1$

$$\frac{f_s + ig_s}{\Omega l} = \left\{ \begin{array}{c} 1 + \left[-1 + 2\beta\xi\left(\eta_1 + \eta_2\right)\right] e^{-W\xi-(x_8+iy_8)\xi} \\ +\frac{iU}{2\Omega l}\left[-1 + 2\beta\xi\left(\eta_3 + \eta_4\right)\right] e^{-W\xi-(x_9+iy_9)\xi+ik\tau} \\ -\frac{iU}{2\Omega l}\left[-1 + 2\beta\xi\left(\eta_5 + \eta_6\right)\right] e^{-W\xi-(x_{10}+iy_{10})\xi-ik\tau} \end{array} \right\}. \tag{4.106}$$

For $U\cos nt$, $k<1$

$$\frac{f_s+ig_s}{\Omega l}=\left\{\begin{array}{c} 1+\left[-1+2\beta\xi\left(\eta_1+\eta_2\right)\right]e^{-W\xi-(x_8+iy_8)\xi} \\ +\frac{U}{2\Omega l}\left[1+2\beta\xi\left(\eta_3+\eta_4\right)\right]e^{-W\xi-(x_9+iy_9)\xi+ik\tau} \\ +\frac{U}{2\Omega l}\left[1+2\beta\xi\left(\eta_6+\eta_7\right)\right]e^{-W\xi-(x_{11}+iy_{11})\xi-ik\tau} \end{array}\right\}. \tag{4.107}$$

For $U\sin nt$, $k<1$

$$\frac{f_s+ig_s}{\Omega l}=\left\{\begin{array}{c} 1+\left[-1+2\beta\xi\left(\eta_1+\eta_2\right)\right]e^{-W\xi-(x_8+iy_8)\xi} \\ +\frac{iU}{2\Omega l}\left[-1+2\beta\xi\left(\eta_3+\eta_4\right)\right]e^{-W\xi-(x_9+iy_9)\xi+ik\tau} \\ -\frac{iU}{2\Omega l}\left[-1+2\beta\xi\left(\eta_6+\eta_7\right)\right]e^{-W\xi-(x_{11}+iy_{11})\xi-ik\tau} \end{array}\right\}, \tag{4.108}$$

where the subscript 's' denotes the steady state situation. Clearly, the steady solutions (4.105) to (4.108) are independent of the initial condition and are periodic in time. For some time after the initiation of the motion, the velocity field contains transients and they gradually disappear in time. The transient solution is obtained by the subtraction of steady state solutions from Eq. (4.91). i.e.

$$\frac{f_t+ig_t}{\Omega l}=\frac{f+ig}{\Omega l}-\left(\frac{f_s+ig_s}{\Omega l}\right). \tag{4.109}$$

In above expression subscript 't' represents the transient solutions. It is seen that for large time, the transient solution (4.109) disappears. Further, expressions (4.105) to (4.108) show the existence of four distinct boundary layers of thicknesses of order $(W+x_j)^{-1}, j=8-10$. It is interesting to note that the associated boundary layers are modified due to the presence of W. These thicknesses decrease with an increase of the suction parameter and the rotation. It is further noted that these thicknesses remain bounded for all values of the frequency of the imposed oscillations. When $k=0, x_8=x_9=x_{10}=x_{11}$ and the four boundary layers coalesces into a single layer of thickness $(W + x_8)^{-1}$. In particular, when $U = 0$, solutions (4.105) to (4.108) reduce to

$$\frac{f_s + ig_s}{\Omega l} = 1 + \left[-1 + 2\beta\xi\left(\eta_1 + \eta_2\right)\right] e^{-W\xi - (x_8 + iy_8)\xi}, \tag{4.110}$$

which for $\beta = 0 = W$ corresponds to Erdogan's result [93].

For resonant case, the steady solutions from Eq. (4.92) is of the following type:

For $U \cos nt$

$$\frac{f_s + ig_s}{\Omega l} = \left\{ \begin{array}{c} 1 + \left[-1 + 2\beta\xi\left(\eta_1 + \eta_2\right)\right] e^{-W\xi - (x_8 + iy_8)\xi} \\ + \frac{U}{2\Omega l}\left[1 + 2\beta\xi\left(\widehat{\eta}_1 + \widehat{\eta}_2\right)\right] e^{-W\xi - (\widehat{x}_5 + i\widehat{y}_5)\xi + i\tau} \\ + \frac{U}{2\Omega l}\left[1 + 4\beta\xi\widehat{\eta}_3\right] e^{-2W\xi - i\tau} \end{array} \right\}. \tag{4.111}$$

For $U \sin nt$

$$\frac{f_s + ig_s}{\Omega l} = \left\{ \begin{array}{c} 1 + \left[-1 + 2\beta\xi\left(\eta_1 + \eta_2\right)\right] e^{-W\xi - (x_8 + iy_8)\xi} \\ + \frac{iU}{2\Omega l}\left[-1 + 2\beta\xi\left(\widehat{\eta}_1 + \widehat{\eta}_2\right)\right] e^{-W\xi - (\widehat{x}_5 + i\widehat{y}_5)\xi + i\tau} \\ - \frac{iU}{2\Omega l}\left[1 + 4\beta\xi\widehat{\eta}_3\right] e^{-2W\xi - i\tau} \end{array} \right\}. \tag{4.112}$$

From above solutions we note that in the presence of suction ($W \neq 0$), the steady solution in the resonant case ($k \to 1$) does not depend on the order of the double limit operation $\tau \to \infty, k \to 1$. In fact, the double limiting procedure with $W \neq 0$ gives

$$Lim_{k\to 1} Lim_{\tau\to\infty}\left(\frac{f + ig}{\Omega l}\right) = Lim_{\tau\to\infty} Lim_{k\to 1}\left(\frac{f + ig}{\Omega l}\right). \tag{4.113}$$

From Eqs. (4.111) and (4.112), it is also noted that steady asymptotic blowing solution for resonant case does not exist i.e. the boundary condition at infinity is not satisfied for blowing. The physical mechanism of non-existence of the steady solution is that the blowing causes thickening of the boundary layer. At sufficiently large distance from the leading edge the boundary layer becomes so thick that it becomes turbulent.

Chapter 5

Unsteady MHD second grade flow due to non-coaxial rotations of an oscillating porous disk and a fluid at infinity

The aim of this chapter is just to provide an answer to the question of finding a meaningful steady blowing solution for resonant frequency in the second grade fluid. An attempt is made here by imposing magnetohydrodynamic problem. It is found that unlike the hydrodynamic problem for blowing and resonance, the steady solution in this chapter satisfies the boundary condition at infinity. Thus, for uniform suction and blowing at the disk, the shear oscillations are confined to Ekman-Hartman layers near the disk for all values of the frequencies including the resonant frequency.

5.1 Mathematical analysis

Here, the problem considered is the same as in the previous chapter except that magnetic field $\mathbf{B}_0$ is applied. The magnetic Reynolds number is taken small and induced magnetic field is neglected. Therefore, the mathematical problem consists of boundary and initial conditions given in Eq. (4.9) and the following equation

$$\beta\frac{\partial^3 H}{\partial\tau\partial\xi^2}-\beta W\frac{\partial^3 H}{\partial\xi^3}+(1-2i\beta)\frac{\partial^2 H}{\partial\xi^2}+2W\frac{\partial H}{\partial\xi}-2\left(s+N_1\right)\frac{\partial H}{\partial\tau}=0, \tag{5.1}$$

Proceeding in a similar manner as in chapter 4 and avoiding the lengthy details of calculations, the suction solutions at this stage for $k\neq 1$ may directly be expressed in the following form

For $U\cos nt$ and $k\lesseqgtr 1$,

$$H(\xi,\tau)=\widetilde{\widetilde{H}}_1(\xi,\tau)e^{i\tau}+\beta\widetilde{\widetilde{H}}_2(\xi,\tau)+\frac{\beta U}{2\Omega l}\left(\widetilde{\widetilde{H}}_3(\xi,\tau)+\widetilde{\widetilde{H}}_4(\xi,\tau)\right). \tag{5.2}$$

For $U\sin nt$ and $k\lesseqgtr 1$

$$H(\xi,\tau)=\widetilde{\widetilde{H}}_5(\xi,\tau)e^{i\tau}+\beta\widetilde{\widetilde{H}}_2(\xi,\tau)+\frac{i\beta U}{2\Omega l}\left(\widetilde{\widetilde{H}}_3(\xi,\tau)-\widetilde{\widetilde{H}}_4(\xi,\tau)\right), \tag{5.3}$$

where

$$\widetilde{\widetilde{H}}_1(\xi,\tau)=\left\{\begin{array}{l}\frac{-e^{-W\xi}}{2}\left(\begin{array}{c}e^{(x_{12}+iy_{12})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_{12}+iy_{12})\sqrt{\frac{\tau}{2}}\right)\\+e^{-(x_{12}+iy_{12})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_{12}+iy_{12})\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\\+\frac{Ue^{-W\xi+ik\tau}}{4\Omega l}\left(\begin{array}{c}e^{(x_{13}+iy_{13})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_{13}+iy_{13})\sqrt{\frac{\tau}{2}}\right)\\+e^{-(x_{13}+iy_{13})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_{13}+iy_{13})\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\\+\frac{Ue^{-W\xi-ik\tau}}{4\Omega l}\left(\begin{array}{c}e^{(x_{14}+iy_{14})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_{14}+iy_{14})\sqrt{\frac{\tau}{2}}\right)\\+e^{-(x_{14}+iy_{14})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_{14}+iy_{14})\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}, \tag{5.4}$$

$$\widetilde{\widetilde{H}}_2(\xi,\tau)=\left\{\begin{array}{c}-\sqrt{\frac{2}{\pi\tau}}\xi\left(\frac{i}{2}-\frac{7W^2}{4}-\frac{W\xi}{\tau}\right)e^{-W\xi-\left(W^2+2N\right)\frac{\tau}{2}-\frac{\xi^2}{2\tau}}+\xi e^{-W\xi+i\tau}\\ \times\left(\begin{array}{c}\left(\eta_2-\eta_8\right)e^{(x_{12}+iy_{12})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+\left(x_{12}+iy_{12}\right)\sqrt{\frac{\tau}{2}}\right)\\ +\left(\eta_2+\eta_8\right)e^{-(x_{12}+iy_{12})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-\left(x_{12}+iy_{12}\right)\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}, \tag{5.5}$$

$$\widetilde{\widetilde{H}}_3(\xi,\tau)=\left\{\begin{array}{c}-\xi\sqrt{\frac{2}{\pi\tau}}\left(q_1-\frac{W\xi}{\tau}\right)e^{-W\xi-\left(W^2+2N\right)\frac{\tau}{2}-\frac{\xi^2}{2\tau}}+\xi e^{-W\xi+i(1+k)\tau}\\ \times\left(\begin{array}{c}\left(\eta_4-\eta_9\right)e^{(x_{13}+iy_{13})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+\left(x_{13}+iy_{13}\right)\sqrt{\frac{\tau}{2}}\right)\\ +\left(\eta_3+\eta_9\right)e^{-(x_{13}+iy_{13})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-\left(x_{13}+iy_{13}\right)\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}, \tag{5.6}$$

$$\widetilde{\widetilde{H}}_4(\xi,\tau)=\left\{\begin{array}{c}-\xi\sqrt{\frac{2}{\pi\tau}}\left(q_2-\frac{W\xi}{\tau}\right)e^{-W\xi-\left(W^2+2N\right)\frac{\tau}{2}-\frac{\xi^2}{2\tau}}+\xi e^{-W\xi-i(k-1)\tau}\\ \times\left(\begin{array}{c}\left(\eta_6-\eta_{10}\right)e^{(x_{14}+iy_{14})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+\left(x_{14}+iy_{14}\right)\sqrt{\frac{\tau}{2}}\right)\\ +\left(\eta_6+\eta_{10}\right)e^{-(x_{14}+iy_{14})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-\left(x_{14}+iy_{14}\right)\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}, \tag{5.7}$$

$$\widetilde{\widetilde{H}}_5(\xi,\tau)=\left\{\begin{array}{c}-\frac{e^{-W\xi}}{2}\left(\begin{array}{c}e^{(x_{12}+iy_{12})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+\left(x_{12}+iy_{12}\right)\sqrt{\frac{\tau}{2}}\right)\\ +e^{-(x_{12}+iy_{12})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-\left(x_{12}+iy_{12}\right)\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\\ -\frac{iU}{4\Omega l}e^{-W\xi+ik\tau}\left(\begin{array}{c}e^{(x_{13}+iy_{13})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+\left(x_{13}+iy_{13}\right)\sqrt{\frac{\tau}{2}}\right)\\ +e^{-(x_{13}+iy_{13})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-\left(x_{13}+iy_{13}\right)\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\\ +\frac{iU}{4\Omega l}e^{-W\xi-ik\tau}\left(\begin{array}{c}e^{(x_{14}+iy_{14})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+\left(x_{14}+iy_{14}\right)\sqrt{\frac{\tau}{2}}\right)\\ +e^{-(x_{14}+iy_{14})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-\left(x_{14}+iy_{14}\right)\sqrt{\frac{\tau}{2}}\right)\end{array}\right)\end{array}\right\}, \tag{5.8}$$

$$\widetilde{\widetilde{H}}_6(\xi,\tau)=\left\{\begin{array}{c} -\frac{e^{-W\xi}}{2}\left(\begin{array}{c} e^{(x_{12}+iy_{12})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_{12}+iy_{12})\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_{12}+iy_{12})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_{12}+iy_{12})\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ +\frac{U}{4\Omega l}e^{-W\xi+ik\tau}\left(\begin{array}{c} e^{(x_{13}+iy_{13})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_{13}+iy_{13})\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_{13}+iy_{13})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_{13}+iy_{13})\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ +\frac{U}{4\Omega l}e^{-W\xi-ik\tau}\left(\begin{array}{c} e^{(x_{14}+iy_{14})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_{14}+iy_{14})\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_{14}+iy_{14})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_{14}+iy_{14})\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \end{array}\right\}, \tag{5.9}$$

$$x_{12}=\left[\frac{\sqrt{(W^2+2N_1)^2+4}+(W+2N_1)^2}{2}\right]^{\frac{1}{2}}, \tag{5.10}$$

$$x_{13}=\left[\frac{\sqrt{(W^2+2N_1)^2+4(k+1)^2}+(W+2N_1)^2}{2}\right]^{\frac{1}{2}}, \tag{5.11}$$

$$x_{14}=\left[\frac{\sqrt{(W^2+2N_1)^2+4(k-1)^2}+(W+2N_1)^2}{2}\right]^{\frac{1}{2}}, \tag{5.12}$$

$$y_{12}=\left[\frac{\sqrt{(W^2+2N_1)^2+4}-(W+2N_1)^2}{2}\right]^{\frac{1}{2}}, \tag{5.13}$$

$$y_{13}=\left[\frac{\sqrt{(W^2+2N_1)^2+4(k+1)^2}-(W+2N_1)^2}{2}\right]^{\frac{1}{2}}, \tag{5.14}$$

$$y_{14}=\left[\frac{\sqrt{(W^2+2N_1)^2+4(k-1)^2}-(W+2N_1)^2}{2}\right]^{\frac{1}{2}}, \tag{5.15}$$

$$\eta_8=(x_{12}+iy_{12})\left(\frac{i}{4}-W^2\right)+\frac{3iW^2}{4(x_{12}+iy_{12})}, \tag{5.16}$$

$$\eta_9=(y_{13}+iy_{13})\left[W^2+\frac{(k+1)i}{4}-\frac{i}{2}\right]-\frac{iW^2(k+3)}{4(x_{13}+iy_{13})}, \tag{5.17}$$

$$\eta_{10}=(x_{14}+iy_{14})\left[W^2-\frac{(k-1)i}{4}-\frac{i}{2}\right]+\frac{iW^2(k-3)}{4(x_{14}+iy_{14})}. \tag{5.18}$$

Now Eqs. (4.87) to (4.90) take the following forms

$$F(\xi,\tau)=\widetilde{\widetilde{H}}_1(\xi,\tau)+\beta e^{-i\tau}\widetilde{\widetilde{H}}_2(\xi,\tau)+\frac{\beta Ue^{-i\tau}}{2\Omega l}\left(\widetilde{\widetilde{H}}_3(\xi,\tau)+\widetilde{\widetilde{H}}_4(\xi,\tau)\right),\ U\cos nt,\quad k\lessgtr 1, \tag{5.19}$$

$$F(\xi,\tau)=\widetilde{\widetilde{H}}_5(\xi,\tau)+\beta e^{-i\tau}\widetilde{\widetilde{H}}_2(\xi,\tau)+\frac{i\beta Ue^{-i\tau}}{2\Omega l}\left(\widetilde{\widetilde{H}}_3(\xi,\tau)-\widetilde{\widetilde{H}}_4(\xi,\tau)\right),\ U\sin nt,\quad k\lessgtr 1. \tag{5.20}$$

With the help of Eq. (2.14), above results become

$$\frac{f+ig}{\Omega l}=\left\{\begin{array}{c} 1+\widetilde{\widetilde{H}}_1(\xi,\tau)+\beta e^{-i\tau}\widetilde{\widetilde{H}}_2(\xi,\tau) \\ +\frac{\beta Ue^{-i\tau}}{2\Omega l}\left(\widetilde{\widetilde{H}}_3(\xi,\tau)+\widetilde{\widetilde{H}}_4(\xi,\tau)\right), U\cos nt, k\lessgtr 1 \\ 1+\widetilde{\widetilde{H}}_5(\xi,\tau)+\beta e_2^{-i\tau}\widetilde{\widetilde{H}}(\xi,\tau) \\ +\frac{i\beta Ue^{-i\tau}}{2\Omega l}\left(\widetilde{\widetilde{H}}_3(\xi,\tau)-\widetilde{\widetilde{H}}_4(\xi,\tau)\right), U\sin nt, k\lessgtr 1 \end{array}\right\}. \tag{5.21}$$

The real part gives $\frac{f}{\Omega l}$ and imaginary part gives $\frac{g}{\Omega l}$. The results for blowing can be obtained by replacing W with $-W_1$

For resonant case $k=1$ and the solutions in this case are given by

$$\frac{f+ig}{\Omega l}=\left\{\begin{array}{c} 1+\widehat{H}_1(\xi,\tau)+\beta e^{-i\tau}\widetilde{\widetilde{H}}_2(\xi,\tau) \\ +\frac{\beta Ue^{-i\tau}}{2\Omega l}\left(\widehat{H}_3(\xi,\tau)+\widehat{H}_4(\xi,\tau)\right), U\cos nt, \end{array}\right\} \tag{5.22}$$

$$\frac{f+ig}{\Omega l}=\left\{\begin{array}{c} 1+\widehat{H}_5(\xi,\tau)+\beta e^{-i\tau}\widetilde{\widetilde{H}}_2(\xi,\tau) \\ +\frac{i\beta Ue^{-i\tau}}{2\Omega l}\left(\widehat{H}_3(\xi,\tau)-\widehat{H}_4(\xi,\tau)\right), U\sin nt, \end{array}\right\} \tag{5.23}$$

where

$$\widehat{H}_1(\xi,\tau)=\left\{\begin{array}{c} \frac{-e^{-W\xi}}{2}\left(\begin{array}{c} e^{(x_{12}+iy_{12})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_{12}+iy_{12})\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_{12}+iy_{12})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_{12}+iy_{12})\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ +\frac{Ue^{-W\xi+i\tau}}{4\Omega l}\left(\begin{array}{c} e^{(\widehat{x}_6+i\widehat{y}_6)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(\widehat{x}_6+i\widehat{y}_6)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\widehat{x}_6+i\widehat{y}_6)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(\widehat{x}_6+i\widehat{y}_6)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ +\frac{Ue^{-W\xi-i\tau}}{4\Omega l}\left(\begin{array}{c} e^{W\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+\widehat{x}_7\sqrt{\frac{\tau}{2}}\right) \\ +e^{-W\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-\widehat{x}_7\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \end{array}\right\}, \quad (5.24)$$

$$\widehat{H}_3(\xi,\tau)=\left\{\begin{array}{c} -\xi\sqrt{\frac{2}{\pi\tau}}\left(\widehat{q}_1-\frac{W\xi}{\tau}\right)e^{-W\xi-\left(W^2+2N\right)\frac{\tau}{2}-\frac{\xi^2}{2\tau}}+\xi e^{-W\xi+2i\tau} \\ \times\left(\begin{array}{c} (\widehat{\eta}_2-\widehat{\eta}_4)\,e^{(\widehat{x}_6+i\widehat{y}_6)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(\widehat{x}_6+i\widehat{y}_6)\sqrt{\frac{\tau}{2}}\right) \\ +(\widehat{\eta}_2+\widehat{\eta}_4)\,e^{-(\widehat{x}_6+i\widehat{y}_6)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(\widehat{x}_6+i\widehat{y}_6)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \end{array}\right\}, \quad (5.25)$$

$$\widehat{H}_4(\xi,\tau)=\left\{\begin{array}{c} -\xi\sqrt{\frac{2}{\pi\tau}}\left(\widehat{q}_2-\frac{W\xi}{\tau}\right)e^{-W\xi-\left(W^2+2N\right)\frac{\tau}{2}-\frac{\xi^2}{2\tau}} \\ +2\widehat{\eta}_3\xi e^{-2W\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-iW\sqrt{\frac{\tau}{2}}\right) \end{array}\right\}, \quad (5.26)$$

$$\widehat{H}_5(\xi,\tau)=\left\{\begin{array}{c} -\frac{e^{-W\xi}}{2}\left(\begin{array}{c} e^{(x_{12}+iy_{12})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(x_{12}+iy_{12})\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(x_{12}+iy_{12})\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(x_{12}+iy_{12})\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ -\frac{iU}{4\Omega l}e^{-W\xi+i\tau}\left(\begin{array}{c} e^{(\widehat{x}_6+i\widehat{y}_6)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(\widehat{x}_6+i\widehat{y}_6)\sqrt{\frac{\tau}{2}}\right) \\ +e^{-(\widehat{x}_6+i\widehat{y}_6)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(\widehat{x}_6+i\widehat{y}_6)\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \\ +\frac{iU}{4\Omega l}e^{-W\xi-i\tau}\left(\begin{array}{c} e^{W\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+\widehat{x}_7\sqrt{\frac{\tau}{2}}\right) \\ +e^{-W\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-\widehat{x}_7\sqrt{\frac{\tau}{2}}\right) \end{array}\right) \end{array}\right\} \quad (5.27)$$

and $\widetilde{\widetilde{H}}_2$ is given through Eq. (4.63). In above expressions

$$\widehat{\eta}_4=(\widehat{x}_6+i\widehat{y}_6)\,W^2-\frac{iW^2}{(\widehat{x}_6+i\widehat{y}_6)}, \quad (5.28)$$

$$\widehat{x}_6=\left[\frac{\sqrt{\left(W^2+2N_1\right)^2+16}+\left(W^2+2\right)N_1}{2}\right]^{\frac{1}{2}}, \quad (5.29)$$

$$\widehat{y}_6 = \left[\frac{\sqrt{(W^2+2N_1)^2+16}-(W^2+2N_1)}{2}\right]^{\frac{1}{2}}. \tag{5.30}$$

$$\widehat{x}_7 = \sqrt{W^2+2N_1}. \tag{5.31}$$

5.2 Graphs and tables

This section includes the table showing the numerical values for the variation of N_1, β and k on $\frac{f}{\Omega l}$ and $\frac{g}{\Omega l}$ in the resonant case. The resonant steady solutions are also sketched for $\epsilon\left(=\frac{U}{2\Omega l}\right)=1$. The table for N_1, β and k on $\frac{f}{\Omega l}$ and $\frac{g}{\Omega l}$ is as follows:

ξ	W	τ	ϵ	k	β	N_1	$\frac{f}{\Omega l}$	$\frac{g}{\Omega l}$
0.5	5	0.5	1	1	0.5	1	0.216477	0.038459
						5	0.414775	0.026184
						10	0.577667	0.015945
						15	0.992641	0.0099332

ξ	W	τ	ϵ	k	N_1	β	$\frac{f}{\Omega l}$	$\frac{g}{\Omega l}$
0.5	5	0.5	1	1	10	0.1	0.913241	0.003189
						0.2	0.829347	0.006378
						0.3	0.745454	0.009567
						0.4	0.661560	0.0127569

ξ	W	τ	ϵ	β	N_1	k	$\frac{f}{\Omega l}$	$\frac{g}{\Omega l}$
0.5	5	0.5	1	0.5	10	1	0.578875	0
						10	0.666184	0.045886
						15	0.772133	0.095678

The results for $\frac{f}{\Omega l}$ and $\frac{g}{\Omega l}$ against ξ are plotted in the following graphs.

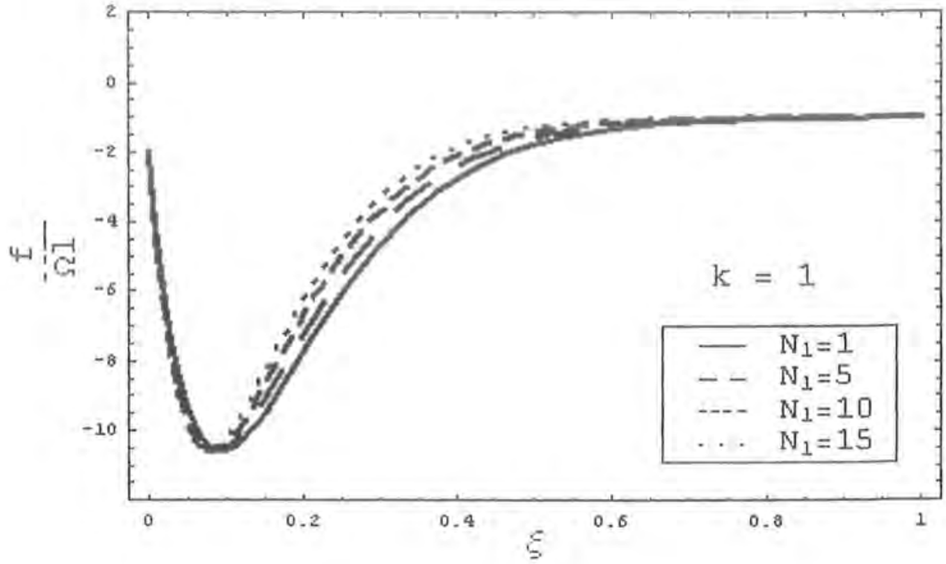

Fig.5.1 The variation of $\frac{f}{\Omega l}$ with ξ for the cosine oscillation at $W = 5, \tau = 0.7, \beta = 0.5,$

$\epsilon = 1$.

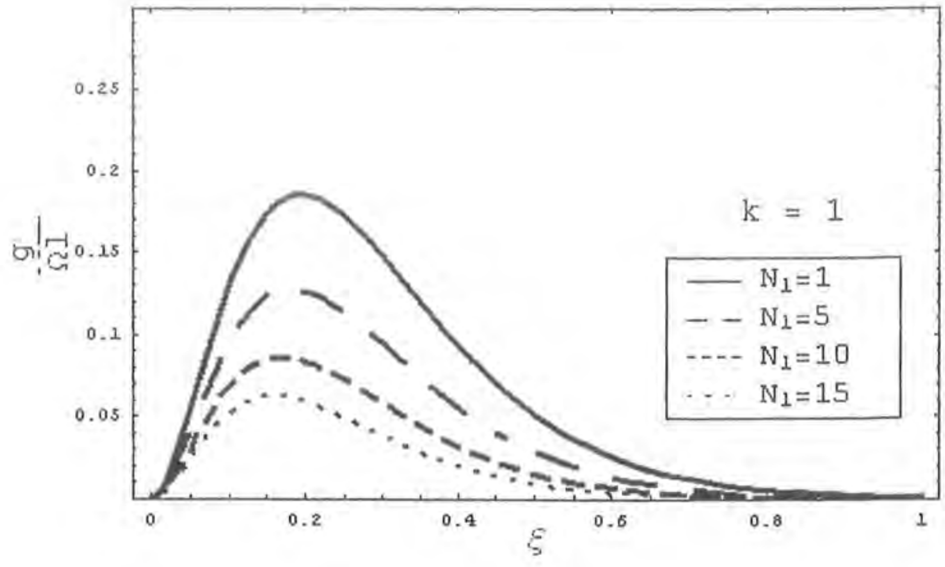

Fig.5.2 The variation of $\frac{g}{\Omega l}$ with ξ for the cosine oscillation at $W = 5, \tau = 0.7, \beta = 0.5,$

$\epsilon = 1$.

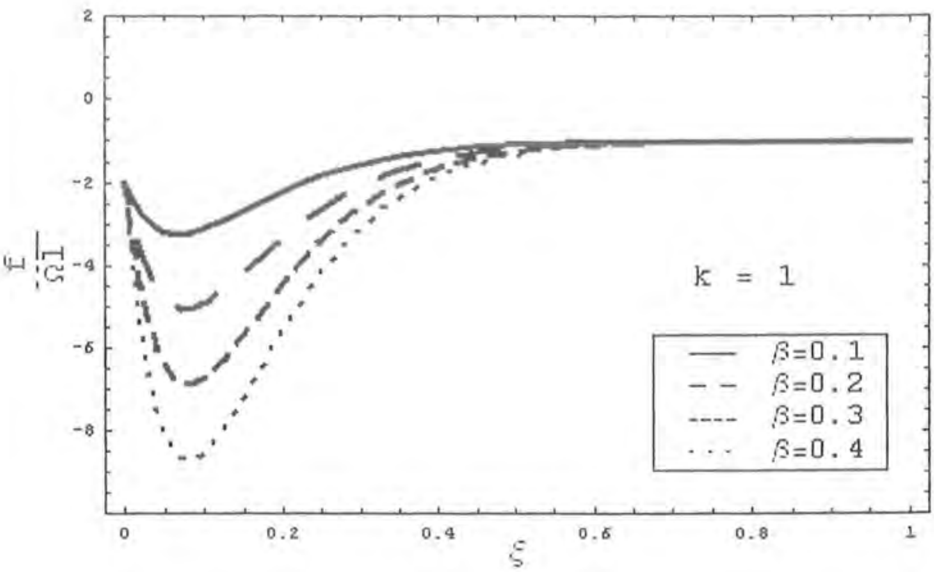

Fig.5.3 The variation of $\frac{f}{\Omega l}$ with ξ for the cosine oscillation at $W = 5, \tau = 0.5, N_1 = 10$, $\epsilon = 1$.

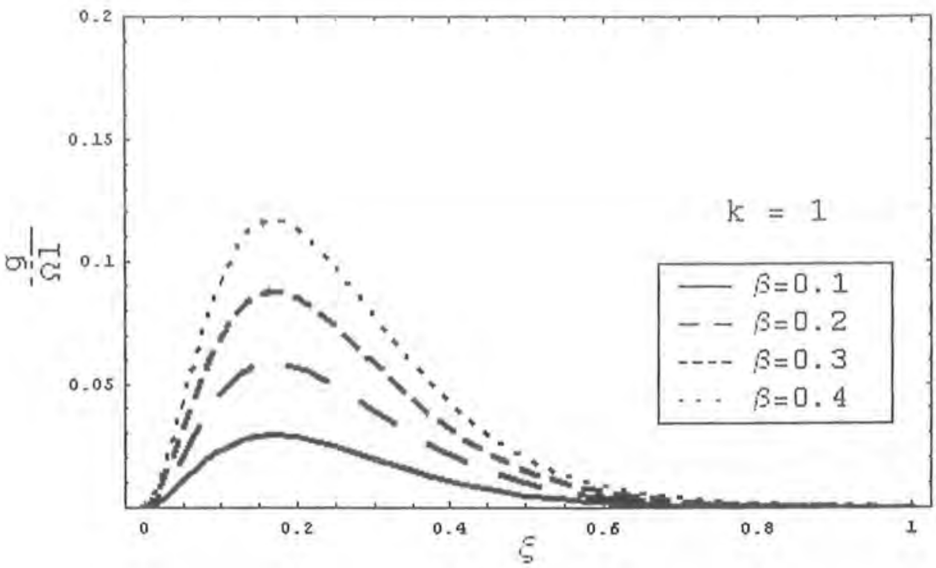

Fig.5.4 The variation of $\frac{g}{\Omega l}$ with ξ for the cosine oscillation at $W = 5, \tau = 0.5, N_1 = 10$, $\epsilon = 1$.

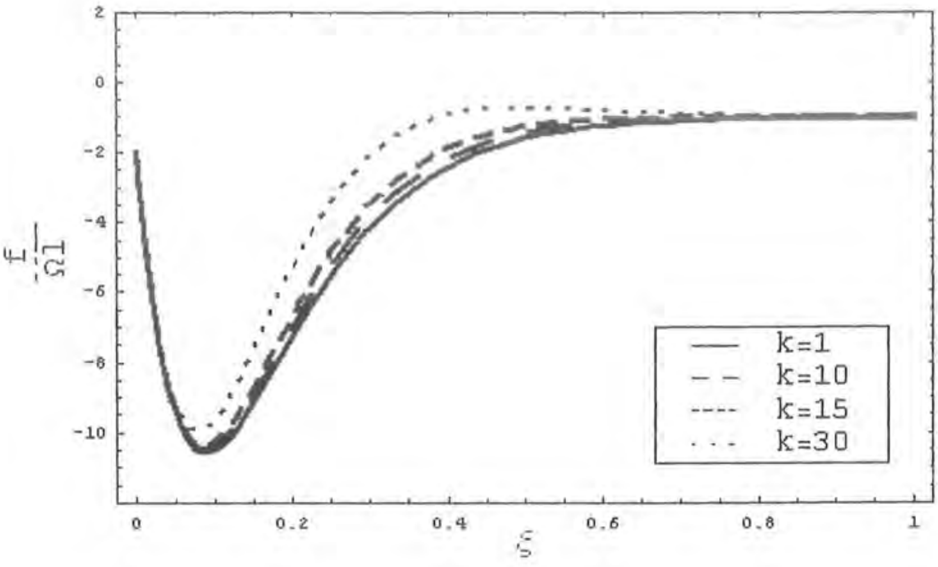

Fig.5.5 The variation of $\frac{f}{\Omega l}$ with ξ for the cosine oscillation at $W = 5$, $\tau = 1$, $N_1 = 5$, $\beta = 0.5$.

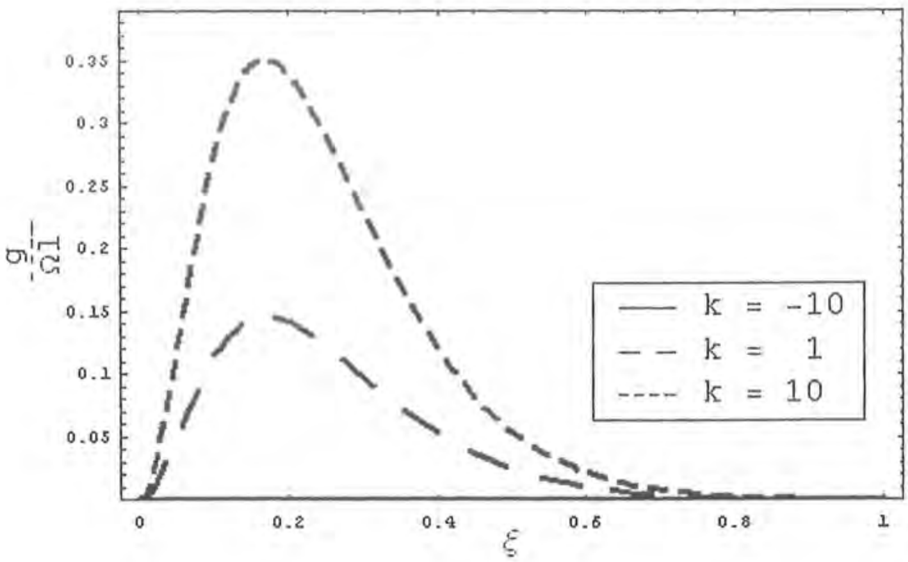

Fig.5.6 The variation of $\frac{g}{\Omega l}$ with ξ for the cosine oscillation at $W = 5$, $\tau = 1$, $N_1 = 5$, $\beta = 0.5$.

5.3 Results and discussion

In order to obtain the steady state solution, we will use the asymptotic form of the complementary error function (3.24).

Here if we let $\tau \to \infty$ in Eq. (5.2), we get for $U \cos nt$, $k \lessgtr 1$

$$\frac{f_s + ig_s}{\Omega l} = \left\{ \begin{array}{c} 1 + \left[-1 + 2\beta\xi\left(\eta_1 + \eta_2\right)\right] e^{-W\xi - (x_{12} + iy_{12})\xi} \\ + \frac{U}{2\Omega l}\left[1 + 2\beta\xi\left(\eta_3 + \eta_4\right)\right] e^{-W\xi - (x_{13} + iy_{13})\xi + ik\tau} \\ + \frac{U}{2\Omega l}\left[1 + 2\beta\xi\left(\eta_5 + \eta_6\right)\right] e^{-W\xi - (x_{14} + iy_{14})\xi - ik\tau} \end{array} \right\}. \quad (5.32)$$

For $U \sin nt$, $k \lessgtr 1$

$$\frac{f_s + ig_s}{\Omega l} = \left\{ \begin{array}{c} 1 + \left[-1 + 2\beta\xi\left(\eta_1 + \eta_2\right)\right] e^{-W\xi - (x_{12} + iy_{12})\xi} \\ + \frac{iU}{2\Omega l}\left[-1 + 2\beta\xi\left(\eta_3 + \eta_4\right)\right] e^{-W\xi - (x_{13} + iy_{13})\xi + ik\tau} \\ - \frac{iU}{2\Omega l}\left[-1 + 2\beta\xi\left(\eta_5 + \eta_6\right)\right] e^{-W\xi - (x_{14} + iy_{14})\xi - ik\tau} \end{array} \right\} \quad (5.33)$$

and transient solutions can be obtained by subtracting steady part of the solution from Eqs. (5.2) and (5.3).

It is pertinent to mention here that the solutions (5.32) and (5.33) satisfy required boundary conditions in Eq. (4.6). Therefore for all values of frequencies the flow can always be determined and has a well-defined boundary layer on the disk for suction and blowing. For suction, the distinct boundary layer thicknesses are of order $(W + x_j)^{-1}$, $j = 12 - 14$. It can be seen clearly that thicknesses of the boundary layers strongly depend upon suction, magnetic and rotation parameters. The boundary layer thicknesses decrease drastically with the increase in rotation, suction and magnetic parameters. The steady flow in suction and blowing cases has a structure of an oscillating layer in which shear oscillations are confined near the disk. An interesting result which emerges from this analysis is that for resonant case $(k=1)$, the steady blowing solutions are given by:

For $U\cos nt$

$$\frac{f_s+ig_s}{\Omega l}=\left\{\begin{array}{c} 1+\left[-1+2\beta\xi\left(\eta_1+\eta_2\right)\right]e^{-W\xi-(x_{12}+iy_{12})\xi} \\ +\frac{U}{2\Omega l}\left[1+2\beta\xi\left(\widehat{\eta}_3+\widehat{\eta}_4\right)\right]e^{-W\xi-(\widehat{x}_6+i\widehat{y}_6)\xi+i\tau} \\ +\frac{U}{2\Omega l}\left[1+4\beta\xi\widehat{\eta}_3\right]e^{-W\xi-\widehat{x}_7\xi-i\tau} \end{array}\right\}. \tag{5.34}$$

For $U\sin nt$

$$\frac{f_s+ig_s}{\Omega l}=\left\{\begin{array}{c} 1+\left[-1+2\beta\xi\left(\eta_1+\eta_2\right)\right]e^{-W\xi-(x_{12}+iy_{12})\xi} \\ +\frac{iU}{2\Omega l}\left[-1+2\beta\xi\left(\widehat{\eta}_3+\widehat{\eta}_4\right)\right]e^{-W\xi-(\widehat{x}_6+i\widehat{y}_6)\xi+i\tau} \\ -\frac{iU}{2\Omega l}\left[1+4\beta\xi\widehat{\eta}_3\right]e^{-W\xi-\widehat{x}_7\xi-i\tau} \end{array}\right\} \tag{5.35}$$

which satisfy the boundary condition at infinity, contradicting the hydrodynamic analysis [94]. In [94], it was shown that the obtained steady solution is valid for all values of the frequencies and suction parameter except for the blowing and resonant frequency $(k=1)$. It is known that the suction and blowing have opposite characteristics on the boundary layer flows. In fact, the suction plays a role to prevent the oscillations from spreading far away from the disk for all values of frequencies. This is due to viscous diffusion. On the other hand, the blowing is responsible to promote the spreading of the oscillations far away from the disk and thus the penetration depth of the oscillations is not finite when $k=1$. Therefore, the meaningful hydrodynamic steady blowing solution for the resonant frequency in not possible. To provide the meaningful steady blowing solution in the resonant case we considered the present magnetohydrodynamic analysis. Here, the solutions (5.34) and (5.35) satisfy the boundary conditions for all values of the frequencies including $k=1$. The unbounded spreading of the oscillations away from the disk is controlled by the external magnetic field and thus the hydromagnetic oscillations have been confined to the ultimate boundary layer. Basically, the diffusive hydromagnetic waves occur in the rotating system. These waves decay within the ultimate steady state boundary layers. In other words, the presence of an external magnetic field expedites the decaying process of diffusive hydromagnetic waves. This fact can also be described through Figs.5.1 − 5.6 as follows:

Figs.5.1 and 5.2 elucidate the variation of Hartman number on the steady blowing solution in the resonant case. It is clearly seen that $\frac{f}{\Omega l}$ increases and $\frac{g}{\Omega l}$ decreases with the increase in N_1. The effect of material parameter of the second grade fluid on the velocities $\frac{f}{\Omega l}$ and $\frac{g}{\Omega l}$ is shown in Figs.5.3 and 5.4, respectively. An increase in β leads to decrease in $\frac{f}{\Omega l}$. The behavior of β has been also seen on the velocity $\frac{g}{\Omega l}$. The behavior of β on $\frac{g}{\Omega l}$ is quite opposite to that of $\frac{f}{\Omega l}$. In order to illustrate the variation of k on $\frac{f}{\Omega l}$ and $\frac{g}{\Omega l}$, we made Figs.5.5 and 5.6. These Figs. indicate that both $\frac{f}{\Omega l}$ and $\frac{g}{\Omega l}$ increases by increasing k. However $\frac{g}{\Omega l}$ increases more as compared to $\frac{f}{\Omega l}$.

It is worth emphasizing to note that mathematical nature and physical content of the solution obtained by three limiting procedures $k \to 1$ $\tau \to \infty$ $N_1 \to 0$ in this case or in the reverse order are different for $W = 0$.

From Eqs. (5.21)

For $U \cos nt$

$$Lim_{k\to 1}\ Lim_{\tau\to\infty} Lim_{N_1\to 0}\left(\frac{f+ig}{\Omega l}\right) = \left\{\begin{array}{c} 1 - e^{-(1+i)\xi} + \frac{U}{2\Omega l} e^{i\tau-\sqrt{2}(1+i)\xi} \\ +\frac{U}{2\Omega l} e^{-i\tau} + \beta\xi i\,(1+i)\, e^{-(1+i)\xi} \end{array}\right\} \quad (5.36)$$

and

$$Lim_{\tau\rightarrow\infty}\ Lim_{k\rightarrow 1} Lim_{N_1\rightarrow 0}\left(\frac{f+ig}{\Omega l}\right)=\left\{\begin{array}{c} 1-e^{-(1+i)\xi}+\frac{U}{2\Omega l}e^{i\tau-\sqrt{2}(1+i)\xi} \\ +\frac{U}{2\Omega l}e^{-i\tau}\operatorname{erf}c\left(\frac{\xi}{2\tau}\right)+\frac{\beta\xi i(1+i)}{2}e^{-(1+i)\xi} \end{array}\right\}. \tag{5.37}$$

For $U\sin nt$

$$Lim_{k\rightarrow 1}\ Lim_{\tau\rightarrow 1} Lim_{N_1\rightarrow 0}\left(\frac{f+ig}{\Omega l}\right)=\left\{\begin{array}{c} 1-e^{-(1+i)\xi}-\frac{iU}{2\Omega l}e^{i\tau-\sqrt{2}(1+i)\xi} \\ +\frac{iU}{2\Omega l}e^{-i\tau}-\beta\xi\left(1+i\right)e^{-(1+i)\xi} \end{array}\right\} \tag{5.38}$$

and

$$Lim_{\tau\rightarrow\infty}\ Lim_{k\rightarrow 1} Lim_{N_1\rightarrow 0}\left(\frac{f+ig}{\Omega l}\right)=\left\{\begin{array}{c} 1-e^{-(1+i)\xi}-\frac{iU}{2\Omega l}e^{i\tau-\sqrt{2}(1+i)\xi} \\ +\frac{iU}{2\Omega l}e^{-i\tau}\operatorname{erf}c\left(\frac{\xi}{2\tau}\right)-\frac{\beta\xi(1+i)}{2}e^{-(1+i)\xi} \end{array}\right\}. \tag{5.39}$$

Clearly the expressions (5.36), (5.37) and (5.38), (5.39) are quite different. It may be noted that Eqs.(5.36) and (5.38) do not satisfy the boundary conditions at infinity and hence are not the meaningful solution. But Eqs. (5.37) and (5.39) satisfy the boundary conditions and hence qualify for the solution. Moreover, the present analysis is more general and the results of several previous studies [45, 49, 50, 93, 94] can be obtained as the limiting cases. In particular this study includes the solution for hydrodynamic flow to a non-coaxial rotation of rigid non-oscillating disk and a viscous fluid by taking $\beta=0=W=U=N_1$ [45]. For hydrodynamic viscous flow of a rigid oscillating disk we put $\beta=0=W=N_1$ [49]. The results for magnetohydrodynamic flow of a second grade fluid and porous non-oscillating disk can be recovered by taking $U=0$ [50]. For viscous magnetohydrodynamic fluid and porous oscillating disk one can recover the results by choosing $\beta=0$ [93]. The results for hydrodynamic flow of a second grade fluid and porous oscillating disk can be obtained by taking $N_1 = 0$ [94].

5.4 Concluding remarks

In this chapter, we have presented a study of an incompressible, magnetohydrodynamic second grade fluid under transversely applied magnetic field. The flow is induced due to non-coaxial rotations of an oscillating porous disk and a fluid at infinity. The problem solved by the method of Laplace transform provides the results for unsteady and steady meaningful solutions in the suction and blowing cases for all values of the frequencies. The effect of second grade fluid parameter and the magnetic field have also been studied. The physical mechanism for the existence of meaningful steady blowing solution in resonant case has been explained. The obtained analytical solutions have been also compared with the previous studies in the literature.

Chapter 6

Generalized second grade flow past a porous plate with heat transfer

An analysis is carried out to study the steady flow and heat transfer characteristics in a non-Newtonian fluid past a uniformly porous plate. A modified model of second grade fluid that has shear dependent viscosity and can predict the normal stress difference is used. In heat transfer, two cases are considered *i.e.* *(i)* constant wall temperature *(ii)* insulated wall. The solution of equation of momentum is obtained using homotopy analysis method (HAM). Expression for temperature is given by solving the energy equation. Emphasis has been laid to study the effects of various emerging parameters on the flow and heat transfer characteristics.

6.1 Development of the flow

Consider the flow of a generalized second grade fluid (in the region $0 \leq y < \infty$) past a porous plate at $y = 0$ and temperature θ_0. The flow far away from the plate is uniform and temperature of the fluid is θ_∞. We select Cartesian coordinate system in which $x-$axis is parallel to the plate and the $y-$axis perpendicular to the plate. The equations which govern the incompressible flow are the continuity (2.2), linear momentum (4.1) and energy equations. The energy equation may be expressed as

$$\rho \frac{de}{dt} = \mathbf{T} \cdot \mathbf{L} - div\mathbf{q} + \rho r, \tag{6.1}$$

where e is the specific internal energy, r is the radiant heating and $\mathbf{q}$ is the heat flux vector. The constitutive equation for Cauchy stress tensor $\mathbf{T}$ in a generalized second grade fluid [95] is

$$\mathbf{T} = -p_1\mathbf{I} + \mu\Pi^{\widetilde{m}/2}\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{6.2}$$

where $\widetilde{m}$ is the material parameter. The second invariant of the symmetric part of the velocity gradient is defined as

$$\Pi = \frac{1}{2} tr\mathbf{A}_1^2. \tag{6.3}$$

It should be noted that for $\widetilde{m} = 0$, the model defined by Eq. (6.2) reduces to that considered by Rajagopal $[96 - 98]$, Rajagopal and Gupta $[99]$, Hayat et al. $[100 - 102]$ and Siddiqui et al. $[103]$. If $\alpha_1 = \alpha_2 = 0$, on the other hand, the generalized power law model [54] is recovered from Eq. (6.2). Moreover, if $\alpha_1 = \alpha_2 = 0$ and $\widetilde{m} = 0$ we are left with the classical model of Navier and Stokes.

We seek the velocity and temperature fields as

$$\mathbf{V} = u(y)\,\mathbf{i} + v(y)\,\mathbf{j}, \tag{6.4}$$

$$\theta = \theta(y), \tag{6.5}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the $x-$ and $y-$directions. It follows from Eq. (2.2) that

$$v(y) = -W_0 = \text{constant} \tag{6.6}$$

in which $W_0 > 0$ is a scale of suction velocity and $W_0 < 0$ is blowing velocity at the plate.

From Eqs. (1.4), (1.5), (6.2), (6.4) and (6.6) we can write

$$T_{xx} = -p_1 + \alpha_2 \left(\frac{du}{dy}\right)^2, \tag{6.7}$$

$$T_{xy} = \mu \left[\left(\frac{du}{dy}\right)^2\right]^{\bar{m}/2} \frac{du}{dy} + \alpha_1 \left(\frac{du}{dy} - W_0 \frac{d^2u}{dy^2}\right), \tag{6.8}$$

$$T_{xz} = 0 = T_{yz}, \tag{6.9}$$

$$T_{yy} = -p_1 + (2\alpha_1 + \alpha_2)\left(\frac{du}{dy}\right)^2, \tag{6.10}$$

$$T_{zz} = -p_1 \tag{6.11}$$

and Eq. (4.1) gives

$$-\rho W_0 \frac{du}{dy} = -\frac{\partial P}{\partial x} + \frac{d}{dy}\left\{\mu \left[\left|\frac{du}{dy}\right|^2\right]^{\bar{m}/2} \frac{du}{dy}\right\} - \alpha_1 W_0 \frac{d^3u}{dy^3}, \tag{6.12}$$

$$0 = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z}, \tag{6.13}$$

where the modified pressure P is

$$P = p_1 - (2\alpha_1 + \alpha_2)\left(\frac{\partial u}{\partial y}\right)^2 \tag{6.14}$$

and Eq. (6.13) indicates that P is not a function of y and z. With a consideration of uniform free stream velocity U_∞ far away from the plate and following Rajagopal and Gupta [99] we have from Eq. (6.13) that

$$\frac{\partial P}{\partial x}=0 \tag{6.15}$$

and hence Eq. (6.12) becomes

$$\frac{d}{dy}\left\{\mu\left[\left|\frac{du}{dy}\right|^2\right]^{\widetilde{m}/2}\frac{du}{dy}\right\}+\rho W_0\frac{du}{dy}-\alpha_1 W_0\frac{d^3u}{dy^3}=0. \tag{6.16}$$

The boundary conditions are

$$u\left(0\right)=0,\ u\to U_\infty \quad \text{as} \quad y\to\infty. \tag{6.17}$$

Since Eq. (6.16) is a third order ordinary differential equation and we thus have one boundary condition less than that necessary to solve Eq. (6.16). While it is possible to augment the boundary conditions based on the asymptotic structures for the velocity field or the stress. We thus have

$$\frac{du}{dy}\to 0 \quad \text{as} \quad y\to\infty \tag{6.18}$$

as there is no shear in the free stream.

Let us introduce the following non-dimensional parameters

$$y^*=\frac{yU_\infty}{\nu},\quad u^*=\frac{u}{U_\infty},\quad W_0^*=\frac{W_0}{U_\infty},\quad \mu^*=\frac{\mu}{\mu_0}. \tag{6.19}$$

The Eq. (6.16) and boundary conditions (6.17) and (6.18) after dropping asterisks take the following form

$$\alpha V_0\frac{d^3u}{dy^3}-W_0\frac{du}{dy}-\mu\Gamma_{\widetilde{m}}\frac{d}{dy}\left\{\left[\left|\frac{du}{dy}\right|^2\right]^{\widetilde{m}/2}\frac{du}{dy}\right\}=0, \tag{6.20}$$

$$u(0)=0,\ u\to 1\ \text{ as }\ y\to\infty,\ \frac{du}{dy}\to 0\ \text{ as }\ y\to\infty, \tag{6.21}$$

where

$$\alpha=\frac{\alpha_1 U_\infty^2}{\rho\nu^2},\quad \Gamma_{\widetilde{m}}=\frac{\mu_0}{\rho U_\infty^2}\left(\frac{U_\infty^2}{\nu}\right)^{\widetilde{m}+1}. \tag{6.22}$$

Eq. (6.1) for the flow under considration becomes

$$k\frac{d^2\theta}{dy^2}+\rho c W_0\frac{d\theta}{dy}=\left\{\alpha_1 W_0\frac{d^2u}{dy^2}-\mu\left[\left(\frac{du}{dy}\right)^2\right]^{\widetilde{m}/2}\frac{du}{dy}\right\}\frac{du}{dy}, \tag{6.23}$$

where c is the fluid specific heat, θ is the temperature and k is the constant thermal conductivity. The equation (6.23) is solved subject to the following boundary conditions:

6.1.1 Case 1: Constant wall temperature

$$\theta(0)=\theta_0,\ \theta(y)\to\theta_\infty\quad\text{as}\quad y\to\infty. \tag{6.24}$$

6.1.2 Case 2: Insulated wall

$$\left.\frac{d\theta}{dy}\right|_{y=0}=0,\ \theta(y)\to\theta_\infty\quad\text{as}\quad y\to\infty. \tag{6.25}$$

6.2 Analytic solutions for integer material parameter ($\widetilde{m}$)

Here, we seek solutions with $du/dy>0$, $\alpha_1>0$ and consider the case that $\widetilde{m}$ is a positive integer. When $\widetilde{m}=1$, Eq. (6.20) becomes

$$\alpha W_0\frac{d^3u}{dy^3}-W_0\frac{du}{dy}-2\mu\Gamma_1\frac{du}{dy}\frac{d^2u}{dy^2}=0. \tag{6.26}$$

When $\widetilde{m}=2$, Eq. (6.20) is

$$\alpha W_0\frac{d^3u}{dy^3}-W_0\frac{du}{dy}-3\mu\Gamma_2\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}=0 \tag{6.27}$$

and for $\widetilde{m}=3$,

$$\alpha W_0\frac{d^3u}{dy^3}-W_0\frac{du}{dy}-4\mu\Gamma_3\left(\frac{du}{dy}\right)^3\frac{d^2u}{dy^2}=0, \tag{6.28}$$

in which Γ_1, Γ_2 and Γ_3 can be taken from Eq. (6.22).

Now to solve the non-linear differential Eqs. (6.26) to (6.28) subject to boundary conditions (6.21), we apply the HAM to give an explicit, uniformly valid and totally analytic solutions.

6.2.1 The zero-order deformation equation

In view of boundary conditions (6.21) and Eq. (6.20), we take

$$u_0(y)=\left(1-e^{-\frac{y}{\sqrt{\alpha}}}\right), \tag{6.29}$$

as the initial guess of $u(y)$ and

$$\mathcal{L}\left[\overline{u}(y;p)\right]=\left(\alpha W_0\frac{\partial^3}{\partial y^3}-W_0\frac{\partial}{\partial y}\right)\overline{u}(y;p) \tag{6.30}$$

as the auxiliary linear operator. Moreover using Eq. (6.20) the non-linear operator is

$$\begin{aligned}\mathcal{N}\left[\overline{u}(y;p)\right] &= \alpha W_0\frac{\partial^3\overline{u}(y;p)}{\partial y^3}-W_0\frac{\partial\overline{u}(y;p)}{\partial y}- \\ &\quad \mu\Gamma_{\widetilde{m}}\frac{\partial}{\partial y}\left\{\left[\left(\frac{\partial\overline{u}(y;p)}{\partial y}\right)^2\right]^{\widetilde{m}/2}\frac{\partial\overline{u}(y;p)}{\partial y}\right\}\end{aligned} \tag{6.31}$$

and the zero-order deformation problem is

$$(1-p)\,\mathcal{L}\left[\overline{u}(y;p)-u_0(y)\right]=p\hbar\mathcal{N}\left[\overline{u}(y;p)\right], \tag{6.32}$$

$$\overline{u}(0;p)=0,\ \overline{u}(y;p)\rightarrow 1 \text{ as } y\rightarrow\infty,\ \frac{\partial\overline{u}(y;p)}{\partial y}\rightarrow 0 \text{ as } y\rightarrow\infty, \tag{6.33}$$

in which p and $\hbar$ are embedding and non-zero auxiliary parameters respectively.

For $p=0$, the solution of Eqs. (6.32) and (6.33) is

$$\overline{u}(y;0)=u_0(y) \tag{6.34}$$

and for $p=1$, Eqs. (6.32) and (6.33) are equivalent to Eqs. (6.20) and (6.21), provided

$$\overline{u}(y;1)=u(y). \tag{6.35}$$

Clearly when p increases from 0 to 1, $\overline{u}(y;p)$ varies from $u_0(y)$ to $u(y)$ governed by Eqs. (6.20) and (6.21). We have great freedom to choose $\hbar$. Assume that $\hbar$ is properly chosen so that the zero-order deformation Eqs. (6.32) and (6.33) have solutions for all $p\in[0,1]$ and thus the term

$$u_k(y)=\frac{1}{k!}\frac{\partial^k\overline{u}(y;p)}{\partial p^k}\bigg|_{p=0}, \tag{6.36}$$

exists for $k\geq 1$. By Taylor's theorem and Eq. (6.34), $\overline{u}(y;p)$ can be expanded in power series of p as

$$\overline{u}(y;p)=u_0(y)+\sum_{k=1}^{+\infty}u_k(y)\,p^k. \tag{6.37}$$

Furthermore, assuming that $\hbar$ is so properly chosen that the above power series is convergent at $p=1$, we have from Eq. (6.35) the solution series

$$u(y)=u_0(y)+\sum_{k=1}^{+\infty}u_k(y). \tag{6.38}$$

6.2.2 The higher-order deformation equations

For brevity, define the vectors

$$\mathbf{u}_k = \{u_0(y), u_1(y), u_2(y), \cdots, u_k(y)\}. \tag{6.39}$$

Differentiating the zero-order deformation problem consisting of Eqs. (6.32) and (6.33) with respect to p we have for $\widetilde{m}=1$ as

$$(1-p)\mathcal{L}[\overline{u}(y;p) - u_0(y)] = p\hbar\left[-W_0\frac{\partial\overline{u}}{\partial y} + \alpha W_0\frac{\partial^3\overline{u}}{\partial y^3} - 2\mu\Gamma_1\frac{\partial\overline{u}}{\partial y}\frac{\partial^2\overline{u}}{\partial y^2}\right], \tag{6.40}$$

$$\overline{u}(0;p) = 0,\ \overline{u}(y;p) \to 1 \ \text{ as } \ y \to \infty. \tag{6.41}$$

Now differentiate (6.40) with respect to p and then setting $p=0$, we obtain the following first order deformation problem

$$\alpha W_0\frac{d^3u_1(y)}{dy^3} - W_0\frac{du_1(y)}{dy} = \hbar\left[-W_0\frac{du_0}{dy} + \alpha W_0\frac{d^3u_0}{dy^3} - 2\mu\Gamma_1\frac{du}{dy}\frac{d^2u_0}{dy^2}\right], \tag{6.42}$$

$$u_1(0) = 0, u_1(y) \to 0 \ \text{ as } \ y \to \infty. \tag{6.43}$$

Now differentiating Eq. (6.40) with respect to p twice and then setting $p=0$ and dividing by 2!, we get the second-order deformation problem as follows

$$\left.\begin{array}{l}\alpha W_0\frac{d^3u_2(y)}{dy^3} - W_0\frac{du_2(y)}{dy} = W_0(1+\hbar)\left(\alpha\frac{d^3u_1(y)}{dy^3} - \frac{du_1(y)}{dy}\right)\\ \quad -2\hbar\mu\Gamma_1\left(\frac{du_0}{dy}\frac{d^2u_1(y)}{dy^2} + \frac{d^2u_0(y)}{dy^2}\frac{du_1(y)}{dy}\right)\end{array}\right\}, \tag{6.44}$$

$$u_2(0) = 0, \quad u_2 \to 0 \ \text{ as } \quad y \to \infty. \tag{6.45}$$

Now again differentiate Eq. (6.40) thrice with respect to p then dividing by 3! and setting $p=0$, we get the third-order deformation problem as follows

$$\left.\begin{array}{l}\alpha W_0\frac{d^3u_3(y)}{dy^3} - W_0\frac{du_3(y)}{dy} = \frac{1}{2}(1+\hbar)W_0\left(\alpha\frac{d^3u_2(y)}{dy^3} - \frac{du_2(y)}{dy}\right)\\ \quad -\hbar\mu\Gamma_1\left(\frac{du_0}{dy}\frac{d^2u_2(y)}{dy^2} + \frac{du_1(y)}{dy}\frac{d^2u_1(y)}{dy^2}\right)\\ \quad -\hbar\mu\Gamma_1\left(\frac{d^2u_0}{dy^2}\frac{du_2(y)}{dy} + \frac{du_1(y)}{dy}\frac{d^2u_1(y)}{dy^2}\right)\end{array}\right\}, \tag{6.46}$$

$$u_3(0) = 0, \ u_3(\infty) = \infty, \tag{6.47}$$

For higher-order deformation equation, we first differentiate Eq.(6.40) $k-$ times with respect to p then divide by $k!$ and set $p = 0$. Here, the high-order deformation problem becomes

$$\mathcal{L}\left[u_k(y) - \chi_k u_{k-1}(y)\right] = \hbar \mathcal{R}_k(\mathbf{u}_{k-1}), \quad k \geq 1, \tag{6.48}$$

$$u_k(0) = 0, \ u_k \to 0 \ \text{ as } \ y \to \infty, \ \frac{du_k}{dy} \to 0 \ \text{ as } \ y \to \infty, \tag{6.49}$$

in which

$$\mathcal{R}_k(\mathbf{u}_{k-1}) = \frac{1}{(k-1)!} \frac{\partial^{k-1} \mathcal{N}\left[\overline{u}(y;p)\right]}{\partial p^{k-1}}\bigg|_{p=0}, \tag{6.50}$$

$$\chi_k = \begin{cases} 1, & k > 1, \\ 0, & k \leq 1. \end{cases} \tag{6.51}$$

We note that $\mathcal{R}_k(\mathbf{u}_{k-1})$ is dependent upon the integer material parameter $\widetilde{m}$. For $\widetilde{m} = 1$, 2 and 3 we respectively have

$$\mathcal{R}_k(\mathbf{u}_{k-1}) = \alpha W_0 u_{k-1}^{'''} - W_0 u_{k-1}^{'} - 2\mu\Gamma_1 \sum_{n=0}^{k-1} u_{k-1-n}^{'} u_n^{''}, \tag{6.52}$$

where primes indicate the derivative with respect to y.

When $\widetilde{m} = 2$ then

$$\mathcal{R}_k(\mathbf{u}_{k-1}) = \alpha W_0 u_{k-1}^{'''} - W_0 u_{k-1}^{'} - 3\mu\Gamma_2 \sum_{n=0}^{k-1} u_{k-1-n}^{'} \sum_{i=0}^{n} u_{n-i}^{'} u_i^{''} \tag{6.53}$$

and for $\widetilde{m} = 3$

$$\mathcal{R}_k(\mathbf{u}_{k-1}) = \alpha W_0 u_{k-1}^{'''} - W_0 u_{k-1}^{'} - 4\mu\Gamma_3 \sum_{n=0}^{k-1} u_{k-1-n}^{'} \sum_{i=0}^{n} u_{n-i}^{'} \sum_{j=0}^{i} u_{i-j}^{'} u_j^{''}. \tag{6.54}$$

6.2.3 Solution expressions when $\widetilde{m}$ is positive integer

Here we solve the linear kth-order deformation Eq. (6.48) with boundary conditions (6.49) upto third-order of approximations. The four term solution of Eqs.(6.20) and (6.21) is

$$u(y) = u_0(y) + u_1(y) + u_2(y) + u_3(y), \tag{6.55}$$

where $u_0(y)$ is given in Eq. (6.29) and solutions of Eqs. (6.42) to (6.45) for $\widetilde{m} = 1$ are

$$u_1(y) = \frac{1}{3\alpha W_0}\hbar\mu\Gamma_1\left(1 - e^{-\frac{y}{\sqrt{\alpha}}}\right)e^{-\frac{y}{\sqrt{\alpha}}}, \tag{6.56}$$

$$u_2(y) = \frac{\hbar\mu\Gamma_1}{18\alpha^2 W_0^2}\left[6\alpha W_0(1+\hbar) - \hbar\mu\Gamma_1\left(1 - 3e^{-\frac{y}{\sqrt{\alpha}}}\right)\right]\left(1 - e^{-\frac{y}{\sqrt{\alpha}}}\right)e^{-\frac{y}{\sqrt{\alpha}}}, \tag{6.57}$$

$$u_3(y) = \frac{\hbar\mu\Gamma_1 e^{-\frac{y}{\sqrt{\alpha}}}}{270\alpha^3 W_0^3}\left\{\begin{array}{c} 26\hbar^2\Gamma_1^2\mu^2\left(1 - e^{-\frac{3y}{\sqrt{\alpha}}}\right) \\ -45\hbar\mu\Gamma_1\left(\hbar\mu\Gamma_1 - 2(1+\hbar)W_0\alpha\right)\left(1 - e^{-\frac{2y}{\sqrt{\alpha}}}\right) \\ +10\left(\begin{array}{c} 9(1+\hbar)^2 W_0^2\alpha^2 \\ -12\hbar(1+\hbar)W_0\alpha\mu\Gamma_1 + 2\hbar^2\Gamma_1^2\mu^2 \end{array}\right)\left(1 - e^{-\frac{y}{\sqrt{\alpha}}}\right) \end{array}\right\}. \tag{6.58}$$

Similarly when $\widetilde{m} = 2$

$$u_1(y) = \frac{1}{8\alpha^{3/2} W_0}\hbar\mu\Gamma_2\left(1 - e^{-\frac{2y}{\sqrt{\alpha}}}\right)e^{-\frac{y}{\sqrt{\alpha}}}, \tag{6.59}$$

$$u_2(y) = \frac{\hbar\mu\Gamma_2}{64\alpha^3 W_0^2}\left[8\alpha^{3/2} W_0(1+\hbar) + 3\hbar\mu e^{-\frac{2y}{\sqrt{\alpha}}}\right]\left(1 - e^{-\frac{2y}{\sqrt{\alpha}}}\right)e^{-\frac{y}{\sqrt{\alpha}}}, \tag{6.60}$$

$$u_3(y)=\frac{\hbar\mu\Gamma_2 e^{-\frac{y}{\sqrt{\alpha}}}}{35840\alpha^{9/2}W_0^3}\left\{\begin{array}{c}351\hbar^2\Gamma_2^2\mu^2\left(1-e^{-\frac{6y}{\sqrt{\alpha}}}\right)+576\hbar^2\Gamma_2\mu^2\left(1-e^{-\frac{5y}{\sqrt{\alpha}}}\right)\\ -210\hbar^2\Gamma_2\left(3+2\Gamma_2\right)\mu^2\left(1-e^{-\frac{4y}{\sqrt{\alpha}}}\right)\\ +112\hbar\left(1+\hbar\right)\left(15+16\Gamma_2\right)\mu\alpha^{3/2}W_0\left(1-e^{-\frac{3y}{\sqrt{\alpha}}}\right)\\ -210\hbar\mu\left\{8\left(1+\hbar\right)\left(1+\Gamma_2\right)\alpha^{3/2}W_0-\hbar\Gamma_2^2\mu\right\}\left(1-e^{-\frac{2y}{\sqrt{\alpha}}}\right)\\ +4480\left(1+\hbar\right)^2\alpha^3W_0^2\left(1-e^{-\frac{y}{\sqrt{\alpha}}}\right)\end{array}\right\} \tag{6.61}$$

and when $\widetilde{m}=3$

$$u_1(y)=\frac{1}{15\alpha^2W_0}\hbar\mu\Gamma_3\left(1-e^{-\frac{3y}{\sqrt{\alpha}}}\right)e^{-\frac{y}{\sqrt{\alpha}}}, \tag{6.62}$$

$$u_2(y)=\frac{\hbar\mu\Gamma_3}{225\alpha^4W_0^2}\left[15\alpha^2W_0\left(1+\hbar\right)+\hbar\mu\Gamma_3\left(1+5e^{-\frac{3y}{\sqrt{\alpha}}}\right)\right]\left(1-e^{-\frac{3y}{\sqrt{\alpha}}}\right)e^{-\frac{y}{\sqrt{\alpha}}}, \tag{6.63}$$

$$u_3(y)=\frac{\hbar\mu\Gamma_3 e^{-\frac{y}{\sqrt{\alpha}}}}{111375\alpha^6W_0^3}\left\{\begin{array}{c}1180\hbar^2\Gamma_3^2\mu^2\left(1-e^{-\frac{9y}{\sqrt{\alpha}}}\right)\\ +165\hbar\mu\Gamma_3\left(30\left(1+\hbar\right)\alpha^2W_0-7\hbar\mu\Gamma_3\right)\left(1-e^{-\frac{6y}{\sqrt{\alpha}}}\right)\\ +33\left(\begin{array}{c}225\left(1+\hbar\right)^2\alpha^4W_0^2+2\hbar^2\Gamma_3^2\mu^2\\ -120\hbar\left(1+\hbar\right)\alpha^2W_0\mu\Gamma_3\end{array}\right)\left(1-e^{-\frac{3y}{\sqrt{\alpha}}}\right)\end{array}\right\}. \tag{6.64}$$

6.3 Heat transfer analysis

Using the non-dimensional parameters defined in Eq. (6.19) and

$$\theta^*=\frac{\theta-\theta_\infty}{\theta_0-\theta_\infty} \tag{6.65}$$

Eq. (6.23) and the boundary conditions (6.24), after dropping the asterisks reduce to

$$\frac{d^2\theta}{dy^2}+P_rW_0\frac{d\theta}{dy}=E_cP_r\left[\alpha W_0\frac{d^2u}{dy^2}-\mu\Gamma_{\widetilde{m}}\left(\left|\frac{du}{dy}\right|\right)^{\widetilde{m}}\frac{du}{dy}\right]\frac{du}{dy}, \tag{6.66}$$

$$\theta(0) = 1,\ \theta(\infty) = 0, \tag{6.67}$$

where the Prandtl P_r and Eckert E_c numbers are

$$P_r = \frac{c\mu}{k},\ E_c = \frac{U_\infty^2}{c(\theta_0 - \theta_\infty)}. \tag{6.68}$$

To non-dimensionalize the boundary conditions (6.25), we define [104]

$$\theta^* = \frac{\theta - \theta_\infty}{\theta_b - \theta_\infty} \tag{6.69}$$

and get

$$\left.\frac{d\theta}{dy}\right|_{y=0} = 0,\ \theta(\infty) = 0, \tag{6.70}$$

where θ_b is the bulk temperature and Eckert number $\widetilde{E}_c$ in this case is

$$\widetilde{E}_c = \frac{U_\infty^2}{c(\theta_b - \theta_\infty)}. \tag{6.71}$$

Making use of Eq.(6.55) into Eq.(6.66) and then solving the resulting equation along-with the boundary conditions (6.67) we have the following expressions

6.3.1 Case I: Constant wall temperature

For $\widetilde{m} = 1$

$$\theta = \left. \begin{array}{c} \left(1 - \sqrt{\alpha} E_c P_r W_0 \widetilde{M}_1\right) e^{-P_r W_0 y} \\ +\alpha E_c P_r \left(\begin{array}{c} \frac{M_1 e^{\frac{-2y}{\sqrt{\alpha}}}}{\beta^* - 2y} + \frac{M_2 e^{\frac{-3y}{\sqrt{\alpha}}}}{\beta^* - 3y} + \frac{M_3 e^{\frac{-4y}{\sqrt{\alpha}}}}{\beta^* - 4y} + \frac{M_4 e^{\frac{-5y}{\sqrt{\alpha}}}}{\beta^* - 5y} \\ + \frac{M_5 e^{\frac{-6y}{\sqrt{\alpha}}}}{\beta^* - 6y} + \frac{M_6 e^{\frac{-7y}{\sqrt{\alpha}}}}{\beta^* - 7y} + \frac{M_7 e^{\frac{-8y}{\sqrt{\alpha}}}}{\beta^* - 8y} + \frac{M_8 e^{\frac{-9y}{\sqrt{\alpha}}}}{\beta^* - 9y} \end{array} \right) \end{array} \right\}. \tag{6.72}$$

For $\widetilde{m}=2$

$$\theta = \left.\begin{array}{c} \left(1-\sqrt{\alpha}E_cP_rW_0\widetilde{M}_2\right)e^{-P_rW_0y} \\ +\alpha E_cP_r\left(\begin{array}{c} \frac{M_9e^{\frac{-2y}{\sqrt{\alpha}}}}{\beta^*-2y}+\frac{M_{10}e^{\frac{-4y}{\sqrt{\alpha}}}}{\beta^*-4y}+\frac{M_{11}e^{\frac{-6y}{\sqrt{\alpha}}}}{\beta^*-6y}+\frac{M_{12}e^{\frac{-8y}{\sqrt{\alpha}}}}{\beta^*-8y} \\ +\frac{M_{13}e^{\frac{-10y}{\sqrt{\alpha}}}}{\beta^*-10y}+\frac{M_{14}e^{\frac{-12y}{\sqrt{\alpha}}}}{\beta^*-12y}+\frac{M_{15}e^{\frac{-14y}{\sqrt{\alpha}}}}{\beta^*-14y}+\frac{M_{16}e^{\frac{-16y}{\sqrt{\alpha}}}}{\beta^*-16y} \\ +\frac{M_{17}e^{\frac{-18y}{\sqrt{\alpha}}}}{\beta^*-18y}+\frac{M_{18}e^{\frac{-20y}{\sqrt{\alpha}}}}{\beta^*-20y} \end{array}\right) \end{array}\right\} \tag{6.73}$$

and for $\widetilde{m}=3$

$$\theta = \left.\begin{array}{c} \left(1-\sqrt{\alpha}E_cP_rW_0\widetilde{M}_3\right)e^{-P_rW_0y} \\ +\alpha E_cP_r\left(\begin{array}{c} \frac{M_{19}e^{\frac{-2y}{\sqrt{\alpha}}}}{\beta^*-2y}+\frac{M_{20}e^{\frac{-5y}{\sqrt{\alpha}}}}{\beta^*-5y}+\frac{M_{21}e^{\frac{-8y}{\sqrt{\alpha}}}}{\beta^*-8y}+\frac{M_{22}e^{\frac{-11y}{\sqrt{\alpha}}}}{\beta^*-11y} \\ +\frac{M_{23}e^{\frac{-14y}{\sqrt{\alpha}}}}{\beta^*-14y}+\frac{M_{24}e^{\frac{-17y}{\sqrt{\alpha}}}}{\beta^*-17y}+\frac{M_{25}e^{\frac{-20y}{\sqrt{\alpha}}}}{\beta^*-20y}+\frac{M_{26}e^{\frac{-23y}{\sqrt{\alpha}}}}{\beta^*-23y} \\ +\frac{M_{27}e^{\frac{-26y}{\sqrt{\alpha}}}}{\beta^*-26y}+\frac{M_{28}e^{\frac{-29y}{\sqrt{\alpha}}}}{\beta^*-29y}+\frac{M_{29}e^{\frac{-32y}{\sqrt{\alpha}}}}{\beta^*-32y}+\frac{M_{30}e^{\frac{-35y}{\sqrt{\alpha}}}}{\beta^*-35y} \end{array}\right) \end{array}\right\}. \tag{6.74}$$

6.3.2 Case II: Insulated wall

For $\widetilde{m}=1$

$$\theta = \left.\begin{array}{c} \frac{\widetilde{M}_4\widetilde{E}_c}{P_r^2W_0^3}\left(1-P_rV_0\right)e^{-P_rW_0y} \\ +\alpha\widetilde{E}_cP_r\left(\begin{array}{c} \frac{M_1e^{\frac{-2y}{\sqrt{\alpha}}}}{\beta^*-2y}+\frac{M_2e^{\frac{-3y}{\sqrt{\alpha}}}}{\beta^*-3y}+\frac{M_3e^{\frac{-4y}{\sqrt{\alpha}}}}{\beta^*-4y}+\frac{M_4e^{\frac{-5y}{\sqrt{\alpha}}}}{\beta^*-5y} \\ +\frac{M_5e^{\frac{-6y}{\sqrt{\alpha}}}}{\beta^*-6y}+\frac{M_6e^{\frac{-7y}{\sqrt{\alpha}}}}{\beta^*-7y}+\frac{M_7e^{\frac{-8y}{\sqrt{\alpha}}}}{\beta^*-8y}+\frac{M_8e^{\frac{-9y}{\sqrt{\alpha}}}}{\beta^*-9y} \end{array}\right) \end{array}\right\}. \tag{6.75}$$

For $\widetilde{m}=2$

$$\theta = \left.\begin{array}{c} \frac{\widetilde{M}_5\widetilde{E}_c}{P_r^2W_0^3}\left(1-P_rW_0\right)e^{-P_rW_0y} \\ +\alpha\widetilde{E}_cP_r\left(\begin{array}{c} \frac{M_9e^{\frac{-2y}{\sqrt{\alpha}}}}{\beta^*-2y}+\frac{M_{10}e^{\frac{-4y}{\sqrt{\alpha}}}}{\beta^*-4y}+\frac{M_{11}e^{\frac{-6y}{\sqrt{\alpha}}}}{\beta^*-6y}+\frac{M_{12}e^{\frac{-8y}{\sqrt{\alpha}}}}{\beta^*-8y} \\ +\frac{M_{13}e^{\frac{-10y}{\sqrt{\alpha}}}}{\beta^*-10y}+\frac{M_{14}e^{\frac{-12y}{\sqrt{\alpha}}}}{\beta^*-12y}+\frac{M_{15}e^{\frac{-14y}{\sqrt{\alpha}}}}{\beta^*-14y}+\frac{M_{16}e^{\frac{-16y}{\sqrt{\alpha}}}}{\beta^*-16y} \\ +\frac{M_{17}e^{\frac{-18y}{\sqrt{\alpha}}}}{\beta^*-18y}+\frac{M_{18}e^{\frac{-20y}{\sqrt{\alpha}}}}{\beta^*-20y} \end{array}\right) \end{array}\right\} \tag{6.76}$$

and for $\widetilde{m} = 3$

$$\theta = \left. \begin{array}{c} \frac{\widetilde{M}_6 \widetilde{E}_c}{P_r^2 W_0^3} \left(1 - P_r W_0\right) e^{-P_r W_0 y} \\ +\alpha \widetilde{E}_c P_r \left(\begin{array}{c} \frac{M_{19} e^{\frac{-2y}{\sqrt{\alpha}}}}{\beta^* - 2y} + \frac{M_{20} e^{\frac{-5y}{\sqrt{\alpha}}}}{\beta^* - 5y} + \frac{M_{21} e^{\frac{-8y}{\sqrt{\alpha}}}}{\beta^* - 8y} + \frac{M_{22} e^{\frac{-11y}{\sqrt{\alpha}}}}{\beta^* - 11y} \\ + \frac{M_{23} e^{\frac{-14y}{\sqrt{\alpha}}}}{\beta^* - 14y} + \frac{M_{24} e^{\frac{-17y}{\sqrt{\alpha}}}}{\beta^* - 17y} + \frac{M_{25} e^{\frac{-20y}{\sqrt{\alpha}}}}{\beta^* - 20y} + \frac{M_{26} e^{\frac{-23y}{\sqrt{\alpha}}}}{\beta^* - 23y} \\ + \frac{M_{27} e^{\frac{-26y}{\sqrt{\alpha}}}}{\beta^* - 26y} + \frac{M_{28} e^{\frac{-29y}{\sqrt{\alpha}}}}{\beta^* - 29y} + \frac{M_{29} e^{\frac{-32y}{\sqrt{\alpha}}}}{\beta^* - 32y} + \frac{M_{30} e^{\frac{-35y}{\sqrt{\alpha}}}}{\beta^* - 35y} \end{array} \right) \end{array} \right\} \tag{6.77}$$

where

$$\beta^* = P_r W_0 \sqrt{\alpha}, \tag{6.78}$$

$$\widetilde{M}_1 = \left(M_1 + M_2 + M_3 + M_4 + M_5 + M_6 + M_7 + M_8\right),$$

$$\widetilde{M}_2 = \left(M_9 + M_{10} + M_{11} + M_{12} + M_{13} + M_{14} + M_{15} + M_{16} + M_{17} + M_{18}\right),$$

$$\widetilde{M}_3 = \left(\begin{array}{c} M_{19} + M_{20} + M_{21} + M_{22} + M_{23} + M_{24} + M_{25} \\ +M_{26} + M_{27} + M_{28} + M_{29} + M_{30} \end{array} \right),$$

$$\widetilde{M}_4 = \left(2M_1 + 3M_2 + 4M_3 + 5M_4 + 6M_5 + 7M_6 + 8M_7 + 9M\right),$$

$$\widetilde{M}_5 = \left(\begin{array}{c} 2M_9 + 4M_{10} + 6M_{11} + 8M_{12} + 10M_{13} + \\ 12M_{14} + 14M_{15} + 16M_{16} + 18M_{17} + 20M_{18} \end{array} \right),$$

$$\widetilde{M}_6 = \left(\begin{array}{c} 2M_{19} + 5M_{20} + 8M_{21} + 11M_{22} + 14M_{23} + 17M_{24} + 20M \\ +23M_{26} + 26M_{27} + 29M_{28} + 32M_{29} + 35M_{30} \end{array} \right),$$

$$M_1 = -\frac{1}{5832 W_0^6 \alpha^{\frac{15}{2}}} \left(\begin{array}{c} 5832 W_0^7 \alpha^7 - 7776 \hbar W_0^6 \alpha^6 \mu \Gamma_1 - 3888 \hbar^2 W_0^6 \alpha^6 \mu \Gamma_1 \\ +3240 \hbar^2 W_0^5 \alpha^5 \mu^2 \Gamma_1^2 + 2592 \hbar^3 W_0^5 \alpha^5 \mu^2 \Gamma_1^2 \\ +648 \hbar^4 W_0^5 \alpha^5 \mu^2 \Gamma_1^2 - 432 \hbar^3 W_0^4 \alpha^4 \mu^3 \Gamma_1^3 \\ -216 \hbar^4 W_0^4 \alpha^4 \mu^3 \Gamma_1^3 + 18 \hbar^4 W_0^3 \alpha^3 \mu^4 \Gamma_1^4 \end{array} \right),$$

$$M_2 = -\frac{1}{5832W_0^6\alpha^{\frac{15}{2}}}\left(\begin{array}{c} 5832W_0^6\alpha^6\mu\Gamma_1 + 23328\hbar W_0^6\alpha^6\mu\Gamma_1 + 11664\hbar^2 W_0^6\alpha^6\mu\Gamma_1 \\ -11664\hbar W_0^5\alpha^5\mu^2\Gamma_1^2 - 29160\hbar^2 W_0^5\alpha^5\mu^2\Gamma_1^2 - 15552\hbar^3 W_0^5\alpha^5\mu^2\Gamma_1^2 \\ -3888\hbar^4 W_0^5\alpha^5\mu^2\Gamma_1^2 + 8748\hbar^2 W_0^4\alpha^4\mu^3\Gamma_1^3 + 14256\hbar^3 W_0^4\alpha^4\mu^3\Gamma_1^3 \\ +5184\hbar^4 W_0^4\alpha^4\mu^3\Gamma_1^3 - 3024\hbar^3 W_0^3\alpha^3\mu^4\Gamma_1^4 - 3672\hbar^4 W_0^3\alpha^3\mu^4\Gamma_1^4 \\ -1296\hbar^5 W_0^3\alpha^3\mu^4\Gamma_1^4 - 216\hbar^6 W_0^3\alpha^3\mu^4\Gamma_1^4 + 486\hbar^4 W_0^2\alpha^2\mu^5\Gamma_1^5 \\ +432\hbar^5 W_0^2\alpha^2\mu^5\Gamma_1^5 + 108\hbar^6 W_0^2\alpha^2\mu^5\Gamma_1^5 - 36\hbar^5 W_0\alpha\mu^6\Gamma_1^6 \\ -18\hbar^6 W_0\alpha\mu^6\Gamma_1^6 + \hbar^6\mu^7\Gamma_1^7 \end{array}\right),$$

$$M_3 = -\frac{1}{5832W_0^6\alpha^{\frac{15}{2}}}\left(\begin{array}{c} 23328\hbar V^5\alpha^5\mu^2\Gamma_1^2 + 44064\hbar^2 V^5\alpha^5\mu^2\Gamma_1^2 + 20736\hbar^3 V^5\alpha^5\mu^2\Gamma_1^2 \\ +5184\hbar^4 V^5\alpha^5\mu^2\Gamma_1^2 - 38880\hbar^2 V^4\alpha^4\mu^3\Gamma_1^3 - 52704\hbar^3 V^4\alpha^4\mu^3\Gamma_1^3 \\ -18576\hbar^4 V^4\alpha^4\mu^3\Gamma_1^3 + 23328\hbar^3 V^3\alpha^3\mu^4\Gamma_1^4 + 24984\hbar^4 V^3\alpha^3\mu^4\Gamma_1^4 \\ +7776\hbar^5 V^3\alpha^3\mu^4\Gamma_1^4 + 1296\hbar^6 V^3\alpha^3\mu^4\Gamma_1^4 - 6048\hbar^4 V^2\alpha^2\mu^5\Gamma_1^5 \\ -5184\hbar^5 V^2\alpha^2\mu^5\Gamma_1^5 - 1296\hbar^6 V^2\alpha^2\mu^5\Gamma_1^5 + 648\hbar^5 V\alpha\mu^6\Gamma_1^6 \\ +324\hbar^6 V\alpha\mu^6\Gamma_1^6 - 24\hbar^6\mu^7\Gamma_1^7 \end{array}\right),$$

$$M_4 = -\frac{1}{5832V^6\alpha^{\frac{15}{2}}}\left(\begin{array}{c} 39852\hbar^2 W_0^4\alpha^4\mu^3\Gamma_1^3 + 50544\hbar^3 W_0^4\alpha^4\mu^3\Gamma_1^3 + 17496\hbar^4 W_0^4\alpha^4\mu^3\Gamma_1^3 \\ -53136\hbar^3 W_0^3\alpha^3\mu^4\Gamma_1^4 - 53784\hbar^4 W_0^3\alpha^3\mu^4\Gamma_1^4 - 15552\hbar^5 W_0^3\alpha^3\mu^4\Gamma_1^4 \\ -2592\hbar^6 W_0^3\alpha^3\mu^4\Gamma_1^4 + 23868\hbar^4 W_0^2\alpha^2\mu^5\Gamma_1^5 + 19440\hbar^5 W_0^2\alpha^2\mu^5\Gamma_1^5 \\ +4860\hbar^6 W_0^2\alpha^2\mu^5\Gamma_1^5 - 4104\hbar^5 W_0\alpha\mu^6\Gamma_1^6 \\ -2052\hbar^6 W_0\alpha\mu^6\Gamma_1^6 + 219\hbar^6\mu^7\Gamma_1^7 \end{array}\right),$$

$$M_5 = -\frac{1}{5832W_0^6\alpha^{\frac{15}{2}}}\left(\begin{array}{c} 37152\hbar^3 W_0^3\alpha^3\mu^4\Gamma_1^4 + 36774\hbar^4 W_0^3\alpha^3\mu^4\Gamma_1^4 + 10368\hbar^5 W_0^3\alpha^3\mu^4\Gamma_1^4 \\ +1728\hbar^6 W_0^3\alpha^3\mu^4\Gamma_1^4 - 37152\hbar^4 W_0^2\alpha^2\mu^5\Gamma_1^5 - 29376\hbar^5 W_0^2\alpha^2\mu^5\Gamma_1^5 \\ -7344\hbar^6 W_0^2\alpha^2\mu^5\Gamma_1^5 + 11088\hbar^5 W_0\alpha\mu^6\Gamma_1^6 \\ +5544\hbar^6 W_0\alpha\mu^6\Gamma_1^6 - 944\hbar^6\mu^7\Gamma_1^7 \end{array}\right),$$

$$M_6 = -\frac{1}{5832W_0^6\alpha^{\frac{15}{2}}}\begin{pmatrix} 19926\hbar^4W_0^2\alpha^2\mu^5\Gamma_1^5 + 15552\hbar^5W_0^2\alpha^2\mu^5\Gamma_1^5 + 3888\hbar^6W_0^2\alpha^2\mu^5\Gamma_1^5 \\ -13284\hbar^5W_0\alpha\mu^6\Gamma_1^6 - 6642\hbar^6W_0\alpha\mu^6\Gamma_1^6 + 1971\hbar^6\mu^7\Gamma_1^7 \end{pmatrix},$$

$$M_7 = \left(-\frac{5832\hbar^5W_0\alpha\mu^6\Gamma_1^6 + 2916\hbar^6W_0\alpha\mu^6\Gamma_1^6 - 1944\hbar^6\mu^7\Gamma_1^7}{5832W_0^6\alpha^{\frac{15}{2}}}\right),$$

$$M_8 = \left(-\frac{\hbar^6\mu^7\Gamma_1^7}{8W_0^6\alpha^{\frac{15}{2}}}\right),$$

$$M_9 = \frac{1}{64W_0^6\alpha^{\frac{7}{2}}}\begin{pmatrix} -64W_0^3\alpha^3 + 32\hbar W_0^2\alpha^{\frac{3}{2}}\mu\Gamma_2 + 16\hbar^2W_0^2\alpha^{\frac{3}{2}}\mu\Gamma_2 \\ -4\hbar^2W_0\mu^2\Gamma_2^2 - 4\hbar^3W_0\mu^2\Gamma_2^2 - \hbar^4W_0\mu^2\Gamma_2^2 \end{pmatrix},$$

$$M_{10} = \frac{1}{64W_0^6\alpha^{\frac{7}{2}}}\begin{pmatrix} -64W_0^2\alpha^{\frac{3}{2}}\mu\Gamma_2 - 192\hbar W_0^2\alpha^{\frac{3}{2}}\mu\Gamma_2 - 96\hbar^2W_0^2\alpha^{\frac{3}{2}}\mu\Gamma_2 \\ +36\hbar^2W_0\mu^2\Gamma_2^2 + 64\hbar W_0\mu^2\Gamma_2^2 + 80\hbar W_0\mu^2\Gamma_2^2 + 80\hbar^2W_0\mu^2\Gamma_2^2 \\ +48\hbar^3W_0\mu^2\Gamma_2^2 + 12\hbar^4W_0\mu^2\Gamma_2^2 - \frac{9\hbar^2\mu^2\Gamma_2^2}{\alpha^{\frac{3}{2}}} - \frac{9\hbar^2\mu^2\Gamma_2^2}{2\alpha^{\frac{3}{2}}} \\ -\frac{24\hbar^2\mu^3\Gamma_2^3}{\alpha^{\frac{3}{2}}} - \frac{24\hbar^3\mu^3\Gamma_2^3}{\alpha^{\frac{3}{2}}} - \frac{6\hbar^4\mu^3\Gamma_2^3}{\alpha^{\frac{3}{2}}} + \frac{4\hbar^3\mu^4\Gamma_2^4}{W_0\alpha^3} + \frac{6\hbar^4\mu^4\Gamma_2^4}{W_0\alpha^3} + \frac{3\hbar^5\mu^4\Gamma_2^4}{W_0\alpha^3} \\ +\frac{\hbar^6\mu^4\Gamma_2^4}{2W_0\alpha^3} - \frac{\hbar^4\mu^5\Gamma_2^5}{4W_0^2\alpha^{\frac{9}{2}}} - \frac{\hbar^5\mu^5\Gamma_2^5}{2W_0^2\alpha^{\frac{9}{2}}} - \frac{3\hbar^6\mu^5\Gamma_2^5}{8W_0^2\alpha^{\frac{9}{2}}} - \frac{\hbar^7\mu^5\Gamma_2^5}{8W_0^2\alpha^{\frac{9}{2}}} - \frac{\hbar^8\mu^5\Gamma_2^5}{64W_0^2\alpha^{\frac{9}{2}}} \end{pmatrix},$$

$$M_{11} = \frac{1}{64W_0^6\alpha^{\frac{7}{2}}}\begin{pmatrix} -90\hbar^2W_0\mu^2\Gamma_2 - 192\hbar W_0\mu^2\Gamma_2^2 - 204\hbar^2W_0\mu^2\Gamma_2^2 \\ -108\hbar^3W_0\mu^2\Gamma_2^2 - 27\hbar^4W_0\mu^2\Gamma_2^2 + \frac{36\hbar^2\mu^3\Gamma_2^2}{\alpha^{\frac{3}{2}}} + \frac{63\hbar^3\mu^3\Gamma_2^2}{\alpha^{\frac{3}{2}}} \\ +\frac{63\hbar^4\mu^3\Gamma_2^2}{2\alpha^{\frac{3}{2}}} - \frac{243\hbar^4\mu^4\Gamma_2^4}{64V\alpha^3} + \frac{144\hbar^2\mu^3\Gamma_2^2}{\alpha^{\frac{3}{2}}} + \frac{144\hbar^3\mu^3\Gamma_2^2}{\alpha^{\frac{3}{2}}} + \frac{36\hbar^4\mu^3\Gamma_2^2}{\alpha^{\frac{3}{2}}} \\ -\frac{27\hbar^3\mu^4\Gamma_2^3}{W_0\alpha^3} - \frac{27\hbar^4\mu^4\Gamma_2^3}{2W_0\alpha^3} - \frac{36\hbar^3\mu^4\Gamma_2^4}{W_0\alpha^3} - \frac{54\hbar^4\mu^4\Gamma_2^4}{W_0\alpha^3} - \frac{27\hbar^5\mu^4\Gamma_2^4}{W_0\alpha^3} - \frac{9\hbar^6\mu^4\Gamma_2^4}{2W_0\alpha^3} \\ +\frac{27\hbar^4\mu^5\Gamma_2^4}{4W_0^2\alpha^{\frac{9}{2}}} + \frac{27\hbar^5\mu^5\Gamma_2^4}{4W_0^2\alpha^{\frac{9}{2}}} + \frac{27\hbar^6\mu^5\Gamma_2^4}{16W_0^2\alpha^{\frac{9}{2}}} + \frac{3\hbar^4\mu^5\Gamma_2^5}{W_0^2\alpha^{\frac{9}{2}}} + \frac{6\hbar^5\mu^5\Gamma_2^5}{W_0^2\alpha^{\frac{9}{2}}} + \frac{9\hbar^6\mu^5\Gamma_2^5}{2W_0^2\alpha^{\frac{9}{2}}} \\ +\frac{3\hbar^7\mu^5\Gamma_2^5}{2W_0^2\alpha^{\frac{9}{2}}} + \frac{3\hbar^8\mu^5\Gamma_2^5}{16W_0^2\alpha^{\frac{9}{2}}} - \frac{9\hbar^5\mu^6\Gamma_2^5}{16W_0^3\alpha^6} - \frac{27\hbar^6\mu^6\Gamma_2^5}{32W_0^3\alpha^6} - \frac{27\hbar^7\mu^6\Gamma_2^5}{64W_0^3\alpha^6} - \frac{9\hbar^8\mu^6\Gamma_2^5}{128W_0^3\alpha^6} \end{pmatrix},$$

$$M_{12} = \frac{1}{64W_0^6\alpha^{\frac{7}{2}}} \begin{pmatrix} -\frac{60\hbar^2\mu^3\Gamma_2^2}{\alpha^{\frac{3}{2}}} - \frac{90\hbar^3\mu^3\Gamma_2^2}{\alpha^{\frac{3}{2}}} - \frac{45\hbar^4\mu^3\Gamma_2^2}{\alpha^{\frac{3}{2}}} \\ +\frac{135\hbar^4\mu^4\Gamma_2^2}{8V\alpha^3} - \frac{216\hbar^2\mu^3\Gamma_2^3}{\alpha^{\frac{3}{2}}} - \frac{216\hbar^3\mu^3\Gamma_2^3}{\alpha^{\frac{3}{2}}} \\ -\frac{54\hbar^4\mu^3\Gamma_2^3}{\alpha^{\frac{3}{2}}} + \frac{216\hbar^3\mu^4\Gamma_2^3}{W_0\alpha^3} + \frac{63\hbar^4\mu^4\Gamma_2^3}{\alpha^3} \\ -\frac{243\hbar^4\mu^5\Gamma_2^3}{32W_0^2\alpha^{\frac{9}{2}}} + \frac{108\hbar^3\mu^4\Gamma_2^4}{W_0\alpha^3} + \frac{162\hbar^4\mu^4\Gamma_2^4}{W_0\alpha^3} \\ +\frac{81\hbar^5\mu^4\Gamma_2^4}{W_0\alpha^3} + \frac{27\hbar^6\mu^4\Gamma_2^4}{2W_0\alpha^3} - \frac{207\hbar^4\mu^5\Gamma_2^4}{4W_0^2\alpha^{\frac{9}{2}}} \\ -\frac{207\hbar^5\mu^5\Gamma_2^4}{4W_0^2\alpha^{\frac{9}{2}}} - \frac{207\hbar^6\mu^5\Gamma_2^4}{16W_0^2\alpha^{\frac{9}{2}}} + \frac{243\hbar^5\mu^6\Gamma_2^4}{64W_0^3\alpha^6} \\ +\frac{243\hbar^6\mu^6\Gamma_2^4}{218W_0^3\alpha^6} - \frac{27\hbar^4\mu^5\Gamma_2^5}{4W_0^2\alpha^{\frac{9}{2}}} - \frac{27\hbar^5\mu^5\Gamma_2^5}{W_0^2\alpha^{\frac{9}{2}}} \\ -\frac{81\hbar^6\mu^5\Gamma_2^5}{4W_0^2\alpha^{\frac{9}{2}}} - \frac{27\hbar^7\mu^5\Gamma_2^5}{4W_0^2\alpha^{\frac{9}{2}}} - \frac{27\hbar^8\mu^5\Gamma_2^5}{32W_0^2\alpha^{\frac{9}{2}}} \\ +\frac{6\hbar^5\mu^6\Gamma_2^5}{W_0^3\alpha^6} + \frac{9\hbar^6\mu^6\Gamma_2^5}{W_0^3\alpha^6} + \frac{6\hbar^7\mu^6\Gamma_2^5}{2W_0^3\alpha^6} \\ +\frac{3\hbar^8\mu^6\Gamma_2^5}{4W_0^3\alpha^6} - \frac{243\hbar^6\mu^7\Gamma_2^5}{512W_0^4\alpha^{\frac{15}{2}}} - \frac{243\hbar^7\mu^7\Gamma_2^5}{512W_0^4\alpha^{\frac{15}{2}}} \\ -\frac{243\hbar^8\mu^7\Gamma_2^5}{2048W_0^4\alpha^{\frac{15}{2}}} - \frac{729\hbar^7\mu^8\Gamma_2^5}{4096W_0^5\alpha^9} - \frac{729\hbar^8\mu^8\Gamma_2^5}{8192W_0^5\alpha^9} \end{pmatrix},$$

$$M_{13} = \frac{1}{64W_0^6\alpha^{\frac{7}{2}}} \begin{pmatrix} -\frac{1125\hbar^4\mu^4\Gamma_2^2}{64W_0\alpha^3} - \frac{135\hbar^3\mu^4\Gamma_2^3}{W_0\alpha^3} - \frac{135\hbar^4\mu^4\Gamma_2^3}{2W_0\alpha^3} \\ +\frac{405\hbar^4\mu^5\Gamma_2^3}{16W_0^2\alpha^{\frac{9}{2}}} - \frac{108\hbar^3\mu^4\Gamma_2^4}{W_0\alpha^3} - \frac{162\hbar^4\mu^4\Gamma_2^4}{W_0\alpha^3} \\ -\frac{81\hbar^5\mu^4\Gamma_2^4}{W_0\alpha^3} - \frac{27\hbar^6\mu^4\Gamma_2^4}{2W_0\alpha^3} + \frac{513\hbar^4\mu^5\Gamma_2^4}{4W_0^2\alpha^{\frac{9}{2}}} \\ +\frac{513\hbar^5\mu^5\Gamma_2^4}{4W_0^2\alpha^{\frac{9}{2}}} + \frac{513\hbar^6\mu^5\Gamma_2^4}{16W_0^2\alpha^{\frac{9}{2}}} - \frac{1539\hbar^5\mu^6\Gamma_2^4}{64W_0^3\alpha^6} \\ -\frac{1539\hbar^6\mu^6\Gamma_2^4}{128W_0^3\alpha^6} + \frac{729\hbar^6\mu^7\Gamma_2^4}{1024W_0^4\alpha^{\frac{15}{2}}} + \frac{27\hbar^4\mu^5\Gamma_2^5}{W_0^2\alpha^{\frac{9}{2}}} \\ +\frac{54\hbar^5\mu^5\Gamma_2^5}{W_0^2\alpha^{\frac{9}{2}}} + \frac{81\hbar^6\mu^5\Gamma_2^5}{2W_0^2\alpha^{\frac{9}{2}}} + \frac{27\hbar^7\mu^5\Gamma_2^5}{2W_0^2\alpha^{\frac{9}{2}}} \\ \frac{27\hbar^8\mu^5\Gamma_2^5}{16W_0^2\alpha^{\frac{9}{2}}} - \frac{189\hbar^5\mu^6\Gamma_2^5}{8W_0^3\alpha^6} - \frac{567\hbar^6\mu^6\Gamma_2^5}{16W_0^3\alpha^6} \\ -\frac{567\hbar^7\mu^6\Gamma_2^5}{32W_0^3\alpha^6} - \frac{189\hbar^8\mu^6\Gamma_2^5}{64W_0^3\alpha^6} + \frac{567\hbar^6\mu^7\Gamma_2^5}{128W_0^4\alpha^{\frac{15}{2}}} \\ +\frac{567\hbar^7\mu^7\Gamma_2^5}{128W_0^4\alpha^{\frac{15}{2}}} + \frac{567\hbar^8\mu^7\Gamma_2^5}{512W_0^4\alpha^{\frac{15}{2}}} - \end{pmatrix},$$

$$M_{14} = \frac{1}{64W_0^6\alpha^{\frac{7}{2}}}\left(\begin{array}{c} -\frac{675\hbar^4\mu^5\Gamma_2^3}{32W_0^2\alpha^{\frac{9}{2}}} - \frac{405\hbar^4\mu^5\Gamma_2^4}{4W_0^2\alpha^{\frac{9}{2}}} - \frac{405\hbar^5\mu^5\Gamma_2^4}{4W_0^2\alpha^{\frac{9}{2}}} \\ -\frac{405\hbar^6\mu^5\Gamma_2^4}{16W_0^2\alpha^{\frac{9}{2}}} + \frac{3105\hbar^5\mu^6\Gamma_2^4}{64W_0^3\alpha^6} + \frac{3105\hbar^6\mu^6\Gamma_2^4}{128W_0^3\alpha^6} \\ -\frac{3645\hbar^6\mu^7\Gamma_2^4}{1024W_0^4\alpha^{\frac{15}{2}}} - \frac{81\hbar^4\mu^5\Gamma_2^5}{4W_0^2\alpha^{\frac{9}{2}}} - \frac{81\hbar^5\mu^5\Gamma_2^5}{2W_0^2\alpha^{\frac{9}{2}}} \\ -\frac{243\hbar^6\mu^5\Gamma_2^5}{8W_0^2\alpha^{\frac{9}{2}}} - \frac{81\hbar^7\mu^5\Gamma_2^5}{8W_0^2\alpha^{\frac{9}{2}}} - \frac{81\hbar^8\mu^5\Gamma_2^5}{64W_0^2\alpha^{\frac{9}{2}}} \\ +\frac{81\hbar^5\mu^6\Gamma_2^5}{2W_0^3\alpha^6} + \frac{243\hbar^6\mu^6\Gamma_2^5}{4W_0^3\alpha^6} + \frac{243\hbar^7\mu^6\Gamma_2^5}{8W_0^3\alpha^6} \\ +\frac{81\hbar^8\mu^6\Gamma_2^5}{16W_0^3\alpha^6} - \frac{3861\hbar^6\mu^7\Gamma_2^5}{256W_0^4\alpha^{\frac{15}{2}}} - \frac{3861\hbar^7\mu^7\Gamma_2^5}{256W_0^4\alpha^{\frac{15}{2}}} \\ -\frac{3861\hbar^8\mu^7\Gamma_2^5}{1024W_0^4\alpha^{\frac{15}{2}}} + \frac{729\hbar^7\mu^8\Gamma_2^5}{512W_0^5\alpha^9} \\ +\frac{729\hbar^8\mu^8\Gamma_2^5}{1024W_0^5\alpha^9} - \frac{6561\hbar^8\mu^9\Gamma_2^5}{262144W_0^6\alpha^{\frac{21}{2}}} \end{array}\right),$$

$$M_{15} = \frac{1}{64W_0^6\alpha^{\frac{7}{2}}}\left(\begin{array}{c} -\frac{2025\hbar^5\mu^6\Gamma_2^4}{64W_0^3\alpha^6} - \frac{2025\hbar^6\mu^6\Gamma_2^4}{128W_0^3\alpha^6} + \frac{6075\hbar^6\mu^7\Gamma_2^4}{1024W_0^4\alpha^{\frac{15}{2}}} \\ -\frac{405\hbar^5\mu^6\Gamma_2^5}{16W_0^3\alpha^6} - \frac{1215\hbar^6\mu^6\Gamma_2^5}{32W_0^3\alpha^6} - \frac{1215\hbar^7\mu^6\Gamma_2^5}{64W_0^3\alpha^6} \\ -\frac{405\hbar^8\mu^6\Gamma_2^5}{128W_0^3\alpha^6} + \frac{2835\hbar^6\mu^7\Gamma_2^5}{128W_0^4\alpha^{\frac{15}{2}}} + \frac{2835\hbar^7\mu^7\Gamma_2^5}{128W_0^4\alpha^{\frac{15}{2}}} \\ +\frac{2835\hbar^8\mu^7\Gamma_2^5}{512W_0^4\alpha^{\frac{15}{2}}} - \frac{8505\hbar^7\mu^8\Gamma_2^5}{2048W_0^3\alpha^9} - \frac{8505\hbar^8\mu^8\Gamma_2^5}{4096W_0^3\alpha^9} \\ +\frac{10935\hbar^8\mu^9\Gamma_2^5}{65536W_0^6\alpha^{\frac{21}{2}}} \end{array}\right),$$

$$M_{16} = \frac{1}{64W_0^6\alpha^{\frac{7}{2}}}\left(\begin{array}{c} -\frac{3375\hbar^6\mu^7\Gamma_2^4}{1024W_0^4\alpha^{\frac{15}{2}}} - \frac{6075\hbar^6\mu^7\Gamma_2^5}{512W_0^4\alpha^{\frac{15}{2}}} - \frac{6075\hbar^7\mu^7\Gamma_2^5}{512W_0^4\alpha^{\frac{15}{2}}} - \frac{6075\hbar^8\mu^7\Gamma_2^5}{2048W_0^4\alpha^{\frac{15}{2}}} \\ +\frac{675\hbar^7\mu^8\Gamma_2^5}{128W_0^3\alpha^9} + \frac{675\hbar^8\mu^8\Gamma_2^5}{256W_0^3\alpha^9} - \frac{54675\hbar^8\mu^9\Gamma_2^5}{131072W_0^6\alpha^{\frac{21}{2}}} \end{array}\right),$$

$$M_{17} = \frac{1}{64W_0^6\alpha^{\frac{7}{2}}}\left(-\frac{10125\hbar^7\mu^8\Gamma_2^5}{4096W_0^5\alpha^9} - \frac{10125\hbar^8\mu^8\Gamma_2^5}{8192W_0^5\alpha^9} + \frac{30375\hbar^8\mu^9\Gamma_2^5}{65536W_0^6\alpha^{\frac{21}{2}}}\right),$$

$$M_{18} = \frac{1}{64W_0^6\alpha^{\frac{7}{2}}}\left(-\frac{50625\hbar^8\mu^9\Gamma_2^5}{16777216W_0^8\alpha^{14}}\right),$$

$$M_{19} = \left(\begin{array}{c} -\frac{W_0}{\sqrt{\alpha}} + \frac{4\hbar\mu\Gamma_3}{15\alpha^{\frac{5}{2}}} + \frac{4\hbar^2\mu\Gamma_3}{15\alpha^{\frac{5}{2}}} - \frac{2\hbar^2\mu^2\Gamma_3^2}{225W_0\alpha^{\frac{9}{2}}} - \frac{4\hbar^3\mu^2\Gamma_3^2}{225W_0\alpha^{\frac{9}{2}}} \\ -\frac{\hbar^4\mu^2\Gamma_3^2}{225W_0\alpha^{\frac{9}{2}}} - \frac{4\hbar^3\mu^3\Gamma_3^3}{3375W_0^2\alpha^{\frac{13}{2}}} - \frac{2\hbar^4\mu^3\Gamma_3^3}{3375W_0^2\alpha^{\frac{13}{2}}} - \frac{\hbar^4\mu^4\Gamma_3^4}{50625W_0^3\alpha^{\frac{17}{2}}} \end{array}\right),$$

$$M_{20} = \left(\begin{array}{c}
-\frac{\mu\Gamma_3}{\alpha^{\frac{5}{2}}} - \frac{8\hbar\mu\Gamma_3}{3\alpha^{\frac{5}{2}}} - \frac{4\hbar^2\mu\Gamma_3}{3\alpha^{\frac{5}{2}}} + \frac{2\hbar\mu^2\Gamma_3^2}{3W_0\alpha^{\frac{9}{2}}} + \frac{47\hbar^2\mu^2\Gamma_3^2}{45W_0\alpha^{\frac{9}{2}}} \\
+\frac{16\hbar^3\mu^2\Gamma_3^2}{45W_0\alpha^{\frac{9}{2}}} + \frac{4\hbar^4\mu^2\Gamma_3^2}{45W_0\alpha^{\frac{9}{2}}} - \frac{7\hbar^2\mu^3\Gamma_3^3}{45W_0^2\alpha^{\frac{13}{2}}} - \frac{16\hbar^3\mu^3\Gamma_3^3}{75W_0^2\alpha^{\frac{13}{2}}} \\
-\frac{14\hbar^4\mu^3\Gamma_3^3}{225W_0^2\alpha^{\frac{13}{2}}} + \frac{8\hbar^3\mu^4\Gamma_3^4}{75W_0^3\alpha^{\frac{17}{2}}} + \frac{284\hbar^4\mu^4\Gamma_3^4}{10125W_0^3\alpha^{\frac{17}{2}}} + \frac{4\hbar^5\mu^4\Gamma_3^4}{225W_0^3\alpha^{\frac{17}{2}}} \\
+\frac{2\hbar^6\mu^4\Gamma_3^4}{675W_0^3\alpha^{\frac{17}{2}}} + \frac{2\hbar^4\mu^5\Gamma_3^5}{3375W_0^4\alpha^{\frac{21}{2}}} - \frac{8\hbar^5\mu^5\Gamma_3^5}{10125W_0^4\alpha^{\frac{21}{2}}} - \frac{2\hbar^6\mu^5\Gamma_3^5}{1125W_0^4\alpha^{\frac{21}{2}}} \\
-\frac{8\hbar^7\mu^5\Gamma_3^5}{10125W_0^4\alpha^{\frac{21}{2}}} - \frac{\hbar^8\mu^5\Gamma_3^5}{10125W_0^4\alpha^{\frac{21}{2}}} - \frac{68\hbar^5\mu^6\Gamma_3^6}{759375W_0^5\alpha^{\frac{25}{2}}} - \frac{26\hbar^6\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} \\
-\frac{8\hbar^7\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} + \frac{4\hbar^8\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} + \frac{2\hbar^9\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} \\
+\frac{\hbar^{10}\mu^6\Gamma_3^6}{759375W_0^5\alpha^{\frac{25}{2}}} - \frac{2\hbar^6\mu^7\Gamma_3^7}{759375W_0^6\alpha^{\frac{29}{2}}} + \frac{8\hbar^7\mu^7\Gamma_3^7}{2278125W_0^6\alpha^{\frac{29}{2}}} \\
+\frac{2\hbar^8\mu^7\Gamma_3^7}{253125W_0^6\alpha^{\frac{29}{2}}} + \frac{8\hbar^9\mu^7\Gamma_3^7}{2278125W_0^6\alpha^{\frac{29}{2}}} + \frac{\hbar^{10}\mu^7\Gamma_3^7}{2278125W_0^6\alpha^{\frac{29}{2}}} \\
+\frac{\hbar^8\mu^8\Gamma_3^8}{34171975W_0^7\alpha^{\frac{33}{2}}} + \frac{4\hbar^9\mu^9\Gamma_3^8}{11390625W_0^7\alpha^{\frac{33}{2}}} + \frac{2\hbar^{10}\mu^8\Gamma_3^8}{34171875W_0^7\alpha^{\frac{33}{2}}} \\
+\frac{7\hbar^8\mu^9\Gamma_3^9}{512578125W_0^8\alpha^{\frac{37}{2}}} + \frac{8\hbar^9\mu^9\Gamma_3^9}{512578125W_0^8\alpha^{\frac{37}{2}}} + \frac{2\hbar^{10}\mu^9\Gamma_3^9}{512578125W_0^8\alpha^{\frac{37}{2}}} \\
+\frac{2\hbar^9\mu^{10}\Gamma_3^{10}}{7688671875W_0^9\alpha^{\frac{41}{2}}} + \frac{\hbar^{10}\mu^{10}\Gamma_3^{10}}{7688671875W_0^9\alpha^{\frac{41}{2}}} + \frac{\hbar^{10}\mu^{11}\Gamma_3^{11}}{576650390625W_0^{10}\alpha^{\frac{45}{2}}}
\end{array} \right),$$

$$M_{21} = \left(\begin{array}{c}
-\frac{8\hbar\mu^2\Gamma_3^2}{3W_0\alpha^{\frac{9}{2}}} - \frac{836\hbar^2\mu^2\Gamma_3^2}{225W_0\alpha^{\frac{9}{2}}} - \frac{256\hbar^3\mu^2\Gamma_3^2}{225W_0\alpha^{\frac{9}{2}}} - \frac{64\hbar^4\mu^2\Gamma_3^2}{225W_0\alpha^{\frac{9}{2}}} \\
+\frac{16\hbar^2\mu^3\Gamma_3^3}{9W_0^2\alpha^{\frac{13}{2}}} + \frac{2128\hbar^3\mu^3\Gamma_3^3}{1125W_0^2\alpha^{\frac{13}{2}}} + \frac{664\hbar^4\mu^3\Gamma_3^3}{1125W_0^2\alpha^{\frac{13}{2}}} - \frac{32\hbar^3\mu^4\Gamma_3^4}{75W_0^3\alpha^{\frac{17}{2}}} \\
-\frac{8648\hbar^4\mu^4\Gamma_3^4}{168755W_0^3\alpha^{\frac{17}{2}}} - \frac{16\hbar^5\mu^4\Gamma_3^4}{75W_0^3\alpha^{\frac{17}{2}}} - \frac{8\hbar^6\mu^4\Gamma_3^4}{225W_0^3\alpha^{\frac{17}{2}}} + \frac{304\hbar^8\mu^8\Gamma_3^8}{34171875W_0^7\alpha^{\frac{33}{2}}} \\
+\frac{128\hbar^4\mu^5\Gamma_3^5}{3375W_0^4\alpha^{\frac{21}{2}}} + \frac{704\hbar^5\mu^5\Gamma_3^5}{10125W_0^4\alpha^{\frac{21}{2}}} + \frac{16\hbar^6\mu^5\Gamma_3^5}{375W_0^4\alpha^{\frac{21}{2}}} + \frac{128\hbar^7\mu^5\Gamma_3^5}{10125W_0^4\alpha^{\frac{21}{2}}} \\
+\frac{16\hbar^8\mu^5\Gamma_3^5}{10125W_0^4\alpha^{\frac{21}{2}}} + \frac{16\hbar^5\mu^6\Gamma_3^6}{30375W_0^5\alpha^{\frac{25}{2}}} - \frac{344\hbar^6\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} - \frac{416\hbar^7\mu^6\Gamma_3^6}{161875W_0^5\alpha^{\frac{25}{2}}} \\
-\frac{176\hbar^8\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} - \frac{8\hbar^9\mu^6\Gamma_3^6}{30375W_0^5\alpha^{\frac{25}{2}}} - \frac{4\hbar^{10}\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} - \frac{32\hbar^6\mu^7\Gamma_3^7}{151875W_0^6\alpha^{\frac{29}{2}}} \\
-\frac{64\hbar^7\mu^7\Gamma_3^7}{253125W_0^6\alpha^{\frac{29}{2}}} - \frac{16\hbar^8\mu^7\Gamma_3^7}{253125W_0^6\alpha^{\frac{29}{2}}} - \frac{32\hbar^7\mu^8\Gamma_3^8}{253125W_0^7\alpha^{\frac{33}{2}}} \\
+\frac{16\hbar^9\mu^8\Gamma_3^8}{2278125W_0^7\alpha^{\frac{33}{2}}} + \frac{8\hbar^{10}\mu^8\Gamma_3^8}{6834375W_0^7\alpha^{\frac{33}{2}}} + \frac{256\hbar^8\mu^9\Gamma_3^9}{512578125W_0^8\alpha^{\frac{37}{2}}} \\
+\frac{64\hbar^9\mu^9\Gamma_3^9}{102515625W_0^8\alpha^{\frac{37}{2}}} + \frac{\hbar^{10}\mu^9\Gamma_3^9}{102515625W_0^8\alpha^{\frac{37}{2}}} + \frac{8\hbar^9\mu^{10}\Gamma_3^{10}}{512578125W_0^9\alpha^{\frac{41}{2}}} \\
+\frac{4\hbar^{10}\mu^{10}\Gamma_3^{10}}{512578125W_0^9\alpha^{\frac{41}{2}}} + \frac{16\hbar^{10}\mu^{11}\Gamma_3^{11}}{115330078125W_0^{10}\alpha^{\frac{45}{2}}}
\end{array} \right),$$

$$M_{22} = \begin{pmatrix} -\frac{163\hbar^2\mu^3\Gamma_3^3}{45W_0^2\alpha^{\frac{13}{2}}} - \frac{2536\hbar^3\mu^3\Gamma_3^3}{675W_0^2\alpha^{\frac{13}{2}}} - \frac{788\hbar^4\mu^3\Gamma_3^3}{675W_0^2\alpha^{\frac{13}{2}}} + \frac{104\hbar^3\mu^4\Gamma_3^4}{45W_0^3\alpha^{\frac{17}{2}}} \\ + \frac{24452\hbar^4\mu^4\Gamma_3^4}{10125W_0^3\alpha^{\frac{17}{2}}} + \frac{64\hbar\mu^4\Gamma_3^4}{45W_0^3\alpha^{\frac{17}{2}}} + \frac{32\hbar^6\mu^4\Gamma_3^4}{225W_0^3\alpha^{\frac{17}{2}}} - \frac{604\hbar^4\mu^5\Gamma_3^5}{1125W_0^4\alpha^{\frac{21}{2}}} \\ - \frac{88\hbar^5\mu^5\Gamma_3^5}{135W_0^4\alpha^{\frac{21}{2}}} - \frac{118\hbar^6\mu^5\Gamma_3^5}{375W_0^4\alpha^{\frac{21}{2}}} - \frac{256\hbar^7\mu^5\Gamma_3^5}{3375W_0^4\alpha^{\frac{21}{2}}} - \frac{32\hbar^8\mu^5\Gamma_3^5}{3375W_0^4\alpha^{\frac{21}{2}}} \\ + \frac{7448\hbar^5\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} + \frac{92\hbar^6\mu^6\Gamma_3^6}{1215W_0^5\alpha^{\frac{25}{2}}} + \frac{6856\hbar^7\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} \\ + \frac{1996\hbar^8\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} + \frac{64\hbar^9\mu^6\Gamma_3^6}{30375W_0^5\alpha^{\frac{25}{2}}} + \frac{32\hbar^{10}\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} \\ - \frac{166\hbar^6\mu^7\Gamma_3^7}{759375W_0^6\alpha^{\frac{29}{2}}} - \frac{1648\hbar^7\mu^7\Gamma_3^7}{759375W_0^6\alpha^{\frac{29}{2}}} - \frac{484\hbar^8\mu^7\Gamma_3^7}{253125W_0^6\alpha^{\frac{29}{2}}} \\ - \frac{104\hbar^9\mu^7\Gamma_3^7}{151875W_0^6\alpha^{\frac{29}{2}}} - \frac{13\hbar^{10}\mu^7\Gamma_3^7}{151875W_0^6\alpha^{\frac{29}{2}}} - \frac{2056\hbar^7\mu^8\Gamma_3^8}{11390625W_0^7\alpha^{\frac{33}{2}}} \\ - \frac{1828\hbar^8\mu^8\Gamma_3^8}{11390625W_0^7\alpha^{\frac{33}{2}}} - \frac{8\hbar^9\mu^8\Gamma_3^8}{151875W_0^7\alpha^{\frac{33}{2}}} - \frac{4\hbar^{10}\mu^8\Gamma_3^8}{455625W_0^7\alpha^{\frac{33}{2}}} \\ + \frac{964\hbar^8\mu^9\Gamma_3^9}{512578125W_0^8\alpha^{\frac{37}{2}}} + \frac{472\hbar^9\mu^9\Gamma_3^9}{102515625W_0^8\alpha^{\frac{37}{2}}} + \frac{118\hbar^{10}\mu^9\Gamma_3^9}{10251562W_0^8\alpha^{\frac{37}{2}}} \\ + \frac{152\hbar^9\mu^{10}\Gamma_3^{10}}{512578125W_0^8\alpha^{\frac{41}{2}}} + \frac{76\hbar^{10}\mu^{10}\Gamma_3^{10}}{512578125W_0^9\alpha^{\frac{41}{2}}} + \frac{53\hbar^{10}\mu^{11}\Gamma_3^{11}}{12814453125W_0^{10}\alpha^{\frac{45}{2}}} \end{pmatrix},$$

$$M_{23} = \begin{pmatrix} -\frac{2144\hbar^3\mu^4\Gamma_3^4}{675W_0^3\alpha^{\frac{17}{2}}} - \frac{6631\hbar^4\mu^4\Gamma_3^4}{2025W_0^3\alpha^{\frac{17}{2}}} - \frac{256\hbar^5\mu^4\Gamma_3^4}{225W_0^3\alpha^{\frac{17}{2}}} - \frac{128\hbar^6\mu^4\Gamma_3^4}{675W_0^3\alpha^{\frac{17}{2}}} \\ + \frac{256\hbar^4\mu^5\Gamma_3^5}{135W_0^4\alpha^{\frac{21}{2}}} + \frac{21056\hbar^5\mu^5\Gamma_3^5}{10125W_0^4\alpha^{\frac{21}{2}}} + \frac{21056\hbar^5\mu^5\Gamma_3^5}{10125W_0^4\alpha^{\frac{21}{2}}} \\ + \frac{208\hbar^6\mu^5\Gamma_3^5}{225W_0^4\alpha^{\frac{21}{2}}} + \frac{2048\hbar^7\mu^5\Gamma_3^5}{10125W_0^4\alpha^{\frac{21}{2}}} + \frac{256\hbar^8\mu^5\Gamma_3^5}{10125W_0^4\alpha^{\frac{21}{2}}} \\ - \frac{62432\hbar^5\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} - \frac{75376\hbar^6\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} - \frac{3721\hbar^7\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} \\ - \frac{9616\hbar^8\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} - \frac{256\hbar^9\mu^6\Gamma_3^6}{30375W_0^5\alpha^{\frac{25}{2}}} - \frac{128\hbar^{10}\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} \\ + \frac{3008\hbar^6\mu^7\Gamma_3^7}{84375W_0^6\alpha^{\frac{29}{2}}} + \frac{109184\hbar^7\mu^7\Gamma_3^7}{2278125W_0^6\alpha^{\frac{29}{2}}} + \frac{630\hbar^8\mu^7\Gamma_3^7}{253125W_0^6\alpha^{\frac{29}{2}}} \\ + \frac{2944\hbar^9\mu^7\Gamma_3^7}{455625W_0^6\alpha^{\frac{29}{2}}} + \frac{368\hbar^{10}\mu^7\Gamma_3^7}{455625W_0^6\alpha^{\frac{29}{2}}} - \frac{14752\hbar^7\mu^8\Gamma_3^8}{34171875W_0^7\alpha^{\frac{33}{2}}} \\ - \frac{37456\hbar^8\mu^8\Gamma_3^8}{34171875W_0^7\alpha^{\frac{33}{2}}} - \frac{1504\hbar^9\mu^8\Gamma_3^8}{2278125W_0^7\alpha^{\frac{33}{2}}} - \frac{752\hbar^{10}\mu^8\Gamma_3^8}{6834375W_0^7\alpha^{\frac{33}{2}}} \\ - \frac{40064\hbar^8\mu^9\Gamma_3^9}{512578125W_0^8\alpha^{\frac{37}{2}}} - \frac{1216\hbar^9\mu^9\Gamma_3^9}{20503125W_0^8\alpha^{\frac{37}{2}}} - \frac{304\hbar^{10}\mu^9\Gamma_3^9}{20503125W_0^8\alpha^{\frac{37}{2}}} \\ + \frac{544\hbar^9\mu^{10}\Gamma_3^{10}}{512578125W_0^9\alpha^{\frac{41}{2}}} + \frac{272\hbar^{10}\mu^{10}\Gamma_3^{10}}{512578125W_0^9\alpha^{\frac{41}{2}}} + \frac{1984\hbar^{10}\mu^{11}\Gamma_3^{11}}{38443359375W_0^{10}\alpha^{\frac{45}{2}}} \end{pmatrix},$$

$$M_{24} = \begin{pmatrix} -\frac{6662\hbar^4\mu^5\Gamma_3^5}{3375W_0^4\alpha^{\frac{21}{2}}} - \frac{21632\hbar^5\mu^5\Gamma_3^5}{10125W_0^4\alpha^{\frac{21}{2}}} - \frac{35\hbar^6\mu^5\Gamma_3^5}{375W_0^4\alpha^{\frac{21}{2}}} \\ -\frac{2048\hbar^7\mu^5\Gamma_3^5}{10125W_0^4\alpha^{\frac{21}{2}}} - \frac{256\hbar^8\mu^5\Gamma_3^5}{10125W_0^4\alpha^{\frac{21}{2}}} + \frac{32636\hbar^5\mu^6\Gamma_3^6}{30375W_0^5\alpha^{\frac{25}{2}}} \\ +\frac{184502\hbar^6\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} + \frac{85376\hbar^7\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} + \frac{21056\hbar^8\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} \\ +\frac{512\hbar^9\mu^6\Gamma_3^6}{30375W_0^5\alpha^{\frac{25}{2}}} + \frac{256\hbar^{10}\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} - \frac{159158\hbar^6\mu^7\Gamma_3^7}{759375W_0^6\alpha^{\frac{29}{2}}} \\ -\frac{2248\hbar^7\mu^7\Gamma_3^7}{9375W_0^6\alpha^{\frac{29}{2}}} - \frac{27974\hbar^8\mu^7\Gamma_3^7}{253125W_0^6\alpha^{\frac{29}{2}}} - \frac{256\hbar^9\mu^7\Gamma_3^7}{10125W_0^6\alpha^{\frac{29}{2}}} \\ -\frac{32\hbar^{10}\mu^7\Gamma_3^7}{10125W_0^6\alpha^{\frac{29}{2}}} + \frac{553912\hbar^7\mu^8\Gamma_3^8}{34171875W_0^7\alpha^{\frac{33}{2}}} + \frac{621676\hbar^8\mu^8\Gamma_3^8}{34171875W_0^7\alpha^{\frac{33}{2}}} \\ +\frac{17236\hbar^9\mu^8\Gamma_3^8}{2278125W_0^7\alpha^{\frac{33}{2}}} + \frac{8618\hbar^{10}\mu^8\Gamma_3^8}{683475W_0^7\alpha^{\frac{33}{2}}} - \frac{113246\hbar^8\mu^8\Gamma_3^9}{512578125W_0V^8\alpha^{\frac{37}{2}}} \\ -\frac{29576\hbar^9\mu^9\Gamma_3^9}{102515625W_0^8\alpha^{\frac{37}{2}}} - \frac{7394\hbar^{10}\mu^9\Gamma_3^9}{102515625W_0^8\alpha^{\frac{37}{2}}} - \frac{9772\hbar^9\mu^{10}\Gamma_3^{10}}{512578125W_0^9\alpha^{\frac{41}{2}}} \\ -\frac{4886\hbar^{10}\mu^{10}\Gamma_3^{10}}{512578125W_0^9\alpha^{\frac{41}{2}}} + \frac{4742\hbar^{10}\mu^{11}\Gamma_3^{11}}{38443359375W_0^{10}\alpha^{\frac{45}{2}}} \end{pmatrix},$$

$$M_{25} = \begin{pmatrix} -\frac{6851628\hbar^5\mu^6\Gamma_3^6}{759375W_0^5\alpha^{\frac{25}{2}}} - \frac{153304\hbar^6\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} - \frac{70144\hbar^7\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} \\ -\frac{17152\hbar^8\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} - \frac{2048\hbar^9\mu^6\Gamma_3^6}{151875W_0^5\alpha^{\frac{25}{2}}} - \frac{1024\hbar^{10}\mu^6\Gamma_3^6}{759375W_0^5\alpha^{\frac{25}{2}}} \\ +\frac{330592\hbar^6\mu^7\Gamma_3^7}{759375W_0^6\alpha^{\frac{29}{2}}} + \frac{361024\hbar^7\mu^7\Gamma_3^7}{759375W_0^6\alpha^{\frac{29}{2}}} + \frac{53296\hbar^8\mu^7\Gamma_3^7}{253125W_0^6\alpha^{\frac{29}{2}}} \\ +\frac{3481624\hbar^9\mu^7\Gamma_3^7}{759375W_0^6\alpha^{\frac{29}{2}}} + \frac{4352\hbar^{10}\mu^7\Gamma_3^7}{759375W_0^6\alpha^{\frac{29}{2}}} - \frac{832928\hbar^7\mu^8\Gamma_3^8}{11390625W_0^7\alpha^{\frac{33}{2}}} \\ -\frac{845072\hbar^8\mu^8\Gamma_3^8}{11390625W_0^7\alpha^{\frac{33}{2}}} - \frac{107152\hbar^9\mu^8\Gamma_3^8}{3796875W_0^7\alpha^{\frac{33}{2}}} - \frac{53576\hbar^{10}\mu^8\Gamma_3^8}{11390625W_0^7\alpha^{\frac{33}{2}}} \\ +\frac{2387968\hbar^8\mu^8\Gamma_3^9}{512578125W_0^8\alpha^{\frac{37}{2}}} + \frac{2049728\hbar^9\mu^9\Gamma_3^9}{512578125W_0^8\alpha^{\frac{37}{2}}} + \frac{512432\hbar^{10}\mu^9\Gamma_3^9}{512578125W_0^8\alpha^{\frac{37}{2}}} \\ -\frac{133136\hbar^9\mu^{10}\Gamma_3^{10}}{2562890625W_0^9\alpha^{\frac{41}{2}}} - \frac{66568\hbar^{10}\mu^{10}\Gamma_3^{10}}{2562890625W_0^9\alpha^{\frac{41}{2}}} - \frac{136736\hbar\hbar^{10}\mu^{11}\Gamma_3^{11}}{64072265625W_0^{10}\alpha^{\frac{45}{2}}} \end{pmatrix},$$

$$M_{26} = \begin{pmatrix} -\frac{46634\hbar^6\mu^7\Gamma_3^7}{151875W_0^6\alpha^{\frac{29}{2}}} - \frac{151424\hbar^7\mu^7\Gamma_3^7}{455625W_0^6\alpha^{\frac{29}{2}}} - \frac{2464\hbar^8\mu^7\Gamma_3^7}{16875W_0^6\alpha^{\frac{29}{2}}} \\ -\frac{14336\hbar^9\mu^7\Gamma_3^7}{455625W_0^6\alpha^{\frac{29}{2}}} - \frac{1792\hbar^{10}\mu^7\Gamma_3^7}{455625W_0^6\alpha^{\frac{29}{2}}} + \frac{862456\hbar^7\mu^8\Gamma_3^8}{6834375W_0^7\alpha^{\frac{33}{2}}} \\ +\frac{848764\hbar^8\mu^8\Gamma_3^8}{6834375W_0^7\alpha^{\frac{33}{2}}} + \frac{104384\hbar^9\mu^8\Gamma_3^8}{2278125W_0^7\alpha^{\frac{33}{2}}} + \frac{52192\hbar^{10}\mu^8\Gamma_3^8}{6834375W_0^7\alpha^{\frac{33}{2}}} \\ -\frac{1757308\hbar^8\mu^9\Gamma_3^9}{102515625W_0^8\alpha^{\frac{37}{2}}} - \frac{1415288\hbar^9\mu^9\Gamma_3^9}{102515625W_0^8\alpha^{\frac{37}{2}}} - \frac{353822\hbar^{10}\mu^9\Gamma_3^9}{102515625W_0^8\alpha^{\frac{37}{2}}} \\ +\frac{136136\hbar^9\mu^{10}\Gamma_3^{10}}{170859375W_0^9\alpha^{\frac{41}{2}}} + \frac{68068\hbar^{10}\mu^{10}\Gamma_3^{10}}{170859375W_0^9\alpha^{\frac{41}{2}}} - \frac{33194\hbar^{10}\mu^{11}\Gamma_3^{11}}{7688671875W_0^{10}\alpha^{\frac{45}{2}}} \end{pmatrix},$$

$$M_{27} = \begin{pmatrix} -\frac{105056\hbar^7\mu^8\Gamma_3^8}{1366875W_0^7\alpha^{\frac{33}{2}}} - \frac{102704\hbar^8\mu^8\Gamma_3^8}{1366875W_0^7\alpha^{\frac{33}{2}}} - \frac{12544\hbar^9\mu^8\Gamma_3^8}{455625W_0^7\alpha^{\frac{33}{2}}} \\ -\frac{6272\hbar^{10}\mu^8\Gamma_3^8}{1366875W_0^7\alpha^{\frac{33}{2}}} + \frac{520576\hbar^8\mu^9\Gamma_3^9}{20503125W_0^8\alpha^{\frac{37}{2}}} + \frac{410816\hbar^9\mu^9\Gamma_3^9}{20503125W_0^8\alpha^{\frac{37}{2}}} \\ +\frac{102704\hbar^{10}\mu^9\Gamma_3^9}{20503125W_0^8\alpha^{\frac{37}{2}}} - \frac{255584\hbar^9\mu^{10}\Gamma_3^{10}}{102515625W_0^9\alpha^{\frac{41}{2}}} - \frac{127792\hbar^{10}\mu^{10}\Gamma_3^{10}}{102515625W_0^9\alpha^{\frac{41}{2}}} \\ +\frac{97216\hbar^{10}\mu^{11}\Gamma_3^{11}}{1537734375W_0^{10}\alpha^{\frac{45}{2}}} \end{pmatrix},$$

$$M_{28} = \begin{pmatrix} -\frac{55909\hbar^8\mu^9\Gamma_3^9}{4100625W_0^8\alpha^{\frac{37}{2}}} - \frac{43904\hbar^9\mu^9\Gamma_3^9}{4100625W_0^8\alpha^{\frac{37}{2}}} - \frac{10976\hbar^{10}\mu^9\Gamma_3^9}{4100625W_0^8\alpha^{\frac{37}{2}}} \\ +\frac{66542\hbar^9\mu^{10}\Gamma_3^{10}}{20503125W_0^9\alpha^{\frac{41}{2}}} + \frac{33271\hbar^{10}\mu^{10}\Gamma_3^{10}}{20503125W_0^9\alpha^{\frac{41}{2}}} - \frac{18179\hbar^{10}\mu^{11}\Gamma_3^{11}}{102515625W_0^{10}\alpha^{\frac{45}{2}}} \end{pmatrix},$$

$$M_{29} = \left(-\frac{19208\hbar^9\mu^{10}\Gamma_3^{10}}{12301875W_0^9\alpha^{\frac{41}{2}}} - \frac{9604\hbar^{10}\mu^{10}\Gamma_3^{10}}{12301875W_0^9\alpha^{\frac{41}{2}}} + \frac{38416\hbar^{10}\mu^{11}\Gamma_3^{11}}{184528125W_0^{10}\alpha^{\frac{45}{2}}} \right),$$

$$M_{30} = \left(-\frac{16807\hbar^{10}\mu^{11}\Gamma_3^{11}}{184528125W_0^{10}\alpha^{\frac{45}{2}}} \right).$$

6.4 Convergence of the solution expression for velocity

The explicit expression given in Eq. (6.55) contains the auxiliary parameter $\hbar$ which gives the convergence region and rate of approximation for the homotopy analysis method. In Fig.6.1, the $\hbar-$curve is plotted for second order approximation for the non-dimensional velocity field u. It is clear from Fig.6.1 that the admissible values for $\hbar$ is $-0.6 \leq \hbar \leq 0$. Our calculations for $\widetilde{m} = 1$ to 3 indicate that the series of the velocity field given in Eq. (6.55) converges in the whole region when $\hbar = -0.5$.

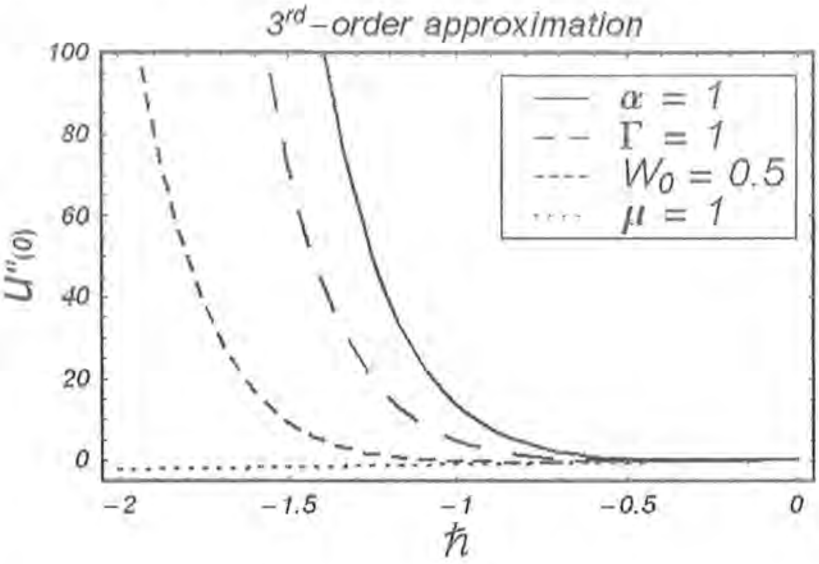

Fig. 6.1 $\hbar-$ curve for the 3rd-order approximation of $u(y)$.

6.5 Discussion of the velocity and temperature profiles

In this section we assign the physical interpretations to velocity and temperature profiles. For that we first discuss the expressions of velocity profiles with the help of graphs. Of particular interest here are the influences of the normal stress coefficient α and the porosity parameter W_0 for three values of $\widetilde{m}$. For velocity profile, Figs.6.2 and 6.3 are plotted when $\widetilde{m} = 1$, Figs.6.4 and 6.5 for $\widetilde{m} = 2$ and Figs.6.6 and 6.7 for $\widetilde{m} = 3$.

In order to examine the effects of α we made Figs. 6.2, 6.4 and 6.6 for $\widetilde{m} = 1$, 2 and 3 respectively. It is noted that large values of α are responsible to increase the velocity boundary layer thickness. Figs.6.3, 6.5 and 6.7 depict the effects of suction and blowing on the velocity. The velocity boundary layer thickness is found to decrease for higher values of suction velocity.

For the case of blowing, it is well known that in the case of Newtonian fluids, there is no solution to the Navier-Stokes equations. However, a solution to the equations of motion in the case of fluid injected into the domain is possible in the case of non-Newtonian fluids. As expected, the blowing causes thickening of the velocity boundary layer and this boundary layer thickness is greater when compared to the case of suction.

It is further seen that the velocities in the three cases of $\widetilde{m}$ have no much difference through the variation of α. However, if we take the variation of W_0 then the velocities are sensitive for the different values of $\widetilde{m}$ which shows the shear-thickening effects of the examined non-Newtonian fluid. Thus, a generalized second grade fluid exhibits the shear-thinning and shear-thickening effects for $\widetilde{m} < 0$ and $\widetilde{m} > 0$, respectively.

Figs.6.8(a) $-$ 6.9(a) illustrate the temperature profiles for $\widetilde{m} = 1$, 2 and 3 respectively in case of constant wall temperature. In Fig.6.8 (a), we observe that an increase in the Eckert number E_c enhances the thermal boundary layer thickness. Also, for large values of Prandtl number, the thermal boundary layer thickness decreases. It can also be seen that increasing the normal stress coefficient has the effect of decreasing the thermal boundary layer thickness. Further, Figs.6.8(b) and 6.9(a) show the similar observation as in the Figs.6.8(a) for E_c, α and P_r.

The temperature profiles corresponding to the case of porous plate subject to blowing can be obtained by changing the sign of W_0. This was done in this analysis and variations of temperature with respect to E_c, α and P_r are found similar to that of suction. Figs.6.9(b) $-$ 6.10(b) depict the temperature distributions for the insulated wall for $\widetilde{m} = 1, 2, 3$ and different values of $\widetilde{E}_c$, α and P_r. Interestingly, it is observed that the behavior for insulated wall is similar to the case of an isothermal wall

(a) Suction ($W_0 = 0.5$) (b) Blowing ($W_0 = -0.5$)

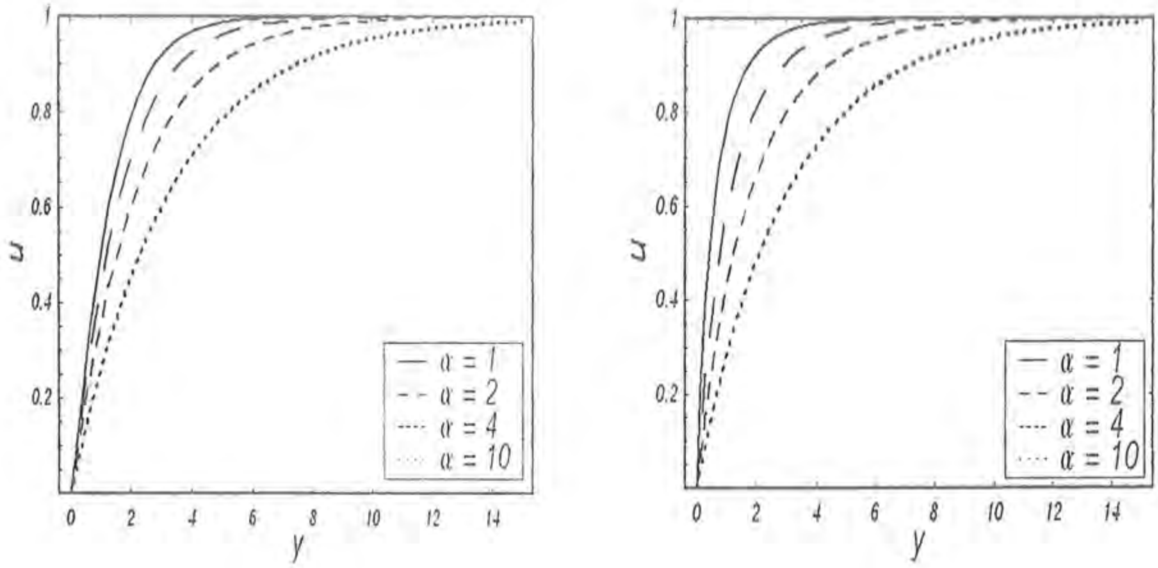

Fig.6.2 Profiles of velocity $u(y)$ for various values of normal stress coefficient α for fixed $\hbar = -0.5$, $\mu = 1$ and $\Gamma_1 = 1$.

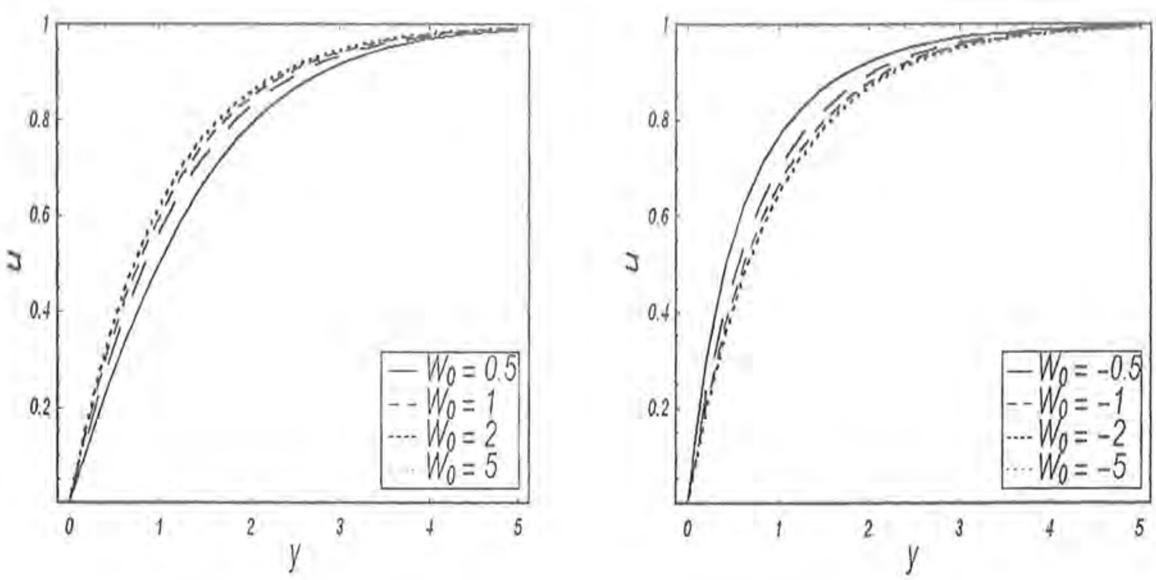

Fig.6.3 Profiles of velocity $u(y)$ for various values of suction (panel a) and blowing (panel b) for fixed $\hbar = -0.5$, $\mu = 1$, $\Gamma_1 = 1$ and $\alpha = 1$.

(a) Suction ($W_0 = 0.5$) (b) Blowing ($W_0 = -0.5$)

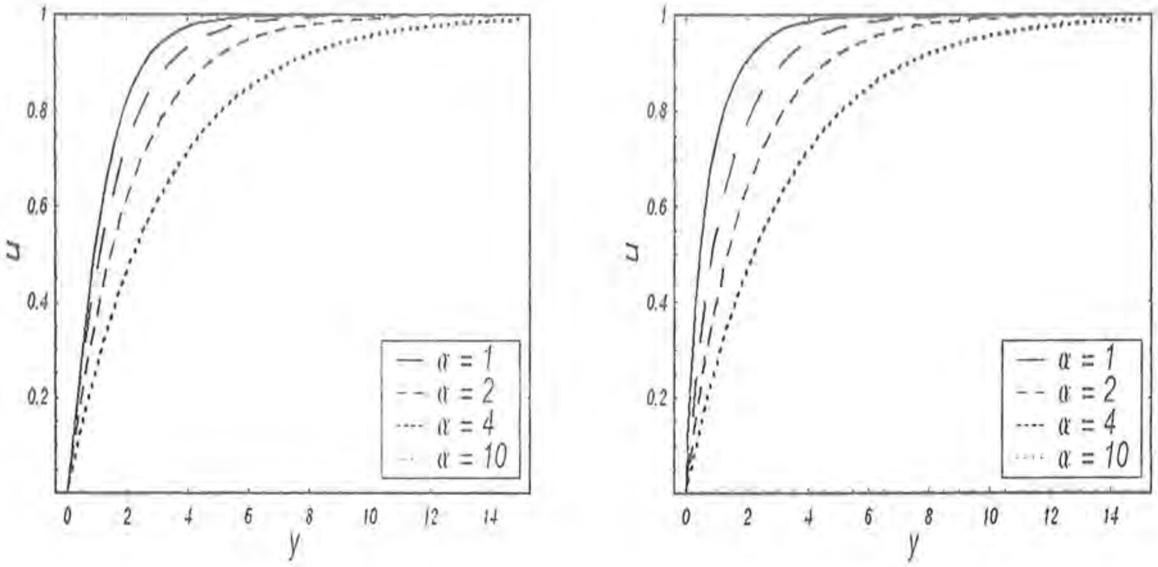

Fig.6.4 Profiles of velocity $u(y)$ for various values of normal stress coefficient α for fixed $\hbar = -0.5$, $\mu = 1$ and $\Gamma_2 = 1.5$.

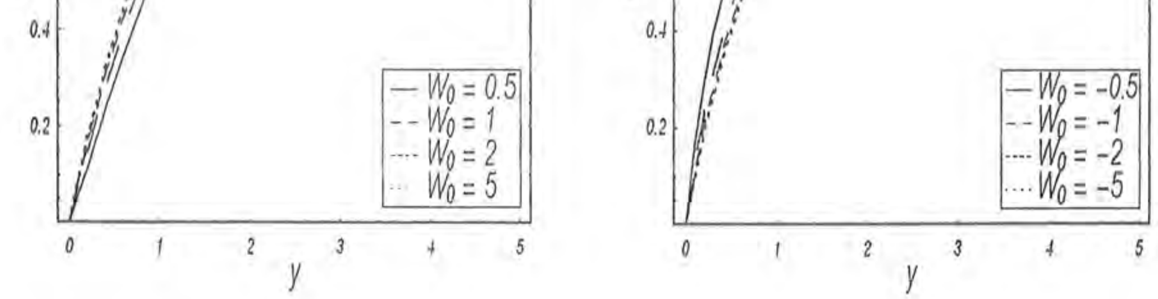

Fig.6.5 Profiles of velocity $u(y)$ for various values of suction (panel a) and blowing (panel b) for fixed $\hbar = -0.5$, $\mu = 1$, $\Gamma_2 = 1.5$ and $\alpha = 1$.

(a) Suction ($W_0 = 0.5$) (b) Blowing ($W_0 = -0.5$)

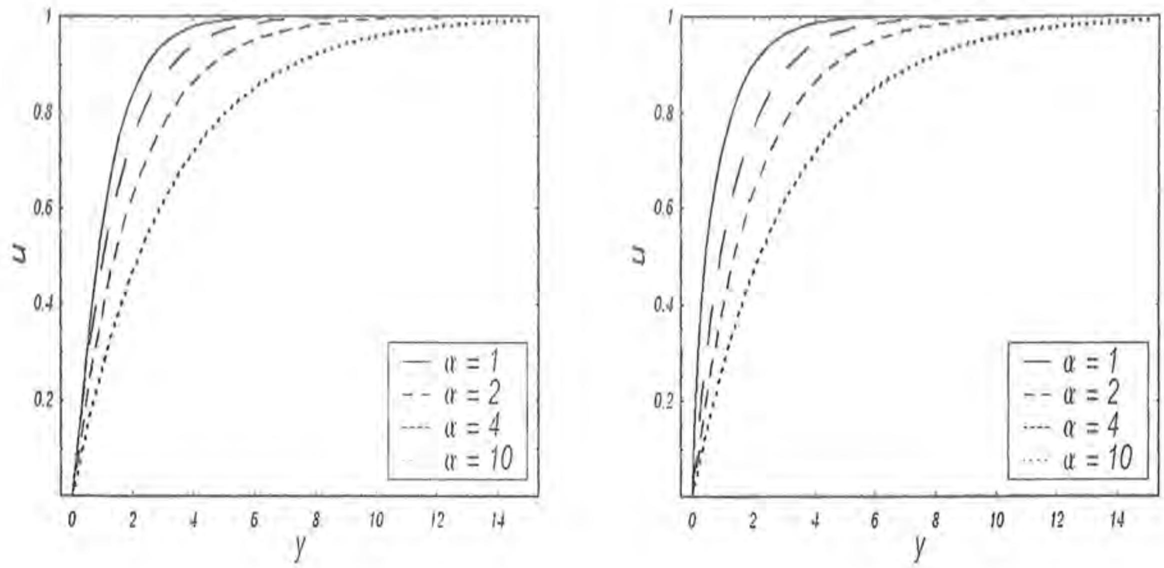

Fig.6.6 Profiles of velocity $u(y)$ for various values of normal stress coefficient α for fixed $\hbar = -0.5$, $\mu = 1$ and $\Gamma_3 = 1.8$.

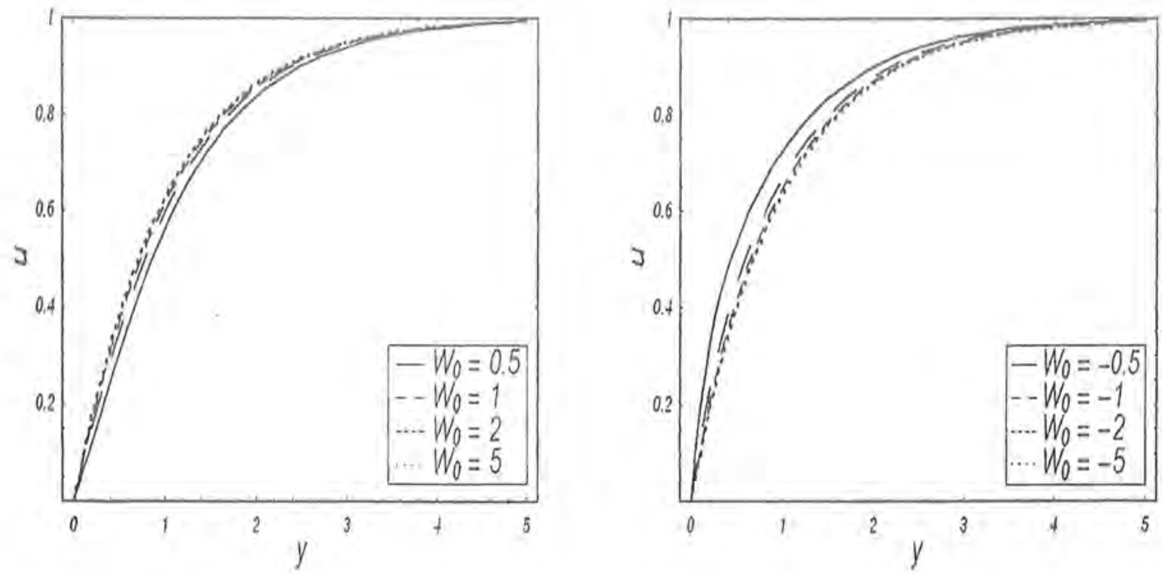

Fig.6.7 Profiles of velocity $u(y)$ for various values of suction (panel a) and blowing (panel b) for fixed $\hbar = -0.5$, $\mu = 1$, $\Gamma_3 = 1.8$ and $\alpha = 1$

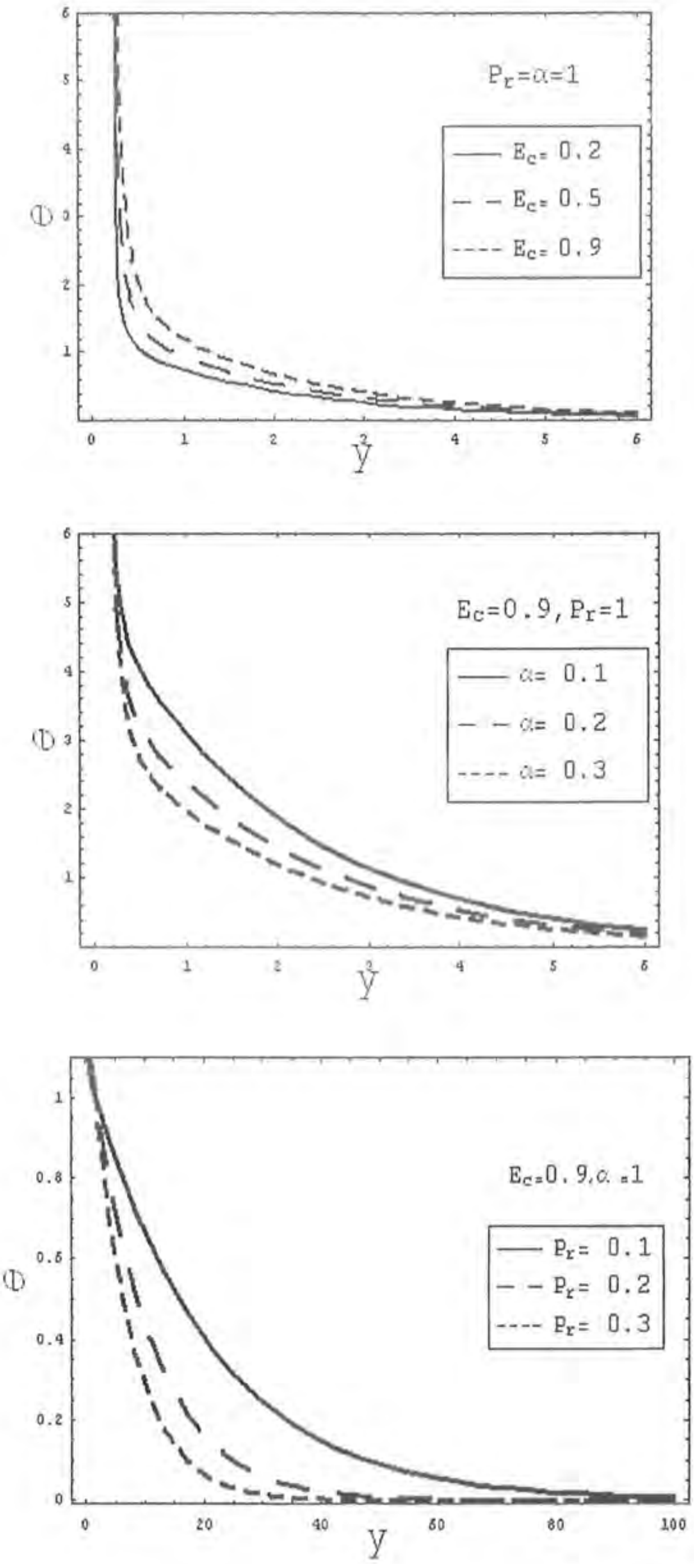

Fig.6.8(a) Profiles of temperature $\theta(y)$ for $\widetilde{m} = 1$, and $\hbar = -0.5$, $\mu = 1$, $\Gamma_1 = 1.5$.

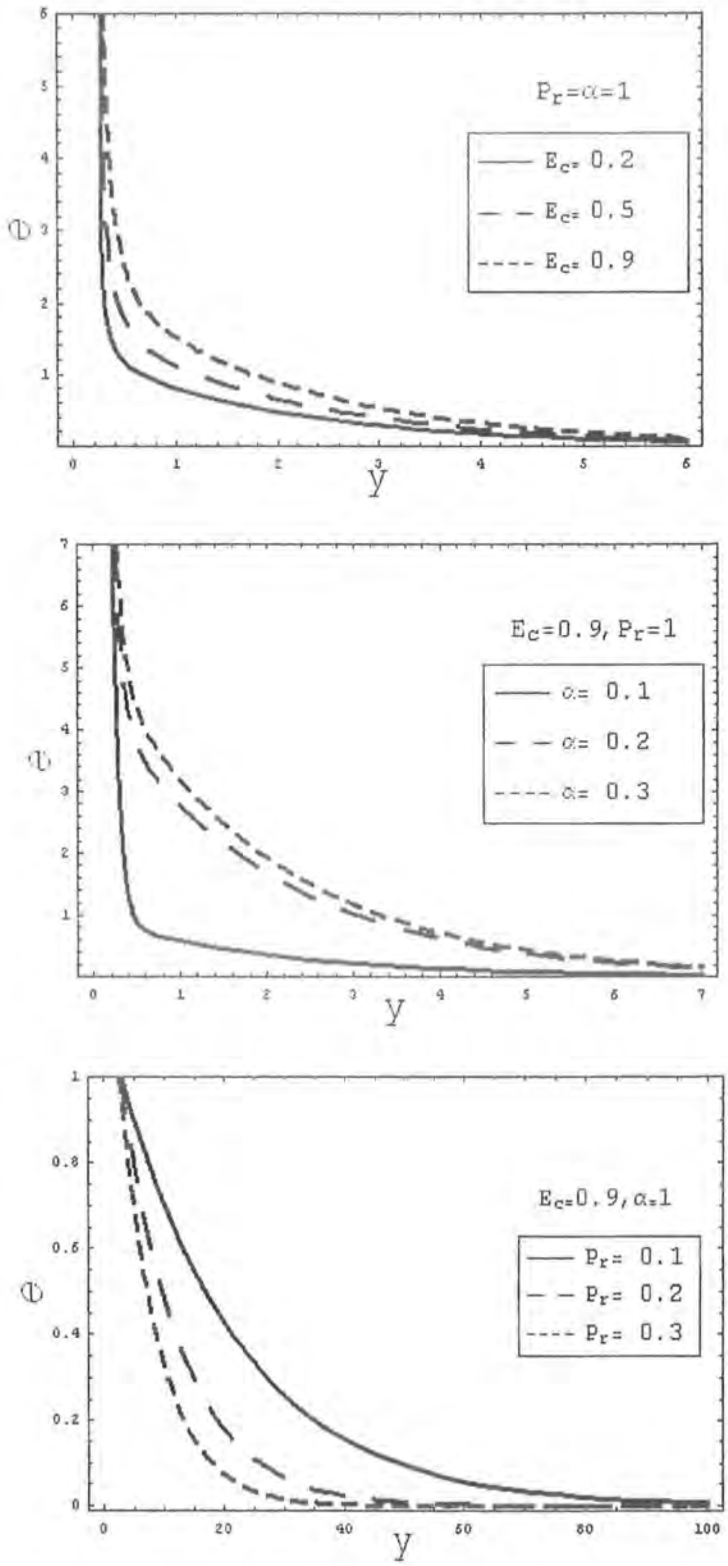

Fig.6.8(b) Profiles of temperature $\theta\,(y)$ for $\widetilde{m}=2$ and $\hbar=-0.5$, $\mu=1$, $\Gamma_2=3$.

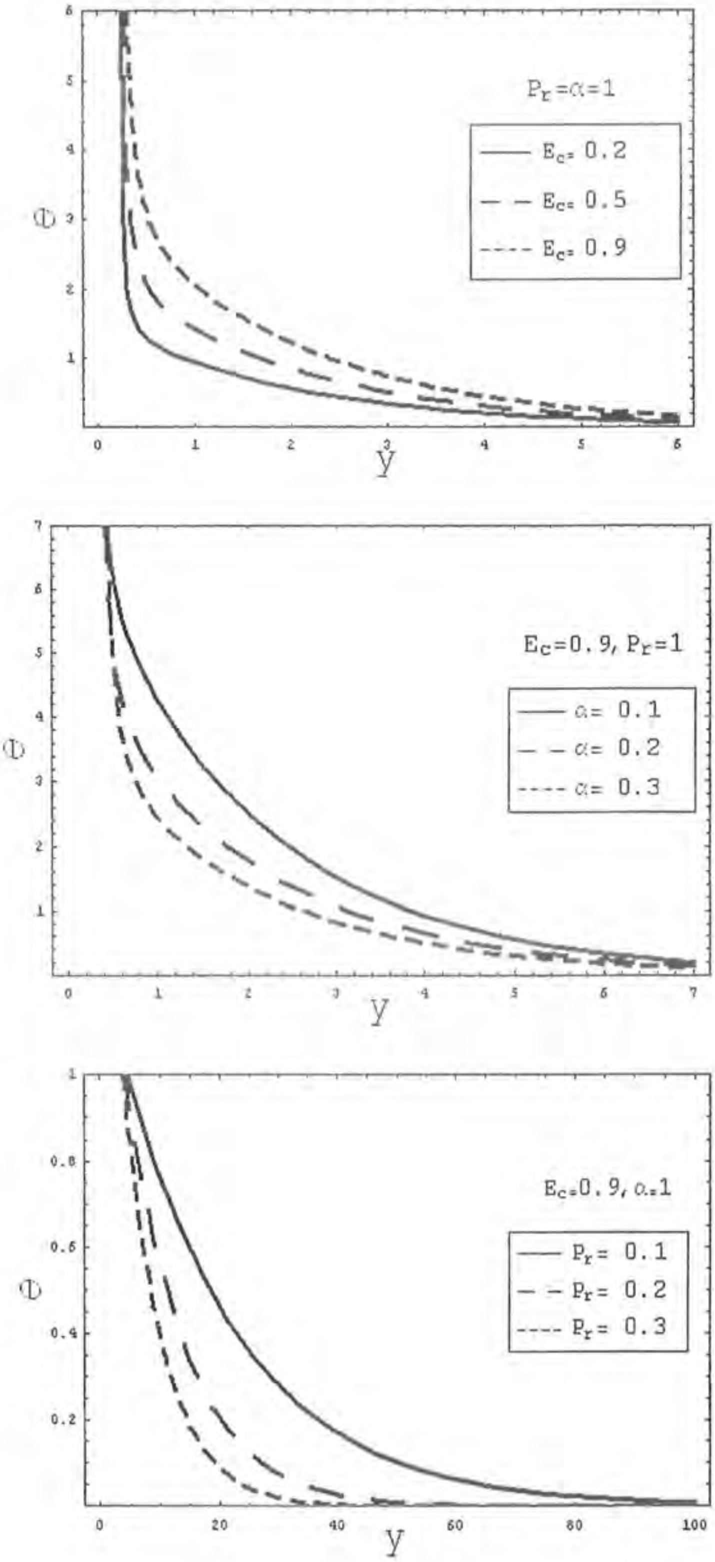

Fig.6.9(a) Profiles of temperature $\theta\left(y\right)$ for $\widetilde{m}=3$ and $\hbar=-0.5,\ \mu=1,\ \Gamma_3=6.$

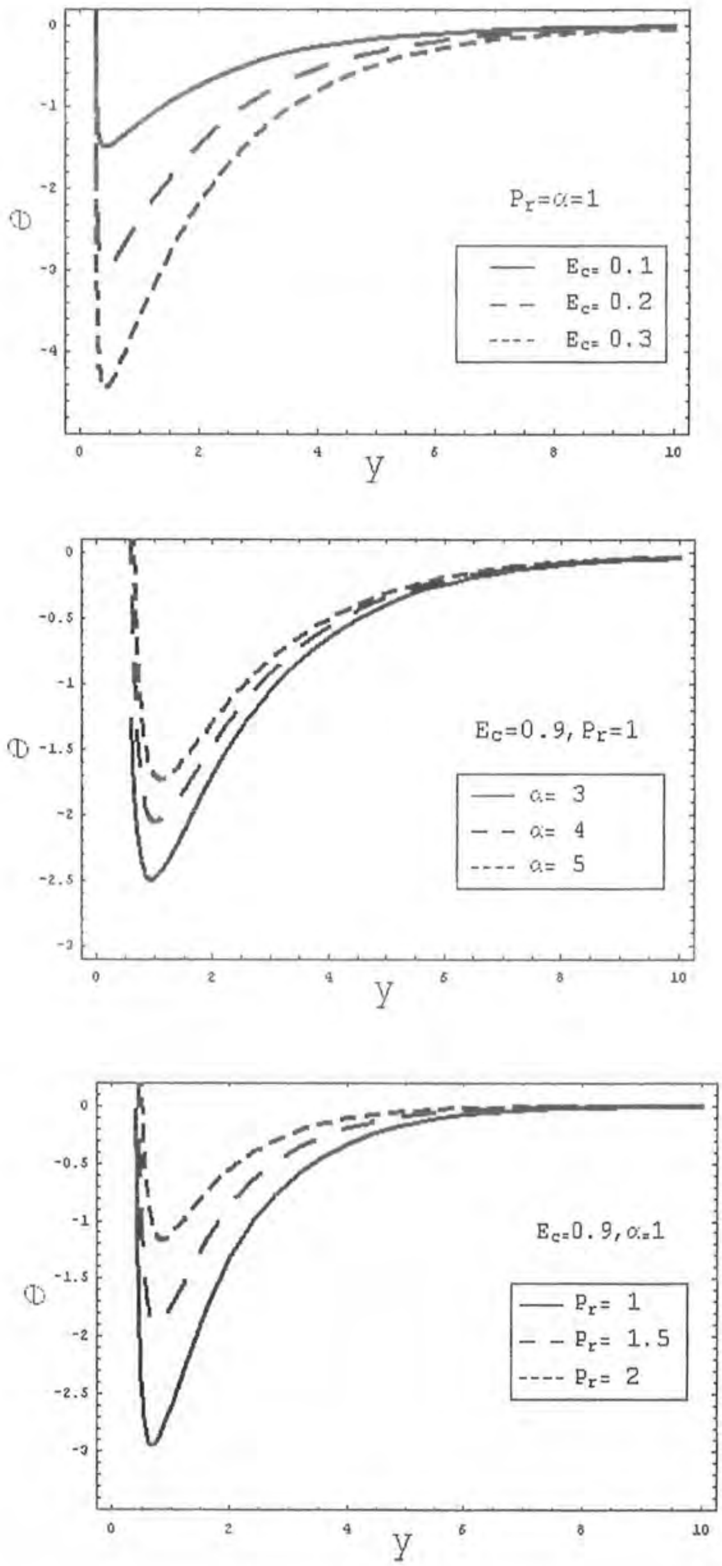

Fig.6.9(b) Profiles of temperature $\theta(y)$ for $\widetilde{m}=1$, and $\hbar=-0.5$, $\mu=1$, $\Gamma_1=1.5$.

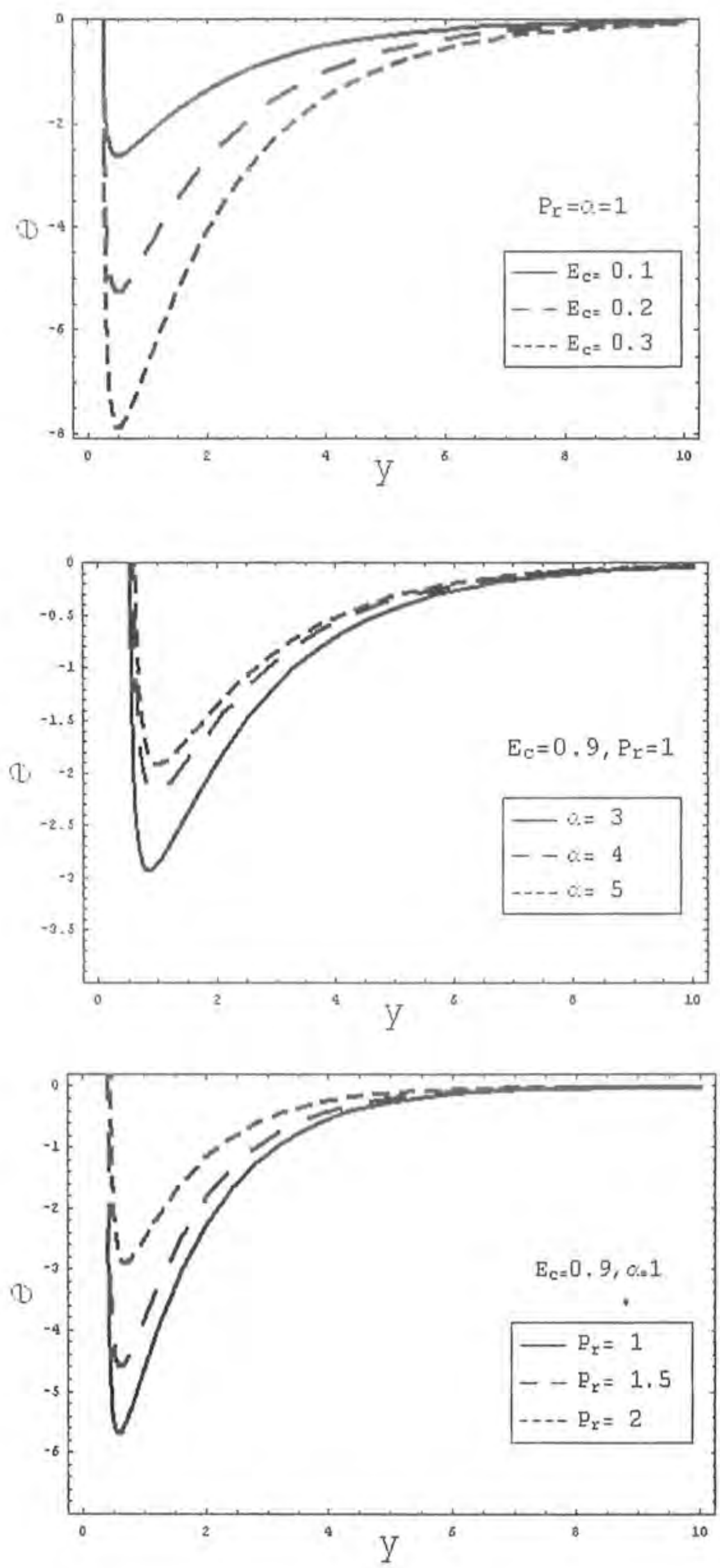

Fig.6.10(a) Profiles of temperature $\theta(y)$ for $\widetilde{m}=2$, and $\hbar=-0.5$, $\mu=1$, $\Gamma_2=3$.

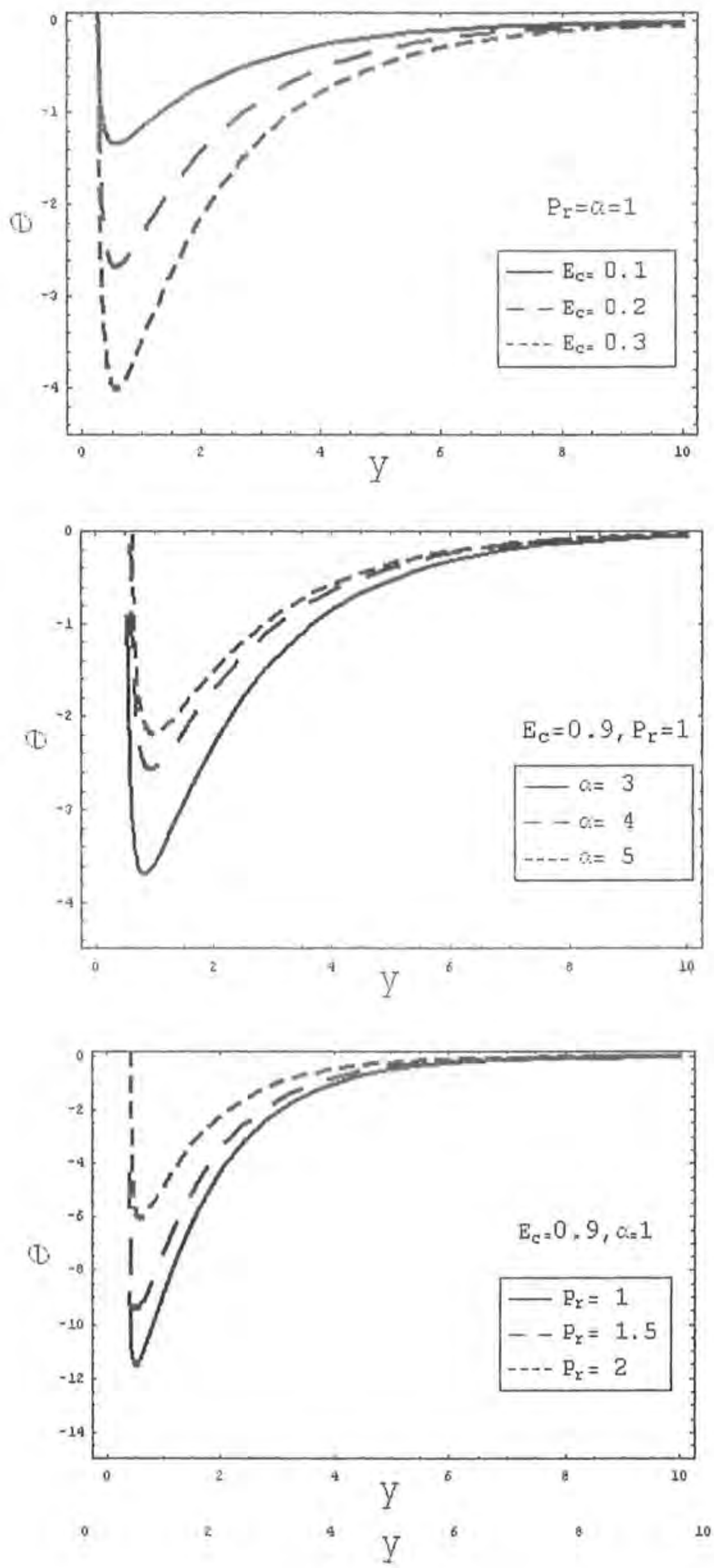

Fig.6.10(b) Profiles of temperature $\theta\,(y)$ for $\widetilde{m} = 3$, and $\hbar = -0.5$, $\mu = 1$, $\Gamma_3 = 6$.

6.6 Concluding remarks

In this work, the velocity and temperature profiles for flow of generalized second grade fluid past a porous plate are obtained analytically. The two interesting cases of constant plate temperature and insulated plate are discussed. A method of homotopy analysis is used to find the velocity profiles. It enables computation of the flow and heat transfer characteristics for any value of the normal stress coefficient of the fluid. The variation of material parameter, normal stress coefficient and suction and blowing on the velocity profiles is illustrated. Further the effect of various parameters such as Eckert number, normal stress coefficient, Prandtl number and the material parameter on the temperature profiles is studied. As a result, the following are determined:

- The considered model of generalized second grade fluids exhibits both shear thinning and shear thickening properties.
- Increase in the normal stress coefficient leads to an increase in the velocity boundary layer thickness.
- Increasing the values of the suction velocity provides decrease in the velocity boundary layer thickness.
- The increase in blowing velocity increases the velocity boundary layer thickness when compared with that of suction velocity.
- The velocity profiles are not much sensitive to variations in α for various values of $\widetilde{m}$.

- The temperature profile changes significantly with Eckert number and thermal boundary layer thickness increases with an increase in Eckert number.
- The temperature profile depends strongly on the normal stress coefficient and Prandtl number. By increasing these parameters, the thermal boundary layer thickness is found to decrease.
- It is worth-mentioning to note that our zeroth order solution (6.29) corresponds to the solutions in the references [97] and [99] when $\mu = 0$. This guarantees the correctness of the presented calculations.

Chapter 7

Channel flow of a third grade fluid using HAM

This chapter presents a study of the steady flow of an incompressible third grade fluid through two parallel porous walls. For the present study, the rate of injection of the fluid at one wall is equal to the rate of suction at the other wall. The flow is induced by applying external pressure gradient. The flow is governed by a third order non-linear boundary value problem. Analytical solution for the velocity is obtained using HAM. A comparison is also made with the existing exact numerical solution [20] of velocity for the various values of the physical parameters. It is found that a proper choice of the auxiliary parameter occurring in HAM solution gives very close results.

7.1 Mathematical formulation

Consider the steady flow of a third grade fluid between two porous walls at $y = a$ and $y = b$. The flow is due to a constant pressure gradient. Also, there is cross flow because of uniform injection of the fluid at lower wall with velocity W_0 and an equal suction at the upper wall. For third grade fluids, we use the constitutive Eq. (1.11) which along with Eqs. (4.1) and (6.6) yields

$$\mu\frac{d^2u}{dy^2} + \alpha_1 W_0\frac{d^3u}{dy^3} - \rho W_0\frac{du}{dy} + 6\beta_3\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} = \frac{dP}{dx}, \tag{7.1}$$

where P is defined in Eq. (6.4)

The boundary conditions are

$$u(a) = u(b) = 0. \tag{7.2}$$

Defining the non-dimensional variables

$$\eta = \frac{y}{b},\ U = -\frac{\mu u}{b^2}\left(\frac{dP}{dx}\right)^{-1} \tag{7.3}$$

and substituting Eq. (7.3) in Eq. (7.1) and boundary conditions (7.2) we obtain

$$KRU''' + U'' - RU' + TU'^2U'' = -1, \tag{7.4}$$

$$U(\sigma) = U(1) = 0, \tag{7.5}$$

in which

$$\sigma = \frac{a}{b},\ R = \frac{\rho W_0 b}{\mu},\ K = \frac{\alpha_1}{\rho b^2},\ T = \frac{6\beta_3 b^2\left(\frac{dP}{dx}\right)^2}{\mu^3} \tag{7.6}$$

and primes denote differentiation with respect to η (incidentally the value of T in [20]–is incorrectly reported).

7.2 Homotopy solution

Now to solve the non-linear ordinary differential Eq. (7.4) subject to boundary conditions (7.5), we apply the homotopy analysis method to give an explicit, uniformly valid and totally analytic solution

7.2.1 Zero-order deformation

In view of Eq. (7.4) and boundary conditions (7.5), we take the initial guess of $U(\eta)$ of the following form

$$U_0 = C_{10} + C_{11} f(\eta) + C_{12} g(\eta) + \frac{\eta}{R} \tag{7.7}$$

which satisfies

$$U_0(\sigma) = U_0(1) = 0, \tag{7.8}$$

and define the auxiliary non-linear operator

$$\mathcal{L}_1 \left[\overline{U}(y;p)\right] = \left(KR\frac{\partial^3}{\partial \eta^3} + \frac{\partial^2}{\partial \eta^2} - R\frac{\partial}{\partial \eta}\right) \overline{U}(y;p). \tag{7.9}$$

In the initial guess (7.7), C_{10}, C_{11} and C_{12} are the constants of integrations

$$f(\eta) = e^{m_3 \eta}, g(\eta) = e^{m_4 \eta}, \tag{7.10}$$

where m_3 and m_4 are the roots of

$$KRm^2 + m - R = 0 \tag{7.11}$$

which are

$$m_3 = \frac{\sqrt{4KR^2+1} - 1}{2KR}, \; m_4 = -\frac{\sqrt{4KR^2+1} + 1}{2KR}. \tag{7.12}$$

For small K (or R) one can write

$$m_3 = R - KR^3 + O(K^2R^5), \tag{7.13}$$

$$m_4 = \frac{1}{KR} - R + +O(KR^3). \tag{7.14}$$

Now the independent solution $f(\eta)$ matches with the solution for a corresponding Newtonian fluid. Thus, from Eqs. (7.7), (7.10), (7.13) and (7.14) we obtain

$$U_0 = C_{10} + C_{11}e^{m_3\eta} + \frac{\eta}{R}, \tag{7.15}$$

where

$$C_{10} = \frac{e^{m_3\sigma} - \sigma e^{m_3}}{R\left(e^{m_3} - e^{m_3\sigma}\right)}, \quad C_{11} = -\frac{1-\sigma}{R\left(e^{m_3} - e^{m_3\sigma}\right)}.$$

This solution is the same as given in reference [105] for a second grade fluid.

We can construct the zeroth-order deformation problem as

$$(1-p)\mathcal{L}_1\left[\overline{U}(\eta;p) - U_0(\eta)\right] = p\hbar\left[\begin{array}{c} KR\frac{\partial^3\overline{U}}{\partial\eta^3} + \frac{\partial^2\overline{U}}{\partial\eta^2} - R\frac{\partial\overline{U}}{\partial\eta} \\ +T\left(\frac{\partial\overline{U}}{\partial\eta}\right)^2\frac{\partial^2\overline{U}}{\partial\eta^2} + 1 \end{array}\right], \tag{7.16}$$

$$\overline{U}(\sigma;p) = 0, \ \overline{U}(1;p) = 0, \tag{7.17}$$

where $p \in [0,1]$ is an embedding parameter and $\hbar$ as an auxiliary parameter. Eqs. (7.16) and (7.17) is

$$\overline{U}(\eta;0) = U_0(\eta) \tag{7.18}$$

and for $p = 1$, the Eqs. (7.16) and (7.17) are equivalent to Eq. (7.4) and (7.5) so that

$$\overline{U}(\eta;1) = U(\eta). \tag{7.19}$$

Clearly when p increases from 0 to 1, $\overline{U}(\eta;p)$ varies (or deforms) from $U_0(\eta)$ to $U(\eta)$ governed by the Eqs. (7.4) and (7.5). We have great freedom to choose $\hbar$. Assume that the deformation $\overline{U}(\eta; p)$ governed by Eqs. (7.16) and (7.17) is smooth enough so that

$$U_0^{[k]}(\eta) = \left. \frac{\partial^k \overline{U}(\eta; p)}{\partial p^k} \right|_{p=0}, \quad k \geq 1. \tag{7.20}$$

By Taylor's theorem and Eq. (7.18), we can write

$$\overline{U}(\eta; p) = U_0(\eta) + \sum_{k=1}^{\infty} U_k(\eta) p^k, \tag{7.21}$$

where

$$U_k(\eta) = \frac{1}{k!} \left. \frac{\partial^k U(\eta; p)}{\partial p^k} \right|_{p=0}. \tag{7.22}$$

Assume that the above series is convergent when $p = 1$, we have, from Eq. (7.21)

$$U(\eta) = U_0(\eta) + \sum_{k=1}^{\infty} U_k(\eta). \tag{7.23}$$

Differentiating Eqs. (7.16) and (7.17) with respect to p and then setting $p = 0$ and using Eqs. (7.18) and (7.20), we have the following first-order deformation problem

$$\mathcal{L}_1[U_1] = \hbar \left[KR \frac{d^3 U_0}{d\eta^3} + \frac{d^2 U_0}{d\eta^2} - R \frac{dU_0}{d\eta} + T \left(\frac{dU_0}{d\eta} \right)^2 \frac{d^2 U_0}{d\eta^2} + 1 \right], \tag{7.24}$$

$$U_1(\sigma) = U_1(1) = 0. \tag{7.25}$$

The solution of the above first-order deformation problem is

$$U_1 = M_{31} + M_{32} e^{m_3 \eta} + M_{33} \eta e^{m_3 \eta} + M_{34} e^{2m_3 \eta} + M_{35} e^{3m_3 \eta}, \tag{7.26}$$

where

$$M_{31} = \frac{e^{(1+\sigma)m_3}}{e^{m_3} - e^{m_3 \sigma}} \left[M_{33}(1-\sigma) + M_{34}\left(e^{m_3} - e^{m_3 \sigma}\right) + M_{35}\left(e^{2m_3} - e^{2m_3 \sigma}\right) \right],$$

$$M_{32} = -\frac{1}{e^{m_3} - e^{m_3 \sigma}} \left[M_{33}\left(e^{m_3} - \sigma e^{m_3 \sigma}\right) + M_{34}\left(e^{2m_3} - e^{2m_3 \sigma}\right) + M_{35}\left(e^{3m_3} - e^{3m_3 \sigma}\right) \right],$$

$$M_{33}=\frac{\hbar TB_0m_3}{R^2\left(2KRm_3+1\right)},\qquad M_{34}=\frac{\hbar TB_0^2m_3}{R\left(3KRm_3+1\right)},\qquad M_{35}=\frac{\hbar TB_0^3m_3^2}{6\left(4KRm_3+1\right)}.$$

Now differentiating Eqs. (7.16) and (7.17) with respect to p twice and setting $p=0$ and using the relations (7.18) and (7.19) we have the second-order deformation problem as

$$\mathcal{L}_1[U_2]=\begin{array}{c}KR\left(1+\hbar\right)\frac{d^3U_1}{d\eta^3}+\left(1+\hbar\right)\frac{d^2U_1}{d\eta^2}-R\left(1+\hbar\right)\frac{dU_1}{d\eta}\\+T\hbar\left\{\left(\frac{dU_0}{d\eta}\right)^2\frac{d^2U_1}{d\eta^2}+2\frac{d^2U_0}{d\eta^2}\frac{dU_0}{d\eta}\frac{dU_1}{d\eta}\right\},\end{array}\tag{7.27}$$

$$U_2\left(\sigma\right)=U_2\left(1\right)=0.\tag{7.28}$$

Solving the above problem one obtains

$$U_2=\left(1+\hbar\right)U_1+\gamma,\tag{7.29}$$

$$\gamma=\left(\begin{array}{c}M_{36}+\left(M_{37}+M_{38}\eta+M_{39}\eta^2\right)e^{m_3\eta}+\left(M_{40}+M_{41}\eta\right)e^{2m_3\eta}\\+\left(M_{42}+M_{43}\eta\right)e^{3m_3\eta}+M_{44}e^{4m_3\eta}+M_{45}e^{5m_3\eta}\end{array}\right),\tag{7.30}$$

$$M_{36}=\frac{e^{(1+\sigma)m_3}}{e^{m_3}-e^{m_3\sigma}}\left(\begin{array}{c}M_{38}\left(1-\sigma\right)+M_{39}\left(1-\sigma^2\right)+M_{40}\left(e^{m_3}-e^{m_3\sigma}\right)\\+M_{41}\left(e^{m_3}-\sigma e^{m_3\sigma}\right)+M_{42}\left(e^{2m_3}-e^{2m_3\sigma}\right)\\+M_{43}\left(e^{2m_3}-e^{2m_3\sigma}\right)+M_{44}\left(e^{3m_3}-e^{3m_3\sigma}\right)+M_{45}\left(e^{4m_3}-e^{4m_3\sigma}\right)\end{array}\right),$$

$$M_{37}=-\frac{1}{e^{m_3}-e^{m_3\sigma}}\left(\begin{array}{c}M_{38}\left(e^{m_3}-\sigma e^{m_3\sigma}\right)+M_{39}\left(e^{m_3}-\sigma^2e^{m_3\sigma}\right)\\+M_{40}\left(e^{2m_3}-e^{2m_3\sigma}\right)+M_{41}\left(e^{2m_3}-\sigma e^{2m_3\sigma}\right)\\+M_{42}\left(e^{3m_3}-e^{3m_3\sigma}\right)+M_{43}\left(e^{3m_3}-\sigma e^{3m_3\sigma}\right)\\+M_{44}\left(e^{4m_3}-e^{4m_3\sigma}\right)+M_{45}\left(e^{5m_3}-e^{5m_3\sigma}\right)\end{array}\right),$$

$$M_{38}=\hbar T\left(\frac{M_{32}m_3}{R^2\left(2KRm_3+1\right)}+\frac{M_{33}\left(KRm_3+1\right)}{R^2\left(2KRm_3+1\right)^2}\right),\ M_{39}=\frac{\hbar TM_{33}m_3}{2R^2\left(2KRm_3+1\right)},$$

$$M_{40}=\hbar T\left(\frac{2m_3C_{20}M_{32}R+2M_{34}}{R^2\left(3KRm_3+1\right)}-\frac{2m_3C_{20}KM_{33}}{\left(3KRm_3+1\right)^2}\right),\ M_{41}=\frac{2m_3C_{20}\hbar TM_{33}}{R\left(3KRm_3+1\right)},$$

$$M_{42} = \hbar T\left(\frac{m_3^2 C_{20}^2 M_{32} R^2 + 4m_3 C_{20} M_{34} R + 3M_{35}}{2R^2\left(4KRm_3+1\right)} + \frac{m_3 C_{20}^2 M_{33}\left(2KRm_3+1\right)}{4\left(4KRm_3+1\right)^2}\right),$$

$$M_{43} = \frac{\hbar T C_{20}^2 M_{33}^2 m_3}{2\left(4KRm_3+1\right)},\quad M_{44} = \frac{2m_3 C_{20}\hbar T\left(m_3 C_{20} M_{34} R + 3M_{35}\right)}{3R\left(5KRm_3+1\right)},$$

$$M_{45} = \frac{3\hbar T C_{20}^2 M_{35}^2 m_3}{4\left(6KRm_3+1\right)}.$$

The three term solution is

$$U = U_0 + U_1 + U_2. \tag{7.31}$$

7.3 Higher order deformation

For higher order deformation equations, we first differentiate Eqs. (7.16) and (7.17) k times with respect to p then dividing by $k!$ and set $p = 0$. Here, the higher order deformation problem becomes

$$\left(1-p\right)\mathcal{L}_1\left[U_k\left(\eta\right) - \chi_k U_{k-1}\left(\eta\right)\right] = \hbar\left[\begin{array}{c} KRU_{k-1}''' + U_{k-1}'' - RU_{k-1}' \\ +T\sum_{j=0}^{k-1} U_{k-1-j}' \sum_{i=0}^{n} U_{j-i}' U_i'' + \chi_k \end{array}\right], \tag{7.32}$$

$$U_k\left(\sigma\right) = U_k\left(0\right) = 0. \tag{7.33}$$

7.4 Convergence of the solution expression for velocity

The explicit, analytic expression (7.31) contains the auxiliary parameter $\hbar$. As pointed out by Liao [71], the convergence region and rate of approximations given by the homotopy analysis method are strongly dependent upon the auxiliary parameter. In

Figs. 7.1 and 7.2 the $\hbar$-curves are plotted to see the range of admissible values for the parameter $\hbar$. It is clear from Figs. 7.1 and 7.2 that the range for the admissible values for $\hbar$ is $-1 \leq \hbar < 0$. And the solution given in Eq. (7.31) converges in the whole region of η, when $\hbar$ is in the neighborhood of -1.0. It is also observed that the interval for the values of $\hbar$ converges to value -1.0 as T increases keeping K and R fixed. It is also clear from Fig. 7.2 that value of $\hbar$ decreases from -1.0 towards 0.0 as K increases.

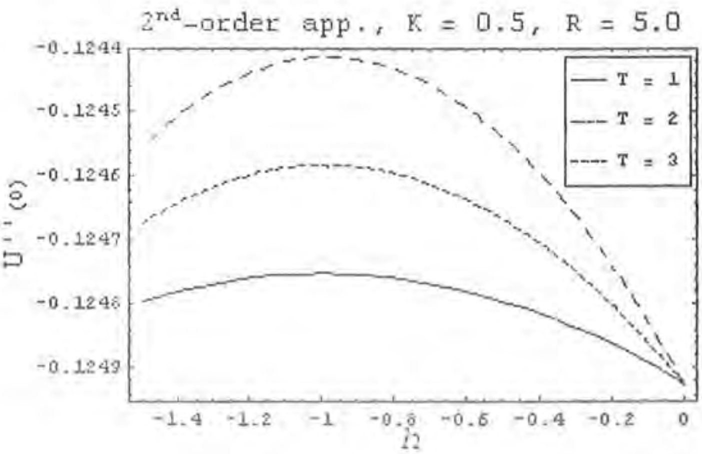

Fig.7.1 $\hbar-$ curve is plotted for the 2nd-order approximation of $U(\eta)$ for increasing T.

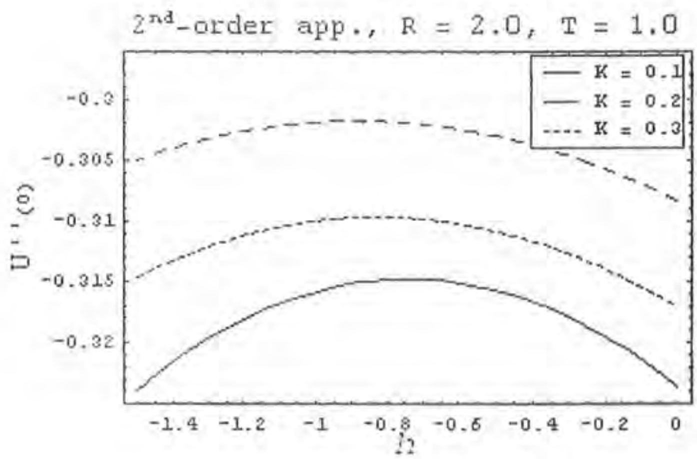

Fig.7. 2 $\hbar-$ curve is plotted for the 2nd-order approximation of $U(\eta)$ for increasing K.

7.5 Results and Discussion

From Table 1, we observe that the optimum value of $\hbar$ depends upon the physical parameters R, K, and T. In particular, we note that if R and K are small then the value of $\hbar$ must be chosen close to -0.83 reduced progressively as the value of T is increased. However, when R and K are moderate to large a value of $\hbar$ close to -0.96 gives the velocity profile very close to the one that was obtained numerically by Ariel [20].

Table 1

Illustrating the variation of $U(1/2)$, the velocity in the middle of the channel, with R, the cross-flow Reynold's number K, the viscoelastic fluid parameter T, the third grade parameter and $\hbar$ the homotopy auxiliary parameter.

R	K	T	$U(1/2)$ Exact numerical solution	$\hbar$	HAM solution
1	0.1	1	0.109334	-0.87	0.109368
		2	0.106976	-0.87	0.106976
		5	0.100651	-0.58	0.101969
	0.2	1	0.102101	-0.85	0.102151
		2	0.099691	-0.73	0.099400
	0.5	1	0.087626	-0.82	0.087673
		2	0.085475	-0.69	0.085471

R	K	T	$U(1/2)$ Exact numerical solution	$\hbar$	HAM solution
2	0.1	0	0.091280		
		1	0.089911	-0.94	0.089912
		2	0.088679	-0.89	0.088692
		5	0.085578	-0.77	0.085708
	0.2	1	0.078179	-0.95	0.078179
		2	0.077282	-0.9	0.077288
		5	0.074993	-0.79	0.075064
	0.5	1	0.060643	-0.94	0.060643
		2	0.060099	-0.93	0.060100
		5	0.058673	-0.83	0.058690
5	0.1	1	0.052089	-0.96	0.052089
		2	0.051974	-0.99	0.051974
		5	0.051638	-0.94	0.051638
		10	0.051102	-0.94	0.051102
		20	0.050117	-0.98	0.050119
	0.2	1	0.041948	-0.99	0.041948
		2	0.041886	-0.96	0.041886
		5	0.041703	-0.94	0.041703
		10	0.041407	-0.97	0.041407
		20	0.040852	-0.93	0.040852

R	K	T	$U(1/2)$ Exact numerical solution	$\hbar$	HAM solution
0.5	1		0.029750	-0.99	0.029750
	2		0.029723	-0.99	0.029723
	5		0.029644	-0.96	0.029644
	10		0.029514	-0.97	0.029514
	20		0.029266	-0.99	0.029266

Here it is worth mentioning that the HAM solution is valid for *all* values of the physical parameters R, K, and T. Therefore, it seems reasonable to assume that the HAM solution holds even for those values of the physical parameters for which Ariel [4] had a problem in obtaining the convergence of the series solution for large values of T. It is believed that as long as the above mentioned guidelines regarding the choice of $\hbar$ are adhered to, the HAM solution *must* provide a reasonably accurate solution of the problem for all values of the parameters. In any case a value of $\hbar$ between -1 and -0.6 would give a result which would not be far off from the exact solution.

In Fig. 7.3, U has been plotted against η for $R = 1$, $T = 5$, and $K = 0.2$ and 0.5. For these values of the parameters, an intelligent guess had to be made for $\hbar$. Looking at Table 1, we expect that for $K = 0.2$, $\hbar$ must be close to -0.5, and for $K = 0.5$, $\hbar$ must be approximately equal to -0.45. It must be remarked here that Ariel [20] was not able to get the convergence of the solution for the above mentioned values of the parameters. Thus HAM offers an attractive alternative for computing the flow of viscoelastic fluids where numerical techniques fail to give the solution for varying reasons.

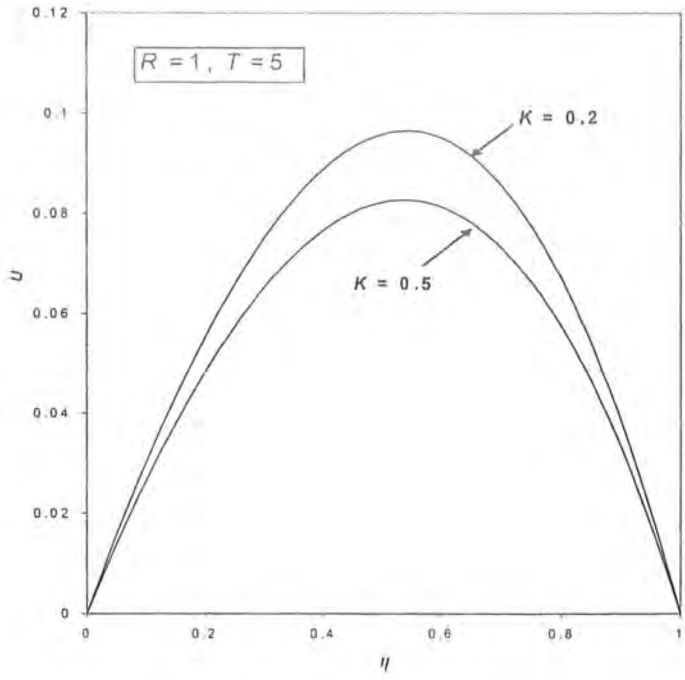

Fig.7.3 Variation of velocity profile for different values of K.

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Unsteady Periodic Flows of a Magnetohydrodynamic Fluid Due to Noncoaxial Rotations of a Porous Disk and a Fluid at Infinity

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Abstract—The unsteady flow of a viscous electrically conducting fluid bounded by a porous disk in the presence of a uniform magnetic field has been studied. Exact analytic solution for the flow generated by arbitrary periodic oscillation of a porous disk is obtained. Some interesting flows caused by certain special oscillations are also examined. Asymptotic analysis is carried out to determine the solutions for the large time. The structure of the velocity boundary layers is discussed physically. In the case of blowing and resonance, the hydromagnetic steady state flow is found to exist.

Keywords—General periodic oscillation, Porous disk, Asymptotic blowing solution, Resonance, Laplace transform.

1. INTRODUCTION

The unsteady flow due to two noncoaxial rotations of a disk and a fluid at infinity has received considerable attention. Rao and Kasiviswanathan [1] presented an exact solution for the unsteady flow in which the eccentric disks execute nontorsional oscillations. Kasiviswanathan and Rao [2] also presented an exact but large time solution of the unsteady Navier-Stokes equations for the flow due to an eccentrically rotating disk oscillating in its own plane and the fluid at infinity. The initial conditions which make the flow two-dimensional are investigated by Erdogan [3,4]. Recently, Erdogan [5] has studied the flow due to noncoaxial rotations of a disk oscillating in its own plane and a fluid rotating at infinity. Rajagopal [6] has considered the flow of a simple fluid in an orthogonal rheometer. Parter and Rajagopal [7] also proved that when the disks rotate with different angular velocities about distinct axes or a common axis, there is a one parameter family of solutions. Further, the flows of Newtonian and non-Newtonian fluids between parallel disks rotating about a common axis has been reviewed by Rajagopal [8].

However, no attempt has been made to discuss the flow for small time due to noncoaxial rotations of a porous oscillating disk and a fluid at infinity. It is apparent from physical considerations that suction prevents the imposed oscillations from spreading far away from the disk

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by viscous diffusion for all values of the frequency parameter. On the contrary, the blowing promotes the spreading of the oscillations far away from the boundary disk, and hence, one of the boundary layers tends to be infinitely thick when the disk is forced to oscillate with the resonant frequency. Thus, oscillatory flows are no longer possible for the case of blowing and resonance. The phenomena of resonance here occurs when the oscillating frequency of the disk is equal to the angular velocity of rotation. Even if suction or blowing is absent, the boundary layers extend throughout the flow field [5]. Thus, it remains to answer the question of finding a meaningful solution for blowing and resonance. An attempt is made to answer this question. The aim of this communication is to discuss the unsteady magnetohydrodynamic flow of an electrically conducting fluid bounded by an infinite nonconducting porous disk with uniform suction or blowing in the presence of a transverse magnetic field. The disk executes arbitrary periodic oscillations. A general periodic oscillation $h(t)$ with period T_0 is considered. The response of oscillations in the flow field can be built up using Fourier series representation. Exact analytical solution (due to arbitrary periodic oscillation) valid for large and small times is obtained. The flow field, due to certain special values of oscillation, is then derived as a special case of the arbitrary periodic oscillations. By using the Laplace transform method, the structure of the steady and the unsteady hydromagnetic boundary layers are examined, including the case of blowing and resonance. It is found that, in the case of blowing and resonance, steady solution satisfies the boundary condition at infinity, which is a deviation from the hydrodynamic situation. Several results of interest are obtained as special cases of the presented analysis.

2. FORMULATION OF THE PROBLEM

Let us consider the unsteady flow of an electrically conducting incompressible fluid. The fluid is bounded by a nonconductive porous disk at $z = 0$. A uniform magnetic field $\mathbf{B}_0$ is imposed, perpendicular to the disk. The axes of rotation, of both the disk and fluid, are assumed to be in the plane $x = 0$, with the distance between the axes being l. The disk and the fluid are initially rotating about the $z' =$ axis with constant angular velocity Ω and at time $t = 0$, the disk and the fluid start to rotate at z and z' axes, respectively, with Ω. For $t > 0$, the disk also oscillates in its own plane with frequency n.

The equations governing the unsteady MHD flow are

$$\frac{d\mathbf{V}}{dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{V} + \frac{1}{\rho}\mathbf{j}\times\mathbf{B}, \tag{1}$$

$$\operatorname{div}\ \mathbf{V} = 0, \tag{2}$$

$$\operatorname{div}\ \mathbf{B} = 0, \tag{3}$$

$$\operatorname{curl}\ \mathbf{B} = \mu_m\mathbf{j}, \tag{4}$$

$$\operatorname{curl}\ \mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{5}$$

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{V}\times\mathbf{B}), \tag{6}$$

where $\mathbf{V} = (u,\ v,\ w)$ is the velocity vector, p is the pressure, ρ is the density, $\mathbf{j}$ is the electric current density, ν is the kinematic viscosity, μ_m is the magnetic permeability, $\mathbf{E}$ is the electric field, and σ is the electric conductivity of the fluid. $\mathbf{B}$ is the total magnetic field so that $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, $\mathbf{b}$ is the induced magnetic field. The magnetic Reynolds number R_m [9] is assumed to be small, as is the case with most of the conducting fluids, and hence, $\mathbf{b}$ is small in comparison with $\mathbf{B}_0$ and is, therefore, not taken into account. The force $\mathbf{j}\times\mathbf{B}$ now becomes $\sigma(\mathbf{V}\times\mathbf{B}_0)\times\mathbf{B}_0$. In addition, $\operatorname{div}\mathbf{j} = 0$ is acquired by using equation (4).

The appropriate boundary and initial conditions for the velocity field are

$$\begin{aligned} &u = -\Omega y + Uh(t), && v = \Omega x, && \text{at } z = 0, && \text{for } t > 0,\\ &u = -\Omega(y - l), && v = \Omega x, && \text{as } z \to \infty, && \text{for all } t,\\ &u = -\Omega(y - l), && v = \Omega x, && \text{at } t = 0, && \text{for } z > 0, \end{aligned} \tag{7}$$

where u and v are the x and y components of the velocity and U is the velocity and the Fourier series representation of $h(t)$ is as follows

$$h(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{iknt}, \tag{8}$$

$$a_k = \frac{1}{T_0} \int_{T_0} h(t) e^{-iknt} \, dt, \tag{9}$$

in which $n = 2\pi/T_0$ and $\{a_k\}$ are the Fourier series coefficients of $h(t)$.

We seek solutions to the equations of motion in the form

$$u = -\Omega y + f(z,t), \qquad v = \Omega x + g(z,t). \tag{10}$$

Making use of equation (10) in equation (1), one obtains

$$\frac{\partial w}{\partial z} = 0. \tag{11}$$

Following [10], we take

$$w = -W_0, \tag{12}$$

in which $W_0 > 0$ is the suction velocity and $W_0 < 0$ is the blowing velocity.

In view of the small R_m, equations (1), (10), and (12) yield

$$\nu \frac{\partial^3 F}{\partial z^3} + W_0 \frac{\partial^2 F}{\partial z^2} - \frac{\partial^2 F}{\partial t \partial z} - \left(i\Omega + \frac{\sigma}{\rho} B_0^2 \right) \frac{\partial F}{\partial z} = 0, \tag{13}$$

with the following boundary and initial conditions

$$\begin{aligned} F(0,t) &= U h(t), \\ F(\infty,t) &= \Omega l, \end{aligned} \tag{14}$$

$$F(z,0) = \Omega l, \tag{15}$$

where

$$F = f + ig. \tag{16}$$

3. EXACT SOLUTION OF THE PROBLEM

Let $F(z,t)$ has a Laplace transform $\Psi(z,s)$. We can take Laplace transform of equation (13) with the initial condition (15), to obtain an ordinary differential equation with the boundary conditions as:

$$\left[\frac{d^3}{dz^3} + \frac{W_0}{\nu} \frac{d^2}{dz^2} - \left(\frac{N+s}{\nu} \right) \frac{d}{dz} \right] \Psi(z,s) = 0, \tag{17}$$

$$\Psi(0,s) = U \sum_{k=\infty}^{\infty} \frac{a_k}{s - ikn}, \tag{18}$$

$$\Psi(z,s) = \frac{\Omega l}{s}, \text{ as } \quad z \to \infty, \tag{19}$$

where

$$N = i\Omega + \sigma \frac{B_0^2}{\rho}. \tag{20}$$

The differential equation (17), with the boundary conditions (18) and (19), has the following solution

$$
\begin{aligned}
\bar{\Psi}(z,s) &= \frac{\Omega l}{s}\left[1-\exp\left\{-\left(\frac{W_0}{2\nu}+\sqrt{\left(\frac{W_0}{2\nu}\right)^2+\left(\frac{N+s}{\nu}\right)}\right)z\right\}\right] \\
&+U\sum_{k=-\infty}^{\infty}\frac{a_k}{s-ikn}\left[1-\exp\left\{-\left(\frac{W_0}{2\nu}+\sqrt{\left(\frac{W_0}{2\nu}\right)^2+\left(\frac{N+s}{\nu}\right)}\right)z\right\}\right].
\end{aligned}
\tag{21}
$$

The Laplace inversion of above equation will complete the solution of the problem. Therefore, by the convolution theorem for Laplace transform, we have that the inverse of equation (21) as

$$
\begin{aligned}
\frac{f}{\Omega l}+i\frac{g}{\Omega l} &= 1-\frac{e^{-\frac{W_0}{2\nu}z}}{2}\left[\begin{array}{l} e^{-z\sqrt{\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho\nu}+\frac{i\Omega}{\nu}}}\operatorname{erf c}\left(\dfrac{z}{2\sqrt{\nu t}}-\sqrt{\left(\dfrac{W_0^2}{4\nu}+\dfrac{\sigma B_0^2}{\rho\nu}+\dfrac{i\Omega}{\nu}\right)t}\right) \\ +e^{z\sqrt{\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho\nu}+\frac{i\Omega}{\nu}}}\operatorname{erf c}\left(\dfrac{z}{2\sqrt{\nu t}}+\sqrt{\left(\dfrac{W_0^2}{4\nu}+\dfrac{\sigma B_0^2}{\rho\nu}+\dfrac{i\Omega}{\nu}\right)t}\right)\end{array}\right] \\
&+\frac{Ue^{-\frac{W_0}{2\nu}z}}{2\Omega l}\sum_{k=-\infty}^{\infty}a_k e^{ik\omega_0 t}\left[\begin{array}{l} e^{-z\sqrt{\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho\nu}+\frac{i\Omega}{\nu}}}\operatorname{erf c}\left(\dfrac{z}{2\sqrt{\nu t}}-\sqrt{\left(\dfrac{W_0^2}{4\nu}+\dfrac{\sigma B_0^2}{\rho\nu}+\dfrac{i\Omega}{\nu}\right)t}\right) \\ +e^{z\sqrt{\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho\nu}+\frac{i\Omega}{\nu}}}\operatorname{erf c}\left(\dfrac{z}{2\sqrt{\nu t}}+\sqrt{\left(\dfrac{W_0^2}{4\nu}+\dfrac{\sigma B_0^2}{\rho\nu}+\dfrac{i\Omega}{\nu}\right)t}\right)\end{array}\right],
\end{aligned}
\tag{22}
$$

where $erfc(\bar{x})$ is the complementary error function defined by

$$
\operatorname{erf c}(\bar{x}) = 1-\operatorname{erf}(\bar{x}) = \int_{\bar{x}}^{\infty} e^{-\tau^2}\,d\tau, \tag{23}
$$

and the real and imaginary parts of equation (22) give $f/\Omega l$ and $g/\Omega l$, respectively. It is better to write equation (22) in the following form

$$
\frac{f}{\Omega l}+i\frac{g}{\Omega l} = H+\frac{Ue^{-\frac{W_0}{2\nu}z}}{2\Omega l}\sum_{k=-\infty}^{\infty}a_k e^{iknt}\left[\begin{array}{l} e^{-\frac{z}{\sqrt{\nu}}(r_k+i\delta_k)}\operatorname{erf c}\left(\dfrac{z}{2\sqrt{\nu t}}-(r_k+i\delta_k)\sqrt{t}\right) \\ +e^{\frac{z}{\sqrt{\nu}}(r_k+i\delta_k)}\operatorname{erf c}\left(\dfrac{z}{2\sqrt{\nu t}}+(r_k+i\delta_k)\sqrt{t}\right)\end{array}\right], \tag{24}
$$

where

$$
H = 1-\frac{e^{-(W_0/2\nu)z}}{2}\left[\begin{array}{l} e^{-(z/\sqrt{\nu})\,(\alpha+i\beta)}\operatorname{erf c}\left(\dfrac{z}{2\sqrt{\nu t}}-(\alpha+i\beta)\sqrt{t}\right) \\ +e^{(z/\sqrt{\nu})\,(\alpha+i\beta)}\operatorname{erf c}\left(\dfrac{z}{2\sqrt{\nu t}}+(\alpha+i\beta)\sqrt{t}\right)\end{array}\right], \tag{25}
$$

$$
\alpha = \left[\frac{1}{2}\left\{\sqrt{\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}\right)^2+\Omega^2}+\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}\right)\right\}\right]^{1/2}, \tag{26}
$$

$$
\beta = \left[\frac{1}{2}\left\{\sqrt{\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}\right)^2+\Omega^2}-\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}\right)\right\}\right]^{1/2}, \tag{27}
$$

$$
r_k = \left[\frac{1}{2}\left\{\sqrt{\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}\right)^2+(\Omega+nk)^2}+\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}\right)\right\}\right]^{1/2}, \tag{28}
$$

$$
\delta_k = \left[\frac{1}{2}\left\{\sqrt{\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}\right)^2+(\Omega+nk)^2}-\left(\frac{W_0^2}{4\nu}+\frac{\sigma B_0^2}{\rho}\right)\right\}\right]^{1/2}. \tag{29}
$$

We note that equation (24) gives the solution for general periodic oscillation of the disk. As a special case of this, the result for different disk oscillations are obtained by an appropriate choice of the Fourier coefficients, which give rise to different disk oscillations. For example, the results for the oscillations

$$e^{int}, \quad \cos nt, \quad \sin nt, \quad \left\{\begin{matrix} 1, & |t| < \dfrac{T_1}{2} \\ 0, & |T_1| < |t| < \dfrac{T_0}{2} \end{matrix}\right\}, \quad \text{and} \quad \sum_{k=-\infty}^{\infty} \delta(t - kT_0),$$

are respectively given as

$$\frac{f_1}{\Omega l} + i\frac{g_1}{\Omega l} = H + \frac{Ue^{-\frac{W_0}{2\nu}z+int}}{2\Omega l}\left[\begin{matrix} e^{-\frac{z}{\sqrt{\nu}}(r_1+i\delta_1)}\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} - (r_1+i\delta_1)\sqrt{t}\right) \\ +e^{\frac{z}{\sqrt{\nu}}(r_1+i\delta_1)}\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} + (r_1+i\delta_1)\sqrt{t}\right) \end{matrix}\right], \tag{30}$$

$$\frac{f_2}{\Omega l} + i\frac{g_2}{\Omega l} = H + \frac{Ue^{-\frac{W_0}{2\nu}z}}{4\Omega l}\left[\begin{matrix} e^{int}\left\{\begin{matrix} e^{-\frac{z}{\sqrt{\nu}}(r_1+i\delta_1)}\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} - (r_1+i\delta_1)\sqrt{t}\right) \\ +e^{\frac{z}{\sqrt{\nu}}(r_1+i\delta_1)}\operatorname{erfc}\left(d\frac{z}{2\sqrt{\nu t}} + (r_1+i\delta_1)\sqrt{t}\right) \end{matrix}\right\} \\ +e^{-int}\left\{\begin{matrix} e^{-\frac{z}{\sqrt{\nu}}(r_{-1}+i\delta_{-1})}\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} - (r_{-1}+i\delta_{-1})\sqrt{t}\right) \\ +e^{\frac{z}{\sqrt{\nu}}(r_{-1}+i\delta_{-1})}\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} + (r_{-1}+i\delta_{-1})\sqrt{t}\right) \end{matrix}\right\} \end{matrix}\right], \tag{31}$$

$$\frac{f_3}{\Omega l} + i\frac{g_3}{\Omega l} = H - \frac{iUe^{-\frac{W_0}{2\nu}z}}{4\Omega l}\left[\begin{matrix} e^{int}\left\{\begin{matrix} e^{-\frac{z}{\sqrt{\nu}}(r_1+i\delta_1)}\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} - (r_1+i\delta_1)\sqrt{t}\right) \\ +e^{\frac{z}{\sqrt{\nu}}(r_1+i\delta_1)}\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} + (r_1+i\delta_1)\sqrt{t}\right) \end{matrix}\right\} \\ -e^{-int}\left\{\begin{matrix} e^{-\frac{z}{\sqrt{\nu}}(r_{-1}+i\delta_{-1})}\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} - (r_{-1}+i\delta_{-1})\sqrt{t}\right) \\ +e^{\frac{z}{\sqrt{\nu}}(r_{-1}+i\delta_{-1})}\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} + (r_{-1}+i\delta_{-1})\sqrt{t}\right) \end{matrix}\right\} \end{matrix}\right], \tag{32}$$

$$\frac{f_4}{\Omega l} + i\frac{g_4}{\Omega l} = H + \frac{Ue^{-\frac{W_0}{2\nu}z}}{2\Omega l}\sum_{k=-\infty}^{\infty}\left(\frac{\sin knT_1}{k\pi}\right)e^{iknt}\left[\begin{matrix} e^{-\frac{z}{\sqrt{\nu}}(r_k+i\delta_k)}\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} - (r_k+i\delta_k)\sqrt{t}\right) \\ +e^{\frac{z}{\sqrt{\nu}}(r_k+i\delta_k)}\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} + (r_k+i\delta_k)\sqrt{t}\right) \end{matrix}\right], \quad k \neq 0, \tag{33}$$

$$\frac{f_5}{\Omega l} + i\frac{g_5}{\Omega l} = H + \frac{Ue^{-\frac{W_0}{2\nu}z}}{2\Omega lT_0}\sum_{k=-\infty}^{\infty} e^{iknt}\left[\begin{matrix} e^{-\frac{z}{\sqrt{\nu}}(r_k+i\delta_k)}\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} - (r_k+i\delta_k)\sqrt{t}\right) \\ +e^{\frac{z}{\sqrt{\nu}}(r_k+i\delta_k)}\operatorname{erfc}\left(\frac{z}{2\sqrt{\nu t}} + (r_k+i\delta_k)\sqrt{t}\right) \end{matrix}\right]. \tag{34}$$

For the case of blowing, we use $W_0 < 0$ in (24)–(34). Thus, solutions (24) and (30)–(34) clearly describe the general features of the unsteady hydromagnetic flow, including the case of uniform suction or blowing, according to $W_0 > 0$ or $W_0 < 0$, respectively.

4. THE ULTIMATE STEADY STATE FLOWS AND THE BOUNDARY LAYERS

Taking $t \to \infty$ in (24) and (30)–(34) and using the asymptotic formula for the complementary error function, we have for suction

$$\frac{f}{\Omega l} + i\frac{g}{\Omega l} = M + \frac{Ue^{-(W_0/2\nu)z}}{\Omega l}\sum_{k=-\infty}^{\infty} a_k e^{iknt-(z/\sqrt{\nu})(r_k+i\delta_k)}, \tag{35}$$

$$\frac{f_1}{\Omega l} + i\frac{g_1}{\Omega l} = M + \frac{Ue^{-(W_0/2\nu)z+int-(z/\sqrt{\nu})(r_1+i\delta_1)}}{\Omega l}, \tag{36}$$

$$\frac{f_2}{\Omega l}+i\frac{g_2}{\Omega l}=M+\frac{Ue^{-(W_0/2\nu)z}}{2\Omega l}\left[e^{int-(z/\sqrt{\nu})(r_1+i\delta_1)}+e^{-int-(z/\sqrt{\nu})(r_{-1}+i\delta_{-1})}\right], \tag{37}$$

$$\frac{f_3}{\Omega l}+i\frac{g_3}{\Omega l}=M-\frac{iUe^{-(W_0/2\nu)z}}{2\Omega l}\left[e^{int-(z/\sqrt{\nu})(r_1+i\delta_1)}-e^{-int-(z/\sqrt{\nu})(r_{-1}+i\delta_{-1})}\right], \tag{38}$$

$$\frac{f_4}{\Omega l}+i\frac{g_4}{\Omega l}=M+\frac{Ue^{-(W_0/2\nu)z}}{\Omega l}\sum_{k=-\infty}^{\infty}\left(\frac{\sin knT_1}{k\pi}\right)e^{iknt-(z/\sqrt{\nu})(r_k+i\delta_k)},\qquad k\neq 0, \tag{39}$$

$$\frac{f_5}{\Omega l}+i\frac{g_5}{\Omega l}=M+\frac{Ue^{-(W_0/2\nu)z}}{\Omega lT_0}\sum_{k=-\infty}^{\infty}e^{iknt-(z/\sqrt{\nu})(r_k+i\delta_k)}, \tag{40}$$

where

$$M=1-\frac{e^{-(W_0/2\nu)z-(z/\sqrt{\nu})\,(\alpha+i\beta)}}{2}.$$

We note that solution (35) is of the oscillatory type. For this flow, thicknesses of the boundary layers are of the order $(W_0/2\nu+\sqrt{\nu}/a)^{-1}$ and $(W_0/2\nu+\sqrt{\nu}/r_k)^{-1}$. It is interesting to note that the thicknesses of the boundary layers decrease with an increase of the magnetic field or suction parameter s and remain bounded for all values of the frequency. When $n=0$, then $\alpha=r_k$, and the distinct boundary layers combines into a single layer of the thickness of the order $(W_0/2\nu+\sqrt{\nu}/a)^{-1}$.

In the case of blowing, the steady state solutions are obtained from equations (24) and (30)–(34) by replacing W_0 by $-W_1$, $(W_1>0)$ and taking the limit $t\to\infty$ and has the form

$$\frac{f}{\Omega l}+i\frac{g}{\Omega l}=\bar{M}+\frac{Ue^{(W_1/2\nu)z}}{\Omega l}\sum_{k=-\infty}^{\infty}a_k e^{iknt-z/\sqrt{\nu}(\bar{r}_k+i\bar{\delta}_k)}, \tag{41}$$

$$\frac{f_1}{\Omega l}+i\frac{g_1}{\Omega l}=\bar{M}+\frac{Ue^{(W_1/2\nu)z+int-z/\sqrt{\nu}(\bar{r}_1+i\bar{\delta}_1)}}{\Omega l}, \tag{42}$$

$$\frac{f_2}{\Omega l}+i\frac{g_2}{\Omega l}=\bar{M}+\frac{Ue^{(W_1/2\nu)z}}{2\Omega l}\left[e^{int-z/\sqrt{\nu}(\bar{r}_1+i\bar{\delta}_1)}+e^{-int-z/\sqrt{\nu}(\bar{r}_1+i\bar{\delta}_1)}\right], \tag{43}$$

$$\frac{f_3}{\Omega l}+i\frac{g_3}{\Omega l}=\bar{M}-\frac{iUe^{(W_1/2\nu)z}}{2\Omega l}\left[e^{int-z/\sqrt{\nu}(\bar{r}_1+i\bar{\delta}_1)}-e^{-int-z/\sqrt{\nu}(\bar{r}_1+i\bar{\delta}_1)}\right], \tag{44}$$

$$\frac{f_4}{\Omega l}+i\frac{g_4}{\Omega l}=\bar{M}+\frac{Ue^{(W_1/2\nu)z}}{\Omega l}\sum_{k=-\infty}^{\infty}\left(\frac{\sin knT_1}{k\pi}\right)e^{iknt-z/\sqrt{\nu}(\bar{r}_k+i\bar{\delta}_k)},\qquad k\neq 0, \tag{45}$$

$$\frac{f_5}{\Omega l}+i\frac{g_5}{\Omega l}=\bar{M}+\frac{Ue^{(W_1/2\nu)z}}{\Omega lT_0}\sum_{k=-\infty}^{\infty}e^{iknt-z/\sqrt{\nu}(\bar{r}_k+i\bar{\delta}_k)}, \tag{46}$$

where

$$\bar{M}=1-\frac{e^{(w_1/2\nu)z-z/\sqrt{\nu}\,(\bar{\alpha}+i\bar{\beta})}}{2},$$

where $\bar{\alpha}$, $\bar{\beta}$, $\bar{r}_k$, and $\bar{\delta}_k$ are obtained by replacing W_0 by $-W_1$ in the expressions (26) to (29), respectively.

In absence of magnetic field $B_0=0$, we note that solutions (41)–(45) satisfy the boundary conditions for all values of frequency n, but the boundary condition at infinity is not satisfied when $n=\Omega$. In other words, when the disk oscillates with the resonant frequency $n=\Omega$, one of the boundary layers becomes infinitely thick. Consequently, the oscillations generated by the disk are no longer confined to the ultimate boundary layers. In a hydromagnetic situation, solutions (41)–(46) satisfy the boundary conditions for all values of frequencies, including the resonant frequency and unbounded spreading of shear oscillations away from the disk in blowing, and resonance is controlled by the external magnetic field (all the boundary layers remain finite for all values of frequencies). Physically, the diffusive hydromagnetic waves exist in the magnetohydrodynamic system. These waves are found to decay within the ultimate steady state boundary layers. The external magnetic field expedites the decay process of these waves.

5. CONCLUDING REMARKS

In this communication, we have studied the oscillating disk problem. The combined efforts of a transverse magnetic field, suction, blowing, and noncoaxial rotation are undertaken. Besides engineering applications, the present analysis of the magnetic oscillating problem possesses also astrophysical and geophysical applications [11]; namely, in connection with the use of such equations for a better understanding of the motion of electrically conducting fluid, such as ionized clouds.

Another important field of application is the electromagnetic propulsion. Basically, an electromagnetic propulsion system consists of a power source, such as a nuclear reactor, a plasma, and a tube through which the plasma is accelerated by electromagnetic forces. The study of such a system, which is closely associated with magnetochemistry, requires a complete understanding of the equation of state and transport properties such as diffusion, shear stress-shear rate relationship, thermal conductivity, electrical conductivity, and radiation. Some of these properties will undoubtedly be influenced by the presence of an external magnetic field which sets the plasma in hydrodynamic motion.

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Flow induced by non-coaxial rotation of a porous disk executing non-torsional oscillations and a second grade fluid rotating at infinity

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Abstract

The purpose of this work is to examine the flow of a fluid bounded by a porous disk. The fluid is assumed to be non-Newtonian (second grade) and incompressible. Such a flow model has great significance not only of its own theoretical interest, but also for application to geophysics and engineering. The governing initial value problem has been solved analytically by using the Laplace transform technique. Explicit expressions for the velocity for steady and unsteady cases have been constructed. The analysis of the obtained results showed that the flow field is appreciably influenced by the material parameter of the second grade fluid, the imposed frequency, rotation and porosity parameters. Several known results of interest are found to follow as particular cases of the solution of the problem considered.

Keywords: Unsteady flow; Second grade fluid; Non-coaxial rotation; Oscillating disk; Porosity

1. Introduction

The possibility of an exact solution of the Navier–Stokes equations for the flow due to non-coaxial rotations of a disk and a fluid at infinity has been implied by Berker [1]. Exact solutions for the flow due to a single disk in a variety of situations have been obtained by number of workers. Coirier [2] has studied the flow due to a disk and a fluid at infinity which are

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rotating non-coaxially at slightly different angular velocities. Exact solutions of three-dimensional Navier–Stokes equations are obtained by Erdogan [3] for the flow due to non-coaxial rotation of a porous disk and a fluid at infinity. Rajagopal [4] has considered the flow of a simple fluid in an orthogonal rheometer. The flows of Newtonian and non-Newtonian fluids between parallel disks rotating about a common axis has been reviewed by Rajagopal [5]. Parter and Rajagopal [6] also discussed that solutions which lack symmetry are possible for swirling flows about a common axis. These asymmetric flows have relevance to the stability of the axi-symmetric flows.

In order to extend the work of Berker [7] to the case of unsteady motions, Rao and Kasiviswanathan [8] considered the flow of a fluid between two eccentric rotating disks for which the streamlines at a given instant are concentric circles in each plane parallel to a fixed plane and each point of the plane is performing non-torsional oscillations. Unsteady flow due to non-coaxial rotations of a disk, executing non-torsional oscillation in its own plane and a fluid at infinity has been investigated by Kasiviswanathan and Rao [9]. The unsteady flow due to eccentric rotations of a disk and fluid at infinity which are impulsively started was examined by Pop [10]. He assumed that flow is two-dimensional, and that both the disk and the fluid at infinity are initially at rest and that they are impulsively started at time zero. It is clearly seen that under the assumed conditions by Pop, the flow does not become two-dimensional, but becomes three-dimensional. The conditions which make the flow two-dimensional have been investigated by Erdogan [11]. More-recently, Erdogan [12] obtained an exact solution for the flow due to non-coaxial rotations of a disk, executing oscillations in its own plane, and a fluid at infinity.

As is known, the Navier–Stokes equations seem to be an inappropriate model for a class of real fluids, called non-Newtonian fluids, which includes viscoelastic liquids with short memory and viscous fluids with polymer additives. In recent years, considerable efforts have been usefully devoted to the study of flow of non-Newtonian fluids because of their practical and fundamental importance associated with many technological applications. A vast amount of literature is now available for various kinds of geometries and for a variety of non-Newtonian fluids. In order to increase the basic understanding of second grade fluids, several authors including Rajagopal [13,14], Rajagopal and Gupta [15], Rajagopal et al. [16], Bandelli and Rajagopal [17], Fetecau and Fetecau [18], Benharbit and Siddiqui [19], Coscia and Galdi [20] and Hayat et al. [21–23] have studied the theory of non-Newtonian fluids in various geometrical configurations. In general, it is not easy to study a non-Newtonian fluid flow problem associated with a complex geometry. Keeping this in mind the aim of this work is twofold. Firstly, the flow of a second grade fluid produced by an oscillating disk which rotates non-coaxially with the fluid at infinity, is examined. For second grade fluid, the proposed constitutive relations are involved and complicated, leading partial differential equations with highly non-linear terms which make the question of well-posedness extremely difficult to address. In contrast, such a question is of fundamental importance, not only from a mathematical point of view, but mainly as an essential test for the underlying physical model. Here, it is shown that the equations of motion have an exact solution. Secondly, to include the effect of porosity by taking into account the porous disk instead of a rigid disk. An analytical solution for large and small times after the start is obtained. The solution for three cases, when the angular velocity is greater than, smaller than or equal to the frequency of oscillation are given. For small times the Laplace transform method is used. For large times, the steady solution is possible for suction and blowing when angular velocity is greater than or smaller than the frequency of oscillation. If the angular velocity is equal to the frequency of oscillations, the system resonates. It is seen that steady blowing solution in resonant case does not satisfy the boundary condition at infinity. The solution of Erdogan [12] can be obtained as a special case of the presented analysis by taking the suction/blowing velocity and material parameter of the second grade fluid to be zero.

2. Flow problem

The flow of an incompressible second grade fluid, neglecting thermal effects and body forces, is governed by

$$\operatorname{div} \mathbf{V} = 0, \tag{1}$$

$$\rho \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \operatorname{div} \mathbf{T}, \tag{2}$$

where the Cauchy stress tensor $\mathbf{T}$ in an incompressible and homogeneous Rivilin–Ericksen fluid of second grade is related to the fluid motion in the following manner [24]

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{3}$$

$$\mathbf{A}_1 = (\operatorname{grad} \mathbf{V}) + (\operatorname{grad} \mathbf{V})^{\mathrm{T}},$$

$$\mathbf{A}_2 = \frac{\mathrm{d}\mathbf{A}_1}{\mathrm{d}t} + \mathbf{A}_1(\operatorname{grad} \mathbf{V}) + (\operatorname{grad} \mathbf{V})^{\mathrm{T}}\mathbf{A}_1$$

Here $\mathbf{V}$ is the velocity vector field, p the fluid pressure, ρ the constant fluid density, μ the constant coefficient of viscosity, $\mathrm{d}/\mathrm{d}t$ the material time derivative and α_1, α_2 the normal stress moduli.

According to Dunn and Fosdick [25], the second grade fluid model is compatible with thermodynamics when the Helmholtz free energy of the fluid is a minimum for the fluid in equilibrium. The fluid model then has general and pleasant boundedness and stability properties. The Clausius–Duhem inequality and the assumption that the Helmholtz free energy is a minimum in equilibrium provide the following restrictions

$$\mu \geqslant 0, \quad \alpha_1 \geqslant 0, \quad \alpha_1 + \alpha_2 = 0.$$

Since we are dealing with second grade fluid flow, the strict inequality holds true. Fosdick and Rajagopal [26] showed that when $\alpha_1 < 0$, the fluid exhibits anomalous behavior that is incompatible with any fluid of rheological interest, and so results in a fluid that is unstable.

We consider a Cartesian coordinate system with the z-axis normal to the porous disk. The axes of rotation, of both the disk and the fluid, are assumed to be in the plane $x = 0$, with the distance between the axes being l. The common angular velocity of disk and the fluid is taken as Ω. We seek a solution for the velocity field of the form

$$u = -\Omega y + f(z,t), \quad v = \Omega x + g(z,t) \tag{4}$$

with the following boundary and initial conditions

$$\begin{aligned} &u = -\Omega y + U\cos nt \quad \text{or} \quad u = -\Omega y + U\sin nt; \quad v = \Omega x, \quad \text{at } z = 0 \quad \text{for } t > 0,\\ &u = -\Omega(y - l), \quad v = \Omega x, \quad \text{as } z \to \infty \quad \text{for all } t,\\ &u = -\Omega(y - l), \quad v = \Omega x, \quad \text{at } t = 0, \quad \text{for } z > 0. \end{aligned} \tag{5}$$

The condition of incompressibility implies that z-component of velocity is equal to $-w_0$. Clearly $w_0 > 0$ is for suction velocity and $w_0 < 0$ is for the blowing velocity. Making use of Eq. (4) in Eq. (2) and eliminating the pressure from the resulting equations give

$$\alpha\frac{\partial^3 F^*}{\partial t\,\partial z^2} - \alpha w_0\frac{\partial^3 F^*}{\partial z^3} + (v - \mathrm{i}\alpha\Omega)\frac{\partial^2 F^*}{\partial z^2} + w_0\frac{\partial F^*}{\partial z} - \frac{\partial F^*}{\partial t} - \mathrm{i}\Omega F^* = -\mathrm{i}\Omega^2 l \tag{6}$$

with the following boundary and initial conditions

$$\begin{aligned} &F^*(0,t) = \Omega l + U\cos nt \quad \text{or} \quad F^*(0,t) = \Omega l + U\sin nt,\\ &F^*(\infty,t) = 0, \quad F^*(0,t) = 0, \end{aligned} \tag{7}$$

where

$$\alpha = \frac{\alpha_1}{\rho}, \quad v = \frac{\mu}{\rho}, \quad F^* = f + \mathrm{i}g. \tag{8}$$

Defining

$$\xi = \sqrt{\frac{\Omega}{2v}}z, \quad \tau = \Omega t, \quad \beta = \frac{\Omega}{v}\alpha, \quad w = \frac{w_0}{\sqrt{2v\Omega}}, \quad F(\xi,\tau) = \frac{F^*}{\Omega l} - 1, \quad H(\xi,\tau) = F(\xi,\tau)\mathrm{e}^{\mathrm{i}\tau} \tag{9}$$

Eq. (6) and conditions (7) become

$$\beta\frac{\partial^3 H}{\partial\tau\,\partial\xi^2} - \beta w\frac{\partial^3 H}{\partial\xi^3} + (1 - 2\mathrm{i}\beta)\frac{\partial^2 H}{\partial\xi^2} + 2w\frac{\partial H}{\partial\xi} - 2\frac{\partial H}{\partial\tau} = 0, \tag{10}$$

$$\begin{aligned} &H(0,\tau) = \mathrm{e}^{\mathrm{i}\tau}\left(-1 + \frac{U}{\Omega l}\cos k\tau\right) \quad \text{or} \quad H(0,\tau) = \mathrm{e}^{\mathrm{i}\tau}\left(-1 + \frac{U}{\Omega l}\sin k\tau\right);\\ &H(\infty,\tau) = 0, \quad H(\xi,0) = 0. \end{aligned} \tag{11}$$

3. Solution of the problem

The Laplace transform pair is defined by the following equations

$$\overline{H}(\xi,s) = \int_0^\infty H\mathrm{e}^{-s\tau}\,\mathrm{d}\tau, \tag{12}$$

$$H(\xi,\tau) = \frac{1}{2\pi\mathrm{i}}\int_{\bar\lambda-\mathrm{i}\infty}^{\bar\lambda+\mathrm{i}\infty}\overline{H}\mathrm{e}^{s\tau}\,\mathrm{d}s, \quad \bar\lambda > 0. \tag{13}$$

In view of the Laplace transform method, the initial-boundary value problem reduces to

$$\beta w\overline{H}''' - (1 - 2\mathrm{i}\beta + \beta s)\overline{H}'' - 2w\overline{H}' + 2s\overline{H} = 0, \tag{14}$$

$$\overline{H}(0,s) = -\frac{1}{s-\mathrm{i}} + \frac{U}{2\Omega l}\left[\frac{1}{s-\mathrm{i}(k+1)} + \frac{1}{s+\mathrm{i}(k-1)}\right], \tag{15}$$

or

$$\overline{H}(0,s) = -\frac{1}{s-\mathrm{i}} + \frac{U}{2\mathrm{i}\Omega l}\left[\frac{1}{s-\mathrm{i}(k+1)} - \frac{1}{s+\mathrm{i}(k-1)}\right], \tag{16}$$

$$\overline{H}(\infty,s) = 0. \tag{17}$$

In Eq. (14), primes denote differentiation with respect to ξ. Eq. (14) is the third-order differential equation for $\beta \neq 0$ and $w \neq 0$ and for $\beta = 0$ or $w = 0$ this reduces to equation governing the Newtonian fluid or no suction respectively. The analysis of the flow of the second grade fluids, in particular, and the viscoelastic fluids, in general, is more challenging mathematically and computationally. The equations of motion are higher order than the Navier–Stokes equations. Also, the order of the differential equation characterizing the flow is more than the number of the available boundary conditions. Thus,the adherence boundary condition is insufficient for determinacy. The standard method used to overcome this difficulty is to expand the solution as a power series in the non-Newtonian parameter treats a singular perturbation as a regular perturbation as the non-Newtonian parameter multiplies the highest order terms in the equation. This is indeed how most of the earlier flow problems were solved since the initial effort of Beard and Walters [27] who considered the two-dimensional flow of a viscoelastic fluid near a stagnation point. One may also refer, for example, to the works of Shrestha [28], Misra and Mohapatra [29], Rajagopal et al. [30], Verma et al. [31] and Erdogan [32] for other problems in various geometries. In the present analysis, the difficulty is also removed by seeking a solution of the following form:

$$\overline{H} = \overline{H}_1 + \beta\overline{H}_2 + \mathrm{O}(\beta^2) \tag{18}$$

which is valid for small values of β only. On substituting Eq. (18) into Eqs. (14)–(17) and then comparing coefficients of equal powers of β one obtains the following systems of differential equations along with the appropriate boundary conditions.

3.1. Zeroth-order system

$$\overline{H}_1'' + 2w\overline{H}_1' - 2s\overline{H}_1 = 0, \tag{19}$$

$$\overline{H}_1(0,s) = -\frac{1}{s-\mathrm{i}} + \frac{U}{2\Omega l}\left(\frac{1}{s-\mathrm{i}(k+1)} + \frac{1}{s+\mathrm{i}(k-1)}\right), \tag{20}$$

or

$$\overline{H}_1(0,s) = -\frac{1}{s-\mathrm{i}} + \frac{U}{2\mathrm{i}\Omega l}\left(\frac{1}{s-\mathrm{i}(k+1)} - \frac{1}{s+\mathrm{i}(k-1)}\right), \tag{21}$$

$$\overline{H}_1(\infty,s) = 0. \tag{22}$$

3.2. First-order system

$$w\overline{H}_1''' - \overline{H}_2'' - (s-2\mathrm{i})\overline{H}_1'' - 2w\overline{H}_2'' + 2s\overline{H}_2 = 0, \tag{23}$$

$$\overline{H}_2(0,s) = 0, \quad \overline{H}_2(\infty,s) = 0. \tag{24}$$

Solving above systems and then using Eq. (18) we arrive at

for $U\cos nt$ and $k > 1$

$$\begin{aligned}\overline{H} = {} & \mathrm{e}^{-m\xi}\left[-\frac{1}{s-\mathrm{i}} + \frac{U}{2\Omega l}\left(\frac{1}{s+\mathrm{i}(k-1)} + \frac{1}{s-\mathrm{i}(k+1)}\right)\right] + \beta\frac{\xi\mathrm{e}^{-m\xi}}{s-\mathrm{i}}\left[-2w^2\sqrt{w^2+2s}\right.\\ & \left. - 2w^3 - 2ws - \frac{s^2}{\sqrt{w^2+2s}} + 2\mathrm{i}\sqrt{w^2+2s} + 2\mathrm{i}w - \frac{2\mathrm{i}s}{\sqrt{w^2+2s}}\right]\\ & - \frac{\beta U\xi\mathrm{e}^{-m\xi}}{2\Omega l}\left[-2w^2\sqrt{w^2+2s} - 2w^3 - 2ws - \frac{s^2}{\sqrt{w^2+2s}} + 2\mathrm{i}\sqrt{w^2+2s} + 2\mathrm{i}w - \frac{2\mathrm{i}s}{\sqrt{w^2+2s}}\right]\\ & \times\left[\frac{1}{s+\mathrm{i}(k-1)} + \frac{1}{s-\mathrm{i}(k+1)}\right].\end{aligned} \tag{25}$$

For $U\sin nt$ and $k > 1$

$$\begin{aligned}\overline{H} = {} & \mathrm{e}^{-m\xi}\left[-\frac{1}{s-\mathrm{i}} + \frac{\mathrm{i}U}{2\Omega l}\left(\frac{1}{s+\mathrm{i}(k-1)} - \frac{1}{s-\mathrm{i}(k+1)}\right)\right] + \beta\frac{\xi\mathrm{e}^{-m\xi}}{s-\mathrm{i}}\left[-2w^2\sqrt{w^2+2s}\right.\\ & \left. - 2w^3 - 2ws - \frac{s^2}{\sqrt{w^2+2s}} + 2\mathrm{i}\sqrt{w^2+2s} + 2\mathrm{i}w - \frac{2\mathrm{i}s}{\sqrt{w^2+2s}}\right]\\ & - \frac{\beta\mathrm{i}U\xi\mathrm{e}^{-m\xi}}{2\Omega l}\left[-2w^2\sqrt{w^2+2s} - 2w^3 - 2ws - \frac{s^2}{\sqrt{w^2+2s}} + 2\mathrm{i}\sqrt{w^2+2s} + 2\mathrm{i}w - \frac{2\mathrm{i}s}{\sqrt{w^2+2s}}\right]\\ & \times\left[\frac{1}{s+\mathrm{i}(k-1)} - \frac{1}{s-\mathrm{i}(k+1)}\right].\end{aligned} \tag{26}$$

For $U\cos nt$ and $k < 1$

$$\begin{aligned}\overline{H} = {} & \left[-\frac{1}{s-\mathrm{i}} + \frac{U}{2\Omega l}\left(\frac{1}{s-\mathrm{i}(1-k)} + \frac{1}{s-\mathrm{i}(k+1)}\right)\right] + \beta\frac{\xi\mathrm{e}^{-m\xi}}{s-\mathrm{i}}\left[-2w^2\sqrt{w^2+2s} - 2w^3\right.\\ & \left. - 2ws - \frac{s^2}{\sqrt{w^2+2s}} + 2\mathrm{i}\sqrt{w^2+2s} + 2\mathrm{i}w - \frac{2\mathrm{i}s}{\sqrt{w^2+2s}}\right]\\ & - \frac{\beta U\xi\mathrm{e}^{-m\xi}}{2\Omega l}\left[-2w^2\sqrt{w^2+2s} - 2w^3 - 2ws - \frac{s^2}{\sqrt{w^2+2s}} + 2\mathrm{i}\sqrt{w^2+2s} + 2\mathrm{i}w - \frac{2\mathrm{i}s}{\sqrt{w^2+2s}}\right]\\ & \times\left[\frac{1}{s-\mathrm{i}(1-k)} + \frac{1}{s-\mathrm{i}(k+1)}\right].\end{aligned} \tag{27}$$

For $U\sin nt$ and $k<1$

$$\overline{H}=\mathrm{e}^{-m\xi}\left[-\frac{1}{s-\mathrm{i}}+\frac{\mathrm{i}U}{2\Omega l}\left(\frac{1}{s-\mathrm{i}(1-k)}-\frac{1}{s-\mathrm{i}(k+1)}\right)\right]+\beta\frac{\xi\mathrm{e}^{-m\xi}}{s-\mathrm{i}}\left[-2w^2\sqrt{w^2+2s}\right.$$
$$\left.-2w^3-2ws-\frac{s^2}{\sqrt{w^2+2s}}+2\mathrm{i}\sqrt{w^2+2s}+2\mathrm{i}w-\frac{2\mathrm{i}s}{\sqrt{w^2+2s}}\right]$$
$$-\frac{\mathrm{i}\beta U\xi\mathrm{e}^{-m\xi}}{2\Omega l}\left[-2w^2\sqrt{w^2+2s}-2w^3-2ws-\frac{s^2}{\sqrt{w^2+2s}}+2\mathrm{i}\sqrt{w^2+2s}+2\mathrm{i}w-\frac{2\mathrm{i}s}{\sqrt{w^2+2s}}\right]$$
$$\times\left[\frac{1}{s-\mathrm{i}(1-k)}-\frac{1}{s-\mathrm{i}(k+1)}\right]. \tag{28}$$

Making use of Eqs. (25)–(28) in Eq. (13) we respectively obtain

for $U\cos nt$ and $k>1$,

$$H(\xi,\tau)=\widetilde{H}_1(\xi,\tau)\mathrm{e}^{\mathrm{i}\tau}+\beta\widetilde{H}_2(\xi,\tau)+\frac{\beta U}{2\Omega l}(\widetilde{H}_3(\xi,\tau)+\widetilde{H}_4(\xi,\tau)). \tag{29}$$

For $U\sin nt$ and $k>1$

$$H(\xi,\tau)=\widetilde{H}_5(\xi,\tau)\mathrm{e}^{\mathrm{i}\tau}+\beta\widetilde{H}_2(\xi,\tau)+\frac{\mathrm{i}\beta U}{2\Omega l}(\widetilde{H}_3(\xi,\tau)-\widetilde{H}_4(\xi,\tau)). \tag{30}$$

For $U\cos nt$ and $k<1$,

$$H(\xi,\tau)=\widetilde{H}_6(\xi,\tau)\mathrm{e}^{\mathrm{i}\tau}+\beta\widetilde{H}_2(\xi,\tau)+\frac{\beta U}{2\Omega l}(\widetilde{H}_3(\xi,\tau)+\widetilde{H}_7(\xi,\tau)). \tag{31}$$

For $U\sin nt$ and $k<1$

$$H(\xi,\tau)=\widetilde{H}_8(\xi,\tau)\mathrm{e}^{\mathrm{i}\tau}+\beta\widetilde{H}_2(\xi,\tau)+\frac{\mathrm{i}\beta U}{2\Omega l}(\widetilde{H}_3(\xi,\tau)-\widetilde{H}_7(\xi,\tau)), \tag{32}$$

where

$$\widetilde{H}_1(\xi,\tau)=\frac{-\mathrm{e}^{-w\xi}}{2}\left[\mathrm{e}^{(a_1+\mathrm{i}b_1)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(a_1+\mathrm{i}b_1)\sqrt{\frac{\tau}{2}}\right)+\mathrm{e}^{-(a_1+\mathrm{i}b_1)\xi}\right.$$
$$\left.\times\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(a_1+\mathrm{i}b_1)\sqrt{\frac{\tau}{2}}\right)\right]+\frac{U\mathrm{e}^{-w\xi+\mathrm{i}k\tau}}{4\Omega l}\left[\mathrm{e}^{(a_2+\mathrm{i}b_2)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(a_2+\mathrm{i}b_2)\sqrt{\frac{\tau}{2}}\right)\right.$$
$$\left.+\mathrm{e}^{-(a_2+\mathrm{i}b_2)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(a_2+\mathrm{i}b_2)\sqrt{\frac{\tau}{2}}\right)\right]+\frac{U\mathrm{e}^{-w\xi-\mathrm{i}k\tau}}{4\Omega l}\left[\mathrm{e}^{(a_3+\mathrm{i}b_3)\xi}\right.$$
$$\left.\times\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(a_3+\mathrm{i}b_3)\sqrt{\frac{\tau}{2}}\right)+\mathrm{e}^{-(a_3+\mathrm{i}b_3)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(a_3+\mathrm{i}b_3)\sqrt{\frac{\tau}{2}}\right)\right], \tag{33}$$

$$\widetilde{H}_2(\xi,\tau)=-\sqrt{\frac{2}{\pi\tau}}\xi\left(\frac{\mathrm{i}}{2}-\frac{7w^2}{4}-\frac{w\xi}{\tau}\right)\mathrm{e}^{-w\xi-w^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}+\xi\mathrm{e}^{-w\xi+\mathrm{i}\tau}\left[(\eta_2-\eta_1)\mathrm{e}^{(a_1+\mathrm{i}b_1)\xi}\right.$$
$$\left.\times\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(a_1+\mathrm{i}b_1)\sqrt{\frac{\tau}{2}}\right)+(\eta_2+\eta_1)\mathrm{e}^{-(a_1+\mathrm{i}b_1)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(a_1+\mathrm{i}b_1)\sqrt{\frac{\tau}{2}}\right)\right], \tag{34}$$

$$\widetilde{H}_3(\xi,\tau) = -\xi\sqrt{\frac{2}{\pi\tau}}\left(p - \frac{w\xi}{\tau}\right)e^{-w\xi - w^2\frac{\tau}{2} - \frac{\xi^2}{2\tau}} + \xi e^{-w\xi + i(1+k)\tau}\left[(\eta_4 - \eta_3)e^{(a_2+ib_2)\xi}\right.$$
$$\left.\times \operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} + (a_2+ib_2)\sqrt{\frac{\tau}{2}}\right) + (\eta_3+\eta_4)e^{-(a_2+ib_2)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} - (a_2+ib_2)\sqrt{\frac{\tau}{2}}\right)\right], \tag{35}$$

$$\widetilde{H}_4(\xi,\tau) = -\xi\sqrt{\frac{2}{\pi\tau}}\left(q - \frac{w\xi}{\tau}\right)e^{-w\xi - w^2\frac{\tau}{2} - \frac{\xi^2}{2\tau}} + \xi e^{-w\xi - i(k-1)\tau}\left[(\eta_6 - \eta_5)e^{(a_3+ib_3)\xi}\right.$$
$$\left.\times \operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} + (a_3+ib_3)\sqrt{\frac{\tau}{2}}\right) + (\eta_6+\eta_5)e^{-(a_3+ib_3)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} - (a_3+ib_3)\sqrt{\frac{\tau}{2}}\right)\right], \tag{36}$$

$$\widetilde{H}_5(\xi,\tau) = -\frac{e^{-w\xi}}{2}\left[e^{(a_1+ib_1)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} + (a_1+ib_1)\sqrt{\frac{\tau}{2}}\right) + e^{-(a_1+ib_1)\xi}\right.$$
$$\left.\times \operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} - (a_1+ib_1)\sqrt{\frac{\tau}{2}}\right)\right] - \frac{iU}{4\Omega l}e^{-w\xi+ik\tau}\left[e^{(a_2+ib_2)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} + (a_2+ib_2)\sqrt{\frac{\tau}{2}}\right)\right.$$
$$\left. + e^{-(a_2+ib_2)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} - (a_2+ib_2)\sqrt{\frac{\tau}{2}}\right)\right] + \frac{iU}{4\Omega l}e^{-w\xi-ik\tau}\left[e^{(a_3+ib_3)\xi}\right.$$
$$\left.\times \operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} + (a_3+ib_3)\sqrt{\frac{\tau}{2}}\right) + e^{-(a_3+ib_3)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} - (a_3+ib_3)\sqrt{\frac{\tau}{2}}\right)\right], \tag{37}$$

$$\widetilde{H}_6(\xi,\tau) = -\frac{e^{-w\xi}}{2}\left[e^{(a_1+ib_1)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} + (a_1+ib_1)\sqrt{\frac{\tau}{2}}\right) + e^{-(a_1+ib_1)\xi}\right.$$
$$\left.\times \operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} - (a_1+ib_1)\sqrt{\frac{\tau}{2}}\right)\right] + \frac{U}{4\Omega l}e^{-w\xi+ik\tau}\left[e^{(a_2+ib_2)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} + (a_2+ib_2)\sqrt{\frac{\tau}{2}}\right)\right.$$
$$\left. + e^{-(a_2+ib_2)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} - (a_2+ib_2)\sqrt{\frac{\tau}{2}}\right)\right] + \frac{U}{4\Omega l}e^{-w\xi-ik\tau}\left[e^{(a_4+ib_4)\xi}\right.$$
$$\left.\times \operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} + (a_4+ib_4)\sqrt{\frac{\tau}{2}}\right) + e^{-(a_4+ib_4)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} - (a_4+ib_4)\sqrt{\frac{\tau}{2}}\right)\right], \tag{38}$$

$$\widetilde{H}_7(\xi,\tau) = -\xi\sqrt{\frac{2}{\pi\tau}}\left(q - \frac{w\xi}{\tau}\right)e^{-w\xi - w^2\frac{\tau}{2} - \frac{\xi^2}{2\tau}} + \xi e^{-w\xi + i(1-k)\tau}\left[(\eta_6 - \eta_7)e^{(a_4+ib_4)\xi}\right.$$
$$\left.\times \operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} + (a_4+ib_4)\sqrt{\frac{\tau}{2}}\right) + (\eta_6+\eta_7)e^{-(a_4+ib_4)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} - (a_4+ib_4)\sqrt{\frac{\tau}{2}}\right)\right], \tag{39}$$

$$\widetilde{H}_8(\xi,\tau) = -\frac{e^{-w\xi}}{2}\left[e^{(a_1+ib_1)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(a_1+ib_1)\sqrt{\frac{\tau}{2}}\right)+e^{-(a_1+ib_1)\xi}\right.$$
$$\left.\times\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(a_1+ib_1)\sqrt{\frac{\tau}{2}}\right)\right]-\frac{iU}{4\Omega l}e^{-w\xi+ik\tau}\left[e^{(a_2+ib_2)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(a_2+ib_2)\sqrt{\frac{\tau}{2}}\right)\right.$$
$$\left.+e^{-(a_2+ib_2)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(a_2+ib_2)\sqrt{\frac{\tau}{2}}\right)\right]+\frac{iU}{4\Omega l}e^{-w\xi-ik\tau}\left[e^{(a_4+ib_4)\xi}\right.$$
$$\left.\times\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(a_4+ib_4)\sqrt{\frac{\tau}{2}}\right)+e^{-(a_4+ib_4)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(a_4+ib_4)\sqrt{\frac{\tau}{2}}\right)\right], \tag{40}$$

$$p = \frac{i(1-k)}{2} - \frac{7w^2}{4}, \tag{41}$$

$$q = \frac{i(1+k)}{2} - \frac{7w^2}{4}, \tag{42}$$

$$a_1 = \left[\frac{\sqrt{w^4+4}+w^2}{2}\right]^{\frac{1}{2}}, \tag{43}$$

$$a_2 = \left[\frac{\sqrt{w^4+4(k+1)^2}+w^2}{2}\right]^{\frac{1}{2}}, \tag{44}$$

$$a_3 = \left[\frac{\sqrt{w^4+4(k-1)^2}+w^2}{2}\right]^{\frac{1}{2}}, \tag{45}$$

$$a_4 = \left[\frac{\sqrt{w^4+4(1-k)^2}+w^2}{2}\right]^{\frac{1}{2}}, \tag{46}$$

$$b_1 = \left[\frac{\sqrt{w^4+4}-w^2}{2}\right]^{\frac{1}{2}}, \tag{47}$$

$$b_2 = \left[\frac{\sqrt{w^4+4(k+1)^2}-w^2}{2}\right]^{\frac{1}{2}}, \tag{48}$$

$$b_3 = \left[\frac{\sqrt{w^4 + 4(k-1)^2} - w^2}{2}\right]^{\frac{1}{2}}, \tag{49}$$

$$b_4 = \left[\frac{\sqrt{w^4 + 4(1-k)^2} - w^2}{2}\right]^{\frac{1}{2}}, \tag{50}$$

$$\eta_1 = (a_1 + \mathrm{i}b_1)\left(\frac{\mathrm{i}}{4} - w^2\right) + \frac{3\mathrm{i}w^2}{4(a_1 + \mathrm{i}b_1)}, \tag{51}$$

$$\eta_2 = -w^3, \tag{52}$$

$$\eta_3 = (a_2 + \mathrm{i}b_2)\left[w^2 + \frac{(k+1)\mathrm{i}}{4} - \frac{\mathrm{i}}{2}\right] - \frac{\mathrm{i}w^2(k+1+2)}{4(a_2 + \mathrm{i}b_2)}, \tag{53}$$

$$\eta_4 = w^3 + \mathrm{i}kw, \tag{54}$$

$$\eta_5 = (a_3 + \mathrm{i}b_3)\left[w^2 - \frac{(k-1)\mathrm{i}}{4} - \frac{\mathrm{i}}{2}\right] + \frac{\mathrm{i}w^2(k-1-2)}{4(a_3 + \mathrm{i}b_3)}, \tag{55}$$

$$\eta_6 = w^3 - \mathrm{i}wk, \tag{56}$$

$$\eta_7 = (a_4 + \mathrm{i}b_4)\left[w^2 + \frac{(1-k)\mathrm{i}}{4} - \frac{\mathrm{i}}{2}\right] - \frac{\mathrm{i}w^2(1-k+2)}{4(a_4 + \mathrm{i}b_4)}. \tag{57}$$

Using

$$F(\xi,\tau) = H(\xi,\tau)\mathrm{e}^{-\mathrm{i}\tau} \tag{58}$$

we can write from Eqs. (29)–(32) as

$$F(\xi,\tau) = \widetilde{H}_1(\xi,\tau) + \beta\mathrm{e}^{-\mathrm{i}\tau}\widetilde{H}_2(\xi,\tau) + \frac{\beta U\mathrm{e}^{-\mathrm{i}\tau}}{2\Omega l}(\widetilde{H}_3(\xi,\tau) + \widetilde{H}_4(\xi,\tau)), \quad U\cos nt,\ k > 1, \tag{59}$$

$$F(\xi,\tau) = \widetilde{H}_5(\xi,\tau) + \beta\mathrm{e}^{-\mathrm{i}\tau}\widetilde{H}_2(\xi,\tau) + \frac{\mathrm{i}\beta U\mathrm{e}^{-\mathrm{i}\tau}}{2\Omega l}(\widetilde{H}_3(\xi,\tau) - \widetilde{H}_4(\xi,\tau)), \quad U\sin nt,\ k > 1, \tag{60}$$

$$F(\xi,\tau) = \widetilde{H}_6(\xi,\tau) + \beta\mathrm{e}^{-\mathrm{i}\tau}\widetilde{H}_2(\xi,\tau) + \frac{\beta U\mathrm{e}^{-\mathrm{i}\tau}}{2\Omega l}(\widetilde{H}_3(\xi,\tau) + \widetilde{H}_7(\xi,\tau)), \quad U\cos nt,\ k < 1, \tag{61}$$

$$F(\xi,\tau) = \widetilde{H}_8(\xi,\tau) + \beta\mathrm{e}^{-\mathrm{i}\tau}\widetilde{H}_2(\xi,\tau) + \frac{\mathrm{i}\beta U\mathrm{e}^{-\mathrm{i}\tau}}{2\Omega l}(\widetilde{H}_3(\xi,\tau) - \widetilde{H}_7(\xi,\tau)), \quad U\sin nt,\ k < 1. \tag{62}$$

In above equations, we give exact solutions of zeroth and first order, but in view of the lack of convergence it is not sure that the obtained solution is close to the full solution.

With the help of Eq. (9), above results become

$$\frac{f+\mathrm{i}g}{\Omega l}=\begin{bmatrix} 1+\widetilde{H}_1(\xi,\tau)+\beta \mathrm{e}^{-\mathrm{i}\tau}\widetilde{H}_2(\xi,\tau)+\frac{\beta U \mathrm{e}^{-\mathrm{i}\tau}}{2\Omega l}(\widetilde{H}_3(\xi,\tau)+\widetilde{H}_4(\xi,\tau)), & U\cos nt, \ k>1 \\ 1+\widetilde{H}_5(\xi,\tau)+\beta \mathrm{e}^{-\mathrm{i}\tau}\widetilde{H}_2(\xi,\tau)+\frac{\mathrm{i}\beta U \mathrm{e}^{-\mathrm{i}\tau}}{2\Omega l}(\widetilde{H}_3(\xi,\tau)-\widetilde{H}_4(\xi,\tau)), & U\sin nt, \ k>1 \\ 1+\widetilde{H}_6(\xi,\tau)+\beta \mathrm{e}^{-\mathrm{i}\tau}\widetilde{H}_2(\xi,\tau)+\frac{\beta U \mathrm{e}^{-\mathrm{i}\tau}}{2\Omega l}(\widetilde{H}_3(\xi,\tau)+\widetilde{H}_7(\xi,\tau)), & U\cos nt, \ k<1 \\ 1+\widetilde{H}_8(\xi,\tau)+\beta \mathrm{e}^{-\mathrm{i}\tau}\widetilde{H}_2(\xi,\tau)+\frac{\mathrm{i}\beta U \mathrm{e}^{-\mathrm{i}\tau}}{2\Omega l}(\widetilde{H}_3(\xi,\tau)-\widetilde{H}_7(\xi,\tau)), & U\sin nt, \ k<1 \end{bmatrix}. \tag{63}$$

The real part gives $\frac{f}{\Omega l}$ and imaginary part gives $\frac{g}{\Omega l}$. The results for blowing can be obtained by replacing w with $-w$.

For resonant case $k=1$ and the solution in this case is given by

$$\begin{aligned} \frac{f+\mathrm{i}g}{\Omega l}&=1+\widehat{H}_1(\xi,\tau)+\beta \mathrm{e}^{-\mathrm{i}\tau}\widetilde{H}_2(\xi,\tau)+\frac{\beta U \mathrm{e}^{-\mathrm{i}\tau}}{2\Omega l}(\widehat{H}_3(\xi,\tau)+\widehat{H}_4(\xi,\tau)), \quad U\cos nt, \\ \frac{f+\mathrm{i}g}{\Omega l}&=1+\widehat{H}_5(\xi,\tau)+\beta \mathrm{e}^{-\mathrm{i}\tau}\widetilde{H}_2(\xi,\tau)+\frac{\mathrm{i}\beta U \mathrm{e}^{-\mathrm{i}\tau}}{2\Omega l}(\widehat{H}_3(\xi,\tau)-\widehat{H}_4(\xi,\tau)), \quad U\sin nt, \end{aligned} \tag{64}$$

where

$$\begin{aligned} \widehat{H}_1(\xi,\tau)={}&\frac{-\mathrm{e}^{-w\xi}}{2}\left[\mathrm{e}^{(a_1+\mathrm{i}b_1)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(a_1+\mathrm{i}b_1)\sqrt{\frac{\tau}{2}}\right)+\mathrm{e}^{-(a_1+\mathrm{i}b_1)\xi}\right.\\ &\left.\times\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(a_1+\mathrm{i}b_1)\sqrt{\frac{\tau}{2}}\right)\right]+\frac{U\mathrm{e}^{-w\xi+\mathrm{i}\tau}}{4\Omega l}\left[\mathrm{e}^{(\hat{a}_2+\mathrm{i}\hat{b}_2)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(\hat{a}_2+\mathrm{i}\hat{b}_2)\sqrt{\frac{\tau}{2}}\right)\right.\\ &\left.+\mathrm{e}^{-(\hat{a}_2+\mathrm{i}\hat{b}_2)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(\hat{a}_2+\mathrm{i}\hat{b}_2)\sqrt{\frac{\tau}{2}}\right)\right]+\frac{U\mathrm{e}^{-w\xi-\mathrm{i}\tau}}{4\Omega l}\left[\mathrm{e}^{w\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+\mathrm{i}w\sqrt{\frac{\tau}{2}}\right)\right.\\ &\left.+\mathrm{e}^{-w\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-\mathrm{i}w\sqrt{\frac{\tau}{2}}\right)\right], \end{aligned}$$

$$\begin{aligned} \widehat{H}_3(\xi,\tau)={}&-\xi\sqrt{\frac{2}{\pi\tau}}\left(\hat{p}-\frac{w\xi}{\tau}\right)\mathrm{e}^{-w\xi-w^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}+\xi\mathrm{e}^{-w\xi+2\mathrm{i}\tau}\left[(\widehat{\eta}_4-\widehat{\eta}_3)\mathrm{e}^{(\hat{a}_2+\mathrm{i}\hat{b}_2)\xi}\right.\\ &\left.\times\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(\hat{a}_2+\mathrm{i}\hat{b}_2)\sqrt{\frac{\tau}{2}}\right)+(\widehat{\eta}_4+\widehat{\eta}_3)\mathrm{e}^{-(\hat{a}_2+\mathrm{i}\hat{b}_2)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(\hat{a}_2+\mathrm{i}\hat{b}_2)\sqrt{\frac{\tau}{2}}\right)\right], \end{aligned}$$

$$\widehat{H}_4(\xi,\tau)=-\xi\sqrt{\frac{2}{\pi\tau}}\left(\hat{q}-\frac{w\xi}{\tau}\right)\mathrm{e}^{-w\xi-w^2\frac{\tau}{2}-\frac{\xi^2}{2\tau}}+2\widehat{\eta}_5\xi\mathrm{e}^{-2w\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-\mathrm{i}w\sqrt{\frac{\tau}{2}}\right),$$

$$\widehat{H}_5(\xi,\tau) = -\frac{e^{-w\xi}}{2}\left[e^{(a_1+ib_1)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(a_1+ib_1)\sqrt{\frac{\tau}{2}}\right)+e^{-(a_1+ib_1)\xi}\right.$$
$$\left.\times\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(a_1+ib_1)\sqrt{\frac{\tau}{2}}\right)\right]-\frac{iU}{4\Omega l}e^{-w\xi+i\tau}\left[e^{(\hat{a}_2+i\hat{b}_2)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+(\hat{a}_2+i\hat{b}_2)\sqrt{\frac{\tau}{2}}\right)\right.$$
$$\left.+e^{-(\hat{a}_2+i\hat{b}_2)\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-(\hat{a}_2+i\hat{b}_2)\sqrt{\frac{\tau}{2}}\right)\right]+\frac{iU}{4\Omega l}e^{-w\xi-i\tau}\left[e^{w\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}+iw\sqrt{\frac{\tau}{2}}\right)\right.$$
$$\left.+e^{-w\xi}\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}}-iw\sqrt{\frac{\tau}{2}}\right)\right]$$

and $\widetilde{H}_2$ is given by Eq. (34). In above expressions

$$\hat{p} = -\frac{7w^2}{4},$$

$$\hat{q} = i - \frac{7w^2}{4},$$

$$\widehat{\eta}_3 = (\hat{a}_2+i\hat{b}_2)w^2 - \frac{iw^2}{(\hat{a}_2+i\hat{b}_2)},$$

$$\widehat{\eta}_4 = w^3 + iw,$$

$$\widehat{\eta}_5 = w^3 - iw,$$

$$\hat{a}_2 = \left[\frac{\sqrt{w^4+16}+w^2}{2}\right]^{\frac{1}{2}},$$

$$\hat{b}_2 = \left[\frac{\sqrt{w^4+16}-w^2}{2}\right]^{\frac{1}{2}}.$$

4. Discussion

The starting solution for the case of suction is

$$H = H_1 + \beta H_2. \tag{65}$$

The above solution describes the general features of the unsteady boundary layer flow in a fluid bounded by a porous disk for small values of β. This solution clearly brings out the contribution due to the material parameter of the second grade fluid. It should be noted that Newtonian solution [12] can be recovered as a special case for $\beta = 0$. We also note that for $\beta = 0 = U$, the velocity field is identical to that of Erdogan [11]. This provides a useful mathematical check.

The solution given by Eq. (63) is general and independent of the assumption of the form of the steady state solution. In order to determine the steady structure of the solution (63) we use the asymptotic formula for the complementary error function i.e. when τ goes to infinity we find that

$$\operatorname{erf}c\left(\frac{\xi}{\sqrt{2\tau}} \pm (a_j + \mathrm{i}b_j)\sqrt{\frac{\tau}{2}}\right) \to (0,2), \quad j = 1,2,3,4$$

and solution (63) takes the following form:

For $U\cos nt$, $k > 1$

$$\frac{f_s + \mathrm{i}g_s}{\Omega l} = \left[1 + [-1 + 2\beta\xi(\eta_1 + \eta_2)]\mathrm{e}^{-w\xi-(a_1+\mathrm{i}b_1)\xi} + \frac{U}{2\Omega l}[1 + 2\beta\xi(\eta_3 + \eta_4)]\mathrm{e}^{-w\xi-(a_2+\mathrm{i}b_2)\xi+\mathrm{i}k\tau} + \frac{U}{2\Omega l}[1 + 2\beta\xi(\eta_5 + \eta_6)]\mathrm{e}^{-w\xi-(a_3+\mathrm{i}b_3)\xi-\mathrm{i}k\tau}\right]. \tag{66}$$

For $U\sin nt$, $k > 1$

$$\frac{f_s + \mathrm{i}g_s}{\Omega l} = \left[1 + [-1 + 2\beta\xi(\eta_1 + \eta_2)]\mathrm{e}^{-w\xi-(a_1+\mathrm{i}b_1)\xi} + \frac{\mathrm{i}U}{2\Omega l}[-1 + 2\beta\xi(\eta_3 + \eta_4)]\mathrm{e}^{-w\xi-(a_2+\mathrm{i}b_2)\xi+\mathrm{i}k\tau} - \frac{\mathrm{i}U}{2\Omega l}[-1 + 2\beta\xi(\eta_5 + \eta_6)]\mathrm{e}^{-w\xi-(a_3+\mathrm{i}b_3)\xi-\mathrm{i}k\tau}\right]. \tag{67}$$

For $U\cos nt$, $k < 1$

$$\frac{f_s + \mathrm{i}g_s}{\Omega l} = \left[1 + [-1 + 2\beta\xi(\eta_1 + \eta_2)]\mathrm{e}^{-w\xi-(a_1+\mathrm{i}b_1)\xi} + \frac{U}{2\Omega l}[1 + 2\beta\xi(\eta_3 + \eta_4)]\mathrm{e}^{-w\xi-(a_2+\mathrm{i}b_2)\xi+\mathrm{i}k\tau} + \frac{U}{2\Omega l}[1 + 2\beta\xi(\eta_6 + \eta_7)]\mathrm{e}^{-w\xi-(a_4+\mathrm{i}b_4)\xi-\mathrm{i}k\tau}\right]. \tag{68}$$

For $U\sin nt$, $k < 1$

$$\frac{f_s + \mathrm{i}g_s}{\Omega l} = \left[1 + [-1 + 2\beta\xi(\eta_1 + \eta_2)]\mathrm{e}^{-w\xi-(a_1+\mathrm{i}b_1)\xi} + \frac{\mathrm{i}U}{2\Omega l}[-1 + 2\beta\xi(\eta_3 + \eta_4)]\mathrm{e}^{-w\xi-(a_2+\mathrm{i}b_2)\xi+\mathrm{i}k\tau} - \frac{\mathrm{i}U}{2\Omega l}[-1 + 2\beta\xi(\eta_6 + \eta_7)]\mathrm{e}^{-w\xi-(a_4+\mathrm{i}b_4)\xi-\mathrm{i}k\tau}\right], \tag{69}$$

where subscript 's' denotes the steady state situation. Clearly, the steady solutions (66)–(69) are independent of the initial condition and are periodic in time. For some time after the initiation of the motion, the velocity field contains transients and they gradually disappear in time. The transient solution is obtained by the subtraction of steady state solutions from Eq. (63). i.e.

$$\frac{f_t + \mathrm{i}g_t}{\Omega l} = \frac{f + \mathrm{i}g}{\Omega l} - \left(\frac{f_s + \mathrm{i}g_s}{\Omega l}\right). \tag{70}$$

In above expression subscript 't' represents the transient solutions. It is seen that for large time, the transient solution (70) disappears. Further, expressions (66)–(69) show the existence of four distinct boundary layers of thicknesses of order $(w + a_j)^{-1}$, $j = 1,2,3,4$. It is interesting to note that the associated boundary layers are modified due to the presence of w. These thicknesses decrease with an increase of the suction parameter and the rotation. It is further noted that these thicknesses remain bounded for all values of the frequency of the imposed oscillations. When $k=0$, $a_1=a_2=a_3=a_4$ and the four boundary layers coalesces into a single layer of thickness $(w+a_1)^{-1}$. In particular, when $U=0$, solutions (66)–(69) reduce to

$$\frac{f_s+\mathrm{i}g_s}{\Omega l}=1+[-1+2\beta\xi(\eta_1+\eta_2)]\mathrm{e}^{-w\xi-(a_1+\mathrm{i}b_1)\xi},$$

which for $\beta=0=w$ corresponds to Erdogan's result [12].

For resonant case, we have from Eq. (64)

For $U\cos nt$

$$\frac{f_s+\mathrm{i}g_s}{\Omega l}=\left[1+[-1+2\beta\xi(\eta_1+\eta_2)]\mathrm{e}^{-w\xi-(a_1+\mathrm{i}b_1)\xi}+\frac{U}{2\Omega l}[1+2\beta\xi(\widehat{\eta}_3+\widehat{\eta}_4)]\mathrm{e}^{-w\xi-(\hat{a}_2+\mathrm{i}\hat{b}_2)\xi+\mathrm{i}\tau}\right.$$
$$\left.+\frac{U}{2\Omega l}[1+4\beta\xi\widehat{\eta}_5]\mathrm{e}^{-2w\xi-\mathrm{i}\tau}\right]. \tag{71}$$

For $U\sin nt$

$$\frac{f_s+\mathrm{i}g_s}{\Omega l}=\left[1+[-1+2\beta\xi(\eta_1+\eta_2)]\mathrm{e}^{-w\xi-(a_1+\mathrm{i}b_1)\xi}+\frac{\mathrm{i}U}{2\Omega l}[-1+2\beta\xi(\widehat{\eta}_3+\widehat{\eta}_4)]\mathrm{e}^{-w\xi-(\hat{a}_2+\mathrm{i}\hat{b}_2)\xi+\mathrm{i}\tau}\right.$$
$$\left.-\frac{\mathrm{i}U}{2\Omega l}[1+4\beta\xi\widehat{\eta}_5]\mathrm{e}^{-2w\xi-\mathrm{i}\tau}\right]. \tag{72}$$

From above solutions we note that in the presence of suction ($w\neq 0$), the steady solution in the resonant case ($k\to 1$) does not depend on the order of the double limit operation $\tau\to\infty$, $k\to 1$. In fact, the double limiting procedure with $w\neq 0$ gives

$$\mathrm{Lim}_{k\to 1}\mathrm{Lim}_{\tau\to\infty}\left(\frac{f+\mathrm{i}g}{\Omega l}\right)=\mathrm{Lim}_{\tau\to\infty}\mathrm{Lim}_{k\to 1}\left(\frac{f+\mathrm{i}g}{\Omega l}\right).$$

From Eqs. (71) and (72), it is also noted that steady asymptotic blowing solution for resonant case does not exist. The physical mechanism of non-existence of the solution is that the blowing causes thickening of the boundary layer. At sufficiently large distance from the leading edge the boundary layer becomes so thick that it becomes turbulent.

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Homotopy Solution for the Channel Flow of a Third Grade Fluid

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Abstract. The solution for the flow of a third grade fluid bounded by two parallel porous plates is given using homotopy analysis method (HAM). A comparison is made with the exact numerical solution for the various values of the physical parameters. It is found that a proper choice of the auxiliary parameter occurring in HAM solution gives very close results.

Key words: Third grade fluid, HAM, porous channel

1. Introduction

In numerous technological applications, the fluids in use do not follow the commonly assumed linear relationship between the stress and the rate of strain at a point. Such fluids have come to be known as non-Newtonian fluids. One of the widely accepted models amongst non-Newtonian fluids is the class of viscoelastic fluids which have their constitutive equations based on sound theoretical foundations. In general, they can have an arbitrary number of parameters-the addition of each parameter aiding in explaining away same features of the flow that could not be explained otherwise.

The earliest model of viscoelastic fluids was proposed by Rivlin and Ericksen [1], which included two parameters α_1 and α_2 in the constitutive equation besides the Newtonian viscosity μ. The inclusion of the parameter α_1, in particular, leads to some spectacular ramifications in the solution of the flow problems. In most of the flows the order of the differential equations governing the motion is raised by at least one, while there is no corresponding increase in the number of boundary conditions. Though there have been several proposals regarding the extra boundary conditions, at the time of writing, there is no consensus amongst the researchers on the acceptability of any of the proposed boundary condition. Under the circumstances, the solutions of the flow problems have to be obtained on the basis of some plausible assumptions regarding the behaviour of the solution for values of α_1 close to zero.

The earlier attempts were mostly centered around the first order perturbation solutions in terms of the non-Newtonian fluid parameters – the classical case being the two dimensional flow near a stagnation point [2]. The arguments advanced for the use of the first order terms are not the higher order terms are neglected in the derivation of the constitutive equations. What the proponents of such arguments overlook is the fact that once the equations of motion have been derived then they ought to be solved in their entirely. The validity of the perturbation solution can only be judged by comparing it with the solution of the full equations of motion without making any assumption on the size of the parameters in the equations of motion. For the aforementioned problem of two dimensional flow near a stagnation point several discrepancies were noted by Serth [3], Teipel [4] and Ariel [5] who obtained the solutions without using the first order perturbation technique.

The solution for the stagnation point flow was facilitated by the fact that the boundary value problem describing the motion, even though it was of higher order, was singular at the boundary. Such an aid is not available when the flow takes place between parallel porous plates. Now there is an increase in the order of differential equation, but the highest derivative is multiplied by α_1. Because of this, a different approach must be chosen that involves the pruning of the spurious solution introduced on account of the α_1-term in the differential equation. Ariel [6] derived the solutions for the flow of a viscoelastic fluid between parallel boundaries when there is an injection of the fluid at one boundary and an equal suction at the other boundary.

The inclusion of some more parameters in the constitutive equations, known as the third grade fluid parameters, makes the model more realistic. However it adds a new dimension in the solution processes of the flow problems, namely, non-linearity. For the flow between parallel plates, Ariel [7] derived an interesting method for computing the flow by seeking the solution in a series of exponential terms. He was able to obtain the solution for a combination of values of physical parameter, but the performance of the series solution degraded sharply as the value of the third grade fluid parameter was increased. This warranted the search of alternate methods for computing the flow of the third grade fluids.

For the last decade, the perturbation methods have received a fresh impetus on account of some pioneering work by Liao [8]. He introduced a newly developed method known as HAM which has led to much improved solutions of several problems in fluid mechanics. These solutions were earlier limited in their applicability on account of the small radius of convergence of the perturbation solution. In HAM the convergence can be dramatically improved by properly choosing the value of an auxiliary parameter $\hbar$, occurring in the solution. As a demonstration of the usefulness of the HAM, one may refer to Liao's solution [9] of the Blasius problem in the form of a series which converges throughout the domain of interest. Recently, Ayub et al. [10] derived the solution of the steady flow of a third grade fluid past a porous plate by using HAM.

In the present paper we apply the HAM to obtain the solution of the laminar flow of a third grade fluid through a flat porous channel when the rate of injection at one wall is equal to the rate of suction at the other wall. The flow is caused by the external pressure gradient. The solution is expressed in terms of the auxiliary parameter $\hbar$, which is then varied to determine its optimum value. For the said values of $\hbar$, the solution obtained in the present paper, it is believed, is applicable for all values of the physical parameters.

2. Mathematical Formulation

Consider the steady flow of a third grade fluid between two porous walls at $y = a$ and $y = b$. The flow is due to a constant pressure gradient. Also, there is cross flow because of uniform injection of the fluid at lower wall with velocity v_0 and an equal suction at the upper wall. For third grade fluids, physical considerations were taken into account by Fosdick and Rajagopal [11] in order to obtain the following form for the constitutive law:

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1, \tag{1}$$

which, when introduced in the equation of conservation of momentum leads to the following equation [7]:

$$\mu\frac{d^2u}{dy^2} + \alpha_1 v_0\frac{d^3u}{dy^3} - \rho v_0\frac{du}{dy} + 6\beta_3\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} = \frac{\partial p}{\partial x}. \tag{2}$$

Moreover, the coefficients μ, α_1, α_2 and β_3 must satisfy the following inequalities:

$$\mu \geq 0, \quad \alpha_1 > 0, \quad \beta_3 \geq 0 \quad \text{and} \quad |\alpha_1 + \alpha_2| \leq \sqrt{24\mu\beta_3}. \tag{3}$$

The boundary conditions are

$$u(a) = u(b) = 0. \tag{4}$$

Defining the non-dimensional variables

$$\eta = \frac{y}{b} \quad \text{and} \quad U = -\frac{\mu u}{b^2}\left(\frac{\partial p}{\partial x}\right)^{-1} \tag{5}$$

and substituting Equation (5) in Equation (2) and boundary conditions (4) we obtain

$$KRU''' + U'' - RU' + TU'^2U'' = -1, \tag{6}$$

$$U(\sigma) = U(1) = 0, \tag{7}$$

in which

$$\sigma = \frac{a}{b}, \quad R = \frac{\rho v_0 b}{\mu}, \quad K = \frac{\alpha_1}{\rho b^2}, \quad T = \frac{6\beta_3 b^2\left(\frac{\partial p}{\partial x}\right)^2}{\mu^3} \tag{8}$$

and primes denote differentiation with respect to η (incidentally the value of T in [7] – is incorrectly reported).

3. Homotopy Solution

In view of Equation (6) and boundary conditions (7), we take the initial guess of $u(\eta)$ of the following form

$$U_0 = A_0 + B_0 f(\eta) + C_0 g(\eta) + \frac{\eta}{R} \tag{9}$$

which satisfies

$$U_0(\sigma) = U_0(1) = 0, \tag{10}$$

and

$$L = KR\frac{d^3}{d\eta^3} + \frac{d^2}{d\eta^2} - R\frac{d}{d\eta} \tag{11}$$

as the auxiliary linear operator. In the initial guess (9), A_0, B_0 and C_0 are the constants of integration and

$$f(\eta) = e^{m\eta}, \quad g(\eta) = e^{\bar{m}\eta} \tag{12}$$

in which m and $\bar{m}$ are the roots of

$$KRm^2 + m - R = 0 \tag{13}$$

and are taken as

$$m = \frac{\sqrt{4KR^2+1}-1}{2KR}, \quad \bar{m} = -\frac{\sqrt{4KR^2+1}+1}{2KR}. \tag{14}$$

For small K (or R) one can write

$$m = R - KR^3 + O(K^2R^5), \tag{15}$$

$$\bar{m} = \frac{1}{KR} - R + +O(KR^3). \tag{16}$$

Now the independent solution $f(\eta)$ matches with the solution for a corresponding Newtonian fluid. Thus, from Equations (9), (10), (15) and (16) we obtain

$$U_0 = A_0 + B_0 e^{m\eta} + \frac{\eta}{R}, \tag{17}$$

where

$$A_0 = \frac{e^{m\sigma} - \sigma e^m}{R(e^m - e^{m\sigma})}, \quad B_0 = -\frac{1-\sigma}{R(e^m - e^{m\sigma})}.$$

This solution was first derived by Ariel [6] for a second grade fluid.

Denoting $\hbar$ as an auxiliary parameter, we can construct the zeroth-order deformation equation as

$$(1-p)L[\bar{U}(\eta;p) - U_0(\eta)] = p\hbar\left[KR\frac{\partial^3\bar{U}}{\partial\eta^3} + \frac{\partial^2\bar{U}}{\partial\eta^2} - R\frac{\partial\bar{U}}{\partial\eta} + T\left(\frac{\partial\bar{U}}{\partial\eta}\right)^2\frac{\partial^2\bar{U}}{\partial\eta^2} + 1\right] \tag{18}$$

with the boundary conditions

$$\bar{U}(\sigma;p) = 0, \quad \bar{U}(1;p) = 0, \tag{19}$$

in which $p \in [0, 1]$ is an embedding parameter. For $p = 0$ the solution of Equations (18) and (19) is

$$\bar{U}(\eta;0) = U_0(\eta) \tag{20}$$

and for $p = 1$, the Equations (18) and (19) are equivalent to Equations (6) and (7) so that

$$\bar{U}(\eta;1) = U(\eta). \tag{21}$$

Clearly when p increases from 0 to 1, $\bar{U}(\eta;p)$ varies (or deforms) from $U_0(\eta)$ to $U(\eta)$ governed by the Equations (6) and (7). We have great freedom to choose $\hbar$. Assume that the deformation $\bar{U}(\eta;p)$ governed by Equations (18) and (19) is smooth enough so that

$$U_0^{[K]}(\eta) = \frac{1}{k!}\frac{\partial^k\bar{U}(\eta;p)}{\partial p^k}\bigg|_{p=0}, \quad k \geq 1. \tag{22}$$

By Taylor's theorem and Equation (20), we can write

$$\bar{U}(\eta;p) = U_0(\eta) + \sum_{k=1}^{\infty} U_k(\eta)p^k, \tag{23}$$

in which

$$U_k(\eta) = \frac{1}{k!}\frac{\partial^k U(\eta;p)}{\partial p^k}\bigg|_{p=0}. \tag{24}$$

Assume that the above series is convergent when $p = 1$, we have, from Equation (21)

$$U(\eta) = U_0(\eta) + \sum_{k=1}^{\infty} U_k(\eta). \tag{25}$$

Differentiating Equations (18) and (19) with respect to p and then setting $p = 0$ and using Equations (20) and (22), we have the following first-order deformation problem

$$LU_1 = \hbar\left[KR\frac{d^3U_0}{d\eta^3} + \frac{d^2U_0}{d\eta^2} - R\frac{dU_0}{d\eta} + T\left(\frac{dU_0}{d\eta}\right)^2\frac{d^2U_0}{d\eta^2} + 1\right], \tag{26}$$

$$U_1(\sigma) = U_1(1) = 0. \tag{27}$$

Solving the above first-order deformation problem one obtains

$$U_1 = A_1 + B_1e^{m\eta} + C_1\eta e^{m\eta} + D_1e^{2m\eta} + E_1e^{3m\eta}, \tag{28}$$

where

$$A_1 = \frac{e^{(1+\sigma)m}}{e^m - e^{m\sigma}}[C_1(1-\sigma) + D_1(e^m - e^{m\sigma}) + E_1(e^{2m} - e^{2m\sigma})],$$

$$B_1 = \frac{1}{e^m - e^{m\sigma}}[C_1(e^m - \sigma e^{m\sigma}) + D_1(e^{2m} - e^{2m\sigma}) + E_1(e^{3m} - e^{3m\sigma})],$$

$$C_1 = \frac{\hbar T B_0 m}{R^2(2KRm+1)}, \quad D_1 = \frac{\hbar T B_0^2 m}{R(3KRm+1)}, \quad E_1 = \frac{\hbar T B_0^3 m^2}{6(4KRm+1)}.$$

Now differentiating Equations (18) and (19) with respect to p twice and setting $p = 0$ and using the relations (20) and (22) we have the second-order deformation problem as

$$LU_2 = \begin{matrix} KR(1+\hbar)\dfrac{d^3U_1}{d\eta^3} + (1+\hbar)\dfrac{d^2U_1}{d\eta^2} - R(1+\hbar)\dfrac{dU_1}{d\eta} \\ +T\hbar\left\{\left(\dfrac{dU_0}{d\eta}\right)^2\dfrac{d^2U_1}{d\eta^2} + 2\dfrac{d^2U_0}{d\eta^2}\dfrac{dU_0}{d\eta}\dfrac{dU_1}{d\eta}\right\}\end{matrix} \tag{29}$$

$$U_2(\sigma) = U_2(1) = 0. \tag{30}$$

The solution of the above problem is

$$U_2 = (1+\hbar)U_1 + W, \tag{31}$$

$$W = \begin{bmatrix} A_2 + (B_2 + C_2\eta + D_2\eta^2)e^{m\eta} + (E_2 + F_2\eta)e^{2m\eta} \\ +(G_2 + H_2\eta)e^{3m\eta} + I_2e^{4m\eta} + J_2e^{5m\eta}\end{bmatrix}, \tag{32}$$

where

$$A_2 = \frac{e^{(1+\sigma)m}}{e^m - e^{m\sigma}} \begin{bmatrix} C_2(1-\sigma) + D_2(1-\sigma^2) + E_2(e^m - e^{m\sigma}) \\ +F_2(e^m - \sigma e^{m\sigma}) + G_2(e^{2m} - e^{2m\sigma}) \\ +H_2(e^{2m} - \sigma e^{2m\sigma}) + I_2(e^{3m} - e^{3m\sigma}) + J_2(e^{4m} - e^{4m\sigma}) \end{bmatrix},$$

$$B_2 = -\frac{1}{e^m - e^{m\sigma}} \begin{bmatrix} C_2(e^m - \sigma e^{m\sigma}) + D_2(e^m - \sigma^2 e^{m\sigma}) \\ +E_2(e^{2m} - e^{2m\sigma}) + F_2(e^{2m} - \sigma e^{2m\sigma}) \\ +G_2(e^{3m} - e^{3m\sigma}) + H_2(e^{3m} - \sigma e^{3m\sigma}) \\ +I_2(e^{4m} - e^{4m\sigma}) + J_2(e^{5m} - e^{5m\sigma}) \end{bmatrix},$$

$$C_2 = \hbar T \left[\frac{B_1 m}{R^2(2KRm+1)} + \frac{C_1(KRm+1)}{R^2(2KRm+1)^2} \right], \quad D_2 = \frac{\hbar T C_1 m}{2R^2(2KRm+1)},$$

$$E_2 = \hbar T \left[\frac{2mB_0B_1R + 2D_1}{R^2(3KRm+1)} - \frac{2mB_0KC_1}{(3KRm+1)^2} \right], \quad F_2 = \frac{2mB_0\hbar T C_1}{R(3KRm+1)},$$

$$G_2 = \hbar T \left[\frac{m^2B_0^2B_1R^2 + 4mB_0D_1R + 3E_1}{2R^2(4KRm+1)} + \frac{mB_0^2C_1(2KRm+1)}{4(4KRm+1)^2} \right],$$

$$H_2 = \frac{\hbar T B_0^2 C_1 m^2}{2(4KRm+1)}, \quad I_2 = \frac{2mB_0\hbar T(mB_0D_1R + 3E_1)}{3R(5KRm+1)}, \quad J_2 = \frac{3\hbar T B_0^2 E_1 m^2}{4(6KRm+1)}.$$

The three term solution is

$$U = U_0 + U_1 + U_2. \tag{33}$$

For higher order deformation equations, we first differentiate (18) and (19) k times with respect to p then dividing by $k!$ and set $p = 0$. Here, the higher order deformation problem becomes

$$(1-p)L[U_k(\eta) - \chi_k U_{k-1}(\eta)] = \hbar \begin{bmatrix} KRU'''_{k-1} + U''_{k-1} - RU'_{k-1} \\ +T\sum_{j=0}^{k-1} U'_{k-1-j} \sum_{i=0}^{n} U'_{j-i}U''_i + \chi_k \end{bmatrix}, \tag{34}$$

$$U_k(\sigma) = U_k(0) = 0, \tag{35}$$

where

$$\chi_k = \begin{cases} 1, & k > 1, \\ 0 & k = 1 \end{cases} \tag{36}$$

4. The Convergence of the Solution

The explicit, analytic expression (33) contains the auxiliary parameter $\hbar$. As pointed out by Liao [8], the convergence region and rate of approximations given by the homotopy analysis method are strongly dependent upon the auxiliary parameter. In Figures 1 and 2 the $\hbar$-curves are plotted to see the range of admissible values for the parameter $\hbar$. It is clear from Figures 1 and 2 that the range for the admissible values for $\hbar$ is $-1 \leq \hbar < 0$. And the solution given in Equation (33) converges in the whole region of η, when $\hbar$ is in the neighborhood of -1.0. It is also observed that the interval for the values of $\hbar$

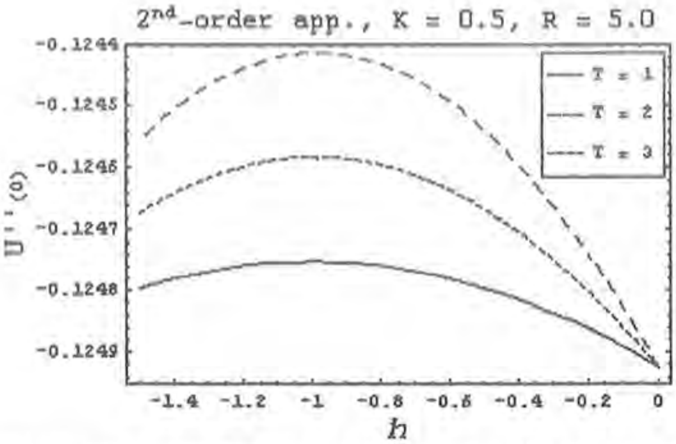

Figure 1. $\hbar$ – curve is plotted for the 2nd-order approximation of $U(\eta)$ for increasing T.

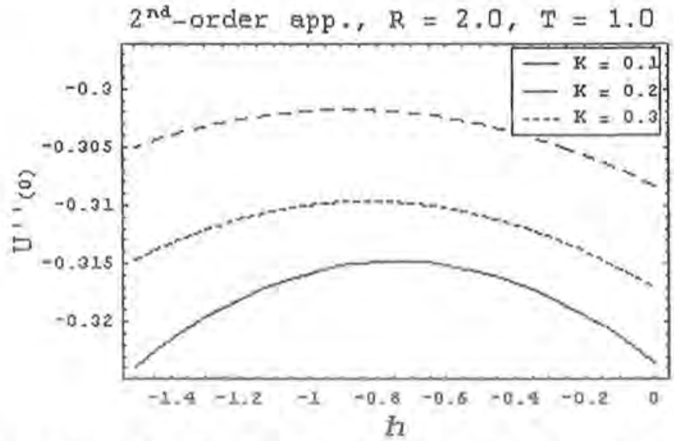

Figure 2. $\hbar$ – curve is plotted for the 2nd-order approximation of $U(\eta)$ for increasing K.

converges to value -1.0 as T increases keeping K and R fixed. It is also clear from Figure 2 that value of $\hbar$ decreases from -1.0 towards 0.0 as K increases.

5. Results and Discussion

From Table 1, we observe that the optimum value of $\hbar$ depends upon the physical parameters R, K, and T. In particular, we note that if R and K are small then the value of $\hbar$ must be chosen close to -0.83 reduced progressively as the value of T is increased. However, when R and K are moderate to large a value of $\hbar$ close to -0.96 gives the velocity profile very close to the one that was obtained numerically by Ariel [7].

Here it is worth mentioning that the HAM solution is valid for *all* values of the physical parameters R, K, and T. Therefore, it seems reasonable to assume that the HAM solution holds even for those values of the physical parameters for which Ariel [7] had a problem in obtaining the convergence of the series solution for large values of T. It is believed that as long as the above mentioned guidelines regarding the choice of $\hbar$ are adhered to, the HAM solution *must* provide a reasonably accurate solution

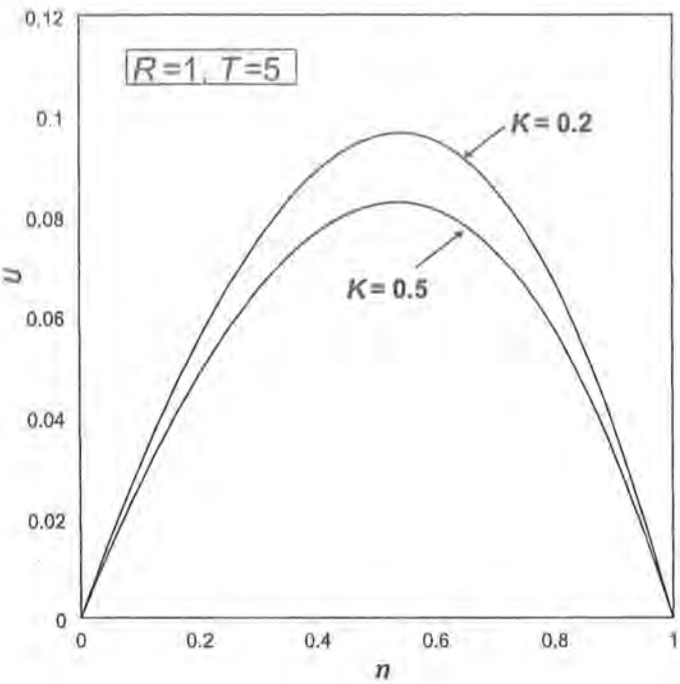

Figure 3. Variation of velocity profile for different values of K.

mentioned values of the parameters. Thus HAM offers an attractive alternative for computing the flow of viscoelastic fluids where numerical techniques fail to give the solution for varing reasons.

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