Some Aspects of Symmetries of Differential Equations and their Connection with the Underlying Geometry

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A Thesis
Submitted in Partial Fulfillment of the
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CERTIFICATE

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We accept this thesis as confirming to the required standard.

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2004

Dedicated

To

My Daughter

Nizalia

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Abstract

In this thesis symmetry methods have been used to solve some differential equations and to find the connection of isometries of some spaces with the symmetries of some related differential (geodesic) equations.

It is proved here that the McVittie solution and its non-static analogue are the only plane symmetric spacetimes with electromagnetic field. The Einstein equations for non-static, shear-free, spherically symmetric, perfect fluid distributions reduce to one second-order non-linear differential equation in the radial parameter. General solution of this equation is obtained in [11] by symmetry analysis. Corrections of some examples of the solution in the earlier work [11], by formulating a general requirement for physical relevance of the solution, are presented.

An algebraic proof that the Lie algebra of generators of the system of n differential equations, $\left(y^{a}\right)'' = 0$, is isomorphic to the Lie algebra of the special linear group of order $(n+2)$, over the real numbers, is provided. A connection between the symmetries of manifolds and their geodesic equations, which are systems of second order ordinary differential equations, is sought through the geodesic equations of maximally symmetric spaces. Since such spaces have either constant positive, constant negative or zero curvature, three cases are considered. It is proved that for a space admitting $so(n+1)$ or $so(n,1)$ as the maximal isometry algebra, the symmetry of the geodesic equations of the space is given by $so(n+1) \oplus d_2$ or $so(n,1) \oplus d_2$ (where d_2 is the 2-dimensional dilation algebra), while for those admitting $so(n) \oplus_s \mathbb{R}^n$ the algebra is $sl(n+2)$. A corresponding

result holds on replacing $so(n)$ by $so(p,q)$ with $p+q=n$. It is conjectured that if the isometry algebra of any underlying space of non-zero curvature is h, then the Lie symmetry algebra of the geodesic equations is given by $h \oplus d_2$ provided that there is no cross-section of zero curvature.

Some results on the Lie symmetry generators of equations with a small parameter and the relationship between symmetries and conservation laws for such equations are used to construct first integrals and Lagrangians for autonomous weakly non-linear systems. An adaptation of a theorem that provides the generators that leave the functional involving a Lagrangian for such equations is presented. A detailed example to illustrate the method is given.

List of Papers from the Thesis

1. Feroze, T., and Kara, A.H., *Group theoretic methods for approximate invariants and Lagrangians for some classes of* $y'' + \varepsilon F(t)y' + y = f(y, y')$, Int. J. Non-linear Mech., **37** (2002) 275-280.

2. Feroze, T., Mahomed, F.M., and Qadir, A., *Non-static, spherically symmetric, shear-free perfect-fluid sloutions of Einstein's equations*, Il Nuovo Cimento B, **118** **(9)** (2003) 895-902.

3. Feroze, T., Mahomed, F.M., and Qadir, A., *The Connection Between Isometries and Symmetries of Geodesic Equations of the Underlying Spaces*, (Submitted for publication).

4. Feroze, T., and Qadir, A., *An Alternate Proof for the Maximal Algebra of Second Order Vector Differential Equations*, (Submitted for publication).

5. Feroze, T., Qadir, A., and Ziad, M., *Uniqueness of the McVittie Solution as Plane Symmetric Sourceless Electromagnetic Field Spacetimes*, (Submitted for publication).

Contents

Chapter 1

Preliminaries

1.1 Introduction

Differential equations play a central role in most scientific and engineering problems, but their role is not limited to these, as they also have significance in other fields. They appear in mathematical models of various physical phenomena and engineering applications. Generally, the differential equations governing some phenomena are non-linear. Solutions of these equations then explain the phenomena and provide answers to these engineering problems. Differential equations have intrinsically interesting properties such as whether or not solutions exist, whether those solutions are unique and whether they are stable under perturbations.

There are many well known methods to find the solutions of different types of differential equations. However, most of the developed methods are for the particular classes of first and second order linear differential equations. For the solution of first order non-linear ordinary differential equations there are some known methods e.g.

Ricatti's and Bernoulli's methods, but these methods deal with a very small class of equations. In fact, one can not classify non-linear differential equations completely, therefore, it is not possible to develop different techniques for different classes of non-linear differential equations. However, one may try symmetry methods for solving different types of differential equations. This method deals with both linear and non-linear differential equations. In this thesis, the relation between the symmetries of the manifold and the symmetries of the geodesic equations of the underlying spaces is investigated. Some applications of symmetry methods in General Relativity are also discussed.

The history of differential equations is quite rich in its development. It is briefly given in the following subsection, before giving some basic definitions and concepts in the remaining sections of this chapter

1.1.1 Brief History of Differential Equations

The history begins in the 17th century, when **Sir Issac Newton** (1642–1727) discovered calculus in 1665-1666. He wrote his work in 1671 in Latin, whose English translation carries the title "*The method of fluxion and infinite series*", which was not published until 1736. His contemporary German mathematician, **Gottfried Wilhelm Leibniz** (1646-1716), developed his calculus and published the infinitesimal methods in *Acta Eruditorum* in two different articles, one in 1682 and the other in 1684. Thus, while Newton's techniques were developed first, Leibniz was the first to publish. Leibniz developed calculus in order to find methods by which discrete infinitesimal quantities could be summed up to calculate the area of a larger whole. Newton, on the other hand, had been occupied with problems of gravitation and planetary motion, and his mathematical methods attempted to understand motion and force in terms of infinitesimal changes with respect to time. Because of these distinct approaches, most modern commentators consider Leibniz as the inventor of integral calculus and Newton as the inventor of differential calculus. Other great mathematicians of this field in that century were the Swiss mathematicians **Jakob** (1654-1705) **and Johann Bernoulli** (1667-1748). They interpreted Leibniz's articles and extended his methods. Leibniz and his student Johann Bernoulli developed calculus into a magnificent tool for solving a variety of problems. In 1690 in a paper published in *Acta Eruditorum*, Jakob Bernoulli showed that the problem of determining the *isochrone* is equivalent to solving a first-order nonlinear differential equation. The isochrone, or curve of constant descent, is the curve along which a particle will descend under gravity from any point to the bottom in exactly the same time, no matter what the starting point. It had been studied by **Huygens** in 1687 and Leibniz in 1689. After finding the differential equation, Bernoulli then solved it by what we now call separation of variables. In 1696 Jakob Bernoulli solved the equation, now called the Bernoulli equation $y' + p(x)y = q(x)y^n$. These pioneers focused only on these special cases and did not generalize them.

In 1712, an Italian mathematician **Iacopo Francesco Ricatti** (1676-1754) introduced the method of solving a particular type of first order non-linear differential equation called the *Ricatti equation*. A Frenchman **Alexis-Claude Clairaut** (1713-1765) studied the differential equations now known as *Clairaut's differential equations* and gave a singular solution in addition to the general integral of the equations. In 1739 and 1740 he published further work on the integral calculus, proving the existence of integrating factors for solving first order differential equations. **Leonhard Euler** (1707-1783), a Swiss mathematician played a major role in the development of this subject. His contribution includes, development of many new functions based on series solutions of special types of differential equations; several methods of lowering the order of an equation; the concept of integrating factor and the theory of linear equations of arbitrary order. In 1739, he developed the method of variation of parameters. In 1799, **Pierre Simon Laplace** (1749-1827) , introduced the ideas of a Laplacian of a function. **Joseph Fourier** (1768-1830) developed the technique of expressing the continuous functions by trigonometric functions. **Taylor**, **D'Almbert**, **Lagrange**, **Legendre** and **Bessel** are the other remarkable mathematicians of this century.

The nineteenth century was very important in the sense that in it, people not only developed new methods but they also thought about the existence of solutions of differential equations. French mathematician **Augustin-Louis Cauchy** (1789-1857) gave this new thought about the existence and the uniqueness of the solution of a partial differential equation. He himself proved the first existence theorem. He invented the method of characteristics, which is important in the analysis and solution of many partial differential equations. As the end of the 19th century approached, the major efforts in differential equations moved into a more theoretical realm. In 1876, **Lipschitz** developed existence theorems for solutions of first order differential equations. **Hermite, Liouville, Riemann, Kovalevsky, Laguerue, Noether,**

Gauss and many other mathematicians developed new techniques for the solution. In the next section we shall see how the Norwegian mathematician **Sophus Lie,** used group theory to find the solutions of differential equations.

1.1.2 Group Theory in Differential Equations

The three main areas that were to give rise to group theory are geometry, number theory and the theory of algebraic equations.

The historical origin of group theory goes back to the work of **Evariste Galois** (1811-1832) in 1831. He was the first who really understood that the algebraic solutions of an equation are related to the structure of a group. Before that groups were mainly studied concretely, in the form of permutations; some aspects of abelian group theory were known in the theory of quadratic forms. Galois solved the ancient problem of finding a formula to solve a polynomial equation of arbitrary degree. Galois realized that roots of a polynomial equation (which form a finite set) have *symmetries*, and that these symmetries form a group. A *symmetry* is a transformation that preserves the form of the figure or symbolic object. For example we say that a ball has spherical symmetry since if we rotate it about an axis through its centre it does not change. Galois proved that there are polynomial equations (quintic and higher order) whose groups are not *solvable* and that the associated equations are not solvable by means of radicals.

Inspired by Galois, Lie tried to use groups for differential equations as Galois had done for polynomial equations. He developed a highly algorithmic method in

1870 for the solution of differential equations. He investigated the role of general transformation theory in classical integration method.

Lie studied groups of continuous transformations in Cartesian space, now known as *Lie groups of transformations* [1]. Such a group depends on continuous parameter(s) and consists of either point transformations (point symmetries) acting on the space of independent and dependent variables, or more generally, contact transformations (contact symmetries) acting on the space including all first derivatives of the dependent variable. These transformations satisfy all the axioms of a group and form a group of point (contact) transformations. (Transformations which map points (x, y) into points (x^*, y^*) are point transformations.) A simple example of 1−parameter group of point transformations is given by the rotations of a circle about its center through a parameter ε [2],

$$(x^*, y^*) = (x\cos\varepsilon - y\sin\varepsilon, x\sin\varepsilon + y\cos\varepsilon)\,, \quad (-\pi \leq \varepsilon < \pi)\,. \tag{1.1}$$

These transformations depend only on one continuous parameter ε and satisfy axioms of a group and, hence, form a 1−parameter group of point transformations. On the other hand, the reflection

$$x^* = -x, \;\; y^* = -y,$$

is a point transformation that does not constitute a 1−parameter group of point transformations. Translational symmetries in the xy−plane

$$x^* = x + \varepsilon_1, \;\; y^* = y + \varepsilon_2, \;\; \varepsilon_1, \varepsilon_2 \in \mathbb{R}, \tag{1.2}$$

is an example of 2−parameter group of point transformations. The set of all translations given in eq.(1.2) depends upon two parameters, ε_1 and ε_2 [3]. By setting ε_1

to zero, we obtain the 1−parameter group of translations in the y direction. Similarly, the 1−parameter group of translations in the x direction is obtained by setting ε_2 to zero. Translations given in eq.(1.2) can be regarded as a composition of the 1−parameter groups of translations parametrized by ε_1 and ε_2. Similarly, symmetries belonging to an $r-$parameter group of point transformations can be regarded as a composition of symmetries from r 1−parameter groups. In this thesis, hereafter, point symmetries will be referred to as symmetries.

If there is an $r-$parameter group of transformations (symmetries) of a differential equation, then on integration symmetries either give the solution of the differential equation or reduce it to a simpler equation. An n^{th} order ordinary differential equation with an $r-$parameter Lie group of transformations can be reduced to an $(n-r)^{th}$ order differential equation. On application to partial differential equations these symmetries reduce the number of independent variables.

The above discussion gives the impression that all differential equations admit symmetries that lead to their solutions. In general, this is not true; for example the differential equation

$$y'' = xy + e^{y'} + e^{-y'}, \tag{1.3}$$

does not admit any symmetry. As symmetries of a differential equation give solution or reduce it to a simpler equation on integration, therefore, if there is no symmetry then the reduction of order or solution cannot be found by this method. In such cases, one must look at the Cauchy criterion for the existence of solutions of the ordinary differential equations. If a solution exists then one can look for other methods to solve the equation.

Symmetries are also related to conservation laws and their relation can be used to find the first integral and Lagrangian of a variational problem. In this thesis, different applications of the Lie point symmetries to solve different differential equations are presented. The plan of the thesis is given below.

1.1.3 Plan of the Thesis

In the remaining sections of this chapter some basic definitions and concepts, namely Manifolds, Lie groups and Lie algebra are given. Then, after introducing Lie derivatives and isometries, the concept of symmetries of differential equations and the related conservation laws are introduced in general.

In Chapter 2, we first investigate whether plane symmetric spacetimes admit sourceless electromagnetic fields or not. In the remaining sections of this chapter, after discussing the non-static, spherically symmetric, shear-free, perfect fluid solutions of the Einstein field equations, some physically acceptable solutions of the Einstein field equations are presented. The symmetry method is used to solve some differential equations appearing in these investigations.

In Chapter 3, a relation between the isometries and the symmetries of the geodesic equations of the maximally symmetric spaces is worked out and presented in detail. In the first section of this chapter the Lie algebra of a second order vector differential equation is discussed. In the next two sections the symmetries of the geodesic equations of maximally symmetric and less symmetric 2-dimensional spaces are presented respectively. The last two sections of this chapter present the symmetries of the full and the reduced system of geodesic equations of maximally symmetric higher dimensional spaces.

In Chapter 4, after describing the algorithm for calculating infinitesimal approximate symmetries, their use to construct the first integral and Lagrangian is shown.

In Chapter 5, after a brief review of the thesis, concluding remarks are given with some ideas for further work in this field.

1.2 Manifolds and Vector Fields

Manifolds are used in various branches of mathematics e.g. differential geometry, relativity theory, Lie group theory, differential equations, theory of complex variables, algebraic geometry etc. They generalize the concepts of curves and surfaces in 3−dimensional spaces. In general a manifold is a space that looks like Euclidean space, but whose global character might be quite different. For example, a cylinder and the Euclidean plane $\mathbb{R}^2$ are different but a very small patch of a cylinder looks like a small patch of $\mathbb{R}^2$. *A real $n-$dimensional manifold is a separable, connected, Housdroff space with a homeomophism from each element of its open cover into* $\mathbb{R}^n$. e.g. S^1 is a manifold as one can cover it with two coordinate systems which over-lap [4]. In both the regions continuity is maintained and in the over-lapping regions there exist coordinate transformations between the coordinate systems.

Let M be a smooth manifold, $\forall p \in M$, the set $T_p(M)$ of all vectors tangent to

M at p, is called the *tangent space*. The coordinate basis of $T_p(M)$ is $\{\partial/\partial x^a\}$ i.e.

$$T_p(M) = <\partial/\partial x^1, \partial/\partial x^2, ..., \partial/\partial x^n> = <\partial/\partial x^a> = <\mathbf{e}_a>, (a = 1, 2, 3, ..., n).$$

The union of all $T_p(M)$ is called the *tangent bundle*, TM i.e. $TM = \bigcup_{p\in M} T_p(M)$. Let N be another smooth manifold and $\Phi : M \to N$ be a smooth function. Then the tangent map $\Phi_* : TM \to TN$, called the *push forward*, denotes the differential, whose restriction $\Phi_*(p)$ to the tangent space $T_p(M)$ is a linear mapping of $T_p(M)$ into $T_{\Phi(p)}(N)$. In terms of local coordinates in M and N, $\Phi_*(p)$ is given by the Jacobian matrix. The dual space $T_p^*(M)$ of the tangent space $T_p(M)$, called the *cotangent space*, is the vector space of all linear functions i.e.

$$T_p^*(M) = \{Y \mid Y : T_p(M) \to \mathbb{R}\}. \tag{1.4}$$

The map $\Phi^* : T^*N \to T^*M$ is known as the *pull back*, where T^*M and T^*N are called *cotangent bundles* and are the unions of cotangent spaces whose elements are *cotangent vectors*. The coordinate basis of $T_p^*(M)$ is $\{dx^a\}$ i.e.

$$T_p^*(M) = <dx^1, dx^2, ..., dx^n> = <dx^a> = <\mathbf{e}^a>. \tag{1.5}$$

The basis of the cotangent space is dual to the basis of the tangent space. i.e.

$$<\mathbf{e}_a, \mathbf{e}^b> = \delta_a^b,$$

where δ_a^b is the *Kronecker delta*.

A *vector field* [5] on M is a function $\mathbf{X} = X^a\partial/\partial x^a$ (using the Einstein summation convention that repeated indices are summed over), which assigns to each point $p \in M$ a tangent vector $\mathbf{X}_p \in T_p(M)$, varying smoothly with p. The tangent vector $\mathbf{X}_p$

is associated with some curve $\alpha_p(t) : (-\varepsilon, \varepsilon) \to M$, which satisfies the condition $\frac{\partial}{\partial t}\alpha_p(t) = \mathbf{X}(\alpha_p(t))$ and $\alpha_p(0) = p$, called the *integral curve* passing through p. There is a maximal integral curve through p in the sense that all the other integral curves are a subset of this integral curve. Considering all the integral curves $\alpha_q(t)$, where q is in some neighborhood $U \subset M$ of p, we define a differentiable mapping $\phi : (-\varepsilon, \varepsilon) \times U \to M$, such that the curve $t \to \phi(t,q)$ is a unique curve which satisfies $\frac{\partial \phi}{\partial t} = \mathbf{X}(\phi(t,q))$ and $\phi(0,q) = q$. If we fix t and vary q we obtain the maps $\phi_t : U \to M$ such that $\phi_t(q) = \phi(t,q)$, called the *local flows* of the vector field $\mathbf{X}$.

1.3 Lie Groups and Lie Algebras

A *Lie group* G is a smooth manifold which is also a group and the mappings $m : G \times G \to G$ and $i : G \to G$ defined by multiplication and inversion $m(x,y) = xy$, $i(x) = x^{-1}$ are smooth [1]. e.g.

- the additive group of $\mathbb{R}^n$ is a Lie group;
- none of the even dimensional spheres, S^{2n}, are Lie groups [4];
- the Dihedral group is not a manifold so it is not a Lie group;
- the compactified complex plane, $\overline{\mathbb{C}}$ is a group under multiplication, it is also a manifold but it is not a Lie group as it is not smooth at the identity. (This may also be seen by noting that $\overline{\mathbb{C}}$ is homeomorphic to S^2.)

Let $T_e(G)$ be the tangent space at the identity of the Lie group and $\mathbf{X} = X^a\partial/\partial x^a$ and $\mathbf{Y} = Y^b\partial/\partial x^b \in T_e(G)$ be two differentiable vector fields. The multiplication defined on $T_e(G)$

$$[\mathrm{X},\mathrm{Y}]_e f = \mathrm{X}_e(\mathrm{Y}f) - \mathrm{Y}_e(\mathrm{X}f), \tag{1.6}$$

where f is a differentiable function on G, is called the *Lie product or Lie bracket or commutator* and the algebraic structure is called the *Lie algebra* [6]. The product $[\mathrm{X},\mathrm{Y}]_e \in T_e(G)$ is also a vector field. A Lie algebra determines the local structures of the group. Thus, two groups will be locally isomorphic if and only if their Lie algebras are isomorphic. The Lie algebra is a finite dimensional algebra. Therefore, the local study of Lie groups is entirely equivalent to the study of certain finite dimensional linear algebraic structures.

A Lie algebra L is a vector space over some field F, equipped with Lie brackets satisfying the properties

$$\begin{aligned} &i.\ \ [\mathrm{X},\mathrm{Y}] = -[\mathrm{Y},\mathrm{X}], \\ &ii.\ \ [\mathrm{X},[\mathrm{Y},\mathrm{Z}]] + [\mathrm{Y},[\mathrm{Z},\mathrm{X}]] + [\mathrm{Z},[\mathrm{X},\mathrm{Y}]] = 0. \end{aligned} \tag{1.7}$$

The above equations show that the Lie brackets are *anti-commutative* and they must satisfy the *Jacobi identity*. Lie algebra is said to be real if F is the field of real numbers and complex if F is the field of complex numbers. The Lie algebra associated with a Lie group is always real.

The group of rigid motion in $\mathbb{R}^2$ preserves distance between any two points [1] in $\mathbb{R}^2$. It is the three-parameter Lie group of transformations of rotation and translations in $\mathbb{R}^2$ given by

$$\begin{aligned} x^* &= x\cos\theta_1 - y\sin\theta_1 + \theta_2, \\ y^* &= x\sin\theta_1 + y\cos\theta_1 + \theta_3, \end{aligned} \tag{1.8}$$

where θ_1, θ_2 and θ_3 are the infinitesimal parameters. Corresponding to each θ_i $(i = 1, 2, 3)$, one obtains an infinitesimal generator of the corresponding Lie algebra. The generators of the Lie algebra, X_i, corresponding to the parameters θ_i, in this case are

$$\mathbf{X}_1 = -y\partial/\partial x + x\partial/\partial y, \quad \mathbf{X}_2 = \partial/\partial x, \quad \mathbf{X}_3 = \partial/\partial y, \tag{1.9}$$

and the corresponding Lie algebra is

$$[\mathbf{X}_1, \mathbf{X}_2] = -\mathbf{X}_3, \quad [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2, \quad [\mathbf{X}_i, \mathbf{X}_k] = 0 \ (otherwise). \tag{1.10}$$

Suppose we have

$$\begin{aligned} \mathbf{X}_1 &= \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}, \\ \mathbf{X}_2 &= \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}, \end{aligned} \tag{1.11}$$

then $\mathbf{X}_1$, $\mathbf{X}_2$ do not form a Lie algebra since $[\mathbf{X}_1, \mathbf{X}_2] = \frac{\partial}{\partial\phi}$. However, settig $\mathbf{X}_3 = \partial/\partial\phi$, $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ do form the above real Lie algebra.

1.4 Lie Derivatives and Isometries

Geometrically, the *Lie derivative* corresponds to a change in the components of a tensor by transporting it from any point to a neighboring point on a curve, in the direction of the tangent vector at that point; it is a differentiation along a curve in a manifold.

Let $p \in M$, $\mathbf{X}$ be a vector field on M and $\phi_t : M \to M$ be the flow of $\mathbf{X}$. Let $\mathbf{Y}$ be another vector field, then

$$\pounds_{\mathbf{X}}\mathbf{Y} = \lim_{t\to 0} \frac{1}{t}\left(\phi_t^*\mathbf{Y}_q - \mathbf{Y}_p\right), \tag{1.12}$$

is the Lie derivative [7] of the vector field $\mathbf{Y}$ along a vector field $\mathbf{X}$. The Lie derivative of any tensorial object $\mathbf{T}$ with respect to a vector field $\mathbf{X}$ on M is denoted by $\pounds_X \mathbf{T}$ and is given by

$$\pounds_{\mathbf{X}} \mathbf{T} = \lim_{t \to 0} \frac{1}{t} \left(\phi_t^* \mathbf{T} - \mathbf{T} \right). \tag{1.13}$$

The tensor $\mathbf{T}$ and $\phi_t^* \mathbf{T}$ have the same valence (r, s) and are both evaluated at the same point p. Therefore, the Lie derivative is also tensor of valence (r, s) at p. The Lie derivative vanishes if the tensors $\mathbf{T}$ and $\phi_t^* \mathbf{T}$ coincide. If

$$\mathbf{T} = g_{ab} dx^a dx^b = \mathbf{g}, \tag{1.14}$$

and

$$\pounds_{\mathbf{K}} \mathbf{g} = 0, \tag{1.15}$$

then $\mathbf{K}$ gives the direction of symmetry for g and is called an *isometry*. The solutions of eq.(1.15) give *Killing vector fields* $\mathbf{K} = k^a \partial / \partial x^a$. In simple words isometries are the transformations that preserve the metric properties of the space.

1.5 Symmetries of Differential Equations

A transformation is said to be the *symmetry transformation* or *symmetry of a differential equation* if it leaves the form of the differential equation invariant. To find the symmetries we take the point transformations that depend upon at least one parameter, $\varepsilon \in \mathbb{R}$, [2]

$$\begin{aligned} x^* &= x^* \left(x, y; \varepsilon \right), \\ y^* &= y^* \left(x, y; \varepsilon \right). \end{aligned} \tag{1.16}$$

Expanding these transformations by Taylor's expansion method at $\varepsilon = 0$ to have

$$\begin{aligned} x^* &= x + \varepsilon \frac{\partial x^*}{\partial \varepsilon} \mid_{\varepsilon=0} + O\left(\varepsilon^2\right) = x + \varepsilon \xi\left(x, y\right) + O\left(\varepsilon^2\right), \\ y^* &= y + \varepsilon \frac{\partial y^*}{\partial \varepsilon} \mid_{\varepsilon=0} + O\left(\varepsilon^2\right) = y + \varepsilon \eta\left(x, y\right) + O\left(\varepsilon^2\right). \end{aligned} \tag{1.17}$$

Or equivalently

$$\begin{aligned} x^* &= x + \varepsilon \mathbf{X} x + O\left(\varepsilon^2\right), \\ y^* &= y + \varepsilon \mathbf{X} y + O\left(\varepsilon^2\right), \end{aligned} \tag{1.18}$$

where

$$\mathbf{X} = \xi\left(x, y\right) \frac{\partial}{\partial x} + \eta\left(x, y\right) \frac{\partial}{\partial y}, \tag{1.19}$$

and the functions ξ and η are defined by

$$\xi\left(x, y\right) = \frac{\partial x^*}{\partial \varepsilon} \mid_{\varepsilon=0}, \text{ and } \eta\left(x, y\right) = \frac{\partial y^*}{\partial \varepsilon} \mid_{\varepsilon=0}. \tag{1.20}$$

The operator $\mathbf{X}$ is called the infinitesimal generator, group operator or Lie operator.

We extend the point transformations to apply to the differential equation

$$E\left(x, y, y', y'', ..., y^{(n)}\right) = 0, \tag{1.21}$$

where $y' = d/dx, y'' = d^2/dx^2, ..., y^{(n)} = d^n/dx^n$. The extended (prolonged) transformations along with eqs.(1.20, 1.21) are

$$\begin{aligned} y'^* &= y'^*\left(x, y; \varepsilon\right) = y + \varepsilon \eta_{,x}\left(x, y, y'\right) + O\left(\varepsilon^2\right) = y' + \mathbf{X}^{[n]} y' + O\left(\varepsilon^2\right), \\ y''^* &= y''^*\left(x, y; \varepsilon\right) = y + \varepsilon \eta_{,xx}\left(x, y, y', y''\right) + O\left(\varepsilon^2\right) = y'' + \mathbf{X}^{[n]} y'' + O\left(\varepsilon^2\right), \\ y^{n*} &= y^{n*}\left(x, y; \varepsilon\right) = y + \varepsilon \eta_{,(n)}\left(x, y, y', ..., y^{(n)}\right) + O\left(\varepsilon^2\right) = y^{(n)} + \mathbf{X}^{[n]} y^{(n)} + O\left(\varepsilon^2\right), \end{aligned} \tag{1.22}$$

where the prolongation coefficients are

$$\begin{aligned} \eta_{,x} &= \frac{d\eta}{dx} - y'\frac{d\xi}{dx}, \\ \eta_{,(n)} &= \frac{d\eta_{(n-1)}}{dx} - y^{(n)}\frac{d\xi}{dx},\ n \geq 2, \end{aligned} \tag{1.23}$$

and the prolonged infinitesimal generator is

$$\mathbf{X}^{[n]} = \xi\left(x, y\right)\frac{\partial}{\partial x} + \eta\left(x, y\right)\frac{\partial}{\partial y} + \eta_{,x}\left(x, y, y'\right)\frac{\partial}{\partial y'} + ... + \eta_{,(n)}\left(x, y, y', ..., y^{(n)}\right)\frac{\partial}{\partial y^{(n)}}. \tag{1.24}$$

For the symmetries of differential equations we use the invariance criterion:

Theorem 1 *An ordinary differential eq.*(1.21) *admits a group of symmetries with generator* $\mathbf{X}^{[n]}$ *if and only if*

$$\mathbf{X}^{[n]}E\mid_{E=0} = 0, \tag{1.25}$$

(i.e. along the solutions of $E = 0$*) holds.*

Proof If a differential equation admits a symmetry then eq.(1.25) holds. Conversely, since

$$E\left(x^*, y^*, y^{*\prime}, y^{*\prime\prime}, ..., y^{*(n)}\right) = 0, \tag{1.26}$$

has to be valid for all values of ε, therefore, differentiating eq.(1.26) we obtain

$$\frac{\partial}{\partial\varepsilon}E\left(x^*, y^*, y^{*\prime}, y^{*\prime\prime}, ..., y^{*(n)}\right) \mid_{\varepsilon=0} = 0, \tag{1.27}$$

$$\frac{\partial E}{\partial x^*}\frac{\partial x^*}{\partial\varepsilon} + \frac{\partial E}{\partial y^*}\frac{\partial y^*}{\partial\varepsilon} + ... + \frac{\partial E}{\partial y^{*(n)}}\frac{\partial y^{*(n)}}{\partial\varepsilon} \mid_{\varepsilon=0} = 0. \tag{1.28}$$

Using the definitions, eqs.(1.20, 1.23) and

$$\frac{\partial E}{\partial x^*}\mid_{\varepsilon=0} = \frac{\partial E}{\partial x},\ \frac{\partial E}{\partial y^*}\mid_{\varepsilon=0} = \frac{\partial E}{\partial y},\ ..., \frac{\partial E}{\partial y^{*(n)}}\mid_{\varepsilon=0} = \frac{\partial E}{\partial y^{(n)}},$$

eq.(1.28) is equivalent to

$$\xi\frac{\partial E}{\partial x}+\eta\frac{\partial E}{\partial y}+\eta_{,x}\frac{\partial E}{\partial y'}+...+\eta_{,(n)}\frac{\partial E}{\partial y^{(n)}}=0,$$

which is equivalent to eq.(1.25).

This invariance criterion gives a system of linear partial differential equations, called the *determining equations.* Solutions of these partial differential equations give the symmetries of the differential equation. These symmetries are closed under the commutator, forming a Lie algebra. Corresponding to each symmetry of the differential equation one can write *characteristic equations* e.g. the symmetries of the differential equation

$$y''=-y+y^{-3},$$

are

$$\begin{aligned}\mathbf{X}_1 &= \frac{\partial}{\partial x},\ \mathbf{X}_2=-\cos 2x\frac{\partial}{\partial x}+y\sin 2x\frac{\partial}{\partial y},\\ \mathbf{X}_3 &= -\sin 2x\frac{\partial}{\partial x}+y\cos 2x\frac{\partial}{\partial y}.\end{aligned}$$

The characteristic equations for $\mathbf{X}_1$

$$\frac{dx}{1}=\frac{dy}{0}=\frac{dy'}{0},$$

give the *invariants* $u=y$, $v=y'$. Then

$$\frac{dv}{du}=\frac{u^{-3}-u}{v},$$

is the reduced equation, which is a separable variable equation. In this way a given nth order ordinary differential equation having $r-$symmetries can be reduced to an ordinary differential equation of order $(n-r)$.

The Lie group of transformations is obtained by solving the *Lie equations*

$$\begin{aligned} \frac{\partial x^*}{\partial \varepsilon} &= \xi(x,y),\ x^*\mid_{\varepsilon=0}= x, \\ \frac{\partial y^*}{\partial \varepsilon} &= \eta(x,y),\ y^*\mid_{\varepsilon=0}= y. \end{aligned} \tag{1.29}$$

The set of symmetries form a Lie algebra. These symmetries permute the integral curves among themselves. Some of them remain invariant under the Lie group of transformations. These integral curves are called *invariant solutions.*

In order to determine symmetries of a system of ordinary differential equations we use the invariance criterion [8]:

Theorem 2 *The system of p ordinary differential equations of order k*

$$E_i\left(s,\mathrm{x},\dot{\mathrm{x}},...\mathrm{x}^{(k)}\right)=0,\ (i=1,2,...,p),$$

admits a symmetry algebra with generator

$$\mathrm{X}=\xi(s,\mathrm{x})\frac{\partial}{\partial s}+\eta^{\alpha}(s,\mathrm{x})\frac{\partial}{\partial \mathrm{x}^{\alpha}}, \tag{1.30}$$

if and only if $\mathrm{X}^{[k]}E_i\mid_{E_i=0}=0$ *holds , where* x *is a point in the underlying* $m-$*dimensional space and* $\dot{\mathrm{x}}$ *is the first derivative of* x *and* $\mathrm{x}^{(k)}$ *is the* k^{th} *order derivative with respect to s and*

$$\mathrm{X}^{[k]}=\xi(s,\mathrm{x})\frac{\partial}{\partial s}+\eta^{\alpha}(s,\mathrm{x})\frac{\partial}{\partial \mathrm{x}^{\alpha}}+\eta^{\alpha}_{,s}(s,\mathrm{x},\dot{\mathrm{x}})\frac{\partial}{\partial \dot{\mathrm{x}}^{\alpha}}+...+\eta^{\alpha}_{,(k)}\left(s,\mathrm{x},...,\mathrm{x}^{(k)}\right)\frac{\partial}{\partial \mathrm{x}^{\alpha(k)}}. \tag{1.31}$$

The prolongation coefficients are

$$\begin{aligned} \eta^{\alpha}_{,s} &= \frac{d\eta^{\alpha}}{ds}-\dot{\mathrm{x}}^{\alpha}\frac{d\xi}{ds}, \\ \eta^{\alpha}_{,(k)} &= \frac{d\eta^{\alpha}_{(k-1)}}{ds}-\mathrm{x}^{\alpha(k)}\frac{d\xi}{ds},\ k\geq 2. \end{aligned} \tag{1.32}$$

Proof The proof of this theorem is similar to that of theorem 1.

Solutions of the resulting partial differential equations give symmetries of the system of equations.

The symmetry transformations of partial differential equations can depend not only on one or several arbitrary parameters, but also on one or several arbitrary functions [2]. In either case, a particular set of transformations

$$x^{*n} = x^{*n}\left(x^i, y^\beta; \varepsilon\right), \qquad y^{*\alpha} = y^{*\alpha}\left(x^i, y^\beta; \varepsilon\right), \tag{1.33}$$

can be chosen that depends on only one parameter $\varepsilon \in \mathbb{R}$ so that they form a group, with $x^{*n} = x^n$ and $y^{*\alpha} = y^\alpha$ for $\varepsilon = 0$. For the corresponding infinitesimal transformations

$$x^{*n} = x^n + \varepsilon \xi^n\left(x^i, y^\beta\right) + O\left(\varepsilon^2\right), \tag{1.34}$$

$$y^{*\alpha} = y^\alpha + \varepsilon \eta^\alpha\left(x^i, y^\beta\right) + O\left(\varepsilon^2\right),$$

where ξ^n and η^α are defined as

$$\xi^n\left(x^i, y^\beta\right) = \frac{\partial x^{*n}}{\partial \varepsilon}\Big|_{\varepsilon=0}, \; \eta^\alpha\left(x^i, y^\beta\right) = \frac{\partial y^{*\alpha}}{\partial \varepsilon}\Big|_{\varepsilon=0}. \tag{1.35}$$

The infinitesimal generator takes the form

$$\mathbf{X} = \xi^n\left(x^i, y^\beta\right)\frac{\partial}{\partial x} + \eta^\alpha\left(x^i, y^\beta\right)\frac{\partial}{\partial y}. \tag{1.36}$$

1.6 Symmetries and Conservation Laws

The Lagrangian, $L(x, y, y')$, of a system is defined as the difference between the kinetic and the potential energy. In any isolated system, energy can be transformed from one kind to another, but the total amount of energy is constant (conserved) e.g. when a block slides over a rough surface, the force of friction gives rise to the heating of block and surface. As a result, mechanical energy is transformed into heat energy, but the total energy is constant.

Theorem 3 *A variational integral defined as*

$$I = \int_V L(x, y, y')dx, \tag{1.37}$$

is said to be invariant under a 1*−parameter Lie group of transformations [8]*

$$x^{i*} = x^{i*}(x, y; \varepsilon),\ y^{\alpha *} = y^{\alpha *}(x, y; \varepsilon), \tag{1.38}$$

with the generator

$$X = \xi^i(x, y)\frac{\partial}{\partial x^i} + \eta^\alpha(x, y)\frac{\partial}{\partial y^\alpha},$$

if

$$\int_{V^*} L(x^*, y^*, y'^*)dx^* = \int_V L(x, y, y')dx. \tag{1.39}$$

Since V is an arbitrary volume and V^* is a volume obtained from V by transformation eq.(1.38).

Proof $L(x, y, y')$ be a first order Lagrangian corresponding to a second order differential equation. The functional

$$I = \int_V L(x, y, y')dx, \tag{1.40}$$

is invariant under the 1−parameter Lie group of transformations with the generator **X** upto gauge $B(x, y)$ if

$$\mathbf{X}(L) + LD(\xi) = DB. \tag{1.41}$$

Invariance of $\int\limits_V L(x,y,y')dx$ upto gauge $B(x,y)$ by eq.(1.17) is

$$\int\limits_{V^*} L(x^*,y^*,y'^*)dx^* = \int\limits_V \left[L(x,y,y') + \varepsilon\frac{dB}{dx}\right] dx. \tag{1.42}$$

Now taking $\frac{d}{dx} = \frac{dx^*}{dx}\frac{d}{dx^*}$, and differentiating eq.(1.42) with respect to x, we have

$$\begin{aligned}
\left(1+\varepsilon\frac{d\xi}{dx}\right) L\left(x^*,y^*,y'^*\right) &= L(x,y,y') + \varepsilon\frac{dB}{dx}, \\
\left(1+\varepsilon\frac{d\xi}{dx}\right) L\left(x+\varepsilon\xi, y+\varepsilon\eta, y'+\varepsilon\eta_{,x}\right) &= L(x,y,y') + \varepsilon\frac{dB}{dx}, \\
\left(1+\varepsilon\frac{d\xi}{dx}\right)\left[L\left(x,y,y'\right) + \varepsilon\frac{\partial L}{\partial x}\xi + \varepsilon\frac{\partial L}{\partial y}\eta + \varepsilon\frac{\partial L}{\partial y'}\eta_{,x}\right] &= L(x,y,y') + \varepsilon\frac{dB}{dx}, \\
\varepsilon\frac{\partial L}{\partial x}\xi + \varepsilon\frac{\partial L}{\partial y}\eta + \varepsilon\frac{\partial L}{\partial y'}\eta_{,x} + \varepsilon\frac{d\xi}{dx}L\left(x,y,y'\right) + O\left(\varepsilon^2\right) &= \varepsilon\frac{dB}{dx}.
\end{aligned}$$

Neglecting $O(\varepsilon^2)$ one can obtain eq.(1.41).

Emmy Noether combined the method of calculus of variation with the theory of Lie groups [9]. She proved that [10]:

Theorem 4 *Corresponding to each symmetry that satisfies the condition of the previous theorem there exists a first integral (conserved vector)* T, *defined by*

$$T^n = L\xi^n + (\eta^\alpha - \xi^m y^\alpha_m)\frac{\partial L}{\partial y^\alpha_n}.$$

Chapter 2

Applications of Symmetry Methods in General Relativity

In classical physics it is assumed that all observers anywhere in the universe, whether moving or not, obtain identical measurements of space and time intervals. There, the space and time are kept separate. Einstein's theory of relativity is a four dimensional unified theory of space and time, where results depend on the relative motions of observers and space and time are treated as a single entity by combining a point in space at an instant of time. Extending the special theory of relativity, where one deals with non-accelerated frames of reference and the phenomena in which the gravitational field is absent, to the general theory of relativity, one takes into account the effect of gravity and that of accelerated frames. Accordingly, the speed of light is considered as a universal constant and the inertial and gravitational masses are treated as equivalent. The latter assumption is called the principle of equivalence.

The presence of matter, characterized by the stress-energy tensor (T_{ab}), effects geometry, represented by the spacetime metric (g_{ab}) and causes a local curvature in the spacetime. Einstein related geometry and matter by the field equations:

$$R_{ab} - \frac{1}{2} g_{ab} R + g_{ab} \Lambda = \kappa T_{ab}, \ (a, b... = 0, 1, 2, 3), \tag{2.1}$$

where κ, R_{ab}, R and Λ are respectively the coupling constant, the Ricci tensor, the Ricci scalar and the cosmological constant; the left hand side giving the geometry and the right hand side represents the matter content of the spacetime.

In this chapter two applications of symmetry methods are presented (section 2.2, 2.4). In section 2.3, all plane symmetric electromagnetic solutions of the Einstein field equations are obtained. In section 2.4 a paper on the application of Noether symmetries of $y'' = f(x) y^n$ to spherically symmetric solutions of Einstein's equation [11] is reviewed. On the basis of this paper, some physically acceptable solutions of the Einstein field equations are found [12], which are presented in section 2.5.

2.1 Plane Symmetric Electromagnetic Solutions of the Einstein Field Equations

To understand the physics and geometry of the spacetimes, one may classify spacetimes by different schemes. One scheme is to impose some symmetry conditions on the metric tensor e.g. by the isometries. This approach not only classifies the spacetimes according to their isometries but also provides a complete list of possible solutions of Einstein's field equations. Here we explore which of the plane symmetric spacetimes admit electromagnetic field. Some work has already been done in this area for cylindrically symmetric static spacetimes [13].

Those spacetimes are said to be plane symmetric that admit the minimal isometry group $SO(2) \otimes_s \mathbb{R}^2$ (where $\otimes_s$ represents semidirect product), such that the group orbits are two dimensional hypersurfaces of zero intrinsic curvature. They are given by the metric [7]

$$ds^2 = e^{2\nu(t,x)}dt^2 - e^{2\lambda(t,x)}dx^2 - e^{2\mu(t,x)}(dy^2 + dz^2), \tag{2.2}$$

where ν, λ and μ are the arbitrary functions of t and x.

It is often convenient to use the null tetrad formalism given by Newman and Penrose [14], to explore some physical aspects of a spacetime. The null tetrad consists of two real null vectors $\mathbf{k}$, $\mathbf{l}$ and two complex conjugate null vectors $\mathbf{m}$, $\overline{\mathbf{m}}$. The null tetrad $\{\mathrm{e}_a\} = (k_a, l_a, m_a, \overline{m}_a)$ can be introduced at every point of the spacetime and the metric tensor in this basis is:

$$g_{ab} = 2k_{(a}l_{b)} - 2m_{(a}\overline{m}_{b)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \tag{2.3}$$

The scalar product of the tetrad vectors vanishes apart from

$$k^a l_a = 1, \quad m^a \overline{m}_a = -1. \tag{2.4}$$

In terms of a coordinate basis, a complex null tetrad $\{\mathrm{e}_a\}$ and its dual basis $\{\mathrm{e}^a\}$ will take the form

$$\mathrm{e}_1 = k^a \partial/\partial x^a, \quad \mathrm{e}_2 = l^a \partial/\partial x^a, \quad \mathrm{e}_3 = m^a \partial/\partial x^a, \quad \mathrm{e}_4 = \overline{m}^a \partial/\partial x^a,$$

$$\mathrm{e}^1 = -k_a dx^a, \quad \mathrm{e}^2 = -l_a dx^a, \quad \mathrm{e}^3 = m_a dx^a, \quad \mathrm{e}^4 = \overline{m}_a dx^a.$$

The Ricci tensor can be decomposed into the trace, R, and the traceless, S_{ab}, parts

$$R_{ab} = S_{ab} + \frac{1}{4} g_{ab} R \tag{2.5}$$

We can write the traceless part of the Ricci tensor as a 3×3 Hermitian matrix, $\phi_{ij} (i, j = 0, 1, 2)$, given by

$$\begin{aligned} \phi_{00} &= \tfrac{1}{2} S_{ab} k^a k^b, \quad \phi_{01} = \tfrac{1}{2} S_{ab} k^a m^b, \quad \phi_{02} = \tfrac{1}{2} S_{ab} m^a m^b, \\ \phi_{11} &= \tfrac{1}{4} S_{ab} (k^a l^b + m^a \overline{m}^b), \quad \phi_{12} = \tfrac{1}{2} S_{ab} l^a m^b, \quad \phi_{22} = \tfrac{1}{2} S_{ab} l^a l^b. \end{aligned} \tag{2.6}$$

For the metric given by eq.(2.2) the complex null tetrad is

$$\begin{aligned} \mathbf{k} &= \tfrac{1}{\sqrt{2}} \left(e^{-\nu} \tfrac{\partial}{\partial t} + e^{-\lambda} \tfrac{\partial}{\partial x} \right), \quad \mathbf{l} = \tfrac{1}{\sqrt{2}} \left(e^{-\nu} \tfrac{\partial}{\partial t} - e^{-\lambda} \tfrac{\partial}{\partial x} \right), \\ \mathbf{m} &= -\tfrac{e^{-\mu}}{\sqrt{2}} \left(\tfrac{\partial}{\partial y} + \iota \tfrac{\partial}{\partial z} \right), \quad \overline{\mathbf{m}} = -\tfrac{e^{-\mu}}{\sqrt{2}} \left(\tfrac{\partial}{\partial y} - \iota \tfrac{\partial}{\partial z} \right). \end{aligned} \tag{2.7}$$

A metric represents a sourceless electromagnetic field if the stress-energy tensor is traceless, i.e. $T = 0$, and it satisfies the positive energy condition, $T_{00} > 0$. The field is null or non-null accordingly as ϕ_{22} or ϕ_{11} are the only non-zero components of ϕ_{ij} [15]. The surviving components of ϕ_{ij} are

$$\begin{aligned} \phi_{00} &= e^{-2\lambda} \left(\nu' \mu' + \lambda' \mu' - \mu'' - \mu'^2 \right) + e^{-2\nu} \left(\dot{\nu} \dot{\mu} + \dot{\lambda} \dot{\mu} - \ddot{\mu} - \dot{\mu}^2 \right) \\ &\quad - e^{-(\lambda+\nu)} \left(\dot{\mu}' + \dot{\mu} \mu' - \nu' \dot{\mu} - \dot{\lambda} \mu' \right), \end{aligned} \tag{2.8}$$

$$\begin{aligned} \phi_{22} &= e^{-2\lambda} \left(\nu' \mu' + \lambda' \mu' - \mu'' - \mu'^2 \right) + e^{-2\nu} \left(\dot{\nu} \dot{\mu} + \dot{\lambda} \dot{\mu} - \ddot{\mu} - \dot{\mu}^2 \right) \\ &\quad + e^{-(\lambda+\nu)} \left(\dot{\mu}' + \dot{\mu} \mu' - \nu' \dot{\mu} - \dot{\lambda} \mu' \right), \end{aligned} \tag{2.9}$$

$$\phi_{11} = \frac{1}{2} \left[e^{-2\nu} \left(\dot{\mu}^2 - \ddot{\lambda} - \dot{\lambda}^2 + \dot{\nu} \dot{\lambda} \right) - e^{-2\lambda} \left(\mu'^2 - \nu'' - \nu'^2 + \nu' \lambda' \right) \right], \tag{2.10}$$

and the trace of the stress-energy tensor is

$$\begin{aligned} T &= 2e^{-2\nu} \left(2 \ddot{\mu} + 3 \dot{\mu}^2 - 2 \dot{\nu} \dot{\mu} + 2 \dot{\lambda} \dot{\mu} + \ddot{\lambda} + \dot{\lambda}^2 - \dot{\nu} \dot{\lambda} \right) \\ &\quad - 2e^{-2\lambda} \left(2\mu'' + 3\mu'^2 + 2\nu' \mu' - 2\lambda' \mu' + \nu'' + \nu'^2 - \nu' \lambda' \right), \end{aligned} \tag{2.11}$$

where '.' and ',' represent partial derivatives with respect to t and x respectively. Therefore, the problem of finding the electromagnetic field solutions is now reduced to solving $T = 0$, $\phi_{00} = 0 = \phi_{22}$ and $\phi_{11} \neq 0$.

In the following sections we first discuss the spacetimes that do not admit sourceless electromagnetic fields and then the spacetimes that do admit them.

2.2 Spacetimes Not Admitting Sourceless Electromagnetic Fields

Plane symmetric spacetimes have been completely classified according to their isometries and metrics or classes of metrics without imposing any restriction on the stress-energy tensor (T_{ab})[16]. It was found that these spacetimes are manifolds that admit the isometry group G_r (where $r = 3, 4, 5, 6, 7$or 10) containing $G_3 \equiv SO(2) \otimes_s \mathbb{R}^2$, as the minimal symmetry inherited by the plane symmetric manifolds. Notice that there do not exist Lorentzian metrics admitting G_8 and G_9 as the maximal isometry groups [17]. Now, we discuss all plane symmetric spacetimes admitting 10, 7, 6, 5 or 3−isometries and then the spacetimes admitting 4 isometries, as this is the only case for which electromagnetic fields exist.

2.2.1 Spacetimes Admitting 10 or 7−Isometries

Only three spacetimes have 10 isometries; De Sitter

$$ds^2 = dt^2 - e^{\frac{2t}{a}}(dx^2 + dy^2 + dz^2), \quad (a = constt. \neq 0), \tag{2.12}$$

anti-De Sitter

$$ds^2 = e^{\frac{2x}{a}}(dt^2 - dy^2 - dz^2) - dx^2, \quad (a = constt. \neq 0), \tag{2.13}$$

and Minkowski

$$ds^2 = dt^2 - (dx^2 + dy^2 + dz^2). \tag{2.14}$$

All of them have zero S_{ab} and hence cannot represent an electromagnetic field.

The 7−isometry cases are the Einstein universe

$$ds^2 = dt^2 - dx^2 - e^{\frac{2t}{a}}(dy^2 + dz^2), \quad (a = constt. \neq 0), \tag{2.15}$$

having the stress-energy tensor and the tetrad components of S_{ab} as

$$\begin{aligned} \kappa T_{00} &= \frac{1}{a^2}, \; \kappa T_{11} = -3\frac{1}{a^2}, \; \kappa T_{22} = -\frac{1}{a^2}e^{\frac{2t}{a}} = \kappa T_{33}, \; T = \frac{6}{a^2}, \\ \phi_{00} &= -\frac{1}{a^2} = \phi_{22} \text{ and } \phi_{11} = \frac{1}{2a^2}, \end{aligned}$$

and the anti-Einstein universe

$$ds^2 = dt^2 - dx^2 - e^{\frac{2x}{a}}(dy^2 + dz^2), \quad (a = constt. \neq 0), \tag{2.16}$$

with the surviving components of the stress-energy tensor and the tetrad components of S_{ab}

$$\begin{aligned} \kappa T_{00} &= -3\frac{1}{a^2}, \; \kappa T_{11} = \frac{1}{a^2}, \; \kappa T_{22} = \frac{1}{a^2}e^{\frac{2x}{a}} = \kappa T_{33}, \; T = -\frac{6}{a^2}, \\ \phi_{00} &= -\frac{1}{a^2} = \phi_{22} \text{ and } \phi_{11} = -\frac{1}{2a^2}. \end{aligned}$$

In both the cases the condition $\phi_{00} = 0$, for a spacetime to have an electromagnetic field does not hold.

2.2.2 Spacetimes Admitting 6−Isometries

There are six 6−isometry cases:

(i) the Bertotti Robinson-like metrics [18]

$$ds^2 = \cosh^2\frac{x}{a}dt^2 - dx^2 - dy^2 - dz^2, \quad (a \neq 0), \tag{2.17}$$

$$ds^2 = \cos^2\frac{x}{a}dt^2 - dx^2 - dy^2 - dz^2, \quad (a \neq 0), \tag{2.18}$$

with the non-zero components $\kappa T_{22} = \frac{1}{a} = \kappa T_{33}$, $T = \frac{2}{a}$, and $\phi_{11} = \frac{1}{2a}$;

$$ds^2 = e^{\frac{2x}{a}}dt^2 - dx^2 - dy^2 - dz^2, \quad (a \neq 0), \tag{2.19}$$

with $\kappa T_{22} = \frac{1}{a^2} = \kappa T_{33}$, $T = \frac{2}{a^2}$, and $\phi_{11} = \frac{1}{2a^2}$;

$$ds^2 = dt^2 - \cosh^2\frac{t}{a}dx^2 - dy^2 - dz^2, \quad (a \neq 0), \tag{2.20}$$

$$ds^2 = dt^2 - \cos^2\frac{t}{a}dx^2 - dy^2 - dz^2, \quad (a \neq 0), \tag{2.21}$$

with $\kappa T_{22} = -\frac{1}{a} = \kappa T_{33}$, $T = -\frac{2}{a}$, and $\phi_{11} = -\frac{1}{2a}$ and

$$ds^2 = dt^2 - e^{\frac{2t}{a}}dx^2 - dy^2 - dz^2, \quad (a \neq 0), \tag{2.22}$$

with $\kappa T_{22} = -\frac{1}{a^2} = \kappa T_{33}$, $T = -\frac{2}{a^2}$, and $\phi_{11} = -\frac{1}{2a^2}$.

From here it can be very easily seen that for all the metrics (2.17−2.22) $\phi_{00} = 0 = \phi_{22}$, and $\phi_{11} \neq 0$, but the trace of the energy-momentum tensor $T \neq 0$. Therefore, the Bertotti-Robinson like metrics do not admit electromagnetic field.

$$(ii) \quad ds^2 = dt^2 - e^{2f(t)}dx^2 - e^{2\frac{x}{a}+2f(t)}(dy^2 + dz^2). \tag{2.23}$$

The non-zero components of the stress-energy tensor and S_{ab} are

$$\kappa T_{00} = 3\left(\dot{f}^2 - \frac{1}{a^2}e^{-2f(t)}\right), \quad \kappa T_{11} = \frac{1}{a^2} - \left(2\,\ddot{f} + 3\,\dot{f}^2\right)e^{2f(t)},$$

$$\kappa T_{22} = \left[\frac{1}{a^2} - \left(2\ddot{f} + 3\dot{f}^2\right)e^{2f(t)}\right]e^{2\frac{x}{a}} = \kappa T_{33},\ T = 6\left(\frac{1}{a^2}e^{-2f(t)} + 2\dot{f}^2 + \ddot{f}\right),$$
$$\phi_{00} = -\left(\frac{1}{a^2}e^{-2f(t)} + \ddot{f}\right) = \phi_{22},\ \phi_{11} = -\frac{1}{2}\left(\frac{1}{a^2}e^{-2f(t)} + \ddot{f}\right).$$

$$(iii)\ ds^2 = e^{2f(x)}dt^2 - dx^2 - e^{2\frac{t}{a}+2f(x)}(dy^2 + dz^2). \tag{2.24}$$

The non-zero components of the stress-energy tensor and S_{ab} are

$$\kappa T_{00} = -\left(2f'' + 3f'^2\right)e^{2f(x)} + a^2,\ \kappa T_{11} = 3\left(f'^2 - \frac{1}{a^2}e^{-2f(x)}\right),$$
$$\kappa T_{22} = \left[\left(2f'' + 3f'^2\right)e^{2f(x)} - \frac{1}{a^2}\right]e^{2\frac{t}{a}} = \kappa T_{33},\ R = 6\left(\frac{1}{a^2}e^{-2f(x)} - 2f'^2 - f''\right),$$
$$\phi_{00} = -\left(\frac{1}{a^2}e^{-2f(x)} + f''\right) = \phi_{22},\ \phi_{11} = \frac{1}{2}\left(\frac{1}{a^2}e^{-2f(x)} + f''\right).$$

Neither of the metrics eq.(2.23) and eq.(2.24) satisfy $\phi_{00} = 0 \neq \phi_{11}$ simultaneously.

$$(iv)\ ds^2 = e^{2\nu(t+x)}dt^2 - e^{2\lambda(t+x)}dx^2 - e^{\frac{-t+x}{a}}\left(dy^2 + dz^2\right),\ (a \neq 0), \tag{2.25}$$

subject to the constraint

$$e^{2(\lambda + \frac{t+x}{2a})} = e^{2(\nu + \frac{t-x}{2a})} - \frac{2}{a}\int e^{2(\nu + \frac{t-x}{2a})}dt. \tag{2.26}$$

The non-vanishing components of the stress energy tensor are

$$\kappa T_{00} = e^{2(\nu-\lambda)}\left(2\lambda' - \frac{3}{4a}\right) + \left(-\lambda' + \frac{1}{4a}\right),\quad \kappa T_{01} = \frac{1}{2a} - \nu' + \lambda',$$
$$\kappa T_{11} = -e^{2(\lambda-\nu)}\left(\nu' + \frac{3}{4a}\right) + \left(\nu' + \frac{1}{4a}\right),$$
$$\kappa T_{22} = \kappa T_{33} = e^{\frac{-t+x}{a}}[e^{-2\lambda}\left(\frac{1}{4a^2} + \frac{1}{2a}\nu' - \frac{1}{2a}\lambda' + \nu'' + \nu'^2 - \nu'\lambda'\right)$$
$$-e^{-2\nu}\left(\frac{1}{4a^2} + \frac{\nu'}{a} + \frac{\lambda'}{a} + \lambda'' + \lambda'^2 - \nu'\lambda'\right)],$$
$$T = 2e^{-2\lambda}\left(\frac{\nu'}{a} - \frac{\lambda'}{a} + \nu'' + \nu'^2 - \nu'\lambda' + \frac{3}{4a^2}\right) - 2e^{-2\nu}\left(\frac{\nu'}{a} - \frac{2}{a}\lambda' + \lambda'' + \lambda'^2 - \nu'\lambda'\right),$$

and the surviving tetrad components of traceless part of Ricci tensor (S_{ab}) are

$$\begin{aligned}
\phi_{00} &= \frac{1}{2a}\left[\left(\nu' + \lambda' - \frac{1}{2a}\right)e^{-2\lambda} - \left(\nu' + \lambda' + \frac{1}{2a}\right)e^{-2\nu} - \left(\nu' - \lambda' - \frac{1}{2a}\right)e^{-(\nu+\lambda)}\right] \\
&= \phi_{22}, \\
\phi_{11} &= \frac{1}{2}\left[e^{-2\nu}\left(\frac{1}{4a^2} - \lambda'' - \lambda'^2 + \nu'\lambda'\right) - e^{-2\lambda}\left(\frac{1}{4a^2} - \nu'' - \nu'^2 + \nu'\lambda'\right)\right].
\end{aligned}$$

For the metric given by eq.(2.25), the constraint eq.(2.26) and the metric condition $\phi_{00} = 0$ yield $\nu = \lambda + \frac{t+x}{2a}$. Therefore, in this case the only possible metric is

$$ds^2 = \left(e^{\frac{t+x}{a}} - 1\right)dt^2 - \left(1 - e^{-\frac{t+x}{a}}\right)dx^2 - e^{\frac{-t+x}{a}}\left(dy^2 + dz^2\right), \ (a \neq 0), \tag{2.27}$$

for which $T \neq 0$.

$$(v) \ \ ds^2 = e^{2\nu(t+x)}dt^2 - e^{2\lambda(t+x)}dx^2 - e^{2\frac{x}{a}}\left(dy^2 + dz^2\right), \ (a \neq 0), \tag{2.28}$$

subject to the constraint

$$e^{2(\lambda - \frac{x}{2a})} = e^{2(\nu - \frac{x}{2a})} - \frac{2}{a}\int e^{2(\nu - \frac{x}{2a})}dt. \tag{2.29}$$

The non-vanishing components of the stress energy tensor are

$$\begin{aligned}
\kappa T_{00} &= -\frac{2}{a}\lambda' + \frac{1}{a^2}, \ \ \kappa T_{01} = -2\nu', \ \ \kappa T_{11} = -e^{2(\lambda-\nu)}\left(\frac{2}{a}\nu' + \frac{3}{a^2}\right), \\
\kappa T_{22} &= -e^{-2(t+\nu)}\left(\frac{1}{a^2} + \frac{\nu'}{a} - \frac{\lambda'}{a} + \lambda'' + \lambda'^2 - \nu'\lambda'\right) = \kappa T_{33}, \\
T &= 2e^{-2\lambda}\left(2\nu' - 2\lambda' + \nu'' + \nu'^2 - \nu'\lambda' + \frac{3}{a}\right) - 2e^{-2\nu}\left(\lambda'' + \lambda'^2 - \nu'\lambda'\right),
\end{aligned}$$

and the surviving tetrad components of traceless part of Ricci tensor (S_{ab}) are

$$\begin{aligned}
\phi_{00} &= \frac{1}{a}\left(-\nu' + \lambda' - \frac{1}{a}\right)e^{-2\nu} - \frac{\nu'}{a}e^{-(\nu+\lambda)} = \phi_{22}, \\
\phi_{11} &= \frac{1}{2}\left[e^{-2\nu}\left(\frac{1}{a^2} - \lambda'' - \lambda'^2 + \nu'\lambda'\right) - e^{-2\lambda}\left(\nu'' - \nu'^2 + \nu'\lambda'\right)\right].
\end{aligned}$$

Finally

$$(vi)\ \ ds^2 = e^{2\nu(t+x)}dt^2 - e^{2\lambda(t+x)}dx^2 - e^{-2\frac{t}{a}}\left(dy^2 + dz^2\right),\ \ (a \neq 0), \tag{2.30}$$

subject to the constraint

$$e^{2(\lambda+\frac{t}{2a})} = e^{2(\nu+\frac{t}{2a})} - \frac{2}{a}\int e^{2(\nu+\frac{t}{2a})}dt. \tag{2.31}$$

The non-vanishing components of the stress energy tensor are

$$\begin{aligned}
\kappa T_{00} &= e^{2(\nu-\lambda)}\left(2\lambda' - \frac{3}{a}\right),\ \ \kappa T_{11} = 2\nu' + \frac{1}{a}, \\
\kappa T_{22} &= e^{2x}[e^{-2\lambda}\left(\frac{1}{a^2} + \frac{\nu'}{a} - \frac{\lambda'}{a} + \nu'' + \nu'^2 - \nu'\lambda'\right), \\
R &= 2e^{-2\lambda}\left(\nu'' + \nu'^2 - \nu'\lambda'\right) - 2e^{-2\nu}\left(2\nu' - 2\lambda' + \lambda'' + \lambda'^2 - \nu'\lambda' + \frac{3}{a}\right),
\end{aligned}$$

and the surviving tetrad components of traceless part of Ricci tensor (S_{ab}) are

$$\begin{aligned}
&\phi_{00} = [\left(\nu' + \lambda' - 1\right)e^{-2\lambda} + \lambda' e^{\nu+\lambda}] = \phi_{22}, \\
&\phi_{11} = -\tfrac{1}{2}\left[e^{-2\nu}\left(\lambda'' + \lambda'^2 - \nu'\lambda'\right) + e^{-2\lambda}\left(1 - \nu'' - \nu'^2 + \nu'\lambda'\right)\right].
\end{aligned}$$

The constraints eq.(2.29) and eq.(2.31) together with the condition $\phi_{00} = 0$, for the metrics eq.(2.28) and eq.(2.30), yield

$$e^{-2\nu} + e^{-2\lambda} + e^{-(\nu+\lambda)} = 0, \tag{2.32}$$

which has no real solution. Hence none of the metrics admitting six isometries yields a sourceless electromagnetic field.

2.2.3 Spacetimes Admitting 5–Isometries

The metrics admitting 5–isometries are

$$(i)\ \ ds^2 = e^{2\frac{x}{a}}dt^2 - dx^2 - e^{2\frac{x}{c}}(dy^2 + dz^2),\ \ (a \neq 0 \neq c,\ \ a \neq c), \tag{2.33}$$

the non-vanishing components of the stress-energy tensor and S_{ab} are

$$\begin{aligned}
\kappa T_{00} &= -3\frac{1}{c^2}e^{2\frac{x}{a}}, \quad \kappa T_{11} = \frac{2}{ac} + \frac{1}{c^2}, \quad \kappa T_{22} = \left(\frac{1}{a^2} + \frac{1}{c^2} + \frac{1}{ac}\right)e^{2\frac{x}{c}} = \kappa T_{33}, \\
T &= -2\left(\frac{2}{ac} + \frac{1}{a^2} + \frac{3}{c^2}\right), \\
\phi_{00} &= \frac{1}{ac} - \frac{1}{c^2} = \phi_{22} \text{ and } \phi_{11} = -\frac{1}{2}\left(\frac{1}{c^2} - \frac{1}{a^2}\right);
\end{aligned}$$

and its non-static-analogue is

$$(ii) \quad ds^2 = dt^2 - e^{2\frac{t}{a}}dx^2 - e^{2\frac{t}{c}}(dy^2 + dz^2), \quad (a \neq 0 \neq c, \ \ a \neq c), \tag{2.34}$$

which has

$$\begin{aligned}
\kappa T_{00} &= \frac{2}{ac} + \frac{1}{c^2}, \quad \kappa T_{11} = -3\frac{1}{c^2}e^{2\frac{t}{a}}, \quad \kappa T_{22} = -\left(\frac{1}{a^2} + \frac{1}{c^2} + \frac{1}{ac}\right)e^{2\frac{t}{c}} = \kappa T_{33}, \\
T &= 2\left(\frac{2}{ac} + \frac{1}{a^2} + \frac{3}{c^2}\right), \\
\phi_{00} &= \frac{1}{ac} - \frac{1}{c^2} = \phi_{22}, \quad \phi_{11} = \frac{1}{2}\left(\frac{1}{c^2} - \frac{1}{a^2}\right).
\end{aligned}$$

Now, $\phi_{00} = \frac{1}{ac} - \frac{1}{c^2} \neq 0$, as $a \neq c$. Therefore, they do not admit a sourceless electromagnetic field. The metrics admitting 4−isometries will be discussed in the next section.

2.2.4 Spacetimes Admitting 3−Isometries

The minimal isometry (i.e. 3−isometries) case is eq.(2.2) with ϕ_{00}, ϕ_{11}, ϕ_{22} and T given by eqs.(2.8−2.11). A spacetime admitting 3−isometries may at most have $\dot{\mu} = 0 = \mu'$, $\dot{\mu} = 0 \neq \mu'$, $\dot{\mu} \neq 0 = \mu'$, or $\dot{\mu} \neq 0 \neq \mu'$. The case with $\dot{\mu} = 0 = \mu'$ gives $T = R_{0101}$, which is the only non-zero Riemann curvature tensor component in this case, so $T \neq 0$ (otherwise, it is a flat space with ten isometries). For the second case,

$\dot{\mu} = 0 \neq \mu'$, we may obtain a spacetime admitting 3−isometries when $\dot{\nu}' \neq 0$, $\lambda = 0$ and $\mu'' \neq 0$. Differentiating $\phi_{00} = 0$ with respect to t, yields

$$\dot{\nu}'\mu' + \nu'\dot{\mu}' - \dot{\mu}'' - 2\mu'\dot{\mu}' = 0,$$

which implies that $\dot{\nu}' = 0$ (as $\dot{\mu} = 0$ in this case), which is a contradiction. Similarly, the third case, $\dot{\mu} \neq 0 = \mu'$, may give a spacetime admitting 3−isometries when $\dot{\lambda}' \neq 0$, $\nu = 0$ and $\ddot{\mu} \neq 0$. Again,

$$\phi_{00} = \dot{\lambda}\dot{\mu} - \ddot{\mu} - \dot{\mu}^2 = 0,$$

The last case gives 3−isometries when:

(a) $\dot{\lambda} \neq 0$, $\dot{\eta} = 0$, $\eta' \neq 0$; (b) $\nu' \neq 0$, $\dot{\eta} \neq 0$, $\eta' = 0$, where $\eta = \mu'^2 e^{-2\lambda} - \dot{\mu}^2 e^{-2\nu}$.

One may consider two sub-cases for both (a) and (b): (α) $\nu \neq \lambda$ and (β) $\nu = \lambda$.

Case $a\,(\alpha)$: For $\nu \neq \lambda$, $\phi_{00} = 0$ implies

$$\nu'\mu' + \lambda'\mu' - \mu'' - \mu'^2 = 0, \tag{2.35}$$

$$\dot{\nu}\,\dot{\mu} + \dot{\lambda}\,\dot{\mu} - \ddot{\mu} - \dot{\mu}^2 = 0, \tag{2.36}$$

$$\nu'\dot{\mu} + \dot{\lambda}\mu' - \dot{\mu}' - \dot{\mu}\mu' = 0. \tag{2.37}$$

Eqs.(2.35) and (2.36) give $(e^{\mu})' = (e^{\mu})^{\cdot}$ which implies that $e^{\mu} = h\,(t + x)$. Using eqs.(2.35) and (2.36), eq.(2.37) reduces to

$$\left(e^{\nu+\lambda}\right)' - \nu' e^{\nu+\lambda} - \dot{\lambda} e^{\nu+\lambda} = 0, \tag{2.38}$$

which instantly gives $\left(e^{\lambda}\right)' = \left(e^{\lambda}\right)^{\cdot}$. Thus, $e^{\lambda} = g\,(t + x)$. Similarly, $e^{\nu} = f\,(t + x)$. For these values of ν, λ and μ, $\dot{\eta} = 0$ and $T = 0$ give $\nu' = \lambda'$ and

$$3\mu'^2 + 2\mu'' + \lambda'' = 0, \tag{2.39}$$

respectively. Substituting the value of $\mu'' + \mu'^2$ from eq.(2.35) in eq.(2.38) yields $\nu' = \mu'$. Hence $\nu' = \lambda' = \mu'$, this gives $\eta' = 0$ which is a contradiction.

Case $a\,(\beta)$: Here $\nu = \lambda$. Therefore, the constraints eqs.(2.8-2.11), respectively reduce to

$$2\nu'\mu' - \mu'' - \mu'^2 + 2\dot{\nu}\dot{\mu} - \ddot{\mu} - \dot{\mu}^2 + \nu'\dot{\mu} + \dot{\lambda}\mu' - \dot{\mu}' - \dot{\mu}\mu' = 0, \tag{2.40}$$

$$2\nu'\mu' - \mu'' - \mu'^2 + 2\dot{\nu}\dot{\mu} - \ddot{\mu} - \dot{\mu}^2 - \nu'\dot{\mu} - \dot{\lambda}\mu' + \dot{\mu}' + \dot{\mu}\mu' = 0, \tag{2.41}$$

$$2\left(\ddot{\mu} - \mu''\right) + 3\left(\dot{\mu}^2 - \mu'^2\right) + \ddot{\nu} - \nu'' = 0. \tag{2.42}$$

Now, we use symmetry method for solving differential equations [8]. On applying the second prolongation of the symmetry generator eq.(1.36)

$$\mathbf{X}^{[2]} = \xi^n\left(x^i, u^\beta\right)\frac{\partial}{\partial x^n} + \eta^\alpha\left(x^i, u^\beta\right)\frac{\partial}{\partial u^\alpha} + \eta^\alpha_n\left(x^i, u^\beta\right)\frac{\partial}{\partial u^\alpha_{,n}} + \eta^\alpha_{nm}\left(x^i, u^\beta\right)\frac{\partial}{\partial u^\alpha_{,nm}},$$

to eq.(2.42) we get

$$2\left(\eta^1_{,tt} - \eta^1_{,xx}\right) + 6\left(\dot{\mu}\eta^1_{,t} - \mu'\eta^1_{,x}\right) + \left(\eta^2_{,tt} - \eta^2_{,xx}\right) = 0, \tag{2.43}$$

where

$$\begin{aligned}
\eta^1_{,t} &= \eta^1_t + \dot{\mu}\left(\eta^1_\mu - \xi^1_t\right) + \dot{\nu}\eta^1_\nu - \mu'\xi^2_t - \dot{\mu}^2\xi^1_\mu - \dot{\nu}\dot{\mu}\xi^1_\nu - \dot{\mu}\mu'\xi^2_\mu - \dot{\nu}\mu'\xi^2_\nu, \\
\eta^1_{,x} &= \eta^1_x + \mu'\left(\eta^1_\mu - \xi^2_x\right) + \nu'\eta^1_\nu - \dot{\mu}\xi^1_x - \mu'^2\xi^2_\mu - \nu'\mu'\xi^2_\nu - \dot{\mu}\mu'\xi^1_\mu - \nu'\dot{\mu}\xi^1_\nu, \\
\eta^1_{,tt} &= \eta^1_{tt} + \dot{\mu}\left(2\eta^1_{t\mu} - \xi^1_{tt}\right) + 2\dot{\nu}\eta^1_{t\nu} - \mu'\xi^2_{tt} + \dot{\mu}^2\eta^1_{\mu\mu} + 2\dot{\nu}\dot{\mu}\eta^1_{\mu\nu} + \dot{\nu}^2\eta^1_{\nu\nu} \\
&\quad -2\dot{\mu}\left(\dot{\mu}\xi^1_{t\mu} + \dot{\nu}\xi^1_{t\nu}\right) - 2\mu'\left(\dot{\mu}\xi^2_{t\mu} + \dot{\nu}\xi^2_{t\nu}\right) - \dot{\mu}^2\left(\dot{\mu}\xi^1_{\mu\mu} + \dot{\nu}\xi^1_{\mu\nu}\right) + \ddot{\nu}\eta^1_\nu \\
&\quad -\dot{\nu}\dot{\mu}\left(\dot{\mu}\xi^1_{\mu\nu} + \dot{\nu}\xi^1_{\nu\mu}\right) - \mu'\left(\dot{\mu}^2\xi^2_{\mu\mu} - \dot{\nu}^2\xi^2_{\nu\nu}\right) - 2\dot{\mu}\dot{\nu}\mu'\xi^2_{\mu\nu} + \ddot{\mu}\left(\eta^1_\mu - 2\xi^1_t\right) \\
&\quad -2\dot{\mu}'\xi^2_t - 3\dot{\mu}\ddot{\mu}\xi^1_\mu - \left(\dot{\mu}\ddot{\nu} + 2\dot{\nu}\ddot{\mu}\right)\xi^1_\nu - \left(\mu'\ddot{\mu} + 2\dot{\mu}\dot{\mu}'\right)\xi^2_\mu - \left(\mu'\ddot{\nu} + 2\dot{\nu}\dot{\mu}'\right)\xi^2_\nu, \\
\eta^1_{,xx} &= \eta^1_{xx} + \mu'\left(2\eta^1_{x\mu} - \xi^2_{xx}\right) + 2\nu'\eta^1_{x\nu} - \dot{\mu}\xi^1_{xx} + \nu'^2\eta^1_{\nu\nu} + 2\nu'\mu'\eta^1_{\mu\nu} + \mu'^2\eta^1_{\mu\mu}
\end{aligned}$$

$$
\begin{aligned}
& -2\dot{\mu}\left(\mu'\xi^1_{x\mu}+\nu'\xi^1_{x\nu}\right)-2\mu'\left(\mu'\xi^2_{x\mu}+\nu'\xi^2_{x\nu}\right)-\dot{\mu}\left(\mu'^2\xi^1_{\mu\mu}+\nu'^2\xi^1_{\nu\nu}\right)\\
& -2\mu'\dot{\mu}\nu'\xi^1_{\mu\nu}-\mu'^2\left(\mu'\xi^2_{\mu\mu}+2\nu'\xi^2_{\mu\nu}\right)-2\nu'^2\mu'\xi^2_{\mu\nu}+\mu''\left(\eta^1_\mu-2\xi^2_x\right)\\
& +\nu''\eta^1_\nu-2\dot{\mu}'\xi^1_x-3\mu'\mu''\xi^2_\mu-\left(\mu'\nu''+2\nu'\mu''\right)\xi^2_\nu-\left(\dot{\mu}\mu''+2\mu'\dot{\mu}'\right)\xi^1_\mu\\
& -\left(\dot{\mu}\nu''+2\nu'\dot{\mu}'\right)\xi^1_\nu,\\
\eta^2_{,tt} = \; & \eta^2_{tt}+\dot{\nu}\left(2\eta^2_{t\nu}-\xi^1_{tt}\right)+2\dot{\mu}\eta^2_{t\mu}-\nu'\xi^2_{tt}+\dot{\mu}^2\eta^2_{\mu\mu}+2\dot{\nu}\dot{\mu}\eta^2_{\mu\nu}+\dot{\nu}^2\eta^2_{\nu\nu}\\
& -2\dot{\nu}\left(\dot{\mu}\xi^1_{t\mu}+\dot{\nu}\xi^1_{t\nu}\right)-2\nu'\left(\dot{\mu}\xi^2_{t\mu}+\dot{\nu}\xi^2_{t\nu}\right)-\dot{\nu}\dot{\mu}\left(\dot{\mu}\xi^1_{\mu\mu}+\dot{\nu}\xi^1_{\mu\nu}\right)\\
& -\dot{\nu}^2\left(\dot{\mu}\xi^1_{\mu\nu}+\dot{\nu}\xi^1_{\nu\nu}\right)-\nu'\dot{\mu}\left(\dot{\mu}\xi^2_{\mu\mu}+\dot{\nu}\xi^2_{\nu\mu}\right)-\nu'\left(\dot{\mu}\dot{\nu}\xi^2_{\nu\mu}+\nu'^2\xi^2_{\nu\mu}\right)+\ddot{\mu}\eta^2_\mu\\
& -2\dot{\nu}'\xi^2_t-3\dot{\nu}\ddot{\nu}\xi^1_\nu+\ddot{\nu}\left(\eta^2_\nu-2\xi^1_t\right)-\left(\nu'\ddot{\nu}+2\dot{\nu}\dot{\nu}'\right)\xi^2_\nu-\left(\nu'\ddot{\mu}+2\dot{\mu}\dot{\nu}'\right)\xi^2_\mu,\\
\eta^2_{,xx} = \; & \eta^2_{xx}+\nu'\left(2\eta^2_{x\nu}-\xi^2_{xx}\right)+2\mu'\eta^2_{x\mu}-\dot{\nu}\xi^1_{xx}+\mu'^2\eta^2_{\mu\mu}+2\nu'\mu'\eta^2_{\mu\nu}+\nu'^2\eta^2_{\nu\nu}\\
& -2\dot{\nu}\left(\mu'\xi^1_{x\mu}+\nu'\xi^1_{x\nu}\right)-2\nu'\left(\mu'\xi^2_{x\mu}+\nu'\xi^2_{x\nu}\right)-\nu'\mu'\left(\mu'\xi^2_{\mu\mu}+2\nu'\xi^2_{\mu\nu}\right)\\
& -\dot{\mu}\nu'\mu'\xi^1_{\mu\nu}-\nu'^2\left(\dot{\nu}\xi^1_{\nu\nu}+\nu'\xi^2_\nu\right)-\dot{\nu}\mu'\left(\mu'\xi^1_{\mu\mu}+\nu'\xi^2_{\nu\nu}\right)+\nu''\left(\eta^2_\nu-2\xi^2_x\right)\\
& +\mu''\eta^2_\mu-2\dot{\nu}'\xi^1_x-3\nu'\nu''\xi^2_\nu-\left(\nu'\mu''+2\mu'\nu''\right)\xi^2_\mu-\left(\dot{\nu}\mu''+2\mu'\dot{\nu}'\right)\xi^1_\mu\\
& -\left(\nu'\nu''+2\dot{\nu}\dot{\nu}'\right)\xi^1_\nu.
\end{aligned}
$$

Substituting $\eta^1_{,t}$, $\eta^1_{,x}$, $\eta^1_{,tt}$, $\eta^1_{,xx}$, $\eta^2_{,tt}$ and $\eta^2_{,xx}$ in eq.(2.43) and comparing the coefficients of different derivative terms of ν and μ we obtain the following system of equations

$$
\begin{aligned}
& \xi^1_\nu=0,\ \xi^1_\mu=0,\ \xi^2_\nu=0,\ \xi^2_\mu=0,\ 2\xi^1_{xx}-2\xi^1_{tt}+6\eta^1_t+2\eta^2_{t\mu}+4\eta^1_{t\mu}=0,\\
& \xi^1_{xx}-\xi^1_{tt}+2\eta^2_{t\nu}+4\eta^1_{t\nu}=0,\ 2\xi^2_{tt}+2\eta^1_{\mu\mu}+\eta^2_{\mu\mu}-2\xi^2_{x\mu}+6\eta^1_\mu-6\xi^2_x=0,\\
& \xi^2_t-\xi^1_x=0,\ \xi^1_{xx}-\xi^1_{tt}+\eta^2_{t\mu}+3\eta^1_t+2\eta^1_{t\mu}=0,\ \xi^1_{xx}-\xi^1_{tt}+4\eta^1_{t\mu}+2\eta^2_{t\nu}=0,\\
& \xi^2_{xx}-\xi^2_{tt}-2\eta^1_{x\mu}-\eta^2_{x\mu}-3\eta^1_x=0,\ 2\eta^1_{\mu\nu}+\eta^2_{\mu\nu}+3\eta^1_\nu=0,\ 2\eta^1_{\nu\nu}+\eta^2_{\nu\nu}=0,\\
& 2\eta^1_{\mu\mu}+\eta^2_{\mu\mu}+6\eta^1_\mu-6\eta^1_\nu-3\eta^2_\nu+6\xi^2_x-6\xi^1_t=0,\ 2\eta^1_\mu+\eta^2_\mu-4\eta^1_\nu-2\eta^2_\nu=0,\\
& 2\eta^1_\mu-4\eta^1_\nu+\eta^2_\mu-2\eta^2_\nu+4\xi^2_x-4\xi^1_t=0,\ 2\eta^1_{tt}+\eta^2_{tt}-2\eta^1_{xx}-\eta^2_{xx}=0.
\end{aligned}
\tag{2.44}
$$

Solving the above system of equations, we get $\xi^1 = \xi^2 = f_1(t+x) + f_2(t+x)$, $\eta^1 = c_1\mu + c_2$ and $\eta^2 = 2c_1(\mu+\nu) + c_3t + c_4$. Substituting these values in the above generator yields

$$\begin{aligned} \mathrm{X}^{[2]} &= [f_1(t+x) + f_2(t+x)]\left(\tfrac{\partial}{\partial t} + \tfrac{\partial}{\partial x}\right) + (c_1\mu + c_2)\tfrac{\partial}{\partial \mu} \\ &+ (2c_1(\mu+\nu) + c_3t + c_4)\tfrac{\partial}{\partial \nu}. \end{aligned} \tag{2.45}$$

Therefore,

$$\frac{dt}{f_1(t+x) + f_2(t+x)} = \frac{dx}{f_1(t+x) + f_2(t+x)}. \tag{2.46}$$

This gives $t = x$, and hence $\phi_{11} = 0$, so the case is not possible.

(*b*) Similarly for the case $\nu' \neq 0$, $\dot{\eta} \neq 0$, $\eta' = 0$, there does not exist any sourceless electromagnetic field.

2.3 Spacetimes Admitting Sourceless Electromagnetic Fields

Now, we consider the 4−isometry case. There are only four such metrics [16]:

$$ds^2 = e^{2\nu(x)}dt^2 - e^{2\lambda(x)}dx^2 - e^{2\mu(x)}\left(dy^2 + dz^2\right), \tag{2.47}$$

$$ds^2 = e^{2\nu(t)}dt^2 - e^{2\lambda(t)}dx^2 - e^{2\mu(t)}\left(dy^2 + dz^2\right), \tag{2.48}$$

$$ds^2 = e^{2\nu(x)}dt^2 - dx^2 - e^{2\frac{x}{c}}(dy^2 + dz^2), \tag{2.49}$$

$$ds^2 = dt^2 - e^{2\lambda(t)}dx^2 - e^{2\frac{t}{c}}(dy^2 + dz^2). \tag{2.50}$$

Here eq.(2.48) and eq.(2.50) are the non-static analogues of eq.(2.47) and eq.(2.49) respectively. For the metric eq.(2.47), one could choose to absorb $\lambda(x)$ in the definition of the variable x. Instead, we find it more convenient to redefine x so as to make

$\mu = \ln \frac{x}{X}$, $(X = constt.)$ which is reminiscent of the coefficient of $d\Omega^2$ for spherically symmetric metrics. Using this choice of redefined x, $\phi_{00} = 0$ yields $\nu' + \lambda' = 0$. The condition $T = 0$ gives

$$T = \nu'' + 2\nu'^2 + 4\frac{\nu'}{x} + \frac{1}{x^2} = 0,$$

$e^{2\nu} = \frac{m}{x} + \frac{q^2}{x^2}$ $(m, q = constt.)$. Then the metric becomes

$$ds^2 = \left(\frac{m}{x} + \frac{q^2}{x^2}\right) dt^2 - \left(\frac{m}{x} + \frac{q^2}{x^2}\right)^{-1} dx^2 - \left(\frac{x}{X}\right)^2 (dy^2 + dz^2), \qquad (2.51)$$

admitting 4−isometries, which is the McVittie solution [19].

Similarly, the second case gives a non-static analogue of the McVittie spacetime [20]

$$ds^2 = \left(\frac{m}{t} - \frac{q^2}{t^2}\right)^{-1} dt^2 - \left(\frac{m}{t} - \frac{q^2}{t^2}\right) dx^2 - \left(\frac{t}{T}\right)^2 (dy^2 + dz^2), \ (T = constt.), \qquad (2.52)$$

also admitting 4-isometries. The non-zero components of the electric and magnetic field intensities for the metrics (2.51) and (2.52) have the relations $2q^2 = x^4 E_1^2 + X^4 H_1^2$ and $2q^2 = t^4 E_1^2 + T^4 H_1^2$ respectively. These metrics have coordinate and essential singularities at x (or t) $= -\frac{q^2}{m}$ and x (or t) $= 0$ respectively. They belong to Segre type $[(1,1),(11)]$ and Petrov type D.

For the metric eq.(2.49),

$$\begin{aligned}
&\kappa T_{00} = -3\tfrac{1}{a^2} e^{2\nu}, \ \kappa T_{11} = \tfrac{1}{a^2} + 2\tfrac{\nu'}{a}, \\
&\kappa T_{22} = \left(\nu'' + \nu'^2 + \tfrac{\nu'}{a} + \tfrac{1}{a^2}\right) e^{2\frac{x}{a}} = \kappa T_{33}, \\
&T = 2\left(\nu'' + \nu'^2 + 2\tfrac{\nu'}{a} + \tfrac{3}{a^2}\right), \\
&\phi_{00} = \left(\tfrac{\dot{\nu}}{a} - \tfrac{1}{a^2}\right) = \phi_{22} \text{ and } \phi_{11} = \tfrac{1}{2}\left(\ddot{\nu} + \dot{\nu}^2 - \tfrac{1}{a^2}\right),
\end{aligned}$$

then the condition $\phi_{00} = 0$ gives $\dot{\nu} = \frac{1}{a}$ and the condition $\phi_{11} \neq 0$ yields $\dot{\nu} \neq 0$. Hence the two conditions are inconsistent. For the metric eq.(2.50) the components of the energy-momentum tensor are

$$\kappa T_{00} = \frac{1}{a^2} + 2\frac{\dot{\lambda}}{a},\ \kappa T_{11} = -3e^{2\lambda}\frac{1}{a^2},$$
$$\kappa T_{22} = e^{2\frac{t}{a}}\left(\ddot{\lambda} + \dot{\lambda}^2 + \frac{\dot{\lambda}}{a} + \frac{1}{a^2}\right) = \kappa T_{33},$$
$$T = -2\left(\ddot{\lambda} + \dot{\lambda}^2 + 2\frac{\dot{\lambda}}{a} + \frac{3}{a^2}\right),$$
$$\phi_{00} = \left(\frac{\dot{\lambda}}{a} - \frac{1}{a^2}\right) = \phi_{22} \text{ and } \phi_{11} = \frac{1}{2}\left(\ddot{\lambda} + \dot{\lambda}^2 - \frac{1}{a^2}\right),$$

one can see very easily that the conditions $\phi_{00} = 0$ and $\phi_{11} \neq 0$ are inconsistent.

In light of the above we can state the following:

Theorem 5 *The only plane symmetric spacetimes with electromagnetic field are the McVittie solution given by metric eq.(2.51) and its non-static analogue given by metric eq.(2.52).*

2.4 Non-Static, Spherically Symmetric, Shear-Free, Perfect Fluid Solutions of Einstein's Equations

Schwarzschild provided the first exact solution [21] of Einstein's equations. He also provided an interior solution which is static [22]. This could be taken as a first approximation to a relativistic description for a star. However, it is of interest to find non-static solutions that can describe stars not in equilibrium. The Einstein equations for non-static, shear-free, spherically symmetric, perfect fluid distributions reduce to one second-order non-linear ordinary differential equation in the radial parameter.

Kustaanheimo and Qvist [23] found that the isotropic condition for a perfect fluid with no shear reduces the Einstein equations to

$$\frac{\partial^2 y}{\partial x^2} = f(x)y^2, \tag{2.53}$$

where $x = r^2$, $y = e^{-\omega(r,t)/2}$. The spherically symmetric metric in isotropic coordinates is

$$ds^2 = e^{\nu(t,r)}dt^2 - e^{\omega(t,r)}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2). \tag{2.54}$$

The function ν is given by $e^{\frac{\nu}{2}} = c\,(t)\,/\omega_t$, where $c\,(t)$ is an arbitrary function of t and the Einstein field eqs.(2.1) in parametric form are

$$\begin{aligned}
\kappa\rho &= -y^4 + 4y^2\left(2y'^2 - yy''\right) + 192\frac{\dot{y}^6}{c^4 y^6}, \\
\kappa p &= y^4 + 4\left(2\frac{y'\dot{y}'y^3}{\dot{y}} - y'^2 y^2\right) - 64\frac{\dot{y}^4}{c^4 y^2}\left(6\frac{\dot{y}^2}{y^4} - 2\frac{\dot{y}\ddot{y}}{y^4} - \frac{\ddot{y}}{y^3} + 2\frac{\dot{c}\ddot{y}}{cy^3}\right), \\
\kappa p &= 2y^4\left(-\frac{\dot{y}''}{\dot{y}} + 3\frac{\dot{y}'^2}{\dot{y}^2} - 2\frac{y'^2}{y^2}\right) - 64\frac{\dot{y}^4}{c^4 y^4}\left(-\frac{\ddot{y}}{y} + 6\frac{\dot{y}^2}{y^2} + 2\frac{\dot{c}\dot{y}}{cy} - 2\frac{\dot{y}^2\ddot{y}}{y^2}\right), \\
0 &= -2\dot{y}'y + \dot{y}y',
\end{aligned}$$

where ρ is the energy density, p is the pressure and κ is the gravitational constant. Solutions of eq.(2.53) have been found by three different approaches; by making ad hoc ansatz for the metric coefficients or for the function $f\,(x)$, by finding those $f\,(x)$ which give only one or two Lie point symmetries or Noether symmetries of the equation and by search of those solutions which have the Painlevē property. (An ordinary differential equation is said to have the Painlevé Property when every solution is single valued, except at the fixed singularities of the coefficients. That is, the Painlevé Property requires that the movable singularities are no worse than poles. A partial differential equation has the Painlevé Property when the solutions of the partial differential equation are single valued about the movable, singularity manifolds.). Specific solutions of this equation were found by Kustaanheimo and Qvist [23] and subsequently others have further analyzed it [24]- [26]. Some exact solutions of (2.53) are also tabulated in [7]. Stephani [27] has used symmetry methods and obtained some solutions but, as he notes there, it is difficult to extract physical information from these solutions as the relevant quantities are only given implicitly.

A general procedure to find all solutions obtainable by symmetry analysis was provided by Wafo Soh and Mahomed [11]. They studied the Noether point symmetries of eq.(2.53). Noether point symmetries are the Lie point symmetries associated with some conservation laws or first integral. A symmetry, $\mathbf{X}$, is a Noether point symmetry corresponding to a Lagrangian, $L(x, y, y')$, of a second order ordinary differential equation of the form $y'' = f(x, y, y')$, if there exists a function $A(x, y)$ such that

$$\mathbf{X}^{[1]}L + \frac{d\xi}{dx}L = \frac{dA}{dx}. \tag{2.55}$$

A second order ordinary differential equation with only one Noether symmetry is integrable by quadratures. Thus, the advantage of studying Noether symmetries corresponding to a Lagrangian of eq.(2.53) is that it is a more fundamental approach for integrability in that a Noether symmetry gives via the Noether theorem an explicit integral which itself is invariant. They found the natural Lagrangian of this equation

$$L = \frac{y'^2}{2} + g(x)\frac{y^3}{3}. \tag{2.56}$$

They gave three cases that recover some previously known results and also give some new results. These cases include solutions that gave the metric coefficients explicitly (directly or parametrically) or implicitly in terms of integrals.

By substituting eq.(2.56) into eq.(2.55) , they obtain

$$\begin{aligned}
\xi &= a(x), \ \eta = \frac{1}{2}a'y + b(x), \\
A &= \frac{1}{4}a''y^2 + b'y + B(x), \\
& y^3\left(ag' + \frac{5}{2}ga'\right) + 3y^2\left(bg - \frac{1}{4}a'''\right) - 3b''y - 3B' = 0,
\end{aligned}$$

the last equation gives

$$\begin{aligned}
ag' + \frac{5}{2}ga' &= 0, \\
4bg - a''' &= 0, \\
b &= \alpha x + \beta, \ B = constt.,
\end{aligned}$$

where α and β are constants. The following cases arise:

Case I: $\alpha = 0 = \beta$, (i.e. $b = 0$) ,

Case II: $\alpha \neq 0 = \beta$, (i.e. $b \neq 0$) ,

Case III: $\alpha \neq 0 \neq \beta$, (i.e. $b \neq 0$) .

Case I gives explicit solutions that have been discussed before and a tabulation of the interesting physical case is given in [7]. In general, the solutions of Cases II and III are given implicitly in terms of integrals and are consequently difficult to interpret physically. There are two explicit one-parameter families of solutions to which they reduce

$$\begin{aligned}
y &= \beta^{1/7}\left(\frac{3}{2}\right)^{-1/7}\left(-\frac{2}{7}\right)^{-3/7}\left[(x+k)^{1/7}\beta^{-2/7}\left(\frac{3}{2}\right)^{-3/14}\left(-\frac{2}{7}\right)^{-1/7} + C(t)\right]^{-2} \\
& (x+k)^{3/7} - \beta^{5/7}\left(\frac{3}{2}\right)^{2/7}\left(-\frac{2}{7}\right)^{-1/7}(x+k)^{1/7}, \qquad (2.57)
\end{aligned}$$

$$f(x) = \beta^{-5/7}\left(\frac{3}{2}\right)^{5/7}\left(-\frac{2}{7}\right)^{15/7}(x+k)^{-15/7}, \tag{2.58}$$

and

$$\begin{aligned} y &= \alpha^{-3/7}\left(\frac{3}{2}\right)^{-1/7}\left(\frac{2}{7}\right)^{-3/7}(\alpha x+)^{4/7}(1-k-k\alpha x)^{3/7} \\ &\times\left\{(1-k\beta-k\alpha x)^{1/7}(\alpha x+\beta)^{-1/7}\alpha^{-1/7}\left(\frac{3}{2}\right)^{-3/14}\left(-\frac{2}{7}\right)^{-1/7}+C(t)\right\}^{-2} \\ &-\alpha^{-1/7}\left(\frac{3}{2}\right)^{2/7}\left(\frac{2}{7}\right)^{-1/7}(\alpha x+\beta)^{6/7}(1-k\beta-k\alpha x)^{1/7}, \qquad (2.59) \\ f(x) &= \alpha^{15/7}\left(\frac{3}{2}\right)^{5/7}\left(\frac{2}{7}\right)^{15/7}(\alpha x+\beta)^{-20/7}(1-k\beta-k\alpha x)^{-15/7}, \qquad (2.60) \end{aligned}$$

where k is arbitrary constants and $C(t)$ is an arbitrary function of time.

The example of new physical solutions obtained by Wafo Soh and Mahomed [11] is erroneous. Einstein's field equations for the metric [7]

$$ds^2 = e^{2\nu(r,t)}dt^2 - e^{2\lambda(r,t)}dr^2 - Y^2(r,t)\,d\Omega^2, \tag{2.61}$$

are

$$\begin{aligned} \kappa_0\mu &= \frac{1}{Y^2} - \frac{2}{Y}e^{-2\lambda}\left(Y'' - Y'\lambda' + \frac{Y'^2}{2Y}\right) + \frac{2}{Y}e^{-2\nu}(\dot{Y}\dot{\lambda} + \frac{\left(\dot{Y}\right)^2}{2Y}), \\ \kappa_0 p &= \frac{-1}{Y^2} + \frac{2}{Y}e^{-2\lambda}(Y'\nu' + \frac{Y'^2}{2Y}) - \frac{2}{Y}e^{-2\nu}(\ddot{Y} + \frac{\dot{Y}^2}{2Y} - \dot{Y}\dot{\nu}), \\ \kappa_0 p &= \frac{1}{Y}[e^{-2\lambda}\{\left(\nu'' + \nu'^2 - \nu'\lambda'\right)Y + Y'' - Y'\nu' - Y'\lambda'\} \\ &\quad -e^{-2\nu}\{(\ddot{\lambda} + \dot{\lambda}^2 - \dot{\nu}\dot{\lambda}\}Y + \ddot{Y} + \dot{Y}\dot{\lambda} - \dot{Y}\dot{\nu}], \\ 0 &= \dot{Y}' - \nu'\dot{Y} - Y'\dot{\lambda}. \end{aligned}$$

Here the 4–velocity vector is

$$u^i = \left(e^{-\nu}, 0, 0, 0\right), \quad i = 0, 1, 2, 3,$$

Therefore, for a solution without acceleration $\nu' = 0$, i.e. $\nu = \nu(t)$. Now for $Y' \neq 0$, we may choose $\nu = 0$ these equations become

$$\kappa_0\mu = \frac{1}{Y^2} - \frac{2}{Y}e^{-2\lambda}\left(Y'' - Y'\lambda' + \frac{Y'^2}{2Y}\right) + \frac{2}{Y}(\dot{Y}\dot{\lambda} + \frac{\left(\dot{Y}\right)^2}{2Y}), \tag{2.62}$$

$$\kappa_0 p = \frac{-1}{Y^2} + \frac{2}{Y}e^{-2\lambda}\frac{Y'^2}{2Y} - \frac{2}{Y}(\ddot{Y} + \frac{\dot{Y}^2}{2Y}), \tag{2.63}$$

$$\kappa_0 p = \frac{1}{Y}[e^{-2\lambda}(Y'' - Y'\lambda') - Y\ddot{\lambda} - \dot{\lambda}^2 Y - \ddot{Y} - \dot{Y}\dot{\lambda}], \tag{2.64}$$

$$\frac{d^2\lambda}{dt^2} = \frac{\ddot{Y}'}{Y'} - \left(\frac{\dot{Y}'}{Y'}\right)^2. \tag{2.65}$$

Substituting eq.(2.65) in eq.(2.62) and eq.(2.64) to obtain

$$\kappa_0\mu = \frac{1}{Y^2} - \frac{2}{Y}e^{-2\lambda}\left(Y'' - Y'\lambda' + \frac{Y'^2}{2Y}\right) + \frac{2}{Y}(\dot{Y}\frac{\dot{Y}'}{Y'} + \frac{\left(\dot{Y}\right)^2}{2Y}), \tag{2.66}$$

$$\kappa_0 p = \frac{1}{Y}[e^{-2\lambda}(Y'' - Y'\lambda') - Y\left(\frac{\ddot{Y}'}{Y'} - \frac{\dot{Y}'^2}{Y'}\right) - \left(\frac{\dot{Y}'}{Y'}\right)^2 Y - \ddot{Y} - \dot{Y}\frac{\dot{Y}'}{Y'}],$$

$$\Rightarrow e^{-2\lambda}(Y'' - Y'\lambda') = \kappa_0 pY + Y\frac{\ddot{Y}'}{Y'}\ddot{Y} + \dot{Y}\frac{\dot{Y}'}{Y'}. \tag{2.67}$$

Substituting eq.(2.67) in eq.(2.66) we get

$$\kappa_0\mu = -3\kappa_0 p - 2\frac{\ddot{Y}'}{Y'} - 4\frac{\ddot{Y}}{Y}. \tag{2.68}$$

The last term of the above equation i.e. $-4\ddot{Y}/Y$ is missing in [7] (in the revised edition this error has been removed). The authors carried over this mistake in their paper. The second error is in the calculations of the example of the solution

$$Y = \rho^{2/3}[a(r)\int^{t} \rho^{-2}(\xi, r)\, d\xi + b(r)]^{2/3},$$

given by Wafo Soh and Mohamed. They take

$$\rho = t^{\frac{-1}{2}},\ a(r) = 2r,\ b(r) = -1.$$

For these values the correct expressions for the metric coefficients satisfying the Einstein field equations along with the conservation of the energy-momentum tensor, $T^{ab};b = 0$ are

$$Y = t^{\frac{-1}{3}}[2r\int^{t}\xi d\xi - 1]^{\frac{2}{3}}, \tag{2.69}$$

$$Y = t^{\frac{-1}{3}}\left(rt^2 - 1\right)^{\frac{2}{3}}, \ \lambda = \frac{1}{2}\ln\frac{4}{9}\frac{\left(\sqrt[3]{t}\right)^{10}}{\left(\sqrt[3]{(rt^2-1)}\right)^2}, \tag{2.70}$$

and the values of ρ and p from eqs.(2.62 − 2.64) are

$$\rho = \frac{1}{\kappa_0 t^2}\frac{3rt^2+1}{rt^2-1}, \tag{2.71}$$

$$p = \frac{-1}{\kappa_0 t^2}. \tag{2.72}$$

The values given by eqs.(2.70 − 2.72) satisfy eq.(2.68).

It is worthwhile to construct valid physical examples for these solutions. We consider only the two explicit solutions to try and construct physically meaningful examples. For this purpose we need to first formulate the criteria for being *physically meaningful*.

2.5 The Physical Criteria

The most important physical criterion is that the density be non-negative everywhere. The next criterion must be that both the density and pressure remain finite everywhere. If they become infinite somewhere, that place should correspond to a curvature singularity and would automatically limit the part of the manifold that can be considered. In other words, the singular point(s) must be cut out from the manifold. Additionally, one would ask that the strong energy condition holds.

There could be more subtle considerations. The spherically symmetric spacetime could be used to represent spheres of perfect fluids of finite size. Such spheres should have zero surface pressure and match with a Schwarzschild exterior. If these conditions cannot be met, the metric would have to go on to describe the whole universe. However, such a description violates the cosmological principle unless the density and pressure are constant everywhere (though they could vary with time). As such, we will require that the metric should admit a zero pressure somewhere and contain a positive density inside that boundary, while also satisfying the strong energy condition.

2.5.1 The First Case Example

Let $u = (x+k)^{1/7}$ in eq.(2.57), then it can be written as

$$y = -\frac{a}{b^2} u C(t) \frac{(2u + C(t))}{(u + C(t))^2}, \tag{2.73}$$

where

$$\begin{aligned} a &= \beta^{1/7} \left(\frac{3}{2}\right)^{-1/7} \left(-\frac{2}{7}\right)^{-3/7}, \\ b &= \beta^{-2/7} \left(\frac{3}{2}\right)^{-3/14} \left(-\frac{2}{7}\right)^{-1/7}, \end{aligned}$$

and $C(t)$ now refers to $bC(t)$. In this case we have the density and pressure given by

$$\begin{aligned}\rho(t,x) &= \frac{3}{8\pi}+\frac{3C^3(t)\,a^2}{98\pi u^{12}\,[u+C(t)]^6\,b^4}[10u^{10}+20u^9C(t)+12u^8C^2(t)\\ &\quad +2u^7C^3(t)+32u^3k+57u^2kC(t)+30ukC^2(t)+5kC^3(t)], \quad (2.74)\end{aligned}$$

$$p(t,x)=\frac{-3}{8\pi}+\frac{C^4(t)\,a^2}{98\pi u^{12}\,[u+C(t)]^5\,b^4}[5u^8+4u^7C(t)+9uk+3kC(t)]. \quad (2.75)$$

Here, β, k and $C(t)$ would be constrained by the requirement that $\rho(t,x)\geq 0$. Further, $\rho(t,x)$ and $p(t,x)$ must not be singular and satisfy the strong energy condition ($T^{\mu}_{\mu}\geq 0$):

$$\begin{aligned}T^{\mu}_{\mu} &= \rho+3p=\frac{-3}{4\pi}+\frac{3C^3(t)a^2}{98\pi u^{12}(u+C(t))^6b^4}[10u^{10}+25u^9C(t)+21u^8C^2(t)\\ &\quad +6u^7C^3(t)+32ku^3+66u^2kC(t)+42ukC^2(t)+8kC^3(t)]. \quad (2.76)\end{aligned}$$

We can keep the energy density positive and satisfy the strong energy condition as there is enough freedom in selecting the constants and the function of time $C(t)$. Clearly, taking $k\geq 0$, and $C(t)>0$ would maintain the requirement that the density remain positive. Also the trace of the stress-energy tensor will remain non-negative by appropriately choosing β.

We investigate the case $k=0$ and consider a non-static sphere of radius R matched to a Schwarzschild exterior geometry. In that case we get the matching condition $p|_{r=R}=0$. The condition on $C(t)$ becomes

$$147b^4u^5\,[u+C(t)]^5-4a^2C^4(t)[5u+4C(t)]=0, \quad (2.77)$$

and the equation of state is

$$\rho-\frac{3}{8\pi}=6\left(p+\frac{3}{8\pi}\right)\frac{5\left(r^{2/7}\right)^2+5r^{2/7}C(t)+C^2(t)}{5r^{2/7}+4C(t)}. \quad (2.78)$$

A solution of eq.(2.77) has the form

$$u(t) = \gamma(t)C(t), \tag{2.79}$$

where $\gamma(t)$ is as yet an arbitrary function and $C(t) > 0$ satisfies

$$C(t) = \left[\frac{4a^2\left(4+5\gamma\left(t\right)\right)}{147b^4\gamma^5\left(t\right)\left(1+\gamma\left(t\right)\right)}\right]^{1/5}. \tag{2.80}$$

Thus the moving boundary is given by

$$R(t) = \gamma^{7/2}(t)\left[\frac{4a^2\left(4+5\gamma\left(t\right)\right)}{147b^4\gamma^5\left(t\right)\left(1+\gamma\left(t\right)\right)^5}\right]^{7/10}. \tag{2.81}$$

Further the strong energy condition, $\rho(t) + 3p(t) \geq 0$, imposes

$$\gamma(t) > 0, \tag{2.82}$$

which avoids the singularities $\gamma(t) = -1$ and $\gamma(t) = 0$.

2.5.2 The Second Case Example

Here we let $u = (\alpha x + \beta)^{1/7}$ in eq.(2.59), then it becomes

$$y = \frac{a}{b^2}u^7 C(t)(1-ku^7)^{1/7}\frac{\left[2\left(1-ku^7\right)^{1/7} - uC\left(t\right)\right]}{\left[-uC\left(t\right) + \left(1-ku^7\right)^{1/7}\right]^2}, \tag{2.83}$$

where $a = \alpha^{-3/7}\left(\frac{3}{2}\right)^{-1/7}\left(\frac{2}{7}\right)^{-3/7}$, $b = \alpha^{-1/7}\left(\frac{3}{2}\right)^{-3/14}\left(\frac{2}{7}\right)^{-1/7}$ and $C(t)$ now refers to $bC(t)$. We specialize and study the situation when $k = 0$. Eq.(2.83) then reduces to

$$y = \frac{a}{b^2}u^7 C(t)\frac{\left[2 - uC\left(t\right)\right]}{\left[1 - uC\left(t\right)\right]^2}. \tag{2.84}$$

The density and pressure are given by

$$\begin{aligned}\rho(t,x) &= \frac{3}{8\pi} + \frac{C^2\left(t\right)}{2\pi a^2 b^6 \alpha\left[1-uC\left(t\right)\right]^6}[-5C(t)u^8 + 10C^2(t)u^9 - 6C^3(t)u^{10} + C^4(t)u^{11} \\ &\quad -268\beta C(t)u + 270\beta C^2(t)u^2 - 120\beta C^3(t)u^3 + 20\beta C^4(t)u^4 + 98\beta],\end{aligned} \tag{2.85}$$

$$p(t,x) = \frac{-3}{8\pi} + \frac{C^2(t)\,a^2\alpha}{98\pi b^4\,[1-uC(t)]^6}[-149C^2(t)u^9 - 9C^3(t)u^{10} - 116C^4(t)u^{11}$$
$$-4\beta\{49 - 147C(t)u + 122C^2(t)u^2 - 81C^3(t)u^3 - 15C^4(t)u^4\}]. \quad (2.86)$$

Here β and $C(t)$ would be constrained by the requirement that $\rho(t,x) \geq 0$ and further that $\rho(t,x)$ and $p(t,x)$ be non-singular and satisfy the strong energy condition.

We match a non-static sphere of radius R to a Schwarzschild exterior. The matching condition $p|_{r=R} = 0$ is invoked. The condition on $C(t)$ is then

$$147b^4(1-uC(t))^6 + 4\alpha a^2C^2(t)[149C^2(t)u^9 + 9C^3(t)u^{10} + 116C^4(t)u^{11}$$
$$+4\{49 - 147C(t)u + 122C^2(t)u^2 - 81C^3(t)u^3 - 15C^4(t)u^4\}] = 0, \quad (2.87)$$

and the equation of state for $a = b^2$ and $\beta = 0$ is

$$\rho - \frac{3}{8\pi} = \left(p + \frac{3}{8\pi}\right)\frac{4.4682}{C(t)\left(\sqrt[7]{r}\right)^2} \quad (2.88)$$
$$\times \frac{\left[-5.0 + 13.429C(t)\left(\sqrt[7]{r}\right)^2 - 10.82C^2(t)\left(\sqrt[7]{r}\right)^4 + 2.4216C^3(t)\left(\sqrt[7]{r}\right)^6\right]}{\left[-149.0 - 12.086C(t)\left(\sqrt[7]{r}\right)^2 - 209.18C^2(t)\left(\sqrt[7]{r}\right)^4\right]}.$$

A solution to eq.(2.87) is

$$u(t) = \gamma(t)/C(t), \quad (2.89)$$

where $\gamma(t)$ is as yet arbitrary and $C(t)$ satisfies

$$C(t) = \left[\frac{\alpha a^2\gamma^9(t)\left[149 + 9\gamma(t) + 116\gamma^2(t)\right]}{98\pi b^4\left(1-\gamma(t)\right)^6}\right]^{1/5} \quad (2.90)$$
$$\times\left[\frac{3}{8\pi} + 4\{49 - 147\gamma(t) + 122\gamma^2(t) - 81\gamma^3(t) - 15\gamma^4(t)\}\right]^{-1/5}.$$

Therefore, the moving boundary is given by

$$R(t) = \alpha^{-1/2}\gamma^{7/2}(t) \times \left[\frac{-\alpha a^2\gamma^9(t)\left[149 + 9\gamma(t) + 116\gamma^2(t)\right]}{98\pi b^4\left(1-\gamma(t)\right)^6}\right]^{-7/10}$$
$$\times\left[\frac{3}{8\pi} + 4\beta\{49 - 147\gamma(t) + 122\gamma^2(t) - 81\gamma^3(t) - 15\gamma^4(t)\}\right]^{7/10}. \quad (2.91)$$

Moreover the strong energy condition, $\rho(t)+3p(t)\geq 0$, for $\gamma(t)>1$ requires that

$$[\tfrac{3}{8\pi}+4\beta\{49-147\gamma(t)+122\gamma^2(t)-81\gamma^3(t)-15\gamma^4(t)\}]\times[5\gamma^8(t)-10\gamma^9(t)$$
$$+6\gamma^{10}(t)-\gamma^{11}(t)+\{268\gamma(t)-270\gamma^2(t)+120\gamma^3(t)-20\gamma^4(t)-98\}]\geq 0. \tag{2.92}$$

Also $R(t)$ in eq.(2.91) is non-negative provided

$$\frac{3}{8\pi}+4\beta\{49-147\gamma(t)+122\gamma^2(t)-81\gamma^3(t)-15\gamma^4(t)\}<0, \tag{2.93}$$

which does not hold for $\beta=0$. However, one can choose a non-zero β so that eq.(2.92) and eq.(2.93) are both satisfied. For example, one can select $\beta=1$ for appropriate γ values.

To summarize, in [11] the authors solved the eq.(2.53) with the help of Noether point symmetry and provided some physically acceptable solutions of the Einstein field equations. We have constructed two classes of exact solutions of the Einstein field equations for non-static, spherically symmetric, shear-free, perfect fluids which could be matched to a Schwarzschild exterior geometry.

Chapter 3

The Connection Between Isometries and Symmetries of Geodesic Equations of the Underlying Spaces

As mentioned earlier, the symmetry properties of a space are characterized by its isometries, which form a Lie algebra [28]. On the other hand, there are methods to determine the point symmetries of any set of differential equations which are often used to solve those equations. These symmetries also form a Lie algebra [2], [9]. In this chapter we investigate and present a connection between the isometries of some spaces and symmetries of the related differential equations. It turns out that the connection lies in the geodesic equations, which give the shortest distance between two points. They are given by

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \tag{3.1}$$

where the dot represents the derivative with respect to the arc length parameter, s and

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}\left(g_{bd,c} + g_{cd,b} - g_{bc,d}\right). \tag{3.2}$$

For our investigation, we determine the symmetries of eq.(3.1) and compare them with the set of isometries of maximally symmetric spaces. The geodesic equations, form a system of ordinary differential equations. Therefore, their symmetries can be obtained by following theorem 2.

In the following sections we first give an alternate method to prove that a second order vector differential equation,

$$(y^a)'' = 0, \ a = 1, 2, 3, ..., n, \tag{3.3}$$

admits $sl(n+2)$, after that we present the symmetries of the geodesic equations of maximally and less symmetric 2−dimensional spaces, then the symmetries of the geodesic equations of maximally symmetric higher dimensional spaces are given.

3.1 The Lie algebra of a Second Order Vector Differential Equation $(y^a)'' = 0$

Eq.(3.3) represents the family of straight lines in $\mathbb{R}^n$. They correspond to the world lines of free particles. The classification of all scalar second order ordinary differential equations, according to the Lie algebra of generators they admit, is complete [29]. Eq.(3.3) for $n = 1$ admits eight Lie symmetries [2], which is the maximum number of symmetries admitted by any second order ordinary differential equation defined on a domain in the plane [29]. This Lie algebra is isomorphic to $sl\,(3)$ [30]. The maximality of the Lie algebra of $y'' = 0$ was proved by Lie, using a geometric argument [31]. The fact that the algebra is $sl\,(3)$ and its $n-$dimensional generalization, is demonstrated in the following section by an elegant and simple algebraic method. This result has also been derived earlier by A. V. Aminova [32], using a different approach.

3.2 The Algebra of $(y^a)'' = 0$

Ibragimov [8] gives the symmetry generators of the 3−dimensional vector equation without appealing to geometry. The operator is a linear combination of the following independent infinitesimal symmetries

$$\begin{aligned} &\mathbf{X}_0 = \tfrac{\partial}{\partial s},\;\; \mathbf{X}_a = \tfrac{\partial}{\partial y^a},\;\; \mathbf{S} = s\tfrac{\partial}{\partial s},\;\; \mathbf{P}^a = y^a\tfrac{\partial}{\partial s},\;\; \mathbf{Q}_a = s\tfrac{\partial}{\partial y^a}, \\ &\mathbf{Y}^a_b = y^a\tfrac{\partial}{\partial y^b},\;\; \mathbf{Z}^0 = s^2\tfrac{\partial}{\partial s} + sy^b\tfrac{\partial}{\partial y^b},\;\; \mathbf{Z}^a = sy^a\tfrac{\partial}{\partial s} + y^a y^b\tfrac{\partial}{\partial y^b}, \end{aligned} \tag{3.4}$$

where we have used the Einstein summation convention that repeated indices are summed over and re-written the indices so that they balance. The symmetries for the scalar equation are

$$\begin{aligned} &\mathbf{X}_0 = \tfrac{\partial}{\partial s},\;\; \mathbf{X}_1 = \tfrac{\partial}{\partial y},\;\; \mathbf{X}_2 = s\tfrac{\partial}{\partial s},\;\; \mathbf{X}_3 = y\tfrac{\partial}{\partial s},\;\; \mathbf{X}_4 = s\tfrac{\partial}{\partial y}, \\ &\mathbf{X}_5 = y\tfrac{\partial}{\partial y},\;\; \mathbf{X}_6 = s^2\tfrac{\partial}{\partial s} + sy\tfrac{\partial}{\partial y},\;\; \mathbf{X}_7 = sy\tfrac{\partial}{\partial s} + y^2\tfrac{\partial}{\partial y}. \end{aligned} \tag{3.5}$$

Eq.(3.5) shows that the scalar (i.e. $n = 1$) equation has eight symmetries and eq.(3.4) shows that the 3−dimensional vector (i.e. $n = 3$) equation has twenty four symmetries. To generalize the result for the $n-$dimensional vector equation we determine the Lie algebra for the 2−dimensional equation. It turns out that the number of infinitesimal generators is fifteen. These are

$$\begin{aligned}
&\mathbf{X}_0 = \tfrac{\partial}{\partial s},\ \mathbf{X}_1 = \tfrac{\partial}{\partial y^1},\ \mathbf{X}_2 = \tfrac{\partial}{\partial y^2},\ \mathbf{X}_3 = s\tfrac{\partial}{\partial s},\ \mathbf{X}_4 = y^1\tfrac{\partial}{\partial s},\ \mathbf{X}_5 = y^2\tfrac{\partial}{\partial s},\\
&\mathbf{X}_6 = s\tfrac{\partial}{\partial y^1},\ \mathbf{X}_7 = s\tfrac{\partial}{\partial y^2},\ \mathbf{X}_8 = y^1\tfrac{\partial}{\partial y^1},\ \mathbf{X}_9 = y^1\tfrac{\partial}{\partial y^2},\ \mathbf{X}_{10} = y^2\tfrac{\partial}{\partial y^1},\\
&\mathbf{X}_{11} = y^2\tfrac{\partial}{\partial y^2},\ \mathbf{X}_{12} = s^2\tfrac{\partial}{\partial s} + sy^1\tfrac{\partial}{\partial y^1} + sy^2\tfrac{\partial}{\partial y^2},\\
&\mathbf{X}_{13} = sy^1\tfrac{\partial}{\partial s} + \left(y^1\right)^2\tfrac{\partial}{\partial y^1} + y^1y^2\tfrac{\partial}{\partial y^2},\ \mathbf{X}_{14} = sy^2\tfrac{\partial}{\partial s} + y^1y^2\tfrac{\partial}{\partial y^1} + \left(y^2\right)^2\tfrac{\partial}{\partial y^2}.
\end{aligned}\tag{3.6}$$

The Lie algebra for this set of generators is $sl\,(4)$. In fact, for $n=3$, the generators in eq.(3.4) correspond to $sl\,(5)$. Therefore, we conjecture the following:

Conjecture 6 *The Lie algebra, for the second order $n-$dimensional vector equation is $sl\,(n+2)$.*

Proof The generators of the Lie algebra for $gl\,(n)$ are [33]

$$\mathbf{Y}^{\mu}_{\nu} = \eta^{\gamma\mu}\frac{\partial}{\partial\eta^{\gamma\nu}},\quad \mu,\nu = 1,2,3,..,n, \tag{3.7}$$

which satisfy the commutation relations

$$[\mathbf{Y}^{\mu}_{\nu},\mathbf{Y}^{\rho}_{\tau}] = \delta^{\rho}_{\nu}Y^{\mu}_{\tau} - \delta^{\mu}_{\tau}Y^{\rho}_{\nu}, \tag{3.8}$$

where δ^{μ}_{ν} is the usual Kronecker delta. Further, setting $\mathbf{Y}^{\alpha}_{\alpha} = 0$ gives the Lie algebra of $sl\,(n)$.

It can be easily verified that the algebra for n dependent variables is eq.(3.4) with $a = 1,2,3,...n$. Now define $y^{\alpha} = s$ for $\alpha = 0$ and y^{a} for $\alpha = a$. Then the generators can be re-written as

$$\mathbf{X}_{\alpha} = \frac{\partial}{\partial y^{\alpha}},\ \ \mathbf{Y}^{\alpha}_{\beta} = y^{\alpha}\frac{\partial}{\partial y^{\beta}},\ \ \mathbf{Z}^{\alpha} = y^{\alpha}y^{\beta}\frac{\partial}{\partial y^{\beta}},\ \ \alpha,\beta = 0,1,2,...,n, \tag{3.9}$$

where $\mathbf{Y}_0^0 = \mathbf{S}$ and $\mathbf{Y}_0^a = \mathbf{P}^a$. Now, further putting

$$\mathbf{Y}_\alpha^{n+1} = \mathbf{X}_\alpha, \ \ \mathbf{Y}_{n+1}^\alpha = -\mathbf{Z}^\alpha, \tag{3.10}$$

we only need to define $\mathbf{Y}_{n+1}^{n+1}$. This may be defined by setting $\mathbf{Y}_\mu^\mu = 0$, where $\mu, \nu = 0, 1, 2, ..., n+1$. Then the generators given by eqs.(3.9) and (3.10) satisfy eq.(3.8). The negative sign in eq.(3.10) is introduced, so that the generators satisfy the required algebra. It is allowable to introduce the negative sign as $-V^\alpha$ will be a generator if V^α is. Hence the maximal symmetry algebra of the second order $n-$dimensional vector differential equation *is* $sl\,(n+2)$.

The proof that the algebra of symmetry generators of $(y^a)'' = 0$, $(a = 1, 2, ..., n)$, is isomorphic to $sl\,(n+2)$ is elegant and simple, only requiring an appropriate *labelling* of the generators. The presentation here has, thus, avoided the tedious methods, usually adopted in classification problems.

3.3 Symmetries of the Geodesic Equations of Maximally Symmetric 2-Dimensional Spaces

There are three kinds of surfaces: those of positive, zero and negative curvature. In 2−dimensions, the surface of positive curvature is a sphere, of zero curvature is locally a plane and of negative curvature is a hyperboloid of one sheet. The metric of the sphere can be written as

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{3.11}$$

The geodesic equations for this metric are

$$\begin{aligned} E_1 :&\quad \ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0, \\ E_2 :&\quad \ddot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0. \end{aligned} \tag{3.12}$$

As the geodesic equations are second order ordinary differential equations, we apply the second prolongation

$$\mathbf{X}^{[2]} = \xi\frac{\partial}{\partial s} + \eta^1\frac{\partial}{\partial\theta} + \eta^2\frac{\partial}{\partial\phi} + \eta^1_{,s}\frac{\partial}{\partial\dot{\theta}} + \eta^2_{,s}\frac{\partial}{\partial\dot{\phi}} + \eta^1_{,ss}\frac{\partial}{\partial\ddot{\theta}} + \eta^2_{,ss}\frac{\partial}{\partial\ddot{\phi}}, \tag{3.13}$$

where ξ, η^1 and η^2 are all functions of s, θ and ϕ; $\eta^1_{,s}$ and $\eta^2_{,s}$ are all functions of $s, \theta, \phi, \dot{\theta}$ and $\dot{\phi}$; and $\eta^1_{,ss}$ and $\eta^2_{,ss}$ are all functions of $s, \theta, \phi, \dot{\theta}, \dot{\phi}, \ddot{\theta}$ and $\ddot{\phi}$, of the symmetry generator given by eq.(1.30) [1, 9], to both the geodesic equations. Then $\mathbf{X}^{[2]}E_1 \mid_{E_1=0=E_2}= 0$ and $\mathbf{X}^{[2]}E_2 \mid_{E_1=0=E_2}= 0$ respectively yield

$$\eta^1_{,ss} - 2\sin\theta\cos\dot{\phi}\eta^2_{,s} - \eta^1\left(\cos^2\theta - \sin^2\theta\right)\dot{\phi}^2 \mid_{E_1=0=E_2}= 0, \tag{3.14}$$

and

$$\eta^2_{,ss} + 2\cot\theta\left(\dot{\phi}\eta^1_{,s} + \dot{\theta}\eta^2_{,s}\right) - 2\dot{\theta}\dot{\phi}\csc^2\theta\eta^1 \mid_{E_1=0=E_2}= 0, \tag{3.15}$$

where, for

$$D = \frac{\partial}{\partial s} + \dot{\theta}\frac{\partial}{\partial\theta} + \dot{\phi}\frac{\partial}{\partial\phi} + \ddot{\theta}\frac{\partial}{\partial\dot{\theta}} + \ddot{\phi}\frac{\partial}{\partial\dot{\phi}}, \tag{3.16}$$

$$\begin{aligned} \eta^1_{,s} &= D\eta^1 - \dot{\theta}D\xi = \eta^1_s + \dot{\theta}\left(\eta^1_\theta - \xi_s\right) - \dot{\theta}^2\xi_\theta + \dot{\phi}\eta^1_\phi - \dot{\theta}\dot{\phi}\xi_\phi, \qquad (3.17) \\ \eta^2_{,s} &= D\eta^2 - \dot{\phi}D\xi = \eta^2_s + \dot{\theta}\eta^2_\theta + \dot{\phi}\left(\eta^2_\phi - \xi_s\right) - \dot{\phi}^2\xi_\phi - \dot{\theta}\dot{\phi}\xi_\theta, \\ \eta^1_{,ss} &= D\eta^1_{,s} - \ddot{\theta}D\xi, \\ &= \eta^1_{ss} + \dot{\theta}\left(2\eta^1_{s\theta} - \xi_{ss}\right) + \dot{\theta}^2\left(\eta^1_{\theta\theta} - 2\xi_{s\theta}\right) - \dot{\theta}^3\xi_{\theta\theta} + 2\dot{\phi}\eta^1_{s\phi} \\ &\quad + \dot{\phi}^2\eta^1_{\phi\phi} + 2\dot{\theta}\dot{\phi}\left(\eta^1_{\theta\phi} - \xi_{s\phi}\right) - 2\dot{\theta}^2\dot{\phi}\xi_{\theta\phi} - \dot{\theta}\dot{\phi}^2\xi_{\phi\phi} \end{aligned}$$

$$+\ddot{\theta}\left(\eta^1_{\theta}-2\xi_s-3\dot{\theta}\xi_{\theta}\right)-2\ddot{\theta}\dot{\phi}\xi_{\phi}+\ddot{\phi}\eta^1_{\phi}-\dot{\theta}\ddot{\phi}\xi_{\phi},$$

$$\begin{aligned}\eta^2_{,ss} &= D\eta^2_{,s}-\ddot{\phi}D\xi,\\ &= \eta^2_{ss}+2\dot{\theta}\eta^2_{s\theta}+\dot{\theta}^2\eta^2_{\theta\theta}+\dot{\phi}\left(2\eta^2_{s\phi}-\xi_{ss}\right)+\dot{\phi}^2\left(\eta^2_{\phi\phi}-2\xi_{s\phi}\right)\\ &\quad -\dot{\phi}^3\xi_{\phi\phi}+2\dot{\theta}\dot{\phi}\left(\eta^2_{\theta\phi}-\xi_{s\theta}\right)-\dot{\theta}^2\dot{\phi}\xi_{\theta\theta}-2\dot{\theta}\dot{\phi}^2\xi_{\theta\phi}+\ddot{\theta}\eta^2_{\theta}\\ &\quad -\ddot{\theta}\dot{\phi}\xi_{\theta}+\ddot{\phi}\left(\eta^2_{\phi}-2\xi_s-3\dot{\phi}\xi_{\phi}\right)-2\dot{\theta}\ddot{\phi}\xi_{\theta}.\end{aligned}$$

Inserting eqs.(3.17) into eq.(3.14) and eq.(3.15) we obtain

$$\begin{aligned}&[\eta^1_{ss}+\dot{\theta}\left(2\eta^1_{\theta s}-\xi_{ss}\right)+2\dot{\phi}\eta^1_{\phi s}+\dot{\theta}^2\left(\eta^1_{\theta\theta}-2\xi_{\theta s}\right)-\xi_{\theta\theta}\dot{\theta}^3\\ &+2\dot{\theta}\dot{\phi}\left(\eta^1_{\theta\phi}-\xi_{\phi s}\right)-2\dot{\theta}^2\dot{\phi}\xi_{\theta\phi}+\dot{\phi}^2\eta^1_{\phi\phi}-\dot{\phi}^2\dot{\theta}\xi_{\phi\phi}+\ddot{\theta}\left(\eta^1_{\theta}-2\xi_s-3\xi_{\phi}\dot{\theta}\right)\\ &-2\ddot{\theta}\dot{\phi}^3\xi_{\phi}+\ddot{\phi}\left(\eta^1_{\phi}-\dot{\theta}\xi_{\phi}\right)]-\sin 2\theta\dot{\phi}[\eta^2_s+\dot{\theta}\eta^2_{\theta}+\dot{\phi}\left(\eta^2_{\phi}-\xi_s\right)-\dot{\theta}\dot{\phi}\xi_{\phi}-\dot{\phi}^2\xi_{\phi}]\\ &-\eta^1\cos 2\theta\dot{\phi}^2\mid_{E_1=0=E_2}=0,\end{aligned}\tag{3.18}$$

$$\begin{aligned}&\eta^2_{ss}+2\dot{\theta}\eta^2_{\theta s}+\dot{\phi}\left(2\eta^2_{\phi s}-\xi_{ss}\right)+\dot{\phi}^2\left(\eta^2_{\phi\phi}-2\xi_{\phi\text{ñ}s}\right)-\dot{\phi}^3\xi_{\phi\phi}+2\dot{\theta}\dot{\phi}\left(\eta^2_{\phi\theta}-\xi_{\theta s}\right)\\ &-\ddot{\theta}^2\dot{\phi}\xi_{\theta\theta}+\dot{\theta}^2\eta^2_{\theta\theta}-2\dot{\theta}\dot{\phi}^2\xi_{\theta\phi}+\ddot{\theta}\left(\eta^2_{\theta}-\dot{\phi}\xi_{\theta}\right)+\ddot{\phi}\left(\eta^2_{\phi}-2\xi_s-3\xi_{\phi}\dot{\phi}\right)\\ &-2\dot{\theta}\ddot{\phi}\xi_{\theta}+2\cot\theta[\dot{\phi}\{\eta^1_s+\dot{\theta}\left(\eta^1_{\theta}-\xi_s\right)-\dot{\theta}^2\xi_{\theta}+\dot{\phi}\eta^1_{\phi}-\dot{\theta}\dot{\phi}\xi_{\phi}\}+\dot{\theta}\{\eta^2_s\\ &+\dot{\phi}\left(\eta^2_{\phi}-\xi_s\right)-\dot{\phi}^2\xi_{\phi}+\dot{\theta}\eta^2_{\theta}-\dot{\theta}\dot{\phi}\xi_{\theta}\}]-2\dot{\theta}\dot{\phi}\eta^1\csc^2\theta\mid_{E_1=0=E_2}=0.\end{aligned}\tag{3.19}$$

Inserting the values of $\ddot{\theta}$ and $\ddot{\phi}$ from the geodesic equations, eq.(3.12), and then comparing the coefficients of the powers of $\dot{\theta}$ and $\dot{\phi}$, we obtain a system of fifteen partial differential equations

$$\xi_{\theta\theta}=0,\ \xi_{\phi\phi}-\frac{1}{2}\sin 2\theta\xi_{\theta}=0,\ \xi_{\theta\phi}-2\cot\theta\xi_{\phi}=0,\tag{3.20}$$

$$\eta^1_{\theta\theta}-2\xi_{\theta s}=0,\ \eta^1_{ss}=0,\tag{3.21}$$

$$\eta^2_{ss}=0,\ \eta^2_{\theta s}+\cot\theta\eta^2_s=0,\ \eta^2_{\theta\theta}+2\cot\theta\eta^2_{\theta}=0,\tag{3.22}$$

$$\eta^1_{\phi s} - \frac{1}{2}\sin 2\theta \eta^2_s = 0, \tag{3.23}$$

$$2\eta^1_{\theta s} - \xi_{ss} = 0, \tag{3.24}$$

$$\eta^1_{\theta\phi} - \xi_{\phi s} - \cot\theta\eta^1_\phi - \frac{1}{2}\sin 2\theta\eta^2_\theta = 0,\ \eta^1_{\phi\phi} + \frac{1}{2}\sin 2\theta\eta^1_\theta - \sin 2\theta\eta^2_\phi - \cos 2\theta\eta^1 = 0, \tag{3.25}$$

$$2\eta^2_{\phi s} + 2\cot\theta\eta^1_s - \xi_{ss} = 0,\ \eta^2_{\phi\phi} + \frac{1}{2}\sin 2\theta\eta^2_\theta + 2\cot\theta\eta^1_\phi - 2\xi_{\phi s} = 0, \tag{3.26}$$

$$\eta^2_{\theta\phi} - \xi_{\theta s} + \cot\theta\eta^1_\theta - \csc^2\theta\eta^1 = 0. \tag{3.27}$$

Eqs.(3.20) yield $\xi = b_1(s)$. Substituting the value of ξ into eqs.(3.21) we get

$$\eta^1 = [f_1(\phi)s + f_2(\phi)]\theta + f_3(\phi)s + f_4(\phi). \tag{3.28}$$

From eqs.(3.22) we obtain

$$\eta^2 = -\cot\theta f_6(\phi)s + f_7(\phi). \tag{3.29}$$

Substituting the value of η^1 and η^2 into eq.(3.23) we get

$$\eta^1 = [As + f_2(\phi)]\theta + Bs + f_4(\phi). \tag{3.30}$$

Then eq.(3.24) yields

$$\xi = As^2 + c_1 s + c_0. \tag{3.31}$$

Now eqs.(3.25) give

$$\xi = c_1 s + c_0, \tag{3.32}$$

$$\eta^1 = c_3\cos\phi + c_4\sin\phi, \tag{3.33}$$

$$\eta^2 = \cot\theta(c_4\cos\phi - c_3\sin\phi) + c_2. \tag{3.34}$$

Therefore, the symmetries are

$$\begin{aligned} X_0 &= \frac{\partial}{\partial s},\ X_1 = s\frac{\partial}{\partial s},\ X_2 = \frac{\partial}{\partial \phi},\ X_3 = \cos\phi\frac{\partial}{\partial \theta} - \cot\theta\sin\phi\frac{\partial}{\partial \phi}, \\ X_4 &= \sin\phi\frac{\partial}{\partial \theta} + \cot\theta\cos\phi\frac{\partial}{\partial \phi}. \end{aligned} \tag{3.35}$$

The Lie algebra for the above generators is $so\,(3) \oplus d_2$ (where $d_2 = < X_0, X_1 >$ is a dilation algebra). The Lie algebra of the symmetries of the geodesic equations for a plane is $sl\,(4)$ and that for a hyperboloid is $so\,(2,1) \oplus d_2$.

3.4 Symmetries of the Geodesic Equations of Less Symmetric 2-Dimensional Spaces

The metric for a less symmetric surface of positive curvature, i.e. the spheroid

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1, \quad (a \neq b)\,, \tag{3.36}$$

is

$$ds^2 = \left(a^2\cos^2\theta + b^2\sin^2\theta\right)d\theta^2 + \sin^2\theta d\phi^2. \tag{3.37}$$

The geodesic equations are

$$E_1 \ : \ \ddot{\theta} + \frac{(b^2 - a^2)\sin\theta\cos\theta}{a^2\cos^2\theta + b^2\sin^2\theta}\dot{\theta}^2 - \frac{a^2\sin\theta\cos\theta}{a^2\cos^2\theta + b^2\sin^2\theta}\dot{\phi}^2 = 0, \tag{3.38}$$

$$E_2 \ : \ \ddot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0. \tag{3.39}$$

Now $X^{[2]}E_1 \mid_{E_1=0=E_2} = 0$ implies

$$\eta^1_{,ss} + 2\dot{\theta}\eta^1_{,s}\frac{(b^2 - a^2)\sin\theta\cos\theta}{a^2\cos^2\theta + b^2\sin^2\theta} - 2\dot{\phi}\eta^2_{,s}\frac{a^2\sin\theta\cos\theta}{a^2\cos^2\theta + b^2\sin^2\theta}$$

$$+\eta^1\left(\frac{\left[\left(b^2-a^2\right)\dot{\theta}^2-a^2\dot{\phi}^2\right]\cos 2\theta}{a^2\cos^2\theta+b^2\sin^2\theta}\right. \quad (3.40)$$

$$\left.-2\left(b^2-a^2\right)\frac{\left[\left(b^2-a^2\right)\dot{\theta}^2-a^2\dot{\phi}^2\right]}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^2}\sin^2\theta\cos^2\theta\right)|_{E_1=0=E_2}=0, \quad (3.41)$$

and $\mathbf{X}^{[2]}E_2\,|_{E_1=0=E_2}=0$ gives eq.(3.15). The determining equations after substituting the geodesic equations are

$$\xi_{\theta\theta}-\xi_\theta\frac{\left(b^2-a^2\right)\sin\theta\cos\theta}{a^2\cos^2\theta+b^2\sin^2\theta}=0,\quad \xi_{\theta\phi}-\cot\theta\xi_\phi=0,$$
$$\xi_{\phi\phi}+\xi_\theta\frac{a^2\sin\theta\cos\theta}{a^2\cos^2\theta+b^2\sin^2\theta}=0, \quad (3.42)$$

$$\eta^1_{ss}=0, \quad (3.43)$$

$$2\eta^1_{\theta s}-\xi_{ss}+2\eta^1_s\frac{\left(b^2-a^2\right)\sin\theta\cos\theta}{a^2\cos^2\theta+b^2\sin^2\theta}=0, \quad (3.44)$$

$$\eta^1_{\phi s}-\eta^2_s\frac{a^2\sin\theta\cos\theta}{a^2\cos^2\theta+b^2\sin^2\theta}=0, \quad (3.45)$$

$$\eta^1_{\theta\phi}-\xi_{\phi s}-\cot\theta\eta^1_\phi+\eta^1_\phi\frac{\left(b^2-a^2\right)\sin\theta\cos\theta}{a^2\cos^2\theta+b^2\sin^2\theta}-\eta^2_\theta\frac{a^2\sin\theta\cos\theta}{a^2\cos^2\theta+b^2\sin^2\theta}=0, \quad (3.46)$$

$$\eta^1_{\theta\theta}-2\xi_{\theta s}+\eta^1_s\frac{\left(b^2-a^2\right)\sin\theta\cos\theta}{a^2\cos^2\theta+b^2\sin^2\theta}+\eta^1\frac{a^2\cos^2\theta-b^2\sin^2\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^2}=0, \quad (3.47)$$

$$\eta^1_{\phi\phi}+\eta^1_\theta\frac{a^2\sin\theta\cos\theta}{a^2\cos^2\theta+b^2\sin^2\theta}-\eta^2_\phi\frac{2a^2\sin\theta\cos\theta}{a^2\cos^2\theta+b^2\sin^2\theta}-a^2\eta^1\frac{a^2\cos^2\theta-b^2\sin^2\theta}{\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^2}=0, \quad (3.48)$$

$$\eta^2_{ss}=0,\quad \eta^2_{\theta s}+\cot\theta\eta^2_s=0, \quad (3.49)$$

$$2\eta^2_{\phi s}-\xi_{ss}+2\eta^1_s\cot\theta=0, \quad (3.50)$$

$$\eta^2_{\theta\theta}-\eta^2_\theta\frac{\left(b^2-a^2\right)\sin\theta\cos\theta}{a^2\cos^2\theta+b^2\sin^2\theta}+2\eta^2_\theta\cot\theta=0, \quad (3.51)$$

$$\eta^2_{\theta\phi}-\xi_{\theta s}-2\eta^1\csc^2\theta+2\eta^1_\theta\cot\theta=0, \quad (3.52)$$

$$\eta^2_{\phi\phi} - 2\xi_{\phi s} + \eta^2_\theta \frac{a^2 \sin\theta\cos\theta}{a^2\cos^2\theta + b^2\sin^2\theta} + 2\eta^1_\phi \cot\theta = 0. \tag{3.53}$$

Eqs.(3.42) give $\xi = b_1(s)$. Eq.(3.43) gives $\eta^1 = a_1(\theta,\phi)\, s + a_2(\theta,\phi)$. Eqs.(3.49) yield

$$\eta^2 = f_1(\phi)\, s \csc\theta + a_4(\theta,\phi), \tag{3.54}$$

and from eq.(3.45) we get

$$f_2(\phi) = 0 = a_{1,\phi}(\phi). \tag{3.55}$$

Therefore,

$$\eta^2 = a_4(\theta,\phi). \tag{3.56}$$

Now eq.(3.52) gives

$$a_1(\phi) = 0, \tag{3.57}$$

using eq.(3.50) we have

$$\xi = c_0 + c_1 s. \tag{3.58}$$

From eqs.(3.46 − 3.48) and (3.51 − 3.53) we have $\eta^1 = 0$ and $\eta^2 = c_3$. Hence the spheroid has three symmetries $\mathbf{X}_0$, $\mathbf{X}_1$ and $\mathbf{X}_2$, generating $so(2) \oplus d_2$.

The geodesic equations of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad (a \neq b \neq c \neq a), \tag{3.59}$$

are

$$\ddot{\theta} + \frac{\left(b^2\sin^2\phi - a^2\cos^2\phi + c^2\right)\sin\theta\cos\theta}{a^2\cos^2\theta\cos^2\phi + b^2\sin^2\phi\cos^2\theta + c^2\sin^2\theta}\dot{\theta}^2$$
$$+2\frac{\left(b^2 - a^2\right)\sin\phi\cos\phi\cos^2\theta}{a^2\cos^2\theta\cos^2\phi + b^2\sin^2\phi\cos^2\theta + c^2\sin^2\theta}\dot{\theta}\dot{\phi} \tag{3.60}$$
$$-\frac{\sin\theta\cos\theta\left(a^2\sin^2\phi + b^2\cos^2\phi\right)}{a^2\cos^2\theta\cos^2\phi + b^2\sin^2\phi\cos^2\theta + c^2\sin^2\theta}\dot{\phi}^2 = 0, \tag{3.61}$$
$$\ddot{\phi} + \frac{\left(a^2 - b^2\right)\sin\phi\cos\phi\cot^2\theta}{a^2\sin^2\phi + b^2\cos^2\phi}\dot{\theta}^2 + 2\cot\theta\dot{\theta}\dot{\phi} + \frac{\left(a^2 - b^2\right)\sin\phi\cos\phi}{a^2\sin^2\phi + b^2\cos^2\phi}\dot{\phi}^2 = 0.$$

These equations have only the two symmetries $\mathbf{X}_0$ and $\mathbf{X}_1$, generating d_2.

For the sphere, the number of symmetries of the geodesic equations was five. This number reduced to three for the spheroid and further to two for the ellipsoid, showing that further reduction of symmetries of a space reduces the original algebra to its corresponding subalgebras. The geodesic equations retain the symmetries along those of the space.

3.5 Symmetries of the Geodesic Equations of Maximally Symmetric Higher Dimensional Spaces

The metric and the geodesic equations for a 3−dimensional space of positive curvature are

$$ds^2 = d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{3.62}$$

$$\begin{aligned}
&E_1 : \ddot{\chi} - \sin\chi\cos\chi\dot{\theta}^2 - \sin\chi\cos\chi\sin^2\theta\dot{\phi}^2 = 0,\\
&E_2 : \ddot{\theta} + \cot\chi\dot{\chi}\dot{\phi} - \sin\theta\cos\theta\dot{\phi}^2 = 0,\\
&E_3 : \ddot{\phi} + 2\cot\chi\dot{\chi}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0.
\end{aligned} \tag{3.63}$$

The symmetry generator to be applied on these geodesic equations is

$$\begin{aligned}
\mathbf{X}^{[2]} &= \xi\frac{\partial}{\partial s} + \eta^1\frac{\partial}{\partial\chi} + \eta^2\frac{\partial}{\partial\theta} + \eta^3\frac{\partial}{\partial\phi} + \eta^1_{,s}\frac{\partial}{\partial\dot{\chi}} + \eta^2_{,s}\frac{\partial}{\partial\dot{\theta}} + \eta^3_{,s}\frac{\partial}{\partial\dot{\phi}}\\
&\quad + \eta^1_{,ss}\frac{\partial}{\partial\ddot{\chi}} + \eta^2_{,ss}\frac{\partial}{\partial\ddot{\theta}} + \eta^3_{,ss}\frac{\partial}{\partial\ddot{\phi}},
\end{aligned} \tag{3.64}$$

where ξ, η^1, η^2 and η^3 are all functions of s, χ, θ and ϕ; $\eta^1_{,s}, \eta^2_{,s}$ and $\eta^3_{,s}$ are all functions of $s, \chi, \theta, \phi, \dot{\chi}, \dot{\theta}$ and $\dot{\phi}$; and $\eta^1_{,ss}, \eta^2_{,ss}$ and $\eta^3_{,ss}$ are all functions of $s, \chi, \theta, \phi, \dot{\chi}, \dot{\theta}, \dot{\phi}, \ddot{\chi}, \ddot{\theta}$

and $\ddot{\phi}$. Then

$$\begin{aligned}
\mathbf{X}^{[2]}E_1 &= \eta^1_{,ss} - \left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)\eta^1\cos 2\chi - \dot{\theta}\eta^2_{,s}\sin 2\chi - \dot{\phi}^2\eta^2\sin 2\chi\sin\theta\cos\theta \\
&\quad -\dot{\phi}\eta^3_{,s}\sin 2\chi\sin^2\theta \mid_{E_1=E_2=E_3=0}= 0, \\
\mathbf{X}^{[2]}E_2 &= \eta^2_{,ss} - \dot{\chi}\dot{\theta}\eta^1\csc^2\chi + 2\dot{\theta}\eta^1_{,s}\cot\chi + 2\dot{\chi}^2\eta^2_{,s}\cot\chi \qquad (3.65) \\
&\quad -\dot{\phi}^2\eta^2\cos 2\theta - \dot{\phi}\eta^3_{,s}\sin 2\theta - \dot{\phi}\eta^3_{,s}\sin 2\chi\sin^2\theta \mid_{E_1=E_2=E_3=0}= 0, \\
\mathbf{X}^{[2]}E_3 &= \eta^3_{,ss} - 2\dot{\chi}\dot{\phi}\eta^1\csc^2\chi + 2\dot{\phi}\eta^1_{,s}\cot\chi + 2\dot{\chi}\eta^3_{,s}\cot\chi \\
&\quad -2\dot{\theta}\dot{\phi}\eta^2\csc^2\theta + 2\dot{\theta}\cot\theta\eta^3_{,s} + 2\dot{\phi}\cot\theta\eta^2_{,s} \mid_{E_1=E_2=E_3=0}= 0.
\end{aligned}$$

The values of $\eta^1_{,s}$, $\eta^2_{,s}$, $\eta^3_{,s}$, $\eta^1_{,ss}$, $\eta^2_{,ss}$ and $\eta^3_{,ss}$ are

$$\eta^1_{,s} = \eta^1_s + \dot{\chi}\left(\eta^1_\chi - \xi_s\right) - \dot{\chi}^2\xi_\chi + \dot{\theta}\eta^1_\theta + \dot{\phi}\eta^1_\phi - \dot{\chi}\dot{\theta}\xi_\theta - \dot{\chi}\dot{\phi}\xi_\phi, \qquad (3.66)$$

$$\eta^2_{,s} = \eta^2_s + \dot{\chi}\eta^2_\chi + \dot{\theta}\left(\eta^2_\theta - \xi_s\right) - \dot{\theta}^2\xi_\theta + \dot{\phi}\eta^2_\phi - \dot{\chi}\dot{\theta}\xi_\chi - \dot{\theta}\dot{\phi}\xi_\phi, \qquad (3.67)$$

$$\eta^3_{,s} = \eta^3_s + \dot{\chi}\eta^3_\chi + \dot{\theta}\eta^3_\theta + \dot{\phi}\left(\eta^3_\phi - \xi_s\right) - \dot{\phi}^2\xi_\phi - \dot{\chi}\dot{\phi}\xi_\chi - \dot{\theta}\dot{\phi}\xi_\theta, \qquad (3.68)$$

$$\begin{aligned}
\eta^1_{,ss} &= \eta^1_{ss} + \dot{\chi}\left(2\eta^1_{s\chi} - \xi_{ss}\right) + \dot{\chi}^2\left(\eta^1_{\chi\chi} - 2\xi_{s\chi}\right) - \dot{\chi}^3\xi_{\chi\chi} + 2\dot{\theta}\eta^1_{s\theta} + \dot{\theta}^2\eta^1_{\theta\theta} \\
&\quad +2\dot{\phi}\eta^1_{s\phi} + \dot{\phi}^2\eta^1_{\phi\phi} + 2\dot{\chi}\dot{\theta}\left(\eta^1_{\chi\theta} - \xi_{s\theta}\right) + 2\dot{\chi}\dot{\phi}\left(\eta^1_{\chi\phi} - \xi_{s\phi}\right) + 2\dot{\theta}\dot{\phi}\eta^1_{\theta\phi} \\
&\quad -\dot{\chi}\dot{\theta}^2\xi_{\theta\theta} - \dot{\chi}\dot{\phi}^2\xi_{\phi\phi} - 2\dot{\chi}^2\dot{\theta}\xi_{\chi\theta} - 2\dot{\chi}^2\dot{\phi}\xi_{\chi\phi} - 2\dot{\chi}\dot{\theta}\dot{\phi}\xi_{\theta\phi} \qquad (3.69) \\
&\quad +\ddot{\chi}\left(\eta^1_\chi - 2\xi_s - 3\dot{\chi}\xi_\chi\right) - 2\ddot{\chi}\left(\dot{\theta}\xi_\theta + \dot{\phi}\xi_\phi\right) + \ddot{\theta}\left(\eta^1_\theta - \dot{\chi}\xi_\theta\right) + \ddot{\phi}\left(\eta^1_\phi - \dot{\chi}\xi_\phi\right),
\end{aligned}$$

$$\begin{aligned}
\eta^2_{,ss} &= \eta^2_{ss} + 2\dot{\chi}\eta^2_{s\chi} + \dot{\chi}^2\eta^2_{\chi\chi} + \dot{\theta}\left(2\eta^2_{s\theta} - \xi_{ss}\right) + \dot{\theta}^2\left(\eta^2_{\theta\theta} - 2\xi_{s\theta}\right) - \dot{\theta}^3\xi_{\theta\theta} + 2\dot{\phi}\eta^2_{s\phi} \\
&\quad +\dot{\phi}^2\eta^2_{\phi\phi} + 2\dot{\chi}\dot{\theta}\left(\eta^2_{\chi\theta} - \xi_{s\chi}\right) + 2\dot{\chi}\dot{\phi}\eta^2_{\chi\phi} + 2\dot{\theta}\dot{\phi}\left(\eta^2_{\theta\phi} - \xi_{s\phi}\right) - \dot{\chi}^2\dot{\theta}\xi_{\chi\chi} \\
&\quad -\dot{\theta}\dot{\phi}^2\xi_{\phi\phi} - \dot{\chi}\dot{\theta}^2\xi_{\chi\theta} - 2\dot{\theta}^2\dot{\phi}\xi_{\theta\phi} - 2\dot{\chi}\dot{\theta}\dot{\phi}\xi_{\chi\phi} + \ddot{\chi}\left(\eta^2_\chi - \dot{\theta}\xi_\theta\right) \\
&\quad +\ddot{\theta}\left(\eta^2_\theta - 2\xi_s - 3\dot{\theta}\xi_\theta\right) - 2\ddot{\theta}\left(\dot{\chi}\xi_\chi + 2\dot{\phi}\xi_\phi\right) + \ddot{\phi}\left(\eta^2_\phi - \dot{\theta}\xi_\phi\right), \qquad (3.70)
\end{aligned}$$

$$\eta^3_{,ss} = \eta^3_{ss} + 2\dot{\chi}\eta^3_{s\chi} + \dot{\chi}^2\eta^3_{\chi\chi} + 2\dot{\theta}\eta^3_{s\theta} + \dot{\theta}^2\eta^3_{\theta\theta} + \dot{\phi}\left(2\eta^3_{s\phi} - \xi_{ss}\right) + \dot{\phi}^2\left(\eta^3_{\phi\phi} - 2\xi_{s\phi}\right)$$
$$-\dot{\phi}^3\xi_{\phi\phi} + 2\dot{\chi}\dot{\theta}\eta^3_{\chi\theta} + 2\dot{\chi}\dot{\phi}\left(\eta^3_{\chi\phi} - \xi_{s\chi}\right) + 2\dot{\theta}\dot{\phi}\left(\eta^3_{\theta\phi} - \xi_{s\theta}\right) - \dot{\chi}^2\dot{\phi}\xi_{\chi\chi}$$
$$-\dot{\theta}^2\dot{\phi}\xi_{\theta\theta} - 2\dot{\chi}\dot{\phi}^2\xi_{\chi\phi} - 2\dot{\theta}\dot{\phi}^2\xi_{\theta\phi} - 2\dot{\chi}\dot{\theta}\dot{\phi}\xi_{\chi\theta} + \ddot{\chi}\left(\eta^3_{\chi} - \dot{\phi}\xi_{\chi}\right)$$
$$+\ddot{\theta}\left(\eta^3_{\theta} - \dot{\phi}\xi_{\theta}\right) + \ddot{\phi}\left(\eta^3_{\theta} - 2\xi_{s} - 3\dot{\phi}\xi_{\phi}\right) - 2\ddot{\phi}\left(\dot{\chi}\xi_{\chi} + 2\dot{\theta}\xi_{\theta}\right). \tag{3.71}$$

Substituting these values in eqs.(3.65) and using the geodesic equations. The following system of 36 partial differential equations, given by eqs.(3.72) – (3.95) (in which one equation number often contains more than one equation), is obtained

$$\begin{aligned}\xi_{\chi\chi} = 0,\ \xi_{\chi\theta} - \cot\chi\xi_{\theta} = 0,\ \xi_{\chi\phi} - \cot\chi\xi_{\phi} = 0,\ \xi_{\theta\theta} + \sin\chi\cos\chi\xi_{\chi} = 0,\\ \xi_{\theta\phi} - \cot\theta\xi_{\phi} = 0,\ \xi_{\phi\phi} + \sin\chi\cos\chi\sin^2\theta\xi_{\chi} + \sin\theta\cos\theta\xi_{\theta} = 0,\end{aligned} \tag{3.72}$$

$$\eta^1_{ss} = 0,\ \eta^1_{\chi\chi} - 2\xi_{s\chi} = 0, \tag{3.73}$$

$$\eta^1_{s\theta} - \sin\chi\cos\chi\eta^2_{s} = 0,\ \eta^1_{\chi\theta} - \xi_{s\theta} - \cot\chi\eta^1_{\theta} - \sin\chi\cos\chi\eta^2_{\chi} = 0, \tag{3.74}$$

$$\eta^1_{\theta\theta} + \sin\chi\cos\chi\eta^1_{\chi} - \cos 2\chi\eta^1 - \sin 2\chi\eta^2_{\theta} = 0, \tag{3.75}$$

$$2\eta^1_{s\chi} - \xi_{ss} = 0, \tag{3.76}$$

$$\eta^1_{\theta\phi} - \cot\chi\eta^1_{\phi} - \sin\chi\cos\chi\eta^2_{\phi} - \sin\chi\cos\chi\sin^2\theta\eta^3_{\theta} = 0, \tag{3.77}$$

$$\eta^1_{\chi\phi} - \xi_{s\phi} - \cot\chi\eta^1_{\phi} - \sin\chi\cos\chi\sin^2\theta\eta^3_{\chi} = 0, \tag{3.78}$$

$$\eta^1_{s\phi} - \sin\chi\cos\chi\sin^2\theta\eta^3_{s} = 0, \tag{3.79}$$

$$\begin{aligned}&\eta^1_{\phi\phi} + \sin\chi\cos\chi\sin^2\theta\eta^1_{\chi} + \sin\theta\cos\theta\eta^1_{\theta} - \cos 2\chi\sin^2\theta\eta^1\\ &- \sin 2\chi\sin^2\theta\eta^3_{\phi} - \sin 2\chi\sin\theta\cos\theta\eta^2 = 0,\end{aligned} \tag{3.80}$$

$$\eta^2_{ss} = 0,\ \eta^2_{s\chi} + \cot\chi\eta^2_{s} = 0,\ \eta^2_{\chi\chi} + 2\cot\chi\eta^2_{\chi} = 0, \tag{3.81}$$

$$\eta^2_{\chi\phi} - \sin\theta\cos\theta\eta^3_\chi = 0, \tag{3.82}$$

$$2\eta^2_{s\theta} + 2\cot\chi\eta^1_s - \xi_{ss} = 0, \tag{3.83}$$

$$\eta^2_{s\phi} - \sin\theta\cos\theta\eta^3_s = 0, \tag{3.84}$$

$$\eta^2_{\chi\theta} - \xi_{s\chi} - \cot\eta^1_\chi - \csc^2\chi\eta^1 = 0, \tag{3.85}$$

$$\eta^2_{\theta\theta} - 2\xi_{s\theta} + \sin\chi\cos\chi\eta^2_\chi + 2\cot\chi\eta^1_\theta = 0, \tag{3.86}$$

$$\eta^2_{\theta\phi} - \xi_{s\phi} - \cot\theta\eta^2_\phi + \cot\chi\eta^1_\phi - \sin\theta\cos\theta\eta^3_\theta = 0, \tag{3.87}$$

$$\eta^2_{\phi\phi} + \sin\chi\cos\chi\sin^2\theta\eta^2_\chi + \sin\theta\cos\theta\eta^2_\theta - \cos 2\theta\eta^2 - \sin 2\theta\eta^3_\phi = 0, \tag{3.88}$$

$$\eta^3_{ss} = 0,\ \eta^3_{s\chi} + \cot\chi\eta^3_s = 0,\ \eta^3_{\chi\chi} + 2\cot\chi\eta^3_\chi = 0, \tag{3.89}$$

$$\eta^3_{\chi\theta} + \cot\theta\eta^3_\chi = 0,\ \eta^3_{\theta\theta} + \sin\chi\cos\chi\eta^3_\chi + 2\cot\theta\eta^3_\theta = 0, \tag{3.90}$$

$$\eta^3_{\theta\phi} - \xi_{s\theta} + \cot\chi\eta^1_\theta - \csc^2\theta\eta^2 + \cot\theta\eta^2_\theta = 0, \tag{3.91}$$

$$\eta^3_{s\theta} + \cot\theta\eta^3_s = 0, \tag{3.92}$$

$$2\eta^3_{s\phi} + 2\cot\chi\eta^1_s + 2\cot\theta\eta^2_s - \xi_{ss} = 0, \tag{3.93}$$

$$\eta^3_{\chi\phi} - \xi_{s\chi} + \cot\chi\eta^1_\chi - \csc^2\chi\eta^1 + \cot\theta\eta^2_\chi = 0, \tag{3.94}$$

$$\eta^3_{\phi\phi} - 2\xi_{s\phi} + \sin\chi\cos\chi\sin^2\theta\eta^3_\chi + \sin\theta\cos\theta\eta^3_\theta + 2\cot\chi\eta^1_\phi + 2\cot\theta\eta^2_\phi = 0. \tag{3.95}$$

Eqs.(3.72) give $\xi = a_1(s)$. From eqs.(3.73)

$$\eta^1 = (f_1(\theta,\phi)s + f_2(\theta,\phi))\chi + f_3(\theta,\phi)s + f_4(\theta,\phi), \tag{3.96}$$

is obtained. Eqs.(3.81) yield

$$\eta^2 = -f_5(\theta,\phi)\cot\chi + f_6(\theta,\phi). \tag{3.97}$$

From eqs.(3.74) we get

$$f_1(\theta,\phi)=k_1(\phi),\ f_2(\theta,\phi)=k_2(\phi),\ f_3(\theta,\phi)=k_3(\phi),\ f_{4,\theta}(\theta,\phi)+f_5(\theta,\phi)=0. \tag{3.98}$$

Therefore, eq.(3.96) becomes

$$\eta^1=(k_1(\phi)s+k_2(\phi))\chi+k_3(\phi)s+f_4(\theta,\phi). \tag{3.99}$$

Now eq.(3.75) yields

$$\begin{aligned}&k_1(\phi)=0,\ k_2(\phi)=0,\ k_3(\phi)=0,\\&f_4(\theta,\phi)=k_4(\phi)\cos\theta+k_5(\phi)\sin\theta,\ f_6(\theta,\phi)=k_6(\phi).\end{aligned} \tag{3.100}$$

Therefore,

$$\begin{aligned}&\eta^1=k_4(\phi)\cos\theta+k_5(\phi)\sin\theta,\\&\eta^2=-\cot\chi(k_4(\phi)\sin\theta-k_5(\phi)\cos\theta)+k_6(\phi),\end{aligned} \tag{3.101}$$

and eq.(3.76) gives

$$\xi=c_1s+c_0. \tag{3.102}$$

From eqs.(3.89) we have

$$\eta^3=-f_7(\theta,\phi)\cot\chi+f_8(\theta,\phi). \tag{3.103}$$

Eqs.(3.90) give

$$\eta^3=-k_7(\phi)\cot\chi\csc\theta-k_8(\phi)\cot\theta+k_9(\phi). \tag{3.104}$$

From eqs.(3.78, 3.82) one can easily obtain

$$k_4(\phi)=c_2,\ k_5'(\phi)+k_7(\phi)=0. \tag{3.105}$$

Therefore,

$$\eta^1 = c_2 \cos\theta + k_5(\phi)\sin\theta,\ \eta^2 = -\cot\chi\left(c_2\sin\theta - k_5(\phi)\cos\theta\right) + k_6(\phi). \quad (3.106)$$

Solving eqs.(3.77, 3.87 and 3.88) yields

$$\begin{aligned}
&k_5(\phi) = c_5\cos\phi + c_6\sin\phi,\ k_6(\phi) = c_3\cos\phi + c_4\sin\phi,\\
&k_7(\phi) = c_5\sin\phi - c_6\cos\phi,\ k_8(\phi) = c_3\sin\phi - c_4\cos\phi,\\
&k_9(\phi) = c_7.
\end{aligned}$$

Hence

$$\eta^1 = c_2\cos\theta + (c_5\cos\phi + c_6\sin\phi)\sin\theta, \quad (3.107)$$

$$\eta^2 = (c_3\cos\phi + c_4\sin\phi) - \cot\chi\left(c_2\sin\theta - (c_5\cos\phi + c_6\sin\phi)\cos\theta\right), \quad (3.108)$$

$$\eta^3 = -(c_5\sin\phi - c_6\cos\phi)\cot\chi\csc\theta - (c_3\sin\phi - c_4\cos\phi)\cot\theta + c_7. \quad (3.109)$$

The extra symmetries of eq.(3.35) are

$$\begin{aligned}
&\mathbf{X}_5 = \cos\theta\frac{\partial}{\partial\chi} - \cot\chi\sin\theta\frac{\partial}{\partial\theta},\\
&\mathbf{X}_6 = \cos\phi\sin\theta\frac{\partial}{\partial\chi} + \cot\chi\cos\phi\cos\theta\frac{\partial}{\partial\theta} - \cot\chi\csc\theta\sin\phi\frac{\partial}{\partial\phi},\\
&\mathbf{X}_7 = \sin\phi\sin\theta\frac{\partial}{\partial\chi} + \cot\chi\sin\phi\cos\theta\frac{\partial}{\partial\theta} + \cot\chi\csc\theta\cos\phi\frac{\partial}{\partial\phi},
\end{aligned} \quad (3.110)$$

and the Lie algebra for these symmetries is $so(4)\oplus d_2$. Similarly, the Lie algebra for the geodesic equations of a hyperhyperboloid is $so(3,1)\oplus d_2$ and that for a hyperplane is $sl(5)$.

The 4−dimensional metric is

$$ds^2 = d\psi^2 + \sin^2\psi\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \quad (3.111)$$

and the geodesic equations are

$$\begin{aligned}
&E_1 : \ddot{\psi} - \sin\psi\cos\psi\dot{\chi}^2 - \sin\psi\cos\psi\sin^2\chi\dot{\theta}^2 - \sin\psi\cos\psi\sin^2\chi\sin^2\theta\dot{\phi}^2 = 0,\\
&E_2 : \ddot{\chi} + 2\cot\psi\dot{\psi}\dot{\chi} - \sin\chi\cos\chi\dot{\theta}^2 - \sin\chi\cos\chi\sin^2\theta\dot{\phi}^2 = 0,\\
&E_3 : \ddot{\theta} + 2\cot\psi\dot{\psi}\dot{\theta} + 2\cot\chi\dot{\chi}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0,\\
&E_4 : \ddot{\phi} + 2\cot\psi\dot{\psi}\dot{\phi} + 2\cot\chi\dot{\chi}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0.
\end{aligned} \tag{3.112}$$

The symmetry generator is

$$\begin{aligned}
\mathbf{X} &= \xi\frac{\partial}{\partial s} + \eta^1\frac{\partial}{\partial\psi} + \eta^2\frac{\partial}{\partial\chi} + \eta^3\frac{\partial}{\partial\theta} + \eta^4\frac{\partial}{\partial\phi} + \eta^1_{,s}\frac{\partial}{\partial\dot{\psi}} + \eta^2_{,s}\frac{\partial}{\partial\dot{\chi}} \qquad (3.113)\\
&\quad + \eta^3_{,s}\frac{\partial}{\partial\dot{\theta}} + \eta^4_{,s}\frac{\partial}{\partial\dot{\phi}} + \eta^1_{,ss}\frac{\partial}{\partial\ddot{\psi}} + \eta^2_{,ss}\frac{\partial}{\partial\ddot{\chi}} + \eta^3_{,ss}\frac{\partial}{\partial\ddot{\theta}} + \eta^4_{,ss}\frac{\partial}{\partial\ddot{\phi}},
\end{aligned}$$

where ξ, η^1, η^2 and η^3 are all functions of s, ψ, χ, θ and ϕ; $\eta^1_{,s}, \eta^2_{,s}$ and $\eta^3_{,s}$ are all functions of $s, \psi, \chi, \theta, \phi, \dot{\psi}, \dot{\chi}, \dot{\theta}$ and $\dot{\phi}$; and $\eta^1_{,ss}, \eta^2_{,ss}$ and $\eta^3_{,ss}$ are all functions of $s, \psi, \chi, \theta, \phi$, $\dot{\psi}, \dot{\chi}, \dot{\theta}, \dot{\phi}, \ddot{\psi}, \ddot{\chi}, \ddot{\theta}$ and $\ddot{\phi}$. Applying this symmetry generator to the geodesic equations gives

$$\begin{aligned}
\mathbf{X}^{[2]}E_1 &= \eta^1_{,ss} - \left(\dot{\chi}^2 + \dot{\theta}^2\sin^2\chi + \dot{\phi}^2\sin^2\chi\sin^2\theta\right)\eta^1\cos 2\psi\\
&\quad - \left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)\eta^2\sin 2\psi\sin\chi\cos\chi - \dot{\phi}^2\eta^3\sin 2\psi\sin^2\chi\sin\theta\cos\theta\\
&\quad - \dot{\chi}\eta^2_{,s}\sin 2\psi - \dot{\theta}\eta^3_{,s}\sin 2\psi\sin^2\chi - \dot{\phi}\eta^4_{,s}\sin 2\psi\sin^2\chi\sin^2\theta \,|_{E_i=0} = 0,
\end{aligned} \tag{3.114}$$

$$\begin{aligned}
\mathbf{X}^{[2]}E_2 &= \eta^2_{,ss} - 2\dot{\psi}\dot{\chi}\eta^1\csc^2\psi + 2\dot{\chi}\eta^1_{,s}\cot\psi + 2\dot{\psi}\eta^2_{,s}\cot\psi - \left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right)\eta^2\cos 2\chi\\
&\quad - \dot{\phi}^2\eta^3\sin 2\chi\sin\theta\cos\theta - \dot{\theta}\eta^3_{,s}\sin 2\chi - \dot{\phi}\eta^4_{,s}\sin 2\chi\sin^2\theta \,|_{E_i=0} = 0,
\end{aligned} \tag{3.115}$$

$$\mathbf{X}^{[2]}E_3 = \eta^3_{,ss} - 2\dot{\psi}\dot{\theta}\eta^1\csc^2\psi + 2\dot{\theta}\eta^1_{,s}\cot\psi + 2\left(\dot{\psi}\cot\psi + \dot{\chi}\cot\chi\right)\eta^3_{,s}$$

$$+2\dot{\theta}\eta^2_{,s}\cot\chi - 2\dot{\chi}\dot{\theta}\eta^2\csc^2\chi - \dot{\phi}^2\cos 2\theta\eta^3 - \dot{\phi}\sin 2\theta\eta^4_{,s}\,|_{E_i=0}= 0, \tag{3.116}$$

$$\begin{aligned} \mathbf{X}^{[2]}E_4 &= \eta^4_{,ss} - 2\dot{\psi}\dot{\phi}\eta^1\csc^2\psi + 2\dot{\phi}\eta^1_{,s}\cot\psi + 2\dot{\phi}\cot\chi\eta^2_{,s} + 2\dot{\phi}\cot\theta\eta^3_{,s} \\ &+2\left(\dot{\psi}\cot\psi + \dot{\chi}\cot\chi + \dot{\theta}\cot\theta\right)\eta^4_{,s} - 2\dot{\chi}\dot{\phi}\eta^2\csc^2\chi - 2\dot{\theta}\dot{\phi}\eta^3\csc^2\theta\,|_{E_i=0}= 0, \end{aligned} \tag{3.117}$$

where $i = 1, 2, 3, 4$. Now the prolongation coefficients take the form

$$\eta^1_{,s} = \eta^1_s + \dot{\psi}\left(\eta^1_\psi - \xi_s\right) + \dot{\chi}\eta^1_\chi + \dot{\theta}\eta^1_\theta + \dot{\phi}\eta^1_\phi - \dot{\psi}^2\xi_\psi - \dot{\psi}\dot{\chi}\xi_\chi - \dot{\psi}\dot{\theta}\xi_\theta - \dot{\psi}\dot{\phi}\xi_\phi, \tag{3.118}$$

$$\eta^2_{,s} = \eta^2_s + \dot{\psi}\eta^2_\psi + \dot{\chi}\left(\eta^2_\chi - \xi_s\right) + \dot{\theta}\eta^2_\theta + \dot{\phi}\eta^2_\phi - \dot{\psi}\dot{\chi}\xi_\psi - \dot{\chi}^2\xi_\chi - \dot{\chi}\dot{\theta}\xi_\theta - \dot{\chi}\dot{\phi}\xi_\phi, \tag{3.119}$$

$$\eta^3_{,s} = \eta^3_s + \dot{\psi}\eta^3_\psi + \dot{\chi}\eta^3_\chi + \dot{\theta}\left(\eta^3_\theta - \xi_s\right) + \dot{\phi}\eta^3_\phi - \dot{\psi}\dot{\theta}\xi_\psi - \dot{\chi}\dot{\theta}\xi_\chi - \dot{\theta}^2\xi_\theta - \dot{\theta}\dot{\phi}\xi_\phi, \tag{3.120}$$

$$\eta^4_{,s} = \eta^4_s + \dot{\psi}\eta^4_\psi + \dot{\chi}\eta^4_\chi + \dot{\theta}\eta^4_\theta + \dot{\phi}\left(\eta^4_\phi - \xi_s\right) - \dot{\psi}\dot{\phi}\xi_\psi - \dot{\chi}\dot{\phi}\xi_\chi - \dot{\theta}\dot{\phi}\xi_\theta - \dot{\phi}^2\xi_\phi, \tag{3.121}$$

$$\begin{aligned} \eta^1_{,ss} &= \eta^1_{ss} + \dot{\psi}\left(2\eta^1_{s\psi} - \xi_{ss}\right) + \dot{\psi}^2\left(\eta^1_{\psi\psi} - 2\xi_{s\psi}\right) - \dot{\psi}^3\xi_{\psi\psi} + 2\dot{\chi}\eta^1_{s\chi} + \dot{\chi}^2\eta^1_{\chi\chi} \\ &+2\dot{\theta}\eta^1_{s\theta} + \dot{\theta}^2\eta^1_{\theta\theta} + 2\dot{\phi}\eta^1_{s\phi} + \dot{\phi}^2\eta^1_{\phi\phi} + 2\dot{\psi}\dot{\chi}\left(\eta^1_{\psi\chi} - \xi_{s\chi}\right) + 2\dot{\psi}\dot{\theta}\left(\eta^1_{\psi\theta} - \xi_{s\theta}\right) \\ &+2\dot{\psi}\dot{\phi}\left(\eta^1_{\psi\phi} - \xi_{s\phi}\right) + 2\dot{\chi}\dot{\theta}\eta^1_{\chi\theta} + 2\dot{\chi}\dot{\phi}\eta^1_{\chi\phi} + 2\dot{\theta}\dot{\phi}\eta^1_{\theta\phi} - 2\dot{\psi}^2\dot{\chi}\xi_{\psi\chi} - 2\dot{\psi}^2\dot{\theta}\xi_{\psi\theta} \\ &-2\dot{\psi}^2\dot{\phi}\xi_{\psi\phi} - \dot{\psi}\dot{\chi}^2\xi_{\chi\chi} - \dot{\psi}\dot{\theta}^2\xi_{\theta\theta} - \dot{\psi}\dot{\phi}^2\xi_{\phi\phi} - 2\dot{\psi}\dot{\chi}\dot{\theta}\xi_{\chi\theta} - 2\dot{\psi}\dot{\chi}\dot{\phi}\xi_{\chi\phi} \\ &-2\dot{\psi}\dot{\theta}\dot{\phi}\xi_{\theta\phi} + \ddot{\psi}\left(\eta^1_\psi - 2\xi_s - 3\dot{\psi}\xi_\psi\right) - 2\ddot{\psi}\left(\dot{\chi}\xi_\chi + \dot{\theta}\xi_\theta + \dot{\phi}\xi_\phi\right) \\ &+\ddot{\chi}\left(\eta^1_\chi - \dot{\psi}\xi_\chi\right) + \ddot{\theta}\left(\eta^1_\theta - \dot{\psi}\xi_\theta\right) + \ddot{\phi}\left(\eta^1_\phi - \dot{\psi}\xi_\phi\right), \end{aligned} \tag{3.122}$$

$$\begin{aligned}
\eta^2_{,ss} &= \eta^2_{ss} + 2\dot{\psi}\eta^2_{s\psi} + \dot{\psi}^2\eta^2_{\psi\psi} + \dot{\chi}\left(2\eta^2_{s\chi} - \xi_{ss}\right) + \dot{\chi}^2\left(\eta^2_{\chi\chi} - 2\xi_{s\chi}\right) - \dot{\chi}^3\xi_{\chi\chi} + 2\dot{\theta}\eta^2_{s\theta} \\
&+ \dot{\theta}^2\eta^2_{\theta\theta} + 2\dot{\phi}\eta^2_{s\phi} + \dot{\phi}^2\eta^2_{\phi\phi} + 2\dot{\psi}\dot{\chi}\left(\eta^2_{\psi\chi} - \xi_{s\psi}\right) + 2\dot{\psi}\dot{\theta}\eta^2_{\psi\theta} + 2\dot{\psi}\dot{\phi}\eta^2_{\psi\phi} \\
&+ 2\dot{\chi}\dot{\theta}\left(\eta^2_{\chi\theta} - \xi_{s\theta}\right) + 2\dot{\chi}\dot{\phi}\left(\eta^2_{\chi\phi} - \xi_{s\phi}\right) + 2\dot{\theta}\dot{\phi}\eta^2_{\theta\phi} - \dot{\psi}^2\dot{\chi}\xi_{\psi\psi} - \dot{\chi}\dot{\theta}^2\xi_{\theta\theta} \\
&- \dot{\chi}\dot{\phi}^2\xi_{\phi\phi} - 2\dot{\psi}\dot{\chi}^2\xi_{\psi\chi} - 2\dot{\chi}^2\dot{\theta}\xi_{\chi\theta} - 2\dot{\chi}^2\dot{\phi}\xi_{\chi\phi} - 2\dot{\psi}\dot{\chi}\dot{\theta}\xi_{\psi\theta} - 2\dot{\psi}\dot{\chi}\dot{\phi}\xi_{\psi\phi} \\
&- 2\dot{\chi}\dot{\theta}\dot{\phi}\xi_{\theta\phi} + \ddot{\psi}\left(\eta^2_{\psi} - \dot{\chi}\xi_{\psi}\right) + \ddot{\chi}\left(\eta^2_{\chi} - 2\xi_s - 3\dot{\chi}\xi_{\chi}\right) - 2\ddot{\chi}\left(\dot{\psi}\xi_{\psi} + \dot{\theta}\xi_{\theta} + \dot{\phi}\xi_{\phi}\right) \\
&+ \ddot{\theta}\left(\eta^2_{\theta} - \dot{\chi}\xi_{\theta}\right) + \ddot{\phi}\left(\eta^2_{\phi} - \dot{\chi}\xi_{\phi}\right), \qquad (3.123)
\end{aligned}$$

$$\begin{aligned}
\eta^3_{,ss} &= \eta^3_{ss} + 2\dot{\psi}\eta^3_{s\psi} + \dot{\psi}^2\eta^3_{\psi\psi} + 2\dot{\chi}\eta^3_{s\chi} + \dot{\chi}^2\eta^3_{\chi\chi} + \dot{\theta}\left(2\eta^3_{s\theta} - \xi_{ss}\right) + \dot{\theta}^2\left(\eta^3_{\theta\theta} - 2\xi_{s\theta}\right) \\
&- \dot{\theta}^3\xi_{\theta\theta} + 2\dot{\phi}\eta^3_{s\phi} + \dot{\phi}^2\eta^3_{\phi\phi} + 2\dot{\psi}\dot{\chi}\eta^3_{\psi\chi} + 2\dot{\psi}\dot{\theta}\left(\eta^3_{\psi\theta} - \xi_{s\psi}\right) + 2\dot{\psi}\dot{\phi}\eta^3_{\psi\phi} \\
&+ 2\dot{\chi}\dot{\theta}\left(\eta^3_{\chi\theta} - \xi_{s\chi}\right) + 2\dot{\chi}\dot{\phi}\eta^3_{\chi\phi} + 2\dot{\theta}\dot{\phi}\left(\eta^3_{\theta\phi} - \xi_{s\phi}\right) - \dot{\psi}^2\dot{\theta}\xi_{\psi\psi} - \dot{\chi}^2\dot{\theta}\xi_{\chi\chi} \\
&- \dot{\theta}\dot{\phi}^2\xi_{\phi\phi} - 2\dot{\psi}\dot{\theta}^2\xi_{\psi\theta} - 2\dot{\chi}\dot{\theta}^2\xi_{\chi\theta} - 2\dot{\theta}^2\dot{\phi}\xi_{\theta\phi} - 2\dot{\psi}\dot{\chi}\dot{\theta}\xi_{\psi\chi} - 2\dot{\psi}\dot{\theta}\dot{\phi}\xi_{\psi\phi} \\
&- 2\dot{\chi}\dot{\theta}\dot{\phi}\xi_{\chi\phi} + \ddot{\psi}\left(\eta^3_{\psi} - \dot{\theta}\xi_{\psi}\right) + \ddot{\chi}\left(\eta^3_{\chi} - \dot{\theta}\xi_{\chi}\right) + \ddot{\theta}\left(\eta^3_{\theta} - 2\xi_s - 3\dot{\theta}\xi_{\theta}\right) \\
&- 2\ddot{\theta}\left(\dot{\psi}\xi_{\psi} + \dot{\chi}\xi_{\chi} + \dot{\phi}\xi_{\phi}\right) + \ddot{\phi}\left(\eta^3_{\phi} - \dot{\theta}\xi_{\phi}\right), \qquad (3.124)
\end{aligned}$$

$$\begin{aligned}
\eta^4_{,ss} &= \eta^4_{ss} + 2\dot{\psi}\eta^4_{s\psi} + \dot{\psi}^2\eta^4_{\psi\psi} + 2\dot{\chi}\eta^4_{s\chi} + \dot{\chi}^2\eta^4_{\chi\chi} + 2\dot{\theta}\eta^4_{s\theta} + \dot{\theta}^2\eta^4_{\theta\theta} + \dot{\phi}\left(2\eta^4_{s\phi} - \xi_{ss}\right) \\
&+ \dot{\phi}^2\left(\eta^4_{\phi\phi} - 2\xi_{s\phi}\right) - \dot{\phi}^3\xi_{\phi\phi} + 2\dot{\psi}\dot{\chi}\eta^4_{\psi\chi} + 2\dot{\psi}\dot{\theta}\eta^4_{\psi\theta} + 2\dot{\psi}\dot{\phi}\left(\eta^4_{\psi\phi} - \xi_{s\psi}\right) \\
&+ 2\dot{\chi}\dot{\theta}\eta^4_{\chi\theta} + 2\dot{\chi}\dot{\phi}\left(\eta^4_{\chi\phi} - \xi_{s\chi}\right) + 2\dot{\theta}\dot{\phi}\left(\eta^4_{\theta\phi} - \xi_{s\theta}\right) - 2\dot{\psi}\dot{\chi}\dot{\phi}\xi_{\psi\chi} - 2\dot{\psi}\dot{\theta}\dot{\phi}\xi_{\psi\theta} \\
&- 2\dot{\chi}\dot{\theta}\dot{\phi}\xi_{\chi\theta} - 2\dot{\psi}\dot{\phi}^2\xi_{\psi\phi} - 2\dot{\chi}\dot{\phi}^2\xi_{\chi\phi} - \dot{\psi}^2\dot{\phi}\xi_{\psi\psi} - \dot{\chi}^2\dot{\phi}\xi_{\chi\chi} - \dot{\theta}^2\dot{\phi}\xi_{\theta\theta} \\
&- 2\dot{\theta}\dot{\phi}^2\xi_{\theta\phi} + \ddot{\psi}\left(\eta^4_{\psi} - \dot{\phi}\xi_{\psi}\right) + \ddot{\chi}\left(\eta^4_{\chi} - \dot{\phi}\xi_{\chi}\right) + \ddot{\theta}\left(\eta^4_{\theta} - \dot{\phi}\xi_{\theta}\right) \\
&+ \ddot{\phi}\left(\eta^4_{\phi} - 2\xi_s - 3\dot{\phi}\xi_{\phi}\right) - 2\ddot{\phi}\left(\dot{\psi}\xi_{\psi} + \dot{\chi}\xi_{\chi} + \dot{\theta}\xi_{\theta}\right). \qquad (3.125)
\end{aligned}$$

Inserting these values in eqs.(3.117) and using the geodesic equations. The following system of 70 partial differential equations, given by eqs.(3.126) – (3.169) (in which one equation number often contains more than one equation), is obtained

$$\begin{aligned}
&\xi_{\psi\psi}=0,\ \ \xi_{\psi\chi}-\cot\psi\xi_{\chi}=0,\ \ \xi_{\psi\theta}-\cot\psi\xi_{\theta}=0,\ \ \xi_{\psi\phi}-\cot\psi\xi_{\phi}=0,\\
&\xi_{\chi\chi}+\sin\psi\cos\psi\xi_{\psi}=0,\ \xi_{\chi\theta}-\cot\chi\xi_{\theta}=0,\ \xi_{\chi\phi}-\cot\chi\xi_{\phi}=0,\\
&\xi_{\theta\phi}-\cot\theta\xi_{\phi}=0,\ \xi_{\theta\theta}+\sin\psi\cos\psi\xi_{\psi}\sin^2\chi+\sin\chi\cos\chi\xi_{\chi}=0,\\
&\xi_{\phi\phi}+\sin\psi\cos\psi\sin^2\chi\sin^2\theta\xi_{\psi}+\sin\chi\cos\chi\sin^2\theta\xi_{\chi}+\sin\theta\cos\theta\xi_{\theta}=0,
\end{aligned}\tag{3.126}$$

$$\eta^1_{ss}=0,\ \eta^1_{\psi\psi}-2\xi_{s\psi}=0,\tag{3.127}$$

$$2\eta^1_{s\psi}-\xi_{ss}=0,\tag{3.128}$$

$$2\eta^1_{s\chi}-\sin2\psi\eta^2_s=0,\tag{3.129}$$

$$\eta^1_{\psi\chi}-\xi_{s\chi}-\cot\psi\eta^1_{\chi}-\sin\psi\cos\psi\eta^2_{\psi}=0,\tag{3.130}$$

$$\eta^1_{\chi\chi}+\sin\psi\cos\psi\eta^1_{\psi}-\sin2\psi\eta^2_{\chi}-\cos2\psi\eta^1=0,\tag{3.131}$$

$$\eta^1_{\psi\theta}-\xi_{s\theta}-\cot\psi\eta^1_{\theta}-\sin\psi\cos\psi\sin^2\chi\eta^3_{\psi}=0,\tag{3.132}$$

$$\eta^1_{\chi\theta}-\cot\chi\eta^1_{\theta}-\sin\psi\cos\psi\eta^2_{\theta}-\sin\psi\cos\psi\sin^2\chi\eta^3_{\chi}=0,\tag{3.133}$$

$$\eta^1_{\psi\phi}-\xi_{s\phi}-\cot\psi\eta^1_{\phi}-\sin\psi\cos\psi\sin^2\chi\sin^2\theta\eta^4_{\psi}=0,\tag{3.134}$$

$$2\eta^1_{s\theta}-\sin2\psi\sin^2\chi\eta^3_s=0,\tag{3.135}$$

$$\eta^1_{s\phi}-\sin2\psi\sin^2\chi\sin^2\theta\eta^4_s=0,\tag{3.136}$$

$$\eta^1_{\chi\phi}-\cot\chi\eta^1_{\phi}-\sin\psi\cos\psi\eta^2_{\phi}-\sin\psi\cos\psi\sin^2\chi\sin^2\theta\eta^4_{\chi}=0,\tag{3.137}$$

$$\begin{aligned}
&\eta^1_{\theta\theta}+\sin\psi\cos\psi\sin^2\chi\eta^1_{\psi}+\sin\chi\cos\chi\eta^1_{\chi}-\sin2\psi\sin^2\chi\eta^3_{\theta}\\
&-\cos2\psi\sin^2\chi\eta^1-\sin2\psi\sin\chi\cos\chi\eta^2=0,
\end{aligned}\tag{3.138}$$

$$\eta^1_{\theta\phi}-\cot\theta\eta^1_{\phi}-\sin\psi\cos\psi\sin^2\chi\eta^3_{\phi}-\sin\psi\cos\psi\sin^2\chi\sin^2\theta\eta^4_{\theta}=0,\tag{3.139}$$

$$\eta^1_{\phi\phi} + \sin\psi\cos\psi\sin^2\chi\sin^2\theta\eta^1_\psi + \sin\chi\cos\chi\sin^2\theta\eta^1_\chi + \sin\theta\cos\theta\eta^1_\theta$$
$$-\sin 2\psi\sin^2\chi\sin^2\theta\eta^4_\phi - \cos 2\psi\sin^2\chi\sin^2\theta\eta^1 \tag{3.140}$$
$$-\sin 2\psi\sin\chi\cos\chi\sin^2\theta\eta^2 - \sin 2\psi\sin^2\chi\sin\theta\cos\theta\eta^3 = 0,$$

$$\eta^2_{ss} = 0,\ \eta^2_{s\psi} + \cot\psi\eta^2_s = 0,\ \eta^2_{\psi\psi} + 2\cot\psi\eta^2_\psi = 0, \tag{3.141}$$

$$\eta^2_{\psi\chi} - \xi_{\psi s} + \cot\psi\eta^1_\psi - \csc^2\psi\eta^1 = 0, \tag{3.142}$$

$$\eta^2_{\chi\chi} - \xi_{s\chi} + 2\cot\psi\eta^1_\chi + \sin\psi\cos\psi\eta^2_\psi = 0, \tag{3.143}$$

$$2\eta^2_{s\chi} + 2\cot\psi\eta^1_s - \xi_{ss} = 0, \tag{3.144}$$

$$2\eta^2_{s\theta} - \sin\chi\cos\chi\eta^3_s = 0,\ 2\eta^2_{s\phi} - \sin 2\chi\sin^2\theta\eta^4_s = 0, \tag{3.145}$$

$$\eta^2_{\chi\theta} - \xi_{s\theta} + \cot\psi\eta^1_\theta - \sin\chi\cos\chi\eta^3_\chi - \cot\chi\eta^2_\theta = 0,$$
$$\eta^2_{\chi\phi} - \xi_{s\phi} - \cot\chi\eta^2_\phi + \cot\psi\eta^1_\phi - \sin\chi\cos\chi\sin^2\theta\eta^4_\chi = 0, \tag{3.146}$$
$$\eta^2_{\psi\phi} - \sin\chi\cos\chi\sin^2\theta\eta^4_\psi = 0,$$

$$\eta^2_{\psi\theta} - \sin\chi\cos\chi\eta^3_\psi = 0,\ \eta^2_{\theta\theta} + \sin\psi\cos\psi\sin^2\chi\eta^2_\psi + \sin\chi\cos\chi\eta^2_\chi - \cos 2\chi\eta^2 - \sin 2\chi\eta^3_\theta = 0, \tag{3.147}$$

$$\eta^2_{\theta\phi} - \sin\chi\cos\chi\eta^3_\phi - \cot\theta\eta^2_\phi - \sin\chi\cos\chi\sin^2\theta\eta^4_\theta = 0, \tag{3.148}$$

$$\eta^2_{\phi\phi} + \sin\psi\cos\psi\sin^2\chi\sin^2\theta\eta^2_\psi + \sin\chi\cos\chi\sin^2\theta\eta^2_\chi - \sin 2\chi\sin^2\theta\eta^4_\phi$$
$$-\cos 2\chi\sin^2\theta\eta^2 - \sin 2\chi\sin\theta\cos\theta\eta^3 = 0, \tag{3.149}$$

$$\eta^3_{ss} = 0,\ \eta^3_{s\psi} + 2\cot\psi\eta^3_s = 0,\ \eta^3_{s\chi} + \cot\chi\eta^3_s = 0,\ \eta^3_{\psi\psi} + 2\cot\psi\eta^3_\psi = 0, \tag{3.150}$$

$$2\eta^3_{s\theta} - \xi_{ss} + 2\cot\psi\eta^1_s + 2\cot\chi\eta^2_s = 0, \tag{3.151}$$

$$\eta^3_{s\phi} - \sin\theta\cos\theta\eta^4_s = 0, \tag{3.152}$$

$$\eta^3_{\psi\theta} - \xi_{s\psi} - \cot\psi\eta^1_\psi + \cot\chi\eta^2_\psi - \csc^2\psi\eta^1 = 0, \tag{3.153}$$

$$\eta^3_{\chi\theta} - \xi_{s\chi} + \cot\psi\eta^1_\chi + \cot\chi\eta^2_\chi - \csc^2\chi\eta^2 = 0, \tag{3.154}$$

$$\eta^3_{\psi\chi}+\cot\chi\eta^3_\psi=0,\ \eta^3_{\chi\chi}+\sin\psi\cos\psi\eta^3_\psi+2\cot\chi\eta^3_\chi=0, \tag{3.155}$$

$$\eta^3_{\psi\phi}-\sin\theta\cos\theta\eta^4_\psi=0, \tag{3.156}$$

$$\eta^3_{\chi\phi}-\sin\theta\cos\theta\eta^4_\chi=0, \tag{3.157}$$

$$\eta^3_{\theta\theta}-2\xi_{s\theta}+\sin\psi\cos\psi\sin^2\chi\eta^3_\psi+\sin\chi\cos\chi\eta^3_\chi+2\cot\psi\eta^1_\theta+2\cot\chi\eta^2_\theta=0, \tag{3.158}$$

$$\eta^3_{\theta\phi}-\xi_{s\phi}+\cot\psi\eta^1_\phi+\cot\chi\eta^2_\phi+\cot\theta\eta^3_\phi-\sin\theta\cos\theta\eta^4_\theta=0, \tag{3.159}$$

$$\begin{aligned}&\eta^3_{\phi\phi}+\sin\psi\cos\psi\sin^2\chi\sin^2\theta\eta^3_\psi+\sin\chi\cos\chi\sin^2\theta\eta^3_\chi+\sin\theta\cos\theta\eta^3_\theta\\&-\sin 2\theta\eta^4_\phi-\cos 2\theta\eta^3=0,\end{aligned} \tag{3.160}$$

$$\eta^4_{ss}=0,\ \eta^4_{s\psi}+\cot\psi\eta^4_s=0,\ \eta^4_{\psi\psi}+2\cot\psi\eta^4_\psi=0, \tag{3.161}$$

$$\eta^4_{\psi\chi}+\cot\chi\eta^4_\psi=0,\ \eta^4_{\psi\theta}+\cot\theta\eta^4_\psi=0,\ \ \eta^4_{\chi\chi}+\sin\psi\cos\psi\eta^4_\psi+2\cot\chi\eta^4_\chi=0, \tag{3.162}$$

$$\eta^4_{\chi\theta}+\cot\theta\eta^4_\chi=0,\ \eta^4_{\theta\theta}+\sin\psi\cos\psi\sin^2\chi\eta^4_\psi+\sin\chi\cos\chi\eta^4_\chi+2\cot\theta\eta^4_\theta=0, \tag{3.163}$$

$$\eta^4_{s\chi}+\cot\chi\eta^4_s=0,\ \eta^4_{s\theta}+\cot\theta\eta^4_s=0, \tag{3.164}$$

$$2\eta^4_{s\phi}-\xi_{ss}+2\cot\psi\eta^1_s+2\cot\chi\eta^2_s+2\cot\theta\eta^3_s=0, \tag{3.165}$$

$$\eta^4_{\psi\phi}-\xi_{s\psi}+\cot\psi\eta^1_\psi+\cot\chi\eta^2_\psi+\cot\theta\eta^3_\psi-\csc^2\psi\eta^1=0, \tag{3.166}$$

$$\eta^4_{\chi\phi}-\xi_{s\chi}+\cot\psi\eta^1_\chi+\cot\chi\eta^2_\chi+\cot\theta\eta^3_\chi-\csc^2\chi\eta^2=0, \tag{3.167}$$

$$\eta^4_{\theta\phi}-\xi_{s\theta}+\cot\psi\eta^1_\theta+\cot\chi\eta^2_\theta+\cot\theta\eta^3_\theta-\csc^2\theta\eta^3=0, \tag{3.168}$$

$$\begin{aligned}&\eta^4_{\phi\phi}-2\xi_{s\phi}+\sin\psi\cos\psi\sin^2\chi\sin^2\theta\eta^4_\psi+\sin\chi\cos\chi\sin^2\theta\eta^4_\chi\\&+\sin\theta\cos\theta\eta^4_\theta+2\cot\psi\eta^1_\phi+2\cot\chi\eta^2_\phi+2\cot\theta\eta^3_\phi=0.\end{aligned} \tag{3.169}$$

Eqs.(3.126) give $\xi=a_1(s)$. Eqs.(3.127) yield

$$\eta^1=\left(g_1(\chi,\theta,\phi)\,s+g_2(\chi,\theta,\phi)\right)\psi+g_3(\chi,\theta,\phi)\,s+g_4(\chi,\theta,\phi)\,. \tag{3.170}$$

From eqs.(3.141) we get

$$\eta^2 = -g_6(\chi,\theta,\phi)\cot\psi + g_7(\chi,\theta,\phi). \tag{3.171}$$

Eqs.(3.128) yield

$$\xi = c_0 + c_1 s, \tag{3.172}$$

$$\eta^1 = g_2(\chi,\theta,\phi)\psi + g_4(\chi,\theta,\phi). \tag{3.173}$$

Eqs.(3.130, 3.142) give

$$\eta^1 = -f_4(\theta,\phi)\sin\chi + f_5(\theta,\phi)\cos\chi, \tag{3.174}$$

$$\eta^2 = -(f_4(\theta,\phi)\cos\chi + f_5\sin\chi)\cot\psi + g_7(\chi,\theta,\phi), \tag{3.175}$$

and from eqs.(3.131, 3.143) we have

$$g_7(\chi,\theta,\phi) = f_7(\theta,\phi), \tag{3.176}$$

therefore,

$$\eta^2 = -(f_4(\theta,\phi)\cos\chi + f_5\sin\chi)\cot\psi + f_7(\theta,\phi), \tag{3.177}$$

is obtained. Eq.(3.150) gives

$$\eta^3 = -g_8(\chi,\theta,\phi)\cot\psi + g_9(\chi,\theta,\phi). \tag{3.178}$$

Now eq.(3.155) yields

$$g_9(\chi,\theta,\phi) = -f_9(\theta,\phi)\cot\chi + f_{10}(\theta,\phi), \tag{3.179}$$

hence

$$\eta^3 = -f_8(\theta,\phi)\cot\psi\csc\chi - f_9(\theta,\phi)\cot\chi + f_{10}(\theta,\phi) \tag{3.180}$$

and from eq.(3.132)

$$\eta^1 = -f_4(\theta,\phi)\sin\chi + k_1(\phi)\cos\chi, \tag{3.181}$$

$$\eta^2 = -(f_4(\theta,\phi)\cos\chi + k_1(\phi)\sin\chi)\cot\psi + f_7(\theta,\phi). \tag{3.182}$$

Eq.(3.147) then gives

$$f_4(\theta,\phi) = k_2(\phi)\cos\theta + k_3(\phi)\sin\theta,\ f_8(\theta,\phi) = -k_2(\phi)\sin\theta + k_3(\phi)\cos\theta,$$

$$f_9(\theta,\phi) = k_5(\phi)\sin\theta - k_6(\phi)\cos\theta + k_7(\phi),\ f_{10}(\theta,\phi) = k_4(\phi),$$

$$\eta^1 = -(k_2(\phi)\cos\theta + k_3(\phi)\sin\theta)\sin\chi + k_1(\phi)\cos\chi, \tag{3.183}$$

$$\eta^2 = -[(k_2(\phi)\cos\theta + k_3(\phi)\sin\theta)\cos\chi + k_1(\phi)\sin\chi]\cot\psi + (k_5(\phi)\cos\theta + k_6(\phi)\sin\theta), \tag{3.184}$$

$$\eta^3 = -(-k_2(\phi)\sin\theta + k_3(\phi)\cos\theta)\cot\psi\csc\chi - (k_5(\phi)\sin\theta - k_6(\phi)\cos\theta + k_7(\phi))\cot\chi + k_4(\phi), \tag{3.185}$$

and from eq.(3.133) we have

$$\eta^3 = -(-k_2(\phi)\sin\theta + k_3(\phi)\cos\theta)\cot\psi\csc\chi - (k_5(\phi)\sin\theta - k_6(\phi)\cos\theta)\cot\chi + k_4(\phi). \tag{3.186}$$

Eq.(3.161) gives

$$\eta^4 = -g_{10}(\chi,\theta,\phi)\cot\psi + g_{11}(\chi,\theta,\phi), \tag{3.187}$$

and so eq.(3.162) implies

$$\eta^4 = -k_8(\phi)\cot\psi\csc\chi\csc\theta - f_{11}(\theta,\phi)\cot\chi + f_{12}(\theta,\phi). \tag{3.188}$$

From eqs.(3.163) we get

$$\eta^4 = -k_8(\phi)\cot\psi\csc\chi\csc\theta - k_9(\phi)\cot\chi\csc\theta - k_{10}(\phi)\cot\theta + k_{11}(\phi). \tag{3.189}$$

Eq.(3.134) gives

$$k_1' = 0,\ k_2' = 0. \tag{3.190}$$

Therefore,

$$\eta^1 = -\left(c_9 \cos\theta + k_3(\phi)\sin\theta\right)\sin\chi + c_8\cos\chi, \tag{3.191}$$

$$\begin{aligned}\eta^2 &= -\left[\left(c_9 \cos\theta + k_3(\phi)\sin\theta\right)\cos\chi + c_8\sin\chi\right]\cot\psi \\ &\quad + k_5(\phi)\cos\theta + k_6(\phi)\sin\theta.\end{aligned} \tag{3.192}$$

From eqs.(3.156) and (3.166) we have

$$k_3 = c_{10}\cos\phi + c_{11}\sin\phi,\ k_8 = -c_{10}\sin\phi + c_{11}\cos\phi. \tag{3.193}$$

Eqs.(3.157) and (3.167) yield

$$k_5 = c_5,\ k_6 = c_6\cos\phi + c_7\sin\phi,\ k_9 = c_6\sin\phi - c_7\cos\phi. \tag{3.194}$$

Eqs.(3.159) and (3.168) give

$$k_4 = c_3\cos\phi + c_4\sin\phi,\ k_{10} = c_3\sin\phi - c_4\cos\phi, \tag{3.195}$$

and eq.(3.160) gives

$$k_{11} = c_2. \tag{3.196}$$

Hence

$$\eta^1 = -\left[c_9\cos\theta + (c_{10}\cos\phi + c_{11}\sin\phi)\sin\theta\right]\sin\chi + c_8\cos\chi, \tag{3.197}$$

$$\begin{aligned}\eta^2 &= -\left[\left(c_9\cos\theta + (c_{10}\cos\phi + c_{11}\sin\phi)\sin\theta\right)\cos\chi + c_8\sin\chi\right]\cot\psi \\ &\quad + c_5\cos\theta + (c_6\cos\phi + c_7\sin\phi)\sin\theta,\end{aligned} \tag{3.198}$$

$$\eta^3 = -\left(-c_9\sin\theta + (c_{10}\cos\phi + c_{11}\sin\phi)\cos\theta\right)\cot\psi\csc\chi$$

$$- (c_5 \sin\theta - (c_6 \cos\phi + c_7 \sin\phi)\cos\theta)\cot\chi + c_3 \cos\phi + c_4 \sin\phi, \tag{3.199}$$

$$\eta^4 = -(-c_{10}\sin\phi + c_{11}\cos\phi)\cot\psi\csc\chi\csc\theta - (c_6\sin\phi - c_7\cos\phi)\cot\chi\csc\theta$$

$$- (c_3\sin\phi - c_4\cos\phi)\cot\theta + c_2. \tag{3.200}$$

The extra symmetries of eq.(3.110) are

$$\begin{aligned}
&\mathbf{X}_8 = \cos\chi\tfrac{\partial}{\partial\psi} - \cot\psi\sin\chi\tfrac{\partial}{\partial\chi},\\
&\mathbf{X}_9 = -\cos\theta\sin\chi\tfrac{\partial}{\partial\psi} - \cot\psi\cos\theta\cos\chi\tfrac{\partial}{\partial\chi} + \cot\psi\csc\chi\sin\theta\tfrac{\partial}{\partial\theta},\\
&\mathbf{X}_{10} = -\cos\phi\sin\theta\sin\chi\tfrac{\partial}{\partial\psi} - \cot\psi\cos\phi\sin\theta\cos\chi\tfrac{\partial}{\partial\chi}\\
&\quad - \cot\psi\csc\chi\cos\phi\cos\theta\tfrac{\partial}{\partial\theta} + \cot\psi\csc\theta\csc\chi\sin\phi\tfrac{\partial}{\partial\phi},\\
&\mathbf{X}_{11} = -\sin\phi\sin\theta\sin\chi\tfrac{\partial}{\partial\psi} - \cot\psi\sin\phi\sin\theta\cos\chi\tfrac{\partial}{\partial\chi}\\
&\quad - \cot\psi\csc\chi\sin\phi\cos\theta\tfrac{\partial}{\partial\theta} - \csc\theta\cot\psi\csc\chi\cos\phi\tfrac{\partial}{\partial\phi}.
\end{aligned} \tag{3.201}$$

These symmetries form the Lie algebra $so(5) \oplus d_2$. Similarly, the Lie algebra of the symmetries of the geodesic equations for a plane is $sl(6)$ (35 symmetries) and that for a hyperboloid is $so(4,1) \oplus d_2$ (12 symmetries).

Notice that when we go from maximally symmetric spaces of zero curvature to maximally symmetric spaces of constant (positive or negative) curvature we lose $n(n-5)/2$ symmetries of the geodesic equations. In case of positive or negative curvature (unlike the case of zero curvature) the symmetries of the parameter s and the symmetries of the other coordinates appear separately, i.e. there is no symmetry in which the affine parameter s appears with any other coordinates.

3.6 Symmetries of the Reduced System of the Geodesic Equations of Maximally Symmetric Higher Dimensional Spaces

Up till now we were introducing an extra dimension and were determining the higher dimensional system of equations. In this section we reduce the system of equations by solving one of them and inserting the solution into the remaining set of equations. In this case the number of symmetries is not the same as those of the original set. This is most easily seen by considering the vector equation $\mathbf{y}'' = 0$ in $n-$dimensions, which has the Lie algebra $sl\,(n+2)$. Now, when we reduce the system to $(n-1)-$dimensions, we get the symmetry algebra $sl\,(n+1)$. Generally, we would expect that the further reduced set will have even lower symmetry. However, this process of reduction may not have followed a canonical procedure from n to $(n-1)$ to $(n-2)$. In that case the system of $(n-2)$ equations may have more symmetries than the system of $(n-1)$ equations. For example the geodesic equations of the sphere can be reduced to a single equation by solving eq.(3.12) and substituting $\dot{\phi} = h\csc^2\theta$ (h is an arbitrary constant) into E_1. Simplification gives

$$\frac{d^2\cot\theta}{d\phi^2} + \cot\theta = 0. \tag{3.202}$$

Taking $y = \cot\theta$ and $\phi = s$ the above equation is same as eq.(4.21) so eq.(3.202) admits an $8-$parameter Lie algebra. Similarly, the geodesic equations of the hyperboloid can be transformed to

$$\frac{d^2\cot\theta}{d\phi^2} - \cot\theta = 0. \tag{3.203}$$

The symmetries of eq.(3.203) also correspond to an 8−parameter Lie algebra.

Our canonical procedure is to reduce the system geometrically, by considering the equatorial plane. For example, for the hypersphere, taking $\psi = \frac{\pi}{2}$, the metric given by eq.(3.111) and the geodesic equations (3.112) reduce to the metric eq.(3.63) and the geodesic equations (3.65) respectively. Similarly, by taking $\chi = \frac{\pi}{2}$ in eqs.(3.63, 3.65), the system of three geodesic equations (3.65) can be reduced into the system of two geodesic equations (3.12). Using this procedure one readily finds the number of symmetries of the reduced system of equations, starting from a space of maximal symmetry.

A very important relation between the isometries (symmetries of manifolds) and the symmetries of the geodesic equations of the maximally symmetric spaces is obtained from the two results, obtained by introducing a new dimension and by reducing the system, using the canonical procedure. This relation can be written in the form of the following:

Theorem 7 *For the spaces with isometry algebra* $so\,(n+1)$, $so\,(n)\oplus_s\mathbb{R}^n$ *and* $so\,(n,1)$, *the Lie algebra for symmetries of the geodesic equations are* $so\,(n+1)\oplus d_2$, $sl\,(n+2)$ *and* $so\,(n,1)\oplus d_2$ *respectively (where* $\oplus_s$ *represents semidirect sum).*

Proof The Lie algebra of the symmetries of the system of geodesic equations of a 2−dimensional sphere, having the isometry algebra $so\,(3)$, is $so\,(3)\oplus d_2$. For a hypersphere of isometry algebra $so\,(4)$, the symmetries of the geodesic equations form $so\,(4)\oplus d_2$ and the symmetries of the geodesic equations of the 4−dimensional hypersphere $so\,(5)$, form the algebra $so\,(5)\oplus d_2$. Now the canonical reduction of the system of differential equations for $n-$dimensional space gives the system for $(n-1)-$dimensional space. (The canonical reduction of the system for $4-$dimensional space gives the system for $3-$dimensional space and the reduction for $3-$dimensional space gives the system for $2-$dimensional space.) Working in reverse, if the algebra for an $n-$dimensional space is $so(n) \oplus d_2$ that for an $(n+1)-$dimensional space gets n more generators of rotation between the new dimension and the previous dimensions. Hence the algebra is $so(n+1) \oplus d_2$. Similarly, for the space of negative curvature it is $so(n,1) \oplus d_2$. In the case of the $n-$dimensional hyperplane, the Lie algebra of symmetries of geodesic equations is $sl(n+2)$ [32].

Chapter 4

Approximate Symmetries

Sometimes differential equations appearing in mathematical modeling are written with terms involving a small parameter called the *perturbed term*. Generally, the perturbed term in a differential equation corresponds to some small error or correction, but the presence of such terms changes the Lie point symmetries significantly. For example, $y''' + 2\varepsilon y y'^5 + 6y = 0$, $y'' + \varepsilon y''' = 0$ and $y''' + 6y = 0$, $y'' = 0$ will have quite different symmetries. In the previous chapter we have seen that the perturbation in the manifold destroys the exact symmetries of the manifold. (A sphere has three symmetries whereas a perturbed sphere i.e. a spheroid has only one symmetry.) Due to the instability of the Lie point symmetries with respect to perturbation of coefficients of differential equations, one finds the *perturbed (approximate) symmetries* for such equations. The method of finding the symmetries of exact differential equation has been discussed in Chapter 1. In this chapter, after discussing the method of determining the symmetries of *approximate differential equations* our work about the conservation laws and Lagrangian of a particular subclass of a class of approximate differential equations,

$$y'' + 2\varepsilon y' + y = f(y, y'), \tag{4.1}$$

is presented [34].

Generally, in order to obtain solutions of perturbed differential equations they are first linearized with respect to the parameter and are written as

$$E \equiv E_0 + \varepsilon E_1 \approx 0, \tag{4.2}$$

where E_0 and E_1 are the exact and the approximate parts of the approximate differential equation E. The Lagrangian corresponding to an approximate differential equation may also have its exact and approximate parts i.e. $L(x, y, y', \varepsilon) = L_0(x, y, y') + \varepsilon L_1(x, y, y')$. Denman discusses the approximate invariants for some classes of eq.(4.1) in references [35, 36], e.g.,

$$y'' + 2\varepsilon y' + y = 0, \tag{4.3}$$

and the damped Duffing equation with $f = -\gamma y^3$. The ideas are then extended to time-dependent invariants [37]. A detailed account of the application of the equations are presented in reference [35]. The Lagrangians constructed in these references are exact, i.e. they considered ε as any constant, e.g. the equation $y'' + 2\varepsilon y' + y = 0$ has Lagrangian $1/2e^{2t}(y'^2 - y^2)$.

In the sequel, we deal with the equations in the above references using the group theoretic approach. First, the equations are regarded as perturbed equations with some small parameter ε. To this end, we utilize the theory developed to determine the first-order *approximate* Lie symmetries of the equations [38, 39]. Then, the first integrals *associated* with these symmetries are determined [40]. Finally, Lagrangians (first-order in ε) are constructed using an extended form of results developed for unperturbed equations [41, 42], i.e. from the symmetries and associated first integrals. Thus, the approximate symmetries are Noether symmetries and the associated first (approximate) integrals are Noether invariants corresponding to the constructed Lagrangians. It should be borne in mind that we maintain the order of the perturbation for the symmetries, first integrals and the Lagrangian — it being first order in this case.

4.1 Algorithm for Calculating Infinitesimal Approximate Symmetries

In this section the method of determining the approximate symmetries and Lagrangian corresponding to eq.(4.2) is given. For this purpose, some pertinent results are reviewed. The presentation here is for ordinary differential equations. However, most of the theory has been generalized to partial differential equations. For eq.(4.2) the infinitesimal generator is

$$\mathrm{X} = \mathrm{X}_0 + \varepsilon \mathrm{X}_1, \tag{4.4}$$

where

$$\mathrm{X}_0 = \xi_0 \frac{\partial}{\partial x} + \eta_0 \frac{\partial}{\partial y} \text{ and } \mathrm{X}_1 = \xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial y}. \tag{4.5}$$

Now the invariance criterion is

Theorem 8 *Eq.(4.2) is approximately invariant under the approximate group of transformations with the generator eq.(4.4) if and only if* $XE \mid_{E\approx 0} \approx O(\varepsilon)$*, equivalently*

$$[X_0E_0 + \varepsilon\,(X_0E_1 + X_1E_0)] \mid_{E\approx 0} \approx O(\varepsilon). \tag{4.6}$$

Proof Substitute eqs.(4.2, 4.4) into the determining equations of theorem 2 and single out the principal part.

Eq.(4.6) gives the determining equation. X is called an *infinitesimal approximate symmetry*, or *approximate operator* admitted by eq.(4.5) [8].

If X_0 is a generator of Lie (point) symmetry of a differential equation

$$E_0 = 0, \tag{4.7}$$

then an *approximate symmetry*, eq.(4.4), of the perturbed differential eq.(4.2) is obtained by solving for X_1 in

$$X_1(E_0)|_{E_0=0} + H = 0, \tag{4.8}$$

where

$$H = \frac{1}{\varepsilon}X_0(E_0 + \varepsilon E_1)|_{E_0+\varepsilon E_1=0}. \tag{4.9}$$

Here E_1 is the perturbation and H is referred to as an auxiliary function. The one-parameter approximate group of transformations

$$x^* = x_0^* + \varepsilon x_1^*,\;\; y^* = y_0^* + \varepsilon y_1^*, \tag{4.10}$$

is obtained from Lie's equations [39].

Furthermore, a first integral $T = T_0 + \varepsilon T_1$ (i.e. $DT = 0$ along the solutions of eq.(4.2)) *associated* with X satisfies [40]

$$\mathrm{X}_0 T_0 = 0, \tag{4.11}$$

$$\mathrm{X}_0 T_1 = -\mathrm{X}_1 T_0. \tag{4.12}$$

For the Lagrangian formulation, more specifically for the inverse problem, we appeal to Noether's theorem. However, we first need to extend theorem 3 given in the first chapter for the perturbed Lagrangian. This extension is presented here in the following form:

Theorem 9 *If $L(x,y,y',\varepsilon) = L_0(x,y,y') + \varepsilon L_1(x,y,y')$ is a first-order Lagrangian corresponding to a second-order ordinary differential eq.(4.2) and the functional $\int_V L dx$ is invariant under the one-parameter group of transformations with approximate Lie symmetry generator $\mathrm{X} = \mathrm{X}_0 + \varepsilon \mathrm{X}_1$, upto gauge $B = B_0 + \varepsilon B_1$, where*

$$\mathrm{X}_0 = \xi_0 \partial/\partial x + \eta_0 \partial/\partial y, \; \mathrm{X}_1 = \xi_1 \partial/\partial x + \eta_1 \partial/\partial y,$$

then

$$\mathrm{X}_0 L_0 + L_0 D\xi_0 = DB_0, \tag{4.13}$$

$$\mathrm{X}_1 L_0 + \mathrm{X}_0 L_1 + L_0 D\xi_1 + L_1 D\xi_0 = DB_1, \tag{4.14}$$

where D is the total differential operator with respect to x, i.e.

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots.$$

Proof Suppose that the transformed variables are $x^* = x + a\xi$ and $y^* = y + a\eta$ (so that $y^{*\prime} = y' + a\zeta^1$ where $\zeta^1 = D\eta - y'D\xi$). Then

$$\int_{\overline{V}} L(x^*, \ldots, \varepsilon)dx^* = \int_V L_0(x, \ldots)dx^* + \varepsilon \int_V L_1(x, \ldots)dx^*. \tag{4.15}$$

Differentiating both sides and expanding the left hand side using a Taylor expansion upto order a, we obtain

$$\mathbf{X}_0 L_0 + L_0 D\xi_0 + \varepsilon \left[\mathbf{X}_1 L_0 + \mathbf{X}_0 L_1 + L_0 D\xi_1 + L_1 D\xi_0\right] = DB_0 + \varepsilon DB_1. \tag{4.16}$$

Separations by powers of ε yield eqs.(4.13, 4.14).

The Noether theorem (theorem 4 of the first chapter) reads as follows:

Theorem 10 *Corresponding to each symmetry* $\mathbf{X} = \xi\partial/\partial x + \eta\partial/\partial y$ *that satisfies the conditions of previous theorem, there exists a first integral* T *(or conservation law* $DT = 0$ *along the solutions of the differential equation) given by*

$$T = B - L\xi - (\eta - y'\xi)\frac{\partial L}{\partial y'}. \tag{4.17}$$

With $T = T_0 + \varepsilon T_1$*, if we separate by powers of* ε *upto order* ε*, we get*

$$T_0 = B_0 - L_0\xi_0 - (\eta_0 - y'\xi_0)\frac{\partial L_0}{\partial y'^2}, \tag{4.18}$$

$$T_1 = B_1 - (L_0\xi_1 + L_1\xi_0) - \left[(\eta_0 - y'\xi_0)\frac{\partial L_1}{\partial y'} + (\eta_1 - y'\xi_1)\frac{\partial L_0}{\partial y'}\right]. \tag{4.19}$$

4.2 Applications

We consider the damped harmonic oscillator equation

$$y'' + y + 2\varepsilon y' = 0, \tag{4.20}$$

which has been analyzed in references [35, 36, 37]. A Lagrangian formulation of the equation and comments on integration by quadratures are given in reference [44]. Herein, however, $\varepsilon = -a/2$ for some constant a and the equation is not considered as a perturbation of $y'' + y = 0$.

It is well known that the unperturbed equation

$$y'' + y = 0, \tag{4.21}$$

admits the 8−dimensional Lie algebra of point symmetry generators given by

$$\begin{aligned}
&\mathrm{X}_0^1 = \tfrac{\partial}{\partial x}, \quad \mathrm{X}_0^2 = y\tfrac{\partial}{\partial y}, \quad \mathrm{X}_0^3 = \sin x\tfrac{\partial}{\partial y}, \\
&\mathrm{X}_0^4 = \cos x\tfrac{\partial}{\partial y}, \quad \mathrm{X}_0^5 = \sin 2x\tfrac{\partial}{\partial x} + y\cos 2x\tfrac{\partial}{\partial y}, \\
&\mathrm{X}_0^6 = \cos 2x\tfrac{\partial}{\partial x} - y\sin 2x\tfrac{\partial}{\partial y}, \quad \mathrm{X}_0^7 = y\cos x\tfrac{\partial}{\partial x} - y^2\sin x\tfrac{\partial}{\partial y}, \\
&\mathrm{X}_0^8 = y\sin x\tfrac{\partial}{\partial x} + y^2\cos x\tfrac{\partial}{\partial y}.
\end{aligned} \tag{4.22}$$

We choose the symmetry $\mathbf{X}_0 = \sin x\partial/\partial y$ to construct $\mathbf{X}_1$ in eq.(4.8). First,

$$H = \frac{1}{\varepsilon}\mathrm{X}_0(y'' + y + 2\varepsilon y')|_{y''+y+2\varepsilon y'=0} = 2\cos x, \tag{4.23}$$

and eq.(4.8) becomes

$$\begin{aligned}
&\eta_{1xx} + y'(2\eta_{1xy} - \xi_{1xx}) + y'^2(\eta_{1yy} - 2\xi_{1xy}) - y'^3\xi_{1yy} \\
&-y''(\eta_{1y} - 2\xi_{1x} - 3y'\xi_{1y}) + \eta_1 + 2\cos x = 0,
\end{aligned} \tag{4.24}$$

along the solutions of eq.(4.20). Using the standard procedure of separation by powers of y', we get

$$\xi_{1yy} = 0, \quad \eta_{1yy} - 2\xi_{1xy} = 0, \tag{4.25}$$

$$\begin{aligned}
&\eta_{1xx} + \eta_1 + y\left(\eta_{1y} - 2\xi_{1x}\right) + 2\cos x = 0, \\
&2\eta_{1xy} - \xi_{1xx} + 3y\xi_{1y} = 0.
\end{aligned} \tag{4.26}$$

Now the eqs.(4.25) give

$$\xi_1 = a(x)y + b(x),$$

and

$$\eta_1 = a'y^2 + c(x)y + d(x),$$

respectively. From eqs.(4.26) we have

$$\begin{aligned} &a'' + a = 0 \Rightarrow a = A\cos x + B\sin x, \\ &b' = 2c, \\ &c'' + 4c = 0 \Rightarrow c = P\cos 2x - Q\sin 2x, \\ &d'' + d + 2\cos x = 0 \Rightarrow d = R\cos x + S\sin x - x\sin x. \end{aligned} \tag{4.27}$$

Putting $A = 1$ and the other constants equal to zero yields a generator from the list in eq.(4.22), i.e. the approximate symmetry $\mathbf{X}$ is a multiple of some symmetry of $y'' + y = 0$. So too with the other constants B, P, Q, R and S. The only 'new' symmetry comes from $d = -x\sin x$, i.e. we have a non-trivial approximate symmetry of eq.(4.3) given by

$$\mathbf{X} = \sin x\partial/\partial y - \varepsilon x\sin x\partial/\partial y. \tag{4.28}$$

We construct the approximate invariant $T = T_0 + \varepsilon T_1$ using eqs.(4.11, 4.12) and $DT = 0$ along the solutions of eq.(4.2). First, $\mathbf{X}_0 T_0 = 0$ implies that

$$(\sin x\partial/\partial y + \cos x\partial/\partial y')\,T_0 = 0, \tag{4.29}$$

which yields the characteristic equation

$$\frac{dx}{0} = \frac{dy}{\sin x} = \frac{dy'}{\cos x} = \frac{dT_0}{0} \tag{4.30}$$

Next, from above equation

$$\frac{dx}{0} = \frac{dy}{\sin x},$$

we obtain the invariant $u = x$ and from

$$\frac{dy}{\sin x} = \frac{dy'}{\cos x} \Rightarrow dy = \tan x dy', \tag{4.31}$$

the invariant is

$$v = y' \tan x - y.$$

Thus, $T_0 = F(u, v)$. The conserved form, in terms of u and v, with y'' replaced by $-y$, is

$$\frac{\partial F}{\partial u} + v \tan u \frac{\partial F}{\partial v} = 0, \tag{4.32}$$

so that

$$du = \frac{dv}{v \tan u} = \frac{dF}{0}. \tag{4.33}$$

Therefore,

$$du = \frac{dv}{v \tan u}, \tag{4.34}$$

gives, $F = G(v/\sec u)$. If we choose G to be the identity function, we get

$$T_0 = y' \sin x - y \cos x. \tag{4.35}$$

From the above equation

$$DT_0 = \sin x(y'' + y), \tag{4.36}$$

so that, for T_1, we need to solve eq.(4.12) simultaneously with $DT_1 = 2y' \sin x$ (since $2y'$ is the coefficient of ε and $\sin x$ must be a part of T_1). Eq.(4.12) becomes

$$\frac{dx}{0} = \frac{dy}{\sin x} = \frac{dy'}{\cos x} = \frac{dT_1}{\sin^2 x}. \tag{4.37}$$

We take

$$\frac{dy}{\sin x} = \frac{dT_1}{\sin^2 x}, \tag{4.38}$$

so that

$$T_1 = y \sin x + J(u, v). \tag{4.39}$$

Now, using $DT_1 = 2y' \sin x$, gives

$$\frac{\partial T_1}{\partial x} + y' \frac{\partial T_1}{\partial y} + y'' \frac{dT_1}{\partial y'} = 2y' \sin x, \tag{4.40}$$

which can be written, in terms of u and v, as

$$\frac{\partial J}{\partial u} + \frac{\partial J}{\partial v}(v - 2\varepsilon y') \tan u = v \cos u. \tag{4.41}$$

As DT_1 is the coefficient of ε in the conserved form of eq.(4.3), the $2\varepsilon y'$ term in eq.(4.41) is of order ε^2 and, hence, is equated to zero. Thus,

$$J = uv \cos u + J_1(u, v), \tag{4.42}$$

say. We choose $J_1 = 0$ so that

$$T_1 = y \sin x + xy' \sin x - yx \cos x. \tag{4.43}$$

We note that

$$DT = \sin x(y'' + y + 2\varepsilon y') + \varepsilon x \sin x(y'' + y), \tag{4.44}$$

and the last term is of order ε^2 and, hence, equated to zero (as $y'' + y$ is of order ε).

We use eqs.(4.18, 4.19) to construct the first order Lagrangian in ε viz., $L = L_0 + \varepsilon L_1$. First, from eq.(4.18) we have

$$y' \sin x - y \cos x = B_0(x, y) - \sin x \frac{\partial L_0}{\partial y'}, \tag{4.45}$$

which is a first order linear ordinary differential equation in L_0 and y' yielding

$$L_0 = -F_0(x,y)\csc x - \frac{1}{2}y'^2 + yy'\cot x - y'B_0\csc x, \tag{4.46}$$

where F_0 is the constant of integration. If we choose the gauge term $B_0 = 0$ and substitute L_0 into the Euler-Lagrange equation

$$F_0 = \frac{1}{2}\cos x\cot x, \tag{4.47}$$

so that

$$L_0 = \frac{1}{2}y^2\cot^2 x - \frac{1}{2}y'^2 + yy'\cot x. \tag{4.48}$$

Eq. (4.19) now becomes

$$y\sin x + xy'\sin x - yx\cos x = B_1(x,y) - \sin x\frac{\partial L_1}{\partial y'} - y'x\sin x + xy\cos x, \tag{4.49}$$

form which we obtain

$$L_1 = -yy' - xy'^2 + 2xyy'\cot x + B_1y'\csc x + F_1(x,y), \tag{4.50}$$

where F_1 is the constant of integration. With $B_1 = 0$, we need to choose, from the Euler-Lagrange equation

$$F_1 = -xy^2 + y^2\cot x, \tag{4.51}$$

so that

$$L_1 = -yy' - xy'^2 + 2xyy'\cot x + y^2\cot x - xy^2, \tag{4.52}$$

and, hence,

$$\left(\frac{d}{dx}\frac{\partial}{\partial y'} - \frac{\partial}{\partial y}\right)(L_0 + \varepsilon L_1) = -(y'' + y + 2\varepsilon y'). \tag{4.53}$$

We can repeat the procedure with the other generators listed in eq.(4.22), e.g. we obtain approximate symmetries of eq.(4.3) given by

$$\mathbf{X}^4 = \cos x\frac{\partial}{\partial y} - \varepsilon x\sin x\frac{\partial}{\partial y}, \tag{4.54}$$

$$\begin{aligned}\mathbf{X}^7 = {} & y\cos x\frac{\partial}{\partial x} - y^2\sin x\frac{\partial}{\partial y} + \varepsilon[\left(-\frac{5}{3}xy\cos x + \int 4x\sin x dx\right)\frac{\partial}{\partial x} + \\ & \left(-\frac{5}{3}y^2(\cos x - x\sin x) + 2y^2\cos x + 2xy\sin x\right)\frac{\partial}{\partial y}].\end{aligned} \tag{4.55}$$

With each generator, we can construct (approximate) invariants and *alternative* Lagrangians. (Alternative Lagrangians for ordinary differential equations have been discussed in [43].)

The Lagrangian obtained above, $L = L_0 + \varepsilon L_1$, differs from the usual Lagrangian

$$L_e = \frac{1}{2}e^{2\varepsilon x}(y'^2 - y^2), \tag{4.56}$$

the latter being an exact Lagrangian. However, as $\mathbf{X}$ is an approximate Noether symmetry and the associated first integral, T, one can use $\mathbf{X}$ and T to reduce the differential equation twice. For an elaborate discussion on the reduction of systems of ordinary differential equations using Noether symmetries, the reader is referred to [42, 44]. Briefly, as $DT = 0$ (up to order ε), $T = constt.$, is a reduced first order form of eq.(4.3) with solution

$$y = c\sin x/(1+\varepsilon x) \equiv c(1-\varepsilon x)\sin x \text{ for } T = 0, \tag{4.57}$$

and

$$y = \sin x/(1+\varepsilon x)\exp\left[T\int dx/\left(\sin x/(1+\varepsilon x)\right)\right], \tag{4.58}$$

otherwise.

We consider the van der Pol oscillator equation

$$y'' - \varepsilon(1 - x^2)y' + y = 0, \tag{4.59}$$

which has also been analyzed numerically [36]. We choose the time translation $X_0 = \partial/\partial x$ of the unperturbed eq.(4.21). The auxiliary function

$$H = \frac{1}{\varepsilon}X_0(y'' - \varepsilon(1 - x^2)y' + y), \tag{4.60}$$

along with eq.(4.59) gives $H = 2xy'$. Substituting this value of H in eq.(4.8), we have

$$X_1(y'' + y)|_{y''+y=0} + 2xy' = 0. \tag{4.61}$$

This yields, amongst other generators being symmetries of the unperturbed equation,

$$X_1 = \frac{1}{2}x\partial/\partial x - \frac{1}{2}x^2 y\partial/\partial y,$$

so that

$$X = \frac{\partial}{\partial x} + \varepsilon\left(\frac{1}{2}x\partial/\partial x - \frac{1}{2}x^2 y\partial/\partial y\right). \tag{4.62}$$

The symmetry condition $X_0T_0 = 0$ where $DT_0 = 0$ along the solutions of eq.(4.21) becomes

$$\frac{\partial}{\partial x}T_0 = 0, \quad y'\frac{\partial T_0}{\partial y} - y\frac{\partial T_0}{\partial y'} = 0, \tag{4.63}$$

and gives the usual first integral

$$T_0 = \frac{1}{2}(y^2 + y'^2). \tag{4.64}$$

Note that

$$DT_0 = y'(y'' + y). \tag{4.65}$$

The Lagrangian, L_0, corresponding to the unperturbed equation is as calculated above. We obtain the 'usual' Lagrangian

$$L_0 = -\frac{1}{2}y^2 + \frac{1}{2}y'^2.$$

We now solve for T_1 in eq.(4.12) simultaneously with $DT_1 = y'$. First, eq.(4.12) becomes

$$\frac{\partial T_1}{\partial x} = -\mathrm{X}_1 T_0 = -\frac{1}{2}x^2y^2 - xyy' - \frac{1}{2}x^2y'^2 - \frac{1}{2}y'^2, \tag{4.66}$$

so that

$$T_1 = \frac{1}{6}x^3y^2 + \frac{1}{2}x^2yy' + \frac{1}{6}x^3y'^2 + \frac{1}{2}xy'^2 + F(y, y'). \tag{4.67}$$

Thus,

$$\frac{\partial T_1}{\partial y} = \frac{1}{3}x^3y + \frac{1}{2}x^2y' + \frac{\partial F}{\partial y}, \tag{4.68}$$

$$\frac{\partial T_1}{\partial y'} = \frac{1}{2}x^2y + \frac{1}{3}x^3y' + 2xy' + \frac{\partial F}{\partial y'}. \tag{4.69}$$

Hence, the condition $DT_1 = y'$, along the solutions of eq.(4.59), becomes

$$y'\frac{\partial F}{\partial y} - y\frac{\partial F}{\partial y'} = -\frac{3}{2}y'^2, \tag{4.70}$$

whose solution is given by the system

$$\frac{dy}{y'} = -\frac{dy'}{y} - \frac{2}{3}\frac{dF}{y'^2}. \tag{4.71}$$

We may then choose

$$F = \frac{1}{2}yy' + \frac{1}{2}(y'^2 + y^2)\arcsin\frac{y}{\sqrt{y'^2 + y^2}}, \tag{4.72}$$

so that T_1 is determined from eq.(4.67) and L_1 is calculated from eq.(4.19) as in the previous example. X and T may also be used for the double reduction of eq.(4.59).

4.3 Approximate Symmetries of $y'' + \varepsilon F(x)\, y' + y = f(y')$

To find the approximate symmetries of the differential equation

$$y'' + \varepsilon F(x)\, y' + y = f(y'), \tag{4.73}$$

we first find the symmetries of the exact differential equation

$$y'' + y = f(y'). \tag{4.74}$$

As it is a second order differential equation so we apply prolonged symmetry generator $\mathbf{X}^{[2]}$, to eq.(4.74), to get

$$\eta_{,xx} - f'(y')\,\eta_{,x} + \eta \mid_{y''+y=f(y')} = 0. \tag{4.75}$$

Since $\eta_{,xx}$ is a third order polynomial in y', we therefore, take

$$f(y') = k_4 + k_3 y' + k_2 y'^2 + k_1 y'^3. \tag{4.76}$$

By substituting the values of $\eta_{,xx}$, $\eta_{,x}$ and $f'(y')$ in eq.(4.75) we have

$$\begin{aligned}
\eta_{xx} + k_3\eta_x + \eta + (k_4 - y)\,\eta_y - 2\,(k_4 - y)\,\xi_x &= 0, \\
\eta_{yy} - 2\xi_{xy} + 2k_3\xi_y + k_2\eta_y + 3k_1\eta_x &= 0, \\
2\eta_{xy} - \xi_{xx} + 3\,(k_4 - y)\,\xi_y + k_3\xi_x + 2k_2\eta_x &= 0, \\
\xi_{yy} + k_2\xi_y + 2k_1\eta_y - k_1\xi_x &= 0.
\end{aligned}$$

This system of partial differential equations is very difficult to solve. Therefore, we try to find the generators of this equation by using the closure property of the Lie algebra. We know that one symmetry of this equation is $X_0 = \partial/\partial x$. Suppose that the other symmetry of eq.(4.74) is

$$X_1 = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y},$$

then

$$[X_0, X_1] = \xi_x\frac{\partial}{\partial x} + \eta_x\frac{\partial}{\partial y} = c_0X_0 + c_1X_1,$$

where $c_1$and c_2 are constants, called the *structure constants*. Hence,

$$\xi_x = c_0 + c_1\xi, \ \eta_x = c_1\eta.$$

Now we have four cases

$$(i)\ c_0 = 0 = c_1, \ \ (ii)\ c_0 = 0 \neq c_1, \ \ (iii)\ c_0 \neq 0 = c_1, \ \ (iv)\ c_0 \neq 0 \neq c_1.$$

For all the four cases the above system of partial differential equations does not give any new symmetry, so we have only one symmetry $X_1 = \partial/\partial x$, for the eq.(4.74), therefore, this case is not interesting.

Chapter 5

Conclusion

In this thesis some aspects of symmetries of differential equations have been discussed. In particular, we have used a symmetry method for solving differential equations arising from the Einstein-Maxwell equations. Some examples of the non-static, spherically symmetric, shear-free, perfect fluid solutions (obtained by Wafo Soh and Mahomed [11]) of Einstein's field equations are constructed. The thesis also presents a relation between the isometries (symmetries of a manifold) and the symmetries of the geodesic equations of the manifold. An algebraic proof that the Lie algebra of generators of the system of n differential equations, $(y^a)'' = 0$, is isomorphic to the Lie algebra of the special linear group of order $(n+2)$, over the real numbers, $sl\,(n+2)$ is obtained. A method is developed to find the first integrals and Lagrangians for autonomous weakly non-linear systems and a detailed example to illustrate the method is given. In this chapter we discuss the results and mention further work that needs to be done.

5.1 Uniqueness of the McVittie Solution

Some exact solutions, e.g. Schwarzschild, Reissner-Nordstrom and Kerr solutions have significance in connection with black holes; the Friedman solution has its importance in cosmology and the plane wave solution is used for resolving the controversies about the existence of gravitational radiation. Therefore, finding exact solutions of the Einstein field equations is one of the major tasks of the theory of General Relativity. Complete classification of plane symmetric metrics according to their isometries was obtained [16]. We investigated whether those solutions admit sourceless electromagnetic fields or not.

In order to find plane symmetric spacetimes admitting electromagnetic fields, we consider all metrics appearing in the classification of plane symmetric spacetimes according to their isometries. It is found that none of the metrics admitting 3, 5, 6, 7 or 10 isometries have electromagnetic fields. There are two metrics in the 4-isometry case which admit sourceless electromagnetic fields. These two are the McVittie solution given by the metric eq.(2.51) and its non-static analogue given by the metric eq.(2.52). It is shown that no other metric admits electromagnetic fields. Hence, we have proved the uniqueness of the McVittie solution, i.e. this is the only plane symmetric solution of the Einstein field equations that admits electromagnetic fields. This method for proving uniqueness, or finding solutions of Einstein's equations, could be extended to other symmetries.

5.2 Physics of Some New Solutions of the Einstein Field Equations

In this section, a discussion on some physically acceptable solutions, obtained by Wafo Soh and Mahomed [11], of the Einstein field equations is presented. These solutions were obtained using Noether symmetry, as Lie point symmetries do not provide a complete classification of eq.(2.53). They found two one-parameter families of solutions in their explicit form. In this thesis, using their results, we have constructed two classes of physical exact solutions of the Einstein field equations for non-static, spherically symmetric, shear-free, perfect fluids which could be matched to a Schwarzschild exterior geometry. The Bianchi identities are automatically satisfied for such solutions and there are infinitely many solutions. They have a curvature singularity at a value of r which depends on t, viz. when $u + C(t) = 0$ or at $u = 0$ for the first case. The latter case could be avoided by taking positive values of a parameter, k, appearing in the solution. This violates our choice of $k = 0$. However, that choice was only made for convenience to obtain a relatively simple expression for $C(t)$ and $R(t)$. We could take any, sufficiently small value of k to make the curvature non-singular at $r = 0$, while not significantly changing the value of $C(t)$ and $R(t)$.

In the second case the singularity is at $uC(t) = \gamma(t) = 1$. Earlier works had ignored the matching conditions, required to give physical meaning to a non-homogeneous, spherically symmetric metric. One can construct infinitely many solutions, but the number of solutions is restricted by the requirement that the solutions be physically reasonable. We looked at two such cases here. The singularity at $u + C(t) = 0$ in the first case, lies outside the sphere. For the second case we must maintain $\gamma(t) > 1$ to avoid the singularity.

5.3 Isometries of the Maximally Symmetric Spaces and Symmetries of their Geodesic Equations

In order to find the connection between symmetries of the geodesic equations and the isometries of the underlying maximally symmetric spaces, we worked out the Lie algebra of the symmetries (Table 1) of the geodesic equations. This Lie algebra is then compared with the Lie algebra of the isometries of the underlying spaces. It is found that for the spaces with isometry algebra $so\,(n+1)$, $so\,(n)\oplus_s\mathbb{R}^n$ and $so\,(n,1)$, the Lie algebra for symmetries of the geodesic equations are $so\,(n+1)\oplus d_2$, $sl\,(n+2)$ and $so\,(n,1)\oplus d_2$ respectively.

For $n-$dimensional positive or negative curvature, the system of n geodesic equations has $2+n(n+1)/2$ symmetries and for zero curvature, the system of n equations has $(n+1)\,(n-1)$ symmetries. The symmetries of the reduced system of the geodesic equations of maximally symmetric higher dimensional spaces are also determined. For the reduced system of $(n-1)$ equations the symmetries are the same as the symmetries of the full system of $(n-1)$ geodesic equations for the $(n-1)-$dimensional space of maximum curvature. The procedure carries through to the single geodesic equation for a surface of constant curvature. Notice that the system reduced to a single equation always has 8 generators but they are different for the three cases. The algebras are also different. For the positive and negative curvatures they are simply related but for the flat case they are not.

k			+	0	−
2−dimensional	Full Set	2 Equations	5	15	5
	Reduced Set	1 Equation	8	8	8
3−dimensional	Full Set	3 Equations	8	24	8
	Reduced Set	2 Equations	5	15	5
		1 Equation	8	8	8
4−dimensional	Full Set	4 Equations	12	35	12
	Reduced Set	3 Equations	8	24	8
		2 Equations	5	15	5
		1 Equation	8	8	8
5−dimensional	Full Set	5 Equations	17	48	17
	Reduced Set	4 Equations	12	35	12
		3 Equations	8	24	8
		2 Equations	5	15	5
		1 Equation	8	8	8

Table 1: The number of symmetries of the geodesic equations for maximally symmetric 2, 3, 4 and 5−dimensional spaces of positive, zero and negative curvatures are given. The symmetries of the full and the reduced sets of equations are given in each case.

5.3.1 Isometries of Some Less Symmetric Spaces and Symmetries of their Geodesic Equations

To extend our work to less symmetric spaces, we determined the symmetries of the geodesic equations of the spheroid and the ellipsoid. These symmetries form Lie algebras $so(n) \oplus d_2$ and d_2 respectively.

As for the sphere, if we introduce extra dimensions to the spheroid then for an $n-$dimensional hyperspheroid, having the isometry algebra $so(n)$, it is expected that the algebra of the symmetries of the system of n geodesic equations is $so(n) \oplus d_2$. Now to reduce the system of n geodesic equations to a system of $(n-1)$ equations, $(n-1)$ equations to $(n-2)$ equations and so on down to a single equation, there must be some canonical procedure for the reduction. We follow essentially the same canonical procedure for the reduction of the geodesic equations of an $n-$dimensional hyperspheroid as for the sphere. Now we cut it in such a way that its cross section is an $(n-1)-$ dimensional hyperspheroid, i.e. we cut orthogonal to the direction of asymmetry, using coordinates so that it corresponds to the equatorial hyperplane. The isometries of this hyperspheroid form the algebra $so(n-1)$ and the symmetries of the $(n-1)$ geodesic equations an $so(n-1) \oplus d_2$.

Following the same canonical procedure, this system of $(n-1)$ equations can be further reduced to a system of $(n-2)$ geodesic equations with symmetry algebra $so(n-2) \oplus d_2$ of an $(n-2)-$ dimensional hyperspheroid with isometry algebra $so(n-2)$and so on. Similarly, one can obtain the algebra of the geodesic equations of a hyperellipsoid and a less symmetric hyper-hyperboloid of one sheet.

For surfaces like a cylinder whose cross-sections have different characters (e.g. surfaces of zero and positive curvatures) the algebra of symmetries of the geodesic equations should be the direct sum of the algebras of symmetries of two different surfaces. However, it must be admitted that is not clear that such a naive procedure would work in general. As such we state the following conjecture.

Conjecture 2: For a space of non-zero curvature with isometry algebra h the symmetry algebra of the geodesic equations is $h \oplus d_2$ provided that there is no cross section of zero curvature. If there is an $m-$dimensional cross-section of zero curvature, M, and the symmetry of the orthogonal subspace, $M^{\perp}$, is h_1, the symmetry algebra of the geodesic equations will be $h_1 \oplus sl\,(m+2)$.

5.3.2 Canonical Reduction of System of Ordinary Differential Equations

It is noted that the reduction of a system of equations by inserting the solution of one equation into others may not be canonical. However, the reduction obtained by geometrical considerations provides a canonical procedure for finding the number of symmetries, at least from maximal symmetry down. It would be useful to find similar methods in more general cases.

5.3.3 The Connection Between Isometries and Symmetries of Higher Order Ordinary Differential Equations

In this paper we have dealt only with geodesic equations, which are systems of second order ordinary differential equations. These equations are essentially homogeneous, linear in highest order term and quadratically non-linear in first order term. The results of this paper can be used to find a similar connection for those higher order systems of ordinary differential equations which it is possible to transform into larger systems of second order ordinary differential equations having the above mentioned properties. It would be interesting to determine what classes of ordinary differential equations can be expressed as geodesic equations and hence related to some manifold by the method outlined above. Again, it would be worth exploring whether a corresponding procedure could be developed for systems of partial differential equations.

5.4 Some Discussion on Approximate Symmetries

An algorithmic method that develops the calculation of symmetry generators and first integrals to Lagrangians was presented in Chapter 4 of this thesis. To illustrate the method some examples were also given. The Lagrangian of the perturbed equation then has approximate Noether symmetries with corresponding approximate Noether invariants (or conservation laws). The procedure allows one to maintain the order of the perturbation parameter as stipulated by the given differential equation, i.e. for all the constructed functions like the Lagrangian.

Some examples may not successfully lend themselves to the symmetry based method of determining the approximate invariants and Lagrangians. For example, the unperturbed equation

$$y'' + y - \gamma y^3 = 0, \tag{5.1}$$

only admits the point symmetry generator $\partial/\partial x$, so that for the related perturbed equation

$$y'' + y - \gamma y^3 - 2\varepsilon y' = 0, \tag{5.2}$$

$H = 0$. The approximate symmetries are then multiples of the symmetries of the exact equation. Thus, the procedure presented here becomes redundant. Furthermore, the perturbed equation (related to the same exact equation given by eq.(5.1))

$$y'' + y - \gamma y^3 - \varepsilon(1 - x^2)y' = 0, \tag{5.3}$$

for which $H = 2xy'$ (i.e. having non-zero H), gives inconsistencies in the calculations for perturbed symmetries. Similar results can be obtain if y^3 is replaced by y'^2 in equations (5.2) and (5.3).

Nevertheless, as the underlying theory is proved for Lie-Bäcklund symmetries, so one can use the algorithmic procedure for symmetries which are not necessarily generators of point transformations. However, 'non-local' symmetries, as discussed in reference [1] require more work.

The methods developed in this thesis to find Lagrangians and the first integrals have been proved [34] only for ordinary differential equations, one can further extend this work to develop techniques for determining the Lagrangians and the first integrals for partial differential equations.

REFERENCES

1 Bluman, G. W. and Kumei, S., *Symmetries and Differential Equations* (Springer-Verlag, 1989).

2 Stephani, H., *Differential Equations — Their Solution Using Symmetries*, (Cambridge University Press,1990).

3 Hydon, P., E., *Symmetry Methods for Differential Equations,* (Cambridge University Press, 1999).

4 Bott, R., *Proceedings of the SRC/LMS Research Symposiumon Representation Theory of Lie Group,* (Cambridge University Press, 1977).

5 Yano,K., *The Theory of Lie Derivatives and its Applicationas*, (North-Holland Publishing company, 1955).

6 Hauser, M., and Schwartz, J. T., *Lie Groups, Lie Algebras (*Gorden and Breach, Science Publisher, 1968*)*.

7 Kramer, D., Stephani, H., MacCallum, M. A. H., and Herlt, E., *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, 1980).

8 Ibragimov, N. H., *Elementary Lie Group Analysis and Ordinary Differential Equations* (John Wiley & Sons Ltd. 1999).

9 Olver, P. J., *Applications of Lie Groups to Differential Equations* (Springer-Verlag, New York, 1993).

10 Noether, E., Nachr. König. Gesell. Wissen., Göttingen, Math.-Phys. Kl. Heft 2, 235. (English translation in *Transport Theory and Statistical Physics,* **1** (**3**) (1971) 186.)

11 Wafo Soh, C. and Mahomed, F. M.,*Class. Quantum Grav.,* **16** (1999) 3553.

12 Feroze, T., Mahomed, F.M., and Qadir, A., Nuovo Cimento B, **118** (**9**) (2003) 895.

13 Safko, J. L., and Witten, L., J. Math. Phys., **12** (1971) 257; Safko, J. L., Phys. Rev. D, **16** (1977) 1678; Liang, C., Gen. Rel. Grav., **27** (1995) 669.

14 Newman, E., and Penrose, R., J. Math. Phys., **3** (1962) 566.

15 Mclnotsh, C. B. G., Foyster, J. M. and Lun, A. W. C., J. Math. Phys., **22** (1981) 2620.

16 Feroze, T., Qadir, A., and Ziad, M., J. Math. Phys., **42** (2001) 4947.

17 Yegorov, I. P., Ph.D. Thesis, Moscow State University (1955); G. Fubini, Atti. R. Accad. Sci. Torino, **38** (1903) 404.

18 Qadir, A., and Ziad, M.,*Proceedings of VI Marcel Grossmann Meeting*, (World Scientific, 1993).

19 McVittie, Proc. Roy. Soc. Lond., **124** (1929) 366.

20 H. Fikar, J. Horsky, Czech. J. Phys., **48** (1998) 113.

21 Schwarzchild, K., Sitzber. Deut. Akad. Wiss. Berlin, Kl. Math.-phys. Tech., (1916) 189.

22 Schwarzchild, K., Sitzber. Deut. Akad. Wiss. Berlin, Kl. Math.-phys. Tech., (1916) 424.

23 Kustaanheimo, P. and Qvist, B., Soc. Sci. Fennica, Commun. Phys.-Math., **XIII** (1948) 12.

24 Leach, P. G. L., Maartens, R. and Maharaj, S. D., Int. J. Non-Linear Mech., **27** (1992) 575.

25 Herlt, H. and Stephani, H., J. Math. Phys., **28** (1992) 11.

26 Stephani, H. and Wolf, T., Class. Quantum Grav., **13** (1996) 1261.

27 Stephani, H., J. Phys. A: Math. Gen., **16** (1983) 3529.

28 Hawking, S. W. and Ellis, G. F. R., *The Large Scale Structure of Space-Time*, (Cambridge University Press, Cambridge, 1973).

29 Mahomed, F. M., Kara, A. H. and Leach, P. G. L., J. Math. Analy. and Applic., **178** (1993) 116.

30 Kara, A. H. and Mahomed, F. M., Int. J. Theoret. Phys., **39** (2000) 23.

31 Lie, S., Math. Ann., **32** (1888) 213.

32 Aminova, A.V., Izv.-Vyssh.-Uchebn.-Zaved.-Mat., **2** (1994) 3; Sbornik Mathematics, **186** (1995) 1711.

33 Qadir, A., Int. J. Theoret. Phys., **14** (1976)74; Ph.D Thesis, London University (1971).

34 Feroze, T., and Kara, A.H., Int. J. Non-linear Mech., **37** (2002) 275.

35 Denman, H. H., Int. J. Non-linear Mech., **29** (1994) 409.

36 Denman, H. H., Int. J. Non-linear Mech., **33** (1998) 301.

37 Denman, H. H., Int. J. Non-linear Mech., **35** (2000) 385.

38 Baikov, V. A., Gazizov, R. K., and Ibragimov, N. H., Journal of Soviet Mathematics, **55** (1) (1991).

39 Baikov, V. A., Gazizov, R. K., and Ibragimov, N. H., in *CRC Handbook of Lie Group Analysis of Differential Equations*, vol **3**, (CRC Press, Boca Raton, Florida 1996).

40 Kara, A. H., Mahomed, F. M. and Ünal, G., Int. J. Theoretical Physics, **38** (**9**) (1999) 2389.

41 Sarlet, W., J. Math. Phys., **19** (1978) 1049.

42 Kara, A. H., Mahomed, F. M. and Vawda, F. E., Lie Groups and Their Applications, **2** (1994) 27.

43 Kara, A. H., Mahomed, F. M., Int. J. Th. Phys., **34** (**11**) (1995) 2209.

44 Olver, P., *Application of Lie groups to differential equations*, Springer-Verlag, New York (1986).

45 Ibragimov, N. H., Kara, A. H. and Mahomed, F. M., Nonlinear Dynamics **15** (**2**) (1998) 115.

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Group theoretic methods for approximate invariants and Lagrangians for some classes of $y'' + \varepsilon F(t)y' + y = f(y, y')$

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Abstract

Some recent results on the Lie symmetry generators of equations with a small parameter and the relationship between symmetries and conservation laws for such equations are used to construct first integrals and Lagrangians for autonomous weakly non-linear systems, $y'' + \varepsilon F(t)y' + y = f(y, y')$. An adaptation of a theorem that provides the point symmetry generators that leave the invariant functional involving a Lagrangian for such equations is presented. A detailed example to illustrate the method is given (and other examples are discussed). The (approximate) symmetry generators, invariants and Lagrangians maintain the perturbation order of the 'small parameter' stipulated in the equation — first order in this case. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Approximate symmetries; First integrals; Invariants; Lagrangians

1. Introduction

In [1,2], Denman discusses the approximate invariants for some classes of

$$y'' + 2\varepsilon y' + y = f(y, y'), \tag{1.1}$$

e.g., $y'' + 2\varepsilon y' + y = 0$ and the damped Duffing equation with $f = -\gamma y^3$. In [3], the ideas are then extended to time-dependent invariants. In [1], a detailed account of the application of the equations is presented. The Lagrangians constructed in these references are exact, i.e., the ε is construed as any constant, for e.g., the equation $y'' + 2\varepsilon y' + y = 0$ has Lagrangian $\frac{1}{2}e^{2\varepsilon t}(y'^2 - y^2)$.

In the sequel, we deal with the equations in the above references using a completely different approach, viz., the group theoretic approach. Firstly, the equations are regarded as perturbed equations with some small parameter ε. To this end, we utilise the theory developed in [4,5] to determine the first-order *approximate* Lie symmetries of the equations. Then, the first integrals *associated* with these symmetries are determined (see [6]). Finally, Lagrangians (first order in ε) are constructed using an extended form of results developed for unperturbed equations (see [7,8]), i.e., from the symmetries and associated first integrals. Thus, the approximate symmetries are Noether symmetries and the associated first integrals (approximate) Noether

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invariants corresponding to the constructed Lagrangians. It should be borne in mind that we maintain the order of the perturbation for the symmetries, first integrals and the Lagrangian — it being first order in this case.

We now review some pertinent results. The presentation here is for ordinary differential equations. However, most of the theory has been generalised to partial differential equations in the references cited.

In [5], it is shown that if X_0 is a generator of Lie (point) symmetry of a differential equation

$$E_0 = 0, \tag{1.2}$$

then an *approximate symmetry*, $X = X_0 + \varepsilon X_1$, of the perturbed differential equation

$$E_0 + \varepsilon E_1 = 0 \tag{1.3}$$

is obtained by solving for X_1 in

$$X_1(E_0)|_{E_0=0} + H = 0, \tag{1.4}$$

where

$$H = \frac{1}{\varepsilon} X_0(E_0 + \varepsilon E_1)|_{E_0+\varepsilon E_1=0} \tag{1.5}$$

(E_1 is the perturbation and H is referred to as an auxiliary function). Furthermore, a first integral $T = T_0 + \varepsilon T_1$ (i.e., $DT = 0$ along the solutions of (1.3)) *associated* with X satisfies (see [6])

$$X_0 T_0 = 0, \tag{1.6a}$$

$$X_0 T_1 = -X_1 T_0. \tag{1.6b}$$

For the Lagrangian formulation, more specifically for the inverse problem, we appeal to Noether's theorem. However, we first need to state the following theorem regarding the invariance of the functional in the variational problem. The proof is straightforward and proceeds in a way similar to the well-known unperturbed case.

Theorem 1. *Suppose $L(t, y, y', \varepsilon) = L_0(t, y, y') + \varepsilon L_1(t, y, y')$ is a first-order Lagrangian corresponding to a second-order ordinary differential equation (1.3). If the functional $\int L\,dt$ is invariant under the one-parameter group of transformations with approximate Lie symmetry generator $X = X_0 + \varepsilon X_1$, where $X_0 = \tau_0 \partial/\partial t + \eta_0 \partial/\partial y$ and $X_1 = \tau_1 \partial/\partial t + \eta_1 \partial/\partial y$ up to gauge $B = B_0 + \varepsilon B_1$, then*

$$X_0 L_0 + L_0 D\tau_0 = DB_0,$$

$$X_1 L_0 + X_0 L_1 + L_0 D\tau_1 + L_1 D\tau_0 = DB_1, \tag{1.7}$$

where D is the total differential operator with respect to t, i.e.,

$$D = \frac{\partial}{\partial t} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots.$$

Proof. We may suppose that the transformed variables are $\bar{t} = t + a\tau$ and $\bar{y} = y + a\eta$ (so that $\bar{y}' = y' + a\zeta^1$ where $\zeta^1 = D\eta - y'D\tau$). Then

$$\int L(\bar{t}, \ldots, \varepsilon)\,d\bar{t} = \int L_0(\bar{t}, \ldots)\,d\bar{t} + \varepsilon \int L_1(\bar{t}, \ldots)\,d\bar{t}.$$

Differentiating both sides and expanding the left-hand side using a Taylor expansion up to order a, we obtain

$$X_0 L_0 + L_0 D\tau_0 + \varepsilon[X_1 L_0 + X_0 L_1 + L_0 D\tau_1 + L_1 D\tau_0] = DB_0 + \varepsilon DB_1. \tag{1.8}$$

Separation by powers of ε yields (1.7). □

In this notation, Noether's theorem [9] reads as follows.

Theorem 2. *Corresponding to each symmetry $X = \tau\partial/\partial t + \eta\partial/\partial y$ that satisfies the conditions of Theorem 1, there exists a first integral T (or conservation law $DT = 0$ along the solutions of the differential equation) given by*

$$T = B - L\tau - (\eta - y'\tau)\frac{\partial L}{\partial y'}. \tag{1.9}$$

Note: With $T = T_0 + \varepsilon T_1$, if we separate by powers of ε up to order ε, we get

$$T_0 = B_0 - L_0\tau_0 - (\eta_0 - y'\tau_0)\frac{\partial L_0}{\partial y'}, \tag{1.10a}$$

$$T_1 = B_1 - (L_0\tau_1 + L_1\tau_0) - \left[(\eta_0 - y'\tau_0)\frac{\partial L_1}{\partial y'} + (\eta_1 - y'\tau_1)\frac{\partial L_0}{\partial y'}\right]. \tag{1.10b}$$

2. Applications

2.1

We consider the DHO equation

$$y'' + y + 2\varepsilon y' = 0, \tag{2.1}$$

which has been analysed in [1–3] (see also a Lagrangian formulation of the equation and comments on integration by quadratures in [10]. Herein, however, $\varepsilon = -\frac{1}{2}a$ for some constant a and the equation is not construed as a perturbation of $y'' + y = 0$.). It is well known that the unperturbed equation $y'' + y = 0$ admits the 8-dimensional Lie algebra of point symmetry generators $sl(3, R)$ given by

$$X_0^1 = \frac{\partial}{\partial t}, \qquad X_0^2 = y\frac{\partial}{\partial y}, \qquad X_0^3 = \sin t\frac{\partial}{\partial y},$$

$$X_0^4 = \cos t\frac{\partial}{\partial y}, \qquad X_0^5 = \sin 2t\frac{\partial}{\partial t} + y\cos 2t\frac{\partial}{\partial y},$$

$$X_0^6 = \cos 2t\frac{\partial}{\partial t} + y\sin 2t\frac{\partial}{\partial y},$$

$$X_0^7 = y\cos t\frac{\partial}{\partial t} - y^2\sin t\frac{\partial}{\partial y},$$

$$X_0^8 = y\sin t\frac{\partial}{\partial t} + y^2\cos t\frac{\partial}{\partial y}. \tag{2.2}$$

We choose the symmetry $X_0 = \sin t\partial/\partial y$ to construct X_1 in (1.4). Firstly,

$$H = \frac{1}{\varepsilon}X_0(y'' + y + 2\varepsilon)\Big|_{y''+y+2\varepsilon y'=0} = 2\cos t$$

and (1.4) becomes

$$\begin{aligned}&\eta_{1_{tt}} + y'(2\eta_{1_{ty}} - \xi_{1_{tt}}) + y'^2(\eta_{1_{yy}} - 2\xi_{1_{ty}}) \\ &\quad - y'^3\xi_{1_{yy}} - y''(\eta_{1_y} - 2\xi_{1_t} - 3y'\xi_{1_y}) \\ &\quad + \eta_1 + 2\cos t = 0\end{aligned} \tag{2.3}$$

along the solutions of $y'' + y = 0$. Using the standard procedure of separation by powers of y', we get

$$\xi_1 = a(t)y + b(t), \qquad \eta_1 = a'y^2 + c(t)y + d(t), \tag{2.4}$$

where

$$a'' + a = 0 \quad (a = A\cos t + B\sin t),$$

$$b' = 2c,$$

$$c'' + 4c = 0 \quad (c = P\cos 2t - Q\sin 2t),$$

$$d'' + d + 2\cos t = 0$$

$$(d = R\cos t + S\sin t - t\sin t). \tag{2.5}$$

Putting $A = 1$ and the other constants equal to zero yields a generator from the list in (2.2), i.e., the approximate symmetry X is a multiple of some symmetry of $y'' + y = 0$. So too with the other constants B, P, Q, R and S. The only 'new' symmetry comes from $d = -t\sin t$, i.e., we have a non-trivial approximate symmetry of (2.1) given by $X = \sin t\partial/\partial y - \varepsilon t\sin t\partial/\partial y$. We now construct the approximate invariant $T = T_0 + \varepsilon T_1$ using (1.6) and $DT = 0$ along the solutions of (1.3). Firstly, $X_0T_0 = 0$ yields the characteristic equation

$$\frac{dt}{0} = \frac{dy}{\sin t} = \frac{dy'}{\cos t} = \frac{dT_0}{0},$$

from which we obtain the invariants $u = t$ and $v = y'\tan t - y$. Thus, $T_0 = F(u, v)$. The conserved form, in terms of u and v and with y'' replaced by $-y$, is

$$\frac{\partial F}{\partial u} + v\tan u\frac{\partial F}{\partial v} = 0,$$

so that $F = G(v/\sec u)$. If we choose G to be the identity function, we get

$$T_0 = y'\sin t - y\cos t. \tag{2.6}$$

Here, $DT_0 = \sin t(y'' + y)$ so that, for T_1, we need to solve (1.6b) simultaneously with $DT_1 = 2y'\sin t$. Eq. (1.6b) becomes

$$\frac{dt}{0} = \frac{dy}{\sin t} = \frac{dy'}{\cos t} = \frac{dT_1}{\sin^2 t},$$

so that $T_1 = y\sin t + J(u, v)$. Now, $DT_1 = 2y'\sin t$, in terms of u and v, is

$$\frac{\partial J}{\partial u} + \frac{\partial J}{\partial v}(v - 2\varepsilon y')\tan u = v\cos u. \tag{2.7}$$

As DT_1 is the coefficient of ε in the conserved form of (2.1), the $2\varepsilon y'$ term in (2.7) is of order ε^2 and, hence, is equated to zero. Thus, $J = uv\cos u + J_1(u,v)$, say. We choose $J_1 = 0$ so that $T_1 = y\sin t + ty'\sin t - yt\cos t$. We note that

$$DT = \sin t(y'' + y + 2\varepsilon y') + \varepsilon t\sin t(y'' + y)$$

and the last term is of order ε^2 and, hence, equated to zero (as $y'' + y$ is of order ε).

We now use (1.10) to construct the first order Lagrangian in ε viz., $L = L_0 + \varepsilon L_1$. Firstly, from (1.10a) we have

$$y'\sin t - y\cos t = B_0(t,y) - L_0 0 - (\sin t - y'0)\frac{\partial L_0}{\partial y'}, \qquad (2.8)$$

which is a first-order linear ordinary differential equation in L_0 and y' yielding

$$L_0 = -F_0(t,y)\csc t - \tfrac{1}{2}y'^2 + yy'\cot t - y'B_0\csc t, \qquad (2.9)$$

where F_0 is the constant of integration. If we choose the gauge term $B_0 = 0$ and substitute L_0 into the Euler–Lagrange equation, $F_0 = \frac{1}{2}\cos t\cot t$ so that $L_0 = \frac{1}{2}y^2\cot^2 t - \frac{1}{2}y'^2 + yy'\cot t$. Eq. (1.10b) now is

$$y\sin t + ty'\sin t - yt\cos t = B_1(t,y) - \sin t\frac{\partial L_1}{\partial y'} - y't\sin t + ty\cos t$$

from which we obtain

$$L_1 = -yy' - ty'^2 + 2tyy'\cot t + B_1 y'\csc t + F_1(t,y), \qquad (2.10)$$

where F_1 is the constant of integration. With $B_1 = 0$, we need to choose, from the Euler–Lagrange equation, $F_1 = -ty^2 + y^2\cot t$ so that $L_1 = -yy' - ty'^2 + 2tyy'\cot t + y^2\cot t - ty^2$ and, hence,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial y'} - \frac{\partial}{\partial y}\right)(L_0 + \varepsilon L_1) = -(y'' + y + 2\varepsilon y').$$

Remarks. (a) We can repeat the procedure with the other generators listed in (2.2), for e.g., we obtain approximate symmetries of (2.1) given by

$$X^4 = \cos t\frac{\partial}{\partial y} - \varepsilon t\sin t\frac{\partial}{\partial y},$$

$$X^7 = y\cos t\frac{\partial}{\partial t} - y^2\sin t\frac{\partial}{\partial y} + \varepsilon\left[\left(-\tfrac{5}{3}ty\cos t + \int 4t\sin t\,\mathrm{d}t\right)\frac{\partial}{\partial t} + \left(-\tfrac{5}{3}y^2(\cos t - t\sin t) + 2y^2\cos t + 2ty\sin t\right)\frac{\partial}{\partial y}\right]. \qquad (2.11)$$

With each, we can construct (approximate) invariants and *alternative* Lagrangians. For a detailed discussion on alternative Lagrangians for ordinary differential equations, see [11].

(b) The Lagrangian obtained above, viz., $L = L_0 + \varepsilon L_1$, differs from the 'usual' Lagrangian $L_e = \frac{1}{2}e^{2\varepsilon t}(y'^2 - y^2)$; the latter being an exact Lagrangian. However, as X is an approximate Noether symmetry and T the associated first integral, one can use the symmetry X and associated first integral T to twice reduce the differential equation. For an elaborate discussion on the reduction of systems of ordinary differential equations using Noether symmetries, the reader is referred to [10] and [8]. Briefly, as $DT = 0$ (up to order ε), $T = c_1$, c_1 being a constant, is a reduced first-order form of (2.1) with solution $y = c_2\sin t/(1+\varepsilon t) \equiv c_2(1-\varepsilon t)\sin t$ for $c_1 = 0$ and $y = \sin t/(1+\varepsilon t)\exp[c_1\int 1/(\sin t(1+\varepsilon t))\,\mathrm{d}t]$, otherwise.

2.2

We now consider, without providing detailed calculations, the van der Pol oscillator equation (of motion)

$$y'' - \varepsilon(1 - t^2)y' + y = 0, \qquad (2.12)$$

which has also been numerically analysed in [2]. We choose the time translation $X_0 = \partial/\partial t$ of the unperturbed equation $y'' + y = 0$. The auxiliary function $H = 1/\varepsilon X_0(y'' - \varepsilon(1-t^2)y' + y)$ along (2.12) so that $H = 2ty'$. Eq. (1.4), viz.,

$$X_1(y'' + y)|_{y''+y=0} + 2ty' = 0,$$

yields, amongst other generators being symmetries of the unperturbed equation, $X_1 = \frac{1}{2}t\partial/\partial t - \frac{1}{2}t^2 y\partial/\partial y$ so that

$$X = \frac{\partial}{\partial t} + \varepsilon(\tfrac{1}{2}t\partial/\partial t - \tfrac{1}{2}t^2 y\partial/\partial y).$$

The associate symmetry condition $X_0 T_0 = 0$ where $DT_0 = 0$ along the solutions of $y'' + y = 0$ becomes

$$\frac{\partial}{\partial t}T_0 = 0, \qquad y'\frac{\partial T_0}{\partial y} - y\frac{\partial T_0}{\partial y'} = 0$$

and yields the usual first integral $T_0 = \frac{1}{2}(y^2 + y'^2)$. Note that $DT_0 = y'(y'' + y)$. The Lagrangian, L_0, corresponding to the unperturbed equation is as calculated above. We obtain the 'usual' Lagrangian $L_0 = -\frac{1}{2}y^2 + \frac{1}{2}y'^2$.

We now solve for T_1 in (1.6b) simultaneously with $DT_1 = y'$. Firstly, (1.6b) becomes

$$\frac{\partial T_1}{\partial t} = -X_1 T_0$$
$$= -\tfrac{1}{2}t^2y^2 - tyy' - \tfrac{1}{2}t^2y'^2 - \tfrac{1}{2}y'^2,$$

so that

$$T_1 = \tfrac{1}{6}t^3y^2 + \tfrac{1}{2}t^2yy' + \tfrac{1}{6}t^3y'^2 + \tfrac{1}{2}ty'^2 + F(y, y'). \qquad (2.13)$$

Thus,

$$\frac{\partial T_1}{\partial y} = \tfrac{1}{3}t^3y + \tfrac{1}{2}t^2y' + \frac{\partial F}{\partial y},$$

$$\frac{\partial T_1}{\partial y'} = \tfrac{1}{2}t^2y + \tfrac{1}{3}t^3y' + 2ty' + \frac{\partial F}{\partial y'}.$$

Hence, the condition $DT_1 = y'$ becomes, along the solutions of (2.12),

$$y'\frac{\partial F}{\partial y} - y\frac{\partial F}{\partial y'} = -\tfrac{3}{2}y'^2$$

whose solution is given by the system

$$\frac{dy}{y'} = -\frac{dy'}{y} = -\frac{2}{3}\frac{dF}{y'^2}.$$

We may then choose

$$F = \tfrac{1}{2}yy' + \tfrac{3}{2}(y'^2 + y^2)\arcsin\frac{y}{\sqrt{y'^2 + y^2}},$$

so that T_1 is determined from (2.13) and L_1 is calculated from (1.10b) as in the previous example.

The remarks made above on double reduction by way X and T are applicable to (2.12) also.

Notes: (a) Some examples may prove to be problematic. For example, the unperturbed equation $y'' + y - \gamma y^3 = 0$ only admits the point symmetry generator $\partial/\partial t$ so that the auxiliary function $H = 0$ for the perturbed equation $y'' + y - \gamma y^3 - 2\varepsilon y' = 0$. The approximate symmetries are, then, multiples of the symmetries of the exact equation. Thus, the procedure presented above becomes redundant. Furthermore, the perturbed equation $y'' + y - \gamma y^3 - \varepsilon(1 - t^2)y' = 0$, for which $H = 2ty'$, presents inconsistencies in the calculations for X_1, viz., we require constants m and n for which $n\cos t - m\sin t = t/\gamma$.

Similar comments can be made for the equation in which y^3 is replaced by y'^2.

(b) Nevertheless, as the underlying theory is proved for Lie–Bäcklund symmetries, one can use the above algorithmic procedure for symmetries which are not necessarily generators of point transformations. However, 'non-local' symmetries, as discussed in [12], require more work.

3. Conclusion

The method presented here is an algorithmic one that develops from the calculation of symmetry generators and first integrals to Lagrangians. The Lagrangians of the perturbed equation then have approximate Noether symmetries with corresponding approximate Noether invariants (or conservation laws). The procedure allows one to maintain the order of the perturbation parameter as stipulated by the given differential equation, i.e., for all the constructed functions like the Lagrangian.

As the necessary theorems used have been generally proved in the literature mentioned above (see also [13,14]), the method may be extended to any nonlinear ordinary and partial differential equation.

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References

[1] H.H. Denman, Approximate invariants and Lagrangians for autonomous, weakly non-linear systems, Int. J. Non-Linear Mech. 29 (1994) 409.

[2] H.H. Denman, Approximate invariants and Lagrangians for autonomous, weakly non-linear systems — II, Int. J. Non-Linear Mech. 33 (1998) 301.

[3] H.H. Denman, Approximate invariants and Lagrangians for autonomous, weakly non-linear systems — III, Int. J. Non-Linear Mech. 35 (3) (2000) 385.

[4] V.A. Baikov, R.K. Gazizov, N.H. Ibragimov, Perturbation methods in group analysis, J. Sov. Math. 55 (1) (1991) 1450.

[5] V.A. Baikov, R.K. Gazizov, N.H. Ibragimov, in: N.H. Ibragimov (Ed.), CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 3, CRC Press, Boca Raton, FL, 1996.

[6] A.H. Kara, F.M. Mahomed, G. Ünal, Approximate symmetries and conservation laws with applications, Int. J. Theoret. Phys. 38 (9) (1999) 2389.

[7] W. Sarlet, Invariance and conservation laws for Lagrangian systems with one degree of freedom, J. Math. Phys. 19 (1978) 1049.

[8] A.H. Kara, F.M. Mahomed, F.E. Vawda, Symmetries of first integrals and solutions of differential equations, Lie Groups Appl. 2 (1994) 27.

[9] E. Noether, Invariante Variationsprobleme, Nachr. König. Gesell. Wissen. Göttingen, Math.-Phys. Kl. Heft 2 (1918) 235 (English translation in Transp. Theory Statist. Phys. 1(3) (1971) 186.)

[10] P. Olver, Application of Lie Groups to Differential Equations, Springer, New York, 1986.

[11] A.H. Kara, F.M. Mahomed, On the classification of first-order Lagrangians on the line, Int. J. Theoret. Phys. 34 (11) (1995) 2209.

[12] G. Bluman, S. Kumei, Symmetries and Differential Equations, Springer, New York, 1986.

[13] N.H. Ibragimov, A.H. Kara, F.M. Mahomed, Lie-Backlund and Noether symmetries with applications, Nonlinear Dyn. 15 (2) (1998) 115.

[14] A.H. Kara, F.M. Mahomed, The relationship between symmetries and conservation laws, Int. J. Theoret. Phys. 39 (1) (2000) 23.

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Non-static, spherically symmetric, shear-free, perfect-fluid solutions of Einstein's equations(*)

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Summary. — The Einstein equations for non-static, shear-free, spherically symmetric, perfect-fluid distributions reduce to one second-order non-linear differential equation in the radial parameter. General solutions of this equation have been obtained in *Class. Quantum Grav.*, **16** (1999) 3553 by symmetry analysis. In this paper, we correct some examples of the solutions in the above-mentioned earlier work by formulating a general requirement for physical relevance of the solutions.

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1. – Introduction

Spherically symmetric solutions of Einstein's equations have always been sought since the time that Schwarzschild provided the first exact solution [1]. He also provided an interior solution which is static [2]. This could be taken as a first approximation to a relativistic description for a star. However, it is of interest to find non-static solutions that can describe stars not in equilibrium. The usual procedure would be to take a given equation of state for a perfect fluid and solve it, using the energy generated in the star as an input. An alternative procedure would be to look for all possible spherically symmetric solutions and pick the one closest to describing the star. Correctly speaking, since stars have intrinsic angular momentum, we should use a Kerr metric. It is a good

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approximation, however, to take the Schwarzschild metric. Again, a star would possess some shear (due to its rotation) but as a simplifying assumption, one takes the fluid to be shear free. Kustaanheimo and Qvist [3] found that the Einstein equations reduce to

$$\frac{\partial^2 y}{\partial x^2} = f(x)y^2 \tag{1}$$

for a perfect fluid with no shear, where $x = r^2$, $y = e^{-\omega(r,t)/2}$ in the Schwarzschild metric in isotropic coordinates

$$ds^2 = e^{\nu(t,r)}dt^2 - e^{\omega(t,r)}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2). \tag{2}$$

The function ν is given by $e^{\nu/2} = c(t)/\omega_t$, where $c(t)$ is an arbitrary function of t and the Einstein field equations in parametric form are

$$\begin{aligned}
\kappa\rho &= -y^4 + 4y^2\left(2y'^2 - yy''\right) + 192\frac{\dot{y}^6}{c^4 y^6},\\
\kappa p &= y^4 + 4\left(2\frac{y'\dot{y}'y^3}{\dot{y}} - y'^2 y^2\right) - 64\frac{\dot{y}^4}{c^4 y^2}\left(6\frac{\dot{y}^2}{y^4} - 2\frac{\dot{y}\ddot{y}}{y^4} - \frac{\ddot{y}}{y^3} + 2\frac{\dot{c}\dot{y}}{cy^3}\right),\\
\kappa p &= 2y^4\left(-\frac{\dot{y}''}{\dot{y}} + 3\frac{\dot{y}'^2}{\dot{y}^2} - 2\frac{y'^2}{y^2}\right) - 64\frac{\dot{y}^4}{c^4 y^4}\left(-\frac{\ddot{y}}{y} + 6\frac{\dot{y}^2}{y^2} + 2\frac{\dot{c}\dot{y}}{cy} - 2\frac{\dot{y}^2\ddot{y}}{y^2}\right),\\
0 &= -2\dot{y}'y + \dot{y}y'.
\end{aligned}$$

Specific solutions of this equation were found by the authors [3] and subsequently others have further analysed it [4-6]. Some exact solutions of (1) are also tabulated in the book of exact solutions of Einstein's field equations [7]. Stephani [8] has used symmetry methods and obtained some solutions but, as he notes there, it is difficult to extract physical information from these solutions as the relevant quantities are only given implicitly.

A general procedure to find all solutions obtainable by symmetry analysis was provided by Wafo Soh and Mahomed [9]. They gave three cases which include solutions that gave the metric coefficients explicitly (directly or parametrically) or implicitly in terms of integrals.

Explicit solutions of Case I have been discussed before and a tabulation of the interesting physical case is given in [7]. In general the solutions of Cases II and III are given implicitly in terms of integrals and are consequently difficult to interpret physically. There are two explicit one-parameter families of solutions to which they reduce

$$\begin{aligned}
y = {}& \beta^{1/7}\left(\frac{3}{2}\right)^{-1/7}\left(-\frac{2}{7}\right)^{-3/7}(x+k)^{3/7}\times\\
&\times\left[(x+k)^{1/7}\beta^{-2/7}\left(\frac{3}{2}\right)^{-3/14}\left(-\frac{2}{7}\right)^{-1/7} + C(t)\right]^{-2} -\\
&- \beta^{5/7}\left(\frac{3}{2}\right)^{2/7}\left(-\frac{2}{7}\right)^{-1/7}(x+k)^{1/7},
\end{aligned} \tag{3}$$

$$f(x) = \beta^{-5/7}\left(\frac{3}{2}\right)^{5/7}\left(-\frac{2}{7}\right)^{15/7}(x+k)^{-15/7}, \tag{4}$$

and

$$(5)\quad y=\alpha^{-3/7}\left(\frac{3}{2}\right)^{-1/7}\left(\frac{2}{7}\right)^{-3/7}(\alpha x+\beta)^{4/7}(1-k\beta-k\alpha x)^{3/7}\times$$
$$\times\left\{(1-k\beta-k\alpha x)^{1/7}(\alpha x+\beta)^{-1/7}\alpha^{-1/7}\left(\frac{3}{2}\right)^{-3/14}\left(-\frac{2}{7}\right)^{-1/7}+C(t)\right\}^{-2}-$$
$$-\alpha^{-1/7}\left(\frac{3}{2}\right)^{2/7}\left(\frac{2}{7}\right)^{-1/7}(\alpha x+\beta)^{6/7}(1-k\beta-k\alpha x)^{1/7},$$

$$(6)\quad f(x)=\alpha^{15/7}\left(\frac{3}{2}\right)^{5/7}\left(\frac{2}{7}\right)^{15/7}(\alpha x+\beta)^{-20/7}(1-k\beta-k\alpha x)^{-15/7}$$

(where α, β, and k are arbitrary constants and $C(t)$ is an arbitrary function of time) and also two parametrically expressed one-parameter families of solutions defined by h and ω as

$$(7)\qquad h(\tau)=\begin{cases}\left(\dfrac{-1}{2(p-q)(\tau-p)^2}+\dfrac{5}{4(p-q)^2(\tau-p)}+\dfrac{15}{4(p-q)^3}\right)\\ \times\dfrac{(\frac{3}{2})^{3/2}}{\sqrt{\tau-q}}+\dfrac{15(\frac{3}{2})^{3/2}}{8(p-q)^{7/2}}\operatorname{arctanh}\dfrac{\sqrt{\tau-q}}{\sqrt{p-q}}, & p>q,\\ \left(\dfrac{-1}{2(p-q)(\tau-p)^2}+\dfrac{5}{4(p-q)^2(\tau-p)}+\dfrac{15}{4(p-q)^3}\right)\\ \times\dfrac{(\frac{3}{2})^{3/2}}{\sqrt{\tau-q}}-\dfrac{15(\frac{3}{2})^{3/2}}{8(p-q)^{7/2}}\arctan\dfrac{\sqrt{\tau-q}}{\sqrt{q-p}}, & p<q,\end{cases}$$

and

$$(8)\qquad \omega=\begin{cases}q+(p-q)\left[\dfrac{(\tau-q)^{1/2}+C(t)\sqrt{p-q}}{\sqrt{p-q}+C(t)(\tau-q)^{1/2}}\right]^2, & p>q,\\ q+(q-p)\left[\dfrac{(\tau-q)^{1/2}-C(t)\sqrt{q-p}}{\sqrt{q-p}+C(t)(\tau-q)^{1/2}}\right]^2, & p<q,\end{cases}$$

where p and q are constants and $C(t)$ is a function of time. The parametric solutions are then given by the formulas in Cases II and III.

The examples of the new physical solutions obtained by Wafo Soh and Mahomed [9] were erroneous. It is worthwhile to construct valid physical examples for these solutions. We consider only the two explicit solutions to try and construct physically meaningful examples. For this purpose we need to first formulate the criteria for being "physically meaningful."

2. – The physical criteria

The most important physical criterion is that the density be non-negative everywhere. The next criterion must be that both the density and pressure remain finite everywhere. If they become infinite somewhere, that place should correspond to a curvature singularity and would automatically limit the part of the manifold that can be considered. In other words, the singular point(s) and all others connected by a path that passes through them must be cut out from the manifold. Additionally, one would ask that the strong-energy condition holds.

There could be more subtle considerations. The spherically symmetric space-time could be used to represent spheres of perfect fluids of finite size. Such spheres should have zero surface pressure and match with a Schwarzschild exterior. If these conditions cannot be met, the metric would have to go on to describe the whole universe. However, such a description violates the cosmological principle unless the density and pressure are constant everywhere (though they could vary with time). As such, we will require that the metric should admit a zero pressure somewhere and contain a positive density inside that boundary, while also satisfying the strong-energy condition.

3. – The first-case example

Let $u=(x+k)^{1/7}$ in eq. (3). Then (3) can be written as

$$y=-\frac{a}{b^2}uC(t)\frac{(2u+C(t))}{(u+C(t))^2}, \tag{9}$$

where $a=\beta^{1/7}(\frac{3}{2})^{-1/7}(-\frac{2}{7})^{-3/7}$, $b=\beta^{-2/7}(\frac{3}{2})^{-3/14}(-\frac{2}{7})^{-1/7}$ and $C(t)$ now refers to $bC(t)$. In this case we have the density and pressure given by

$$\rho(t,x)=\frac{3}{8\pi}+\frac{3C^3(t)a^2}{98\pi u^{12}(u+C(t))^6b^4}\left[10u^{10}+20u^9C(t)+\right.$$
$$+12u^8C^2(t)+2u^7C^3(t)+32u^3k+57u^2kC(t)+30ukC^2(t)+$$
$$\left.+5kC^3(t)\right], \tag{10}$$

$$p(t,x)=\frac{-3}{8\pi}+\frac{C^4(t)a^2}{98\pi u^{12}(u+C(t))^5b^4}\left[5u^8+4u^7C(t)+9uk+\right.$$
$$\left.+3kC(t)\right]. \tag{11}$$

Here β, k and $C(t)$ would be constrained by the requirement that $\rho(t,x)\geq 0$. Further, $\rho(t,x)$ and $p(t,x)$ must not be singular and satisfy the strong-energy condition ($T^\mu_\mu\geq 0$):

$$T^\mu_\mu=\rho+3p=\frac{-3}{4\pi}+\frac{3C^3(t)a^2}{98\pi u^{12}(u+C(t))^6b^4}\left[10u^{10}+25u^9C(t)+\right.$$
$$+21u^8C^2(t)+6u^7C^3(t)+32ku^3+66u^2kC(t)+$$
$$\left.+42ukC^2(t)+8kC^3(t)\right]. \tag{12}$$

We can keep the energy density positive and satisfy the strong-energy condition as there is enough freedom in selecting the constants and the function of time $C(t)$. Clearly, taking $k\geq 0$, and $C(t)>0$ would maintain the requirement that the density remain positive. Also the trace of the stress-energy will remain non-negative by appropriately choosing β.

We investigate the case $k=0$ and consider a non-static sphere of radius R matched to a Schwarzschild exterior geometry. In that case we get the matching condition $p|_{r=R}=0$. The condition on $C(t)$ becomes

$$147b^4u^5[u+C(t)]^5-4a^2C^4(t)[5u+4C(t)]=0, \tag{13}$$

and the equation of state is

$$\rho-\frac{3}{8\pi}=6\left(p+\frac{3}{8\pi}\right)\frac{5\left(r^{2/7}\right)^2+5r^{2/7}C\left(t\right)+C^2\left(t\right)}{5r^{2/7}+4C\left(t\right)}.$$

A solution to (13) has the form

$$u(t)=\gamma(t)C(t), \tag{14}$$

where $\gamma(t)$ is as yet an arbitrary function and $C(t)>0$ satisfies

$$C(t)=\left[\frac{4a^2(4+5\gamma(t))}{147b^4\gamma^5(t)(1+\gamma(t))^5}\right]^{1/5}. \tag{15}$$

Thus the moving boundary is given by

$$R(t)=\gamma^{7/2}(t)\left[\frac{4a^2(4+5\gamma(t))}{147b^4\gamma^5(t)(1+\gamma(t))^5}\right]^{7/10}. \tag{16}$$

Further the strong-energy condition, $\rho(t)+3p(t)\geq 0$, imposes

$$\gamma(t)>0, \tag{17}$$

which avoids the singularities $\gamma(t)=-1$ and $\gamma(t)=0$.

4. – The second-case example

Here we let $u=(\alpha x+\beta)^{1/7}$ in eq. (5). Then it becomes

$$y=\frac{a}{b^2}u^7C(t)(1-ku^7)^{1/7}\frac{[2(1-ku^7)^{1/7}-uC(t)]}{[-uC(t)+(1-ku^7)^{1/7}]^2}, \tag{18}$$

where $a=\alpha^{-3/7}(\frac{3}{2})^{-1/7}(\frac{2}{7})^{-3/7}$, $b=\alpha^{-1/7}(\frac{3}{2})^{-3/14}(\frac{2}{7})^{-1/7}$ and $C(t)$ now refers to $bC(t)$. We specialise and study the situation when $k=0$. Equation (18) reduces to

$$y=\frac{a}{b^2}u^7C(t)\frac{[2-uC(t)]}{[1-uC(t)]^2}. \tag{19}$$

[3] KUSTAANHEIMO P. and QVIST B., *Soc. Sci. Fennica, Commun. Phys.-Math*, **XIII** (1948) 12.
[4] LEACH P. G. L., MAARTENS R. and MAHARAJ S. D., *Int. J. Non-Linear Mech.*, **27** (1992) 575.
[5] HERLT H. and STEPHANI H., *J. Math. Phys.*, **28** (1992) 11.
[6] STEPHANI H. and WOLF T., *Class. Quantum Grav.*, **13** (1996) 1261.
[7] KRAMER D., STEPHANI H., MACCALLUM M. and HERLT E., *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge) 1980.
[8] STEPHANI H., *J. Phys. A*, **16** (1983) 3529.
[9] WAFO SOH C. and MAHOMED F. M., *Class. Quantum Grav.*, **16** (1999) 3553.