Deformed Spheres in General Relativity and Beyond

Rehana Rahim

Department of Mathematics

Quaid-i-Azam University

Islamabad, Pakistan

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Deformed Spheres in General Relativity and Beyond

By

Rehana Rahim

Supervised

By

Dr. Khalid Saifullah

Department of Mathematics

Quaid-i-Azam University

Islamabad, Pakistan

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Deformed Spheres in General Relativity and Beyond

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Quaid-i-Azam University, Islamabad, in partial fulfillment of
the requirement for the degree of

Doctor of Philosophy

in

Mathematics

by

Rehana Rahim

Department of Mathematics

Quaid-i-Azam University

Islamabad, Pakistan

2020

Certificate of Approval

This is to certify that the research work presented in this thesis entitled **Deformed spheres in general relativity and beyond** was conducted by **Ms. Rehana Rahim** under the supervision of **Prof. Khalid Saifullah**. No part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the Department of Mathematics, Quaid-i-Azam University, Islamabad in partial fulfillment of the requirements for the degree of Doctor of Philosophy in field of Mathematics from Department of Mathematics, Quaid-i-Azam University Islamabad, Pakistan.

Student Name: **Rehana Rahim** Signature: Rehana Rahim

External committee:

a) **External Examiner 1** Signature: [signature] 15/01/2020
Name: **Prof. Azad Akhtar Siddiqui**
Designation: Professor
Office Address: School of Natural Sciences, National University of Sciences and Technology, H-12, Islamabad.

b) **External Examiner 2** Signature: [signature]
Name: **Prof. Jameel Un Nabi**
Designation: Professor
Office Address: Faculty of Engineering Sciences, Ghulam Ishaq Khan Institute of Engineering Sciences and Technology, Topi, Swabi.

c) **Internal Examiner** Signature: K. Saifullah
Name: **Prof. Khalid Saifullah**
Designation: Professor
Office Address: Department of Mathematics, Quaid-i-Azam University, Islamabad.

Supervisor Name Signature: K. Saifullah
Prof. Khalid Saifullah

Name of Dean/ HOD Signature: Nadeem

Prof. Sohail Nadeem

Deformed Spheres in General Relativity and Beyond

By

Rehana Rahim

CERTIFICATE

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

We accept this dissertation as conforming to the required standard.

1. ______________________
Prof. Sohail Nadeem
(Chairman)

2. ______________________
Prof. Khalid Saifullah
(Supervisor)

3. ______________________
Prof. Azad Akhtar Siddiqui
School of Natural Sciences,
National University of Sciences and Technology,
H-12, Islamabad.

4. ______________________
Prof. Jameel Un Nabi
Faculty of Engineering Sciences,
Ghulam Ishaq Khan Institute of Engineering Sciences and Technology,
Topi, Swabi.

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
2020

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Rehana Rahim

Name of Student: **Rehana Rahim**

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To my family

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Abstract

Currently, Einstein's general theory of relativity (GR) provides the best description for the phenomena called gravity. But it is not the only theory that does the job. There is the version given by Newton also. This version describes gravity as the force between the objects. Such a force depends on masses of the objects involved and also on the distance from each other. In GR, gravity is not a force. It is the curvature of the spacetime resulting due to the presence of the matter. The gravitational field as described by GR is a manifestation that space is the curved Riemannian one instead of the flat Minkowski. The gravitational field gets geometrized in GR, which is a tensor theory of the gravitational field instead of scalar one (Newtonian theory is a scalar theory). The gravitational field is represented in terms of the metric tensor of the Riemannian space, its source being the matter tensor. The components of the matter tensor source the gravitational field in an elegant way determined by the Einstein filed equations (EFEs).

Continuous progress is being made in finding the solutions of the EFEs. Schwarzschild, Reissner-Nordström, Kerr and Kerr-Newman spacetimes are the simplest vacuum solutions of EFEs that describe the black holes. Reissner-Nordström is the charged generalization of the Schwarzschild solution, both being spherically symmetric. Kerr metric is the rotating generalization of the Schwarzschild metric. Introduction of the charge in the Kerr metric gives the Kerr-Newman metric. Kerr and Kerr-Newman spacetimes are axially symmetric. In the limit of mass being vanished, they reduce to Minkowski metric in spheroidal coordinates. Spheroids are the geometric objects which we can take as deformed spheres.

Apart from the research and interest in GR, there has been a growing interest in alternate theories of gravity. One such theory is the Chern-Simons (CS) theory. The action of this theory consists of the usual Einstein-Hilbert term and a new parity violating four-dimensional correction. Two kinds of formulations exist in CS theory, namely dynamical and non-dynamical. Black hole solutions have been developed in both the cases. Our interest as regards to this thesis is the spacetime which has been developed in the former formulation.

The solutions beyond GR can also be formulated by another method. Such method involves the model independent parameterization of the metric. The metric thus obtained must describe the black hole solution in any theory of gravity. The possible deviations from the Kerr spacetime are measured by the deviation parameters.

The detailed outline of the thesis is as follows: Chapter 1 is about the preliminaries. In Chapter 2, the Misner-Sharp mass is generalized for the spheroidal geometry. Misner-Sharp mass is a type of quasilocal mass that previously worked only in the spherically symmetric spacetimes. It also gives the location of the marginally outer trapped surface in such spacetimes. The Misner-Sharp mass is extended for spheroids within GR and

the location of marginally outer trapped surface is determined in this new setting. The parameter which gives deviation from spherical geometry is kept small throughout the analysis. In quantum physics, the energy density which defines the Misner-Sharp mass (and ADM mass, named after Richard Arnowitt, Stanley Deser and Charles Misner) becomes a quantum observable and one could conjecture that the gravitational radius admits a similar description. The gravitational radius is made a quantum mechanical operator which acts on the "horizon wave function". The horizon wave function is given by the quantum state of the source. The horizon quantum mechanics has been extended to the case of spheroidal sources at the end of the chapter.

The next two chapters deal with the spacetimes in the alternate theories of gravity. Chapter 3 involves spacetime in dynamical CS theory. This spacetime is valid in slow rotation approximation and small coupling constant. The effects of the CS coupling constant on some physical phenomena e.g. quasilocal mass, particle motion and energy extraction process are studied.

Johannsen and Psaltis developed a rotating deformed Kerr-like metric in an alternate theory of gravity other than GR. It is obtained by applying Newman-Janis algorithm to a deformed Schwarzschild metric. Motivated by this spacetime, a charged analogue of the Johannsen-Psaltis metric is developed in Chapter 4. Here the seed metric is taken as the Reissner-Nordström spacetime. The new metric is studied for the event and Killing horizons, the latter are also represented graphically. Lorentz violating regions are analyzed by the determinant of the charged version of the Johannsen-Psaltis metric. Analysis of the closed time-like curves are also included in this chapter. Considering the motion of a particle on the equatorial plane, we obtain its energy and angular momentum. Location of the circular photon orbits and innermost stable circular orbits are also determined.

The Chapter 5 contains the summary and conclusion of the thesis.

List of Publications

As publication is one of the requirements of the Higher Education Commission of Pakistan, we give here a list of publications from the thesis:

- R. Casadio, A. Giusti and R. Rahim, Horizon quantum mechanics for spheroidal sources, Europhysics Letters **121** (2018) 60004.
- R. Rahim and K. Saifullah, Charging the Johannsen-Psaltis spacetime, Annals of Physics **405** (2019) 220.
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- R. Rahim and K. Saifullah, Energy extraction and the centre of mass energy in slowly rotating Chern-Simons black hole, arXiv: 1906.05632 (2019).
- R. Rahim and K. Saifullah, Effect of charge and deformation parameter on energy extraction in charged non-Kerr black holes, International Journal of Modern Physics D **28** (2019) 2040012.
- R. Rahim and K. Saifullah, The Hawking mass for an ellipsoidal surface, Modern Physics Letters A, Doi 10.1142/S021773232050073X (2020).

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Chapter 1

Introduction

1.1 Gravity

Gravity is one of the basic interactions of nature others being electromagnetic, strong and weak nuclear forces. Each of these forces has its own peculiar nature. Comparison of strength of these fundamental forces shows that gravity is weakest of them all (nearly 10^{29} times weaker than weak force, 10^{36} times weaker than the electromagnetic force and 10^{38} times weaker than the strong force). Despite this weakness in strength, gravity has certain properties that make it distinguished from others. It is a long range attractive interaction. Unlike strong and weak interactions, there isn't a length scale fixing the range for gravitational interactions. It cannot be shielded. Negative gravitational charges don't exist to neutralize the positive ones, therefore the gravitational interaction cannot be screened. Gravity is a universal interaction among all masses according to the Newtonian gravitational theory and among all energies in the relativistic theory of gravity as

$$E = mc^2, \tag{1.1}$$

where E, m and c denote energy, particle's mass and speed of light respectively.

Gravitational physics finds its applications from macroscopic to microscopic scale. On the macroscopic scale, it is connected to astrophysics and cosmology and on the opposite side of the scale it is linked to particle physics and quantum physics. Universe's organization is done by gravity (despite being weak in strength) on the large distance measures of cosmology and astrophysics. Such distances are outside the range of applicability of strong and weak nuclear forces. We could try the electromagnetic force as the dominant force on this scale. It could be of long range provided there existed large scale things having some net electric charge. Our universe is electrically neutral, and the electromagnetic interaction due to its strength (stronger than gravitational interaction) will neutralize some large scale net charge sharply. This leaves gravity to take control over the immense distances of the universe.

The electromagnetic, strong and weak nuclear forces have been combined into a single force called the Grand Unified Theory (GUT). Union of gravitational and the other three forces has been termed as the Theory of Everything (ToE). Formulation of such a ToE is amongst the important unresolved problems in physics.

It is quite obvious that this amazing gravitational interaction or gravity needs to be explained mathematically. It must be represented by some equations that govern its behavior. Currently gravity is best described by the theory of General Relativity (GR) given by Einstein in 1915. It is important to briefly describe Newton's universal law of gravity and see the difference between two theories. Newton's universal law of gravity describes gravity as a force that acts on physical objects. This force depends directly on their masses and is inversely proportional to the square of the distance between them. Mathematically it is given as

$$F = \frac{Gm_1m_2}{r^2}, \tag{1.2}$$

where G is the gravitational constant having value $6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$, m_1, m_2 are the masses of the objets and r denotes their distance from each other. Gravitational force as described by Newton is an instantaneous force. The force experienced by an object depends on the position of the second object at same instant. This is not acceptable by the Special Theory of Relativity, presented by Einstein in 1905, as this theory restricts that nothing can move faster than light.

Einstein's pursuit for a relativistic version of a gravitational theory did not just give some new formulation of relativistic gravitational field, but changed our whole concept of space, time and gravity. He changed the previous concepts of space and time as separate entities into a single continuum called the spacetime. According to his theory of GR, the presence of matter causes the spacetime to curve or bend. The paths along which all objects fall are straight paths in that curved spacetime. According to Newton, sun applies force of gravity on earth and motion of earth around the sun is due to this force. GR explains the same situation in this way: presence of sun causes the spacetime to curve. Earth just follows the straight paths in such curved environment. Geometry defines gravity.

GR deals with two important aspects:
(i) Effects of curvature on motion of matter,
(ii) Effects of matter on the curvature of spacetime.
John Wheeler has summarized these points in the form "matter tells space how to

curve, and space tells matter how to move". It is important for the interpretation of many astrophysical objects such as black holes, pulars, end of a star, for the big bang and our universe.

In the language of mathematics, GR is given by a set of coupled partial differential equations known as the Einstein field equations (EFEs) which are given as follows

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1.3}$$

where $R_{\mu\nu}, R, g_{\mu\nu}, \Lambda, G, c, T_{\mu\nu}$ represent the Ricci tensor, Ricci scalar, metric tensor, cosmological constant, Newton's gravitational constant, speed of light in vacuum and the energy-momentum tensor, respectively. The first two terms on the left hand side are collectively known as the Einstein tensor represented by the symbol $G_{\mu\nu}$. Thus the EFEs can also be written as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{1.4}$$

In 4 dimensions the total number of EFEs is 10. Thus we have a system of 10 coupled nonlinear partial differential equations. The terms on the left of Eq. (1.3) denote the geometry of the spacetime under consideration and on the right we have the terms which are representative of the matter. The Ricci tensor is a symmetric tensor of rank 2 and is a contraction of the Riemann tensor which is given by

$$R^{\alpha}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\lambda\gamma}\Gamma^{\lambda}_{\beta\delta} - \Gamma^{\alpha}_{\lambda\delta}\Gamma^{\lambda}_{\beta\gamma}, \tag{1.5}$$

where $\Gamma^{\alpha}_{\beta\delta}$ denotes the Christoffel symbols having the mathematical form as

$$\Gamma^{\alpha}_{\beta\delta} = \frac{1}{2}g^{\alpha\sigma}\left(g_{\beta\sigma,\delta} + g_{\delta\sigma,\beta} - g_{\beta\delta,\sigma}\right), \tag{1.6}$$

where $g^{\alpha\sigma}$ is the inverse of the metric tensor. By contracting the first and third indices of the Riemann tensor i.e. $R^{\alpha}_{\beta\gamma\delta} = R^{\alpha}_{\beta\alpha\delta}$, one gets the Ricci tensor. The trace of the Ricci tensor gives the Ricci scalar i.e. $R = R^{\alpha}_{\alpha}$.

The energy-momentum tensor is a symmetric second rank tensor describing the matter distribution at each event of the spacetime. In 4 dimensions its matrix form is written as

$$\begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}. \tag{1.7}$$

The physical quantities represented by these components are as follows:
T^{00} describes energy density of the matter,
T^{0i} denotes energy flux $\times c^{-1}$ in the i-direction,
T^{i0} denotes momentum density $\times c$ in the i-direction,
T^{ij} represents flow of the i-component of momentum per unit area in the j-direction,
where i and j are indices representing the spatial coordinates.

The EFEs contain metric tensor and its partial derivatives. So they are basically a set of coupled partial differential equations to be solved for the metric tensor. These solutions can be exact or non-exact. Here we describe some of the exact vacuum solutions of EFEs in GR that describe black holes. Before going into these solutions, we first describe what black holes are and how they are formed.

1.2 Black holes

Black holes are the most important predications of GR. A black hole can be defined as a region from which nothing can escape not even light. This is due to very strong gravitational field. It is formed when an object having mass M shrinks to the size which is less than its gravitational radius denoted by $R_H = 2GM/c^2$.

The research for properties of black holes is being done rapidly. Considerable progress has been made on understanding the properties of black holes, their astrophysical aspects and details of the different physical phenomena. The connections between the black hole theory with other apparently different concepts such as thermodynamics, quantum theory and information theory are also being studied. The interest in black holes has led to the existence of a new branch of physics termed as the black hole physics.

A black hole is formed due to the gravitational collapse of a massive star. A star is acted upon by attractive gravitational forces and expanding forces of gases which are heated due to nuclear reactions and as a result emit energy. Such a process is termed as thermonuclear burning. The life journey of a star starts because of gravitational collapse of a cloud of interstellar gas which has hydrogen and helium as its main constituents. Compressional heating increases temperature of the core to that level at which the thermonuclear reactions start, converting hydrogen to helium and emitting energy as a consequence. The star then comes to a steady state where the energy produced in the process of thermonuclear burning is being balanced with energy released in the form of radiations. There comes a stage when this balance is disturbed. This happens when a good quantity of hydrogen has been utilized and not enough nuclear fuel is left to balance energy being given as radiations. The contraction due to gravity restarts. The core's temperature is again increased by compressional heating for thermonuclear reactions but this time helium burns. A considerable quantity of helium gets consumed in this burning, core shrinks and a new phase of thermonuclear burning begins. Now what will happen when there is no nuclear fuel to burn? There are two cases that happen in such a situation. One is that the final object is an equilibrium star in which gravitational force is being balanced by some nonthermal pressure source. The second case is that the star continues to collapse, thus leading to the formation of black holes. Many kinds of nonthermal pressures exist. There is electron Fermi pressure created due to the fact that no two electrons can occupy the same quantum state (Pauli's exclusion principle). Similar Fermi pressures for neutrons and protons also exist. Repulsive nuclear forces also contribute to nonthermal pressures. A star supported against gravity by electron Fermi pressure is termed as the white dwarf. Stars supported by neutron Fermi pressure and nuclear forces are known as neutron stars. Neutron stars and white dwarfs are more dense and smaller as compared to ordinary stars. White dwarfs and neutron stars have upper bound for their masses known as the Chandrasekhar limit and the Tolman-Oppenheimer-Volkoff limit (or TOV limit) respectively. Chandrasekhar limit is about 1.4 solar mass ($2.765 \times 10^{30} kg$). It is the limit for the mass above which the electron degeneracy pressure in the core of a star is not enough to equalize the star's inward gravitational force thus leading the white dwarf into further gravitational collapse into a neutron star or a black hole. The TOV limit is the maximum mass of the cold non-spinning neutron star. The analysis from gravitational waves produced by the merger of neutron stars places the limit to be near 2.17 solar masses [1]. Previous studies suggested limit at approximately 1.5 to 3.0 solar masses, for an initial stellar mass of 15 to 20 solar masses [2]. For the neutron star having mass greater than this limit, it continuous to the path of gravitational collapse into much more denser black holes.

Below we describe mathematical forms of a few vacuum solutions of EFEs that correspond to black holes [3].

The Schwarzschild spacetime is the simplest static vacuum solution of EFEs. Mathematically it is represented as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{1.8}$$

where M is the mass of the black hole. This solution was given by Karl Schwarzschild in 1916 just one year after the development of EFEs.

The Reissner-Nordström spacetime is the charged generalization of the Schwarzschild spacetime and is given as

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \quad (1.9)$$

Here black hole's charge is represented by Q. Setting $Q = 0$, one gets the Schwarzschild spacetime (1.8). The non-vanshing components of the Maxwell tensor $F^{\mu\nu}$ are

$$F^{01} = -F^{10} = \frac{Q}{r^2}.$$

The rotating generalization of the Schwarzschild spacetime is the Kerr metric. It is stationary and axisymmetric. In Boyer-Lindquist coordinates, its mathematical expression is given by

$$ds^2 = -\left[1 - \frac{2Mr}{\Sigma}\right] dt^2 - \frac{4aMr\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \sin^2\theta \Big[r^2 + a^2 + \frac{2a^2 Mr \sin^2\theta}{\Sigma}\Big] d\phi^2, \quad (1.10)$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 + a^2 - 2Mr.$$

Here a denotes spin of the black hole. Setting $a = 0$, one gets the Schwarzschild spacetime.

The Kerr-Newman spacetime is the charged generalization of the Kerr metric as well as rotating generalization of the Reissner-Nordström spacetime. It is also stationary, axisymmetric and vacuum solution of EFEs. In Boyer-Lindquist coordinates, its mathematical expression is given by

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma} + \frac{Q^2}{\Sigma}\right) dt^2 - \frac{2a(2Mr - Q^2)\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \sin^2\theta \left(r^2 + a^2 + \frac{a^2(2Mr - Q^2)\sin^2\theta}{\Sigma}\right) d\phi^2, \quad (1.11)$$

where Q and a denote the charge and spin of the black hole, respectively. Here Σ is same as in the Kerr metric whereas Δ modifies to $r^2 + a^2 - 2Mr + Q^2$.

1.3 The horizons

1.3.1 Event horizon

For a black hole, an event horizon can be described as the boundary (horizon) between its inside and its outside. For someone outside the event horizon, he cannot know about the happenings inside. If we consider a black hole as a ball, then its surface constitutes an event horizon. Nothing can escape from that surface.

Mathematically, it is a null surface having null geodesics as its generators. These geodesics are trapped inside the surface.

1.3.2 Killing horizon

The Killing equation is given by

$$\nabla_a k_b + \nabla_b k_a = 0, \tag{1.12}$$

where ∇_a above is representing the covariant derivative. Solutions of the Killing equation are called the Killing vectors. Killing horizon is a null hypersurface which is every where tangent to the Killing vector field k^a. The Killing vectors are null vectors on the Killing horizon.

The relationship between the event and Killing horizons can be established by the following theorem [4]

Theorem 1.1. *Let a stationary, asymptotically flat spacetime contain a black hole and be a solution of Einstein equations with matter satisfying suitable hyperbolic equations. Then the event horizon is a Killing horizon.*

The Killing vector in such a case can be written as

$$k^a = t^a + \Omega\phi^a. \tag{1.13}$$

This equation is a linear combination of Killing vectors related to time symmetry t^a and rotational symmetry ϕ^a for some constant Ω. The condition of k^a being null gives

$$g_{33}\left(\Omega^2 + 2\Omega\frac{g_{03}}{g_{33}} + \frac{g_{00}}{g_{33}}\right) = 0. \tag{1.14}$$

The solutions for the above equation are

$$\Omega_{\pm} = -\frac{g_{03}}{g_{33}} \pm \sqrt{\frac{g_{03}^2}{g_{33}^2} - \frac{g_{00}}{g_{33}}}. \tag{1.15}$$

The angular velocity $\Omega_{\pm}$ approaches the constant value g_{03}/g_{33} on approaching event horizon which is angular velocity of ZAMO (zero angular momentum observer) on horizon and it must be single valued. This takes place when

$$g_{03}^2 - g_{00}g_{33} = 0. \tag{1.16}$$

The radius at which the above equation holds is known as the Killing horizon. In general, the event and Killing horizons do not coincide for stationary spacetimes.

1.3.3 Trapped surfaces and apparent horizon

Trapped surface is an important concept in GR given by Penrose [5]. For a 2-dimensional surface in a 4-dimsnsional spacetime, there exist two null directions which are normal to the surface at each point. Let the outgoing and ingoing null normals to the surface be denoted by $\boldsymbol{l}$ and $\boldsymbol{n}$ respectively. We denote the corresponding expansion scalars by Θ_l and Θ_n. Trapped surface is then defined [5] as a compact, orientable and space-like 2-surface for which $\Theta_l \Theta_n > 0$. Let us have the set of all such trapped surfaces. The apparent horizon is then the boundary of these surfaces.

A surface is called marginally outer trapped surface (MOTS) if the expansion of outgoing null geodesics vanishes [4, 6, 7, 8, 9, 10] i.e. $\Theta_l = 0$.

In general, the expansion scalars associated with outgoing and ingoing geodesics are, respectively, given by

$$\Theta_{\boldsymbol{\ell}} = q^{\mu\nu} \nabla_\mu l_\nu, \qquad \Theta_{\boldsymbol{n}} = q^{\mu\nu} \nabla_\mu n_\nu, \tag{1.17}$$

where $\mu, \nu = 0, \ldots, 3$ and

$$q_{\mu\nu} = g_{\mu\nu} + l_\mu \, n_\nu + n_\mu \, l_\nu, \tag{1.18}$$

represents the metric induced by the spacetime metric $g_{\mu\nu}$ on the 2-dimensional space-like surface formed by spatial foliations of the null hypersurface generated by the outgoing null tangent vector $\boldsymbol{\ell}$ and the ingoing null tangent vector $\boldsymbol{n}$. This 2-dimensional metric is purely spatial and has the following properties

$$q_{\mu\nu} \, \ell^\mu = q_{\mu\nu} \, n^\mu = 0, \quad q^\mu{}_\mu = 2, \quad q^\mu{}_\lambda \, q^\lambda{}_\nu = q^\mu{}_\nu, \tag{1.19}$$

where $q^\mu{}_\nu$ represents the projection operator onto the 2-space orthogonal to $\boldsymbol{\ell}$ and $\boldsymbol{n}$.

Given these definitions, it is evident that the study of marginally trapped surfaces in any realistic system is a very complex topic, and determining their existence and location is in general possible only by means of numerical methods.

1.4 From spheres to spheroids

Both Schwarzschild and Reissner-Nordström spacetimes are spherically symmetric. Here the surfaces of constant t and constant r are spheres. In fact, for such surfaces, Eqs. (1.8) and (1.9) reduce to

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{1.20}$$

which is the line element for a sphere of radius r. Now, consider the Kerr black hole given in Eq. (1.10). Taking its limit as $M \to 0$, we get

$$g_{\mu\nu} = -dt^2 + \frac{r^2 + a^2\cos^2\theta}{a^2 + r^2} dr^2 + \left(r^2 + a^2\cos^2\theta\right) \mathrm{d}\theta^2 + \left(a^2 + r^2\right)\sin^2\theta \, \mathrm{d}\phi^2. \tag{1.21}$$

The above equation represents Minkowski spacetime in spheroidal coordinates. It can also be written as

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{1.22}$$

where

$$x = \sqrt{r^2 + a^2}\sin\theta\cos\phi, \quad y = \sqrt{r^2 + a^2}\sin\theta\sin\phi, \quad z = r\cos\theta. \tag{1.23}$$

Here a can be considered as deviation from the spherical geometry. The usual spherical polar coordinates are recovered when $a = 0$.

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta. \tag{1.24}$$

To determine the spatial part of Eq. (1.22), we determine the differentials of Eq. (1.23) as

$$dx = \frac{r}{\sqrt{r^2 + a^2}}\sin\theta\cos\phi dr + \sqrt{r^2 + a^2}\cos\theta\cos\phi d\theta - \sqrt{r^2 + a^2}\sin\theta\sin\phi d\phi, \tag{1.25}$$

$$dy = \frac{r}{\sqrt{r^2 + a^2}}\sin\theta\sin\phi dr + \sqrt{r^2 + a^2}\cos\theta\sin\phi d\theta + \sqrt{r^2 + a^2}\sin\theta\cos\phi d\phi, \tag{1.26}$$

$$dz = \cos\theta dr - r\sin\theta d\theta, \tag{1.27}$$

which give

$$dx^2 + dy^2 + dz^2 = \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2} dr^2 + \left(r^2 + a^2\cos^2\theta\right) \mathrm{d}\theta^2 + \left(r^2 + a^2\right)\sin^2\theta \, \mathrm{d}\phi^2. \tag{1.28}$$

For $a^2 > 0$, the above line element describes a prolate spheroid, which extends more along the axis of symmetry than on the equatorial plane (see the yellow surface in Figure. 1.1). In order to describe an oblate spheroid, which is flatter along the axis of symmetry (see the red surface in Figure. 1.1), we can simply consider the mapping

$a \to i\, a$ (so that $a^2 \to -a^2$). For $a = 0$, one recovers the usual sphere (represented in green in Figure. 1.1).

As with the Kerr case, we can also see what happens to the Kerr-deSitter spacetime when mass vanishes. The Kerr-deSitter spacetime is

$$ds^2 = -\frac{\Delta_r}{I^2\Sigma}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\sin^2\theta\Delta_\theta}{I^2\Sigma}(adt - (r^2+a^2)d\phi)^2 + \frac{\Sigma}{\Delta_r}dr^2 + \frac{\Sigma}{\Delta_\theta}d\theta^2, \quad (1.29)$$

where

$$\begin{aligned}
\Delta_r &= \big(1 - \frac{\Lambda r^2}{3}\big)\big(r^2 + a^2\big) - 2Mr,\\
\Delta_\theta &= \big(1 + \frac{\Lambda a^2\cos^2\theta}{3}\big),\\
I &= 1 + \frac{\Lambda a^2}{3},\\
\Sigma &= r^2 + a^2\cos^2\theta.
\end{aligned}$$

When $M \to 0$, the Kerr-deSitter metric takes the form [11]

$$\begin{aligned}
ds^2 = \Sigma\Big[\frac{dr^2}{\big(1-\frac{\Lambda r^2}{3}\big)\big(r^2+a^2\big)} + \frac{d\theta^2}{\Delta_\theta}\Big] + \frac{\sin^2\theta\Delta_\theta}{\Sigma I^2}\Big[adt - (r^2+a^2)d\phi\Big]^2 \\
- \frac{(1-\frac{\Lambda r^2}{3})(r^2+a^2)}{\Sigma I^2}\Big[dt - a\sin^2\theta d\phi\Big]^2. \quad (1.30)
\end{aligned}$$

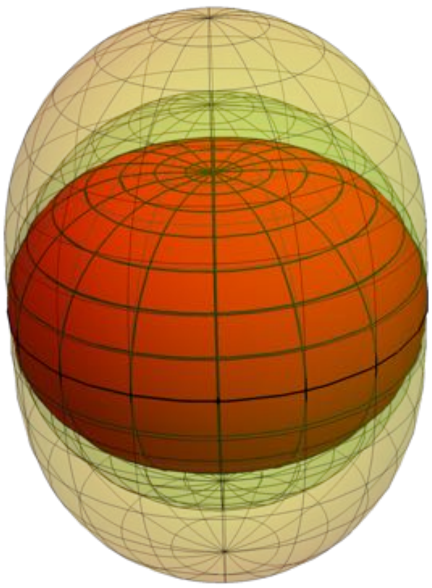


FIGURE 1.1: Spheroids: prolate spheroid with $a^2 > 0$ (in yellow) compared to oblate spheroid with $a^2 < 0$ (in red) and to the reference sphere $a^2 = 0$ (in green).

1.5 Modified Kerr spacetimes

Extremely massive, rotating black holes are believed to be present at the centre of most of galaxies. The gravitational field outside such black holes is important not only for the evolution of the captured compact objects but is also a source of gravitational waves discovered recently [12]. A qualitative description of such a gravitational field is provided by the Kerr metric in GR. According to the no-hair theorem [13, 14], Kerr metric is the only axisymmetric, stationary, asymptotically flat, vacuum solution of EFEs that is regular outside the event horizon and is specified by mass and spin.

GR has been put to test extensively in numerous ways. One such test is to see that the black holes are described by the Kerr solution. This can be done in many ways [15, 16, 17]. The tests in the weak gravitational field regime can depend on the parameterized post-Newtonian approach [18]. On the contrary, one must model the black hole metric in terms of parametric deviations from the Kerr spacetime in order to take the test in the strong gravitational regime.

Many modified Kerr spacetimes are available in literature. Such modified forms include the Kerr black hole as the limiting case so that, if the deviations are set to zero, these reduce to the Kerr metric. Two asymptotically flat metrics describing the superposition of the Kerr solution with an arbitrary static vacuum Weyl field which differ in their angular momentum distributions were developed in Ref. [19]. Collins et al. discussed bumpy black holes, objects which are almost black holes but not quite general relativity's black holes having multipoles that deviate slightly from black hole solution [20]. Glampedakis et al. formulated the quasi-Kerr metric [21]. It is constructed by adding leading order deviations in the Kerr metric. These deviations appear in spacetime's quadrupole moment. The slowly rotating black hole in dynamical Chern-Simons (CS) gravity, to leading order in the coupling constant has been derived independently in Refs. [22, 23]. Vigeland et al. gave two model independent parametric deviations from Kerr metric [24]. One of them is formulated from generalization of the quasi-Kerr and bumpy metrics. The second one is built from the perturbations of the Kerr sapcetime in Lewis-Papapetrou form. Johannsen and Psaltis [25] constructed a Kerr like metric which depends on the set of free parameters in addition to mass and spin by using Newman-Janis algorithm [26]. It has also been generalized to include the electric charge Q [27].

In this thesis, we have focused on the metrics developed in [22, 23, 25, 27]. Below we describe the mathematical forms of the spacetimes given there.

In the slow rotation approximation $a << M$, the Kerr metric (1.10) takes the form

$$ds^2_{SK} = -\Big[U(r) + \frac{2a^2 M\cos^2\theta}{r^3}\Big]dt^2 - \frac{4aM\sin^2\theta}{r}dtd\phi + \frac{1}{U(r)^2}\Big[U(r) - \frac{a^2}{r^2}\Big(1 - U(r)\cos^2\theta\Big)\Big]dr^2 + \Sigma d\theta^2 + \sin^2\theta\Big[r^2 + a^2 + \frac{2a^2 M\sin^2\theta}{r}\Big]d\phi^2,$$

where $U(r) = 1 - 2M/r$ and the terms upto $O(a^2)$ are retained. The solution corresponding to the CS term is given as [22]

$$ds^2 = ds^2_{SK} + \frac{5\gamma^2 a\sin^2\theta}{4kr^4}\left[1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\right]dtd\phi, \tag{1.31}$$

where γ is the CS coupling constant. The equation for the scalar field φ is

$$\varphi = \left[\frac{5}{2} + \frac{5M}{r} + \frac{9M^2}{r^2}\right]\frac{\gamma a\cos\theta}{4Mr^2}. \tag{1.32}$$

We note that the off-diagonal term which results in a weakened dragging effect has the coupling constant contribution to $O(a\gamma^2)$.

The Johannsen and Psaltis metric is given by [25]

$$\begin{aligned} ds^2 &= -(1 + P(r,\theta))\Big(1 - \frac{2Mr}{\Sigma}\Big)dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}(1 + P(r,\theta))dtd\phi + \Sigma d\theta^2 \\ &+ \frac{\Sigma(1 + P(r,\theta))}{\Delta + a^2\sin^2\theta P(r,\theta)}dr^2 + \Big[\sin^2\theta(r^2 + a^2 + \frac{2a^2 Mr\sin^2\theta}{\Sigma}) \\ &+ P(r,\theta)\frac{a^2\sin^4\theta(\Sigma + 2Mr)}{\Sigma}\Big]d\phi^2, \end{aligned} \tag{1.33}$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 + a^2 - 2Mr$ and $P(r,\theta)$ has the general expression

$$P(r,\theta) = \sum_{k=0}^{\infty}\Big(\epsilon_{2k} + \epsilon_{2k+1}\frac{Mr}{\Sigma}\Big)\Big(\frac{M^2}{\Sigma}\Big)^k. \tag{1.34}$$

This metric has been constructed by applying the Newman-Janis algorithm on a deformed Schwarzschild metric. It is a solution of some unknown field equations which are different from the EFEs due to the presence of the function $P(r,\theta)$. The infinite parameters are constrained by the imposing the conditions of the asymptotic flatness and consistency with the observational weak field constrains on the deviations from the Kerr spacetime. In the simplest case ϵ_3 is the only non-zero deviation parameter.

The charged generalization of the above metric has the form [27]

$$\begin{aligned} ds^2 &= -(1+P(r,\theta))\Big(1-\frac{2Mr}{\Sigma}+\frac{Q^2}{\Sigma}\Big)dt^2 - \frac{2a(2Mr-Q^2)\sin^2\theta}{\Sigma}(1+P(r,\theta))dtd\phi \\ &\quad +\frac{\Sigma(1+P(r,\theta))}{\Delta+a^2\sin^2\theta P(r,\theta)}dr^2+\Sigma d\theta^2+\Big[\sin^2\theta\Big(r^2+a^2+\frac{a^2(2Mr-Q^2)\sin^2\theta}{\Sigma}\Big) \\ &\quad +P(r,\theta)\frac{a^2\sin^4\theta(\Sigma+2Mr-Q^2)}{\Sigma}\Big]d\phi^2, \end{aligned} \tag{1.35}$$

where Δ and Σ are have the same expressions as in case of Kerr-Newman metric. It is also constructed by applying the Newman-Janis algorithm on a deformed Reissner-Nordström spacetime (this construction has been discussed in detail in Chapter 4). By setting $Q = 0$, we obtain the Johannsen and Psaltis metric.

1.6 Geodesic equation

In Euclidean geometry, the shortest possible path between two points is a straight line. But this is not the case in GR. Here the shortest possible path between two points is a curve called the geodesic. For light particles i.e. photons, they are called null geodesics and for massive particles, the name is time-like geodesics. In this thesis, both kind of geodesics have been employed. Here we describe a derivation of the geodesics equation.

Let us denote the curve by x^μ and parameterize it by some arbitrary parameter say λ. This x^μ is required to be the shortest curve between two points i.e. for such a curve the arc-length s has the smallest value for some starting point P_i and final point P_f. This gives

$$s = \int_{P_i}^{P_f} ds = extremum, \tag{1.36}$$

where $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. Eq. (1.36) can be rewritten as

$$s = \int_{\lambda_i}^{\lambda_f} \frac{ds}{d\lambda}d\lambda = \int_{\lambda_i}^{\lambda_f} \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda = extremum, \tag{1.37}$$

where λ_i and λ_f are values of the parameter λ at the points P_i and P_f respectively. Eq. (1.37) has the form of Hamilton's principle having Lagrangian

$$\mathcal{L} = \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = \sqrt{A}, \tag{1.38}$$

and, the time t being replaced by λ. Here overdot denotes derivative with respect to the parameter λ. Therefore, the geodesics satisfy the Euler-Lagrange equation given by

$$\frac{d}{d\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^\alpha} - \frac{\partial\mathcal{L}}{\partial x^\alpha} = 0. \tag{1.39}$$

By substituting the values, the above equation takes the form

$$\frac{d}{d\lambda}\frac{g_{\mu\alpha}\dot{x}^{\mu}}{\sqrt{A}} - \frac{g_{\mu\nu,\alpha}\dot{x}^{\mu}\dot{x}^{\nu}}{2\sqrt{A}} = 0. \tag{1.40}$$

So far we have not said any thing about the parameter λ. To make things easier, λ can be taken to be proportional to the arc-length s. This gives A to be constant. This assumption gives a simplified form of Eq. (1.40) as

$$\frac{\partial g_{\mu\alpha}}{\partial\lambda}\dot{x}^{\mu} + g_{\mu\alpha}\frac{\partial\dot{x}^{\mu}}{\partial\lambda} - \frac{g_{\mu\nu,\alpha}\dot{x}^{\mu}\dot{x}^{\nu}}{2} = 0. \tag{1.41}$$

Using

$$g_{\mu\alpha,\lambda} = g_{\mu\alpha,\beta}\dot{x}^{\beta}, \tag{1.42}$$

$$g_{\mu\alpha,\beta}\dot{x}^{\mu}\dot{x}^{\beta} = \frac{(g_{\mu\alpha,\beta} + g_{\beta\alpha,\mu})}{2}\dot{x}^{\mu}\dot{x}^{\beta}, \tag{1.43}$$

Eq. (1.41) takes the form

$$\frac{d^2x^{\sigma}}{d\lambda^2} + \frac{1}{2}g^{\sigma\alpha}\Big(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}\Big)\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = 0. \tag{1.44}$$

Thus the final form of the geodesic equation is

$$\frac{d^2x^{\sigma}}{d\lambda^2} + \Gamma^{\sigma}_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = 0. \tag{1.45}$$

The above equation shows that geodesic equation is a second order ordinary differential equation involving the Christoffel symbol which might not always be in a simple form. This leads to difficulty in constructing the equation and also in its solution.

1.7 Mass in General Relativity

Eq. (1.2) describes the gravitational force between two masses which are distance r apart. In the same way, the electromagnetic force between two charges Q_1 and Q_2 distance r apart is given as

$$F = \frac{Q_1Q_2}{4\pi\varepsilon_0 r^2}, \tag{1.46}$$

where ε_0 is permittivity of the free space. This equation is known as Coulomb's law. Another aspect of electromagnetic theory is that we can determine the total charge enclosed by a closed surface S by Gauss law written as

$$Q = \int_V \rho_q dV = \varepsilon_0 \oint_S \mathbf{E}.d\mathbf{S}, \tag{1.47}$$

where V denotes the volume of the surface and $\mathbf{E}$ is the electric field. The above equation shows that the total charge contained inside the surface, given by volume integral of charge density ρ_q, is given by surface integral of electric flux over the whole surface. In this way, electric charge is described by the electric field. The information about the electric charge can be obtained by determining the electric field.

The similarities between the gravitational force law and Coulomb's law might lead one to write a formula for the total mass enclosed by some surface similar to the Gauss law i.e.

$$M = \int_V \rho_m dV = \oint_S W(g_{\mu\nu}, \partial g_{\mu\nu}) dS, \tag{1.48}$$

where ρ_m is mass density and W is some function of metric tensor $g_{\mu\nu}$ and its first order derivative $\partial g_{\mu\nu}$. But such a result for total mass has not been determined yet. In this regard, problem lies with the mass density.

Einstein's theory of relativity altered many previous concepts including how we see mass in the presence of gravitational field. He gave an equation relating mass and energy given in Eq. (1.1) which sets mass and energy equivalent to each other. Everything possesses energy that includes the gravitational field too. Thus the path to generalize the Gauss law to gravitation is not simple and easy. One must keep in mind that gravitational field itself has the mass.

The structure of EFEs is reflective of their non-linear characteristic. A field that is described by non-linear equations shows the property of self interactions. The fields which are governed by linear equations follow the superposition principle. Electromagnetic field is one such example. This superposition characteristic leads to the fact that field being produced due to two sources is sum of the fields produced by those sources individually. For example, if we know the relationship between a point charged source and the electric field produced by it, then we can infer charge of that field configuration. The same cannot be done for the gravitational field as it does not follow superposition principle.

Einstein's equivalence principle sets gravitational force equivalent to forces applied in an accelerating frame of reference. It means (locally) an observer cannot decide if he is on a gravitational body's surface or the surface is accelerating. Thus a reference frame can always be found where the gravitational field (at a point) vanishes. This leads to the fact that gravitational field is non-local in nature. The connection between gravitational field and mass density cannot be found as in the case of electromagnetic field. For such a connection, one needs to specify mass at each point of a spacetime which is not possible.

Next we ask ourself if the energy density cannot be determined locally then what is the solution in such a situation? In such situations, we use quasilocal approach i.e. mass is determined for some finite regions of spacetime. There exists the concept of total energy of an isolated asymptotically flat spacetimes. These are the ADM (Arnowitt-Deser-Misner) mass [28] (defined at spatial infinity) and Bondi-Sachs mass [29] at null infinity. Many definitions for quailocal mass have been proposed, each being applicable in different situations. There does not exist an expression which can serve as a unique definition for this quasilocal mass. In fact, in 1982 a list of unsolved problems in classical general relativity was given by Penrose [30]. The first problem on the list is "find a suitable quasilocal definition of energy-momentum". The few definitions of quasilocal mass are: Brown-York energy [31], the Misner-Sharp mass [32], the Komar mass [33], the Bartnik mass [34], the Hawking mass [35], Hawking-Hayward mass [35, 36], the Geroch mass [37] and the Penrose mass [38]. This thesis mainly involves Misner-Sharp mass and Hawking mass in spheroidal geometry [39, 40, 41].

There are certain properties that a good definition of a quasilocal mass must satisfy [42]:
(i) A point in a spacetime has zero mass. It means that mass should become zero when the surface contracts to a point.
(ii) A metric 2-sphere in a Minkowski space also has zero mass.
(iii) For special case of spherical symmetry, there exists a particular mass function which should be a limit of any definition of quasilocal mass in spherical symmetry.
(iv) A quasilocal mass must give ADM mass and Bondi-Sachs mass at spatial and null infinity, respectively.
(v) For a 2-surface S completely contained inside another surface S', the mass enclosed by S' is greater than or equal to the corresponding expression for S.

1.8 Black holes as source of energy and acceleration

The study of motion of particles around the black holes is an interesting and important problem in astrophysics. Such studies are not only helpful in understanding the geometrical structure of spacetime but also give insight for high energy phenomenon happening in the vicinity of these objects like formation of jets and acceleration disks.

One way to use the black hole as an energy source is the Penrose process [43]. The key point in this process is that the rotational energy associated with a black hole is not present inside its event horizon but in ergosphere which lies outside the event horizon. Things get dragged in the ergosphere of the rotating spacetime. The process can be pictured as follows: consider that matter falls into the ergosphere and it splits into two pieces once it enters the ergoshphere. The momenta of the pieces is arranged so that one piece escapes to infinity and the other one moves into the event horizon and is kind of lost. It can be made possible that the matter fragment that had left for infinity has more energy than the original matter and the other fragment carries negative energy. Since angular momentum is conserved, this leads to saying that more amount of energy can be taken out from a black hole than the amount initially provided. The source of the energy difference is black hole itself which loses some amount of angular momentum in the process which gets converted into energy that is extractable. The maximum amount of energy which a single particle gains is 20.7% [3]. By performing the process over and over again causes a black hole to lose all of its rotational energy, thus taking the form of a non rotating black hole e.g. the Schwarzschild.

Rotating black holes can also act as particle accelerators. A mechanism called BSW (called after Bañados, Silk and West) mechanism has been suggested in Ref. [44] by considering particle collision. It was shown that if two neutral particles collide at the horizon of extremal Kerr spacetime, then the centre of mass energy, E_{CM}, in such a situation could be infinite.

Chapter 2

The marginally trapped surfaces in spheroidal spacetimes

2.1 Introduction

According to GR, black holes are portions of Lorentzian manifolds characterised by the existence of an event horizon, from within which no signals can ever escape. In more general gravitating systems, the local counterpart of the event horizon is given by a marginally outer trapped surface (MOTS) [6, 7], which can be naively understood as the location where the escape velocity equals the speed of light at a given instant. If the system approaches an asymptotically static regime, the outermost MOTS should then become the future event horizon, like it happens in the very simple Oppenheimer-Snyder model [45].

For the particular case of a spherically symmetric self-gravitating source, one can employ the gravitational radius, and the equivalent Misner-Sharp mass function. The spherically symmetric line element can be written in the form

$$\mathrm{d}s^2 = g_{\mu\nu}(x^k)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + r^2(x^k)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{2.1}$$

where $x^k = (x^0, x^1)$ parameterize surfaces of constant angular coordinates ϕ and θ. For the metric (2.1), the gradient $\nabla_\mu r$ is orthogonal to surfaces having constant area given by $\mathcal{A} = 4\,\pi\,r^2$, and one finds that the product [8, 46]

$$\Theta_{\boldsymbol{\ell}}\,\Theta_{\boldsymbol{n}} \propto g^{\mu\nu}\,\nabla_\mu r\,\nabla_\nu r, \tag{2.2}$$

precisely vanishes on marginally trapped surfaces. Moreover, if we set [1] here $x^0 = t$ and $x^1 = r$, EFEs yield the solution

$$g^{11} = 1 - \frac{r_{\rm H}(t,r)}{r}, \tag{2.3}$$

where

$$r_{\rm H}(t,r) = 2\, m(t,r), \tag{2.4}$$

is known as the gravitational radius determined by the Misner-Sharp mass function [32]

$$m(t,r) = 4\,\pi \int_0^r \bar{r}^2\, \rho(t,\bar{r})\, {\rm d}\bar{r}, \tag{2.5}$$

where $\rho = \rho(t,r)$ is the matter density. According to Eq. (2.2), a MOTS exists where $g^{11} = 0$, or where Eq. (2.4) satisfies

$$r_{\rm H}(t,r) = r, \tag{2.6}$$

for $r > 0$. If the source is surrounded by the vacuum, the Misner-Sharp mass asymptotically approaches the ADM mass of the source, $m(t, r \to \infty) = M$, and the gravitational radius likewise becomes the Schwarzschild radius $R_{\rm H} = 2\,M$. To summarise, the relevant properties of the Misner-Sharp mass (2.5) are that
i) it only depends on the source energy density,
ii) it allows one to locate the (time dependent) MOTS via Eq. (2.4).

In this chapter the classical analysis of marginally trapped surfaces will be generalized to the systems with a slightly spheroidal symmetry. Moreover, since it is hardly possible to describe analytically such systems if they evolve in time, we shall consider static configurations as simple case studies. In particular, we shall deform a static spherically symmetric spacetime, and study location of marginally trapped surfaces perturbatively in the deformation parameter. In this respect, it is worth stressing that the assumption of staticity will ultimately lead to matter distributions which break some of the energy conditions. The cases presented here are therefore only intended to serve as toy models, whose purpose is to shed some light on the possible relation between these small perturbations and a mass function.

Explicit expressions will be given for the deformed de Sitter spacetime. We shall also study the case of a spheroidal spacetime containing source whose energy and pressure depart from such a symmetry. In both the cases, we will see that the location of marginally

[1] We shall use the coordinates $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$, throughout the thesis. Moreover, we have set $G = c = 1$.

trapped surfaces is given by surfaces of symmetry, and can therefore be found by computing the Misner-Sharp mass on the reference unperturbed (spherically symmetric) spacetime.

2.2 Static spheroidal sources

In this section, we will investigate how the particular description for static spherically symmetric systems extends to the case in which the symmetry is associated with (slightly) spheroidal surfaces. We start from the spherically symmetric metric given in Eq. (2.1), with r the areal radius constant on the 2-spheres of symmetry, and assume that the time dependence is negligible. EFEs then yield the solution (2.3), in which the now time independent Misner-Sharp mass $m = m(r)$ is determined by a static density $\rho = \rho(r)$ according to Eq. (2.5). It will also be assumed that the matter source also contains a (isotropic) pressure term, such that the Tolman-Oppenheimer-Volkov equation of hydrostatic equilibrium is satisfied [47]. We then change to (prolate or oblate) spheroidal coordinates and consider a localized source of spheroidal radius $r = r_0$, say with mass M_0, surrounded by a fluid with the energy density $\rho = \rho(r)$.

The central source only serves the purpose to avoid discussing coordinate singularities at $r = 0$. In the interesting portion of space $r > r_0$, the metric $g_{\mu\nu}$ is given as

$$\begin{aligned} g_{\mu\nu} = &- h(r,\theta;a)\,\mathrm{d}t^2 + \frac{1}{h(r,\theta;a)}\left(\frac{r^2 + a^2\cos^2\theta}{a^2 + r^2}\right)\mathrm{d}r^2 + \left(r^2 + a^2\cos^2\theta\right)\mathrm{d}\theta^2 \\ &+ \sin^2\theta\,\left(a^2 + r^2\right)\mathrm{d}\phi^2, \end{aligned} \tag{2.7}$$

where $h = h(r,\theta;a)$ is a function to be determined. The surfaces of constant r now represent ellipsoids of revolution, or spheroids, on which the density is constant. Therefore, by spheriodal symmetry, we mean dependence solely on the spheriodal radial coordinate r.

For $a^2 > 0$, the above metric can describe the spacetime outside a prolate spheroidal source and for $a^2 < 0$, it describes the spacetime outside an oblate spheroidal source. It is also important to remark that a spacetime equipped with the metric represented by Eq. (2.7) admits two trivial Killing vectors, namely $\boldsymbol{\partial_t}$ and $\boldsymbol{\partial_\phi}$. Furthermore, one can easily see that the vanishing of $g_{00} = -h(r,\theta;a)$ gives the location of the Killing horizons for spacetimes belonging to this class [2].

For consistency, the energy-momentum tensor $T_{\mu\nu}$ of the source can be inferred from the EFEs represented in Eq. (1.4). Here we put the cosmological constant equal to

[2] In a more general, time dependent spacetime, no such Killing structure would of course exist.

zero for further analysis. The component of EFEs which is our main concern is the 00 component because it relates the Einstein tensor to density. For the analysis, energy density is restricted to be spheroidally symmetric, so that we have the 00 component of the EFEs in the form

$$G^0{}_0 = -8\,\pi\,\rho(r), \tag{2.8}$$

and we will therefore assume that the necessary pressure terms are present in order to maintain equilibrium. In order to solve Eq. (2.8), we change variable from the azimuthal angle θ to $x \equiv \cos\theta$. This transforms $d\theta^2$ to

$$d\theta^2 = \frac{dx^2}{1-x^2}. \tag{2.9}$$

After making the required substitutions, the line element reads

$$\begin{aligned} \mathrm{d}s^2 = &- h(r,x;a)\,\mathrm{d}t^2 + \frac{1}{h(r,x;a)}\left(\frac{r^2+a^2\,x^2}{r^2+a^2}\right)\mathrm{d}r^2 + \frac{r^2+a^2\,x^2}{1-x^2}\,\mathrm{d}x^2 \\ &+ \left(r^2+a^2\right)\left(1-x^2\right)\mathrm{d}\phi^2. \end{aligned} \tag{2.10}$$

Given the symmetry of the system, we can restrict the analysis to the upper half spatial volume $1 \geq x \geq 0$ corresponding to $0 \leq \theta \leq \pi/2$. We then find

$$\begin{aligned} G^0{}_0 \;=\; & \frac{1}{4\,(r^2+a^2x^2)^3\,h^2}\Big\{4\left[r^4+a^2r^2(4\,x^2-1)+a^4\,x^2\,(1+x^2)\right]h^3 \\ & +3\,(1-x^2)(r^2+a^2x^2)^2\,(\partial_x h)^2 \\ & +2\,(r^2+a^2x^2)\,h\left[x\left(2\,r^2+a^2\left\{3\,x^2-1\right\}\right)\partial_x h+(x^2-1)\,(r^2+a^2\,x^2)\,\partial_x^2 h\right] \\ & +2\,h^2\big[2\left(a^2r^2\left\{1-4\,x^2\right\}-a^4\,x^2\left\{1+x^2\right\}-r^4\right) \\ & +r\left(r^2+a^2x^2\right)\left(2\,r^2+a^2\left\{1+x^2\right\}\right)\partial_r h\big]\Big\}, \end{aligned} \tag{2.11}$$

so that Eq. (2.8) appears to be a rather convoluted differential equation for the unknown metric fuction $h = h(r,x;a)$.

For completeness, we also show the remaining (non-vanishing) components of the Einstein tensor, namely

$$G^1{}_1 = \frac{r^2\,(h-1)}{(r^2+a^2\,x^2)^2}, \tag{2.12}$$

$$G^2{}_2 = \frac{x\left[\left(r^2+a^2\,x^2\right)\partial_x h + 2\,a^2\,x\,(h-1)\,h\right] + r\,\left(r^2+a^2\,x^2\right)\,h\,\partial_r h}{2\,(r^2+a^2\,x^2)^2\;h}, \tag{2.13}$$

and

$$
\begin{aligned}
G^3{}_3 \;=\; & \frac{1}{4\,(r^2+a^2\,x^2)^3\,h^2}\left\{4\,a^2\,h^3\left[a^2x^2+r^2\,(2\,x^2-1)\right]+3\,(1-x^2)(r^2+a^2x^2)^2\,(\partial_x h)^2\right.\\
& +2\,(r^2+a^2x^2)\,h\left[x\left(r^2+a^2\left\{2\,x^2-1\right\}\right)\partial_x h+(x^2-1)\,(r^2+a^2\,x^2)\,\partial_x^2 h\right]\\
& \left.+2\,h^2\left[2\,a^2\left(r^2-x^2\left\{a^2+2r^2\right\}\right)+r\left(r^2+a^2\,x^2\right)\left(r^2+a^2\right)\partial_r h\right]\right\}, \qquad (2.14)
\end{aligned}
$$

from which the complete stress energy momentum can be easily obtained for a generic metric of the form (2.7) from EFEs (1.4).

We proceed by considering small departures from spherical symmetry, parameterised by $a^2 \ll r_0^2$, and expand all expressions up to order a^2. In particular, the energy density must have the form

$$\rho \simeq \rho_{(0)}(r)+a^2\,\rho_{(2)}(r). \qquad (2.15)$$

There are no x dependent terms in this expression as it was assumed that density is function of r only. For the unknown metric function $h(r,x)$, the perturbed form is

$$
\begin{aligned}
h &\simeq h_{(00)}(r)+a^2\left[h_{(20)}(r)+x^2\,h_{(22)}(r)\right],\\
&= 1-\frac{2\,m_{(00)}(r)}{r}-2\,a^2\,\frac{m_{(20)}(r)+x^2\,m_{(22)}(r)}{r}, \qquad (2.16)
\end{aligned}
$$

where we introduced a Misner-Sharp mass function $m_{(00)}$, like in Eq. (2.5), for the zero order term and corrective terms $m_{(2i)}$ at order a^2 (with $i=0,2$ representing the polynomial order in x). In fact, at zero order in a, Eq. (2.8) reads

$$G_{(0)}{}^0{}_0=\frac{2\,m'_{(00)}(r)}{r^2}=8\,\pi\,\rho_{(0)}(r), \qquad (2.17)$$

with primes denoting derivatives with respect to r. The relation (2.5) gives the solution $m_{(00)}$.

At first order in a^2, the component of the Einstein tensor in Eq. (2.11) contains two terms,

$$G_{(2)}{}^0{}_0(r,x)=F(r)+x^2\,L(r), \qquad (2.18)$$

where $F(r)$ and $L(r)$ do not depend on x. Since ρ does not depend on x by construction, we must have $L(r)=0$, which yields

$$m'_{(22)}+\left(1-\frac{2\,m_{(00)}}{r}\right)^{-1}\frac{3\,m_{(22)}}{r}-\frac{3}{2\,r^2}\left(m'_{(00)}-\frac{5\,m_{(00)}}{3\,r}\right)=0. \qquad (2.19)$$

Finally, we are left with

$$F(r) = -\,\frac{2\,m'_{(20)}}{r^2} - \frac{m'_{(00)}}{r^4} + \frac{3\,m_{(00)}}{r^5} + \frac{2\,m_{(22)}}{r^3}\left(1 - \frac{2\,m_{(00)}}{r}\right)^{-1},$$
$$= -\,8\,\pi\,\rho_{(2)}, \tag{2.20}$$

in which $m_{(00)}$ is determined by Eq. (2.17) and $m_{(22)}$ by Eq. (2.19). Eq. (2.20) can then be used to determine $m_{(20)}$.

Once the metric function $h = h(r, x; a)$ is obtained, one can determine the locations of marginally trapped surfaces from the expansions of null geodesics defined in Eq. (1.17). It will then be interesting to compare the result with the solutions of the generalised Eq. (2.4), namely

$$2\,m(r_{\rm H}, x; a) = r_{\rm H}(x), \tag{2.21}$$

where

$$m(r, x; a) \simeq m_{(00)}(r) + a^2\left[m_{(20)}(r) + x^2\,m_{(22)}(r)\right], \tag{2.22}$$

is now the extended Misner-Sharp mass. We also note that Eq. (2.21) is equivalent to

$$h(r_{\rm H}, x; a) = 0, \tag{2.23}$$

which will be checked below with a specific example.

We can just anticipate that we expect the location of the MOTS to respect the spheroidal symmetry of the system and be thus given by the spheroidal deformation of the isotropic horizon obtained in the limit $a \to 0$.

2.3 Application to specific spheroidal sources

In order to proceed and find more explicit results, the above general construction will be applied to specific examples. The first example is of the spheroidally deformed de Sitter metric in which the density is independent of θ while in the second (more complicated metric) density has both radial and angular dependence.

2.3.1 Slightly spheroidal de Sitter

We start this case with the assumption of an inner core of radius $r = r_0$ and mass M_0, which is here surrounded by a fluid with energy density

$$\rho(r) = \rho_{(0)}(r) = \frac{\alpha^2}{4\,\pi\,r}, \tag{2.24}$$

where $r > r_0$, and α is a constant independent of a (so that $\rho_{(2)} = 0$). From Eq. (2.17), we obtain

$$m_{(00)} = M_0 + \frac{\alpha^2\,(r^2 - r_0^2)}{2}, \tag{2.25}$$

which of course holds for $r > r_0$. We further set $\alpha^2\,r_0^2 \simeq M_0$, so that

$$m_{(00)}(r) \simeq \frac{\alpha^2\,r^2}{2}. \tag{2.26}$$

This case admits a MOTS when $2\,m_{(00)}(r) = r$, that is

$$r_{\rm H} = \alpha^{-2}, \tag{2.27}$$

which is just the usual horizon for the isotropic de Sitter space. After substituting the above value of $m_{(00)}(r)$ in Eq. (2.19), it takes the form

$$m'_{(22)} + \frac{3\,m_{(22)}}{(1 - \alpha^2\,r)\,r} - \frac{\alpha^2}{4\,r} = 0. \tag{2.28}$$

The general solution of the above first order ordinary differential equation is

$$\begin{aligned} m_{(22)}(r) = \frac{1 - \alpha^2\,r}{8\,\alpha^4\,r^3}\,\big\{1 - (1 - \alpha^2\,r)\,\big[4 - 8\,\alpha^4\,(1 - \alpha^2\,r)\,C_{(22)} \\ +2\,(1 - \alpha^2\,r)\,\ln(1 - \alpha^2\,r)\big]\big\}\,, \end{aligned} \tag{2.29}$$

with $C_{(22)}$ an integration constant. For $r \simeq \alpha^{-2}$, the general solution reduces to

$$m_{(22)}(r) = \frac{\alpha^2}{8}\,\left(1 - \alpha^2\,r\right) + o[(1 - \alpha^2\,r)^2]. \tag{2.30}$$

We can then determine $m_{(20)}$ from Eq. (2.20), which, on employing the above expansion for $m_{(22)}$, reads

$$m'_{(20)} \simeq \frac{3\,\alpha^2}{8\,r}, \tag{2.31}$$

and yields

$$m_{(20)}(r) \simeq C_{(20)} + \frac{3\,\alpha^2}{8}\,\ln(\alpha^2 r), \tag{2.32}$$

with $C_{(20)}$ another integration constant.

We then set $C_{(20)} = C_{(22)} = 0$ for simplicity. The unknowns in the structure of the extended Misner-Sharp mass function given in Eq. (2.22) and also in the unknown function h in Eq. (2.16) are given in Eqs. (2.26), (2.30), (2.32). After substituting $m_{(00)}$, $m_{(20)}$ and $m_{(22)}$, the extended Misner-Sharp mass function takes the form

$$m(r, x; a) \simeq \frac{\alpha^2\,r^2}{2} + \frac{a^2\,\alpha^2}{8}\,\left[(1 - \alpha^2\,r)\,x^2 + 3\,\ln(\alpha^2 r)\right], \tag{2.33}$$

for $r \simeq \alpha^{-2}$. Also, the expression of h is

$$h(r,x;a) \simeq \left(1-\alpha^2\,r\right) - \frac{a^2\,\alpha^2}{4\,r}\left[3\,\log(\alpha^2\,r) + x^2\,(1-\,\alpha^2\,r)\right]. \tag{2.34}$$

The condition (2.23) then admits two separate solutions, namely

$$r_{\rm H}^{(1)} \simeq \alpha^{-2}, \tag{2.35a}$$

and

$$r_{\rm H}^{(2)}(x) \simeq \frac{a^2\,\alpha^2\,(9-2\,x^2)}{3\,a^2\,\alpha^4-8} \sim a^2\,\alpha^2\,\frac{2\,x^2-9}{8}, \tag{2.35b}$$

the latter of which is clearly negative for $\alpha^2\,a \ll 1$ (since $0 \le x^2 \le 1$). Therefore, we expect that there exists a horizon, whose location $r_{\rm H}^{(1)} \simeq r_{\rm H}$ is given exactly by the original (spherically symmetric) solution (2.27) for the unperturbed spacetime. This expectation will have to be confirmed from the study of expansions of null geodesics on $r_{\rm H}^{(1)} \simeq r_{\rm H}$, but we should also add that this calculation does not imply uniqueness and more marginally (outer) trapped surfaces could in principle develop.

2.3.1.1 Marginally trapped surfaces

Let us denote by S the surface defined by $r = r_{\rm H}^{(1)} \simeq \alpha^{-2}$. In the limit of small spheroidal deformation (i.e., for $\alpha^2\,a \ll 1$), one can easily obtain the tangent vectors to the outgoing and ingoing null geodesics on S, from the conditions that they are light-like, $\boldsymbol{\ell}^2 = \boldsymbol{n}^2 = 0$, and the normalization $\boldsymbol{\ell}\cdot\boldsymbol{n} = -1$.

To determine these null vectors, we need to study in some details the null geodesics for the metric (2.10) with h given in Eq. (2.34). Since the g_{xx} component of the metric (2.10) is not well defined at $x = 1$ ($\theta = 0$), it will be more convenient to work with the metric in the form given originally in Eq. (2.7). The Lagrangian is given by

$$2\,\mathcal{L} = g_{\mu\nu}\,\frac{dx^\mu}{d\lambda}\,\frac{dx^\nu}{d\lambda}, \tag{2.36}$$

where λ is the affine parameter along the geodesic. For the metric (2.7), it can be written as

$$\begin{aligned} 2\,\mathcal{L} = &-h(r,\theta;a)\,\dot{t}^2 + \frac{r^2+a^2\cos^2\theta}{r^2+a^2}\,\frac{\dot{r}^2}{h(r,\theta;a)} + \left(r^2+a^2\cos^2\theta\right)\dot{\theta}^2 \\ &+ \left(r^2+a^2\right)\sin^2\theta\,\dot{\phi}^2, \end{aligned} \tag{2.37}$$

where again overdot represents the derivative with respect to the parameter λ. Since t and ϕ are cyclic variables, one has the conserved conjugate momenta

$$p_t = -h(r,\theta;a)\,\dot{t} = -E, \tag{2.38}$$

$$p_\phi = \left(r^2 + a^2\right)\sin^2\theta\,\dot{\phi} = J, \tag{2.39}$$

with constants E and J representing energy and angular momentum of the particle, respectively. In particular one can always set $\dot{\phi} \sim J = 0$.

For purely radial geodesics to exist about S, the equation of motion for $\theta = \theta(\lambda)$ with $J = 0$, which reads

$$\begin{aligned} &2\,h^2\left[\left(r^2 + a^2\,\cos^2\theta\right)\ddot{\theta} + 2\,r\,\dot{r}\,\dot{\theta} - a^2\cos\theta\,\sin\theta\,\dot{\theta}^2\right] - E^2\,\partial_\theta h \\ &= \left[2\,a^2\,h\,\cos\theta\,\sin\theta + \left(r^2 + a^2\,\cos^2\theta\right)\partial_\theta h\right]\left(h\,\dot{\theta}^2 - \frac{E^2}{r^2 + a^2\,\cos^2\theta}\right), \end{aligned} \tag{2.40}$$

must admit solutions with $\theta(\lambda) = \theta_0$ and (at least locally) constant. We then notice that

$$\partial_\theta h \simeq \frac{a^2\,\alpha^2}{2\,r}\left(1 - \alpha^2\,r\right)\sin\theta\,\cos\theta, \tag{2.41}$$

so that Eq. (2.40) is trivially satisfied for $\theta = 0$ or $\theta = \pi$ (corresponding to a motion along the axis of symmetry) and for $\theta = \pi/2$ (motion on the equatorial plane). Moreover, for a general value of the angular coordinate θ, Eq. (2.41) ensures that $\partial_\theta h = h = 0$ on S (since this surface is defined by $\alpha^2\,r = 1$) and Eq. (2.40) again reduces to an identity on S. This shows that radial null geodesics exist everywhere in a neighbourhood of S and can be straightforwardly used to determine that S is indeed a MOTS.

After substituting

$$\dot{\theta} = 0, \quad J = 0, \quad \mathcal{L} = 0, \tag{2.42}$$

the Lagrangian (2.37) takes the form

$$2\,\mathcal{L} = \frac{1}{h(r,\theta;a)}\left(\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}\,\dot{r}^2 - E^2\right) = 0, \tag{2.43}$$

where we used the equation of motion (2.38). From Eq. (2.43), we then obtain

$$\frac{\dot{r}}{E} = \pm\sqrt{\frac{r^2 + a^2}{r^2 + a^2\cos^2\theta}}. \tag{2.44}$$

By using Eqs. (2.38) and (2.44), we can write the two normalized null tangent vectors as

$$\boldsymbol{\ell} = \frac{1}{2}\,\boldsymbol{\partial_t} + \frac{h(r,\theta;a)}{2}\sqrt{\frac{r^2 + a^2}{r^2 + a^2\,\cos^2\theta}}\,\boldsymbol{\partial_r}, \tag{2.45}$$

$$\boldsymbol{n} = \frac{1}{h(r,\theta;a)}\,\boldsymbol{\partial_t} - \sqrt{\frac{r^2+a^2}{r^2+a^2\,\cos^2\theta}}\,\boldsymbol{\partial_r}. \tag{2.46}$$

Now we determine the induced metric $q_{\mu\nu}$ on S. The non zero components are

$$q_{22} = r^2 + a^2\cos^2\theta, \tag{2.47}$$

$$q_{33} = \left(r^2+a^2\right)\sin^2\theta. \tag{2.48}$$

From Eq. (1.17) one can then conclude that

$$\Theta_{\boldsymbol{n}} = -\frac{r\left(2\,r^2+a^2+a^2\,\cos^2\theta\right)}{\sqrt{(r^2+a^2)(r^2+a^2\,\cos^2\theta)^3}}, \tag{2.49}$$

which is always negative, and $\Theta_{\boldsymbol{\ell}}$ is calculated as

$$\Theta_{\boldsymbol{\ell}} = -h(r,\theta;a)\,\frac{\Theta_{\boldsymbol{n}}}{2}, \tag{2.50}$$

which is positive for $r > r_{\rm H}^{(1)}$ and vanishes on the surface S, thus confirming our initial conjecture.

2.3.1.2 Misner-Sharp mass

In the example considered in this section, we have found two results:

i) The location of the MOTS is given by the same value of the radial coordinate as for the isotropic case (with $a = 0$). In particular, we have seen that

$$\Theta_{\boldsymbol{\ell}} = -h(r,\theta;a)\,\frac{\Theta_{\boldsymbol{n}}}{2}\ , \tag{2.51}$$

for all angles θ.

ii) The second result is that $h(r_{\rm H},\theta;a) = 0$ exactly where the spherically symmetric $h(r_{\rm H},\theta;a=0) = 0$.

Putting the two results together, we then find that

$$2\,m(r_{\rm H},\theta;a) = 2\,m(r_{\rm H}) = r_{\rm H}\ , \tag{2.52}$$

where $m(r) = m(r,\theta;a=0)$. We can therefore conjecture that the relevant mass function for determining the location of MOTS's in (slightly) spheroidal systems is given by the Misner-Sharp mass computed according to Eq. (2.5) on the reference isotropic spacetime. This conjecture is somewhat reminiscent of the property of the original Misner-Sharp mass that is given by the volume integral over the flat reference space.

2.3.2 A non-spheroidal source

The previous example had density that did not depend on the angle θ but the function h showed both radial and angular dependence i.e.

$$\rho = \rho(r), \quad h = h(r, \theta). \tag{2.53}$$

Now we modify our case by taking density to be of the form $\rho = \rho(r, \theta)$ and $h = h(r)$.

In this section, we want to consider the more complex case of a localised source of spheroidal radius $r = r_0$, with mass M_0 and charge Q, surrounded by its static electric field, with energy-momentum tensor $T^{(Q)}_{\mu\nu}$, and a suitable (electrically neutral) fluid. We are again not interested in the inner structure of the central source, but only in the portion of space for $r > r_0$, where we assume the metric to be of the form [48] given in Eq. (2.7) with

$$h(r, \theta) = h(r) = 1 - \frac{2\,M}{r} + \frac{Q^2}{r^2}. \tag{2.54}$$

The metric thus is of the form

$$\begin{aligned} \mathrm{d}s^2 = &- \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} \left(\frac{r^2 + a^2 \cos^2\theta}{r^2 + a^2}\right) \mathrm{d}r^2 \\ &+ \left(r^2 + a^2 \cos^2\theta\right) \mathrm{d}\theta^2 + \left(r^2 + a^2\right) \sin^2\theta \, \mathrm{d}\phi^2. \end{aligned} \tag{2.55}$$

It is clear that the deformation parameter a now measures deviations from the spherically symmetric Reissner-Nordström metric. In this respect, it is worth stressing that such a deformation should be regarded as a simple example of a (most likely) unstable intermediate configuration [13] within the framework of a dynamical gravitational collapse.

One can easily compute the corresponding energy-momentum tensor $T_{\mu\nu}$ by means of the EFEs. In particular, the (only) relevant component of the energy-momentum tensor, as far as our argument is concerned, is T^0_0. For completesness we give all the non-vanshing components of EFEs as follows

$$\begin{aligned} 8\pi T^0{}_0 = &- \frac{a^2\,M\left[\left(\cos^2\theta - 3\right)\left(r^2 + a^2\,\cos^2\theta\right) + 4\left(r^2 + a^2\right)\cos^2\theta\right]}{r\left(r^2 + a^2\cos^2\theta\right)^3} \\ &- \frac{Q^2}{\left(r^2 + a^2\,\cos^2\theta\right)^3}\left[2\left(r^2 + a^2\right) - r^2 - a^2\,\cos^2\theta\right], \end{aligned} \tag{2.56}$$

$$8\pi T^1{}_2 = \frac{a^2(Q^2 - Mr)(a^2 + r^2)\cos\theta\sin\theta}{r^3(r^2 + a^2\cos^2\theta)^2}, \tag{2.57}$$

$$8\pi T^1{}_1 = -\frac{Q^2(2r^2+3a^2+a^2\cos 2\theta)}{2r^2(r^2+a^2\cos^2\theta)^2} + \frac{a^2 M(3+\cos 2\theta)}{2r(r^2+a^2\cos^2\theta)^2}, \tag{2.58}$$

$$\begin{aligned} 8\pi T^2{}_2 &= \frac{Q^2(3a^4+9a^2r^2+2r^4+3a^4\cos 2\theta + a^2r^2\cos 2\theta)}{2r^4(r^2+a^2\cos^2\theta)^2} \\ &\quad - \frac{a^2 M(2a^2+7r^2+\cos 2\theta(2a^2+r^2))}{2r^3(r^2+a^2\cos^2\theta)^2}, \end{aligned} \tag{2.59}$$

$$\begin{aligned} 8\pi T^3{}_3 = {} & \frac{Q^2}{8r^4(r^2+a^2\cos^2\theta)^3}\Big[4a^2(3a^4+8a^2r^2+5r^4)\cos 2\theta + (3a^2+2r^2)(3a^4 \\ & + 8a^2r^2+4r^4+a^4\cos 4\theta)\Big] - \frac{a^2 M}{8r^3(r^2+a^2\cos^2\theta)^3}\Big[6a^4+23a^2r^2 \\ & + 12r^4 + 4(2a^4+6a^2r^2+5r^4)\cos 2\theta + a^2\cos 4\theta(2a^2+r^2)\Big]. \end{aligned} \tag{2.60}$$

Eqs. (2.56)-(2.60) show that this energy-momentum tensor has separate contributions proportional to the charge Q^2 and the mass M, respectively,

$$T^\mu_\nu = M\, T^{(M)\mu}{}_\nu + Q^2\, T^{(Q)\mu}{}_\nu, \tag{2.61}$$

which will be analyzed separately.

For the part of the energy-momentum tensor associated to M, we consider an anisotropic fluid form,

$$M\, T^{(M)\mu}{}_\nu = (\varrho + p)\, u^\mu\, u_\nu + p\, \delta^\mu_\nu + \Pi^\mu_\nu, \tag{2.62}$$

where ϱ is the energy density of the fluid, p the radial pressure, $\boldsymbol{u}$ the time-like 4-velocity of the fluid and $\Pi^{\mu\nu}$ is the traceless pressure tensor orthogonal to $\boldsymbol{u}$,

$$\Pi^\mu{}_\mu = \Pi_{\mu\nu}\, u^\nu = 0. \tag{2.63}$$

Since the system is static, we can take

$$\boldsymbol{u} = \left(1 - \frac{2\,M}{r} + \frac{Q^2}{r^2}\right)^{-1/2} \boldsymbol{\partial}_t. \tag{2.64}$$

Thus we can write the T^0_0 component (our main concern) as

$$8\pi T^0_0 = -\varrho + Q^2\, T^{(Q)0}{}_0. \tag{2.65}$$

To analyze the effects of the deviation a^2 on the electrostatic field, we expand the electrostatic part of the Eqs. (2.56)-(2.60) in powers of a^2 to obtain

$$8\pi\, {T^{(Q)}}^0{}_0 \simeq -\frac{Q^2}{r^4} + \frac{2a^2Q^2\cos 2\theta}{r^6} - \frac{3a^4Q^2\cos^2\theta(-1+3\cos 2\theta)}{2r^8} + o(a^6), \tag{2.66}$$

$$8\pi\, {T^{(Q)}}^1{}_2 \simeq \frac{a^2Q^2\cos\theta\sin\theta}{r^5} - \frac{a^4Q^2\sin 4\theta}{4r^7} + o(a^6), \tag{2.67}$$

$$8\pi\, {T^{(Q)}}^1{}_1 \simeq -\frac{Q^2}{r^4} - \frac{a^2Q^2\sin^2\theta}{r^6} - \frac{a^4Q^2\cos^2\theta(-3+\cos 2\theta)}{2r^8} + o(a^6), \tag{2.68}$$

$$8\pi\, {T^{(Q)}}^2{}_2 \simeq \frac{Q^2}{r^4} - \frac{a^2Q^2(-7+\cos 2\theta)}{2r^6} + \frac{a^4Q^2\cos^2\theta(-9+\cos 2\theta)}{2r^8} + o(a^6), \tag{2.69}$$

$$8\pi\, {T^{(Q)}}^3{}_3 \simeq \frac{Q^2}{r^4} + \frac{a^2Q^2(2+\cos 2\theta)}{r^6} - \frac{a^4Q^2\cos^2\theta(1+7\cos 2\theta)}{2r^8} + o(a^6). \tag{2.70}$$

We only wish to discuss what happens for small deviation from spherical symmetry, and will therefore assume $a^2 \ll Q^2 \ll r_0^2$. As a further simplification, we also take $Q^2 \ll M^2$, so that the Reissner-Nordström spacetime we deform is far from the extremal configuration and admits the two horizons

$$r_\pm = M \pm \sqrt{M^2 - Q^2}, \tag{2.71}$$

such that $h(r_\pm) = 0$.

If we then put together all the previous assumptions, in every expansion we will be allowed to neglect terms of order $a^2\, Q^2$ and higher. At leading order in a^2, we get

$$\varrho \simeq \frac{a^2\, M}{16\,\pi\, r^5}\,(5\,\cos 2\theta - 1)\,, \tag{2.72}$$

$${T^{(Q)}}^0{}_0 \simeq -\frac{1}{8\,\pi\, r^4}, \tag{2.73}$$

and the *total* energy density ρ, up to order a^2, is given by

$$\rho(r) \simeq \frac{1}{8\,\pi}\left[\frac{Q^2}{r^4} + \frac{a^2 M}{2\,r^5}\,(5\cos 2\theta - 1)\right], \tag{2.74}$$

from which one can easily see that the electrostatic contribution is constant on spheroids of constant r, whereas the contribution proportional to M is not. The electrostatic contribution falls within the treatment of the previous sections, and we are here particularly interested in analyzing the effects of the latter.

2.3.2.1 Marginally trapped surfaces

We denote by S the surfaces defined by $r = r_\pm$. Again as in the previous case, one then finds that the tangent vector to the outgoing null geodesics on S by first evaluating the geodesics through Lagrangian. The Lagrangian for metric (2.55) takes the form

$$
\begin{aligned}
2\,\mathcal{L} = & -h(r)\,\dot{t}^2 + \frac{r^2 + a^2\cos^2\theta}{r^2+a^2}\,\frac{\dot{r}^2}{h(r)} \\
& + \left(r^2 + a^2\cos^2\theta\right)\dot{\theta}^2 + \left(r^2+a^2\right)\sin^2\theta\,\dot{\phi}^2,
\end{aligned}
\tag{2.75}
$$

with $h = h(r)$ given in Eq. (2.54). Since t and ϕ are cyclic variables for the above Lagrangian we still have the conserved momenta

$$
p_t = -h(r)\,\dot{t} = -E, \tag{2.76}
$$

$$
p_\phi = \left(r^2+a^2\right)\sin^2\theta\,\dot{\phi} = J. \tag{2.77}
$$

We can again set $\dot{\phi} \sim J = 0$.

Whether the spacetime at hand admits radial geodesics can be determined from the dynamical equation for $\theta = \theta(\lambda)$ with $J = 0$, that is

$$
\left(r^2 + a^2\cos^2\theta\right)\ddot{\theta} = -2\,r\,\dot{r}\,\dot{\theta} - \frac{a^2\sin\theta\cos\theta}{r^2+a^2}\,\frac{\dot{r}^2}{h(r)} + a^2\sin\theta\cos\theta\,\dot{\theta}^2. \tag{2.78}
$$

In particular, upon setting $\dot{\theta} = 0$, we obtain

$$
\ddot{\theta} = -\frac{a^2\sin\theta\cos\theta}{\left(r^2+a^2\right)\left(r^2+a^2\cos^2\theta\right)}\,\frac{\dot{r}^2}{h(r)}, \tag{2.79}
$$

which yields $\ddot{\theta} = 0$ on the equatorial plane (at $\theta = \pi/2$) and along the axis of symmetry (at $\theta = 0$ or $\theta = \pi$). For a general angular coordinate θ, however, $\ddot{\theta}$ diverges for $r \to r_\pm$, unless $\dot{r}^2/h \simeq 0$ on S. Of course, the latter condition must be satisfied by outgoing null geodesics if S is indeed a horizon. We moreover note that $r = r(\lambda)$, with $\dot{\theta} = \dot{\phi} = 0$, can be obtained from $\mathcal{L} = 0$ and reads

$$
\dot{r}^2 = \frac{r^2+a^2}{r^2+a^2\cos^2\theta}\,h^2(r)\,\dot{t}^2 = \frac{r^2+a^2}{r^2+a^2\cos^2\theta}\,E^2, \tag{2.80}
$$

so that $\dot{r}^2 \propto h^2(r) = 0$ on S implies that $E = 0$ and $\dot{t}$ is arbitrary for $r = r_\pm$.

Let us compare with the behavior of null geodesics in the Schwarzschild spacetime, for which the Lagrangian is given by

$$
2\,\mathcal{L} = -U(r)\,\dot{t}^2 + \frac{\dot{r}^2}{U(r)} + r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right), \tag{2.81}
$$

with $U(r) = 1 - 2\,M/r$. One immediately finds the conserved quantities

$$U(r)\,\dot{t} = E, \tag{2.82}$$

$$r^2\,\sin^2\theta\,\dot{\phi} = J, \tag{2.83}$$

and the equation for $\theta = \theta(\lambda)$ then reads [3]

$$\ddot{\theta} = -\frac{2\,\dot{\theta}\,\dot{r}}{r} + \frac{J^2\,\cos\theta}{r^2\,\sin\theta}. \tag{2.84}$$

This equation clearly admits as solution $\ddot{\theta} = \dot{\theta} = 0$ for $\dot{\phi} \sim J = 0$. Hence, $\mathcal{L} = 0$ yields the radial equation of motion

$$\dot{r} = \pm E. \tag{2.85}$$

From Eq. (2.82), it is easy to see that for $U = 0$ one can set $E = \dot{r} = 0$ and therefore $r = 2\,M$ and constant is a solution. We conclude that $\dot{t}$ is arbitrary for null geodesics trapped on the surface S, as above.

Finally, we remark that radial geodesics with $E \neq 0$ do not show any pathology in Schwarzschild, since $\ddot{\theta}$ in Eq. (2.84) remains finite on S. The same geodesics should satisfy Eq. (2.79) in the metric Eq. (2.55), namely

$$\ddot{\theta} = -\frac{a^2\sin\theta\cos\theta}{(r^2 + a^2\cos^2\theta)^2}\,\frac{E^2}{h(r)}, \tag{2.86}$$

which instead diverges on S (with the exception of the equatorial plane and the symmetry axis).

From Eqs. (2.76) and (2.80), we can write two null orthonormal vectors as follows

$$\boldsymbol{\ell} = \frac{1}{2}\,\boldsymbol{\partial}_t + \frac{h(r)}{2}\,\sqrt{\frac{r^2 + a^2}{r^2 + a^2\,\cos^2\theta}}\,\boldsymbol{\partial}_r, \tag{2.87}$$

$$\boldsymbol{n} = \frac{1}{h}\,\boldsymbol{\partial}_t - \sqrt{\frac{r^2 + a^2}{r^2 + a^2\,\cos^2\theta}}\,\boldsymbol{\partial}_r. \tag{2.88}$$

Here again the 2-dimensional metric $q^{\mu\nu}$ is diagonal having components $q^{22} = g^{22}$ and $q^{33} = g^{33}$. Here, g^{22} and g^{33} are inverses of g_{22} and g_{33} components of the metric tensor respectively. The outgoing null expansion in this case is

$$\Theta_{\boldsymbol{\ell}} = -h(r)\,\frac{r\,\left(2\,r^2 + a^2 + a^2\cos^2\theta\right)}{\sqrt{(r^2 + a^2)(r^2 + a^2\,\cos^2\theta)^3}}, \tag{2.89}$$

which again vanishes on S.

[3] We do not employ the freedom to rotate the reference frame so that the geodesic motion occurs on the equatorial plane $\theta = \pi/2$ precisely for the purpose of comparing with the spheroidal case.

2.4 Hawking (-Hayward) mass

The general Hawking-Hayward mass [35, 36] is defined as the surface integral

$$m_H = \frac{A^{1/2}}{32\,\pi^{3/2}} \int_S v \left[\mathcal{R} + \Theta_+ \Theta_- - \frac{\sigma^+_{\alpha\beta}\,\sigma_-^{\alpha\beta}}{2} - 2\,\omega_\alpha\,\omega^\alpha \right], \tag{2.90}$$

where $\mathcal{R}$ denotes the induced Ricci scalar on the 2-surface S, $\Theta_{(\pm)}$ and $\sigma^{(\pm)}_{\alpha\beta}$ denote the expansion scalars and shear tensors of a pair of outgoing and ingoing null geodesic congruences from the surface S, respectively, ω_α is the projection onto S of the commutator of the null normal vectors to S, v is the volume 2-form on S and A the area of S. For the metric (2.55), one immediately finds that ω_α is of order a^2 and, since we are considering all expressions only up to order a^2, the last term can be dropped. The Hawking-Hayward mass then reduces to the Hawking mass, which we are now going to determine.

According to the general contracted Gauss equation [36]

$$\mathcal{R} + \Theta_+\,\Theta_- - \frac{1}{2}\,\sigma^+_{\alpha\beta}\,\sigma_-^{\alpha\beta} = q^{\alpha\mu}\,q^{\beta\delta}\,R_{\alpha\beta\mu\delta}, \tag{2.91}$$

where $q^{\alpha\mu}$ as defined in Chapter 1, is the induced metric on the 2-surface S having expression

$$q^{\alpha\mu} = g^{\alpha\mu} + l^\alpha\,n^\mu + l^\mu\,n^\alpha. \tag{2.92}$$

On expanding in powers of a^2,

$$q^{\alpha\mu} \simeq q_0^{\alpha\mu} + a^2 q_1^{\alpha\mu}, \tag{2.93}$$

$$R_{\alpha\beta\mu\delta} \simeq R_{\alpha\beta\mu\delta(0)} + a^2 R_{\alpha\beta\mu\delta(1)}, \tag{2.94}$$

we obtain

$$\begin{aligned} q^{\alpha\mu}\,q^{\beta\delta}\,R_{\alpha\beta\mu\delta} \simeq{}& q_0^{\alpha\mu}\,q_0^{\beta\delta}\,R_{\alpha\beta\mu\delta(0)} + a^2\,q_0^{\alpha\mu}\,q_0^{\beta\delta}\,R_{\alpha\beta\mu\delta(1)} \\ &+ a^2 \left(q_1^{\alpha\mu}\,q_0^{\beta\delta} + q_0^{\alpha\mu}\,q_1^{\beta\delta} \right) R_{\alpha\beta\mu\delta(0)}. \end{aligned} \tag{2.95}$$

The components of $q^{\alpha\mu}$ are determined from Eqs. (2.92) and (2.93), which give

$$\begin{aligned} q_0^{\alpha\mu} + a^2 q_1^{\alpha\mu} \simeq{}& g_0^{\alpha\mu} + a^2\,g_1^{\alpha\mu} + (l_0^\alpha + a^2\,l_1^\alpha)(n_0^\mu + a^2 n_1^\mu) \\ &+ (l_0^\mu + a^2 l_1^\mu)(n_0^\alpha + a^2 n_1^\alpha), \end{aligned} \tag{2.96}$$

where $g_0^{\alpha\mu}$ and $g_1^{\alpha\mu}$ are zeroth and first order terms of the metric tensor respectively, and similarly for l_0^α, l_1^α and n_0^α, n_1^α. For the unperturbed Reissner-Nordström spacetime

$l_0^\alpha = n_0^\alpha = 0$, for $\alpha = 2$ and 3, so that, up to order a^2, we have

$$q_0^{22} + a^2\, q_1^{22} = g_0^{22} + a^2 g_1^{22} + 2\, a^4\, l_1^2\, n_1^2 \simeq g_0^{22} + a^2\, g_1^{22},$$
$$q_0^{33} + a^2\, q_1^{33} = g_0^{33} + a^2 g_1^{33} + 2\, a^4\, l_1^3\, n_1^3 \simeq g_0^{33} + a^2\, g_1^{33}.$$

In particular, for the metric (2.55), we find

$$q^{22} \simeq \frac{1}{r^2} - a^2\, \frac{\cos^2\theta}{r^4}. \tag{2.97}$$

Similarly,

$$q^{33} \simeq \frac{1}{r^2 \sin^2\theta} - \frac{a^2}{r^4 \sin^2\theta}. \tag{2.98}$$

Eq. (2.95) now reduces to

$$\begin{aligned} q^{\alpha\mu}\, q^{\beta\delta}\, R_{\alpha\beta\mu\delta} \simeq & 2\left[q_0^{22}\, q_0^{33}\, R_{2323(0)} + a^2\, q_0^{22}\, q_0^{33}\, R_{2323(1)} \right. \\ & \left. + a^2 \left(q_1^{22}\, q_0^{33} + q_0^{22}\, q_1^{33}\right) R_{2323(0)}\right], \end{aligned} \tag{2.99}$$

where

$$\begin{aligned} R_{2323} &= \frac{\sin^2\theta (r^2 + a^2)(2Mr - Q^2)}{r^2 + a^2 \cos^2\theta} \\ &\simeq \left(2Mr - Q^2\right)\sin^2\theta + \frac{2a^2 M \sin^4\theta}{r}. \end{aligned} \tag{2.100}$$

By using Eqs. (2.97)-(2.100), the contracted Gauss equation (2.91) takes the form

$$\begin{aligned} q^{\alpha\mu}\, q^{\beta\delta}\, R_{\alpha\beta\mu\delta} &= \mathcal{R} + \theta_+\, \theta_- - \frac{1}{2}\, \sigma^+_{\alpha\beta}\, \sigma_-^{\alpha\beta} \\ &= \frac{2\left(2Mr - Q^2\right)}{r^4} - \frac{8a^2 M \cos^2\theta}{r^5}. \end{aligned} \tag{2.101}$$

Since $q_{22} = g_{22} = (r^2 + a^2\cos^2\theta)$ and $q_{33} = g_{33} = (r^2 + a^2)\sin^2\theta$, the volume 2-form

$$\begin{aligned} v &= \sqrt{\det(q_{\alpha\beta})}\, d\theta\, d\phi, \\ &\simeq \left(r^2 + a^2\, \frac{3 + \cos 2\theta}{4}\right) \sin\theta\, d\theta\, d\phi. \end{aligned} \tag{2.102}$$

The area of S is then given by

$$A = \int_S v \simeq 4\,\pi \left(r^2 + \frac{2}{3}\, a^2\right). \tag{2.103}$$

The Hawking mass is finally obtained from Eq. (2.90) with Eqs. (2.101) and (2.103), which yields

$$m_{\mathrm{H}}(r) \simeq M \left(1 + \frac{a^2}{3\,r^2}\right) - \frac{Q^2}{2\,r}. \tag{2.104}$$

We are next going to recover this result in a different way.

2.5 Adapted Misner-Sharp mass

The Misner-Sharp mass (2.5) is properly defined only for spherically symmetric spacetimes. One could generalise it by integrating the matter density on the spatial volume inside surfaces of symmetry, which are given by spheroids in the present case. In other words, we replace Eq. (2.5) with

$$m(r) = M_0 + 2\pi \int_0^{\pi} \int_{r_0}^{r} \sqrt{\varpi(\bar{r},\theta)}\, \rho(\bar{r},\theta)\, \mathrm{d}\bar{r}\, \mathrm{d}\theta, \tag{2.105}$$

where we recall that $r = r_0$ is the coordinate of the inner core, and $\varpi = (r^2 + a^2\cos^2\theta)^2 \sin^2\theta$ denotes the determinant of the flat 3-metric in spheroidal coordinates,

$$\varpi_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j = \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}\, \mathrm{d}r^2 + \left(r^2 + a^2\cos^2\theta\right) \mathrm{d}\theta^2 + \left(r^2 + a^2\right) \sin^2\theta\, \mathrm{d}\phi^2. \tag{2.106}$$

Eq. (2.105) then yields

$$m(r) \simeq M_0 - \frac{Q^2}{2}\left(\frac{1}{r} - \frac{1}{r_0}\right) + \frac{a^2\, M}{3}\left(\frac{1}{r^2} - \frac{1}{r_0^2}\right). \tag{2.107}$$

For $r \to \infty$, the above expression should equal the total ADM mass M, that is

$$M \simeq M_0 - \frac{M\, a^2}{3\, r_0^2} + \frac{Q^2}{2\, r_0}. \tag{2.108}$$

This allows us to express r_0 and M_0 so that Eq. (2.107) becomes

$$m(r) \simeq M \left(1 + \frac{a^2}{3\,r^2}\right) - \frac{Q^2}{2\,r} = m_{\mathrm{H}}(r), \tag{2.109}$$

from which the isotropic Misner-Sharp mass is obtained by taking $a^2 \to 0$.

This calculation therefore shows that, at least for spheroidal spacetimes like (2.55), one can expect the Hawking mass function evaluated on surfaces of symmetry to equal the adapted Misner-Sharp function evaluated inside volumes bounded by the same surfaces of symmetry.

2.6 Misner-Sharp and ADM mass

We should finally recall that the Misner-Sharp mass for the isotropic Reissner-Nordström spacetime is given by (see Section 2.5)

$$m(r) \simeq M - \frac{Q^2}{2\,r}, \tag{2.110}$$

and the condition $h(r_\pm) = 0$ that yields the horizons (2.71) can indeed be given in the form of Eq. (2.4), that is $2\,m(r_\pm) = r_\pm$. This implies that the results of the above analysis for the metric (2.55) do not really differ from those for the de Sitter spacetime in Section 2.3.1, and the isotropic Misner-Sharp mass remains a precious indicator of the location of horizons. In this perspective, it actually appears just like an accident that the asymptotic ADM mass computed for the isotropic reference spacetime (obtained by setting $a^2 = 0$) also determines the location of the horizons.

The conjecture that the isotropic Misner-Sharp mass determines the location of slightly spheroidal horizons nonetheless remains somewhat surprising, if one considers that the above isotropic Misner-Sharp mass $m = m(r)$ does not coincide with the Misner-Sharp mass adapted to the surfaces of symmetry of the spheroidal geometry. As shown above it coincides with the Hawking quasilocal mass for the system.

2.7 Horizon quantum mechanics of deformation parameter

In quantum physics, the energy density which defines the Misner-Sharp mass (and ADM mass M) becomes a quantum observable and the gravitational radius is expected to give a similar description. The horizon quantum mechanics (HQM) was in fact proposed [49, 50, 51, 52] for describing the "fuzzy" Schwarzschild (or gravitational) radius of a localised quantum source, by essentially lifting Eq. (2.4) to a quantum constraint acting on the state vectors of matter and the gravitational radius. In this respect, the HQM is different from most other attempts in which the gravitational degrees of freedom of the horizon, or of the black hole solution, are instead quantised independently of the state of the source. It however follows that, in order to extend the HQM to non-spherical systems, we need to identify a mass function from which the location of a MOTS, $r = r_{\rm H}$, can be uniquely determined and which depends only on the state of the matter source, like the Misner-Sharp mass (2.5) for isotropic sources. The latter property is crucial in a perspective in which one would eventually like to recover the geometric properties of spacetimes from the quantum state of the whole matter-gravity system.

The adapted Misner-Sharp mass and flat 3-dimensional metric are given in Eqs. (2.105) and (2.106) respectively. It is easy to see that we can split the general volume measure into two contributions, and similarly the adapted Misner-Sharp mass inside spheroids can also be split as

$$\tilde{m}(r) = \int \mu_0\, \rho + a^2 \int \mu_2\, \rho, \tag{2.111}$$

where $\rho = \rho(r)$ is the energy density, $\mu_0(r;\bar{r},\bar{\theta}) = \bar{r}^2 \sin\bar{\theta}\, \Theta(r-\bar{r})\, \mathrm{d}\bar{r}\, \mathrm{d}\bar{\theta}$ is the flat volume measure for spherical domains, and $\mu_2(r;\bar{r},\bar{\theta}) = \cos^2\bar{\theta}\, \sin\bar{\theta}\, \Theta(r-\bar{r})\, \mathrm{d}\bar{r}\, \mathrm{d}\bar{\theta}$ its deformation.

At the quantum level, if we wish to tell a spheroidal horizon from a spherical one, from a local perspective [50], we must analyse the modification to the energy spectrum of the source induced by the spheroidal deformation. As before, we will only consider small perturbations with respect to a spherical system, and replace the adapted Misner-Sharp (2.111) with the expectation value of the local Hamiltonian describing the quantum nature of the source,

$$\hat{H}(r) = \hat{H}^{(0)}(r) + \frac{a^2}{\Lambda^2}\, \hat{H}^{(2)}(r). \tag{2.112}$$

In practice, this quantum prescription implies the following replacements

$$\int \mu_0\, \rho \quad \overset{\text{HQM}}{\longrightarrow} \quad \langle\, \hat{H}^{(0)}(r)\, \rangle, \tag{2.113}$$

$$\int \mu_2\, \rho \quad \overset{\text{HQM}}{\longrightarrow} \quad \Lambda^{-2}\, \langle\, \hat{H}^{(2)}(r)\, \rangle, \tag{2.114}$$

where $\Lambda \simeq r_0$ is such that $\zeta \equiv a/\Lambda \ll 1$. One can then try and infer the structure of the spectrum $\sigma(\hat{H}(r)) = \{E_\alpha(r) \,|\, \alpha \in \mathcal{I}\}$, with $\mathcal{I}$ a discrete set of labels (due to the localised nature of the source), from the spectrum of the corresponding spherical system $\sigma(\hat{H}^{(0)}(r)) = \left\{E_\alpha^{(0)}(r) \,|\, \alpha \in \mathcal{I}\right\}$ using the standard perturbation theory. We expand the solution of the eigenvalue problem for $\hat{H}(r)$ as a Taylor series in ϵ, namely

$$E_\alpha(r) = E_\alpha^{(0)}(r) + \zeta^2\, E_\alpha^{(2)}(r) + \dots, \tag{2.115a}$$

and, omitting the radial dependence for the sake of brevity, we write the eigenvectors as

$$|\, E_\alpha\, \rangle = |\, E_\alpha^{(0)}\, \rangle + \zeta^2\, |\, E_\alpha^{(2)}\, \rangle + \dots\ . \tag{2.115b}$$

At order ζ^2, the perturbative solution is given by

$$E_\alpha \simeq E_\alpha^{(0)} + \zeta^2\, \langle\, E_\alpha^{(0)}\ |\ \hat{H}^{(2)}(r)\ |\ E_\alpha^{(0)}\, \rangle, \tag{2.116a}$$

and

$$| E_\alpha \rangle \simeq | E_\alpha^{(0)} \rangle + \zeta^2 \sum_{\beta\neq\alpha} \frac{\langle E_\beta^{(0)} \mid \hat{H}^{(2)}(r) \mid E_\alpha^{(0)} \rangle}{E_\alpha^{(0)} - E_\beta^{(0)}} | E_\beta^{(0)} \rangle. \tag{2.116b}$$

The above expressions allow us to introduce a characterization of the deformation parameter a^2 in a purely quantum framework.

In particular, Eq. (2.116b) implies

$$\frac{a^2}{\Lambda^2} \simeq \frac{E_\alpha - E_\alpha^{(0)}}{\langle E_\alpha^{(0)} \mid \hat{H}^{(2)}(r) \mid E_\alpha^{(0)} \rangle}. \tag{2.117}$$

Moreover, given a quantum state for the source, we can express it either using the deformed spectrum $\sigma(\hat{H}(r))$,

$$| \psi \rangle = \sum_\alpha C_\alpha(r) \, | E_\alpha \rangle, \tag{2.118}$$

or the isotropic spectrum $\sigma(\hat{H}^{(0)}(r))$,

$$| \psi \rangle = \sum_\alpha C_\alpha^{(0)}(r) \, | E_\alpha^{(0)} \rangle, \tag{2.119}$$

and Eq. (2.116b) yields

$$C_\alpha(r) - C_\alpha^{(0)}(r) \simeq \frac{a^2}{\Lambda^2} \sum_{\beta\neq\alpha} \frac{\langle E_\beta^{(0)} \mid \hat{H}^{(2)}(r) \mid E_\alpha^{(0)} \rangle}{E_\alpha^{(0)} - E_\beta^{(0)}} C_\beta^{(0)}(r). \tag{2.120}$$

We now recall that the spectrum of the gravitational radius operator is related to the source through the Hamiltonian constraint [49, 50]

$$\left(\hat{R}_{\rm H}(r) - \frac{2\,\ell_{\rm p}}{m_{\rm p}} \hat{H}(r) \right) | \Psi \rangle = 0, \tag{2.121}$$

where $| \Psi \rangle = \sum_\alpha \mathcal{C}_{\alpha,\beta} | E_\alpha \rangle | R_{{\rm H},\beta} \rangle$, and one therefore finds $\mathcal{C}_{\alpha,\beta} = C_\alpha(r)\,\delta_{\alpha,\beta}$. One can then infer how the spheroidal deformation affects the spectrum of the gravitational radius and the form of the horizon wave function. Indeed, by means of the Hamiltonian constraint one can immediately see that

$$R_{{\rm H},\alpha} \simeq R_{{\rm H},\alpha}^{(0)} + \zeta^2 \, \langle R_{{\rm H},\alpha}^{(0)} \, | \, \hat{R}_{\rm H}^{(2)}(r) \, | \, R_{{\rm H},\alpha}^{(0)} \rangle, \tag{2.122}$$

with $\hat{R}_{\rm H}^{(2)}(r) \equiv 2\,\ell_{\rm p}\,\hat{H}^{(2)}(r)/m_{\rm p}$. Furthermore, recalling that the state of the geometry in this language can be obtained by tracing away the contribution of the source from the general state $|\,\Psi\,\rangle$, one obtains [49, 50]

$$|\,\psi_{\rm H}\,\rangle = \sum_\alpha C_\alpha(r)\,|\,R_{{\rm H},\alpha}\,\rangle, \tag{2.123}$$

and the horizon wave function is finally given by

$$\begin{aligned} \psi_{\rm H}(R_{{\rm H},\alpha}) &= \langle\, R^{(0)}_{{\rm H},\alpha}\,|\,\psi_{\rm H}\,\rangle = C_\alpha(r), \\ &\simeq C^{(0)}_\alpha(r) + \frac{a^2}{\Lambda^2}\sum_{\beta\neq\alpha}\frac{\langle\, R^{(0)}_{{\rm H},\beta}\,|\,\hat{H}^{(2)}(r)\,|\,R^{(0)}_{{\rm H},\alpha}\,\rangle}{R^{(0)}_{{\rm H},\alpha}-R^{(0)}_{{\rm H},\beta}}\,C^{(0)}_\beta(r). \end{aligned} \tag{2.124}$$

It will be interesting to apply the above treatment to specific cases, and also include rotation [53].

Chapter 3

Energy extraction and the centre of mass energy in slowly rotating Chern-Simons black holes

3.1 Introduction

One of the many interesting theories that modify gravity is the Chern-Simons (CS) theory [54, 55]. The action for a CS gravity is written as [56]

$$S = k\int\sqrt{-g}Rdx^4 + \frac{\gamma}{4}\int dx^4\sqrt{-g}\varphi^*RR - \frac{1}{2}\int\sqrt{-g}(\nabla\varphi)^2dx^4, \tag{3.1}$$

where $k = 1/16\pi$, φ is a scalar field, g is the determinant of the metric tensor $g_{\mu\nu}$, $R = g^{\lambda\alpha}R_{\lambda\alpha}$ is the Ricci scalar with $R_{\lambda\alpha}$ being the Ricci tensor, ${}^*RR$ is the Pontryagin density defined as

$${}^*RR = {}^*R^{\mu\rho\sigma}_{\nu}R^{\nu}_{\mu\rho\sigma}, \tag{3.2}$$

where the dual Riemann tensor is

$${}^*R^{\mu\rho\sigma}_{\nu} = \frac{1}{2}\varepsilon^{\rho\sigma\delta\gamma}R^{\mu}_{\nu\delta\gamma}. \tag{3.3}$$

Here $\varepsilon^{\rho\sigma\delta\gamma}$ represents the 4 dimensional Levi-Civita tensor. The first term is the standard Einstein-Hilbert action, the second term is the CS correction term and the third term is the scalar field term. CS theory has two types of independent theoretical formulations, namely, dynamical and non-dynamical. In the non-dynamical theory, the CS scalar is *a priori* prescribed function with the evolution equation becoming a differential constraint on the space of allowed solutions. On the other hand, the dynamical theory has CS scalar as a dynamical field having its own evolution equation and energy-momentum tensor. In recent years there has been an increasing interest in the dynamical CS theory.

In Refs. [57, 58, 59, 60], non-dynamical black hole solutions have been developed. The solution in Refs. [57, 58] is determined by employing far-field approximation and φ (the CS scalar field) being linearly proportional to the asymptotic time coordinate t. This solution is stationary but not axisymmetric and gives correction to the frame dragging effect. The slow rotation approximation was employed to obtain a rotating black hole solution [59]. An exact rotating solution in non-dynamical theory is given in Ref. [60] which is stationary and axisymmetric for arbitrary φ. In dynamical theory, the solution in slow rotation approximation and small coupling constant has been discussed in Chapter 1 (Eqs. (1.31)-(1.32)). The solution was extended to include the terms for second order in spin parameter in Ref. [61]. The solution in slow rotation approximation upto the 5^{th} order in spin parameter has also been found [62]. The assumption of slow rotation was relaxed in Ref. [63] where CS scalar field induced by a (rapidly) rotating black hole (Kerr metric) in the dynamical theory was considered.

In this chapter, we investigate black holes under slow rotation approximation in dynamical CS theory and study their different properties. Our focus is to see how the coupling constant of CS theory effects different physical phenomena e.g. quasilocal mass, particle motion and energy extraction process. We will show that the Hawking mass and efficiency of the Penrose process do not depend on the coupling constant, whereas, it is interesting to note that the centre of mass energy shows a behavior that is dependent on the coupling constant. All the computations are done to the second order in the spin parameter and to the order $a\gamma^2$ in the coupling constant where a is the spin parameter and γ is CS coupling constant.

This chapter is organized as follows. In Section 3.2 we will discuss Hawking mass in CS theory and study its relation to the coupling constant. Section 3.3 provides an analytical expression for the centre of mass energy for two colliding neutral particles as a function of CS coupling constant. In Section 3.4 energy extraction through Penrose process is discussed.

3.2 Hawking mass

In this section we consider the Hawking mass. Let S be a spacelike 2-surface having area A. Consider a null tetrad denoted by $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ on S. Here l^μ and n^μ respectively represent outgoing and the ingoing future directed null vectors orthogonal to S and

$m^\mu, \bar{m}^\mu$ are the tangent vectors to S. On S, the Hawking mass is given by [35]

$$m_H(S) = \sqrt{\frac{A}{(4\pi)^3}}\ \left[2\pi + \int_S \rho\acute{\rho} dS\right]. \tag{3.4}$$

In the above expression ρ and $\acute{\rho}$ denote the spin coefficients of the Newman-Penrose formalism [64]. They determine the expansions of the outgoing and ingoing null cones.

Hawking mass on the event horizons of Reissner-Nordström and Kerr black holes is discussed in Ref. [65]. We discuss Hawking mass for spacetime (1.31) for the regions outside the event horizon.

First we review briefly the null geodesics for metric (1.31) [56, 66]. The geodesics for a spacetime geometry can be determined by Hamilton-Jacobi equation

$$\frac{\partial \hat{S}}{\partial \lambda} = -\frac{1}{2} g^{\mu\nu} \frac{\partial \hat{S}}{\partial x^\mu} \frac{\partial \hat{S}}{\partial x^\nu}, \tag{3.5}$$

where $\hat{S}$ is the Jacobi action. Let

$$\hat{S} = \frac{m^2\lambda}{2} - Et + J\phi + \hat{S}_r(r) + \hat{S}_\theta(\theta). \tag{3.6}$$

For null geodesics, we take particle's mass m to be zero. Substitution of Eq. (3.6) and the values of the inverse metric in Eq. (3.5) leads to a separable equation of the form

$$\begin{aligned}\Delta\left(\frac{d\hat{S}_r}{dr}\right)^2 + \left(\frac{d\hat{S}_\theta}{d\theta}\right)^2 + E^2\left(\frac{r^3}{2M - r} + \frac{4a^2M^2}{(r-2M)^2}\right) - E^2a^2\cos^2\theta \\ + J^2\csc^2\theta + J^2 - \frac{J^2a^2}{r(r-2M)} + \frac{4aMJE}{(r-2M)} - \frac{20a\pi JE\gamma^2}{r^3(r-2M)}\left(1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\right),\end{aligned} \tag{3.7}$$

where $\Delta = r^2 + a^2 - 2Mr$.
Eq. (3.7) is separable in variables r and θ and can be set equal to some separation constant K. Simplification leads to

$$\frac{1}{E^2}\left(\frac{d\hat{S}_\theta}{d\theta}\right)^2 = \eta + a^2\cos^2\theta - \xi^2\cot^2\theta = \Theta(\theta), \tag{3.8}$$

and

$$\begin{aligned}\frac{\Delta}{E^2}\left(\frac{d\hat{S}_r}{dr}\right)^2 = -\eta - \xi^2 + \frac{\xi^2a^2}{r^2 - 2Mr} - \frac{4aM\xi}{r - 2M} + \frac{r^3}{r-2M} - \frac{4a^2M^2}{(r-2M)^2} \\ + \frac{20a\pi\xi\gamma^2}{r^3(r-2M)}\left(1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\right),\end{aligned} \tag{3.9}$$

where we have denoted $\eta = K/E^2$ and $\xi = J/E$. Multiplying the above equation by Δ and further simplification leads to the equation

$$\frac{\Delta^2}{E^2}\left(\frac{d\hat{S}_r}{dr}\right)^2 = r^4 - \big(r^2 - 2Mr\big)(\eta + \xi^2) - 4aMr\xi + \frac{20a\pi\xi\gamma^2}{r^2}\left(1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\right) + a^2\big(r^2 + 2Mr - \eta\big) = R(r). \tag{3.10}$$

In order to determine the equations involving the derivatives of t and ϕ, we determine the Lagrangian $\mathcal{L}$ of the system, which for the metric (1.31) is

$$2\mathcal{L} = g_{00}\frac{dx^0}{d\lambda}\frac{dx^0}{d\lambda} + 2g_{03}\frac{dx^0}{d\lambda}\frac{dx^3}{d\lambda} + g_{11}\frac{dx^1}{d\lambda}\frac{dx^1}{d\lambda} + g_{22}\frac{dx^2}{d\lambda}\frac{dx^2}{d\lambda} + g_{33}\frac{dx^3}{d\lambda}\frac{dx^3}{d\lambda}. \tag{3.11}$$

The conserved conjugate momenta p_t and p_ϕ are

$$p_t = g_{00}\dot{t} + g_{03}\dot{\phi} = -E \ , \tag{3.12}$$

$$p_\phi = g_{03}\dot{t} + g_{33}\dot{\phi} = J \ . \tag{3.13}$$

By solving the above equations, we thus obtain

$$\dot{t} = \frac{g_{33}E + Jg_{03}}{g_{03}^2 - g_{33}g_{00}}, \tag{3.14}$$

$$\dot{\phi} = -\frac{g_{03}E + Jg_{00}}{g_{03}^2 - g_{33}g_{00}}. \tag{3.15}$$

Substituting the values in the above equations we get

$$\dot{t} = \frac{rE}{r - 2M} - \frac{2aML}{r^2(r - 2M)} + \frac{10a\pi\gamma^2 L}{r^5\big(r - 2M\big)} - \frac{2Ma^2E\big(r\cos^2\theta + 2M\sin^2\theta\big)}{r^2\big(r - 2M\big)^2}, \tag{3.16}$$

$$\dot{\phi} = \frac{J}{r^2\sin^2\theta} + \frac{2aME}{r^2\big(r - 2M\big)} - \frac{10a\pi\gamma^2 E}{r^5\big(r - 2M\big)} + \frac{a^2J\big(M - r + M\cos 2\theta\big)}{r^4\sin^2\theta\big(r - 2M\big)}. \tag{3.17}$$

In terms of η and ξ, the above expressions take the form

$$\frac{\dot{t}}{E} = \frac{r}{r - 2M} - \frac{2aM\xi}{r^2(r - 2M)} + \frac{10a\pi\gamma^2\xi}{r^5(r - 2M)} - \frac{2Ma^2}{r^2(r - 2M)^2}(r\cos^2\theta + 2M\sin^2\theta), \tag{3.18}$$

$$\frac{\dot{\phi}}{E} = \frac{\xi}{r^2\sin^2\theta} + \frac{2aM}{r^2(r - 2M)} - \frac{10a\pi\gamma^2}{r^5(r - 2M)} + \frac{a^2\xi(M - r + M\cos 2\theta)}{r^4\sin^2\theta(r - 2M)}. \tag{3.19}$$

The expressions for $\dot{r}$ and $\dot{\theta}$ are determined from the relations $d\hat{S}/dr = p_r = \dot{r}g_{11}$ and $d\hat{S}/d\theta = p_\theta = g_{22}\dot{\theta}$ respectively, where p_r and p_θ are the conjugate momenta. Thus we have

$$\frac{\dot{r}}{E} = \pm\frac{1}{\Delta g_{11}}\sqrt{R}, \tag{3.20}$$

$$\frac{\dot{\theta}}{E} = \pm\frac{\sqrt{\Theta}}{g_{22}}. \tag{3.21}$$

We can write $\Theta(\theta) = \eta + (\xi - a)^2 - \csc^2\theta(\xi - a\sin^2\theta)^2$. For $\Theta \geqslant 0$, we have

$$(\eta + \xi^2 - a\xi) \geqslant 0, \tag{3.22}$$

where the equality holds for

$$\xi = a\sin^2\theta, \tag{3.23}$$

$$\theta = \theta_0 = constant. \tag{3.24}$$

In this case we have

$$\eta = -a^2\cos^4\theta. \tag{3.25}$$

With these values of ξ and η, the equations of motion take the form

$$\dot{t} = \frac{r}{r-2M} - \frac{2Ma^2}{r\big(r-2M\big)^2}, \tag{3.26}$$

$$\dot{r} = \pm 1, \tag{3.27}$$

$$\dot{\theta} = 0, \tag{3.28}$$

$$\dot{\phi} = \frac{a}{r\big(r-2M\big)} - \frac{10a\pi\gamma^2}{r^5\big(r-2M\big)}\left(1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\right), \tag{3.29}$$

where we have taken $E = 1$ in the above equations. From Eqs. (3.26)-(3.29) we can write outgoing and ingoing null 4-vectors as

$$l^\mu = \left(\frac{r}{r-2M} - \frac{2Ma^2}{r\big(r-2M\big)^2}, 1, 0, \frac{a}{r\big(r-2M\big)} - \frac{10a\pi\gamma^2}{r^5\big(r-2M\big)}\left[1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\right]\right),$$

$$n^\mu = \left(\frac{1}{2} + \frac{a^2\sin^2\theta}{2r^2}, -\frac{\big(r-2M\big)}{2r} + \frac{a^2\big(r-2M\big)\cos^2\theta}{2r^3} - \frac{a^2}{2r^2}, 0, \frac{a}{2r^2} - \frac{5a\pi\gamma^2}{r^6}\left[1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\right]\right),$$

where we have multiplied the ingoing null 4-vector by $(r - 2m)/2r + a^2/2r^2 - a^2(r - 2M)\cos^2\theta/2r^3$ for the orthogonality condition $\mathbf{l.n} = \mathbf{-1}$ to hold and called it n^μ. To determine complex null 4-vector, we will employ certain properties that a null tetrad and a metric tensor satisfy in Newman-Penrose formalism. Let us denote the complex null vector by **m**. In the component form it is represented as

$$m^\mu = (A, B, C, D), \tag{3.30}$$

and its complex conjugate as $\bar{m}^\mu = (\bar{A}, \bar{B}, \bar{C}, \bar{D})$. The null vectors m^μ and $\bar{m}^\mu$ satisfy the condition $\mathbf{m}.\bar{\mathbf{m}} = 1$. In terms of the null tetrad $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$, the inverse metric tensor can be written as [64]

$$g^{\mu\nu} = -l^\mu n^\nu - l^\nu n^\mu + m^\mu \bar{m}^\nu + m^\nu \bar{m}^\mu. \tag{3.31}$$

By employing this and the orthogonality conditions $m_\mu m^\mu = 0 = l_\mu m^\mu$, one can determine the components of the null vector m^μ and its complex conjugate $\bar{m}^\mu$. The expression of m^μ is given as

$$m^\mu = \left(\frac{ia\sin\theta}{\sqrt{2}r} + \frac{a^2\cos\theta\sin\theta}{\sqrt{2}r^2}, 0, \frac{1}{\sqrt{2}}\left[\frac{1}{r} - \frac{ia\cos\theta}{r^2} - \frac{a^2\cos^2\theta}{r^3}\right], \frac{i}{\sqrt{2}r\sin\theta} + \frac{a\cos\theta}{\sqrt{2}r^2\sin\theta} - \frac{ia^2\cos^2\theta}{\sqrt{2}r^3\sin\theta} \right).$$

By replacing i by $-i$, $\bar{m}^\mu$ can be obtained. The null vectors $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ will be used to determine the spin coefficients ρ and ρ' appearing in the expression of the Hawking mass. The formula for ρ is $\nabla_\mu l_\nu m^\nu \bar{m}^\mu$ and for ρ', it is $\nabla_\mu n_\nu \bar{m}^\nu m^\mu$, where ∇_μ denotes the covariant derivative. By substituting the values, we obtain the expressions for ρ and ρ' as follows

$$\rho = \frac{1}{r} + \frac{ia\cos\theta}{r^2} - \frac{a^2\cos^2\theta}{r^3}, \tag{3.32}$$

$$\rho' = -\frac{\left(r-2M\right)}{2r^2} - \frac{ia\cos\theta\left(r-2M\right)}{2r^3} + \frac{a^2}{2r^4}\left[2r\cos^2\theta - 4M\cos^2\theta - r\right]. \tag{3.33}$$

From these expressions, it is clear that **l** and **n** are not orthogonal to S. For this purpose the coordinate system needs to be rotated such that

$$m^0 = 0 = m^1. \tag{3.34}$$

Then ρ and $\acute{\rho}$ become real. Two rotations are performed for this purpose. First we do a type II rotation [3]

$$n^\mu \to n^\mu, \quad m^\mu \to m^\mu + \beta_1 n^\mu, \quad \bar{m}^\mu \to \bar{m}^\mu + \bar{\beta}_1 n^\mu, \quad l^\mu \to l^\mu + \bar{\beta}_1 m^\mu + \beta_1 \bar{m}^\mu + \beta_1\bar{\beta}_1 n^\mu, \tag{3.35}$$

and then a type I rotation [3]

$$l^\mu \to l^\mu, \quad m^\mu \to m^\mu + \beta_2 l^\mu, \quad \bar{m}^\mu \to \bar{m}^\mu + \bar{\beta}_2 l^\mu, \quad n^\mu \to n^\mu + \bar{\beta}_2 m^\mu + \beta_2 \bar{m}^\mu + \beta_2 \bar{\beta}_2 l^\mu, \tag{3.36}$$

where β_1 and β_2 are complex functions. These two rotations lead to new rotated tetrad given as

$$l^\mu \to l^\mu + \bar{\beta}_1 m^\mu + \beta_1 \bar{m}^\mu + \beta_1 \bar{\beta}_1 n^\mu, \tag{3.37}$$

$$n^\mu \to n^\mu + \bar{\beta}_2 \left(m^\mu + \beta_1 n^\mu\right) + \beta_2 \left(\bar{m}^\mu + \bar{\beta}_1 n^\mu\right) + \beta_2 \bar{\beta}_2 \left(l^\mu + \bar{\beta}_1 m^\mu + \beta_1 \bar{m}^\mu + \beta_1 \bar{\beta}_1 n^\mu\right), \tag{3.38}$$

$$m^\mu \to m^\mu(1 + \beta_2 \bar{\beta}_1) + \beta_2 \beta_1 \bar{m}^\mu + n^\mu(\beta_1 + \beta_2 \beta_1 \bar{\beta}_1) + \beta_2 l^\mu. \tag{3.39}$$

From Eq. (3.39) the conditions $m^0 = 0 = m^1$ give two equations

$$\beta_2 = -\frac{\beta_1 n^1}{l^1 + \beta_1 \bar{\beta}_1 n^1}, \tag{3.40}$$

$$n^1 \beta_1^2 \bar{m}^0 + \beta_1(n^1 l^0 - n^0 l^1) - m^0 l^1 = 0. \tag{3.41}$$

Eq. (3.41) is quadratic in the parameter β_1 and has two solutions

$$\begin{aligned} \beta_{1a} = & \frac{\left(ir + a\cos\theta\right)\csc\theta}{2\sqrt{2}ar^2\left(r - 2M\right)^2} \Bigg\{ 8\left(2M - r\right)r^3 + a^2\left(4M^2 + 4Mr + 5r^2\right) \\ & - a^2\left(4M^2 + 4Mr - 3r^2\right)\cos 2\theta \Bigg\}, \end{aligned} \tag{3.42}$$

$$\beta_{1b} = -\frac{a\left(ir + a\cos\theta\right)\sin\theta}{\sqrt{2}r^2}. \tag{3.43}$$

The γ-independent behavior of β_1 is due to the fact that only l^3 and n^3 components of the null vectors l^μ and n^μ depend on γ which are not involved in the equation for β_1. In the Schwarzschild spacetime the spin coefficients are real, therefore, the null tetrad need not be rotated. Both β_1 and β_2 are zero there. From Eq. (3.42) it is clear that the limit $a \to 0$ does not correspond to a physical situation and must be ignored. But β_{1b} tends to zero as $a \to 0$ giving the Schwarzschild result. Therefore, we select this value of β_1. With this choice of β_1, the expression for β_2 given in Eq. (3.40) becomes

$$\beta_2 = \frac{-i\left(r - 2M\right)a\sin\theta}{2\sqrt{2}r^2} - \frac{a^2\left(r - 2M\right)\cos\theta\sin\theta}{2\sqrt{2}r^3}. \tag{3.44}$$

By using Eqs. (3.43)-(3.44), Eqs. (3.37)-(3.39) take the form

$$l^\mu = \Bigg(\frac{r}{r-2M} - \frac{2Ma^2}{r\Big(r-2M\Big)^2} - \frac{3a^2\sin^2\theta}{4r^2}, 1 - \frac{a^2\Big(r-2M\Big)\sin^2\theta}{4r^3}, 0, \frac{2Ma}{r^2\Big(r-2M\Big)} - \frac{10a\pi\gamma^2}{r^5\Big(r-2M\Big)}\Big[1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\Big]\Bigg), \tag{3.45}$$

$$n^\mu = \Bigg(\frac{1}{2} + \frac{\Big(3r+2M\Big)a^2\sin^2\theta}{8r^3}, -\frac{r-2M}{2r} + a^2\Big[- \frac{\Big(r-2M\Big)^2\sin^2\theta}{8r^4} + \frac{\cos^2\theta\Big(r-2M\Big)}{2r^3} - \frac{1}{2r^2}\Big], 0, \frac{Ma}{r^3} - \frac{5a\pi\gamma^2}{r^6}\Big[1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\Big]\Bigg), \tag{3.46}$$

$$m^\mu = \Bigg(0, 0, \frac{1}{\sqrt{2}}\Big[\frac{1}{r} - \frac{ia\cos\theta}{r^2} - \frac{a^2\cos^2\theta}{r^3}\Big], \frac{1}{\sqrt{2}}\Big[\frac{i}{r\sin\theta} + \frac{a\cot\theta}{r^2} - \frac{ia^2\cos^2\theta}{r^3\sin\theta} - \frac{ia^2\sin\theta\Big(r+2M\Big)}{2r^4}\Big]\Bigg). \tag{3.47}$$

The spin coefficients ρ and ρ' as determined from the above new null tetrad are

$$\rho = \frac{1}{r} + \frac{a^2}{4r^4}\Bigg[- r\cos^2\theta - 3r - 4M + 4M\cos^2\theta \Bigg], \tag{3.48}$$

$$\rho' = -\frac{1}{2}\frac{\Big(r-2M\Big)}{r^2} + \frac{a^2\Big(r-4M\Big)}{2r^4} - \frac{a^2\sin^2\theta}{8r^5}\Big(7r^2 + 16M^2 - 22Mr\Big). \tag{3.49}$$

Consider the surface defined by $r =$constant and $t =$constant. The induced metric for the surface in this case has the components given by g_{22} and g_{33} of the metric in Eq. (1.31). The surface area element is

$$dS = \sqrt{g_{22}g_{33}}d\theta d\phi = \sin\theta\sqrt{\Sigma\Big(r^2 + a^2 + \frac{2a^2M\sin^2\theta}{r}\Big)}d\theta d\phi, \tag{3.50}$$

where $\Sigma = r^2 + a^2\cos^2\theta$. The surface area is $A = \int_0^\pi\int_0^{2\pi} dS$. From Eqs. (3.48)-(3.50) it is clear that ρ, ρ' and the area element are independent of the CS coupling constant γ. As a result, the Hawking mass is independent of γ and is given by

$$m_H = M - \frac{M^2a^2}{r^3}. \tag{3.51}$$

This shows the dependence of Hawking mass on the mass M of the black hole, spin parameter a and radius r upto order a^2. This result matches with the Hawking mass for the Kerr metric [68]. It also shows that Hawking mass is independent of the CS coupling constant γ. Here it is important to mention that upto the order $a\gamma^2$, the scalar field φ

does not contribute to the total energy of the spacetime [22]. This is also evident from Eq. (3.51) which shows the γ independnt behavior.

3.3 The centre of mass energy

In this section, we calculate the centre of mass energy E_{CM} for the collision of two neutral particles with equal masses i.e. $m_1 = m_2 = m_0$ in the vicinity of the slowly rotating Chern-Simons black hole. The particles are moving from infinity with equal energies $E_1/m_1 = E_2/m_2 = 1$ towards the black hole having different angular momenta J_1 and J_2. The motion and the collision of the particles takes place in the equatorial plane ($\theta = \pi/2$). The expression for the E_{CM} given by Bañados, Silk and West (BSW) [44] is

$$\frac{E_{CM}^2}{2m_0^2} = 1 - g_{\mu\nu} u_1^\mu u_2^\nu, \tag{3.52}$$

where $u_1^\mu = (\dot{t}_1, \dot{r}_1, \dot{\theta}_1, \dot{\phi}_1)$ and $u_2^\mu = (\dot{t}_2, \dot{r}_2, \dot{\theta}_2, \dot{\phi}_2)$ represent the 4-velocity of the first and second particle, respectively, with the overdot representing the derivative with respect to the proper time τ. This formula is valid both for curved and flat spacetimes. In the equatorial plane $\dot{\theta}_1 = \dot{\theta}_2 = 0$. By Giving variation 0-3 to indices μ and ν in Eq. (3.52), one obtains

$$\frac{E_{CM}^2}{2m_0^2} = 1 - \Big(g_{00}\dot{t}_1 + g_{03}\dot{\phi}_1\Big)\dot{t}_2 - g_{11}\dot{r}_1\dot{r}_2 - \dot{\phi}_2\Big(g_{03}\dot{t}_1 + g_{33}\dot{\phi}_1\Big). \tag{3.53}$$

The time-like geodesics for a particle of mass m are [66, 67]

$$\frac{dt}{d\tau} = \frac{r\varepsilon}{r-2M} - \frac{2a\pounds M}{r^2\Big(r-2M\Big)} + \frac{10a\pi\pounds\gamma^2}{r^5\Big(r-2M\Big)}\left[1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\right] - \frac{4\varepsilon M^2 a^2}{r^2\Big(r-2M\Big)^2}, \tag{3.54}$$

$$\frac{d\phi}{d\tau} = \frac{\pounds}{r^2} + \frac{2a\varepsilon M}{r^2\Big(r-2M\Big)} - \frac{10a\pi\varepsilon\gamma^2}{r^5\Big(r-2M\Big)}\left[1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\right] - \frac{\pounds a^2}{r^3\Big(r-2M\Big)}, \tag{3.55}$$

$$\begin{aligned}\Big(\frac{dr}{d\tau}\Big)^2 = {}& \varepsilon^2 + \frac{\Big(2M-r\Big)\Big(\pounds^2+r^2\Big)}{r^3} - \frac{4aM\varepsilon\pounds}{r^3} + \frac{20a\pi\gamma^2}{r^6}\varepsilon\pounds\left[1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\right] \\ & + a^2\Big(\frac{-r+\varepsilon^2\Big(r+2M\Big)}{r^3}\Big),\end{aligned} \tag{3.56}$$

where τ is the proper time, $\varepsilon = E/m$, and $\pounds = J/m$ denote the specific energy and angular momentum of the particle per unit mass, respectively. These equations are velocity components of a particle of mass m. After substituting the values of the components of the 4-velocities of particles from Eqs. (3.54)-(3.56), we obtain the expression for the E_{CM}^2 as

$$\frac{E_{CM}^2}{2m_0^2}=1-\frac{\pounds_1\pounds_2}{r^2}+\frac{r}{r-2M}-\frac{\sqrt{S_1S_2}}{r^2\Big(r-2M\Big)}+a\Bigg[-14Mr^5+189M^2\gamma^2\pi+120Mr\gamma^2\pi+70\pi r^2\gamma^2\Bigg]$$
$$\times\left[\frac{\pounds_1+\pounds_2}{7r^7\Big(r-2M\Big)}-\frac{2\pounds_1^2\pounds_2M+2\pounds_2^2\pounds_1M-\pounds_1^2\pounds_2r-\pounds_2^2\pounds_1r+2\pounds_1r^2M+2\pounds_2r^2M}{7r^7\Big(r-2M\Big)\sqrt{S_1S_2}}\right]$$
$$+\frac{a^2}{2r^4\Big(r-2M\Big)^2}\Bigg[-8M^2r^2+2\pounds_1\pounds_2\Big(r^2-2Mr\Big)-\frac{8\pounds_1\pounds_2M^2r^2\Big(r-2M\Big)}{\sqrt{S_1S_2}}$$
$$+r\sqrt{S_1S_2}\Big[\frac{2\pounds_1^2Mr^2\Big(2M-r\Big)+8M^3r^3+\pounds_1^4\Big(r-2M\Big)^2}{S_1^2}$$
$$+\frac{2\pounds_2^2Mr^2\Big(2M-r\Big)+8M^3r^3+\pounds_2^4\Big(r-2M\Big)^2}{S_2^2}\Big]\Bigg],$$

where $\pounds_1$ and $\pounds_2$ are the angular momenta of the particles, $S_1=2Mr^2-\pounds_1^2\,(r-2M)$, $S_2=2Mr^2-\pounds_2^2\,(r-2M)$. For $a=0=\gamma$, we recover the expression for the centre of mass energy for the Schwarzschild metric. The above expression shows that the centre of mass energy depends on rotation and the coupling constant γ^2. The $r\to\infty$ limit of the above expression gives $E_{CM}=2m_0$, which is the same as if the particles are colliding in a flat spacetime. The event horizon of the slowly rotating CS black hole is at $r_H=r_{H(Kerr)}$ where $r_{H(Kerr)}$ is the event horizon of the Kerr metric [22]. To the required order in the spin parameter, the event horizon can be written as $r_H\simeq 2M-a^2/2M$. From Eq. (**??**), we see that the centre of mass energy is finite at $r=r_H$ for finite $\pounds_1$ and $\pounds_2$ in this slow rotation limit. As the expression shows that it might diverge at $r=2M$, we take limit $r\to 2M$, and get the expression

$$E_{CM}(r\to 2M)=$$
$$\frac{m_0}{32\sqrt{7}M^4}\Bigg[1792M^6\Big[\Big(\pounds_1-\pounds_2\Big)^2+16M^2\Big]+a\Big(\pounds_1-\pounds_2\Big)^2\Big(\pounds_1+\pounds_2\Big)\Big(448M^4-709\pi\gamma^2\Big)$$
$$+28a^2M^2\Big(5\pounds_1^2+6\pounds_1\pounds_2+5\pounds_2^2-16M^2\Big)\Big(\pounds_1-\pounds_2\Big)^2\Bigg]^{1/2},$$

which is also finite for finite values $\pounds_1$ and $\pounds_2$. The profiles of the centre of mass energy have been plotted in Figures (3.1) and (3.2). From the graphs it is clear that the centre of mass energy increases with increase in the coupling constant γ and rotation parameter a. Variation between different curves is obvious for small values of radius but as the radius r increases, all curves merge together.

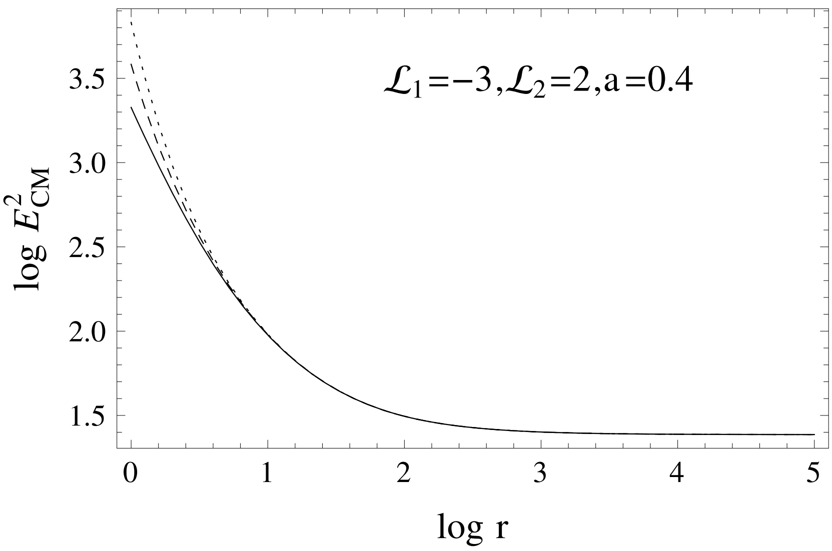


FIGURE 3.1: Graph showing radial dependence of the centre of mass energy for some values of γ, when $a = 0.4$, $\mathcal{L}_1 = -3$ and $\mathcal{L}_2 = 2$. In the solid curve $\gamma = 0$, dashed curve is for $\gamma = 0.2$ and the dotted curve is for $\gamma = 0.3$.

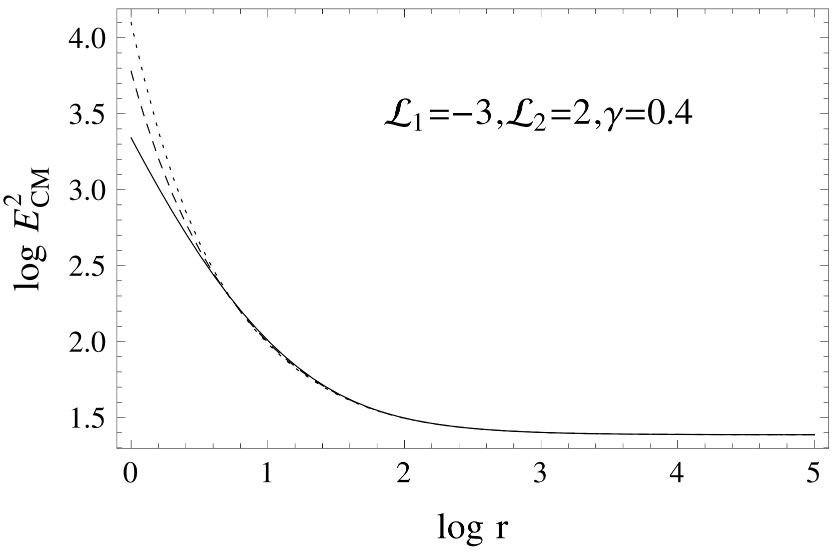


FIGURE 3.2: Graph showing radial dependence of the centre of mass energy for some values of a, when $\gamma = 0.4$, $\mathcal{L}_1 = -3$ and $\mathcal{L}_2 = 2$. In the solid curve $a = 0$, dashed curve is for $a = 0.2$ and the dotted curve is for $a = 0.4$.

3.4 The Penrose process

As described in Chapter 1, Penrose [43] proposed a mechanism for extraction of energy from a rotating black hole. It is based on the existence of negative energy orbits in the ergosphere. Consider a positive energy particle moving along a time-like geodesic into the ergosphere. The particle decays into two photons, one crossing the event horizon and the other escaping to infinity. The photon crossing the event horizon carries negative energy while the other one carries more positive energy than the initial particle. It is assumed that such a decay occurs at the turning point of the equatorial radial geodesics where $\dot{r} = 0$, then we have from Eq. (3.56)

$$\left[r\left(r^2+a^2\right)+2Ma^2\right]E^2-2aJE\left\{2M-\frac{10\pi\gamma^2}{r^3}\left[1+\frac{12M}{7r}+\frac{27M^2}{10r^2}\right]\right\}-J^2\left(r-2M\right)-m^2r\Delta=0. \tag{3.57}$$

The above equation is quadratic both in E and J, so we solve for both. The solution in terms of E leads to

$$E = \frac{1}{[r(r^2+a^2)+2Ma^2]}\left\{J\left[2aM - \frac{10a\pi\gamma^2}{r^3}\left(1+\frac{12M}{7r}+\frac{27M^2}{10r^2}\right)\right] \pm \left[J^2z + J^2r^2\Delta + m^2\Delta r(r(r^2+a^2)+2Ma^2)\right]^{\frac{1}{2}}\right\}, \tag{3.58}$$

and that for the angular momentum is given by

$$J = \frac{1}{r-2M}\left\{-E\left[2aM-\frac{10a\pi\gamma^2}{r^3}\left(1+\frac{12M}{7r}+\frac{27M^2}{10r^2}\right)\right]\pm\sqrt{E^2z+E^2r^2\Delta-m^2\Delta(r^2-2Mr)}\right\}, \tag{3.59}$$

where

$$z = \frac{100a^2\pi^2\gamma^4}{r^6}\left[1+\frac{12M}{7r}+\frac{27M^2}{10r^2}\right]^2 - \frac{40a^2M\pi\gamma^2}{r^3}\left[1+\frac{12M}{7r}+\frac{27M^2}{10r^2}\right], \tag{3.60}$$

and the following identity was used for simplification

$$r^2\Delta - 4Ma^2 = \left[r^2(r^2+a^2)+2Ma^2r\right]\left(1-\frac{2M}{r}\right). \tag{3.61}$$

Only positive sign is selected in Eq. (3.58) because we want the particle to have future directed 4-momentum. For the particle to have negative energy we must have

$$J < 0,$$

and

$$\left[r^2(r^2+a^2)+2Ma^2r\right]\left\{J^2(1-\frac{2M}{r})+m^2\Delta\right\} < 0.$$

The last equation is same as in Kerr spacetime as the terms involving the CS coupling parameter get canceled out in the simplification. Thus the conditions for the particle to have negative energy are same as for the Kerr spacetime [3] i.e.

$$E < 0 \Longleftrightarrow J < 0,$$

and

$$r - 2M < -\frac{m^2\Delta r}{J^2}.$$

Consider the scenario where a particle of mass m decays into two photons, one of which goes inside the event horizon and other one escapes to infinity. The photon which crosses the event horizon has negative energy and the other photon carries more energy than the initial particle. Let $E^{(0)}, E^{(1)}$ and $E^{(2)}$ denote the energies of the initial particle and photons respectively and $J^{(0)}, J^{(1)}$ and $J^{(2)}$ are their angular momenta. Let us take $m = 1 = E^{(0)}$ and $m = 0$ for the initial particle and photons respectively. The angular momentum of these particles can be obtained from Eq. (3.59)

$$J^{(0)} = \frac{1}{r-2M}\Bigg\{-2aM + \frac{10a\pi\gamma^2}{r^3}\Big[1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\Big] + \sqrt{z + 2Mr\Delta}\Bigg\} = \alpha^{(0)}, \quad (3.62)$$

$$J^{(1)} = -\frac{1}{r-2M}\Bigg\{2aM - \frac{10a\pi\gamma^2}{r^3}\Big[1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\Big] + \sqrt{z + r^2\Delta}\Bigg\}E^{(1)} = \alpha^{(1)}E^{(1)}, \quad (3.63)$$

$$J^{(2)} = -\frac{1}{r-2M}\Bigg\{2aM - \frac{10a\pi\gamma^2}{r^3}\Big[1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\Big] - \sqrt{z + r^2\Delta}\Bigg\}E^{(2)} = \alpha^{(2)}E^{(2)} . \quad (3.64)$$

According to the law of conservation of energy and angular momentum

$$E^{(0)} = E^{(1)} + E^{(2)}, \quad (3.65)$$

and

$$J^{(0)} = J^{(1)} + J^{(2)} = \alpha^{(0)} = \alpha^{(1)}E^{(1)} + \alpha^{(2)}E^{(2)}. \quad (3.66)$$

Solving the above system of equations for $E^{(1)}$ and $E^{(2)}$ gives

$$\begin{aligned} E^{(1)} &= -\frac{1}{2}\Bigg[\sqrt{\frac{z + 2Mr\Delta}{z + r^2\Delta}} - 1\Bigg] \\ &\simeq -\frac{1}{2}\Bigg[\sqrt{\frac{2M}{r}} - 1\Bigg] + O(a^2\gamma^2), \\ E^{(2)} &= \frac{1}{2}\Bigg[\sqrt{\frac{z + 2Mr\Delta}{z + r^2\Delta}} + 1\Bigg] \\ &\simeq \frac{1}{2}\Bigg[\sqrt{\frac{2M}{r}} + 1\Bigg] + O(a^2\gamma^2), \end{aligned}$$

which are the same as in Kerr metric as the terms involving the CS coupling constant are of higher order than $a\gamma^2$, so they are neglected . The gain in energy is

$$\Delta E = \frac{1}{2}[\sqrt{2M/r} - 1] = -E^{(1)}. \quad (3.67)$$

The efficiency of the energy extraction by the Penrose process is given by

$$\eta = \frac{E^{(0)} + \Delta E}{E^{(0)}} = \frac{1}{2}\Big(1 + \sqrt{\frac{2M}{r}}\Big). \tag{3.68}$$

For maximum efficiency one must consider the situation where the radial distance is minimum. Therefore we consider the situation where $r = r_H$. For Kerr metric the maximum efficiency is found to be $\eta_{Kerr} = 1.207$ (which corresponds to $a = M$) [3]. Since we are dealing with slow rotation approximation, we cannot have $a = M$, therefore for the metric in Eq. (1.31) the maximaum efficiency is less than η_{Kerr} and is independent of CS coupling constant γ.

Chapter 4

Charging the Johannsen-Psaltis spacetime

4.1 Introduction

The rotating solutions are not easy to find in GR. The van Stockum rotating solution [69] was developed two decades later and it took more than forty years to discover the rotating solution of Kerr. The Schwarzschild and Reissner-Nordström are solutions of the EFEs in the spherically symmetric and static setting. The Kerr metric is the rotating generalization of the Schwarzschild solution and Kerr-Newman spacetime is the rotating generalization of the Reissner-Nordström solution as well as Kerr's charged generalization. The Newman-Janis algorithm [26] was used to develop exterior rotating solutions but later was applied for developing rotating interior metrics which were matched to the exterior Kerr [70]. This algorithm is based on complex coordinate transformation. It introduces rotation into the static spacetimes. Kerr and Kerr-Newman spacetimes have been derived via Newman-Janis algorithm by using the Schwarzschild and Reissner-Nordström spacetimes as seed metrics respectively [26, 71].

Johannsen-Psaltis spacetime [25] has been introduced in Chapter 1 (Eq. (1.33)). It is derived by applying the Newman-Janis algorithm to a deformed Schwarzschild metric. It is a stationary, axisymmetric, asymptotically flat vacuum solution of some unknown field equations differing from the Einstein field equations due to the presence of function $P(r, \theta)$. In addition to the spin and mass parameters, this metric contains at least one parameter which measures the potential deviations from the Kerr spacetime. Setting such deviation parameters to zero, one recovers the Kerr metric. It could be the ad-hoc starting point for the investigation of deviations from general relativity [72]. In case of modified theories of gravity, Newman-Janis formulation usually does not relate static solutions with corresponding stationary solutions but can be applied to modified forms of the Schwarzschild spacetime [73].

Most known astrophysical compact objects are either electrically neutral or slightly charged. The Kerr-Newman metric is an exact electro vacuum solution of the Einstein-Maxwell system of equations. It is an ideal model for the interaction between the electromagnetic field and the gravitoelectric and gravitomagnetic components of gravity. Therefore, it is significant both conceptually and theoretically.

Keeping in view the general relativity's Kerr-Newman metric, we look for a charged spacetime in the alternate theories of gravity. In the case of vector-tensor theories of alternate gravity, an exact rotating solution has been developed recently [74]. Charged rotating solutions in the limit of small Weyl corrections was developed in Ref. [75].

The main motivation for this chapter comes from the Johannsen-Psaltis spacetime. In this chapter we take the deformed Reissner-Nordström spacetime as the seed metric and construct the charged generalization of the Johannsen-Psaltis spacetime.

The outline of the chapter is as follows: The charged metric is developed in Section 4.2. In Section 4.3 the event horizon and Killing horizon have been discussed. In Section 4.4 we analyze our metric for the existence of Lorentz violation regions and closed time-like curves. In Section 4.5 innermost stable circular orbits (ISCO) and circular photon orbits are computed.

4.2 The construction

The Reissner-Nordström metric is

$$g_{\mu\nu} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + \sin^2\theta r^2 d\phi^2, \quad (4.1)$$

where M and Q denote mass and charge of the central object, respectively. The 4-potential for the above metric is

$$A_\mu = \left(-\frac{Q}{r}, 0, 0, 0\right). \quad (4.2)$$

The $(t-r)$ sector is modified by multiplying the corresponding component by the expression of the form $1 + P(r)$ where $P(r)$ is given by [25]

$$P(r) = \sum_{k=0}^{\infty} \epsilon_k \left(\frac{M}{r}\right)^k. \quad (4.3)$$

The deformed Reissner-Nordström spacetime thus takes the form

$$g_{\mu\nu} = h(r)(1 + P(r))dt^2 + h(r)^{-1}(1 + P(r))dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{4.4}$$

where $h(r) = 1-2M/r+Q^2/r^2$. Setting $\epsilon_k = 0$ gives the Reissner-Nordström spacetime. The next step is to change from (t, r, θ, ϕ) to the Eddington-Finkelstein coordinates (u', r', θ', ϕ') where

$$du' = dt - \frac{dr}{h}, \tag{4.5}$$

$$r = r', \quad \theta = \theta', \quad \phi = \phi'. \tag{4.6}$$

Using the above equations, Eq. (4.4) takes the form

$$ds^2 = -h\Big(1 + P(r)\Big)du^2 - 2\Big(1 + P(r)\Big)dudr + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{4.7}$$

having inverse

$$g^{\mu\nu} = -\frac{2}{\Big(1 + P(r)\Big)}dudr + \frac{h}{\Big(1 + P(r)\Big)}dr^2 + \frac{1}{r^2}d\theta^2 + \frac{1}{r^2\sin^2\theta}d\phi^2, \tag{4.8}$$

where we have removed the primes. The above inverse metric can also be expressed in the Newman-Penrose formalism [64] as

$$g^{\mu\nu} = -l^\mu n^\nu - l^\nu n^\mu + m^\mu \overline{m}^\nu + m^\nu \overline{m}^\mu, \tag{4.9}$$

where $(l^\mu, n^\mu, m^\mu, \overline{m}^\mu)$ are the null vectors. For the metric in Eq. (4.8) these are given as

$$l^\mu = \Big(0, 1, 0, 0\Big), \quad n^\mu = \frac{1}{\Big(1 + P(r)\Big)}\Big(1, \frac{-h}{2}, 0, 0\Big), \tag{4.10}$$

$$m^\mu = \frac{1}{\sqrt{2}r}\Big(0, 0, 1, \frac{i}{\sin\theta}\Big), \quad \overline{m}^\mu = \frac{1}{\sqrt{2}r}\Big(0, 0, 1, -\frac{i}{\sin\theta}\Big). \tag{4.11}$$

Next, we consider r to be complex and re-write the above null vectors as

$$l^\mu = \Big(0, 1, 0, 0\Big), \quad n^\mu = \frac{1}{\Big(1 + P(r, \overline{r})\Big)}\Big(1, \frac{-h(r, \overline{r})}{2}, 0, 0\Big), \tag{4.12}$$

$$m^\mu = \frac{1}{\sqrt{2}r}\Big(0, 0, 1, \frac{i}{\sin\theta}\Big), \quad \overline{m}^\mu = \frac{1}{\sqrt{2}\overline{r}}\Bigg(0, 0, 1, -\frac{i}{\sin\theta}\Bigg), \tag{4.13}$$

where bar denotes the complex conjugate and the expressions for $h(r,\bar{r})$ and $P(r,\bar{r})$ are

$$h(r,\bar{r}) = 1 - M\left(\frac{1}{r} + \frac{1}{\bar{r}}\right) + \frac{Q^2}{r\bar{r}}, \quad P(r,\bar{r}) = \sum_{k=0}^{\infty}\left(\epsilon_{2k} + \epsilon_{2k+1}\frac{M}{2}\left(\frac{1}{r} + \frac{1}{\bar{r}}\right)\right)\left(\frac{M^2}{r\bar{r}}\right)^k. \tag{4.14}$$

Using the transformation

$$u' = u - ia\cos\theta, \quad r' = r + ia\cos\theta, \tag{4.15}$$

$$\theta = \theta', \quad \phi = \phi', \tag{4.16}$$

the null vectors in Eqs. (4.12)-(4.14) take the form

$$l^\mu = \Big(0,1,0,0\Big), \quad n^\mu = \frac{1}{\Big(1 + P(r',\theta')\Big)}\Big(1, \frac{-h(r',\theta')}{2}, 0, 0\Big), \tag{4.17}$$

$$m^\mu = \frac{1}{\sqrt{2}r'}\Big(ia\sin\theta', -ia\sin\theta', 1, \frac{i}{\sin\theta'}\Big), \tag{4.18}$$

$$\overline{m}^\mu = \frac{1}{\sqrt{2}\bar{r}'}\Big(-ia\sin\theta', ia\sin\theta', 1, -\frac{i}{\sin\theta'}\Big). \tag{4.19}$$

Here the expressions for $h(r',\theta'), P(r',\theta')$ are

$$h(r',\theta') = 1 - \frac{2Mr'}{\Sigma} + \frac{Q^2}{\Sigma}, \tag{4.20}$$

$$P(r',\theta') = \sum_{k=0}^{\infty}\left(\epsilon_{2k} + \epsilon_{2k+1}\frac{Mr'}{\Sigma}\right)\left(\frac{M^2}{\Sigma}\right)^k, \tag{4.21}$$

where Σ is given by

$$\Sigma = r'^2 + a^2\cos^2\theta'. \tag{4.22}$$

Using Eqs. (4.17)-(4.22), the metric tensor $g_{\mu\nu}$ is written in terms of the coordinates (u', r', θ', ϕ') as

$$g_{00} = -h\left(1+P\right), \quad g_{01} = -\left(1+P\right), \tag{4.23}$$

$$g_{03} = a\left(1+P\right)\left(h-1\right)\sin^2\theta', \quad g_{13} = a\left(1+P\right)\sin^2\theta', \tag{4.24}$$

$$g_{22} = \Sigma, \quad g_{33} = \sin^2\theta'\left[\Sigma - a^2\left(1+P\right)\left(h-2\right)\sin^2\theta'\right]. \tag{4.25}$$

In order to remove the off-diagonal terms g_{01} and g_{13}, we use the transformation [25, 76]

$$du' = dt - W(r',\theta')dr, \quad d\phi' = d\phi - N(r',\theta')dr, \tag{4.26}$$

$$r = r', \quad \theta = \theta', \tag{4.27}$$

where

$$W(r',\theta') = \frac{g_{01}g_{33} - g_{03}g_{13}}{g_{00}g_{33} - g_{03}^2}, \quad N(r',\theta') = \frac{g_{00}g_{13} - g_{01}g_{03}}{g_{00}g_{33} - g_{03}^2}. \tag{4.28}$$

This leads to the metric tensor given by

$$\begin{aligned} ds^2 &= -(1+P(r,\theta))\Big(1-\frac{2Mr}{\Sigma}+\frac{Q^2}{\Sigma}\Big)dt^2-\frac{2a(2Mr-Q^2)\sin^2\theta}{\Sigma}(1+P(r,\theta))dtd\phi \\ &\quad +\frac{\Sigma(1+P(r,\theta))}{\Delta+a^2\sin^2\theta P(r,\theta)}dr^2+\Sigma d\theta^2+\Big[\sin^2\theta\Big(r^2+a^2+\frac{a^2(2Mr-Q^2)\sin^2\theta}{\Sigma}\Big) \\ &\quad +P(r,\theta)\frac{a^2\sin^4\theta(\Sigma+2Mr-Q^2)}{\Sigma}\Big]d\phi^2, \end{aligned} \tag{4.29}$$

where $\Sigma = r^2+a^2\cos^2\theta$, $\Delta = r^2+a^2-2Mr+Q^2$ and $P(r,\theta)$ has the general expression given in Eq. (4.21) with $r = r'$ and $\theta = \theta'$. Setting $Q = 0$ gives the Johannsen-Psaltis metric. With $P = 0$, the Kerr-Newman spacetime is recovered.

The function $P(r,\theta)$ contains infinite number of parameters. The first two parameters ϵ_0 and ϵ_1 are set to zero by requiring that the metric must be asymptotically flat and the next parameter ϵ_2 is constrained at 10^{-4} by weak field tests of general relativity in the parameterized post-Newtonian approach. Thus ϵ_2 can also be set to zero. As in the case of Ref. [25], we set $\epsilon_k = 0$ for $k > 3$, which leads to $P(r,\theta)$ as

$$P(r,\theta) = \frac{\epsilon_3 M^3 r}{\Sigma^2}, \tag{4.30}$$

which is the same as in Ref. [25]. Since ϵ_3 is the only retained parameter, we drop the subscript 3 in the rest of the chapter and represent it as ϵ. Applying the Newman-Janis algorithm on the potential A_μ in Eq. (4.2) leads to

$$A_\mu = \left(-\frac{Qr}{\Sigma}, \frac{Qr}{\Delta+a^2P\sin^2\theta}, 0, \frac{aQr\sin^2\theta}{\Sigma}\right). \tag{4.31}$$

Here the A_r component depends on θ. So it cannot be gauged away. This makes the electromagnetic potential in Eq. (4.31) different from Kerr-Newman's potential. Expanding the A_r component in terms of powers of the deviation parameter ϵ, we obtain

$$A_r = \frac{Qr}{\Delta}\big(1-\frac{a^2\epsilon M^3 r\sin^2\theta}{\Sigma^2}+\text{higher order terms}\big). \tag{4.32}$$

From the second term we see that the electromagnetic potential depends on mass M. Thus we can consider the deviation parameter as a coupling between the gravitational and electromagnetic fields.

The Maxwell tensor can be computed from the equation

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \tag{4.33}$$

Using the potential in Eq. (4.31), the nonzero components of the Maxwell tensor are

$$F_{tr} = -\frac{Q\left(r^2 - a^2\cos^2\theta\right)}{\Sigma^2}, \quad F_{t\theta} = \frac{a^2 Q r \sin 2\theta}{\Sigma^2}, \tag{4.34}$$

$$F_{r\theta} = -\frac{a^2 M^3 Q r^2 \epsilon \sin\theta\cos\theta\left(a^2\cos 2\theta - 3a^2 - 2r^2\right)}{\Sigma^3(\Delta + a^2 P\sin^2\theta)^2}, \tag{4.35}$$

$$F_{\phi r} = -\frac{aQ\sin^2\theta\left(a^2\cos^2\theta - r^2\right)}{\Sigma^2}, \quad F_{\phi\theta} = -\frac{aQr\left(a^2 + r^2\right)\sin 2\theta}{\Sigma^2}. \tag{4.36}$$

From these components we see that the only component different from the Kerr-Newman's is the $F_{r\theta}$ component. The metric tensor in Eq. (4.29) with $P(r,\theta)$ as in Eq. (4.30) and the Maxwell tensor components (Eqs. (4.34)-(4.36)) do not satisfy the usual Einstein-Maxwell equations. So, we assume that our metric is an electro vacuum solution to some unknown field equations which are different from the Einstein-Maxwell equations for nonzero $P(r,\theta)$.

4.3 The horizons

The event horizon (if it exists) is located at a radius $r_{hor} = H(\theta)$, where $H(\theta)$ is the solution of the equation [77]

$$g^{11} - 2g^{12}\left(\frac{dH}{d\theta}\right) + g^{22}\left(\frac{dH}{d\theta}\right)^2 = 0. \tag{4.37}$$

As $g^{12} = 0$ for the spacetime under consideration, so Eq. (4.37) reduces to

$$g^{11} + g^{22}\left(\frac{dH}{d\theta}\right)^2 = 0. \tag{4.38}$$

For the case of $\theta = 0, \pi$ and $\theta = \pi/2$ the above equation shortens to $g^{11} = 0$. For the pole $\theta = 0$ or $\theta = \pi$, $g^{11} \propto \Delta$ having root $r = H_{KN} = r_+$ where H_{KN} is the Kerr-Newman's event horizon. This is possible if $\epsilon \neq -(2Mr_+ - Q^2)^2/M^3 r_+$ at which the denominator of g^{11} vanishes. Here we will expand the function $H(\theta)$ [77], first around $\theta = 0$ and then the same analysis will be done for $\theta = \pi/2$. Let $\theta = 0 + \delta\theta$. In this case $r = H(\delta\theta)$ is given by

$$H(\delta\theta) = H_{KN} + \delta\theta\frac{dH}{d\theta}\mid_{\theta=0} + \frac{\delta\theta^2}{2}\frac{d^2H}{d\theta^2}\mid_{\theta=0}. \tag{4.39}$$

Putting $r = H(\theta)$ in Eq. (4.38) and expanding in terms of $\theta = \delta\theta$, we obtain for the first order in $\delta\theta$,

$$\frac{\left(\frac{dH}{d\theta}\mid_{\theta=0}\right)^2}{(2MH_{KN}-Q^2)} + \delta\theta\Bigg[-2\left(\frac{dH}{d\theta}\mid_{\theta=0}\right)^3 H_{KN} + 2\frac{dH}{d\theta}\mid_{\theta=0}\Big(2MH_{KN} - Q^2\Big)\Bigg(\frac{\sqrt{-a^2+M^2-Q^2}}{1+\frac{\epsilon M^3 H_{KN}}{(2MH_{KN}-Q^2)^2}} + \frac{d^2H}{d\theta^2}\mid_{\theta=0}\Bigg)\Bigg] = 0. \tag{4.40}$$

From the above equation it can be concluded that

$$\frac{dH}{d\theta}\mid_{\theta=0} = 0. \tag{4.41}$$

For the second order in $\delta\theta^2$, we obtain the equation for $\frac{d^2H}{d\theta^2}\mid_{\theta=0}$ as

$$\left(\frac{d^2H}{d\theta^2}\mid_{\theta=0}\right)^2\left(1+\frac{\epsilon M^3 H_{KN}}{(2MH_{KN}-Q^2)^2}\right) + \frac{a^2\epsilon M^3 H_{KN}}{(2MH_{KN}-Q^2)^2} + \sqrt{-a^2+M^2-Q^2}\frac{d^2H}{d\theta^2}\mid_{\theta=0} = 0. \tag{4.42}$$

For the existence of null surface it is required that $\frac{d^2H}{d\theta^2}\mid_{\theta=0}$ must have a real solution. This requirement leads to a lower and upper bound for ϵ as

$$\epsilon^{min-pole} = -\frac{b_1}{2M^3H_{KN}} - \frac{b_1\sqrt{M^2-Q^2}}{2M^3aH_{KN}}, \tag{4.43}$$

$$\epsilon^{max-pole} = -\frac{b_1}{2M^3H_{KN}} + \frac{b_1\sqrt{M^2-Q^2}}{2M^3aH_{KN}}, \tag{4.44}$$

where $b_1 = 2MH_{KN} - Q^2$.

When $\theta = \pi/2$, the equation $g^{11} = 0$ gives

$$r^2 + a^2 - 2Mr + Q^2 + \frac{a^2\epsilon M^3}{r^3} = 0. \tag{4.45}$$

At the equatorial plane, the maximum positive value of the parameter ϵ for the real solution of Eq. (4.45) is given by

$$\epsilon^{max-eq} = \frac{1}{3125a^2M^2}\Bigg[150(b_2+15)a^4 - 20a^2\left(8(7b_2+40)M^2 - 15(b_2+15)Q^2\right) + 1024(b_2+4)M^4 - 160(7b_2+40)M^2Q^2 + 150(b_2+15)Q^4\Bigg], \tag{4.46}$$

where $b_2 = \sqrt{16 - \frac{15a^2}{M^2} - \frac{15Q^2}{M^2}}$. Thus the null surface at the equatorial plane exists for the deviation parameter in the range $\epsilon < \epsilon^{max-eq}$. By comparing maximum values of ϵ at the equatorial plane and the pole, it is found that $\epsilon^{max-pole} < \epsilon^{max-eq}$.

For the case of the equatorial plane, it is assumed that $\theta = \pi/2 + \delta\theta$ so that $H(\theta)$ is given by

$$H(\pi/2 + \delta\theta) = H_{eq} + \delta\theta \frac{dH}{d\theta} |_{\theta=\pi/2} + \frac{\delta\theta^2}{2} \frac{d^2 H}{d\theta^2} |_{\theta=\pi/2}, \tag{4.47}$$

where H_{eq} is the radius of the null surface at $\theta = \pi/2$. Substituting $r = H(\pi/2 + \delta\theta)$ in Eq. (4.38), and expanding in terms of $\delta\theta$, we find that upto first order in $\delta\theta$, $\frac{dH}{d\theta} |_{\theta=\pi/2}=$ 0. In the second order in $\delta\theta$, the real solution for $\frac{d^2H}{d\theta^2} |_{\theta=\pi/2}$ exists for all ϵ in the range $0 < \epsilon < \epsilon^{max-pole}$ but same cannot be said for the range $\epsilon^{min-pole} < \epsilon < 0$.

4.3.1 The Killing horizon

A Killing horizon is a null hypersurface on which there is a null Killing vector field. For an axisymmetric and stationary spacetime, the Killing horizon is given by

$$g_{00}g_{33} - g_{03}^2 = 0, \tag{4.48}$$

which for metric (4.29) becomes $(1 + P)(\Delta + a^2 P \sin^2\theta)\sin^2\theta = 0$. When $\theta = 0$ or π, we have

$$g_{00}g_{33} - g_{03}^2 \propto \left(r^2 + a^2 - 2Mr + Q^2\right)\left[1 + \frac{\epsilon M^3 r}{(r^2 + a^2)^2}\right]. \tag{4.49}$$

As in the case of event horizon at the pole, the Killing horizon at the pole coincides with the Kerr-Newman event horizon. From Eq. (4.49) the value of ϵ comes out to be

$$\epsilon^{kil-pol} = -\frac{16}{3\sqrt{3}}\left(\frac{a}{M}\right)^3. \tag{4.50}$$

For the case of the equatorial plane, Eq. (4.48) takes the form

$$(1 + \frac{\epsilon M^3}{r^3})(r^2 + a^2 - 2Mr + Q^2 + \frac{a^2 \epsilon M^3}{r^3}) = 0. \tag{4.51}$$

The second factor in the above equation is same as in Eq. (4.45) having two solutions for $\epsilon < \epsilon^{max-eq}$ and single solution for negative ϵ.
In the extremal case when $a^2 + q^2 > M$, Eq. (4.51) has solutions for negative deviation parameter only. At the pole Eq. (4.49) has two solutions for $\epsilon < \epsilon^{kil-pol}$, one solution for $\epsilon = \epsilon^{kil-pol}$ and no solution otherwise.
The Killing horizon for different positive values of ϵ are shown in Figures 4.1 to 4.3. In these figures, we have taken $a = 0.7, Q = 0.6$ and $M = 1$. In Figure 4.1, $\epsilon = 0.389 < \epsilon^{max-eq}$, the outer and inner Killing horizons have spherical topology. In Figure 4.2 $\epsilon = 0.396 \approx \epsilon^{max-eq}$, they merge at the equatorial plane. In Figure 4.3 $\epsilon = 0.4 > \epsilon^{max-eq}$, disjoint Killing horizons appear above and below the equatorial plane.
Figures 4.4 and 4.5 represent the Killing horizons for negative ϵ. In Figure 4.4 the values

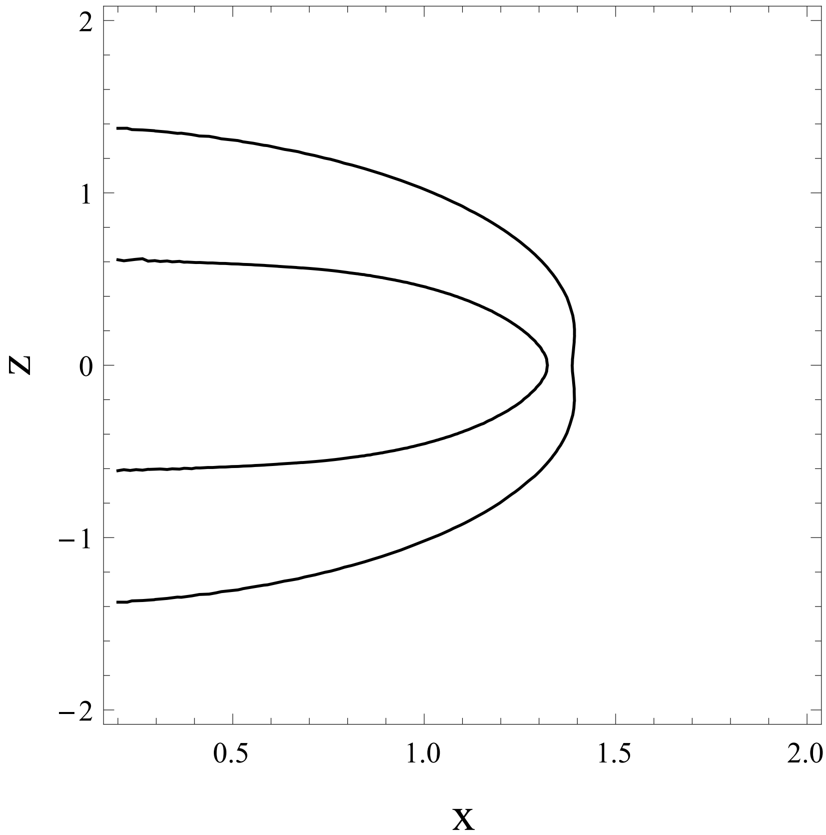


FIGURE 4.1: Killing horizon for $\epsilon = 0.389, a = 0.7, Q = 0.6$ and $M = 1$.

of a and Q are such that $a^2 + Q^2 < M^2$. Here the outer and inner Killing horizons are in the form of spherical surface. In Figure 4.5, a and Q are such that $a^2 + Q^2 > M^2$. The Killing horizon is in the shape of toroidal surface in such case.

4.3.2 The event horizon of the linearized charged Johannsen-Psaltis metric

The charged Johannsen-Psaltis metric can be considered as a small perturbation of the Kerr-Newman spacetime in terms of the deviation parameter ϵ. The modifications in the Kerr-Newman metric components upto the first order in ϵ are

$$h_{00} = -\frac{M^3 r \left(\Sigma - 2Mr + Q^2\right)}{\Sigma^3}, \quad h_{11} = \frac{M^3 r \left(\Sigma - 2Mr + Q^2\right)}{\Sigma\Delta^2}, \tag{4.52}$$

$$h_{22} = 0, \quad h_{03} = \frac{aM^3 r \sin^2\theta \left(Q^2 - 2Mr\right)}{\Sigma^3}, \tag{4.53}$$

$$h_{33} = \frac{a^2 M^3 r \epsilon \sin^4\theta \left(2Mr - Q^2 + \Sigma\right)}{\Sigma^3}. \tag{4.54}$$

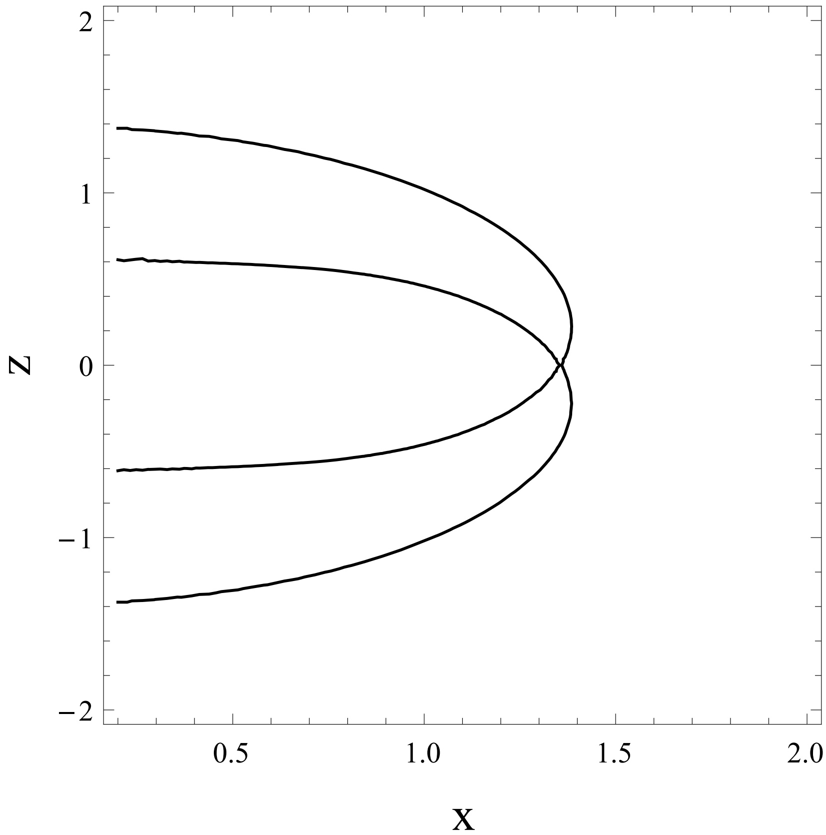


FIGURE 4.2: Killing horizon for $\epsilon = 0.396, a = 0.7, Q = 0.6$ and $M = 1$.

The event horizon in this case can be determined from the equation [77]

$$g^{11}_{KN}\left(1 - \epsilon g^{11}_{KN} h_{11}\right) = 0, \tag{4.55}$$

where g^{11}_{KN} is the inverse of g_{11} component of the Kerr-Newman metric and h_{11} is given in Eq. (4.52). The radius of the event horizon in this case has the form

$$r_H = H_{KN}\left(1 + \chi\epsilon\right), \tag{4.56}$$

where χ measures deviation from the Kerr-Newman event horizon H_{KN}. Substituting Eq. (4.56) in Eq. (4.55), and linearizing in terms of ϵ, we obtain

$$\chi = -\frac{a^2 M^3 \sin^2\theta}{2\sqrt{M^2 - a^2 - Q^2}(2MH_{KN} - Q^2 - a^2\sin^2\theta)^2}. \tag{4.57}$$

Thus the event horizon is located at the radius

$$r_H = H_{KN}\left(1 - \frac{\epsilon a^2 M^3 \sin^2\theta}{2\sqrt{M^2 - a^2 - Q^2}(2MH_{MN} - Q^2 - a^2\sin^2\theta)^2}\right). \tag{4.58}$$

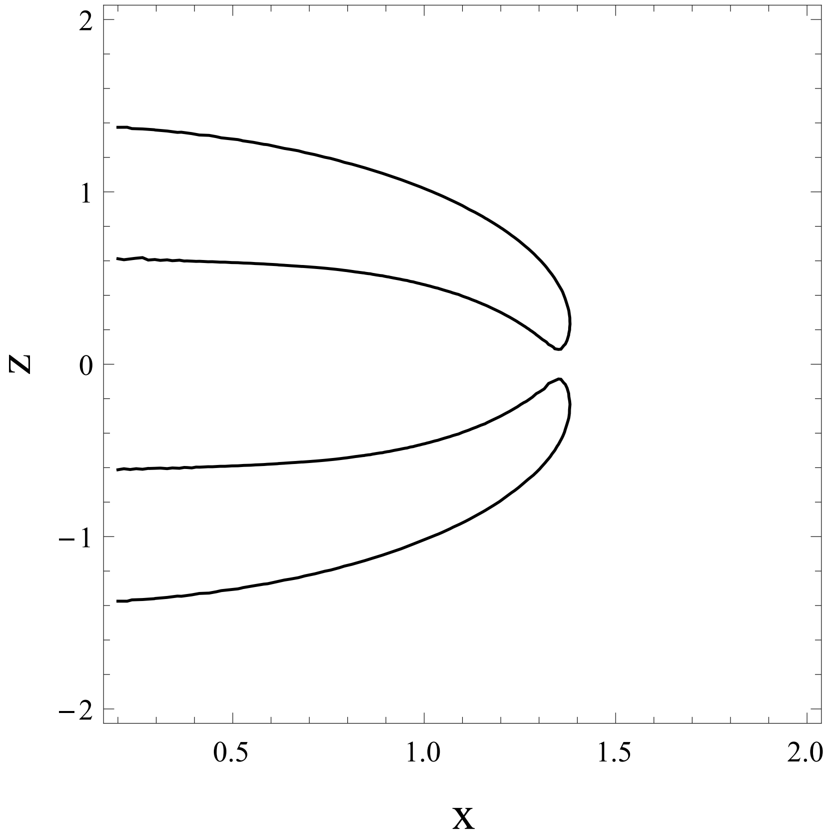


FIGURE 4.3: Killing horizon for $\epsilon = 0.4, a = 0.7, Q = 0.6$ and $M = 1$.

The Kretschmann scalar for the linearized form of (4.29) diverges at $r = 0$, therefore it represents a black hole for all values of ϵ.

4.4 Lorentz violations, closed time-like curves and regions of validity

The metric (4.29) has the following determinant

$$det(g_{\mu\nu}) = -\frac{\sin^2\theta}{64\Sigma^2}\left[3a^4+8a^2r^2+8r^4+8\epsilon M^3 r+4a^2(2r^2+a^2)\cos 2\theta+a^4\cos 4\theta\right]^2, \quad (4.59)$$

which is independent of the charge Q. It is semi-definite, negative and vanishes at two values of radii for $\epsilon < -(2Mr_+ - Q^2)^2/M^3 r_+$ which coincide with the location of the Killing horizon. So, the charged spacetime does not contain any Lorentz violating region. By using g_{33} component of metric (4.29), we plot the closed time-like curves and combine them with the Killing horizons to show their location. These plots are shown in Figure 4.6. Also from g_{33} component, we can determine the upper bound for ϵ at the equatorial

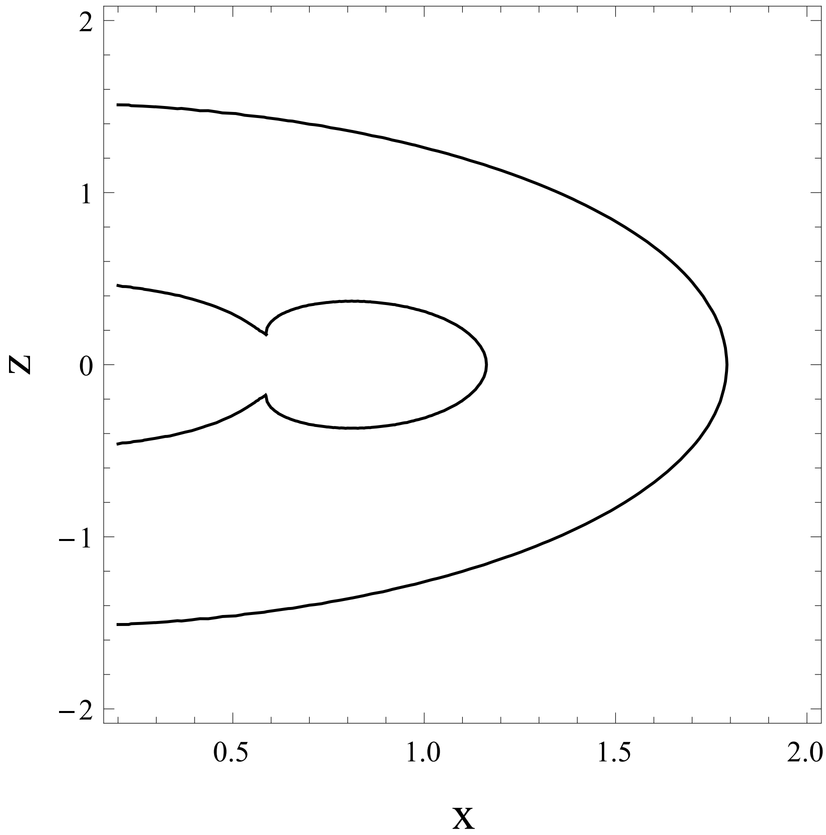


FIGURE 4.4: Killing horizon for $a = 0.8, Q = 0.3, \epsilon = -0.6$ and $M = 1$.

plane and denote it with symbol ϵ^{CTC} and has the following expression

$$\epsilon^{CTC} = -\frac{r^3\left(r^2\left(r^2+a^2\right)-a^2\left(Q^2-2Mr\right)\right)}{a^2M^3\left(r(2M+r)-Q^2\right)}, \tag{4.60}$$

while at the equatorial plane (4.45) gives the value of ϵ as

$$\epsilon^{hor} = -\frac{r^3\left(r^2+a^2+Q^2-2Mr\right)}{a^2M^3}, \tag{4.61}$$

where symbol ϵ^{hor} shows that this is the value at the horizon for the equatorial plane. It is clear that $\epsilon^{CTC} < \epsilon^{hor}$ indicting that for the equatorial plane the regions of closed time-like curves lie inside the inner Killing horizon. In the top panel of the Figure 4.6, Q and a are such that $a^2 + Q^2 < M^2$, resulting in an inner and outer Killing horizon and the closed time-like region lying inside the inner Killing horizon. In the lower panel $a^2 + Q^2 > M^2$, resulting in the toroidal shaped horizon surrounding the closed time-like region. So we conclude that the charged Johannsen-Psaltis metric does not contain closed time-like curves outside the central object.

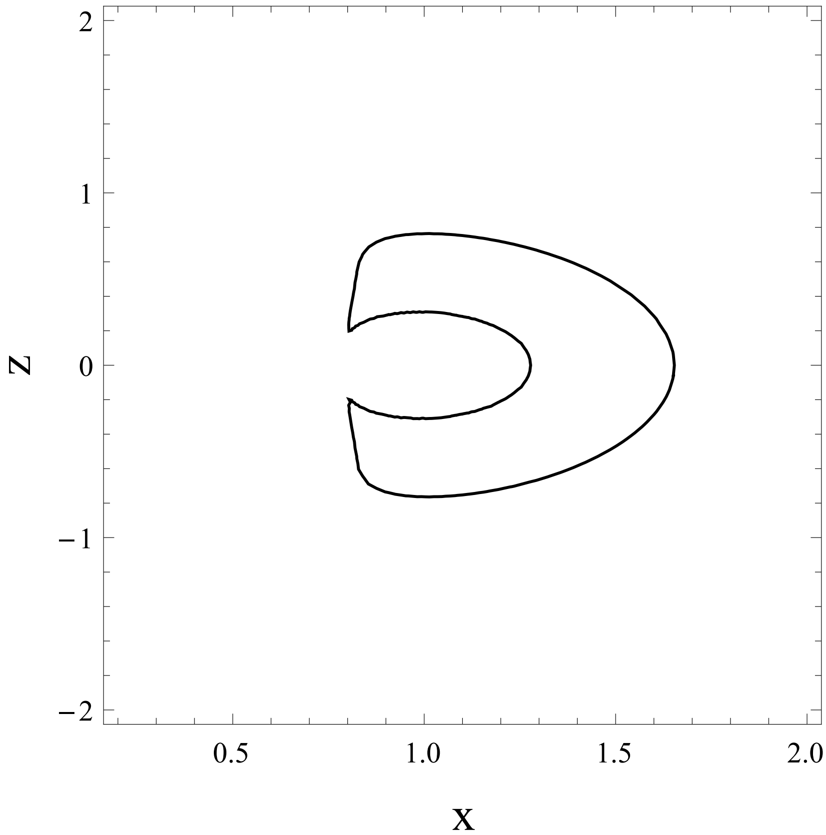


FIGURE 4.5: Killing horizon for $a = 0.96, Q = 0.43, \epsilon = -0.6$ and $M = 1$.

4.5 Innermost stable circular orbits and the circular photon orbits

Rotation of the central object has significant impact on the motion of particles in GR. Consider a massive particle moving around a black hole. There exists a minimum radius at which the stable circular motion is possible. This defines the innermost stable circular orbits or the ISCO.

In this section, particle motion on the equatorial plane is considered. First particle's angular momentum and energy are computed which are later employed to determine the ISCO and circular photon orbits.

The radial equation of motion is determined by solving the equation

$$p_\alpha p^\alpha = -m^2, \tag{4.62}$$

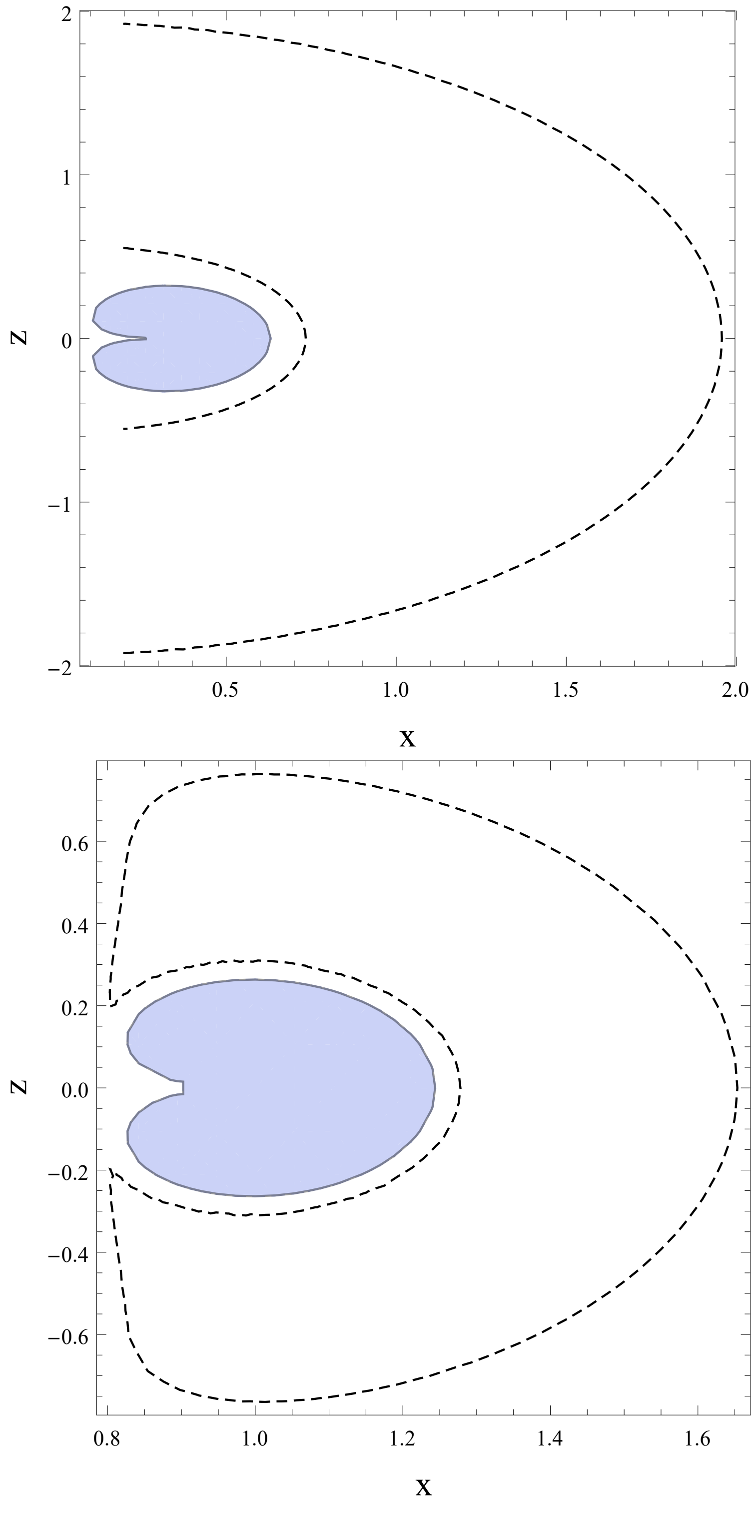


FIGURE 4.6: The Killing horizon and closed time-like curves for some values of ϵ, a and Q. Mass M has been set to 1. On the top $a = 0.3, Q = 0.2, \epsilon = -0.3$. On the bottom $a = 0.96, Q = 0.43, \epsilon = -0.6$. The dashed curves denote the inner and outer Killing horizons and the solid region shows the closed time-like curve.

for the radial momentum. Here p_α is the 4-momentum of the particle of mass m. The radial equation of motion, so obtained is

$$(\frac{dr}{d\tau})^2 = R(r) = -\frac{1}{g_{11}}(g^{00}E^2 + g^{33}J^2 - 2EJg^{03} + m^2), \tag{4.63}$$

where τ denotes the proper time. Here E is the energy of the particle and J denotes particle's angular momentum. These two quantities can be obtained by solving the equations

$$R(r) = 0, \tag{4.64}$$

$$\frac{dR(r)}{dr} = 0. \tag{4.65}$$

This gives

$$\frac{E}{m} = \sqrt{\frac{P_1 + P_2}{P_3}}, \tag{4.66}$$

where

$$\begin{aligned} P_1 = {} & 12a^4M^3r^5\epsilon\Big(Mr - Q^2\Big)\Big(M^3\epsilon + r^3\Big)^4 + a^2r^8\Big(M^3\epsilon + r^3\Big)^2\Big[Q^4\Big(-7M^6\epsilon^2 + 10M^3r^3\epsilon \\ & + 8r^6\Big) + 2Q^2r\Big(16M^7\epsilon^2 - 9M^6r\epsilon^2 - 16M^4r^3\epsilon + 6M^3r^4\epsilon - 14Mr^6 + 6r^7\Big) \\ & + Mr^2\Big(-40M^7\epsilon^2 + 48M^6r\epsilon^2 - 15M^5r^2\epsilon^2 + 16M^4r^3\epsilon - 6M^2r^5\epsilon + 20Mr^6 - 12r^7\Big)\Big], \end{aligned} \tag{4.67}$$

$$\begin{aligned} P_2 = {} & 2\Big[2\Big(r^{26} \mp P_4\Big) - 48M^{12}r^{14}\epsilon^3 + 68M^{11}r^{15}\epsilon^3 - 32M^{10}r^{16}\epsilon^3 + 5M^9r^{17}(\epsilon - 24)\epsilon^2 \\ & + 168M^8r^{18}\epsilon^2 - 78M^7r^{19}\epsilon^2 + 12M^6r^{20}(\epsilon - 8)\epsilon + 132M^5r^{21}\epsilon - 60M^4r^{22}\epsilon \\ & + Q^6r^{11}\Big(M^3\epsilon + r^3\Big)^2\Big(7M^3\epsilon + 4r^3\Big) - 3M^3r^{23}(8 - 3\epsilon) + 32M^2r^{24} \\ & - Q^4r^{12}\Big(M^3\epsilon + r^3\Big)^2\Big(40M^4\epsilon - 19M^3r\epsilon + 22Mr^3 - 10r^4\Big) \\ & + Q^2r^{13}(2M - r)\Big(M^3\epsilon + r^3\Big)^2\Big(38M^4\epsilon - 17M^3r\epsilon + 20Mr^3 - 8r^4\Big) - 14Mr^{25}\Big], \end{aligned} \tag{4.68}$$

$$\begin{aligned} P_3 = {} & r^{13}\Big[8a^2\Big(Q^2 - Mr\Big)\Big(M^3\epsilon + r^3\Big)^2\Big(5M^3\epsilon + 2r^3\Big) + r^3\Big(Q^2\Big(7M^3\epsilon + 4r^3\Big) \\ & + r\Big(-12M^4\epsilon + 5M^3r\epsilon - 6Mr^3 + 2r^4\Big)\Big)^2\Big], \end{aligned} \tag{4.69}$$

$$P_4 = \Big[a^2 r^{10}\Big(Q^2 - Mr\Big)^2\Big(M^3\epsilon + r^3\Big)^6\Big(a^2\Big(M^3\epsilon + r^3\Big) + r^3\Big(-2Mr + Q^2 + r^2\Big)\Big)^2$$
$$\Big(9a^2M^6\epsilon^2 + 16M^4r^4\epsilon - 2q^2\Big(5M^3r^3\epsilon + 2r^6\Big) - 6M^3r^5\epsilon + 4Mr^7\Big)\Big]^{\frac{1}{2}}. \quad (4.70)$$

The angular momentum is

$$\frac{J}{m} = \pm\frac{1}{\sqrt{P_3}\beta}\Bigg[\Big(\mp 2a^2r^6(Q^6 + Q^2r^2M(8M - 5r) + rQ^4(2r - 5M)$$
$$+ M^2r^3(3r - 4M))(r^3 + \epsilon M^3)^4 + a^4r^3(Q^2 - Mr)(r^3 + \epsilon M^3)^4(2Mr^4 - 4M^4r\epsilon$$
$$+ Q^2(-2r^3 + M^3\epsilon)) + P_4\Big)\sqrt{P_1 + P_2}\Bigg], \quad (4.71)$$

where

$$\beta = ar^3(Mr - Q^2)(M^3\epsilon + r^3)^4(2r^3\left(r(r - 2M) + Q^2\right)^2 + a^2(Q^2(2r^3 - M^3\epsilon)$$
$$+ Mr(4M^3\epsilon - 3M^2r\epsilon - 2r^3))). \quad (4.72)$$

Here the upper sign indicates that the particle corotates with the black hole while the lower sign refers to the opposite situation.

In the case $\epsilon = 0$, the above expressions for the E and J reduce to the ones obtained for the Kerr-Newman metric [78]

$$\frac{E}{m} = \frac{r^2 - 2Mr + Q^2 \pm a(Mr - q^2)^{1/2}}{r\sqrt{(r^2 - 3Mr + 2Q^2 \pm 2a(Mr - Q^2)^{1/2})}}, \quad (4.73)$$
$$\frac{J}{m} = \pm\frac{(Mr - Q^2)^{1/2}(r^2 + a^2 \mp 2a(Mr - Q^2)^{1/2}) \mp aQ^2}{r\sqrt{(r^2 - 3Mr + 2Q^2 \pm 2a(Mr - Q^2)^{1/2})}}. \quad (4.74)$$

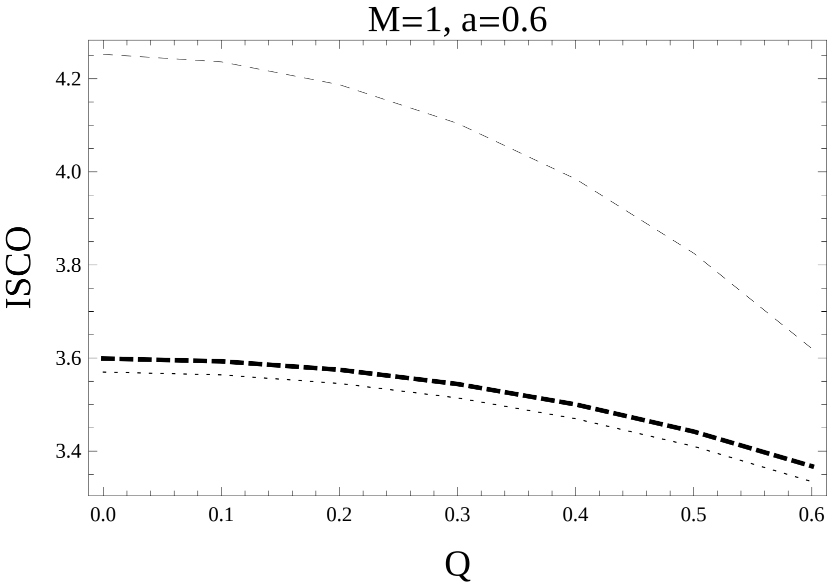
Taking $Q = 0 = \epsilon$ gives E and J for the Kerr metric [79]

$$\frac{E}{m} = \frac{r^{3/2} - 2Mr^{1/2} \pm aM^{1/2}}{r^{3/4}\sqrt{r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2}}}, \quad (4.75)$$
$$\frac{J}{m} = \pm\frac{M^{1/2}(r^2 \mp 2aM^{1/2}r^{1/2} + a^2)}{r^{3/4}\sqrt{r^{3/2} - 3Mr^{1/2} \pm 2aM^{1/2}}}. \quad (4.76)$$

The ISCO is determined by numerically solving the equation

$$\frac{dE}{dr} = 0. \quad (4.77)$$

In Figure 4.7, radial dependence of ISCO is shown for some values of the parameters.



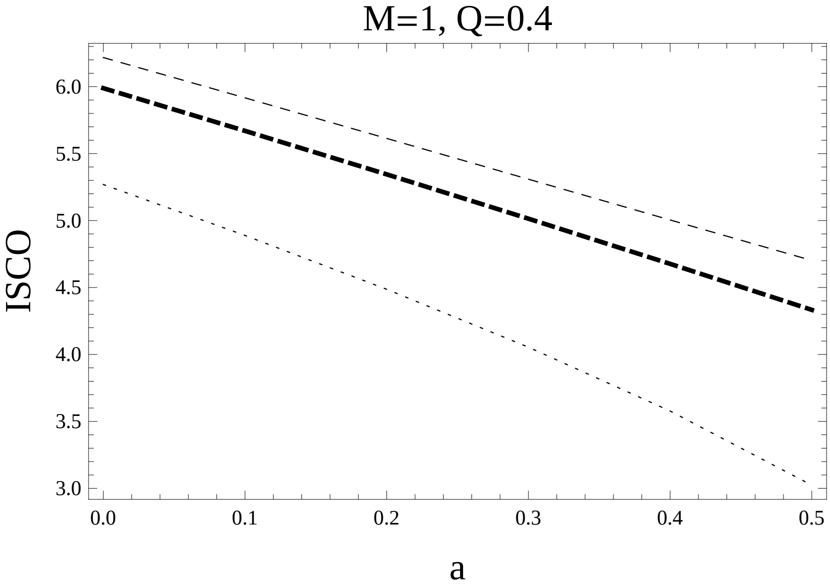


FIGURE 4.7: The Inner most stable circular orbits for some values of ϵ, a and Q. Mass M has been set to 1. In the upper graph $a = 0.6$ with varying values of charge Q. Starting from the upper curve, the values of ϵ are $-1, 1, 2$ respectively. The lower graph has been drawn for $Q = 0.4$ with various values of a. Starting from the upper line, the values of ϵ are $-2, -1, 2$ respectively.

The circualr photon orbits are located where

$$\frac{E}{m} \to \infty, \quad \frac{J}{m} \to \infty. \tag{4.78}$$

In Figure 4.8, radial profile of the circular photon orbit is shown. As in the ISCO case, increasing the value of ϵ decreases the radius of the photon orbits.

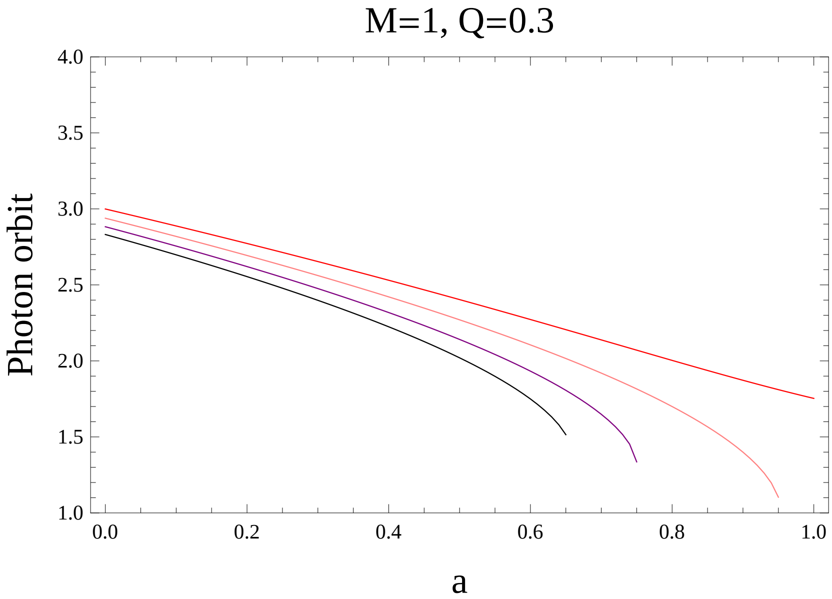


FIGURE 4.8: The circular photon orbits for some values of ϵ and a. Mass M has been set to 1 and Q is taken as $Q = 0.3$. Red plot is for $\epsilon = -1$, pink for $\epsilon = 0$, purple for $\epsilon = 1$, and black for $\epsilon = 2$.

Chapter 5

Summary and conclusion

Gravity is currently best described by the Einstein general theory of relativity which has stood the test of time but only in weak gravitational field regime. In the strong gravitational field, one needs to look for alternate theories. In this thesis, we have focused on the spacetimes in both GR and the alternate theories of gravity.

The Chapter 1 contains the preliminaries. In the Chapter 2, we considered small spheroidal deformations of static isotropic systems and studied how the MOTS is correspondingly deformed. Our main motivation for this investigation is to generalise the HQM [49, 50, 51, 52] beyond the spherical symmetry, for which we need a way to locate the horizon from quantities solely determined from the quantum state of the source.

In this chapter, first, we studied a slightly spheroidal de Sitter space generated by an energy density $\rho \propto 1/r$, for which we provided a precise characterization of the structure of the MOTS. From this analysis one can infer that the location of a MOTS appears to be (analytically) the same function of the radial coordinate r for both the isotropic and slightly spheroidal systems (where r is constant on surfaces of the respective symmetries). Moreover, the discussion suggests that the appropriate generalization of the Misner-Sharp mass, for determining the location of these hypersurfaces for small a^2, is the the usual Misner-Sharp mass computed for the isotropic space (that is, for $a^2 = 0$). In Section 2.3.2, we considered a spheroidal deformation of the Reissner-Nordström metric for which a similar result is found for the location of the MOTS, although the energy-momentum tensor of the corresponding source also contains a non-spheroidal component. This suggests that the general situation is very rich.

Finally, it is important to remark again that, despite the classical instability [13] of the last example, it is still possible that such a configuration appears as an intermediate step during the collapse that leads to the formation of a black hole. Finally, in the last section we have qualitatively discussed the perturbation induced on the spectrum of the isotropic source by these small spheroidal deformations within the framework of quantum perturbation theory. This standard procedure allowed us to describe the effects of the deformation parameter a on the spectrum of the operator associated with the gravitational radius of the system (see Eq. 2.122), as well as on the local notion of the horizon wave function (2.124), which represents the most important outcome of the HQM formalism.

In Chapter 3, we discussed black holes in the modified gravity theory. One of the modified gravity theories is the Chern-Simons gravity theory having two independent formulations, namely, the non-dynamical theory and dynamical theory. Black hole solutions have been developed in both the formulations but our focus in this chapter is the solution of the CS gravity given in Ref. [22]. The solution has been developed under the assumptions of small rotation parameter and small coupling constant. Setting the coupling constant equal to zero, the solution reduces to Kerr in small rotation approximation. Therefore, our focus in this chapter is to study the effects of the coupling constant in different physical situations. First we studied the Hawking mass outside the event horizon of the black hole and obtained an exact value. Our mathematical work shows that the m_H does not depend on γ. It just depends on the mass and spin parameter like in Kerr's case. Next, based on the BSW mechanism, we studied the dependence of the coupling constant on the E_{CM} for two neutral colliding particles of equal masses. The graphs are drawn for different values of the coupling constant and rotation parameter. The graphical results show that both a and γ cause an increase in the centre of mass energy. Next we studied the energy extraction through Penrose process and found that the efficiency of the process is independent of the CS coupling constant γ but is not exactly equal to the efficiency of the Kerr case due to the assumption of the small rotation approximation.

One of the modified Kerr spacetimes is the Johannsen-Psaltis [25] metric. This metric takes the deformed Schwarszchild spacetime as the starting point and applies Newman-Janis algorithm to obtain a rotating spacetime. In Chapter 4, we have extended the approach to obtain the charged analogue of the Johannsen-Psaltis metric. This has been done by taking the Reissner-Nordström spacetime as the seed metric and applying Newman-Janis algorithm to it. This leads to a spacetime which is stationary, axisymmetric and asymptotically flat. It gives the Johannsen-Psaltis metric on setting charge equal to zero. Setting the deviation parameter $\epsilon = 0$ yields the Kerr-Newman metric. The electromagnetic 4-potential contains the deviation parameter ϵ in its radial component, thus inducing θ dependence in it. Thus A_r component cannot be gauged away, so we have a nonzero A_r making the 4-potential A_μ different from the case of Kerr-Newman. Further analysis of A_r revels its mass M dependence through the deviation parameter ϵ which can be considered as a coupling between the gravitational and the electromagnetic fields.

The event horizon of the charged Johannsen-Psaltis spacetime has been studied. Here horizon is taken as a general function [77] of θ, $H(\theta)$, so that horizon is $r = H(\theta)$. The equation which determines $H(\theta)$ is given by (4.38). Next, the Killing horizon is discussed. The graphs are drawn for various values of parameters M, Q, a, ϵ. The graphs show that for $\epsilon < \epsilon^{max-pole}$, the outer and inner horizons have spherical topology while for $\epsilon \approx \epsilon^{max-pole}$ Killing horizons intersect at the equatorial plane. In case of $\epsilon > \epsilon^{max-pole}$, disjoint Killing horizons appear above and below the equatorial plane. For negative ϵ, the inner and outer Killing horizons are in the shape of the spherical surface for $a^2+Q^2 < M$ while the shape changes to a toroidal surface if $a^2 + Q^2 > M$. From the determinant of the charged Johannsen-Psaltis metric we find that it is independent of the Lorentz violating regions. The plots of the closed time-like curve and the Killing horizons show that it is surrounded by the Killing horizon. Therefore, the charged Johannsen-Psaltis metric does not contain closed time-like curves outside of the central body.

Considering the charged Johannsen-Psaltis metric as a perturbation of the Kerr-Newman metric upto first order in ϵ, we find that its Kretschmann scalar diverges at $r = 0$, therefore it represents a black hole for all values of ϵ.

The analysis of the ISCO and circular photon orbits shows the dependence on ϵ. Both show decreasing behavior with the increase of ϵ.

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