Some Studies on Roughness of Bipolar Fuzzy and Bipolar Soft Substructures

Ph. D Thesis

By

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Quaid-i-Azam University
Islamabad, Pakistan
2020

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IN

MATHEMATICS

Supervised

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2020

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Some Studies on Roughness of Bipolar Fuzzy and Bipolar Soft Substructures

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A thesis Submitted in Partial Fulfillment of the requirement for the degree of doctor of philosophy in mathematics.

We accept this thesis as conforming to the required standard

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This is to certify that the research work presented in this thesis entitled **"Some Studies on Roughness of Bipolar Fuzzy and Bipolar Soft Substructures"** was conducted by Ms. **Nosheen Malik** under the supervision of **Prof. Dr. Muhammad Shabir**. No part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the Department of Mathematics, Quaid-i-Azam University, Islamabad in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Field of Mathematics from Department of Mathematics, Quaid-i-Azam University Islamabad, Pakistan.

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DEDICATION

THIS RESEARCH HAS BEEN DEDICATED TO MY PARENTS, HONORABLE TEACHERS, HUSBAND AND CHILDREN

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Introduction

Theories of rough sets [41, 42], fuzzy sets [48] and soft sets [37] are the most eminent and dynamic mathematical tools for modeling various types of data with uncertainty. Uncertainty and vagueness is often faced in the data assembled and studied for various purposes. The classical mathematical tools are not always convenient to describe such aspects of the real world problems. These theories are presented to handle the uncertainty in data in order to construct a productive mathematical model. The fuzzy set theory reflects the uncertain knowledge in a very fruitful way by grading the elements of the universe on the basis of their characteristics; where grades are assigned from the interval [0, 1].

The rough set theory acquires a completely different mathematical approach towards the uncertainty and ambiguity of the data. In this theory, the level of definiteness of the information associated to a set of objects is interpreted by two definable sets, called the lower and upper rough approximations, subject to the available information.

Another problem in data analysis is to pinpoint the objects from a universe of discourse (a referential or a universe of objects) possessing a particular property or an attribute. This problem is magically ironed out by the soft set theory. Soft sets are defined with the help of a mapping, but entirely different in nature from that of a fuzzy set. This is a set-valued mapping which associates to each attribute a set of objects (instead of a single value or a number) from a universe of discourse pertaining the property of that attribute. Both of these theories, i.e., the fuzzy set theory and the soft set theory diminished the gap between the classical mathematical methodologies and the vague data of the real world.

Bipolarity of the information is also an essential aspect of data while modeling the real world problems. Bipolarity reveals the positive and negative aspects of the data. The positive part demonstrates the preferred information or the feasible data, while the negative part analyzes the inadmissible or implausible data. Bipolar information, therefore, increases the modeling and reasoning capabilities in all domains.

In this thesis, we have hybridized the theory of rough sets with the theory of fuzzy sets and the theory of soft sets endowed with the bipolar information in three different directions. We have introduced the notions of the rough bipolar fuzzy sets, the rough bipolar soft sets and the rough fuzzy bipolar soft sets by defining the roughness in the bipolar fuzzy sets [29, 50], the bipolar soft sets [40] and the fuzzy bipolar soft sets [45]. This is done by developing the lower and upper rough approximations of these sets using the approach adopted by Pawlak [41, 42], who partitioned the universe of objects into granules (classes) of objects. We have also explored some characterizations of the rough bipolar fuzzy sets, the rough bipolar soft sets and the rough fuzzy bipolar soft sets. We have also developed the similarity relations, the accuracy measures and the roughness measures for these newly presented notions.

The theory of semigroups is a substantial part of algebra and this theory is incomplete without the study of ideals. The theory of semigroups and the ideals in semigroups are also amalgamated with rough sets, along with the bipolar fuzzy sets, the bipolar soft sets and the fuzzy bipolar soft sets in this work. In this thesis, we have also defined and discussed the notions of different rough (left, right, two-sided, interior, bi-) ideals in the bipolar fuzzy semigroups, the bipolar soft semigroups and the fuzzy bipolar soft semigroups.

The basic purpose to build the rough set theory, the fuzzy set theory and the soft set theory was to model many real world problems more efficiently. So, these theories have a great practicality in many areas of data analysis. An important application of these theories is the development of many decision making techniques. We have developed different decision making techniques using the rough approximations of the bipolar fuzzy sets, the bipolar soft sets and the fuzzy bipolar soft sets. We have also designed the algorithms for those techniques, accompanied by suitable examples. These algorithms also support different group decision making (GDM) problems when there is a group of decision makers having different opinions who intend to arrive at a single decision.

Chapter-wise study

This thesis consists of seven chapters which are briefly described below.

Chapter 1 reviews previous work related to the fuzzy set theory, the soft set theory and the rough set theory. The bipolar fuzzy sets, the bipolar soft sets and the fuzzy bipolar soft sets are also discussed. Some basic definitions related to the ideals and fuzzy ideals in semigroups are also reviewed.

Chapter 2 presents the rough bipolar fuzzy sets by defining the lower and upper rough approximations of the bipolar fuzzy sets in the Pawlak approximation space. The notion is further explained by exploring its structural properties. Based on these approximations, some similarity relations between the bipolar fuzzy sets are presented. The accuracy measure and the roughness measure for the lower and upper rough approximations of the bipolar fuzzy sets are also provided. The practical application of the proposed model in decision analysis is demonstrated with an algorithm for a group decision making problem, supported by an example.

In Chapter 3, the concept of roughness developed in Chapter 2 is applied to the bipolar fuzzy ideals in semigroups and some properties of the rough bipolar fuzzy semigroups are investigated. The rough bipolar fuzzy (left, right, two-sided) ideals, the rough bipolar fuzzy interior ideals and the rough bipolar fuzzy bi-ideals are also defined and an overview of the properties of these ideals is presented.

Chapter 4 presents the rough bipolar soft sets by defining the lower and upper rough approximations of the bipolar fuzzy sets in the Pawlak approximation space. The structural properties of these approximations are explored and some similarity relations between the bipolar soft sets based on these approximations, are presented. The practicality of the rough approximations of the bipolar soft sets in decision making techniques is demonstrated in two different directions by designing two different algorithms accompanied by suitable examples.

In Chapter 5, the notions of bipolar soft semigroups and the bipolar soft subsemigroups are presented. The concept of rough bipolar soft sets developed in the Chapter 4 is infused with the theory of semigroups and the rough bipolar soft (left, right, two-sided) ideals, the rough bipolar soft interior ideals and the rough bipolar soft bi-ideals are also defined and some characterizations of these ideals are examined.

Chapter 6 presents the rough fuzzy bipolar soft sets by defining the lower and upper rough approximations of the fuzzy bipolar soft sets in the Pawlak approximation space. The notion is further explored by studying its structural properties. Based on these approximations, some similarity relations between the fuzzy bipolar soft sets are presented. The accuracy measure and the roughness measure for the lower and upper rough approximations of the fuzzy bipolar soft sets are also provided. The practical application of the proposed model in decision analysis is demonstrated with an algorithm, supported by an example.

In Chapter 7, the notions of fuzzy bipolar soft semigroups and the fuzzy bipolar soft subsemigroups are presented. The fuzzy bipolar soft ideals, the fuzzy bipolar soft interior ideals and the fuzzy bipolar soft bi-ideals are also defined and discussed. The concept of rough fuzzy bipolar soft sets developed in the Chapter 4 is applied to the theory of fuzzy bipolar soft semigroups, the fuzzy bipolar soft ideals, the fuzzy bipolar soft interior ideals and the fuzzy bipolar soft bi-ideals. An overview of some characterizations of these ideals is also presented.

Contents

Chapter 1

Preliminaries

1.1 Introduction

Uncertainty and vagueness is often faced in the data assembled and studied for various purposes. The classical mathematical tools are not always convenient to demonstrate such aspects of the real world problems. The most eminent and dynamic mathematical tools for modeling various types of data with uncertainty are the theories of rough sets, fuzzy sets and soft sets. These theories are presented to handle the uncertainty in data in order to formulate a productive mathematical model. For this purpose, Zadeh [65] developed fuzzy set theory. This theory has tremendous applicability in both, mathematics and computer sciences, for example, fuzzy logic, fuzzy automata, decision making, medical science and engineering; see [3, 14, 22, 23, 25, 30, 35, 36, 40, 54]. This theory proved to be very successful to administer the ambiguity observed in many types of data. The fuzzy sets work with the help of a mapping which associates a degree to each object in the universe of discourse. This degree belongs to the interval [0, 1] and exhibits the measure of presence of a particular property or characteristic in the objects. These properties are mostly uncertain to be completely present or completely absent in the objects. For example, the set of *emotional* persons or the set of *good* players. Thus, the fuzzy sets better reflect the real knowledge about the objects.

Another problem in data analysis is to pinpoint the objects from a universe of discourse (a referential or a universe of objects) possessing a particular property or an attribute. Molodtsov [48] ironed out this problem magically by introducing the soft sets in 1999. Soft sets are also defined with the help of a mapping, but entirely different in nature from that of a fuzzy set. This is a set-valued mapping which associates to each attribute a set of objects (instead of a single object or a number) from a universe of discourse pertaining that particular attribute. Some important operations on the soft sets were defined by Ali et al. [9]. Many other researchers have worked on the soft sets in different directions. For instance, [6, 11, 12, 13, 21, 27, 42, 43, 46, 56, 57, 69]. Soft sets are also combined with fuzzy sets to build new concepts; see [7, 10, 29, 28, 41, 44, 50]. Both, the fuzzy sets and the soft sets administer the problems of uncertainty and imprecision. These theories diminished the gap between the classical mathematical methodologies and the vague data of the real world.

A recent trend in contemporary information processing emphasizes on bipolar information: both from a knowledge representation point of view and from a processing and reasoning one. Bipolarity of the information is an essential aspect of data while modeling the real world problems in many sciences. Bipolarity reveals the positive and negative aspects of the data. The positive part demonstrates the preferred information or the feasible data, while the negative part analyzes the inadmissible or implausible data. Bipolar information, therefore, increases the modeling and reasoning capabilities in all domains. The bipolar representation of the information was introduced by Dubois and Prade [26]. Zhang [67] equipped the fuzzy sets with bipolarity and presented the bipolar fuzzy sets. Lee [38] defined a few elementary operations on the bipolar fuzzy sets. Bipolar fuzzy sets were also studied by some other authors; see [4, 5, 20, 34]. Bipolar soft sets were initiated by Shabir and Naz [61].

The rough set theory [52, 53] presented by Pawlak, acquires a completely different mathematical approach towards the uncertainty and ambiguity of the data. This theory provides an efficient strategy to tackle the uncertain and doubtful data. In this theory, the rough approximations of a set of objects interpret the level of definiteness of the information associated to the objects, subject to the available information. These approximations make us able to speculate the exactness or uncertainty in the data. This theory is raised by many authors; see [8, 15, 17, 19, 32, 37, 64, 68]. Rough sets are also crossbred with soft sets and fuzzy sets to present innovative and practical concepts; see [17, 18, 24, 29, 37, 45, 49].

1.2 Fuzzy sets

Zadeh [65] generalized the crisp sets to the fuzzy sets (abbreviated as FSs). An FS f_U in a universe U ($\neq \phi$) is described with the help of a mapping $f_U : U \rightarrow [0, 1]$ which associates a value $f_U(u)$ to each object u of the set U. This value portrays the extent to which an object u satisfies the property of f_U. The value $f_U(u)$ is known as the belongingness grade of the object u and the mapping f_U is known as the belongingness map of U. The FS f_U is non-empty if f_U is not a zero map. Let U be a universe of discourse (obviously non-empty) and let the collection of all FSs in U be symbolized by $F_z(U)$. Then, the formal definitions of operations on FSs, as established by Zadeh, are given below.

Definition 1.2.1 *For the FSs f_U and g_U in U, we say that f_U is subset of g_U, that is, $f_U \subseteq g_U$ if and only if $f_U(u) \leq g_U(u)$ for each $u \in U$.*

Clearly, $f_U = g_U$ if and only if $f_U \subseteq g_U$ and $g_U \subseteq f_U$.

Definition 1.2.2 *The null FS in U is defined by the mapping $\emptyset_U : U \longrightarrow [0,1]$, such that, $\emptyset_U(u) = 0$ for each $u \in U$.*

Definition 1.2.3 *The whole FS in U is defined by the mapping $I_U : U \longrightarrow [0,1]$, such that, $I_U(u) = 1$ for each $u \in U$.*

Definition 1.2.4 *The union and intersection of the FSs f_U and g_U in U are defined as:*

$$(f_U \cup g_U)(u) = f_U(u) \vee g_U(u) \qquad \text{for each } u \in U,$$

$$(f_U \cap g_U)(u) = f_U(u) \wedge g_U(u) \qquad \text{for each } u \in U.$$

Definition 1.2.5 *The compliment of an FS f_U in U is symbolized by f'_U and defined as:*

$$f'_U(u) = 1 - f_U(u) \qquad \text{for each } u \in U.$$

Definition 1.2.6 *An FS f_U in U is taken to be constant in U, if and only if the belongingness map $f_U : U \longrightarrow [0,1]$ is a constant function.*

Example 1.2.7 *Consider a group $U = \{p_{15}, p_{20}, p_{28}, p_{35}, p_{46}, p_{60}\}$ of six persons of same height, where each $p_y \in U$ is 'y' years old. Let f_U be an FS in U, describing "how young a person is". Then, f_U is defined as:*

U	p_{15}	p_{20}	p_{28}	p_{35}	p_{46}	p_{60}
$f_U(p_y)$	1	1	0.7	0.5	0.3	0

Let g_U be another FS in U, describing "how tall a person is", defined as:

U	p_{15}	p_{20}	p_{28}	p_{35}	p_{46}	p_{60}
$g_U(p_y)$	0.8	0.8	0.8	0.8	0.8	0.8

Then, g_U is a constant FS in U.

1.3 Bipolar fuzzy sets

Zhang [67] enriched the FSs with the bipolar information and presented the idea of the bipolar FSs (BFSs). These sets are capable to handle the fuzziness, as well as, the bipolarity (the degrees of positivity and negativity) in the data. In the BFSs, the belongingness degrees are expressed by a pair of belongingness maps.

Definition 1.3.1 *[67] A BFS λ in a non-empty universe U is defined as:*

$$\lambda = \{(u, \lambda^P(u), \lambda^N(u)) : u \in U\},$$

where $\lambda^P : U \longrightarrow [0,1]$ and $\lambda^N : U \longrightarrow [-1,0]$ are the positive belongingness map and the negative belongingness map, respectively.

The value $\lambda^P(u)$ of the positive belongingness map denotes the degree of fulfilment of an object u to the property of the BFS λ, while the value $\lambda^N(u)$ of the negative belongingness map denotes the degree of fulfilment of the object u to some implicit counter property of λ. The object u is taken to be irrelevant to the property of λ, if $\lambda^P(u) = 0 = \lambda^N(u)$. The set of all BFSs in U is symbolized by $BFS(U)$. We can write $\lambda(u) = (\lambda^P(u), \lambda^N(u))$ for $(u, \lambda^P(u), \lambda^N(u)) \in \lambda$. Lee [38] defined some elementary operations on the BFSs, which are given below.

Definition 1.3.2 *Let $\lambda, \nu \in BFS(U)$. Then, λ is contained in ν, that is, $\lambda \subseteq \nu$, if $\lambda^P(u) \leq \nu^P(u)$ and $\lambda^N(u) \geq \nu^N(u)$ for each $u \in U$. Clearly, $\lambda = \nu$ if and only if $\lambda \subseteq \nu$ and $\nu \subseteq \lambda$.*

Definition 1.3.3 *The null BFS in U is symbolized by $\widehat{\emptyset} = (\widehat{\emptyset}^P, \widehat{\emptyset}^N)$, such that, $\widehat{\emptyset}^P(u) = 0$ and $\widehat{\emptyset}^N(u) = -1$ for each $u \in U$. Thus, $\widehat{\emptyset}(u) = (0, -1)$ for each $u \in U$.*

Definition 1.3.4 *The whole BFS in U is symbolized by $\widehat{I} = (\widehat{I}^P, \widehat{I}^N)$, such that, $\widehat{I}^P(u) = 1$ and $\widehat{I}^N(u) = 0$ for each $u \in U$. Thus, $\widehat{I}(u) = (1, 0)$ for each $u \in U$.*

Definition 1.3.5 *Let $\lambda, \nu \in BFS(U)$. The union and intersection of λ and ν are the BFSs in U defined as follows:*

$$\lambda \cup \nu = \{(u, \lambda^P(u) \vee \nu^P(u), \lambda^N(u) \wedge \nu^N(u)) : u \in U\},$$

$$\lambda \cap \nu = \{(u, \lambda^P(u) \wedge \nu^P(u), \lambda^N(u) \vee \nu^N(u)) : u \in U\}.$$

We write $\lambda \cup \nu = (\lambda^P \cup \nu^P, \lambda^N \cap \nu^N)$ and $\lambda \cap \nu = (\lambda^P \cap \nu^P, \lambda^N \cup \nu^N)$.

Definition 1.3.6 *The compliment λ' of $\lambda \in BFS(U)$ is given by*

$$\lambda' = \{(u, 1 - \lambda^P(u), -1 - \lambda^N(u)) : u \in U\}.$$

We write $\lambda' = ((\lambda')^P, (\lambda')^N)$.

Definition 1.3.7 *A BFS λ in U is said to be a constant BFS in U, if and only if the belongingness maps $\lambda^P : U \longrightarrow [0,1]$ and $\lambda^N : U \longrightarrow [-1,0]$ are constant functions in U.*

Example 1.3.8 *Consider the data of Example 1.2.7, in which, the FS f_U in U, approximates the degree to which a person is young. But, the degree to which a person is old, may not be approximated by the compliment of f_U. For instance, $f_U(p_{35}) = 0.5$. But, the degree of p_{35} for "being old" is not $1 - f_U(p_y) = 0.5$, as a person aging 35 years, may be considered as a middle aged person, but not old. A BFS λ in U, can better define the degrees of the persons for "being young" and "being old", as below.*

U	p_{15}	p_{20}	p_{28}	p_{35}	p_{46}	p_{60}
$\lambda(p_y)$	$(1,0)$	$(1,0)$	$(0.7,0)$	$(0.5,-0.1)$	$(0.3.-0.3)$	$(0.-0.8)$

Another BFS ν in U, describing the degrees of the persons for "being tall" and "being short", is defined as:

U	p_{15}	p_{20}	p_{28}	p_{35}	p_{46}	p_{60}
$\nu(p_y)$	$(0.8,0)$	$(0.8,0)$	$(0.8,0)$	$(0.8,0)$	$(0.8,0)$	$(0.8,0)$

Then, ν is a constant BFS in U.

1.4 Soft sets

The theory of soft sets, a state-of-the-art device to handle the ambiguities in mathematical models, was initiated by Molodtsov [48]. The foundation of this theory is the conjecture that each set containing objects in the universe U is accompanied by a set $\breve{E}$ of attributes (characteristics or properties that the objects of U may possess) for U. Molodtsov's soft sets point out all the objects in U, which own some particular attributes $e \in \breve{E}$, employing a set-valued mapping. This mapping assigns to each attribute $e \in \breve{E}$, a subset of U comprising of those objects which possess the property e. Let $P(U)$ express the power set of U. Then, the formal definition of a soft set over U is presented below.

Definition 1.4.1 *[48] A soft set over U is expressed by $(\xi; \breve{A})$, where $\breve{A} \subseteq \breve{E}$ and $\xi : \breve{A} \to P(U)$ is a set-valued mapping.*

A soft set $(\xi; \breve{A})$ over U, thus, associates to each parameter $e \in \breve{A}$ a subset $\xi(e)$ of U. This subset $\xi(e)$ contains the objects of U having the property e (according to ξ). The elements of $\xi(e)$ may be called the $e-$approximate objects of $(\xi; \breve{A})$. These sets cover the problem (of uncertainty) that the objects pertaining a particular property often differ according to the opinions of different persons. For instance, if $(\xi_1; \breve{A})$ and $(\xi_2; \breve{A})$ are two soft sets over U, describing the opinions of two different persons, then $\xi_1(e)$ and $\xi_2(e)$ may contain deferent objects for same parameter $e \in \breve{A}$. Some necessary operations on the soft sets are discussed in [9] and [42].

1.5 Bipolar soft sets

The bipolar soft sets (BSSs) were proposed by Shabir and Naz [61] in 2009. These sets are built to distinguish between the preferred and adverse sides of data. The preferred part demonstrates the feasible information, while the adverse part analyzes the inadmissible or implausible data. A BSS is obtained by defining two set-valued mappings from two sets of attributes to the power set of U. One mapping is from the set $\breve{E}$ having positive attributes of the objects of U, while, the other is from the attribute set $\neg\breve{E}$ having the attributes implicitly adverse to those of $\breve{E}$. The set $\neg\breve{E}$ is pronounced and defined as the "counter" set of $\breve{E}$.

Definition 1.5.1 *[61] A BSS over U is symbolized by $\eth = (\xi, \psi; \breve{A})$, where $\breve{A} \subseteq \breve{E}$ and ξ, ψ are set-valued mappings given by $\xi : \breve{A} \to P(U)$ and $\psi : \neg\breve{A} \to P(U)$ such that $\xi(e) \cap \psi(\neg e) = \phi$ for each $e \in \breve{A}$.*

Thus, a BSS $(\xi, \psi; \breve{A})$ can be obtained by merging two soft sets $(\xi, \breve{A})$ and $(\psi, \neg\breve{A})$ together, such that $\xi(e)$ and $\psi(\neg e)$ are disjoint for each $e \in \breve{A}$. Here, $\neg\breve{A}$ is the *counter set of* $\breve{A}$ and $\neg\breve{A} \subseteq \neg\breve{E}$. The restriction $\xi(e) \cap \psi(\neg e) = \phi$ is applied as a consistency restraint. $\xi(e)$ denotes the objects in U having a property $e \in \breve{A}$ and $\psi(\neg e)$ denotes the objects in U having a property $\neg e$, opposite to e. We denote the set containing all BSSs over U by $BSS(U)$. It is worth noting that an object lacking a property e, may not have the opposite property $\neg e$. So, we may have $\psi(\neg e) \neq U - \xi(e)$ for some $e \in \breve{E}$. This difference is named as the degree of reluctance, which occurs due to the inadequate knowledge or hesitancy in deciding for an object to have an attribute e or $\neg e$. The BSSs beautifully highlight such objects neither possessing the property e, nor $\neg e$. These sets can be better perceived from the following simple example.

Example 1.5.2 *Suppose that $U = \{q_i; i = 1, 2, 3, 4, 5\}$ is a universe containing five houses and $\breve{E} = \{e_1 =$costly, $e_2 =$attractive, $e_3 =$wooden, $e_4 =$in natural surroundings, $e_5 =$properly maintained$\}$ is a set of possible attributes for U. Let the "counter" set of $\breve{E}$ be $\neg\breve{E} = \{\neg e_1 =$cheap, $\neg e_2 =$dull, $\neg e_3 =$not wooden, $\neg e_4 =$in urban area, $\neg e_5 =$not maintained$\}$. Take a BSS $(\xi, \psi; \breve{A})$ expressing the "attractiveness of houses" that Mr. X intends to purchase. Here, $(\xi, \psi; \breve{A})$ points out the costly or cheep houses, attractive or dull houses and so on, according to Mr. X and $\breve{A} = \{e_1, e_2, e_3\} \subseteq \breve{E}$ contains the attributes of interest of Mr. X. We can construct the BSS $(\xi, \psi; \breve{A})$ completely as:*

$\xi(e_1) = \{q_1, q_3, q_4\}$, $\xi(e_2) = \{q_2, q_3, q_5\}$, $\xi(e_3) = \{q_4\}$ *and*

$\psi(\neg e_1) = \{q_2, q_5\}$, $\psi(\neg e_2) = \phi$, $\psi(\neg e_3) = \{q_1, q_2, q_3, q_5\}$.

This BSS $(\xi, \psi; \breve{A})$ can be seen in Figure 1.1 below.

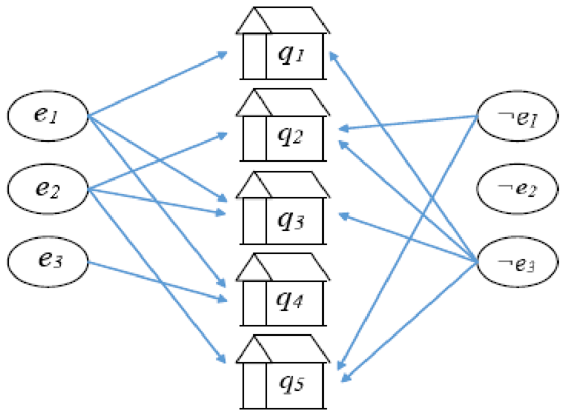

Figure 1.1: BSS $(\xi, \psi; \breve{A})$

The BSS $(\xi, \psi; \breve{A})$ can also be expressed in tabular form by putting the (i, j)th entry a_{ij} of the table as:

$$a_{ij} = \begin{cases} 1 & \text{if } q_j \in \xi(e_i) \\ -1 & \text{if } q_j \in \psi(\neg e_i) \\ 0 & \text{otherwise} \end{cases}$$

Hence, the tabular expression of the BSS $(\xi, \psi; \breve{A})$ is shown in Table 1.1.

$(\xi, \psi; \breve{A})$	q_1	q_2	q_3	q_4	q_5
e_1	1	−1	1	1	−1
e_2	0	1	1	0	1
e_3	−1	−1	−1	1	−1

Table 1.1: BSS $(\xi, \psi; \breve{A})$

Note that the houses q_1 and q_4 are not attractive, but at the same time, they are not dull, as well. Here, $\{q_1, q_4\}$ is the degree of reluctance of the BSS $(\xi, \psi; \breve{A})$ for $e_2 =$attractive.

Definition 1.5.3 *[61] For any two BSSs $ð_1 = (\xi_1, \psi_1; \breve{A}_1)$, $ð_2 = (\xi_2, \psi_2; \breve{A}_2) \in BSS(U)$, we say that $ð_1$ is a BS subset of $ð_2$, symbolized by $ð_1 \widetilde{\subseteq} ð_2$ if*

1) $\breve{A}_1 \subseteq \breve{A}_2$,

2) $\xi_1(e) \subseteq \xi_2(e)$ and $\psi_1(\neg e) \supseteq \psi_2(\neg e)$ for each $e \in \breve{A}_1$.

Two BSSs $ð_1, ð_2 \in BSS(U)$ are equal if and only if each of them is a BS subset of the other.

Definition 1.5.4 *[61] The relative null BSS over U is $(\Theta, \mathcal{U}; \breve{A}) \in BSS(U)$, where $\Theta : \breve{A} \to P(U)$ and $\mathcal{U} : \neg\breve{A} \to P(U)$ are given by $\Theta(e) = \phi$ and $\mathcal{U}(\neg e) = U$ for each $e \in \breve{A}$. We denote $(\Theta, \mathcal{U}; \breve{A})$ by $\Theta_{\breve{A}}$.*

Definition 1.5.5 *[61] The relative whole BSS over U is* $(\mathcal{U}, \Theta; \breve{A}) \in BSS(U)$, *where* $\mathcal{U} : \breve{A} \to P(U)$ *and* $\Theta : \neg\breve{A} \to P(U)$ *are given by* $\mathcal{U}(e) = U$ *and* $\Theta(\neg e) = \phi$ *for each* $e \in \breve{A}$. *We denote* $(\mathcal{U}, \Theta; \breve{A})$ *by* $\mathcal{U}_{\breve{A}}$.

Definition 1.5.6 *[61] Let* $\eth_1 = (\xi_1, \psi_1; \breve{A}_1)$, $\eth_2 = (\xi_2, \psi_2; \breve{A}_2) \in BSS(U)$. *Then, their union and intersection are expressed as follows:*

1. *The extended union of* $\eth_1$ *and* $\eth_2$ *is a BSS*

$$\eth_1 \widetilde{\sqcup}_\varepsilon \eth_2 = (\xi_1 \widetilde{\cup}_\varepsilon \xi_2, \psi_1 \widetilde{\cap}_\varepsilon \psi_2; \breve{A}_1 \cup \breve{A}_2)$$

over U, where $\xi_1 \widetilde{\cup}_\varepsilon \xi_2 : \breve{A}_1 \cup \breve{A}_2 \to P(U)$ *is expressed as:*

$$(\xi_1 \widetilde{\cup}_\varepsilon \xi_2)(e) = \begin{cases} \xi_1(e) & \text{if } e \in \breve{A}_1 - \breve{A}_2 \\ \xi_2(e) & \text{if } e \in \breve{A}_2 - \breve{A}_1 \\ \xi_1(e) \cup \xi_2(e) & \text{if } e \in \breve{A}_1 \cap \breve{A}_2 \end{cases}$$

and $\psi_1 \widetilde{\cap}_\varepsilon \psi_2 : \neg(\breve{A}_1 \cup \breve{A}_2) \to P(U)$ *is expressed as:*

$$(\psi_1 \widetilde{\cap}_\varepsilon \psi_2)(\neg e) = \begin{cases} \psi_1(\neg e) & \text{if } \neg e \in (\neg\breve{A}_1) - (\neg\breve{A}_2) \\ \psi_2(\neg e) & \text{if } \neg e \in (\neg\breve{A}_2) - (\neg\breve{A}_1) \\ \psi_1(\neg e) \cap \psi_2(\neg e) & \text{if } \neg e \in \neg(\breve{A}_1 \cap \breve{A}_2) \end{cases}$$

2. *The extended intersection of* $\eth_1$ *and* $\eth_2$ *is a BSS*

$$\eth_1 \widetilde{\sqcap}_\varepsilon \eth_2 = (\xi_1 \widetilde{\cap}_\varepsilon \xi_2, \psi_1 \widetilde{\cup}_\varepsilon \psi_2; \breve{A}_1 \cup \breve{A}_2)$$

over U, where $\xi_1 \widetilde{\cap}_\varepsilon \xi_2 : \breve{A}_1 \cup \breve{A}_2 \to P(U)$ *is expressed as:*

$$(\xi_1 \widetilde{\cap}_\varepsilon \xi_2)(e) = \begin{cases} \xi_1(e) & \text{if } e \in \breve{A}_1 - \breve{A}_2 \\ \xi_2(e) & \text{if } e \in \breve{A}_2 - \breve{A}_1 \\ \xi_1(e) \cap \xi_2(e) & \text{if } e \in \breve{A}_1 \cap \breve{A}_2 \end{cases}$$

and $\psi_1 \widetilde{\cup}_\varepsilon \psi_2 : \neg(\breve{A}_1 \cup \breve{A}_2) \to P(U)$ *is expressed as:*

$$(\psi_1 \widetilde{\cup}_\varepsilon \psi_2)(\neg e) = \begin{cases} \psi_1(\neg e) & \text{if } \neg e \in (\neg\breve{A}_1) - (\neg\breve{A}_2) \\ \psi_2(\neg e) & \text{if } \neg e \in (\neg\breve{A}_2) - (\neg\breve{A}_1) \\ \psi_1(\neg e) \cup \psi_2(\neg e) & \text{if } \neg e \in \neg(\breve{A}_1 \cap \breve{A}_2) \end{cases}$$

3. *The restricted union of* $\eth_1$ *and* $\eth_2$ *is a BSS*

$$\eth_1 \widetilde{\sqcup}_r \eth_2 = (\xi_1 \widetilde{\cup}_r \xi_2, \psi_1 \widetilde{\cap}_r \psi_2; \breve{A}_1 \cap \breve{A}_2)$$

over U, where $\xi_1 \widetilde{\cup}_r \xi_2 : \breve{A}_1 \cap \breve{A}_2 \to P(U)$ *is expressed as* $(\xi_1 \widetilde{\cup}_r \xi_2)(e) = \xi_1(e) \cup \xi_2(e)$ *for each* $e \in \breve{A}_1 \cap \breve{A}_2$, *and* $\psi_1 \widetilde{\cap}_r \psi_2 : \neg(\breve{A}_1 \cap \breve{A}_2) \to P(U)$ *is expressed as* $(\psi_1 \widetilde{\cap}_r \psi_2)(\neg e) = \psi_1(\neg e) \cap \psi_2(\neg e)$ *for each* $\neg e \in \neg(\breve{A}_1 \cap \breve{A}_2)$, *provided* $\breve{A}_1 \cap \breve{A}_2 \neq \phi$.

4. *The restricted intersection of* $\eth_1$ *and* $\eth_2$ *is a BSS*

$$\eth_1 \widetilde{\sqcap}_r \eth_2 = (\xi_1 \widetilde{\sqcap}_r \xi_2, \psi_1 \widetilde{\sqcup}_r \psi_2; \breve{A}_1 \cap \breve{A}_2)$$

over U, *where* $\xi_1 \widetilde{\sqcap}_r \xi_2 : \breve{A}_1 \cap \breve{A}_2 \to P(U)$ *is expressed as* $(\xi_1 \widetilde{\sqcap}_r \xi_2)(e) = \xi_1(e) \cap \xi_2(e)$ *for each* $e \in \breve{A}_1 \cap \breve{A}_2$, *and* $\psi_1 \widetilde{\sqcup}_r \psi_2 : \neg(\breve{A}_1 \cap \breve{A}_2) \to P(U)$ *is expressed as* $(\psi_1 \widetilde{\sqcup}_r \psi_2)(\neg e) = \psi_1(\neg e) \cup \psi_2(\neg e)$ *for each* $\neg e \in \neg(\breve{A}_1 \cap \breve{A}_2)$, *provided* $\breve{A}_1 \cap \breve{A}_2 \neq \phi$.

Definition 1.5.7 *[61] The compliment of a BSS* $\eth = (\xi, \psi; \breve{A})$ *over* U *is a BSS* $\eth^c = (\xi^c, \psi^c; \breve{A})$ *over* U, *where* $\xi^c(e) = \psi(\neg e)$ *and* $\psi^c(\neg e) = \xi(e)$ *for each* $e \in \breve{A}$.

1.6 Fuzzy bipolar soft sets

Fuzzy bipolar soft sets (FBSSs) were presented by Naz and Shabir [51]. Similar to the BSSs, an FBSS is also constructed by employing two mappings. One mapping is from the set $\breve{E}$ of attributes to the fuzzy power set $\digamma_z(U)$, which approximates the degree of presence of the attributes in the objects of U. While, the other is from $\neg\breve{E}$ to $\digamma_z(U)$, which approximates the degree of presence of the implicit counter attributes in the objects of U. Following the discussion in [51], we present some definitions and examples.

Definition 1.6.1 *A triplet* $\omega = (\xi, \psi; \breve{A})$ *is called an FBSS over* U, *where* $\breve{A} \subseteq \breve{E}$ *and* ξ, ψ *are mappings given by* $\xi : \breve{A} \to \digamma_z(U)$ *and* $\psi : \neg\breve{A} \to \digamma_z(U)$ *such that*

$$0 \leq \xi(e)(u) + \psi(\neg e)(u) \leq 1$$

for each $e \in \breve{A}$ *and for each* $u \in U$, *where* $\neg\breve{A}$ *stands for the "counter" set of* $\breve{A}$.

Here, $\xi(e)$ and $\psi(\neg e)$ represent FSs in U. The value $\xi(e)(u)$ denotes the degree of presence of a property e in an object u of U, while $\psi(\neg e)(u)$ denotes the degree of presence of the adverse property $\neg e$ in u. The restriction $0 \leq \xi(e)(x) + \psi(\neg e)(x) \leq 1$ is applied as a consistency restraint. We symbolize the set containing all FBSSs over U by $FBSS(U)$.

Example 1.6.2 *Consider the universe* $U = \{q_1, q_2, q_3, q_4, q_5\}$ *of houses and the attribute sets* $\breve{E}$ *and* $\neg\breve{E}$, *as in Example 1.5.2. The BSS defined in that example identifies the houses with attributes* e *or* $\neg e$. *We define, here, an FBSS* $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$ *over* U, *describing the degree to which these houses pertain these attributes. Assume that* $\breve{A}_1 = \{e_1, e_2, e_3\}$ *and that Mr. X assigns the belongingness values* $\{0.7, 0.6, 0.8, 0.5, 0.6\}$

and $\{0.2, 0.3, 0.1, 0.5, 0.3\}$*, as shown in Figure 1.2, to the houses in* U *for the attribute* e_1*, describing the degrees of how costly and how cheap are the houses, respectively.*

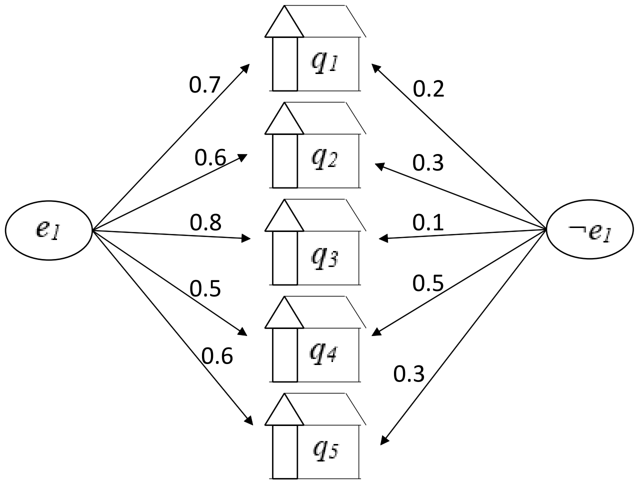

Figure 1.2: Belongingness values of ω_1 *for* e_1

Then, $\xi_1(e_1)$ *and* $\psi_1(\neg e_1)$ *are the FSs in* U *given below.*

$\xi_1(e_1) = \{q_1/0.7,\ q_2/0.6,\ q_3/0.8,\ q_4/0.5,\ q_5/0.6\}$

$\psi_1(\neg e_1) = \{q_1/0.2,\ q_2/0.3,\ q_3/0.1,\ q_4/0.5,\ q_5/0.3\}$

In the same way, we assume:

$\xi_1(e_2) = \{q_1/0.8,\ q_2/0.7,\ q_3/0.8,\ q_4/0.6,\ q_5/0.6\}$

$\psi_1(\neg e_2) = \{q_1/0.1,\ q_2/0.1,\ q_3/0.2,\ q_4/0.2,\ q_5/0.3\}$

$\xi_1(e_3) = \{q_1/0.4,\ q_2/0.6,\ q_3/0.4,\ q_4/0.6,\ q_5/0.5\}$

$\psi_1(\neg e_3) = \{q_1/0.5,\ q_2/0.2,\ q_3/0.5,\ q_4/0.4,\ q_5/0.5\}$

This FBSS can also be represented in tabular form by setting the entry against e_i *and* q_j *as* (a_{ij}, b_{ij})*, where* $a_{ij} = \xi(e_i)(q_j)$ *and* $b_{ij} = \psi(\neg e_i)(q_j)$*. Hence, the tabular representation of* $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$ *is given by Table 1.2.*

ω_1	q_1	q_2	q_3	q_4	q_5
e_1	$(0.7, 0.2)$	$(0.6, 0.3)$	$(0.8, 0.1)$	$(0.5, 0.5)$	$(0.6, 0.3)$
e_2	$(0.8, 0.1)$	$(0.7, 0.1)$	$(0.8, 0.2)$	$(0.6, 0.2)$	$(0.6, 0.3)$
e_3	$(0.4, 0.5)$	$(0.6, 0.2)$	$(0.4, 0.5)$	$(0.6, 0.4)$	$(0.5, 0.5)$

Table 1.2: Table of FBSS $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$

Definition 1.6.3 *For any two FBSSs* $\omega_1 = (\xi_1, \psi_1; \breve{A}_1), \omega_2 = (\xi_2, \psi_2; \breve{A}_2) \in FBSS(U)$*, we say that* ω_1 *is an FBS subset of* ω_2*, symbolized by* $\omega_1 \widetilde{\subseteq} \omega_2$*, if*

1) $\breve{A}_1 \subseteq \breve{A}_2$*,*

2) $\xi(e) \subseteq \xi_1(e)$ *and* $\psi(\neg e) \supseteq \psi_1(\neg e)$ *for each* $e \in \breve{A}_1$*.*

Two FBSSs ω_1 *and* ω_2 *over* U *are equal if and only if each of them is an FBS subset of the other.*

Definition 1.6.4 *The relative null FBSS is* $(\Phi, \tilde{U}; \breve{A}) \in FBSS(U)$*, symbolized by* $\Phi_{\breve{A}}$*, where* $\Phi(e) = \emptyset_U$ *and* $\tilde{U}(\neg e) = I_U$ *for each* $e \in \breve{A}$*.*

Definition 1.6.5 *The relative whole FBSS is* $(\tilde{U}, \Phi; \breve{A}) \in FBSS(U)$*, symbolized by* $\tilde{U}_{\breve{A}}$*, where* $\tilde{U}(e) = I_U$ *and* $\Phi(\neg e) = \emptyset_U$ *for each* $e \in \breve{A}$*.*

Definition 1.6.6 *Let* $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$*,* $\omega_2 = (\xi_2, \psi_2; \breve{A}_2) \in FBSS(U)$*. Then, their unions and intersections are defined as under:*

1. *The extended union of* ω_1 *and* ω_2 *is an FBSS*

$$\omega_1 \widetilde{\sqcup}_\varepsilon \omega_2 = (\xi_1 \widetilde{\cup}_\varepsilon \xi_2, \psi_1 \widetilde{\cap}_\varepsilon \psi_2; \breve{A}_1 \cup \breve{A}_2)$$

over U*, where* $\xi_1 \widetilde{\cup}_\varepsilon \xi_2 : \breve{A}_1 \cup \breve{A}_2 \to P(U)$ *is given by:*

$$(\xi_1 \widetilde{\cup}_\varepsilon \xi_2)(e) = \begin{cases} \xi_1(e) & \text{if } e \in \breve{A}_1 - \breve{A}_2 \\ \xi_2(e) & \text{if } e \in \breve{A}_2 - \breve{A}_1 \\ \xi_1(e) \cup \xi_2(e) & \text{if } e \in \breve{A}_1 \cap \breve{A}_2 \end{cases}$$

and $\psi_1 \widetilde{\cap}_\varepsilon \psi_2 : \neg(\breve{A}_1 \cup \breve{A}_2) \to P(U)$ *is given by:*

$$(\psi_1 \widetilde{\cap}_\varepsilon \psi_2)(\neg e) = \begin{cases} \psi_1(\neg e) & \text{if } \neg e \in (\neg\breve{A}_1) - (\neg\breve{A}_2) \\ \psi_2(\neg e) & \text{if } \neg e \in (\neg\breve{A}_2) - (\neg\breve{A}_1) \\ \psi_1(\neg e) \cap \psi_2(\neg e) & \text{if } \neg e \in \neg(\breve{A}_1 \cap \breve{A}_2) \end{cases}$$

2. *The extended intersection of* ω_1 *and* ω_2 *is an FBSS*

$$\omega_1 \widetilde{\sqcap}_\varepsilon \omega_2 = (\xi_1 \widetilde{\cap}_\varepsilon \xi_2, \psi_1 \widetilde{\cup}_\varepsilon \psi_2; \breve{A}_1 \cup \breve{A}_2)$$

over U*, where* $\xi_1 \widetilde{\cap}_\varepsilon \xi_2 : \breve{A}_1 \cup \breve{A}_2 \to P(U)$ *is given by:*

$$(\xi_1 \widetilde{\cap}_\varepsilon \xi_2)(e) = \begin{cases} \xi_1(e) & \text{if } e \in \breve{A}_1 - \breve{A}_2 \\ \xi_2(e) & \text{if } e \in \breve{A}_2 - \breve{A}_1 \\ \xi_1(e) \cap \xi_2(e) & \text{if } e \in \breve{A}_1 \cap \breve{A}_2 \end{cases}$$

and $\psi_1 \widetilde{\cup}_\varepsilon \psi_2 : \neg(\breve{A}_1 \cup \breve{A}_2) \to P(U)$ *is given by:*

$$(\psi_1 \widetilde{\cup}_\varepsilon \psi_2)(\neg e) = \begin{cases} \psi_1(\neg e) & \text{if } \neg e \in (\neg\breve{A}_1) - (\neg\breve{A}_2) \\ \psi_2(\neg e) & \text{if } \neg e \in (\neg\breve{A}_2) - (\neg\breve{A}_1) \\ \psi_1(\neg e) \cup \psi_2(\neg e) & \text{if } \neg e \in \neg(\breve{A}_1 \cap \breve{A}_2) \end{cases}$$

3. *The restricted union of* ω_1 *and* ω_2 *is an FBSS*

$$\omega_1 \widetilde{\sqcup}_r \omega_2 = (\xi_1 \widetilde{\cup}_r \xi_2, \psi_1 \widetilde{\cap}_r \psi_2; \breve{A}_1 \cap \breve{A}_2)$$

over U*, where* $(\xi_1 \widetilde{\cup}_r \xi_2)(e) = \xi_1(e) \cup \xi_2(e)$ *for each* $e \in \breve{A}_1 \cap \breve{A}_2$ *and* $(\psi_1 \widetilde{\cap}_r \psi_2)(\neg e) = \psi_1(\neg e) \cap \psi_2(\neg e)$ *for each* $\neg e \in \neg(\breve{A}_1 \cap \breve{A}_2)$*, provided* $\breve{A}_1 \cap \breve{A}_2 \neq \phi$*.*

4. *The restricted intersection of* ω_1 *and* ω_2 *is an FBSS*

$$\omega_1 \widetilde{\sqcap}_r \omega_2 = (\xi_1 \widetilde{\sqcap}_r \xi_2, \psi_1 \widetilde{\cup}_r \psi_2; \breve{A}_1 \cap \breve{A}_2)$$

over U, *where* $(\xi_1 \widetilde{\sqcap}_r \xi_2)(e) = \xi_1(e) \cap \xi_2(e)$ *for each* $e \in \breve{A}_1 \cap \breve{A}_2$ *and* $(\psi_1 \widetilde{\cup}_r \psi_2)(\neg e) = \psi_1(\neg e) \cup \psi_2(\neg e)$ *for each* $\neg e \in \neg(\breve{A}_1 \cap \breve{A}_2)$, *provided* $\breve{A}_1 \cap \breve{A}_2 \neq \phi$.

Definition 1.6.7 *The compliment of an FBSS* $\omega = (\xi, \psi; \breve{A}) \in FBSS(U)$ *is an FBSS* $\omega^c = (\xi^c, \psi^c; \breve{A})$, *where* $\xi^c(e) = \psi(\neg e)$ *and* $\psi^c(\neg e) = \xi(e)$ *for each* $e \in \breve{A}$.

Definition 1.6.8 *An FBSS* $\omega = (\xi, \psi; \breve{A})$ *over* U *is said to be constant, if and only if* $\xi(e)$ *and* $\psi(\neg e)$ *are constant FSs in* U *for each* $e \in \breve{A}$.

1.7 Rough sets

The rough set theory [52] implements a systematic procedure for dealing with vagueness in data due to a situation with doubtful information or inadequate knowledge. The foundation of this theory is the conjecture that every object in the universe of discourse pertains some sort of (exact or vague) information (data). Objects characterized by the same information are indiscernible. If the indiscernible objects are taken to be related to each other, then, an indiscernibility relation is obtained which partitions the object in the universe. The infrastructure of the rough set theory stands on such indiscernibility relations. The Pawlak approximation space (shortly written as P-apx space) is the space $(U, \Re)$, where $\Re$ is an equivalence relation (shortly written as eqv-rel) defined on the universe U. The relation $\Re$ serves as an indiscernibility relation because it defines a partition $U/\Re$ of the universe U into the equivalence classes (eqv-classes) due to indiscernible objects of U. These eqv-classes of $\Re$ are the main constituents of the rough sets. The eqv-class of the element $u \in U$ under the relation $\Re$, is symbolized by $[u]_\Re$ (or by $[u]$, for convenience). On a subset X of U, the relation $\Re$ defines the following two operators.

$$\overline{X} = \{u \in U : [u]_\Re \cap X \neq \phi\},$$
$$\underline{X} = \{u \in U : [u]_\Re \subseteq X\}.$$

The subsets $\overline{X}$ and $\underline{X}$ of U, designated to the subset X of U, are called the upper and lower approximations of X, subject to the relation $\Re$, respectively. Note that, both $\overline{X}$ and $\underline{X}$ are unions of disjoint classes in $U/\Re$. Moreover;

$Pos_\Re X = \underline{X}$,
$Neg_\Re X = U - \overline{X}$,

$Bnd_\Re X = \overline{X} - \underline{X}$

are called the positive, negative and boundary regions (the associated regions) of X in U. The information about X depicted by these regions is as follows:

- $x \in Pos_{\Re}X$ means that X definitely contains x.
- $x \in Neg_{\Re}X$ means that X certainly does not contain x.
- $x \in Bnd_{\Re}X$ means that X may or may not contain x.

Definition 1.7.1 *[54] Take a P-apx space* $(U, \Re)$*. A subset* $X \subseteq U$ *is definable if* $\underline{X} = \overline{X}$*; otherwise* X *is known as a rough set.*

That is, $X \subseteq U$ is rough if $Bnd_{\Re}X \neq \phi$. By this, we mean that a subset $X \subseteq U$ is rough if there are some objects in U, whose belongingness in X is doubtful. Thus, rough sets have a completely different approach towards the uncertainty of data. Using the upper and lower approximations, one can judge how accurate is the information attached to the objects. This can be better understood in the next example.

Example 1.7.2 *Consider a set* U *containing fifteen balls of same size in a bag, out of which, three are blue, five are green and seven are black. We define a relation on the collection of balls in the way such that two balls are related if these are of same color. Surely, this relation turns to be an eqv-rel on* U *which partitions* U *into three eqv-classes. Suppose that we pick out a sample* X *of four balls without seeing and after picking up we see that three selected balls are blue and one is green. The sets* U*,* X*,* $\underline{X}$ *and* $\overline{X}$ *as well as the associated regions of* X *can be seen in Figure 1.3.*

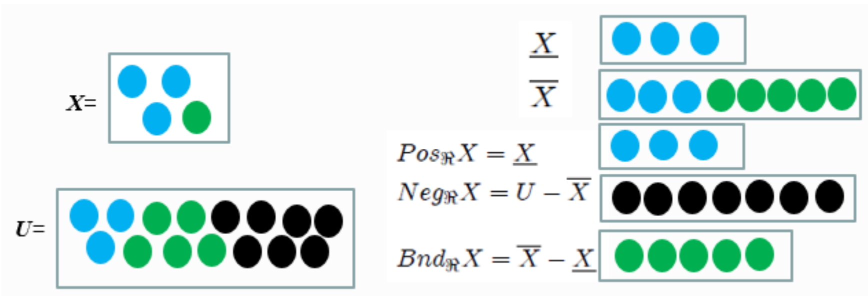

Figure 1.3: Set X and associated sets.

The information about the balls in set X *and its associated sets depicted from Figure 1.3 can be interpreted in the following way.*

- *The lower approximation* $\underline{X}$*, which is also the positive region of* X*, contains the balls (blue balls) of* U *which have definitely gone to sample* X*.*

- *The upper approximation* $\overline{X}$ *contains the balls (blue and green balls) of* U *which have possibly gone to sample* X*.*
- *The difference* $\overline{X} - \underline{X}$ *contains exactly those balls (green balls), such that, the selection of each of them in the sample* X *(deciding that which green ball has gone to the sample) is doubtful. These balls comprise the boundary region of* X*.*
- *The black balls have, surely, no chance to be selected in the sample* X*. These balls comprise the negative region of* X*.*

The following theorem shows some important characteristics of the Pawlak's rough sets.

Theorem 1.7.3 *[53] Take a P-apx space* $(U, \Re)$ *and let* $M, N \subseteq U$*. Then, the subsequent asservations are true.*

1. $\underline{M} \subseteq M \subseteq \overline{M}$;
2. $\underline{\phi} = \phi = \overline{\phi}$;
3. $\underline{U} = U = \overline{U}$;
4. $\underline{(\underline{M})} = \underline{M} = \overline{(\underline{M})}$;
5. $\underline{(\overline{M})} = \overline{M} = \overline{(\overline{M})}$;
6. $(\underline{M})' = \overline{(M')}$;
7. $(\overline{M})' = \underline{(M')}$;
8. $\underline{M \cap N} = \underline{M} \cap \underline{N}$;
9. $\overline{M \cap N} \subseteq \overline{M} \cap \overline{N}$;
10. $\underline{M \cup N} \supseteq \underline{M} \cup \underline{N}$;
11. $\overline{M \cup N} = \overline{M} \cup \overline{N}$;
12. $M \subseteq N$ *implies that* $\underline{M} \subseteq \underline{N}$ *and* $\overline{M} \subseteq \overline{N}$*.*

1.8 Ideals in semigroups

This section reviews some definitions of ideals in semigroups and their characterizations. Recall that, a semigroup comprises of a non-empty set on which a binary operation is defined, satisfying the associative law. Throughout this work, Υ will denote a semigroup, unless and otherwise specified. Take M $(\neq \phi) \subseteq \Upsilon$.

- A subsemigroup of Υ is a set $M \subseteq \Upsilon$ such that $ab \in M$ for each $a, b \in M$ (that is, $MM \subseteq M$).
- A left ideal of Υ is a set $M \subseteq \Upsilon$ such that $xa \in M$ for each $a \in M$ and $x \in \Upsilon$ (that is, $\Upsilon M \subseteq M$).
- A right ideal of Υ is a set $M \subseteq \Upsilon$ such that $ax \in M$ for each $a \in M$ and $x \in \Upsilon$ (that is, $M\Upsilon \subseteq M$).
- M is an ideal of Υ if it is right, as well as, left ideal of Υ.
- An interior ideal of Υ is a set $M \subseteq \Upsilon$ such that $xay \in M$ for each $a \in M$ and $x, y \in \Upsilon$ (that is, $\Upsilon M \Upsilon \subseteq M$).
- A bi-ideal of Υ is a subsemigroup $M \subseteq \Upsilon$ such that $axb \in M$ for each $a, b \in M$ and $x \in \Upsilon$ (that is, $M\Upsilon M \subseteq M$).

A congruence relation (written as cng-rel) $\Re$ on a semigroup Υ is an eqv-rel $\Re$ on Υ which is right and left compatible (that is, $(m, n) \in \Re$ implies that $(am, an), (ma, na) \in \Re$ for each $a, m, n \in \Upsilon$). Let $[m]_\Re$ represent the $\Re$-cng-class of $m \in \Upsilon$. For a cng-rel $\Re$ on Υ, we generally have $[m]_\Re[n]_\Re \subseteq [mn]_\Re$ for each $m, n \in \Upsilon$. A cng-rel $\Re$ on Υ is complete, if $[m]_\Re[n]_\Re = [mn]_\Re$ for each $m, n \in \Upsilon$. This can be observed in the subsequent example.

Example 1.8.1 *Let $\Upsilon = \{s, t, u, v\}$ represent a semigroup whose table of binary operation is given below.*

	s	t	u	v
s	s	t	u	v
t	t	t	u	v
u	u	u	u	v
v	v	v	v	u

We take two cng-rels $\Re_1$ and $\Re_2$ on Υ, such that $\Re_1$ defines the cng-classes $\{s\}, \{t\}$ and $\{u, v\}$, while, $\Re_2$ defines the cng-classes $\{s\}, \{t, u\}$ and $\{v\}$. Observe that, $[x]_{\Re_1}[y]_{\Re_1} = [xy]_{\Re_1}$ for each $x, y \in \Upsilon$. That is, $\Re_1$ is a complete cng-rel on Υ. While, $[v]_{\Re_2}[v]_{\Re_2} \subsetneq [vv]_{\Re_2}$ for $v \in \Upsilon$, because $[v]_{\Re_2} = \{v\}$, so, $[v]_{\Re_2}[v]_{\Re_2} = \{vv\} = \{u\}$ and $[vv]_{\Re_2} = [u]_{\Re_2} = \{t, u\}$. This means, that, $\Re_2$ is not a complete cng-rel on Υ.

1.9 Bipolar fuzzy ideals in semigroups

The bipolar fuzzy ideals (BF-ids) in semigroups were defined by Kim et al. [34]. Yaqoob [63] studied BF-ids in LA-semigroups. This section reviews some definitions about the BF-ids, BF left ideals (BFl-ids), BF right ideals (BFr-ids), BF interior ideals (BFi-ids) and BF bi-ideals (BFb-ids) in a semigroup.

Definition 1.9.1 *[63] Let Υ be a semigroup and let $\lambda, \mu \in BFS(\Upsilon)$. The composition $\lambda \circ \mu$ of the BFSs λ and μ in Υ is defined as:*

$$\lambda \circ \mu = (\lambda^P \circ \mu^P, \lambda^N \circ \mu^N),$$

where

$$(\lambda^P \circ \mu^P)(s) = \begin{cases} \bigvee\limits_{s=mn} (\lambda^P(m) \wedge \mu^P(n)) & \text{if } s = mn \text{ for some } m, n \in \Upsilon \\ 0 & \text{otherwise} \end{cases}$$

and

$$(\lambda^N \circ \mu^N)(s) = \begin{cases} \bigwedge\limits_{s=mn} (\lambda^N(m) \vee \mu^N(n)) & \text{if } s = mn \text{ for some } m, n \in \Upsilon \\ 0 & \text{otherwise} \end{cases}$$

for each $s \in \Upsilon$.

Definition 1.9.2 *[34] A BFS λ in Υ is called a BF subsemigroup of Υ if for each $m, n \in \Upsilon$,*

$$\lambda^P(mn) \geq \lambda^P(m) \wedge \lambda^P(n) \text{ and } \lambda^N(mn) \leq \lambda^N(m) \vee \lambda^N(n).$$

Definition 1.9.3 *[34] A BFS λ in Υ is called a BFl-id (or BFr-id) of Υ if $\lambda^P(mn) \geq \lambda^P(n)$ and $\lambda^N(mn) \leq \lambda^N(n)$ (or $\lambda^P(mn) \geq \lambda^P(m)$ and $\lambda^N(mn) \leq \lambda^N(n)$) for each $m, n \in \Upsilon$.*

A BFS λ in Υ is called a BF-id of Υ if it is both, a BFl-id and a BFr-id of Υ, that is, $\lambda^P(mn) \geq \lambda^P(m) \vee \lambda^P(n)$ and $\lambda^N(mn) \leq \lambda^N(m) \wedge \lambda^N(n)$ for each $m, n \in \Upsilon$.

Definition 1.9.4 *[63] A BFS λ in Υ is called a BFi-id of Υ if for each $s, t, u \in \Upsilon$,*

$$\lambda^P(stu) \geq \lambda^P(t) \text{ and } \lambda^N(stu) \leq \lambda^N(t).$$

Definition 1.9.5 *[34] A BF subsemigroup λ of Υ is called a BFb-id of Υ if for each $s, t, u \in \Upsilon$,*

$$\lambda^P(stu) \geq \lambda^P(s) \wedge \lambda^P(u) \text{ and } \lambda^N(stu) \leq \lambda^N(s) \vee \lambda^N(u).$$

Example 1.9.6 *Recall the semigroup $\Upsilon = \{s, t, u, v\}$ as established in Example 1.8.1. Take some BFSs in Υ, defined below.*

$\lambda_1 = \{(s, 0.3, -0.4), (t, 0.4, -0.3), (u, 0.6, -0.1), (v, 0.6, -0.1)\}$,

$\lambda_2 = \{(s, 0.2, -0.2), (t, 0.4, -0.4), (u, 0.5, -0.5), (v, 0.5, -0.5)\}$,

$\lambda_3 = \{(s, 0.3, -0.1), (t, 0.4, -0.2), (u, 0.7, -0.2), (v, 0.7, -0.2)\}$,

$\lambda_4 = \{(s, 0.1, -0.1), (t, 0.3, -0.3), (u, 0.4, -0.4), (v, 0.4, -0.4)\}$.

Simple calculations confirm that λ_1 is a BF subsemigroup, λ_2 is a BFl-id, λ_3 is a BFi-id and λ_4 is a BFb-id of Υ.

Chapter 2

Rough bipolar fuzzy sets

2.1 Introduction

Roughness in FSs is studied by many researchers; see [17, 18, 24, 29, 49]. Zhang [67] enriched the FSs with the bipolar information in 1994 and presented the concept of the bipolar fuzzy sets (BFSs). Later in 2000, Lee [38] also discussed the BFSs. These sets are able to handle the fuzziness, as well as, the bipolarity (the degrees of positivity and negativity) in the data. We have investigated roughness in BFSs using the concept of roughness furnished by Pawlak [52]. We have defined rough BFSs (written as RBFSs), which are the approximations of the BFSs in a P-apx space. We have also studied some characterizations of the RBFSs. Some similarity relations between the BFSs are defined and discussed by applying their rough BF approximations (written as RBF-apxes) in Section 2.4. Another interesting direction in this chapter is the uncertainty measures, such as accuracy measure and roughness measure for the RBF-apxes of the BFSs. Earlier in 1996, Banarjee and Pal [15] provided a roughness measure for the FSs using α-cuts on the FSs. The roughness measures for the BFSs using the approach of Banarjee and Pal, are defined and discussed in Section 2.5. These are the measures which provide an estimation to investigate how accurate are the RBF-apxes of the BFSs. Throughout this work, U is a non-empty universe and $\Re$ is an eqv-rel on U.

Different applications of the bipolar and m-polar FSs are discussed in [4, 5, 20]. Decision making techniques are another important application of the BFSs. We present, in the last section, a group decision making (GDM) problem and solve it using the RBF-apxes of the BFSs. An algorithm is also designed to solve this GDM problem, supported by a suitable example.

2.2 Rough bipolar fuzzy sets

We present and describe the RBFSs in this section. The RBFSs are the approximations of the BFSs in a P-apx space. We also give the interpretations of the RBF-apxes of the BFSs.

Definition 2.2.1 *Take a P-apx space $(U, \Re)$ and let $\lambda \in BFS(U)$. The lower and upper RBF-apxes of λ with respect to $(U, \Re)$ are the BFSs $\underline{\Re}(\lambda)$ and $\overline{\Re}(\lambda)$ in U, respectively, defined by*

$$\underline{\Re}(\lambda) = \{(u, \bigwedge_{y \in [u]_\Re} \lambda^P(y), \bigvee_{y \in [u]_\Re} \lambda^N(y)) : u \in U\}, \tag{2.1}$$

$$\overline{\Re}(\lambda) = \{(u, \bigvee_{y \in [u]_\Re} \lambda^P(y), \bigwedge_{y \in [u]_\Re} \lambda^N(y)) : u \in U\}. \tag{2.2}$$

If $\underline{\Re}(\lambda) = \overline{\Re}(\lambda)$, then, λ is said to be $\Re-$definable; otherwise, λ is an RBFS in U.

Let $\underline{\Re}(\lambda)(u)$ and $\overline{\Re}(\lambda)(u)$ be symbolized by $(\underline{\lambda^P(u)}, \underline{\lambda^N(u)})$ and $(\overline{\lambda^P(u)}, \overline{\lambda^N(u)})$, respectively. Then, the information about the object u interpreted by these RBF-apxes is as follows:

- The degree of definite fulfilment of u to the property of λ is given by $\underline{\lambda^P(u)}$.
- The degree of definite fulfilment of u to the counter property of λ is given by $\underline{\lambda^N(u)}$.
- The degree of possible fulfilment of u to the property of λ is given by $\overline{\lambda^P(u)}$.
- The degree of possible fulfilment of u to the counter property of λ is given by $\overline{\lambda^N(u)}$.

The difference $\overline{\lambda^P(u)} - \underline{\lambda^P(u)}$ of possible and definite fulfilment measures the uncertain fulfilment of u to the property of λ. Similarly, $\underline{\lambda^N(u)} - \overline{\lambda^N(u)}$ measures the uncertain fulfilment of u to the counter property of λ.

Definition 2.2.2 *A BFS λ in U is referred to be classwise constant under an eqv-rel $\Re$ on U if and only if $\lambda(u) = \lambda(u')$, whenever, $u' \in [u]_\Re$ for $u, u' \in U$.*

It is easy to note that the null BFS $\widehat{\emptyset}$, the whole BFS $\widehat{I}$ and the constant BFSs are classwise constant under each eqv-rel on U. Also, the lower RBF-apx $\underline{\Re}(\lambda)$ and the upper RBF-apx $\overline{\Re}(\lambda)$ of λ are classwise constant under $\Re$.

For the illustration of the RBF-apxes, we give a simple example.

Example 2.2.3 *Let $U = \{m_i; i = 1, ..., 7\}$ be a set containing some food items such that m_1, m_2 are cakes, m_3, m_4 are some chilli products and m_5, m_6, m_7 are pies. Let $\Re$ be a binary relation on U such that $(m_i, m_j) \in \Re$ if and only if m_i and m_j are of same type. Then, $\Re$ defines classes $\{m_1, m_2\}$, $\{m_3, m_4\}$ and $\{m_5, m_6, m_7\}$. Consider a BFS λ in U, defined as:*

$$\begin{aligned} \lambda &= \{(m_1, 0.7, -0.2), (m_2, 0.7, -0.2), (m_3, 0.1, -0.9), (m_4, 0.1, -0.9), \\ &\quad (m_5, 0.5, -0.4), (m_6, 0.5, -0.4), (m_7, 0.5, -0.4)\}. \end{aligned}$$

It can be clearly seen, that, λ is classwise constant under $\Re$ and that the lower and upper RBF-apxes of λ under $\Re$ are same. That is,

$$\begin{aligned} \underline{\Re}(\lambda) &= \{(m_1, 0.7, -0.2), (m_2, 0.7, -0.2), (m_3, 0.1, -0.9), (m_4, 0.1, -0.9), \\ &\quad (m_5, 0.5, -0.4), (m_6, 0.5, -0.4), (m_7, 0.5, -0.4)\} \\ &= \overline{\Re}(\lambda). \end{aligned}$$

This indicates, that, λ is $\Re-$definable. Let μ be another BFS in U, describing the sweetness in the food items, as below.

$$\begin{aligned} \mu &= \{(m_1, 1, 0), (m_2, 0.8, -0.18), (m_3, 0.1, -0.8), (m_4, 0, -0.9), \\ &\quad (m_5, 0.4, -0.4), (m_6, 0.5, -0.2), (m_7, 0.4, -0.35)\}. \end{aligned}$$

The RBF-apxes of μ are calculated by Definition 2.2.1 as:

$$\begin{aligned} \underline{\Re}(\mu) &= \{(m_1, 0.8, 0), (m_2, 0.8, 0), (m_3, 0, -0.8), (m_4, 0, -0.8), \\ &\quad (m_5, 0.4, -0.2), (m_6, 0.4, -0.2), (m_7, 0.4, -0.2)\}, \quad (2.3) \\ \overline{\Re}(\mu) &= \{(m_1, 1, -0.18), (m_2, 1, -0.18), (m_3, 0.1, -0.9), (m_4, 0.1, -0.9), \\ &\quad (m_5, 0.5, -0.4), (m_6, 0.5, -0.4), (m_7, 0.5, -0.4)\}. \quad (2.4) \end{aligned}$$

Equations 2.3 and 2.4 show that, $\underline{\Re}(\mu) \neq \overline{\Re}(\mu)$, rather, $\underline{\Re}(\mu) \subseteq \mu \subseteq \overline{\Re}(\mu)$. So, μ is an RBFS in U. The upper RBF-apx $\overline{\Re}(\mu)$ demonstrates that the degree of possible fulfilment of cakes (m_1 and m_2) to the trait of λ (that is, the degree of possible sweetness in cakes) is 1.0. *But, $\underline{\Re}(\mu)$ interprets that the degree of definite sweetness in the cakes is* 0.8. *Similarly, the degree of possible sourness in the cakes is* −0.18, *but the definite sourness in them is* 0.

2.3 Characterizations of rough bipolar fuzzy sets

This section studies some characterizations of the RBF-apxes of the BFSs.

Lemma 2.3.1 *Take a P-apx space $(U, \Re)$. Then, every classwise constant BFS in U is $\Re-$definable.*

Proof. Take a classwise constant BFS $\lambda \in BFS(U)$ and $u \in U$. Then $\lambda^P(y) = \lambda^P(u)$ and $\lambda^N(y) = \lambda^N(u)$ for each $y \in [u]_\Re$. Then

$$\bigwedge_{y\in[u]_\Re} \lambda^P(y) = \bigvee_{y\in[u]_\Re} \lambda^P(y)$$

and

$$\bigvee_{y\in[u]_\Re} \lambda^N(y) = \bigwedge_{y\in[u]_\Re} \lambda^N(y).$$

Thus, we have

$$\begin{aligned} \underline{\Re}(\lambda) &= \{(u, \bigwedge_{y\in[u]_\Re} \lambda^P(y), \bigvee_{y\in[u]_\Re} \lambda^N(y)) : u \in U\} \\ &= \{(u, \bigvee_{y\in[u]_\Re} \lambda^P(y), \bigwedge_{y\in[u]_\Re} \lambda^N(y)) : u \in U\} = \overline{\Re}(\lambda). \end{aligned}$$

Which shows, that, λ is $\Re-$definable. ■

Corollary 2.3.2 *Take a P-apx space $(U, \Re)$. Then, every constant BFS in U is $\Re-$definable.*

Proof. The proof follows from Lemma 2.3.1, as every constant BFS in U can be considered as a classwise constant BFS in U, under any eqv-rel $\Re$ defined on U. ■

Some basic properties of the RBFSs are given below.

Theorem 2.3.3 *Take a P-apx space $(U, \Re)$ and let $\mu, \nu \in BFS(U)$. Then, the subsequent asservations hold.*

1. $\underline{\Re}(\mu) \subseteq \mu \subseteq \overline{\Re}(\mu)$,
2. $\underline{\Re}(\widehat{I}) = \widehat{I} = \overline{\Re}(\widehat{I})$,
3. $\underline{\Re}(\widehat{\emptyset}) = \widehat{\emptyset} = \overline{\Re}(\widehat{\emptyset})$,
4. $\underline{\Re}(\underline{\Re}(\mu)) = \underline{\Re}(\mu) = \overline{\Re}(\underline{\Re}(\mu))$,
5. $\underline{\Re}(\overline{\Re}(\mu)) = \overline{\Re}(\mu) = \overline{\Re}(\overline{\Re}(\mu))$,
6. $\overline{\Re}(\mu') = (\underline{\Re}(\mu))'$,
7. $\underline{\Re}(\mu') = (\overline{\Re}(\mu))'$,
8. $\underline{\Re}(\mu \cap \nu) = \underline{\Re}(\mu) \cap \underline{\Re}(\nu)$,

9. $\underline{\Re}(\mu \cup \nu) \supseteq \underline{\Re}(\mu) \cup \underline{\Re}(\nu)$,

10. $\overline{\Re}(\mu \cup \nu) = \overline{\Re}(\mu) \cup \overline{\Re}(\nu)$,

11. $\overline{\Re}(\mu \cap \nu) \subseteq \overline{\Re}(\mu) \cap \overline{\Re}(\nu)$,

12. *$\mu \subseteq \nu$ implies that $\underline{\Re}(\mu) \subseteq \underline{\Re}(\nu)$ and $\overline{\Re}(\mu) \subseteq \overline{\Re}(\nu)$.*

Proof. (1) The proof follows from Definitions 1.3.2 and 2.2.1.

(2-3) The whole BFS $\widehat{I}$ and the null BFS $\widehat{\emptyset}$ in U are constant. Hence, $\widehat{I}$ and $\widehat{\emptyset}$ are $\Re-$definable by Corollary 2.3.2. That is,

$$\begin{aligned} \underline{\Re}(\widehat{I}) &= \widehat{I} = \overline{\Re}(\widehat{I}), \\ \underline{\Re}(\widehat{\emptyset}) &= \widehat{\emptyset} = \overline{\Re}(\widehat{\emptyset}). \end{aligned}$$

(4-5) The lower RBF-apx $\underline{\Re}(\mu)$ and the upper RBF-apx $\overline{\Re}(\mu)$ of μ are classwise constant under $\Re$. Hence, $\underline{\Re}(\mu)$ and $\overline{\Re}(\mu)$ are $\Re-$definable by Lemma 2.3.1. That is,

$$\begin{aligned} \underline{\Re}(\underline{\Re}(\mu)) &= \underline{\Re}(\mu) = \overline{\Re}(\underline{\Re}(\mu)), \\ \underline{\Re}(\overline{\Re}(\mu)) &= \overline{\Re}(\mu) = \overline{\Re}(\overline{\Re}(\mu)). \end{aligned}$$

(6) From Definition 1.3.6, we obtain

$$\begin{aligned} \overline{\Re}(\mu') &= \{(x, \bigvee_{y\in[x]} (\mu')^P(y), \bigwedge_{y\in[x]} (\mu')^N(y)) : x \in U\} \\ &= \{(x, \bigvee_{y\in[x]} (1 - \mu^P(y)), \bigwedge_{y\in[x]} (-1 - \mu^N(y))) : x \in U\} \\ &= \{(x, 1 - \bigwedge_{y\in[x]} \mu^P(y), -1 - \bigvee_{y\in[x]} \mu^N(y)) : x \in U\} = (\underline{\Re}(\mu))'. \end{aligned}$$

(7) Analogous to the proof of (6).

(8) From Definition 1.3.5, we obtain

$$\begin{aligned} & \underline{\Re}(\mu \cap \nu) \\ = \; & \{(x, \bigwedge_{y\in[x]} (\mu^P(y) \wedge \nu^P(y)), \bigvee_{y\in[x]} (\mu^N(y) \vee \nu^N(y))) : x \in U\} \\ = \; & \{(x, (\bigwedge_{y\in[x]} \mu^P(y)) \wedge (\bigwedge_{y\in[x]} \nu^P(y)), (\bigvee_{y\in[x]} \mu^N(y)) \vee (\bigvee_{y\in[x]} \nu^N(y))) : x \in U\} \\ = \; & \underline{\Re}(\mu) \cap \underline{\Re}(\nu). \end{aligned}$$

(9) Again from Definition 1.3.5, we obtain

$$\begin{aligned} & \underline{\Re}(\mu \cup \nu) \\ = \; & \{(x, \bigwedge_{y\in[x]} (\mu^P(y) \vee \nu^P(y)), \bigvee_{y\in[x]} (\mu^N(y) \wedge \nu^N(y))) : x \in U\} \\ \supseteq \; & \{(x, (\bigwedge_{y\in[x]} \mu^P(y)) \vee (\bigwedge_{y\in[x]} \nu^P(y)), (\bigvee_{y\in[x]} \mu^N(y)) \wedge (\bigvee_{y\in[x]} \nu^N(y))) : x \in U\} \\ = \; & \underline{\Re}(\mu) \cup \underline{\Re}(\nu). \end{aligned}$$

(10) Analogous to the proof of (8).

(11) Analogous to the proof of (9).

(12) Since $\mu \subseteq \nu$, so, $\mu^P(z) \leq \nu^P(z)$ and $\mu^N(z) \geq \nu^N(z)$ for each $z \in U$. Then we obtain

$$\begin{aligned}\underline{\Re}(\mu) &= \{(y, \underset{z\in[y]}{\wedge}\mu^P(z), \underset{z\in[y]}{\vee}\mu^N(z)) : y \in U\} \\ &\subseteq \{(y, \underset{z\in[y]}{\wedge}\nu^P(z), \underset{z\in[y]}{\vee}\nu^N(z)) : y \in U\} = \underline{\Re}(\nu).\end{aligned}$$

Thus, $\underline{\Re}(\mu) \subseteq \underline{\Re}(\nu)$. Similarly, $\overline{\Re}(\mu) \subseteq \overline{\Re}(\nu)$. ∎

The assertions $(8-11)$ of Theorem 2.3.3 hold for any number of BFSs in U. Thus, we have the subsequent asservations for any indexing set I.

Theorem 2.3.4 *Take a P-apx space* $(U, \Re)$ *and let* $\{\nu_i : i \in I\} \subseteq BFS(U)$. *Then, the subsequent asservations hold.*

1. $\underline{\Re}(\underset{i\in I}{\cap}\nu_i) = \underset{i\in I}{\cap}\underline{\Re}(\nu_i)$,
2. $\underline{\Re}(\underset{i\in I}{\cup}\nu_i) \supseteq \underset{i\in I}{\cup}\underline{\Re}(\nu_i)$,
3. $\overline{\Re}(\underset{i\in I}{\cup}\nu_i) = \underset{i\in I}{\cup}\overline{\Re}(\nu_i)$,
4. $\overline{\Re}(\underset{i\in I}{\cap}\nu_i) \subseteq \underset{i\in I}{\cap}\overline{\Re}(\nu_i)$.

Proof. (1) Take $\nu_i \in BFS(U)$, where $i \in I$. Then, we have

$$\underset{i\in I}{\cap}\nu_i = \{(u, \underset{i\in I}{\wedge}\nu_i^P(u), \underset{i\in I}{\vee}\nu_i^N(u)) : u \in U\}.$$

By using Equation 2.1, we get

$$\begin{aligned}&\underline{\Re}(\underset{i\in I}{\cap}\nu_i) \\ =\ & \{(u, \underset{y\in[u]_\Re}{\wedge}(\underset{i\in I}{\wedge}\nu_i^P(y)), \underset{y\in[u]_\Re}{\vee}(\underset{i\in I}{\vee}\nu_i^N(y))) : u \in U\} \\ =\ & \{(u, \underset{i\in I}{\wedge}(\underset{y\in[u]_\Re}{\wedge}\nu_i^P(y)), \underset{i\in I}{\vee}(\underset{y\in[u]_\Re}{\vee}\nu_i^N(y))) : u \in U\} \\ =\ & \underset{i\in I}{\cap}\underline{\Re}(\nu_i).\end{aligned}$$

(2) For $\nu_i \in BFS(U)$, where $i \in I$, we have

$$\underset{i\in I}{\cup}\nu_i = \{(u, \underset{i\in I}{\vee}\nu_i^P(u), \underset{i\in I}{\wedge}\nu_i^N(u)) : u \in U\}.$$

By using Equation 2.1 and Definition 1.3.2, we get

$$\begin{aligned} & \underline{\Re}(\bigcup_{i\in I}\nu_i) \\ = \ & \{(u, \bigwedge_{y\in[u]_\Re}(\bigvee_{i\in I}\nu_i^P(y)), \bigvee_{y\in[u]_\Re}(\bigwedge_{i\in I}\nu_i^N(y))) : u \in U\} \\ \supseteq \ & \{(u, \bigvee_{i\in I}(\bigwedge_{y\in[u]_\Re}\nu_i^P(y)), \bigwedge_{i\in I}(\bigvee_{y\in[u]_\Re}\nu_i^N(y))) : u \in U\} \\ = \ & \bigcup_{i\in I}\underline{\Re}(\nu_i). \end{aligned}$$

(3) Analogous to the proof of (1).

(4) Analogous to the proof of (2). ■

The inclusion in the assertions 9 and 11 of the Theorem 2.3.3 may be proper. This can be observed in the subsequent example.

Example 2.3.5 *Consider the collection $U = \{m_i; i = 1, ..., 7\}$ of food items and the relation $\Re$, as described in Example 2.2.3. We take two BFS ν and ω in U, defined as:*

$$\begin{aligned} \nu &= \{(m_1, 0.3, -0.4), (m_2, 0.4, -0.5), (m_3, 0.6, -0.3), (m_4, 0.2, -0.5), \\ &\quad (m_5, 0.5, -0.5), (m_6, 0.6, -0.2), (m_7, 0.8, -0.1)\}, \\ \omega &= \{(m_1, 0.6, -0.2), (m_2, 0.3, -0.3), (m_3, 0.8, -0.4), (m_4, 0.4, -0.5), \\ &\quad (m_5, 0.1, -0.7), (m_6, 0.9, 0), (m_7, 0.2, -0.8)\}. \end{aligned}$$

The lower RBF-apxes of ν and ω, as well as their union $\underline{\Re}(\nu) \cup \underline{\Re}(\omega)$, are given as:

$$\begin{aligned} \underline{\Re}(\nu) &= \{(m_1, 0.3, -0.4), (m_2, 0.3, -0.4), (m_3, 0.2, -0.3), (m_4, 0.2, -0.3), \\ &\quad (m_5, 0.5, -0.1), (m_6, 0.5, -0.1), (m_7, 0.5, -0.1)\}, \\ \underline{\Re}(\omega) &= \{(m_1, 0.3, -0.2), (m_2, 0.3, -0.2), (m_3, 0.4, -0.4), (m_4, 0.4, -0.4), \\ &\quad (m_5, 0.1, 0), (m_6, 0.1, 0), (m_7, 0.1, 0)\}, \\ \underline{\Re}(\nu) \cup \underline{\Re}(\omega) &= \{(m_1, 0.3, -0.4), (m_2, 0.3, -0.4), (m_3, 0.4, -0.4), (m_4, 0.4, -0.4), \\ &\quad (m_5, 0.5, -0.1), (m_6, 0.5, -0.1), (m_7, 0.5, -0.1)\}. \qquad (2.5) \end{aligned}$$

Now we calculate $\nu \cup \omega$ and its lower RBF-apx as:

$$\begin{aligned} \nu \cup \omega &= \{(m_1, 0.6, -0.4), (m_2, 0.4, -0.5), (m_3, 0.8, -0.4), (m_4, 0.4, -0.5), \\ &\quad (m_5, 0.5, -0.7), (m_6, 0.9, -0.2), (m_7, 0.8, -0.8)\}, \\ \underline{\Re}(\nu \cup \omega) &= \{(m_1, 0.4, -0.4), (m_2, 0.4, -0.4), (m_3, 0.4, -0.4), (m_4, 0.4, -0.4), \\ &\quad (m_5, 0.5, -0.2), (m_6, 0.5, -0.2), (m_7, 0.5, -0.2)\}. \qquad (2.6) \end{aligned}$$

Equations 2.5 and 2.6 verify that the inclusion (9) of Theorem 2.3.3 is proper. That is,

$$\underline{\Re}(\nu \cup \omega) \supsetneq \underline{\Re}(\nu) \cup \underline{\Re}(\omega).$$

To observe the proper inclusion in (11) of Theorem 2.3.3, we consider the BFS μ *defined in Example 2.2.3 and calculate* $\mu \cap \nu$ *and* $\overline{\Re}(\mu \cap \nu)$ *as:*

$$\begin{aligned} \mu \cap \nu &= \{(m_1, 0.3, 0), (m_2, 0.4, -0.18), (m_3, 0.1, -0.3), (m_4, 0, -0.5), \\ &\quad (m_5, 0.4, -0.4), (m_6, 0.5, -0.2), (m_7, 0.4, -0.1)\}, \\ \overline{\Re}(\mu \cap \nu) &= \{(m_1, 0.4, -0.18), (m_2, 0.4, -0.18), (m_3, 0.1, -0.5), (m_4, 0.1, -0.5), \\ &\quad (m_5, 0.5, -0.4), (m_6, 0.5, -0.4), (m_7, 0.5, -0.4)\}. \end{aligned} \tag{2.7}$$

The upper RBF-apx of ν *is calculated as:*

$$\begin{aligned} \overline{\Re}(\nu) &= \{(m_1, 0.4, -0.5), (m_2, 0.4, -0.5), (m_3, 0.6, -0.5), (m_4, 0.6, -0.5), \\ &\quad (m_5, 0.8, -0.5), (m_6, 0.8, -0.5), (m_7, 0.8, -0.5)\}. \end{aligned} \tag{2.8}$$

$\overline{\Re}(\mu) \cap \overline{\Re}(\nu)$ *is calculated from Equations 2.4 and 2.8, as:*

$$\begin{aligned} \overline{\Re}(\mu) \cap \overline{\Re}(\nu) &= \{(m_1, 0.4, -0.5), (m_2, 0.4, -0.5), (m_3, 0.1, -0.5), (m_4, 0.1, -0.5), \\ &\quad (m_5, 0.5, -0.4), (m_6, 0.5, -0.4), (m_7, 0.5, -0.4)\}. \end{aligned} \tag{2.9}$$

The Equations 2.7 and 2.9 verify that the inclusion in (11) of Theorem 2.3.3 is proper. That is,

$$\overline{\Re}(\mu \cap \nu) \subsetneq \overline{\Re}(\mu) \cap \overline{\Re}(\nu).$$

Theorem 2.3.6 *Let* $\Re$ *and* σ *be two eqv-rels on* U*, such that,* $\Re \subseteq \sigma$*. Then,* $\underline{\sigma}(\mu) \subseteq \underline{\Re}(\mu)$ *and* $\overline{\Re}(\mu) \subseteq \overline{\sigma}(\mu)$ *for each* $\mu \in BFS(U)$*.*

Proof. Take any $\mu \in BFS(U)$. Since $\Re \subseteq \sigma$, we have $[x]_\Re \subseteq [x]_\sigma$ for each $x \in U$. Thus,

$$\bigwedge_{y \in [x]_\sigma} \mu^P(y) \leq \bigwedge_{y \in [x]_\Re} \mu^P(y)$$

and

$$\bigvee_{y \in [x]_\sigma} \mu^N(y) \geq \bigvee_{y \in [x]_\Re} \mu^N(y)$$

for each $x \in U$. So, Definition 1.3.2 gives $\underline{\sigma}(\mu) \subseteq \underline{\Re}(\mu)$.
Similarly, $\overline{\Re}(\mu) \subseteq \overline{\sigma}(\mu)$. ■

Corollary 2.3.7 *Let* $\Re$ *and* σ *be two eqv-rels on a non-empty set* U*. Then, the subsequent asservations hold for each* $\lambda \in BFS(U)$*.*

1. $\overline{(\Re \cap \sigma)}(\lambda) \subseteq \overline{\Re}(\lambda) \cap \overline{\sigma}(\lambda)$,
2. $\underline{(\Re \cap \sigma)}(\lambda) \supseteq \underline{\Re}(\lambda) \cup \underline{\sigma}(\lambda)$.

Proof. (1) Let $\Re$ and σ be two eqv-rels on U. Then, $\Re \cap \sigma$ is also an eqv-rel on U. It is also clear that $\Re \cap \sigma \subseteq \Re$ and $\Re \cap \sigma \subseteq \sigma$. By Theorem 2.3.6, we have

$$\begin{aligned}\overline{(\Re \cap \sigma)}(\lambda) &\subseteq \overline{\Re}(\lambda), \\ \overline{(\Re \cap \sigma)}(\lambda) &\subseteq \overline{\sigma}(\lambda)\end{aligned}$$

for any $\lambda \in BFS(U)$. This proves, that,

$$\overline{(\Re \cap \sigma)}(\lambda) \subseteq \overline{\Re}(\lambda) \cap \overline{\sigma}(\lambda)$$

for each $\lambda \in BFS(U)$.

(2) Analogous to the proof of (1). ■

Theorem 2.3.8 *Take a P-apx space* $(U, \Re)$ *and let* $\lambda \in BFS(U)$. *Then, the subsequent asservations are equivalent.*

1. $\overline{\Re}(\lambda) \subseteq \lambda$
2. $\lambda \subseteq \underline{\Re}(\lambda)$
3. λ *is* $\Re-$*definable.*

Proof. (1)⇒(2) Take any $\lambda \in BFS(U)$. Then, Theorem 2.3.3 gives,

$$\lambda \subseteq \overline{\Re}(\lambda) = \underline{\Re}(\overline{\Re}(\lambda)). \tag{2.10}$$

But, assumption (1), that is $\overline{\Re}(\lambda) \subseteq \lambda$, implies that,

$$\underline{\Re}(\overline{\Re}(\lambda)) \subseteq \underline{\Re}(\lambda). \tag{2.11}$$

Combining Expressions 2.10 and 2.11, we get $\lambda \subseteq \underline{\Re}(\lambda)$.

(2)⇒(3) For $\lambda \in BFS(U)$, assume that $\lambda \subseteq \underline{\Re}(\lambda)$. But, $\underline{\Re}(\lambda) \subseteq \lambda$. So, $\lambda = \underline{\Re}(\lambda)$. Which implies that,

$$\overline{\Re}(\lambda) = \overline{\Re}(\underline{\Re}(\lambda)) = \underline{\Re}(\lambda).$$

Thus, λ is $\Re-$definable.

(3)⇒(1) Obvious. ■

Proposition 2.3.9 *Take a P-apx space* $(U, \Re)$.

1. *An* $\Re-$*definable BFS in* U *is constant, if* $\Re$ *is the universal binary relation on* U.
2. *Each BFS* μ *in* U *is* $\Re-$*definable, if* $\Re$ *is the identity relation on* U.

Proof. (1) Let $\mu \in BFS(U)$ be $\Re-$definable. Then $\underline{\Re}(\mu) = \overline{\Re}(\mu)$. Which gives $\underline{\Re}(\mu)(u) = \overline{\Re}(\mu)(u)$. Hence,

$$(\bigwedge_{y \in [u]} \mu^P(y), \bigvee_{y \in [u]} \mu^N(y)) = (\bigvee_{y \in [u]} \mu^P(y), \bigwedge_{y \in [u]} \mu^N(y))$$

for each $u \in U$. We have $[u] = U$ for each $u \in U$, since $\Re = U \times U$. So, the above equation gives

$$\bigwedge_{y \in U} \mu^P(y) = \bigvee_{y \in U} \mu^P(y) \text{ and } \bigwedge_{y \in U} \mu^N(y) = \bigvee_{y \in U} \mu^N(y).$$

This clearly indicates that μ is a constant BFS in U.

(2) Straightforward. ■

2.4 Similarity relations associated with RBF approximations

This section presents a few binary relations between the BFSs based on their RBF-apxes and investigate their properties.

Definition 2.4.1 *Take a P-apx space $(U, \Re)$and let $\mu, \nu \in BFS(U)$. Then, we define*
$\mu \simeq_{\Re} \nu$ *if and only if* $\underline{\Re}(\mu) = \underline{\Re}(\nu)$,
$\mu \eqsim_{\Re} \nu$ *if and only if* $\overline{\Re}(\mu) = \overline{\Re}(\nu)$,
$\mu \approx_{\Re} \nu$ *if and only if* $\underline{\Re}(\mu) = \underline{\Re}(\nu)$ *and* $\overline{\Re}(\mu) = \overline{\Re}(\nu)$.

Clearly, $\mu \approx_{\Re} \nu$ if and only if $\mu \simeq_{\Re} \nu$ and $\mu \eqsim_{\Re} \nu$. The relations $\simeq_{\Re}$, $\eqsim_{\Re}$ and $\approx_{\Re}$ may be called lower RBF similarity relation, upper RBF similarity relation and RBF similarity relation, respectively.

Proposition 2.4.2 *The relations $\simeq_{\Re}$, $\eqsim_{\Re}$ and $\approx_{\Re}$ are eqv-rels on $BFS(U)$.*

Proof. Straightforward. ■

Theorem 2.4.3 *Take a P-apx space $(U, \Re)$. Then, the subsequent asservations hold for each $\mu, \nu, \lambda, \omega \in BFS(U)$.*

1. $\mu \eqsim_{\Re} \nu$ *if and only if* $\mu \eqsim_{\Re} (\mu \cup \nu)_{\Re} \eqsim \nu$,
2. $\mu \eqsim_{\Re} \nu$ *and* $\lambda \eqsim_{\Re} \omega$ *imply that* $(\mu \cup \lambda) \eqsim_{\Re} (\nu \cup \omega)$,
3. $\mu \subseteq \nu$ *and* $\nu \eqsim_{\Re} \widehat{\emptyset}$ *imply that* $\mu \eqsim_{\Re} \widehat{\emptyset}$,
4. $\mu \subseteq \nu$ *and* $\mu \eqsim_{\Re} \widehat{I}$ *imply that* $\nu \eqsim_{\Re} \widehat{I}$,

5. *If* $(\mu \cup \nu) \eqsim_{\Re} \widehat{\emptyset}$, *then,* $\mu \eqsim_{\Re} \widehat{\emptyset}$ *and* $\nu \eqsim_{\Re} \widehat{\emptyset}$,

6. *If* $(\mu \cap \nu) \eqsim_{\Re} \widehat{I}$, *then,* $\mu \eqsim_{\Re} \widehat{I}$ *and* $\nu \eqsim_{\Re} \widehat{I}$.

Proof. (1) Let $\mu \eqsim_{\Re} \nu$. Then $\overline{\Re}(\mu) = \overline{\Re}(\nu)$. From Theorem 2.3.3, we have

$$\overline{\Re}(\mu \cup \nu) = \overline{\Re}(\mu) \cup \overline{\Re}(\nu) = \overline{\Re}(\mu) = \overline{\Re}(\nu).$$

So, $\mu \eqsim_{\Re} (\mu \cup \nu) \eqsim_{\Re} \nu$.

Converse holds by the transitivity of the relation $\eqsim_{\Re}$.

(2) Let $\mu \eqsim_{\Re} \nu$ and $\lambda \eqsim_{\Re} \omega$. Then, $\overline{\Re}(\mu) = \overline{\Re}(\nu)$ and $\overline{\Re}(\lambda) = \overline{\Re}(\omega)$. From Theorem 2.3.3, we have

$$\begin{aligned} \overline{\Re}(\mu \cup \lambda) &= \overline{\Re}(\mu) \cup \overline{\Re}(\lambda) \\ &= \overline{\Re}(\nu) \cup \overline{\Re}(\omega) = \overline{\Re}(\nu \cup \omega). \end{aligned}$$

So, $(\mu \cup \lambda) \eqsim (\nu \cup \omega)$.

(3) $\nu \eqsim_{\Re} \widehat{\emptyset}$ implies that $\overline{\Re}(\nu) = \overline{\Re}(\widehat{\emptyset})$. Also $\mu \subseteq \nu$ implies that,

$$\overline{\Re}(\mu) \subseteq \overline{\Re}(\nu) = \overline{\Re}(\widehat{\emptyset}) = \widehat{\emptyset} \subseteq \overline{\Re}(\mu).$$

Thus, $\overline{\Re}(\mu) = \overline{\Re}(\widehat{\emptyset})$. Which shows that $\mu \eqsim_{\Re} \widehat{\emptyset}$.

(4) $\mu \eqsim_{\Re} \widehat{I}$ implies that $\overline{\Re}(\mu) = \overline{\Re}(\widehat{I})$. Also $\mu \subseteq \nu$ implies that,

$$\overline{\Re}(\nu) \supseteq \overline{\Re}(\mu) = \overline{\Re}(\widehat{I}) = \widehat{I} \supseteq \overline{\Re}(\nu).$$

Thus, $\overline{\Re}(\nu) = \overline{\Re}(\widehat{I})$. Which shows that $\nu \eqsim_{\Re} \widehat{I}$.

(5) This follows from (3).

(6) This follows from (4). ■

Theorem 2.4.4 *Take a P-apx space* $(U, \Re)$. *Then, the subsequent asservations hold for each* $\mu, \nu, \lambda, \omega \in BFS(U)$.

1. $\mu \simeq_{\Re} \nu$ *if and only if* $\mu \simeq_{\Re} (\mu \cap \nu) \simeq_{\Re} \nu$,

2. $\mu \simeq_{\Re} \nu$ *and* $\lambda \simeq_{\Re} \omega$ *imply that* $(\mu \cap \lambda) \simeq_{\Re} (\nu \cap \omega)$,

3. $\mu \subseteq \nu$ *and* $\nu \simeq_{\Re} \widehat{\emptyset}$ *imply that* $\mu \simeq_{\Re} \widehat{\emptyset}$,

4. $\mu \subseteq \nu$ *and* $\mu \simeq_{\Re} \widehat{I}$ *imply that* $\nu \simeq_{\Re} \widehat{I}$,

5. *If* $(\mu \cup \nu) \simeq_{\Re} \widehat{\emptyset}$, *then,* $\mu \simeq_{\Re} \widehat{\emptyset}$ *and* $\nu \simeq_{\Re} \widehat{\emptyset}$,

6. *If* $(\mu \cap \nu) \simeq_{\Re} \widehat{I}$, *then,* $\mu \simeq_{\Re} \widehat{I}$ *and* $\nu \simeq_{\Re} \widehat{I}$.

Proof. Parallel to the proof of Theorem 2.4.3. ∎

Theorem 2.4.5 *Take a P-apx space $(U, \Re)$. Then, the subsequent asservations hold for each $\mu, \nu \in BFS(U)$.*

1. $\mu \subseteq \nu$ *and* $\nu \approx_{\Re} \widehat{\emptyset}$ *imply that* $\mu \approx_{\Re} \widehat{\emptyset}$,
2. $\mu \subseteq \nu$ *and* $\mu \approx_{\Re} \widehat{I}$ *imply that* $\nu \approx_{\Re} \widehat{I}$,
3. *If* $(\mu \cup \nu) \approx_{\Re} \widehat{\emptyset}$, *then,* $\mu \approx_{\Re} \widehat{\emptyset}$ *and* $\nu \approx_{\Re} \widehat{\emptyset}$,
4. *If* $(\mu \cap \nu) \approx_{\Re} \widehat{I}$, *then,* $\mu \approx_{\Re} \widehat{I}$ *and* $\nu \approx_{\Re} \widehat{I}$.

Proof. This directly follows from Definition 2.4.1, Theorems 2.4.3 and 2.4.4. ∎

2.5 Accuracy measures for BFSs

An important application of the RBF-apxes of the BFSs is, that, these approximations provide a scheme to investigate how accurately the belongingness maps of a BFSs describe the objects. We introduce the degree of accuracy and the degree of roughness for the positive and negative belongingness maps of the BFSs, separately. For this purpose, we first define the $\alpha-$level cuts of a BFS and describe their basic properties.

Definition 2.5.1 *Let $\lambda \in BFS(U)$. For $0 \lneqq \alpha \leq 1$, the $\alpha-$level P-cut (positive cut) of λ is symbolized by λ_α and defined as:*

$$\lambda_\alpha = \{u \in U : \lambda^P(u) \geq \alpha\}. \tag{2.12}$$

Definition 2.5.2 *Let $\lambda \in BFS(U)$. For $0 \lneqq \alpha \leq 1$, the $\alpha-$level N-cut (negative cut) of λ is symbolized by λ^α and defined as:$\lambda^\alpha = \{u \in U : \lambda^N(u) \leq -\alpha\}$.*

Lemma 2.5.3 *Let $\lambda, \mu \in BFS(U)$ and $0 \lneqq \alpha \leq 1$. Then, $\lambda \subseteq \mu$ implies the following.*

1. $\lambda_\alpha \subseteq \mu_\alpha$
2. $\lambda^\alpha \subseteq \mu^\alpha$.

Proof. Direct outcome of Definitions 2.5.1 and 2.5.2. ∎

Lemma 2.5.4 *Let $\lambda \in BFS(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then, the following hold.*

1. $\lambda_\alpha \subseteq \lambda_\beta$
2. $\lambda^{1-\beta} \subseteq \lambda^{1-\alpha}$.

Proof. Direct outcomes of Definitions 2.5.1 and 2.5.2. ■

Note that $\underline{\Re}(\lambda_\alpha)$ is the lower approximation of the crisp set λ_α, while, $\underline{\Re}(\lambda)_\alpha$ is the $\alpha-$level P-cut of the lower RBF-apx $\underline{\Re}(\lambda)$ of the BFS λ in U. Thus, we conclude the following, with the help of Definitions 2.2.1, 2.5.1 and 2.5.2.

$$\begin{aligned}
\underline{\Re}(\lambda)_\alpha &= \{u \in U : \underline{\Re}(\lambda)^P(u) \geq \alpha\} \\
&= \{u \in U : \bigwedge_{y\in[u]_\Re} \lambda^P(y) \geq \alpha\}, \\
\overline{\Re}(\lambda)_\alpha &= \{u \in U : \bigvee_{y\in[u]_\Re} \lambda^P(y) \geq \alpha\}, \\
\underline{\Re}(\lambda)^\alpha &= \{u \in U : \bigvee_{y\in[u]_\Re} \lambda^N(y) \leq -\alpha\}, \\
\overline{\Re}(\lambda)^\alpha &= \{u \in U : \bigwedge_{y\in[u]_\Re} \lambda^N(y) \leq -\alpha\}.
\end{aligned}$$

Lemma 2.5.5 *Let $\lambda \in BFS(U)$ and $0 \lneqq \alpha \leq 1$. Then, the subsequent assertions hold.*

1. $\underline{\Re}(\lambda_\alpha) = \underline{\Re}(\lambda)_\alpha$,
2. $\overline{\Re}(\lambda_\alpha) = \overline{\Re}(\lambda)_\alpha$,
3. $\underline{\Re}(\lambda^\alpha) = \underline{\Re}(\lambda)^\alpha$,
4. $\overline{\Re}(\lambda^\alpha) = \overline{\Re}(\lambda)^\alpha$.

Proof. (1) Let $\lambda \in BFS(U)$ and $0 \lneqq \alpha \leq 1$. For the crisp set λ_α, we have

$$\begin{aligned}
\underline{\Re}(\lambda_\alpha) &= \{u \in U : [u]_\Re \subseteq \lambda_\alpha\} \\
&= \{u \in U : y \in \lambda_\alpha \text{ for each } y \in [u]_\Re\} \\
&= \{u \in U : \lambda^P(y) \geq \alpha \text{ for each } y \in [u]_\Re\} \\
&= \{u \in U : \bigwedge_{y\in[u]_\Re} \lambda^P(y) \geq \alpha\} \\
&= \underline{\Re}(\lambda)_\alpha.
\end{aligned}$$

The remaining parts can be verified in the same manner. ■

Now, we define the degree of accuracy and the degree of roughness for the positive and negative belongingness maps of a BFS in U.

Definition 2.5.6 *Take a P-apx space $(U, \Re)$. The degree of accuracy for the positive belongingness map of the BFS $\lambda \in BFS(U)$, with respect to the parameters α, β such that, $0 \lneqq \beta \leq \alpha \leq 1$, is expressed as:*

$$\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) = \frac{|\underline{\Re}(\lambda_\alpha)|}{|\overline{\Re}(\lambda_\beta)|}.$$

The degree of roughness for the positive belongingness map of $\lambda \in BFS(U)$, with respect to the parameters α, β such that, $0 \lneqq \beta \leq \alpha \leq 1$, is expressed as:

$$\rho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) = 1 - \mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda).$$

Definition 2.5.7 *Take a P-apx space $(U, \Re)$. The degree of accuracy for the negative belongingness map of the BFS $\lambda \in BFS(U)$, with respect to the parameters α, β such that, $0 \lneqq \beta \leq \alpha \leq 1$, is expressed as:*

$$\mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) = \frac{\left|\underline{\Re}(\lambda^{1-\beta})\right|}{\left|\overline{\Re}(\lambda^{1-\alpha})\right|}.$$

The degree of roughness for the negative belongingness map of $\lambda \in BFS(U)$, with respect to the parameters α, β such that, $0 \lneqq \beta \leq \alpha \leq 1$, is expressed as:

$$\varrho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) = 1 - \mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda).$$

Notice that, $\underline{\Re}(\lambda_\alpha)$ (or $\overline{\Re}(\lambda_\beta)$) comprises of the objects of U having α (or β) as the least degree of definite (or possible) fulfilment for λ. Equivalently, $\underline{\Re}(\lambda_\alpha)$ (or $\overline{\Re}(\lambda_\beta)$) may be viewed as union of the eqv-classes of U having the degree of fulfilment atleast α (or β) in the lower (or upper) RBF-apx of λ. Therefore, the parameters α and β serve as the thresholds of definite and possible fulfilment of the objects of U for λ, respectively. Hence, $\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$ may be interpreted as the degree to which the positive belongingness map of λ is accurate, constrained to the threshold parameters α and β. Similarly, $\mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$ denotes the degree to which the negative belongingness map of λ is accurate, constrained to the threshold parameters α and β. In other words, $\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$ and $\mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$ describe how accurate are the positive and the negative belongingness maps of the BFS λ, respectively. We explain these degrees in the subsequent example.

Example 2.5.8 *Consider the set $U = \{m_i; i = 1, ..., 7\}$, the relation $\Re$ and the BFS $\mu \in BFS(U)$, as in Example 2.2.3. That is, $\Re$ defines classes $\{m_1, m_2\}$, $\{m_3, m_4\}$, $\{m_5, m_6, m_7\}$ and*

$$\begin{aligned} \mu &= \{(m_1, 1, 0), (m_2, 0.8, -0.18), (m_3, 0.1, -0.8), (m_4, 0, -0.9), \\ &\quad (m_5, 0.4, -0.4), (m_6, 0.5, -0.2), (m_7, 0.4, -0.35)\}. \end{aligned}$$

Take $\beta = 0.3$ and $\alpha = 0.4$. Then, $\alpha-$level P-cuts $\mu_{0.3}$ and $\mu_{0.4}$ are calculated by using Equation 2.12, as follows.

$$\mu_{0.3} = U - \{m_4\} = \mu_{0.4}.$$

$\alpha-$level N-cuts $\mu^{0.6}$ and $\mu^{0.7}$ are calculated by using Equation **??***, as follows.*

$$\mu^{0.6} = \{m_3, m_4\} = \mu^{0.7}.$$

The degree of accuracy for the positive belongingness map of μ is calculated by using Definition 2.5.6, as follows.

$$\begin{aligned}\underline{\mathfrak{R}}(\mu_{0.4}) &= \{m_1, m_2, m_5, m_6, m_7\},\\ \overline{\mathfrak{R}}(\mu_{0.3}) &= \{m_1, m_2, m_3, m_4, m_5, m_6, m_7\},\\ \mathfrak{Dp}^{\mathfrak{R}}_{\langle\alpha,\beta\rangle}(\mu) &= \frac{|\underline{\mathfrak{R}}(\mu_{0.4})|}{|\overline{\mathfrak{R}}(\mu_{0.3})|} = \frac{5}{7} = 0.714.\end{aligned}$$

While, the degree of accuracy for the negative belongingness map of μ is calculated by using Definition 2.5.7, as follows.

$$\begin{aligned}\underline{\mathfrak{R}}(\mu^{0.7}) &= \{m_3, m_4\} = \overline{\mathfrak{R}}(\mu^{0.6}),\\ \mathfrak{Dn}^{\mathfrak{R}}_{\langle\alpha,\beta\rangle}(\mu) &= \frac{|\underline{\mathfrak{R}}(\mu^{0.7})|}{|\overline{\mathfrak{R}}(\mu^{0.6})|} = \frac{2}{2} = 1.0.\end{aligned}$$

Hence, the positive belongingness map μ^P of μ describes the sweetness in food items accurate upto the degree 0.714. *While the negative belongingness map μ^N of μ describes the sourness in food items accurately (upto the degree* 1.0*).*

Theorem 2.5.9 *Take a P-apx space $(U, \mathfrak{R})$, $\lambda \in BFS(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then, $0 \leq \mathfrak{Dp}^{\mathfrak{R}}_{\langle\alpha,\beta\rangle}(\lambda) \leq 1$ and $0 \leq \mathfrak{Dn}^{\mathfrak{R}}_{\langle\alpha,\beta\rangle}(\lambda) \leq 1$.*

Proof. Take a BFS $\lambda \in BFS(U)$ and the parameters α, β be such that, $0 \lneqq \beta \leq \alpha \leq 1$. To prove $0 \leq \mathfrak{Dp}^{\mathfrak{R}}_{\langle\alpha,\beta\rangle}(\lambda) \leq 1$, we show that $|\underline{\mathfrak{R}}(\lambda_\alpha)| \leq |\overline{\mathfrak{R}}(\lambda_\beta)|$. Using Lemma 2.5.4, we have $\lambda_\alpha \subseteq \lambda_\beta$. Now, Theorem 1.7.3 gives that,

$$\underline{\mathfrak{R}}(\lambda_\alpha) \subseteq \overline{\mathfrak{R}}(\lambda_\alpha) \subseteq \overline{\mathfrak{R}}(\lambda_\beta).$$

So, $|\underline{\mathfrak{R}}(\lambda_\alpha)| \leq |\overline{\mathfrak{R}}(\lambda_\beta)|$, or the ratio $\frac{|\underline{\mathfrak{R}}(\lambda_\alpha)|}{|\overline{\mathfrak{R}}(\lambda_\beta)|}$ fluctuates between 0 and 1. Which certainly yields

$$0 \leq \mathfrak{Dp}^{\mathfrak{R}}_{\langle\alpha,\beta\rangle}(\lambda) \leq 1.$$

Similarly, it can be shown, that, $0 \leq \mathfrak{Dn}^{\mathfrak{R}}_{\langle\alpha,\beta\rangle}(\lambda) \leq 1$. ■

Corollary 2.5.10 *For the P-apx space $(U, \mathfrak{R})$, $\lambda \in BFS(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$, we have, $0 \leq \rho^{\mathfrak{R}}_{\langle\alpha,\beta\rangle}(\lambda) \leq 1$ and $0 \leq \varrho^{\mathfrak{R}}_{\langle\alpha,\beta\rangle}(\lambda) \leq 1$.*

Proof. Definitions 2.5.6, 2.5.7 and Theorem 2.5.9 verify these asservations directly. ■

Theorem 2.5.11 *Take a P-apx space $(U, \mathfrak{R})$, $\lambda \in BFS(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$.*

1. *If α stands fixed, then $\mathfrak{Dp}^{\mathfrak{R}}_{\langle\alpha,\beta\rangle}(\lambda)$ and $\mathfrak{Dn}^{\mathfrak{R}}_{\langle\alpha,\beta\rangle}(\lambda)$ increase with the increase in β.*

2. *If β stands fixed, then $\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$ and $\mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$ decrease with the increase in α.*

Proof. (1) Let α stand fixed, $0 \lneqq \beta_1 \leq \beta_2 \leq 1$ and $\lambda \in BFS(U)$. Using Lemma 2.5.4, we have $\lambda_{\beta_2} \subseteq \lambda_{\beta_1}$. Theorem 1.7.3 gives $\overline{\Re}(\lambda_{\beta_2}) \subseteq \overline{\Re}(\lambda_{\beta_1})$. That is,

$$\left|\overline{\Re}(\lambda_{\beta_2})\right| \leq \left|\overline{\Re}(\lambda_{\beta_1})\right|.$$

This implies that,

$$\frac{|\underline{\Re}(\lambda_\alpha)|}{\left|\overline{\Re}(\lambda_{\beta_1})\right|} \leq \frac{|\underline{\Re}(\lambda_\alpha)|}{\left|\overline{\Re}(\lambda_{\beta_2})\right|}.$$

That is, $\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta_1\rangle}(\lambda) \leq \mathfrak{Dp}^{\Re}_{\langle\alpha,\beta_2\rangle}(\lambda)$. This verifies that $\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$ increases with the increase in β. In the same manner, it can be shown that $\mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$ increases with the increase in β.

(2) Parallel to the proof of (1). ■

Corollary 2.5.12 *Take a P-apx space $(U,\Re)$, $\lambda \in BFS(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$.*

1. *If α stands fixed, then $\rho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$ and $\varrho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$ decrease with the increase in β.*

2. *If β stands fixed, then $\rho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$ and $\varrho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$ increase with the increase in α.*

Proof. Definitions 2.5.6, 2.5.7 and Theorem 2.5.11 verify these asservations directly. ■

Theorem 2.5.13 *Take a P-apx space $(U,\Re)$, $\lambda,\mu \in BFS(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then, $\lambda \subseteq \mu$ implies the subsequent asservations.*

1. $\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) \leq \mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$, *whenever,* $\overline{\Re}(\lambda_\beta) = \overline{\Re}(\mu_\beta)$,

2. $\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) \geq \mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$, *whenever,* $\underline{\Re}(\lambda_\alpha) = \underline{\Re}(\mu_\alpha)$,

3. $\mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) \leq \mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$, *whenever,* $\overline{\Re}(\lambda^{1-\alpha}) = \overline{\Re}(\mu^{1-\alpha})$,

4. $\mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) \geq \mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$, *whenever,* $\underline{\Re}(\lambda^{1-\beta}) = \underline{\Re}(\mu^{1-\beta})$.

Proof. (1) Let $0 \lneqq \beta \leq \alpha \leq 1$, $\lambda \subseteq \mu$ and $\overline{\Re}(\lambda_\beta) = \overline{\Re}(\mu_\beta)$ for $\lambda,\mu \in BFS(U)$. Lemma 2.5.3 gives, that, $\lambda_\alpha \subseteq \mu_\alpha$. Theorem 1.7.3 gives that, $\underline{\Re}(\lambda_\alpha) \subseteq \underline{\Re}(\mu_\alpha)$, that is, $|\underline{\Re}(\lambda_\alpha)| \leq |\underline{\Re}(\mu_\alpha)|$. Which implies,

$$\frac{|\underline{\Re}(\lambda_\alpha)|}{\left|\overline{\Re}(\lambda_\beta)\right|} \leq \frac{|\underline{\Re}(\mu_\alpha)|}{\left|\overline{\Re}(\mu_\beta)\right|}.$$

Hence proved, that, $\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) \leq \mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$.

The remaining parts can be proved in the same manner. ■

Corollary 2.5.14 *Take a P-apx space* $(U, \Re)$, $\lambda, \mu \in BFS(U)$ *and* $0 \lneqq \beta \leq \alpha \leq 1$. *Then,* $\lambda \subseteq \mu$ *implies the subsequent asservations.*

1. $\rho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) \geq \rho^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$, *whenever,* $\overline{\Re}(\lambda_\beta) = \overline{\Re}(\mu_\beta)$,
2. $\rho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) \leq \rho^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$, *whenever,* $\underline{\Re}(\lambda_\alpha) = \underline{\Re}(\mu_\alpha)$,
3. $\varrho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) \geq \varrho^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$, *whenever,* $\overline{\Re}(\lambda^{1-\alpha}) = \overline{\Re}(\mu^{1-\alpha})$,
4. $\varrho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) \leq \varrho^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$, *whenever,* $\underline{\Re}(\lambda^{1-\beta}) = \underline{\Re}(\mu^{1-\beta})$.

Proof. Definitions 2.5.6, 2.5.7 and Theorem 2.5.13 verify these asservations directly. ■

Theorem 2.5.15 *Take a P-apx space* $(U, \Re)$, $\lambda \in BFS(U)$ *and* $0 \lneqq \beta \leq \alpha \leq 1$. *If* σ *is an eqv-rel on* U, *containing* $\Re$. *Then, the subsequent assertions hold.*

1. $\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) \geq \mathfrak{Dp}^{\sigma}_{\langle\alpha,\beta\rangle}(\lambda)$
2. $\mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) \leq \mathfrak{Dn}^{\sigma}_{\langle\alpha,\beta\rangle}(\lambda)$.

Proof. 1) Let $\lambda \in BFS(U)$ and let $\Re$ and σ be two eqv-rels on U, such that, $\Re \subseteq \sigma$. Theorem 2.3.6 states that $\underline{\Re}(\lambda) \supseteq \underline{\sigma}(\lambda)$ and $\overline{\Re}(\lambda) \subseteq \overline{\sigma}(\lambda)$. Using Lemma 2.5.3, we get $\underline{\Re}(\lambda)_\alpha \supseteq \underline{\sigma}(\lambda)_\alpha$ and $\overline{\Re}(\lambda)_\beta \subseteq \overline{\sigma}(\lambda)_\beta$. Lemma 2.5.5 gives

$$\begin{aligned} |\underline{\Re}(\lambda_\alpha)| &= |\underline{\Re}(\lambda)_\alpha| \geq |\underline{\sigma}(\lambda)_\alpha| = |\underline{\sigma}(\lambda_\alpha)|, \\ |\overline{\Re}(\lambda_\beta)| &= |\underline{\Re}(\lambda)_\beta| \leq |\underline{\sigma}(\lambda)_\beta| = |\underline{\sigma}(\lambda_\beta)|. \end{aligned}$$

Which implies,

$$\frac{|\underline{\Re}(\lambda_\alpha)|}{|\overline{\Re}(\lambda_\beta)|} \geq \frac{|\underline{\sigma}(\lambda_\alpha)|}{|\overline{\sigma}(\lambda_\beta)|}.$$

That is, $\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) \geq \mathfrak{Dp}^{\sigma}_{\langle\alpha,\beta\rangle}(\lambda)$.

2) It can be verified in the same manner as (1). ■

Corollary 2.5.16 *Take a P-apx space* $(U, \Re)$, $\lambda \in BFS(U)$ *and* $0 \lneqq \beta \leq \alpha \leq 1$. *If* σ *is an eqv-rel on* U *containing* $\Re$. *Then, the subsequent assertions hold.*

1. $\rho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) \leq \rho^{\sigma}_{\langle\alpha,\beta\rangle}(\lambda)$
2. $\varrho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) \geq \varrho^{\sigma}_{\langle\alpha,\beta\rangle}(\lambda)$.

Proof. Definitions 2.5.6, 2.5.7 and Theorem 2.5.15 verify these asservations directly. ■

Theorem 2.5.17 *Take a P-apx space* $(U, \Re)$*,* $\lambda, \mu \in BFS(U)$ *and* $0 \lneqq \beta \leq \alpha \leq 1$*. Then,* $\lambda \simeq_{\Re} \mu$ *implies the subsequent asservations.*

1. $\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda \cap \mu) \geq \mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$,
2. $\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda \cap \mu) \geq \mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$,
3. $\mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda \cap \mu) \geq \mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$,
4. $\mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda \cap \mu) \geq \mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$.

Proof. (1) Let $0 \lneqq \beta \leq \alpha \leq 1$ and $\lambda, \mu \in BFS(U)$ be such that $\lambda \simeq_{\Re} \mu$. By Definition 2.4.1, $\underline{\Re}(\lambda) = \underline{\Re}(\mu)$. Then, Theorem 2.4.4 implies $\underline{\Re}(\lambda \cap \mu) = \underline{\Re}(\lambda)$. This gives $\underline{\Re}(\lambda \cap \mu)_{\alpha} = \underline{\Re}(\lambda)_{\alpha}$. That is,

$$|\underline{\Re}((\lambda \cap \mu)_{\alpha})| = |\underline{\Re}(\lambda_{\alpha})| . \tag{2.13}$$

On the other hand, $\lambda \cap \mu \subseteq \lambda$. Which implies that, $\overline{\Re}(\lambda \cap \mu) \subseteq \overline{\Re}(\lambda)$, that is, $\overline{\Re}(\lambda \cap \mu)_{\beta} \subseteq \overline{\Re}(\lambda)_{\beta}$. This gives

$$\left|\overline{\Re}((\lambda \cap \mu)_{\beta})\right| \leq \left|\overline{\Re}(\lambda_{\beta})\right| . \tag{2.14}$$

Expressions 2.13 and 2.14 yield the following.

$$\frac{|\underline{\Re}((\lambda \cap \mu)_{\alpha})|}{\left|\overline{\Re}((\lambda \cap \mu)_{\beta})\right|} \geq \frac{|\underline{\Re}(\lambda_{\alpha})|}{\left|\overline{\Re}(\lambda_{\beta})\right|}.$$

This proves that $\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda \cap \mu) \geq \mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$.

The remaining parts can be proved in the same manner. ∎

Corollary 2.5.18 *Take a P-apx space* $(U, \Re)$*,* $\lambda, \mu \in BFS(U)$ *and* $0 \lneqq \beta \leq \alpha \leq 1$*. Then,* $\lambda \simeq_{\Re} \mu$ *implies the subsequent asservations.*

1. $\rho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda \cap \mu) \leq \rho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$,
2. $\rho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda \cap \mu) \leq \rho^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$,
3. $\varrho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda \cap \mu) \leq \varrho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$,
4. $\varrho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda \cap \mu) \leq \varrho^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$.

Proof. Definitions 2.5.6, 2.5.7 and Theorem 2.5.17 verify these asservations directly. ∎

Theorem 2.5.19 *Take a P-apx space* $(U, \Re)$*,* $\lambda, \mu \in BFS(U)$ *and* $0 \lneqq \beta \leq \alpha \leq 1$*. Then,* $\lambda \eqsim_{\Re} \mu$ *implies the subsequent asservations.*

1. $\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda\cup\mu)\geq\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$,
2. $\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda\cup\mu)\geq\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$,
3. $\mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda\cup\mu)\geq\mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$,
4. $\mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda\cup\mu)\geq\mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$.

Proof. Parallel to the proof of the Theorem 2.5.17. ■

Corollary 2.5.20 *Take a P-apx space $(U,\Re)$, $\lambda,\mu\in BFS(U)$ and $0\lneqq\beta\leq\alpha\leq1$. Then, $\lambda\approx_{\Re}\mu$ implies the subsequent asservations.*

1. $\rho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda\cup\mu)\leq\rho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$,
2. $\rho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda\cup\mu)\leq\rho^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$,
3. $\varrho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda\cup\mu)\leq\varrho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)$,
4. $\varrho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda\cup\mu)\leq\varrho^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$.

Proof. Definitions 2.5.6, 2.5.7 and Theorem 2.5.19 verify these asservations directly. ■

Theorem 2.5.21 *Take a P-apx space $(U,\Re)$, $\lambda,\mu\in BFS(U)$ and $0\lneqq\beta\leq\alpha\leq1$. Then, $\lambda\approx_{\Re}\mu$ implies the subsequent asservations.*

1. $\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)=\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$,
2. $\mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)=\mathfrak{Dn}^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$.

Proof. (1) Let $0\lneqq\beta\leq\alpha\leq1$ and $\lambda,\mu\in BFS(U)$ be such that, $\lambda\approx_{\Re}\mu$. From Definition 2.4.1, we have $\underline{\Re}(\lambda)=\underline{\Re}(\mu)$ and $\overline{\Re}(\lambda)=\overline{\Re}(\mu)$. Then, Lemma 2.5.5 implies, that, $\underline{\Re}(\lambda_\alpha)=\underline{\Re}(\mu_\alpha)$ and $\overline{\Re}(\lambda_\beta)=\overline{\Re}(\mu_\beta)$. That is, $|\underline{\Re}(\lambda_\alpha)|=|\underline{\Re}(\mu_\alpha)|$ and $\left|\overline{\Re}(\lambda_\beta)\right|=\left|\overline{\Re}(\mu_\beta)\right|$. This yields the following.

$$\frac{|\underline{\Re}(\lambda_\alpha)|}{\left|\overline{\Re}(\lambda_\beta)\right|}=\frac{|\underline{\Re}(\mu_\alpha)|}{\left|\overline{\Re}(\mu_\beta)\right|}.$$

This verifies, that, $\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\lambda)=\mathfrak{Dp}^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$.

(2) Analogous to the proof of (1). ■

Corollary 2.5.22 *Take a P-apx space $(U,\Re)$, $\lambda,\mu\in BFS(U)$ and $0\lneqq\beta\leq\alpha\leq1$. Then, $\lambda\approx_{\Re}\mu$ implies the subsequent asservations.*

1. $\rho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) = \rho^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$,
2. $\varrho^{\Re}_{\langle\alpha,\beta\rangle}(\lambda) = \varrho^{\Re}_{\langle\alpha,\beta\rangle}(\mu)$.

Proof. Definitions 2.5.6, 2.5.7 and Theorem 2.5.21 verify these asservations directly. ∎

2.6 Application of RBF approximations

Decision making is a major area to be conferred in almost all kinds of data analysis. Researchers and experts use their knowledge to design algorithms in order to find a wise and best decision. Many algorithms are designed by the researchers, in this regard, to find a best decision [46, 55, 66]. All those algorithms provide the decision to choose the best object. But, in some circumstances, the best decision becomes difficult to be taken and one has to look for another better option. So, it will always be advantageous if the poor decision becomes apparent, in order to avoid the poor decision, as well. There may also be a group of more than one decision makers, who desire to arrive at a single decision. We present, in this section, an efficient computational algorithm to obtain the best, as well as, the poor decisions, for the group decision making (GDM) problems by using the RBF-apxes. We consider a GDM problem in which a group of decision makers wishes to decide for a best and a worst object, keeping in view the degree of positivity, as well as, the degree of negativity examined in the objects. Let the set containing objects be symbolized by $A = \{a_j : 1 \leq j \leq k\}$ and the set containing the BFSs describing the assessments of m decision makers be symbolized by $\Omega = \{\omega_i : 1 \leq i \leq m\}$. The information about the objects a_j, provided by each ω_i, is represented by a table (called the table of Ω) with $(i, j)th$ entry as $\omega_i(a_j) = (x_{ij}, y_{ij})$, where the value x_{ij} denotes the degree of positivity and y_{ij} denotes the degree of negativity in a_j towards the property of ω_i. First, we assign the indiscernibility grades to each object under consideration relative to each BFS ω_i. After that, we define the indiscernibility relations between the objects.

Definition 2.6.1 *The indiscernibility grades G_{ij} corresponding to each object $a_j \in A$ and each BFS $\omega_i \in \Omega$, are given by*

$$G_{ij} = \begin{cases} P & \text{if } x_{ij} \gneqq |y_{ij}| \\ N & \text{if } x_{ij} \lneqq |y_{ij}| \\ O & \text{if } x_{ij} = |y_{ij}| \end{cases} \tag{2.15}$$

where $(x_{ij}, y_{ij}) = \omega_i(a_j)$.

The indiscernibility grades depict the following information about the objects.

- If $G_{ij} = P$, the object a_j has positive belongingness value x_{ij} higher than the negative belongingness value $|y_{ij}|$, with respect to ω_i.
- If $G_{ij} = N$, the object a_j has negative belongingness value $|y_{ij}|$ higher than the positive belongingness value x_{ij}, with respect to ω_i.
- If $G_{ij} = O$, the object a_j has positive belongingness x_{ij} equal to the negative belongingness $|y_{ij}|$, with respect to ω_i.

Now we give the concept of indiscernibility relations on A associated with the BFSs in Ω. We say that two objects a_j and a_k are indiscernible, written as $a_j \sim a_k$, if and only if they have same grades for each ω_i. Thus, when we say that the objects a_j and a_k are indiscernible, it means that, either both the objects have positivity higher than the negativity, or both the objects have negativity higher than the positivity, or both the objects have equal measures of positivity and negativity. The indiscernibility relation $\Re$ between the objects of A is defined as:

$$\Re = \{(a_j, a_k) \in A \times A : a_j \sim a_k\}. \tag{2.16}$$

Surely, $\Re$ is an eqv-rel on A.

Now, we proceed to the decision values d_j for the objects. For the eqv-rel $\Re$, we write $\underline{\Re}(\omega_i)(a_j) = (\underline{x_{ij}}, \underline{y_{ij}})$ and $\overline{\Re}(\omega_i)(a_j) = (\overline{x_{ij}}, \overline{y_{ij}})$. Denote

$$\underline{z_j} = \sum_{i=1}^{m} (\underline{x_{ij}} - \left|\underline{y_{ij}}\right|), \tag{2.17}$$

$$\overline{z_j} = \sum_{i=1}^{m} (\overline{x_{ij}} - |\overline{y_{ij}}|). \tag{2.18}$$

Then, $\underline{z_j}$ represents the definite fulfilment of the object a_j, while, $\overline{z_j}$ represents the maximum possible fulfilment of the object a_j, towards all decision makers $\{\omega_i : 1 \leq i \leq m\}$. Thus, the uncertain (or doubtful) fulfilment of a_j is given by the difference $\overline{z_j} - \underline{z_j}$. The decision will be taken on the basis of the decision parameter, defined below.

Definition 2.6.2 *The decision parameter D has the values d_j corresponding to each object $a_j \in A$, given by*

$$d_j = \underline{z_j} + \overline{z_j}. \tag{2.19}$$

This value gives the definite fulfilment $\underline{z_j}$ of the object a_j, a double weightage than to the uncertain (doubtful) fulfilment $\overline{z_j} - \underline{z_j}$, because we have

$$d_j = \underline{z_j} + \overline{z_j} = 2\underline{z_j} - \underline{z_j} + \overline{z_j},$$

or,

$$d_j = 2\underline{z_j} + (\overline{z_j} - \underline{z_j}). \tag{2.20}$$

We can rewrite Equation 2.20 as:

$$d_j = 2\sum_{i=1}^{m}(\underline{x_{ij}} - \left|\underline{y_{ij}}\right|) + \sum_{i=1}^{m}((\overline{x_{ij}} - |\overline{y_{ij}}|) - (\underline{x_{ij}} - \left|\underline{y_{ij}}\right|)) \tag{2.21}$$

From Equation 2.21, it is clear that the higher the definite positive fulfilment $\underline{x_{ij}}$ of a_j, the larger the value d_j. Also, the higher the definite negative fulfilment $\left|\underline{y_{ij}}\right|$ of a_j, the smaller the value d_j. In this way, we identify the poor objects having lowest value of d_j. These are the objects with high definite negative fulfilment, according to ω_i. Hence, our algorithm has the following main advantages.

- It manipulates technically the fuzziness of the data enriched with the bipolarity of information.
- It accommodates the assessments of any (finite) number of decision makers about any (finite) number of objects.
- It gives double weightage to the definite fulfilment of the objects, than to the uncertain fulfilment.
- It yields a wise decision, containing the best, as well as, the poor decision, so that, one can sidestep the poor decision.

Main steps of the algorithm are as follows.

Algorithm 2.6.3 *The algorithm to decide for the best and poor objects in* A, *is given below.*

Step I: *Input the BFSs* Ω.

Step II: *Find out the eqv-rel* $\Re$ *on* A, *using Formula 2.16.*

Step III: *Evaluate* $\underline{\Re}(\omega_i)$ *and the values* $\underline{z_j}$ *using Equations 2.1 and 2.17, respectively.*

Step IV: *Evaluate* $\overline{\Re}(\omega_i)$ *and the values* $\overline{z_j}$ *using Equations 2.2 and 2.18, respectively.*

Step V: *Find the decision values* d_j *for each object* $a_j \in A$, *using Formula 2.19.*

Step VI: *Construct the decision table with rows of* A *and the decision parameter* D *in the descending order with respect to the values of* D. *Choose* p *and* q, *so that* $d_p = \max_j d_j$ *and* $d_q = \min_j d_j$. *Then,* a_p *is the best optimal object, while,* a_q *is the poor object to be decided. If* p *has more than one values, any one of* a_p*'s can be selected.*

The flow chart of Algorithm 2.6.3 is shown in Figure 2.1.

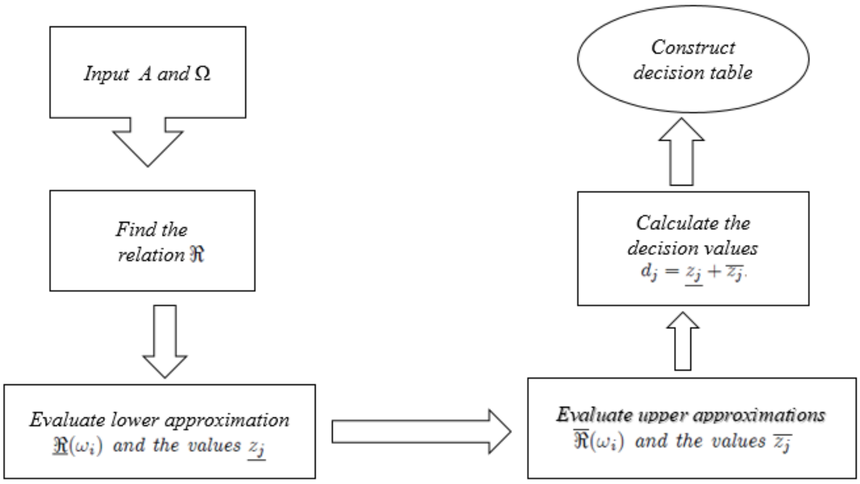

Figure 2.1: Flow chart of Algorithm 2.6.3.

Now we consider an illustration of the Algorithm 2.6.3, as follows.

Example 2.6.4 *The GDM problem.*

Let $A = \{a_i; i = 1, ..., 6\}$ be a set containing some similar products and a company X wishes to decide for one product to manufacture. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ be a set containing the BFSs describing the assessments of four independent experts, who are assigned by the company to decide in the favour of a single product. Here, we have $1 \leq i \leq 4$ and $1 \leq j \leq 6$.

Step I:

The information about the objects a_j, provided by each ω_i, is represented in the Table 2.1, where the $(i, j)th$ entry $\omega_i(a_j) = (x_{ij}, y_{ij})$ describes the assessment of ω_i about the product a_j. The value x_{ij} represents the degree to which the product a_j is suitable for being manufactured and the value y_{ij} represents the degree to which a_j is not favorable for production, according to the assessment of the expert ω_i.

$\eth$	a_1	a_2	a_3	a_4	a_5	a_6
ω_1	$(0.6, -0.3)$	$(0.7, -0.4)$	$(0.5, -0.5)$	$(0.6, -0.6)$	$(0.7, -0.3)$	$(0.4, -0.4)$
ω_2	$(0.6, -0.4)$	$(0.5, -0.5)$	$(0.6, -0.3)$	$(0.4, -0.5)$	$(0.5, -0.5)$	$(0.3, -0.4)$
ω_3	$(0.7, -0.2)$	$(0.6, -0.3)$	$(0.5, -0.2)$	$(0.5, -0.5)$	$(0.6, -0.4)$	$(0.4, -0.4)$
ω_4	$(0.5, -0.4)$	$(0.3, -0.3)$	$(0.5, -0.5)$	$(0.5, -0.3)$	$(0.4, -0.4)$	$(0.5, -0.3)$

Table 2.1: Table of Ω

Step II:

For the relation $\Re$, first we assign the indiscernibility grades to each object, corresponding to each BFS $\omega_i \in \Omega$, using Formula 2.15, as in Table 2.2.

A	a_1	a_2	a_3	a_4	a_5	a_6
G_{1j}	P	P	O	O	P	O
G_{2j}	P	O	P	N	O	N
G_{3j}	P	P	P	O	P	O
G_{4j}	P	O	O	P	O	P

Table 2.2: Assignment of indiscernibility grades G_{ij}

From Table 2.2, it can be clearly seen that, a_2 and a_5 got the same grades, while, a_4 and a_6 received the same grades. So, the Formula 2.16 leads to the following eqv-rel on A.

$$\begin{aligned}\Re &= \{(a_1,a_1),(a_2,a_2),(a_3,a_3),(a_4,a_4),(a_5,a_5),(a_6,a_6),\\ &\quad (a_2,a_5),(a_5,a_2),(a_4,a_6),(a_6,a_4)\}.\end{aligned}$$

This relation serves as our key tool to evaluate the RBF-apxes of ω_i.

Step III:

The lower RBF-apxes $\underline{\Re}(\omega_i)$ of each ω_i and the values $\underline{z_j}$ are evaluated in Table 2.3, by using the Equation 2.1 and Equation 2.17, respectively.

$\underline{\Re}(\omega_i)$	a_1	a_2	a_3	a_4	a_5	a_6
$\underline{\Re}(\omega_1)$	$(0.6,-0.3)$	$(0.7,-0.3)$	$(0.5,-0.5)$	$(0.4,-0.4)$	$(0.7,-0.3)$	$(0.4,-0.4)$
$\underline{\Re}(\omega_2)$	$(0.6,-0.4)$	$(0.5,-0.5)$	$(0.6,-0.3)$	$(0.3,-0.4)$	$(0.5,-0.5)$	$(0.3,-0.4)$
$\underline{\Re}(\omega_3)$	$(0.7,-0.2)$	$(0.6,-0.3)$	$(0.5,-0.2)$	$(0.4,-0.4)$	$(0.6,-0.3)$	$(0.4,-0.4)$
$\underline{\Re}(\omega_4)$	$(0.5,-0.4)$	$(0.3,-0.3)$	$(0.5,-0.5)$	$(0.5,-0.3)$	$(0.3,-0.3)$	$(0.5,-0.3)$
$\underline{z_j}$	1.1	0.7	0.6	0.1	0.7	0.1

Table 2.3: Calculations of $\underline{\Re}(\omega_i)$ and $\underline{z_j}$

Step IV:

The upper RBF-apxes $\overline{\Re}(\omega_i)$ of each ω_i and the values $\overline{z_j}$ are evaluated in Table 2.4, by using the Equation 2.2 and Equation 2.18, respectively.

$\overline{\Re}(\omega_i)$	a_1	a_2	a_3	a_4	a_5	a_6
$\overline{\Re}(\omega_1)$	$(0.6,-0.3)$	$(0.7,-0.4)$	$(0.5,-0.5)$	$(0.6,-0.6)$	$(0.7,-0.4)$	$(0.6,-0.6)$
$\overline{\Re}(\omega_2)$	$(0.6,-0.4)$	$(0.5,-0.5)$	$(0.6,-0.3)$	$(0.4,-0.5)$	$(0.5,-0.5)$	$(0.4,-0.5)$
$\overline{\Re}(\omega_3)$	$(0.7,-0.2)$	$(0.6,-0.4)$	$(0.5,-0.2)$	$(0.5,-0.5)$	$(0.6,-0.4)$	$(0.5,-0.5)$
$\overline{\Re}(\omega_4)$	$(0.5,-0.4)$	$(0.4,-0.4)$	$(0.5,-0.5)$	$(0.5,-0.3)$	$(0.4,-0.4)$	$(0.5,-0.3)$
$\overline{z_j}$	1.1	0.5	0.6	0.1	0.5	0.1

Table 2.4: Calculations of $\overline{\Re}(\omega_i)$ and $\overline{z_j}$

Step V:

The decision values d_j for each $a_j \in A$ are determined in Table 2.5, using Formula 2.19.

A	a_1	a_2	a_3	a_4	a_5	a_6
$\underline{z_j}$	1.1	0.7	0.6	0.1	0.7	0.1
$\overline{z_j}$	1.1	0.5	0.6	0.1	0.5	0.1
d_j	2.2	1.2	1.2	0.2	1.2	0.2

Table 2.5: Calculations of decision values d_j

Step VI:

Now we construct our decision table by placing the set A in first row and the decision parameter D in the second row. The table is rearranged in the descending order with respect to the values of D. Decision table is given by the Table 2.6.

A	a_1	a_2	a_3	a_5	a_4	a_6
D	2.2	1.2	1.2	1.2	0.2	0.2

Table 2.6: Decision table

We get $\max_j d_j = d_1 = 2.2$ and $\min_j d_j = d_4 = d_6 = 0.2$. Hence $p = 1$ and $q = 4, 6$. Thus, the product a_1 is the best decision, while a_4 and a_6 are the poor selections. So, the most favorable item which the company can manufacture is a_1. But, in any case, it should not go for the products a_4 or a_6.

Chapter 3

Rough bipolar fuzzy ideals in semigroups

3.1 Introduction

The notion of RBFSs defined in Chapter 2 handles the vagueness and uncertainty, as well as, bipolarity in data which can not be avoided in many real life problems. The concepts of roughness, fuzziness and bipolarity are also correlated to the semigroups in different manners by many authors to study the data having the structure of semigroups. Kuroki [36] presented the concept of fuzzy semigroup in 1991. Ideal theory in semigroups is correlated to the FSs by many authors; see [1, 2, 31, 33, 56, 58, 59, 60]. Rough ideals in semigroups were initiated by Kuroki [37] in 1997. After that, Maji et al. [41] applied the concept of roughness to the fuzzy semigroups. Later on, rough fuzzy ideals in semigroups are studied in [52]. BF-ids in semigroups were presented by Kim et al. [34]. The direction of this chapter is to extend this work and to the present the RBF-ids of semigroups. In this chapter, we have discussed the BFSs and BF-ids in a semigroup. We have also studied the roughness in the BF subsemigroups with the help of a cng-rel defined on the semigroup and investigated some properties of the RBF subsemigroup. RBF-id, RBFi-id and RBFb-id in the semigroups are defined and discussed in this chapter.

3.2 Bipolar fuzzy sets in semigroups

In this section, we link the BF subsemigroups to the subsemigroups of a semigroup and the BFl-ids (BFr-ids, BF-ids, BFi-id and BFb-id in semigroups to the left (right, two-sided, interior, bi-) ideals of semigroups, using the $\alpha-$level P-cuts and $\alpha-$level N-cuts of the BFSs as defined in Section 2.5. Throughout this chapter, Υ is a semigroup and $\Re$ is a cng-rel on Υ. Recall the Definition 2.5.1 of the $\alpha-$level P-cuts and the Definition 2.5.2 of the $\alpha-$level N-cuts of a BFS λ in U, that are,

$$\begin{aligned}\lambda_\alpha &= \{u \in U : \lambda^P(u) \geq \alpha\},\\ \lambda^\alpha &= \{u \in U : \lambda^N(u) \leq -\alpha\},\end{aligned}$$

respectively, for each $0 \lneqq \alpha \leq 1$.

Theorem 3.2.1 *A BFS λ in Υ is a BF subsemigroup of Υ if and only if λ_α and λ^α, if non-empty, are subsemigroups of Υ, for each $\alpha \in [0,1]$.*

Proof. Let λ be a BF subsemigroup of Υ, and let $a, b \in \lambda_\alpha$. Then, $\lambda^p(a) \geqslant \alpha$ and $\lambda^p(b) \geqslant \alpha$. Since λ is a BF subsemigroup of Υ, so

$$\lambda^p(ab) \geqslant \lambda^p(a) \wedge \lambda^p(b) \geqslant \alpha.$$

Which implies, $ab \in \lambda_\alpha$. So, λ_α is a subsemigroup of Υ for each α. Similarly λ^α is a subsemigroup of Υ for each α.

Conversely, let λ_α and λ^α be non-empty subsemigroups of Υ for each $\alpha \in [0,1]$, and let $a, b \in \Upsilon$. Denote $\lambda^p(a) \wedge \lambda^p(b)$ by $\alpha_\circ \in [0,1]$. Then surely, $\lambda^p(a), \lambda^p(b) \geqslant \alpha_\circ$, and so $a, b \in \lambda_{\alpha_\circ}$. But $\lambda_{\alpha_\circ}$ is a subsemigroup of Υ, so $ab \in \lambda_{\alpha_\circ}$. Which yields $\lambda^p(ab) \geqslant \alpha_\circ$. That is,

$$\lambda^p(ab) \geqslant \lambda^p(a) \wedge \lambda^p(b). \tag{3.1}$$

Now, denote $\lambda^N(a) \vee \lambda^N(b)$ by $-\alpha_1$, where $\alpha_1 \in [0,1]$. Then $\lambda^N(a), \lambda^N(b) \leqslant -\alpha_1$, and so $a, b \in \lambda^{\alpha_1}$. But λ^{α_1} is a subsemigroup of Υ, so $ab \in \lambda^{\alpha_1}$. Which yields $\lambda^N(ab) \leqslant -\alpha_1$. That is,

$$\lambda^N(ab) \leqslant \lambda^N(a) \vee \lambda^N(b). \tag{3.2}$$

Assertions 3.1 and 3.2 prove that λ is a BF subsemigroup of Υ. ■

Theorem 3.2.2 *A BFS λ in Υ is a BFl-id (BFr-id, BF-id) of Υ if and only if λ_α and λ^α, if non-empty, are left (right, two-sided) ideals of Υ, for each $\alpha \in [0,1]$.*

Proof. Let λ be a BFl-id of Υ and let $x \in \Upsilon$. Then, for any $\alpha \in [0,1]$ and for each $a \in \lambda_\alpha$, we have

$$\lambda^p(xa) \geqslant \lambda^p(a) \geqslant \alpha.$$

Which implies, $xa \in \lambda_\alpha$ for each $x \in \Upsilon$ and $a \in \lambda_\alpha$. That is, λ_α is a left ideal of Υ for each $\alpha \in [0,1]$. Now, for each $a \in \lambda^\alpha$, we have

$$\lambda^N(xa) \leqslant \lambda^N(a) \leqslant -\alpha.$$

Which implies that, $xa \in \lambda^{\alpha}$ for each $x \in \Upsilon$ and $a \in \lambda^{\alpha}$. That is, λ^{α} is also a left ideal of Υ for each $\alpha \in [0,1]$.

Conversely, let λ_{α} and λ^{α} be non-empty left ideals of Υ for each $\alpha \in [0,1]$ and let $a, b \in \Upsilon$. Denote $\lambda^{p}(b)$ by $\alpha_{\circ} \in [0,1]$. Then surely, $b \in \lambda_{\alpha_{\circ}}$. Thus $ab \in \lambda_{\alpha_{\circ}}$, as $\lambda_{\alpha_{\circ}}$ is a left ideal of Υ. Which gives $\lambda^{p}(ab) \geqslant \alpha_{\circ}$. That is,

$$\lambda^{p}(ab) \geqslant \lambda^{p}(b). \tag{3.3}$$

Now, denote $\lambda^{N}(b)$ by $-\alpha_1$, where $\alpha_1 \in [0,1]$. Then $b \in \lambda^{\alpha_1}$ and λ^{α_1} is a left ideal of Υ. Thus, $ab \in \lambda^{\alpha_1}$. Which gives $\lambda^{N}(ab) \leqslant -\alpha_1$. That is,

$$\lambda^{N}(ab) \leqslant \lambda^{N}(b). \tag{3.4}$$

The expressions 3.3 and 3.4 prove that λ is a BFl-id of Υ.

Similar is the proof when λ is a BFr-id or a BF-id of Υ. ■

Theorem 3.2.3 *A BFS λ in Υ is a BFi-id of Υ if and only if λ_{α} and λ^{α}, if non-empty, are interior ideals of Υ, for each $\alpha \in [0,1]$.*

Proof. Let λ be a BFi-id of Υ and let $x, y \in \Upsilon$. Then, for any $\alpha \in [0,1]$ and for each $a \in \lambda_{\alpha}$, we have

$$\lambda^{p}(xay) \geqslant \lambda^{p}(a) \geqslant \alpha.$$

Which implies, $xay \in \lambda_{\alpha}$ for each $x, y \in \Upsilon$ and $a \in \lambda_{\alpha}$. That is, λ_{α} is an interior ideal of Υ for each $\alpha \in [0,1]$. Now, for each $a \in \lambda^{\alpha}$, we have

$$\lambda^{N}(xay) \leqslant \lambda^{N}(a) \leqslant -\alpha.$$

Which implies that, $xay \in \lambda^{\alpha}$ for each $x, y \in \Upsilon$ and $a \in \lambda^{\alpha}$. That is, λ^{α} is also an interior ideal of Υ for each $\alpha \in [0,1]$.

Conversely, let λ_{α} and λ^{α} be non-empty interior ideals of Υ for each $\alpha \in [0,1]$ and let $a, b, c \in \Upsilon$. Denote $\lambda^{p}(b)$ by $\alpha_{\circ} \in [0,1]$. Then surely, $b \in \lambda_{\alpha_{\circ}}$. Thus $abc \in \lambda_{\alpha_{\circ}}$, as $\lambda_{\alpha_{\circ}}$ is interior ideal of Υ. Which gives $\lambda^{p}(abc) \geqslant \alpha_{\circ}$. That is,

$$\lambda^{p}(abc) \geqslant \lambda^{p}(b). \tag{3.5}$$

Now, denote $\lambda^{N}(b)$ by $-\alpha_1$, where $\alpha_1 \in [0,1]$. Surely, $b \in \lambda^{\alpha_1}$ and λ^{α_1} is an interior ideal of Υ. Thus, $abc \in \lambda^{\alpha_1}$ for each $a, c \in \Upsilon$. Which gives $\lambda^{N}(abc) \leqslant -\alpha_1$. That is,

$$\lambda^{N}(abc) \leqslant \lambda^{N}(b). \tag{3.6}$$

The expressions 3.5 and 3.6 prove that λ is a BFi-id of Υ. ■

Theorem 3.2.4 *A BFS λ in Υ is a BFb-id of Υ if and only if λ_α and λ^α, if non-empty, are bi-ideals of Υ, for each $\alpha \in [0,1]$.*

Proof. Let λ be a BFb-id of Υ. Then, λ is also a BF subsemigroup of Υ. Hence, λ_α and λ^α, if non-empty, are subsemigroups of Υ, for each $\alpha \in [0,1]$, by Theorem 3.2.1. Now, let $a, c \in \lambda_\alpha$. Then, $\lambda^p(a) \geqslant \alpha$ and $\lambda^p(c) \geqslant \alpha$. Since λ is a BFb-id of Υ, so for each $b \in \Upsilon$, we have

$$\lambda^p(abc) \geqslant \lambda^p(a) \wedge \lambda^p(c) \geqslant \alpha.$$

Which implies $abc \in \lambda_\alpha$ for each $a, c \in \lambda_\alpha$ and $b \in \Upsilon$. So, λ_α is a bi-ideal of Υ for each α. Similarly, λ^α is a bi-ideal of Υ for each α.

Conversely, let λ_α and λ^α be non-empty bi-ideals of Υ for each $\alpha \in [0,1]$, and let $a, b, c \in \Upsilon$. Denote $\lambda^p(a) \wedge \lambda^p(c)$ by $\alpha_\circ \in [0,1]$. Then surely, $\lambda^p(a), \lambda^p(c) \geqslant \alpha_\circ$, and so $a, c \in \lambda_{\alpha_\circ}$. But, $\lambda_{\alpha_\circ}$ is a bi-ideal of Υ. So $abc \in \lambda_{\alpha_\circ}$ for each $b \in \Upsilon$. Which yields $\lambda^p(abc) \geqslant \alpha_\circ$. Or,

$$\lambda^p(abc) \geqslant \lambda^p(a) \wedge \lambda^p(c). \tag{3.7}$$

Now, denote $\lambda^N(a) \vee \lambda^N(c)$ by $-\alpha_1$, where $\alpha_1 \in [0,1]$. Then $\lambda^N(a), \lambda^N(c) \leqslant -\alpha_1$, and so $a, c \in \lambda^{\alpha_1}$. But λ^{α_1} is a bi-ideal of Υ. So $abc \in \lambda^{\alpha_1}$ for each $b \in \Upsilon$. Which yields $\lambda^N(abc) \leqslant -\alpha_1$. Or,

$$\lambda^N(abc) \leqslant \lambda^N(a) \vee \lambda^N(c). \tag{3.8}$$

Assertions 3.7 and 3.8 prove that λ is a BFb-id of Υ. ■

3.3 Rough bipolar fuzzy sets in semigroups

The RBFSs in semigroups are defined with the help of lower and upper RBF-apxes of the BFSs in the semigroup Υ, on which a cng-rel $\Re$ is defined. These approximations are defined and discussed in this section. We also present the RBF subsemigroup of Υ.

Definition 3.3.1 *Let $\Re$ be a cng-rel on Υ and let $\lambda \in BFS(\Upsilon)$. The lower and upper RBF-apxes of λ under the relation $\Re$, are the BFSs $\underline{\Re}(\lambda)$ and $\overline{\Re}(\lambda)$ in Υ, respectively, defined for each $s \in \Upsilon$ as:*

$$\begin{aligned} \underline{\Re}(\lambda) &= \{(s, \underline{\Re}\lambda^P(s), \underline{\Re}\lambda^N(s)) : s \in \Upsilon\}, \\ \overline{\Re}(\lambda) &= \{(s, \overline{\Re}\lambda^P(s), \overline{\Re}\lambda^N(s)) : s \in \Upsilon\}, \end{aligned}$$

where,

$$\begin{aligned} \underline{\Re}\lambda^P(s) &= \bigwedge_{t \in [s]_\Re} \lambda^P(t), \qquad \underline{\Re}\lambda^N(s) = \bigvee_{t \in [s]_\Re} \lambda^N(t), \\ \overline{\Re}\lambda^P(s) &= \bigvee_{t \in [s]_\Re} \lambda^P(t), \qquad \overline{\Re}\lambda^N(s) = \bigwedge_{t \in [s]_\Re} \lambda^N(t). \end{aligned}$$

If $\underline{\Re}(\lambda) = \overline{\Re}(\lambda)$*, then,* λ *is said to be* $\Re$*-definable; otherwise,* λ *is an RBFS in* Υ*.*

In Chapter 2, some characterizations of RBFSs in a non-empty set U having an eqv-rel $\Re$ were presented. These characterizations are also valid when the set U is replaced by the semigroup Υ and the eqv-rel on U is replaced by a cng-rel on Υ. So the results in Chapter 2 also hold for the lower and upper RBF-apxes of the BFSs in Υ, given in the Definition 3.3.1. Some other results are as follows.

Theorem 3.3.2 *Let* $\Re$ *be a cng-rel on* Υ*. Then,*

$$\overline{\Re}(\lambda) \circ \overline{\Re}(\nu) \subseteq \overline{\Re}(\lambda \circ \nu)$$

holds for each $\lambda, \nu \in BFS(\Upsilon)$*.*

Proof. Since $\Re$ is a cng-rel on Υ, so $[x]_\Re[y]_\Re \subseteq [xy]_\Re$ for each $x, y \in \Upsilon$. Let $\lambda, \nu \in BFS(\Upsilon)$. We have

$$\begin{aligned}
\overline{\Re}(\lambda) \circ \overline{\Re}(\nu) &= (\overline{\Re}\lambda^P \circ \overline{\Re}\nu^P, \overline{\Re}\lambda^N \circ \overline{\Re}\nu^N), \\
\overline{\Re}(\lambda \circ \nu) &= (\overline{\Re}(\lambda^P \circ \nu^P), \overline{\Re}(\lambda^N \circ \nu^N)).
\end{aligned}$$

Take any $s \in \Upsilon$. If some $x, y \in \Upsilon$ exist, such that $s = xy$, then we have

$$\begin{aligned}
(\overline{\Re}\lambda^P \circ \overline{\Re}\nu^P)(s) &= \bigvee_{s=xy} (\overline{\Re}\lambda^P(x) \wedge \overline{\Re}\nu^P(y)) \\
&= \bigvee_{s=xy} ((\bigvee_{a\in[x]_\Re} \lambda^P(a)) \wedge (\bigvee_{b\in[y]_\Re} \nu^P(b))) \\
&= \bigvee_{s=xy} (\bigvee_{a\in[x]_\Re,\, b\in[y]_\Re} (\lambda^P(a) \wedge \nu^P(b))) \\
&\leq \bigvee_{s=xy} (\bigvee_{ab\in[xy]_\Re} (\lambda^P(a) \wedge \nu^P(b))), \qquad \text{since } ab \in [x]_\Re[y]_\Re \subseteq [xy]_\Re \\
&= \bigvee_{ab\in[s]_\Re} (\lambda^P(a) \wedge \nu^P(b)), \qquad \text{since } xy = s \\
&= \bigvee_{t\in[s]_\Re,\, t=ab} (\lambda^P(a) \wedge \nu^P(b)) \\
&= \bigvee_{t\in[s]_\Re} (\bigvee_{t=ab} (\lambda^P(a) \wedge \nu^P(b))) \\
&= \bigvee_{t\in[s]_\Re} (\lambda^P \circ \nu^P)(t) = \overline{\Re}(\lambda^P \circ \nu^P)(s).
\end{aligned}$$

Otherwise;

$$(\overline{\Re}\lambda^P \circ \overline{\Re}\nu^P)(s) = 0 \leq \overline{\Re}(\lambda^P \circ \nu^P)(s).$$

Similarly, for each $s \in \Upsilon$, we have

$$(\overline{\Re}\lambda^N \circ \overline{\Re}\nu^N)(s) \geq \overline{\Re}(\lambda^N \circ \nu^N)(s).$$

Thus, by Definition 1.3.2, we have

$$\overline{\Re}(\lambda) \circ \overline{\Re}(\nu) \subseteq \overline{\Re}(\lambda \circ \nu)$$

for each $\lambda, \nu \in BFS(\Upsilon)$. ■

Theorem 3.3.3 *Let $\Re$ be a complete cng-rel on Υ. Then,*

$$\underline{\Re}(\lambda) \circ \underline{\Re}(\nu) \subseteq \underline{\Re}(\lambda \circ \nu)$$

holds for each $\lambda, \nu \in BFS(\Upsilon)$.

Proof. Since $\Re$ is a complete cng-rel on Υ, so $[x]_{\Re}[y]_{\Re} = [xy]_{\Re}$ for each $x, y \in \Upsilon$. Let $\lambda, \nu \in BFS(\Upsilon)$. We have

$$\begin{aligned} \underline{\Re}(\lambda) \circ \underline{\Re}(\nu) &= (\underline{\Re}\lambda^P \circ \underline{\Re}\nu^P, \underline{\Re}\lambda^N \circ \underline{\Re}\nu^N), \\ \underline{\Re}(\lambda \circ \nu) &= (\underline{\Re}(\lambda^P \circ \nu^P), \underline{\Re}(\lambda^N \circ \nu^N)). \end{aligned}$$

Take any $s \in \Upsilon$. If some $x, y \in \Upsilon$ exist, such that $s = xy$, then we have

$$\begin{aligned} (\underline{\Re}\lambda^P \circ \underline{\Re}\nu^P)(s) &= \bigvee_{s=xy} (\underline{\Re}\lambda^P(x) \wedge \underline{\Re}\nu^P(y)) \\ &= \bigvee_{s=xy} ((\bigwedge_{a\in[x]_{\Re}} \lambda^P(a)) \wedge (\bigwedge_{b\in[y]_{\Re}} \nu^P(b))) \\ &= \bigvee_{s=xy} (\bigwedge_{a\in[x]_{\Re},\, b\in[y]_{\Re}} (\lambda^P(a) \wedge \nu^P(b))) \\ &\leq \bigvee_{s=xy} (\bigwedge_{a\in[x]_{\Re},\, b\in[y]_{\Re}} \bigvee_{ab=t_1t_2} (\lambda^P(t_1) \wedge \nu^P(t_2))), \qquad \text{where } t_1, t_2 \in \Upsilon \\ &= \bigvee_{s=xy} (\bigwedge_{a\in[x]_{\Re},\, b\in[y]_{\Re}} (\lambda^P \circ \nu^P)(ab)) \\ &= \bigvee_{s=xy} (\bigwedge_{ab\in[xy]_{\Re}} (\lambda^P \circ \nu^P)(ab)), \qquad \text{since } ab \in [x]_{\Re}[y]_{\Re} = [xy]_{\Re} \\ &= \bigvee_{s=xy} (\underline{\Re}(\lambda^P \circ \nu^P)(xy)) = \underline{\Re}(\lambda^P \circ \nu^P)(s). \end{aligned}$$

Otherwise;

$$(\underline{\Re}\lambda^P \circ \underline{\Re}\nu^P)(s) = 0 \leq \underline{\Re}(\lambda^P \circ \nu^P)(s).$$

Similarly, for each $s \in \Upsilon$, we have

$$(\underline{\Re}\lambda^N \circ \underline{\Re}\nu^N)(s) \geq \underline{\Re}(\lambda^N \circ \nu^N)(s).$$

Thus, by Definition 1.3.2, we get

$$\underline{\Re}(\lambda) \circ \underline{\Re}(\nu) \subseteq \underline{\Re}(\lambda \circ \nu)$$

for each $\lambda, \nu \in BFS(\Upsilon)$. ■

Definition 3.3.4 *A BFS λ in Υ is a lower (or upper) RBF subsemigroup of Υ, if $\underline{\Re}(\lambda)$ (or $\overline{\Re}(\lambda)$) is a BF subsemigroup of Υ, for the cng-rel $\Re$ on Υ.*

A BFS λ in Υ, which is both, lower and upper RBF subsemigroup of Υ, is called an RBF subsemigroup of Υ.

Theorem 3.3.5 *Let $\Re$ be a cng-rel on Υ. Then, each BF subsemigroup of Υ is an upper RBF subsemigroup of Υ.*

Proof. Take a BF subsemigroup λ of Υ. Then, $\lambda^P(ab) \geq \lambda^P(a) \wedge \lambda^P(b)$ and $\lambda^N(ab) \leq \lambda^N(a) \vee \lambda^N(b)$ for each $a, b \in \Upsilon$. Now, for each $x, y \in \Upsilon$, we have

$$\begin{aligned}
\overline{\Re}\lambda^P(xy) &= \bigvee_{s\in[xy]_\Re} \lambda^P(s) \\
&\geq \bigvee_{s\in[x]_\Re[y]_\Re} \lambda^P(s), \qquad \text{since } [x]_\Re[y]_\Re \subseteq [xy]_\Re \\
&= \bigvee_{ab\in[x]_\Re[y]_\Re} \lambda^P(ab), \qquad \text{where } s = ab \\
&\geq \bigvee_{a\in[x]_\Re,\ b\in[y]_\Re} (\lambda^P(a) \wedge \lambda^P(b)) \\
&= (\bigvee_{a\in[x]_\Re} \lambda^P(a)) \wedge (\bigvee_{b\in[y]_\Re} \lambda^P(b)) \\
&= \overline{\Re}\lambda^P(x) \wedge \overline{\Re}\lambda^P(y)
\end{aligned}$$

and

$$\begin{aligned}
\overline{\Re}\lambda^N(xy) &= \bigwedge_{s\in[xy]_\Re} \lambda^N(s) \\
&\leq \bigwedge_{s\in[x]_\Re[y]_\Re} \lambda^N(s), \qquad \text{since } [x]_\Re[y]_\Re \subseteq [xy]_\Re \\
&= \bigwedge_{ab\in[x]_\Re[y]_\Re} \lambda^N(ab), \qquad \text{where } s = ab \\
&\leq \bigwedge_{a\in[x]_\Re,\ b\in[y]_\Re} (\lambda^N(a) \vee \lambda^N(b)) \\
&= (\bigwedge_{a\in[x]_\Re} \lambda^N(a)) \vee (\bigwedge_{b\in[y]_\Re} \lambda^N(b)) \\
&= \overline{\Re}\lambda^N(x) \vee \overline{\Re}\lambda^N(y).
\end{aligned}$$

This verifies that $\overline{\Re}(\lambda)$ is a BF subsemigroup of Υ. Therefore, λ is an upper RBF subsemigroup of Υ. ■

The converse statement of Theorem 3.3.5 is invalid generally, as exhibited in the next example.

Example 3.3.6 *The table of binary operation on a semigroup $\Upsilon = \{a, b, c, d\}$ is given below.*

	a	b	c	d
a	a	b	b	d
b	b	b	b	d
c	b	b	b	d
d	d	d	d	d

Consider a binary relation $\Re = \{(a,a),(b,b),(c,c),(d,d),(b,d),(d,b)\}$ *on* Υ. *Then* $\Re$ *is a cng-rel on* Υ, *defining the cng-classes* $\{a\},\{c\},\{b,d\}$. *We take a BFS* λ *in* Υ, *as below.*

$\lambda = \{(a, 0.4, -0.1), (b, 0.3, -0.2), (c, 0.4, -0.2), (d, 0.5, -0.3)\}$.

The upper RBF-apx of λ *is calculated as:*

$\overline{\Re}(\lambda) = \{(a, 0.4, -0.1), (b, 0.5, -0.3), (c, 0.4, -0.2), (d, 0.5, -0.3)\}$.

Simple calculations verify that $\overline{\Re}(\lambda)$ *is a BF subsemigroup of* Υ. *But,* λ *is not a BF subsemigroup of* Υ, *as:*

$$\begin{aligned} \lambda^P(cc) &= \lambda^P(b) = 0.3 \\ &\ngeq \lambda^P(c) \wedge \lambda^P(c) = 0.4. \end{aligned}$$

Theorem 3.3.7 *Let* $\Re$ *be a complete cng-rel on* Υ. *Then, each BF subsemigroup of* Υ *is a lower RBF subsemigroup of* Υ.

Proof. Let $\Re$ be a complete cng-rel on Υ and λ be a BF subsemigroup of Υ. Now, for each $x, y \in \Upsilon$, we obtain

$$\begin{aligned} \underline{\Re}\lambda^P(xy) &= \bigwedge_{s \in [xy]_\Re} \lambda^P(s) \\ &= \bigwedge_{s \in [x]_\Re [y]_\Re} \lambda^P(s), \qquad \text{since } [x]_\Re [y]_\Re = [xy]_\Re \\ &= \bigwedge_{ab \in [x]_\Re [y]_\Re} \lambda^P(ab), \qquad \text{where } s = ab \\ &\geq \bigwedge_{a \in [x]_\Re,\ b \in [y]_\Re} (\lambda^P(a) \wedge \lambda^P(b)) \\ &= (\bigwedge_{a \in [x]_\Re} \lambda^P(a)) \wedge (\bigwedge_{b \in [y]_\Re} \lambda^P(b)) \\ &= \underline{\Re}\lambda^P(x) \wedge \underline{\Re}\lambda^P(y) \end{aligned}$$

and

$$\begin{aligned} \underline{\Re}\lambda^N(xy) &= \bigvee_{s \in [xy]_\Re} \lambda^N(s) \\ &= \bigvee_{s \in [x]_\Re [y]_\Re} \lambda^N(s), \qquad \text{since } [x]_\Re [y]_\Re = [xy]_\Re \\ &= \bigvee_{ab \in [x]_\Re [y]_\Re} \lambda^N(ab), \qquad \text{where } s = ab \\ &\leq \bigvee_{a \in [x]_\Re,\ b \in [y]_\Re} (\lambda^N(a) \vee \lambda^N(b)) \\ &= (\bigvee_{a \in [x]_\Re} \lambda^N(a)) \vee (\bigvee_{b \in [y]_\Re} \lambda^N(b)) \\ &= \underline{\Re}\lambda^N(x) \vee \underline{\Re}\lambda^N(y). \end{aligned}$$

This verifies that $\underline{\Re}(\lambda)$ is a BF subsemigroup of Υ. Therefore, λ is a lower RBF subsemigroup of Υ. ■

The converse statement of Theorem 3.3.7 is invalid generally, as exhibited in the next example.

Example 3.3.8 *Let $\Upsilon = \{s, t, u, v\}$ represent a semigroup whose table of binary operation is given below.*

	s	t	u	v
s	s	t	u	v
t	t	t	u	v
u	u	u	u	v
v	v	v	v	u

Consider a cng-rel $\Re$ on Υ, defining cng-classes $\{s\}, \{t\}$ and $\{u, v\}$. Then, $\Re$ is a complete cng-rel on Υ. We take a BFS λ in Υ, as below.

$\lambda = \{(s, 0.3, -0.4), (t, 0.4, -0.3), (u, 0.6, -0.2), (v, 0.8, -0.1)\}$.

The lower RBF-apx $\underline{\Re}(\lambda)$ of λ is calculated as:

$\underline{\Re}(\lambda) = \{(s, 0.3, -0.4), (t, 0.4, -0.3), (u, 0.6, -0.1), (v, 0.6, -0.1)\}$.

Simple calculations verify that $\underline{\Re}(\lambda)$ is a BF subsemigroup of Υ. But, λ is not a BF subsemigroup of Υ, as:

$$\begin{aligned} \lambda^P(vv) &= \lambda^P(u) = 0.6 \\ &\ngeq \lambda^P(v) \wedge \lambda^P(v) = 0.8. \end{aligned}$$

Next example shows that Theorem 3.3.7 is invalid if the cng-rel $\Re$ is not complete. Next example is established to verify this fact.

Example 3.3.9 *Recall the semigroup $\Upsilon = \{a, b, c, d\}$ and the cng-rel $\Re$ on Υ, as established in Example 3.3.6. Then, $\Re$ is not complete. We take a BFS λ in Υ, as below.*

$\lambda = \{(a, 0.4, -0.2), (b, 0.6, -0.4), (c, 0.4, -0.1), (d, 0.3, -0.3)\}$.

Then, λ is a BF subsemigroup of Υ. The RBF-apxes of λ are calculated by Definition 3.3.1 as:

$\overline{\Re}(\lambda) = \{(a, 0.4, -0.2), (b, 0.6, -0.4), (c, 0.4, -0.1), (d, 0.6, -0.4)\}$,

$\underline{\Re}(\lambda) = \{(a, 0.4, -0.2), (b, 0.3, -0.3), (c, 0.4, -0.1), (d, 0.3, -0.3)\}$.

Simple calculations verify that $\overline{\Re}(\lambda)$ is also a BF subsemigroup of Υ, while, $\underline{\Re}(\lambda)$ is not a BF subsemigroup of Υ, as

$$\begin{aligned} \underline{\Re}\lambda^P(ac) &= \underline{\Re}\lambda^P(b) = 0.3 \\ &\ngeq \underline{\Re}\lambda^P(a) \wedge \underline{\Re}\lambda^P(c) = 0.4. \end{aligned}$$

3.4 Rough bipolar fuzzy ideals in semigroups

This section presents the notions of the RBFl-id, RBFr-id, RBF-id, RBFi-id and RBFb-id of Υ. We also explore some of their characteristics.

Definition 3.4.1 *Let $\Re$ be a cng-rel on Υ and let $\lambda \in BFS(\Upsilon)$. Then, λ is a lower RBFl-id (RBFr-id, RBF-id) of Υ, if $\underline{\Re}(\lambda)$ is a BFl-id (BFr-id, BF-id) of Υ.*

Definition 3.4.2 *Let $\Re$ be a cng-rel on Υ and let $\lambda \in BFS(\Upsilon)$. Then, λ is an upper RBFl-id (RBFr-id, RBF-id) of Υ, if $\overline{\Re}(\lambda)$ is a BFl-id (BFr-id, BF-id) of Υ.*

A BFS λ in Υ, which is both, lower and upper RBFl-id (RBFr-id, RBF-id) of Υ, is called an RBFl-id (RBFr-id, RBF-id) of Υ.

Theorem 3.4.3 *Let $\Re$ be a cng-rel on Υ. Then,*

$$\overline{\Re}(\lambda \circ \nu) \subseteq \overline{\Re}(\lambda) \cap \overline{\Re}(\nu)$$

holds for each BFr-id λ and BFl-id ν of Υ.

Proof. Take a BFr-id λ and a BFl-id ν of Υ. We obtain

$$\begin{aligned} \overline{\Re}(\lambda \circ \nu) &= (\overline{\Re}(\lambda^P \circ \nu^P), \overline{\Re}(\lambda^N \circ \nu^N)), \\ \overline{\Re}(\lambda) \cap \overline{\Re}(\nu) &= (\overline{\Re}\lambda^P \cap \overline{\Re}\nu^P, \overline{\Re}\lambda^N \cup \overline{\Re}\nu^N). \end{aligned}$$

Now, for each $s \in \Upsilon$, we obtain

$$\begin{aligned} \overline{\Re}(\lambda^P \circ \nu^P)(s) &= \bigvee_{t \in [s]_\Re} (\lambda^P \circ \nu^P)(t) \\ &= \bigvee_{t \in [s]_\Re} \bigvee_{t=ab} (\lambda^P(a) \wedge \nu^P(b)) \\ &\leq \bigvee_{t \in [s]_\Re} \bigvee_{t=ab} (\lambda^P(ab) \wedge \nu^P(ab)), \text{ since } \lambda \text{ is a BFr-id} \\ &\qquad\qquad \text{and } \nu \text{ is a BFl-id of } \Upsilon. \\ &= \bigvee_{t \in [s]_\Re} (\lambda^P(t) \wedge \nu^P(t)) \\ &\leq \bigvee_{t \in [s]_\Re} \bigvee_{t' \in [s]_\Re} (\lambda^P(t) \wedge \nu^P(t')) \\ &= (\bigvee_{t \in [s]_\Re} \lambda^P(t)) \wedge (\bigvee_{t' \in [s]_\Re} \nu^P(t')) \\ &= \overline{\Re}\lambda^P(s) \wedge \overline{\Re}\nu^P(s) \\ &= (\overline{\Re}\lambda^P \cap \overline{\Re}\nu^P)(s). \end{aligned}$$

Similarly, for each $s \in \Upsilon$, we obtain

$$\overline{\Re}(\lambda^N \circ \nu^N)(s) \geq (\overline{\Re}\lambda^N \cup \overline{\Re}\nu^N)(s).$$

Thus, Definition 1.3.2 gives

$$\overline{\Re}(\lambda \circ \nu) \subseteq \overline{\Re}(\lambda) \cap \overline{\Re}(\nu)$$

for each BFr-id λ and BFl-id ν of Υ. ■

Theorem 3.4.4 *Let $\Re$ be a cng-rel on Υ. Then, each BFl-id (BFr-id, BF-id) of Υ is an upper RBFl-id (RBFr-id, RBF-id) of Υ.*

Proof. Take a BFl-id λ of Υ. Then, $\lambda^P(ab) \geq \lambda^P(b)$ and $\lambda^N(ab) \leq \lambda^N(b)$ for each $a, b \in \Upsilon$. Now, for each $x, y \in \Upsilon$, we obtain

$$\begin{aligned}
\overline{\Re}\lambda^P(xy) &= \bigvee_{s\in[xy]_\Re} \lambda^P(s) \\
&\geq \bigvee_{s\in[x]_\Re[y]_\Re} \lambda^P(s), && \text{since } [x]_\Re[y]_\Re \subseteq [xy]_\Re \\
&= \bigvee_{ab\in[x]_\Re[y]_\Re} \lambda^P(ab), && \text{where } s = ab \\
&\geq \bigvee_{a\in[x]_\Re,\ b\in[y]_\Re} \lambda^P(b) \\
&= \bigvee_{b\in[y]_\Re} \lambda^P(b) \\
&= \overline{\Re}\lambda^P(y)
\end{aligned}$$

and

$$\begin{aligned}
\overline{\Re}\lambda^N(xy) &= \bigwedge_{s\in[xy]_\Re} \lambda^N(s) \\
&\leq \bigwedge_{s\in[x]_\Re[y]_\Re} \lambda^N(s), && \text{since } [x]_\Re[y]_\Re \subseteq [xy]_\Re \\
&= \bigwedge_{ab\in[x]_\Re[y]_\Re} \lambda^N(ab), && \text{where } s = ab \\
&\leq \bigwedge_{a\in[x]_\Re,\ b\in[y]_\Re} \lambda^N(b) \\
&= \bigwedge_{b\in[y]_\Re} \lambda^N(b) \\
&= \overline{\Re}\lambda^N(y).
\end{aligned}$$

This verifies that $\overline{\Re}(\lambda)$ is a BFl-id of Υ. Therefore, λ is an upper RBFl-id of Υ. Similarly, the cases of BFr-id and BF-id of Υ can be verified. ■

The converse statement of Theorem 3.4.4 is invalid generally, as exhibited in the next example.

Example 3.4.5 *Let $\Upsilon = \{k, l, m, n\}$ represent a semigroup whose table of binary operation is given below.*

	k	l	m	n
k	k	k	k	n
l	k	l	k	n
m	k	k	m	n
n	n	n	n	n

Let $\Re$ be a cng-rel on Υ, defining cng-classes $\{k, l, n\}$ and $\{m\}$. We take a BFS λ in Υ, as:

$\lambda = \{(k, 0.5, -0.1), (l, 0.7, -0.1), (m, 0.6, -0.1), (n, 0.8, -0.1)\}$.

The upper RBF-apx of λ is calculated as:

$\overline{\Re}(\lambda) = \{(k, 0.8, -0.1), (l, 0.8, -0.1), (m, 0.6, -0.1), (n, 0.8, -0.1)\}$.

Simple calculations verify that $\overline{\Re}(\lambda)$ is a BFl-id of Υ. But, λ is not a BFl-id of Υ, as:

$$\begin{aligned} \lambda^P(lm) &= \lambda^P(k) = 0.5 \\ &\ngeq \lambda^P(m) = 0.6. \end{aligned}$$

Theorem 3.4.6 *Let $\Re$ be a complete cng-rel on Υ. Then, each BFl-id (BFr-id, BF-id) of Υ is a lower RBFl-id (RBFr-id, RBF-id) of Υ.*

Proof. Let $\Re$ be a complete cng-rel on Υ and λ be a BFl-id of Υ. Now, for each $x, y \in \Upsilon$, we obtain

$$\begin{aligned} \underline{\Re}\lambda^P(xy) &= \bigwedge_{s \in [xy]_\Re} \lambda^P(s) \\ &= \bigwedge_{s \in [x]_\Re [y]_\Re} \lambda^P(s), \qquad \text{since } [x]_\Re [y]_\Re = [xy]_\Re \\ &= \bigwedge_{ab \in [x]_\Re [y]_\Re} \lambda^P(ab), \qquad \text{where } s = ab \\ &\geq \bigwedge_{a \in [x]_\Re,\ b \in [y]_\Re} \lambda^P(b) \\ &= \bigwedge_{b \in [y]_\Re} \lambda^P(b) \\ &= \underline{\Re}\lambda^P(y) \end{aligned}$$

and

$$\begin{aligned}
\underline{\Re}\lambda^N(xy) &= \bigvee_{s\in[xy]_\Re} \lambda^N(s) \\
&= \bigvee_{s\in[x]_\Re[y]_\Re} \lambda^N(s), && \text{since } [x]_\Re[y]_\Re = [xy]_\Re \\
&= \bigvee_{ab\in[x]_\Re[y]_\Re} \lambda^N(ab), && \text{where } s = ab \\
&\leq \bigvee_{a\in[x]_\Re,\ b\in[y]_\Re} \lambda^N(b) \\
&= \bigvee_{b\in[y]_\Re} \lambda^N(b) \\
&= \underline{\Re}\lambda^N(y).
\end{aligned}$$

This verifies that $\underline{\Re}(\lambda)$ is a BFl-id of Υ. Therefore, λ is a lower RBFl-id of Υ. Similarly, the cases of BFr-id and the BF-id of Υ can be verified. ∎

The converse statement of Theorem 3.4.6 is invalid generally, as exhibited in the next example.

Example 3.4.7 *Recall the semigroup* $\Upsilon = \{s, t, u, v\}$ *and the complete cng-rel* $\Re$ *on* Υ*, as established in Example 3.3.8. We take a BFS* λ *in* Υ*, as below.*

$\lambda = \{(s, 0.2, -0.2), (t, 0.4, -0.4), (u, 0.5, -0.6), (v, 0.6, -0.5)\}$.

The lower RBF-apx $\underline{\Re}(\lambda)$ *of* λ *is calculated as:*

$\underline{\Re}(\lambda) = \{(s, 0.2, -0.2), (t, 0.4, -0.4), (u, 0.5, -0.5), (v, 0.5, -0.5)\}$.

Simple calculations verify that $\underline{\Re}(\lambda)$ *is a BFl-id of* Υ*. But,* λ *is not a BFl-id of* Υ*, as:*

$$\begin{aligned}
\lambda^P(vv) &= \lambda^P(u) = 0.5 \\
&\ngeq \lambda^P(v) = 0.6.
\end{aligned}$$

Next example shows that Theorem 3.4.6 is invalid if the cng-rel $\Re$ is not complete. Next example is established to verify this fact.

Example 3.4.8 *Let* $\Upsilon = \{a, b, c, d\}$ *be the semigroup as established in Example 3.3.6, on which, we take a cng-rel* $\Re = \{(a, a), (b, b), (c, c), (d, d), (b, c), (c, b)\}$*. Then,* $\Re$ *defines the cng-classes* $\{a\}, \{b, c\}, \{d\}$ *and* $\Re$ *is not complete. We take a BFS* λ *in* Υ*, as below.*

$\lambda = \{(a, 0.5, -0.4), (b, 0.6, -0.6), (c, 0.4, -0.5), (d, 0.7, -0.7)\}$.

Then, λ *is a BFl-id of* Υ*. The RBF-apxes of* λ *are calculated by Definition 3.3.1 as:*

$\overline{\Re}(\lambda) = \{(a, 0.5, -0.4), (b, 0.6, -0.6), (c, 0.6, -0.6), (d, 0.7, -0.7)\}$,

$\underline{\Re}(\lambda) = \{(a, 0.5, -0.4), (b, 0.4, -0.5), (c, 0.4, -0.5), (d, 0.7, -0.7)\}$.

Simple calculations verify that $\overline{\Re}(\lambda)$ *is also a BFl-id of* Υ, *while,* $\underline{\Re}(\lambda)$ *is not a BFl-id of* Υ, *as*

$$\begin{aligned} \underline{\Re}\lambda^P(ba) &= \underline{\Re}\lambda^P(b) = 0.4 \\ &\ngeq \underline{\Re}\lambda^P(a) = 0.5. \end{aligned}$$

Definition 3.4.9 *Let* $\Re$ *be a cng-rel on* Υ *and let* $\lambda \in BFS(\Upsilon)$. *Then,* λ *is a lower (or upper) RBFi-id of* Υ, *if* $\underline{\Re}(\lambda)$ *(or* $\overline{\Re}(\lambda)$*) is a BFi-id of* Υ.

A BFS λ in Υ, which is both, lower and upper RBFi-id of Υ, is called an RBFi-id of Υ.

Theorem 3.4.10 *Take a cng-rel* $\Re$ *on* Υ. *Then, each BFi-id of* Υ *is an upper RBFi-id of* Υ.

Proof. Take a BFi-id λ of Υ. Then $\lambda^P(abc) \geq \lambda^P(b)$ and $\lambda^N(abc) \leq \lambda^N(b)$ for each $a, b, c \in \Upsilon$. Now, for each $x, w, y \in \Upsilon$, we have

$$\begin{aligned} \overline{\Re}\lambda^P(xwy) &= \bigvee_{s\in[xwy]_\Re} \lambda^P(s) \\ &\geq \bigvee_{s\in[x]_\Re[w]_\Re[y]_\Re} \lambda^P(s), && \text{since } [x]_\Re[w]_\Re[y]_\Re \subseteq [xwy]_\Re \\ &= \bigvee_{abc\in[x]_\Re[w]_\Re[y]_\Re} \lambda^P(abc), && \text{where } s = abc \\ &\geq \bigvee_{b\in[w]_\Re} \lambda^P(b) \\ &= \overline{\Re}\lambda^P(w) \end{aligned}$$

and

$$\begin{aligned} \overline{\Re}\lambda^N(xwy) &= \bigwedge_{s\in[xwy]_\Re} \lambda^N(s) \\ &\leq \bigwedge_{s\in[x]_\Re[w]_\Re[y]_\Re} \lambda^N(s), && \text{since } [x]_\Re[w]_\Re[y]_\Re \subseteq [xwy]_\Re \\ &= \bigwedge_{abc\in[x]_\Re[w]_\Re[y]_\Re} \lambda^N(abc), && \text{where } s = abc \\ &\leq \bigwedge_{b\in[w]_\Re} \lambda^N(b) \\ &= \overline{\Re}\lambda^N(w). \end{aligned}$$

This verifies that, $\overline{\Re}(\lambda)$ is a BFi-id of Υ. Therefore, λ is an upper RBFi-id of Υ. ■

The converse statement of Theorem 3.4.10 is invalid generally, as exhibited in the next example.

Example 3.4.11 *Recall the semigroup* $\Upsilon = \{k, l, m, n\}$ *and the cng-rel* $\Re$ *on* Υ, *as established in Example 3.4.5. We take a BFS* λ *in* Υ, *defined as below.*

$\lambda = \{(k, 0.7, -0.2), (l, 0.8, -0.2), (m, 0.4, -0.2), (n, 0.9, -0.2)\}$.

The upper RBF-apx of λ *is calculated as:*

$\overline{\Re}(\lambda) = \{(k, 0.9, -0.2), (l, 0.9, -0.2), (m, 0.4, -0.2), (n, 0.9, -0.2)\}$.

Simple calculations verify that $\overline{\Re}(\lambda)$ *is a BFi-id of* Υ. *But,* λ *is not a BFi-id of* Υ, *as:*

$$\begin{aligned}\lambda^P(klm) &= \lambda^P(k) = 0.7 \\ &\ngeq \lambda^P(l) = 0.8.\end{aligned}$$

Theorem 3.4.12 *Let* $\Re$ *be a complete cng-rel on* Υ. *Then, each BFi-id of* Υ *is a lower RBFi-id of* Υ.

Proof. Let $\Re$ be a complete cng-rel on Υ and λ be a BFi-id of Υ. Now, for each $x, w, y \in \Upsilon$, we have

$$\begin{aligned}\underline{\Re}\lambda^P(xwy) &= \bigwedge_{s\in[xwy]_\Re} \lambda^P(s) \\ &= \bigwedge_{s\in[x]_\Re[w]_\Re[y]_\Re} \lambda^P(s), \qquad \text{since } [x]_\Re[w]_\Re[y]_\Re = [xwy]_\Re \\ &= \bigwedge_{abc\in[x]_\Re[w]_\Re[y]_\Re} \lambda^P(abc), \qquad \text{where } s = abc \\ &\geq \bigwedge_{b\in[w]_\Re} \lambda^P(b) \\ &= \underline{\Re}\lambda^P(w)\end{aligned}$$

and

$$\begin{aligned}\underline{\Re}\lambda^N(xwy) &= \bigvee_{s\in[xwy]_\Re} \lambda^N(s) \\ &= \bigvee_{s\in[x]_\Re[w]_\Re[y]_\Re} \lambda^N(s), \qquad \text{since } [x]_\Re[w]_\Re[y]_\Re = [xwy]_\Re \\ &= \bigvee_{abc\in[x]_\Re[w]_\Re[y]_\Re} \lambda^N(abc), \qquad \text{where } s = abc \\ &\leq \bigvee_{b\in[w]_\Re} \lambda^N(b) \\ &= \underline{\Re}\lambda^N(w).\end{aligned}$$

This verifies that $\underline{\Re}(\lambda)$ is a BFi-id of Υ. Therefore, λ is a lower RBFi-id of Υ. ■

The converse statement of Theorem 3.4.12 is invalid generally, as exhibited in the next example.

Example 3.4.13 *Recall the semigroup $\Upsilon = \{s, t, u, v\}$ and the complete cng-rel $\Re$ on Υ, as established in Example 3.3.8. We take a BFS λ in Υ, as below.*

$\lambda = \{(s, 0.3, -0.1), (t, 0.4, -0.2), (u, 0.8, -0.3), (v, 0.7, -0.2)\}$.

The lower RBF-apx $\underline{\Re}(\lambda)$ of λ is calculated as:

$\underline{\Re}(\lambda) = \{(s, 0.3, -0.1), (t, 0.4, -0.2), (u, 0.7, -0.2), (v, 0.7, -0.2)\}$.

Simple calculations verify that $\underline{\Re}(\lambda)$ is a BFi-id of Υ. But, λ is not a BFi-id of Υ, as:

$$\begin{aligned} \lambda^P(tuv) &= \lambda^P(v) = 0.7 \\ &\ngeq \lambda^P(u) = 0.8. \end{aligned}$$

Next example shows that Theorem 3.4.12 is invalid if the cng-rel $\Re$ is not complete. Next example is established to verify this fact.

Example 3.4.14 *Let $\Upsilon = \{a, b, c, d\}$ be the semigroup as established in Example 3.3.6 and $\Re$ be the cng-rel on Υ as in Example 3.4.8, which is not complete and defines the cng-classes $\{a\}, \{b, c\}, \{d\}$. We take a BFS λ in Υ, as below.*

$\lambda = \{(a, 0.3, -0.4), (b, 0.4, -0.5), (c, 0.2, -0.2), (d, 0.7, -0.7)\}$.

Then, λ is a BFi-id of Υ. The $\Re$-RBF-apxes of λ are calculated by Definition 3.3.1 as:

$\overline{\Re}(\lambda) = \{(a, 0.3, -0.4), (b, 0.4, -0.5), (c, 0.4, -0.5), (d, 0.7, -0.7)\}$,

$\underline{\Re}(\lambda) = \{(a, 0.3, -0.4), (b, 0.2, -0.2), (c, 0.2, -0.2), (d, 0.7, -0.7)\}$.

Simple calculations verify that $\overline{\Re}(\lambda)$ is also a BFi-id of Υ, while, $\underline{\Re}(\lambda)$ is not a BFi-id of Υ, as

$$\begin{aligned} \underline{\Re}\lambda^P(bac) &= \underline{\Re}\lambda^P(b) = 0.2 \\ &\ngeq \underline{\Re}\lambda^P(a) = 0.3. \end{aligned}$$

Definition 3.4.15 *Take a cng-rel $\Re$ on Υ and let $\lambda \in BFS(\Upsilon)$. Then, λ is a lower (or upper) RBFb-id of Υ, if $\underline{\Re}(\lambda)$ (or $\overline{\Re}(\lambda)$) is a BFb-id of Υ.*

A BFS λ in Υ, which is both, lower and upper RBFb-id of Υ, is called an RBFb-id of Υ.

Theorem 3.4.16 *Take a cng-rel $\Re$ on Υ. Then, each BFb-id of Υ is an upper RBFb-id of Υ.*

Proof. Take a BFb-id λ of Υ. Then, λ is also a BF subsemigroup of Υ. Which implies by Theorem 3.3.5, that, $\overline{\Re}(\lambda) = (\overline{\Re}(\lambda^P), \overline{\Re}(\lambda^N))$ is a BF subsemigroup of Υ.

Now, for each $x, w, y \in \Upsilon$, we have

$$\begin{aligned}
\overline{\Re}\lambda^P(xwy) &= \bigvee_{s\in[xwy]_\Re} \lambda^P(s) \\
&\geq \bigvee_{s\in[x]_\Re[w]_\Re[y]_\Re} \lambda^P(s), \qquad \text{since } [x]_\Re[w]_\Re[y]_\Re \subseteq [xwy]_\Re \\
&= \bigvee_{abc\in[x]_\Re[w]_\Re[y]_\Re} \lambda^P(abc), \qquad \text{where } s = abc \\
&\geq \bigvee_{a\in[x]_\Re,\ b\in[w]_\Re,\ c\in[y]_\Re} (\lambda^P(a) \wedge \lambda^P(c)) \\
&= (\bigvee_{a\in[x]_\Re} \lambda^P(a)) \wedge (\bigvee_{c\in[y]_\Re} \lambda^P(c)) \\
&= \overline{\Re}\lambda^P(x) \wedge \overline{\Re}\lambda^P(y)
\end{aligned}$$

and

$$\begin{aligned}
\overline{\Re}\lambda^N(xwy) &= \bigwedge_{s\in[xwy]_\Re} \lambda^N(s) \\
&\leq \bigwedge_{s\in[x]_\Re[w]_\Re[y]_\Re} \lambda^N(s), \qquad \text{since } [x]_\Re[w]_\Re[y]_\Re \subseteq [xwy]_\Re \\
&= \bigwedge_{abc\in[x]_\Re[w]_\Re[y]_\Re} \lambda^N(abc), \qquad \text{where } s = abc \\
&\leq \bigwedge_{a\in[x]_\Re,\ b\in[w]_\Re,\ c\in[y]_\Re} (\lambda^N(a) \vee \lambda^N(c)) \\
&= (\bigwedge_{a\in[x]_\Re} \lambda^N(a)) \vee (\bigwedge_{c\in[y]_\Re} \lambda^N(c)) \\
&= \overline{\Re}\lambda^N(x) \vee \overline{\Re}\lambda^N(y).
\end{aligned}$$

This verifies that, $\overline{\Re}(\lambda)$ is a BFb-id of Υ. Therefore, λ is an upper RBFb-id of Υ. ■

The converse statement of Theorem 3.4.16 is invalid generally, as exhibited in the next example.

Example 3.4.17 *Recall the semigroup $\Upsilon = \{k, l, m, n\}$ and the cng-rel $\Re$ on Υ, as established in Example 3.4.5. We take a BFS λ in Υ, defined as below.*

$\lambda = \{(k, 0.4, -0.3), (l, 0.3, -0.3), (m, 0.1, -0.3), (n, 0.2, -0.3)\}$.

The upper RBF-apx of λ is calculated as:

$\overline{\Re}(\lambda) = \{(k, 0.4, -0.3), (l, 0.4, -0.3), (m, 0.1, -0.3), (n, 0.4, -0.3)\}$.

Simple calculations verify that $\overline{\Re}(\lambda)$ is a BFb-id of Υ. But, λ is not a BFb-id of Υ, as:

$$\begin{aligned}
\lambda^P(knl) &= \lambda^P(n) = 0.2 \\
&\not\geq \lambda^P(k) \wedge \lambda^P(l) = 0.3.
\end{aligned}$$

Theorem 3.4.18 *Let $\Re$ be a complete cng-rel on Υ. Then, each BFb-id of Υ is a lower RBFb-id of Υ.*

Proof. Let $\Re$ be a complete cng-rel on Υ and λ be a BFb-id of Υ. Then, λ is also a BF subsemigroup of Υ. Which implies by Theorem 3.3.7, that, $\underline{\Re}(\lambda) = (\underline{\Re}(\lambda^P), \underline{\Re}(\lambda^N))$ is a BF subsemigroup of Υ. Now, for each $x, w, y \in \Upsilon$, we have

$$\begin{aligned}
\underline{\Re}\lambda^P(xwy) &= \bigwedge_{s\in[xwy]_\Re} \lambda^P(s) \\
&= \bigwedge_{s\in[x]_\Re[w]_\Re[y]_\Re} \lambda^P(s), \qquad \text{since } [x]_\Re[w]_\Re[y]_\Re = [xwy]_\Re \\
&= \bigwedge_{abc\in[x]_\Re[w]_\Re[y]_\Re} \lambda^P(abc), \qquad \text{where } s = abc \\
&\geq \bigwedge_{a\in[x]_\Re,\ b\in[w]_\Re,\ c\in[y]_\Re} (\lambda^P(a) \wedge \lambda^P(c)) \\
&= (\bigwedge_{a\in[x]_\Re} \lambda^P(a)) \wedge (\bigwedge_{c\in[y]_\Re} \lambda^P(c)) \\
&= \underline{\Re}\lambda^P(x) \wedge \underline{\Re}\lambda^P(y)
\end{aligned}$$

and

$$\begin{aligned}
\underline{\Re}\lambda^N(xwy) &= \bigvee_{s\in[xwy]_\Re} \lambda^N(s) \\
&= \bigvee_{s\in[x]_\Re[w]_\Re[y]_\Re} \lambda^N(s), \qquad \text{since } [x]_\Re[w]_\Re[y]_\Re = [xwy]_\Re \\
&= \bigvee_{abc\in[x]_\Re[w]_\Re[y]_\Re} \lambda^N(abc), \qquad \text{where } s = abc \\
&\leq \bigvee_{a\in[x]_\Re,\ b\in[w]_\Re,\ c\in[y]_\Re} (\lambda^N(a) \vee \lambda^N(c)) \\
&= (\bigvee_{a\in[x]_\Re} \lambda^N(a)) \vee (\bigvee_{c\in[y]_\Re} \lambda^N(c)) \\
&= \underline{\Re}\lambda^N(x) \vee \underline{\Re}\lambda^N(y).
\end{aligned}$$

This verifies that $\underline{\Re}(\lambda)$ is a BFb-id of Υ. Therefore, λ is a lower RBFb-id of Υ. ∎

The converse statement of the Theorem 3.4.18 is invalid generally, as exhibited in the next example.

Example 3.4.19 *Recall the semigroup $\Upsilon = \{s, t, u, v\}$ and the complete cng-rel $\Re$ on Υ, as established in Example 3.3.8. We take a BFS λ in Υ, as below.*

$\lambda = \{(s, 0.1, -0.1), (t, 0.3, -0.3), (u, 0.4, -0.4), (v, 0.5, -0.5)\}$.

The lower RBF-apx $\underline{\Re}(\lambda)$ of λ is calculated as:

$\underline{\Re}(\lambda) = \{(s, 0.1, -0.1), (t, 0.3, -0.3), (u, 0.4, -0.4), (v, 0.4, -0.4)\}$.

Simple calculations verify that $\underline{\Re}(\lambda)$ is a BFb-id of Υ. But, λ is not a BFb-id of Υ, as:

$$\begin{aligned}
\lambda^P(vsv) &= \lambda^P(u) = 0.4 \\
&\ngeq \lambda^P(v) \wedge \lambda^P(v) = 0.5.
\end{aligned}$$

Next example shows that Theorem 3.4.18 is invalid if the cng-rel $\Re$ is not complete. Next example is established to verify this fact.

Example 3.4.20 *Let $\Upsilon = \{a, b, c, d\}$ be the semigroup as established in Example 3.3.6 and $\Re$ be the cng-rel on Υ, which defines the cng-classes $\{a\}, \{b, c\}, \{d\}$. Then, $\Re$ is not complete. We take a BFS λ in Υ, as below.*

$\lambda = \{(a, 0.3, -0.2), (b, 0.4, -0.3), (c, 0.2, -0.1), (d, 0.5, -0.4)\}$.

Then, λ is a BFb-id of Υ. The $\Re$-RBF-apxes of λ are calculated by Definition 3.3.1 as:

$\overline{\Re}(\lambda) = \{(a, 0.3, -0.2), (b, 0.4, -0.3), (c, 0.4, -0.3), (d, 0.7, -0.7)\}$,

$\underline{\Re}(\lambda) = \{(a, 0.3, -0.2), (b, 0.2, -0.1), (c, 0.2, -0.1), (d, 0.7, -0.7)\}$.

Simple calculations verify that $\overline{\Re}(\lambda)$ is also a BFb-id of Υ, while, $\underline{\Re}(\lambda)$ is not a BFb-id of Υ, as

$$\begin{aligned} \underline{\Re}\lambda^P(aba) &= \underline{\Re}\lambda^P(b) = 0.2 \\ &\ngeq \underline{\Re}\lambda^P(a) \wedge \underline{\Re}\lambda^P(a) = 0.3. \end{aligned}$$

Chapter 4

Rough bipolar soft sets

4.1 Introduction

Rough sets and soft sets are both meant for managing the data confronted by ambiguity and vagueness. Many authors have worked on the soft sets and their parameters; see [6, 13, 21, 69]. Roughness in soft sets is defined by Feng et al. in [29]. Ali also discussed the rough soft sets in [7]. In this chapter, we extend this concept to the BSSs and define the rough bipolar soft sets (RBSSs) using the RBS approximations (RBS-apxes) of the BSSs in a P-apx space. Some characterizations of these RBS-apxes are explored in Section 4.3. Some similarity relations on the set containing BSSs in the universe of discourse are defined in Section 4.4 with the help of the lower and upper RBS-apxes of the BSSs.

As mentioned earlier, the soft sets are built to characterize the objects according to some particular attributes. Due to this quality, the soft sets are very useful and applicable in many types of data analysis. Specially, they have great applicability in decision analysis. Applications of different soft structures in decision making techniques are discussed in [28, 43, 46, 47, 55, 61, 66]. In the last section of this chapter, we apply the RBS-apxes of the BSSs to develop two interesting but important decision making techniques. One is to decide between some objects pertaining some particular attributes or their counter attributes; the other is to decide between the attributes which are affecting some particular objects. We also design the algorithms for those applications and demonstrate the steps of algorithms by suitable examples.

4.2 Rough bipolar soft sets

Definition 4.2.1 *Take a P-apx space* $(U, \Re)$ *and let* $\eth = (\xi, \psi; \breve{A}) \in BSS(U)$*. The lower and upper RBS-apxes of* $\eth$ *with respect to* $(U, \Re)$ *are the BSSs symbolized by*

$\underline{\eth}_{\Re} = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ *and* $\overline{\eth}^{\Re} = (\overline{\xi}^{\Re}, \overline{\psi}^{\Re}; \breve{A})$*, respectively, where* $\underline{\xi}_{\Re}$*,* $\overline{\xi}^{\Re}$ *are defined as:*

$$\underline{\xi}_{\Re}(e) = \{u \in U : [u]_{\Re} \subseteq \xi(e)\},$$

$$\overline{\xi}^{\Re}(e) = \{u \in U : [u]_{\Re} \cap \xi(e) \neq \phi\}$$

for each $e \in \breve{A}$*, and* $\underline{\psi}_{\Re}$*,* $\overline{\psi}^{\Re}$ *are defined as:*

$$\underline{\psi}_{\Re}(\neg e) = \{u \in U : [u]_{\Re} \cap \psi(\neg e) \neq \phi\},$$

$$\overline{\psi}^{\Re}(\neg e) = \{u \in U : [u]_{\Re} \subseteq \psi(\neg e)\}$$

for each $\neg e \in \neg\breve{A}$*. If* $\underline{\eth}_{\Re} = \overline{\eth}^{\Re}$*, then,* $\eth$ *is said to be* $\Re-$*definable; otherwise,* $\eth$ *is an RBSS over* U*.*

We claim that these RBS-apxes of a BSS are also BSSs.

Claim 4.2.2 *Let* $\eth = (\xi, \psi; \breve{A})$ *be a BSS. Then,* $\xi(e) \cap \psi(\neg e) = \phi$ *for each* $e \in \breve{A}$*, that is,* $\xi(e) \subseteq U - \psi(\neg e) = \psi'(\neg e)$*. Take the lower RBS-apx* $\underline{\eth}_{\Re} = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ *of* $\eth$*. If* $u \in \underline{\xi}_{\Re}(e) \cap \underline{\psi}_{\Re}(\neg e)$ *for some* $e \in \breve{A}$*, then,* $u \in \underline{\xi}_{\Re}(e)$ *and* $u \in \underline{\psi}_{\Re}(\neg e)$*. Which implies,* $[u]_{\Re} \subseteq \xi(e) \subseteq \psi'(\neg e)$ *and* $[u]_{\Re} \cap \psi(\neg e) \neq \phi$*. That is,* $[u]_{\Re} \subseteq \psi'(\neg e)$ *and* $[u]_{\Re} \cap \psi(\neg e) \neq \phi$ *which is not possible. Thus,* $\underline{\xi}_{\Re}(e) \cap \underline{\psi}_{\Re}(\neg e) = \phi$ *and hence,* $\underline{\eth}_{\Re}$ *is a BSS over* U*. Similarly, the upper RBS-apx of* $\eth$ *is a BSS over* U*.*

The interpretation of these RBS-apxes of $\eth$, that is, the information about an object u, interpreted by the RBS-apxes of $\eth$, is as follows:

- $u \in \underline{\xi}_{\Re}(e)$ depicts that u definitely possesses the attribute e.
- $u \in \overline{\psi}^{\Re}(\neg e)$ depicts that u definitely possesses the attribute $\neg e$.
- $u \in \overline{\xi}^{\Re}(e) - \underline{\xi}_{\Re}(e)$ depicts that u probably has the attribute e.
- $u \in \underline{\psi}_{\Re}(\neg e) - \overline{\psi}^{\Re}(\neg e)$ depicts that u probably has the attribute $\neg e$.
- $u \in (\overline{\xi}^{\Re}(e))'$ depicts that u definitely does not have the attribute e.
- $u \in (\underline{\psi}_{\Re}(\neg e))'$ depicts that u definitely does not have the attribute $\neg e$.

Here, $(\overline{\xi}^{\Re}(e))' = U - \overline{\xi}^{\Re}(e)$ and $(\underline{\psi}_{\Re}(\neg e))' = U - \underline{\psi}_{\Re}(\neg e)$.

Example 4.2.3 *Suppose that* $U = \{u_i; i = 1, 2, ..., 6\}$ *is a universe containing six objects and* $\breve{E} = \{e_i; i = 1, 2, ..., 5\}$ *is a set of possible attributes for* U*. Consider a BSS* $\eth_1 = (\xi_1, \psi_1; \breve{A}_1)$ *over* U*, with* $\breve{A}_1 = \{e_1, e_2, e_3\}$ *defined as:*

$\xi_1(e_1) = \{u_1, u_3, u_4, u_6\}, \xi_1(e_2) = \{u_2, u_3, u_5\}, \xi_1(e_3) = \{u_3, u_4, u_5\}$,
$\psi_1(\neg e_1) = \{u_2, u_5\}, \psi_1(\neg e_2) = \phi, \psi_1(\neg e_3) = \{u_2, u_6\}$.

Take an eqv-rel $\Re$ *on* U*, defining classes* $\{u_1, u_2, u_3\}$, $\{u_4, u_5\}$ *and* $\{u_6\}$*. Then, the lower RBS-apx of* $ð_1$ *is* $\underline{ð_1}_{\Re} = (\underline{\xi_1}_{\Re}, \underline{\psi_1}_{\Re}; \breve{A}_1)$*, where* $\underline{\xi_1}_{\Re}$ *and* $\underline{\psi_1}_{\Re}$ *are calculated by using Definition 4.2.1 as:*

$\underline{\xi_1}_{\Re}(e_1) = \{u_6\}, \underline{\xi_1}_{\Re}(e_2) = \phi, \underline{\xi_1}_{\Re}(e_3) = \{u_4, u_5\}$,
$\underline{\psi_1}_{\Re}(\neg e_1) = U - \{u_6\}, \underline{\psi_1}_{\Re}(\neg e_2) = \phi, \underline{\psi_1}_{\Re}(\neg e_3) = \{u_1, u_2, u_3, u_6\}$.

The upper RBS-apx of $ð_1$ *is* $\overline{ð_1}^{\Re} = (\overline{\xi_1}^{\Re}, \overline{\psi_1}^{\Re}; \breve{A}_1)$*, calculated as:*
$\overline{\xi_1}^{\Re}(e_1) = U, \overline{\xi_1}^{\Re}(e_2) = \overline{\xi_1}^{\Re}(e_3) = U - \{u_6\}$,
$\overline{\psi_1}^{\Re}(\neg e_1) = \phi, \overline{\psi_1}^{\Re}(\neg e_2) = \phi, \overline{\psi_1}^{\Re}(\neg e_3) = \{u_6\}$.

Notice that $\underline{ð_1}_{\Re} \neq \overline{ð_1}^{\Re}$*, so* $ð_1$ *is an RBSS. Also* $\underline{\xi_1}_{\Re}(e) \subseteq \xi_1(e) \subseteq \overline{\xi_1}^{\Re}(e)$ *and* $\underline{\psi_1}_{\Re}(\neg e) \supseteq \psi_1(\neg e) \supseteq \overline{\psi_1}^{\Re}(\neg e)$ *for each* $e \in \breve{A}_1$*. This verifies that* $\underline{ð_1}_{\Re} \,\widetilde{\subseteq}\, ð_1 \,\widetilde{\subseteq}\, \overline{ð_1}^{\Re}$*. From these RBS-apxes, it can be concluded that the object* u_6 *definitely possesses the attribute* e_1*, but the other objects may or may not possess* e_1*. Similarly, the objects* u_4 *and* u_5 *definitely possess the attribute* e_3*, while the objects* u_1, u_2 *and* u_3 *may or may not possess* e_3*. But, the object* u_6 *definitely possesses the attribute* $\neg e_3$*.*

4.3 Characterizations of rough bipolar soft sets

Theorem 4.3.1 *Take a P-apx space* $(U, \Re)$ *and let* $ð = (\xi, \psi; \breve{A}) \in BSS(U)$*. Then, the subsequent asservations hold.*

1. $\underline{ð}_{\Re} \,\widetilde{\subseteq}\, ð \,\widetilde{\subseteq}\, \overline{ð}^{\Re}$,
2. $\underline{\Theta_{\breve{A}}}_{\Re} = \Theta_{\breve{A}} = \overline{\Theta_{\breve{A}}}^{\Re}$,
3. $\underline{\mathcal{U}_{\breve{A}}}_{\Re} = \mathcal{U}_{\breve{A}} = \overline{\mathcal{U}_{\breve{A}}}^{\Re}$,
4. $\underline{\underline{ð}_{\Re}}_{\Re} = \underline{ð}_{\Re} = \overline{(\underline{ð}_{\Re})}^{\Re}$,
5. $\underline{(\overline{ð}^{\Re})}_{\Re} = \overline{ð}^{\Re} = \overline{(\overline{ð}^{\Re})}^{\Re}$,
6. $\overline{ð^c}^{\Re} = (\underline{ð}_{\Re})^c$,
7. $\underline{ð^c}_{\Re} = \left(\overline{ð}^{\Re}\right)^c$.

Proof. (1) Obvious.

(2) The null BSS $\Theta_{\breve{A}} = (\Theta, \mathcal{U}; \breve{A})$ has the lower and upper RBS-apxes symbolized by $\underline{\Theta_{\breve{A}}}_{\Re} = (\underline{\Theta}_{\Re}, \underline{\mathcal{U}}_{\Re}; \breve{A})$ and $\overline{\Theta_{\breve{A}}}^{\Re} = (\overline{\Theta}^{\Re}, \overline{\mathcal{U}}^{\Re}; \breve{A})$. We have for each $e \in \breve{A}$

$$\begin{aligned}
\underline{\Theta}_{\Re}(e) &= \{u \in U : [u]_{\Re} \subseteq \Theta(e)\} \\
&= \{u \in U : [u]_{\Re} \subseteq \phi\} = \phi = \Theta(e), \\
\overline{\Theta}^{\Re}(e) &= \{u \in U : [u]_{\Re} \cap \Theta(e) \neq \phi\} \\
&= \{u \in U : [u]_{\Re} \cap \phi \neq \phi\} = \phi = \Theta(e), \\
\underline{\mathcal{U}}_{\Re}(\neg e) &= \{u \in U : [u]_{\Re} \cap \mathcal{U}(\neg e) \neq \phi\} \\
&= \{u \in U : [u]_{\Re} \cap U \neq \phi\} = U = \mathcal{U}(\neg e), \\
\overline{\mathcal{U}}^{\Re}(\neg e) &= \{u \in U : [u]_{\Re} \subseteq \mathcal{U}(\neg e)\} \\
&= \{u \in U : [u]_{\Re} \subseteq U\} = U = \mathcal{U}(\neg e).
\end{aligned}$$

Thus, Definition 1.5.3 implies that,

$$\underline{\Theta_{\breve{A}}}_{\Re} = \Theta_{\breve{A}} = \overline{\Theta_{\breve{A}}}^{\Re}.$$

(3) Analogous to the proof of (2).

(4) The lower RBS-apx of the BSS $\eth = (\xi, \psi; \breve{A})$ is symbolized by $\underline{\eth}_{\Re} = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$. Now, the lower and upper RBS-apxes of $\underline{\eth}_{\Re}$ are symbolized by $\underline{\underline{\eth}_{\Re}}_{\Re} = (\underline{\underline{\xi}_{\Re}}_{\Re}, \underline{\underline{\psi}_{\Re}}_{\Re}; \breve{A})$ and $\overline{(\underline{\eth}_{\Re})}^{\Re} = (\overline{\underline{\xi}_{\Re}}^{\Re}, \overline{\underline{\psi}_{\Re}}^{\Re}; \breve{A})$, respectively. Note that, for each $e \in \breve{A}$, we have

$$\underline{\xi}_{\Re}(e) = \{u \in U : [u]_{\Re} \subseteq \xi(e)\} = \bigcup_{[u]_{\Re} \subseteq \xi(e)} [u]_{\Re}.$$

That is, $\underline{\xi}_{\Re}(e)$ is union of the eqv-classes of U contained in $\xi(e)$. So,

$$\underline{\underline{\xi}_{\Re}}_{\Re}(e) = \{u \in U : [u]_{\Re} \subseteq \underline{\xi}_{\Re}(e)\} = \underline{\xi}_{\Re}(e)$$

and

$$\overline{\underline{\xi}_{\Re}}^{\Re}(e) = \{u \in U : [u]_{\Re} \cap \underline{\xi}_{\Re}(e) \neq \phi\} = \underline{\xi}_{\Re}(e)$$

for each $e \in \breve{A}$. Similarly,

$$\underline{\underline{\psi}_{\Re}}_{\Re}(\neg e) = \underline{\psi}_{\Re}(\neg e) = \overline{\underline{\psi}_{\Re}}^{\Re}(\neg e)$$

for each $\neg e \in \neg\breve{A}$. Thus, from Definition 1.5.3

$$\underline{\underline{\eth}_{\Re}}_{\Re} = \underline{\eth}_{\Re} = \overline{(\underline{\eth}_{\Re})}^{\Re}.$$

(5) Analogous to the proof of (4).

(6) We have $\overline{\eth^c}^{\Re} = \left(\overline{\xi^c}^{\Re}, \overline{\psi^c}^{\Re}; \breve{A}\right)$ and $(\underline{\eth}_{\Re})^c = ((\underline{\xi}_{\Re})^c, (\underline{\psi}_{\Re})^c; \breve{A})$, where $\xi^c(e) = \psi(\neg e)$ and $\psi^c(\neg e) = \xi(e)$ for each $e \in \breve{A}$. So, we get

$$\begin{aligned} \overline{\xi^c}^{\Re}(e) &= \{u \in U : [u]_{\Re} \cap \xi^c(e) \neq \phi\} = \{u \in U : [u]_{\Re} \cap \psi(\neg e) \neq \phi\} \\ &= \underline{\psi}_{\Re}(\neg e) = (\underline{\xi}_{\Re})^c(e) \end{aligned}$$

and

$$\begin{aligned} \overline{\psi^c}^{\Re}(\neg e) &= \{u \in U : [u]_{\Re} \subseteq \psi^c(\neg e)\} = \{u \in U : [u]_{\Re} \subseteq \xi(e)\} \\ &= \underline{\xi}_{\Re}(e) = (\underline{\psi}_{\Re})^c(\neg e) \end{aligned}$$

for each $e \in \breve{A}$. Which yields by Definition 1.5.3, that, $\overline{\eth^c}^{\Re} = (\underline{\eth}_{\Re})^c$.

(7) Analogous to the proof of (6). ■

Proposition 4.3.2 *Take a P-apx space $(U, \Re)$. Then, the subsequent asservations hold for each $\eth_1 = (\xi_1, \psi_1; \breve{A}_1), \eth_2 = (\xi_2, \psi_2; \breve{A}_2) \in BSS(U)$ and $e \in \breve{A}_1 \cap \breve{A}_2$.*

1. $\underline{\xi_1}_{\Re}(e) \cup \underline{\xi_2}_{\Re}(e) \subseteq \underline{\xi_1 \widetilde{\cup}_r \xi_2}_{\Re}(e)$,
2. $\overline{\xi_1}^{\Re}(e) \cup \overline{\xi_2}^{\Re}(e) = \overline{\xi_1 \widetilde{\cup}_r \xi_2}^{\Re}(e)$,
3. $\underline{\xi_1}_{\Re}(e) \cap \underline{\xi_2}_{\Re}(e) = \underline{\xi_1 \widetilde{\cap}_r \xi_2}_{\Re}(e)$,
4. $\overline{\xi_1}^{\Re}(e) \cap \overline{\xi_2}^{\Re}(e) \supseteq \overline{\xi_1 \widetilde{\cap}_r \xi_2}^{\Re}(e)$,
5. $\underline{\psi_1}_{\Re}(\neg e) \cup \underline{\psi_2}_{\Re}(\neg e) = \underline{\psi_1 \widetilde{\cup}_r \psi_2}_{\Re}(\neg e)$,
6. $\overline{\psi_1}^{\Re}(\neg e) \cup \overline{\psi_2}^{\Re}(\neg e) \subseteq \overline{\psi_1 \widetilde{\cup}_r \psi_2}^{\Re}(\neg e)$,
7. $\underline{\psi_1}_{\Re}(\neg e) \cap \underline{\psi_2}_{\Re}(\neg e) \supseteq \underline{\psi_1 \widetilde{\cap}_r \psi_2}_{\Re}(\neg e)$,
8. $\overline{\psi_1}^{\Re}(\neg e) \cap \overline{\psi_2}^{\Re}(\neg e) = \overline{\psi_1 \widetilde{\cap}_r \psi_2}^{\Re}(\neg e)$.

Proof. (1) For the BSSs $\eth_1 = (\xi_1, \psi_1; \breve{A}_1)$ and $\eth_2 = (\xi_2, \psi_2; \breve{A}_2)$, take $e \in \breve{A}_1 \cap \breve{A}_2$. Then, we obtain

$$\begin{aligned} & \underline{\xi_1}_{\Re}(e) \cup \underline{\xi_2}_{\Re}(e) \\ = & \{u \in U : [u]_{\Re} \subseteq \xi_1(e)\} \cup \{u \in U : [u]_{\Re} \subseteq \xi_2(e)\} \\ \subseteq & \{u \in U : [u]_{\Re} \subseteq (\xi_1(e) \cup \xi_2(e))\} \\ = & \{u \in U : [u]_{\Re} \subseteq (\xi_1 \widetilde{\cup}_r \xi_2)(e)\} \\ = & \underline{\xi_1 \widetilde{\cup}_r \xi_2}_{\Re}(e). \end{aligned}$$

Which proves assertion (1).

(2) Again, for the BSSs $\eth_1 = (\xi_1, \psi_1; \breve{A}_1)$, $\eth_2 = (\xi_2, \psi_2; \breve{A}_2)$ and for $e \in \breve{A}_1 \cap \breve{A}_2$, we obtain

$$\begin{aligned} & \overline{\xi_1}^{\Re}(e) \cup \overline{\xi_2}^{\Re}(e) \\ = & \ \{u \in U : [u]_{\Re} \cap \xi_1(e) \neq \phi\} \cup \{u \in U : [u]_{\Re} \cap \xi_2(e) \neq \phi\} \\ = & \ \{u \in U : [u]_{\Re} \cap (\xi_1(e) \cup \xi_2(e)) \neq \phi\} \\ = & \ \{u \in U : [u]_{\Re} \cap (\xi_1 \widetilde{\cup}_r \xi_2)(e) \neq \phi\} \\ = & \ \overline{\xi_1 \widetilde{\cup}_r \xi_2}^{\Re}(e). \end{aligned}$$

Which proves assertion (2).

In the same way, one can verify assertions (3-8). ■

Theorem 4.3.3 *Take a P-apx space $(U, \Re)$. Then, the subsequent asservations are true for each $\eth_1 = (\xi_1, \psi_1; \breve{A}_1), \eth_2 = (\xi_2, \psi_2; \breve{A}_2) \in BSS(U)$.*

1. $\eth_1 \widetilde{\subseteq} \eth_2$ *implies that* $\underline{\eth_1}_{\Re} \widetilde{\subseteq} \underline{\eth_2}_{\Re}$ *and* $\overline{\eth_1}^{\Re} \widetilde{\subseteq} \overline{\eth_2}^{\Re}$,
2. $\underline{\eth_1 \widetilde{\sqcap}_r \eth_2}_{\Re} = \underline{\eth_1}_{\Re} \widetilde{\sqcap}_r \underline{\eth_2}_{\Re}$,
3. $\underline{\eth_1 \widetilde{\sqcup}_r \eth_2}_{\Re} \widetilde{\supseteq} \underline{\eth_1}_{\Re} \widetilde{\sqcup}_r \underline{\eth_2}_{\Re}$,
4. $\overline{\eth_1 \widetilde{\sqcap}_r \eth_2}^{\Re} \widetilde{\subseteq} \overline{\eth_1}^{\Re} \widetilde{\sqcap}_r \overline{\eth_2}^{\Re}$,
5. $\overline{\eth_1 \widetilde{\sqcup}_r \eth_2}^{\Re} = \overline{\eth_1}^{\Re} \widetilde{\sqcup}_r \overline{\eth_2}^{\Re}$.

Proof. (1) Assume that $\eth_1 \widetilde{\subseteq} \eth_2$. Then $\xi_1(e) \subseteq \xi_2(e)$ and $\psi_1(\neg e) \supseteq \psi_2(\neg e)$ for each $e \in \breve{A}_1$, where $\breve{A}_1 \subseteq \breve{A}_2$. Which yields

$$\begin{aligned} \underline{\xi_1}_{\Re}(e) &= \{u \in U : [u]_{\Re} \subseteq \xi_1(e)\} \\ &\subseteq \{u \in U : [u]_{\Re} \subseteq \xi_2(e)\} = \underline{\xi_2}_{\Re}(e), \end{aligned}$$

$$\begin{aligned} \underline{\psi_1}_{\Re}(\neg e) &= \{u \in U : [u]_{\Re} \cap \psi_1(\neg e) \neq \phi\} \\ &\supseteq \{u \in U : [u]_{\Re} \cap \psi_2(\neg e) \neq \phi\} = \underline{\psi_2}_{\Re}(\neg e) \end{aligned}$$

for each $e \in \breve{A}_1$. Thus, from Definition 1.5.3 , $\underline{\eth_1}_{\Re} \widetilde{\subseteq} \underline{\eth_2}_{\Re}$.

Similarly, it is verified that $\overline{\eth_1}^{\Re} \widetilde{\subseteq} \overline{\eth_2}^{\Re}$.

(2) From Proposition 4.3.2, we have

$$\begin{aligned} \underline{\eth_1 \widetilde{\sqcap}_r \eth_2}_{\Re} &= (\underline{\xi_1 \widetilde{\cap}_r \xi_2}_{\Re}, \underline{\psi_1 \widetilde{\cup}_r \psi_2}_{\Re}; \breve{A}_1 \cap \breve{A}_2) \\ &= (\underline{\xi_1}_{\Re} \widetilde{\cap}_r \underline{\xi_2}_{\Re}, \underline{\psi_1}_{\Re} \widetilde{\cup}_r \underline{\psi_2}_{\Re}; \breve{A}_1 \cap \breve{A}_2) \\ &= \underline{\eth_1}_{\Re} \widetilde{\sqcap}_r \underline{\eth_2}_{\Re}. \end{aligned}$$

(3) Again, from Proposition 4.3.2, we have

$$\begin{aligned}\underline{\eth_1 \widetilde{\sqcup}_r \eth_2}_{\Re} &= (\underline{\xi_1 \widetilde{\cup}_r \xi_2}_{\Re}, \underline{\psi_1 \widetilde{\cap}_r \psi_2}_{\Re}; \breve{A}_1 \cap \breve{A}_2) \\ &\widetilde{\supseteq} (\underline{\xi_1}_{\Re} \widetilde{\cup}_r \underline{\xi_2}_{\Re}, \underline{\psi_1}_{\Re} \widetilde{\cap}_r \underline{\psi_2}_{\Re}; \breve{A}_1 \cap \breve{A}_2) \\ &= \underline{\eth_1}_{\Re} \widetilde{\sqcup}_r \underline{\eth_2}_{\Re}.\end{aligned}$$

(4-5) These can be verified in the same way as (2-3) above. ■

The subsequent example shows the inclusions in (3) and (4) of the Theorem 4.3.3 may be proper.

Example 4.3.4 *Consider the set $U = \{u_i; i = 1, 2, ..., 6\}$ and the BSS $\eth_1$ as in Example 4.2.3. To observe the proper inclusion in (3) and (4) of the Theorem 4.3.3, take another BSS $\eth_2 = (\xi_2, \psi_2; \breve{A}_2)$ over U, where $\breve{A}_2 = \{e_1, e_4\}$ and ξ_2 and ψ_2 are given as below.*

$$\xi_2(e) = \begin{cases} \{u_1, u_3, u_5\} & \text{if } e = e_1 \\ \{u_1, u_2, u_5, u_6\} & \text{if } e = e_4 \end{cases} \qquad \psi_2(\neg e) = \begin{cases} \{u_2\} & \text{if } \neg e = \neg e_1 \\ \{u_4\} & \text{if } \neg e = \neg e_4 \end{cases}$$

The lower RBS-apx $\underline{\eth_2}_{\Re} = (\underline{\xi_2}_{\Re}, \underline{\psi_2}_{\Re}; \breve{A}_2)$ of $\eth_2$ is calculated as:

$$\underline{\xi_2}_{\Re}(e) = \begin{cases} \phi & \text{if } e = e_1 \\ \{u_6\} & \text{if } e = e_4 \end{cases} \qquad \underline{\psi_2}_{\Re}(\neg e) = \begin{cases} \{u_1, u_2, u_3\} & \text{if } \neg e = \neg e_1 \\ \{u_4, u_5\} & \text{if } \neg e = \neg e_4 \end{cases}$$

and the upper RBS-apx $\overline{\eth_2}^{\Re} = (\overline{\xi_2}^{\Re}, \overline{\psi_2}^{\Re}; \breve{A}_2)$ of $\eth_2$ is calculated as:

$$\overline{\xi_2}^{\Re}(e) = \begin{cases} \{u_1, u_2, u_3, u_4, u_5\} & \text{if } e = e_1 \\ U & \text{if } e = e_4 \end{cases} \qquad \overline{\psi_2}^{\Re}(\neg e) = \begin{cases} \phi & \text{if } \neg e = \neg e_1 \\ \phi & \text{if } \neg e = \neg e_4 \end{cases}$$

The restricted unions and intersections are calculated for the attributes $\breve{A}_1 \cap \breve{A}_2 = \{e_1\}$. First we show the proper inclusion in (3) of the Theorem 4.3.3. For this, we calculate the restricted union $\eth_1 \widetilde{\sqcup}_r \eth_2 = (\xi_1 \widetilde{\cup}_r \xi_2, \psi_1 \widetilde{\cap}_r \psi_2; \breve{A}_1 \cap \breve{A}_2)$ as:

$(\xi_1 \widetilde{\cup}_r \xi_2)(e_1) = \xi_1(e_1) \cup \xi_2(e_1) = \{u_1, u_3, u_4, u_5, u_6\}$,
$(\psi_1 \widetilde{\cap}_r \psi_2)(\neg e_1) = \psi_1(\neg e_1) \cap \psi_2(\neg e_1) = \{u_2\}$.

The lower RBS-apx $\underline{\eth_1 \widetilde{\sqcup}_r \eth_2}_{\Re} = (\underline{\xi_1 \widetilde{\cup}_r \xi_2}_{\Re}, \underline{\psi_1 \widetilde{\cap}_r \psi_2}_{\Re}; \breve{A}_1 \cap \breve{A}_2)$ of $\eth_1 \widetilde{\sqcup}_r \eth_2$ is calculated as:

$(\underline{\xi_1 \widetilde{\cup}_r \xi_2}_{\Re})(e_1) = \{u_4, u_5, u_6\}$,
$(\underline{\psi_1 \widetilde{\cap}_r \psi_2}_{\Re})(\neg e_1) = \{u_1, u_2, u_3\}$.

The restricted union $\underline{\eth_1}_{\Re} \widetilde{\sqcup}_r \underline{\eth_2}_{\Re} = (\underline{\xi_1}_{\Re} \widetilde{\cup}_r \underline{\xi_2}_{\Re}, \underline{\psi_1}_{\Re} \widetilde{\cap}_r \underline{\psi_2}_{\Re}; \breve{A}_1 \cap \breve{A}_2)$ is calculated as:

$(\underline{\xi_1}_{\Re} \widetilde{\cup}_r \underline{\xi_2}_{\Re})(e_1) = \underline{\xi_1}_{\Re}(e_1) \cup \underline{\xi_2}_{\Re}(e_1) = \{u_6\}$,
$(\underline{\psi_1}_{\Re} \widetilde{\cap}_r \underline{\psi_2}_{\Re})(\neg e_1) = \underline{\psi_1}_{\Re}(\neg e_1) \cap \underline{\psi_2}_{\Re}(\neg e_1) = \{u_1, u_2, u_3\}$.

This verifies the proper inclusion in (3), that is, $\underline{\eth_1 \widetilde{\sqcup}_r \eth_2}_{\Re} \widetilde{\supsetneq} \underline{\eth_1}_{\Re} \widetilde{\sqcup}_r \underline{\eth_2}_{\Re}$. *To verify the proper inclusion in (4) of the Theorem 4.3.3, we calculate the restricted intersection* $\eth_1 \widetilde{\sqcap}_r \eth_2 = (\xi_1 \widetilde{\sqcap}_r \xi_2, \psi_1 \widetilde{\sqcup}_r \psi_2; \breve{A}_1 \cap \breve{A}_2)$ *as:*

$(\xi_1 \widetilde{\sqcap}_r \xi_2)(e_1) = \xi_1(e_1) \cap \xi_2(e_1) = \{u_1, u_3\}$,

$(\psi_1 \widetilde{\sqcup}_r \psi_2)(\neg e_1) = \psi_1(\neg e_1) \cup \psi_2(\neg e_1) = \{u_2, u_5\}$.

The upper RBS-apx $\overline{\eth_1 \widetilde{\sqcap}_r \eth_2}^{\Re} = (\overline{\xi_1 \widetilde{\sqcap}_r \xi_2}^{\Re}, \overline{\psi_1 \widetilde{\sqcup}_r \psi_2}^{\Re}; \breve{A}_1 \cap \breve{A}_2)$ *of* $\eth_1 \widetilde{\sqcap}_r \eth_2$ *is calculated as:*

$(\overline{\xi_1 \widetilde{\sqcap}_r \xi_2}^{\Re})(e_1) = \{u_1, u_2, u_3\}$,

$(\overline{\psi_1 \widetilde{\sqcup}_r \psi_2}^{\Re})(\neg e_1) = \phi$.

The restricted intersection $\overline{\eth_1}^{\Re} \widetilde{\sqcap}_r \overline{\eth_2}^{\Re} = (\overline{\xi_1}^{\Re} \widetilde{\sqcap}_r \overline{\xi_2}^{\Re}, \overline{\psi_1}^{\Re} \widetilde{\sqcup}_r \overline{\psi_2}^{\Re}; \breve{A}_1 \cap \breve{A}_2)$ *is calculated as:*

$(\overline{\xi_1}^{\Re} \widetilde{\sqcap}_r \overline{\xi_2}^{\Re})(e_1) = \overline{\xi_1}^{\Re}(e_1) \cap \overline{\xi_2}^{\Re}(e_1) = \{u_1, u_2, u_3, u_4, u_5\}$,

$(\overline{\psi_1}^{\Re} \widetilde{\sqcup}_r \overline{\psi_2}^{\Re})(\neg e_1) = \overline{\psi_1}^{\Re}(\neg e_1) \cup \overline{\psi_2}^{\Re}(\neg e_1) = \phi$.

This shows the proper inclusion in (4), that is, $\overline{\eth_1 \widetilde{\sqcap}_r \eth_2}^{\Re} \widetilde{\subsetneq} \overline{\eth_1}^{\Re} \widetilde{\sqcap}_r \overline{\eth_2}^{\Re}$.

The following proposition points out the $\Re-$definable BSSs over U, when $\Re$ is identity or universal binary relation on U.

Proposition 4.3.5 *Take a P-apx space* $(U, \Re)$.

1. *If* $\Re$ *is the identity relation on* U, *then each BSS over* U *is* $\Re-$*definable.*

2. *If* $\Re$ *is the universal binary relation on* U, *then the* $\Re-$*definable BSSs are* $\mathcal{U}_{\breve{A}}$ *and* $\Theta_{\breve{A}}$, *where* $\breve{A} \subseteq \breve{E}$.

Proof. Straightforward. ■

Proposition 4.3.6 *Take a P-apx space* $(U, \Re)$ *and let* $\eth = (\xi, \psi; \breve{A}) \in BSS(U)$. *Then, the subsequent statements are equivalent.*

1. $\overline{\eth}^{\Re} \widetilde{\subseteq} \eth$,

2. $\eth \widetilde{\subseteq} \underline{\eth}_{\Re}$,

3. $\eth$ is $\Re-$definable.

Proof. (1)⇒(2) Assume that $\overline{\eth}^{\Re} \widetilde{\subseteq} \eth$. From Theorem 4.3.3, we have $\underline{(\overline{\eth}^{\Re})}_{\Re} \widetilde{\subseteq} \underline{\eth}_{\Re}$. Then, Theorem 4.3.1 yields

$$\eth \widetilde{\subseteq} \overline{\eth}^{\Re} = \underline{(\overline{\eth}^{\Re})}_{\Re} \widetilde{\subseteq} \underline{\eth}_{\Re}$$

(2)⇒(3) Assume that $\eth \mathrel{\widetilde{\subseteq}} \underline{\eth}_{\Re}$. But we have $\underline{\eth}_{\Re} \mathrel{\widetilde{\subseteq}} \eth$. So, $\eth = \underline{\eth}_{\Re}$. This gives

$$\overline{\eth}^{\Re} = \overline{(\underline{\eth}_{\Re})}^{\Re} = \underline{\eth}_{\Re}$$

Thus, $\eth$ is $\Re-$definable.

(3)⇒(1) Obvious. ■

Theorem 4.3.7 *Take a P-apx space $(U, \Re)$ and let σ be an eqv-rel on U, such that, $\Re \subseteq \sigma$. Then, $\underline{\eth}_{\sigma} \mathrel{\widetilde{\subseteq}} \underline{\eth}_{\Re}$ and $\overline{\eth}^{\Re} \mathrel{\widetilde{\subseteq}} \overline{\eth}^{\sigma}$ for each $\eth \in BSS(U)$.*

Proof. Let $\eth = (\xi, \psi; \breve{A}) \in BSS(U)$ for some $\breve{A} \subseteq \breve{E}$. Since $\Re \subseteq \sigma$, we have $[u]_{\Re} \subseteq [u]_{\sigma}$ for each $u \in U$. Thus,

$$\begin{aligned} \underline{\xi}_{\sigma}(e) &= \{u \in U : [u]_{\sigma} \subseteq \xi(e)\} \\ &\subseteq \{u \in U : [u]_{\Re} \subseteq \xi(e)\} = \underline{\xi}_{\Re}(e) \end{aligned}$$

for each $e \in \breve{A}$. Similarly, $\underline{\psi}_{\sigma}(\neg e) \supseteq \underline{\psi}_{\Re}(\neg e)$ for each $\neg e \in \neg\breve{A}$. Hence, $\underline{\eth}_{\sigma} \mathrel{\widetilde{\subseteq}} \underline{\eth}_{\Re}$. In the same way, it can be verified that $\overline{\eth}^{\Re} \mathrel{\widetilde{\subseteq}} \overline{\eth}^{\sigma}$. ■

4.4 Similarity relations associated with RBS approximations

This section establishes some binary relations between the BSSs based on their RBS-apxes and investigate their properties.

Definition 4.4.1 *Take a P-apx space $(U, \Re)$ and let $\eth_1, \eth_2 \in BSS(U)$. Then, we define the following notions.*

- $\eth_1 \simeq_{\Re} \eth_2$ *if and only if* $\underline{\eth_1}_{\Re} = \underline{\eth_2}_{\Re}$.
- $\eth_1 \eqsim_{\Re} \eth_2$ *if and only if* $\overline{\eth_1}^{\Re} = \overline{\eth_2}^{\Re}$.
- $\eth_1 \approx_{\Re} \eth_2$ *if and only if* $\underline{\eth_1}_{\Re} = \underline{\eth_2}_{\Re}$ *and* $\overline{\eth_1}^{\Re} = \overline{\eth_2}^{\Re}$.

These relations may be termed as the lower RBS similarity relation, upper RBS similarity relation and the RBS similarity relation, respectively. Obviously, $\eth_1$ and $\eth_2$ are RBS similar if and only if they are both, lower and upper RBS similar.

Note that if $\eth_1 \simeq_{\Re} \eth_2$, then, $\underline{\eth_1}_{\Re} = \underline{\eth_2}_{\Re}$. Which means that

$$(\underline{\xi_1}_{\Re}, \underline{\psi_1}_{\Re}; \breve{A}_1) = (\underline{\xi_2}_{\Re}, \underline{\psi_2}_{\Re}; \breve{A}_2).$$

That is, $\breve{A}_1 = \breve{A}_2$. Same is the case when $\eth_1 \eqsim_{\Re} \eth_2$ and $\eth_1 \approx_{\Re} \eth_2$. So, any two lower RBS similar, upper RBS similar or RBS similar BSSs over U have same set of attributes. This means that their restricted and extended intersections and unions coincide.

Proposition 4.4.2 *The relations* $\simeq_{\Re}$, $\eqsim_{\Re}$ *and* $\approx_{\Re}$ *are eqv-rels on* $BSS(U)$.

Proof. Straightforward. ■

Theorem 4.4.3 *Take a P-apx space* $(U, \Re)$. *Then, the subsequent asservations hold for each* $\eth_i = (\xi_i, \psi_i; \breve{A}_i) \in BSS(U)$; $i = 1, 2, 3, 4$.

1. $\eth_1 \eqsim_{\Re} \eth_2$ *if and only if* $\eth_1 \eqsim_{\Re} (\eth_1 \widetilde{\sqcup}_r \eth_2) \eqsim_{\Re} \eth_2$;
2. $\eth_1 \eqsim_{\Re} \eth_2$ *and* $\eth_3 \eqsim_{\Re} \eth_4$ *imply that* $(\eth_1 \widetilde{\sqcup}_r \eth_3) \eqsim_{\Re} (\eth_2 \widetilde{\sqcup}_r \eth_4)$, *provided that* $\breve{A}_1 \cap \breve{A}_3 \neq \phi$;
3. $\eth_1 \widetilde{\subseteq} \eth_2$ *and* $\eth_2 \eqsim_{\Re} \Theta_{\breve{A}_2}$ *imply that* $\eth_1 \eqsim_{\Re} \Theta_{\breve{A}_1}$;
4. $\eth_1 \widetilde{\subseteq} \eth_2$ *and* $\eth_1 \eqsim_{\Re} \mathcal{U}_{\breve{A}_1}$ *imply that* $\eth_2 \eqsim_{\Re} \mathcal{U}_{\breve{A}_2}$, *provided that* $\breve{A}_1 = \breve{A}_2$.

Proof. (1) Let $\eth_1 \eqsim_{\Re} \eth_2$. Then, $\overline{\eth_1}^{\Re} = \overline{\eth_2}^{\Re}$ and $\breve{A}_1 = \breve{A}_2$. From Theorem 4.3.3, we have

$$\overline{\eth_1 \widetilde{\sqcup}_r \eth_2}^{\Re} = \overline{\eth_1}^{\Re} \widetilde{\sqcup}_r \overline{\eth_2}^{\Re} = \overline{\eth_1}^{\Re} = \overline{\eth_2}^{\Re}.$$

So, $\eth_1 \eqsim_{\Re} (\eth_1 \widetilde{\sqcup}_r \eth_2) \eqsim_{\Re} \eth_2$.

Converse holds by the transitivity of the relation $\eqsim_{\Re}$.

(2) Let $\eth_1 \eqsim_{\Re} \eth_2$ and $\eth_3 \eqsim_{\Re} \eth_4$, with $\breve{A}_1 \cap \breve{A}_3 = \breve{A}_2 \cap \breve{A}_4 \neq \phi$. Then, $\overline{\eth_1}^{\Re} = \overline{\eth_2}^{\Re}$ and $\overline{\eth_3}^{\Re} = \overline{\eth_4}^{\Re}$ with $\breve{A}_1 = \breve{A}_2$ and $\breve{A}_3 = \breve{A}_4$. From Theorem 4.3.3, we have

$$\overline{\eth_1 \widetilde{\sqcup}_r \eth_3}^{\Re} = \overline{\eth_1}^{\Re} \widetilde{\sqcup}_r \overline{\eth_3}^{\Re} = \overline{\eth_2}^{\Re} \widetilde{\sqcup}_r \overline{\eth_4}^{\Re} = \overline{\eth_2 \widetilde{\sqcup}_r \eth_4}^{\Re}.$$

Thus, $(\eth_1 \widetilde{\sqcup}_r \eth_3) \eqsim_{\Re} (\eth_2 \widetilde{\sqcup}_r \eth_4)$.

(3) We have $\eth_1 \widetilde{\subseteq} \eth_2$ and $\eth_2 \eqsim_{\Re} \Theta_{\breve{A}_2}$. Which implies that $\breve{A}_1 \subseteq \breve{A}_2$ and

$$\overline{\eth_1}^{\Re} \widetilde{\subseteq} \overline{\eth_2}^{\Re} = \overline{\Theta_{\breve{A}_2}}^{\Re} = \Theta_{\breve{A}_2}.$$

Restricting the attribute set of $\Theta_{\breve{A}_2}$ to $\breve{A}_1$, we get $\overline{\eth_1}^{\Re} \widetilde{\subseteq} \Theta_{\breve{A}_1}$. But $\Theta_{\breve{A}_1} \widetilde{\subseteq} \overline{\eth_1}^{\Re}$, so $\overline{\eth_1}^{\Re} = \Theta_{\breve{A}_1} = \overline{\Theta_{\breve{A}_1}}$. Which shows that $\eth_1 \eqsim_{\Re} \Theta_{\breve{A}_1}$.

(4) $\eth_1 \eqsim_{\Re} \mathcal{U}_{\breve{A}_1}$ implies that $\overline{\eth_1}^{\Re} = \overline{\mathcal{U}_{\breve{A}_1}}^{\Re} = \mathcal{U}_{\breve{A}_1}$. By $\breve{A}_1 = \breve{A}_2$ we have $\mathcal{U}_{\breve{A}_1} = \mathcal{U}_{\breve{A}_2}$. Clearly, $\eth_2 \widetilde{\subseteq} \mathcal{U}_{\breve{A}_2}$. Then, $\eth_1 \widetilde{\subseteq} \eth_2$ implies that

$$\overline{\eth_2}^{\Re} \widetilde{\subseteq} \overline{\mathcal{U}_{\breve{A}_2}}^{\Re} = \mathcal{U}_{\breve{A}_2} = \mathcal{U}_{\breve{A}_1} = \overline{\eth_1}^{\Re} \widetilde{\subseteq} \overline{\eth_2}^{\Re}$$

Which yields $\overline{\eth_2}^{\Re} = \overline{\mathcal{U}_{\breve{A}_2}}^{\Re}$. Hence, $\eth_2 \eqsim_{\Re} \mathcal{U}_{\breve{A}_2}$. ■

Theorem 4.4.4 *Take a P-apx space* $(U, \Re)$. *Then, the subsequent asservations hold for each* $\eth_i = (\xi_i, \psi_i; \breve{A}_i) \in BSS(U)$; $i = 1, 2, 3, 4$.

1. $\eth_1 \simeq_{\Re} \eth_2$ *if and only if* $\eth_1 \simeq_{\Re} (\eth_1 \widetilde{\sqcap}_r \eth_2) \simeq_{\Re} \eth_2$;

2. $\eth_1 \simeq_{\Re} \eth_2$ *and* $\eth_3 \simeq_{\Re} \eth_4$ *imply that* $(\eth_1 \widetilde{\sqcap}_r \eth_3) \simeq_{\Re} (\eth_2 \widetilde{\sqcap}_r \eth_4)$*, provided that* $\breve{A}_1 \cap \breve{A}_3 \neq \phi$;

3. $\eth_1 \widetilde{\subseteq} \eth_2$ *and* $\eth_2 \simeq_{\Re} \Theta_{\breve{A}_2}$ *imply that* $\eth_1 \simeq_{\Re} \Theta_{\breve{A}_1}$;

4. $\eth_1 \widetilde{\subseteq} \eth_2$ *and* $\eth_1 \simeq_{\Re} \mathcal{U}_{\breve{A}_1}$ *imply that* $\eth_2 \simeq_{\Re} \mathcal{U}_{\breve{A}_2}$*, provided that* $\breve{A}_1 = \breve{A}_2$.

Proof. Parallel to the proof of Theorem 4.4.3. ∎

Theorem 4.4.5 *Take a P-apx space* $(U, \Re)$*. Then, the subsequent asservations hold for each* $\eth_i = (\xi_i, \psi_i; \breve{A}_i) \in BSS(U)$; $i = 1, 2$.

1. $\eth_1 \widetilde{\subseteq} \eth_2$ *and* $\eth_2 \approx_{\Re} \Theta_{\breve{A}_2}$ *imply that* $\eth_1 \approx_{\Re} \Theta_{\breve{A}_1}$;

2. $\eth_1 \widetilde{\subseteq} \eth_2$ *and* $\eth_1 \approx_{\Re} \mathcal{U}_{\breve{A}_1}$ *imply that* $\eth_2 \approx_{\Re} \mathcal{U}_{\breve{A}_2}$*, provided that* $\breve{A}_1 = \breve{A}_2$.

Proof. This directly follows from Definition 4.4.1 and Theorems 4.4.3, 4.4.4. ∎

4.5 Applications of RBS approximations

Decision making is a major area to be conferred in almost all kinds of data analysis. There is often a desire to decide for the optimum object. But sometimes, a decision between the attributes also needs to be made. We propose algorithms for both situations by applying the concept of RBS-apxes of a BSS. Let the set of attributes and the set containing objects be symbolized by $\breve{E} = \{e_i : 1 \leq i \leq m\}$ and $U = \{u_j : 1 \leq j \leq n\}$, respectively. The BSSs describing the assessment of the decision maker about the objects is $\eth = (\xi, \psi; \breve{E})$. We use, in this section, the representation of the BSS $\eth$ as given in Table 1.1 of Example 1.5.2. Recall that, the $(i, j)th$ entry a_{ij} in the table of $\eth$ represents the information about the object u_j provided by $\eth$ for the attribute $e_i \in \breve{E}$.

4.5.1 Deciding between the attributes

There are many situations in which one needs to decide between some attributes or parameters possessed by some objects, or when one is trying to find an attribute having maximum effect on the objects. The algorithm presented here helps to decide for an attribute which is causing maximum effect on the objects. First, we define an indiscernibility parameter $\mathcal{P}$ for the objects u_j of U. Then, we come to the indiscernibility relation on U.

Definition 4.5.1 *The indiscernibility parameter $\mathcal{P}$, having the values p_j corresponding to each object $u_j \in U$, is defined by*

$$p_j = \sum_{i=1}^{m} a_{ij}$$

This parameter depicts the difference between the number of positive attributes e_i and the number of negative attributes $\neg e_i$ possessed by the object $u_j \in U$. Now we come to the indiscernibility relation on U associated with the BSS $\eth$. We say that two objects u_j and u_k are indiscernible, written as $u_j \sim u_k$, if and only if they have same indiscernibility value. That is, $u_j \sim u_k$ if and only if $p_j = p_k$. The indiscernibility relation $\Re$ between the objects of U is established as:

$$\Re = \{(u_j, u_k) \in U \times U : u_j \sim u_k\}. \tag{4.1}$$

Surely, $\Re$ is an eqv-rel on U. Now, denote $\underline{\alpha_i} = \sum_{j=1}^{n} \underline{a_{ij}}$ and $\overline{\alpha_i} = \sum_{j=1}^{n} \overline{a_{ij}}$, where $\underline{a_{ij}}$ is the $(i,j)th$ entry in the table of $\underline{\eth}_{\Re}$ and $\overline{a_{ij}}$ is the $(i,j)th$ entry in the table of $\overline{\eth}^{\Re}$.

Definition 4.5.2 *The decision parameter D has the values d_i for each $e_i \in \breve{E}$, given by*

$$d_i = \underline{\alpha_i} + \overline{\alpha_i}.$$

Algorithm 4.5.3 *The algorithm to decide for the best attribute in $\breve{E}$ is as follows:*

1. *Input the BSS $\eth = (\xi, \psi; \breve{E})$.*

2. *Find the eqv-rel $\Re$ on the set U of objects, using Formula 4.1.*

3. *Evaluate $\underline{\eth}_{\Re}$ and $\overline{\eth}^{\Re}$ for the BSS $\eth$ using the eqv-rel $\Re$. Find the values $\underline{\alpha_i}$ and $\overline{\alpha_i}$.*

4. *Find the decision values d_i for each attribute $e_i \in \breve{E}$ using Definition 4.5.2.*

5. *Construct the decision table having columns of $\breve{E}$ and the decision parameter D rearranged in descending order with respect to the values d_i. Choose k, so that, $d_k = \max_i d_i$. Then, e_k is the best optimal attribute.*

The flow chart of Algorithm 4.5.3 is shown in Figure 4.1.

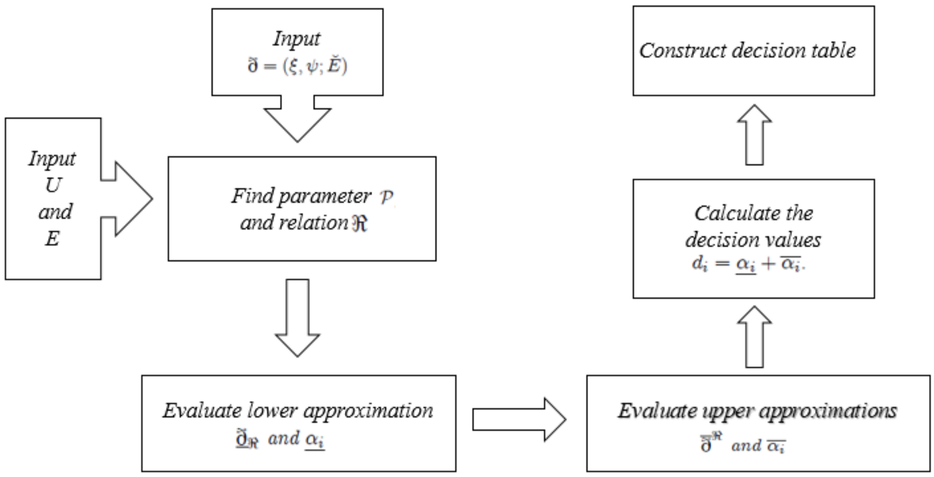

Figure 4.1: Flow chart of Algorithm 4.5.3

For the illustration, we apply this algorithm to an example.

Example 4.5.4 *We discuss the situation of a city X where many young citizens are suffering from the cardiac attack, while it is usually considered to be a problem of old age. Dr. Y, a cardiologist, is trying to search why this is occurring in early age so frequently in the city. For this purpose, he takes a sample of heart patients of age group (30-50 years) admitted in different hospitals of the city. The common risk factors causing heart attacks are* $\breve{E} = \{e_1 =$ *smoking,* $e_2 =$ *heavy drinking of alcohol,* $e_3 =$ *diabetic,* $e_4 =$ *high cholesterol level,* $e_5 =$ *sedentary life style,* $e_6 =$ *high blood pressure*$\}$*, which will serve as the attribute set for U. The "counter" set of* $\breve{E}$ *may be taken as* $\neg\breve{E} = \{\neg e_1 =$ *no smoking,* $\neg e_2 =$ *no drinking of alcohol,* $\neg e_3 =$ *non-diabetic,* $\neg e_4 =$ *normal cholesterol level,* $\neg e_5 =$ *healthy life style,* $\neg e_6 =$ *normal blood pressure*$\}$*. To give an understanding of the procedure, we take a small sample* $U = \{t_1, t_2, ..., t_8\}$ *of eight patients.*

1. *The BSS* $ð = (\xi, \psi; \breve{E})$ *describing the history and examination report of patients*

under consideration taken by Dr. Y, is given in Table 4.1.

$\eth$	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
e_1	1	−1	1	1	1	1	−1	−1
e_2	−1	1	1	−1	−1	1	1	1
e_3	1	1	−1	1	0	−1	−1	1
e_4	0	1	0	1	−1	−1	−1	0
e_5	−1	1	0	−1	−1	1	1	−1
e_6	1	−1	1	1	0	−1	1	−1

Table 4.1: The BSS $(\xi, \psi; \breve{E})$

2. *The values of the indiscernibility parameter* $\mathcal{P}$*, corresponding to the objects are calculated in Table 4.2.*

$\eth$	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8
e_1	1	−1	1	1	1	1	−1	−1
e_2	−1	1	1	−1	−1	1	1	1
e_3	1	1	−1	1	0	−1	−1	1
e_4	0	1	0	1	−1	−1	−1	0
e_5	−1	1	0	−1	−1	1	1	−1
e_6	1	−1	1	1	0	−1	1	−1
$\mathcal{P}$	1	2	2	2	−2	0	0	−1

Table 4.2: Calculation of values of $\mathcal{P}$

This table immediately gives the eqv-rel $\Re$ *on* U*, dividing* U *into eqv-classes* $\{t_1\}$, $\{t_2, t_3, t_4\}$, $\{t_5\}$, $\{t_6, t_7\}$ *and* $\{t_8\}$.

3. *Find* $\underline{\eth}_{\Re}$ *and* $\underline{\alpha_i}$ *in Table 4.3 as:*

$\underline{\eth}_{\Re}$	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	$\underline{\alpha_i}$
e_1	1	−1	−1	−1	1	−1	−1	−1	−4
e_2	−1	−1	−1	−1	−1	1	1	1	−2
e_3	1	−1	−1	−1	0	−1	−1	1	−3
e_4	0	0	0	0	−1	−1	−1	0	−3
e_5	−1	−1	−1	−1	−1	1	1	−1	−4
e_6	1	−1	−1	−1	0	−1	−1	−1	−5

Table 4.3: Calculation of $\underline{\eth}_{\Re}$ *and* $\underline{\alpha_i}$

Table 4.4 gives $\overline{\eth}^{\Re}$ *and* $\overline{\alpha_i}$ *as:*

$\overline{\eth}^{\Re}$	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	$\overline{\alpha_i}$
e_1	1	1	1	1	1	1	1	−1	6
e_2	−1	1	1	1	−1	1	1	1	4
e_3	1	1	1	1	0	−1	−1	1	3
e_4	0	1	1	1	−1	−1	−1	0	0
e_5	−1	1	1	1	−1	1	1	1	2
e_6	1	1	1	1	0	1	1	1	5

Table 4.4: Calculation of $\overline{\eth}^{\Re}$ *and* $\overline{\alpha_i}$

4. *The decision values* d_i *for the attributes are calculated in Table 4.5.*

5. *Table 4.6 is the decision table of* $\eth$.

$\breve{E}$	$\overline{\alpha_i}$	$\underline{\alpha_i}$	d_i
e_1	6	−4	2
e_2	4	−2	2
e_3	3	−3	0
e_4	0	−3	−3
e_5	2	−4	−2
e_6	5	−5	0

Table 4.5: Calculation of d_i

$\breve{E}$	D
e_1	2
e_2	2
e_3	0
e_6	0
e_5	−2
e_4	−3

Table 4.6: Decision table of $\eth$

We get $\max_i d_i = d_1 = d_2 = 2$ *and hence* $k = 1, 2$. *Thus, Dr. Y comes to the result that the most dominant reason of so frequent cardiac attacks in younger generation of the city X is heavy smoking* (e_1) *and heavy drinking of alcohol* (e_2). *This indicates the excessive use of alcoholic drinks and heavy smoking in the city, which is to be controlled on first preference in order to overcome the cardiac attacks in younger generation.*

4.5.2 Deciding between the objects

While deciding in the favour of an object from a given set containing objects, sometimes one may be unable to take the best decision, even when the best decision is known. So, it will always be helpful if the worst decision also becomes visible. We design an algorithm which provides the best, as well as, the worst decision. As already discussed, the RBS-apxes of a BSS demonstrate the definite or the uncertain presence of an attribute in an object. So, the decision made with the help of these RBS-apxes is more reliable and refined, as compared to the decision made by just adding the values a_{ij}, as in [61]. Let $\breve{A} = \{e_i : 1 \leq i \leq m_o\}$ be the set of choice attributes or the attributes of interest for the decision problem and $\eth = (\xi, \psi; \breve{A})$ be the BSS describing the assessment of the decision maker about the objects. The indiscernibility parameter $\mathcal{P}$ for the objects is given in Definition 4.5.1 and the indiscernibility relation $\Re$ on U is found by Formula 4.1. For the relation $\Re$, we take the $(i, j)th$ entry in the table of $\underline{\eth}_{\Re}$ as $\underline{a_{ij}}$ and the $(i, j)th$ entry in the table of $\overline{\eth}^{\Re}$ as $\overline{a_{ij}}$. Denote

$$\underline{p_j} = \sum_{i=1}^{m_o} \underline{a_{ij}} \text{ and } \overline{p_j} = \sum_{i=1}^{m_o} \overline{a_{ij}}$$

Now we proceed to the decisions values for the objects.

Definition 4.5.5 *The decision parameter D' has the values d'_j corresponding to each object $u_j \in U$, given by*

$$d'_j = \underline{p_j} + \overline{p_j}$$

If $|\{e_i : u_j \in \xi(e_i)\}|$ and $|\{\neg e_i : u_j \in \psi(\neg e_i)\}|$ represent the number of positive attributes e_i and the number of negative attributes $\neg e_i$ possessed by u_j , respectively, then, we can write

$$p_j = \sum_{i=1}^{m_o} a_{ij} = |\{e_i : u_j \in \xi(e_i)\}| - |\{\neg e_i : u_j \in \psi(\neg e_i)\}|$$

Similarly,

$$\begin{aligned} \underline{p_j} &= \left|\{e_i : u_j \in \underline{\xi}_{\Re}(e_i)\}\right| - \left|\{\neg e_i : u_j \in \underline{\psi}_{\Re}(\neg e_i)\}\right|, \\ \overline{p_j} &= \left|\{e_i : u_j \in \overline{\xi}^{\Re}(e_i)\}\right| - \left|\{\neg e_i : u_j \in \overline{\psi}^{\Re}(\neg e_i)\}\right|, \end{aligned}$$

where $\left|\{e_i : u_j \in \underline{\xi}_{\Re}(e_i)\}\right|$ is the number of attributes e_i surely possessed by u_j,
$\left|\{e_i : u_j \in \overline{\xi}^{\Re}(e_i)\}\right|$ is the number of attributes e_i probably possessed by u_j,
$\left|\{\neg e_i : u_j \in \underline{\psi}_{\Re}(\neg e_i)\}\right|$ is the number of attributes $\neg e_i$ probably possessed by u_j,
$\left|\{\neg e_i : u_j \in \overline{\psi}^{\Re}(\neg e_i)\}\right|$ is the number of attributes $\neg e_i$ surely possessed by u_j.
So, d'_j becomes

$$\begin{aligned} d'_j &= \left|\{e_i : u_j \in \underline{\xi}_{\Re}(e_i)\}\right| - \left|\{\neg e_i : u_j \in \underline{\psi}_{\Re}(\neg e_i)\}\right| \\ &\quad + \left|\{e_i : u_j \in \overline{\xi}^{\Re}(e_i)\}\right| - \left|\{\neg e_i : u_j \in \overline{\psi}^{\Re}(\neg e_i)\}\right|. \end{aligned}$$

Note that,

$$\{e_i : u_j \in \underline{\xi}_{\Re}(e_i)\} \subseteq \{e_i : u_j \in \overline{\xi}^{\Re}(e_i)\}$$

and

$$\{\neg e_i : u_j \in \overline{\psi}^{\Re}(\neg e_i)\} \subseteq \{\neg e_i : u_j \in \underline{\psi}_{\Re}(\neg e_i)\}.$$

So, the attributes e_i (or $\neg e_i$) whose presence in a particular object is definite, are automatically counted twice in the decision values for that object, while the attributes with uncertain presence are counted once. On the other hand, the algorithm in [61] treats all the attributes equally, regardless of whether the presence of these attributes in the objects is definite or uncertain. The algorithm presented in this paper helps to find a better decision in a natural way. Hence, this algorithm has two main advantages over the algorithm presented in [61].

- It provides the best, as well as, the worst decision, so that, one can avoid taking the worst decision.
- Using the RBS-apxes, the attribute whose presence in an object is definite, are given double weightage as compared to the attributes whose presence is not certain.

Algorithm 4.5.6 *The algorithm to decide for the best object in U is, as follows:*

1. *Input the set of choice attributes $\breve{A} \subseteq \breve{E}$.*
2. *Input the BSS $\eth = (\xi, \psi; \breve{A})$ and find the values of the parameter $\mathcal{P}$.*
3. *Find the eqv-rel $\Re$ on the set U of objects, using Formula 4.1.*
4. *Evaluate $\underline{\eth}_{\Re}$ and $\overline{\eth}^{\Re}$ for the BSS $\eth$ under the eqv-rel $\Re$. Find the values $\underline{p_j}$ and $\overline{p_j}$.*
5. *Find the decision values d'_j corresponding to each object $u_j \in U$.*
6. *Construct the decision table having rows of U and the decision parameter D'. Rearrange the table in descending order with respect to the values d'_j. Choose k and l, so that $d'_k = \max_j d'_j$ and $d'_l = \min_j d'_j$. Then u_k is the best optimal object, while u_l is the worst optimal object to be decided.*

The flow chart of Algorithm 4.5.6 is shown in Figure 4.2.

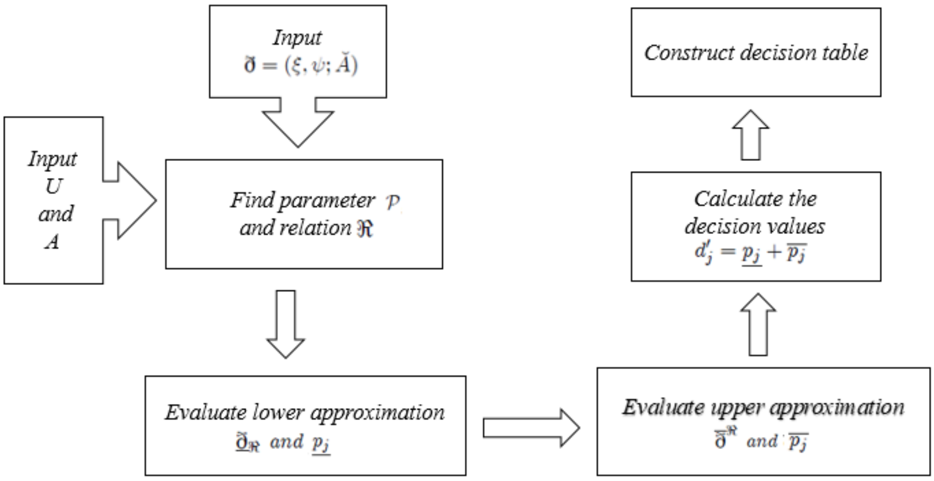

Figure 4.2: Flow chart of Algorithm 4.5.6

As an illustration, we apply this algorithm to an example.

Example 4.5.7 *Let $U = \{c_i; i = 1, 2, ..., 7\}$ be a set containing some construction companies considered by Mrs. Z for the construction of his home and consider the attribute set as $\breve{E} = \{e_1 =$strong structure, $e_2 =$innovative designs, $e_3 =$good reputation, $e_4 =$competitive pricing, $e_5 =$having own crew, $e_6 =$skilled crew, $e_7 =$high quality materials, $e_8 =$decisiveness, $e_9 =$flexibility, $e_{10} =$well organized$\}$ and $\neg\breve{E} = \{\neg e_1 =$weak structure, $\neg e_2 =$traditional designs, $\neg e_3 =$ill reputation, $\neg e_4 =$high pricing, $\neg e_5 =$not having own crew, $\neg e_6 =$unskilled crew, $\neg e_7 =$low quality materials, $\neg e_8 =$indecisive, $e_9 =$rigidity, $e_{10} =$disorganized$\}$. Let the "Quality Analysis of construction work" be described by a BSS $\eth = (\xi, \psi; \breve{A})$ with $\breve{A} = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ by*

$\xi(e_1) = \{c_3, c_4, c_7\}$, $\xi(e_2) = \{c_1, c_2, c_4, c_6\}$, $\xi(e_3) = \{c_3, c_4, c_5, c_7\}$, $\xi(e_4) = \{c_3, c_5\}$, $\xi(e_5) = \{c_1, c_2, c_3, c_4, c_6, c_7\}$, $\xi(e_6) = \{c_1, c_3, c_4, c_6, c_7\}$

and $\psi(\neg e_1) = \{c_2, c_6\}$, $\psi(\neg e_2) = \{c_3, c_7\}$, $\psi(\neg e_3) = \{c_2, c_6\}$, $\psi(\neg e_4) = \{c_1, c_4, c_7\}$, $\psi(\neg e_5) = \{c_5\}$, $\psi(\neg e_6) = \{c_2, c_5\}$.

1. *Input the choice attributes $\breve{A} = \{e_1, e_2, e_3, e_4, e_5, e_6\}$.*

2. *The BSS* $\eth$ *is presented and the parameter* $\mathcal{P}$ *is evaluated in Table 4.7.*

$\eth = (\xi, \psi; \breve{A})$	c_1	c_2	c_3	c_4	c_5	c_6	c_7
e_1	0	−1	1	1	0	−1	1
e_2	1	1	−1	1	0	1	−1
e_3	0	−1	1	1	1	−1	1
e_4	−1	0	1	−1	1	0	−1
e_5	1	1	1	1	−1	1	1
e_6	1	−1	1	1	−1	1	1
$\mathcal{P}$	2	−1	4	4	0	1	2

Table 4.7: The BSS $\eth$ *and parameter* $\mathcal{P}$

3. *From Table 4.7, we immediately get the eqv-rel*

$$\begin{aligned}\Re \;=\; &\{(c_1,c_1),(c_2,c_2),(c_3,c_3),(c_4,c_4),(c_5,c_5),(c_6,c_6),(c_7,c_7),\\ &(c_1,c_7),(c_7,c_1),(c_3,c_4),(c_4,c_3)\}.\end{aligned}$$

This relation divides U *into eqv-classes* $\{c_1, c_7\}$, $\{c_3, c_4\}$, $\{c_2\}$, $\{c_5\}$ *and* $\{c_6\}$.

4. *Find* $\underline{\eth}_{\Re}$ *and* $\underline{p_j}$ *in Table 4.8 as:*

$\underline{\eth}_{\Re}$	c_1	c_2	c_3	c_4	c_5	c_6	c_7
e_1	0	−1	1	1	0	−1	0
e_2	−1	1	−1	−1	0	1	−1
e_3	0	−1	1	1	1	−1	0
e_4	−1	0	−1	−1	1	0	−1
e_5	1	1	1	1	−1	1	1
e_6	1	−1	1	1	−1	1	1
$\underline{p_j}$	0	−1	2	2	0	1	0

Table 4.8: Calculation of $\underline{\eth}_{\Re}$ *and* $\underline{p_j}$

Table 4.9 gives $\overline{\eth}^{\Re}$ *and* $\overline{p_j}$ *as:*

$\overline{\eth}^{\Re}$	c_1	c_2	c_3	c_4	c_5	c_6	c_7
e_1	1	−1	1	1	0	−1	1
e_2	1	1	1	1	0	1	1
e_3	1	−1	1	1	1	−1	1
e_4	−1	0	1	1	1	0	−1
e_5	1	1	1	1	−1	1	1
e_6	1	−1	1	1	−1	1	1
$\overline{p_j}$	4	−1	6	6	0	1	4

Table 4.9: Calculation of $\overline{\eth}^{\Re}$ *and* $\overline{p_j}$

5. *The decision values* d'_j *for the companies* c_j *are calculated in Table 4.10.*

6. *Table 4.11 is the decision table.*

U	c_1	c_2	c_3	c_4	c_5	c_6	c_7
$\underline{p_j}$	0	-1	2	2	0	1	0
$\overline{p_j}$	4	-1	6	6	0	1	4
d'_j	4	-2	8	8	0	2	4

Table 4.10: Calculation of d'_j

U	c_3	c_4	c_1	c_7	c_6	c_5	c_2
D'	8	8	4	4	2	0	-2

Table 4.11: Decision table of $\eth$

We get $\max_j d'_j = d'_3 = d'_4 = 8$ *and* $\min_j d'_j = d'_2 = -2$. *Hence* $k = 3, 4$ *and* $l = 2$. *Thus, the companies* c_3 *and* c_4 *are the best selections, while* c_2 *is the worst selection. So, Mrs. Z can decide between any one of* c_3 *and* c_4 *for the construction of her house. But, in any case, she must not go for* c_2.

Chapter 5

Rough bipolar soft ideals over semigroups

5.1 Introduction

Semigroups are a substantial part of algebra and the study of semigroups is incomplete without the study of ideals. The theory of semigroups and the ideals in semigroups were amalgamated with rough sets and soft sets by many authors in many ways. For instance, the rough ideals in semigroups were first discussed by Kuroki [37] and the soft ideals over semigroups were initiated by Ali et al. [12]. Motivated by this idea, we continue the work of Chapter 4 in the direction of the BS subsemigroups and bipolar soft ideals (BS-ids) over semigroups. We further define and discuss the notions of the BS subsemigroups, BS left ideal (BSl-id), BS right ideal (BSr-id), BS interior ideal (BSi-id) and BS bi-ideal (BSb-id) over a semigroup. The roughness in the BSSs and the BS subsemigroups under a cng-rel defined on the semigroup are also studied. We further present the rough BS left ideal (RBSl-id), rough BS right ideal (RBSr-id), rough BS interior ideal (RBSi-id) and rough BS bi-ideal (RBSb-id) over a semigroup by defining the lower RBSl-id, RBSr-id, RBSi-id and RBSb-id and the upper RBSl-id, RBSr-id, RBSi-id and RBSb-id over a semigroup and investigate some of their basic properties.

5.2 Bipolar soft sets over semigroups

The BSSs in semigroups are constructed by hybridizing the RBS-apxes of the BSSs with the semigroups. Throughout this work, Υ is a semigroup and $\breve{E}$ is the set of attributes for Υ. Recall that a BSS over Υ is given by $\eth = (\xi, \psi; \breve{A})$, where $\breve{A} \subseteq \breve{E}$ and ξ, ψ are mappings given by $\xi : \breve{A} \to P(\Upsilon)$ and $\psi : \neg\breve{A} \to P(\Upsilon)$ with the consistency restraint $\xi(e) \cap \psi(\neg e) = \phi$ for each $e \in \breve{A}$. The set containing all BSSs over Υ is denoted by $BSS(\Upsilon)$.

Definition 5.2.1 *Let* $\eth_1 = (\xi_1, \psi_1; \breve{A}_1), \eth_2 = (\xi_2, \psi_2; \breve{A}_2) \in BSS(\Upsilon)$ *for a semigroup* Υ. *The product of* $\eth_1$ *and* $\eth_2$ *is a BSS* $\eth_1 \widehat{*}\, \eth_2 = (\xi_1 * \xi_2, \psi_1 * \psi_2; \breve{A}_1 \cap \breve{A}_2)$ *over* Υ, *where*

$$\begin{aligned}(\xi_1 * \xi_2)(e) &= \xi_1(e)\xi_2(e),\\ (\psi_1 * \psi_2)(\neg e) &= (\psi_1'(\neg e)\psi_2'(\neg e))'\end{aligned}$$

for each $e \in \breve{A}$.

Here, $\psi_1'(\neg e)$ denotes the crisp compliment $\Upsilon - \psi_1(\neg e)$ of $\psi_1(\neg e)$.

Definition 5.2.2 *A BSS* $\eth$ *over a semigroup* Υ *is a BS subsemigroup over* Υ *if and only if* $\eth \,\widehat{*}\, \eth \,\widetilde{\subseteq}\, \eth$.

Theorem 5.2.3 *A BSS* $\eth = (\xi, \psi; \breve{A}) \in BSS(\Upsilon)$ *over a semigroup* Υ *is a BS subsemigroup over* Υ *if and only if* $\xi(e)$ *and* $\psi'(\neg e)$ *are subsemigroups of* Υ *for each* $e \in \breve{A}$.

Proof. Let $\eth = (\xi, \psi; \breve{A})$ be a BS subsemigroup over Υ. Then, $\eth \,\widehat{*}\, \eth \,\widetilde{\subseteq}\, \eth$. That is,

$$(\xi * \xi, \psi * \psi; \breve{A}) \,\widetilde{\subseteq}\, (\xi, \psi; \breve{A}).$$

This gives $(\xi * \xi)(e) \subseteq \xi(e)$ and $(\psi * \psi)(\neg e) \supseteq \psi(\neg e)$ for each $e \in \breve{A}$. Which yields $\xi(e)\xi(e) \subseteq \xi(e)$ and $(\psi'(\neg e)\psi'(\neg e))' \supseteq \psi(\neg e)$, that is, $\psi'(\neg e)\psi'(\neg e) \subseteq \psi'(\neg e)$ for each $e \in \breve{A}$. Hence, $\xi(e)$ and $\psi'(\neg e)$ are subsemigroups of Υ for each $e \in \breve{A}$.

Converse follows by reversing the above steps. ■

Theorem 5.2.4 *Let* $\eth_1$ *and* $\eth_2$ *be any two BS subsemigroups over a semigroup* Υ. *Then, their extended intersection* $\eth_1 \widetilde{\sqcap}_\varepsilon \eth_2$ *and the restricted intersection* $\eth_1 \widetilde{\sqcap}_r \eth_2$ *are also BS subsemigroups over* Υ.

Proof. Let $\eth_1 = (\xi_1, \psi_1; \breve{A}_1)$ and $\eth_2 = (\xi_2, \psi_2; \breve{A}_2)$ be BS subsemigroups over Υ. From Theorem 5.2.3, $\xi_1(e_1)$, $\xi_2(e_2)$, $\psi_1'(\neg e_1)$ and $\psi_2'(\neg e_2)$ are subsemigroups of Υ for each $e_1 \in \breve{A}_1$ and $e_2 \in \breve{A}_2$. The extended intersection of $\eth_1$ and $\eth_2$ is

$$\eth_1 \widetilde{\sqcap}_\varepsilon \eth_2 = (\xi_1 \widetilde{\sqcap}_\varepsilon \xi_2, \psi_1 \widetilde{\sqcup}_\varepsilon \psi_2; \breve{A}_1 \cup \breve{A}_2).$$

The case is obvious when $e \in \breve{A}_1 - \breve{A}_2$ or $e \in \breve{A}_2 - \breve{A}_1$. Take $e \in \breve{A}_1 \cap \breve{A}_2$. Then, $\xi_1(e) \cap \xi_2(e)$ and $\psi_1'(\neg e) \cap \psi_2'(\neg e)$ are subsemigroups of Υ. But,

$$\psi_1'(\neg e) \cap \psi_2'(\neg e) = (\psi_1(\neg e) \cup \psi_2(\neg e))'.$$

Thus, $\xi_1(e) \cap \xi_2(e)$ and $(\psi_1(\neg e) \cup \psi_2(\neg e))'$ are subsemigroups of Υ for each $e \in \breve{A}_1 \cap \breve{A}_2$. Hence, $ð_1 \widetilde{\sqcap}_\varepsilon ð_2$ is a BS subsemigroup over Υ. Similar is the proof of the restricted intersection $ð_1 \widetilde{\sqcap}_r ð_2$. ■

Note that, if $ð_1 = (\xi_1, \psi_1; \breve{A}_1)$ and $ð_2 = (\xi_2, \psi_2; \breve{A}_2)$ are BS subsemigroups over Υ and $\breve{A}_1 \cap \breve{A}_2 = \phi$, then $ð_1 \widetilde{\sqcup}_\varepsilon ð_2$ is surely a BS subsemigroup over Υ. But generally, the restricted (or extended) union of two BS subsemigroups over Υ may not be a BS subsemigroup over Υ. This is established in the subsequent example.

Example 5.2.5 *Let $\Upsilon = \{1, a, b, c\}$ represent a semigroup whose table of binary operation is given below.*

	1	a	b	c
1	1	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	b	a	1

Let $\hat{E} = \{e_i : i = 1, 2, 3, 4\}$. We take two BS subsemigroups $ð_1 = (\xi_1, \psi_1; \breve{A}_1)$ and $ð_2 = (\xi_2, \psi_2; \breve{A}_2)$ over Υ with $\breve{A}_1 = \{e_1, e_2\}$ and $\breve{A}_2 = \{e_1, e_2, e_3\}$, defined below.

$\xi_1(e_1) = \{1, c\}$, $\xi_1(e_2) = \{a\}$,

$\psi_1(\neg e_1) = \phi$, $\psi_1(\neg e_2) = \{c\}$,

$\xi_2(e_1) = \{b\} = \xi_2(e_2)$, $\xi_2(e_3) = \{1, a, b\}$,

$\psi_2(\neg e_1) = \{a, c\}$, $\psi_2(\neg e_2) = \{c\} = \psi_2(\neg e_3)$.

The restricted union $ð_1 \widetilde{\sqcup}_r ð_2 = (\xi_1 \widetilde{\cup}_r \xi_2, \psi_1 \widetilde{\cap}_r \psi_2; \breve{A}_1 \cap \breve{A}_2)$ is calculated for $\breve{A}_1 \cap \breve{A}_2 = \{e_1, e_2\}$, as below.

$(\xi_1 \widetilde{\cup}_r \xi_2)(e_1) = \{1, b, c\}$, $(\xi_1 \widetilde{\cup}_r \xi_2)(e_2) = \{a, b\}$,

$(\psi_1 \widetilde{\cap}_r \psi_2)(\neg e_1) = \phi$, $(\psi_1 \widetilde{\cap}_r \psi_2)(\neg e_2) = \{c\}$.

We find that $b, c \in (\xi_1 \widetilde{\cup}_r \xi_2)(e_1)$. But, $cb = a \notin (\xi_1 \widetilde{\cup}_r \xi_2)(e_1)$. So, $ð_1 \widetilde{\sqcup}_r ð_2$ (and similarly $ð_1 \widetilde{\sqcup}_\varepsilon ð_2$) is not a BS subsemigroup over Υ.

5.3 Bipolar soft ideals over semigroups

In this section, we define the BSl-ids, BSr-ids, BS-ids, BSi-ids and BSb-ids over the semigroup Υ. Some characterizations of these ideals are also discussed.

Definition 5.3.1 *A BSS $ð = (\xi, \psi; \breve{A})$ over Υ is a BSl-id (BSr-id) over Υ if and only if $\mathcal{U}_{\breve{A}} \mathbin{\widehat{*}} ð \mathbin{\widetilde{\subseteq}} ð$ (or $ð \mathbin{\widehat{*}} \mathcal{U}_{\breve{A}} \mathbin{\widetilde{\subseteq}} ð$).*

A BSS $ð \in BSS(\Upsilon)$ is a BS-id (BS ideal) over Υ if it is both, BSl-id and BSr-id over Υ. Recall that $\mathcal{U}_{\breve{A}} = (\mathcal{U}, \Theta; \breve{A}) \in BSS(\Upsilon)$ is the relative whole BSS over Υ, where $\mathcal{U}(e) = \Upsilon$ and $\Theta(\neg e) = \phi$ for each $e \in \breve{A}$.

Theorem 5.3.2 *A BSS* $\eth = (\xi, \psi; \breve{A})$ *over* Υ *is a BSl-id* (*BSr-id, BS-id*) *over* Υ *if and only if* $\xi(e)$ *and* $\psi'(\neg e)$ *are left* (*right, two-sided*) *ideals of* Υ *for each* $e \in \breve{A}$.

Proof. Let $\eth = (\xi, \psi; \breve{A})$ be a BSl-id over Υ. Then, $\mathcal{U}_{\breve{A}} \mathbin{\widehat{*}} \eth \mathrel{\widetilde{\subseteq}} \eth$. That is, $(\mathcal{U} * \xi, \Theta * \psi; \breve{A}) \mathrel{\widetilde{\subseteq}} (\xi, \psi; \breve{A})$. This gives

$$(\mathcal{U} * \xi)(e) = \mathcal{U}(e)\xi(e) = \Upsilon.\xi(e) \subseteq \xi(e)$$

and

$$\begin{aligned}(\Theta * \psi)(\neg e) &= (\Theta'(\neg e)\psi'(\neg e))' = (\phi'\psi'(\neg e))' \\ &= (\Upsilon\ \psi'(\neg e))' \supseteq \psi(\neg e)\end{aligned}$$

for each $e \in \breve{A}$. Which yields that, $\Upsilon\xi(e) \subseteq \xi(e)$ and $\Upsilon\psi'(\neg e) \subseteq \psi'(\neg e)$ for each $e \in \breve{A}$. Hence, proved that $\xi(e)$ and $\psi'(\neg e)$ are left ideals of Υ for each $e \in \breve{A}$.

Converse follows by reversing the above steps.

Similarly, the cases of the BSr-ids and the BS-id can be verified. ■

Theorem 5.3.3 *Let* $\eth_1$ *and* $\eth_2$ *be any two BSl-ids* (*BSr-ids, BS-ids*) *over* Υ. *Then, their extended intersection* $\eth_1 \widetilde{\sqcap}_\varepsilon \eth_2$ *and the restricted intersection* $\eth_1 \widetilde{\sqcap}_r \eth_2$ *are also BSl-ids* (*BSr-ids, BS-ids*) *over* Υ.

Proof. Parallel to the proof of Theorem 5.2.4. ■

Theorem 5.3.4 *Let* $\eth_1$ *and* $\eth_2$ *be any two BSl-ids* (*BSr-ids, BS-ids*) *over* Υ. *Then, their extended union* $\eth_1 \widetilde{\sqcup}_\varepsilon \eth_2$ *and the restricted union* $\eth_1 \widetilde{\sqcup}_r \eth_2$ *are also BSl-ids* (*BSr-ids, BS-ids*) *over* Υ.

Proof. Let $\eth_1 = (\xi_1, \psi_1; \breve{A}_1)$ and $\eth_2 = (\xi_2, \psi_2; \breve{A}_2)$ be BSl-ids over Υ. From Theorem 5.3.2, $\xi_1(e_1)$, $\xi_2(e_2)$, $\psi_1'(\neg e_1)$ and $\psi_2'(\neg e_2)$ are left ideals of Υ for each $e_1 \in \breve{A}_1$ and $e_2 \in \breve{A}_2$. The extended union of $\eth_1$ and $\eth_2$ is

$$\eth_1 \widetilde{\sqcup}_\varepsilon \eth_2 = (\xi_1 \widetilde{\cup}_\varepsilon \xi_2, \psi_1 \widetilde{\cap}_\varepsilon \psi_2; \breve{A}_1 \cup \breve{A}_2).$$

The case is obvious when $e \in \breve{A}_1 - \breve{A}_2$ or $e \in \breve{A}_2 - \breve{A}_1$. Take $e \in \breve{A}_1 \cap \breve{A}_2$. Then, $\xi_1(e) \cup \xi_2(e)$ and $\psi_1'(\neg e) \cup \psi_2'(\neg e)$ are left ideals of Υ. But,

$$\psi_1'(\neg e) \cup \psi_2'(\neg e) = (\psi_1(\neg e) \cap \psi_2(\neg e))'.$$

Thus, $(\xi_1 \widetilde{\cup}_\varepsilon \xi_2)(e)$ and $(\psi_1 \widetilde{\cap}_\varepsilon \psi_2)'(\neg e)$ are left ideals of Υ for each $e \in \breve{A}_1 \cap \breve{A}_2$. Hence, $\eth_1 \widetilde{\sqcup}_\varepsilon \eth_2$ is a BSl-id over Υ.

Similarly, the cases of the BSr-ids, BS-ids and the restricted union $\eth_1 \widetilde{\sqcup}_r \eth_2$ can be verified. ■

Theorem 5.3.5 *Let Υ be a semigroup. Then, the following assertion holds for each BSr-id* $\eth_1$ *and BSl-id* $\eth_2$ *over* Υ.

$$\eth_1 \widehat{*}\, \eth_2 \,\widetilde{\subseteq}\, \eth_1 \widetilde{\sqcap}_r \eth_2.$$

Proof. Let $\eth_1 = (\xi_1, \psi_1; \breve{A}_1)$ be a BSr-id and $\eth_2 = (\xi_2, \psi_2; \breve{A}_2)$ be a BSl-id over Υ. We have

$$\begin{aligned} \eth_1 \widehat{*}\, \eth_2 &= (\xi_1 * \xi_2, \psi_1 * \psi_2; \breve{A}_1 \cap \breve{A}_2), \\ \eth_1 \widetilde{\sqcap}_r \eth_2 &= (\xi_1 \widetilde{\cap}_r \xi_2, \psi_1 \widetilde{\cup}_r \psi_2; \breve{A}_1 \cap \breve{A}_2). \end{aligned}$$

From Theorem 5.3.2, $\xi_1(e)$ and $\psi_1'(\neg e)$ are right ideals of Υ for each $e \in \breve{A}_1$, while $\xi_2(e)$ and $\psi_2'(\neg e)$ are left ideals of Υ for each $e \in \breve{A}_2$. Then, for each $e \in \breve{A}_1 \cap \breve{A}_2$, we have

$$\begin{aligned} (\xi_1 * \xi_2)(e) &= \xi_1(e)\xi_2(e) \\ &\subseteq \xi_1(e) \cap \xi_2(e) = (\xi_1 \widetilde{\cap}_r \xi_2)(e) \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \psi_1'(\neg e)\psi_2'(\neg e) &\subseteq \psi_1'(\neg e) \cap \psi_2'(\neg e) \\ &= (\psi_1(\neg e) \cup \psi_2(\neg e))'. \end{aligned} \tag{5.2}$$

Equation 5.2 gives

$$\begin{aligned} (\psi_1 * \psi_2)(\neg e) &= (\psi_1'(\neg e)\psi_2'(\neg e))' \\ &\supseteq \psi_1(\neg e) \cup \psi_2(\neg e) \\ &= (\psi_1 \widetilde{\cup}_r \psi_2)(\neg e). \end{aligned} \tag{5.3}$$

The expressions 5.1 and 5.3 yield that,

$$\eth_1 \widehat{*}\, \eth_2 \,\widetilde{\subseteq}\, \eth_1 \widetilde{\sqcap}_r \eth_2$$

for each BSr-id $\eth_1$ and BSl-id $\eth_2$ over Υ. ■

Corollary 5.3.6 *Let Υ be a semigroup. Then, for each BSr-id* $\eth_1$ *and BSl-id* $\eth_2$ *over* Υ*, the subsequent assertion hold.*

$$\eth_1 \widehat{*}\, \eth_2 \,\widetilde{\subseteq}\, \eth_1 \widetilde{\sqcap}_\varepsilon \eth_2.$$

Proof. This is verified directly from Theorem 5.3.5, as $\eth_1 \widetilde{\sqcap}_r \eth_2 \,\widetilde{\subseteq}\, \eth_1 \widetilde{\sqcap}_\varepsilon \eth_2$. ■

Definition 5.3.7 *A BSS* $\eth = (\xi, \psi; \breve{A})$ *over* Υ *is a BSi-id (BS interior ideal) over* Υ *if and only if* $\mathcal{U}_{\breve{A}} \,\widehat{*}\, \eth \,\widehat{*}\, \mathcal{U}_{\breve{A}} \,\widetilde{\subseteq}\, \eth$.

Theorem 5.3.8 *A BSS $\eth = (\xi, \psi; \breve{A})$ over Υ is a BSi-id over Υ if and only if $\xi(e)$ and $\psi'(\neg e)$ are interior ideals of Υ for each $e \in \breve{A}$.*

Proof. Let $\eth = (\xi, \psi; \breve{A})$ be a BSi-id over Υ. Then, $\mathcal{U}_{\breve{A}} \mathbin{\widehat{*}} \eth \mathbin{\widehat{*}} \mathcal{U}_{\breve{A}} \mathrel{\widetilde{\subseteq}} \eth$. Which gives

$$(\mathcal{U} * \xi * \mathcal{U}, \Theta * \psi * \Theta; \breve{A}) \mathrel{\widetilde{\subseteq}} (\xi, \psi; \breve{A}).$$

That is, $(\mathcal{U} * \xi * \mathcal{U})(e) \subseteq \xi(e)$ and $(\Theta * \psi * \Theta)(\neg e) \supseteq \psi(\neg e)$ for each $e \in \breve{A}$. This yields $\mathcal{U}(e)\xi(e)\mathcal{U}(e) \subseteq \xi(e)$ and $(\Theta'(\neg e)\psi'(\neg e)\Theta'(\neg e))' \supseteq \psi(\neg e)$ for each $e \in \breve{A}$. Thus, we have $\Upsilon\xi(e)\Upsilon \subseteq \xi(e)$ and $\Upsilon\psi'(\neg e)\Upsilon \subseteq \psi'(\neg e)$ for each $e \in \breve{A}$. Hence, proved that $\xi(e)$ and $\psi'(\neg e)$ are interior ideals of Υ for each $e \in \breve{A}$.

Converse follows by reversing the above steps. ■

Theorem 5.3.9 *Let $\eth_1$ and $\eth_2$ be any two BSi-ids over Υ. Then, their extended intersection $\eth_1 \widetilde{\sqcap}_\varepsilon \eth_2$ and the restricted intersection $\eth_1 \widetilde{\sqcap}_r \eth_2$ are BSi-ids over Υ.*

Proof. Parallel to the proof of Theorem 5.2.4. ■

Theorem 5.3.10 *Let $\eth_1$ and $\eth_2$ be any two BSi-ids over Υ. Then, their extended union $\eth_1 \widetilde{\sqcup}_\varepsilon \eth_2$ and the restricted union $\eth_1 \widetilde{\sqcup}_r \eth_2$ are also the BSi-ids over Υ.*

Proof. Parallel to the proof of Theorem 5.3.4. ■

Definition 5.3.11 *A BS subsemigroup $\eth = (\xi, \psi; \breve{A})$ over Υ is a BSb-id (BS bi-ideal) over Υ if and only if $\eth \mathbin{\widehat{*}} \mathcal{U}_{\breve{A}} \mathbin{\widehat{*}} \eth \mathrel{\widetilde{\subseteq}} \eth$.*

Theorem 5.3.12 *A BSS $\eth = (\xi, \psi; \breve{A})$ over Υ is a BSb-id over Υ if and only if $\xi(e)$ and $\psi'(\neg e)$ are bi-ideals of Υ for each $e \in \breve{A}$.*

Proof. Let $\eth = (\xi, \psi; \breve{A})$ be a BSb-id over Υ. Then, $\eth$ is a BS subsemigroup over Υ. From Theorem 5.2.3, $\xi(e)$ and $\psi'(\neg e)$ are subsemigroups of Υ for each $e \in \breve{A}$. Since $\eth$ is a BSb-id over Υ, so, $\eth \mathbin{\widehat{*}} \mathcal{U}_{\breve{A}} \mathbin{\widehat{*}} \eth \mathrel{\widetilde{\subseteq}} \eth$. Which gives

$$(\xi * \mathcal{U} * \xi, \psi * \Theta * \psi; \breve{A}) \mathrel{\widetilde{\subseteq}} (\xi, \psi; \breve{A}).$$

That is, $(\xi * \mathcal{U} * \xi)(e) \subseteq \xi(e)$ and $(\psi * \Theta * \psi)(\neg e) \supseteq \psi(\neg e)$ for each $e \in \breve{A}$. This yields $\xi(e)\mathcal{U}(e)\xi(e) \subseteq \xi(e)$ and $(\psi'(\neg e)\Theta'(\neg e)\psi'(\neg e))' \supseteq \psi(\neg e)$ for each $e \in \breve{A}$. Thus, we have $\xi(e)\Upsilon\xi(e) \subseteq \xi(e)$ and $\psi'(\neg e)\Upsilon\psi'(\neg e) \subseteq \psi'(\neg e)$ for each $e \in \breve{A}$. Hence, proved that $\xi(e)$ and $\psi'(\neg e)$ are bi-ideals of Υ for each $e \in \breve{A}$.

Converse follows by reversing the above steps. ■

Theorem 5.3.13 *Let $\eth_1$ and $\eth_2$ be any two BSb-ids over Υ. Then, their extended intersection $\eth_1 \widetilde{\sqcap}_\varepsilon \eth_2$ and the restricted intersection $\eth_1 \widetilde{\sqcap}_r \eth_2$ are also BSb-ids over Υ.*

Proof. Parallel to the proof of Theorem 5.2.4. ■

The extended and restricted unions of $\eth_1$ and $\eth_2$ are not necessarily BSb-ids over Υ, because these are not BS subsemigroups over Υ, as shown in Example 5.2.5.

5.4 Rough bipolar soft sets over semigroups

The rough BSSs (or RBSSs) are defined with the help of lower and upper RBS-apxes of a BSS over Υ, on which a cng-rel $\Re$ is defined. These approximations are defined in this section. The RBS subsemigroups over Υ are also discussed.

Definition 5.4.1 *Take a cng-rel $\Re$ on Υ and let $\eth = (\xi, \psi; \breve{A}) \in BSS(\Upsilon)$. The lower and upper RBS-apxes of $\eth$ with respect to $(\Upsilon, \Re)$ are the BSSs symbolized by $\underline{\eth}_{\Re} = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ and $\overline{\eth}^{\Re} = (\overline{\xi}^{\Re}, \overline{\psi}^{\Re}; \breve{A})$, respectively, where $\underline{\xi}_{\Re}$, $\overline{\xi}^{\Re}$ are defined as:*

$$\underline{\xi}_{\Re}(e) = \{u \in \Upsilon : [u]_{\Re} \subseteq \xi(e)\},$$

$$\overline{\xi}^{\Re}(e) = \{u \in \Upsilon : [u]_{\Re} \cap \xi(e) \neq \phi\}$$

for each $e \in \breve{A}$, and $\underline{\psi}_{\Re}$, $\overline{\psi}^{\Re}$ are defined as:

$$\underline{\psi}_{\Re}(\neg e) = \{u \in \Upsilon : [u]_{\Re} \cap \psi(\neg e) \neq \phi\},$$

$$\overline{\psi}^{\Re}(\neg e) = \{u \in \Upsilon : [u]_{\Re} \subseteq \psi(\neg e)\}$$

for each $\neg e \in \neg\breve{A}$. $\eth$ is $\Re-$definable if $\underline{\eth}_{\Re} = \overline{\eth}^{\Re}$; otherwise, $\eth$ is an RBSS over Υ.

In Chapter 4, some characterizations of the RBSSs over a non-empty set U having an eqv-rel $\Re$ were presented. These characterizations are also valid when the set U is replaced by the semigroup Υ and the eqv-rel on U is replaced by a cng-rel on Υ. So the results in Chapter 4 also hold for the lower and upper RBS-apxes of the BSSs over Υ, given in the Definition 4.2.1.

Theorem 5.4.2 *Take a cng-rel $\Re$ on Υ and $\eth_1, \eth_2 \in BSS(\Upsilon)$. Then, we have*

$$\overline{\eth_1}^{\Re} \widehat{*}\, \overline{\eth_2}^{\Re} \,\widetilde{\subseteq}\, \overline{\eth_1 \widehat{*}\, \eth_2}^{\Re}.$$

Proof. Let $\eth_1 = (\xi_1, \psi_1; \breve{A}_1)$, $\eth_2 = (\xi_2, \psi_2; \breve{A}_2) \in BSS(\Upsilon)$. We have

$$\begin{aligned} \overline{\eth_1}^{\Re} \widehat{*}\, \overline{\eth_2}^{\Re} &= (\overline{\xi_1}^{\Re} * \overline{\xi_2}^{\Re}, \overline{\psi_1}^{\Re} * \overline{\psi_2}^{\Re}; \breve{A}_1 \cap \breve{A}_2), \\ \overline{\eth_1 \widehat{*}\, \eth_2}^{\Re} &= (\overline{\xi_1 * \xi_2}^{\Re}, \overline{\psi_1 * \psi_2}^{\Re}; \breve{A}_1 \cap \breve{A}_2). \end{aligned}$$

Take $e \in \breve{A}_1 \cap \breve{A}_2$ and $s \in (\overline{\xi_1}^{\Re} * \overline{\xi_2}^{\Re})(e) = \overline{\xi_1}^{\Re}(e)\overline{\xi_2}^{\Re}(e)$. Then, $s = ab$ for some $a \in \overline{\xi_1}^{\Re}(e)$ and $b \in \overline{\xi_2}^{\Re}(e)$. Which gives $[a]_{\Re} \cap \xi_1(e) \neq \phi$ and $[b]_{\Re} \cap \xi_2(e) \neq \phi$. Let $c \in [a]_{\Re} \cap \xi_1(e)$ and $d \in [b]_{\Re} \cap \xi_2(e)$. Then, $cd \in [a]_{\Re}[b]_{\Re} \subseteq [ab]_{\Re}$ and also $cd \in \xi_1(e)\xi_2(e)$, as, $c \in \xi_1(e)$ and $d \in \xi_2(e)$. This yields $[a]_{\Re}[b]_{\Re} \cap \xi_1(e)\xi_2(e) \neq \phi$. That is, $[ab]_{\Re} \cap (\xi_1 * \xi_2)(e) \neq \phi$. So, $ab = s \in (\overline{\xi_1 * \xi_2}^{\Re})(e)$. Hence, we get

$$(\overline{\xi_1}^{\Re} * \overline{\xi_2}^{\Re})(e) \subseteq (\overline{\xi_1 * \xi_2}^{\Re})(e) \tag{5.4}$$

for each $e \in \breve{A}_1 \cap \breve{A}_2$. Now, let $x \in (\overline{\psi_1 * \psi_2}^{\Re})(\neg e)$. Then,

$$[x]_{\Re} \subseteq (\psi_1 * \psi_2)(\neg e) = (\psi'_1(\neg e)\psi'_2(\neg e))'.$$

This means that,

$$[x]_{\Re} \cap \psi'_1(\neg e)\psi'_2(\neg e) = \phi. \tag{5.5}$$

We claim that, $x \notin (\overline{\psi_1}^{\Re}(\neg e))'(\overline{\psi_2}^{\Re}(\neg e))'$. As if, $x \in (\overline{\psi_1}^{\Re}(\neg e))'(\overline{\psi_2}^{\Re}(\neg e))'$, then there exist $a \in (\overline{\psi_1}^{\Re}(\neg e))'$ and $b \in (\overline{\psi_2}^{\Re}(\neg e))'$, such that $x = ab$. That is, $a \notin \overline{\psi_1}^{\Re}(\neg e)$ and $b \notin \overline{\psi_2}^{\Re}(\neg e)$. Which means that, $[a]_{\Re} \nsubseteq \psi_1(\neg e)$ and $[b]_{\Re} \nsubseteq \psi_2(\neg e)$. Let $c \in [a]_{\Re}$, $c \notin \psi_1(\neg e)$, $d \in [b]_{\Re}$ and $d \notin \psi_2(\neg e)$. Then, $cd \in [a]_{\Re}[b]_{\Re} \subseteq [ab]_{\Re} \subseteq [x]_{\Re}$ and $cd \in (\psi_1(\neg e))'(\psi_2(\neg e))'$. Therefore, $[x]_{\Re} \cap (\psi_1(\neg e))'(\psi_2(\neg e))' \neq \phi$. Which contradicts Equation 5.5. Hence our claim is true, that is,

$$x \in ((\overline{\psi_1}^{\Re}(\neg e))'(\overline{\psi_2}^{\Re}(\neg e))')' = (\overline{\psi_1}^{\Re} * \overline{\psi_2}^{\Re})(\neg e).$$

So, we have

$$(\overline{\psi_1 * \psi_2}^{\Re})(\neg e) \subseteq (\overline{\psi_1}^{\Re} * \overline{\psi_2}^{\Re})(\neg e) \tag{5.6}$$

for each $e \in \breve{A}_1 \cap \breve{A}_2$. The assertions 5.4 and 5.6 prove that $\overline{\eth_1}^{\Re} \widehat{*}\, \overline{\eth_2}^{\Re} \widetilde{\subseteq}\, \overline{\eth_1 \widehat{*}\, \eth_2}^{\Re}$. ■

Corollary 5.4.3 *Let* Υ *be a semigroup and* $\{\eth_i : 1 \leq i \leq n\} \subset BSS(\Upsilon)$. *Then, for a cng-rel* $\Re$ *on* Υ, *we have*

$$\widehat{*}_{i=1}^{n} \overline{\eth_i}^{\Re} \widetilde{\subseteq}\, \overline{\widehat{*}_{i=1}^{n} \eth_i}^{\Re}.$$

Here, $\widehat{*}_{i=1}^{n} \eth_i$ denotes the finite product $\eth_1 \,\widehat{*}\, \eth_2 \,\widehat{*} ... \widehat{*}\, \eth_n$.

Theorem 5.4.4 *For a semigroup* Υ, *a complete cng-rel* $\Re$ *on* Υ *and* $\eth_1, \eth_2 \in BSS(\Upsilon)$, *we have*

$$\underline{\eth_1}_{\Re} \widehat{*}\, \underline{\eth_2}_{\Re} \widetilde{\subseteq}\, \underline{\eth_1 \,\widehat{*}\, \eth_2}_{\Re}.$$

Proof. Let $\eth_1 = (\xi_1, \psi_1; \breve{A}_1)$, $\eth_2 = (\xi_2, \psi_2; \breve{A}_2) \in BSS(\Upsilon)$. We have

$$\begin{aligned} \underline{\eth_1}_{\Re} \widehat{*}\, \underline{\eth_2}_{\Re} &= (\underline{\xi_1}_{\Re} * \underline{\xi_2}_{\Re}, \underline{\psi_1}_{\Re} * \underline{\psi_2}_{\Re}; \breve{A}_1 \cap \breve{A}_2), \\ \underline{\eth_1 \widehat{*}\, \eth_2}_{\Re} &= (\underline{\xi_1 * \xi_2}_{\Re}, \underline{\psi_1 * \psi_2}_{\Re}; \breve{A}_1 \cap \breve{A}_2). \end{aligned}$$

Take $e \in \breve{A}_1 \cap \breve{A}_2$ and let $s \in (\underline{\xi_1}_{\Re} * \underline{\xi_2}_{\Re})(e) = \underline{\xi_1}_{\Re}(e)\underline{\xi_2}_{\Re}(e)$. Then, we can write $s = ab$ for some $a \in \underline{\xi_1}_{\Re}(e)$ and $b \in \underline{\xi_2}_{\Re}(e)$. Which gives $[a]_{\Re} \subseteq \xi_1(e)$ and $[b]_{\Re} \subseteq \xi_2(e)$. Then, $[a]_{\Re}[b]_{\Re} \subseteq \xi_1(e)\xi_2(e)$. Since $\Re$ is complete cng-rel, so, $[a]_{\Re}[b]_{\Re} = [ab]_{\Re}$. Which gives $[ab]_{\Re} \subseteq \xi_1(e)\xi_2(e)$. That is, $[s]_{\Re} = [ab]_{\Re} \subseteq (\xi_1 * \xi_2)(e)$. So, $s \in (\underline{\xi_1 * \xi_2}_{\Re})(e)$. Hence

$$(\underline{\xi_1}_{\Re} * \underline{\xi_2}_{\Re})(e) \subseteq (\underline{\xi_1 * \xi_2}_{\Re})(e) \tag{5.7}$$

for each $e \in \breve{A}_1 \cap \breve{A}_2$. Now, let $x \in (\underline{\psi_1 * \psi_2}_{\Re})(\neg e)$. Then, $[x]_{\Re} \cap (\psi_1 * \psi_2)(\neg e) \neq \phi$. That is,

$$[x]_{\Re} \cap (\psi_1'(\neg e)\psi_2'(\neg e))' \neq \phi. \tag{5.8}$$

We claim that,

$$x \in (\underline{\psi_1}_{\Re} * \underline{\psi_2}_{\Re})(\neg e) = ((\underline{\psi_1}_{\Re}(\neg e))'(\underline{\psi_2}_{\Re}(\neg e))')'.$$

As if, $x \notin ((\underline{\psi_1}_{\Re}(\neg e))'(\underline{\psi_2}_{\Re}(\neg e))')'$,that is, $x \in (\underline{\psi_1}_{\Re}(\neg e))'(\underline{\psi_2}_{\Re}(\neg e))'$. Then, we can write $x = yz$, for some $y \in (\underline{\psi_1}_{\Re}(\neg e))'$ and $z \in (\underline{\psi_2}_{\Re}(\neg e))'$. So, $[y]_{\Re} \cap \psi_1(\neg e) = \phi$ and $[z]_{\Re} \cap \psi_2(\neg e) = \phi$. That is, $[y]_{\Re} \subseteq \psi_1'(\neg e)$ and $[z]_{\Re} \subseteq \psi_2'(\neg e)$. Thus, $[y]_{\Re}[z]_{\Re} \subseteq \psi_1'(\neg e)\psi_2'(\neg e)$. Since $\Re$ is complete, so, $[y]_{\Re}[z]_{\Re} = [yz]_{\Re} = [x]_{\Re}$. Which gives $[x]_{\Re} \subseteq \psi_1'(\neg e)\psi_2'(\neg e)$. This contradicts Equation 5.8. So, our claim is true. That is, $x \in (\underline{\psi_1}_{\Re} * \underline{\psi_2}_{\Re})(\neg e)$. So

$$(\underline{\psi_1 * \psi_2}_{\Re})(\neg e) \subseteq (\underline{\psi_1}_{\Re} * \underline{\psi_2}_{\Re})(\neg e) \tag{5.9}$$

for each $e \in \breve{A}_1 \cap \breve{A}_2$. The assertions 5.7 and 5.9 prove that $\underline{\eth_1}_{\Re} \mathbin{\widehat{*}} \underline{\eth_2}_{\Re} \mathrel{\widetilde{\subseteq}} \underline{\eth_1 \widehat{*} \eth_2}_{\Re}$. ■

Corollary 5.4.5 *For a semigroup Υ, a complete cng-rel $\Re$ on Υ and $\{\eth_i : 1 \leq i \leq n\} \subset BSS(\Upsilon)$, we have*

$$\widehat{*}_{i=1}^{n} \underline{\eth_i}_{\Re} \mathrel{\widetilde{\subseteq}} \underline{\widehat{*}_{i=1}^{n} \eth_i}_{\Re}.$$

Theorem 5.4.6 *Take a cng-rel $\Re$ on Υ. Then, the following assertion holds for each BSr-id $\eth_1$ and BSl-id $\eth_2$ over Υ.*

$$\overline{\eth_1 \widehat{*}\, \eth_2}^{\Re} \mathrel{\widetilde{\subseteq}} \overline{\eth_1}^{\Re} \widetilde{\sqcap}_r \overline{\eth_2}^{\Re}.$$

Proof. Let $\eth_1 = (\xi_1, \psi_1; \breve{A}_1)$ be a BSr-id and $\eth_2 = (\xi_2, \psi_2; \breve{A}_2)$ be a BSl-id over Υ. Then $\xi_1(e)$ and $\psi_1'(\neg e)$ are right ideals for each $e \in \breve{A}_1$, while $\xi_2(e)$ and $\psi_2'(\neg e)$ are left ideals of Υ, for each $e \in \breve{A}_2$. We have

$$\begin{aligned} \overline{\eth_1 \widehat{*}\, \eth_2}^{\Re} &= (\overline{\xi_1 * \xi_2}^{\Re}, \overline{\psi_1 * \psi_2}^{\Re}; \breve{A}_1 \cap \breve{A}_2), \\ \overline{\eth_1}^{\Re} \widetilde{\sqcap}_r \overline{\eth_2}^{\Re} &= (\overline{\xi_1}^{\Re} \widetilde{\cap}_r \overline{\xi_2}^{\Re}, \overline{\psi_1}^{\Re} \widetilde{\cup}_r \overline{\psi_2}^{\Re}; \breve{A}_1 \cap \breve{A}_2). \end{aligned}$$

Take $e \in \breve{A}_1 \cap \breve{A}_2$ and $s \in (\overline{\xi_1 * \xi_2}^{\Re})(e)$. Then $[s]_{\Re} \cap (\xi_1 * \xi_2)(e) \neq \phi$, that is, $[s]_{\Re} \cap \xi_1(e)\xi_2(e) \neq \phi$. Let $t \in [s]_{\Re} \cap \xi_1(e)\xi_2(e)$. Then $t \in [s]_{\Re}$ and $t \in \xi_1(e)\xi_2(e)$. We can write $t = ab$, where $a \in \xi_1(e)$ and $b \in \xi_2(e)$. Now $ab = t \in \xi_1(e)$, as $\xi_1(e)$ is right ideal of Υ and $ab = t \in \xi_2(e)$, as $\xi_2(e)$ is left ideal of $\hat{S}$. So, $t \in [s]_{\Re} \cap \xi_1(e)$ and $t \in [s]_{\Re} \cap \xi_2(e)$. Which means that,

$$[s]_{\Re} \cap (\xi_1)(e) \neq \phi \neq [s]_{\Re} \cap (\xi_2)(e).$$

That is, $s \in \overline{\xi_1}^{\Re}(e) \cap \overline{\xi_2}^{\Re}(e) = (\overline{\xi_1}^{\Re} \widetilde{\sqcap}_r \overline{\xi_2}^{\Re})(e)$. So, we have

$$(\overline{\xi_1 * \xi_2}^{\Re})(e) \subseteq (\overline{\xi_1}^{\Re} \widetilde{\sqcap}_r \overline{\xi_2}^{\Re})(e) \tag{5.10}$$

for each $e \in \breve{A}_1 \cap \breve{A}_2$. Now, let $x \in (\overline{\psi_1}^{\Re} \widetilde{\cup}_r \overline{\psi_2}^{\Re})(\neg e) = (\overline{\psi_1}^{\Re})(\neg e) \cup (\overline{\psi_2}^{\Re})(\neg e)$. Then, $x \in (\overline{\psi_1}^{\Re})(\neg e)$ or $x \in (\overline{\psi_2}^{\Re})(\neg e)$. Which means that,

$$[x]_{\Re} \subseteq \psi_1(\neg e) \text{ or } [x]_{\Re} \subseteq \psi_2(\neg e). \tag{5.11}$$

We claim that $[x]_{\Re} \cap \psi_1'(\neg e)\psi_2'(\neg e) = \phi$. As if there is some $y \in [x]_{\Re} \cap \psi_1'(\neg e)\psi_2'(\neg e)$. Then, $y = cd$ for some $c \in \psi_1'(\neg e)$ and $d \in \psi_2'(\neg e)$. But $\psi_1'(\neg e)$ is a right ideal and $\psi_2'(\neg e)$ is a left ideal of Υ. So, $y = cd \in \psi_1'(\neg e)$ and $y = cd \in \psi_2'(\neg e)$. Which yields $y \in [x]_{\Re} \cap \psi_1'(\neg e)$ and $y \in [x]_{\Re} \cap \psi_2'(\neg e)$. This contradicts the assertion 5.11. Thus $[x]_{\Re} \cap \psi_1'(\neg e)\psi_2'(\neg e) = \phi$. This gives $[x]_{\Re} \subseteq (\psi_1'(\neg e)\psi_2'(\neg e))'$. Which means that, $x \in (\overline{\psi_1 * \psi_2}^{\Re})(\neg e)$. Thus, we have

$$(\overline{\psi_1}^{\Re} \widetilde{\cup}_r \overline{\psi_2}^{\Re})(\neg e) \subseteq (\overline{\psi_1 * \psi_2}^{\Re})(\neg e) \tag{5.12}$$

for each $e \in \breve{A}_1 \cap \breve{A}_2$. The assertions 5.10 and 5.12 prove that $\overline{\eth_1 \widehat{*} \eth_2}^{\Re} \widetilde{\subseteq} \overline{\eth_1}^{\Re} \widetilde{\sqcap}_r \overline{\eth_2}^{\Re}$. ■

Theorem 5.4.7 *Let $\Re$ be a complete cng-rel on Υ. Then, the following assertion holds for each BSr-id $\eth_1$ and BSl-id $\eth_2$ over Υ.*

$$\underline{\eth_1 \widehat{*} \eth_2}_{\Re} \widetilde{\subseteq} \underline{\eth_1}_{\Re} \widetilde{\sqcap}_r \underline{\eth_2}_{\Re}.$$

Proof. Parallel to the proof of Theorem 5.4.6. ■

Definition 5.4.8 *Take a cng-rel $\Re$ on Υ and let $\eth \in BSS(\Upsilon)$. Then, $\eth$ is a lower (or upper) RBS subsemigroup over Υ, if $\underline{\eth}_{\Re}$ (or $\overline{\eth}^{\Re}$) is a BS subsemigroup over Υ.*

A BSS $\eth = (\xi, \psi; \breve{A})$ over Υ, which is both, lower and upper RBS subsemigroup over Υ, is called an RBS subsemigroup over Υ.

Theorem 5.4.9 *Each BS subsemigroup over Υ is an upper RBS subsemigroup over Υ.*

Proof. Let $\eth \in BSS(\Upsilon)$ be a BS subsemigroup over Υ. Then, $\eth \widehat{*} \eth \widetilde{\subseteq} \eth$. From Theorems 4.3.3 and 5.4.2, we have

$$\overline{\eth}^{\Re} \widehat{*} \overline{\eth}^{\Re} \widetilde{\subseteq} \overline{\eth \widehat{*} \eth}^{\Re} \widetilde{\subseteq} \overline{\eth}^{\Re}.$$

This verifies that $\overline{\eth}^{\Re}$ is a BS subsemigroup over Υ. Therefore, $\eth$ is an upper RBS subsemigroup over Υ. ■

The converse statement of the Theorem 5.4.9 is invalid generally, as exhibited in the next example.

Example 5.4.10 *Let $\Upsilon = \{a,b,c,d\}$ represent a semigroup whose table of binary operation is given below.*

	a	b	c	d
a	a	b	b	d
b	b	b	b	d
c	b	b	b	d
d	d	d	d	d

Let $\hat{E} = \{e_i : i = 1,2,3,4\}$. Take a cng-rel $\Re$ on Υ, defining the cng-classes $\{a\},\{b,d\}$ and $\{c\}$. We take a BSS $\eth = (\xi,\psi;\breve{A})$ over Υ with $\breve{A} = \{e_1,e_2\}$, defined below.

$\xi(e_1) = \{c,d\}$, $\xi(e_2) = \{b,c\}$,
$\psi(\neg e_1) = \{a\}$, $\psi(\neg e_2) = \{a,d\}$.

Note that $\eth$ is not a BS subsemigroup over Υ because $c \in \xi(e_1)$, but $cc = b \notin \xi(e_1)$. The upper RBS-apx $\overline{\Re}(\eth) = (\overline{\xi}^{\Re}, \overline{\psi}^{\Re}; \breve{A})$ of $\eth$ under the relation $\Re$ is calculated as:

$\overline{\xi}^{\Re}(e_1) = \{b,c,d\}$, $\overline{\xi}^{\Re}(e_2) = \{b,c,d\}$,
$\overline{\psi}^{\Re}(\neg e_1) = \{a\}$, $\overline{\psi}^{\Re}(\neg e_2) = \{a\}$.

Simple calculations verify that $\overline{\Re}(\eth)$ is a BS subsemigroup over Υ. So, $\eth$ is not a BS subsemigroup over Υ, although, it is an upper RBS subsemigroup over Υ.

Theorem 5.4.11 *Let $\Re$ be a complete cng-rel on Υ. Then, each BS subsemigroup over Υ is a lower RBS subsemigroup over Υ.*

Proof. Let $\eth$ be a BS subsemigroup over Υ. Then, $\eth \,\hat{*}\, \eth \,\tilde{\subseteq}\, \eth$. From Theorems 4.3.3 and 5.4.4, we have

$$\underline{\eth}_{\Re} \,\hat{*}\, \underline{\eth}_{\Re} \,\tilde{\subseteq}\, \underline{\eth \,\hat{*}\, \eth}_{\Re} \,\tilde{\subseteq}\, \underline{\eth}_{\Re}.$$

This verifies that $\underline{\eth}_{\Re}$ is a BS subsemigroup over Υ. Therefore, $\eth$ is a lower RBS subsemigroup over Υ. ■

The converse statement of the Theorem 5.4.11 is invalid generally, as exhibited in the next example.

Example 5.4.12 *Let $\Upsilon = \{s,t,u,v\}$ represent a semigroup whose table of binary operation is given below.*

	s	t	u	v
s	s	t	u	v
t	t	t	u	v
u	u	u	u	v
v	v	v	v	u

Let $\hat{E} = \{e_i : i = 1, 2, ..., 5\}$ *and* $\Re$ *be a complete cng-rel over* Υ, *defining cng-classes* $\{s\}, \{t\}$ *and* $\{u, v\}$. *We take a BSS* $\eth = (\xi, \psi; \breve{A})$ *over* Υ *with* $\breve{A} = \{e_1\}$, *defined below.*

$\xi(e_1) = \{t, v\}, \ \ \psi(\neg e_1) = \{s\}$.

Note that $\eth$ *is not a BS subsemigroup over* Υ *because* $v \in \xi(e_1)$, *but* $vv = u \notin \xi(e_1)$. *The lower RBS-apx* $\underline{\Re}(\eth) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ *of* $\eth$ *under* $\Re$ *is calculated as:*

$\underline{\xi}_{\Re}(e_1) = \{t\}, \ \ \underline{\psi}_{\Re}(\neg e_1) = \{s\}$.

Simple calculations verify that $\underline{\Re}(\eth)$ *is a BS subsemigroup over* Υ. *So,* $\eth$ *is not a BS subsemigroups over* Υ, *although, it is a lower RBS subsemigroup over* Υ.

Theorem 5.4.11 is invalid if the cng-rel $\Re$ is not complete. Next example is established to verify this fact.

Example 5.4.13 *Recall the semigroup* $\Upsilon = \{a, b, c, d\}$, *the attribute set* $\breve{E}$ *and the cng-rel* $\Re$ *as established in Example 5.4.10. Then,* $\Re$ *is not complete. We take a BS subsemigroup* $\eth = (\xi, \psi; \breve{A})$ *over* Υ *with* $\breve{A} = \{e_2\}$, *defined below.*

$\xi(e_2) = \{b, c\}, \quad \psi(\neg e_2) = \{a\}$.

The lower RBS-apx $\underline{\Re}(\eth) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ *of* $\eth$ *under* $\Re$ *is calculated as:*

$\underline{\xi}_{\Re}(e_2) = \{c\}, \quad \underline{\psi}_{\Re}(\neg e_2) = \{a\}$.

We find that $\underline{\Re}(\eth)$ *is not BS subsemigroup over* Υ, *because* $c \in \underline{\xi}_{\Re}(e_2)$, *but we have* $cc = b \notin \underline{\xi}_{\Re}(e_2)$. *So,* $\eth$ *is not a lower RBS subsemigroup over* Υ.

5.5 Rough bipolar soft ideals over semigroups

This section establishes the notions of the RBSl-ids, RBSr-ids, RBS-ids, RBSi-ids and RBSb-ids over the semigroup Υ and examines their basic properties.

Definition 5.5.1 *Take a cng-rel* $\Re$ *on* Υ *and let* $\eth \in BSS(\Upsilon)$. *Then,* $\eth$ *is a lower (resp. upper) RBSl-id (RBSr-id, RBS-id) over* Υ, *if* $\underline{\eth}_{\Re}$ *(resp.* $\overline{\eth}^{\Re}$*) is a BSl-id (BSr-id, BS-id) over* Υ.

A BSS $\eth$ in Υ, which is both, lower and upper RBSl-id (RBSr-id, RBS-id) over Υ, is called an RBSl-id (RBSr-id, RBS-id) over Υ.

Theorem 5.5.2 *Take a cng-rel* $\Re$ *on* Υ. *Then, each BSl-id (BSr-id, BS-id) over* Υ *is an upper RBSl-id (RBSr-id, RBS-id) over* Υ.

Proof. Let $\eth \in BSS(\Upsilon)$ be a BSl-id over Υ. Then, $\mathcal{U}_{\breve{A}} \widehat{*} \, \eth \,\widetilde{\subseteq}\, \eth$. From Theorem 4.3.3, we have $\overline{\mathcal{U}_{\breve{A}} \widehat{*} \, \eth}^{\Re} \widetilde{\subseteq} \, \overline{\eth}^{\Re}$. Now, from Theorems 4.3.1 and 5.4.2, we have

$$\begin{aligned} \mathcal{U}_{\breve{A}} \widehat{*} \, \overline{\eth}^{\Re} &= \overline{\mathcal{U}_{\breve{A}}}^{\Re} \widehat{*} \, \overline{\eth}^{\Re} \\ &\widetilde{\subseteq} \, \overline{\mathcal{U}_{\breve{A}} \widehat{*} \, \eth}^{\Re} \,\widetilde{\subseteq}\, \overline{\eth}^{\Re}. \end{aligned}$$

This verifies that $\overline{\eth}^{\Re}$ is a BSl-id over Υ. Therefore, $\eth$ is an upper RBSl-id over Υ. Similarly, the cases of BSr-ids and the BS-ids over Υ can be verified. ■

The converse statement of the Theorem 5.5.2 is invalid generally, as exhibited in the next example.

Example 5.5.3 *Let $\Upsilon = \{k, l, m, n\}$ represent a semigroup whose table of binary operation is given below.*

	k	l	m	n
k	k	k	k	n
l	k	l	k	n
m	k	k	m	n
n	n	n	n	n

Let $\hat{E} = \{e_1, e_2, e_3\}$ and $\Re$ be a cng-rel over Υ, defining cng-classes $\{k, l, n\}$ and $\{m\}$. We take a BSS $\eth = (\xi, \psi; \breve{A})$ over Υ with $\breve{A} = \{e_1, e_3\}$, defined below.

$\xi(e_1) = \{k, l\}$, $\xi(e_3) = \{l, n\}$,

$\psi(\neg e_1) = \{m\}$, $\psi(\neg e_3) = \{m\}$.

Note that $\eth$ is not a BSl-id over Υ because $k \in \xi(e_1)$, but $nk = n \notin \xi(e_1)$. The upper RBS-apx $\overline{\Re}(\eth) = (\overline{\xi}^{\Re}, \overline{\psi}^{\Re}; \breve{A})$ of $\eth$ under the relation $\Re$ is calculated as:

$\overline{\xi}^{\Re}(e_1) = \{k, l, n\}$, $\overline{\xi}^{\Re}(e_3) = \{k, l, n\}$,

$\overline{\psi}^{\Re}(\neg e_1) = \{m\}$, $\overline{\psi}^{\Re}(\neg e_3) = \{m\}$.

Simple calculations verify that $\overline{\Re}(\eth)$ is a BSl-id over Υ. So, $\eth$ is not a BSl-id over Υ, although, it is an upper RBSl-id over Υ.

Theorem 5.5.4 *Let $\Re$ be a complete cng-rel on Υ. Then, each BSl-id (BSr-id, BS-id) over Υ is a lower RBSl-id (RBSr-id, RBS-id) over Υ.*

Proof. Let $\eth$ be a BSl-id over Υ. Then, $\mathcal{U}_{\breve{A}} \widehat{*} \, \eth \,\widetilde{\subseteq}\, \eth$. From Theorem 4.3.3, we have $\underline{\mathcal{U}_{\breve{A}} \widehat{*} \, \eth}_{\Re} \widetilde{\subseteq} \, \underline{\eth}_{\Re}$. Now, from Theorems 4.3.1 and 5.4.4, we have

$$\begin{aligned} \mathcal{U}_{\breve{A}} \widehat{*} \, \underline{\eth}_{\Re} &= \underline{\mathcal{U}_{\breve{A}}}_{\Re} \widehat{*} \, \underline{\eth}_{\Re} \\ &\widetilde{\subseteq} \, \underline{\mathcal{U}_{\breve{A}} \widehat{*} \, \eth}_{\Re} \,\widetilde{\subseteq}\, \underline{\eth}_{\Re}. \end{aligned}$$

This verifies that $\underline{\eth}_{\Re}$ is a BSl-id over Υ. Therefore, $\eth$ is a lower RBSl-id over Υ. Similarly, the cases of BSr-ids and BS-ids over Υ can be verified. ■

The converse statement of the Theorem 5.5.4 is invalid generally, as exhibited in the next example.

Example 5.5.5 *Recall the semigroup* $\Upsilon = \{k, l, m, n\}$ *and the attribute set* $\hat{E}$ *as established in Example 5.5.3. Take a complete cng-rel* $\Re$ *on* Υ, *defining classes* $\{k, l, m\}$ *and* $\{n\}$. *We take a BSS* $\eth = (\xi, \psi; \breve{A})$ *over* Υ *with* $\breve{A} = \{e_2, e_3\}$, *defined below.*

$\xi(e_2) = \{l, m, n\}$, $\xi(e_3) = \{l, n\}$,

$\psi(\neg e_2) = \{k\}$, $\psi(\neg e_3) = \{m\}$.

Note that $\eth$ *is not a BSl-id over* Υ *because* $m \in \xi(e_2)$, *but* $lm = k \notin \xi(e_2)$. *The lower RBS-apx* $\underline{\Re}(\eth) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ *of* $\eth$ *under* $\Re$ *is calculated as:*

$\underline{\xi}_{\Re}(e_2) = \{n\} = \underline{\xi}_{\Re}(e_3)$,

$\underline{\psi}_{\Re}(\neg e_2) = \{k, l, m\} = \underline{\psi}_{\Re}(\neg e_3)$.

Simple calculations verify that $\underline{\Re}(\eth)$ *is a BSl-id over* Υ. *So,* $\eth$ *is not a BSl-id over* Υ, *although, it is a lower RBSl-id over* Υ.

Theorem 5.5.4 is invalid if the cng-rel $\Re$ is not complete. Next example is established to verify this fact.

Example 5.5.6 *Recall the semigroup* $\Upsilon = \{s, t, u, v\}$ *and the attribute set* $\breve{E}$ *as established in Example 5.4.12. Take a cng-rel* $\Re$ *on* Υ, *defining the cng-classes* $\{s\}, \{t, u\}$, *and* $\{v\}$. *Then,* $\Re$ *is not complete. We take a BSl-id* $\eth = (\xi, \psi; \breve{A})$ *over* Υ *with* $\breve{A} = \{e_4\}$, *defined below.*

$\xi(e_4) = \{u, v\}$, $\psi(\neg e_4) = \{s\}$.

The lower RBS-apx $\underline{\Re}(\eth) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ *of* $\eth$ *under* $\Re$ *is calculated as:*

$\underline{\xi}_{\Re}(e_4) = \{v\}$, $\underline{\psi}_{\Re}(\neg e_4) = \{s\}$.

We find that $\underline{\Re}(\eth)$ *is not BSl-id over* Υ, *because* $v \in \underline{\xi}_{\Re}(e_4)$, *but we have* $vv = u \notin \underline{\xi}_{\Re}(e_4)$. *So,* $\eth$ *is not a lower RBSl-id over* Υ.

Definition 5.5.7 *Take a cng-rel* $\Re$ *on* Υ *and let* $\eth \in BSS(\Upsilon)$. *Then,* $\eth$ *is a lower (or upper) RBSi-id over* Υ, *if* $\underline{\eth}_{\Re}$ *(or* $\overline{\eth}^{\Re}$*) is a BSi-id over* Υ.

A BSS $\eth$ over Υ, which is both, lower and upper RBSi-id over Υ, is called an RBSi-id over Υ.

Theorem 5.5.8 *Take a cng-rel* $\Re$ *on* Υ. *Then, each BSi-id over* Υ *is an upper RBSi-id over* Υ.

Proof. Let $\eth$ be a BSi-id over Υ. Then, $\mathcal{U}_{\breve{A}} \widehat{*}\, \eth \,\widehat{*}\, \mathcal{U}_{\breve{A}} \mathrel{\widetilde{\subseteq}} \eth$. From Theorem 4.3.3, we have $\overline{\mathcal{U}_{\breve{A}} \widehat{*}\, \eth \,\widehat{*}\, \mathcal{U}_{\breve{A}}}^{\Re} \mathrel{\widetilde{\subseteq}} \overline{\eth}^{\Re}$. Now, from Theorem 4.3.1 and Corollary 5.4.3, we have

$$\begin{aligned} \mathcal{U}_{\breve{A}} \widehat{*}\, \overline{\eth}^{\Re} \,\widehat{*}\, \mathcal{U}_{\breve{A}} &= \overline{\mathcal{U}_{\breve{A}}}^{\Re} \,\widehat{*}\, \overline{\eth}^{\Re} \,\widehat{*}\, \overline{\mathcal{U}_{\breve{A}}}^{\Re} \\ &\mathrel{\widetilde{\subseteq}} \overline{\mathcal{U}_{\breve{A}} \widehat{*}\, \eth \,\widehat{*}\, \mathcal{U}_{\breve{A}}}^{\Re} \mathrel{\widetilde{\subseteq}} \overline{\eth}^{\Re}. \end{aligned}$$

This verifies that $\overline{\eth}^{\Re}$ is a BSi-id over Υ. Therefore, $\eth$ is an upper RBSi-id over Υ. ■

The converse statement of the Theorem 5.5.8 is invalid generally, as exhibited in the next example.

Example 5.5.9 *Recall the semigroup $\Upsilon = \{k, l, m, n\}$, the attribute set $\hat{E}$ and the cng-rel $\Re$ on Υ, as established in Example 5.5.3. We take a BSS $\eth = (\xi, \psi; \breve{A})$ over Υ with $\breve{A} = \{e_2, e_3\}$, defined below.*

$\xi(e_2) = \{l, n\}$, $\xi(e_3) = \{k, l\}$,

$\psi(\neg e_2) = \{m\}$, $\psi(\neg e_3) = \{m\}$.

Note that $\eth$ is not a BSi-id over Υ as $l \in \xi(e_2)$, but $mlm = k \notin \xi(e_2)$. The upper RBS-apx $\overline{\Re}(\eth) = (\overline{\xi}^{\Re}, \overline{\psi}^{\Re}; \breve{A})$ of $\eth$ under the relation $\Re$ is calculated as:

$\overline{\xi}^{\Re}(e_2) = \{k, l, n\} = \overline{\xi}^{\Re}(e_3)$,

$\overline{\psi}^{\Re}(\neg e_2) = \{m\} = \overline{\psi}^{\Re}(\neg e_3)$.

Simple calculations verify that $\overline{\Re}(\eth)$ is a BSi-id over Υ. So, $\eth$ is not a BSi-id over Υ, although, it is an upper RBSi-id over Υ.

Theorem 5.5.10 *Let $\Re$ be a complete cng-rel on Υ. Then, each BSi-id over Υ is a lower RBSi-id over Υ.*

Proof. Let $\eth$ be a BSi-id over Υ. Then, $\mathcal{U}_{\breve{A}} \widehat{*}\, \eth \,\widehat{*}\, \mathcal{U}_{\breve{A}} \widetilde{\subseteq} \eth$. From Theorem 4.3.3, we have $\underline{\mathcal{U}_{\breve{A}} \widehat{*}\, \eth \,\widehat{*}\, \mathcal{U}_{\breve{A}}}_{\Re} \widetilde{\subseteq} \underline{\eth}_{\Re}$. Now, from Theorem 4.3.1 and Corollary 5.4.5, we have

$$\begin{aligned} \mathcal{U}_{\breve{A}} \widehat{*}\, \underline{\eth}_{\Re} \widehat{*}\, \mathcal{U}_{\breve{A}} &= \underline{\mathcal{U}_{\breve{A}}}_{\Re} \widehat{*}\, \underline{\eth}_{\Re} \widehat{*}\, \underline{\mathcal{U}_{\breve{A}}}_{\Re} \\ &\widetilde{\subseteq} \underline{\mathcal{U}_{\breve{A}} \widehat{*}\, \eth \,\widehat{*}\, \mathcal{U}_{\breve{A}}}_{\Re} \widetilde{\subseteq} \underline{\eth}_{\Re}. \end{aligned}$$

This verifies that $\underline{\eth}_{\Re}$ is a BSi-id over Υ. Therefore, $\eth$ is a lower RBSi-id over Υ. ■

The converse statement of the Theorem 5.5.10 is invalid generally, as exhibited in the next example.

Example 5.5.11 *Recall the semigroup $\Upsilon = \{k, l, m, n\}$, the set $\hat{E}$ and the complete cng-rel $\Re$ over Υ, as established in Example 5.5.5. We take a BSS $\eth = (\xi, \psi; \breve{A})$ over Υ with $\breve{A} = \{e_1, e_2\}$ defined below.*

$\xi(e_1) = \{m, n\}$, $\xi(e_2) = \{l, n\}$,

$\psi(\neg e_1) = \{l\}$, $\psi(\neg e_2) = \{k\}$.

Note that $\eth$ is not a BSi-id over Υ because $m \in \xi(e_1)$, but $kml = k \notin \xi(e_1)$. The lower RBS-apx $\underline{\Re}(\eth) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ of $\eth$ under $\Re$ is calculated as:

$\underline{\xi}_{\Re}(e_1) = \{n\} = \underline{\xi}_{\Re}(e_2)$,

$\underline{\psi}_{\Re}(\neg e_1) = \{k, l, m\} = \underline{\psi}_{\Re}(\neg e_2)$.

Simple calculations verify that $\underline{\Re}(\eth)$ is a BSi-id over Υ. So, $\eth$ is not a BSi-id over Υ, although, it is a lower RBSi-id over Υ.

Theorem 5.5.10 is invalid if the cng-rel $\Re$ is not complete. Next example is established to verify this fact.

Example 5.5.12 *Recall the semigroup* $\Upsilon = \{k, l, m, n\}$ *and the attribute set* $\breve{E}$ *as established in Example 5.5.3. Take the cng-rel* $\Re$ *on* Υ *defining the classes* $\{k, m, n\}$ *and* $\{l\}$*. Then,* $\Re$ *is not complete. We take a BSi-id* $\eth = (\xi, \psi; \breve{A})$ *over* Υ *with* $\breve{A} = \{e_2, e_3\}$*, defined below.*

$\xi(e_2) = \{k, l, n\}$, $\xi(e_3) = \{n\}$,

$\psi(\neg e_2) = \psi(\neg e_3) = \{m\}$.

The lower RBS-apx $\underline{\Re}(\eth) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ *of* $\eth$ *under* $\Re$ *is calculated as:*

$\underline{\xi}_{\Re}(e_2) = \{l\}$, $\underline{\xi}_{\Re}(e_3) = \phi$,

$\underline{\psi}_{\Re}(\neg e_2) = \{k, m, n\} = \underline{\psi}_{\Re}(\neg e_3)$.

We find that $\underline{\Re}(\eth)$ *is not BSi-id over* Υ*, because* $l \in \underline{\xi}_{\Re}(e_2)$*, but we have* $klm = k \notin \underline{\xi}_{\Re}(e_2)$*. So,* $\eth$ *is not a lower RBSi-id over* Υ*.*

Definition 5.5.13 *Take a cng-rel* $\Re$ *on* Υ *and let* $\eth \in BSS(\Upsilon)$*. Then,* $\eth$ *is a lower (or upper) RBSb-id over* Υ*, if* $\underline{\eth}_{\Re}$ *(or* $\overline{\eth}^{\Re}$*) is a BSb-id over* Υ*.*

A BSS $\eth$ over Υ, which is both, lower and upper RBSb-id over Υ, is called an RBSb-id over Υ.

Theorem 5.5.14 *Take a cng-rel* $\Re$ *on* Υ*. Then, each BSb-id over* Υ *is an upper RBSb-id over* Υ*.*

Proof. Let $\eth \in BSS(\Upsilon)$ be a BSb-id over Υ. Then, $\eth$ is BS subsemigroup over Υ and $\eth \,\widehat{*}\, \mathcal{U}_{\breve{A}} \widehat{*}\, \eth \mathrel{\widetilde{\subseteq}} \eth$. So, $\overline{\eth}^{\Re}$ is a BS subsemigroup over Υ from Theorem 5.4.9 and $\overline{\eth \,\widehat{*}\, \mathcal{U}_{\breve{A}} \widehat{*}\, \eth}^{\Re} \mathrel{\widetilde{\subseteq}} \overline{\eth}^{\Re}$ from Theorem 4.3.3. Now, from Theorem 4.3.1 and Corollary 5.4.3, we have

$$\begin{aligned} \overline{\eth}^{\Re} \,\widehat{*}\, \mathcal{U}_{\breve{A}} \widehat{*}\, \overline{\eth}^{\Re} &= \overline{\eth}^{\Re} \,\widehat{*}\, \overline{\mathcal{U}_{\breve{A}}}^{\Re} \widehat{*}\, \overline{\eth}^{\Re} \\ &\mathrel{\widetilde{\subseteq}} \overline{\eth \,\widehat{*}\, \mathcal{U}_{\breve{A}} \widehat{*}\, \eth}^{\Re} \mathrel{\widetilde{\subseteq}} \overline{\eth}^{\Re}. \end{aligned}$$

This verifies that $\overline{\eth}^{\Re}$ is a BSb-id over Υ. Therefore, $\eth$ is an upper RBSb-id over Υ. ■

The converse statement of the Theorem 5.5.14 is invalid generally, as exhibited in the next example.

Example 5.5.15 *Recall the semigroup* $\Upsilon = \{k, l, m, n\}$*, the attribute set* $\hat{E}$ *and the cng-rel* $\Re$ *over* Υ *as established in Example 5.5.3. We take a BSS* $\eth = (\xi, \psi; \breve{A})$ *over* Υ *with* $\breve{A} = \{e_1, e_2\}$*, defined below.*

$\xi(e_1) = \{l, k\}$, $\xi(e_2) = \{l, n\}$,
$\psi(\neg e_1) = \{m\} = \psi(\neg e_2)$.

Note that $\eth$ *is not a BSb-id over* Υ *because* $l \in \xi(e_1)$, *but* $lnl = n \notin \xi(e_1)$. *The upper RBS-apx* $\overline{\Re}(\eth) = (\overline{\xi}^{\Re}, \overline{\psi}^{\Re}; \breve{A})$ *of* $\eth$ *under the relation* $\Re$ *is calculated as:*
$\overline{\xi}^{\Re}(e_1) = \{l, k, n\} = \overline{\xi}^{\Re}(e_2)$,
$\overline{\psi}^{\Re}(\neg e_1) = \{m\} = \overline{\psi}^{\Re}(\neg e_2)$.

Simple calculations verify that $\overline{\Re}(\eth)$ *is a BSb-id over* Υ. *So,* $\eth$ *is not a BSb-id over* Υ, *although, it is an upper RBSb-id over* Υ.

Theorem 5.5.16 *Let* $\Re$ *be a complete cng-rel on* Υ. *Then, each BSb-id over* Υ *is a lower RBSb-id over* Υ.

Proof. Let $\eth$ be a BSb-id over Υ. Then, $\eth$ is BS subsemigroup over Υ and $\eth \widehat{*} \mathcal{U}_{\breve{A}} \widehat{*} \eth \,\widetilde{\subseteq}\, \eth$. So, $\underline{\eth}_{\Re}$ is a BS subsemigroup over Υ from Theorem 5.4.11 and $\underline{\eth \widehat{*} \mathcal{U}_{\breve{A}} \widehat{*} \eth}_{\Re} \,\widetilde{\subseteq}\, \underline{\eth}_{\Re}$ from Theorem 4.3.3. Now, from Theorem 4.3.1 and Corollary 5.4.5, we have

$$\begin{aligned} \underline{\eth}_{\Re} \widehat{*} \mathcal{U}_{\breve{A}} \widehat{*} \underline{\eth}_{\Re} &= \underline{\eth}_{\Re} \widehat{*} \underline{\mathcal{U}_{\breve{A}}}_{\Re} \widehat{*} \underline{\eth}_{\Re} \\ &\widetilde{\subseteq} \underline{\eth \widehat{*} \mathcal{U}_{\breve{A}} \widehat{*} \eth}_{\Re} \,\widetilde{\subseteq}\, \underline{\eth}_{\Re}. \end{aligned}$$

This verifies that $\underline{\eth}_{\Re}$ is a BSb-id over Υ. Therefore, $\eth$ is a lower RBSb-id over Υ. ∎

The converse statement of the Theorem 5.5.16 is invalid generally, as exhibited in the next example.

Example 5.5.17 *Recall the semigroup* $\Upsilon = \{s, t, u, v\}$, *the attribute set* $\hat{E}$ *and the complete cng-rel* $\Re$ *over* Υ, *as established in Example 5.4.12. We take a BSS* $\eth = (\xi, \psi; \breve{A})$ *over* Υ *with* $\breve{A} = \{e_5\}$, *defined below.*
$\xi(e_5) = \{s, u\}$, $\psi(\neg e_5) = \{t, v\}$.

Note that $\eth$ *is not a BSb-id over* Υ *because* $s, u \in \xi(e_5)$, *but* $svu = v \notin \xi(e_5)$. *The lower RBS-apx* $\underline{\Re}(\eth) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ *of* $\eth$ *under* $\Re$ *is calculated as:*
$\underline{\xi}_{\Re}(e_5) = \{s\}$, $\underline{\psi}_{\Re}(\neg e_5) = \{t, u, v\}$.

Simple calculations verify that $\underline{\Re}(\eth)$ *is a BSb-id over* Υ. *So,* $\eth$ *is not a BSb-id over* Υ, *although, it is a lower RBSb-id over* Υ.

Theorem 5.5.16 is invalid if the cng-rel $\Re$ is not complete. Next example is established to verify this fact.

Example 5.5.18 *Recall the semigroup* $\Upsilon = \{k, l, m, n\}$, *the attribute set* $\breve{E}$ *as taken in Example 5.5.3. Take the cng-rel* $\Re$ *on* Υ *defining the classes* $\{k, m, n\}$ *and* $\{l\}$.

Then, $\Re$ *is not complete. We take a BSb-id* $\eth = (\xi, \psi; \breve{A})$ *over* Υ *with* $\breve{A} = \{e_1, e_3\}$*, defined below.*

$\xi(e_1) = \{k, l, n\}$, $\xi(e_3) = \{m\}$,

$\psi(\neg e_1) = \{m\}$, $\psi(\neg e_3) = \{k, l, n\}$.

The lower RBS-apx $\underline{\Re}(\eth) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ *of* $\eth$ *under* $\Re$ *is calculated as:*

$\underline{\xi}_{\Re}(e_1) = \{l\}$, $\underline{\xi}_{\Re}(e_3) = \phi$,

$\underline{\psi}_{\Re}(\neg e_1) = \{k, m, n\}$, $\underline{\psi}_{\Re}(\neg e_3) = \Upsilon$.

$\underline{\Re}(\eth)$ *is not BSb-id over* Υ*, because* $l \in \underline{\xi}_{\Re}(e_1)$*, but* $lml = k \notin \underline{\xi}_{\Re}(e_2)$*. So,* $\eth$ *is not a lower RBSb-id over* Υ*.*

Chapter 6

Rough fuzzy bipolar soft sets

6.1 Introduction

Fuzzy bipolar soft sets (FBSSs) are built to manage the fuzziness, as well as bipolarity of the data with respect to multiple characteristics at a single platform. These sets are presented by Naz and Shabir [51]. In this chapter, we aim to explore the roughness in FBSSs and to initiate the notion of the rough FBSSs (RFBSSs) over the universe U of discourse. The RFBSSs are defined with the help of the lower and upper RFBS-apxes of the FBSSs in the P-apx space. Roughness in different soft and fuzzy structures is defined by many researchers. Rough FSs are studied in [18, 24, 29]. Malik and Shabir [45] studied rough BFSs. Rough soft sets are discussed in [7, 29]. We have presented the RFBSSs in Section 6.2. Some characterizations of the RFBSSs are studied in Section 6.3 and some similarity relations on the set containing FBSSs over U are defined in Section 6.4 with the help of their RFBS-apxes. Another exotic feature of this chapter is the uncertainty measures, such as accuracy measure and roughness measure for the RFBS-apxes of the FBSSs. Earlier in 1996, Banarjee and Pal [15] provided a roughness measure for the FSs using α-cuts on the FSs. The roughness measures for the FBSSs using the approach of Banarjee and Pal, are defined and discussed in Section 6.5. These are the measures which provide an estimation to investigate how accurate are the RFBS-apxes of the FBSSs.

As mentioned earlier, the FBSSs are built to handle fuzziness, as well as bipolarity of the data with respect to multiple attributes. Due to this quality, these sets have great practicality and applicability in decision making techniques, which is an important application of the FBSSs. Applicability of the rough sets, FSs and soft sets in decision analysis are discussed in [28, 43, 45, 46, 47, 55, 61, 66]. The last section of this chapter presents an application of the RFBS-apxes of the FBSSs. An algorithm is also designed for that application, supported by a suitable example to illustrate the steps of the algorithm.

6.2 Rough fuzzy bipolar soft sets

Roughness in the FBSSs using an eqv-rel $\Re$ on the universe U ($\neq \phi$) possessing an attribute set $\breve{E}$ is discussed in this section by defining the upper and lower RFBS-apxes of FBSSs in the P-apx space $(U, \Re)$.

Definition 6.2.1 *Take a P-apx space* $(U, \Re)$ *and* $\omega = (\xi, \psi; \breve{A}) \in FBSS(U)$. *The upper and lower RFBS-apxes of* ω *in* $(U, \Re)$ *are the FBSSs* $\overline{\Re}(\omega) = (\overline{\xi}^{\Re}, \overline{\psi}^{\Re}; \breve{A})$ *and* $\underline{\Re}(\omega) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$, *respectively, where* $\underline{\xi}_{\Re}(e)$, $\overline{\xi}^{\Re}(e)$, $\underline{\psi}_{\Re}(\neg e)$, $\overline{\psi}^{\Re}(\neg e)$ *are FSs in* U, *defined by*

$$\begin{aligned}
\underline{\xi}_{\Re}(e)(u) &= \underline{\xi(e)}_{\Re}(u) = \bigwedge_{y \in [u]_{\Re}} \xi(e)(y), \\
\overline{\xi}^{\Re}(e)(u) &= \overline{\xi(e)}^{\Re}(u) = \bigvee_{y \in [u]_{\Re}} \xi(e)(y), \\
\underline{\psi}_{\Re}(\neg e)(u) &= \overline{\psi(\neg e)}^{\Re}(u) = \bigvee_{y \in [u]_{\Re}} \psi(\neg e)(y), \\
\overline{\psi}^{\Re}(\neg e)(u) &= \underline{\psi(\neg e)}_{\Re}(u) = \bigwedge_{y \in [u]_{\Re}} \psi(\neg e)(y)
\end{aligned}$$

for each $e \in \breve{A}$ *and for each* $u \in U$. *If* $\underline{\Re}(\omega) = \overline{\Re}(\omega)$, *then,* ω *is said to be* $\Re-$*definable; otherwise,* ω *is an RFBSS over* U.

The interpretation of these RFBS-apxes of ω, that is, the information about an object u of U, depicted by the above defined FSs is as follows.

- $\underline{\xi}_{\Re}(e)(u)$ indicates the degree to which u definitely has the property e.
- $\overline{\xi}^{\Re}(e)(u)$ indicates the degree to which u probably has the property e.
- $\underline{\psi}_{\Re}(\neg e)(u)$ indicates the degree to which u probably has the property opposite to e.
- $\overline{\psi}^{\Re}(\neg e)(u)$ indicates the degree to which u definitely has the property opposite to e.

Definition 6.2.2 *Let* $\Re$ *be an eqv-rel on the universe* U. *an FBSS* $\omega = (\xi, \psi; \breve{A})$ *over* U *is referred to be classwise constant under* $\Re$ *if* $\xi(e)(u) = \xi(e)(u')$ *and* $\psi(\neg e)(u) = \psi(\neg e)(u')$, *whenever,* $u' \in [u]_{\Re}$ *for* $u, u' \in U$.

It is easy to note that the relative null FBSS $\Phi_{\breve{A}}$, the relative whole FBSS $\tilde{U}_{\breve{A}}$ and the constant FBSS are all classwise constant under each eqv-rel $\Re$ on U. The subsequent example presents a classwise constant FBSS. The RFBS-apxes of an arbitrary FBSS over U are also explained in this example.

Example 6.2.3 *Let $U = \{q_1, q_2, q_3, q_4, q_5\}$ contain five houses and $\breve{E} = \{e_1 =$costly, $e_2 =$attractive, $e_3 =$wooden, $e_4 =$in natural surroundings, $e_5 =$properly maintained$\}$ be a set of attributes for U. Let the counter set of $\breve{E}$ be $\neg\breve{E} = \{\neg e_1 =$cheap, $\neg e_2 =$dull, $\neg e_3 =$not wooden, $\neg e_4 =$in urban area, $\neg e_5 =$not maintained$\}$. Let the house q_1 be in some locality A, the houses q_2 and q_3 be in a locality B and the houses q_4 and q_5 be in a locality C. We define a binary relation $\Re$ on U, such that, two houses are related in $\Re$ if and only if they are in same locality. Then, $\Re$ is an eqv-rel on U, defining the eqv-classes $\{q_1\}, \{q_2, q_3\}$ and $\{q_4, q_5\}$. We construct an FBSS $\omega = (\xi, \psi; \breve{A})$ over U, describing the assessment of Mr. A, who intends to buy a house, preferring the attribute set $\breve{A} = \{e_2, e_5\}$. Let the FBSS ω be defined as follows.*

$\xi(e_2) = \{q_1/0.8,\ q_2/0.6,\ q_3/0.6,\ q_4/0.5,\ q_5/0.5\}$,
$\psi(\neg e_2) = \{q_1/0.1,\ q_2/0.3,\ q_3/0.3,\ q_4/0.4,\ q_5/0.4\}$,
$\xi(e_5) = \{q_1/0.5,\ q_2/0.4,\ q_3/0.4,\ q_4/0.6,\ q_5/0.6\}$,
$\psi(\neg e_5) = \{q_1/0.5,\ q_2/0.4,\ q_3/0.4,\ q_4/0.4,\ q_5/0.4\}$.

Then, ω is a classwise constant FBSS over U. Now, consider another FBSS $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$ over U with $\breve{A}_1 = \{e_1, e_2, e_3\}$, which demonstrates the assessment of some other person about the houses of U. Let ω_1 be defined as given below.

$\xi_1(e_1) = \{q_1/0.7,\ q_2/0.6,\ q_3/0.8,\ q_4/0.5,\ q_5/0.6\}$,
$\psi_1(\neg e_1) = \{q_1/0.2,\ q_2/0.3,\ q_3/0.1,\ q_4/0.5,\ q_5/0.3\}$,
$\xi_1(e_2) = \{q_1/0.8,\ q_2/0.7,\ q_3/0.8,\ q_4/0.6,\ q_5/0.6\}$,
$\psi_1(\neg e_2) = \{q_1/0.1,\ q_2/0.1,\ q_3/0.2,\ q_4/0.2,\ q_5/0.3\}$,
$\xi_1(e_3) = \{q_1/0.4,\ q_2/0.6,\ q_3/0.4,\ q_4/0.6,\ q_5/0.5\}$,
$\psi_1(\neg e_3) = \{q_1/0.5,\ q_2/0.2,\ q_3/0.5,\ q_4/0.4,\ q_5/0.5\}$.

The lower RFBS-apx $\underline{\Re}(\omega_1) = (\underline{\xi_1}_{\Re}, \underline{\psi_1}_{\Re}; \breve{A}_1)$ of ω_1 is calculated as below.

$\underline{\xi_1}_{\Re}(e_1) = \{q_1/\bigwedge_{y\in[q_1]} \xi_1(e_1)(y), q_2/\bigwedge_{y\in[q_2]} \xi_1(e_1)(y), ..., q_5/\bigwedge_{y\in[q_5]} \xi_1(e_1)(y)\}$
$= \{q_1/\bigwedge_{y=q_1} \xi_1(e_1)(y), q_2/\bigwedge_{y=q_2,q_3} \xi_1(e_1)(y), ..., q_5/\bigwedge_{y=q_4,q_5} \xi_1(e_1)(y)\}$
$= \{q_1/0.7, q_2/(0.6 \wedge 0.8), q_3/(0.6 \wedge 0.8), q_4/(0.5 \wedge 0.6), q_5/(0.5 \wedge 0.6)\}$
$= \{q_1/0.7, q_2/0.6, q_3/0.6, q_4/0.5, q_5/0.5\}$.

In the same way, the following FSs are calculated for each $e \in \breve{A}_1$.

$\underline{\xi_1}_{\Re}(e_2) = \{q_1/0.8, q_2/0.7, q_3/0.7, q_4/0.6, q_5/0.6\}$,
$\underline{\xi_1}_{\Re}(e_3) = \{q_1/0.4, q_2/0.4, q_3/0.4, q_4/0.5, q_5/0.5\}$,
$\underline{\psi_1}_{\Re}(\neg e_1) = \{q_1/0.2, q_2/0.3, q_3/0.3, q_4/0.5, q_5/0.5\}$,
$\underline{\psi_1}_{\Re}(\neg e_2) = \{q_1/0.1, q_2/0.2, q_3/0.2, q_4/0.3, q_5/0.3\}$,

$\underline{\psi_1}_{\Re}(\neg e_3) = \{q_1/0.5, q_2/0.5, q_3/0.5, q_4/0.5, q_5/0.5\}$.

For the upper RFBS-apx $\overline{\Re}(\omega_1) = (\overline{\xi_1}^{\Re}, \overline{\psi_1}^{\Re}; \breve{A}_1)$ *of* ω_1*, the FSs* $\overline{\xi_1}^{\Re}(e_i)$ *and* $\overline{\psi_1}^{\Re}(\neg e_i)$ *are calculated for* $i = 1, 2, 3$ *as below.*

$\overline{\xi_1}^{\Re}(e_1) = \{q_1/0.7, q_2/0.8, q_3/0.8, q_4/0.6, q_5/0.6\}$,
$\overline{\xi_1}^{\Re}(e_2) = \{q_1/0.8, q_2/0.8, q_3/0.8, q_4/0.6, q_5/0.6\}$,
$\overline{\xi_1}^{\Re}(e_3) = \{q_1/0.4, q_2/0.6, q_3/0.6, q_4/0.6, q_5/0.6\}$,
$\overline{\psi_1}^{\Re}(\neg e_1) = \{q_1/0.2, q_2/0.1, q_3/0.1, q_4/0.3, q_5/0.3\}$,
$\overline{\psi_1}^{\Re}(\neg e_2) = \{q_1/0.1, q_2/0.1, q_3/0.1, q_4/0.2, q_5/0.2\}$,
$\overline{\psi_1}^{\Re}(\neg e_3) = \{q_1/0.5, q_2/0.2, q_3/0.2, q_4/0.4, q_5/0.4\}$.

By comparing belongingness values of the above FSs, one can easily see that $\underline{\xi_1}_{\Re}(e) \subseteq \xi_1(e) \subseteq \overline{\xi_1}^{\Re}(e)$ *and* $\underline{\psi_1}_{\Re}(e) \supseteq \psi_1(e) \supseteq \overline{\psi_1}^{\Re}(e)$ *for each* $e \in \breve{A}_1$*. This verifies* $\underline{\Re}(\omega_1) \,\widetilde{\subseteq}\, \omega_1 \,\widetilde{\subseteq}\, \overline{\Re}(\omega_1)$*, by using Definition 1.6.3.*

6.3 Characterizations of rough fuzzy bipolar soft sets

Lemma 6.3.1 *Let* $\Re$ *be an eqv-rel defined on* U*. Then, every classwise constant FBSS over* U *is* $\Re-$*definable.*

Proof. Take a classwise constant FBSS $\omega = (\xi, \psi; \breve{A}) \in FBSS(U)$. The lower and upper RFBS-apxes of ω under $\Re$ are $\underline{\Re}(\omega) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ and $\overline{\Re}(\omega) = (\overline{\xi}^{\Re}, \overline{\psi}^{\Re}; \breve{A})$, respectively. For $e \in \breve{A}$ and $u \in U$, let $\xi(e)(u) = c_{e,u}$, where $c_{e,u} \in [0, 1]$ is a constant. Then, for any $e \in \breve{A}$ and $u \in U$, we have

$$\underline{\xi}_{\Re}(e)(u) = \bigwedge_{y \in [u]_{\Re}} \xi(e)(y) = \bigwedge_{y \in [u]_{\Re}} c_{e,u} = c_{e,u}$$

and

$$\overline{\xi}^{\Re}(e)(u) = \bigvee_{y \in [u]_{\Re}} \xi(e)(y) = \bigvee_{y \in [u]_{\Re}} c_{e,u} = c_{e,u}.$$

Which shows that, $\underline{\xi}_{\Re}(e)(u) = \overline{\xi}^{\Re}(e)(u)$ for each $e \in \breve{A}$ and $u \in U$. Similarly, we have $\underline{\psi}_{\Re}(\neg e)(u) = \overline{\psi}^{\Re}(\neg e)(u)$ for each $\neg e \in \neg\breve{A}$ and $u \in U$. This proves, that, $\underline{\Re}(\omega) = \overline{\Re}(\omega)$. Thus, ω is $\Re-$definable. ■

Corollary 6.3.2 *Let* $\Re$ *be an arbitrary eqv-rel defined on a non-empty finite set* U*. Then, every constant FBSS over* U *is* $\Re-$*definable.*

Proof. Proof can be deduced from Lemma 6.3.1, as every constant FBSS over U can be considered as a classwise constant FBSS over U, under any eqv-rel $\Re$ defined on U. ■

Theorem 6.3.3 *Take a P-apx space* $(U, \Re)$ *and let* $\omega = (\xi, \psi; \breve{A}) \in FBSS(U)$. *Then, the subsequent asservations are true.*

1. $\underline{\Re}(\omega) \tilde{\subseteq} \omega \tilde{\subseteq} \overline{\Re}(\omega)$,
2. $\underline{\Re}(\Phi_{\breve{A}}) = \Phi_{\breve{A}} = \overline{\Re}(\Phi_{\breve{A}})$,
3. $\underline{\Re}(\tilde{U}_{\breve{A}}) = \tilde{U}_{\breve{A}} = \overline{\Re}(\tilde{U}_{\breve{A}})$,
4. $\underline{\Re}(\underline{\Re}(\omega)) = \underline{\Re}(\omega) = \overline{\Re}(\underline{\Re}(\omega))$,
5. $\underline{\Re}(\overline{\Re}(\omega)) = \overline{\Re}(\omega) = \overline{\Re}(\overline{\Re}(\omega))$,
6. $\overline{\Re}(\omega^c) = (\underline{\Re}(\omega))^c$,
7. $\underline{\Re}(\omega^c) = \left(\overline{\Re}(\omega)\right)^c$.

Proof. (1) Obvious by the Definition 6.2.1.

(2) The relative null FBSS $\Phi_{\breve{A}} = (\Phi, \tilde{U}; \breve{A})$ over U is constant. Hence $\Phi_{\breve{A}}$ is $\Re-$definable, by Corollary 6.3.2. That is,

$$\underline{\Re}(\Phi_{\breve{A}}) = \Phi_{\breve{A}} = \overline{\Re}(\Phi_{\breve{A}}).$$

(3) The relative whole FBSS $\tilde{U}_{\breve{A}} = (\tilde{U}, \Phi; \breve{A})$ over U is constant. Hence $\tilde{U}_{\breve{A}}$ is $\Re-$definable. That is,

$$\underline{\Re}(\tilde{U}_{\breve{A}}) = \tilde{U}_{\breve{A}} = \overline{\Re}(\tilde{U}_{\breve{A}}).$$

(4) The lower RFBS-apx of the FBSS $\omega = (\xi, \psi; \breve{A})$ is symbolized by $\underline{\Re}(\omega) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$. Since, $\underline{\Re}(\omega)$ is a classwise constant FBSS, so, $\underline{\Re}(\omega)$ is $\Re-$definable by Lemma 6.3.1. Which proves that,

$$\underline{\Re}(\underline{\Re}(\omega)) = \underline{\Re}(\omega) = \overline{\Re}(\underline{\Re}(\omega)).$$

(5) Analogous to the proof of (4).

(6) By using Definitions 1.6.7 and 6.2.1, the FBSSs $\overline{\Re}(\omega^c)$ and $(\underline{\Re}(\omega))^c$ are described as $\overline{\Re}(\omega^c) = (\overline{\xi^c}^{\Re}, \overline{\psi^c}^{\Re}; \breve{A})$ and $(\underline{\Re}(\omega))^c = ((\underline{\xi}_{\Re})^c, (\underline{\psi}_{\Re})^c; \breve{A})$. Notice that,

$$\begin{aligned} \overline{\xi^c}^{\Re}(e)(u) &= \bigvee_{y \in [u]_\Re} \xi^c(e)(y) = \bigvee_{y \in [u]_\Re} \psi(\neg e)(y) \\ &= \underline{\psi}_{\Re}(\neg e)(u) = (\underline{\xi}_{\Re})^c(e)(u) \end{aligned}$$

and

$$\begin{aligned} \overline{\psi^c}^{\Re}(\neg e)(u) &= \bigwedge_{y \in [u]_\Re} \psi^c(\neg e)(y) = \bigwedge_{y \in [u]_\Re} \xi(e)(y) \\ &= \underline{\xi}_{\Re}(e)(u) = (\underline{\psi}_{\Re})^c(\neg e)(u) \end{aligned}$$

hold for each $e \in \breve{A}$ and for each $u \in U$. Which immediately gives $\overline{\Re}(\omega^c) = (\underline{\Re}(\omega))^c$.

(7) Analogous to the proof of (6). ■

Lemma 6.3.4 *Take a P-apx space $(U, \Re)$. Then, the subsequent asservations are true for any $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$, $\omega_2 = (\xi_2, \psi_2; \breve{A}_2) \in FBSS(U)$ and for any $e \in \breve{A}_1 \cup \breve{A}_2$.*

1. $(\overline{\xi_1}^{\Re} \widetilde{\cup}_\varepsilon \overline{\xi_2}^{\Re})(e) = (\overline{\xi_1 \widetilde{\cup}_\varepsilon \xi_2}^{\Re})(e)$,
2. $(\overline{\xi_1}^{\Re} \widetilde{\cap}_\varepsilon \overline{\xi_2}^{\Re})(e) \supseteq (\overline{\xi_1 \widetilde{\cap}_\varepsilon \xi_2}^{\Re})(e)$,
3. $(\underline{\xi_1}_{\Re} \widetilde{\cup}_\varepsilon \underline{\xi_2}_{\Re})(e) \subseteq (\underline{\xi_1 \widetilde{\cup}_\varepsilon \xi_2}_{\Re})(e)$,
4. $(\underline{\xi_1}_{\Re} \widetilde{\cap}_\varepsilon \underline{\xi_2}_{\Re})(e) = (\underline{\xi_1 \widetilde{\cap}_\varepsilon \xi_2}_{\Re})(e)$,
5. $(\overline{\psi_1}^{\Re} \widetilde{\cup}_\varepsilon \overline{\psi_2}^{\Re})(\neg e) \subseteq (\overline{\psi_1 \widetilde{\cup}_\varepsilon \psi_2}^{\Re})(\neg e)$,
6. $(\overline{\psi_1}^{\Re} \widetilde{\cap}_\varepsilon \overline{\psi_2}^{\Re})(\neg e) = (\overline{\psi_1 \widetilde{\cap}_\varepsilon \psi_2}^{\Re})(\neg e)$,
7. $(\underline{\psi_1}_{\Re} \widetilde{\cup}_\varepsilon \underline{\psi_2}_{\Re})(\neg e) = (\underline{\psi_1 \widetilde{\cup}_\varepsilon \psi_2}_{\Re})(\neg e)$,
8. $(\underline{\psi_1}_{\Re} \widetilde{\cap}_\varepsilon \underline{\psi_2}_{\Re})(\neg e) \supseteq (\underline{\psi_1 \widetilde{\cap}_\varepsilon \psi_2}_{\Re})(\neg e)$.

Proof. (1) Let $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$, $\omega_2 = (\xi_2, \psi_2; \breve{A}_2) \in FBSS(U)$. The case is obvious for $e \in \breve{A}_1 - \breve{A}_2$ or $e \in \breve{A}_2 - \breve{A}_1$. For $e \in \breve{A}_1 \cap \breve{A}_2$ and $u \in U$, we have,

$$\begin{aligned}
(\overline{\xi_1}^{\Re} \widetilde{\cup}_\varepsilon \overline{\xi_2}^{\Re})(e)(u) &= \overline{\xi_1}^{\Re}(e)(u) \vee \overline{\xi_2}^{\Re}(e)(u) \\
&= (\bigvee_{y \in [u]_\Re} \xi_1(e)(y)) \vee (\bigvee_{y \in [u]_\Re} \xi_2(e)(y)) \\
&= \bigvee_{y \in [u]_\Re} (\xi_1(e)(y) \vee \xi_2(e)(y)) \\
&= \bigvee_{y \in [u]_\Re} (\xi_1 \widetilde{\cup}_\varepsilon \xi_2)(e)(y) \\
&= (\overline{\xi_1 \widetilde{\cup}_\varepsilon \xi_2}^{\Re})(e)(u).
\end{aligned}$$

Which shows that,

$$(\overline{\xi_1}^{\Re} \widetilde{\cup}_\varepsilon \overline{\xi_2}^{\Re})(e) = (\overline{\xi_1 \widetilde{\cup}_\varepsilon \xi_2}^{\Re})(e).$$

(2) Again, for $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$, $\omega_2 = (\xi_2, \psi_2; \breve{A}_2) \in FBSS(U)$, the case is obvious when $e \in (\breve{A}_1 - \breve{A}_2) \cup (\breve{A}_2 - \breve{A}_1)$. For $e \in \breve{A}_1 \cap \breve{A}_2$ and $u \in U$, we have,

$$\begin{aligned}
(\overline{\xi_1}^{\Re} \widetilde{\cap}_\varepsilon \overline{\xi_2}^{\Re})(e)(u) &= \overline{\xi_1}^{\Re}(e)(u) \wedge \overline{\xi_2}^{\Re}(e)(u) \\
&= (\bigvee_{y \in [u]_\Re} \xi_1(e)(y)) \wedge (\bigvee_{y \in [u]_\Re} \xi_2(e)(y)) \\
&\geq \bigvee_{y \in [u]_\Re} (\xi_1(e)(y) \wedge \xi_2(e)(y)) \\
&= \bigvee_{y \in [u]_\Re} (\xi_1 \widetilde{\cap}_\varepsilon \xi_2)(e)(y) \\
&= (\overline{\xi_1 \widetilde{\cap}_\varepsilon \xi_2}^{\Re})(e)(u).
\end{aligned}$$

Which shows that,

$$(\overline{\xi_1}^{\Re}\widetilde{\sqcap}_\varepsilon\overline{\xi_2}^{\Re})(e) \supseteq (\overline{\xi_1\widetilde{\sqcap}_\varepsilon\xi_2}^{\Re})(e).$$

(3-8) Analogous to the proof of (1) and (2). ■

Theorem 6.3.5 *Take a P-apx space $(U, \Re)$. Then, the subsequent asservations are true for any $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$, $\omega_2 = (\xi_2, \psi_2; \breve{A}_2) \in FBSS(U)$.*

1. $\omega_1 \widetilde{\subseteq} \omega_2$ *implies that* $\underline{\Re}(\omega_1) \widetilde{\subseteq} \underline{\Re}(\omega_2)$ *and* $\overline{\Re}(\omega_1) \widetilde{\subseteq} \overline{\Re}(\omega_2)$
2. $\underline{\Re}(\omega_1\widetilde{\sqcap}_\varepsilon\omega_2) = \underline{\Re}(\omega_1)\widetilde{\sqcap}_\varepsilon\underline{\Re}(\omega_2)$,
3. $\underline{\Re}(\omega_1\widetilde{\sqcap}_r\omega_2) = \underline{\Re}(\omega_1)\widetilde{\sqcap}_r\underline{\Re}(\omega_2)$,
4. $\underline{\Re}(\omega_1\widetilde{\sqcup}_\varepsilon\omega_2) \widetilde{\supseteq} \underline{\Re}(\omega_1)\widetilde{\sqcup}_\varepsilon\underline{\Re}(\omega_2)$,
5. $\underline{\Re}(\omega_1\widetilde{\sqcup}_r\omega_2) \widetilde{\supseteq} \underline{\Re}(\omega_1)\widetilde{\sqcup}_r\underline{\Re}(\omega_2)$,
6. $\overline{\Re}(\omega_1\widetilde{\sqcap}_\varepsilon\omega_2) \widetilde{\subseteq} \overline{\Re}(\omega_1)\widetilde{\sqcap}_\varepsilon\overline{\Re}(\omega_2)$,
7. $\overline{\Re}(\omega_1\widetilde{\sqcap}_r\omega_2) \widetilde{\subseteq} \overline{\Re}(\omega_1)\widetilde{\sqcap}_r\overline{\Re}(\omega_2)$,
8. $\overline{\Re}(\omega_1\widetilde{\sqcup}_\varepsilon\omega_2) = \overline{\Re}(\omega_1)\widetilde{\sqcup}_\varepsilon\overline{\Re}(\omega_2)$,
9. $\overline{\Re}(\omega_1\widetilde{\sqcup}_r\omega_2) = \overline{\Re}(\omega_1)\widetilde{\sqcup}_r\overline{\Re}(\omega_2)$.

Proof. (1) Given that $\omega_1 \widetilde{\subseteq} \omega_2$, that is, $(\xi_1, \psi_1; \breve{A}_1) \widetilde{\subseteq} (\xi_2, \psi_2; \breve{A}_2)$. Then, $\xi_1(e)$, $\xi_2(e)$, $\psi_1(\neg e)$, $\psi_2(\neg e)$ are FSs in U, such that $\xi_1(e) \subseteq \xi_2(e)$ and $\psi_1(\neg e) \supseteq \psi_2(\neg e)$ for each $e \in \breve{A}_1$, where $\breve{A}_1 \subseteq \breve{A}_2$. This yields

$$\underline{\xi_1}_{\Re}(e) = \underline{\xi_1(e)}_{\Re} \subseteq \underline{\xi_2(e)}_{\Re} = \underline{\xi_2}_{\Re}(e)$$

and

$$\underline{\psi_1}_{\Re}(\neg e) = \overline{\psi_1(\neg e)}^{\Re} \supseteq \overline{\psi_2(\neg e)}^{\Re} = \underline{\psi_2}_{\Re}(\neg e)$$

for each $e \in \breve{A}_1$. Thus, $\underline{\Re}(\omega_1) \widetilde{\subseteq} \underline{\Re}(\omega_2)$. Similarly, one can verify, that, $\overline{\Re}(\omega_1) \widetilde{\subseteq} \overline{\Re}(\omega_2)$.

(2) The FBSSs $\underline{\Re}(\omega_1)\widetilde{\sqcap}_\varepsilon\underline{\Re}(\omega_2)$ and $\underline{\Re}(\omega_1\widetilde{\sqcap}_\varepsilon\omega_2)$ are described as

$$\underline{\Re}(\omega_1)\widetilde{\sqcap}_\varepsilon\underline{\Re}(\omega_2) = (\underline{\xi_1}_{\Re}\widetilde{\sqcap}_\varepsilon\underline{\xi_2}_{\Re}, \underline{\psi_1}_{\Re}\widetilde{\sqcup}_\varepsilon\underline{\psi_2}_{\Re}; \breve{A}_1 \cup \breve{A}_2)$$

and

$$\underline{\Re}(\omega_1\widetilde{\sqcap}_\varepsilon\omega_2) = (\underline{\xi_1\widetilde{\sqcap}_\varepsilon\xi_2}_{\Re}, \underline{\psi_1\widetilde{\sqcup}_\varepsilon\psi_2}_{\Re}; \breve{A}_1 \cup \breve{A}_2)$$

Now, Lemma 6.3.4 states that,

$$(\underline{\xi_1\widetilde{\sqcap}_\varepsilon\xi_2}_{\Re})(e) = (\underline{\xi_1}_{\Re}\widetilde{\sqcap}_\varepsilon\underline{\xi_2}_{\Re})(e)$$

and

$$(\underline{\psi_1 \widetilde{\cup}_\varepsilon \psi_2}_{\Re})(\neg e) = (\underline{\psi_1}_{\Re} \widetilde{\cup}_\varepsilon \underline{\psi_2}_{\Re})(\neg e)$$

hold for each $e \in \breve{A}_1 \cup \breve{A}_2$. The above equations assert that,

$$\underline{\Re}(\omega_1 \widetilde{\sqcap}_\varepsilon \omega_2) = \underline{\Re}(\omega_1) \widetilde{\sqcap}_\varepsilon \underline{\Re}(\omega_2).$$

(3) This is deduced from (2).

(4) The FBSSs $\underline{\Re}(\omega_1 \widetilde{\sqcup}_\varepsilon \omega_2)$ and $\underline{\Re}(\omega_1) \widetilde{\sqcup}_\varepsilon \underline{\Re}(\omega_2)$ are described as

$$\underline{\Re}(\omega_1 \widetilde{\sqcup}_\varepsilon \omega_2) = (\underline{\xi_1 \widetilde{\cup}_\varepsilon \xi_2}_{\Re}, \underline{\psi_1 \widetilde{\cap}_\varepsilon \psi_2}_{\Re}; \breve{A}_1 \cup \breve{A}_2)$$

and

$$\underline{\Re}(\omega_1) \widetilde{\sqcup}_\varepsilon \underline{\Re}(\omega_2) = (\underline{\xi_1}_{\Re} \widetilde{\cup}_\varepsilon \underline{\xi_2}_{\Re}, \underline{\psi_1}_{\Re} \widetilde{\cap}_\varepsilon \underline{\psi_2}_{\Re}; \breve{A}_1 \cup \breve{A}_2).$$

Lemma 6.3.4 states that the expressions

$$(\underline{\xi_1 \widetilde{\cup}_\varepsilon \xi_2}_{\Re})(e) \supseteq (\underline{\xi_1}_{\Re} \widetilde{\cup}_\varepsilon \underline{\xi_2}_{\Re})(e)$$

and

$$(\underline{\psi_1 \widetilde{\cap}_\varepsilon \psi_2}_{\Re})(\neg e) \subseteq (\underline{\psi_1}_{\Re} \widetilde{\cap}_\varepsilon \underline{\psi_2}_{\Re})(\neg e)$$

hold for each $e \in \breve{A}_1 \cup \breve{A}_2$. Which prove that,

$$\underline{\Re}(\omega_1 \widetilde{\sqcup}_\varepsilon \omega_2) \widetilde{\supseteq} \underline{\Re}(\omega_1) \widetilde{\sqcup}_\varepsilon \underline{\Re}(\omega_2).$$

(5) This expression is deduced from (4).

(6-9) The proof is same as the proof of (2-5). ■

The proper inclusions (4-7) of the above theorem may be proper, as exhibited in the subsequent example.

Example 6.3.6 *Consider the universe U of five houses, the relation $\Re$ on U, the attribute sets $\breve{E}$ and $\neg\breve{E}$ and the FBSS ω_1, as established in Example 6.2.3. Take another FBSS $\omega_2 = (\xi_2, \psi_2; \breve{A}_2)$ over U, with $\breve{A}_2 = \{e_1, e_2\}$ as given below.*

$\xi_2(e_1) = \{q_1/0.6,\ q_2/0.6,\ q_3/0.7,\ q_4/0.7,\ q_5/0.6\}$,

$\psi_2(\neg e_1) = \{q_1/0.3,\ q_2/0.2,\ q_3/0,\ q_4/0.2,\ q_5/0.3\}$,

$\xi_2(e_2) = \{q_1/0.7,\ q_2/0.6,\ q_3/0.6,\ q_4/0.7,\ q_5/0.6\}$,

$\psi_2(\neg e_2) = \{q_1/0.1,\ q_2/0.2,\ q_3/0.1,\ q_4/0.1,\ q_5/0.2\}$.

The lower RFBS-apx $\underline{\Re}(\omega_2) = (\underline{\xi_2}_{\Re}, \underline{\psi_2}_{\Re}; \breve{A}_2)$ of ω_2 is evaluated as below.

$\underline{\xi_2}_{\Re}(e_1) = \{q_1/0.6,\ q_2/0.6,\ q_3/0.6,\ q_4/0.6,\ q_5/0.6\}$,

$\underline{\xi_2}_{\Re}(e_2) = \{q_1/0.7,\ q_2/0.6,\ q_3/0.6,\ q_4/0.6,\ q_5/0.6\}$,

$\underline{\psi_2}_{\Re}(\neg e_1) = \{q_1/0.3,\ q_2/0.2,\ q_3/0.2,\ q_4/0.3,\ q_5/0.3\}$,

$\underline{\psi_2}_{\Re}(\neg e_2) = \{q_1/0.1,\ q_2/0.2,\ q_3/0.2,\ q_4/0.2,\ q_5/0.2\}$.

For the upper RFBS-apx $\overline{\Re}(\omega_2) = (\overline{\xi_2}^{\Re}, \overline{\psi_2}^{\Re}; \breve{A}_2)$ *of* ω_2*, the FSs* $\overline{\xi_2}^{\Re}(e_i)$ *and* $\overline{\psi_2}^{\Re}(\neg e_i)$ *are calculated for* $i = 1, 2$ *as below.*

$\overline{\xi_2}^{\Re}(e_1) = \{q_1/0.6, q_2/0.7, q_3/0.7, q_4/0.7, q_5/0.7\}$,
$\overline{\xi_2}^{\Re}(e_2) = \{q_1/0.7, q_2/0.6, q_3/0.6, q_4/0.6, q_5/0.6\}$,
$\overline{\psi_2}^{\Re}(\neg e_1) = \{q_1/0.3, q_2/0, q_3/0, q_4/0.2, q_5/0.2\}$,
$\overline{\psi_2}^{\Re}(\neg e_2) = \{q_1/0.1, q_2/0.1, q_3/0.1, q_4/0.1, q_5/0.1\}$.

To observe the proper inclusion in (5), the restricted union $\omega_1 \widetilde{\sqcup}_r \omega_2 = (\xi_1 \widetilde{\cup}_r \xi_2, \psi_1 \widetilde{\cap}_r \psi_2; \breve{A}_1 \cap \breve{A}_2)$ *of the FBSSs* ω_1 *and* ω_2 *is calculated as:*

$(\xi_1 \widetilde{\cup}_r \xi_2)(e_1) = \{q_1/0.7,\ q_2/0.6,\ q_3/0.8,\ q_4/0.7,\ q_5/0.6\}$,
$(\xi_1 \widetilde{\cup}_r \xi_2)(e_2) = \{q_1/0.8,\ q_2/0.7,\ q_3/0.8,\ q_4/0.7,\ q_5/0.6\}$,
$(\psi_1 \widetilde{\cap}_r \psi_2)(\neg e_1) = \{q_1/0.2,\ q_2/0.2,\ q_3/0,\ q_4/0.2,\ q_5/0.3\}$,
$(\psi_1 \widetilde{\cap}_r \psi_2)(\neg e_2) = \{q_1/0.1,\ q_2/0.1,\ q_3/0.1,\ q_4/0.1,\ q_5/0.2\}$.

Now, the FSs $(\underline{\xi_1 \widetilde{\cup}_r \xi_2}_{\Re})(e)$ *and* $(\underline{\psi_1 \widetilde{\cap}_r \psi_2}_{\Re})(\neg e)$ *of the FBSS* $\underline{\Re}(\omega_1 \widetilde{\sqcup}_r \omega_2)$ *described by* $(\underline{\xi_1 \widetilde{\cup}_r \xi_2}_{\Re}, \underline{\psi_1 \widetilde{\cap}_r \psi_2}_{\Re}; \breve{A}_1 \cap \breve{A}_2)$ *are evaluated for* $e \in \breve{A}_1 \cap \breve{A}_2$*, as below.*

$(\underline{\xi_1 \widetilde{\cup}_r \xi_2}_{\Re})(e_1) = \{q_1/0.7,\ q_2/0.6,\ q_3/0.6,\ q_4/0.6,\ q_5/0.6\}$,
$(\underline{\xi_1 \widetilde{\cup}_r \xi_2}_{\Re})(e_2) = \{q_1/0.8,\ q_2/0.7,\ q_3/0.7,\ q_4/0.6,\ q_5/0.6\}$,
$(\underline{\psi_1 \widetilde{\cap}_r \psi_2}_{\Re})(\neg e_1) = \{q_1/0.2,\ q_2/0.2,\ q_3/0.2,\ q_4/0.3,\ q_5/0.3\}$,
$(\underline{\psi_1 \widetilde{\cap}_r \psi_2}_{\Re})(\neg e_2) = \{q_1/0.1,\ q_2/0.1,\ q_3/0.1,\ q_4/0.2,\ q_5/0.2\}$.

The restricted union $\underline{\Re}(\omega_1) \widetilde{\sqcup}_r \underline{\Re}(\omega_2) = (\underline{\xi_1}_{\Re} \widetilde{\cup}_r \underline{\xi_2}_{\Re}, \underline{\psi_1}_{\Re} \widetilde{\cap}_r \underline{\psi_2}_{\Re}, \breve{A}_1 \cap \breve{A}_2)$ *is calculated below.*

$(\underline{\xi_1}_{\Re} \widetilde{\cup}_r \underline{\xi_2}_{\Re})(e_1) = \{q_1/0.7,\ q_2/0.6,\ q_3/0.6,\ q_4/0.6,\ q_5/0.6\}$,
$(\underline{\xi_1}_{\Re} \widetilde{\cup}_r \underline{\xi_2}_{\Re})(e_2) = \{q_1/0.8,\ q_2/0.7,\ q_3/0.7,\ q_4/0.6,\ q_5/0.6\}$,
$(\underline{\psi_1}_{\Re} \widetilde{\cap}_r \underline{\psi_2}_{\Re})(\neg e_1) = \{q_1/0.2,\ q_2/0.2,\ q_3/0.2,\ q_4/0.3,\ q_5/0.3\}$,
$(\underline{\psi_1}_{\Re} \widetilde{\cap}_r \underline{\psi_2}_{\Re})(\neg e_2) = \{q_1/0.1,\ q_2/0.2,\ q_3/0.2,\ q_4/0.2,\ q_5/0.2\}$.

Notice that,

$$(\underline{\psi_1 \widetilde{\cap}_r \psi_2}_{\Re})(\neg e_2)(q_2) \lneqq (\underline{\psi_1}_{\Re} \widetilde{\cap}_r \underline{\psi_2}_{\Re})(\neg e_2)(q_2).$$

This immediately yields the proper inclusion in (5). That is,

$$\underline{\Re}(\omega_1 \widetilde{\sqcup}_r \omega_2) \supsetneqq \underline{\Re}(\omega_1) \widetilde{\sqcup}_r \underline{\Re}(\omega_2).$$

Similarly, one can observe the proper inclusion in (4). That is,

$$\underline{\Re}(\omega_1 \widetilde{\sqcup}_\varepsilon \omega_2) \supsetneqq \underline{\Re}(\omega_1) \widetilde{\sqcup}_\varepsilon \underline{\Re}(\omega_2).$$

Next, we verify the proper inclusion in (7) of the Theorem 6.3.5. The restricted intersection $\omega_1 \widetilde{\sqcap}_r \omega_2 = (\xi_1 \widetilde{\cap}_r \xi_2, \psi_1 \widetilde{\cup}_r \psi_2; \breve{A}_1 \cap \breve{A}_2)$ *of* ω_1 *and* ω_2 *is calculated as:*

$(\xi_1 \widetilde{\cap}_r \xi_2)(e_1) = \{q_1/0.6,\ q_2/0.6,\ q_3/0.7,\ q_4/0.5,\ q_5/0.6\}$,
$(\xi_1 \widetilde{\cap}_r \xi_2)(e_2) = \{q_1/0.7,\ q_2/0.6,\ q_3/0.6,\ q_4/0.6,\ q_5/0.6\}$,

$(\psi_1\widetilde{\cup}_r\psi_2)(\neg e_1) = \{q_1/0.3,\ q_2/0.3,\ q_3/0.1,\ q_4/0.5,\ q_5/0.3\}$,
$(\psi_1\widetilde{\cup}_r\psi_2)(\neg e_2) = \{q_1/0.1,\ q_2/0.2,\ q_3/0.2,\ q_4/0.2,\ q_5/0.3\}$.

Now, the FBSS $\overline{\Re}(\omega_1\widetilde{\sqcap}_r\omega_2) = (\overline{\xi_1\widetilde{\sqcap}_r\xi_2}^{\Re}, \overline{\psi_1\widetilde{\cup}_r\psi_2}^{\Re}; \breve{A}_1 \cap \breve{A}_2)$ *is evaluated below for* $e \in \breve{A}_1 \cap \breve{A}_2$.

$(\overline{\xi_1\widetilde{\sqcap}_r\xi_2}^{\Re})(e_1) = \{q_1/0.6,\ q_2/0.7,\ q_3/0.7,\ q_4/0.6,\ q_5/0.6\}$,
$(\overline{\xi_1\widetilde{\sqcap}_r\xi_2}^{\Re})(e_2) = \{q_1/0.7,\ q_2/0.6,\ q_3/0.6,\ q_4/0.6,\ q_5/0.6\}$,
$(\overline{\psi_1\widetilde{\cup}_r\psi_2}^{\Re})(\neg e_1) = \{q_1/0.3,\ q_2/0.1,\ q_3/0.1,\ q_4/0.3,\ q_5/0.3\}$,
$(\overline{\psi_1\widetilde{\cup}_r\psi_2}^{\Re})(\neg e_2) = \{q_1/0.1,\ q_2/0.2,\ q_3/0.2,\ q_4/0.2,\ q_5/0.2\}$.

The restricted intersection $\overline{\Re}(\omega_1)\widetilde{\sqcap}_r\overline{\Re}(\omega_2) = (\overline{\xi_1}^{\Re}\widetilde{\sqcap}_r\overline{\xi_2}^{\Re}, \overline{\psi_1}^{\Re}\widetilde{\cup}_r\overline{\psi_2}^{\Re}; \breve{A}_1\cap\breve{A}_2)$ *is calculated below.*

$(\overline{\xi_1}^{\Re}\widetilde{\sqcap}_r\overline{\xi_2}^{\Re})(e_1) = \{q_1/0.6,\ q_2/0.7,\ q_3/0.7,\ q_4/0.6,\ q_5/0.6\}$,
$(\overline{\xi_1}^{\Re}\widetilde{\sqcap}_r\overline{\xi_2}^{\Re})(e_2) = \{q_1/0.7,\ q_2/0.6,\ q_3/0.6,\ q_4/0.6,\ q_5/0.6\}$,
$(\overline{\psi_1}^{\Re}\widetilde{\cup}_r\overline{\psi_2}^{\Re})(\neg e_1) = \{q_1/0.3,\ q_2/0.1,\ q_3/0.1,\ q_4/0.3,\ q_5/0.3\}$,
$(\overline{\psi_1}^{\Re}\widetilde{\cup}_r\overline{\psi_2}^{\Re})(\neg e_2) = \{q_1/0.1,\ q_2/0.1,\ q_3/0.1,\ q_4/0.1,\ q_5/0.1\}$.

Notice that,

$$(\overline{\psi_1\widetilde{\cup}_r\psi_2}^{\Re})(\neg e_2)(q_2) \gneqq (\overline{\psi_1}^{\Re}\widetilde{\cup}_r\overline{\psi_2}^{\Re})(\neg e_2)(q_2).$$

Which immediately shows the proper inclusion in (7). That is,

$$\overline{\Re}(\omega_1\widetilde{\sqcap}_r\omega_2) \,\widetilde{\subsetneq}\, \overline{\Re}(\omega_1)\widetilde{\sqcap}_r\overline{\Re}(\omega_2).$$

Similarly, one can observe the proper inclusion in (6). That is,

$$\overline{\Re}(\omega_1\widetilde{\sqcap}_\varepsilon\omega_2) \,\widetilde{\subsetneq}\, \overline{\Re}(\omega_1)\widetilde{\sqcap}_\varepsilon\overline{\Re}(\omega_2).$$

Theorem 6.3.7 *Take a P-apx space* $(U, \Re)$ *and let* $\omega \in FBSS(U)$. *Then, the subsequent asservations are equivalent.*

1. $\overline{\Re}(\omega) \,\widetilde{\subseteq}\, \omega$,
2. $\omega \,\widetilde{\subseteq}\, \underline{\Re}(\omega)$,
3. ω *is* $\Re-$*definable.*

Proof. (1)⇒(2) Assume that $\overline{\Re}(\omega) \,\widetilde{\subseteq}\, \omega$. From Theorem 6.3.5, we have $\underline{\Re}(\overline{\Re}(\omega)) \,\widetilde{\subseteq}\, \underline{\Re}(\omega)$. Then, Theorem 6.3.3 yields

$$\omega \,\widetilde{\subseteq}\, \overline{\Re}(\omega) = \underline{\Re}(\overline{\Re}(\omega)) \,\widetilde{\subseteq}\, \underline{\Re}(\omega).$$

Thus, $\overline{\Re}(\omega) \,\widetilde{\subseteq}\, \omega$.

(2)⇒(3) Assume that $\omega \,\widetilde{\subseteq}\, \underline{\Re}(\omega)$. From Theorem 6.3.3, we have $\underline{\Re}(\omega) \,\widetilde{\subseteq}\, \omega$. So, $\omega = \underline{\Re}(\omega)$. This gives $\overline{\Re}(\omega) = \overline{\Re}(\underline{\Re}(\omega)) = \underline{\Re}(\omega)$. Thus, ω is $\Re-$definable.

(3)⇒(1) Obvious. ■

Proposition 6.3.8 *Take a P-apx space* $(U, \Re)$.

1. *Let* $\Re$ *be the identity relation on* U. *Then, each FBSS over* U *is* $\Re-$*definable.*
2. *Let* $\Re$ *be the universal binary relation on* U *and* $\omega \in FBSS(U)$ *be* $\Re-$*definable. Then,* ω *is a constant FBSS over* U.

Proof. (1) Let $\Re$ be the identity relation on U. Then, each eqv-class is singleton subset of U. Which implies that each FBSS over U is classwise constant. Hence, each FBSS over U is $\Re-$definable, by Lemma 6.3.1.

(2) Let $\Re = U \times U$ and let $\omega = (\xi, \psi; \breve{A}) \in FBSS(U)$ be $\Re-$definable. Then, $\underline{\Re}(\omega) = \overline{\Re}(\omega)$. Which gives $(\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A}) = (\overline{\xi}^{\Re}, \overline{\psi}^{\Re}; \breve{A})$. That is, $\underline{\xi}_{\Re}(e)(u) = \overline{\xi}^{\Re}(e)(u)$ and $\underline{\psi}_{\Re}(\neg e)(u) = \overline{\psi}^{\Re}(\neg e)(u)$ for each $e \in \breve{A}$ and for each $u \in U$. This yields

$$\bigwedge_{y \in [u]_{\Re}} \xi(e)(y) = \bigvee_{y \in [u]_{\Re}} \xi(e)(y)$$

and

$$\bigvee_{y \in [u]_{\Re}} \psi(\neg e)(y) = \bigwedge_{y \in [u]_{\Re}} \psi(\neg e)(y)$$

for each $e \in \breve{A}$ and for each $u \in U$. Since, $[u]_{\Re} = U$ for each $u \in U$, so, we get

$$\bigwedge_{y \in U} \xi(e)(y) = \bigvee_{y \in U} \xi(e)(y)$$

and

$$\bigvee_{y \in U} \psi(\neg e)(y) = \bigwedge_{y \in U} \psi(\neg e)(y)$$

for each $e \in \breve{A}$ and for each $y \in U$. Which clearly shows that, ω is a constant FBSS over U. ■

Theorem 6.3.9 *Take a P-apx space* $(U, \Re)$. *Take another eqv-rel* σ *on* U, *such that,* $\Re \subseteq \sigma$. *Then* $\underline{\sigma}(\omega) \tilde{\subseteq} \underline{\Re}(\omega)$ *and* $\overline{\Re}(\omega) \tilde{\subseteq} \overline{\sigma}(\omega)$ *for any FBSS* ω *over* U.

Proof. Take $\omega = (\xi, \psi; \breve{A}) \in FBSS(U)$ for any $\breve{A} \subseteq \breve{E}$. Since $\Re \subseteq \sigma$, we have $[u]_{\Re} \subseteq [u]_{\sigma}$ for each $u \in U$. Thus, we get

$$\begin{aligned} \underline{F}_{\sigma}(e)(u) &= \bigwedge_{y \in [u]_{\sigma}} \xi(e)(y) \\ &\leq \bigwedge_{y \in [u]_{\Re}} \xi(e)(y) = \underline{\xi}_{\Re}(e)(u) \end{aligned}$$

for each $u \in U$ and for each $e \in \breve{A}$. Hence, $\underline{\xi}_{\sigma}(e) \subseteq \underline{\xi}_{\Re}(e)$ for each $e \in \breve{A}$. Similarly, $\underline{\psi}_{\sigma}(\neg e) \supseteq \underline{\psi}_{\Re}(\neg e)$ for each $\neg e \in \neg\breve{A}$. Thus, $\underline{\sigma}(\omega) \tilde{\subseteq} \underline{\Re}(\omega)$. In the same way, one can verify, that, $\overline{\Re}(\omega) \tilde{\subseteq} \overline{\sigma}(\omega)$. ■

6.4 Similarity relations associated with RFBS approximations

This section establishes some binary relations between the FBSSs based on their RFBS-apxes and investigate their properties.

Definition 6.4.1 *Take a P-apx space* $(U, \Re)$. *We define the following binary relations for* $\omega_1, \omega_2 \in FBSS(U)$,

$\omega_1 \simeq_{\Re} \omega_2$ *if and only if* $\underline{\Re}(\omega_1) = \underline{\Re}(\omega_2)$;

$\omega_1 \eqsim_{\Re} \omega_2$ *if and only if* $\overline{\Re}(\omega_1) = \overline{\Re}(\omega_2)$;

$\omega_1 \approx_{\Re} \omega_2$ *if and only if* $\underline{\Re}(\omega_1) = \underline{\Re}(\omega_2)$ *and* $\overline{\Re}(\omega_1) = \overline{\Re}(\omega_2)$.

We may term these relations as the lower RFBS similarity relation, upper RFBS similarity relation and the RFBS similarity relation, respectively. Obviously, ω_1 and ω_2 are RFBS similar if and only if they are both, lower and upper RFBS similar.

Proposition 6.4.2 *The relations* $\simeq_{\Re}$, $\eqsim_{\Re}$ *and* $\approx_{\Re}$ *are eqv-rels on* $FBSS(U)$.

Proof. Straightforward. ■

Theorem 6.4.3 *Take a P-apx space* $(U, \Re)$. *Then, the subsequent asservations hold for* $\{\omega_i = (\xi_i, \psi_i, \breve{A}_i) : i = 1, 2, 3, 4\} \subseteq FBSS(U)$.

1. $\omega_1 \eqsim_{\Re} \omega_2$ *if and only if* $\omega_1 \eqsim_{\Re} (\omega_1 \widetilde{\sqcup}_{\varepsilon} \omega_2) \eqsim_{\Re} \omega_2$,
2. $\omega_1 \eqsim_{\Re} \omega_2$ *and* $\omega_3 \eqsim_{\Re} \omega_4$ *imply that* $(\omega_1 \widetilde{\sqcup}_{\varepsilon} \omega_3) \eqsim_{\Re} (\omega_2 \widetilde{\sqcup}_{\varepsilon} \omega_4)$,
3. $\omega_1 \mathrel{\widetilde{\subseteq}} \omega_2$ *and* $\omega_2 \eqsim_{\Re} \Phi_{\breve{A}_2}$ *imply that* $\omega_1 \eqsim_{\Re} \Phi_{\breve{A}_1}$,
4. $\omega_1 \mathrel{\widetilde{\subseteq}} \omega_2$ *and* $\omega_1 \eqsim_{\Re} \tilde{U}_{\breve{A}_1}$ *imply that* $\omega_2 \eqsim_{\Re} \tilde{U}_{\breve{A}_2}$, *provided that* $\breve{A}_1 = \breve{A}_2$,
5. $(\omega_1 \widetilde{\sqcup}_{\varepsilon} \omega_2) \eqsim_{\Re} \Phi_{\breve{A}_1 \cup \breve{A}_2}$ *if and only if* $\omega_1 \eqsim_{\Re} \Phi_{\breve{A}_1}$ *and* $\omega_2 \eqsim_{\Re} \Phi_{\breve{A}_2}$,
6. $(\omega_1 \widetilde{\sqcap}_{\varepsilon} \omega_2) \eqsim_{\Re} \tilde{U}_{\breve{A}_1 \cup \breve{A}_2}$ *implies that* $\omega_1 \eqsim_{\Re} \tilde{U}_{\breve{A}_1}$ *and* $\omega_2 \eqsim_{\Re} \tilde{U}_{\breve{A}_2}$.

Proof. (1) Let $\omega_1 \eqsim_{\Re} \omega_2$. Then, $\overline{\Re}(\omega_1) = \overline{\Re}(\omega_2)$. From Theorem 6.3.5, we get

$$\overline{\Re}(\omega_1 \widetilde{\sqcup}_{\varepsilon} \omega_2) = \overline{\Re}(\omega_1) \widetilde{\sqcup}_{\varepsilon} \overline{\Re}(\omega_2) = \overline{\Re}(\omega_1) = \overline{\Re}(\omega_2).$$

So, $\omega_1 \eqsim_{\Re} (\omega_1 \widetilde{\sqcup}_{\varepsilon} \omega_2) \eqsim_{\Re} \omega_2$.

Converse holds by transitivity of the relation $\eqsim_{\Re}$.

(2) Given that $\omega_1 \eqsim_{\Re} \omega_2$ and $\omega_3 \eqsim_{\Re} \omega_4$. Then, $\overline{\Re}(\omega_1) = \overline{\Re}(\omega_2)$ and $\overline{\Re}(\omega_3) = \overline{\Re}(\omega_4)$. From Theorem 6.3.5, we get

$$\begin{aligned} \overline{\Re}(\omega_1 \widetilde{\sqcup}_\varepsilon \omega_3) &= \overline{\Re}(\omega_1)\widetilde{\sqcup}_\varepsilon \overline{\Re}(\omega_3) \\ &= \overline{\Re}(\omega_2)\widetilde{\sqcup}_\varepsilon \overline{\Re}(\omega_4) = \overline{\Re}(\omega_2 \widetilde{\sqcup}_\varepsilon \omega_4). \end{aligned}$$

Thus, $(\omega_1 \widetilde{\sqcup}_\varepsilon \omega_3) \eqsim_{\Re} (\omega_2 \widetilde{\sqcup}_\varepsilon \omega_4)$.

(3) Given that, $\omega_2 \eqsim_{\Re} \Phi_{\breve{A}_2}$. Which implies $\overline{\Re}(\omega_2) = \overline{\Re}(\Phi_{\breve{A}_2}) = \Phi_{\breve{A}_2}$. Also $\omega_1 \widetilde{\subseteq} \omega_2$ implies that $\overline{\Re}(\omega_1) \widetilde{\subseteq} \overline{\Re}(\omega_2) = \Phi_{\breve{A}_2}$. Restricting the attribute set of $\Phi_{\breve{A}_2}$ to $\breve{A}_1 \subseteq \breve{A}_2$, we get $\overline{\Re}(\omega_1) \widetilde{\subseteq} \Phi_{\breve{A}_1}$. But, $\Phi_{\breve{A}_1} \widetilde{\subseteq} \overline{\Re}(\omega_1)$. So, $\overline{\Re}(\omega_1) = \Phi_{\breve{A}_1} = \overline{\Re}(\Phi_{\breve{A}_1})$. Which shows that, $\omega_1 \eqsim_{\Re} \Phi_{\breve{A}_1}$.

(4) $\omega_1 \eqsim_{\Re} \tilde{U}_{\breve{A}_1}$ implies that $\overline{\Re}(\omega_1) = \overline{\Re}(\tilde{U}_{\breve{A}_1}) = \tilde{U}_{\breve{A}_1} = \tilde{U}_{\breve{A}_2}$, as $\breve{A}_1 = \breve{A}_2$. Also given that, $\omega_1 \widetilde{\subseteq} \omega_2$. So, we get

$$\begin{aligned} \overline{\Re}(\omega_2) \widetilde{\subseteq} \overline{\Re}(\tilde{U}_{\breve{A}_2}) &= \tilde{U}_{\breve{A}_2} = \tilde{U}_{\breve{A}_1} \\ &= \overline{\Re}(\omega_1) \widetilde{\subseteq} \overline{\Re}(\omega_2). \end{aligned}$$

This gives $\overline{\Re}(\omega_2) = \overline{\Re}(\tilde{U}_{\breve{A}_2})$. Hence, $\omega_2 \eqsim_{\Re} \tilde{U}_{\breve{A}_2}$.

(5) Let $\omega_1 \eqsim_{\Re} \Phi_{\breve{A}_1}$ and $\omega_2 \eqsim_{\Re} \Phi_{\breve{A}_2}$. Then, we have $\overline{\Re}(\omega_1) = \overline{\Re}(\Phi_{\breve{A}_1}) = \Phi_{\breve{A}_1}$ and $\overline{\Re}(\omega_2) = \overline{\Re}(\Phi_{\breve{A}_2}) = \Phi_{\breve{A}_2}$. From Theorem 6.3.5, we get

$$\begin{aligned} \overline{\Re}(\omega_1 \widetilde{\sqcup}_\varepsilon \omega_2) &= \overline{\Re}(\omega_1)\widetilde{\sqcup}_\varepsilon \overline{\Re}(\omega_2) = \Phi_{\breve{A}_1}\widetilde{\sqcup}_\varepsilon \Phi_{\breve{A}_2} \\ &= \Phi_{\breve{A}_1 \cup \breve{A}_2} = \overline{\Re}(\Phi_{\breve{A}_1 \cup \breve{A}_2}). \end{aligned}$$

Thus, $(\omega_1 \widetilde{\sqcup}_\varepsilon \omega_2) \eqsim_{\Re} \Phi_{\breve{A}_1 \cup \breve{A}_2}$. Converse follows from (3).

(6) This assertion follows from (4). ■

Note that in (1) and (2) of Theorem 6.4.3, $\omega_1 \eqsim_{\Re} \omega_2$ means that $\overline{\Re}(\omega_1) = \overline{\Re}(\omega_2)$. Which indicates $\breve{A}_1 = \breve{A}_2$ by using Definition 1.6.3. Thus, the attribute sets of RFBS similar (lower, upper or both) FBSSs are same. Hence, their restricted and extended unions, as well as intersections coincide. Same is the case when $\omega_1 \simeq_{\Re} \omega_2$ or $\omega_1 \approx_{\Re} \omega_2$.

Theorem 6.4.4 *Take a P-apx space $(U, \Re)$. Then, the subsequent asservations hold for $\{\omega_i = (\xi_i, \psi_i, \breve{A}_i) : i = 1, 2, 3, 4\} \subseteq FBSS(U)$.*

1. $\omega_1 \simeq_{\Re} \omega_2$ *if and only if* $\omega_1 \simeq_{\Re} (\omega_1 \widetilde{\sqcap}_\varepsilon \omega_2) \simeq_{\Re} \omega_2$,
2. $\omega_1 \simeq_{\Re} \omega_2$ *and* $\omega_3 \simeq_{\Re} \omega_4$ *imply that* $(\omega_1 \widetilde{\sqcap}_\varepsilon \omega_3) \simeq_{\Re} (\omega_2 \widetilde{\sqcap}_\varepsilon \omega_4)$,
3. $\omega_1 \widetilde{\subseteq} \omega_2$ *and* $\omega_2 \simeq_{\Re} \Phi_{\breve{A}_2}$ *imply that* $\omega_1 \simeq_{\Re} \Phi_{\breve{A}_1}$,
4. $\omega_1 \widetilde{\subseteq} \omega_2$ *and* $\omega_1 \simeq_{\Re} \tilde{U}_{\breve{A}_1}$ *imply that* $\omega_2 \simeq_{\Re} \tilde{U}_{\breve{A}_2}$, *provided that* $\breve{A}_1 = \breve{A}_2$,

5. $(\omega_1 \widetilde{\sqcup}_\varepsilon \omega_2) \simeq_\Re \Phi_{\breve{A}_1 \cup \breve{A}_2}$ *implies that* $\omega_1 \simeq_\Re \Phi_{\breve{A}_1}$ *and* $\omega_2 \simeq_\Re \Phi_{\breve{A}_2}$,

6. $(\omega_1 \widetilde{\sqcap}_\varepsilon \omega_2) \simeq_\Re \tilde{U}_{\breve{A}_1 \cup \breve{A}_2}$ *if and only if* $\omega_1 \simeq_\Re \tilde{U}_{\breve{A}_1}$ *and* $\omega_2 \simeq_\Re \tilde{U}_{\breve{A}_2}$.

Proof. Parallel to the proof of Theorem 6.4.3. ■

Theorem 6.4.5 *Take a P-apx space* $(U, \Re)$*. Then, the subsequent asservations hold for* $\{\omega_i = (\xi_i, \psi_i; \breve{A}_i) : i = 1, 2, 3, 4\} \subseteq FBSS(U)$.

1. $\omega_1 \approx_\Re \omega_2$ *if and only if* $\omega_1 \eqsim_\Re (\omega_1 \widetilde{\sqcup}_\varepsilon \omega_2) \eqsim_\Re \omega_2$ *and* $\omega_1 \simeq_\Re (\omega_1 \widetilde{\sqcap}_\varepsilon \omega_2) \simeq_\Re \omega_2$,

2. $\omega_1 \widetilde{\subseteq} \omega_2$ *and* $\omega_2 \approx_\Re \Phi_{\breve{A}_2}$ *imply that* $\omega_1 \approx_\Re \Phi_{\breve{A}_1}$,

3. $\omega_1 \widetilde{\subseteq} \omega_2$ *and* $\omega_1 \approx_\Re \tilde{U}_{\breve{A}_1}$ *imply that* $\omega_2 \approx_\Re \tilde{U}_{\breve{A}_2}$, *provided that* $\breve{A}_1 = \breve{A}_2$,

4. $(\omega_1 \widetilde{\sqcup}_\varepsilon \omega_2) \approx_\Re \Phi_{\breve{A}_1 \cup \breve{A}_2}$ *implies that* $\omega_1 \approx_\Re \Phi_{\breve{A}_1}$ *and* $\omega_2 \approx_\Re \Phi_{\breve{A}_2}$,

5. $(\omega_1 \widetilde{\sqcap}_\varepsilon \omega_2) \approx_\Re \tilde{U}_{\breve{A}_1 \cup \breve{A}_2}$ *implies that* $\omega_1 \approx_\Re \tilde{U}_{\breve{A}_1}$ *and* $\omega_2 \approx_\Re \tilde{U}_{\breve{A}_2}$.

Proof. It can be directly deduced from Theorems 6.4.3 and 6.4.4. ■

6.5 Accuracy measures for FBSSs

An important application of the RFBS-apxes of the FBSSs is, that, these approximations provide a scheme to investigate how accurately the belongingness maps of a FBSS describe the objects. We introduce the degree of accuracy and the degree of roughness for the positive and negative belongingness maps of the FBSSs, separately. For this purpose, we first define the $\alpha-$level cuts of a FBSS and describe their basic properties.

Definition 6.5.1 *Let* $\omega = (\xi, \psi; \breve{A}) \in FBSS(U)$*. For* $0 \lneqq \alpha \leq 1$*, the* $\alpha-$*level P-cut (positive cut) of* ω *relative to the attribute* $e \in \breve{A}$ *is symbolized by* $\omega_{\langle e, \alpha \rangle}$ *and defined as:*

$$\omega_{\langle e, \alpha \rangle} = \{u \in U : \xi(e)(u) \geq \alpha\}. \tag{6.1}$$

Definition 6.5.2 *Let* $\omega = (\xi, \psi; \breve{A}) \in FBSS(U)$*. For* $0 \lneqq \alpha \leq 1$*, the* $\alpha-$*level N-cut (negative cut) of* ω *relative to the attribute* $e \in \breve{A}$ *is symbolized by* $\omega^{\langle e, \alpha \rangle}$ *and defined as:*

$$\omega^{\langle e, \alpha \rangle} = \{u \in U : \psi(\neg e)(u) \leq \alpha\}. \tag{6.2}$$

Lemma 6.5.3 *Let* $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$, $\omega_2 = (\xi_2, \psi_2; \breve{A}_2) \in FBSS(U)$ *and* $0 \lneqq \alpha \leq 1$*. Then,* $\omega_1 \subseteq \omega_2$ *implies that,* $\omega_{1\langle e, \alpha \rangle} \widetilde{\subseteq} \omega_{2\langle e, \alpha \rangle}$ *and* $\omega_1^{\langle e, \alpha \rangle} \widetilde{\subseteq} \omega_2^{\langle e, \alpha \rangle}$ *for each* $e \in \breve{A}_1$.

Proof. It can be directly deduced from Definitions 6.5.1 and 6.5.2. ■

Lemma 6.5.4 *Let* $\omega = (\xi, \psi; \breve{A}) \in FBSS(U)$ *and* $0 \lneqq \beta \leq \alpha \leq 1$. *Then, we have* $\omega_{\langle e,\alpha\rangle} \mathrel{\widetilde{\subseteq}} \omega_{\langle e,\beta\rangle}$ *and* $\omega^{\langle e,\beta\rangle} \mathrel{\widetilde{\subseteq}} \omega^{\langle e,\alpha\rangle}$ *for each* $e \in \breve{A}$.

Proof. It can be directly deduced from Definitions 6.5.1 and 6.5.2. ■

With the help of Definitions 6.2.1, 6.5.1 and 6.5.2, we conclude the following statements for $\omega = (\xi, \psi; \breve{A})$.

$$\begin{aligned}
\underline{\Re}(\omega)_{\langle e,\alpha\rangle} &= \{u \in U : \underline{\xi}_{\Re}(e)(u) \geq \alpha\} \\
&= \{u \in U : \bigwedge_{y\in[u]_\Re} \xi(e)(y) \geq \alpha\}, \\
\overline{\Re}(\omega)_{\langle e,\alpha\rangle} &= \{u \in U : \bigvee_{y\in[u]_\Re} \xi(e)(y) \geq \alpha\}, \\
\underline{\Re}(\omega)^{\langle e,\alpha\rangle} &= \{u \in U : \bigvee_{y\in[u]_\Re} \psi(\neg e)(y) \leq \alpha\}, \\
\overline{\Re}(\omega)^{\langle e,\alpha\rangle} &= \{u \in U : \bigwedge_{y\in[u]_\Re} \psi(\neg e)(y) \leq \alpha\}.
\end{aligned}$$

Lemma 6.5.5 *Let* $\Re$ *be an eqv-rel on* U *and* $\omega = (\xi, \psi; \breve{A}) \in FBSS(U)$. *Then, the subsequent asservations hold for each* $e \in \breve{A}$ *and* $0 \lneqq \alpha \leq 1$.

1. $\underline{\Re}(\omega_{\langle e,\alpha\rangle}) = \underline{\Re}(\omega)_{\langle e,\alpha\rangle}$,
2. $\overline{\Re}(\omega_{\langle e,\alpha\rangle}) = \overline{\Re}(\omega)_{\langle e,\alpha\rangle}$,
3. $\underline{\Re}(\omega^{\langle e,\alpha\rangle}) = \underline{\Re}(\omega)^{\langle e,\alpha\rangle}$,
4. $\overline{\Re}(\omega^{\langle e,\alpha\rangle}) = \overline{\Re}(\omega)^{\langle e,\alpha\rangle}$.

Proof. (1) For the crisp set $\omega_{\langle e,\alpha\rangle}$, we have the following.

$$\begin{aligned}
\underline{\Re}(\omega_{\langle e,\alpha\rangle}) &= \{u \in U : [u]_\Re \subseteq \omega_{\langle e,\alpha\rangle}\} \\
&= \{u \in U : \xi(e)(y) \geq \alpha \text{ for each } y \in [u]_\Re\} \\
&= \{u \in U : \bigwedge_{y\in[u]_\Re} \xi(e)(y) \geq \alpha\} \\
&= \underline{\Re}(\omega)_{\langle e,\alpha\rangle}.
\end{aligned}$$

The remaining parts can be verified in the same manner. ■

Now, we define the degree of accuracy and the degree of roughness for the positive and negative belongingness maps of an FBSS over U.

Definition 6.5.6 *Take a P-apx space $(U, \Re)$. The degree of accuracy for the positive belongingness map of $\omega = (\xi, \psi; \breve{A}) \in FBSS(U)$, relative to the attribute $e \in \breve{A}$ and the parameters α, β satisfying $0 \lneqq \beta \leq \alpha \leq 1$, is expressed as:*

$$\mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega) = \frac{\left|\underline{\Re}(\omega_{\langle e,\alpha\rangle})\right|}{\left|\overline{\Re}(\omega_{\langle e,\beta\rangle})\right|}.$$

The degree of roughness for the positive belongingness map of ω relative to the attribute $e \in \breve{A}$ and the parameters α, β satisfying $0 \lneqq \beta \leq \alpha \leq 1$, is expressed as:

$$\rho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega) = 1 - \mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega).$$

Definition 6.5.7 *Take a P-apx space $(U, \Re)$. The degree of accuracy for the negative belongingness map of $\omega = (\xi, \psi; \breve{A}) \in FBSS(U)$, relative to the attribute $e \in \breve{A}$ and the parameters α, β satisfying $0 \lneqq \beta \leq \alpha \leq 1$, is expressed as:*

$$\mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega) = \frac{\left|\underline{\Re}(\omega^{\langle e,\beta\rangle})\right|}{\left|\overline{\Re}(\omega^{\langle e,\alpha\rangle})\right|}.$$

The degree of roughness for the negative belongingness map of ω relative to the attribute $e \in \breve{A}$ and the parameters α, β satisfying $0 \lneqq \beta \leq \alpha \leq 1$, is expressed as:

$$\varrho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega) = 1 - \mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega).$$

Notice that, $\underline{\Re}(\omega_{\langle e,\alpha\rangle})$ (or $\overline{\Re}(\omega_{\langle e,\beta\rangle})$) comprises of the objects of U having α (or β) as the least degree of definite (or possible) fulfilment towards the attribute e in ω. Equivalently, $\underline{\Re}(\omega_{\langle e,\alpha\rangle})$ (or $\overline{\Re}(\omega_{\langle e,\beta\rangle})$) may be viewed as union of the eqv-classes of U having the degree of fulfilment atleast α (or β) in the lower (upper) RFBS-apx of ω. Therefore, the parameters α and β serve as the thresholds of definite and possible fulfilment of the objects of U towards the attribute e in ω, respectively. Hence, $\mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega)$ may be interpreted as the degree to which the positive belongingness map of ω is accurate, constrained to the threshold parameters α and β. Similarly, $\mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega)$ denotes the degree to which the negative belongingness map of ω is accurate, constrained to the threshold parameters α and β. In other words, $\mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega)$ and $\mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega)$ describe how accurate are the positive and the negative belongingness maps of the FBSSs, respectively. We explain these degrees in the subsequent example.

Example 6.5.8 *Consider the set $U = \{q_i : i = 1, 2, ..., 7\}$, the relation $\Re$ defining eqv-classes $\{q_1\}, \{q_2, q_3\}$ and $\{q_4, q_5\}$ and the FBSS $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$ over U as in Example 6.2.3. Take $\alpha = 0.6$, $\beta = 0.3$, and $e = e_1$. Then, $\alpha-$level P-cuts $\omega_{\langle e_1,0.6\rangle}$ and $\omega_{\langle e_1,0.3\rangle}$ are calculated using Equation 6.1, as follows.*

$$\begin{aligned} \omega_{\langle e_1,0.6\rangle} &= \{q_1, q_2, q_3, q_5\}, \\ \omega_{\langle e_1,0.3\rangle} &= \{q_1, q_2, q_3, q_4, q_5\}. \end{aligned}$$

The $\alpha-$level N-cuts $\omega^{\langle e_1,0.6\rangle}$ and $\omega^{\langle e_1,0.3\rangle}$ are calculated using Equation 6.2, as follows.

$$\begin{aligned}\omega^{\langle e_1,0.6\rangle} &= \{q_1,q_2,q_3,q_4,q_5\},\\ \omega^{\langle e_1,0.3\rangle} &= \{q_1,q_2,q_3,q_5\}.\end{aligned}$$

The degree of accuracy for the positive belongingness map of ω relative to $e_1 \in \breve{A}_1$ is calculated using Definition 6.5.6, as follows.

$$\begin{aligned}\underline{\Re}(\omega_{\langle e_1,0.6\rangle}) &= \{q_1,q_2,q_3\},\\ \overline{\Re}(\omega_{\langle e_1,0.3\rangle}) &= \{q_1,q_2,q_3,q_4,q_5\},\\ \mathfrak{Dp}^{\Re}_{\langle e_1,0.6,0.3\rangle}(\omega) &= \frac{\left|\underline{\Re}(\omega_{\langle e_1,0.6\rangle})\right|}{\left|\overline{\Re}(\omega_{\langle e_1,0.3\rangle})\right|} = \frac{3}{5} = 0.6.\end{aligned}$$

While, the degree of accuracy for the negative belongingness map of ω relative to $e_1 \in \breve{A}_1$ is calculated by using Definition 6.5.7, as follows.

$$\begin{aligned}\underline{\Re}(\omega^{\langle e_1,0.3\rangle}) &= \{q_1,q_2,q_3\},\\ \overline{\Re}(\omega^{\langle e_1,0.6\rangle}) &= \{q_1,q_2,q_3,q_4,q_5\},\\ \mathfrak{Dn}^{\Re}_{\langle e_1,0.6,0.3\rangle}(\omega) &= \frac{\left|\underline{\Re}(\omega^{\langle e_1,0.3\rangle})\right|}{\left|\overline{\Re}(\omega^{\langle e_1,0.6\rangle})\right|} = \frac{3}{5} = 0.6.\end{aligned}$$

Hence, both (positive and negative) belongingness maps of ω describes the expensiveness or cheapness of houses accurate upto the degree 0.6.

Theorem 6.5.9 *Take a P-apx space $(U,\Re)$, $\omega \in FBSS(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then, $0 \leq \mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega) \leq 1$ and $0 \leq \mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega) \leq 1$ for each $e \in \breve{A}$.*

Proof. Take an FBSS $\omega = (\xi,\psi;\breve{A}) \in FBSS(U)$ and the parameters α,β satisfying $0 \lneqq \beta \leq \alpha \leq 1$. To prove $0 \leq \mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega) \leq 1$, we show that

$$\left|\underline{\Re}(\omega_{\langle e,\alpha\rangle})\right| \leq \left|\overline{\Re}(\omega_{\langle e,\beta\rangle})\right|$$

for each $e \in \breve{A}$. From Lemma 6.5.4, we have $\omega_{\langle e,\alpha\rangle} \subseteq \omega_{\langle e,\beta\rangle}$. Now, Theorem 1.7.3 gives that,

$$\underline{\Re}(\omega_{\langle e,\alpha\rangle}) \subseteq \overline{\Re}(\omega_{\langle e,\alpha\rangle}) \subseteq \overline{\Re}(\omega_{\langle e,\beta\rangle}).$$

So, $\left|\underline{\Re}(\omega_{\langle e,\alpha\rangle})\right| \leq \left|\overline{\Re}(\omega_{\langle e,\beta\rangle})\right|$, or the ratio $\frac{\left|\underline{\Re}(\omega_{\langle e,\alpha\rangle})\right|}{\left|\overline{\Re}(\omega_{\langle e,\beta\rangle})\right|}$ fluctuates between 0 and 1. Which certainly yields

$$0 \leq \mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega) \leq 1.$$

Similarly, one can verify that, $0 \leq \mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega) \leq 1$ for each $e \in \breve{A}$. ■

Corollary 6.5.10 *For the P-apx space* $(U,\Re)$, $\omega \in FBSS(U)$ *and* $0 \lneqq \beta \leq \alpha \leq 1$, *we have,* $0 \leq \rho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega) \leq 1$ *and* $0 \leq \varrho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega) \leq 1$ *for each* $e \in \breve{A}$.

Proof. Definitions 6.5.6, 6.5.7 and Theorem 6.5.9 certify these asservations directly. ■

Theorem 6.5.11 *Take a P-apx space* $(U,\Re)$, $\omega = (\xi,\psi;\breve{A}) \in FBSS(U)$, $e \in \breve{A}$ *and* $0 \lneqq \beta \leq \alpha \leq 1$.

1. *If* α *stands fixed, then* $\mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega)$ *and* $\mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega)$ *increase with the increase in* β.
2. *If* β *stands fixed, then* $\mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega)$ *and* $\mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega)$ *decrease with the increase in* α.

Proof. (1) Let α stand fixed and let $0 \lneqq \beta_1 \leq \beta_2 \leq 1$. For any $e \in \breve{A}$, we have $\omega_{\langle e,\beta_2\rangle} \subseteq \omega_{\langle e,\beta_1\rangle}$ from Lemma 6.5.4. This gives $\overline{\Re}(\omega_{\langle e,\beta_2\rangle}) \subseteq \overline{\Re}(\omega_{\langle e,\beta_1\rangle})$. That is, $\left|\overline{\Re}(\omega_{\langle e,\beta_2\rangle})\right| \leq \left|\overline{\Re}(\omega_{\langle e,\beta_1\rangle})\right|$. Which implies that,

$$\frac{\left|\underline{\Re}(\omega_{\langle e,\alpha\rangle})\right|}{\left|\overline{\Re}(\omega_{\langle e,\beta_1\rangle})\right|} \leq \frac{\left|\underline{\Re}(\omega_{\langle e,\alpha\rangle})\right|}{\left|\overline{\Re}(\omega_{\langle e,\beta_2\rangle})\right|}.$$

That is, $\mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta_1\rangle}(\omega) \leq \mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta_2\rangle}(\omega)$. This verifies that, $\mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega)$ increases with the increase in β. In the same manner, one can verify that, $\mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega)$ increases with the increase in β.

(2) Analogous to the proof of (1). ■

Corollary 6.5.12 *Take a P-apx space* $(U,\Re)$, $\omega = (\xi,\psi;\breve{A}) \in FBSS(U)$, $e \in \breve{A}$ *and* $0 \lneqq \beta \leq \alpha \leq 1$.

1. *If* α *stands fixed, then* $\rho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega)$ *and* $\varrho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega)$ *decrease with the increase in* β.
2. *If* β *stands fixed, then* $\rho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega)$ *and* $\varrho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega)$ *increase with the increase in* α.

Proof. Definitions 6.5.6, 6.5.7 and Theorem 6.5.11 verify these asservations directly. ■

Theorem 6.5.13 *Take a P-apx space* $(U,\Re)$ *and* ω_1, $\omega_2 \in FBSS(U)$. *Then,* $\omega_1 \widetilde{\subseteq} \omega_2$ *implies the subsequent asservations for each* $e \in \breve{A}_1$ *and* $0 \lneqq \beta \leq \alpha \leq 1$.

1. $\mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1) \leq \mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_2)$, *whenever,* $\overline{\Re}(\omega_{1\langle e,\beta\rangle}) = \overline{\Re}(\omega_{2\langle e,\beta\rangle})$,
2. $\mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1) \geq \mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_2)$, *whenever,* $\underline{\Re}(\omega_{1\langle e,\alpha\rangle}) = \underline{\Re}(\omega_{2\langle e,\alpha\rangle})$,

3. $\mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1) \leq \mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_2)$, *whenever,* $\overline{\Re}(\omega_1^{\langle e,\alpha\rangle}) = \overline{\Re}(\omega_2^{\langle e,\alpha\rangle})$,

4. $\mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1) \geq \mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_2)$, *whenever,* $\underline{\Re}(\omega_1^{\langle e,\beta\rangle}) = \underline{\Re}(\omega_2^{\langle e,\beta\rangle})$.

Proof. (1) Let $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$, $\omega_2 = (\xi_2, \psi_2; \breve{A}_2)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Given that $\omega_1 \widetilde{\subseteq} \omega_2$ and $\overline{\Re}(\omega_{1\langle e,\beta\rangle}) = \overline{\Re}(\omega_{2\langle e,\beta\rangle})$. Lemma 6.5.3 gives, that, $\omega_{1\langle e,\alpha\rangle} \subseteq \omega_{2\langle e,\alpha\rangle}$. Theorem 1.7.3 gives that, $\underline{\Re}(\omega_{1\langle e,\alpha\rangle}) \subseteq \underline{\Re}(\omega_{2\langle e,\alpha\rangle})$, or, $\left|\underline{\Re}(\omega_{1\langle e,\alpha\rangle})\right| \leq \left|\underline{\Re}(\omega_{2\langle e,\alpha\rangle})\right|$. Which implies that,

$$\frac{\left|\underline{\Re}(\omega_{1\langle e,\alpha\rangle})\right|}{\left|\overline{\Re}(\omega_{1\langle e,\beta\rangle})\right|} \leq \frac{\left|\underline{\Re}(\omega_{2\langle e,\alpha\rangle})\right|}{\left|\overline{\Re}(\omega_{2\langle e,\beta\rangle})\right|}.$$

Hence proved, that, $\mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1) \leq \mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_2)$ for each $e \in \breve{A}_1$.

The remaining parts can be proved in the same manner. ■

Corollary 6.5.14 *Take a P-apx space* $(U, \Re)$ *and* ω_1, $\omega_2 \in FBSS(U)$. *Then,* $\omega_1 \widetilde{\subseteq} \omega_2$ *implies the subsequent asservations for each* $e \in \breve{A}_1$ *and* $0 \lneqq \beta \leq \alpha \leq 1$.

1. $\rho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1) \geq \rho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_2)$, *whenever,* $\overline{\Re}(\omega_{1\langle e,\beta\rangle}) = \overline{\Re}(\omega_{2\langle e,\beta\rangle})$,

2. $\rho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1) \leq \rho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_2)$, *whenever,* $\underline{\Re}(\omega_{1\langle e,\alpha\rangle}) = \underline{\Re}(\omega_{2\langle e,\alpha\rangle})$,

3. $\varrho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1) \geq \varrho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_2)$, *whenever,* $\overline{\Re}(\omega_1^{\langle e,\alpha\rangle}) = \overline{\Re}(\omega_2^{\langle e,\alpha\rangle})$,

4. $\varrho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1) \leq \varrho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_2)$, *whenever,* $\underline{\Re}(\omega_1^{\langle e,\beta\rangle}) = \underline{\Re}(\omega_2^{\langle e,\beta\rangle})$.

Proof. Definitions 6.5.6, 6.5.7 and Theorem 6.5.13 certify these asservations directly. ■

Theorem 6.5.15 *Take a P-apx space* $(U, \Re)$, $\omega \in FBSS(U)$ *and* $0 \lneqq \beta \leq \alpha \leq 1$. *If* σ *is an eqv-rel on* U *containing* $\Re$, *then,* $\mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega) \geq \mathfrak{Dp}^{\sigma}_{\langle e,\alpha,\beta\rangle}(\omega)$ *and* $\mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega) \leq \mathfrak{Dn}^{\sigma}_{\langle e,\alpha,\beta\rangle}(\omega)$ *for each* $e \in \breve{A}$.

Proof. Let $\Re$ and σ be two eqv-rels on U, satisfying $\Re \subseteq \sigma$. Theorem 6.3.9 states that $\underline{\Re}(\omega) \widetilde{\supseteq} \underline{\sigma}(\omega)$ and $\overline{\Re}(\omega) \widetilde{\subseteq} \overline{\sigma}(\omega)$ for any $\omega \in FBSS(U)$. From Lemma 6.5.3, we get $\underline{\Re}(\omega)_{\langle e,\alpha\rangle} \supseteq \underline{\sigma}(\omega)_{\langle e,\alpha\rangle}$ and $\overline{\Re}(\omega)_{\langle e,\beta\rangle} \subseteq \overline{\sigma}(\omega)_{\langle e,\beta\rangle}$ for each $e \in \breve{A}$. Lemma 6.5.5 gives

$$\left|\underline{\Re}(\omega_{\langle e,\alpha\rangle})\right| = \left|\underline{\Re}(\omega)_{\langle e,\alpha\rangle}\right| \geq \left|\underline{\sigma}(\omega)_{\langle e,\alpha\rangle}\right| = \left|\underline{\sigma}(\omega_{\langle e,\alpha\rangle})\right|$$

and

$$\left|\overline{\Re}(\omega_{\langle e,\beta\rangle})\right| = \left|\underline{\Re}(\omega)_{\langle e,\beta\rangle}\right| \leq \left|\underline{\sigma}(\omega)_{\langle e,\beta\rangle}\right| = \left|\underline{\sigma}(\omega_{\langle e,\beta\rangle})\right|.$$

This implies the following for each $e \in \breve{A}$.

$$\frac{\left|\underline{\Re}(\omega_{\langle e,\alpha\rangle})\right|}{\left|\overline{\Re}(\omega_{\langle e,\beta\rangle})\right|} \geq \frac{\left|\underline{\sigma}(\omega_{\langle e,\alpha\rangle})\right|}{\left|\overline{\sigma}(\omega_{\langle e,\beta\rangle})\right|}.$$

That is, $\mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega) \geq \mathfrak{Dp}^{\sigma}_{\langle e,\alpha,\beta\rangle}(\omega)$ for each $e \in \breve{A}$. On the same lines, one can verify, that, $\mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega) \leq \mathfrak{Dn}^{\sigma}_{\langle e,\alpha,\beta\rangle}(\omega)$ for each $e \in \breve{A}$. ■

Corollary 6.5.16 *Take a P-apx space $(U, \Re)$, $\omega \in FBSS(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. If σ is an eqv-rel on U containing $\Re$, then, $\rho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega) \leq \rho^{\sigma}_{\langle e,\alpha,\beta\rangle}(\omega)$ and $\varrho^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega) \geq \varrho^{\sigma}_{\langle ee,\alpha,\beta\rangle}(\omega)$ for each $e \in \breve{A}$.*

Proof. Definitions 6.5.6, 6.5.7 and Theorem 6.5.15 certify these asservations directly. ■

Theorem 6.5.17 *Take a P-apx space $(U, \Re)$, $0 \lneqq \beta \leq \alpha \leq 1$ and $\omega_1 = (\xi_1, \psi_1; \breve{A})$, $\omega_2 = (\xi_2, \psi_2; \breve{A}) \in FBSS(U)$, such that, $\omega_1 \simeq_{\Re} \omega_2$. Then, the subsequent asservations hold for each $e \in \breve{A}$.*

1. $\mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1 \widetilde{\sqcap}_{\varepsilon} \omega_2) \geq \mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1) \vee \mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_2)$,
2. $\mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1 \widetilde{\sqcap}_{\varepsilon} \omega_2) \geq \mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1) \vee \mathfrak{Dn}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_2)$.

Proof. (1) Let $0 \lneqq \beta \leq \alpha \leq 1$ and $\omega_1 \simeq_{\Re} \omega_2$. Then, $\underline{\Re}(\omega_1) = \underline{\Re}(\omega_2)$. Theorem 6.4.4 implies that, $\underline{\Re}(\omega_1 \widetilde{\sqcap}_{\varepsilon} \omega_2) = \underline{\Re}(\omega_1)$. This gives $\underline{\Re}(\omega_1 \widetilde{\sqcap}_{\varepsilon} \omega_2)_{\langle e,\alpha\rangle} = \underline{\Re}(\omega_1)_{\langle e,\alpha\rangle}$. That is,

$$\left|\underline{\Re}((\omega_1 \widetilde{\sqcap}_{\varepsilon} \omega_2)_{\langle e,\alpha\rangle})\right| = \left|\underline{\Re}(\omega_{1\langle e,\alpha\rangle})\right|. \tag{6.3}$$

On the other hand, $\omega_1 \widetilde{\sqcap}_{\varepsilon} \omega_2 \widetilde{\subseteq} \omega_1$. Which implies that, $\overline{\Re}(\omega_1 \widetilde{\sqcap}_{\varepsilon} \omega_2) \widetilde{\subseteq} \overline{\Re}(\omega_1)$. That is, $\overline{\Re}(\omega_1 \widetilde{\sqcap}_{\varepsilon} \omega_2)_{\langle e,\beta\rangle} \subseteq \overline{\Re}(\omega_1)_{\langle e,\beta\rangle}$. This gives

$$\left|\overline{\Re}((\omega_1 \widetilde{\sqcap}_{\varepsilon} \omega_2)_{\langle e,\beta\rangle})\right| \leq \left|\overline{\Re}(\omega_{1\langle e,\beta\rangle})\right|. \tag{6.4}$$

Equation 6.3 and Equation 6.4 yield the following.

$$\frac{\left|\underline{\Re}((\omega_1 \widetilde{\sqcap}_{\varepsilon} \omega_2)_{\langle e,\alpha\rangle})\right|}{\left|\overline{\Re}((\omega_1 \widetilde{\sqcap}_{\varepsilon} \omega_2)_{\langle e,\beta\rangle})\right|} \geq \frac{\left|\underline{\Re}(\omega_{1\langle e,\alpha\rangle})\right|}{\left|\overline{\Re}(\omega_{1\langle e,\beta\rangle})\right|}.$$

This proves that $\mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1 \widetilde{\sqcap}_{\varepsilon} \omega_2) \geq \mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1)$ for each $e \in \breve{A}$. Similarly, we have $\mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1 \widetilde{\sqcap}_{\varepsilon} \omega_2) \geq \mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_2)$ for each $e \in \breve{A}$. Which proves the following for each $e \in \breve{A}$.

$$\mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1 \widetilde{\sqcap}_{\varepsilon} \omega_2) \geq \mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_1) \vee \mathfrak{Dp}^{\Re}_{\langle e,\alpha,\beta\rangle}(\omega_2).$$

(2) Analogous to the proof of (1). ■

Corollary 6.5.18 *Take a P-apx space $(U, \Re)$, $0 \lneqq \beta \leq \alpha \leq 1$ and $\omega_1 = (\xi_1, \psi_1; \breve{A})$, $\omega_2 = (\xi_2, \psi_2; \breve{A}) \in FBSS(U)$, such that, $\omega_1 \simeq_{\Re} \omega_2$. Then, the subsequent asservations hold for each $e \in \breve{A}$.*

1. $\rho^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_1 \widetilde{\sqcap}_{\varepsilon} \omega_2) \leq \rho^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_1) \wedge \rho^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_2)$,
2. $\varrho^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_1 \widetilde{\sqcap}_{\varepsilon} \omega_2) \leq \varrho^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_1) \wedge \varrho^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_2)$.

Proof. Definitions 6.5.6, 6.5.7 and Theorem 6.5.17 verify these asservations directly. ■

Theorem 6.5.19 *Take a P-apx space* $(U, \mathfrak{R})$, $0 \lneqq \beta \leq \alpha \leq 1$ *and* $\omega_1 = (\xi_1, \psi_1; \breve{A})$, $\omega_2 = (\xi_2, \psi_2; \breve{A}) \in FBSS(U)$, *such that,* $\omega_1 \eqsim_{\mathfrak{R}} \omega_2$. *Then, the subsequent asservations hold for each* $e \in \breve{A}$.

1. $\mathfrak{Dp}^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_1 \widetilde{\sqcup}_{\varepsilon} \omega_2) \geq \mathfrak{Dp}^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_1) \vee \mathfrak{Dp}^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_2)$,
2. $\mathfrak{Dn}^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_1 \widetilde{\sqcup}_{\varepsilon} \omega_2) \geq \mathfrak{Dn}^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_1) \vee \mathfrak{Dn}^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_2)$.

Proof. Parallel to the proof of the Theorem 6.5.17. ■

Corollary 6.5.20 *Take a P-apx space* $(U, \mathfrak{R})$, $0 \lneqq \beta \leq \alpha \leq 1$ *and* $\omega_1 = (\xi_1, \psi_1; \breve{A})$, $\omega_2 = (\xi_2, \psi_2; \breve{A}) \in FBSS(U)$, *such that,* $\omega_1 \eqsim_{\mathfrak{R}} \omega_2$. *Then, the subsequent asservations hold for each* $e \in \breve{A}$.

1. $\rho^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_1 \widetilde{\sqcup}_{\varepsilon} \omega_2) \leq \rho^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_1) \wedge \rho^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_2)$,
2. $\varrho^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_1 \widetilde{\sqcup}_{\varepsilon} \omega_2) \leq \varrho^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_1) \wedge \varrho^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_2)$.

Proof. Definitions 6.5.6, 6.5.7 and Theorem 6.5.19 verify these asservations directly. ■

Theorem 6.5.21 *Take a P-apx space* $(U, \mathfrak{R})$, $0 \lneqq \beta \leq \alpha \leq 1$ *and* $\omega_1 = (\xi_1, \psi_1; \breve{A})$, $\omega_2 = (\xi_2, \psi_2; \breve{A}) \in FBSS(U)$, *such that,* $\omega_1 \approx_{\mathfrak{R}} \omega_2$. *Then, the subsequent asservations hold for each* $e \in \breve{A}$.

1. $\mathfrak{Dp}^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_1) = \mathfrak{Dp}^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_2)$,
2. $\mathfrak{Dn}^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_1) = \mathfrak{Dn}^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_2)$.

Proof. (1) Let $0 \lneqq \beta \leq \alpha \leq 1$ and $\omega_1 \approx_{\mathfrak{R}} \omega_2$. Then, $\underline{\mathfrak{R}}(\omega_1) = \underline{\mathfrak{R}}(\omega_2)$ and $\overline{\mathfrak{R}}(\omega_1) = \overline{\mathfrak{R}}(\omega_2)$. Lemma 6.5.5 implies, that, $\underline{\mathfrak{R}}(\omega_{1\langle e,\alpha\rangle}) = \underline{\mathfrak{R}}(\omega_{2\langle e,\alpha\rangle})$ and $\overline{\mathfrak{R}}(\omega_{1\langle e,\beta\rangle}) = \overline{\mathfrak{R}}(\omega_{2\langle e,\beta\rangle})$. This yields the following for each $e \in \breve{A}$.

$$\frac{\left|\underline{\mathfrak{R}}(\omega_{1\langle e,\alpha\rangle})\right|}{\left|\overline{\mathfrak{R}}(\omega_{1\langle e,\beta\rangle})\right|} = \frac{\left|\underline{\mathfrak{R}}(\omega_{2\langle e,\alpha\rangle})\right|}{\left|\overline{\mathfrak{R}}(\omega_{2\langle e,\beta\rangle})\right|}.$$

This verifies, that, $\mathfrak{Dp}^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_1) = \mathfrak{Dp}^{\mathfrak{R}}_{\langle e,\alpha,\beta\rangle}(\omega_2)$ for each $e \in \breve{A}$.
(2) Analogous to the proof of (1). ■

Corollary 6.5.22 *Take a P-apx space* $(U, \Re)$, $0 \lneqq \beta \leq \alpha \leq 1$ *and* $\omega_1 = (\xi_1, \psi_1; \breve{A})$, $\omega_2 = (\xi_2, \psi_2; \breve{A}) \in FBSS(U)$ *be such that* $\omega_1 \approx_{\Re} \omega_2$. *Then, the subsequent asservations hold for each* $e \in \breve{A}$.

1. $\rho^{\Re}_{\langle e,\alpha,\beta \rangle}(\omega_1) = \rho^{\Re}_{\langle e,\alpha,\beta \rangle}(\omega_2)$,
2. $\varrho^{\Re}_{\langle e,\alpha,\beta \rangle}(\omega_1) = \varrho^{\Re}_{\langle e,\alpha,\beta \rangle}(\omega_2)$.

Proof. Definitions 6.5.6, 6.5.7 and Theorem 6.5.21 verify these asservations directly. ■

6.6 Application of RFBS approximations

Decision making is a major area to be conferred in almost all kinds of data analysis. It is often required to decide for the optimum object in U. But sometimes, one may be unable to make the best decision, even when the best decision is known. In that case, it may be helpful if the worst decision also becomes visible. We propose an algorithm which provides the best, as well as, the worst decision. With the help of this algorithm, one can avoid making the worst decision as well. Let U be the sets of objects under consideration, $\breve{E}$ be the set of attributes for U and let $\breve{A} = \{e_i : 1 \leq i \leq n\}$ be a subset of $\breve{E}$ containing the attributes of interest or the choice attributes. The information about the objects is represented by an FBSS $\omega = (\xi, \psi; \breve{A})$ in tabular form, whose $(i,j)^{th}$ entry (a_{ij}, b_{ij}) depicts the information about the object $u_j \in U$ relative to the attribute $e_i \in \breve{A}$. First we assign the indiscernibility grades to each object and then define the indiscernibility relation with the help of indiscernibility grades associated with ω.

Definition 6.6.1 *The indiscernibility grades* G_{ij} *corresponding to each object* $u_j \in U$ *and each attribute* $e_i \in \breve{A}$ *are given by*

$$G_{ij} = \begin{cases} P & \textit{if } a_{ij} \gneqq b_{ij} \\ N & \textit{if } a_{ij} \lneqq b_{ij} \\ O & \textit{if } a_{ij} = b_{ij} \end{cases} \tag{6.5}$$

The indiscernibility grades depict the following information about the objects.

- If $G_{ij} = P$, the object u_j has positive belongingness value a_{ij} higher than the negative belongingness value b_{ij}, relative to e_i.
- If $G_{ij} = N$, the object u_j has negative belongingness value b_{ij} higher than the positive belongingness value a_{ij}, relative to e_i.

- If $G_{ij} = O$, the object u_j has positive belongingness a_{ij} equal to the negative belongingness b_{ij}, relative to e_i.

Now we give the concept of indiscernibility relation on U associated with the FBSS ω. We say that two objects u_j and u_k are indiscernible, written as $u_j \sim u_k$, if and only if they have same grades for each e_i. Thus, when we say that the objects u_j and u_k are indiscernible, it means that, either both the objects have positivity higher than the negativity, or both the objects have negativity higher than the positivity, or both the objects have equal measures of positivity and negativity. The indiscernibility relation $\Re$ between the objects of U is constructed as:

$$\Re = \{(u_j, u_k) \in U \times U : u_j \sim u_k\}. \tag{6.6}$$

Surely, $\Re$ is an eqv-rel on U.

Definition 6.6.2 *The indiscernibility parameter N has the values n_j corresponding to each object $u_j \in U$, given by*

$$n_j = \sum_{i=1}^{n}(a_{ij} - b_{ij}).$$

The parameter N represents the difference between the degree of positivity and the degree of negativity for each object u_j, cumulative to all parameters. In the same way, the values of the parameter $\underline{N}_{\Re}$ can be calculated by $\underline{n_j}_{\Re} = \sum_{i=1}^{n}(\underline{a_{ij}}_{\Re} - \underline{b_{ij}}_{\Re})$ and the parameter $\overline{N}^{\Re}$ by $\overline{n_j}^{\Re} = \sum_{i=1}^{n}(\overline{a_{ij}}^{\Re} - \overline{b_{ij}}^{\Re})$, where $(\underline{a_{ij}}_{\Re}, \underline{b_{ij}}_{\Re})$ and $(\overline{a_{ij}}^{\Re}, \overline{b_{ij}}^{\Re})$ are the $(i,j)th$ entries in the tables of $\underline{\Re}(\omega)$ and $\overline{\Re}(\omega)$, respectively. Here, $\underline{n_j}_{\Re}$ represents the definite fulfilment of the object u_j, while, $\overline{n_j}^{\Re}$ represents the maximum possible fulfilment of the object u_j, towards ω. Thus, the uncertain (or doubtful) fulfilment of u_j is given by the difference $\overline{n_j}^{\Re} - \underline{n_j}_{\Re}$.The table of ω is consistent if and only if $\Re \subseteq IND(N)$, where $IND(N)$ is the eqv-rel on U, dividing U into the classes having same values n_j. Now, we proceed to the decision values d_j for the objects.

Definition 6.6.3 *The values d_j of the decision parameter D for each object $u_j \in U$, given by*

$$d_j = \underline{n_j}_{\Re} + \overline{n_j}^{\Re}.$$

This value gives the definite fulfilment $\underline{n_j}_{\Re}$ of the object u_j, a double weightage than to the uncertain (doubtful) fulfilment $\overline{n_j}^{\Re} - \underline{n_j}_{\Re}$, because we have

$$d_j = \underline{n_j}_{\Re} + \overline{n_j}^{\Re} = 2\underline{n_j}_{\Re} - \underline{n_j}_{\Re} + \overline{n_j}^{\Re},$$

or,

$$d_j = 2\underline{n_j}_{\Re} + (\overline{n_j}^{\Re} - \underline{n_j}_{\Re}). \tag{6.7}$$

We can rewrite Equation 6.7 as:

$$d_j = 2\sum_{i=1}^{n}(\underline{a_{ij}}_{\Re} - \underline{b_{ij}}_{\Re}) + \sum_{i=1}^{n}((\overline{a_{ij}}^{\Re} - \overline{b_{ij}}^{\Re}) - (\underline{a_{ij}}_{\Re} - \underline{b_{ij}}_{\Re})) \tag{6.8}$$

From Equation 6.8, it is clear that the higher the definite positive fulfilment $\underline{a_{ij}}$ of u_j, the larger the value d_j. Also, the higher the definite negative fulfilment $\underline{b_{ij}}$ of u_j, the smaller the value d_j. In this way, we identify the poor objects having lowest value of d_j. These are the objects with high definite negative fulfilment to ω. Hence, our algorithm has the following main advantages.

- It manipulates technically the fuzziness of the data enriched with the bipolarity of information.
- It accommodates the opinions about the objects with respect to any (finite) number of attributes.
- It gives double weightage to the definite fulfilment of the objects, than to the uncertain fulfilment.
- It yields a wise decision, containing the best, as well as, the poor decision, so that, one can sidestep the poor decision.

Main steps of the algorithm are as follows.

Algorithm 6.6.4 *The algorithm to decide for the best and the worst object in U is as follows.*

1. *Input the set of choice attributes $\breve{A} \subseteq \breve{E}$.*
2. *Input the FBSS $\omega = (\xi, \psi; \breve{A})$.*
3. *Construct the indiscernibility relation $\Re$ on U and find the values of the indiscernibility parameter N. Check the consistency of the table of ω.*
4. *Evaluate $\underline{\Re}(\xi, \psi; \breve{A})$ and $\overline{\Re}(\xi, \psi; \breve{A})$ for the FBSS $(\xi, \psi; \breve{A})$ using the indiscernibility relation $\Re$ defined in Formula 6.6. Also find the values $\underline{n_j}_{\Re}$ and $\overline{n_j}^{\Re}$.*
5. *Find the decision values $d_j = \underline{n_j}_{\Re} + \overline{n_j}^{\Re}$ for each object $u_j \in U$.*
6. *Construct the decision table having columns of U and the decision parameter D only, by rearranging in the descending order with respect to the decision values d_j. Choose k and l, so that $d_k = \max_j d_j$ and $d_l = \min_j d_j$. Then u_k is the best optimal object, while u_l is the worst optimal object to be decided.*

The flow chart of Algorithm 6.6.4 is shown in Figure 6.1.

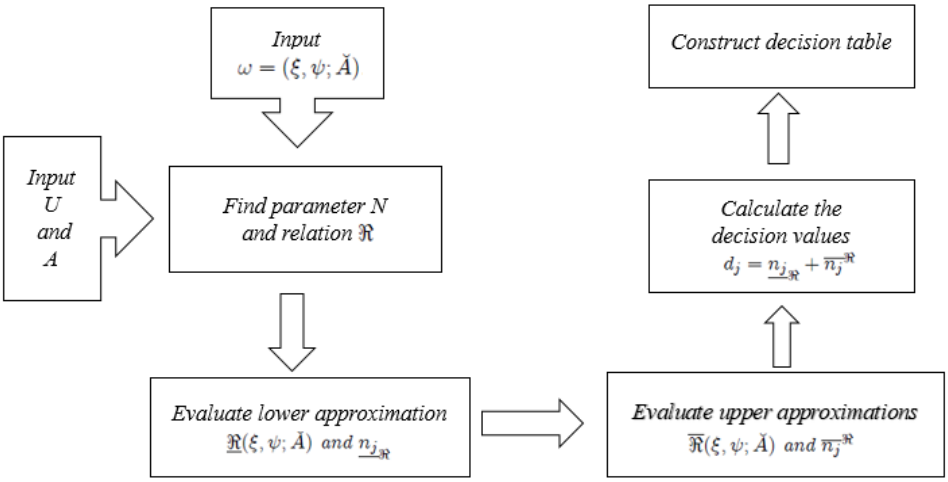

Figure 6.1: Flow chart of Algorithm 6.6.4

As an illustration, we apply this algorithm to an example.

Example 6.6.5 *Take a collection $U = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ of some construction companies considered by Mrs. X for the construction of her home and consider the attribute set $\check{E} = \{e_1$ =strong structure, e_2 =innovative designs, e_3 =high quality materials, e_4 =good reputation, e_5 =well organized, e_6 =competitive pricing, e_7 =having own crew, e_8 =decisiveness, e_9 =flexibility, e_{10} =skilled crew$\}$ and $\neg\check{E} = \{\neg e_1$ =weak structure, $\neg e_2$ =traditional designs, $\neg e_3$ =low quality materials, $\neg e_4$ =ill reputation, $\neg e_5$ =disorganized, $\neg e_6$ =high pricing, $\neg e_7$ =not having own crew, $\neg e_8$ =indecisive, $\neg e_9$ =rigidity, $\neg e_{10}$ =unskilled crew$\}$. Let the "Quality Analysis" of construction work be described by an FBSS $\omega = (\xi, \psi; \breve{A})$ given in Table 6.1.*

1. *Input $\breve{A} = \{e_1, e_2, e_4, e_6, e_7, e_{10}\}$.*

2. *Input the FBSS $\omega = (\xi, \psi; \breve{A})$ as in Table 6.1.*

ω	c_1	c_2	c_3	c_4	c_5	c_6
e_1	$(0.6, 0.2)$	$(0.5, 0.5)$	$(0.6, 0.3)$	$(0.3, 0.5)$	$(0.6, 0.2)$	$(0.4, 0.4)$
e_2	$(0.6, 0.4)$	$(0.5, 0.4)$	$(0.6, 0.2)$	$(0.7, 0.3)$	$(0.5, 0.5)$	$(0.3, 0.4)$
e_4	$(0.7, 0.1)$	$(0.4, 0.4)$	$(0.6, 0.2)$	$(0.3, 0.5)$	$(0.5, 0.4)$	$(0.4, 0.4)$
e_6	$(0.5, 0.5)$	$(0.6, 0.3)$	$(0.4, 0.5)$	$(0.6, 0.3)$	$(0.4, 0.5)$	$(0.5, 0.4)$
e_7	$(0.4, 0.5)$	$(0.3, 0.6)$	$(0.6, 0.2)$	$(0.7, 0.2)$	$(0.6, 0.4)$	$(0.4, 0.4)$
e_{10}	$(0.7, 0.1)$	$(0.6, 0.3)$	$(0.5, 0.3)$	$(0.5, 0.4)$	$(0.4, 0.5)$	$(0.3, 0.5)$

Table 6.1: FBSS $\omega = (\xi, \psi; \breve{A})$

3. *The indiscernibility grades are assigned to* $c_j \in U$ *and the values of the indiscernibility parameter* N *are calculated in Table 6.2.*

ω	c_1	c_2	c_3	c_4	c_5	c_6
e_1	P	O	P	N	P	O
e_2	P	P	P	P	O	N
e_4	P	O	P	N	P	O
e_6	O	P	N	P	N	P
e_7	N	N	P	P	P	O
e_{10}	P	P	P	P	N	N
N	1.7	0.4	1.6	0.9	0.5	−0.2

Table 6.2: Calculations of G_{ij} *and the values of* N

We find that

$$\begin{aligned} \Re &= \{(c_1,c_1),(c_2,c_2),(c_3,c_3),(c_4,c_4),(c_5,c_5),(c_6,c_6)\} \\ &= IND(N). \end{aligned}$$

Which indicates that the table of ω *is consistent.*

4. *Since* $\Re$ *is the identity relation on* U, *so,* ω *is* $\Re-$*definable by Theorem 6.3.8. That is,* $\underline{\Re}(\omega) = \overline{\Re}(\omega)$. *This gives* $\underline{n_j}_{\Re} = n_j = \overline{n_j}^{\Re}$ *for each* $c_j \in U$.

5. *The values* $d_j = \underline{n_j}_{\Re} + \overline{n_j}^{\Re} = 2n_j$ *of the decision parameter* D *for each* $c_j \in U$, *are evaluated in the Table 6.3.*

U	c_1	c_2	c_3	c_4	c_5	c_6
e_1	$(0.6, 0.2)$	$(0.5, 0.5)$	$(0.6, 0.3)$	$(0.3, 0.5)$	$(0.6, 0.2)$	$(0.4, 0.4)$
e_2	$(0.6, 0.4)$	$(0.5, 0.4)$	$(0.6, 0.2)$	$(0.7, 0.3)$	$(0.5, 0.5)$	$(0.3, 0.4)$
e_4	$(0.7, 0.1)$	$(0.4, 0.4)$	$(0.6, 0.2)$	$(0.3, 0.5)$	$(0.5, 0.4)$	$(0.4, 0.4)$
e_6	$(0.5, 0.5)$	$(0.6, 0.3)$	$(0.4, 0.5)$	$(0.6, 0.3)$	$(0.4, 0.5)$	$(0.5, 0.4)$
e_7	$(0.4, 0.5)$	$(0.3, 0.6)$	$(0.6, 0.2)$	$(0.7, 0.2)$	$(0.6, 0.4)$	$(0.4, 0.4)$
e_{10}	$(0.7, 0.1)$	$(0.6, 0.3)$	$(0.5, 0.3)$	$(0.5, 0.4)$	$(0.4, 0.5)$	$(0.3, 0.5)$
N	1.7	0.4	1.6	0.9	0.5	−0.2
D	3.4	0.8	3.2	1.8	1.0	−0.4

Table 6.3: Calculation of the decision parameter D

6. *Table 6.4 is the decision table.*

U	D
c_1	3.4
c_3	3.2
c_4	1.8
c_5	1.0
c_2	0.8
c_6	−0.4

Table 6.4: The decision table of ω

We get $\max_j d_j = d_1 = 3.4$ *and* $\min_j d_j = d_6 = -0.4$. *Hence* $k = 1$ *and* $l = 6$. *Thus, the company* c_1 *is the best selection. If Mrs. X could not make a deal with* c_1 *for some reason, then,* c_3 *will be the second best decision. But, in any case, she must not go for* c_6.

Chapter 7

Rough fuzzy bipolar soft ideals over semigroups

7.1 Introduction

Rough ideals in semigroups were first discussed by Kuroki [37] in 1997. Soft ideals over semigroups were initiated by Ali et al. [12]. Fuzzy ideals in semigroups were discussed by many authors; see [1, 2, 31, 33, 56, 58, 59, 60]. Later on, Yang [62] presented the fuzzy soft ideals over semigroups. These concepts motivate the idea of continuing the work of Chapter 6 in the direction of the semigroups and ideals in semigroups. We define and discuss the notion of the FBS subsemigroups, FBS left ideals (FBSl-ids), FBS right ideals (FBSr-ids), FBS two-sided ideals (FBS-ids), FBS interior ideals (FBSi-ids) and FBS bi-ideals (FBSb-ids) over semigroups. The roughness in FBSSs and FBS subsemigroups under a cng-rel defined on the semigroup is also studied. We further present the concept of roughness in FBSl-ids, FBSr-ids, FBS-ids, FBSi-ids and FBSb-ids over a semigroup by defining the lower and upper RFBS-apxes of these ideals over a semigroup and investigate some of their basic properties.

7.2 Fuzzy bipolar soft sets over semigroups

The fuzzy bipolar soft (FBS) subsemigroups are constructed by hybridizing the RFBS-apxes of the FBSSs with the semigroups. Throughout this chapter, Υ is a semigroup, $\breve{E}$ is the set of attributes for Υ and $\Re$ is a cng-rel on Υ. Recall that an FBSS over a semigroup Υ is symbolized by $\omega = (\xi, \psi; \breve{A})$, where $\breve{A} \subseteq \breve{E}$ and ξ, ψ are mappings given by $\xi : \breve{A} \to \digamma_z(\Upsilon)$ and $\psi : \neg\breve{A} \to \digamma_z(\Upsilon)$ with a consistency restraint

$$0 \leq \xi(e)(a) + \psi(\neg e)(a) \leq 1 \tag{7.1}$$

for each $e \in \breve{A}$ and for each $a \in \Upsilon$. We denote the set containing all FBSSs over Υ by $FBSS(\Upsilon)$.

Definition 7.2.1 *For any two FBSSs $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$, $\omega_2 = (\xi_2, \psi_2; \breve{A}_2)$ over Υ, the composition $\omega_1 \widehat{\circ}\, \omega_2$ of ω_1 and ω_2 is the FBSS $(\xi_1 \circ \xi_2, \psi_1 \circ \psi_2; \breve{A}_1 \cap \breve{A}_2)$ over Υ, where*

$$(\xi_1 \circ \xi_2)(e)(a) = \begin{cases} \bigvee\limits_{a=bc} (\xi_1(e)(b) \wedge \xi_2(e)(c)) & \text{if } a = bc \text{ for some } b, c \in \Upsilon \\ 0 & \text{otherwise} \end{cases}$$

and

$$(\psi_1 \circ \psi_2)(\neg e)(a) = \begin{cases} \bigwedge\limits_{a=bc} (\psi_1(\neg e)(b) \vee \psi_2(\neg e)(c)) & \text{if } a = bc \text{ for some } b, c \in \Upsilon \\ 0 & \text{otherwise} \end{cases}$$

for each $e \in \breve{A}_1 \cap \breve{A}_2$ and for each $a \in \Upsilon$.

Definition 7.2.2 *An FBSS $\omega = (\xi, \psi; \breve{A})$ over Υ is called an FBS subsemigroup over Υ, if for each $a, b \in \Upsilon$ and for each $e \in \breve{A}$, we have*

$$\begin{aligned} \xi(e)(ab) &\geqslant \xi(e)(a) \wedge \xi(e)(b), \\ \psi(\neg e)(ab) &\leqslant \psi(\neg e)(a) \vee \psi(\neg e)(b). \end{aligned}$$

Example 7.2.3 *Let $\Upsilon = \{a, b, c, d\}$ represent a semigroup whose table of binary operation is given below.*

	a	b	c	d
a	a	b	b	d
b	b	b	b	d
c	b	b	b	d
d	d	d	d	d

Let $\hat{E} = \{e_i : i = 1, 2, 3, 4\}$ and let $\omega = (\xi, \psi; \breve{A})$ be an FBSS over Υ with $\breve{A} = \{e_1, e_3\}$, such that:

$\xi(e_1) = \{a/0.2,\ b/0.6,\ c/0.6,\ d/0.5\}$,
$\psi(\neg e_1) = \{a/0.7,\ b/0.1,\ c/0.2,\ d/0.4\}$,
$\xi(e_3) = \{a/0.3,\ b/0.7,\ c/0.3,\ d/0.4\}$,
$\psi(\neg e_3) = \{a/0.6,\ b/0.2,\ c/0.5,\ d/0.5\}$.

Simple calculations verify that ω is an FBS subsemigroup over Υ.

Theorem 7.2.4 *Let ω_1 and ω_2 be any two FBS subsemigroups over Υ. Then, their restricted intersection $\omega_1 \widetilde{\sqcap}_r \omega_2$ and extended intersection $\omega_1 \widetilde{\sqcap}_\varepsilon \omega_2$ are also FBS subsemigroups over Υ.*

Proof. Take FBS subsemigroups $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$ and $\omega_2 = (\xi_2, \psi_2; \breve{A}_2)$ over Υ. Their extended intersection is given by $\omega_1 \widetilde{\sqcap}_\varepsilon \omega_2 = (\xi_1 \widetilde{\sqcap}_\varepsilon \xi_2, \psi_1 \widetilde{\sqcup}_\varepsilon \psi_2; \breve{A}_1 \cup \breve{A}_2)$. Take any $a, b \in \Upsilon$. Then, the following cases arise.

Case I:

Let $e \in \breve{A}_1 \cap \breve{A}_2$. Since ω_1 and ω_2 are FBS subsemigroups over Υ, so we have the following for each $a, b \in \Upsilon$.

$$\begin{aligned}(\xi_1 \widetilde{\sqcap}_\varepsilon \xi_2)(e)(ab) &= \xi_1(e)(ab) \wedge \xi_2(e)(ab) \\ &\geqslant (\xi_1(e)(a) \wedge \xi_1(e)(b)) \wedge (\xi_2(e)(a) \wedge \xi_2(e)(b)) \\ &= (\xi_1(e)(a) \wedge \xi_2(e)(a)) \wedge (\xi_1(e)(b) \wedge \xi_2(e)(b)) \\ &= (\xi_1 \widetilde{\sqcap}_\varepsilon \xi_2)(e)(a) \wedge (\xi_1 \widetilde{\sqcap}_\varepsilon \xi_2)(e)(b).\end{aligned}$$

Similarly,

$$(\psi_1 \widetilde{\sqcup}_\varepsilon \psi_2)(\neg e)(ab) \leqslant (\psi_1 \widetilde{\sqcup}_\varepsilon \psi_2)(\neg e)(a) \vee (\psi_1 \widetilde{\sqcup}_\varepsilon \psi_2)(\neg e)(b)$$

for each $a, b \in \Upsilon$ and $e \in \breve{A}_1 \cap \breve{A}_2$.

Case II:

Let $e \in A_1 - A_2$. Then, for each $a, b \in \Upsilon$, we have

$$\begin{aligned}(\xi_1 \widetilde{\sqcap}_\varepsilon \xi_2)(e)(ab) &= \xi_1(e)(ab) \\ &\geqslant \xi_1(e)(a) \wedge \xi_1(e)(b) \\ &= (\xi_1 \widetilde{\sqcap}_\varepsilon \xi_2)(e)(a) \wedge (\xi_1 \widetilde{\sqcap}_\varepsilon \xi_2)(e)(b).\end{aligned}$$

Similarly,

$$(\psi_1 \widetilde{\sqcup}_\varepsilon \psi_2)(\neg e)(ab) \leqslant (\psi_1 \widetilde{\sqcup}_\varepsilon \psi_2)(\neg e)(a) \vee (\psi_1 \widetilde{\sqcup}_\varepsilon \psi_2)(\neg e)(b)$$

for each $a, b \in \Upsilon$ and $e \in A_1 - A_2$.

Same is the case when $e \in A_2 - A_1$.

Thus, by Definition 7.2.2, it is proved that the extended intersection $\omega_1 \widetilde{\sqcap}_\varepsilon \omega_2$ is an FBS subsemigroups over Υ. In the same way, it can be shown that the restricted intersection $\omega_1 \widetilde{\sqcap}_r \omega_2$ is an FBS subsemigroup over Υ. ■

The extended or restricted union of ω_1 and ω_2 may not be an FBS subsemigroup over Υ. This is shown in the subsequent example.

Example 7.2.5 *Let $\Upsilon = \{k, l, m, n\}$ represent a semigroup whose table of binary operation is given below.*

	k	l	m	n
k	k	k	k	n
l	k	l	k	n
m	k	k	m	n
n	n	n	n	n

Let $\hat{E} = \{e_1, e_2, e_3\}$. *We take two FBS subsemigroups* $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$ *and* $\omega_2 = (\xi_2, \psi_2; \breve{A}_2)$ *over* Υ *with* $\breve{A}_1 = \{e_1, e_3\}$ *and* $\breve{A}_2 = \{e_1, e_2\}$, *defined as below.*

$\xi_1(e_1) = \{k/0.2,\ l/0.6,\ m/0.1,\ n/0.5\}$,
$\psi_1(\neg e_1) = \{k/0.4,\ l/0.2,\ m/0.6,\ n/0.5\}$,
$\xi_1(e_3) = \{k/0.9,\ l/0.5,\ m/0.3,\ n/0.1\}$,
$\psi_1(\neg e_3) = \{k/0.1,\ l/0.4,\ m/0.5,\ n/0.8\}$,
$\xi_2(e_1) = \{k/0.3,\ l/0.2,\ m/0.7,\ n/0.4\}$,
$\psi_2(\neg e_1) = \{k/0.5,\ l/0.6,\ m/0.1,\ n/0.5\}$,
$\xi_2(e_2) = \{k/0.7,\ l/0.6,\ m/0.5,\ n/0.4\}$,
$\psi_2(\neg e_2) = \{k/0.3,\ l/0.4,\ m/0.5,\ n/0.6\}$.

Simple calculations verify that $\omega_1 \tilde{\sqcap}_\varepsilon \omega_2$ *and* $\omega_1 \tilde{\sqcap}_r \omega_2$ *are FBS subsemigroups over* Υ. *The restricted union* $\omega_1 \tilde{\sqcup}_r \omega_2 = (\xi_1 \tilde{\cup}_r \xi_2, \psi_1 \tilde{\cap}_r \psi_2; \breve{A}_1 \cap \breve{A}_2)$ *is calculated below.*

$(\xi_1 \tilde{\cup}_r \xi_2)(e_1) = \{k/0.3,\ l/0.6,\ m/0.7,\ n/0.5\}$,
$(\psi_1 \tilde{\cap}_r \psi_2)(\neg e_1) = \{k/0.4,\ l/0.2,\ m/0.1,\ n/0.5\}$.

We find that

$$\begin{aligned}(\xi_1 \tilde{\cup}_r \xi_2)(e_1)(lm) &= (\xi_1 \tilde{\cup}_r \xi_2)(e_1)(k) = 0.3 \\ &\ngeqq (\xi_1 \tilde{\cup}_r \xi_2)(e_1)(l) \wedge (\xi_1 \tilde{\cup}_r \xi_2)(e_1)(m) = 0.6.\end{aligned}$$

So, $\omega_1 \tilde{\sqcup}_r \omega_2$ *(and similarly* $\omega_1 \tilde{\sqcup}_\varepsilon \omega_2$*) is not an FBS subsemigroups over* Υ.

Note that $\omega_1 \tilde{\sqcup}_\varepsilon \omega_2$ is trivially an FBS subsemigroups over Υ if $\breve{A}_1 \cap \breve{A}_2 = \phi$. Recall the Definition 6.5.1 of the $\alpha-$level P-cuts and the Definition 6.5.2 of the $\alpha-$level N-cuts of ω relative to the attribute $e \in \breve{A}$, defined respectively as:

$$\begin{aligned}\omega_{\langle e,\alpha\rangle} &= \{u \in \Upsilon : \xi(e)(u) \geq \alpha\}, \\ \omega^{\langle e,\alpha\rangle} &= \{u \in \Upsilon : \psi(\neg e)(u) \leq \alpha\}\end{aligned}$$

for each $0 \lneqq \alpha \leq 1$.

Theorem 7.2.6 *An FBSS* $\omega = (\xi, \psi; \breve{A})$ *over* Υ *is an FBS subsemigroup over* Υ *if and only if* $\omega_{\langle e,\alpha\rangle}$ *and* $\omega^{\langle e,\alpha\rangle}$, *if non-empty, are subsemigroups of* Υ, *for each* $\alpha \in [0, 1]$ *and for each* $e \in \breve{A}$.

Proof. Let $\omega = (\xi, \psi; \breve{A})$ be an FBS subsemigroup over Υ. Take any $e \in \breve{A}$ and $a, b \in \omega_{\langle e,\alpha\rangle}$. Then, $\xi(e)(a) \geqslant \alpha$ and $\xi(e)(b) \geqslant \alpha$. Since ω is an FBS subsemigroup of Υ, so

$$\xi(e)(ab) \geqslant \xi(e)(a) \wedge \xi(e)(b) \geqslant \alpha.$$

Which implies $ab \in \omega_{\langle e,\alpha\rangle}$. So, $\omega_{\langle e,\alpha\rangle}$ is a subsemigroup of Υ for each $\alpha \in [0, 1]$ and for each $e \in \breve{A}$. Similarly, $\omega^{\langle e,\alpha\rangle}$ is a subsemigroup of Υ for each $\alpha \in [0, 1]$ and for each $e \in \breve{A}$.

Conversely, let $\omega_{\langle e,\alpha\rangle}$ and $\omega^{\langle e,\alpha\rangle}$ be non-empty subsemigroups of Υ for each $\alpha \in [0,1]$ and for each $e \in \breve{A}$. Take any $a, b \in \Upsilon$ and denote $\xi(e)(a) \wedge \xi(e)(b)$ by $\alpha_e \in [0,1]$. Surely, $\xi(e)(a), \xi(e)(b) \geqslant \alpha_e$, and so $a, b \in \omega_{\langle e,\alpha_e\rangle}$. But $\omega_{\langle e,\alpha_e\rangle}$ is a subsemigroup of Υ. So $ab \in \omega_{\langle e,\alpha_e\rangle}$. Which yields $\xi(e)(ab) \geqslant \alpha_e$. That is,

$$\xi(e)(ab) \geqslant \xi(e)(a) \wedge \xi(e)(b). \tag{7.2}$$

Now, denote $\psi(\neg e)(a) \vee \psi(\neg e)(b)$ by $\alpha_{\neg e}$, where $\alpha_{\neg e} \in [0,1]$. Then, $\psi(\neg e)(a), \psi(\neg e)(b) \leqslant \alpha_{\neg e}$, and so $a, b \in \omega^{\langle e,\alpha_{\neg e}\rangle}$. But $\omega^{\langle e,\alpha_{\neg e}\rangle}$ is a subsemigroup of Υ. So $ab \in \omega^{\langle e,\alpha_{\neg e}\rangle}$. Which yields $\psi(\neg e)(ab) \leqslant \alpha_{\neg e}$. That is,

$$\psi(\neg e)(ab) \leqslant \psi(\neg e)(a) \vee \psi(\neg e)(b). \tag{7.3}$$

Assertions 7.2 and 7.3 combine to prove that ω is an FBS subsemigroup over Υ. ■

7.3 Fuzzy bipolar soft ideals over semigroups

We construct and confer, in this section, the FBSl-ids, FBSr-ids, FBS-ids, FBSi-ids and FBSb-ids over the semigroup Υ.

Definition 7.3.1 *An FBSS $\omega = (\xi, \psi; \breve{A})$ over Υ is called an FBSl-id over Υ, if for each $a, b \in \Upsilon$ and for each $e \in \breve{A}$, we have*

$$\begin{aligned} \xi(e)(ab) &\geqslant \xi(e)(b), \\ \psi(\neg e)(ab) &\leqslant \psi(\neg e)(b). \end{aligned}$$

Definition 7.3.2 *An FBSS $\omega = (\xi, \psi; \breve{A})$ over Υ is called an FBSr-id over Υ, if for each $a, b \in \Upsilon$ and for each $e \in \breve{A}$, we have*

$$\begin{aligned} \xi(e)(ab) &\geqslant \xi(e)(a), \\ \psi(\neg e)(ab) &\leqslant \psi(\neg e)(a). \end{aligned}$$

Definition 7.3.3 *An FBSS $\omega = (\xi, \psi; \breve{A})$ over Υ is called an FBS-id over Υ, if it is both, an FBSl-id and an FBSr-id over Υ. That is, for each $a, b \in \Upsilon$ and for each $e \in \breve{A}$, we have*

$$\begin{aligned} \xi(e)(ab) &\geqslant \xi(e)(a) \vee \xi(e)(b), \\ \psi(\neg e)(ab) &\leqslant \psi(\neg e)(a) \wedge \psi(\neg e)(b). \end{aligned}$$

Theorem 7.3.4 *The extended intersection $\omega_1 \widetilde{\sqcap}_\varepsilon \omega_2$ and the restricted intersection $\omega_1 \widetilde{\sqcap}_r \omega_2$ of any two FBSl-ids (FBSr-ids, FBS-ids) ω_1 and ω_2 over a semigroup Υ are FBSl-ids (FBSr-ids, FBS-ids) over Υ.*

Proof. Let $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$, $\omega_2 = (\xi_2, \psi_2; \breve{A}_2)$ be any two FBSl-ids over Υ. Their extended intersection is given by $\omega_1 \widetilde{\sqcap}_\varepsilon \omega_2 = (\xi_1 \widetilde{\cap}_\varepsilon \xi_2, \psi_1 \widetilde{\cup}_\varepsilon \psi_2; \breve{A}_1 \cup \breve{A}_2)$. Take any $a, b \in \Upsilon$. Then, the following cases arise.

Case I:

Let $e \in \breve{A}_1 \cap \breve{A}_2$. Since ω_1 and ω_2 are FBSl-ids over Υ, so, we have the following for each $a, b \in \Upsilon$.

$$\begin{aligned} (\xi_1 \widetilde{\cap}_\varepsilon \xi_2)(e)(ab) &= \xi_1(e)(ab) \wedge \xi_2(e)(ab) \\ &\geqslant \xi_1(e)(b) \wedge \xi_2(e)(b) \\ &= (\xi_1 \widetilde{\cap}_\varepsilon \xi_2)(e)(b). \end{aligned}$$

Similarly,

$$(\psi_1 \widetilde{\cup}_\varepsilon \psi_2)(\neg e)(ab) \leqslant (\psi_1 \widetilde{\cup}_\varepsilon \psi_2)(\neg e)(b)$$

for each $a, b \in \Upsilon$ and $e \in \breve{A}_1 \cap \breve{A}_2$.

Case II:

Let $e \in A_1 - A_2$. Then, for each $a, b \in \Upsilon$, we have

$$\begin{aligned} (\xi_1 \widetilde{\cap}_\varepsilon \xi_2)(e)(ab) &= \xi_1(e)(ab) \\ &\geqslant \xi_1(e)(b) = (\xi_1 \widetilde{\cap}_\varepsilon \xi_2)(e)(b). \end{aligned}$$

Similarly,

$$(\psi_1 \widetilde{\cup}_\varepsilon \psi_2)(\neg e)(ab) \leqslant (\psi_1 \widetilde{\cup}_\varepsilon \psi_2)(\neg e)(b)$$

for each $a, b \in \Upsilon$ and $e \in A_1 - A_2$.

Same is the case when $e \in A_2 - A_1$.

Thus, be Definition 7.3.1, it is proved that the extended intersection $\omega_1 \widetilde{\sqcap}_\varepsilon \omega_2$ is an FBSl-id over Υ. In the same way, it can be shown that the restricted intersection $\omega_1 \widetilde{\sqcap}_r \omega_2$ is an FBSl-id over Υ and that, $\omega_1 \widetilde{\sqcap}_\varepsilon \omega_2$ and $\omega_1 \widetilde{\sqcap}_r \omega_2$ are FBSr-ids and FBS-ids over Υ. ■

Theorem 7.3.5 *Let Υ be a semigroup and ω_1, ω_2 be two FBSl-ids (FBSr-ids, FBS-ids) over Υ. Then, their extended union $\omega_1 \widetilde{\sqcup}_\varepsilon \omega_2$ and restricted union $\omega_1 \widetilde{\sqcup}_r \omega_2$ are also FBSl-ids (FBSr-ids, FBS-ids) over Υ.*

Proof. Let $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$, $\omega_2 = (\xi_2, \psi_2; \breve{A}_2)$ be any two FBSl-ids over Υ. Their extended union is given by $\omega_1 \widetilde{\sqcup}_\varepsilon \omega_2 = (\xi_1 \widetilde{\cup}_\varepsilon \xi_2, \psi_1 \widetilde{\cap}_\varepsilon \psi_2; \breve{A}_1 \cup \breve{A}_2)$. Take any $a, b \in \Upsilon$. Then, the following cases arise.

Case I:

Let $e \in \breve{A}_1 \cap \breve{A}_2$. Since ω_1 and ω_2 are FBSl-ids over Υ, so, we have the following for each $a, b \in \Upsilon$.

$$\begin{aligned}(\xi_1 \widetilde{\cup}_\varepsilon \xi_2)(e)(ab) &= \xi_1(e)(ab) \vee \xi_2(e)(ab)\\ &\geqslant \xi_1(e)(b) \vee \xi_2(e)(b)\\ &= (\xi_1 \widetilde{\cup}_\varepsilon \xi_2)(e)(b).\end{aligned}$$

Similarly,

$$(\psi_1 \widetilde{\cap}_\varepsilon \psi_2)(\neg e)(ab) \leqslant (\psi_1 \widetilde{\cap}_\varepsilon \psi_2)(\neg e)(b)$$

for each $a, b \in \Upsilon$ and $e \in \breve{A}_1 \cap \breve{A}_2$.

Case II:

Let $e \in A_1 - A_2$. Then, for each $a, b \in \Upsilon$, we have

$$\begin{aligned}(\xi_1 \widetilde{\cup}_\varepsilon \xi_2)(e)(ab) &= \xi_1(e)(ab)\\ &\geqslant \xi_1(e)(b) = (\xi_1 \widetilde{\cup}_\varepsilon \xi_2)(e)(b).\end{aligned}$$

Similarly,

$$(\psi_1 \widetilde{\cap}_\varepsilon \psi_2)(\neg e)(ab) \leqslant (\psi_1 \widetilde{\cap}_\varepsilon \psi_2)(\neg e)(b)$$

for each $a, b \in \Upsilon$ and $e \in A_1 - A_2$.

Same is the case when $e \in A_2 - A_1$.

Thus, be Definition 7.3.1, it is proved that the extended union $\omega_1 \widetilde{\sqcup}_\varepsilon \omega_2$ is an FBSl-id over Υ. In the same way, it can be shown that the restricted union $\omega_1 \widetilde{\sqcup}_r \omega_2$ is an FBSl-id over Υ and that, $\omega_1 \widetilde{\sqcup}_\varepsilon \omega_2$ and $\omega_1 \widetilde{\sqcup}_r \omega_2$ are FBSr-ids and FBS-ids over Υ. ■

Theorem 7.3.6 *Let Υ be a semigroup. Then, for each FBSr-id ω_1 and FBSl-id ω_2 over Υ, the following assertion hold.*

$$\omega_1 \widehat{\circ}\, \omega_2 \mathrel{\widetilde{\sqsubseteq}} \omega_1 \widetilde{\sqcap}_r \omega_2.$$

Proof. Let $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$ be an FBSr-id and $\omega_2 = (\xi_2, \psi_2; \breve{A}_2)$ be an FBSl-id over Υ. We have

$$\begin{aligned}\omega_1 \widehat{\circ}\, \omega_2 &= (\xi_1 \circ \xi_2, \psi_1 \circ \psi_2; \breve{A}_1 \cap \breve{A}_2),\\ \omega_1 \widetilde{\sqcap}_r \omega_2 &= (\xi_1 \widetilde{\cap}_r \xi_2, \psi_1 \widetilde{\cup}_r \psi_2; \breve{A}_1 \cap \breve{A}_2).\end{aligned}$$

Take any $s \in \Upsilon$. If there exist elements $a, b \in \Upsilon$, such that $s = ab$, then for each $e \in \breve{A}_1 \cap \breve{A}_2$, we obtain

$$\begin{aligned}(\xi_1 \circ \xi_2)(e)(s) &= \bigvee_{s=ab} (\xi_1(e)(a) \wedge \xi_2(e)(b))\\ &\leq \bigvee_{s=ab} (\xi_1(e)(ab) \wedge \xi_2(e)(ab)), && \text{since } \omega_1 \text{ is FBSr-id}\\ & && \text{and } \omega_2 \text{ is FBSl-id over } \Upsilon.\\ &= \xi_1(e)(s) \wedge \xi_2(e)(s), && \text{since } ab = s\\ &= (\xi_1 \widetilde{\cap}_r \xi_2)(e)(s).\end{aligned}$$

Otherwise
$$(\xi_1 \circ \xi_2)(e)(s) = 0 \leq (\xi_1 \widetilde{\sqcap}_r \xi_2)(e)(s)$$
for each $e \in \breve{A}_1 \cap \breve{A}_2$. Similarly, for each $s \in \Upsilon$ and for each $e \in \breve{A}_1 \cap \breve{A}_2$, we have
$$(\psi_1 \circ \psi_2)(\neg e)(s) \geq (\psi_1 \widetilde{\sqcup}_r \psi_2)(\neg e)(s).$$
Thus, proved that,
$$\omega_1 \widehat{\circ}\, \omega_2 \,\widetilde{\subseteq}\, \omega_1 \widetilde{\sqcap}_r \omega_2$$
for each FBSr-id ω_1 and FBSl-id ω_2 over Υ. ■

Corollary 7.3.7 *For each FBSr-id ω_1 and FBSl-id ω_2 over Υ, the following assertion hold.*
$$\omega_1 \widehat{\circ}\, \omega_2 \,\widetilde{\subseteq}\, \omega_1 \widetilde{\sqcap}_\varepsilon \omega_2$$

Proof. Theorem 7.3.6 verifies it directly, as $\omega_1 \widetilde{\sqcap}_r \omega_2 \,\widetilde{\subseteq}\, \omega_1 \widetilde{\sqcap}_\varepsilon \omega_2$. ■

Theorem 7.3.8 *An FBSS $\omega = (\xi, \psi; \breve{A})$ over the semigroup Υ is an FBSl-id (FBSr-id, FBS-id) over Υ if and only if $\omega_{\langle e,\alpha\rangle}$ and $\omega^{\langle e,\alpha\rangle}$, if non-empty, are left (right, two-sided) ideals of Υ, for each $\alpha \in [0,1]$ and for each $e \in \breve{A}$.*

Proof. Let $\omega = (\xi, \psi; \breve{A})$ be an FBSl-id over Υ. Take any $e \in \breve{A}$, $x \in \Upsilon$ and $a \in \omega_{\langle e,\alpha\rangle}$. Then, $\xi(e)(a) \geqslant \alpha$. Since ω is an FBSl-id over Υ, so
$$\xi(e)(xa) \geqslant \xi(e)(a) \geqslant \alpha.$$
Which implies $xa \in \omega_{\langle e,\alpha\rangle}$. So, $\omega_{\langle e,\alpha\rangle}$ is a left ideal of Υ for each $\alpha \in [0,1]$ and for each $e \in \breve{A}$. Similarly, $\omega^{\langle e,\alpha\rangle}$ is a left ideal of Υ for each $\alpha \in [0,1]$ and for each $e \in \breve{A}$.

Conversely, let $\omega_{\langle e,\alpha\rangle}$ and $\omega^{\langle e,\alpha\rangle}$ be non-empty left ideals of Υ for each $\alpha \in [0,1]$ and for each $e \in \breve{A}$. Take any $a, b \in \Upsilon$ and denote $\xi(e)(b)$ by $\alpha_e \in [0,1]$. Then surely, $b \in \omega_{\langle e,\alpha_e\rangle}$. But $\omega_{\langle e,\alpha_e\rangle}$ is a left ideal of Υ. So $ab \in \omega_{\langle e,\alpha_e\rangle}$. Which yields $\xi(e)(ab) \geqslant \alpha_e$. That is,
$$\xi(e)(ab) \geqslant \xi(e)(b). \tag{7.4}$$
Now, denote $\psi(\neg e)(b)$ by $\alpha_{\neg e}$, where $\alpha_{\neg e} \in [0,1]$. Then, $\psi(\neg e)(b) \leqslant \alpha_{\neg e}$, and so $b \in \omega^{\langle e,\alpha_{\neg e}\rangle}$. But $\omega^{\langle e,\alpha_{\neg e}\rangle}$ is a left ideal of Υ. So $ab \in \omega^{\langle e,\alpha_{\neg e}\rangle}$. Which yields $\psi(\neg e)(ab) \leqslant \alpha_{\neg e}$. That is,
$$\psi(\neg e)(ab) \leqslant \psi(\neg e)(b). \tag{7.5}$$

Assertions 7.4 and 7.5 prove that ω is an FBSl-id over Υ.

A similar proof follows when ω is an FBSr-id or an FBS-id over Υ. ■

Definition 7.3.9 *An FBSS $\omega = (\xi, \psi; \breve{A})$ over Υ is called an FBSi-id over Υ if for each $a, b, c \in \Upsilon$ and for each $e \in \breve{A}$, we have*

$$\begin{aligned} \xi(e)(abc) &\geqslant \xi(e)(b), \\ \psi(\neg e)(abc) &\leqslant \psi(\neg e)(b). \end{aligned}$$

Theorem 7.3.10 *The extended intersection $\omega_1 \widetilde{\sqcap}_\varepsilon \omega_2$ and the restricted intersection $\omega_1 \widetilde{\sqcap}_r \omega_2$ of any two FBSi-ids ω_1 and ω_2 over a semigroup Υ are FBSi-ids over Υ.*

Proof. Parallel to the proof of Theorem 7.3.4. ■

Theorem 7.3.11 *The extended union $\omega_1 \widetilde{\sqcup}_\varepsilon \omega_2$ and the restricted union $\omega_1 \widetilde{\sqcup}_r \omega_2$ of any two FBSi-ids ω_1 and ω_2 over a semigroup Υ are FBSi-ids over Υ.*

Proof. Parallel to the proof of Theorem 7.3.5. ■

Theorem 7.3.12 *An FBSS $\omega = (\xi, \psi; \breve{A})$ over the semigroup Υ is an FBSi-id over Υ if and only if $\omega_{\langle e,\alpha\rangle}$ and $\omega^{\langle e,\alpha\rangle}$, if non-empty, are interior ideals of Υ, for each $\alpha \in [0,1]$ and for each $e \in \breve{A}$.*

Proof. Let $\omega = (\xi, \psi; \breve{A}) \in FBSS(\Upsilon)$ be an FBSi-id over Υ. Take any $e \in \breve{A}$, $a \in \omega_{\langle e,\alpha\rangle}$ and $x, y \in \Upsilon$. Then, $\xi(e)(a) \geqslant \alpha$. Since ω is an FBSi-id over Υ, so

$$\xi(e)(xay) \geqslant \xi(e)(a) \geqslant \alpha.$$

Which implies $xay \in \omega_{\langle e,\alpha\rangle}$. So, $\omega_{\langle e,\alpha\rangle}$ is an interior ideal of Υ for each $\alpha \in [0,1]$ and for each $e \in \breve{A}$. Similarly, $\omega^{\langle e,\alpha\rangle}$ is an interior ideal of Υ for each $\alpha \in [0,1]$ and for each $e \in \breve{A}$.

Conversely, let $\omega_{\langle e,\alpha\rangle}$ and $\omega^{\langle e,\alpha\rangle}$ be non-empty interior ideals of Υ for each $\alpha \in [0,1]$ and for each $e \in \breve{A}$. Take any $a, b, c \in \Upsilon$ and denote $\xi(e)(b)$ by $\alpha_e \in [0,1]$. Then surely, $b \in \omega_{\langle e,\alpha_e\rangle}$. But $\omega_{\langle e,\alpha_e\rangle}$ is an interior ideal of Υ. So $abc \in \omega_{\langle e,\alpha_e\rangle}$. Which yields $\xi(e)(abc) \geqslant \alpha_e$. That is,

$$\xi(e)(abc) \geqslant \xi(e)(b). \tag{7.6}$$

Now, denote $\psi(\neg e)(b)$ by $\alpha_{\neg e}$, where $\alpha_{\neg e} \in [0,1]$. Then, $\psi(\neg e)(b) \leqslant \alpha_{\neg e}$, and so $b \in \omega^{\langle e,\alpha_{\neg e}\rangle}$. But $\omega^{\langle e,\alpha_{\neg e}\rangle}$ is an interior ideal of Υ. So, $abc \in \omega^{\langle e,\alpha_{\neg e}\rangle}$. Which yields $\psi(\neg e)(abc) \leqslant \alpha_{\neg e}$. That is,

$$\psi(\neg e)(abc) \leqslant \psi(\neg e)(b). \tag{7.7}$$

The expressions 7.6 and 7.7 prove that ω is an FBSi-id over Υ. ■

Definition 7.3.13 *An FBS subsemigroup $\omega = (\xi, \psi; \breve{A})$ over a semigroup Υ is called an FBSb-id over Υ if for each $a, b, c \in \Upsilon$ and for each $e \in \breve{A}$, we have*

$$\begin{aligned} \xi(e)(abc) &\geqslant \xi(e)(a) \wedge \xi(e)(c), \\ \psi(\neg e)(abc) &\leqslant \psi(\neg e)(a) \vee \psi(\neg e)(c). \end{aligned}$$

Theorem 7.3.14 *The extended intersection $\omega_1 \widetilde{\sqcap}_\varepsilon \omega_2$ and the restricted intersection $\omega_1 \widetilde{\sqcap}_r \omega_2$ of the FBSb-ids ω_1 and ω_2 over a semigroup Υ are FBSb-ids over Υ.*

Proof. Parallel to the proof of Theorem 7.2.4. ■

The extended or restricted union of ω_1 and ω_2 may not be FBSb-ids over Υ, because these are not FBS subsemigroups over Υ, as shown in Example 7.2.5.

Theorem 7.3.15 *An FBSS $\omega = (\xi, \psi; \breve{A})$ over the semigroup Υ is an FBSb-id over Υ if and only if $\omega_{\langle e,\alpha\rangle}$ and $\omega^{\langle e,\alpha\rangle}$, if non-empty, are bi-ideals of Υ, for each $e \in \breve{A}$ and for each $\alpha \in [0, 1]$.*

Proof. Let $\omega = (\xi, \psi; \breve{A})$ be an FBSb-id over Υ. Then, ω is also an FBS subsemigroup over Υ. So, $\omega_{\langle e,\alpha\rangle}$ and $\omega^{\langle e,\alpha\rangle}$ are subsemigroups of Υ, for each $e \in \breve{A}$ and for each $\alpha \in [0, 1]$ by Theorem 7.2.6. Now take any $e \in \breve{A}$, $b \in \Upsilon$ and $a, c \in \omega_{\langle e,\alpha\rangle}$. Then surely, $\xi(e)(a) \geqslant \alpha$ and $\xi(e)(c) \geqslant \alpha$. Since ω is an FBSb-id over Υ, so

$$\xi(e)(abc) \geqslant \xi(e)(a) \wedge \xi(e)(c) \geqslant \alpha.$$

Which implies $abc \in \omega_{\langle e,\alpha\rangle}$. So, $\omega_{\langle e,\alpha\rangle}$ is a bi-ideal of Υ for each $\alpha \in [0, 1]$ and for each $e \in \breve{A}$. Similarly, $\omega^{\langle e,\alpha\rangle}$ is a bi-ideal of Υ for each $\alpha \in [0, 1]$ and for each $e \in \breve{A}$.

Conversely, let $\omega_{\langle e,\alpha\rangle}$ and $\omega^{\langle e,\alpha\rangle}$ be non-empty bi-ideals of Υ for each $\alpha \in [0, 1]$ and for each $e \in \breve{A}$. By Theorem 7.2.6, ω is an FBS subsemigroup over Υ. Now take any $a, b, c \in \Upsilon$ and denote $\xi(e)(a) \wedge \xi(e)(c)$ by $\alpha_e \in [0, 1]$. Surely, $\xi(e)(a), \xi(e)(c) \geqslant \alpha_e$, and so $a, c \in \omega_{\langle e,\alpha_e\rangle}$. But $\omega_{\langle e,\alpha_e\rangle}$ is a bi-ideal of Υ. So $abc \in \omega_{\langle e,\alpha_e\rangle}$. Which yields $\xi(e)(abc) \geqslant \alpha_e$. That is,

$$\xi(e)(abc) \geqslant \xi(e)(a) \wedge \xi(e)(c). \tag{7.8}$$

Now, denote $\psi(\neg e)(a) \vee \psi(\neg e)(c)$ by $\alpha_{\neg e}$, where $\alpha_{\neg e} \in [0, 1]$. Then $\psi(\neg e)(a), \psi(\neg e)(c) \leqslant \alpha_{\neg e}$, and so $a, c \in \omega^{\langle e,\alpha_{\neg e}\rangle}$. But $\omega^{\langle e,\alpha_{\neg e}\rangle}$ is a bi-ideal of Υ. So $abc \in \omega^{\langle e,\alpha_{\neg e}\rangle}$ for each $b \in \Upsilon$. Which yields $\psi(\neg e)(abc) \leqslant \alpha_{\neg e}$. That is,

$$\psi(\neg e)(abc) \leqslant \psi(\neg e)(a) \vee \psi(\neg e)(c). \tag{7.9}$$

Assertions 7.8 and 7.9 prove that ω is an FBSb-id over Υ. ■

7.4 Rough fuzzy bipolar soft sets over semigroups

The RFBSSs are defined using the lower and upper RFBS-apxes of an FBSS over Υ, on which a cng-rel $\Re$ is defined. These approximations are defined in this section. The RFBS subsemigroups over Υ are also defined and some characterizations are investigated.

Definition 7.4.1 *The lower and upper RFBS-apxes of an FBSS* $\omega = (\xi, \psi; \breve{A}) \in FBSS(\Upsilon)$ *under the cng-rel* $\Re$ *are the FBSSs* $\underline{\Re}(\omega) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ *and* $\overline{\Re}(\omega) = (\overline{\xi}^{\Re}, \overline{\psi}^{\Re}; \breve{A})$ *over* Υ*, respectively, where* $\underline{\xi}_{\Re}(e)$*,* $\overline{\xi}^{\Re}(e)$*,* $\underline{\psi}_{\Re}(\neg e)$*,* $\overline{\psi}^{\Re}(\neg e)$ *are FSs in* Υ*, defined by*

$$\begin{aligned}
\underline{\xi}_{\Re}(e)(u) &= \underline{\xi(e)}_{\Re}(u) = \bigwedge_{y\in[u]_{\Re}} \xi(e)(y), \\
\overline{\xi}^{\Re}(e)(u) &= \overline{\xi(e)}^{\Re}(u) = \bigvee_{y\in[u]_{\Re}} \xi(e)(y), \\
\underline{\psi}_{\Re}(\neg e)(u) &= \overline{\psi(\neg e)}^{\Re}(u) = \bigvee_{y\in[u]_{\Re}} \psi(\neg e)(y), \\
\overline{\psi}^{\Re}(\neg e)(u) &= \underline{\psi(\neg e)}_{\Re}(u) = \bigwedge_{y\in[u]_{\Re}} \psi(\neg e)(y)
\end{aligned}$$

for each $e \in \breve{A}$ *and for each* $u \in \Upsilon$*. If* $\underline{\Re}(\omega) = \overline{\Re}(\omega)$*, then,* ω *is said to be* $\Re-$*definable; otherwise,* ω *is an RFBSS over* Υ*.*

In Chapter 6, some characterizations of the RFBSSs over a non-empty set U having an eqv-rel $\Re$ were presented. These characterizations are also valid when the set U is replaced by the semigroup Υ and the eqv-rel on U is replaced by a cng-rel on Υ. So the results in Chapter 6 also hold for the lower and upper RFBS-apxes of the FBSSs over Υ, given in the Definition 7.4.1.

Theorem 7.4.2 *For the cng-rel* $\Re$ *on* Υ *and for each* $\omega_1, \omega_2 \in FBSS(\Upsilon)$*, the following holds.*

$$\overline{\Re}(\omega_1)\ \widehat{\circ}\ \overline{\Re}(\omega_2) \ \widetilde{\subseteq}\ \overline{\Re}(\omega_1 \widehat{\circ}\ \omega_2)$$

Proof. Since $\Re$ is a cng-rel on Υ, so $[x]_{\Re}[y]_{\Re} \subseteq [xy]_{\Re}$ for each $x, y \in \Upsilon$. For any $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$, $\omega_2 = (\xi_2, \psi_2; \breve{A}_2) \in FBSS(\Upsilon)$, we have

$$\begin{aligned}
\overline{\Re}(\omega_1)\ \widehat{\circ}\ \overline{\Re}(\omega_2) &= (\overline{\xi_1}^{\Re} \circ \overline{\xi_2}^{\Re}, \overline{\psi_1}^{\Re} \circ \overline{\psi_2}^{\Re}; \breve{A}_1 \cap \breve{A}_2), \\
\overline{\Re}(\omega_1 \widehat{\circ}\ \omega_2) &= (\overline{\xi_1 \circ \xi_2}^{\Re}, \overline{\psi_1 \circ \psi_2}^{\Re}; \breve{A}_1 \cap \breve{A}_2).
\end{aligned}$$

Take any $s \in \Upsilon$. If some $x, y \in \Upsilon$ exist, such that $s = xy$, then we have the following for each $e \in \breve{A}_1 \cap \breve{A}_2$.

$$\begin{aligned}
(\overline{\xi_1}^{\Re} \circ \overline{\xi_2}^{\Re})(e)(s) &= \bigvee_{s=xy}(\overline{\xi_1}^{\Re}(e)(x) \wedge \overline{\xi_2}^{\Re}(e)(y)) \\
&= \bigvee_{s=xy}((\bigvee_{a\in[x]_\Re} \xi_1(e)(a)) \wedge (\bigvee_{b\in[y]_\Re} \xi_2(e)(b))) \\
&= \bigvee_{s=xy}(\bigvee_{a\in[x]_\Re,\ b\in[y]_\Re}(\xi_1(e)(a) \wedge \xi_2(e)(b))) \\
&\leq \bigvee_{s=xy}(\bigvee_{ab\in[xy]_\Re}(\xi_1(e)(a) \wedge \xi_2(e)(b))), \quad \text{since } ab \in [x]_\Re[y]_\Re \subseteq [xy]_\Re \\
&= \bigvee_{ab\in[s]_\Re}(\xi_1(e)(a) \wedge \xi_2(e)(b)), \qquad \text{since } xy = s \\
&= \bigvee_{t\in[s]_\Re,\ t=ab}(\xi_1(e)(a) \wedge \xi_2(e)(b)) \\
&= \bigvee_{t\in[s]_\Re}(\bigvee_{t=ab}(\xi_1(e)(a) \wedge \xi_2(e)(b))) \\
&= \bigvee_{t\in[s]_\Re}(\xi_1 \circ \xi_2)(e)(t) = \overline{\Re}(\xi_1 \circ \xi_2)(e)(s).
\end{aligned}$$

Otherwise we have

$$(\overline{\xi_1}^{\Re} \circ \overline{\xi_2}^{\Re})(e)(s) = 0 \leq \overline{\xi_1 \circ \xi_2}^{\Re}(e)(s)$$

for each $e \in \breve{A}_1 \cap \breve{A}_2$. Similarly, for each $s \in \Upsilon$ and for each $e \in \breve{A}_1 \cap \breve{A}_2$, we have

$$(\overline{\psi_1}^{\Re} \circ \overline{\psi_2}^{\Re})(\neg e)(s) \geq \overline{\psi_1 \circ \psi_2}^{\Re}(\neg e)(s).$$

Hence, proved that, $\overline{\Re}(\omega_1) \mathbin{\widehat{\circ}} \overline{\Re}(\omega_2) \mathrel{\widetilde{\subseteq}} \overline{\Re}(\omega_1 \mathbin{\widehat{\circ}} \omega_2)$. ■

Theorem 7.4.3 *Let $\Re$ be a complete cng-rel on Υ. Then, the following holds for each $\omega_1, \omega_2 \in FBSS(\Upsilon)$.*

$$\underline{\Re}(\omega_1) \mathbin{\widehat{\circ}} \underline{\Re}(\omega_2) \mathrel{\widetilde{\subseteq}} \underline{\Re}(\omega_1 \mathbin{\widehat{\circ}} \omega_2)$$

Proof. Since $\Re$ is a complete cng-rel on Υ, so $[x]_\Re[y]_\Re = [xy]_\Re$ for each $x, y \in \Upsilon$. Let $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$, $\omega_2 = (\xi_2, \psi_2; \breve{A}_2) \in FBSS(\Upsilon)$. We have

$$\begin{aligned}
\underline{\Re}(\omega_1) \mathbin{\widehat{\circ}} \underline{\Re}(\omega_2) &= (\underline{\xi_1}_{\Re} \circ \underline{\xi_2}_{\Re}, \underline{\psi_1}_{\Re} \circ \underline{\psi_2}_{\Re}; \breve{A}_1 \cap \breve{A}_2), \\
\underline{\Re}(\omega_1 \mathbin{\widehat{\circ}} \omega_2) &= (\underline{\xi_1 \circ \xi_2}_{\Re}, \underline{\psi_1 \circ \psi_2}_{\Re}; \breve{A}_1 \cap \breve{A}_2).
\end{aligned}$$

Take any $s \in \Upsilon$. If some $x, y \in \Upsilon$ exist, such that $s = xy$, then for each $e \in \breve{A}_1 \cap \breve{A}_2$, we have

$$\begin{aligned}
(\underline{\xi_1}_{\Re} \circ \underline{\xi_2}_{\Re})(e)(s) &= \bigvee_{s=xy} (\underline{\xi_1}_{\Re}(e)(x) \wedge \underline{\xi_2}_{\Re}(e)(y)) \\
&= \bigvee_{s=xy} ((\bigwedge_{a\in[x]_\Re} \xi_1(e)(a)) \wedge (\bigwedge_{b\in[y]_\Re} \xi_2(e)(b))) \\
&= \bigvee_{s=xy} (\bigwedge_{a\in[x]_\Re,\ b\in[y]_\Re} (\xi_1(e)(a) \wedge \xi_2(e)(b))) \\
&\leq \bigvee_{s=xy} (\bigwedge_{a\in[x]_\Re,\ b\in[y]_\Re} \bigvee_{ab=t_1t_2} (\xi_1(e)(t_1) \wedge \xi_2(e)(t_2))), \quad \text{where } t_1, t_2 \in \Upsilon \\
&= \bigvee_{s=xy} (\bigwedge_{a\in[x]_\Re,\ b\in[y]_\Re} (\xi_1 \circ \xi_2)(e)(ab)) \\
&= \bigvee_{s=xy} (\bigwedge_{ab\in[xy]_\Re} (\xi_1 \circ \xi_2)(e)(ab)), \quad \text{since } ab \in [x]_\Re[y]_\Re = [xy]_\Re \\
&= \bigvee_{s=xy} (\underline{\xi_1 \circ \xi_2}_{\Re}(e)(xy)) = \underline{\xi_1 \circ \xi_2}_{\Re}(e)(s).
\end{aligned}$$

Otherwise;

$$(\underline{\xi_1}_{\Re} \circ \underline{\xi_2}_{\Re})(e)(s) = 0 \leq \underline{\xi_1 \circ \xi_2}_{\Re}(e)(s)$$

for each $e \in \breve{A}_1 \cap \breve{A}_2$. Similarly, for each $s \in \Upsilon$ and for each $e \in \breve{A}_1 \cap \breve{A}_2$, we have

$$(\underline{\psi_1}_{\Re} \circ \underline{\psi_2}_{\Re})(\neg e)(s) \geq \underline{\psi_1 \circ \psi_2}_{\Re}(\neg e)(s).$$

Hence, proved that $\underline{\Re}(\omega_1) \,\widehat{\circ}\, \underline{\Re}(\omega_2) \,\widetilde{\subseteq}\, \underline{\Re}(\omega_1 \widehat{\circ}\, \omega_2)$. ■

Theorem 7.4.4 *For the cng-rel $\Re$ on Υ, the FBSr-id ω_1 and the FBSl-id ω_2 over Υ, the subsequent asservations hold.*

1. $\overline{\Re}(\omega_1 \widehat{\circ} \omega_2) \,\widetilde{\subseteq}\, \overline{\Re}(\omega_1) \widetilde{\sqcap}_r \overline{\Re}(\omega_2)$,

2. $\underline{\Re}(\omega_1 \widehat{\circ} \omega_2) \,\widetilde{\subseteq}\, \underline{\Re}(\omega_1) \widetilde{\sqcap}_r \underline{\Re}(\omega_2)$.

Proof. (1) Take an FBSr-id $\omega_1 = (\xi_1, \psi_1; \breve{A}_1)$ and an FBSl-id $\omega_2 = (\xi_2, \psi_2; \breve{A}_2)$ over Υ. We have

$$\begin{aligned}
\overline{\Re}(\omega_1 \widehat{\circ} \omega_2) &= (\overline{\xi_1 \circ \xi_2}^{\Re}, \overline{\psi_1 \circ \psi_2}^{\Re}; \breve{A}_1 \cap \breve{A}_2), \\
\overline{\Re}(\omega_1) \widetilde{\sqcap}_r \overline{\Re}(\omega_2) &= (\overline{\xi_1}^{\Re} \widetilde{\cap}_r \overline{\xi_2}^{\Re}, \overline{\psi_1}^{\Re} \widetilde{\cup}_r \overline{\psi_2}^{\Re}; \breve{A}_1 \cap \breve{A}_2).
\end{aligned}$$

.

Now, for each $s \in \Upsilon$ and for each $e \in \breve{A}_1 \cap \breve{A}_2$, we have

$$\begin{aligned}
\overline{\xi_1 \circ \xi_2}^{\Re}(e)(s) &= \bigvee_{t\in[s]_\Re} (\xi_1 \circ \xi_2)(e)(t) \\
&= \bigvee_{t\in[s]_\Re} \bigvee_{t=ab} (\xi_1(e)(a) \wedge \xi_2(e)(b)) \\
&\leq \bigvee_{t\in[s]_\Re} \bigvee_{t=ab} (\xi_1(e)(ab) \wedge \xi_2(e)(ab)), \quad \text{since } \omega_1 \text{ is FBSr-id} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{and } \omega_2 \text{ is FBSl-id} \\
&= \bigvee_{t\in[s]_\Re} (\xi_1(e)(t) \wedge \xi_2(e)(t)) \\
&\leq \bigvee_{t\in[s]_\Re} \bigvee_{t'\in[s]_\Re} (\xi_1(e)(t) \wedge \xi_2(e)(t')) \\
&= (\bigvee_{t\in[s]_\Re} \xi_1(e)(t)) \wedge (\bigvee_{t'\in[s]_\Re} \xi_2(e)(t')) \\
&= \overline{\xi_1}^{\Re}(e)(s) \wedge \overline{\xi_2}^{\Re}(e)(s) \\
&= (\overline{\xi_1}^{\Re} \widetilde{\cap}_r \overline{\xi_2}^{\Re})(e)(s).
\end{aligned}$$

Similarly, for each $s \in \Upsilon$ and for each $e \in \breve{A}_1 \cap \breve{A}_2$, we have

$$\overline{\psi_1 \circ \psi_2}^{\Re}(\neg e)(s) \geq (\overline{\psi_1}^{\Re} \widetilde{\cup}_r \overline{\psi_2}^{\Re})(\neg e)(s).$$

Thus, proved, that,

$$\overline{\Re}(\omega_1 \widehat{\circ} \omega_2) \widetilde{\subseteq} \overline{\Re}(\omega_1) \widetilde{\sqcap}_r \overline{\Re}(\omega_2).$$

(2) Analogous to the proof of (1). ■

Definition 7.4.5 *An FBSS ω over Υ is a lower (or upper) RFBS subsemigroup over Υ, if $\underline{\Re}(\omega)$ (or $\overline{\Re}(\omega)$) is an FBS subsemigroup over Υ.*

An FBSS $\omega = (\xi, \psi; \breve{A})$ over Υ, which is both, lower and upper RFBS subsemigroup over Υ, is called an RFBS subsemigroup over Υ.

Theorem 7.4.6 *Each FBS subsemigroup over Υ is an upper RFBS subsemigroup over Υ.*

Proof. Take an FBS subsemigroup $\omega = (\xi, \psi; \breve{A})$ over Υ. Then, we have $\xi(e)(ab) \geq \xi(e)(a) \wedge \xi(e)(b)$ and $\psi(\neg e)(ab) \leq \psi(\neg e)(a) \vee \psi(\neg e)(b)$ for each $a, b \in \Upsilon$ and for each

$e \in \breve{A}$. Now, for $x, y \in \Upsilon$ and $e \in \breve{A}$, we have

$$\begin{aligned}
\overline{\xi}^{\Re}(e)(xy) &= \bigvee_{s\in[xy]_{\Re}} \xi(e)(s) \\
&\geq \bigvee_{s\in[x]_{\Re}[y]_{\Re}} \xi(e)(s), \qquad \text{since } [x]_{\Re}[y]_{\Re} \subseteq [xy]_{\Re} \\
&= \bigvee_{ab\in[x]_{\Re}[y]_{\Re}} \xi(e)(ab), \qquad \text{where } s = ab \\
&\geq \bigvee_{a\in[x]_{\Re},\, b\in[y]_{\Re}} (\xi(e)(a) \wedge \xi(e)(b)) \\
&= (\bigvee_{a\in[x]_{\Re}} \xi(e)(a)) \wedge (\bigvee_{b\in[y]_{\Re}} \xi(e)(b)) \\
&= \overline{\xi}^{\Re}(e)(x) \wedge \overline{\xi}^{\Re}(e)(y)
\end{aligned}$$

and

$$\begin{aligned}
\overline{\psi}^{\Re}(\neg e)(xy) &= \bigwedge_{s\in[xy]_{\Re}} \psi(\neg e)(s) \\
&\leq \bigwedge_{s\in[x]_{\Re}[y]_{\Re}} \psi(\neg e)(s), \qquad \text{since } [x]_{\Re}[y]_{\Re} \subseteq [xy]_{\Re} \\
&= \bigwedge_{ab\in[x]_{\Re}[y]_{\Re}} \psi(\neg e)(ab), \qquad \text{where } s = ab \\
&\leq \bigwedge_{a\in[x]_{\Re},\, b\in[y]_{\Re}} (\psi(\neg e)(a) \vee \psi(\neg e)(b)) \\
&= (\bigwedge_{a\in[x]_{\Re}} \psi(\neg e)(a)) \vee (\bigwedge_{b\in[y]_{\Re}} \psi(\neg e)(b)) \\
&= \overline{\psi}^{\Re}(\neg e)(x) \vee \overline{\psi}^{\Re}(\neg e)(y).
\end{aligned}$$

This verifies that $\overline{\Re}(\omega)$ is an FBS subsemigroup over Υ. Therefore, ω is an upper RFBS subsemigroup over Υ. ■

The converse statement of Theorem 7.4.6 is invalid generally, as exhibited in the next example.

Example 7.4.7 *Recall the semigroup $\Upsilon = \{a, b, c, d\}$ and the attribute set $\breve{E}$, as established in Example 7.2.3. Take a binary relation $\Re$ on Υ, defining classes $\{a\}, \{b, c\}$ and $\{d\}$. Then, $\Re$ is a cng-rel on Υ. We take an FBSS $\omega = (\xi, \psi; \breve{A})$ over Υ with $\breve{A} = \{e_1, e_2\}$, defined below.*

$\xi(e_1) = \{a/0.1,\ b/0.3,\ c/0.4,\ d/0.6\}$,

$\psi(\neg e_1) = \{a/0.5,\ b/0.5,\ c/0.4,\ d/0.1\}$,

$\xi(e_2) = \{a/0.2,\ b/0.5,\ c/0.4,\ d/0.9\}$,

$\psi(\neg e_2) = \{a/0.7,\ b/0.4,\ c/0.6,\ d/0.1\}$.

The upper RFBS-apx $\overline{\Re}(\omega) = (\overline{\xi}^{\Re}, \overline{\psi}^{\Re}; \breve{A})$ of ω under $\Re$ is calculated as:

$\overline{\xi}^{\Re}(e_1) = \{a/0.1,\ b/0.4,\ c/0.4,\ d/0.6\}$,

$\overline{\psi}^{\Re}(\neg e_1) = \{a/0.5,\ b/0.4,\ c/0.4,\ d/0.1\}$,

$\overline{\xi}^{\Re}(e_2) = \{a/0.2,\ b/0.5,\ c/0.5,\ d/0.9\}$,

$\overline{\psi}^{\Re}(\neg e_2) = \{a/0.7,\ b/0.4,\ c/0.4,\ d/0.1\}$.

Simple calculations verify that $\overline{\Re}(\omega)$ *is an FBS subsemigroup over* Υ. *But, we find that,*

$$\begin{aligned}\xi(e_1)(cc) &= \xi(e_1)(b) = 0.3\\ &\ngeq \xi(e_1)(c) \wedge \xi(e_1)(c) = 0.4.\end{aligned}$$

So, ω *is not an FBS subsemigroup over* Υ. *Although, it is an upper RFBS subsemigroup over* Υ.

Theorem 7.4.8 *Let* $\Re$ *be a complete cng-rel on* Υ. *Then, each FBS subsemigroup over* Υ *is a lower RFBS subsemigroup over* Υ.

Proof. Let $\omega = (\xi, \psi; \breve{A})$ be an FBS subsemigroup over Υ. Now, for each $x, y \in \Upsilon$ and for each $e \in \breve{A}$, we have

$$\begin{aligned}\underline{\xi}_{\Re}(e)(xy) &= \bigwedge_{s\in[xy]_{\Re}} \xi(e)(s)\\ &= \bigwedge_{s\in[x]_{\Re}[y]_{\Re}} \xi(e)(s), \qquad \text{since } [x]_{\Re}[y]_{\Re} = [xy]_{\Re}\\ &= \bigwedge_{ab\in[x]_{\Re}[y]_{\Re}} \xi(e)(ab), \qquad \text{where } s = ab\\ &\geq \bigwedge_{a\in[x]_{\Re},\ b\in[y]_{\Re}} (\xi(e)(a) \wedge \xi(e)(b))\\ &= (\bigwedge_{a\in[x]_{\Re}} \xi(e)(a)) \wedge (\bigwedge_{b\in[y]_{\Re}} \xi(e)(b))\\ &= \underline{\xi}_{\Re}(e)(x) \wedge \underline{\xi}_{\Re}(e)(y)\end{aligned}$$

and

$$\begin{aligned}\underline{\psi}_{\Re}(\neg e)(xy) &= \bigvee_{s\in[xy]_{\Re}} \psi(\neg e)(s)\\ &= \bigvee_{s\in[x]_{\Re}[y]_{\Re}} \psi(\neg e)(s), \qquad \text{since } [x]_{\Re}[y]_{\Re} = [xy]_{\Re}\\ &= \bigvee_{ab\in[x]_{\Re}[y]_{\Re}} \psi(\neg e)(ab), \qquad \text{where } s = ab\\ &\leq \bigvee_{a\in[x]_{\Re},\ b\in[y]_{\Re}} (\psi(\neg e)(a) \vee \psi(\neg e)(b))\\ &= (\bigvee_{a\in[x]_{\Re}} \psi(\neg e)(a)) \vee (\bigvee_{b\in[y]_{\Re}} \psi(\neg e)(b))\\ &= \underline{\psi}_{\Re}(\neg e)(x) \vee \underline{\psi}_{\Re}(\neg e)(y).\end{aligned}$$

This verifies that $\underline{\Re}(\omega)$ is an FBS subsemigroup over Υ. Therefore, ω is a lower RFBS subsemigroup over Υ. ■

The converse statement of the Theorem 7.4.8 is invalid generally, as exhibited in the next example.

Example 7.4.9 *Let $\Upsilon = \{s, t, u, v\}$ represent a semigroup whose table of binary operation is given below.*

	s	t	u	v
s	s	t	u	v
t	t	t	u	v
u	u	u	u	v
v	v	v	v	u

Let $\hat{E} = \{e_i; i = 1, 2, ..., 5\}$ and let $\Re$ be a cng-rel over Υ, defining cng-classes $\{s\}, \{t\}$ and $\{u, v\}$. Then, $\Re$ is a complete cng-rel over Υ. We take an FBSS $\omega = (\xi, \psi; \breve{A})$ over Υ with $\breve{A} = \{e_1\}$, defined below.

$\xi(e_1) = \{s/0.4,\ t/0.5,\ u/0.5,\ v/0.6\}$,
$\psi(\neg e_1) = \{s/0.3,\ t/0.4,\ u/0.4,\ v/0.3\}$,
$\xi(e_2) = \{s/0.6,\ t/0.7,\ u/0.8,\ v/0.9\}$,
$\psi(\neg e_2) = \{s/0.1,\ t/0.2,\ u/0.1,\ v/0.1\}$.

The lower RFBS-apxes $\underline{\Re}(\omega) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ of ω under $\Re$ are calculated as:

$\underline{\xi}_{\Re}(e_1) = \{s/0.4,\ t/0.5,\ u/0.5,\ v/0.5\}$,
$\underline{\psi}_{\Re}(\neg e_1) = \{s/0.3,\ t/0.4,\ u/0.4,\ v/0.4\}$,
$\underline{\xi}_{\Re}(e_2) = \{s/0.6,\ t/0.7,\ u/0.8,\ v/0.8\}$,
$\underline{\psi}_{\Re}(\neg e_2) = \{s/0.1,\ t/0.2,\ u/0.1,\ v/0.1\}$.

Simple calculations verify that $\underline{\Re}(\omega)$ is an FBS subsemigroup over Υ. But, we find that,

$$\begin{aligned} \xi(e_1)(vv) &= \xi(e_1)(u) = 0.5 \\ &\ngeq \xi(e_1)(v) \wedge \xi(e_1)(v) = 0.6. \end{aligned}$$

So, ω is not an FBS subsemigroups over Υ, although, it is a lower RFBS subsemigroup over Υ.

Theorem 7.4.8 is invalid if the cng-rel $\Re$ is not complete. Next example is established to verify this fact.

Example 7.4.10 *Recall the semigroup $\Upsilon = \{a, b, c, d\}$ and the attribute set $\breve{E}$, as established in Example 7.2.3. The binary relation $\Re$ defining the classes $\{a\}, \{c\}, \{b, d\}$ is a cng-rel on Υ and $\Re$ is not complete. We take an FBS subsemigroup $\omega = (\xi, \psi; \breve{A})$ over Υ with $\breve{A} = \{e_1, e_2\}$, defined below.*

$\xi(e_1) = \{a/0.4,\ b/0.6,\ c/0.4,\ d/0.3\}$,
$\psi(\neg e_1) = \{a/0.2,\ b/0.4,\ c/0.5,\ d/0.3\}$,
$\xi(e_2) = \{a/0.1,\ b/0.2,\ c/0.1,\ d/0.3\}$,

$\psi(\neg e_2) = \{a/0.4,\ b/0.3,\ c/0.5,\ d/0.6\}$.

The lower RFBS-apxes $\underline{\Re}(\omega) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ *of* ω *under* $\Re$ *are calculated as:*

$\underline{\xi}_{\Re}(e_1) = \{a/0.4,\ b/0.3,\ c/0.4,\ d/0.3\}$,

$\underline{\psi}_{\Re}(\neg e_1) = \{a/0.2,\ b/0.4,\ c/0.5,\ d/0.4\}$,

$\underline{\xi}_{\Re}(e_2) = \{a/0.1,\ b/0.2,\ c/0.1,\ d/0.2\}$,

$\underline{\psi}_{\Re}(\neg e_2) = \{a/0.4,\ b/0.6,\ c/0.5,\ d/0.6\}$.

We find that

$$\begin{aligned} \underline{\xi}_{\Re}(e_1)(ac) &= \underline{\xi}_{\Re}(e_1)(b) = 0.3 \\ &\not\geq \underline{\xi}_{\Re}(e_1)(a) \wedge \underline{\xi}_{\Re}(e_1)(c) = 0.4 \end{aligned}$$

So, $\underline{\Re}(\omega)$ *is not FBS subsemigroups over* Υ, *that is,* ω *is not lower RFBS subsemigroup over* Υ.

7.5 Rough fuzzy bipolar soft ideals over semigroups

We establish and elaborate, in this section, the notions of the RFBS-ids, RFBSi-ids and RFBSb-ids over Υ. Some characterizations of the lower and upper RFBS-ids, lower and upper RFBSi-ids and lower and upper RFBSb-ids over Υ are also discussed.

Definition 7.5.1 *An FBSS* ω *over* Υ *is a lower (resp. upper) RFBSl-id* (*RFBSr-id, RFBS-id*) *over* Υ, *if* $\underline{\Re}(\omega)$ *(resp.* $\overline{\Re}(\omega)$*) is an FBSl-id* (*FBSr-id, FBS-id*) *over* Υ.

An FBSS ω over Υ is called an RFBSl-id (RFBSr-id, RFBS-id) if it is both, lower and upper RFBSl-id (RFBSr-id, RFBS-id) over Υ.

Theorem 7.5.2 *Each FBSl-id* (*FBSr-id, FBS-id*) *over* Υ *is an upper RFBSl-id* (*RFBSr-id, RFBS-id*) *over* Υ.

Proof. Take an FBSl-id $\omega = (\xi, \psi; \breve{A})$ over Υ. Then, $\xi(e)(ab) \geq \xi(e)(b)$ and $\psi(\neg e)(ab) \leq \psi(\neg e)(b)$ for each $a, b \in \Upsilon$ and for each $e \in \breve{A}$. Now, for each $x, y \in \Upsilon$ and for each $e \in \breve{A}$, we have

$$\begin{aligned} \overline{\xi}^{\Re}(e)(xy) &= \bigvee_{s\in[xy]_{\Re}} \xi(e)(s) \\ &\geq \bigvee_{s\in[x]_{\Re}[y]_{\Re}} \xi(e)(s), && \text{since } [x]_{\Re}[y]_{\Re} \subseteq [xy]_{\Re} \\ &= \bigvee_{ab\in[x]_{\Re}[y]_{\Re}} \xi(e)(ab), && \text{where } s = ab \\ &\geq \bigvee_{a\in[x]_{\Re},\ b\in[y]_{\Re}} \xi(e)(b) \\ &= \bigvee_{b\in[y]_{\Re}} \xi(e)(b) \\ &= \overline{\xi}^{\Re}(e)(y) \end{aligned}$$

and

$$\begin{aligned}
\overline{\psi}^{\Re}(\neg e)(xy) &= \bigwedge_{s\in[xy]_{\Re}} \psi(\neg e)(s) \\
&\leq \bigwedge_{s\in[x]_{\Re}[y]_{\Re}} \psi(\neg e)(s), && \text{since } [x]_{\Re}[y]_{\Re} \subseteq [xy]_{\Re} \\
&= \bigwedge_{ab\in[x]_{\Re}[y]_{\Re}} \psi(\neg e)(ab), && \text{where } s = ab \\
&\leq \bigwedge_{a\in[x]_{\Re},\, b\in[y]_{\Re}} \psi(\neg e)(b) \\
&= \bigwedge_{b\in[y]_{\Re}} \psi(\neg e)(b) \\
&= \overline{\psi}^{\Re}(\neg e)(y).
\end{aligned}$$

This verifies that $\overline{\Re}(\omega)$ is an FBSl-id over Υ. Therefore, ω is an upper RFBSl-id over Υ. Similarly, the cases of FBSr-id and FBS-id over Υ can be verified. ■

The converse statement of the Theorem 7.5.2 is invalid generally, as exhibited in the next example.

Example 7.5.3 *Recall the semigroup* $\Upsilon = \{k, l, m, n\}$ *and* $\hat{E} = \{e_1, e_2, e_3\}$ *as established in Example 7.2.5. Let* $\Re$ *be a cng-rel over* Υ*, defining cng-classes* $\{k, l, n\}$ *and* $\{m\}$*. We take an FBSS* $\omega = (\xi, \psi; \breve{A})$ *over* Υ *with* $\breve{A} = \{e_1, e_3\}$*, defined below.*

$\xi(e_1) = \{k/0.1,\ l/0.2,\ m/0.1,\ n/0.4\}$,
$\psi(\neg e_1) = \{k/0.5,\ l/0.6,\ m/0.7,\ n/0.4\}$,
$\xi(e_3) = \{k/0.7,\ l/0.6,\ m/0.1,\ n/0.8\}$,
$\psi(\neg e_3) = \{k/0.2,\ l/0.3,\ m/0.3,\ n/0.1\}$.

The upper RFBS-apx $\overline{\Re}(\omega) = (\overline{\xi}^{\Re}, \overline{\psi}^{\Re}; \breve{A})$ *of* ω *under* $\Re$ *is calculated as:*

$\overline{\xi}^{\Re}(e_1) = \{k/0.4,\ l/0.4,\ m/0.1,\ n/0.4\}$,
$\overline{\psi}^{\Re}(\neg e_1) = \{k/0.4,\ l/0.4,\ m/0.7,\ n/0.4\}$,
$\overline{\xi}^{\Re}(e_3) = \{k/0.8,\ l/0.8,\ m/0.1,\ n/0.8\}$,
$\overline{\psi}^{\Re}(\neg e_3) = \{k/0.1,\ l/0.1,\ m/0.3,\ n/0.1\}$.

Simple calculations verify that $\overline{\Re}(\omega)$ *is an FBSl-id over* Υ*. But, we find that,*

$$\begin{aligned}
\xi(e_1)(ml) &= \xi(e_1)(k) = 0.1 \\
&\ngeq \xi(e_1)(l) = 0.2.
\end{aligned}$$

So, ω *is not an FBSl-id over* Υ*, although, it is an upper RFBSl-id over* Υ*.*

Theorem 7.5.4 *Let* $\Re$ *be a complete cng-rel on* Υ*. Then, each FBSl-id (FBSr-id, FBS-id) over* Υ *is a lower RFBSl-id (RFBSr-id, RFBS-id) over* Υ*.*

Proof. Let $\omega = (\xi, \psi; \breve{A})$ be an FBSl-id over Υ. Now, for each $x, y \in \Upsilon$ and for each $e \in \breve{A}$, we have

$$\begin{aligned}
\underline{\xi}_{\Re}(e)(xy) &= \bigwedge_{s\in[xy]_{\Re}} \xi(e)(s) \\
&= \bigwedge_{s\in[x]_{\Re}[y]_{\Re}} \xi(e)(s), && \text{since } [x]_{\Re}[y]_{\Re} = [xy]_{\Re} \\
&= \bigwedge_{ab\in[x]_{\Re}[y]_{\Re}} \xi(e)(ab), && \text{where } s = ab \\
&\geq \bigwedge_{a\in[x]_{\Re},\ b\in[y]_{\Re}} \xi(e)(b) \\
&= \bigwedge_{b\in[y]_{\Re}} \xi(e)(b) \\
&= \underline{\xi}_{\Re}(e)(y)
\end{aligned}$$

and

$$\begin{aligned}
\underline{\psi}_{\Re}(\neg e)(xy) &= \bigvee_{s\in[xy]_{\Re}} \psi(\neg e)(s) \\
&= \bigvee_{s\in[x]_{\Re}[y]_{\Re}} \psi(\neg e)(s), && \text{since } [x]_{\Re}[y]_{\Re} = [xy]_{\Re} \\
&= \bigvee_{ab\in[x]_{\Re}[y]_{\Re}} \psi(\neg e)(ab), && \text{where } s = ab \\
&\leq \bigvee_{a\in[x]_{\Re},\ b\in[y]_{\Re}} \psi(\neg e)(b) \\
&= \bigvee_{b\in[y]_{\Re}} \psi(\neg e)(b) \\
&= \underline{\psi}_{\Re}(\neg e)(y).
\end{aligned}$$

This verifies that $\underline{\Re}(\omega)$ is an FBSl-id over Υ. Therefore, ω is a lower RFBSl-id over Υ. Similarly, the cases of FBSr-id and FBS-id over Υ can be verified. ■

The converse statement of the Theorem 7.5.4 is invalid generally, as exhibited in the next example.

Example 7.5.5 *Recall the semigroup* $\Upsilon = \{s, t, u, v\}$, *the attribute set* $\hat{E}$ *and the complete cng-rel* $\Re$ *over* Υ, *as established in Example 7.4.9. We take an FBSS* $\omega = (\xi, \psi; \breve{A})$ *over* Υ *with* $\breve{A} = \{e_2, e_3\}$, *defined below.*

$\xi(e_2) = \{s/0.1,\ t/0.2,\ u/0.6,\ v/0.3\}$,
$\psi(\neg e_2) = \{s/0.7,\ t/0.6,\ u/0.3,\ v/0.3\}$,
$\xi(e_3) = \{s/0.2,\ t/0.4,\ u/0.6,\ v/0.8\}$,
$\psi(\neg e_3) = \{s/0.5,\ t/0.4,\ u/0.3,\ v/0.2\}$.

The lower RFBS-apxes $\underline{\Re}(\omega) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ *of* ω *under* $\Re$ *are calculated as:*

$\underline{\xi}_{\Re}(e_2) = \{s/0.1,\ t/0.2,\ u/0.3,\ v/0.3\}$,
$\underline{\psi}_{\Re}(\neg e_2) = \{s/0.7,\ t/0.6,\ u/0.3,\ v/0.3\}$,
$\underline{\xi}_{\Re}(e_3) = \{s/0.2,\ t/0.4,\ u/0.6,\ v/0.6\}$,

$\underline{\psi}_{\Re}(\neg e_3) = \{s/0.5,\ t/0.4,\ u/0.3,\ v/0.3\}$.
Simple calculations verify that $\underline{\Re}(\omega)$ *is an FBSl-id over* Υ. *But, we find that,*

$$\begin{aligned}\xi(e_2)(uv) &= \xi(e_2)(v) = 0.3\\ &\not\geq \xi(e_2)(u) = 0.6.\end{aligned}$$

So, ω *is not an FBSl-id over* Υ, *although, it is a lower RFBSl-id over* Υ.

Theorem 7.5.4 is invalid if the cng-rel $\Re$ is not complete. Next example is established to verify this fact.

Example 7.5.6 *Recall the semigroup* $\Upsilon = \{a, b, c, d\}$ *and the attribute set* $\breve{E}$, *as established in Example 7.2.3. Take a cng-rel* $\Re$ *on* Υ, *defining the cng-classes* $\{a\}, \{b, c\}$, $\{d\}$, *then* $\Re$ *is not complete. We take an FBSl-id* $\omega = (\xi, \psi; \breve{A})$ *over* Υ *with* $\breve{A} = \{e_1, e_3\}$, *defined below.*
$\xi(e_1) = \{a/0.3,\ b/0.4,\ c/0.2,\ d/0.6\}$,
$\psi(\neg e_1) = \{a/0.6,\ b/0.3,\ c/0.4,\ d/0.1\}$,
$\xi(e_3) = \{a/0.5,\ b/0.6,\ c/0.5,\ d/0.7\}$,
$\psi(\neg e_3) = \{a/0.5,\ b/0.3,\ c/0.4,\ d/0.1\}$.
The lower RFBS-apxes $\underline{\Re}(\omega) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ *of* ω *under* $\Re$ *are calculated as:*
$\underline{\xi}_{\Re}(e_1) = \{a/0.3,\ b/0.2,\ c/0.2,\ d/0.6\}$,
$\underline{\psi}_{\Re}(\neg e_1) = \{a/0.6,\ b/0.4,\ c/0.4,\ d/0.1\}$,
$\underline{\xi}_{\Re}(e_3) = \{a/0.5,\ b/0.5,\ c/0.5,\ d/0.7\}$,
$\underline{\psi}_{\Re}(\neg e_3) = \{a/0.5,\ b/0.4,\ c/0.4,\ d/0.1\}$.
We find that

$$\begin{aligned}\underline{\xi}_{\Re}(e_1)(ba) &= \underline{\xi}_{\Re}(e_1)(b) = 0.2\\ &\not\geq \underline{\xi}_{\Re}(e_1)(a) = 0.3.\end{aligned}$$

So, $\underline{\Re}(\omega)$ *is not an FBSl-id over* Υ, *that is,* ω *is not lower RFBSl-id over* Υ.

Definition 7.5.7 *An FBSS* ω *over* Υ *is a lower (or upper) RFBSi-id over* Υ, *if* $\underline{\Re}(\omega)$ *(or* $\overline{\Re}(\omega)$*) is an FBSi-id over* Υ.

An FBSS ω over Υ, which is both, lower and upper RFBSi-id over Υ, is called an RFBSi-id over Υ.

Theorem 7.5.8 *Each FBSi-id over* Υ *is an upper RFBSi-id over* Υ.

Proof. Take an FBSi-id $\omega = (\xi, \psi; \breve{A})$ over Υ. Then $\xi(e)(abc) \geq \xi(e)(b)$ and $\psi(\neg e)(abc) \leq \psi(\neg e)(b)$ for each $a, b, c \in \Upsilon$. Now, for each $x, w, y \in \Upsilon$ and for each $e \in \breve{A}$, we have

$$\begin{aligned}
\overline{\xi}^{\Re}(e)(xwy) &= \bigvee_{s \in [xwy]_{\Re}} \xi(e)(s) \\
&\geq \bigvee_{s \in [x]_{\Re}[w]_{\Re}[y]_{\Re}} \xi(e)(s), && \text{since } [x]_{\Re}[w]_{\Re}[y]_{\Re} \subseteq [xwy]_{\Re} \\
&= \bigvee_{abc \in [x]_{\Re}[w]_{\Re}[y]_{\Re}} \xi(e)(abc), && \text{where } s = abc \\
&\geq \bigvee_{b \in [w]_{\Re}} \xi(e)(b) \\
&= \overline{\xi}^{\Re}(e)(w)
\end{aligned}$$

and

$$\begin{aligned}
\overline{\psi}^{\Re}(\neg e)(xwy) &= \bigwedge_{s \in [xwy]_{\Re}} \psi(\neg e)(s) \\
&\leq \bigwedge_{s \in [x]_{\Re}[w]_{\Re}[y]_{\Re}} \psi(\neg e)(s), && \text{since } [x]_{\Re}[w]_{\Re}[y]_{\Re} \subseteq [xwy]_{\Re} \\
&= \bigwedge_{abc \in [x]_{\Re}[w]_{\Re}[y]_{\Re}} \psi(\neg e)(abc), && \text{where } s = abc \\
&\leq \bigwedge_{b \in [w]_{\Re}} \psi(\neg e)(b) \\
&= \overline{\psi}^{\Re}(\neg e)(w).
\end{aligned}$$

This verifies that $\overline{\Re}(\omega)$ is an FBSi-id over Υ. Therefore, ω is an upper RFBSi-id over Υ. ■

The converse statement of the Theorem 7.5.8 is invalid generally, as exhibited in the next example.

Example 7.5.9 *Recall the semigroup $\Upsilon = \{k, l, m, n\}$, the attribute set $\hat{E}$ for Υ and the cng-rel $\Re$ over Υ, as established in Example 7.5.3. We take an FBSS $\omega = (\xi, \psi; \breve{A})$ over Υ with $\breve{A} = \{e_2, e_3\}$, defined below.*

$\xi(e_2) = \{k/0.65,\ l/0.5,\ m/0.6,\ n/0.75\}$,
$\psi(\neg e_2) = \{k/0.4,\ l/0.3,\ m/0.3,\ n/0.15\}$,
$\xi(e_3) = \{k/0.92,\ l/0.84,\ m/0.81,\ n/0.99\}$,
$\psi(\neg e_3) = \{k/0,\ l/0.1,\ m/0.11,\ n/0\}$.
The upper RFBS-apx $\overline{\Re}(\omega) = (\overline{\xi}^{\Re}, \overline{\psi}^{\Re}; \breve{A})$ of ω under $\Re$ is calculated as:
$\overline{\xi}^{\Re}(e_2) = \{k/0.75,\ l/0.75,\ m/0.6,\ n/0.75\}$,
$\overline{\psi}^{\Re}(\neg e_2) = \{k/0.15,\ l/0.15,\ m/0.3,\ n/0.15\}$,
$\overline{\xi}^{\Re}(e_3) = \{k/0.99,\ l/0.99,\ m/0.81,\ n/0.99\}$,
$\overline{\psi}^{\Re}(\neg e_3) = \{k/0,\ l/0,\ m/0.11,\ n/0\}$.

Simple calculations verify that $\overline{\Re}(\omega)$ *is an FBSi-id over* Υ*. But, we find that,*

$$\begin{aligned}\psi(\neg e_2)(llm) &= \psi(\neg e_2)(k) = 0.4\\ &\nleq \psi(\neg e_2)(l) = 0.3.\end{aligned}$$

So, ω *is not an FBSi-id over* Υ*, although, it is an upper RFBSi-id over* Υ*.*

Theorem 7.5.10 *Let* $\Re$ *be a complete cng-rel on* Υ*. Then, each FBSi-id over* Υ *is a lower RFBSi-id over* Υ*.*

Proof. Let $\omega = (\xi, \psi; \breve{A})$ be an FBSi-id over Υ. Now, for each $x, w, y \in \Upsilon$ and for each $e \in \breve{A}$, we have

$$\begin{aligned}\underline{\xi}_{\Re}(e)(xwy) &= \bigwedge_{s\in[xwy]_{\Re}} \xi(e)(s)\\ &= \bigwedge_{s\in[x]_{\Re}[w]_{\Re}[y]_{\Re}} \xi(e)(s), \qquad \text{since } [x]_{\Re}[w]_{\Re}[y]_{\Re} = [xwy]_{\Re}\\ &= \bigwedge_{abc\in[x]_{\Re}[w]_{\Re}[y]_{\Re}} \xi(e)(abc), \qquad \text{where } s = abc\\ &\geq \bigwedge_{b\in[w]_{\Re}} \xi(e)(b)\\ &= \underline{\xi}_{\Re}(e)(w)\end{aligned}$$

and

$$\begin{aligned}\underline{\psi}_{\Re}(\neg e)(xwy) &= \bigvee_{s\in[xwy]_{\Re}} \psi(\neg e)(s)\\ &= \bigvee_{s\in[x]_{\Re}[w]_{\Re}[y]_{\Re}} \psi(\neg e)(s), \qquad \text{since } [x]_{\Re}[w]_{\Re}[y]_{\Re} = [xwy]_{\Re}\\ &= \bigvee_{abc\in[x]_{\Re}[w]_{\Re}[y]_{\Re}} \psi(\neg e)(abc), \qquad \text{where } s = abc\\ &\leq \bigvee_{b\in[w]_{\Re}} \psi(\neg e)(b)\\ &= \underline{\psi}_{\Re}(\neg e)(w).\end{aligned}$$

This verifies that $\underline{\Re}(\omega)$ is an FBSi-id over Υ. Therefore, ω is a lower RFBSi-id over Υ. ■

The converse statement of the Theorem 7.5.10 is invalid generally, as exhibited in the next example.

Example 7.5.11 *Recall the semigroup* $\Upsilon = \{s, t, u, v\}$*, the attribute set* $\hat{E}$ *and the complete cng-rel* $\Re$ *over* Υ*, as established in Example 7.4.9. We take an FBSS* $\omega = (\xi, \psi; \breve{A})$ *over* Υ *with* $\breve{A} = \{e_1, e_3\}$*, defined below.*

$\xi(e_1) = \{s/0.2,\ t/0.4,\ u/0.6,\ v/0.8\}$,

$\psi(\neg e_1) = \{s/0.6,\ t/0.5,\ u/0.3,\ v/0.2\}$,

$\xi(e_3) = \{s/0.4,\ t/0.5,\ u/0.5,\ v/0.6\}$,
$\psi(\neg e_3) = \{s/0.1,\ t/0.1,\ u/0.1,\ v/0.1\}$.
The lower RFBS-apxes $\underline{\Re}(\omega) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ *of* ω *under* $\Re$ *are calculated as:*
$\underline{\xi}_{\Re}(e_1) = \{s/0.2,\ t/0.4,\ u/0.6,\ v/0.6\}$,
$\underline{\psi}_{\Re}(\neg e_1) = \{s/0.6,\ t/0.5,\ u/0.3,\ v/0.3\}$,
$\underline{\xi}_{\Re}(e_3) = \{s/0.4,\ t/0.5,\ u/0.5,\ v/0.5\}$,
$\underline{\psi}_{\Re}(\neg e_3) = \{s/0.1,\ t/0.1,\ u/0.1,\ v/0.1\}$.
Simple calculations verify that $\underline{\Re}(\omega)$ *is an FBSi-id over* Υ. *But, we find that,*

$$\begin{aligned} \xi(e_1)(vvs) &= \xi(e_1)(u) = 0.6 \\ &\ngeq \xi(e_1)(v) = 0.8. \end{aligned}$$

So, ω *is not an FBSi-id over* Υ*, although, it is a lower RFBSi-id over* Υ*.*

Theorem 7.5.10 is invalid if the cng-rel $\Re$ is not complete. Next example is established to verify this fact.

Example 7.5.12 *Recall the semigroup* $\Upsilon = \{a, b, c, d\}$ *and* $\breve{E}$ *as established in Example 7.2.3. Take the cng-rel* $\Re$ *on* Υ *from Example 7.5.6. Which is not complete, and defines the cng-classes* $\{a\}, \{b, c\}, \{d\}$*. We take an FBSi-id* $\omega = (\xi, \psi; \breve{A})$ *over* Υ *with* $\breve{A} = \{e_2, e_3\}$*, defined below.*
$\xi(e_2) = \{a/0.3,\ b/0.4,\ c/0.2,\ d/0.5\}$,
$\psi(\neg e_2) = \{a/0.3,\ b/0.1,\ c/0.2,\ d/0\}$,
$\xi(e_3) = \{a/0.4,\ b/0.6,\ c/0.5,\ d/0.7\}$,
$\psi(\neg e_3) = \{a/0.6,\ b/0.2,\ c/0.4,\ d/0\}$.
The lower RFBS-apxes $\underline{\Re}(\omega) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ *of* ω *under* $\Re$ *are calculated as:*
$\underline{\xi}_{\Re}(e_2) = \{a/0.3,\ b/0.2,\ c/0.2,\ d/0.5\}$,
$\underline{\psi}_{\Re}(\neg e_2) = \{a/0.3,\ b/0.2,\ c/0.2,\ d/0\}$,
$\underline{\xi}_{\Re}(e_3) = \{a/0.4,\ b/0.5,\ c/0.5,\ d/0.7\}$,
$\underline{\psi}_{\Re}(\neg e_3) = \{a/0.6,\ b/0.4,\ c/0.4,\ d/0\}$.
We find that

$$\begin{aligned} \underline{\xi}_{\Re}(e_2)(bac) &= \underline{\xi}_{\Re}(e_2)(b) = 0.2 \\ &\ngeq \underline{\xi}_{\Re}(e_2)(a) = 0.3. \end{aligned}$$

So, $\underline{\Re}(\omega)$ *is not an FBSi-id over* Υ*, that is,* ω *is not lower RFBSi-id over* Υ*.*

Definition 7.5.13 *An FBSS* ω *over* Υ *is a lower (or upper) RFBSb-id over* Υ*, if* $\underline{\Re}(\omega)$ *(or* $\overline{\Re}(\omega)$*) is an FBSb-id over* Υ*.*

An FBSS ω over Υ, which is both, lower and upper RFBSb-id over Υ, is called a RFBSb-id over Υ.

Theorem 7.5.14 *For the cng-rel $\Re$ on Υ, each FBSb-id over Υ is an upper RFBSb-id over Υ.*

Proof. Take an FBSb-id $\omega = (\xi, \psi; \breve{A})$ over Υ. Then, ω is also an FBS subsemigroup over Υ. Which implies by Theorem 7.4.6, that, $\overline{\Re}(\omega) = (\overline{\xi}^{\Re}, \overline{\psi}^{\Re}; \breve{A})$ is an FBS subsemigroup over Υ. Now, for each $x, w, y \in \Upsilon$ and for each $e \in \breve{A}$, we have

$$\begin{aligned}
\overline{\xi}^{\Re}(e)(xwy) &= \bigvee_{s\in[xwy]_{\Re}} \xi(e)(s) \\
&\geq \bigvee_{s\in[x]_{\Re}[w]_{\Re}[y]_{\Re}} \xi(e)(s), \qquad \text{since } [x]_{\Re}[w]_{\Re}[y]_{\Re} \subseteq [xwy]_{\Re} \\
&= \bigvee_{abc\in[x]_{\Re}[w]_{\Re}[y]_{\Re}} \xi(e)(abc), \qquad \text{where } s = abc \\
&\geq \bigvee_{a\in[x]_{\Re},\ b\in[w]_{\Re},\ c\in[y]_{\Re}} (\xi(e)(a) \wedge \xi(e)(c)) \\
&= (\bigvee_{a\in[x]_{\Re}} \xi(e)(a)) \wedge (\bigvee_{c\in[y]_{\Re}} \xi(e)(c)) \\
&= \overline{\xi}^{\Re}(e)(x) \wedge \overline{\xi}^{\Re}(e)(y)
\end{aligned}$$

and

$$\begin{aligned}
\overline{\psi}^{\Re}(\neg e)(xwy) &= \bigwedge_{s\in[xwy]_{\Re}} \psi(\neg e)(s) \\
&\leq \bigwedge_{s\in[x]_{\Re}[w]_{\Re}[y]_{\Re}} \psi(\neg e)(s), \qquad \text{since } [x]_{\Re}[w]_{\Re}[y]_{\Re} \subseteq [xwy]_{\Re} \\
&= \bigwedge_{abc\in[x]_{\Re}[w]_{\Re}[y]_{\Re}} \psi(\neg e)(abc), \qquad \text{where } s = abc \\
&\leq \bigwedge_{a\in[x]_{\Re},\ b\in[w]_{\Re},\ c\in[y]_{\Re}} (\psi(\neg e)(a) \vee \psi(\neg e)(c)) \\
&= (\bigwedge_{a\in[x]_{\Re}} \psi(\neg e)(a)) \vee (\bigwedge_{c\in[y]_{\Re}} \psi(\neg e)(c)) \\
&= \overline{\psi}^{\Re}(\neg e)(x) \vee \overline{\psi}^{\Re}(\neg e)(y).
\end{aligned}$$

This verifies that $\overline{\Re}(\omega)$ is an FBSb-id over Υ. Therefore, ω is an upper RFBSb-id over Υ. ∎

The converse statement of the Theorem 7.5.14 is invalid generally, as exhibited in the next example.

Example 7.5.15 *Recall the semigroup $\Upsilon = \{k, l, m, n\}$, the attribute set $\hat{E}$ for Υ and the cng-rel $\Re$ over Υ, as established in Example 7.5.3. We take an FBSS $\omega = (\xi, \psi; \breve{A})$ over Υ with $\breve{A} = \{e_1, e_2\}$, defined below.*

$\xi(e_1) = \{k/0.76,\ l/0.75,\ m/0.71,\ n/0.78\}$,

$\psi(\neg e_1) = \{k/0.13,\ l/0.15,\ m/0.16,\ n/0.11\}$,
$\xi(e_2) = \{k/0.85,\ l/0.82,\ m/0.83,\ n/0.8\}$,
$\psi(\neg e_2) = \{k/0.12,\ l/0.14,\ m/0.15,\ n/0.1\}$.
The upper RFBS-apx $\overline{\Re}(\omega) = (\overline{\xi}^{\Re}, \overline{\psi}^{\Re}; \breve{A})$ *of* ω *under* $\Re$ *is calculated as:*
$\overline{\xi}^{\Re}(e_1) = \{k/0.78,\ l/0.78,\ m/0.71,\ n/0.78\}$,
$\overline{\psi}^{\Re}(\neg e_1) = \{k/0.11,\ l/0.11,\ m/0.16,\ n/0.11\}$,
$\overline{\xi}^{\Re}(e_2) = \{k/0.85,\ l/0.85,\ m/0.83,\ n/0.85\}$,
$\overline{\psi}^{\Re}(\neg e_2) = \{k/0.1,\ l/0.1,\ m/0.15,\ n/0.1\}$.
Simple calculations verify that $\overline{\Re}(\omega)$ *is an FBSb-id over* Υ. *But, we find that,*

$$\begin{aligned} \xi(e_2)(knl) &= \xi(e_2)(n) = 0.8 \\ &\ngeq \xi(e_2)(k) \wedge \xi(e_2)(l) = 0.82. \end{aligned}$$

So, ω *is not an FBSb-id over* Υ, *although, it is an upper RFBSb-id over* Υ.

Theorem 7.5.16 *Let* $\Re$ *be a complete cng-rel on* Υ. *Then, each FBSb-id over* Υ *is a lower RFBSb-id over* Υ.

Proof. Let $\omega = (\xi, \psi; \breve{A})$ be an FBSb-id over Υ. Then, ω is also an FBS subsemigroup over Υ. Which implies by Theorem 7.4.8, that, $\underline{\Re}(\omega) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ is an FBS subsemigroup over Υ. Now, for each $x, w, y \in \Upsilon$ and for each $e \in \breve{A}$, we have

$$\begin{aligned} \underline{\xi}_{\Re}(e)(xwy) &= \bigwedge_{s \in [xwy]_{\Re}} \xi(e)(s) \\ &= \bigwedge_{s \in [x]_{\Re}[w]_{\Re}[y]_{\Re}} \xi(e)(s), \qquad \text{since } [x]_{\Re}[w]_{\Re}[y]_{\Re} = [xwy]_{\Re} \\ &= \bigwedge_{abc \in [x]_{\Re}[w]_{\Re}[y]_{\Re}} \xi(e)(abc), \qquad \text{where } s = abc \\ &\geq \bigwedge_{a \in [x]_{\Re},\ b \in [w]_{\Re},\ c \in [y]_{\Re}} (\xi(e)(a) \wedge \xi(e)(c)) \\ &= (\bigwedge_{a \in [x]_{\Re}} \xi(e)(a)) \wedge (\bigwedge_{c \in [y]_{\Re}} \xi(e)(c)) \\ &= \underline{\xi}_{\Re}(e)(x) \wedge \underline{\xi}_{\Re}(e)(y) \end{aligned}$$

and

$$\begin{aligned} \underline{\psi}_{\Re}(\neg e)(xwy) &= \bigvee_{s \in [xwy]_{\Re}} \psi(\neg e)(s) \\ &= \bigvee_{s \in [x]_{\Re}[w]_{\Re}[y]_{\Re}} \psi(\neg e)(s), \qquad \text{since } [x]_{\Re}[w]_{\Re}[y]_{\Re} = [xwy]_{\Re} \\ &= \bigvee_{abc \in [x]_{\Re}[w]_{\Re}[y]_{\Re}} \psi(\neg e)(abc), \qquad \text{where } s = abc \\ &\leq \bigvee_{a \in [x]_{\Re},\ b \in [w]_{\Re},\ c \in [y]_{\Re}} (\psi(\neg e)(a) \vee \psi(\neg e)(c)) \\ &= (\bigvee_{a \in [x]_{\Re}} \psi(\neg e)(a)) \vee (\bigvee_{c \in [y]_{\Re}} \psi(\neg e)(c)) \\ &= \underline{\psi}_{\Re}(\neg e)(x) \vee \underline{\psi}_{\Re}(\neg e)(y). \end{aligned}$$

This verifies that $\underline{\Re}(\omega)$ is an FBSb-id over Υ. Therefore, ω is a lower RFBSb-id over Υ. ■

The converse statement of the Theorem 7.5.16 is invalid generally, as exhibited in the next example.

Example 7.5.17 *Recall the semigroup $\Upsilon = \{s, t, u, v\}$, the attribute set $\hat{E}$ and the complete cng-rel $\Re$ over Υ, as established in Example 7.4.9. We take an FBSS $\omega = (\xi, \psi; \breve{A})$ over Υ with $\breve{A} = \{e_2\}$, defined below.*
$\xi(e_2) = \{s/0.1,\ t/0.1,\ u/0.4,\ v/0.2\}$,
$\psi(\neg e_2) = \{s/0.6,\ t/0.5,\ u/0.3,\ v/0.3\}$.
The lower RFBS-apxes $\underline{\Re}(\omega) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ of ω under $\Re$ are calculated as:
$\underline{\xi}_{\Re}(e_2) = \{s/0.1,\ t/0.1,\ u/0.2,\ v/0.2\}$,
$\underline{\psi}_{\Re}(\neg e_2) = \{s/0.6,\ t/0.5,\ u/0.3,\ v/0.3\}$.
Simple calculations verify that $\underline{\Re}(\omega)$ is an FBSb-id over Υ. But, we find that,

$$\begin{aligned}\xi(e_2)(uvu) &= \xi(e_2)(v) = 0.2\\ &\ngeq \xi(e_2)(u) \wedge \xi(e_2)(u) = 0.4.\end{aligned}$$

So, ω is not an FBSb-id over Υ, although, it is a lower RFBSb-id over Υ.

Theorem 7.5.16 is invalid if the cng-rel $\Re$ is not complete. Next example is established to verify this fact.

Example 7.5.18 *Recall the semigroup $\Upsilon = \{a, b, c, d\}$, the attribute set $\breve{E}$ and the cng-rel $\Re$ on Υ (which is not complete), as taken in Example 7.5.12. Now, take an FBSb-id $\omega = (\xi, \psi; \breve{A})$ over Υ with $\breve{A} = \{e_1, e_4\}$, defined below.*
$\xi(e_1) = \{a/0.7,\ b/0.8,\ c/0.6,\ d/0.9\}$,
$\psi(\neg e_1) = \{a/0.2,\ b/0.1,\ c/0.3,\ d/0.1\}$,
$\xi(e_4) = \{a/0.3,\ b/0.5,\ c/0.4,\ d/0.6\}$,
$\psi(\neg e_4) = \{a/0.4,\ b/0.3,\ c/0.4,\ d/0.2\}$.
The lower RFBS-apxes $\underline{\Re}(\omega) = (\underline{\xi}_{\Re}, \underline{\psi}_{\Re}; \breve{A})$ of ω under $\Re$ are calculated as:
$\underline{\xi}_{\Re}(e_1) = \{a/0.7,\ b/0.6,\ c/0.6,\ d/0.9\}$,
$\underline{\psi}_{\Re}(\neg e_1) = \{a/0.2,\ b/0.3,\ c/0.3,\ d/0.1\}$,
$\underline{\xi}_{\Re}(e_4) = \{a/0.3,\ b/0.4,\ c/0.4,\ d/0.6\}$,
$\underline{\psi}_{\Re}(\neg e_4) = \{a/0.4,\ b/0.4,\ c/0.4,\ d/0.2\}$.
We find that

$$\begin{aligned}\underline{\xi}_{\Re}(e_1)(aba) &= \underline{\xi}_{\Re}(e_1)(b) = 0.6\\ &\ngeq \underline{\xi}_{\Re}(e_1)(a) \wedge \underline{\xi}_{\Re}(e_1)(a) = 0.7.\end{aligned}$$

So, $\underline{\Re}(\omega)$ is not an FBSb-id over Υ, that is, ω is not lower RFBSb-id over Υ.

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List of abbreviations

BF	Bipolar fuzzy
BFb-id	Bipolar fuzzy bi-ideal
BF-id	Bipolar fuzzy ideal
BFi-id	Bipolar fuzzy interior ideal
BFl-id	Bipolar fuzzy left ideal
BFr-id	Bipolar fuzzy right ideal
BFS	Bipolar fuzzy set
BS	Bipolar soft
BSb-id	Bipolar soft bi-ideal
BS-id	Bipolar soft ideal
BSi-id	Bipolar soft interior ideal
BSl-id	Bipolar soft left ideal
BSr-id	Bipolar soft right ideal
BSS	Bipolar soft set
Cng-class	Congruence class
Cng-rel	Congruence relation
Eqv-class	Equivalence class
Eqv-rel	Equivalence relation
FBS	Fuzzy bipolar soft
FBSb-id	Fuzzy bipolar soft bi-ideal
FBS-id	Fuzzy bipolar soft ideal
FBSi-id	Fuzzy bipolar soft interior ideal
FBSl-id	Fuzzy bipolar soft left ideal
FBSr-id	Fuzzy bipolar soft right ideal
FBSS	Fuzzy bipolar soft set

FS	Fuzzy set
GDM	Group decision making
P-apx	Pawlak approximation
RBF	Rough bipolar fuzzy
RBF-apx	Rough bipolar fuzzy approximation
RBFb-id	Rough bipolar fuzzy bi-ideal
RBF-id	Rough bipolar fuzzy ideal
RBFi-id	Rough bipolar fuzzy interior ideal
RBFl-id	Rough bipolar fuzzy left ideal
RBFr-id	Rough bipolar fuzzy right ideal
RBFS	Rough bipolar fuzzy set
RBS	Rough bipolar soft
RBS-apx	Rough bipolar soft approximation
RBSb-id	Rough bipolar soft bi-ideal
RBS-id	Rough bipolar soft ideal
RBSi-id	Rough bipolar soft interior ideal
RBSl-id	Rough bipolar soft left ideal
RBSr-id	Rough bipolar soft right-ideal
RBSS	Rough bipolar soft set
RFBS	Rough fuzzy bipolar set
RFBS-apx	Rough fuzzy bipolar soft approximation
RFBSb-id	Rough fuzzy bipolar soft bi-ideal
RFBS-id	Rough fuzzy bipolar soft ideal
RFBSi-id	Rough fuzzy bipolar soft interior ideal
RFBSl-id	Rough fuzzy bipolar soft left ideal
RFBSr-id	Rough fuzzy bipolar soft right ideal
RFBSS	Rough fuzzy bipolar soft set

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