A study of roughness in different algebraic structures

By

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Department of Mathematics
Quaid-I-Azam University
Islamabad, Pakistan
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A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

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Supervised By
Dr. Waqas Mahmood

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2020

Certificate of Approval

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A study of roughness in different algebraic structures

By

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Contents

Introduction

Groups, rings and modules are the fundamental concepts of abstract algebra. The study of groups started in 18th century and gradually developed in various decades, initiated by very famous mathematicians Cauchy, Abel, Galois and Lagrange. The other fundamental concepts of abstract algebra (*e.g.* rings and modules) are the generalizations of group. Rings and modules are the central notions of Commutative algebra and Homological algebra. The development of these mathematical theories has been greatly influenced by numerous problems and ideas occurring naturally in algebraic number theory and algebraic geometry, algebraic topology. Many books have been written on these fundamental concepts. For systematic evaluation of groups, rings and modules, we refer the reader to [14, 5].

However, most of the classical mathematical methods and solutions do not address the imprecision and uncertainties in given information. A genius mind L. A. Zadeh said:

"The closer one looks at a real world problem, the fuzzier becomes its solution."

In 1965, Zadeh [46] formulated the concept of fuzzy set theory to describe the vagueness and imprecision containing most of the data. Fuzzy sets handel such situations by attributing a degree to which a certain object belong to a set. Fuzzy sets has a wide range of applications in applied sciences such as computer sciences, management sciences, control sciences, robotics, artificial intelligence, pattern recognition and operation research etc. In its trajectory of stupendous growth, it has also come to include the theory of fuzzy algebra and for the past several decades, several researchers have been working on the fuzzy commutative algebra.

Rosenfeld [36] inspired the fuzzification of algebraic structures and introduced the notion of fuzzy subgroups in 1971. Pan defined fuzzy quotient modules and fuzzy exact sequences (see [33]). A systematic description of fuzzy commutative algebra by Mordeson and Malik appeared in [31], and Wang et al. [44] where one can find detailed theoretical study on various algebraic structures.

In 1982, Pawlak [34] projected a light on incredible theory of rough sets. Rough set theory helps in management of uncertainties and imprecision containing raw data. It has been successfully implemented to various real life problems. For instance, data mining, pattern recognition, artificial intelligence and decision support system are most of the auspicious amongst to it's numerous applications. Hence, this theory grabbed attention of several researchers owed to its valuable features. Since it's inception, the question of applications of rough set theory to various algebraic systems has been addressed by a great number of researchers. For instance, Biswas and Nanda [11] were initiated the study of roughness in groups and introduced the notion of rough subgroups.

Kuroki and Wang [22] proposed the notion of lower and upper approximation spaces in groups based on normal subgroups and normal fuzzy subgroups. In [26], Mahmood et al. studied the notion of roughness in fuzzy subgroups based on a congruence relation and explore a relationship between the approximation spaces of two different groups by maneuvering a group homomorphism. In [25], the same authors investigated the concept of roughness in quotient groups and established several homomorphism theorems between lower approximations. In [16], Davvaz and Mahdavipour studied the concept of roughness in modules and also gave some related properties. Since, Pawlak approximation space based on an equivalence relation, a too restrictive condition for many practical applications. In order to cope with this issue, there has been a lot of researchs on the extensions of Pawlak's rough sets [34], in literature (see [15, 23, 42, 43, 45, 47, 35]). In these papers, authors defined the lower and upper approximation spaces of a subset on the universe of discourse by using an arbitrary binary relation, a set valued map, a soft binary relation, tolerance relation and similarity relation to investigate the essential properties of classical rough sets [34]. In [35], Qian et al. extended the notion of Pawlak's rough sets to multi-granulation rough sets (MGRS) based on mutli equivalence relations on the universe of discourse.

On the other hand, the concept of soft set theory was introduced by Molodtsov [30] in 1999 to deal with the complexity of data containing uncertainties. It has many useful applications in diverse areas of game theory, the smoothness of functions and measurement theory. The fundamental operations of soft sets was studied by several number of authors (see [1, 37, 2, 12, 29]). Later on, many authors were engaged to found its applications in different algebraic structures. Aktaş and Çağman [1] initiated the concept of soft groups and derived some of it's basic properties. Later on, the concept of soft groups was further studied by Sezgin and Atagün [38], and Aslam and Qurashi [4]. The concept of soft-intersection groups was investigated by Çağman et al. [12] and further studied by Kaygisiz [20, 21]. Feng et al. applied the soft relations to semigroup theory in [18].

Although the theories of soft sets, fuzzy sets and rough sets are very conflicting in nature. However, many authors have been established a linkage among these theories (see [3, 17, 19, 24, 28, 39, 27, 40, 41]).

This thesis consisting of five chapters inclusively arranged in the following manner. In chapter one we give some preliminary ideas which will be helpful to our research work. To the motivation of Cagman et al.'s work [12], we have studied the concept of roughness in soft sets and find it's applications on soft-intersection groups in chapter 2. In this chapter, we also construct a relationship among the soft approximation spaces of two different groups via group homomorphisms. Chapter 3 is devoted to the study of roughness in modules of fractions by using a special kind of an equivalence relation on its submodules. Hence, we introduce the notion of lower and upper approximations in modules of fractions. Some fundamental results related to these approximation spaces are studied. Furthermore, we develop several connections between their approximation spaces of two different modules of fractions by utilizing the concept of module homomorphisms. In chapter 4, we introduce the notion of fuzzy modules of fractions and study the concept of roughness in fuzzy modules of fractions. For this, we introduce a pair of fuzzy approximation spaces based on soft modules of fractions in view of multi-granulation rough sets. Some essential properties relating to the fuzzy approximation spaces are investigated. Finally, chapter 5 extended the classical concept of rough subgroups studied by Kuroki and Wang [22], and a connection between lower and upper approximations via group homomorphism investigated by Mahmood et al. [26] to soft set theory. In this regard, the concept of soft lower and upper approximation spaces is introduced in groups by manipulating normal soft groups. Based on the soft image and soft pre-image of a normal soft group, some connections between the soft approximation spaces are built by maneuvering group homomorphisms.

Chapter 1

Preliminaries

In this chapter, some basic notions of commutative algebra, fuzzy sets, rough sets and soft sets are presented. Several kinds of operations and results of soft sets, fuzzy sets and rough sets are given, which will be useful for our research work. For details we refer [1, 2, 5, 12, 14, 29, 30, 31, 34, 44].

1.1 Rings and modules of fractions

In this section, we recall some basic definitions and necessary results of rings and modules of fractions. All the material in this section is taken from [5] and [14].

Definition 1.1.1. *[14, page 223] A non-empty set L is called a ring, if :*

(i) $(L, +)$ *is an abelian group.*

(ii) Multiplication is associative: $(l_1 l_2)_3 = l_1(l_2 l_3)$ *for all* $l_1, l_2, l_3 \in L$.

(iii) Distributive laws: $l_1(l_2 + l_3) = l_1 l_2 + l_1 l_3$ *and* $(l_1 + l_2)l_3 = l_1 l_3 + l_2 l_3$ *for all* $l_1, l_2, l_3 \in L$.

(a) The ring L is called commutative, if multiplication is commutative: $l_1 l_2 = l_2 l_1$ *for all* $l_1, l_2 \in L$.

(b) *The ring L is said to have an identity element, if there is an element $1 \in L$ such that $l1 = 1l = l$ for all $l \in L$. We'll denote it by 1_L.*

The additive identity of L will be represented by 0_L and the additive inverse of $l \in L$ will be denoted by $-l$.

Throughout this work, L will be symbolized as a commutative ring with the multiplicative identity element 1_L.

Definition 1.1.2. *[5, page 2] An element $l \in L$ is called unit, if $lm = 1_L$ for some $m \in L$. Then the element m is said to be an inverse of $l \in L$ and will be denoted by l^{-1}. We shall represent the collection of all units of L by $U(L)$.*

Definition 1.1.3. *[5, page 2] A non-empty subset K of L is called*

(1) *a subring, if*

  (i) *K is subgroup of the abelian group $(L, +)$,*

  (ii) *K is subsemiring of the semiring $(L, \cdot)$,*

  (iii) *$1_L \in K$.*

(2) *an ideal, if*

  (i) *K is a subring of $(L, +, \cdot)$.*

  (ii) *$lK \subseteq K$ for all $l \in L$.*

Definition 1.1.4. *[14, page 239] Let L and L' be two commutative rings with 1_L and $1_{L'}$ respectively. Then, a map $\psi : L \to L'$ is called a ring homomorphism, if:*

(i) *$\psi(l + l') = \psi(l) + \psi(l')$.*

(ii) *$\psi(ll') = \psi(l)\psi(l')$.*

for all $l, l' \in L$.

Definition 1.1.5. *[5, page 36] A non-empty subset $\mathcal{D}$ of L is called multiplicative closed subset (MCS) of L, if:*

(i) $1_L \in \mathcal{D}$.

(ii) $d_1 d_2 \in \mathcal{D}$ *for all* $d_1, d_2 \in \mathcal{D}$.

Let $\mathcal{D}$ be a MCS of L. The relation $\equiv$ on $L \times \mathcal{D}$ is defined as follows:

$$(l_1, d_1) \equiv (l_2, d_2) \Leftrightarrow (l_1 d_2 - l_2 d_1)d = 0_L \text{ for some } d \in \mathcal{D}.$$

which is an equivalence relation. The equivalence class of $(l_1, d_1) \in L \times \mathcal{D}$ with respect to $\equiv$ is denoted by $\frac{l_1}{d_1} = \{(l_2, d_2) \in L \times \mathcal{D} : (l_1, d_1) \equiv (l_2, d_2)\}$. The set of all equivalence classes is denoted by:

$$\mathcal{D}^{-1}L = \{\frac{l}{d} : l \in L, d \in \mathcal{D}\}$$

Then $\mathcal{D}^{-1}L$ is a commutative ring with identity and binary operations "addition" and "multiplication" defined as follows:

$$\frac{l_1}{d_1} + \frac{l_2}{d_2} = \frac{l_1 d_2 + l_2 d_1}{d_1 d_2} \text{ and } \frac{l_1}{d_1} . \frac{l_2}{d_2} = \frac{l_1 l_2}{d_1 d_2}$$

The ring $\mathcal{D}^{-1}L$ is called the ring of fractions of L with respect to $\mathcal{D}$.

Definition 1.1.6. *[5, page 17] A commutative group $(Z, +)$ is called an $L-$module or a module over L, if the map:*

$$\cdot : L \times Z \longrightarrow Z; (l, z) \mapsto l \cdot z,$$

satisfies the following properties:

(1) $l_1 \cdot (z_1 + z_2) = l_1 \cdot z_1 + l_1 \cdot z_2$,

(2) $(l_1 + l_2) \cdot z_1 = l_1 \cdot z_1 + l_2 \cdot z_1$,

(3) $l_1 \cdot (l_2 \cdot z) = (l_1 l_2) \cdot z$,

(4) $1_L \cdot z = z$,

for all $l_1,\ l_2 \in L$ *and* $z_1,\ z_2 \in Z$.

Throughout this thesis, Z will be denoting a module over L.

Definition 1.1.7. *[5, page 18] A non-empty subset* W *of* Z *is called a submodule, if it fulfils the following two properties:*

(1) W *is an additive subgroup of* Z.

(2) $l \cdot w \in W$, *for any* $l \in L$, $w \in W$.

If $\{W_i : 1 \leq i \leq n\}$ is a family of submodules of Z and K is an ideal of L, then the following sets are submodules of Z:

$$\cap_{i=1}^{n} W_i = \{z \in Z : z \in W_i, 1 \leq i \leq n\}.$$

$$\sum_{i=1}^{n} W_i = \{\sum_{i=1}^{n} w_i : w_i \in W_i, 1 \leq i \leq n\}.$$

$$KW_1 = \{\sum_{i=1}^{n} l_i \cdot z_i : l_i \in K, z_i \in W_1, n \in \mathbb{N}\}.$$

Definition 1.1.8. *[5, page 18] Let* Z_1 *and* Z_2 *be* L*-modules. Then:*

(1) *a map* $\varphi : Z_1 \longrightarrow Z_2$ *is called an* $L-$*linear map (or a module homomorphism), if it satisfies the following conditions:*

(1) $\varphi(z + z') = \varphi(z) + \varphi(z')$,

(2) $\varphi(l \cdot z) = l \cdot \varphi(z)$,

for all $z, z' \in Z$ *and* $l \in L$.

(2) The kernel of φ is the set $Ker\varphi = \{z \in Z_1 : \varphi(z) = 0_{Z_2}\}$.

Let $\mathcal{D}$ be a MCS and Z be an L-module. Then, there exists a well-known equivalence relation "$\sim$" on the set $Z \times \mathcal{D}$, defined by:

$$(z, d) \sim (z', d') \Leftrightarrow w \cdot (d' \cdot z - d \cdot z') = 0, \text{ for some } w \in \mathcal{D}.$$

The equivalence class of (z, d) is denoted by $\frac{z}{d}$. Consider the following set of equivalence classes:

$$\mathcal{D}^{-1}Z = \{\frac{z}{d} : z \in Z, d \in \mathcal{D}\}.$$

Note that $\mathcal{D}^{-1}Z$ is an $\mathcal{D}^{-1}L-$module with the following addition and scalar multiplication:

$$\frac{z_1}{d_1} + \frac{z_2}{d_2} = \frac{z_1 d_2 + z_2 d_1}{d_1 d_2} \text{ and } \frac{l}{d} \cdot \frac{z_1}{d_1} = \frac{l z_1}{d d_1}$$

$\frac{z_1}{d_1}, \frac{z_2}{d_2} \in \mathcal{D}^{-1}Z$ and $\frac{l}{d} \in \mathcal{D}^{-1}L$.

Lemma 1.1.9. *[5, Page 38] If $\varphi : Z \to Z'$ is an $L-$linear map, then it induces the following $\mathcal{D}^{-1}L-$linear map:*

$$\mathcal{D}^{-1}\varphi : \mathcal{D}^{-1}Z \to \mathcal{D}^{-1}Z', \quad \frac{z}{d} \mapsto \frac{\varphi(z)}{d}.$$

Definition 1.1.10. *Suppose that $\emptyset \neq L_1 \subseteq L$, $\emptyset \neq \mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{D}^{-1}Z$, then define the following sets:*

$$L_1\mathcal{V}_1 = \{\sum_{i=1}^{n} l_i v_i : l_i \in L_1, v_i \in \mathcal{V}_1 \textit{ and } n \in \mathbb{N}\},$$

$$\mathcal{V}_1 + \mathcal{V}_2 = \{v_1 + v_2 : v_1 \in \mathcal{V}_1, v_2 \in \mathcal{V}_2\} \textit{ and}$$

$$\mathcal{D}^{-1}L_1 = \{\frac{l}{d} : l \in L_1, d \in \mathcal{D}\}.$$

It is clear that the set $L_1\mathcal{V}_1$ is closed under addition.

Lemma 1.1.11. *If $\emptyset \neq Z_i \subseteq Z$ for all $i = 1, 2$ and $\emptyset \neq L_1 \subseteq L$. Then:*

(1) $\mathcal{D}^{-1}(Z_1 \cup Z_2) = \mathcal{D}^{-1}Z_1 \cup \mathcal{D}^{-1}Z_2$.

(2) $\mathcal{D}^{-1}(Z_1 \cap Z_2) \subseteq \mathcal{D}^{-1}Z_1 \cap \mathcal{D}^{-1}Z_2$.

(3) $\mathcal{D}^{-1}(Z_1 + Z_2) \subseteq \mathcal{D}^{-1}Z_1 + \mathcal{D}^{-1}Z_2$.

(4) $\mathcal{D}^{-1}(L_1Z_1) \subseteq (\mathcal{D}^{-1}L_1)(\mathcal{D}^{-1}Z_1)$. *Equality holds, if either* L_1 *is an ideal of* L *or* Z_1 *is a submodule of* Z.

Proof. All the claims are easy to proof. □

Corollary 1.1.12. *[5, Corollary 3.4] Let* Z_1 *and* Z_2 *be submodules of* Z. *For any ideal* L_1 *of* L, *the following conditions hold:*

(1) $\mathcal{D}^{-1}Z_1$ *and* $\mathcal{D}^{-1}(L_1Z_1)$ *are submodules of* $\mathcal{D}^{-1}L-$*module* $\mathcal{D}^{-1}Z$.

(2) $\mathcal{D}^{-1}(Z_1 \cap Z_2) = \mathcal{D}^{-1}Z_1 \cap \mathcal{D}^{-1}Z_2$.

(3) $\mathcal{D}^{-1}(Z_1 + Z_2) = \mathcal{D}^{-1}Z_1 + \mathcal{D}^{-1}Z_2$.

(4) $\mathcal{D}^{-1}(L_1Z_1) = (\mathcal{D}^{-1}L_1)(\mathcal{D}^{-1}Z_1)$.

1.2 Fuzzy sets

In this section, we will recall some basic notions in fuzzy set theory. In the sequel, $\mathcal{U}$ will be representing as an initial universe and $\mathcal{F}(\mathcal{U})$ will be representing as the family of all fuzzy sets of $\mathcal{U}$.

Definition 1.2.1. *[46] A fuzzy set* μ *in* $\mathcal{U}$ *(or sometimes called a fuzzy subset of* $\mathcal{U}$*), is defined by a membership function:*

$$\mu : \mathcal{U} \to [0, 1].$$

For each object $\mathfrak{u} \in \mathcal{U}$, *the membership value* $\mu(\mathfrak{u})$ *describes the grade or degree of membership to which* $\mathfrak{u} \in \mathcal{U}$ *belongs to the fuzzy set* μ.

Definition 1.2.2. *[46] Let* $\mu_1, \mu_2 \in \mathcal{F}(\mathcal{U})$*. Then, their union and intersection are denoted and defined by Zadeh componentwise, as follows:*

(i) $(\mu_1 \cup \mu_2)(\mathfrak{z}) = \mu_1(\mathfrak{z}) \vee \mu_2(\mathfrak{z})$.

(ii) $(\mu_1 \cap \mu_2)(\mathfrak{z}) = \mu_1(\mathfrak{z}) \wedge \mu_2(\mathfrak{z})$.

for all $\mathfrak{z} \in \mathcal{U}$*. Here,* $\wedge$ *denotes the infimum value and* $\vee$ *denotes the supremum value.*

Definition 1.2.3. *[32] Let* $\mathcal{U} = Z$ *be an* $L-$*module and* $\mu \in \mathcal{F}(Z)$*. Then,* μ *is called a fuzzy submodule of* Z*, if*

(i) $\mu(0_Z) = 1$,

(ii) $\mu(z_1 - z_2) \geq \mu(z_1) \wedge \mu(z_2)$,

(iii) $\mu(lz_1) \geq \mu(z_1)$

for all $z_1, z_2 \in Z$, $l \in L$*. The set of all fuzzy submodules of* Z *will be denoted by* $\mathcal{FS}(Z)$*.*

Definition 1.2.4. *[33, Definition 1.1] Let* Z_i *be* $L-$*modules and* $\mu_i \in \mathcal{FS}(Z_i)$*, for all* $i = 1, 2$*. Then,* $\widetilde{\varphi} : \mu_1 \to \mu_2$ *is called a fuzzy module homomorphism, if*

(i) $\varphi : Z_1 \to Z_2$ *is an* $L-$*linear map.*

(ii) $\mu_2(\varphi(z)) \geq \mu_1(z)$*, for all* $z \in Z_1$*.*

1.3 Rough sets

In this section, we furnish some basic ideas and important results about rough sets. For detailed study, we refer to the following paper [34].

Definition 1.3.1. *[34] Let* $\mathcal{U}$ *be an initial universe and* θ *be an equivalence relation on* $\mathcal{U}$*. Then, a pair* $(\mathcal{U}, \theta)$ *is called a Pawlak approximation space. For any non-empty subset* $\mathfrak{X}$ *of* $\mathcal{U}$*, the lower and upper approximations of* $\mathfrak{X}$ *are defined as:*

$$\underline{\mathfrak{X}}_\theta = \{\mathfrak{u} \in \mathcal{U} : [\mathfrak{u}]_\theta \subseteq \mathfrak{X}\} \text{ and } \overline{\mathfrak{X}}^\theta = \{\mathfrak{u} \in \mathcal{U} : [\mathfrak{u}]_\theta \cap \mathfrak{X} \neq \emptyset\},$$

where $[\mathfrak{u}]_\theta$ *represents the equivalence class of* $\mathfrak{u} \in \mathcal{U}$ *with respect to the equivalence relation* θ*. The set* $BR(\mathfrak{X}) = \overline{\mathfrak{X}}^\theta \setminus \underline{\mathfrak{X}}_\theta$*, is called the boundary region of* $\mathfrak{X}$*. If* $BR(\mathfrak{X}) \neq \emptyset$*, then* $\mathfrak{X}$ *is called a rough set. Otherwise* $\mathfrak{X}$ *is a crisp set.*

Proposition 1.3.2. *[34] Let* $(\mathcal{U}, \theta)$ *be an approximation space and* $\emptyset \neq \mathfrak{X}_1, \mathfrak{X}_2 \subseteq \mathcal{U}$*. Then:*

(1) $\underline{\emptyset}_\theta = \overline{\emptyset}^\theta = \emptyset$.

(2) $\underline{\mathcal{U}}_\theta = \overline{\mathcal{U}}^\theta = \mathcal{U}$.

(3) $\underline{\mathfrak{X}_1}_\theta \subseteq \mathfrak{X}_1 \subseteq \overline{\mathfrak{X}_1}^\theta$.

(4) $\underline{\mathfrak{X}_1^c}_\theta = (\overline{\mathfrak{X}_1}^\theta)^c$, $\overline{\mathfrak{X}_1^c}^\theta = (\underline{\mathfrak{X}_1}_\theta)^c$.

(5) $\underline{\underline{\mathfrak{X}_1}_\theta}_\theta = \overline{\underline{\mathfrak{X}_1}_\theta}^\theta = \underline{\mathfrak{X}_1}_\theta$.

(6) $\overline{\overline{\mathfrak{X}_1}^\theta}^\theta = \underline{\overline{\mathfrak{X}_1}^\theta}_\theta = \overline{\mathfrak{X}_1}^\theta$.

(7) $\underline{\mathfrak{X}_1 \cap \mathfrak{X}_2}_\theta = \underline{\mathfrak{X}_1}_\theta \cap \underline{\mathfrak{X}_2}_\theta$.

(8) $\underline{\mathfrak{X}_1 \cup \mathfrak{X}_2}_\theta \supseteq \underline{\mathfrak{X}_1}_\theta \cup \underline{\mathfrak{X}_2}_\theta$.

(9) $\overline{\mathfrak{X}_1 \cup \mathfrak{X}_2}^\theta = \overline{\mathfrak{X}_1}^\theta \cup \overline{\mathfrak{X}_2}^\theta$.

(10) $\overline{\mathfrak{X}_1 \cap \mathfrak{X}_2}^\theta \subseteq \overline{\mathfrak{X}_1}^\theta \cap \overline{\mathfrak{X}_2}^\theta$.

(11) If $\mathfrak{X}_1 \subseteq \mathfrak{X}_2$*, then* $\underline{\mathfrak{X}_1}_\theta \subseteq \underline{\mathfrak{X}_2}_\theta$.

(12) If $\mathfrak{X}_1 \subseteq \mathfrak{X}_2$*, then* $\overline{\mathfrak{X}_1}^\theta \subseteq \overline{\mathfrak{X}_2}^\theta$.

where, $\mathfrak{X}_1^c$ *denotes the complement of* $\mathfrak{X}_1$*.*

In literature, many extensions of Pawlak's rough sets were modified by a numerous scientists to an arbitrary binary relation, a soft binary relation, a reflexive relation, a tolerance relation, multi-equivalence relations and a set-valued map instead of an equivalence relation (see [19, 15, 42, 43, 45, 47, 35]).

1.4 Soft sets

In this section, we review some fundamental concepts about soft sets, soft groups, normal soft groups, soft-intersection groups and normal soft-intersection groups. From now on, Æ will be indicating as a set of parameters of the universe $\mathcal{U}$. The parameters are usually attributes or characteristics of $\mathcal{U}$ which may be words or sentences. The power set of $\mathcal{U}$ is represented by $\mathcal{P}(\mathcal{U})$.

Definition 1.4.1. *[30] A pair* (ϖ, Λ) *is called a soft set over* $\mathcal{U}$*, where*

$$\varpi : \Lambda \to \mathcal{P}(\mathcal{U}),$$

is a set-valued function and $\Lambda \subseteq \text{Æ}$*. The set of all soft sets of Æ over* $\mathcal{U}$ *will be denoted by* $\mathcal{S}(\mathcal{U})_{\text{Æ}}$*.*

To explain this concept, let us give an example in the following:

Example 1.4.2. *Let* $\mathcal{U} = \{a, b, c, d\}$ *be the set of calculators and* $\text{Æ} = \{p, c, o, f\}$ *be the set of parameters of these calculators, where*

(p) denotes the price (should be low)

(c) denotes the coverage (should be large)

(o) denotes the outlook (should be beautiful)

(f) *denotes the functions (more are required)*

Suppose that Mr. X wants to purchase some calculators. To define a soft set (ϖ, Λ)*, where* $\Lambda \subseteq Æ$*, means to categories the calculators with respect to its parameters. Let* $\Lambda = Æ$*. Then, define a soft set* (ϖ, Λ) *as follows:*

$$\varpi(x) = \begin{cases} \{a, b, d\}, & \text{if } x = o \\ \{a, c, d\}, & \text{if } x = c \\ \{a, d\}, & \text{if } x = f \\ \{b\}, & \text{if } x = p \end{cases}$$

for all $x \in \Lambda$*. So, the soft set* (ϖ, Λ) *describes that:*

beautiful calculators=$\{a,b,d\}$*, calculators having large coverage=*$\{a,c,d\}$*, calculators with more functions=*$\{a,d\}$*, low price calculators=*$\{b\}$

Definition 1.4.3. *[2] Let* $(\varpi_1, \Lambda_1), (\varpi_2, \Lambda_2) \in \mathcal{S}(\mathcal{U})_{Æ}$*. Then,* (ϖ_1, Λ_1) *is called a soft subset of* (ϖ_2, Λ_2)*, written as* $(\varpi_1, \Lambda_1)\widetilde{\subseteq}(\varpi_2, \Lambda_2)$*, if:*

(1) $\Lambda_1 \subseteq \Lambda_2$ *and*

(2) $\varpi_1(k) \subseteq \varpi_2(k)$ *for all* $k \in \Lambda_1$.

$(\varpi_1, \Lambda_1), (\varpi_2, \Lambda_2) \in \mathcal{S}(\mathcal{U})_{Æ}$ *are said to be equal, if* $(\varpi_1, \Lambda_1)\widetilde{\subseteq}(\varpi_2, \Lambda_2)$ *and* $(\varpi_2, \Lambda_2)\widetilde{\subseteq}(\varpi_1, \Lambda_1)$*, denoted by* $(\varpi_1, \Lambda_1)\widetilde{=}(\varpi_2, \Lambda_2)$.

Definition 1.4.4. *[2, Definition 3.3,3.7] Let* $(\varpi_1, \Lambda_1), (\varpi_2, \Lambda_2) \in \mathcal{S}(\mathcal{U})_{Æ}$*. Then, their restricted intersection* $(\varpi_1, \Lambda_1) \Cap (\varpi_2, \Lambda_2) = (\vartheta_1, \Lambda_3)$ *and restricted union* $(\varpi_1, \Lambda_1) \cup_R (\varpi_2, \Lambda_2) = (\vartheta_2, \Lambda_3)$ *are defined as follows:*

(1) $\vartheta_1(\mathfrak{k}) = \varpi_1(\mathfrak{k}) \cap \varpi_2(\mathfrak{k})$ *for all* $\mathfrak{k} \in \Lambda_3$,

(2) $\vartheta_2(\mathfrak{k}) = \varpi_1(\mathfrak{k}) \cup \varpi_2(\mathfrak{k})$ *for all* $\mathfrak{k} \in \Lambda_3$ *and*

where $\Lambda_3 = \Lambda_1 \cap \Lambda_2 \neq \emptyset$*.*

Definition 1.4.5. *[2, Definition 3.5] Let* $(\varpi, \Lambda) \in \mathcal{S}(\mathcal{U})_{Æ}$*. Then,* (ϖ, Λ) *is said to be:*

(1) a relative null soft set if $\varpi(k) = \emptyset$ *for each* $k \in \Lambda$ *and is denoted by* $\widetilde{\aleph}_{\Lambda}$*.*

(2) a relative whole soft set if $\varpi(k) = \mathcal{U}$ *for each* $k \in \Lambda$ *and is denoted by* $\widetilde{\mathcal{U}}_{\Lambda}$*.*

We will denote the relative null soft set $\widetilde{\aleph}_{Æ}$ *by* $\widetilde{\aleph}$ *and relative whole soft set* $\widetilde{\mathcal{U}}_{Æ}$ *by* $\widetilde{\mathcal{U}}$*.*

Now assume that $\mathcal{U} = \mathcal{G}$, where $\mathcal{G}$ denotes a multiplicative group.

Definition 1.4.6. *[1, Definition 13] A soft set* $(\varpi, \Lambda) \in \mathcal{S}(\mathcal{G})_{Æ}$ *is called a soft group over* $\mathcal{G}$*, if* $\varpi(k)$ *is a subgroup of* $\mathcal{G}$ *for each parameter* $k \in \Lambda$*.*

Definition 1.4.7. *[1, 38, 4] A soft group* $(\varpi, \Lambda) \in \mathcal{S}(\mathcal{G})_{Æ}$ *is called a normal soft group over* $\mathcal{G}$*, if* $\varpi(k)$ *is a normal subgroup of* $\mathcal{G}$ *for each parameter* $k \in \Lambda$*.*

Definition 1.4.8. *[4, Definition 6.1] Let* (ϖ_1, Λ_1) *and* (ϖ_2, Λ_2) *be two soft groups over* $\mathcal{G}$*. Then, the restricted soft product of* (ϖ_1, Λ_1) *and* (ϖ_2, Λ_2) *is denoted by the soft set* $(\varpi_3, \Lambda_3) = (\varpi_1, \Lambda_1)\hat{\circ}(\varpi_2, \Lambda_2)$*, where* $\Lambda_3 = \Lambda_1 \cap \Lambda_2$*, and defined by* $\varpi_3(\mathfrak{k}) = \varpi_1(\mathfrak{k}) \cdot \varpi_2(\mathfrak{k})$ *for all* $\mathfrak{k} \in \Lambda_3$*.*

Principally, soft sets are of two types. First, the soft sets are with fixed set of parameters and second is with distinct types of parameters. In the sequel, we shall discuss the case of fixed set of parameters. For more details, see [12, 20, 21]. From now onwards, Æ will be fixed and $\mathcal{U}$ will varies. In this case, the basic notions about soft sets are defined in the following:

Definition 1.4.9. *[12, Definition 1] Let* $\emptyset \neq \Lambda \subseteq Æ$*. Then, a soft set* τ_{Λ} *over* $\mathcal{U}$ *is defined by the following set-valued mapping:*

$$\tau_{\Lambda} : Æ \to \mathcal{P}(\mathcal{U}) \text{ such that } \tau_{\Lambda}(\mathfrak{p}) = \emptyset, \text{ for all } \mathfrak{p} \notin \Lambda.$$

This soft set is also signified by:

$$\tau_{\Lambda} = \{(\mathfrak{p}, \tau_{\Lambda}(\mathfrak{p})) : \mathfrak{p} \in Æ, \tau_{\Lambda}(\mathfrak{p}) \in \mathcal{P}(\mathcal{U})\}.$$

The set of all soft sets of Æ over $\mathcal{U}$ will be denoted by $\mathcal{S}(Æ)_{\mathcal{U}}$.

Definition 1.4.10. *[12, Definition 14] Suppose that $\theta : \Lambda_1 \to \Lambda_2$ is a function and $\tau_{\Lambda_1}, \tau_{\Lambda_2} \in \mathcal{S}(Æ)_{\mathcal{U}}$. Then, the soft image of τ_{Λ_1} under θ is defined as:*

$$\theta(\tau_{\Lambda_1})(\mathfrak{e}_2) = \begin{cases} \cup_{\mathfrak{e}_2=\theta(\mathfrak{e}_1)} \tau_{\Lambda_1}(\mathfrak{e}_1), & \text{if } \theta^{-1}(\mathfrak{e}_2) \neq \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}$$

for all $\mathfrak{e}_2 \in \Lambda_2$. The soft pre-image of τ_{Λ_2} under θ is defined by:

$$\theta^{-1}(\tau_{\Lambda_2})(\mathfrak{e}_1) = \tau_{\Lambda_2}(\theta(\mathfrak{e}_1)) \text{ for all } \mathfrak{e}_1 \in \Lambda_1.$$

Clearly, $\theta(\tau_{\Lambda_1}), \theta^{-1}(\tau_{\Lambda_2}) \in \mathcal{S}(Æ)_{\mathcal{U}}$.

Lemma 1.4.11. *[21, Lemma 3.43] With the above notion, we have $\theta(\theta^{-1}(\tau_{\Lambda_2})) \subseteq \tau_{\Lambda_2}$. In particular, if θ is surjective then $\theta(\theta^{-1}(\tau_{\Lambda_2})) = \tau_{\Lambda_2}$.*

Definition 1.4.12. *[12, Definition 6] Let Æ $= \mathcal{G}$ and $\tau_{\mathcal{G}} \in \mathcal{S}(\mathcal{G})_{\mathcal{U}}$. Then, $\tau_{\mathcal{G}}$ is called a soft-intersection group over $\mathcal{U}$, if*

(1) $\tau_{\mathcal{G}}(g_1) \cap \tau_{\mathcal{G}}(g_2) \subseteq \tau_{\mathcal{G}}(g_1 g_2)$,

(2) $\tau_{\mathcal{G}}(g_1^{-1}) = \tau_{\mathcal{G}}(g_1)$,

for all $g_1, g_2 \in \mathcal{G}$.

The family of all soft-intersection groups of $\mathcal{G}$ over $\mathcal{U}$ will be denoted by $\mathcal{SI}(\mathcal{G})_{\mathcal{U}}$.

Theorem 1.4.13. *[12, Theorem 1] Let $\tau_{\mathcal{G}} \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$. Then, $\tau(g) \subseteq \tau(e)$ for all $g \in \mathcal{G}$.*

Definition 1.4.14. *[12, Theorem 8] Let $\tau_{\mathcal{G}} \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$ and $\tau_{\mathcal{H}}$ be a soft-intersection subgroup of $\tau_{\mathcal{G}}$, where $\mathcal{H}$ is a subgroup of $\mathcal{G}$. Then, $\tau_{\mathcal{H}}$ is called a normal soft-intersection subgroup of $\tau_{\mathcal{G}}$ over $\mathcal{U}$, if any of the following equivalent conditions holds:*

(1) $\tau_{\mathcal{H}}(g_1g_2) = \tau_{\mathcal{H}}(g_2g_1)$,

(2) $\tau_{\mathcal{H}}(g_1g_2g_1^{-1}) = \tau_{\mathcal{H}}(g_2)$,

(3) $\tau_{\mathcal{H}}(g_1g_2g_1^{-1}) \subseteq \tau_{\mathcal{H}}(g_2)$,

(4) $\tau_{\mathcal{H}}(g_1g_2g_1^{-1}) \supseteq \tau_{\mathcal{H}}(g_2)$ *and*

for all $g_1, g_2 \in \mathcal{G}$

Definition 1.4.15. *[20, Definition 3.20] Let* $\tau_1, \tau_2 \in \mathcal{S}(\mathcal{G})_{\mathcal{U}}$. *Then, soft product of* τ_1 *and* τ_2 *is defined as follows:*

$$(\tau_1 * \tau_2)(x) = \cup_{x=yz}\{\tau_1(y) \cap \tau_2(z)\}, \text{ where } x, y, z \in \mathcal{G}.$$

Definition 1.4.16. *Let* $\tau_1, \tau_2 \in \mathcal{S}(\mathcal{G})_{\mathcal{U}}$. *Then, soft composition of* τ_1 *and* τ_2 *is defined as follows:*

$$(\tau_1 \cdot \tau_2)(x) = \cap_{x=yz}\{\tau_1(y) \cap \tau_2(z)\}, \text{ where } x, y, z \in \mathcal{G}.$$

Chapter 2

Applications of roughness in soft-intersection groups

This chapter consists of 2 sections inclusively. In section 1, the notion of roughness on soft sets will be defined. Some fundamental results of this roughness for soft-intersection groups will be proved. In the next section, we create a connection between soft approximation spaces of two different groups via group homomorphisms. Throughout this chapter, $Æ = \mathcal{G}$ will be denoting a multiplicative group with identity e as a set of attributes of $\mathcal{U}$, and $\mathcal{N}$, $\mathcal{M}$ will be denoting as normal subgroups of $\mathcal{G}$. All the results in this chapter are taken from paper [7].

2.1 The lower and upper soft approximations in soft-intersection groups

Definition 2.1.1. *Let $\tau \in \mathcal{S}(\mathcal{G})_{\mathcal{U}}$. Then, by the rough approximation in an approximation space $(\tau, \mathcal{G}, \mathcal{N}, \mathcal{U})$ we mean a mapping $Apr : \mathcal{S}(\mathcal{G})_{\mathcal{U}} \to (\mathcal{S}(\mathcal{G}))_{\mathcal{U}} \times (\mathcal{S}(\mathcal{G}))_{\mathcal{U}}$:*

$$Apr(\tau) = (\underline{\tau}_{\mathcal{N}}, \overline{\tau}^{\mathcal{N}}),$$

where two operators $\underline{\tau}_{\mathcal{N}}$ *and* $\overline{\tau}^{\mathcal{N}}$ *are defined as:*

$$\underline{\tau}_{\mathcal{N}}(x) = \cap_{y \in x\mathcal{N}} \tau(y) \text{ and}$$

$$\overline{\tau}^{\mathcal{N}}(x) = \cup_{y \in x\mathcal{N}} \tau(y),$$

for all $x \in \mathcal{G}$*. Clearly,* $\underline{\tau}_{\mathcal{N}}$ *and* $\overline{\tau}^{\mathcal{N}}$ *are soft sets of* $\mathcal{G}$ *over* $\mathcal{U}$*. The operators* $\underline{\tau}_{\mathcal{N}}$ *and* $\overline{\tau}^{\mathcal{N}}$ *are called the lower and upper approximations of* τ *with respect to* $\mathcal{N}$ *respectively.*

Proposition 2.1.2. *Let* $\tau \in \mathcal{S}(\mathcal{G})_{\mathcal{U}}$*. Then, the following conditions are hold:*

(1) $\underline{\tau}_{\mathcal{N}} \subseteq \tau \subseteq \overline{\tau}^{\mathcal{N}}$.

(2) If $\mathcal{N} \subseteq \mathcal{M}$*, then* $\underline{\tau}_{\mathcal{M}} \subseteq \underline{\tau}_{\mathcal{N}}$ *and* $\overline{\tau}^{\mathcal{N}} \subseteq \overline{\tau}^{\mathcal{M}}$.

(3) $\underline{\tau}_{\mathcal{N}} \cup \underline{\tau}_{\mathcal{M}} \subseteq \underline{\tau}_{\mathcal{N} \cap \mathcal{M}}$.

(4) $\overline{\tau}^{\mathcal{N} \cap \mathcal{M}} \subseteq \overline{\tau}^{\mathcal{N}} \cap \overline{\tau}^{\mathcal{M}}$.

(5) $\underline{\tau}_{\mathcal{N}\mathcal{M}} \subseteq \underline{\tau}_{\mathcal{N}}$.

(6) $\overline{\tau}^{\mathcal{N}} \subseteq \overline{\tau}^{\mathcal{N}\mathcal{M}}$.

Proof. All claims are easy to prove by using the Definition 2.1.1.

□

Theorem 2.1.3. *If* $\tau \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$*, then* $\overline{\tau}^{\mathcal{N}}, \underline{\tau}_{\mathcal{N}} \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$.

Proof. Let $t \in \overline{\tau}^{\mathcal{N}}(x_1) \cap \overline{\tau}^{\mathcal{N}}(x_2)$, where $x_1, x_2 \in \mathcal{G}$. Then, there exists $y_i \in x_i\mathcal{N}$ such that $t \in \tau(y_i)$ for all $i = 1, 2$. Since $\tau \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$ and $t \in \tau(y_1) \cap \tau(y_2)$, it follows that $t \in \tau(y_1y_2)$. As $y_1y_2 \in (x_1\mathcal{N})(x_2\mathcal{N}) = x_1x_2\mathcal{N}$, it implies that $t \in \overline{\tau}^{\mathcal{N}}(x_1x_2)$. This proves the following inclusion:

$$\overline{\tau}^{\mathcal{N}}(x_1) \cap \overline{\tau}^{\mathcal{N}}(x_2) \subseteq \overline{\tau}^{\mathcal{N}}(x_1x_2) \text{ for all } x_1, x_2 \in \mathcal{G}.$$

Now, suppose that $t \in \overline{\tau}^{\mathcal{N}}(x_1)$ then there exists $y_1 \in x_1\mathcal{N}$ such that $t \in \tau(y_1)$. It implies that $y_1^{-1} \in x_1^{-1}\mathcal{N}$ and $t \in \tau(y_1^{-1})$. Recall that $\tau(y_1^{-1}) = \tau(y_1)$. So, $\overline{\tau}^{\mathcal{N}}(x_1) \subseteq \overline{\tau}^{\mathcal{N}}(x_1^{-1})$. Similarly, the reverse inclusion can be proved. Hence, $\overline{\tau}^{\mathcal{N}}$ is a soft-intersection group.
Suppose that $x_1, x_2 \in \mathcal{G}$ and $t \in \underline{\tau}_{\mathcal{N}}(x_1) \cap \underline{\tau}_{\mathcal{N}}(x_2)$. By Definition 2.1.1 of $\underline{\tau}_{\mathcal{N}}$, it follows that $t \in \tau(y_i)$ for all $y_i \in x_i\mathcal{N}$, $i = 1, 2$. Since $\tau \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$ and $t \in \tau(y_1) \cap \tau(y_2)$ for all $y_1 \in x_1\mathcal{N}$ and $y_2 \in x_2\mathcal{N}$. Then:

$$t \in \tau(y_1y_2) \text{ for all } y_1 \in x_1\mathcal{N} \text{ and } y_2 \in x_2\mathcal{N}. \tag{2.1.1}$$

Let $y \in x_1x_2\mathcal{N}$ be an arbitrary element. Since $(x_1\mathcal{N})(x_2\mathcal{N}) = x_1x_2\mathcal{N}$, then it can be assumed that $y = y_1y_2$, where $y_1 \in x_1\mathcal{N}$ and $y_2 \in x_2\mathcal{N}$. Using Equation (2.1.1), we get $t \in \tau(y_1y_2) = \tau(y)$ for all $y \in x_1x_2\mathcal{N}$. By Definition 2.1.1, it follows that $t \in \underline{\tau}_{\mathcal{N}}(x_1x_2)$. Hence, the following inclusion is proved:

$$\underline{\tau}_{\mathcal{N}}(x_1) \cap \underline{\tau}_{\mathcal{N}}(x_2) \subseteq \underline{\tau}_{\mathcal{N}}(x_1x_2) \text{ for all } x_1, x_2 \in G.$$

With the similar arguments made above, it can be proved that $\underline{\tau}_{\mathcal{N}}(x_1) = \underline{\tau}_{\mathcal{N}}(x_1^{-1})$ for all $x_1 \in \mathcal{G}$. Hence, $\underline{\tau}_{\mathcal{N}}$ is a soft-intersection group. □

Corollary 2.1.4. *Let $\tau \in \mathcal{S}(\mathcal{G})_{\mathcal{U}}$. Then, $\overline{\tau}^{\mathcal{N}} \cap \overline{\tau}^{\mathcal{M}}$, $\underline{\tau}_{\mathcal{N}} \cap \underline{\tau}_{\mathcal{M}} \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$.*

Proof. These results are immediate consequence of Theorem 2.1.3 and [12, Thoerem 6]. □

In [12, Example 5], the authors proved that the union of two soft-intersection groups is not a soft-intersection group. In the next result, it is shown that $\underline{\tau}_{\mathcal{N}} \cup \underline{\tau}_{\mathcal{M}}$ is a soft-intersection group.

Theorem 2.1.5. *Let $\tau \in \mathcal{S}(\mathcal{G})_{\mathcal{U}}$. Then, $\underline{\tau}_{\mathcal{N}} \cup \underline{\tau}_{\mathcal{M}} \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$.*

Proof. Let $\mathfrak{t} \in (\underline{\tau}_{\mathcal{N}} \cup \underline{\tau}_{\mathcal{M}})(\mathfrak{x}_1) \cap (\underline{\tau}_{\mathcal{N}} \cup \underline{\tau}_{\mathcal{M}})(\mathfrak{x}_2)$, where $\mathfrak{x}_1$, $\mathfrak{x}_2 \in \mathcal{G}$. Then, $\mathfrak{t} \in (\underline{\tau}_{\mathcal{N}} \cup \underline{\tau}_{\mathcal{M}})(\mathfrak{x}_i) =$

$\tau_{\mathcal{N}}(\mathfrak{x}_i) \cup \tau_{\mathcal{M}}(\mathfrak{x}_i)$ for all $i = 1, 2$. It implies that

$$\mathfrak{t} \in \tau_{\mathcal{N}}(\mathfrak{x}_1) \text{ or } \mathfrak{t} \in \tau_{\mathcal{M}}(\mathfrak{x}_1) \text{ and } \mathfrak{t} \in \tau_{\mathcal{N}}(\mathfrak{x}_2) \text{ or } \mathfrak{t} \in \tau_{\mathcal{M}}(\mathfrak{x}_2).$$

In order to prove that $\mathfrak{t} \in (\tau_{\mathcal{N}} \cup \tau_{\mathcal{M}})(\mathfrak{x}_1\mathfrak{x}_2)$, consider the following cases:

Case – I : If $\mathfrak{t} \in \tau_{\mathcal{N}}(\mathfrak{x}_1)$ and $\mathfrak{t} \in \tau_{\mathcal{N}}(\mathfrak{x}_2)$. Then $\mathfrak{t} \in \tau_{\mathcal{N}}(\mathfrak{x}_1) \cap \tau_{\mathcal{N}}(\mathfrak{x}_2) \subseteq \tau_{\mathcal{N}}(\mathfrak{x}_1\mathfrak{x}_2)$, since $\tau_{\mathcal{N}} \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$ (see Theorem 2.1.3). This proves that:

$$\mathfrak{t} \in \tau_{\mathcal{N}}(\mathfrak{x}_1\mathfrak{x}_2) \cup \tau_{\mathcal{M}}(\mathfrak{x}_1\mathfrak{x}_2) = (\tau_{\mathcal{N}} \cup \tau_{\mathcal{M}})(\mathfrak{x}_1\mathfrak{x}_2).$$

Similarly, if $\mathfrak{t} \in \tau_{\mathcal{M}}(\mathfrak{x}_1)$ and $\mathfrak{t} \in \tau_{\mathcal{M}}(\mathfrak{x}_2)$, then $\mathfrak{t} \in (\tau_N \cup \tau_{\mathcal{M}})(\mathfrak{x}_1\mathfrak{x}_2)$.

Case – II : If $\mathfrak{t} \in \tau_{\mathcal{N}}(\mathfrak{x}_1)$ and $\mathfrak{t} \in \tau_{\mathcal{M}}(\mathfrak{x}_2)$, then $\mathfrak{t} \in \tau(y_1)$ and $\mathfrak{t} \in \tau(y_2)$ for all $y_1 \in \mathfrak{x}_1\mathcal{N}$ and $y_2 \in \mathfrak{x}_2\mathcal{M}$. Using $\tau \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$, we obtain:

$$\mathfrak{t} \in \tau(y_1) \cap \tau(y_2) \subseteq \tau(y_1y_2) \text{ for all } y_1 \in \mathfrak{x}_1\mathcal{N} \text{ and } y_2 \in \mathfrak{x}_2\mathcal{M}. \tag{2.1.2}$$

Now, let $y \in \mathfrak{x}_1\mathfrak{x}_2\mathcal{M}$. Then, y can be written as $y = \mathfrak{x}_1\mathfrak{x}_2 m$ for some $m \in \mathcal{M}$. Take $y_1 = \mathfrak{x}_1 e \in \mathfrak{x}_1\mathcal{N}$ and $y_2 = \mathfrak{x}_2 m \in \mathfrak{x}_2\mathcal{M}$, then $y = y_1y_2$. It implies that $\mathfrak{t} \in \tau(y_1y_2) = \tau(y)$ for all $y \in \mathfrak{x}_1\mathfrak{x}_2\mathcal{M}$ (see Equation (2.1.2)). Hence, by Definition 2.1.1 we have $\mathfrak{t} \in \tau_{\mathcal{M}}(\mathfrak{x}_1\mathfrak{x}_2)$. Then,

$$\mathfrak{t} \in \tau_{\mathcal{N}}(\mathfrak{x}_1\mathfrak{x}_2) \cup \tau_{\mathcal{M}}(\mathfrak{x}_1\mathfrak{x}_2) = (\tau_{\mathcal{N}} \cup \tau_{\mathcal{M}})(\mathfrak{x}_1\mathfrak{x}_2).$$

Similarly, if $\mathfrak{t} \in \tau_{\mathcal{M}}(\mathfrak{x}_1)$ and $\mathfrak{t} \in \tau_{\mathcal{N}}(\mathfrak{x}_2)$ then similar arguments can be employed to prove $\mathfrak{t} \in (\tau_{\mathcal{N}} \cup \tau_{\mathcal{M}})(\mathfrak{x}_1\mathfrak{x}_2)$.

Therefore, from all cases it can be seen that $(\tau_{\mathcal{N}} \cup \tau_{\mathcal{M}})(\mathfrak{x}_1) \cap (\tau_{\mathcal{N}} \cup \tau_{\mathcal{M}})(\mathfrak{x}_2)$ is a subset of $(\tau_{\mathcal{N}} \cup \tau_{\mathcal{M}})(\mathfrak{x}_1\mathfrak{x}_2)$.

Note that, $(\tau_{\mathcal{N}} \cup \tau_{\mathcal{M}})(\mathfrak{x}_1) = (\tau_{\mathcal{N}} \cup \tau_{\mathcal{M}})(\mathfrak{x}_1^{-1})$ for all $\mathfrak{x}_1 \in \mathcal{G}$. Recall that $\tau \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$. Hence, $\tau_{\mathcal{N}} \cup \tau_{\mathcal{M}}$ is a soft-intersection group. □

In general, $\overline{\tau}^{\mathcal{N}} \cup \overline{\tau}^{\mathcal{M}}$ is not a soft-intersection group (see next Example).

Example 2.1.6. *Let $\mathcal{G} = \mathbb{Z}_6$ and $\mathcal{U}$ be any set. Define $\tau : \mathcal{G} \to \mathcal{P}(\mathcal{U})$ as follows:*

$$\tau(\mathfrak{g}) = \begin{cases} \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \mathfrak{A}_3 \cup \mathfrak{A}_4, & \text{if } \mathfrak{g} = \overline{0}; \\ \mathfrak{A}_1 \cup \mathfrak{A}_3, & \text{if } \mathfrak{g} = \overline{3}; \\ \mathfrak{A}_1, & \text{if } \mathfrak{g} = \overline{1}, \overline{5}; \\ \mathfrak{A}_1 \cup \mathfrak{A}_2, & \text{if } \mathfrak{g} = \overline{2}, \overline{4}; \end{cases}$$

for all $\mathfrak{g} \in \mathcal{G}$, where $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ and $\mathfrak{A}_4$ are any subsets of $\mathcal{U}$. It can be proved that $\tau \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$. Let $\mathcal{M} = \{\overline{0}, \overline{2}, \overline{4}\}$ and $\mathcal{N} = \{\overline{0}, \overline{3}\}$. It follows that:

$$\overline{\tau}^{\mathcal{M}}(\overline{3}) = \tau(\overline{5}) = \mathfrak{A}_1 \cup \mathfrak{A}_3 \text{ and } \overline{\tau}^{\mathcal{M}}(\overline{2}) = \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \mathfrak{A}_3 \cup \mathfrak{A}_4.$$

$$\overline{\tau}^{\mathcal{N}}(\overline{3}) = \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \mathfrak{A}_3 \cup \mathfrak{A}_4 \text{ and } \overline{\tau}^{\mathcal{N}}(\overline{2}) = \tau(\overline{5}) = \mathfrak{A}_1 \cup \mathfrak{A}_2.$$

Let $\mathfrak{g}_1 = \overline{2}$ and $\mathfrak{g}_2 = \overline{3}$, it follows that:

$$(\overline{\tau}^{\mathcal{M}} \cup \overline{\tau}^{\mathcal{N}})(\mathfrak{g}_1 + \mathfrak{g}_2) = \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \mathfrak{A}_3.$$

$$(\overline{\tau}^{\mathcal{M}} \cup \overline{\tau}^{\mathcal{N}})(\mathfrak{g}_2) = \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \mathfrak{A}_3 \cup \mathfrak{A}_4.$$

$$(\overline{\tau}^{\mathcal{M}} \cup \overline{\tau}^{\mathcal{N}})(\mathfrak{g}_1) = \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \mathfrak{A}_3 \cup \mathfrak{A}_4.$$

Hence, if $\mathfrak{A}_4 \neq \emptyset$, then:

$$(\overline{\tau}^{\mathcal{M}} \cup \overline{\tau}^{\mathcal{N}})(\mathfrak{g}_1) \cap (\overline{\tau}^{\mathcal{M}} \cup \overline{\tau}^{\mathcal{N}})(\mathfrak{g}_2) \nsubseteq (\overline{\tau}^{\mathcal{M}} \cup \overline{\tau}^{\mathcal{N}})(\mathfrak{g}_1 + \mathfrak{g}_2).$$

Therefore, $\overline{\tau}^{\mathcal{M}} \cup \overline{\tau}^{\mathcal{N}}$ is not a soft-intersection group.

Theorem 2.1.7. *If τ is a normal soft-intersection group, then $\overline{\tau}^{\mathcal{N}}$ and $\underline{\tau}_{\mathcal{N}}$ are also.*

Proof. By Theorem 2.1.3, we only prove the normality condition. Let $t \in \overline{\tau}^{\mathcal{N}}(x)$. Then, there exists $z \in x\mathcal{N}$ such that $t \in \tau(z)$. It implies that:

$$t \in \tau(z) = \tau(yzy^{-1}), \text{ where } yzy^{-1} \in yxy^{-1}\mathcal{N}.$$

Here, we used that τ is a normal soft-intersection group. Hence, $t \in \overline{\tau}^{\mathcal{N}}(yxy^{-1})$ for all $y \in \mathcal{G}$. This proves that $\overline{\tau}^{\mathcal{N}}(x) \subseteq \overline{\tau}^{\mathcal{N}}(yxy^{-1})$ for all $x, y \in \mathcal{G}$. Hence, $\overline{\tau}^{\mathcal{N}}(x)$ is a normal soft-intersection group.

To prove the normality of $\underline{\tau}_{\mathcal{N}}$, suppose that $t \in \underline{\tau}_{\mathcal{N}}(x)$. Then $t \in \tau(z)$ for all $z \in x\mathcal{N}$. Let $y \in \mathcal{G}$ and $w \in yxy^{-1}\mathcal{N}$. Then, $y^{-1}wy \in x\mathcal{N}$. As τ is a normal soft-intersection group, it implies that:

$$t \in \tau(y^{-1}wy) = \tau(y^{-1}w(y^{-1})^{-1}) = \tau(w).$$

Hence, $t \in \underline{\tau}_{\mathcal{N}}(yxy^{-1})$ for all $y \in \mathcal{G}$. This proves that $\underline{\tau}_{\mathcal{N}}(x) \subseteq \underline{\tau}_{\mathcal{N}}(yxy^{-1})$ for all $y \in \mathcal{G}$. So, $\underline{\tau}_{\mathcal{N}}(x)$ is a normal soft-intersection group. □

Theorem 2.1.8. *Let $\tau \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$. Then, the following statements hold:*

(1) $\tau * \tau = \tau$.

(2) $\tau \cdot \tau \subseteq \tau$ *and* $\tau \cdot \tau \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$.

(3) If τ is a normal soft-intersection group, then $\tau \cdot \tau$ is also.

Proof. (1) Let $x \in \mathcal{G}$ and $t \in (\tau * \tau)(x)$. By Definition 1.4.15 of $\tau * \tau$, we have:

$$t \in \tau(y) \cap \tau(z) \text{ for some } y, z \in \mathcal{G} \text{ such that } x = yz.$$

Due to the assumption on τ, it follows that $t \in \tau(yz) = \tau(x)$. Hence, $(\tau * \tau)(x) \subseteq \tau(x)$ for all $x \in \mathcal{G}$.

Conversely, assume that $t \in \tau(x)$. By Theorem 1.4.13, it follows that

$$t \in \tau(x) = \tau(x) \cap \tau(e) \text{ such that } x = xe.$$

Hence, $t \in (\tau * \tau)(x)$. This proves the equality $\tau * \tau = \tau$.
(2) By Definition 1.4.12 of soft-intersection group, it follows that $\tau(y) \cap \tau(z) \subseteq \tau(yz)$ for all $y, z \in \mathcal{G}$. Let $x \in \mathcal{G}$ such that $x = yz$. Then, by Definition 1.4.16 of soft composition:

$$(\tau \cdot \tau)(x) \subseteq \tau(y) \cap \tau(z) \subseteq \tau(yz) = \tau(x).$$

Now, we prove that $\tau \cdot \tau \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$. Suppose that $t \in (\tau \cdot \tau)(x_1) \cap (\tau \cdot \tau)(x_2)$. Then, for all $y_1, z_1, y_2, z_2 \in \mathcal{G}$ such that $x_1 = y_1 z_1$, and $x_2 = y_2 z_2$, we have:

$$t \in \tau(y_1) \cap \tau(z_1) \text{ and } t \in \tau(y_2) \cap \tau(z_2). \tag{2.1.3}$$

Assume that $x_1 x_2 = yz$, for some $y, z \in \mathcal{G}$. Take, $y_1 = y$, $z_1 = y^{-1}x_1$, $y_2 = x_2 z$ and $z_2 = z^{-1}$. Then, $x_1 = y_1 z_1$, and $x_2 = y_2 z_2$. Hence:

$$t \in \tau(y) \cap \tau(y^{-1}x_1) \text{ and } t \in \tau(x_2 z) \cap \tau(z), \text{ see Equation (2.1.3).}$$

In particular, $t \in \tau(y) \cap \tau(z)$. This proves that $t \in (\tau \cdot \tau)(x_1 x_2)$.
Now, if $t \in (\tau \cdot \tau)(x_1)$, then $t \in \tau(y_1) \cap \tau(z_1)$ for all $y_1, z_1 \in G$ such that $x_1 = y_1 z_1$. Let $x_1^{-1} = yz$. Then, $x_1 = z^{-1}y^{-1}$. It implies that

$$t \in \tau(y^{-1}) \cap \tau(z^{-1}).$$

Since τ is a soft-intersection group, then $t \in \tau(y) \cap \tau(z)$. This proves that $(\tau \cdot \tau)(x_1) \subseteq (\tau \cdot \tau)(x_1^{-1})$. Similarly, other inclusion can be proved. Therefore, $\tau \cdot \tau \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$.
(3) Let τ be a normal soft-intersection group and $x, y \in \mathcal{G}$. If $t \in (\tau \cdot \tau)(xyx^{-1})$, $t \in$

$\tau(y_1) \cap \tau(z_1)$ for all $y_1, z_1 \in \mathcal{G}$ such that $xyx^{-1} = y_1 z_1$. Let $y = uv$ with $u, v \in \mathcal{G}$. Then, $xyx^{-1} = xuvx^{-1} = (xux^{-1})(xvx^{-1})$. It implies that

$$t \in \tau(xux^{-1}) \cap \tau(xvx^{-1}).$$

But τ is normal, then $t \in \tau(u) \cap \tau(v)$. This proves that $t \in (\tau \cdot \tau)(y)$ and $(\tau \cdot \tau)(xyx^{-1}) \subseteq (\tau \cdot \tau)(y)$. $\square$

The converse in Theorem 2.1.8 (1) does not hold in general.

Example 2.1.9. *Let $\mathcal{G} = \mathbb{Z}_6$ and $\mathcal{U} = \mathbb{Z}$. The soft set $\tau : \mathcal{G} \to \mathcal{P}(\mathcal{U})$ is defined as:*

$$\tau(\mathfrak{g}) = \begin{cases} \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}, & \text{if } \mathfrak{g} = \overline{0}; \\ \{0, 2, 4, 6, 8, 10, 12\}, & \text{if } \mathfrak{g} = \overline{1}, \overline{5}; \\ \{1, 3, 4, 6, 7\}, & \text{if } \mathfrak{g} = \overline{2}, \overline{4}; \\ \{0, 2, 3, 6, 9, 11\}, & \text{if } \mathfrak{g} = \overline{3}; \end{cases}$$

for all $\mathfrak{g} \in \mathcal{G}$. Then $\tau \notin \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$ because of $\tau(\overline{1} + \overline{2}) \not\supseteq \tau(\overline{1}) \cap \tau(\overline{2})$. Also, it is obvious that $(\tau * \tau)(\mathfrak{g}) = \tau(\mathfrak{g})$ for all $\mathfrak{g} \in \mathcal{G}$.

The next Example shows that the inclusion is strict in Theorem 2.1.8 (2).

Example 2.1.10. *Let $\mathcal{G} = \mathbb{Z}_3$ and $\mathcal{U} = \mathbb{Z}_2$. Define $\tau \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$ as follows:*

$$\tau(\mathfrak{g}) = \begin{cases} \mathcal{U}, & \text{if } \mathfrak{g} = \overline{0}; \\ \{\overline{0}\}, & \text{if } \mathfrak{g} = \overline{1}, \overline{2}. \end{cases}$$

for all $\mathfrak{g} \in \mathcal{G}$. Then,

$$\tau(\overline{0}) = \mathcal{U} \nsubseteq (\tau \cdot \tau)(\overline{0}) = \{\overline{0}\}.$$

Hence, $\tau \nsubseteq \tau \cdot \tau$.

In the following Example, it is shown that if $\tau_1, \tau_2 \in \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$ then $\tau_1 \cdot \tau_2 \notin \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$ and $\tau_1 * \tau_2 \notin \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$.

Example 2.1.11. *If $\mathcal{G} = \mathbb{Z}_6$ and $\mathcal{U} = S_3$, consider the following soft-intersection groups τ_1 and τ_2 over $\mathcal{U}$:*

$$\tau_1(\mathfrak{g}) = \begin{cases} \mathcal{U}, & \textit{if } \mathfrak{g} = \overline{0}; \\ \{(12), (13), (132)\}, & \textit{if } \mathfrak{g} = \overline{1}, \overline{5}; \\ \{(12), (13), (23), (123), (132)\}, & \textit{if } \mathfrak{g} = \overline{2}, \overline{4}; \\ \{e, (12), (13), (132)\}, & \textit{if } \mathfrak{g} = \overline{3}; \end{cases}$$

$$\tau_2(\mathfrak{g}) = \begin{cases} \mathcal{U}, & \textit{if } \mathfrak{g} = \overline{0}; \\ \{e, (12), (13), (23)\}, & \textit{if } \mathfrak{g} = \overline{1}, \overline{5}; \\ \emptyset, & \textit{if } \mathfrak{g} = \overline{2}, \overline{3}, \overline{4}; \end{cases}$$

for all $\mathfrak{g} \in \mathcal{G}$. Now, we calculate $\tau_1 * \tau_2$ and $\tau_1 \cdot \tau_2$ as follows:

$$(\tau_1 * \tau_2)(\mathfrak{g}) = \begin{cases} \mathcal{U}, & \textit{if } \mathfrak{g} = \overline{0}; \\ \{(12), (13)\}, & \textit{if } \mathfrak{g} = \overline{1}, \overline{4}, \overline{5}; \\ \{e, (12), (13)\}, & \textit{if } \mathfrak{g} = \overline{3}; \\ \{(12), (13), (23)\}, & \textit{if } \mathfrak{g} = \overline{2}; \end{cases}$$

$$(\tau_1 \cdot \tau_2)(\mathfrak{g}) = \begin{cases} \emptyset, & \textit{if } \mathfrak{g} = \overline{0}, \overline{2}, \overline{3}, \overline{4}; \\ \{(12), (13)\}, & \textit{if } \mathfrak{g} = \overline{1}, \overline{5}; \end{cases}$$

for all $\mathfrak{g} \in \mathcal{G}$. Note that $(\tau_1 * \tau_2)(\overline{2}) \neq (\tau_1 * \tau_2)(\overline{4})$ and $(\tau_1 \cdot \tau_2)(\overline{1}) \nsubseteq (\tau_1 \cdot \tau_2)(\overline{0})$. Hence, by Theorem 1.4.13 and Definition 1.4.12, it follows that $\tau_1 * \tau_2 \notin \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$ and $\tau_1 \cdot \tau_2 \notin \mathcal{SI}(\mathcal{G})_{\mathcal{U}}$.

2.2 A connection between lower and upper approximations of soft sets

In this section, we develop a relationship between the lower and upper approximation spaces of soft sets, described in the previous section. In this section, $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ will be considered as a group homomorphism between two groups, and $\mathcal{N}$ and $\mathcal{M}$ will be denoting as normal subgroups of $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. It is well-known that:

(1) $\phi^{-1}(\mathcal{M})$ is a normal subgroup of $\mathcal{G}_1$ and $\ker\phi \subseteq \phi^{-1}(\mathcal{M})$.

(2) If ϕ is an epimorphism, then $\phi(\mathcal{N})$ is a normal subgroup of $\mathcal{G}_2$.

Theorem 2.2.1. *Let $\tau_2 \in \mathcal{S}(\mathcal{G}_2)_{\mathcal{U}}$ and $x \in \mathcal{G}_1$. Then, the following implications are true:*

$$t \in \underline{\tau_2}_{\mathcal{M}}(\phi(x)) \Longrightarrow t \in \underline{(\phi^{-1}(\tau_2))}_{\phi^{-1}(\mathcal{M})}(x).$$

$$t \in \overline{(\phi^{-1}(\tau_2))}^{\phi^{-1}(\mathcal{M})}(x) \Longrightarrow t \in \overline{\tau_2}^{\mathcal{M}}(\phi(x)).$$

Proof. To prove the first implication, assume that $t \in \underline{\tau_2}_{\mathcal{M}}(\phi(x))$. Then,

$$t \in \tau(z) \text{ for all } z \in \phi(x)\mathcal{M}. \tag{2.2.1}$$

Now, let $y \in x\phi^{-1}(\mathcal{M})$. Since ϕ is a group homomorphism, it implies that $\phi(y) \in \phi(x)\phi(\phi^{-1}(\mathcal{M})) \subseteq \phi(x)\mathcal{M}$. From Equation (2.2.1), it follows that $t \in \tau_2(\phi(y)) = \phi^{-1}(\tau_2)(y)$ for all $y \in x\phi^{-1}(\mathcal{M})$. Therefore,

$$t \in \underline{(\phi^{-1}(\tau_2))}_{\phi^{-1}(\mathcal{M})}(x) = \cap_{y\in x\phi^{-1}(\mathcal{M})}[\phi^{-1}(\tau_2)(y)].$$

Now, we prove the second implication. Let $t \in \overline{(\phi^{-1}(\tau_2))}^{\phi^{-1}(\mathcal{M})}(x)$. Then, there exists some $y \in x\phi^{-1}(\mathcal{M})$ such that $t \in \tau_2(\phi(y)) = \phi^{-1}(\tau_2)(y)$. Then,

$$t \in \tau_2(\phi(y)), \text{ such that } \phi(y) \in \phi(x)\phi(\phi^{-1}(\mathcal{M})) \subseteq \phi(x)\mathcal{M}.$$

Hence, $t \in \overline{\tau}^{\mathcal{M}}(\phi(x))$. $\square$

Theorem 2.2.2. *With the previous notion, suppose that ϕ is an epimorphism. Then the following implications hold:*

$$t \in \underline{(\phi^{-1}(\tau_2))}_{\phi^{-1}(\mathcal{M})}(x) \Longrightarrow t \in \underline{\tau_2}_{\mathcal{M}}(\phi(x)).$$

$$t \in \overline{\tau_2}^{\mathcal{M}}(\phi(x)) \Longrightarrow t \in \overline{(\phi^{-1}(\tau_2))}^{\phi^{-1}(\mathcal{M})}(x).$$

Proof. Assume that ϕ is an epimorphism. We claim that:

$$\text{if } z \in \phi(x)\mathcal{M}, \text{ then } y \in x\phi^{-1}(\mathcal{M}) \text{ for some } y \in \mathcal{G}_1 \text{ such that } \phi(y) = z.$$

Note that z can be written as $z = \phi(x)m$, for some $m \in \mathcal{M}$. By surjectivity of ϕ, it follows that $\phi(y) = z$ and $\phi(n) = m$, where $y, n \in \mathcal{G}_1$. So,

$$\phi(y) = \phi(x)\phi(n) = \phi(xn), \text{ since } \phi \text{ is a homomorphism.}$$

This implies that $y^{-1}(xn) \in \ker\phi$. Since $\ker\phi \subseteq \phi^{-1}(\mathcal{M})$, then $y^{-1}(xn) \in \phi^{-1}(\mathcal{M})$. As $n \in \phi^{-1}(\mathcal{M})$, it implies that $x^{-1}y \in \phi^{-1}(\mathcal{M})$. Then, $y \in x\phi^{-1}(\mathcal{M})$. This prove the claim.
Let $t \in \underline{(\phi^{-1}(\tau_2))}_{\phi^{-1}(\mathcal{M})}(x)$. By Definition 2.1.1, we have:

$$t \in \phi^{-1}(\tau_2)(y) \text{ for all } y \in x\phi^{-1}(\mathcal{M}). \tag{2.2.2}$$

Let $z \in \phi(x)\mathcal{M}$ be an arbitrary element. By the above claim, it follows that $y \in x\phi^{-1}(\mathcal{M})$, for some $y \in \mathcal{G}_1$ such that $\phi(y) = z$. Therefore,

$$t \in \tau_2(\phi(y)) = \phi^{-1}(\tau_2)(y), \text{ (see Equation (2.2.2)).}$$

Then, $t \in \tau_2(z)$ for all $z \in \phi(x)\mathcal{M}$. This proves the following result:

$$t \in \underline{(\phi^{-1}(\tau_2))}_{\phi^{-1}(\mathcal{M})}(x) \Longrightarrow t \in \underline{\tau_2}_{\mathcal{M}}(\phi(x)).$$

Now, assume that $t \in \overline{\tau_2}^{\mathcal{M}}(\phi(x))$. Then $t \in \tau_2(z)$, for some $z \in \phi(x)\mathcal{M}$. Again, by the above claim it follows that $y \in x\phi^{-1}(\mathcal{M})$ for some $y \in \mathcal{G}_1$ such that $\phi(y) = z$. Then, $t \in \tau_2(z) = \tau_2(\phi(y)) = \phi^{-1}(\tau_2)(y)$. Therefore, $t \in \overline{(\phi^{-1}(\tau_2))}^{\phi^{-1}(\mathcal{M})}(x)$. □

Note that if ϕ is not an epimorphism, then the claims in Theorem 2.2.2 are not true (see Examples 2.2.3 and 2.2.4).

Example 2.2.3. *Suppose that $\mathcal{G}_1 = \mathbb{Z}_4$ and $\mathcal{G}_2 = \mathbb{Z}_6$. Let $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ be defined as $\overline{a} \mapsto \widehat{3a}$. Then, ϕ is a homomorphism but not onto. Take $\mathcal{U} = \{u_1, u_2, u_3\}$ and define a soft set $\tau_2 : \mathbb{Z}_6 \to \mathcal{P}(\mathcal{U})$ as:*

$$\tau_2(\mathfrak{g}) = \begin{cases} \mathcal{U}, & \text{if } \mathfrak{g} = \widehat{0}; \\ \{u_1, u_2\}, & \text{if } \mathfrak{g} = \widehat{1}, \widehat{3}; \\ \{u_1\}, & \text{if } \mathfrak{g} = \widehat{2}, \widehat{4}; \\ \emptyset, & \text{if } \mathfrak{g} = \widehat{5}; \end{cases}$$

for all $\mathfrak{g} \in \mathcal{G}_2$. Consider the normal subgroup $\mathcal{M} = \{\widehat{0}, \widehat{2}, \widehat{4}\}$ of G_2. By simple calculations, one can see that:

$$\phi^{-1}(\tau_2)(\mathfrak{e}) = \begin{cases} \mathcal{U}, & \text{if } \mathfrak{e} = \overline{0}, \overline{2}; \\ \{u_1, u_2\}, & \text{if } \mathfrak{e} = \overline{1}, \overline{3}; \end{cases} \text{ and } \underline{\tau_2}_M(\phi(\mathfrak{e})) = \begin{cases} \{u_1\}, & \text{if } \phi(\mathfrak{e}) = \widehat{0}, \widehat{2}; \\ \emptyset, & \text{if } \phi(\mathfrak{e}) = \widehat{1}, \widehat{3}; \end{cases}$$

for all $\mathfrak{e} \in \mathcal{G}_1$. Note that

$$\underline{\phi^{-1}(\tau_2)}_{\phi^{-1}(\mathcal{M})}(\overline{0}) = \mathcal{U}, \text{ and } \underline{\phi^{-1}(\tau_2)}_{\phi^{-1}(\mathcal{M})}(\overline{1}) = \{u_1, u_2\}.$$

Hence, $u_2, u_3 \in \underline{\phi^{-1}(\tau_2)}_{\phi^{-1}(\mathcal{M})}(\overline{0})$ and $u_2, u_3 \notin \underline{\tau_2}_{\mathcal{M}}(\phi(\overline{0}))$.

Example 2.2.4. *Suppose that $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{U}$, $\mathcal{M}$ and ϕ are same as in Example 2.2.3. Consider the following soft set:*

$$\tau_2(\mathfrak{g}) = \begin{cases} \{u_1\}, & \text{if } \mathfrak{g} = \widehat{0}; \\ \{u_2\}, & \text{if } \mathfrak{g} = \widehat{1}; \\ \{u_3\}, & \text{if } \mathfrak{g} = \widehat{2}, \widehat{3}; \\ \emptyset, & \text{if } \mathfrak{g} = \widehat{4}, \widehat{5}; \end{cases}$$

for all $\mathfrak{g} \in \mathcal{G}_2$. Then, it follows that

$$\phi^{-1}(\tau_2)(\mathfrak{e}) = \tau_2(\phi(\mathfrak{e})) = \begin{cases} \{u_1\}, & \text{if } \mathfrak{e} = \overline{0}, \overline{2}; \\ \{u_3\}, & \text{if } \mathfrak{e} = \overline{1}, \overline{3}; \end{cases}$$

for all $\mathfrak{e} \in \mathcal{G}_1$. Since, $\phi^{-1}(\mathcal{M}) = \{\overline{0}, \overline{2}\}$. Then,

$$\overline{\phi^{-1}(\tau_2)}^{\phi^{-1}(\mathcal{M})}(\overline{0}) = \{u_1\} \text{ and } \overline{\tau_2}^{\mathcal{M}}(\phi(\overline{0})) = \{u_1, u_3\}.$$

This proves that $u_3 \in \overline{\tau_2}^{\mathcal{M}}(\phi(\overline{0}))$ and $u_3 \notin \overline{\phi^{-1}(\tau_2)}^{\phi^{-1}(\mathcal{M})}(\overline{0})$.

Theorem 2.2.5. *Let $\tau_1 \in \mathcal{S}(\mathcal{G}_2)_{\mathcal{U}}$, $x \in \mathcal{G}_1$ and ϕ be an epimorphism. Then, the following implication is true:*

$$t \in \underline{\tau_1}_{\mathcal{N}}(x) \Longrightarrow t \in \underline{\phi(\tau_1)}_{\phi(\mathcal{N})}(\phi(x))$$

Further, if we assume that ϕ is one-one then the converse is also true.

Proof. Let $x \in \mathcal{G}_1$ and $t \in \underline{\tau_1}_{\mathcal{N}}(x)$. Then, $t \in \tau_1(y)$ for all $y \in x\mathcal{N}$. Note that $\phi(x)\phi(\mathcal{N}) = \phi(x\mathcal{N})$, since ϕ is a group homomorphism. Let $u \in \phi(x)\phi(\mathcal{N})$. Then, $u = \phi(xn)$, for some $n \in \mathcal{N}$. As $xn \in x\mathcal{N}$, it follows that $t \in \tau_1(xn)$. So,

$$t \in \phi(\tau_1)(u) = \cup_{u=\phi(y), y \in \mathcal{G}_1} \tau_1(y), \text{ for all } u \in \phi(x)\phi(\mathcal{N}).$$

By Definition 2.1.1, we have $t \in \underline{\phi(\tau_1)}_{\phi(\mathcal{N})}(\phi(x))$.
Conversely, assume that ϕ is one-one. If $t \in \underline{\phi(\tau_1)}_{\phi(\mathcal{N})}(\phi(x))$, then $t \in \phi(\tau_1)(u)$ for all $u \in \phi(x)\phi(\mathcal{N})$. By Definition 1.4.10 of $\phi(\tau_1)(u)$, it follows that:

$$t \in \cup_{u=\phi(v),v\in\mathcal{G}_1}\tau_1(v) \text{ for all } u \in \phi(x)\phi(\mathcal{N}). \tag{2.2.3}$$

Let $y \in x\mathcal{N}$. Then, $y = xn$ for some $n \in \mathcal{N}$. Hence, $\phi(y) = \phi(x)\phi(n) \in \phi(x)\phi(\mathcal{N})$. Then

$$t \in \cup_{\phi(y)=\phi(v),v\in\mathcal{G}_1}\tau_1(v), \text{see Equation (2.2.3).}$$

Due to the assumption on ϕ, for $\phi(v) = \phi(y)$, it implies that $v = y$. Then, $t \in \tau_1(y)$, for all $y \in x\mathcal{N}$. Hence, $t \in \underline{\tau_1}_{\mathcal{N}}(x)$. □

The next Example shows that if ϕ is not one-one, then the converse does not hold in Theorem 2.2.5.

Example 2.2.6. *Suppose that $\mathcal{U} = \mathbb{Z}$, $\mathcal{G}_1 = \mathbb{Z}_4$ and $\mathcal{G}_2 = \mathbb{Z}_2$. Consider the epimorphism of groups $\phi : \mathcal{G}_1 \to \mathcal{G}_2$:*

$$\phi(\mathfrak{g}) = \begin{cases} \overline{0}, & \text{if } \mathfrak{g} = \overline{0}, \overline{2}; \\ \overline{1}, & \text{if } \mathfrak{g} = \overline{1}, \overline{3}; \end{cases}$$

for all $\mathfrak{g} \in \mathcal{G}_1$. Define a soft-intersection group $\tau_1 : \mathcal{G}_1 \to \mathcal{P}(\mathcal{U})$ as follows:

$$\tau_1(\mathfrak{g}) = \begin{cases} \{1,2,3\}, & \text{if } \mathfrak{g} = \overline{0}; \\ \{1\}, & \text{if } \mathfrak{g} = \overline{1}, \overline{3}; \\ \{1,2\}, & \text{if } \mathfrak{g} = \overline{2}; \end{cases}$$

for all $\mathfrak{g} \in \mathcal{G}_1$*. Take* $\mathcal{N} = \ker(\phi)$*, then we get:*

$$\underline{\tau_1}_{\mathcal{N}}(\mathfrak{g}) = \begin{cases} \{1,2\}, & \textit{if } \mathfrak{g} = \overline{0}, \overline{2}; \\ \{1\}, & \textit{if } \mathfrak{g} = \overline{1}, \overline{3}; \end{cases} \textit{ and } \phi(\tau_1)(\mathfrak{e}) = \begin{cases} \{1,2,3\}, & \textit{if } \mathfrak{e} = \widehat{0}; \\ \{1\}, & \textit{if } \mathfrak{e} = \widehat{1}; \end{cases}$$

for all $\mathfrak{g} \in \mathcal{G}_1$ *and* $\mathfrak{e} \in \mathcal{G}_2$*. Since* $\phi(\mathcal{N}) = \{\overline{0}\}$*, then it follows that:*

$$\underline{\phi(\tau_1)}_{\phi(\mathcal{N})}(\phi(\overline{0})) = \{1,2,3\} \textit{ and } \underline{\phi(\tau_1)}_{\phi(\mathcal{N})}(\phi(\overline{1})) = \{1\}.$$

Hence, $3 \in \underline{\phi(\tau_1)}_{\phi(\mathcal{N})}(\phi(\overline{0}))$*, but* $3 \notin \underline{\tau_1}_{\mathcal{N}}(\overline{0})$*.*

Theorem 2.2.7. *With the same assumptions as in Theorem 2.2.5, then the following statement is true:*

$$t \in \overline{\tau_1}^{\mathcal{N}}(x) \Longrightarrow t \in \overline{\phi(\tau_1)}^{\phi(\mathcal{N})}(\phi(x))$$

In addition, if $\ker(\phi) \subseteq \mathcal{N}$ *then the converse is also true.*

Proof. Let $x \in \mathcal{G}_1$ and $t \in \overline{\tau_1}^{\mathcal{N}}(x)$. By Definition 2.1.1, there exists $y \in xN$ such that $t \in \tau_1(y)$. This implies that

$$t \in \phi(\tau_1)(\phi(y)) = \cup_{\phi(y)=\phi(z)} \tau_1(z).$$

Let $y = xn$, where $n \in \mathcal{N}$. Then, $\phi(y) = \phi(x)\phi(n) \in \phi(x)\phi(\mathcal{N})$. Hence, $t \in \overline{\phi(\tau_1)}^{\phi(\mathcal{N})}(\phi(x))$. Conversely, assume that $\ker(\phi) \subseteq \mathcal{N}$ and $t \in \overline{\phi(\tau_1)}^{\phi(\mathcal{N})}(\phi(x))$. Then $t \in \phi(\tau_1)(u)$, for some $u \in \phi(x)\phi(\mathcal{N}) = \phi(x\mathcal{N})$. This implies that

$$t \in \cup_{u=\phi(y), y \in \mathcal{G}_1} \tau_1(y) \text{ and } u = \phi(xn), \text{ where } n \in \mathcal{N}.$$

Suppose that $t \in \tau_1(y)$ such that $u = \phi(y)$. Then, $\phi(y) = \phi(xn)$ and hence $y^{-1}xn \in \ker(\phi) \subseteq \mathcal{N}$. Then, $y^{-1}x \in \mathcal{N}$ or $y \in x\mathcal{N}$. So, $t \in \overline{\tau_1}^{\mathcal{N}}(x)$. □

The converse in Theorem 2.2.7 become invalid, if $\ker(\phi) \nsubseteq \mathcal{N}$.

Example 2.2.8. *Define* $\phi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ *as* $(a,b) \mapsto a$ *for all* $a, b \in \mathbb{Z}_2$. *Then* ϕ *is an epimorphism with* $\ker\phi = \{(\overline{0},\overline{0}),(\overline{0},\overline{1})\}$. *Suppose that* $\mathcal{N} = \{(\overline{0},\overline{0}),(\overline{1},\overline{0})\}$ *and* $\mathcal{U} = \{x,y,z,w\}$. *Note that* $\ker\phi \nsubseteq \mathcal{N}$. *The soft set* $\tau_1 : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathcal{P}(\mathcal{U})$ *is defined as follows:*

$$\tau_1((\mathfrak{g}_1,\mathfrak{g}_2)) = \begin{cases} \mathcal{U}, & \text{if } (\mathfrak{g}_1,\mathfrak{g}_2) = (\overline{0},\overline{0}); \\ \{x\}, & \text{if } (\mathfrak{g}_1,\mathfrak{g}_2) = (\overline{0},\overline{1}); \\ \{x,y\}, & \text{if } (\mathfrak{g}_1,\mathfrak{g}_2) = (\overline{1},\overline{0}); \\ \{z\}, & \text{if } (\mathfrak{g}_1,\mathfrak{g}_2) = (\overline{1},\overline{1}); \end{cases}$$

for all $(\mathfrak{g}_1,\mathfrak{g}_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$. *It implies that:*

$$\phi(\tau_1)(\mathfrak{g}_2) = \mathcal{U} \text{ for all } \mathfrak{g}_2 \in \mathbb{Z}_2.$$

Since $\phi(\mathcal{N}) = \{\overline{0},\overline{1}\}$, *then we have:*

$$\overline{\tau_1}^{\mathcal{N}}((\overline{1},\overline{1})) = \{x,z\} \text{ and } \overline{\phi(\tau_1)}^{\phi(\mathcal{N})}(\phi((\overline{1},\overline{1}))) = \mathcal{U}.$$

It follows that $y \in \overline{\phi(\tau_1)}^{\phi(\mathcal{N})}(\phi((\overline{1},\overline{1})))$ *and* $y \notin \overline{\tau_1}^{\mathcal{N}}((\overline{1},\overline{1}))$.

Proposition 2.2.9. *Let* $\tau_1 \in \mathcal{S}(\mathcal{G}_1)_{\mathcal{U}}$. *Then:*

(1) If ϕ *is an epimorphism and* $t \in \underline{\tau_1}_{\phi^{-1}(\mathcal{M})}(x)$, *then* $t \in \underline{\phi(\tau_1)}_{\mathcal{M}}(\phi(x))$.

(2) If ϕ *is one-one and* $t \in \underline{\phi(\tau_1)}_{\mathcal{M}}(\phi(x))$, *then* $t \in \underline{\tau_1}_{\phi^{-1}(\mathcal{M})}(x)$.

Proof. (1) Let ϕ be an epimorphism and $\mathcal{N} = \phi^{-1}(\mathcal{M})$. Then, $\phi(\mathcal{N}) = \phi(\phi^{-1}(\mathcal{M})) = \mathcal{M}$. Hence, by Theorem 2.2.5 the result can be easily deduced.

(2) Assume that ϕ is one-one and $t \in \underline{\phi(\tau_1)}_{\mathcal{M}}(\phi(x))$. It follows that $t \in \phi(\tau_1)(u)$ for all $u \in \phi(x)\mathcal{M}$. Then

$$t \in \cup_{u=\phi(v), v\in\mathcal{G}_1} \tau_1(v) \text{ for all } u \in \phi(x)\mathcal{M}. \tag{2.2.4}$$

Let $y \in x\phi^{-1}(\mathcal{M})$ be an arbitrary element. Then, $\phi(y) \in \phi(x)\phi(\phi^{-1}(\mathcal{M})) \subseteq \phi(x)M$, it implies that

$$t \in \cup_{\phi(y)=\phi(v), v \in \mathcal{G}_1} \tau_1(v), \text{ see Equation (2.2.4).}$$

Note that if $\phi(y) = \phi(v)$, then $y = v$ (since ϕ is one-one). Consequently, $t \in \tau_1(y)$ for all $y \in x\phi^{-1}(\mathcal{M})$. Thus, $t \in \underline{\tau_1}_{\phi^{-1}(\mathcal{M})}(x)$. □

The following Examples illustrates that result 2.2.9 is not true, in general.

Example 2.2.10. *Consider the following group homomorphism from S_3 to $\mathbb{Z}_6$:*

$$\phi(\mathfrak{g}) = \begin{cases} \overline{0}, & \text{if } \mathfrak{g} = e, (123), (132); \\ \overline{3}, & \text{if } \mathfrak{g} = (12), (13), (23); \end{cases}$$

for all $\mathfrak{g} \in S_3$. Clearly, ϕ is not onto. Suppose that $\mathcal{U} = \{p, q, r\}$. The soft set $\tau_1 : S_3 \to P(\mathcal{U})$ is defined as:

$$\tau_1(\mathfrak{g}) = \begin{cases} \mathcal{U}, & \text{if } \mathfrak{g} = e, (132); \\ \{p\}, & \text{if } \mathfrak{g} = (123); \\ \emptyset, & \text{otherwise;} \end{cases}$$

for all $\mathfrak{g} \in S_3$. If $\mathcal{M} = \{\overline{0}, \overline{2}, \overline{4}\}$, then

$$\phi^{-1}(\mathcal{M}) = \{e, (123), (132)\}.$$

The soft image $\phi(\tau_1)(\mathfrak{g}')$ is calculated as follows:

$$\phi(\tau_1)(\mathfrak{g}') = \begin{cases} \mathcal{U}, & \text{if } \mathfrak{g}' = \overline{0}; \\ \emptyset, & \text{otherwise;} \end{cases}$$

for all $\mathfrak{g}' \in \mathbb{Z}_6$*. It follows that*

$$\underline{\tau_1}_{\phi^{-1}(\mathcal{M})}(e) = \{p\} \text{ and } \underline{\phi(\tau_1)}_{\mathcal{M}}(\phi(e)) = \emptyset.$$

This proves that $p \in \underline{\tau_1}_{\phi^{-1}(\mathcal{M})}(e)$ *and* $p \notin \underline{\phi(\tau_1)}_{\mathcal{M}}(\phi(e))$*.*

Example 2.2.11. *Let* $\mathcal{U} = \{l, m, n\}$*,* $\mathcal{G}_1 = \{1, -1, i, -i\}$ *and* $\mathcal{G}_2 = \{1, -1\}$*. Then, the map* $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ *defined by:*

$$\phi(\mathfrak{g}) = \begin{cases} 1, & \text{if } \mathfrak{g} = 1, -1; \\ -1, & \text{if } \mathfrak{g} = i, -i; \end{cases}$$

is a group homomorphism. Define a soft set $\tau_1 : \mathcal{G}_1 \to \mathcal{P}(\mathcal{U})$ *as:*

$$\tau_1(\mathfrak{g}) = \begin{cases} \mathcal{U}, & \text{if } \mathfrak{g} = 1; \\ \{l\}, & \text{if } \mathfrak{g} = i, -i; \\ \{l, m\}, & \text{if } \mathfrak{g} = -1; \end{cases}$$

for all $\mathfrak{g} \in \mathcal{G}_1$*. If* $\mathcal{M}$ *is the trivial normal subgroup of* $\mathcal{G}_2$*, then* $\phi^{-1}(\mathcal{M}) = \{1, -1\}$*. It follows that:*

$$\phi(\tau_1)(\mathfrak{g}') = \begin{cases} \mathcal{U}, & \text{if } \mathfrak{g}' = 1; \\ \{l\}, & \text{if } \mathfrak{g}' = -1. \end{cases}$$

$$\underline{\phi(\tau_1)}_{\mathcal{M}}(\phi(1)) = \mathcal{U} \text{ and } \underline{\tau_1}_{\phi^{-1}(\mathcal{M})}(1) = \{l, m\}.$$

We see that $n \in \underline{\phi(\tau_1)}_{\mathcal{M}}(\phi(1))$ *and* $n \notin \underline{\tau_1}_{\phi^{-1}(\mathcal{M})}(1)$*.*

Corollary 2.2.12. *With the same notion as in Theorem 2.2.7, the following conditions are equivalent:*

(1) $t \in \overline{\tau_1}^{\phi^{-1}(\mathcal{M})}(x)$*.*

(2) $t \in \overline{\phi(\tau_1)}^{\mathcal{M}}(\phi(x))$*.*

Proof. Let ϕ be an epimorphism and $\mathcal{N} = \phi^{-1}(\mathcal{M})$. Then, $\phi(\phi^{-1}(\mathcal{M})) = \mathcal{M} = \phi(\mathcal{N})$ and $\ker(\phi) \subseteq \mathcal{N}$. The result follows from Theorem 2.2.7. $\square$

Theorem 2.2.13. *Let ϕ be an epimorphism and $\tau_2 \in \mathcal{S}(\mathcal{G}_2)_{\mathcal{U}}$. Then, the following statements are true:*

$$t \in \underline{\tau_2}_{\phi(\mathcal{N})}(\phi(x)) \Longleftrightarrow t \in \underline{\phi^{-1}(\tau_2)}_{\mathcal{N}}(x).$$

$$t \in \overline{\tau_2}^{\phi(\mathcal{N})}(\phi(x)) \Longleftrightarrow t \in \overline{\phi^{-1}(\tau_2)}^{\mathcal{N}}(x).$$

Proof. If ϕ is an epimorphism, then $\phi(\phi^{-1}(\tau_2)) = \tau_2$, see Lemma 1.4.11. By Theorems 2.2.5 and 2.2.7, the following implications hold:

$$t \in \underline{\phi^{-1}(\tau_2)}_{\mathcal{N}}(x) \Longrightarrow t \in \underline{\tau_2}_{\phi(\mathcal{N})}(\phi(x)).$$

$$t \in \overline{\phi^{-1}(\tau_2)}^{\mathcal{N}}(x) \Longrightarrow t \in \overline{\tau_2}^{\phi(\mathcal{N})}(\phi(x)).$$

To prove the converse, suppose that $t \in \underline{\tau_2}_{\phi(\mathcal{N})}(\phi(x))$. Then

$$t \in \tau_2(z) \text{ for all } z \in \phi(x)\phi(\mathcal{N}).$$

Let $y \in x\mathcal{N}$. Then, $\phi(y) \in \phi(x)\phi(\mathcal{N})$. It follows that:

$$t \in \tau_2(\phi(y)) = \phi^{-1}(\tau_2)(y) \text{ for all } y \in x\mathcal{N}.$$

Hence, $t \in \underline{\phi^{-1}(\tau_2)}_{\mathcal{N}}(x)$. This completes the proof of first statement.

Now, suppose that $t \in \overline{\tau_2}^{\phi(\mathcal{N})}(\phi(x))$. Then

$$t \in \tau_2(z) \text{ for some } z \in \phi(x)\phi(\mathcal{N}) = \phi(x\mathcal{N}).$$

Let $z = \phi(xn)$, for some $n \in \mathcal{N}$ and $y = xn$. Then,

$$t \in \tau_2(z) = \tau_2(\phi(y)) = \phi^{-1}(\tau_2)(y) \text{ with } y \in x\mathcal{N}.$$

Hence, $t \in \overline{\phi^{-1}(\tau_2)}^{\mathcal{N}}(x)$. $\square$

Chapter 3

A study roughness in modules of fractions

This chapter containing 2 sections. In section 1, a new concept of roughness in modules of fractions will be defined. Section 2 is devoted to built up a relationship among the approximation spaces of two different modules of fractions by using the module homomorphisms. In this chapter, L will be denoting a commutative ring with multiplicative identity element 1_L, Z an $L-$module and $\mathcal{D}$ denotes a MCS in L. This chapter is taken from paper [13].

3.1 Roughness in modules of fractions

In this section, we introduce the notion of lower and upper approximation spaces in modules of fractions with respect to its submodules. Important results interconnected with the presented notion are studied in detail.

Definition 3.1.1. *For an L-submodule W of Z, define a relation $\theta_{\mathcal{D}^{-1}W}$ on $\mathcal{D}^{-1}W$ as follows:*

$$\frac{z}{d}\theta_{\mathcal{D}^{-1}W}\frac{z'}{d'} \Leftrightarrow \frac{z}{d} = \frac{u}{v} \cdot \frac{z'}{d'} \text{ for some } \frac{u}{v} \in U(\mathcal{D}^{-1}L),$$

where $U(\mathcal{D}^{-1}L)$ denotes the set of all unit elements of $\mathcal{D}^{-1}L$. Since L is a commutative ring with identity, then one can verify that $\theta_{\mathcal{D}^{-1}W}$ is an equivalence relation on $D^{-1}W$.

For any $z \in W$ and $d \in \mathcal{D}$, the equivalence class of $\frac{w}{d} \in \mathcal{D}^{-1}W$ will be denoted as:

$$[\frac{z}{d}]_{\theta_{\mathcal{D}^{-1}W}} = \{\frac{w}{v} \in \mathcal{D}^{-1}W : w \in W, v \in \mathcal{D} \text{ and } \frac{z}{d}\theta_{\mathcal{D}^{-1}W}\frac{w}{v}\} = \{\frac{l}{m} \cdot \frac{z}{d} : \frac{l}{m} \in U(\mathcal{D}^{-1}L)\}.$$

Lemma 3.1.2. *Let W_i, $i = 1, 2$ be submodules of Z and K be an ideal of L. Then, the following assertions hold:*

(1) $[\frac{z_1}{d_1} + \frac{z_2}{d_2}]_{\theta_{\mathcal{D}^{-1}W_1+\mathcal{D}^{-1}W_2}} \subseteq [\frac{z_1}{d_1}]_{\theta_{\mathcal{D}^{-1}W_1}} + [\frac{z_2}{d_2}]_{\theta_{\mathcal{D}^{-1}W_2}}$,

(2) $\frac{l}{m} \cdot [\frac{z_1}{d_1}]_{\theta_{\mathcal{D}^{-1}W_1}} = [\frac{lz_1}{md_1}]_{\theta_{\mathcal{D}^{-1}W_1}}$,

(3) $[\frac{u \cdot z_1}{vd_1}]_{\theta_{\mathcal{D}^{-1}(KW_1)}} \subseteq [\frac{u}{v}]_{\theta_{\mathcal{D}^{-1}K}} \cdot [\frac{z_1}{d_1}]_{\theta_{\mathcal{D}^{-1}W_1}}$,

(4) $[\frac{x}{y}]_{\theta_{\mathcal{D}^{-1}W_1}} = [\frac{x}{y}]_{\theta_{\mathcal{D}^{-1}W_2}} = [\frac{x}{y}]_{\theta_{\mathcal{D}^{-1}W_1}} \cap [\frac{x}{y}]_{\theta_{\mathcal{D}^{-1}W_2}} = [\frac{x}{y}]_{\theta_{\mathcal{D}^{-1}(W_1 \cap W_2)}}$,

for all $\frac{l}{m} \in \mathcal{D}^{-1}L$, $\frac{u}{v} \in \mathcal{D}^{-1}K$, $\frac{x}{y} \in \mathcal{D}^{-1}(W_1 \cap W_2)$, $\frac{z_1}{d_1} \in \mathcal{D}^{-1}W_1$, $\frac{z_2}{d_2} \in \mathcal{D}^{-1}W_2$.

Proof. It is effortless in view of Lemma 1.1.11. □

In Example 3.1.3, it is revealed that $\theta_{\mathcal{D}^{-1}W}$ is not a congruence relation. Hence, the reverse inclusion in Lemma 3.1.2 (1) is not true.

Example 3.1.3. *Consider $L = \mathbb{Z}_4$ and $\mathcal{D} = \{\overline{1}, \overline{3}\}$, then $\mathcal{D}^{-1}L = \{\overline{0}, \frac{\overline{1}}{\overline{1}}, \frac{\overline{2}}{\overline{1}}, \frac{\overline{3}}{\overline{1}}\}$ and $U(\mathcal{D}^{-1}L) = \{\frac{\overline{1}}{\overline{1}}, \frac{\overline{3}}{\overline{1}}\}$. It follows that:*

$$[\overline{0}]_{\theta_{\mathcal{D}^{-1}L}} = \{\overline{0}\}, \quad [\frac{\overline{1}}{\overline{1}}]_{\theta_{\mathcal{D}^{-1}L}} = [\frac{\overline{3}}{\overline{1}}]_{\theta_{\mathcal{D}^{-1}L}} = \{\frac{\overline{1}}{\overline{1}}, \frac{\overline{3}}{\overline{1}}\} \textit{ and } [\frac{\overline{2}}{\overline{1}}]_{\theta_{\mathcal{D}^{-1}L}} = \{\frac{\overline{2}}{\overline{1}}\}.$$

Note that: $[\frac{\overline{1}}{\overline{1}}]_{\theta_{\mathcal{D}^{-1}L}} + [\frac{\overline{3}}{\overline{1}}]_{\theta_{\mathcal{D}^{-1}L}} = \{\overline{0}, \frac{\overline{2}}{\overline{1}}\}$ and $[\frac{\overline{1}}{\overline{1}} + \frac{\overline{3}}{\overline{1}}]_{\theta_{\mathcal{D}^{-1}L}} = \{\overline{0}\}$. This clearly shows that $[\frac{z_1}{d_1}]_{\theta_{\mathcal{D}^{-1}W_1}} + [\frac{z_2}{d_2}]_{\theta_{\mathcal{D}^{-1}W_2}} \nsubseteq [\frac{z_1}{d_1} + \frac{z_2}{d_2}]_{\theta_{\mathcal{D}^{-1}W_1+\mathcal{D}^{-1}W_2}}$ for all $\frac{z_1}{d_1}, \frac{z_2}{d_2} \in \mathcal{D}^{-1}Z$.

Definition 3.1.4. *Let X be a non-empty subset of Z. If W is a submodule Z. Then, by the rough approximation in the approximation space $(\mathcal{D}^{-1}W, \theta_{\mathcal{D}^{-1}W})$ means there is a mapping $Apr : P(\mathcal{D}^{-1}W) \to P(\mathcal{D}^{-1}W) \times P(\mathcal{D}^{-1}W)$:*

$$Apr(\mathcal{D}^{-1}X) = (\underline{\mathcal{D}^{-1}X}_{\mathcal{D}^{-1}W}, \overline{\mathcal{D}^{-1}X}^{\mathcal{D}^{-1}W}),$$

where

$$\underline{\mathcal{D}^{-1}X}_{\mathcal{D}^{-1}W} = \{\frac{w}{d} \in \mathcal{D}^{-1}W : [\frac{w}{d}]_{\theta_{\mathcal{D}^{-1}W}} \subseteq \mathcal{D}^{-1}X\} \text{ and}$$
$$\overline{\mathcal{D}^{-1}X}^{\mathcal{D}^{-1}W} = \{\frac{w}{d} \in \mathcal{D}^{-1}W : [\frac{w}{d}]_{\theta_{\mathcal{D}^{-1}W}} \cap \mathcal{D}^{-1}X \neq \emptyset\}.$$

The sets $\underline{\mathcal{D}^{-1}X}_{\mathcal{D}^{-1}W}$ and $\overline{\mathcal{D}^{-1}X}^{\mathcal{D}^{-1}W}$ are called lower and upper approximations of $\mathcal{D}^{-1}X$ in the approximation space $(\mathcal{D}^{-1}W, \theta_{\mathcal{D}^{-1}W})$ respectively.

Remark 3.1.5. *Note that $\underline{\mathcal{D}^{-1}X}_{\mathcal{D}^{-1}W} \subseteq \mathcal{D}^{-1}X$. If $X \subseteq W$, then:*

$$\mathcal{D}^{-1}X \subseteq \overline{\mathcal{D}^{-1}X}^{\mathcal{D}^{-1}W}.$$

Hence, both the lower and upper approximations of a non-empty set $\mathcal{D}^{-1}X$ can be empty sets simultaneously, see Examples 3.1.11 and 3.1.13.

The Lemma 3.1.6 is the generalized form of Lemma 3.10 of [16].

Lemma 3.1.6. *Let W_i, $i = 1, 2$ be submodules of Z and K be an ideal of L. Suppose that $W_1 \subseteq W_2$ and $X_1 \subseteq X_2 \subseteq Z$. Then:*

$$\underline{\mathcal{D}^{-1}X_1}_{\mathcal{D}^{-1}W_1} \subseteq \underline{\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}W_2} \text{ and } \overline{\mathcal{D}^{-1}X_1}^{\mathcal{D}^{-1}W_1} \subseteq \overline{\mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}W_2}.$$

Proof. It is an easy consequence of the Definition 3.1.4 of approximation spaces. □

If X is a submodule of Z, then the approximation spaces of $\mathcal{D}^{-1}X$ do not yield any new information.

Theorem 3.1.7. *If X and W are submodules of Z, then*

$$\underline{\mathcal{D}^{-1}X}_{\mathcal{D}^{-1}W} = \mathcal{D}^{-1}X = \overline{\mathcal{D}^{-1}X}^{\mathcal{D}^{-1}W}.$$

Proof. Let $\frac{w}{d} \in \overline{\mathcal{D}^{-1}X}^{\mathcal{D}^{-1}W}$. By Definition 3.1.4, $[\frac{w}{d}]_{\theta_{\mathcal{D}^{-1}W}} \cap \mathcal{D}^{-1}X \neq \emptyset$. There exists $\frac{w'}{d'} \in \mathcal{D}^{-1}W$ such that $\frac{w'}{d'} \in [\frac{w}{d}]_{\theta_{\mathcal{D}^{-1}W}}$ and $\frac{w'}{d'} \in \mathcal{D}^{-1}X$. It implies that:

$$[\frac{w}{d}]_{\theta_{\mathcal{D}^{-1}W}} = [\frac{w'}{d'}]_{\theta_{\mathcal{D}^{-1}W}}.$$

Since $\frac{l}{m} \cdot \frac{w'}{d'} \in \mathcal{D}^{-1}X$ for some $\frac{l}{m} \in U(\mathcal{D}^{-1}L)$, then $[\frac{w'}{d'}]_{\theta_{\mathcal{D}^{-1}W}} \subseteq \mathcal{D}^{-1}X$. Thus, $\frac{w}{d} \in \underline{\mathcal{D}^{-1}X}_{\mathcal{D}^{-1}W}$. This completes the proof. □

In the following Theorem, it is shown that if $\mathfrak{X} \subseteq Z$ is submodule, then the lower and upper approximations in Davvaz and Mahdavipour's [16] case also do not give us any new information.

Theorem 3.1.8. *Let $\mathfrak{A}$ and $\mathfrak{X}$ be two submodules of Z. Then,*

$$\underline{Apr}_{\mathfrak{A}}(\mathfrak{X}) = \mathfrak{X} = \overline{Apr}^{\mathfrak{A}}(\mathfrak{X})$$

Proof.

$$\underline{Apr}_{\mathfrak{A}}(\mathfrak{X}) = \{\mathfrak{z} \in Z : (\mathfrak{z} + \mathfrak{A}) \subseteq \mathfrak{X}\}, \overline{Apr}^{\mathfrak{A}}(\mathfrak{X}) = \{\mathfrak{z} \in Z : (\mathfrak{z} + A) \cap \mathfrak{X} \neq \emptyset\}$$

By (1) of Proposition 3.2 of [16],

$$\underline{Apr}_{\mathfrak{A}}(\mathfrak{X}) \subseteq \mathfrak{X} \subseteq \overline{Apr}^{\mathfrak{A}}(\mathfrak{X})$$

Let $\mathfrak{x} \in \overline{Apr}^{\mathfrak{A}}(\mathfrak{X})$. Then, $\mathfrak{y} \in (\mathfrak{z} + \mathfrak{A}) \cap \mathfrak{X}$ for some $\mathfrak{y} \in Z$. Since $\mathfrak{y} = \mathfrak{z} + \mathfrak{a}$, for some $\mathfrak{a} \in \mathfrak{A}$ and $\mathfrak{y} \in \mathfrak{X}$. But $\mathfrak{X}$ is a submodule of Z, hence $\mathfrak{y} = \mathfrak{z} + \mathfrak{a} \in \mathfrak{X}$. It follows that $\mathfrak{z} + \mathfrak{A} \subseteq \mathfrak{X}$. Thus

$\mathfrak{x} \in \underline{Apr}_{\mathfrak{A}}(\mathfrak{X})$.

Or

$\mathfrak{y} \in (\mathfrak{z} + \mathfrak{A}) \cap \mathfrak{X}$ for some $\mathfrak{y} \in Z$ imply that $\mathfrak{y} \in \mathfrak{z} + \mathfrak{A}$ and $\mathfrak{y} \in \mathfrak{X}$. Hence, $\mathfrak{y} + \mathfrak{A} = \mathfrak{z} + \mathfrak{A}$ and $\mathfrak{X}$ is a submodule of Z, thus $\mathfrak{y} + \mathfrak{A} \subseteq \mathfrak{X}$ proves that $\mathfrak{z} + \mathfrak{A} \subseteq \mathfrak{X}$. Thus $\mathfrak{z} \in \underline{Apr}_{\mathfrak{A}}(\mathfrak{X})$. □

Proposition 3.1.9. *Let K be an ideal of L and W a submodule of Z. For any non-empty subsets X and Y of L and Z respectively, we have:*

$$\underline{\mathcal{D}^{-1}(XY)}_{\mathcal{D}^{-1}(KW)} \subseteq \underline{(\mathcal{D}^{-1}X)(\mathcal{D}^{-1}Y)}_{\mathcal{D}^{-1}(KW)}.$$

$$(\underline{\mathcal{D}^{-1}X}_{\mathcal{D}^{-1}K})(\underline{\mathcal{D}^{-1}Y}_{\mathcal{D}^{-1}W}) \subseteq \underline{(\mathcal{D}^{-1}X)(\mathcal{D}^{-1}Y)}_{\mathcal{D}^{-1}(KW)}.$$

In addition, if X is an ideal of L or Y is a submodule of Z, then:

$$\underline{\mathcal{D}^{-1}(XY)}_{\mathcal{D}^{-1}(KW)} = \underline{(\mathcal{D}^{-1}X)(\mathcal{D}^{-1}Y)}_{\mathcal{D}^{-1}(KW)}.$$

Proof. We will only prove the second containment, see Lemma 1.1.11(4). Suppose that $z \in (\underline{\mathcal{D}^{-1}X}_{\mathcal{D}^{-1}K})(\underline{\mathcal{D}^{-1}Y}_{\mathcal{D}^{-1}W})$. Then, $z = \sum_{i=1}^{n} \frac{a_i}{b_i} \cdot \frac{z_i}{d_i}$, where $\frac{a_i}{b_i} \in \underline{\mathcal{D}^{-1}X}_{\mathcal{D}^{-1}K}$ and $\frac{z_i}{d_i} \in \underline{\mathcal{D}^{-1}Y}_{\mathcal{D}^{-1}W}$ for all $i = 1, \ldots, n$. By definition of the lower approximation, $[\frac{a_i}{b_i}]_{\theta_{\mathcal{D}^{-1}K}} \subseteq \mathcal{D}^{-1}X$ and $[\frac{z_i}{d_i}]_{\theta_{\mathcal{D}^{-1}W}} \subseteq \mathcal{D}^{-1}Y$ for all $i = 1, \ldots, n$. Then, $z \in \mathcal{D}^{-1}(KW)$ such that:

$$[\frac{a_i}{b_i} \cdot \frac{z_i}{d_i}]_{\theta_{\mathcal{D}^{-1}(KW)}} \subseteq [\frac{a_i}{b_i}]_{\theta_{\mathcal{D}^{-1}K}} \cdot [\frac{z_i}{d_i}]_{\theta_{\mathcal{D}^{-1}W}} \subseteq (\mathcal{D}^{-1}X)(\mathcal{D}^{-1}Y), \text{ for all } i = 1, \ldots, n, \text{ see Lemma 3.1.2.}$$

Since $(\mathcal{D}^{-1}X)(\mathcal{D}^{-1}Y)$ is closed under addition, then Lemma 3.1.2(1) implies that:

$$[\sum_{i=1}^{n} \frac{a_i}{b_i} \cdot \frac{z_i}{d_i}]_{\theta_{\mathcal{D}^{-1}(KW)}} \subseteq \sum_{i=1}^{n} [\frac{a_i}{b_i} \cdot \frac{z_i}{d_i}]_{\theta_{\mathcal{D}^{-1}(KW)}} \subseteq (\mathcal{D}^{-1}X)(\mathcal{D}^{-1}Y).$$

Hence, $z = \sum_{i=1}^{n} \frac{a_i}{b_i} \cdot \frac{z_i}{d_i} \in \underline{(\mathcal{D}^{-1}X)(\mathcal{D}^{-1}Y)}_{\mathcal{D}^{-1}(KW)}$. □

Remark 3.1.10. *In Proposition 3.6 of [16], only second inclusion hold of above result.*

Example 3.1.11. (1) *Let* $L = \mathbb{Z}_6$*,* $K = \{\overline{0}, \overline{3}\}$ *and* $\mathcal{D} = \{\overline{1}, \overline{2}, \overline{4}\}$*. Then*

$$\mathcal{D}^{-1}L = \{\overline{0}, \frac{\overline{1}}{\overline{1}}, \frac{\overline{1}}{\overline{2}}\} \text{ and } KL = K.$$

Also, $\mathcal{D}^{-1}K = \mathcal{D}^{-1}(KL) = (\mathcal{D}^{-1}K)(\mathcal{D}^{-1}L) = \{\overline{0}\}$*. Since* $U(\mathcal{D}^{-1}L) = \{\frac{\overline{1}}{\overline{1}}, \frac{\overline{1}}{\overline{2}}\}$*, then:*

$$[\overline{0}]_{\theta_{\mathcal{D}^{-1}L}} = [\overline{0}]_{\theta_{\mathcal{D}^{-1}K}} = \{\overline{0}\} \text{ and } [\frac{\overline{1}}{\overline{1}}]_{\theta_{\mathcal{D}^{-1}L}} = [\frac{\overline{1}}{\overline{2}}]_{\theta_{\mathcal{D}^{-1}L}} = \{\frac{\overline{1}}{\overline{1}}, \frac{\overline{1}}{\overline{2}}\}.$$

If $X = \{\overline{2}\}$ *and* $Y = \{\overline{3}\}$*, then* $\mathcal{D}^{-1}X = \{\frac{\overline{1}}{\overline{1}}, \frac{\overline{1}}{\overline{2}}\}$*,* $\mathcal{D}^{-1}Y = \{\overline{0}\}$*,* $XY = \mathcal{D}^{-1}(XY) = \{\overline{0}\}$ *and* $(\mathcal{D}^{-1}X)(\mathcal{D}^{-1}Y) = \{\overline{0}\}$*. This implies that:*

$$\underline{\mathcal{D}^{-1}X}_{\mathcal{D}^{-1}K} = (\underline{\mathcal{D}^{-1}X}_{\mathcal{D}^{-1}K})(\underline{\mathcal{D}^{-1}Y}_{\mathcal{D}^{-1}L}) = \emptyset, \quad \underline{\mathcal{D}^{-1}Y}_{\mathcal{D}^{-1}L} = \{\overline{0}\} \text{ and}$$

$$\underline{(\mathcal{D}^{-1}X)(\mathcal{D}^{-1}Y)}_{\mathcal{D}^{-1}(KL)} = \{\overline{0}\}.$$

This proves that $\underline{(\mathcal{D}^{-1}X)(\mathcal{D}^{-1}Y)}_{\mathcal{D}^{-1}(KL)}$ *is not a subset of* $(\underline{\mathcal{D}^{-1}X}_{\mathcal{D}^{-1}K})(\underline{\mathcal{D}^{-1}Y}_{\mathcal{D}^{-1}L})$ *.*

Proposition 3.1.12. *With the same assumptions as in Proposition 3.1.9, the following inclusion hold:*

$$\overline{\mathcal{D}^{-1}(XY)}^{\mathcal{D}^{-1}(KW)} \subseteq \overline{(\mathcal{D}^{-1}X)(\mathcal{D}^{-1}Y)}^{\mathcal{D}^{-1}(KW)}.$$

If we assume in addition that $X \subseteq K$*,* $Y \subseteq W$ *and either* X *is an ideal of* L *or* Y *is a submodule of* Z*, then:*

$$\overline{\mathcal{D}^{-1}(XY)}^{\mathcal{D}^{-1}(KW)} = \overline{(\mathcal{D}^{-1}X)(\mathcal{D}^{-1}Y)}^{\mathcal{D}^{-1}(KW)} \subseteq (\overline{\mathcal{D}^{-1}X}^{\mathcal{D}^{-1}K})(\overline{\mathcal{D}^{-1}Y}^{\mathcal{D}^{-1}W}).$$

Proof. By Lemma 1.1.11, it follows that

$$\overline{\mathcal{D}^{-1}(XY)}^{\mathcal{D}^{-1}(KW)} \subseteq \overline{(\mathcal{D}^{-1}X)(\mathcal{D}^{-1}Y)}^{\mathcal{D}^{-1}(KW)}. \tag{3.1.1}$$

Now, let $X \subseteq K$ and $Y \subseteq W$. Also, assume that either X is an ideal of L or Y is a submodule of Z. Then,

$$\overline{\mathcal{D}^{-1}(XY)}^{\mathcal{D}^{-1}(KW)} = \overline{(\mathcal{D}^{-1}X)(\mathcal{D}^{-1}Y)}^{\mathcal{D}^{-1}(KW)}, \text{ see Lemma 1.1.11.}$$

To prove the other inclusion, suppose that $z \in \overline{\mathcal{D}^{-1}(XY)}^{\mathcal{D}^{-1}(KW)}$. Then, there exists $y \in \mathcal{D}^{-1}(KW)$ such that $y \in [x]_{\theta_{\mathcal{D}^{-1}(KW)}} \cap \mathcal{D}^{-1}(XY)$. Note that y can be written as $y = \sum_{i=1}^{n}[\frac{x_i}{d_i}][\frac{y_i}{w_i}]$ with $x_i \in X$ and $y_i \in Y$. Also, $x = \frac{a}{b}y$, for some $\frac{a}{b} \in U(\mathcal{D}^{-1}L)$. It follows that

$$z = \sum_{i=1}^{n}[\frac{a}{b}\frac{x_i}{d_i}][\frac{y_i}{w_i}] = \sum_{i=1}^{n}[\frac{x_i}{d_i}][\frac{a}{b}\frac{y_i}{w_i}] \in (\mathcal{D}^{-1}X)(\mathcal{D}^{-1}Y) \subseteq (\overline{D^{-1}X}^{\mathcal{D}^{-1}K})(\overline{\mathcal{D}^{-1}Y}^{\mathcal{D}^{-1}W}).$$

(see Remark 3.1.5). Hence, $\overline{\mathcal{D}^{-1}(XY)}^{\mathcal{D}^{-1}(KW)} \subseteq (\overline{\mathcal{D}^{-1}X}^{\mathcal{D}^{-1}K})(\overline{\mathcal{D}^{-1}Y}^{\mathcal{D}^{-1}W})$, see Equation (3.1.1). □

The following Example shows that the inclusion in Proposition 3.1.12 is strict.

Example 3.1.13. *Suppose that L, K, $\mathcal{D}$, X and Y are same as in Example 3.1.11. Since $X = \{\overline{2}\} \nsubseteq K = \{\overline{0}, \overline{3}\}$ and neither X is an ideal of L nor Y is submodule of Z. Thus*

$$\overline{\mathcal{D}^{-1}X}^{\mathcal{D}^{-1}K} = (\overline{\mathcal{D}^{-1}X}^{\mathcal{D}^{-1}K})(\overline{\mathcal{D}^{-1}Y}^{\mathcal{D}^{-1}L}) = \emptyset.$$

$$\overline{\mathcal{D}^{-1}Y}^{\mathcal{D}^{-1}L} = \{\overline{0}\} \text{ and } \overline{(\mathcal{D}^{-1}X)(\mathcal{D}^{-1}Y)}^{\mathcal{D}^{-1}(KL)} = \{\overline{0}\}.$$

Hence, it follows that:

$$\overline{(\mathcal{D}^{-1}X)(\mathcal{D}^{-1}Y)}^{\mathcal{D}^{-1}(KL)} \nsubseteq (\overline{\mathcal{D}^{-1}X}^{\mathcal{D}^{-1}K})(\overline{\mathcal{D}^{-1}Y}^{\mathcal{D}^{-1}L}).$$

The following result is same as the (11) and (12) of Proposition 3.2 of [16].

Proposition 3.1.14. *Suppose that W is an submodule of Z. For $\emptyset \neq X_1, X_2 \subseteq Z$, the following conditions are true:*

(1) $\underline{\mathcal{D}^{-1}(X_1 \cup X_2)}_{\mathcal{D}^{-1}W} = \underline{\mathcal{D}^{-1}X_1 \cup \mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}W} \supseteq \underline{\mathcal{D}^{-1}X_1}_{\mathcal{D}^{-1}W} \cup \underline{\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}W}$.

(2) $\overline{\mathcal{D}^{-1}(X_1 \cup X_2)}^{\mathcal{D}^{-1}W} = \overline{\mathcal{D}^{-1}X_1 \cup \mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}W} = \overline{\mathcal{D}^{-1}X_1}^{\mathcal{D}^{-1}W} \cup \overline{\mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}W}$.

Proof. By Lemma's 1.1.11 and 3.1.6, the following results are true:

$$\underline{\mathcal{D}^{-1}(X_1 \cup X_2)}_{\mathcal{D}^{-1}W} = \underline{\mathcal{D}^{-1}X_1 \cup \mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}W} \supseteq \underline{\mathcal{D}^{-1}X_1}_{\mathcal{D}^{-1}W} \cup \underline{\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}W} \text{ and}$$

$$\overline{\mathcal{D}^{-1}(X_1 \cup X_2)}^{\mathcal{D}^{-1}W} = \overline{\mathcal{D}^{-1}X_1 \cup \mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}W} \supseteq \overline{\mathcal{D}^{-1}X_1}^{\mathcal{D}^{-1}W} \cup \overline{\mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}W}.$$

Now, we prove that $\overline{\mathcal{D}^{-1}X_1 \cup \mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}W}$ is a subset of $\overline{\mathcal{D}^{-1}X_1}^{\mathcal{D}^{-1}W} \cup \overline{\mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}W}$. Consider the element $\frac{z}{d} \in \overline{\mathcal{D}^{-1}X_1 \cup \mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}W}$. Then, $[\frac{z}{d}]_{\theta_{\mathcal{D}^{-1}W}} \cap (\mathcal{D}^{-1}X_1 \cup \mathcal{D}^{-1}X_2) \neq \emptyset$. It follows that:

$$([\frac{z}{d}]_{\theta_{\mathcal{D}^{-1}W}} \cap \mathcal{D}^{-1}X_1) \cup ([\frac{z}{d}]_{\theta_{\mathcal{D}^{-1}W}} \cap \mathcal{D}^{-1}X_2) \neq \emptyset.$$

Hence, $\frac{z}{d} \in \overline{\mathcal{D}^{-1}X_1}^{\mathcal{D}^{-1}W} \cup \overline{\mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}W}$. □

The following Example shows that $\underline{\mathcal{D}^{-1}X_1 \cup \mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}W}$ is not a subset of $\underline{\mathcal{D}^{-1}X_1}_{\mathcal{D}^{-1}W} \cup \underline{\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}W}$.

Example 3.1.15. *Let $L = \mathbb{Z}_8$ and $\mathcal{D} = \{\overline{1}, \overline{3}\}$. Then, $\mathcal{D}^{-1}L = \{\overline{0}, \frac{\overline{1}}{\overline{1}}, \frac{\overline{1}}{\overline{3}}, \frac{\overline{2}}{\overline{1}}, \frac{\overline{2}}{\overline{3}}, \frac{\overline{4}}{\overline{1}}, \frac{\overline{5}}{\overline{1}}, \frac{\overline{5}}{\overline{3}}\}$ and $U(\mathcal{D}^{-1}L) = \{\frac{\overline{1}}{\overline{1}}, \frac{\overline{1}}{\overline{3}}, \frac{\overline{5}}{\overline{1}}, \frac{\overline{5}}{\overline{3}}\}$. The equivalence classes with respect to $\theta_{\mathcal{D}^{-1}L}$ are:*

$$[\overline{0}]_{\theta_{\mathcal{D}^{-1}L}} = \{\overline{0}\}, \quad [\frac{\overline{2}}{\overline{1}}]_{\theta_{\mathcal{D}^{-1}L}} = \{\frac{\overline{2}}{\overline{1}}, \frac{\overline{2}}{\overline{3}}\}, \quad [\frac{\overline{4}}{\overline{1}}]_{\theta_{\mathcal{D}^{-1}L}} = \{\frac{\overline{4}}{\overline{1}}\} \text{ and } [\frac{\overline{1}}{\overline{3}}]_{\theta_{\mathcal{D}^{-1}L}} = \{\frac{\overline{1}}{\overline{1}}, \frac{\overline{1}}{\overline{3}}, \frac{\overline{5}}{\overline{1}}, \frac{\overline{5}}{\overline{3}}\}.$$

Take $X_1 = \{\overline{3}, \overline{4}\}$ and $X_2 = \{\overline{5}\}$, then $\mathcal{D}^{-1}X_1 = \{\frac{\overline{1}}{\overline{1}}, \frac{\overline{3}}{\overline{1}}, \frac{\overline{4}}{\overline{1}}\}$, $\mathcal{D}^{-1}X_1 \cup \mathcal{D}^{-1}X_2 = \{\frac{\overline{1}}{\overline{1}}, \frac{\overline{1}}{\overline{3}}, \frac{\overline{4}}{\overline{1}}, \frac{\overline{5}}{\overline{1}}, \frac{\overline{5}}{\overline{3}}\}$ and $\mathcal{D}^{-1}X_2 = \{\frac{\overline{5}}{\overline{1}}, \frac{\overline{5}}{\overline{3}}\}$. By Lemma 1.1.11, it can be obtained:

$$\underline{\mathcal{D}^{-1}X_1}_{\mathcal{D}^{-1}L} = \{\frac{\overline{4}}{\overline{1}}\}, \quad \underline{\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}L} = \emptyset \text{ and}$$

$$\underline{\mathcal{D}^{-1}X_1 \cup \mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}L} = \underline{\mathcal{D}^{-1}(X_1 \cup X_2)}_{\mathcal{D}^{-1}L} = \{\frac{\overline{1}}{\overline{1}}, \frac{\overline{1}}{\overline{3}}, \frac{\overline{4}}{\overline{1}}, \frac{\overline{5}}{\overline{1}}, \frac{\overline{5}}{\overline{3}}\}.$$

Therefore, $\underline{\mathcal{D}^{-1}X_1 \cup \mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}L} \nsubseteq \underline{\mathcal{D}^{-1}X_1}_{\mathcal{D}^{-1}L} \cup \underline{\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}L}$.

The following result provides the generalized form of (9) and (10) of Proposition 3.2, Corollary 3.1 and Proposition 3.12 of [16].

Proposition 3.1.16. *With the previous notion, suppose that W_i, $i = 1, 2$ are submodules of Z. Then:*

(1) $\underline{\mathcal{D}^{-1}(X_1 \cap X_2)}_{\mathcal{D}^{-1}(W_1 \cap W_2)} \subseteq \underline{\mathcal{D}^{-1}X_1 \cap \mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}(W_1 \cap W_2)} = \underline{\mathcal{D}^{-1}X_1}_{\mathcal{D}^{-1}W_1} \cap \underline{\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}W_2}$.

(2) $\overline{\mathcal{D}^{-1}(X_1 \cap X_2)}^{\mathcal{D}^{-1}(W_1 \cap W_2)} \subseteq \overline{\mathcal{D}^{-1}X_1 \cap \mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}(W_1 \cap W_2)} \subseteq \overline{\mathcal{D}^{-1}X_1}^{\mathcal{D}^{-1}W_1} \cap \overline{\mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}W_2}$.

Proof. Note that $\mathcal{D}^{-1}(X_1 \cap X_2) \subseteq \mathcal{D}^{-1}X_1 \cap \mathcal{D}^{-1}X_2 \subseteq \mathcal{D}^{-1}X_i$, for all $i = 1, 2$ (see Lemma 1.1.11). By Lemma 3.1.6,

$$\begin{aligned} &\underline{\mathcal{D}^{-1}(X_1 \cap X_2)}_{\mathcal{D}^{-1}(W_1 \cap W_2)} \subseteq \underline{\mathcal{D}^{-1}X_1 \cap \mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}(W_1 \cap W_2)} \subseteq \underline{\mathcal{D}^{-1}X_1}_{\mathcal{D}^{-1}W_1} \cap \underline{\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}W_2}. \\ &\overline{\mathcal{D}^{-1}(X_1 \cap X_2)}^{\mathcal{D}^{-1}(W_1 \cap W_2)} \subseteq \overline{\mathcal{D}^{-1}X_1 \cap \mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}(W_1 \cap W_2)} \subseteq \overline{\mathcal{D}^{-1}X_1}^{\mathcal{D}^{-1}W_1} \cap \overline{\mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}W_2}. \end{aligned} \tag{3.1.2}$$

Now, suppose that $z \in \underline{\mathcal{D}^{-1}X_1}_{\mathcal{D}^{-1}W_1} \cap \underline{\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}W_2}$. Then, $z \in \mathcal{D}^{-1}W_1$ and $z \in \mathcal{D}^{-1}W_2$ such that $[z]_{\theta_{\mathcal{D}^{-1}W_1}} \subseteq \mathcal{D}^{-1}X_1$ and $[z]_{\theta_{\mathcal{D}^{-1}W_2}} \subseteq \mathcal{D}^{-1}X_2$. By Lemma 3.1.2 and Corollary 1.1.12, it implies that:

$$z \in \mathcal{D}^{-1}W_1 \cap \mathcal{D}^{-1}W_2 = \mathcal{D}^{-1}(W_1 \cap W_2) \text{ such that } [z]_{\theta_{\mathcal{D}^{-1}(W_1 \cap W_2)}} \subseteq \mathcal{D}^{-1}X_1 \cap \mathcal{D}^{-1}X_2.$$

Consequently, we get $z \in \underline{\mathcal{D}^{-1}X_1 \cap \mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}(W_1 \cap W_2)}$. Form Equation (3.1.2), the following equality holds:

$$\underline{\mathcal{D}^{-1}X_1 \cap \mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}(W_1 \cap W_2)} = \underline{\mathcal{D}^{-1}X_1}_{\mathcal{D}^{-1}W_1} \cap \underline{\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}W_2}.$$

□

In general, $\underline{\mathcal{D}^{-1}(X_1 \cap X_2)}_{\mathcal{D}^{-1}(W_1 \cap W_2)} \neq \underline{\mathcal{D}^{-1}X_1 \cap \mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}(W_1 \cap W_2)}$, and $\overline{\mathcal{D}^{-1}(X_1 \cap X_2)}^{\mathcal{D}^{-1}(W_1 \cap W_2)} \neq \overline{\mathcal{D}^{-1}X_1 \cap \mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}(W_1 \cap W_2)}$.

Example 3.1.17. *Consider L with $\mathcal{D}$ of Example 3.1.11. Let $X_1 = \{\overline{0}, \overline{1}\}$ and $X_2 = \{\overline{0}, \overline{2}, \overline{3}, \overline{4}\}$. Then, $X_1 \cap X_2 = \{\overline{0}\}$. It implies that*

$$\mathcal{D}^{-1}X_1 = \mathcal{D}^{-1}X_2 = \mathcal{D}^{-1}L.$$

This proves that $\mathcal{D}^{-1}X_1 \cap \mathcal{D}^{-1}X_2 = \mathcal{D}^{-1}L$ and $\mathcal{D}^{-1}(X_1 \cap X_2) = \{\overline{0}\}$. It can be easily deduced that:

$$\underline{\mathcal{D}^{-1}X_1 \cap \mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}L} = \overline{\mathcal{D}^{-1}X_1 \cap \mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}L} = \mathcal{D}^{-1}L \text{ and}$$

$$\underline{\mathcal{D}^{-1}(X_1 \cap X_2)}_{\mathcal{D}^{-1}L} = \overline{\mathcal{D}^{-1}(X_1 \cap X_2)}^{\mathcal{D}^{-1}L} = \{\overline{0}\}.$$

It is important to note that the result 3.1.18 is a generalized form of the results 3.8, 3.9 and 3.13 of [16].

Proposition 3.1.18. *With the same notion as in Proposition 3.1.16, the following statements hold:*

(1) $\underline{\mathcal{D}^{-1}(X_1 + X_2)}_{\mathcal{D}^{-1}(W_1 + W_2)} \subseteq \underline{\mathcal{D}^{-1}X_1 + \mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}(W_1 + W_2)}$.

(2) $\underline{\mathcal{D}^{-1}X_1}_{\mathcal{D}^{-1}W_1} + \underline{\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}W_2} \subseteq \underline{\mathcal{D}^{-1}X_1 + \mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}(W_1 + W_2)}$.

(3) $\overline{\mathcal{D}^{-1}(X_1 + X_2)}^{\mathcal{D}^{-1}(W_1 + W_2)} \subseteq \overline{\mathcal{D}^{-1}X_1 + \mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}(W_1 + W_2)}$.

Proof. By Lemma's 1.1.11 and 3.1.6, (1) and (3) are obvious. We prove only (2). Assume that $z \in \underline{\mathcal{D}^{-1}X_1}_{\mathcal{D}^{-1}W_1} + \underline{\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}W_2}$, then

$$z = \frac{w_1}{d_1} + \frac{w_2}{d_2}, \text{ for some } \frac{w_1}{d_1} \in \underline{\mathcal{D}^{-1}X_1}_{\mathcal{D}^{-1}W_1} \text{ and } \frac{w_2}{d_2} \in \underline{\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}W_2}.$$

Note that $\frac{w_1}{d_1} \in \mathcal{D}^{-1}W_1$ such that $[\frac{w_1}{d_1}]_{\theta_{\mathcal{D}^{-1}W_1}} \subseteq \mathcal{D}^{-1}X_1$ and $\frac{w_2}{d_2} \in \mathcal{D}^{-1}W_2$ such that $[\frac{w_2}{d_2}]_{\theta_{\mathcal{D}^{-1}W_2}} \subseteq \mathcal{D}^{-1}X_2$. Consequently, $z = \frac{w_1}{d_1} + \frac{w_2}{d_2} \in \mathcal{D}^{-1}W_1 + \mathcal{D}^{-1}W_2$ such that $[\frac{w_1}{d_1}]_{\theta_{\mathcal{D}^{-1}W_1}} + [\frac{w_2}{d_2}]_{\theta_{\mathcal{D}^{-1}W_2}} \subseteq$

$\mathcal{D}^{-1}X_1+\mathcal{D}^{-1}X_2$ (see Corollary 1.1.12). Using Lemma 3.1.2(1), we obtain:

$$[z]_{\theta_{\mathcal{D}^{-1}W_1+\mathcal{D}^{-1}W_2}} = [\frac{w_1}{d_1}+\frac{w_2}{d_2}]_{\theta_{\mathcal{D}^{-1}W_1+\mathcal{D}^{-1}W_2}} \subseteq \mathcal{D}^{-1}X_1+\mathcal{D}^{-1}X_2.$$

Hence, $z \in \underline{\mathcal{D}^{-1}X_1+\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}(W_1+W_2)}$. □

In the following Example, it is proved that the inclusions in Proposition 3.1.18 are strict. Also, $\overline{\mathcal{D}^{-1}X_1}^{\mathcal{D}^{-1}W_1}+\overline{\mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}W_2} \nsubseteq \overline{\mathcal{D}^{-1}X_1+\mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}(W_1+W_2)}$.

Example 3.1.19. *Suppose L and $\mathcal{D}$ as used in Example 3.1.15. Assume that $K=L$ and $K'=\{\overline{0},\overline{4}\}$, then $K+K'=K$. Then:*

$$\mathcal{D}^{-1}K'=\{\overline{0},\frac{\overline{4}}{\overline{1}}\},\quad \mathcal{D}^{-1}K=\mathcal{D}^{-1}L \text{ and } \mathcal{D}^{-1}K+\mathcal{D}^{-1}K'=\mathcal{D}^{-1}(K+K')=\mathcal{D}^{-1}K.$$

Suppose that $X_1=\{\overline{2}\}$, $X_2=\{\overline{3},\overline{4}\}$, then $X_1+X_2=\{\overline{5},\overline{6}\}$. It implies that:

$$\mathcal{D}^{-1}X_1=\{\frac{\overline{2}}{\overline{1}},\frac{\overline{2}}{\overline{3}}\},\mathcal{D}^{-1}X_2=\{\frac{\overline{1}}{\overline{1}},\frac{\overline{1}}{\overline{3}},\frac{\overline{4}}{\overline{1}}\},\mathcal{D}^{-1}(X_1+X_2)=\{\frac{\overline{2}}{\overline{1}},\frac{\overline{2}}{\overline{3}},\frac{\overline{5}}{\overline{1}},\frac{\overline{5}}{\overline{3}}\} \text{ and}$$

$$\mathcal{D}^{-1}X_1+\mathcal{D}^{-1}X_2=\{\frac{\overline{1}}{\overline{1}},\frac{\overline{2}}{\overline{1}},\frac{\overline{2}}{\overline{3}},\frac{\overline{1}}{\overline{3}},\frac{\overline{5}}{\overline{1}},\frac{\overline{5}}{\overline{3}}\}$$

By Definition 3.1.4, it can be seen that:

$$\underline{\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}K'}=\overline{\mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}K'}=\{\frac{\overline{4}}{\overline{1}}\},\quad \underline{\mathcal{D}^{-1}X_1}_{\mathcal{D}^{-1}K}=\overline{\mathcal{D}^{-1}X_1}^{\mathcal{D}^{-1}K}=\underline{\mathcal{D}^{-1}(X_1+X_2)}_{\mathcal{D}^{-1}K}=\{\frac{\overline{2}}{\overline{1}},\frac{\overline{2}}{\overline{3}}\},$$

$$\overline{\mathcal{D}^{-1}(X_1+X_2)}^{\mathcal{D}^{-1}K}=\underline{\mathcal{D}^{-1}X_1+\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}K}=\overline{\mathcal{D}^{-1}X_1+\mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}K}=\{\frac{\overline{1}}{\overline{1}},\frac{\overline{1}}{\overline{3}},\frac{\overline{2}}{\overline{1}},\frac{\overline{2}}{\overline{3}},\frac{\overline{5}}{\overline{1}},\frac{\overline{5}}{\overline{3}}\} \text{ and}$$

$$\underline{\mathcal{D}^{-1}X_1}_{\mathcal{D}^{-1}K}+\underline{\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}K'}=\overline{\mathcal{D}^{-1}X_1}^{\mathcal{D}^{-1}K}+\overline{\mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}K'}=\{\frac{\overline{2}}{\overline{1}},\frac{\overline{2}}{\overline{3}}\}.$$

Hence, $\underline{\mathcal{D}^{-1}(X_1+X_2)}_{\mathcal{D}^{-1}K} \nsupseteq \underline{\mathcal{D}^{-1}X_1+\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}K'}$, $\overline{\mathcal{D}^{-1}X_1+\mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}K} \nsubseteq \overline{\mathcal{D}^{-1}X_1}^{\mathcal{D}^{-1}K}+\overline{\mathcal{D}^{-1}X_2}^{\mathcal{D}^{-1}K'}$ *and* $\underline{\mathcal{D}^{-1}X_1}_{\mathcal{D}^{-1}K}+\underline{\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}K'} \nsupseteq \underline{\mathcal{D}^{-1}X_1+\mathcal{D}^{-1}X_2}_{\mathcal{D}^{-1}K}$.

Now, if we take $X_1 = \{\overline{1}\}$ *and continuing with same* X_2*, then* $\mathcal{D}^{-1}X_1 = \{\frac{\overline{1}}{\overline{1}}, \frac{\overline{1}}{\overline{3}}\}$, $X_1 + X_2 = \{\overline{4}, \overline{5}\}$ *and hence:*

$$\mathcal{D}^{-1}(X_1 + X_2) = \{\frac{\overline{4}}{\overline{1}}, \frac{\overline{5}}{\overline{1}}, \frac{\overline{5}}{\overline{3}}\} \text{ and } \mathcal{D}^{-1}X_1 + \mathcal{D}^{-1}X_2 = \{\frac{\overline{2}}{\overline{1}}, \frac{\overline{2}}{\overline{3}}, \frac{\overline{4}}{\overline{1}}, \frac{\overline{5}}{\overline{1}}, \frac{\overline{5}}{\overline{3}}\}$$

It follows us that:

$$\overline{D^{-1}(X_1 + X_2)}^{D^{-1}K} = \{\frac{\overline{1}}{\overline{1}}, \frac{\overline{1}}{\overline{3}}, \frac{\overline{4}}{\overline{1}}, \frac{\overline{5}}{\overline{1}}, \frac{\overline{5}}{\overline{3}}\} \text{ and } \overline{D^{-1}X_1 + D^{-1}X_2}^{D^{-1}K} = \{\frac{\overline{1}}{\overline{1}}, \frac{\overline{1}}{\overline{3}}, \frac{\overline{2}}{\overline{1}}, \frac{\overline{2}}{\overline{3}}, \frac{\overline{4}}{\overline{1}}, \frac{\overline{5}}{\overline{1}}, \frac{\overline{5}}{\overline{3}}\}$$

Therefore, $\overline{D^{-1}(X_1 + X_2)}^{D^{-1}K} \not\supseteq \overline{D^{-1}X_1 + D^{-1}X_2}^{D^{-1}K}$.

3.2 Lower and upper approximations via $S^{-1}R$-linear maps

In this section, we assume that $\varphi : Z \longrightarrow Z'$ is a module homomorphism. By Lemma 1.1.9, the induced map $\mathcal{D}^{-1}\varphi : \mathcal{D}^{-1}Z \longrightarrow \mathcal{D}^{-1}Z'$; $\frac{z}{d} \mapsto \frac{\varphi(z)}{d}$ is also a module homomorphism. If W and W' are submodules of Z and Z' respectively. Then, it is well-known that $\varphi(W)$ and $\varphi^{-1}(W')$ are submodules of Z' and Z respectively.

Lemma 3.2.1. *With the same notion followed and* $\frac{z}{d} \in \mathcal{D}^{-1}W$*, the following statement is true:*

$$\frac{z'}{d'} \in [\frac{z}{d}]_{\theta_{\mathcal{D}^{-1}W}} \Longrightarrow \mathcal{D}^{-1}\varphi(\frac{z'}{d'}) \in [\mathcal{D}^{-1}\varphi(\frac{z}{d})]_{\theta_{\mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}W)}}.$$

In addition, if $\mathcal{D}^{-1}\varphi$ *is one-one then the converse is also true.*

Proof. The following implication can be proved in view of linear property of $\mathcal{D}^{-1}\varphi$:

$$\frac{z'}{d'} \in [\frac{z}{d}]_{\theta_{\mathcal{D}^{-1}W}} \Longrightarrow \mathcal{D}^{-1}\varphi(\frac{z'}{d'}) \in [\mathcal{D}^{-1}\varphi(\frac{z}{d})]_{\theta_{\mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}W)}}.$$

Conversely, assume that $\mathcal{D}^{-1}\varphi$ is one-one and $\mathcal{D}^{-1}\varphi(\frac{z'}{d'}) \in [\mathcal{D}^{-1}\varphi(\frac{z}{d})]_{\theta_{\mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}W)}}$. By Definition of $\mathcal{D}^{-1}\varphi$, it implies that:

$$\mathcal{D}^{-1}\varphi(\frac{z}{d}) = \frac{\varphi(z)}{d} = \frac{l}{m}\cdot\frac{\varphi(z')}{d'} = \frac{\varphi(l\cdot z')}{md'} = \mathcal{D}^{-1}\varphi(\frac{l\cdot z'}{md'}), \text{ for some } \frac{l}{m}\in U(\mathcal{D}^{-1}L).$$

Since $\mathcal{D}^{-1}\varphi$ is injective, it follows that $\frac{z}{d} = \frac{l}{m}\cdot\frac{z'}{d'}$. This proves the required result. □

The following Example shows that the converse of Lemma 3.2.1 is not true, if $\mathcal{D}^{-1}\varphi$ is not one-one.

Example 3.2.2. *Let us consider $L = \mathbb{Z}_4$. Define $\varphi : L \longrightarrow L$ as follows:*

$$\varphi(x) = \begin{cases} \overline{0}, & \text{if } x = \overline{0},\ \overline{2} \\ \overline{2}, & \text{if } x = \overline{1},\ \overline{3} \end{cases}$$

Then, φ is a module homomorphism. Take $\mathcal{D} = \{\overline{1}, \overline{3}\}$, then $\mathcal{D}^{-1}L = \{\overline{0}, \frac{\overline{1}}{\overline{1}}, \frac{\overline{2}}{\overline{1}}, \frac{\overline{3}}{\overline{1}}\}$. By Definition of $\mathcal{D}^{-1}\varphi$, we obtain:

$$\mathcal{D}^{-1}\varphi(\frac{l}{m}) = \begin{cases} \overline{0}, & \text{if } \frac{l}{m} = \overline{0},\ \frac{\overline{2}}{\overline{1}} \\ \frac{\overline{2}}{\overline{1}}, & \text{if } \frac{l}{m} = \frac{\overline{1}}{\overline{1}},\ \frac{\overline{3}}{\overline{1}} \end{cases}$$

Since, $\mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}L) = \{\overline{0}, \frac{\overline{2}}{\overline{1}}\}$. By Example 3.1.3, it follows that:

$$\mathcal{D}^{-1}\varphi(\overline{0}) \in [\mathcal{D}^{-1}\varphi(\frac{\overline{2}}{\overline{1}})]_{\theta_{\mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}L)}} \text{ and } \overline{0} \notin [\frac{\overline{2}}{\overline{1}}]_{\theta_{\mathcal{D}^{-1}L}}.$$

Example 3.2.3. *Since there exists a one-one ring homomorphism $\varphi : \mathbb{Z}_5 \to \mathbb{Z}_{10}$; $\overline{x} \mapsto \widehat{6x}$. Then $\mathbb{Z}_{10}$ becomes a $\mathbb{Z}_5-$module under the scalar multiplication defined as follows:*

$$\overline{l}.\widehat{z} = \varphi(\overline{l}).\widehat{z} = \widehat{6lz}, \text{ for all } \overline{l} \in \mathbb{Z}_5 \text{ and } \widehat{z} \in \mathbb{Z}_{10}.$$

Assume that $\mathcal{D} = \{\overline{1}, \overline{2}, \overline{3}, \overline{4}\}$. Let $U = \mathcal{D}^{-1}\mathbb{Z}_5 = \{v_1, v_2, v_3, v_4, v_5\}$ and $U' = \mathcal{D}^{-1}\mathbb{Z}_{10} =$

$\{u_1, u_2, u_3, u_4, u_5\}$ *be two universal sets of stores, where*

$$v_1 = \overline{0}, v_2 = \overline{\frac{1}{1}}, v_3 = \overline{\frac{1}{2}}, v_4 = \overline{\frac{1}{3}}, v_5 = \overline{\frac{1}{4}} \text{ and } u_1 = \widehat{0}, u_2 = \widehat{\frac{1}{1}}, u_3 = \widehat{\frac{1}{2}}, u_4 = \widehat{\frac{1}{3}}, u_5 = \widehat{\frac{1}{4}}.$$

Suppose that $a, b \in U$ *(resp.* $a', b' \in U'$*) have the same value of attributes, if* $(a, b) \in \theta_U$ *(resp.* $(a', b') \in \theta_{U'}$*). Let* $A = \{\mathfrak{E}, \mathfrak{Q}, \mathfrak{L}, \mathfrak{P}\}$ *be the subset of attributes, where* $\mathfrak{E}$*=Empowerment of sales personnel,* $\mathfrak{Q}$*=Perceived quality of merchandisers,* $\mathfrak{L}$*=High traffic location,* $\mathfrak{P}$*=Store profit or loss. Consider the following information system* (U', A)*:*

Table 1

Information System (U', A)

	$\mathfrak{E}$	$\mathfrak{Q}$	$\mathfrak{L}$	$\mathfrak{P}$
u_1	*med.*	*good*	*yes*	*loss*
u_2	*high*	*good*	*no*	*profit*
u_3	*high*	*good*	*no*	*profit*
u_4	*high*	*good*	*no*	*profit*
u_5	*high*	*good*	*no*	*profit*

Since, $\mathcal{D}^{-1}\varphi$ *is one-one. By Lemma 3.2.1, the following information system* (U, A) *can be* *deduced from Table 1:*

Table 2

Information System (U, A)

	$\mathfrak{E}$	$\mathfrak{Q}$	$\mathfrak{L}$	$\mathfrak{P}$
v_1	*med.*	*good*	*yes*	*loss*
v_2	*high*	*good*	*no*	*profit*
v_3	*high*	*good*	*no*	*profit*
v_4	*high*	*good*	*no*	*profit*
v_5	*high*	*good*	*no*	*profit*

Theorem 3.2.4. *With the above notion, suppose that* $X \subseteq Z$ *and* $X' \subseteq Z'$*. Then the following implications are true:*

$$\frac{z}{d} \in \overline{\mathcal{D}^{-1}X}^{(\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}W')} \Longrightarrow \mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \overline{\mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}X)}^{\mathcal{D}^{-1}W'}$$

$$\frac{z}{d} \in \overline{(\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}X')}^{(\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}W')} \Longrightarrow \mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \overline{\mathcal{D}^{-1}X'}^{\mathcal{D}^{-1}W'}$$

Proof. Let $\frac{z}{d} \in \overline{\mathcal{D}^{-1}X}^{V}$, where $V = (\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}W')$. Then, $\frac{z}{d} \in V$ such that $[\frac{z}{d}]_{\theta_V} \cap \mathcal{D}^{-1}X \neq \emptyset$. There exists $\frac{z'}{d'} \in V$ such that $\frac{z'}{d'} \in [\frac{z}{d}]_{\theta_V}$ and $\frac{z'}{d'} \in \mathcal{D}^{-1}X$. By Lemma 3.2.1, $\mathcal{D}^{-1}\varphi(\frac{z'}{d'}) \in [\mathcal{D}^{-1}\varphi(\frac{z}{d})]_{\theta_{\mathcal{D}^{-1}\varphi(V)}}$ and $\mathcal{D}^{-1}\varphi(\frac{z'}{d'}) \in \mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}X)$. Since, $\mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \mathcal{D}^{-1}W'$. By Lemma 3.1.2(4), it follows that:

$$[\mathcal{D}^{-1}\varphi(\frac{z}{d})]_{\theta_{\mathcal{D}^{-1}\varphi(V)}} = [\mathcal{D}^{-1}\varphi(\frac{z}{d})]_{\theta_{\mathcal{D}^{-1}W'}} \text{ and } \mathcal{D}^{-1}\varphi(\frac{z'}{d'}) \in [\mathcal{D}^{-1}\varphi(\frac{z}{d})]_{\theta_{\mathcal{D}^{-1}W'}} \cap \mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}X).$$

Hence, $\mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \overline{\mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}X)}^{\mathcal{D}^{-1}W'}$. This proves the first implication. The second implication can be proved on the same lines. □

In Theorem 3.2.5, the converse of above result is proved.

Theorem 3.2.5. *With the same assumptions as in Theorem 3.2.4, we have:*

$$\mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \overline{\mathcal{D}^{-1}X'}^{\mathcal{D}^{-1}W'} \Longrightarrow \frac{z}{d} \in \overline{(\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}X')}^{(\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}W')}$$

In addition, if $\mathcal{D}^{-1}X$ is an additive subgroup of $\mathcal{D}^{-1}Z$ with $Ker(\mathcal{D}^{-1}\varphi) \subseteq \mathcal{D}^{-1}X$, then

$$\mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \overline{\mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}X)}^{\mathcal{D}^{-1}W'} \Longrightarrow \frac{z}{d} \in \overline{\mathcal{D}^{-1}X}^{(\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}W')}$$

Proof. Let $\mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \overline{\mathcal{D}^{-1}X'}^{\mathcal{D}^{-1}W'}$. By definition of the upper approximation $\mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \mathcal{D}^{-1}W'$ such that $[\mathcal{D}^{-1}\varphi(\frac{z}{d})]_{\theta_{\mathcal{D}^{-1}W'}} \cap \mathcal{D}^{-1}X' \neq \emptyset$. It implies that $\frac{n}{t} \in [\mathcal{D}^{-1}\varphi(\frac{z}{d})]_{\theta_{\mathcal{D}^{-1}W'}}$ and $\frac{n}{t} \in \mathcal{D}^{-1}X'$, for some $\frac{n}{t} \in \mathcal{D}^{-1}W'$. Then,

$$\frac{n}{t} = \frac{l}{m}\mathcal{D}^{-1}\varphi(\frac{z}{d}) = \mathcal{D}^{-1}\varphi(\frac{lz}{md}), \text{ for some } \frac{l}{m} \in U(\mathcal{D}^{-1}L).$$

Hence, $\frac{lz}{md} \in (\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}X')$. Note that $\frac{z}{d} \in V$ and $\frac{lz}{md} \in [\frac{z}{d}]_{\theta_V}$, where $V = (\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}W')$. Thus, $\frac{lz}{md} \in [\frac{z}{d}]_{\theta_V} \cap (\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}X')$. This proves the first implication.

Now, assume that $\mathcal{D}^{-1}X$ is an additive subgroup of $\mathcal{D}^{-1}Z$ with $Ker(\mathcal{D}^{-1}\varphi) \subseteq \mathcal{D}^{-1}X$ and $\mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \overline{\mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}X)}^{\mathcal{D}^{-1}W'}$, then $\mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \mathcal{D}^{-1}W'$ such that $[\mathcal{D}^{-1}\varphi(\frac{z}{d})]_{\theta_{\mathcal{D}^{-1}W'}} \cap \mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}X) \neq \emptyset$. There exists $\frac{n}{t} \in \mathcal{D}^{-1}W'$ such that $\frac{n}{t} \in [\mathcal{D}^{-1}\varphi(\frac{z}{d})]_{\theta_{\mathcal{D}^{-1}W'}}$ and $\frac{n}{t} \in \mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}X)$. So,

$$\frac{n}{t} = \frac{l}{m}\mathcal{D}^{-1}\varphi(\frac{z}{d}) = \mathcal{D}^{-1}\varphi(\frac{lz}{md}), \text{ for some } \frac{l}{m} \in U(\mathcal{D}^{-1}L).$$

Assume that $\mathcal{D}^{-1}\varphi(\frac{lz}{md}) = \mathcal{D}^{-1}\varphi(\frac{x}{t})$, for some $\frac{x}{t} \in \mathcal{D}^{-1}X$. It implies that $\frac{lz}{md} - \frac{x}{t} \in Ker(\mathcal{D}^{-1}\varphi) \subseteq \mathcal{D}^{-1}X$. By the assumption on $\mathcal{D}^{-1}X$, we have $\frac{lz}{md} \in [\frac{z}{d}]_{\theta_V} \cap \mathcal{D}^{-1}X$. Hence, $\frac{z}{d} \in \overline{\mathcal{D}^{-1}X}^{V}$. □

An illustration of the second implication in Theorem 3.2.5 is made in following Example.

Example 3.2.6. *Let $L = Z = \mathbb{Z}_{24}$, $Z' = \mathbb{Z}_{10}$. Define a map $\varphi : \mathbb{Z}_{24} \to \mathbb{Z}_{10}$; $\bar{l} \mapsto \widehat{5l}$, i.e.,*

$$\varphi(\bar{l}) = \begin{cases} \widehat{0}, & \text{if } \bar{l} = \bar{0}, \bar{2}, \bar{4}, \bar{6}, ..., \overline{22} \\ \widehat{5}, & \text{if } \bar{l} = \bar{1}, \bar{3}, \bar{5}, \bar{7}, ..., \overline{23} \end{cases}$$

Then φ is a ring homomorphism. Hence, $\mathbb{Z}_{10}$ is a $\mathbb{Z}_{24}-$module over the scalar multiplication defined as:

$$\bar{l}.\widehat{z} = \varphi(\bar{l}).\widehat{z} = \widehat{5lz}$$

for all $\bar{l} \in \mathbb{Z}_{24}, \widehat{z} \in \mathbb{Z}_{10}$. For $\mathcal{D} = \{\bar{1}, \bar{9}\}$, we have:

$$\mathcal{D}^{-1}\mathbb{Z}_{24} = \{\bar{0}, \frac{\bar{1}}{\bar{1}}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{3}}{\bar{1}}, \frac{\bar{4}}{\bar{1}}, \frac{\bar{5}}{\bar{1}}, \frac{\bar{6}}{\bar{1}}, \frac{\bar{7}}{\bar{1}}\} \text{ and } \mathcal{D}^{-1}\mathbb{Z}_{10} = \{\widehat{0}, \frac{\widehat{1}}{\bar{1}}\}$$

By Lemma 1.1.9, the map $\mathcal{D}^{-1}\varphi : \mathcal{D}^{-1}\mathbb{Z}_{24} \to \mathcal{D}^{-1}\mathbb{Z}_{10}$ is defined as follows:

$$\mathcal{D}^{-1}\varphi(\frac{z}{d}) = \begin{cases} \widehat{0}, & \text{if } \frac{z}{d} = \bar{0}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{4}}{\bar{1}}, \frac{\bar{6}}{\bar{1}} \\ \frac{\widehat{1}}{\bar{1}}, & \text{if } \frac{z}{d} = \frac{\bar{1}}{\bar{1}}, \frac{\bar{3}}{\bar{1}}, \frac{\bar{5}}{\bar{1}}, \frac{\bar{7}}{\bar{1}} \end{cases}$$

for all $\frac{z}{d} \in \mathcal{D}^{-1}\mathbb{Z}_{24}$. Let $W' = \{\widehat{0}, \widehat{2}, \widehat{4}, \widehat{8}\}$, then $\mathcal{D}^{-1}W' = \{\widehat{0}\}$ and $(\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}W') = \{\bar{0}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{4}}{\bar{1}}, \frac{\bar{6}}{\bar{1}}\} = V$ (say). Since $U(\mathcal{D}^{-1}\mathbb{Z}_{24}) = \{\frac{\bar{1}}{\bar{1}}, \frac{\bar{3}}{\bar{1}}, \frac{\bar{5}}{\bar{1}}, \frac{\bar{7}}{\bar{1}}\}$. Then,

$$[\widehat{0}]_{\theta_{\mathcal{D}^{-1}W'}} = \{\widehat{0}\}, [\bar{0}]_{\theta_V} = \{\bar{0}\}, [\frac{\bar{2}}{\bar{1}}]_{\theta_V} = [\frac{\bar{6}}{\bar{1}}]_{\theta_V} = \{\frac{\bar{2}}{\bar{1}}, \frac{\bar{6}}{\bar{1}}\} \text{ and } [\frac{\bar{4}}{\bar{1}}]_{\theta_V} = \{\frac{\bar{4}}{\bar{1}}\}$$

Assume that $X = \{\bar{1}, \bar{2}, \bar{3}\}$, then $\mathcal{D}^{-1}X = \{\frac{\bar{1}}{\bar{1}}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{3}}{\bar{1}}\}$ and $(\mathcal{D}^{-1}\varphi)(\mathcal{D}^{-1}X) = \{\widehat{0}, \frac{\widehat{1}}{\bar{1}}\}$. Hence:

$$\overline{\mathcal{D}^{-1}X}^{V} = \{\frac{\bar{2}}{\bar{1}}, \frac{\bar{6}}{\bar{1}}\} \text{ and } \overline{(\mathcal{D}^{-1}\varphi)(\mathcal{D}^{-1}X)}^{\mathcal{D}^{-1}W'} = \{\widehat{0}\}$$

Note that $\mathcal{D}^{-1}\varphi(\bar{0}) = \mathcal{D}^{-1}\varphi(\frac{\bar{4}}{\bar{1}}) = \widehat{0} \in \overline{(\mathcal{D}^{-1}\varphi)(\mathcal{D}^{-1}X)}^{\mathcal{D}^{-1}W'}$ and $\bar{0}, \frac{\bar{4}}{\bar{1}} \notin \overline{\mathcal{D}^{-1}X}^{V}$.

Theorem 3.2.7. *Suppose that $\emptyset \neq X' \subseteq Z'$ and W' is a submodule of Z'. Then the following* *claims are true:*

$$\frac{z}{d} \in \underline{(\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}X')}_{(\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}W')} \Longrightarrow \mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \underline{\mathcal{D}^{-1}X'}_{\mathcal{D}^{-1}W'}.$$

$$\frac{z}{d} \in \underline{\mathcal{D}^{-1}X}_{(\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}W')} \Longrightarrow \mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \underline{\mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}X)}_{\mathcal{D}^{-1}W'}.$$

Proof. Let $V = (\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}W')$ and $\frac{z}{d} \in V$. Note that $\mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \mathcal{D}^{-1}W'$. If $\frac{n}{t}$ is an arbitrary element of $[\mathcal{D}^{-1}\varphi(\frac{z}{d})]_{\theta_{\mathcal{D}^{-1}W'}}$. Then there exists $\frac{l}{m} \in U(\mathcal{D}^{-1}L)$ such that

$$\frac{n}{t} = \frac{l}{m} \cdot \mathcal{D}^{-1}\varphi(\frac{z}{d}) = \mathcal{D}^{-1}\varphi(\frac{lz}{md}). \tag{3.2.1}$$

Firstly, suppose that $\frac{z}{d} \in \underline{(\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}X')}_V$. Since, $\frac{lz}{md} \in [\frac{z}{d}]_{\theta_V} \subseteq (\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}X')$. It follows that $\frac{n}{t} = \mathcal{D}^{-1}\varphi(\frac{lz}{md}) \in \mathcal{D}^{-1}X'$ (see Equation 3.2.1). Thus, $[\mathcal{D}^{-1}\varphi(\frac{z}{d})]_{\theta_{\mathcal{D}^{-1}W'}} \subseteq \mathcal{D}^{-1}X'$. The proof of the other claim is on the similar lines. □

In the next result, the converse of Theorem 3.2.7 is proved.

Theorem 3.2.8. *With the previous notion, the following statement holds:*

$$\mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \underline{\mathcal{D}^{-1}X'}_{\mathcal{D}^{-1}W'} \Longrightarrow \frac{z}{d} \in \underline{(\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}X')}_{(\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}W')}$$

In addition, if $\mathcal{D}^{-1}X$ is an additive subgroup of $\mathcal{D}^{-1}Z$ with $Ker(\mathcal{D}^{-1}\varphi) \subseteq \mathcal{D}^{-1}X$, then:

$$\mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \underline{\mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}X)}_{\mathcal{D}^{-1}W'} \Longrightarrow \frac{z}{d} \in \underline{\mathcal{D}^{-1}X}_{(\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}W')}$$

Proof. Let $V = (\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}W')$ and $\frac{z}{d} \in V$. Suppose that $\frac{z'}{d'} \in [\frac{z}{d}]_{\theta_V}$ is an arbitrary element. From Lemma 3.2.1, it follows that $\mathcal{D}^{-1}\varphi(\frac{z'}{d'}) \in [\mathcal{D}^{-1}\varphi(\frac{z}{d})]_{\theta_{\mathcal{D}^{-1}\varphi(V)}}$. Since, $\mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \mathcal{D}^{-1}W' \cap \mathcal{D}^{-1}\varphi(V)$. It implies that

$$\mathcal{D}^{-1}\varphi(\frac{z'}{d'}) \in [\mathcal{D}^{-1}\varphi(\frac{z}{d})]_{\theta_{\mathcal{D}^{-1}\varphi(V)}} = [\mathcal{D}^{-1}\varphi(\frac{z}{d})]_{\theta_{\mathcal{D}^{-1}W'}}, \text{ see Lemma 3.1.2.} \tag{3.2.2}$$

Now, assume that $\mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \underline{\mathcal{D}^{-1}X'}_{\mathcal{D}^{-1}W'}$. By Definition of the lower approximation, $[\mathcal{D}^{-1}\varphi(\frac{z}{d})]_{\theta_{\mathcal{D}^{-1}W'}}$ is a subset of $\mathcal{D}^{-1}X'$. By Equation (3.2.2), it follows that $\frac{z'}{d'} \in (\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}X')$. This proves that

$$[\frac{z}{d}]_{\theta_V} \subseteq (\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}X') \text{ and } \frac{z}{d} \in \underline{(\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}X')}_V.$$

The second implication can be proved using same methodology as used in proof of Theorem 3.2.5. □

Theorem 3.2.9. *Let $\emptyset \neq X \subseteq Z$ and $\emptyset \neq X' \subseteq Z'$. If W is a submodule of Z and $\frac{z}{d} \in \mathcal{D}^{-1}W$, then following statements hold:*

$$\frac{z}{d} \in \overline{(\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}X')}^{\mathcal{D}^{-1}W} \Longleftrightarrow \mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \overline{\mathcal{D}^{-1}X'}^{\mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}W)}$$

$$\frac{z}{d} \in \overline{\mathcal{D}^{-1}X}^{\mathcal{D}^{-1}W} \Longrightarrow \mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \overline{\mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}X)}^{\mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}W)}$$

Further, if $\mathcal{D}^{-1}X$ is an additive subgroup of $\mathcal{D}^{-1}Z$ with $Ker(\mathcal{D}^{-1}\varphi) \subseteq \mathcal{D}^{-1}X$. Then, the converse of second statement can also be proved.

Proof. This proof is analogous to the proof of Theorems 3.2.4 and 3.2.5. □

Theorem 3.2.10. *With the same notion as in Theorem 3.2.9, the following assertions hold:*

$$\frac{z}{d} \in \underline{(\mathcal{D}^{-1}\varphi)^{-1}(\mathcal{D}^{-1}X')}_{\mathcal{D}^{-1}W} \Longleftrightarrow \mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \underline{\mathcal{D}^{-1}X'}_{\mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}W)}$$

$$\frac{z}{d} \in \underline{\mathcal{D}^{-1}X}_{\mathcal{D}^{-1}W} \Longrightarrow \mathcal{D}^{-1}\varphi(\frac{z}{d}) \in \underline{\mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}X)}_{\mathcal{D}^{-1}\varphi(\mathcal{D}^{-1}W)}$$

Moreover, if $\mathcal{D}^{-1}X$ is an additive subgroup of $\mathcal{D}^{-1}Z$ with $Ker(\mathcal{D}^{-1}\varphi) \subseteq \mathcal{D}^{-1}X$. Then, the converse of second assertion hold.

Proof. This proof is parallel to the proof of Theorems 3.2.7 and 3.2.8. □

Chapter 4

Fuzzy modules of fractions and roughness

This chapter consists of 3 sections. In section 1, we define the notion of soft modules of fractions. In section 2, we introduce a new notion of fuzzy modules of fractions and some related results. Finally, section 3 is devoted to the study of roughness in fuzzy modules of fractions in view of multi-granulation roughness by utilizing the concept of soft modules of fractions. All the time in this chapter, we assume that L is a commutative ring with a multiplicative identity 1_L, $\mathcal{D}$ is a MCS in L, an $\mathcal{D}^{-1}L-$module $\mathcal{D}^{-1}Z$ as a universal set with additive identity $\frac{\theta}{s}$, where θ is an additive identity of Z and Æ as a finite set of parameters. This chapter is taken from paper [9].

4.1 Soft modules of fractions

In this section, we define the notion of soft modules of fractions and study some related results.

Definition 4.1.1. *A soft set (ξ, Λ) over $\mathcal{D}^{-1}Z$ is called a soft module of fractions, if $\xi(r)$ is a submodule of $\mathcal{D}^{-1}Z$ for each $r \in \Lambda$, where $\Lambda \subseteq$ Æ.*

Proposition 4.1.2. *Let* (ξ_1, Λ_1) *and* (ξ_2, Λ_2) *be two soft modules of fractions over* $\mathcal{D}^{-1}Z$*. Then, their restricted intersection* (ζ, Λ_3) *is also a soft module of fractions over* $D^{-1}Z$*, where* $(\zeta, \Lambda_3) = (\xi_1, \Lambda_1) \Cap (\xi_2, \Lambda_2)$ *and* $\Lambda_3 = \Lambda_1 \cap \Lambda_2 \neq \emptyset$ *(see Definition 3.3 of [2]).*

Proof. Since $\xi_1(r)$ and $\xi_2(r)$ are submodules of $\mathcal{D}^{-1}Z$ for each $r \in \Lambda_3$. By Definition 3.3 of [2], $\zeta(r) = \xi_1(r) \cap \xi_2(r)$ is a submodule of $D^{-1}Z$ for all $r \in \Lambda_3$ (see Page 19 of [5]). Thus (ζ, Λ_3) is also a soft module of fractions over $\mathcal{D}^{-1}Z$. □

Definition 4.1.3. *Let* (ξ_1, Λ_1) *and* (ξ_2, Λ_2) *be two soft modules of fractions over* $\mathcal{D}^{-1}Z$*. The* sum *of* (ξ_1, Λ_1) *and* (ξ_2, Λ_2) *is denoted by* $(\zeta, \Lambda_3) = (\xi_1, \Lambda_1) + (\xi_2, \Lambda_2)$*, where* $\Lambda_3 = \Lambda_1 \times \Lambda_2$*, and is defined by* $\zeta(r) = \xi_1(r_1) + \xi_2(r_2)$ *for all* $r \in \Lambda_3$*, where* $r = (r_1, r_2)$*.*

Proposition 4.1.4. *Let* (ξ_1, Λ_1) *and* (ξ_2, Λ_2) *be two soft modules of fractions over* $\mathcal{D}^{-1}Z$*. Then, their sum* $(\zeta, \Lambda_3) = (\xi_1, \Lambda_1) + (\xi_2, \Lambda_2)$*, where* $\Lambda_3 = \Lambda_1 \times \Lambda_2$ *is also a soft module of fractions over* $\mathcal{D}^{-1}Z$*.*

Proof. Since $\xi_1(r_1)$ and $\xi_2(r_2)$ are submodules of $\mathcal{D}^{-1}Z$ for all $r_1 \in \Lambda_1$, $r_2 \in \Lambda_2$. Since the sum of two submodules is again a submodule, therefore $\zeta(r) = \xi_1(r_1) + \xi_2(r_2)$ is a submodule of $\mathcal{D}^{-1}Z$ for all $r = (r_1, r_2) \in \Lambda_3$ (see Page 19 of [5]). Hence, (ζ, Λ_3) where $\Lambda_3 = \Lambda_1 \times \Lambda_2$ is also a soft module of fractions over $D^{-1}Z$. □

Definition 4.1.5. *Let* (ξ, Λ_1) *be a soft ideal of fractions over* $\mathcal{D}^{-1}L$ *and* (η, Λ_2) *be a soft module of fractions over* $\mathcal{D}^{-1}Z$*. The* soft product *of* (ξ, Λ_1) *and* (η, Λ_2) *is denoted by* $(\delta, \Lambda_3) = (\xi, \Lambda_1) \times (\eta, \Lambda_2)$*, where* $\Lambda_3 = \Lambda_1 \times \Lambda_2$*, and is defined by* $\delta(r) = \xi(r_1).\eta(r_2)$ *for all* $r \in \Lambda_3$*, where* $r = (r_1, r_2)$*.*

Proposition 4.1.6. *If* (ξ, Λ_1) *is a soft ideal of fractions over* $\mathcal{D}^{-1}L$ *and* (η, Λ_2) *is a soft module of fractions over* $\mathcal{D}^{-1}Z$*. Then, their soft product* (δ, Λ_3) *is also a soft module of fractions over* $\mathcal{D}^{-1}Z$*.*

Proof. Since $\xi(r_1)$ is an ideal of $\mathcal{D}^{-1}L$ and $\eta(r_2)$ is a submodule of $\mathcal{D}^{-1}Z$, for all $r_1 \in \Lambda_1$, $r_2 \in \Lambda_2$. Since the product of an ideal with a submodule is a submodule (see Page 19 of [5]).

Therefore, $\delta(r) = \xi(r_1).\eta(r_2)$ is also a submodule of $\mathcal{D}^{-1}Z$ for all $r = (r_1, r_2) \in \Lambda_3$. Hence, (δ, Λ_3) is also a soft module over $\mathcal{D}^{-1}Z$. □

4.2 Fuzzy modules of fractions

In this section, we introduce an important notion of fuzzy modules of fractions, and investigate some fundamental results. Moreover, some isomorphisms are established via fuzzy homomorphisms.

Definition 4.2.1. *Let μ be a fuzzy submodule of Z. Define a fuzzy set $\mathcal{D}^{-1}\mu : \mathcal{D}^{-1}Z \to [0,1]$ as follows:*

$$\mathcal{D}^{-1}\mu(\frac{z}{d}) = \vee\{\mu(nt) : \frac{n}{t} = \frac{z}{d},\ where\ \frac{n}{t} \in \mathcal{D}^{-1}Z\}$$

for all $\frac{z}{d} \in \mathcal{D}^{-1}Z$. We denote the set of all fuzzy sets of $\mathcal{D}^{-1}Z$ by $\mathcal{F}(\mathcal{D}^{-1}Z)$ and fuzzy submodules of $\mathcal{D}^{-1}Z$ by $\mathcal{FS}(\mathcal{D}^{-1}Z)$.

Theorem 4.2.2. *Let μ be a fuzzy submodule of Z. Then, $\mathcal{D}^{-1}\mu \in \mathcal{F}(\mathcal{D}^{-1}Z)$ is a fuzzy module of fractions of $\mathcal{D}^{-1}Z$.*

Proof. First, we shall show that the map given in Definition 4.2.1 is well-defined. Let $\frac{m}{s}, \frac{n}{t} \in \mathcal{D}^{-1}Z$ be such that $\frac{m}{s} = \frac{n}{t}$. We claim that

$$\mathcal{D}^{-1}\mu(\frac{m}{s}) = \mathcal{D}^{-1}\mu(\frac{n}{t}). \tag{4.2.1}$$

Suppose that $\mathcal{D}^{-1}\mu(\frac{m}{s}) = \mu(au)$ for some $\frac{a}{u} \in \mathcal{D}^{-1}Z$ such that $\frac{a}{u} = \frac{m}{s}$. By our assumption, we have $\frac{a}{u} = \frac{n}{t}$. Thus

$$\begin{aligned}\mathcal{D}^{-1}\mu(\frac{n}{t}) &= \vee\{\mu(xw) : \frac{x}{w} = \frac{n}{t}\}\\ &\geq \mu(au)\\ &= \mathcal{D}^{-1}\mu(\frac{m}{s}).\end{aligned}$$

This proves that $\mathcal{D}^{-1}\mu(\frac{n}{t}) \geq \mathcal{D}^{-1}\mu(\frac{m}{s})$. Similarly, it can be proved that $\mathcal{D}^{-1}\mu(\frac{n}{t}) \leq \mathcal{D}^{-1}\mu(\frac{m}{s})$. Thus, our claim in Equation 4.2.1 is true. Now, it is remaining to prove that $\mathcal{D}^{-1}\mu$ is a fuzzy submodule of $\mathcal{D}^{-1}Z$. For this, we have to show that $\mathcal{D}^{-1}\mu$ fulfills the conditions in Definition 4.1.8 of [31]. Using the Definition 4.2.1 of $\mathcal{D}^{-1}\mu$, we have:

$$\begin{aligned}\mathcal{D}^{-1}\mu(\frac{\theta}{s}) &= \vee\{\mu(nt) : \frac{n}{t} = \frac{\theta}{s}\}\\ &= \vee\{\mu(nt) : nsu = \theta \text{ for some } u \in \mathcal{D}\}\\ &= 1.\end{aligned}$$

Let $\frac{z_1}{d_1}, \frac{z_2}{d_2} \in \mathcal{D}^{-1}Z$. Assume that

$$\mathcal{D}^{-1}\mu(\frac{z_1}{d_1}) = \mu(a_1b_1), D^{-1}\mu(\frac{z_2}{d_2}) = \mu(a_2b_2) \text{ such that } \frac{z_1}{d_1} = \frac{a_1}{b_1}, \frac{z_2}{d_2} = \frac{a_2}{b_2}. \tag{4.2.2}$$

Then $\frac{z_1}{d_1} + \frac{z_2}{d_2} = \frac{a_1}{b_1} + \frac{a_2}{b_2}$, that is, $\frac{z_1d_2+z_2d_1}{d_1d_2} = \frac{a_1b_2+a_2b_1}{b_1b_2}$. From Definition 4.2.1, it follows that:

$$\begin{aligned}\mathcal{D}^{-1}\mu(\frac{z_1}{d_1} + \frac{z_2}{d_2}) &= \mathcal{D}^{-1}\mu(\frac{z_1d_2 + z_2d_1}{d_1d_2})\\ &= \vee\{\mu(xw) : \frac{x}{w} = \frac{z_1d_2 + z_2d_1}{d_1d_2}, \frac{x}{w} \in \mathcal{D}^{-1}Z\}\\ &\geq \mu((a_1b_2 + a_2b_1)b_1b_2)\\ &= \mu(a_1b_1b_2^2 + a_2b_1^2b_2)\\ &\geq \mu(a_1b_1b_2^2) \wedge \mu(a_2b_1^2b_2)\\ &\geq \mu(a_1b_1) \wedge \mu(a_2b_2)\\ &= \mathcal{D}^{-1}\mu(\frac{z_1}{d_1}) \wedge \mathcal{D}^{-1}\mu(\frac{z_2}{d_2}),\end{aligned}$$

(see Equation 4.2.2). Furthermore, let $\frac{l}{m} \in \mathcal{D}^{-1}L$, then

$$\mathcal{D}^{-1}\mu(\frac{l}{m}.\frac{z_1}{d_1}) = \mathcal{D}^{-1}\mu(\frac{lz_1}{md_1}) = \vee\{\mu(xz) : \frac{x}{z} = \frac{lz_1}{md_1}\} \geq \mu(a_1b_1) = \mathcal{D}^{-1}\mu(\frac{z_1}{d_1}),$$

(Using Equation 4.2.2.) Hence, $\mathcal{D}^{-1}\mu$ is a fuzzy submodule of $\mathcal{D}^{-1}Z$. Thus $\mathcal{D}^{-1}\mu \in \mathcal{F}(\mathcal{D}^{-1}Z)$ is a fuzzy module of fractions. $\square$

Lemma 4.2.3. *Let $\mu_1, \mu_2 \in \mathcal{F}(Z)$ be such that $\mu_1 \subseteq \mu_2$. Then, $D^{-1}\mu_1 \subseteq D^{-1}\mu_2$.*

Proof. The proof is straightforward in view of Definition 4.2.1. $\square$

Theorem 4.2.4. *Let μ_1 and μ_2 be two fuzzy submodules of L-modules Z_1 and Z_2 respectively. Suppose that $\widetilde{\varphi} : \mu_1 \to \mu_2$ is a fuzzy $L-$module homomorphism. Then, there is a fuzzy $\mathcal{D}^{-1}L$-module homomorphism*

$$\widetilde{\mathcal{D}^{-1}\varphi} : \mathcal{D}^{-1}\mu_1 \to \mathcal{D}^{-1}\mu_2.$$

Proof. It is well-known that if $\varphi : Z_1 \to Z_2$ is an $L-$module homomorphism, then there is an $\mathcal{D}^{-1}L-$module homomorphism $\mathcal{D}^{-1}\varphi : \mathcal{D}^{-1}Z_1 \to \mathcal{D}^{-1}Z_2$ defined by $\frac{z}{d} \mapsto \frac{\varphi(z)}{d}$ (see Page 38 of [5]). Let $\mathcal{D}^{-1}\mu_1(\frac{z}{d}) = \mu_1(nt)$ be such that $\frac{n}{t} = \frac{z}{d}$. Then, $\frac{\varphi(z)}{d} = \frac{\varphi(n)}{t}$. Hence,

$$\begin{aligned}
\mathcal{D}^{-1}\mu_2(\mathcal{D}^{-1}\varphi(\frac{z}{d})) &= \mathcal{D}^{-1}\mu_2(\frac{\varphi(z)}{d}) \\
&= \vee\{\mu_2(au) : \frac{a}{u} = \frac{\varphi(z)}{d}\} \\
&\geq \mu_2(\varphi(n)t) = \mu_2(\varphi(nt)) \\
&\geq \mu_1(nt) \\
&= \mathcal{D}^{-1}\mu_1(\frac{z}{d}),
\end{aligned}$$

since $\widetilde{\varphi}$ is a fuzzy $L-$module homomorphism (see Definition 1.1 of [33]). Thus, $\widetilde{D^{-1}\varphi}$: $\mathcal{D}^{-1}\mu_1 \to \mathcal{D}^{-1}\mu_2$ is a fuzzy $\mathcal{D}^{-1}L-$module homomorphism. This completes the proof. $\square$

Theorem 4.2.5. *Let $\mu \in \mathcal{F}(L)$ and $\nu \in \mathcal{F}(Z)$. Then,*

$$\mathcal{D}^{-1}(\mu.\nu) = (\mathcal{D}^{-1}\mu).(\mathcal{D}^{-1}\nu).$$

Proof. Let $\frac{z}{d} \in \mathcal{D}^{-1}Z$ be such that

$$((\mathcal{D}^{-1}\mu).(\mathcal{D}^{-1}\nu))(\frac{z}{d}) = \mathcal{D}^{-1}\mu(\frac{x_1}{t_1}) \wedge \mathcal{D}^{-1}\nu(\frac{x_2}{t_2}) = \mu(a_1b_1) \wedge \nu(a_2b_2). \tag{4.2.3}$$

where $\frac{a_1}{b_1}, \frac{x_1}{t_1} \in \mathcal{D}^{-1}L, \frac{a_2}{b_2}, \frac{x_2}{t_2} \in \mathcal{D}^{-1}Z$ such that $\frac{x_1}{t_1}.\frac{x_2}{t_2} = \frac{z}{d}$ and $\frac{a_1}{b_1} = \frac{x_1}{t_1}, \frac{a_2}{b_2} = \frac{x_2}{t_2}$. From Definition 4.2.1 and Definition 4.1.6 of [31], we have:

$$\begin{aligned}
\mathcal{D}^{-1}(\mu.\nu)(\frac{z}{d}) &= \vee\{(\mu.\nu)(nt) : \frac{n}{t} = \frac{z}{d}\} \\
&= \vee \vee \{\mu(a) \wedge \nu(b) : a \in L, b \in Z, ab = nt, \frac{n}{t} = \frac{z}{d}\} \\
&\geq \vee\{\mu(a) \wedge \nu(b) : \frac{ab}{t^2} = \frac{a}{t}.\frac{b}{t} = \frac{n}{t} = \frac{z}{d}\} \\
&\geq \mu(a_1b_1) \wedge \nu(a_2b_2) \\
&= \mathcal{D}^{-1}\mu(\frac{x_1}{t_1}) \wedge D^{-1}\nu(\frac{x_2}{t_2}) \\
&= ((\mathcal{D}^{-1}\mu).(\mathcal{D}^{-1}\nu))(\frac{z}{d}),
\end{aligned}$$

(see Equation 4.2.3). Thus $(\mathcal{D}^{-1}\mu).(\mathcal{D}^{-1}\nu) \subseteq \mathcal{D}^{-1}(\mu.\nu)$. Similarly, the reverse containment can be proved. This completes the proof. □

Theorem 4.2.6. *Let $\mu_1, \mu_2 \in \mathcal{F}(Z)$ and $\nu \in \mathcal{F}(Z')$. Then:*

(i) $\mathcal{D}^{-1}\mu_1 + \mathcal{D}^{-1}\mu_2 \subseteq \mathcal{D}^{-1}(\mu_1 + \mu_2)$.

(ii) $\mathcal{D}^{-1}(\mu_1 \cap \mu_2) = \mathcal{D}^{-1}\mu_1 \cap \mathcal{D}^{-1}\mu_2$.

(iii) There is a fuzzy $\mathcal{D}^{-1}L$-module isomorphism

$$\mathcal{D}^{-1}(\mu_1 \oplus \nu) \to \mathcal{D}^{-1}\mu_1 \oplus \mathcal{D}^{-1}\nu.$$

Proof. (i) It follows from Lemma 4.2.3 and Proposition 2.1.7 of [31].

(ii) It is clear that $\mu_1 \cap \mu_2 \subseteq \mu_i$ for all $i = 1, 2$. It follows from Lemma 4.2.3 that

$$\mathcal{D}^{-1}(\mu_1 \cap \mu_2) \subseteq \mathcal{D}^{-1}\mu_1 \cap \mathcal{D}^{-1}\mu_2$$

Conversely, assume that $\frac{z}{d} \in \mathcal{D}^{-1}Z$ and $(\mathcal{D}^{-1}\mu_1 \cap \mathcal{D}^{-1}\mu_2)(\frac{z}{d}) = \mu_1(x_1t_1) \wedge \mu_2(x_2t_2)$ such that $\frac{z}{d} = \frac{x_1}{t_1} = \frac{x_2}{t_2}$. Then:

$$\begin{aligned}\mathcal{D}^{-1}(\mu_1 \cap \mu_2)(\frac{z}{d}) &= \vee\{(\mu_1 \cap \mu_2)(\frac{x}{t}) : \frac{x}{t} = \frac{z}{d}\} \\ &\geq \mu_1(x_1t_1) \wedge \mu_2(x_2t_2) \\ &= (\mathcal{D}^{-1}\mu_1 \cap \mathcal{D}^{-1}\mu_2)(\frac{z}{d}).\end{aligned}$$

(iii) By Proposition 42 (5) of [5], we only need to show that

$$(\mathcal{D}^{-1}\mu_1 \oplus \mathcal{D}^{-1}\nu)(\mathcal{D}^{-1}\varphi(\frac{(z,n)}{d})) = \mathcal{D}^{-1}(\mu_1 \oplus \nu)(\frac{(z,n)}{d}), \tag{4.2.4}$$

for all $\frac{(z,n)}{d} \in \mathcal{D}^{-1}(Z \oplus Z')$. Assume that

$$\mathcal{D}^{-1}(\mu_1 \oplus \nu)(\frac{(z,n)}{d}) = (\mu_1 \oplus \nu)((x,y)w) = \mu_1(xw) \wedge \nu(yw), \tag{4.2.5}$$

where $\frac{(z,n)}{d}, \frac{(x,y)}{w} \in \mathcal{D}^{-1}(Z \oplus Z')$ such that $\frac{(x,y)}{w} = \frac{(z,n)}{d}$. By Definition 1.1.12 of [31] and

Definition 4.2.1, we have:

$$\begin{aligned}(\mathcal{D}^{-1}\mu_1 \oplus \mathcal{D}^{-1}\nu)(\mathcal{D}^{-1}\varphi(\frac{(z,n)}{d})) &= (\mathcal{D}^{-1}\mu_1 \oplus \mathcal{D}^{-1}\nu)(\frac{z}{d},\frac{n}{d})\\ &= \mathcal{D}^{-1}\mu_1(\frac{z}{d}) \wedge \mathcal{D}^{-1}\nu(\frac{n}{d})\\ &= \vee\{\mu_1(at_1) \wedge \nu(bt_2) : \frac{a}{t_1} = \frac{z}{d}, \frac{b}{t_2} = \frac{n}{d}\}\\ &\geq \mu_1(xw) \wedge \nu(yw)\\ &= (\mu_1 \oplus \nu)((x,y)w)\\ &= \mathcal{D}^{-1}(\mu_1 \oplus \nu)(\frac{(z,n)}{d}),\end{aligned}$$

(see Equation 4.2.5). Hence, $(\mathcal{D}^{-1}\mu_1 \oplus \mathcal{D}^{-1}\nu)(\mathcal{D}^{-1}\varphi(\frac{(z,n)}{d})) \geq \mathcal{D}^{-1}(\mu_1 \oplus \nu)(\frac{(z,n)}{d})$ for all $\frac{(z,n)}{d} \in \mathcal{D}^{-1}(Z \oplus Z')$. Similarly, the reverse containment can be proved. Hence, our claim 4.2.4 is true. This completes the proof. □

4.3 The lower and upper approximations in fuzzy modules of fractions

In this last section, we have introduced the notion of fuzzy modules of fractions. Since a soft module of fractions is a parameterized family of submodules of $\mathcal{D}^{-1}Z$ (see Definition 4.1.1). Using this concept, we approximate a fuzzy set in modules of fractions in the sense of multi-granulation rough sets and hence obtain a new hybrid model, namely multi-granulation soft rough fuzzy sets (MGSR-fuzzy sets) in modules of fractions. In this way, we obtain a pair of fuzzy sets in modules of fractions, viz. fuzzy lower approximation space and fuzzy upper approximation space based on submodules of the parameterized family of modules of fractions.

Definition 4.3.1. *Let $\mathcal{D}^{-1}\mu \in \mathcal{F}(\mathcal{D}^{-1}Z)$ and (ξ, Λ) be a soft module of fractions over $\mathcal{D}^{-1}Z$, where $\Lambda = \{e_1, e_2\} \subseteq Æ$. Then, the approximation space $(\mathcal{D}^{-1}Z, \mathcal{D}^{-1}\mu, \xi, \Lambda)$ is called a multi-*

granulation soft rough fuzzy approximation space in $\mathcal{D}^{-1}Z$. Define the fuzzy lower approximation space $\underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda)}$ and the fuzzy upper approximation space $\overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)}$ for $\mathcal{D}^{-1}\mu \in \mathcal{F}(\mathcal{D}^{-1}Z)$ as follows:

$$\underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda)}(\frac{z}{d}) = \wedge\{\mathcal{D}^{-1}\mu(\frac{x}{y}) : \frac{x}{y} \in [\frac{z}{d}]_{\xi(e_i)},\ \text{where } e_i \in \Lambda, i = 1,2\} \text{ and}$$

$$\overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)}(\frac{z}{d}) = \vee\{\mathcal{D}^{-1}\mu(\frac{x}{y}) : \frac{x}{y} \in [\frac{z}{d}]_{\xi(e_i)},\ \text{where } e_i \in \Lambda, i = 1,2\},$$

for all $\frac{z}{d} \in \mathcal{D}^{-1}Z$, where $[\frac{z}{d}]_{\xi(e_i)} = \frac{z}{d} + \xi(e_i)$ denotes the cosets of $\xi(e_i)$ for each $e_i \in \Lambda$ in $\mathcal{D}^{-1}Z$. It is clear that these $\underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda)}$ and $\overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)}$ are fuzzy sets of $\mathcal{D}^{-1}Z$.

Theorem 4.3.2. *Let (ξ,Λ) be a soft module of fractions over $\mathcal{D}^{-1}Z$. Suppose that $\mathcal{D}^{-1}\mu \in \mathcal{FS}(\mathcal{D}^{-1}Z)$. Then,*

(1) $\underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda)} = \underline{\mathcal{D}^{-1}\mu}_{\xi(e_1)} \cap \underline{\mathcal{D}^{-1}\mu}_{\xi(e_2)}$.

(2) $\overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)} = \overline{\mathcal{D}^{-1}\mu}^{\xi(e_1)} \cap \overline{\mathcal{D}^{-1}\mu}^{\xi(e_2)}$.

(3) $\underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda)} \subseteq \mathcal{D}^{-1}\mu \subseteq \overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)}$.

(4) $\underline{(\mathcal{D}^{-1}\mu)^c}_{(\xi,\Lambda)} = (\overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)})^c$.

(5) $\overline{(\mathcal{D}^{-1}\mu)^c}^{(\xi,\Lambda)} = (\underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda)})^c$.

(6) $\underline{\hat{0}}_{(\xi,\Lambda)} = \overline{\hat{0}}^{(\xi,\Lambda)} = \hat{0}$.

(7) $\underline{\hat{1}}_{(\xi,\Lambda)} = \overline{\hat{1}}^{(\xi,\Lambda)} = \hat{1}$.

(8) $\underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda)} = \underline{\underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda)}}_{(\xi,\Lambda)}$.

(9) $\overline{\underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda)}}^{(\xi,\Lambda)} \supseteq \underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda)}$.

(10) $\overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)} = \overline{\overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)}}^{(\xi,\Lambda)}$.

(11) $\overline{\underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda)}}^{(\xi,\Lambda)} \subseteq \overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)}$.

where $\hat{0}$ *and* $\hat{1}$ *represents the constant fuzzy sets in* $\mathcal{D}^{-1}Z$ *which maps each element of the universe* $\mathcal{D}^{-1}Z$ *to* 0 *and* 1*, respectively.*

Proof. All the claims are easy to prove by using the Definition 4.3.1 of fuzzy lower and upper approximation spaces. □

Theorem 4.3.3. *Let* (ξ, Λ_1) *and* (η, Λ_2) *be two soft module of fractions over* $\mathcal{D}^{-1}Z$*. Then, for any* $\mathcal{D}^{-1}\mu, \mathcal{D}^{-1}\nu \in \mathcal{F}(\mathcal{D}^{-1}Z)$*, we have:*

(1) $\underline{\mathcal{D}^{-1}\mu \cap \mathcal{D}^{-1}\nu}_{\xi(e)} = \underline{\mathcal{D}^{-1}\mu}_{\xi(e)} \cap \underline{\mathcal{D}^{-1}\nu}_{\xi(e)}$ *for all* $e \in \Lambda_1$.

(2) $\overline{\mathcal{D}^{-1}\mu \cup \mathcal{D}^{-1}\nu}^{\xi(e)} = \overline{\mathcal{D}^{-1}\mu}^{\xi(e)} \cup \overline{\mathcal{D}^{-1}\nu}^{\xi(e)}$ *for all* $e \in \Lambda_1$.

(3) $\underline{\mathcal{D}^{-1}\mu \cap \mathcal{D}^{-1}\nu}_{(\xi,\Lambda_1)} = [\underline{\mathcal{D}^{-1}\mu}_{\xi(e_1)} \cap \underline{\mathcal{D}^{-1}\nu}_{\xi(e_1)}] \cap [\underline{\mathcal{D}^{-1}\mu}_{\xi(e_2)} \cap \underline{\mathcal{D}^{-1}\nu}_{\xi(e_2)}]$.

(4) $\overline{\mathcal{D}^{-1}\mu \cup \mathcal{D}^{-1}\nu}^{(\xi,\Lambda_1)} = [\overline{\mathcal{D}^{-1}\mu}^{\xi(e_1)} \cup \overline{\mathcal{D}^{-1}\nu}^{\xi(e_1)}] \cap [\overline{\mathcal{D}^{-1}\mu}^{\xi(e_2)} \cup \overline{\mathcal{D}^{-1}\nu}^{\xi(e_2)}]$.

(5) *If* $\mathcal{D}^{-1}\mu \subseteq \mathcal{D}^{-1}\nu$*, then* $\underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda_1)} \subseteq \underline{\mathcal{D}^{-1}\nu}_{(\xi,\Lambda_1)}$.

(6) *If* $\mathcal{D}^{-1}\mu \subseteq \mathcal{D}^{-1}\nu$*, then* $\overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda_1)} \subseteq \overline{\mathcal{D}^{-1}\nu}^{(\xi,\Lambda_1)}$.

(7) $\overline{\mathcal{D}^{-1}\mu \cap \mathcal{D}^{-1}\nu}^{(\xi,\Lambda_1)} \subseteq \overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda_1)} \cap \overline{\mathcal{D}^{-1}\nu}^{(\xi,\Lambda_1)}$.

(8) $\underline{\mathcal{D}^{-1}\mu \cup \mathcal{D}^{-1}\nu}_{(\xi,\Lambda_1)} \supseteq \underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda_1)} \cup \underline{\mathcal{D}^{-1}\nu}_{(\xi,\Lambda_1)}$.

(9) *If* $\mathcal{D}^{-1}\mu \subseteq \mathcal{D}^{-1}\nu$ *and* $(\xi, \Lambda_1)\widetilde{\subseteq}(\eta, \Lambda_2)$*, then* $\underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda_1)} \subseteq \underline{\mathcal{D}^{-1}\nu}_{(\eta,\Lambda_2)}$.

(10) *If* $\mathcal{D}^{-1}\mu \subseteq \mathcal{D}^{-1}\nu$ *and* $(\xi, \Lambda_1)\widetilde{\subseteq}(\eta, \Lambda_2)$*, then* $\overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda_1)} \subseteq \overline{\mathcal{D}^{-1}\nu}^{(\eta,\Lambda_2)}$.

Proof. All claims are easy to prove by using Definition 4.3.1 and Theorem 4.3.2. □

The restricted intersection of two soft modules of fractions is a soft module of fractions (see Proposition 4.1.2). Thus, the relationships between fuzzy approximation spaces of intersection and intersection of fuzzy approximations are established in Theorem 4.3.4.

Theorem 4.3.4. *Let (ξ_1, Λ_1) and (ξ_2, Λ_2) be two soft modules of fractions over $\mathcal{D}^{-1}Z$. If $\mathcal{D}^{-1}\mu_1, \mathcal{D}^{-1}\mu_2 \in \mathcal{F}(\mathcal{D}^{-1}Z)$, then*

(1) $\overline{\mathcal{D}^{-1}(\mu_1 \cap \mu_2)}^{(\eta,\Lambda_3)} = \overline{\mathcal{D}^{-1}\mu_1 \cap \mathcal{D}^{-1}\mu_2}^{(\eta,\Lambda_3)} \subseteq \overline{\mathcal{D}^{-1}\mu_1}^{(\xi_1,\Lambda_1)} \cap \overline{\mathcal{D}^{-1}\mu_2}^{(\xi_2,\Lambda_2)}$ *and*

(2) $\underline{\mathcal{D}^{-1}(\mu_1 \cap \mu_2)}_{(\eta,\Lambda_3)} = \underline{\mathcal{D}^{-1}\mu_1 \cap \mathcal{D}^{-1}\mu_2}_{(\eta,\Lambda_3)} = \underline{\mathcal{D}^{-1}\mu_1}_{(\xi_1,\Lambda_1)} \cap \underline{\mathcal{D}^{-1}\mu_2}_{(\xi_2,\Lambda_2)}$,

where $(\eta, \Lambda_3) = (\xi_1, \Lambda_1) \Cap (\xi_2, \Lambda_2)$ and $\Lambda_3 = \Lambda_1 \cap \Lambda_2 \neq \emptyset$ (see Definition 3.3 of [2]).

Proof. Since $(\eta, \Lambda_3)\widetilde{\subseteq}(\xi_i, \Lambda_i)$ and $\mathcal{D}^{-1}\mu_1 \cap \mathcal{D}^{-1}\mu_2 \subseteq \mathcal{D}^{-1}\mu_i$ for all $i = 1, 2$. By Theorem 4.3.3 (9) and (10), we have:

$$\overline{\mathcal{D}^{-1}(\mu_1 \cap \mu_2)}^{(\eta,\Lambda_3)} = \overline{\mathcal{D}^{-1}\mu_1 \cap \mathcal{D}^{-1}\mu_2}^{(\eta,\Lambda_3)} \subseteq \overline{\mathcal{D}^{-1}\mu_1}^{(\xi_1,\Lambda_1)} \cap \overline{\mathcal{D}^{-1}\mu_2}^{(\xi_2,\Lambda_2)} \text{ and}$$

$$\underline{\mathcal{D}^{-1}(\mu_1 \cap \mu_2)}_{(\eta,\Lambda_3)} = \underline{\mathcal{D}^{-1}\mu_1 \cap \mathcal{D}^{-1}\mu_2}_{(\eta,\Lambda_3)} \subseteq \underline{\mathcal{D}^{-1}\mu_1}_{(\xi_1,\Lambda_1)} \cap \underline{\mathcal{D}^{-1}\mu_2}_{(\xi_2,\Lambda_2)}.$$

To prove the converse of (2), let $\frac{\mathfrak{z}}{\mathfrak{d}} \in \mathcal{D}^{-1}Z$ be such that

$$(\underline{\mathcal{D}^{-1}\mu_1}_{(\xi_1,\Lambda_1)} \cap \underline{\mathcal{D}^{-1}\mu_2}_{(\xi_2,\Lambda_2)})(\frac{\mathfrak{z}}{\mathfrak{d}}) = \underline{\mathcal{D}^{-1}\mu_1}_{(\xi_1,\Lambda_1)}(\frac{\mathfrak{z}}{\mathfrak{d}}) \wedge \underline{\mathcal{D}^{-1}\mu_2}_{(\xi_2,\Lambda_2)}(\frac{\mathfrak{z}}{\mathfrak{d}}) = \mathcal{D}^{-1}\mu_1(\frac{x_1}{y_1}) \wedge \mathcal{D}^{-1}\mu_2(\frac{x_2}{y_2}), \tag{4.3.1}$$

where $\frac{x_1}{y_1}, \frac{x_2}{y_2} \in \mathcal{D}^{-1}Z$ such that $\frac{x_1}{y_1} \in [\frac{\mathfrak{z}}{\mathfrak{d}}]_{\xi_1(a_i)}, \frac{x_2}{y_2} \in [\frac{\mathfrak{z}}{\mathfrak{d}}]_{\xi_2(b_i)}$, for all $a_i \in \Lambda_1$, $b_i \in \Lambda_2, i = 1, 2$. By Definitions 4.3.1, 4.2.1 and Definition 1.1.5 of [31], we have:

$$\begin{aligned}
\underline{\mathcal{D}^{-1}\mu_1 \cap \mathcal{D}^{-1}\mu_2}_{(\eta,\Lambda_3)}(\frac{\mathfrak{z}}{\mathfrak{d}}) &= \wedge\{(\mathcal{D}^{-1}\mu_1 \cap \mathcal{D}^{-1}\mu_2)(\frac{\mathfrak{x}}{\mathfrak{y}}) : \frac{\mathfrak{x}}{\mathfrak{y}} \in [\frac{\mathfrak{z}}{\mathfrak{d}}]_{\eta(c_i)}, c_i \in \Lambda_3 = \Lambda_1 \cap \Lambda_2, i = 1, 2\} \\
&= \wedge\{\mathcal{D}^{-1}\mu_1(\frac{\mathfrak{x}}{\mathfrak{y}}) \wedge \mathcal{D}^{-1}\mu_2(\frac{\mathfrak{x}}{\mathfrak{y}}) : \frac{\mathfrak{x}}{\mathfrak{y}} \in [\frac{\mathfrak{z}}{\mathfrak{d}}]_{\xi_1(c_i)}, \frac{\mathfrak{x}}{\mathfrak{y}} \in [\frac{\mathfrak{z}}{\mathfrak{d}}]_{\xi_2(c_i)} \\
&\quad \text{for all } c_i \in \Lambda_1, c_i \in \Lambda_2, i = 1, 2\} \\
&\leq \mathcal{D}^{-1}\mu_1(\frac{x_1}{y_1}) \wedge \mathcal{D}^{-1}\mu_2(\frac{x_2}{y_2}) \\
&= (\underline{\mathcal{D}^{-1}\mu_1}_{(\xi_1,\Lambda_1)} \cap \underline{\mathcal{D}^{-1}\mu_2}_{(\xi_2,\Lambda_2)})(\frac{\mathfrak{z}}{\mathfrak{d}}),
\end{aligned}$$

(see Equation 4.3.1). Thus $\underline{\mathcal{D}^{-1}\mu_1 \cap \mathcal{D}^{-1}\mu_2}_{(\eta,\Lambda_3)} \subseteq \underline{\mathcal{D}^{-1}\mu_1}_{(\xi_1,\Lambda_1)} \cap \underline{\mathcal{D}^{-1}\mu_2}_{(\xi_2,\Lambda_2)}$. This completes the proof. $\square$

To illustrate that the inclusion in (2) of above Theorem cannot be replaced with equality, we consider the following Example.

Example 4.3.5. *Let* $Z = L = \mathbb{Z}_{10}$ *and* $\mathcal{D} = \{\bar{1}, \bar{3}, \bar{9}\}$. *Then,*

$$\mathcal{D}^{-1}\mathbb{Z}_{10} = \{\bar{0}, \frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}, \frac{\bar{1}}{\bar{9}}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}, \frac{\bar{2}}{\bar{9}}, \frac{\bar{3}}{\bar{1}}, \frac{\bar{4}}{\bar{9}}, \frac{\bar{5}}{\bar{1}}\},$$

where $\frac{\bar{1}}{\bar{1}} = \frac{\bar{3}}{\bar{3}} = \frac{\bar{9}}{\bar{9}}$, $\frac{\bar{1}}{\bar{9}} = \frac{\bar{9}}{\bar{1}} = \frac{\bar{7}}{\bar{3}}$, $\frac{\bar{2}}{\bar{1}} = \frac{\bar{6}}{\bar{3}} = \frac{\bar{8}}{\bar{9}}$, $\frac{\bar{2}}{\bar{3}} = \frac{\bar{6}}{\bar{9}} = \frac{\bar{4}}{\bar{1}}$, $\frac{\bar{3}}{\bar{1}} = \frac{\bar{7}}{\bar{9}} = \frac{\bar{9}}{\bar{3}}$, $\frac{\bar{4}}{\bar{3}} = \frac{\bar{8}}{\bar{1}} = \frac{\bar{2}}{\bar{9}}$, $\frac{\bar{4}}{\bar{9}} = \frac{\bar{8}}{\bar{3}} = \frac{\bar{6}}{\bar{1}}$, $\frac{\bar{5}}{\bar{1}} = \frac{\bar{5}}{\bar{3}} = \frac{\bar{5}}{\bar{9}}$. *Define two fuzzy submodules* $\mu, \nu \in \mathcal{F}(Z)$ *as follows:*

$$\mu(\mathfrak{l}) = \begin{cases} 1, & \text{if } \mathfrak{l} = \bar{0}, \bar{2}, \bar{4}, \bar{6} \\ 0.5, & \text{if } \mathfrak{l} = \bar{1}, \bar{3}, \bar{5}, \bar{7} \end{cases} \quad \text{and} \quad \nu(\mathfrak{l}) = \begin{cases} 1, & \text{if } \mathfrak{l} = \bar{0}, \bar{5} \\ 0.6, & \text{otherwise} \end{cases}$$

for all $\mathfrak{l} \in Z$. *Using the Definition 4.2.1, we obtain*

$$\mathcal{D}^{-1}\mu(\frac{\mathfrak{l}}{t}) = \begin{cases} 1, & \text{if } \mathfrak{l} = \bar{0}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}, \frac{\bar{2}}{\bar{9}}, \frac{\bar{4}}{\bar{9}} \\ 0.5, & \text{if } \mathfrak{l} = \frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}, \frac{\bar{1}}{\bar{9}}, \frac{\bar{5}}{\bar{1}} \end{cases} \quad \text{and} \quad \mathcal{D}^{-1}\nu(\frac{\mathfrak{l}}{t}) = \begin{cases} 1, & \text{if } \mathfrak{l} = \bar{0}, \frac{\bar{5}}{\bar{1}} \\ 0.6, & \text{otherwise} \end{cases}$$

for all $\frac{\mathfrak{l}}{t} \in \mathcal{D}^{-1}Z$. *By using the Definition 1.1.5 of [31], we get the following fuzzy module of fractions:*

$$(\mathcal{D}^{-1}\mu \cap \mathcal{D}^{-1}\nu)(\frac{\mathfrak{l}}{t}) = \begin{cases} 1, & \text{if } \frac{\mathfrak{l}}{t} = \bar{0} \\ 0.6, & \text{if } \frac{\mathfrak{l}}{t} = \frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}, \frac{\bar{2}}{\bar{9}}, \frac{\bar{4}}{\bar{9}} \\ 0.5, & \text{if } \frac{\mathfrak{l}}{t} = \frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}, \frac{\bar{1}}{\bar{9}}, \frac{\bar{5}}{\bar{1}} \end{cases}$$

for all $\frac{\mathfrak{l}}{t} \in \mathcal{D}^{-1}Z$. *Define two soft modules of fractions* (ξ_1, Λ_1) *and* (ξ_2, Λ_2) *over* $\mathcal{D}^{-1}Z$ *as* *follows:*

$$\xi_1(\alpha)=\begin{cases}\{\bar{0},\frac{\bar{5}}{\bar{1}}\}, & \text{if } \alpha=e_1\\ \mathcal{D}^{-1}\mathbb{Z}_{10}, & \text{if } \alpha=e_2\end{cases}\quad \text{and} \quad \xi_2(\beta)=\begin{cases}\{\bar{0},\frac{\bar{2}}{\bar{1}},\frac{\bar{2}}{\bar{3}},\frac{\bar{2}}{\bar{9}},\frac{\bar{4}}{\bar{9}}\}, & \text{if } \beta=e_1\\ \{\bar{0},\frac{\bar{5}}{\bar{1}}\}, & \text{if } \beta=e_2\end{cases}$$

for all $\alpha,\beta\in\Lambda_1=\Lambda_2=\{e_1,e_2\}$. By Definition 3.3 of [2], we obtain the following soft module of fractions:

$$\eta(\gamma)=\begin{cases}\{\bar{0}\}, & \text{if } \alpha=e_1\\ \{\bar{0},\frac{\bar{5}}{\bar{1}}\}, & \text{if } \alpha=e_2\end{cases}$$

for all $\gamma\in\Lambda_3=\Lambda_1\cap\Lambda_2$. From Definition 4.3.1 of fuzzy upper approximation space, we obtain:

$$\overline{(\mathcal{D}^{-1}\mu\cap\mathcal{D}^{-1}\nu)}^{(\eta,\Lambda_3)}(\frac{\mathfrak{l}}{t})=(\mathcal{D}^{-1}\mu\cap\mathcal{D}^{-1}\nu)(\frac{\mathfrak{l}}{t}),\ \text{for all } \frac{\mathfrak{l}}{t}\in\mathcal{D}^{-1}Z.$$

Also,

$$\overline{\mathcal{D}^{-1}\mu}^{(\xi_1,\Lambda_1)}(\frac{\mathfrak{l}}{t})=1 \ \text{and} \quad \overline{\mathcal{D}^{-1}\nu}^{(\xi_2,\Lambda_2)}(\frac{\mathfrak{l}}{t})=\mathcal{D}^{-1}\nu(\frac{\mathfrak{l}}{t}) \ \text{for all } \frac{\mathfrak{l}}{t}\in\mathcal{D}^{-1}Z.$$

Thus $\overline{(\mathcal{D}^{-1}\mu\cap\mathcal{D}^{-1}\nu)}^{(\eta,\Lambda_3)}(\frac{\mathfrak{l}}{t})=(\mathcal{D}^{-1}\mu\cap\mathcal{D}^{-1}\nu)(\frac{\mathfrak{l}}{t})\ngeq\overline{\mathcal{D}^{-1}\mu}^{(\xi_1,\Lambda_1)}(\frac{\mathfrak{l}}{t})\wedge\overline{\mathcal{D}^{-1}\nu}^{(\xi_2,\Lambda_2)}(\frac{\mathfrak{l}}{t}).$

From Proposition 4.1.4, we know that the sum of two soft modules of fractions is a soft module of fractions. Hence, we consider the lower and upper approximations of fuzzy sets with respect to sum of two soft modules of fractions in the following two results.

Theorem 4.3.6. *Let (ξ_1,Λ_1) and (ξ_2,Λ_2) be two soft modules of fractions over $\mathcal{D}^{-1}Z$. Suppose that $\mathcal{D}^{-1}\mu_1,\mathcal{D}^{-1}\mu_2\in\mathcal{F}(\mathcal{D}^{-1}Z)$. Then*

(1) $\overline{\mathcal{D}^{-1}\mu_1}^{(\xi_1,\Lambda_1)}+\overline{\mathcal{D}^{-1}\mu_2}^{(\xi_2,\Lambda_2)}\subseteq\overline{\mathcal{D}^{-1}\mu_1+\mathcal{D}^{-1}\mu_2}^{(\eta,\Lambda_3)}\subseteq\overline{\mathcal{D}^{-1}(\mu_1+\mu_2)}^{(\eta,\Lambda_3)}$ *and*

(2) $\underline{\mathcal{D}^{-1}\mu_1}_{(\xi_1,\Lambda_1)}+\underline{\mathcal{D}^{-1}\mu_2}_{(\xi_2,\Lambda_2)}=\underline{\mathcal{D}^{-1}\mu_1+\mathcal{D}^{-1}\mu_2}_{(\eta,\Lambda_3)}\subseteq\underline{\mathcal{D}^{-1}(\mu_1+\mu_2)}_{(\eta,\Lambda_3)},$

where $(\eta,\Lambda_3)=(\xi_1,\Lambda_1)+(\xi_2,\Lambda_2)$, $\Lambda_3=\Lambda_1\times\Lambda_2$ (see Definition 4.1.3).

Proof. By Theorem 4.3.3 (6) and (i) of 4.2.6,

$$\overline{\mathcal{D}^{-1}\mu_1 + \mathcal{D}^{-1}\mu_2}^{(\eta,\Lambda_3)} \subseteq \overline{\mathcal{D}^{-1}(\mu_1+\mu_2)}^{(\eta,\Lambda_3)}, \quad \underline{\mathcal{D}^{-1}\mu_1 + \mathcal{D}^{-1}\mu_2}_{(\eta,\Lambda_3)} \subseteq \underline{D^{-1}(\mu_1+\mu_2)}_{(\eta,\Lambda_3)}$$

To prove $\overline{\mathcal{D}^{-1}\mu_1}^{(\xi_1,\Lambda_1)} + \overline{\mathcal{D}^{-1}\mu_2}^{(\xi_2,\Lambda_2)} \subseteq \overline{\mathcal{D}^{-1}\mu_1 + \mathcal{D}^{-1}\mu_2}^{(\eta,\Lambda_3)}$. Assume that $\frac{z}{d} \in \mathcal{D}^{-1}Z$ such that

$$(\overline{\mathcal{D}^{-1}\mu_1}^{(\xi_1,\Lambda_1)} + \overline{\mathcal{D}^{-1}\mu_2}^{(\xi_2,\Lambda_2)})(\frac{z}{d}) = \overline{\mathcal{D}^{-1}\mu_1}^{(\xi_1,\Lambda_1)}(\frac{z_1}{d_1}) \wedge \overline{\mathcal{D}^{-1}\mu_2}^{(\xi_2,\Lambda_2)}(\frac{z_2}{d_2}) = \mathcal{D}^{-1}\mu_1(\frac{x_1}{y_1}) \wedge \mathcal{D}^{-1}\mu_2(\frac{x_2}{y_2}), \tag{4.3.2}$$

where $\frac{z_1}{d_1}, \frac{z_2}{d_2}, \frac{x_1}{y_1}, \frac{x_2}{y_2} \in \mathcal{D}^{-1}Z$ such that $\frac{z_1}{d_1} + \frac{z_2}{d_2} = \frac{z}{d}$ and $\frac{x_1}{y_1} \in [\frac{z_1}{d_1}]_{\xi_1(a_i)}, \frac{x_2}{y_2} \in [\frac{z_2}{d_2}]_{\xi_2(b_i)}$, $a_i \in \Lambda_1, b_i \in \Lambda_2, i = 1,2$. It implies that $\frac{x_1}{y_1} + \frac{x_2}{y_2} \in [\frac{z_1}{d_1}]_{\xi_1(a_i)} + [\frac{z_2}{d_2}]_{\xi_2(b_i)} = [\frac{z_1}{d_1} + \frac{z_2}{d_2}]_{\xi_1(a_i)+\xi_2(b_i)} = [\frac{z}{d}]_{\eta(c_i)}, c_i = (a_i, b_i) \in \Lambda_3, i = 1,2$. From Definition 4.1.1 of [31] and Definitions 4.3.1, 4.2.1,

$$\begin{aligned}
\overline{\mathcal{D}^{-1}\mu_1 + \mathcal{D}^{-1}\mu_2}^{(\eta,\Lambda_3)}(\frac{m}{s}) &= \vee\{(\mathcal{D}^{-1}\mu_1 + \mathcal{D}^{-1}\mu_2)(\frac{x}{y}) : \frac{x}{y} \in [\frac{z}{d}]_{\eta(c_i)}, c_i = (a_i, b_i) \in \Lambda_3, i = 1,2\} \\
&= \vee\vee\{\mathcal{D}^{-1}\mu_1(\frac{a_1}{b_1}) \wedge \mathcal{D}^{-1}\mu_2(\frac{a_2}{b_2}) : \frac{x}{y} = \frac{a_1}{b_1} + \frac{a_2}{b_2}, \frac{x}{y} \in [\frac{z}{d}]_{\eta(c_i)}, c_i \in \Lambda_3\} \\
&\geq \mathcal{D}^{-1}\mu_1(\frac{x_1}{y_1}) \wedge \mathcal{D}^{-1}\mu_2(\frac{x_2}{y_2}) \\
&= \overline{\mathcal{D}^{-1}\mu_1}^{(\xi_1,\Lambda_1)}(\frac{z_1}{d_1}) \wedge \overline{\mathcal{D}^{-1}\mu_2}^{(\xi_2,\Lambda_2)}(\frac{z_2}{d_2}) \\
&= (\overline{\mathcal{D}^{-1}\mu_1}^{(\xi_1,\Lambda_1)} + \overline{\mathcal{D}^{-1}\mu_2}^{(\xi_2,\Lambda_2)})(\frac{z}{d}),
\end{aligned}$$

(see Equation 4.3.2). This proves that $\overline{\mathcal{D}^{-1}\mu_1}^{(\xi_1,\Lambda_1)} + \overline{\mathcal{D}^{-1}\mu_2}^{(\xi_2,\Lambda_2)} \subseteq \overline{\mathcal{D}^{-1}\mu_1 + \mathcal{D}^{-1}\mu_2}^{(\eta,\Lambda_3)}$.

(2) To prove $\underline{\mathcal{D}^{-1}\mu_1}_{(\xi_1,\Lambda_1)} + \underline{\mathcal{D}^{-1}\mu_2}_{(\xi_2,\Lambda_2)} = \underline{\mathcal{D}^{-1}\mu_1 + \mathcal{D}^{-1}\mu_2}_{(\eta,\Lambda_3)}$, let

$$(\underline{\mathcal{D}^{-1}\mu_1}_{(\xi_1,\Lambda_1)} + \underline{\mathcal{D}^{-1}\mu_2}_{(\xi_2,\Lambda_2)})(\frac{z}{d}) = \underline{\mathcal{D}^{-1}\mu_1}_{(\xi_1,\Lambda_1)}(\frac{z_1}{d_1}) \wedge \underline{\mathcal{D}^{-1}\mu_2}_{(\xi_2,\Lambda_2)}(\frac{z_2}{d_2}) = \mathcal{D}^{-1}\mu_1(\frac{x_1}{y_1}) \wedge \mathcal{D}^{-1}\mu_2(\frac{x_2}{y_2}), \tag{4.3.3}$$

where $\frac{x_1}{y_1}, \frac{z_1}{d_1}, \frac{x_2}{y_2}, \frac{z_2}{d_2} \in \mathcal{D}^{-1}Z$ such that $\frac{x_1}{y_1} \in [\frac{z_1}{d_1}]_{\xi_1(a_i)}, \frac{x_2}{y_2} \in [\frac{z_2}{d_2}]_{\xi_2(b_i)}$, $a_i \in \Lambda_1, b_i \in \Lambda_2, i = 1,2$

and $\frac{z}{d} = \frac{z_1}{d_1} + \frac{z_2}{d_2}$. It implies that $\frac{x_1}{y_1} + \frac{x_2}{y_2} \in [\frac{z_1}{d_1} + \frac{z_2}{d_2}]_{\eta(c_i)} = [\frac{z}{d}]_{\eta(c_i)}$, $c_i \in \Lambda_3, i = 1, 2$. Thus

$$\begin{aligned}\underline{\mathcal{D}^{-1}\mu_1 + \mathcal{D}^{-1}\mu_2}_{(\eta,\Lambda_3)}(\frac{z}{d}) &= \wedge\{(\mathcal{D}^{-1}\mu_1 + \mathcal{D}^{-1}\mu_2)(\frac{x}{y}) : \frac{x}{y} \in [\frac{z}{d}]_{\eta(c_i)}, c_i \in \Lambda_3, i = 1, 2\} \\ &= \wedge \vee \{\mathcal{D}^{-1}\mu_1(\frac{a_1}{b_1}) \wedge \mathcal{D}^{-1}\mu_2(\frac{a_2}{b_2}) : \frac{x}{y} = \frac{a_1}{b_1} + \frac{a_2}{b_2}, \frac{x}{y} \in [\frac{z}{d}]_{\eta(c_i)}, \\ &c_i \in \Lambda_3, i = 1, 2\} \\ &\leq \mathcal{D}^{-1}\mu_1(\frac{x_1}{y_1}) \wedge \mathcal{D}^{-1}\mu_2(\frac{x_2}{y_2}) \\ &= \underline{\mathcal{D}^{-1}\mu_1}_{(\xi_1,\Lambda_1)}(\frac{z_1}{d_1}) \wedge \underline{\mathcal{D}^{-1}\mu_2}_{(\xi_2,\Lambda_2)}(\frac{z_2}{d_2}) \\ &= (\underline{\mathcal{D}^{-1}\mu_1}_{(\xi_1,\Lambda_1)} + \underline{\mathcal{D}^{-1}\mu_2}_{(\xi_2,\Lambda_2)})(\frac{z}{d}),\end{aligned}$$

(see Equation 4.3.3). This shows that $\underline{\mathcal{D}^{-1}\mu_1}_{(\xi_1,\Lambda_1)} + \underline{\mathcal{D}^{-1}\mu_2}_{(\xi_2,\Lambda_2)} \supseteq \underline{\mathcal{D}^{-1}\mu_1 + \mathcal{D}^{-1}\mu_2}_{(\eta,\Lambda_3)}$. Similarly, the reverse inclusion can be proved. This completes the proof. □

To prove the inclusion in (1) of above Theorem strictly holds, the following Example is given.

Example 4.3.7. *Consider Z, L and $\mathcal{D}$ same as in Example 4.3.5. Define two fuzzy submodules $\mu, \nu \in \mathcal{F}(Z)$ as follows:*

$$\mu(\mathfrak{l}) = \begin{cases} 1, & \text{if } \mathfrak{l} = \bar{0}, \bar{2}, \bar{4}, \bar{6} \\ 0.5, & \text{if } \mathfrak{l} = \bar{1}, \bar{3}, \bar{5}, \bar{7} \end{cases} \quad \text{and} \quad \nu(\mathfrak{l}) = \begin{cases} 1, & \text{if } \mathfrak{l} = \bar{0} \\ 0.6, & \text{otherwise} \end{cases}$$

for all $\mathfrak{l} \in Z$. Using the Definition of $\mathcal{D}^{-1}\mu$ and $\mathcal{D}^{-1}\nu$ given in 4.2.1, we obtain

$$\mathcal{D}^{-1}\mu(\frac{\mathfrak{l}}{t}) = \begin{cases} 1, & \text{if } \mathfrak{l} = \bar{0}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}, \frac{\bar{2}}{\bar{9}}, \frac{\bar{4}}{\bar{9}} \\ 0.5, & \text{if } \mathfrak{l} = \frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}, \frac{\bar{1}}{\bar{9}}, \frac{\bar{5}}{\bar{1}} \end{cases} \quad \text{and} \quad \mathcal{D}^{-1}\nu(\mathfrak{l}) = \begin{cases} 1, & \text{if } \mathfrak{l} = \bar{0} \\ 0.6, & \text{otherwise} \end{cases}$$

for all $\frac{\mathfrak{l}}{t} \in \mathcal{D}^{-1}Z$. By easy calculations of sum of two fuzzy submodules as given in Definition

4.1.1 of [31], we obtain:

$$(\mathcal{D}^{-1}\mu + \mathcal{D}^{-1}\nu)(\frac{\mathfrak{l}}{t}) = \begin{cases} 1, & \textit{if } \frac{\mathfrak{l}}{t} = \bar{0}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}, \frac{\bar{2}}{\bar{9}}, \frac{\bar{4}}{\bar{9}} \\ 0.6, & \textit{if } \frac{\mathfrak{l}}{t} = \frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}, \frac{\bar{1}}{\bar{9}}, \frac{\bar{3}}{\bar{1}}, \frac{\bar{5}}{\bar{1}} \end{cases}$$

for all $\frac{\mathfrak{l}}{t} \in \mathcal{D}^{-1}\mathbb{Z}_{10}$. Define two soft modules of fractions (ξ_1, Λ_1) and (ξ_2, Λ_2) over $\mathcal{D}^{-1}\mathbb{Z}_{10}$ as follows, where $\Lambda_1 = \Lambda_2 = \{e_1, e_2\}$:

$$\xi_1(\alpha) = \begin{cases} \{\bar{0}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}, \frac{\bar{2}}{\bar{9}}, \frac{\bar{4}}{\bar{9}}\}, & \textit{if } \alpha = e_1 \\ \{\bar{0}, \frac{\bar{5}}{\bar{1}}\}, & \textit{if } \alpha = e_2 \end{cases} \quad \textit{and} \quad \xi_2(\beta) = \begin{cases} \{\bar{0}, \frac{\bar{5}}{\bar{1}}\}, & \textit{if } \beta = e_1 \\ \mathcal{D}^{-1}\mathbb{Z}_{10}, & \textit{if } \beta = e_2 \end{cases}$$

for all $\alpha \in \Lambda_1$, $\beta \in \Lambda_2$. From Definition 4.1.3 of the sum of two soft modules of fractions, we have

$$\eta((\alpha, \beta)) = \begin{cases} \{\bar{0}, \frac{\bar{5}}{\bar{1}}\}, & \textit{if } (\alpha, \beta) = (e_2, e_1) \\ \mathcal{D}^{-1}\mathbb{Z}_{10}, & \textit{if } (\alpha, \beta) = (e_1, e_1), (e_1, e_2), (e_2, e_2) \end{cases}$$

for all $(\alpha, \beta) \in \Lambda_3$. Using the Definition 4.3.1 of upper approximation, we have:

$$\overline{(\mathcal{D}^{-1}\mu + \mathcal{D}^{-1}\nu)}^{(\eta,\Lambda_3)}(\frac{\mathfrak{l}}{t}) = 1 \textit{ and } \quad \overline{\mathcal{D}^{-1}\mu}^{(\xi_1,\Lambda_1)}(\frac{\mathfrak{l}}{t}) = \mathcal{D}^{-1}\mu(\frac{\mathfrak{l}}{t}), \quad \overline{\mathcal{D}^{-1}\nu}^{(\xi_2,\Lambda_2)}(\frac{\mathfrak{l}}{t}) = \mathcal{D}^{-1}\nu(\frac{\mathfrak{l}}{t}),$$

for all $\frac{\mathfrak{l}}{t} \in \mathcal{D}^{-1}\mathbb{Z}_{10}$. Thus, it is clear that $\overline{(\mathcal{D}^{-1}\mu + \mathcal{D}^{-1}\nu)}^{(\eta,\Lambda_3)}(\frac{\mathfrak{l}}{t}) \nleq \overline{\mathcal{D}^{-1}\mu}^{(\xi_1,\Lambda_1)}(\frac{\mathfrak{l}}{t}) + \overline{\mathcal{D}^{-1}\nu}^{(\xi_2,\Lambda_2)}(\frac{\mathfrak{l}}{t})$.

Definition 4.3.8. *Let (ξ, Λ) be a soft module of fractions over $\mathcal{D}^{-1}Z$. Then, a fuzzy module of fractions $\mathcal{D}^{-1}\mu$ of $\mathcal{D}^{-1}Z$ is called a fuzzy lower (resp. upper) rough module of fractions, if $\underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda)}$ (resp. $\overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)}$) is a fuzzy module of fractions of $\mathcal{D}^{-1}Z$. Moreover, a fuzzy module of fractions $\mathcal{D}^{-1}\mu$ of $\mathcal{D}^{-1}Z$ is called a fuzzy rough module of fractions, if it is both fuzzy lower and upper rough module of fractions of $\mathcal{D}^{-1}Z$.*

Theorem 4.3.9. *Let (ξ, Λ) be a soft module of fractions over $\mathcal{D}^{-1}Z$ and $\mu \in \mathcal{F}(Z)$ be a fuzzy submodule of Z. Then, $\overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)}$ is a fuzzy module of fractions of $\mathcal{D}^{-1}Z$.*

Proof. By Definition 4.3.1 of the upper approximation,

$$\overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)}(\frac{\theta}{s}) = \vee\{\mathcal{D}^{-1}\mu(\frac{x}{y}) : \frac{x}{y} \in [\frac{\theta}{s}]_{\xi(a_i)} \text{ where } a_i \in \Lambda, i = 1, 2\}$$
$$= 1.$$

Recall that $\mathcal{D}^{-1}\mu$ is a fuzzy submodule of $\mathcal{D}^{-1}Z$. Let $\frac{z_1}{d_1}, \frac{z_2}{d_2} \in \mathcal{D}^{-1}Z$. By our assumption and Definition 4.3.1, we have:

$$\begin{aligned}
\overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)}(\frac{z_1}{d_1} + \frac{z_2}{d_2}) &= \vee\{\mathcal{D}^{-1}\mu(\frac{x}{y}) : \frac{x}{y} \in [\frac{z_1}{d_1} + \frac{z_2}{d_2}]_{\xi(a_i)}, a_i \in \Lambda, i = 1, 2\} \\
&= \vee\{\mathcal{D}^{-1}\mu(\frac{x}{y}) : \frac{x}{y} \in [\frac{z_1}{d_1}]_{\xi(a_i)} + [\frac{z_2}{d_2}]_{\xi(a_i)}, a_i \in \Lambda, i = 1, 2\} \\
&= \vee\{\mathcal{D}^{-1}\mu(\frac{x}{y}) : \frac{x}{y} = \frac{x_1}{y_1} + \frac{x_2}{y_2} \text{ where } \frac{x_1}{y_1} \in [\frac{z_1}{d_1}]_{\xi(a_i)}, \frac{x_2}{y_2} \in [\frac{z_2}{d_2}]_{\xi(a_i)}, \\
&\quad \text{and } a_i \in \Lambda, i = 1, 2\} \\
&= \vee\{\mathcal{D}^{-1}\mu(\frac{x_1}{y_1} + \frac{x_2}{y_2}) : \frac{x_1}{y_1} \in [\frac{z_1}{d_1}]_{\xi(a_i)}, \frac{x_2}{y_2} \in [\frac{z_2}{d_2}]_{\xi(a_i)}, a_i \in \Lambda, i = 1, 2\} \\
&\geq \vee\{\mathcal{D}^{-1}\mu(\frac{x_1}{y_1}) \wedge \mathcal{D}^{-1}\mu(\frac{x_2}{y_2}) : \frac{x_1}{y_1} \in [\frac{z_1}{d_1}]_{\xi(a_i)}, \frac{x_2}{y_2} \in [\frac{z_2}{d_2}]_{\xi(a_i)}, a_i \in \Lambda, i = 1, 2\} \\
&= (\vee\{\mathcal{D}^{-1}\mu(\frac{x_1}{y_1}) : \frac{x_1}{y_1} \in [\frac{z_1}{d_1}]_{\xi(a_i)}, a_i \in \Lambda, i = 1, 2\}) \wedge \\
&\quad (\vee\{\mathcal{D}^{-1}\mu(\frac{x_2}{y_2}) : \frac{x_2}{y_2} \in [\frac{z_2}{d_2}]_{\xi(a_i)}, a_i \in \Lambda, i = 1, 2\}) \\
&= \overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)}(\frac{z_1}{d_1}) \wedge \overline{D^{-1}\mu}^{(\xi,\Lambda)}(\frac{z_2}{d_2}).
\end{aligned}$$

Now let $\frac{l}{m} \in \mathcal{D}^{-1}L$. Then,

$$\begin{aligned}
\overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)}(\frac{l}{m}.\frac{z_1}{d_1}) &= \vee\{\mathcal{D}^{-1}\mu(\frac{x}{y}) : \frac{x}{y} \in [\frac{l}{m}.\frac{z_1}{d_1}]_{\xi(a_i)}, a_i \in \Lambda, i = 1, 2\} \\
&= \vee\{\mathcal{D}^{-1}\mu(\frac{x}{y}) : \frac{x}{y} \in \frac{l}{m}.[\frac{z_1}{d_1}]_{\xi(a_i)}, a_i \in \Lambda, i = 1, 2\} \\
&= \vee\{\mathcal{D}^{-1}\mu(\frac{l}{m}.\frac{p}{q}) : \frac{x}{y} = \frac{l}{m}.\frac{p}{q} \text{ where } \frac{p}{q} \in [\frac{z_1}{d_1}]_{\xi(a_i)}, a_i \in \Lambda, i = 1, 2\} \\
&\geq \vee\{\mathcal{D}^{-1}\mu(\frac{p}{q}) : \frac{p}{q} \in [\frac{z_1}{d_1}]_{\xi(a_i)}, a_i \in \Lambda, i = 1, 2\} \\
&= \overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)}(\frac{z_1}{d_1}).
\end{aligned}$$

Thus $\overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)}(\frac{l}{m}.\frac{z_1}{d_1}) \geq \overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)}(\frac{z_1}{d_1})$. Hence, $\overline{\mathcal{D}^{-1}\mu}^{(\xi,\Lambda)}$ is a fuzzy submodule of $\mathcal{D}^{-1}Z$. □

Remark 4.3.10. *Note that, the fuzzy lower approximation $\underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda)}$ of a fuzzy module of fractions $\mathcal{D}^{-1}\mu$ of $\mathcal{D}^{-1}Z$ does not a fuzzy module of fractions of $\mathcal{D}^{-1}Z$. To illustrate this, we consider the following Example.*

Example 4.3.11. *Let $L = Z = \mathbb{Z}_8$, $\mathcal{D} = \{\overline{1}, \overline{3}\}$. Then*

$$\mathcal{D}^{-1}\mathbb{Z}_8 = \{\overline{0}, \frac{\overline{1}}{\overline{1}}, \frac{\overline{1}}{\overline{3}}, \frac{\overline{2}}{\overline{1}}, \frac{\overline{2}}{\overline{3}}, \frac{\overline{4}}{\overline{1}}, \frac{\overline{5}}{\overline{1}}, \frac{\overline{5}}{\overline{3}}\},$$

where $\frac{\overline{1}}{\overline{3}} = \frac{\overline{3}}{\overline{1}}$, $\frac{\overline{2}}{\overline{1}} = \frac{\overline{6}}{\overline{3}}$, $\frac{\overline{2}}{\overline{3}} = \frac{\overline{6}}{\overline{1}}$, $\frac{\overline{4}}{\overline{1}} = \frac{\overline{4}}{\overline{3}}$, $\frac{\overline{5}}{\overline{1}} = \frac{\overline{7}}{\overline{3}}$, $\frac{\overline{5}}{\overline{3}} = \frac{\overline{7}}{\overline{1}}$. Define a fuzzy submodule of Z as follows:

$$\mu(\mathfrak{l}) = \begin{cases} 1, & \text{if } \mathfrak{l} = \overline{0} \\ 0.2, & \text{otherwise} \end{cases}$$

for all $\mathfrak{l} \in Z$. From Definition 4.2.1, we obtain the following fuzzy submodule of $\mathcal{D}^{-1}Z$:

$$\mathcal{D}^{-1}\mu(\frac{\mathfrak{l}}{t}) = \begin{cases} 1, & \text{if } \frac{\mathfrak{l}}{t} = \overline{0} \\ 0.2, & \text{otherwise} \end{cases}$$

for all $\frac{l}{t} \in \mathcal{D}^{-1}Z$. *Consider a soft module of fractions* (ξ, Λ), *where* $\Lambda = \{e_1, e_2\}$ *defined as follows:*

$$\xi(\alpha) = \begin{cases} \{\overline{0}, \frac{\overline{2}}{\overline{1}}, \frac{\overline{2}}{\overline{3}}, \frac{\overline{4}}{\overline{1}}\}, & \text{if } \alpha = e_1 \\ \mathcal{D}^{-1}\mathbb{Z}_8, & \text{if } \alpha = e_2 \end{cases}$$

for all $\alpha \in \Lambda$. *By simple computations, we get:*

$$\underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda)}(\frac{l}{t}) = 0.2 \text{ for all } \frac{l}{t} \in \mathcal{D}^{-1}Z.$$

Since $\underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda)}(\overline{0}) \neq 1$. *Hence, first condition of Definition 4.1.8 of [31] fails. Thus* $\underline{\mathcal{D}^{-1}\mu}_{(\xi,\Lambda)}$ *is not a fuzzy submodule of* $\mathcal{D}^{-1}Z$.

In the following result, a connection between the fuzzy lower and upper approximation spaces of product of a fuzzy set in L with a fuzzy set in Z and the product of the fuzzy lower and upper approximations of fuzzy sets with respect to a soft module of fractions is made.

Theorem 4.3.12. *Let* (ξ, Λ) *be a soft submodule over* $\mathcal{D}^{-1}Z$. *Assume that* $\mu \in \mathcal{F}(L)$ *and* $\nu \in \mathcal{F}(Z)$. *Then,*

(1) $\overline{\mathcal{D}^{-1}(\mu.\nu)}^{(\xi,\Lambda)} = \overline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}^{(\xi,\Lambda)} = \mathcal{D}^{-1}\mu.\overline{\mathcal{D}^{-1}\nu}^{(\xi,\Lambda)}$.

(2) $\mathcal{D}^{-1}\mu.\underline{\mathcal{D}^{-1}\nu}_{(\xi,\Lambda)} \subseteq \underline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}_{(\xi,\Lambda)} = \underline{\mathcal{D}^{-1}(\mu.\nu)}_{(\xi,\Lambda)}$.

Proof. By Theorem 4.2.5 and Theorem 4.3.3 (5) and (6),

$$\overline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}^{(\xi,\Lambda)} = \mathcal{D}^{-1}\mu.\overline{\mathcal{D}^{-1}\nu}^{(\xi,\Lambda)}, \underline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}_{(\xi,\Lambda)} = \underline{\mathcal{D}^{-1}(\mu.\nu)}_{(\xi,\Lambda)}.$$

(1) Let $\frac{z}{d} \in \mathcal{D}^{-1}Z$ be such that

$$\overline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}^{(\xi,\Lambda)}(\frac{z}{d}) = (\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu)(\frac{x}{y}) = \mathcal{D}^{-1}\mu(\frac{u}{v}) \wedge \mathcal{D}^{-1}\nu(\frac{w}{z}), \tag{4.3.4}$$

where $\frac{x}{y}, \frac{w}{z} \in \mathcal{D}^{-1}Z$, $\frac{u}{v} \in \mathcal{D}^{-1}L$ such that $\frac{x}{y} \in [\frac{z}{d}]_{\xi(a_i)}$ for all $a_i \in \Lambda, i = 1, 2$ and $\frac{x}{y} = \frac{u}{v}.\frac{w}{z}$. By

Definition 4.1.6 of [31] and Definition 4.3.1, we have:

$$
\begin{aligned}
(\mathcal{D}^{-1}\mu.\overline{\mathcal{D}^{-1}\nu}^{(\xi,\Lambda)})(\frac{z}{d}) &= \vee\{\mathcal{D}^{-1}\mu(\frac{a}{b}) \wedge \overline{\mathcal{D}^{-1}\nu}^{(\xi,\Lambda)}(\frac{p}{q}) : \frac{a}{b}.\frac{p}{q} = \frac{z}{d}, \frac{a}{b} \in \mathcal{D}^{-1}L, \frac{p}{q} \in \mathcal{D}^{-1}Z\} \\
&= \vee\vee\{\mathcal{D}^{-1}\mu(\frac{a}{b}) \wedge \mathcal{D}^{-1}\nu(\frac{n}{t}) : \frac{n}{t} \in [\frac{p}{q}]_{\xi(a_i)}, a_i \in \Lambda, i = 1,2 \text{ and} \\
&\frac{a}{b}.\frac{p}{q} = \frac{z}{d}, \frac{a}{b} \in \mathcal{D}^{-1}L, \frac{p}{q}, \frac{n}{t} \in \mathcal{D}^{-1}Z\} \\
&= \vee\vee\{\mathcal{D}^{-1}\mu(\frac{a}{b}) \wedge \mathcal{D}^{-1}\nu(\frac{n}{t}) : \frac{a}{b}.\frac{n}{t} \in [\frac{z}{d}]_{\xi(a_i)}, a_i \in \Lambda, i = 1,2\} \\
&\geq \mathcal{D}^{-1}\mu(\frac{u}{v}) \wedge \mathcal{D}^{-1}\nu(\frac{w}{z}) \\
&= (\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu)(\frac{x}{y}) \\
&= \overline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}^{(\xi,\Lambda)}(\frac{z}{d}),
\end{aligned}
$$

(see Equation 4.3.4). Thus $\mathcal{D}^{-1}\mu.\overline{\mathcal{D}^{-1}\nu}^{(\xi,\Lambda)} \supseteq \overline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}^{(\xi,\Lambda)}$. Conversely, assume that

$$
(\mathcal{D}^{-1}\mu.\overline{\mathcal{D}^{-1}\nu}^{(\xi,\Lambda)})(\frac{z}{d}) = \mathcal{D}^{-1}\mu(\frac{u}{v}) \wedge \overline{\mathcal{D}^{-1}\nu}^{(\xi,\Lambda)}(\frac{p}{q}) = \mathcal{D}^{-1}\mu(\frac{u}{v}) \wedge \mathcal{D}^{-1}\nu(\frac{a}{b}), \tag{4.3.5}
$$

where $\frac{u}{v} \in \mathcal{D}^{-1}L, \frac{a}{b}, \frac{p}{q} \in \mathcal{D}^{-1}Z$ such that $\frac{u}{v}.\frac{p}{q} = \frac{z}{d}$ and $\frac{a}{b} \in [\frac{p}{q}]_{\xi(a_i)}$, $a_i \in \Lambda, i = 1,2$. It follows that $\frac{u}{v}.\frac{a}{b} \in [\frac{u}{v}.\frac{p}{q}]_{\xi(a_i)} = [\frac{z}{d}]_{\xi(a_i)}$. Thus

$$
\begin{aligned}
\overline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}^{(\xi,\Lambda)}(\frac{z}{d}) &= \vee\{(\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu)(\frac{x}{y}) : \frac{x}{y} \in [\frac{z}{d}]_{\xi(a_i)}, a_i \in \Lambda, i = 1,2\} \\
&= \vee\vee\{\mathcal{D}^{-1}\mu(\frac{x_1}{y_1}) \wedge \mathcal{D}^{-1}\nu(\frac{x_2}{y_2}) : \frac{x}{y} = \frac{x_1}{y_1}.\frac{x_2}{y_2} \in [\frac{z}{d}]_{\xi(a_i)} \\
&\text{for all } a_i \in \Lambda, i = 1,2, \frac{x_1}{y_1} \in \mathcal{D}^{-1}L, \frac{x_2}{y_2} \in \mathcal{D}^{-1}Z\} \\
&\geq \mathcal{D}^{-1}\mu(\frac{u}{v}) \wedge \mathcal{D}^{-1}\nu(\frac{a}{b}) \\
&= \mathcal{D}^{-1}\mu(\frac{u}{v}) \wedge \overline{\mathcal{D}^{-1}\nu}^{(\xi,\Lambda)}(\frac{p}{q}) \\
&= (\mathcal{D}^{-1}\mu.\overline{\mathcal{D}^{-1}\nu}^{(\xi,\Lambda)})(\frac{z}{d}),
\end{aligned}
$$

(see Equation 4.3.5). Hence, $\mathcal{D}^{-1}\mu.\overline{\mathcal{D}^{-1}\nu}^{(\xi,\Lambda)} \subseteq \overline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}^{(\xi,\Lambda)}$. In the similar way, it can be proved that $\mathcal{D}^{-1}\mu.\underline{\mathcal{D}^{-1}\nu}_{(\xi,\Lambda)} \subseteq \underline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}_{(\xi,\Lambda)}$. This completes the proof. □

In the following Example it is shown that the sign of inequality in (2) of above result cannot be replaced with the sign of equality.

Example 4.3.13. *Suppose that L, Z and $\mathcal{D}$ are same as in Example 4.3.5. Assume that $\mu, \nu \in \mathcal{F}(Z)$ are two fuzzy submodules same as in Example 4.3.5. Using the Definition 4.1.6 of [31], we get*

$$(\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu)(\frac{\mathfrak{l}}{t}) = \begin{cases} 1, & \text{if } \frac{\mathfrak{l}}{t} = \bar{0} \\ 0.6, & \text{if } \frac{\mathfrak{l}}{t} = \frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}, \frac{\bar{2}}{\bar{9}}, \frac{\bar{4}}{\bar{9}} \\ 0.5, & \text{if } \frac{\mathfrak{l}}{t} = \frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}, \frac{\bar{1}}{\bar{9}}, \frac{\bar{3}}{\bar{1}}, \frac{\bar{5}}{\bar{1}} \end{cases}$$

for all $\frac{\mathfrak{l}}{t} \in \mathcal{D}^{-1}\mathbb{Z}_{10}$. Define a soft module of fractions (ξ, Λ), where $\Lambda = \{e_1, e_2\}$ as follows:

$$\xi(\alpha) = \begin{cases} \{\bar{0}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}, \frac{\bar{2}}{\bar{9}}, \frac{\bar{4}}{\bar{9}}\}, & \text{if } \alpha = e_1 \\ S^{-1}\mathbb{Z}_{10}, & \text{if } \alpha = e_2 \end{cases}$$

for all $\alpha \in \Lambda$. By Definition of lower approximation, we have

$$\underline{(\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu)}_{(\xi,\Lambda)}(\frac{\mathfrak{l}}{t}) = \begin{cases} 0.6, & \text{if } \frac{\mathfrak{l}}{t} = \bar{0}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}, \frac{\bar{2}}{\bar{9}}, \frac{\bar{4}}{\bar{9}} \\ 0.5, & \text{if } \frac{\mathfrak{l}}{t} = \frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}, \frac{\bar{1}}{\bar{9}}, \frac{\bar{3}}{\bar{1}}, \frac{\bar{5}}{\bar{1}} \end{cases} \quad \text{and} \quad \underline{\mathcal{D}^{-1}\nu}_{(\xi,\Lambda)}(\frac{\mathfrak{l}}{t}) = \begin{cases} 0.6, & \text{if } \frac{\mathfrak{l}}{t} = \bar{0}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{2}}{\bar{3}}, \frac{\bar{2}}{\bar{9}}, \frac{\bar{4}}{\bar{9}} \\ 0.5, & \text{if } \frac{\mathfrak{l}}{t} = \frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}, \frac{\bar{1}}{\bar{9}}, \frac{\bar{3}}{\bar{1}}, \frac{\bar{5}}{\bar{1}} \end{cases}$$

for all $\frac{\mathfrak{l}}{t} \in \mathcal{D}^{-1}\mathbb{Z}_{10}$. On the other hand,

$$(\mathcal{D}^{-1}\mu.\underline{\mathcal{D}^{-1}\nu}_{(\xi,\Lambda)})(\frac{\mathfrak{l}}{t}) = \begin{cases} 0.6, & \text{if } \frac{\mathfrak{l}}{t} = \bar{0}, \frac{\bar{2}}{\bar{3}}, \frac{\bar{2}}{\bar{9}}, \frac{\bar{4}}{\bar{9}} \\ 0.5, & \text{if } \frac{\mathfrak{l}}{t} = \frac{\bar{1}}{\bar{1}}, \frac{\bar{1}}{\bar{3}}, \frac{\bar{2}}{\bar{1}}, \frac{\bar{1}}{\bar{9}}, \frac{\bar{5}}{\bar{1}} \end{cases}$$

for all $\frac{\mathfrak{l}}{t} \in \mathcal{D}^{-1}\mathbb{Z}_{10}$. One can easily see that $(\mathcal{D}^{-1}\mu.\underline{\mathcal{D}^{-1}\nu}_{(\xi,\Lambda)})(\frac{\bar{2}}{\bar{1}}) = 0.5 \leq \underline{(\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu)}_{(\xi,\Lambda)}(\frac{\bar{2}}{\bar{1}}) =$

0.6. *Thus* $(\mathcal{D}^{-1}\mu.\underline{\mathcal{D}^{-1}\nu}_{(\xi,\Lambda)})(\frac{l}{t}) \leq \underline{(\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu)}_{(\xi,\Lambda)}(\frac{l}{t})$ *for all* $\frac{l}{t} \in \mathcal{D}^{-1}\mathbb{Z}_{10}$. *Therefore,* $\mathcal{D}^{-1}\mu.\underline{\mathcal{D}^{-1}\nu}_{(\xi,\Lambda)} \not\supseteq \underline{(\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu)}_{(\xi,\Lambda)}$.

Since the product of a soft ideal with a soft module of fractions is a soft module of fractions (see Proposition 4.1.6). Thus, we consider the approximations of the product of a fuzzy set of $\mathcal{D}^{-1}L$ with fuzzy set of $\mathcal{D}^{-1}Z$ in the result 4.3.14.

Theorem 4.3.14. *Let* (ξ_1, Λ_1) *be a soft ideal over* $\mathcal{D}^{-1}L$ *and* (ξ_2, Λ_2) *be a soft module of fractions over* $\mathcal{D}^{-1}Z$. *Assume that* $\mathcal{D}^{-1}\mu \in \mathcal{F}(\mathcal{D}^{-1}L)$ *and* $\mathcal{D}^{-1}\nu \in \mathcal{F}(\mathcal{D}^{-1}Z)$. *Then*

(1) $\overline{\mathcal{D}^{-1}(\mu.\nu)}^{(\eta,\Lambda_3)} = \overline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}^{(\eta,\Lambda_3)} = \overline{\mathcal{D}^{-1}\mu}^{(\xi_1,\Lambda_1)}.\overline{\mathcal{D}^{-1}\nu}^{(\xi_2,\Lambda_2)}$ *and*

(2) $\underline{\mathcal{D}^{-1}(\mu.\nu)}_{(\eta,\Lambda_3)} = \underline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}_{(\eta,\Lambda_3)} = \underline{\mathcal{D}^{-1}\mu}_{(\xi_1,\Lambda_1)}.\underline{\mathcal{D}^{-1}\nu}_{(\xi_2,\Lambda_2)}$,

where $(\eta, \Lambda_3) = (\xi_1, \Lambda_1) \times (\xi_2, \Lambda_2)$, $\Lambda_3 = \Lambda_1 \times \Lambda_2$ *(see Definition 4.1.5).*

Proof. By Theorem 4.2.5 and Theorem 4.3.3 (5), (6),

$$\overline{\mathcal{D}^{-1}(\mu.\nu)}^{(\eta,\Lambda_3)} = \overline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}^{(\eta,\Lambda_3)}, \underline{\mathcal{D}^{-1}(\mu.\nu)}_{(\eta,\Lambda_3)} = \underline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}_{(\eta,\Lambda_3)}.$$

(1) Let $\frac{z}{d} \in \mathcal{D}^{-1}Z$ be such that

$$(\overline{\mathcal{D}^{-1}\mu}^{(\xi_1,\Lambda_1)}.\overline{\mathcal{D}^{-1}\nu}^{(\xi_2,\Lambda_2)})(\frac{z}{d}) = \overline{\mathcal{D}^{-1}\mu}^{(\xi_1,\Lambda_1)}(\frac{p_1}{q_1}) \wedge \overline{\mathcal{D}^{-1}\nu}^{(\xi_2,\Lambda_2)}(\frac{p_2}{q_2}) = \mathcal{D}^{-1}\mu(\frac{m}{n}) \wedge \mathcal{D}^{-1}\nu(\frac{p}{q}), \tag{4.3.6}$$

where $\frac{p_1}{q_1}, \frac{m}{n} \in \mathcal{D}^{-1}L$, $\frac{p_2}{q_2}, \frac{p}{q} \in \mathcal{D}^{-1}Z$ such that $\frac{p_1}{q_1}.\frac{p_2}{q_2} = \frac{z}{d}$ and $\frac{m}{n} \in [\frac{p_1}{q_1}]_{\xi_1(a_i)}$, $\frac{p}{q} \in [\frac{p_2}{q_2}]_{\xi_2(b_i)}$, $a_i \in \Lambda_1, b_i \in \Lambda_2, i = 1, 2$. It follows that, $\frac{m}{n}.\frac{p}{q} \in [\frac{p_1}{q_1}.\frac{p_2}{q_2}]_{\xi_1(a_i).\xi_2(b_i)} = [\frac{z}{d}]_{\eta(a_i,b_i)=\eta(c_i)}, c_i \in \Lambda_3, i = 1, 2.$

Thus

$$\begin{aligned}
\overline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}^{(\eta,\Lambda_3)}(\frac{z}{d}) &= \vee\{(\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu)(\frac{x}{y}) : \frac{x}{y} \in [\frac{z}{d}]_{\eta(c_i)}, c_i \in \Lambda_3, i = 1, 2\} \\
&= \vee \vee \{\mathcal{D}^{-1}\mu(\frac{x_1}{y_1}) \wedge \mathcal{D}^{-1}\nu(\frac{x_2}{y_2}) : \frac{x}{y} = \frac{x_1}{y_1}.\frac{x_2}{y_2}, \frac{x}{y} \in [\frac{z}{d}]_{\eta(c_i)}, c_i \in \Lambda_3, i = 1, 2\} \\
&\geq \mathcal{D}^{-1}\mu(\frac{m}{n}) \wedge \mathcal{D}^{-1}\nu(\frac{p}{q}) \\
&= \overline{\mathcal{D}^{-1}\mu}^{(\xi_1,\Lambda_1)}(\frac{p_1}{q_1}) \wedge \overline{\mathcal{D}^{-1}\nu}^{(\xi_2,\Lambda_2)}(\frac{p_2}{q_2}) \\
&= (\overline{\mathcal{D}^{-1}\mu}^{(\xi_1,\Lambda_1)}.\overline{\mathcal{D}^{-1}\nu}^{(\xi_2,\Lambda_2)})(\frac{z}{d}),
\end{aligned}$$

(see Equation 4.3.6). Thus $\overline{\mathcal{D}^{-1}\mu}^{(\xi_1,\Lambda_1)}.\overline{\mathcal{D}^{-1}\nu}^{(\xi_2,\Lambda_2)} \subseteq \overline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}^{(\eta,\Lambda_3)}$.

(2) Let $\frac{z}{d} \in \mathcal{D}^{-1}Z$ be such that

$$\underline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}_{(\eta,\Lambda_3)}(\frac{z}{d}) = (\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu)(\frac{x}{y}) = \mathcal{D}^{-1}\mu(\frac{x_1}{y_1}) \wedge \mathcal{D}^{-1}\nu(\frac{x_2}{y_2}), \tag{4.3.7}$$

where $\frac{x_1}{y_1} \in \mathcal{D}^{-1}L$, $\frac{x}{y}, \frac{x_2}{y_2} \in \mathcal{D}^{-1}Z$ such that $\frac{x}{y} \in [\frac{z}{d}]_{\eta(c_i)}$, $c_i \in \Lambda_3, i = 1, 2$ and $\frac{x}{y} = \frac{x_1}{y_1}.\frac{x_2}{y_2}$. By

Definition 4.1.6 of [31] and Definition 4.3.1, we have:

$$
\begin{aligned}
(\underline{\mathcal{D}^{-1}\mu}_{(\xi_1,\Lambda_1)}.\underline{\mathcal{D}^{-1}\nu}_{(\xi_2,\Lambda_2)})(\frac{z}{d}) &= \vee\{\underline{\mathcal{D}^{-1}\mu}_{(\xi_1,\Lambda_1)}(\frac{u_1}{v_1}) \wedge \underline{\mathcal{D}^{-1}\nu}_{(\xi_2,\Lambda_2)})(\frac{u_2}{v_2}) : \\
&\frac{u_1}{v_1}.\frac{u_2}{v_2} = \frac{z}{d}, \frac{u_1}{v_1} \in \mathcal{D}^{-1}L, \frac{u_2}{v_2} \in \mathcal{D}^{-1}Z\} \\
&= \vee\{(\wedge\{\mathcal{D}^{-1}\mu(\frac{m}{n}) : \frac{m}{n} \in [\frac{u_1}{v_1}]_{\xi_1(a_i)}, a_i \in \Lambda_1, i = 1,2\})\wedge \\
&(\wedge\{\mathcal{D}^{-1}\nu(\frac{p}{q}) : \frac{p}{q} \in [\frac{u_2}{v_2}]_{\xi_2(b_i)}, b_i \in \Lambda_2, i = 1,2\}), \\
&\frac{u_1}{v_1}.\frac{u_2}{v_2} = \frac{z}{d}, \frac{u_1}{v_1} \in \mathcal{D}^{-1}L, \frac{u_2}{v_2} \in \mathcal{D}^{-1}Z\} \\
&= \vee \wedge \{\mathcal{D}^{-1}\mu(\frac{m}{n}) \wedge \mathcal{D}^{-1}\nu(\frac{p}{q}) : \frac{m}{n} \in [\frac{u_1}{v_1}]_{\xi_1(a_i)}, \frac{p}{q} \in [\frac{u_2}{v_2}]_{\xi_2(b_i)}, \\
&a_i \in \Lambda_1, b_i \in \Lambda_2, i = 1,2, \frac{u_1}{v_1}.\frac{u_2}{v_2} = \frac{z}{d}.\frac{u_1}{v_1} \in \mathcal{D}^{-1}L, \frac{u_2}{v_2} \in \mathcal{D}^{-1}Z\} \\
&= \wedge \vee \{\mathcal{D}^{-1}\mu(\frac{m}{n}) \wedge \mathcal{D}^{-1}\nu(\frac{p}{q}) : \frac{m}{n} \in [\frac{u_1}{v_1}]_{\xi_1(a_i)}, \frac{p}{q} \in [\frac{u_2}{v_2}]_{\xi_2(b_i)}, \\
&a_i \in \Lambda_1, b_i \in \Lambda_2, i = 1,2, \frac{u_1}{v_1}.\frac{u_2}{v_2} = \frac{z}{d}, \frac{u_1}{v_1} \in \mathcal{D}^{-1}L, \frac{u_2}{v_2} \in \mathcal{D}^{-1}Z\} \\
&\leq \wedge \vee \{\mathcal{D}^{-1}\mu(\frac{m}{n}) \wedge \mathcal{D}^{-1}\nu(\frac{p}{q}) : \frac{m}{n}.\frac{p}{q} \in [\frac{u_1}{v_1}.\frac{u_2}{v_2}]_{\xi_1(a_i).\xi_2(b_i)} = [\frac{z}{d}]_{\eta(a_i,b_i)} = \\
&[\frac{z}{d}]_{\eta(c_i)}, c_i = (a_i,b_i) \in \Lambda_3 = \Lambda_1 \times \Lambda_2, i = 1,2, \frac{u_1}{v_1} \in \mathcal{D}^{-1}L, \frac{u_2}{v_2} \in \mathcal{D}^{-1}Z\} \\
&\leq \mathcal{D}^{-1}\mu(\frac{x_1}{y_1}) \wedge \mathcal{D}^{-1}\nu(\frac{x_2}{y_2}) \\
&= (\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu)(\frac{x}{y}) \\
&= \underline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}_{(\eta,\Lambda_3)}(\frac{z}{d}),
\end{aligned}
$$

(see Equation 4.3.7). Thus $\underline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}_{(\eta,\Lambda_3)} \supseteq \underline{\mathcal{D}^{-1}\mu}_{(\xi_1,\Lambda_1)}.\underline{\mathcal{D}^{-1}\nu}_{(\xi_2,\Lambda_2)}$. In the similar manner, it can be proved that $\underline{\mathcal{D}^{-1}\mu}_{(\xi_1,\Lambda_1)}.\underline{\mathcal{D}^{-1}\nu}_{(\xi_2,\Lambda_2)} \subseteq \underline{\mathcal{D}^{-1}\mu.\mathcal{D}^{-1}\nu}_{(\eta,\Lambda_3)}$. This completes the proof. □

Chapter 5

New types of soft rough sets in groups based on normal soft groups

In this chapter, we have defined the notion of roughness in groups in a different way using normal soft groups. This chapter consisting of 2 sections inclusively. In section 1, we have defined the concept of soft lower and upper approximation spaces in groups and study some intrinsic properties related to the soft approximation spaces. In section 2, we developed some relationships among the soft approximation spaces of two different groups by utilizing group homomorphisms. In this chapter, $\mathcal{G}$ will be denoting a multiplicative group with identity element $1_{\mathcal{G}}$. This chapter is the part of our paper [10].

5.1 Lower and upper approximations in groups via normal soft groups

In this section, we initiate the notion of soft rough sets over groups. We introduce the notion of soft lower approximation space and soft upper approximation space over groups by using a normal soft group and study some related properties. All through this section, $\emptyset \neq \Lambda \subseteq Æ$ where Æ is a finite set of parameters.

Definition 5.1.1. *Let (ϖ, Λ) be a normal soft group over $\mathcal{G}$ and $\emptyset \neq X \subseteq \mathcal{G}$. Define the soft lower approximation space $(\underline{X}_{\varpi}, \Lambda)$ and the soft upper approximation space $(\overline{X}^{\varpi}, \Lambda)$ of X with respect to $\varpi(\mathfrak{e})$ corresponding to each parameter $\mathfrak{e} \in \Lambda$ as follows:*

$$\underline{X}_{\varpi}(\mathfrak{e}) = \{\mathfrak{g} \in \mathcal{G} : \mathfrak{g}\varpi(\mathfrak{e}) \subseteq X\} \text{ and } \overline{X}^{\varpi}(\mathfrak{e}) = \{\mathfrak{g} \in \mathcal{G} : \mathfrak{g}\varpi(\mathfrak{e}) \cap X \neq \emptyset\},$$

for all $\mathfrak{e} \in \Lambda$. Intuitively, $\underline{X}_{\varpi} : \Lambda \to P(\mathcal{G})$ and $\overline{X}^{\varpi} : \Lambda \to P(\mathcal{G})$ are soft sets over $\mathcal{G}$.

From the Definition 5.1.1, one can easily obtain the following properties of the soft approximation spaces.

Proposition 5.1.2. *Let (ϖ_1, Λ_1) and (ϖ_2, Λ_2) be two normal soft groups over $\mathcal{G}$. Suppose that $\emptyset \neq X_1, X_2 \subseteq \mathcal{G}$. Then:*

(1) $\underline{X_1}_{\varpi_1}(\mathfrak{e}) \subseteq X_1 \subseteq \overline{X_1}^{\varpi_1}(\mathfrak{e})$, *for all* $\mathfrak{e} \in \Lambda$.

(2) If $X_1 \subseteq X_2$ *and* $\Lambda_1 \subseteq \Lambda_2$, *then* $(\underline{X_1}_{\varpi_1}, \Lambda_1)\widetilde{\subseteq}(\underline{X_1}_{\varpi_2}, \Lambda_2)$.

(3) If $X_1 \subseteq X_2$ *and* $\Lambda_1 \subseteq \Lambda_2$, *then* $(\overline{X_1}^{\varpi_1}, \Lambda_1)\widetilde{\subseteq}(\overline{X_2}^{\varpi_2}, \Lambda_2)$.

(4) $(\underline{X_1}_{\varpi_1}, \Lambda_1) \cup_{\mathfrak{R}} (\underline{X_2}_{\varpi_1}, \Lambda_1)\widetilde{\subseteq}(\underline{X_1 \cup X_2}_{\varpi_1}, \Lambda_1)$.

(5) $(\overline{X_1 \cap X_2}^{\varpi_1}, \Lambda_1)\widetilde{\subseteq}(\overline{X_1}^{\varpi_1}, \Lambda_1) \Cap (\overline{X_2}^{\varpi_1}, \Lambda_1)$.

(6) $(\overline{X_1 \cup X_2}^{\varpi_1}, \Lambda_1)\widetilde{=}(\overline{X_1}^{\varpi_1}, \Lambda_1) \cup_{\mathfrak{R}} (\overline{X_2}^{\varpi_1}, \Lambda_1)$.

(7) $(\underline{X_1 \cap X_2}_{\varpi_1}, \Lambda_1)\widetilde{=}(\underline{X_1}_{\varpi_1}, \Lambda_1) \Cap (\underline{X_2}_{\varpi_1}, \Lambda_1)$.

(8) If $(\varpi_1, \Lambda_1)\widetilde{\subseteq}(\varpi_2, \Lambda_2)$, *then* $(\overline{X_1}^{\varpi_1}, \Lambda_1)\widetilde{\subseteq}(\overline{X_1}^{\varpi_2}, \Lambda_2)$.

(9) If $(\varpi_1, \Lambda_1)\widetilde{\subseteq}(\varpi_2, \Lambda_2)$, *then* $(\underline{X_1}_{\varpi_1}, \Lambda_1)\widetilde{\supseteq}(\underline{X_1}_{\varpi_2}, \Lambda_2)$.

(10) $(\underline{X_1}_{\vartheta_1}, \Lambda_3)\widetilde{\supseteq}(\underline{X_1}_{\varpi_1}, \Lambda_1) \Cap (\underline{X_1}_{\varpi_2}, \Lambda_2)$.

(11) $(\overline{X_1}^{\vartheta_1}, \Lambda_3)\widetilde{\subseteq}(\overline{X_1}^{\varpi_1}, \Lambda_1) \Cap (\overline{X_1}^{\varpi_2}, \Lambda_2)$.

Here, $(\varpi_1, \Lambda_1) \Cap (\varpi_2, \Lambda_2) = (\vartheta_1, \Lambda_3)$ *and* $\Lambda_3 = \Lambda_1 \cap \Lambda_2 \neq \emptyset$ *(see Definition 1.4.4).*

Proof. The proof is obvious. □

The following Examples illustrate that inclusions in (4), (5), (10) and (11) of Proposition 5.1.2 are strict.

Example 5.1.3. *Let* $\mathcal{G} = \mathcal{D}_4 =< u, v : u^4 = v^2 = e, vu = u^3 v >$ *be the dihedral group and* $\Lambda = \{\mathfrak{a}_1, \mathfrak{a}_2\}$. *Consider a set valued function* $\varpi : \Lambda \to P(\mathcal{G})$ *defined as:*

$$\varpi(\mathfrak{e}) = \begin{cases} \{e, u, u^2, u^3\}, & \text{if } \mathfrak{e} = \mathfrak{a}_1, \\ \{e, u^2, v, u^2 v\}, & \text{if } \mathfrak{e} = \mathfrak{a}_2, \end{cases}$$

for all $\mathfrak{e} \in \Lambda$. *Then* (ϖ, Λ) *is a normal soft group over* $\mathcal{G}$. *Assume that* $X_1 = \{u, v, uv, u^2 v\}$ *and* $X_2 = \{e, u^2, u^3, u^3 v\}$, *then* $X_1 \cup X_2 = \mathcal{G}$. *By simple calculations:*

$$\underline{X_1}_{\varpi}(\mathfrak{a}_1) = \underline{X_2}_{\varpi}(\mathfrak{a}_1) = \emptyset \text{ and } \underline{X_1 \cup X_2}_{\varpi}(\mathfrak{a}_1) = \mathcal{G}.$$

This shows that, $\underline{X_1 \cup X_2}_{\varpi}(\mathfrak{a}_1) \nsubseteq \underline{X_1}_{\varpi}(\mathfrak{a}_1) \cup \underline{X_2}_{\varpi}(\mathfrak{a}_1)$. *Continuing with the same* X_1 *and* $X_2 = \{v, u^3\}$, *we have:*

$$\overline{X_1}^{\varpi}(\mathfrak{a}_2) = \overline{X_2}^{\varpi}(\mathfrak{a}_2) = \mathcal{G} \text{ and } \overline{X_1 \cap X_2}^{\varpi}(\mathfrak{a}_2) = \{e, v, u^2, u^2 v\}.$$

It implies that $\overline{X_1}^{\varpi}(\mathfrak{a}_2) \cap \overline{X_2}^{\varpi}(\mathfrak{a}_2) \nsubseteq \overline{X_1 \cap X_2}^{\varpi}(\mathfrak{a}_2)$.

Example 5.1.4. *Let* $\mathcal{G} = \mathbf{V}_4 = \{1_{\mathcal{G}}, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ *be the Klien-4 group and* $\Lambda = \{l, m\}$. *Consider the following normal soft groups* $\varpi_1 : \Lambda \to P(\mathcal{G})$ *and* $\varpi_2 : \Lambda \to P(\mathcal{G})$ *defined by:*

$$\varpi_1(\mathfrak{e}) = \begin{cases} \{1_{\mathcal{G}}, \mathfrak{a}\}, & \text{if } \mathfrak{e} = l, \\ \{1_{\mathcal{G}}, \mathfrak{c}\}, & \text{if } \mathfrak{e} = m, \end{cases} \quad \text{and } \varpi_2(\mathfrak{e}) = \begin{cases} \{1_{\mathcal{G}}, \mathfrak{b}\}, & \text{if } \mathfrak{e} = l, \\ \{1_{\mathcal{G}}\}, & \text{if } \mathfrak{e} = m, \end{cases}$$

for all $\mathfrak{e} \in \Lambda$. *By Definition 1.4.4,* $\vartheta_1 : \Lambda \to P(\mathcal{G})$ *is obtained as:*

$$\vartheta_1(\mathfrak{e}) = \{1_{\mathcal{G}}\}, \textit{ for all } \mathfrak{e} \in \Lambda.$$

Assume that $X = \{e, c\}$. *Using the Definition 5.1.1 of soft approximation spaces, the following results can be obtained:*

$$\underline{X}_{\varpi_1}(\mathfrak{e}) = \underline{X}_{\varpi_2}(\mathfrak{e}) = \emptyset, \quad \underline{X}_{\vartheta_1}(\mathfrak{e}) = X, \quad \overline{X}^{\varpi_1}(\mathfrak{e}) = \overline{X}^{\varpi_2}(\mathfrak{e}) = \mathcal{G} \textit{ and } \overline{X}^{\vartheta_1}(\mathfrak{e}) = X.$$

Hence, $\underline{X}_{\vartheta_1}(\mathfrak{e}) \nsubseteq \underline{X}_{\varpi_1}(\mathfrak{e}) \cap \underline{X}_{\varpi_2}(\mathfrak{e})$ *and* $\overline{X}^{\varpi_1}(\mathfrak{e}) \cap \overline{X}^{\varpi_2}(\mathfrak{e}) \nsubseteq \overline{X}^{\vartheta_1}(\mathfrak{e})$.

Note that, Theorem 5.1.5 illustrates that if X is a subgroup of $\mathcal{G}$, then the soft approximation spaces of a subgroup do not provide us any new information with a non-null lower soft approximation space.

Theorem 5.1.5. *Let* X *be a subgroup of* $\mathcal{G}$ *such that* $\underline{X}_{\varpi}(\mathfrak{e}) \neq \emptyset$, *for some* $\mathfrak{e} \in \Lambda$. *Then, following equalities hold:*

$$\underline{X}_{\varpi}(\mathfrak{e}) = X = \overline{X}^{\varpi}(\mathfrak{e}).$$

for all $\mathfrak{e} \in \Lambda$.

Proof. We claim that $1_{\mathcal{G}} \in \underline{X}_{\varpi}(\mathfrak{e})$, $\mathfrak{e} \in \Lambda$. By assumption on $\underline{X}_{\varpi}(\mathfrak{e})$, there exists $g \in \underline{X}_{\varpi}(\mathfrak{e})$. By Proposition 5.1.2 (1), it follows that $g \in X$. Now:

$$\varpi(\mathfrak{e}) = g^{-1}.g\varpi(\mathfrak{e}) \subseteq X.X = X.$$

Recall that X is a subgroup. Thus, $1_{\mathcal{G}} \in \underline{X}_{\varpi}(\mathfrak{e})$ and this proves the claim.

Note that $\underline{X}_{\varpi}(\mathfrak{e}) \subseteq X \subseteq \overline{X}^{\varpi}(\mathfrak{e})$, for all $\mathfrak{e} \in \Lambda$ (see Proposition 5.1.2 (1)). Let $x \in \overline{X}^{\varpi}(\mathfrak{e})$, where $\mathfrak{e} \in \Lambda$. Then, there exists $u \in \mathcal{G}$ such that $u \in x\varpi(\mathfrak{e}) \cap X$. It yields that $u\varpi(\mathfrak{e}) = x\varpi(\mathfrak{e})$ and $u \in X$. We claim that $u\varpi(\mathfrak{e}) \subseteq X$. Since $\varpi(\mathfrak{e}) \subseteq X$, then $u\varpi(\mathfrak{e}) \subseteq uX = X$. This proves the claim. Hence, $x \in \underline{X}_{\varpi}(\mathfrak{e})$ for all $\mathfrak{e} \in \Lambda$. Therefore, $\underline{X}_{\varpi}(\mathfrak{e}) = X = \overline{X}^{\varpi}(\mathfrak{e})$, for all $\mathfrak{e} \in \Lambda$. □

The following Theorem shows that the Propositions from 3.1-3.4 in [22] are weak.

Theorem 5.1.6. *Let $\mathcal{W}$ and $\mathcal{N}$ be normal subgroups of $\mathcal{G}$. Suppose that $\underline{\mathcal{W}}_{\mathcal{N}} \neq \emptyset$. Then,*

$$\underline{\mathcal{W}}_{\mathcal{N}} = \mathcal{W} = \overline{\mathcal{W}}^{\mathcal{N}}$$

Proof. Since $\underline{\mathcal{W}}_{\mathcal{N}} \subseteq \mathcal{W} \subseteq \overline{\mathcal{W}}^{\mathcal{N}}$ (see Proposition 2.1 (1) of [22]). From hypothesis $\underline{\mathcal{W}}_{\mathcal{N}} \neq \emptyset$, there exists $q \in \mathcal{G}$ such that $q \in \underline{\mathcal{W}}_{\mathcal{N}}$. By Proposition 2.1 (1) in [22], we have $q \in \mathcal{W}$. Hence,

$$\mathcal{N} = 1_{\mathcal{G}}\mathcal{N} = (q.q^{-1})\mathcal{N} = (q\mathcal{N}).(q^{-1}\mathcal{N}) \subseteq \mathcal{W}.\mathcal{W} \subseteq \mathcal{W} \tag{5.1.1}$$

Let $x \in \overline{\mathcal{W}}^{\mathcal{N}}$. From Definition of upper approximation in [22] at page 204, we have $y \in x\mathcal{N} \cap \mathcal{W}$ for some $y \in \mathcal{G}$. Hence, Equation 5.1.1 imply that

$$y\mathcal{N} = (y.1_{\mathcal{G}})\mathcal{N} = y\mathcal{N}.e\mathcal{N} \subseteq y\mathcal{N}.\mathcal{W} \subseteq \mathcal{W}$$

But $y\mathcal{N} = x\mathcal{N}$. Thus $x\mathcal{N} \subseteq \mathcal{W}$. Hence, $x \in \underline{\mathcal{W}}_{\mathcal{N}}$. This completes the proof. □

In the rest of this chapter, (ϖ, Λ) will be denoting a normal soft group over $\mathcal{G}$. In order to discover the relationship between soft approximation spaces of the product of two subsets of $\mathcal{G}$ and restricted soft product of approximations of these sets, the following results are presented.

Proposition 5.1.7. *Let $\emptyset \neq X_1, X_2 \subseteq \mathcal{G}$. Then,*

(1) $(\overline{X_1X_2}^{\varpi}, \Lambda) \widetilde{=} (\overline{X_1}^{\varpi}, \Lambda) \hat{\circ} (\overline{X_2}^{\varpi}, \Lambda)$.

(2) $(\underline{X_1}_{\varpi}, \Lambda) \hat{\circ} (\underline{X_2}_{\varpi}, \Lambda) \widetilde{\subseteq} (\underline{X_1X_2}_{\varpi}, \Lambda)$.

where $\hat{\circ}$ represents the restricted soft product of two soft groups (see Definition 1.4.8).

Proof. (1) Suppose that $x \in \overline{X_1}^{\varpi}(\mathfrak{e}) \cdot \overline{X_2}^{\varpi}(\mathfrak{e})$ and $\mathfrak{e} \in \Lambda$, then $x = uv$ for some $u \in \overline{X_1}^{\varpi}(\mathfrak{e})$ and $v \in \overline{X_2}^{\varpi}(\mathfrak{e})$. There exist $x_1, x_2 \in \mathcal{G}$ such that $x_1 \in u\varpi(\mathfrak{e}) \cap X_1$ and $x_2 \in v\varpi(\mathfrak{e}) \cap X_2$.

This implies that $x_1x_2 \in uv\varpi(\mathfrak{e})$ and $x_1x_2 \in X_1X_2$. This proves that $x_1x_2 \in uv\varpi(\mathfrak{e}) \cap X_1X_2$. Hence, $x = uv \in \overline{X_1X_2}^{\varpi}(\mathfrak{e})$ for all $\mathfrak{e} \in \Lambda$.

Conversely, assume that $z \in \overline{X_1X_2}^{\varpi}(\mathfrak{e})$ with $\mathfrak{e} \in \Lambda$. Hence, $g \in z\varpi(\mathfrak{e}) \cap X_1X_2$ for some $g \in \mathcal{G}$. From hypothesis, $z \in g\varpi(\mathfrak{e})$ and $g = l_1l_2$ for some $l_1 \in X_1$ and $l_2 \in X_2$. Thus, $z \in (l_1l_2)\varpi(\mathfrak{e}) = (l_1\varpi(\mathfrak{e}))(l_2\varpi(\mathfrak{e}))$. Let $z = p_1p_2$, for some $p_1 \in l_1\varpi(\mathfrak{e})$ and $p_2 \in l_2\varpi(\mathfrak{e})$. Then, $l_1 \in p_1\varpi(\mathfrak{e})$ and $l_2 \in p_2\varpi(\mathfrak{e})$. Thus, $l_1 \in p_1\varpi(\mathfrak{e}) \cap X_1$ and $l_2 \in p_2\varpi(\mathfrak{e}) \cap X_2$ which yields that $p_1 \in \overline{X_1}^{\varpi}(\mathfrak{e})$ and $p_2 \in \overline{X_2}^{\varpi}(\mathfrak{e})$. Hence, $z = p_1p_2 \in \overline{X_1}^{\varpi}(\mathfrak{e}) \cdot \overline{X_2}^{\varpi}(\mathfrak{e})$ for all $\mathfrak{e} \in \Lambda$.

(2) Let $z \in \underline{X_1}_{\varpi}(\mathfrak{e}) \cdot \underline{X_2}_{\varpi}(\mathfrak{e})$, where $\mathfrak{e} \in \Lambda$. Then, $z = uv$ for some $u \in \underline{X_1}_{\varpi}(\mathfrak{e})$ and $v \in \underline{X_2}_{\varpi}(\mathfrak{e})$. Hence, $u\varpi(\mathfrak{e}) \subseteq X_1$ and $v\varpi(\mathfrak{e}) \subseteq X_2$. It implies that $(uv)\varpi(\mathfrak{e}) = (u\varpi(\mathfrak{e}))(v\varpi(\mathfrak{e})) \subseteq X_1X_2$. This proves that $z = uv \in \underline{X_1X_2}_{\varpi}(\mathfrak{e})$, for all $\mathfrak{e} \in \Lambda$. □

The following Example illustrates that the sign of soft inclusion $\widetilde{\subseteq}$ in above Proposition (2) cannot be replaced with the sign of soft equality $\widetilde{=}$.

Example 5.1.8. *Let $\mathcal{G} = \mathcal{S}_3$ and $\Lambda = \{m, n\}$. Define a normal soft group (ϖ, Λ) over G as follows:*

$$\varpi(\mathfrak{e}) = \begin{cases} \mathcal{S}_3, & \text{if } \mathfrak{e} = m, \\ \{e, (123), (132)\}, & \text{if } \mathfrak{e} = n \end{cases}$$

for all $\mathfrak{e} \in \Lambda$. Assume that $X_1 = \{e, (12)\}$ and $X_2 = \{(12), (13), (23)\}$. Then $X_1X_2 = \mathcal{G}$. By Definition 5.1.1 of soft lower approximation, we have:

$$\underline{X_1X_2}_{\varpi}(n) = \mathcal{G}, \underline{X_1}_{\varpi}(n) = \emptyset \text{ and } \underline{X_2}_{\varpi}(n) = X_2.$$

This shows that $\underline{X_1}_{\varpi}(n) \cdot \underline{X_2}_{\varpi}(n) \not\supseteq \underline{X_1X_2}_{\varpi}(n)$.

The following Lemma 5.1.9 will be helpful to prove some more useful results in the sequel.

Lemma 5.1.9. *Let $\emptyset \neq X \subseteq \mathcal{G}$. Then, $\overline{X}^{\varpi}(\mathfrak{e}) = X.\varpi(\mathfrak{e})$ for all $\mathfrak{e} \in \Lambda$.*

Proof. Let $x \in \overline{X}^{\varpi}(\mathfrak{e})$, $\mathfrak{e} \in \Lambda$. Then, there exists $g \in \mathcal{G}$ such that $g \in x.\varpi(\mathfrak{e}) \cap X$. According to our assumption on (ϖ, Λ), $x \in g\varpi(\mathfrak{e}) \subseteq X.\varpi(\mathfrak{e})$ for all $\mathfrak{e} \in \Lambda$. Thus $\overline{X}^{\varpi}(\mathfrak{e}) \subseteq X.\varpi(\mathfrak{e})$ for all $\mathfrak{e} \in \Lambda$.

Conversely, assume that $g \in X.\varpi(\mathfrak{e})$ for $\mathfrak{e} \in \Lambda$. Then $g = l.u$ for some $l \in X$ and $u \in \varpi(\mathfrak{e})$. It follows that $l = gu^{-1} \in g\varpi(\mathfrak{e})$ and $l \in X$. Therefore, $l \in g\varpi(\mathfrak{e}) \cap X$. Thus $g \in \overline{X}^{\varpi}(\mathfrak{e})$ for $\mathfrak{e} \in \Lambda$. This completes the proof. □

Since the restricted soft product of two normal soft groups is a normal soft group (see Corollary 6.10 of [4]). Hence, the following results are presented:

Proposition 5.1.10. *Let (υ_1, Ξ_1) and (υ_2, Ξ_2) be two normal soft groups over $\mathcal{G}$, where $\emptyset \neq \Xi_1, \Xi_2 \subseteq$ Æ. Assume that $\emptyset \neq X \subseteq \mathcal{G}$, then:*

(1) $(\overline{X}^{\upsilon_1}, \Xi_1)\hat{\circ}(\overline{X}^{\upsilon_2}, \Xi_2)\widetilde{=}(\overline{X}^{\varepsilon}, \Xi_3)\hat{\circ}(\overline{X}^{\varepsilon}, \Xi_3)$.

(2) If X is a subgroup of $\mathcal{G}$. Then

$$(\overline{X}^{\varepsilon}, \Xi_3)\widetilde{=}(\overline{X}^{\upsilon_1}, \Xi_1)\hat{\circ}(\overline{X}^{\upsilon_2}, \Xi_2)\widetilde{=}(\overline{X}^{\upsilon_2}, \Xi_2)\hat{\circ}(\upsilon_1, \Xi_1)\widetilde{=}(\overline{X}^{\upsilon_1}, \Xi_1)\hat{\circ}(\upsilon_2, \Xi_2),$$

where $(\varepsilon, \Lambda_3) = (\upsilon_1, \Xi_1)\hat{\circ}(\upsilon_2, \Xi_2)$ and $\Xi_3 = \Xi_1 \cap \Xi_2 \neq \emptyset$ (See Definition 6.1 [4]).

Proof. (1) Let $\mathfrak{e} \in \Xi_3$. From Lemma 5.1.9 and Definition 6.1, Theorem 6.6 and and Corollary

6.9 of [4], we have:

$$\begin{aligned}
\overline{X}^{\upsilon_1}(\mathfrak{e})\overline{X}^{\upsilon_2}(\mathfrak{e}) &= (X\upsilon_1(\mathfrak{e}))(X\upsilon_2(\mathfrak{e}))\\
&= (X\upsilon_1(\mathfrak{e})\upsilon_1(\mathfrak{e}))(X\upsilon_2(\mathfrak{e})\upsilon_2(\mathfrak{e}))\\
&= (X\upsilon_1(\mathfrak{e})\upsilon_1(\mathfrak{e}))(\upsilon_2(\mathfrak{e})X)\upsilon_2(\mathfrak{e})\\
&= X\upsilon_1(\mathfrak{e})(\upsilon_1(\mathfrak{e})\upsilon_2(\mathfrak{e}))X\upsilon_2(\mathfrak{e})\\
&= X\upsilon_1(\mathfrak{e})(\upsilon_2(\mathfrak{e})\upsilon_1(\mathfrak{e}))X\upsilon_2(\mathfrak{e})\\
&= (X\upsilon_1(\mathfrak{e})\upsilon_2(\mathfrak{e}))(X\upsilon_1(\mathfrak{e})\upsilon_2(\mathfrak{e}))\\
&= (X\varepsilon(\mathfrak{e}))(X\varepsilon(\mathfrak{e}))\\
&= \overline{X}^{\varepsilon}(\mathfrak{e})\overline{X}^{\varepsilon}(\mathfrak{e})
\end{aligned}$$

Hence, $(\overline{X}^{\upsilon_1}, \Xi_1)\hat{\circ}(\overline{X}^{\upsilon_2}, \Xi_2)\widetilde{=}(\overline{X}^{\varepsilon}, \Xi_3)\hat{\circ}(\overline{X}^{\varepsilon}, \Xi_3)$.

(2) By Lemma 5.1.9, $\overline{X}^{\upsilon_1}(\mathfrak{e})\upsilon_2(\mathfrak{e}) = X\upsilon_1(\mathfrak{e})\upsilon_2(\mathfrak{e}) = X\varepsilon(\mathfrak{e}) = \overline{X}^{\varepsilon}(\mathfrak{e})$. Similarly, $\overline{X}^{\upsilon_2}(\mathfrak{e})\upsilon_1(\mathfrak{e}) = \overline{X}^{\varepsilon}(\mathfrak{e})$. Therefore, $\overline{X}^{\varepsilon}(\mathfrak{e}) = \overline{X}^{\upsilon_2}(\mathfrak{e})\upsilon_1(\mathfrak{e}) \cap \overline{X}^{\upsilon_1}(\mathfrak{e})\upsilon_2(\mathfrak{e})$ for all $\mathfrak{e} \in \Xi_3$. Thus,

$$(\overline{X}^{\varepsilon}, \Xi_3)\widetilde{=}(\overline{X}^{\upsilon_2}, \Xi_2)\hat{\circ}(\upsilon_1, \Xi_1) \Cap (\overline{X}^{\upsilon_1}, \Xi_1)\hat{\circ}(\upsilon_2, \Xi_2).$$

This completes the proof. □

It notable that, (2) in above Proposition is inconsistent with Proposition 3.5 and 3.6 of [22].

5.2 Connection between soft lower and upper approximation spaces

Let $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ be a group homomorphism. In this section, (ϖ_1, Λ_1) and (ϖ_2, Λ_2) will represent the normal soft groups over $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively, where $\emptyset \neq \Lambda_i \subseteq$ Æ for all

$i = 1, 2$. In the following, the soft image and soft pre-image of (ϖ_1, Λ_1) and (ϖ_2, Λ_2) are defined under ϕ respectively.

Definition 5.2.1. *Let ϕ be as defined above. Then,*

(i) The soft image $\phi(\varpi_1) : \Lambda_1 \to P(\mathcal{G}_2)$ of (ϖ_1, Λ_1) is defined as $\phi(\varpi_1)(\mathfrak{w}) = \phi(\varpi_1(\mathfrak{w}))$, for all $\mathfrak{w} \in \Lambda_1$.

(ii) The soft pre-image $\phi^{-1}(\varpi_2) : \Lambda_2 \to P(\mathcal{G}_1)$ of (ϖ_2, Λ_2) is defined as $\phi^{-1}(\varpi_2)(\mathfrak{e}) = \phi^{-1}(\varpi_2(\mathfrak{e}))$, for all $\mathfrak{e} \in \Lambda_2$.

Lemma 5.2.2. *The soft pre-image $(\phi^{-1}(\varpi_2), \Lambda_2)$ of (ϖ_2, Λ_2) is a normal soft group over $\mathcal{G}_1$ and $\ker \phi \subseteq \phi^{-1}(\varpi_2)(\mathfrak{e})$, for all $\mathfrak{e} \in \Lambda_2$.*

Proof. Following the same steps as in [38, Proposition 9], the proof is elementary. □

Theorem 5.2.3. *With the above notion, let $\emptyset \neq X_1 \subseteq \mathcal{G}_1$, $\emptyset \neq X_2 \subseteq \mathcal{G}_2$ and $\mathfrak{e} \in \Lambda_2$. Then, the following implications are true:*

$$x \in \overline{X_1}^{\phi^{-1}(\varpi_2)}(\mathfrak{e}) \Rightarrow \phi(x) \in \overline{\phi(X_1)}^{\varpi_2}(\mathfrak{e}).$$

$$x \in \overline{\phi^{-1}(X_2)}^{\phi^{-1}(\varpi_2)}(\mathfrak{e}) \Rightarrow \phi(x) \in \overline{X_2}^{\varpi_2}(\mathfrak{e}).$$

Moreover, if ϕ is onto then the converse of above statements also hold.

Proof. Let $x \in \overline{X_1}^{\phi^{-1}(\varpi_2)}(\mathfrak{e})$. By Definition 5.1.1, there exists $g \in \mathcal{G}_1$ such that $g \in x \cdot \phi^{-1}[\varpi_2(\mathfrak{e})] \cap X_1$. Since ϕ is a homomorphism and $\phi(\phi^{-1}[Z]) \subseteq Z$, for any $Z \subseteq \mathcal{G}_2$. It follows that

$$\phi(g) \in \phi(x \cdot \phi^{-1}[\varpi_2(\mathfrak{e})] \cap X_1) \subseteq \phi(x \cdot \phi^{-1}[\varpi_2(\mathfrak{e})]) \cap \phi(X_1) \subseteq \phi(x)\varpi_2(\mathfrak{e}) \cap \phi(X_1).$$

This completes the proof of first implication. The second implication can be proved by following the same methodology.

For converse, suppose that ϕ is onto and $\phi(x) \in \overline{\phi(X_1)}^{\varpi_2}(\mathfrak{e})$. There exists $g' \in \mathcal{G}_2$ such that $g' \in \phi(x)\varpi_2(\mathfrak{e}) \cap \phi(X_1)$. It concludes that $g' = \phi(x) \cdot u = \phi(y)$, for some $u \in \varpi_2(\mathfrak{e})$ and $y \in X_1$. Since ϕ is onto, assume that $u = \phi(v)$, where $v \in \mathcal{G}_1$. Then, $\phi(y) = \phi(x)\phi(v) = \phi(xv)$. By Lemma 5.2.2, it follows that $y^{-1}(xv) \in \ker\phi \subseteq \phi^{-1}(\varpi_2(\mathfrak{e}))$ and hence $x^{-1}y \in \phi^{-1}(\varpi_2(\mathfrak{e}))$. Then, $y \in x\phi^{-1}(\varpi_2(\mathfrak{e})) \cap X_1$. This proves the first implication. The other implication can be proved along the similar lines. □

Theorem 5.2.4. *Fix the notion of Theorem 5.2.3, assume that ϕ is onto. Then the following statements hold:*

$$x \in \underline{\phi^{-1}(X_2)}_{\phi^{-1}(\varpi_2)}(\mathfrak{e}) \Rightarrow \phi(x) \in \underline{X_2}_{\varpi_2}(\mathfrak{e}).$$

$$x \in \underline{X_1}_{\phi^{-1}(\varpi_2)}(\mathfrak{e}) \Rightarrow \phi(x) \in \underline{\phi(X_1)}_{\varpi_2}(\mathfrak{e}).$$

Proof. Since ϕ is onto, then $\phi(\phi^{-1}(\varpi_2(\mathfrak{e}))) = \varpi_2(\mathfrak{e})$. Hence, $x(\phi^{-1}(\varpi_2)(\mathfrak{e})) \subseteq \phi^{-1}(X_2)$ implies that $\phi(x)\varpi_2(\mathfrak{e}) \subseteq X_2$. Also, $x(\phi^{-1}(X_2)(\mathfrak{e})) \subseteq X_1$ implies that $\phi(x)\varpi_2(\mathfrak{e}) \subseteq \phi(X_1)$. So, the claims are proved. □

In the Example 5.2.5, it is shown that the assertions of Theorem 5.2.4 fail if ϕ is not onto.

Example 5.2.5. *Let $\mathcal{G}_1 = \mathcal{S}_3$ and $\mathcal{G}_2 = \mathbb{Z}_6$. Define a group homomorphism $\phi : \mathcal{S}_3 \to \mathbb{Z}_6$ by:*

$$\phi(x) = \begin{cases} \overline{0}, & \textit{if } x = e, (123), (132), \\ \overline{3}, & \textit{if } x = (12), (13), (23), \end{cases}$$

for all $x \in \mathcal{G}_1$. One can see that ϕ is not onto. Assume that $\Lambda_2 = \mathbb{Z}_6$. Define a normal soft group $\varpi_2 : \Lambda_2 \to P(\mathcal{G}_2)$ as follows:

$$\varpi_2(\mathfrak{e}) = \begin{cases} \mathbb{Z}_6, & \textit{if } \mathfrak{e} = \overline{0}, \\ \{\overline{0}\}, & \textit{if } \mathfrak{e} = \overline{1}, \overline{5}, \\ \{\overline{0}, \overline{3}\}, & \textit{if } \mathfrak{e} = \overline{2}, \overline{4}, \\ \{\overline{0}, \overline{2}, \overline{4}\}, & \textit{if } \mathfrak{e} = \overline{3}, \end{cases}$$

for all $\mathfrak{e} \in \Lambda_2$*. By Definition 5.2.1 of* $\phi^{-1}(\varpi_2)$*, we get:*

$$\phi^{-1}(\varpi_2)(\mathfrak{e}) = \begin{cases} \mathcal{S}_3, & \text{if } \mathfrak{e} = \overline{0}, \overline{2}, \overline{4}, \\ \{e, (123), (132)\}, & \text{if } \mathfrak{e} = \overline{1}, \overline{5}, \overline{3}, \end{cases}$$

for all $\mathfrak{e} \in \Lambda_2$*. Suppose that* $X_2 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$*,* $\phi^{-1}(X_2) = \mathcal{S}_3$*. By simple calculations, it follows that:*

$$\underline{\phi^{-1}(X_2)}_{\phi^{-1}(\varpi_2)}(\overline{0}) = \mathcal{S}_3 \text{ and } \underline{X_2}_{\varpi_2}(\overline{0}) = \emptyset.$$

Now, assume that $X_1 = \{(12), (13), (23), (123)\}$ *then* $\phi(X_1) = \{\overline{0}, \overline{3}\}$*. From Definition 5.1.1,*

$$\underline{X_1}_{\phi^{-1}(\varpi_2)}(\overline{3}) = \{(12), (13), (23)\} \text{ and } \underline{\phi(X_1)}_{\varpi_2}(\overline{3}) = \emptyset.$$

In the next result, converse of Theorem 5.2.4 is proved under some conditions.

Theorem 5.2.6. *With the same assumptions as in Theorem 5.2.3, the following assertion holds:*

$$\phi(x) \in \underline{X_2}_{\varpi_2}(\mathfrak{e}) \Rightarrow x \in \underline{\phi^{-1}(X_2)}_{\phi^{-1}(\varpi_2)}(\mathfrak{e}).$$

Further, if X_1 *is a subgroup of* $\mathcal{G}_1$ *such that* $\ker\phi \subseteq X_1$*. Then:*

$$\phi(x) \in \underline{\phi(X_1)}_{\varpi_2}(\mathfrak{e}) \Rightarrow x \in \underline{X_1}_{\phi^{-1}(\varpi_2)}(\mathfrak{e}).$$

Proof. Let $\phi(x) \in \underline{X_2}_{\varpi_2}(\mathfrak{e})$. By Definition 5.1.1, it follows that $\phi(x)\varpi_2(\mathfrak{e}) \subseteq X_2$. Suppose that $v \in \phi^{-1}(\varpi_2(\mathfrak{e}))$. Then, $\phi(v) \in \varpi_2(\mathfrak{e})$. It induces that $\phi(xv) = \phi(x)\phi(v) \in X_2$. Then $xv \in \phi^{-1}(X_2)$. This proves that $x\phi^{-1}(\varpi_2(\mathfrak{e})) \subseteq \phi^{-1}(X_2)$ and $x \in \underline{\phi^{-1}(X_2)}_{\phi^{-1}(\varpi_2)}(\mathfrak{e})$.

Now, assume that $\phi(x) \in \underline{\phi(X_1)}_{\varpi_2}(\mathfrak{e})$ then $\phi(x)\varpi_2(\mathfrak{e}) \subseteq \phi(X_1)$. It implies that

$$\phi(x)u \in \phi(X_1), \text{ for all } u \in \varpi_2(\mathfrak{e}). \tag{5.2.1}$$

Let $l \in x(\phi^{-1}(\varpi_2)(\mathfrak{e}))$ be an arbitrary element. Then, $l = xv$ such that $v \in \phi^{-1}(\varpi_2)(\mathfrak{e}) = \phi^{-1}(\varpi_2(\mathfrak{e}))$. It implies that $\phi(v) \in \varpi_2(\mathfrak{e})$ and $\phi(l) = \phi(xv) \in \phi(X_1)$, see Equation (5.2.1). So, it can be written as $\phi(xv) = \phi(y)$ for some $y \in X_1$. Then, $y^{-1}(xv) \in \ker\phi \subseteq X_1$. It yields that $l = xv \in X$. Hence, $x\phi^{-1}(\varpi_2(\mathfrak{e})) \subseteq X_1$. □

To build a relationship between soft approximation spaces with respect to soft-image $(\phi(\kappa_1), \Lambda_1)$, the following Lemma 5.2.7 will be used.

Lemma 5.2.7. *If ϕ is onto, then the soft image $(\phi(\varpi_1), \Lambda_1)$ of (ϖ_1, Λ_1) is a normal soft group over $\mathcal{G}_2$.*

Proof. The proof is analogous to the proof of [38, Proposition 2.1]. □

Remark 5.2.8. *In the view of Lemma 5.2.7, ϕ will be taken as an onto homomorphism in the remaining results of this section.*

Theorem 5.2.9. *With the same notion as in Theorem 5.2.3, let $\mathfrak{w} \in \Lambda_1$. Then:*

$$x \in \overline{X_1}^{\varpi_1}(\mathfrak{w}) \Rightarrow \phi(x) \in \overline{\phi(X_1)}^{\phi(\varpi_1)}(\mathfrak{w}).$$

$$x \in \overline{\phi^{-1}(X_2)}^{\varpi_1}(\mathfrak{w}) \Rightarrow \phi(x) \in \overline{X_2}^{\phi(\varpi_1)}(\mathfrak{w}).$$

Proof. This proof is parallel to the proof of Theorem 5.2.3. □

The converse of Theorem 5.2.9 is proved in the following result with some conditions.

Theorem 5.2.10. *With the previous notion, suppose that $\ker\phi \subseteq \varpi_1(\mathfrak{w})$. Then:*

$$\phi(x) \in \overline{\phi(X_1)}^{\phi(\varpi_1)}(\mathfrak{w}) \Rightarrow x \in \overline{X_1}^{\varpi_1}(\mathfrak{w}).$$

$$\phi(x) \in \overline{X_2}^{\phi(\varpi_1)}(\mathfrak{w}) \Rightarrow x \in \overline{\phi^{-1}(X_2)}^{\varpi_1}(\mathfrak{w}).$$

Proof. Let $\phi(x) \in \overline{\phi(X_1)}^{\phi(\varpi_1)}(\mathfrak{w})$. Then, there exists $g \in \mathcal{G}_2$ such that $g \in \phi(x)(\phi(\varpi_1)(\mathfrak{w})) \cap \phi(X_1)$. It suggest that $g \in \phi(x)(\phi(\varpi_1(\mathfrak{w})))$ and $g \in \phi(X_1)$. Therefore, $g = \phi(x)\phi(u) = \phi(xu) = \phi(y)$ for some $u \in \varpi_1(\mathfrak{w})$ and $y \in X_1$. Then $y^{-1}(xu) \in \ker\phi \subseteq \varpi_1(\mathfrak{w})$. Since (ϖ_1, Λ_1) is a normal soft group, it implies that $y^{-1}x \in \varpi_1(\mathfrak{w})$ and $y \in x\varpi_1(\mathfrak{w})$. This proves that $y \in x\varpi_1(\mathfrak{w}) \cap X_1 \neq \emptyset$ and $x \in \overline{X_1}^{\varpi_1}(\mathfrak{w})$. Similarly, the second implication can be proved. □

Example 5.2.11. *Let $\mathcal{G}_1 = \mathcal{C}_4 = \{\pm 1, \pm i\}$ and $\mathcal{G}_2 = \mathcal{C}_2 = \{\pm 1\}$ such that $i^2 = -1$. Define an onto group homomorphism $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ as:*

$$\phi(\mathfrak{g}) = \begin{cases} 1, & \text{if } \mathfrak{g} = 1, -1 \\ -1, & \text{if } \mathfrak{g} = i, -i \end{cases}$$

for all $\mathfrak{g} \in \mathcal{G}_1$. Let $\Lambda_1 = \{1, -1\}$ be a subset of $\mathcal{G}_1$. Consider a normal soft group $\varpi_1 : \Lambda_1 \to P(\mathcal{G}_1)$ with $\varpi_1(1) = \varpi_1(-1) = \{1\}$. Then,

$$(\phi(\varpi_1))(1) = (\phi(\varpi_1))(-1) = \{1\}.$$

Note that $\ker\phi = \{1, -1\} \nsubseteq \varpi_1(\mathfrak{w})$, for all $\mathfrak{w} \in \Lambda_1$. Let $X_1 = \{1, -1, i\}$. Then, $\phi(X_1) = \mathcal{G}_2$. By simple calculations, it can be proved that:

$$\overline{\phi(X_1)}^{\phi(\varpi_1)}(1) = \overline{\phi(X_1)}^{\phi(\varpi_1)}(-1) = \{1, -1\} \text{ and } \overline{X_1}^{\varpi_1}(1) = \overline{X_1}^{\varpi_1}(-1) = \{1, -1, i\}$$

It is clear that, $\phi(-i) = -1 \in \overline{\phi(X_1)}^{\phi(\varpi_1)}(\mathfrak{w})$ but $-i \notin \overline{X_1}^{\varpi_1}(\mathfrak{w})$, for all $\mathfrak{w} \in \Lambda_1$.

Theorem 5.2.12. *With the same notion as in Theorem 5.2.9, the following implications hold:*

$$x \in \underline{\phi^{-1}(X_2)}_{\varpi_1}(\mathfrak{w}) \Leftrightarrow \phi(x) \in \underline{X_2}_{\phi(\varpi_1)}(\mathfrak{w}).$$

$$x \in \underline{X_1}_{\varpi_1}(\mathfrak{w}) \Rightarrow \phi(x) \in \underline{\phi(X_1)}_{\phi(\varpi_1)}(\mathfrak{w}).$$

The converse of second implication is also true, if X_1 *is a subgroup of* $\mathcal{G}_1$ *and* $\ker\phi \subseteq X_1$.

Proof. This can be proved following the same methodology as in Theorems 5.2.4 and 5.2.6. □

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