Boundedness of p-Adic Integral Operators On Function Spaces

By

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT

OF THE REQUIREMENT FOR THE DEGREE OF

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IN

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Supervised By

Dr. Amjad Hussain

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2020

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Boundedness of p-Adic Integral Operators on Function Spaces

By

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We accept this thesis as conforming to the required standard

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Abstract

In this thesis, we focus on the boundedness of p-adic integral operators on p-adic function spaces. We prove the boundedness of p-adic generalized Hausdorff operator on weighted p-adic Herz and Morrey-type spaces. We also take into account the boundedness of commutator of same operator by considering symbol function either from central BMO spaces or Lipschitz spaces defined on the local field $\mathbb{Q}_p^n$. Also, we introduce p-adic analog of fractional Hausdorff operator and prove weak and strong type estimates for it and related commutators. We compute sharp weak bounds for Hardy operator and its adjoint operator. Furthermore, we establish the boundedness of weighted multilinear p-adic Hardy operators on product of Herz-type spaces. Most of these results (in the form of chapter in this thesis) has been published and remaining is under review in renowned journals of mathematical science.

Contents

Preface

The aim of this thesis is to study p-adic integral operators on p-adic function spaces. Our main results include the boundedness of some integral operators like the p-adic matrix Hausdorff operator, p-adic fractional Hausdorff operator, p-adic Hardy type operators and weighted multilinear p-adic Hardy operator on function spaces.

In Chapter 2, we come up with the boundedness of Hausdorff operator, defined by means of linear transformation A, on the weighted p-adic Morrey and weighted p-adic Herz type spaces. Also, by imposing some special conditions on A, we discuss the sharpness of the results presented in this Chapter. The contents of this Chapter has been published in [60].

In Chapter 3, we investigate the boundedness of commutators of matrix Hausdorff operator on the weighted p-adic Herz-Morrey space with the symbol functions in weighted central BMO and Lipschitz spaces. In addition, a result showing boundedness of Hausdorff operator on weighted p-adic λ-central BMO spaces is provided as well. The contents of this Chapter has been published in [95].

In Chapter 4, we prove the weak and strong boundedness of fractional Hausdorff operator and its commutator on weighted p-adic Lorentz spaces. The boundedness of commutators is made possible when the symbol function b is taken from Lipschitz class of functions. The contents of this Chapter has been published on arXiv [94] and submitted for publication in well reputed journal of mathematics.

In Chapter 5, we consider continuity properties of another important averaging operator defined on p-adic field, namely the p-adic Hardy operator. We computed the sharp weak bounds for p-adic Hardy-type operators on the weighted p-adic Lebesgue spaces $L^q(w : \mathbb{Q}_p^n)(1 < q < \infty)$. The contents of this Chapter has been published in [61].

Our study in Chapter 6 adds to and extends the results of Chapter 5 in two ways. Firstly, we prove the weak boundedness of p-adic fractional Hardy-type operators on the Lebesgue space $L^1(w : \mathbb{Q}_p^n)$, and in case of Hardy operator obtain the sharpness of weak bounds. Secondly, we give an intermediate result showing the sharp weak bounds for fractional Hardy operator on p-adic central Morrey space. The contents of this Chapter has been submitted for publication [63].

In Chapter 7, we consider the multilinear version of p-adic weighted Hardy operator. We studied the weighted boundedness of weighted multilinear p-adic Hardy operator on the product of Herz-type spaces. The contents of this Chapter are online on research gate [62] and will be submitted soon to some mathematical science journal.

Naqash Sarfraz
Islamabad, Pakistan
August 10, 2020

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Chapter 1
Introduction to Some Integral Operators on p-adic Linear Spaces

1.1 Introduction

Over the years, p-adic analysis has gained much attention because of its numerous applications in different fields of science, especially, in the field of mathematical physics (see, for instance, [4, 5, 22, 98]). Back in 1989, in the book [100], Vladimirov et al. proposed a formulation of p-adic quantum mechanics and introduced p-adic stochastic process, p-adic pseudo-differential operators and p-adic quantum theory. Most importantly, a systematic reconstruction of the well known p-adic Schwartz theory of distributions was made in this very book. Since the appearance of this monograph, the study of harmonic analysis on p-adic fields have taken the interest of many researchers which resulted in various generalizations in operator theory and function spaces. It has now become a key tool to describe the power decay law, the logarithmic decay law [5] and Kohlrausch-Williams-Watts law. It proved itself a natural base for development of various models of ultrametric diffusion energy landscape [4]. It also attracted a great deal of interest towards quantum mechanics [100], theoretical biology [23], quantum gravity [2, 8], string theory [89, 99] and spin glass theory [3, 90]. In [4], it was shown that the p-adic analysis may be efficiently applied both to relaxation in complex speed systems and processes combined with the relaxation of a complex environment. p-adic analysis has a vital role in p-adic pseudo-differential equations and stochastic process, see for example [68, 100]. Besides, in this day and age many researchers have shown plenty of interest in the study of harmonic and wavelet analysis, for instance, see [44, 45, 68].

Here, in this thesis, our aim is to study some integral operators on function spaces defined on the p-adic field. Therefore, in order to fulfill our objective, a brief introduction of such a field is mandatory at this stage.

1.2 Preliminaries

Suppose a symbol $\mathbb{Q}$ denotes the field of rational numbers. The absolute value $|x|$ of $x \in \mathbb{Q}$ satisfies the below properties:
(i) $|x| \geq 0$, $|x| = 0$ iff $x = 0$,
(ii) $|xy| = |x||y|$,
(iii) $|x + y| \leq |x| + |y|$.
Therefore, the function $|\cdot| : \mathbb{Q} \to \mathbb{R}$ is termed as norm. Differently, let p be a prime number, if a non-zero rational number x can be written in the form

$$x = \frac{s}{t}p^k,$$

where the integer $k = k(x) \in \mathbb{Z}$ and $s, t \in \mathbb{Z}$ are not multiples of p, then the function:

$$|\cdot|_p : \mathbb{Q} \setminus \{0\} \to \mathbb{R},$$

defined as:

$$|x|_p = \begin{cases} 0 & \text{if } x = 0, \\ p^{-k} & \text{if } x \neq 0. \end{cases}$$

satisfies all the axioms of a field norm with an additional property that:

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}, \tag{1.2.1}$$

and is commonly known as p-adic norm.

The field of p-adic numbers, denoted by $\mathbb{Q}_p$, is the completion of rational numbers with respect to the p-adic norm $|\cdot|_p$. A p-adic number $x \in \mathbb{Q}_p$ can be written in the formal power series as (see [100]):

$$x = p^\gamma \sum_{k=0}^{\infty} \alpha_k p^k, \tag{1.2.2}$$

where $\alpha_i, k \in \mathbb{Z}, \alpha_0 \neq 0, \alpha_i \in \{0, 1, 2, ..., p-1\}, i = 1, 2, \cdots$. The p-adic norm ensures the convergence of series (1.2.2) in $\mathbb{Q}_p$, because $|p^\gamma \beta_i p^i|_p \leq p^{-\gamma - i}$

The n-dimensional vector space $\mathbb{Q}_p^n$, $n \geq 1$, consists of tuples $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where $x_j \in \mathbb{Q}_p$ and $j = 1, 2, \ldots, n$. The norm on this space is given by

$$|\mathbf{x}|_p = \max_{1 \leq j \leq n} |x_j|_p.$$

In non-Archimedean geometry, the ball and and its boundary are defined, respectively, as:

$$B_k(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x} - \mathbf{a}|_p \leq p^k\},\ S_k(\mathbf{a}) = \{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x} - \mathbf{a}|_p = p^k\}.$$

We denote $B_k(\mathbf{0}) = B_k$ and $S_k(\mathbf{0}) = S_k$ for convenience. Additionally, for every $\mathbf{a}_0 \in \mathbb{Q}_p^n$, $\mathbf{a}_0 + B_k = B_k(\mathbf{a}_0)$ and $\mathbf{a}_0 + S_k = S_k(\mathbf{a}_0)$.

The geometry of space $\mathbb{Q}_p^n$ is different from the geometry of space $\mathbb{R}^n$, which follows from the non-Archimedean property. In particular, the Archimedean axiom is not true in $\mathbb{Q}_p$. For two different balls in non-Archimedean geometry, either they are disjoint or one is contained in the other.

The local compactness and commutativity of the group $\mathbb{Q}_p^n$ under addition implies the existence of Haar measure $d\mathbf{x}$ on $\mathbb{Q}_p^n$, such that

$$\int_{B_0} d\mathbf{x} = |B_0| = 1,$$

where the notation $|B|$ refers to the Haar measure of a measurable subset B of $\mathbb{Q}_p^n$. Also, it is not difficult to show that $|B_k(\mathbf{a})| = p^{nk}$, $|S_k(\mathbf{a})| = p^{nk}(1 - p^{-n})$, for any $\mathbf{a} \in \mathbb{Q}_p^n$.

1.3 Some p-adic Function Spaces

Next, we highlight some definitions of p-adic weighted function spaces which will be used in the upcoming chapters. Taking the weight functions to be equal to unity, we get the definitions of p-adic function spaces. Therefore, we might give the definition of weight function first.

Definition 1.3.1 *([16]) Let $\beta \in \mathbb{R}$. The set of all nonnegative locally integrable function $w(x)$ on $\mathbb{Q}_p^n$ is represented by $\mathbb{W}_\beta$ and undergoes the below properties:*
(a) $0 < w(\mathbf{x})$ *a.e.*,
(b) $\infty > \int_{S_0} w(\mathbf{x})d\mathbf{x}$,
(c) $w(t\mathbf{x}) = |t|_p^\beta w(\mathbf{x})$, *for all* $\mathbf{x} \in \mathbb{Q}_p^n$ *and* $t \in \mathbb{Q}_p \backslash \{0\}$.

A symbol $w(E)$ defines the weighted measure of a measurable subset $E \subset \mathbb{Q}_p^n$, that is $w(E) = \int_E w(x)dx$. Clearly a weight $w(\mathbf{x}) \in \mathbb{W}_\beta$ needs not to be necessarily locally integrable function. Importantly, if $w(\mathbf{x}) = |\mathbf{x}|_p^\beta$, then $w(\mathbf{x}) \in \mathbb{W}_\beta$ but $w(\mathbf{x}) \in L^1_{\text{loc}}(\mathbb{Q}_p^n)$ if and only if $\beta > -n$.

Definition 1.3.2 *Suppose* $w(\mathbf{x})$ *is a weight function on* $\mathbb{Q}_p^n$. *The p-adic space* $L^q(w;\mathbb{Q}_p^n)$, $(0 < q < \infty)$ *is the set of all measurable functions* $f\colon \mathbb{Q}_p^n \to \mathbb{R}$ *satisfying*

$$\|f\|_{L^q(w;\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |f(\mathbf{x})|^q w(\mathbf{x}) d\mathbf{x}\right)^{\frac{1}{q}} < \infty.$$

Now, we define weighted p-adic weak Lebesgue space as

Definition 1.3.3 *Suppose* $w(\mathbf{x})$ *is a weight function on* $\mathbb{Q}_p^n$. *A measurable function* f *is in weighted weak Lebesgue space* $L^{q,\infty}(w,\mathbb{Q}_p^n)$ *if:*

$$\|f\|_{L^{q,\infty}(w,\mathbb{Q}_p^n)} = \sup_{\lambda>0} \lambda w\Big(\{\mathbf{x}\in\mathbb{Q}_p^n : |f(\mathbf{x})| > \lambda\}\Big)^{1/q} < \infty,$$

where

$$w(\{\mathbf{x}\in\mathbb{Q}_p^n : |f(\mathbf{x})| > \lambda\}) = \int_{\{\mathbf{x}\in\mathbb{Q}_p^n : |f(\mathbf{x})|>\lambda\}} w(\mathbf{x}) d\mathbf{x}.$$

When $w = 1$, the above space is reduced to weak Lebesgue space defined on p-adic linear space [103].

1.3.1 Morrey-type Spaces on $\mathbb{Q}_p^n$

Definition 1.3.4 *Suppose* w *is a weight function on* $\mathbb{Q}_p^n$, $\lambda \geq -\frac{1}{q}$, *where* $1 \leq q < \infty$. *The p-adic space* $L^{q,\lambda}(w;\mathbb{Q}_p^n)$ *is the set of all measurable functions* $f\colon \mathbb{Q}_p^n \to \mathbb{R}$ *satisfying*

$$L^{q,\lambda}(w;\mathbb{Q}_p^n) = \{f \in L^q_{\mathrm{loc}}(w;\mathbb{Q}_p^n) : \| f \|_{L^{q,\lambda}(w;\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{L^{q,\lambda}(w;\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z},\mathbf{a}\in\mathbb{Q}_p^n} \left(\frac{1}{w(B_\gamma(\mathbf{a}))^{\lambda q+1}} \int_{B_\gamma(\mathbf{a})} |f(\mathbf{x})|^q w(\mathbf{x}) d\mathbf{x}\right)^{1/q}.$$

Obviously $L^{q,-1/q}(w;\mathbb{Q}_p^n) = L^q(w;\mathbb{Q}_p^n)$, $L^{q,0}(w;\mathbb{Q}_p^n) = L^\infty(w;\mathbb{Q}_p^n)$. When $\lambda > 0$, the Morrey space $L^{q,\lambda}(w;\mathbb{Q}_p^n)$ is reduced to $\{0\}$. So, in this thesis, we always choose λ from the interval $-1/q \leq \lambda < 0$.

Definition 1.3.5 *Suppose* w *is a weight function on* $\mathbb{Q}_p^n$, $\lambda \geq -\frac{1}{q}$, *where* $1 \leq q < \infty$. *The p-adic space* $\dot{B}^{q,\lambda}(w;\mathbb{Q}_p^n)$ *is the set of all measurable functions* $f\colon \mathbb{Q}_p^n \to \mathbb{R}$ *and is defined as*

$$\dot{B}^{q,\lambda}(w;\mathbb{Q}_p^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{Q}_p^n) : \|f\|_{\dot{B}^{q,\lambda}(w;\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{\dot{B}^{q,\lambda}(w;\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{w(B_\gamma)^{\lambda q+1}} \int_{B_\gamma} |f(\mathbf{x})|^q w(\mathbf{x}) d\mathbf{x}\right)^{1/q}.$$

Obviously $L^{q,\lambda}(w;\mathbb{Q}_p^n) \subset \dot{B}^{q,\lambda}(w;\mathbb{Q}_p^n)$ and $\dot{B}^{q,-1/q}(w;\mathbb{Q}_p^n) = L^q(w;\mathbb{Q}_p^n)$. Also, it is interesting to note that $\dot{B}^{q,\lambda}(w;\mathbb{Q}_p^n) = \{0\}$ when $\lambda < -1/q$. When $w = 1$ the weighted central Morrey space in p-adic field is central Morrey space in p-adic field which is defined by

Definition 1.3.6 *Suppose* $1 \leq q < \infty$ *and suppose* $-\frac{1}{q} \leq \lambda < 0$. *A function* $f \in L^p_{loc}(\mathbb{Q}_p^n)$ *is said to belong to central Morrey spaces* $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ *if*

$$\dot{B}^{q,\lambda}(\mathbb{Q}_p^n) = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} |f(\mathbf{x})|^q d\mathbf{x} \right)^{1/q} < \infty.$$

When $\lambda = -1/q$, then $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n) = L^q(\mathbb{Q}_p^n)$. Evidently, $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ is reduced to $\{0\}$ whenever $\lambda < -1/q$.

Definition 1.3.7 *Suppose* $-\frac{1}{q} \leq \lambda < 0$, *where* $\infty > q \geq 1$. *The p-adic space* $W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ *is the set of all measurable functions* $f\colon \mathbb{Q}_p^n \to \mathbb{R}$ *satisfying*

$$W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n) = \{f : \|f\|_{W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} |B_\gamma|_H^{-\lambda-1/q} \|f\|_{WL^q(B_\gamma)},$$

and $\|f\|_{WL^q(B_\gamma)}$ is the local p-adic L^q-norm of $f(x)$ restricted to the ball B_γ as

$$\|f\|_{WL^q(B_\gamma)} = \sup_{\lambda>0} |\{\mathbf{x} \in B_\gamma : |f(\mathbf{x})| > \lambda\}|^{1/q}.$$

Obviously for $\lambda = -1/q$, we have $W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n) = L^{q,\infty}(\mathbb{Q}_p^n)$ is a p-adic weak L^q space. Also, $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n) \subseteq W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ for $-1/q < \lambda < 0$, where $1 \leq q < \infty$.

1.3.2 Herz-type Spaces on $\mathbb{Q}_p^n$

Definition 1.3.8 *Suppose* $w(\mathbf{x})$ *is a weight function on* $\mathbb{Q}_p^n$, $\alpha \in \mathbb{R}$ *and* $0 < l, q < \infty$ *then weighted homogeneous p-adic Herz space* $K_q^{\alpha,l}(w;\mathbb{Q}_p^n)$ *is defined as below:*

$$K_q^{\alpha,l}(w;\mathbb{Q}_p^n) = \{f \in L^q_{\text{loc}}(w;\mathbb{Q}_p^n\backslash\{0\}) : \|f\|_{K_q^{\alpha,l}(w;\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha,l}(w;\mathbb{Q}_p^n)} = \left(\sum_{k=-\infty}^{\infty} w(B_k)^{\alpha l/n} \|f\chi_k\|^l_{L^q(w;\mathbb{Q}_p^n)} \right)^{1/l}. \tag{1.3.1}$$

Since Herz spaces are considered as a generalization of power weighted Lebesgue spaces. Therefore, we have $K_q^{0,q}(w;\mathbb{Q}_p^n) = L^q(w;\mathbb{Q}_p^n)$.

When $w = 1$, the weighted p-adic Herz space is just a p-adic Herz space which can be define as:

$$K_q^{\alpha,l}(\mathbb{Q}_p^n) = \{f \in L^q(\mathbb{Q}_p^n) : \|f\|_{K_q^{\alpha,l}(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha,l}(\mathbb{Q}_p^n)} = \left(\sum_{k=-\infty}^{\infty} p^{k\alpha l} \|f\chi_k\|_{L^q(\mathbb{Q}_p^n)}^l \right)^{1/q}.$$

Definition 1.3.9 *Suppose $w(\mathbf{x})$ is a weight function on $\mathbb{Q}_p^n$, α is a real number, $0 < q, l < \infty$ and $\lambda \in \mathbb{R}^+$ then the weighted homogeneous p-adic Morrey-Herz space $MK_{l,q}^{\alpha,\lambda}(w;\mathbb{Q}_p^n)$ is defined as:*

$$MK_{l,q}^{\alpha,\lambda}(w;\mathbb{Q}_p^n) = \{f \in L_{loc}^q(w, \mathbb{Q}_p^n \backslash \{0\}) : \|f\|_{MK_{l,q}^{\alpha,\lambda}(w;\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{MK_{l,q}^{\alpha,\lambda}(w;\mathbb{Q}_p^n)} = \sup_{k_0 \in \mathbb{Z}} w(B_{k_0})^{-\lambda/n} \left(\sum_{k=-\infty}^{k_0} w(B_k)^{\alpha l/n} \|f\chi_k\|_{L^q(w;\mathbb{Q}_p^n)}^l \right)^{1/l}. \tag{1.3.2}$$

Weighted p-adic Morrey-Herz space is reduced to p-adic Morrey-Herz space if $w = 1$ which can be defined as:

$$M\dot{K}_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n) = \{f \in L_{loc}^q(\mathbb{Q}_p^n \setminus \{0\}) : \|f\|_{M\dot{K}_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{M\dot{K}_{l,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)} = \sup_{k_0 \in \mathbb{Z}} p^{-k_0\lambda} \left(\sum_{k=-\infty}^{\infty} p^{k\alpha l} \|f\chi_k\|_{L^q(\mathbb{Q}_p^n)}^l \right)^{1/l}.$$

A close observation of (1.3.1) and (1.3.2) reveals that by setting $\lambda = 0$ in (1.3.2) we get the following equality $MK_{l,q}^{\alpha,0}(w;\mathbb{Q}_p^n) = K_q^{\alpha,l}(w;\mathbb{Q}_p^n)$.

1.3.3 BMO-type Spaces on $\mathbb{Q}_p^n$

Definition 1.3.10 *[21] The weighted p-adic space $BMO(w,\mathbb{Q}_p^n)$ satisfies*

$$\|f\|_{BMO(w,\mathbb{Q}_p^n)} = \sup_B \frac{1}{w(B)} \int_B |f(x) - f_B| w(\mathbf{x}) d\mathbf{x} < \infty, \tag{1.3.3}$$

where supremum is taken over all balls B of $\mathbb{Q}_p^n$ and

$$f_B = \frac{1}{|B|} \int_B f(\mathbf{x}) d\mathbf{x}. \tag{1.3.4}$$

If $w = 1$, we get the p-adic space $BMO(\mathbb{Q}_p^n)$, see [91]. Now, we turn towards the λ-central BMO space.

Definition 1.3.11 *Suppose $\lambda < \frac{1}{n}$ and $\infty > q > 1$. The p-adic space $CMO^{q,\lambda}(w, \mathbb{Q}_p^n)$ is defined as:*

$$\|f\|_{CMO^{q,\lambda}(w,\mathbb{Q}_p^n)} = \sup_{\gamma \in Z} \left(\frac{1}{w(B_\gamma)^{\lambda q+1}} \int_{B_\gamma} |f(\mathbf{x}) - f_{B_\gamma}|^q w(\mathbf{x}) d\mathbf{x} \right)^{\frac{1}{q}} < \infty, \tag{1.3.5}$$

where

$$f_{B_\gamma} = \frac{1}{|B_\gamma|} \int_{B_\gamma} f(\mathbf{x}) d\mathbf{x}. \tag{1.3.6}$$

Remark: When $\lambda = 0$, the space $CMO^{q,\lambda}(w, \mathbb{Q}_p^n)$ is just reduced to $CMO^q(w, \mathbb{Q}_p^n)$ satisfying

$$\|f\|_{CMO^q(w,\mathbb{Q}_p^n)} = \sup_{\gamma \in Z} \left(\frac{1}{w(B_\gamma)} \int_{B_\gamma} |f(\mathbf{x}) - f_{B_\gamma}|^q w(\mathbf{x}) d\mathbf{x} \right)^{1/q} < \infty.$$

Definition 1.3.12 *Suppose $\delta \in \mathbb{R}^+$. The Lipschitz space $\Lambda_\delta(\mathbb{Q}_p^n)$ is the space of all measurable functions f on $\mathbb{Q}_p^n$, such that:*

$$\|f\|_{\Lambda_\delta(\mathbb{Q}_p^n)} = \sup_{\mathbf{x},\mathbf{h} \neq \mathbf{0} \in \mathbb{Q}_p^n} \frac{|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})|}{|\mathbf{h}|_p^\delta} < \infty.$$

The distribution function of $f \in \mathbb{Q}_p^n$ with a measure $w(\mathbf{x})d\mathbf{x}$ is defined as:

$$\mu_f^w(\lambda) = w\{\mathbf{x} \in \mathbb{Q}_p^n : |f(\mathbf{x})| > \lambda\}.$$

The decreasing rearrangement of f with respect to measure $w(\mathbf{x})d\mathbf{x}$ is as follows:

$$f^w(t) = \inf\{\lambda > 0 : \mu_f^w(\lambda) \leq t\}, t \in \mathbb{R}^+.$$

Definition 1.3.13 *The weighted Lorentz space $L^{q,s}(w, \mathbb{Q}_p^n)$ in p-adic field is the collection of all functions f so that $\|f\|_{L^{q,s}(w,\mathbb{Q}_p^n)} < \infty$, where*

$$\|f\|_{L^{q,s}(w,\mathbb{Q}_p^n)} = \begin{cases} \left(\frac{s}{q} \int_0^\infty [t^{1/q} f^w(t)]^s \frac{dt}{t} \right), & \text{if } 1 \leq s < \infty, \\ \sup_{t>0} t^{1/q} f^w(t), & \text{if } s = \infty. \end{cases}$$

For $f \in L^{q,s}(w, \mathbb{Q}_p^n)$ with $0 < s < \infty,\ 0 < q \leq r < \infty$, clearly

$$\|f\|_{L^{r,s}(w,\mathbb{Q}_p^n)} \leq C\|f\|_{L^{q,s}(w,\mathbb{Q}_p^n)}. \tag{1.3.7}$$

If an operator T is bounded from $L^{q,1}(w, \mathbb{Q}_p^n)$ into $L^{r,\infty}(w, \mathbb{Q}_p^n)$, then T is of weak type (q, r). From (1.3.7), evidently,

$$w\{\mathbf{x} \in \mathbb{Q}_p^n : |Tf(\mathbf{x})| > \lambda\} \leq C\lambda^{-r}\|f\|^r_{L^q(w,\mathbb{Q}_p^n)},\ 1 \leq q \leq r < \infty,$$

justifies weak type (q, r) for T.

In Chapter 4, we shall come across the situation where the following Marcinkiewicz theorem will be helpful for us.

Theorem 1.3.14 *Suppose* $w(\mathbf{x}) = |\mathbf{x}|_p^\alpha,\ \alpha > -n,\ 1 \leq q' < q_0,\ 1 \leq r', r_0, r' \neq r_0,$ $\vartheta \in (0, 1)$ *and*

$$\frac{1}{q} = (1-\vartheta)/q' + \vartheta/q_0, \frac{1}{r} = (1-\vartheta)/r' + \vartheta/r_0.$$

If T *is of weak type* (q_0, r_0) *and* (q', r'), *then* T *is bounded from* $L^{q,s}(|\mathbf{x}|_p^\alpha, \mathbb{Q}_p^n)$ *into* $L^{r,s}(|\mathbf{x}|_p^\alpha, \mathbb{Q}_p^n)$, *for all* $1 \leq s < \infty$.

1.4 Introduction to Some Integral Operators

In this thesis, some p-adic integral operators of our interest include the Hardy operator, the fractional Hardy operator, the weighted Hardy operator, the matrix Hausdorff operator and the fractional Hausdorff operator. A brief introduction to these operators define on different underlying spaces is given in the next few subsections.

1.4.1 Hardy-type Operators on $\mathbb{R}^n$

The operator

$$\mathcal{H}f(x) = \frac{1}{x}\int_0^x f(y)dy, \quad x > 0, \tag{1.4.1}$$

was introduced by Hardy in [49] which satisfies the following inequality:

$$\|\mathcal{H}f\|_{L^q(\mathbb{R}^+)} \leq \frac{q}{q-1}\|f\|_{L^q(\mathbb{R}^+)}, \quad 1 < q < \infty. \tag{1.4.2}$$

It was also shown that the constant $q/(q-1)$ appearing in (1.4.2) is optimal. Knowing its fundamental importance in analysis, Faris in [27] and Christ and Grafakos in [15] proposed an extensions of (1.4.1) and its adjoint to the n-dimensional Euclidian space $\mathbb{R}^n$ of which the equivalent forms are:

$$Hf(\mathbf{x}) = \frac{1}{|\mathbf{x}|^n}\int_{|\mathbf{y}|\leq|\mathbf{x}|} f(\mathbf{y})d\mathbf{y}, \quad H^*f(\mathbf{x}) = \int_{|\mathbf{y}|>|\mathbf{x}|}\frac{f(\mathbf{y})}{|\mathbf{y}|^n}d\mathbf{y},\ \mathbf{x}\in\mathbb{R}^n\setminus\{\mathbf{0}\}. \quad (1.4.3)$$

In addition, the corresponding norms of H and H^* were computed in [15] which were same as that of corresponding one dimensional Hardy operators.

Since the appearance of monographs cited above, the study of optimal bounds for Hardy type operators on function spaces took the global attention and a good number of research publications were produced in this direction. For instance, the authors in [32] and [89] investigated this problem on power weighted Lebesgue and Morrey spaces. Similar results on other function spaces including weak Lebesgue and Campanato spaces were reported in [114]. Finally, the sharp estimates for Hardy operator on higher dimensional p-adic product space were obtained in [83].

Let us turn our discussion towards optimal bounds for fractional Hardy type operators H_α and H^*_α (see [33]) which are obtained by replacing $|\cdot|^n$ with $|\cdot|^{n+\alpha}$ $(0\leq\alpha<n)$ in each of the components of (1.4.3). In 1930, Bliss [7] worked out the following inequality for one-dimensional fractional Hardy operator:

$$\|H_\alpha f\|_{L^q(\mathbb{R}_+)} \leq C_{sharp}\|f\|_{L^p(\mathbb{R}_+)},$$

where $0<\alpha<1<p<\infty$, $\frac{1}{p}-\frac{1}{q}=\alpha$ and $C_{sharp}=\left(\frac{p'}{q}\right)^{1/q}\left(\frac{1}{q\alpha}\ B\left(\frac{1}{q\alpha},\frac{1}{q'\alpha}\right)\right)^{1/q-1/p}$. The problem of sharp bounds for H_α on higher dimensional space is recently addressed in [115]. Mizuta et al. [85] obtained the optimal bounds for H_α in the framework of Banach function spaces. Recently, in [86], they computed the sharp constant for the fractional Hardy operator on variable exponent Lebesgue spaces. Besides, the weak type optimal bounds for the fractional Hardy and adjoint Hardy operators are of special interest to many researcher, see for instance [36, 37, 38, 112]. In this context, the book [24] and [48] are probably one of the most books for Hardy type inequalities. For more details about weak bounds of Hardy type operators see some late publications including [11, 36, 112].

In what follows the weighted Hardy operator which was defined in [10] is given by

$$H_\psi(f)(\mathbf{x}) = \int_0^1 f(t\mathbf{x})\psi(t)dt, \quad \mathbf{x}\in\mathbb{R}^n,$$

where $\psi:[0,1]\to[0,\infty)$ is a measurable function. Interestingly, when $\psi\equiv 1$ and $n=1$, the weighted Hardy operator is reduced to classical Hardy operator given in (1.4.1). Xiao [110] proved the boundedness of H_ψ on $L^p(\mathbb{R}^n), 1 \le p \le \infty$ and $BMO(\mathbb{R}^n)$. Moreover, he worked out corresponding operator norms as well. Fu et al. [29] showed that H_ψ is bounded on central Morrey spaces as well as on λ-central BMO spaces by imposing size conditions on ψ. In addition, in the same paper, they sharpen these size conditions on ψ. Some results showing boundedness of H_ψ on other function spaces were presented in [28, 79].

The weighted multilinear Hardy operator on Euclidean space was defined by Fu et al. [31] and is given by

$$H_\psi^m(f_1, \cdots, f_m)(\mathbf{x}) = \int_0^1 \cdots \int_0^1 f_1(\mathbf{t}_1\mathbf{x}) \cdots f_m(\mathbf{t}_m\mathbf{x})\psi(\mathbf{t}_1, \cdots, \mathbf{t}_m)d\mathbf{t}_1 \cdots d\mathbf{t}_m, \quad \mathbf{x} \in \mathbb{R}^n,$$

where ψ is a nonnegative function defined on $[0,1] \times \cdots \times [0,1]$. In the same paper, authors studied the boundedness of the very operator on the product of Lebesgue spaces and central Morrey spaces. Moreover, the boundedness of weighted multilinear Hardy operators on the product of Herz type spaces was shown in [41]. Finally, we remark that the celebrated work on weighted Hardy and integral inequalities can be found in the works by the authors in [65, 67, 69].

1.4.2 Hardy-type Operators on $\mathbb{Q}_p^n$

The Hardy operator H^p and its adjoint $H^{p,*}$ on the p-adic field was first time defined by Fu et al. in [34] and obtained their optimal bounds on $L^q(\mathbb{Q}_p^n)$. Central Morrey space estimates for Hardy type operators along with their commutators on the p-adic field are established in [109]. In [80], the authors turned towards higher dimensional product spaces and computed the optimal estimates of Hardy type operators on p-adic field. Finally, the n-dimensional fractional p-adic Hardy type operators are defined and studied in [105], which for $f \in L_{loc}(\mathbb{Q}_p^n)$ and $0 \le \alpha < \infty$, are given as:

$$H_\alpha^p f(\mathbf{x}) = \frac{1}{|\mathbf{x}|_p^{n-\alpha}} \int_{|\mathbf{y}|_p \le |\mathbf{x}|_p} f(\mathbf{y})d\mathbf{y}, \quad H_\alpha^{p,*} f(\mathbf{x}) = \int_{|\mathbf{y}|_p > |\mathbf{x}|_p} \frac{f(\mathbf{y})}{|\mathbf{y}|_p^{n-\alpha}} d\mathbf{y}, \ \mathbf{x} \in \mathbb{Q}_p^n \setminus \{\mathbf{0}\}.$$

When $\alpha = 0$, we obtain p-adic Hardy type operators, see [38] for more details.

The weighted p-adic Hardy operator was defined and studied in [91]. For p-adic numbers' field $\mathbb{Q}_p$ and $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : 1 \ge |x|_p\}$, if $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$, then p-adic weighted Hardy operator can be defined as:

$$H_\psi^p f(x) = \int_{\mathbb{Z}_p^*} f(tx)\psi(t)dt,$$

where ψ is a nonnegative function defined on $\mathbb{Z}_p^*$. In a same paper [91], it was shown that H_ψ^p is bounded on $L^q(\mathbb{Q}_p^n)$, $\infty > q > 1$ as well as on $BMO(\mathbb{Q}_p^n)$. Later in 2013, authors in [16] established the boundedness of H^p_ψ on Herz type spaces on p-adic field. Furthermore, weighted p-adic Hardy operator along with its commutator on p-adic central Morrey spaces have been discussed in [108].

In what follows weighted multilinear p-adic Hardy operator was extensively studied in the near past, see for example [18, 80, 81] and the reference therein. The weighted multilinear Hardy operator on the p-adic field was defined in [81]:

Definition 1.4.1 *Let $\mathbf{x} \in \mathbb{Q}_p^n$, $m \in \mathbb{N}$ and $f_1, \cdots, f_m$ be nonnegative measurable functions on $\mathbb{Q}_p^n$, then*

$$H_\psi^{p,m}(f_1, \cdots, f_m)(\mathbf{x}) = \int_{\mathbb{Z}_p^*} \cdots \int_{\mathbb{Z}_p^*} f_1(\mathbf{t}_1\mathbf{x}) \cdots f_m(\mathbf{t}_m\mathbf{x})\psi(\mathbf{t}_1, \cdots, \mathbf{t}_m)d\mathbf{t}_1 \cdots d\mathbf{t}_m,$$

where ψ is a nonnegative measurable function on $\mathbb{Z}_p^* \times \cdots \times \mathbb{Z}_p^*$.

In the same paper, the authors computed the optimal estimates for weighted multilinear Hardy operator on p-adic field on the product of p-adic Lebesgue spaces along with Morrey type spaces.

1.4.3 Hausdorff Operators on $\mathbb{R}^n$

Hausdorff summability methods (Hausdorff operators) contributed a lot in the study of one dimensional Fourier analysis, more specifically, in the study of summability properties of classical Fourier series. Back in 1917, Hurwitz and Silverman [54] studied number of methods related to the Hausdorff summability. The real breakthrough came in 1921, when Hausdorff [49] not only rediscovered the same summability problems but also associated them with a familiar moment problem for a finite interval. The later publications in this topic include [6, 46, 90, 104]. In the manuscript [96], Siskakis started working on composition operators as well as on Cesáro mean in H^p spaces. His nice brief proof for H^1 boundedness of Cesáro operator in [97] was an important result in the context of developing theory of Hausdorff operator. In [39] and [40] extension of results of [96] and [97] was made to the Fourier transform setting on the real line. The research article [39] was the gateway for Móricz and other authors to attempt a more general averaging operator than Cesáro.

Recent attention towards studying Hausdorff operators on function spaces is a consequence of the paper [75] by Liflyand and Mórecz. In the paper, they proved that the operator

$$H_\Phi f(x) = \int_0^\infty \frac{\Phi(t)}{t} f\left(\frac{x}{t}\right) dt, \tag{1.4.4}$$

is bounded from $H^1(\mathbb{R})$ into $H^1(\mathbb{R})$. The ensuing manuscript [73] extended these results on the real Hardy spaces $H^p(\mathbb{R})$, $0 < p < 1$, by employing some smoothness conditions on the kernel function Φ. Also, the article [76] on Hausdorff operator was written in a quick succession of [75].

Besides its summability properties, the operator is considered as the generalization of the operators like the Cesàro operator, Hardy type operators. These are the properties of H_Φ that served to encourage researchers to study it in high dimensional Euclidian space $\mathbb{R}^n$. In this perspective an extension of (1.4.4) was made in [70] and is given by

$$\mathcal{H}_{\Phi,A}f(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{t})f(A(\mathbf{t})\mathbf{x})d\mathbf{t}, \tag{1.4.5}$$

where $\Phi \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $A(\mathbf{t})$ is a square matrix with $\det A(\mathbf{t}) \neq 0$ almost everywhere in the support Φ.

Here, it may be quite appropriate to outline some developments in the theory of Hausdorff operator during the recent years. A survey reveals that much of the literature on Hausdorff operator is focused on its boundedness on the Hardy spaces H^1 [14, 64, 70, 75, 108] and a little on H^p, $0 < p < 1$, [13, 74, 73]. Without going into the detailed history, we mention a few recent publications which include [9, 12, 13, 14, 72, 26, 56, 64, 70, 72, 74, 92, 108, 113]. The matrix Hausdorff operator was defined in [12]:

$$H_{\Phi,A}f(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{\Phi(\mathbf{t})}{|\mathbf{t}|^n} f(A(\mathbf{t})\mathbf{x})d\mathbf{t}. \tag{1.4.6}$$

If $A(\mathbf{t}) = \text{diag}[1/|\mathbf{t}|, ..., 1/|\mathbf{t}|]$, then we get another definition of Hausdorff operator which was defined in the same paper [12]:

$$H_\Phi f(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{\Phi(\mathbf{t})}{|\mathbf{t}|^n} f\left(\frac{\mathbf{x}}{|\mathbf{t}|}\right) d\mathbf{t}. \tag{1.4.7}$$

In [12], atomic decomposition of Hardy spaces was employed for the boundedness of Hausdorff operators in these spaces. Another important development made in [12] was the introduction of rough Hausdorff operator defined by

$$\widetilde{H}_{\Phi,\Omega}f(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{\Phi(\mathbf{x}/|\mathbf{y}|)}{|\mathbf{y}|^n}\Omega(\mathbf{y}')f(\mathbf{y})d\mathbf{y}, \tag{1.4.8}$$

where the restriction $\Omega|_{\mathcal{S}^{n-1}}$ of Ω on the unit sphere is integrable with respect to the normed Lebesgue measure dy and Φ is a radial function defined on $\mathbb{R}^+$. If $\Omega = 1$, we get the following operator:

$$\widetilde{H}_\Phi f(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{\Phi(\mathbf{x}/|\mathbf{y}|)}{|\mathbf{y}|^n} f(\mathbf{y})d\mathbf{y}, \tag{1.4.9}$$

which is also focused to study in the same paper [12].

An extension of the operator in (1.4.9) is the fractional Hausdorff operator which was studied by Lin and Sun [78] and is given as:

$$H_{\Phi,\beta}(f)(\mathbf{x}) = \int_{\mathbb{R}^n} \frac{\Phi(|\mathbf{x}|/|\mathbf{y}|)}{|\mathbf{y}|^{n-\beta}} f(\mathbf{y})d\mathbf{y}, \quad 0 \leq \beta < n. \tag{1.4.10}$$

It is important to mention here that if the kernel function Φ in (1.4.10) is selected smartly then $H_{\Phi,\beta}$ is reduced to the some classical operators like Hardy operator, adjoint Hardy operator, Hardy-Littlewood-Pólya operator and Cesáro operator. Weak and strong estimates of two kinds of multilinear fractional Hausdorff operator on Lebesgue space were studied by Fan and Zhao in [26]. Similar estimates for the commutators of fractional Hausdorff operator were reported in [56]. Gao and Zhao [38] obtained the optimal estimates for fractional Hausdorff operators.

1.4.4 Hausdorff Operators on $\mathbb{Q}_p^n$

The study of Hausdorff operator on p-adic function spaces is solely due to Volosivets [102, 101]. Suppose that $\Phi(\mathbf{t})$ is measurable on $\mathbb{Q}_p^n$, $n \times n$ matrix $A(\mathbf{t})$ is nonsingular almost everywhere and has the continuous components $a_{ij}(\mathbf{t}) : \mathbb{Q}_p^n \to \mathbb{Q}_p$ and $f(\mathbf{x})$ is continuous on $\mathbb{Q}_p^n$. Then the Hausdorff operator is as follows

$$H_{\Phi,A}(f)(\mathbf{x}) = \int_{\mathbb{Q}_p^n} \Phi(\mathbf{t}) f(A(\mathbf{t})\mathbf{x}) d\mathbf{t}.$$

whenever Lebesgue integral exists.

In [102], Volosivets studied the boundedness of $H_{\Phi,A}$ in the real Hardy spaces $H^1(\mathbb{Q}_p^n)$, the matrix $A(\mathbf{t})$ was restricted to $A(\mathbf{t}) = a(\mathbf{t})E$, where E is the identity matrix. Subsequently, in [101], by replacing the condition on the form of the matrix A with the constraints on its norm in $\mathbb{Q}_p^n$, he obtained the boundedness of $H_{\Phi,A}$ in $H^1(\mathbb{Q}_p^n)$ and $BMO(\mathbb{Q}_p^n)$.

1.5 Our Contribution to the Theory of Hardy-type Operators

We contributed to the theory of p-adic Hardy-type operators in many ways. Firstly, we obtain the boundedness of p-adic matrix Hausdorff operator on p-adic Morrey and Herz-type spaces. Also, under some special conditions on the norm of the matrix, we proved sharpness of these results. Secondly, we not only defined the commutators of p-adic matrix Hausdorff operator on p-adic field:

$$H^b_{\Phi,A}(f)(\mathbf{x}) = b(\mathbf{x})H_{\Phi,A}(f)(\mathbf{x}) - H_{\Phi,A}(bf)(\mathbf{x}), \tag{1.5.1}$$

but also introduced the p-adic analog of fractional Hausdorff operator in the following form:

$$H_{\Phi,\beta}(f)(\mathbf{x}) = \int_{\mathbb{Q}_p^n} \frac{\Phi(\mathbf{x}|\mathbf{y}|_p)}{|\mathbf{y}|_p^{n-\beta}} f(\mathbf{y})d\mathbf{y}, \quad 0 \leq \beta < n, \tag{1.5.2}$$

where $|y|_p$ is considered to be equal to some power of p as an element of $\mathbb{Q}_p$. We also define and studied the commutators generated by fractional Hausdorff operator and the Lipschitz function. Finally, we remark that if $\beta = 0$, we get the another type of Hausdorff operator and is as follows

$$H_{\Phi}(f)(\mathbf{x}) = \int_{\mathbb{Q}_p^n} \frac{\Phi(\mathbf{x}|\mathbf{y}|_p)}{|\mathbf{y}|_p^{n}} f(\mathbf{y})d\mathbf{y}, \tag{1.5.3}$$

which was recently studied in [103]. Also, we computed optimal weak bounds for p-adic Hardy type operators on weighted p-adic Lebesgue spaces and weighted p-adic central Morrey spaces. Finally, we acquired sharp size stipulations on the kernal function ψ such that weighted p-adic Hardy operator is bounded on product of Herz type spaces.

1.5.1 References of Contribution

The contribution is cited in the reference list which include [60, 61, 62, 63, 94, 95].

Chapter 2
The Hausdorff Operator on Weighted p-adic Morrey and Herz-type Spaces

2.1 Introduction

In this Chapter, we will study Hausdorff operator on p-adic function spaces of power weighted type including Lebesgue spaces, Morrey spaces, and Herz type spaces. Contrary to [101, 102], we employ a different methodology to prove our results. In addition, by imposing some conditions on the norm of the matrix A, we discuss the sharpness of our results as well. These results are important in the sense that $H_{\Phi,A}$ is a p-adic Hardy-Littlewood operator and p-adic modified Hardy operator if $A(\mathbf{t})$ is diagonal matrix and Φ is suitably chosen. Secondly, the analysis presented in this paper can be used to prove the boundedness of $H_{\Phi,A}$ and its commutators on other weighted function spaces of p-adic nature, see for example [95].

In the remaining of this section, we suggest some basic definitions and notation, as well as some useful lemmas. We prove the main theorems stating the boundedness of $H_{\Phi,A}$ on weighted p-adic Herz-type spaces in the next Section. However, the boundedness of $H_{\Phi,A}$ on weighted p-adic Morrey-type spaces is proved in the last Section.

Lemma 2.1.1 *([101]) Suppose D is an $n \times n$ matrix with entries $d_{ij} \in \mathbb{Q}_p$. Then the norm of D, regarded as an operator from $\mathbb{Q}_p^n$ to $\mathbb{Q}_p^n$, is*

$$\|D\| = \max_{1\leq i\leq n} \max_{1\leq j\leq n} |d_{ij}|_p.$$

The action of operator D with matrix $\{d_{ij}\}_{i,j=1}^n$ on $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{Q}_p^n$ can be expressed as

$$D\mathbf{x} = \left(\sum_{j=1}^{n} d_{1j}x_j, ..., \sum_{j=1}^{n} d_{nj}x_j \right).$$

Lemma 2.1.2 *Suppose that an $n \times n$ matrix D with entries $d_{ij} \in \mathbb{Q}_p$ is invertible. Then*

$$\|D^{-1}\|^n \geq |\det D^{-1}|_p \geq \|D\|^{-n} \tag{2.1.1}$$

Proof. By virtue of (1.2.1) and Lemma 2.1.1, we have

$$|D\mathbf{x}|_p \leq \max_{1\leq i\leq n} \left| \sum_{j=1}^{n} d_{ij}x_j \right|_p \leq \max_{1\leq i\leq n} \max_{1\leq j\leq n} |d_{ij}|_p |x_j|_p \leq \|D\| |\mathbf{x}|_p$$

for any $\mathbf{x} \in \mathbb{Q}_p^n$. Then, by replacing $\mathbf{x}$ with $D^{-1}\mathbf{x}$, we have

$$\|D\|^{-1}|\mathbf{x}|_p \leq |D^{-1}\mathbf{x}|_p \leq \|D^{-1}\| |\mathbf{x}|_p. \tag{2.1.2}$$

Thus,

$$\begin{aligned} |\{\mathbf{x} \in \mathbb{Q}_p^n : \|D\|^{-1}|\mathbf{x}|_p \leq 1\}| &\geq |\{\mathbf{x} \in \mathbb{Q}_p^n : |D^{-1}\mathbf{x}|_p \leq 1\}| \\ &\geq |\{\mathbf{x} \in \mathbb{Q}_p^n : \|D^{-1}\| |\mathbf{x}|_p \leq 1\}|, \end{aligned}$$

which leads to (2.1.1) without any essential difficulty, since $|B_0| = 1$.

Lemma 2.1.3 *([21]) Let $w \in \mathbb{W}_\beta$, $\gamma \in \mathbb{Z}$ and $\beta > -n$. Then we have*

$$w(B_\gamma) = p^{(n+\beta)\gamma} \quad \textit{and} \quad w(S_\gamma) = p^{(n+\beta)\gamma} \cdot w(S_0)$$

Here and in the sequel, for brevity's sake, we use the following sign:

$$G(D, \delta) = \begin{cases} \|D\|^\delta & \text{if } \delta > 0, \\ \|D^{-1}\|^{-\delta} & \text{if } \delta \leq 0, \end{cases} \tag{2.1.3}$$

where δ is a real number and D is a nonsingular matrix.
Evidently:

$$G(D, \beta(1/p + 1/q)) = G(d, \beta/p)G(d, \beta/q), \tag{2.1.4}$$

where $p, q \in \mathbb{Z}^+$.

Lemma 2.1.4 *Let $\beta > -n$, $w(\mathbf{x}) = |\mathbf{x}|_p^\beta$, D be any invertible matrix and $\mathbf{x} \in \mathbb{Q}_p^n$, then*

$$w(D\mathbf{x}) \leq \begin{cases} \|D\|^\beta w(\mathbf{x}) & \text{if } \beta > 0, \\ \|D^{-1}\|^{-\beta} w(\mathbf{x}) & \text{if } \beta \leq 0, \end{cases}$$
$$= G(D, \beta) w(\mathbf{x}).$$

Proof. The definition of $w(\mathbf{x})$ and (2.1.2) will do world of good to prove a lemma.

Lemma 2.1.5 *Suppose $\beta > -n$, $w(\mathbf{x}) = |\mathbf{x}|_p^\beta$ and D is any invertible matrix, then we have*

$$(i) \qquad w(DB_k(\mathbf{a})) \leq G(D, \beta)|\det D|_p w(B_k(\mathbf{a})),$$

$$(ii) \qquad w(B_{k+\log_p \|D\|}) = w(S_0)\frac{p^{(n+\beta)}}{p^{(n+\beta)} - 1}\|D\|^{n+\beta} w(B_k).$$

Proof.(i) Since,

$$\begin{aligned} w(DB_k(\mathbf{a})) &= \int_{DB_k(\mathbf{a})} |\mathbf{x}|_p^\beta d\mathbf{x} \\ &= \int_{B_k(\mathbf{a})} |D\mathbf{z}|_p^\beta |\det D|_p d\mathbf{z} \\ &= |\det D|_p \int_{B_k(\mathbf{a})} w(D\mathbf{z}) d\mathbf{z}. \end{aligned}$$

The proof is completed courtesy Lemma 2.1.4.

(ii) Similarly,

$$\begin{aligned} w(B_{k+\log_p \|D\|}) &= \sum_{j=-\infty}^{\log_p \|D\|} w(S_{k+j}) \\ &= w(S_0) \sum_{j=-\infty}^{\log_p \|D\|} p^{(n+\beta)(k+j)} \\ &= w(S_0)\frac{p^{(n+\beta)}}{p^{(n+\beta)} - 1}\|D\|^{n+\beta} w(B_k), \end{aligned}$$

where we employed Lemma 2.1.3 at the second and third step.

2.2 Bounds of Hausdorff Operator on Weighted p-adic Herz-type Spaces

The current section consists of results on the boundedness of p-adic Hausdorff operator on p-adic Herz type spaces. The results and their proofs are as under:

Theorem 2.2.1 *Let $1 \leq q, l < \infty,\ 0 \leq \lambda > \alpha,\ \beta > -n$ and $w(\mathbf{x}) = |\mathbf{x}|_p^{\beta}$, then $H_{\Phi,A}$ is bounded on $MK_{l,q}^{\alpha,\lambda}(w;\mathbb{Q}_p^n)$ and satisfy the following inequality*

$$\begin{aligned}&\|H_{\Phi,A}\|_{MK_{l,q}^{\alpha,\lambda}(w;\mathbb{Q}_p^n)}\\ &\leq C\|f\|_{MK_{l,q}^{\alpha,\lambda}(w;\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}|\det A^{-1}(\mathbf{t})|_p^{1/q}\|A(\mathbf{t})\|^{(\lambda-\alpha)(n+\beta)/n}G(A^{-1}(\mathbf{t}),\beta/q)|\Phi(\mathbf{t})|d\mathbf{t}.\end{aligned}$$

Proof. For $k \in \mathbb{Z}$, use of Minkowski's inequality, p-adic change of variables and Lemma 2.1.4 give

$$\begin{aligned}&\|(H_{\Phi,A})\chi_k f\|_{L^q(w;\mathbb{Q}_p^n)}\\ &= \left(\int_{S_k}\left|\int_{\mathbb{Q}_p^n} f(A(\mathbf{t})\mathbf{x})\Phi(\mathbf{t})d\mathbf{t}\right|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}\\ &\leq \int_{\mathbb{Q}_p^n}\left(\int_{S_k}|f(A(\mathbf{t})\mathbf{x})|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}|\Phi(\mathbf{t})|d\mathbf{t}\\ &= \int_{\mathbb{Q}_p^n}\left(\int_{A(\mathbf{t})S_k}|f(\mathbf{x})|^q w(A^{-1}(\mathbf{t})\mathbf{x})|\det A^{-1}(\mathbf{t})|_p d\mathbf{x}\right)^{1/q}|\Phi(\mathbf{t})|d\mathbf{t}\\ &= \int_{\mathbb{Q}_p^n}\|f\chi_{A(\mathbf{t})S_k}\|_{L^q(w;\mathbb{Q}_p^n)}|\det A^{-1}(\mathbf{t})|_p^{1/q}G(A^{-1}(\mathbf{t}),\beta/q)|\Phi(\mathbf{t})|d\mathbf{t}, \qquad (2.2.1)\end{aligned}$$

where $A(\mathbf{t})S_k$ denotes the set: $\{\mathbf{x}:\ A^{-1}(\mathbf{t})\mathbf{x} \in S_k\}$. Hence, by definition of S_k and (2.1.2), Evidently,

$$\|A(\mathbf{t})\|^{-1}|\mathbf{x}|_p \leq |A^{-1}(\mathbf{t})\mathbf{x}|_p = p^k.$$

Making use of the condition $1 \leq q < \infty$, one has

$$\begin{aligned}&\|f\chi_{A(\mathbf{t})S_k)}\|_{L^q(w;\mathbb{Q}_p^n}\\ &\leq \left(\int_{|\mathbf{x}|_p\leq\|A(\mathbf{t})\|p^k}|f(\mathbf{x})|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q} \leq C\sum_{j=-\infty}^{\log_p\|A(\mathbf{t})\|}\|f\chi_{k+j}\|_{L^q(w;\mathbb{Q}_p^n)}. \qquad (2.2.2)\end{aligned}$$

Inequalities (2.2.1) and (2.2.2) together yield

$$\begin{aligned}&\|(H_{\Phi,A}f)\chi_k\|_{L^q(w;\mathbb{Q}_p^n)}\\ &\leq C\int_{\mathbb{Q}_p^n}\sum_{j=-\infty}^{\log_p\|A(\mathbf{t})\|}\|f\chi_{k+j}\|_{L^q(w;\mathbb{Q}_p^n)}|\det A^{-1}(\mathbf{t})|_p^{1/q}G(A^{-1}(\mathbf{t}),\beta/q)|\Phi(\mathbf{t})|d\mathbf{t}, \qquad (2.2.3)\end{aligned}$$

Hence, by means of Minkowski inequality and (2.2.3), we have

$$\begin{aligned}
&\|H_{\Phi,A}f\|_{MK^{\alpha,\lambda}_{l,q}(w;\mathbb{Q}_p^n)}\\
&= \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda(n+\beta)/n}\left\{\sum_{k=-\infty}^{k_0}\left(p^{k\alpha(n+\beta)/n}\,\|(H_{\Phi,A}f)\,\chi_k\|_{L^q(w;\mathbb{Q}_p^n)}\right)^l\right\}^{1/l}\\
&\leq C\int_{\mathbb{Q}_p^n}|\det A^{-1}(\mathbf{t})|_p^{1/q}G(A^{-1}(\mathbf{t}),\beta/q)|\Phi(\mathbf{t})|\\
&\times \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda(n+\beta)/n}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{j=-\infty}^{\log_p\|A(\mathbf{t})\|}p^{k\alpha(n+\beta)/n}\,\|f\chi_{k+j}\|_{L^q(w;\mathbb{Q}_p^n)}\right)^l\right\}^{1/l}d\mathbf{t}\\
&\leq C\int_{\mathbb{Q}_p^n}|\det A^{-1}(\mathbf{t})|_p^{1/q}G(A^{-1}(\mathbf{t}),\beta/q)|\Phi(\mathbf{t})|\sum_{j=-\infty}^{\log_p\|A(\mathbf{t})\|}p^{j(\lambda-\alpha)(n+\beta)/n}\\
&\times \sup_{k_0\in\mathbb{Z}} p^{-(k_0+j)\lambda(n+\beta)/n}\left\{\sum_{k=-\infty}^{k_0+j}p^{k\alpha(n+\beta)l/n}\,\|f\chi_k\|^l_{L^q(w;\mathbb{Q}_p^n)}\right\}^{1/l}d\mathbf{t}. \qquad (2.2.4)
\end{aligned}$$

Since, $\alpha<\lambda$, therefore

$$\sum_{j=-\infty}^{\log_p\|A(\mathbf{t})\|}p^{j(\lambda-\alpha)(n+\beta)/n}=\frac{\|A(\mathbf{t})\|^{(\lambda-\alpha)(n+\beta)/n}}{1-p^{(\alpha-\lambda)(n+\beta)/n}}. \qquad (2.2.5)$$

Substituting the value of this sum into (2.2.4), we get

$$\begin{aligned}
&\|H_{\Phi,A}f\|_{MK^{\alpha,\lambda}_{l,q}(w;\mathbb{Q}_p^n)}\\
&\leq C\|f\|_{MK^{\alpha,\lambda}_{l,q}(w;\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}|\det A^{-1}(\mathbf{t})|_p^{1/q}G(A^{-1}(\mathbf{t}),\beta/q)\|A(\mathbf{t})\|^{(\lambda-\alpha)(n+\beta)/n}|\Phi(\mathbf{t})|d\mathbf{t}.
\end{aligned}$$

Having slightly strict conditions on Φ and the norm of the matrix A, we get the sharp result as below:

Theorem 2.2.2 *Let* $1\leq q,l<\infty,\ 0\leq\lambda,\ \alpha<\lambda<\alpha+n/q,\ \beta>-n,\ w(\mathbf{x})=|\mathbf{x}|_p^\beta$ *and* Φ *be a non-negative function. Let there exists a constant* C_0 *free from* t *in such a way that* $\|A(\mathbf{t})\|^{-1}\geq\frac{1}{C_0}\|A^{-1}(\mathbf{t})\|$ *for every* $\mathbf{t}\in\mathrm{supp}(\Phi)$*, then* $H_{\Phi,A}$ *is bounded on* $MK^{\alpha,\lambda}_{l,q}(w;\mathbb{Q}_p^n)$ *if and only if*

$$\int_{\mathbb{Q}_p^n}\|A(\mathbf{t})\|^{(\lambda-\alpha-n/q)(n+\beta)/n}\Phi(\mathbf{t})d\mathbf{t}<\infty.$$

Proof. The sufficient part can be obtained from Theorem 2.2.1, so we are only interested in necessary part. If $\|A(\mathbf{t})\|^{-1} \geq \frac{1}{C_0}\|A^{-1}(\mathbf{t})\|$, then the inequality (2.1.1) reduces to

$$\|A(\mathbf{t})\|^{-n} \simeq |\det A^{-1}(\mathbf{t})|_p \simeq \|A^{-1}(\mathbf{t})\|^n, \tag{2.2.6}$$

and we divide the remaining proof into couple of cases which are as follows.
Case I: $\lambda \neq 0$.
For present case we choose

$$f_0(\mathbf{x}) = |\mathbf{x}|_p^{(\lambda-\alpha-n/q)(n+\beta)/n}.$$

Obviously, $f_0 \in L^q_{\rm loc}(w;\mathbb{Q}_p^n\backslash\{0\})$, therefore, by Lemma 2.1.3, we have

$$\|(f_0)\chi_k\|^q_{L^q(w;\mathbb{Q}_p^n)} = \int_{S_k} |\mathbf{x}|_p^{(\lambda-\alpha-n/q)(n+\beta)q/n} w(\mathbf{x})d\mathbf{x} = p^{k(\lambda-\alpha)(n+\beta)q/n} w(S_0).$$

Also,

$$\begin{aligned}\|f_0\|_{MK^{\alpha,\lambda}_{l,q}(w;\mathbb{Q}_p^n)} &= \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda(n+\beta)/n}\left\{\sum_{k=-\infty}^{k_0}\left(p^{k\alpha(n+\beta)/n}\|(f_0)\chi_k\|_{L^q(w;\mathbb{Q}_p^n)}\right)^l\right\}^{1/l}\\ &= w(S_0)^{1/q}\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda(n+\beta)/n}\left\{\sum_{k=-\infty}^{k_0} p^{k\lambda(n+\beta)l/n}\right\}^{1/l}\\ &= w(S_0)^{1/q}\frac{p^{\lambda(n+\beta)/n}}{\left(p^{\lambda(n+\beta)l/n}-1\right)^{1/l}} < \infty.\end{aligned}$$

Also, with a stipulation $\lambda < \alpha + n/q$, we get

$$\begin{aligned}H_{\Phi,A}f_0(\mathbf{x}) &= \int_{\mathbb{Q}_p^n} |A(\mathbf{t})\mathbf{x}|_p^{(\lambda-\alpha-n/q)(n+\beta)/n}\Phi(\mathbf{t})d\mathbf{t}\\ &\geq |\mathbf{x}|_p^{(\lambda-\alpha-n/q)(n+\beta)/n}\int_{\mathbb{Q}_p^n}\|A(\mathbf{t})\|^{(\lambda-\alpha-n/q)(n+\beta)/n}\Phi(\mathbf{t})d\mathbf{t}\\ &= f_0(\mathbf{x})\int_{\mathbb{Q}_p^n}\|A(\mathbf{t})\|^{(\lambda-\alpha-n/q)(n+\beta)/n}\Phi(\mathbf{t})d\mathbf{t}.\end{aligned}$$

Since $H_{\Phi,A}$ is bounded on $MK^{\alpha,\lambda}_{l,q}(w;\mathbb{Q}_p^n)$, it follows that

$$\int_{\mathbb{Q}_p^n}\|A(\mathbf{t})\|^{(\lambda-\alpha-n/q)(n+\beta)/n}\Phi(\mathbf{t})d\mathbf{t} \leq \|H_{\Phi,A}\|_{MK^{\alpha,\lambda}_{l,q}(w;\mathbb{Q}_p^n)\longrightarrow MK^{\alpha,\lambda}_{l,q}(w;\mathbb{Q}_p^n)} < \infty.$$

Case II: If $\lambda = 0$.
For the current case, the space $MK^{\alpha,\lambda}_{l,q}(w;\mathbb{Q}_p^n)$ reduces to $K_q^{\alpha,l}(w;\mathbb{Q}_p^n)$ which is the

weighted Herz space. In view of the condition $\alpha + n/q > 0$, we choose

$$f_\epsilon(\mathbf{x}) = |\mathbf{x}|_p^{-(\alpha+n/q)(n+\beta)/n-\epsilon/q}\chi_{\{|\mathbf{x}|_p\geq 1\}} \tag{2.2.7}$$

Hence, for $k \geq 0$, we have

$$\|(f_\epsilon)\chi_k\|^q_{L^q(w;\mathbb{Q}_p^n)} = \int_{S_k} |\mathbf{x}|_p^{-(\alpha+n/q)(n+\beta)q/n-\epsilon} w(\mathbf{x})d\mathbf{x} = p^{-k\alpha(n+\beta)q/n-k\epsilon}w(S_0).$$

Also,

$$\begin{aligned}\|f_\epsilon\|_{\dot{K}_q^{\alpha,l}(w;\mathbb{Q}_p^n)} &= \left\{\sum_{k=0}^{\infty}\left(p^{k\alpha(n+\beta)/n}\|(f_\epsilon)\chi_k\|_{L^q(w;\mathbb{Q}_p^n)}\right)^l\right\}^{1/l}\\ &= w(S_0)^{1/q}\left\{\sum_{k=0}^{\infty}p^{-k\epsilon l/q}\right\}^{1/l} = w(S_0)^{1/q}\frac{p^{\epsilon/q}}{(p^{\epsilon l/q}-1)^{1/l}} < \infty.\end{aligned}$$

On the other hand, we get

$$\begin{aligned}H_{\Phi,A}f_\epsilon(\mathbf{x}) &= \int_{\mathbb{Q}_p^n} f_\epsilon(A(\mathbf{t})\mathbf{x})\chi_{\{|A(\mathbf{t})\mathbf{x}|_p>1\}}\Phi(\mathbf{t})d\mathbf{t}\\ &\geq \left(\int_{\|A(\mathbf{t})\|\geq\frac{1}{|\mathbf{x}|_p}} \|A(\mathbf{t})\|^{-(\alpha+n/q)(n+\beta)/n-\epsilon/q}\Phi(\mathbf{t})d\mathbf{t}\right)|\mathbf{x}|_p^{-(\alpha+n/q)(n+\beta)/n-\epsilon/q},\end{aligned}$$

which suggest that

$$\begin{aligned}H_{\Phi,A}f_\epsilon(\mathbf{x}) &= \left\{\sum_{k=-\infty}^{\infty} p^{k\alpha(n+\beta)l/n}\|(H_{\Phi,A}f_\epsilon)\chi_k\|^l_{L^q(w;\mathbb{Q}_p^n)}\right\}^{1/l}\\ &\geq \left\{\sum_{k=-\infty}^{\infty} p^{k\alpha(n+\beta)l/n}\left(\int_{S_k}|\mathbf{x}|_p^{-(\alpha+n/q)(n+\beta)q/n-\epsilon}\right.\right.\\ &\quad\times\left.\left.\left|\int_{\|A(\mathbf{t})\|\geq\frac{1}{|\mathbf{x}|_p}}\|A(\mathbf{t})\|^{-(\alpha+n/q)(n+\beta)/n-\epsilon/q}\Phi(\mathbf{t})d\mathbf{t}\right|^q w(\mathbf{x})d\mathbf{x}\right)^{l/q}\right\}^{1/l}\\ &\geq w(S_0)^{1/q}\left\{\sum_{k=-\infty}^{\infty}p^{-k\epsilon l/q}\left(\int_{\|A(\mathbf{t})\|\geq p^{-k}}\|A(\mathbf{t})\|^{-(\alpha+n/q)(n+\beta)/n-\epsilon/q}\Phi(\mathbf{t})d\mathbf{t}\right)^l\right\}^{1/l}.\end{aligned}$$

Let us take $\epsilon = p^{-m}$ then $|\epsilon|_p = p^m$. Therefore, for $0 \leq m \leq k$ we have

$$\begin{aligned}&\|H_{\Phi,A}f_\epsilon\|_{\dot{K}_q^{\alpha,l}(w;\mathbb{Q}_p^n)}\\ &\geq w(S_0)^{1/q}\int_{\|A(\mathbf{t})\|\geq\epsilon}\|A(\mathbf{t})\|^{-(\alpha+n/q)(n+\beta)/n-\epsilon/q}\Phi(\mathbf{t})d\mathbf{t}\left\{\sum_{k=m}^{\infty}p^{-k\epsilon l/q}\right\}^{1/l}\\ &\geq \epsilon^{\epsilon/q}\|f_\epsilon\|_{\dot{K}_q^{\alpha,l}(w;\mathbb{Q}_p^n)}\int_{\|A(\mathbf{t})\|\geq\epsilon}\|A(\mathbf{t})\|^{-(\alpha+n/q)(n+\beta)/n-\epsilon/q}\Phi(\mathbf{t})d\mathbf{t}.\end{aligned}$$

Letting $\epsilon \to 0^+$, we get

$$\int_{\mathbb{Q}_p^n} \|A(\mathbf{t})\|^{-(\alpha+n/q)(n+\beta)/n}\Phi(\mathbf{t})d\mathbf{t} \leq \|H_{\Phi,A}\|_{K_q^{\alpha,l}(w;\mathbb{Q}_p^n)\longrightarrow K_q^{\alpha,l}(w;\mathbb{Q}_p^n)} < \infty.$$

Here, it is important to note that we cannot take $\alpha = \lambda = 0$ in Theorems 2.2.1 and 2.2.2 and therefore cannot deduce results for Lebesgue space $L^q(w;\mathbb{Q}_p^n)$. The reason is that in case $\alpha = \lambda = 0$, the series (2.2.5) will be no longer convergent.

2.3 Bounds of Hausdorff Operator on Weighted Morrey-type Spaces

In a present section, we show the boundedness of $H_{\Phi,A}$ on Morrey spaces $L^{q,\lambda}(w;\mathbb{Q}_p^n)$, Moreover, the results for Lebesgue space $L^q(w;\mathbb{Q}_p^n)$ can be deduced from the Theorems of this section by assigning special values to the parameters.

Theorem 2.3.1 *Suppose* $0 > \lambda \geq -1/q$ $\infty > q > 1$ *and* $w(\mathbf{x}) = |\mathbf{x}|_p^{\beta}$, *then* $H_{\Phi,A}$ *is bounded on* $L^{q,\lambda}(w;\mathbb{Q}_p^n)$ *and satisfy the following inequality*

$$\begin{aligned}&\|H_{\Phi,A}f\|_{L^{q,\lambda}(w;\mathbb{Q}_p^n)}\\ &\leq \|f\|_{L^{q,\lambda}(w;\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n} |\det A(\mathbf{t})|_p^{\lambda} G(A(\mathbf{t}),\beta(\lambda+1/q))G(A^{-1}(\mathbf{t}),\beta/q)|\Phi(\mathbf{t})|d\mathbf{t}.\end{aligned}$$

Proof. Let $f \in L^{q,\lambda}(w;\mathbb{Q}_p^n)$, then an application of Minkowski's inequality, p-adic change of variables and Lemma 2.1.4, yields

$$\begin{aligned}&\left(\frac{1}{w(B_\gamma(\mathbf{a}))^{\lambda q+1}}\int_{B_\gamma(\mathbf{a})}|H_{\Phi,A}f(\mathbf{x})|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}\\
&=\left(\frac{1}{w(B_\gamma(\mathbf{a}))^{\lambda q+1}}\int_{B_\gamma(\mathbf{a})}\left|\int_{\mathbb{Q}_p^n} f(A(\mathbf{t})\mathbf{x})\Phi(\mathbf{t})d\mathbf{t}\right|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}\\
&\leq \int_{\mathbb{Q}_p^n}\left(\frac{1}{w(B_\gamma(\mathbf{a}))^{\lambda q+1}}\int_{B_\gamma(\mathbf{a})}|f(A(\mathbf{t})\mathbf{x})|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}|\Phi(\mathbf{t})|dt\\
&\leq \int_{\mathbb{Q}_p^n}\left(\frac{1}{w(B_\gamma(\mathbf{a}))^{\lambda q+1}}\int_{A(\mathbf{t})B_\gamma(\mathbf{a})}|f(\mathbf{x})|^q w(A^{-1}(\mathbf{t})\mathbf{x})|\det A^{-1}(\mathbf{t})|_p d\mathbf{x}\right)^{1/q}|\Phi(\mathbf{t})|dt\\
&=\int_{\mathbb{Q}_p^n}|\det A^{-1}(\mathbf{t})|_p^{1/q}G(A^{-1}(\mathbf{t}),\beta/q)|\Phi(\mathbf{t})|\\
&\qquad\times\left(\frac{1}{w(B_\gamma(\mathbf{a}))^{\lambda q+1}}\int_{A(\mathbf{t})B_\gamma(\mathbf{a})}|f(\mathbf{x})|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}d\mathbf{t}. \qquad (2.3.1)\end{aligned}$$

Making use of the condition $1+\lambda q\geq 0$ and Lemma 2.1.5(i), we have

$$\begin{aligned}&\left(\frac{1}{w(B_\gamma(\mathbf{a}))^{\lambda q+1}}\int_{B_\gamma(\mathbf{a})}|H_{\Phi,A}f(\mathbf{x})|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}\\ &\leq \int_{\mathbb{Q}_p^n}\left(\frac{1}{w(A(\mathbf{t})B_\gamma(\mathbf{a}))^{\lambda q+1}}\int_{A(\mathbf{t})B_\gamma(\mathbf{a})}|f(\mathbf{x})|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}\\ &\quad\times(|\det A(\mathbf{t})|_p G(A(\mathbf{t}),\beta))^{\lambda+1/q}|\det A^{-1}(\mathbf{t})|_p^{1/q}G(A^{-1}(\mathbf{t}),\beta/q)|\Phi(\mathbf{t})|d\mathbf{t}\\ &\leq \|f\|_{L^{q,\lambda}(w;\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}|\det A(\mathbf{t})|_p^{\lambda}G(A(\mathbf{t}),\beta(\lambda+1/q))G(A^{-1}(\mathbf{t}),\beta/q)|\Phi(\mathbf{t})|d\mathbf{t}.\end{aligned}$$

In the next Theorem, we sharpen the result by imposing special conditions on Φ and on the norm of matrix $A(\mathbf{t})$.

Theorem 2.3.2 *Suppose* $0>\lambda>-1/q$ *and* $\infty>q>1$, $-n<\beta$, $w(\mathbf{x})=|\mathbf{x}|_p^{\beta}$ *and* Φ *is a non-negative function. Suppose there exists a constant* C_0 *free from* $\mathbf{t}$ *in such a way that* $\|A(\mathbf{t})\|^{-1}\geq\frac{1}{C_0}\|A^{-1}(\mathbf{t})\|$ *for every* $\mathbf{t}\in\operatorname{supp}(\Phi)$, *then* $H_{\Phi,A}$ *is bounded on* $L^{q,\lambda}(w;\mathbb{Q}_p^n)$ *if and only if*

$$\int_{\mathbb{Q}_p^n}\|A(\mathbf{t})\|^{(n+\beta)\lambda}\Phi(\mathbf{t})d\mathbf{t}<\infty.$$

Proof. The sufficient part follows from Theorem 2.3.1, thus we will only prove the necessary part. If $\|A(\mathbf{t})\|^{-1}\geq\frac{1}{C_0}\|A^{-1}(\mathbf{t})\|$, then the inequality (2.1.1) gives us (2.2.6). Here, we consider the cases $-\frac{1}{q}<\lambda<0$ and $\lambda=-\frac{1}{q}$, separately.
Case I: $-1/q<\lambda<\infty$.
We consider $f_0(\mathbf{x})=|\mathbf{x}|_p^{(n+\beta)\lambda}$, then by Lemma 2.4 in [21], $f_0\in L^{q,\lambda}(w;\mathbb{Q}_p^n)$ and $\|f_0\|_{L^{q,\lambda}(w;\mathbb{Q}_p^n)}>0$. Hence, we have

$$\begin{aligned}H_{\Phi,A}f_0&=\int_{\mathbb{Q}_p^n}|A(\mathbf{t})\mathbf{x}|_p^{(n+\beta)\lambda}\Phi(\mathbf{t})d\mathbf{t}\\ &\succeq|\mathbf{x}|_p^{(n+\beta)\lambda}\int_{\mathbb{Q}_p^n}\|A(\mathbf{t})\|^{(n+\beta)\lambda}\Phi(\mathbf{t})d\mathbf{t}\\ &=f_0(\mathbf{x})\int_{\mathbb{Q}_p^n}\|A(\mathbf{t})\|^{(n+\beta)\lambda}\Phi(\mathbf{t})d\mathbf{t}.\end{aligned}$$

Therefore,

$$\int_{\mathbb{Q}_p^n}\|A(\mathbf{t})\|^{(n+\beta)\lambda}\Phi(\mathbf{t})d\mathbf{t}\leq\|H_{\Phi,A}\|_{L^{q,\lambda}(w;\mathbb{Q}_p^n)\longrightarrow L^{q,\lambda}(w;\mathbb{Q}_p^n)}.$$

Case II: $\lambda = -1/q$.
In this case weighted Morrey space $L^{q,\lambda}(w;\mathbb{Q}_p^n)$ on p-adic field reduces to weighted p-adic Lebesgue space $L^q(w;\mathbb{Q}_p^n)$. By taking $\alpha = 0$ in (2.2.7) we choose

$$f_\epsilon(x) = |\mathbf{x}|_p^{-(n+\beta)/q-\epsilon/q}\chi_{\{|\mathbf{x}|_p\geq 1\}}.$$

In addition, taking $l = q$ and following the same procedure as followed in the proof of Theorem 2.2.2(Case II), it is easy to conclude that

$$\int_{\mathbb{Q}_p^n} \|A(\mathbf{t})\|^{-(n+\beta)/q}\Phi(\mathbf{t})d\mathbf{t} \leq \|H_{\Phi,A}\|_{L^q(w;\mathbb{Q}_p^n)\longrightarrow L^q(w;\mathbb{Q}_p^n)}.$$

Next, we give the following result for the boundedness of $H_{\Phi,A}$ on p-adic central Morrey space $\dot{B}^{q,\lambda}(w;\mathbb{Q}_p^n)$:

Theorem 2.3.3 *Suppose* $\infty > q > 1$, $0 > \lambda > -1/q$ *and* $w(\mathbf{x}) = |\mathbf{x}|_p^\beta$, *then* $H_{\Phi,A}$ *is bounded on* $\dot{B}^{q,\lambda}(w;\mathbb{Q}_p^n)$ *and satisfy the following inequality*

$$\|H_{\Phi,A}\|_{\dot{B}^{q,\lambda}(w;\mathbb{Q}_p^n)} \leq \|f\|_{\dot{B}^{q,\lambda}(w;\mathbb{Q}_p^n)} w(S_0)^{\lambda+1/q}\left(\frac{p^{(n+\beta)}}{p^{(n+\beta)}-1}\right)^{\lambda+1/q}$$
$$\times \int_{\mathbb{Q}_p^n} |\det A^{-1}(\mathbf{t})|_p^{1/q}G(A^{-1}(\mathbf{t}),\beta/q)\|A(\mathbf{t})\|^{(n+\beta)(\lambda+1/q)}\Phi(\mathbf{t})d\mathbf{t}.$$

Proof. Let $f \in \dot{B}^{q,\lambda}(w;\mathbb{Q}_p^n)$. In this case $B_\gamma(a) = B_\gamma$, therefore, we infer from (2.3.1) that

$$\left(\frac{1}{w(B_\gamma)^{\lambda q+1}}\int_{B_\gamma} |H_{\Phi,A}f(\mathbf{x})|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}$$
$$= \int_{\mathbb{Q}_p^n}\left(\frac{1}{w(B_\gamma)^{\lambda q+1}}\int_{A(\mathbf{t})B_\gamma} |f(\mathbf{x})|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q} |\det A^{-1}(\mathbf{t})|_p^{1/q}G(A^{-1}(\mathbf{t}),\beta/q)\Phi(\mathbf{t})d\mathbf{t}.$$

In view of the condition that $1 + \lambda q \geq 0$ and Lemma 2.1.5 (ii), we get

$$\left(\frac{1}{w(B_\gamma)^{\lambda q+1}}\int_{B_\gamma} |H_{\Phi,A}f(\mathbf{x})|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}$$
$$\leq \left(\frac{w(S_0)p^{(n+\beta)}}{p^{(n+\beta)}-1}\right)^{\lambda+1/q}\int_{\mathbb{Q}_p^n} |\det A^{-1}(\mathbf{t})|_p^{1/q}G(A^{-1}(\mathbf{t}),\beta/q)\|A(\mathbf{t})\|^{(n+\beta)(\lambda+1/q)}\Phi(\mathbf{t})$$
$$\times\left(\frac{1}{w(B_{\gamma+\log_p\|A(\mathbf{t})\|})^{\lambda q+1}}\int_{B_{\gamma+\log_p\|A(\mathbf{t})\|}} |f(\mathbf{x})|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q} d\mathbf{t}$$
$$\leq w(S_0)^{\lambda+1/q}\left(\frac{p^{(n+\beta)}}{p^{(n+\beta)}-1}\right)^{\lambda+1/q}\|f\|_{\dot{B}_w^{q,\lambda}(\mathbb{Q}_p^n)}$$
$$\times \int_{\mathbb{Q}_p^n} |\det A^{-1}(\mathbf{t})|_p^{1/q}G(A^{-1}(\mathbf{t}),\beta/q)\|A(\mathbf{t})\|^{(n+\beta)(\lambda+1/q)}\Phi(\mathbf{t})d\mathbf{t}.$$

Finally , the following theorem is stated without proof. The proof can easily be obtained from the standard analysis presented in this Chapter.

Theorem 2.3.4 *Suppose* $0 > \lambda > -1/q$ *and* $\infty > q > 1$, $-n > \beta$ *and* Φ *is a non-negative function. Suppose there is a constant* C_0 *free from* $\mathbf{t}$ *in such a way that* $\|A(\mathbf{t})\|^{-1} \geq \frac{1}{C_0}\|A^{-1}(\mathbf{t})\|$ *for every* $\mathbf{t} \in \mathrm{supp}(\Phi)$*, then* $H_{\Phi,A}$ *is bounded on* $\dot{B}^{q,\lambda}(w;\mathbb{Q}_p^n)$ *if and only if*

$$\int_{\mathbb{Q}_p^n} \|A(\mathbf{t})\|^{(n+\beta)\lambda}\Phi(\mathbf{t})d\mathbf{t} < \infty.$$

Chapter 3
Estimates for p-adic Hausdorff Operator and Commutators

3.1 Introduction

The boundedness properties of commutator operators is an important aspect of harmonic analysis as these are useful in the study of characterization of function spaces and regularity theory of partial differential equations. The commutators of Hausdorff operator $H_{\Phi,A}$ with locally integrable function b were defined in (1.5.1). The boundedness of the analog of $H^b_{\Phi,A}$ on $\mathbb{R}^n$ and its special case, when $A(\mathbf{t})$ is diagonal, were discussed in [55, 56, 57, 59, 93, 106]. However, this topic still needs further considerations in the sense of its boundedness on p-adic function spaces.

In the current Chapter, we mainly discuss the boundedness of $H^b_{\Phi,A}$ on p-adic weighted Herz type spaces when b is either from $CMO^{q_2}(w,\mathbb{Q}_p^n)$ or $\Lambda_\delta(\mathbb{Q}_p^n)$. In addition an intermediate result stating the boundedness of matrix Hausdorff operator on p-adic field on $CMO^{q,\lambda}(w,\mathbb{Q}_p^n)$ spaces will be given at first.

3.2 Hausdorff Operator on p-adic $CMO^{q,\lambda}(w,\mathbb{Q}_p^n)$

Having discussed the basic theory on p-adic function spaces and special result regarding the matrix Hausdorff operator in Chapters 1 and 2, now we come up with the following theroem:

Theorem 3.2.1 *Suppose $1/n > \lambda > 0$, $\infty > q > 1$, $-n < \beta$ and $w(\mathbf{x}) = |\mathbf{x}|_p^\beta$, then $H_{\Phi,A}$ is bounded on $CMO^{q,\lambda}(w,\mathbb{Q}_p^n)$ and satisfies the following inequality:*

$$\|H_{\Phi,A}\|_{CMO^{q,\lambda}(w,\mathbb{Q}_p^n)} \leq K_1\|f\|_{CMO^{q,\lambda}(w,\mathbb{Q}_p^n)},$$

where

$$K_1 = \int_{\mathbb{Q}_p^n} |\det A(\mathbf{t})|_p^{\lambda} G(A^{-1}(\mathbf{t}), \beta/q) G(A(\mathbf{t}), \beta(\lambda + 1/q)) |\Phi(\mathbf{t})| d\mathbf{t}.$$

Proof. Suppose $f \in CMO^{q,\lambda}(w,\mathbb{Q}_p^n)$. By means of Fubini theorem and p-adic change of variables we have:

$$\begin{aligned}
\left(H_{\Phi,A} f\right)_{B_\gamma} &= \frac{1}{|B_\gamma|}\int_{B_\gamma}\left(\int_{\mathbb{Q}_p^n} f(A(\mathbf{t})\mathbf{x})\Phi(\mathbf{t})d\mathbf{t}\right)d\mathbf{x}\\
&= \int_{\mathbb{Q}_p^n}\left(\frac{1}{|B_\gamma|}\int_{B_\gamma} f(A(\mathbf{t})\mathbf{x})d\mathbf{x}\right)\Phi(\mathbf{t})d\mathbf{t}\\
&= \int_{\mathbb{Q}_p^n}\left(\frac{1}{|B_\gamma|}\int_{A(\mathbf{t})B_\gamma} f(\mathbf{x})d\mathbf{x}\right)|\det A^{-1}(\mathbf{t})|_p\Phi(\mathbf{t})d\mathbf{t}\\
&= \int_{\mathbb{Q}_p^n}\left(\frac{1}{|A(\mathbf{t})B_\gamma|}\int_{A(\mathbf{t})B_\gamma} f(\mathbf{x})d\mathbf{x}\right)\Phi(\mathbf{t})d\mathbf{t}\\
&= \int_{\mathbb{Q}_p^n} f_{A(\mathbf{t})B_\gamma}\Phi(\mathbf{t})d\mathbf{t}.
\end{aligned}$$

Using Minkowski's inequality, Lemma 2.1.4 and Lemma 2.1.5(i) with $\mathbf{a} = 0$, we are down to:

$$\begin{aligned}
&\left(\frac{1}{w(B_\gamma)^{\lambda q+1}}\int_{B_\gamma}|H_{\Phi,A}f(\mathbf{x}) - (H_{\Phi,A})_{B_\gamma}|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}\\
&= \left(\frac{1}{w(B_\gamma)^{\lambda q+1}}\int_{B_\gamma}\left|\int_{\mathbb{Q}_p^n}(f(A(\mathbf{t})\mathbf{x}) - f_{A(\mathbf{t})B_\gamma})\Phi(\mathbf{t})d\mathbf{t}\right|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}\\
&\leq \int_{\mathbb{Q}_p^n}\left(\frac{1}{w(B_\gamma)^{\lambda q+1}}\int_{B_\gamma}\left|f(A(\mathbf{t})\mathbf{x}) - f_{A(\mathbf{t})B_\gamma}\right|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}|\Phi(\mathbf{t})|d\mathbf{t}\\
&\leq \int_{\mathbb{Q}_p^n}|\det A^{-1}(\mathbf{t})|_p^{1/q}G(A^{-1}(\mathbf{t}),\beta/q)\\
&\qquad\times\left(\frac{1}{w(B_\gamma)^{1+\lambda q}}\int_{A(\mathbf{t})B_\gamma}\left|f(\mathbf{x}) - f_{A(\mathbf{t})B_\gamma}\right|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}|\Phi(\mathbf{t})|d\mathbf{t}
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbb{Q}_p^n}|\det A(\mathbf{t})|_p^{\lambda}G(A^{-1}(\mathbf{t}),\beta/q)G(A(\mathbf{t}),\beta(\lambda+1/q))\\
&\quad\times\left(\frac{1}{w(A(\mathbf{t})B_\gamma)^{\lambda q+1}}\int_{A(\mathbf{t})B_\gamma}\left|f(\mathbf{x}) - f_{A(\mathbf{t})B_\gamma}\right|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}|\Phi(\mathbf{t})|d\mathbf{t}\\
&\leq \|f\|_{CMO^{q,\lambda}(w,\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}|\det A(\mathbf{t})|_p^{\lambda}G(A^{-1}(\mathbf{t}),\beta/q)G(A(\mathbf{t}),\beta(\lambda+1/q))|\Phi(\mathbf{t})|d\mathbf{t}.
\end{aligned}$$

Thus, we completed the proof of Theorem 3.2.1.

3.3 Weighted $CBMO$ Estimates for $H^b_{\Phi,A}$ on Weighted Herz-Morrey Spaces

The present section demonstrates that commutators of Hausdorff operator are bounded when $b \in CMO^q(w, \mathbb{Q}_p^n)$.

Theorem 3.3.1 *Suppose* $1 \le l, q, q_1, q_2 < \infty$, $1/q + 1/q_1 = 1/q_2$, $\alpha_1 = \alpha_2 + n/q$, $0 \le \lambda > \alpha_1$, $\beta > -n$ *and* $w(\mathbf{x}) = |\mathbf{x}|_p^\beta$. *Assume that* $b \in CMO^q(w, \mathbb{Q}_p^n)$ *and*

$$\varphi(\mathbf{t}) = |\det A^{-1}(\mathbf{t})|_p^{1/q_1} G(A^{-1}(\mathbf{t}), \beta/q_1) \max\left\{1, G(A^{-1}(\mathbf{t}), \beta/q) G(A(\mathbf{t}), \beta/q)\right\} |\Phi(\mathbf{t})|.$$

Then the commutator operator $H^b_{\Phi,A}$ *is bounded from* $MK^{\alpha_1,\lambda}_{l,q_1}(w, \mathbb{Q}_p^n)$ *to* $MK^{\alpha_2,\lambda}_{l,q_2}(w, \mathbb{Q}_p^n)$ *and satisfies the inequality:*

$$\|H^b_{\Phi,A} f\|_{MK^{\alpha_2,\lambda}_{l,q_2}(w,\mathbb{Q}_p^n)} \le CK_2 \|b\|_{CMO^{q_2}(w,\mathbb{Q}_p^n)} \|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}$$

where

$$\begin{aligned} K_2 = &\int_{\|A(\mathbf{t})\| \le 1} \|A(\mathbf{t})\|^{(\lambda-\alpha_1)(n+\beta)/n} \left(\frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p} + \log_p \frac{p}{\|A(\mathbf{t}\|} \right) \varphi(\mathbf{t}) d\mathbf{t} \\ &+ \int_{\|A(\mathbf{t})\| > 1} \|A(\mathbf{t})\|^{(\lambda-\alpha_1)(n+\beta)/n} \left(\frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p} + \log_p(p\|A(\mathbf{t})\|) \right) \varphi(\mathbf{t}) d\mathbf{t}. \end{aligned}$$

Proof. Let $f \in MK^{\alpha_1,\lambda}_{l,q_1}(w, \mathbb{Q}_p^n)$ and $b \in CMO^q(w, \mathbb{Q}_p^n)$,

$$\begin{aligned} \|(H^b_{\Phi,A} f)\chi_k\|_{L^{q_2}(w,\mathbb{Q}_p^n)} = &\left(\int_{S_k} \left| \int_{\mathbb{Q}_p^n} (b(\mathbf{x}) - b(A(\mathbf{t})\mathbf{x})) f(A(\mathbf{t})\mathbf{x}) \Phi(\mathbf{t}) d\mathbf{t} \right|^{q_2} w(\mathbf{x}) d\mathbf{x} \right)^{1/q_2} \\ \le &\int_{\mathbb{Q}_p^n} \left(\int_{S_k} |(b(\mathbf{x}) - b(A(\mathbf{t})\mathbf{x})) f(A(\mathbf{t})\mathbf{x})|^{q_2} w(\mathbf{x}) d\mathbf{x} \right)^{1/q_2} |\Phi(\mathbf{t})| d\mathbf{t} \end{aligned}$$

$$\begin{aligned} \le &\int_{\mathbb{Q}_p^n} \left(\int_{S_k} |(b(\mathbf{x}) - b_{B_k}) f(A(\mathbf{t})\mathbf{x})|^{q_2} w(\mathbf{x}) d\mathbf{x} \right)^{1/q_2} |\Phi(\mathbf{t})| d\mathbf{t} \\ &+ \int_{\mathbb{Q}_p^n} \left(\int_{S_k} |(b_{B_k} - b_{A(\mathbf{t})B_k}) f(A(\mathbf{t})\mathbf{x})|^{q_2} w(\mathbf{x}) d\mathbf{x} \right)^{1/q_2} |\Phi(\mathbf{t})| d\mathbf{t} \\ &+ \int_{\mathbb{Q}_p^n} \left(\int_{S_k} |(b(A(\mathbf{t})\mathbf{x}) - b_{A(\mathbf{t})B_k}) f(A(\mathbf{t})\mathbf{x})|^{q_2} w(\mathbf{x}) d\mathbf{x} \right)^{1/q_2} |\Phi(\mathbf{t})| d\mathbf{t} \\ = &I + II + III. \end{aligned}$$

By Hölder's inequality, p-adic change of variables and Lemma 2.1.4, we estimate I as below:

$$
\begin{aligned}
I &\leq \int_{\mathbb{Q}_p^n}\left(\int_{S_k}|b(\mathbf{x})-b_{B_k}|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}\left(\int_{S_k}|f(A(\mathbf{t})\mathbf{x})|^{q_1}w(\mathbf{x})d\mathbf{x}\right)^{1/q_1}|\Phi(\mathbf{t})|d\mathbf{t}\\
&\leq \int_{\mathbb{Q}_p^n}\left(\int_{B_k}|b(\mathbf{x})-b_{B_k}|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}\\
&\quad\times\left(\int_{A(\mathbf{t})S_k}|f(\mathbf{x})|^{q_1}|\det A^{-1}(\mathbf{t})|_p G(A^{-1}(\mathbf{t}),\beta)w(\mathbf{x})d\mathbf{x}\right)^{1/q_1}|\Phi(\mathbf{t})|d\mathbf{t}\\
&\leq \int_{\mathbb{Q}_p^n}|\det A^{-1}(\mathbf{t})|_p^{1/q_1}G(A^{-1}(\mathbf{t}),\beta/q_1)\\
&\quad\times\left(\int_{B_k}|b(\mathbf{x})-b_{B_k}|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}\left(\int_{A(\mathbf{t})S_k}|f(\mathbf{x})|^{q_1}w(\mathbf{x})d\mathbf{x}\right)^{1/q_1}|\Phi(\mathbf{t})|d\mathbf{t}\\
&\leq w(B_k)^{1/q}\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\\
&\quad\times\int_{\mathbb{Q}_p^n}|\det A^{-1}(\mathbf{t})|_p^{1/q_1}G(A^{-1}(\mathbf{t}),\beta/q_1)\|f\chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}|\Phi(\mathbf{t})|d\mathbf{t}. \qquad (3.3.1)
\end{aligned}
$$

Similarly for III, first making p-adic change of variables and then making use of Hölder's inequality to have:

$$
\begin{aligned}
III &\leq \int_{\mathbb{Q}_p^n}\left(\int_{S_k}|(b(A(\mathbf{t})\mathbf{x})-b_{A(\mathbf{t})B_k})f(A(\mathbf{t})\mathbf{x})|^{q_2}w(\mathbf{x})d\mathbf{x}\right)^{1/q_2}|\Phi(\mathbf{t})|d\mathbf{t}\\
&= \int_{\mathbb{Q}_p^n}\det A^{-1}(\mathbf{t})|_p^{1/q_2}G(A^{-1}(\mathbf{t}),\beta/q_2)|\\
&\quad\times\left(\int_{A(\mathbf{t})S_k}|(b(\mathbf{x})-b_{A(\mathbf{t})B_k})f(\mathbf{x})|^{q_2}w(\mathbf{x})d\mathbf{x}\right)^{1/q_2}|\Phi(\mathbf{t})|d\mathbf{t}\\
&\leq \int_{\mathbb{Q}_p^n}\det A^{-1}(\mathbf{t})|_p^{1/q_2}G(A^{-1}(\mathbf{t}),\beta/q_2)|\\
&\quad\times\left(\int_{A(\mathbf{t})S_k}|b(\mathbf{x})-b_{A(\mathbf{t})B_k}|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}\left(\int_{A(\mathbf{t})S_k}|f(\mathbf{x})|^{q_1}w(\mathbf{x})d\mathbf{x}\right)^{1/q_1}|\Phi(\mathbf{t})|d\mathbf{t}\\
&\leq \int_{\mathbb{Q}_p^n}\det A^{-1}(\mathbf{t})|_p^{1/q_2}G(A^{-1}(\mathbf{t}),\beta/q_2)|\\
&\quad\times w(A(\mathbf{t})B_k)^{1/q}\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\|f\chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}|\Phi(\mathbf{t})|d\mathbf{t}.
\end{aligned}
$$

From (2.1.4) ,$1/q + 1/q_1 = 1/q_2$, and Lemma 2.1.5, the above inequality becomes:

$$\begin{aligned} III \leq & \|b\|_{CMO^q(w,\mathbb{Q}_p^n)} \int_{\mathbb{Q}_p^n} |\det A^{-1}(\mathbf{t})|_p^{1/q_2} G(A^{-1}(\mathbf{t}), \beta/q_1) G(A^{-1}(\mathbf{t}), \beta/q)| \\ & \times G(A(\mathbf{t}), \beta/q)| \det A(\mathbf{t})|_p^{1/q} w(B_k)^{1/q} \|f\chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)} |\Phi(\mathbf{t})| d\mathbf{t} \\ = & w(B_k)^{1/q} \|b\|_{CMO^q(w,\mathbb{Q}_p^n)} \int_{\mathbb{Q}_p^n} |\det A^{-1}(\mathbf{t})|_p^{1/q_1} G(A^{-1}(\mathbf{t}), \beta/q_1) \\ & \times G(A^{-1}(\mathbf{t}), \beta/q) G(A(\mathbf{t}), \beta/q) \|f\chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)} |\Phi(\mathbf{t})| d\mathbf{t}. \end{aligned} \tag{3.3.2}$$

The estimation of II is more of a same to that of I and III except that in this case, additionally, we have to bound the term $|b_{B_k} - b_{A(\mathbf{t})B_k}|$. Therefore, in this case, we will make use of the Hölder's inequality, p-adic change of variables and Lemma 2.1.3 to have:

$$\begin{aligned} II \leq & \int_{\mathbb{Q}_p^n} \left(\int_{S_k} |f(A(\mathbf{t})\mathbf{x})|^{q_1} w(\mathbf{x}) d\mathbf{x} \right)^{1/q_1} \left(\int_{S_k} w(\mathbf{x}) d\mathbf{x} \right)^{1/q} |b_{B_k} - b_{A(\mathbf{t})B_k}| |\Phi(\mathbf{t})| d\mathbf{t} \\ \leq & w(S_0)^{1/q} w(B_k)^{1/q} \\ & \times \int_{\mathbb{Q}_p^n} |\det A^{-1}(t)|_p^{1/q_1} G(A^{-1}(\mathbf{t}), \beta/q_1) \|f\chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)} |b_{B_k} - b_{A(\mathbf{t})B_k}| |\Phi(\mathbf{t})| d\mathbf{t}. \end{aligned}$$

Next, if $\|A(\mathbf{t})\| > 1$, then there exists an integer $0 \leq j$ such that:

$$p^j < \|A(\mathbf{t})\| \leq p^{j+1}.$$

Therefore,

$$|b_{B_k} - b_{A(\mathbf{t})B_k}| \leq \sum_{i=0}^{j} |b_{B_{k+i}} - b_{B_{k+i+1}}| + |b_{B_{k+j+1}} - b_{A(\mathbf{t})B_k}|.$$

Hölder's Inequality together with the definition of $CMO^q(w, \mathbb{Q}_p^n)$ yields:

$$\begin{aligned} |b_{B_{k+i}} - b_{B_{k+i+1}}| \leq & \frac{1}{|B_{k+i}|} \int_{B_{k+i}} |b(\mathbf{x}) - b_{B_{k+i+1}}| d\mathbf{x} \\ \leq & \frac{C}{|B_{k+i+1}|} \int_{B_{k+i+1}} |b(\mathbf{x}) - b_{B_{k+i+1}}| d\mathbf{x} \\ \leq & \frac{C}{|B_{k+i+1}|} \left(\int_{B_{k+i+1}} |b(\mathbf{x}) - b_{B_{k+i+1}}|^q w(\mathbf{x}) d\mathbf{x} \right)^{1/q} \\ & \times \left(\int_{B_{k+i+1}} w(\mathbf{x})^{q'/q} \right)^{1/q'} \end{aligned}$$

$$\begin{aligned}
&\le C\frac{w(B_{k+i+1})^{1/q}}{|B_{k+i+1}|}\left(\int_{B_{k+i+1}}|\mathbf{x}|_p^{-\beta q'/q}\right)^{1/q'}\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\\
&\le C\frac{p^{(n+\beta)(k+i+1)/q}}{p^{(k+i+1)n}}p^{(k+i+1)(-\beta/q+n/q')}\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\\
&=C\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}.
\end{aligned}$$

The other term can be handled in a similar fashion as below:

$$\begin{aligned}
|b_{B_{k+j+1}}-b_{A(\mathbf{t})B_k}| \le&\frac{1}{|A(\mathbf{t})B_k|}\int_{A(\mathbf{t})B_k}|b(\mathbf{x})-b_{B_{k+j+1}}|d\mathbf{x}\\
\le&\frac{1}{|A(\mathbf{t})B_k|}\left(\int_{B_{k+j+1}}|b(\mathbf{x})-b_{B_{k+j+1}}|^q w(\mathbf{x})d\mathbf{x}\right)^{1/q}\\
&\times\left(\int_{B_{k+j+1}}w(\mathbf{x})^{-q'/q}\right)^{1/q'}\\
\le&\frac{w(B_{k+j+1})^{1/q}}{|A(\mathbf{t})B_k|}\left(\int_{B_{k+j+1}}|\mathbf{x}|_p^{-\beta q'/q}d\mathbf{x}\right)^{1/q'}\\
&\times\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\\
\le&\frac{p^{(n+\beta)(k+j+1)/q}}{|\det A(\mathbf{t})|_p p^{kn}}p^{(k+j+1)(-\beta/q+n/q')}\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\\
=&\frac{p^{(j+1)n}}{|\det A(\mathbf{t})|_p}\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\\
\le&C\frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p}\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}.
\end{aligned}$$

Therefore, for $\|A(\mathbf{t})\|>1$

$$\begin{aligned}
|b_{B_k}-b_{A(\mathbf{t})B_k}|&\le C\left(j+1+\frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p}\right)\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\\
&\le C\left\{\log_p(\|A(\mathbf{t})p\|)+\frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p}\right\}\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}.
\end{aligned}$$

When $\|A(\mathbf{t})\|\le 1$, a similar argument yields:

$$|b_{B_k}-b_{A(\mathbf{t})B_k}|\le C\left\{\log_p\frac{p}{\|A(\mathbf{t})\|}+\frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p}\right\}\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}.$$

Therefore,

$$\begin{aligned}
II\le& Cw(B_k)^{1/q}\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}|\det A^{-1}(\mathbf{t})|_p^{1/q_1}G(A^{-1}(\mathbf{t}),\beta/q_1)|\\
&\times\left(\frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p}+1+\max\left\{\log_p\|A(\mathbf{t})\|,\log_p\frac{1}{\|A(\mathbf{t})\|}\right\}\right)\|f\chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}|\Phi(\mathbf{t})|d\mathbf{t}.
\end{aligned}$$

Finally, we combine the estimates for I,II and III to have:

$$\begin{aligned}
&\|(H^b_{\Phi,A}f)\chi_k\|_{L^{q_2}(w,\mathbb{Q}_p^n)}\\
&\leq Cw(B_k)^{1/q}\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\\
&\times\int_{\mathbb{Q}_p^n}\left(\frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p}+1+\max\left\{\log_p\|A(\mathbf{t})\|,\log_p\frac{1}{\|A(\mathbf{t})\|}\right\}\right)\|f\chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\varphi(\mathbf{t})d\mathbf{t}.
\end{aligned}$$

In order to avoid repetition of the same factor in the subsequent calculations, we let:

$$\psi(\mathbf{t})=\left(\frac{\|A(\mathbf{t})\|^n}{|\det A(\mathbf{t})|_p}+1+\max\left\{\log_p(\|A\mathbf{t})\|,\log_p\frac{1}{\|A(\mathbf{t})\|}\right\}\right)\varphi(\mathbf{t})$$

Also, we take:

$$\begin{aligned}
\|f\chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}&=\left(\int_{A(\mathbf{t})S_k}|f(\mathbf{x})|^{q_1}d\mathbf{x}\right)^{1/q_1}\\
&\leq\left(\int_{|\mathbf{x}|_p\leq\|A(\mathbf{t})\|p^k}|f(\mathbf{x})|^{q_1}d\mathbf{x}\right)^{1/q_1}\\
&\leq C\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}\|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}. \qquad (3.3.3)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|(H^b_{\Phi,A}f)\chi_k\|_{L^{q_2}(w,\mathbb{Q}_p^n)}\\
&\leq Cw(B_k)^{1/q}\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}\|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\psi(\mathbf{t})d\mathbf{t}. \qquad (3.3.4)
\end{aligned}$$

Now, using Morrey-Herz space's definition, the inequality (3.3.4), Minkowski's inequality and a stipulation $\alpha_1=\alpha_2+n/q$, we have:

$$\begin{aligned}
&\|H^b_{\Phi,A}f\|_{MK^{\alpha_2,\lambda}_{l,q_2}(w,\mathbb{Q}_p^n)}\\
&=\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda(n+\beta)/n}\left(\sum_{k=-\infty}^{k_0}p^{k\alpha_2(n+\beta)l/n}\|(H^b_{\Phi,A}f)\chi_k\|^l_{L^q(w,\mathbb{Q}_p^n)}\right)^{1/l}\\
&\leq C\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda(n+\beta)/n}\\
&\times\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}p^{k(\alpha_2+n/q)(n+\beta)/n}\|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^l\right\}^{1/l}\psi(\mathbf{t})d\mathbf{t}\\
&\leq C\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}\sup_{k_0\in\mathbb{Z}}p^{-k_0\lambda(n+\beta)/n}\\
&\times\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}p^{k\alpha_1(n+\beta)/n}\|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^l\right\}^{1/l}\psi(\mathbf{t})d\mathbf{t}
\end{aligned}$$

$$\leq C\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}p^{m(\lambda-\alpha_1)(n+\beta)/n}$$
$$\times\sup_{k_0\in\mathbb{Z}}p^{-(k_0+m)\lambda(n+\beta)/n}\left(\sum_{k=-\infty}^{k_0+m}p^{k\alpha_1(n+\beta)l/n}\|f\chi_k\|^l_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^{1/l}\psi(\mathbf{t})d\mathbf{t}.$$

Since $\alpha_1<\lambda$, as a consequence

$$\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}p^{m(\lambda-\alpha_1)(n+\beta)/n}=\frac{\|A(\mathbf{t})\|^{(\lambda-\alpha_1)(n+\beta)/n}}{1-p^{(\alpha_1-\lambda)(n+\beta)/n}}. \tag{3.3.5}$$

Hence,

$$\begin{aligned}\|H^b_{\Phi,A}f\|_{MK^{\alpha_2,\lambda}_{l,q_2}(w,\mathbb{Q}_p^n)}\leq &C\|b\|_{CMO^q(w,\mathbb{Q}_p^n)}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}\\ &\times\int_{\mathbb{Q}_p^n}\|A(\mathbf{t})\|^{(\lambda-\alpha_1)(n+\beta)/n}\psi(\mathbf{t})d\mathbf{t}.\end{aligned}$$

Therefore, we conclude Theorem 3.3.1.

3.4 Lipschitz estimates for $H^b_{\Phi,A}$ on weighted p-adic Herz-Morrey spaces

In the current section we will show that the boundedness of commutator of p-adic matrix Hausdorff operator on p-adic Morrey-Herz space holds by taking $b\in\Lambda_\delta(\mathbb{Q}_p^n)$.

Theorem 3.4.1 *Let* $1\leq q_2\leq q_1<\infty$, $0<l,\delta<\infty, 1/r=1/q_2-1/q_1$, $\beta>-n$ *and* $w(\mathbf{x})=|\mathbf{x}|_p^\beta$, $\alpha_1=\alpha_2+n\delta/(n+\beta)+n(1/q_2-1/q_1)$, $0\leq\lambda>\alpha_1$ *and* $b\in\Lambda_\delta(\mathbb{Q}_p^n)$. *Then the commutator operator* $H^b_{\Phi,A}$ *is bounded from* $MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)$ *to* $MK^{\alpha_2,\lambda}_{l,q_2}(w,\mathbb{Q}_p^n)$ *and satisfies the inequality:*

$$\|H^b_{\Phi,A}f\|_{MK^{\alpha_2,\lambda}_{l,q_2}(w,\mathbb{Q}_p^n)}\leq CK_3\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)},$$

where

$$K_3=\int_{\mathbb{Q}_p^n}\|A(\mathbf{t})\|^{(n+\beta)(\lambda/n-\alpha_1/n)}\max\{1,\|A(\mathbf{t})\|^\delta\}|\det A^{-1}(\mathbf{t})|_p^{1/q_1}G(A^{-1}(\mathbf{t}),\beta/q_1)\psi(\mathbf{t})d\mathbf{t}.$$

Proof. Let $f \in MK_{l,q_1}^{\alpha_1,\lambda}(w,\mathbb{Q}_p^n)$, $b \in \Lambda_\delta(\mathbb{Q}_p^n)$. Using the Minkowski's inequality and the Holder's inequality to have:

$$\begin{aligned}
&\|(H_{\Phi,A}^b f)\chi_k\|_{L^{q_2}(w,\mathbb{Q}_p^n)} \\
&= \left[\int_{S_k}\left|\int_{\mathbb{Q}_p^n} f(A(\mathbf{t})\mathbf{x})(b(\mathbf{x})-b(A(\mathbf{t})\mathbf{x}))\Phi(\mathbf{t})d\mathbf{t}\right|^{q_2} w(\mathbf{x})d\mathbf{x}\right]^{1/q_2} \\
&\le \int_{\mathbb{Q}_p^n}\left[\int_{S_k}\left|f(A(\mathbf{t})\mathbf{x})(b(\mathbf{x})-b(A(\mathbf{t})\mathbf{x}))\right|^{q_2} w(\mathbf{x})d\mathbf{x}\right]^{1/q_2}|\Phi(\mathbf{t})|d\mathbf{t} \\
&\le \int_{\mathbb{Q}_p^n}\left[\int_{S_k}|f(A(\mathbf{t})\mathbf{x})|^{q_1}w(\mathbf{x})d\mathbf{x}\right]^{1/q_1} \\
&\quad\times\left[\int_{S_k}|b(\mathbf{x})-b(A(\mathbf{t})\mathbf{x})|^r w(\mathbf{x})d\mathbf{x}\right]^{1/r}|\Phi(\mathbf{t})|d\mathbf{t}, \qquad (3.4.1)
\end{aligned}$$

where $1/r = 1/q_2 - 1/q_1$. It follows from the definition of Lipschitz space $\Lambda_\delta(\mathbb{Q}_p^n)$ that

$$\begin{aligned}
|b(\mathbf{x})-b(A(\mathbf{t})\mathbf{x})| &\le C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}|\mathbf{x}-A(\mathbf{t})\mathbf{x}|_p^\delta \\
&\le C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\max\{|\mathbf{x}|_p,|A(\mathbf{t})\mathbf{x}|_p\}^\delta \\
&\le p^{k\delta}C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\max\{1,\|A(\mathbf{t})\|^\delta\}, \qquad (3.4.2)
\end{aligned}$$

for each $\mathbf{x} \in S_k$ and for almost everywhere $\mathbf{t} \in \mathbb{Q}_p^n$.

By p-adic change of variables, Lemma 2.1.4, inequality (3.4.2) and according to Lemma (2.1.3) $w(S_k)w(B_0) = w(B_k)w(S_0)$, inequality (3.4.1) assumes the following form

$$\begin{aligned}
&\|(H_{\Phi,A}^b f)\chi_k\|_{L^{q_2}(w,\mathbb{Q}_p^n)} \\
&\le C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n} w(B_k)^{\delta/(n+\beta)+1/r}\|f\chi_{A(\mathbf{t})S_k}\|_{L^{q_1}(w,\mathbb{Q}_p^n)} \\
&\quad\times\max\{1,\|A(\mathbf{t})\|^\delta\}|\det A^{-1}(\mathbf{t})|_p^{1/q_1}G(A^{-1}(\mathbf{t}),\beta/q_1)|\Phi(\mathbf{t})|d\mathbf{t}. \qquad (3.4.3)
\end{aligned}$$

Furthermore, in view of inequality (3.3.3) and $1/r = 1/q_2 - 1/q_1$, we get:

$$\begin{aligned}
&\|(H_{\Phi,A}^b f)\chi_k\|_{L^{q_2}(w,\mathbb{Q}_p^n)} \\
&\le C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}w(B_k)^{\delta/(n+\beta)+1/q_2-1/q_1}\int_{\mathbb{Q}_p^n}\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}\|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}_p^n)} \\
&\quad\times\max\{1,\|A(\mathbf{t})\|^\delta\}|\det A^{-1}(\mathbf{t})|_p^{1/q_1}G(A^{-1}(\mathbf{t}),\beta/q_1)|\Phi(\mathbf{t})|d\mathbf{t}. \qquad (3.4.4)
\end{aligned}$$

The factor $\max\{1,\|A(\mathbf{t})\|^\delta\}|\det A^{-1}(\mathbf{t})|_p^{1/q_1}G(A^{-1}(\mathbf{t}),\beta/q_1)|\Phi(\mathbf{t})|$ repeats itself many times in the remaining proof of this theorem, so we let it be denoted by $\phi(\mathbf{t})$. With

this we break our proof in two cases which are given by:
Case 1: $0<\lambda$, in the present case we first evaluate the inner norm $\|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}$ as below:

$$\begin{aligned}
\|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}_p^n)} &\leq w(B_{k+m})^{-\alpha_1/n}\left[\sum_{j=-\infty}^{k+m} w(B_j)^{\alpha_1 l/n}\|f\chi_j\|^l_{L^{q_1}(w,\mathbb{Q}_p^n)}\right]^{1/l}\\
&= w(B_{k+m})^{-\alpha_1/n}w(B_{k+m})^{\lambda/n}\\
&\times w(B_{k+m})^{-\lambda/n}\left(\sum_{j=-\infty}^{k+m} w(B_j)^{\alpha_1 l/n}\|f\chi_j\|^l_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^{1/l}\\
&= w(B_{k+m})^{(\lambda-\alpha_1)/n}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}\\
&\leq p^{(k+m)(n+\beta)(\lambda-\alpha_1)/n}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}. \qquad (3.4.5)
\end{aligned}$$

Next, by virtue of equation (3.3.5), the inequality (3.4.4) becomes

$$\begin{aligned}
&\|(H^b_{\Phi,A}f)\chi_k\|_{L^{q_2}(w,\mathbb{Q}_p^n)}\\
&\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}p^{k(n+\beta)(\delta/(n+\beta)+1/q_2-1/q_1+(\lambda-\alpha_1)/n)}\\
&\quad\times\int_{\mathbb{Q}_p^n}\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}p^{m(n+\beta)(\lambda-\alpha_1)/n}\phi(\mathbf{t})d\mathbf{t}\\
&\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}p^{k(n+\beta)(\delta/(n+\beta)+1/q_2-1/q_1+(\lambda-\alpha_1)/n)}\\
&\quad\times\int_{\mathbb{Q}_p^n}\|A(\mathbf{t})\|^{(n+\beta)(\lambda-\alpha_1)/n}\phi(\mathbf{t})d\mathbf{t}.
\end{aligned}$$

Therefore, by definition of Morrey-Herz space and $\alpha_1=\alpha_2+n\delta/(n+\beta)+n(1/q_2-1/q_1)$, we have:

$$\begin{aligned}
&\|H^b_{\Phi,A}f\|_{MK^{\alpha_2,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}\\
&\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}\sup_{k_0\in\mathbb{Z}}p^{-k_0(n+\beta)\lambda/n}\\
&\quad\times\Bigg[\sum_{k=-\infty}^{k_0}p^{kl(n+\beta)(\alpha_2/n+\delta/(n+\beta)+1/q_2-1/q_1+(\lambda-\alpha_1)/n)}\\
&\quad\times\left(\int_{\mathbb{Q}_p^n}\|A(\mathbf{t})\|^{(n+\beta)(\lambda-\alpha_1)/n}\phi(\mathbf{t})d\mathbf{t}\right)^l\Bigg]^{1/l}\\
&\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}\|A(\mathbf{t})\|^{(n+\beta)(\lambda-\alpha_1)/n}\phi(\mathbf{t})d\mathbf{t}\\
&\quad\times\sup_{k_0\in\mathbb{Z}}p^{-k_0(n+\beta)\lambda/n}\Bigg[\sum_{k=-\infty}^{k_0}p^{kl(n+\beta)\lambda/n}\Bigg]^{1/l}\\
&\leq C\frac{p^{(n+\beta)\lambda/n}}{(p^{l(n+\beta)\lambda/n}-1)^{1/l}}\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}\|A(\mathbf{t})\|^{(n+\beta)(\lambda-\alpha_1)/n}\phi(\mathbf{t})d\mathbf{t}\\
&\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{MK^{\alpha_1,\lambda}_{l,q_1}(w,\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}\|A(\mathbf{t})\|^{(n+\beta)(\lambda-\alpha_1)/n}\phi(\mathbf{t})d\mathbf{t}, \qquad (3.4.6)
\end{aligned}$$

substituting back the value of $\phi(\mathbf{t})$ we get the desired result.
Case 2: When $l\in[1,\infty)$ and $\lambda=0$
In this case Morrey-Herz spaces are reduced to the Herz spaces. It is clear that:

$$\begin{aligned}
&\|(H^b_{\Phi,A}f)\chi_k\|_{L^{q_2}(w,\mathbb{Q}_p^n)}\\
&\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}p^{k(n+\beta)(\delta/(n+\beta)+1/q_2-1/q_1)}\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}\|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\phi(\mathbf{t})d\mathbf{t}. \qquad (3.4.7)
\end{aligned}$$

Hence, by using the Minkowski's inequality and $\alpha_1=\alpha_2+n\delta/(n+\beta)+n(1/q_2-1/q_1)$, we obtain:

$$\begin{aligned}
&\|H^b_{\Phi,A}f\|_{K^{\alpha_2}_{l,q_2}(w,\mathbb{Q}_p^n)}\\
&\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\Bigg\{\sum_{k=-\infty}^{+\infty}p^{k(n+\beta)\alpha_2 l/n}\\
&\quad\times\Bigg[\int_{\mathbb{Q}_p^n}p^{k(n+\beta)(\delta/(n+\beta)+1/q_2-1/q_1)}\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}\|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\phi(\mathbf{t})d\mathbf{t}\Bigg]^l\Bigg\}^{1/l}\\
&\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}\Bigg\{\sum_{k=-\infty}^{+\infty}\Bigg(\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}p^{k(n+\beta)\alpha_1/n}\|f\chi_{k+m}\|_{L^{q_1}(w,\mathbb{Q}_p^n)}\Bigg)^l\Bigg\}^{1/l}\phi(\mathbf{t})d\mathbf{t}
\end{aligned}$$

$$\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\int_{\mathbb{Q}_p^n}\sum_{m=-\infty}^{\log_p\|A(\mathbf{t})\|}p^{-m(n+\beta)\alpha_1/n}\left(\sum_{k=-\infty}^{+\infty}p^{kl(n+\beta)\alpha_1/n}\|f\chi_k\|^l_{L^{q_1}(w,\mathbb{Q}_p^n)}\right)^{1/l}\phi(\mathbf{t})d\mathbf{t}$$

$$\leq C\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{K^{\alpha_1}_{l,q_1}(w,\mathbb{Q}_p^n)}$$

$$\times\int_{\mathbb{Q}_p^n}\max\{1,\|A(\mathbf{t})\|^\delta\}|\det A^{-1}(\mathbf{t})|_p^{1/q_1}G(A^{-1}(\mathbf{t}),\beta/q_1)\|A(\mathbf{t})\|^{-(n+\beta)\alpha_1/n}|\Phi(\mathbf{t})|d\mathbf{t}. \quad (3.4.8)$$

By equations (3.4.6) and (3.4.8), we get the proof.

Chapter 4
Weak-Type Estimates of p-adic Fractional Hausdorff Operators

4.1 Introduction

In this Chapter, among variety of Hausdorff operators, we choose to study the fractional Hausdorff operator and related commutators on weak type spaces. These spaces include weak Lebesgue spaces and central Morrey spaces. Using Marcinkiewicz type interpolation theorem, we also give strong type estimates for these operators on Lebesgue spaces. However, in the case of central Morrey space we establish the weak estimates for a special case of fractional Hausdorff operator.

The first section contains results on the said boundedness of Hausdorff operator and the subsequent section comprises of similar results for the commutators of the same operator. Finally, at the very end we prove the boundedness of Hausdorff operator on weak p-adic central Morrey space.

4.2 Lebesgue Space Estimates for p-adic Fractional Hausdorff Operator

We first give the weak type boundedness result for the fractional Hausdorff operator following its proof.

Theorem 4.2.1 *Suppose* $n > \beta \geq 0$, $\infty > q, r \geq 1$, $-n < \min\{\alpha, \gamma\}$ *and* $w(\mathbf{x}) = |\mathbf{x}|_p^{\alpha}$. *If* Φ *is radial function,* $\frac{n+\alpha}{q} - \beta = \frac{n+\gamma}{r}$ *and*

$$\mathcal{A}^{q'}(\psi, q) = \int_0^{\infty} |\psi(t)|^{q'} t^{(n+\alpha)(q'-1)-\beta q'-1} dt < \infty, \tag{4.2.1}$$

then

$$\|H_{\Phi,\beta}(f)\|_{L^{r,\infty}(|\mathbf{x}|_p^{\gamma},\mathbb{Q}_p^n)} \leq K_2\|f\|_{L^q(|\mathbf{x}|_p^{\alpha},\mathbb{Q}_p^n)},$$

where

$$K_2 = C\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{1/r}(1-p^{-n})^{1/q'}\mathcal{A}(\psi,q).$$

Proof. We first consider

$$\begin{aligned} H_{\Phi,\beta}f(\mathbf{x}) &= \int_{\mathbb{Q}_p^n}\frac{\Phi(\mathbf{x}|\mathbf{y}|_p)}{|\mathbf{y}|_p^{n-\beta}w(\mathbf{y})^{1/q}}f(\mathbf{y})w(\mathbf{y})^{1/q}d\mathbf{y} \\ &= \int_{\mathbb{Q}_p^n}\frac{\Phi(\mathbf{x}|\mathbf{y}|_p)}{|\mathbf{y}|_p^{n-\beta+\alpha/q}}f(\mathbf{y})w(\mathbf{y})^{1/q}d\mathbf{y}. \end{aligned}$$

Applying Hölder's inequality at the outset to have

$$\begin{aligned} |H_{\Phi,\beta}f(\mathbf{x})| \leq& \left\{\int_{\mathbb{Q}_p^n}\left|\frac{\Phi(\mathbf{x}|\mathbf{y}|_p)}{|\mathbf{y}|_p^{n-\beta+\alpha/q}}\right|^{q'}d\mathbf{y}\right\}^{1/q'}\left\{\int_{\mathbb{Q}_p^n}|f(\mathbf{y})|^q|\mathbf{y}|_p^{\alpha}d\mathbf{y}\right\}^{1/q} \\ =&\left\{\int_{\mathbb{Q}_p^n}\left|\frac{\Phi(\mathbf{x}|\mathbf{y}|_p)}{|\mathbf{y}|_p^{n-\beta+\alpha/q}}\right|^{q'}d\mathbf{y}\right\}^{1/q'}\|f\|_{L^q(|\mathbf{x}|_p^{\alpha};\mathbb{Q}_p^n)}. \end{aligned} \tag{4.2.2}$$

Sine Φ is radial function then

$$\begin{aligned} \int_{\mathbb{Q}_p^n}\frac{|\Phi(\mathbf{x}|\mathbf{y}|_p)|^{q'}}{|\mathbf{y}|_p^{(n-\beta+\alpha/q)q'}}d\mathbf{y} =& \sum_{k\in\mathbb{Z}}\int_{S_k}\frac{|\psi(p^{l-k})|^{q'}}{p^{k(n-\beta+\alpha/q)q'}}d\mathbf{y} \\ =&(1-p^{-n})p^{l((n+\alpha)(1-q')+\beta q')}\sum_{k\in\mathbb{Z}}|\psi(p^{l-k})|^{q'}p^{(l-k)((n+\alpha)(q'-1)-\beta q')-1+1} \\ \leq& C(1-p^{-n})|\mathbf{x}|_p^{-((n+\alpha)(q'-1)-\beta q')}\int_0^{\infty}|\psi(t)|^{q'}t^{(n+\alpha)(q'-1)-\beta q'-1}dt \\ =&C(1-p^{-n})|\mathbf{x}|_p^{-((n+\alpha)(q'-1)-\beta q')}\mathcal{A}^{q'}(\psi,q). \end{aligned}$$

Therefore, by the stipulated condition $\frac{n+\alpha}{q}-\beta=\frac{n+\gamma}{r}$, (5.2.1) becomes:

$$\begin{aligned} |H_{\Phi,\beta}f(\mathbf{x})| \leq& C(1-p^{-n})^{1/q'}\mathcal{A}(\psi,q)|\mathbf{x}|_p^{-(n+\alpha)/q+\beta}\|f\|_{L^q(|\mathbf{x}|_p^{\alpha},\mathbb{Q}_p^n)} \\ =&C(1-p^{-n})^{1/q'}\mathcal{A}(\psi,q)|\mathbf{x}|_p^{-(n+\gamma)/r}\|f\|_{L^q(|\mathbf{x}|_p^{\alpha};\mathbb{Q}_p^n)}. \end{aligned}$$

Let $C_1 = C(1-p^{-n})^{1/q'}\mathcal{A}(\psi,q)\|f\|_{L^q(|\mathbf{x}|_p^{\alpha},\mathbb{Q}_p^n)}$, then for $\lambda>0$,

$$\{\mathbf{x}\in\mathbb{Q}_p^n : |H_{\Phi,\beta}f(\mathbf{x})|>\lambda\}\subset\{\mathbf{x}\in\mathbb{Q}_p^n : |\mathbf{x}|_p\leq(C_1/\lambda)^{r/n+\gamma}\}.$$

Therefore,

$$\begin{aligned}
\|H_{\Phi,\beta}f(\mathbf{x})\|_{L^{r,\infty}(|x|_p^\gamma,\mathbb{Q}_p^n)} \leq& \sup_{\lambda>0}\lambda\bigg(\int_{\mathbb{Q}_p^n}\chi_{\left\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{x}|_p<(C_1/\lambda)^{r/n+\gamma}\right\}}(\mathbf{x})|\mathbf{x}|_p^\gamma d\mathbf{x}\bigg)^{1/r}\\
=&\sup_{\lambda>0}\lambda\bigg(\int_{|\mathbf{x}|_p<(C_1/\lambda)^{r/n+\gamma}}|\mathbf{x}|_p^\gamma d\mathbf{x}\bigg)^{1/r}\\
=&\sup_{\lambda>0}\lambda\bigg(\sum_{j=-\infty}^{\log_p(C_1/\lambda)^{r/n+\gamma}}\int_{S_j}p^{j\gamma}d\mathbf{x}\bigg)^{1/r}\\
=&\bigg(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\bigg)^{1/r}C_1\\
=&C\bigg(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\bigg)^{1/r}(1-p^{-n})^{1/q'}\mathcal{A}(\psi,q)\|f\|_{L^q(|\mathbf{x}|_p^\alpha,\mathbb{Q}_p^n)}\\
=&K_2\|f\|_{L^q(|\mathbf{x}|_p^\alpha,\mathbb{Q}_p^n)}.
\end{aligned}$$

Hence, we weak type estimate of $H_{\Phi,\beta}$ have been obtained.

Next, using Marcinkiewicz interpolation theorem, we give strong type estimates for $H_{\Phi,\beta}$.

Theorem 4.2.2 *Suppose $n > \beta \geq 0$, $\infty > q, r \geq 1$. Let also $-n < \min\{\alpha,\gamma\}$, $w(\mathbf{x}) = |\mathbf{x}|_p^\alpha, \alpha > -n$. If Φ is radial function, let $\frac{n+\alpha}{q} - \beta = \frac{n+\gamma}{r}$ and equation (4.2.1) is valid for $q \pm \epsilon$ instead of q, then*

$$\|H_{\Phi,\beta}f\|_{L^{r,s}(|\mathbf{x}|_p^\alpha,\mathbb{Q}_p^n)} \preceq \|f\|_{L^{q,s}(|\mathbf{x}|_p^\alpha,\mathbb{Q}_p^n)}.$$

Proof. Since $1 < q, r < \infty$, so we can find ϵ such that $1 < q-\epsilon$ and $1 < r-\epsilon$. Then, by Theorem (4.2.1) $H_{\Phi,\beta}$ has weak types $(q-\epsilon, r-\epsilon)$ and $(q+\epsilon, r+\epsilon)$. Desired result is acquired by using Theorem 1.3.14.

4.3 Lipschitz Estimates for the Commutator Operator

Similar to previous section, this section contains weak and strong boundedness results for the commutator operator. We prove the former one first.

Theorem 4.3.1 *Let $1 < q < r < \infty$, $\min\{\alpha,\gamma\} > -n$, $(\beta+\delta) - \frac{n+\alpha}{q} = -\frac{n+\gamma}{r}$, and $w(\mathbf{x}) = |\mathbf{x}|_p^\alpha$. If Φ is a radial function, $b \in \Lambda_\delta(\mathbb{Q}_p^n)$, $0 < \delta < 1$ and*

$$K_{3,q} = C\int_0^\infty \psi^{q'}(t)t^{(q'-1)(n+\alpha)-\beta q'-1}\max(1,t^{-\delta q'})dt < \infty, \tag{4.3.1}$$

then

$$\|H^b_{\Phi,\beta}f\|_{L^{r,\infty}(|\mathbf{x}|^\alpha_p,\mathbb{Q}^n_p)} \leq K_4\|f\|_{L^q(|\mathbf{x}|^\alpha_p,\mathbb{Q}^n_p)}, \tag{4.3.2}$$

where

$$K_4 = K_{3,q}\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{1/r}(1-p^{-n})\|b\|_{\Lambda_\delta(\mathbb{Q}^n_p)}.$$

Proof. By definition of Lipschitz space, we have

$$\begin{aligned}|H^b_{\Phi,\beta}(f)(\mathbf{x})| \leq &\left|\int_{\mathbb{Q}^n_p}\frac{\Phi(\mathbf{x}|\mathbf{y}|_p)}{|\mathbf{y}|_p^{n-\beta+\alpha/q}}(b(\mathbf{x})-b(\mathbf{0}))f(\mathbf{y})|\mathbf{y}|_p^{\alpha/q}d\mathbf{y}\right|\\ &+\left|\int_{\mathbb{Q}^n_p}\frac{\Phi(\mathbf{x}|\mathbf{y}|_p)}{|\mathbf{y}|_p^{n-\beta+\alpha/q}}(b(\mathbf{y})-b(\mathbf{0}))f(\mathbf{y})|\mathbf{y}|_p^{\alpha/q}d\mathbf{y}\right|\end{aligned}$$

$$\begin{aligned}\leq&\|b\|_{\Lambda_\delta(\mathbb{Q}^n_p)}|\mathbf{x}|_p^\delta\int_{\mathbb{Q}^n_p}\frac{\Phi(\mathbf{x}|\mathbf{y}|_p)}{|\mathbf{y}|_p^{n-\beta+\alpha/q}}f(\mathbf{y})|\mathbf{y}|_p^{\alpha/q}d\mathbf{y}\\ &+\|b\|_{\Lambda_\delta(\mathbb{Q}^n_p)}\int_{\mathbb{Q}^n_p}\frac{\Phi(\mathbf{x}|\mathbf{y}|_p)}{|\mathbf{y}|_p^{n-\beta+\alpha/q-\delta}}f(\mathbf{y})|\mathbf{y}|_p^{\alpha/q}d\mathbf{y}\\ =&I_1+I_2.\end{aligned}$$

We evaluate I_2 first. Use of Hölder's inequality gives:

$$\begin{aligned}I_2\leq&\|b\|_{\Lambda_\delta(\mathbb{Q}^n_p)}\left\{\int_{\mathbb{Q}^n_p}\left|\frac{\Phi(\mathbf{x}|\mathbf{y}|_p)}{|\mathbf{y}|_p^{n-\beta+\alpha/q-\delta}}\right|^{q'}d\mathbf{y}\right\}^{1/q'}\left\{\int_{\mathbb{Q}^n_p}|f(\mathbf{y})|^q|\mathbf{y}|^\alpha_p d\mathbf{y}\right\}^{1/q}\\ =&\|b\|_{\Lambda_\delta(\mathbb{Q}^n_p)}\left\{\int_{\mathbb{Q}^n_p}\left|\frac{\Phi(\mathbf{x}|\mathbf{y}|_p)}{|\mathbf{y}|_p^{n-\beta+\alpha/q-\delta}}\right|^{q'}d\mathbf{y}\right\}^{1/q'}\|f\|_{L^q(|\mathbf{x}|^\alpha_p,\mathbb{Q}^n_p)}.\end{aligned} \tag{4.3.3}$$

If $|\mathbf{x}|_p = p^l, l\in\mathbb{Z}$, then repeating the same process as in Theorem (4.2.1), we arrive at:

$$\begin{aligned}\int_{\mathbb{Q}^n_p}\frac{|\Phi(\mathbf{x}|\mathbf{y}|_p)|^{q'}}{|\mathbf{y}|_p^{(n-\beta+\alpha/q-\delta)q'}}d\mathbf{y} =&\sum_{k\in\mathbb{Z}}\int_{S_k}\frac{|\psi(p^{l-k})|^{q'}}{p^{k(n-\beta+\alpha/q-\delta)q'}}d\mathbf{y}\\ =&(1-p^{-n})\sum_{k\in\mathbb{Z}}\psi^{q'}(p^{l-k})p^{(l-k)((q'-1)(n+\alpha)-(\beta+\delta)q'-1+1)}\\ &\times|\mathbf{x}|_p^{-((n+\alpha)(q'-1)-(\beta+\delta)q')}\\ \leq&C(1-p^{-n})\int_0^\infty\psi^{q'}(t)t^{(q'-1)(n+\alpha)-(\beta+\delta)q'-1}dt\\ &\times|\mathbf{x}|_p^{-((n+\alpha)(q'-1)-(\beta+\delta)q')}.\end{aligned}$$

Making use of above value, (5.2.2) becomes:

$$I_2 \leq C(1-p^{-n})\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}|\mathbf{x}|_p^{-(n+\alpha)/q+(\beta+\delta)}$$
$$\times\left(\int_0^\infty \psi^{q'}(t)t^{(q'-1)(n+\alpha)-(\beta+\delta)q'-1}dt\right)^{1/q'}\|f\|_{L^q(|\mathbf{x}|_p^\alpha,\mathbb{Q}_p^n)}.$$

In order to estimate I_1, we observe that it differ from I_2 by a factor δ in the power of $|\mathbf{y}|_p$. Thus, from (4.3.4), we infer that

$$I_1 \leq C(1-p^{-n})\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}|\mathbf{x}|_p^{-(n+\alpha)/q+(\beta+\delta)}$$
$$\times\left(\int_0^\infty \psi^{q'}(t)t^{(q'-1)(n+\alpha)-\beta q'-1}dt\right)^{1/q'}\|f\|_{L^q(|\mathbf{x}|_p^\alpha,\mathbb{Q}_p^n)}.$$

In view of the condition that $(\beta+\delta)-\frac{n+\alpha}{q}=-\frac{n+\gamma}{r}$, and the inequality (4.3.1), we obtain

$$|H^b_{\Phi,\beta}(f)(\mathbf{x})| \leq K_{3,q}(1-p^{-n})\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}|\mathbf{x}|_p^{-(n+\gamma)/r}\|f\|_{L^q(|\mathbf{x}|_p^\alpha,\mathbb{Q}_p^n)}$$

Let

$$C_3 = K_{3,q}(1-p^{-n})\|b\|_{\Lambda_\delta(\mathbb{Q}_p^n)}\|f\|_{L^q(|\mathbf{x}|_p^\alpha,\mathbb{Q}_p^n)},$$

then for all $\lambda > 0$, we have

$$\{\mathbf{x}\in\mathbb{Q}_p^n : |H^b_{\Phi,\beta}f(\mathbf{x})| > \lambda\} \subset \{\mathbf{x}\in\mathbb{Q}_p^n : |\mathbf{x}|_p \leq (C_3/\lambda)^{r/n+\gamma}\}.$$

Ultimately,

$$\begin{aligned}\|H^b_{\Phi,\beta}f(\mathbf{x})\|_{L^{r,\infty}(|\mathbf{x}|_p^\gamma,\mathbb{Q}_p^n)} \leq& \sup_{\lambda>0}\lambda\left(\int_{\mathbb{Q}_p^n}\chi_{\left\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{x}|_p<(C_3/\lambda)^{r/n+\gamma}\right\}}(\mathbf{x})|\mathbf{x}|_p^\gamma d\mathbf{x}\right)^{1/r}\\ =&\sup_{\lambda>0}\lambda\left(\int_{|\mathbf{x}|_p<(C_3/\lambda)^{r/n+\gamma}}|\mathbf{x}|_p^\gamma d\mathbf{x}\right)^{1/r}\\ =&\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{1/r}C_3\\ =&K_{3,q}\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{1/r}(1-p^{-n})\|b\|_{\Lambda_\delta}\|f\|_{L^q(|\mathbf{x}|_p^\alpha,\mathbb{Q}_p^n)}\\ =&K_4\|f\|_{L^q(|\mathbf{x}|_p^\alpha,\mathbb{Q}_p^n)}.\end{aligned}$$

Next, we will show that the strong type estimates also hold for $H^b_{\Phi,\beta}$.

Theorem 4.3.2 *Let $1 < q < r < \infty$, $0 < \delta < 1$, $\min\{\alpha,\gamma\} > -n$, $-\frac{n+\gamma}{r} = (\beta+\delta)-\frac{n+\alpha}{q}$, $w(\mathbf{x}) = |\mathbf{x}|_p^\alpha$. If Φ is radial function, $b\in\Lambda_\delta(\mathbb{Q}_p^n)$ and equation (4.3.1) is true for $q\pm\epsilon$ instead of q where $0\leq\epsilon<\epsilon_0$, then*

$$\|H^b_{\Phi,\beta}f\|_{L^{r,s}(|\mathbf{x}|_p^\alpha,\mathbb{Q}_p^n)} \preceq \|f\|_{L^{q,s}(|\mathbf{x}|_p^\alpha,\mathbb{Q}_p^n)}.$$

Proof. Since $q, r \in (1, \infty)$, so $q \in (1, n/\delta)$, we can find $0 \leq \epsilon < \epsilon_0$ such that $q_1 = q - \epsilon \in (1, n/\delta)$ and $q_2 = q + \epsilon \in (1, n/\delta)$. Also, we can choose r_1 and r_2 such that $r_1 < r < r_2$ which satisfies

$$(\beta + \delta) - \frac{n + \alpha}{q_i} = -\frac{n + \gamma}{r_i}, \ i = 1, 2.$$

Using Theorem 4.3.1, we have:

$$\|H^b_{\Phi,\beta} f\|_{L^{r_i,\infty}(|\mathbf{x}|^\gamma_p, \mathbb{Q}^n_p)} \preceq \|f\|_{L^{q_i}(|\mathbf{x}|^\alpha_p, \mathbb{Q}^n_p)}.$$

But the equality $1/q = \vartheta/q_1 + (1 - \vartheta)/q_2$ implies a similar equality $1/r = \vartheta/r_1 + (1 - \vartheta)/r_2$. Required result is obtained by using Theorem 1.3.14.

4.4 Hausdorff Operator on Weak p-adic Central Morrey Space

Theorem 4.4.1 *Let $-1/q \leq \lambda < 0$. and let $1 \leq q < \infty$. Let Φ be a radial function, that is $\Phi(\mathbf{x}) = \psi(|\mathbf{x}|_p)$, where ψ is defined in all $p^k, k \in \mathbb{Z}$ and $f \in \dot{B}^{q,\lambda}(\mathbb{Q}^n_p)$, then*

$$\|H_\Phi f\|_{W\dot{B}^{q,\lambda}(\mathbb{Q}^n_p)} \leq K_1 (1 - p^{-n})^{1/q'} \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}^n_p)},$$

where $K_1 = C \int_0^\infty \psi(t) t^{-n\lambda - 1} dt$.

Proof. Similar to the previous results, we decompose the integral as:

$$\begin{aligned} H_\Phi f(\mathbf{x}) &= \int_{\mathbb{Q}^n_p} \frac{\Phi(\mathbf{x}|\mathbf{y}|_p)}{|\mathbf{y}|^n_p} f(\mathbf{y}) d\mathbf{y} \\ &= \sum_{k \in \mathbb{Z}} \int_{S_k} \frac{\Phi(\mathbf{x}|\mathbf{y}|_p)}{|\mathbf{y}|^n_p} f(\mathbf{y}) d\mathbf{y}. \end{aligned}$$

Applying Hölder's inequality to get

$$\begin{aligned} |H_\Phi f(\mathbf{x})| \leq & \sum_{k \in \mathbb{Z}} \left(\left(\int_{S_k} \frac{|\Phi(\mathbf{x}|\mathbf{y}|_p)|^{q'}}{|\mathbf{y}|^{nq'}_p} d\mathbf{y} \right)^{1/q'} \left(\int_{S_k} |f(\mathbf{y})|^q d\mathbf{y} \right)^{1/q} \right) \\ \leq & \sum_{k \in \mathbb{Z}} \left(\left(\int_{S_k} \frac{|\Phi(\mathbf{x}|\mathbf{y}|_p)|^{q'}}{|\mathbf{y}|^{nq'}_p} d\mathbf{y} \right)^{1/q'} \left(\int_{B_k} |f(\mathbf{y})|^q d\mathbf{y} \right)^{1/q} \right) \\ \leq & \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}^n_p)} \sum_{k \in \mathbb{Z}} |B_k|^{1/q + \lambda}_H \left(\int_{S_k} \frac{|\Phi(\mathbf{x}|\mathbf{y}|_p)|^{q'}}{|\mathbf{y}|^{nq'}_p} d\mathbf{y} \right)^{1/q'}. \end{aligned} \tag{4.4.1}$$

If $|\mathbf{x}|_p = p^l, l \in \mathbb{Z}$, then $\Phi(\mathbf{x}|\mathbf{y}|_p) = \psi(p^{l-k})$, then we take:

$$\begin{aligned}\sum_{k\in\mathbb{Z}} |B_k|_H^{1/q+\lambda}\Big(\int_{S_k} \frac{|\Phi(\mathbf{x}|\mathbf{y}|_p)|^{q'}}{|\mathbf{y}|_p^{nq'}} d\mathbf{y}\Big)^{1/q'} &= \sum_{k\in\mathbb{Z}} p^{kn(1/q+\lambda)}\Big(\int_{S_k} \frac{|\psi(p^{l-k})|^{q'}}{p^{knq'}} d\mathbf{y}\Big)^{1/q'} \\ &= (1-p^{-n})^{1/q'} \sum_{k\in\mathbb{Z}} |\psi(p^{l-k})| p^{kn\lambda} \\ &= (1-p^{-n})^{1/q'} p^{ln\lambda} \sum_{k\in\mathbb{Z}} |\psi(p^{l-k})| p^{(l-k)(-n\lambda)-1+1} \\ &\le C(1-p^{-n})^{1/q'} p^{ln\lambda} \int_0^\infty \psi(t) t^{-n\lambda-1} dt \\ &= C(1-p^{-n})^{1/q'} |\mathbf{x}|_p^{n\lambda} \int_0^\infty \psi(t) t^{-n\lambda-1} dt,\end{aligned}$$

where we majorized at the penultimate step and last step is courtesy of $|\mathbf{x}|_p = p^l$.

Letting

$$K_1 = C\int_0^\infty \psi(t) t^{-n\lambda-1} dt$$

and substituting into the inequality (4.4.1), we get

$$|H_\Phi| \le K_1(1-p^{-n})^{1/q'} |\mathbf{x}|_p^{n\lambda} \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}.$$

Further, let $A = K_1(1-p^{-n})^{1/q'}\|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}$. Since $\lambda < 0$, we have

$$\begin{aligned}\|H_\Phi f\|_{W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} &\le \sup_{\gamma\in\mathbb{Z}} \sup_{t>0} t|B_\gamma|_H^{-\lambda-1/q} |\{\mathbf{x} \in B_\gamma : A|\mathbf{x}|_p^{n\lambda} > t\}|^{1/q} \\ &= \sup_{\gamma\in\mathbb{Z}} \sup_{t>0} t|B_\gamma|_H^{-\lambda-1/q} |\{|\mathbf{x}|_p \le p^\gamma : |\mathbf{x}|_p < (t/A)^{1/n\lambda}\}|^{1/q}.\end{aligned}$$

If $\gamma \le \log_p(t/A)^{1/n\lambda}$, then for $\lambda < 0$, we obtain

$$\begin{aligned}&\sup_{t>0} \sup_{\gamma\le\log_p(t/A)^{1/n\lambda}} t|B_\gamma|_H^{-\lambda-1/q} |\{|\mathbf{x}|_p \le p^\gamma : |\mathbf{x}|_p < (t/A)^{1/n\lambda}\}|^{1/q} \\ =&\sup_{t>0} \sup_{\gamma\le\log_p(t/A)^{1/n\lambda}} t|B_\gamma|_H^{-\lambda} \\ =&\sup_{t>0} \sup_{\gamma\le\log_p(t/A)^{1/n\lambda}} tp^{-n\gamma\lambda}\end{aligned}$$

$$
\begin{aligned}
&\sup_{t>0} \sup_{\gamma \le \log_p (t/A)^{1/n\lambda}} t|B_\gamma|_H^{-\lambda-1/q} |\{|\mathbf{x}|_p \le p^\gamma : |\mathbf{x}|_p < (t/A)^{1/n\lambda}\}|^{1/q} \\
=&\sup_{t>0} \sup_{\gamma \le \log_p (t/A)^{1/n\lambda}} t|B_\gamma|_H^{-\lambda} \\
=&\sup_{t>0} \sup_{\gamma \le \log_p (t/A)^{1/n\lambda}} tp^{-n\gamma\lambda} \\
=&A \\
=&K_1(1-p^{-n})^{1/q'} |\|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

Now, if $\gamma > \log_p (t/A)^{1/n\lambda}$, then for $\lambda \ge -1/q$, we have:

$$
\begin{aligned}
&\sup_{t>0} \sup_{\gamma > \log_p (t/A)^{1/n\lambda}} t|B_\gamma|_H^{-\lambda-1/q} |\{|\mathbf{x}|_p \le p^\gamma : |\mathbf{x}|_p < (t/A)^{1/n\lambda}\}|^{1/q} \\
=&\sup_{t>0} \sup_{\gamma > \log_p (t/A)^{1/n\lambda}} tp^{\gamma n(-\lambda-1/q)} ||\mathbf{x}|_p < (t/A)^{1/n\lambda}|^{1/q} \\
=&\sup_{t>0} \sup_{\gamma > \log_p (t/A)^{1/n\lambda}} tp^{\gamma n(-\lambda-1/q)} (t/A)^{1/\lambda q} \\
=&A \\
=&K_1(1-p^{-n})^{1/q'} \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

Therefore,

$$
\|H_\Phi f\|_{W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \le K_1(1-p^{-n})^{1/q'} |\|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}.
$$

Chapter 5
Optimal Weak Type Estimates for p-adic Hardy Operators

5.1 Introduction

A brief history regarding the boundedness of Hardy type operators and their optimal bounds on function spaces has been given in Chapter 1. In this Chapter, we will auquire the optimal weak bounds for fractional p-adic Hardy operator and its adjoint operator. Our results include all feasible cases of power weighted weak type estimate for p-adic Hardy type operators. Our method of proving main results involves a frequent use the following formula:

$$\int_{\mathbb{Q}_p^n} f(\mathbf{x})w(\mathbf{x})d\mathbf{x} = \sum_{\gamma\in Z}\int_{S_\gamma} f(\mathbf{x})w(\mathbf{x})d\mathbf{x}.$$

However, the optimality of the bounds is obtained by employing the idea of use of power function given in [110].

From this point forward, the notation $L^q(|x|_p^\rho)$ will be reserved for $L^q(|x|_p^\rho, \mathbb{Q}_p^n)$ and $L^{q,\infty}(|x|^\rho)$ stand for $L^{q,\infty}(|x|^\rho, \mathbb{Q}_p^n)$. The next section comprises of results showing optimal weak bounds for p-adic Hardy operator while the last section includes similar results for its adjoint operator.

5.2 Sharp Weak-tye Estimates for p-adic Hardy Operators

This section considers the problem of obtaining optimal weak bounds for H_α^p and H^p. In this regard our main results and corollaries are as under:

Theorem 5.2.1 *Let* $1 < q < \frac{n+\beta}{\alpha},\ 1 \le r < \infty,\ 0 < \alpha \le \frac{\beta}{q-1} < n$. *If* $\frac{n+\beta}{q} - \alpha = \frac{n+\gamma}{r}$, *then*

$$\|H_\alpha^p\|_{L^q(|\mathbf{x}|_p^\beta)\to L^{r,\infty}(|x|_p^\gamma)} = \left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{1/r}\left(\frac{1-p^{-n}}{1-p^{\frac{\beta}{q-1}-n}}\right)^{1/q'}.$$

Corollary 5.2.2 *Let* $1 < q < \frac{n}{\alpha},\ 1 \le r < \infty$. *If* $\frac{n}{q} - \alpha = \frac{n}{r}$, *then*

$$\|H_\alpha^p\|_{L^q(\mathbb{Q}_p^n)\to L^{r,\infty}(\mathbb{Q}_p^n)} = 1.$$

Theorem 5.2.3 *Let* $1 < q < \infty,\ 1 \le r < \infty$. *If* $\frac{n+\beta}{q} = \frac{n+\gamma}{r}$, *then*

$$\|H^p\|_{L^q(|\mathbf{x}|_p^\beta)\to L^{r,\infty}(|x|_p^\gamma)} = \left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{1/r}\left(\frac{1-p^{-n}}{1-p^{\frac{\beta}{q-1}-n}}\right)^{1/q'}.$$

Corollary 5.2.4 *Let* $1 < q < \infty$, *then*

$$\|H^p\|_{L^q(\mathbb{Q}_p^n)\to L^{q,\infty}(\mathbb{Q}_p^n)} = 1.$$

Since the proofs of Theorems 5.2.1 and 5.2.3 follows similar pattern, we only prove Theorem 5.2.1.

Proof of Theorem 5.2.1: Employing Hölder's inequality at the initial stage to get:

$$\begin{aligned}|H_\alpha^p f(\mathbf{x})| =& \left|\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{y}|_p\le|\mathbf{x}|_p} f(\mathbf{y})d\mathbf{y}\right| \\ \le& |\mathbf{x}|_p^{\alpha-n}\left(\int_{|\mathbf{y}|_p\le|\mathbf{x}|_p}|f(\mathbf{y})|^q|\mathbf{y}|_p^\beta d\mathbf{y}\right)^{1/q}\left(\int_{|\mathbf{y}|_p\le|\mathbf{x}|_p}|\mathbf{y}|_p^{-\beta q'/q}d\mathbf{y}\right)^{1/q'}. \quad (5.2.1)\end{aligned}$$

Easy computation leads to

$$\begin{aligned}\left(\int_{|\mathbf{y}|_p\le|\mathbf{x}|_p}|\mathbf{y}|_p^{-\beta q'/q}d\mathbf{y}\right)^{1/q'} =& \left(\sum_{j=-\infty}^{\log_p|\mathbf{x}|_p}\int_{S_j}p^{-j\beta q'/q}d\mathbf{y}\right)^{1/q'} \\ =& (1-p^{-n})^{1/q'}\left(\sum_{j=-\infty}^{\log_p|\mathbf{x}|_p}p^{j(n-\frac{\beta}{q-1})}\right)^{1/q'} \\ =& \left(\frac{1-p^{-n}}{1-p^{\frac{\beta}{q-1}-n}}\right)^{1/q'}|\mathbf{x}|_p^{\frac{n}{q'}-\frac{\beta}{q}}, \quad (5.2.2)\end{aligned}$$

where, at the second step, the convergence of the series is by virtue of the condition that $\frac{\beta}{q-1} < n$. In view of the condition $\frac{n+\beta}{q} - \alpha = \frac{n+\gamma}{r}$ and (5.2.2), the inequality (5.2.1) assumes the following form:

$$
\begin{aligned}
|H_\alpha^p f(\mathbf{x})| \leq & \left(\frac{1-p^{-n}}{1-p^{\frac{\beta}{q-1}-n}}\right)^{1/q'} \|f\|_{L^q(|\mathbf{x}|_p^\beta)} |\mathbf{x}|_p^{\alpha-\frac{n+\beta}{q}} \\
= & \left(\frac{1-p^{-n}}{1-p^{\frac{\beta}{q-1}-n}}\right)^{1/q'} \|f\|_{L^q(|\mathbf{x}|_p^\beta)} |\mathbf{x}|_p^{-\frac{n+\gamma}{r}}.
\end{aligned}
$$

Next, we let $C_1^f = \left(\frac{1-p^{-n}}{1-p^{\frac{\beta}{q-1}-n}}\right)^{1/q'} \|f\|_{L^q(|\mathbf{x}|_p^\beta)}$, then for any $\lambda > 0$, we get

$$
\{\mathbf{x} \in \mathbb{Q}_p^n : |H_\alpha^p f(\mathbf{x})| > \lambda\} \subset \left\{\mathbf{x} \in \mathbb{Q}_p^n : |\mathbf{x}|_p < \left(\frac{C_1^f}{\lambda}\right)^{\frac{r}{n+\gamma}}\right\}.
$$

Thus,

$$
\begin{aligned}
\|H_\alpha^p f\|_{L^{r,\infty}(|x|_p^\gamma)} = & \sup_{\lambda>0} \lambda \left(\int_{\mathbb{Q}_p^n} \chi_{\{\mathbf{x}\in\mathbb{Q}_p^n : |H_\alpha^p f(\mathbf{x})|>\lambda\}}(\mathbf{x}) |\mathbf{x}|_p^\gamma d\mathbf{x}\right)^{1/r} \\
\leq & \sup_{\lambda>0} \lambda \left(\int_{\mathbb{Q}_p^n} \chi_{\left\{\mathbf{x}\in\mathbb{Q}_p^n : |\mathbf{x}|_p < \left(C_1^f/\lambda\right)^{r/(n+\gamma)}\right\}}(\mathbf{x}) |\mathbf{x}|_p^\gamma d\mathbf{x}\right)^{1/r} \\
= & \sup_{\lambda>0} \lambda \left(\int_{|\mathbf{x}|_p < \left(C_1^f/\lambda\right)^{r/(n+\gamma)}} |\mathbf{x}|_p^\gamma d\mathbf{x}\right)^{1/r} \\
= & \sup_{\lambda>0} \lambda \left(\sum_{j=-\infty}^{\log_p \left(C_1^f/\lambda\right)^{r/(n+\gamma)}} \int_{S_j} |\mathbf{x}|_p^\gamma d\mathbf{x}\right)^{1/r} \\
= & (1-p^{-n})^{1/r} \sup_{\lambda>0} \lambda \left(\sum_{j=-\infty}^{\log_p \left(C_1^f/\lambda\right)^{r/(n+\gamma)}} p^{j(n+\gamma)} d\mathbf{x}\right)^{1/r} \\
= & \left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{1/r} C_1^f \\
= & \left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{1/r} \left(\frac{1-p^{-n}}{1-p^{\frac{\beta}{q-1}-n}}\right)^{1/q'} \|f\|_{L^q(|\mathbf{x}|_p^\beta)}.
\end{aligned}
$$

On the other hand, suppose

$$
f_0(\mathbf{x}) = |\mathbf{x}|_p^{-\frac{\beta}{q-1}} \chi_{\{\mathbf{x}\in\mathbb{Q}_p^n : |\mathbf{x}|_p \leq 1\}}(\mathbf{x}),
$$

then

$$\begin{aligned}\|f_0\|_{L^q(|\mathbf{x}|_p^\beta)} =&\left(\int_{|\mathbf{x}|_p\leq 1}|\mathbf{x}|_p^{-\frac{\beta}{q-1}}d\mathbf{x}\right)^{1/q}\\ =&(1-p^{-n})^{1/q}\left(\sum_{j=-\infty}^{0}p^{j(n-\frac{\beta}{q-1})}\right)\\ =&\left(\frac{1-p^{-n}}{1-p^{\frac{\beta}{q-1}-n}}\right)^{1/q}. \end{aligned} \tag{5.2.3}$$

Also,

$$\begin{aligned}H_\alpha^p f_0(\mathbf{x}) =&\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p}f_0(\mathbf{y})d\mathbf{y}\\ =&\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p}|\mathbf{y}|_p^{-\frac{\beta}{q-1}}\chi_{\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{y}|_p\leq 1\}}(\mathbf{y})d\mathbf{y}\\ =&\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\begin{cases}\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p}|\mathbf{y}|_p^{-\frac{\beta}{q-1}}d\mathbf{y}, & |\mathbf{x}|_p\leq 1;\\ \int_{|\mathbf{y}|_p\leq 1}|\mathbf{y}|_p^{-\frac{\beta}{q-1}}d\mathbf{y}, & |\mathbf{x}|_p> 1.\end{cases}\end{aligned}$$

By necessary splitting of integration domains as done in (5.2.2) and (5.2.3), we get

$$H_\alpha^p f_0(\mathbf{x}) = \frac{1-p^{-n}}{1-p^{\frac{\beta}{q-1}-n}}\begin{cases}|\mathbf{x}|_p^{\alpha-\frac{\beta}{q-1}}, & |\mathbf{x}|_p\leq 1;\\ |\mathbf{x}|_p^{\alpha-n}, & |\mathbf{x}|_p>1.\end{cases}$$

Let $C_2=\frac{1-p^{-n}}{1-p^{\frac{\beta}{q-1}-n}}$, then

$$\{\mathbf{x}\in\mathbb{Q}_p^n:|H_\alpha^p f_0(\mathbf{x})|>\lambda\} =\{|\mathbf{x}|_p\leq 1:|\mathbf{x}|_p^{\alpha-\frac{\beta}{q-1}}C_2>\lambda\}\cup\{|\mathbf{x}|_p>1:|\mathbf{x}|_p^{\alpha-n}C_2>\lambda\}.$$

When $0<\lambda<C_2$, with consideration $0<\alpha\leq\frac{\beta}{q-1}<n$ to have

$$\begin{aligned}\{\mathbf{x}\in\mathbb{Q}_p^n:|H_\alpha^p f_0(\mathbf{x})|>\lambda\} =&\{|\mathbf{x}|_p\leq 1\}\cup\{|\mathbf{x}|_p>1:|\mathbf{x}|_p^{\alpha-n}>\lambda/C_2\}\\ =&\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{x}|_p<(C_2/\lambda)^{1/(n-\alpha)}\}.\end{aligned}$$

Therefore, for $n-\alpha>0<n+\gamma$, we have

$$\begin{aligned}\|H^p_\alpha f_0\|_{L^{r,\infty}(|\mathbf{x}|^\gamma_p)} &= \sup_{0<\lambda<C_2}\lambda\bigg(\int_{\mathbb{Q}_p^n}\chi_{\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{x}|_p<(C_2/\lambda)^{1/(n-\alpha)}\}}(\mathbf{x})|\mathbf{x}|_p^\gamma d\mathbf{x}\bigg)^{1/r}\\ &= \sup_{0<\lambda<C_2}\lambda\bigg(\int_{|\mathbf{x}|_p<(C_2/\lambda)^{1/(n-\alpha)}}|\mathbf{x}|_p^\gamma d\mathbf{x}\bigg)^{1/r}\\ &=\bigg(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\bigg)^{1/r}\sup_{0<\lambda<C_2}\lambda\bigg(\frac{C_2}{\lambda}\bigg)^{(n+\gamma)/r(n-\alpha)}\\ &=\bigg(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\bigg)^{1/r}C_2\\ &=\bigg(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\bigg)^{1/r}\bigg(\frac{1-p^{-n}}{1-p^{\frac{\beta}{q-1}-n}}\bigg)^{1/q'}\|f_0\|_{L^q(|\mathbf{x}|_p^\beta)}.\end{aligned}$$

When $\lambda\geq C_2$ and $\alpha=\frac{\beta}{q-1}$, we have

$$\{\mathbf{x}\in\mathbb{Q}_p^n:|H^p_\alpha(f_0)(\mathbf{x})|>\lambda\}=\emptyset.$$

Also, if $\lambda\geq C_2$ and $\alpha<\frac{\beta}{q-1}$, then

$$\{\mathbf{x}\in\mathbb{Q}_p^n:|H^p_\alpha(f_0)(\mathbf{x})|>\lambda\}=\bigg\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{x}|_p<(C_2/\lambda)^{\frac{1}{\frac{\beta}{q-1}-\alpha}}\bigg\}.$$

The condition $\frac{\beta}{q-1}<n$ implies the inequality: $\frac{\beta}{q-1}<\frac{n+\beta}{q}$, which together with the condition $\frac{n+\beta}{q}-\alpha=\frac{n+\gamma}{r}$, yields $\frac{\beta}{q-1}-\alpha<\frac{n+\gamma}{r}$. Therefore,

$$\begin{aligned}\|H^p_\alpha f_0\|_{L^{r,\infty}(|\mathbf{x}|^\gamma_p)} &= \sup_{\lambda\geq C_2}\lambda\bigg(\int_{|\mathbf{x}|_p<(C_2/\lambda)^{\frac{1}{-\alpha+\beta/(q-1)}}}|\mathbf{x}|_p^\gamma d\mathbf{x}\bigg)^{1/r}\\ &=\bigg(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\bigg)^{1/r}\sup_{\lambda\geq C_2}C_2^{\left(\frac{n+\gamma}{r}\right)\left(\frac{\beta}{q-1}-\alpha\right)^{-1}}\lambda^{1-\left(\frac{n+\gamma}{r}\right)\left(\frac{\beta}{q-1}-\alpha\right)^{-1}}\\ &=\bigg(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\bigg)^{1/r}C_2\\ &=\bigg(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\bigg)^{1/r}\bigg(\frac{1-p^{-n}}{1-p^{\frac{\beta}{q-1}-n}}\bigg)^{1/q'}\|f_0\|_{L^q(|\mathbf{x}|_p^\beta)}.\end{aligned}$$

Hence,

$$\|H^p_\alpha f\|_{L^q(|\mathbf{x}|_p^\beta)\to L^{r,\infty}(|\mathbf{x}|_p^\gamma)}=\bigg(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\bigg)^{1/r}\bigg(\frac{1-p^{-n}}{1-p^{\frac{\beta}{q-1}-n}}\bigg)^{1/q'}.$$

5.3 Sharp Weak-type Estimates for Adjoint p-adic Hardy Operators

Likewise this section contains the results having sharp weak bounds for adjoint Hardy operator. The first theorem gives the operator norm of fractional p-adic adjoint Hardy operator:

Theorem 5.3.1 *Let* $1 < q < \frac{n+\beta}{\alpha},\ 1 \leq r < \infty$. *If* $\frac{n+\beta}{q} - \alpha = \frac{n+\gamma}{r}$, *then*

$$\|H_\alpha^{p,*}\|_{L^q(|\mathbf{x}|_p^\beta)\to L^{r,\infty}(|\mathbf{x}|_p^\gamma)} = \left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{1/r}\left(\frac{1-p^{-n}}{p^{(\frac{n+\gamma}{r})q'}-1}\right)^{1/q'}$$

An obvious corollary of Theorem 5.3.1 is achieved by taking $\beta = \gamma = 0$, as given below:

Corollary 5.3.2 *Let* $1 < q < \frac{n}{\alpha},\ 1 \leq r < \infty$. *If* $\frac{n}{q} - \alpha = \frac{n}{r}$, *then*

$$\|H_\alpha^{p,*}\|_{L^q(\mathbb{Q}_p^n)\to L^{r,\infty}(\mathbb{Q}_p^n)} = \left(\frac{1-p^{-n}}{p^{\frac{nq'}{r}}-1}\right)^{1/q'}$$

The next Theorem 5.3.3 and its corollary give the operator norms of p-adic adjoint Hardy operator:

Theorem 5.3.3 *Let* $1 < q < \infty,\ 1 \leq r < \infty$. *If* $\frac{n+\beta}{q} = \frac{n+\gamma}{r}$, *then*

$$\|H^{p,*}\|_{L^q(|\mathbf{x}|_p^\beta)\to L^{r,\infty}(|\mathbf{x}|_p^\gamma)} = \left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{1/r}\left(\frac{1-p^{-n}}{p^{(\frac{n+\beta}{r})q'}-1}\right)^{1/q'}$$

Corollary 5.3.4 *Let* $1 < q < \infty$, *then*

$$\|H^{p,*}\|_{L^q(\mathbb{Q}_p^n)\to L^{q,\infty}(\mathbb{Q}_p^n)} = \left(\frac{1-p^{-n}}{p^{\frac{n}{q-1}}-1}\right)^{1/q'}$$

As in the previous section, we will prove only Theorem 5.3.1, the proof of other result can be obtained similarly.

Proof of Theorem 5.3.1: By Hölder's inequality:

$$\begin{aligned}|H_\alpha^{p,*}f(\mathbf{x})| =& \left|\int_{|\mathbf{y}|_p>|\mathbf{x}|_p}\frac{f(\mathbf{y})}{|\mathbf{y}|_p^{n-\alpha}}d\mathbf{y}\right| \\ \leq& \left(\int_{|\mathbf{y}|_p>|\mathbf{x}|_p}|f(\mathbf{y})||\mathbf{y}|_p^\beta d\mathbf{y}\right)^{1/q}\left(\int_{|\mathbf{y}|_p>|\mathbf{x}|_p}|\mathbf{y}|_p^{(\alpha-n-\frac{\beta}{q})q'}d\mathbf{y}\right)^{1/q'}. \quad (5.3.1)\end{aligned}$$

Now, the inequality (5.3.1) possesses two factors and the factor on the right side is computed as:

$$\begin{aligned}
\left(\int_{|\mathbf{y}|_p>|\mathbf{x}|_p}|\mathbf{y}|_p^{(\alpha-n-\frac{\beta}{q})q'}d\mathbf{y}\right)^{1/q'} &=\left(\sum_{j=\log_p|x|_p+1}^{\infty}\int_{S_j}p^{jq'(\alpha-n-\frac{\beta}{q})}\right)^{1/q'}\\
&=(1-p^{-n})^{1/q'}\left(\sum_{j=\log_p|x|_p+1}^{\infty}p^{jq'(\alpha-\frac{n+\beta}{q})}\right)^{1/q'}\\
&=\left(\frac{1-p^{-n}}{p^{(\frac{n+\beta}{q}-\alpha)q'}-1}\right)^{1/q'}|\mathbf{x}|_p^{\alpha-\frac{n+\beta}{q}}. \qquad (5.3.2)
\end{aligned}$$

By virtue of the condition $\frac{n+\beta}{q}-\alpha=\frac{n+\gamma}{r}$ and (5.3.2), we rewrite the inequality (5.3.1) as below:

$$|H_\alpha^{p,*}f(\mathbf{x})|\leq\left(\frac{1-p^{-n}}{p^{(\frac{n+\gamma}{r})q'}-1}\right)^{1/q'}|\mathbf{x}|_p^{-\frac{n+\gamma}{r}}\|f\|_{L^q(|\mathbf{x}|_p^\beta)}=C_3^f|\mathbf{x}|_p^{-\frac{n+\gamma}{r}},$$

where

$$C_3^f=\left(\frac{1-p^{-n}}{p^{(\frac{n+\gamma}{r})q'}-1}\right)^{1/q'}\|f\|_{L^q(|\mathbf{x}|_p^\beta)}.$$

Now,

$$\{\mathbf{x}\in\mathbb{Q}_p^n:|H_\alpha^{p,*}f(\mathbf{x})|>\lambda\}\subset\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{x}|_p<(C_3^f/\lambda)^{\frac{r}{n+\gamma}}\}.$$

Thus,

$$\begin{aligned}
\|H_\alpha^{p,*}f\|_{L^{r,\infty}(|\mathbf{x}|_p^\gamma)}&\leq\sup_{\lambda>0}\lambda\left(\int_{\mathbb{Q}_p^n}\chi_{\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{x}|_p<(C_3^f/\lambda)^{r/(n+\gamma)}\}}(\mathbf{x})|\mathbf{x}|_p^\gamma d\mathbf{x}\right)^{1/r}\\
&=\sup_{\lambda>0}\lambda\left(\int_{|\mathbf{x}|_p<(C_3^f/\lambda)^{r/(n+\gamma)}}|\mathbf{x}|_p^\gamma d\mathbf{x}\right)^{1/r}\\
&=\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{1/r}C_3^f\\
&=\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{1/r}\left(\frac{1-p^{-n}}{p^{(\frac{n+\gamma}{r})q'}-1}\right)^{1/q'}\|f\|_{L^q(|\mathbf{x}|_p^\beta)}.
\end{aligned}$$

To prove the optimality of the constant, we let

$$f_0(\mathbf{x})=|\mathbf{x}|_p^{\frac{\alpha-n-\beta}{q-1}}\chi_{\{|\mathbf{x}|_p>1\}},$$

then making use of the condition $q < \frac{n+\beta}{\alpha}$, we obtain

$$\begin{aligned}
\|f_0\|_{L^q(|\mathbf{x}|_p^\beta)} =& \left(\int_{|\mathbf{x}|_p>1} |\mathbf{x}|_p^{\beta+(\alpha-n-\beta)\frac{q}{q-1}} d\mathbf{x}\right)^{1/q} \\
=& \left(\sum_{j=1}^{\infty}\int_{S_j} p^{j\left(\beta+(\alpha-n-\beta)\frac{q}{q-1}\right)} d\mathbf{x}\right)^{1/q} \\
=& (1-p^{-n})^{1/q}\left(\sum_{j=1}^{\infty} p^{\frac{j\alpha}{q-1}\left(q-\frac{n+\beta}{\alpha}\right)}\right)^{1/q} \\
=& \left(\frac{1-p^{-n}}{p^{(\frac{n+\gamma}{r})q'}-1}\right)^{1/q}.
\end{aligned} \tag{5.3.3}$$

On the other hand,

$$\begin{aligned}
H_\alpha^{p,*} f_0(\mathbf{x}) &= \int_{|\mathbf{y}|_p>|\mathbf{x}|_p} \frac{f_0(y)}{|\mathbf{y}|_p^{n-\alpha}} d\mathbf{y} \\
&= \int_{|\mathbf{y}|_p>|\mathbf{x}|_p} |\mathbf{y}|_p^{(\alpha-n)\frac{q}{q-1}-\frac{\beta}{q-1}} \chi_{\{\mathbf{y}\in\mathbb{Q}_p^n:|\mathbf{y}|_p>1\}}(\mathbf{y}) d\mathbf{y} \\
&= \begin{cases} \int_{|\mathbf{y}|_p>|\mathbf{x}|_p} |\mathbf{y}|_p^{(\alpha-n)\frac{q}{q-1}-\frac{\beta}{q-1}} d\mathbf{y}, & |\mathbf{x}|_p>1 \\ \int_{|\mathbf{y}|_p>1} |\mathbf{y}|_p^{(\alpha-n)\frac{q}{q-1}-\frac{\beta}{q-1}} d\mathbf{y}, & |\mathbf{x}|_p \le 1, \end{cases}
\end{aligned}$$

which in view of equations (5.3.2) and (5.3.3) yields

$$H_\alpha^{p,*} f_0(\mathbf{x}) = \frac{1-p^{-n}}{p^{(\frac{n+\gamma}{r})q'}-1} \begin{cases} |\mathbf{x}|_p^{-(\frac{n+\gamma}{r})q'}, & |\mathbf{x}|_p>1 \\ 1, & |\mathbf{x}|_p\le 1 \end{cases}$$

Let $C_4 = \frac{1-p^{-n}}{p^{(\frac{n+\gamma}{r})q'}-1}$, then

$$\{\mathbf{x}\in\mathbb{Q}_p^n : |H_\alpha^{p,*} f_0(\mathbf{x})|>\lambda\} = \{|\mathbf{x}|_p\le 1 : C_4>\lambda\}\cup\{|\mathbf{x}|_p>1 : C_4|\mathbf{x}|_p^{-(\frac{n+\gamma}{r})q'}>\lambda\}.$$

Obviously for $\lambda > C_4$, we have

$$\{\mathbf{x}\in\mathbb{Q}_p^n : |H_\alpha^{p,*} f_0(\mathbf{x})|>\lambda\} = \emptyset,$$

and for $\lambda < C_4$

$$\{\mathbf{x}\in\mathbb{Q}_p^n : |H_\alpha^{p,*} f_0(\mathbf{x})|>\lambda\} = \{\mathbf{x}\in\mathbb{Q}_p^n : |\mathbf{x}|_p < (C_4/\lambda)^{\frac{r}{(n+\gamma)q'}}\}.$$

Therefore,

$$
\begin{aligned}
\|H_\alpha^{p,*}f_0\|_{L^{r,\infty}(|\mathbf{x}|_p^\gamma)} &= \sup_{0<\lambda<C_4}\lambda\Bigg(\int_{|\mathbf{x}|_p<(C_4/\lambda)^{\frac{r}{(n+\gamma)q'}}}|\mathbf{x}|_p^\gamma d\mathbf{x}\Bigg)^{1/r}\\
&= \sup_{0<\lambda<C_4}\lambda\Bigg(\sum_{j=-\infty}^{\log_p(C_4/\lambda)^{\frac{r}{(n+\gamma)q'}}}\int_{S_j}|\mathbf{x}|_p^\gamma d\mathbf{x}\Bigg)^{1/r}\\
&=(1-p^{-n})^{1/r}\sup_{0<\lambda<C_4}\lambda\Bigg(\sum_{j=-\infty}^{\log_p(C_4/\lambda)^{\frac{r}{(n+\gamma)q'}}}p^{j(n+\gamma)}\Bigg)^{1/r}\\
&=\bigg(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\bigg)^{1/r}\sup_{0<\lambda<C_4}\lambda\bigg(\frac{C_4}{\lambda}\bigg)^{1/q'}\\
&=\bigg(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\bigg)^{1/r}C_4\\
&=\bigg(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\bigg)^{1/r}\bigg(\frac{1-p^{-n}}{p^{(\frac{n+\gamma}{r})q'}-1}\bigg)^{1/q'}\|f_0\|_{L^q(|\mathbf{x}|_p^\beta)}.
\end{aligned}
$$

Hence,

$$
\|H_\alpha^{p,*}\|_{L^q(|\mathbf{x}|_p^\beta)\to L^{r,\infty}(|\mathbf{x}|_p^\gamma)}=\bigg(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\bigg)^{1/r}\bigg(\frac{1-p^{-n}}{p^{(\frac{n+\gamma}{r})q'}-1}\bigg)^{1/q'}
$$

Remark In this Chapter, all the theorems are proved under the condition that $1<q<\infty$. Therefore, there arise a problem regarding the weak type operator norm of p-adic Hardy operator when $q=1$, which we will address in the next Chapter.

Chapter 6
Sharp Weak Bounds for p-adic Hardy Operators on p-adic Linear Spaces

6.1 Introduction

This Chapter is a continuation of a couple of results about p-adic Hardy-type operators presented in Chapter 5, where we have obtained optimal weak bounds for these operators on weighted p-adic Lebesgue spaces $L^q(|x|_p^\rho, \mathbb{Q}_p^n),\ 1 < q < \infty$. However, when $q = 1$, the results of Chapter 5 remain no more true. Therefore, there arise a question regarding the weak type endpoint estimates for fractional Hardy type operators on p-adic field. Here, in this Chapter, we will answer this question.

In this Chapter, weak boundedness of p-adic fractional Hardy operator and its adjoint operator is established on weighted p-adic Lebesgue space $L^q(w, \mathbb{Q}_p^n)$ at the endpoint $q = 1$. In some cases, we obtain sharp constants for these boundedness inequalities. Moreover, we obtain the optimal weak estimates for p-adic Hardy operator on p-adic central Morrey space.

6.2 Endpoint Estimates for p-adic Fractional Hardy Operator

Here, we show that the weak bound for p-adic fractional Hardy type operators on p-adic weighted weak Lebesgue space $L^{q,\infty}(|\mathbf{x}|_p^\beta)$ at $q = 1$ is not sharp. However, if we take $\beta = 0$ then it becomes sharp and the same is proved in the results of the current section.

Theorem 6.2.1 *Suppose* $-n<\gamma$, $-n+\alpha<\beta$ *and* $\gamma>\beta-\alpha$. *If* $f\in L^1(|\mathbf{x}|_p^{\beta})$, *then*

$$\|H_\alpha^p\|_{L^1(|\mathbf{x}|_p^{\beta})\to L^{(n+\gamma)/(n-\alpha+\beta),\infty}(|\mathbf{x}|_p^{\gamma})}\leq\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha+\beta)/(n+\gamma)}.$$

Proof. Since

$$\begin{aligned}|H_\alpha^p f(\mathbf{x})|=&\left|\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p}f(\mathbf{y})d\mathbf{y}\right|\\=&\left|\frac{1}{|\mathbf{x}|_p^{n-\alpha}}\int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p}f(\mathbf{y})|\mathbf{y}|_p^{\beta}|\mathbf{y}|_p^{-\beta}d\mathbf{y}\right|\\\leq&|\mathbf{x}|_p^{-(n-\alpha+\beta)}\|f\|_{L^1(|\mathbf{x}|_p^{\beta})}.\end{aligned}\tag{6.2.1}$$

Let $C_1=\|f\|_{L^1(|\mathbf{x}|_p^{\beta})}$, then

$$\{\mathbf{x}\in\mathbb{Q}_p^n:|H_\alpha^p f(\mathbf{x})|>\lambda\}\subset\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{x}|_p<(C_1/\lambda)^{1/(n-\alpha+\beta)}\}.$$

Thus,

$$\begin{aligned}&\|H_\alpha^p f\|_{L^{(n+\gamma)/(n-\alpha+\beta),\infty}(|x|_p^{\gamma})}\\=&\sup_{\lambda>0}\lambda\bigg(\int_{\mathbb{Q}_p^n}\chi_{\{\mathbf{x}\in\mathbb{Q}_p^n:|H_\alpha^p f(\mathbf{x})|>\lambda\}}(\mathbf{x})|\mathbf{x}|_p^{\gamma}d\mathbf{x}\bigg)^{(n-\alpha+\beta)/(n+\gamma)}\\\leq&\sup_{\lambda>0}\lambda\bigg(\int_{\mathbb{Q}_p^n}\chi_{\left\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{x}|_p<\left(C_1/\lambda\right)^{1/(n-\alpha+\beta)}\right\}}(\mathbf{x})|\mathbf{x}|_p^{\gamma}d\mathbf{x}\bigg)^{(n-\alpha+\beta)/(n+\gamma)}\\=&\sup_{\lambda>0}\lambda\bigg(\int_{|\mathbf{x}|_p<\left(C_1/\lambda\right)^{1/(n-\alpha+\beta)}}|\mathbf{x}|_p^{\gamma}d\mathbf{x}\bigg)^{(n-\alpha+\beta)/(n+\gamma)}\end{aligned}$$

$$\begin{aligned}=&\sup_{\lambda>0}\lambda\bigg(\sum_{j=-\infty}^{\log_p\left(C_1/\lambda\right)^{1/(n-\alpha+\beta)}}\int_{S_j}|\mathbf{x}|_p^{\gamma}d\mathbf{x}\bigg)^{(n-\alpha+\beta)/(n+\gamma)}\\=&(1-p^{-n})^{(n-\alpha+\beta)/(n+\gamma)}\sup_{\lambda>0}\lambda\bigg(\sum_{j=-\infty}^{\log_p\left(C_1/\lambda\right)^{1/(n-\alpha+\beta)}}p^{j(n+\gamma)}d\mathbf{x}\bigg)^{(n-\alpha+\beta)/(n+\gamma)}\\=&\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha+\beta)/(n+\gamma)}\sup_{\lambda>0}\lambda\left(\frac{C_1}{\lambda}\right)\\=&\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha+\beta)/(n+\gamma)}\|f\|_{L^1(|\mathbf{x}|_p^{\beta})}.\end{aligned}\tag{6.2.2}$$

Theorem 6.2.2 *Let $0 < \alpha < n$ and $n + \gamma > 0$. If $f \in L^1(\mathbb{Q}_p^n)$, then*

$$\|H_\alpha^p\|_{L^1(\mathbb{Q}_p^n)\to L^{(n+\gamma)/(n-\alpha),\infty}(|\mathbf{x}|_p^\gamma)} = \left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)}.$$

Proof. Taking $\beta = 0$ in Theorem 6.2.1, we infer from inequality (6.2.2) that

$$\|H_\alpha^p f\|_{L^{(n+\gamma)/(n-\alpha),\infty}(|\mathbf{x}|_p^\gamma)} \leq \left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)} \|f\|_{L^1(\mathbb{Q}_p^n)}. \quad (6.2.3)$$

Conversely, let

$$f_0(\mathbf{x}) = \chi_{\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{x}|_p\leq 1\}}(\mathbf{x}),$$

then

$$\|f_0\|_{L^1(\mathbb{Q}_p^n)} = 1.$$

Also,

$$\begin{aligned} H_\alpha^p f_0(\mathbf{x}) =& \frac{1}{|\mathbf{x}|_p^{n-\alpha}} \int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p} f_0(\mathbf{y})d\mathbf{y} \\ =& \frac{1}{|\mathbf{x}|_p^{n-\alpha}} \int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p} \chi_{\{\mathbf{x}\in\mathbb{Q}_p^n:|\mathbf{y}|_p\leq 1\}}(\mathbf{y})d\mathbf{y} \\ =& \frac{1}{|\mathbf{x}|_p^{n-\alpha}} \begin{cases} \int_{|\mathbf{y}|_p\leq|\mathbf{x}|_p} d\mathbf{y}, & |\mathbf{x}|_p \leq 1; \\ \int_{|\mathbf{y}|_p\leq 1} d\mathbf{y}, & |\mathbf{x}|_p > 1. \end{cases} \end{aligned}$$

Since $|B_{\log_p |\mathbf{x}|_p}|_H = |\mathbf{x}|_p^n |B_0|_H$, therefore,

$$H_\alpha^p f_0(\mathbf{x}) = \begin{cases} |\mathbf{x}|_p^\alpha, & |\mathbf{x}|_p \leq 1; \\ |\mathbf{x}|_p^{\alpha-n}, & |\mathbf{x}|_p > 1. \end{cases}$$

Now,

$$\{\mathbf{x} \in \mathbb{Q}_p^n : |H_\alpha^p f_0(\mathbf{x})| > \lambda\} = \{|\mathbf{x}|_p \leq 1 : |\mathbf{x}|_p^\alpha > \lambda\} \cup \{|\mathbf{x}|_p > 1 : |\mathbf{x}|_p^{\alpha-n} > \lambda\}.$$

Since $0 < \alpha < n$, therefore, when $\lambda \geq 1$, then

$$\{\mathbf{x} \in \mathbb{Q}_p^n : |H_\alpha^p f_0(\mathbf{x})| > \lambda\} = \emptyset,$$

and when $0 < \lambda < 1$, then

$$\{\mathbf{x} \in \mathbb{Q}_p^n : |H_\alpha^p f_0(\mathbf{x})| > \lambda\} = \{\mathbf{x} \in \mathbb{Q}_p^n : (\lambda)^{1/n} < |\mathbf{x}|_p < (1/\lambda)^{1/n-\alpha}\}.$$

Ultimately we are down to:

$$
\begin{aligned}
&\|H^p_\alpha f_0\|_{L^{(n+\gamma)/(n-\alpha))),\infty}(|\mathbf{x}|_p^\gamma)}\\
=&\sup_{0<\lambda<1}\lambda\bigg(\int_{\mathbb{Q}_p^n}\chi_{\{\mathbf{x}\in\mathbb{Q}_p^n:(\lambda)^{1/\alpha}<|\mathbf{x}|_p<(1/\lambda)^{1/(n-\alpha)}\}}(\mathbf{x})|\mathbf{x}|_p^\gamma d\mathbf{x}\bigg)^{(n-\alpha)/(n+\gamma)}\\
=&\sup_{0<\lambda<1}\lambda\bigg(\int_{(\lambda)^{1/\alpha}<|\mathbf{x}|_p<(1/\lambda)^{1/(n-\alpha)}}|\mathbf{x}|_p^\gamma d\mathbf{x}\bigg)^{(n-\alpha)/(n+\gamma)}\\
=&(1-p^{-n})^{(n-\alpha)/(n+\gamma)}\sup_{0<\lambda<1}\lambda\bigg(\sum_{j=\log_p\lambda^{1/\alpha}+1}^{\log_p\lambda^{1/(\alpha-n)}}p^{j(n+\gamma)}\bigg)^{(n-\alpha)/(n+\gamma)}\\
=&(1-p^{-n})^{(n-\alpha)/(n+\gamma)}\sup_{0<\lambda<1}\lambda\bigg(\frac{p^{(\log_p\lambda^{1/\alpha}+1)(n+\gamma)}-p^{(\log_p\lambda^{1/(\alpha-n)}+1)(n+\gamma)}}{1-p^{(n+\gamma)}}\bigg)^{(n-\alpha)/(n+\gamma)}
\end{aligned}
$$

$$
\begin{aligned}
=&(1-p^{-n})^{(n-\alpha)/(n+\gamma)}\sup_{0<\lambda<1}\lambda\bigg(\frac{\lambda^{(n+\gamma)/\alpha}-\lambda^{(n+\gamma)/(\alpha-n)}}{p^{-(n+\gamma)}-1}\bigg)^{(n-\alpha)/(n+\gamma)}\\
=&(1-p^{-n})^{(n-\alpha)/(n+\gamma)}\sup_{0<\lambda<1}\bigg(\frac{1-\lambda^{(n+\gamma)/\alpha}\lambda^{(n+\gamma)/(n-\alpha)}}{1-p^{-(n+\gamma)}}\bigg)^{(n-\alpha)/(n+\gamma)}\\
=&\bigg(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\bigg)^{(n-\alpha)/(n+\gamma)}\sup_{0<\lambda<1}\Big(1-\lambda^{(n+\gamma)/\alpha}\lambda^{(n+\gamma)/(n-\alpha)}\Big)^{(n-\alpha)/(n+\gamma)}\\
=&\bigg(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\bigg)^{(n-\alpha)/(n+\gamma)}\\
=&\bigg(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\bigg)^{(n-\alpha)/(n+\gamma)}\|f_0\|_{L^1(\mathbb{Q}_p^n)}.
\end{aligned}
\tag{6.2.4}
$$

We thus conclude from (6.2.3) and (6.2.4) that

$$
\|H^p_\alpha\|_{L^1(\mathbb{Q}_p^n)\to L^{(n+\gamma)/(n-\alpha),\infty}(|\mathbf{x}|_p^\gamma)}=\bigg(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\bigg)^{1/q}.
$$

6.3 Endpoint Estimates for p-adic Adjoint Fractional Hardy Operator

This Section contains results regarding weak endpoint estimates for p-adic adjoint fractional Hardy operator $H^{p,*}_\alpha$ on Lebesgue space. However, contrary to the results of previous section, we are unable to find sharp bounds even in the un-weighted case.

Theorem 6.3.1 *Suppose* $-n+\alpha<\beta$, $-n<\gamma$ *and* $\gamma>\beta-\alpha$. *If* $f\in L^1(|\mathbf{x}|_p^\beta)$, *then*

$$\|H_\alpha^{p,*}\|_{L^1(|\mathbf{x}|_p^\beta)\to L^{(n+\gamma)/(n-\alpha+\beta),\infty}(|\mathbf{x}|_p^\gamma)}\leq\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(-\alpha+n+\beta)/(\gamma+n)}.$$

Proof. Obviously,

$$\begin{aligned}|H_\alpha^{p,*}f(\mathbf{x})|=&\left|\int_{|\mathbf{y}|_p>|\mathbf{x}|_p}\frac{f(\mathbf{y})}{|\mathbf{y}|_p^{n-\alpha}}d\mathbf{y}\right|\\=&\left|\int_{|\mathbf{y}|_p>|\mathbf{x}|_p}f(\mathbf{y})|\mathbf{y}|_p^{\alpha-(n+\beta)}|\mathbf{y}|_p^\beta d\mathbf{y}\right|.\end{aligned}$$

Since $\alpha-n<\beta$ and $|\mathbf{y}|_p>|\mathbf{x}|_p$, we have:

$$|H_\alpha^{p,*}f(\mathbf{x})|\leq|\mathbf{x}|_p^{-(n-\alpha+\beta)}\int_{|\mathbf{y}|_p>|\mathbf{x}|_p}|f(\mathbf{y})||\mathbf{y}|_p^\beta d\mathbf{y}. \tag{6.3.1}$$

Notice that the right hand sides of inequalities (6.2.1) and (6.3.1) are same, therefore, applying the definition of $L^q(|\mathbf{x}|_p^\gamma)$ and following the steps as followed in establishing the inequality (6.2.2), we obtain

$$\|H_\alpha^{p,*}f\|_{L^{(n+\gamma)/(n-\alpha+\beta),\infty}(|x|_p^\gamma)}\leq\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(-\alpha+n+\beta)/(\gamma+n)}\|f\|_{L^1(|\mathbf{x}|_p^\beta)}. \tag{6.3.2}$$

Theorem 6.3.2 *Let* $0<\alpha<n$ *and* $n+\gamma>0$. *If* $f\in L^1(\mathbb{Q}_p^n)$, *then*

$$\|H_\alpha^{p,*}\|_{L^1(\mathbb{Q}_p^n)\to L^{(n+\gamma)/(n-\alpha),\infty}(|\mathbf{x}|_p^\gamma)}\simeq\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)}.$$

Proof. We take $\beta=0$ in Theorem 6.3.1, then from the inequality (6.3.2), we get

$$\|H_\alpha^{p,*}\|_{L^1(\mathbb{Q}_p^n)\to L^{(n+\gamma)/(n-\alpha),\infty}(|\mathbf{x}|_p^\gamma)}\leq\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)}. \tag{6.3.3}$$

Conversely let $H_\alpha^{p,*}$ be bounded from $L^1(\mathbb{Q}_p^n)$ to $L^{(n+\gamma)/(n-\alpha),\infty}(|\mathbf{x}|_p^\gamma)$. Let $f_0(\mathbf{x})=|\mathbf{x}|_p^{-(\alpha+n)}\chi_{\{|\mathbf{x}|_p>1\}}(\mathbf{x})$, then

$$\begin{aligned}\|f_0\|_{L^1(\mathbb{Q}_p^n)}=&\int_{|\mathbf{x}|_p>1}|\mathbf{x}|_p^{-n-\alpha}d\mathbf{x}\\=&(1-p^{-n})\sum_{j=1}^{\infty}p^{-j\alpha}\\=&p^{-\alpha}\frac{1-p^{-n}}{1-p^{-\alpha}}.\end{aligned} \tag{6.3.4}$$

Also,

$$H_\alpha^{p,*}f_0(\mathbf{x}) = \int_{|\mathbf{y}|_p>|\mathbf{x}|_p} \frac{f_0(\mathbf{y})}{|\mathbf{y}|_p^{n-\alpha}}\chi_{\{|\mathbf{y}|_p>1\}}(\mathbf{y})d\mathbf{y}$$
$$= \int_{|\mathbf{y}|_p>|\mathbf{x}|_p} |\mathbf{y}|_p^{-2n}\chi_{\{|\mathbf{y}|_p>1\}}(\mathbf{y})d\mathbf{y}.$$

Hence,

$$H_\alpha^{p,*}f_0(\mathbf{x}) = \begin{cases} \int_{|\mathbf{y}|_p>|\mathbf{x}|_p} |\mathbf{y}|_p^{-2n}d\mathbf{y}, & |\mathbf{x}|_p > 1; \\ \int_{|\mathbf{y}|_p>1} |\mathbf{y}|_p^{-2n}d\mathbf{y}, & |\mathbf{x}|_p \le 1. \end{cases} \tag{6.3.5}$$

Let us first consider

$$\int_{|\mathbf{y}|_p>|\mathbf{x}|_p} |\mathbf{y}|_p^{-2n}d\mathbf{y} = \sum_{j=\log_p|\mathbf{x}|_p+1}^{\infty} \int_{S_j} p^{-2jn}d\mathbf{y}$$
$$=(1-p^{-n})\sum_{j=\log_p|\mathbf{x}|_p+1}^{\infty} p^{-jn}$$
$$=p^{-n}|\mathbf{x}|_p^{-n}.$$

By a similar pattern as followed in (6.3.4), we obtain

$$\int_{|\mathbf{y}|_p>1} |\mathbf{y}|_p^{-2n}d\mathbf{y} = p^{-n}.$$

Therefore, equation (6.3.5) takes the following shape

$$H_\alpha^{p,*}f_0(\mathbf{x}) = p^{-n}\begin{cases} |\mathbf{x}|_p^{-n}, & |\mathbf{x}|_p > 1; \\ 1, & |\mathbf{x}|_p \le 1. \end{cases}$$

Let

$$C_2 = p^{-n},$$

then

$$\{\mathbf{x}\in\mathbb{Q}_p^n : |H_\alpha^{p,*}f_0(\mathbf{x})| > \lambda\} = \{|\mathbf{x}|_p \le 1 : C_2 > \lambda\} \cup \{|\mathbf{x}|_p > 1 : C_2|\mathbf{x}|_p^{-n} > \lambda\}.$$

Evidently, if $\lambda > C_2$, then

$$\{\mathbf{x}\in\mathbb{Q}_p^n : |H_\alpha^{p,*}f_0(\mathbf{x})| > \lambda\} = \emptyset.$$

On the other side if $\lambda \le C_2$, then

$$\{\mathbf{x}\in\mathbb{Q}_p^n : |H_\alpha^{p,*}f_0(\mathbf{x})| > \lambda\} = \{\mathbf{x}\in\mathbb{Q}_p^n : |\mathbf{x}|_p < (C_2/\lambda)^{1/n}\}.$$

Ultimately, we are lead to

$$
\begin{aligned}
&\|H_{\alpha}^{p,*}f_0\|_{L^{(n+\gamma)/(n-\alpha),\infty}(|\mathbf{x}|_p^{\gamma})} \\
&\leq \sup_{0<\lambda<1} \lambda\left(\int_{\mathbb{Q}_p^n} \chi_{\{|\mathbf{x}|_p<(C_2/\lambda)^{1/n}\}}(\mathbf{x})|\mathbf{x}|_p^{\gamma}d\mathbf{x}\right)^{(n-\alpha)/(n+\gamma)} \\
&= \sup_{0<\lambda<C_2} \lambda\left(\int_{|\mathbf{x}|_p<(C_2/\lambda)^{1/(\beta+n)}} |\mathbf{x}|_p^{\gamma}\right)^{(n-\alpha)/(n+\gamma)} \\
&= \sup_{0<\lambda<C_2} \lambda\left(\sum_{j=-\infty}^{\log_p(C_2/\lambda)^{1/n}} \int_{S_j} p^{j\gamma}d\mathbf{x}\right)^{(n-\alpha)/(n+\gamma)} \\
&= (1-p^{-n})^{(n-\alpha)/(n+\gamma)} \sup_{0<\lambda<C_2} \lambda\left(\sum_{j=-\infty}^{\log_p(C_2/\lambda)^{1/n}} p^{j(n+\gamma)}\right)^{(n-\alpha)/(n+\gamma)} \\
&= \left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)} \sup_{0<\lambda<C_2} \lambda\left(\frac{C_2}{\lambda}\right)^{(n-\alpha)/n} \\
&= \left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)} C_2^{(n-\alpha)/n} \sup_{0<\lambda<C_2} \lambda^{1-(n-\alpha)/n} \\
&= \left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)} C_2 \\
&= \left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)} \frac{p^{\alpha}-1}{p^n-1}\|f_0\|_{L^1(\mathbb{Q}_p^n)}.
\end{aligned}
$$

Since $H_{\alpha}^{p,*}$ is bounded from $L^1(\mathbb{Q}_p^n)$ to $L^{(n+\gamma)/(n-\alpha),\infty}(|\mathbf{x}|_p^{\gamma})$, therefore,

$$
\left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)} \frac{p^{\alpha}-1}{p^n-1} \leq \|H_{\alpha}^{p,*}\|_{L^1(\mathbb{Q}_p^n)\to L^{(n+\gamma)/(n-\alpha),\infty}(|\mathbf{x}|_p^{\gamma})}. \tag{6.3.6}
$$

From the inequalities (6.3.3) and (6.3.6), we conclude that

$$
\|H_{\alpha}^{p,*}\|_{L^1(\mathbb{Q}_p^n)\to L^{(n+\gamma)/(n-\alpha),\infty}(|\mathbf{x}|_p^{\gamma})} \simeq \left(\frac{1-p^{-n}}{1-p^{-(n+\gamma)}}\right)^{(n-\alpha)/(n+\gamma)}.
$$

6.4 Optimal Weak Bounds for p-adic Hardy Operator on p-adic Morrey-type Spaces

The present section investigates the weak boundedness of p-adic Hardy operator on p-adic central Morrey spaces. Most importantly, we have acquired the optimal constant.

Theorem 6.4.1 *Let $-1/q \leq \lambda < 0$ and $1 \leq q < \infty$, and if $f \in \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$, then*

$$\|H\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)\to W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} = 1.$$

Proof. Applying Hölder's inequality:

$$\begin{aligned}|H^p f(\mathbf{x})| \leq& \frac{1}{|\mathbf{x}|_p^n}\left(\int_{B(0,|\mathbf{x}|_p)} |f(\mathbf{y})|^q d\mathbf{y}\right)^{1/q}\left(\int_{B(0,|\mathbf{x}|_p)} d\mathbf{y}\right)^{1/q'}\\ =&|\mathbf{x}|_p^{n\lambda}\|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}.\end{aligned}$$

Let $C = \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}$. Since $\lambda < 0$, then

$$\begin{aligned}\|Hf\|_{W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \leq& \sup_{\gamma\in\mathbb{Z}}\sup_{t>0} t|B_\gamma|_H^{-\lambda-1/q}\big|\{\mathbf{x}\in B_\gamma : C|\mathbf{x}|_p^{n\lambda} > t\}\big|^{1/q}\\ =&\sup_{\gamma\in\mathbb{Z}}\sup_{t>0} t|B_\gamma|_H^{-\lambda-1/q}\big|\{|\mathbf{x}|_p \leq p^\gamma : |\mathbf{x}|_p < (t/C)^{1/n\lambda}\}\big|^{1/q}.\end{aligned}$$

If $\gamma \leq \log_p(t/C)^{1/n\lambda}$, then for $\lambda < 0$, we obtain

$$\begin{aligned}&\sup_{t>0}\sup_{\gamma\leq\log_p(t/C)^{1/n\lambda}} t|B_\gamma|_H^{-\lambda-1/q}\big|\{|\mathbf{x}|_p \leq p^\gamma : |\mathbf{x}|_p < (t/C)^{1/n\lambda}\}\big|^{1/q}\\ &\leq \sup_{t>0}\sup_{\gamma\leq\log_p(t/C)^{1/n\lambda}} tp^{-\gamma n\lambda}\\ &= C\\ &= \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}.\end{aligned}$$

If $\gamma > \log_p(t/C)^{1/n\lambda}$, then for $\lambda + 1/q > 0$, we get

$$\begin{aligned}&\sup_{t>0}\sup_{\gamma>\log_p(t/C)^{1/n\lambda}} t|B_\gamma|_H^{-\lambda-1/q}|\{|\mathbf{x}|_p \leq p^\gamma : |\mathbf{x}|_p < (t/C)^{1/n\lambda}\}|^{1/q}\\ &\leq \sup_{t>0}\sup_{\gamma>\log_p(t/C)^{1/n\lambda}} tp^{-\gamma n(\lambda+1/q)}(t/C)^{1/q\lambda}\\ &= C\\ &= \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}.\end{aligned}$$

Therefore,

$$\|Hf\|_{W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \leq \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)},$$

which implies that

$$\|H\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)\to W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \leq 1. \tag{6.4.1}$$

To show that the constant 1 is optimal, we employ the idea of use of power function given in [110]. Hence, we suppose

$$f_0(\mathbf{x}) = \chi_{\{|\mathbf{x}|_p \leq 1\}}(\mathbf{x}),$$

then,

$$\|f_0\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} \chi_{\{|\mathbf{x}|_p\leq 1\}}(\mathbf{x}) d\mathbf{x} \right)^{1/q}.$$

Since $\lambda < 0$, thus if $\gamma < 0$ then

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ \gamma<0}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_\gamma} d\mathbf{x} \right)^{1/q} = \sup_{\substack{\gamma\in\mathbb{Z}\\ \gamma<0}} p^{-n\gamma\lambda} = 1.$$

Since $\lambda + 1/q > 0$, thus if $\gamma \geq 0$ then

$$\sup_{\substack{\gamma\in\mathbb{Z}\\ \gamma\geq 0}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda q}} \int_{B_0} d\mathbf{x} \right)^{1/q} = \sup_{\substack{\gamma\in\mathbb{Z}\\ \gamma\geq 0}} p^{-n\gamma(\lambda+1/q)} = 1.$$

Therefore,

$$\|f_0\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} = 1.$$

Moreover,

$$H^p f_0(\mathbf{x}) = \begin{cases} 1, & |\mathbf{x}|_p \leq 1; \\ |\mathbf{x}|_p^{-n}, & |\mathbf{x}|_p > 1, \end{cases}$$

which implies that $|H^p f_0(\mathbf{x})| \leq 1$.

Next, in order to construct weak central Morrey norm we take following couple of cases:

Case 1. When $0 \geq \gamma$, then

$$\|Hf_0\|_{WL^q(B_\gamma)} = \sup_{0<t\leq 1} t|\{\mathbf{x} \in B_\gamma : 1 > t\}|^{1/q} = p^{n\gamma/q},$$

and

$$\|Hf_0\|_{W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} = \sup_{\gamma<0} |B_\gamma|_H^{-\lambda-1/q} \|f\|_{WL^q(B_\gamma)} = \sup_{\gamma<0} p^{-n\gamma\lambda} = 1 = \|f_0\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}.$$

Case 2. When $0 < \gamma$, we have

$$\|Hf_0\|_{WL^q(B_\gamma)} = \sup_{0<t\leq 1} t|\{\mathbf{x} \in B_\mathbf{0} : 1 > t\} \cup \{1 \leq |\mathbf{x}|_p < p^\gamma : |\mathbf{x}|_p^{-n} > t\}|^{1/q}.$$

For further analysis this case is divided into two more cases:
Case 2(a). If $1 < \gamma < \log_p t^{-1/n}$, then

$$\|Hf_0\|_{WL^q(B_\gamma)} = \sup_{0<t\leq 1} t\{1 + p^{n\gamma} - 1\}^{1/q} = \sup_{0<t\leq 1} tp^{n\gamma/q}.$$

Case 2(b). If $1 < \log_p t^{-1/n} < \gamma$, then:

$$\|Hf_0\|_{WL^q(B_\gamma)} = \sup_{0<t\leq 1} t(1 + t^{-1} - 1)^{1/q} = \sup_{0<t\leq 1} t^{1-1/q}.$$

Now, for $-1/q \leq \lambda < 0$, and $1 \leq q < \infty$, from case 2(a) and 2(b), we obtain

$$\begin{aligned}
&\|Hf_0\|_{W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \\
&= \max\left\{ \sup_{0<t\leq 1} \sup_{1<\gamma\leq \log_p(1/t)^{1/n}} tp^{-n\gamma\lambda}, \sup_{0<t\leq 1} \sup_{1<\log_p(1/t)^{1/n}<\gamma} t^{1-1/q}p^{-n\gamma(\lambda+1/q)} \right\} \\
&= \max\left\{ \sup_{0<t\leq 1} t^{1+\lambda}, \sup_{0<t\leq 1} t^{1+\lambda} \right\} \\
&= 1 = \|f_0\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}.
\end{aligned}$$

Hence, using the $(\dot{B}^{q,\lambda}(\mathbb{Q}_p^n), W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n))$ boundedness of H in each case, we get

$$1 \leq \|H\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)\to W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}. \tag{6.4.2}$$

Finally, combining inequalities (6.4.1) and (6.4.2), we arrive at:

$$\|H\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)\to W\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} = 1.$$

Chapter 7
Boundedness of Weighted Multilinear p-adic Hardy Operator on Herz-Type Spaces

7.1 Introduction

In the last two Chapters, we mainly focussed on the multidimensional case of p-adic Hardy-type operators. In this Chapter, we consider another variant of Hardy operator, namely, the weighted Hardy operator. Like Hardy operator, the weighted Hardy operator has a long bibliographic history of boundedness results on function spaces. Since, we are concerned with p-adic analog of multilinear weighted Hardy operator, therefore, in the remaining of this thesis, we confine ourselves within this context.

Multilinear operators are studied in analysis because of their natural appearance in numerous physical phenomenons and their purpose is not merely to generalize the theory of linear operators. A reader can see the papers [25, 31, 42] and the references therein for better understanding of multilinear operators. The purpose of this Chapter is to study the boundedness of $H_{\psi}^{p,m}$ on the product of p-adic Herz spaces and p-adic Morrey-Herz spaces. The corresponding operator norms are also acquired.

7.2 Boundedness of $H_{\psi}^{p,m}$ on the Product of Herz Spaces

In the current section we throw light on the boundedness of $H_{\psi}^{p,m}$ on the product of p-adic Herz spaces. Furthermore, norm of the very operator is attained as well. We state and prove our first result.

Theorem 7.2.1 *Suppose $\alpha, \alpha_1, \alpha_2, \cdots, \alpha_m$ is any arbitrary real numbers, $1 < p, p_1, \cdots, p_m, q, q_1, \cdots, q_m < \infty$ and let also $\alpha_1 + \alpha_2 + \cdots + \alpha_m = \alpha$, $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = \frac{1}{p}$, $\frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m} = \frac{1}{q}$, then $H_{\psi}^{p,m}$ is bounded from $\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{Q}_p^n) \times \cdots \times \dot{K}_{q_m}^{\alpha_m,p_m}(\mathbb{Q}_p^n)$ to $\dot{K}_{q}^{\alpha,p}(\mathbb{Q}_p^n)$ if*

$$\int_{\mathbb{Z}_p^*} \cdots \int_{\mathbb{Z}_p^*} |\mathbf{t}_1|_p^{-(\alpha_1+n/q_1)} \cdots |\mathbf{t}_m|_p^{-(\alpha_m+n/q_m)} \psi(\mathbf{t}_1, \cdots, \mathbf{t}_m) d\mathbf{t}_1 \cdots d\mathbf{t}_m < \infty. \tag{7.2.1}$$

Conversely, if $q_1, q_2, \cdots, q_m = mq$, $p_1, p_2, \cdots, p_m = mp$ and $H_{\psi}^{p,m}$ is bounded from $\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{Q}_p^n) \times \cdots \times \dot{K}_{q_m}^{\alpha_m,p_m}(\mathbb{Q}_p^n)$ to $\dot{K}_{q}^{\alpha,p}(\mathbb{Q}_p^n)$ then (7.2.1) holds. Furthermore,

$$\|H_{\psi}^{p,m}\|_{\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{Q}_p^n)\times\cdots\times\dot{K}_{q_m}^{\alpha_m,p_m}(\mathbb{Q}_p^n)\to\dot{K}_{q}^{\alpha,p}(\mathbb{Q}_p^n)} = \int_{\mathbb{Z}_p^*} \cdots \int_{\mathbb{Z}_p^*} |\mathbf{t}_1|_p^{-(\alpha_1+n/q_1)} \ldots |\mathbf{t}_m|_p^{-(\alpha_m+n/q_m)} \times \psi(\mathbf{t}_1, \cdots, \mathbf{t}_m) d\mathbf{t}_1 \cdots d\mathbf{t}_m. \tag{7.2.2}$$

Proof. We will prove the Theorem only for $m = 2$ which will work for every $m \in \mathbb{N}$. As $1/q = 1/q_1 + 1/q_2$ use of Minkowski's inequality and Hölder's inequality give:

$$\begin{aligned}
\|H_{\psi}^{p,2}(f_1, f_2)\chi_k\|_{L^q(\mathbb{Q}_p^n)} &= \left(\int_{S_k} \left| \int\int_{\mathbb{Z}_p^*} f_1(\mathbf{t}_1\mathbf{x}) f_2(\mathbf{t}_2\mathbf{x}) \psi(\mathbf{t}_1, \mathbf{t}_2) d\mathbf{t}_1 d\mathbf{t}_2 \right|^q \right)^{1/q} \\
&\leq \int\int_{\mathbb{Z}_p^*} \left(\int_{S_k} \left| f_1(\mathbf{t}_1\mathbf{x}) f_2(\mathbf{t}_2\mathbf{x}) \right|^q \right)^{1/q} \psi(\mathbf{t}_1, \mathbf{t}_2) d\mathbf{t}_1 d\mathbf{t}_2 \\
&\leq \int\int_{\mathbb{Z}_p^*} \left(\int_{S_k} |f_1(\mathbf{t}_1\mathbf{x})|^{q_1} d\mathbf{x} \right)^{1/q_1} \left(\int_{S_k} |f_2(\mathbf{t}_2\mathbf{x})|^{q_2} d\mathbf{x} \right)^{1/q_2} \\
&\quad \times \psi(\mathbf{t}_1, \mathbf{t}_2) d\mathbf{t}_1 d\mathbf{t}_2 \\
&= \int\int_{\mathbb{Z}_p^*} \left(\int_{\mathbf{t}_1 S_k} |f_1(\mathbf{x})|^{q_1} d\mathbf{x} \right)^{1/q_1} \left(\int_{\mathbf{t}_2 S_k} |f_2(\mathbf{x})|^{q_2} d\mathbf{x} \right)^{1/q_2} \\
&\quad \times |\mathbf{t}_1|_p^{-n/q_1} |\mathbf{t}_2|_p^{-n/q_2} \psi(\mathbf{t}_1, \mathbf{t}_2) d\mathbf{t}_1 d\mathbf{t}_2.
\end{aligned}$$

Now for each $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{Z}_p^*$, nonnegative integers m, l exists such that $|\mathbf{t}_1|_p = p^{-m}$ and $|\mathbf{t}_2|_p = p^{-l}$. Therefore, we easily have:

$$\|H_{\psi}^{p,2}(f_1,f_2)\chi_k\|_{L^q(\mathbb{Q}_p^n)} \leq \int\int_{\mathbb{Z}_p^*}\Big(\|f_1\chi_{k-m}\|_{L^{q_1}(\mathbb{Q}_p^n)}.\|f_2\chi_{k-l}\|_{L^{q_2}(\mathbb{Q}_p^n)}\Big) \\ \times |\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-n/q_2}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2.$$

Hence, by means of Minkowski's inequality and Hölder's inequality together with $1/p_1 + 1/p_2 = 1/p$ and $\alpha = \alpha_1 + \alpha_2$, we get:

$$\begin{aligned}
&\|H_{\psi}^{p,2}(f_1,f_2)\|_{\dot{K}_q^{\alpha,p}(\mathbb{Q}_p^n)}\\
=&\Big(\sum_{k=-\infty}^{\infty} p^{k\alpha p}\|H_{\psi}^{p,2}(f_1,f_2)\chi_k\|^p_{L^q(\mathbb{Q}_p^n)}\Big)^{1/p}\\
\leq&\Big(\sum_{k=-\infty}^{\infty} p^{k\alpha p}\Big(\int\int_{\mathbb{Z}_p^*}\Big(\|f_1\chi_{k-m}\|_{L^{q_1}(\mathbb{Q}_p^n)}.\|f_2\chi_{k-l}\|_{L^{q_2}(\mathbb{Q}_p^n)}\Big)\\
&\times |\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-n/q_2}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2\Big)^p\Big)^{1/p}\\
\leq&\int\int_{\mathbb{Z}_p^*}\Big(\sum_{k=-\infty}^{\infty} p^{k\alpha p}\Big(\|f_1\chi_{k-m}\|_{L^{q_1}(\mathbb{Q}_p^n)}.\|f_2\chi_{k-l}\|_{L^{q_2}(\mathbb{Q}_p^n)}\Big)^p\Big)^{1/p}\\
&\times |\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-n/q_2}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2\\
\leq&\int\int_{\mathbb{Z}_p^*}\Big(\sum_{k=-\infty}^{\infty} p^{k\alpha_1 p_1}\|f_1\chi_{k-m}\|^{p_1}_{L^{q_1}(\mathbb{Q}_p^n)}\Big)^{1/p_1}\\
&\times \Big(\sum_{k=-\infty}^{\infty} p^{k\alpha_2 p_2}\|f_2\chi_{k-l}\|^{p_2}_{L^{q_2}(\mathbb{Q}_p^n)}\Big)^{1/p_2}|\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-n/q_2}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2\\
\leq&\int\int_{\mathbb{Z}_p^*}\Big(\sum_{k=-\infty}^{\infty} p^{k\alpha_1 p_1}\|f_1\chi_{k}\|^{p_1}_{L^{q_1}(\mathbb{Q}_p^n)}\Big)^{1/p_1}\\
&\times \Big(\sum_{k=-\infty}^{\infty} p^{k\alpha_2 p_2}\|f_2\chi_{k}\|^{p_2}_{L^{q_2}(\mathbb{Q}_p^n)}\Big)^{1/p_2}|\mathbf{t}_1|_p^{-(\alpha_1+n/q_1)}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2)}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2\\
\leq&\|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{Q}_p^n)}\|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{Q}_p^n)}\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-(\alpha_1+n/q_1)}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2)}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\|H_{\psi}^{p,2}(f_1,f_2)\chi_k\|_{\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{Q}_p^n)\times\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{Q}_p^n)\to\dot{K}_q^{\alpha,p}(\mathbb{Q}_p^n)}\\
&\leq\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-(\alpha_1+n/q_1)}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2)}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2. \qquad (7.2.3)
\end{aligned}$$

Conversely, let $H_{\psi}^{p,2}$ be bounded from $\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{Q}_p^n) \times \dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{Q}_p^n)$ to $\dot{K}_{q}^{\alpha,p}(\mathbb{Q}_p^n)$. For $0<\epsilon<1$, we let

$$f_1(\mathbf{x}) = \begin{cases} 0 & \text{if } |\mathbf{x}|_p < 1, \\ |\mathbf{x}|_p^{-\alpha_1-(n/q_1)-\epsilon} & \text{if } |\mathbf{x}|_p \geq 1, \end{cases}$$

$$f_2(\mathbf{x}) = \begin{cases} 0 & \text{if } |\mathbf{x}|_p < 1, \\ |\mathbf{x}|_p^{-\alpha_2-(n/q_2)-\epsilon} & \text{if } |\mathbf{x}|_p \geq 1. \end{cases}$$

It is quite evident that $f_1\chi_k = f_2\chi_k = 0$ for $k<0$. Our interest lies only for $k \geq 0$. So, we proceed as follows.

$$\|f_1\chi_k\|_{L^{q_1}(\mathbb{Q}_p^n)} = \left(\int_{S_k} |\mathbf{x}|_p^{-(\alpha_1+(n/q_1)+\epsilon)q_1}\right)^{1/q_1} = (1-p^{-n})^{1/q_1} p^{-k(\alpha_1+\epsilon)}.$$

In a similar fashion, we have:

$$\|f_2\chi_k\|_{L^{q_2}(\mathbb{Q}_p^n)} = (1-p^{-n})^{1/q_2} p^{-k(\alpha_2+\epsilon)}.$$

Hence,

$$\begin{aligned} \|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{Q}_p^n)} =& \left(\sum_{k=-\infty}^{\infty} p^{k\alpha_1 p_1} \|f_1\chi_k\|_{L^{q_1}(\mathbb{Q}_p^n)}^{p_1}\right)^{1/p_1} \\ =&(1-p^{-n})^{1/q_1}\left(\sum_{k=0}^{\infty} p^{k\alpha_1 p_1} p^{-k(\alpha_1+\epsilon)p_1}\right)^{1/p_1} \\ =&(1-p^{-n})^{1/q_1}\left(\sum_{k=0}^{\infty} p^{-kp_1\epsilon}\right)^{1/p_1} \\ =&(1-p^{-n})^{1/q_1}\frac{p^{\epsilon}}{p^{p_1\epsilon}-1}. \end{aligned}$$

Similarly,

$$\|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{Q}_p^n)} =(1-p^{-n})^{1/q_2}\frac{p^{\epsilon}}{p^{p_2\epsilon}-1}.$$

It is obvious to see that when $|\mathbf{x}|_p < 1$ then $H_{\psi}^{p,2}(f_1,f_2) = 0$. We will evaluate the case as:

$$\begin{aligned} H_{\psi}^{p,2}(f_1,f_2) =& \int\int_{|\mathbf{x}|_p^{-1}\leq|\mathbf{t}|_p\leq 1} f_1(\mathbf{t}_1\mathbf{x})(\mathbf{t}_2\mathbf{x})\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2 \\ =&|\mathbf{x}|_p^{-(\alpha_1+n/q+2\epsilon)}\int\int_{|\mathbf{x}|_p^{-1}\leq|\mathbf{t}|_p\leq 1} |\mathbf{t}_1|_p^{-\alpha_1-n/q_1-\epsilon}|\mathbf{t}_2|_p^{-\alpha_1-n/q_2-\epsilon}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2. \end{aligned}$$

Now, for $k \le 0$, we have $H_\psi^{p,2}(f_1, f_2)\chi_k = 0$. By the definition of Herz space, we have:

$$\|H_\psi^{p,2}(f_1,f_2)\chi_k\|^p_{\dot{K}_q^{\alpha,p}(\mathbb{Q}_p^n)} = \sum_{k=0}^{\infty} p^{k\alpha p}\|H_\psi^{p,2}(f_1,f_2)\chi_k\|^p_{L^q(\mathbb{Q}_p^n)}$$

$$\begin{aligned}
=&\sum_{k=0}^{\infty} p^{k\alpha p}\bigg\{\int_{S_k}(|\mathbf{x}|_p^{-(\alpha+n/q+2\epsilon)}\\
&\times\int\int_{p^{-k}\le|\mathbf{t}|_p\le 1}|\mathbf{t}_1|_p^{-\alpha_1-n/q_1-\epsilon}|\mathbf{t}_2|_p^{-\alpha_2-n/q_2-\epsilon}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2)^q d\mathbf{x}\bigg\}^{p/q}\\
=&(1-p^{-n})^{p/q}\sum_{k=0}^{\infty} p^{k\alpha p}(p^{-k(\alpha+2\epsilon)p})\\
&\times\left(\int\int_{p^{-k}\le|\mathbf{t}|_p\le 1}|\mathbf{t}_1|_p^{-\alpha_1-n/q_1-\epsilon}|\mathbf{t}_2|_p^{-\alpha_2-n/q_2-\epsilon}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2\right)^p.
\end{aligned}$$

Next, for any $l \le k$, we get

$$\begin{aligned}
&\|H_\psi^{p,2}(f_1,f_2)\chi_k\|_{\dot{K}_q^{\alpha,p}(\mathbb{Q}_p^n)}\\
\ge&(1-p^{-n})^{1/q}\left(\sum_{k=l}^{\infty} p^{-2k\epsilon p}\right)^{1/p}\\
&\times\left(\int\int_{p^{-l}\le|\mathbf{t}|_p\le 1}|\mathbf{t}_1|_p^{-\alpha_1-n/q_1-\epsilon}|\mathbf{t}_2|_p^{-\alpha_2-n/q_2-\epsilon}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2\right)\\
=&(1-p^{-n})^{1/q}\left(\sum_{k=0}^{\infty} p^{-2k\epsilon p}\right)^{1/p}\\
&\times\left(p^{-2l\epsilon}\int\int_{p^{-l}\le|\mathbf{t}|_p\le 1}|\mathbf{t}_1|_p^{-\alpha_1-n/q_1-\epsilon}|\mathbf{t}_2|_p^{-\alpha_2-n/q_2-\epsilon}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2\right)\\
=&(1-p^{-n})^{1/q}\frac{p^{2\epsilon}}{(p^{2\epsilon p}-1)^{1/p}}\\
&\times\left(p^{-2l\epsilon}\int\int_{p^{-l}\le|\mathbf{t}|_p\le 1}|\mathbf{t}_1|_p^{-\alpha_1-n/q_1-\epsilon}|\mathbf{t}_2|_p^{-\alpha_2-n/q_2-\epsilon}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2\right).
\end{aligned}$$

Since $q_1 = q_2 = 2q$, $1/p = 1/p_1 + 1/p_2$ and $p_1 = p_2 = 2p$, we have:

$$\begin{aligned}
&\|H_\psi^{p,2}(f_1,f_2)\chi_k\|_{\dot{K}_q^{\alpha,p}(\mathbb{Q}_p^n)}\\
\ge&\|f_1\|_{\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{Q}_p^n)}\|f_2\|_{\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{Q}_p^n)}\\
&\times\left(p^{-2l\epsilon}\int\int_{p^{-l}\le|\mathbf{t}|_p\le 1}|\mathbf{t}_1|_p^{-\alpha_1-n/q_1-\epsilon}|\mathbf{t}_2|_p^{-\alpha_2-n/q_2-\epsilon}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2\right).
\end{aligned}$$

We take $\epsilon = p^{-l}$, then letting $l \to \infty$, we have $\epsilon \to 0$. Ultimately, we get

$$\|H_{\psi}^{p,2}(f_1, f_2)\chi_k\|_{\dot{K}_{q_1}^{\alpha_1,p_1}(\mathbb{Q}_p^n)\times\dot{K}_{q_2}^{\alpha_2,p_2}(\mathbb{Q}_p^n)\to\dot{K}_{q}^{\alpha,p}(\mathbb{Q}_p^n)}$$
$$\geq \int\int_{\mathbb{Z}_p^*} |\mathbf{t}_1|_p^{-(\alpha_1+n/q_1)}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2)}\psi(\mathbf{t}_1, \mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2. \tag{7.2.4}$$

From (7.2.3) and (7.2.4), we get the required proof.

7.3 Boundedness of $H_{\psi}^{p,m}$ on the Product of Morrey-Herz Spaces

In this Section, the operator $H_{\psi}^{p,m}$ is proved to be bounded on the product of Morrey-Herz spaces. Similar to previous Section, norm of the operator is computed in this case also. The outset of this Section is the main result which is stated and proved as below.

Theorem 7.3.1 *Suppose $\alpha, \alpha_1, \alpha_2, \cdots, \alpha_m$ is any arbitrary real numbers, $1 < p, p_1, \cdots, p_m, q, q_1, \cdots, q_m < \infty$ and let also $\alpha_1 + \alpha_2 + \cdots + \alpha_m = \alpha$, $\lambda, \lambda_1, \cdots, \lambda_m > 0$, $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = \frac{1}{p}$, $\frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_m} = \frac{1}{q}$, $\lambda_1 + \cdots + \lambda_m = \lambda$, then $H_{\psi}^{p,m}$ is bounded from $M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n) \times \cdots \times M\dot{K}_{p_m,q_m}^{\alpha_m,\lambda_m}(\mathbb{Q}_p^n)$ to $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)$ if*

$$\int_{\mathbb{Z}_p^*}\cdots\int_{\mathbb{Z}_p^*} |\mathbf{t}_1|_p^{-(\alpha_1+n/q_1-\lambda_1)}\cdots|\mathbf{t}_m|_p^{-(\alpha_m+n/q_m-\lambda_m)}\psi(\mathbf{t}_1, \cdots, \mathbf{t}_m)d\mathbf{t}_1\cdots d\mathbf{t}_m < \infty. \tag{7.3.1}$$

Conversely, if $q_1, q_2, ...q_m = mq$, $p_1, p_2, \cdots, p_m = mp$, $\alpha_1 = \cdots = \alpha_m = (1/m)\alpha$, $\lambda_1 = \cdots = \lambda_m = (1/m)\lambda$ and $H_{\psi}^{p,m}$ is bounded from $M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n)\times\cdots\times M\dot{K}_{p_m,q_m}^{\alpha_m,\lambda_m}(\mathbb{Q}_p^n)$ to $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)$ then (7.3.1) holds. Furthermore,

$$\|H_{\psi}^{p,m}\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n)\times\cdots\times M\dot{K}_{p_m,q_m}^{\alpha_m,\lambda_m}(\mathbb{Q}_p^n)\to M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)}$$
$$= \int_{\mathbb{Z}_p^*}\cdots\int_{\mathbb{Z}_p^*} |\mathbf{t}_1|_p^{-(\alpha_1+n/q_1-\lambda_1)}\cdots|\mathbf{t}_m|_p^{-(\alpha_m+n/q_m-\lambda_m)}\psi(\mathbf{t}_1, \cdots, \mathbf{t}_m)d\mathbf{t}_1\cdots d\mathbf{t}_m. \tag{7.3.2}$$

Proof. From the previous theorem, we have:

$$\|H_{\psi}^{p,2}(f_1, f_2)\chi_k\|_{L^q(\mathbb{Q}_p^n)} \leq \int\int_{\mathbb{Z}_p^*}\Big(\|f_1\chi_{k-m}\|_{L^{q_1}(\mathbb{Q}_p^n)}.\|f_2\chi_{k-l}\|_{L^{q_2}(\mathbb{Q}_p^n)}\Big)$$
$$\times |\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-n/q_2}\psi(\mathbf{t}_1, \mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2.$$

For $1/p = 1/p_1 + 1/p_2$, $\alpha = \alpha_1 + \alpha_2$ and $\lambda = \lambda_1 + \lambda_2$. Applying Hölder's inequality together with Minkowki's inequality, we are down to:

$$\|H_{\psi}^{p,2}(f_1, f_2)\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)} = \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} p^{k\alpha p}\|H_{\psi}^{p,2}(f_1, f_2)\chi_k\|_{L^q(\mathbb{Q}_p^n)}^p\Big)^{1/p}$$

$$
\begin{aligned}
&\leq \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0} p^{k\alpha p}\Bigg(\int\int_{\mathbb{Z}_p^*}\|f_1\chi_{k-m}\|_{L^{q_1}(\mathbb{Q}_p^n)}\cdot\|f_2\chi_{k-l}\|_{L^{q_2}(\mathbb{Q}_p^n)}\\
&\quad\times|\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-n/q_2}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1d\mathbf{t}_2\Bigg)^p\Bigg)^{1/p}\\
&\leq \int\int_{\mathbb{Z}_p^*}\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0} p^{k\alpha p}\Big(\|f_1\chi_{k-m}\|_{L^{q_1}(\mathbb{Q}_p^n)}\cdot\|f_2\chi_{k-l}\|_{L^{q_2}(\mathbb{Q}_p^n)}\Big)^p\Bigg)^{1/p}\\
&\quad\times|\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-n/q_2}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1d\mathbf{t}_2\\
&\leq \int\int_{\mathbb{Z}_p^*}\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0} p^{k\alpha_1p_1}\|f_1\chi_{k-m}\|_{L^{q_1}(\mathbb{Q}_p^n)}^{p_1}\Bigg)^{1/p_1}\\
&\quad\times\Bigg(\sum_{k=-\infty}^{k_0} p^{k\alpha_2p_2}\|f_2\chi_{k-l}\|_{L^{q_2}(\mathbb{Q}_p^n)}^{p_2}\Bigg)^{1/p_2}|\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-n/q_2}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1d\mathbf{t}_2\\
&\leq \int\int_{\mathbb{Z}_p^*}\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda_1}\Bigg(\sum_{k=-\infty}^{k_0} p^{k\alpha_1p_1}\|f_1\chi_{k-m}\|_{L^{q_1}(\mathbb{Q}_p^n)}^{p_1}\Bigg)^{1/p_1}\\
&\quad\times\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda_2}\Bigg(\sum_{k=-\infty}^{k_0} p^{k\alpha_2p_2}\|f_2\chi_{k-l}\|_{L^{q_2}(\mathbb{Q}_p^n)}^{p_2}\Bigg)^{1/p_2}\\
&\quad\times|\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-n/q_2}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1d\mathbf{t}_2\\
&\leq \int\int_{\mathbb{Z}_p^*}\sup_{k_0\in\mathbb{Z}} p^{-(k_0-m)\lambda_1}\Bigg(\sum_{k=-\infty}^{k_0} p^{k\alpha_1p_1}\|f_1\chi_{k}\|_{L^{q_1}(\mathbb{Q}_p^n)}^{p_1}\Bigg)^{1/p_1}\\
&\quad\times\sup_{k_0\in\mathbb{Z}} p^{-(k_0-l)\lambda_2}\Bigg(\sum_{k=-\infty}^{k_0} p^{k\alpha_2p_2}\|f_2\chi_{k}\|_{L^{q_2}(\mathbb{Q}_p^n)}^{p_2}\Bigg)^{1/p_2}\\
&\quad\times|\mathbf{t}_1|_p^{-(\alpha_1+n/q_1-\lambda_1)}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2-\lambda_2)}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1d\mathbf{t}_2\\
&\leq \int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-(\alpha_1+n/q_1-\lambda_1)}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2-\lambda_2)}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1d\mathbf{t}_2\\
&\quad\times\|f_1\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{M\dot{K}_{p_2,q_2}^{\alpha_2,\lambda_2}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

Conversely, we define

$$f_1(\mathbf{x}) = |\mathbf{x}|_p^{-(\alpha_1+n/q_1-\lambda_1)}, \quad \mathbf{x}\in\mathbb{Q}_p^n,$$

$$f_2(\mathbf{x}) = |\mathbf{x}|_p^{-(\alpha_2+n/q_2-\lambda_2)}, \quad \mathbf{x}\in\mathbb{Q}_p^n.$$

When $\alpha_1 \neq \lambda_1$ and $\alpha_2 \neq \lambda_2$, we arrive at:

$$
\begin{aligned}
\|f_1\chi_k\|_{L^{q_1}(\mathbb{Q}_p^n)} =&\Bigg(\int_{S_k}|\mathbf{x}|_p^{-(\alpha_1+n/q_1-\lambda_1)q_1}\Bigg)^{1/q_1}\\
=&(1-p^{-n})^{1/q_1}p^{k(\lambda_1-\alpha_1)}.
\end{aligned}
$$

Also, it is not difficult to obtain

$$\|f_1\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\mathbb{Q}_p^n)} = (1-p^{-n})^{1/q_1}\frac{p^{\lambda_1}}{(p^{p_1\lambda_1}-1)^{1/p_1}}.$$

Similarly, we can get

$$\|f_2\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\mathbb{Q}_p^n)} = (1-p^{-n})^{1/q_2}\frac{p^{\lambda_2}}{(p^{p_2\lambda_2}-1)^{1/p_2}}.$$

For $\lambda = \lambda_1 + \lambda_2$, $\alpha = \alpha_1 + \alpha_2$ and $1/q = 1/q_1 + 1/q_2$, we get

$$H^{p,2}_{\psi}(f_1, f_2)(\mathbf{x}) = |\mathbf{x}|_p^{-(\alpha+n/q-\lambda)}\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-(\alpha_1+n/q_1-\lambda_1)}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2-\lambda_2)}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1d\mathbf{t}_2.$$

Ultimately, we have:

$$\begin{aligned}
&\|H^{p,2}_{\psi}(f_1, f_2)\chi_k\|^p_{L^q(\mathbb{Q}_p^n)}\\
=&\bigg(\int_{S_k}|\mathbf{x}|_p^{-(\alpha+n/q-\lambda)q}\bigg(\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-(\alpha_1+n/q_1-\lambda_1)}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2-\lambda_2)}\\
&\times\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1d\mathbf{t}_2\bigg)^q d\mathbf{x}\bigg)^{p/q}\\
=&\bigg(\int_{S_k}|\mathbf{x}|_p^{-(\alpha+n/q-\lambda)q}d\mathbf{x}\bigg)^{p/q}\bigg(\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-(\alpha_1+n/q_1-\lambda_1)}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2-\lambda_2)}\\
&\times\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1d\mathbf{t}_2\bigg)^p\\
=&(1-p^{-n})^{p/q}p^{-k(\alpha-\lambda)p}\bigg(\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-(\alpha_1+n/q_1-\lambda_1)}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2-\lambda_2)}\\
&\times\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1d\mathbf{t}_2\bigg)^p.
\end{aligned}$$

Since $\lambda_1 = \lambda_2 = (1/2)\lambda$, $p_1 = p_2 = 2p$ and $q_1 = q_2 = 2q$, we have:

$$\begin{aligned}
\|H_\psi^{p,2}(f_1,f_2)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{Q}_p^n)} =& \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0} p^{k\alpha p}\|H_\psi^{p,2}(f_1,f_2)\chi_k\|^p_{L^q(\mathbb{Q}_p^n)}\Bigg)^{1/p}\\
=&(1-p^{-n})^{1/q}\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0} p^{k\alpha p}p^{-k(\alpha-\lambda)p}\\
&\times\Bigg(\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-(\alpha_1+n/q_1-\lambda_1)}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2-\lambda_2)}\\
&\times\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1d\mathbf{t}_2\Bigg)^p\Bigg)^{1/p}\\
=&(1-p^{-n})^{1/q}\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{\infty} p^{k\lambda p}\Bigg)^{1/p}\\
&\times\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-(\alpha_1+n/q_1-\lambda_1)}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2-\lambda_2)}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1d\mathbf{t}_2\\
=&(1-p^{-n})^{1/q}\frac{p^\lambda}{(p^{\lambda p}-1)^{1/p}}\\
&\times\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-(\alpha_1+n/q_1-\lambda_1)}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2-\lambda_2)}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1d\mathbf{t}_2\\
=&\|f_1\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\mathbb{Q}_p^n)}\|f_2\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\mathbb{Q}_p^n)}\\
&\times\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-(\alpha_1+n/q_1-\lambda_1)}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2-\lambda_2)}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1d\mathbf{t}_2.
\end{aligned}$$

When $\alpha_1 = \lambda_1$ and $\alpha_2 = \lambda_2$, we have

$$\|f_1\chi_k\|^{q_1}_{L^{q_1}(\mathbb{Q}_p^n)} = \|f_2\chi_k\|^{q_2}_{L^{q_2}(\mathbb{Q}_p^n)} = \int_{S_k}|\mathbf{x}|_p^{-n}d\mathbf{x} = (1-p^{-n})^{1/q_1}.$$

It is not hard to see that

$$\|f_1\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\mathbb{Q}_p^n)} = p^{\lambda_1}(1-p^{-n})^{1/q_1}\frac{1}{(p^{\lambda_1 p^1}-1)^{1/p_1}},$$

$$\|f_2\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\mathbb{Q}_p^n)} = p^{\lambda_2}(1-p^{-n})^{1/q_2}\frac{1}{(p^{\lambda_2 p^2}-1)^{1/p_2}}.$$

So,

$$H_\psi^{p,2}(f_1,f_2)(\mathbf{x}) = |\mathbf{x}|_p^{-n/q}\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-n/q_2}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1d\mathbf{t}_2.$$

Now, we have:

$$\|H_\psi^{p,2}(f_1,f_2)\chi_k\|_{L^{q_1}(\mathbb{Q}_p^n)} = (1-p^{-n})^{1/q}\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-n/q_2}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2.$$

Thus,

$$\begin{aligned}\|H_\psi^{p,2}(f_1,f_2)\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)} &= \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} p^{k\alpha p}\|H_\psi^{p,2}(f_1,f_2)\chi_k\|_{L^q(\mathbb{Q}_p^n)}^p\right)^{1/p}\\ &=(1-p^{-n})^{1/q}\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} p^{k\alpha p}\right)^{1/p}\\ &\quad\times\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-n/q_2}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2\end{aligned}$$

As $\lambda_1=\alpha_1$ and $\lambda_2=\alpha_2$, we get $\lambda=\alpha$. Hence

$$\begin{aligned}\|H_\psi^{p,2}(f_1,f_2)\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)} &=p^{\lambda}(1-p^{-n})^{1/q}\frac{1}{(p^{\lambda p}-1)^{1/p}}\\ &\quad\times\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-n/q_2}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2\\ &=\|f_1\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{M\dot{K}_{p_2,q_2}^{\alpha_2,\lambda_2}(\mathbb{Q}_p^n)}\\ &\quad\times\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-n/q_2}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2.\end{aligned}$$

This shows (7.3.2) is also valid in this particular case.
Furthermore, when either $\alpha_1=\lambda_1$ or $\alpha_2=\lambda_2$ holds, we assume former holds but the later doesn't, then on the basis of previous computations we have:

$$\|f_1\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n)} = p^{\lambda_1}(1-p^{-n})^{1/q_1}\frac{1}{(p^{\lambda_1 p_1}-1)^{1/p_1}},$$

$$\|f_2\|_{M\dot{K}_{p_2,q_2}^{\alpha_2,\lambda_2}(\mathbb{Q}_p^n)} = p^{\lambda_2}(1-p^{-n})^{1/q_2}\frac{1}{(p^{\lambda_2 p_2}-1)^{1/p_2}}.$$

We definitely have the following representation

$$H_\psi^{p,2}(f_1,f_2)(\mathbf{x}) = |\mathbf{x}|_p^{-(\alpha_2+n/q-\lambda_2)}\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2-\lambda_2)}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2.$$

Taking norm on both sides, one has

$$\begin{aligned}\|H_\psi^{p,2}(f_1,f_2)\chi_k\|_{L^q(\mathbb{Q}_p^n)} &=(1-p^{-n})^{1/q}p^{-k(\alpha_2-\lambda_2)}\\ &\quad\times\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2-\lambda_2)}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2.\end{aligned}$$

At the very end, we obtain

$$\begin{aligned}
\|H_{\psi}^{p,2}(f_1,f_2)\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)} =& \sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} p^{k\alpha p}\|H_{\psi}^{p,2}(f_1,f_2)\chi_k\|_{L^q(\mathbb{Q}_p^n)}^p\right)^{1/p} \\
=&(1-p^{-n})^{1/q}\sup_{k_0\in\mathbb{Z}} p^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} p^{k\alpha p}p^{-(\alpha_2-\lambda_2 kp)}\right)^{1/p} \\
&\times\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2-\lambda_2)}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2 \\
=&(1-p^{-n})^{1/q}p^{\lambda}\frac{1}{(p^{\lambda p}-1)^{1/p}} \\
&\times\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2-\lambda_2)}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2.
\end{aligned}$$

Since $\alpha_1=\alpha_2=(1/2)\alpha$, $p_1=p_2=2p$, $q_1=q_2=2q$ and $\lambda_1=\lambda_2=(1/2)\lambda$, we get:

$$\begin{aligned}
\|H_{\psi}^{p,2}(f_1,f_2)\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{Q}_p^n)} =&\|f_1\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\mathbb{Q}_p^n)}\|f_2\|_{M\dot{K}_{p_2,q_2}^{\alpha_2,\lambda_2}(\mathbb{Q}_p^n)} \\
&\times\int\int_{\mathbb{Z}_p^*}|\mathbf{t}_1|_p^{-n/q_1}|\mathbf{t}_2|_p^{-(\alpha_2+n/q_2-\lambda_2)}\psi(\mathbf{t}_1,\mathbf{t}_2)d\mathbf{t}_1 d\mathbf{t}_2.
\end{aligned}$$

In this case (7.3.2) also holds, so we conclude the proof.

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