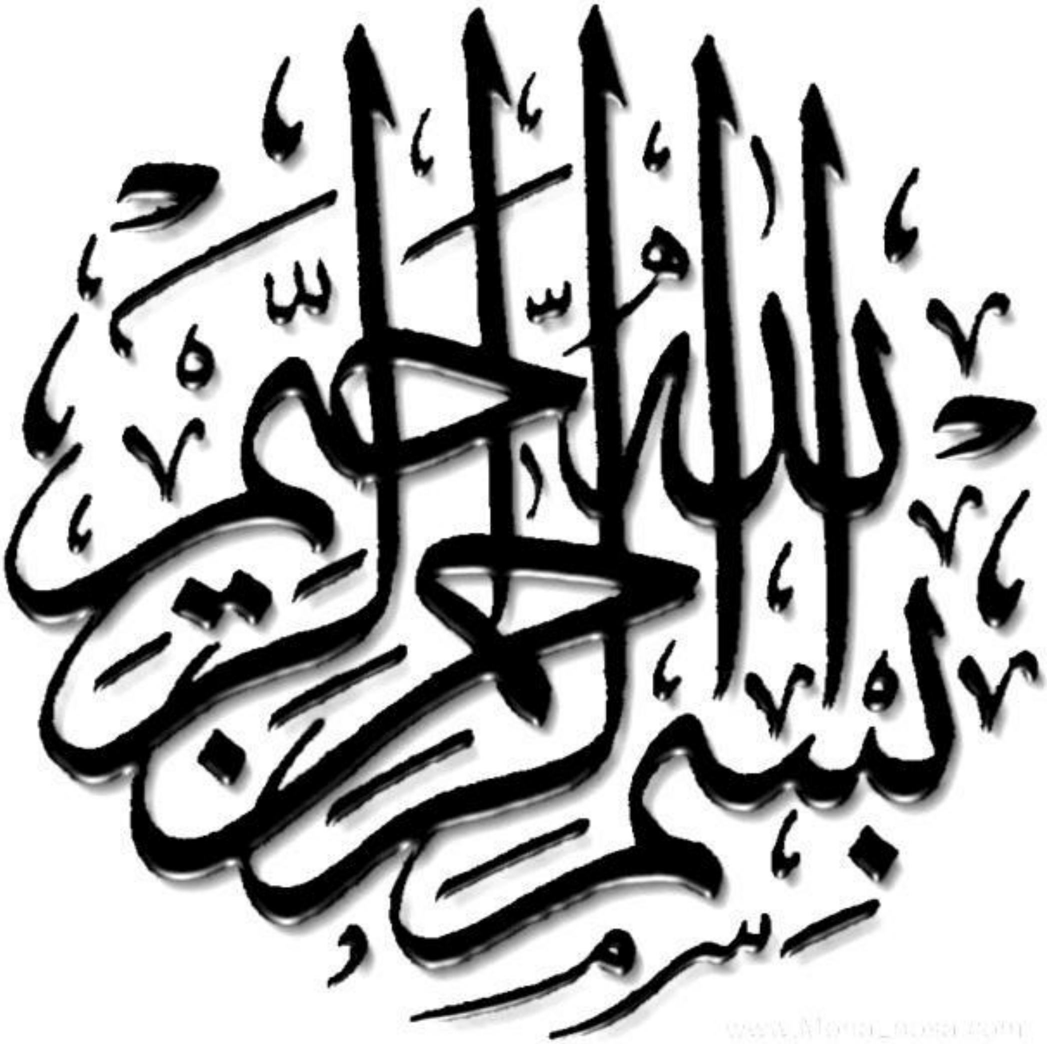

In the Name of Allah,
The Most Gracious, The Most Merciful.

Some Studies in Rough Set Theory Based on Soft Relations and Their Applications

Ph. D Thesis

By

Rani Sumaira Kanwal

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
2020

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A THESIS SUBMITTED IN THE PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

Supervised

By

Prof. Dr. Muhammad Shabir

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan

2020

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I, **Rani Sumaira Kanwal** hereby state that my PhD thesis titled "**Some Studies in Rough Set Theory Based on Soft Relations and Their Applications"** is my own work and has not been submitted previously by me for taking any degree from Quaid-i-Azam University Islamabad, Pakistan or anywhere else in the country/world.

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Some Studies in Rough Set Theory Based on Soft Relations and Their Applications

By

Rani Sumaira Kanwal

Certificate

A thesis Submitted in Partial Fulfillment of the requirement for the degree of doctor of philosophy in mathematics.

We accept this thesis as conforming to the required standard

1. ____________________
Prof. Dr. Muhammad Shabir
(Supervisor)

2. ____________________
Prof. Dr. Sohail Nadeem
(Chairman)

3. ____________________
Prof. Dr. Akbar Azam
Department of Mathematics,
COMSATS University, Park Road, Chak Shahzad ,
Islamabad.
(External Examiner I)

4. ____________________
Dr. Tahir Mahmood
Assistant Professor
Department of Mathematics and Statistics, International Islamic University, Islamabad.
(External Examiner II)

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
2020

Certificate of Approval

This is to certify that the research work presented in this thesis entitled **"Some Studies in Rough Set Theory Based on Soft Relations and Their Applications"** was conducted by Mrs. **Rani Sumaira Kanwal** under the supervision of **Prof. Dr. Muhammad Shabir**. No part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the Department of Mathematics, Quaid-i-Azam University, Islamabad in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Field of Mathematics from Department of Mathematics, Quaid-i-Azam University Islamabad, Pakistan.

Student Name: **Rani Sumaira Kanwal** Signature: ____________

External committee:

a) **External Examiner 1**: Signature: ____________
Name: **Dr. Akbar Azam**
Designation: Professor
Office Address: Department of Mathematics COMSATS University, Park Road, Chak Shahzad, Islamabad. Pakistan

b) **External Examiner 2**: Signature: ____________
Name: **Dr. Tahir Mahmood**
Designation: Assistant Professor
Office Address: Department of Mathematics and Statistics, International Islamic University, Islamabad. Pakistan

c) **Internal Examiner** : Signature: ____________
Name: **Dr. Muhammad Shabir**
Designation: Professor
Office Address: Department of Mathematics, Quaid-i-Azam University, Islamabad. Pakistan

Supervisor Name: Signature: ____________
Prof. Dr. Muhammad Shabir

Name of Dean/ HOD: Signature: ____________

Prof. Dr. Sohail Nadeem

Dedicated
to

The Holy Prophet Muhammad
(Peace be upon Him)

Contents

0.1 Acknowledgment

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May Almighty Allah shower His blessings and prosperity on all who assisted me in any way during the completion of my thesis.

Rani Sumaira Kanwal

0.2 Research Profile

The thesis is based on the following research papers.

1. Reduction of an information system, Soft Computing 24 (2020) 10801-10813.
2. Approximation of ideals in semigroups by soft relations, Journal of Intelligent and Fuzzy Systems, 35(3) (2018) 3895-3908.
3. Roughness of ideals in semigroups by soft relations, (submitted).
4. Approximation of a fuzzy set by soft relation and corresponding decision making, (submitted).
5. Rough approximation of a fuzzy set in semigroups based on soft relations, Computational and Applied Mathematics, (2019) 38:89.
6. Approximation of a soft set by soft binary relations and corresponding decision making, (submitted).
7. Approximation of soft ideals by soft relations in semigroups, Journal of Intelligent and Fuzzy Systems 37 (2019) 7977-7989.

0.3 Introduction

Traditional tools for reasoning, computing and formal modeling are crisp, precise and deterministic in character. However, many practical problems under different fields have uncertainties. Traditional mathematical tools can never be used due to many uncertainties existing in these problems. Traditionally, Probability theory is considered as a useful tool to handle uncertainty. A very basic requirement for its application is that a system must be stochastically stable. To establish it, a large number of trials is required. So, a lot of time will be spent on it. In today's fast pace life, humans have scarcity of time. This means some non-traditional ways must be sought to tackle this problem linked with uncertainty. For extraction of helpful information, specialists are dedicated step by step. They have pulled in light of a legitimate concern for specialists and experts in different fields of science and innovation. In this regard, Zadeh made a very nice attempt and introduced the theory of fuzzy sets [99]. Fuzzy sets not only cope with uncertainty but also have ability to translate human linguistic terms mathematically. Actually it was a great leap, and paced development many times. Now there are many generalizations of fuzzy sets and they are perceived and useful to manage uncertainty but every one of these speculations inherents certain challenges as Molodtsov have demonstrated [66].

Molodtsov [66] showed up the soft set theory out of the blue for the first time as the key notion as another numerical instrument meant for to handle uncertainty. This new hypothesis is free from the troubles related with officially existing techniques. Here an appropriate number of parameters are available, which makes it possible. Moreover, soft sets contain many operations which are very handy to deal with various types of situations. Numerous creators [8, 66, 67] gave a few activities in soft set theory and hypothesis. The utilization of soft set hypothesis in a decision making issue is talked about by Maji et al. [65]. Same creators also expanded classical soft sets to fuzzy soft sets [63]. Roy and Maji displayed a method of question affirmation from an uncertain multi-eyewitness information and connected it to decision based issues in light of fuzzy soft sets. Notion of parametric reduction in soft sets have been contemplated by many authors [21, 7, 64]. Numerous scientists have contemplated the utilizations of soft sets in alternate point of views [6, 19, 36].

In soft set theory, many authors [5, 8, 59, 66, 67] described several operations. There is a quick development for soft set theory with applications now-a-days [28].

One more theory, which tackles uncertainty in a non-traditional manner is called rough set theory, presented by Pawlak [69, 73, 77, 78]. It is another numerical way to deal with vague or dubious information. Pragmatic applications like information disclosure, information investigation, machine learning, surmised grouping, struggle examination is resolved in various zones. Subsequently, one of the fundamental headings of research in rough set theory is normally the speculation of the Pawlak rough set theory. In light of soft binary relations, rough set theory of Pawlak can be seen as a unique instance of soft rough sets. Since its inception, it fascinated researchers and scholars. The original rough set theory has potential to propose solutions of many problems even now. Important use of rough set theory is that it helps to reduce the data without losing useful information. Maji et al. [65] began the possibility of soft sets applications in decision making. In this initial level work, lamentably mistakes were called regard for as in [21] by Chen et al. The point of view displayed in [65] is dismissed by them. Konga et al. in [51] demonstrated some odd conditions which happens when method of parametric reduction in the event of soft sets gave in [21] is connected. So the idea of normal parametric diminishment is presented by them.

The rough set theory is basically an augmentation of the set theory described by inadequate and deficient data [69, 70, 74]. The idea of rough set is persuaded by useful needs especially in characterization and concept formation with deficient data [75]. It is not the same as and corresponding to different generalizations, such as multisets and fuzzy sets [26, 75]. In this new emerging theory, there has been a rich interest. The successful applications of rough set models have shown their benefits in many problems [15, 16, 17, 18, 20, 23, 32, 59, 70, 71, 72, 76, 77, 78, 79, 94, 95, 105, 106, 107].

The roughness in algebraic structures is discussed by many scholars. In algebra, Iwinski initiated the study of roughness [34]. Kuroki studied roughness in semigroups [52]. Rough groups, rough subgroups and rough ideals of rings are discussed by Biswas and Nanda, and Davvaz, respectively, [14], [22]. By using algebraic and fuzzy algebraic structures, T-roughness is discussed by Liu in [52], Qurashi and Shabir in [81, 82, 83], Mahmood et al. in [62], Akram et al. in [1] and Pomykala in [80]. Based on pseudoorder in ordered semigroups, Shabir and Irshad presented roughness [88]. The properties of rough sets is discussed by Ali et al in hemirings [9]. In [112], the generalized rough set is defined by Zhu based on binary relation. Moreover, recently, Akram et al. presented Neutrosophic soft rough graphs with applications in [3] and Akram et al. investigated some decision making methods in [2, 4].

Maji et al. [65] began the possibility of soft sets applications in decision making. In this initial level work, lamentably mistakes were called regard for as in [21] by Chen et al. The point of view displayed in [65] is dismissed by them. Konga et al. in [51] demonstrated some odd conditions which happen when method of parametric reduction in the event of soft sets gave in [21] is connected. Many researchers have proposed a roughness measure for fuzzy sets by mass assignment [10, 11]. Banerjee and Pal in [12] have presented a roughness measure of a fuzzy set. The idea of fuzziness is generally utilized in the theory of formal languages and automata. Numerous scientists utilized this idea for the generalization of algebra.

Initially, a fuzzy set was defined by Zadeh in [99]. Fuzzy set theory has given an important scientific and mathematical tool to the description of those frameworks which are unreasonably perplexing or uncertain. The fuzzy set theory is well established in [49].

In [56], fuzzy semigroup was defined by Kuroki. The concepts of fuzzy subgroups and fuzzy ordered semigroups were innovated by Rosenfeld and Kehayopulu and Tsingelis, respectively in [84] and [45, 46]. Different types of fuzzy ideals in semigroups are discussed by Kuroki [53, 54, 55].

Although rough and fuzzy set theory are two prominent notions to study uncertainty, unpredictability and vagueness yet these theories are distinct in nature. It can be combined in a good manner to solve many problems. Theory of fuzzy sets proposes an exceptionally decent way to deal with vagueness. In 1990, Dubois and Prade [26], presented the concepts of rough fuzzy sets and fuzzy rough sets.

Soft binary relation is a generalization of ordinary binary relations on a soft set. Single binary relation is addressed by rough approximations in rough set theory. Different binary relations can be treated in light of soft binary relations through rough approximations. Rough set theory of Pawlak, in the sense of soft binary relations can also be seen.

This thesis contains an investigation of rough approximations in the light of soft binary relations by obtaining two soft sets. We approximate a set, a fuzzy set and a soft set to get two sets of soft sets, two sets of fuzzy soft sets and two sets of soft sets, respectively. These are called the upper approximation and lower approximation with respect to the aftersets and foresets, respectively.

The above mentioned concepts will be applied on semigroups. The approximations of substructures , fuzzy substructures and soft substructures of semigroups are presented along with suitable examples.

The lower and the upper approximations with soft equivalence relations are discussed. The concept of soft equivalence relation is another way to present an information table. By soft reflexive, soft symmetric and soft transitive binary relations, some properties are viewed.

Fuzzy topologies and soft topologies induced by soft relations are presented. Moreover, Similarity relations associated with soft binary relations are given The idea of parametric reduction by a soft binary relation is inspected. Different decision making methods are also given along with algorithms and applications with respect to the aftersets and the foresets.

0.4 Chapter-wise Study

Seven chapters make this thesis.

Chapter one consists of introductory nature concepts, needed for the consequent chapters.

Chapter two represents the notion of soft relations. We direct an investigation through soft binary relations in rough approximations by obtaining two soft sets. With the help of the aftersets and the foresets, we can approximate a set to get two sets of soft sets. The related concepts of soft equivalence relations are also given. The concept of soft equivalence relation is another way to present an information table. This setup provides an opportunity to apply operations available in soft sets to extract new knowledge from information tables. By taking soft reflexive, soft symmetric and soft transitive binary relations, some properties are viewed. It is talked about for any subset X, there is a related fuzzy subset respecting to every parameter. Finally, the idea of parametric reduction by a soft binary relation is inspected. A procedure has been displayed in the end.

In chapter three, we applied the concepts of chapter two on semigroups and the approximations of substructures of semigroups are studied along with examples. The problems of homomorphism have been talked about.

In chapter four, by using the aftersets and the foresets, we approximate a fuzzy set to get two sets of fuzzy soft sets. The decision making by fuzzy set is provided ahead. The decision method algorithm is also given.

In the chapter five, we applied the concepts of chapter four on semigroups and approximations of fuzzy substructures of semigroups are discussed along with examples.

In the chapter six, for obtaining two sets of soft sets, a soft set is approximated by the aftersets and the foresets. Soft topologies induced by soft reflexive relations are presented. Moreover, Similarity relations associated with soft binary relations are given. A decision making method is given along with algorithms with respect to the aftersets and the foresets and an application is provided in the end.

In chapter seven, we applied the concepts of chapter six on semigroups and approximations of soft substructures of semigroups are presented along with examples.

Chapter 1

Preliminaries

This chapter contains some ideas concerning with an information system, semigroups, rough sets, fuzzy sets, soft sets and fuzzy soft sets which are valuable for consequent chapters.

In the first section, some fundamental definitions about an information system are recalled. The definition of a semigroup and its substructures are presented in the second section along with some basic examples. In the third section, the notion of rough set is presented along with some examples and basic results. Moreover, the classification of a rough set is also done in this section. Soft sets are introduced in section four. The containment, equality, union, intersection, complement and product of soft sets are given. Moreover, some results of soft sets in a semigroup are also presented to form a basis of other chapters. Some basic results about fuzzy set theory are introduced in section five. Some operations in fuzzy sets are also given here. Fuzzy substructures are presented in this section along with examples. In the last section, fuzzy soft sets and its substructures are presented. First, fuzzy soft sets operations are given and fuzzy soft substructures related to semigroups are presented.

Now, some basic and useful ideas are given. U represents a non-empty finite set unless expressed otherwise throughout this chapter.

1.1 Information systems: Definitions and examples

Definition 1.1.1 *A binary relation J from U to W is a subset of $U \times W$, where U and W are sets.*

J is a binary relation on U, if $U = W$.

Definition 1.1.2 *If J represents a binary relation on U, then J is said to be*

(1) *Reflexive, if $(z, z) \in J$ for all $z \in U$.*

(2) *Symmetric, if $(z, t) \in J \Rightarrow (t, z) \in J$ for all $z, t \in U$.*

(3) *Transitive, if $(z, l) \in J$ and $(l, t) \in J \Rightarrow (z, t) \in J$ for all $z, l, t \in U$.*

Definition 1.1.3 *A binary relation J is an equivalence relation if it is*

(1) *Reflexive*

(2) *Symmetric*

(3) *Transitive.*

A set is partitioned by each equivalence relation into disjoint classes.

Definition 1.1.4 *A pair (U, A), where A is a non-empty finite set of attributes and U is a non-empty finite set of objects, is an information system.*

Definition 1.1.5 *[73] An associated equivalence relation for each subset of attributes $B \subseteq A$, can be defined by $IN_d(B) = \{(m, n) \in U \times U : \text{for every } \beta \in B, \beta(m) = \beta(n)\}$ and $IN_d(B) = \cap_{\beta \in B} IN_d(\beta)$, where A is set of attributes and U is a universal set.*

Definition 1.1.6 *[73] Let $\mathbf{J}$ be a family of equivalence relations and $\rho \in \mathbf{J}$. If $IN_d(\mathbf{J}) = IN_d(\mathbf{J} - \rho)$, then ρ is dispensable, otherwise indispensable in $\mathbf{J}$. If each $\rho \in \mathbf{J}$ is indispensable in $\mathbf{J}$, then the family $\mathbf{J}$ is independent, otherwise dependent. If $\mathbf{J}$ is independent and $R \subseteq \mathbf{J}$, then R is independent. If S is independent and $S \subset \mathbf{J}$, then S is called a reduct of $\mathbf{J}$.*

Definition 1.1.7 *[73] The $COR(\mathbf{J})$ is defined as $COR(\mathbf{J}) = \cap RDC(\mathbf{J})$, where $RDC(\mathbf{J})$ is the family of all reducts of $\mathbf{J}$.*

Definition 1.1.8 *[73] A pair $K_R = (U, A)$ is a knowledge representation system, where $\phi \neq U$ is a finite set and $\phi \neq A$ is a finite set of primitive parameters. Every primitive parameter $\rho \in A$ can be defined by a total function $\rho : U \to V$, where V is the set of values of ρ.*

If $U = \{m_1, m_2, ..., m_n\}$ is a universe, $A = \{\rho_1, \rho_2, ..., \rho_m\}$ a set of attributes, $V = \cup_{i=1}^{m} V_i$, then a triplet (U, A, V) is called an information system or an information table where V is values of the attribute ρ_i. Moreover ρ_i is a total function $\rho_i : U \to V$. Let $K_R = (U, A)$ be a knowledge representation system or a KR system and let $C, D \subset A$ be condition and decision parameters. A decision table is a KR system with recognized condition and decision parameters, denoted by $T = (U, A, C, D)$.

1.2 Semigroups: Definitions and examples

Here, some detail about semigroups is presented.

Definition 1.2.1 *A semigroup is a set $S \neq \phi$ having a binary operation $''\cdot''$ which is associative.*

Definition 1.2.2 *The product XY for two subsets X and Y of S in a semigroup S can be defined as*

$$XY = \{xy : x \text{ belongs to } X, \ y \text{ belongs to } Y\}.$$

Definition 1.2.3 *The Cartesian product $S \times T$ of semigroups S and T is a semigroup if we define $(s, t)(s', t') = (ss', tt')$ for all $t, t' \in T$ and $s, s' \in S$.*

Definition 1.2.4 *If $xy \in X$ for all $x, y \in X$, then $\phi \neq X$ of a semigroup S is said to be a subsemigroup of S.*

Definition 1.2.5 *If $SX \subseteq X$ $(XS \subseteq X)$, then $\phi \neq X \subseteq S$ (where S is a semigroup), is a left (right) ideal of a semigroup S.*

Example 1.2.6 *Let $S = [0, 1]$, $(S, \star)$ is a semigroup if the binary operation on S is defined as:*

$a \star b = max\ \{a, b\}$ for all $a, b \in S$.

Consider the subset $A = [0, 0.5]$ of S, then A is a subsemigroup of S. And, consider the subset $A = [0.5, 1]$ of S, this is an ideal of S.

Definition 1.2.7 *If $SXS \subseteq X$, then $\phi \neq X \subseteq S$ is called an interior ideal of S for a semigroup S.*

Every ideal is an interior ideal but the converse is not true which is shown in the following example.

Example 1.2.8 *Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table*

$\bullet$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

Here $A = \{a, c\}$ is an interior ideal of S, but neither left nor right ideal of S.

Definition 1.2.9 *If $XSX \subseteq X$, then a subsemigroup X of a semigroup S is called a bi-ideal of S.*

Every one-sided ideal is a bi-ideal but the converse is not true which is shown in the following example.

Example 1.2.10 *Consider the semigroup of Example 1.2.8. Here $A = \{a, c\}$ is a bi-ideal of S but neither left nor right ideal of S.*

Throughout this thesis, we shall denote a subsemigroup, left ideal, right ideal, ideal, bi-ideal and interior ideal by SS, LIL, RIL, IL BIL and IIL, respectively.

1.3 Rough sets: Definitions and examples

Pawlak at first proposed the theory of rough sets. It was utilized to deal with imprecision and deficiency in data frameworks.

In this section, we will give a few ideas identified with rough set theory. Moreover, some examples are added to demonstrate these concepts.

[67] If $U \neq \phi$ is a finite set and J is an equivalence relation on U, then (U, J) is called an approximation space. If $M \subseteq U$ consists of union of some equivalence classes of U, then M is definable. Otherwise it is not definable. If M is not definable, then we can approximate it by the lower and upper approximations which are two definable subsets of M as the following

$$\begin{aligned} \underline{J}(M) &= \cup\{[m]_J : [m]_J \subseteq M\} \text{ and} \\ \overline{J}(M) &= \cup\{[m]_J : [m]_J \cap M \neq \phi\}. \end{aligned}$$

Definition 1.3.1 *[69] The upper approximation of a set M with respect to J is the set of all objects which can be for certain classified as M with respect to J (are possibly M in view of J).*

From the different representations of an equivalence relation, we obtain three constructive definitions of lower approximation

1. Element based definition

$$\underline{J}(M) = \{m \in U : [m]_J \subseteq M\},$$

2. Granule based definition

$$\underline{J}(M) = \bigcup_{[m]_J \subseteq M} [m]_J,$$

3. Subsystem based definition

$$\underline{J}(M) = \bigcup\{A \in U/J : A \subseteq M\},$$

where $[m]_J = \{n : mJn\}$.

Definition 1.3.2 *[69] The upper approximation of a set M with respect to J is the set of all objects which can be possibly classified as M with respect to J (are possibly M in view of J).*

From the different representations of an equivalence relation, we obtain three constructive definitions of upper approximation

1. Element based definition

$$\overline{J}(M) = \{m \in U : [m]_J \cap M \neq \emptyset\},$$

2. Granule based definition

$$\overline{J}(M) = \bigcup_{[m]_J \cap M \neq \emptyset} [m]_J,$$

3. Subsystem based definition

$$\overline{J}(M) = \bigcap \{A \in U/J : A \cap M \neq \emptyset\},$$

where $[m]_J = \{n : mJn\}$.

A rough set is the pair $(\underline{J}(M), \overline{J}(M))$. Boundary region is represented by the set $\overline{J}(M) - \underline{J}(M)$. Clearly, if $\underline{J}(M) = \overline{J}(M)$, then M is definable and $\overline{J}(M) - \underline{J}(M) = \phi$.

Definition 1.3.3 *[96] A subset M of U represents a crisp set when its boundary region is empty, i.e., $\underline{J}(M) = \overline{J}(M)$.*

The universe U can be separated into three disjoint regions, by using the lower and upper approximations of a set $M \subseteq U$.

(1) the positive region $(\mathcal{POS})_J(M) = \underline{J}(M)$;

(2) the negative region $(\mathcal{NEG})_J(M) = U - \overline{J}(M) = (\overline{J}(M))^c$;

(3) the boundary region $(\mathcal{BND})_J(M) = \overline{J}(M) - \underline{J}(M)$.

Example 1.3.4 *Consider a set $U = \{1, 2, 3, 4, 5, 6\}$ as a universal set. Define J as an equivalence relation such that, for the equivalence relation J on U :*

$$1J1, \quad 2J2, \quad 2J3, \quad 3J2, \quad 3J3, \quad 4J4, \quad 5J6, \quad 6J5, \quad 5J5, \quad 6J6$$

The equivalence relation induces four equivalence classes, which are the subsets $C_1 = \{1\}$, $C_2 = \{2, 3\}$, $C_3 = \{4\}$, $C_4 = \{5, 6\}$, here we want to characterize the set $D = \{3, 4, 5\}$ with respect to J. For this we have

$\underline{J}(D) = \{4\} = C_3$

$\overline{J}(D) = \{2, 3, 4, 5, 6\} = C_2 \cup C_3 \cup C_4$.

Proposition 1.3.5 *[67] Let J be an equivalence relation on a set U. If M and N are subsets of U, then the given assertions are valid:*

(1) $\underline{J}(M) \subseteq M \subseteq \overline{J}(M)$

(2) $M \subseteq N \Rightarrow \underline{J}(M) \subseteq \underline{J}(N)$

(3) $M \subseteq N \Rightarrow \overline{J}(M) \subseteq \overline{J}(N)$

(4) $\underline{J}(M \cap N) = \underline{J}(M) \cap \underline{J}(N)$

(5) $\underline{J}(M \cup N) \supseteq \underline{J}(M) \cup \underline{J}(N)$

(6) $\overline{J}(M \cup N) = \overline{J}(M) \cup \overline{J}(N)$

(7) $\overline{J}(M \cap N) \subseteq \overline{J}(M) \cap \overline{J}(N)$.

Example 1.3.6 *Let (U, J) be an approximation space, and J be an equivalence relation, where $U = \{m_1, m_2, m_3, \ldots, m_8\}$. Consider the following equivalence classes:*

$$\mathcal{E}_1 = \{m_1, m_4, m_8\}, \ \mathcal{E}_2 = \{m_2, m_5, m_7\}, \ \mathcal{E}_3 = \{m_3\}, \ \mathcal{E}_4 = \{m_6\}.$$

Let $M = \{m_3, m_5\}$ and $N = \{m_3, m_6\}$

$\underline{J}(M) = \{m_3\}$ *and* $\overline{J}(M) = \{m_2, m_3, m_5, m_7\}$

$\underline{J}(N) = \{m_3, m_6\}$ *and* $\overline{J}(N) = \{m_3, m_6\}$

So $J(M) = (\{m_3\},\{m_2,m_3,m_5,m_7\})$ *is a rough set and* $J(N)$ *is a crisp set.*

Example 1.3.7 *[96] Consider a universe consisting of three elements* $U = \{1, 2, 3\}$ *and an equivalence relation* J *on* U*:*

$$1J1, \quad 2J2, \quad 1J3, \quad 3J1, \quad 3J3$$

The equivalence relation induces two equivalence classes $[1]_J = [3]_J = \{1, 3\}$, $[2]_J = \{2\}$, *now*

$$P(U) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, U\}$$

is the set of all subsets of U. The following table summarizes the lower and upper approximations, the positive, negative and boundary regions for all subsets of U*.*

M	$\underline{J}(M)$	$\overline{J}(M)$	$POS(M)$	$NEG(M)$	$BND(M)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	U	$\emptyset$
$\{1\}$	$\emptyset$	$\{1,3\}$	$\emptyset$	$\{2\}$	$\{1,3\}$
$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{1,3\}$	$\emptyset$
$\{3\}$	$\emptyset$	$\{1,3\}$	$\emptyset$	$\{2\}$	$\{1,3\}$
$\{1,2\}$	$\{2\}$	U	$\{2\}$	$\emptyset$	$\{1,3\}$
$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{2\}$	$\emptyset$
$\{2,3\}$	$\{2\}$	U	$\{2\}$	$\emptyset$	$\{1,3\}$
U	U	U	U	$\emptyset$	$\emptyset$

The above table shows

$$\begin{aligned} \underline{J}(\{1\}) &\neq \overline{J}(\{1\}) \\ \underline{J}(\{3\}) &\neq \overline{J}(\{3\}) \\ \underline{J}(\{1,2\}) &\neq \overline{J}(\{1,2\}) \\ \underline{J}(\{2,3\}) &\neq \overline{J}(\{2,3\}) \end{aligned}$$

So $\{1\}$, $\{3\}$, $\{1, 2\}$, $\{2, 3\}$ *are rough sets with respect to* J, *and* $\{2\}$, $\{1, 3\}$ *are crisp sets with respect to* J*.*

Throughout the thesis, we shall denote an equivalence relation by an $E-$relation.

1.4 Soft sets and soft substructures

Molodtsov [66] showed up the soft set theory for the first time as the key notion to handle uncertainty. Soft sets have many operations which are very handy to deal with various types of situations. Many authors [5, 8, 60, 66, 67] described several operations in soft set theory. This theory speaks about that every collection of objects in the universe U can be accompanied by a set E of attributes (Characteristics or parameters) for U.

Let $P(U)$ denotes the power set of U and for $A, B \subseteq E$, where E is a universe set of parameters, some basic definitions associated with soft sets are discussed.

Definition 1.4.1 *[67] Define $J : A \to P(U)$. Then (J, A) is called a soft set over U.*

For the illustration of soft sets, Molodtsov gave many concrete examples. One of them is presented here.

Example 1.4.2 *Suppose that $U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\}$ is a universe containing six houses and* $A = \left\{ \begin{array}{c} e_1 = \text{ expensive, } e_2 = \text{ beautiful, } e_3 = \text{ wooden,} \\ e_4 = \text{ in green surroundings, } e_5 = \text{ in good repair} \end{array} \right\}$

Consider a soft set (J, A) which describes the "attractiveness of houses" that Mr. X wants to purchase. Here the soft set (J, A) points out the expensive houses, beautiful houses, wooden houses and so on, according to Mr. X. Thus, $J(e_1)$ represents the subset of U comprising of all the beautiful houses in U.

We can define the soft set (J, A) completely as

$$\begin{aligned} J(e_1) &= \{h_1, h_3, h_4, h_6\}, J(e_2) = \{h_2, h_3, h_5\}, \\ J(e_3) &= \{h_4, h_6\}, J(e_4) = \{h_2, h_3, h_4, h_5\} \\ \text{and } J(e_5) &= \{h_1, h_2, h_3, h_4, h_5\}. \end{aligned}$$

This table represents a soft set (J, A) as

(J, A)	h_1	h_2	h_3	h_4	h_5	h_6	h_7
e_1	1	0	1	1	0	1	0
e_2	0	1	1	0	1	0	0
e_3	0	0	0	1	0	1	0
e_4	0	1	1	1	1	0	0
e_5	1	1	1	1	1	0	0

In this table, if $h_j \in J(e_i)$, then $a_{ij} = 1$, otherwise $a_{ij} = 0$. where a_{ij} is the (i, j) th entry.

Now, some necessary operations of soft sets are discussed.

Definition 1.4.3 *[8] Over a common universe U, for two soft sets (J, A) and (L, B) , if $A \subseteq B$ and $J(e) \subseteq L(e)$ for all $e \in A$, then (J, A) is a soft subset of (L, B) and is denoted by $(J, A) \subseteq (L, B)$.*

Definition 1.4.4 *[8] Over a common universe U, two soft sets (J, A) and (L, B) are said to be soft equal if $(J, A) \subseteq (L, B)$ and $(L, B) \subseteq (J, A)$.*

Definition 1.4.5 *[8] Let A be the set of parameters, U be an initial universe set.*

(a) If $J(a) = \phi$ for all $a \in A$, then (J, A) is called a relative null soft set with the parameter set A, denoted by ϕ_A.

(b) If $L(e) = U$ for all $e \in A$, then (L, A) is called a relative whole soft set with the parameter set A, denoted by A_U.

Definition 1.4.6 *[8] Over the common universe U, the union of two soft sets (J, A) and (L, A) is the soft set (H, A), for all $e \in A$ such that $H(e) = J(e) \cup L(e)$.*

Definition 1.4.7 *[8] Over the common universe U, the intersection of two soft sets (J, A) and (L, A) is the soft set (H, A) for all $e \in A$ such that $H(e) = J(e) \cap L(e)$.*

Definition 1.4.8 *[8] The product of two soft sets (J, A) and (L, A) over a universe U is the soft set (JL, A) such that $JL(e) = J(e)L(e)$ for all $e \in A$.*

Definition 1.4.9 *[8] The relative complement of a soft set* (J, A) *is denoted by* $(J, A)^r$ *and is defined by* $(J, A)^r = (J^r, A)$ *where* $J^r : A \to P(U)$ *is a mapping given by* $J^r(e) = U - J(e)$ *for all* $e \in A$.

Definition 1.4.10 *Let* (J, A) *be a soft set over* S *and* S *be a semigroup. Then*

(1) *If* $J(e)$ *is a SS of* S *for all* $e \in A$ *with* $J(e) \neq \phi$, *then* (J, A) *over* S *is called a soft SS over* S.

(2) *If* $J(e)$ *is an IL of* S *for all* $e \in A$ *with* $J(e) \neq \phi$, (J, A) *over* S *is called a soft IL over* S.

(3) *If* $J(e)$ *is a BIL of* S *for all* $e \in A$ *with* $J(e) \neq \phi$, *then* (J, A) *over* S *is a soft BIL over* S.

(4) *If* $J(e)$ *is an IIL of* S *for all* $e \in A$ *with* $J(e) \neq \phi$, (J, A) *over* S *is said to be a soft IIL over* S.

Example 1.4.11 *Let* $S = \{a, b, c, d, e\}$ *be a semigroup with the following multiplication table:*

$\bullet$	a	b	c	d	e
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
e	d	d	c	d	c

and $A = \{e_1, e_2\}$. *Define* (J, A), *a soft set over* S *by*

$$J(e_1) = \{a, b, c, d\}, \ J(e_2) = \{b, c, d\}.$$

Here (J, A) *is a soft SS of* S. *Also* (J, A) *is a soft LIL of* S.

Example 1.4.12 *Let* $S = \{a, b, c\}$ *be a semigroup with the following multiplication table:*

$\bullet$	a	b	c
a	a	a	c
b	a	b	c
c	a	c	c

and $A = \{e_1, e_2\}$. *Define* (J, A), *a soft set over* S *by*

$$J(e_1) = \{a, b, c\}, \ J(e_2) = \{a, c\}.$$

Here (J, A) *is a soft IIL of* S.

Example 1.4.13 *Let* $S = \{a, b, c, d\}$ *be a semigroup with the following multiplication table:*

$\bullet$	a	b	c	d
a	a	a	a	d
b	a	b	a	d
c	a	a	c	d
d	d	d	d	d

and $A = \{e_1, e_2\}$. *Define* (J, A), *a soft set over* S *by*

$$J(e_1) = \{a, b, c, d\}, \ J(e_2) = \{a, d\}.$$

Here (J, A) *is a soft BIL of* S.

Throughout this thesis, we shall denote a soft SS, soft LIL, soft RIL, soft BIL and soft IIL by SSS, $SLIL$, $SRIL$, $SBIL$ and $SIIL$, respectively.

1.5 Fuzzy sets and fuzzy substructures

Theory of fuzzy sets proposes an exceptionally decent way to deal with vagueness.

Fuzzy set theory, introduced by Zadeh in [99], has given an important scientific and mathematical tool to the description of those frameworks which are unreasonably perplexing or uncertain.

For convenience, we shall denote a fuzzy subset, fuzzy subsemigroup, fuzzy left ideal, fuzzy right ideal, fuzzy ideal, fuzzy bi-ideal and fuzzy interior ideal by FS, FSS, $FLIL$, $FRIL$, FIL, $FBIL$ and $FIIL$, respectively, throughout this thesis.

Definition 1.5.1 *A FS,* λ *in* U *is defined by a mapping* $\lambda : U \rightarrow [0, 1]$. *A FS,* $\lambda : U \longrightarrow [0,1]$ *is non-empty if* λ *is not a zero map.*

The value $\lambda(x)$ is known as the membership grade of the object x and the mapping λ is known as the membership function of U.

The families of all subsets and FS in U are denoted by $P(U)$ and $\mathrm{F}(U)$, respectivley.

Definition 1.5.2 *Let λ_1 and λ_2 be two FSs in U. Then $\lambda_1 \leq \lambda_2$ if and only if $\lambda_1(u) \leq \lambda_2(u)$ for all $u \in U$. Moreover, $\lambda_1 = \lambda_2$ if and only if $\lambda_1 \leq \lambda_2$ and $\lambda_1 \geq \lambda_2$.*

Definition 1.5.3 *A FS, λ in U is called a null FS if $\lambda(u) = 0$ for all $u \in U$. A FS, λ is called a whole FS in U if $\lambda(u) = 1$ for all $u \in U$.*

Definition 1.5.4 *A FS, λ in U is said to be a constant FS in U if and only if $\lambda : U \longrightarrow [0,1]$ is a constant function.*

We usually denote null FS by 0 and whole FS by 1.

Definition 1.5.5 *Intersection, union, and complement of FSs are given below:*

$$\begin{aligned} \lambda(x) \wedge \mu(x) &= (\lambda \cap \mu)(x) \\ \lambda(x) \vee \mu(x) &= (\lambda \cup \mu)(x) \end{aligned}$$

$$\lambda^c(x) = 1 - \lambda(x), \text{ where } \lambda, \mu \in \mathrm{F}(U) \text{ and } x \in U.$$

More generally, if $\{f_i : i \in I\}$ is a family of FSs of U, then the union and intersection are defined as

$$\begin{aligned} (\cup_i f_i)(x) &= \vee_i (f_i(x)) \text{ for all } x \in U. \\ (\cap_i f_i)(x) &= \wedge_i (f_i(x)) \text{ for all } x \in U. \end{aligned}$$

Definition 1.5.6 *For a number $\alpha \in (0,1]$, the α-cut or α-level set of a FS, λ in U is $\lambda_\alpha = \{x \in U : \lambda(x) \geq \alpha\}$ which is a subset of U.*

Definition 1.5.7 *If $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$ for all $x, y \in S$, then a FS, λ in a semigroup S is called a FSS of S.*

Definition 1.5.8 *If $\lambda(xy) \geq \lambda(y)$ $(\lambda(xy) \geq \lambda(x))$ for all $x, y \in S$, then a FS, λ in a semigroup S is called a FLIL $(FRIL)$ of S.*

Definition 1.5.9 *If $\lambda(xay) \geq \lambda(a)$ for all $x, y, a \in S$, then a FS, λ in a semigroup S is called a FIIL of S.*

Definition 1.5.10 *If $\lambda(xyz) \geq \lambda(x) \wedge \lambda(z)$ for all $x, y, z \in S$, then a FSS, λ in a semigroup S is called a FBIL of S.*

Example 1.5.11 *Let $S = \{a, b, c, d, e\}$ be a semigroup with the following multiplication table:*

$\bullet$	a	b	c	d	e
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
e	d	d	c	d	c

Define $\lambda : S \to [0, 1]$ by

	a	b	c	d	e
λ	0.5	0.5	0.3	1	0.1

Here, λ represents a FSS of S. Also λ represents a FLIL of S.

Example 1.5.12 *Let $S = \{0, x, y, z\}$ be a semigroup with the following multiplication table:*

$\bullet$	0	x	y	z
0	0	0	0	0
x	0	0	0	0
y	0	0	0	x
z	0	0	x	y

Define $\lambda : S \to [0, 1]$ by

	0	x	y	z
λ	0.7	0.3	0.7	0.3

Here, λ represents a FIIL of S.

Example 1.5.13 *Let $S = \{a, b, c, d\}$ be a semigroup with the following multiplication table:*

$\bullet$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

Define $\lambda : S \to [0, 1]$ by

	a	b	c	d
λ	0.7	0.3	0.7	0.3

Here, λ represents a $FBIL$ of S.

1.6 Fuzzy soft sets and fuzzy soft substructures

Now, we give some results related to a fuzzy soft set which will be vital in the thesis.

Definition 1.6.1 *[85] A pair (J, A) is called a fuzzy soft set over U if J is a mapping given by $J : A \to \text{Ғ}(U)$ and $A \subseteq E$ (the set of parameters). Thus, $J(e)$ is a fuzzy set in U for all $e \in A$. Hence, a fuzzy soft set over U is a collection of fuzzy sets in U.*

Definition 1.6.2 *[85] Over a common universe U, for two fuzzy soft sets (J, A) and (G, B), we say that (J, A) is a fuzzy soft subset of (G, B) if (1) $J(e)$ is a fuzzy set of $G(e)$ for all $e \in A$ and (2) $A \subseteq B$.*

Definition 1.6.3 *[85] Two fuzzy soft sets (J, A) and (G, B) over a common universe U are said to be fuzzy soft equal if (J, A) is a fuzzy soft subset of (G, B) and (G, B) is a fuzzy soft subset of (J, A).*

Definition 1.6.4 *[85] Over the common universe U, the union of two fuzzy soft sets (J, A) and (G, A) is the fuzzy soft set (H, A) for all $e \in A$ such that $H(e) = J(e) \vee G(e)$.*

Definition 1.6.5 *[85] Over the common universe U, the intersection of two fuzzy soft sets (J, A) and (G, A) is the fuzzy soft set (H, A) for all $e \in A$ such that $H(e) = J(e) \wedge G(e)$.*

Definition 1.6.6 *[97] Let S be a semigroup and let (J, A) be a fuzzy soft set over S. Then*

(1) If $J(e)$ is a FSS of S for all $e \in A$ with $J(e) \neq 0$, then (J, A) is called a fuzzy SSS over S.

(2) If $J(e)$ is a FIL(resp. FLIL, FRIL) of S for all $e \in A$ with $J(e) \neq 0$, then a fuzzy soft set (J, A) over a semigroup S is called a fuzzy SIL (resp. fuzzy SLIL, fuzzy SRIL) over S.

(3) If $J(e)$ is a FIIL of S for all $e \in A$ with $J(e) \neq 0$, then a fuzzy soft set (J, A) over a semigroup S is called a fuzzy SIIL over S.

(4) If $J(e)$ is a FBIL of S for all $e \in A$ with $J(e) \neq 0$, then a fuzzy soft set (J, A) over a semigroup S is called a fuzzy SBIL over S.

Example 1.6.7 *Let $S = \{1, 2, 3, 4\}$ be a semigroup with the following multiplication table:*

$\bullet$	1	2	3	4
1	1	2	3	4
2	2	2	2	2
3	3	3	3	3
4	4	3	2	1

Define a fuzzy soft set (J, A), where $A = \{e_1, e_2\}$ by

	1	2	3	4
$J(e_1)$	0.8	0.7	0.5	0.1
$J(e_2)$	0.9	0.8	0.6	0.2

Here, $J(e_1)$ and $J(e_2)$ are FSS of S. Therefore, (J, A) is a fuzzy SSS of S.

Also $J(e_1)$ and $J(e_2)$ are FLILs of S. Therefore, (J, A) is a fuzzy SLIL of S.

Example 1.6.8 *Consider the semigroup of Example 1.5.12.*

Define a fuzzy soft set (J, A), where $A = \{e_1, e_2\}$ by

	0	x	y	z
$J(e_1)$	0.7	0.3	0.7	0.3
$J(e_2)$	0.7	0.4	0.7	0

Here, $J(e_1)$ *and* $J(e_2)$ *are* $FIIL$ *of* S*. Therefore,* (J, A) *is a fuzzy* $SIIL$ *of* S*.*

Example 1.6.9 *Consider the semigroup of Example 1.5.13.*

Define a fuzzy soft set (J, A)*, where* $A = \{e_1, e_2\}$ *by*

	a	b	c	d
$J(e_1)$	0.7	0.3	0.7	0.3
$J(e_2)$	0.7	0.4	0.7	0

Here, $J(e_1)$ *and* $J(e_2)$ *are FBIL of* S*. Therefore,* (J, A) *is a fuzzy SBIL of* S*.*

Throughout this thesis, we shall denote a fuzzy SSS, fuzzy $SLIL$, fuzzy $SRIL$, fuzzy $SBIL$ and fuzzy $SIIL$ by $FSSS$, $FSLIL$, $FSRIL$, $FSBIL$ and $FSIIL$, respectively.

Chapter 2

Reduction of an information system

In this chapter, we present an investigation of soft binary relations and some of their properties. Furthermore, a soft equivalence relation offers ascent to a fuzzy set for every parameter and a fuzzy set related with the soft equivalence relation is discussed. Soft similarity relations have also been examined. Finally, a parametric decrease have been talked about in soft rough sets. At last, an application is displayed toward the conclusion to explain this work.

2.1 Soft Binary Relations

This section represents the notion of soft binary relation from a set U to a set W. Some basic concepts, characterizations and related properties with regard to soft binary relation are proposed here. Throughout this chapter, a soft binary relation is denoted by an $SBRE$.

Definition 2.1.1 *If (J, A) is a soft set over $U \times W$, that is $J : A \to P(U \times W)$, then (J, A) is said to be an SBRE from U to W, where $A \subseteq E$ (parameters set).*

In fact (J, A) is a parameterized collection of binary relations from U to W. That is, we have a binary relation $J(e)$ from U to W for each parameter $e \in A$. In what follows, we shall denote the collection of all soft binary relations from U to W by $\eta_{Br}(U, W)$.

Definition 2.1.2 *If $N \subseteq W$, then we can define two soft sets over U, by*

$$\begin{aligned} \underline{J}^N(e) &= \{u \in U : \phi \neq uJ(e) \subseteq N\} \text{ and} \\ \overline{J}^N(e) &= \{u \in U : uJ(e) \cap N \neq \phi\} \end{aligned}$$

where $uJ(e) = \{w \in W : (u, w) \in J(e)\}$ for each $e \in A$.

Moreover, $\underline{J}^N : A \to P(U)$ and $\overline{J}^N : A \to P(U)$ and we say (U, W, J) a Generalized soft approximation space.

In order to explain this concept, the following example is given.

Example 2.1.3 *Let $U = \{x_1, x_2, x_3, x_4\}$ and $W = \{b, c, w\}$ and the set of attributes be $A = \{e_1, e_2\}$.*

Define $J : A \to P(U \times W)$ by

$$\begin{aligned} J(e_1) &= \{(x_1, b), (x_1, c), (x_2, w)\} \\ \text{and } J(e_2) &= \{(x_2, b), (x_2, w), (x_4, b)\}. \end{aligned}$$

Let $N = \{b, w\} \subseteq W$. Then

$$x_1J(e_1) = \{b, c\}, \; x_2J(e_1) = \{w\}, \; x_3J(e_1) = \phi, \; x_4J(e_1) = \phi$$

and

$$x_1J(e_2) = \phi, \; x_2J(e_2) = \{b, w\}, \; x_3J(e_2) = \phi, \; x_4J(e_2) = \{b\}.$$

Therefore,

$$\begin{aligned} \underline{J}^N(e_1) &= \{x_2\}, \; \underline{J}^N(e_2) = \{x_2, x_4\}, \\ \overline{J}^N(e_1) &= \{x_1, x_2\}, \overline{J}^N(e_2) = \{x_2, x_4\}. \end{aligned}$$

Theorem 2.1.4 *Let (U, W, J) be a Generalized soft approximation space and $J : A \to P(U \times W)$ be an SBRE from U to W. For $N_1, N_2 \subseteq W$, the following properties hold:*

(1) $N_1 \subseteq N_2 \Rightarrow \underline{J}^{N_1} \subseteq \underline{J}^{N_2}$

(2) $N_1 \subseteq N_2 \Rightarrow \overline{J}^{N_1} \subseteq \overline{J}^{N_2}$

(3) $\underline{J}^{N_1} \cap \underline{J}^{N_2} = \underline{J}^{N_1 \cap N_2}$

(4) $\overline{J}^{N_1} \cap \overline{J}^{N_2} \supseteq \overline{J}^{N_1 \cap N_2}$

(5) $\underline{J}^{N_1} \cup \underline{J}^{N_2} \subseteq \underline{J}^{N_1 \cup N_2}$

(6) $\overline{J}^{N_1} \cup \overline{J}^{N_2} = \overline{J}^{N_1 \cup N_2}$

(7) $\underline{J}^{W}(e) \subseteq U$ *for all* $e \in A$ *and if* $uJ(e) \neq \phi$ *for all* $u \in U$, *then* $\underline{J}^{W}(e) = U$ *for all* $e \in A$.

(8) $\overline{J}^{W}(e) \subseteq U$ *for all* $e \in A$ *and if* $uJ(e) \neq \phi$ *for all* $u \in U$, *then* $\overline{J}^{W}(e) = U$ *for all* $e \in A$.

(9) $\underline{J}^{N} = \left(\overline{J}^{N^c}\right)^c$

(10) $\overline{J}^{N} = \left(\underline{J}^{N^c}\right)^c$.

Proof. (1) Let $u \in \underline{J}^{N_1}(e)$ for $e \in A$. Then $\phi \neq uJ(e) \subseteq N_1$. As $N_1 \subseteq N_2$, we have $\phi \neq uJ(e) \subseteq N_2$. Thus $u \in \underline{J}^{N_2}(e)$. Hence, $\underline{J}^{N_1} \subseteq \underline{J}^{N_2}$.

(2) Let $u \in \overline{J}^{N_1}(e)$ for $e \in A$. Then $uJ(e) \cap N_1 \neq \phi$. As $N_1 \subseteq N_2$, we have

$uJ(e) \cap N_2 \neq \phi$. Thus $u \in \overline{J}^{N_2}(e)$. Hence, $\overline{J}^{N_1} \subseteq \overline{J}^{N_2}$.

(3) Using (1) and the fact that $N_1 \cap N_2 \subseteq N_1, N_2$, we have $\underline{J}^{N_1 \cap N_2} \subseteq \underline{J}^{N_1}$, $\underline{J}^{N_2}$

so $\underline{J}^{N_1 \cap N_2} \subseteq \underline{J}^{N_1} \cap \underline{J}^{N_2}$. Conversely, let $u \in \underline{J}^{N_1}(e) \cap \underline{J}^{N_2}(e)$ for $e \in A \Rightarrow u \in \underline{J}^{N_1}(e)$ and $u \in \underline{J}^{N_2}(e) \Rightarrow uJ(e) \subseteq N_1$ and $uJ(e) \subseteq N_2 \Rightarrow uJ(e) \subseteq N_1 \cap N_2$. $\Rightarrow u \in \underline{J}^{N_1 \cap N_2}(e) \Rightarrow \underline{J}^{N_1} \cap \underline{J}^{N_2} \subseteq \underline{J}^{N_1 \cap N_2}$. Hence, $\underline{J}^{N_1} \cap \underline{J}^{N_2} = \underline{J}^{N_1 \cap N_2}$.

(4) Using (2) and the fact that $N_1 \cap N_2 \subseteq N_1$, N_2, we have $\overline{J}^{N_1 \cap N_2} \subseteq \overline{J}^{N_1}$, $\overline{J}^{N_2}$

$\Rightarrow \overline{J}^{N_1 \cap N_2} \subseteq \overline{J}^{N_1} \cap \overline{J}^{N_2}$.

(5) Since N_1, $N_2 \subseteq N_1 \cup N_2$, so by using part (1), we get $\underline{J}^{N_1}$, $\underline{J}^{N_2} \subseteq \underline{J}^{N_1 \cup N_2}$ and so $\underline{J}^{N_1} \cup \underline{J}^{N_2} \subseteq \underline{J}^{N_1 \cup N_2}$.

(6) Since N_1, $N_2 \subseteq N_1 \cup N_2$, so by using part (2), we get $\overline{J}^{N_1}$, $\overline{J}^{N_2} \subseteq \overline{J}^{N_1 \cup N_2}$ implies $\overline{J}^{N_1} \cup \overline{J}^{N_2} \subseteq \overline{J}^{N_1 \cup N_2}$. Conversely, let $u \in \overline{J}^{N_1 \cup N_2}(e)$ for $e \in A$.

$\Rightarrow uJ(e) \cap (N_1 \cup N_2) \neq \phi \Rightarrow uJ(e) \cap N_1 \neq \phi$ or $uJ(e) \cap N_2 \neq \phi$.

$\Rightarrow u \in \overline{J}^{N_1}(e)$ or $u \in \overline{J}^{N_2}(e) \Rightarrow u \in \left(\overline{J}^{N_1} \cup \overline{J}^{N_2}\right)(e) \Rightarrow \overline{J}^{N_1 \cup N_2} \subseteq \overline{J}^{N_1} \cup \overline{J}^{N_2}$.

Hence, $\overline{J}^{N_1 \cup N_2} = \overline{J}^{N_1} \cup \overline{J}^{N_2}$.

(7) Since $\underline{J}^{W}(e)=\{u\in U: uJ(e)\subseteq W\}\subseteq U$ for $e\in A$, because

$$uJ(e)=\{w\in W\colon (u,w)\in J(e)\}\subseteq W.$$

If $uJ(e)\neq\phi$ for all $u\in U$, then $\underline{J}^{W}(e)=U$ for all $e\in A$.

(8) By definition, $\overline{J}^{W}(e)=\{u\in U: uJ(e)\cap W\neq\phi\}\subseteq U$. Moreover, if $uJ(e)\neq\phi$ for every $u\in U$, then $\overline{J}^{W}=U$.

(9) Let $u\in\underline{J}^{N}(e)$ for $e\in A\Leftrightarrow\phi\neq uJ(e)\subseteq N\Leftrightarrow uJ(e)\cap N^{c}=\phi\Leftrightarrow u\notin\overline{J}^{N^{c}}(e)$ $\Leftrightarrow u\in\left(\overline{J}^{N^{c}}(e)\right)^{c}$. Hence, $\underline{J}^{N}=\left(\overline{J}^{N^{c}}\right)^{c}$.

(10) By (9), $\underline{J}^{N}=\left(\overline{J}^{N^{c}}\right)^{c}$. Therefore, $\underline{J}^{N^{c}}=\left(\overline{J}^{(N^{c})^{c}}\right)^{c}\Rightarrow\underline{J}^{N^{c}}=\left(\overline{J}^{N}\right)^{c}$. Hence, $\left(\underline{J}^{N^{c}}\right)^{c}=\overline{J}^{N}$. ■

It is demonstrated by the following example that equality is not valid in (4), (5) and (8) in general.

Example 2.1.5 *Consider $U=\{m_1,m_2,m_3,m_4,m_5\}$ is a collection of five mobile phones as the universal set. These mobile phones are classified by attributes age and color represented by $A=\{e_1,e_2\}$. Let $W=\{$new, used, old, black, white$\}$ be represented by $W=\{n,u,o,b,w\}$.*

Define a relation $J: A\to P(W\times U)$ by

$$\begin{aligned} J(e_1) &= \{(n,m_1),(n,m_2),(o,m_3),(o,m_4),(u,m_5)\} \text{ and}\\ J(e_2) &= \{(b,m_2),(b,m_3),(w,m_1),(w,m_4),(w,m_5)\}. \end{aligned}$$

Now, $nJ(e_1)=\{m_1,m_2\}$, $u(e_1)=\{m_5\}$, $oJ(e_1)=\{m_3,m_4\}$,

and $bJ(e_2)=\{m_2,m_3\}$, $wJ(e_2)=\{m_1,m_4,m_5\}$.

Let $N_1=\{m_1,m_2,m_3\}\subseteq U$ and $N_2=\{m_2,m_4,m_5\}\subseteq U$. Then $N_1\cap N_2=\{m_2\}$

and $N_1\cup N_2=\{m_1,m_2,m_3,m_4,m_5\}$. Therefore,

$\overline{J}^{N_1\cap N_2}(e_1)=\{n\}$, $\underline{J}^{N_1\cup N_2}(e_1)=\{n,u,o\}$, $\overline{J}^{N_1}(e_1)=\{n,o\}$,

$\overline{J}^{N_2}(e_1)=\{n,o,u\}$, $\underline{J}^{N_1}(e_1)=\{n\}$, $\underline{J}^{N_2}(e_1)=\{u\}$.

Hence, $\overline{J}^{N_1}(e_1)\cap\overline{J}^{N_2}(e_1)=\{n,o\}\nsubseteq\{n\}=\overline{J}^{N_1\cap N_2}(e_1)$, $\underline{J}^{N_1\cup N_2}(e_1)=\{n,o,u\}\nsubseteq\{n,u\}=\underline{J}^{N_1}(e_1)\cup\underline{J}^{N_2}(e_1)$ and $\overline{J}^{U}(e_1)=\{n,o,u\}\neq W$.

Proposition 2.1.6 *Let (U, W, J) be a generalized soft approximation space. Let $\{N_i\}$ be an arbitrary family of subsets of W. Then*

(1) $\underline{J}^{\cap_{i\in I}N_i} = \cap_{i\in I}\underline{J}^{N_i}$

(2) $\overline{J}^{\cap_{i\in I}N_i} \subseteq \cap_{i\in I}\overline{J}^{N_i}$.

Proof. (1) Let $u \in \underline{J}^{\cap_{i\in I}N_i}(e) \Leftrightarrow \phi \neq uJ(e) \subseteq \cap_{i\in I}N_i \Leftrightarrow \phi \neq uJ(e) \subseteq N_i$ for all $i \in I \Leftrightarrow u \in \underline{J}^{N_i}(e)$ for all $i \in I \Leftrightarrow u \in \cap_{i\in I}\underline{J}^{N_i}(e)$. Hence, $\underline{J}^{\cap_{i\in I}N_i} = \cap_{i\in I}\underline{J}^{N_i}$.

(2) Let $u \in \overline{J}^{\cap_{i\in I}N_i}(e) \Rightarrow uJ(e) \cap (\cap_{i\in I}N_i) \neq \phi \Rightarrow uJ(e) \cap N_i \neq \phi$ for all $i \in I$.

$\Rightarrow u \in \overline{J}^{N_i}(e)$ for all $i \in I \Rightarrow u \in \cap_{i\in I}\overline{J}^{N_i}(e)$. Hence, $\overline{J}^{\cap_{i\in I}N_i} \subseteq \cap_{i\in I}\overline{J}^{N_i}$. ■

Definition 2.1.7 *If (J, A) is a soft set over $U \times U$, then (J, A) is called an SBRE on U.*

In fact (J, A) is a parameterized collection of binary relations on U. That is, we have a binary relation $J(e)$ on U for each parameter $e \in A$.

Definition 2.1.8 *An SBRE (J, A) on U is said to be a soft reflexive relation on U if $J(e)$ is a reflexive relation on U for all $e \in A$.*

Definition 2.1.9 *[27] An SBRE (J, A) on a set S is said to be soft reflexive if $(a, a) \in J(e)$ for all $a \in S$ and $e \in A$.*

In this case, each $uJ(e)$ is non-empty and $u \in uJ(e)$. The approximation operators have additional properties with respect to soft reflexive binary relation as follows:

Theorem 2.1.10 *Let $J : A \to P(U \times U)$ be a soft reflexive relation on U. For $\phi \neq N \subseteq U$, the properties below hold:*

(1) $\underline{J}^N(e) \subseteq N$

(2) $N \subseteq \overline{J}^N(e)$

(3) $\underline{J}^\phi = \phi = \overline{J}^\phi$

(4) $\overline{J}^W(e) = U$ *for all* $e \in A$.

Proof. (1) Let $u \in \underline{J}^N(e)$. Then $\phi \neq uJ(e) \subseteq N$. But $u \in uJ(e)$, therefore $u \in N$. Therefore $\underline{J}^N(e) \subseteq N$.

(2) Let $u \in N$. Then $uJ(e) \neq \phi$. As $u \in uJ(e) \cap N$, so $uJ(e) \cap N \neq \phi$. It follows $u \in \overline{J}^N(e)$. Hence, $N \subseteq \overline{J}^N(e)$.

(3) It is direct.

(4) By definition, $\overline{J}^W(e) = \{u \in U : uJ(e) \cap W \neq \phi\}$. As $uJ(e) \neq \phi$ for every $u \in U$, therefore, $\overline{J}^W = U$. ■

Definition 2.1.11 *An SBRE (J, A) on U is a soft symmetric relation on U if $J(e)$ is a symmetric relation on U for all $e \in A$.*

The approximation operators have additional properties with respect to soft symmetric binary relation as follows:

Lemma 2.1.12 *If (J, A) is a soft symmetric relation on U, then $v \in uJ(e)$ implies $u \in vJ(e)$.*

Theorem 2.1.13 *Let $J : A \to P(U \times U)$ be a soft symmetric relation on U. For $\phi \neq N \subseteq U$, the properties below are valid:*

(1) $\overline{J}^{\left(\underline{J}^N(e)\right)}(e) \subseteq N$

(2) $N \subseteq \underline{J}^{\left(\overline{J}^N(e)\right)}(e)$ *for all $e \in A$.*

Proof. (1) *Let $u \in \overline{J}^{\left(\underline{J}^N(e)\right)}(e)$. If $uJ(e) \neq \phi$ for all $e \in A$, then $uJ(e) \cap \underline{J}^N(e) \neq \phi$ so there exists atleast one $u_1 \in uJ(e) \cap \underline{J}^N(e)$. This implies $u_1 \in uJ(e)$ and $u_1 \in \underline{J}^N(e)$. Now, $u_1 \in \underline{J}^N(e)$ implies $u_1 J(e) \subseteq N$. Also $u_1 \in uJ(e)$ and the relation is soft symmetric implies $u \in u_1 J(e)$. Thus, $u \in u_1 J(e) \subseteq N$. It follows $u \in N$. Therefore, $\overline{J}^{\left(\underline{J}^N(e)\right)}(e) \subseteq N$.*

(2) *Let $u \in N$. If $u_1 \in uJ(e)$, then $u \in u_1 J(e)$, because the relation is soft symmetric. It is clear that $u \in u_1 J(e) \cap N$, so $u_1 J(e) \cap N \neq \phi$. It means that $u_1 \in \overline{J}^N(e) \Rightarrow uJ(e) \subseteq \overline{J}^N(e)$ implies $u \in \underline{J}^{\left(\overline{J}^N(e)\right)}(e)$. Therefore, $N \subseteq \underline{J}^{\left(\overline{J}^N(e)\right)}(e)$.* ■

Definition 2.1.14 *An SBRE (J, A) on U is a soft transitive relation on U if $J(e)$ is a transitive relation on U for all $e \in A$.*

The approximation operators have additional properties with respect to soft transitive binary relation as described below:

Theorem 2.1.15 *Let $J : A \to P(U \times U)$ be a soft transitive relation from U to U. For $N \subseteq U$, $\overline{J}^{\left(\overline{J}^N(e)\right)} \subseteq \overline{J}^N$ for all $e \in A$.*

Proof. Let $u \in \overline{J}^{\left(\overline{J}^N(e)\right)}(e)$. It follows $uJ(e) \cap \overline{J}^N(e) \neq \phi$ so there exists atleast one $u_1 \in uJ(e) \cap \overline{J}^N(e)$ such that $u_1 \in uJ(e)$ and $u_1 \in \overline{J}^N(e)$. Now $u_1 \in \overline{J}^N(e)$ implies that $u_1 J(e) \cap N \neq \phi$. So there exists atleast one $x \in u_1 J(e) \cap N$ such that $x \in u_1 J(e)$ and $x \in N$. But $u_1 \in uJ(e)$ implies $(u, u_1) \in J(e)$ and $x \in u_1 J(e)$ implies $(u_1, x) \in J(e)$. Since the relation is soft transitive so $(u, x) \in J(e)$. It follows that $x \in uJ(e)$. Therefore, $x \in uJ(e) \cap N$. This implies $uJ(e) \cap N \neq \phi$.

Therefore $u \in \overline{J}^N(e)$. Thus, $\overline{J}^{\left(\overline{J}^N(e)\right)}(e) \subseteq \overline{J}^N(e)$. Hence, $\overline{J}^{\left(\overline{J}^N(e)\right)} \subseteq \overline{J}^N$. ■

Theorem 2.1.16 *If an SBRE, (J, A) on U is soft reflexive and soft transitive, then for any subset $N \subseteq U$, the following property holds:*
$\overline{J}^{\left(\overline{J}^N(e)\right)} = \overline{J}^N$ for all $e \in A$.

Proof. Since it is soft transitive so by previous theorem
$\overline{J}^{\left(\overline{J}^N(e)\right)} \subseteq \overline{J}^N$. It is also soft reflexive, therefore $N \subseteq \overline{J}^N(e)$. By using Theorem 2.1.4(2), $\overline{J}^N(e) \subseteq \overline{J}^{\left(\overline{J}^N(e)\right)}(e)$. Hence, $\overline{J}^{\left(\overline{J}^N(e)\right)} = \overline{J}^N$. ■

Definition 2.1.17 *An SBRE, (J, A) on U is a soft equivalence relation on U if it is soft reflexive, soft symmetric and soft transitive relation on U.*

From now, a soft equivalence relation is represented by an SE-relation and soft equivalence classes by SE-classes. And, an equivalence relation is represented by $E-$relation and equivalence classes by $E-$classes.

Definition 2.1.18 *An SBRE, (J, A) on U is an SE-relation on U if $J(e)$ for all $e \in A$, is an $E-$relation on U.*

Theorem 2.1.19 *Every information system (U, A, V) can be represented by a soft equivalence relation and vice versa.*

Proof. Let (U, A, V) be an information system. Then every attribute $e \in A$ induces a function from U to V. That is $e : U \to V$ defined as $e(x) = v_e \in V_e$. Where V_e is a set of all values associated with e, and $\bigcup\limits_{e\in A} V_e = V$. Now define a soft relation (J, A) on U as

$$J(e) = \{(x, y) \in U \times U : e(x) = e(y)\} \text{ for all } e \in A.$$

Clearly, $J(e)$ is an equivalence relation for all $e \in A$. Thus (J, A) is a soft equivalence relation on U. Conversely, Let (J, A) be a $SE-$relation on U. Then $J(e)$ partitions the set U into equivalence classes for all $e \in A$. To each class obtained by $J(e)$ a value v_e can be associated. Let the collection of all v_e is denoted by V_e. Now put $\bigcup\limits_{e\in A} V_e = V$. Thus a soft equivalence relation (J, A) can be represented by an information system (U, A, V). ■

If (J, A) is an SE-relation on U, then each $J(e)$ is an $E-$relation over U. Thus, $J(e)$ partitions the set U into $E-$classes $uJ(e)$.

To elaborate this concept, an example is added.

Example 2.1.20 *Let $U = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6\}$ be a set where $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and $A = \{e_1, e_2, e_3, e_4, e_5\}$ be the set of attributes.*

Define an SE-relation $J : A \to P(U \times U)$ for each parameter $e \in A$.

The following SE-classes are obtained for each of the SE-relation.

For $J(e_1)$, the SE-classes $uJ(e_1)$ are $\{\pi_1, \pi_3\}, \{\pi_2, \pi_4, \pi_5, \pi_6\}$.

For $J(e_2)$, the SE-classes $uJ(e_2)$ are $\{\pi_1, \pi_3, \pi_6\}, \{\pi_2, \pi_4, \pi_5\}$.

For $J(e_3)$, the SE-classes $uJ(e_3)$ are $\{\pi_1, \pi_2, \pi_4, \pi_5\}, \{\pi_3\}, \{\pi_6\}$.

For $J(e_4)$, the SE-classes $uJ(e_4)$ are $\{\pi_1, \pi_2, \pi_4, \pi_5, \pi_6\}, \{\pi_3\}$.

For $J(e_5)$, the SE-classes $uJ(e_5)$ is $\{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6\}$.

A soft indiscernibility relation is obtained by the intersecting all the SE-relations induced by parameters represented as $IN_d(J, A) = \cap_{e_i \in A} J(e_i) = \omega$

In above example, the partition of U obtained by soft indiscernibility relation $IN_d(J, A)$ is $\{\pi_1\}, \{\pi_2, \pi_4, \pi_5\}, \{\pi_3\}$ and $\{\pi_6\}$. It is evident that for each $J(e_i)$ ($SE-$ relation) where $i = 1, 2, 3, 4, 5$, $(U, J(e_i))$ gives an approximation space. Also, (U, ω) is an approximation space.

In view of the above example, any subset M of U can be approximated by the SE-relation $J(e_i)$. The SE-class determined by the SE-relation $J(e_i)$ is denoted by

$uJ(e_i)$. The parameterized collection of subsets, denoted by $\left(\underline{J}^M, A\right)$, is defined as $\underline{J}^M(e_i) = \cup_{u\in M}\{uJ(e_i) : uJ(e_i) \subseteq M\}$ for all $e_i \in A$, is said to be soft lower approximation of M with respect to SE-relation (J, A). The parameterized collection of subsets, denoted by $\left(\overline{J}^M, A\right)$, is defined as $\overline{J}^M(e_i) = \cup_{u\in M}\{uJ(e_i) : uJ(e_i) \cap M \neq \phi\}$ for all $e_i \in A$, is said to be soft upper approximation of M with respect to SE-relation (J, A).

The soft set $\left(BJ^M, A\right)$ defined by $BJ^M(e_i) = \overline{J}^M(e_i) - \underline{J}^M(e_i)$ for all $e_i \in A$ is called soft boundary region of M, with respect to SE-relation (J, A). A subset M of U is called totally rough with respect to SE-relation (J, A) if $BJ^M(e_i) \neq \phi$ for all $e_i \in A$. A subset M of U is said to be partly definable with respect to SE-relation (J, A) if $BJ^M(e_i) = \phi$ for some $e_i \in A$. A subset M of U is called totally definable with respect to SE-relation (J, A) if $BJ^M(e_i) = \phi$ for all $e_i \in A$.

Proposition 2.1.21 *For the SE-relation (J, A) on U and for $M, N \subseteq U$,*

(1) $\left(\underline{J}^M, A\right) \subseteq \left(\overline{J}^M, A\right)$

(2) $\left(\underline{J}^\phi, A\right) = \left(\overline{J}^\phi, A\right) = \phi$, $\left(\underline{J}^U, A\right) = \left(\overline{J}^U, A\right) = U$

(3) $M \subseteq N \Rightarrow \left(\underline{J}^M, A\right) \subseteq \left(\underline{J}^N, A\right)$

(4) $M \subseteq N \Rightarrow \left(\overline{J}^M, A\right) \subseteq \left(\overline{J}^N, A\right)$

(5) $\left(\underline{J}^M, A\right) \cap \left(\underline{J}^N, A\right) = \left(\underline{J}^{M\cap N}, A\right)$

(6) $\left(\overline{J}^M, A\right) \cup \left(\overline{J}^N, A\right) = \left(\overline{J}^{M\cup N}, A\right)$

(7) $\left(\underline{J}^M, A\right) \cup \left(\underline{J}^N, A\right) \subseteq \left(\underline{J}^{M\cup N}, A\right)$

(8) $\left(\overline{J}^M, A\right) \cap \left(\overline{J}^N, A\right) \supseteq \left(\overline{J}^{M\cap N}, A\right)$

(9) $\left(\overline{J}^{M^c}, A\right) = \left(\underline{J}^M, A\right)^c$.

(10) $\left(\underline{J}^{M^c}, A\right) = \left(\overline{J}^M, A\right)^c$.

Proof. (1) Let $u \in \underline{J}^M(e) \Rightarrow uJ(e) \subseteq M \Rightarrow uJ(e) \cap M \neq \phi \Rightarrow u \in \overline{J}^M(e)$. Hence, $\left(\underline{J}^M, A\right) \subseteq \left(\overline{J}^M, A\right)$.

(2) Straightforward.

(3) Let $u \in \underline{J}^M(e)$. Then $uJ(e) \subseteq M$. As $M \subseteq N$, we have $uJ(e) \subseteq N$. Thus $u \in \underline{J}^N(e)$. Hence, $\left(\underline{J}^M, A\right) \subseteq \left(\underline{J}^N, A\right)$.

(4) Let $u \in \overline{J}^M(e)$. Then $uJ(e) \cap M \neq \phi$. As $M \subseteq N$, we have $uJ(e) \cap N \neq \phi$. Thus $u \in \overline{J}^N(e)$. Hence, $\left(\overline{J}^M, A\right) \subseteq \left(\overline{J}^N, A\right)$.

(5)Using part (3) and the fact that $M \cap N \subseteq M, N$, we have $\underline{J}^{M\cap N} \subseteq \underline{J}^M$, $\underline{J}^N$ so $\underline{J}^{M\cap N} \subseteq \underline{J}^M \cap \underline{J}^N$.

Hence, $\left(\underline{J}^{M\cap N}, A\right) \subseteq \left(\underline{J}^M, A\right) \cap \left(\underline{J}^N, A\right)$.

Conversely, let $u \in \underline{J}^M(e) \cap \underline{J}^N(e) \Rightarrow u \in \underline{J}^M(e)$ and $u \in \underline{J}^N(e)$

$\Rightarrow uJ(e) \subseteq M$ and $uJ(e) \subseteq N \Rightarrow uJ(e) \subseteq M \cap N$

$\Rightarrow u \in \underline{J}^M(e) \Rightarrow \left(\underline{J}^M, A\right) \cap \left(\underline{J}^N, A\right) \subseteq \left(\underline{J}^{M\cap N}, A\right)$.

Hence, $\left(\underline{J}^{M\cap N}, A\right) = \left(\underline{J}^M, A\right) \cap \left(\underline{J}^N, A\right)$.

(6) Since $M, N \subseteq M \cup N$ so by using part (4), we get $\overline{J}^{M\cup N} \supseteq \overline{J}^M, \overline{J}^N$ which implies $\overline{J}^{M\cup N} \supseteq \overline{J}^M \cup \overline{J}^N$. Hence, $\left(\overline{J}^M, A\right) \cup \left(\overline{J}^N, A\right) \subseteq \left(\overline{J}^{M\cup N}, A\right)$.

Conversely, let $u \in \overline{J}^{M\cup N}(e)$. Then $uJ(e) \cap (M \cup N)(e) \neq \phi \Rightarrow uJ(e) \cap M \neq \phi$ or $uJ(e) \cap N \neq \phi$

$\Rightarrow u \in \overline{J}^M(e)$ or $u \in \overline{J}^N(e) \Rightarrow u \in \left(\overline{J}^M \cup \overline{J}^N\right)(e)$

$\Rightarrow \overline{J}^{M\cup N} \subseteq \overline{J}^M \cup \overline{J}^N \Rightarrow \left(\overline{J}^{M\cup N}, A\right) \subseteq \left(\overline{J}^M, A\right) \cup \left(\overline{J}^N, A\right)$.

Hence, $\left(\overline{J}^M, A\right) \cup \left(\overline{J}^N, A\right) = \left(\overline{J}^{M\cup N}, A\right)$.

(7) Since $M, N \subseteq M \cup N$, so by using part (3), we get $\underline{J}^M$, $\underline{J}^N \subseteq \underline{J}^{M\cup N}$ and so $\underline{J}^M \cup \underline{J}^N \subseteq \underline{J}^{M\cup N}$. Hence, $\left(\underline{J}^M, A\right) \cup \left(\underline{J}^N, A\right) \subseteq \left(\underline{J}^{M\cup N}, A\right)$.

(8) Using part (4) and the fact that $M \cap N \subseteq M, N$, we have $\overline{J}^{M\cap N} \subseteq \overline{J}^M, \overline{J}^N$

$\Rightarrow \overline{J}^{M\cap N} \subseteq \overline{J}^M \cap \overline{J}^N$. Hence, $\left(\overline{J}^M, A\right) \cap \left(\overline{J}^N, A\right) \supseteq \left(\overline{J}^{M\cap N}, A\right)$.

(9) Let $u \in \underline{J}^M(e) \Leftrightarrow uJ(e) \subseteq M \Leftrightarrow uJ(e) \cap M \neq \phi \Leftrightarrow uJ(e) \cap M^c = \phi$

$\Leftrightarrow u \notin \overline{J}^{M^c}(e) \Leftrightarrow u \in \left(\overline{J}^{M^c}(e)\right)^c$. Hence,$\left(\overline{J}^{M^c}, A\right) = \left(\underline{J}^M, A\right)^c$.

(10) By part (9), $\left(\overline{J}^{M^c}, A\right) = \left(\underline{J}^M, A\right)^c$. Therefore, $\left(\underline{J}^{M^c}, A\right)^c = \left(\overline{J}^{(M^c)^c}, A\right)$

$\Rightarrow \left(\underline{J}^{M^c}, A\right)^c = \left(\overline{J}^M, A\right)$. Hence, $\left(\underline{J}^{M^c}, A\right) = \left(\overline{J}^M, A\right)^c$. ■

It is demonstrated by the following example that equality is not valid in (7) and (8) in general.

Example 2.1.22 *Let $U = \{\pi_1, \pi_2, \pi_3, \pi_4\}$ be a set where $E = \{e_1, e_2, e_3, e_4, e_5\}$ and $A = \{e_1, e_2, e_3\}$ be the set of attributes. Define a SE-relation $J : A \to P(U \times U)$ for each $e \in A$. Then $E-$classes for $J(e_1)$ are $\{\pi_1, \pi_4\}$ and $\{\pi_2, \pi_3\}$, for $J(e_2)$ are $\{\pi_1, \pi_2, \pi_4\}$ and $\{\pi_3\}$ and for $J(e_3)$ are $\{\pi_1\}, \{\pi_2, \pi_3\}$ and $\{\pi_4\}$.*

Let $M = \{\pi_1, \pi_3\}$ and $N = \{\pi_1, \pi_2\}$ so $\left(\underline{J}^M, A\right)$ can be represented as

$\underline{J}^M (e_1) = \phi$, $\underline{J}^M (e_2) = \{\pi_3\}$, $\underline{J}^M (e_3) = \{\pi_1\}$.

And $\left(\underline{J}^N, A\right)$ can be represented as $\underline{J}^N (e_1) = \phi$, $\underline{J}^N (e_2) = \phi$, $\underline{J}^N (e_3) = \{\pi_1\}$.

Now, $M \cup N = \{\pi_1, \pi_2, \pi_3\}$ and $\left(\underline{J}^{M\cup N}, A\right)$ can be represented as

$\underline{J}^{M\cup N} (e_1) = \{\pi_2, \pi_3\}$, $\underline{J}^{M\cup N} (e_2) = \{\pi_3\}$, $\underline{J}^{M\cup N} (e_3) = \{\pi_1, \pi_2, \pi_3\}$.

Evidently, $\left(\underline{J}^{M\cup N}, A\right) \neq \left(\underline{J}^M, A\right) \cup \left(\underline{J}^N, A\right)$.

Now, $\overline{J}^M (e_1) = U$, $\overline{J}^M (e_2) = U$, $\overline{J}^M (e_3) = \{\pi_1, \pi_2, \pi_3\}$.

And $\left(\overline{J}^N, A\right)$ can be represented as

$\overline{J}^N (e_1) = U$, $\overline{J}^N (e_1) = \{\pi_1, \pi_2, \pi_4\}$, $\overline{J}^N (e_1) = \{\pi_1, \pi_2, \pi_3\}$.

Now, $M \cap N = \{\pi_1\}$ and $\left(\overline{J}^{M\cap N}, A\right)$ can be represented as

$\overline{J}^{M\cap N} (e_1) = \{\pi_1, \pi_4\}$, $\overline{J}^{M\cap N} (e_2) = \{\pi_1, \pi_2, \pi_4\}$, $\overline{J}^{M\cap N} (e_3) = \{\pi_1\}$.

Evidently, $\left(\overline{J}^{M\cap N}, A\right) \neq \left(\overline{J}^M, A\right) \cap \left(\overline{J}^N, A\right)$.

2.2 Fuzziness associated with SE−relation

In this section, we discuss for any subset M of U, there is an associated FS, λ_i of U for each $e_i \in A$: further we can also find a FS associated with a subset M of U for $IN_d (J, A) = \omega$.

For an SE-relation (J, A) over U, each $J(e_i)$ where $e_i \in A$, gives a partition of U. Hence, a FS, λ_i is defined of U for each $J(e_i)$.

Let $\lambda_i : U \to [0, 1]$ be defined as $\lambda_i (u) = \mid uJ(e_i) \cap M \mid / \mid uJ(e_i) \mid$.

Also, we can find a FS, η of U for $IN_d (J, A) = \omega$. It can be defined in the similar fashion as explained above, that is $\eta : U \to [0, 1]$ defined as $\eta (u) = (\mid u\omega \cap M \mid / \mid u\omega \mid)$.

In the following example, we see that for any subset of the universe U, there is a FS

of U for each parameter. Hence, we have a fuzzy soft set.

Example 2.2.1 11 *books to be arranged from left to right on a shelf. Here* $U = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}\}$ *and* $A = \{e_1, e_2, e_3, e_4, e_5\}$ *is a set of parameters where* e_1 *denotes the binding of books,* e_2 *the publishing years of books,* e_3 *the color of book binding,* e_4 *the category in which Mathematics books are placed,* e_5 *the level of books. We further characterize these parameters as follows:*

- *The binding of books includes hard binding, paper binding, leather binding and without binding.*
- *The publishing year of books includes 2003-2005, 2006-2009, 2010-2014, 2015-2016 and 2017.*
- *The Color of book binding is red, yellow, blue, black and green.*
- *The mathematics books are fluid mechanics, ring theory and discrete mathematics.*
- *The level of book is Intermediate, BS, Masters and M.phil.*

Define an SE-relation $J : A \to P(U \times U)$ *for each* $e \in A$*. The SE-classes for each of the SE-relation is obtained as follows:*

For $J(e_1)$*, the SE-classes* $uJ(e_1)$ *are* $\{b_1, b_{10}\}, \{b_2, b_4, b_6, b_7\}, \{b_3, b_5, b_8, b_9\}, \{b_{11}\}.$

For $J(e_2)$*, the SE-classes* $uJ(e_2)$ *are* $\{b_1\}, \{b_2, b_{11}\}, \{b_4, b_7\}, \{b_3, b_5, b_8, b_9\}, \{b_6, b_{10}\}.$

For $J(e_3)$*, the SE-classes* $uJ(e_3)$ *are* $\{b_1\}, \{b_2\}, \{b_3, b_4, b_5, b_7, b_8, b_9, b_{10}\}, \{b_6\}, \{b_{11}\}.$

For $J(e_4)$*, the SE-classes* $uJ(e_4)$ *are* $\{b_2\}, \{b_3, b_4, b_5, b_7, b_8, b_9, b_{11}\}, \{b_1, b_6, b_{10}\}.$

For $J(e_5)$*, the SE-classes* $uJ(e_5)$ *are* $\{b_{10}\}, \{b_6\}, \{b_1, b_2, b_3, b_4, b_5, b_7, b_8, b_9\}, \{b_{11}\}.$

Further SE-classes determined by $IN_d(J, A) = \cap_{i=1}^{5} J(e_i) = \omega$ *are* $\{b_1\}, \{b_2\}, \{b_6\}, \{b_{10}\}, \{b_{11}\}, \{b_4, b_7\}, \{b_3, b_5, b_8, b_9\}.$

Let $M = \{b_3, b_4, b_5, b_7, b_9\}$*. Then*

$\underline{J}^M(e_1) = \phi$, $\underline{J}^M(e_2) = \{b_4, b_7\}$, $\underline{J}^M(e_3) = \phi$, $\underline{J}^M(e_4) = \phi$, $\underline{J}^M(e_5) = \phi$.

And,

$\overline{J}^M(e_1) = \{b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9\}$, $\overline{J}^M(e_2) = \{b_3, b_4, b_5, b_7, b_8, b_9\}$

$\overline{J}^M(e_3) = \{b_3, b_4, b_5, b_7, b_8, b_9, b_{10}\}$, $\overline{J}^M(e_4) = \{b_3, b_4, b_5, b_7, b_8, b_9, b_{11}\}$

$\overline{J}^M(e_5) = \{b_1, b_2, b_3, b_4, b_5, b_7, b_8, b_9\}$

Now, $\omega^M = \{b_4, b_7\}$ *and* $\overline{\omega}^M = \{b_3, b_4, b_5, b_7, b_8, b_9\}$.

Therefore, for each parameter e_i *where* $i = 1, 2, 3, 4, \ldots$, *fuzzy subsets of* U *for* $M = \{b_3, b_4, b_5, b_7, b_9\}$ *are given below:*

For $J(e_1)$, *the FS,* λ_1 *is*

$\left\{\frac{0}{b_1}, \frac{0.5}{b_2}, \frac{0.7}{b_3}, \frac{0.5}{b_4}, \frac{0.7}{b_5}, \frac{0.5}{b_6}, \frac{0.5}{b_7}, \frac{0.7}{b_8}, \frac{0.7}{b_9}, \frac{0}{b_{10}}, \frac{0}{b_{11}}\right\}$.

For $J(e_2)$, *the FS,* λ_2 *is*

$\left\{\frac{0}{b_1}, \frac{0}{b_2}, \frac{0.7}{b_3}, \frac{1}{b_4}, \frac{0.7}{b_5}, \frac{0}{b_6}, \frac{1}{b_7}, \frac{0.7}{b_8}, \frac{0.7}{b_9}, \frac{0}{b_{10}}, \frac{0}{b_{11}}\right\}$.

For $J(e_3)$, *the FS,* λ_3 *is*

$\left\{\frac{0}{b_1}, \frac{0}{b_2}, \frac{0.7}{b_3}, \frac{0.7}{b_4}, \frac{0.7}{b_5}, \frac{0}{b_6}, \frac{0.7}{b_7}, \frac{0.7}{b_8}, \frac{0.7}{b_9}, \frac{0.7}{b_{10}}, \frac{0}{b_{11}}\right\}$.

For $J(e_4)$, *the FS,* λ_4 *is*

$\left\{\frac{0}{b_1}, \frac{0}{b_2}, \frac{0.7}{b_3}, \frac{0.7}{b_4}, \frac{0.7}{b_5}, \frac{0}{b_6}, \frac{0.7}{b_7}, \frac{0.7}{b_8}, \frac{0.7}{b_9}, \frac{0}{b_{10}}, \frac{0.7}{b_{11}}\right\}$.

For $J(e_5)$, *the FS,* λ_5 *is*

$\left\{\frac{0.6}{b_1}, \frac{0.6}{b_2}, \frac{0.6}{b_3}, \frac{0.6}{b_4}, \frac{0.6}{b_5}, \frac{0}{b_6}, \frac{0.6}{b_7}, \frac{0.6}{b_8}, \frac{0.6}{b_9}, \frac{0}{b_{10}}, \frac{0}{b_{11}}\right\}$.

Further for ω, *FS,* η *is*

$\left\{\frac{0}{b_1}, \frac{0}{b_2}, \frac{0.7}{b_3}, \frac{1}{b_4}, \frac{0.7}{b_5}, \frac{0}{b_6}, \frac{1}{b_7}, \frac{0.7}{b_8}, \frac{0.7}{b_9}, \frac{0}{b_{10}}, \frac{0}{b_{11}}\right\}$.

The arrangement of the books for $J(e_1)$ *is* $b_1 = b_{10} = b_{11} \lneq b_2 = b_4 = b_6 = b_7 \lneq b_3 = b_5 = b_8 = b_9$.

The arrangement of the books for $J(e_2)$ *is* $b_1 = b_2 = b_6 = b_{10} = b_{11} \lneq b_3 = b_5 = b_8 = b_9 \lneq b_4 = b_7$.

The arrangement of the books for $J(e_3)$ *is* $b_1 = b_2 = b_6 = b_{11} \lneq b_3 = b_4 = b_5 = b_7 = b_8 = b_9 = b_{10}$.

The arrangement of the books for $J(e_4)$ *is* $b_1 = b_2 = b_6 = b_{10} \lneq b_3 = b_4 = b_5 = b_7 = b_8 = b_9 = b_{11}$.

The arrangement of the books for $J(e_5)$ *is* $b_6 = b_{10} = b_{11} \lneq b_1 = b_2 = b_3 = b_4 = b_5 = b_7 = b_8 = b_9$.

The arrangement of the books for ω *is* $b_1 = b_2 = b_6 = b_{10} = b_{11} \lneq b_3 = b_5 = b_8 = b_9 \lneq b_4 = b_7$.

Hence, the books b_3, b_5, b_8 and b_9 will not be placed adjacent to each other in the start.

Moreover, if $M = U$, then $\lambda_i(u) = 1$ and if $M = \phi$, then $\lambda_i(u) = 0$ for all $u \in U$.

2.3 Similarity relations associated with soft binary relations

In this section, some soft binary relations between two crisp sets are defined based on their rough approximations and their properties are investigated.

Definition 2.3.1 *Let (U, J) be a Generalized soft approximation space. Define the following binary relations on $P(W)$, for $N_1, N_2 \subseteq W$.*

$N_1 \backsimeq N_2$ *if and only if* $\underline{J}^{N_1} = \underline{J}^{N_2}$

$N_1 \eqsim N_2$ *if and only if* $\overline{J}^{N_1} = \overline{J}^{N_2}$

$N_1 \approx N_2$ *if and only if* $\underline{J}^{N_1} = \underline{J}^{N_2}$ *and* $\overline{J}^{N_1} = \overline{J}^{N_2}$.

These binary relations are named as the lower similarity relation , upper similarity relation and similarity relation, respectively. Obviously, $\underline{J}^{N}$ and $\overline{J}^{N}$ are similar if and only if they are both lower and upper similar.

Proposition 2.3.2 *The relations $\backsimeq$, $\eqsim$ and $\approx$ are E-relations.*

Proof. It is direct. ■

Theorem 2.3.3 *Let (U, J) be a Generalized soft approximation space and $J : A \rightarrow P(U \times U)$ be a soft reflexive binary relation on U. For $N_i \subseteq U$ for $i = 1, 2, 3, 4$, the following assertions hold:*

(1) $N_1 \eqsim N_2$ *if and only if* $N_1 \eqsim (N_1 \cup N_2) \eqsim N_2$

(2) $N_1 \eqsim N_2$ *and* $N_3 \eqsim N_4$ *imply that* $(N_1 \cup N_3) \eqsim (N_2 \cup N_4)$

(3) $N_1 \subseteq N_2$ *and* $N_2 \eqsim \phi$ *imply that* $N_1 \eqsim \phi$

(4) $(N_1 \cup N_2) \eqsim \phi$ *if and only if* $N_1 \eqsim \phi$ *and* $N_2 \eqsim \phi$

Proof. (1) Let $N_1 \eqsim N_2$. Then $\overline{J}^{N_1} = \overline{J}^{N_2}$. By Theorem 2.1.4(6), we get $\overline{J}^{N_1 \cup N_2} = \overline{J}^{N_1} \cup \overline{J}^{N_2} = \overline{J}^{N_1} = \overline{J}^{N_2}$ so $N_1 \eqsim (N_1 \cup N_2) \eqsim N_2$. Converse holds by transitivity of the relation $\eqsim$.

(2) Given that $N_1 \eqsim N_2$ and $N_3 \eqsim N_4$. Then $\overline{J}^{N_1} = \overline{J}^{N_2}$ and $\overline{J}^{N_3} = \overline{J}^{N_4}$.

By Theorem 2.1.4(6), we get $\overline{J}^{N_1 \cup N_3} = \overline{J}^{N_1} \cup \overline{J}^{N_3} = \overline{J}^{N_2} \cup \overline{J}^{N_4} = \overline{J}^{N_2 \cup N_4}$. Thus, $(N_1 \cup N_3) \eqsim (N_2 \cup N_4)$.

(3) Given $N_2 \eqsim \phi$. This implies $\overline{J}^{N_2} = \overline{J}^{\phi}$.

Also, $N_1 \subseteq N_2 \Rightarrow \overline{J}^{N_1} \subseteq \overline{J}^{N_2} = \overline{J}^{\phi}$. It follows that $\overline{J}^{N_1} \subseteq \overline{J}^{\phi}$ but $\overline{J}^{\phi} \subseteq \overline{J}^{N_1}$. Therefore, $\overline{J}^{N_1} = \overline{J}^{\phi} \Rightarrow N_1 \eqsim \phi$.

(4) Let $N_1 \eqsim \phi$ and $N_2 \eqsim \phi$. Then $\overline{J}^{N_1} = \overline{J}^{\phi}$ and $\overline{J}^{N_2} = \overline{J}^{\phi}$. By Theorem 2.1.4(6), we get $\overline{J}^{N_1 \cup N_2} = \overline{J}^{N_1} \cup \overline{J}^{N_2} = \overline{J}^{\phi} \cup \overline{J}^{\phi} = \overline{J}^{\phi}$.

Thus, $(N_1 \cup N_2) \eqsim \phi$. Converse follows from (3) . ■

Theorem 2.3.4 *Let (U, J) be a Generalized soft approximation space and $J : A \rightarrow P(U \times U)$ be a soft reflexive relation on U. For $N_i \subseteq U$ for $i = 1, 2, 3, 4$, the assertions given below hold:*

(1) $N_1 \backsimeq N_2$ *if and only if* $N_1 \backsimeq (N_1 \cap N_2) \backsimeq N_2$

(2) $N_1 \backsimeq N_2$ *and* $N_3 \backsimeq N_4$ *imply that* $(N_1 \cap N_3) \backsimeq (N_2 \cap N_4)$

(3) $N_1 \subseteq N_2$ *and* $N_2 \backsimeq \phi$ *imply that* $N_1 \backsimeq \phi$

(4) $(N_1 \cup N_2) \backsimeq \phi$ *if and only if* $N_1 \backsimeq \phi$ *and* $N_2 \backsimeq \phi$

Proof. The verification is like the evidence of Theorem 2.3.3. ■

Theorem 2.3.5 *Let (U, J) be a Generalized soft approximation space and $J : A \rightarrow P(U \times U)$ be a soft reflexive relation on U. For $N_i \subseteq U$ for $i = 1, 2, 3, 4$, the properties are valid:*

(1) $N_1 \approx N_2$ *if and only if* $N_1 \eqsim (N_1 \cup N_2) \eqsim N_2$ *and* $N_1 \backsimeq (N_1 \cap N_2) \backsimeq N_2$.

(2) $N_1 \backsimeq N_2$ *and* $N_3 \backsimeq N_4$ *imply that* $(N_1 \cap N_3) \cup (N_2 \cap N_4)$

(3) $N_1 \subseteq N_2$ *and* $N_2 \approx \phi$ *imply that* $N_1 \approx \phi$

(4) $(N_1 \cup N_2) \approx \phi$ *if and only if* $N_1 \approx \phi$ *and* $N_2 \approx \phi$

Proof. It is an immediate consequence of Theorem 2.3.3 and 2.3.4. ■

2.4 Parametric reduction

The attributes reduction is designed to keep the classification ability of conditional parameters relative to the decision parameters in rough set theory. The decision values are computed by conditional parameters and the parametric reduction is designed to offer minimal subset of the conditional parameters set to keep the optimal choice objects. Parametric reduction in soft and rough set theory play a vital role and it saves time and expenses on several tests.

Moreover, in soft sets reduction of condition parameters is a meaningful concept [21]. The condition parametric reduction is possible while it does not disturb the ability of classification of decision parameters (in original data). Intention of study is to classify ability of $\sum(\pi_i, \pi_j)$ instead of its values. And classification attitude of decision parameter will be viewed by exiting some condition parameters from decision table. For this purpose, the example presented by Maji et al. in [65] and Chen et al in [21], is analyzed here. Here parameters of an $SBRE$ are reduced one by one in order to minimize the data and handle information easily.

Example 2.4.1 *Considering* $U = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6\}$*, a collection of six houses where* $E = \left\{ \begin{array}{c} \text{expensive, beautiful, wooden, cheap, in green surroundings,} \\ \text{modern, in good repair, in bad repair} \end{array} \right\}$ *is a parametric set. Suppose Mr. X wants to purchase a house on the following parametric set* $P = \{\text{beautiful, wooden, cheap, in green surroundings, in good repair}\}$*. Symbolically,* $P = \{e_1, e_2, e_3, e_4, e_5\}$*. Let* $A = \{e_1, e_2, e_3, e_4, e_5, d\}$, $C = \{e_1, e_2, e_3, e_4, e_5\}$ *and* $D = \{d\}$ *where* C *and* D *are the sets of condition parameters and decision parameters, respectively.*

Define a SE*-relation* $J : A \rightarrow P(U \times U)$ *for each* $e \in A$*.*

The SE*-classes are obtained for each of the* SE*-relation as follows:*

For $J(e_1)$*, the* SE*-classes* $uJ(e_1)$ *are* $\{\pi_1, \pi_3\}, \{\pi_2, \pi_4, \pi_5, \pi_6\}$.

For $J(e_2)$*, the* SE*-classes* $uJ(e_2)$ *are* $\{\pi_1, \pi_3, \pi_6\}, \{\pi_2, \pi_4, \pi_5\}$.

For $J(e_3)$*, the* SE*-classes* $uJ(e_3)$ *are* $\{\pi_1, \pi_2, \pi_4, \pi_5\}, \{\pi_3\}, \{\pi_6\}$.

For $J(e_4)$, *the SE-classes* $uJ(e_4)$ *are* $\{\pi_1, \pi_2, \pi_4, \pi_5, \pi_6\}, \{\pi_3\}$.

For $J(e_5)$, *the SE-classes* $uJ(e_5)$ *is* $\{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6\}$.

For decision-making, a decision parameter d is adjoined with table 1*, where* $\sum_{k=1}^{5}(\pi_i, \pi_j)_k$ *is the value of* (π_i, π_j) *pair corresponding to* $J(e_k)$*. Those objects are placed adjacent to each other which have the same values* $\sum(\pi_i, \pi_j)$ *so we reset the table and elements which have the same value, are put side by side to each other and elements of different values are separated so we have Table* 3 *which is called original classification table.*

Our task is to find without distorting the classification ability of the parameter d, it is hoped to find the minimum number of condition parameters for decision making without distorting the ability of classification of parameter d.

If we kill e_5 *from Table* 3*, we get Table* 4*. It is obvious that deletion of* e_1 *affects the classification of d different from that in Table* 3*. Therefore* e_1 *is a core parameter. If we delete* e_2 *from Table* 3*, we get another core parameter. If we proceed in the same way, we find a set of core parameters* $\{e_1, e_2, e_3, e_4\}$*. But elimination of* e_5 *does not affect the classification capacity of decision parameter d, so* e_5 *is dispensible in Table* 4 *and it is the condition parameter. Hence, Table* 4 *gives the same classification with least condition parameters.*

Here, Table 1 *represents SE−relation representation of houses under consideration*

Table 2 *represents SE−relation after adjoining decision parameter d*

Table 3 *represents SE-relation after the rearrangement of the decision parameter d*

Table 4 *represents SE−relation after eliminating the condition parameter* e_5.

Table 1

	e_1	e_2	e_3	e_4	e_5
(π_1,π_1)	1	1	1	1	1
(π_1,π_2)	0	0	1	1	1
(π_1,π_3)	1	1	0	0	1
(π_1,π_4)	0	0	1	1	1
(π_1,π_5)	0	0	1	1	1
(π_1,π_6)	0	1	0	1	1
(π_2,π_1)	0	0	1	1	1
(π_2,π_2)	1	1	1	1	1
(π_2,π_3)	0	0	0	0	1
(π_2,π_4)	1	1	1	1	1
(π_2,π_5)	1	1	1	1	1
(π_2,π_6)	1	0	0	1	1
(π_3,π_1)	1	1	0	0	1
(π_3,π_2)	0	0	0	0	1
(π_3,π_3)	1	1	1	1	1
(π_3,π_4)	0	0	0	0	1
(π_3,π_5)	0	0	0	0	1
(π_3,π_6)	0	1	0	0	1
(π_4,π_1)	0	0	1	1	1
(π_4,π_2)	1	1	1	1	1
(π_4,π_3)	0	0	0	0	1
(π_4,π_4)	1	1	1	1	1
(π_4,π_5)	1	1	1	1	1
(π_4,π_6)	1	0	0	1	1
(π_5,π_1)	0	0	1	1	1
(π_5,π_2)	1	1	1	1	1
(π_5,π_3)	0	0	0	0	1
(π_5,π_4)	1	1	1	1	1
(π_5,π_5)	1	1	1	1	1
(π_5,π_6)	1	0	0	1	1
(π_6,π_1)	0	1	0	1	1
(π_6,π_2)	1	0	0	1	1
(π_6,π_3)	0	1	0	0	1
(π_6,π_4)	1	0	0	1	1
(π_6,π_5)	1	0	0	1	1
(π_6,π_6)	1	1	1	1	1

Table 2

	e_1	e_2	e_3	e_4	e_5	d
(π_1,π_1)	1	1	1	1	1	5
(π_1,π_2)	0	0	1	1	1	3
(π_1,π_3)	1	1	0	0	1	3
(π_1,π_4)	0	0	1	1	1	3
(π_1,π_5)	0	0	1	1	1	3
(π_1,π_6)	0	1	0	1	1	3
(π_2,π_1)	0	0	1	1	1	3
(π_2,π_2)	1	1	1	1	1	5
(π_2,π_3)	0	0	0	0	1	1
(π_2,π_4)	1	1	1	1	1	5
(π_2,π_5)	1	1	1	1	1	5
(π_2,π_6)	1	0	0	1	1	3
(π_3,π_1)	1	1	0	0	1	3
(π_3,π_2)	0	0	0	0	1	1
(π_3,π_3)	1	1	1	1	1	5
(π_3,π_4)	0	0	0	0	1	1
(π_3,π_5)	0	0	0	0	1	1
(π_3,π_6)	0	1	0	0	1	2
(π_4,π_1)	0	0	1	1	1	3
(π_4,π_2)	1	1	1	1	1	5
(π_4,π_3)	0	0	0	0	1	1
(π_4,π_4)	1	1	1	1	1	5
(π_4,π_5)	1	1	1	1	1	5
(π_4,π_6)	1	0	0	1	1	3
(π_5,π_1)	0	0	1	1	1	3
(π_5,π_2)	1	1	1	1	1	5
(π_5,π_3)	0	0	0	0	1	1
(π_5,π_4)	1	1	1	1	1	5
(π_5,π_5)	1	1	1	1	1	5
(π_5,π_6)	1	0	0	1	1	3
(π_6,π_1)	0	1	0	1	1	3
(π_6,π_2)	1	0	0	1	1	3
(π_6,π_3)	0	1	0	0	1	2
(π_6,π_4)	1	0	0	1	1	3
(π_6,π_5)	1	0	0	1	1	3
(π_6,π_6)	1	1	1	1	1	5

Table 3

	e_1	e_2	e_3	e_4	e_5	d
(π_1,π_1)	1	1	1	1	1	5
(π_2,π_2)	1	1	1	1	1	5
(π_2,π_4)	1	1	1	1	1	5
(π_2,π_5)	1	1	1	1	1	5
(π_3,π_3)	1	1	1	1	1	5
(π_4,π_2)	1	1	1	1	1	5
(π_4,π_4)	1	1	1	1	1	5
(π_4,π_5)	1	1	1	1	1	5
(π_5,π_2)	1	1	1	1	1	5
(π_5,π_4)	1	1	1	1	1	5
(π_5,π_5)	1	1	1	1	1	5
(π_6,π_6)	1	1	1	1	1	5
(π_1,π_2)	0	0	1	1	1	3
(π_1,π_3)	1	1	0	0	1	3
(π_1,π_4)	0	0	1	1	1	3
(π_1,π_5)	0	0	1	1	1	3
(π_1,π_6)	0	1	0	1	1	3
(π_2,π_1)	0	0	1	1	1	3
(π_2,π_6)	1	0	0	1	1	3
(π_3,π_1)	1	1	0	0	1	3
(π_4,π_1)	0	0	1	1	1	3
(π_4,π_6)	1	0	0	1	1	3
(π_5,π_1)	0	0	1	1	1	3
(π_5,π_6)	1	0	0	1	1	3
(π_6,π_1)	0	1	0	1	1	3
(π_6,π_2)	1	0	0	1	1	3
(π_6,π_4)	1	0	0	1	1	3
(π_6,π_5)	1	0	0	1	1	3
(π_3,π_6)	0	1	0	0	1	2
(π_6,π_3)	0	1	0	0	1	2
(π_2,π_3)	0	0	0	0	1	1
(π_3,π_2)	0	0	0	0	1	1
(π_3,π_4)	0	0	0	0	1	1
(π_3,π_5)	0	0	0	0	1	1
(π_4,π_3)	0	0	0	0	1	1
(π_5,π_3)	0	0	0	0	1	1

Table 4

	e_1	e_2	e_3	e_4	d_{e_5}
(π_1,π_1)	1	1	1	1	4
(π_2,π_2)	1	1	1	1	4
(π_2,π_4)	1	1	1	1	4
(π_2,π_5)	1	1	1	1	4
(π_3,π_3)	1	1	1	1	4
(π_4,π_2)	1	1	1	1	4
(π_4,π_4)	1	1	1	1	4
(π_4,π_5)	1	1	1	1	4
(π_5,π_2)	1	1	1	1	4
(π_5,π_4)	1	1	1	1	4
(π_5,π_5)	1	1	1	1	4
(π_6,π_6)	1	1	1	1	4
(π_1,π_2)	0	0	1	1	2
(π_1,π_3)	1	1	0	0	2
(π_1,π_4)	0	0	1	1	2
(π_1,π_5)	0	0	1	1	2
(π_1,π_6)	0	1	0	1	2
(π_2,π_1)	0	0	1	1	2
(π_2,π_6)	1	0	0	1	2
(π_3,π_1)	1	1	0	0	2
(π_4,π_1)	0	0	1	1	2
(π_4,π_6)	1	0	0	1	2
(π_5,π_1)	0	0	1	1	2
(π_5,π_6)	1	0	0	1	2
(π_6,π_1)	0	1	0	1	2
(π_6,π_2)	1	0	0	1	2
(π_6,π_4)	1	0	0	1	2
(π_6,π_5)	1	0	0	1	2
(π_3,π_6)	0	1	0	0	1
(π_6,π_3)	0	1	0	0	1
(π_2,π_3)	0	0	0	0	0
(π_3,π_2)	0	0	0	0	0
(π_3,π_4)	0	0	0	0	0
(π_3,π_5)	0	0	0	0	0
(π_4,π_3)	0	0	0	0	0
(π_5,π_3)	0	0	0	0	0

Therefore, an algorithm for the selection of a house is provided with least condition parameters.

An algorithm to select a house:

$Mr.$ X selected the house according to this algorithm listed as follows:

(1) Input the SE-relation (J, E).

(2) Input the condition parameter C which is a subset of E.

(3) At last column of the table, input decision parameter $d = \sum (\pi_i, \pi_j)$ achieved by condition parameters in the table.

(4) Input is re-settled by putting the objects having the same value side by side to each other for parameter d.

(5) Differentiate those objects which have different values of d.

(6) Identify the core parameters.

(7) By eliminating all the dispensible parameters one by one, an output is obtained which gives a table with minimum number of condition parameters which has the same ability of classification of d as in original table with d.

Now considering the example presented by authors in [24]. All the parameters are only condition parameters here. For decision making, one or more decision parameters are required. The decision parameter should be $\sum_{k=1}^{7} (\pi_i, \pi_j)_k$, as mentioned in [24].

Example 2.4.2 *Let* $A = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, d\}$, $C = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ *and* $D = \{d\}$ *where* C *and* D *be the set of condition parameters and decision parameters, respectively.*

Define an SE*-relation* $J : A \to P(U \times U)$ *for each* $e \in A$.

The following SE*-classes are obtained for each of the* SE*-relation.*

For $J(e_1)$, *the* SE*-classes* $uJ(e_1)$ *are* $\{\pi_1, \pi_3\}, \{\pi_2, \pi_4, \pi_5, \pi_6\}$.

For $J(e_2)$, *the* SE*-classes* $uJ(e_2)$ *are* $\{\pi_1, \pi_3, \pi_6\}, \{\pi_2, \pi_4, \pi_5\}$.

For $J(e_3)$, *the* SE*-classes* $uJ(e_3)$ *are* $\{\pi_1, \pi_2, \pi_4, \pi_5\}, \{\pi_3\}, \{\pi_6\}$.

For $J(e_4)$, *the* SE*-classes* $uJ(e_4)$ *are* $\{\pi_1, \pi_2, \pi_4, \pi_5, \pi_6\}, \{\pi_3\}$.

For $J(e_5)$, *the* SE*-classes* $uJ(e_5)$ *are* $\{\pi_1\}$ *and* $\{\pi_2, \pi_3, \pi_4, \pi_5, \pi_6\}$.

For $J(e_6)$, *the SE−class* $uJ(e_6)$ *is* $\{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6\}$.

For $J(e_7)$, *the SE-classes* $uJ(e_7)$ *are* $\{\pi_1\}$ *and* $\{\pi_2, \pi_4, \pi_5\}, \{\pi_3\}, \{\pi_6\}$.

Repeating the same procedure as in previous example, if we kill any of $e_1, e_2, e_3, e_4, e_5, e_7$, *then classification pattern of d changes. So* e_6 *is dispensible parameter here. Therefore elimination of* e_6 *does not affect the classification pattern of d.*

Selection of a car:

The selection of a suitable car to buy is not an easy task. Suppose a person $Mr.$ X wants to select a car from the alternatives $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}$. Let $U = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}\}$ be the universe of ten different cars and let $A = \{e_1, e_2, e_3\} \subseteq E$ be the set of attributes, where e_1 stands for Price, e_2 stands for color and e_3 stands for car brands.

We further characterize these parameters as follows:

• The Price of a car includes under 30 lacs, between $31-35$ lacs and between $36-40$ lacs.

• The car brand includes Honda Accord, Audi, Mercedes benz and bmw.

•The Color of a car includes black, white and silver.

Define an SE-relation $J : A \rightarrow P(U \times U)$ for each $e \in A$ which explains the requirement of the car which a person $Mr.$ X is going to buy. The following SE-classes for each of the SE-relation are obtained as follows:

For $J(e_1)$, the SE-classes $uJ(e_1)$ are $\{\gamma_1, \gamma_{10}\}, \{\gamma_2, \gamma_4, \gamma_6, \gamma_7\}, \{\gamma_3, \gamma_5, \gamma_8, \gamma_9\}$ which means that price of cars γ_1 and γ_{10} is under 30 lacs, price of cars $\gamma_2, \gamma_4, \gamma_6$ and γ_7 is between $31-35$ lacs and price of cars $\gamma_3, \gamma_5, \gamma_8$ and γ_9 is between $36-40$ lacs.

For $J(e_2)$, the SE-classes $uJ(e_2)$ are $\{\gamma_1\}, \{\gamma_2\}, \{\gamma_3, \gamma_4, \gamma_5, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}\}, \{\gamma_6\}$ which represents that car brand of the car γ_1 is Honda Accord, car brand of the car γ_2 is Audi, car brand of the cars $\gamma_3, \gamma_4, \gamma_5, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}$ is Mercedes benz and car brand of the car γ_6 is bmw.

For $J(e_3)$, the SE-classes $uJ(e_3)$ are $\{\gamma_{10}\}, \{\gamma_6\}, \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_7, \gamma_8, \gamma_9\}$ which represents the color of the cars $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_7, \gamma_8, \gamma_9$ is black, the color of car j_{10} is white and the color of car j_6 is silver.

Further SE-classes determined by $IN_d(J, A) = \cap_{i=1}^{3} J(e_i) = \omega$ are

$$\{\gamma_1\},\ \{\gamma_2\},\ \{\gamma_6\},\ \{\gamma_{10}\},\ \{\gamma_4, \gamma_7\},\ \{\gamma_3, \gamma_5, \gamma_8, \gamma_9\}.$$

Let $M = \{\gamma_1, \gamma_2, \gamma_{10}\}$ be a subset of U consisting of those cars which are most favorite for $Mr.\ X$, then $\underline{J}^M(e_1) = \{\gamma_1, \gamma_{10}\}$, $\underline{J}^M(e_2) = \{\gamma_2\}$, $\underline{J}^M(e_3) = \{\gamma_{10}\}$. And, $\overline{J}^M(e_1) = \{\gamma_1, \gamma_2, \gamma_4, \gamma_6, \gamma_7, \gamma_{10}\}$, $\overline{J}^M(e_2) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}\}$, $\overline{J}^M(e_3) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}\}$.

Now, $\underline{J}^M = \{\gamma_1, \gamma_2, \gamma_{10}\}$ and $\overline{J}^M = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}\}$. Therefore, for each parameter e_i where $i = 1, 2, 3, 4, ...,$ FS of U for $M = \{\gamma_1, \gamma_2, \gamma_{10}\}$ are given below:

For $J(e_1)$, FS, λ_1 is $\left\{\frac{1}{\gamma_1}, \frac{0.25}{\gamma_2}, \frac{0}{\gamma_3}, \frac{0.25}{\gamma_4}, \frac{0}{\gamma_5}, \frac{0.25}{\gamma_6}, \frac{0.25}{\gamma_7}, \frac{0}{\gamma_8}, \frac{0}{\gamma_9}, \frac{1}{\gamma_{10}}\right\}$.

For $J(e_2)$, FS, λ_2 is $\left\{\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{0.1}{\gamma_3}, \frac{0.1}{\gamma_4}, \frac{0.1}{\gamma_5}, \frac{0}{\gamma_6}, \frac{0.1}{\gamma_7}, \frac{0.1}{\gamma_8}, \frac{0.1}{\gamma_9}, \frac{0.1}{\gamma_{10}}\right\}$.

For $J(e_3)$, FS, λ_3 is $\left\{\frac{0.25}{\gamma_1}, \frac{0.25}{\gamma_2}, \frac{0.25}{\gamma_3}, \frac{0.25}{\gamma_4}, \frac{0.25}{\gamma_5}, \frac{0}{\gamma_6}, \frac{0.6}{\gamma_7}, \frac{0.25}{\gamma_8}, \frac{0.25}{\gamma_9}, \frac{1}{\gamma_{10}}\right\}$.

Further for ω, FS η is $\left\{\frac{1}{\gamma_1}, \frac{1}{\gamma_2}, \frac{0}{\gamma_3}, \frac{0}{\gamma_4}, \frac{0}{\gamma_5}, \frac{0}{\gamma_6}, \frac{0}{\gamma_7}, \frac{0}{\gamma_8}, \frac{0}{\gamma_9}, \frac{1}{\gamma_{10}}\right\}$.

It is obvious that $Mr.\ X$ will select the car γ_1 which is under 30 lacs, brand is Honda Accord and white in color.

Remark 2.4.3 *The parametric reduction method which is introduced here is close to the parametric reduction in rough sets. Here the parameters are ordered one by one rather than a subset of parameters all in all. Additionally, speciality of this method over other methods is that we reduce the parameters of an $SBRE$ one by one rather than the parameters of a soft set. If there should arise an occurrence of vast information, it is proficiently utilized.*

Chapter 3

Approximation of ideals in semigroups by soft relations

In this chapter, aftersets and foresets are utilized to approximate a set. This results in the formation of two sets which are soft sets. We call them the lower approximation and the upper approximation in the sense of aftersets and the foresets. The behaviour of these concepts are applied to semigroups. Further, approximations of substructures of semigroups are studied. For better understanding, examples are addded to explain the concepts in the chapter. Homomorphic images and their relations under semigroup homomorphism are studied in the last.

3.1 Approximation by soft relations

Soft relations from one semigroup to another semigroup are applied in this section. This is done to approximate a set in two different ways. We take a subset of S_2, the resulting approximation is the subset of S_1 with respect to afterset. On the other hand if a subset of S_1 is taken then the resulting approximation is the subset of S_2. In the last of this section, we used soft compatible relation to approximate the subsets of two different semigroups and proved some results.

If we take $S_1 = S_2 = S$ in Definition 2.1.2 and $Y \subseteq S$, then $\left(\underline{J}^Y, A\right)$, $\left(\overline{J}^Y, A\right)$, $\left({}^Y\underline{J}, A\right)$

and $\left({}^{Y}\overline{J}, A\right)$ are soft sets over S defined as

$$\begin{aligned}\underline{J}^{Y}(e) &= \{s \in S : sJ(e) \subseteq Y\} \\ \overline{J}^{Y}(e) &= \{s \in S : sJ(e) \cap Y \neq \phi\} \text{ and}\end{aligned}$$

$$\begin{aligned}{}^{Y}\underline{J}(e) &= \{s \in S : J(e)\, s \subseteq Y\} \\ {}^{Y}\overline{J}(e) &= \{s \in S : J(e)\, s \cap Y \neq \phi\}\end{aligned}$$

for all $e \in A$, where $sJ(e) = \{t \in S : (s,t) \in J(e)\}$ where $sJ(e)$ is called the afterset of s and $J(e)\, s = \{t \in S : (t,s) \in J(e)\}$ and is called the foreset of s for each $e \in A$.

Generally $sJ(e) \neq J(e)\, s$ and so $\underline{J}^{Y}(e) \neq {}^{Y}\underline{J}(e)$ and $\overline{J}^{Y}(e) \neq {}^{Y}\overline{J}(e)$. However, if $J(e)$ is a symmetric relation, then they are equal.

Example 3.1.1 *Let $S = \{1,2,3\}$ be a non-empty set and $A = \{e_1, e_2\}$. Define $J : A \to P(S \times S)$ by $J(e_1) = \{(1,1), (2,2), (3,3), (1,2)\}$ and*

$J(e_2) = \{(1,1), (2,2), (3,3), (2,3)\}$. Thus $1J(e_1) = \{1,2\}$, $2J(e_1) = \{2\}$, $3J(e_1) = \{3\}$, $1J(e_2) = \{1\}$, $2J(e_2) = \{2,3\}$, $3J(e_2) = \{3\}$ and $J(e_1)1 = \{1\}$, $J(e_1)2 = \{1,2\}$, $J(e_1)3 = \{3\}$, $J(e_2)1 = \{1\}$, $J(e_2)2 = \{2\}$, $J(e_2)3 = \{2,3\}$. Now if we take $Y = \{1,3\}$ then $\underline{J}^{Y}(e_1) = \{3\}$, $\underline{J}^{Y}(e_2) = \{1,3\}$, $\overline{J}^{Y}(e_1) = \{1,3\}$, $\overline{J}^{Y}(e_2) = \{1,2,3\}$ and ${}^{Y}\underline{J}(e_1) = \{1,3\}$, ${}^{Y}\underline{J}(e_2) = \{1\}$, ${}^{Y}\overline{J}(e_1) = \{1,3\}$, ${}^{Y}\overline{J}(e_2) = \{1,3\}$. This shows that $\left(\underline{J}^{Y}, A\right) \neq \left({}^{Y}\underline{J}, A\right)$ and $\left(\overline{J}^{Y}, A\right) \neq \left({}^{Y}\overline{J}, A\right)$.

Definition 3.1.2 *An SBRE, (J, A) from a semigroup S_1 to a semigroup S_2 is called soft compatible if (a,b), $(c,d) \in J(e) \Rightarrow (ac, bd) \in J(e)$ for all $a, c \in S_1$ and $b, d \in S_2$ and $e \in A$.*

Example 3.1.3 *Let $S_1 = \{1,2,3\}$ and $S_2 = \{a,b,c\}$ be two semigroups. The operations on both the semigroups are shown in the tables below:*

$\bullet$	1	2	3
1	1	2	3
2	1	2	3
3	1	2	3

$\bullet$	a	b	c
a	a	a	c
b	a	b	c
c	a	c	c

and $A = \{e_1, e_2\}$. Define $J : A \to P(S_1 \times S_2)$ by

$$J(e_1) = \{(1,a),(2,b),(3,c),(1,b),(2,a),(1,c)\}$$

and

$$J(e_2) = \{(1,a),(2,b),(3,c),(2,c),(2,a)\}.$$

Then (J,A) is a soft compatible relation.

Now, in the next the shortend-form of a soft compatible relation will be $SCRE$ throughout the thesis.

Example 3.1.4 *Let $S = \{1,2,3\}$ be a semigroup. Then the operation on S in the table is as follows:*

•	1	2	3
1	1	2	3
2	1	2	3
3	1	2	3

and $A = \{e_1, e_2\}$. Define $J : A \to P(S \times S)$ by

$J(e_1) = \{(1,1),(2,2),(3,3),(1,2)\}$ and $J(e_2) = \{(1,1),(2,2),(3,3),(2,3)\}$. Then (J,A) is an $SCRE$ and soft reflexive relation on S.

If (J,A) is an $SCRE$ from S_1 to S_2, even then $aJ(e).bJ(e) \subseteq (ab)J(e)$ for $a,b \in S_1$, indeed if $x \in aJ(e)$ and $y \in bJ(e)$, then $(a,x) \in J(e)$ and $(b,y) \in J(e)$. By compatibility of (J,A), $(ab,xy) \in J(e)$ that is $xy \in (ab)J(e)$ for all $x,y \in S_2$. Similarly, $J(e)x.J(e)y \subseteq J(e)(xy)$ for $x,y \in S_2$.

The following examples shows that in general $aJ(e).bJ(e) \neq (ab)J(e)$ and $J(e)x.J(e)y \neq J(e)(xy)$.

Example 3.1.5 *Let $S_1 = \{a,b,c,d\}$ and $S_2 = \{1,2,3,4\}$ be two semigroups. The operations on both the semigroups are shown in the tables below*

•	a	b	c	d
a	a	a	a	d
b	a	b	a	d
c	a	a	c	d
d	d	d	d	d

•	1	2	3	4
1	1	2	3	4
2	2	2	2	2
3	3	3	3	3
4	4	3	2	1

and $A = \{e_1, e_2\}$. *Define* $J : A \to P(S_1 \times S_2)$ *by*

$$J(e_1) = \left\{ \begin{array}{c} (a,1), (b,2), (c,3), (d,4), (a,2), (a,3), \\ (b,4), (a,4), (b,1), (d,1), (d,2), (d,3) \end{array} \right\}$$

and

$$J(e_2) = \{(a,2), (a,3), (b,2), (b,3), (c,2), (c,3), (d,2), (d,3)\}.$$

Then J *is an SCRE from the semigroup* S_1 *to the semigroup* S_2.

$$aJ(e_1) = \{1,2,3,4\}, \ bJ(e_1) = \{1,2,4\}, \ cJ(e_1) = \{3\} \ \textit{and} \ dJ(e_1) = \{1,2,3,4\}.$$

Also,

$$aJ(e_2) = \{2,3\}, \ bJ(e_2) = \{2,3\}, \ cJ(e_2) = \{2,3\} \ \textit{and} \ dJ(e_2) = \{2,3\}.$$

But

$$aJ(e_1).cJ(e_1) = \{2,3\} \neq \{1,2,3,4\} = (ac)\, J(e_1).$$

Now,

$$J(e_1)1 = \{a,b,d\}, \ J(e_1)2 = \{a,b,d\}, \ J(e_1)3 = \{a,c,d\} \ \textit{and} \ J(e_1)4 = \{a,b,d\}.$$

Also,

$$\begin{aligned} J(e_2)1 &= \phi, \ J(e_2)2 = \{a,b,c,d\}, \\ J(e_2)3 &= \{a,b,c,d\} \ \textit{and} \ J(e_2)4 = \phi. \end{aligned}$$

But

$$J(e_1)3.J(e_1)1 = \phi \neq \{a,b,c,d\} = J(e_1)\,(31).$$

In general, if (J, A) is an $SCRE$ on a semigroup S, then $aJ(e).bJ(e) \subseteq (ab)\,J(e)$, indeed if $x \in aJ(e)$ and $y \in bJ(e)$ then $(a,x) \in J(e)$ and $(b,y) \in J(e)$. By compatibility of (J, A), $(ab, xy) \in J(e)$ that is $xy \in (ab)\,J(e)$. Similarly $J(e)\,a.J(e)\,b \subseteq J(e)\,(ab)$. The following example shows that in general $aJ(e).bJ(e) \neq (ab)\,J(e)$ and $J(e)\,a.J(e)\,b \neq J(e)\,(ab)$.

Example 3.1.6 *Let* $S = \{a, b, c\}$ *be a semigroup with the multiplication table as below:*

$\bullet$	a	b	c
a	a	a	c
b	a	b	c
c	a	c	c

and $A = \{e_1, e_2\}$. *Define* $J : A \to P(S \times S)$ *by*

$$J(e_1) = \left\{ \begin{array}{c} (a,a), (b,b), (c,c), (a,b), \\ (a,c), (b,a), (c,a) \end{array} \right\}$$

and $J(e_2) = \{(a,a), (b,b), (c,c)\}$. *Then* J *is a soft reflexive and SCRE on* S.

$aJ(e_1) = \{a, b, c\}, bJ(e_1) = \{a, b\}$ *and* $cJ(e_1) = \{a, c\}$. *Also,* $aJ(e_2) = \{a\}$, $bJ(e_2) = \{b\}$ *and* $cJ(e_2) = \{c\}$. *But* $cJ(e_1)aJ(e_1) = \{a, c\} \neq \{a, b, c\} = (ca)\, J(e)$. *On the other hand,* $J(e_1)a = \{a, b, c\}, J(e_1)b = \{a, b\}$ *and* $J(e_1)c = \{a, c\}$. *Also,* $J(e_2)a = \{a\}$, $J(e_2)b = \{b\}$ *and* $J(e_2)c = \{c\}$. *But* $J(e_1)cJ(e_1)a = \{a, c\}$

$\neq \{a, b, c\} = J(e)\,(ca)$.

Definition 3.1.7 *An SCRE,* (J, A) *from a semigroup* S_1 *to a semigroup* S_2 *is called soft complete relation respecting to the aftersets if* $aJ(e).bJ(e) = (ab)\,J(e)$ *for all* $a, b \in S_1$ *and* $e \in A$ *and is called soft complete relation respecting to the foresets if* $J(e)\,a.J(e)\,b = J(e)\,(ab)$ *for all* $a, b \in S_2$ *and* $e \in A$.

Now, in the next the shortend-form of a soft complete relation will be SC_mR throughout the thesis.

Neither SC_mR respecting to the aftersets implies SC_mR respecting to the foresets nor SC_mR respecting to the foresets implies SC_mR respecting to the aftersets.

Example 3.1.8 *Consider the semigroup of Example 3.1.5 and* $A = \{e_1, e_2\}$. *Define* $J : A \to P(S_1 \times S_2)$ *by*

$$J(e_1) = \{(a,2), (a,3), (b,2), (b,3), (c,2), (c,3), (d,2), (d,3)\} \text{ and}$$

$$J(e_2) = \{(a,2), (b,2), (c,2), (d,2)\}.$$

Then J *is an* SC_mR *from the semigroup* S_1 *to the semigroup* S_2 *with respect to the aftersets.*

$$aJ(e_2) = \{2,3\},\ bJ(e_2) = \{2,3\},\ cJ(e_2) = \{2,3\} \text{ and } dJ(e_2) = \{2,3\}.$$

Also,

$$aJ(e_2) = \{2\},\ bJ(e_2) = \{2\},\ cJ(e_2) = \{2\} \text{ and } dJ(e_2) = \{2\}.$$

Simple calculations show that J *is an* SC_mR *with respect to the aftersets.*

Example 3.1.9 *Consider the semigroup of Example 3.1.5 and* $A = \{e_1, e_2\}$. *Define* $J : A \to P(S_1 \times S_2)$ *by*

$$J(e_1) = \{(a,1), (a,2), (a,3), (a,4), (d,1), (d,2), (d,3), (d,4)\}$$

and

$$J(e_2) = \{(a,1), (a,2), (a,3), (a,4)\}.$$

Then J *is an* SC_mR *from the semigroup* S_1 *to the semigroup* S_2 *with respect to the foresets.*

$$J(e_2)1 = \{a, d\}, \ J(e_2)2 = \{a, d\}, \ J(e_2)3 = \{a, d\} \ \text{and } J(e_2)4 = \{a, d\}.$$

Also,

$$J(e_2)1 = \{a\}, \ J(e_2)2 = \{a\}, \ J(e_2)3 = \{a\} \ \text{and } J(e_2)4 = \{a\}.$$

Simple calculations show that J *is an* SC_mR *with respect to the foresets.*

Example 3.1.10 *Consider the semigroup of Example 3.1.6. Define* $J : A \to P(S \times S)$ *by* $J(e_1) = \{(a,a), (b,b), (c,c), (a,b), (a,c)\}$ *and* $J(e_2) = \{(a,a), (b,b), (c,c)\}$. *Then* J *is a soft reflexive and* $SCRE$ *on* S.

$J(e_1)a = \{a\}, J(e_1)b = \{a,b\}$ *and* $J(e_1)c = \{a,c\}$. *Also,* $J(e_2)a = \{a\}$, $J(e_2)b = \{b\}$ *and* $J(e_2)c = \{c\}$. *Simple calculations show that* (J, A) *is a* SC_mR *with respect to the foresets. But this one is not* SC_mR *respecting to the aftersets because* $aJ(e_1) = \{a,b,c\}, bJ(e_1) = \{b\}$ *and* $cJ(e_1) = \{c\}$ *and* $cJ(e_1).aJ(e_1) = \{c\}.\{a,b,c\} = \{a,c\} \neq (ca)\,J(e_1) = \{a,b,c\}$.

Theorem 3.1.11 *Let* (J, A) *and* (Z, A) *be SBRE from non-empty sets* S_1 *to* S_2 *and* X_1, X_2 *be non-empty subsets of* S_2. *Then the following hold:*

(1) $X_1 \subseteq X_2 \Rightarrow \underline{J}^{X_1}(e) \subseteq \underline{J}^{X_2}(e)$ *for all* $e \in A$;

(2) $X_1 \subseteq X_2 \Rightarrow \overline{J}^{X_1}(e) \subseteq \overline{J}^{X_2}(e)$ *for all* $e \in A$;

(3) $\left(\underline{J}^{X_1}, A\right) \cap \left(\underline{J}^{X_2}, A\right) = \left(\underline{J}^{X_1 \cap X_2}, A\right)$;

(4) $\left(\overline{J}^{X_1}, A\right) \cap \left(\overline{J}^{X_2}, A\right) \supseteq \left(\overline{J}^{X_1 \cap X_2}, A\right)$;

(5) $\left(\underline{J}^{X_1}, A\right) \cup \left(\underline{J}^{X_2}, A\right) \subseteq \left(\underline{J}^{X_1 \cup X_2}, A\right)$;

(6) $\left(\overline{J}^{X_1}, A\right) \cup \left(\overline{J}^{X_2}, A\right) = \left(\overline{J}^{X_1 \cup X_2}, A\right)$;

(7) $(J, A) \subseteq (Z, A)$ *implies* $\left(\underline{J}^{X_1}, A\right) \supseteq \left(\underline{Z}^{X_1}, A\right)$;

(8) $(J, A) \subseteq (Z, A)$ *implies* $\left(\overline{J}^{X_1}, A\right) \subseteq \left(\overline{Z}^{X_1}, A\right)$.

Proof. Straightforward. ■

Theorem 3.1.12 *Let* (J, A) *and* (Z, A) *be SBRE from non-empty sets* S_1 *to* S_2 *and* Y_1, Y_2 *be non-empty subsets of* S_1. *Then the following hold:*

(1) $Y_1 \subseteq Y_2 \Rightarrow {}^{Y_1}\underline{J}(e) \subseteq {}^{Y_2}\underline{J}(e)$ *for all* $e \in A$;

(2) $Y_1 \subseteq Y_2 \Rightarrow {}^{Y_1}\overline{J}(e) \subseteq {}^{Y_2}\overline{J}(e)$ *for all* $e \in A$;

(3) $\left({}^{Y_1}\underline{J}, A\right) \cap \left({}^{Y_2}\underline{J}, A\right) = \left({}^{Y_1 \cap Y_2}J, A\right)$;

(4) $\left({}^{Y_1}\overline{J}, A\right) \cap \left({}^{Y_2}\overline{J}, A\right) \supseteq \left({}^{Y_1 \cap Y_2}\overline{J}, A\right)$;

(5) $\left({}^{Y_1}\underline{J}, A\right) \cup \left({}^{Y_2}\underline{J}, A\right) \subseteq \left({}^{Y_1 \cup Y_2}\underline{J}, A\right)$;

(6) $\left({}^{Y_1}\overline{J}, A\right) \cup \left({}^{Y_2}\overline{J}, A\right) = \left({}^{Y_1 \cup Y_2}\overline{J}, A\right)$;

(7) $(J, A) \subseteq (Z, A)$ *implies* $\left({}^{Y_1}\underline{J}, A\right) \supseteq \left({}^{Y_1}\underline{Z}, A\right)$;

(8) $(J, A) \subseteq (Z, A)$ *implies* $\left({}^{Y_1}\overline{J}, A\right) \subseteq \left({}^{Y_1}\overline{Z}, A\right)$.

Proof. Straightforward. ■

Corollary 3.1.13 *Let* (J, A) *and* (Z, A) *be soft reflexive relations on a non-empty set* S *and* X, Y *be non-empty subsets of* S. *Then the following hold:*

(1) $\underline{J}^X(e) \subseteq X \subseteq \overline{J}^X(e)$ *for all* $e \in A$;

(2) $X \subseteq Y \Rightarrow \underline{J}^X(e) \subseteq \underline{J}^Y(e)$ *for all* $e \in A$;

(3) $X \subseteq Y \Rightarrow \overline{J}^X(e) \subseteq \overline{J}^Y(e)$ *for all* $e \in A$;

(4) $\left(\underline{J}^X, A\right) \cap \left(\underline{J}^Y, A\right) = \left(\underline{J}^{X \cap Y}, A\right)$;

(5) $\left(\overline{J}^X, A\right) \cap \left(\overline{J}^Y, A\right) \supseteq \left(\overline{J}^{X \cap Y}, A\right)$;

(6) $\left(\underline{J}^X, A\right) \cup \left(\underline{J}^Y, A\right) \subseteq \left(\underline{J}^{X \cup Y}, A\right)$;

(7) $\left(\overline{J}^X, A\right) \cup \left(\overline{J}^Y, A\right) = \left(\overline{J}^{X \cup Y}, A\right)$;

(8) $(J, A) \subseteq (Z, A)$ *implies* $\left(\underline{J}^X, A\right) \supseteq \left(\underline{Z}^X, A\right)$;

(9) $(J, A) \subseteq (Z, A)$ *implies* $\left(\overline{J}^X, A\right) \subseteq \left(\overline{Z}^X, A\right)$.

Proof. Straightforward. ■

Corollary 3.1.14 *Let (J, A) and (Z, A) be soft reflexive relations on a non-empty set S and X, Y be non-empty subsets of S. Then the following hold:*

(1) ${}^{X}\underline{J}(e) \subseteq X \subseteq {}^{X}\overline{J}(e)$ *for all* $e \in A$;

(2) $X \subseteq Y \Rightarrow {}^{X}\underline{J}(e) \subseteq {}^{Y}\underline{J}(e)$ *for all* $e \in A$;

(3) $X \subseteq Y \Rightarrow {}^{X}\overline{J}(e) \subseteq {}^{Y}\overline{J}(e)$ *for all* $e \in A$;

(4) $\left({}^{X}\underline{J}, A\right) \cap \left({}^{Y}\underline{J}, A\right) = \left({}^{X\cap Y}\underline{J}, A\right)$;

(5) $\left({}^{X}\overline{J}, A\right) \cap \left({}^{Y}\overline{J}, A\right) \supseteq \left({}^{X\cap Y}\overline{J}, A\right)$;

(6) $\left({}^{X}\underline{J}, A\right) \cup \left({}^{Y}\underline{J}, A\right) \subseteq \left({}^{X\cup Y}\underline{J}, A\right)$;

(7) $\left({}^{X}\overline{J}, A\right) \cup \left({}^{Y}\overline{J}, A\right) = \left({}^{X\cup Y}\overline{J}, A\right)$;

(8) $(J, A) \subseteq (Z, A)$ *implies* $\left({}^{X}\underline{J}, A\right) \supseteq \left({}^{X}\underline{Z}, A\right)$;

(9) $(J, A) \subseteq (Z, A)$ *implies* $\left({}^{X}\overline{J}, A\right) \subseteq \left({}^{X}\overline{Z}, A\right)$.

Proof. Straightforward. ■

Theorem 3.1.15 *Let U and W be non-empty sets. Let (J, A) and (Z, A) be SBRE from U to W. If $\emptyset \neq X \subseteq W$, then*

(1) $\left(\left(\overline{J \cap Z}\right)^{X}, A\right) \subseteq \left(\overline{J}^{X}, A\right) \cap \left(\overline{Z}^{X}, A\right)$.

(2) $\left(\left(\underline{J \cap Z}\right)^{X}, A\right) \supseteq \left(\underline{J}^{X}, A\right) \cup \left(\underline{Z}^{X}, A\right)$.

Proof. The proof with similar arguments of parts (7) and (8) of Theorem 3.1.11 are obtained. ■

Now in the next Example, we show that there does not exist equality in above results.

Example 3.1.16 *Let $U = \{a, b, c, d, e\}$ and $W = \{1, 2, 3, 4, 5\}$ and $A = \{e_1, e_2\}$. Define $J : A \to P(U \times W)$ and $Z : A \to P(U \times W)$ by*

$$J(e_1) = \left\{ \begin{array}{c} (a,1), (b,2), (c,3), (d,4), (e,5), (b,1), (c,5), \\ (b,5), (d,3), (d,5), (d,1), (e,1) \end{array} \right\}$$

$$J(e_2) = \{(a,1), (b,2), (c,3), (d,4), (e,5)\},$$

$$Z(e_1) = \{(a,1), (b,2), (c,3), (d,4), (e,5), (b,1)\} \text{ and}$$

$$Z(e_2) = \{(a,1),(b,2),(c,3),(d,4),(e,5),(b,3)\}.$$

Therefore,

$$(J \cap Z)(e_1) = \{(a,1),(b,2),(c,3),(d,4),(e,5),(b,1)\}$$

and

$$(J \cap Z)(e_2) = \{(a,1),(b,2),(c,3),(d,4),(e,5)\}.$$

Now,

$$\begin{aligned} aJ(e_1) &= \{1\},\ bJ(e_1) = \{1,2,5\},\ cJ(e_1) = \{3,5\}, \\ dJ(e_1) &= \{1,3,4,5\} \text{ and } eJ(e_1) = \{1,5\} \end{aligned}$$

and

$$\begin{aligned} aZ(e_1) &= \{1\},\ bZ(e_1) = \{1,2\},\ cZ(e_1) = \{3\}, \\ dZ(e_1) &= \{4\} \text{ and } eZ(e_1) = \{2,5\}. \end{aligned}$$

Also,

$$\begin{aligned} a(J \cap Z)(e_1) &= \{1\},\ b(J \cap Z)(e_1) = \{1,2\},\ c(J \cap Z)(e_1) = \{3\}, \\ d(J \cap Z)(e_1) &= \{4\} \text{ and } e(J \cap Z)(e_1) = \{5\}. \end{aligned}$$

Let $X = \{1,2\}$. Then

$$\overline{J}^X(e_1) = \{a,b,d,e\},\ \overline{Z}^X(e_1) = \{a,b,e\} \text{ and } \left(\overline{J \cap Z}\right)^X(e_1) = \{a,b\}.$$

This shows that

$$\overline{J}^X(e_1) \cap \overline{Z}^X(e_1) = \{a,b,e\} \neq \{a,b\} = \left(\overline{J \cap Z}\right)^X(e_1).$$

Now, let $X = \{5\}$. Then $\underline{J}^X(e_1) = \phi$, $\underline{Z}^X(e_1) = \phi$ and $(\underline{J \cap Z})^X(e_1) = \{e\}$. This shows that

$$\underline{J}^X(e_1) \cup \underline{Z}^X(e_1) = \phi \neq \{e\} = (\underline{J \cap Z})^X(e_1).$$

Theorem 3.1.17 *Let U and W be non-empty sets. Let (J, A) and (Z, A) be SBRE from U to W. $\emptyset \neq Y \subseteq U$, then*

(1) $\left({}^{Y}\left(\overline{J \cap Z}\right), A\right) \subseteq \left({}^{Y}\overline{J}, A\right) \cap \left({}^{Y}\overline{Z}, A\right)$.

(2) $\left({}^{Y}\left(\underline{J \cap Z}\right), A\right) \supseteq \left({}^{Y}\underline{J}, A\right) \cup \left({}^{Y}\underline{Z}, A\right)$.

Proof. By using (7) and (8) of Theorem 3.1.12, similar proof can be obtained. ■

It is observed in the next Example that equality does not hold in above results.

Example 3.1.18 *Let* $U = \{a, b, c, d, e\}$ *and* $W = \{1, 2, 3, 4, 5\}$ *and* $A = \{e_1, e_2\}$. *Define* $J : A \to P(U \times W)$ *and* $Z : A \to P(U \times W)$ *by*

$$J(e_1) = \left\{ \begin{array}{c} (a,1), (b,2), (c,3), (d,4), (e,5), (b,1), (c,5), \\ (b,5), (d,3), (d,5), (d,1), (e,1)\} \end{array} \right\},$$

$$J(e_2) = \{(a,1), (b,2), (c,3), (d,4), (e,5)\},$$

$$Z(e_1) = \{(a,1), (b,2), (c,3), (d,4), (e,5), (b,1), (a,5)\} \text{ and}$$

$$Z(e_2) = \{(a,1), (b,2), (c,3), (d,4), (e,5), (b,3)\}.$$

Therefore,

$$(J \cap Z)(e_1) = \{(a,1), (b,2), (c,3), (d,4), (e,5), (b,1)\}$$

and

$$(J \cap Z)(e_2) = \{(a,1), (b,2), (c,3), (d,4), (e,5)\}.$$

Now,

$$\begin{aligned} J(e_1)1 &= \{a, b, e\},\ J(e_1)2 = \{b\},\ J(e_1)3 = \{c, d\}, \\ J(e_1)4 &= \{d\} \text{ and } J(e_1)5 = \{b, c, d, e\}, \end{aligned}$$

and

$$\begin{aligned} Z(e_1)1 &= \{a, b\},\ Z(e_1)2 = \{b\},\ 3Z(e_1) = \{c\}, \\ Z(e_1)4 &= \{d\} \text{ and } Z(e_1)5 = \{a, e\}. \end{aligned}$$

Also,

$$\begin{aligned} (J \cap Z)(e_1)1 &= \{a, b\},\ (J \cap Z)(e_1)2 = \{b\},\ (J \cap Z)(e_1)3 = \{c\}, \\ (J \cap Z)(e_1)4 &= \{d\} \text{ and } (J \cap Z)(e_1)5 = \{e\}. \end{aligned}$$

Let $Y = \{a, b\}$. *Then* ${}^{Y}\overline{J}(e_1) = \{1, 2, 5\}, {}^{Y}\overline{Z}(e_1) = \{1, 2, 5\}$ *and* ${}^{Y}\left(\overline{J \cap Z}\right)(e_1) = \{1, 2\}$. *This shows that*

$${}^{Y}\overline{J}(e_1) \cap^{Y} \overline{Z}(e_1) = \{1, 2, 5\} \neq \{1, 2\} =^{Y} \left(\overline{J \cap Z}\right)(e_1).$$

Now, let $Y = \{e\}$. *Then* ${}^{Y}\underline{J}(e_1) = \phi$, ${}^{Y}\underline{Z}(e_1) = \phi$ *and* ${}^{Y}\left(\underline{J \cap Z}\right)(e_1) = \{5\}$. *This shows that*

$${}^{Y}\underline{J}(e_1) \cup^{Y} \underline{Z}(e_1) = \phi \neq \{5\} = {}^{Y}\left(\underline{J \cap Z}\right)(e_1).$$

Corollary 3.1.19 *Let* (J, A) *and* (Z, A) *be soft reflexive relations on a non-empty set* S. *If* $\emptyset \neq X \subseteq S$, *then*

(1) $\left(\left(\overline{J \cap Z}\right)^{X}, A\right) \subseteq \left(\overline{J}^{X}, A\right) \cap \left(\overline{Z}^{X}, A\right)$.

(2) $\left(\left(\underline{J \cap Z}\right)^{X}, A\right) \supseteq \left(\underline{J}^{X}, A\right) \cup \left(\underline{Z}^{X}, A\right)$.

(3) $\left({}^{X}\left(\overline{J \cap Z}\right), A\right) \subseteq \left({}^{X}\overline{J}, A\right) \cap \left({}^{X}\overline{Z}, A\right)$.

(4) $\left({}^{X}\left(\underline{J \cap Z}\right), A\right) \supseteq \left({}^{X}\underline{J}, A\right) \cup \left({}^{X}\underline{Z}, A\right)$.

The following example shows that the equality does not hold in above results.

Example 3.1.20 *Let* $U = \{a, d, e\}$ *and* $A = \{e_1, e_2\}$. *Define* $J : A \to P(U \times U)$ *and* $Z : A \to P(U \times U)$ *by* $J(e_1) = \{(a, a), (d, d), (e, e), (a, d), (d, a)\}$, $J(e_2) = \{(a, a), (d, d), (e, e)\}$, $Z(e_1) = \{(a, a), (d, d), (e, e), (a, e), (e, a)\}$ *and*

$Z(e_2) = \{(a, a), (d, d), (e, e)\}$. *Then* $(J \cap Z)(e_1) = \{(a, a), (d, d), (e, e)\}$ *and*

$(J \cap Z)(e_2) = \{(a, a), (d, d), (e, e)\}$. *Now* $aJ(e_1) = \{a, d\}, dJ(e_1) = \{a, d\}$ *and* $eJ(e_1) = \{e\}$

and $aZ(e_1) = \{a, e\}, dZ(e_1) = \{d\}$ *and* $eZ(e_1) = \{a, e\}$.

Also, $a(J \cap Z)(e_1) = \{a\}$, $d(J \cap Z)(e_1) = \{d\}$ *and* $e(J \cap Z)(e_1) = \{e\}$. *Let* $X = \{d, e\}$. *Then* $\overline{J}^{X}(e_1) = \{a, d, e\}, \overline{Z}^{X}(e_1) = \{a, d, e\}$ *and* $\left(\overline{J \cap Z}\right)^{X}$

$(e_1) = \{d, e\}$. *This shows that* $\overline{J}^{X}(e_1) \cap \overline{Z}^{X}(e_1) = \{a, d, e\} \neq \{d, e\}$

$= \left(\overline{J \cap Z}\right)^{X}(e_1)$. *Now, let* $Y = \{a\}$. *Then* $\underline{J}^{Y}(e_1) = \phi, \underline{Z}^{Y}(e_1) = \phi$ *and* $\left(\underline{J \cap Z}\right)^{Y}(e_1) = \{a\}$. *This shows that* $\underline{J}^{Y}(e_1) \cup \underline{Z}^{Y}(e_1) = \phi \neq \{a\} = \left(\underline{J \cap Z}\right)^{Y}(e_1)$.

On the other hand, $J(e_1)a = \{a, d\}, J(e_1)d = \{a, d\}$ and $J(e_1)e = \{e\}$ and $Z(e_1)a = \{a, e\}, Z(e_1)d = \{d\}$ and $Z(e_1)e = \{a, e\}$.

Also, $(J \cap Z)(e_1)a = \{a\}, (J \cap Z)(e_1)d = \{d\}$ and $(J \cap Z)(e_1)e = \{e\}$. Then ${}^{X}\overline{J}(e_1) = \{a, d, e\}, {}^{X}\overline{Z}(e_1) = \{a, d, e\}$ and ${}^{X}\left(\overline{J \cap Z}\right)(e_1) = \{d, e\}$. This shows that ${}^{X}\overline{J}(e_1) \cap {}^{X}\overline{Z}(e_1) = \{a, d, e\} \neq \{d, e\} = {}^{X}\left(\overline{J \cap Z}\right)(e_1)$. Also, ${}^{Y}\underline{J}(e_1) = \phi$, ${}^{Y}\underline{Z}(e_1) = \phi$ and ${}^{Y}\left(\underline{J \cap Z}\right)(e_1) = \{a\}$. This shows that ${}^{Y}\underline{J}(e_1) \cup {}^{Y}\underline{Z}(e_1) = \phi \neq \{a\} = {}^{Y}\left(\underline{J \cap Z}\right)(e_1)$.

Theorem 3.1.21 *Let* (J, A) *be an SCRE from a semigroup* S_1 *to* S_2. *For any non-empty subsets* X_1 *and* X_2 *of* S_2, $\overline{J}^{X_1}(e).\overline{J}^{X_2}(e) \subseteq \overline{J}^{X_1 X_2}(e)$ *for all* $e \in A$.

Proof. Let $u \in \overline{J}^{X_1}(e).\overline{J}^{X_2}(e)$. Then $u = x_1x_2$ for some $x_1 \in \overline{J}^{X_1}(e)$ and $x_2 \in \overline{J}^{X_2}(e)$. This implies that $x_1J(e) \cap X_1 \neq \phi$ and $x_2J(e) \cap X_2 \neq \phi$, so their exist elements $a, b \in S_2$ such that $a \in x_1J(e) \cap X_1$ and $b \in x_2J(e) \cap X_2$. Thus $a \in x_1J(e)$, $b \in x_2J(e)$, $a \in X_1$ and $b \in X_2$. Now, $(x_1, a) \in J(e)$ and $(x_2, b) \in J(e)$ implies that $(x_1x_2, ab) \in J(e)$, that is $ab \in x_1x_2J(e)$. Also, $ab \in X_1X_2 \subseteq S_2$, therefore, $ab \in x_1x_2J(e) \cap X_1X_2$. Hence, $u = x_1x_2 \in \overline{J}^{X_1X_2}(e)$. ■

Theorem 3.1.22 *Let (J, A) be an SCRE from a semigroup S_1 to S_2. For any non-empty subsets Y_1 and Y_2 of S_1, ${}^{Y_1}\overline{J}(e).{}^{Y_2}\overline{J}(e) \subseteq {}^{Y_1Y_2}\overline{J}(e)$ for all $e \in A$.*

Proof. With the similar arguments as in the above Theorem, the proof is obtained. ■

If we take a soft reflexive relation and $SCRE$ on a semigroup S, then the following corollaries are proceeded.

Corollary 3.1.23 *Let (J, A) be a soft reflexive and $SCRE$ on a semigroup S. Then for any non-empty subsets X and Y of S, $\overline{J}^{X}(e).\overline{J}^{Y}(e) \subseteq \overline{J}^{XY}(e)$ for all $e \in A$.*

Corollary 3.1.24 *Let (J, A) be a soft reflexive and $SCRE$ on a semigroup S. Then for any non-empty subsets X and Y of S, ${}^{X}\overline{J}(e).{}^{Y}\overline{J}(e) \subseteq {}^{XY}\overline{J}(e)$ for all $e \in A$.*

Theorem 3.1.25 *Let (J, A) be an SC_mR from a semigroup S_1 to S_2 (with respect to the aftersets). Then for any non-empty subsets X_1 and X_2 of S_2, $\underline{J}^{X_1}(e).\underline{J}^{X_2}(e) \subseteq \underline{J}^{X_1X_2}(e)$ for all $e \in A$.*

Proof. First we consider that $\underline{J}^{X}(e)$ and $\underline{J}^{X_2}(e)$ are non-empty and $u \in \underline{J}^{X_1}(e).\underline{J}^{Y}(e)$. Then $u = x_1x_2$ for some $x_1 \in \underline{J}^{X_1}(e)$ and $x_2 \in \underline{J}^{X_2}(e)$. This implies that $\phi \neq x_1J(e) \subseteq X_1$ and $\phi \neq x_2J(e) \subseteq X_2$. As $x_1x_2J(e) = x_1J(e).x_2J(e) \subseteq X_1X_2$, we have $x_1x_2 \in \underline{\delta}^{X_1X_2}(e)$. Hence, $\underline{J}^{X_1}(e).\underline{J}^{X_2}(e) \subseteq \underline{J}^{X_1X_2}(e)$. ■

Theorem 3.1.26 *Let (J, A) be an SC_mR from a semigroup S_1 to S_2 (with respect to the foresets). Then for any non-empty subsets Y_1 and Y_2 of S_1, ${}^{Y_1}\underline{J}(e).{}^{Y_2}\underline{J}(e) \subseteq {}^{Y_1Y_2}\underline{J}(e)$ for all $e \in A$.*

Proof. The proof is simple. ■

With the similar arguments, the following corollaries are proceeded.

Corollary 3.1.27 *Let (J,A) be a soft reflexive relation and SC_mR with respect to aftersets on a semigroup S. Then for any non-empty subsets X and Y of S, $\underline{J}^X(e).\underline{J}^Y(e) \subseteq \underline{J}^{XY}(e)$ for all $e \in A$.*

Corollary 3.1.28 *Let (J,A) be a soft reflexive and SC_mR with respect to foresets on a semigroup S. Then for any non-empty subsets X and Y of S, ${}^X\underline{J}(e).{}^Y\underline{J}(e) \subseteq {}^{XY}\underline{J}(e)$ for all $e \in A$.*

Example 3.1.29 *Consider the Example 3.1.3,*

$$\begin{aligned} 1J(e_1) &= \{a,b,c\},\ 2J(e_1)=\{a,b\},\ 3J(e_1)=\{c\}, \\ 1J(e_2) &= \{a\},\ 2J(e_2)=\{b,c\} \text{ and } 3J(e_2)=\{c\}. \end{aligned}$$

Let $X_1=\{a\}$ and $X_2=\{b,c\}$ be non-empty subsets of S_2. Then $\overline{J}^{X_1}(e_1)=\{1\}$ and $\overline{J}^{X_2}(e_1)=\{2,3\}$. Now, $X_1X_2=\{a,c\}$ and

$$\overline{J}^{X_1X_2}(e_1)=\{1,2,3\}\neq\{2,3\}=\{1\}\{2,3\}=\overline{J}^{X_1}(e_1)\overline{J}^{X_2}(e_1).$$

Example 3.1.30 *Consider the Example 3.1.3,*

$$\begin{aligned} J(e_1)a &= \{1,2\},\ J(e_1)b=\{1,2\},\ J(e_1)c=\{1,3\}, \\ J(e_2)a &= \{1,2\},\ J(e_2)b=\{2\} \text{ and } J(e_2)c=\{2,3\}. \end{aligned}$$

Let $Y_1=\{3\}$ and $Y_2=\{1\}$ be non-empty subsets of S_1. Then ${}^{Y_1}\overline{J}(e_1)=\{c\}$ and ${}^{Y_2}\overline{J}(e_1)=\{a,b,c\}$. Now, $Y_1Y_2=\{1\}$ and

$${}^{Y_1Y_2}\overline{J}(e_1)=\{a,b,c\}\nsubseteq\{a,c\}=\{c\}\{a,b,c\}={}^{Y_1}\overline{J}(e_1).{}^{Y_2}\overline{J}(e_1).$$

Example 3.1.31 *Consider the semigroup of Example 3.1.5 and $A=\{e_1,e_2\}$. Define $J:A\to P(S_1\times S_2)$ by*

$$J(e_1)=\{(a,2),(a,3),(b,2),(b,3),(c,2),(c,3),(d,2),(d,3)\}$$

and

$$J(e_2)=\{(a,2),(b,2),(c,2),(d,2)\}.$$

Then with respect to the aftersets, (J,A) is a SC_mR.

$$aJ(e_2)=\{2,3\},\ bJ(e_2)=\{2,3\},\ cJ(e_2)=\{2,3\} \text{ and } dJ(e_2)=\{2,3\}.$$

Also,

$$aJ(e_2) = \{2\}, \; bJ(e_2) = \{2\}, \; cJ(e_2) = \{2\} \text{ and } dJ(e_2) = \{2\}.$$

Let $X_1 = \{4\}$ *and* $X_2 = \{1, 2, 3\}$ *be non-empty subsets of* S_2. *Then* $\underline{J}^{X_1}(e_1) = \phi$ *and* $\underline{J}^{X_2}(e_1) = \{a, b, c, d\}$. *Now,* $X_1X_2 = \{2, 3, 4\}$ *and*

$$\underline{J}^{X_1X_2}(e_1) = \{a, b, c, d\} \nsubseteq \phi = \phi\{a, b, c, d\} = \underline{J}^{X_1}(e_1)\underline{J}^{X_2}(e_1).$$

Example 3.1.32 *Consider the semigroup of Example 3.1.5 and* $A = \{e_1, e_2\}$. *Define* $J : A \to P(S_1 \times S_2)$ *by*

$$\begin{aligned} J(e_1) &= \{(a,1),(a,2),(a,3),(a,4),(d,1),(d,2),(d,3),(d,4)\} \text{ and} \\ J(e_2) &= \{(d,1),(d,2),(d,3),(d,4)\}. \end{aligned}$$

Then J *is a* SC_mR *respecting to the aftersets from the semigroup* S_1 *to the semigroup* S_2.

$$\begin{aligned} J(e_2)1 &= \{a, d\}, \; J(e_2)2 = \{a, d\}, \; J(e_2)3 = \{a, d\} \text{ and } J(e_2)4 = \{a, d\}. \\ \text{Also, } J(e_2)1 &= \{d\}, \; J(e_2)2 = \{d\}, \; J(e_2)3 = \{d\} \text{ and } J(e_2)4 = \{d\}. \end{aligned}$$

Let $Y_1 = \{a, c, d\}$ *and* $Y_2 = \{b\}$ *be non-empty subsets of* S_1. *Then* ${}^{Y_1}\underline{J}(e_1) = \{1, 2, 3, 4\}$ *and* ${}^{Y_2}\underline{J}(e_1) = \phi$. *Now,* $Y_1Y_2 = \{a, d\}$ *and*

$${}^{Y_1Y_2}\underline{J}(e_1) = \{1, 2, 3, 4\} \nsubseteq \phi = \{1, 2, 3, 4\}\phi =^{Y_1} \underline{J}(e_1).^{Y_2}\underline{J}(e_1).$$

Example 3.1.33 *Consider the semigroup of Example 3.1.6. Let* $A = \{e_1, e_2\}$. *Define* $J : A \to P(S \times S)$ *by* $J(e_1) = \{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c)\}$ *and* $J(e_2) = \{(a, a), (b, b), (c, c), (a, b)\}$. *Then* J *is a soft reflexive and soft compatible relation on* S. *Now,* $aJ(e_1) = \{a, b, c\}, bJ(e_1) = \{b, c\}, cJ(e_1) = \{c\}$, $aJ(e_2) = \{a, b\}, bJ(e_2) = \{b\}$ *and* $cJ(e_2) = \{c\}$. *Let* $X = \{a\}$ *and* $Y = \{c\}$ *be non-empty subsets of* S. *Then* $\overline{J}^X(e_1) = \{a\}$ *and* $\overline{J}^Y(e_1) = \{a, b, c\}$. *Now,* $XY = \{c\}$ *and* $\overline{J}^{XY}(e_1) = \{a, b, c\} \neq \{a, c\} = \{a\}\{a, b, c\} = \overline{J}^X(e_1)\overline{J}^Y(e_1)$. *On the other hand,* $\underline{J}^X(e_1) = \phi$ *and* $\underline{J}^Y(e_1) = \{c\}$. *Now,* $XY = \{c\}$ *and* $\underline{J}^{XY}(e_1) = \{c\} \neq \phi = \phi\{c\} = \underline{J}^X(e_1)\underline{J}^Y(e_1)$.

Example 3.1.34 *Let* $S = \{a, b, c, d\}$ *be a semigroup. The operation on* S *is as follows:*

$\bullet$	a	b	c	d
a	a	a	a	d
b	a	b	a	d
c	a	a	c	d
d	d	d	d	d

and $A = \{e_1, e_2\}$. *Define* $J : A \to P(S \times S)$ *by*

$$J(e_1) = \left\{ \begin{array}{c} (a,a),(b,b),(c,c),(a,b),(a,c),(b,c), \\ (c,b),(c,a),(b,a) \end{array} \right\} \text{ and}$$

$J(e_2) = \{(a,a),(b,b),(c,c),(d,d)\}$. *Then* (J,A) *is a soft reflexive and* SC_mR *(respecting to aftersets and foresets) on* S. *Now,* $aJ(e_1) = \{a,b,c\}, bJ(e_1) = \{a,b,c\}$, $cJ(e_1) = \{a,b,c\}$ *and* $dJ(e_1) = \{d\}$. *Also,* $aJ(e_2) = \{a\}, bJ(e_2) = \{b\}$, $cJ(e_2) = \{c\}$ *and* $dJ(e_2) = \{d\}$. *Let* $X = \{a,b,c\}$ *and* $Y = \{b,c\}$ *be non-empty subsets of* S. *Then* $\underline{J}^X(e_1) = \{a,b,c\}$ *and* $\underline{J}^Y(e_1) = \phi$. *Now,* $XY = \{a,b,c\}$ *and* $\underline{J}^{XY}(e_1) = \{a,b,c\} \neq \phi = \{a,b,c\}\phi = \underline{J}^X(e_1)\underline{J}^X(e_1)$. *On the other hand,* $J(e_1)a = \{a,b,c\}, J(e_1)b = \{a,b,c\}$, $J(e_1)c = \{a,b,c\}$ *and* $J(e_1)d = \{d\}$. *Also,* $J(e_2)a = \{a\}, J(e_2)b = \{b\}$, $J(e_2)c = \{c\}$ *and* $J(e_2)d = \{d\}$. *Then* ${}^X\underline{J}(e_1) = \{a,b,c\}$ *and* ${}^Y\underline{J}(e_1) = \phi$. *Now,* $XY = \{a,b,c\}$ *and* ${}^{XY}\underline{J}(e_1) = \{a,b,c\} \neq \phi = \{a,b,c\}\phi = {}^X\underline{J}(e_1).{}^X\underline{J}(e_1)$.

Definition 3.1.35 *Let* (J,A) *and* (P,A) *be* $SBRE$ *on a non-empty set* S. *Then the product* $(J \circ P, A)$ *of* (J,A) *and* (P,A) *is an* $SBRE$ *on* S *defined as follows:*

$(J \circ P)(e) = \{(x,y) \in S \times S$ *:if there exists* $a \in S$ *such that* $(x,a) \in J(e)$ *and* $(a,y) \in P(e)\}$, *for all* $e \in A$.

Lemma 3.1.36 *Let* (J,A) *and* (P,A) *be* $SBRE$ *on a semigroup* S. *Then*

(1) $(J \circ P, A)$ is soft reflexive if (J,A) and (P,A) are soft reflexive.

(2) $(J \circ P, A)$ is $SCRE$ if (J,A) and (P,A) are $SCRE$.

Proof. (1) Obvious.

(2) Let $(x,y),(a,b) \in (J \circ P)(e)$. Then there are $c,d \in S$ such that $(x,c),(a,d) \in J(e)$ and $(c,y),(d,b) \in P(e)$. Now, $(xa,cd) \in J(e)$ and $(cd,yb) \in P(e) \Rightarrow (xa,yb) \in (J \circ P)(e)$. This implies that $(J \circ P, A)$ is $SCRE$. $\blacksquare$

Example 3.1.37 *For semigroup $S = \{a, b, c, d, e\}$ with multiplication table is as follows:*

$\bullet$	a	b	c	d	e
a	a	e	c	d	e
b	a	b	c	d	e
c	a	e	c	d	e
d	a	e	c	d	e
e	a	e	c	d	e

and $A = \{e_1, e_2\}$. Define J, $P : A \rightarrow P(S \times S)$ by

$$\begin{aligned} J(e_1) &= \left\{ \begin{array}{c} (a,a),(b,b),(c,c),(d,d), \\ (e,e),(a,d),(c,e) \end{array} \right\}, \\ P(e_1) &= \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,e)\}, \\ J(e_2) &= \{(a,a),(b,b),(c,c),(d,d),(e,e)\} \text{ and} \\ P(e_2) &= \{(a,a),(b,b),(c,c),(d,d),(e,e)\}. \end{aligned}$$

Then (J, A) and (P, A) are soft reflexive and $SCRE$ on S. Now,

$$\begin{aligned} (J \circ P)(e_1) &= \left\{ \begin{array}{c} (a,a),(b,b),(c,c),(d,d), \\ (e,e),(a,e),(a,d),(c,e) \end{array} \right\} \text{ and} \\ (J \circ P)(e_2) &= \{(a,a),(b,b),(c,c),(d,d),(e,e)\}. \end{aligned}$$

Thus,

$$\begin{aligned} aJ(e_1) &= \{a,d\}, bJ(e_1) = \{b\}, cJ(e_1) = \{c,e\}, \\ dJ(e_1) &= \{d\}, eJ(e_1) = \{e\}, \\ aP(e_1) &= \{a,e\}, bP(e_1) = \{b\}, cP(e_1) = \{c\}, \\ dP(e_1) &= \{d\}, eP(e_1) = \{e\} \text{ and} \end{aligned}$$

$$\begin{aligned} a(J \circ \beta)(e_1) &= \{a,d,e\}, b(J \circ \beta)(e_1) = \{b\}, \\ c(J \circ \beta)(e_1) &= \{c,e\}, \\ d(J \circ \beta)(e_1) &= \{d\}, e(J \circ \beta)(e_1) = \{e\}. \end{aligned}$$

Let $X = \{a, b, c\}$ be a non-empty subset of S. Then $\overline{J}^X(e_1) = \{a, b, c\}, \overline{P}^X(e_1) = \{a, b, c\}$ and $\left(\overline{J \circ P}\right)^X(e_1) = \{a, b, c\}$. But $\overline{J}^X(e_1) \cdot \overline{P}^X(e_1) = \{a, b, c, e\} \nsubseteq \{a, b, c\} = \left(\overline{J \circ P}\right)^X(e_1)$.

On the other hand,

$$\begin{aligned} J(e_2)a &= \{a\}, J(e_2)b = \{b\}, J(e_2)c = \{c\}, J(e_2)d \\ &= \{d\}, J(e_2)e = \{e\}, \\ P(e_2)a &= \{a\}, P(e_2)b = \{b\}, P(e_2)c = \{c\}, P(e_2)d \\ &= \{d\}, P(e_2)e = \{e\} \text{ and} \end{aligned}$$

$$\begin{aligned} (J \circ P)(e_2)a &= \{a\}, (J \circ P)(e_2)b = \{b\}, (J \circ P)(e_2)c \\ &= \{c\}, \\ (J \circ P)(e_2)d &= \{d\}, (J \circ P)(e_2)e = \{e\}. \end{aligned}$$

Let $\emptyset \neq X = \{a, b, c\} \subseteq S$. Then ${}^{X}\overline{J}(e_2) = \{a, b, c\}, {}^{X}\overline{P}(e_2) = \{a, b, c\}$ and ${}^{X}\left(\overline{J \circ P}\right)(e_2) = \{a, b, c\}$. But $\overline{J}^{X}(e_2) \cdot \overline{P}^{X}(e_2) = \{a, b, c, e\} \nsubseteq \{a, b, c\} = \left(\overline{J \circ P}\right)^{X}(e_2)$.

However, if X is a subsemigroup (SS) of S then, the following Theorem is presented:

Theorem 3.1.38 *Let (J, A) and (P, A) be soft reflexive and SCRE on a semigroup S. For a SS, X of S, $\overline{J}^{X}(e) . \overline{P}^{X}(e) \subseteq \left(\overline{J \circ P}\right)^{X}(e)$ for all $e \in A$.*

Proof. Let $z \in \overline{J}^{X}(e) . \overline{P}^{X}(e)$. Then $z = xy$ for some $x \in \overline{J}^{X}(e)$ and $y \in \overline{P}^{X}(e)$. This implies that $xJ(e) \cap X \neq \phi$ and $yP(e) \cap X \neq \phi$, so their exist $a, b \in S$ such that $a \in xJ(e) \cap X$ and $b \in yP(e) \cap X$. Thus $a \in xJ(e)$, $b \in yP(e)$, $a \in X$ and $b \in X$. Since X is a SS of S, we have $ab \in X$. Since (J, A) and (P, A) are soft reflexive and $SCRE$, we have $(x, a), (y, y) \in J(e)$ and $(a, a), (y, b) \in P(e) \Rightarrow (xy, ay) \in J(e)$ and $(ay, ab) \in P(e)$. This implies that $(xy, ab) \in (J \circ P)(e) \Rightarrow ab \in xy(J \circ P)(e)$. Thus, $ab \in xy(J \circ P)(e) \cap X \Rightarrow z = xy \in \left(\overline{J \circ P}\right)^{X}(e)$. Hence, $\overline{J}^{X} . \overline{P}^{X} \subseteq \left(\overline{J \circ P}\right)^{X}$. ■

The following theorem has similar proof as above.

Theorem 3.1.39 *Let (J, A) and (P, A) be soft reflexive and SCRE on a semigroup S. For a SS, X of S, ${}^{X}\overline{J}(e) . {}^{Y}\overline{P}(e) \subseteq^{XY} \left(\overline{J \circ P}\right)(e)$ for all $e \in A$.*

Now in the next Example, we show that there does not exist equality in above results.

Example 3.1.40 *Consider the Example 3.1.37, and take $X = \{e\}$ a SS of S. Then $\overline{J}^{X}(e_1) = \{c, e\}, \overline{P}^{X}(e_1) = \{a, e\}$ and $\left(\overline{J \circ P}\right)^{X}(e_1) = \{a, c, e\}$. But $\left(\overline{J \circ P}\right)^{X}(e_1) = \{a, c, e\} \nsubseteq \{a, e\} = \overline{J}^{X}(e_1) \cdot \overline{P}^{X}(e_1)$.*

Also, if $Y = \{a\}$, then ${}^{Y}\overline{J}(e_1) = \{a, d\}, {}^{Y}\overline{P}(e_1) = \{a, e\}$ and ${}^{Y}\left(\overline{J \circ P}\right)(e_1) = \{a, d, e\}$. But ${}^{Y}\left(\overline{J \circ P}\right)(e_1) = \{a, d, e\} \nsubseteq \{a, e\} = {}^{Y}\overline{J}(e_1) \cdot {}^{Y}\overline{P}(e_1)$.

3.2 Approximation of ideals in semigroups

Now the aftersets and foresets are applied to $SCRE$ with $J(e) \neq \phi$ for all $e \in A$ in the following to approximate substructures of semigroups. It is observed that upper approximation of a SS (LIL, RIL, BIL, IIL) of a semigroup is a SSS ($SLIL$, $SRIL$, $SBIL$, $SIIL$) of the semigroup. Examples are proposed to verify that the converse is not true. While SC_mR are needed to find out lower approximation of a SS (LIL, RIL, BIL, IIL) of a semigroup and are SSS ($SLIL$, $SRIL$, $SBIL$, $SIIL$) of the semigroup.

Definition 3.2.1 *Let (J, A) be an $SBRE$ from a semigroup S_1 to S_2. If the upper approximation $\left(\overline{J}^X, A\right)$ is a SS of S_1 for $\emptyset \neq X \subseteq S_2$, then X is said to be the generalized upper SSS of S_1 respecting to the aftersets. The set X is said to be the generalized upper $SLIL$ ($SRIL$, SIL) of S_1 respecting to the aftersets if $\left(\overline{J}^X, A\right)$ is a LIL (RIL, IL) of S_1.*

Definition 3.2.2 *Let (J, A) be an $SBRE$ from a semigroup S_1 to S_2. If the upper approximation $\left({}^{Y}\overline{J}, A\right)$ is a SS of S_2 for $\emptyset \neq Y \subseteq S_1$, then Y is said to be the generalized upper SSS of S_2 respecting to the foresets. The set Y is said to be the generalized upper $SLIL$ ($SRIL$, SIL) of S_2 with respect to the foresets if $\left({}^{Y}\overline{J}, A\right)$ is a LIL (RIL, IL) of S_2.*

Theorem 3.2.3 *Let (J, A) be an $SCRE$ from a semigroup S_1 to S_2. Then*

(1) *If X is a SS of S_2, then X is a generalized upper SSS of S_1 respecting to the aftersets.*

(2) *If Y is a SS of S_1, then Y is a generalized upper SSS of S_2 respecting to the foresets.*

(3) *If X is a LIL(RIL, IL) of S_2, then X is a generalized upper $SLIL$($SRIL$, SIL) of S_1 respecting to the aftersets.*

(4) *If Y is a LIL(RIL, IL) of S_1, then Y is a generalized upper $SLIL$($SRIL$, SIL) of S_2 respecting to the foresets.*

Proof. (1) Suppose X is a SS of S_2. If $\phi \neq \overline{J}^X(e)$ for all $e \in A$. Then by Theorem 3.1.21, $\overline{J}^X(e).\overline{J}^X(e) \subseteq \overline{J}^{XX}(e) \subseteq \overline{J}^X(e)$, that is $\overline{J}^X(e)$ is a SS of S_2 for all $e \in A$ and so X is a generalized upper SSS of S_1 respecting to the aftersets.

(2) The proof is a routine verification and is similar to part (1).

(3) Suppose X is a LIL of S_2. As we know that $\overline{J}^{S_2}(e) = S_1$ for all $e \in A$, we have from Theorem 3.1.21 $S_1\overline{J}^X(e) = \overline{J}^{S_2}(e).\overline{J}^X(e) \subseteq \overline{J}^{S_2X}(e) \subseteq \overline{J}^X(e)$. Hence $\overline{J}^X(e)$ is a $SLIL$ of S_2 and so X is a generalized upper $SLIL$ of S_1 respecting to the aftersets.

(4) The proof is a routine verification and is similar to part (3). ■

Now in the next Example, we show that there does not exist converse in above Theorem.

Example 3.2.4 *Let $S_1 = \{a, b, c, d, e\}$ and $S_2 = \{1, 2, 3, 4, 5\}$ be two semigroups with the multiplication tables as follows:*

$\bullet$	a	b	c	d	e
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
e	d	d	c	d	c

$\bullet$	1	2	3	4	5
1	1	5	3	4	5
2	1	2	3	4	5
3	1	5	3	4	5
4	1	5	3	4	5
5	1	5	3	4	5

Let $A = \{e_1, e_2\}$. Define $J : A \to P(S_1 \times S_2)$ by

$$J(e_1) = \left\{ \begin{array}{c} (a,1),(b,2),(c,3),(d,4),(e,5),(b,1), \\ (c,5),(b,5),(d,3),(d,5),(d,1) \end{array} \right\} \text{ and}$$

$$J(e_2) = \left\{ \begin{array}{c} (a,1),(b,2),(c,3),(d,4),(e,5),(b,1), \\ (c,5),(b,5),(d,3),(d,5),(d,1),(b,3) \end{array} \right\}.$$

Then (J, A) is an $SCRE$ from the semigroups S_1 to S_2. Now,

$$\begin{aligned} aJ(e_1) &= \{1\}, \; bJ(e_1) = \{1,2,5\}, \; cJ(e_1) = \{3,5\}, \\ dJ(e_1) &= \{1,3,4,5\}, \; eJ(e_1) = \{5\} \text{ and} \end{aligned}$$

$$\begin{aligned} aJ(e_2) &= \{1\}\,,\ bJ(e_2) = \{1,2,3,5\}\,,\ cJ(e_2) = \{3,5\}\,, \\ dJ(e_2) &= \{1,3,4,5\}\,,\ eJ(e_2) = \{5\}\,. \end{aligned}$$

Also,

$$\begin{aligned} J(e_1)1 &= \{a,b,d\}\,,\ J(e_1)2 = \{b\}\,,\ J(e_1)3 = \{c,d\}\,, \\ J(e_1)4 &= \{d\}\,,\ J(e_1)5 = \{b,c,d,e\} \text{ and} \end{aligned}$$

$$\begin{aligned} J(e_2)1 &= \{a,b,d\}\,,\ J(e_2)2 = \{b\}\,,\ J(e_2)3 = \{b,c,d\}\,, \\ J(e_2)4 &= \{d\}\,,\ J(e_2)5 = \{b,c,d,e\}\,. \end{aligned}$$

(1) Let $X = \{1,2,3\}$. Then X is not SS of S_2 as $\{1,2,3\}\{1,2,3\} = \{1,2,3,5\} \nsubseteq \{1,2,3\}$ but $\overline{J}^X(e_1) = \{a,b,c,d\}$ and $\overline{J}^X(e_2) = \{a,b,c,d\}$ are SSs of S_1. Hence, X is a generalized upper SSS of S_1 respecting to the aftersets.

(2) Let $Y = \{a,b,c\}$. Then Y is not a SS of S_1 as $\{a,b,c\}\{a,b,c\} = \{b,c,d\} \nsubseteq \{a,b,c\}$ but ${}^Y\overline{J}(e_1) = \{1,2,3,5\}$ and ${}^Y\overline{J}(e_2) = \{1,2,3,5\}$ are SSs of S_2. Hence, Y is a generalized upper SSS of S_2 respecting to the foresets.

(3) Let $X = \{1,2,3\}$. Then X is not a LIL of S_2 as $\{1,2,3,4,5\}\{1,2,3\} = \{1,2,3,5\} \nsubseteq \{1,2,3\}$ but $\overline{J}^X(e_1) = \{a,b,c,d\}$ and $\overline{J}^X(e_2) = \{a,b,c,d\}$ are LILs of S_1. Hence, X is a generalized upper $SLIL$ of S_2 respecting to the aftersets.

(4) Let $Y = \{d,e\}$. Then Y is not a LIL of S_1 as $\{a,b,c,d,e\}\{d,e\} = \{c,d\} \nsubseteq \{d,ea\}$but ${}^Y\overline{J}(e_1) = \{1,3,4,5\}$ and ${}^Y\overline{J}(e_2) = \{1,3,4,5\}$ are LILs of S_2. Hence, Y is a generalized upper $SLIL$ of S_1 respecting to the foresets.

Following corollary is proceeded by considering soft reflexive and $SCRE$ on a semigroup S.

Corollary 3.2.5 *Let (J,A) be a soft reflexive and $SCRE$ on a semigroup S. Then*

(1) *If X is a SS of S, then X is a generalized upper SSS of S respecting to the aftersets.*

(2) *If X is a SS of S, then X is a generalized upper SSS of S respecting to the foresets.*

(3) *If X is a LIL(RIL, IL) of S, then X is a generalized upper $SLIL$ ($SRIL$, SIL) of S respecting to the aftersets.*

(4) *If X is a $LIL(RIL,\ IL)$ of S, then X is a generalized upper $SLIL(SRIL,\ SIL)$ of S respecting to the foresets.*

In the next Example, we show that there does not exist converse in above Theorem.

Example 3.2.6 *Let $S = \{a, b, c, d, e\}$ be a semigroup with the multiplication table as follows:*

$\bullet$	a	b	c	d	e
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
e	d	d	c	d	c

Let $A = \{e_1, e_2\}$. Define $J : A \rightarrow P(S \times S)$ by

$$J(e_1) = \left\{ \begin{array}{c} (a,a), (b,b), (c,c), (d,d), (e,e), \\ (a,b), (a,c), (a,d), (b,a), (b,c), \\ (b,d), (d,a), (d,b), (d,c), (e,c) \end{array} \right\} \text{ and}$$

$$J(e_2) = \left\{ \begin{array}{c} (a,a), (b,b), (c,c), (d,d), (e,e), \\ (a,b), (a,c), (a,d), (b,a), (b,c), \\ (b,d), (d,a), (d,b), (d,c) \end{array} \right\}.$$

Then (J, A) is a soft reflexive and $SCRE$ on S. Now,

$$\begin{aligned} aJ(e_1) &= \{a, b, c, d\}, bJ(e_1) = \{a, b, c, d\}, cJ(e_1) \\ &= \{c\}, dJ(e_1) = \{a, b, c, d\}, eJ(e_1) = \{c, e\} \end{aligned}$$

and

$$\begin{aligned} aJ(e_2) &= \{a, b, c, d\}, bJ(e_2) = \{a, b, c, d\}, cJ(e_2) \\ &= \{c\}, dJ(e_2) = \{a, b, c, d\}, eJ(e_2) = \{e\}. \end{aligned}$$

Also,

$$\begin{aligned} J(e_1)a &= \{a,b,d\}, J(e_1)b = \{a,b,d\}, J(e_1)c \\ &= \{a,b,c,d,e\}, J(e_1)d = \{a,b,d\}, \\ J(e_1)e &= \{e\} \text{ and} \end{aligned}$$

$$\begin{aligned} J(e_2)a &= \{a,b,d\}, J(e_2)b = \{a,b,d\}, J(e_2)c \\ &= \{a,b,c,d\}, J(e_2)d = \{a,b,d\}, J(e_2)e \\ &= \{e\}. \end{aligned}$$

(1) Let $X = \{a,b,c\}$. Then X is not a SS of S as $\{a,b,c\}\{a,b,c\} = \{b,c,d\} \nsubseteq \{a,b,c\}$ but $\overline{J}^X(e_1) = \{a,b,c,d,e\}$ and $\overline{J}^X(e_2) = \{a,b,c,d\}$ are SSs of S and thus $\left(\overline{J}^X, A\right)$ is a SSS of S. Hence, X is a generalized upper SSS of S respecting to the aftersets.

(2) Let $X = \{a,b,c\}$. Then X is not a LIL of S as $\{a,b,c,d,e\}\{a,b,c\} = \{b,c,d\} \nsubseteq \{a,b,c\}$ but $\overline{J}^X(e_1) = \{a,b,c,d,e\}$ and $\overline{J}^X(e_2) = \{a,b,c,d\}$ are LILs of S and thus $\left(\overline{J}^X, A\right)$ is a $SLIL$ of S. Hence, X is a generalized upper $SLIL$ of S respecting to the aftersets.

(3) Let $X = \{a,b,c\}$. Then X is not a SS of S as $\{a,b,c\}\{a,b,c\} = \{b,c,d\} \nsubseteq \{a,b,c\}$ but ${}^X\overline{J}(e_1) = \{a,b,c,d\}$ and ${}^X\overline{J}(e_2) = \{a,b,c,d\}$ are SSs of S and thus $\left(\overline{J}^X, A\right)$ is a SSS of S. Hence, X is a generalized upper SSS of S respecting to the foresets.

(4) Let $X = \{a,b,c\}$. Then X is not a LIL of S as $\{a,b,c,d,e\}\{a,b,c\} = \{b,c,d\} \nsubseteq \{a,b,c\}$but ${}^X\overline{J}(e_1) = \{a,b,c,d\}$ and ${}^X\overline{J}(e_2) = \{a,b,c,d\}$ are $LILs$ of S and thus $\left(\overline{J}^X, A\right)$ is a $SLIL$ of S. Hence, X is a generalized upper $SLIL$ of S respecting to the foresets.

Definition 3.2.7 *Let (J, A) be an $SCRE$ from a semigroup S_1 to S_2. A non-empty subset X of S_2 is said to be generalized lower SSS of S_1 respecting to the aftersets if $\left(\underline{J}^X, A\right)$ is a SS of S_1. The set X is said to be generalized lower $SLIL$($SRIL$, SIL) of S_1 respecting to the aftersets if $\left(\underline{J}^X, A\right)$ is a LIL(RIL, IL) of S_1.*

Definition 3.2.8 *Let (J, A) be an $SCRE$ from a semigroup S_1 to S_2. A non-empty subset Y of S_1 is said to be generalized lower SSS of S_2 respecting to the foresets if $\left({}^Y\underline{J}, A\right)$ is a SS of S_2. The set Y is said to be generalized lower $SLIL$($SRIL$, SIL) of S_2 respecting to the foresets if $\left({}^Y\underline{J}, A\right)$ is a LIL(RIL, IL) of S_2.*

Example 3.2.9 *Consider the Example 3.2.4 and take* $Y = \{1, 3, 5\}$*. Then* Y *is a* LIL *of* S_2 *but* $\underline{J}^Y(e_1) = \{a, c, e\}$ *is not a* LIL *of* S_1 *as* $\{a, b, c, d, e\}\{a, c, e\} = \{b, c, d\} \nsubseteq \{a, c, e\}$.

It is shown in the above Example that if (J, A) is a $SCRE$ from a semigroup S_1 to S_2 and Y is a LIL of S_2, then $\left(\underline{J}^Y, A\right)$ is not a $SLIL$ of S_1.

Example 3.2.10 *Consider the Example 3.2.6 and take* $Y = \{c, d, e\}$*. Then* Y *is a* LIL *of* S *but* $\underline{J}^Y(e_1) = \{c\}$ *is not a* LIL *of* S *as* $\{a, b, c, d, e\}\{c\} = \{c, d\} \nsubseteq \{c\}$*. Similarly,* ${}^Y\underline{J}(e_1) = \{e\}$ *is not a* LIL *of* S *as* $\{a, b, c, d, e\}\{e\} = \{c, d\} \nsubseteq \{e\}$.

In the above example, we have shown that if (J, A) is a soft reflexive and $SCRE$ on a semigroup S and Y is a LIL of S even then $\left(\underline{J}^Y, A\right)$ and $\left({}^Y\underline{J}, A\right)$ are not $SLIL$s of S. However, we have the following.

Theorem 3.2.11 *Let* (J, A) *be an* SC_mR *from a semigroup* S_1 *to* S_2 *respecting to the aftersets. Then*

(1) *If* X *is a* SS *of* S_2*, then* X *is a generalized lower* SSS *of* S_1 *respecting to the aftersets.*

(2) *If* X *is a* $LIL(RIL,\ IL)$ *of* S_2*, then* X *is a generalized lower* $SLIL(SRIL,\ SIL)$ *of* S_1 *respecting to the aftersets.*

Proof. (1) Suppose that X is a SS of S_2. It follows from Theorem 3.1.25 and Theorem 3.1.11(1), $\underline{J}^X(e)\,.\underline{J}^X(e) \subseteq \underline{J}^{XX}(e) \subseteq \underline{J}^X(e)$. Therefore, $\underline{J}^X$ is a SS of S_2. Hence, X is a generalized lower SSS of S_1 respecting to the aftersets.

(2) Suppose that X is a LIL of S_2. It follows from Theorem 3.1.25 and Theorem 3.1.11(1), $S_2\underline{J}^X(e) = \underline{J}^{S_2}(e)\,.\underline{J}^X(e) \subseteq \underline{J}^{S_2X}(e) \subseteq \underline{J}^X(e)$. Therefore, $\underline{J}^X$ is a LIL of S_2. Hence, X is a generalized lower $SLIL(SRIL,\ SIL)$ of S_1 respecting to the aftersets.

The remaining cases can be proved likewise. ■

If we take a soft reflexive relation and SC_mR on a semigroup S, then the following corollary is proceeded.

Corollary 3.2.12 *Let (J, A) be a soft reflexive and SC_mR respecting to aftersets on S. Then*

(1) $\left(\underline{J}^X, A\right)$*, if it is non-empty, is a SSS of S provided X is a SS of S.*

(2) $\left(\underline{J}^X, A\right)$*, if it is non-empty, is a $SLIL(SRIL,\ SIL)$ of S provided X is a $LIL(RIL,$ $IL)$ of S.*

Theorem 3.2.13 *Let (J, A) be an SC_mR from a semigroup S_1 to S_2 with respect to the foresets. Then*

(1) *If Y is a SS of S_1, then Y is a generalized lower SSS of S_2 respecting to the foresets.*

(2) *If Y is a $LIL(RIL,\ IL)$ of S_1, then Y is a generalized lower $SLIL(SRIL,\ SIL)$ of S_2 respecting to the foresets.*

If we take a soft reflexive relation and SC_mR on a semigroup S, then the following corollary is proceeded.

Corollary 3.2.14 *Let (J, A) be a soft reflexive and SC_mR on S with respect to the foresets. Then*

(1) $\left({}^X\underline{J}, A\right)$*, if it is non-empty, is a SSS of S provided X is a SS of S .*

(2) $\left({}^X\underline{J}, A\right)$*, if it is non-empty, is a $SLIL(SRIL,\ SIL)$ of S provided X is a $LIL(RIL,$ $IL)$ of S.*

The Parts of Theorem 3.2.11 and 3.2.13 do not hold in general as shown below.

Example 3.2.15 *Consider the semigroup of Example 3.1.5 and $A = \{e_1, e_2\}$. Define $J : A \to P(S_1 \times S_2)$ by*

$$\begin{aligned} J(e_1) &= \{(a,2),(a,3),(b,2),(b,3),(c,2),(c,3),(d,2),(d,3)\} \text{ and} \\ J(e_2) &= \{(a,2),(b,2),(c,2),(d,2)\}. \end{aligned}$$

Then J is an SC_mR from the semigroup S_1 to the semigroup respecting to the aftersets.

$$\begin{aligned} aJ(e_2) &= \{2,3\},\ bJ(e_2) = \{2,3\},\ cJ(e_2) = \{2,3\} \text{ and } dJ(e_2) = \{2,3\}. \text{ Also,} \\ aJ(e_2) &= \{2\},\ bJ(e_2) = \{2\},\ cJ(e_2) = \{2\} \text{ and } dJ(e_2) = \{2\}. \end{aligned}$$

(1) *Let* $X = \{2,3,4\}$ *be a non-empty subset of* S_2 *which is not a* SS *of* S_2 *as* $\{2,3,4\}\{2,3,4\} = \{1,2,3\} \nsubseteq \{2,3,4\}$ *but* $\underline{J}^X(e_1) = \{a,b,c,d\}$ *is a* SS *of* S_1*. Hence,* X *is a generalized lower* SSS *of* S_1*.*

(3) *Let* $X = \{2,3,4\}$ *be a non-empty subset of* S_2 *which is not a* LIL *of* S_2 *as* $\{1,2,3,4\}\{2,3,4\} = \{1,2,3,4\} \nsubseteq \{2,3,4\}$ *but* $\underline{J}^X(e_1) = \{a,b,c,d\}$ *is a* LIL *of* S_1*. Hence,* X *is a generalized lower* $SLIL$ *of* S_1*.*

Now, Define $J : A \to P(S_1 \times S_2)$ *by*

$$\begin{aligned} J(e_1) &= \{(d,1),(d,2),(d,3),(d,4)\} \text{ and} \\ J(e_2) &= \{(a,1),(a,2),(a,3),(a,4),(d,1),(d,2),(d,3),(d,4)\}. \end{aligned}$$

Then J *is an* SC_mR *from the semigroup* S_1 *to the semigroup* S_2 *respecting to the foresets.*

$$\begin{aligned} J(e_2)1 &= \{d\},\ J(e_2)2 = \{d\},\ J(e_2)3 = \{d\} \text{ and } J(e_2)4 = \{d\}. \text{ Also,} \\ J(e_2)1 &= \{a,d\},\ J(e_2)2 = \{a,d\},\ J(e_2)3 = \{a,d\} \text{ and } J(e_2)4 = \{a,d\}. \end{aligned}$$

(2) *Let* $Y = \{b,c,d\}$ *be a non-empty subset of* S_1 *which is not a* SS *of* S_1 *as* $\{b,c,d\}\{b,c,d\} = \{a,b,c,d\} \nsubseteq \{b,c,d\}$ *but* ${}^Y\underline{J}(e_1) = \{1,2,3,4\}$ *is a* SS *of* S_2*. Hence,* Y *is a generalized lower* SSS *of* S_2*.*

(4) *Let* $Y = \{b,c,d\}$ *be a non-empty subset of* S_1 *which is not a* LIL *of* S_1 *as* $\{a,b,c,d\}\{b,c,d\} = \{a,b,c,d\} \nsubseteq \{b,c,d\}$ *but* ${}^Y\underline{J}(e_1) = \{1,2,3,4\}$ *is a* LIL *of* S_2*. Hence,* Y *is a generalized lower* $SLIL$ *of* S_2*.*

Example 3.2.16 *Consider the semigroup of Example 3.1.34, and* $A = \{e_1, e_2\}$*. Define* $J : A \to P(S \times S)$ *by*

$$J(e_1) = \left\{ \begin{array}{c} (a,a),(b,b),(c,c),(d,d),(a,b), \\ (a,c),(b,c),(c,b),(c,a),(b,a) \end{array} \right\} \text{ and}$$

$$J(e_2) = \{(a,a),(b,b),(c,c),(d,d),(a,b),(b,a)\}.$$

Then J *is a soft reflexive and* SC_mR *on* S *respecting to the aftersets and foresets. Now,* $aJ(e_1) = \{a,b,c\}, bJ(e_1) = \{a,b,c\}$*,* $cJ(e_1) = \{a,b,c\}$ *and* $dJ(e_1) = \{d\}$*. Also,* $aJ(e_2) = \{a,b\}, bJ(e_2) = \{a,b\}$*,* $cJ(e_2) = \{c\}$ *and* $dJ(e_2) = \{d\}$*. And,* $J(e_1)a = \{a,b,c\}, J(e_1)b = \{a,b,c\}$*,* $J(e_1)c = \{a,b,c\}$ *and* $J(e_1)d = \{d\}$*. Also,* $J(e_2)a = \{a,b\}, J(e_2)b = \{a,b\}$*,* $J(e_2)c = \{c\}$ *and* $J(e_2)d = \{d\}$*. Let* $X = \{b,c,d\}$*. Then* X *is not a* SS *of* S *as* $\{b,c,d\}\{b,c,d\} = \{a,b,c,d\} \nsubseteq \{b,c,d\}$ *but* $\underline{J}^X(e_1) = \{d\}$*,*

$\underline{J}^X(e_2) = \{c, d\}$ *and* ${}^X\underline{J}(e_1) = \{d\}$, ${}^X\underline{J}(e_2) = \{c, d\}$ *are SSs of* S *and thus* $\left(\underline{J}^X, A\right)$ *is a SSS of* S. *Hence,* X *is a generalized lower SSS of* S.

Theorem 3.2.17 *Let* (J, A) *be an SBRE from a semigroup* S_1 *to* S_2. *Then for any RIL,* X_1 *and LIL,* X_2 *of* S_2, $\overline{J}^{X_1X_2} \subseteq \overline{J}^{X_1} \cap \overline{J}^{X_2}$.

Proof. Suppose that X_1 is a RIL and X_2 a LIL of S_2, so by definition $X_1X_2 \subseteq X_1S_2 \subseteq X_1$ and $X_1X_2 \subseteq S_2X_2 \subseteq X_2$ which implies that $X_1X_2 \subseteq X_1 \cap X_2$. It follows from Theorem 3.1.11 $(3), (5)$, $\overline{J}^{X_1X_2}(e) \subseteq \overline{J}^{X_1 \cap X_2}(e) \subseteq \overline{J}^{X_1}(e) \cap \overline{J}^{X_2}(e)$. Hence, $\overline{J}^{X_1X_2} \subseteq \overline{J}^{X_1} \cap \overline{J}^{X_2}$. ■

Theorem 3.2.18 *Let* (J, A) *be an SBRE from a semigroup* S_1 *to* S_2. *Then for any RIL,* Y_1 *and LIL,* Y_2 *of* S_1, ${}^{Y_1Y_2}\overline{J} \subseteq {}^{Y_1}\overline{J} \cap {}^{Y_2}\overline{J}$.

Proof. The proof is similar to the proof of above Theorem. ■

Corollary 3.2.19 *Let* (J, A) *be a soft reflexive relation on a semigroup* S. *Then for any RIL,* X *and LIL,* Y *of* S, $\overline{J}^{XY} \subseteq \overline{J}^X \cap \overline{J}^Y$.

Corollary 3.2.20 *Let* (J, A) *be a soft reflexive relation on a semigroup* S. *Then for any RIL,* X *and LIL,* Y *of* S, ${}^{XY}\overline{J} \subseteq {}^X\overline{J} \cap {}^Y\overline{J}$.

Theorem 3.2.21 *Let* (J, A) *be an SBRE from a semigroup* S_1 *to* S_2. *Then for any RIL,* X_1 *and LIL,* X_2 *of* S_2, $\underline{J}^{X_1X_2} \subseteq \underline{J}^{X_1} \cap \underline{J}^{X_2}$.

Proof. Suppose that X_1 is a RIL and X_2 a LIL of S_2, so by definition $X_1X_2 \subseteq X_1S_2 \subseteq X_1$ and $X_1X_2 \subseteq S_2X_2 \subseteq X_2$ which implies that $X_1X_2 \subseteq X_1 \cap X_2$. It follows from Theorem 3.1.11 $(2), (4)$, $\underline{J}^{X_1X_2}(e) \subseteq \underline{J}^{X_1 \cap X_2}(e) = \underline{J}^{X_1}(e) \cap \underline{J}^{X_2}(e)$. Hence, $\underline{J}^{X_1X_2} \subseteq \underline{J}^{X_1} \cap \underline{J}^{X_2}$. ■

Theorem 3.2.22 *Let* (J, A) *be an SBRE from a semigroup* S_1 *to* S_2. *Then for any RIL,* Y_1 *and LIL,* Y_2 *of* S_2, ${}^{Y_1Y_2}\underline{J} \subseteq {}^{Y_1}\underline{J} \cap {}^{Y_2}\underline{J}$.

Proof. The proof is simple and is similar to above Theorem. ■

Corollary 3.2.23 *Let (J, A) be a soft reflexive relation on a semigroup S. Then for any RIL, X and LIL, Y of S, $\underline{J}^{XY} \subseteq \underline{J}^{X} \cap \underline{J}^{Y}$.*

Corollary 3.2.24 *Let (J, A) be a soft reflexive relation on a semigroup S. Then for any RIL, X and LIL, Y of S, ${}^{XY}\underline{J} \subseteq {}^{X}\underline{J} \cap {}^{Y}\underline{J}$.*

Definition 3.2.25 *Let X be a non-empty subset of S_2 and (J, A) be an $SBRE$ from a semigroup S_1 to S_2. Then X is said to be generalized lower (upper) $SIIL$ of S_1 respecting to the aftersets if $(\underline{J}^{X}, A)$ $\left(\text{respectively } \left(\overline{J}^{X}, A\right)\right)$ is an IIL of S_1.*

Definition 3.2.26 *Let Y be a non-empty subset of S_1 and (J, A) be an $SBRE$ from a semigroup S_1 to S_2. Then Y is said to be generalized lower (upper) $SIIL$ of S_2 respecting to the foresets if $({}^{Y}\underline{J}, A)$ $\left(\text{respectively } ({}^{Y}\overline{J}, A)\right)$ is an IIL of S_2.*

Theorem 3.2.27 *Let (J, A) be an $SCRE$ from a semigroup S_1 to S_2. If X is an IIL of S_2, then X is a generalized upper $SIIL$ of S_1 respecting to the aftersets.*

Proof. As X is an IIL of S_2, so $S_2XS_2 \subseteq X$. It follows from Theorem 3.1.21 that $S_1\overline{J}^{X}(e)\, S_1 = \overline{J}^{S_2}(e)\, .\overline{J}^{X}(e)\, .\overline{J}^{S_2}(e) \subseteq \overline{J}^{S_2XS_2}(e) \subseteq \overline{J}^{X}(e)$. Hence, $\left(\overline{J}^{X}, A\right)$ is a $SIIL$ of S_1. Then X is said to be generalized upper $SIIL$ of S_1 respecting to the aftersets. ■

It is found in the accompanying Example that converse of above Theorem is not true.

Example 3.2.28 *Consider the semigroups of Example 3.1.3 and $A = \{e_1, e_2\}$. Define $J : A \to P(S_1 \times S_2)$ by*

$$
\begin{aligned}
J(e_1) &= \{(1,a),(2,b),(3,c),(1,b),(2,a),(1,c),(3,a)\} \text{ and} \\
J(e_2) &= \{(1,a),(2,b),(3,c),(2,c),(2,a)\}.
\end{aligned}
$$

Then (J, A) is an $SCRE$ from the semigroup S_1 to the semigroup S_2. Now,

$$
\begin{aligned}
1J(e_1) &= \{a,b,c\},\ 2J(e_1) = \{a,b\} \text{ and } 3J(e_1) = \{a,c\}, \\
1J(e_2) &= \{a\},\ 2J(e_2) = \{a,b,c\} \text{ and } 3J(e_2) = \{c\}.
\end{aligned}
$$

Now, $X = \{a\}$ is not an IIL of S_2 as $\{a,b,c\}\{a\}\{a,b,c\} = \{a,c\} \nsubseteq \{a\}$ but $\overline{J}^{X}(e_1) = \{1,2,3\}$ is an IIL of S_1. Hence X is a generalized upper $SIIL$ of S_1 respecting to the aftersets.

Theorem 3.2.29 *Let (J, A) be an $SCRE$ from a semigroup S_1 to S_2. If Y is an IIL of S_1, then Y is generalized upper $SIIL$ of S_2 respecting to the foresets.*

Proof. The proof is obtained in a similar way from above Theorem. ■

It is found in the following Example that converse of above Theorem is not true.

Example 3.2.30 *Consider the semigroups of Example 3.1.3 and soft relation of above example,*

$$\begin{aligned} J(e_1)a &= \{1,2,3\},\ J(e_1)b = \{1,2\},\ J(e_1)c = \{1,3\}, \\ J(e_2)a &= \{1,2\},\ J(e_2)b = \{2\} \text{ and } J(e_2)c = \{2,3\}. \end{aligned}$$

Now, $Y = \{1\}$ is not an IIL of S_1 as $\{1,2,3\}\{1\}\{1,2,3\} = \{1,2,3\} \nsubseteq \{1\}$ but ${}^{Y}\overline{J}(e_1) = \{a, b, c\}$ is an IIL of S_2. Hence Y is a generalized upper $SIIL$ of S_2 respecting to the foresets.

If we take a soft reflexive relation and $SCRE$ on a semigroup S, then the following corollaries are proceeded.

Corollary 3.2.31 *Let (J, A) be a soft reflexive and $SCRE$ on a semigroup S. If X is an IIL of S, then $\left(\overline{J}^{X}, A\right)$, is a $SIIL$ of S.*

Corollary 3.2.32 *Let (J, A) be a soft reflexive and $SCRE$ on a semigroup S. If X is an IIL of S, then $\left({}^{X}\overline{J}, A\right)$, is a generalized upper $SIIL$ of S.*

The converse of above Theorems is not true.

Example 3.2.33 *Consider the semigroup of Example 3.1.6, Let $A = \{e_1, e_2\}$. Define $J : A \to P(S \times S)$ by*

$$\begin{aligned} J(e_1) &= \{(a,a), (b,b), (c,c), (a,b), (a,c), (b,c)\} \\ \text{and } J(e_2) &= \{(a,a), (b,b), (c,c), (a,b), (a,c)\}. \end{aligned}$$

Then (J, A) is a soft reflexive and $SCRE$ on S. Now

$$\begin{aligned} aJ(e_1) &= \{a,b,c\}, bJ(e_1) = \{b,c\}, cJ(e_1) = \{c\}, \\ \text{and } aJ(e_2) &= \{a,b,c\}, bJ(e_2) = \{b\}, cJ(e_2) = \{c\}. \end{aligned}$$

Now, $X = \{c\}$ *is not an* IIL *of* S *as* $\{a,b,c\}\{c\}\{a,b,c\} = \{a,c\} \nsubseteq \{c\}$ *but* $\overline{J}^{X}(e_1) = \{a,b,c\}$ *and* $\overline{J}^{X}(e_2) = \{a,c\}$ *are* $IILs$ *of* S *and thus* $\left(\overline{J}^{X}, A\right)$ *is a* $SIIL$ *of* S*. Hence* X *is a generalized upper* $SIIL$ *of* S *respecting to the aftersets. On the other hand,*

$$J(e_1)a = \{a\}, J(e_1)b = \{a,b\}, J(e_1)c = \{a,b,c\},$$

and

$$J(e_2)a = \{a\}, J(e_2)b = \{a,b\}, J(e_2)c = \{a,c\}.$$

Now, $X = \{a\}$ *is not an* IIL *of* S *as* $\{a,b,c\}\{a\}\{a,b,c\} = \{a,c\} \nsubseteq \{a\}$ *but* ${}^{X}\overline{J}(e_1) = \{a,b,c\}$ *and* ${}^{X}\overline{J}(e_2) = \{a,b,c\}$ *are* $IILs$ *of* S *and thus* $\left(\overline{J}^{X}, A\right)$ *is a* $SIIL$ *of* S*. Hence* X *is a generalized upper* $SIIL$ *of* S *respecting to the foresets.*

Theorem 3.2.34 *Let* (J, A) *be an* SC_mR *from a semigroup* S_1 *to* S_2*. If* X *is an* IIL *of* S_2*, then* X *is a generalized lower* $SIIL$ *of* S_1 *respecting to the aftersets.*

Proof. As X is an IIL of S_2, so $S_2XS_2 \subseteq X$. Then by Theorem 3.1.11 (2) and Theorem 3.1.25, $S_1.\underline{J}^{X}(e)\,S_1 = \underline{J}^{S_2}(e)\,.\underline{J}^{X}(e)\,.\underline{J}^{S_2}(e) \subseteq \underline{J}^{S_2XS_2}(e) \subseteq \underline{J}^{X}(e)$. Hence, $\left(\underline{J}^{X}, A\right)$ is a $SIIL$ of S_1. Thus, X is a generalized lower $SIIL$ of S_1 respecting to the aftersets. ■

Example 3.2.35 *Consider the semigroup of Example 3.1.5 and* $A = \{e_1, e_2\}$*. Define* $J : A \to P(S_1 \times S_2)$ *by*

$$\begin{aligned} J(e_1) &= \{(a,2),(a,3),(b,2),(b,3),(c,2),(c,3),(d,2),(d,3)\} \text{ and} \\ J(e_2) &= \{(a,2),(b,2),(c,2),(d,2)\}. \end{aligned}$$

Then J *is an* SC_mR *respecting to the aftersets from the semigroup* S_1 *to the semigroup* S_2*.*

$$\begin{aligned} aJ(e_2) &= \{2,3\},\ bJ(e_2) = \{2,3\},\ cJ(e_2) = \{2,3\} \text{ and } dJ(e_2) = \{2,3\}. \text{ Also,} \\ aJ(e_2) &= \{2\},\ bJ(e_2) = \{2\},\ cJ(e_2) = \{2\} \text{ and } dJ(e_2) = \{2\}. \end{aligned}$$

Let $X = \{2,3,4\}$ *be not an* IIL *of* S_2 *as* $\{1,2,3,4\}\{2,3,4\}\{1,2,3,4\} = \{1,2,3,4\} \nsubseteq \{2,3,4\}$ *but* $\underline{J}^{X}(e_1) = \{a,b,c,d\}$ *and* $\underline{J}^{X}(e_2) = \{a,b,c,d\}$ *are* $IILs$ *of* S_1*. Hence,* X *is a generalized lower* $SIIL$ *of* S_1 *respecting to the aftersets.*

Theorem 3.2.36 *Let* J *be a* $SCRE$ *from a semigroup* S_1 *to* S_2*. If* Y *is an* IIL *of* S_1*, then* Y *is a generalized lower* $SIIL$ *of* S_2 *respecting to the foresets.*

Proof. The proof is simple and is similar to above Theorem ■

Example 3.2.37 *Consider the semigroup of example 3.1.5 and* $A = \{e_1, e_2\}$*. Define* $J : A \to P(S_1 \times S_2)$ *by*

$$\begin{aligned} J(e_1) &= \{(d,1),(d,2),(d,3),(d,4)\} \text{ and} \\ J(e_2) &= \{(a,1),(a,2),(a,3),(a,4),(d,1),(d,2),(d,3),(d,4)\}. \end{aligned}$$

Then J *is a* SC_mR *respecting to the aftersets from the semigroup* S_1 *to the semigroup* S_2*.*

$$\begin{aligned} J(e_2)1 &= \{d\},\ J(e_2)2 = \{d\},\ J(e_2)3 = \{d\} \text{ and } J(e_2)4 = \{d\}. \text{ Also,} \\ J(e_2)1 &= \{a,d\},\ J(e_2)2 = \{a,d\},\ J(e_2)3 = \{a,d\} \text{ and } J(e_2)4 = \{a,d\}. \end{aligned}$$

Let $Y = \{b,c,d\}$ *be not an* IIL *of* S_1 *as* $\{a,b,c,d\}\{b,c,d\}\{a,b,c,d\} = \{a,b,c,d\} \nsubseteq \{b,c,d\}$ *but* ${}^{Y}\underline{J}(e_1) = \{1,2,3,4\}$ *and* ${}^{Y}\underline{J}(e_2) = \{1,2,3,4\}$ *are* $IILs$ *of* S_2*. Hence,* Y *is a generalized lower* $SIIL$ *of* S_2 *respecting to the foresets.*

If we take a soft reflexive relation and SC_mR on a semigroup S, then the following corollaries are proceeded.

Corollary 3.2.38 *Let* (J, A) *be a soft reflexive and* SC_mR *on a semigroup* S *respecting to aftersets. If* X *is an* IIL *of* S*, then* $\left(\underline{J}^{X}, A\right)$*, if it is non-empty, is a* $SIIL$ *of* S*.*

Corollary 3.2.39 *Let* (J, A) *be a soft reflexive and* SC_mR *on a semigroup* S *respecting to foresets. If* X *is an* IIL *of* S*, then* $\left({}^{X}\underline{J}, A\right)$*, if it is non-empty, is a* $SIIL$ *of* S*.*

Example 3.2.40 *Consider the semigroup of Example 3.1.34, and* $A = \{e_1, e_2\}$*. Define* $J : A \to P(S \times S)$ *by*

$$J(e_1) = \left\{ \begin{array}{c} (a,a),(b,b),(c,c),(d,d),(a,b), \\ (a,c),(b,c),(c,b),(c,a),(b,a) \end{array} \right\} \text{ and } J(e_2) = \{(a,a),(b,b),(c,c),(d,d)\}.$$

Then (J, A) *is a soft reflexive and* SC_mR *on* S *respecting to the aftersets. Now,* $aJ(e_1) = \{a,b,c\}, bJ(e_1) = \{a,b,c\}$*,* $cJ(e_1) = \{a,b,c\}$ *and* $dJ(e_1) = \{d\}$*. Also,* $aJ(e_2) = \{a\}, bJ(e_2) = \{b\}$*,* $cJ(e_2) = \{c\}$ *and* $dJ(e_2) = \{d\}$*.And,* $J(e_1)a = \{a,b,c\}, J(e_1)b =$

$\{a,b,c\}$, $J(e_1)c = \{a,b,c\}$ and $J(e_1)d = \{d\}$. Also, $J(e_2)a = \{a\}, J(e_2)b = \{b\}$, $J(e_2)c = \{c\}$ and $J(e_2)d = \{d\}$. Let $X = \{b,d\}$. Then X is not an IIL of S as $\{a,b,c,d\}\{b,d\}\{a,b,c,d\} = \{a,b,d\} \nsubseteq \{b,d\}$but $\underline{J}^X(e_1) = \{d\}$, $\underline{J}^X(e_2) = \{d\}$ and ${}^X\underline{J}(e_1) = \{d\}$, ${}^X\underline{J}(e_2) = \{d\}$ are IILs of S and thus $\left(\underline{J}^X, A\right)$ is a $SIIL$ of S. Hence, X is a generalized lower $SIIL$ of S.

Definition 3.2.41 *Let X be a non-empty subset of S_2 and (J,A) an $SBRE$ from a semigroup S_1 to S_2. Then X is said to be generalized lower (upper) $SBIL$ of S_1 respecting to the aftersets if $\left(\underline{J}^X, A\right)$ $\left(\text{respectively } \left(\overline{J}^X, A\right)\right)$ is a BIL of S_1.*

Definition 3.2.42 *Let Y be a non-empty subset of S_1 and (J,A) an $SBRE$ from a semigroup S_1 to S_2. Then Y is said to be generalized lower (upper) $SBIL$ of S_2 respecting to the foresets if $\left({}^Y\underline{J}, A\right)$ $\left(\text{respectively } \left({}^Y\overline{J}, A\right)\right)$ is a BIL of S_2.*

Theorem 3.2.43 *Let (J,A) be an $SCRE$ from a semigroup S_1 to S_2. Then every BIL, X of S_2 is a generalized upper $SBIL$ of S_1 respecting to the aftersets.*

Proof. Let X be a BIL of S_2. It follows from 3.2.3(1), $\left(\overline{J}^X, A\right)$ is a SSS of S_2. By Theorem 3.1.11 (2) and Theorem 3.1.21, $\overline{J}^X(e)\, S_1 \overline{J}^X(e) = \overline{J}^X(e)\, .\overline{J}^{S_2}(e)\, .\overline{J}^X(e) \subseteq \overline{J}^{XS_2X}(e) \subseteq \overline{J}^X(e)$. Hence, $\left(\overline{J}^X, A\right)$ is a $SBIL$ of S_1. Thus, X is a generalized upper $SBIL$ of S_1. ■

Example 3.2.44 *Consider the semigroups and soft relations of Example 3.2.4. Then $X = \{1,2,3\}$ is not a BIL of S_2 as $\{1,2,3\}\{1,2,3,4,5\}\{1,2,3\} = \{1,2,3,5\} \nsubseteq \{1,2,3\}$ but $\overline{J}^X(e_1) = \{a,b,c,d\}$ and $\overline{J}^X(e_2) = \{a,b,c,d\}$ are BILs of S_1. Hence, X is a generalized upper $SBIL$ of S_1 respecting to the aftersets.*

Theorem 3.2.45 *Let (J,A) be an $SCRE$ from a semigroup S_1 to S_2. Then every BIL, Y of S_1 is a generalized upper $SBIL$ of S_2 respecting to the foresets.*

Proof. The proof is simple and is similar to above Theorem. ■

Example 3.2.46 *Consider the semigroups and soft relations of example 3.2.4.*

Then $Y = \{a,b,c\}$ is not a BIL of S_1 as $\{a,b,c\}\{a,b,c,d,e\}\{a,b,c\} = \{b,c,d\} \nsubseteq \{a,b,c\}$ but ${}^Y\overline{J}(e_1) = \{1,2,3,5\}$ and ${}^Y\overline{J}(e_1) = \{1,2,3,5\}$ are BILs of S_2. Hence, Y is a generalized upper $SBIL$ of S_2 respecting to the foresets.

If we take a soft reflexive and $SCRE$ on a semigroup S, then the following corollaries are proceeded.

Corollary 3.2.47 *Let (J, A) be a soft reflexive and $SCRE$ on a semigroup S. Then every BIL, X of S is a generalized upper $SBIL$ of S respecting to the aftersets.*

Corollary 3.2.48 *Let (J, A) be a soft reflexive and $SCRE$ on a semigroup S. Then every BIL, X of S is a generalized upper $SBIL$ of S respecting to the foresets.*

Example 3.2.49 *Consider the semigroup of Example 3.2.6, Let $A = \{e_1, e_2\}$. Define $J : A \rightarrow P(S \times S)$ by*

$$J(e_1) = \left\{ \begin{array}{c} (a,a),(b,b),(c,c),(d,d),(e,e), \\ (a,b),(a,c),(a,d),(b,a),(b,c), \\ (b,d),(d,a),(d,b),(d,c),(e,c) \end{array} \right\} \text{ and}$$

$$J(e_2) = \left\{ \begin{array}{c} (a,a),(b,b),(c,c),(d,d),(e,e), \\ (a,b),(a,c),(a,d),(b,a),(b,c), \\ (b,d),(d,a),(d,b),(d,c) \end{array} \right\}.$$

Then (J, A) is a soft reflexive and $SCRE$ on S. Now,

$$\begin{aligned} aJ(e_1) &= \{a,b,c,d\}, bJ(e_1) = \{a,b,c,d\}, \\ cJ(e_1) &= \{c\}, dJ(e_1) = \{a,b,c,d\}, \\ eJ(e_1) &= \{c,e\} \text{ and} \end{aligned}$$

$$\begin{aligned} aJ(e_2) &= \{a,b,c,d\}, bJ(e_2) = \{a,b,c,d\}, cJ(e_2) \\ &= \{c\}, dJ(e_2) = \{a,b,c,d\}, eJ(e_2) = \{e\}. \end{aligned}$$

Also,

$$\begin{aligned} J(e_1)a &= \{a,b,d\}, J(e_1)b = \{a,b,d\}, \\ J(e_1)c &= \{a,b,c,d,e\}, J(e_1)d = \{a,b,d\}, \\ J(e_1)e &= \{e\} \text{ and} \end{aligned}$$

$$\begin{aligned} J(e_2)a &= \{a,b,d\}, J(e_2)b = \{a,b,d\}, \\ J(e_2)c &= \{a,b,c,d\}, J(e_2)d = \{a,b,d\}, \\ J(e_2)e &= \{e\}. \end{aligned}$$

Let $X = \{a,b,c\}$. Then X is not a BIL of S as $\{a,b,c\}\{a,b,c,d,e\}\{a,b,c\} = \{b,c,d\} \nsubseteq \{a,b,c\}$ but $\overline{J}^X(e_1) = \{a,b,c,d,e\}$ and $\overline{J}^X(e_2) = \{a,b,c,d\}$ are BILs of S and thus $\left(\overline{J}^X, A\right)$ is a $SBIL$ of S. Hence, X is a generalized upper $SBIL$ of S respecting to the aftersets. Similarly, ${}^X\overline{J}(e_1) = \{a,b,c,d\}$ and ${}^X\overline{J}(e_2) = \{a,b,c,d\}$ are BILs of S and thus $\left({}^X\overline{J}, A\right)$ is a $SBIL$ of S. Hence, X is a generalized upper $SBIL$ of S respecting to the foresets.

Theorem 3.2.50 *Let (J,A) be an SC_mR respecting to the aftersets from a semigroup S_1 to S_2. Then every BIL, X of S_2 is a generalized lower $SBIL$ of S_1 respecting to the aftersets.*

Proof. Let X be a BIL of S_2. It follows from 3.2.11(1), $\left(\underline{J}^X, A\right)$ is a SSS of S_2. By Theorem 3.1.11(2) and Theorem 3.1.25, $\underline{J}^X(e).S_1.\underline{J}^X(e) = \underline{J}^X(e).\underline{J}^{S_2}(e).\underline{J}^X(e) \subseteq \underline{J}^{XS_2X}(e) \subseteq \underline{J}^X(e)$. Hence, $\left(\underline{J}^X, A\right)$ is a $SBIL$ of S_1. Hence, X is a generalized lower $SBIL$ of S_1. ■

Example 3.2.51 *Consider the semigroup of Example 3.1.5 and $A = \{e_1, e_2\}$. Define $J : A \to P(S_1 \times S_2)$ by*

$$\begin{aligned} J(e_1) &= \{(a,2),(a,3),(b,2),(b,3),(c,2),(c,3),(d,2),(d,3)\} \text{ and} \\ J(e_2) &= \{(a,2),(b,2),(c,2),(d,2)\}. \end{aligned}$$

Then J is an SC_mR respecting to the aftersets from the semigroup S_1 to the semigroup S_2. Now,

$$\begin{aligned} aJ(e_1) &= \{2,3\}, \ bJ(e_1) = \{2,3\}, \ cJ(e_1) = \{2,3\} \text{ and } dJ(e_1) = \{2,3\}. \text{ Also,} \\ aJ(e_2) &= \{2\}, \ bJ(e_2) = \{2\}, \ cJ(e_2) = \{2\} \text{ and } dJ(e_2) = \{2\}. \end{aligned}$$

Then $X = \{2,3,4\}$ is not a BIL of S_2 as $\{2,3,4\}\{1,2,3,4\}\{2,3,4\} = \{1,2,3,4\} \nsubseteq \{2,3,4\}$ but $\underline{J}^X(e_1) = \{a,b,c,d\}$ and $\underline{J}^X(e_2) = \{a,b,c,d\}$ is a BIL of S_1. Hence, X is a generalized lower $SBIL$ of S_1 respecting to the aftersets.

Theorem 3.2.52 *Let (J,A) be an SC_mR respecting to the foresets from a semigroup S_1 to S_2. Then every BIL, Y of S_1 is a generalized lower $SBIL$ of S_2 respecting to the foresets.*

Proof. The proof is simple and is similar to Theorem. ■

Example 3.2.53 *Consider the semigroup of Example 3.1.5 and $A = \{e_1, e_2\}$. Define $J : A \to P(S_1 \times S_2)$ by*

$$
\begin{aligned}
J(e_1) &= \{(d,1),(d,2),(d,3),(d,4)\} \text{ and} \\
J(e_2) &= \{(a,1),(a,2),(a,3),(a,4),(d,1),(d,2),(d,3),(d,4)\}.
\end{aligned}
$$

Then J is an SC_mR respecting to the foresets from the semigroup S_1 to the semigroup S_2.

$$
\begin{aligned}
J(e_2)1 &= \{d\},\ J(e_2)2 = \{d\},\ J(e_2)3 = \{d\} \text{ and } J(e_2)4 = \{d\}. \text{ Also,} \\
J(e_2)1 &= \{a,d\},\ J(e_2)2 = \{a,d\},\ J(e_2)3 = \{a,d\} \text{ and } J(e_2)4 = \{a,d\}.
\end{aligned}
$$

Then $Y = \{b,c,d\}$ be not a BIL of S_1 as $\{b,c,d\}\{a,b,c,d\}\{b,c,d\} = \{a,b,c,d\} \nsubseteq \{b,c,d\}$ but ${}^{Y}\underline{J}(e_1) = \{1,2,3,4\}$ is a BIL of S_2. Hence, Y is a generalized upper $SBIL$ of S_2 respecting to the foresets.

If we take a soft reflexive and SC_mR on a semigroup S, then the following corollaries are proceeded.

Corollary 3.2.54 *Let (J,A) be a soft reflexive and SC_mR on a semigroup S respecting to aftersets. Then every BIL, X of S is a generalized lower $SBIL$ of S with respect to the aftersets.*

Corollary 3.2.55 *Let (J,A) be a soft reflexive and SC_mR on a semigroup S with respect to foresets. Then every BIL, X of S is a generalized lower $SBIL$ of S respecting to the foresets.*

Example 3.2.56 *Consider the semigroup of Example 3.1.34, and $A = \{e_1, e_2\}$. Define $J : A \to P(S \times S)$ by*

$$\begin{aligned} J(e_1) &= \left\{ \begin{array}{c} (a,a),(b,b),(c,c),(d,d),(a,b), \\ (a,c),(b,c),(c,b),(c,a),(b,a) \end{array} \right\} \text{ and} \\ J(e_2) &= \{(a,a),(b,b),(c,c),(d,d),(c,b), \\ &\quad (c,a),(b,a),(a,b),(a,c),(b,c)\}. \end{aligned}$$

Then (J, A) is a soft reflexive and SC_mR respecting to the aftersets on S. Now,

$$\begin{aligned} aJ(e_1) &= \{a,b,c\}, bJ(e_1) = \{a,b,c\}, cJ(e_1) \\ &= \{a,b,c\} \text{ and } dJ(e_1) = \{d\}. \\ \text{Also, } aJ(e_2) &= \{a,b,c\}, bJ(e_2) = \{a,b,c\}, cJ(e_2) \\ &= \{a,b,c\} \text{ and } dJ(e_2) = \{d\}. \end{aligned}$$

And,

$$\begin{aligned} J(e_1)a &= \{a,b,c\}, J(e_1)b = \{a,b,c\}, J(e_1)c \\ &= \{a,b,c\} \text{ and } J(e_1)d = \{d\}. \\ \text{Also, } J(e_2)a &= \{a,b,c\}, J(e_2)b = \{a,b,c\}, J(e_2)c \\ &= \{a,b,c\} \text{ and } J(e_2)d = \{d\}. \end{aligned}$$

Let $X = \{c,d\}$ be not a BIL of S as $\{c,d\}\{a,b,c,d\}\{c,d\} = \{a,c,d\} \nsubseteq \{c,d\}$ but $\underline{J}^X(e_1) = \{d\}$ and $\underline{J}^X(e_2) = \{d\}$ are BILs of S and thus $\left(\underline{J}^X, A\right)$ is a $SBIL$ of S. Hence, X is a generalized lower $SBIL$ of S respecting to the aftersets. Similarly, ${}^X\underline{J}(e_1) = \{d\}$ and ${}^X\underline{J}(e_2) = \{d\}$ are BILs of S and thus $\left({}^X\underline{J}, A\right)$ is a $SBIL$ of S. Hence, X is a generalized lower $SBIL$ of S respecting to the foresets.

3.3 Problems of Homomorphisms

A mapping $\gamma : S \to T$, where S and T are semigroups is said to be a homomorphism if $\gamma(ab) = \gamma(a)\gamma(b)$ for all $a, b \in S$. If γ is a homomorphism from a semigroup S to a semigroup T, then kernel of γ is defined as $\kappa = \{(a,b) \in S \times S : \gamma(a) = \gamma(b)\}$. It is well known that kernel of γ is a congruence relation on S. Let S and T be semigroups and $\{\gamma_e : S \to T \text{ for } e \in A\}$ be the collection of homomorphisms. Then we have a soft congruence relation (κ, A) on S, where $\kappa(e) =$ kernel of γ_e.

Theorem 3.3.1 *Let S and T be two semigroups and γ_e be homomorphism from S to T, for each $e \in A$. If X is a non-empty subset of S, then $\gamma_e\left(\overline{\kappa}^X(e)\right) = \gamma_e(X)$ where (A, κ) is the soft congruence relation on S.*

Proof. Since by Theorem 3.1.13(1), $X \subseteq \overline{\kappa}^X(e)$, it follows $\gamma_e(X) \subseteq \gamma_e\left(\overline{\kappa}^X(e)\right)$. For the converse inclusion, let $y \in \gamma_e\left(\overline{\kappa}^X(e)\right)$. Then there exists an element $x \in \overline{\kappa}^X(e)$ such that $\gamma_e(x) = y$. Now, $x \in \overline{\kappa}^X(e)$ implies that there exists $a \in S$ such that $a \in x\kappa(e) \cap X$, that is $a \in x\kappa(e)$ and $a \in X$. Thus $(x, a) \in \kappa(e)$, this implies that $\gamma_e(a) = \gamma_e(x)$. Then $y = \gamma_e(x) = \gamma_e(a) \in \gamma_e(X)$, and so $\gamma_e\left(\overline{\kappa}^X(e)\right) \subseteq \gamma_e(X)$. Hence $\gamma_e\left(\overline{\kappa}^X(e)\right) = \gamma_e(X)$. ■

If we have a soft relation on a semigroup T and a homomorphism from a semigroup S to a semigroup T, then we can construct a soft relation on S as follows:

Lemma 3.3.2 *Let $\gamma : S \to T$ be an onto homomorphism and (J_2, A) be a soft relation on T. Define $J_1(e) = \{(s,t) \in S \times S : (\gamma(s), \gamma(t)) \in J_2(e)\}$ for all $e \in A$. Then the following hold:*

(1) (J_1, A) is a soft relation on S.

(2) (J_1, A) is soft reflexive if (J_2, A) is soft reflexive.

(3) (J_1, A) is $SCRE$ if (J_2, A) is a $SCRE$.

(4) (J_1, A) is SC_mR respecting to aftersets (respecting to foresets) if (J_2, A) is a SC_mR respecting to aftersets (respecting to foresets) and γ is one one.

(5) $\gamma\left(\overline{J}_1^X(e)\right) = \overline{J}_2^{\gamma(X)}(e)$ for $X \subseteq S$ and $e \in A$.

(6) $\gamma\left(\underline{J}_1^X(e)\right) \subseteq \underline{J}_2^{\gamma(X)}(e)$ and if γ is one one, then $\gamma\left(\underline{J}_1^X(e)\right) = \underline{J}_2^{\gamma(X)}(e)$ for $e \in A$.

Proof. (1) and (2) are obvious.

(3) Let $(a,b), (c,d) \in J_1(e)$.

Then $(\gamma(a), \gamma(b)), (\gamma(c), \gamma(d)) \in J_2(e)$. As (J_2, A) is soft compatible, we have $(\gamma(a)\gamma(c), \gamma(b)\gamma(d)) \in J_2(e)$. This implies that $(\gamma(ac), \gamma(bd))$

$\in J_2(e)$. Thus $(ac, bd) \in J_1(e)$. This shows that (J_1, A) is a soft compatible relation on S.

(4) Obviously $aJ_1(e).bJ_1(e) \subseteq (ab)J_1(e)$. Conversely, assume that $x \in (ab)J_1(e)$. Then $(ab, x) \in J_1(e) \Longrightarrow (\gamma(ab), \gamma(x)) \in J_2(e) \Rightarrow$

$(\gamma(a)\gamma(b), \gamma(x)) \in J_2(e)$. This implies that $\gamma(x) \in (\gamma(a)\gamma(b)) J_2(e) =$

$\gamma(a) J_2(e) .\gamma(b) J_2(e)$. Thus there exist $t_1 \in \gamma(a) J_2(e)$ and $t_2 \in \gamma(b) J_2(e)$ such that

$\gamma(x) = t_1 t_2$. As γ is onto, we have $s_1, s_2 \in S$ such that $\gamma(s_1) = t_1$ and $\gamma(s_2) = t_2$ and $\gamma(x) = t_1 t_2 = \gamma(s_1)\gamma(s_2) = \gamma(s_1 s_2)$. As γ is one one, we have $x = s_1 s_2$. Now, $t_1 \in \gamma(a) J_2(e) \Rightarrow \gamma(s_1) \in \gamma(a) J_2(e) \Rightarrow (\gamma(a), \gamma(s_1)) \in J_2(e) \Rightarrow (a, s_1) \in J_1(e)$.

This implies that $s_1 \in aJ_1(e)$. Similarly, $s_2 \in bJ_1(e)$. Now, $x = s_1 s_2 \in aJ_1(e) .bJ_1(e)$. This implies that $(ab) J_1(e) \subseteq aJ_1(e) .bJ_1(e)$. Hence, $(ab) J_1(e) = aJ_1(e) .bJ_1(e)$. The proof of parenthesis part is similar to the proof of this.

(5) Let $b \in \gamma\left(\overline{J}_1^X\right)(e)$. Then there exists $a \in \overline{J}_1^X(e)$ such that $\gamma(a) = b$ and $aJ_1(e) \cap X \neq \phi$. This implies there exists $x \in aJ_1(e) \cap X$. Now, $(a, x) \in J_1(e) \Rightarrow (\gamma(a), \gamma(x)) \in J_2(e) \Rightarrow \gamma(x) \in \gamma(a) J_2(e)$. Also, $\gamma(x) \in \gamma(X)$. Thus $\gamma(a) J_2(e) \cap \gamma(X) \neq \phi \Rightarrow b = \gamma(a) \in \overline{J}_2^{\gamma(X)}$, that is $\gamma\left(\overline{J}_1^X(e)\right) \subseteq \overline{J}_2^{\gamma(X)}(e)$. Conversely, let $b \in \overline{J}_2^{\gamma(X)}(e)$. This implies that $bJ_2(e) \cap \gamma(X) \neq \phi$. So there exists $a \in bJ_2(e) \cap \gamma(X)$, that is $(b, a) \in J_2(e)$ and $a \in \gamma(X)$. Since γ is onto so there exist $x \in X$ and $s \in S$ such that $a = \gamma(x)$ and $b = \gamma(s)$. Thus $(\gamma(s), \gamma(x)) = (b, a) \in J_2(e) \Rightarrow (s, x) \in J_1(e)$. This implies $x \in sJ_1(e) \cap X$, so we have $s \in \overline{J}_1^X(e)$, that is $b = \gamma(s) \in \gamma\left(\overline{J}_1^X(e)\right)$. Thus $\overline{J}_2^{\gamma(X)}(e) \subseteq \gamma\left(\overline{J}_1^X(e)\right)$. Hence $\gamma\left(\overline{J}_1^X(e)\right) = \overline{J}_2^{\gamma(X)}(e)$.

(6) Let $b \in \gamma\left(\underline{J}_1^X(e)\right)$. Then there exists $a \in \underline{J}_1^X(e)$ such that $\gamma(a) = b$, and $aJ_1(e) \subseteq X$. It can be easily shown that if $aJ_1(e) \subseteq X$ then $\gamma(a)J_2(e) \subseteq \gamma(X)$. This implies that $b = \gamma(a) \in \underline{J}_2^{\gamma(X)}(e)$. Now, suppose that γ is one one and let $b \in \underline{J}_2^{\gamma(X)}(e)$. Then there exists a unique $a \in S$ such that $\gamma(a) = b$ and $\gamma(a) J_2(e) \subseteq \gamma(X)$. As γ is an isomorphism so $\gamma(aJ_1(e)) = \gamma(a) J_2(e)$ and as $\gamma(a) J_2(e) \subseteq \gamma(X)$ we have $aJ_1(e) \subseteq X$. This implies $a \in \underline{J}_1^X(e)$ and $b = \gamma(a) \in \gamma\left(\underline{J}_1^X(e)\right)$. Hence, $\gamma\left(\underline{J}_1^X\right) = \underline{J}_2^{\gamma(X)}$. ■

Theorem 3.3.3 *Let γ be a surjective homomorphism from a semigroup S to a semigroup T and (J_2, A) be a soft reflexive and SCRE respecting to aftersets (with respect to foresets) on T. Let $X \subseteq S$. Then $\overline{J}_1^X(e)$ $\left({}^X\overline{J}_1(e)\right)$is an IL of S if and only if $\overline{J}_2^{\gamma(X)}(e)$ $\left({}^X\overline{J}_1(e)\right)$ is an IL of T.*

Proof. Let $\overline{J}_1^X(e)$ be IL of S. Then,

$$S\overline{J}_1^X(e) \subseteq \overline{J}_1^X(e) \Rightarrow \gamma\left(S\overline{J}_1^X(e)\right) \subseteq \gamma\left(\overline{J}_1^X(e)\right)$$

$\Rightarrow \gamma(S)\left(\overline{J}_1^X(e)\right) \subseteq \gamma\left(\overline{J}_1^X(e)\right)$. Since γ is an epimorphism, we have by above Lemma 3.3.2, $T\overline{J}_2^{\gamma(X)}(e) \subseteq \overline{J}_2^{\gamma(X)}(e)$. Similarly, $\overline{J}_2^{\gamma(X)}(e)T \subseteq \overline{J}_2^{\gamma(X)}(e)$. Thus, $\overline{J}_2^{\gamma(X)}(e)$ is an IL of T. Conversely, suppose that $\overline{J}_2^{\gamma(X)}(e)$ is an IL of T. Now by Lemma 3.3.2, $\gamma\left(\overline{J}_1^X(e)\right) = \overline{J}_2^{\gamma(X)}(e)$. Let $y \in \overline{J}_1^X(e)$ and $s \in S$. This shows that $\gamma(s) \in T$ and $\gamma(y) \in \overline{J}_2^{\gamma(X)}(e)$. Since $\overline{J}_2^{\gamma(X)}(e)$ is an IL so $\gamma(s)\gamma(y) \in \overline{J}_2^{\gamma(X)}(e)$, that is $\gamma(sy) \in \overline{J}_2^{\gamma(X)}(e)$. Thus $\gamma(sy)J_2(e) \cap \gamma(X) \neq \phi$. This implies there exist $z \in X$ such that $\gamma(z) \in \gamma(sy)J_2(e)$, that is $(\gamma(sy), \gamma(z)) \in J_2(e)$ and so $(sy, z) \in J_1(e)$. Thus $z \in syJ_1(e) \cap X$. Hence $sy \in \overline{J}_1^X(e)$. Similarly, we can show that $ys \in \overline{J}_1^X(e)$. This shows that $\overline{J}_1^X(e)$ is an IL of S. Similarly, we can prove the parenthesis case. ■

Theorem 3.3.4 *Let γ be an isomorphism between the semigroups S and T and (J_2, A) be a soft reflexive and SC_mR on T respecting to aftersets (respecting to soft foresets). Let $X \subseteq S$. Consider $J_1(e) = \{(a,b) \in S \times S : (\gamma(a), \gamma(b)) \in J_2(e)\}$, for all $e \in A$. Then $\underline{J}_1^X(e)$ $\left({}^X\underline{J}_1(e)\right)$ is an IL of S if and only if $\underline{J}_2^{\gamma(X)}(e)$ $\left({}^{\gamma(X)}\underline{J}_2(e)\right)$ is an IL of T.*

Proof. It follows from Lemma 3.3.2, $\gamma\left(\underline{J}_1^X(e)\right) = \underline{J}_2^{\gamma(X)}(e)$. The theorem can be proved on the same line as the proof of above theorem. ■

Chapter 4

Approximation of a fuzzy set by soft relation

This chapter presents an investigation of soft binary relations and some of their properties. Two kinds of fuzzy topologies induced by soft reflexive relations are investigated. Soft similarity relations have also been examined. We introduce the degree of accuracy for membership functions of fuzzy sets respecting to the aftersets and foresets. A decision making problem on a fuzzy set is also presented.

4.1 Approximations by soft binary relations

This section defines the approximations of a FS by an $SBRE$. Some related properties are proposed here.

Definition 4.1.1 *Let (J, A) be an $SBRE$ from U to W and λ be a FS in W. Then we define two fuzzy soft sets over U with respect to the aftersets by*

$$\underline{J}^{\lambda}(e)(u) = \begin{cases} \wedge_{a' \in uJ(e)} \lambda\left(a'\right) & \text{if } uJ(e) \neq \phi \\ 0 & \text{if } uJ(e) = \phi \end{cases}$$

and

$$\overline{J}^{\lambda}(e)(u) = \begin{cases} \vee_{a' \in uJ(e)} \lambda\left(a'\right) & \text{if } uJ(e) \neq \phi \\ 0 & \text{if } uJ(e) = \phi \end{cases}$$

where $uJ(e) = \{w \in W : (u, w) \in J(e)\}$ and is called the afterset of u for $u \in U$ and

$e \in A$.

Definition 4.1.2 *Let δ be a FS in U, we define two fuzzy soft sets over W with respect to the foresets by*

$$^{\delta}\underline{J}(e)(w) = \begin{cases} \wedge_{a' \in J(e)w} \delta\left(a'\right) & \text{if } J(e)w \neq \phi \\ 0 & \text{if } J(e)w = \phi \end{cases}$$

and

$$^{\delta}\overline{J}(e)(w) = \begin{cases} \vee_{a' \in J(e)w} \delta\left(a'\right) & \text{if } J(e)w \neq \phi \\ 0 & \text{if } J(e)w = \phi \end{cases}$$

where $J(e)w = \{u \in U : (u, w) \in J(e)\}$ and is called the foreset of w for $w \in W$ and $e \in A$.

Moreover, $\underline{J}^{\lambda} : A \to \mathrm{F}(U)$, $\overline{J}^{\lambda} : A \to \mathrm{F}(U)$ and $^{\delta}\underline{J} : A \to \mathrm{F}(W)$, $^{\delta}\overline{J} : A \to \mathrm{F}(W)$ and we say (U, W, J) a Generalized Soft Approximation Space.

In order to explain these concepts, the following example is given.

Example 4.1.3 *Suppose that Mr. X wants to buy a shirt for his own use. Let $U = \{\text{the set of all shirts designs}\} = \{d_1, d_2, d_3, d_4, d_5, d_6\}$ and $W = \{\text{the colors of all designs}\} = \{c_1, c_2, c_3, c_4\}$ and the set of attributes be $A = \{e_1, e_2, e_3\} = \{\text{the set of stores near his house}\}$.*

Define $J : A \to P(U \times W)$ by

$$\begin{aligned} J(e_1) &= \left\{ \begin{array}{c} (d_1, c_1), (d_1, c_2), (d_1, c_3), (d_2, c_2), (d_2, c_4), \\ (d_4, c_2), (d_4, c_3), (d_5, c_3), (d_5, c_4), (d_6, c_1) \end{array} \right\}, \\ J(e_2) &= \{(d_1, c_3), (d_2, c_3), (d_4, c_1), (d_5, c_1), (d_6, c_2), (d_6, c_3)\} \text{ and} \\ J(e_3) &= \{(d_3, c_3), (d_3, c_1), (d_2, c_4), (d_5, c_3), (d_5, c_4)\}. \end{aligned}$$

represents the relation between designs and colors available on store e_i for $1 \leq i \leq 3$. Then

$$\begin{aligned} d_1 J(e_1) &= \{c_1, c_2, c_3\},\ d_2 J(e_1) = \{c_2, c_4\},\ d_3 J(e_1) = \phi, \\ d_4 J(e_1) &= \{c_2, c_3\},\ d_5 J(e_1) = \{c_3, c_4\},\ d_6 J(e_1) = \{c_1\} \text{ and} \\ d_1 J(e_2) &= \{c_3\},\ d_2 J(e_2) = \{c_3\},\ d_3 J(e_2) = \phi, \\ d_4 J(e_2) &= \{c_1\},\ d_5 J(e_2) = \{c_1\},\ d_6 J(e_2) = \{c_2, c_3\} \text{ and} \\ d_1 J(e_3) &= \phi,\ d_2 J(e_3) = \{c_4\},\ d_3 J(e_3) = \{c_1, c_3\}, \\ d_4 J(e_3) &= \phi,\ d_5 J(e_3) = \{c_3, c_4\},\ d_6 J(e_3) = \phi \end{aligned}$$

where $d_i J(e_j)$ *represents the color of the design* d_i *available on store* e_j*.*

And

$$\begin{aligned} J(e_1)\,c_1 &= \{d_1, d_6\},\ J(e_1)\,c_2 = \{d_1, d_2, d_4\}, \\ J(e_1)\,c_3 &= \{d_1, d_4, d_5\},\ J(e_1)\,c_4 = \{d_2, d_5\},\ \textit{and} \\ J(e_2)\,c_1 &= \{d_4, d_5\},\ J(e_2)\,c_2 = \{d_6\}, \\ J(e_2)\,c_3 &= \{d_1, d_2\},\ J(e_2)\,c_4 = \phi,\ \textit{and} \\ J(e_3)\,c_1 &= \{d_3\},\ J(e_3)\,c_2 = \phi, \\ J(e_3)\,c_3 &= \{d_3, d_5\},\ J(e_3)\,c_4 = \{d_2, d_5\} \end{aligned}$$

where $J(e_j)\,c_i$ *represents the design of the color* c_i *available on store* e_j*.*

$$\begin{aligned} \textit{Define } \lambda &: W \to [0,1] \textit{ which represents the preference of the} \\ & \quad \textit{color given by } Mr.X \textit{ such that} \\ \lambda(c_1) &= 0.9,\ \lambda(c_2) = 0.8,\ \lambda(c_3) = 0.4,\ \lambda(c_4) = 0 \textit{ and} \\ \textit{Define } \delta &: U \to [0,1] \textit{ which represents the preference of the} \\ & \quad \textit{design given by } Mr.X \textit{ such that} \\ \delta(d_1) &= 1,\ \delta(d_2) = 0.7,\ \delta(d_3) = 0.5,\ \delta(d_4) = 0.1, \\ \delta(d_5) &= 0,\ \delta(d_6) = 0.4. \end{aligned}$$

Therefore, the lower and upper approximations (respecting to the aftersets as well as foresets) are

	d_1	d_2	d_3	d_4	d_5	d_6
$\underline{J}^{\lambda}(e_1)$	0.4	0	0	0.4	0	0.9
$\overline{J}^{\lambda}(e_1)$	0.9	0.8	0	0.8	0.4	0.9
$\underline{J}^{\lambda}(e_2)$	0.4	0.4	0	0.9	0.9	0.4
$\overline{J}^{\lambda}(e_2)$	0.4	0.4	0	0.9	0.9	0.8
$\underline{J}^{\lambda}(e_3)$	0	0	0.4	0	0	0
$\overline{J}^{\lambda}(e_3)$	0	0	0.9	0	0.4	0

and

	c_1	c_2	c_3	c_4
${}^{\delta}\underline{J}(e_1)$	0.4	0.1	0	0
${}^{\delta}\overline{J}(e_1)$	1	1	1	0.7
${}^{\delta}\underline{J}(e_2)$	0	0.4	0.7	0
${}^{\delta}\overline{J}(e_2)$	0.1	0.4	1	0
${}^{\delta}\underline{J}(e_3)$	0.5	0	0	0
${}^{\delta}\overline{J}(e_3)$	0.5	0	0.5	0.7

Hence, $\underline{J}^{\lambda}(e_i)(d)$ *gives the degree of definite fulfilment of the objects of* $dJ(e_i)$ *to* λ *on store* e_i *and* $\overline{J}^{\lambda}(e_i)(d)$ *gives the degree of possible fulfilment of the objects of* $dJ(e_i)$ *to* λ *on store* e_i *for* $1 \leq i \leq 3$ *with respect to aftersets. Similarly,* ${}^{\delta}\underline{J}(e_i)(c)$ *gives the degree of definite fulfilment of the objects of* $J(e_i)\,c$ *to* δ *on store* e_i *and* ${}^{\delta}\overline{J}(e_i)(c)$ *gives the degree of possible fulfilment of the objects of* $J(e_i)\,c$ *to* δ *on store* e_i *for* $1 \leq i \leq 3$ *with respect to foresets.*

Theorem 4.1.4 *Let* (U, W, J) *be a Generalized Soft Approximation Space, that is* $J : A \rightarrow P(U \times W)$ *be an SBRE from* U *to* W*. For* $\lambda, \lambda_1, \lambda_2 \in \mathrm{F}(W)$*, the following properties for lower and upper approximations respecting to the aftersets hold:*

(1) $\lambda_1 \leq \lambda_2 \Rightarrow \underline{J}^{\lambda_1} \leq \underline{J}^{\lambda_2}$

(2) $\lambda_1 \leq \lambda_2 \Rightarrow \overline{J}^{\lambda_1} \leq \overline{J}^{\lambda_2}$

(3) $\underline{J}^{\lambda_1} \cap \underline{J}^{\lambda_2} = \underline{J}^{\lambda_1 \cap \lambda_2}$

(4) $\overline{J}^{\lambda_1} \cap \overline{J}^{\lambda_2} \geq \overline{J}^{\lambda_1 \cap \lambda_2}$

(5) $\underline{J}^{\lambda_1} \cup \underline{J}^{\lambda_2} \leq \underline{J}^{\lambda_1 \cup \lambda_2}$

(6) $\overline{J}^{\lambda_1} \cup \overline{J}^{\lambda_2} = \overline{J}^{\lambda_1 \cup \lambda_2}$

(7) $\underline{J}^{1}(e)(u) = 1$ *for all* $e \in A$ *if* $uJ(e) \neq \phi$.

(8) $\overline{J}^{1}(e)(u) = 1$ *for all* $e \in A$ *if* $uJ(e) \neq \phi$.

(9) $\underline{J}^{\lambda} = \left(\overline{J}^{\lambda^c}\right)^c$ *if* $uJ(e) \neq \phi$.

(10) $\overline{J}^{\lambda} = \left(\underline{J}^{\lambda^c}\right)^c$ *if* $uJ(e) \neq \phi$.

(11) $\underline{J}^{0} = 0 = \overline{J}^{0}$

Proof. For $u \in U$, we have two cases: (i) If $uJ(e) = \phi$ and (ii) If $uJ(e) \neq \phi$. If $uJ(e) = \phi$, then all the above parts are trivial. So we consider only the case when $uJ(e) \neq \phi$.

(1) Since $\lambda_1 \leq \lambda_2$, so $\underline{J}^{\lambda_1}(e)(u) = \wedge_{a' \in uJ(e)} \lambda_1 \left(a'\right) \leq \wedge_{a' \in uJ(e)} \lambda_2 \left(a'\right)$

$= \underline{J}^{\lambda_2}(e)(u)$. Hence, $\underline{J}^{\lambda_1} \leq \underline{J}^{\lambda_2}$.

(2) Since $\lambda_1 \leq \lambda_2$, so $\overline{J}^{\lambda_1}(e)(u) = \vee_{a' \in uJ(e)} \lambda_1 \left(a'\right) \leq \vee_{a' \in uJ(e)} \lambda_2 \left(a'\right)$

$= \overline{J}^{\lambda_2}(e)(u)$. Hence, $\overline{J}^{\lambda_1} \leq \overline{J}^{\lambda_2}$.

(3) Consider $\left(\underline{J}^{\lambda_1} \cap \underline{J}^{\lambda_2}\right)(e)(u) = \underline{J}^{\lambda_1}(e)(u) \wedge \underline{J}^{\lambda_2}(e)(u)$

$= \left(\wedge_{a \in uJ(e)} \lambda_1(a)\right) \wedge \left(\wedge_{a \in uJ(e)} \lambda_2(a)\right) = \wedge_{a \in uJ(e)} \left(\lambda_1(a) \wedge \lambda_2(a)\right)$

$= \wedge_{a \in uJ(e)} \left(\lambda_1 \wedge \lambda_2\right)(a) = \underline{J}^{\lambda_1 \cap \lambda_2}(e)(u)$. Hence, $\underline{J}^{\lambda_1} \cap \underline{J}^{\lambda_2} = \underline{J}^{\lambda_1 \cap \lambda_2}$.

(4) Consider $\left(\overline{J}^{\lambda_1} \cap \overline{J}^{\lambda_2}\right)(e)(u) = \overline{J}^{\lambda_1}(e)(u) \wedge \overline{J}^{\lambda_2}(e)(u)$

$= \left(\vee_{a \in uJ(e)} \lambda_1(a)\right) \wedge \left(\vee_{a \in uJ(e)} \lambda_2(a)\right) \geq \vee_{a \in uJ(e)} \left(\lambda_1 \wedge \lambda_2\right)(a) = \overline{J}^{\lambda_1 \cap \lambda_2}(e)(u)$. Hence, $\overline{J}^{\lambda_1} \cap \overline{J}^{\lambda_2} \geq \overline{J}^{\lambda_1 \cap \lambda_2}$.

(5) Consider $\left(\underline{J}^{\lambda_1} \cup \underline{J}^{\lambda_2}\right)(e)(u) = \underline{J}^{\lambda_1}(e)(u) \vee \underline{J}^{\lambda_2}(e)(u)$

$= \left(\wedge_{a \in uJ(e)} \lambda_1(a)\right) \vee \left(\wedge_{b \in uJ(e)} \lambda_2(b)\right) = \wedge_{a,b \in uJ(e)} \left(\lambda_1(a) \vee \lambda_2(b)\right)$

$\leq \wedge_{a \in uJ(e)} \left(\lambda_1 \vee \lambda_2\right)(a) = \underline{J}^{\lambda_1 \cup \lambda_2}(e)(u)$. Hence, $\underline{J}^{\lambda_1} \cup \underline{J}^{\lambda_2} \leq \underline{J}^{\lambda_1 \cup \lambda_2}$.

(6) Consider $\left(\overline{J}^{\lambda_1} \cup \overline{J}^{\lambda_2}\right)(e)(u) = \overline{J}^{\lambda_1}(e)(u) \vee \overline{J}^{\lambda_2}(e)(u)$

$= \left(\vee_{a \in uJ(e)} \lambda_1(a)\right) \vee \left(\vee_{a \in uJ(e)} \lambda_2(a)\right) = \vee_{a \in uJ(e)} \left(\lambda_1(a) \vee \lambda_2(a)\right)$

$= \vee_{a \in uJ(e)} \left(\lambda_1 \vee \lambda_2\right)(a) = \overline{J}^{\lambda_1 \cup \lambda_2}(e)(u)$. Hence, $\overline{J}^{\lambda_1} \cup \overline{J}^{\lambda_2} = \overline{J}^{\lambda_1 \cup \lambda_2}$.

(7) Consider $\underline{J}^{1}(e)(u) = \wedge_{a \in uJ(e)} 1(a) = \wedge_{a \in uJ(e)} (1) = 1$, because $uJ(e) \neq \phi$.

(8) Consider $\overline{J}^{1}(e)(u) = \vee_{a \in uJ(e)} 1(a) = \vee_{a \in uJ(e)} (1) = 1$, because $uJ(e) \neq \phi$.

(9) Consider $\overline{J}^{\lambda^c}(e)(u) = \vee_{a \in uJ(e)} \lambda^c(a) = \vee_{a \in uJ(e)} \left(1 - \lambda(a)\right) = \left(\wedge_{a \in uJ(e)} \lambda(a)\right)^c$

$= \left(\underline{J}^{\lambda}(e)(u)\right)^c$. Therefore, $\left(\overline{J}^{\lambda^c}(e)(u)\right)^c = \underline{J}^{\lambda}(e)(u)$. Hence, $\underline{J}^{\lambda} = \left(\overline{J}^{\lambda^c}\right)^c$.

(10) By part (9), $\underline{J}^{\lambda} = \left(\overline{J}^{\lambda^c}\right)^c$, therefore, $\underline{J}^{\lambda^c} = \left(\overline{J}^{(\lambda^c)^c}\right)^c \Rightarrow \underline{J}^{\lambda^c} = \left(\overline{J}^{\lambda}\right)^c$.

Hence, $\overline{J}^{\lambda} = \left(\underline{J}^{\lambda^c}\right)^c$.

(11) Straightforward. ■

Theorem 4.1.5 *Let* (U, W, J) *be a Generalized Soft Approximation Space, that is* $J : A \to P(U \times W)$ *be an SBRE from* U *to* W*. For* $\delta, \delta_1, \delta_2 \in \mathrm{F}(U)$*, the following properties for lower and upper approximations respecting to the foresets hold:*

(1) $\delta_1 \leq \delta_2 \Rightarrow {}^{\delta_1}\underline{J} \leq {}^{\delta_2}\underline{J}$

(2) $\delta_1 \leq \delta_2 \Rightarrow {}^{\delta_1}\overline{J} \leq {}^{\delta_2}\overline{J}$

(3) ${}^{\delta_1}\underline{J} \cap {}^{\delta_2}\underline{J} = {}^{\delta_1 \cap \delta_2}\underline{J}$

(4) ${}^{\delta_1}\overline{J} \cap {}^{\delta_2}\overline{J} \geq {}^{\delta_1 \cap \delta_2}\overline{J}$

(5) ${}^{\delta_1}\underline{J} \cup {}^{\delta_2}\underline{J} \leq {}^{\delta_1 \cup \delta_2}\underline{J}$

(6) ${}^{\delta_1}\overline{J} \cup {}^{\delta_2}\overline{J} = {}^{\delta_1 \cup \delta_2}\overline{J}$

(7) ${}^{1}\underline{J}(e)(u) = 1$ *for all* $e \in A$ *if* $J(e)w \neq \phi$.

(8) ${}^{1}\overline{J}(e)(u) = 1$ *for all* $e \in A$ *if* $J(e)w \neq \phi$.

(9) ${}^{\delta}\underline{J} = \left({}^{\delta^c}\overline{J}\right)^c$ *if* $J(e)w \neq \phi$.

(10) ${}^{\delta}\overline{J} = \left({}^{\delta^c}\underline{J}\right)^c$ *if* $J(e)w \neq \phi$.

(11) ${}^{0}\underline{J} = 0 = {}^{0}\overline{J}$

Proof. Straightforward. ∎

It is demonstrated by the following example that equality is not valid in (4) and (5) assertions of above Theorems in general.

Example 4.1.6 *Consider* $U = \{n, u, o, b, w\}$ *and* $W = \{m_1, m_2, m_3, m_4\}$*. Let* $A = \{e_1, e_2\}$ *be the set of attributes. Define* $J : A \to P(U \times W)$ *by*

$$\begin{aligned} J(e_1) &= \{(n, m_1), (n, m_2), (o, m_3), (o, m_4), (u, m_1), (o, m_2), (n, m_3), (u, m_4)\}, \\ J(e_2) &= \{(b, m_3), (b, m_1), (b, m_2), (w, m_1), (w, m_3), (w, m_4)\}. \end{aligned}$$

Now,

$$\begin{aligned} nJ(e_1) &= \{m_1, m_2, m_3\}, \ uJ(e_1) = \{m_1, m_4\}, \ oJ(e_1) = \{m_2, m_3, m_4\}, \\ bJ(e_1) &= \phi, \ wJ(e_1) = \phi \end{aligned}$$

and

$$\begin{aligned} nJ(e_2) &= \phi \ , u(e_2) = \phi \ , oJ(e_2) = \phi, \\ bJ(e_2) &= \{m_1, m_2, m_3\}, wJ(e_2) = \{m_1, m_3, m_4\}. \end{aligned}$$

Moreover,

$$\begin{aligned} J(e_1)\,m_1 &= \{n,u\}\,,\ J(e_1)\,m_2=\{n,o\}\,,\ J(e_1)\,m_3=\{n,o\}\,,\\ J(e_1)\,m_4 &= \{o,u\} \end{aligned}$$

and

$$\begin{aligned} J(e_2)\,m_1 &= \{b,w\}\,,\ J(e_2)\,m_2=\{b\}\,,\ J(e_2)\,m_3=\{b,w\}\,,\\ J(e_2)\,m_4 &= \{w\}\,. \end{aligned}$$

Define $\lambda_1,\ \lambda_2,\ \lambda_1\cap\lambda_2,\ \lambda_1\cup\lambda_2 : W\to[0,1]$ *by*

	m_1	m_2	m_3	m_4
λ_1	0.1	0	0.5	0.4
λ_2	0.2	1	0.3	0.6
$\lambda_1\cap\lambda_2$	0.1	0	0.3	0.4
$\lambda_1\cup\lambda_2$	0.2	1	0.5	0.6

And

	n	u	o	b	w
δ_1	0.1	0.5	0.3	0.6	0.8
δ_2	0	0.1	0.4	1	0.7
$\delta_1\cap\delta_2$	0	0.1	0.3	0.6	0.7
$\delta_1\cup\delta_2$	0.1	0.5	0.4	1	0.8

Therefore,

	$(e_1)\,(o)$
$\overline{J}^{\lambda_1}$	0.5
$\overline{J}^{\lambda_2}$	1
$\underline{J}^{\lambda_1}$	0
$\underline{J}^{\lambda_2}$	0.3
$\overline{J}^{\lambda_1\cap\lambda_2}$	0.4
$\underline{J}^{\lambda_1\cup\lambda_2}$	0.5

and

	$(e_1)\,(m_4)$
${}^{\delta_1}\overline{J}$	0.5
${}^{\delta_2}\overline{J}$	0.4
${}^{\delta_1}\underline{J}$	0.3
${}^{\delta_2}\underline{J}$	0.1
${}^{\delta_1\cap\delta_2}\overline{J}$	0.3
${}^{\delta_1\cup\delta_2}\underline{J}$	0.4

Hence,

$$\begin{aligned} \overline{J}^{\lambda_1}(e_1)\,(o)\cap\overline{J}^{\lambda_2}(e_1)\,(o) &= 0.5\nleq 0.4=\overline{J}^{\lambda_1\cap\lambda_2}(e_1)\,(o)\ \ \textit{and}\\ \underline{J}^{\lambda_1\cup\lambda_2}(e_1)\,(o) &= 0.5\nleq 0.3=\underline{J}^{\lambda_1}(e_1)\,(o)\cup\underline{J}^{\lambda_2}(e_1)\,(o)\,. \end{aligned}$$

And

$$\begin{aligned} {}^{\delta_1}\overline{J}(e_1)(m_4) \cap {}^{\delta_2}\overline{J}(e_1)(m_4) &= 0.4 \nleq 0.3 = {}^{\delta_1\cap\delta_2}\overline{J}(e_1)(m_4) \ \text{and} \\ {}^{\delta_1\cup\delta_2}\underline{J}(e_1)(m_4) &= 0.4 \nleq 0.3 = {}^{\delta_1}\underline{J}(e_1)(m_4) \cup {}^{\delta_2}\underline{J}(e_1)(m_4). \end{aligned}$$

Theorem 4.1.7 *Let (J, A) and (Z, A) be two SBRE from non-empty sets U to W and λ_1, λ_2 be non-empty FS of W. Then the following hold:*

(1) $(J, A) \subseteq (Z, A)$ *implies* $\left(\underline{J}^{\lambda_1}, A\right) \supseteq \left(\underline{Z}^{\lambda_1}, A\right)$;

(2) $(J, A) \subseteq (Z, A)$ *implies* $\left(\overline{J}^{\lambda_1}, A\right) \subseteq \left(\overline{Z}^{\lambda_1}, A\right)$.

Proof. Straightforward. ■

Theorem 4.1.8 *Let (J, A) and (Z, A) be two SBRE from non-empty sets U to W and Y_1, Y_2 be non-empty subsets of U. Then the following hold:*

(7) $(J, A) \subseteq (Z, A)$ *implies* $\left({}^{\delta_1}\underline{J}, A\right) \supseteq \left({}^{\delta_1}\underline{Z}, A\right)$;

(8) $(J, A) \subseteq (Z, A)$ *implies* $\left({}^{\delta_1}\overline{J}, A\right) \subseteq \left({}^{\delta_1}\overline{Z}, A\right)$.

Proof. Straightforward. ■

Theorem 4.1.9 *Let (J, A) and (Z, A) be SBRE from non-empty sets U to W and λ be a FS of W. Then*

(1) $\left(\left(\overline{J \cap Z}\right)^{\lambda}, A\right) \subseteq \left(\overline{J}^{\lambda}, A\right) \cap \left(\overline{Z}^{\lambda}, A\right)$.

(2) $\left(\left(\underline{J \cap Z}\right)^{\lambda}, A\right) \supseteq \left(\underline{J}^{\lambda}, A\right) \cup \left(\underline{Z}^{\lambda}, A\right)$.

Proof. It follows from parts (1) and (2) of Theorem 4.1.7. ■

It is found in the accompanying Example that equality is not valid in above Theorem.

Example 4.1.10 *Let $U = \{a, b, c, d, e\}$ and $W = \{1, 2, 3, 4, 5\}$ and $A = \{e_1, e_2\}$. Define $J : A \to P(U \times W)$ and $Z : A \to P(U \times W)$ by*

$$J(e_1) = \left\{ \begin{array}{c} (a,1),(b,2),(c,3),(d,4),(e,5),(b,1),(c,5), \\ (b,5),(d,3),(d,5),(d,1),(e,1) \end{array} \right\}$$

$$J(e_2) = \{(a,1),(b,2),(c,3),(d,4),(e,5)\},$$

$$Z(e_1) = \{(a,1),(b,2),(c,3),(d,4),(e,5),(b,1)\} \textit{ and}$$

$$Z(e_2) = \{(a,1),(b,2),(c,3),(d,4),(e,5),(b,3)\}.$$

Therefore,

$$(J \cap Z)(e_1) = \{(a,1),(b,2),(c,3),(d,4),(e,5),(b,1)\}$$

and

$$(J \cap Z)(e_2) = \{(a,1),(b,2),(c,3),(d,4),(e,5)\}.$$

Now,

$$\begin{aligned} aJ(e_1) &= \{1\},\ bJ(e_1) = \{1,2,5\},\ cJ(e_1) = \{3,5\}, \\ dJ(e_1) &= \{1,3,4,5\} \textit{ and } eJ(e_1) = \{1,5\} \end{aligned}$$

and

$$\begin{aligned} aZ(e_1) &= \{1\},\ bZ(e_1) = \{1,2\},\ cZ(e_1) = \{3\}, \\ dZ(e_1) &= \{4\} \textit{ and } eZ(e_1) = \{2,5\}. \end{aligned}$$

Also,

$$\begin{aligned} a(J \cap Z)(e_1) &= \{1\},\ b(J \cap Z)(e_1) = \{1,2\},\ c(J \cap Z)(e_1) = \{3\}, \\ d(J \cap Z)(e_1) &= \{4\} \textit{ and } e(J \cap Z)(e_1) = \{5\}. \end{aligned}$$

Define $\lambda_1 : W \longrightarrow [0,1]$ *by*

	1	2	3	4	5
λ_1	0.1	0.5	0	0.7	0

Then λ_1 is a FS of W but

	a	b	c	d	e
$\overline{J}^{\lambda_1}(e_1)$	0.1	0.5	0	0.7	0.1
$\overline{Z}^{\lambda_1}(e_1)$	0.1	0.5	0	0.7	0.5
$\overline{(J \cap Z)}^{\lambda_1}(e_1)$	0.1	0.5	0	0.7	0

This shows that

$$\overline{J}^{\lambda_1}(e_1) \cap \overline{Z}^{\lambda_1}(e_1) \neq \left(\overline{J \cap Z}\right)^{\lambda_1}(e_1).$$

Now, Define $\lambda_2 : W \longrightarrow [0,1]$ by

	1	2	3	4	5
λ_2	0.2	0.7	1	0.6	1

Then λ_2 is a FS of W but

	a	b	c	d	e
$\underline{J}^{\lambda_2}(e_1)$	0.1	0.5	0	0.7	0.1
$\underline{Z}^{\lambda_2}(e_1)$	0.1	0.5	0	0.7	0.5
$(\underline{J \cap Z})^{\lambda_2}(e_1)$	0.1	0.5	0	0.7	0

This shows that

$$\underline{J}^{\lambda_2}(e_1) \cup \underline{Z}^{\lambda_2}(e_1) \neq (\underline{J \cap Z})^{\lambda_2}(e_1).$$

Theorem 4.1.11 *Let (J,A) and (Z,A) be SBRE from non-empty sets U to W. If δ is a FS of U, then*

(1) $\left({}^{\delta}\left(\overline{J \cap Z}\right), A\right) \subseteq \left({}^{\delta}\overline{J}, A\right) \cap \left({}^{\delta}\overline{Z}, A\right).$

(2) $\left({}^{\delta}\left(\underline{J \cap Z}\right), A\right) \supseteq \left({}^{\delta}\underline{J}, A\right) \cup \left({}^{\delta}\underline{Z}, A\right).$

Proof. It follows from parts (1) and (2) of Theorem 4.1.8. ■

It is found in the accompanying Example that equality is not valid in above Theorem.

Example 4.1.12 *Let $U = \{a,b,c,d,e\}$ and $W = \{1,2,3,4,5\}$ and $A = \{e_1, e_2\}$. Define $J : A \to P(U \times W)$ and $Z : A \to P(U \times W)$ by*

$$J(e_1) = \left\{ \begin{array}{c} (a,1),(b,2),(c,3),(d,4),(e,5),(b,1),(c,5), \\ (b,5),(d,3),(d,5),(d,1),(e,1)\} \end{array} \right\},$$

$$J(e_2) = \{(a,1),(b,2),(c,3),(d,4),(e,5)\},$$

$$Z(e_1) = \{(a,1),(b,2),(c,3),(d,4),(e,5),(b,1),(a,5)\} \text{ and}$$

$$Z(e_2) = \{(a,1),(b,2),(c,3),(d,4),(e,5),(b,3)\}.$$

Therefore,

$$(J \cap Z)(e_1) = \{(a,1),(b,2),(c,3),(d,4),(e,5),(b,1)\}$$

and

$$(J \cap Z)(e_2) = \{(a,1), (b,2), (c,3), (d,4), (e,5)\}.$$

Now,

$$\begin{aligned} J(e_1)1 &= \{a,b,e\},\ J(e_1)2 = \{b\},\ J(e_1)3 = \{c,d\}, \\ J(e_1)4 &= \{d\} \textit{ and } J(e_1)5 = \{b,c,d,e\}, \end{aligned}$$

and

$$\begin{aligned} Z(e_1)1 &= \{a,b\},\ Z(e_1)2 = \{b\},\ 3Z(e_1) = \{c\}, \\ Z(e_1)4 &= \{d\} \textit{ and } Z(e_1)5 = \{a,e\}. \end{aligned}$$

Also,

$$\begin{aligned} (J \cap Z)(e_1)1 &= \{a,b\},\ (J \cap Z)(e_1)2 = \{b\},\ (J \cap Z)(e_1)3 = \{c\}, \\ (J \cap Z)(e_1)4 &= \{d\} \textit{ and } (J \cap Z)(e_1)5 = \{e\}. \end{aligned}$$

Define $\delta_1 : U \longrightarrow [0,1]$ *by*

	a	b	c	d	e
δ_1	0.3	0	0.4	0.5	0

Then δ_1 is a FS of U but

	1	2	3	4	5
${}^{\delta_1}\overline{J}(e_1)$	0.3	0	0.5	0.5	0.5
${}^{\delta_1}\overline{Z}(e_1)$	0.3	0	0.4	0.5	0.3
${}^{\delta_1}\overline{(J \cap Z)}(e_1)$	0.3	0	0.4	0.5	0

This shows that

$${}^{\delta_1}\overline{J}(e_1) \cap^{\delta_1} \overline{Z}(e_1) \neq^{\delta_1} \left(\overline{J \cap Z}\right)(e_1).$$

Now, Define $\delta_2 : U \longrightarrow [0,1]$ by

	a	b	c	d	e
δ_2	0.5	1	0.7	0.1	1

Then δ_2 is a FS of U but

	1	2	3	4	5
${}^{\delta_2}\underline{J}\,(e_1)$	0.5	1	0.1	0.1	0.1
${}^{\delta_2}\underline{Z}\,(e_1)$	0.5	1	0.7	0.1	0.5
${}^{\delta_2}\,(\underline{J \cap Z})\,(e_1)$	0.5	1	0.7	0.1	1

This shows that

$$^{\delta_2}\underline{J}\,(e_1) \cup^{\delta_2} \underline{Z}\,(e_1) \neq \; ^{\delta_2}\,(\underline{J \cap Z})\,(e_1)\,.$$

Theorem 4.1.13 *Let (U, W, J) be a Generalized Soft Approximation Space. For $\{i \in I : \lambda_i\} \subseteq$ $\text{Ꞙ}\,(W)$, the following properties for lower and upper approximations respecting to the aftersets hold:*

(1) $\underline{J}^{(\cap_{i\in I}\lambda_i)}\,(e) = \cap_{i\in I}\underline{J}^{\lambda_i}\,(e)$ *for all* $e \in A$

(2) $\underline{J}^{(\cup_{i\in I}\lambda_i)}\,(e) \supseteq \cup_{i\in I}\underline{J}^{\lambda_i}\,(e)$ *for all* $e \in A$

(3) $\overline{J}^{(\cup_{i\in I}\lambda_i)}\,(e) = \cup_{i\in I}\overline{J}^{\lambda_i}\,(e)$ *for all* $e \in A$

(4) $\overline{J}^{(\cap_{i\in I}\lambda_i)}\,(e) \subseteq \cap_{i\in I}\overline{J}^{\lambda_i}\,(e)$ *for all* $e \in A$.

Proof. (1) Take $\lambda_i \in \text{Ꞙ}\,(U)$, where $i \in I$. Then we have

$\underline{J}^{(\cap_{i\in I}\lambda_i)}\,(e)\,(u) = \wedge_{a\in uJ(e)}(\wedge_{i\in I}\lambda_i)(a) = \wedge_{i\in I}(\wedge_{a\in uJ(e)}\lambda_i(a)) = \cap_{i\in I}\underline{J}^{\lambda_i}\,(e)\,(u)$ for all $e \in A$ and $u \in U$.

(2) Take $\lambda_i \in \text{Ꞙ}\,(U)$, where $i \in I$. Then we have

$\underline{J}^{(\cup_{i\in I}\lambda_i)}\,(e)\,(u) = \wedge_{a\in uJ(e)}(\vee_{i\in I}\lambda_i)(a) \geq \vee_{i\in I}\left(\wedge_{a\in uJ(e)}\lambda_i\,(a)\right) = \cup_{i\in I}\underline{J}^{\lambda_i}\,(e)\,(u)$ for all $e \in A$ and $u \in U$.

(3) It has a comparable proof to the proof of (1).

(4) It has a comparable proof to the proof of (2). ■

Theorem 4.1.14 *Let (U, W, J) be a Generalized Soft Approximation Space. For $\{i \in I : \delta_i\} \subseteq$ $\text{Ꞙ}\,(U)$, the following properties for lower and upper approximations respecting to the foresets hold:*

(1) ${}^{(\cap_{i\in I}\delta_i)}\underline{J}\,(e) = \cap_{i\in I}\;{}^{\delta_i}\underline{J}\,(e)$ *for all* $e \in A$

(2) ${}^{(\cup_{i\in I}\delta_i)}\underline{J}\,(e) \supseteq \cup_{i\in I}\;{}^{\delta_i}\underline{J}\,(e)$ *for all* $e \in A$

(3) ${}^{(\cup_{i\in I}\delta_i)}\overline{J}\,(e) = \cup_{i\in I}\;\overline{J}^{\delta_i}\,(e)$ *for all* $e \in A$

(4) ${}^{(\cap_{i\in I}\delta_i)}\overline{J}\,(e) \subseteq \cap_{i\in I}\ \overline{J}^{\delta_i}\,(e)$ *for all* $e \in A$.

Proof. It has a comparable proof to the proof of above Theorem. ■

If (J, A) is a soft reflexive relation then each $uJ(e)$ (resp. $J(e)u$) is non-empty and $u \in uJ(e)$ (resp. $u \in J(e)u$). It is not necessary that $uJ(e) = J(e)u$.

If (J, A) is a soft $E-$relation n U, then $uJ(e) = J(e)u$ and $\{uJ(e) : u \in U\}$ is a partition of U. Also, in this case, ${}^{\lambda}\underline{J}\,(e) = \underline{J}^{\lambda}\,(e)$ and ${}^{\lambda}\overline{J}\,(e) = \overline{J}^{\lambda}\,(e)$ for all $\lambda \in Ғ\,(U)$.

The approximation operators have additional properties with respect to soft reflexive binary relation as follows:

Theorem 4.1.15 *For* $\lambda \in Ғ\,(U)$, *the following properties for lower and upper approximations with respect to the aftersets hold:*

(1) $\underline{J}^{\lambda}\,(e) \leq \lambda$ *for all* $e \in A$

(2) $\lambda \leq \overline{J}^{\lambda}\,(e)$ *for all* $e \in A$

(3) $\underline{J}^{\lambda}\,(e) \leq \overline{J}^{\lambda}\,(e)$ *for all* $e \in A$.

Proof. For $u \in U$,

(1) Consider $\underline{J}^{\lambda}\,(e)\,(u) = \wedge_{a\in uJ(e)}\lambda\,(a) \leq \lambda\,(u)$, because $u \in uJ\,(e)$, therefore $\underline{J}^{\lambda}\,(e)\,(u) \leq \lambda\,(u)$. Hence, $\underline{J}^{\lambda}\,(e) \leq \lambda$.

(2) Consider $\lambda\,(u) \leq \vee_{a\in uJ(e)}\lambda\,(a) = \overline{J}^{\lambda}\,(e)\,(u)$. Hence, $\lambda \leq \overline{J}^{\lambda}\,(e)$.

(3) It directly follows from (1) and (2). ■

Theorem 4.1.16 *For* $\delta \in Ғ\,(U)$, *the following properties for lower and upper approximations with respect to the foresets hold:*

(1) ${}^{\delta}\underline{J}\,(e) \leq \delta$ *for all* $e \in A$

(2) $\delta \leq {}^{\delta}\overline{J}\,(e)$ *for all* $e \in A$

(3) ${}^{\delta}\underline{J}\,(e) \leq {}^{\delta}\overline{J}\,(e)$ *for all* $e \in A$.

Proof. It has a comparable proof to the proof of above Theorem. ■

Theorem 4.1.17 *Let (J,A) and (Z,A) be two soft reflexive relations on a non-empty set U such that $J(e) \subseteq Z(e)$ for all $e \in A$. Then $\underline{Z}^{\mu}(e) \leq \underline{J}^{\mu}(e)$ and $\overline{J}^{\mu}(e) \leq \overline{Z}^{\mu}(e)$ for all $\mu \in \digamma(U)$ and for all $e \in A$ with respect to the aftersets.*

Proof. Let $\mu \in \digamma(U)$. Since $J(e) \subseteq Z(e)$, we have $uJ(e) \subseteq uZ(e)$ for all $u \in U$ and $e \in A$. Therefore, $\wedge_{a \in uJ(e)} \mu(a) \geq \wedge_{a \in uZ(e)} \mu(a)$ and $\vee_{a \in uJ(e)} \mu(a) \leq \vee_{a \in uZ(e)} \mu(a)$ for all $u \in U$. By definition 4.1.1, $\underline{Z}^{\mu}(e) \leq \underline{J}^{\mu}(e)$ and $\overline{J}^{\mu}(e) \leq \overline{Z}^{\mu}(e)$ with respect to the aftersets. ■

Theorem 4.1.18 *Let (J,A) and (Z,A) be two soft reflexive relations on a non-empty set U such that $J(e) \subseteq Z(e)$ for all $e \in A$. Then ${}^{\mu}\underline{Z}(e) \leq {}^{\mu}\underline{J}(e)$ and ${}^{\mu}\overline{J}(e) \leq {}^{\mu}\overline{Z}(e)$ for all $\mu \in \digamma(U)$ and for all $e \in A$ with respect to foresets.*

Proof. It has a comparable proof to above theorem. ■

Corollary 4.1.19 *Let (J,A) and (Z,A) be two soft reflexive relations on a non-empty set U. Then the following assertions hold for all $\lambda \in \digamma(U)$ and for all $e \in A$ with respect to the aftersets.*

(1) $\overline{(J \cap Z)}^{\lambda}(e) \leq \overline{J}^{\lambda}(e) \cap \overline{Z}^{\lambda}(e)$

(2) $\underline{(J \cap Z)}^{\lambda}(e) \geq \underline{J}^{\lambda}(e) \cup \underline{Z}^{\lambda}(e)$.

Proof. (1) Let (J,A) and (Z,A) be two soft reflexive relations on a non-empty set U. Then $(J \cap Z, A)$ is also a soft reflexive relation on U. Also, $(J \cap Z)(e) \subseteq J(e)$ and $(J \cap Z) \subseteq Z(e)$. By Theorem 4.1.17, $\overline{(J \cap Z)}^{\lambda}(e) \leq \overline{J}^{\lambda}(e)$ and $\overline{(J \cap Z)}^{\lambda}(e) \leq \overline{Z}^{\lambda}(e)$ for any $\lambda \in \digamma(U)$. This proves that $\overline{(J \cap Z)}^{\lambda}(e) \leq \overline{J}^{\lambda}(e) \cap \overline{Z}^{\lambda}(e)$ for all $\lambda \in \digamma(U)$ and for all $e \in A$.

(2) This can be proved as (1). ■

Corollary 4.1.20 *Let (J,A) and (Z,A) be two soft reflexive relations on a non-empty set U. Then the following assertions hold for all $\delta \in \digamma(U)$ and for all $e \in A$ with respect to the foresets.*

(1) ${}^{\delta}\overline{(J \cap Z)}(e) \leq {}^{\delta}\overline{J}(e) \cap {}^{\delta}\overline{Z}(e)$

(2) ${}^{\delta}\underline{(J \cap Z)}(e) \geq {}^{\delta}\underline{J}(e) \cup {}^{\delta}\underline{Z}(e)$.

Proof. It has a comaprable proof as above Theorem. ■

4.2 Fuzzy topologies induced by soft reflexive relations

This section investigates two kinds of fuzzy topologies induced by soft reflexive relations and related results are also considered.

Definition 4.2.1 *[68] A family $T \subseteq \mathrm{F}(U)$ of FS on U is called a fuzzy topology for U if it satisfies the three axioms given below:*

(1) $0, 1 \in T$.

(2) $\forall \lambda,\ \mu \in T \Rightarrow \lambda \cap \mu \in T$.

(3) $\lambda_j \in T$ *for all* $j \in \mathbf{J} \Longrightarrow \cup_{j \in J} \lambda_j \in T$.

The pair (U, T) is called a fuzzy topological space. The elements of T are called fuzzy open sets.

Theorem 4.2.2 *If (J, A) is a soft reflexive relation on U, then $T_e = \{\lambda \in \mathrm{F}(U) : \underline{J}^{\lambda}(e) = \lambda\ \}$ is a fuzzy topology on U for each $e \in A$.*

Proof. (1) Take $e \in A$. By Theorem 4.1.4, $\underline{J}^{0}(e) = 0$ and $\underline{J}^{1}(e) = 1$. This implies that $0, 1 \in T_e$.

(2) Let $\lambda, \delta \in T_e$. This implies that $\underline{J}^{\lambda}(e) = \lambda$ and $\underline{J}^{\delta}(e) = \delta$.

Now, by using Theorem 4.1.4, $\underline{J}^{\lambda \cap \delta}(e) = \underline{J}^{\lambda}(e) \cap \underline{J}^{\delta}(e) = \lambda \cap \delta$. This implies that $\lambda \cap \delta \in T_e$.

(3) Let $\lambda_j \in T_e$ for $j \in \mathbf{J}$. This implies that $\underline{J}^{\lambda_j}(e) = \lambda_j$ for $j \in J$. Since the relation is soft reflexive, so by Theorem 4.1.15, $\underline{J}^{\cup_{j \in J}\lambda_j}(e) \leq \cup_{j \in J}\lambda_j$. Since, $\lambda_j \leq \cup_{j \in J}\ \lambda_j$, by using Theorem 4.1.4, $\underline{J}^{\lambda_j}(e) \leq \underline{J}^{\cup_{j \in J}\lambda_j}(e)$. This implies $\cup_{j \in J}\underline{J}^{\lambda_j}(e) \leq \underline{J}^{\cup_{j \in J}\lambda_j}(e)$. This implies $\cup_{j \in J}\lambda_j \leq \underline{J}^{\cup_{j \in J}\lambda_j}(e)$. Therefore, $\underline{J}^{\cup_{j \in J}\lambda_j}(e) = \cup_{j \in J}\lambda_j$. Hence, $\cup_{j \in J}\lambda_j \in T_e$. ■

Theorem 4.2.3 *If (J, A) is a soft reflexive relation on U, then $T_e^{'} = \{\mu \in \mathrm{F}(U) : {}^{\mu}\underline{J}(e) = \mu\ \}$ is a fuzzy topology on U for $e \in A$.*

Proof. It has a comparable proof to the proof of above theorem. ■

Remark 4.2.4 *In the above two theorems, corresponding to each $e \in A$, we constructed two fuzzy topologies on U. If we define $\underline{T_e} = \{\lambda \in \mathrm{F}(U) : \underline{J}^{\lambda}(e) = \lambda$ for all*

$e \in A\}$, *then* $\underline{T_e}$ *is a fuzzy topology on* U *and* $\underline{T_e} = \cap_{e \in A} T_e$. *Similarly, if we define* $\underline{T_e'} = \{\mu \in \digamma(U) : {}^{\mu}\underline{J}(e) = \mu$ *for all* $e \in A\}$, *then* $\underline{T_e'}$ *is a fuzzy topology on* U *and* $\underline{T_e'} = \cap_{e \in A} T_e'$.

Definition 4.2.5 *Let* (J, A) *be a soft reflexive relation over* U. *Define a binary relation* R_J *on* U *by* $xR_Jy \Leftrightarrow xJ(e)y$ *for some* $e \in A$ *where* $x, y \in U$. *Then* R_J *is called the binary relation induced by the soft binary relation* (J, A).

Remark 4.2.6 (J, A) *is a soft reflexive relation over* $U \Rightarrow R_J$ *is a reflexive relation over* U.

Theorem 4.2.7 *Let* (J, A) *be an SBRE over* U *and* R_J *be the induced binary relation by* (J, A) *over* U. *For* $\lambda_1, \lambda_2 \in \digamma(U)$, *the following properties for lower and upper approximations respecting to the aftersets hold:*

(1) $\lambda_1 \leq \lambda_2 \Rightarrow \underline{R_J}(\lambda_1) \leq \underline{R_J}(\lambda_2)$

(2) $\lambda_1 \leq \lambda_2 \Rightarrow \overline{R_J}(\lambda_1) \leq \overline{R_J}(\lambda_2)$

(3) $\underline{R_J}(\lambda_1) \cap \underline{R_J}(\lambda_2) = \underline{R_J}(\lambda_1 \cap \lambda_2)$

(4) $\overline{R_J}(\lambda_1) \cap \overline{R_J}(\lambda_2) \geq \overline{R_J}(\lambda_1 \cap \lambda_2)$

(5) $\underline{R_J}(\lambda_1) \cup \underline{R_J}(\lambda_2) \leq \underline{R_J}(\lambda_1 \cup \lambda_2)$

(6) $\overline{R_J}(\lambda_1) \cup \overline{R_J}(\lambda_2) = \overline{R_J}(\lambda_1 \cup \lambda_2)$.

Proof. Straightforward. ■

Theorem 4.2.8 *Let* (J, A) *be an SBRE over* U *and* R_J *be the induced binary relation by* (J, A) *over* U. *For* $\delta_1, \delta_2 \in \digamma(U)$, *the following properties for lower and upper approximations respecting to the foresets hold:*

(1) $\delta_1 \leq \delta_2 \Rightarrow (\delta_1)\underline{R_J} \leq (\delta_2)\underline{R_J}$

(2) $\delta_1 \leq \delta_2 \Rightarrow (\delta_1)\overline{R_J} \leq (\delta_2)\overline{R_J}$

(3) $(\delta_1)\underline{R_J} \cap (\delta_2)\underline{R_J} = (\delta_1 \cap \delta_2)\underline{R_J}$

(4) $(\delta_1)\overline{R_J} \cap (\delta_2)\overline{R_J} \geq (\delta_1 \cap \delta_2)\overline{R_J}$

(5) $(\delta_1)\underline{R_J} \cup (\delta_2)\underline{R_J} \leq (\delta_1 \cup \delta_2)\underline{R_J}$

(6) $(\delta_1)\overline{R_J} \cup (\delta_2)\overline{R_J} = (\delta_1 \cup \delta_2)\overline{R_J}$.

Proof. Straightforward. ■

Theorem 4.2.9 *If (J, A) is a soft reflexive relation on U, then $T_{R_J} = \{\lambda \in \text{Ғ}(U) : \underline{R_J}(\lambda) = \lambda\ \}$ is a fuzzy topology on U with respect to the aftersets.*

Proof. (1) By Theorem 4.2.7, $\underline{R_J}(0) = 0$ and $\underline{R_J}(1) = 1$. This implies $0, 1 \in T_{R_J}$.

(2) Let $\lambda, \delta \in T_{R_J}$. This implies $\underline{R_J}(\lambda) = \lambda$ and $\underline{R_J}(\delta) = \delta$.

Now, by using Theorem 4.2.7, $\underline{R_J}(\lambda \cap \delta) = \underline{R_J}(\lambda) \cap \underline{R_J}(\delta) = \lambda \cap \delta$. This implies $\lambda \cap \delta \in T_{R_J}$.

(3) Let $\lambda_j \in T_{R_J}$ for $j \in \mathbf{J}$. This implies $\underline{R_J}(\lambda_j) = \lambda_j$ for $j \in \mathbf{J}$. By definition $\underline{R_J}(\cup_{j \in J}\lambda_j) = \cup_{j \in J}\lambda_j$. Hence, $\cup_{j \in J}\lambda_j \in T_{R_J}$. ■

Theorem 4.2.10 *If (J, A) is a soft reflexive relation on U, then $T'_{R_J} = \{\mu \in \text{Ғ}(U) : (\mu)\underline{R_J} = \mu\ \}$ is a fuzzy topology on U.*

Proof. It has a comaparable proof to above Theorem 4.2.9. ■

4.3 Similarity relations associated with soft binary relations

In this section, some binary relations between FS are defined based on their rough approximations and their properties are investigated.

Definition 4.3.1 *Let (J, A) be a soft reflexive relation over U. For $\lambda_1, \lambda_2 \in \text{Ғ}(U)$, we define*

$\lambda_1 \backsimeq_A \lambda_2$ if and only if $\underline{J}^{\lambda_1} = \underline{J}^{\lambda_2}$

$\lambda_1 \eqsim_A \lambda_2$ if and only if $\overline{J}^{\lambda_1} = \overline{J}^{\lambda_2}$

$\lambda_1 \approx_A \lambda_2$ if and only if $\underline{J}^{\lambda_1} = \underline{J}^{\lambda_2}$ and $\overline{J}^{\lambda_1} = \overline{J}^{\lambda_2}$.

Definition 4.3.2 *Let (J, A) be an $SBRE$ over U. For $\delta_1, \delta_2 \in \text{Ғ}(U)$, we define*

$\delta_1 \backsimeq_F \delta_2$ if and only if ${}^{\delta_1}\underline{J} = {}^{\delta_2}\underline{J}$

$\delta_1 \eqsim_F \delta_2$ if and only if ${}^{\delta_1}\overline{J} = {}^{\delta_2}\overline{J}$

$\delta_1 \approx_F \delta_2$ *if and only if* ${}^{\delta_1}\underline{J} = {}^{\delta_2}\underline{J}$ *and* ${}^{\delta 1}\overline{J} = {}^{\delta_2}\overline{J}$.

These binary relations are named as the lower fuzzy similarity relation , upper fuzzy similarity relation and fuzzy similarity relation, respectively. Obviously, λ_1 and λ_2 are similar if and only if they are both lower and upper similar and δ_1 and δ_2 are similar if and only if they are both lower and upper similar.

Proposition 4.3.3 *The relations* $\backsimeq_A$, $\eqsim_A$ *and* $\approx_A$ *are* $E-$*relations on* $\mathrm{F}(U)$.

Proof. Straightforward. ■

Proposition 4.3.4 *The relations* $\backsimeq_F$, $\eqsim_F$ *and* $\approx_F$ *are* $E-$*relations on* $\mathrm{F}(U)$.

Proof. Straightforward. ■

Theorem 4.3.5 *Let* (J, A) *be a soft reflexive relation on* U. *For* $\lambda_i \in \mathrm{F}(U)$ *where* $i = 1, 2, 3, 4$ *the following assertions hold:*

(1) $\lambda_1 \eqsim_A \lambda_2$ *if and only if* $\lambda_1 \eqsim_A (\lambda_1 \cup \lambda_2) \eqsim_A \lambda_2$

(2) $\lambda_1 \eqsim_A \lambda_2$ *and* $\lambda_3 \eqsim_A \lambda_4$ *imply that* $(\lambda_1 \cup \lambda_3) \eqsim_A (\lambda_2 \cup \lambda_4)$

(3) $\lambda_1 \leq \lambda_2$ *and* $\lambda_2 \eqsim_A 0$ *imply that* $\lambda_1 \eqsim_A 0$

(4) $(\lambda_1 \cup \lambda_2) \eqsim_A 0$ *if and only if* $\lambda_1 \eqsim_A 0$ *and* $\lambda_2 \eqsim_A 0$

(5) $\lambda_1 \leq \lambda_2$ *and* $\lambda_1 \eqsim_A 1$ *imply that* $\lambda_2 \eqsim_A 1$

(6) *If* $(\lambda_1 \cap \lambda_2) \eqsim_A 1$ *then,* $\lambda_1 \eqsim_A 1$ *and* $\lambda_2 \eqsim_A 1$.

Proof. (1) Let $\lambda_1 \eqsim_A \lambda_2$. Then $\overline{J}^{\lambda_1} = \overline{J}^{\lambda_2}$. By Theorem 4.1.4(6), we get $\overline{J}^{\lambda_1 \cup \lambda_2} = \overline{J}^{\lambda_1} \cup \overline{J}^{\lambda_2} = \overline{J}^{\lambda_1} = \overline{J}^{\lambda_2}$ so $\lambda_1 \eqsim_A (\lambda_1 \cup \lambda_2) \eqsim_A \lambda_2$. Converse holds by transitivity of the relation $\eqsim_A$.

(2) Given that $\lambda_1 \eqsim_A \lambda_2$ and $\lambda_3 \eqsim_A \lambda_4$. Then $\overline{J}^{\lambda_1} = \overline{J}^{\lambda_2}$ and $\overline{J}^{\lambda_3} = \overline{J}^{\lambda_4}$.

By Theorem 4.1.4(6), we get $\overline{J}^{\lambda_1 \cup \lambda_3} = \overline{J}^{\lambda_1} \cup \overline{J}^{\lambda_3} = \overline{J}^{\lambda_2} \cup \overline{J}^{\lambda_4} = \overline{J}^{\lambda_2 \cup \lambda_4}$. Thus, $(\lambda_1 \cup \lambda_3) \eqsim_A (\lambda_2 \cup \lambda_4)$.

(3) Given $\lambda_2 \eqsim_A 0$. This implies $\overline{J}^{\lambda_2} = \overline{J}^{0}$.

Also, $\lambda_1 \leq \lambda_2 \Rightarrow \overline{J}^{\lambda_1} \subseteq \overline{J}^{\lambda_2} = \overline{J}^{0}$. It follows that $\overline{J}^{\lambda_1} \subseteq \overline{J}^{0}$ but $\overline{J}^{0} \subseteq \overline{J}^{\lambda_1}$. Therefore, $\overline{J}^{\lambda_1} = \overline{J}^{0} \Rightarrow \lambda_1 \eqsim_A 0$.

(4) Let $\lambda_1 \eqsim_A 0$ and $\lambda_2 \eqsim_A 0$. Then $\overline{J}^{\lambda_1} = \overline{J}^{0}$ and $\overline{J}^{\lambda_2} = \overline{J}^{0}$. By Theorem 4.1.4(6), we get $\overline{J}^{\lambda_1 \cup \lambda_2} = \overline{J}^{\lambda_1} \cup \overline{J}^{\lambda_2} = \overline{J}^{0} \cup \overline{J}^{0} = \overline{J}^{0}$.

Thus, $(\lambda_1 \cup \lambda_2) \eqsim_A 0$. Converse follows from (3).

(5) Given $\lambda_1 \eqsim_A 1$. This implies $\overline{J}^{\lambda_1} = \overline{J}^{1}$.

Also, $\lambda_1 \leq \lambda_2 \Rightarrow \overline{J}^{\lambda_2} \supseteq \overline{J}^{\lambda_1} = \overline{J}^{1} = 1 \supseteq \overline{J}^{\lambda_2}$. Therefore, $\overline{J}^{\lambda_2} = \overline{J}^{1} \Rightarrow \lambda_2 \eqsim_A 1$.

(6) It follows from (5). ■

Theorem 4.3.6 *Let (J, A) be a soft reflexive relation on U. For $\delta_i \in \mathbb{F}(U)$ where $i = 1, 2, 3, 4$ the following assertions hold:*

(1) $\delta_1 \eqsim_F \delta_2$ *if and only if* $\delta_1 \eqsim_F (\delta_1 \cup \delta_2) \eqsim_F \delta_2$

(2) $\delta_1 \eqsim_F \delta_2$ *and* $\delta_3 \eqsim_F \delta_4$ *imply that* $(\delta_1 \cup \delta_3) \eqsim_F (\delta_2 \cup \delta_4)$

(3) $\delta_1 \leq \delta_2$ *and* $\delta_2 \eqsim_F 0$ *imply that* $\delta_1 \eqsim_F 0$

(4) $(\delta_1 \cup \delta_2) \eqsim_F 0$ *if and only if* $\delta_1 \eqsim_F 0$ *and* $\delta_2 \eqsim_F 0$

(5) $\delta_1 \leq \delta_2$ *and* $\delta_1 \eqsim_F 1$ *imply that* $\delta_2 \eqsim_F 1$

(6) *If* $(\delta_1 \cap \delta_2) \eqsim_F 1$ *then,* $\delta_1 \eqsim_F 1$ *and* $\delta_2 \eqsim_F 1$.

Proof. It has a comaparable proof as the proof of above Theorem 4.3.5. ■

Theorem 4.3.7 *Let (J, A) be a soft reflexive relation on U. For $\lambda_i \in \mathbb{F}(U)$ where $i = 1, 2, 3, 4$ the following assertions hold:*

(1) $\lambda_1 \backsimeq_A \lambda_2$ *if and only if* $\lambda_1 \backsimeq_A (\lambda_1 \cap \lambda_2) \backsimeq_A \lambda_2$

(2) $\lambda_1 \backsimeq_A \lambda_2$ *and* $\lambda_3 \backsimeq_A \lambda_4$ *imply that* $(\lambda_1 \cap \lambda_3) \backsimeq_A (\lambda_2 \cap \lambda_4)$

(3) $\lambda_1 \leq \lambda_2$ *and* $\lambda_2 \backsimeq_A 0$ *imply that* $\lambda_1 \backsimeq_A 0$

(4) $(\lambda_1 \cup \lambda_2) \backsimeq_A 0$ *if and only if* $\lambda_1 \backsimeq_A 0$ *and* $\lambda_2 \backsimeq_A 0$

(5) $\lambda_1 \leq \lambda_2$ *and* $\lambda_1 \backsimeq_A 1$ *imply that* $\lambda_2 \backsimeq_A 1$

(6) *If* $(\lambda_1 \cap \lambda_2) \backsimeq_A 1$ *then,* $\lambda_1 \backsimeq_A 1$ *and* $\lambda_2 \backsimeq_A 1$.

Proof. The verification is like the evidence of Theorem 4.3.5. ■

Theorem 4.3.8 *Let (J, A) be a soft reflexive relation on U. For $\delta_i \in \text{₣}(U)$ where $i = 1, 2, 3, 4$ the following assertions hold:*

(1) $\delta_1 \simeq_F \delta_2$ *if and only if* $\delta_1 \simeq_F (\delta_1 \cap \delta_2) \simeq_F \delta_2$

(2) $\delta_1 \simeq_F \delta_2$ *and* $\delta_3 \simeq_F \delta_4$ *imply that* $(\delta_1 \cap \delta_3) \simeq_F (\delta_2 \cap \delta_4)$

(3) $\delta_1 \leq \delta_2$ *and* $\delta_2 \simeq_F 0$ *imply that* $\delta_1 \simeq_F 0$

(4) $(\delta_1 \cup \delta_2) \simeq_F 0$ *if and only if* $\delta_1 \simeq_F 0$ *and* $\delta_2 \simeq_F 0$

(5) $\delta_1 \leq \delta_2$ *and* $\delta_1 \simeq_F 1$ *imply that* $\delta_2 \simeq_F 1$

(6) *If* $(\delta_1 \cap \delta_2) \simeq_F 1$ *then,* $\delta_1 \simeq_F 1$ *and* $\delta_2 \simeq_F 1$.

Proof. It has a comaparable proof as Theorem 4.3.7. ■

Theorem 4.3.9 *Let (J, A) be a soft reflexive relation on U. For $\lambda_i \in \text{₣}(U)$ where $i = 1, 2$ the assertions below hold:*

(1) $\lambda_1 \leq \lambda_2$ *and* $\lambda_2 \approx_A 0$ *imply that* $\lambda_1 \approx_A 0$

(2) $\lambda_1 \leq \lambda_2$ *and* $\lambda_1 \approx_A 1$ *imply that* $\lambda_2 \approx_A 1$

(3) $(\lambda_1 \cup \lambda_2) \approx_A 0$, *then* $\lambda_1 \approx_A 0$ *and* $\lambda_2 \approx_A 0$

(4) $(\lambda_1 \cap \lambda_2) \approx_A 1$, *then* $\lambda_1 \approx_A 1$ *and* $\lambda_2 \approx_A 1$

(5) $\lambda_1 \approx_A \lambda_2$ *if and only if* $\lambda_1 \eqsim_A (\lambda_1 \cup \lambda_2) \eqsim_A \lambda_2$ *and* $\lambda_1 \simeq_A (\lambda_1 \cap \lambda_2) \simeq_A \lambda_2$.

Proof. It is an immediate consequence of Theorems 4.3.5 and 4.3.7. ■

Theorem 4.3.10 *Let (J, A) be a soft reflexive relation on U. For $\delta_i \in \text{₣}(U)$ where $i = 1, 2, 3, 4$ the following assertions hold:*

(1) $\delta_1 \leq \delta_2$ *and* $\delta_2 \approx_F 0$ *imply that* $\delta_1 \approx_F 0$

(2) $\delta_1 \leq \delta_2$ *and* $\delta_1 \approx_F 1$ *imply that* $\delta_2 \approx_F 1$

(3) $(\delta_1 \cup \delta_2) \approx_F 0$, *then* $\delta_1 \approx_F 0$ *and* $\delta_2 \approx_F 0$

(4) $(\delta_1 \cap \delta_2) \approx_F 1$, *then* $\delta_1 \approx_F 1$ *and* $\delta_2 \approx_F 1$

(5) $\delta_1 \approx_F \delta_2$ *if and only if* $\delta_1 \eqsim_F (\delta_1 \cup \delta_2) \eqsim_F \delta_2$ *and* $\delta_1 \simeq_F (\delta_1 \cap \delta_2) \simeq_F \delta_2$.

Proof. It is an immediate consequence of Theorems 4.3.6 and 4.3.8. ■

4.4 Accuracy measures

The approximation of FSs presents a method to investigate how accurately the membership functions of FSs describe the objects. In this section, we introduce the degree of accuracy and the degree of roughness for membership functions of FSs respecting to the aftersets and foresets. For this purpose, $\alpha-$level cuts of FSs are defined and present some of its properties.

Definition 4.4.1 *Let U be a non-empty universe and $\lambda \in \mathrm{F}(U)$. For $0 \lneqq \alpha \leq 1$, the $\alpha-$level cut of λ is denoted by $\lambda_\alpha = \{u \in U : \lambda(u) \geq \alpha\}$.*

Lemma 4.4.2 *Let U be a non-empty universe and λ, $\mu \in \mathrm{F}(U)$. For $0 \lneqq \alpha \leq 1$, $\lambda \leq \mu$ implies that $\lambda_\alpha \subseteq \mu_\alpha$.*

Proof. It is an immediate consequence of Definition 4.4.1. ■

Lemma 4.4.3 *Let U be a non-empty universe and $\lambda \in \mathrm{F}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then $\lambda_\alpha \subseteq \lambda_\beta$.*

Proof. The proof is a direct consequence of Definition 4.4.1. ■

$\underline{J}^{(\lambda_\alpha)}$ is the lower approximation of the crisp set λ_α while $\left(\underline{J}^\lambda(e)\right)_\alpha$ is the $\alpha-$level cut of $\underline{J}^\lambda(e)$ with respect to the aftersets. Therefore,

$$\begin{aligned}\left(\underline{J}^\lambda(e)\right)_\alpha &= \left\{u \in U : \underline{J}^\lambda(e)(u) \geq \alpha\right\} \\ &= \left\{u \in U : \wedge_{a \in uJ(e)} \lambda(a) \geq \alpha\right\} \text{ and} \\ \left(\overline{J}^\lambda(e)\right)_\alpha &= \left\{u \in U : \vee_{a \in uJ(e)} \lambda(a) \geq \alpha\right\} \text{ for all } e \in A.\end{aligned}$$

Similarly, for $\delta \in \mathrm{F}(U)$, we have

$$\begin{aligned}\left({}^\delta\underline{J}(e)\right)_\alpha &= \left\{u \in U : {}^\delta\underline{J}(e)(u) \geq \alpha\right\} \\ &= \left\{u \in U : \wedge_{a \in J(e)u} \delta(a) \geq \alpha\right\} \text{ and} \\ \left({}^\delta\overline{J}(e)\right)_\alpha &= \left\{u \in U : \vee_{a \in J(e)u} \delta(a) \geq \alpha\right\} \text{ for all } e \in A.\end{aligned}$$

with respect to the foresets.

Lemma 4.4.4 *Let (J, A) be a soft reflexive relation on a non-empty universe U, $\lambda \in \text{Ғ}(U)$ and $0 \lneqq \alpha \leq 1$. Then, the following assertions hold with respect to the aftersets:*

(1) $\underline{J}^{(\lambda_\alpha)}(e) = \left(\underline{J}^\lambda(e)\right)_\alpha$ *for all* $e \in A$.

(2) $\overline{J}^{(\lambda_\alpha)}(e) = \left(\overline{J}^\lambda(e)\right)_\alpha$ *for all* $e \in A$.

Proof. (1) Let $\lambda \in \text{Ғ}(U)$ and $0 \lneqq \alpha \leq 1$. For the crisp set λ_α, we have

$$\begin{aligned} \underline{J}^{(\lambda_\alpha)}(e) &= \{u \in U : uJ(e) \subseteq \lambda_\alpha\} \\ &= \{u \in U : \lambda(a) \geq \alpha \text{ for all } a \in uJ(e)\} \\ &= \left\{u \in U : \wedge_{a \in uJ(e)} \lambda(a) \geq \alpha\right\} \\ &= \left(\underline{J}^\lambda(e)\right)_\alpha \text{ for all } e \in A. \end{aligned}$$

(2) It can be verified in the similar way as (1). ■

Lemma 4.4.5 *Let (J, A) be a soft reflexive relation on a non-empty universe U, $\delta \in \text{Ғ}(U)$ and $0 \lneqq \alpha \leq 1$. Then, the following assertions hold with respect to the foresets:*

(1) ${}^{(\delta_\alpha)}\underline{J}(e) = \left({}^\delta\underline{J}(e)\right)_\alpha$ *for all* $e \in A$.

(2) ${}^{(\delta_\alpha)}\overline{J}(e) = \left({}^\delta\overline{J}(e)\right)_\alpha$ *for all* $e \in A$.

Proof. It has a comaparable proof as the proof of above lemma. ■

Now, we define the degree of accuracy and the degree of roughness for membership functions of a FS, in a non-empty finite universe.

Definition 4.4.6 *Let (J, A) be a soft reflexive relation on a non-empty universe U. The degree of accuracy for the membership of $\lambda \in \text{Ғ}(U)$, with respect to α, β such that $0 \lneqq \beta \leq \alpha \leq 1$ and with respect to the aftersets, is denoted and defined as*

$\gamma^J_{(\alpha,\beta)}(\lambda)(e_i) = \left|\underline{J}^{(\lambda_\alpha)}(e_i)\right| / \left|\overline{J}^{(\lambda_\beta)}(e_i)\right|$ *for all* $e_i \in A$.

Similarly, the degree of accuracy for the membership of $\delta \in \text{Ғ}(U)$, with respect to α, β such that $0 \lneqq \beta \leq \alpha \leq 1$ and with respect to the foresets, is denoted and defined as

${}_{(\alpha,\beta)}\gamma^J(\delta)(e_i) = \left|{}^{(\delta_\alpha)}\underline{J}(e_i)\right| / \left|{}^{(\delta_\beta)}\overline{J}(e_i)\right|$ *for all* $e_i \in A$.

The degree of roughness for the membership of $\lambda \in \text{Ғ}(U)$, with respect to α, β such that $0 \lneqq \beta \leq \alpha \leq 1$ and with respect to the aftersets, is denoted and defined as

$\rho^J_{(\alpha,\beta)}(\lambda)(e_i) = 1 - \gamma^J_{(\alpha,\beta)}(\lambda)(e_i)$ *for all* $e_i \in A$.

Similarly, the degree of roughness for the membership of $\delta \in \mathrm{F}(U)$*, with respect to* α*,* β *such that* $0 \lneqq \beta \leq \alpha \leq 1$ *and with respect to the foresets, is denoted and defined as*

${}_{(\alpha,\beta)}\rho^J(\delta)(e_i) = 1 - {}_{(\alpha,\beta)}\gamma^J(\delta)(e_i)$ *for all* $e_i \in A$.

Note that, in case of $SE-$relation, the concept of the foresets and aftersets coincides. Further, $\underline{J}^{(\lambda_\alpha)}(e)$ or $\overline{J}^{(\lambda_\beta)}(e)$ comprise of the objects of U having α or β as the least degree of definite or possible fulfilment in λ for all $e \in A$. Equivalently, $\underline{J}^{(\lambda_\alpha)}(e)$ or $\overline{J}^{(\lambda_\beta)}(e)$ can be seen as the union of the $SE-$classes of U having degree of fulfilment atleast α or β in the lower or upper fuzzy approximation of λ with respect to the aftersets. Therefore, α and β serve as the thresholds of definite and possible fulfilment of the objects of α or β in λ, respectively. Hence, $\gamma^J_{(\alpha,\beta)}(\lambda)(e)$ may be interpreted as the degree to which the membership functions of λ is accurate, constrained to the thresholds α and β. In other words, $\gamma^J_{(\alpha,\beta)}(\lambda)(e)$ describes how accurate are the membership functions of the FSs with respect to the aftersets. These degrees are illustrated in the following example.

Example 4.4.7 *Let* $U = \{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}\}$ *be a collection of trees of different types and* $A = \{e_1, e_2, e_3, e_4\}$ *be a set of parameters such that* e_1 *stands for the attribute Height,* e_2 *stands for Age,* e_3 *stands for Fruitibility and* e_4 *stands for the Thickness. Define a* $SE-$*relation* $J : A \rightarrow P(U \times U)$ *for each* $e \in A$*. The corresponding* $SE-$*class for each of the* $SE-$*relation is obtained as follows:*

For $J(e_1)$*, the SE−classes* $tJ(e_1)$ *are* $\{t_1, t_{10}\}$, $\{t_2, t_4, t_6, t_7\}$, $\{t_3, t_5, t_8, t_9\}$, $\{t_{11}\}$.

For $J(e_2)$*, the SE−classes* $tJ(e_2)$ *are* $\{t_1\}$, $\{t_2, t_{11}\}$, $\{t_4, t_7\}$, $\{t_3, t_5, t_8, t_9\}$, $\{t_6, t_{10}\}$.

For $J(e_3)$*, the SE−classes* $tJ(e_3)$ *are* $\{t_1\}$, $\{t_2\}$, $\{t_3, t_4, t_5, t_7, t_8, t_9, t_{10}\}$, $\{t_6\}$, $\{t_{11}\}$.

For $J(e_4)$*, the SE−classes* $tJ(e_4)$ *are* $\{t_{10}\}$, $\{t_6\}$, $\{t_1, t_2, t_3, t_4, t_5, t_7, t_8, t_9\}$, $\{t_{11}\}$.

Define a FS, $\lambda : U \rightarrow [0, 1]$ *by*

$$\begin{aligned}
\lambda(t_1) &= 0.9,\ \lambda(t_2) = 0.6,\ \lambda(t_3) = 0.3,\ \lambda(t_4) = 0, \\
\lambda(t_5) &= 0.2,\ \lambda(t_6) = 0.4,\ \lambda(t_7) = 0.6,\ \lambda(t_8) = 0.8, \\
\lambda(t_9) &= 1,\ \lambda(t_{10}) = 0,\ \lambda(t_{11}) = 1.
\end{aligned}$$

Take $\alpha = 0.7$ *and* $\beta = 0.6$*. Then* $\alpha-$*level cuts* $\lambda_{0.6}$ *and* $\lambda_{0.7}$ *are calculated as*

$\lambda_{0.6} = \{t_1, t_2, t_7, t_8, t_9, t_{11}\}$

$\lambda_{0.7} = \{t_1, t_8, t_9, t_{11}\}$.

Now,

$$\begin{aligned} \underline{J}^{(\lambda_{0.7})}(e_1) &= \{t_{11}\}, \ \underline{J}^{(\lambda_{0.7})}(e_2) = \{t_1\}, \\ \underline{J}^{(\lambda_{0.7})}(e_3) &= \{t_1, t_{11}\}, \ \underline{J}^{(\lambda_{0.7})}(e_4) = \{t_{11}\}. \end{aligned}$$

And,

$$\begin{aligned} \overline{J}^{(\lambda_{0.6})}(e_1) &= \{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, t_{10}, t_{11}\}, \\ \overline{J}^{(\lambda_{0.6})}(e_2) &= \{t_1, t_2, t_3, t_4, t_5, t_7, t_8, t_9, t_{11}\}, \\ \overline{J}^{(\lambda_{0.6})}(e_3) &= \{t_1, t_2, t_3, t_4, t_5, t_7, t_8, t_9, t_{10}, t_{11}\}, \\ \overline{J}^{(\lambda_{0.6})}(e_4) &= \{t_1, t_2, t_3, t_4, t_5, t_7, t_8, t_9, t_{11}\}. \end{aligned}$$

The degree of accuracy for the membership of λ is calculated as

$\gamma^J_{(\alpha,\beta)}(\lambda)(e_1) = \left|\underline{J}^{(\lambda_{0.7})}(e_1)\right| / \left|\overline{J}^{(\lambda_{0.6})}(e_1)\right| = 1/11 = 0.091$

$\gamma^J_{(\alpha,\beta)}(\lambda)(e_2) = \left|\underline{J}^{(\lambda_{0.7})}(e_2)\right| / \left|\overline{J}^{(\lambda_{0.6})}(e_2)\right| = 1/9 = 0.111$

$\gamma^J_{(\alpha,\beta)}(\lambda)(e_3) = \left|\underline{J}^{(\lambda_{0.7})}(e_3)\right| / \left|\overline{J}^{(\lambda_{0.6})}(e_3)\right| = 1/5 = 0.200$

$\gamma^J_{(\alpha,\beta)}(\lambda)(e_4) = \left|\underline{J}^{(\lambda_{0.7})}(e_4)\right| / \left|\overline{J}^{(\lambda_{0.6})}(e_4)\right| = 1/9 = 0.111.$

Hence, $\gamma^J_{(\alpha,\beta)}(\lambda)(e_i)$ shows the degree to which the membership function of λ is accurate constrained to the parameters α and β for $i = 1, 2, 3, 4$ with respect to aftersets. Similarly, we can calculate with respect to foresets.

Theorem 4.4.8 *Let (J, A) be a soft reflexive relation on a non-empty universe U, $\lambda \in \mathrm{F}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then, $0 \leq \gamma^J_{(\alpha,\beta)}(\lambda)(e) \leq 1$ for all $e \in A$ respecting to the aftersets.*

Proof. For a FS, $\lambda \in \mathrm{F}(U)$ and α, $\beta \in [0, 1]$ such that $0 \lneqq \beta \leq \alpha \leq 1$. By using Lemma 4.4.3, $\lambda_\alpha \subseteq \lambda_\beta$. Now, $\underline{J}^{(\lambda_\alpha)}(e) \subseteq \overline{J}^{(\lambda_\alpha)}(e) \leq \overline{J}^{(\lambda_\beta)}(e)$. Thus $\left|\underline{J}^{(\lambda_\alpha)}(e)\right| \leq \left|\overline{J}^{(\lambda_\beta)}(e)\right|$ so the ratio $\left|\underline{J}^{(\lambda_\alpha)}(e) / \overline{J}^{(\lambda_\beta)}(e)\right|$ fluctuates between 0 and 1 which yields certainly $0 \leq \gamma^J_{(\alpha,\beta)}(\lambda)(e) \leq 1$ for all $e \in A$. ■

Corollary 4.4.9 *Let (J, A) be a soft reflexive relation on a non-empty universe U, $\lambda \in \mathrm{F}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then, $0 \leq \rho^{J}_{(\alpha,\beta)}(\lambda)(e) \leq 1$ for all $e \in A$ respecting to the aftersets.*

Proof. It is an immediate consequence of Definition 4.4.6 and Theorem 4.4.8. ■

Theorem 4.4.10 *Let (J, A) be a soft reflexive relation on a non-empty universe U, $\lambda \in \mathrm{F}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$.*

(1) *If α stands fixed, then $\gamma^{J}_{(\alpha,\beta)}(\lambda)(e)$ increases with the increase in β.*

(2) *If β stands fixed, then $\gamma^{J}_{(\alpha,\beta)}(\lambda)(e)$ decreases with the increase in α for all $e \in A$.*

Proof. (1) Let α stands fixed and let $0 \lneqq \beta_1 \leq \beta_2 \leq 1$. Using Lemma 4.4.3, we have $\lambda_{\beta_2} \subseteq \lambda_{\beta_1}$. Then $\overline{J}^{(\lambda_{\beta_2})}(e) \subseteq \overline{J}^{(\lambda_{\beta_1})}(e)$ and $\left|\overline{J}^{(\lambda_{\beta_2})}(e)\right| \leq \left|\overline{J}^{(\lambda_{\beta_1})}(e)\right|$. This implies that

$\left|\underline{J}^{(\lambda_\alpha)}(e)\right| / \left|\overline{J}^{(\lambda_{\beta_1})}(e)\right| \leq \left|\underline{J}^{(\lambda_\alpha)}(e)\right| / \left|\overline{J}^{(\lambda_{\beta_2})}(e)\right|$. That is $\gamma^{J}_{(\alpha,\beta_1)}(\lambda)(e) \leq \gamma^{J}_{(\alpha,\beta_2)}(\lambda)(e)$. This shows that $\gamma^{J}_{(\alpha,\beta)}(\lambda)(e)$ increases with the increase in β for all $e \in A$.

(2) It has a comaparable proof as the proof of (1). ■

Corollary 4.4.11 *Let (J, A) be a soft reflexive relation on a non-empty universe U, $\lambda \in \mathrm{F}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$.*

(1) *If α stands fixed, then $\rho^{J}_{(\alpha,\beta)}(\lambda)(e)$ decreases with the increase in β.*

(2) *If β stands fixed, then $\rho^{J}_{(\alpha,\beta)}(\lambda)(e)$ increases with the increase in α for all $e \in A$.*

Proof. It is an immediate consequence of Definition 4.4.6 and Theorem 4.4.10. ■

Theorem 4.4.12 *Let (J, A) be a soft reflexive relation on a non-empty universe U, λ, $\mu \in \mathrm{F}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then, $\lambda \leq \mu$ implies the following assertions for all $e \in A$ and respecting to the aftersets.*

(1) $\gamma^{J}_{(\alpha,\beta)}(\lambda)(e) \leq \gamma^{J}_{(\alpha,\beta)}(\mu)(e)$, *whenever* $\overline{J}^{(\lambda_\beta)}(e) = \overline{J}^{(\mu_\beta)}(e)$

(2) $\gamma^{J}_{(\alpha,\beta)}(\lambda)(e) \geq \gamma^{J}_{(\alpha,\beta)}(\mu)(e)$, *whenever* $\underline{J}^{(\lambda_\alpha)}(e) = \underline{J}^{(\mu_\alpha)}(e)$.

Proof. (1) Let $0 \lneqq \beta \leq \alpha \leq 1$ and λ, $\mu \in \mathrm{F}(U)$ be such that $\lambda \leq \mu$. Then $\lambda_\alpha \subseteq \mu_\alpha$. Thus, $\underline{J}^{(\lambda_\alpha)}(e) \subseteq \underline{J}^{(\mu_\alpha)}(e)$ that is $\left|\underline{J}^{(\lambda_\alpha)}(e)\right| \leq \left|\underline{J}^{(\mu_\alpha)}(e)\right|$. This implies that

$\left|\underline{J}^{(\lambda_\alpha)}(e)\right| / \left|\overline{J}^{(\lambda_\beta)}(e)\right| \leq \left|\underline{J}^{(\mu_\alpha)}(e)\right| / \left|\overline{J}^{(\mu_\beta)}(e)\right|$. Hence, $\gamma^J_{(\alpha,\beta)}(\lambda)(e) \leq \gamma^J_{(\alpha,\beta)}(\mu)(e)$.

(2) It has a comaparable proof as the proof of (1). ■

Corollary 4.4.13 *Let (J, A) be a soft reflexive relation on a non-empty universe U, λ, $\mu \in \text{₣}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then, $\lambda \leq \mu$ implies the following assertions for all $e \in A$ respecting to the aftersets.*

(1) $\rho^J_{(\alpha,\beta)}(\lambda)(e) \geq \rho^J_{(\alpha,\beta)}(\mu)(e)$, *whenever* $\overline{J}^{(\lambda_\beta)}(e) = \overline{J}^{(\mu_\beta)}(e)$

(2) $\rho^J_{(\alpha,\beta)}(\lambda)(e) \leq \rho^J_{(\alpha,\beta)}(\mu)(e)$, *whenever* $\underline{J}^{(\lambda_\alpha)}(e) = \underline{J}^{(\mu_\alpha)}(e)$.

Proof. It is an immediate consequence of Definition 4.4.6 and Theorem 4.4.12. ■

Theorem 4.4.14 *Let (J, A) be a soft reflexive relation on a non-empty universe U, $\lambda \in \text{₣}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. If (Z, A) is a soft reflexive relation on U such that $J(e) \subseteq Z(e)$. Then, $\gamma^J_{(\alpha,\beta)}(\lambda)(e) \geq \gamma^Z_{(\alpha,\beta)}(\lambda)(e)$ for all $e \in A$ respecting to the aftersets.*

Proof. Let $\lambda \in \text{₣}(U)$ and let (J, A) and (Z, A) be two soft reflexive relations on U such that $J(e) \subseteq Z(e)$ for all $e \in A$. Then $\underline{J}^\lambda(e) \geq \underline{Z}^\lambda(e)$ and $\overline{J}^\lambda(e) \leq \overline{Z}^\lambda(e)$. Also, $\underline{J}^{(\lambda_\alpha)}(e) \supseteq \underline{Z}^{(\lambda_\alpha)}(e)$ and $\overline{J}^{(\lambda_\beta)}(e) \subseteq \overline{Z}^{(\lambda_\beta)}(e)$. By lemma 4.4.4,

$$\begin{aligned}
\left|\underline{J}^{(\lambda_\alpha)}(e)\right| &= \left|\left(\underline{J}^\lambda(e)\right)_\alpha\right| \geq \left|\left(\underline{Z}^\lambda(e)\right)_\alpha\right| = \left|\underline{Z}^{(\lambda_\alpha)}(e)\right| \text{ and} \\
\left|\overline{J}^{(\lambda_\beta)}(e)\right| &= \left|\left(\overline{J}^\lambda(e)\right)\beta\right| \leq \left|\left(\overline{Z}^\lambda(e)\right)_\beta\right| = \left|\overline{Z}^{(\lambda_\beta)}(e)\right|.
\end{aligned}$$

Rearranging and dividing the above two equations, we get $\left|\underline{J}^{(\lambda_\alpha)}(e)\right| / \left|\overline{J}^{(\lambda_\beta)}(e)\right| \geq \left|\underline{Z}^{(\lambda_\alpha)}(e)\right| / \left|\overline{Z}^{(\lambda_\beta)}(e)\right|$. Hence, $\gamma^J_{(\alpha,\beta)}(\lambda)(e) \geq \gamma^Z_{(\alpha,\beta)}(\lambda)(e)$ for all $e \in A$. ■

Corollary 4.4.15 *Let (J, A) be a soft reflexive relation on a non-empty universe U, $\lambda \in \text{₣}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. If (Z, A) is a soft reflexive relation on U such that $J(e) \subseteq Z(e)$. Then, $\rho^J_{(\alpha,\beta)}(\lambda)(e) \geq \rho^Z_{(\alpha,\beta)}(\lambda)(e)$ for all $e \in A$ respecting to the aftersets.*

Proof. It is an immediate consequence of Definition 4.4.6 and Theorem 4.4.14. ■

Theorem 4.4.16 *Let (J, A) be a soft reflexive relation on a non-empty universe U, λ, $\mu \in \text{Ŧ}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then, $\lambda \simeq_A \mu$ implies the following assertions for all $e \in A$ with respect to the aftersets are true.*

(1) $\gamma^J_{(\alpha,\beta)}(\lambda \cap \mu)(e) \geq \gamma^J_{(\alpha,\beta)}(\lambda)(e)$

(2) $\gamma^J_{(\alpha,\beta)}(\lambda \cap \mu)(e) \geq \gamma^J_{(\alpha,\beta)}(\mu)(e)$.

Proof. (1) Let λ, $\mu \in \text{Ŧ}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$ such that $\lambda \simeq_A \mu$. By definition 4.3.1, $\underline{J}^{\lambda}(e) = \underline{J}^{\mu}(e)$. Now by Theorem 4.3.7, $\underline{J}^{\lambda\cap\mu}(e) = \underline{J}^{\lambda}(e)$. Therefore, $\underline{J}^{(\lambda\cap\mu)_\alpha}(e) = \underline{J}^{(\lambda)_\alpha}(e)$. Therefore, $\left|\underline{J}^{(\lambda\cap\mu)_\alpha}(e)\right| = \left|\underline{J}^{(\lambda)_\alpha}(e)\right|$. On the other hand, $\lambda \cap \mu \leq \lambda$ which implies $(\lambda \cap \mu)_\beta \subseteq \lambda_\beta$ that is $\overline{J}^{(\lambda\cap\mu)_\beta}(e) \leq \overline{J}^{(\lambda)_\beta}(e)$. Therefore, $\left|\overline{J}^{(\lambda\cap\mu)_\beta}(e)\right| \leq \left|\overline{J}^{(\lambda)_\beta}(e)\right|$. Hence, by re-setting, we get $\left|\underline{J}^{(\lambda\cap\mu)_\alpha}(e)\right| / \left|\overline{J}^{(\lambda\cap\mu)_\beta}(e)\right| \geq \left|\underline{J}^{(\lambda)_\alpha}(e)\right| / \left|\overline{J}^{(\lambda)_\beta}(e)\right|$. Hence, $\gamma^J_{(\alpha,\beta)}(\lambda \cap \mu)(e) \geq \gamma^J_{(\alpha,\beta)}(\lambda)(e)$ for all $e \in A$.

(2) This can be proved in the same manner as (1). ■

Corollary 4.4.17 *Let (J, A) be a soft reflexive relation on a non-empty universe U, λ, $\mu \in \text{Ŧ}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then, $\lambda \simeq_A \mu$ implies the following assertions for all $e \in A$ with respect to the aftersets are true.*

(1) $\rho^J_{(\alpha,\beta)}(\lambda \cap \mu)(e) \geq \rho^J_{(\alpha,\beta)}(\lambda)(e)$

(2) $\rho^J_{(\alpha,\beta)}(\lambda \cap \mu)(e) \geq \rho^J_{(\alpha,\beta)}(\mu)(e)$.

Proof. The proof is direct consequence of Definition 4.4.6 and Theorem 4.4.16. ■

Theorem 4.4.18 *Let (J, A) be a soft reflexive relation on a non-empty universe U, λ, $\mu \in \text{Ŧ}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then, $\lambda \eqsim_A \mu$ implies the following assertions for all $e \in A$ with respect to the aftersets are true.*

(1) $\gamma^J_{(\alpha,\beta)}(\lambda \cup \mu)(e) \geq \gamma^J_{(\alpha,\beta)}(\lambda)(e)$

(2) $\gamma^J_{(\alpha,\beta)}(\lambda \cup \mu)(e) \geq \gamma^J_{(\alpha,\beta)}(\mu)(e)$.

Proof. It has a comaparable proof as the proof of Theorem 4.4.16. ■

Corollary 4.4.19 *Let (J, A) be a soft reflexive relation on a non-empty universe U, λ, $\mu \in \text{Ŧ}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then, $\lambda \eqsim_A \mu$ implies the following assertions for all $e \in A$ respecting to the aftersets are true.*

(1) $\rho^J_{(\alpha,\beta)}(\lambda \cup \mu)(e) \geq \rho^J_{(\alpha,\beta)}(\lambda)(e)$

(2) $\rho^J_{(\alpha,\beta)}(\lambda \cup \mu)(e) \geq \rho^J_{(\alpha,\beta)}(\mu)(e)$.

Proof. The proof is direct consequence of Definition 4.4.6 and Theorem 4.4.18. ■

Theorem 4.4.20 *Let (J, A) be a soft reflexive relation on a non-empty universe U, λ, $\mu \in \mathrm{F}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then, $\lambda \approx_A \mu$ implies that $\gamma^J_{(\alpha,\beta)}(\lambda)(e) = \gamma^J_{(\alpha,\beta)}(\mu)(e)$ for all $e \in A$ with respect to the aftersets.*

Proof. Let $0 \lneqq \beta \leq \alpha \leq 1$ and λ, $\mu \in \mathrm{F}(U)$ such that $\lambda \approx_A \mu$. By definition 4.3.1, $\underline{J}^{\lambda}(e) = \underline{J}^{\mu}(e)$ and $\overline{J}^{\lambda}(e) = \overline{J}^{\mu}(e)$. By Lemma 4.4.4, $\underline{J}^{(\lambda_\alpha)}(e) = \underline{J}^{(\mu_\alpha)}(e)$ and $\overline{J}^{(\lambda_\beta)}(e) = \overline{J}^{(\mu_\beta)}(e)$, that is $\left|\underline{J}^{(\lambda_\alpha)}(e)\right| = \left|\underline{J}^{(\mu_\alpha)}(e)\right|$ and $\left|\overline{J}^{(\lambda_\beta)}(e)\right| = \left|\overline{J}^{(\mu_\beta)}(e)\right|$. This yields $\left|\underline{J}^{(\lambda_\alpha)}(e)\right| / \left|\overline{J}^{(\lambda_\beta)}(e)\right| = \left|\underline{J}^{(\mu_\alpha)}(e)\right| / \left|\overline{J}^{(\mu_\beta)}(e)\right|$. Hence, $\gamma^J_{(\alpha,\beta)}(\lambda)(e) = \gamma^J_{(\alpha,\beta)}(\mu)(e)$ for all $e \in A$. ■

Corollary 4.4.21 *Let (J, A) be a soft reflexive relation on a non-empty universe U, λ, $\mu \in \mathrm{F}(U)$ and $0 \lneqq \beta \leq \alpha \leq 1$. Then, $\lambda \approx_A \mu$ implies that $\rho^J_{(\alpha,\beta)}(\lambda)(e) = \rho^J_{(\alpha,\beta)}(\mu)(e)$ for all $e \in A$ respecting to the aftersets.*

Proof. The proof is a direct consequence of Definition 4.4.6 and Theorem 4.4.20. ■

Note: Similarly we can prove the results corresponding to foresets.

4.5 Decision making

Soft set was defined by Molodtsov and applied in decision making problems by (Maji et al. [64, 65]). The evaluation of all the decision parameters is involved by the results of soft sets depending on decision making. Moreover, for the evaluation of decision parameters, there is not a uniform criterion generally (Feng et al. [29]). So, there are limitations on previous decision approaches. On fuzzy soft set theory, decision making is done by the authors Roy and Maji in [85]. The limitations in Roy and Maji's method [85] is overcomed by Feng's et al. [29].

Depending on fuzzy soft rough set theory, the decision making methods by soft binary relations are proposed in this section. This approach helps to use data provided by decision makers and further information is not required. Hence, the results should avoid the paradox results.

The existing approaches to decision making problems based on fuzzy soft set are mainly focussed on the choice value c_i (Roy and Maji [85]) of the membership degree about the parameter set for the given object in universe U and the score of an object o_i according to the comparison table. Then select the object of the universe U with maximum choice value c_i or maximum score as the optimum decision.

As the rough lower approximation and upper approximation are two most closed to the approximated set of the universe. Therefore, we obtain two most closed values $\underline{J}^{\lambda}(e_i)(x_j)$ and $\overline{J}^{\lambda}(e_i)(x_j)$ with respect to the aftersets to the decision alternative $x_j \in U$ of the universe U by the fuzzy soft lower and upper approximations of the fuzzy set λ. So, we redefine the choice value γ_i for the decision alternative x_i on the universe U with respect to the aftersets as follows:

$$\gamma_j = \sum_{i=1}^{n} \underline{J}^{\lambda}(e_i)(x_j) + \sum_{i=1}^{n} \overline{J}^{\lambda}(e_i)(x_j), x_j \in U.$$

Taking the object $x_j \in U$ in universe U with the maximum choice value γ_j as the optimum decision for the given decision making problem and by taking the object $x_j \in U$ in universe U with the minimum choice value γ_j as the worst decision for the given decision making problem. In general, if there exist two or more object $x_j \in U$ with the same maximum (minimum) choice value γ_j, then take one of them random as the optimum (worst) decision for the given decision making problem.

Algorithm 1:

An algorithm for the approach to a decision making problem is presented here with respect to aftersets. The decision algorithm is as follows:

(1) Compute the lower fuzzy soft set approximation $\underline{J}^{\lambda}$ and upper fuzzy soft set approximation $\overline{J}^{\lambda}$ of the fuzzy set λ respecting to the aftersets.

(2) Compute the sum of lower approximation $\sum_{i=1}^{n} \underline{J}^{\lambda}(e_i)(x_j)$ and the sum of upper approximation $\sum_{i=1}^{n} \overline{J}^{\lambda}(e_i)(x_j)$ corresponding to each i with respect to the aftersets.

(3) Compute the choice value $\gamma_j = \sum_{i=1}^{n} \underline{J}^{\lambda}(e_i)(x_j) + \sum_{i=1}^{n} \overline{J}^{\lambda}(e_i)(x_j)$, $x_j \in U$ with

respect to the aftersets.

(4) The best decision is $x_k \in U$ if $\gamma_k = \max_j \gamma_j$, $j = 1, 2, ..., |U|$.

(5) The worst decision is $x_k \in U$ if $\gamma_k = \min_j \gamma_j$, $j = 1, 2, ..., |U|$.

(6) If k has more than one value, then any one of x_k may be chosen.

Algorithm 2:

Here we present an algorithm for the approach to a decision making problem with respect to foresets. The decision algorithm is as follows:

(1) Compute the lower fuzzy soft set approximation ${}^{\delta}\underline{J}$ and upper fuzzy soft set approximation ${}^{\delta}\overline{J}$ of the fuzzy set δ with respect to the foresets.

(2) Compute the sum of lower approximation $\sum_{i=1}^{n} {}^{\delta}\underline{J}(e_i)(x_j)$ and the sum of upper approximation $\sum_{i=1}^{n} {}^{\delta}\overline{J}(e_i)(x_j)$ corresponding to each i with respect to the foresets.

(3) Compute the choice value $\gamma_j' = \sum_{i=1}^{n} {}^{\delta}\underline{J}(e_i)(x_j) + \sum_{i=1}^{n} {}^{\delta}\overline{J}(e_i)(x_j)$, $x_j \in U$ with respect to the foresets.

(4) The best decision is $x_k \in U$ if $\gamma_k' = \max_j \gamma_j'$, $j = 1, 2, ..., |U|$.

(5) The worst decision is $x_k \in U$ if $\gamma_k' = \min_j \gamma_j'$, $j = 1, 2, ..., |U|$.

(6) If k has more than one value, then any one of x_k may be chosen.

4.5.1 An Application of the Decision Making Approach

This subsection shows the steps of decision making proposed in this section by using an example.

Example 4.5.1 *Consider the soft relations of Example 4.1.3 again, where a person wants to select a shirt out of six shirt designs and four shirt colors.*

$$\begin{aligned}
\text{Define } \lambda &: W \to [0,1] \text{ which represents the preference of the} \\
&\quad \text{color given by } Mr.\ X \text{ such that} \\
\lambda(c_1) &= 0.3,\ \lambda(c_2) = 0.1,\ \lambda(c_3) = 0,\ \lambda(c_4) = 0.5 \text{ and} \\
\text{Define } \delta &: U \to [0,1] \text{ which represents the preference of the} \\
&\quad \text{color given by } Mr.\ X \text{ such that} \\
\delta(d_1) &= 1,\ \delta(d_2) = 0.7,\ \delta(d_3) = 0.5, \\
\delta(d_4) &= 0.1,\ \delta(d_5) = 0,\ \delta(d_6) = 0.4.
\end{aligned}$$

Consider the following table after applying the above algorithm.

Table : The results of decision algorithm with respect to the aftersets

	$\underline{J}^{\lambda}(e_1)$	$\underline{J}^{\lambda}(e_2)$	$\underline{J}^{\lambda}(e_3)$	$\overline{J}^{\lambda}(e_1)$	$\overline{J}^{\lambda}(e_2)$	$\overline{J}^{\lambda}(e_3)$	*Choice value* γ_j
d_1	0	0	0	0.3	0	0	0.3
d_2	0.1	0	0.5	0.5	0	0.5	1.6
d_3	0	0	0	0	0	0.3	0.3
d_4	0	0.3	0	0.1	0.3	0	0.7
d_5	0	0.3	0	0.5	0.3	0.5	1.6
d_6	0.3	0	0	0.3	0.1	0	0.7

And,

Table : The results of decision algorithm with respect to the foresets

	${}^{\delta}\underline{J}(e_1)$	${}^{\delta}\underline{J}(e_2)$	${}^{\delta}\underline{J}(e_3)$	${}^{\delta}\overline{J}(e_1)$	${}^{\delta}\overline{J}(e_2)$	${}^{\delta}\overline{J}(e_3)$	*Choice value* γ_j'
c_1	0.4	0	0.5	1	0.1	0.5	2.5
c_2	0.1	0.4	0	1	0.4	0	1.9
c_3	0	0.7	0	1	1	0.5	3.2
c_4	0	0	0	0.7	0	0.7	1.4

Here the choice value $\gamma_j = \sum_{i=1}^{3} \underline{J}^{\lambda}(e_i)(x_j) + \sum_{i=1}^{3} \overline{J}^{\lambda}(e_i)(x_j)$ *is calculated with respect to the aftersets and the choice value* $\gamma_j' = \sum_{i=1}^{3} {}^{\delta}\underline{J}(e_i)(x_j) + \sum_{i=1}^{3} {}^{\delta}\overline{J}(e_i)(x_j)$ *is calculated with respect to the foresets.*

It is clear that the maximum choice value is $\gamma_k = 1.6 = \gamma_2 = \gamma_5$, *scored by the shirt of designs* d_2 *and* d_5 *and the decision is in favour of selecting the shirt of design* d_2 *or* d_5. *Moreover, the shirts of designs* d_1 *and* d_3 *are totally ignored. Hence, Mr. X will choose the shirt of design* d_2 *or* d_5 *for his personal use and he will not select the shirts of design* d_1 *and* d_3 *with respect to the aftersets. Similarly, the maximum choice value is* $\gamma_k^{'} = 3.2 = \gamma_3$, *scored by the shirt of color* c_3 *and the decision is in favour of selecting the shirt of color* c_3. *Moreover, the shirt of color* c_4 *is totally ignored. Hence, Mr. X will choose the shirt of color* c_3 *for his personal use and he will not select the shirt of color* c_4 *with respect to foresets.*

Chapter 5

Rough approximation of a fuzzy set in semigroups based on soft relations

In the present chapter, with reference to the aftersets and foresets, a new approach is being presented. This way gives two sets named as fuzzy soft sets. Further, two approximations such as upper and lower are obtained while using the aftersets and foresets. For better understanding, these concepts are applied on semigroups. Moreover, two approximations such as upper and lower fuzzy substructures of semigroups are studied.

5.1 Approximation of ideals in semigroups

Approximations of FSS ($FLIL$, $FRIL$, $FBIL$, $FIIL$) of a semigroup are presented with the help of $SCRE$. This is proceeded with the help of aftersets and the foresets. It is noticed that the two approximations such as upper and lower of a fuzzy substructures like FSS ($FLIL$, $FRIL$, $FBIL$, $FIIL$) of a semigroup are fuzzy soft substructures of the semigroup. During this process a SC_mR, aftersets and foresets are utilized. To verify our results some examples will be presented.

Definition 5.1.1 *Let (J, A) be an $SBRE$ from a semigroup S_1 to a semigroup S_2. For any non-empty FS, λ of S_2, if the upper approximation $\left(\overline{J}^{\lambda}, A\right)$ is a FSS of*

S_1, then λ is said to be generalized upper $FSSS$ of S_1 respecting to the aftersets. If $\left(\overline{J}^{\lambda}, A\right)$ is a $FLIL$ ($FRIL$, FIL) of S_1, then the FS, λ is said to be generalized upper $FSLIL$ ($FSRIL$, $FSIL$) of S_1 respecting to aftersets.

Definition 5.1.2 *Let (J, A) be an $SBRE$ from a semigroup S_1 to a semigroup S_2. For any non-empty FS, δ of S_1, if the upper approximation $\left({}^{\delta}\overline{J}, A\right)$ is a FSS of S_2, then δ is said to be generalized upper $FSSS$ of S_2 respecting to the foresets. If $\left({}^{\delta}\overline{J}, A\right)$ is a $FLIL$ ($FRIL$, FIL) of S_2, then FS, δ is said to be generalized upper $FSLIL$ ($FSRIL$, $FSIL$) of S_2 respecting to the foresets,*

Next, some results related to FSS, $FLIL$, $FRIL$ of a semigroup are presented for upper approximations.

Theorem 5.1.3 *Let (J, A) be an $SCRE$ from a semigroup S_1 to a semigroup S_2.*

(1) Then λ is a generalized upper $FSSS$ of S_1 respecting to the aftersets, if λ is a FSS of S_2.

(2) Then δ is a generalized upper $FSSS$ of S_2 respecting to the foresets, if δ is a FSS of S_1.

(3) Then λ is a generalized upper $FSLIL$ ($FSRIL$, $FSIL$) of S_1 respecting to the aftersets, if λ is a $FLIL$ ($FRIL$, FIL) of S_2.

(4) Then δ is a generalized upper $FSLIL$ ($FSRIL$, $FSIL$) of S_2 respecting to the foresets if δ is a $FLIL$ ($FRIL$, FIL) of S_1.

Proof. (1) Suppose λ is a FSS of S_2. For $x, y \in S_1$,

$$\begin{aligned} \overline{J}^{\lambda}(e)(x) \wedge \overline{J}^{\lambda}(e)(y) &= \left(\vee_{p \in xJ(e)} \lambda(p)\right) \wedge \left(\vee_{q \in yJ(e)} \lambda(q)\right) \\ &= \vee_{p \in xJ(e)} \vee_{q \in yJ(e)} \left(\lambda(p) \wedge \lambda(q)\right) \\ &\leq \vee_{p \in xJ(e)} \vee_{q \in yJ(e)} \left(\lambda(pq)\right) \\ &\leq \vee_{pq \in (xy)J(e)} \left(\lambda(pq)\right) \\ &= \vee_{a' \in (xy)J(e)} \lambda\left(a'\right) \\ &= \overline{J}^{\lambda}(e)(xy). \end{aligned}$$

Hence, $\overline{J}^{\lambda}(e)$ is a FSS of S_1 for all $e \in A$ and so λ is a generalized upper $FSSS$ of S_1 respecting to the aftersets.

(2) The proof is simple like the proof of part (1).

(3) Suppose λ is a $FLIL$ of S_2. For $x, y \in S_1$,

$$\begin{aligned}\overline{J}^{\lambda}(e)(y) &= \vee_{q \in yJ(e)} \lambda(q) \\ &\leq \vee_{p \in xJ(e)} \vee_{q \in yJ(e)} \lambda(pq) \\ &\leq \vee_{pq \in (xy)J(e)} (\lambda(pq)) \\ &= \vee_{a' \in (xy)J(e)} \lambda\left(a'\right) \\ &= \overline{J}^{\lambda}(e)(xy).\end{aligned}$$

Hence, $\overline{J}^{\lambda}(e)$ is a $FLIL$ of S_1 for all $e \in A$ and so λ is a generalized upper $FSLIL$ of S_1 respecting to the aftersets.

(4) The proof is simple like part (3).

Likewise, other cases can be proved. ■

It is followed by the next Example that the converse of the parts of above Theorem do not hold in general.

Example 5.1.4 *For two semigroups $S_1 = \{a, b, c, d, e\}$ and $S_2 = \{1, 2, 3, 4, 5\}$ with the multiplication tables as follows:*

$\bullet$	a	b	c	d	e
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
e	d	d	c	d	c

$\bullet$	1	2	3	4	5
1	1	5	3	4	5
2	1	2	3	4	5
3	1	5	3	4	5
4	1	5	3	4	5
5	1	5	3	4	5

Let $A = \{e_1, e_2\}$. Define $J : A \to P(S_1 \times S_2)$ by

$$J(e_1) = \left\{ \begin{array}{c} (a,1), (b,2), (c,3), (d,4), (e,5), (b,1), \\ (c,5), (b,5), (d,3), (d,5), (d,1) \end{array} \right\} \textit{ and}$$

$$J(e_2) = \left\{ \begin{array}{c} (a,1), (b,2), (c,3), (d,4), (e,5), (b,1), \\ (c,5), (b,5), (d,3), (d,5), (d,1), (b,3) \end{array} \right\}.$$

Then (J, A) is an $SCRE$ from the semigroup S_1 to S_2 with $J(e) \neq \phi$. Now,

$$\begin{aligned} aJ(e_1) &= \{1\},\ bJ(e_1) = \{1,2,5\},\ cJ(e_1) = \{3,5\}, \\ dJ(e_1) &= \{1,3,4,5\},\ eJ(e_1) = \{5\} \text{ and} \end{aligned}$$

$$\begin{aligned} aJ(e_2) &= \{1\},\ bJ(e_2) = \{1,2,3,5\},\ cJ(e_2) = \{3,5\}, \\ dJ(e_2) &= \{1,3,4,5\},\ eJ(e_2) = \{5\}. \end{aligned}$$

Also,

$$\begin{aligned} J(e_1)1 &= \{a,b,d\},\ J(e_1)2 = \{b\},\ J(e_1)3 = \{c,d\}, \\ J(e_1)4 &= \{d\},\ J(e_1)5 = \{b,c,d,e\} \text{ and} \end{aligned}$$

$$\begin{aligned} J(e_2)1 &= \{a,b,d\},\ J(e_2)2 = \{b\},\ J(e_2)3 = \{b,c,d\}, \\ J(e_2)4 &= \{d\},\ J(e_2)5 = \{b,c,d,e\}. \end{aligned}$$

(1) Define $\lambda : S_2 \longrightarrow [0,1]$ by

	1	2	3	4	5
λ	0.5	0.4	0.3	1	0.1

Then λ is not a FSS and $FLIL$ of S_2 as $\lambda(1.2) = \lambda(5) = 0.1 \ngeq \lambda(1) \wedge \lambda(2)$ but

	a	b	c	d	e
$\overline{J}^{\lambda}(e_1)$	0.5	0.5	0.3	1	0.1
$\overline{J}^{\lambda}(e_2)$	0.5	0.5	0.3	1	0.1

Clearly, $\overline{J}^{\lambda}(e_1)$ and $\overline{J}^{\lambda}(e_2)$ are FSSs and $FLILs$ of S_1. Hence, λ is a generalized upper $FSSS$ and $FSLIL$ of S_1 respecting to the aftersets.

(2) Define $\delta : S_1 \longrightarrow [0,1]$ by

	a	b	c	d	e
δ	0.2	0.7	0.8	0	0.9

Then δ is not a FSS and $FLIL$ of S_1 as $\delta(c.b) = \delta(d) = 0 \ngeq 0.7 = \delta(c) \wedge \delta(b)$ but

	1	2	3	4	5
$^{\delta}\overline{J}(e_1)$	0.7	0.7	0.8	0	0.9
$^{\delta}\overline{J}(e_2)$	0.7	0.7	0.8	0	0.9

Clearly, $^{\delta}\overline{J}(e_1)$ and $^{\delta}\overline{J}(e_2)$ are FSSs and $FLIL$ of S_2. Hence, δ is a generalized upper $FSSS$ and $FSLIL$ of S_2 respecting to the foresets.

Definition 5.1.5 *Let (J, A) be an $SCRE$ from a semigroup S_1 to a semigroup S_2. For a non-empty FS, λ of S_2, if $(\underline{J}^{\lambda}, A)$ is a FSS of S_1, then λ is said to be generalized lower $FSSS$ of S_1 respecting to the aftersets. If $(\underline{J}^{\lambda}, A)$ is a $FLIL(FRIL,\ FIL)$ of S_1, then FS, λ is said to be generalized lower $FSLIL$ $(FSRIL,\ FSIL)$ of S_1 respecting to the aftersets.*

Definition 5.1.6 *Let (J, A) be an $SCRE$ from a semigroup S_1 to a semigroup S_2. For a non-empty FS, δ of S_1, if $(^{\delta}\underline{J}, A)$ is a FSS of S_2, then δ is said to be generalized lower $FSSS$ of S_2 respecting to the foresets. If $(^{\delta}\underline{J}, A)$ is a $FLIL(FRIL,\ FIL)$ of S_2, then FS, δ is said to be generalized lower $FSLIL$ $(FSRIL,\ FSIL)$ of S_2 respecting to the foresets.*

Example 5.1.7 *Consider the semigroups and soft relations of Example 5.1.4,*

Define $\lambda : S_2 \longrightarrow [0, 1]$ by

	1	2	3	4	5
λ	0.7	0.7	0.8	0	0.9

Then λ is a $FLIL$ of S_2 and

	a	b	c	d	e
$\overline{J}^{\lambda}(e_1)$	0.7	0.7	0.8	0	0.9

But, $\overline{J}^{\lambda}(e_1)$ is not a $FLIL$ of S_1 as $\overline{J}^{\lambda}(e_1)(ac) = \overline{J}^{\lambda}(e_1)(d) = 0 \ngeq 0.7 = \overline{J}^{\lambda}(e_1)(a) \wedge \overline{J}^{\lambda}(e_1)(c)$.

In the above example, we have shown that if (J, A) is a $SCRE$ from the semigroup S_1 to S_2 and λ is a $FLIL$ of S_2 even then $\left(\underline{J}^{\lambda}, A\right)$ is not a $FSLIL$ of S_1. Moreover, some results related to lower approximations for FSS ($FLIL$, $FRIL$) of a semigroup respecting to aftersets are presented as follows.

Theorem 5.1.8 *Let (J, A) be an SC_mR respecting to the aftersets from a semigroup S_1 to a semigroup S_2.*

(1) *Then λ is a generalized lower $FSSS$ of S_1 respecting to the aftersets, if λ is a FSS of S_2.*

(2) *Then λ is a generalized lower $FSLIL$ ($FSRIL$, $FSIL$) of S_1 respecting to the aftersets, if λ is a $FLIL$ ($FRIL$, FIL) of S_2.*

Proof. (1) Let $x, y \in S_1$ and λ be a FSS of S_2. Then

$$\begin{aligned}
\underline{J}^{\lambda}(e)(xy) &= \wedge_{a' \in (xy)J(e)} \lambda\left(a'\right) \\
&= \wedge_{a' \in xJ(e).yJ(e)} \lambda\left(a'\right) \\
&= \wedge_{p \in xJ(e),\ q \in yJ(e)} \lambda(pq) \\
&\geq \wedge_{p \in xJ(e)} \wedge_{q \in yJ(e)} \left(\lambda(p) \wedge \lambda(q)\right) \\
&\geq \left(\wedge_{p \in xJ(e)} \lambda(p)\right) \wedge \left(\wedge_{q \in yJ(e)} \lambda(q)\right) \\
&= \underline{J}^{\lambda}(e)(x) \wedge \underline{J}^{\lambda}(e)(y).
\end{aligned}$$

Hence, $\underline{J}^{\lambda}(e)$ is a FSS of S_1 for all $e \in A$ and so λ is a generalized lower $FSSS$ of S_1 respecting to the aftersets.

(2) Let λ be a $FLIL$ of S_2. Then for $x, y \in S_1$, we have,

$$\begin{aligned}
\underline{J}^{\lambda}(e)(xy) &= \wedge_{a' \in (xy)J(e)} \lambda\left(a'\right) \\
&= \wedge_{a' \in xJ(e).yJ(e)} \lambda\left(a'\right) \\
&= \wedge_{p \in xJ(e),\ q \in yJ(e)} \lambda(pq) \\
&\geq \wedge_{q \in yJ(e)} \lambda(q) \\
&= \underline{J}^{\lambda}(e)(y).
\end{aligned}$$

Hence, $\underline{J}^{\lambda}(e)$ is a $FLIL$ of S_1 for all $e \in A$ and so λ is a generalized lower $FSLIL$ of S_1 respecting to the aftersets. ■

Now, results related to lower approximations interms of FSS ($FLIL$, $FRIL$) of a semigroup respecting to foresets are being given.

Theorem 5.1.9 *Let (J, A) be an SC_mR respecting to the foresets from a semigroup S_1 to a semigroup S_2.*

(1) Then δ is a generalized lower $FSSS$ of S_2 respecting to the foresets, if δ is a FSS of S_1.

(2) Then δ is a generalized lower $FSLIL$ ($FSRIL$, $FSIL$) of S_2 respecting to the foresets, if δ is a $FLIL$ ($FRIL$, FIL) of S_1.

Proof. The proof is simple like the proof of Theorem 5.1.8. ■

Example 5.1.10 *For two semigroups $S_1 = \{a, b, c, d\}$ and $S_2 = \{1, 2, 3, 4\}$ with the multiplication tables as follows:*

$\bullet$	a	b	c	d
a	a	a	a	d
b	a	b	a	d
c	a	a	c	d
d	d	d	d	d

$\bullet$	1	2	3	4
1	1	2	3	4
2	2	2	2	2
3	3	3	3	3
4	4	3	2	1

and $A = \{e_1, e_2\}$. Define $J : A \to P(S_1 \times S_2)$ by

$$\begin{aligned} J(e_1) &= \{(a,2),(a,3),(b,2),(b,3),(c,2),(c,3),(d,2),(d,3)\} \text{ and} \\ J(e_2) &= \{(a,2),(b,2),(c,2),(d,2)\}. \end{aligned}$$

Then J is an SC_mR respecting to the aftersets. from the semigroup S_1 to the semigroup S_2.

$$\begin{aligned} aJ(e_2) &= \{2,3\},\ bJ(e_2) = \{2,3\},\ cJ(e_2) = \{2,3\} \text{ and } dJ(e_2) = \{2,3\}. \text{ Also,} \\ aJ(e_2) &= \{2\},\ bJ(e_2) = \{2\},\ cJ(e_2) = \{2\} \text{ and } dJ(e_2) = \{2\}. \end{aligned}$$

(1) *Define $\lambda : S_2 \longrightarrow [0, 1]$ by*

	1	2	3	4
λ	0.2	0.4	0.6	0.8

Then λ *is not a* FSS *and* $FLIL$ *of* S_2 *as* $\lambda(4.3) = \lambda(2) = 0.4 \ngeq 0.6 = \lambda(4) \wedge \lambda(3)$ *but*

	a	b	c	d
$\underline{J}^{\lambda}(e_1)$	0.4	0.4	0.4	0.4
$\underline{J}^{\lambda}(e_2)$	0.4	0.4	0.4	0.4

Clearly, $\underline{J}^{\lambda}(e_1)$ *and* $\underline{J}^{\lambda}(e_2)$ *are* $FSSs$ *and* $FLILs$ *of* S_1*. Hence,* λ *is a generalized lower* $FSSS$ *and* $FSLILs$ *of* S_1 *respecting to the aftersets.*

Define $J : A \to P(S_1 \times S_2)$ *by*

$$\begin{aligned} J(e_1) &= \{(d,1),(d,2),(d,3),(d,4)\} \text{ and} \\ J(e_2) &= \{(a,1),(a,2),(a,3),(a,4),(d,1),(d,2),(d,3),(d,4)\}. \end{aligned}$$

Then J *is an* SC_mR *respecting to the foresets from the semigroup* S_1 *to the semigroup* S_2*.*

$$\begin{aligned} J(e_2)1 &= \{d\},\ J(e_2)2 = \{d\},\ J(e_2)3 = \{d\} \text{ and } J(e_2)4 = \{d\}. \text{ Also,} \\ J(e_2)1 &= \{a,d\},\ J(e_2)2 = \{a,d\},\ J(e_2)3 = \{a,d\} \text{ and } J(e_2)4 = \{a,d\}. \end{aligned}$$

(1) *Define* $\delta : S_1 \longrightarrow [0,1]$ *by*

	a	b	c	d
δ	0.1	0.3	0.5	0.7

Then δ *is not a* FSS *and* FLI *of* S_1 *as* $\delta(cb) = \delta(a) = 0.1 \ngeq 0.3 = \delta(c) \wedge \delta(b)$ *but*

	1	2	3	4
${}^{\delta}\underline{J}(e_1)$	0.7	0.7	0.7	0.7
${}^{\delta}\underline{J}(e_2)$	0.1	0.1	0.1	0.1

Clearly, ${}^{\delta}\underline{J}(e_1)$ *and* ${}^{\delta}\underline{J}(e_2)$ *are* $FSSs$ *and* $FLIs$ *of* S_2*. Hence,* δ *is a generalized lower* $FSSS$ *and* $FSLIs$ *of* S_2 *respecting to the foresets.*

Theorem 5.1.11 *Let* (J, A) *be an* $SBRE$ *from a semigroup* S_1 *to a semigroup* S_2*. Then for any* $FRIL$*,* λ_1 *and* $FLIL$ λ_2 *of* S_2*,* $\overline{J}^{\lambda_1\lambda_2} \subseteq \overline{J}^{\lambda_1} \cap \overline{J}^{\lambda_2}$*.*

Proof. Suppose that λ_1 is a $FRIL$ and λ_2 a $FLIL$ of S_2, so by definition $\lambda_1\lambda_2 \subseteq \lambda_1 \cap \lambda_2$. It follows from Theorem 4.1.4 (part (2) and (4)), $\overline{J}^{\lambda_1\lambda_2}(e) \subseteq \overline{J}^{\lambda_1\cap\lambda_2}(e) \subseteq \overline{J}^{\lambda_1}(e) \cap \overline{J}^{\lambda_2}(e)$. Hence, $\overline{J}^{\lambda_1\lambda_2} \subseteq \overline{J}^{\lambda_1} \cap \overline{J}^{\lambda_2}$. ■

The next Theorem has similar proof as Theorem 5.1.11.

Theorem 5.1.12 *Let (J, A) be an SBRE from a semigroup S_1 to a semigroup S_2. Then for any FRIL, δ_1 and FLIL δ_2 of S_1, ${}^{\delta_1\delta_2}\overline{J} \subseteq {}^{\delta_1}\overline{J} \cap^{\delta_2} \overline{J}$.*

Theorem 5.1.13 *Let (J, A) be an SBRE from a semigroup S_1 to a semigroup S_2. Then for any $FRIL$, λ_1 and $FLIL$, λ_2 of S_2, $\underline{J}^{\lambda_1\lambda_2} \subseteq \underline{J}^{\lambda_1} \cap \underline{J}^{\lambda_2}$.*

Proof. Suppose that λ_1 is a $FRIL$ and λ_2 a $FLIL$ of S_2 , so by definition $\lambda_1\lambda_2 \subseteq \lambda_1 \cap \lambda_2$. It follows from Theorem 4.1.4 (part (1) and (3)), $\underline{J}^{\lambda_1\lambda_2}(e) \subseteq \underline{J}^{\lambda_1\cap\lambda_2}(e) = \underline{J}^{\lambda_1}(e) \cap \underline{J}^{\lambda_2}(e)$. Hence, $\underline{J}^{\lambda_1\lambda_2} \subseteq \underline{J}^{\lambda_1} \cap \underline{J}^{\lambda_2}$. ■

The next Theorem has similar proof as Theorem 5.1.13.

Theorem 5.1.14 *Let (J, A) be an SBRE from a semigroup S_1 to a semigroup S_2. Then $FRIL$, δ_1 and $FLIL$, δ_2 of S_2, ${}^{\delta_1\delta_2}\underline{J} \subseteq {}^{\delta_1}\underline{J} \cap {}^{\delta_2}\underline{J}$.*

Discussion related to $FIIL$s of a semigroup are presented below.

Definition 5.1.15 *Let (J, A) be an SBRE from a semigroup S_1 to a semigroup S_2 and λ be a non-empty FS of S_2. Then λ is said to be generalized lower (upper) $FSIIL$ of S_1 respecting to the aftersets if $\left(\underline{J}^{\lambda}, A\right)$ $\left(\text{respectively } \left(\overline{J}^{\lambda}, A\right)\right)$ is a FIIL of S_1.*

Definition 5.1.16 *Let δ be a non-empty FS of S_1 and (J, A) be an SBRE from a semigroup S_1 to a semigroup S_2. Then δ is said to be generalized lower (upper) $FSIIL$ of S_2 respecting to the foresets if $\left({}^{\delta}\underline{J}, A\right)$ $\left(\text{respectively } \left({}^{\delta}\overline{J}, A\right)\right)$ is a FIIL of S_2.*

Theorem 5.1.17 *Let (J, A) be an SCRE respecting to the aftersets from a semigroup S_1 to a semigroup S_2. Then λ is a generalized upper FSIIL of S_1 respecting to the aftersets, if λ is a FIIL of S_2.*

Proof. Suppose λ is a $FIIL$ of S_2. For $x, a, y \in S_1$,

$$\begin{aligned}
\overline{J}^{\lambda}(e)(a) &= \vee_{q \in aJ(e)} \lambda(q) \\
&\leq \vee_{p \in xJ(e)} \vee_{q \in aJ(e)} \vee_{r \in yJ(e)} \lambda(pqr) \\
&\leq \vee_{(pqr) \in (xay)J(e)} \lambda\left(a'\right) \\
&= \vee_{a' \in (xay)J(e)} \lambda\left(a'\right) \\
&= \overline{J}^{\lambda}(e)(xay).
\end{aligned}$$

Hence, $\overline{J}^{\lambda}(e)$ is a $FIIL$ of S_1 for all $e \in A$ and so λ is a generalized upper $FSIIL$ of S_1 respecting to the aftersets. ∎

It is found in the accompanying Example that converse to above Theorem is not precise.

Example 5.1.18 *For two semigroups $S_1 = \{1, 2, 3\}$ and $S_2 = \{a, b, c\}$ with the multiplication tables as follows:*

$\bullet$	1	2	3
1	1	2	3
2	1	2	3
3	1	2	3

$\bullet$	a	b	c
a	a	a	c
b	a	b	c
c	a	c	c

and $A = \{e_1, e_2\}$. Define $J : A \to P(S_1 \times S_2)$ by

$$\begin{aligned}
J(e_1) &= \{(1,a),(2,b),(3,c),(1,b),(2,a),(1,c),(3,a)\} \text{ and} \\
J(e_2) &= \{(1,a),(1,b),(1,c),(2,a),(2,b),(3,c)\}.
\end{aligned}$$

Then (J, A) is an $SCRE$ from S_1 to S_2. Now,

$$\begin{aligned}
1J(e_1) &= \{a,b,c\},\ 2J(e_1) = \{a,b\} \text{ and } 3J(e_1) = \{a,c\}, \\
1J(e_2) &= \{a,b,c\},\ 2J(e_2) = \{a,b\} \text{ and } 3J(e_2) = \{c\}.
\end{aligned}$$

Define $\lambda : S_2 \longrightarrow [0,1]$ by

	a	b	c
λ	0	0.1	0.1

Then λ is not a $FIIL$ of S_2 as $\lambda(bca) = \lambda(a) = 0 \ngeq 0.1 = \lambda(c)$ but

	1	2	3
$\overline{J}^{\lambda}(e_1)$	0.1	0.1	0.1
$\overline{J}^{\lambda}(e_2)$	0.1	0.1	0.1

Clearly, $\overline{J}^{\lambda}(e_1)$ and $\overline{J}^{\lambda}(e_2)$ are $FIIL$s of S_1. Hence, λ is a generalized upper $FSIIL$ of S_1 respecting to the aftersets.

Next upper approximations are being related to $FIIL$ of a semigroup concerning foresets.

Theorem 5.1.19 *Let (J, A) be an $SCRE$ respecting to the foresets from a semigroup S_1 to a semigroup S_2. Then δ is a generalized upper $FSIIL$ of S_2 respecting to the foresets, if δ is a $FIIL$ of S_1.*

Proof. It has similar proof as Theorem 5.1.17. ■

Moreover the next example shows that the converse of above Theorem is not true.

Example 5.1.20 *Consider the Example 5.1.18,*

$$\begin{aligned} J(e_1)a &= \{1,2,3\},\ J(e_1)b = \{1,2\},\ J(e_1)c = \{1,3\}, \\ J(e_2)a &= \{1,2\},\ J(e_2)b = \{1,2\} \text{ and } J(e_2)c = \{1,2,3\}. \end{aligned}$$

Define $\delta : S_1 \longrightarrow [0,1]$ *by*

	1	2	3
δ	0	0.1	0.1

Then δ is not a $FIIL$ of S_1 as $\delta(231) = \delta(1) = 0 \ngeq 0.1 = \delta(3)$ but

	a	b	c
${}^{\delta}\overline{J}(e_1)$	0.1	0.1	0.1
${}^{\delta}\overline{J}(e_2)$	0.1	0.1	0.1

Clearly, ${}^{\delta}\overline{J}(e_1)$ and ${}^{\delta}\overline{J}(e_2)$ are $FIIL$s of S_2. Hence, δ is a generalized upper $FSIIL$ of S_2 respecting to the foresets.

Now, lower approximations are described interms of $FIIL$ of a semigroup respecting to aftersets.

Theorem 5.1.21 *Let (J, A) be an SC_mR respecting to the aftersets from a semigroup S_1 to a semigroup S_2. Then λ is a generalized lower $FSIIL$ of S_1 respecting to the aftersets, if λ is a $FIIL$ of S_2.*

Proof. Let λ be a $FIIL$ of S_2. For $x, a, y \in S_1$,

$$\begin{aligned} \underline{J}^{\lambda}(e)(xay) &= \wedge_{a' \in (xay)J(e)} \lambda\left(a'\right) \\ &= \wedge_{a' \in xJ(e).aJ(e).yJ(e)} \lambda\left(a'\right) \\ &= \wedge_{p \in xJ(e),\ q \in aJ(e),\ r \in yJ(e)} \lambda(pqr) \\ &\geq \wedge_{q \in aJ(e)} \lambda(q) \\ &= \underline{J}^{\lambda}(e)(a). \end{aligned}$$

Hence, $\underline{J}^{\lambda}(e)$ is a $FIIL$ of S_1 for all $e \in A$ and so λ is a generalized lower $FSIIL$ of S_1 respecting to the aftersets. ■

Example 5.1.22 *Consider the Example 5.1.10 and $A = \{e_1, e_2\}$. Define $J : A \rightarrow P(S_1 \times S_2)$ by*

$$\begin{aligned} J(e_1) &= \{(a,2),(a,3),(b,2),(b,3),(c,2),(c,3),(d,2),(d,3)\} \text{ and} \\ J(e_2) &= \{(a,2),(b,2),(c,2),(d,2)\}. \end{aligned}$$

Then J is an SC_mR respecting to the aftersets from S_1 to S_2.

$$\begin{aligned} aJ(e_2) &= \{2,3\},\ bJ(e_2) = \{2,3\},\ cJ(e_2) = \{2,3\} \text{ and } dJ(e_2) = \{2,3\}. \text{ Also,} \\ aJ(e_2) &= \{2\},\ bJ(e_2) = \{2\},\ cJ(e_2) = \{2\} \text{ and } dJ(e_2) = \{2\}. \end{aligned}$$

Define $\lambda : S_2 \longrightarrow [0, 1]$ by

	1	2	3	4
λ	0.4	0.6	0.8	1

Then λ is not a $FIIL$ of S_2 as $\lambda(143) = \lambda(2) = 0.6 \ngeq 1 = \lambda(4)$ but

	a	b	c	d
$\underline{J}^{\lambda}(e_1)$	0.6	0.6	0.6	0.6
$\underline{J}^{\lambda}(e_2)$	0.6	0.6	0.6	0.6

Clearly, $\underline{J}^{\lambda}(e_1)$ and $\underline{J}^{\lambda}(e_2)$ are $FIIL$s of S_1. Hence, λ is a generalized lower $FSIIL$ of S_1 respecting to the aftersets.

Now, $FIIL$ of a semigroup respecting to foresets for lower approximations are presented below.

Theorem 5.1.23 *Let (J, A) be an SC_mR respecting to the foresets from a semigroup S_1 to a semigroup S_2. Then δ is a generalized lower $FSIIL$ of S_2 respecting to the foresets, if δ is a $FIIL$ of S_1.*

Proof. The proof is simple and is similar to Theorem 5.1.21. ■

Example 5.1.24 *Considering the Example 5.1.10 and $A = \{e_1, e_2\}$. Define $J : A \rightarrow P(S_1 \times S_2)$ by*

$$\begin{aligned} J(e_1) &= \{(d,1), (d,2), (d,3), (d,4)\} \text{ and} \\ J(e_2) &= \{(a,1), (a,2), (a,3), (a,4), (d,1), (d,2), (d,3), (d,4)\}. \end{aligned}$$

Then J is an SC_mR respecting to the foresets from the semigroup S_1 to the semigroup S_2.

$$\begin{aligned} J(e_2)1 &= \{d\}, \ J(e_2)2 = \{d\}, \ J(e_2)3 = \{d\} \text{ and } J(e_2)4 = \{d\}. \text{ Also,} \\ J(e_2)1 &= \{a,d\}, \ J(e_2)2 = \{a,d\}, \ J(e_2)3 = \{a,d\} \text{ and } J(e_2)4 = \{a,d\}. \end{aligned}$$

Define $\delta : S_1 \longrightarrow [0,1]$ by

	a	b	c	d
δ	0	0.3	0.5	0.7

Then δ is not a $FIIL$ of S_1 as $\delta(cba) = \delta(a) = 0 \ngeq 0.3 = \delta(b)$ but

	1	2	3	4
${}^{\delta}\underline{J}(e_1)$	0.7	0.7	0.7	0.7
${}^{\delta}\underline{J}(e_2)$	0	0	0	0

Clearly, ${}^{\delta}\underline{J}(e_1)$ and ${}^{\delta}\underline{J}(e_2)$ are $FIIL$s of S_2. Hence, δ is a generalized lower $FSIIL$ of S_2 respecting to the foresets.

Now, we discuss $FBIL$s of a semigroup.

Definition 5.1.25 *Let λ be a non-empty FS of S_2 and (J, A) an SBRE from a semigroup S_1 to a semigroup S_2. Then λ is said to be generalized lower (upper) FSBIL of S_1 respecting to the aftersets if $(\underline{J}^{\lambda}, A)$ $\left(\text{respectively } \left(\overline{J}^{\lambda}, A\right)\right)$ is a FBIL of S_1.*

Definition 5.1.26 *Let δ be a non-empty FS of S_1 and (J, A) an SBRE from a semigroup S_1 to a semigroup S_2. Then δ is said to be generalized lower (upper) $FSBIL$ of S_2 respecting to the foresets if $\left({}^{\delta}\underline{J}, A\right)$ $\left(\text{respectively } \left({}^{\delta}\overline{J}, A\right)\right)$ is a fuzzy $FBIL$ of S_2.*

Now, discussion about upper approximations for $FBIL$ of a semigroup respecting to aftersets are presented here.

Theorem 5.1.27 *Let (J, A) be an $SCRE$ from a semigroup S_1 to a semigroup S_2. Then every $FBIL$, λ of S_2 is a generalized upper $FSBIL$ of S_1 respecting to the aftersets.*

Proof. Let λ be a $FBIL$ of S_2. Obviously, λ is a FSS of S_2, therefore by Theorem 5.1.3, $\overline{J}^{\lambda}(e)$ is a FSS of S_1 for all $e \in A$. For $x, a, y \in S_1$,

$$\begin{aligned}
\overline{J}^{\lambda}(e)(x) \wedge \overline{J}^{\lambda}(e)(y) &= \left(\vee_{p \in xJ(e)} \lambda(p)\right) \wedge \left(\vee_{q \in yJ(e)} \lambda(q)\right) \\
&= \vee_{p \in xJ(e)} \vee_{q \in yJ(e)} \lambda(p) \wedge \lambda(q) \\
&\leq \vee_{p \in xJ(e)} \vee_{r \in yJ(e)} \vee_{q \in zJ(e)} (\lambda(prq)) \\
&\leq \vee_{prq \in (xyz)J(e)} \lambda(prq) \\
&= \vee_{a' \in (xyz)J(e)} \lambda\left(a'\right) \\
&= \overline{J}^{\lambda}(e)(xay).
\end{aligned}$$

Hence, $\overline{J}^{\lambda}(e)$ is a $FBIL$ of S_1 for all $e \in A$ and so λ is a generalized upper $FSBIL$ of S_1 respecting to the aftersets. ■

Example 5.1.28 *Considering the Example 5.1.4,* *Define* $\lambda : S_2 \longrightarrow [0,1]$ *by*

	1	2	3	4	5
λ	0.7	0.6	0.2	0	0.1

Then λ is not a $FBIL$ of S_2 as $\lambda(152) = \lambda(5) = 0.1 \ngeq 0.6 = \lambda(1) \wedge \lambda(2)$but

	a	b	c	d	e
$\overline{J}^{\lambda}(e_1)$	0.7	0.7	0.2	0.7	0.1
$\overline{J}^{\lambda}(e_2)$	0.7	0.7	0.2	0.7	0.1

Clearly, $\overline{J}^{\lambda}(e_1)$ and $\overline{J}^{\lambda}(e_2)$ are $FBILs$ of S_1. Hence, λ is a generalized upper $FSBIL$ of S_1 respecting to the aftersets.

Now, discussion about upper approximations for $FBIL$ of a semigroup respecting to foresets are presented here

Theorem 5.1.29 *Let (J, A) be an SCRE from a semigroup S_1 to a semigroup S_2 for all $e \in A$. Then every FBIL, δ of S_1 is a generalized upper FSBIL of S_2 respecting to the foresets.*

Proof. The proof is simple and is similar to Theorem 5.1.27. ■

Example 5.1.30 *Considering the Example 5.1.4.*

Define $\delta : S_1 \longrightarrow [0,1]$ *by*

	a	b	c	d	e
δ	1	0.5	0.7	0.9	0.2

Then δ is not a $FBIL$ of S_1 as $\delta(cae) = \delta(d) = 0.9 \ngeq 0.2 = \delta(c) \wedge \delta(e)$ but

	1	2	3	4	5
${}^{\delta}\overline{J}(e_1)$	1	0.5	0.9	0.9	0.9
${}^{\delta}\overline{J}(e_2)$	1	0.5	0.9	0.9	0.9

Clearly, ${}^{\delta}\overline{J}(e_1)$ and ${}^{\delta}\overline{J}(e_2)$ are $FBILs$ of S_2. Hence, δ is a generalized upper $FSBIL$ of S_2 respecting to the foresets.

Some discussion about lower approximations for $FBIL$ of a semigroup respecting to aftersets are presented here.

Theorem 5.1.31 *Let (J, A) be an SC_mR respecting to the aftersets from a semigroup S_1 to a semigroup S_2. Then every $FBIL$, λ of S_2 is a generalized lower $FSBIL$ of S_1 respecting to the aftersets.*

Proof. Let λ be a $FBIL$ of S_2. Obviously, λ is a FSS of S_2, therefore by Theorem 5.1.8, $\underline{J}^{\lambda}(e)$ is a FSS of S_1 for all $e \in A$. For $x, a, y \in S_1$,

$$\begin{aligned}
\underline{J}^{\lambda}(e)(xay) &= \wedge_{a' \in (xay)J(e)} \lambda\left(a'\right) \\
&= \wedge_{a' \in xJ(e).aJ(e).yJ(e)} \lambda\left(a'\right) \\
&= \wedge_{p \in xJ(e),\ q \in aJ(e),\ r \in yJ(e)} \lambda(pqr) \\
&\geq \left(\wedge_{p \in xJ(e)} \lambda(p)\right) \wedge \left(\wedge_{r \in yJ(e)} \lambda(r)\right) \\
&= \underline{J}^{\lambda}(e)(x) \wedge \underline{J}^{\lambda}(e)(y).
\end{aligned}$$

Hence, $\underline{J}^{\lambda}(e)$ is a $FBIL$ of S_1 for all $e \in A$ and so λ is a generalized lower $FSBIL$ of S_1 respecting to the aftersets. ■

Example 5.1.32 *Considering the Example 5.1.10 and $A = \{e_1, e_2\}$. Define $J : A \to P(S_1 \times S_2)$ by*

$$\begin{aligned}
J(e_1) &= \{(a,2), (a,3), (b,2), (b,3), (c,2), (c,3), (d,2), (d,3)\} \text{ and} \\
J(e_2) &= \{(a,2), (b,2), (c,2), (d,2)\}.
\end{aligned}$$

Then J is a SC_mR respecting to the aftersets from the semigroup S_1 to S_2. Now,

$$\begin{aligned}
aJ(e_1) &= \{2,3\},\ bJ(e_1) = \{2,3\},\ cJ(e_1) = \{2,3\} \text{ and } dJ(e_1) = \{2,3\}. \text{ Also,} \\
aJ(e_2) &= \{2\},\ bJ(e_2) = \{2\},\ cJ(e_2) = \{2\} \text{ and } dJ(e_2) = \{2\}.
\end{aligned}$$

Define $\lambda : S_2 \longrightarrow [0,1]$ by

	1	2	3	4
λ	0.7	0.2	0.1	0.4

Then λ is not a $FBIL$ of S_2 as $\lambda(312) = \lambda(3) = 0.1 \ngeq 0.2 = \lambda(3) \wedge \lambda(2)$ but

	a	b	c	d
$\underline{J}^{\lambda}(e_1)$	0.1	0.1	0.1	0.1
$\underline{J}^{\lambda}(e_2)$	0.2	0.2	0.2	0.2

Clearly, $\underline{J}^{\lambda}(e_1)$ and $\underline{J}^{\lambda}(e_2)$ are $FBILs$ of S_1. Hence, λ is a generalized lower $FSBIL$ of S_1 respecting to the aftersets.

Further, discussion about lower approximations for $FBIL$ of a semigroup respecting to the foresets are presented here.

Theorem 5.1.33 *Let (J, A) be an SC_mR respecting to the foresets from a semigroup S_1 to a semigroup S_2. Then FBIL, δ of S_1 is a generalized lower FSBIL of S_2 respecting to the foresets.*

Proof. The proof is simple and is similar to Theorem 5.1.31. ■

Example 5.1.34 *Considering the Example 5.1.10 and $A = \{e_1, e_2\}$. Define $J : A \to P(S_1 \times S_2)$ by*

$$\begin{aligned} J(e_1) &= \{(d,1),(d,2),(d,3),(d,4)\} \text{ and} \\ J(e_2) &= \{(a,1),(a,2),(a,3),(a,4),(d,1),(d,2),(d,3),(d,4)\}. \end{aligned}$$

Then J is an SC_mR respecting to the foresets from a semigroup S_1 to a semigroup S_2.

$$\begin{aligned} J(e_2)1 &= \{d\},\ J(e_2)2 = \{d\},\ J(e_2)3 = \{d\} \text{ and } J(e_2)4 = \{d\}. \text{ Also,} \\ J(e_2)1 &= \{a,d\},\ J(e_2)2 = \{a,d\},\ J(e_2)3 = \{a,d\} \text{ and } J(e_2)4 = \{a,d\}. \end{aligned}$$

Define $\delta : S_1 \longrightarrow [0,1]$ by

	a	b	c	d
δ	0.1	0.8	0.6	0.7

Then δ is not a $FBIL$ of S_1 as $\delta(cab) = \delta(a) = 0.1 \ngeq \delta(c) \wedge \delta(b)$ but

	1	2	3	4
${}^{\delta}\underline{J}(e_1)$	0.7	0.7	0.7	0.7
${}^{\delta}\underline{J}(e_2)$	0.1	0.1	0.1	0.1

Clearly, ${}^{\delta}\underline{J}(e_1)$ and ${}^{\delta}\underline{J}(e_2)$ are $FBILs$ of S_2. Hence, δ is a generalized lower $FSBIL$ of S_2 respecting to the foresets.

Chapter 6

Approximation of a soft set by soft relation

In this chapter, some fundamental thoughts identified with rough sets and soft sets are given. Two kinds of soft topologies induced by soft reflexive relations are investigated. Soft similarity relations have also been examined. A decision making problem is given based on a soft set.

6.1 Approximations by Soft Binary Relations

This section presents soft set approximations by an $SBRE$ from a set U to a set W. Some related properties are proposed here.

Definition 6.1.1 *Let (J, A) be an $SBRE$ from U to W and $G : A \to P(W)$ be a soft set in W. Then we define two soft sets over U, by*

$$\begin{aligned}\underline{J}^{G}(e) &= \{u \in U : \phi \neq uJ(e) \subseteq G(e)\} \text{ and} \\ \overline{J}^{G}(e) &= \{u \in U : uJ(e) \cap G(e) \neq \phi\}\end{aligned}$$

where $uJ(e) = \{w \in W : (u, w) \in J(e)\}$ and is called the afterset of u, for $e \in A$ and $u \in U$.

Moreover, $\underline{J}^{G} : A \to P(U)$ and $\overline{J}^{G} : A \to P(U)$.

Definition 6.1.2 *Let (J, A) be an SBRE from U to W and $L : A \to P(U)$ be a soft set in U, two soft sets in W are defined by*

$$\begin{aligned} {}^{L}\underline{J}(e) &= \{w \in W : \phi \neq J(e)\, w \subseteq L(e)\} \text{ and} \\ {}^{L}\overline{J}(e) &= \{w \in W : J(e)\, w \cap L(e) \neq \phi\} \end{aligned}$$

where $J(e)\, w = \{u \in U : (u, w) \in J(e)\}$ and is called foreset of w, for each $w \in W$ and $e \in A$.

Moreover, ${}^{L}\underline{J} : A \to P(W)$ and ${}^{L}\overline{J} : A \to P(W)$.

In order to explain these concepts, the following example is given.

Example 6.1.3 *Suppose that Mr. X wants to buy a shirt for his own use. Let $U = \{$the set of all shirts designs$\} = \{d_1, d_2, d_3, d_4, d_5, d_6\}$ and $W = \{$the colors of all designs$\} = \{c_1, c_2, c_3, c_4\}$ and the set of attributes be $A = \{e_1, e_2, e_3\} = \{$the set of stores near his house$\}$.*

Define $J : A \to P(U \times W)$ by

$$\begin{aligned} J(e_1) &= \left\{ \begin{array}{c} (d_1, c_1), (d_1, c_2), (d_1, c_3), (d_2, c_2), (d_2, c_4), \\ (d_4, c_2), (d_4, c_3), (d_5, c_3), (d_5, c_4), (d_6, c_1) \end{array} \right\}, \\ J(e_2) &= \{(d_1, c_3), (d_2, c_3), (d_4, c_1), (d_5, c_1), (d_6, c_2), (d_6, c_3)\} \text{ and} \\ J(e_3) &= \{(d_3, c_3), (d_3, c_1), (d_2, c_4), (d_5, c_3), (d_5, c_4)\}. \end{aligned}$$

represents the relation between designs and colors available on store e_i for $1 \leq i \leq 3$. Then

$$\begin{aligned} d_1 J(e_1) &= \{c_1, c_2, c_3\},\ d_2 J(e_1) = \{c_2, c_4\},\ d_3 J(e_1) = \phi, \\ d_4 J(e_1) &= \{c_2, c_3\},\ d_5 J(e_1) = \{c_3, c_4\},\ d_6 J(e_1) = \{c_1\} \text{ and} \\ d_1 J(e_2) &= \{c_3\},\ d_2 J(e_2) = \{c_3\},\ d_3 J(e_2) = \phi, \\ d_4 J(e_2) &= \{c_1\},\ d_5 J(e_2) = \{c_1\},\ d_6 J(e_2) = \{c_2, c_3\} \text{ and} \\ d_1 J(e_3) &= \phi,\ d_2 J(e_3) = \{c_4\},\ d_3 J(e_3) = \{c_1, c_3\}, \\ d_4 J(e_3) &= \phi,\ d_5 J(e_3) = \{c_3, c_4\},\ d_6 J(e_3) = \phi. \end{aligned}$$

where $d_i J(e_j)$ represents the color of the design d_i available on store e_j.

And

$$\begin{aligned}
J(e_1)\,c_1 &= \{d_1, d_6\},\ J(e_1)\,c_2 = \{d_1, d_2, d_4\},\\
J(e_1)\,c_3 &= \{d_1, d_4, d_5\},\ J(e_1)\,c_4 = \{d_2, d_5\},\ \textit{and}\\
J(e_2)\,c_1 &= \{d_4, d_5\},\ J(e_2)\,c_2 = \{d_6\},\\
J(e_2)\,c_3 &= \{d_1, d_2, d_6\},\ J(e_2)\,c_4 = \phi,\ \textit{and}\\
J(e_3)\,c_1 &= \{d_3\},\ J(e_3)\,c_2 = \phi,\\
J(e_3)\,c_3 &= \{d_3, d_5\},\ J(e_3)\,c_4 = \{d_2, d_5\}.
\end{aligned}$$

where $J(e_j)\,c_i$ represents the design of the color c_i available on store e_j.

$$\begin{aligned}
\textit{Define } G &: A \to P(W) \textit{ which represents the preference of the}\\
& \quad \textit{color given by Mr. X such that}\\
G(e_1) &= \{c_1, c_2\},\ G(e_2) = \{c_2, c_3\},\ G(e_3) = \{c_1, c_3, c_4\}\ \textit{ and}\\
\textit{Define } H &: A \to P(U) \textit{ which represents the preference of the}\\
& \quad \textit{design given by Mr. X such that}\\
H(e_1) &= \{d_1, d_3, d_6\},\ H(e_2) = \{d_1, d_5\},\ H(e_3) = \{d_3, d_4, d_5, d_6\}.
\end{aligned}$$

Therefore, for each parameter the lower $\underline{J}^G(e)$ and upper $\overline{J}^G(e)$ (respecting with aftersets as well as foresets) are

	d_1	d_2	d_3	d_4	d_5	d_6
$\underline{J}^G(e_1)$	0	0	0	0	0	1
$\overline{J}^G(e_1)$	1	1	0	1	0	1
$\underline{J}^G(e_2)$	1	1	0	0	0	1
$\overline{J}^G(e_2)$	1	1	0	0	0	1
$\underline{J}^G(e_3)$	0	1	1	0	0	1
$\overline{J}^G(e_3)$	0	1	1	0	1	0

and

	c_1	c_2	c_3	c_4
${}^H\underline{J}(e_1)$	1	0	0	0
${}^H\overline{J}(e_1)$	1	1	1	0
${}^H\underline{J}(e_2)$	0	0	0	0
${}^H\overline{J}(e_2)$	1	0	1	0
${}^H\underline{J}(e_3)$	1	0	1	0
${}^H\overline{J}(e_3)$	1	0	1	1

Hence, $\underline{J}^G(e_i)$ *gives the degree of definite fulfilment of the objects of* $dJ(e_i)$ *to* G *on store* e_i *and* $\overline{J}^G(e_i)$ *gives the degree of possible fulfilment of the objects of* $dJ(e_i)$ *to* G *on store* e_i *for* $1 \leq i \leq 3$ *respecting to aftersets. Similarly,* ${}^H\underline{J}(e_i)$ *gives the degree of definite fulfilment of the objects of* $J(e_i)c$ *to* H *on store* e_i *and* ${}^H\overline{J}(e_i)$ *gives the degree of possible fulfilment of the objects of* $J(e_i)c$ *to* H *on store* e_i *for* $1 \leq i \leq 3$ *respecting to the foresets.*

Theorem 6.1.4 *Let* (J, A) *be an SBRE from* U *to* W*, that is* $J : A \rightarrow P(U \times W)$*. For* $G_1 : A \rightarrow P(W)$ *and* $G_2 : A \rightarrow P(W)$*, the properties given below for lower and upper approximations respecting to the aftersets hold:*

(1) $G_1(e) \subseteq G_2(e) \Rightarrow \underline{J}^{G_1}(e) \subseteq \underline{J}^{G_2}(e)$

(2) $G_1(e) \subseteq G_2(e) \Rightarrow \overline{J}^{G_1}(e) \subseteq \overline{J}^{G_2}(e)$

(3) $\underline{J}^{G_1}(e) \cap \underline{J}^{G_2}(e) = \underline{J}^{G_1 \cap G_2}(e)$

(4) $\overline{J}^{G_1}(e) \cap \overline{J}^{G_2}(e) \supseteq \overline{J}^{G_1 \cap G_2}(e)$

(5) $\underline{J}^{G_1}(e) \cup \underline{J}^{G_2}(e) \subseteq \underline{J}^{G_1 \cup G_2}(e)$

(6) $\overline{J}^{G_1}(e) \cup \overline{J}^{G_2}(e) = \overline{J}^{G_1 \cup G_2}(e)$

(7) $\underline{J}^{\mathbf{W}}(e) \subseteq U$ *for all* $e \in A$ *and if for all* $u \in U$, $uJ(e) \neq \phi$ *then* $\underline{J}^{\mathbf{W}}(e) = U$

(8) $\overline{J}^{\mathbf{W}}(e) \subseteq U$ *for all* $e \in A$ *and if* $uJ(e) \neq \phi$ *for all* $u \in U$*, then* $\overline{J}^{\mathbf{W}}(e) = U$*, where* $\mathbf{W} : A \rightarrow P(W)$ *such that* $\mathbf{W}(e) = W$ *for all* $e \in A$*.*

(9) $\underline{J}^{G_1}(e) = \left(\overline{J}^{\left(G_1^c\right)}\right)^c(e)$

(10) $\overline{J}^{G_1}(e) = \left(\underline{J}^{\left(G_1^c\right)}\right)^c(e)$.

Proof. (1) Let $u \in \underline{J}^{G_1}(e)$. Then $\phi \neq uJ(e) \subseteq G_1(e)$. As $G_1(e) \subseteq G_2(e)$, we have $\phi \neq uJ(e) \subseteq G_2(e)$. Thus $u \in \underline{J}^{G_2}(e)$. Hence, $\underline{J}^{G_1}(e) \subseteq \underline{J}^{G_2}(e)$.

(2) Let $u \in \overline{J}^{G_1}(e)$. Then $uJ(e) \cap G_1(e) \neq \phi$. As $G_1(e) \subseteq G_2(e)$, we have $uJ(e) \cap G_2(e) \neq \phi$. Thus $u \in \overline{J}^{G_2}(e)$. Hence, $\overline{J}^{G_1}(e) \subseteq \overline{J}^{G_2}(e)$.

(3) Using part (1) and the fact that $G_1(e) \cap G_2(e) \subseteq G_1(e), G_2(e)$, we have $\underline{J}^{G_1 \cap G_2}(e) \subseteq \underline{J}^{G_1}(e)$, $\underline{J}^{G_2}(e)$ so $\underline{J}^{G_1 \cap G_2}(e) \subseteq \underline{J}^{G_1}(e) \cap \underline{J}^{G_2}(e)$. For the reverse inclusion, let $u \in \underline{J}^{G_1}(e) \cap \underline{J}^{G_2}(e) \Rightarrow u \in \underline{J}^{G_1}(e)$ and $u \in \underline{J}^{G_2}(e)$

$\Rightarrow uJ(e) \subseteq G_1(e)$ and $uJ(e) \subseteq G_2(e) \Rightarrow uJ(e) \subseteq G_1(e) \cap G_2(e)$

$\Rightarrow u \in \underline{J}^{G_1 \cap G_2}(e) \Rightarrow \underline{J}^{G_1}(e) \cap \underline{J}^{G_2}(e) \subseteq \underline{J}^{G_1 \cap G_2}(e)$. Hence, $\underline{J}^{G_1}(e) \cap \underline{J}^{G_2}(e) = \underline{J}^{G_1 \cap G_2}(e)$.

(4) Using part (2) and the fact that $G_1(e) \cap G_2(e) \subseteq G_1(e), G_2(e)$, we have
$\overline{J}^{G_1 \cap G_2}(e) \subseteq \overline{J}^{G_1}(e), \overline{J}^{G_2}(e) \Rightarrow \overline{J}^{G_1 \cap G_2}(e) \subseteq \overline{J}^{G_1}(e) \cap \overline{J}^{G_2}(e)$.

(5) Since $G_1(e), G_2(e) \subseteq G_1(e) \cup G_2(e)$, so by using part (1), we get
$\underline{J}^{G_1}(e), \underline{J}^{G_2}(e) \subseteq \underline{J}^{G_1 \cup G_2}(e)$ and so $\underline{J}^{G_1}(e) \cup \underline{J}^{G_2}(e) \subseteq \underline{J}^{G_1 \cup G_2}(e)$.

(6) Since $G_1(e), G_2(e) \subseteq G_1(e) \cup G_2(e)$, so by using part (2), we get
$\overline{J}^{G_1}(e), \overline{J}^{G_2}(e) \subseteq \overline{J}^{G_1 \cup G_2}(e)$ which implies $\overline{J}^{G_1}(e) \cup \overline{J}^{G_2}(e) \subseteq \overline{J}^{G_1 \cup G_2}(e)$.

For the reverse inclusion, let $u \in \overline{J}^{G_1 \cup G_2}(e) \Rightarrow uJ(e) \cap (G_1(e) \cup G_2(e)) \neq \phi \Rightarrow uJ(e) \cap G_1(e) \neq \phi$ or $uJ(e) \cap G_2(e) \neq \phi \Rightarrow u \in \overline{J}^{G_1}(e)$ or $u \in \overline{J}^{G_2}(e)$
$\Rightarrow u \in \left(\overline{J}^{G_1}(e) \cup \overline{J}^{G_2}\right)(e) \Rightarrow \overline{J}^{G_1 \cup G_2}(e) \subseteq \overline{J}^{G_1}(e) \cup \overline{J}^{G_2}(e)$. Hence, $\overline{J}^{G_1 \cup G_2}(e) = \overline{J}^{G_1}(e) \cup \overline{J}^{G_2}(e)$.

(7) By definition, $\underline{J}^{\mathbf{W}}(e) = \{u \in U : \phi \neq uJ(e) \subseteq \mathbf{W}(e)\} \subseteq U$, because $uJ(e) = \{w \in W : (u, w) \in J(e)\} \subseteq W$. If $uJ(e) \neq \phi$ for all $u \in U$, then $\underline{J}^{\mathbf{W}}(e) = U$.

(8) By definition, $\overline{J}^{\mathbf{W}}(e) = \{u \in U : uJ(e) \cap \mathbf{W}(e) \neq \phi\} \subseteq U$. Moreover, If $uJ(e) \neq \phi$ for every $u \in U$, then $\overline{J}^{\mathbf{W}}(e) = U$.

(9) Let $u \in \underline{J}^{G_1}(e) \Leftrightarrow \phi \neq uJ(e) \subseteq G_1(e) \Leftrightarrow uJ(e) \cap G_1^c(e) = \phi$
$\Leftrightarrow u \notin \overline{J}^{G_1^c}(e) \Leftrightarrow u \in \left(\overline{J}^{(G_1^c)}\right)^c(e)$. Hence, $\underline{J}^{G_1}(e) = \left(\overline{J}^{(G_1^c)}\right)^c(e)$.

(10) By part (9), $\underline{J}^{G_1}(e) = \left(\overline{J}^{(G_1^c)}\right)^c(e)$, therefore $\underline{J}^{G_1^c}(e) = \left(\overline{J}^{(G_1^c)^c}\right)^c(e)$.

$\Rightarrow \underline{J}^{G_1^c}(e) = \left(\overline{J}^{G_1}\right)^c(e)$. Hence, $\left(\underline{J}^{(G_1^c)}\right)^c(e) = \overline{J}^{G_1}(e)$. ■

It is demonstrated by the following example that equality is not valid in (4) and (5) in general.

Example 6.1.5 *Consider $W = \{m_1, m_2, m_3, m_4\}$ is a collection of four mobile phones as the universal set. These mobile phones are classified by attributes age and color represented by $A = \{e_1, e_2\}$. Let $U = \{$new, used, old, black, white$\}$ be represented by $U = \{n, u, o, b, w\}$.*

Define a relation $J : A \to P(U \times W)$ *by*

$$J(e_1) = \{(n, m_1), (n, m_2), (o, m_3), (o, m_4), (u, m_5)\}$$

and

$$J(e_2) = \{(b, m_2), (b, m_3), (w, m_1), (w, m_4), (w, m_5)\}.$$

Now, $nJ(e_1) = \{m_1, m_2\}$, $u(e_1) = \{m_5\}$, $oJ(e_1) = \{m_3, m_4\}$ *and* $bJ(e_2) = \{m_2, m_3\}$, $wJ(e_2) = \{m_1, m_4, m_5\}$.

Define $G_1 : A \to P(W)$ *by* $G_1(e_1) = \{m_1, m_2, m_3\}$ *and* $G_1(e_2) = \{m_1, m_3\}$.

And,

Define $G_2 : A \to P(W)$ *by* $G_2(e_1) = \{m_2, m_4, m_5\}$ *and* $G_2(e_2) = \{m_1, m_4\}$.

Then $(G_1 \cap G_2)(e_1) = G_1(e_1) \cap G_2(e_1) = \{m_2\}$ *and* $(G_1 \cup G_2)(e_1) = G_1(e_1) \cup G_2(e_1) = \{m_1, m_2, m_3, m_4, m_5\}$. *Therefore,*

$\overline{J}^{G_1 \cap G_2}(e_1) = \{n\}$, $\underline{J}^{G_1 \cup G_2}(e_1) = \{n, u, o, b, w\}$,

$\overline{J}^{G_1}(e_1) = \{n, o\}$, $\overline{J}^{G_2}(e_1) = \{n, u, o\}$, $\underline{J}^{G_1}(e_1) = \{n\}$, $\underline{J}^{G_2}(e_1) = \{u\}$.

Hence, $\overline{J}^{G_1}(e) \cap \overline{J}^{G_2}(e) \nsubseteq \overline{J}^{G_1 \cap G_2}(e)$

$\underline{J}^{G_1 \cup G_2}(e) \nsubseteq \underline{J}^{G_1}(e_1) \cup \underline{J}^{G_2}(e_1)$.

Theorem 6.1.6 *Let* (J, A) *be an SBRE from* U *to* W, *that is* $J : A \to P(U \times W)$. *For* $H_1 : A \to P(U)$ *and* $H_2 : A \to P(U)$, *the following properties respecting to foresets hold:*

(1) $H_1(e) \subseteq H_2(e) \Rightarrow {}^{H_1}\underline{J}(e) \subseteq {}^{H_2}\underline{J}(e)$

(2) $H_1(e) \subseteq H_2(e) \Rightarrow {}^{H_1}\overline{J}(e) \subseteq {}^{H_2}\overline{J}(e)$

(3) ${}^{H_1}\underline{J}(e) \cap {}^{H_2}\underline{J}(e) = {}^{H_1 \cap H_2}\underline{J}(e)$

(4) ${}^{H_1}\overline{J}(e) \cap {}^{H_2}\overline{J}(e) \supseteq {}^{H_1 \cap H_2}\overline{J}(e)$

(5) ${}^{H_1}\underline{J}(e) \cup {}^{H_2}\underline{J}(e) \subseteq {}^{H_1 \cup H_2}\underline{J}(e)$

(6) ${}^{H_1}\overline{J}(e) \cup {}^{H_2}\overline{J}(e) = {}^{H_1 \cup H_2}\overline{J}(e)$

(7) ${}^{U}\underline{J}\,(e) \subseteq W$ *for all* $e \in A$ *and if* $uJ\,(e) \neq \phi$ *for all* $u \in U$, *then* ${}^{U}\underline{J}\,(e) = U$

(8) ${}^{U}\overline{J}\,(e) \subseteq W$ *for all* $e \in A$ *and if* $uJ\,(e) \neq \phi$ *for all* $u \in U$, *then* ${}^{U}\overline{J}\,(e) = U$

(9) ${}^{H_1}\underline{J}\,(e) = \left({}^{(H_1^c)}\overline{J}\right)^c(e)$

(10) ${}^{H_1}\overline{J}\,(e) = \left({}^{(H_1^c)}\underline{J}\right)^c(e)\,.$

Proof. The proof is obtained in a similar way from 6.1.4. ■

Theorem 6.1.7 *Let* (J, A) *and* (K, A) *be two SBRE from a non-empty set* U *to a non-empty set* W *and let* (G_1, A) *and* (G_2, A) *be two soft sets over* W. *Then the following assertions hold:*

(1) $(J, A) \subseteq (K, A)$ *implies* $\left(\underline{J}^{G_1}, A\right) \supseteq \left(\underline{K}^{G_1}, A\right)$;

(2) $(J, A) \subseteq (K, A)$ *implies* $\left(\overline{J}^{G_1}, A\right) \subseteq \left(\overline{K}^{G_1}, A\right)$.

Theorem 6.1.8 *Let* (J, A) *and* (K, A) *be two SBRE from a non-empty set* U *to a non-empty set* W *and let* (L, A) *be a soft set over* U. *Then the assertions following hold:*

(1) $(J, A) \subseteq (K, A)$ *implies* $\left({}^{L}\underline{J}, A\right) \supseteq \left({}^{L}\underline{K}, A\right)$;

(2) $(J, A) \subseteq (K, A)$ *implies* $\left({}^{L}\overline{J}, A\right) \subseteq \left({}^{L}\overline{K}, A\right)$.

Theorem 6.1.9 *Let* (J, A) *and* (K, A) *be two SBRE from a non−empty set* U *to a non-empty set* W. *If* (G, A) *is a soft set over* W, *then*

(1) $\left(\left(\overline{J \cap K}\right)^G, A\right) \subseteq \left(\overline{J}^G, A\right) \cap \left(\overline{K}^G, A\right)$.

(2) $\left(\left(\underline{J \cap K}\right)^G, A\right) \supseteq \left(\underline{J}^G, A\right) \cup \left(\underline{K}^G, A\right)$.

Proof. The results are proceeded by parts (1) and (2) of Theorem 6.1.7. ■

Observed that converse of above results is not valid. Now we discuss it in the next example.

Example 6.1.10 *Let* $U = \{a, b, c, d, e\}$ *and* $W = \{1, 2, 3, 4, 5\}$ *and* $A = \{e_1, e_2\}$. *Define* $J : A \to P(U \times W)$ *and* $K : A \to P(U \times W)$ *by*

$$J(e_1) = \left\{ \begin{array}{c} (e,5)\,,(d,4)\,,(a,1)\,,(c,3)\,,(b,1)\,,(c,5)\,, \\ (b,5)\,,(d,3)\,,(d,5)\,,(d,1)\,,(e,1)\,,(b,2) \end{array} \right\}$$

$$J(e_2) = \{(b,2), (e,5), (a,1), (d,4), (c,3)\},$$

$$K(e_1) = \{(e,2), (a,1), (e,5), (b,1), (d,4), (c,3), (b,2)\} \text{ and}$$

$$K(e_2) = \{(b,2), (a,1), (d,4), (e,5), (b,3), (c,3)\}.$$

Therefore,

$$(J \cap K)(e_1) = \{(b,2), (a,1), (c,3), (d,4), (e,5), (b,1)\}$$

and

$$(J \cap K)(e_2) = \{(a,1), (d,4), (e,5), (c,3), (b,2)\}.$$

Now,

$$\begin{aligned} aJ(e_1) &= \{1\},\ bJ(e_1) = \{1,2,5\},\ cJ(e_1) = \{3,5\}, \\ dJ(e_1) &= \{1,3,4,5\} \text{ and } eJ(e_1) = \{1,5\} \end{aligned}$$

and

$$\begin{aligned} aK(e_1) &= \{1\},\ bK(e_1) = \{1,2\},\ cK(e_1) = \{3\}, \\ dK(e_1) &= \{4\} \text{ and } eK(e_1) = \{2,5\}. \end{aligned}$$

Also,

$$\begin{aligned} a(J \cap K)(e_1) &= \{1\},\ b(J \cap K)(e_1) = \{1,2\},\ c(J \cap K)(e_1) = \{3\}, \\ d(J \cap K)(e_1) &= \{4\} \text{ and } e(J \cap K)(e_1) = \{5\}. \end{aligned}$$

Define (G_1, A), *a soft set over* W *by* $G_1(e_1) = \{1,2\}$ *and* $G_1(e_2) = \{2,4,5\}$.

Then

$$\overline{J}^{G_1}(e_1) = \{a,b,d,e\},\ \overline{K}^{G_1}(e_1) = \{a,b,e\} \text{ and } \left(\overline{J \cap K}\right)^{G_1}(e_1) = \{a,b\}.$$

This shows that

$$\overline{J}^{G_1}(e_1) \cap \overline{K}^{G_1}(e_1) = \{a,b,e\} \neq \{a,b\} = \left(\overline{J \cap K}\right)^{G_1}(e_1).$$

Now,

Define (G_2, A), *a soft set over* W *by* $G_2(e_1) = \{5\}$ *and* $G_2(e_2) = \{1,3,4\}$.

Then $\underline{J}^{G_2}(e_1) = \phi$, $\underline{K}^{G_2}(e_1) = \phi$ *and* $(\underline{J \cap K})^{G_2}(e_1) = \{e\}$. *This shows that*

$$\underline{J}^{G_2}(e_1) \cup \underline{K}^{G_2}(e_1) = \phi \neq \{e\} = (\underline{J \cap K})^{G_2}(e_1).$$

Theorem 6.1.11 *Let (J, A) and (K, A) be two SBRE from a non-empty set U to a non-empty set W. If (L, A) is a soft set over U, then*

(1) $\left({}^{L}\left(\overline{J \cap K}\right), A\right) \subseteq \left({}^{L}\overline{J}, A\right) \cap \left({}^{L}\overline{K}, A\right)$.

(2) $\left({}^{L}\left(\underline{J \cap K}\right), A\right) \supseteq \left({}^{L}\underline{J}, A\right) \cup \left({}^{L}\underline{K}, A\right)$.

Proof. The results follows from parts (1) and (2) of Theorem 6.1.8. ■

The counter parts are not valid as in the example below:

Example 6.1.12 *Let $U = \{a, b, c, d, e\}$ and $W = \{1, 2, 3, 4, 5\}$ and $A = \{e_1, e_2\}$. Define $J : A \to P(U \times W)$ and $K : A \to P(U \times W)$ by*

$$J(e_1) = \left\{ \begin{array}{c} (a,1), (b,2), (e,5), (b,1), (c,5), (b,5), \\ (d,3), (d,5), (c,3), (d,4), (d,1), (e,1)\} \end{array} \right\},$$

$$J(e_2) = \{(a,1), (c,3), (d,4), (b,2), (e,5)\},$$

$$K(e_1) = \{(a,1), (c,3), (d,4), (b,2), (e,5), (b,1), (a,5)\} \text{ and}$$

$$K(e_2) = \{(a,1), (c,3), (b,2), (e,5), (b,3), (d,4)\}.$$

Therefore,

$$(J \cap K)(e_1) = \{(a,1), (d,4), (e,5), (b,2), (b,1), (c,3)\}$$

and

$$(J \cap K)(e_2) = \{(d,4), (a,1), (c,3), (e,5), (b,2)\}.$$

Now,

$$\begin{aligned} J(e_1)1 &= \{a, b, d, e\},\ J(e_1)2 = \{b\},\ J(e_1)3 = \{c, d\}, \\ J(e_1)4 &= \{d\} \text{ and } J(e_1)5 = \{b, c, d, e\}, \end{aligned}$$

and

$$\begin{aligned} K(e_1)1 &= \{a, b\},\ K(e_1)2 = \{b\},\ 3K(e_1) = \{c\}, \\ K(e_1)4 &= \{d\} \text{ and } K(e_1)5 = \{a, e\}. \end{aligned}$$

Also,

$$\begin{aligned} (J \cap K)(e_1)1 &= \{a, b\},\ (J \cap K)(e_1)2 = \{b\},\ (J \cap K)(e_1)3 = \{c\}, \\ (J \cap K)(e_1)4 &= \{d\} \text{ and } (J \cap K)(e_1)5 = \{e\}. \end{aligned}$$

Define (L_1, A), *a soft set over* U *by* $L_1(e_1) = \{a, b\}$ *and* $L_1(e_2) = \{c, d\}$.

Then ${}^{L_1}\overline{J}(e_1) = \{1,2,5\}$, ${}^{L_1}\overline{K}(e_1) = \{1,2,5\}$ *and* ${}^{L_1}\left(\overline{J \cap K}\right)(e_1) = \{1,2\}$. *This shows that*

$$ {}^{L_1}\overline{J}(e_1) \cap {}^{L_1}\overline{K}(e_1) = \{1,2,5\} \neq \{1,2\} = {}^{L_1}\left(\overline{J \cap K}\right)(e_1). $$

Now,

Define (L_2, A), *a soft set over* U *by* $L_2(e_1) = \{e\}$ *and* $L_2(e_2) = \{a, b\}$.

Then ${}^{L_2}\underline{J}(e_1) = \phi$, ${}^{L_2}\underline{K}(e_1) = \phi$ *and* ${}^{L_2}\left(\underline{J \cap K}\right)(e_1) = \{5\}$. *This shows that*

$$ {}^{L_2}\underline{J}(e_1) \cup {}^{L_2}\underline{K}(e_1) = \phi \neq \{5\} = {}^{L_2}\left(\underline{J \cap K}\right)(e_1). $$

Proposition 6.1.13 *Let* (J, A) *be an SBRE from* U *to* W. *Let* $G_i : A \to P(W)$ *for* $i \in I$ *be an arbitrary family of soft subsets of* W. *Then the following properties hold with respect to the aftersets:*

(1) $\underline{J}^{\cap_{i\in I} G_i}(e) = \cap_{i\in I}\underline{J}^{G_i}(e)$

(2) $\overline{J}^{\cap_{i\in I} G_i}(e) \subseteq \cap_{i\in I}\overline{J}^{G_i}(e)$.

Proof. (1) Let $u \in \underline{J}^{\cap_{i\in I} G_i}(e) \Leftrightarrow \phi \neq uJ(e) \subseteq \cap_{i\in I} G_i(e) \Leftrightarrow \phi \neq uJ(e) \subseteq G_i(e)$ for all $i \in I \Leftrightarrow u \in \underline{J}^{G_i}(e)$ for all $i \in I \Leftrightarrow u \in \cap_{i\in I}\underline{J}^{G_i}(e)$. Hence, $\underline{J}^{\cap_{i\in I} G_i}(e) = \cap_{i\in I}\underline{J}^{G_i}(e)$.

(2) Let $u \in \overline{J}^{\cap_{i\in I} G_i}(e) \Rightarrow uJ(e) \cap (\cap_{i\in I} G_i(e)) \neq \phi \Rightarrow uJ(e) \cap G_i(e) \neq \phi$ for all $i \in I \Rightarrow u \in \overline{J}^{G_i}(e)$ for all $i \in I \Rightarrow u \in \cap_{i\in I}\overline{J}^{G_i}(e)$. Hence, $\overline{J}^{\cap_{i\in I} G_i}(e) \subseteq \cap_{i\in I}\overline{J}^{G_i}(e)$.
■

Proposition 6.1.14 *Let* (J, A) *be an SBRE from* U *to* W. *Let* $H_i : A \to P(U)$ *be an arbitrary family of soft subsets of* U. *Then the properties below are valid with respect to the aftersets:*

(1) ${}^{\cap_{i\in I} H_i}\underline{J}(e) = \cap_{i\in I}\, {}^{H_i}\underline{J}(e)$

(2) ${}^{\cap_{i\in I} H_i}\overline{J}(e) \subseteq \cap_{i\in I}\, {}^{H_i}\overline{J}(e)$.

If (J, A) is a soft reflexive relation, then each $uJ(e)$ (resp. $J(e)u$) is non-empty and $u \in uJ(e)$ (resp. $u \in J(e)u$). It is not necessary that $uJ(e) = J(e)u$. The approximation operators have additional properties with respect to soft reflexive binary relation as follows:

Theorem 6.1.15 *Let (J, A) be a soft reflexive relation on U. For a soft subset $G : A \to P(U)$, the following properties hold respecting to the aftersets:*

(1) $\underline{J}^G(e) \subseteq G(e)$

(2) $G(e) \subseteq \overline{J}^G(e)$

(3) $\underline{J}^\phi(e) = \phi(e) = \overline{J}^\phi(e)$

(4) $\overline{J}^W(e) = U$ *for all* $e \in A$.

Proof. (1) Let $u \in \underline{J}^G(e)$. Then $\phi \neq uJ(e) \subseteq G(e)$. But $u \in uJ(e)$, therefore $u \in G(e)$. Therefore, $\underline{J}^G(e) \subseteq G(e)$.

(2) Let $u \in G(e)$. Then $u \in uJ(e) \cap G(e)$, so $uJ(e) \cap G(e) \neq \phi$. This implies that $u \in \overline{J}^G(e)$. Therefore, $G(e) \subseteq \overline{J}^G(e)$.

(3) It is direct.

(4) By definition, $\overline{J}^W(e) = \{u \in U : uJ(e) \cap W(e) \neq \phi\}$. As $uJ(e) \neq \phi$ for every $u \in U$, therefore, $\overline{J}^W(e) = U$. ■

Theorem 6.1.16 *Let (J, A) be a soft reflexive relation on U. For a soft subset $H : A \to P(U)$, the following properties hold respecting to the foresets:*

(1) ${}^H\underline{J}(e) \subseteq H(e)$

(2) $H(e) \subseteq {}^H\overline{J}(e)$

(3) ${}^\phi\underline{J}(e) = \phi(e) = {}^\phi\overline{J}(e)$

(4) ${}^U\overline{J}(e) = W$ *for all* $e \in A$.

The approximation operators have additional properties respecting to soft symmetric binary relation as follows:

Lemma 6.1.17 *If (J, A) is a soft symmetric relation on U, then $v \in uJ(e)$ implies $u \in vJ(e)$.*

Proof. Straightforward. ■

Theorem 6.1.18 *Let $J : A \to P(U \times U)$ be a soft symmetric relation on U. For a soft subset $G : A \to P(U)$, the following properties hold:*

(1) $G(e) \supseteq \overline{J}^{\left(\underline{J}^{G}(e)\right)}(e)$

(2) $G(e) \subseteq \underline{J}^{\left(\overline{J}^{G}(e)\right)}(e)$

Proof. (1) *Let* $u \in \overline{J}^{\left(\underline{J}^{G}(e)\right)}(e)$ *for* $uJ(e) \neq \phi$ *for all* $e \in A$. *It proceeds* $uJ(e) \cap \underline{J}^{G}(e) \neq \phi$ *so there exists atleast one* $u_1 \in uJ(e) \cap \underline{J}^{G}(e)$. *This implies* $u_1 \in uJ(e)$ *and* $u_1 \in \underline{J}^{G}(e)$. *Now,* $u_1 \in \underline{J}^{G}(e)$ *implies* $u_1 J(e) \subseteq G(e)$. *Also* $u_1 \in uJ(e)$ *and the relation is soft symmetric so* $u \in u_1 J(e)$. *Thus,* $u \in u_1 J(e) \subseteq G(e)$. *Therefore,* $u \in G(e)$. *Therefore,* $\overline{J}^{\left(\underline{J}^{G}(e)\right)}(e) \subseteq G(e)$. *Hence,* $\overline{J}^{\left(J^{G}(e)\right)} \subseteq G$.

(2) *Let* $u \in G(e)$. *If* $u_1 \in uJ(e)$, *then* $u \in u_1 J(e)$, *because the relation is soft symmetric. It is clear that* $u \in u_1 J(e) \cap G(e)$, *so* $u_1 J(e) \cap G \neq \phi$. *It means that* $u_1 \in \overline{J}^{G}(e) \Rightarrow uJ(e) \subseteq \overline{J}^{G}(e)$ *implies* $u \in \underline{J}^{\left(\overline{J}^{G}(e)\right)}(e)$. *Therefore,* $\underline{J}^{\left(\overline{J}^{G}(e)\right)}(e) \supseteq G(e)$. *Therefore,* $G \subseteq \underline{J}^{\left(\overline{J}^{G}(e)\right)}$. ■

Theorem 6.1.19 *Let* $J : A \to P(U \times U)$ *be a soft symmetric relation on* U. *For a soft subset* $H : A \to P(U)$, *the following properties hold respecting to foresets:*

(1) ${}^{\left({}^{H}\underline{J}(e)\right)}\overline{J}(e) \subseteq H(e)$

(2) $H(e) \subseteq^{\left({}^{H}\overline{J}(e)\right)} \underline{J}(e)$

The approximation operators have additional properties with respect to soft transitive binary relation as follows:

Theorem 6.1.20 *Let* $J : A \to P(U \times U)$ *be a soft transitive relation on* U. *For a soft subset* $G : A \to P(U)$, *the following property hold respecting to aftersets:* $\overline{J}^{\left(\overline{J}^{G}(e)\right)}(e) \subseteq \overline{J}^{G}(e)$.

Proof. Let $u \in \overline{J}^{\left(\overline{J}^{G}(e)\right)}(e)$. This implies $uJ(e) \cap \overline{J}^{G}(e) \neq \phi$ so there exists atleast one $u_1 \in uJ(e) \cap \overline{J}^{G}(e)$ such that $u_1 \in uJ(e)$ and $u_1 \in \overline{J}^{G}(e)$. Now $u_1 \in \overline{J}^{G}(e)$ implies that $u_1 J(e) \cap G(e) \neq \phi$. So there exists atleast one $x \in u_1 J(e) \cap G(e)$ such that $x \in u_1 J(e)$ and $x \in G(e)$. But $u_1 \in uJ(e)$ implies $(u, u_1) \in J(e)$ and $x \in u_1 J(e)$ implies $(u_1, x) \in J(e)$. Since the relation is soft transitive so $(u, x) \in J(e)$. It follows that $x \in uJ(e)$. Therefore, $x \in uJ(e) \cap G(e)$. This implies $uJ(e) \cap G(e) \neq \phi$. Therefore $u \in \overline{J}^{G}(e)$. Thus, $\overline{J}^{\left(\overline{J}^{G}(e)\right)}(e) \subseteq \overline{J}^{G}(e)$. Hence, $\overline{J}^{\left(\overline{J}^{G}(e)\right)} \subseteq \overline{J}^{G}(e)$. ■

Theorem 6.1.21 *Let* $J : A \to P(U \times U)$ *be a soft transitive relation on* U. *For a soft subset* $H : A \to P(U)$, *the following property holds respecting to foresets:*

$^{\left(^{H}\overline{J}(e)\right)}\overline{J}\,(e) \subseteq {}^{H}\overline{J}\,(e)\,.$

Proof. Straightforward. ■

Theorem 6.1.22 *If an SBRE, $J : A \to P(U \times U)$ on U is soft reflexive and soft transitive, then for any soft subset $G : A \to P(U)$, the following property holds respecting to afterset:*

$\overline{J}^{\left(\overline{J}^{G}(e)\right)} = \overline{J}^{G}\,(e)\,.$

Proof. Since it is soft transitive so by previous result $\overline{J}^{\left(\overline{J}G(e)\right)}(e) \subseteq \overline{J}^{G}\,(e)$. It is also soft reflexive therefore $G\,(e) \subseteq \overline{J}^{G}\,(e)$. By using Theorem 6.1.4(2), $\overline{J}^{G}\,(e) \subseteq \overline{J}^{\left(\overline{J}^{G}(e)\right)}(e)$. Hence, $\overline{J}^{\left(\overline{J}^{G}(e)\right)}(e) = \overline{J}^{G}\,(e)$. ■

Theorem 6.1.23 *If an SBRE, $J : A \to P(U \times U)$ on U is soft reflexive and soft transitive, then for any soft subset $H : A \to P(U)$, the following property holds respecting to foreset:*

$^{\left(^{H}\overline{J}(e)\right)}\overline{J} = {}^{H}\overline{J}\,(e)\,.$

Proof. Straightforward. ■

If (J, A) is a $SE-$relation on U, then each $J\,(e)$ is an $E-$relation on U. Thus, $J\,(e)$ partitions the set U into $E-$classes $uJ\,(e)$. In this case, $uJ\,(e) = J\,(e)\,u$ and $\{uJ\,(e) : u \in U\}$ is a partition of U. Also, in this case, $= {}^{G}\underline{J}\,(e) = \underline{J}^{G}\,(e)$ and ${}^{G}\overline{J}\,(e) = \overline{J}^{G}\,(e)\,.$

To elaborate this concept, consider the next example:

Example 6.1.24 *Let $U = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6\}$ be a set where $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and $A = \{e_1, e_2, e_3, e_4, e_5\}$ be a set of attributes.*

Define an $SE-$relation $J : A \to P(U \times U)$ for each parameter $e \in A$.

The following $SE-$classes are obtained for each of the $SE-$relation.

For $J\,(e_1)$, the $SE-$classes $uJ\,(e_1)$ are $\{\pi_1, \pi_3\}, \{\pi_2, \pi_4, \pi_5, \pi_6\}$.

For $J\,(e_2)$, the $SE-$classes $uJ\,(e_2)$ are $\{\pi_1, \pi_3, \pi_6\}, \{\pi_2, \pi_4, \pi_5\}$.

For $J\,(e_3)$, the $SE-$classes $uJ\,(e_3)$ are $\{\pi_1, \pi_2, \pi_4, \pi_5\}, \{\pi_3\}, \{\pi_6\}$.

For $J(e_4)$, the $SE-$classes $uJ(e_4)$ are $\{\pi_1,\pi_2,\pi_4,\pi_5,\pi_6\},\{\pi_3\}$.

For $J(e_5)$, the $SE-$classes $uJ(e_5)$ are $\{\pi_1,\pi_2,\pi_3,\pi_4,\pi_5,\pi_6\}$.

A soft indiscernibility relation is obtained by the intersection of all the $E-$relations induced by parameters represented as $IN_d(J,A)=\cap_{e_i\in A}J(e_i)=\eta$

In above example, the partition of U obtained by soft indiscernibility relation $IN_d(J,A)$ is $\{\pi_1\},\{\pi_2,\pi_4,\pi_5\},\{\pi_3\}$ and $\{\pi_6\}$. It is evident that for each $J(e_i)$ ($SE-$relation) where $i=1,2,3,4,5$, $(U,J(e_i))$ gives an approximation space. Also, (U,η) is an approximation space.

Recall that if (J,A) is a $SE-$relation on U and (G_1,A) is a soft set over U, then

$$\begin{aligned}\underline{J}^G(e) &= \{u\in U: uJ(e)\subseteq G(e)\} \text{ and}\\ \overline{J}^G(e) &= \{u\in U: uJ(e)\cap G(e)\neq\phi\} \text{ for all } e\in A.\end{aligned}$$

The soft set $\left(BJ^{G_1},A\right)$ defined as $BJ^{G_1}(e)=\overline{J}^{G_1}(e)-\underline{J}^{G_1}(e)$ for all $e\in A$ is named soft boundary region of G_1, respecting to $SE-$relation (J,A). A soft subset G_1 of U is called totally rough respecting to $SE-$relation (J,A) if $BJ^{G_1}(e)\neq\phi$ for all $e\in A$. A soft subset G_1 of U is said to be partly definable with respect to $SE-$relation (J,A) if $BJ^{G_1}(e)=\phi$ for some $e\in A$. A soft subset G_1 of U is called totally definable with respect to $SE-$relation (J,A) if $BJ^{G_1}(e)=\phi$ for all $e\in A$.

Proposition 6.1.25 *For the $SE-$relation (J,A) on U and for soft sets G_1 and G_2 over U, the assertions given below hold:*

(1) $\left(\underline{J}^{G_1},A\right)\subseteq\left(\overline{J}^{G_1},A\right)$

(2) $\left(\underline{J}^{\phi},A\right)=\left(\overline{J}^{\phi},A\right)=\phi$, $\left(\underline{J}^{U},A\right)=\left(\overline{J}^{U},A\right)=U$

(3) $G_1(e)\subseteq G_2(e)\Rightarrow\left(\underline{J}^{G_1},A\right)\subseteq\left(\underline{J}^{G_2},A\right)$

(4) $G_1(e)\subseteq G_2(e)\Rightarrow\left(\overline{J}^{G_1},A\right)\subseteq\left(\overline{J}^{G_2},A\right)$

(5) $\left(\underline{J}^{G_1},A\right)\cap\left(\underline{J}^{G_2},A\right)=\left(\underline{J}^{G_1\cap G_2},A\right)$

(6) $\left(\overline{J}^{G_1},A\right)\cup\left(\overline{J}^{G_2},A\right)=\left(\overline{J}^{G_1\cup G_2},A\right)$

(7) $\left(\underline{J}^{G_1},A\right)\cup\left(\underline{J}^{G_2},A\right)\subseteq\left(\underline{J}^{G_1\cup G_2},A\right)$

(8) $\left(\overline{J}^{G_1},A\right)\cap\left(\overline{J}^{G_2},A\right)\supseteq\left(\overline{J}^{G_1\cap G_2},A\right)$

(9) $\left(\overline{J}^{G_1^c}, A\right)^c = \left(\underline{J}^{G_1}, A\right).$

(10) $\left(\underline{J}^{G_1^c}, A\right)^c = \left(\overline{J}^{G_1}, A\right).$

Proof. (1) Let $u \in \underline{J}^{G_1}(e) \Rightarrow uJ(e) \subseteq G_1(e) \Rightarrow u \in \overline{J}^{G_1}(e)$. Hence, $\left(\underline{J}^{G_1}, A\right) \subseteq \left(\overline{J}^{G_1}, A\right).$

(2) Straightforward.

(3) Let $u \in \underline{J}^{G_1}(e)$. Then $uJ(e) \subseteq G_1(e)$. As $G_1(e) \subseteq G_2(e)$, we have $uJ(e) \subseteq G_2(e)$. Thus $u \in \underline{J}^{G_2}(e)$. Hence, $\left(\underline{J}^{G_1}, A\right) \subseteq \left(\underline{J}^{G_2}, A\right).$

(4) Let $u \in \overline{J}^{G_1}(e)$. Then $uJ(e) \cap G_1(e) \neq \phi$. As $G_1(e) \subseteq G_2(e)$, we have $uJ(e) \cap G_2(e) \neq \phi$. Thus $u \in \overline{J}^{G_2}(e)$. Hence, $\left(\overline{J}^{G_1}, A\right) \subseteq \left(\overline{J}^{G_2}, A\right).$

(5)Using part (3) and the fact that $G_1(e) \cap G_2(e) \subseteq G_1(e), G_2(e)$, we have $\underline{J}^{G_1 \cap G_2}(e) \subseteq \underline{J}^{G_1}(e)$, $\underline{J}^{G_2}(e)$ so $\underline{J}^{G_1 \cap G_2}(e) \subseteq \underline{J}^{G_1}(e) \cap \underline{J}^{G_2}(e)$.

Hence, $\left(\underline{J}^{G_1 \cap G_2}, A\right) \subseteq \left(\underline{J}^{G_1}, A\right) \cap \left(\underline{J}^{G_2}, A\right).$

Conversely, let $u \in \underline{J}^{G_1}(e) \cap \underline{J}^{G_2}(e) \Rightarrow u \in \underline{J}^{G_1}(e)$ and $u \in \underline{J}^{G_2}(e)$

$\Rightarrow uJ(e) \subseteq G_1(e)$ and $uJ(e) \subseteq G_2(e) \Rightarrow uJ(e) \subseteq G_1(e) \cap G_2(e)$

$\Rightarrow u \in \underline{J}^{G_1}(e) \Rightarrow \left(\underline{J}^{G_1}, A\right) \cap \left(\underline{J}^{G_2}, A\right) \subseteq \left(\underline{J}^{G_1 \cap G_2}, A\right).$

Hence, $\left(\underline{J}^{G_1 \cap G_2}, A\right) = \left(\underline{J}^{G_1}, A\right) \cap \left(\underline{J}^{G_2}, A\right).$

(6) Since $G_1(e), G_2(e) \subseteq G_1(e) \cup G_2(e)$ so by using part (4), we get $\overline{J}^{G_1}(e)$, $\overline{J}^{G_2}(e) \subseteq \overline{J}^{G_1 \cup G_2}(e)$ which implies $\overline{J}^{G_1}(e) \cup \overline{J}^{G_2}(e) \subseteq \overline{J}^{G_1 \cup G_2}(e)$. Hence, $\left(\overline{J}^{G_1}, A\right) \cup \left(\overline{J}^{G_2}, A\right) \subseteq \left(\overline{J}^{G_1 \cup G_2}, A\right).$

For the reverse inclusion, let $u \in \overline{J}^{G_1 \cup G_2}(e)$.

$\Rightarrow uJ(e) \cap (G_1 \cup G_2)(e) \neq \phi \Rightarrow uJ(e) \cap G_1(e) \neq \phi$ or $uJ(e) \cap G_2(e) \neq \phi$

$\Rightarrow u \in \overline{J}^{G_1}(e)$ or $u \in \overline{J}^{G_2}(e) \Rightarrow u \in \left(\overline{J}^{G_1} \cup \overline{J}^{G_2}\right)(e)$

$\Rightarrow \overline{J}^{G_1 \cup G_2}(e) \subseteq \overline{J}^{G_1}(e) \cup \overline{J}^{G_2}(e) \Rightarrow \left(\overline{J}^{G_1 \cup G_2}, A\right) \subseteq \left(\overline{J}^{G_1}, A\right) \cup \left(\overline{J}^{G_2}, A\right).$

Hence, $\left(\overline{J}^{G_1}, A\right) \cup \left(\overline{J}^{G_2}, A\right) = \left(\overline{J}^{G_1 \cup G_2}, A\right).$

(7) Since $G_1(e), G_2(e) \subseteq G_1(e) \cup G_2(e)$, so by using part (3), we get $\underline{J}^{G_1}(e)$, $\underline{J}^{G_2}(e) \subseteq \underline{J}^{G_1 \cup G_2}(e)$ and so $\underline{J}^{G_1}(e) \cup \underline{J}^{G_2}(e) \subseteq \underline{J}^{G_1 \cup G_2}(e)$. Hence, $\left(\underline{J}^{G_1}, A\right) \cup \left(\underline{J}^{G_2}, A\right) \subseteq \left(\underline{J}^{G_1 \cup G_2}, A\right).$

(8) Using part (4) and the fact that $G_1(e), G_2(e) \supseteq G_1(e) \cap G_2(e)$, we have $\overline{J}^{G_1}(e)$, $\overline{J}^{G_2}(e) \supseteq \overline{J}^{G_1 \cap G_2}(e) \Rightarrow \overline{J}^{G_1}(e) \cap \overline{J}^{G_2}(e) \supseteq \overline{J}^{G_1 \cap G_2}(e)$.

Hence, $\left(\overline{J}^{G_1}, A\right) \cap \left(\overline{J}^{G_2}, A\right) \supseteq \left(\overline{J}^{G_1 \cap G_2}, A\right)$.

(9) Let $u \in \underline{J}^{G_1}(e) \Leftrightarrow uJ(e) \subseteq G_1(e) \Leftrightarrow uJ(e) \cap G_1^c(e)$

$= \phi \Leftrightarrow u \notin \overline{J}^{G_1^c}(e) \Leftrightarrow u \in \left(\overline{J}^{G_1^c}(e)\right)^c$. Hence, $\left(\overline{J}^{G_1^c}, A\right)^c = \left(\underline{J}^{G_1}, A\right)$.

(10) By part (9), $\left(\overline{J}^{G_1^c}, A\right)^c = \left(\underline{J}^{G_1}, A\right)$, therefore, $\left(\overline{J}^{(G_1^c)^c}, A\right)^c = \left(\underline{J}^{G_1^c}, A\right)$

$\Rightarrow \left(\overline{J}^{G_1}, A\right)^c = \left(\underline{J}^{G_1^c}, A\right)$.Hence, $\left(\underline{J}^{G_1^c}, A\right)^c = \left(\overline{J}^{G_1}, A\right)$. ■

It is demonstrated that the equality is not valid in (7) and (8) in general.

Example 6.1.26 *Let $U = \{\pi_1, \pi_2, \pi_3, \pi_4\}$ be a set where $E = \{e_1, e_2, e_3, e_4, e_5\}$ and $A = \{e_1, e_2, e_3\}$ be a set of attributes. Define a SE−relation $J : A \to P(U \times U)$ for each $e \in A$, such that the E−classes for $J(e_1)$ are $\{\pi_1, \pi_4\}$ and $\{\pi_2, \pi_3\}$, for $J(e_2)$ are $\{\pi_1, \pi_2, \pi_4\}$ and $\{\pi_3\}$ and for $J(e_3)$ are $\{\pi_1\}, \{\pi_2, \pi_3\}$ and $\{\pi_4\}$.*

$$\begin{aligned} \text{Define } G_1 &: A \to P(U) \text{ such that} \\ G_1(e_1) &= \{\pi_1, \pi_3\},\ G_1(e_2) = \{\pi_1, \pi_2\},\ G_1(e_3) = \{\pi_1, \pi_4\}. \end{aligned}$$

And,

$$\begin{aligned} \text{Define } G_2 &: A \to P(U) \text{ such that} \\ G_2(e_1) &= \{\pi_1, \pi_2\},\ G_2(e_2) = \{\pi_2, \pi_3\},\ G_2(e_3) = \{\pi_2, \pi_3, \pi_4\}, \end{aligned}$$

so $\left(\underline{J}^{G_1}, A\right)$ can be represented as

$$\underline{J}^{G_1}(e_1) = \phi,\ \underline{J}^{G_1}(e_2) = \{\pi_3\},\ \underline{J}^{G_1}(e_3) = \{\pi_2, \pi_3, \pi_4\}.$$

And $\left(\underline{\eta}^{G_2}, A\right)$ can be represented as

$$\underline{J}^{G_2}(e_1) = U,\ \underline{J}^{G_2}(e_1) = \{\pi_1, \pi_2, \pi_4\},\ \underline{J}^{G_2}(e_1) = \{\pi_1, \pi_2, \pi_3\}.$$

Now,

$$\begin{aligned} \text{Define } G_1 \cup G_2 &: A \to P(U) \text{ such that} \\ G_1 \cup G_2(e_1) &= \{\pi_1, \pi_2, \pi_3\},\ G_1 \cup G_2(e_2) = \{\pi_1, \pi_2, \pi_3\}, \\ G_1 \cup G_2(e_3) &= \{\pi_1, \pi_2, \pi_3, \pi_4\}, \end{aligned}$$

and $\left(\underline{J}^{G_1 \cap G_2}, A\right)$ *can be represented as*

$$\underline{J}^{G_1 \cup G_2}(e_1) = \{\pi_2, \pi_3\}, \ \underline{J}^{G_1 \cup G_2}(e_2) = \{\pi_3\}, \ \underline{J}^{G_1 \cup G_2}(e_3) = \{\pi_1, \pi_2, \pi_3, \pi_4\}.$$

Evidently, $\left(\underline{J}^{G_1 \cup G_2}, A\right) \neq \left(\underline{J}^{G_1}, A\right) \cup \left(\underline{J}^{G_2}, A\right).$

Now, $\left(\overline{J}^{G_1}, A\right)$ *can be represented as*

$$\overline{J}^{G_1}(e_1) = U, \ \overline{J}^{G_1}(e_2) = \{\pi_1, \pi_2, \pi_4\}, \ \overline{J}^{G_1}(e_3) = \{\pi_1, \pi_4\}.$$

And $\left(\overline{J}^{G_2}, A\right)$ *can be represented as*

$$\overline{J}^{G_2}(e_1) = U, \ \overline{J}^{G_2}(e_1) = U, \ \overline{J}^{G_2}(e_1) = \{\pi_2, \pi_3, \pi_4\}.$$

Now,

$$\begin{aligned} \text{Define } G_1 \cap G_2 &: A \to P(U) \text{ such that} \\ G_1 \cap G_2(e_1) &= \{\pi_1\}, \ G_1 \cap G_2(e_2) = \{\pi_2\}, \ G_1 \cap G_2(e_3) = \{\pi_4\}, \end{aligned}$$

and $\left(\overline{J}^{G_1 \cap G_2}, A\right)$ *can be represented as*

$$\overline{J}^{G_1 \cap G_2}(e_1) = \{\pi_1, \pi_4\}, \ \overline{J}^{G_1 \cap G_2}(e_2) = \{\pi_2, \pi_3\}, \ \overline{J}^{G_1 \cap G_2}(e_3) = \{\pi_4\}.$$

Evidently, $\left(\overline{J}^{G_1 \cap G_2}, A\right) \neq \left(\overline{J}^{G_1}, A\right) \cap \left(\overline{J}^{G_2}, A\right).$

6.2 Soft topologies induced by soft reflexive relations

In this section, we investigate two kinds of soft topologies induced by soft reflexive relation.

Definition 6.2.1 *[89] A family* $T \subseteq \mathbf{S}(U)$ *of soft sets in* U *is called a soft topology over* U *if it satisfies the following three axioms:*

(1) ϕ, $\mathbf{U} \in T$, *where* $\phi : A \to P(U)$ *and* $\mathbf{U} : A \to P(U)$ *are defined as* $\phi(e) = \phi$ *and* $\mathbf{U}(e) = U$ *for all* $e \in A$.

(2) $\forall\ (G_1, A), (G_2, A) \in T \Rightarrow (G_1 \cap G_2, A) \in T$.

(3) $\forall\ (G_j, A)_{j \in J} \in T \Longrightarrow \cup_{j \in J}(G_j, A) \in T$.

The pair (U, T, A) *is called a soft topological space over* U. *The elements of* T *are called soft open sets.*

Theorem 6.2.2 *If (J, A) is a soft reflexive relation on U, then*

$$T_{J(e)} = \{G : A \to P(U) : \underline{J}^G(e) = G(e)\, for all e \in A\}$$

is a soft topology on U with respect to the aftersets.

Proof. (1) Take $e \in A$, by Theorem 6.1.25, $\underline{J}^{\phi}(e) = \phi(e)$ and $\underline{J}^U(e) = U$ for all $e \in A$. It proceeds ϕ, $U \in T_{J(e)}$.

(2) Let $(G_1, A), (G_2, A) \in T_{J(e)}$ be two soft sets. This implies $\underline{J}^{G_1}(e) = G_1(e)$ and $\underline{J}^{G_2}(e) = G_2(e)$. Now, by using Theorem 6.1.25, $\underline{J}^{G_1 \cap G_2}(e) = \underline{J}^{G_1}(e) \cap \underline{J}^{G_2}(e)$

$= G_1(e) \cap G_2(e) = (G_1 \cap G_2)(e)$. This implies $(G_1 \cap G_2, A) \in T_{J(e)}$.

(3) Let $(G_j, A) \in T_{J(e)}$ for $j \in \mathbf{J}$. This implies $\underline{J}^{G_j}(e) = G_j(e)$ for $j \in \mathbf{J}$. Now, by using Theorem 6.1.25, $\underline{J}^{\cup_{j\in\mathbf{J}} G_j}(e) \supseteq \cup_{j\in\mathbf{J}} \underline{J}^{G_j}(e)$. Therefore, $\underline{J}^{\cup_{j\in\mathbf{J}} G_j}(e) \supseteq \cup_{j\in\mathbf{J}} G_j(e)$. Since, $G_j(e) \subseteq \cup_{j\in\mathbf{J}} G_j(e)$ for $j \in \mathbf{J}$. By using Theorem 6.1.25, $\underline{J}^{G_j}(e) \subseteq \underline{J}^{\cup_{j\in\mathbf{J}} G_j}(e)$. This implies $\cup_{j\in\mathbf{J}} \underline{J}^{G_j}(e) \subseteq \underline{J}^{\cup_{j\in\mathbf{J}} G_j}(e)$. This implies $\cup_{j\in\mathbf{J}} G_j(e) \subseteq \underline{J}^{\cup_{j\in\mathbf{J}} G_j}(e)$. Therefore, $\underline{J}^{\cup_{j\in\mathbf{J}} G_j}(e) = \cup_{j\in\mathbf{J}} G_j(e)$. Hence, $(\cup G_j, A) \in T_{J(e)}$. ■

Theorem 6.2.3 *If (J, A) is a soft reflexive relation on U, then*

$$T'_{J(e)} = \{L : A \to P(U) :^L \underline{J}(e) = L(e)\, for all e \in A\}$$

is a soft topology on U respecting to the foresets.

Proof. In a like way, the proof is obtained from above Theorem. ■

Definition 6.2.4 *Let (J, A) be a soft reflexive relation over U. Define a binary relation R_J on U by $xR_J y \Leftrightarrow xJ(e)y$ for some $e \in A$ where $x, y \in U$. Then R_J is called the binary relation induced by soft binary relation (J, A).*

Theorem 6.2.5 *Let (J, A) be an SBRE on U and R_J be the induced binary relation by (J, A). For two soft sets $G_1 : A \to P(U)$ and $G_2 : A \to P(U)$, the following properties, respecting to the aftersets hold:*

(1) $G_1(e) \subseteq G_2(e) \Rightarrow \underline{R_J} G_1(e) \subseteq \underline{R_J} G_2(e)$

(2) $G_1(e) \subseteq G_2(e) \Rightarrow \overline{R_J} G_1(e) \subseteq \overline{R_J} G_2(e)$

(3) $\underline{R_J} G_1(e) \cap \underline{R_J} G_2(e) = \underline{R_J}(G_1 \cap G_2)(e)$

(4) $\overline{R_J}G_1(e) \cap \overline{R_J}G_2(e) \supseteq \overline{R_J}(G_1 \cap G_2)(e)$

(5) $\underline{R_J}G_1(e) \cup \underline{R_J}G_2(e) \subseteq \underline{R_J}(G_1 \cup G_2)(e)$

(6) $\overline{R_J}G_1(e) \cup \overline{R_J}G_2(e) = \overline{R_J}(G_1 \cup G_2)(e)$.

Proof. Straightforward. ■

Theorem 6.2.6 *Let (J, A) be an SBRE on U and R_J be the induced binary relation by (J, A). For two soft sets $L_1 : A \to P(U)$ and $L_2 : A \to P(U)$, the following properties respecting to the foresets hold:*

(1) $L_1(e) \subseteq L_2(e) \Rightarrow L_1\underline{R_J}(e) \subseteq L_2\underline{R_J}(e)$

(2) $L_1(e) \subseteq L_2(e) \Rightarrow L_1\overline{R_J}(e) \subseteq L_2\overline{R_J}(e)$

(3) $L_1\underline{R_J}(e) \cap L_2\underline{R_J}(e) = (L_1 \cap L_2)\underline{R_J}(e)$

(4) $L_1\overline{R_J}(e) \cap L_2\overline{R_J}(e) \supseteq (L_1 \cap L_2)\overline{R_J}(e)$

(5) $L_1\underline{R_J}(e) \cup L_2\underline{R_J}(e) \subseteq (L_1 \cup L_2)\underline{R_J}(e)$

(6) $L_1\overline{R_J}(e) \cup L_2\overline{R_J}(e) = (L_1 \cup L_2)\overline{R_J}(e)$.

Proof. Straightforward. ■

Theorem 6.2.7 *If (J, A) is a soft reflexive relation on U, then*

$$T_{R_J} = \{G : A \to P(U) : \underline{R_J}G(e) = G(e)\, forall e \in A\}$$

is a soft topology on U respecting to the aftersets.

Proof. (1) By Theorem 6.1.25, $\underline{R_J}\phi(e) = \phi(e)$ and $\underline{R_J}U(e) = U$ for all $e \in A$. It follows ϕ, $U \in T_{R_J(e)}$.

(2) Let $(G_1, A), (G_2, A) \in T_{R_J(e)}$ be two soft sets. This implies $\underline{R_J}G_1(e) = G_1(e)$ and $\underline{R_J}G_2(e) = G_2(e)$ for all $e \in A$. Now, by using Theorem 6.1.25,

$$\underline{R_J}(G_1 \cap G_2)(e) = \underline{R_J}G_1(e) \cap \underline{R_J}G_2(e) = G_1(e) \cap G_2(e) = (G_1 \cap G_2)(e).$$

This implies $(G_1 \cap G_2, A) \in T_{R_J(e)}$ for all $e \in A$.

(3) Let $G_j \in T_{R_J(e)}$ for $j \in \mathbf{J}$. This implies $\underline{R_J}G_j(e) = G_j(e)$ for all $e \in A$. Now, by using Theorem 6.2.5, $\underline{R_J}(\cup_{j \in J}G_j)(e) = \cup_{j \in J}\underline{R_J}G_j(e) = \cup_{j \in J}G_j(e)$. ■

Theorem 6.2.8 *If (J,A) is a soft reflexive relation on U, then*

$$T'_{R_J} = \{L : A \to P(U) :^{L} \underline{R_J}(e) = L(e)\, forall e \in A\}$$

is a soft topology on U respecting to the foresets.

Proof. With the similar argument, the proof is obtained from Theorem 4.2.9. ■

6.3 Soft similarity relations associated with soft binary relations

In this section, binary relations between soft sets are defined based on their rough approximations and investigate their properties.

Definition 6.3.1 *Let (J,A) be a soft reflexive relation on U. Define the following binary relations on $\mathbf{S}(U)$ by*

$(G_1,A) \backsimeq_A (G_2,A)$ *if and only if* $\left({}^{G_1}\underline{J}, A\right) = \left(\underline{J}^{G_2}, A\right)$

$(G_1,A) \precapprox_A (G_2,A)$ *if and only if* $\left({}^{G_1}\overline{J}, A\right) = \left(\overline{J}^{G_2}, A\right)$

$(G_1,A) \approx_A (G_2,A)$ *if and only if* $\left({}^{G_1}\underline{J}, A\right) = \left(\underline{J}^{G_2}, A\right)$ *and* $\left({}^{G_1}\overline{J}, A\right) = \left(\overline{J}^{G_2}, A\right)$.

Definition 6.3.2 *Let (J,A) be a soft reflexive relation on U. Define the following soft binary relations on $\mathbf{S}(U)$ by*

$(L_1,A) \simeq_F (L_2,A)$ *if and only if* $\left({}^{L_1}\underline{J}, A\right) = \left({}^{L_2}\underline{J}, A\right)$

$(L_1,A) \precapprox_F (L_2,A)$ *if and only if* $\left({}^{L_1}\overline{J}, A\right) = \left({}^{L_2}\overline{J}, A\right)$

$(L_1,A) \approx_F (L_2,A)$ *if and only if* $\left({}^{L_1}\underline{J}, A\right) = \left({}^{L_2}\underline{J}, A\right)$ *and* $\left({}^{L_1}\overline{J}^{G_1}, A\right) = \left({}^{L_2}\overline{J}, A\right)$.

These soft binary relations may be called the lower soft similarity relation , upper soft similarity relation and soft similarity relation, respectively. Obviously, $\underline{J}^{G}(e)$ and $\overline{J}^{G}(e)$ are soft similar if and only if they are both lower and upper soft similar.

Proposition 6.3.3 *The relations $\backsimeq_A$, $\precapprox_A$ and $\approx_A$ are equivalence relations on U.*

Proof. Straightforward. ■

Proposition 6.3.4 *The relations $\simeq_F$, $\eqsim_F$ and $\approx_F$ are $E-$relations on U.*

Proof. The proof is simple. ■

Next some results related are given.

Theorem 6.3.5 *Let (J, A) be a soft reflexive relation on U. Define soft sets (G_i, A) over U for $i = 1, 2, 3, 4,$ then the properties below are valid respecting to aftersets:*

(1) $(G_1, A) \eqsim_A (G_2, A)$ *if and only if* $(G_1, A) \eqsim_A ((G_1, A) \cup (G_2, A)) \eqsim_A (G_2, A)$

(2) $(G_1, A) \eqsim_A (G_2, A)$ *and* $(G_3, A) \eqsim_A (G_4, A)$ *imply that* $((G_1, A) \cup (G_3, A)) \eqsim_A ((G_2, A) \cup G_4 (e))$

(3) $(G_1, A) \subseteq (G_2, A)$ *and* $(G_2, A) \eqsim_A (\phi, A)$ *imply that* $(G_1, A) \eqsim_A (\phi, A)$

(4) $((G_1, A) \cup (G_2, A)) \eqsim_A (\phi, A)$ *if and only if* $(G_1, A) \eqsim_A (\phi, A)$ *and* $(G_2, A) \eqsim_A (\phi, A)$.

Proof. (1) Let $(G_1, A) \eqsim_A (G_2, A)$. Then $\overline{J}^{G_1}(e) = \overline{J}^{G_2}(e)$. By Theorem 6.1.25(6), we get $\overline{J}^{G_1 \cup G_2}(e) = \overline{J}^{G_1}(e) \cup \overline{J}^{G_2}(e) = \overline{J}^{G_1}(e) = \overline{J}^{G_2}(e)$ so $(G_1, A) \eqsim_A ((G_1, A) \cup (G_2, A)) \eqsim_A (G_2, A)$. Converse holds by transitivity of the relation $\eqsim_A$.

(2) Given that $(G_1, A) \eqsim_A (G_2, A)$ and $(G_3, A) \eqsim_A (G_4, A)$. Then $\overline{J}^{G_1}(e) = \overline{J}^{G_2}(e)$ and $\overline{J}^{G_3}(e) = \overline{J}^{G_4}(e)$.

By Theorem 6.1.25(6), we get $\overline{J}^{G_1 \cup G_3}(e) = \overline{J}^{G_1}(e) \cup \overline{J}^{G_3}(e) = \overline{J}^{G_2}(e) \cup \overline{J}^{G_4}(e) = \overline{J}^{G_2 \cup G_4}(e)$. Thus, $((G_1, A) \cup (G_3, A)) \eqsim_A ((G_2, A) \cup (G_4, A))$.

(3) Given $(G_2, A) \eqsim_A (\phi, A)$. This implies $\overline{J}^{G_2}(e) = \overline{J}^{\phi}(e)$.

Also, $(G_1, A) \subseteq (G_2, A) \Rightarrow \overline{J}^{G_1}(e) \subseteq \overline{J}^{G_2}(e) = \overline{J}^{\phi}(e)$. It follows that $\overline{J}^{G_1}(e) \subseteq \overline{J}^{\phi}(e)$ but $\overline{J}^{\phi}(e) \subseteq \overline{J}^{G_1}(e)$. Therefore, $\overline{J}^{G_1}(e) = \overline{J}^{\phi}(e) \Rightarrow (G_1, A) \eqsim_A (\phi, A)$.

(4) Let $(G_1, A) \eqsim_A (\phi, A)$ and $(G_2, A) \eqsim_A (\phi, A)$. Then $\overline{J}^{G_1}(e) = \overline{J}^{\phi}(e) = (\phi, A)$ and $\overline{J}^{G_2}(e) = \overline{J}^{\phi}(e)$. By Theorem 6.1.25(6), we get $\overline{J}^{G_1 \cup G_2}(e) = \overline{J}^{G_1}(e) \cup \overline{J}^{G_2}(e) = \overline{J}^{\phi}(e) \cup \overline{J}^{\phi}(e) = \overline{J}^{\phi}(e)$.

Thus, $((G_1, A) \cup (G_2, A)) \eqsim_A (\phi, A)$. Converse follows from (3). ■

Theorem 6.3.6 *Let (J, A) be a soft reflexive relation on U. Define soft sets (L_i, A) over U for $i = 1, 2, 3, 4,$ then the properties given below are valid respecting to foresets:*

(1) $(L_1, A) \eqsim_F (L_2, A)$ *if and only if* $(L_1, A) \eqsim_F ((L_1, A) \cup (L_2, A)) \eqsim_F (L_2, A)$

(2) $(L_1, A) \eqsim_F (L_2, A)$ *and* $(L_3, A) \eqsim_F (L_4, A)$ *imply that* $((L_1, A) \cup (L_3, A)) \eqsim_F ((L_2, A) \cup (L_4, A))$

(3) $(L_1, A) \subseteq (L_2, A)$ *and* $(L_2, A) \eqsim_F (\phi, A)$ *imply that* $(L_1, A) \eqsim_F (\phi, A)$

(4) $((L_1, A) \cup (L_2, A)) \eqsim_F (\phi, A)$ *if and only if* $(L_1, A) \eqsim_F (\phi, A)$ *and* $(L_2, A) \eqsim_F (\phi, A)$.

Proof. The verification is like the evidence of Theorem 6.3.5. ■

Theorem 6.3.7 *Let (J, A) be a soft reflexive relation on U. Define soft sets (G_i, A) over U for $i = 1, 2, 3, 4$, then the properties below are valid respecting to aftersets:*

(1) $(G_1, A) \backsimeq_A (G_2, A)$ *if and only if* $(G_1, A) \backsimeq_A ((G_1, A) \cap (G_2, A)) \backsimeq_A (G_2, A)$

(2) $(G_1, A) \backsimeq_A (G_2, A)$ *and* $(G_3, A) \backsimeq_A (G_4, A)$ *imply that* $((G_1, A) \cap (G_3, A)) \backsimeq_A ((G_2, A) \cap (G_4, A))$

(3) $(G_1, A) \subseteq (G_2, A)$ *and* $(G_2, A) \backsimeq_A (\phi, A)$ *imply that* $(G_1, A) \backsimeq_A (\phi, A)$

(4) $((G_1, A) \cup (G_2, A)) \backsimeq_A (\phi, A)$ *if and only if* $(G_1, A) \backsimeq_A (\phi, A)$ *and* $(G_2, A) \backsimeq_A (\phi, A)$.

Proof. The verification is like the evidence of Theorem (7). ■

Theorem 6.3.8 *Let (J, A) be a soft reflexive relation on U. Define soft sets (L_i, A) over U for $i = 1, 2, 3, 4$, then the properties below are valid respecting to foresets:*

(1) $(L_1, A) \simeq_F (L_2, A)$ *if and only if* $(L_1, A) \simeq_F ((L_1, A) \cap (L_2, A)) \simeq_F (L_2, A)$

(2) $(L_1, A) \simeq_F (L_2, A)$ *and* $(L_3, A) \simeq_F (L_4, A)$ *imply that* $((L_1, A) \cap (L_3, A)) \simeq_F ((L_2, A) \cap (L_4, A))$

(3) $(L_1, A) \subseteq (L_2, A)$ *and* $(L_2, A) \simeq_F (\phi, A)$ *imply that* $(L_1, A) \simeq_F (\phi, A)$

(4) $((L_1, A) \cup (L_2, A)) \simeq_F (\phi, A)$ *if and only if* $(L_1, A) \simeq_F (\phi, A)$ *and* $(L_2, A) \simeq_F (\phi, A)$.

Proof. The proof is obtained in a similar way from Theorem 6.3.7. ■

Theorem 6.3.9 *Let (J, A) be a soft reflexive relation on U. Define soft sets (G_i, A) over U for $i = 1, 2, 3, 4$, then the properties below are valid respecting to aftersets:*

(1) $(G_1, A) \approx_A (G_2, A)$ if and only if $(G_1, A) \eqsim_A ((G_1, A) \cup (G_2, A)) \eqsim_A (G_2, A)$ and $(G_1, A) \backsimeq_A ((G_1, A) \cap (G_2, A)) \backsimeq_A (G_2, A)$.

(2) $(G_1, A) \subseteq (G_2, A)$ and $(G_2, A) \approx_A (\phi, A)$ imply that $(G_1, A) \approx_A (\phi, A)$

(3) $((G_1, A) \cup (G_2, A)) \approx_A (\phi, A)$ if and only if $(G_1, A) \approx_A (\phi, A)$ and $(G_2, A) \approx_A (\phi, A)$.

Proof. It is simple consequence of Theorems 6.3.5 and 6.3.7. ■

The next results are simple.

Theorem 6.3.10 *Let (J, A) be a soft reflexive relation on U. Define soft sets (L_i, A) over U for $i = 1, 2, 3, 4$, then the properties below are valid with respect to foresets:*

(1) $(L_1, A) \approx_F (L_2, A)$ if and only if $(L_1, A) \eqsim_F ((L_1, A) \cup (L_2, A)) \eqsim_F (L_2, A)$ and $(L_1, A) \simeq_F ((L_1, A) \cap (L_2, A)) \simeq_F (L_2, A)$.

(2) $(L_1, A) \subseteq (L_2, A)$ and $(L_2, A) \approx_F (\phi, A)$ imply that $(L_1, A) \approx_F (\phi, A)$

(3) $((L_1, A) \cup (L_2, A)) \approx_F (\phi, A)$ if and only if $(L_1, A) \approx_F (\phi, A)$ and $(L_2, A) \approx_F (\phi, A)$.

Proof. It is like the proof of Theorem 6.3.9. ■

6.4 Approach towards Decision making

Depending on soft rough set theory, the decision making methods by soft binary relations are proposed in this section. This approach helps to use data provided by decision makers and further information is not required. Hence, the results should avoid the paradox results.

We obtain two values $\underline{J}^G(e_i)$ and $\overline{J}^G(e_i)$ which are most closed with respect to the aftersets by the soft lower and upper approximations of the soft set G. Therefore, the choice value γ_i is redefined with respect to the aftersets as follows:

$$\gamma_i = \sum_{j=1}^{n} \underline{d}_{ij} + \sum_{j=1}^{n} \overline{d_{ij}}$$

In a decision making problem, the maximum choice value γ_i is the optimum decision for the object $x_i \in U$ and the minimum choice value γ_i is the worst decision for the object $x_i \in U$. For the given decision making problem, if the same maximum choice value γ_i belongs to two or more objects $x_i \in U$, then take one of them as the optimum decision randomly.

Algorithm 1:

An approach to a decision making problem in the form of an algorithm with respect to the aftersets is provided below. The decision algorithm is as follows:

(1) Compute the lower soft set approximation $\underline{J}^G$ and upper soft set approximation $\overline{J}^G$ of a soft set G with respect to the aftersets.

(2) Corresponding to each $x_i \in U$, we calculate $\underline{d}_{ij}$ which is 0 if $x_i \notin \underline{J}^G(e_j)$ and is 1 if $x_i \in \underline{J}^G(e_j)$. Similarly, we calculate $\overline{d_{ij}}$ which is 0 if $x_i \notin \overline{J}^G(e_j)$ and is 1 if $x_i \in \overline{J}^G(e_j)$.

(3) Compute the choice value $\gamma_i = \sum_{j=1}^{n} \underline{d}_{ij} + \sum_{j=1}^{n} \overline{d}_{ij}$ with respect to the aftersets.

(4) The best decision is $x_k \in U$ if $\gamma_k = \max_i \gamma_i$, $i = 1, 2, ..., |U|$.

(5) The worst decision is $x_k \in U$ if $\gamma_k = \min_i \gamma_i$, $i = 1, 2, ..., |U|$.

(6) If the value of k is more than one, then we can chose any one of x_k.

Algorithm 2:

An approach to a decision making problem in the form of an algorithm respecting to the foresets is provided below. The decision algorithm is as follows:

(1) Compute the lower soft set approximation ${}^H\underline{J}$ and upper soft set approximation ${}^H\overline{J}$ of a soft set H with respect to the foresets.

(2) Corresponding to each $x_i \in U$, we calculate $\underline{c}_{ij}$ which is 0 if $x_i \notin {}^H\underline{J}(e_j)$ and is 1 if $x_i \in {}^H\underline{J}(e_j)$. Similarly, we calculate $\overline{c_{ij}}$ which is 0 if $x_i \notin {}^H\overline{J}(e_j)$ and is 1 if $x_i \in {}^H\overline{J}(e_j)$.

(3) Compute the choice value $\gamma_i' = \sum_{j=1}^{n} \underline{c}_{ij} + \sum_{j=1}^{n} \overline{c}_{ij}$ with respect to the foresets.

(4) The best decision is $x_k \in U$ if $\gamma_k^{'} = \max_i \gamma_i^{'}$, $i = 1, 2, ..., |U|$.

(5) The worst decision is $x_k \in U$ if $\gamma_k^{'} = \min_i \gamma_i^{'}$, $i = 1, 2, ..., |U|$.

(6) If the value of k is more than one, then we can chose any one of x_k.

6.4.1 An application of the decision making approach

By an example in this subsection, an application of the decision making approach is given.

Example 6.4.1 *Consider the soft relations of Example 6.1.3 again, where a person wants to select a shirt out of six shirt designs and four shirt colors.*

$$\begin{aligned} \text{Define } G &: A \to P(W) \text{ which represents the preference of the} \\ &\quad \text{color given by } Mr.\ X \text{ such that} \\ G(e_1) &= \{c_1, c_4\},\ G(e_2) = \{c_2, c_5\},\ G(e_3) = \{c_2, c_3, c_4\} \text{ and} \\ \text{Define } H &: A \to P(U) \text{ which represents the preference of the} \\ &\quad \text{design given by } Mr.\ X \text{ such that} \\ H(e_1) &= \{d_2, d_3, d_6\},\ H(e_2) = \{d_1, d_3\},\ H(e_3) = \{d_1, d_2, d_5, d_6\}. \end{aligned}$$

Consider the following table after applying the above algorithm.

Table : The results of decision algorithm with respect to aftersets

	$\underline{d}_{i1}$	$\underline{d}_{i2}$	$\underline{d}_{i3}$	$\overline{d}_{i1}$	$\overline{d}_{i2}$	$\overline{d}_{i3}$	*Choice value* γ_i
d_1	0	0	1	1	0	0	2
d_2	0	0	1	1	0	1	3
d_3	1	1	0	0	0	1	3
d_4	0	0	1	0	0	0	1
d_5	0	0	1	1	0	1	3
d_6	1	0	1	1	1	0	4

And,

Table : The results of decision algorithm with respect to foresets

	$\underline{c}_{i1}$	$\underline{c}_{i2}$	$\underline{c}_{i3}$	$\overline{c}_{i1}$	$\overline{c}_{i2}$	$\overline{c}_{i3}$	Choice value $\gamma_i^{'}$
c_1	0	0	0	1	1	0	2
c_2	0	0	1	1	1	0	3
c_3	0	0	0	0	1	1	2
c_4	0	1	1	1	0	1	4

Here the choice value $\gamma_i = \sum_{j=1}^{3} \underline{d}_{ij} + \sum_{j=1}^{3} \overline{d}_{ij}$ *is calculated with respect to the aftersets and the choice value* $\gamma_i^{'} = \sum_{j=1}^{3} \underline{c}_{ij} + \sum_{j=1}^{3} \overline{c_{ij}}$ *is calculated with respect to the foresets.*

The shirt of designs d_6 *scores the maximum choice value* $\gamma_k = 4 = \gamma_6$, *and the decision is in favour of the shirt of design* d_6 *for selection. Moreover, the shirts of designs* d_4 *are totally ignored. Hence, Mr. X will choose the shirt of design* d_6 *for his personal use and he will not select the shirt of design* d_4 *with respect to the aftersets. Similarly, the shirt of color* c_4 *scores the maximum choice value* $\gamma_k^{'} = 4 = \gamma_4$ *and the decision is in favour of the shirt of color* c_4 *for selection. Moreover, the shirts of color* c_1 *and* c_3 *are totally ignored. Hence, Mr. X will choose the shirt of color* c_4 *for his personal use and he will not select the shirts of color* c_1 *and* c_3 *with respect to the foresets.*

Chapter 7

Approximation of soft ideals by soft relations in semigroups

In the last chapter, we applied the concepts of chapter six on semigroups and approximations of SSSs, $SLIL$ ($SRIL$), $SIIL$s and $SBIL$s of semigroups are studied. Moreover, for the illustration of the concepts, some examples are considered.

7.1 Approximation by soft relations

Theorem 7.1.1 *Let (J, A) be an SCRE from S_1 to S_2 (S_1 and S_2 are semigroups). For any two soft sets (G_1, A) and (G_2, A) over S_2 and for all $e \in A$, $\overline{J}^{G_1}(e) . \overline{J}^{G_2}(e) \subseteq \overline{J}^{G_1 G_2}(e)$.*

Proof. Let $u \in \overline{J}^{G_1}(e) . \overline{J}^{G_2}(e)$. Then $u = g_1 g_2$ for some $g_1 \in \overline{J}^{G_1}(e)$ and $g_2 \in \overline{J}^{G_2}(e)$. It follows $g_1 J(e) \cap G_1(e)$ and $g_2 J(e) \cap G_2(e)$ are non-empty, so their exist elements $a, b \in S_2$ such that $a \in g_1 J(e) \cap G_1(e)$ and $b \in g_2 J(e) \cap G_2(e)$. Thus $a \in g_1 J(e)$, $b \in g_2 J(e)$, $a \in G_1(e)$ and $b \in G_2(e)$. Now, $(g_1, a) \in J(e)$ and $(g_2, b) \in J(e)$ implies that $(g_1 g_2, ab) \in J(e)$, that is $ab \in g_1 g_2 J(e)$. Also, $ab \in G_1(e) G_2(e)$, therefore, $ab \in G_1(e) G_2(e) \cap g_1 g_2 J(e)$. Hence, $u = g_1 g_2 \in \overline{J}^{G_1 G_2}(e)$. ■

The proof of next theorem is a routine verification and hence can be obtained from above theorem.

Theorem 7.1.2 *Let (J, A) be an SCRE from S_1 to S_2 (S_1 and S_2 are semigroups).*

For any two soft sets (L_1, A) *and* (L_2, A) *over* S_1, ${}^{L_1}\overline{J}(e) . {}^{L_2}\overline{J}(e) \subseteq {}^{L_1L_2}\overline{J}(e)$ *for all* $e \in A$.

Theorem 7.1.3 *Let* (J, A) *be an* SC_mR *respecting to the aftersets from* S_1 *to* S_2 *(*S_1 *and* S_2 *are semigroups). For any two soft sets* (G_1, A) *and* (G_2, A) *over* S_2, $\underline{J}^{G_1}(e).\underline{J}^{G_2}(e) \subseteq \underline{J}^{G_1G_2}(e)$ *for all* $e \in A$.

Proof. First we consider that $\underline{J}^{G_1}(e)$ and $\underline{J}^{G_2}(e)$ are non-empty and $u \in \underline{J}^{G_1}(e) . \underline{J}^{G_2}(e)$. Then $u = g_1g_2$ for some $g_1 \in \underline{J}^{G_1}(e)$ and $g_2 \in \underline{J}^{G_2}(e)$. It shows $G_1(e) \supseteq g_1J(e) \neq \phi$ and $G_2(e) \supseteq g_2J(e) \neq \phi$. As $g_1g_2J(e) = g_1J(e) . g_2J(e) \subseteq G_1(e)G_2(e)$, we have $u = g_1g_2 \in \underline{J}^{G_1G_2}(e)$. Hence, $\underline{J}^{G_1G_2}(e) \supseteq \underline{J}^{G_1}(e).\underline{J}^{G_2}(e)$. Now, if one of $\underline{J}^{G_1}(e)$ and $\underline{J}^{G_2}(e)$ is empty then $\phi = \underline{J}^{G_1}(e) . \underline{J}^{G_2}(e) \subseteq \underline{J}^{G_1G_2}(e)$. ■

The proof of next theorem is a routine verification and hence can be obtained from above theorem.

Theorem 7.1.4 *Let* (J, A) *be an* SC_mR *respecting to the foresets from* S_1 *to* S_2 *(*S_1 *and* S_2 *are semigroups). For any two soft sets* (L_1, A) *and* (L_2, A) *over* S_1, ${}^{L_1}\underline{J}(e).{}^{L_2}\underline{J}(e) \subseteq {}^{L_1L_2}\underline{J}(e)$ *for all* $e \in A$.

Example 7.1.5 *For two semigroups* $S_1 = \{1, 2, 3\}$ *and* $S_2 = \{a, b, c\}$ *with the multiplication tables as follows:*

•	1	2	3
1	1	2	3
2	1	2	3
3	1	2	3

•	a	b	c
a	a	a	c
b	a	b	c
c	a	c	c

and $A = \{e_1, e_2\}$. *Define* $J : A \to P(S_1 \times S_2)$ *by*

$$J(e_1) = \{(1, a), (3, c), (2, c), (2, a), (2, b)\}$$

and

$$J(e_2) = \{(1, b), (1, a), (3, c), (2, a), (1, c), (2, b)\}.$$

Then (J, A) *is an SCRE from the semigroup* S_1 *to the semigroup* S_2.

$$\begin{aligned} 1J(e_1) &= \{a\},\ 2J(e_1) = \{a, b, c\} \text{ and } 3J(e_1) = \{c\} \\ 1J(e_2) &= \{a, b, c\},\ 2J(e_2) = \{a, b\},\ 3J(e_2) = \{c\}. \end{aligned}$$

Define (G_1, A) *and* (G_2, A)*, two soft sets over* S_2 *by*

$$G_1(e_1) = \{a\}, G_1(e_2) = \{a,b\} \text{ and } G_2(e_1) = \{b,c\}, G_2(e_2) = \{a,c\}.$$

Then $\overline{J}^{G_1}(e_1) = \{1,2\}$ *and* $\overline{J}^{G_2}(e_1) = \{2,3\}$. *Now,* $G_1(e_1)G_2(e_1) = \{a,c\}$ *and*

$$\overline{J}^{G_1G_2}(e_1) = \{1,2,3\} \neq \{2,3\} = \{1,2\}\{2,3\} = \overline{J}^{G_1}(e_1)\overline{J}^{G_2}(e_1).$$

Example 7.1.6 *Consider the semigroups and soft relations of Example 7.1.5,*

$$J(e_1)a = \{1,2\},\ J(e_1)b = \{2\} \text{ and } J(e_1)c = \{2,3\}$$
$$J(e_2)a = \{1,2\},\ J(e_2)b = \{1,2\},\ J(e_2)c = \{1,3\}.$$

Define (L_1, A) *and* (L_2, A)*, two soft sets over* S_1 *by*

$$L_1(e_1) = \{1,2\}, L_1(e_2) = \{3\} \text{ and } L_2(e_1) = \{2,3\}, L_2(e_2) = \{1\}.$$

Then, ${}^{L_1}\overline{J}(e_2) = \{c\}$ *and* ${}^{L_2}\overline{J}(e_2) = \{a,b,c\}$. *Now,* $L_1(e_2)L_2(e_2) = \{1\}$ *and*

$${}^{L_1L_2}\overline{J}(e_2) = \{a,b,c\} \nsubseteq \{a,c\} = \{c\}\{a,b,c\} = {}^{L_1}\overline{J}(e_2).{}^{L_2}\overline{J}(e_2).$$

Example 7.1.7 *For two semigroups,* $S_1 = \{a,b,c,d\}$ *and* $S_2 = \{1,2,3,4\}$ *with the multiplication tables as follows:*

•	a	b	c	d
a	a	a	a	d
b	a	b	a	d
c	a	a	c	d
d	d	d	d	d

•	1	2	3	4
1	1	2	3	4
2	2	2	2	2
3	3	3	3	3
4	4	3	2	1

and $A = \{e_1, e_2\}$. *Define* $J : A \to P(S_1 \times S_2)$ *by*

$$J(e_1) = \{(a,2),(b,2),(c,2),(c,3),(d,2),(b,3),(d,3),(a,3)\}$$

and

$$J(e_2) = \{(d,2),(a,2),(b,2),(c,2)\}.$$

Then (J, A) *is an* SC_mR *respecting to the aftersets from the semigroup* S_1 *to the semigroup* S_2.

$$aJ(e_1) = \{2,3\},\ bJ(e_1) = \{2,3\},\ cJ(e_1) = \{2,3\} \text{ and } dJ(e_1) = \{2,3\}.$$

Also,

$$aJ(e_2)=\{2\}\,,\ bJ(e_2)=\{2\}\,,\ cJ(e_2)=\{2\}\ \text{ and } dJ(e_2)=\{2\}\,.$$

$$\text{Define } (G_1,A) \text{ and } (G_2,A)\,, \text{ two soft sets over } S_2 \text{ by}$$

$$G_1(e_1) = \{4\}\,, G_1(e_2)=\{2,3\} \text{ and } G_2(e_1)=\{1,2,3\}\,, G_2(e_2)=\{2,4\}\,.$$

Then $\underline{J}^{G_1}(e_1)=\phi$ *and* $\underline{J}^{G_2}(e_1)=\{a,b,c,d\}$*. Now,* $G_1(e_1)\,G_2(e_1)=\{2,3,4\}$ *and*

$$\underline{J}^{G_1G_2}(e_1)=\{a,b,c,d\}\nsubseteq\phi=\phi\{a,b,c,d\}=\underline{J}^{G_1}(e_1)\underline{J}^{G_2}(e_1).$$

Example 7.1.8 *Consider the semigroups of Example 7.1.7*

and $A=\{e_1,e_2\}$*. Define* $J:A\to P(S_1\times S_2)$ *by*

$$\begin{aligned} J(e_1) &= \{(a,1)\,,(a,2)\,,(a,3)\,,(d,4),(a,4)\,,(d,1)\,,(d,3)\,,(d,2)\} \text{ and} \\ J(e_2) &= \{(d,1)\,,(d,3)\,,(d,4)\,,(d,2)\}. \end{aligned}$$

Then (J,A) *is an* SC_mR *from* S_1 *to* S_2 *respecting to the aftersets.*

$$\begin{aligned} J(e_1)1 &= \{a,d\}\,,\ J(e_1)2=\{a,d\}\,,\ J(e_1)3=\{a,d\} \text{ and } J(e_1)4=\{a,d\}\,. \\ \text{Also, } J(e_2)1 &= \{d\}\,,\ J(e_2)2=\{d\}\,,\ J(e_2)3=\{d\} \text{ and } J(e_2)4=\{d\}\,. \end{aligned}$$

$$\text{Define } (L_1,A) \text{ and } (L_2,A)\,, \text{ two soft sets over } S_1 \text{ by}$$

$$L_1(e_1) = \{a,c,d\}\,, L_1(e_2)=\{a\} \text{ and } L_2(e_1)=\{b\}\,, L_2(e_2)=\{a,b\}\,.$$

Then ${}^{L_1}\underline{\delta}(e_1)=\{1,2,3,4\}$ *and* ${}^{L_2}\underline{\delta}(e_1)=\phi$*. Now,* $L_1(e_1)\,L_2(e_1)=\{a,d\}$ *and*

$${}^{L_1L_2}\underline{J}(e_1)=\{1,2,3,4\}\nsubseteq\phi=\{1,2,3,4\}\phi={}^{L_1}\ \underline{J}(e_1).{}^{L_2}\underline{J}(e_1).$$

7.2 Approximation of soft ideals in semigroups

By an $SCRE$, (J,A) and $J(e)\neq\phi$ for all $e\in A$, we approximate a SS (LIL, RIL, BIL, IIL) of a semigroup respecting to the aftersets (resp. respecting to the foresets). We show that upper approximation of a SSS ($SLIL$, $SRIL$, $SBIL$, $SIIL$) of a semigroup is a SSS ($SLIL$, $SRIL$, $SBIL$, $SIIL$) of the semigroup and give counter examples. We also show that lower approximation of a SSS ($SLIL$, $SRIL$, $SBIL$, $SIIL$) of a semigroup by an SC_mR is a SSS ($SLIL$, $SRIL$, $SBIL$, $SIIL$) of the semigroup and give counter examples. Throughout the remaining paper, (J,A) is a soft relation from S_1 to S_2 (S_1 and S_2 are semigroups) and $uJ(e) \neq \phi$ for all $u \in S_1$ and $e \in A$ and $J(e)w \neq \phi$ for all $w \in S_2$ and $e \in A$.

Definition 7.2.1 *Let (J,A) be an $SBRE$ from S_1 to S_2 (S_1 and S_2 are semigroups). If the upper approximation $\left(\overline{J}^G, A\right)$ is a SSS of S_1 for any soft set (G,A) over S_2, then (G,A) is said to be generalized upper SSS of S_1 respecting to the aftersets. The soft set (G,A) is called generalized upper $SLIL$ ($SRIL$, SIL) of S_1 respecting to the aftersets if $\left(\overline{J}^G, A\right)$ is a $SLIL$ ($SRIL$, SIL) of S_1.*

Definition 7.2.2 *Let (J,A) be an $SBRE$ from S_1 to S_2 (S_1 and S_2 are semigroups). If the upper approximation $\left({}^L\overline{J}, A\right)$ is a SSS of S_2 for any soft set (L,A) over S_1, then (L,A) is said to be generalized upper SSS of S_2 respecting to the foresets. The soft set (L,A) is called generalized upper $SLIL$ ($SRIL$, SIL) of S_2 respecting to the foresets if $\left({}^L\overline{J}, A\right)$ is a $SLIL$ ($SRIL$, SIL) of S_2.*

Theorem 7.2.3 *Let (J,A) be an $SCRE$ from S_1 to S_2 (S_1 and S_2 are semigroups). Then*

(1) *If (G,A) is a SSS of S_2, then (G,A) is a generalized upper SSS of S_1 respecting to the aftersets.*

(2) *If (L,A) is a SSS of S_1, then (L,A) is a generalized upper SSS of S_2 respecting to the foresets.*

(3) *If (G,A) is a $SLIL$ ($SRIL$, SIL) of S_2, then (G,A) is a generalized upper $SLIL$ ($SRIL$, SIL) of S_1 respecting to the aftersets.*

(4)*If (L,A) is a $SLIL$ ($SRIL$, SIL) of S_1, then (L,A) is a generalized upper $SLIL$ ($SRIL$, SIL) of S_2 respecting to the foresets.*

Proof. (1) Let (G,A) be a SSS of S_2. If $\phi \neq \overline{J}^G(e)$ for $e \in A$. Then by Theorem 7.1.1, $\overline{J}^G(e) . \overline{J}^G(e) \subseteq \overline{J}^{GG}(e) \subseteq \overline{J}^G(e)$, that is $\overline{J}^G(e)$ is a SS of S_1 for $e \in A$ and so (G,A) is a generalized upper SSS of S_1 respecting to the aftersets.

(2) With similar arguments the proof is obtained from part (1).

(3) Suppose (G,A) is a $SLIL$ of S_2. As we know that $\overline{J}^{S_2}(e) = S_1$ for all $e \in A$. We have from Theorem 7.1.1, $S_1\overline{J}^G(e) = \overline{J}^{S_2}(e) . \overline{J}^G(e) \subseteq \overline{J}^{S_2G}(e)$

$\subseteq \overline{J}^{G}(e)$. Hence $\overline{J}^{G}(e)$ is a LI of S_1 and so (G, A) is a generalized upper $SLIL$ of S_1 respecting to the aftersets.

(4) This part has a routine and similar verification to part (3).

The other cases can be demonstrated comparatively. ■

It is found in the accompanying Example that converse of above Theorem is not true.

Example 7.2.4 *For two semigroups $S_1 = \{a, b, c, d, e\}$ and $S_2 = \{1, 2, 3, 4, 5\}$ with the multiplication tables as follows:*

$\bullet$	a	b	c	d	e
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
e	d	d	c	d	c

$\bullet$	1	2	3	4	5
1	1	5	3	4	5
2	1	2	3	4	5
3	1	5	3	4	5
4	1	5	3	4	5
5	1	5	3	4	5

Let $A = \{e_1, e_2\}$. Define $J : A \to P(S_1 \times S_2)$ by

$$J(e_1) = \left\{ \begin{array}{c} (a,1), (c,3), (e,5), (b,1), (d,4), \\ (c,5), (b,5), (d,3), (b,2), (d,5), (d,1) \end{array} \right\} \text{ and}$$

$$J(e_2) = \left\{ \begin{array}{c} (a,1), (b,2), (c,3), (e,5), (b,1), \\ (c,5), (b,5), (d,5), (d,4), (d,3), (d,1), (b,3) \end{array} \right\}.$$

Then (J, A) is an $SCRE$ from the semigroups S_1 to the semigroup S_2. Now,

$$\begin{aligned} aJ(e_1) &= \{1\}, \ bJ(e_1) = \{1,2,5\}, \ cJ(e_1) = \{3,5\}, \\ dJ(e_1) &= \{1,3,4,5\}, \ eJ(e_1) = \{5\} \text{ and} \end{aligned}$$

$$\begin{aligned} aJ(e_2) &= \{1\}, \ bJ(e_2) = \{1,2,3,5\}, \ cJ(e_2) = \{3,5\}, \\ dJ(e_2) &= \{1,3,4,5\}, \ eJ(e_2) = \{5\}. \end{aligned}$$

Also,

$$\begin{aligned} J(e_1)1 &= \{a,b,d\}, \ J(e_1)2 = \{b\}, \ J(e_1)3 = \{c,d\}, \\ J(e_1)4 &= \{d\}, \ J(e_1)5 = \{b,c,d,e\} \text{ and} \end{aligned}$$

$$J(e_2)1 = \{a,b,d\}, \; J(e_2)2 = \{b\}, \; J(e_2)3 = \{b,c,d\},$$
$$J(e_2)4 = \{d\}, \; J(e_2)5 = \{b,c,d,e\}.$$

Define (G,A), a soft set over S_2 by $G(e_1) = \{1,2,3\}$ and $G(e_2) = \{2,3,4\}$ and define (L,A), a soft set over S_1 by $L(e_1) = \{a,b,c\}$ and $L(e_2) = \{a,b,e\}$.

(1) (G,A) is not a SSS of S_2 as $\{1,2,3\}\{1,2,3\} = \{1,2,3,5\} \nsubseteq \{1,2,3\}$ and $\{2,3,4\}\{2,3,4\} = \{2,3,4,5\} \nsubseteq \{2,3,4\}$ but $\overline{J}^G(e_1) = \{a,b,c,d\}$ and $\overline{J}^G(e_2) = \{a,b,c,d\}$ are SSs of S_1. Hence, (G,A) is a generalized upper SSS of S_1 respecting to the aftersets.

(2) (L,A) is not a SSS of S_1 as $\{a,b,c\}\{a,b,c\} = \{b,c,d\} \nsubseteq \{a,b,c\}$ and $\{a,b,e\}\{a,b,e\} = \{b,c,d\} \nsubseteq \{a,b,e\}$ but ${}^L\overline{J}(e_1) = \{1,2,3,5\}$ and ${}^L\overline{J}(e_2) = \{1,2,3,5\}$ are SSs of S_2. Hence, (L,A) is a generalized upper SSS of S_2 respecting to the foresets.

(3) (G,A) is not a $SLIL$ of S_2 as $\{1,2,3,4,5\}\{1,2,3\} = \{1,2,3,5\} \nsubseteq \{1,2,3\}$ and $\{1,2,3,4,5\}\{2,3,4\} = \{2,3,4,5\} \nsubseteq \{2,3,4\}$ but $\overline{J}^G(e_1) = \{a,b,c,d\}$ and $\overline{J}^G(e_2) = \{b,c,d\}$ are LILs of S_1. Hence, (G,A) is a generalized upper $SLIL$ of S_1 respecting to the aftersets.

(4) (L,A) is not a $SLIL$ of S_1 as $\{a,b,c,d,e\}\{a,b,c\} = \{b,c,d\} \nsubseteq \{a,b,c\}$ and $\{a,b,c,d,e\}\{a,b,e\} = \{b,c,d\} \nsubseteq \{a,b,e\}$ but ${}^L\overline{J}(e_1) = \{1,2,3,5\}$ and ${}^L\overline{J}(e_2) = \{1,2,3,5\}$ are $LILs$ of S_2. Hence, (L,A) is a generalized upper $SLIL$ of S_2 respecting to the foresets.

Definition 7.2.5 *Let (J,A) be a $SCRE$ from S_1 to S_2 (S_1 and S_2 are semigroups). A soft set (G,A) over S_2 is said to be generalized lower SSS of S_1 respecting to the aftersets if $\left(\underline{J}^G, A\right)$ is a SSS of S_1. The soft set (G,A) is called generalized lower $SLIL$ ($SRIL$, SIL) of S_1 respecting to the aftersets if $\left(\underline{J}^G, A\right)$ is a $SLIL$ ($SRIL$, SIL) of S_1.*

Definition 7.2.6 *Let (J,A) be a $SCRE$ from S_1 to S_2 (S_1 and S_2 are semigroups). A soft set (L,A) over S_1 is said to be generalized lower SSS of S_2 respecting to the foresets if $\left({}^L\underline{J}, A\right)$ is a SSS of S_2. The soft set (L,A) is called generalized lower $SLIL$ ($SRIL$, SIL) of S_2 respecting to the foresets if $\left({}^L\underline{J}, A\right)$ is a $SLIL$ ($SRIL$, SIL) of S_2.*

Example 7.2.7 *Consider the Example 7.2.4 and Define (G,A), a soft set over S_2 by $G(e_1) = \{1,3,5\}$ and $G(e_2) = \{1,2\}$. Then (G,A) is a $SLIL$ of S_2 but $\underline{J}^G(e_1) = \{a,c,e\}$ is not a LIL of S_1 as $\{a,b,c,d,e\}\{a,c,e\} = \{b,c,d\} \nsubseteq \{a,c,e\}$.*

It is verified in the above example, that if (J, A) is a $SCRE$ from S_1 to S_2 and (G, A) is a $SLIL$ of S_2 even then $\left(\underline{J}^L, A\right)$ is not a LIL of S_1. However, the next theorem is proceeded.

Theorem 7.2.8 *Let (J, A) be an SC_mR respecting to the aftersets from S_1 to S_2 (S_1 and S_2 are semigroups). Then*

(1) If (G, A) is a SSS of S_2, then (G, A) is a generalized lower SSS of S_1 respecting to the aftersets.

(2) If (G, A) is a $SLIL$ ($SRIL$, SIL) of S_2, then (G, A) is a generalized lower $SLIL$ ($SRIL$, SIL) of S_1 respecting to the aftersets.

Proof. (1) Let (G, A) be a SSS of S_2. If $\phi \neq \underline{J}^G(e)$ for $e \in A$. Then by Theorem 7.1.3 and Theorem 6.1.7(1), $\underline{J}^G(e) . \underline{J}^G(e) \subseteq \underline{J}^{GG}(e) \subseteq \underline{J}^G(e)$. Therefore, $\left(\underline{J}^G, A\right)$ is a SSS of S_1. Hence, (G, A) is a generalized lower SSS of S_1 respecting to aftersets.

(2) Suppose that G is a $SLIL$ of S_2. If $\phi \neq \underline{J}^G(e)$ for $e \in A$. Then by Theorem 7.1.3 and Theorem 6.1.7(1), $S_1 \underline{J}^G(e) = \underline{J}^{S_2}(e) . \underline{J}^G(e) \subseteq \underline{J}^{S_2 G}(e) \subseteq$

$\underline{J}^G(e)$. Therefore, $\left(\underline{J}^G, A\right)$ is a $SLIL$ of S_1. Hence, (G, A) is a generalized lower $SLIL$ of S_1 respecting to the aftersets.

The rest of the cases can be demonstrated comparably. ■

The proof of next theorem is a routine verification and hence can be obtained from above theorem.

Theorem 7.2.9 *Let (J, A) be an SC_mR respecting to the foresets from S_1 to S_2 (S_1 and S_2 are semigroups). Then*

(1) If (L, A) is a SSS of S_1, then (L, A) is a generalized lower SSS of S_2 respecting to the foresets.

(2) If (L, A) is a $SLIL$ ($SRIL$, SIL) of S_1, then (L, A) is a generalized lower $SLIL$ ($SRIL$, SIL) of S_2 respecting to the foresets.

The converse of parts of Theorems 7.2.8 and 7.2.9 do not hold generally as shown by following example.

Example 7.2.10 *Consider the Example 7.1.7 and $A = \{e_1, e_2\}$. Define $J : A \rightarrow$*

$P(S_1 \times S_2)$ by

$$J(e_1) = \{(a,2),(a,3),(b,3),(c,2),(c,3),(b,2),(d,2),(d,3)\} \text{ and}$$
$$J(e_2) = \{(a,2),(c,2),(d,2),(b,2)\}.$$

Then (J, A) is an SC_mR from S_1 to S_2 respecting to the aftersets.

$$aJ(e_2) = \{2,3\},\ bJ(e_2) = \{2,3\},\ cJ(e_2) = \{2,3\} \text{ and } dJ(e_2) = \{2,3\}. \text{ Also,}$$
$$aJ(e_2) = \{2\},\ bJ(e_2) = \{2\},\ cJ(e_2) = \{2\} \text{ and } dJ(e_2) = \{2\}.$$

Define (G, A), a soft set over S_2 by $G(e_1) = \{2,3,4\}$ and $G(e_2) = \{2,4\}$.

(1) *(G, A) is not a SSS of S_2 as $\{2,3,4\}\{2,3,4\} = \{1,2,3\} \nsubseteq \{2,3,4\}$ and $\{2,4\}\{2,4\} = \{1,2,3\} \nsubseteq \{2,4\}$ but $\underline{J}^G(e_1) = \{a,b,c,d\}$ and $\underline{J}^G(e_2) = \{a,b,c,d\}$, which show that $(\underline{J}^G, A)$ is a SSS of S_1. Hence, (G, A) is a generalized lower SSS of S_1 respecting to the aftersets.*

(2) *(G, A) is not a $SLIL$ of S_2 as $\{1,2,3,4\}\{2,3,4\} = \{1,2,3,4\} \nsubseteq \{2,3,4\}$ and $\{1,2,3,4\}\{2,4\} = \{1,2,3,4\} \nsubseteq \{2,4\}$ but $\underline{J}^G(e_1) = \{a,b,c,d\}$ and $\underline{J}^G(e_2) = \{a,b,c,d\}$, which show that $(\underline{J}^G, A)$ is a $SLIL$ of S_1. Hence, (G, A) is a generalized lower $SLIL$ of S_1 respecting to the aftersets.*

Now, Define $J : A \to P(S_1 \times S_2)$ by

$$J(e_1) = \{(d,1),(d,3),(d,4),(d,2)\} \text{ and}$$
$$J(e_2) = \{(a,1),(a,2),(a,4),(d,1),(d,3),(a,3),(d,4),(d,2)\}.$$

Then (J, A) is an SC_mR from S_1 to S_2 respecting to the foresets.

$$J(e_2)1 = \{d\},\ J(e_2)2 = \{d\},\ J(e_2)3 = \{d\} \text{ and } J(e_2)4 = \{d\}. \text{ Also,}$$
$$J(e_2)1 = \{a,d\},\ J(e_2)2 = \{a,d\},\ J(e_2)3 = \{a,d\} \text{ and } J(e_2)4 = \{a,d\}.$$

Define (L, A), a soft set over S_1 by $L(e_1) = \{b,c,d\}$ and $L(e_2) = \{b,c,d\}$.

(1) *(L, A) is not a SSS of S_1 as $\{b,c,d\}\{b,c,d\} = \{a,b,c,d\} \nsubseteq \{b,c,d\}$ but ${}^L\underline{J}(e_1) = \{1,2,3,4\}$ and ${}^L\underline{J}(e_2) = \{1,2,3,4\}$, which show that $(\underline{J}^G, A)$ is a SSS of S_2. Hence, (L, A) is a generalized lower SSS of S_2 respecting to the foresets.*

(2) *(L, A) is not a $SLIL$ of S_1 as $\{a,b,c,d\}\{b,c,d\} = \{a,b,c,d\} \nsubseteq \{b,c,d\}$ but ${}^L\underline{J}(e_1) = \{1,2,3,4\}$ and ${}^L\underline{J}(e_2) = \{1,2,3,4\}$, which show that $(\underline{J}^G, A)$ is a $SLIL$ of S_2. Hence, (L, A) is a generalized lower $SLIL$ of S_2 respecting to the foresets.*

Theorem 7.2.11 *Let (J, A) be an SBRE from S_1 to S_2 (S_1 and S_2 are semigroups). Then $\overline{J}^{G_1G_2} \subseteq \overline{J}^{G_1} \cap \overline{J}^{G_2}$ for any SRIL, (G_1, A) and SLIL, (G_2, A) of S_2.*

Proof. Suppose that (G_1, A) is a $SRIL$ and (G_2, A) a $SLIL$ of S_2, so by definition $G_1(e) \supseteq G_1(e) S_2 \supseteq G_1(e) G_2(e)$ and $G_2(e) \supseteq S_2G_2(e) \supseteq G_1(e) G_2(e)$ which implies that $G_1(e) \cap G_2(e) \supseteq G_1(e) G_2(e)$. It follows from Theorem 6.1.7 (3), (5), $\overline{J}^{G_1G_2}(e) \subseteq \overline{J}^{G_1 \cap G_2}(e) \subseteq \overline{J}^{G_1}(e) \cap \overline{J}^{G_2}(e)$. Hence, $\overline{J}^{G_1G_2} \subseteq \overline{J}^{G_1} \cap \overline{J}^{G_2}$. ■

The proof of the next theorem is a routine verification and hence can be obtained from above theorem.

Theorem 7.2.12 *Let (J, A) be an SBRE from S_1 to S_2 (S_1 and S_2 are semigroups). Then ${}^{L_1L_2}\overline{J} \subseteq {}^{L_1}\overline{J} \cap {}^{L_2}\overline{J}$ for any SRIL, (L_1, A) and soft SLIL, (L_2, A) of S_1.*

Theorem 7.2.13 *Let (J, A) be an SBRE from S_1 to S_2 (S_1 and S_2 are semigroups). Then $\underline{J}^{G_1G_2} \subseteq \underline{J}^{G_1} \cap \underline{J}^{G_2}$ for any SRIL, (G_1, A) and SLIL, (G_2, A) of S_2.*

Proof. Let (G_1, A) be a $SRIL$ and (G_2, A) a $SLIL$ of S_2, so by definition $G_1(e) G_2(e) \subseteq G_1(e) S_2 \subseteq G_1(e)$ and $G_1(e) G_2(e) \subseteq S_2G_2(e) \subseteq G_2(e)$ which implies that $G_1(e) G_2(e) \subseteq G_1(e) \cap G_2(e)$. It follows from Theorem 6.1.7 (2), (4), $\underline{J}^{G_1G_2}(e) \subseteq \underline{J}^{G_1 \cap G_2}(e) = \underline{J}^{G_1}(e) \cap \underline{J}^{G_2}(e)$. Hence, $\underline{J}^{G_1G_2} \subseteq \underline{J}^{G_1} \cap \underline{J}^{G_2}$. ■

The proof of the next theorem is a routine verification and hence can be obtained from above theorem.

Theorem 7.2.14 *Let (J, A) be an SBRE from S_1 to S_2 (S_1 and S_2 are semigroups). Then ${}^{L_1L_2}\underline{J} \subseteq {}^{L_1}\underline{J} \cap {}^{L_2}\underline{J}$ for any SRIL, (L_1, A) and SLIL, (L_2, A) of S_2.*

Next, the approximations of $SIILs$ in semigroups are described and discussed.

Definition 7.2.15 *Let (G, A) be a soft set over S_2 and (J, A) an SBRE from S_1 to S_2 (S_1 and S_2 are semigroups). Then (G, A) is said to be generalized lower (upper) SIIL of S_1 respecting to the aftersets if $(\underline{J}^G, A)$ $\left(\text{respectively } \left(\overline{J}^G, A\right)\right)$ is a SIIL of S_1.*

Definition 7.2.16 *Let (J, A) be an SBRE from S_1 to S_2 (S_1 and S_2 are semigroups) and (L, A) be a soft set over S_1. Then (L, A) is said to be a generalized lower (upper)*

SIIL of S_2 *respecting to the foresets if* $\left({}^{L}\underline{J}, A\right)$ $\left(\text{respectively } \left({}^{L}\overline{J}, A\right)\right)$ *is a SIIL of* S_2.

Theorem 7.2.17 *Let* (J, A) *be an SCRE from* S_1 *to* S_2 *(*S_1 *and* S_2 *are semigroups). If* (G, A) *is a SIIL of* S_2*, then* (G, A) *is a generalized upper SIIL of* S_1 *respecting to the aftersets.*

Proof. As (G, A) is a $SIIL$ of S_2, it follows from Theorem 7.1.1 that $S_1\overline{J}^{G}(e)\,S_1 = \overline{J}^{S_2}(e)\,.\overline{J}^{G}(e)\,.\overline{J}^{S_2}(e) \subseteq \overline{J}^{S_2GS_2}(e) \subseteq \overline{J}^{G}(e)$. Therefore, $\left(\overline{J}^{G}, A\right)$ is a $SIIL$ of S_1. Hence, (G, A) is a generalized upper $SIIL$ of S_1 respecting to the aftersets. ■

It is found in the accompanying Example that converse of above Theorem is not true.

Example 7.2.18 *Consider the semigroups of Example 7.1.5 and* $A = \{e_1, e_2\}$*. Define* $J : A \to P(S_1 \times S_2)$ *by*

$$\begin{aligned} J(e_1) &= \{(1,a),(3,c),(1,b),(2,a),(2,b),(1,c),(3,a)\} \text{ and} \\ J(e_2) &= \{(1,a),(2,b),(3,c),(2,c),(2,a)\}. \end{aligned}$$

Then (J, A) *is a SCRE from the semigroup* S_1 *to the semigroup* S_2*. Now,*

$$\begin{aligned} 1J(e_1) &= \{a,b,c\},\ 2J(e_1) = \{a,b\} \text{ and } 3J(e_1) = \{a,c\}, \\ 1J(e_2) &= \{a\},\ 2J(e_2) = \{a,b,c\} \text{ and } 3J(e_2) = \{c\}. \end{aligned}$$

Now, define (G, A)*, a soft set over* S_2 *by* $G(e_1) = \{a\}$ *and* $G(e_2) = \{a\}$*, which is not a SIIL of* S_2 *as* $\{a,b,c\}\{a\}\{a,b,c\} = \{a,c\} \nsubseteq \{a\}$ *but* $\overline{J}^{G}(e_1) = \{1,2,3\}$ *and* $\overline{J}^{G}(e_2) = \{1,2,3\}$*, which show that* $\left(\overline{J}^{G}, A\right)$ *is a SIIL of* S_1*. Hence,* (G, A) *is a generalized upper SIIL of* S_1 *respecting to the aftersets.*

The next theorem has a routine verification.

Theorem 7.2.19 *Let* (J, A) *be an SCRE from* S_1 *to* S_2 *(*S_1 *and* S_2 *are semigroups). If* (L, A) *is a SIIL of* S_1*, then* (L, A) *is a generalized upper SIIL of* S_2 *respecting to the foresets.*

The example below describes that the counter part of above theorem is not valid generally.

Example 7.2.20 *Consider the Example 7.1.5 and soft relation of example 7.2.18,*

$$\begin{aligned} J(e_1)a &= \{1,2,3\},\ J(e_1)b = \{1,2\},\ J(e_1)c = \{1,3\}, \\ J(e_2)a &= \{1,2\},\ J(e_2)b = \{2\} \text{ and } J(e_2)c = \{2,3\}. \end{aligned}$$

Now, define (L,A), *a soft set over* S_1 *by* $L(e_1) = \{1\}$ *and* $L(e_2) = \{2,3\}$ *which is not a* $SIIL$ *of* S_1 *as* $\{1,2,3\}\{1\}\{1,2,3\} = \{1,2,3\} \nsubseteq \{1\}$ *and* $\{1,2,3\}\{2,3\}\{1,2,3\} = \{1,2,3\} \nsubseteq \{2,3\}$ *but* ${}^{L}\overline{J}(e_1) = \{a,b,c\}$ *and* ${}^{L}\overline{J}(e_2) = \{a,b,c\}$, *which shows that* $\left({}^{L}\overline{J},A\right)$ *is a* $SIIL$ *of* S_2. *Hence* (L,A) *is a generalized upper* $SIIL$ *of* S_2 *respecting to the foresets.*

Theorem 7.2.21 *Let* (J,A) *be an* SC_mR *respecting to the aftersets from* S_1 *to* S_2 *(*S_1 *and* S_2 *are semigroups). If* (G,A) *is a* $SIIL$ *of* S_2*, then* (G,A) *is a generalized lower* $SIIL$ *of* S_1 *respecting to the aftersets.*

Proof. As (G,A) is a $SIIL$ of S_2, we have by Theorem 6.1.7 (2) and Theorem 7.1.3, $S_1\underline{J}^{G}(e)\,S_1 = \underline{J}^{S_2}(e)\,.\underline{J}^{G}(e)\,.\underline{J}^{S_2}(e) \subseteq \underline{J}^{S_2GS_2}(e) \subseteq \underline{J}^{G}(e)$. Hence, $\left(\underline{J}^{G},A\right)$ is a $SIIL$ of S_1. Thus, (G,A) is a generalized lower $SIIL$ of S_1 respecting to the aftersets. ■

Example 7.2.22 *Consider the semigroup of Example 7.1.7 and* $A = \{e_1,e_2\}$*. Define* $J : A \to P(S_1 \times S_2)$ *by*

$$\begin{aligned} J(e_1) &= \{(a,2),(a,3),(b,3),(c,2),(c,3),(d,2),(b,2),(d,3)\} \text{ and} \\ J(e_2) &= \{(a,2),(c,2),(d,2),(b,2)\}. \end{aligned}$$

Then (J,A) *is an* SC_mR *respecting to the aftersets from* S_1 *to* S_2.

$$\begin{aligned} aJ(e_2) &= \{2,3\},\ bJ(e_2) = \{2,3\},\ cJ(e_2) = \{2,3\} \text{ and } dJ(e_2) = \{2,3\}. \text{ Also,} \\ aJ(e_2) &= \{2\},\ bJ(e_2) = \{2\},\ cJ(e_2) = \{2\} \text{ and } dJ(e_2) = \{2\}. \end{aligned}$$

Define (G,A), *a soft set over* S_2 *by* $G(e_1) = \{2,3,4\}$ *and* $G(e_2) = \{2,4\}$ *which is not a* $SIIL$ *of* S_2 *as* $\{1,2,3,4\}\{2,3,4\}\{1,2,3,4\} = \{1,2,3,4\} \nsubseteq \{2,3,4\}$ *and* $\{1,2,3,4\}\{2,4\}\{1,2,3,4\} = \{1,2,3,4\} \nsubseteq \{2,4\}$ *but* $\underline{J}^{G}(e_1) = \{a,b,c,d\}$ *and* $\underline{J}^{G}(e_2) = \{a,b,c,d\}$, *which show that* $\left(\underline{J}^{G},A\right)$ *is a* $SIIL$ *of* S_1. *Hence,* (G,A) *is a generalized lower* $SIIL$ *of* S_1 *respecting to the aftersets.*

The proof of the next theorem is a routine verification.

Theorem 7.2.23 *Let (J, A) be an SC_mR respecting to the foresets from S_1 to S_2 (S_1 and S_2 are semigroups). If (L, A) is a $SIIL$ of S_1, then (L, A) is a generalized lower $SIIL$ of S_2 respecting to the foresets.*

Example 7.2.24 *Consider the semigroup of Example 7.1.7 and $A = \{e_1, e_2\}$. Define $J : A \to P(S_1 \times S_2)$ by*

$$\begin{aligned} J(e_1) &= \{(d,1),(d,2),(d,3),(d,4)\} \text{ and} \\ J(e_2) &= \{(a,1),(a,2),(a,3),(a,4),(d,1),(d,2),(d,3),(d,4)\}. \end{aligned}$$

Then (J, A) is an SC_mR respecting to the foresets from S_1 to S_2.

$$\begin{aligned} J(e_2)1 &= \{d\},\ J(e_2)2 = \{d\},\ J(e_2)3 = \{d\} \text{ and } J(e_2)4 = \{d\}. \text{ Also,} \\ J(e_2)1 &= \{a,d\},\ J(e_2)2 = \{a,d\},\ J(e_2)3 = \{a,d\} \text{ and } J(e_2)4 = \{a,d\}. \end{aligned}$$

Define (L, A), a soft set over S_1 by $L(e_1) = \{b, c, d\}$ and $L(e_2) = \{b, c, d\}$ which is not a $SIIL$ of S_1 as $\{a, b, c, d\}\{b, c, d\}\{a, b, c, d\} = \{a, b, c, d\} \nsubseteq \{b, c, d\}$ but ${}^L\underline{J}(e_1) = \{1, 2, 3, 4\}$ and ${}^L\underline{J}(e_2) = \{1, 2, 3, 4\}$ is an IIL of S_2. Hence, (L, A) is a generalized lower $SIIL$ of S_2 respecting to the foresets.

Now, we describe the approximations in $SBILs$ of semigroups.

Definition 7.2.25 *Let (J, A) be an $SBRE$ from S_1 to S_2 (S_1 and S_2 are semigroups) and (G, A) be a soft set over S_2. Then (G, A) is said to be generalized lower (upper) $SBIL$ of S_1 respecting to the aftersets if $(\underline{J}^G, A)$ $\left(\text{respectively } \left(\overline{J}^G, A\right)\right)$ is a $SBIL$ of S_1.*

Definition 7.2.26 *Let (J, A) be an $SBRE$ from S_1 to S_2 (S_1 and S_2 are semigroups) and (L, A) be a soft set over S_1. Then (L, A) is said to be a generalized lower (upper) $SBIL$ of S_2 respecting to the foresets if $({}^L\underline{J}, A)$ (respectively $({}^L\overline{J}, A)$) is a $SBIL$ of S_2.*

Theorem 7.2.27 *Let (J, A) be an $SCRE$ from S_1 to S_2 (S_1 and S_2 are semigroups). Then every $SBIL$ (G, A) of S_2 is a generalized upper $SBIL$ of S_1 respecting to the aftersets.*

Proof. Let (G,A) be a $SBIL$ of S_2. It follows from Theorem 7.2.3(1), $\left(\overline{J}^G, A\right)$ is a SSS of S_2. By Theorem 6.1.7 (3) and Theorem 7.1.1, $\overline{J}^G(e)\, S_1 \overline{J}^G(e) = \overline{J}^G(e) . \overline{J}^{S_2}(e) . \overline{J}^G(e) \subseteq \overline{J}^{GS_2G}(e) \subseteq \overline{J}^G(e)$. Hence, $\left(\overline{J}^G, A\right)$ is a $SBIL$ of S_1. Thus, (G,A) is a generalized upper $SBIL$ of S_1. ■

Example 7.2.28 *Consider the semigroups and soft relation of Example 7.2.4. Define* (G,A), *a soft set over* S_2 *by* $G(e_1) = \{1,2,3\}$ *and* $G(e_2) = \{1,2\}$ *which is not a* $SBIL$ *of* S_2 *as* $\{1,2,3\}\{1,2,3,4,5\}\{1,2,3\} = \{1,2,3,5\} \nsubseteq \{1,2,3\}$ *and* $\{1,2\}\{1,2,3,4,5\}\{1,2\} = \{1,2,5\} \nsubseteq \{1,2\}$ *but* $\overline{J}^G(e_1) = \{a,b,c,d\}$ *and* $\overline{J}^G(e_2) = \{a,b,d\}$, *which show that* $\left(\overline{J}^G, A\right)$ *is a* $SBIL$ *of* S_1. *Hence,* (G,A) *is a generalized upper* $SBIL$ *of* S_1 *respecting to the aftersets.*

The next theorem has a routine verification.

Theorem 7.2.29 *Let* (J,A) *be an* $SCRE$ *from* S_1 *to* S_2 *(*S_1 *and* S_2 *are semigroups). Then every* $SBIL$, (L,A) *of* S_1 *is a generalized upper* $SBIL$ *of* S_2 *respecting to the foresets.*

Example 7.2.30 *Consider the semigroups and soft relations of example 7.2.4. Define* (L,A), *a soft set over* S_1 *by* $L(e_1) = \{a,b,c\}$ *and* $L(e_2) = \{a,b\}$ *which is not a* $SBIL$ *of* S_1 *as* $\{a,b,c\}\{a,b,c,d,e\}\{a,b,c\} = \{b,c,d\} \nsubseteq \{a,b,c\}$ *and* $\{a,b\}\{a,b,c,d,e\}\{a,b\} = \{b,d\} \nsubseteq \{a,b\}$ *but* ${}^L\overline{J}(e_1) = \{1,5,3,2\}$ *and* ${}^L\overline{J}(e_2) = \{1,5,3,2\}$, *which show that* $\left({}^L\overline{J}, A\right)$ *is a* $SBIL$ *of* S_2. *Hence,* (L,A) *is a generalized upper* $SBIL$ *of* S_2 *respecting to the foresets.*

Theorem 7.2.31 *Let* (J,A) *be an* SC_mR *respecting to the aftersets from* S_1 *to* S_2 *(*S_1 *and* S_2 *are semigroups). Then every* $SBIL$, (G,A) *of* S_2 *is a generalized lower* $SBIL$ *of* S_1 *respecting to the aftersets.*

Proof. Let (G,A) be a $SBIL$ of S_2. It follows from Theorem 7.2.8(2), $\left(\underline{J}^G, A\right)$ is a SSS of S_2. By Theorem 3.1.11(2) and Theorem 7.1.1, $\underline{J}^G(e) . S_1 . \underline{J}^G(e) = \underline{J}^G(e) . \underline{J}^{S_2}(e) . \underline{J}^G(e) \subseteq \underline{J}^{GS_2G}(e) \subseteq \underline{J}^G(e)$. Hence, $\left(\underline{J}^G, A\right)$ is a $SBIL$ of S_1. Hence, (G,A) is a generalized lower $SBIL$ of S_1. ■

Example 7.2.32 *Consider the semigroup of Example 7.1.7 and* $A = \{e_1, e_2\}$. *Define*

$J : A \to P(S_1 \times S_2)$ by

$$J(e_1) = \{(a,2),(b,2),(c,2),(a,3),(c,3),(d,2),(b,3),(d,3)\} \text{ and}$$
$$J(e_2) = \{(d,2),(a,2),(b,2),(c,2)\}.$$

Then (J,A) is an SC_mR respecting to the aftersets from S_1 to S_2. Now,

$$aJ(e_1) = \{2,3\},\ bJ(e_1) = \{2,3\},\ cJ(e_1) = \{2,3\} \text{ and } dJ(e_1) = \{2,3\}. \text{ Also,}$$
$$aJ(e_2) = \{2\},\ bJ(e_2) = \{2\},\ cJ(e_2) = \{2\} \text{ and } dJ(e_2) = \{2\}.$$

Define (G,A), a soft set over S_2 by $G(e_1) = \{2,3,4\}$ and $G(e_2) = \{2,4\}$ which is not a $SBIL$ of S_2 as $\{2,3,4\}\{1,2,3,4\}\{2,3,4\} = \{1,2,3,4\} \nsubseteq \{2,3,4\}$ and $\{2,4\}\{1,2,3,4\}\{2,4\} = \{1,2,3,4\} \nsubseteq \{2,4\}$ but $\underline{J}^G(e_1) = \{a,c,b,d\}$ and $\underline{J}^G(e_2) = \{a,c,b,d\}$, which show that $\left(\underline{J}^G, A\right)$ is a $SBIL$ of S_1. Hence, (G,A) is a generalized lower $SBIL$ of S_1 respecting to the aftersets.

The next theorem has a routine verification.

Theorem 7.2.33 *Let (J,A) be an SC_mR respecting to the foresets from S_1 to S_2 (S_1 and S_2 are semigroups). Then every $SBIL$, (L,A) of S_1 is a generalized lower $SBIL$ of S_2 respecting to the foresets.*

Example 7.2.34 *Consider the semigroup of Example 7.1.7 and $A = \{e_1, e_2\}$. Define $J : A \to P(S_1 \times S_2)$ by*

$$J(e_1) = \{(d,1),(d,3),(d,4),(d,2)\} \text{ and}$$
$$J(e_2) = \{(a,1),(a,2),(a,4),(d,1),(a,3),(d,3),(d,2),(d,4)\}.$$

Then (J,A) is an SC_mR respecting to the foresets from S_1 to S_2.

$$J(e_2)1 = \{d\},\ J(e_2)2 = \{d\},\ J(e_2)3 = \{d\} \text{ and } J(e_2)4 = \{d\}. \text{ Also,}$$
$$J(e_2)1 = \{a,d\},\ J(e_2)2 = \{a,d\},\ J(e_2)3 = \{a,d\} \text{ and } J(e_2)4 = \{a,d\}.$$

Define (L,A), a soft set over S_1 by $L(e_1) = \{b,c,d\}$ and $L(e_2) = \{b,c,d\}$ which is not a $SBIL$ of S_1 as $\{b,c,d\}\{a,b,c,d\}\{b,c,d\} = \{a,b,c,d\} \nsubseteq \{b,c,d\}$ but ${}^L\underline{J}(e_1) = \{1,2,3,4\}$ and ${}^L\underline{J}(e_1) = \{1,2,3,4\}$, which show that $\left({}^L\underline{J}, A\right)$ is a $SBIL$ of S_2. Hence, (L,A) is a generalized upper $SBIL$ of S_2 respecting to the foresets.

Conclusion

This investigation was dedicated to the discourse of soft binary relations. Some fundamental ideas with respect to soft binary relations were proposed. All these ideas are fundamental supporting structures for innovative work on soft set theory. The novelty of this work is that how the number of parameters for a soft equivalence relation can be lessened to the minimum without aggravating its original classification ability. A conclusion can be drawn with minimum parameters by utilizing a soft equivalence relation. It has been talked about that parametric decrease can be made in this new setting. Another algorithm is exhibited for the parametric lessening proposed by a soft binary relation. Moreover, two kinds of fuzzy topologies and two kinds of soft topologies induced by soft reflexive relations are investigated. An approach to decision making problem is presented depending upon on a fuzzy set.

With the inspiration of concrete thoughts introduced, an investigation on the theoretical parts of these generalized ideas is more valuable and need more consideration. An expansion of this work is

- An endeavor can be made toward this path by concentrating on the theoretical establishment of these generalized ideas which are very valuable instruments.

- Additionally, investigation of the axiomatization of the approximation operators is a fascinating issue to be address.

- Besides and in this way, more positive arrangement in reality is acquired in basic decision making problems.

- We trust that within the near future, the idea of roughness using soft binary relations will be connected with other algebraic structures and it is our desire this work would fill in as a foundation for additional examination of the semigroup theory.

- The subject of this work might be stretched out to further results under different environments of soft binary relations.

- In future, we will use a soft tolerance relation to handle this concept in a different way engaging any algebraic structure.

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