On the Continuity of Some Integral Operators and Commutators

By

Amna Ajaib

Department of Mathemtics
Quaid-i-Azam University
Islamabad, Pakistan

2020

On the Continuity of Some Integral Operators and Commutators

By

Amna Ajaib

Supervised By

Dr. Amjad Hussain

Department of Mathemtics Quaid-i-Azam University Islamabad, Pakistan

2020

On the Continuity of Some Integral Operators and Commutators

By

Amna Ajaib

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT

OF THE REQUIREMENT FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

Supervised By

Dr. Amjad Hussain

Department of Mathemtics
Quaid-i-Azam University
Islamabad, Pakistan

2020

Abstract

In this thesis, we consider the problem of boundedness of different Hausdorff operators and their commutators on function spaces. However, from the point of view of applications and complexity, the major part of this thesis is devoted to prove the boundedness of commutator operators generated by Hausdorff operator with locally integrable functions on different function spaces. The function spaces include L^p function spaces, Morrey type function spaces, Herz type function spaces and Triebel Lizorkin type function spaces with the Euclidean space $\mathbb{R}^n$ or the Heisenberg group $\mathbb{H}^n$ as underlying spaces. We also consider the weighted boundedness of Hausdorff operator and commutators on these function spaces defined on $\mathbb{R}^n$ or $\mathbb{H}^n$. Since Hardy integral operators are special cases of Hausdorff operator, therefore, we also include some results regarding weighted boundedness of commutators of Hardy operators on Morrey type spaces. Almost all work has been published in well reputed mathematical science journals and is enlisted in the reference section of this thesis.

Contents

Preface

In this thesis, we will study integral operators including Hausdorff operator and Hardy-type operators on function spaces. This study require the use of some basic concepts of measure theory and integration like for example Lebesgue measure and Lebesgue integrals. We only incidently considered these topics and we will assume in this thesis that the reader is familiar with such basic concepts of measure theory and integration.

The first Chapter contains a brief history and recent development of Hausdorff operators. In this Chapter, starting from one dimensional Hausdorff operator, we recall the definitions of high dimensional matrix Hausdorff operators, rough Hausdorff operators and multilinear fractional Hausdorff operators. Several other integral operators which are special cases of Hausdorff operator are also come into discussion with in the content of this chapter.

In the second chapter, we first give Lipschitz estimates for the commutators of matrix Hausdorff operator on Lebesgue, Morrey, Herz and Triebel-Lizorkin space. Then we give central BMO-estimates for the same commutators on Herz-type spaces. Here, we give answer to an open question regarding Lebesgue space boundedness of the commutators of Hausdorff operator. The results of this chapter has been published in [71].

Keeping in view the importance of weighted theory of function spaces and the work on Hausdorff operators in the past, we developed new results on the boundedness of the matrix Hausdorff operator on weighted central Morrey space in Chapter 3. Employing some special conditions on the norm of the matrix and weight functions, sharp bounds for these results are also computed. Moreover, similar boundedness results for the commutators of the matrix Hausdorff operator are established as well. The results of this chapter has been published in [70].

In chapter four, our objective is to provide some weak type inequalities for fractional Hausdorff operator and related commutators. Since Hardy operators are special cases of fractional Hausdorff operators so, therefore, remaining of this chapter is devoted to obtain sufficient conditions for the boundedness of commutators of Hardy operator on weighted central Morrey spaces when the symbol functions belongs to weighted Lipschitz space.

Chapter five is about the boundedness of Hausdorff operator and commutators on weighted Herz space with the Heisenberg group as underlining space. The first part is about the boundedness of Hausdorff operator on Herz-type spaces while in the second part we established weighted $CBMO$ estimates for the commutator operators. The significance of our results lies in the fact that Herz-type spaces are used in the characterization of multiplier on the Hardy spaces and in the study of certain kind of partial differential equations (PDE's). Some of the results of this chapter has been published in [72].

Amna Ajaib
Islamabad, Pakistan
October 12, 2020

Acknowledgment

All the deepest sense of gratitude to Almighty Allah, the most Companionate and Merciful, Who is Omnipotent and Omnipresent, and has divulged His knowledge to man. May His peace and blessings be upon His prophet Muhammad (PBUH).

I am grateful to Quaid-I-Azam University where I got educational zenith and would like to express gratitude to my supervisor Dr. Amjad Hussain for his benevolent help and encouragement in my research work throughout my stay at this institute. I am really thankful to my father M. Ajaib Malik for his support, prayers and confidence without which I would not be able to achieve the success. I am also very thankful to my lovely friends Ms. Sana Chaudhary and Mr. Ali Awais. Here, I cannot forget my daughters Raafia and Rahma who beard the hardships with patience, without their cooperation the completion of this work would not have been possible. I would also like to thank all my friends and family for their support, encouragement and prayers as they always motivated me and kept me moving. With the people I have mentioned, I would never have gone this much far.

Chapter 1
Introduction

There are several methods to check the convergence of a number series. Among them the Cesàro and Hausdorff summability methods are of fundamental importance. A review paper by Liflyand [85] reveals that the origin of Hausdorff summability method can be traced back to Hurwitz and Silverman [68], who studied a family of methods within the classical framework of number series. However, after the seminal work by Hausdorff [60], who actually developed the summability methods by associating them with the moment problem for a finite interval, the topic was retaken by several authors including Garabedian [47], Hardy [59], Powell and Shah [108], Hille and Tamarkin [62] and Ustina [124].

The historic development of the Hausdorff operator from Hausdorff summability to its present forms is best described in the review articles [14, 87]. In summary, modern theory of Hausdorff operators comprises of two settings, complex analysis settings due to Siskakis [118, 119] and Fourier transform setting due to Georgakis [48] and more specifically due to Liflyand and Móricz [49, 90]. In this chapter, we shall briefly describe the theory of Hausdorff operators in the later settings starting from one-dimensional Hausdorff operators to multi-dimensional one.

1.1 The One Dimensional Hausdorff Operators

To study the continuity properties of Hausdorff operators on function spaces, an appropriate point for opening discussion is the one dimensional Hausdorff operator:

$$h_\Phi f(x) = \int_0^\infty f(\frac{x}{t})\frac{\Phi(t)}{t}dt, \qquad x \in \mathbb{R}, \tag{1.1.1}$$

where $\Phi \in L^1(\mathbb{R})$. Also, in the interest of convenience, it is usually assumed that functions f are initially in Schwartz class $\mathcal{S}(\mathbb{R})$. The operator firstly appeared in [90] in a thoughtful manner, where the authors studied its boundedness on the real Hardy spaces $H^1(\mathbb{R})$. Sharpness of the main result in [90] was shown in [32] which was then negated by the authors of [17] by improving the main theorem in [90]. The exact norm of h_Φ on $H^1(\mathbb{R})$ was computed in [66]. Later works by authors in [80, 89, 96] extended the results provided in [90] to the Hardy spaces $H^p(\mathbb{R}),\ 0 < p < 1$. In [92], it was shown that the Hausdorff operator has commuting relations with Hilbert transforms. Finally, the boundedness of the bivariate and multivariate versions of (1.1.1) on the product Hardy spaces were established in [91] and [67], respectively. Recently, in [11], a novel idea of proving the boundedness of h_Φ on the Hardy space $H^p(\mathbb{R})$ when $1 \geq p > 0$; was employed to extend some known boundedness results regarding Hausdorff operator on Hardy spaces. A change of variables in (1.1.1) results in an equivalent form of the Hausdorff operator [130]:

$$h_\Phi f(x) = \int_0^\infty \frac{\Phi(\frac{x}{t})}{t} f(t)dt, \qquad x \in \mathbb{R}, \tag{1.1.2}$$

of which the weighted boundedness in Lebesgue space with constant exponent and variable exponent is given in [1] and [5], respectively.

Recent interest in the study of Hausdorff operator is due to the fact that there are many popular and important operators in analysis which become special cases of h_Φ, if the function Φ is suitably chosen. Like, for example, the one dimensional Hardy operator:

$$hf(x) = x^{-1}\int_0^x f(t)dt, \qquad x > 0, \tag{1.1.3}$$

is obtained from h_Φ if one chooses $\Phi(t) = t^{-1}\chi_{(1,\infty)}(t)$ in (1.1.2). Alternatively, if $\Phi(t)$ is taken to be equal to $\chi_{(0,1)}(t)$ in (1.1.2) then we obtain adjoint Hardy operator given by:

$$h^* f(x) = \int_x^\infty t^{-1} f(t)dt. \tag{1.1.4}$$

Here it is worth mentioning that the Hardy-Littlewood-Pólya operator p which can be defined as:

$$pf(x) = h^* f(x) + hf(x),$$

which implies that the operator p is also a special case of Hausdorff operator. Another variant of Hausdorff operator is the weighted Hardy operator:

$$u_\psi f(x) = \int_0^1 \psi(t) f(tx)dt, \quad x \in \mathbb{R}, \tag{1.1.5}$$

which is obtained from (1.1.2) by selecting:

$$\Phi(t) = t^{-1}\psi(1/t)\chi_{(1,\infty)}(t),$$

where $\psi : [0,1] \to [0,\infty]$, is a measurable function. Here, we remark that the Cesàro operator is also a subcase of Hausdorff operator, see [80] for more details.

Kuang Jichang, in [82, 83], introduced a new generalization of Hausdorff operator in the form:

$$\mathcal{H}_{\Phi} f(x) = \int_0^{\infty} \frac{\Phi(t)}{t} f(g(t)x)dt, \tag{1.1.6}$$

where $g : (0,\infty) \to (0,\infty)$ is a monotonic function and $\Phi : (0,\infty) \to (0,\infty)$ is a locally integrable function. He obtained the condition on Φ which was necessary and sufficient for the continuity of $\mathcal{H}_{\Phi}$ on power weighted Herz-type spaces. Also, he showed that the operator (1.1.1) can be obtained from $\mathcal{H}_{\Phi}$ if one chooses $g(t) = 1/t$. In case $x \in \mathbb{R}$ and $g(t) = 1/t$, some q-inequalities for Hausdorff operator were obtained in [55]. Finally, our theory of one dimensional Hausdorff operator ends up with the inclusion of the paper [25] in which Daher and Saadi defined and studied the Dunki-Hausdorff operator on the real Hardy spaces $H^1_{\alpha}(\mathbb{R})$.

1.2 The Matrix Hausdorff Operators

In [84], Lerner and Liflyand gave the following extension of h_{Φ} to Euclidean space $\mathbb{R}^n$ for $n \geq 2$

$$H_{\Phi,A} f(x) = \int_{\mathbb{R}^n} f(xA(y))\Phi(y)dy, \tag{1.2.1}$$

where $A(y)$ is an $n \times n$ matrix satisfying non-singularity conditions almost everywhere in the support of a fixed integrable function Φ. Taking into consideration the duality of H^1 and BMO, the authors in [84] showed that $H_{\Phi,A}$ is bounded on Hardy spaces. Subsequently, similar boundedness of $H_{\Phi,A}$ was reconsidered in [86] using atomic decomposition of Hardy spaces. Also, results of [84, 86] on the $H^1(\mathbb{R}^n)$ boundedness of the Hausdorff operators are generalized to the case of locally compact groups in [102, 103]. Recently, Liflyand and Miyachi [88] extended these results on $H^p(\mathbb{R}^n)$ spaces with $0 < p < 1$. In [104], a spectral representation for multidimensional normal Hausdorff operator is given. Actually, before Lerner and Liflyand results, Brown and Móricz defined the multivariate Hausdorff operator $H(\mu, c, A)$ acting on Borel measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ by setting

$$H(\mu, c, A)f(x) = \int_{\mathbb{R}^n} f(xA(y))c(x)d\mu(y), \tag{1.2.2}$$

where μ is a σ-finite complex measure defined on the Borel measurable subsets of $\mathbb{R}^n$, $c : \mathbb{R}^n \to \mathbb{C}$ is a Borel measurable function which is nonzero μ-a.e., and $A := [a_{ij}]$

is a $n \times n$ matrix whose entries $a_{ij} : \mathbb{R}^n \to \mathbb{C}$ are Borel measurable functions and such that A is nonsingular μ-a.e. By defining the operator in this way they obtained the $L^p(\mathbb{R}^n)$ boundedness of $H(\mu, c, A)$. In fact, $H_{\Phi,A}f(x)$ can be considered as a special case of $H(\mu, c, A)$ when μ is absolutely continuous. It should be noted that, unlike $H_{\Phi,A}f(x)$, the boundedness of $H(\mu, c, A)$ on Hardy spaces cannot be obtained by using the duality of H^1 and BMO. However, when the matrix A is diagonal, the boundedness of $H(\mu, c, A)$ on Hardy and BMO spaces was shown in [131]. Latterly, Anderson [3] studied the boundedness properties of the operator:

$$H_\mu f(x) = \int_{\mathbb{R}} f(xy) d\mu(y), \ x \in \mathbb{R}^n, \tag{1.2.3}$$

and its formal adjoint operator:

$$H_\mu^* f(x) = \int_{\mathbb{R}} f(x/y)|t|^{-n} d\mu(y), \ x \in \mathbb{R}^n, \tag{1.2.4}$$

on $L^p(\mathbb{R}^n)$, $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$. These results were then extended to the Herz-type spaces in [44] and rearrangement-invariant function spaces in [64]. The above cited publications are important as their results are the first attempts to study the high dimensional Hausdorff operators on function spaces.

In 2012, Chen et al. [12] modified the form of (1.2.1) by replacing the kernel function $\Phi(y)$ with $|y|^{-n}\Phi(y)$:

$$H_{\Phi,A} f(x) = \int_{\mathbb{R}^n} f(xA(y)) \frac{\Phi(y)}{|y|^n} dy. \tag{1.2.5}$$

As a subcase, when $A(y) = \text{diag}[1/|\text{y}|, 1/|\text{y}|, ..., 1/|\text{y}|]$, they give another definition of n-dimensional Hausdorff operator:

$$H_\Phi f(x) = \int_{\mathbb{R}^n} f(\frac{x}{|y|}) \frac{\Phi(y)}{|y|^n} dy. \tag{1.2.6}$$

Their results include the boundedness of Hausdorff operators on Hardy spaces, local Hardy spaces, Herz and Herz-type Hardy spaces with a conclusion that these operators have better performance on Herz-type Hardy spaces than their performance on Hardy spaces. In the same year, with different co-authors, Chen et al. [16] extended the problem of boundedness of $H_{\Phi,A}$ to the product of Hardy type spaces. The boundedness results regarding Hausdorff operators on $H^1(\mathbb{R}^n)$ were improved in [19]. The continuity of (1.2.5) on Morrey spaces [9, 10], Hardy-Morrey spaces [26], rectangularly defined spaces [27], Block spaces [63], Triebel-Lizorkin-type spaces [93] and Campanato spaces [115, 135] has also been discussed in the recent past. Similarly, some results regarding the boundedness of H_Φ can be found in [65, 73, 139, 140, 141].

On the other hand, weighted norm inequalities for Hausdorff operators on function spaces have recently been reported in the literature which include boundedness of Hausdorff operator on on power weighted Hardy spaces [18, 112], weighted central Morrey space [70], weighted Herz-type Hardy spaces [114], weighted central Morrey and weighted Herz space on Heisenberg group [117, 116] and on weighted Lorentz spaces [121]. The two weighted inequalities for $H_{\Phi,A}$ on Herz-type Hardy spaces with power weights have also been obtained in [21].

In the same way, the study of commutators to integral operators is important as it has many applications in the theory of partial differential equations and in characterizing function spaces. An attempt has been made in [71] to discuss the continuity of commutator operator $H_{\Phi,A}$, defined by:

$$H^b_{\Phi,A}f(x) = \int_{\mathbb{R}^n} (b(x) - b(xA(y)))f(xA(y))\frac{\Phi(y)}{|y|^n}dy, \tag{1.2.7}$$

on function spaces when the symbol function b are either from Lipschitz space or central BMO space. However, when the matrix $A(y)$ is diagonal, we get the commutators of H_Φ which were studied in [69, 74] and [132]. Finally, the weighted norm inequalities for the operator defined in (1.2.7), when b belongs to weighted central mean oscillation space, were recently reported in [70].

A nice contribution from Russian mathematician to the theory of Hausdorff operator is the p-adic analogs of matrix Hausdorff operator given in [125]:

$$\mathcal{H}_{\varphi,A}f(x) = \int_{\mathbb{Q}_p^n} f(A(t)x)\varphi(t)d\mu(t), \tag{1.2.8}$$

where μ is the Haar measure on $\mathbb{Q}_p^n$. In the paper [125], the author gave a sufficient condition on φ for the boundedness of $\mathcal{H}_{\varphi,A}$ on p-adic Hardy and BMO spaces. The BMO-type estimates on different p-adic function spaces were established in [126] for the following Hausdorff operator:

$$\mathcal{H}_\varphi f(x) = \int_{B_0} \varphi(t)f(tx)d\mu(t), \quad x \in \mathbb{Q}_p^n, \tag{1.2.9}$$

with B_0 is the unit ball of $\mathbb{Q}_p^n$. The same author, in [127], obtained the sharp conditions on the size of φ such that the operator $\mathcal{H}_\varphi$ and it adjoint operator:

$$\mathcal{H}^*_\varphi f(x) = \int_{B_0} f(t^{-1}x)|t|_p^{-1}\varphi(t)d\mu(t), \quad x \in \mathbb{Q}_p^n, \tag{1.2.10}$$

are bounded on Morrey and Herz spaces.

The p-adic matrix Hausdorff operator was reconsidered in [128], where by imposing conditions on both the determinant and the norm of the matrix, its boundedness was shown on Hardy, BMO and Hölder spaces. Finally, by generalizing the result presented in [126, 127], the author in [128] proved the two-sided sharp estimate for the norm of Hausdorff operator on the Herz-type spaces. For Hausdorff operator of general type defined on $\mathbb{Q}_p^n$, Bandaliyev and Volosivets [6] gave sufficient conditions of its boundedness in weighted Lebesgue and grand Lebesgue spaces. Sharpness of some of these conditions was also established in the same paper. The p-adic analogues of (1.2.3) and (1.2.4) were discussed in [129]. In [75], the authors came up with the continuity of the Hausdorff operator defined on $\mathbb{Q}_p^n$ on the weighted p-adic Morrey and Herz type spaces with power weights. Also, by imposing some specific restrictions on the norm of the matrix A, they proved that these boundedness results are sharp.

The commutators of p-adic matrix Hausdorff operators were define and studied in [76]. Perhaps the last article we found, before submission of this thesis, on the boundedness of Hausdorff operator is [23] in which the authors studied the operator on variable exponent Morrey-Herz spaces.

1.3 The Rough Hausdorff Operators

Another important development made in [12] was the introduction of rough Hausdorff operator defined by

$$\widetilde{H}_{\Phi,\Omega}f(x) = \int_{\mathbb{R}^n} \Omega(y')f(y)\frac{\Phi(x/|y|)}{|y|^n}dy, \tag{1.3.1}$$

where Φ is a radial function defined on $\mathbb{R}^+$, and $\Omega(y')$ is an integrable function defined on the unit sphere $\mathbb{S}^{n-1}$. Here and in the sequel, if $\Omega = 1$, we denote $\widetilde{H}_{\Phi,1} = \widetilde{H}_\Phi$. Immediately after the appearance of [12], two different extensions of the operator $\widetilde{H}_\Phi$ to the multilinear case were made in [15] and their bounds on Lebesgue spaces and Herz spaces were established. Operator norm for the multilinear Hausdorff operator were computed in [37]. Meanwhile, the sharp constants on the product of Lebesgue spaces for the same operator were obtained in [134]. q-analysis of the results of [134] was made in [31]. The sharp Strong and weak type (p,p) estimates for linear and multilinear Hausdorff operator with the Heisenberg group as underlying spaces were established in [54]. The authors, in [41], gave some conditions on Φ which were sufficient for the continuity of three types of Hausdorff operators on the Lebesgue spaces with power weights.

In [113], the author studied the operator $\widetilde{H}_\Phi$ on power weighted Hardy spaces and found that it is bounded on $H^p_{|.|^\alpha}(\mathbb{R}^n)$ if the kernal function Φ is the Poisson function or the Guass function. The similar boundedness of $\widetilde{H}_\Phi$ on weighted Herz-type Hardy spaces, using their atomic decomposition, was proved in [143].

Keeping in view the importance of commutator operator, Hussain and Ahmed [69] defined the commutators of $\widetilde{H}_{\Phi,\Omega}$ by

$$\widetilde{H}^b_{\Phi,\Omega}f(x) = b(x)\widetilde{H}^b_{\Phi,\Omega}(f)(x) - \widetilde{H}_{\Phi,\Omega}(bf)(x). \tag{1.3.2}$$

Gao and Jia [39] studied the boundedness of $\widetilde{H}^b_\Phi = \widetilde{H}^b_{\Phi,1}$ in Lebesgue, Herz and Morrey-Herz spaces taking the symbol functions b either from central-BMO or Lipschitz spaces. Moreover, central BMO estimates for $\widetilde{H}^b_\Phi$ were studied in [74]. Weak type Lipschitz estimates for such commutators were established in [69]. In the same paper, using Marcinkiewicz interpolation theorem, it was shown that strong type Lipschitz estimates for $\widetilde{H}^b_{\Phi,\Omega}$ also hold. The boundedness of the rough type Hausdorff operator on weighted function spaces was established in [120]. Chuong et al., in [24], provided the sufficient and necessary conditions for the boundedness of $\widetilde{H}_{\Phi,\Omega}$ on weighted Herz-type and Morrey-type spaces. Moreover, they also established the weighted boundedness of $\widetilde{H}^b_{\Phi,\Omega}$ on these spaces. The study, in [22], undertook the weighted boundedness of multilinear p-adic rough-type Hausdorff operator on the product spaces and also established the boundedness for the commutators of same operators with symbols in central-BMO space.

1.4 The Fractional Hausdorff Operators

After the appearance of matrix and rough Hausdorff operator, it was natural to define and study the fractional type Hausdorff operator. The gap was filled in [96] by studying strong and weak type boundedness for the fractional Hausdorff operator:

$$H_{\Phi,\gamma}f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|y|^{n-\gamma}}\Phi(|x|/|y|)dy, \qquad 0 \le \gamma < n, \tag{1.4.1}$$

in Lebesgue spaces. Note that the kernel function Φ is taken radial while proving main results in [96], however, the case of general non-radial function Φ was subsequently considered in [30]. Necessary and sufficient conditions for the continuity of $H_{\Phi,\gamma}$ with non-radial function Φ on power weighted Lebesgue spaces were provided in [40]. Roughness in the fractional Hausdorff was defined and studied in [38]. The paper [43] establishes weak type estimates for fractional Hausdorff operator with a conclusion that it is bounded from Hardy spaces to weak Lebesgue spaces. Although, multilinear fractional Hausdorff operators were defined in [30] but their sharp constant on Lebesgue spaces were fixed in [133].

The problem of boundedness of fractional Hausdorff operator in Hardy spaces $H^p(\mathbb{R}^n)$ was initiated by Chen et al. [13]. Wu and Fan [136] continued the work of [13] in the setting of Homogenous groups and proved the continuity of the Hausdorff-Gauss and the Hausdorff-Poisson operators in Hardy spaces. Also, in the setting of Heisenberg group, characterization of some functions was made in [136] for which the fractional Hausdorff operator is bounded on BMO space, power weighted L^p spaces and Hardy spaces, respectively.

Similar to the definitions of commutators of matrix and rough Hausdorff operators, the commutators of $H_{\Phi,\gamma}$ cab be defined as:

$$H^b_{\Phi,\gamma}f(x) = \int_{\mathbb{R}^n} \frac{\Phi(|x|/|y|)}{|y|^{n-\gamma}}(b(x)-b(y))f(y)dy, \qquad 0 \leq \gamma < n. \tag{1.4.2}$$

Such commutators were defined and studied in [73, 110, 123].

1.5 Our Contribution to the Theory

We contributed to the theory of Hausdorff operators in many ways. Firstly, we defined the commutators of matrix Hausdorff operator for the first time and obtained its boundedness on Morrey, Herz and Triebel-Lizorkin spaces. Also, we gave answer to an open question regarding Lebesgue space estimates for the commutators of the matrix Hausdorff operators. Secondly, keeping in view the importance of weighted theory of function spaces we developed new results on the continuity of the matrix Hausdorff operators and their commutators on weighted central Morrey spaces. Also, by employing some specific conditions on the weight functions and on the norm of the matrix, sharp bounds for these results are also computed. Thirdly, we established some weak type inequalities for fractional Hausdorff operator and related commutators. Finally, we not only showed the boundedness of matrix operator on weighted type Herz space with the Heisenberg group as underlining space but also obtain similar results for its commutators.

1.5.1 References of Contribution

The contribution is cited in the reference list which include [70, 71, 72].

Chapter 2
Commutators of Matrix Hausdorff Operator on Function Spaces

2.1 Introduction

The study of boundedness properties of commutators operators on function spaces is considered an important problem in harmonic analysis. The study can be utilized for characterizing the function spaces, in the well posedness problems of solution to partial differential equations (PDEs) and in the regularity theory to special kind of PDEs. Therefore, the study of boundedness results for the commutators of Hausdorff operators is as important as the study of Hausdorff operators itself. In an exploratory research, one can find very few papers discussing the boundedness of commutators of various Hausdorff operators [39, 70, 73, 74, 132], except that of $H_{\Phi,A}$. Recently, in [69], we defined the commutators of matrix Hausdorff operator $H_{\Phi,A}$ and locally integrable function g as:

$$H^g_{\Phi,A}(f)(x) = g(x)H_{\Phi,A}(f)(x) - H_{\Phi,A}(gf)(x),$$

and constructed weighted estimates for it on central Morrey spaces, when $g \in C\dot{M}O(\mathbb{R}^n)$. Also, we raised an open question regarding $L^p(\mathbb{R}^n)$ boundedness of $H^g_{\Phi,A}$. Here, we give partially positive answer to this question by establishing $L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ estimates for the symbol functions belonging to the Lipschitz class of functions. However, in the case $g \in C\dot{M}O(\mathbb{R}^n)$, the question of $L^p(\mathbb{R}^n)$ boundedness of $H^g_{\Phi,A}$ still remains open.

In this chapter, our aim is to establish the Lipschitz estimates for $H^g_{\Phi,A}$ on Lebesgue, Morrey and Herz-type spaces and thus generalize some results presented in [43, 70]. In addition, when $g \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, we obtain $L^p \to \dot{F}^{\beta,\infty}_p$ boundedness for $H^g_{\Phi,A}$, where $\dot{F}^{\beta,\infty}_p$ is the homogeneous Triebel-Lizorkin space. Also, we estimate $H^g_{\Phi,A}$ on Herz-type spaces when $g \in C\dot{M}O(\mathbb{R}^n)$. The significance of our results lies in the fact that Herz-type spaces are used in the characterization of multiplier on the Hardy spaces [4] and in the study of certain kind of PDEs [100].

This chapter is organized as follows. The second section contains some basic definitions and notations likewise some necessary lemmas which will be used in the succeeding sections of this chapter. Lipschitz estimates for $H^g_{\Phi,A}$ are established in the third section. While results regarding central-BMO estimates of $H^g_{\Phi,A}$ on Herz-type spaces are stated and proved in the last section.

2.2 Some Definitions and Lemmas

In 1938, Morrey [105] studied second order parabolic and elliptic PDEs along with their local behavior and introduced a function space what is called Morrey space.

Definition 2.2.1 *Suppose* $1 \leq p < \infty$, $0 \leq \lambda \leq n$. *The Morrey space* $L^{p,\lambda}(\mathbb{R}^n)$ *is defined as:*

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{ g \in L^p_{\text{loc}}(\mathbb{R}^n) : \|g\|_{L^{p,\lambda}(\mathbb{R}^n)} < \infty \right\},$$

satisfying

$$\|g\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{r>0, x_0 \in \mathbb{R}^n} \left(\frac{1}{r^\lambda} \int_{Q(x_0,r)} |g(x)|^p dx \right)^{1/p},$$

where x_0 *is the center and* r *is the side length of the cube* $Q = Q(x_0, r)$ *along the coordinate axes.*

It is easy to see that $L^p(\mathbb{R}^n) = L^{p,0}(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. If $n < \lambda$, then we have $L^{p,\lambda}(\mathbb{R}^n) = 0$. Hence, we consider, in this paper, the case $0 < \lambda < n$.

For $k \in \mathbb{Z}$, we denote $C_k = B_k/B_{k-1}$ where $B_k := \{x \in \mathbb{R}^n : |x| < 2^k\}$. Then we will consider the following definition of homogeneous Herz space.

Definition 2.2.2 *([138, 109, 51]) Suppose* $0 < p, q < \infty$, $\alpha \in \mathbb{R}$. *The Herz space* $\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ *is the set:*

$$\dot{K}^{\alpha,p}_q(\mathbb{R}^n) := \left\{ g \in L^q_{\text{loc}}(\mathbb{R}^n/\{0\}) : \|g\|_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|g\|_{\dot{K}^{\alpha,p}_q(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|g\|^p_{L^q(C_k)} \right\}^{1/p}.$$

It is not difficult to verify that $L^p(\mathbb{R}^n) = \dot{K}_p^{0,p}(\mathbb{R}^n)$. Hence, Herz space is reduced to Lebesgue space $L^p(\mathbb{R}^n)$ when the indices attain some specific values.

Similarly the definition of homogenous Herz-Morrey space $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$ can be stated as below.

Definition 2.2.3 *Suppose* $0 < p, q < \infty, \alpha \in \mathbb{R}, 0 \leq \lambda$. *The Herz-Morrey space* $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)$ *is the set:*

$$M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n) := \left\{ g \in L^q_{\text{loc}}(\mathbb{R}^n/\{0\}) : \|g\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|g\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|g\|^p_{L^q(C_k)} \right\}^{1/p}.$$

Obviously, $M\dot{K}_{p,q}^{\alpha,0}(\mathbb{R}^n) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $L^{q,\lambda}(\mathbb{R}^n) \subset M\dot{K}_{q,q}^{0,\lambda}(\mathbb{R}^n)$.

For the continuity of commutator operators in function spaces having central nature, one usually looks for a corresponding function class to which the symbol function b belongs and which has BMO-type behavior at the origin. Having such a property an appropriate function space is the homogeneous central mean oscillation space $C\dot{M}O^q(\mathbb{R}^n)$ defined below.

Definition 2.2.4 *([2]) Suppose* $\infty > q > 1$. *A function* $g \in L^q_{loc}(\mathbb{R}^n)$ *is said to belongs to the central-BMO space* $C\dot{M}O^q(\mathbb{R}^n)$ *if*

$$\|g\|_{C\dot{M}O^q(\mathbb{R}^n)} = \sup_{R>0} \left(\frac{1}{|B(0,R)|} \int_{B(0,R)} |g(x) - g_{B(0,R)}|^q dx \right)^{1/q} < \infty,$$

where $|B(0,R)|$ *is the measure of* $B(0,R)$ *and* $g_B = \frac{1}{|B|} \int_B g(x)dx$ *is the average of* g *over* B.

For detailed study of $C\dot{M}O^q(\mathbb{R}^n)$ space we refer the interested reader to [2, 99].

Obviously, $BMO(\mathbb{R}^n) \subset C\dot{M}O^q(\mathbb{R}^n)$ for $1 \leq q < \infty$. However, the two spaces differ in their properties. For example $C\dot{M}O^q(\mathbb{R}^n)$ depends on q and $C\dot{M}O^q(\mathbb{R}^n) \subset C\dot{M}O^p(\mathbb{R}^n)$, $1 \leq p < q < \infty$. Therefore, there is no analogy of John-Nirenberg inequality of $BMO(\mathbb{R}^n)$ for $C\dot{M}O^q(\mathbb{R}^n)$. The function space $BMO(\mathbb{R}^n)$ is the mean oscillation function space satisfying the following norm condition:

$$\|g\|_{BMO(\mathbb{R}^n)} = \sup_B |B|^{-1} \int_B |g(x) - g_B| dx.$$

Definition 2.2.5 *([107]) Suppose $\beta \in (0,1)$. The space with the norm condition:*

$$\|g\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} = \sup_{x,h\in\mathbb{R}^n} \frac{|g(x+h)-g(x)|}{|h|^\beta} < \infty.$$

is called Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$.

Next, for a non-singular matrix E, we consider the following definition of norm

$$\|E\| = \sup_{x\in\mathbb{R}^n, x\neq 0} \frac{|Ex|}{|x|}, \tag{2.2.1}$$

which implies that

$$\|E\|^{-n} \leq |\det(E^{-1})| \leq \|E^{-1}\|^n. \tag{2.2.2}$$

Finally, the fractional maximal function M_β of order β with $0 \leq \beta < n$ have the form:

$$M_\beta g(x) = \sup_{Q\ni x} |Q|^{\frac{\beta}{n}-1} \int_Q |g(y)|dy,$$

where the supremum is taken over all cubes Q containing x. Notice that when $\beta = 0$, we obtain the usual Hardy Littlewood maximal function $M = M_0$.

This finishes the streak of definitions concerning function spaces, norm of a matrix and maximal function. We are now in position to state some lemmas which will be helpful in proving main results.

Lemma 2.2.6 *([107]) Suppose $0 < \beta < 1$ and $f \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, and let Q be any cube in $\mathbb{R}^n$, then*

$$\sup_{x\in Q} |f_Q - f(x)| \leq C|Q|^{\frac{\beta}{n}} \|f\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)},$$

where $|Q|$ denotes the measure of the cube Q and $f_Q = \frac{1}{|Q|}\int_Q f(x)dx$.

Lemma 2.2.7 *Suppose $0 < \beta < 1$ and the symbol function b is in $\dot{\Lambda}_\beta(\mathbb{R}^n)$, then*

$$\begin{aligned} &M(H^b_{\Phi,A} f)(x) \\ &\leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \max\{1, |\det A^{-1}(y)|^{\beta/n}\}(1+\|A(y)\|^\beta) M_\beta(f)(A(y)x)dy. \end{aligned}$$

Proof: Let us consider a cube $Q \subset \mathbb{R}^n$ in such a way that $x \in Q$ and

$$\begin{aligned}\frac{1}{|Q|}\int_Q |H^b_{\Phi,A}f(z)|dz =& \frac{1}{|Q|}\int_Q\left|\int_{\mathbb{R}^n}\frac{\Phi(y)}{|y|^n}(b(z)-b(A(y)z))f(A(y)z)dy\right|dz\\ &\leq \frac{1}{|Q|}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\int_Q |(b(z)-b(A(y)z))f(A(y)z)|\,dzdy\\ &\leq \frac{1}{|Q|}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\int_Q |(b(z)-b_Q)f(A(y)z)|\,dzdy\\ &+\frac{1}{|Q|}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\int_Q \left|(b_Q-b_{A(y)Q})f(A(y)z)\right|dzdy\\ &+\frac{1}{|Q|}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\int_Q \left|(b(A(y)z)-b_{A(y)Q})f(A(y)z)\right|dzdy\\ &= I_1+I_2+I_3.\end{aligned}$$

In the approximation of I_1, we use Lemma 2.2.6 to obtain

$$\begin{aligned}I_1 &\leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\left(\frac{1}{|Q|^{1-\beta/n}}\int_Q |f(A(y)z)|dz\right)dy\\ &\leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\left(\frac{1}{|A(y)Q|^{1-\beta/n}}\int_{A(y)Q}|f(z)|dz\right)|\det A^{-1}(y)|^{\beta/n}dy\\ &\leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{\beta/n}M_\beta(f)(A(y)x)dy.\end{aligned}$$

In order to estimate I_2, we have to approximate $|b_Q - b_{A(y)Q}|$. For $0<\beta<1$, we obtain

$$\begin{aligned}|b_Q-b_{A(y)Q}| &\leq \frac{1}{|Q|}\int_Q |b(z)-b_{A(y)Q}|dz\\ &\leq \frac{1}{|Q|}\frac{1}{|A(y)Q|}\int_Q\int_{A(y)Q}|b(z)-b(t)|dtdz\\ &\leq \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\left(\frac{1}{|Q|}\int_Q |x|^\beta dz+\frac{1}{|A(y)Q|}\int_{A(y)Q}|t|^\beta dt\right)\\ &\leq C|Q|^{\beta/n}\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}(1+\|A(y)\|^\beta)\end{aligned}$$

Therefore,

$$\begin{aligned}I_2 &\leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}(1+\|A(y)\|^\beta)\left(\frac{1}{|Q|^{1-\beta/n}}\int_Q |f(xA(y))|dz\right)dy\\ &\leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}(1+\|A(y)\|^\beta)|\det A^{-1}(y)|^{\beta/n}M_\beta(f)(A(y)x)dy.\end{aligned}$$

It now turns to estimate I_3. By virtue of Lemma 2.2.6, the estimation of I_3 reduces to

$$\begin{aligned}
I_3 &= \frac{1}{|Q|}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\int_Q \left|(b(A(y)z)-b_{A(y)Q})f(A(y)z)\right| dzdy \\
&= \frac{1}{|Q|}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A(y)|^{-1}\int_{A(y)Q}|b(z)-b_{A(y)Q}||f(z)|dzdy \\
&\le C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\left(\frac{1}{|A(y)Q|^{1-\beta/n}}\int_{A(y)Q}|f(z)|dz\right)dy \\
&\le C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}M_\beta(f)(A(y)x)dy.
\end{aligned}$$

We combine the estimates of I_i $(i=1,2,3)$, to have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |H^b_{\Phi,A}f(z)|dz \\
&\le C\|b\|_{\dot{\Lambda}(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{\Phi(y)}{|y|^n}\max\{1,|\det A^{-1}(y)|^{\beta/n}\}(1+\|A(y)\|^\beta)M_\beta(f)(A(y)x).
\end{aligned}$$

Finally, if one takes the supremum over cubes Q in such a way that $x\in Q$, then the result become obvious.

Lemma 2.2.8 *([8]) Suppose $\beta\in(0,1)$, $1<p<n/\beta$, $n-\beta p>\lambda>0$ and $1/p-\beta/(n-\lambda)=1/q$. Then M_β is bounded operator from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.*

Lemma 2.2.9 *([101]) Let $0<p_1\le\infty, p_1\le p_2\le\infty$, $1<q_1<n/\beta$ and $\beta/n=1/q_1-1/q_2$ with $\beta\in(0,1)$. If $n-n/q_1>\alpha>-n/q_1+\beta$, then*

$$\|M_\beta f\|_{\dot{K}^{\alpha,p_2}_{q_2}(\mathbb{R}^n)}\le C\|f\|_{\dot{K}^{\alpha,p_1}_{q_1}(\mathbb{R}^n)}.$$

Lemma 2.2.10 *([53]) Let $0<p_1\le\infty, p_1\le p_2\le\infty, 1<q_1<n/\beta$ and $\beta/n=1/q_1-1/q_2$ with $\beta\in(0,1)$. For $\lambda>0$, if $n-n/q_1>\alpha-\lambda>-n/q_1+\beta$, then*

$$\|M_\beta f\|_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2}(\mathbb{R}^n)}\le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1}(\mathbb{R}^n)}.$$

Lemma 2.2.11 *([107]) Let $\infty>p>1>\beta>0$, then*

$$\|g\|_{F^{\beta,\infty}_p(\mathbb{R}^n)}\approx\left\|\sup_{Q\ni x}|Q|^{-(1+\beta/n)}\int_Q|g(z)-g_Q|dz\right\|_{L^p(\mathbb{R}^n)}.$$

2.3 Lipschitz Estimates for $H^b_{\Phi,A}$ on Function Spaces

As we stated in the introduction, this section is centered on obtaining estimates for $H^b_{\Phi,A}$ on function spaces. In this regard, our main results are as below.

2.3.1 Main Results

Theorem 2.3.1 *Let $\beta \in (0,1)$, $n-\beta p > \lambda > 0$, $n/\beta > p > 1$, and $1/p-\beta/(n-\lambda) = 1/q$. If $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then*

$$\|H^b_{\Phi,A} f\|_{L^{q,\lambda}(\mathbb{R}^n)} \leq CK_1 \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|f\|_{L^{p,\lambda}(\mathbb{R}^n)},$$

where K_1 is

$$\int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \max\{|\det A^{-1}(y)|^{1/q-\lambda/q}, |\det A^{-1}(y)|^{\beta/n+1/q-\lambda/q}\}(1+\|A(y)\|^\beta)dy.$$

Theorem 2.3.2 *Let $\beta \in (0,1)$, $n/\beta > p > 1$, and $1/p-\beta/n = 1/q$. If $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then*

$$\|H^b_{\Phi,A} f\|_{L^{q}(\mathbb{R}^n)} \leq CK_2 \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|f\|_{L^{p}(\mathbb{R}^n)},$$

where K_2 is

$$\int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \max\{|\det A^{-1}(y)|^{1/q}, |\det A^{-1}(y)|^{1/p}\}(1+\|A(y)\|^\beta)dy,$$

Theorem 2.3.3 *Suppose $0 < p_1 \leq \infty, p_1 \leq p_2 \leq \infty, 1 < q_1 < n/\beta$ and $\beta/n = 1/q_1 - 1/q_2$ with $\beta \in (0,1)$. For $\lambda > 0$, if $n-n/q_1 > \alpha-\lambda > -n/q_1+\beta$, and $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then*

$$\|H^b_{\Phi,A} f\|_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2}(\mathbb{R}^n)} \leq CK_3 \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1}(\mathbb{R}^n)},$$

where K_3 is

$$\int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \max\{|\det A^{-1}(y)|^{1/q_2}, |\det A^{-1}(y)|^{1/q_1}\}(1+\|A(y)\|^\beta)G_{\alpha,\lambda}(y)dy,$$

and

$$G_{\alpha,\lambda}(y) = \begin{cases} 1+\log_2(\|A(y)\|\|A^{-1}(y)\|), & \alpha=\lambda, \\ \|A^{-1}(y)\|^{\alpha-\lambda}, & \alpha>\lambda, \\ \|A(y)\|^{\lambda-\alpha}, & \alpha<\lambda. \end{cases} \tag{2.3.1}$$

Theorem 2.3.4 *Suppose $0 < p_1 \leq \infty, p_1 \leq p_2 \leq \infty$, $1 < q_1 < n/\beta$ and $\beta/n = 1/q_1 - 1/q_2$ with $\beta \in (0,1)$. If $n-n/q_1 > \alpha > -n/q_1+\beta$, and $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$, then*

$$\|H^b_{\Phi,A} f\|_{\dot{K}^{\alpha,p_2}_{q_2}(\mathbb{R}^n)} \leq CK_4 \|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)} \|f\|_{\dot{K}^{\alpha,p_1}_{q_1}(\mathbb{R}^n)},$$

where K_4 *is*

$$\int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \max\{|\det A^{-1}(y)|^{1/q_2}, |\det A^{-1}(y)|^{1/q_1}\}(1+\|A(y)\|^{\beta})\widetilde{G}_{\alpha}(y)dy,$$

and

$$\widetilde{G}_{\alpha}(y) = \begin{cases} 1+\log_2(\|A(y)\|\|A^{-1}(y)\|), & \alpha = 0, \\ \|A^{-1}(y)\|^{\alpha}, & \alpha > 0, \\ \|A(y)\|^{-\alpha}, & \alpha < 0. \end{cases} \tag{2.3.2}$$

Theorem 2.3.5 *Let* $\beta \in (0,1)$, $p \in (1,\infty)$ *and* $b \in \dot{\Lambda}_{\beta}(\mathbb{R}^n)$, *then*

$$\|H^b_{\Phi,A} f\|_{F^{\beta,\infty}_p(\mathbb{R}^n)} \leq CK_5\|b\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)},$$

where K_5 *is*

$$\int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} \max\{|\det A(y)|^{-1/p}, |\det A(y)|^{1+\beta/n-1/p}\}(1+\|A(y)\|^{\beta})dy.$$

2.3.2 Proof of the Main Results

A frequent use of Lemma 2.2.7 in proving our main results for this section force us to use the following notation

$$\phi(y) = \frac{|\Phi(y)|}{|y|^n} \max\{1, |\det A^{-1}(y)|^{\beta/n}\}(1+\|A(y)\|^{\beta}),$$

for our convenience.

Proof of Theorem 2.3.1: In view of Lemma 2.2.7 and the Minkowski inequality, we have

$$\|M(H^b_{\Phi,A} f)(\cdot)\|_{L^{q,\lambda}(\mathbb{R}^n)} \leq C\|b\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \phi(y)\|M_{\beta}(f)(A(y)\cdot)\|_{L^{q,\lambda}(\mathbb{R}^n)}dy.$$

Using the scaling argument and the fact that $|H^b_{\Phi,A} f(x)| \leq M(H^b_{\Phi,A} f)(x)$ *a.e.*, we get

$$\|H^b_{\Phi,A} f\|_{L^{q,\lambda}(\mathbb{R}^n)} \leq C\|b\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)}\|M_{\beta} f\|_{L^{p,\lambda}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \phi(y)|\det A^{-1}(y)|^{1/q-\lambda/q}dy.$$

Lastly, an application of Lemma 2.2.8 leads to the desired result.

Proof of Theorem 2.3.2: Following the similar procedure as followed in proving Theorem 2.3.1, the proof of this Theorem can be easily obtained.

Proof of Theorem 2.3.3: Making use of Lemma 2.2.7 and the Minkowski inequality, one has

$$
\begin{aligned}
&\|M(H^b_{\Phi,A}f)(\cdot)\|_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2}(\mathbb{R}^n)}\\
&\leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\phi(y)\|M_\beta f(A(y)\cdot)\|_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2}(\mathbb{R}^n)}dy\\
&= C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\phi(y)\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}2^{k\alpha p}\|M_\beta f(A(y)\cdot)\|^p_{L^q(C_k)}\right\}^{1/p}dy\\
&= C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\phi(y)|\det A^{-1}(y)|^{1/q_2}\\
&\qquad\times\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}2^{k\alpha p}\|M_\beta f(\cdot)\|^p_{L^q(A(y)C_k)}\right\}^{1/p}dy. \qquad (2.3.3)
\end{aligned}
$$

In order to estimate $\|M_\beta f(\cdot)\|_{L^q(A(y)C_k)}$, we follow the method used in [117]. Thus, by definition of C_k and (5.4.15), we can write

$$A(y)C_k\subset\{x:\|A^{-1}(y)\|^{-1}2^{k-1}<|x|<\|A(y)\|2^k\}.$$

Next, for any y in the support of Φ there exist an integer l such that

$$2^l<\|A^{-1}(y)\|^{-1}<2^{l+1}. \qquad (2.3.4)$$

Furthermore, the relation $\|A^{-1}(y)\|^{-1}\leq\|A(y)\|$ implies the existence of non-negative integer m such that

$$2^{l+m}<\|A(y)\|<2^{l+m+1}. \qquad (2.3.5)$$

Inequalities (5.4.18) and (5.4.19) define the bounds for m, that is

$$\log_2(\|A(y)\|\|A^{-1}(y)\|/2)<m<\log_2(2\|A(y\|\|A^{-1}(y))\|),$$

and lead us to have

$$A(y)C_k\subset\{x:2^{l+k-1}<|x|<2^{k+l+m+1}\}.$$

Hence,

$$\|M_\beta f(\cdot)\|_{L^{q_2}(A(y)C_k)}\leq\sum_{j=l}^{l+m+1}\|M_\beta f(\cdot)\|_{L^{q_2}(C_{k+j})}. \qquad (2.3.6)$$

Incorporating the inequality (5.4.20) into (2.3.3), we obtain

$$
\begin{aligned}
&\|M(H^b_{\Phi,A}f)(\cdot)\|_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2}(\mathbb{R}^n)}\\
&\leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\phi(y)|\det A^{-1}(y)|^{1/q_2}\\
&\quad\times\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{j=l}^{l+m+1}2^{k\alpha}\|M_\beta f(\cdot)\|_{L^{q_2}(C_{k+j})}\right)^p\right\}^{1/p}dy\\
&\leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\phi(y)|\det A^{-1}(y)|^{1/q_2}\\
&\quad\times\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\sum_{j=l}^{l+m+1}2^{-j\alpha}\left\{\sum_{k=-\infty}^{k_0+j}2^{k\alpha p}\|M_\beta f(\cdot)\|^p_{L^{q_2}(C_k)}\right\}^{1/p}dy\\
&\leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\|M_\beta f\|_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2}(\mathbb{R}^n)}\int_{\mathbb{R}^n}\phi(y)|\det A^{-1}(y)|^{1/q_2}\sum_{j=l}^{l+m+1}2^{j(\lambda-\alpha)}dy. \qquad (2.3.7)
\end{aligned}
$$

Now, for $\alpha=\lambda$, we have

$$
\sum_{j=l}^{l+m+1}2^{j(\lambda-\alpha)}=m+2\leq C(1+\log_2(\|A(y)\|\|A^{-1}(y)\|)), \qquad (2.3.8)
$$

and otherwise

$$
\begin{aligned}
\sum_{j=l}^{l+m+1}2^{j(\lambda-\alpha)}&=2^{l(\lambda-\alpha)}\frac{1-2^{(\lambda-\alpha)(m+2)}}{1-2^{(\lambda-\alpha)}}\\
&\leq C\begin{cases}\|A^{-1}(y)\|^{\alpha-\lambda}, & \alpha>\lambda,\\ \|A(y)\|^{\lambda-\alpha}, & \alpha<\lambda.\end{cases} \qquad (2.3.9)
\end{aligned}
$$

Hence, (2.3.7), (2.3.8) and (2.3.9) together yield

$$
\begin{aligned}
&\|M(H^b_{\Phi,A}f)(\cdot)\|_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2}(\mathbb{R}^n)}\\
&\leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\|M_\beta f\|_{M\dot{K}^{\alpha,\lambda}_{p_2,q_2}(\mathbb{R}^n)}\int_{\mathbb{R}^n}\phi(y)|\det A^{-1}(y)|^{1/q_2}G_{\alpha,\lambda}(y)dy.
\end{aligned}
$$

Making use of the fact that $|H^b_{\Phi,A}f(x)|\leq M(H^b_{\Phi,A}f)(x)$ *a.e.* and Lemma 2.2.10 we get the desired result.

Proof of Theorem 2.3.4: An argument similar to one used in proving Theorem 2.3.3, results in

$$
\begin{aligned}
&\|M(H^b_{\Phi,A}f)(\cdot)\|_{\dot{K}^{\alpha,p_2}_{q_2}(\mathbb{R}^n)} \\
&\leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\|M_\beta f\|_{\dot{K}^{\alpha,p_2}_{q_2}(\mathbb{R}^n)}\int_{\mathbb{R}^n}\phi(y)|\det A^{-1}(y)|^{1/q_2}\widetilde{G}_\alpha(y)dy.
\end{aligned}
$$

However, in contrast with Theorem 2.3.3, here we use Lemma 2.2.9 to fulfill the assertion made in the statement of this Theorem.

Proof of Theorem 2.3.5: For $x \in Q$, we have

$$
\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q |H^b_{\Phi,A}f(z) - (H^b_{\Phi,A})_Q|dz \\
&\leq \frac{2}{|Q|^{1+\beta/n}}\int_Q |H^b_{\Phi,A}f(z)|dz \\
&\leq \frac{1}{|Q|^{1+\beta/n}}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\int_Q |(b(z)-b_Q)f(A(y)z)|\,dzdy \\
&+ \frac{1}{|Q|^{1+\beta/n}}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\int_Q \left|(b_Q - b_{A(y)Q})f(A(y)z)\right|dzdy \\
&+ \frac{1}{|Q|^{1+\beta/n}}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\int_Q \left|(b(A(y)z) - b_{A(y)Q})f(A(y)z)\right|dzdy \\
&=: J_1 + J_2 + J_3.
\end{aligned}
\tag{2.3.10}
$$

Comparing J_i $(i = 1, 2, 3)$ with I_i $(i = 1, 2, 3)$ estimated in proving Lemma 2.2.7, one can easily estimate J_1, J_2 and J_3 by adjusting the factor $|Q|^{\beta/n}$. Hence, we have

$$
J_1 \leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}M(f)(A(y)x)dy.
$$

$$
J_2 \leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}(1+\|A(y)\|^\beta)M(f)(A(y)x)dy.
$$

$$
J_3 \leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A(y)|^{1+\beta/n}M(f)(A(y)x)dy.
$$

In view of these estimates, inequality (2.3.10) assumes the following form

$$
\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}}\int_Q |H^b_{\Phi,A}f(z) - (H^b_{\Phi,A})_Q|dz \\
&\leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}(1+\|A(y)\|^\beta)\max\{1, |\det A(y)|^{1+\beta/n}\}M(f)(A(y)x)dy.
\end{aligned}
$$

Applying $L^p(\mathbb{R}^n)$ norm on both sides and using Lemma 2.2.11, we obtain

$$\begin{aligned}&\|H^b_{\Phi,A}f\|_{\dot{F}^{\beta,\infty}_p(\mathbb{R}^n)} \leq C\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\\&\quad\times\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}(1+\|A(y)\|^\beta)\max\{1,|\det A(y)|^{1+\beta/n}\}\|Mf(A(y)\cdot)\|_{L^p(\mathbb{R}^n)}dy.\end{aligned}$$

Finally by scaling argument and boundedness of M on $L^p(\mathbb{R}^n)$, (see [49]) we have

$$\|H^b_{\Phi,A}f\|_{F^{\beta,\infty}_p(\mathbb{R}^n)} \leq CK_5\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)},$$

which is as required.

2.4 Central-BMO Estimates for $H^b_{\Phi,A}$ on Herz-type Space

2.4.1 Main Results

Theorem 2.4.1 *Let $1 < p, q_1, q_2 < \infty, 1/q_2 = 1/q + 1/q_1, \lambda > 0$ and $\alpha_2 \in \mathbb{R}$. If $\alpha_1 = n/q + \alpha_2$ and $b \in C\dot{M}O^q(\mathbb{R}^n)$, then*

$$\|H^b_{\Phi,A}f\|_{M\dot{K}^{\alpha_2,\lambda}_{p,q_2}(\mathbb{R}^n)} \leq CK_6\|b\|_{C\dot{M}O^q(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{R}^n)},$$

where

$$\begin{aligned}K_6 = &\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}G_{\alpha_1,\lambda}(y)\\&\times\left(\log\frac{1}{\|A(y\|}\chi_{\{\|A(y)\|<1\}}+\log\|A(y\|\|\chi_{\{\|A(y)\|\geq 1\}}+\frac{\|A(y)\|^n}{|\det A(y)|}\right)dy,\end{aligned}$$

and $G_{\alpha_1,\lambda}(y)$ is the same function as given in (2.3.1) with α is replaced by α_1.

Theorem 2.4.2 *Let $1 < p, q_1, q_2 < \infty, 1/q_2 = 1/q+1/q_1$ and $\alpha_2 \in \mathbb{R}$. If $\alpha_1 = n/q+\alpha_2$ and $b \in C\dot{M}O^q(\mathbb{R}^n)$, then*

$$\|H^b_{\Phi,A}f\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{R}^n)} \leq CK_7\|b\|_{C\dot{M}O^q(\mathbb{R}^n)}\|f\|_{\dot{K}^{\alpha_1,p}_{q_1}(\mathbb{R}^n)},$$

where

$$\begin{aligned}K_7 = &\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\widetilde{G}_{\alpha_1}(y)\\&\times\left(\log\frac{1}{\|A(y\|}\chi_{\{\|A(y)\|<1\}}+\log\|A(y\|\|\chi_{\{\|A(y)\|\geq 1\}}+\frac{\|A(y)\|^n}{|\det A(y)|}\right)dy,\end{aligned}$$

and $\widetilde{G}_{\alpha_1}(y)$ is the same function as given in (2.3.2) with α is replaced by α_1.

2.4.2 Proof of Main Results

Proof of Theorem 2.4.1. Here, we decompose $\|H^b_{\Phi,A}f\|_{L^{q_2}(C_k)}$ as:

$$\begin{aligned}
\|H^b_{\Phi,A}f\|_{L^{q_2}(C_k)} &= \left(\int_{C_k}\left|\int_{\mathbb{R}^n}\frac{\Phi(y)}{|y|^n}(b(x)-b(A(y)x))f(A(y)x)dy\right|^{q_2}dx\right)^{1/q_2}\\
&\leq \int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\left(\int_{C_k}|(b(x)-b(A(y)x))f(A(y)x)|^{q_2}\,dx\right)^{1/q_2}dy\\
&\leq \int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\left(\int_{C_k}|(b(x)-b_{B_k})f(A(y)x)|^{q_2}\,dx\right)^{1/q_2}dy\\
&+ \int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\left(\int_{C_k}\left|(b_{B_k}-b_{A(y)B_k})f(A(y)x)\right|^{q_2}dx\right)^{1/q_2}dy\\
&+ \int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\left(\int_{C_k}\left|(b(A(y)x)-b_{A(y)B_k})f(A(y)x)\right|^{q_2}dx\right)^{1/q_2}dy\\
&= L_1+L_2+L_3,
\end{aligned}$$

By Hölder inequality and change of variables it is simple to have

$$\begin{aligned}
L_1 &\leq \int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\left(\int_{C_k}|b(x)-b_{B_k}|^q\,dx\right)^{1/q}\left(\int_{C_k}|f(A(y)x)|^{q_1}\,dx\right)^{1/q_1}dy\\
&\leq |B_k|^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\|f\|_{L^{q_1}(A(y)C_k)}dy.
\end{aligned}$$

In order to estimate L_2, we rewrite it as

$$\begin{aligned}
L_2 &= \int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_2}\|f\|_{L^{q_2}(A(y)C_k)}\left|b_{B_k}-b_{A(y)B_k}\right|dy\\
&\leq \int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_2}|A(y)B_k|^{1/q}\|f\|_{L^{q_1}(A(y)C_k)}\left|b_{B_k}-b_{A(y)B_k}\right|dy\\
&= |B_k|^{1/q}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\|f\|_{L^{q_1}(A(y)C_k)}\left|b_{B_k}-b_{A(y)B_k}\right|dy\\
&= |B_k|^{1/q}\int_{\|A(y)\|<1}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\|f\|_{L^{q_1}(A(y)C_k)}\left|b_{B_k}-b_{A(y)B_k}\right|dy\\
&\quad + |B_k|^{1/q}\int_{\|A(y)\|\geq 1}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\|f\|_{L^{q_1}(A(y)C_k)}\left|b_{B_k}-b_{A(y)B_k}\right|dy\\
&=: |B_k|^{1/q}\left(L_{21}+L_{22}\right). \qquad (2.4.1)
\end{aligned}$$

Thus, for $\|A(y)\| < 1$, we have

$$L_{21} = \sum_{j=0}^{\infty} \int_{2^{-j-1} \leq \|A(y)\| < 2^{-j}} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|f\|_{L^{q_1}(A(y)C_k)} \times \left\{ \sum_{i=1}^{j} |b_{2^{-i}B_k} - b_{2^{-i+1}B_k}| + |b_{2^{-j}B_k} - b_{A(y)B_k}| \right\} dy.$$

A use of Hölder inequality yields

$$\begin{aligned} |b_{2^{-i}B_k} - b_{2^{-i+1}B_k}| &\leq \frac{1}{|2^{-i}B_k|} \int_{2^{-i}B_k} |b(y) - b_{2^{-i+1}B_k}| dy \\ &\leq \frac{|2^{-j}B_k|}{|2^{-i}B_k|} \|b\|_{C\dot{M}O^q(\mathbb{R}^n)} \\ &\leq C\|b\|_{C\dot{M}O^q(\mathbb{R}^n)}. \end{aligned}$$

Similarly,

$$\begin{aligned} |b_{2^{-j}B_k} - b_{A(y)B_k}| &\leq \frac{1}{|A(y)B_k|} \int_{A(y)B_k} |b(y) - b_{2^{-j}B_k}| dy \\ &\leq \frac{2^{-jn}}{|\det A(y)||2^{-j}B_k|} \int_{2^{-j}B_k} |b(y) - b_{2^{-j}B_k}| dy \\ &\leq C \frac{\|A(y)\|^n}{|\det A(y)|} \|b\|_{C\dot{M}O^q(\mathbb{R}^n)}. \end{aligned}$$

Thus, we have

$$\begin{aligned} L_{21} &\leq C\|b\|_{C\dot{M}O^q(\mathbb{R}^n)} \\ &\times \sum_{j=0}^{\infty} \int_{2^{-j-1} \leq \|A(y)\| < 2^{-j}} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|f\|_{L^{q_1}(A(y)C_k)} \left\{ j + \frac{\|A(y)\|^n}{|\det A(y)|} \right\} dy \\ &\leq C\|b\|_{C\dot{M}O^q(\mathbb{R}^n)} \\ &\quad \times \int_{\|A(y)\| < 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|f\|_{L^{q_1}(A(y)C_k)} \left(\log \frac{1}{\|A(y\|} + \frac{\|A(y)\|^n}{|\det A(y)|} \right) dy. \end{aligned}$$

Similar arguments result in the following estimation of L_{22}.

$$\begin{aligned} L_{22} &\leq C\|b\|_{C\dot{M}O^q(\mathbb{R}^n)} \\ &\quad \times \int_{\|A(y)\| \geq 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|f\|_{L^{q_1}(A(y)C_k)} \left(\log \|A(y\| + \frac{\|A(y)\|^n}{|\det A(y)|} \right) dy. \end{aligned}$$

Having these estimates of L_{21} and L_{22}, (4.4.2) assumes the following form

$$\begin{aligned}L_2 &\le C|B_k|^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{R}^n)}\\ &\times\Bigg[\int_{\|A(y)\|<1}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\|f\|_{L^{q_1}(A(y)C_k)}\left(\log\frac{1}{\|A(y\|}+\frac{\|A(y)\|^n}{|\det A(y)|}\right)dy\\ &+\int_{\|A(y)\|\ge 1}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\|f\|_{L^{q_1}(A(y)C_k)}\left(\log\|A(y\|+\frac{\|A(y)\|^n}{|\det A(y)|}\right)dy\Bigg].\end{aligned}$$

It remains to estimate L_3. For this purpose we take advantage of Hölder inequality to obtain

$$\begin{aligned}&\|(b(A(y)\cdot)-b_{A(y)B_k})f(A(y)\cdot)\|_{L^{q_2}(C_k)}\\ &=\left(\int_{C_k}\left|(b(A(y)x)-b_{A(y)B_k})f(A(y)x)\right|^{q_2}dx\right)^{1/q_2}\\ &=|\det A^{-1}(y)|^{1/q_2}\left(\int_{A(y)C_k}\left|(b(x)-b_{A(y)B_k})f(x)\right|^{q_2}dx\right)^{1/q_2}\\ &\le|\det A^{-1}(y)|^{1/q_2}\|b-b_{A(y)B_k}\|_{L^q(A(y)C_k)}\|f\|_{L^{q_1}(A(y)C_k)}\\ &\le|\det A^{-1}(y)|^{1/q_2}|A(y)B_k|^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{R}^n)}\|f\|_{L^{q_1}(A(y)C_k)}\\ &=|B_k|^{1/q}|\det A^{-1}(y)|^{1/q_1}\|b\|_{C\dot{M}O^q(\mathbb{R}^n)}\|f\|_{L^{q_1}(A(y)C_k)}.\end{aligned}$$

Hence,

$$L_3\le|B_k|^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\|f\|_{L^{q_1}(A(y)C_k)}dy.$$

Combining the estimates for L_1, L_2, and L_3 we obtain

$$\begin{aligned}\|H^b_{\Phi,A}f\|_{L^{q_2}(C_k)}&\le C|B_k|^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\|f\|_{L^{q_1}(A(y)C_k)}\\ &\left(\log\frac{1}{\|A(y\|}\chi_{\{\|A(y)\|<1\}}+\log\|A(y\|\|\chi_{\{\|A(y)\|\ge 1\}}+\frac{\|A(y)\|^n}{|\det A(y)|}\right)dy.\end{aligned}$$

Again, to make our calculation convenient, we use the following notation

$$\begin{aligned}\varphi(y)=&\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\\ &\times\left(\log\frac{1}{\|A(y\|}\chi_{\{\|A(y)\|<1\}}+\log\|A(y\|\|\chi_{\{\|A(y)\|\ge 1\}}+\frac{\|A(y)\|^n}{|\det A(y)|}\right).\end{aligned}$$

Thus, we rewrite above inequality as

$$\|H^b_{\Phi,A}f\|_{L^{q_2}(C_k)}\le C|B_k|^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{R}^n)}\int_{\mathbb{R}^n}\varphi(y)\|f\|_{L^{q_1}(A(y)C_k)}dy. \tag{2.4.2}$$

Still we have to approximate $\|f\|_{L^{q_1}(A(y)C_k)}$. For this end we infer from (5.4.20) that

$$\|f\|_{L^{q_1}(A(y)C_k)} \le \sum_{j=l}^{l+m+1} \|f\|_{L^{q_1}(C_{k+j})}.$$

By this inequality, (2.4.2) becomes

$$\|H^b_{\Phi,A}f\|_{L^{q_2}(C_k)} \le C|B_k|^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{R}^n)} \int_{\mathbb{R}^n} \varphi(y) \sum_{j=l}^{l+m+1} \|f\|_{L^{q_1}(C_{k+j})}dy. \tag{2.4.3}$$

Next by definition of Herz-Morrey spaces $M\dot{K}^{\alpha_2,\lambda}_{p,q_2}(\mathbb{R}^n)$, (3.3.7), Minkowski inequality and the condition $\alpha_1 = \alpha_2 + n/q$, we get

$$\begin{aligned}
&\|H^b_{\Phi,A}f\|_{M\dot{K}^{\alpha_2,\lambda}_{p,q_2}(\mathbb{R}^n)}\\
&= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0} 2^{k\alpha_2 p}\|H^b_{\Phi,A}f\|^p_{L^{q_2}(C_k)}\right\}^{1/p}\\
&\le C\|b\|_{C\dot{M}O^q(\mathbb{R}^n)} \int_{\mathbb{R}^n}\varphi(y)\\
&\quad\times \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{j=l}^{l+m+1} 2^{k(\alpha_2+n/q)}\|f\|_{L^{q_1}(C_{k+j})}\right)^p\right\}^{1/p} dy\\
&\le C\|b\|_{C\dot{M}O^q(\mathbb{R}^n)} \int_{\mathbb{R}^n}\varphi(y)\\
&\quad\times \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\sum_{j=l}^{l+m+1} 2^{-j\alpha_1}\left\{\sum_{k=-\infty}^{k_0+j} 2^{k\alpha_1 p}\|f\|^p_{L^{q_1}(C_k)}\right\}^{1/p} dy\\
&\le C\|b\|_{C\dot{M}O^q(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\alpha_1,\lambda}_{p,q_2}(\mathbb{R}^n)} \int_{\mathbb{R}^n}\varphi(y)\sum_{j=l}^{l+m+1} 2^{j(\lambda-\alpha_1)}dy\\
&\le C\|b\|_{C\dot{M}O^q(\mathbb{R}^n)}\|f\|_{M\dot{K}^{\alpha_1,\lambda}_{p,q_2}(\mathbb{R}^n)} \int_{\mathbb{R}^n}\varphi(y)G_{\alpha_1,\lambda}(y)dy,
\end{aligned}$$

where $G_{\alpha_1,\lambda}(y)$ is the same function as given in (2.3.1) with α is replaced by α_1. Thus, we finish the proof.

Proof of Theorem 2.4.2. Since, the proofs of Theorem 2.4.1 and 2.4.2 are symmetrical. So, by definition of Herz space $\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{R}^n)$, (3.3.7), Minkowski inequality and the condition $\alpha_1 = \alpha_2 + n/q$, we get

$$\|H^b_{\Phi,A}f\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{R}^n)} \le C\|b\|_{C\dot{M}O^q(\mathbb{R}^n)}\|f\|_{\dot{K}^{\alpha_1,p}_{q_1}(\mathbb{R}^n)} \int_{\mathbb{R}^n}\varphi(y)\widetilde{G}_{\alpha_1}(y)dy,$$

where $\widetilde{G}_{\alpha_1}(y)$ is the same function as given in (2.3.2) with α is replaced by α_1. Thus the proof is over.

Chapter 3
Estimates for Hausdorff Operator and Commutators on Weighted Central Morrey Space

3.1 Introduction

In recent years, the Hausdorff operator has gained much attention. This is mainly because of seminal work done by Liflyand and Móricz in [89] and Lerner and Liflyand in [84]. After the appearance of above cited monographs [84, 89], it was natural to study, refine and extend the existing results on relevant function spaces. A number of significant studies have been undertaken in this regard like for example boundedness of one and multidimensional Hausdorff operators on Hardy, L^p and BMO spaces [3, 13, 17, 66, **?**, 89]. Besides, many authors have contributed a lot towards obtaining new estimates on other function spaces. Among them we choose to refer to the papers [12, 14, 43, 63, 64, 65, 66] and the references therein.

On the other hand weighted norm inequalities for Hausdorff operators on function spaces have recently been reported in the literature which include continuity of Hausdorff operator in Hardy spaces [112, 113] with power weights, weighted Hardy spaces associated with Herz spaces [114] and in Herz space with Muckenhoupt weights on the Heisenberg group [117].

The purpose of this chapter is twofold. Firstly, motivated by works in [113, 117], we give estimates for matrix Hausdorff operator on weighted central Morrey space. In addition, under some assumption on $A(y)$, we work out operator norm for $H_{\Phi,A}$ on power weighted central Morrey spaces. Secondly, we try to fill the gap to existing theory of the commutator of Hausdorff operators by defining new type of commutators in (1.2.7) and establishing the weighted estimates for such commutator operators.

More precisely, under some assumptions on $A(y)$, we give necessary and sufficient condition on the function Φ such that $H^b_{\Phi,A}$ is bounded on power weighted central Morrey spaces.

In the very next section, some notations and definitions will be introduced along with some necessary lemmas to be used in the subsequent sections of this chapter. Our main results regarding continuity of matrix operator in weighted-type central Morrey spaces are stated and proved in the third section. Finally, the last section is devoted to obtain weighted estimates of commutators of the same operator.

3.2 Notations and Definitions

Having in hand the definitions of Morrey space (Definition 2.2.1) and central-BMO space (Definition 2.2.4), we now give the definition of central Morrey space. If we take $g_{B(0,R)} = 0$ in the Definition 2.2.4, then the residue space is known as central Morrey space $\dot{M}^{q,\lambda}$ introduced in [2] with norm condition:

$$\|g\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n)} = \sup_{R>0}\left(|B(0,R)|^{-(1+\lambda q)}\int_{B(0,R)}|g(y)|^q dy\right)^{1/q} < \infty.$$

Muckenhoupt [106] firstly introduced the theory of A_p weights while studying Hardy-Littlewood maximal functions on weighted L^p spaces. Any nonnegative function $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ can be taken as a weight. The notation $w(E)$ will serve to denote weighted measure of a given subset E of $\mathbb{R}^n$, that is $w(E) = \int_E w(x)dx$. Also, by p' we mean to consider conjugate index of p, satisfying $1/p + 1/p' = 1$.

Definition 3.2.1 *([46, 114]) A weight w is said to belong to the Muckenhoupt class A_p, $1 < p < \infty$, if there exist a positive constant C such that for every ball $B \subset \mathbb{R}^n$,*

$$\left(|B|^{-1}\int_B w(y)dy\right)\left(|B|^{-1}\int_B w(y)^{-1/(p-1)}dy\right)^{p-1} \leq C.$$

Also, $w \in A_1$ if there exists a positive constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\left(|B|^{-1}\int_B w(y)dy\right) \leq C \operatorname*{essinf}_{y\in B} w(y).$$

For $p = \infty$, we define $A_\infty = \bigcup_{1\leq p<\infty} A_p$.

Definition 3.2.2 *([46]) A weight w is said to belong to the reverse Hölder class RH_r if there exist a fixed positive constant C and $r > 1$ such that for every ball $B \subset \mathbb{R}^n$,*

$$\left(|B|^{-1}\int_B w^r(y)dy\right)^{1/r} \leq C|B|^{-1}\int_B w(y)dy.$$

It is also known for $s > p$ that $A_p \subset A_s$ and that if w is in A_p, then w is in A_q for some $1 < q < p < \infty$. The infimum of all q's such that $w \in A_q$ is denoted by q_w and is known as the critical index for w. In addition, for $r > 1$, if $w \in RH_r$, then for some $\epsilon > 0$ one has $w \in RH_{r+\epsilon}$. We therefore use the notation r_w to denote the critical index of w for the reverse Hölder condition i.e. $r_w \equiv \sup\{r > 1 : w \in RH_r\}$.

A special class of Muckenhoupt A_p weights is $|y|^\alpha$, the power weighted function. It is accepted that $|y|^\alpha \in A_1$ if and only if $-n < \alpha \leq 0$. Moreover, for $0 < \alpha < \infty$, $|y|^\alpha \in \cap_{(n+\alpha)/n<p<\infty} A_p$, where $\frac{n+\alpha}{n}$ is known as the critical index for $|y|^\alpha$.

Here we state some Propositions regarding A_p weights which will be helpful in obtaining weighted estimates for Hausdorff operator and their commutators.

Proposition 3.2.3 *([46]) Suppose $w \in A_p \cap RH_r$, $r > 1$ and $p \geq 1$. Then there exist two nonzero positive constants C_1 and C_2 such that*

$$C_1 \left(\frac{|D|}{|B|}\right)^p \leq \frac{w(D)}{w(B)} \leq C_2 \left(\frac{|D|}{|B|}\right)^{(r-1)/r},$$

for any measurable subset D of the ball B. In general, for a constant $1 < \lambda$,

$$w(B(x_0, \lambda R) \leq \lambda^{np} w(B(x_0, R)).$$

Proposition 3.2.4 *([114]) Suppose $g \in L^1(\mathbb{R}^n)$ be a nonnegative function. If $w \in A_p$, $1 \leq p$, then*

$$|B(x_0, R)|^{-1} \int_{B(x_0,R)} g(y)dy \leq C \left(\frac{1}{w(B(x_0, R))} \int_{B(x_0,R)} g^p(y)w(y)dy\right)^{1/p}.$$

Let w be a weight function on $\mathbb{R}^n$, for any measurable set $E \subset \mathbb{R}^n$, the Lebesgue space with weights $L^p(E; w)$ is the space of all functions satisfying

$$\|g\|_{L^p(E;w)} = \left(\int_E |g(y)|^p w(y)dy\right)^{1/p} < \infty.$$

Komori and Shirai [81], in 2009, introduced weighted Morrey space and studied the properties of some classical operators in this space. Here, we only define the central Morrey space with weights.

Definition 3.2.5 *Let w be a weight function on $\mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $\infty > q \geq 1$. Then the weighted central Morrey space $\dot{M}^{q,\lambda}(\mathbb{R}^n; w)$ can be defined as:*

$$\dot{M}^{q,\lambda}(\mathbb{R}^n; w) = \left\{g \in L^q_{\text{loc}}(\mathbb{R}^n; w) : \|g\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n;w)} < \infty\right\},$$

where

$$\|g\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n;w)} = \sup_{R>0} \left(w(B(0, R))^{-(1+\lambda q)} \int_{B(0,R)} |g(y)|^q w(y)dy\right)^{1/q}.$$

Some recent work discussing these spaces include [56, 57, 58]. Next, the weighted central mean oscillation space can be defined as follows:

Definition 3.2.6 *Suppose w be a weight function and $1 < q < \infty$. Then, a function $g \in L^q_{\mathrm{loc}}(\mathbb{R}^n; w)$ is in the weighted central mean oscillation space $C\dot{M}O^q(\mathbb{R}^n; w)$ if*

$$\|g\|_{C\dot{M}O^q(\mathbb{R}^n;w)} = \sup_{R>0} \left(w(B(0,R))^{-1} \int_{B(0,R)} |g(y) - g_B|^q w(y)dy \right)^{1/q} < \infty.$$

In the sequel, the notation $A \preceq B$ shall imply the existence of a positive constant C such that $A \leq CB$ and the notation $A \simeq B$ shall imply the existence of two positive constants C and c such that $cB \leq A \leq CB$. Moreover, we will denote a weight from Muckenhoupt A_p class by w. However, when the weight is reduced to the power function, we will denote it by v that is $v(x) = |x|^\alpha$.

Proposition 3.2.7 *Let α be a real number, D is any nonsingular matrix and $x \in \mathbb{R}^n$, then we have the following results*
(i)

$$v(Dx) \preceq \begin{cases} \|D\|^\alpha v(x) & \text{if } \alpha > 0, \\ \|D^{-1}\|^{-\alpha} v(x) & \text{if } \alpha \leq 0; \end{cases}$$

(ii)

$$v(B(0, \|D\|R)) = \|D\|^{n+\alpha} v(B(0,R)).$$

Proof. The proof of this Lemma follows from the definition of $v(x)$ and (2.2.1).

Henceforth, for the sake of convenience, we will denote $B(0,R)$ by B.

3.3 Bounds for $H_{\Phi,A}$ on Weighted Central Morrey Space

Present section is reserved for the proofs of results on the boundedness of $H_{\Phi,A}$ on weighted central Morrey space.

3.3.1 Main Results

Here are the main results for the present section.

Theorem 3.3.1 *Suppose $1 \leq q_1, q_2 < \infty$, $\lambda < 0$. Suppose also that w belongs to A_1 class of weights with r_w as its critical index for the reverse Hölder condition and let $q_1 > q_2 r_w/(r_w - 1)$.*
Then for any $1 < \delta < r_w$

$$\|H_{\Phi,A} f\|_{\dot{M}^{q_2,\lambda}(\mathbb{R}^n;w)} \preceq K_1 \|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)},$$

where

$$K_1 = \int_{\|A(y)\|<1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A(y)\|^{n\lambda + n/q_1} dy + \int_{\|A(y)\|\geq 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A(y)\|^{n/q_1 + n\lambda(\delta-1)/\delta} dy.$$

In case, if general weights are replaced by power function, then we obtain the following theorem.

Theorem 3.3.2 *Suppose that $-1/q \leq \lambda < 0$, and $1 \leq q < \infty$.*
(i) If $0 < \alpha < \infty$,

$$\|H_{\Phi,A} f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n;v)} \preceq K_2 \|f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,v)},$$

where

$$K_2 = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} |\det A^{-1}(y)|^{1/q} \|A(y)\|^{(n+\alpha)(\lambda+1/q)} \|A^{-1}(y)\|^{\alpha/q} dy.$$

(ii) If $-n < \alpha \leq 0$, then

$$\|H_{\Phi,A} f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n;v)} \preceq K_3 \|f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,v)},$$

where

$$K_3 = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} |\det A^{-1}(y)|^{1/q} \|A(y)\|^{n(\lambda+1/q)+\alpha\lambda} dy.$$

Especially, in case $\|A(y)\|^{-1}$ and $\|A^{-1}(y)\|$ are comparable, then the result can be sharpened as below.

Theorem 3.3.3 *Let $1 \leq q < \infty$, $-1/q \leq \lambda < 0$, $-n < \alpha < \infty$, and Φ be a non-negative function. Suppose that there exists a constant C independent of all essential variables such that $\|A^{-1}(y)\| \leq C\|A(y)\|^{-1}$ for all $y \in \operatorname{supp}(\Phi)$, then the necessary and sufficient condition for $H_{\Phi,A}$ to be bounded on $\dot{M}^{q,\lambda}(\mathbb{R}^n; v)$ is that*

$$K_4 = \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n} |\|A(y)\|^{(n+\alpha)\lambda} dy < \infty.$$

Remark Let $A(x)=\mathrm{diag}[1/\mu_1(x),...,1/\mu_n(x)]$ for $\mu_i(x)\neq 0$ $(i=1,2,...,n)$. Define

$$m(x)=\min\{|\mu_1(x)|,...,|\mu_n(x)|\},\ \ M(x)=\max\{|\mu_1(x)|,...,|\mu_n(x)|\}.$$

For a constant $C\geq 1$ independent of y if $M(x)\leq Cm(x)$, then it can be easily verified that $A(x)$ satisfies the assumptions of Theorem 3.3.3.

3.3.2 Proofs of the Main Results

Proof of Theorem 3.3.1. For a fixed ball $B\subset\mathbb{R}^n$, by Minkowski inequality

$$\begin{aligned}\|H_{\Phi,A}f\|_{L^{q_2}(B;w)} &= \left(\int_B\left|\int_{\mathbb{R}^n}\frac{\Phi(y)}{|y|^n}f(A(y)x)dy\right|^{q_2}w(x)dx\right)^{1/q_2}\\ &\leq \int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\left(\int_B|f(A(y)x)|^{q_2}w(x)dx\right)^{1/q_2}dy. \qquad (3.3.1)\end{aligned}$$

In view of the condition $q_1>q_2r_w/(r_w-1)$, there exist $1<r<r_w$ such that $q_1=q_2r'=q_2r/(r-1)$. An application of Hölder inequality and reverse Hölder condition yield

$$\begin{aligned}\|f(A(y)\cdot)\|_{L^{q_2}(B;w)} &\leq \left(\int_B|f(A(y)x)|^{q_1}dx\right)^{1/q_1}\left(\int_B w(x)^r dx\right)^{1/(rq_2)}\\ &\preceq |\det A^{-1}(y)|^{1/q_1}|B|^{-1/q_1}w(B)^{1/q_2}\left(\int_{A(y)B}|f(x)|^{q_1}dx\right)^{1/q_1}.\end{aligned}$$

By virtue of Proposition 3.2.4, one can have

$$\begin{aligned}&\left(\int_{A(y)B}|f(x)|^{q_1}dx\right)^{1/q_1}\\ &\preceq |B(0,\|A(y)\|R)|^{1/q_1}\left(\frac{1}{w(B(0,\|A(y)\|R))}\int_{B(0,\|A(y)\|R)}|f(x)|^{q_1}w(x)dx\right)^{1/q_1}\\ &\preceq \|A(y)\|^{n/q_1}|B(0,R)|^{1/q_1}w(B(0,\|A(y)\|R))^{\lambda}\|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)}, \qquad (3.3.2)\end{aligned}$$

which suggest that

$$\begin{aligned}&\|f(A(y)\cdot)\|_{L^{q_2}(B;w)}\\ &\preceq |\det A^{-1}(y)|^{1/q_1}\|A(y)\|^{n/q_1}w(B)^{1/q_2}w(B(0,\|A(y)\|R))^{\lambda}\|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)}. \qquad (3.3.3)\end{aligned}$$

We thus conclude from (5.4.1) and (5.4.2) that

$$
\begin{aligned}
&\|H_{\Phi,A}f\|_{\dot{M}^{q_2,\lambda}(\mathbb{R}^n;w)} \\
&\preceq \|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)} \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A(y)\|^{n/q_1} \left(\frac{w(B(0,\|A(y)\|R))}{w(B(0,R))}\right)^{\lambda} dy \\
&\preceq \|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)} \\
&\times \left(\int_{\|A(y)\|<1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A(y)\|^{n/q_1} \left(\frac{w(B(0,\|A(y)\|R))}{w(B(0,R))}\right)^{\lambda} dy \right. \\
&\left. + \int_{\|A(y)\|\geq 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A(y)\|^{n/q_1} \left(\frac{w(B(0,\|A(y)\|R))}{w(B(0,R))}\right)^{\lambda} dy \right).
\end{aligned} \tag{3.3.4}
$$

Since $\lambda < 0$, Proposition 5.2.3 implies that, if $\|A(y)\| < 1$,

$$
\left(\frac{w(B(0,\|A(y)\|R))}{w(B(0,R))}\right)^{\lambda} \preceq \left(\frac{|B(0,\|A(y)\|R)|}{|B(0,R)|}\right)^{\lambda} = \|A(y)\|^{n\lambda}, \tag{3.3.5}
$$

and if $\|A(y)\| \geq 1$,

$$
\left(\frac{w(B(0,\|A(y)\|R))}{w(B(0,R))}\right)^{\lambda} \preceq \left(\frac{|B(0,\|A(y)\|R)|}{|B(0,R)|}\right)^{\lambda(\delta-1)/\delta} = \|A(y)\|^{n\lambda(\delta-1)/\delta}, \tag{3.3.6}
$$

for any $1 < \delta < r_w$.

Therefore, from (5.4.3)-(5.4.5) it is easy to see that, for any $1 < \delta < r_w$,

$$
\begin{aligned}
&\|H_{\Phi,A}f\|_{\dot{M}^{q_2,\lambda}(\mathbb{R}^n;w)} \\
&\preceq \|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)} \left(\int_{\|A(y)\|<1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A(y)\|^{n\lambda+n/q_1} dy \right. \\
&\left. + \int_{\|A(y)\|\geq 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A(y)\|^{n/q_1+n\lambda(\delta-1)/\delta} dy \right).
\end{aligned}
$$

The proof is completed.

Proof of Theorem 3.3.2. In view of the Minkowski inequality, change of variables and Proposition 3.2.7, we have

$$\begin{aligned}
&\left(\frac{1}{v(B(0,R))^{1+\lambda q}}\int_{B(0,R)}|H_{\Phi,A}f|^q v(x)dx\right)^{1/q}\\
&\preceq v(B(0,R))^{-(\lambda+1/q)}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\left(\int_B|f(A(y)x)|^q\,v(x)dx\right)^{1/q}dy\\
&\simeq v(B(0,R))^{-(\lambda+1/q)}\\
&\times\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q}\left(\int_{A(y)B}|f(x)|^q\,v(A^{-1}(y)x)dx\right)^{1/q}dy\\
&\preceq\|f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,v)}\\
&\times\begin{cases}\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q}\|A(y)\|^{(n+\alpha)(\lambda+1/q)}\|A^{-1}(y)\|^{\alpha/q}dy & \text{if } \alpha>0,\\ \int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q}\|A(y)\|^{n(\lambda+1/q)+\alpha\lambda}dy & \text{if } \alpha\le 0.\end{cases}
\end{aligned}$$

Therefore, we conclude that

$$\|H_{\Phi,A}\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,v)\to\dot{M}^{q,\lambda}(\mathbb{R}^n,v)}\preceq\begin{cases}K_2 & \text{if } \alpha>0,\\ K_3 & \text{if } \alpha\le 0.\end{cases}$$

Thus we finish the proof.

Proof of Theorem 3.3.3. If $\|A^{-1}(y)\|\preceq\|A(y)\|^{-1}$, we infer from (5.4.15) that

$$\|A(y)\|^{-n}\simeq|\det A^{-1}(y)|\simeq\|A^{-1}(y)\|^n. \tag{3.3.7}$$

Here we will prove the necessary part of the Theorem 3.3.3 as the sufficient part can easily be obtained from Theorem 3.3.2. We divide the proof into the below two cases.
Case 1. If $\infty>\lambda>-1/q$.
In this situation, we select $f_0\in\dot{M}^{p,\lambda}(\mathbb{R}^n;v)$ such that $f_0(x)=|x|^{(n+\alpha)\lambda}$, then

$$\|f_0\|_{\dot{M}^{p,\lambda}(\mathbb{R}^n;v)}=|S^{n-1}|^{-\lambda}(n+\alpha)^\lambda(1+\lambda q)^{-1/q},$$

where $|S^{n-1}|$ is the Lebesgue measure of unit sphere S^{n-1}.

On the other side, making use of the condition that $0<(n+\alpha)\lambda$, we obtain

$$\begin{aligned}
H_{\Phi,A}f_0(x)&=\int_{\mathbb{R}^n}\frac{\Phi(y)}{|y|^n}|A(y)x|^{(n+\alpha)\lambda}dy\\
&\succeq|x|^{(n+\alpha)\lambda}\int_{\mathbb{R}^n}\frac{\Phi(y)}{|y|^n}\|A(y)\|^{(n+\alpha)\lambda}dy,
\end{aligned}$$

this implies that

$$\|H_{\Phi,A}\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n,v)\to\dot{M}^{q,\lambda}(\mathbb{R}^n,v)} \succeq \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n}\|A(y)\|^{(n+\alpha)\lambda}dy,$$

which is as required.
Case 2. If $\lambda = -1/q$, then for $0 < \epsilon < 1$, we take

$$f_\epsilon(x) = |x|^{-(n+\alpha)/q-\epsilon}\chi_{\{|x|>1\}}$$

A simple computation yields $\|f_\epsilon\|^q_{L^q(\mathbb{R}^n;v)} = \frac{|S^{n-1}|}{\epsilon q}$. Now, by definition

$$\begin{aligned} H_{\Phi,A}(f_\epsilon)(x) &= \int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n}|A(y)x|^{-(n+\alpha)/q-\epsilon}\chi_{\{|A(y)x|>1\}}dy \\ &\succeq \left(\int_{\|A(y)\|\succeq 1/|x|} \frac{\Phi(y)}{|y|^n}\|A(y)\|^{-(n+\alpha)/q-\epsilon}\right)|x|^{-(n+\alpha)/q-\epsilon} \end{aligned}$$

Now,

$$\begin{aligned} &\|H_{\Phi,A}(f_\epsilon)\|^q_{L^q(\mathbb{R}^n,v)} \\ &\succeq \int_{|x|>1}\left(|x|^{-(n+\alpha)/q-\epsilon}\int_{\|A(y)\|\succeq 1/|x|}\frac{\Phi(y)}{|y|^n}\|A(y)\|^{-(n+\alpha)/q-\epsilon}\right)^q v(x)dx \\ &\succeq \int_{|x|>\frac{1}{\epsilon}}|x|^{-n-\epsilon q}dx\left(\int_{\|A(y)\|\succeq\epsilon}\frac{\Phi(y)}{|y|^n}\|A(y)\|^{-(n+\alpha)/q-\epsilon}dy\right)^q \\ &= \left(\int_{\|A(y)\|\succeq\epsilon}\frac{\Phi(y)}{|y|^n}\|A(y)\|^{-(n+\alpha)/q-\epsilon}dy\right)^q(\epsilon^\epsilon)^q\|f_\epsilon\|^q_{L^q(\mathbb{R}^n,v)}, \end{aligned}$$

by letting $\epsilon \to 0$, we have

$$\|H_\Phi\|_{L^q(\mathbb{R}^n,v)\to L^q(\mathbb{R}^n,v)} \succeq \int_{\mathbb{R}^n}\frac{\Phi(y)}{|y|^n}\|A(y)\|^{-(n+\alpha)/q}dy.$$

With this we complete the proof.

3.4 Bounds for $H^b_{\Phi,A}$ on Weighted Central Morrey Space

3.4.1 Main Results

Below are our main results for this section

Theorem 3.4.1 *Let* $\infty > q \geq 1$, $1 \leq s < q_1 < \infty$, $1/q_1 + 1/q_2 = 1/s$, *and* $0 > \lambda$. *Suppose also that* w *belongs to* A_1 *class of weights with* r_w *as its critical index for the reverse Hölder condition and let* $s > qr_w/(r_w - 1)$.
Then for any $1 < \delta < r_w$

$$\|H^b_{\Phi,A}f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n;w)} \preceq K_5 \|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)} \|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)},$$

where

$$\int_{\|A(y)\|\geq 1} \frac{|\Phi(y)||\det A^{-1}(y)|^{1/q_1}}{|y|^n \|A(y)\|^{-n/q_1 - n\lambda(\delta-1)/\delta}} \left(1 + \frac{|\det A^{-1}(y)|^{1/q_2}}{\|A(y)\|^{-n/q_2}}\right) \max\left\{\log 2\|A(y\|, \frac{\|A(y)\|^n}{|\det A(y)|}\right\} dy$$
$$+ \int_{\|A(y)\|<1} \frac{|\Phi(y)||\det A^{-1}(y)|^{1/q_1}}{|y|^n \|A(y)\|^{-n\lambda - n/q_1}} \left(1 + \frac{|\det A^{-1}(y)|^{1/q_2}}{\|A(y)\|^{-n/q_2}}\right) \max\left\{\log \frac{2}{\|A(y\|}, \frac{\|A(y)\|^n}{|\det A(y)|}\right\} dy$$
$$= K_5.$$

Instead of general weights, when dealing with power weights, we have the following results.

Theorem 3.4.2 *Suppose* $1 \leq q < q_1 < \infty$, $1/q_1 + 1/q_2 = 1/q$, $0 > \lambda > -1/q$. *Then*
(i) If $0 < \alpha < \infty$,

$$\|H^b_{\Phi,A}f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n;v)} \preceq K_6 \|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,v)} \|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,v)},$$

where

$$K_6 = \int_{\mathbb{R}^n} \frac{|\Phi(y)||\det A^{-1}(y)|^{\frac{1}{q_1}} \|A^{-1}(y)\|^{\frac{\alpha}{q_1}}}{|y|^n \|A(y)\|^{-(n+\alpha)(\lambda + \frac{1}{q_1})}} \left(1 + \frac{|\det A^{-1}(y)|^{\frac{1}{q_2}} \|A^{-1}(y)\|^{\frac{\alpha}{q_2}}}{\|A(y)\|^{-\frac{n+\alpha}{q_2}}}\right)$$
$$\left(\max\left\{\log \frac{2}{\|A(y\|}, \frac{\|A(y)\|^n}{|\det A(y)|}\right\} \chi_{\{\|A(y)\|<1\}} + \max\left\{\log 2\|A(y\|, \frac{\|A(y)\|^n}{|\det A(y)|}\right\} \chi_{\{\|A(y)\|\geq 1\}}\right) dy.$$

(ii) If $-n < \alpha \leq 0$, *then*

$$\|H^b_{\Phi,A}f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n;v)} \preceq K_7 \|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,v)} \|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,v)},$$

where

$$K_7 = \int_{\mathbb{R}^n} \frac{|\Phi(y)||\det A^{-1}(y)|^{\frac{1}{q_1}}}{|y|^n \|A(y)\|^{-n(\lambda + \frac{1}{q_1}) - \alpha\lambda}} \left(1 + \frac{|\det A^{-1}(y)|^{\frac{1}{q_2}}}{\|A(y)\|^{-\frac{n}{q_2}}}\right)$$
$$\left(\max\left\{\log \frac{2}{\|A(y\|}, \frac{\|A(y)\|^n}{|\det A(y)|}\right\} \chi_{\{\|A(y)\|<1\}} + \max\left\{\log 2\|A(y\|, \frac{\|A(y)\|^n}{|\det A(y)|}\right\} \chi_{\{\|A(y)\|\geq 1\}}\right) dy.$$

More specially, in case $\|A(y)\|^{-1}$ and $\|A^{-1}(y)\|$ are comparable, then the result can be sharpened by decomposing $H^b_{\phi,A}$ as follows

$$H^{b,1}_{\Phi,A}f = \int_{\|A(y)\|<1} \frac{\Phi(y)}{|y|^n}(b(x) - b(A(y)x))f(A(y)x)dy,$$

$$H^{b,2}_{\Phi,A}f = \int_{\|A(y)\|\geq 1} \frac{\Phi(y)}{|y|^n}(b(x) - b(A(y)x))f(A(y)x)dy.$$

Theorem 3.4.3 *Suppose* $1 < q < q_1 < \infty$, $1/q_1 + 1/q_2 = 1/q$, $0 > \lambda > -1/q$, *and* Φ *be a nonnegative function. If there exists a positive constant* C *independent of all essential variables such that* $\|A^{-1}(y)\| \leq C\|A(y)\|^{-1}$ *for all* $y \in \mathrm{supp}(\Phi)$. *In addition, if* $\Phi(y)/|y|^n$ *is integrable then*
(i) $H^{b,1}_{\Phi,A}$ *is bounded from* $\dot{M}^{q_1,\lambda}(\mathbb{R}^n; v)$ *to* $\dot{M}^{q,\lambda}(\mathbb{R}^n; v)$ *if and only if*

$$K_8 = \int_{\|A(y)\|<1} \frac{\Phi(y)}{|y|^n} |\|A(y)\|^{(n+\alpha)\lambda} \log \frac{2}{\|A(y)\|} dy < \infty.$$

(ii) $H^{b,2}_{\Phi,A}$ *is bounded from* $\dot{M}^{q_1,\lambda}(\mathbb{R}^n; v)$ *to* $\dot{M}^{q,\lambda}(\mathbb{R}^n; v)$ *if and only if*

$$K_9 = \int_{\|A(y)\|\geq 1} \frac{\Phi(y)}{|y|^n} |\|A(y)\|^{(n+\alpha)\lambda} \log 2\|A(y)\| dy < \infty.$$

Remark. Note that, from the preceding Theorem, one cannot deduce $L^p(\mathbb{R}^n; v)$ boundedness for the commutator operator by taking $\lambda = -1/p$, just in the case of Theorem 3.3.3. This raises an open question regarding L^p boundedness of $H^b_{\Phi,A}$ which will be answered later.

3.4.2 Proofs of the Main Results

Proof of Theorem 3.4.1. As before we fix a ball $B \subset \mathbb{R}^n$. Using Minkowski inequality, we obtain

$$\begin{aligned}
&\|H^b_{\Phi,A}f\|_{L^q(B;w)} \\
&= \left(\int_B \left|\int_{\mathbb{R}^n} \frac{\Phi(y)}{|y|^n}(b(x)-b(A(y)x))f(A(y)x)dy\right|^q w(x)dx\right)^{1/q} \\
&\le \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n}\left(\int_B |(b(x)-b(A(y)x))f(A(y)x)|^q\, w(x)dx\right)^{1/q} dy \\
&\le \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n}\left(\int_B |(b(x)-b_B)f(A(y)x)|^q\, w(x)dx\right)^{1/q} dy \\
&+ \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n}\left(\int_B \left|(b_B-b_{A(y)B})f(A(y)x)\right|^q w(x)dx\right)^{1/q} dy \\
&+ \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n}\left(\int_B \left|(b(A(y)x)-b_{A(y)B})f(A(y)x)\right|^q w(x)dx\right)^{1/q} dy \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Let us start estimating I_1. For this purpose, we first compute the inner norm $\|(b(\cdot)-b_B)f(A(y)\cdot)\|_{L^q(B;w)}$. The condition $s > qr_w/(r_w-1)$ implies that there is $1 < r < r_w$ such that $s = qr' = qr/(r-1)$. By Hölder inequality and reverse Hölder condition, we have

$$\begin{aligned}
&\|(b_B-b(\cdot))f(A(y)\cdot)\|_{L^q(B;w)} \\
&\le \left(\int_B |(b(x)-b_B)f(A(y)x)|^s\, dx\right)^{1/s}\left(\int_B w(x)^r dx\right)^{1/(rq)} \\
&\preceq |B|^{-1/s}w(B)^{1/q}\left(\int_B |(b(x)-b_B)f(A(y)x)|^s dx\right)^{1/s}.
\end{aligned}$$

In view of the condition $1/s = 1/q_1 + 1/q_2$, we have

$$\begin{aligned}
&\|(b(\cdot)-b_B)f(A(y)\cdot)\|_{L^q(B;w)} \\
&\preceq |B|^{-1/s}w(B)^{1/q}\left(\int_B |b(x)-b_B|^{q_2}dx\right)^{1/q_2}\left(\int_B |f(A(y)x)|^{q_1}dx\right)^{1/q_1} \\
&\preceq |\det A^{-1}(y)|^{1/q_1}|B|^{-1/s}w(B)^{1/q} \\
&\times \left(\int_B |b(x)-b_B|^{q_2}dx\right)^{1/q_2}\left(\int_{A(y)B} |f(x)|^{q_1}dx\right)^{1/q_1}.
\end{aligned} \tag{3.4.1}$$

By virtue of Proposition 3.2.4, it becomes simple to get that

$$\left(\int_B |b(y)-b_B|^{q_2}dy\right)^{1/q_2} \preceq |B|^{1/q_2}\|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)}. \tag{3.4.2}$$

Substituting result from inequalities (3.3.2) and (3.4.2) into (3.4.1), one has

$$\begin{aligned}&\|(b(\cdot)-b_B)f(A(y)\cdot)\|_{L^q(B;w)}\\&\preceq w(B)^{1/q}|\det A^{-1}(y)|^{1/q_2}\|A(y)\|^{n/q_1}w(A(y)B)^{\lambda}\|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)}\|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}I_1 \preceq{}& w(B)^{\lambda+1/q}\|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)}\|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)}\\&\times\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\|A(y)\|^{n/q_1}\left(\frac{w(B(0,\|A(y)\|R))}{w(B(0,R))}\right)^{\lambda}dy. \quad (3.4.3)\end{aligned}$$

Making use of the inequalities (5.4.4) and (5.4.5) into (3.4.3), we get

$$\begin{aligned}I_1 \preceq{}& w(B)^{\lambda+1/q}\|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)}\|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)}\\&\left(\int_{\|A(y)\|<1}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\|A(y)\|^{n\lambda+n/q_1}dy\right.\\&\left.+\int_{\|A(y)\|\geq 1}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\|A(y)\|^{n\lambda(\delta-1)/\delta+n/q_1}dy\right).\end{aligned}$$

Now, it turns to estimate I_2, which can be written as

$$I_2=\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\|f(A(y)\cdot)\|_{L^q(B;w)}\left|b_B-b_{A(y)B}\right|dy. \quad (3.4.4)$$

Here, the indices q and s bear the same relationship as observed between q_1 and q_2 of Theorem 3.3.1. Therefore, we infer from (5.4.2) that

$$\begin{aligned}&\|f(A(y)\cdot)\|_{L^q(B;w)}\\&\preceq|\det A^{-1}(y)|^{1/s}\|A(y)\|^{n/s}w(B(0,R))^{1/q}w(B(0,\|A(y)\|R))^{\lambda}\|f\|_{\dot{M}^{s,\lambda}(\mathbb{R}^n,w)}.\end{aligned}$$

Applying the Hölder inequality to s/q_1 and s/q_2, we have

$$\begin{aligned}&\|f(A(y)\cdot)\|_{L^q(B;w)}\\&\preceq|\det A^{-1}(y)|^{1/s}\|A(y)\|^{n/s}w(B)^{1/q}w(B(0,\|A(y)\|R))^{\lambda}\|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)}. \quad (3.4.5)\end{aligned}$$

With the help of (3.4.5), inequality (3.4.4) assumes the following form

$$\begin{aligned}I_2 \preceq{}& w(B)^{\lambda+1/q}\|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)}\\&\times\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/s}\|A(y)\|^{n/s}\frac{w(B(0,\|A(y)\|R))^{\lambda}}{w(B(0,R))^{\lambda}}\left|b_B-b_{A(y)B}\right|dy.\end{aligned}$$

For $\lambda<0$, the inequalities (5.4.4) and (5.4.5) help us to obtain

$$\begin{aligned}
I_2 \preceq & w(B)^{\lambda+1/q}\|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)} \\
& \left(\int_{\|A(y)\|<1}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/s}\|A(y)\|^{n\lambda+n/s}\left|b_B-b_{A(y)B}\right|dy\right. \\
& \left.+\int_{\|A(y)\|\geq 1}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/s}\|A(y)\|^{n\lambda(\delta-1)/\delta+n/s}\left|b_B-b_{A(y)B}\right|dy\right) \\
& =: w(B)^{\lambda+1/q}\|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)}\left(I_{21}+I_{22}\right). \qquad (3.4.6)
\end{aligned}$$

For our convenience, we denote

$$\|A(y)\|^{n(\lambda+1/s)}\phi(y)=|\det A^{-1}(y)|^{1/s}\frac{|\Phi(y)|}{|y|^n}.$$

Moreover, for $\|A(y)\|<1$, there exist $j\in\mathbb{Z}$ such that $2^{-j-1}\leq\|A(y)\|<2^{-j}$. Thus

$$I_{21}=\int_{\|A(y)\|<1}\phi(y)\left\{\sum_{i=1}^{j}|b_{2^{-i}B}-b_{2^{-i+1}B}|+|b_{2^{-j}B}-b_{A(y)B}|\right\}dy.$$

Since $w\in A_1$, using Proportion 3.2.4, it becomes simple to get that

$$\begin{aligned}
|b_{2^{-i}B}-b_{2^{-i+1}B}| &\leq \frac{1}{|2^{-i}B|}\int_{2^{-i}B}|b(y)-b_{2^{-i+1}B}|dy \\
&\leq \frac{2^n}{|2^{-i+1}B|}\int_{2^{-i+1}B}|b(y)-b_{2^{-i+1}B}|dy \\
&\preceq \|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)}.
\end{aligned}$$

Similarly, Proposition 3.2.4 again helps us to have

$$\begin{aligned}
|b_{2^{-j}B}-b_{A(y)B}| &\leq \frac{1}{|A(y)B|}\int_{A(y)B}|b(y)-b_{2^{-j}B}|dy \\
&\leq \frac{2^{-jn}}{|\det A(y)|2^{-j}B|}\int_{2^{-j}B}|b(y)-b_{2^{-j}B}|dy \\
&\preceq \frac{\|A(y)\|^n}{|\det A(y)|}\|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)}.
\end{aligned}$$

$|b_{2^{-j}B,w}-b_{A(y)B}|\preceq\|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)}$ and thus

$$\begin{aligned}
I_{21} &\preceq \|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)}\sum_{j=0}^{\infty}\int_{2^{-j-1}\leq\|A(y)\|<2^{-j}}\phi(y)\left\{j+\frac{\|A(y)\|^n}{|\det A(y)|}\right\}dy \\
&\preceq \|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)}\sum_{j=0}^{\infty}\int_{2^{-j-1}\leq\|A(y)\|<2^{-j}}\phi(y)\left\{\log 2^j+\frac{\|A(y)\|^n}{|\det A(y)|}\right\}dy \\
&\preceq \|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)}\times \\
&\int_{\|A(y)\|<1}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/s}\|A(y)\|^{n(\lambda+1/s)}\max\left\{\log\frac{2}{\|A(y\|},\frac{\|A(y)\|^n}{|\det A(y)|}\right\}dy.
\end{aligned}$$

Following the same procedure as followed in bounding I_{21}, we estimate I_{22} as

$$I_{22} \preceq \|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)} \times \int_{\|A(y)\|\geq 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/s} \|A(y)\|^{n(\lambda(\delta-1)/\delta+1/s)} \max\left\{\log 2\|A(y\|, \frac{\|A(y)\|^n}{|\det A(y)|}\right\} dy.$$

Incorporating the estimates of I_{21} and I_{22} into (3.4.6), we obtain

$$\begin{aligned} I_2 \preceq & w(B)^{\lambda+1/q} \|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)} \|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)} \\ & \left(\int_{\|A(y)\|\geq 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/s} \|A(y)\|^{n(\lambda(\delta-1)/\delta+1/s)} \max\left\{\log 2\|A(y\|, \frac{\|A(y)\|^n}{|\det A(y)|}\right\} dy \right. \\ & \left. + \int_{\|A(y)\|<1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/s} \|A(y)\|^{n(\lambda+1/s)} \max\left\{\log \frac{2}{\|A(y\|}, \frac{\|A(y)\|^n}{|\det A(y)|}\right\} dy \right). \end{aligned}$$

It remains to approximate I_3. For this purpose, we infer from (3.4.1) that

$$\begin{aligned} & \|(b(A(y)\cdot) - b_{A(y)B}) f(A(y)\cdot)\|_{L^q(B;w)} \\ & \preceq |\det A^{-1}(y)|^{1/q_1} |B|^{-1/s} w(B)^{1/q} \\ & \times \left(\int_B |b(A(y)x) - b_{A(y)B}|^{q_2} dx \right)^{1/q_2} \left(\int_{A(y)B} |f(x)|^{q_1} dx \right)^{1/q_1}. \end{aligned} \tag{3.4.7}$$

Making use of the Proposition 3.2.4, one can obtain

$$\begin{aligned} & \left(\int_B |b(A(y)x) - b_{A(y)B}|^{q_2} dx \right)^{1/q_2} \\ & = |\det A^{-1}(y)|^{1/q_2} \left(\int_{A(y)B} |b(x) - b_{A(y)B}|^{q_2} dx \right)^{1/q_2} \\ & = |\det A^{-1}(y)|^{1/q_2} \|A(y)\|^{n/q_2} |B(0,R)|^{1/q_2} \|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)}. \end{aligned} \tag{3.4.8}$$

In view of (3.3.2), (3.4.7) and (3.4.8), it becomes simple to get that

$$\begin{aligned} & \|(b(A(y)\cdot) - b_{A(y)B}) f(A(y)\cdot)\|_{L^q(B;w)} \\ & \preceq w(B)^{1/q} |\det A^{-1}(y)|^{1/s} \|A(y)\|^{n/s} w(A(y)B)^{\lambda} \|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)} \|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)}. \end{aligned}$$

We thus obtain

$$\begin{aligned} I_3 \preceq & w(B)^{\lambda+1/q} \|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)} \|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)} \\ & \times \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/s} \|A(y)\|^{n/s} \left(\frac{w(B(0,\|A(y)\|R))}{w(B(0,R))} \right)^{\lambda} dy. \end{aligned}$$

Finally, inequalities (5.4.4) and (5.4.5) help us to have

$$\begin{aligned}I_3 \preceq & w(B)^{\lambda+1/q}\|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)}\|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)} \\ & \left(\int_{\|A(y)\|<1}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/s}\|A(y)\|^{n\lambda+n/s}dy\right. \\ & \left.+\int_{\|A(y)\|\geq 1}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/s}\|A(y)\|^{n\lambda(\delta-1)/\delta+n/s}dy\right),\end{aligned}$$

for any $1<\delta<r_w$.

Incorporating the upper bounds for I_i $(i=1,2,3)$, we obtain

$$\|H^b_{\Phi,A}f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n;w)} \preceq K_5\|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,w)}\|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,w)}.$$

This completes the proof.

Proof of Theorem 3.4.2. (i) As in the previous Theorem

$$\|H^b_{\Phi,A}f\|_{L^q(B;v)} \leq \sum_{i=1}^{3} J_i,$$

where J_i $(i=1,2,3)$, assume the form of I_1, I_2 and I_3, respectively, but with $w(\cdot)$ is replaced by $v(\cdot)$.

An application of Hölder inequality and change of variables yield

$$\begin{aligned}J_1 \leq & \int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\left(\int_B |b(x)-b_B|^{q_2}\,v(x)dx\right)^{1/q_2}\left(\int_B |f(A(y)x)|^{q_1}\,v(x)dx\right)^{1/q_1} dy \\ \leq & v(B)^{1/q_2}\|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,v)} \\ & \times \int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\left(\int_{A(y)B}|f(x)|^{q_1}\,v(A^{-1}(y)x)dx\right)^{1/q_1} dy.\end{aligned}$$

In view of Proposition 3.2.7 it becomes simple to have that

$$\begin{aligned}J_1 \preceq & v(B)^{\lambda+1/q}\|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,v)}\|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,v)} \\ & \times\int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\|A(y)\|^{(n+\alpha)(\lambda+1/q_1)}\|A^{-1}(y)\|^{\alpha/q_1}dy.\end{aligned}$$

The expression for J_2 is written as

$$J_2 = \int_{\mathbb{R}^n}\frac{|\Phi(y)|}{|y|^n}\|f(A(y)\cdot)\|_{L^q(B;v)}\left|b_B-b_{A(y)B}\right|dy. \tag{3.4.9}$$

In order to estimate J_2. We first compute $\|f(A(y)\cdot)\|_{L^q(B;v)}$. For this purpose a change of variables following the Hölder inequality and Proposition 3.2.7 give us

$$\begin{aligned}
&\|f(A(y)\cdot)\|_{L^q(B;v)}\\
&=\left(\int_B |f(A(y)x)|^q\, v(x)dx\right)^{1/q}\\
&=|\det A^{-1}(y)|^{1/q}\left(\int_{A(y)B} |f(x)|^q\, v(A^{-1}(y)x)dx\right)^{1/q}\\
&\preceq |\det A^{-1}(y)|^{1/q}\|A^{-1}(y)\|^{\alpha/q}\|f\|_{L^q(A(y)B;v)}\\
&\preceq |\det A^{-1}(y)|^{1/q}\|A^{-1}(y)\|^{\alpha/q}\|f\|_{L^{q_1}(A(y)B;v)}v(A(y)B)^{1/q_2}\\
&\preceq v(B)^{\lambda+1/q}|\det A^{-1}(y)|^{1/q}\|A(y)\|^{(n+\alpha)(\lambda+1/q)}\|A^{-1}(y)\|^{\alpha/q}\|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,v)}.
\end{aligned}$$

So, therefore (3.4.9) becomes

$$\begin{aligned}
J_2 \preceq\, & v(B)^{\lambda+1/q}\|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,v)}\\
&\times \int_{\mathbb{R}^n}\frac{|\Phi(y)||\det A^{-1}(y)|^{1/q}\|A^{-1}(y)\|^{\alpha/q}}{|y|^n\|A(y)\|^{-(n+\alpha)(\lambda+1/q)}}\left|b_B-b_{A(y)B}\right|dy.
\end{aligned}$$

By denoting $\psi(y)=\frac{|\Phi(y)||\det A^{-1}(y)|^{1/q}\|A^{-1}(y)\|^{\alpha/q}}{|y|^n\|A(y)\|^{-(n+\alpha)(\lambda+1/q)}}$, we decompose J_2 as

$$\begin{aligned}
J_2 \preceq\, & v(B)^{\lambda+1/q}\|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,v)}\\
&\left(\int_{\|A(y)\|<1}\psi(y)\left|b_B-b_{A(y)B}\right|dy+\int_{\|A(y)\|\geq 1}\psi(y)\left|b_B-b_{A(y)B}\right|dy\right)\\
&= v(B)^{\lambda+1/q}\|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n)}(J_{21}+J_{22}).
\end{aligned}$$

Again, we arrive at the same point as reached in (3.4.6) with $w(\cdot)$ is replaced by $v(\cdot)$. Therefore, performing in a way similar to that point forward we estimate J_2 as

$$\begin{aligned}
J_2 \preceq\, & v(B)^{\lambda+1/q}\|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,v)}\|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,v)}\\
&\left(\int_{\|A(y)\|<1}\frac{|\Phi(y)||\det A^{-1}(y)|^{1/q}\|A^{-1}(y)\|^{\alpha/q}}{|y|^n\|A(y)\|^{-(n+\alpha)(\lambda+1/q)}}\left\{\log\frac{2}{\|A(y\|}+\frac{\|A(y)\|^n}{|\det A(y)|}\right\}dy\right.\\
&\left.+\int_{\|A(y)\|\geq 1}\frac{|\Phi(y)||\det A^{-1}(y)|^{1/q}\|A^{-1}(y)\|^{\alpha/q}}{|y|^n\|A(y)\|^{-(n+\alpha)(\lambda+1/q)}}\left\{\log 2\|A(y\|+\frac{\|A(y)\|^n}{|\det A(y)|}\right\}dy\right).
\end{aligned}$$

It remains to estimate J_3. For this purpose we proceed as follows

$$\begin{aligned}
&\|(b(A(y)\cdot) - b_{A(y)B})f(A(y)\cdot)\|_{L^q(B;v)} \\
&= \left(\int_B \left|(b(A(y)x) - b_{A(y)B})f(A(y)x)\right|^q v(x)dx\right)^{1/q} \\
&= |\det A^{-1}(y)|^{1/q} \left(\int_{A(y)B} \left|(b(x) - b_{A(y)B})f(x)\right|^q v(A^{-1}(y)x)dx\right)^{1/q} \\
&\preceq |\det A^{-1}(y)|^{1/q} \|A^{-1}(y)\|^{\alpha/q} \|(b(\cdot) - b_{A(y)B})f(\cdot)\|_{L^q(A(y)B;v)} \\
&\preceq |\det A^{-1}(y)|^{1/q} \|A^{-1}(y)\|^{\alpha/q} \|b - b_{A(y)B}\|_{L^{q_2}(A(y)B;v)} \|f\|_{L^{q_1}(A(y)B;v)} \\
&\preceq v(B)^{\lambda+1/q} \|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,v)} \|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,v)} \\
&\times \ |\det A^{-1}(y)|^{1/q} \|A(y)\|^{(n+\alpha)(\lambda+1/q)} \|A^{-1}(y)\|^{\alpha/q}.
\end{aligned}$$

Hence,

$$\begin{aligned}
J_3 &\preceq v(B)^{\lambda+1/q} \|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,v)} \|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,v)} \\
&\quad \times \int_{\mathbb{R}^n} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q} \|A(y)\|^{(n+\alpha)(\lambda+1/q)} \|A^{-1}(y)\|^{\alpha/q} dy.
\end{aligned}$$

Coupling the estimates of J_i $(i = 1, 2, 3)$, we obtain

$$\|H^b_{\Phi,A} f\|_{\dot{M}^{q,\lambda}(\mathbb{R}^n;v)} \preceq K_6 \|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,v)} \|f\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,v)}.$$

Thus assertion made in part (i) is fulfilled.

(ii) Using Proposition 3.2.7 along with an argument as given above, the proof of this part becomes simpler. We thus finish the proof.

Proof of Theorem 3.4.3. (i) If $\|A^{-1}(y)\| \preceq \|A(y)\|^{-1}$, then (3.3.7) is valid. The sufficient part of Theorem 3.4.3 can be easily obtained from Theorem 3.4.2. Next we will show the necessary part.

For $-1/q < \lambda < 0$, choose $f_0(x) = |x|^{(n+\alpha)\lambda}$. It is simple to get that $f_0 \in \dot{M}^{q_1,\lambda}(\mathbb{R}^n; v)$ and $\|f_0\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n;v)} = |S^{n-1}|^{-\lambda}(n+\alpha)^\lambda(1+\lambda q_1)^{-1/q_1}$. Assume that $H^{b,1}_{\phi,A}$ is continuous from $\dot{M}^{q_1,\lambda}$ to $\dot{M}^{q,\lambda}$ for all $b \in \|b\|_{C\dot{M}O^{q_2}(\mathbb{R}^n,v)}$. Taking $b_0 = \log|x|$, then by Lemma 2.3 in [20], $b \in C\dot{M}O^{q_2}(\mathbb{R}^n, v)$. Noting that $(n+\alpha)\lambda < 0$, we have

$$\begin{aligned}
H^{b_0,1}_{\Phi,A} f_0(x) &= \int_{\|A(y)\|<1} \frac{\Phi(y)}{|y|^n} |A(y)x|^{(n+\alpha)\lambda} \log\left(\frac{|A(y)x|}{|x|}\right)^{-1} dy \\
&\succeq f_0(x) \int_{\|A(y)\|<1} \frac{\Phi(y)}{|y|^n} \|A(y)\|^{(n+\alpha)\lambda} \log\frac{1}{\|A(y)\|} dy.
\end{aligned}$$

Hence,

$$\|H^{b_0,1}_{\Phi,A}\|_{\dot{M}^{q_1,\lambda}(\mathbb{R}^n,v)\to\dot{M}^{q,\lambda}(\mathbb{R}^n,v)} \succeq \int_{\|A(y)\|<1} \frac{\Phi(y)}{|y|^n}\|A(y)\|^{(n+\alpha)\lambda}\log\frac{1}{\|A(y)\|}dy.$$

Therefore, we obtain

$$\int_{\|A(y)\|<1} \frac{\Phi(y)}{|y|^n}\|A(y)\|^{(n+\alpha)\lambda}\log\frac{1}{\|A(y)\|}dy<\infty. \tag{3.4.10}$$

Contrarily,

$$\begin{aligned}&\int_{\|A(y)\|\leq 1/2} \frac{\Phi(y)}{|y|^n}\|A(y)\|^{(n+\alpha)\lambda}dy\\ &\leq \int_{\|A(y)\|\leq 1/2} \frac{\Phi(y)}{|y|^n}\|A(y)\|^{(n+\alpha)\lambda}\log\frac{1}{\|A(y)\|}dy.\end{aligned} \tag{3.4.11}$$

Since $\Phi(y)/|y|^n$ is integrable and $(n+\alpha)\lambda<0$, therefore

$$\int_{1/2\leq\|A(y)\|\leq 1} \frac{\Phi(y)}{|y|^n}\|A(y)\|^{(n+\alpha)\lambda}dy\leq\infty. \tag{3.4.12}$$

From (3.4.11) and (3.4.12), we get

$$\int_{\|A(y)\|\leq 1} \frac{\Phi(y)}{|y|^n}\|A(y)\|^{(n+\alpha)\lambda}dy<\infty. \tag{3.4.13}$$

It is important to note that

$$\begin{aligned}K_8=\log 2&\int_{\|A(y)\|\leq 1} \frac{\Phi(y)}{|y|^n}\|A(y)\|^{(n+\alpha)\lambda}dy\\ &+\int_{\|A(y)\|\leq 1} \frac{\Phi(y)}{|y|^n}\|A(y)\|^{(n+\alpha)\lambda}\log\frac{1}{\|A(y)\|}dy.\end{aligned}$$

Then, combining (3.4.10) and (3.4.13), we have $K_8<\infty$. This proves part (i) of Theorem 3.4.3.

(ii) In this case we replace $b_0(x)$ by $\log\frac{1}{|x|}$, then by a similar argument as given above the proof can be obtained easily.

Chapter 4
Weighted Estimates for Hardy Type Operators and Commutators

4.1 Introduction

Suppose g be a Lebesgue measurable on Euclidean space $\mathbb{R}^n$ with g^* be decreasing rearrangement of g i.e.

$$g^*(s) = \inf\{\gamma > 0 : d_g(\gamma) \leq s\}, \quad s \in [0, \infty),$$

where $d_g(\gamma)$ denotes the distribution function of g, given by:

$$d_g(\gamma) = |\{y \in \mathbb{R}^n : |g(y)| > \gamma\}|.$$

The Lorentz space $L^{p,q}(\mathbb{R}^n)$ is the set:

$$L^{p,q}(\mathbb{R}^n) = \left\{g : \|g\|_{L^{p,q}(\mathbb{R}^n)} < \infty\right\},$$

where for $0 < q, p \leq \infty$,

$$\|g\|_{L^{p,q}(\mathbb{R}^n)} = \begin{cases} \left(\int_0^\infty \left(s^{\frac{1}{p}} g^*(s)\right)^q \frac{ds}{s}\right)^{\frac{1}{q}} & \text{if } q < \infty, \\ \sup\limits_{s>0} s^{\frac{1}{p}} g^*(s) & \text{if } q = \infty. \end{cases}$$

In case $q = \infty$, it is obvious that $L^{p,\infty}(\mathbb{R}^n) =$ weak $L^p(\mathbb{R}^n)$. Also, if $p = q$ then the space $L^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. It is important to note that, for $1 < p, q < \infty$, the dual space of $L^{p,q}(\mathbb{R}^n)$ is the space $L^{p',q'}(\mathbb{R}^n)$. See [50] for more details regarding the duals of Lorentz spaces.

In recent years Morrey spaces found applications to many classes of PDE's and influence mathematical analysis in many ways. We defined these spaces in chapter 2, Definition 2.2.1. The pre-dual of Morrey space is the block space $B^{p,\mu}(\mathbb{R}^n)$ introduced in [7]. In order to define $B^{p,\mu}(\mathbb{R}^n)$ we first give the following definition:

Definition 4.1.1 *(see, [7]) Let* $1 \leq p < \infty$ *and* $n > \mu \geq 0$. *A measurable function* b *is said to a* (p, μ) *block if* $\mathrm{supp(b) = B(x_0, r), \quad x_0 \in \mathbb{R}^n, \quad r > 0}$, *and*

$$\|b\|_{L^p(\mathbb{R}^n)} \leq r^{-\frac{\mu}{p}}.$$

The space $B^{p,\mu}(\mathbb{R}^n)$ can be defined as:

$$B^{p,\mu}(\mathbb{R}^n) = \left\{ \sum_{k=1}^{\infty} \mu_k b_k : \sum_{k=1}^{\infty} |\mu_k| < \infty \text{ and } b_k \text{ is a } (p, \mu) \text{ block} \right\},$$

having norm

$$\|g\|_{B^{p,\mu}(\mathbb{R}^n)} = \inf \left\{ \sum_{k=1}^{\infty} |\mu_k| : g = \sum_{k=1}^{\infty} \mu_k b_k \right\},$$

where infimum is taken over all such decompositions of g.

The duality between block and Morrey spaces is further extending in [63], where the Hölder's inequality for $B^{p',\mu}(\mathbb{R}^n)$ and $M^{p,\mu}(\mathbb{R}^n)$ is provided in the form of following lemma:

Lemma 4.1.2 *Suppose* $\infty > p > 1$ *and* $n > \mu \geq 0$, *then*

$$\int_{\mathbb{R}^n} |h(y)g(y)|dy \leq \|h\|_{M^{p,\mu}(\mathbb{R}^n)} \|g\|_{B^{p',\mu}(\mathbb{R}^n)}.$$

In this chapter, we shall prove that fractional Hausdorff type operator is continuous from Morrey space to the weak Lebesgue space with power weights and from Lorentz space to the same without weights. Furthermore, these boundedness results can be used to prove Lipschitz type estimates for H^b_Φ. Finally, in the last section, we shall obtain weighted Lipschitz estimates for the Hardy operator's commutators in weighted central Morrey spaces.

4.2 Weak Type Inequalities for the Fractional Hausdorff Operator

Let $t \in \mathbb{R} \backslash \{0\}$ and for any measurable function f on $\mathbb{R}^n$, let D_t denotes the dilation operator i.e.

$$(D_t f)(y) = f(y/t), \quad x \in \mathbb{R}^n,$$

then the following lemma is from [63].

Lemma 4.2.1 *Let* $\infty > p > 1$ *and* $n > \mu \geq 0$. *Then*

$$\|D_t f\|_{B^{p,\mu}(\mathbb{R}^n)} = |t|^{\frac{n+\mu}{p}} \|f\|_{B^{p,\mu}(\mathbb{R}^n)}.$$

Above lemma is very useful in proving results of this chapter stated below.

Theorem 4.2.2 *Suppose* $n > \alpha \geq 0$ *and*

$$\Psi_\alpha(y) = \Phi\left(\frac{1}{|y|}\right)|y|^{\alpha-n}.$$

Assume that $1 < p, q < \infty,\ 0 < \mu < 1,\ n > (p-1)\mu + \alpha p$ *and* $1/p - (\alpha+\mu)/(n+\mu) = 1/q$. *If* Φ *is a radial function then*

$$\|H_{\Phi,\alpha}f\|_{L^{q,\infty}(|x|^\mu)} \leq \left(\frac{|S^{n-1}|}{n+\mu}\right)^{1/q} \|\Psi_\alpha\|_{B^{p',\mu}(\mathbb{R}^n)}\|f\|_{M^{p,\mu}(\mathbb{R}^n)}.$$

Proof. Let

$$\Psi_\alpha(y) = \frac{\Phi(1/|y|)}{|y|^{n-\alpha}}.$$

then

$$\Phi(|x|/|y|)|y|^{\alpha-n} = |x|^{-n+\alpha}\Psi_\alpha\left(y/|x|\right).$$

Next, with the help of Lemma 4.1.2 and 4.2.1, one has

$$\begin{aligned}
|H_{\Phi,\alpha}f(x)| &\leq \int_{\mathbb{R}^n}\left|\frac{\Phi(|x|/|y|)}{|y|^{n-\alpha}}f(y)\right|dy \\
&= |x|^{-n+\alpha}\int_{\mathbb{R}^n}\left|\Psi_\alpha\left(\frac{y}{|x|}\right)f(y)\right|dy \\
&\leq |x|^{-n+\alpha}\left\|\Psi_\alpha\left(\frac{\cdot}{|x|}\right)\right\|_{B^{p',\mu}(\mathbb{R}^n)}\|f\|_{M^{p,\mu}(\mathbb{R}^n)} \\
&= |x|^{(\alpha+\mu)-(n+\mu)/p}\|\Psi_\alpha\|_{B^{p',\mu}(\mathbb{R}^n)}\|f\|_{M^{p,\mu}(\mathbb{R}^n)} \\
&= |x|^{-(n+\mu)/q}\|\Psi_\alpha\|_{B^{p',\mu}(\mathbb{R}^n)}\|f\|_{M^{p,\mu}(\mathbb{R}^n)},
\end{aligned}$$

where, in the last equation, we have used the condition $1/p - (\alpha+\mu)/(n+\mu) = 1/q$.

For convenience we use the following notation $C = \|\Psi_\alpha\|_{B^{p',\mu}(\mathbb{R}^n)}\|f\|_{M^{p,\mu}(\mathbb{R}^n)}$, then we have

$$\begin{aligned}
\|H_{\Phi,\alpha}f\|_{L^{q,\infty}(|x|^\mu)} &\leq \sup_{\gamma>0}\gamma\left(\int_{\mathbb{R}^n}\chi_{\{C|x|^{-\frac{n+\mu}{q}}>\gamma\}}(x)|x|^\mu dx\right)^{1/q} \\
&\leq \sup_{\gamma>0}\gamma\left(|S^{n-1}|\int_0^{(C\gamma^{-1})^{\frac{q}{n+\mu}}} r^{n+\mu-1}dr\right)^{1/q} \\
&= C\left(\frac{|S^{n-1}|}{n+\mu}\right)^{1/q}.
\end{aligned}$$

Thus we conclude that

$$\|H_{\Phi,\alpha}f\|_{L^{q,\infty}(|x|^\mu)} \leq \left(\frac{|S^{n-1}|}{n+\mu}\right)^{1/q} \|\Psi_\alpha\|_{B^{p',\mu}(\mathbb{R}^n)}\|f\|_{M^{p,\mu}(\mathbb{R}^n)}.$$

An obvious corollary of the above result can be stated as:

Corollary 4.2.3 *Under same conditions as stated in Theorem 4.4.1 with the exception that now* $\alpha = 0$, *then we have*

$$\|\widetilde{H}_\Phi f\|_{L^{q,\infty}(|x|^\mu)} \leq \left(\frac{|S^{n-1}|}{n+\mu}\right)^{1/q} \|\Psi_0\|_{B^{p',\mu}(\mathbb{R}^n)}\|f\|_{M^{p,\mu}(\mathbb{R}^n)}.$$

In the proof of next theorem, we employ duality between Lorentz spaces $L^{p,q}(\mathbb{R}^n)$ is the space $L^{p',q'}(\mathbb{R}^n)$.

Theorem 4.2.4 *Let* $0 \leq \alpha < n$ *with*

$$\phi_\alpha(x') = \left\||\cdot|^{\alpha-n}\Phi(x'/|\cdot|)\right\|_{L^{p',q'}(\mathbb{R}^n)}.$$

Assume that $1 < r, q, p < \infty,\ r > \max\{\mathrm{p},\mathrm{q}\},\ \mathrm{n} > \alpha\mathrm{p}$ *and* $1/p - \alpha/n = 1/r$. *If* Φ *is a radial function and* $\phi_\alpha(\cdot) \in L^\infty(S^{n-1})$, *then*

$$\|H_{\Phi,\alpha}f\|_{L^{r,\infty}(\mathbb{R}^n)} \leq \left(\frac{|S^{n-1}|}{n}\right)^{1/r} \|\phi_\alpha\|_{L^\infty(S^{n-1})}\|f\|_{L^{p,q}(\mathbb{R}^n)}.$$

Proof. Using duality, we obtain

$$\begin{aligned}|H_{\Phi,\alpha}f(x)| &\leq \int_{\mathbb{R}^n}\left|\frac{\Phi(x/|y|)}{|y|^{n-\alpha}}f(y)\right|dy \\ &\leq \|h_x(\cdot)\|_{L^{p',q'}(\mathbb{R}^n)}\|f\|_{L^{p,q}(\mathbb{R}^n)},\end{aligned}$$

where $h_x(y) = \frac{\Phi(x/|y|)}{|y|^{n-\alpha}}$. Now, it becomes simple to show that

$$\begin{aligned}\|h_x(\cdot)\|_{L^{p',q'}(\mathbb{R}^n)} &= \left\|\frac{\Phi(x/|\cdot|)}{|\cdot|^{n-\alpha}}\right\|_{L^{p',q'}(\mathbb{R}^n)} \\ &= |x|^{\alpha-n}\left\|\Phi\left(x'\frac{|x|}{|\cdot|}\right)\frac{|x|^{n-\alpha}}{|\cdot|^{n-\alpha}}\right\|_{L^{p',q'}(\mathbb{R}^n)} \\ &= |x|^{\alpha-n/p}\left\|\frac{\Phi(x'/|\cdot|)}{|\cdot|^{n-\alpha}}\right\|_{L^{p',q'}(\mathbb{R}^n)},\end{aligned}$$

where we have used the dilation property of Lorentz spaces $L^{p',q'}(\mathbb{R}^n)$. Making use of the condition $1/p - \alpha/n = 1/r$, we obtain

$$|H_{\Phi,\alpha}f(x)| \leq |x|^{-n/r}\|\phi_\alpha(\cdot)\|_{L^\infty(S^{n-1})}\|f\|_{L^{p,q}(\mathbb{R}^n)}.$$

For convenience we fix $C_1 = \|\phi_\alpha(\cdot)\|_{L^\infty(S^{n-1})}\|f\|_{L^{p,q}(\mathbb{R}^n)}$ and compute:

$$\begin{aligned}|\{x \in \mathbb{R}^n : |H_{\Phi,\alpha}f(x)| > \gamma\}| &\leq |\{x \in \mathbb{R}^n : C_1|x|^{-\frac{n}{r}} > \gamma\}| \\ &\leq \left|\left\{x \in \mathbb{R}^n : |x| < \left(C_1\gamma^{-1}\right)^{\frac{r}{n}}\right\}\right| \\ &= \frac{|S^{n-1}|}{n}\left(C_1\gamma^{-1}\right)^r.\end{aligned}$$

We thus obtain

$$\|H_{\Phi,\alpha}f\|_{L^{r,\infty}(\mathbb{R}^n)} \leq \left(\frac{|S^{n-1}|}{n}\right)^{1/r}\|\phi_\alpha\|_{L^\infty(S^{n-1})}\|f\|_{L^{p,q}(\mathbb{R}^n)}.$$

When $\alpha = 0$, we obtain the estimate for $\widetilde{H}_\Phi$ that can be stated in the form of following corollary.

Corollary 4.2.5 *Let*

$$\phi_0(x') = \left\|\frac{\Phi(x'/|\cdot|)}{|\cdot|^n}\right\|_{L^{p',q'}(\mathbb{R}^n)}.$$

Assume that $1 < p, q < \infty$, *If* Φ *is a radial function and* $\phi_0(\cdot) \in L^\infty(S^{n-1})$, *then*

$$\|\widetilde{H}_\Phi f\|_{L^{p,\infty}(\mathbb{R}^n)} \leq \left(\frac{|S^{n-1}|}{n}\right)^{1/p}\|\phi_0\|_{L^\infty(S^{n-1})}\|f\|_{L^{p,q}(\mathbb{R}^n)}.$$

Thus the Hausdorff operator $\widetilde{H}_\Phi$ maps $L^{p,q}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$.

4.3 Lipschitz Estimates for the Commutators of Hausdorff Operator $\widetilde{H}_\Phi$

In this section we will show that $\widetilde{H}^b_\Phi$ posses similar boundedness results as followed by fractional Hausdorff operator $\widetilde{H}_{\Phi,\alpha}$ in Theorem 5.4.1 and 5.4.2, when b belongs to homogeneous Lipschitz space $\dot{\mu}_\beta(\mathbb{R}^n)$.

Theorem 4.3.1 *Let* $1 > \beta > 0$ *and*

$$\Psi_0(y) = |y|^n\Phi(1/|y|), \Psi_\beta(y) = |y|^{n-\beta}\Phi(1/|y|).$$

Suppose $1 < p, q < \infty,\ n - \beta p > (p-1)\mu$ *and* $1/q = 1/p - (\beta+\mu)/(n+\mu)$. *If* Φ *is a radial function and* $b \in \dot{\mu}_\beta(\mathbb{R}^n)$ *then*

$$\|\widetilde{H}^b_\Phi f\|_{L^{q,\infty}(|x|^\mu)} \leq \left(\frac{|S^{n-1}|}{n+\mu}\right)^{1/q} C_{\Psi_0,\Psi_\beta}\|b\|_{\dot{\mu}_\beta(\mathbb{R}^n)}\|f\|_{M^{p,\mu}(\mathbb{R}^n)}$$

where

$$C_{\Psi_0,\Psi_\beta} = max\left\{\|\Psi_0\|_{B^{p',\mu}(\mathbb{R}^n)}, \|\Psi_\beta\|_{B^{p',\mu}(\mathbb{R}^n)}\right\}.$$

Proof. Consider

$$
\begin{aligned}
|\widetilde{H}^b_\Phi f(x)| &\le \|b\|_{\dot{\mu}_\beta(\mathbb{R}^n)}|x|^\beta \int_{\mathbb{R}^n}\left|\frac{\Phi(|x|/|y|)}{|y|^n}f(y)\right|dy \\
&+ \|b\|_{\dot{\mu}_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\left|\frac{\Phi(|x|/|y|)}{|y|^{n-\beta}}f(y)\right|dy \\
&=: \|b\|_{\dot{\mu}_\beta(\mathbb{R}^n)}(|x|^\beta I_1 + I_2).
\end{aligned}
$$

Next, using computation made in Theorem 4.4.1, we obtain

$$I_2 \le |x|^{(\beta+\mu)-(n+\mu)/p}\|\Psi_\beta\|_{B^{p',\mu}(\mathbb{R}^n)}\|f\|_{M^{p,\mu}(\mathbb{R}^n)}.$$

Similarly, we estimate I_1 as:

$$I_1 \le |x|^{\mu-(n+\mu)/p}\|\Psi_0\|_{B^{p',\mu}(\mathbb{R}^n)}\|f\|_{M^{p,\mu}(\mathbb{R}^n)}.$$

Incorporating estimates of I_i $(i = 1, 2)$, one can get

$$|\widetilde{H}^b_\Phi f(x)| \le \|b\|_{\dot{\mu}_\beta(\mathbb{R}^n)}|x|^{-(n+\mu)/q}C_{\Psi_0,\Psi_\beta}\|f\|_{M^{p,\mu}(\mathbb{R}^n)}.$$

We omit the remaining proof as it is the replica of the proof of Theorem 4.4.1.

Adopting the procedure followed in Theorem 5.4.2 and Theorem 4.3.1, the next Theorem can be proved easily.

Theorem 4.3.2 *Suppose* $1 > \beta > 0$ *and*

$$\phi_0(x') = \left\|\frac{\Phi(x'/|\cdot|)}{|\cdot|^n}\right\|_{L^{p',q'}(\mathbb{R}^n)}, \ \phi_\beta(x') = \left\|\frac{\Phi(x'/|\cdot|)}{|\cdot|^{n-\beta}}\right\|_{L^{p',q'}(\mathbb{R}^n)}.$$

Assume that $1 < r, q, p < \infty$, $\max\{\mathrm{q}, \mathrm{p}\} < \mathrm{r}$, $\beta\mathrm{p} < \mathrm{n}$ *and* $1/r = 1/p - \beta/n$. *If* Φ *is a radial function,* $\phi_0(\cdot), \phi_\beta(\cdot) \in L^\infty(S^{n-1})$ *and* $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *then*

$$\|\widetilde{H}^b_\Phi f\|_{L^{r,\infty}(\mathbb{R}^n)} \le \left(\frac{|S^{n-1}|}{n}\right)^{1/r} C_{\phi_0,\phi_\beta}\|b\|_{\dot{\Lambda}_\beta(\mathbb{R}^n)}\|f\|_{L^{p,q}(\mathbb{R}^n)},$$

where

$$C_{\phi_0,\phi_\beta} = max\left\{\|\phi_0\|_{L^\infty(S^{n-1})}, \|\phi_\beta\|_{L^\infty(S^{n-1})}\right\}.$$

4.4 Weighted Lipschitz Estimates for the Commutators of Hardy Operator

For locally integrable function b we consider here the following two commutators of high-dimensional Hardy operators:

$$H_b f = b(Hf) - H(bf),$$

and

$$H_b^*f = b(H^*f) - H^*(bf).$$

Fu et al [34] studied the continuity of these operators in Lebesgue spaces. Furthermore, it was proved in [35] and [122] that the commutators of Hardy operators can be used to characterize some function spaces. Besides, Gao and Wang [42] established weighted estimates for H_b and H_b^* on weighted Lebesgue space. Further extension of these weighted estimates for the commutators of rough Hardy operator (see [36] for definition) was made in [111]. However, two weight inequalities for these commutator operators were obtained in [94].

Here, our objective is to provide some sufficient conditions in support of the continuity of commutators of Hardy operator on weighted central Morrey spaces when the symbol functions belongs to weighted Lipschitz space. Thus, one of the main results can be stated as:

Theorem 4.4.1 *Let $w \in A_1$, $b \in Lip_{\beta,w}$, $0<\beta<1$, $1<q<\infty$, $-\frac{1}{q}<\mu<\mu<0$ and $\mu=\mu+\frac{\beta}{n}$, then H_b and H_b^* are bounded from $\dot{M}^{q,\mu}(w)$ to $\dot{M}^{q,\mu}(w^{1-q})$.*

The function space of our interest, other than the spaces defined in previous chapters, is the weighted Lipschitz space defined in [45] by the condition

$$\sup_{B\in\mathbb{R}^n} w(B)^{-\beta/n}\left(w(B)^{-1}\int_B |g(y)-g_B|^q w(y)^{1-q}dy\right)^{1/q} \le C<\infty,$$

where $0<\beta<1\le p\le\infty$ and $w\in A_\infty$. The smallest constant C fulfilling the condition above is considered as the norm of g in this space which is denoted by $\|g\|_{Lip^q_{\beta,w}}$. When $w=1$, the space $Lip_{\beta,w}=Lip^1_{\beta,w}$ reduces to the classical Lipschitz space Lip_β. If $w\in A_1$, then it was proved in [45] that for any $1<q\le\infty$, the norms $\|g\|_{Lip^q_{\beta,w}}$ are equivalent for various values of q, i,e. $\|g\|_{Lip^q_{\beta,w}}\sim\|g\|_{Lip_{\beta,w}}$. Here, we need some useful lemmas to proceed further.

Lemma 4.4.2 *([95]) Suppose $w\in A_1$, and $b\in Lip_{\beta,w}$, there exist a $C>0$ in such a way that for $i,j\in\mathbb{Z}$ with $i>j$,*

$$|b_{B_i}-b_{B_j}|\le C(i-j)\|b\|_{Lip_{\beta,w}}w(B_i)^{\beta/n}\frac{w(B_j)}{|B_j|}.$$

Lemma 4.4.3 *([95]) Let $w\in A_1$, then for any $1\le p<\infty$,*

$$\int_B w(y)^{1-p'}dy\le C|B|^{p'}w(B)^{1-p'},$$

where $1/p+1/p'=1$.

Proof of Theorem 4.4.1.

Suppose $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $C_k = B_k \backslash B_{k-1}$. Assume $B(0,R) = B_{k_0}$ for $k_0 \in \mathbb{Z}$. Here, we want to establish following two inequalities:

$$\left(\frac{1}{w(B_{k_0})^{1+q\mu}} \int\limits_{B_{k_0}} |H_b f(x)|^q w(x)^{1-q} dx \right)^{1/q} \le C\|b\|_{Lip_{\beta,w}} \|f\|_{\dot{M}^{q,\mu}(w)}, \tag{4.4.1}$$

$$\left(\frac{1}{w(B_{k_0})^{1+q\mu}} \int\limits_{B_{k_0}} |H_b^* f(x)|^q w(x)^{1-q} dx \right)^{1/q} \le C\|b\|_{Lip_{\beta,w}} \|f\|_{\dot{M}^{q,\mu}(w)}. \tag{4.4.2}$$

In order to construct (5.3.14), we consider

$$\begin{aligned}
\int\limits_{B_{k_0}} |H_b f(x)|^q w(x)^{1-q} dx &= \int\limits_{B_{k_0}} \left| \frac{1}{|x|^n} \int\limits_{|y|<|x|} (b(x) - b(y))\, f(y) dy \right|^q w(x)^{1-q} dx \\
&\le \sum_{k=-\infty}^{k_0} \int\limits_{C_k} \left| \frac{1}{|x|^n} \int\limits_{B_k} (b(x) - b(y))\, f(y) dy \right|^q w(x)^{1-q} dx \\
&\le C \sum_{k=-\infty}^{k_0} \int\limits_{C_k} \left| \frac{1}{|x|^n} \sum_{i=-\infty}^{k} \int\limits_{C_i} |b(x) - b_{B_k}|\, |f(y)| dy \right|^q w(x)^{1-q} dx \\
&+ C \sum_{k=-\infty}^{k_0} \int\limits_{C_k} \left| \frac{1}{|x|^n} \sum_{i=-\infty}^{k} \int\limits_{C_i} |b(y) - b_{B_k}|\, |f(y)| dy \right|^q w(x)^{1-q} dx \\
&=: I_1 + I_2.
\end{aligned}$$

In view of the fact that $A_1 \subset A_q$, Lemma 4.4.3 and the Hölder's inequality, we get

$$\begin{aligned}
\int\limits_{C_i} |f(y)| dy &\le \left(\int\limits_{B_i} |f(y)|^q w(y) dy \right)^{1/q} \left(\int\limits_{B_i} w^{-1/(q-1)}(y) dy \right)^{(q-1)/q} \\
&\le C w(B_i)^{\mu} |B_i| \|f\|_{\dot{M}^{q,\mu}(w)}.
\end{aligned} \tag{4.4.3}$$

So by Proposition 5.2.3 and the inequality (5.4.3), we estimate I_1 as

$$\begin{aligned}
I_1 &\leq C\sum_{k=-\infty}^{k_0} 2^{-knq}\int_{C_k} |b(x)-b_{B_k}|^q\, w(x)^{1-q}dx \left|\sum_{i=-\infty}^{k}\int_{C_i} |f(y)|dy\right|^q \\
&\leq C\|b\|^q_{Lip_{\beta,w}}\|f\|^q_{\dot{M}^{q,\mu}(w)}\sum_{k=-\infty}^{k_0} w(B_k)^{1+q\beta/n}\left|\sum_{i=-\infty}^{k} 2^{n(i-k)}w(B_i)^{\mu}\right|^q \\
&\leq C\|b\|^q_{Lip_{\beta,w}}\|f\|^q_{\dot{M}^{q,\mu}(w)}\sum_{k=-\infty}^{k_0} w(B_k)^{1+q\mu}\left|\sum_{i=-\infty}^{k} 2^{n(i-k)(1+\mu)}\right|^q \\
&\leq C\|b\|^q_{Lip_{\beta,w}}\|f\|^q_{\dot{M}^{q,\mu}(w)} w(B_{k_0})^{1+q\mu}\sum_{k=-\infty}^{k_0} 2^{n\delta(k-k_0)(1+\mu q)} \\
&\leq C\|b\|^q_{Lip_{\beta,w}}\|f\|^q_{\dot{M}^{q,\mu}(w)} w(B_{k_0})^{1+q\mu},
\end{aligned}$$

where we have used the condition $\mu+\beta/n=\mu$, and the fact that $-1/q<\mu<\mu<0$.

Next, we approximate I_2 by decomposing it as below

$$\begin{aligned}
I_2 &\leq C\sum_{k=-\infty}^{k_0} 2^{-knq}\int_{C_k}\left|\sum_{i=-\infty}^{k}\int_{C_i} |b(y)-b_{B_i}|\,|f(y)|dy\right|^q w(x)^{1-q}dx \\
&\quad + C\sum_{k=-\infty}^{k_0} 2^{-knq}\int_{C_k}\left|\sum_{i=-\infty}^{k}\int_{C_i} |b_{B_k}-b_{B_i}|\,|f(y)|dy\right|^q w(x)^{1-q}dx \\
&=: I_{21}+I_{22}.
\end{aligned}$$

We shall deal with each of I_{21} and I_{22}, seperatly. First we have to establish an inequality similar to (5.4.3). By Hölder's inequality

$$\int_{C_i} |b(y)-b_{B_i}||f(y)|dy \leq Cw(B_i)^{1+\mu}\|b\|_{Lip_{\beta,w}}\|f\|_{\dot{M}^{q,\mu}(w)}. \tag{4.4.4}$$

With the help of inequality (5.4.4) , Proposition 5.2.3 and Lemma 4.4.3, I_{21} reduces to

$$\begin{aligned}
I_{21} &\le C\|b\|^q_{Lip_{\beta,w}}\|f\|^q_{\dot{M}^{q,\mu}(w)}\sum_{k=-\infty}^{k_0} 2^{-knq}\int\limits_{C_k} w(x)^{1-q}dx\left|\sum_{i=-\infty}^{k} w(B_i)^{1+\mu}\right|^q \\
&\le C\|b\|^q_{Lip_{\beta,w}}\|f\|^q_{\dot{M}^{q,\mu}(w)}\sum_{k=-\infty}^{k_0} w(B_k)^{1-q}\left|\sum_{i=-\infty}^{k} w(B_i)^{1+\mu}\right|^q \\
&\le C\|b\|^q_{Lip_{\beta,w}}\|f\|^q_{\dot{M}^{q,\mu}(w)}\sum_{k=-\infty}^{k_0} w(B_k)^{1+q\mu}\left|\sum_{i=-\infty}^{k} 2^{n\delta(i-k)(1+\mu)}\right|^q \\
&\le C\|b\|^q_{Lip_{\beta,w}}\|f\|^q_{\dot{M}^{q,\mu}(w)} w(B_{k_0})^{1+q\mu}\sum_{k=-\infty}^{k_0} 2^{n\delta(k-k_0)(1+q\mu)} \\
&\le C\|b\|^q_{Lip_{\beta,w}}\|f\|^q_{\dot{M}^{q,\mu}(w)} w(B_{k_0})^{1+q\mu},
\end{aligned}$$

where the convergence of above series is due to the fact that $-1/q < \mu < 0$.

It remains to estimate I_{22}. For this purpose we again get help from Lemmas 5.2.3-4.4.3 and the inequality (5.4.3) to obtain

$$\begin{aligned}
I_{22} &\le C\|b\|^q_{Lip_{\beta,w}}\|f\|^q_{\dot{M}^{q,\mu}(w)}\sum_{k=-\infty}^{k_0} w(B_k)^{1-q+q\beta/n}\left|\sum_{i=-\infty}^{k} (k-i)w(B_i)^{1+\mu}\right|^q \\
&\le C\|b\|^q_{Lip_{\beta,w}}\|f\|^q_{\dot{M}^{q,\mu}(w)}\sum_{k=-\infty}^{k_0} w(B_k)^{1+q\mu}\left|\sum_{i=-\infty}^{k} (k-i)2^{n\delta(i-k)(1+\mu)}\right|^q \\
&\le C\|b\|^q_{Lip_{\beta,w}}\|f\|^q_{\dot{M}^{q,\mu}(w)} w(B_{k_0})^{1+q\mu}\sum_{k=-\infty}^{k_0} 2^{n\delta(k-k_0)(1+q\mu)} \\
&\le C\|b\|^q_{Lip_{\beta,w}}\|f\|^q_{\dot{M}^{q,\mu}(w)} w(B_{k_0})^{1+q\mu}.
\end{aligned}$$

Finally, we combine the estimates for I_1, I_{21} and I_{22} to have the inequality (5.3.14).

Now, we proceed to establish (5.4.2). For this purpose, we consider

$$\begin{aligned}\int_{B_{k_0}} |H_b^* f(x)|^q w(x)^{1-q} dx &= \int_{B_{k_0}} \left| \int_{|y|\geq|x|} \frac{(b(x)-b(y))}{|y|^n} f(y)dy \right|^q w(x)^{1-q} dx \\ &\leq C \int_{B_{k_0}} \left| \int_{2^{nk_0}\geq|y|\geq|x|} \frac{(b(x)-b(y))}{|y|^n} f(y)dy \right|^q w(x)^{1-q} dx \\ &+ C \int_{B_{k_0}} \left| \int_{|y|>2^{nk_0}} \frac{(b(x)-b(y))}{|y|^n} f(y)dy \right|^q w(x)^{1-q} dx \\ &=: J + J'.\end{aligned}$$

The computation of upper bounds for J is much similar to that for (5.3.14). However, estimation of J' needs more computational work. Analysis similar to H_b indicates

$$\begin{aligned}J &\leq C \int_{B_{k_0}} \left| \frac{1}{|x|^n} \int_{|y|<2^{nk_0}} |(b(x)-b(y))f(y)|dy \right|^q w(x)^{1-q} dx \\ &\leq C \sum_{k=-\infty}^{k_0} \int_{C_k} \left| \frac{1}{|x|^n} \sum_{i=-\infty}^{k} \int_{C_i} |b(x)-b(y)f(y)|dy \right|^q w(x)^{1-q} dx \\ &\leq C\|b\|^q_{Lip_{\beta,w}} \|f\|^q_{\dot{M}^{q,\mu}(w)} w(B_{k_0})^{1+q\mu}.\end{aligned}$$

In computing upper bound for J', we proceed as below

$$\begin{aligned}J' &\leq C \int_{B_{k_0}} \left| \sum_{k=k_0}^{\infty} \int_{C_k} \frac{|b(x)-b_{B_{k_0}}|}{|y|^n} |f(y)|dy \right|^q w(x)^{1-q} dx \\ &+ C \int_{B_{k_0}} \left| \sum_{k=k_0}^{\infty} \int_{C_k} \frac{|b(y)-b_{B_{k_0}}|}{|y|^n} |f(y)|dy \right|^q w(x)^{1-q} dx \\ &=: J_1' + J_2'.\end{aligned}$$

We use inequality (5.4.3) and Proposition 5.2.3 for the analysis of J_1'. So that we have

$$\begin{aligned} J_1' &\le C \int\limits_{B_{k_0}} |b(x)-b_{B_{k_0}}|^q w(x)^{1-q} dx \left| \sum_{k=k_0}^{\infty} \int\limits_{C_k} \frac{|f(y)|}{|y|^n} dy \right|^q \\ &\le C\|b\|^q_{Lip_{\beta,w}} \|f\|^q_{\dot{M}^{q,\mu}(w)} w(B_{k_0})^{1+q\beta/n} \left| \sum_{k=k_0}^{\infty} w(B_k)^{\mu} \right|^q \\ &\le C\|b\|^q_{Lip_{\beta,w}} \|f\|^q_{\dot{M}^{q,\mu}(w)} w(B_{k_0})^{1+q\mu} \left| \sum_{k=k_0}^{\infty} 2^{n\delta(k-k_0)\mu} \right|^q \\ &\le C\|b\|^q_{Lip_{\beta,w}} \|f\|^q_{\dot{M}^{q,\mu}(w)} w(B_{k_0})^{1+q\mu}. \end{aligned}$$

To get the boundedness of J_2', we need the following decomposition

$$\begin{aligned} J_2' &\le C \int\limits_{B_{k_0}} \left| \sum_{k=k_0}^{\infty} \int\limits_{C_k} \frac{|b(y)-b_{B_k}|}{|y|^n} |f(y)| dy \right|^q w(x)^{1-q} dx \\ &+ C \int\limits_{B_{k_0}} \left| \sum_{k=k_0}^{\infty} \int\limits_{C_k} \frac{|b_{B_k}-b_{B_{k_0}}|}{|y|^n} |f(y)| dy \right|^q w(x)^{1-q} dx \\ &=: J_{21}' + J_{22}'. \end{aligned}$$

We first compute J_{21}'. To do this, we use Proposition 5.2.3, Lemma 4.4.3 and the inequality (5.4.4) to obtain

$$\begin{aligned} J_{21}' &\le C \int\limits_{B_{k_0}} \left| \sum_{k=k_0}^{\infty} 2^{-kn} \int\limits_{C_k} |b(y)-b_{B_k}||f(y)| dy \right|^q w(x)^{1-q} dx \\ &\le C\|b\|_{Lip_{\beta,w}} \|f\|_{\dot{M}^{q,\mu}(w)} \int\limits_{B_{k_0}} w(x)^{1-q} dx \left| \sum_{k=k_0}^{\infty} 2^{-kn} w(B_k)^{1+\mu} \right|^q \\ &\le C\|b\|_{Lip_{\beta,w}} \|f\|_{\dot{M}^{q,\mu}(w)} |B_{k_0}|^q w(B_{k_0})^{1-q} \left| \sum_{k=k_0}^{\infty} 2^{-kn} w(B_k)^{1+\mu} \right|^q \\ &\le C\|b\|_{Lip_{\beta,w}} \|f\|_{\dot{M}^{q,\mu}(w)} w(B_{k_0})^{1+q\mu} \left| \sum_{k=k_0}^{\infty} 2^{n(k-k_0)\mu} \right|^q \\ &\le C\|b\|_{Lip_{\beta,w}} \|f\|_{\dot{M}^{q,\mu}(w)} w(B_{k_0})^{1+q\mu}. \end{aligned}$$

The last objective is to analyze the integral J'_{22}. For this we use Lemmas 5.2.3-4.4.3 and the inequality (5.4.3) to have

$$
\begin{aligned}
J'_{22} &\leq C \int\limits_{B_{k_0}} w(x)^{1-q} dx \left| \sum_{k=k_0}^{\infty} 2^{-kn} |b_{B_k} - b_{B_{k_0}}| \int\limits_{C_k} |f(y)| dy \right|^q \\
&\leq C \|b\|_{Lip_{\beta,w}} \|f\|_{\dot{M}^{q,\mu}(w)} |B_{k_0}|^q w(B_{k_0})^{1-q} \left| \sum_{k=k_0}^{\infty} (k-k_0) w(B_k)^{\mu+\beta/n} \frac{w(B_{k_0})}{|B_{k_0}|} \right|^q \\
&\leq C \|b\|_{Lip_{\beta,w}} \|f\|_{\dot{M}^{q,\mu}(w)} w(B_{k_0}) \left| \sum_{k=k_0}^{\infty} (k-k_0) w(B_k)^{\mu} \right|^q \\
&\leq C \|b\|_{Lip_{\beta,w}} \|f\|_{\dot{M}^{q,\mu}(w)} w(B_{k_0})^{1+q\mu} \left| \sum_{k=k_0}^{\infty} (k-k_0) 2^{n\delta(k-k_0)\mu} \right|^q \\
&\leq C \|b\|_{Lip_{\beta,w}} \|f\|_{\dot{M}^{q,\mu}(w)} w(B_{k_0})^{1+q\mu}.
\end{aligned}
$$

By incorporating the estimates of J, J'_1, J'_{21} and J'_{22}, we get (5.4.2). With this we finish the proof.

Chapter 5
Weighted Estimates for Hausdorff Operators and Commutators on Heisenberg Group

5.1 Introduction

Besides the Euclidean space $\mathbb{R}^n$, the matrix Hausdorff operator can be defined on p-adic linear space $\mathbb{Q}_p^n$, which is, under addition, a locally compact commutative group (see, for instance, [125, 126]) and on the Heisenberg group $\mathbb{H}^n$ [103, 117, 137]. Since, we are mainly concerned with the study of Hausdorff operators commutators defined on the Heisenberg group $\mathbb{H}^n$, therefore, it is mandatory to introduce this group briefly and the definition of matrix Hausdorff operator on it first.

With underlying manifold $\mathbb{R}^{2n} \times \mathbb{R}$, the Heisenberg group $\mathbb{H}^n$, under the law of non-commutative multiplication

$$\begin{aligned} x \cdot y &= (x_1, x_2, ..., x_{2n+1}) \cdot (y_1, y_2, ..., y_{2n+1}) \\ &= \left(x_1 + y_1, ..., x_{2n} + y_{2n}, x_{2n+1} + y_{2n+1} + 2\sum_{j=1}^{n} (y_j x_{n+j} - x_j y_{n+j}) \right). \end{aligned}$$

is a Lie group. This definition suggests that for $z \in \mathbb{H}^n$,

$$z \cdot -z = 0 \text{ and } z \cdot 0 = z.$$

Therefore, the identity and inverse elements of $\mathbb{H}^n$ are same as as that of $\mathbb{R}^{2n+1}$ space. The vector fields

$$\begin{aligned} Z_j &= \frac{\partial}{\partial z_j} + 2z_{n+j}\frac{\partial}{\partial z_{2n+1}}, \quad 1 \leq j \leq n, \\ Z_{n+j} &= \frac{\partial}{\partial z_{n+j}} - 2z_j\frac{\partial}{\partial z_{2n+1}}, \quad 1 \leq j \leq n, \\ Z_{2n+1} &= \frac{\partial}{\partial z_{2n+1}}. \end{aligned}$$

form the basis for this Lie algebra. The only non-vanishing commutator relations satisfied by these vector fields are

$$[Z_j, Z_{n+j}] = -4Z_{2n+1}, \quad 1 \leq j \leq n.$$

The dilation, on $\mathbb{H}^n$, is defined as

$$\delta_t(z_1, z_2, ..., z_{2n}, z_{2n+1}) = (tz_1, tz_2, tz_{2n}, t^2 z_{2n+1}), \quad t > 0.$$

Also, the Haar measure on $\mathbb{H}^n$ corresponds to the usual Lebesgue measure on $\mathbb{R}^{2n} \times \mathbb{R}^1$. Thus, for any measurable set $E \subset \mathbb{H}^n$, its measure is denoted by $|E|$. Moreover,

$$|\delta_t(E)| = t^Q|E|, \qquad d(\delta_t z) = t^Q dz,$$

where $Q = 2n + 2$ is the so-called homogeneous dimension of $\mathbb{H}^n$.

The group $\mathbb{H}^n$ is endowed with the norm:

$$|z|_h = \left[\left(\sum_{i=1}^{2n} z_i^2\right)^2 + z_{2n+1}^2\right]^{1/4},$$

and the Heisenberg distance d, generated by this norm is:

$$d(s,t) = d(t^{-1}s, 0) = |t^{-1}s|_h.$$

Notice that d satisfies triangular inequality and is left-invariant i.e.

$$d(r \cdot s, r \cdot t) = d(s,t), \quad \forall\ r, s, t \in \mathbb{H}^n.$$

The ball and sphere on $\mathbb{H}^n$, for $t > 0$ and $x \in \mathbb{H}^n$, can be defined as

$$B(z,t) = \{y \in \mathbb{H}^n : d(z,y) < t\},$$

and

$$S(z,t)=\{y\in\mathbb{H}^n : d(z,y)=t\},$$

respectively. To compute the measure of this ball on $\mathbb{H}^n$, we proceed as below

$$|B(z,t)|=|B(0,t)|=\Omega_Q t^Q,$$

where $\Omega_Q=|B(0,1)|$, is a function of n only, is the volume of the unit ball . Also, area of the unit sphere $S(0,1)$ in $\mathbb{H}^n$ is $w_Q=Q\Omega_Q$. For further readings on Heisenberg group, the book by Folland and Stein [33] and the works by authors in [52, 77, 78] are standard references.

Now, we are in position to define the operator and relevant commutators on the Heisenberg group $\mathbb{H}^n$. Suppose $\Phi\in L^1_{\text{loc}}(\mathbb{H}^n)$. The Hausdorff operators on $\mathbb{H}^n$ assume the following form (see [54] and [117]):

$$\begin{aligned}T_\Phi g(x) &= \int_{\mathbb{H}^n}\frac{\Phi(\delta_{|z|_h^{-1}}x)}{|z|_h^Q}g(z)dz,\\ T_{\Phi,A}f(x) &= \int_{\mathbb{H}^n}\frac{\Phi(z)}{|z|_h^Q}g(A(z)x)dz,\end{aligned}$$

where the function $A(z)$ is a matrix-valued function, and we assume that $\det A(z)\neq 0$ almost every where in the support of Φ. Also, we define the commutators $T^b_{\Phi,A}$ of $T_{\Phi,A}$ with locally integrable function b as

$$T^b_{\Phi,A}(g)=bT_{\Phi,A}(g)-T_{\Phi,A}(bg). \tag{5.1.1}$$

In this chapter, we study the boundedness of T_Φ, $T_{\Phi,A}$ and $T^b_{\Phi,A}$ on the Herz-type spaces with both Muckenhoupt and power weights with the Heisenberg group as underlying space. The next section contains some basic definitions and notations likewise some necessary propositions which will be used in the succeeding sections. In Section 3, we give weighted estimates for T_Φ on Herz-type spaces. Finally, the last section is reserved for the study of $T^b_{\Phi,A}$ on weighted Herz space.

5.2 Some Definitions and Notations

The Muckenhoupt weighted function theory introduced in [106] was well studied in the later work by García-Cuerva et al. [46]. An extension of this theory, in the settings of Heisenberg group $\mathbb{H}^n$, was provided in [52] and studied in [77, 78]. A function w on $\mathbb{H}^n$ can be given the role of a weight if $w\in L^1_{\text{loc}}(\mathbb{H}^n)$ and is non-negative. The notation $w(E)$ will be reserved for weighted measure of $E\subset\mathbb{H}^n$, that is $w(E)=\int_E w(z)dz$. Also, if $1/p+1/p'=1$, then p and p' will be called mutually conjugate indices.

Definition 5.2.1 *It is said that w is in the Muckenhoupt class $A_p(\mathbb{H}^n)$, $\infty > p > 1$, if there exists a $C > 0$ in such a way that for every ball $B \subset \mathbb{H}^n$,*

$$\left(|B|^{-1}\int_B w(z)dz\right)\left(|B|^{-1}\int_B w(z)^{-p'/p}dz\right)^{p/p'} \leq C.$$

Also, $w \in A_1$ implies the existence of a constant $C > 0$ in such a way that for every ball $B \subset \mathbb{H}^n$,

$$\left(|B|^{-1}\int_B w(z)dz\right) \leq C \operatorname*{essinf}_{z\in B} w(z).$$

When $p = \infty$, we define $A_\infty = \bigcup_{1\leq p<\infty} A_p$.

According to Proposition 2.2 in [117], we have $A_p(\mathbb{H}^n) \subset A_q(\mathbb{H}^n)$, for $\infty > q > p \geq 1$, and if $w \in A_p(\mathbb{H}^n), \infty > p > 1$, then for $\epsilon > 0$ there exist $p - \epsilon > 1$ such that $w \in A_{p-\epsilon}(\mathbb{H}^n)$. Therefore, we may take $q_w := \inf\{q > 1 : w \in A_q\}$ as the critical index of w.

Definition 5.2.2 *It is said that w is in reverse Hölder class $RH_r(\mathbb{H}^n)$, if there exists a fixed constant $C > 0$ and $r > 1$, such that*

$$\left(|B|^{-1}\int_B w^r(z)dz\right)^{1/r} \leq C|B|^{-1}\int_B w(z)dz.$$

holds for every ball $B \subset \mathbb{H}^n$. In [78], it was proved that $w \in A_\infty(\mathbb{H}^n)$ if and only if there exist some $1 < r$ such that $w \in RH_r(\mathbb{H}^n)$. In addition, if $w \in RH_r(\mathbb{H}^n)$, $r > 1$, then for some $\epsilon > 0$ we have $w \in RH_{r+\epsilon}(\mathbb{H}^n)$. We therefore use $r_w := \sup\{r > 1 : w \in RH_r(\mathbb{H}^n)\}$ to denote the critical index of w for the reverse Hölder condition.

A particular case of Muckenhoupt $A_p(\mathbb{H}^n)$ weights is the power weight function $|x|_h^\alpha$. From Proposition 2.3 in [117], for $x \in \mathbb{H}^n$, we have $|x|_h^\alpha \in A_1(\mathbb{H}^n)$ if and only if $-Q < \alpha \leq 0$. Also, $|z|_h \in A_p(\mathbb{H}^n)$ for $\infty > p > 1$ if and only if $-Q < \alpha < Q(p-1)$. In view of these observation , it is easy to see that for $0 < \alpha < \infty$,

$$|x|_h^\alpha \in \bigcap_{\frac{Q+\alpha}{Q}<p<\infty} A_p(\mathbb{H}^n),$$

where $(Q + \alpha)/Q$ is known as the critical index of $|x|_h^\alpha$.

The following two Propositions, proved in [117], concerning $A_p(\mathbb{H}^n)$ weights will be useful in establishing weighted estimates for $T^b_{\Phi,A}$ on Herz-type spaces on $\mathbb{H}^n$.

Proposition 5.2.3 *Suppose $w \in RH_r \cap A_p(\mathbb{H}^n)$, $1 \leq p$ and $1 < r$. Then there exist constants $C_1, C_2 > 0$ such that*

$$C_1 \left(\frac{|D|}{|B|} \right)^p \leq \frac{w(D)}{w(B)} \leq C_2 \left(\frac{|D|}{|B|} \right)^{(r-1)/r},$$

for any $D \subseteq B$, where D is measurable. In general, for any $1 < \lambda$,

$$w(B(x_0, \lambda R) \leq \lambda^{Qp} w(B(x_0, R)).$$

Proposition 5.2.4 *If $w \in A_p(\mathbb{H}^n), 1 \leq p < \infty$, then for any $f \in L^1_{\text{loc}}(\mathbb{H}^n)$ and any ball $B \subset \mathbb{H}^n$*

$$|B|^{-1} \int_B |f(z)| dz \leq C \left(w(B)^{-1} \int_B |f(z)|^p w(z) dz \right)^{1/p}.$$

Let $E \subseteq \mathbb{H}^n$ be measurable, then the weighted Lebesgue space $L^p(E; w)$ contains the functions f satisfying

$$\|g\|_{L^p(E;w)} = \left(\int_E |g(z)|^p w(z) dz \right)^{1/p} < \infty,$$

where $\infty > p \geq 1$ and w is a weight function on $\mathbb{H}^n$. In case $p = \infty$, one has $L^\infty(\mathbb{H}^n; w) = L^\infty(\mathbb{H}^n)$ and $\|f\|_{L^\infty(\mathbb{H}^n;w)} = \|f\|_{L^\infty(\mathbb{H}^n)}$.

Let $E_k = B_k / B_{k-1}$, where $B_k := \{z \in \mathbb{H}^n : |z|_h < 2^k\}$, for $k \in \mathbb{Z}$. Then the homogeneous weighted Herz space in the setting of Heisenberg group can be defined as below.

Definition 5.2.5 *Let $\alpha \in \mathbb{R}, \infty > p, q > 0$, and w is a weight function on $\mathbb{H}^n$. The homogeneous weighted Herz space $\dot{K}_q^{\alpha,p}(\mathbb{H}^n)$ is the set*

$$\dot{K}_q^{\alpha,p}(\mathbb{H}^n; w) := \left\{ g \in L^q_{\text{loc}}(\mathbb{H}^n / \{0\}; w) : \|g\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;w)} < \infty \right\},$$

where

$$\|g\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;w)} = \left\{ \sum_{k=-\infty}^{\infty} w(B_k)^{\alpha p/Q} \|g\|^p_{L^p(E_k;w)} \right\}^{1/p}.$$

When $w = 1$, we obtain $\dot{K}_q^{\alpha,p}(\mathbb{H}^n)$ introduced in [97]. More details on Herz spaces along with their application can be seen in [28, 29, 61, 98].

Definition 5.2.6 *Suppose $\alpha \in \mathbb{R}, \infty > p, q > 0, 0 \leq \lambda$ and w is a weight function on $\mathbb{H}^n$. The homogeneous weighted Herz-Morrey space $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{H}^n; w)$ is defined by*

$$M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{H}^n; w) := \left\{ g \in L^q_{\text{loc}}(\mathbb{H}^n / \{0\}; w) : \|g\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{H}^n;w)} < \infty \right\},$$

where

$$\|g\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{H}^n;w)} = \sup_{k_0\in\mathbb{Z}} w(B_{k_0})^{-\lambda/Q} \left\{ \sum_{k=-\infty}^{k_0} w(B_k)^{\alpha p/Q} \|g\|^p_{L^q(C_k;w)} \right\}^{1/p}.$$

Obviously, $M\dot{K}^{\alpha,0}_{p,q}(\mathbb{H}^n;w) = \dot{K}^{\alpha,p}_q(\mathbb{H}^n;w)$.

Definition 5.2.7 *Suppose $1 < q < \infty$ and w be a weight function on $\mathbb{H}^n$. Then, it is said that a function $g \in L^q_{loc}(\mathbb{H}^n;w)$ belongs to the weighted CBMO space $C\dot{M}O^q(\mathbb{H}^n;w)$ if*

$$\|g\|_{C\dot{M}O^q(\mathbb{H}^n;w)} = \sup_{R>0} \left(w(B(0,R))^{-1} \int_{B(0,R)} |g(z) - g_B|^q w(z)dz \right)^{1/q} < \infty,$$

where

$$g_B = |B(0,r)|^{-1} \int_{B(0,r)} g(z)dz. \tag{5.2.1}$$

Recently, weighted boundedness of matrix Hausdorff operators and their commutators defined on different underlying spaces are established in [18, 70, 75, 76, 112, 114, 116, 117, 121].

Lemma 5.2.8 *([117]) Suppose that the $(2n+1)\times(2n+1)$ matrix M is invertible. Then*

$$\|M\|^{-Q} \leq |\det M^{-1}| \leq \|M^{-1}\|^Q, \tag{5.2.2}$$

where

$$\|M\| = \sup_{x\in\mathbb{H}^n, x\neq 0} \frac{|Mx|_h}{|x|_h}. \tag{5.2.3}$$

Also, when A_p weights are reduced to the power function, we shall use the notation $v(\cdot)$ instead of $w(\cdot)$, that is $v(\cdot) = |\cdot|^\beta_h$. In that case, an easy computation results in:

$$v(B_k) = \int_{|x|_h\leq 2^k} |x|^\beta_h\, dx = \omega_Q 2^{k(Q+\beta)}/(\beta+Q). \tag{5.2.4}$$

Moreover, in case of boundedness of $T^b_{\Phi,A}$ on power weighted Herz space, we shall frequently use the piecewise defined function G :

$$G(M,\delta\beta) = \begin{cases} \|M\|^{\delta\beta} & \text{if } \beta > 0, \\ \|M^{-1}\|^{-\delta\beta} & \text{if } \beta \leq 0, \end{cases}$$

where M is any invertible matrix, $\alpha \in \mathbb{R}$ and $\delta \in \mathbb{R}^+$. Obviously, G satisfies:

$$G(M,\beta(1/q+1/p)) = G(M,\beta/q)G(M,\beta/p), \tag{5.2.5}$$

where $p, q \in \mathbb{Z}^+$.

Proposition 5.2.9 *Suppose that the* $(2n+1)\times(2n+1)$ *matrix* M *is invertible. Let* $\beta > -n,\ v(x) = |x|_h^\beta$ *and* $x \in \mathbb{H}^n$, *then*

$$v(Mx) \le \begin{cases} \|M\|^\beta v(x) & \text{if } \beta > 0, \\ \|M^{-1}\|^{-\beta} v(x) & \text{if } \beta \le 0, \end{cases}$$
$$= G(M,\beta)v(x).$$

From this point forward, we will use an obvious notation $\lambda B(0,R) = B(0,\lambda R)$, for $\lambda > 0$.

5.3 Estimates for T_Φ on Herz-Morrey Space

As we stated in the introduction, this section is centered on obtaining estimates for T_Φ on weighted Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{H}^n;w)$. In this regard, our main results are contained in the following subsection.

5.3.1 Main Results

Theorem 5.3.1 *Let* $1 \le q_1, q_2 < \infty, 1 \le p < \infty, \lambda > \max\{0,\alpha_1\}, \alpha_2 \in \mathbb{R}$. *Let* Φ *be a radial function and* $1/q_1 + \alpha_1/Q = 1/q_2 + \alpha_2/Q$. *Let* $q_1 > q_2 s r_w/(r_w - 1)$, *where* $w \in A_s(\mathbb{H}^n), 1 \le s < \infty$, *with the critical index* r_w *for the reverse Hölder condition.*
(i) If $1/q_1 + \alpha_1/Q \ge 0$, *then we have for any* $1 < \delta < r_w$,

$$\|T_\Phi f\|_{M\dot{K}^{\alpha_2,\lambda}_{p,q_2}(\mathbb{H}^n;w)} \preceq K_1^\lambda \|f\|_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{H}^n;w)},$$

where

$$K_1^\lambda = \int_0^1 \frac{|\Phi(t)|}{t} t^{(\alpha_1+Q/q_1)(\delta-1)/\delta - s\lambda} dt + \int_1^\infty \frac{|\Phi(t)|}{t} t^{s(\alpha_1+Q/q_1)-\lambda(\delta-1)/\delta} dt.$$

(ii) If $1/q_1 + \alpha_1/Q < 0$, *then we have for any* $1 < \delta < r_w$,

$$\|T_\Phi f\|_{M\dot{K}^{\alpha_2,\lambda}_{p,q_2}(\mathbb{H}^n;w)} \preceq K_2^\lambda \|f\|_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{H}^n;w)},$$

where

$$K_2^\lambda = \int_0^1 \frac{|\Phi(t)|}{t} t^{s(\alpha_1-\lambda+Q/q_1)} dt + \int_1^\infty \frac{|\Phi(t)|}{t} t^{(\alpha_1-\lambda+Q/q_1)(\delta-1)/\delta} dt.$$

When $\lambda = 0$, we obtain the following estimates for T_Φ on $\dot{K}^{\alpha,p}_q(\mathbb{H}^n;w)$.

Theorem 5.3.2 *Let $1 \leq p < \infty, 1 \leq q_1, q_2 < \infty, \alpha_1 < 0, \alpha_2 \in \mathbb{R}$. Let Φ be a radial function and $1/q_1 + \alpha_1/Q = 1/q_2 + \alpha_2/Q$. Let $q_1 > q_2 s r_w/(r_w - 1)$, where $w \in A_s(\mathbb{H}^n), 1 \leq s < \infty$, with the critical index r_w for the reverse Hölder condition.*
(i) If $1/q_1 + \alpha_1/Q \geq 0$, then we have for any $1 < \delta < r_w$,

$$\|T_\Phi f\|_{\dot{K}_{q_2}^{\alpha_2,p}(\mathbb{H}^n;w)} \preceq K_1^0 \|f\|_{\dot{K}_{q_1}^{\alpha_1,p}(\mathbb{H}^n;w)}.$$

(ii) If $1/q_1 + \alpha_1/Q < 0$, then we have for any $1 < \delta < r_w$,

$$\|T_\Phi f\|_{\dot{K}_{q_2}^{\alpha_2,p}(\mathbb{H}^n;w)} \preceq K_2^0 \|f\|_{\dot{K}_{q_1}^{\alpha_1,p}(\mathbb{H}^n;w)}.$$

By replacing Muckenhoupt weights with power weights, one can have sharp result as below.

Theorem 5.3.3 *Suppose $\alpha \in \mathbb{R}, \lambda \geq 0, \infty > p, q <\geq 1, -Q < \beta < \infty$ and suppose Φ be a radial and non-negative function. If*

$$\int_0^\infty \frac{\Phi(t)}{t} t^{(1+\beta/Q)(\alpha-\lambda+Q/q)} dt < \infty, \tag{5.3.1}$$

then T_Φ is bounded on $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{H}^n, |\cdot|^\beta)$.

Conversely, assume that T_Φ is bounded on power weighted Herz-Morrey space $M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{H}^n, |\cdot|^\beta)$. If $\lambda = 0$ or $0 < \lambda < \alpha$, then (5.3.1) is true. Moreover,

$$\|T_\Phi\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{H}^n,|\cdot|_h^\beta) \to M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{H}^n,|\cdot|_h^\beta)} \simeq \int_0^\infty \frac{\Phi(t)}{t} t^{(1+\beta/Q)(\alpha-\lambda+Q/q)} dt. \tag{5.3.2}$$

5.3.2 Proof of the Main Results

Since, proofs of Theorems 5.3.1 and 5.3.2 involves similar steps and arguments, therefore, we will prove Theorem 5.3.1 and Theorem 5.3.3, only.

Proof of Theorem 5.3.1: We need to compute $\|T_\Phi f\|_{L^{q_2}(C_k;w)}$ first. The condition $q_1 > q_2 s r_w/(r_w - 1)$ implies that there exists $r \in (1, r_w)$ such that $q_1 = q_2 s r'$. By virtue of the Hölder inequality and the reverse Hölder condition, we obtain

$$\begin{aligned} \|T_\Phi f(x)\|_{L^{q_2}(C_k;w)} &\leq \left(\int_{C_k} |T_\Phi f(x)|^{q_1/s} dx \right)^{s/q_1} \left(\int_{C_k} w(x)^r dx \right)^{1/(rq_2)} \\ &\preceq |B_k|^{-1/(q_2 r')} w(B_k)^{1/q_2} \left(\int_{C_k} |T_\Phi f(x)|^{q_1/s} dx \right)^{s/q_1}. \end{aligned} \tag{5.3.3}$$

Following [54], we rewrite T_Φ as

$$T_\Phi f(x) = \int_0^\infty \frac{\Phi(t)}{t} \int_{|y'|_h=1} f(\delta_{t^{-1}|x|_h} y') dy' dt. \tag{5.3.4}$$

Using (5.3.4) and the Minkowski inequality, we have

$$\begin{aligned} &\left(\int_{C_k} |T_\Phi f(x)|^{q_1/s} dx \right)^{s/q_1} \\ &= \left(\int_{C_k} \left| \int_0^\infty \frac{\Phi(t)}{t} \int_{|y'|_h=1} f(\delta_{t^{-1}|x|_h} y') dy' dt \right|^{q_1/s} dx \right)^{s/q_1} \\ &\preceq \int_0^\infty \frac{|\Phi(t)|}{t} \left(\int_{C_k} \left| \int_{|y'|_h=1} f(\delta_{t^{-1}|x|_h} y') dy' \right|^{q_1/s} dx \right)^{s/q_1} dt. \end{aligned} \tag{5.3.5}$$

Next, Hölder inequality, polar decomposition and change of variables, yield

$$\begin{aligned} &\int_0^\infty \frac{|\Phi(t)|}{t} \left(\int_{C_k} \left| \int_{|y'|_h=1} f(\delta_{t^{-1}|x|_h} y') dy' \right|^{q_1/s} dx \right)^{s/q_1} dt \\ &\preceq \int_0^\infty \frac{|\Phi(t)|}{t} \left(\int_{C_k} \int_{|y'|_h=1} |f(\delta_{t^{-1}|x|_h} y')|^{q_1/s} dy' dx \right)^{s/q_1} dt \\ &\simeq \int_0^\infty \frac{|\Phi(t)|}{t} \left(\int_{2^{k-1}}^{2^k} \int_{|y'|_h=1} |f(\delta_{t^{-1}\tau} y')|^{q_1/s} dy' \tau^{Q-1} d\tau \right)^{s/q_1} dt \\ &\simeq \int_0^\infty \frac{|\Phi(t)|}{t} t^{Qs/q_1} \left(\int_{t^{-1}C_k} |f(y)|^{q_1/s} dy \right)^{s/q_1} dt, \end{aligned} \tag{5.3.6}$$

where $t^{-1}C_k$ denotes the set

$$\{x : tx \in C_k\}.$$

Making use of Proposition 5.2.4, it becomes simple to see that

$$\left(\int_{t^{-1}C_k} |f(y)|^{q_1/s} dy \right)^{s/q_1} \preceq \frac{|B(0, 2^k t^{-1})|^{s/q_1}}{w(B(0, 2^k t^{-1}))^{1/q_1}} \|f\|_{L^{q_1}(B(0,2^k t^{-1});w)}. \tag{5.3.7}$$

Therefore, we infer from (5.3.3–5.3.7) that

$$\|T_\Phi(f)\|_{L^{q_2}(C_k;w)} \preceq \int_0^\infty \frac{|\Phi(t)|}{t} \frac{w(B(0,2^k))^{1/q_2}}{w(B(0,2^k t^{-1}))^{1/q_1}} \|f\|_{L^{q_1}(B(0,2^k t^{-1});w)} dt.$$

Now, by definition and the Minkowski inequality

$$\begin{aligned}
&\|T_\Phi(f)\|_{M\dot{K}^{\alpha_2,\lambda}_{p,q_2}(\mathbb{H}^n;w)} \\
&= \sup_{k_0\in\mathbb{Z}} w(B_{k_0})^{-\lambda/Q}\left\{\sum_{k=-\infty}^{k_0} w(B_k)^{\alpha_2 p/Q}\,\|T_\Phi f\|^p_{L^q(C_k;w)}\right\}^{1/p} \\
&\preceq \int_0^\infty \frac{|\Phi(t)|}{t}\sup_{k_0\in\mathbb{Z}} w(B_{k_0})^{-\lambda/Q} \\
&\quad\times\left\{\sum_{k=-\infty}^{k_0}\left(\frac{w(B_k)^{1/q_2+\alpha_2/Q}}{w(B(0,2^k t^{-1}))^{1/q_1}}\|f\|_{L^{q_1}(B(0,2^k t^{-1});w)}\right)^p\right\}^{1/p} dt \\
&\preceq \sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}} \frac{|\Phi(t)|}{t}\sup_{k_0\in\mathbb{Z}} w(B_{k_0})^{-\lambda/Q} \\
&\quad\times\left\{\sum_{k=-\infty}^{k_0}\left(\frac{w(B_k)^{1/q_2+\alpha_2/Q}}{w(B_{k-j})^{1/q_1}}\|f\|_{L^{q_1}(B_{k-j};w)}\right)^p\right\}^{1/p} dt \\
&\preceq \sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}} \frac{|\Phi(t)|}{t}\sup_{k_0\in\mathbb{Z}} w(B_{k_0})^{-\lambda/Q}\left\{\sum_{k=-\infty}^{k_0}\left[\left(\frac{w(B_k)}{w(B_{k-j})}\right)^{1/q_1+\alpha_1/Q}\right.\right. \\
&\quad\left.\left.\times\sum_{l=j}^{\infty}\left(\frac{w(B_{k-j})}{w(B_{k-l})}\right)^{\alpha_1/Q}\left(\sum_{i=-\infty}^{k-l} w(B_i)^{\alpha_1 p/Q}\|f\|^p_{L^{q_1}(C_i;w)}\right)^{1/p}\right]^p\right\}^{1/p} dt, \qquad (5.3.8)
\end{aligned}$$

where we have used the condition $1/q_1+\alpha_1/Q = 1/q_2+\alpha_2/Q$.

Again, by definition of weighted Herz-Morrey space, the inequality (5.3.8) assumes the following form

$$\begin{aligned}
&\|T_\Phi(f)\|_{M\dot{K}^{\alpha_2,\lambda}_{p,q_2}(\mathbb{H}^n;w)} \\
&\preceq \|f\|_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{H}^n;w)}\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}\frac{|\Phi(t)|}{t}\sup_{k_0\in\mathbb{Z}}\left(\frac{w(B_{k_0-j})}{w(B_{k_0})}\right)^{\lambda/Q} \\
&\qquad\times\left\{\sum_{k=-\infty}^{k_0}\left[\left(\frac{w(B_k)}{w(B_{k-j})}\right)^{1/q_1+\alpha_1/Q}\right.\right. \\
&\qquad\left.\left.\times\left(\frac{w(B_{k-j})}{w(B_{k_0-j})}\right)^{\lambda/Q}\sum_{l=j}^{\infty}\left(\frac{w(B_{k-j})}{w(B_{k-l})}\right)^{(\alpha_1-\lambda)/Q}\right]^p\right\}^{1/p} dt. \qquad (5.3.9)
\end{aligned}$$

Since $\lambda>\max\{0,\alpha_1\}$, $j\le l$ and $k\le k_0$, by Proposition 5.2.4, we have

$$\left(\frac{w(B_{k-j})}{w(B_{k-l})}\right)^{(\alpha_1-\lambda)/Q} \preceq \left(\frac{|B_{k-j}|}{|B_{k-l}|}\right)^{(\alpha_1-\lambda)(\delta-1)/(\delta Q)} = 2^{(l-j)(\alpha_1-\lambda)(\delta-1)/\delta}, \qquad (5.3.10)$$

and

$$\left(\frac{w(B_{k-j})}{w(B_{k_0-j})}\right)^{\lambda/Q} \preceq \left(\frac{|B_{k-j}|}{|B_{k_0-j}|}\right)^{\lambda(\delta-1)/(\delta Q)} = 2^{\lambda(k-k_0)(\delta-1)/\delta}, \tag{5.3.11}$$

also, if $j \geq 0$,

$$\left(\frac{w(B_{k_0-j})}{w(B_{k_0})}\right)^{\lambda/Q} \preceq \left(\frac{|B_{k_0-j}|}{|B_{k_0}|}\right)^{\lambda(\delta-1)/(\delta Q)} = 2^{-j\lambda(\delta-1)/\delta}, \tag{5.3.12}$$

and if $j < 0$,

$$\left(\frac{w(B_{k_0-j})}{w(B_{k_0})}\right)^{\lambda/Q} \preceq \left(\frac{|B_{k_0-j}|}{|B_{k_0}|}\right)^{s\lambda/Q} = 2^{-js\lambda} \tag{5.3.13}$$

for any $1 < \delta < r_w$.

When $1/q_1 + \alpha_1/Q \geq 0$, then from Proposition 5.2.4, we have, if $j < 0$,

$$\left(\frac{w(B_k)}{w(B_{k-j})}\right)^{1/q_1+\alpha_1/Q} \preceq \left(\frac{|B_k|}{|B_{k-j}|}\right)^{(1/q_1+\alpha_1/Q)(\delta-1)/\delta} = 2^{j(Q/q_1+\alpha_1)(\delta-1)/\delta}, \tag{5.3.14}$$

and if $j \geq 0$,

$$\left(\frac{w(B_k)}{w(B_{k-j})}\right)^{1/q_1+\alpha_1/Q} \preceq \left(\frac{|B_k|}{|B_{k-j}|}\right)^{s(1/q_1+\alpha_1/Q)} = 2^{js(Q/q_1+\alpha_1)}, \tag{5.3.15}$$

for any $1 < \delta < r_w$.

When $1/q_1 + \alpha_1/Q < 0$, then it is easy to see from Proposition 5.2.4 that, if $j < 0$,

$$\left(\frac{w(B_k)}{w(B_{k-j})}\right)^{1/q_1+\alpha_1/Q} \preceq \left(\frac{|B_k|}{|B_{k-j}|}\right)^{s(1/q_1+\alpha_1/Q)} = 2^{js(Q/q_1+\alpha_1)}, \tag{5.3.16}$$

and if $j \geq 0$,

$$\left(\frac{w(B_k)}{w(B_{k-j})}\right)^{1/q_1+\alpha_1/Q} \preceq \left(\frac{|B_k|}{|B_{k-j}|}\right)^{(1/q_1+\alpha_1/Q)(\delta-1)/\delta} = 2^{j(Q/q_1+\alpha_1)(\delta-1)/\delta}, \tag{5.3.17}$$

for any $1 < \delta < r_w$.

Therefore, if $1/q_1+\alpha_1/Q\geq 0$, then for any $1<\delta<r_w$, inequalities (5.3.9–5.3.15) help us to obtain

$$\begin{aligned}
&\|T_\Phi(f)\|_{M\dot{K}^{\alpha_2,\lambda}_{p,q_2}(\mathbb{H}^n;w)}\\
&\preceq \|f\|_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{H}^n;w)}\left(\sum_{j=-\infty}^{-1}\int_{2^j}^{2^{j+1}}\frac{|\Phi(t)|}{t}2^{j(Q/q_1+\alpha_1)(\delta-1)/\delta-js\lambda}\right.\\
&\quad\times\sup_{k_0\in\mathbb{Z}}\left\{\sum_{k=-\infty}^{k_0}2^{p\lambda(k-k_0)(\delta-1)/\delta}\left(\sum_{l=j}^{\infty}2^{(l-j)(\alpha_1-\lambda)(\delta-1)/\delta}\right)^p\right\}^{1/p}dt\\
&\quad+\sum_{j=0}^{\infty}\int_{2^j}^{2^{j+1}}\frac{|\Phi(t)|}{t}2^{js(Q/q_1+\alpha_1)-j\lambda(\delta-1)/\delta}\\
&\quad\left.\times\sup_{k_0\in\mathbb{Z}}\left\{\sum_{k=-\infty}^{k_0}2^{p\lambda(k-k_0)(\delta-1)/\delta}\left(\sum_{l=j}^{\infty}2^{(l-j)(\alpha_1-\lambda)(\delta-1)/\delta}\right)^p\right\}^{1/p}dt\right)\\
&\preceq K_1^\lambda\|f\|_{M\dot{K}^{\alpha_1,\lambda}_{p,q_1}(\mathbb{H}^n;w)}.
\end{aligned}$$

With this the proof of Theorem 5.4.1(i) is completed.

By a similar argument as above, when $1/q_1+\alpha_1/Q<0$, part (ii) of Theorem 5.4.1 can be proved using (5.3.9–5.3.13) and (5.3.16–5.3.17).

Proof of Theorem 5.3.3: Computing the norm of T_Φ on the power weighted Herz-Morrey space involves a similar first step as is taken in the proof of previous Theorem. Therefore, by (5.3.4) and the Minkowski inequality we have

$$\begin{aligned}
\|T_\Phi f\|_{L^q(C_k,|\cdot|_h^\beta)} &= \left(\int_{C_k}\left|\int_0^\infty\frac{\Phi(t)}{t}\int_{|y'|_h=1}f(\delta_{t^{-1}|x|_h}y')dy'dt\right|^q|x|_h^\beta dx\right)^{1/q}\\
&\leq \int_0^\infty\frac{\Phi(t)}{t}\left(\int_{C_k}\left|\int_{|y'|_h=1}f(\delta_{t^{-1}|x|_h}y')dy'\right|^q|x|_h^\beta dx\right)^{1/q}dt.
\end{aligned}$$

By Hölder inequality, polar decomposition and change of variables, we obtain

$$\begin{aligned}
\|T_\Phi f\|_{L^q(C_k,|\cdot|_h^\beta)} &\preceq \int_0^\infty\frac{\Phi(t)}{t}\left(\int_{2^{k-1}}^{2^k}\int_{|y'|_h=1}|f(\delta_{t^{-1}\tau}y')|^q\,dy'\tau^{Q+\beta-1}d\tau\right)^{1/q}dt\\
&\preceq \int_0^\infty\frac{\Phi(t)}{t}t^{(Q+\beta)/q}\,\|f\|_{L^q(t^{-1}C_k,|\cdot|_h^\beta)}\,dt.
\end{aligned}$$

Let $j_0\in\mathbb{Z}$, be such that $2^{j_0-1}<t^{-1}\leq 2^{j_0}$. Then $t^{-1}C_k$ is contained in two adjacent annulus C_{k+j_0-1} and C_{k+j_0}. Therefore,

$$\|T_\Phi f\|_{L^q(C_k,|\cdot|_h^\beta)}\preceq\int_0^\infty\frac{\Phi(t)}{t}t^{(Q+\beta)/q}\sum_{l=j_0-1}^{j_0}\|f\|_{L^q(C_{k+l},|\cdot|_h^\beta)}\,dt. \tag{5.3.18}$$

For $1 \leq p < \infty$, the inequality (5.3.18) following the Minkowski inequality yields

$$\begin{aligned}
&\|T_\Phi f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{H}^n,|\cdot|_h^\beta)} \\
&\simeq \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda(1+\beta/Q)} \left\{ \sum_{k=-\infty}^{k_0} \left(2^{k\alpha(1+\beta/Q)} \|T_\Phi f\|_{L^q(C_k;|\cdot|_h^\beta)} \right)^p \right\}^{1/p} \\
&\preceq \int_0^\infty \frac{\Phi(t)}{t} t^{(Q+\beta)/q} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda(1+\beta/Q)} \\
&\quad \times \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{l=j_0}^{j_0+1} 2^{k\alpha(1+\beta/Q)} \|f\|_{L^q(C_{k+l},|\cdot|_h^\beta)} \right)^p \right\}^{1/p} dt \\
&\preceq \int_0^\infty \frac{\Phi(t)}{t} t^{(Q+\beta)/q} \sum_{l=j_0}^{j_0+1} 2^{l(\lambda-\alpha)(1+\beta/Q)} dt \\
&\quad \times \sup_{k_0\in\mathbb{Z}} 2^{-\lambda(k_0+l)(1+\beta/Q)} \left\{ \sum_{k=-\infty}^{k_0+l} 2^{k\alpha p(1+\beta/Q)} \|f\|^p_{L^q(C_k,|\cdot|_h^\beta)} \right\}^{1/p} \\
&\preceq \|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{H}^n,|\cdot|_h^\beta)} \int_0^\infty \frac{\Phi(t)}{t} t^{(1+\beta/Q)(\alpha-\lambda+Q/q)} dt. \qquad (5.3.19)
\end{aligned}$$

To prove the converse, we assume that T_Φ is bounded on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{H}^n, |\cdot|_h^\beta)$. With this we split our problem in two cases:
Case I: $0 < \lambda < \alpha$
In this case, we select $f_0 \in L^q_{\mathrm{loc}}(\mathbb{H}^n, |\cdot|_h^\beta)$, such that

$$f_0(x) = |x|_h^{-(1+\beta/Q)(\alpha-\lambda+Q/q)},$$

then it is easy to see that

$$\|f_0\|_{L^q(C_k,|\cdot|_h^\beta)} \simeq 2^{-k(\alpha-\lambda)(1+\beta/Q)} \left(\frac{2^{q(\alpha-\lambda)(1+\beta/Q)} - 1}{q(\alpha-\lambda)(1+\beta/Q)} \right)^{1/q}.$$

Therefore,

$$\begin{aligned}
&\|f_0\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{H}^n,|\cdot|_h^\beta)} \\
&\simeq \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda(1+\beta/Q)} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p(1+\beta/Q)} \|f_0\|^p_{L^q(C_k,|\cdot|_h^\beta)} \right\}^{1/p} \\
&\simeq \left(\frac{2^{q(\alpha-\lambda)(1+\beta/Q)} - 1}{q(\alpha-\lambda)(1+\beta/Q)} \right)^{1/q} \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda(1+\beta/Q)} \left\{ \sum_{k=-\infty}^{k_0} 2^{pk\lambda(1+\beta/Q)} \right\}^{1/p} \\
&= \left(\frac{2^{q(\alpha-\lambda)(1+\beta/Q)} - 1}{q(\alpha-\lambda)(1+\beta/Q)} \right)^{1/q} \frac{2^{\lambda(1+\beta/Q)}}{(2^{\lambda(1+\beta/Q)} - 1)^{1/p}} < \infty.
\end{aligned}$$

On the other hand, a change of variables following the polar decomposition yields

$$T_\Phi(f_0)(x) = \int_{\mathbb{H}^n} \frac{\Phi(\delta_{|y|_h^{-1}}x)}{|y|_h^Q}|y|_h^{-(1+\beta/Q)(\alpha-\lambda+Q/q)}dy$$
$$\simeq \int_0^\infty \frac{\Phi(t^{-1}|x|_h)}{t}t^{-(1+\beta/Q)(\alpha-\lambda+Q/q)}dt$$

$$= f_0(x)\int_0^\infty \frac{\Phi(t)}{t}t^{(1+\beta/Q)(\alpha-\lambda+Q/q)}dt.$$

According to our assumption T_Φ is bounded on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{H}^n, |\cdot|_h^\beta)$, therefore,

$$\|T_\Phi\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{H}^n,|\cdot|_h^\beta)\to M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{H}^n,|\cdot|_h^\beta)} \succeq \int_0^\infty \frac{\Phi(t)}{t}t^{(1+\beta/Q)(\alpha-\lambda+Q/q)}dt. \tag{5.3.20}$$

Finally, inequalities (5.3.19) and (5.3.20) imply that

$$\|T_\Phi\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{H}^n,|\cdot|_h^\beta)\to M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{H}^n,|\cdot|_h^\beta)} \simeq \int_0^\infty \frac{\Phi(t)}{t}t^{(1+\beta/Q)(\alpha-\lambda+Q/q)}dt.$$

Case II: $\lambda = 0$
In this case, our assumption reduces to the boundedness of T_Φ on $\dot{K}^{\alpha,p}_q(\mathbb{H}^n, |\cdot|_h^\beta)$. To prove the converse relation let us consider

$$f_\epsilon(x) = |x|_h^{-(1+\beta/Q)(\alpha+Q/q)-\epsilon/q}\chi_{\{|x|_h>1\}}.$$

Then for $k > 0$, an easy computation leads to

$$\|f_\epsilon\|_{L^q(C_k,|\cdot|_h^\beta)} \simeq 2^{-k\alpha(1+\beta/Q)-k\epsilon/q}\left(\frac{2^{q\alpha(1+\beta/Q)+\epsilon}-1}{q\alpha(1+\beta/Q)+\epsilon}\right)^{1/q},$$

which gives that

$$\|f_\epsilon\|_{\dot{K}^{\alpha,p}_q(\mathbb{H}^n,|\cdot|_h^\beta)} \simeq \frac{1}{(2^{\epsilon p/q}-1)^{1/p}}\left(\frac{2^{q\alpha(1+\beta/Q)+\epsilon}-1}{q\alpha(1+\beta/Q)+\epsilon}\right)^{1/q} < \infty. \tag{5.3.21}$$

By decomposing T_Φ into polar coordinates and changing variables, we obtain

$$T_\Phi(f_\epsilon)(x) = \int_{\mathbb{H}^n} \frac{\Phi(\delta_{|y|_h^{-1}}x)}{|y|_h^Q}|y|_h^{-(1+\beta/Q)(\alpha+Q/q)-\epsilon/q}\chi_{\{|y|_h>1\}}(y)dy$$
$$\simeq \int_1^\infty \frac{\Phi(t^{-1}|x|_h)}{t}t^{-(1+\beta/Q)(\alpha+Q/q)-\epsilon/q}dt$$
$$= |x|^{-(1+\beta/Q)(\alpha+Q/q)-\epsilon/q}\int_0^{|x|_h} \frac{\Phi(t)}{t}t^{(1+\beta/Q)(\alpha+Q/q)+\epsilon/q}\chi_{\{|x|_h>1\}}(x)dt.$$

This suggests that $(T_\phi f_\epsilon)\chi_{C_k} = 0$ for $k \leq 0$. Thus, for $k > 0$, one has

$$\begin{aligned}\|T_\Phi(f_\epsilon)\|_{L^q(C_k,|\cdot|^\beta)} &= \left(\int_{C_k} |T_\Phi f_\epsilon(x)|^q |x|^\beta dx\right)^{1/q} \\ &\simeq \left(\int_{C_k} |x|^{-(1+\beta/Q)(q\alpha+Q)-\epsilon+\beta}\right. \\ &\quad \left.\times \int_0^{|x|_h} \frac{\Phi(t)}{t} t^{(1+\beta/Q)(\alpha+Q/q)+\epsilon/q} dt dx\right)^{1/q}.\end{aligned}$$

For any $\epsilon > 0$, there exist a $m \in \mathbb{Z}$ such that $2^{-m} \leq \epsilon < 2^{-m+1}$. Therefore for $m \leq k$, we have

$$\begin{aligned}&\|T_\Phi(f_\epsilon)\|_{L^q(C_k,|\cdot|^\beta)} \\ &\succeq 2^{-k\alpha(1+\beta/Q)-k\epsilon/q}\left(\frac{2^{q\alpha(1+\beta/Q)+\epsilon}-1}{q\alpha(1+\beta/Q)+\epsilon}\right)^{1/q} \int_0^{2^{m-1}} \frac{\Phi(t)}{t} t^{(1+\beta/Q)(\alpha+Q/q)+\epsilon/q} dt \qquad (5.3.22)\end{aligned}$$

Now, the definition of power weighted Herz space and the the inequality (5.3.22) help us to have

$$\begin{aligned}&\|T_\Phi(f_\epsilon)\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n,|\cdot|^\beta)} \\ &\simeq \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p(1+\beta/Q)} \|T_\Phi(f_\epsilon)\|^p_{L^q(C_k,|\cdot|^\beta)}\right)^{1/p} \\ &\succeq \left(\frac{2^{q\alpha(1+\beta/Q)+\epsilon}-1}{q\alpha(1+\beta/Q)+\epsilon}\right)^{1/q} \left(\sum_{k=m}^{\infty} 2^{-kp\epsilon/q}\right)^{1/p} \int_0^{1/(2\epsilon)} \frac{\Phi(t)}{t} t^{(1+\beta/Q)(\alpha+Q/q)+\epsilon/q} dt \\ &= \|f_\epsilon\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;|\cdot|_h^\beta)} \epsilon^{\epsilon/q} 2^{\epsilon/q} \int_0^{1/(2\epsilon)} \frac{\Phi(t)}{t} t^{(1+\beta/Q)(\alpha+Q/q)+\epsilon/q} dt,\end{aligned}$$

where the last equality is by virtue of (5.3.21).

Letting $\epsilon \to 0$, we obtain

$$\|T_\Phi\|_{\dot{K}_q^{\alpha,P}(\mathbb{H}^n;|\cdot|_h^\beta)\to \dot{K}_q^{\alpha,P}(\mathbb{H}^n;|\cdot|_h^\beta)} \succeq \int_0^\infty \frac{\Phi(t)}{t} t^{(1+\beta/Q)(\alpha+Q/q)} dt. \qquad (5.3.23)$$

Finally, (5.3.19) and (5.3.23) mean that

$$\|T_\Phi\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n;|\cdot|_h^\beta)\to \dot{K}_q^{\alpha,p}(\mathbb{H}^n;|\cdot|_h^\beta)} \simeq \int_0^\infty \frac{\Phi(t)}{t} t^{(1+\beta/Q)(\alpha+Q/q)} dt.$$

Hence, we finish the proof of Theorem 5.3.3.

5.4 Estimates for $T^b_{\Phi,A}$ on Weighted Herz Space

This section contains the main results of the boundedness of $T^b_{\Phi,A}$ and the relevant proofs.

5.4.1 Main Results

Theorem 5.4.1 *Suppose* $1 \le p, q, q_1, q_2 \le \infty$ *and* $\alpha_1, \alpha_2 \in \mathbb{R}$ *with* $\alpha_1 < 0$. *Suppose that* $1/s = 1/q_1 + 1/q$ *and* $\alpha_1/Q + 1/q_1 = \alpha_2/Q + 1/q_2$. *In addition, let* $w \in A_1$ *with the critical index* r_w *for the reverse Hölder condition and* $s > q_2 r_w/(r_w - 1)$.
(i) If $1/q_1 + \alpha_1/Q \ge 0$, *then for any* $1 < \delta < r_w$,

$$\left\|T^b_{\Phi,A} f\right\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)} \le K_1 \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)} \|f\|_{\dot{K}^{\alpha_1,p}_{q_1}(\mathbb{H}^n;w)},$$

where

$$\begin{aligned} K_1 = &\int_{\|A(y)\|<1} \frac{|\Phi(y)|}{|y|_h^Q} \left(1 + |\det A^{-1}(y)|^{1/q} \|A(y)\|^{Q/q}\right) \\ &\times |\det A^{-1}(y)|^{1/q_1} \|A(y)\|^{-\alpha_1} \log \frac{2}{\|A(y)\|} dy \\ &+ \int_{\|A(y)\|\ge 1} \frac{|\Phi(y)|}{|y|_h^Q} \left(1 + |\det A^{-1}(y)|^{1/q} \|A(y)\|^{Q/q}\right) \\ &\times |\det A^{-1}(y)|^{1/q_1} \|A(y)\|^{Q/q_1 - (\alpha_1 + Q/q_1)(\delta-1)/\delta} \log 2\|A(y)\| dy. \end{aligned}$$

(ii) If $\alpha_1/Q + 1/q_1 < 0$, *then for any* $1 < \delta < r_w$

$$\left\|T^b_{\Phi,A}\right\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)} \le K_2 \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)} \|f\|_{\dot{K}^{\alpha_1,p}_{q_1}(\mathbb{H}^n;w)}.$$

where

$$\begin{aligned} K_2 = &\int_{\|A(y)\|\ge 1} \frac{|\Phi(y)|}{|y|_h^Q} \left(1 + |\det A^{-1}(y)|^{1/q} \|A(y)\|^{Q/q}\right) \\ &\times |\det A^{-1}(y)|^{1/q_1} \|A(y)\|^{-\alpha_1} \log 2\|A(y)\| dy \\ &+ \int_{\|A(y)\|<1} \frac{|\Phi(y)|}{|y|_h^Q} \left(1 + |\det A^{-1}(y)|^{1/q} \|A(y)\|^{Q/q}\right) \\ &\times |\det A^{-1}(y)|^{1/q_1} \|A(y)\|^{Q/q_1 - (\alpha_1 + Q/q_1)(\delta-1)/\delta} \log \frac{2}{\|A(y)\|} dy. \end{aligned}$$

When general weights are reduced to power weights, then the next theorem is:

Theorem 5.4.2 *Suppose* $1 \le p < \infty$, $1 < q, q_1, q_2 < \infty$ *and* $\beta > -n$. *If* $1/q_2 = 1/q + 1/q_1$ *and* $1/q + \alpha_2/Q = \alpha_1/Q$, *then we have*

$$\|T^b_{\Phi,A}\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;v)} \preceq K_3\|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)}\|f\|_{\dot{K}^{\alpha_1,p}_{q_1}(\mathbb{H}^n;v)}.$$

where K_3 *is*

$$K_3 = \begin{cases} \int_{\mathbb{H}^n} \Theta(y)\left(1+\log_2\left(\|A^{-1}(y)\|\|A(y)\|\right)\right)dy, & \textit{if} \quad \alpha_1 = 0, \\ \int_{\mathbb{H}^n} \Theta(y) G\big(A^{-1}(y), \alpha_1(Q+\beta)/Q\big)dy, & \textit{if} \quad \alpha_1 \neq 0. \end{cases}$$

and

$$\Theta(y) = \frac{|\Phi(y)|}{|y|_h^Q}|\det A^{-1}(y)|^{1/q_1}\left(\log\frac{2}{\|A(y)\|}\chi_{\{\|A(y)\|<1\}} + \log 2\|A(y)\|\chi_{\{\|A(y)\|\geq 1\}}\right)$$
$$\times G\big(A^{-1}(y), \beta/q_1\big)\left(1+|\det A^{-1}(y)|^{1/q}G\big(A^{-1}(y),\beta/q\big)\|A(y)\|^{(Q+\beta)/q}\right).$$

5.4.2 Proof of Theorem 5.4.1

Here, we have to show that

$$\left\{\sum_{k=-\infty}^{\infty} w(B_k)^{\alpha_2 p/Q}\left\|T^b_{\Phi,A}f\right\|^p_{L^{q_2}(E_k,w)}\right\}^{1/p} \preceq \|f\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)}.$$

By the Minkowski inequality and necessary splitting, an upper bound for the inner norm $\|T^b_{\Phi,A}f\|^p_{L^{q_2}(E_k,w)}$ can be obtained as:

$$\begin{aligned}
&\left\|\big(T^b_{\Phi,A}f\big)\right\|_{L^{q_2}(E_k;w)} \\
&= \left\|\left(\int_{\mathbb{H}^n}\frac{\Phi(y)}{|y|_h^Q}(b(x)-b(A(y)x))f(A(y)x)dy\right)\right\|_{L^{q_2}(E_k;w)} \\
&\leq \int_{\mathbb{H}^n}\frac{\Phi(y)}{|y|_h^Q}\|(b(x)-b(A(y)x))f(A(y)x)\|_{L^{q_2}(E_k;w)}dy \\
&\leq \int_{\mathbb{H}^n}\frac{\Phi(y)}{|y|_h^Q}\|(b(x)-b_{B_k})f(A(y)x)\|_{L^{q_2}(E_k;w)}dy \\
&\quad + \int_{\mathbb{H}^n}\frac{\Phi(y)}{|y|_h^Q}\|(b(A(y)x)-b_{\|A(y)\|B_k})f(A(y)x)\|_{L^{q_2}(E_k;w)}dy \\
&\quad + \int_{\mathbb{H}^n}\frac{\Phi(y)}{|y|_h^Q}\|(b_{B_k}-b_{\|A(y)\|B_k})f(A(y)x)\|_{L^{q_2}(E_k;w)}dy \\
&= I_1 + I_2 + I_3. \qquad (5.4.1)
\end{aligned}$$

While targeting I_1, we first compute $\|(b(x)-b(A(y)x))f(A(y)x)\|_{L^{q_2}(E_k;w)}$. The condition $s > q_2 r_w/(r_w-1)$ implies that there exist $1 < r < r_w$ such that $s = q_2 r'$. Therefore, by the Hölder inequality and the reverse Hölder condition, we have

$$\begin{aligned}
&\| \left(b(\cdot)-b_{B_k}\right) f(A(y)\cdot)\|_{L^{q_2}(E_k;w)} \\
&= \left(\int_{E_k} |(b(x)-b_{B_k})f(A(y)x)|^s \, dx\right)^{1/s} \left(\int_{E_k} w(x)^r dx\right)^{1/rq_2} \\
&\preceq |B_k|^{-1/s} w(B_k)^{1/q_2} \|(b(\cdot)-b_{B_k})f(A(y)\cdot)\|_{L^s(E_k)}. \qquad (5.4.2)
\end{aligned}$$

Next, using the condition $1/s = 1/q + 1/q_1$, we can have

$$\|(b(\cdot)-b_{B_k})f(A(y)\cdot)\|_{L^s(E_k)} \le \|b(\cdot)-b_{B_k}\|_{L^q(B_k)} \|f(A(y)\cdot)\|_{L^{q_1}(B_k)}. \qquad (5.4.3)$$

In second factor, on the right side of the inequality (5.4.3), a change of variables along with Proposition 5.2.4 yields

$$\begin{aligned}
\|f(A(y)\cdot)\|_{L^{q_1}(B_k)} &= |\det A^{-1}(y)|^{1/q_1} \left(\int_{A(y)B_k} |f(x)|^{q_1} dx\right)^{1/q_1} \\
&\preceq |\det A^{-1}(y)|^{1/q_1} |B(0, 2^k \|A(y)\|)|^{1/q_1} \\
&\quad \times \left(\frac{1}{w(B(0,2^k\|A(y)\|))} \int_{B(0,2^k\|A(y)\|)} |f(x)|^{q_1} w(x) dx\right)^{1/q_1} \\
&\preceq \left(|\det A^{-1}(y)| \|A(y)\|^Q |B_k|\right)^{1/q_1} \\
&\quad \times w(\|A(y)\|B_k)^{-1/q_1} \|f\|_{L^{q_1}(\|A(y)\|B_k;w)}. \qquad (5.4.4)
\end{aligned}$$

Similarly, the other factor on the right hand of the inequality (5.4.3), in view of Proposition 5.2.4, gives

$$\|b(\cdot)-b_{B_k}\|_{L^q(B_k)} \preceq \ |B_k|^{1/q} \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}. \qquad (5.4.5)$$

Inequalities (5.4.2–5.4.5) together yield

$$\begin{aligned}
&\|(b(\cdot)-b_{B_k})f(A(y)\cdot)\|_{L^{q_2}(E_k;w)} \\
&\preceq \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)} \|f\|_{L^{q_1}(\|A(y)\|B_k;w)} \\
&\quad \times \left(|\det A^{-1}(y)| \|A(y)\|^Q\right)^{1/q_1} \frac{w(B_k)^{1/q_2}}{w(\|A(y)\|B_k)^{1/q_1}}.
\end{aligned}$$

Hence, we obtain the following estimate for I_1 :

$$\begin{aligned}
I_1 \preceq \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)} &\int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \left(|\det A^{-1}(y)| \|A(y)\|^Q\right)^{1/q_1} \\
&\times \frac{w(B_k)^{1/q_2}}{w(\|A(y)\|B_k)^{1/q_1}} \|f\|_{L^{q_1}(\|A(y)\|B_k;w)} dy
\end{aligned}$$

Next, we fix to estimate I_2, which is given by

$$I_2 \;=\; \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \left\|(b(A(y)\cdot) - b_{\|A(y)\|B_k})f(A(y)\cdot)\right\|_{L^{q_2}(E_k;w)} dy$$

Since $s = q_2 r'$, therefore we infer from (5.4.2) that

$$\begin{aligned}
&\left\|\left(b(A(y)\cdot) - b_{\|A(y)\|B_k}\right) f(A(y)\cdot)\right\|_{L^{q_2}(E_k;w)} \\
&\preceq |B_k|^{-1/s} w(B_k)^{1/q_2} \|(b(A(y)\cdot) - b_{\|A(y)\|B_k})f(A(y)\cdot)\|_{L^s(E_k)}. \qquad (5.4.6)
\end{aligned}$$

Applying the change of variables formula, Proposition 5.2.4 and Hölder's inequality, we have

$$\begin{aligned}
&\left\|(b(A(y).) - b_{\|A(y)\|B_k})f(A(y).)\right\|_{L^s(E_k)} \\
&= |\det A^{-1}(y)|^{1/s} \left(\int_{A(y)B_k} |(b(x) - b_{\|A(y)\|B_k})f(x)|^s dx\right)^{1/s} \\
&\preceq |\det A^{-1}(y)|^{1/s} |\|A(y)\|B_k|^{1/s} \\
&\quad \times \left(\frac{1}{w(\|A(y)\|B_k)} \int_{\|A(y)\|B_k} |(b(x) - b_{\|A(y)\|B_k})f(x)|^s w(x) dx\right)^{1/s} \\
&\preceq |\det A^{-1}(y)|^{1/s} |B_k|^{1/s} \|A(y)\|^{Q1/s} w(\|A(y)\|B_k)^{-1/s} \\
&\quad \times \left(\int_{\|A(y)\|B_k} |b(x) - b_{\|A(y)\|B_k}|^q w(x) dx\right)^{1/q} \left(\int_{\|A(y)\|B_k} |f(x)|^{q_1} w(x) dx\right)^{1/q_1} \\
&\preceq |\det A^{-1}(y)|^{1/s} |B_k|^{1/s} \|A(y)\|^{Q1/s} w(\|A(y)\|B_k)^{-1/q_1} \\
&\quad \times \|f\|_{L^{q_1}(\|A(y)\|B_k;w)} \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}. \qquad (5.4.7)
\end{aligned}$$

By virtue of (5.4.6) and (5.4.7), the expression for I_2 assumes the following form:

$$\begin{aligned}
I_2 \preceq \|b\|_{C\dot{M}O^q(\mathbb{H}^n,w)} &\int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \left(|\det A^{-1}(y)| \|A(y)\|^Q\right)^{1/s} \\
&\times \frac{w(B_k)^{1/q_2}}{w(\|A(y)\|B_k)^{1/q_1}} \|f\|_{L^{q_1}(\|A(y)\|B_k;w)} dy.
\end{aligned}$$

Now, the estimation of I_3, given by

$$I_3 = \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \|f(A(y)\cdot)\|_{L^{q_2}(E_k)} |b_{B_k} - b_{\|A(y)\|B_k}| dy,$$

requires the bounds for $\|f(A(y)\cdot)\|_{L^{q_2}(E_k)}$ and $|b_{B_k} - b_{\|A(y)\|B_k}|$. First we consider $\|f(A(y)\cdot)\|_{L^{q_2}(E_k,w)}$. In view of the condition $s = q_2 r'$, we use the Hölder's inequality and the reverse Hölder condition to obtain

$$\begin{aligned}\|f(A(y)\cdot)\|_{L^{q_2}(E_k,w)} &\leq \left(\int_{B_k} |f(A(y)x)|^{q_2} w(x)dx\right)^{1/q_2}\\ &\leq \left(\int_{B_k} |f(A(y)x)|^{s}dx\right)^{1/s} \left(\int_{B_k} w(x)^r dx\right)^{1/rq_2}\\ &\preceq |B_k|^{-1/s} w(B_k)^{1/q_2} \|f(A(y)\cdot)\|_{L^s(B_k)}. \qquad (5.4.8)\end{aligned}$$

Furthermore, the condition $1/s = 1/q + 1/q_1$ and the inequality (5.4.4) help us to write

$$\begin{aligned}\|f(A(y)\cdot)\|_{L^s(B_k)} &= |B_k|^{1/q} \|f(A(y)\cdot)\|_{L^{q_1}(B_k)}\\ &\preceq |B_k|^{1/s} \left(|\det A^{-1}(y)| \|A(y)\|^Q\right)^{1/q_1}\\ &\quad \times w(\|A(y)\|B_k)^{-1/q_1} \|f\|_{L^{q_1}(\|A(y)\|B_k;w)}. \qquad (5.4.9)\end{aligned}$$

We combine the inequalities (5.4.8) and (5.4.9) to substitute the result in the expression for I_3, which now becomes

$$\begin{aligned}I_3 \preceq \int_{\mathbb{H}^n} &\frac{|\Phi(y)|}{|y|_h^Q} \left(|\det A^{-1}(y)| \|A(y)\|^Q\right)^{1/q_1}\\ &\times \frac{w(B(0,2^k))^{1/q_2}}{w(\|A(y)\|B_k))^{1/q_1}} \|f\|_{L^{q_1}(w(\|A(y)\|B_k),w)} |b_{B_k} - b_{\|A(y)\|B_k}| dy.\end{aligned}$$

Now, it turns to bound $|b_{B_k} - b_{\|A(y)\|B_k}|$. For this purpose, we split the integral as below:

$$\begin{aligned}I_3 &\preceq \int_{\|A(y)\|<1} |b_{B_k} - b_{\|A(y)\|B_k}| \Psi(y) dy + \int_{\|A(y)\|\geq 1} |b_{B_k} - b_{\|A(y)\|B_k}| \Psi(y) dy\\ &= I_{31} + I_{32},\end{aligned}$$

where, for the convenience's sake, we used the following notation:

$$\Psi(y) = \frac{|\Phi(y)|}{|y|_h^Q} \left(|\det A^{-1}(y)| \|A(y)\|^Q\right)^{1/q_1} \frac{w(B(0,2^k))^{1/q_2}}{w(\|A(y)\|B_k))^{1/q_1}} \|f\|_{L^{q_1}(\|A(y)\|B_k;w)}.$$

Further decomposition of integral for I_{31} results in:

$$\begin{aligned}I_{31} = \sum_{j=0}^{\infty} \int_{2^{-j-1}\leq \|A(y)\| < 2^{-j}} &\Psi(y)\\ &\left\{ \sum_{i=1}^{j} |b_{2^{-i}B_k} - b_{2^{-i+1}B_k}| + |b_{2^{-j}B_k} - b_{\|A(y)\|B_k}| \right\} dy.\end{aligned}$$

The first term inside the curly brackets can be approximated using Proposition 5.2.4, that is

$$\begin{aligned}
&|b_{2^{-i}B_k} - b_{2^{-i+1}B_k}|\\
&\le \frac{1}{|2^{-i}B_k|}\int_{2^{-i}B_k}|b(y)-b_{2^{-i+1}B_k}|dy\\
&\le \frac{1}{w(2^{-i}B_k)}\int_{2^{-i}B_k}|b(y)-b_{2^{-i+1}B_k}|w(y)dy\\
&\le \frac{1}{w(2^{-i}B_k)}\left(\int_{2^{-i+1}B_k}|b(y)-b_{2^{-i+1}B_k}|^q w(y)dy\right)^{\frac{1}{q}}\left(\int_{2^{-i+1}B_k}w(y)dy\right)^{1/q'}\\
&\le \frac{w(2^{-i+1}B_k)}{w(2^{-i}B_k)}\left(\frac{1}{w(2^{-i+1}B_k)}\int_{2^{-i+1}B_k}|b(y)-b_{2^{-i+1}B_k}|^q w(y)dy\right)^{\frac{1}{q}}\\
&\preceq \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}\,.
\end{aligned}$$

Similarly, for second term inside the curly brackets in the expression of I_{31}, we have

$$|b_{2^{-j}B_k} - b_{\|A(y)\|B_k}| \preceq \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}.$$

Therefore, we finish the estimation of I_{31} by writing

$$\begin{aligned}
I_{31} &\preceq \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}\sum_{j=0}^{\infty}\int_{2^{-j-1}\le\|A(y)\|<2^{-j}}\Psi(y)(j+1)dy\\
&\preceq \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}\int_{\|A(y)\|<1}\Psi(y)\log\frac{2}{\|A(y)\|}dy.
\end{aligned}$$

In a similar fashion, the integral I_{32} gives us

$$\begin{aligned}
I_{32} &= \int_{\|A(y)\|\ge 1}\Psi(y)|b_{B_k} - b_{\|A(y)\|B_k}|dy.\\
&= \sum_{j=0}^{\infty}\int_{2^j\le\|A(y)\|<2^{j+1}}\Psi(y)\Big\{\sum_{i=1}^{j}|b_{2^iB_k}-b_{2^{i+1}B_k}| + |b_{2^{j+1}B_k} - b_{\|A(y)\|B_k}|\Big\}dy\\
&\preceq \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}\int_{\|A(y)\|\ge 1}\Psi(y)\,\log 2\|A(y)\|dy.
\end{aligned}$$

A combination of expressions for I_1, I_2, I_{31} and I_{32}, gives

$$\begin{aligned}
&\left\|T^b_{\Phi,A}f\right\|_{L^{q_2}(E_k;w)}\\
&\preceq \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}\\
&\times\int_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}\left(|\det A^{-1}(y)|\|A(y)\|^Q\right)^{1/q_1}\left(1+|\det A^{-1}(y)|^{1/q}\|A(y)\|^{Q/q}\right)\\
&\times\frac{w(B(0,2^k))^{1/q_2}}{w(\|A(y)\|B_k))^{1/q_1}}\|f\|_{L^{q_1}(\|A(y)\|B_k;w)}\max\Big\{\log\frac{2}{\|A(y)\|},\ \log(2\|A(y)\|)\Big\}dy.
\end{aligned}$$

Keeping in view the definition of the Herz space, factors containing the index k in the expression of $\Psi(y)$ are important. Therefore, to proceed further and to avoid repetition of unimportant factors relative to the Herz space, we have to modify and rename the expression for Ψ. Hence, in the remaining of this paper we shall use the following notation:

$$\tilde{\Psi}(y) = \frac{|\Phi(y)|}{|y|_h^Q}\left(|\det A^{-1}(y)|\|A(y)\|^Q\right)^{1/q_1}$$
$$\times\left(1+|\det A^{-1}(y)|^{1/q}\|A(y)\|^{Q/q}\right)\max\left\{\log\frac{2}{\|A(y)\|},\ \log 2\|A(y)\|\right\}.$$

Then,

$$\left\|T^b_{\Phi,A}f\right\|_{L^{q_2}(E_k;w)}$$
$$\leq \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}\int_{\mathbb{H}^n}\tilde{\Psi}(y)\frac{w(B_k)^{1/q_2}}{w(\|A(y)\|B_k)^{1/q_1}}\|f\|_{L^{q_1}(\|A(y)\|B_k;w)}dy.$$

Finally, we take into consideration the definition of Herz space and employ the Minkowski inequality to have

$$\begin{aligned}\left\|T^b_{\Phi,A}f\right\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)} &= \left\{\sum_{k=-\infty}^{\infty} w(B_k)^{\frac{\alpha_2 p}{Q}}\left\|T^b_{\Phi,A}f\right\|^p_{L^{q_2}(E_k;w)}\right\}^{1/p}\\ &\preceq \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}\int_{\mathbb{H}^n}\tilde{\Psi}(y)\\ &\times\left\{\left(\sum_{k=-\infty}^{\infty}\frac{w(B_k)^{\alpha_2/Q+1/q_2}}{w(\|A(y)\|B_k)^{1/q_1}}\|f\|_{L^{q_1}(\|A(y)\|B_k;w)}\right)^p\right\}^{1/p} dy \end{aligned} \tag{5.4.10}$$

Comparing inequality (5.4.10) with the inequality (3.9) in [117], we found that the term inside the curly brackets is same in both of these inequalities, the only difference lies in the integrands outside the curly brackets along with a constant multiple $\|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}$ outside the integral. Therefore, the inequality (5.4.10) can be written as:

$$\begin{aligned}&\left\|T^b_{\Phi,A}f\right\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)}\\ &\preceq \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}\sum_{j=-\infty}^{\infty}\int_{2^{j-1}<\|A(y)\|\leq 2^j}\tilde{\Psi}(y)\\ &\times\left\{\sum_{k=-\infty}^{\infty}\left[\left(\frac{w(B_k)}{w(B_{k+j})}\right)^{\alpha_1/Q+1/q_1}\right.\right.\\ &\times\left.\left.\sum_{l=-\infty}^{j}\left(\frac{w(B_{k+j})}{w(B_{k+l})}\right)^{\alpha_1/Q}w(B_{k+l})^{\alpha_1/Q}\|f\|_{L^{q_1}(E_{k+l};w)}\right]^p\right\}^{1/p}dy,\end{aligned} \tag{5.4.11}$$

where the condition $\alpha_1/Q + 1/q_1 = \alpha_2/Q + 1/q_2$ is utilized in obtaining the last inequality.

Under the stated condition that $\alpha_1 < 0$ and $l \le j$, we use Proposition 5.2.3 to have

$$\left(\frac{w(B_{k+j})}{w(B_{k+l})}\right)^{\alpha_1/Q} \preceq \left(\frac{|B_{k+j}|}{|B_{k+l}|}\right)^{\alpha_1(\delta-1)/(Q\delta)} = 2^{(j-l)\alpha_1(\delta-1)/\delta}. \tag{5.4.12}$$

for any $1 < \delta < r_w$.

In view of Proposition 5.2.3, if $\alpha_1/Q + 1/q_1 \ge 0$, then

$$\left(\frac{w(B_k)}{w(B_{k+j})}\right)^{\alpha_1/Q+1/q_1} \preceq \begin{cases} 2^{-jQ(\alpha_1/Q+1/q_1)}, & \text{if } \quad j \le 0, \\ 2^{-jQ(\alpha_1/Q+1/q_1)(\delta-1)/\delta}, & \text{if } \quad j > 0, \end{cases} \tag{5.4.13}$$

and if $\alpha_1/Q + 1/q_1 < 0$, then

$$\left(\frac{w(B_k)}{w(B_{k+j})}\right)^{\alpha_1/Q+1/q_1} \preceq \begin{cases} 2^{jQ(\alpha_1/Q+1/q_1)(\delta-1)/\delta}, & \text{if } \quad j \le 0, \\ 2^{jQ(\alpha_1/Q+1/q_1)}, & \text{if } \quad j > 0, \end{cases} \tag{5.4.14}$$

for any $1 < \delta < r_w$.

Thus, for $\alpha_1/Q + 1/q_1 \ge 0$, from inequalities (5.4.11–5.4.13), for any $1 < \delta < r_w$, we have

$$\begin{aligned}
&\left\|T^b_{\Phi,A} f\right\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)} \\
&\preceq \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)} \sum_{j=-\infty}^{0} \int_{2^{j-1}<\|A(y)\|\le 2^j} \tilde{\Psi}(y)\|A(y)\|^{-\alpha_1-Q/q_1} \\
&\quad \times \sum_{l=-\infty}^{j} 2^{\alpha_1(j-l)(\delta-1)/\delta}\left\{\sum_{k=-\infty}^{\infty} w(B_{k+l})^{\alpha_1 p/Q}\|f\|^p_{L^{q_1}(E_{k+l};w)}\right\}^{1/p} dy \\
&\quad + \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)} \sum_{j=1}^{\infty} \int_{2^{j-1}<\|A(y)\|\le 2^j} \tilde{\Psi}(y)\|A(y)\|^{(\alpha_1+Q/q_1)(\delta-1)/\delta} \\
&\quad \times \sum_{l=-\infty}^{j} 2^{\alpha_1(j-l)(\delta-1)/\delta}\left\{\sum_{k=-\infty}^{\infty} w(B_{k+l})^{\alpha_1 p/Q}\|f\|^p_{L^{q_1}(E_{k+l};w)}\right\}^{1/p} dy.
\end{aligned}$$

Replacing $\tilde{\Psi}(y)$ with its value in the above inequality, we get

$$\begin{aligned}
&\left\|T^b_{\Phi,A}f\right\|_{\dot{K}^{\alpha_2,p}_{q_2}(\mathbb{H}^n;w)}\\
&\preceq \|b\|_{C\dot{M}O^q(\mathbb{H}^n;w)}\|f\|_{\dot{K}^{\alpha_1,p}_{q_1}(\mathbb{H}^n;w)}\\
&\quad\times\Bigg\{\int_{\|A(y)\|<1}\frac{|\Phi(y)|}{|y|_h^Q}\left(1+|\det A^{-1}(y)|^{1/q}\|A(y)\|^{Q/q}\right)\\
&\quad\times|\det A^{-1}(y)|^{1/q_1}\|A(y)\|^{-\alpha_1}\log\frac{2}{\|A(y)\|}dy\\
&\quad+\int_{\|A(y)\|\ge 1}\frac{|\Phi(y)|}{|y|_h^Q}\left(1+|\det A^{-1}(y)|^{1/q}\|A(y)\|^{Q/q}\right)\\
&\quad\times|\det A^{-1}(y)|^{1/q_1}\|A(y)\|^{Q/q_1-(\alpha_1+Q/q_1)(\delta-1)/\delta}\log(2\|A(y)\|)dy\Bigg\}.
\end{aligned}$$

This completes the proof of Theorem 5.4.1(i).

Similarly, when $\alpha_1/Q+1/q_1<0$, by using inequalities (5.4.11), (5.4.12) and (5.4.14), the second part of Theorem 5.4.1 can be proved easily. Hence, we complete the proof of Theorem 5.4.1.

5.4.3 Proof of Theorem 5.4.2

Following the proof of Theorem 5.4.1, we write:

$$\left\|T^b_{\Phi,A}\right\|_{L^{q_2}(E_k;v)}\le J_1+J_2+J_3,$$

where $J_1, J_2,$ and J_3 are as $I_1, I_2,$ and I_3 in the previous theorem with $w(\cdot)$ is replaced by $v(\cdot)=|\cdot|_h^\alpha$. Then by using the Hölder inequality and change of variables, we obtain

$$\begin{aligned}
J_1&\le\int_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}\Big(\int_{E_k}|b(x)-b_{B_k}|^qv(x)dx\Big)^{1/q}\Big(\int_{E_k}|f(A(y)x)|^{q_1}v(x)dx\Big)^{1/q_1}dy\\
&\le\ v(B_k)^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)}\\
&\quad\times\int_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}|\det A^{-1}(y)|^{1/q_1}\Big(\int_{A(y)E_k}|f(z)|^{q_1}v(A^{-1}(y)z)dz\Big)^{1/q_1}dy.
\end{aligned}$$

Using Proposition 5.2.9, we get

$$\begin{aligned}
J_1 &\le v(B_k)^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)}\int_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}|\det A^{-1}(y)|^{1/q_1}\\
&\quad\times\left(\int_{A(y)E_k}|f(x)|^{q_1}G\big(A^{-1}(y),\beta/q_1\big)v(x)dx\right)^{1/q_1}dy\\
&\le v(B_k)^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)}\\
&\quad\times\int_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}|\det A^{-1}(y)|^{1/q_1}G\big(A^{-1}(y),\beta/q_1\big)\|f\|_{L^{q_1}(\|A(y)\|E_k;v)}dy.
\end{aligned}$$

Next, the expression for J_2 can be written as:

$$J_2=\int_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}\big\|(b(A(y)\cdot)-b_{\|A(y)\|B_k})f(A(y)\cdot)\big\|_{L^{q_2}(E_k;v)}dy. \tag{5.4.15}$$

Changing variables and using the condition $q_2/q+q_2/q_1=1$, we get

$$\begin{aligned}
&\Big\|(b(A(y)x)-b_{\|A(y)\|B_k})(f(A(y).))\Big\|_{L^{q_2}(E_k;v)}\\
&=\left(\int_{E_k}\big|(b(A(y)x)-b_{\|A(y)\|B_k})f(A(y)x)\big|^{q_2}v(x)dx\right)^{1/q_2}\\
&=|\det A^{-1}(y)|^{1/q_2}G\big(A^{-1}(y),\beta/q_2\big)\\
&\quad\times\left(\int_{A(y)E_k}\big|(b(x)-b_{\|A(y)\|B_k})f(x)\big|^{q_2}v(x)dx\right)^{1/q_2}\\
&\le|\det A^{-1}(y)|^{1/q_2}G\big(A^{-1}(y),\beta/q_2\big)\\
&\quad\times\left(\int_{\|A(y)\|B_k}\big|b(x)-b_{\|A(y)\|B_k}\big|^{q}v(x)dx\right)^{1/q}\left(\int_{A(y)E_k}|f(x)|^{q_1}v(x)dx\right)^{1/q_1}\\
&=|\det A^{-1}(y)|^{1/q_2}G\big(A^{-1}(y),\beta/q_2\big)\\
&\quad\times v(\|A(y)\|B_k)^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)}\|f\|_{L^{q_1}(A(y)E_k;v)}.
\end{aligned}\tag{5.4.16}$$

It is easy to see that $v(\|A(y)\|B_k)=\|A(y)\|^{Q+\beta}v(B_k)$. Using property (5.2.5) and (5.4.16), the inequality (5.4.15) becomes:

$$\begin{aligned}
J_2&=v(B_k)^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)}\int_{\mathbb{H}^n}\frac{|\Phi(y)|}{|y|_h^Q}|\det A^{-1}(y)|^{1/q_2}\\
&\quad\times G\big(A^{-1}(y),\beta/q\big)G\big(A^{-1}(y),\beta/q_1\big)\|A(y)\|^{(Q+\beta)/q}\|f\|_{L^{q_1}(A(y)E_k;v)}dy.
\end{aligned}$$

It remains to estimates J_3. A change of variables following the Hölder inequality and Proposition 5.2.9 gives us

$$\begin{aligned} J_3 &= \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \Big\|(b_{B_k} - b_{\|A(y)\|B_k}) f(A(y)\cdot)\Big\|_{L^{q_2}(E_k;v)} dy \\ &= \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} \|f(A(y)x)\|_{L^{q_2}(E_k;v)} \left|b_{B_k} - b_{\|A(y)\|B_k}\right| dy \\ &\leq \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} |\det A^{-1}(y)|^{1/q_2} G\big(A^{-1}(y), \beta/q_2\big) \\ &\quad \times v(\|A(y)\|B_k)^{1/q} \|f\|_{L^{q_1}(A(y)E_k;v)} \left|b_{B_k} - b_{\|A(y)\|B_k}\right| dy \end{aligned}$$

Next, if $\|A(y)\| < 1$, then there exists an integer $j \geq 0$, such that

$$2^{-j-1} \leq \|A(y)\| < 2^{-j}.$$

Therefore,

$$\begin{aligned} \left|b_{B_k} - b_{\|A(y)\|B_k}\right| &\leq \sum_{i=1}^{j} |b_{2^{-i}B_k} - b_{2^{-i+1}B_k}| + |b_{2^{-j}B_k} - b_{A(y)B_k}| \\ &\leq \|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)} \log \frac{2}{\|A(y)\|}. \end{aligned}$$

Similarly, for $\|A(y)\| \geq 1$, we have

$$\left|b_{B_k} - b_{\|A(y)\|B_k}\right| \leq \|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)} \log 2\|A(y)\|.$$

Hence

$$\begin{aligned} J_3 &\preceq v(B_k)^{1/q} \|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)} \\ &\quad \times \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} |\det A^{-1}(y)|^{1/q_1} G\big(A^{-1}(y), \beta/q_2\big) G\big(A^{-1}(y), \beta/q\big) \|A(y)\|^{(Q+\beta)/q} \\ &\quad \times \left(\log \frac{2}{\|A(y)\|} \chi_{\{\|A(y)\|<1\}} + \log 2\|A(y)\| \chi_{\{\|A(y)\| \geq 1\}} \right) \|f\|_{L^{q_1}(A(y)E_k;v)} dy. \end{aligned}$$

Thus combining J_1, J_2 and J_3, we get

$$\begin{aligned} &\|T^b_{\Phi,A}\|_{L^{q_2}(E_k;v)} \\ &\preceq v(B_k)^{1/q} \|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)} \int_{\mathbb{H}^n} \frac{|\Phi(y)|}{|y|_h^Q} |\det A^{-1}(y)|^{1/q_1} \\ &\quad \times G\big(A^{-1}(y), \beta/q_1\big) \left(1 + |\det A^{-1}(y)|^{1/q} G\big(A^{-1}(y), \beta/q\big) \|A(y)\|^{(Q+\beta)/q}\right) \\ &\quad \times \left(\log \frac{2}{\|A(y)\|} \chi_{\{\|A(y)\|<1\}} + \log 2\|A(y)\| \chi_{\{\|A(y)\| \geq 1\}} \right) \|f\|_{L^{q_1}(A(y)E_k;v)} dy. \quad (5.4.17) \end{aligned}$$

For the approximation of $\|f(\cdot)\|_{L^q(A(y)C_k)}$, we consider the method used in [117]. Hence, the definition of E_k and (5.2.2) implies that

$$A(y)E_k \subset \{x : \|A^{-1}(y)\|^{-1}2^{k-1} < |x|_h < \|A(y)\|2^k\}.$$

Now, there exists an integer l such that for any $y \in \operatorname{supp}(\Phi)$, we have

$$2^l < \|A^{-1}(y)\|^{-1} < 2^{l+1}. \tag{5.4.18}$$

Finally, the inequality $\|A^{-1}(y)\|^{-1} \leq \|A(y)\|$ implies that there exists a non-negative integer m satisfying:

$$2^{l+m} < \|A(y)\| < 2^{l+m+1}. \tag{5.4.19}$$

We infer from (5.4.18) and (5.4.19) that:

$$\log_2(\|A(y)\|\|A^{-1}(y)\|/2) < m < \log_2(2\|A(y\|\|A^{-1}(y))\|).$$

Therefore,

$$A(y)E_k \subset \{x : 2^{l+k-1} < |x|_h < 2^{k+l+m+1}\}.$$

Hence,

$$\|f\|_{L^{q_1}(A(y)E_k;v)} \leq \sum_{j=l}^{l+m+1} \|f\|_{L^{q_1}(E_{k+j};v)}. \tag{5.4.20}$$

Incorporating the inequality (5.4.20) into (5.4.17), we obtain

$$\|T^b_{\Phi,A}\|_{L^{q_2}(E_k;v)} \leq v(B_k)^{1/q}\|b\|_{C\dot{M}O^q(\mathbb{H}^n;v)} \int_{\mathbb{H}^n} \Theta(y) \sum_{j=l}^{l+m+1} \|f\|_{L^{q_1}(E_{k+j};v)}dy, \tag{5.4.21}$$

where

$$\Theta(y) = \frac{|\Phi(y)|}{|y|_h^Q}|\det A^{-1}(y)|^{1/q_1}\left(\log\frac{2}{\|A(y)\|}\chi_{\{\|A(y)\|<1\}} + \log 2\|A(y)\|\chi_{\{\|A(y)\|\geq 1\}}\right)$$
$$\times G\big(A^{-1}(y), \beta/q_1\big)\Big(1 + |\det A^{-1}(y)|^{1/q}G\big(A^{-1}(y), \beta/q\big)\|A(y)\|^{(Q+\beta)/q}\Big)$$

Using the Minkowski inequality and the condition $1/q+\alpha_2/Q=\alpha_1/Q$, yield

$$\begin{aligned}
&\big\|T^b_{\Phi,A}\big\|_{\dot K^{\alpha_2,p}_{q_2}(\mathbb{H}^n;v)}\\
&\preceq \|b\|_{C\dot MO^q(\mathbb{H}^n;v)}\\
&\quad\times\left\{\sum_{k=-\infty}^{\infty}\left(v(B_k)^{1/q+\alpha_2/Q}\int_{\mathbb{H}^n}\Theta(y)\sum_{j=l}^{l+m+1}\|f\|_{L^{q_1}(E_{k+j};v)}dy\right)^p\right\}^{1/p}\\
&\preceq \|b\|_{C\dot MO^q(\mathbb{H}^n;v)}\\
&\quad\times\int_{\mathbb{H}^n}\Theta(y)\sum_{j=l}^{l+m+1}v(B_{-j})^{\alpha_1/Q}\left\{\sum_{k=-\infty}^{\infty}\left(v(B_{k+j})^{\alpha_1/Q}\|f\|_{L^{q_1}(E_{k+j};v)}\right)^p\right\}^{1/p}dy\\
&\preceq \|b\|_{C\dot MO^q(\mathbb{H}^n;v)}\|f\|_{\dot K^{\alpha_1,p}_{q_1}(\mathbb{H}^n;v)}\int_{\mathbb{H}^n}\Theta(y)\sum_{j=l}^{l+m+1}v(B_{-j})^{\alpha_1/Q}dy.
\end{aligned}$$

It is easy to see that

$$\sum_{j=l}^{l+m+1}v(B_{-j})^{\alpha_1/Q}\simeq\sum_{j=l}^{l+m+1}2^{-j\alpha_1(Q+\beta)/Q}.$$

Next, for $\alpha_1=0$,

$$\sum_{j=l}^{l+m+1}2^{-j\alpha_1(Q+\beta)/Q}=m+2\preceq 1+\log_2\left(\|A^{-1}(y)\|\|A(y)\|\right),$$

and for $\alpha_1\neq 0$,

$$\begin{aligned}
\sum_{j=l}^{l+m+1}2^{-j\alpha_1(Q+\beta)/Q}&\simeq 2^{-l\alpha_1(Q+\beta)/Q}\\
&\preceq\begin{cases}\|A^{-1}(y)\|^{\alpha_1(Q+\beta)/Q}, & \text{if}\quad \alpha_1>0,\\ \|A(y)\|^{-\alpha_1(Q+\beta)/Q}, & \text{if}\quad \alpha_1<0,\end{cases}\\
&=G\big(A^{-1}(y),\alpha_1(Q+\beta)/Q\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|T^b_{\Phi,A}\|_{\dot K^{\alpha_2,p}_{q_2}(\mathbb{H}^n;v)}\\
&\preceq \|b\|_{C\dot MO^q(\mathbb{H}^n;v)}\|f\|_{\dot K^{\alpha_1,p}_{q_1}(\mathbb{H}^n;v)}\\
&\quad\times\begin{cases}\int_{\mathbb{H}^n}\Theta(y)\left(1+\log_2\left(\|A^{-1}(y)\|\|A(y)\|\right)\right)dy, & \text{if}\quad\alpha_1=0,\\ \int_{\mathbb{H}^n}\Theta(y)G\big(A^{-1}(y),\alpha_1(Q+\beta)/Q\big)dy, & \text{if}\quad\alpha_1\neq 0.\end{cases}\\
&=K_3\|b\|_{C\dot MO^q(\mathbb{H}^n;v)}\|f\|_{\dot K^{\alpha_1,p}_{q_1}(\mathbb{H}^n;v)}.
\end{aligned}$$

Thus we complete the proof of Theorem 5.4.2.

Bibliography

[1] D.R. Aliyeva, *On boundedness of Hardy type integral operator in weighted Lebesgue spaces,* Caspian J. Appl. Math. 6 (2018), 35-41.

[2] J. Alvares, M. Guzman-Partida, J. Lakey, *Spaces of bounded λ-cenetral mean ocsillation,morrey spaces, and λ-central Carleson measure,* Collect. Math. 51 (2000), 1-47

[3] K.F. Andersen, *Boundedness of Hausdorff operators on $L^p(\mathbb{R}^n)$, $H^1(\mathbb{R}^n)$ and $BMO(\mathbb{R}^n)$*, Acta Sci. Math. (Szeged) 69 (2003), 409-418.

[4] A. Baernstein and E. Sawyer, *Embedding and multiplier theorems for $H^p(\mathbb{R}^n)$*, Mem. Amer. Math. Soc. 53 (1985), 1-82.

[5] R. Bandaliyev and P. Górka, *Hausdorff operator in Lebesgue spaces*, Math. Inequal. Appl. 22 (2019), 657-676.

[6] R. Bandaliyev and S.S. Volosivets, *Hausdorff operator on weighted Lebesgue and grand Lebesgue p-adic spaces,* p-Adic Numb. Ultrametric Anal. Appl. 11 (2019), 114-122.

[7] O. Blasco, A. Ruiz and L. Vega, *Non interpolation in Morrey-Campanato and block spaces,* Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. 28 (1999), 31-40.

[8] V. Burenkov, H. Guliyev and V. Guliyev, *Necessary and sufficient conditions for the bounndedness of fractional maximal operators in local Morrey type spaces*, J. Comput. Appl. Math. 208 (2007), 280-301.

[9] V. Burenkov and E. Liflyand, *On the boundedness of Hausdorff operators on Morrey-type spaces*, Eurasian Math. J. 8 (2017), 97–104.

[10] V. Burenkov and E. Liflyand, *Hausdorff operators onMorrey-type spaces*, Kyoto J. Math. (2019), 14 pages.

[11] J. Chen, J. Dai, D. Fan and X. Zhu, *Boundedness of Hausdorff operators on Lebesgue spaces and Hardy spaces,* Sci. China Math. 61 (2018), 1647-1664.

[12] J. Chen, D. Fan and J. Li, *Hausdorff operators on function spaces*, Chin. Ann. Math. 33 (2012). 537-556.

[13] J. Chen, D. Fan, X. Lin and J. Ruan, *The fractional Hausdorff operators on the Hardy space* $H^p(\mathbb{R}^n)$, Anal. Math, 42 (2016), 1-17.

[14] J. Chen, D. Fan and S. wang, *Hausdorff operators on Euclidean spaces* , Appl. Math. 28 (2013), 548-564.

[15] J. Chen, D. Fan and C. Zhang, *Multilinear Hausdorff operators and their best contants*, Acta. Math. Sinica 28 (2012), 1521-1530.

[16] J. Chen, D. Fan and C. Zhang, *Boundedness of Hausdorff operators on some product Hardy type spaces*, Appl. Math. J. Chinese Univ. 27 (2012), 114-126.

[17] J. Chen, D. Fan and X. Zhu, *The Hausdorff operatores on Hardy spaces* $H^1(\mathbb{R}^1)$, Acta Math. Hungar. 150 (2016), 142-152.

[18] J. Chen, S. He and X. Zhu, *Boundedness of Hausdorff operators on the power weighted Hardy spaces*, Appl. Math. 32 (2017), 462-476.

[19] J. Chen and X. Zhu, *Boundedness of multidimensional Hausdorff operators on* $H^1(\mathbb{R}^n)$, J. Math. Anal. Appl. 409 (2014), 428-434.

[20] N. M. Chuong and H. D. Hung, *Bounds of weighted Hardy-Cesáro operators on weighted Lebesgue and BMO spaces,* Int. Trans. Special Funct. 25 (2014), 697-710.

[21] N.M. Chuong, D.V. Duong and K.H. Dung, *Two-weighted inequalities for Hausdorff operators in Herz-type Hardy spaces*, Math. Notes 106, 2019, 20-37.

[22] N.M. Chuong, D.V. Duong and K.H. Dung, *Some estimates for p-sdic rough multilinear Hausdorff operators and commutators on weighted Morrey-Herz type spaces,* Russian J. Math. Phy. 26 (2019), 9-31.

[23] N.M. Chuong, D.V. Duong and K.H. Dung, *Multilinear Hausdorff operator on variable exponent Morrey-Herz type spaces*, Int. Trans. Special Funct. 31 (2020), 62-86.

[24] N.M. Chuong, D.V. Duong and N.D. Duyet, *Weighted Morrey-Herz space estimates for rough Hausdorff operator and its commutators*, J. Pseudo-Differ. Oper. https://doi.org/10.1007/s11868-019-00306-1.

[25] R. Daher and F. Saadi, *The Dunkl-Hausdorff operator is bounded on the real Hardy space $H^1_\alpha(\mathbb{R})$,* Int. Trans. Special Funct. 30 (2019), 882-892.

[26] B.M. Damtew, *Boundedness of multidimensional Hausdorff operator on Hardy-Morrey and Besov-Morrey spaces*, J. Inequal. Appl. (2016): 293, 11 pages.

[27] C. Espinoza-Villalva, M. Guzmán-Partida, *Continuity of Hardy type operators on rectangularly defined spaces*, J. Math. Anal. Appl. 436 (2016), 29-38.

[28] D. Fan, D. Yang, *Herz-type Hardy spaces on Vilenkin groups and their applications,* Sci. China Ser. A, 43 (2000), 481-494.

[29] D. Fan, D. Yang, *The weighted Herz-type Hardy spaces $h\dot{K}^{\alpha,p}_q(w_1, w_2)$,* Approx. Theory Appl. (N.S.), 13 (1997), 19-41.

[30] D. Fan and F. Zhao, *Multilinearf fractional Hausdorff operators,* Acta Math. Sinica, 30 (2014), 1407-1421.

[31] D. Fan and F. Zhao, *Sharp constants for multivariate Hausdorff q-inequalities,* J. Aust. Math. Soc. 106 (2018), 274-286.

[32] D. Fan and X. Lin, *Hausdorff operator on real Hardy spaces,* Analysis 34 (2014), 319-337.

[33] G. Folland and E. Stein, *Hardy spaces on homogeneous groups*, Math. Notes 28 (1982), Princeton Univ. Press, Princeton.

[34] Z. Fu, Z. Liu, S. Lu and H. Wang, *Characterization for commutators of n-dimensional fractional Hardy operators,* Sci. China Ser. A: Math. 50 (2007), 1418-1426.

[35] Z. Fu, and S. Lu, *A Characterization of $\lambda-$central BMO spaces,* Front. Math. China. 8 (2013), 229-238.

[36] Z. Fu, S. Lu, and F. Zhao, *Commutators of n-dimensional rough Hardy operators,* Sci. China Math. 54 (2011), 95-104.

[37] G. Gao and Y. Fan, *Sharp bounds for multilinear Hausdorff operators,* Acta Math Sin Chinese Ser. 58, (2015), 1-8.

[38] G. Gao and A. Hussain, *(L^p, L^q)-boundedness of Hausdorff operators with power weight on Euclidean spaces,* Anal. Theory Appl. 31, (2015), 101-108.

[39] G. Gao and H. Jia, *Boundedness of commutators of high dimentional Hausdorff operators*, J Function spaces Appl. 2012 (2012): 54120

[40] G. Gao, X. Wu, and W. Guo, *Some results for Hausdorff operators,* Math. Inequal. Appl. 18 (2015), 155-168.

[41] G. Gao, X. Wu, A. Hussain and G. Zhao, *Some esimates for Hausdorff operators,* J. Math. Inequal. 9 (2015), 641-651.

[42] G. Gao and M. Wang, *Weighted estimates for commutators of higher dimensional Hardy operators,* Appl. Math. J. Chin. Univ. Ser. A, 27 (2012), 104-111.

[43] G. Gao and F. Zhao,*Sharp weak bounds for Hausdorff operators*, Anul. Math. 41 (2015), 163-173.

[44] G. Gao and Y. Zhong, *Some inequalities for Hausdorff operators, Math.* Ineq. Appl. 17 (2014), 1061-1078.

[45] J. García-Cuerva, *Weighted H^p spaces,* Dissert. Math. 162 (1979).

[46] J. García-Cuerva and J. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland, Amsterdam, (1985).

[47] H. Garabedian, *Hausdorff Matrices,* Amer. Math. Monthly. 46 (1939), 390–410.

[48] C. Georgakis, *The Hausdorff mean of a Fourier-Stieltjes transform,* Proc. Am. Math. Soc. **116** (1992), 465–471.

[49] D. Giang and F. Móricz, *The Cesàro operator is bounded on the Hardy space H^1,* Acta Sci. Math. **61** (1995), 535–544.

[50] L. Grafakos, *Classical Fourier Analysis, second edition, Graduate Texts in Mathematics, 249,* Springer New York, (2009)

[51] L. Grafakos, X. Li, D. Yang, *Bilinear operators on Herz-type Hardy spaces,* Trans. Amer. Math. Soc., 350 (1998), 1249-1275.

[52] V.S. Guliev, *Two-weighted L^p-inequalities for singular integral opearator on Heisenberg groups,* Georgian Math. J. 1 (1994), 367-376.

[53] Y. Guo, *Boundedness of some operators on non-homogeneous spaces, Master Degree Dissertation,* Master Degree Dissertation, Beijing Normal University.

[54] J. Guo, L. Sun and F. Zhao, *Hausdorff operators on the Heisenberg group,* Acta Math. Sinca Engl Ser. 31, (2015), 1703-1714.

[55] J. Guo and F. Zhao, *Some q-inequalities for Hausdorff operators*, Front. Math. China, 12 (2017), 879-889.

[56] F. Gürbüz,*Some estimates for generalized commutators of rough fractional maximal and integral operators on generalized weighted Morrey spaces,* Canad. Math. Bull., 60 (2017), 131-145.

[57] F. Gürbüz, *Multi-sublinear operators generated by multilinear fractional integral operators and commutators on the product generalized local Morrey spaces,* Adv. Math. (China), 47 (2018), 855-880.

[58] F. Gürbüz, *Generalized weighted Morrey estimates for Marcinkiewicz integrals with rough kernel associated with Schrödinger operator and their commutators,* Chin. Ann. Math. Ser. B, 41 (2020), 77-98.

[59] G. Hardy, *Divergent series,* Clarendon Press, Oxford, 1949.

[60] F. Hausdorff, *Summationsmethoden und Momentfolgen I,* Math. Z. **9** (1921), 74–109.

[61] E. HernÁndez, D. Yang, *Interpolation of Herz-type Hardy spaces,* Illinois J. Math., 42 (1998), 564-581.

[62] E. Hille, J.D. Tamarkin, *On the summability of Fourier series, III* , Math. Ann. 108 (1933), 525-577.

[63] K.P. Ho, *Hardy Littlewood-pólya inequalities and Hausdorff operator on Block spaces*, Math. Inequal. Appl. 19 (2016), 697-707.

[64] K.P. Ho, *Hardy inequality and Hausdorff operators on rearrangement-invariant Morrey spaces,* Publ. Math, Debrecen, 88 (2016), 201-215.

[65] K.P. Ho, *Dilation operators and integral operators on amalgam space* (Lp, lq), Ricerche Mat. 68 (2019), 661-677.

[66] H.D. Hung, L.D. Ky and T.T. Quang, *Norm of Hausdorrf operator on the real hardy space* $H^1(\mathbb{R})$, Complex Anal. Oper. Theory. 12 (2018), 235-245

[67] D.Q. Huy and L.D. Ky, *The multi-parameter Hausdorff operators on H^1 and L^p*, Math. Inequal. Appl. 21 (2018), 497-510.

[68] W.A. Hurwitz, L. L. Silverman, *The consistence and equivalance of certain definitions of summability,* Trans. Amer. Math. Soc. 18 (1917), 1-20.

[69] A. Hussain and M. Ahmad, *Weak and strong type estimates for the commutators of Huasdorff operators,* Math. inequal. App. 20 (2017), 49-56.

[70] A. Hussain and A. Ajaib, *Some weighted inequalities for Hausdorff operators and commutators,* J. Inequal. Appl.(2018): 6, 19 pages.

[71] A. Hussain and A. Ajaib, *Some results for the commutators of generalized Hausdorff operator,* J. Math. Inequal. 13 (2019), 1129-1146.

[72] A. Hussain and A. Ajaib, *Weighted $CBMO$ estimates for commutators of matrix Hausdorff operator on Heisenberg group,* Open Math. (2020), (accepted).

[73] A. Hussain and G. Gao, *Multidimensional Hausdorff operators and commutators on Herz-type spaces,* J. Ineq. Appl. (2013) (2013): 594.

[74] A. Hussain and G. Gao, *Some new estimates for the commutators of n-dimensional Hausdorff operator,* Appl. Math. 29 (2014), 139-150.

[75] A. Hussain and N. Sarfraz, *The Hausdorff operator on weighted p-Adic Morrey and Herz type spaces,* p-Adic Numb. Ultrametric Anal. Appl. 11 (20109), 151-162.

[76] A. Hussain and N. Sarfraz, *Estimates for the commutators of p-Adic Hausdorff operator on Herz-Morrey spaces,* Mathematics, 7 (2019), 15 pages.

[77] T. Hytönen, C. Pérez and E. Rela, *Sharp reverse Hölder property for A_∞ weights on spaces of homogeneous type,* J. Funct. Anal. 263 (2012), 3883-3899.

[78] S. Indratno, D. Maldonado and S. Silwal, *A visual formalism for weights satisfying reverse inequalities,* Expo. Math. 33 (2015), 1-29.

[79] J. Journé, *Calderón-Zygmund operators, Pseudo-differential operators and the Cauchy integral of Calderón,* Lect. Notes Math. 994 (1983) 1-127.

[80] Y. Kanjin, *The Hausdorff operators on the real Hardy spaces $H^p(\mathbb{R})$*, Studia Math. 148 (2001), 37-45.

[81] Y. Komori and S. Shirai, *Weighted Morrey spaces and a single integral operator,* Math. Nachr. 282 (2009) 219-231.

[82] J.C. Kuang, *Generalized Hausdorff operators on weighted Morrey-Herz spaces,* Acta Math. Sinica (Chin.Ser.) 55 (2012), 895-902.

[83] J.C. Kuang, *Generalized Hausdorff operators on weighted Herz spaces*, Matematički Vesnik, 66 (2014), 19-32.

[84] A. Lerner, E. Liflyand, *Multidimensional Hausdorff operator on the real Hardy space*, J. Austr. Math. Soc. 83 (2007), 79-86.

[85] E. Liflyand, Open problems on Hausdorff operators, In: Complex Analysis and Potential Theory, Proc. Conf. Satellite to ICM 2006, Gebze, Turkey, 8-14 Sept. 2006; Eds. T. Aliyev Azeroglu and P.M. Tamrazov; World Sci., 2007, 280-285.

[86] E. Liflyand, *Boundeness of multidimensional Hausdorff operators on* $H^1(\mathbb{R}^n)$, Acta Sci. Math(Szeged), 74 (2008), 845-851.

[87] E. Liflyand, *Hausdorff operators on Hardy spaces,* Eurasian math. J. 4 (2013), 101-141.

[88] E. Liflyand and A. Miyachi, *Boundedness of Multidimensional Hausdorff operators in* H^p *spaces,* $0 < p < 1$, Trans. Amer. Math. Soc., 371 (2018), 4793-4814.

[89] E. Liflyand and A. Miyachi, *Boundedness of the Hausdorff operators in* H^p *spaces,* $0 < p < 1$, Stud. Math. 194 (2009), 279-292.

[90] E. Liflyand and F. Móricz, *The Hausdorrf operator is bounded on the real Hardy spaces* $H^1(\mathbb{R})$, Proc. Amer. Math. Soc. 128 (2000), 1391-1396.

[91] E. Liflyand and F. Móricz , *The multi-parameter Hausdorff operators is bounded on the product hardy space* $H^1(\mathbb{R} \times \mathbb{R})$, Analysis 21 (2001), 107-118.

[92] E. Liflyand and F. Móricz, *Commuting relations for Hausdorff operators and Hilbert tranforms on real Hardy spaces,* Acta Math. Hungar. 97 (2002), 133-143.

[93] J. Li and M. Wang, *Boundedness of Hausdorff operators on Triebel-Lizorkin-type spaces,* Appl. Math. J. Chinese Uni. (Ser. A) 27 (2012), 465-471.

[94] W. Li, T. Zhang and L. Xue, *Two-weight inequalities for the Hardy operator and commutators,* J. Math. Inequal. 9 (2015), 653-664.

[95] Y. Lin, Z. Liu and M. Song, *Lipschitz estimates for commutators of singular integral operators on weighted Herz spaces,* Jordan J. Math. Stat. 3 (2010) 53-64.

[96] X. Lin and L. Sun, *Some estimates on the Hausdorff operator,* Acta Sci. Math.(Szeged) 78 (2012), 669-681.

[97] M.J. Liu and S.Z. Lu, *The continuity of some operators on Herz-type Hardy spaces on the Heisenberg group,* Taiwanese J. Math. 16 (2012), 151-164.

[98] S. Lu, D. Yang, *Multiplier theorems for Herz type Hardy spaces,* Proc. Amer. Math. Soc., 126 (1998), 3337-3346.

[99] S. Lu and D. Yang, *The central BMO space and Littlewood operators,* Approx. theory Appl. 11 (1995), 72-94.

[100] S. Lu and D. Yang, *Herz-type Sobolev and Bessel potential spaces and their applications,* Sci. China Ser. A 40 (1997), 113-129.

[101] S. Lu, D. Yang and G. Hu, *Herz Type Spaces and Their Application,* Science Press-Beijing, China, 2008.

[102] A.R. Mirotin, *Boundedness of Hausdorff operators on real Hardy spaces H^1 over locally compact groups,* J. Inequal. Appl., 2019, 473, 519-533.

[103] A. R. Mirotin, *Addendum to "Boundedness of Hausdorff operators on real Hardy spaces H^1 over locally compact groups" [J. Math. Anal. Appl. 473 (2019), 519-533],* J. Math. Anal. Appl. 479 (2019), 872-874.

[104] A.R. Mirotin, *On the description of multidimensional normal Hausdorff operators on Lebesgue spaces,* Forum Math. 2019; aop, doi.org/10.1515/forum-2019-0097.

[105] C. Morrey, *On the solutions of quasi-linear elliptic partial differential equations,* Trans. Amer. Math. Soc. 43 (1938), 126-166.

[106] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function,* Trans. Amer. Math. soc. 165 (1972), 207-226.

[107] M. Paluszynski, *Characterization of Besov spaces via the commutator operator of Coifman, Rochberg and Weiss,* Ind. Univ. Math. J. 44 (1995), 1-18.

[108] R.E. Powell, S.M. Shah, *Summability theory and its applications,* Van Nostrand Reinhold Co. , London, (1972).

[109] M.A. Ragusa, *Homogeneous Herz spaces and regularity results,* Nonlinear Anal., 71 (2009), e1909-e1914.

[110] Z.X. Ren and S.P Tao, *Boundedness of higher order commutators of n-dimensional fractional Hausdorff operators,* J. Jilin Uni. (ScienceEdition) 51, 2013, 1-8.

[111] Z.X. Ren and S.P. Tao, *Weighted estimates for commutators of $n-$dimensional rough Hardy operators,* J. Function Spaces Appl. 2013 Article ID 568202, 13 pages.

[112] J. Ruan and D. Fan, *Hausdorff operators on the power weighted Hardy spaces,* J. Math. Anal. Appl. 433 (2016), 31-48.

[113] J. Ruan and D. Fan, *Hausdorff type operators on the power weighted Hardy spaces $H^p_{|.|^\alpha}(\mathbb{R}^n)$*, Math. Nachr. 290 (2017), 2388-2400.

[114] J. Ruan and D. Fan, *Hausdorff operators on the weighted Herz-type Hardy spaces,* Math. Inequal. Appl. 19 (2016), 565-587.

[115] J. Ruan, D. Fan and H. Li, *Hausdorff operators on Morrey spaces and Campanato spaces,* Czech. Math. J. (2019)

[116] J. Ruan, D. Fan and Q.Y. Wu, *Weighted Morrey estimates for Hausdorff operator and its commutator on the Heisenberg group,* Math. Inequal. Appl. (2019), 22, 307-329.

[117] J. Ruan and D. Fan and Q.Y. Wu, *Weighted Herz space estimates for the Hausdorff operators on the Heisenberg group,* Banach J. Math. Anal. 11 (2017), 513-535.

[118] A. G. Siskakis, *Composition operators and Cesàro operator on H^p*, J. London Math. Soc. 36 (1987), 153-164.

[119] A. G. Siskakis, *The Cesàro operator is bounded on H^1*, Proc. Am. Math. Soc. 110 (1990), 461-462.

[120] S.P Tao and Z.X. Ren, *Weighted estimates for commutators of n-dimensional Hausdorff operators,* Acta Math. Sinica, (Chinese Ser.) 57, (2014), 1-14.

[121] Q.X. Sun , D. Fan and H.L. Li, *Hausdorff operators on weighted Lorentz spaces,* Korean J. Math. 26 (2018), 103-127.

[122] S. Shi, and S.Lu, *Characterization of the central Campanato space via the commutator operator of Hardy type,* J. Math. Anal. Appl. 429 (2015), 713-732.

[123] J. Sun and X.L. Shi, *Some estimates for commutators of Hausdorff operators,* J. Inequal. Appl. 2015 (2015):409.

[124] F. Ustina, *Lebesgue constant for double Hausdorff means,* Bull. Austral. Math. Soc. 31 (1986), 17-22.

[125] S.S. Volosivets, *Multidimensinal Hausdorff operator on p-Adic Fields,* p-Adic Numb. Ultrametric Anal. Appl. 2 (2010), 252-259.

[126] S.S. Volosivets, *Hausdorff operator of special kind on p-adic fields and BMO-type spaces,* p-Adic Numb. Ultrametric Anal. Appl. 3 (2011), 149-156.

[127] S.S. Volosivets, *Hausdorff operator of special kind in Morrey and Herz p-adic spaces,* p-Adic Numb. Ultrametric Anal. Appl. 4 (2012), 222-230.

[128] S.S. Volosivets, *Hausdorff operators on p-adic linear spaces and their properties in Hardy, BMO, and Hölder spaces,* Math. Notes 92 (2013), 382-391.

[129] S.S. Volosivets, *Hausdorff operators defined by measures on p-adic Linear spaces and their norms,* p-Adic Numb. Ultrametric Anal. Appl. 7 (2015), 238-244.

[130] F. Weisz, *The boundedness of the Hausdorff operator on multi-dimensional Hardy spaces*, Analysis 24 (2004), 183-195.

[131] F. Weisz, *Multivariate Hausdorff operators on the spaces* $H^1(\mathbb{R}^n)$ *and* $BMO(\mathbb{R}^n)$, Anal. Math. 31 (2005), 31-41.

[132] X. Wu, *Necessary and sufficient conditions for generalized Hausdorff operators and commutators,* Anals Funct. Anal. 6 (2015), 60-72.

[133] X. Wu, *Best constants for a class of Hausdorff operators on Lebesgue spaces,* Adv. Math. (China) 46 (2017), 1-8.

[134] X. Wu and J. Chen, *Best constants for Hausdorff operators on n-dimensional product spaces,* Sci. China Math. 57 (2014), 569-578.

[135] X. Wu, G. Gao and X. Yu, *The boundedness of Hausdorff operators on Campanato spaces,* Acta Math. Sinica (Chinese Series) 57 (2014), 1-8.

[136] Q. Wu and D. Fan, *Hardy space estimates of Hausdorff operators on the Heisenberg group,* Nonlinear Anal. 164 (2017) 135-154.

[137] Q. Wu and Z. Fu, *Boundedness of Hausdorff operators on Hardy spaces in the Heisenberg group,* Banach J. Math. Anal. 12 (2018), 909-934.

[138] D. Yang, S. Lu. G. Hu, Herz Type Spaces and Their Applications, Science Press, Beijing, 2008, vi+232 pp.

[139] G.P. Zhao and W.C. Guo, *Hausdorff operators on Sobolev spaces* $W^{k,1}$, Integ. Transf. Special Funct. 30 (2019), 97-111.

[140] G.P. Zhao and Q. Lou, *Hausdorff operators on modulation spaces* $M^s_{p,p}$, J. Function Spaces, (2018), 7 pages.

[141] G.P. Zhao, D. Fan, and W.C. Guo, *Hausdorff operators on modulation and Wiener amalgam spaces,* Ann. Funct. Anal. 9 (2018), 398-412.

[142] G.H. Zhang, Q.Q. Li and Q.Y. Wu, *The weighted* L^p *and* BMO *estimates for fractional Hausdorff operators on the Heisenberg group,* J. Funct. Spaces. (2020), Article ID 5247420, 9 pages.

[143] S.C. Zou and X.R. Zhu, *The boundness of Hausdorf operators on weighted Herz-type Hardy spaces,* J. Zhejiang Normal Uni. (Nat. Sci.) 40 (2017), 1-7.