Some Contributions to Semihypergroups

QUAID-I-AZAM UNIVERSITY

ISLAMABAD

Ph.D. Thesis

By

Tariq Mahmood

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
2011

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A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
IN
MATHEMATICS

Supervised By

Prof. Dr. Muhammad Shabir

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
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CERTIFICATE

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We accept this thesis as conforming to the required standard

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Tariq Mahmood
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DEDICATED

To

My Parents and Family

Whom prayers and support have always been a source of great inspiration to me

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Introduction

In 1934 during the 8^{th} congress of Scandinavian Mathematicians, F. Marty introduced the concept of hypercompositional structures [61]. Since the beginning, the Hyperstructure Theory had applications to several domains. Marty, who introduced the hypergroups in 1934, applied them to groups, algebraic functions and rational fractions. New applications to groups were also found among others by Eaton, Ore, Krasner, Utumi, Drbohlav, Harrison, Roth, Mocker, Sureau and Haddad.

In the 1940's, Prenowitz represented several kinds of Geometries (Projective, Descriptive, Spherical) as hypergroups, and later, with Jantociak, founded Geometries on Join Spaces, a special hypergroups, which in the decades were shown to be a useful instrument in the study of several branches of mathematics: graphs, hypergraphs, binary relations, fuzzy sets and rough sets (see [13]).

In 1965, Lofti A. Zadeh introduced the notion of a fuzzy subset of a set as a method for representing uncertainty. It provoked, at first (and as expected), a strong negative reaction from some influential scientists and mathematicians, many of whom turned openly hostile. However, despite the controversy, the subject also attracted the attention of other mathematicians and in the following years, the field grew enormously, finding applications in areas as diverse as washing machines to handwritting recognition. In its trajectory of stupendous growth, it has also come to include the theory of fuzzy algebra and for the past several decades, several researchers have been working on concepts like fuzzy semigroup, fuzzy groups, fuzzy rings, fuzzy modules, fuzzy semirings, fuzzy near-rings and so on. Fuzzy ideals in semigroups have been first studied by N. Kuroki [48-55], later by other authors as well [2,3,4,58,66,67,68,76-81,83,84]. In 1996, Corsini introduced Join Spaces with Fuzzy Sets. These structures have been studied again by Corsini, Leoreanu, Tofan. The ideas of associating a hyperstructure with a fuzzy set and of considering algebraic structures endowed with a fuzzy structure, have been brought foward by Zahedi, Ameri, Borzooei, Hasankhani, Davvaz, Bolurian, Corsini, Leoreanu, Cristea, Kehagias and others[13].

It is known that Fuzzy Sets are a powerful tool in several applied sciences and so, in view of the above correspondence, hyperstructures are as well. A short review of the theory of hyperstructures appear in [12, 87]. A recent book "Applications of Hyperstructure Theory" by Corsini and Leoreanu contains a wealth of applications. Via this book, they presented some of the numerous applications of algebraic structures, especailly to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and ruogh sets, automata, criptography, codes, median algebras, relation algebras, artificial intelligence and probablities. Fuzzy algebra was first considered in 1971 by Rosenfeld [73], where many basic notions of fuzzy group were presented. Then other researchers did some more work on fuzzy groups, ideals, modules, vector spaces etc.

In 1980, Corsini gave the idea of semihypergroups [13]. Davvaz introduced hyperideals, fuzzy hyperideals and intuitionistic fuzzy hyperideals in semihypergroups[20]. Davvaz also gave the concept of congruences on semihypergroups [18]. Hedayati worked on semihypergroups and relations [28]. Hasankhani had some work on ideals in a semihypergroup and green's relations [27].

This work in algebra is concerned with the crisp as well as fuzzy approach to study some algebraic properties of semihypergroups. Our approach of study is based on the following points:

Our first approach is to define and use prime hyperideals, semiprime hperideals, m-hypersystem, p-hypersystem, fuzzy prime and fuzzy semiprime hyperideals to study the basic properties of some classes of semihypergroups.

Secondly, we use the hyperideals and fuzzy hyperideals to characterize semisimple semihypergroups.

Thirdly, we give the concept of bi-hyperideals, prime and semiprime bi-hyperideals as well as fuzzy bi-hyperideals and fuzzy prime and semiprime bi-hyperideals. Furthermore, we give some characterizations of regular and intra-regular semihypergroups in terms of these notions.

Our fourth objective in this project is to introduce the concepts of $(\in, \in \vee q)$-fuzzy left (right) hyperideals, $(\in, \in \vee q)$-fuzzy quasi-hyperideal, $(\in, \in \vee q)$-fuzzy interior hyperideal and $(\in, \in \vee q)$-fuzzy bi-hyperideals of semihypergroups and characterize semihypergroups in terms of these notions.

The fifth approach is to generalize the concepts of $(\in, \in \vee q)$-fuzzy subsemihypergroup, $(\in, \in \vee q)$-fuzzy hyperideal, $(\in, \in \vee q)$-fuzzy inteior hyperideal, $(\in, \in \vee q)$-fuzzy bi-hyperideal, and $(\in, \in \vee q)$-fuzzy quasi hyperideal and define $(\in, \in \vee q_k)$-fuzzy subsemihypergroup, $(\in, \in \vee q_k)$-fuzzy hyperideal, $(\in, \in \vee q_k)$-fuzzy inteior hyperideal, $(\in, \in \vee q_k)$-fuzzy bi-hyperideal, and $(\in, \in \vee q_k)$-fuzzy quasi hyperideal of a semihypergroup H and study some basic properties. Also we characterize regular and intra-regular semihypergroups using these notions.

Our sixth approach is to generalize the concept of $(\in, \in \vee q_k)$-fuzzy hyperideals of a semihypergroup H and we define $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) hyperideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior hyperideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal of a semihypergroup H and study some basic properties. Also we characterize regular and intra-regular semihypergroups using these notions.

Chapterwise study of this Thesis contains the following sections:

Throughout this thesis, which contains six chapters, H will denote a semihypergroup, unless otherwise stated. Chapter one, which is of introductory nature provides basic definitions and reviews some of the background materials which are needed for subsequent chapters. In chapter two, we give some basic properties of hyperideals, fuzzy hyperideals and characterize semisimple semihypergroups. In chapter three, we give the concept of bi-hyperideals, prime and semiprime bi-hyperideals of semihypergroups and characterize regular and intra-regular semihypergroups in terms of bi-hyperideals, prime and semiprime bi-hyperideals. We also characterize semihypergroups in terms of prime and semiprime bi-hyperideals. In chapter four, we define the concepts of $(\in, \in \vee q)$-fuzzy left (right) hyperideals, $(\in, \in \vee q)$-fuzzy quasi-hyperideal, $(\in, \in \vee q)$-fuzzy interior hyperideal and $(\in, \in \vee q)$-fuzzy bi-hyperideals of semihypergroups and characterize semihypergroups in terms of these notions. In chapter five, we generalize the concepts of $(\in, \in \vee q)$-fuzzy subsemihypergroup, $(\in, \in \vee q)$-fuzzy hyperideal, $(\in, \in \vee q)$-fuzzy inteior hyperideal, $(\in, \in \vee q)$-fuzzy bi-hyperideal, and $(\in, \in \vee q)$-fuzzy quasi hyperideal and define $(\in, \in \vee q_k)$-fuzzy subsemihypergroup, $(\in, \in \vee q_k)$-fuzzy hyperideal, $(\in, \in \vee q_k)$-fuzzy inteior hyperideal, $(\in, \in \vee q_k)$-fuzzy bi-hyperideal, and $(\in, \in \vee q_k)$-fuzzy quasi hyperideal of a semihypergroup H and study some basic properties. Also we characterize regular and intra-regular semihypergroups using these notions. In chapter six, we generalize the concept of $(\in, \in \vee q_k)$-fuzzy hyperideals of a semihypergroup H and define $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) hyperideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior hyperideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal of a semihypergroup H and study some basic properties. Also we characterize regular and intra-regular semihypergroups using these notions.

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Chapter 1

Elementary theory of semihypergroups

In this chapter, we give the definitions and results from the theory of hyperstructures and semihypergroups which are basic for this subject. The main results of this chapter are taken from [13], [19], [27].

1.1 Hyperstructures

The most important notions and results, obtained on Hyperstructures Theory, are presented here. For more details, see [13]

Let H be a non-empty set and the set of all non-empty subsets of H denoted by $\mathcal{P}^*(H)$.

1.1.1 Definition (cf. [13]).

A *n-hyperoperation* on H is a map $f : H^n \longrightarrow \mathcal{P}^*(H)$. The number n is called *arity* of f.

1.1.2 Definition (cf. [13]).

A set H, endowed with a family $\Gamma$of hyperoperations, is called a *hyperstructure* (or a *multivalued algebra*).

1.1.3 Definition (cf. [13]).

If Γ is a singleton, that is $\Gamma = \{f\}$ where the arity of f is 2, then the hyperstructure is called a *hypergroupoid*.

Usually, the hyperoperation is denoted by " $\circ$ " and the image of the pair (a, b) of H^2 is denoted by $a \circ b$ and called the *hyperproduct* of a and b.

If A and B are non-empty subsets of H, then $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b$.

1.2 Semihypergroups

1.2.1 Definition (cf. [13]).

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if

$$(a \circ b) \circ c = a \circ (b \circ c) \text{ for all } a, b, c \in H.$$

If a semihypergroup $(H, \circ)$ satisfies the reproduction axiom, $a \circ H = H \circ a$, for all $a \in H$, then it is called a *hypergroup*.

A non-empty subset S of a semihypergroup H is called a subsemihypergroup of H if for all $x, y \in S$, $x \circ y \subseteq S$ [13]

We shall write $A \circ x$ instead of $A \circ \{x\}$ and $x \circ A$ for $\{x\} \circ A$.

If a semihypergroup H contains an element e with the property that, for all $x \in H$, $x \in x \circ e$ (resp. $x \in e \circ x$), we say that e is a right (resp. left) identity of H. If $x \circ e = \{x\}$ (resp. $e \circ x = \{x\}$), for all x in H, then e is called scalar right (resp. left) identity in H.

1.2.2 Definition

In [27], it is defined that if $A \in P^*(H)$ then A is called,

(i) a right hyperideal in H if

$$x \in A \Longrightarrow x \circ y \subseteq A, \forall y \in H$$

(ii) a left hyperideal in H if

$$x \in A \Longrightarrow y \circ x \subseteq A, \forall y \in H$$

(iii) a hyperideal in H if it is both a left and a right hyperideal in H.

Also in the same paper it is defined that the principal right (left) hyperideals and principal hyperideal for every $a \in H$ are

$$< a >_r = (a \circ H) \cup \{a\} \quad (< a >_l = (H \circ a) \cup \{a\})$$

and

$$< a >_t = ((H \circ a) \circ H) \cup Ha \cup aH.$$

If H contains an identity element, then $< a >_r = a \circ H$, $< a >_l = H \circ a$ and $< a >_t = H \circ a \circ H$.

1.2.3 Lemma

If I and J are hyperideals of H then $I \cap J$, $I \cup J$ and $I \circ J$ are also hyperideals of H.

1.3 Fuzzy hyperideals in semihypergroups

Let X be a non-empty set. A fuzzy subset λ of X is a function from X into a unit closed interval $[0, 1]$, that is $\lambda : X \to [0, 1]$. If λ and μ are two fuzzy subsets of X then $\lambda \leq \mu$ means that $\lambda(x) \leq \mu(x)$ for all $x \in X$. The fuzzy subsets $\lambda \wedge \mu$ and $\lambda \vee \mu$ of X are called intersection and union of λ and μ, respectively and are defined as follows

$$(\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x)$$
$$(\lambda \vee \mu)(x) = \lambda(x) \vee \mu(x)$$

for all $x \in X$.

If A is a non-empty subset of X, then we shall denote the characteristic function of A by λ_A.

If λ and μ are fuzzy subsets of a semihypergroup H then their product $\lambda \circ \mu$ is a fuzzy subset of H defined by

$$(\lambda \circ \mu)(x) = \begin{cases} \bigvee_{x \in y \circ z} \{\lambda(y) \wedge \mu(z)\} & \text{if there exist } y, z \in H \text{ such that } x \in y \circ z. \\ 0 & \text{otherwise.} \end{cases}$$

1.3.1 Definition

A subset A of H is called *idempotent* if $A \circ A = A$.

Let H be a semihypergroup. By a fuzzy subset λ of H, we mean a mapping $\lambda : H \longrightarrow [0,1]$.

For any fuzzy subset λ of H and for any $t \in [0,1]$, the set

$$U(\lambda; t) = \{x \in H | \lambda(x) \geq t\}$$

is called a *level subset* of λ.

For any two fuzzy subsets λ and μ of H, $\lambda \leq \mu$ means that, for all $x \in H$, $\lambda(x) \leq \mu(x)$.

For $x \in H$, define

$$A_x = \{(y,z) \in H \times H : x \in y \circ z\}$$

For any two fuzzy subsets λ and μ of H, define

$$\lambda \circ \mu : H \longrightarrow [0,1], x \longmapsto \lambda \circ \mu(x) := \begin{cases} \bigvee_{(y,z) \in A_x} \min\{\lambda(y), \mu(z)\} & \text{if } A_x \neq \emptyset \\ 0 & \text{if } A_x = \emptyset \end{cases}$$

For a non-empty family of fuzzy subsets $\{\lambda_i\}_{i \in I}$, of a semihypergroup H, the fuzzy subsets $\bigvee_{i \in I} \lambda_i$ and $\bigwedge_{i \in I} \lambda_i$ of H are defined as follows:

$$\bigvee_{i \in I} \lambda_i : H \longrightarrow [0,1], x \longmapsto \left(\bigvee_{i \in I} \lambda_i\right)(x) := \sup_{i \in I} \{\lambda_i(x)\}$$

and

$$\bigwedge_{i \in I} \lambda_i : H \longrightarrow [0,1], x \longmapsto \left(\bigwedge_{i \in I} \lambda_i\right)(x) := \inf_{i \in I} \{\lambda_i(x)\}.$$

If I is a finite set, say $I = \{1, 2, 3, ..., n\}$, then clearly

$$\left(\bigvee_{i\in I}\lambda_i\right)(x) = \max\{\lambda_1(x), \lambda_2(x), ..., \lambda_n(x)\}$$

and $$\left(\bigwedge_{i\in I}\lambda_i\right)(x) = \min\{\lambda_1(x), \lambda_2(x), ..., \lambda_n(x)\}$$

1.3.2 Lemma

Let $\lambda_1, \lambda_2, \lambda_3$ be fuzzy subsets of a set A, then $\lambda_1 \vee (\lambda_2 \wedge \lambda_3) = (\lambda_1 \vee \lambda_2) \wedge (\lambda_1 \vee \lambda_3)$ and $\lambda_1 \wedge (\lambda_2 \vee \lambda_3) = (\lambda_1 \wedge \lambda_2) \vee (\lambda_1 \wedge \lambda_3)$.

Proof. Straightforward. □

If $A \subseteq H$, then the characteristic function λ_A of A is the fuzzy subset of H, defined as follows:

$$\lambda_A : H \longrightarrow [0,1], x \longmapsto \lambda_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

1.3.3 Lemma

Let λ, μ, ν be fuzzy subsets of a semihypergroup H such that $\mu \leq \nu$ then $\lambda \circ \mu \leq \lambda \circ \nu$ and $\mu \circ \lambda \leq \nu \circ \lambda$.

1.3.4 Proposition

If A, B are subsets of a semihypergroup H, then $A \subseteq B$ if and only if $\lambda_A \leq \lambda_B$.

Proof. Straightforward. □

1.3.5 Corollary

Let A, B be subsets of a set X, then $A = B$ if and only if $\lambda_A = \lambda_B$.

1.3.6 Proposition

Let A, B be subsets of a set X, then $\lambda_{A\cap B} = \lambda_A \wedge \lambda_B$.

Proof. Straightforward. □

1.3.7 Proposition

Let $(H,\circ)$ be a semihypergroup and A, B be subsets of H. Then $\lambda_A \circ \lambda_B = \lambda_{A\circ B}$.

Proof. Let $x \in H$. If $x \notin A \circ B$, then

$$\lambda_{A\circ B}(x) = 0 \qquad \text{(i)}$$

This means that there does not exist $y \in A$ and $z \in B$ such that $x \in y \circ z$.
If $A_x = \emptyset$ then

$$(\lambda_A \circ \lambda_B)(x) = 0 \qquad \text{(ii)}$$

If $A_x \neq \emptyset$ and $(y,z) \in A_x$ then $x \in y \circ z$. Then $y \notin A$ or $z \notin B$. Thus either $\lambda_A(y) = 0$ or $\lambda_B(z) = 0$. So we have, $\min\{\lambda_A(y), \lambda_B(z)\} = 0$. Hence $(\lambda_A \circ \lambda_B)(x) = 0$.

Let $x \in A \circ B$, then $\lambda_{A\circ B}(x) = 1$. Thus $x \in a \circ b$, for some $a \in A$ and $b \in B$, so $(a,b) \in A_x$. Since $A_x \neq \emptyset$, we have

$$\begin{aligned}(\lambda_A \circ \lambda_B)(x) &= \bigvee_{(y,z)\in A_x} \{\lambda_A(y) \wedge \lambda_B(z)\} \\ &\geq \min\{\lambda_A(a), \lambda_B(b)\} = 1.\end{aligned}$$

Thus $(\lambda_A \circ \lambda_B)(x) = 1$. Hence $\lambda_A \circ \lambda_B = \lambda_{A\circ B}$. □

1.3.8 Proposition

Let $\lambda_1, \lambda_2, \lambda_3$ are fuzzy subsets of H, then

(i) $\lambda_1 \circ (\lambda_2 \vee \lambda_3) = (\lambda_1 \circ \lambda_2) \vee (\lambda_1 \circ \lambda_3)$; $(\lambda_2 \vee \lambda_3) \circ \lambda_1 = (\lambda_2 \circ \lambda_1) \vee (\lambda_3 \circ \lambda_1)$.

(ii) $\lambda_1 \circ (\lambda_2 \wedge \lambda_3) \leq (\lambda_1 \circ \lambda_2) \wedge (\lambda_1 \circ \lambda_3)$; $(\lambda_2 \wedge \lambda_3) \circ \lambda_1 \leq (\lambda_2 \circ \lambda_1) \wedge (\lambda_3 \circ \lambda_1)$.

Proof. (i) Let $x \in H$, if $A_x = \emptyset$, then
$(\lambda_1 \circ (\lambda_2 \vee \lambda_3))(x) = 0 = (\lambda_1 \circ \lambda_2)(x) \vee (\lambda_1 \circ \lambda_3)(x)$.
If $X_x \neq \emptyset$, then

$$\begin{aligned}(\lambda_1 \circ (\lambda_2 \vee \lambda_3))(x) &= \bigvee_{(y,z)\in A_x} \{\lambda_1(y) \wedge (\lambda_2 \vee \lambda_3)(z)\} \\ &= \bigvee_{(y,z)\in A_x} \{\lambda_1(y) \wedge (\lambda_2(z) \vee \lambda_3(z))\} \\ &= \bigvee_{(y,z)\in A_x} \{(\lambda_1(y) \wedge \lambda_2(z)) \vee (\lambda_1(y) \wedge \lambda_3(z))\} \\ &= \bigvee_{(y,z)\in A_x} \{(\lambda_1(y) \wedge \lambda_2(z)\} \vee \bigvee_{(y,z)\in A_x}\{ (\lambda_1(y) \wedge \lambda_3(z)\} \\ &= (\lambda_1 \circ \lambda_2)(x) \vee (\lambda_1 \circ \lambda_3)(x) \\ &= ((\lambda_1 \circ \lambda_2) \vee (\lambda_1 \circ \lambda_3))(x).\end{aligned}$$

Hence $\lambda_1 \circ (\lambda_2 \vee \lambda_3) = (\lambda_1 \circ \lambda_2) \vee (\lambda_1 \circ \lambda_3)$.

Similarly, we can prove that $(\lambda_2 \vee \lambda_3) \circ \lambda_1 = (\lambda_2 \circ \lambda_1) \vee (\lambda_3 \circ \lambda_1)$.

(ii) Suppose $A_x = \emptyset$, then $(\lambda_1 \circ (\lambda_2 \wedge \lambda_3))(x) = 0 = (\lambda_1 \circ \lambda_2)(x) \wedge (\lambda_1 \circ \lambda_3)(x)$ implies that $\lambda_1 \circ (\lambda_2 \wedge \lambda_3) \leq (\lambda_1 \circ \lambda_2) \wedge (\lambda_1 \circ \lambda_3)$.

If $A_x \neq \emptyset$, then

$$\begin{aligned}
(\lambda_1 \circ (\lambda_2 \wedge \lambda_3))(x) &= \bigvee_{(y,z)\in A_x} \{\lambda_1(y) \wedge (\lambda_2 \wedge \lambda_3)(z)\} \\
&= \bigvee_{(y,z)\in A_x} \{(\lambda_1(y) \wedge \lambda_2(z)) \wedge (\lambda_1(y) \wedge \lambda_3(z))\} \\
&= \bigvee_{(y,z)\in A_x} \{(\lambda_1(y) \wedge \lambda_2(z)) \wedge (\lambda_1(y) \wedge \lambda_3(z))\} \\
&\leq \bigvee_{(y,z)\in A_x} \{ \lambda_1(y) \wedge \lambda_2(z)\} \wedge \bigvee_{(y,z)\in A_x} \{ \lambda_1(y) \wedge \lambda_3(z)\} \\
&= (\lambda_1 \circ \lambda_2)(x) \wedge (\lambda_1 \circ \lambda_3)(x).
\end{aligned}$$

Therefore, $\lambda_1 \circ (\lambda_2 \wedge \lambda_3) \leq (\lambda_1 \circ \lambda_2) \wedge (\lambda_1 \circ \lambda_3)$.

Similarly we can show that $(\lambda_2 \wedge \lambda_3) \circ \lambda_1 \leq (\lambda_2 \circ \lambda_1) \wedge (\lambda_3 \circ \lambda_1)$. □

1.3.9 Definition (cf. [20])

Let μ be a fuzzy subset of H, then μ is called:

i) a *fuzzy right hyperideal* of H if $\mu(x) \leq \inf_{\alpha \in x \circ y} \{\mu(\alpha)\}$, for every $x, y \in H$;

ii) a *fuzzy left hyperideal* of H if $\mu(y) \leq \inf_{\alpha \in x \circ y} \{\mu(\alpha)\}$, for every $x, y \in H$;

iii) a *fuzzy hyperideal* of H (or fuzzy two-sided hyperideal) if it is both a fuzzy left hyperideal and a fuzzy right hyperideal.

1.3.10 Lemma (cf. [20])

Let μ be a fuzzy hyperideal of H, then

$$\max\{\mu(x_1), ..., \mu(x_n)\} \leq \inf_{\alpha \in x_1 \circ x_2 \circ ... x_n} \{\mu(\alpha)\}, \ \forall x_1, x_2, ..., x_n \in H.$$

1.3.11 Proposition (cf. [20])

A non-empty subset A of H is a hyperideal of H if and only if the characteristic function λ_A of A is a fuzzy hyperideal of H.

1.3.12 Corollary(cf. [20])

Let H be a semihypergroup and μ be a fuzzy hyperideal of H and n is a positive integer. Then for every $\alpha \in x^n$ we have $\mu(x) \leq \mu(\alpha)$.

1.3.13 Theorem(cf. [20])

Let H be a semihypergroup and μ be a fuzzy subset of H. Then μ is a fuzzy right (left, two-sided) hyperideal of H if and only if for every $t \in [0,1], \mu_t \neq \emptyset$ is a right (left, two-sided) hyperideal.

1.3.14 Corollary(cf. [20])

Let μ be a fuzzy right (left, two-sided) hyperideal of a semihypergroup H. Then two level right (left, two-sided) hyperideals μ_{t_1} and μ_{t_2} (with $t_1 < t_2$) of μ are equal if and only if there is no $x \in H$ such that $t_1 \leq \mu(x) < t_2$.

1.3.15 Corollary(cf. [20])

Let H be a semihypergroup and μ be a fuzzy right (left, two-sided) hyperideal of H. If $\mathrm{Im}(\mu) = \{t_1, ..., t_n\}$, where $t_1 < \cdot < t_n$, then the family of right (left, two-sided) hyperideals $\mu_{t_i} (i = 1, ..., n)$ consitutes all the level right (left, two-sided) hyperideals of μ.

1.3.16 Corollary(cf. [20])

Each subset I of H may be regarded as a fuzzy subset by identifying it with its characteristic function λ_I. If I is any non-empty subset of H, then I is a right (left, two-sided) hyperideal if and only if λ_I is a fuzzy right (left, two-sided) hyperideal.

Let H be a semihypergroup. Then for every $a \in H$ we put

$$\begin{aligned}(a)H &= \{a\} \cup (\bigcup_{x \in H} a \cdot x), \\ H(a) &= \{a\} \cup (\bigcup_{x \in H} x \cdot a), \\ H(a)H &= (a)H \cup H(a) \cup (\bigcup_{x,y \in H} x \cdot a \cdot y).\end{aligned}$$

1.3.17 Proposition

A fuzzy subset λ of a semihypergroup H is a fuzzy left (right) hyperideal of H if and only if for each $t \in [0,1]$, $U(\lambda;t) \neq \phi$ is a left (right) hyperideal of H, respectively.

Proof. Suppose λ be a fuzzy left hyperideal of H and $x \in U(\lambda;t)$ and $y \in H$. Then $\lambda(x) \geq t$. Since λ is a fuzzy left hyperideal of H, so $\lambda(x) \leq \inf_{\alpha \in y \circ x} \{\lambda(\alpha)\}$ for every $y \in H$. Hence $\lambda(\alpha) \geq t$ for all $\alpha \in y \circ x$, this implies $\alpha \in U(\lambda;t)$ that is $y \circ x \subseteq U(\lambda;t)$. Hence $U(\lambda;t)$ is a hyperideal of H.

Conversely, assume that $U(\lambda;t) \neq \emptyset$ is a left hyperideal of H. Let $x \in H$ such that $\lambda(x) > \inf_{\alpha \in y \circ x} \{\lambda(\alpha)\}$ for all $y \in H$. Select $t \in [0,1]$ such that $\lambda(x) = t > \inf_{\alpha \in y \circ x} \{\lambda(\alpha)\}$. Then $x \in U(\lambda;t)$ but $y \circ x \not\subseteq U(\lambda;t)$, a contradiction. Hence $\lambda(x) \leq \inf_{\alpha \in y \circ x} \{\lambda(\alpha)\}$, that is λ is a fuzzy left hyperideal of H. □

1.3.18 Proposition

Let λ, μ be fuzzy hyperideals of H, then $\lambda \wedge \mu$ and $\lambda \vee \mu$ are fuzzy hyperideals of H.

Proof. Let λ and μ be fuzzy hyperideals of a semihypergroup H. Then for $x, y, z \in H$, $\inf_{x \in y \circ z} \lambda(x) \geq \lambda(z)$ and $\inf_{x \in y \circ z} \mu(x) \geq \mu(z)$.

Now, for each $x \in y \circ z$,

$$\begin{aligned} (\lambda \wedge \mu)(x) &= \lambda(x) \wedge \mu(x) \\ &\geq \lambda(z) \wedge \mu(z) \\ &= (\lambda \wedge \mu)(z). \end{aligned}$$

Hence, $\inf_{x \in y \circ z} (\lambda \wedge \mu)(x) \geq (\lambda \wedge \mu)(z)$. Thus $\lambda \wedge \mu$ is a fuzzy left hyperideal of H.

Similarly we can show that $\lambda \wedge \mu$ is a fuzzy right hyperideal of H. Thus $\lambda \wedge \mu$ is a fuzzy hyperideal of H.

Similarly we can show that $\lambda \vee \mu$ is a fuzzy hyperideal of H. □

1.3.19 Lemma

A fuzzy subset λ of H is a fuzzy left (resp. right) hyperideal of H if and only if $\lambda_H \circ \lambda \leq \lambda$ (resp. $\lambda \circ \lambda_H \leq \lambda$).

Proof. Let λ be a fuzzy left hyperideal of H and $x \in H$, then

$$\begin{aligned} \lambda_H \circ \lambda(x) &= \bigvee_{x \in y \circ z} \{\lambda_H(y) \wedge \lambda(z)\} \\ &= \bigvee_{x \in y \circ z} \{\lambda(z)\} \quad (\because \lambda_H(y) = 1) \\ &\leq \bigvee_{x \in y \circ z} \lambda(x) \quad \text{since } \lambda(z) \leq \inf_{\alpha \in y \circ z} \lambda(\alpha) \leq \lambda(\alpha) \text{ for each } \alpha \in y \circ z. \\ &= \lambda(x). \end{aligned}$$

Hence, $\lambda_H \circ \lambda(x) \leq \lambda(x)$.

Conversely, suppose that $\lambda_H \circ \lambda \leq \lambda$. We show that λ is a fuzzy left hyperideal of H. For $x \in H$,

$$\begin{aligned} \lambda(x) &\geq \lambda_H \circ \lambda(x) \\ &= \bigvee_{x \in y \circ z} \{\lambda_H(y) \wedge \lambda(z)\} \text{ for } y, z \in H \\ &= \bigvee_{x \in y \circ z} \{\lambda(z)\}, \text{ (because } \lambda_H(y) = 1) \\ &\geq \lambda(z) \text{ for each } z \text{ such that } x \in y \circ z. \end{aligned}$$

Thus $\inf_{x \in y \circ z} \lambda(x) \geq \lambda(z)$. Hence λ is a fuzzy left hyperideal of H.
Similarly, we can prove the case of fuzzy right hyperideal. □

1.3.20 Lemma

If λ is a fuzzy left hyperideal and μ a fuzzy right hyperideal of H, then $\lambda \circ \mu$ is a fuzzy hyperideal of H and $\lambda \circ \mu \leq \lambda \wedge \mu$.

Proof. Let λ be a fuzzy left hyperideal and μ a fuzzy right hyperideal of H, then

$$\begin{aligned} \lambda_H \circ (\lambda \circ \mu) &= (\lambda_H \circ \lambda) \circ \mu \\ &\leq \lambda \circ \mu \text{ (by Lemma 1.3.19)} \end{aligned}$$

Hence $\lambda \circ \mu$ is a fuzzy left hyperideal of H.
Also,

$$\begin{aligned} (\lambda \circ \mu) \circ \lambda_H &= \lambda \circ (\mu \circ \lambda_H) \\ &\leq \lambda \circ \mu \text{ (by Lemma 1.3.19)} \end{aligned}$$

So, $\lambda \circ \mu$ is a fuzzy right hyperideal of H.
Thus $\lambda \circ \mu$ is a fuzzy hyperideal of H.
Let $x \in H$. If $A_x = \emptyset$, then $(\lambda \circ \mu)(x) = 0 \leq (\lambda \wedge \mu)(x)$.
If $A_x \neq \emptyset$, then $(\lambda \circ \mu)(x) = \bigvee_{(y,z) \in A_x} \min\{\lambda(y), \mu(z)\}$.

For $(y, z) \in A_x$ we have $x \in y \circ z$, so we have $\lambda(x) \geq \inf_{\alpha \in y \circ z} \lambda(\alpha) \geq \lambda(y)$ and $\mu(x) \geq \inf_{\alpha \in y \circ z} \mu(\alpha) \geq \mu(z)$. Hence

$$\begin{aligned} (\lambda \circ \mu)(x) &= \bigvee_{(y,z) \in A_x} \min\{\lambda(y), \mu(z)\} \\ &\leq \min\{\lambda(x), \mu(x)\} \\ &= (\lambda \wedge \mu)(x). \end{aligned}$$

Thus, $\lambda \circ \mu \leq \lambda \wedge \mu$. □

Chapter 2

Semihypergroups characterized by their fuzzy hyperideals

Hyperideals play an important role in studying the structure of semihypergroups. In this chapter, we characterize some classes of semihypergroups by the properties of their fuzzy left (resp. right) ideals. In [20], Davvaz introduced the concept of fuzzy hyperideals in a semihypergroup. We extend this idea to prime (semiprime) hyperideals of a semihypergroup.

In this chapter, we define prime (semiprime) hyperideals and prime (semiprime) fuzzy hyperideals of semihypergroups. We characterize semihypergroups in terms of their prime (semiprime) hyperideals and prime (semiprime) fuzzy hyperideals.

In [4], Ahsan et al. have shown that a semigroup S is semisimple if and only if each fuzzy ideal of S is semiprime if and only if each fuzzy ideal of S is the intersection of prime fuzzy ideals of S which contain it.

Here we give parallel characterizations for semihypergroups in crisp as well as in fuzzy form and prove that a semihypergroup H is semisimple if and only if each hyperideal of H is semiprime if and only if each hyperideal of H is the intersection of prime hyperideals of H which contain it. We also extend this property of semihypergroups in fuzzy context and prove that a semihypergroup H is semisimple if and only if for all fuzzy hyperideals λ and μ of H, $\lambda \circ \mu = \lambda \wedge \mu$ if and only if every fuzzy hyperideal of H is semiprime if and only if each fuzzy hyperideal of H is the intersection of prime fuzzy hyperideals of H which contain it.

2.1 Prime and Semiprime hyperideals

Throughout this chapter H will denote a semihypergroup.

2.1.1 Definition

A hyperideal I of H is called *maximal* if there is no proper hyperideal J of H such that $I \subset J \subset H$.

2.1.2 Definition

An element $x \in H$ is called a zero element of H if for every $y \in H, x \circ y = y \circ x = x$.

If H does not contain a zero element then a hyperideal I of H is called *minimal* if there is no proper hyperideal J of H such that $J \subset I$.

2.1.3 Definition

A hyperideal I of a semihypergroup H is called *prime hyperideal* of H if for all hyperideals A, B of H,

$$A \circ B \subseteq I \text{ implies that either } A \subseteq I \text{ or } B \subseteq I.$$

2.1.4 Definition

A non-empty subset M of a semihypergroup H is called an *m-hypersystem* if for all $a, b \in M$, there exists an $h \in H$ such that $a \circ h \circ b \cap M \neq \emptyset$.

2.1.5 Definition

An ideal I of a semihypergroup H is called *semiprime* if for all hyperideals A of H, $A \circ A \subseteq I$ implies $A \subseteq I$.

2.1.6 Definition

A non-empty subset A of a semihypergroup H is called a *p-hypersystem* if for each $a \in A$, there exists an $h \in H$ such that $a \circ h \circ a \cap A \neq \emptyset$.

2.1.7 Definition

Let H be a semihypergroup. An element $a \in H$ is called *idempotent* if $a \in a \circ a$. A hyperideal I of a semihypergroup H is called idempotent if $I \circ I = I$.

2.1.8 Definition

A semihypergroup H is called *semisimple* if every hyperideal of H is idempotent.

2.1.9 Definition

A semihypergroup H is called *fully prime* (*fully semiprime*) if every hyperideal of H is prime (semiprime).

It is obvious from the definitions that every prime hyperideal is a semiprime hyperideal and every m-hypersystem is a p-hypersystem.

2.1.10 Proposition

Let H be a semihypergroup with identity. Then the following conditions are equivalent.

(1) I is a prime hyperideal;

(2) $a \circ H \circ b \subseteq I$ if and only if $a \in I$ or $b \in I$;

(3) Let $a, b \in H$ be such that $(H \circ a \circ H) \circ (H \circ b \circ H) \subseteq I$. Then $a \in I$ or $b \in I$, where $(H \circ a \circ H)$ is the principal hyperideal of H generated by a and $(H \circ b \circ H)$ is the principal hyperideal of H generated by b.

Proof. (1) $\Rightarrow$ (2) Let $a, b \in H$ and set $\acute{I} = a \circ H \circ b$. If $a \in I$ or $b \in I$ then $\acute{I} \subseteq I$, because I is an ideal of the semihypergroup H.

Conversely, assume that $H \circ a \circ H$ and $H \circ b \circ H$ are the principal hyperideals of H generated by a and b, respectively. If $a \circ H \circ b \subseteq I$, then

$$H \circ (a \circ H \circ b) \circ H \subseteq I \text{ (becuase } I \text{ is a hyperideal of } H).$$

Hence

$$(H \circ a \circ H) \circ (H \circ b \circ H) \subseteq I$$

$$\Longrightarrow H \circ a \circ H \subseteq I \text{ or } H \circ b \circ H \subseteq I \text{ (because } I \text{ is a prime hyperideal)}$$

This implies that either $a \in I$ or $b \in I$.

(2) $\Rightarrow$ (3) Let a and b be elements of a semihypergroup H such that $(H \circ a \circ H) \circ (H \circ b \circ H) \subseteq I$. As $a \circ H \circ b \subseteq (H \circ a \circ H) \circ (H \circ b \circ H) \subseteq I \Rightarrow a \circ H \circ b \subseteq I$. Thus by (2), either $a \in I$ or $b \in I$.

(3) $\Rightarrow$ (1) Let A and B be hyperideals of the semihypergroup H such that $A \circ B \subseteq I$. Let $A \nsubseteq I$ and $a \in A$ such that $a \notin I$. Let b be any arbitrary element of B. Then $(H \circ a \circ H) \circ (H \circ b \circ H) \subseteq A \circ B \subseteq I$. By (3), either $a \in I$ or $b \in I$. As $a \notin I$ so $b \in I$, i.e. $B \subseteq I$. Hence, $A \circ B \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$. □

2.1.11 Corollary

A hyperideal I of a semihypergroup H with identity is prime if and only if $H \backslash I$ is an m-hypersystem.

Proof. Let I be a prime hyperideal of H and $a, b \in H \backslash I$. Suppose there does not exist $h \in H$ such that $a \circ h \circ b \cap H \backslash I \neq \emptyset$, that is, for each $h \in H$, $a \circ h \circ b \cap H \backslash I = \emptyset$. This implies $a \circ H \circ b \subseteq I$. By Proposition 2.1.10, either $a \in I$ or $b \in I$, which is a contradiction. Hence there exists an $h \in H$ such that

$$a \circ h \circ b \cap H \backslash I \neq \emptyset.$$

Conversely, let $H \backslash I$ be an m-hypersystem. Let $a, b \in H$ be such that $a \circ H \circ b \subseteq I$. If $a \notin I$ and $b \notin I$, then $a, b \in H \backslash I$, and since $H \backslash I$ is an m-hypersystem, so there exists an $h \in H$ such that $a \circ h \circ b \cap H \backslash I \neq \emptyset$ i.e. $a \circ H \circ b \nsubseteq I$, which is a contradiction. Hence either $a \in I$ or $b \in I$. □

2.1.12 Proposition

Let H be a semihypergroup with scalar identity. Then every maximal hyperideal of H is prime.

Proof. Let P be a maximal hyperideal of H. Suppose A, B be hyperideals of H such that $A \circ B \subseteq P$. Suppose that $A \nsubseteq P$ then $A \cup P = H$. As $e \in H$ so $e \in A \cup P$. Since $e \notin P$ so $e \in A$. Thus $A = H$. Now $B = H \circ B = A \circ B \subseteq P$. Hence P is prime. □

2.1.13 Proposition

If I is a hyperideal of a semihypergroup H and J a hyperideal of H minimal among those hyperideals of H which properly contain I, then $K = \{x \in H : x \circ J \subseteq I\}$ is a prime hyperideal of H.

Proof. First we show that K is a hyperideal of H. Let $x \in K$ and $h \in H$. Then

$$(h \circ x) \circ J = h \circ (x \circ J) \subseteq H \circ I \subseteq I \quad \text{(since } I \text{ is a hyperideal of } H)$$

$$\Rightarrow h \circ x \subseteq K.$$

Again,

$$(x \circ h) \circ J = x \circ (h \circ J) \subseteq x \circ J \subseteq I \quad \text{(because } J \text{ is a hyperideal of } H)$$

$$\Rightarrow x \circ h \subseteq K.$$

Hence K is a hyperideal of H. Now we prove that K is prime. Suppose A, B are hyperideals of H such that $A \circ B \subseteq K$ but $B \nsubseteq K$. As $A \circ B \subseteq K$ and $B \nsubseteq K$, we have

$$(A \circ B) \circ J \subseteq I \text{ (by definition of } K) \text{ and } B \circ J \nsubseteq I.$$

Therefore,

$$I \subseteq I \cup B \circ J \subseteq J \quad \text{(since } J \supset I \text{ and } B \circ J \subseteq J).$$

By minimality of J, we get,

$$I \cup B \circ J = J.$$

Now,

$$\begin{aligned} A \circ J &= A \circ (I \cup B \circ J) \\ &= A \circ I \cup A \circ B \circ J \\ &\subseteq I \text{ (because } A \circ I \subseteq I \text{ and } (A \circ B) \circ J \subseteq I\text{)}. \end{aligned}$$

Hence, $A \subseteq K$ and thus K is prime. □

2.1.14 Proposition

Let H be a semihypergroup with identity. Then the following statements are equivalent for a hyperideal I of H.

(1) I is a semiprime hyperideal;

(2) $a \circ H \circ a \subseteq I$ if and only if $a \in I$.

Proof. (1) $\Rightarrow$ (2) Let $a \in H$ and set $I' = a \circ H \circ a$. If $a \in I$ then

$$I' \subseteq I \text{ (since } I \text{ is a hyperideal of } H).$$

Conversely, let $H \circ a \circ H$ be the principal hyperideal of H generated by a. If $a \circ H \circ a \subseteq I$, then

$$H \circ (a \circ H \circ a) \circ H \subseteq I \text{ (because } I \text{ is a hyperideal of } H)$$

$$\Longrightarrow (H \circ a \circ H) \circ (H \circ a \circ H) \subseteq I \text{ (since } H \text{ is a semihypergroup)}.$$

$$\Longrightarrow H \circ a \circ H \subseteq I \text{ (because } I \text{ is semiprime)}$$

$$\Longrightarrow a \in I.$$

(2) $\Rightarrow$ (1) Let A be a hyperideal of a semihypergroup H such that $A \circ A \subseteq I$. Let $a \in A$. Then

$$a \in H \circ a \circ H \text{ and } H \circ a \circ H \subseteq A \text{ (as } A \text{ is a hyperideal of } H).$$

Also,

$$a \circ H \circ a \subseteq H \circ (a \circ H \circ a) \circ H \subseteq (H \circ a \circ H) \circ (H \circ a \circ H) \subseteq A \circ A \subseteq I.$$

So, by (2), $a \in I$. Hence $A \subseteq I$. □

2.1.15 Corollary

An ideal I of a semihypergroup H is semiprime if and only if $H \backslash I$ is a p-hypersystem.

Proof. Let I be a prime hyperideal of H and $a \in H \backslash I$. Suppose there does not exist $h \in H$ such that $a \circ h \circ a \cap H \backslash I \neq \emptyset$, that is, for each $h \in H$, $a \circ h \circ a \cap H \backslash I = \emptyset$. This implies $a \circ H \circ a \subseteq I$. By Proposition 2.1.14, $a \in I$, which is a contradiction. Hence there exists an $h \in H$ such that

$$a \circ h \circ a \cap H \backslash I \neq \emptyset.$$

Conversely, assume that $H \backslash I$ is a p-hypersystem. Let $a \in H$ be such that $a \circ H \circ a \subseteq I$. If $a \notin I$, then $a \in H \backslash I$. So, there exists an $h \in H$ such that $a \circ h \circ a \cap H \backslash I \neq \emptyset$,

$$\Rightarrow a \circ H \circ a \nsubseteq I, \text{ which is a contradiction.}$$

Hence $a \in I$, i.e. I is semiprime. □

2.1.16 Theorem

A semihypergroup H is fully semiprime if and only if H is semisimple.

Proof. Let H be a semisimple semihypergroup and P, I be hyperideals of H such that $I \circ I \subseteq P$. Then $I \subseteq P$, (because H is semisimple). Hence P is a semiprime hyperideal of H.

Conversely, assume that H is a fully semiprime semihypergroup. Let I be a hyperideal of H. Then $I \circ I$ is also a hyperideal of H. As $I \circ I \subseteq I \circ I$, therefore $I \subseteq I \circ I$ (because $I \circ I$ is a semiprime hyperideal of H). But $I \circ I \subseteq I$ always. Hence $I \circ I = I$, i.e. each hyperideal of H is idempotent. Thus H is semisimple. □

2.1.17 Theorem

A semihypergroup H is fully prime if and only if H is semisimple and the set of hyperideals of H is totally ordered under inclusion.

Proof. Let H be a fully prime semihypregroup. Then H is a fully semiprime semihypergroup. Thus by Theorem 2.1.16, H is semisimple. Suppose P, Q be hyperideals of H. Since $P \circ Q \subseteq P \cap Q$, so either $P \subseteq P \cap Q$ or $Q \subseteq P \cap Q$, i.e. either $P \subseteq Q$ or $Q \subseteq P$. Thus the set of hyperideals of H is totally ordered.

Conversely, assume that H is a semisimple semihypergroup and the set of hyperideals of H is totally ordered under inclusion. Let I, J, P be hyperideals of H with $I \circ J \subseteq P$. Since the set of hyperideals of H is totally ordered under inclusion, so either $I \subseteq J$ or $J \subseteq I$. Assume that $I \subseteq J$, then

$I = I \circ I \subseteq I \circ J \subseteq P$. Hence $I \subseteq P$ and so P is a prime hyperideal of H. □

2.2 Prime and Semiprime fuzzy hyperideals

In this section we define prime fuzzy and semiprime fuzzy hyperideals and characterize semihypergroups in terms of these fuzzy hyperideals.

2.2.1 Definition

Let λ be a fuzzy subset of H. Then λ is called a prime fuzzy hyperideal of H if for all fuzzy hyperideals μ, ν of H, $\mu \circ \nu \leq \lambda$ implies $\mu \leq \lambda$ or $\nu \leq \lambda$.

2.2.2 Definition

Let λ be a fuzzy subset of H. Then λ is called a semiprime fuzzy hyperideal of H if for every fuzzy hyperideal μ of H, $\mu \circ \mu \leq \lambda$ implies $\mu \leq \lambda$.

2.2.3 Proposition

Let I be a hyperideal of a semihypergroup H, then the following hold:

(1) I is a prime hyperideal of H if and only if the characteristic function λ_I of I is a prime fuzzy hyperideal of H.

(2) I is a semiprime hyperideal of H if and only if the characteristic function λ_I of I is a semiprime fuzzy hyperideal of H.

Proof. (1) Suppose I is a prime hyperideal of H. Then by Proposition 1.3.11, λ_I is a fuzzy hyperideal of H. Let μ and ν be any fuzzy hyperideals of H, such that $\mu \circ \nu \leq \lambda_I$ but $\mu \nleq \lambda_I$ and $\nu \nleq \lambda_I$. Then there exist $y, z \in H$ such that $\mu(y) \neq 0$ and $\nu(z) \neq 0$, but $\lambda_I(y) = 0$ and $\lambda_I(z) = 0$. Then $y, z \notin I$. Since I is a prime hyperideal of H, we have $y \circ z \nsubseteq I$. Hence there exists $x \in y \circ z$ such that $x \notin I$. This implies $\lambda_I(x) = 0$. Hence $(\mu \circ \nu)(x) = 0$. Since $\mu(y) \neq 0$ and $\nu(z) \neq 0$, we have $\min\{\mu(y), \nu(z)\} \neq 0$. Since $x \in y \circ z$ so $(y, z) \in A_x$. Since $A_x \neq \emptyset$ we have, $(\mu \circ \nu)(x) = \bigvee\limits_{(y,z) \in A_x} \min\{\mu(y), \nu(z)\} \neq 0$. Thus $(\mu \circ \nu)(x) > 0$, a contradiction. Hence $\mu \circ \nu \leq \lambda_I$ implies that $\mu \leq \lambda_I$ or $\nu \leq \lambda_I$. Thus λ_I is a prime fuzzy hyperideal of H.

Conversely, suppose that λ_I is a prime fuzzy hyperideal of H. Let A, B be any hyperideals of H such that $A \circ B \subseteq I$. Then by Proposition 1.3.11, λ_A, λ_B are fuzzy hyperideals of H. Since $A \circ B \subseteq I$, so by Proposition 1.3.4, $\lambda_{A \circ B} \leq \lambda_I$. Thus by Proposition 1.3.7, $\lambda_{A \circ B} = \lambda_A \circ \lambda_B \leq \lambda_I$, so we have $\lambda_A \leq \lambda_I$ or $\lambda_B \leq \lambda_I$. By Proposition 1.3.4, we get $A \subseteq I$ or $B \subseteq I$. Hence I is a prime hyperideal of H.

Similarly we can prove part (2). □

2.2.4 Proposition

Let $\{\lambda_i : i \in I\}$ be a family of prime fuzzy hyperideals of a semihypergroup H. Then $\bigwedge\limits_{i \in I} \lambda_i$ is a semiprime fuzzy hyperideal of H.

Proof. Straightforward. □

2.2.5 Definition

Let H be a semihypergroup. Then for $a \in H$ and $t \in (0, 1]$, the fuzzy subset a_t of H is defined by

$$a_t(x) = \begin{cases} t \text{ if } x = a \\ 0 \text{ if } x \neq a \end{cases}$$

and is called a fuzzy point of H with support a and value t.

2.2.6 Definition

Let λ be a fuzzy subset of H. The fuzzy right (left) hyperideal of H generated by λ is the smallest fuzzy right (left) hyperideal of H containing λ.

2.2.7 Definition

A fuzzy hyperideal λ of H is called idempotent if $\lambda \circ \lambda = \lambda$. A semihypergroup is called fuzzy semisimple if all its fuzzy hyperideals are idempotent.

2.2.8 Definition

A fuzzy hyperideal λ of a semihypergroup H is called maximal if there does not exist any proper fuzzy hyperideal μ of H such that $\lambda < \mu$.

2.2.9 Lemma

A fuzzy subset λ of a semihypergroup H is a fuzzy left (right) hyperideal of H if and only if $\lambda_H \circ \lambda \leq \lambda$ $(\lambda \circ \lambda_H \leq \lambda)$.

Proof. Let λ be a fuzzy left hyperideal of H and $x \in H$. Then

$$\begin{aligned}
(\lambda_H \circ \lambda)(x) &= \bigvee_{x \in y \circ z} \{\lambda_H(y) \wedge \lambda(z)\} \\
&= \bigvee_{x \in y \circ z} \{\lambda(z)\} \quad (\because \lambda_H(y) = 1) \\
&\leq \bigvee_{x \in y \circ z} \lambda(x), \text{ because } \lambda(z) \leq \inf_{\alpha \in y \circ z} \{\lambda(\alpha)\} \leq \lambda(\alpha) \text{ for each } \alpha \in y \circ z. \\
&= \lambda(x).
\end{aligned}$$

Hence, $(\lambda_H \circ \lambda)(x) \leq \lambda(x)$.

Conversely, suppose that $\lambda_H \circ \lambda \leq \lambda$. We show that λ is a fuzzy left hyperideal of H. Let $x \in H$. Then

$$\begin{aligned}
\lambda(x) &\geq (\lambda_H \circ \lambda)(x) \\
&= \bigvee_{x \in y \circ z} \{\lambda_H(y) \wedge \lambda(z)\} \\
&= \bigvee_{x \in y \circ z} \{\lambda(z)\}, \text{ (because } \lambda_H(a) = 1) \\
&\geq \lambda(z), \text{ for each } z \text{ such that } x \in y \circ z.
\end{aligned}$$

Thus $\inf_{x \in y \circ z} \lambda(x) \geq \lambda(z)$. Hence λ is a fuzzy left hyperideal of H.
Similarly we can prove the case of fuzzy right hyperideal of H. □

2.2.10 Lemma

If λ is a fuzzy left hyperideal and μ a fuzzy right hyperideal of a semihypergroup H, then $\lambda \circ \mu$ is a fuzzy hyperideal of H.

Proof. Consider

$$\begin{aligned} \lambda_H \circ (\lambda \circ \mu) &= (\lambda_H \circ \lambda) \circ \mu \\ &\leq \lambda \circ \mu \text{ (by Lemma 2.2.9).} \end{aligned}$$

Hence $\lambda \circ \mu$ is a fuzzy left hyperideal of H.
Now,

$$\begin{aligned} (\lambda \circ \mu) \circ \lambda_H &= \lambda \circ (\mu \circ \lambda_H) \\ &\leq \lambda \circ \mu \text{ (by Lemma 2.2.9).} \end{aligned}$$

So, $\lambda \circ \mu$ is a fuzzy right hyperideal of H.
Thus, $\lambda \circ \mu$ is a fuzzy hyperideal of H. □

2.2.11 Lemma

Let a_t be a fuzzy point of a semihypergroup H. Then the fuzzy left (right) hyperideal of H generated by a_t is $l_{a_t}(\xi_{a_t})$ defined by

$$l_{a_t}(x) = \begin{cases} t \text{ if } x \in Ha \\ 0 \text{ if } x \notin Ha \end{cases}$$

and

$$\zeta_{a_t} = \begin{cases} t \text{ if } x \in aH \\ 0 \text{ if } x \notin aH \end{cases}$$

Proof. For $x, y \in H$ if $x \circ y \subseteq Ha$, then for each $\alpha \in x \circ y, l_{a_t}(\alpha) = t \geq l_{a_t}(y)$. If $x \circ y \nsubseteq Ha$ then $y \notin Ha$. Because if $y \in Ha$ then $y = a$ or there exists an $h \in H$ such that $y \in h \circ a$. If $y = a$ then $x \circ y = x \circ a \subseteq Ha$, a contradiction and if $y \in h \circ a$ then $x \circ y \subseteq x \circ (h \circ a) = (x \circ h) \circ a \subseteq Ha$, again a contradiction. Thus $y \notin Ha$. Thus $l_{a_t}(y) = 0 \leq l_{a_t}(\alpha)$ for all $\alpha \in x \circ y$. Hence in any case, $l_{a_t}(\alpha) \geq l_{a_t}(y)$ for each $\alpha \in x \circ y$. Thus $l_{a_t}(y) \leq \inf\limits_{\alpha \in x \circ y} l_{a_t}(\alpha)$. So l_{a_t} is a fuzzy left hyperideal of H. By definition of l_{a_t}, we find that $a_t \leq l_{a_t}$. If λ is a fuzzy left hyperideal of H containing a_t and if $x \in Ha$ then as $t = a_t(a) \leq \lambda(a)$ implies that $t \leq \lambda(a) \leq \lambda(\beta)$, for $\beta \in h \circ a$ and $h \in H$. Which implies that $\lambda(x) \geq t = l_{a_t}(x)$. If $x \notin Ha$, then $l_{a_t}(x) = 0 \leq \lambda(x)$. So $l_{a_t} \leq \lambda$. Thus l_{a_t} is a fuzzy left hyperideal of H generated by a_t. □

2.2.12 Corollary

$l_{a_t} \circ \lambda_H$ and $\lambda_H \circ \zeta_{a_t}$ are fuzzy hyperideals of H generated by a_t.

Proof. By Lemma 2.2.10, $l_{a_t} \circ \lambda_H$ is a fuzzy hyperideal of H. As,

$$\begin{aligned}(l_{a_t} \circ \lambda_H)(x) &= \bigvee_{x \in y \circ z} \{l_{a_t}(y) \wedge \lambda_H(z)\} \\ &= \bigvee_{x \in y \circ z} \{l_{a_t}(y)\} \quad (\because \lambda_H(z) = 1) \\ &= \begin{cases} t \text{ if } x \in Ha \\ 0 \text{ otherwise.} \end{cases}\end{aligned}$$

Also it is clear that $a_t \leq l_{a_t} \circ \lambda_H$. If μ is a fuzzy hyperideal of H containing a_t then $l_{a_t} \leq \mu$ implies $l_{a_t} \circ \lambda_H \leq \mu \circ \lambda_H \leq \mu$. Thus $l_{a_t} \circ \lambda_H$ is a fuzzy hyperideal of H containing a_t. Similarly $\lambda_H \circ \zeta_{a_t}$ is a fuzzy hyperideal of H containing a_t. □

2.2.13 Lemma

If λ is a fuzzy left (right) hyperideal of a semihypergroup H and a_t, b_s are fuzzy points of H such that $a_t \circ \lambda_H \circ b_s \leq \lambda$ then $l_{a_t} \circ l_{b_s} \leq \lambda$ $\left(\zeta_{a_t} \circ \zeta_{b_s} \leq \lambda\right)$.

Proof. Suppose λ is a fuzzy left hyperideal of H and $a_t \circ \lambda_H \circ b_s \leq \lambda$. Then $\lambda_H \circ a_t \circ \lambda_H \circ b_s \leq \lambda_H \circ \lambda \leq \lambda$.

Now

$$\begin{aligned}\lambda_H \circ a_t(x) &= \bigvee_{x \in y \circ z} \{\lambda_H(y) \wedge a_t(z)\} \\ &= \bigvee_{x \in y \circ z} \{a_t(z)\}, \quad (\because \lambda_H(y) = 1) \\ &= \begin{cases} t \text{ if } x \in Ha \\ 0 \text{ otherwise} \end{cases}\end{aligned}$$

Similarly, $$\lambda_H \circ b_s(x) = \begin{cases} s \text{ if } x \in Hb \\ 0 \text{ otherwise} \end{cases}$$

Thus $\lambda_H \circ a_t = l_{a_t}$ and $\lambda_H \circ b_s = l_{b_s}$.
Hence $l_{a_t} \circ l_{b_s} = \lambda_H \circ a_t \circ \lambda_H \circ b_s \leq \lambda$. □

2.2.14 Proposition

Let λ be a fuzzy hyperideal of a semihypergroup H. Then the following statements are equivalent:

(i) λ is a fuzzy prime hyperideal of a semihypergroup H;

(ii) $a_t \circ \lambda_H \circ b_s \leq \lambda$ if and only if $a_t \in \lambda$ or $b_s \in \lambda$, for every fuzzy point a_t, b_s of H;

(iii) If a_t and b_s are fuzzy points of H such that $\nu_{a_t} \circ \nu_{b_s} \leq \lambda$ then either $a_t \in \lambda$ or $b_s \in \lambda$,where ν_{a_t} is the fuzzy hyperideal of H generated by a_t.

Proof. (i)$\Longrightarrow$(ii). Suppose $a_t \circ \lambda_H \circ b_s \leq \lambda$. Then by Lemma 2.2.13, $l_{a_t} \circ l_{b_s} \leq \lambda$. Now

$$\begin{aligned} l_{a_t} \circ \lambda_H \circ l_{b_s} \circ \lambda_H &= l_{a_t} \circ (\lambda_H \circ l_{b_s}) \circ \lambda_H \\ &\leq l_{a_t} \circ l_{b_s} \circ \lambda_H \quad (\because \lambda_H \circ l_{b_s} \leq l_{b_s}) \\ &\leq \lambda \circ \lambda_H \qquad (\because l_{a_t} \circ l_{b_s} \leq \lambda) \\ &\leq \lambda. \end{aligned}$$

But $l_{a_t} \circ \lambda_H$ and $l_{b_s} \circ \lambda_H$ are fuzzy hyperideals of H generated by a_t and b_s respectively. Thus either $l_{a_t} \circ \lambda_H \leq \lambda$ or $l_{b_s} \circ \lambda_H \leq \lambda$. This implies either $a_t \in \lambda$ or $b_s \in \lambda$.

Conversely, suppose that $a_t \in \lambda$ or $b_s \in \lambda$ then $a_t \circ \lambda_H \circ b_s \leq \lambda$.

(ii)$\Longrightarrow$(iii). Let ν_{a_t} and ν_{b_s} be fuzzy hyperideals of H generated by a_t and b_s respectively such that $\nu_{a_t} \circ \nu_{b_s} \leq \lambda$. Now $a_t \leq \nu_{a_t}$ and $b_s \leq \nu_{b_s}$ implies that $a_t \circ \lambda_H \leq \nu_{a_t} \circ \lambda_H \leq \nu_{a_t}$. Thus $a_t \circ \lambda_H \circ b_s \leq \nu_{a_t} \circ \nu_{b_s}$. If $\nu_{a_t} \circ \nu_{b_s} \leq \lambda$, then $a_t \circ \lambda_H \circ b_s \leq \lambda$. So either $a_t \in \lambda$ or $b_s \in \lambda$ (by (ii)).

(iii)$\Longrightarrow$(i). Let μ and ν be fuzzy hyperideals of H such that $\mu \circ \nu \leq \lambda$. Suppose $\mu \nleq \lambda$ implies that there exists $x \in H$ such that $\mu(x) \nleq \lambda(x)$. Let $\mu(x) = t \in (0,1]$. Then $x_t \leq \mu$. Let $y_s \in \nu$ then $\nu_{x_t} \circ \nu_{y_s} \leq \mu \circ \nu \leq \lambda$, so by (iii), either $x_t \in \lambda$ or $y_s \in \lambda$, but $x_t \notin \lambda$ so $y_s \in \lambda$. Hence $\nu \leq \lambda$. □

2.2.15 Proposition

The following statements are equivalent for a fuzzy hyperideal λ of a semihypergroup H;

(i) λ is a semiprime fuzzy hyperideal;

(ii) $a_t \circ \lambda_H \circ a_t \leq \lambda$ if and only if $a_t \in \lambda$.

Proof. (i) $\Longrightarrow$ (ii). Let $a_t \circ \lambda_H \circ a_t \leq \lambda$. Then by Lemma 2.2.13, $l_{a_t} \circ l_{a_t} \leq \lambda$.

Now

$$\begin{aligned} l_{a_t} \circ (\lambda_H \circ l_{a_t}) \circ \lambda_H &\leq l_{a_t} \circ l_{a_t} \circ \lambda_H \\ &\leq \lambda \circ \lambda_H \\ &\leq \lambda. \end{aligned}$$

This implies that $(l_{a_t} \circ \lambda_H) \circ (l_{a_t} \circ \lambda_H) \leq \lambda$. But $l_{a_t} \circ \lambda_H$ is a fuzzy hyperideal of H generated by a_t and λ is a semiprime fuzzy hyperideal, so $l_{a_t} \circ \lambda_H \leq \lambda$ implies

$a_t \in \lambda$. If $a_t \in \lambda$ then the fuzzy hyperideal of H generated by a_t is contained in λ i.e. $l_{a_t} \circ \lambda_H \leq \lambda$. Thus

$$a_t \circ \lambda_H \circ a_t \leq l_{a_t} \circ \lambda_H \leq \lambda.$$

$(ii) \Longrightarrow (i)$. Let μ be a fuzzy hyperideal of H such that $\mu \circ \mu \leq \lambda$. Let $a_t \in \mu$. Then

$$a_t \circ \lambda_H \leq \mu \circ \lambda_H \leq \mu$$

$$a_t \circ \lambda_H \circ a_t \leq \mu \circ a_t \leq \mu \circ \mu \leq \lambda$$

implies $a_t \in \lambda$. Hence $\mu = \bigvee_{a_t \in \mu} a_t \leq \lambda$. Thus λ is a semiprime fuzzy hyperideal of H. □

2.2.16 Theorem

A semihypergroup H is fully fuzzy semiprime if and only if H is fuzzy semisimple.

Proof. Let H be a fuzzy semisimple semihypergroup and λ be a fuzzy hyperideal of H. If for a hyperideal μ of H, $\mu \circ \mu \leq \lambda$, then $\mu \leq \lambda$ (since H is fuzzy semisimple). Hence λ is a fuzzy semiprime hyperideal of H. Thus H is fully fuzzy semiprime.

Conversely, let H be a fully fuzzy semiprime semihypergroup. Let μ be a fuzzy hyperideal of H. Then $\mu \circ \mu$ is also a fuzzy hyperideal of H. As $\mu \circ \mu \leq \mu \circ \mu$ implies $\mu \leq \mu \circ \mu$ (because $\mu \circ \mu$ is a fuzzy semiprime hyperideal of H). But $\mu \circ \mu \leq \mu$ always holds. Hence $\mu \circ \mu = \mu$. Thus, each fuzzy hyperideal of H is idempotent. So H is semisimple. □

2.2.17 Theorem

A semihypergroup H is fully fuzzy prime if and only if H is a fuzzy semisimple and the set of fuzzy hyperideals of H is totally ordered under inclusion.

Proof. Suppose H is fully fuzzy prime, so H is fully fuzzy semiprime. Thus by Theorem 2.2.16 H is fuzzy semisimple. Now, suppose that λ, ν are fuzzy hyperideals of H. Since $\lambda \circ \nu \leq \lambda \wedge \nu$ and $\lambda \wedge \nu$ is a fuzzy hyperideal of H, so is a fuzzy prime hyperideal. Thus either $\lambda \leq \lambda \wedge \nu$ or $\nu \leq \lambda \wedge \nu$, this implies either $\lambda \leq \nu$ or $\nu \leq \lambda$.

Conversely, let H be a fuzzy semisimple semihypergroup and the set of fuzzy hyperideals of H is totally ordered under inclusion. Let μ, ν and λ be fuzzy hyperideals of H such that $\mu \circ \nu \leq \lambda$. Since the set of fuzzy hyperideals of H is totally ordered under inclusion, so either $\mu \leq \nu$ or $\nu \leq \mu$. Assume that $\mu \leq \nu$. Now $\mu = \mu \circ \mu \leq \mu \circ \nu \leq \lambda$. Hence $\mu \leq \lambda$ so λ is a fuzzy prime hyperideal of H. □

2.3 Semisimple Semihypergroups in terms of Hyperideals and Fuzzy Hyperideals

In this section we characterize semisimple semihypergroups in terms of hyperideals and also in terms of fuzzy hyperideals.

Recall that a semihypergroup H is called *semisimple*, if for each $h \in H$ there exist $x, y, z \in H$ such that $h \in x \circ h \circ y \circ h \circ z$.

2.3.1 Theorem

Let $(H, \circ)$ be a semihypergroup with identity. Then the following conditions are equivalent:

(1) *H is semisimple;*
(2) *$A \cap B = A \circ B$, for all hyperideals A and B of H;*
(3) *$A = A \circ A$, for every hyperideal A of H;*
(4) $< a >=< a > \circ < a >$.

Proof. (1) $\Longrightarrow$ (2) Let $a \in A \cap B$. Then $a \in A$ and $a \in B$. Since H is semisimple, there exist $x, y, z \in H$ such that

$$a \in x \circ a \circ y \circ a \circ z = (x \circ a \circ y) \circ (a \circ z) \subseteq A \circ B.$$

Thus $A \cap B \subseteq A \circ B$.

On the other hand $A \circ B \subseteq A$ (because A is a hyperideal of H) and $A \circ B \subseteq B$ (because B is a hyperideal of H), so we have $A \circ B \subseteq A \cap B$. Hence, $A \cap B = A \circ B$.

(2) $\Longrightarrow$ (3) Take $B = A$, then by hypothesis $A \cap A = A \circ A$. This implies that $A = A \circ A$.

(3) $\Longrightarrow$ (4) Obvious.

(4) $\Longrightarrow$ (1) As $a \in < a >=< a > \circ < a >$, so

$$\begin{aligned} a &\in (H \circ a \circ H) \circ (H \circ a \circ H) \\ &= H \circ a \circ (H \circ H) \circ a \circ H \\ &\subseteq H \circ a \circ H \circ a \circ H. \end{aligned}$$

This implies that $a \in x \circ a \circ y \circ a \circ z$ for some $x, y, z \in H$. Hence H is semisimple. □

Now we characterize semihypergroups in terms of prime, semiprime and irreducible hyperideals.

2.3.2 Definition

A hyperideal A of a semihypergroup H is called *irreducible* if for all hyperideals B, C of H, $B \cap C = A$ implies $B = A$ or $C = A$.

2.3.3 Proposition

A hyperideal A of a semihypergroup H is prime if and only if A is semiprime and irreducible.

Proof. Let A be a prime hyperideal of H. Then clearly A is semiprime. Let B and C be any hyperideals of H such that $B \cap C = A$. Since $B \circ C \subseteq B \cap C = A$ and A is a prime hyperideal of H, so $B \subseteq A$ or $C \subseteq A$. On the other hand $A \subseteq B$ and $A \subseteq C$ (since $B \cap C = A$). Hence $B = A$ or $C = A$.

Conversely, let A be an irreducible semiprime hyperideal of H. Let B and C be any hyperideals of H such that $B \circ C \subseteq A$. Since $(B \cap C) \circ (B \cap C) \subseteq B \circ C \subseteq A$ and A is semiprime, so $B \cap C \subseteq A$. But $A \cup (B \cap C) = (A \cup B) \cap (A \cup C) = A$ and A is irreducible, so we have, $A \cup B = A$ or $A \cup C = A$. Hence $B \subseteq A$ or $C \subseteq A$. Thus A is prime. □

2.3.4 Proposition

Let I be a hyperideal of H and $a \in H$ such that $a \notin I$. Then there exists an irreducible hyperideal A of H such that $I \subseteq A$ and $a \notin A$.

Proof. Let Ω be the collection of all hyperideals of H which contain I but do not contain "a". Then Ω is non-empty, because $I \in \Omega$. The collection Ω is partially ordered under inclusion. As every totally ordered subset of Ω is bounded above, so by Zorn's Lemma, there exists a maximal element, say, A in Ω. We show that A is an irreducible hyperideal of H. Let C and D be two hyperideals of H such that $C \cap D = A$. If both C and D properly contain A then $a \in C$ and $a \in D$. Now $a \in C \cap D = A$, which is a contradiction Hence $C = A$ or $D = A$, that is A is irreducible. □

In the next theorem we characterize those semihypergroups in which each hyperideal is semiprime.

2.3.5 Theorem

Let H be a semihypergroup. Then the following conditions are equivalent:

(1) H is semisimple,

(2) $A \cap B = A \circ B$, for all hyperideals A and B of H,

(3) $A = A \circ A$, for all hyperideal A of H,

(4) Each hyperideal of H is semiprime,

(5) Each hyperideal of H is the intersection of prime hyperideals of H which contain it.

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) Follow from the Theorem 2.3.1.

(3) $\Longrightarrow$ (4) Let A and I be hyperideals of H such that $A \circ A \subseteq I$. By hypothesis $A \circ A = A$, so $A \subseteq I$. Hence each hyperideal of H is semiprime.

(4) $\Longrightarrow$ (5) Let A be a hyperideal of H. Then obviously A is contained in the intersection of all irreducible hyperideals of H which contain A. If $a \notin A$, then by Proposition 2.3.4, there exists an irreducible hyperideal of H which contains A but does not contain a. Hence A is the intersection of all irreducible hyperideals of H which contain it. By hypothesis, each hyperideal of H is semiprime, so each hyperideal of H is the intersection of all irreducible semiprime hyperideals of H which contain it. By Proposition 2.3.3, each irreducible semiprime hyperideal of H is prime. Hence each hyperideal of H is the intersection of prime hyperideals of H which contain it.

(5) $\Longrightarrow$ (3) Let A be a proper hyperideal of H. Then $A \circ A$ is a hyperideal of H. By hypothesis,

$$A \circ A = \bigcap_{\alpha}\{A_\alpha : A_\alpha \text{ are prime hyperideal of } H \text{ containing } A \circ A\}.$$

This implies that $A \circ A \subseteq A_\alpha$ for each α. Since each A_α is prime, so $A \subseteq A_\alpha$ for each α and hence $A \subseteq \bigcap_{\alpha} A_\alpha = A \circ A$. But $A \circ A \subseteq A$ always. Hence $A \circ A \subseteq A$. $\square$

Next proposition shows that in a semisimple semihypergroup the concepts of prime hyperideal and irreducible hyperideal coincide.

2.3.6 Proposition

Let T be a hyperideal of a semisimple semihypergroup H. Then the following statements are equivalent:

(1) *T is a prime hyperideal of H;*

(2) *T is an irreducible hyperideal of H.*

Proof. (1) $\Longrightarrow$ (2) Suppose T is a prime hyperideal of H. Let A, B be any hyperideals of H such that $A \cap B = T$. Then $T \subseteq A$ and $T \subseteq B$. Since $A \cap B \supseteq A \circ B$, so $A \circ B \subseteq T$. Since T is prime, so $A \subseteq T$ or $B \subseteq T$. Thus $A = T$ or $B = T$.

(2) $\Longrightarrow$ (1) Suppose T is an irreducible hyperideal of H. Let A, B be any hyperideals of H such that $A \circ B \subseteq T$. Since H is semisimple, so we have $A \cap B = A \circ B \subseteq T$. Then $(A \cap B) \cup T = T$. But $(A \cap B) \cup T = (A \cup T) \cap (B \cup T)$. Hence $(A \cup T) \cap (B \cup T) = T$. Since T is irreducible, so $A \cup T = T$ or $B \cup T = T$. Thus we have $A \subseteq T$ or $B \subseteq T$. $\square$

2.3.7 Theorem

The following conditions are equivalent for a semihypergroup H:

(1) *Each hyperideal of H is prime,*

(2) *H is semisimple and the set of hyperideals of H is a chain.*

Proof. (1) $\Longrightarrow$ (2) Suppose each hyperideal of H is prime. Then by Theorem 2.3.5, H is semisimple. Let A, B be hyperideals of H, then $A \circ B \subseteq A \cap B$. By hypothesis each hyperideal of H is prime, so $A \cap B$ is prime. Thus $A \subseteq A \cap B$ or $B \subseteq A \cap B$, that is $A \subseteq B$ or $B \subseteq A$.

(2) $\Longrightarrow$ (1) Suppose H is semisimple and the set of hyperideals of H is a chain. Let A, B, C be hyperideals of H such that $A \circ B \subseteq C$. Since H is semisimple, so $A \circ B = A \cap B$. Since set of hyperideals is a chain, so either $A \subseteq B$ or $B \subseteq A$, that is either $A \cap B = A$ or $A \cap B = B$. Thus either $A \subseteq C$ or $B \subseteq C$. $\square$

2.3.8 Example

Let $(S, \cdot)$ be any semigroup. Define a hyperoperation $\circ$ on S by:

$$a \circ b = \{a, b, ab\} \text{ for all } a, b \in S.$$

Then for all $a, b, c \in S$, we have

$$\begin{aligned} a \circ (b \circ c) &= a \circ \{b, c, bc\} = (a \circ b) \cup (a \circ c) \cup (a \circ (bc)) \\ &= \{a, b, ab\} \cup \{a, c, ac\} \cup \{a, bc, a(bc)\} \\ &= \{a, b, c, ab, ac, bc, a(bc)\} \\ (a \circ b) \circ c &= \{a, b, ab\} \circ c = (a \circ c) \cup (b \circ c) \cup ((ab) \circ c) \\ &= \{a, c, ac\} \cup \{b, c, bc\} \cup \{ab, c, (ab)c\} \\ &= \{a, b, c, ab, ac, bc, (ab)c\} \end{aligned}$$

Thus $a \circ (b \circ c) = (a \circ b) \circ c$. This implies $(S, \circ)$ is a semihypergroup.

The only hyperideal of such a semihypergroup S is S itself. Since $S \circ S = S$, so $(S, \circ)$ is semisimple.

2.3.9 Example

Let $(S, \leq, \cdot)$ be any ordered semigroup. Define a hyperoperation $\circ$ on S by:

$$a \circ b = \{x \in S : x \leq ab\} = (ab] \text{ for all } a, b \in S.$$

Then for all $a, b, c \in S$, we claim that $a \circ (b \circ c) = (a(bc)]$. Let $t \in a \circ (b \circ c)$, then $t \in a \circ x$ for some $x \in (b \circ c)$. This implies $t \leq ax$ and $x \leq bc$. Hence $t \leq a(bc)$, so $a \circ (b \circ c) \subseteq (a(bc)]$. Let $s \in (a(bc)]$, then $s \leq a(bc)$. So $s \in a \circ (bc) \subseteq \bigcup_{x \in b \circ c} a \circ x = a \circ (b \circ c)$. Thus $(a(bc)] \subseteq a \circ (b \circ c)$. Consequently $(a(bc)] = a \circ (b \circ c)$. Similarly, we can show that $((ab)c] = (a \circ b) \circ c$. Hence $(a \circ b) \circ c = a \circ (b \circ c)$. Thus $(S, \circ)$ is a semihypergroup.

Consider the ordered semigroup $S = \{a, b, c, d, e\}$ with the following multiplication table and order relation

$\cdot$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

$\leq = \{(a, a), (a, c), (a, d), (a, e), (b, b), (b, d), (b, e), (c, c), (c, e), (d, d), (d, e), (e, e)\}$

Then the hyperoperation $\circ$ is defined in the following table

$\circ$	a	b	c	d	e
a	a	$\{a,b,d\}$	a	$\{a,b,d\}$	$\{a,b,d\}$
b	a	b	a	$\{a,b,d\}$	$\{a,b,d\}$
c	a	$\{a,b,d\}$	$\{a,c\}$	$\{a,b,d\}$	$\{a,b,c,d,e\}$
d	a	$\{a,b,d\}$	a	$\{a,b,d\}$	$\{a,b,d\}$
e	a	$\{a,b,d\}$	$\{a,c\}$	$\{a,b,d\}$	$\{a,b,c,d,e\}$

Then $(S,\circ)$ is a semihypergroup and the only hyperideals of S are $\{a,b,d\}$ and S. Both the hyperideals are idempotent, so $(S,\circ)$ is a semisimple semihypergroup. Also both the hyperideals are prime.

2.3.10 Example

Consider the ordered semigroup $S = \{a,b,c,d\}$ with the following multiplication table and order relation

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq = \{(a,a),(b,b),(c,c),(d,d),(a,b)\}$$

Then the hyperoperation $\circ$ on S is defined by the following table

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	$\{a,b\}$	a
d	a	a	$\{a,b\}$	$\{a,b\}$

Then $(S,\circ)$ is a semihypergroup and the hyperideals of $(S,\circ)$ are $\{a\},\{a,b\},\{a,b,c\},\{a,b,d\},S$. No hyperideal is prime or semiprime. But the proper hyperideals $\{a,b,c\},\{a,b,d\}$ are irreducible.

2.3.11 Example

Consider the ordered semigroup $S = \{a,b,c,d\}$ with the following multiplication table and order relation

·	a	b	c	d
a	a	a	a	a
b	a	b	c	a
c	a	a	a	a
d	a	d	a	a

$$\leq = \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(a,d)\}$$

Then the hyperoperation $\circ$ on S is defined by the following table

$\circ$	a	b	c	d
a	a	a	a	a
b	a	$\{a,b\}$	$\{a,c\}$	a
c	c	a	a	a
d	a	$\{a,d\}$	a	a

Then $(S,\circ)$ is a semihypergroup and the hyperideals of $(S,\circ)$ are $\{a\},\{a,c\},\{a,d\},\{a,c,d\},S$. The hyperideal $\{a,c,d\}$ is prime and all other hyperideals are neither prime nor semiprime.

Next, we characterize those semihypergroups for which each fuzzy hyperideal is semiprime.

2.3.12 Theorem

A semihypergroup H is semisimple if and only if for all fuzzy hyperideals λ and μ of H, $\lambda\circ\mu=\lambda\wedge\mu$.

Proof. Let λ and μ be fuzzy hyperideals of the semisimple semihypergroup H and $a\in H$. Then there exist $x,y,z\in H$ such that $a\in x\circ a\circ y\circ a\circ z=(x\circ a\circ y)\circ(a\circ z)$. So for each $\alpha\in x\circ a\circ y$ and $\beta\in a\circ z$, $(\alpha,\beta)\in X_a$, i.e. $a\in\alpha\circ\beta\subseteq(x\circ a\circ y)\circ(a\circ z)$, we have $X_a\neq\emptyset$. Hence

$$\begin{aligned}(\lambda\circ\mu)(a) &= \bigvee_{(\alpha,\beta)\in X_a}\min\{\lambda(\alpha),\mu(\beta)\}\\ &\geq \min\{\lambda(\alpha),\mu(\beta)\}.\end{aligned}$$

As λ and μ are fuzzy hyperideals of H, we have $\lambda(\alpha)\geq\lambda(\alpha_1)\geq\lambda(a)$, for each $\alpha\in\alpha_1\circ y$ and $\alpha_1\in x\circ a$ and $\mu(\beta)\geq\mu(a)$, for every $\beta\in a\circ z$.

Hence $\min\{\lambda(\alpha),\mu(\beta)\}\geq\min\{\lambda(a),\mu(a)\}$.

Thus $(\lambda\circ\mu)(a)\geq(\lambda\wedge\mu)(a)$.

On the other hand, by Lemma 1.3.20, we have $(\lambda\circ\mu)(a)\leq(\lambda\wedge\mu)(a)$.

Hence, $(\lambda\circ\mu)(a)=(\lambda\wedge\mu)(a)$.

Conversely, assume that $\lambda\circ\mu=\lambda\wedge\mu$. Let A,B be hyperideals of H. Then by Proposition 1.3.11, λ_A and λ_B are fuzzy hyperideals of H. Hence by hypothesis, $\lambda_A\circ\lambda_B=\lambda_A\wedge\lambda_B$. By Proposition 1.3.7, $\lambda_A\circ\lambda_B=\lambda_{A\circ B}$, thus $\lambda_{A\circ B}=\lambda_A\wedge\lambda_B=\lambda_{A\cap B}$ which implies $A\cap B=A\circ B$. Thus by Theorem 2.3.1, H is semisimple. □

2.3.13 Corollary

A semihypergroup H is semisimple if and only if for each fuzzy hyperideal λ of H, we have $\lambda \circ \lambda = \lambda$.

Proof. Let λ be a fuzzy hyperideal of a semisimple semihypergroup H. Then by Theorem 2.3.12, $\lambda \circ \lambda = \lambda \wedge \lambda = \lambda$.

Conversely, assume that $\lambda \circ \lambda = \lambda$ for each fuzzy hyperideal λ of H. Let A be a hyperideal of H. Then by Proposition 1.3.11, λ_A is a fuzzy hyperideal of H. By hypothesis, $\lambda_A \circ \lambda_A = \lambda_A$. By Proposition 1.3.7, $\lambda_A \circ \lambda_A = \lambda_{A \circ A} = \lambda_A$. Hence $A = A \circ A$. Thus by Theorem 2.3.1, H is semisimple. □

2.3.14 Definition

A fuzzy hyperideal λ of a semihypergroup H is called an *irreducible fuzzy hyperideal* if for each fuzzy hyperideals μ and ν of H, $\mu \wedge \nu = \lambda$ implies $\mu = \lambda$ or $\nu = \lambda$.

2.3.15 Proposition

A non-empty subset I of a semihypergroup H is an irreducible hyperideal of H if and only if the characteristic function λ_I of I is an irreducible fuzzy hyperideal of H.

Proof. Suppose I is an irreducible hyperideal of H. Then λ_I, the characteristic function of I is a fuzzy hyperideal of H. Let μ and ν be any fuzzy hyperideals of H such that $\mu \wedge \nu = \lambda_I$ with $\mu \neq \lambda_I$ and $\nu \neq \lambda_I$. Then there exist $x, y \in H$ such that $\mu(x) \neq 0$ and $\nu(y) \neq 0$ but $\lambda_I(x) = 0$ and $\lambda_I(y) = 0$. Hence $x \notin I$ and $y \notin I$. Since I is an irreducible hyperideal of H, we have $< x > \cap < y > \neq I$. Thus there exists $a \in < x > \cap < y >$ such that $a \notin I$. Hence $\lambda_I(a) = 0$. Thus $(\mu \wedge \nu)(a) = 0$. Since $\mu(x) \neq 0$ and $\nu(y) \neq 0$, we have $\min\{\mu(x), \nu(y)\} \neq 0$.

As $a \in < x > = x \cup H \circ x \cup x \circ H \cup H \circ x \circ H$, therefore $a = x$ or there exist $h, h_1 \in H$ such that $a \in h \circ x$ or $a \in x \circ h$ or $a \in h \circ x \circ h_1$.

If $a = x$ then $\mu(a) = \mu(x)$.

If $a \in h \circ x$ then $\mu(a) \geq \mu(x)$ because μ is a fuzzy left hyperideal of H.

If $a \in x \circ h$ then $\mu(a) \geq \mu(x)$ because μ is a fuzzy right hyperideal of H.

If $a \in h \circ x \circ h_1 = \bigcup_{g \in h \circ x} g \circ h_1$ then $a \in g_1 \circ h_1$ for some $g_1 \in h \circ x$ and since μ is a fuzzy (two-sided) hyperideal of H,

$$\mu(a) \geq \mu(g_1) \geq \mu(x).$$

Also $a \in < y > = y \cup H \circ y \cup y \circ H \cup H \circ y \circ H$. Then $a = y$ or there exist $h, h_1 \in H$, such that $a \in h \circ y$ or $a \in y \circ h$ or $a \in h \circ y \circ h_1$.

If $a = y$ then $\nu(a) = \nu(y)$.

If $a \in h \circ y$ then $\nu(a) \geq \nu(y)$ because ν is a fuzzy left hyperideal of H.

If $a \in y \circ h$ then $\nu(a) \geq \nu(y)$ because ν is a fuzzy right hyperideal of H.

If $a \in h \circ y \circ h_1 = \bigcup_{i \in h \circ y} i \circ h_1$ then for each $\beta \in i \circ h_1$,

$$\nu(a) \geq \nu(\beta) \geq \nu(i) \geq \nu(y).$$

Hence, $\min\{\mu(a), \nu(a)\} \geq \min\{\mu(x), \nu(y)\} \neq 0$ i.e. $(\mu \wedge \nu)(a) \neq 0$ which is a contradiction. Thus $\mu \wedge \nu = \lambda_I$ implies that $\mu = \lambda_I$ or $\nu = \lambda_I$.

Conversely, assume that λ_I is an irreducible fuzzy hyperideal of H. Let A, B be any hyperideals of H such that $A \cap B = I$. Then λ_A and λ_B are fuzzy hyperideals of H and $\lambda_{A \cap B} = \lambda_I$. Since $\lambda_{A \cap B} = \lambda_A \wedge \lambda_B$, so $\lambda_A \wedge \lambda_B = \lambda_I$. Thus by hypothesis, $\lambda_A = \lambda_I$ or $\lambda_B = \lambda_I$. Hence $A = I$ or $B = I$. Thus I is an irreducible hyperideal of H.

□

2.3.16 Proposition

A fuzzy hyperideal λ of a semihypergroup H is prime fuzzy hyperideal if and only if λ is semiprime and irreducible fuzzy hyperideal.

Proof. Let λ be a prime fuzzy hyperideal of H. Then λ is semiprime. Let μ and ν be any fuzzy hyperideals of H such that $\mu \wedge \nu = \lambda$. Since $\mu \circ \nu \leq \mu \wedge \nu = \lambda$ and λ is a prime fuzzy hyperideal of H, we have $\mu \leq \lambda$ or $\nu \leq \lambda$. On the other hand $\mu \wedge \nu = \lambda$ implies that $\lambda \leq \mu$ and $\lambda \leq \nu$. Hence $\mu = \lambda$ or $\nu = \lambda$.

Conversely, let λ be an irreducible semiprime fuzzy hyperideal of H. Let μ and ν be any fuzzy hyperideals of H such that $\mu \circ \nu \leq \lambda$. Since $(\mu \wedge \nu) \circ (\mu \wedge \nu) \leq \mu \circ \nu \leq \lambda$ and λ is semiprime, so we have $\mu \wedge \nu \leq \lambda$. By Lemma 1.3.2, we have $\lambda \vee (\mu \wedge \nu) = (\lambda \vee \mu) \wedge (\lambda \vee \nu)$. But $\lambda \vee (\mu \wedge \nu) = \lambda$. Thus $(\lambda \vee \mu) \wedge (\lambda \vee \nu) = \lambda$. Since λ is irreducible, so we have $\lambda \vee \mu = \lambda$ or $\lambda \vee \nu = \lambda$. Hence $\mu \leq \lambda$ or $\nu \leq \lambda$. Thus λ is a prime fuzzy hyperideal of H.

□

2.3.17 Proposition

Let λ be a fuzzy hyperideal of a semihypergroup H with $\lambda(a) = t$, where $a \in H$ and $t \in (0, 1]$. Then there exists an irreducible fuzzy hyperideal μ of H such that $\lambda \leq \mu$ and $\mu(a) = t$.

Proof. Let $X = \{\nu : \nu$ is a fuzzy hyperideal of H, $\nu(a) = t$ and $\lambda \leq \nu\}$. Then $X \neq \emptyset$, because $\lambda \in X$. The collection X is partially ordered under inclusion. Suppose Y is a totally ordered subset of X, say $Y = \{\nu_i : i \in I\}$. Then by Proposition 1.3.18, $(\bigvee_{i \in I} \nu_i)$ is a fuzzy hyperideal of H. As $\lambda \leq \nu_i$ for each $i \in I$, so $\lambda \leq \bigvee_{i \in I} \nu_i$. Also $\left(\bigvee_{i \in I} \nu_i\right)(a) = \bigvee_{i \in I} (\nu_i(a)) = t$. Thus $\bigvee_{i \in I} \nu_i$ is the least upper bound of Y. So by Zorn's Lemma, there exists a fuzzy hyperideal μ of H which is maximal with respect to the property that $\lambda \leq \mu$ and $\mu(a) = t$.

Now we show that μ is an irreducible fuzzy hyperideal of H. Suppose $\mu = \mu_1 \wedge \mu_2$, where μ_1 and μ_2 are fuzzy hyperideals of H. Then $\mu \leq \mu_1$ and $\mu \leq \mu_2$. We claim that $\mu = \mu_1$ or $\mu = \mu_2$. Suppose, on contrary that $\mu \neq \mu_1$ and $\mu \neq \mu_2$. Since μ is maximal with respect to the property that $\mu(a) = t$ and $\mu \neq \mu_1$ and $\mu \neq \mu_2$, it follows that $\mu_1(a) \neq t$ and $\mu_2(a) \neq t$. Hence $t = \mu(a) = (\mu_1 \wedge \mu_2)(a) \neq t$, which is a contradiction. Hence either $\mu = \mu_1$ or $\mu = \mu_2$. Thus μ is an irreducible fuzzy hyperideal of H. □

2.3.18 Theorem

Let H be a semihypergroup. Then the following conditions are equivalent:

(1) *H is semisimple,*

(2) *$\lambda \circ \mu = \lambda \wedge \mu$, for every fuzzy hyperideals λ and μ of H,*

(3) *$\lambda \circ \lambda = \lambda$, for all fuzzy hyperideal λ of H,*

(4) *Each fuzzy hyperideal of H is semiprime,*

(5) *Each fuzzy hyperideal of H is the intersection of prime fuzzy hyperideals of H which contain it.*

Proof. (1) $\Longleftrightarrow$ (2) Follows from the Theorem 2.3.12.

(1) $\Longleftrightarrow$ (3) Follows from the Corollary 2.3.13.

(3) $\Longrightarrow$ (4) Let λ and μ be fuzzy hyperideals of H such that $\lambda \circ \lambda \leq \mu$. By hypothesis $\lambda \circ \lambda = \lambda$. So $\lambda \leq \mu$. Hence each fuzzy hyperideal of H is semiprime.

(4) $\Longrightarrow$ (5) Let λ be a proper fuzzy hyperideal of H and $\{\lambda_i : i \in I\}$ the collection of all irreducible fuzzy hyperideals of H which contain λ. Proposition 2.3.17, guarantees the existence of such fuzzy hyperideals. Hence $\lambda \leq \bigwedge_{i\in I} \lambda_i$. Let $x \in H$. Then by Proposition 2.3.17, there exists an irreducible fuzzy hyperideal λ_α of H such that $\lambda \leq \lambda_\alpha$ and $\lambda(x) = \lambda_\alpha(x)$. Thus $\lambda_\alpha \in \{\lambda_i : i \in I\}$. Hence $\bigwedge_{i\in I} \lambda_i \leq \lambda_\alpha$. So $\bigwedge_{i\in I} \lambda_i(x) \leq \lambda_\alpha(x) = \lambda(x)$. Thus $\bigwedge_{i\in I} \lambda_i \leq \lambda_\alpha$. Consequently, $\bigwedge_{i\in I} \lambda_i = \lambda$. By hypothesis, each fuzzy hyperideal of H is semiprime, so each fuzzy hyperideal of H is the intersection of all irreducible fuzzy semiprime hyperideals of H which contain it.. By Proposition 2.3.16, each fuzzy irreducible semiprime hyperideal is prime, therefore each fuzzy hyperideal is the intersection of all prime fuzzy hyperideals of H which contain it.

(5) $\Longrightarrow$ (3) Let λ be a proper fuzzy hyperideal of H. Then $\lambda \circ \lambda$ is also a fuzzy hyperideal of H. Since λ is fuzzy hyperideal of H, so $\lambda \circ \lambda \leq \lambda$. By hypothesis, we have $\lambda \circ \lambda = \bigwedge_{i\in I} \lambda_i$ where λ_i are prime fuzzy hyperideals of H. Thus $\lambda \circ \lambda \leq \lambda_i$ for all $i \in I$. Hence $\lambda \leq \lambda_i$ for all $i \in I$. Thus $\lambda \leq \bigwedge_{i\in I} \lambda_i = \lambda \circ \lambda$. Hence $\lambda \circ \lambda = \lambda$. □

2.3.19 Proposition

Let H be a semisimple semihypergroup and λ be a fuzzy hyperideal of H. Then the following are equivalent:

(1) *λ is a prime fuzzy hyperideal of H,*

(2) *λ is an irreducible fuzzy hyperideal of H.*

Proof. (1) $\Longrightarrow$ (2) Suppose λ is a prime fuzzy hyperideal of H. Let μ, ν be fuzzy hyperideals of H such that $\mu \wedge \nu = \lambda$. Since $\mu \circ \nu \leq \mu \wedge \nu \leq \lambda$, so $\mu \leq \lambda$ or $\nu \leq \lambda$. On the other hand, $\mu \wedge \nu = \lambda$ implies that $\lambda \leq \mu$ and $\lambda \leq \nu$. Thus $\mu = \lambda$ or $\nu = \lambda$.

(2) $\Longrightarrow$ (1) Suppose λ is an irreducible fuzzy hyperideal of H. Let μ, ν be fuzzy hyperideals of H such that $\mu \circ \nu \leq \lambda$. Since H is semisimple, so by Theorem 2.3.12, $\mu \circ \nu = \mu \wedge \nu$. Hence $\mu \wedge \nu \leq \lambda$. Thus $(\mu \wedge \nu) \vee \lambda = \lambda$. But, $(\mu \wedge \nu) \vee \lambda = (\mu \vee \lambda) \wedge (\nu \vee \lambda)$, so $(\mu \vee \lambda) \wedge (\nu \vee \lambda) = \lambda$. Since λ is an irreducible fuzzy hyperideal of H, so we have $\mu \vee \lambda = \lambda$ or $\nu \vee \lambda = \lambda$. Hence $\mu \leq \lambda$ or $\nu \leq \lambda$. □

2.3.20 Theorem

Let H be a semihypergroup. Then every fuzzy hyperideal of H is prime if and only if the set of fuzzy hyperideals of H form a chain and H is semisimple.

Proof. Suppose that every fuzzy hyperideal of H is prime. Then by Theorem 2.3.12, H is semisimple.

Let λ, μ be any fuzzy hyperideals of H, then $\lambda \wedge \mu$ is a fuzzy hyperideal of H. By hypothesis, $\lambda \wedge \mu$ is a prime fuzzy hyperideal of H. Since $\lambda \circ \mu \leq \lambda \wedge \mu$, so $\lambda \leq \lambda \wedge \mu$ or $\mu \leq \lambda \wedge \mu$. Hence $\lambda \leq \mu$ or $\mu \leq \lambda$.

Conversely, assume that H is semisimple semihypergroup and the set of fuzzy hyperideals of H form a chain. Let λ, μ and ν be any fuzzy hyperideals of H such that $\mu \circ \nu \leq \lambda$. Since the set of fuzzy hyperideals of H form a chain, so $\mu \leq \nu$ or $\nu \leq \mu$. If $\mu \leq \nu$ then $\mu = \mu \circ \mu \leq \mu \circ \nu \leq \lambda$. If $\nu \leq \mu$ then $\nu = \nu \circ \nu \leq \mu \circ \nu \leq \lambda$. Thus every fuzzy hyperideal of H is prime. □

2.3.21 Example

Consider the semihypergroup given in Example 2.3.8. By Proposition 1.3.17, the only fuzzy hyperideals of S are the constant functions. Since S is semisimple so every fuzzy hyperideal is idempotent. Since the set of fuzzy hyperideals of S is a chain, so every fuzzy hyperideal is prime.

2.3.22 Example

Consider the semihypergroup S given in Example 2.3.9. By Proposition 1.3.17, the only fuzzy hyperideals of S are of the form $\lambda(a) = \lambda(b) = \lambda(d) \geq \lambda(c) = \lambda(e)$. Since S is semisimple so every fuzzy hyperideal is semiprime and also idempotent, that is $\lambda \circ \lambda = \lambda$.

Now consider the fuzzy hyperideals

$$\lambda(a) = \lambda(b) = \lambda(d) = 0.6 \text{ and } \lambda(c) = \lambda(e) = 0.2$$
$$\mu(a) = \mu(b) = \mu(d) = 0.5 \text{ and } \mu(c) = \mu(e) = 0.3$$
$$\nu(a) = \nu(b) = \nu(d) = 0.55 \text{ and } \nu(c) = \nu(e) = 0.25$$

then $\lambda \circ \mu = \lambda \wedge \mu$ because S is semisimple. $(\lambda \wedge \mu)(a) = (\lambda \wedge \mu)(b) = (\lambda \wedge \mu)(d) = 0.5$ and $(\lambda \wedge \mu)(c) = (\lambda \wedge \mu)(e) = 0.2$. Thus $\lambda \circ \mu = \lambda \wedge \mu \leq \nu$ but neither $\lambda \leq \nu$ nor $\mu \leq \nu$. Hence ν is not prime fuzzy hyperideal of S.

Chapter 3

Regular and intra-regular semihypergroups

This chapter consists of three sections. In section 1, we define quasi-hyperideal, bi-hyperideal, fuzzy quasi-hyperideal and fuzzy bi-hyperideal of a semihypergroup. We also prove some results using these notions. In section 2, we define intra-regular semihypergroup and characterize regular and intra-regular semihypergroup in terms of their bi-hyperideals and fuzzy bi-hyperideals. We prove that a semihypergroup H is both regular and intra-regular if and only if for every bi-hyperideal B of H, $B \circ B = B$. Equivalently, we prove that a semihypergroup H is both regular and intra-regular if and only if for all bi-hyperideals B_1 and B_2 of H we have, $B_1 \cap B_2 = B_1 \circ B_2 \cap B_2 \circ B_1$. We extended this property of bi-hyperideals of semihypergroups and prove that a semihypergroup H is both regular and intra-regular if and only if for every fuzzy bi-hyperideal λ of H we have, $\lambda \circ \lambda = \lambda$. Equivalently, we prove that a semihypergroup H is both regular and intra-regular if and only if for every fuzzy bi-hyperideals λ and μ of H we have, $\lambda \wedge \mu = \lambda \circ \mu \wedge \mu \circ \lambda$. In this section, we have shown that H is regular if and only if for every fuzzy bi-hyperideal λ of H, $\lambda \circ 1 \circ \lambda = \lambda$. In section 3, we define prime bi-hyperideal, strongly prime bi-hyperideal, semiprime bi-hyperideal, irreducible bi-hyperideal and strongly irreducible bi-hyperideal of a semihypergroup. We also define, prime fuzzy bi-hyperideal, strongly prime fuzzy bi-hyperideal, semi-prime fuzzy bi-hyperideal, irreducible fuzzy bi-hyperideal and strongly irreducible fuzzy bi-hyperideal of a semihypergroup. We characterize semihypergroups using these notions.

3.1 Quasi-hyperideals and Bi-hyperideals

In this section, we define quasi-hyperideals, bi-hyperideals, fuzzy quasi hyperideals and fuzzy bi-hyperideals of a semihypergroup H. We characterize regular and intra-regular semihypergroups in terms of bi-hyperideals and fuzzy bi-hyperideals.

3.1.1 Definition

A non-empty subset Q of a semihypergroup H is called a quasi-hyperideal of H if $Q \circ H \cap H \circ Q \subseteq Q$.

3.1.2 Definition

A subsemihypergroup B of a semihypergroup H is called a bi-hyperideal of H if $B \circ H \circ B \subseteq B$.

Throughout this section we denote hyperideal (resp. bi-hyperideal, left hyperideal and quasi-hyperideal) generated by a by $I(a)$ (resp. $B(a)$, $L(a)$ and $Q(a)$). We have $I(a) = \{a\} \cup a \circ H \cup H \circ a \cup H \circ a \circ H, L(a) = \{a\} \cup H \circ a, B(a) = \{a\} \cup a \circ a \cup a \circ H \circ a$ and $Q(a) = \{a\} \cup (H \circ a \cap a \circ H)$.

If a semihypergroup H contains identity element, then $I(a) = H \circ a \circ H, L(a) = H \circ a, B(a) = a \circ H \circ a$ and $Q(a) = H \circ a \cap a \circ H$.

Every one sided hyperideal of a semihypergroup is a quasi-hyperideal and every quasi-hyperideal is a bi-hyperideal. But the converse is not true.

3.1.3 Example

Every left (right) hyperideal of a semihypergroup is a quasi-hyperideal but the converse is not true which is shown in this example.

Consider the semihypergroup $H = \{a, b, c, d\}$ with the hyperoperation given by the table

$\circ$	a	b	c	d
a	a	a	a	a
b	a	$\{a, b\}$	$\{a, c\}$	a
c	a	a	$\{a, b\}$	a
d	a	$\{a, d\}$	a	a

$Q = \{a, b\}$ is a quasi-hyperideal of H but neither left nor right hyperideal of H.

3.1.4 Definition

A fuzzy subset λ of a semihypergroup H is called a fuzzy subsemihypergroup of H if for every $\alpha \in x \circ y$,

$$\inf_{\alpha \in x \circ y} \{\lambda(\alpha)\} \geq \min\{\lambda(x), \lambda(y)\} \text{ for all } x, y \in H.$$

3.1.5 Definition

A fuzzy subset λ of a semihypergroup H is called a fuzzy quasi-hyperideal of H if

$$(\lambda \circ 1) \wedge (1 \circ \lambda) \leq \lambda.$$

3.1.6 Definition

Let H be a semihypergroup. A fuzzy subsemihypergroup λ of H is called a fuzzy bi-hyperideal of H if

$$\inf_{\alpha \in x \circ y \circ z} \lambda(\alpha) \geq \min\{\lambda(x), \lambda(z)\} \text{ for all } x, y, z \in H.$$

3.1.7 Lemma

Let A be a non-empty subset of a semihypergroup H. Then A is a subsemihypergroup of H if and only if the characteristic function λ_A of A is a fuzzy subsemihypergroup of H.

Proof. Straightforward. □

3.1.8 Proposition

A non-empty subset B of a semihypergroup H is a bi-hyperideal of H if and only if the characteristic function λ_B of B is a fuzzy bi-hyperideal of H.

Proof. Suppose that B is a bi-hyperideal of H. It follows from Lemma 3.1.7 that λ_B is a fuzzy subsemihypergroup of H. Let x, y and z be any elements of H. If $x, y \in B$, then since $\lambda_B(x) = \lambda_B(y) = 1$ and since for every $\alpha \in x \circ y \circ z \subseteq B \circ H \circ B \subseteq B$, we have

$$\lambda_B(\alpha) = 1 = \min\{\lambda_B(x), \lambda_B(z)\}.$$

Thus

$$\inf_{\alpha \in x \circ y \circ z} \lambda_B(\alpha) = 1 = \min\{\lambda_B(x), \lambda_B(z)\}.$$

Thus

$$\inf_{\alpha \in x \circ y \circ z} \lambda_B(\alpha) = 1 = \min\{\lambda_B(x), \lambda_B(z)\}.$$

If $x \notin B$ or $z \notin B$, then $\lambda_B(x) = 0$ or $\lambda_B(z) = 0$ and so, we have

$$\lambda_B(\alpha) \geq 0 = \min\{\lambda_B(x), \lambda_B(z)\}.$$

Thus

$$\inf_{\alpha \in x \circ y \circ z} \lambda_B(\alpha) \geq \min\{\lambda_B(x), \lambda_B(z)\}.$$

Hence, λ_B is a fuzzy bi-hyperideal of H.

Conversely, assume that λ_B is a fuzzy bi-hyperideal of H. Then it follows from Lemma 3.1.7 that B is a subsemihypergroup of H. Let $\alpha \in B \circ H \circ B$, then there exist $x, z \in B$ and $y \in H$ such that $\alpha \in x \circ y \circ z$. Since

$$\begin{aligned} \inf_{\alpha \in x \circ y \circ z} \lambda_B(\alpha) &\geq \min\{\lambda_B(x), \lambda_B(z)\} \\ &= \min\{1, 1\} \\ &= 1 \end{aligned}$$

Hence for each $\alpha \in x \circ y \circ z$, we have $\lambda_B(\alpha) = 1$, and so $\alpha \in B$. Thus $B \circ H \circ B \subseteq B$. Therefore, B is a bi-hyperideal of H. □

3.1.9 Proposition

A non-empty subset Q of a semihypergroup H is a quasi-hyperideal of H if and only if the characteristic function λ_Q of Q is a fuzzy quasi-hyperideal of H.

Proof. Suppose Q is a quasi-hyperideal of a semihypergroup H and λ_Q be the charateristic function of Q. Let a be any element of H. If $a \in Q$, then

$$((\lambda_Q \circ 1) \wedge (1 \circ \lambda_Q))(a) \leq 1 = \lambda_Q(a).$$

If $a \notin Q$, then $\lambda_Q(a) = 0$. On the other hand, assume that

$$((\lambda_Q \circ 1) \wedge (1 \circ \lambda_Q))(a) = 1.$$

Then

$$\bigvee_{a \in x \circ y} \{\lambda_Q(x) \wedge 1(y)\} = (\lambda_Q \circ 1)(a) = 1$$

and

$$\bigvee_{a \in x \circ y} \{1(x) \wedge \lambda_Q(y)\} = (1 \circ \lambda_Q)(a) = 1.$$

This implies that there exist elements b, c, d and e of H with $a \in b \circ c$ and $a \in d \circ e$ such that $\lambda_Q(b) = 1$ and $\lambda_Q(e) = 1$. Hence $a \in b \circ c \subseteq Q \circ H$ and $a \in d \circ e \subseteq H \circ Q$ that is $a \in Q \circ H \cap H \circ Q \subseteq Q$, which contradicts that $a \notin Q$. Then we have

$$(\lambda_Q \circ 1) \wedge (1 \circ \lambda_Q) \leq \lambda_Q$$

and λ_Q is a fuzzy quasi-hyperideal of a semihypergroup H.

Conversely, let λ_Q be a fuzzy quasi-hyperideal of a semihypergroup H. Let a be any element of $Q \circ H \cap H \circ Q$. Then there exist elements s and t of H and elements b and c of Q such that $a \in b \circ s$ and $a \in t \circ c$. Thus we have

$$\begin{aligned}(\lambda_Q \circ 1)(a) &= \bigvee_{a \in x \circ y} \{\lambda_Q(x) \wedge 1(y)\} \\ &\geq \{\lambda_Q(b) \wedge 1(s)\} \\ &= 1 \wedge 1 \\ &= 1,\end{aligned}$$

and so $(\lambda_Q \circ 1)(a) = 1$.

Similarly, we have

$$(1 \circ \lambda_Q)(a) = 1.$$

Hence

$$\begin{aligned}\lambda_Q(a) &\geq ((\lambda_Q \circ 1) \wedge (1 \circ \lambda_Q))(a) \\ &= 1 \wedge 1 \\ &= 1.\end{aligned}$$

Thus $a \in Q$ and so $Q \circ H \cap H \circ Q \subseteq Q$. Therefore, Q is a quasi-hyperideal of H. $\square$

3.1.10 Lemma

Every one-sided hyperideal of a semihypergroup H is a bi-hyperideal of H.

Proof. Straightforward. $\square$

3.1.11 Lemma

Every fuzzy one-sided hyperideal of a semihypergroup H is a fuzzy bi-hyperideal of H.

Proof. Let μ be a fuzzy left hyperideal of a semihypergroup H and $x, y, z \in H$. Then

$$\inf_{\alpha \in x \circ y} \{\mu(\alpha)\} \geq \mu(y) \geq \min\{\mu(x), \mu(y)\}.$$

Thus μ is a fuzzy subsemihypergroup of H.
Let $\alpha \in x \circ y \circ z$. Then there exists $\beta \in x \circ y$ such that $\alpha \in \beta \circ z \subseteq x \circ y \circ z$. Now

$$\inf_{\alpha \in x \circ y \circ z} \{\mu(\alpha)\} \geq \{\mu(z)\} \geq \min\{\mu(x), \mu(z)\}$$

Thus μ is a fuzzy bi-hyperideal of H.
Similarly, if μ is a fuzzy right hyperideal of H then it is a fuzzy bi-hyperideal of H. $\square$

3.1.12 Lemma

The intersection of any family of bi-hyperideals of a semihypergroup H is either empty or a bi-hyperideal of H.

Proof. Straightforward. $\square$

3.1.13 Lemma

The product of two bi-hyperideals of a semihypergroup H is a bi-hyperideal.

Proof. Let B_1, B_2 be bi-hyperideals of a semihypergroup H. Then

$$\begin{aligned}(B_1 \circ B_2) \circ (B_1 \circ B_2) &= (B_1 \circ B_2 \circ B_1) \circ B_2 \\ &\subseteq (B_1 \circ H \circ B_1) \circ B_2 \\ &\subseteq B_1 \circ B_2.\end{aligned}$$

Thus, $B_1 \circ B_2$ is a subsemihypergroup of H.

Also

$$\begin{aligned}(B_1 \circ B_2) \circ H \circ (B_1 \circ B_2) &= (B_1 \circ (B_2 \circ H) \circ B_1) \circ B_2 \\ &\subseteq (B_1 \circ H \circ B_1) \circ B_2 \\ &\subseteq B_1 \circ B_2.\end{aligned}$$

Thus, $B_1 \circ B_2$ is a bi-hyperideal of H. □

3.1.14 Proposition

Let λ be a fuzzy bi-hyperideal of a semihypergroup H. Then
(1) $\lambda \circ \lambda \leq \lambda$;
(2) $\lambda \circ 1 \circ \lambda \leq \lambda$.

Proof. (1) Let λ be a fuzzy bi-hyperideal of a semihypergroup H. Then for each x, $y, z \in H$ if $x \notin y \circ z$ then

$$(\lambda \circ \lambda)(x) = 0 \leq \lambda(x).$$

If $x \in y \circ z$, then $(\lambda \circ \lambda)(x) = \bigvee_{x \in y \circ z} \min\{\lambda(y), \lambda(z)\}$. As λ is fuzzy subsemihypergroup of H, so for each $x \in y \circ z$,

$$\lambda(x) \geq \min\{\lambda(y), \lambda(z)\}, \text{ for all } y, z \in H.$$

Hence $(\lambda \circ \lambda)(x) = \bigvee_{x \in y \circ z} \min\{\lambda(y), \lambda(z)\} \leq \lambda(x)$. Thus, $(\lambda \circ \lambda)(x) \leq \lambda(x)$.

(2) Let $x \in H$. If $A_x = \emptyset$, then $(\lambda \circ 1 \circ \lambda)(x) = 0 \leq \lambda(x)$. Let $A_x \neq \emptyset$, then

$$\begin{aligned}(\lambda \circ 1 \circ \lambda)(x) &= \bigvee_{(y,z) \in A_x} \min\{\lambda(y), (1 \circ \lambda)(z)\} \\ &= \bigvee_{(y,z) \in A_x} \min\{\lambda(y), \bigvee_{(p,q) \in A_z} \min\{1(p), \lambda(q)\}\} \\ &= \bigvee_{(y,z) \in A_x} \bigvee_{(p,q) \in A_z} \min\{\lambda(y), \min\{1, \lambda(q)\}\} \\ &= \bigvee_{(y,z) \in A_x} \bigvee_{(p,q) \in A_z} \min\{\lambda(y), \lambda(q)\}.\end{aligned}$$

As $(y, z) \in A_x \Longrightarrow x \in y \circ z$ and $(p, q) \in A_z \Longrightarrow z \in p \circ q$. Thus $x \in y \circ z \subseteq y \circ (p \circ q)$. Since λ is a fuzzy bi-ideal of H, we have

$$\inf_{x \in y \circ p \circ q} \lambda(x) \geq \min\{\lambda(y), \lambda(q)\} \text{ for all } y, p, q \in H.$$

Thus we have

$$\bigvee_{(y,z) \in A_x} \bigvee_{(p,q) \in A_z} \min\{\lambda(y), \lambda(q)\} \leq \bigvee_{(y,z) \in A_x} \bigvee_{(p,q) \in A_z} \lambda(x) = \lambda(x).$$

Therefore, $(\lambda \circ 1 \circ \lambda)(x) \leq \lambda(x)$. □

3.1.15 Lemma

Let λ and μ be fuzzy bi-hyperideals of a semihypergroup H. Then $\lambda \circ \mu$ is a fuzzy bi-hyperideal of H.

Proof. By Lemma 1.3.3,

$$\begin{aligned}(\lambda \circ \mu) \circ (\lambda \circ \mu) &= (\lambda \circ \mu \circ \lambda) \circ \mu \\ &\leq (\lambda \circ 1 \circ \lambda) \circ \mu \\ &\leq \lambda \circ \mu \text{ by Proposition 3.1.14.}\end{aligned}$$

Let $a, b, c, d \in H$. Then for each $x \in a \circ b$ and $z \in c \circ d$, we have

$$\begin{aligned}(\lambda \circ \mu)(x) \wedge (\lambda \circ \mu)(z) &= \left[\bigvee_{(a,b)\in A_x} \{\lambda(a) \wedge \mu(b)\}\right] \wedge \left[\bigvee_{(c,d)\in A_z} \{\lambda(c) \wedge \mu(d)\}\right] \\ &= \bigvee_{(a,b)\in A_x} \bigvee_{(c,d)\in A_z} [\{\lambda(a) \wedge \mu(b)\} \wedge \{\lambda(c) \wedge \mu(d)\}] \\ &= \bigvee_{(a,b)\in A_x} \bigvee_{(c,d)\in A_z} [\{\lambda(a) \wedge \lambda(c)\} \wedge \{\mu(b) \wedge \mu(d)\}] \\ &= \bigvee_{(a,b)\in A_x} \bigvee_{(c,d)\in A_z} [\{\lambda(a) \wedge \lambda(c) \wedge \mu(b)\} \wedge \mu(d)] \\ &\leq \bigvee_{(a,b)\in A_x} \bigvee_{(c,d)\in A_z} [\{\lambda(a) \wedge \lambda(c)\} \wedge \mu(d)].\end{aligned}$$

As $x \in a \circ b$ and $z \in c \circ d$ so for $y \in H$ and for every $e \in b \circ y$ and $f \in a \circ e \circ c$ we have $g \in f \circ d \subseteq x \circ y \circ z \subseteq (a \circ b) \circ y \circ (c \circ d) = (a \circ (b \circ y) \circ c) \circ d$. Since λ is a fuzzy bi-hyperideal of H, so we have

$$\inf_{f\in a\circ e\circ c} \lambda(f) \geq \{\lambda(a) \wedge \lambda(c)\}.$$

Thus

$$\begin{aligned}\bigvee_{(a,b)\in A_x} \bigvee_{(c,d)\in A_z} [\{\lambda(a) \wedge \lambda(c)\} \wedge \mu(d)] &\leq \bigvee_{(f,d)\in A_g} [\lambda(f) \wedge \mu(d)] \\ &= (\lambda \circ \mu)(g).\end{aligned}$$

Thus for each $g \in f \circ d \subseteq x \circ y \circ z$, we have $(\lambda \circ \mu)(g) \geq (\lambda \circ \mu)(x) \wedge (\lambda \circ \mu)(z)$ this implies

$$\inf_{g\in x\circ y\circ z} (\lambda \circ \mu)(g) \geq (\lambda \circ \mu)(x) \wedge (\lambda \circ \mu)(z).$$

Therefore $\lambda \circ \mu$ is a fuzzy bi-hyperideal of H. □

3.2 Regular and intra-regular semihypergroups

In this section we define intra-regular semihypergroup and characterize regular semihypergroups and intra-regular semihypergroups in terms of their bi-hyperideals and fuzzy bi-hyperideals.

3.2.1 Definition[10]

A semihypergroup H is called regular, if for each $a \in H$ there exists $x \in H$ such that $a \in a \circ x \circ a$.

3.2.2 Definition

A semihypergroup H is called intra-regular, if for every $a \in H$ there exist $x, y \in H$ such that $a \in x \circ a \circ a \circ y$.

3.2.3 Example

Let $(S, \cdot)$ be any semigroup. Define a hyperoperation $\circ$ on S by:

$$a \circ b = \{a, b, ab\} \text{ for all } a, b \in S.$$

Then for all $a, b, c \in S$, we have

$$\begin{aligned} a \circ (b \circ c) &= a \circ \{b, c, bc\} = (a \circ b) \cup (a \circ c) \cup (a \circ (bc)) \\ &= \{a, b, ab\} \cup \{a, c, ac\} \cup \{a, bc, a(bc)\} \\ &= \{a, b, c, ab, ac, bc, a(bc)\} \\ (a \circ b) \circ c &= \{a, b, ab\} \circ c = (a \circ c) \cup (b \circ c) \cup ((ab) \circ c) \\ &= \{a, c, ac\} \cup \{b, c, bc\} \cup \{ab, c, (ab)c\} \\ &= \{a, b, c, ab, ac, bc, (ab)c\} \end{aligned}$$

Thus $a \circ (b \circ c) = (a \circ b) \circ c$. This implies $(S, \circ)$ is a semihypergroup.

Since $a \in a \circ a = \{a, a^2\}$, so $a \in a \circ (a \circ a) = \{a, a^2, a^3\}$. Also $a \in a \circ (a \circ a) \circ a = \{a, a^2, a^3, a^4\}$. Thus $(S, \circ)$ is a regular as well as intra-regular semihypergroup.

3.2.4 Proposition

The following conditions for a semihypergroup H are equivalent:

(1) H is regular.

(2) For every right hyperideal R and left hyperideal L of H, $R \circ L = R \cap L$.

(3) $R(a) \circ L(a) = R(a) \cap L(a)$, for each $a \in H$, where $R(a)$ is the right hyperideal of H generated by "a" and $L(a)$ is the left hyperideal of H generated by "a".

Proof. (1)⇒(2). Let H be a regular semihypergroup and R, L be right and left hyperideals of H respectively. Then for each $x \in R$ and $y \in L$,

$$\begin{aligned} x \circ y &\subseteq R \text{ (because } R \text{ is a right hyperideal)} \\ \text{and } x \circ y &\subseteq L \text{ (because } L \text{ is left hyperideal)} \\ &\Rightarrow x \circ y \subseteq R \cap L, \text{ for each } x \in R \text{ and } y \in L \\ &\Rightarrow \bigcup_{x \in R\, y \in L} x \circ y \subseteq R \cap L \\ &\Rightarrow R \circ L \subseteq R \cap L. \qquad \text{(i)} \end{aligned}$$

For the reverse inclusion, let $x \in R \cap L$. Then $x \in R$ and $x \in L$. Since H is regular, so there exists $y \in H$ such that

$$\begin{aligned} x &\in x \circ y \circ x = (x \circ y) \circ x \\ &\subseteq R \circ L \quad (\text{since } x \circ y \subseteq R) \\ &\Rightarrow x \in R \circ L \\ &\Rightarrow R \cap L \subseteq R \circ L. \qquad \text{(ii)} \end{aligned}$$

Hence, from (i) and (ii), we have,

$$R \circ L = R \cap L.$$

(2)⇒(3). Obvious.

(3)⇒(1). Let $a \in H$, $R(a)$ be the principal right hyperideal generated by "a" and $L(a)$ be the principal left hyperideal generated by "a". Then

$$a \in R(a) \cap L(a) = R(a) \circ L(a).$$

This implies $a \in (a \circ H \cup \{a\}) \circ (H \circ a \cup \{a\})$.

$$\Rightarrow a \in x \circ y \text{ where } x \in a \circ H \cup \{a\} \text{ and } y \in H \circ a \cup \{a\}.$$

Thus we have four cases:

Case I. If $x = a, y = a$ then $a \in a \circ a$ implies $a \in a \circ (a \circ a)$.

Case II. If $x = a, y \in h \circ a$ for some $h \in H$ then $a \in a \circ (h \circ a)$.

Case III. If $x \in a \circ h$ for some $h \in H$ and $y = a$ then $a \in (a \circ h) \circ a$.

Case IV. If $x \in a \circ h, y \in h_1 \circ a$ for some $h, h_1 \in H$ then $a \in (a \circ h) \circ (h_1 \circ a)$ implies $a \in a \circ (h \circ h_1) \circ a$. Thus $a \in a \circ h_2 \circ a$, for some $h_2 \in h \circ h_1 \subseteq H$.

Hence H is regular.

□

3.2.5 Theorem

The following assertions are equivalent for a semihypergroup H.

(1) H is regular.

(2) For every bi-hyperideal B and left hyperideal L of H, $B \cap L \subseteq B \circ L$.

(3) $B(a) \circ L(a) \subseteq B(a) \cap L(a)$, for each $a \in H$, where $B(a)$ is the bi-hyperideal generated by "a" and $L(a)$ is the left hyperideal generated by "a".

Proof. (1)$\Longrightarrow$(2). Let H be a regular semihypergroup, B be a bi-hyperideal and L be a left hyperideal of H. Suppose $a \in B \cap L$. Then $a \in B$ and $a \in L$. Since H is regular so there exists $h \in H$ such that $a \in a \circ h \circ a$. Then

$$\begin{aligned} a &\in a \circ h \circ a \subseteq a \circ h \circ (a \circ h \circ a) \\ &= (a \circ h \circ a) \circ (h \circ a) \subseteq B \circ L. \end{aligned}$$

This implies $B \cap L \subseteq B \circ L$.

(2)$\Rightarrow$(3). Straightforward.

(3)$\Rightarrow$(1). Since each right hyperideal is a bi-hyperideal si by Proposition 3.2.4, H is regular. □

3.2.6 Theorem

Let H be a semihypergroup with identity. Then the following statements are equivalent:

(1) H is regular.

(2) $B = B \circ H \circ B$.

(3) $B(a) = B(a) \circ H \circ B(a)$, for each $a \in H$.

Proof. (1)$\Longrightarrow$(2). Suppose H is a regular semihypergroup with identity, B a bi-hyperideal of H and $a \in B$. Since H is regular so there exists $x \in H$ such that

$$a \in a \circ x \circ a \subseteq B \circ H \circ B.$$

On the other hand, as B is a bi-hyperideal of H so we have

$$B \circ H \circ B \subseteq B.$$

Thus $B \circ H \circ B = B$.

(2)$\Longrightarrow$(3). Obvious.

(3)$\Longrightarrow$(1). Let $B(a)$ be the bi-hyperideal generated by a. Then

$$\begin{aligned} a &\in B(a) = B(a) \circ H \circ B(a) \\ &= (a \circ H \circ a) \circ H \circ (a \circ H \circ a) \\ &= a \circ H \circ (a \circ H \circ a) \circ H \circ a \\ &\subseteq a \circ H \circ a \end{aligned}$$

Thus $a \in a \circ x \circ a$ for some $x \in H$. Hence H is regular. □

3.2.7 Theorem

A semihypergroup H is regular if and only if $\lambda \circ 1 \circ \lambda = \lambda$ for every fuzzy bi-hyperideal λ of H.

Proof. Let H be a regular semihypergroup, λ a fuzzy bi-hyperideal of H and $a \in H$. As H is regular, there exists $h \in H$ such that $a \in a \circ h \circ a \subseteq a \circ h \circ (a \circ h \circ a) = a \circ (h \circ a \circ h \circ a)$. Thus there exists $\beta \in h \circ a \circ h \circ a$ such that $a \in a \circ \beta$. Therefore

$$\begin{aligned}(\lambda \circ 1 \circ \lambda)(a) &= \bigvee_{a \in x \circ y} \min\{\lambda(x), (1 \circ \lambda)(y)\} \\ &\geq \min\{\lambda(a), (1 \circ \lambda)(\beta)\}.\end{aligned}$$

Now, since $\beta \in h \circ a \circ h \circ a$, so there exists some $\gamma \in h \circ a \circ h$ such that $\beta \in \gamma \circ a$, so

$$\begin{aligned}(1 \circ \lambda)(\beta) &= \bigvee_{\beta \in s \circ t} \min\{1(s), \lambda(t)\} \\ &\geq \min\{1(\gamma), \lambda(a)\} \\ &= \min\{1, \lambda(a)\} \\ &= \lambda(a).\end{aligned}$$

Thus $(\lambda \circ 1 \circ \lambda)(a) \geq \min\{\lambda(a), \lambda(a)\} = \lambda(a)$. On the other hand, by Proposition 3.1.14, we have $\lambda \circ 1 \circ \lambda \leq \lambda$. Thus $\lambda \circ 1 \circ \lambda = \lambda$.

Conversely, assume that $\lambda \circ 1 \circ \lambda = \lambda$ for every fuzzy bi-hyperideal λ of H. Let B be a bi-hyperideal of H. Then by Proposition 3.1.8 λ_B is a fuzzy bi-hyperideal of H. By hypothesis $\lambda_B \circ 1 \circ \lambda_B = \lambda_B$. By Proposition 1.3.7 $\lambda_B \circ 1 \circ \lambda_B = \lambda_{B \circ H \circ B}$. Thus $\lambda_{B \circ H \circ B} = \lambda_B$. Hence $B \circ H \circ B = B$. Therefore by Theorem 3.2.6 H is regular. □

3.2.8 Theorem

The following statements are equivalent for a semihypergroup H:

(1) H is both regular and intra-regular.

(2) $B = B \circ B$ for every bi-hyperideal B of H.

(3) $Q = Q \circ Q$ for every quasi-hyperideal Q of H.

(4) $B_1 \cap B_2 = B_1 \circ B_2 \cap B_2 \circ B_1$ for all bi-hyperideals B_1, B_2 of H.

(5) $R \cap L \subseteq R \circ L \cap L \circ R$ for every right hyperideal R and every left hyperideal L of H.

(6) $R(a) \cap L(a) \subseteq R(a) \circ L(a) \cap L(a) \circ R(a)$ for every $a \in H$.

Proof. (1)$\Longrightarrow$(2). Suppose H is a regular and intra-regular semihypergroup. Let B be a bi-hyperideal of H and $a \in B$. Since H is both regular and intra-regular, there exist $x, y, z \in H$ such that $a \in a \circ x \circ a$ and $a \in y \circ a \circ a \circ z$. Then we have

$$\begin{aligned} a &\in a \circ x \circ a \subseteq a \circ x \circ a \circ x \circ a \\ &\subseteq (a \circ x) \circ y \circ a \circ a \circ z \circ (x \circ a) = (a \circ x \circ y \circ a) \circ (a \circ z \circ x \circ a) \\ &\subseteq (B \circ H \circ B) \circ (B \circ H \circ B) \subseteq B \circ B. \end{aligned}$$

$\Rightarrow a \in B \circ B$.

Since $B \circ B \subseteq B$ always, so $B = B \circ B$.

(2)$\Longrightarrow$(3). Obvious.

(3)$\Longrightarrow$(4). Let B_1 and B_2 be bi-hyperideals of H. Then by Lemma 3.1.12, $B_1 \cap B_2$ is a bi-hyperideal of H. By (2), we have

$$B_1 \cap B_2 = (B_1 \cap B_2) \circ (B_1 \cap B_2) \subseteq B_1 \circ B_2.$$

Similarly, we can prove that $B_1 \cap B_2 \subseteq B_2 \circ B_1$. Thus $B_1 \cap B_2 \subseteq B_1 \circ B_2 \cap B_2 \circ B_1$. On the other hand, by Lemma 3.1.13, $B_1 \circ B_2$ and $B_2 \circ B_1$ are bi-hyperideals of H, so by Lemma 3.1.12, $B_1 \circ B_2 \cap B_2 \circ B_1$ is a bi-hyperideal of H. Since every quasi-hyperideal is a bi-hyperideal so by (2), we have

$$\begin{aligned} B_1 \circ B_2 \cap B_2 \circ B_1 &= (B_1 \circ B_2 \cap B_2 \circ B_1) \circ (B_1 \circ B_2 \cap B_2 \circ B_1) \\ &\subseteq (B_1 \circ B_2) \circ (B_2 \circ B_1) = B_1 \circ (B_2 \circ B_2) \circ B_1 \\ &\subseteq B_1 \circ H \circ B_1 \subseteq B_1. \end{aligned}$$

Similarly, we can prove that $B_1 \circ B_2 \cap B_2 \circ B_1 \subseteq B_2$. Therefore $B_1 \cap B_2 = B_1 \circ B_2 \cap B_2 \circ B_1$.

(4)$\Longrightarrow$(5). Let R and L be right and left hyperideals of H, respectively. Then by Lemma 3.1.10, these are bi-hyperideals of H. The assertion follows by (4).

(5)$\Longrightarrow$(6). Obvious.

(6)$\Longrightarrow$(1). By hypothesis $R(a) \cap L(a) \subseteq R(a) \circ L(a) \cap L(a) \circ R(a)$. This implies $R(a) \cap L(a) \subseteq R(a) \circ L(a)$. But $R(a) \circ L(a) \subseteq R(a) \cap L(a)$ always. Hence $R(a) \cap L(a) = R(a) \circ L(a)$. Thus by Proposition 3.2.4, H is regular.

Let $a \in R(a) \cap L(a) \subseteq R(a) \circ L(a) \cap L(a) \circ R(a)$. Then

$$\begin{aligned} a &\in L(a) \circ R(a) = (\{a\} \cup H \circ a) \circ (\{a\} \cup a \circ H) \\ &= (a \circ a) \cup a \circ a \circ H \cup H \circ a \circ a \cup H \circ a \circ a \circ H \\ &= (a \circ a) \cup a \circ a \circ H \cup H \circ a \circ a \circ H. \end{aligned}$$

Thus $a \in a \circ a$ or $a \in a \circ a \circ H$ or $a \in H \circ a \circ a \circ H$.

If $a \in a \circ a$, then $a \in a \circ a \subseteq (a \circ a) \circ (a \circ a)$.

If $a \in a \circ a \circ H$, then $a \in a \circ a \circ h$ for some $h \in H$. Hence $a \in a \circ (a \circ a \circ h) \circ h = a \circ (a \circ a) \circ (h \circ h)$.

If $a \in H \circ a \circ a \circ H$, then for some $x, y \in H$.

$$a \in x \circ a \circ a \circ y.$$

This shows that H is intra-regular. □

3.2.9 Theorem

Let H be a semihypergroup. Then the following statements are equivalent:

(1) H is both regular and intra-regular.

(2) $\lambda \circ \lambda = \lambda$ for every fuzzy bi-hyperideal λ of H.

(3) $\lambda \wedge \mu = \lambda \circ \mu \wedge \mu \circ \lambda$ for all fuzzy bi-hyperideals λ and μ of H.

Proof. (1)$\Longrightarrow$(2). Suppose H is a regular as well as intra-regular semihypergroup and λ a fuzzy bi-hyperideal of H. Since H is regular and intra-regular so there exist $x, y, z \in H$ such that $a \in a \circ x \circ a$ and $a \in y \circ a \circ a \circ z$. Thus

$$a \in a\circ x\circ a \subseteq a\circ x\circ a\circ x\circ a \subseteq a\circ x\circ(y\circ a\circ a\circ z)\circ x\circ a = (a\circ x\circ y\circ a)\circ(a\circ z\circ x\circ a).$$

Then for some $p \in a \circ x \circ y \circ a, q \in a \circ z \circ x \circ a$ we have $a \in p \circ q$, that is $(p,q) \in A_a$. Since $A_a \neq \emptyset$, we have

$$\begin{aligned}(\lambda\circ\lambda)(a) &= \bigvee_{(p,q)\in A_a}\{\lambda(p)\wedge\lambda(q)\}\\ &\geq \{\lambda(p)\wedge\lambda(q)\}.\end{aligned}$$

As λ is a fuzzy bi-hyperideal of H we have

$$\inf_{p\in a\circ x\circ y\circ a}\lambda(p) \geq \min\{\lambda(a),\lambda(a)\} = \lambda(a),$$

and

$$\inf_{q\in a\circ z\circ x\circ a}\lambda(q) \geq \min\{\lambda(a),\lambda(a)\} = \lambda(a).$$

Thus

$$\begin{aligned}(\lambda\circ\lambda)(a) &\geq \{\lambda(p)\wedge\lambda(q)\}\\ &\geq \{\lambda(a)\wedge\lambda(a)\} = \lambda(a).\end{aligned}$$

Thus $\lambda \leq \lambda \circ \lambda$. By Proposition 3.1.14, we have $\lambda \circ \lambda \leq \lambda$. Thus $\lambda \circ \lambda = \lambda$.

(2)$\Longrightarrow$(3). Let λ and μ be fuzzy bi-hyperideals of H. Then $\lambda \wedge \mu$ is a fuzzy bi-hyperideal of H. By (2), we have

$$\lambda\wedge\mu = (\lambda\wedge\mu)\circ(\lambda\wedge\mu) \leq \lambda\circ\mu.$$

Similarly, we can prove that $\lambda \wedge \mu \leq \mu \circ \lambda$. Thus $\lambda \wedge \mu \leq \lambda \circ \mu \wedge \mu \circ \lambda$. For the reverse inclusion, by Lemma 3.1.15, $\lambda \circ \mu$ and $\mu \circ \lambda$ are fuzzy bi-hyperideals of H and so, $\lambda \circ \mu \wedge \mu \circ \lambda$ is a fuzzy bi-hyperideal of H. By (2), we have

$$\begin{aligned}\lambda\circ\mu\wedge\mu\circ\lambda &= (\lambda\circ\mu\wedge\mu\circ\lambda)\circ(\lambda\circ\mu\wedge\mu\circ\lambda)\\ &\leq \lambda\circ\mu\circ\mu\circ\lambda = \lambda\circ(\mu\circ\mu)\circ\lambda\\ &= \lambda\circ\mu\circ\lambda \text{ (as } \mu\circ\mu=\mu \text{ by (2) above)}\\ &\leq \lambda\circ 1\circ\lambda\\ &= \lambda \text{ (as } \lambda\circ 1\circ\lambda=\lambda \text{ by Theorem 3.2.7).}\end{aligned}$$

Hence $\lambda \circ \mu \wedge \mu \circ \lambda \leq \lambda$. Similarly, we can prove that $\lambda \circ \mu \wedge \mu \circ \lambda \leq \mu$. Thus $\lambda \circ \mu \wedge \mu \circ \lambda \leq \lambda \wedge \mu$. Therefore $\lambda \circ \mu \wedge \mu \circ \lambda = \lambda \wedge \mu$.

(3)$\Longrightarrow$(1). Let B_1 and B_2 be bi-hyperideals of H. Then by Proposition 3.1.8, λ_{B_1} and λ_{B_2} are fuzzy bi-hyperideals of H. By hypothesis

$$\lambda_{B_1} \cap \lambda_{B_2} = \lambda_{B_1} \circ \lambda_{B_2} \cap \lambda_{B_2} \circ \lambda_{B_1}.$$

Then by Proposition 1.3.6 and 1.3.7, we have

$$\lambda_{B_1 \cap B_2} = \lambda_{B_1 \circ B_2} \wedge \lambda_{B_2 \circ B_1} = \lambda_{B_1 \circ B_2 \cap B_2 \circ B_1}.$$

This implies that

$$B_1 \cap B_2 = B_1 \circ B_2 \cap B_2 \circ B_1.$$

Hence by Theorem 3.2.8, H is both regular and intra-regular. □

3.2.10 Theorem

Let H be a semihypergroup. Then the following statements are equivalent:

(1) H is both regular and intra-regular.

(2) $B \cap L \subseteq B \circ L \circ B$ for every bi-hyperideal B and every left hyperideal L of H.

(3) $Q \cap L \subseteq Q \circ L \circ Q$ for every quasi-hyperideal Q and every left hyperideal L of H.

(4) $Q(a) \cap L(a) \subseteq Q(a) \circ L(a) \circ Q(a)$ for every $a \in H$.

Proof. (1)$\Longrightarrow$(2). Suppose H is a regular as well as intra-regular semihypergroup, B a bi-hyperideal and L a left hyperideal of H. Let $a \in B \cap L$. Then $a \in B$ and $a \in L$. Since H is regular, there exists $x \in H$ such that $a \in a \circ x \circ a \subseteq a \circ x \circ a \circ x \circ a$. Also H is intra-regular so there exist $y, z \in H$ such that $a \in y \circ a \circ a \circ z$. Thus, we have

$$\begin{aligned} a &\in a \circ x \circ (y \circ a \circ a \circ z) \circ x \circ a = a \circ (x \circ y \circ a) \circ (a \circ z \circ x \circ a) \\ &\subseteq B \circ (H \circ L) \circ (B \circ H \circ B) \subseteq B \circ L \circ B. \end{aligned}$$

This implies $a \in B \circ L \circ B$.

(2)$\Longrightarrow$(3)$\Longrightarrow$(4). These assertions are obvious.

(4)$\Longrightarrow$(1). Let $Q(a)$ be the quasi-hyperideal and $L(a)$ be the left hyperideal generated by a, respectively. Then

$$\begin{aligned} a &\in Q(a) \cap L(a) \subseteq Q(a) \circ L(a) \circ Q(a) \\ &= (\{a\} \cup (H \circ a \cap a \circ H)) \circ (\{a\} \cup H \circ a) \circ (\{a\} \cup (H \circ a \cap a \circ H)) \\ &= ((\{a\} \cup H \circ a) \cap (\{a\} \cup a \circ H)) \circ (\{a\} \cup H \circ a) \circ ((\{a\} \cup H \circ a) \cap (\{a\} \cup a \circ H)) \\ &\subseteq (\{a\} \cup a \circ H) \circ (\{a\} \cup H \circ a) \circ (\{a\} \cup H \circ a) \\ &= (a \circ a \circ a) \cup (a \circ H \circ a). \end{aligned}$$

Then $a \in a \circ a \circ a$ or for some $h \in H$, $a \in a \circ h \circ a$. Thus H is regular.
Again

$$\begin{aligned} a &\in Q(a) \cap L(a) \subseteq Q(a) \circ L(a) \circ Q(a) \\ &= ((\{a\} \cup H \circ a) \cap (\{a\} \cup a \circ H)) \circ (\{a\} \cup H \circ a) \circ ((\{a\} \cup H \circ a) \cap (\{a\} \cup a \circ H)) \\ &\subseteq (\{a\} \cup H \circ a) \circ (\{a\} \cup H \circ a) \circ (\{a\} \cup a \circ H) \\ &= (a \circ a \circ a \cup a \circ a \circ a \circ H \cup a \circ H \circ a \circ a \cup a \circ H \circ a \circ a \circ H \cup H \circ a \circ a \circ a \\ &\quad \cup H \circ a \circ a \circ a \circ H \cup H \circ a \circ H \circ a \circ a \cup H \circ a \circ H \circ a \circ a \circ H) \\ &\subseteq a \circ a \circ a \cup a \circ a \circ a \circ H \cup a \circ H \circ a \circ a \cup H \circ a \circ a \circ a \cup H \circ a \circ a \circ H. \end{aligned}$$

Then $a \in a \circ a \circ a$ or $a \in a \circ a \circ a \circ x$ or $a \in a \circ x \circ a \circ a$ or $a \in x \circ a \circ a \circ a$ or $a \in x \circ a \circ a \circ y$ for some $x, y \in H$.

If $a \in a \circ a \circ a$ then H is regular. If $a \in a \circ a \circ a \circ x$ or $a \in a \circ x \circ a \circ a$ or $a \in x \circ a \circ a \circ a$ or $a \in x \circ a \circ a \circ y$ then H is intra-regular. □

3.2.11 Theorem

A semihypergroup H is both regular and intra-regular if and only if for every fuzzy bi-hyperideal λ for every fuzzy left hyperideal μ of H, we have

$$\lambda \wedge \mu \leq \lambda \circ \mu \circ \lambda.$$

Proof. Suppose that H is both regular and intra-regular semihypergroup, λ a fuzzy bi-hyperideal and μ a fuzzy left hyperideal of H. Since H is both regular and intra-regular semihypergroup, there exist $x, y, z \in H$ such that $a \in a \circ x \circ a$ and $a \in y \circ a \circ a \circ z$. Thus

$$a \in a \circ x \circ a \subseteq a \circ x \circ a \circ x \circ a \subseteq a \circ x \circ (y \circ a \circ a \circ z) \circ x \circ a = a \circ ((x \circ y \circ a) \circ (a \circ z \circ x \circ a)).$$

Then for each $r \in x \circ y \circ a$, $s \in a \circ z \circ x \circ a$ and for every $p \in r \circ s$, we have $a \in a \circ p$. Now

$$\begin{aligned} (\lambda \circ \mu \circ \lambda)(a) &= \bigvee_{a \in a \circ p} \{\lambda(a) \wedge (\mu \circ \lambda)(p)\} \\ &\geq \{\lambda(a) \wedge (\mu \circ \lambda)(p)\} \\ &= \left\{\lambda(a) \wedge \bigvee_{p \in r \circ s} \{\mu(r) \wedge \lambda(s)\}\right\} \\ &\geq \{\lambda(a) \wedge \{\mu(r) \wedge \lambda(s)\}\}. \end{aligned}$$

As λ is a fuzzy bi-hyperideal and μ is a fuzzy left hyperideal of H, so we have $\inf_{s \in a \circ z \circ x \circ a} \lambda(s) \geq \min\{\lambda(a), \lambda(a)\} = \lambda(a)$ and $\inf_{r \in x \circ y \circ a} \mu(r) \geq \mu(a)$. Thus

$$\begin{aligned} (\lambda \circ \mu \circ \lambda)(a) &\geq \{\lambda(a) \wedge \{\mu(r) \wedge \lambda(s)\}\} \\ &\geq \{\lambda(a) \wedge \{\mu(a) \wedge \lambda(a)\}\} \\ &= \{\lambda(a) \wedge \mu(a)\} = (\lambda \wedge \mu)(a). \end{aligned}$$

Thus $(\lambda \wedge \mu)(a) \leq (\lambda \circ \mu \circ \lambda)(a)$.

Conversely, assume that $\lambda \wedge \mu \leq \lambda \circ \mu \circ \lambda$, for every fuzzy bi-hyperideal λ and every fuzzy left hyperideal μ of H. To prove H is both regular and intra-regular semihypergroup, by Theorem 3.2.10, it is enough to prove that

$$Q(a) \cap L(a) \subseteq Q(a) \circ L(a) \circ Q(a) \text{ for every } a \in H.$$

Let $x \in Q(a) \cap L(a)$. Since $Q(a)$ is the quasi-hyperideal and $L(a)$ is the left hyperideal of H, generated by a, respectively. Then by Proposition 3.1.9, $\lambda_{Q(a)}$ is a fuzzy quasi-hyperideal and $\lambda_{L(a)}$ is a fuzzy left hyperideal of H. By hypothesis, we have

$$(\lambda_{Q(a)} \wedge \lambda_{L(a)})(x) \leq (\lambda_{Q(a)} \circ \lambda_{L(a)} \circ \lambda_{Q(a)})(x).$$

As $x \in Q(a)$ and $x \in L(a)$, we have $\lambda_{Q(a)}(x) = 1$ and $\lambda_{L(a)}(x) = 1$. Thus

$$(\lambda_{Q(a)} \circ \lambda_{L(a)} \circ \lambda_{Q(a)})(x) = 1.$$

But by Proposition 1.3.7, $\lambda_{Q(a)} \circ \lambda_{L(a)} \circ \lambda_{Q(a)} = \lambda_{Q(a) \circ L(a) \circ Q(a)}$.
Thus $\lambda_{Q(a) \circ L(a) \circ Q(a)}(x) = 1$ this implies $x \in Q(a) \circ L(a) \circ Q(a)$. □

3.3 Prime and Semiprime bi-hyperideals

In this section, we define prime bi-hyperideal, strongly prime bi-hyperideal, semiprime bi-hyperideal, irreducible bi-hyperideal and strongly irreducible bi-hyperideal of a semihypergroup. We also define, prime fuzzy bi-hyperideal, strongly prime fuzzy bi-hyperideal, semiprime fuzzy bi-hyperideal, irreducible fuzzy bi-hyperideal and strongly irreducible fuzzy bi-hyperideal of a semihypergroup. We characterize semihypergroups using these notions.

3.3.1 Definition

A bi-hyperideal B of a semihypergroup H is called *prime (resp. semiprime)* if $B_1 \circ B_2 \subseteq B$ (resp. $B_1 \circ B_1 \subseteq B$) implies $B_1 \subseteq B$ or $B_2 \subseteq B$ (resp. $B_1 \subseteq B$) for all bi-hyperideals B_1, B_2 of H.

3.3.2 Definition

A bi-hyperideal B of a semihypergroup H is called *strongly prime* if $B_1 \circ B_2 \cap B_2 \circ B_1 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for every bi-hyperideals B_1, B_2 of H.

3.3.3 Definition

A fuzzy bi-hyperideal λ of a semihypergroup H is called *prime (resp. strongly prime, semiprime)* if $\mu \circ \nu \leq \lambda$ (resp. $\mu \circ \nu \wedge \nu \circ \mu \leq \lambda, \mu \circ \mu \leq \lambda$) implies $\mu \leq \lambda$ or $\nu \leq \lambda$ (resp. $\mu \leq \lambda$ or $\nu \leq \lambda$, $\mu \leq \lambda$) for all fuzzy bi-hyperideals μ and ν of H.

3.3.4 Remark

1. Every strongly prime bi-hyperideal of a semihypergroup is a prime bi-hyperideal but the converse is not true.

2. Every prime bi-hyperideal of a semihypergroup is a semiprime bi-hyperideal but the converse is not true.

3.3.5 Example

Consider the semihypergroup $H = \{a, b, c, d, e\}$ given by the following table

$\circ$	a	b	c	d	e
a	a	$\{a,b,d\}$	a	$\{a,b,d\}$	$\{a,b,d\}$
b	a	b	a	$\{a,b,d\}$	$\{a,b,d\}$
c	a	$\{a,b,d\}$	$\{a,c\}$	$\{a,b,d\}$	$\{a,b,c,d,e\}$
d	a	$\{a,b,d\}$	a	$\{a,b,d\}$	$\{a,b,d\}$
e	a	$\{a,b,d\}$	$\{a,c\}$	$\{a,b,d\}$	$\{a,b,c,d,e\}$

The bi-hyperideals of H are $\{a\}, \{a,c\}, \{a,b,d\}$ and H. The bi-hyperideal $\{a\}$ is semiprime but is not prime bi-hyperideal, because $\{a,b,d\} \circ \{a,c\} = \{a\}$ but $\{a,b,d\} \nsubseteq \{a\}$ and $\{a,c\} \nsubseteq \{a\}$.

3.3.6 Example

Consider the semihypergroup $H = \{0, a, b, c\}$ given by the following table

$\circ$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

Every subset of S containing 0 is a bi-hyperideal of S. Also $A \circ B = A$ for all bi-hyperideals A, B of S. Thus every bi-hyperideal of S is prime but $\{0,a\}$ is not strongly prime, because $\{0,a,b\} \circ \{0,a,c\} \cap \{o,a,c\} \circ \{0,a,b\} = \{0,a\}$ but $\{0,a,b\} \nsubseteq \{0,a\}$ and $\{0,a,c\} \nsubseteq \{0,a\}$.

3.3.7 Proposition

A subset B of a semihypergroup H is a prime bi-hyperideal of H if and only if the characteristic function λ_B of B is a prime fuzzy bi-hyperideal of H.

Proof. Let B be a prime fuzzy bi-hyperideal of a semihypergroup H and λ_B be the characteristic function of B. By the Proposition 3.1.8, λ_B is a fuzzy bi-hyperideal of H. Let μ and ν be fuzzy bi-hyperideals of H such that $\mu \circ \nu \leq \lambda_B$, with $\mu \nleq \lambda_B$

and $\nu \not\leq \lambda_B$. Then there exist $x, y \in H$ such that $\mu(x) \neq 0$ and $\nu(y) \neq 0$ but $\lambda_B(x) = 0 = \lambda_B(y)$. So $x \notin B$ and $y \notin B$. Since B is a prime bi-hyperideal of H, so $B(x) \circ B(y) \not\subseteq B$. Hence there exists $a \in B(x) \circ B(y)$ such that $a \notin B$. So we have $\lambda_B(a) = 0$ and hence $(\mu \circ \nu)(a) = 0$. Since $\mu(x) \neq 0 \neq \nu(y)$ therefore

$$\min\{\mu(x), \nu(y)\} \neq 0.$$

Since $a \in B(x) \circ B(y)$, so there exist $x_1 \in B(x)$ and $y_1 \in B(y)$ such that $a \in x_1 \circ y_1$. Thus

$$\begin{aligned}(\mu \circ \nu)(a) &= \bigvee_{a \in x_1 \circ y_1} \min\{\mu(x_1), \nu(y_1)\} \\ &\geq \min\{\mu(x_1), \nu(y_1)\}.\end{aligned}$$

Since $x_1 \in B(x) = \{x\} \cup x \circ x \cup x \circ H \circ x$. Therefore $x_1 = x$ or $x_1 \in x \circ x$ or for some $h \in H, x_1 \in x \circ h \circ x$.

If $x_1 = x$ then $\mu(x_1) = \mu(x)$. If $x_1 \in x \circ x$ then

$$\inf_{x_1 \in x \circ x} \mu(x_1) \geq \min\{\mu(x), \mu(x)\} = \mu(x).$$

If $x_1 \in x \circ h \circ x$ then since μ is a fuzzy bi-hyperideal of H so

$$\inf_{x_1 \in x \circ h \circ x} \mu(x_1) \geq \min\{\mu(x), \mu(x)\} = \mu(x).$$

Also $y_1 \in B(y) = \{y\} \cup y \circ y \cup y \circ H \circ y$ implies that either $y_1 = y$ or $y_1 \in y \circ y$ or for some $h_1 \in H$, $y_1 \in y \circ h_1 \circ y$.

If $y_1 = y$ then $\nu(y_1) = \nu(y)$. If $y_1 \in y \circ y$ then

$$\inf_{y_1 \in y \circ y} \nu(y_1) \geq \min\{\nu(y), \nu(y)\} = \nu(y).$$

If $y_1 \in y \circ h_1 \circ y$ then as ν is a fuzzy bi-hyperideal of H so

$$\inf_{y_1 \in y \circ h_1 \circ y} \nu(y_1) \geq \min\{\nu(y), \nu(y)\} = \nu(y).$$

Thus,

$$\min\{\mu(x_1), \nu(y_1)\} \geq \min\{\mu(x), \nu(y)\} \neq 0.$$

$\Rightarrow (\mu \circ \nu)(a) \geq 0$, which is a contradiction.

Therefore for any fuzzy bi-hyperideals μ, ν of H, $\mu \circ \nu \leq \lambda_B$ implies $\mu \leq \lambda_B$ or $\nu \leq \lambda_B$.

Conversely, assume that B is a bi-hyperideal of H and let B_1, B_2 are bi-hyperideals of H such that $B_1 \circ B_2 \subseteq B$. Then by Proposition 1.3.4, we have $\lambda_{B_1 \circ B_2} \leq \lambda_B$. By Proposition 1.3.7, $\lambda_{B_1 \circ B_2} = \lambda_{B_1} \circ \lambda_{B_2}$, so we have $\lambda_{B_1} \circ \lambda_{B_2} \leq \lambda_B$. As λ_B is a prime fuzzy bi-hyperideal of H, we have $\lambda_{B_1} \leq \lambda_B$ or $\lambda_{B_2} \leq \lambda_B$. Thus by Proposition 1.3.4, we have $B_1 \subseteq B$ or $B_2 \subseteq B$. □

3.3.8 Proposition

A subset B of a semihypergroup H is a semiprime bi-hyperideal of H if and only if the characteristic function λ_B of B is a semiprime fuzzy bi-hyperideal of H.

Proof. The proof is similar to the proof of Proposition 3.3.7. □

3.3.9 Proposition

A subset B of a semihypergroup H is a strongly prime bi-hyperideal of H if and only if the characteristic function λ_B of B is a strongly prime fuzzy bi-hyperideal of H.

Proof. Suppose B is a strongly prime bi-hyperideal of a semihypergroup H and λ_B is the characteristic function of B. By Proposition 3.1.8, λ_B is a fuzzy bi-hyperideal of H. Let μ and ν be fuzzy bi-hyperideals of H such that $\mu \circ \nu \wedge \nu \circ \mu \leq \lambda_B$, but $\mu \nleq \lambda_B$ and $\nu \nleq \lambda_B$. Then there exist $x, y \in H$ such that $\mu(x) \neq 0$ and $\nu(y) \neq 0$ but $\lambda_B(x) = 0$ and $\lambda_B(y) = 0$. So $x \notin B$ and $y \notin B$. Since B is a strongly prime bi-hyperideal of H, so $B(x) \circ B(y) \cap B(y) \circ B(x) \nsubseteq B$. Hence there exists $a \in B(x) \circ B(y) \cap B(y) \circ B(x)$ such that $a \notin B$. So we have $\lambda_B(a) = 0$ and hence $(\mu \circ \nu)(a) = 0$ and $(\nu \circ \mu)(a) = 0$. Since $\mu(x) \neq 0$ and $\nu(y) \neq 0$ so

$$\min\{\mu(x), \nu(y)\} \neq 0.$$

Since $a \in B(x) \circ B(y) \cap B(y) \circ B(x)$, therefore there exist $x_1, x_2 \in B(x)$ and $y_1, y_2 \in B(y)$ such that $a \in x_1 \circ y_1$ and $a \in y_2 \circ x_2$. Thus

$$\begin{aligned} (\mu \circ \nu)(a) &= \bigvee_{a \in x_1 \circ y_1} \min\{\mu(x_1), \nu(y_1)\} \\ &\geq \min\{\mu(x_1), \nu(y_1)\}. \end{aligned}$$

Since $x_1 \in B(x) = \{x\} \cup x \circ x \cup x \circ H \circ x$, therefore either $x_1 = x$ or $x_1 \in x \circ x$ or for some $h \in H, x_1 \in x \circ h \circ x$.

If $x_1 = x$, then $\mu(x_1) = \mu(x)$. If $x_1 \in x \circ x$, then

$$\inf_{x_1 \in x \circ x} \mu(x_1) \geq \min\{\mu(x), \mu(x)\} = \mu(x).$$

If $x_1 \in x \circ h \circ x$, then since μ is a fuzzy bi-hyperideal of H so

$$\inf_{x_1 \in x \circ h \circ x} \mu(x_1) \geq \min\{\mu(x), \mu(x)\} = \mu(x).$$

Also $y_1 \in B(y) = \{y\} \cup y \circ y \cup y \circ H \circ y$ implies that either $y_1 = y$ or $y_1 \in y \circ y$ or for some $h_1 \in H$, $y_1 \in y \circ h_1 \circ y$. If $y_1 = y$, then $\nu(y_1) = \nu(y)$. If $y_1 \in y \circ y$, then

$$\inf_{y_1 \in y \circ y} \nu(y_1) \geq \min\{\nu(y), \nu(y)\} = \nu(y).$$

If $y_1 \in y \circ h_1 \circ y$, then as ν is a fuzzy bi-hyperideal of H so

$$\inf_{y_1 \in y \circ h_1 \circ y} \nu(y_1) \geq \min\{\nu(y), \nu(y)\} = \nu(y).$$

Thus,

$$\min\{\mu(x_1), \nu(y_1)\} \geq \min\{\mu(x), \nu(y)\} \neq 0.$$

$\Rightarrow (\mu \circ \nu)(a) \geq 0$, which is a contradiction.

Similarly for $a \in y_2 \circ x_2$ we get $(\nu \circ \mu)(a) \geq 0$, which is a contradiction.

Thus for any fuzzy bi-hyperideals μ, ν of H, $\mu \circ \nu \wedge \nu \circ \mu \leq \lambda_B$, we have $\mu \leq \lambda_B$ or $\nu \leq \lambda_B$.

Conversely, assume that B is a bi-hyperideal of H and let B_1, B_2 are bi-hyperideals of H such that $B_1 \circ B_2 \cap B_2 \circ B_1 \subseteq B$. Then by Proposition 1.3.4, we have $\lambda_{B_1 \circ B_2 \cap B_2 \circ B_1} \leq \lambda_B$. By Proposition 1.3.6, $\lambda_{B_1 \circ B_2 \cap B_2 \circ B_1} = \lambda_{B_1} \circ \lambda_{B_2} \wedge \lambda_{B_2} \circ \lambda_{B_1}$. So we have $\lambda_{B_1} \circ \lambda_{B_2} \wedge \lambda_{B_2} \circ \lambda_{B_1} \leq \lambda_B$. As λ_B is a strongly prime fuzzy bi-hyperideal of H, we have $\lambda_{B_1} \leq \lambda_B$ or $\lambda_{B_2} \leq \lambda_B$. Thus by Proposition 1.3.4, we have $B_1 \subseteq B$ or $B_2 \subseteq B$. □

3.3.10 Proposition

Let $\{B_i : i \in I\}$ be a family of prime bi-hyperideals of a semihypergroup H. Then $\bigcap_{i \in I} B_i$ is a semiprime bi-hyperideal of H.

Proof. Straightforward □

3.3.11 Proposition

Let $\{\lambda_i : i \in I\}$ be a family of prime fuzzy bi-hyperideals of a semihypergroup H. Then $\bigwedge_{i \in I} \lambda_i$ is a semiprime fuzzy bi-hyperideal of H.

Proof. Straightforward. □

3.3.12 Definition

Let B be a bi-hyperideal of a semihypergroup H. Then B is called an irreducible (resp. strongly irreducible) if for any bi-hyperideals B_1, B_2 of H we have, $B_1 \cap B_2 = B$ (resp. $B_1 \cap B_2 \subseteq B$) implies $B_1 = B$ or $B_2 = B$ (resp. $B_1 \subseteq B$ or $B_2 \subseteq B$).

It is obvious that every strongly irreducible bi-hyperideal of a semihypergroup H is irreducible but the converse is not true.

3.3.13 Lemma

Let B be a bi-hyperideal of a semihypergroup H and a be any element of H such that $a \notin B$. Then there exists an irreducible bi-hyperideal A of H such that $B \subseteq A$ and $a \notin A$.

Proof. Let Ω be the collection of all bi-hyperideals of H which contain B but do not contain "a". Then Ω is non-empty, because $B \in \Omega$. The collection Ω is a partially ordered set under inclusion. As every totally ordered subset of Ω is bounded above, so by Zorn's Lemma, there exists a maximal element, say, A in Ω. We show that A is an irreducible bi-hyperideal of H. Let C and D be two bi-hyperideals of H such that $C \cap D = A$. If both C and D properly contain A then $a \in C$ and $a \in D$. Hence $a \in C \cap D = A$, this contradicts the fact that $a \notin A$. So $C = A$ or $D = A$ i.e. A is irreducible. □

3.3.14 Proposition

Every strongly irreducible semiprime bi-hyperideal of a semihypergroup H is strongly prime bi-hyperideal of H.

Proof. Let B be a strongly irreducible semiprime bi-hyperideal of a semihypergroup H. Let B_1, B_2 be any bi-hyperideals of H such that $B_1 \circ B_2 \cap B_2 \circ B_1 \subseteq B$. Since $(B_1 \cap B_2) \circ (B_1 \cap B_2) \subseteq B_1 \circ B_2$ and $(B_1 \cap B_2) \circ (B_1 \cap B_2) \subseteq B_2 \circ B_1$. Thus $(B_1 \cap B_2) \circ (B_1 \cap B_2) \subseteq (B_1 \circ B_2) \cap (B_2 \circ B_1) \subseteq B$. Since B is semiprime bi-hyperideal of H, we have $B_1 \cap B_2 \subseteq B$. Since B is strongly irreducible, we have $B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is a strongly prime bi-hyperideal of H. □

3.3.15 Definition

A fuzzy bi-hyperideal λ of a semihypergroup H is called an irreducibl (resp. strongly irreducible) fuzzy bi-hyperideal of H, if $\mu \wedge \nu = \lambda$ (resp. $\mu \wedge \nu \leq \lambda$) implies $\mu = \lambda$ or $\nu = \lambda$ (resp. $\mu \leq \lambda$ or $\nu \leq \lambda$) for every fuzzy bi-hyperideals μ, ν of H.

3.3.16 Proposition

Every strongly irreducible semiprime fuzzy bi-hyperideal of a semihypergroup H is a strongly prime fuzzy bi-hyperideal of H.

Proof. Let λ be a strongly irreducible semiprime fuzzy bi-hyperideal of a semihypergroup H. Suppose μ, ν are any fuzzy bi-hyperideals of H such that $\mu \circ \nu \wedge \nu \circ \mu \leq \lambda$. As $\mu \wedge \nu$ is a fuzzy bi-hyperideal of H and $(\mu \wedge \nu) \circ (\nu \wedge \mu) \leq \mu \circ \nu$, $(\mu \wedge \nu) \circ (\nu \wedge \mu) \leq \nu \circ \mu$. Thus $(\mu \wedge \nu) \circ (\nu \wedge \mu) \leq \mu \circ \nu \wedge \nu \circ \mu \leq \lambda$. Since λ is semiprime, we have $\mu \wedge \nu \leq \lambda$. Since λ is strongly irreducible, we have $\mu \leq \lambda$ or $\nu \leq \lambda$. Thus λ is a strongly prime fuzzy bi-hyperideal of H. □

3.3.17 Proposition

Let λ be a fuzzy bi-hyperideal of a semihypergroup H with $\lambda(a) = t$, where $a \in H$ and $t \in (0, 1]$. Then there exists an irreducible fuzzy bi-hyperideal μ of H such that $\lambda \leq \mu$ and $\mu(a) = t$.

Proof. Let $X = \{\nu \mid \nu$ is a fuzzy bi-hyperideal of H with $\nu(a) = t$ and $\lambda \leq \nu\}$. Then $X \neq \emptyset$, because $\lambda \in X$. The collection X is partially ordered set under inclusion. Let Y be any totally ordered subset of X, say $Y = \{\nu_i \mid i \in I\}$. Let $x, y, z \in H$ and for every $\alpha \in x \circ y$,

$$\begin{aligned} \inf_{\alpha \in x \circ y} (\bigvee_{i \in I} \nu_i)(\alpha) &= \bigvee_{i \in I} \inf_{\alpha \in x \circ y} \nu_i(\alpha) \\ &\geq \bigvee_{i \in I} \{\nu_i(x) \wedge \nu_i(y)\} \\ &= \bigvee_{i \in I} (\nu_i(x)) \wedge \bigvee_{i \in I} (\nu_i(y)) \\ &= \bigvee_{i \in I} (\nu_i)(x) \wedge \bigvee_{i \in I} (\nu_i)(y). \end{aligned}$$

Hence $\bigvee_{i \in I} \nu_i$ is a fuzzy subsemihypergroup of H.
Also for each $\beta \in x \circ y \circ z$,

$$\begin{aligned} \inf_{\beta \in x \circ y \circ z} (\bigvee_{i \in I} \nu_i)(\beta) &= \bigvee_{i \in I} \inf_{\beta \in x \circ y \circ z} (\nu_i)(\beta) \\ &\geq \bigvee_{i \in I} \{\nu_i(x) \wedge \nu_i(z)\} \\ &= \bigvee_{i \in I} (\nu_i(x)) \wedge \bigvee_{i \in I} (\nu_i(z)) \\ &= \bigvee_{i \in I} (\nu_i)(x) \wedge \bigvee_{i \in I} (\nu_i)(z). \end{aligned}$$

Hence $\bigvee_{i \in I} \nu_i$ is a fuzzy bi-hyperideal of H.

As $\lambda \leq \nu_i$ for each $i \in I$, so $\lambda \leq \bigvee_{i \in I} \nu_i$. Also $(\bigvee_{i \in I} \nu_i)(a) = \bigvee_{i \in I} \nu_i(a) = t$. Thus $\bigvee_{i \in I} \nu_i$ is the least upper bound of Y. By Zorn's Lemma, there exists a fuzzy bi-hyperideal μ of H which is maximal with respect to the property that $\lambda \leq \mu$ and $\mu(a) = t$. Now, we show that μ is an irreducible fuzzy bi-hyperideal of H. Suppose that f, g are fuzzy bi-hyperideals of H such that $f \wedge g = \mu$. Thus $\mu < f$ and $\mu < g$. We claim that either $\mu = f$ or $\mu = g$. On contrary, we suppose that $\mu \neq f$ and $\mu \neq g$. Since μ is maximal with respect to the property that $\mu(a) = t$ and since $\mu \neq f$ and $\mu \neq g$, it follows that $f(a) \neq t$ and $g(a) \neq t$. Hence $t = \mu(a) = (f \wedge g)(a) \neq t$, which is a contradiction. Hence either $\mu = f$ or $\mu = g$. Thus μ is an irreducible fuzzy bi-hyperideal of H. □

3.3.18 Theorem

Let H be a semihypergroup. Then the following statements are equivalent:

(1) H is both regular and intra-regular.

(2) $B = B \circ B$ for every bi-hyperideal B of H.

(3) $B_1 \cap B_2 = B_1 \circ B_2 \cap B_2 \circ B_1$ for all bi-hyperideals B_1, B_2 of H.

(4) Each bi-hyperideal of H is semiprime.

(5) Every bi-hyperideal of a semihypergroup H is the intersection of irreducible semiprime bi-hyperideals of H which contain it.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) This is proved in Theorem 3.2.8.

(3) $\Rightarrow$ (4). Let B_1 and B be bi-hyperideals of H such that $B_1 \circ B_1 \subseteq B$. By hypothesis

$$\begin{aligned} B_1 &= B_1 \cap B_1 \\ &= B_1 \circ B_1 \cap B_1 \circ B_1 \\ &= B_1 \circ B_1 \subseteq B. \end{aligned}$$

Thus $B_1 \subseteq B$ and hence every bi-hyperideal of H is semiprime.

(4) $\Rightarrow$ (5). Let B be a proper bi-hyperideal of H. Then B is contained in the intersection of all irreducuible bi-hyperideals of H which contain B. By Lemma 3.3.13, there exist such irreducible bi-hyperideals. If $a \notin B$ then there exists an irreducible bi-hyperideal of H which contains B but does not contain a. Hence B is the intersection of all irreducible bi-hyperideals of H which contain it. By hypothesis each bi-hyperideal of H is semiprime, so each bi-hyperideal of H is the intersection of irreducible semiprime bi-hyperideals of H which contain it.

(5) $\Rightarrow$ (2). Let B be a proper bi-hyperideal of H. If $B \circ B = H$ then B is idempotent, that is, $B \circ B = B$. If $B \circ B \neq H$ then $B \circ B$ is a proper bi-hyperideal of H and by hypothesis,

$$B \circ B = \bigcap_{\alpha}\{B_\alpha \mid B_\alpha \text{ is irreducible semiprime bi-hyperideal of } H \text{ containing } B \circ B\}.$$

This implies $B \circ B \subseteq B_\alpha$ for all α. Since every B_α is semiprime, therefore $B \subseteq B_\alpha$ for all α and so $B \subseteq \bigcap_{\alpha} B_\alpha = B \circ B$. Hence each bi-hyperideal of H is idempotent. □

3.3.19 Theorem

For a semihypergroup H, the following statements are equivalent:

(1) H is both regular and intra-regular.

(2) $\lambda \circ \lambda = \lambda$ for every fuzzy bi-hyperideal λ of H.

(3) $\lambda \wedge \mu = \lambda \circ \mu \wedge \mu \circ \lambda$ for all fuzzy bi-hyperideals λ, μ of H.

(4) Each fuzzy bi-hyperideal of H is fuzzy semiprime.

(5) Every proper fuzzy bi-hyperideal of H is the intersection of irreducible fuzzy semiprime bi-hyperideals of H which contain it.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) From Theorem 3.2.9.

(3) $\Rightarrow$ (4). Let λ, μ be any fuzzy bi-hyperideals of H such that $\lambda \circ \lambda \leq \mu$. By hypothesis,

$$\begin{aligned} \lambda &= \lambda \wedge \lambda \\ &= \lambda \circ \lambda \wedge \lambda \circ \lambda \\ &= \lambda \circ \lambda. \end{aligned}$$

Thus $\lambda \leq \mu$. Hence each fuzzy bi-hyperideal of H is fuzzy semiprime.

(4) $\Rightarrow$ (5). Let λ be a proper fuzzy bi-hyperideal of H and $\{\lambda_i : i \in I\}$ be the collection of all irreducible fuzzy bi-hyperideal of H which contain λ. By Proposition 3.3.17, this collection is non-empty. Hence $\lambda \leq \wedge_{i\in I}\lambda_i$. Let $a \in H$, then there exists an irreducible fuzzy bi-hyperideal λ_α of H such that $\lambda \leq \lambda_\alpha$ and $\lambda(a) = \lambda_\alpha(a)$. Thus $\lambda_\alpha \in \{\lambda_i : i \in I\}$. Hence $\wedge_{i\in I}\lambda_i \leq \lambda_\alpha$. So, $\wedge_{i\in I}\lambda_i(a) = \lambda_\alpha(a) = \lambda(a)$. Thus $\wedge_{i\in I}\lambda_i \leq \lambda$. Consequently $\wedge_{i\in I}\lambda_i = \lambda$. By hypothesis, each fuzzy bi-hyperideal of H is fuzzy semiprime. So each fuzzy bi-hyperideal of H is the intersection of irreducible fuzzy semiprime bi-hyperideals of H which contain it.

(5) $\Rightarrow$ (2). Let λ be a fuzzy bi-hyperideal of H. Then $\lambda \circ \lambda$ is also a fuzzy bi-hyperideal of H. Since λ is a fuzzy subsemihypergroup of H, so $\lambda \circ \lambda \leq \lambda$. By hypothesis $\lambda \circ \lambda = \wedge_{i\in I}\lambda_i$ where λ_i are irreducible fuzzy semiprime bi-hyperideals of H. Thus $\lambda \circ \lambda \leq \lambda_i$ for all $i \in I$. Hence $\lambda \leq \lambda_i$ for all $i \in I$, because λ_i are semiprime. Thus $\lambda \leq \wedge_{i\in I}\lambda_i = \lambda \circ \lambda$. Hence $\lambda \circ \lambda = \lambda$. □

In the following result we study a relationship between strongly irreducible hyperideals and strongly prime bi-hyperideals of semihypergroups.

3.3.20 Proposition

Let H be both regular and intra-regular semihypergroup and B be a bi-hyperideal of H. Then the following statements are equivalent:

(1) B is strongly irreducible.

(2) B is strongly prime.

Proof. (1) $\Rightarrow$ (2). Assume that B is a bi-hyperideal of a regular and intra-regular semihypergroup H and B_1, B_2 be any bi-hyperideals of H such that $B_1 \circ B_2 \cap B_2 \circ B_1 \subseteq B$. Since H is both regular and intra-regular, by Theorem 3.2.8, we have $B_1 \circ B_2 \cap B_2 \circ B_1 = B_1 \cap B_2$. Thus $B_1 \cap B_2 \subseteq B$. Since B is strongly irreducible, we have either $B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is strongly prime.

(2) $\Rightarrow$ (1). Let B be a strongly prime bi-hyperideal of H and suppose B_1, B_2 be any bi-hyperideals of H such that $B_1 \cap B_2 \subseteq B$. As $B_1 \circ B_2 \cap B_2 \circ B_1 = B_1 \cap B_2$, and B is strongly prime we have $B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is strongly irreducible. □

3.3.21 Proposition

Let H be both regular and intra-regular semihypergroup. Then the following statements are equivalent:

(1) Every fuzzy bi-hyperideal of H is strongly irreducible.

(2) Every fuzzy bi-hyperideal of H is strongly prime.

Proof. (1) $\Rightarrow$ (2). Let H be both regular and intra-regular semihypergroup and λ be a strongly irreducible fuzzy bi-hyperideal of H. Let μ, ν be fuzzy bi-hyperideals of H such that $\mu \circ \nu \wedge \nu \circ \mu \leq \lambda$. Since H is both regular and intra-regular, by Theorem 3.2.9, we have $\mu \circ \nu \wedge \nu \circ \mu = \mu \wedge \nu$. Thus $\mu \wedge \nu \leq \lambda$. Since λ is strongly irreducible, so either $\mu \leq \lambda$ or $\nu \leq \lambda$. Thus λ is strongly prime fuzzy bi-hyperideal of H.

(2) $\Rightarrow$ (1). Suppose λ is a strongly prime fuzzy bi-hyperideal of H and μ, ν be fuzzy bi-hyperideals of H such that $\mu \circ \nu \leq \lambda$. As $\mu \circ \nu \wedge \nu \circ \mu \leq \mu \circ \nu \leq \lambda$. Since λ is strongly prime, so either $\mu \leq \lambda$ or $\nu \leq \lambda$. Thus λ is strongly irreducible. □

3.3.22 Proposition

Each bi-hyperideal of a semihypergroup H is strongly prime if and only if H is both regular and intra-regular and the set of bi-hyperideals of H is totally ordered under inclusion.

Proof. Suppose that each bi-hyperideal of a semihypergroup H is strongly prime. Then each bi-hyperideal of H is semiprime. Thus by Theorem 3.3.18, H is both regular and intra-regular. To prove that the set of bi-hyperideals of H is totally ordered under inclusion, let B_1, B_2 be any bi-hyperideals of H. Then by Theorem 3.3.18, $B_1 \circ B_2 \cap B_2 \circ B_1 = B_1 \cap B_2$. As each bi-hyperideal of H is strongly prime, so $B_1 \cap B_2$ is strongly prime. Hence either $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$. If $B_1 \subseteq B_1 \cap B_2$ then $B_1 \subseteq B_2$ and if $B_2 \subseteq B_1 \cap B_2$, then $B_2 \subseteq B_1$.

Conversely, let H be both regular and intra-regular and the set of bi-hyperideals of H is totally ordered under inclusion. Suppose B is any arbitrary bi-hyperideal of H and let B_1, B_2 be any bi-hyperideals of H such that $B_1 \circ B_2 \cap B_2 \circ B_1 \subseteq B$. Since H is both regular and intra-regular, by Theorem 3.2.8, $B_1 \circ B_2 \cap B_2 \circ B_1 = B_1 \cap B_2$. Thus $B_1 \cap B_2 \subseteq B$. Since the set of bi-hyperideals of H is totally ordered under inclusion, so either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Thus $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Therefore either $B_1 \subseteq B$ or $B_2 \subseteq B$ and hence B is strongly prime bi-hyperideal of H. □

3.3.23 Proposition

Each fuzzy bi-hyperideal of a semihypergroup H is strongly prime if and only if H is both regular and intra-regular and the set of fuzzy bi-hyperideals of H is totally ordered under inclusion.

Proof. Suppose that each fuzzy bi-hyperideal of a semihypergroup H is strongly prime. Then each fuzzy bi-hyperideal of H is semiprime. Thus by Theorem 3.3.19, H is both regular and intra-regular. To prove that the set of fuzzy bi-hyperideals of H is totally ordered under inclusion, let μ, ν be any fuzzy bi-hyperideals of H. Then by Theorem 3.3.19, $\lambda \wedge \mu = \lambda \circ \mu \wedge \mu \circ \lambda$. As each fuzzy bi-hyperideal of H is strongly prime, so $\lambda \wedge \mu$ is strongly prime. Hence either $\lambda \leq \lambda \wedge \mu$ or $\mu \leq \lambda \wedge \mu$. If $\lambda \leq \lambda \wedge \mu$ then $\lambda \leq \mu$ and if $\mu \leq \lambda \wedge \mu$ then $\mu \leq \lambda$.

Conversely, assume that H is both regular and intra-regular and the set of fuzzy bi-hyperideals of H is totally ordered under inclusion. Let λ be any arbitrary fuzzy bi-hyperideal of H and let μ, ν be any fuzzy bi-hyperideals of H such that $\mu \circ \nu \wedge \nu \circ \mu \leq \lambda$. Since H is both regular and intra-regular, by Theorem 3.3.19, $\mu \circ \nu \wedge \nu \circ \mu = \mu \wedge \nu$. Thus $\mu \wedge \nu \leq \lambda$. Since the set of fuzzy bi-hyperideals of H is totally ordered under inclusion, so either $\mu \leq \nu$ or $\nu \leq \mu$. Thus $\mu \wedge \nu = \mu$ or $\mu \wedge \nu = \nu$. Therefore either $\mu \leq \lambda$ or $\nu \leq \lambda$ and hence λ is strongly prime fuzzy bi-hyperideal of H. □

3.3.24 Lemma

Let H be a semihypergroup and the set of bi-hyperideals of H is totally ordered under inclusion. Then the following assertions are equivalent:

(1) H is both regular and intra-regular.

(2) Each bi-hyperideal of H is prime.

Proof. (1) $\Rightarrow$ (2). Let H be both regular and intra-regular semihypergroup and B be a bi-hyperideal of H. Suppose B_1, B_2 be any arbitrary bi-hyperideals of H such that $B_1 \circ B_2 \subseteq B$. Since the set of bi-hyperideals of H is totally ordered under inclusion, so $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. If $B_1 \subseteq B_2$ then $B_1 \circ B_1 \subseteq B_1 \circ B_2 \subseteq B$. By Theorem 3.3.18, B is semiprime, so $B_1 \subseteq B$. Hence B is prime.

(2) $\Rightarrow$ (1). Assume that every bi-hyperideals of H is prime. Since the set of bi-hyperideals of H is totally ordered under inclusion, so the concepts of prime and strongly prime bi-hyperideals coincide. Therefore by Theorem 3.3.18, H is both regular and intra-regular. □

3.3.25 Proposition

If the set of fuzzy bi-hyperideals of a semihypergroup H is totally ordered under inclusion, then H is both regular and intra-regular if and only if each fuzzy bi-hyperideal of H is prime.

Proof. Suppose H is both regular and intra-regular semihypergroup and λ is any fuzzy bi-hyperideal of H. Let μ, ν be any arbitrary fuzzy bi-hyperideals of H such that $\mu \circ \nu \leq \lambda$. Since the set of fuzzy bi-hyperideals of H is totally ordered under inclusion, so either $\mu \leq \nu$ or $\nu \leq \mu$. If $\mu \leq \nu$ then $\mu \circ \mu \leq \mu \circ \nu \leq \lambda$. By Theorem 3.3.19, λ is semiprime, so $\mu \leq \lambda$. Similarly we can prove that $\nu \leq \lambda$. Hence λ is prime.

Conversely, suppose that every fuzzy bi-hyperideal of H is prime. Since the set of fuzzy bi-hyperideals of H is totally ordered under inclusion, so the concepts of prime fuzzy bi-hyperideals and strongly prime fuzzy bi-hyperideals coincide. Therefore by Theorem 3.3.19, H is both regular and intra-regular. □

3.3.26 Theorem

For a semihypergroup H, the following statements are equivalent:

(1) The set of bi-hyperideals of H is totally ordered under inclusion.

(2) Each bi-hyperideal of H is strongly irreducible.

(3) Each bi-hyperideal of H is irreducible.

Proof. (1) $\Rightarrow$ (2). Let B be a bi-hyperideal of H and B_1, B_2 be any arbitrary bi-hyperideals of H such that $B_1 \cap B_2 \subseteq B$. Since the set of bi-hyperideals of H is totally ordered under inclusion, so either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Thus either $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Hence $B_1 \cap B_2 \subseteq B$ implies either $B_1 \subseteq B$ or $B_2 \subseteq B$. This show that B is strongly irreducible.

(2) $\Rightarrow$ (3). Let B be a bi-hyperideal of H and B_1, B_2 be any arbitrary bi-hyperideals of H such that $B_1 \cap B_2 = B$. Then $B \subseteq B_1$ and $B \subseteq B_2$. By hypothesis, either $B_1 \subseteq B$ or $B_2 \subseteq B$. Hence, either $B_1 = B$ or $B_2 = B$. Thus B is irreducible.

(3) $\Rightarrow$ (1). Let B_1 and B_2 be any bi-hyperideals of H. Then $B_1 \cap B_2$ is a bi-hyperideal of H. So by hypothesis, either $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$, that is, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Hence the set of bi-hyperideals of H is totally ordered under inclusion. □

3.3.27 Theorem

For a semihypergroup H, the following statements are equivalent:

(1) The set of fuzzy bi-hyperideals of H is totally ordered under inclusion.

(2) Each fuzzy bi-hyperideal of H is strongly irreducible.

(3) Each fuzzy bi-hyperideal of H is irreducible.

Proof. (1) $\Rightarrow$ (2). Let λ be a fuzzy bi-hyperideal of H and μ, ν be any arbitrary fuzzy bi-hyperideals of H such that $\mu \wedge \nu \leq \lambda$. Since the set of fuzzy bi-hyperideals of H is totally ordered under inclusion, so either $\mu \leq \nu$ or $\nu \leq \mu$. Thus either $\mu \wedge \nu = \mu$ or $\mu \wedge \nu = \nu$. Hence $\mu \wedge \nu \leq \lambda$ implies either $\mu \leq \lambda$ or $\nu \leq \lambda$. Thus λ is strongly irreducible.

(2) $\Rightarrow$ (3). Let λ be a fuzzy bi-hyperideal of H and μ, ν be any arbitrary fuzzy bi-hyperideals of H such that $\mu \wedge \nu = \lambda$. Then $\lambda \leq \mu$ and $\lambda \leq \nu$. By hypothesis, either $\mu \leq \lambda$ or $\nu \leq \lambda$. Hence, either $\mu = \lambda$ or $\nu = \lambda$. Thus λ is irreducible.

(3) $\Rightarrow$ (1). Let μ and ν be any fuzzy bi-hyperideals of H. Then $\mu \wedge \nu$ is a bi-hyperideal of H. So by hypothesis, either $\mu = \mu \wedge \nu$ or $\nu = \mu \wedge \nu$, that is, either $\mu \leq \nu$ or $\nu \leq \mu$. Hence the set of fuzzy bi-hyperideals of H is totally ordered under inclusion. □

Chapter 4

Semihypergroups characterized by $(\in, \in \vee q)$-fuzzy hyperideals

After the introduction of fuzzy subgroup by Rosenfeld [73], many researchers generalized this concept. Using the notion of "belongingness" and "quasicoincidence" of fuzzy points and fuzzy sets, Bhakat and Das [8], introduced a new type of fuzzy subgroups, the $(\in, \in \vee q)$-fuzzy subgroups. Different authors applied this concept to define (α, β)-fuzzy substructures of algebraic structures (see [30], [33], [46]).

In this chapter, we introduce the concepts of $(\in, \in \vee q)$-fuzzy left (right) hyperideals, $(\in, \in \vee q)$-fuzzy quasi-hyperideal, $(\in, \in \vee q)$-fuzzy interior hyperideal and $(\in, \in \vee q)$-fuzzy bi-hyperideals of semihypergroups and characterize semihypergroups in terms of these notions.

4.1 (α, β)-fuzzy hyperideals

Let H be a semihypergroup. A fuzzy subset λ of H of the form

$$\lambda(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is called a *fuzzy point* with support x and value t and is denoted by x_t. A fuzzy point x_t is said to *belong to* (resp. *quasi-coincident* with) a fuzzy set λ, written as $x_t \in \lambda$ (resp. $x_t q \lambda$) if $\lambda(x) \geq t$ (resp. $\lambda(x) + t > 1$). In this case $x_t \in \vee q \lambda$ (resp. $x_t \in \wedge q \lambda$) means that $x_t \in \lambda$ or $x_t q \lambda$ (resp. $x_t \in \lambda$ and $x_t q \lambda$). The symbol $x_t \overline{\alpha} \lambda$ means $x_t \alpha \lambda$ does not hold.

In what follows let H denote a semihypergroup and α, β any one of $\in$, q, $\in \vee$q, $\in \wedge$q unless otherwise specified.

Let λ be a fuzzy subset of H such that $\lambda(x) \leq 0.5$ for all $x \in H$. Let $x \in H$ and $t \in (0,1]$ be such that $x_t \in \wedge q \lambda$. Then $\lambda(x) \geq t$ and $\lambda(x) + t \geq 1$. It follows that $1 < \lambda(x) + t < \lambda(x) + \lambda(x) = 2\lambda(x)$. This implies that $\lambda(x) > 0.5$. Hence $\{x_t | x_t \in \wedge q \lambda\} = \emptyset$. Thus the case $\alpha = \in \wedge q$ is omitted in the following definition.

4.1.1 Definition

A fuzzy subset λ of H is called an $(\in, \in \vee q)$-fuzzy subsemihypergroup of H if for all $x, y \in H$ and $t, r \in (0,1]$ the following condition holds

$$x_t \in \lambda, y_r \in \lambda \longrightarrow (z)_{\min\{t,r\}} \in \vee q \lambda, \text{ for each } z \in x \circ y.$$

4.1.2 Theorem

Let A be a subsemihypergroup of H and λ a fuzzy subset in H defined by

$$\lambda(x) = \begin{cases} \geq 0.5 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) A is a subsemihypergroup of H if and only if λ is a $(q, \in \vee q)$-fuzzy subsemihypergroup of H.

(2) A is a subsemihypergroup of H if and only if λ is an $(\in, \in \vee q)$-fuzzy subsemihypergroup of H.

Proof. (1) Let $x, y \in H$ and $t, r \in (0, 1]$ be such that $x_t, y_r q\lambda$. Then $\lambda(x) + t > 1$ and $\lambda(y) + r > 1$. This implies $x, y \in A$ Since A is a subsemihypergroup of H, we have $x \circ y \subseteq A$. Thus for every $z \in x \circ y, \lambda(z) \geq 0.5$. If $\min\{t, r\} \leq 0.5$, then $\lambda(z) \geq \min\{t, r\}$ and so $(z)_{\min\{t,r\}} \in \lambda$. If $\min\{t, r\} > 0.5$, then $\lambda(z) + \min\{t, r\} > 0.5 + 0.5 = 1$ and so $(z)_{\min\{t,r\}} q\lambda$. Therefore $(z)_{\min\{t,r\}} \in \vee q\lambda$.

Conversely, assume that λ is a $(q, \in \vee q)$-fuzzy subsemihypergroup of H and $x, y \in A$. Then $\lambda(x) \geq 0.5, \lambda(y) \geq 0.5$ that is $x_{0.5} q\lambda$ and $y_{0.5} q\lambda$. Now by hypothesis, $z_{0.5} \in \vee q\lambda$ for every $z \in x \circ y$. If $z_{0.5} \in \lambda$ then $\lambda(z) \geq 0.5$ and so $z \in A$. If $z_{0.5} q\lambda$ then $\lambda(z) + 0.5 > 1$ implies $\lambda(z) > 0.5$. Thus $z \in A$. Hence $x \circ y \subseteq A$, that is, A is a subsemihypergroup of H.

(2) Let $x, y \in H$ and $t, r \in (0, 1]$ be such that $x_t, y_r \in \lambda$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq r$. Thus $\lambda(x) \geq 0.5$ and $\lambda(y) \geq 0.5$. This implies $x, y \in A$. Since A is a subsemihypergroup of H, we have $x \circ y \subseteq A$. Thus for every $z \in x \circ y, \lambda(z) \geq 0.5$. If $\min\{t, r\} \leq 0.5$, then $\lambda(z) \geq \min\{t, r\}$ and so $(z)_{\min\{t,r\}} \in \lambda$. If $\min\{t, r\} > 0.5$, then $\lambda(z) + \min\{t, r\} > 0.5 + 0.5 = 1$ and so $(z)_{\min\{t,r\}} q\lambda$. Therefore $(z)_{\min\{t,r\}} \in \vee q\lambda$.

Conversely, assume that λ is a $(\in, \in \vee q)$-fuzzy subsemihypergroup of H and $x, y \in A$. Then $\lambda(x) \geq 0.5, \lambda(y) \geq 0.5$ that is $x_{0.5} \in \lambda$ and $y_{0.5} \in \lambda$. Now by hypothesis, $z_{0.5} \in \vee q\lambda$ for every $z \in x \circ y$. If $z_{0.5} \in \lambda$ then $\lambda(z) \geq 0.5$ and so $z \in A$. If $z_{0.5} q\lambda$ then $\lambda(z) + 0.5 > 1$ implies $\lambda(z) > 0.5$. Thus $z \in A$. Hence $x \circ y \subseteq A$, that is, A is a subsemihypergroup of H. □

4.1.3 Theorem

Let λ be a fuzzy subset of H. Then λ is an $(\in, \in \vee q)$-fuzzy subsemihypergroup of H if and only if $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), 0.5\}$.

Proof. Let λ be an $(\in, \in \vee q)$-fuzzy subsemihypergroup of H. On the contrary, assume that there exist $x, y \in H$ such that $\inf\limits_{z \in x \circ y} \{\lambda(z)\} < \min\{\lambda(x), \lambda(y), 0.5\}$. Then there exists $z \in x \circ y$ such that $\lambda(z) < \min\{\lambda(x), \lambda(y), 0.5\}$. Choose $t \in (0, 1]$ such that

$\lambda(z) < t \leq \min\{\lambda(x), \lambda(y), 0.5\}$. Then $x_t \in \lambda$ and $y_t \in \lambda$ but $\lambda(z) < t$ and $\lambda(z) + t \leq 0.5 + 0.5 = 1$, so $z_t \overline{\in \vee q} \lambda$, which is a contradiction.

Hence $\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), 0.5\}$.

Conversely, assume that $\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), 0.5\}$. Let $x_t \in \lambda$ and $y_r \in \lambda$ for $t, r \in (0, 1]$. Then $\lambda(x) > t$ and $\lambda(y) > r$. Now

$$\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), 0.5\} \geq \min\{t, r, 0.5\}.$$

If $t \wedge r > 0.5$, then $\inf_{z \in x \circ y} \{\lambda(z)\} \geq 0.5$. So for every $z \in x \circ y, \lambda(z) + t \wedge r > 0.5 + 0.5 = 1$, which implies that $(z)_{\min\{t,r\}} \in \vee q\lambda$. If $t \wedge r \leq 0.5$, then $\inf_{z \in x \circ y} \{\lambda(z)\} \geq t \wedge r$. So $(z)_{\min\{t,r\}} \in \lambda$. Thus $(z)_{\min\{t,r\}} \in \vee q\lambda$. Therefore λ is an $(\in, \in \vee q)$-fuzzy subsemihypergroup of H. □

4.1.4 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in, \in \vee q)$-fuzzy subsemihypergroup of H if and only if $U(\lambda; t) (\neq \emptyset)$ is a subsemihypergroup of H for all $t \in (0, 0.5]$.

Proof. Let λ be an $(\in, \in \vee q)$-fuzzy subsemihypergroup of H and $x, y \in U(\lambda; t)$ for some $t \in (0, 0.5]$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq t$. It follows from Theorem 4.1.3 that $\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), 0.5\} \geq \min\{t, 0.5\} = t$. Thus for every $z \in x \circ y$,

$\lambda(z) \geq t$ and so $z \in U(\lambda; t)$ that is $x \circ y \subseteq U(\lambda; t)$. Hence $U(\lambda; t)$ is a subsemihypergroup of H.

Conversely, assume that $U(\lambda; t) (\neq \emptyset)$ is a subsemihypergroup of H for all $t \in (0, 0.5]$. Suppose that there exist $x, y \in H$ such that

$$\inf_{z \in x \circ y} \{\lambda(z)\} < \min\{\lambda(x), \lambda(y), 0.5\}.$$

This implies there exists $z \in x \circ y$ such that $\lambda(z) < \min\{\lambda(x), \lambda(y), 0.5\}$. Choose $t \in (0, 0.5]$ such that $\lambda(z) < t < \min\{\lambda(x), \lambda(y), 0.5\}$. Then $x, y \in U(\lambda; t)$ but $z \notin U(\lambda; t)$ i.e. $x \circ y \nsubseteq U(\lambda; t)$, which contradicts our hypothesis.

Hence $\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), 0.5\}$ and so λ is an $(\in, \in \vee q)$-fuzzy subsemihypergroup of H. □

4.1.5 Definition

Let $(H, \circ)$ be a semihypergroup and λ be a fuzzy subset of H. Then λ is called an (α, β)-fuzzy left (resp. right) hyperideal of H if for all $t \in (0, 1]$ and for all $x, y \in H$, we have

(I_1) $y_t \alpha \lambda \longrightarrow z_t \beta \lambda$ for each $z \in x \circ y$ (resp. for each $z \in y \circ x$).

A fuzzy subset λ of H is called an (α, β)-fuzzy hyperideal of H if it is both (α, β)-fuzzy left hyperideal and (α, β)-fuzzy right hyperideal of H.

4.1.6 Theorem

For a fuzzy subset λ of H, the condition (I_2) is equivalent to the conditions (I_3), where (I_2) and (I_3) are given be low:

(I_2) For all $x, y, z \in H$, $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq \lambda(y)$ (resp. $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq \lambda(x)$).

(I_3) For all $x, y \in H$ and $t \in (0, 1]$, $y_t \in \lambda \longrightarrow z_t \in \lambda$ for each $z \in x \circ y$ (resp. for each $z \in y \circ x$).

Proof. $(I_2) \longrightarrow (I_3)$ Let $x, y \in H$ and $t \in (0, 1]$ be such that $y_t \in \lambda$. Then $\lambda(y) \geq t$. By (I_2), we have $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq \lambda(y) \geq t$. It follows that $z_t \in \lambda$ for each $z \in x \circ y$.

$(I_3) \longrightarrow (I_2)$ Let $x, y \in H$. If $\lambda(y) = 0$, then $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq \lambda(y)$. If $\lambda(y) \neq 0$, then $y_{\lambda(y)} \in \lambda$. Thus by hypothesis $z_{\lambda(y)} \in \lambda$ for each $z \in x \circ y$. Thus it follows that $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq \lambda(y)$. □

4.1.7 Theorem

A fuzzy subset λ of a semihypergroup H is a fuzzy left (resp. right) hyperideal of H if and only if λ is an $(\in, \in)$-fuzzy left (resp. right) hyperideal of H.

Proof. Suppose λ is a fuzzy left hyperideal of H. Let $x, y \in H$ and $t \in (0, 1]$ be such that $y_t \in \lambda$. Then $\lambda(y) \geq t$. By hypothesis $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq \lambda(y)$. Thus $\lambda(z) \geq t$ for all $z \in x \circ y$, that is $z_t \in \lambda$ for all $z \in x \circ y$. Hence λ is an $(\in, \in)$-fuzzy left hyperideal of H.

Conversely, assume that λ is an $(\in, \in)$-fuzzy left hyperideal of H. Let $x, y \in H$. If $\lambda(y) = 0$, then $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq 0 = \lambda(y)$. Suppose $\lambda(y) \neq 0$. Then $y_{\lambda(y)} \in \lambda$. Thus by hypothesis $z_{\lambda(y)} \in \lambda$ for all $z \in x \circ y$, that is $\lambda(z) \geq \lambda(y)$ for all $z \in x \circ y$. This implies that $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq \lambda(y)$. Hence λ is a fuzzy left hyperideal of H. □

4.1.8 Theorem

Every $(\in, \in)$-fuzzy left (resp. right) hyperideal of H is an $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideal.

Proof. Straighforward. □

The converse of the above Theorem is not true in general as shown in the Example 4.1.15.

4.1.9 Theorem

Every $(\in \vee q, \in \vee q)$-fuzzy left (resp. right) hyperideal is an $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideal.

Proof. Let λ be an $(\in \vee q, \in \vee q)$-fuzzy left hyperideal of H. Let $x, y \in H$ and $t \in (0,1]$ be such that $y_t \in \lambda$. Then $y_t \in \vee q\lambda$ and hence $z_t \in \vee q\lambda$ for every $z \in x \circ y$. □

4.1.10 Theorem

Let λ be an (α, β)-fuzzy left (resp. right) hyperideal of H. Then the set $\lambda_0 = \{x \in H | \lambda(x) > 0\}$ is a left (resp. right) hyperideal of H.

Proof. Let λ be an (α, β)-fuzzy left hyperideal of H and $x, y \in H$ be such that $y \in \lambda_0$. Assume that $\lambda(z) = 0$ for some $z \in x \circ y$. If $\alpha \in \{\in, \in \vee q\}$ then $y_{\lambda(y)}\alpha\lambda$ but there exists $z \in x \circ y, z_{\lambda(z)}\overline{\beta}\lambda$ for $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Note that $y_1 q\lambda$ but there exists $z \in x \circ y, z_1\overline{\beta}\lambda$ for $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Hence $\lambda(z) > 0$ for each $z \in x \circ y$, that is, $z \in \lambda_0$. Consequently, λ_0 is a left hyperideal of H. Similarly we can prove that λ_0 is a right hyperideal of H. □

4.1.11 Theorem

Let I be a non-empty subset of a semihypergroup H. Define a fuzzy subset λ of H by

$$\lambda(x) = \begin{cases} t \geq 0.5 & \text{if } x \in I \\ 0 & \text{if } x \notin I. \end{cases}$$

Then λ is an $(\alpha, \in \vee q)$-fuzzy left (resp. right) hyperideal of H if and only if I is a left (resp. right) hyperideal of H.

Proof. Case I $\alpha = \in$. Let λ be an $(\in, \in \vee q)$-fuzzy left hyperideal of H and $y \in I$. Then $\lambda(y) \geq 0.5$. By hypothesis

$$\inf_{z \in x \circ y} \{\lambda(z)\} \geq \lambda(y) \wedge 0.5 = 0.5.$$

Thus $\lambda(z) \geq 0.5$ for all $z \in x \circ y$, that is $x \circ y \subseteq I$. Hence I is a left hyperideal of H.

Conversely, assume that I is a left hyperideal of H. Let $x, y \in H$. If $\lambda(y) = 0$, then $\inf_{z \in x \circ y} \{\lambda(z)\} \geq 0 = \lambda(y) \wedge 0.5$.

If $\lambda(y) \geq 0.5$, then $y \in I$. By hypothesis $x \circ y \subseteq I$, so $\lambda(z) \geq 0.5$ for every $z \in x \circ y$. Thus

$$\inf_{z \in x \circ y} \{\lambda(z)\} \geq 0.5 = \lambda(y) \wedge 0.5.$$

Hence λ is an $(\in, \in \vee q)$-fuzzy left hyperideal of H.

Case II $\alpha = q$. Let λ be an $(q, \in \vee q)$-fuzzy left hyperideal of H. Let $y \in I$. Then $\lambda(y) \geq 0.5$. Thus $y_{0.51} q\lambda$. Hence by hypothesis $z_{0.51} \in \vee q\lambda$ for every $z \in x \circ y$. If $z_{0.51} \in \lambda$ then $z \in I$. If $z_{0.5} q\lambda$ then also $z \in I$. Hence $x \circ y \subseteq I$. This shows that I is a left hyperideal of H.

Coversely, assume that I is a left hyperideal of H. Let $x, y \in H$ be such that $y_t q\lambda$. Then $\lambda(y)+t > 1$. This implies $\lambda(y) \geq 0.5$ that is $y \in I$. By hypothesis $x \circ y \subseteq I$. Thus $\lambda(z) \geq 0.5$. If $t \leq 0.5$ then $\lambda(z) \geq t$ and so $z_t \in \lambda$. If $t > 0.5$, then $\lambda(z) + t > 1$, that is $z_t q\lambda$. Hence $z_t \in \vee q\lambda$ for every $z \in x \circ y$. This shows that λ is an $(q, \in \vee q)$-fuzzy left hyperideal of H.

Case III $\alpha = \in \vee q$. The proof is similar to the proofs of above parts. □

4.1.12 Corollary

Let I be a non-empty subset of a semihypergroup H. Then I is a left (resp. right) hyperideal of H if and only if the characteristic function of I is an $(\alpha, \in \vee q)$-fuzzy left (resp. right) hyperideal of H.

4.1.13 Theorem

Every $(q, \in \vee q)$-fuzzy left (resp. right) hyperideal of H is an $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideal.

Proof. Suppose λ is a $(q, \in \vee q)$-fuzzy left hyperideal of H. Let $x, y \in H$ and $t \in \lambda(z)(0, 1]$ be such that $x_t \in \lambda$. Then $\lambda(x) \geq t$. Suppose there exists $z \in y \circ x$ such that $z_t \overline{\in \vee q}\lambda$. Then $\lambda(z) < t$ and $\lambda(z) + t \leq 1$. This implies $2\lambda(z) < \lambda(z) + t \leq 1$, that is $\lambda(z) < 0.5$. Now $\lambda(z) < t \leq \lambda(x)$. Thus $\lambda(z) < \min\{\lambda(x), 0.5\}$. This implies

$$1 - \min\{\lambda(x), 0.5\} < 1 - \lambda(z) \Longrightarrow \max\{1 - \lambda(x), 0.5\} < 1 - \lambda(z).$$

□

Select $r \in (0, 1]$ such that $\max\{1 - \lambda(x), 0.5\} < r \leq 1 - \lambda(z)$. Then $r > 0.5, r > 1 - \lambda(x)$ and $r \leq 1 - \lambda(z)$. This implies $\lambda(x) + r > 1$, that is $x_t q\lambda$. As $\lambda(z) + r \leq 1$, so $z_r \overline{q}\lambda$. Also $\lambda(z) \leq 1 - r < r$, because $r > 0.5$. This implies $z_r \overline{\in}\lambda$. Hence $z_r \overline{\in \vee q}\lambda$. Which is a contradiction. Hence $z_t \in \vee q\lambda$ for all $z \in y \circ x$, that is λ is an $(\in, \in \vee q)$-fuzzy left hyperideal of H.

4.1.14 Remark

Every $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideal of H is not a $(q, \in \vee q)$-fuzzy left (resp. right) hyperideal.

4.1.15 Example

Consider the semihypergroup $H = \{a, b, c, d, e\}$

$\circ$	a	b	c	d	e
a	a	$\{a, b, d\}$	a	$\{a, b, d\}$	$\{a, b, d\}$
b	a	b	a	$\{a, b, d\}$	$\{a, b, d\}$
c	a	$\{a, b, d\}$	$\{a, c\}$	$\{a, b, d\}$	$\{a, b, c, d, e\}$
d	a	$\{a, b, d\}$	a	$\{a, b, d\}$	$\{a, b, d\}$
e	a	$\{a, b, d\}$	$\{a, c\}$	$\{a, b, d\}$	$\{a, b, c, d, e\}$

The only hyperideals of H are $\{a, b, d\}$ and H.
Define a fuzzy subset λ of H as follows:

$$\lambda(a) = 0.8,\ \lambda(b) = 0.7,\ \lambda(c) = 0.3,\ \lambda(d) = 0.5,\ \lambda(e) = 0.3.$$

Then

$$U(\lambda; t) = \begin{cases} H & \text{if } 0 < t \leq 0.3 \\ \{a, b, d\} & \text{if } 0.3 < t \leq 0.5 \\ \{a, b\} & \text{if } 0.5 < t \leq 0.7 \\ \{a\} & 0.7 < t \leq 0.8 \\ \emptyset & \text{if } 0.8 < t \end{cases}$$

Then

(1) λ is an $(\in, \in \vee q)$-fuzzy hyperideal of H.

(2) λ is not an $(\in, \in)$-fuzzy hyperideal of H, because $a_{0.7} \in \lambda$ but $d \in a \circ b$ and $d_{0.7} \overline{\in} \lambda$.

(3) λ is not an $(\in, q)$-fuzzy hyperideal of H, because $b_{0.2} \in \lambda$ but $d \in a \circ b$ and $d_{0.2} \overline{q} \lambda$.

(4) λ is not an $(\in, \in \wedge q)$-fuzzy hyperideal of H.

4.1.16 Definition

A fuzzy subset λ of a semihypergroup H is called an (α, β)-fuzzy generalized bi-hyperideal of H if for all $x, y, z \in H$ and $t, r \in (0, 1]$ the following condition holds;

$$x_t \alpha \lambda \text{ and } z_r \alpha \lambda \longrightarrow (w)_{\min\{t,r\}} \beta \lambda \text{ for every } w \in x \circ y \circ z.$$

4.1.17 Theorem

Let B be a non-empty subset of H and λ be a fuzzy subset in H defined by

$$\lambda(x) = \begin{cases} \geq 0.5 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) *B is a generalized bi-hyperideal of H if and only if λ is a $(q, \in \vee q)$-fuzzy generalized bi-hyperideal of H.*

(2) *B is a generalized bi-hyperideal of H if and only if λ is a $(\in, \in \vee q)$-fuzzy generalized bi-hyperideal of H.*

Proof. The proof is similar to the proof of Theorem 4.1.2. □

4.1.18 Definition

A fuzzy subset λ of a semihypergroup H is called an (α, β)-fuzzy interior hyperideal of H if for all $x, y, z \in H$ and $t, r \in (0, 1]$ the following conditions hold;

(1) $x_t \alpha \lambda$ and $y_r \alpha \lambda \longrightarrow (z)_{\min\{t,r\}} \beta \lambda$ for every $z \in x \circ y$,

(2) $a_t \alpha \lambda \longrightarrow (w)_{\min\{t,r\}} \beta \lambda$ for every $w \in x \circ a \circ y$.

4.1.19 Theorem

Let A be a non-empty subset of H and λ be a fuzzy subset in H such that

$$\lambda(x) = \begin{cases} \geq 0.5 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) *If A is an interior hyperideal of H then λ is a $(q, \in \vee q)$-fuzzy interoir hyperideal of H.*

(2) *A is an interior hyperideal of H if and only if λ is an $(\in, \in \vee q)$-fuzzy interoir hyperideal of H.*

Proof. The proof is similar to the proof of Theorem 4.1.2. □

4.1.20 Definition

A fuzzy subset λ of H is called an (α, β)-fuzzy bi-hyperideal of H, where $\alpha \neq \in \wedge q$, if for all $x, y, z \in H$ and for all $t, r \in (0, 1]$ it satisfies:

(*i*) $x_t, y_r \alpha \lambda \longrightarrow (z)_{\min\{t,r\}} \beta \lambda$ for every $z \in x \circ y$.

(*ii*) $x_t, z_r \alpha \lambda \longrightarrow (w)_{\min\{t,r\}} \beta \lambda$ for every $w \in x \circ y \circ z$.

4.1.21 Theorem

Every $(\in, \in)$-fuzzy bi-hyperideal is an $(\in, \in \vee q)$-fuzzy bi-hyperideal.

Proof. Straightforward. □

4.1.22 Theorem

Every $(\in \vee q, \in \vee q)$-fuzzy bi-hyperideal is $(\in, \in \vee q)$-fuzzy bi-hyperideal.

Proof. Straightforward. □

4.2 $(\in, \in \vee q)$-fuzzy hyperideals

Every (α, β)-fuzzy left (resp. right) hyperideal of a semihypergroup H is an $(\alpha, \in \vee q)$-fuzzy left (resp. right) hyperideal of H. Theorems 4.1.8, 4.1.9 and 4.1.13 shows that $(\alpha, \in \vee q)$-fuzzy left (resp. right) hyperideal of a semihypergroup H is an $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideal of H. Thus in the theory of (α, β)-fuzzy left (resp. right) hyperideals of H, $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideals play central role.

In this section we study $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideals of a semihypergroup H.

4.2.1 Proposition

For a fuzzy subset λ of a semihypergroup H, the condition (I_4) is equivalent to (I_5), where (I_4) and (I_5) are as follows:

(I_4) For all $x, y \in H$ and $t \in (0,1], y_t \in \lambda \longrightarrow z_t \in \lambda$ for each $z \in x \circ y$ (resp. for every $z \in y \circ x$).

(I_5) For all $x, y \in H$, $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq \{\lambda(y) \wedge 0.5\}$ (resp. $\inf\limits_{z \in y \circ x} \{\lambda(z)\} \geq \{\lambda(x) \wedge 0.5\}$)).

Proof. $(I_4) \longrightarrow (I_5)$ Let $x, y \in H$ be such that

$$\inf_{z \in x \circ y} \{\lambda(z)\} < \{\lambda(y) \wedge 0.5\}.$$

Then there exists $z \in x \circ y$ such that $\lambda(z) < \lambda(y) \wedge 0.5$. If $\lambda(y) < 0.5$, then $\lambda(z) < \lambda(y) < 0.5$. Thus $y_{\lambda(y)} \in \lambda$ but $z_{\lambda(y)} \overline{\in} \wedge \overline{q}\lambda$ which is a contradiction. If $\lambda(y) \geq 0.5$, then $\lambda(z) < 0.5$. Thus $y_{0.5} \in \lambda$ but $z_{0.5} \overline{\in} \wedge \overline{q}\lambda$, again a contradiction. Hence

$$\inf_{z \in x \circ y} \{\lambda(z)\} \geq \lambda(y) \wedge 0.5.$$

$(I_5) \longrightarrow (I_4)$ Let $x, y \in H$, and and $t \in (0,1]$ be such that $y_t \in \lambda$. Then $\lambda(y) \geq t$. By hypothesis,

$$\inf_{z \in x \circ y} \{\lambda(z)\} \geq \lambda(y) \wedge 0.5 \geq t \wedge 0.5.$$

This implies that $\lambda(z) \geq t \wedge 0.5$ for every $z \in x \circ y$. If $t > 0.5$, then $\lambda(z) \geq 0.5$. This implies that $\lambda(z) + t > 0.5 + 0.5 = 1$, that is $z_t q \lambda$ for every $z \in x \circ y$. If $t \leq 0.5$, then $\lambda(z) \geq$ for every $z \in x \circ y$. This implies $z_t \in \lambda$ for every $z \in x \circ y$. Hence $z_t \in \vee q \lambda$ for every $z \in x \circ y$. □

4.2.2 Corollary

A fuzzy subset λ of a semihypergroup H is an $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideal of H if and only if it satisfies condition (I_5).

Now, we characterize $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideals by their level sets.

4.2.3 Theorem

Let λ be a fuzzy subset of a semihypergroup H. Then λ is an $(\in, \in \vee q)$-fuzzy left(resp. right) hyperideal of H if and only if $U(\lambda;t)(\neq \emptyset)$ is a left (resp. right) hyperideal of H for all $t \in (0, 0.5]$.

Proof. Let λ be an $(\in, \in \vee q)$-fuzzy left hyperideal of H and $t \in (0, 0.5]$. Let $x \in H$ and $y \in U(\lambda;t)$. Then $\lambda(y) \geq t$. By hypothesis $\inf\limits_{z \in x o y} \{\lambda(z)\} \geq \lambda(y) \wedge 0.5 \geq t \wedge 0.5 = t$, so $z \in U(\lambda;t)$ for every $z \in xoy$. Thus $xoy \subseteq U(\lambda;t)$. Hence $U(\lambda;t)$ is a left hyperideal of H.

Conversely, assume that $U(\lambda;t) \neq \emptyset$ are left hyperideals of H for all $t \in (0, 0.5]$. Let $x, y \in H$ be such that $\inf\limits_{z \in x o y} \{\lambda(z)\} < \lambda(y) \wedge 0.5$. Then there exists $z \in x \circ y$ such that $\lambda(z) < \lambda(y) \wedge 0.5$. Select $t \in (0, 0.5]$ such that $\lambda(z) < t \leq \lambda(y) \wedge 0.5$. Then $y \in U(\lambda;t)$ but $z \notin U(\lambda;t)$, that is $x \circ y \nsubseteq U(\lambda;t)$. Which is a contradiction. Hence $\inf\limits_{z \in x o y} \{\lambda(z)\} \geq \lambda(y) \wedge 0.5$. □

It is clear from Proposition 1.3.17 that a fuzzy subset λ of a semihypergroup H is a fuzzy left (resp. right) hyperideal of H if and only if $U(\lambda;t)(\neq \emptyset)$ is a left (resp. right) hyperideal of H for all $t \in (0,1]$ and from Theorem 4.2.3, λ is an $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideal of H if and only if $U(\lambda;t)(\neq \emptyset)$ is a left (resp. right) hyperideal of H for all $t \in (0, 0.5]$.

For any fuzzy subset λ of a semihypergroup H and $t \in (0,1]$ we denote by:

$$Q(\lambda;t) := \{x \in H | x_t q \lambda\} \text{ and } [\lambda]_t := \{x \in H | x_t \in \vee q \lambda\}.$$

Obviously $[\lambda]_t = U(\lambda;t) \cup Q(\lambda;t)$.

We call $[\lambda]_t$ an $(\in \vee q)$-level set of λ and $Q(\lambda;t)$ a q-level set of λ.

In Theorem 4.2.3, we gave a characterization of $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideal of H by using level subsets. Now, we give another characterization of $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideals by using $[\lambda]_t$.

4.2.4 Theorem

Let λ be a fuzzy subset of a semihypergroup H. Then λ is an $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideal of H if and only if $[\lambda]_t$ is a left (resp. right) hyperideal of H for all $t \in (0,1]$.

Proof. Let λ be an $(\in, \in \vee q)$-fuzzy left hyperideal of H. Let $x \in H$ and $y \in [\lambda]_t$ for $t \in (0,1]$. Then $y_t \in \vee q \lambda$, that is $\lambda(y) \geq t$ or $\lambda(y) + t > 1$. Since λ is an $(\in, \in \vee q)$-fuzzy left hyperideal of H, we have

$$\inf_{z \in x o y} \{\lambda(z)\} \geq \{\lambda(y) \wedge 0.5\}.$$

Case 1. Let $\lambda(y) \geq t$. If $t > 0.5$, then $\inf_{z\in x\circ y} \{\lambda(z)\} \geq \{\lambda(y) \wedge 0.5\} = 0.5$ and so

$$\lambda(z) + t > 0.5 + 0.5 = 1,$$

hence for every $z \in x \circ y$, $z_t q\lambda$. If $t \leq 0.5$, then $\inf_{z\in x\circ y} \{\lambda(z)\} \geq \{\lambda(y) \wedge 0.5\} \geq t$, and hence $z_t \in \lambda$.

Case 2. $\lambda(y) + t > 1$. If $t > 0.5$, then

$$\inf_{z\in x\circ y} \{\lambda(z)\} \geq \{\lambda(y) \wedge 0.5\} > 1 - t \wedge 0.5 = 1 - t,$$

that is, $\inf_{z\in x\circ y} \{\lambda(z)\} + t > 1$ and thus $z_t q\lambda$. If $t \leq 0.5$, then

$$\inf_{z\in x\circ y} \{\lambda(z)\} \geq \{\lambda(y) \wedge 0.5\} > 1 - t \wedge 0.5 = 0.5 \geq t,$$

and so $z_t \in \lambda$.

Hence $z_t \in \vee \mathrm{q}\lambda$. Consequently, $z_t \in \vee \mathrm{q}\lambda$ for every $z \in x \circ y$. Thus $z \in [\lambda]_t$ for every $z \in x \circ y$. Hence $[\lambda]_t$ is a left hyperideal of H.

Conversely, let λ be a fuzzy subset of H and $[\lambda]_t$ be a left hyperideal of H for every $t \in (0, 1]$. Let $x, y \in H$ be such that $\inf_{z\in x\circ y} \{\lambda(z)\} < \{\lambda(y) \wedge 0.5\}$ for some $t \in (0, 0.5]$. This implies there exists $z \in x \circ y$ such that $\lambda(z) < \lambda(y) \wedge 0.5$. Select $t \in (0, 1]$ such that $\lambda(z) < t \leq \lambda(y) \wedge 0.5$. Then $y_t \in \lambda$ but $z_t \overline{\in \vee q}\lambda$, that is $y \in [\lambda]_t$ but $z \notin [\lambda]_t$. This is a contradiction. Hence, $\inf_{z\in x\circ y} \{\lambda(z)\} \geq \{\lambda(y) \wedge 0.5\}$ for all $x, y \in H$. This shows that λ is an $(\in, \in \vee \mathrm{q})$-fuzzy left hyperideal of H. □

From Theorem 4.2.3 and 4.2.4, we see that if λ is an $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideal of H then, $U(\lambda; t)$ and $[\lambda]_t$ are left (resp. right) hyperideals of H for all $t \in (0, 0.5]$, but $Q(\lambda; t)$ is not a left hyperideal of H for $t \in (0, 0.5]$ in general as shown in the following example.

4.2.5 Example

Consider the Example 4.1.15. Then $Q(\lambda; 0.3)=\{a\}$ is not a hyperideal of H.

4.2.6 Definition

Let H be a semihypergroup and λ, μ are fuzzy subsets of H. Then the 0.5-product of λ and μ is defined by:

$$(\lambda \circ_{0.5} \mu)(x) := \begin{cases} (\lambda \circ \mu) \wedge 0.5 & \text{if } A_x \neq \emptyset \\ 0 & \text{if } A_x = \emptyset, \end{cases}$$

where $A_x = \{(y, z) \in H \times H : x \in y \circ z\}$.

We also define $\lambda \cap_{0.5} \mu$ by $(\lambda \cap_{0.5} \mu)(x) = \min\{\lambda(x), \mu(x), 0.5\}$ for all $x \in H$.

4.2.7 Proposition

If $(H, \circ)$ is a semihypergroup and $\lambda, \mu, \nu, \delta$ are fuzzy subsets of H such that $\lambda \subseteq \nu$ and $\mu \subseteq \delta$. Then $\lambda \circ_{0.5} \mu \subseteq \nu \circ_{0.5} \delta$.

Proof. Straightforward. □

4.2.8 Lemma

Let H be a semihypergroup. If λ and μ are $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideals of H. Then $\lambda \cap_{0.5} \mu$ is an $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideal of H.

Proof. Let λ and μ be $(\in, \in \vee \mathrm{q})$-fuzzy left hyperideals of H. Let $x, y \in H$ Then

$$\begin{aligned}
\inf_{z \in x \circ y} (\lambda \cap_{0.5} \mu)(z) &= \inf_{z \in x \circ y} (\min\{\lambda(z), \mu(z), 0.5\}) \\
&= (\inf_{z \in x \circ y} \{\lambda(z)\}) \wedge (\inf_{z \in x \circ y} \{\mu(z)\}) \wedge 0.5 \\
&\geq \min\{\min\{\lambda(y), 0.5\}, \min\{\mu(y), 0.5\}, 0.5\} \\
&= \min\{\lambda(y), \mu(y), 0.5\} \\
&= (\lambda \cap_{0.5} \mu)(y).
\end{aligned}$$

Hence $\lambda \cap_{0.5} \mu$ is an $(\in, \in \vee \mathrm{q})$-fuzzy left hyperideal of H. □

4.2.9 Theorem

Let H be a semihypergroup. If λ is an $(\in, \in \vee q)$-fuzzy right hyperideal and μ is an $(\in, \in \vee q)$-fuzzy left hyperideal of H, respectively. Then $\lambda \circ_{0.5} \mu \leq \lambda \cap_{0.5} \mu$.

Proof. Let λ be an $(\in, \in \vee \mathrm{q})$-fuzzy right hyperideal and μ be an $(\in, \in \vee \mathrm{q})$-fuzzy left hyperideal of H and $x \in H$. If $A_x = \emptyset$, then $(\lambda \circ_{0.5} \mu)(x) = 0 \leq (\lambda \cap_{0.5} \mu)(x)$. If $A_x \neq \emptyset$, then

$$(\lambda \circ_{0.5} \mu)(x) := \bigvee_{(y,z) \in A_x} \min\{\lambda(y), \mu(z), 0.5\}.$$

Since $x \in y \circ z$, and λ is an $(\in, \in \vee \mathrm{q})$-fuzzy right hyperideal and μ an $(\in, \in \vee \mathrm{q})$-fuzzy left hyperideal of H, we have

$$\inf_{x \in y \circ z} \lambda(x) \geq \min\{\lambda(y), 0.5\}$$

and

$$\inf_{x \in y \circ z} \mu(x) \geq \min\{\mu(z), 0.5\}.$$

Thus, $\min\{\lambda(y), \mu(z), 0.5\} \leq \min\{\lambda(x), \mu(x), 0.5\} = (\lambda \cap_{0.5} \mu)(x)$. This implies

$$(\lambda \circ_{0.5} \mu)(x) = \inf_{(y,z) \in A_x} \min\{\lambda(y), \mu(z), 0.5\} \leq (\lambda \cap_{0.5} \mu)(x).$$

□

4.2.10 Theorem

If λ is an $(\in, \in \vee q)$-fuzzy left hyperideal and μ is an $(\in, \in \vee q)$-fuzzy right hyperideal of H then $\lambda \circ \mu$ is an $(\in, \in \vee q)$-fuzzy hyperideal of H.

Proof. Let λ be an $(\in, \in \vee q)$-fuzzy left hyperideal and μ is an $(\in, \in \vee q)$-fuzzy right hyperideal of H and $x, y \in H$. Then

$$\begin{aligned}(\lambda \circ \mu)(y) \wedge 0.5 &= \left(\bigvee_{(p,q) \in A_y} \{\lambda(p) \wedge \mu(q)\} \right) \wedge 0.5 \\ &= \bigvee_{(p,q) \in A_y} \{\lambda(p) \wedge \mu(q) \wedge 0.5\} \\ &= \bigvee_{(p,q) \in A_y} \{(\lambda(p) \wedge 0.5) \wedge (\mu(q) \wedge 0.5)\}.\end{aligned}$$

If $y \in p \circ q$, then $x \circ y \subseteq x \circ (p \circ q) = (x \circ p) \circ q$. Since λ is an $(\in, \in \vee q)$-fuzzy left hyperideal of H, so

$$\inf_{a \in x \circ p} \{\lambda(a)\} \geq \min\{\lambda(p), 0.5\}.$$

Thus

$$\begin{aligned}(\lambda \circ \mu)(y) \wedge 0.5 &= \bigvee_{(p,q) \in A_y} \{\lambda(p) \wedge 0.5 \wedge \mu(q)\} \\ &\leq \bigvee_{z \in x \circ y \subseteq a \circ q} \{\lambda(a) \wedge \mu(q)\} \text{ because } \inf_{a \in x \circ p} \{\lambda(a)\} \geq \min\{\lambda(p), 0.5\} \\ &= \bigvee_{z \in a \circ q} \{\lambda(a) \wedge \mu(q)\} \\ &= (\lambda \circ \mu)(z), \text{ for every } z \in x \circ y \subseteq a \circ q.\end{aligned}$$

So

$$\min\{(\lambda \circ \mu)(y), 0.5\} \leq (\lambda \circ \mu)(z), \text{ for every } z \in x \circ y.$$

Similarly we can show that for every $z \in x \circ y, (\lambda \circ \mu)(z) \geq \min\{(\lambda \circ \mu)(x), 0.5\}$. Thus $\lambda \circ \mu$ is an $(\in, \in \vee q)$-fuzzy two-sided hyperideal of H. □

4.2.11 Lemma

The union of any family of $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideals of H is an $(\in, \in \vee q)$-fuzzy left (resp. right) hyperideal of H.

Proof. Let $\{\lambda_i\}_{i \in I}$ be a family of $(\in, \in \vee q)$-fuzzy left hyperideals of H and $x, y \in H$. Then $\inf\limits_{z \in x \circ y} (\bigvee\limits_{i \in I} \lambda_i)(z)) = \bigvee\limits_{i \in I} \{ \inf\limits_{z \in x \circ y} (\lambda_i(z))\}$.

(Since each λ_i is an $(\in, \in \vee q)$-fuzzy left hyperideal of H, so $\inf\limits_{z\in x\circ y}\{\lambda_i\}(z) \geq \min\{\lambda_i(y), 0.5\}$ for all $i \in I$.)

Thus

$$\begin{aligned}\inf_{z\in x\circ y}\{\left(\bigvee_{i\in I}\lambda_i\right)(z)\} &= \bigvee_{i\in I}\{\inf_{z\in x\circ y}\{\lambda_i\}(z)\} \\ &\geq \bigvee_{i\in I}\{\lambda_i(y)\wedge 0.5\} \\ &= \left(\bigvee_{i\in I}\lambda_i(y)\right)\wedge 0.5 \\ &= \left(\bigvee_{i\in I}\lambda_i\right)(y)\wedge 0.5.\end{aligned}$$

Hence $\bigvee\limits_{i\in I}\lambda_i$ is an $(\in, \in \vee q)$-fuzzy left hyperideal of H. □

4.2.12 Definition

An $(\in, \in \vee q)$-fuzzy subsemihypergroup λ of a semihypergroup H is called an $(\in, \in \vee q)$-fuzzy bi-hyperideal of H if for all $x, y, z \in H$ and $t, r \in (0, 1]$ the following condition holds;

$$x_t \in \lambda \text{ and } z_r \in \lambda \longrightarrow (w)_{\min\{t,r\}} \in \vee q\lambda \text{ for every } w \in x \circ y \circ z.$$

4.2.13 Theorem

Let B be a non-empty subset of H and λ be a fuzzy subset in H defined by

$$\lambda(x) = \begin{cases} \geq 0.5 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) B is a bi-hyperideal of H if and only if λ is a $(q, \in \vee q)$-fuzzy bi-hyperideal of H.

(2) B is a bi-hyperideal of H if and only if λ is a $(\in, \in \vee q)$-fuzzy bi-hyperideal of H.

Proof. The proof is similar to the proof of Theorem 4.1.2. □

4.2.14 Corollary

(1) If a non-empty subset B of a semihypergroup H is a bi-hyperideal of H then the characteristic function of B is a $(q, \in \vee q)$-fuzzy bi-hyperideal of H.

(2) A non-empty subset B of a semihypergroup H is a bi-hyperideal of H if and only if λ_B is an $(\in, \in \vee q)$-fuzzy bi-hyperideal of H.

4.2.15 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in, \in \vee q)$-fuzzy bi-hyperideal of H if and only if it satisfies the following conditions,

(1) $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), 0.5\}$ *for all* $x, y \in H$,

(2) $\inf\limits_{w \in x \circ y \circ z} \{\lambda(w)\} \geq \min\{\lambda(x), \lambda(z), 0.5\}$ *for all* $x, y, z \in H$.

Proof. The proof is similar to the proof of Theorem 4.1.3. □

4.2.16 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in, \in \vee q)$-fuzzy bi-hyperideal of H if and only if $U(\lambda; t)(\neq \emptyset)$ is a bi-hyperideal of H for all $t \in (0, 0.5]$.

Proof. The proof is similar to the proof of Theorem 4.1.4. □

4.2.17 Definition

A fuzzy subset λ of a semihypergroup H is called an $(\in, \in \vee q)$-fuzzy generalized bi-hyperideal of H if for all $x, y, z \in H$ and $t, r \in (0, 1]$ the following condition holds;

$$x_t \in \lambda \text{ and } z_r \in \lambda \longrightarrow (w)_{\min\{t,r\}} \in \vee q\lambda \text{ for every } w \in x \circ y \circ z.$$

4.2.18 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in, \in \vee q)$-fuzzy generalized bi-hyperideal of H if and only if it satisfies the following condition,

$$\inf_{w \in x \circ y \circ z} \{\lambda(w)\} \geq \min\{\lambda(x), \lambda(z), 0.5\} \text{ for all } x, y, z \in H.$$

Proof. The proof is similar to the proof of Theorem 4.1.3. □

4.2.19 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in, \in \vee q)$-fuzzy generalized bi-hyperideal of H if and only if $U(\lambda; t)(\neq \emptyset)$ is a generalized bi-hyperideal of H for all $t \in (0, 0.5]$.

Proof. The proof is similar to the proof of Theorem 4.1.4. □

It is clear that every $(\in, \in \vee q)$-fuzzy bi-hyperideal of a semihypergroup H is an $(\in, \in \vee q)$-fuzzy generalized bi-hyperideal of H. The next example shows that the $(\in, \in \vee q)$-fuzzy generalized bi-hyperideal of H is not necessarily an $(\in, \in \vee q)$-fuzzy bi-hyperideal of H.

4.2.20 Example

Consider the semihypergroup $H = \{a, b, c, d\}$ with the following table:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	$\{a, b\}$	a
d	a	a	$\{a, b\}$	$\{a, b\}$

Define a fuzzy subset λ of H by

$$\lambda(a) = 0.8, \quad \lambda(b) = 0, \quad \lambda(c) = 0.7, \quad \lambda(d) = 0.$$

Then, we have

$$U(\lambda; t) = \begin{cases} \{a, c\} & \text{if } 0 < t \leq 0.7, \\ \{a\} & \text{if } 0.7 < t \leq 0.8, \\ \emptyset & \text{if } 0.8 < t. \end{cases}$$

Thus by Theorem 4.2.19, λ is an $(\in, \in \vee q)$-fuzzy generalized bi-hyperideal of H but λ is not an $(\in, \in \vee q)$-fuzzy bi-hyperideal of H, because $U(\lambda; 0.6) = \{a, c\}$ is a generalized bi-hyperideal of H but not a bi-hyperideal of H.

4.2.21 Lemma

Every $(\in, \in \vee q)$-fuzzy generalized bi-hyperideal of a regular semihypergroup H is an $(\in, \in \vee q)$-fuzzy bi-hyperideal of H.

Proof. Let λ be an $(\in, \in \vee q)$-fuzzy generalized bi-hyperideal of H and $a, b \in H$. Then there exists $x \in H$ such that $b \in b \circ x \circ b$. Thus we have $a \circ b \subseteq a \circ (b \circ x \circ b) = a \circ (b \circ x) \circ b$. So there exists $u \in b \circ x$, such that $z \in a \circ b \subseteq a \circ u \circ b$. Thus

$$\inf_{z \in a \circ b} \{\lambda(z)\} \geq \inf_{z \in a \circ u \circ b} \{\lambda(z)\} \geq \min\{\lambda(a), \lambda(b), 0.5\}.$$

This shows that λ is an $(\in, \in \vee q)$-fuzzy subsemihypergroup of H and so λ is an $(\in, \in \vee q)$-fuzzy bi-hyperideal of H. □

4.2.22 Definition

A fuzzy subset λ of a semihypergroup H is called an $(\in, \in \vee q)$-fuzzy interior hyperideal of H if for all $x, y, z \in H$ and $t, r \in (0, 1]$ the following conditions hold;

(1) $x_t \in \lambda$ and $y_t \in \lambda \longrightarrow (z)_{\min\{t,r\}} \in \vee q\lambda$ for every $z \in x \circ y$,

(2) $a_t \in \lambda \longrightarrow (w)_{\min\{t,r\}} \in \vee q\lambda$ for every $w \in x \circ a \circ y$.

4.2.23 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in, \in \vee q)$-fuzzy interior hyperideal of H if and only if it satisfies the following conditions,

(1) $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), 0.5\}$ *for all* $x, y \in H$.

(2) $\inf\limits_{w \in x \circ a \circ y} \{\lambda(w)\} \geq \min\{\lambda(a), 0.5\}$ *for all* $a, x, y \in H$.

Proof. The proof is similar to the proof of Theorem 4.1.3. □

4.2.24 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in, \in \vee q)$-fuzzy interior hyperideal of H if and only if $U(\lambda; t)(\neq \emptyset)$ is an interior hyperideal of H for all $t \in (0, 0.5]$.

Proof. The proof is similar to the proof of Theorem 4.1.4. □

4.2.25 Definition

A fuzzy subset λ of a semihypergroup H is called an $(\in, \in \vee q)$-fuzzy quasi-hyperideal of H, if it satisfies,

$$\lambda(x) \geq \min\{(1 \circ \lambda)(x), (\lambda \circ 1)(x), 0.5\}.$$

4.2.26 Theorem

Let λ be an $(\in, \in \vee q)$-fuzzy quasi-hyperideal of H. Then the set $\lambda_0 = \{x \in H \mid \lambda(x) > 0\}$ is a quasi-hyperideal of H.

Proof. In order to show that λ_0 is a quasi-hyperidealof H, we have to show that $H \circ \lambda_0 \cap \lambda_0 \circ H \subseteq \lambda_0$. Let $a \in H \circ \lambda_0 \cap \lambda_0 \circ H$. This means that $a \in H \circ \lambda_0$ and $a \in \lambda_0 \circ H$. So $a \in s \circ x$ and $a \in y \circ t$ for some $s, t \in H$ and $x, y \in \lambda_0$. Thus $\lambda(x) > 0$ and $\lambda(y) > 0$.

Since

$$\begin{aligned}(1 \circ \lambda)(a) &= \bigvee_{a \in s \circ x} \{1(s) \wedge \lambda(x)\} \\ &\geq \{1(s) \wedge \lambda(x)\} \\ &= \{1 \wedge \lambda(x)\} \\ &= \lambda(x).\end{aligned}$$

Similarly $(\lambda \circ 1)(a) \geq \lambda(y)$.

Thus

$$\begin{aligned}\lambda(a) &\geq \min\{(1 \circ \lambda)(a), (\lambda \circ 1)(a), 0.5\} \\ &\geq \min\{\lambda(x), \lambda(y), 0.5\} \\ &> 0 \text{ because } \lambda(x) > 0 \text{ and } \lambda(y) > 0.\end{aligned}$$

Thus $a \in \lambda_0$. Hence λ_0 is a quasi-hyperidealof H. □

4.2.27 Remark

Every fuzzy quasi-hyperideal of H is an $(\in, \in \vee q)$-fuzzy quasi-hyperideal of H but the converse is not true.

4.2.28 Lemma

A non-empty subset Q of a semihypergroup H is a quasi-hyperideal of H if the characteristic function λ_Q of Q is an $(\in, \in \vee q)$-fuzzy quasi-hyperideal of H.

Proof. Suppose Q is a quasi-hyperideal of H and λ_Q is the characteristic function of Q. Let $x \in H$. If $x \notin Q$ then $x \notin H \circ Q$ or $x \notin Q \circ H$. Thus $(1 \circ \lambda_Q)(x) = 0$ or $(\lambda_Q \circ 1)(x) = 0$ and so $\min\{(1 \circ \lambda_Q)(x), (\lambda_Q \circ 1)(x), 0.5\} = 0 = \lambda_Q(x)$. If $x \in Q$ then $\lambda_Q(x) = 1 \geq \min\{(1 \circ \lambda_Q)(x), (\lambda_Q \circ 1)(x), 0.5\}$. Hence λ_Q is an $(\in, \in \vee q)$-fuzzy quasi-hyperideal of H.

Conversely, suppose that λ_Q is an $(\in, \in \vee q)$-fuzzy quasi-hyperideal of H. Let $a \in H \circ Q \cap Q \circ H$. Then there exists $b, c \in H$ and $x, y \in Q$ such that $a \in b \circ x$ and $a \in y \circ c$. Then

$$\begin{aligned}(1 \circ \lambda_Q)(a) &= \bigvee_{a \in b \circ x} \{1(b) \wedge \lambda_Q(x)\} \\ &\geq \{1(b) \wedge \lambda_Q(x)\} \\ &= 1 \wedge 1 \\ &= 1.\end{aligned}$$

So $(1 \circ \lambda_Q)(a) = 1$. Similarly $(\lambda_Q \circ 1)(a) = 1$.

Hence $\lambda_Q(a) \geq \min\{(1 \circ \lambda_Q)(a), (\lambda_Q \circ 1)(a), 0.5\} = 0.5$. Thus $\lambda_Q(a) = 1$, which implies that $a \in Q$. Hence $H \circ Q \cap Q \circ H \subseteq Q$, that is Q is a quasi-hyperideal of H. □

4.2.29 Theorem

Every $(\in, \in \vee q)$-fuzzy left (right) hyperideal λ of H is an $(\in, \in \vee q)$-fuzzy quasi-hyperideal of H.

Proof. Suppose λ is an $(\in, \in \vee q)$-fuzzy quasi-hyperideal of H. Let $x \in H$, then

$$(1 \circ \lambda)(x) = \bigvee_{x \in y \circ z} \{1(y) \wedge \lambda(z)\} = \bigvee_{x \in y \circ z} \lambda(z).$$

This implies that

$$\begin{aligned}(1\circ\lambda)(x)\wedge 0.5 &= \left(\bigvee_{x\in y\circ z}\lambda(z)\right)\wedge 0.5\\ &= \bigvee_{x\in y\circ z}\{\lambda(z)\wedge 0.5\}\\ &\leq \bigvee_{x\in y\circ z}\lambda(x) \quad \text{(because } \lambda \text{ is an } (\in,\in\vee q)\text{-fuzzy left hyperideal of } H.)\\ &= \lambda(x).\end{aligned}$$

Thus $(1\circ\lambda)(x)\wedge 0.5\leq\lambda(x)$. Hence $\lambda(x)\geq(1\circ\lambda)(x)\wedge 0.5\geq\min\{(1\circ\lambda)(x),(\lambda\circ 1)(x),0.5\}$. Thus λ is an $(\in,\in\vee q)$-fuzzy quasi-hyperideal of H. □

4.2.30 Lemma

Every $(\in,\in\vee q)$-fuzzy quasi-hyperideal of a semihypergroup H is an $(\in,\in\vee q)$-fuzzy bi-hyperideal of H.

Proof. Suppose λ is an $(\in,\in\vee q)$-fuzzy quasi-hyperideal of semihypergroup H. Let $x,y\in H$. Now for every $\alpha\in x\circ y$, we have

$$\begin{aligned}\lambda(\alpha) &\geq \min\{(1\circ\lambda)(\alpha),(\lambda\circ 1)(\alpha),0.5\}\\ &= \left[\bigvee_{\alpha\in x\circ y}\{1(x)\wedge\lambda(y)\}\right]\wedge\left[\bigvee_{\alpha\in x\circ y}\{\lambda(x)\wedge 1(y)\}\right]\wedge 0.5\\ &\geq [1(x)\wedge\lambda(y)]\wedge[\lambda(x)\wedge 1(y)]\wedge 0.5\\ &\geq [1\wedge\lambda(y)\wedge\lambda(x)\wedge 1]\wedge 0.5\\ &= \lambda(y)\wedge\lambda(x)\wedge 0.5.\end{aligned}$$

So $\inf\limits_{\alpha\in x\circ y}\{\lambda(\alpha)\}\geq\min\{\lambda(x),\lambda(y),0.5\}$ for all $x,y\in H$.
Also for all $x,y,z\in H$ for every $w\in x\circ y\circ z$, we have

$$\begin{aligned}\lambda(w) &\geq \min\{(1\circ\lambda)(w),(\lambda\circ 1)(w),0.5\}\\ &= \left[\bigvee_{w\in x\circ y\circ z}\{1(x)\wedge\lambda(z)\}\right]\wedge\left[\bigvee_{w\in x\circ y\circ z}\{\lambda(x)\wedge 1(z)\}\right]\wedge 0.5\\ &\geq \{1(x)\wedge\lambda(z)\}\wedge\{\lambda(x)\wedge 1(z)\}\wedge 0.5\\ &\geq \{1\wedge\lambda(z)\}\wedge\{\lambda(x)\wedge 1\}\wedge 0.5\\ &= \lambda(z)\wedge\lambda(x)\wedge 0.5.\end{aligned}$$

So $\inf\limits_{w\in x\circ y\circ z}\{\lambda(w)\}\geq\min\{\lambda(x),\lambda(z),0.5\}$ for all $x,y,z\in H$. Thus λ is an $(\in,\in\vee q)$-fuzzy bi-hyperideal of H. □

The following example shows that the converse of above Lemma is not true.

4.2.31 Example

Consider the semihypergroup $H = \{a, b, c, d\}$ with the following table:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	$\{a, b\}$
d	a	a	$\{a, b\}$	$\{a, b, c\}$

One can easily check that $\{a, c\}$ is a bi-hyperideal of H but not a quasi-hyperideal of H.

Define a fuzzy subset λ of H by

$$\lambda(a) = 0.8 = \lambda(c), \qquad \lambda(b) = 0 = \lambda(d).$$

Then λ is an $(\in, \in \vee q)$-fuzzy bi-hyperideal of H but not an $(\in, \in \vee q)$-fuzzy quasi-hyperideal of H. Because $(1 \circ \lambda)(b) = 0.8 = (\lambda \circ 1)(b)$ but $\lambda(b) = 0 \ngeq \min\{(1 \circ \lambda)(b), (\lambda \circ 1)(b), 0.5\}$.

4.2.32 Lemma

Every $(\in, \in \vee q)$-fuzzy hyperideal of a semihypergroup H is an $(\in, \in \vee q)$-fuzzy interior hyperideal of H.

Proof. Let λ be an $(\in, \in \vee q)$-fuzzy hyperideal of H. Now

$$\inf_{\alpha \in x \circ y} \{\lambda(\alpha)\} \geq \min\{\lambda(y), 0.5\} \geq \min\{\lambda(x), \lambda(y), 0.5\}.$$

So $\inf_{\alpha \in x \circ y} \{\lambda(\alpha)\} \geq \min\{\lambda(x), \lambda(y), 0.5\}$.

Also for all $x, a, y \in H$ and for every $w \in x \circ a \circ y = x \circ (a \circ y)$. Thus for each $z \in a \circ y$, we have $w \in x \circ z$. So $\inf_{w \in x \circ a \circ y} \{\lambda(w)\} = \inf_{w \in x \circ z} \{\lambda(w)\} \geq \min\{\lambda(z), 0.5\} \geq \min\{\lambda(a), 0.5\}$ because for each $z \in a \circ y$, $\inf_{z \in a \circ y} \{\lambda(z)\} \geq \min\{\lambda(a), 0.5\}$.

Thus

$$\inf_{w \in x \circ a \circ y} \{\lambda(w)\} \geq \min\{\lambda(a), 0.5\}.$$

Hence λ is an $(\in, \in \vee q)$-fuzzy interior hyperideal of H. □

4.2.33 Lemma

The intersection of any family of $(\in, \in \vee q)$-fuzzy interior hyperideals of a semihypergroup H is an $(\in, \in \vee q)$-fuzzy interior hyperideal of H.

Proof. Straightforward. □

4.2.34 Proposition

Let λ be an $(\in, \in \vee q)$-fuzzy bi-hyperideal of a semihypergroup H. Then $\lambda \circ_{0.5} \lambda \leq \lambda$.

Proof. Let λ be an $(\in, \in \vee$q)-fuzzy bi-hyperideal of H. If $A_a = \emptyset$, then $(\lambda \circ_{0.5} \lambda)(a) = 0 \leq \lambda(a)$. If $A_a \neq \emptyset$, then

$$\begin{aligned}(\lambda \circ_{0.5} \lambda)(a) &= \bigvee_{(y,z)\in A_a} \min\{\lambda(y), \lambda(z), 0.5\} \\ &\leq \bigvee_{(y,z)\in A_a} \lambda(a) = \lambda(a).\end{aligned}$$

□

4.2.35 Lemma

Let λ, μ be fuzzy subsets of a semihypergroup H. Then $\lambda \circ_{0.5} \mu \leq 1 \circ_{0.5} \mu$ (resp. $\lambda \circ_{0.5} \mu \leq \lambda \circ_{0.5} 1$).

Proof. Straightforward. □

4.2.36 Proposition

Let λ be an $(\in, \in \vee q)$-fuzzy bi-hyperideal of a semihypergroup H. Then $\lambda \circ_{0.5} 1 \circ_{0.5} \lambda \leq \lambda$.

Proof. Let $a, y, z \in H$. If $A_a = \emptyset$, then $(\lambda \circ_{0.5} 1 \circ_{0.5} \lambda)(a) = 0 \leq \lambda(a)$. If $A_a \neq \emptyset$, then

$$\begin{aligned}(\lambda \circ_{0.5} 1 \circ_{0.5} \lambda)(a) &= \bigvee_{(y,z)\in A_a} \min\{\lambda(y), (1 \circ_{0.5} \lambda)(z), 0.5\} \\ &= \bigvee_{(y,z)\in A_a} \min\{\lambda(y), \bigvee_{(r,s)\in A_z} \min\{1(r), \lambda(s), 0.5\}, 0.5\} \text{ for each } z \in r \circ s, \\ &= \bigvee_{(y,z)\in A_a} \bigvee_{(r,s)\in A_z} \min\{\lambda(y), 1, \lambda(s), 0.5\} \\ &= \bigvee_{(y,z)\in A_a} \bigvee_{(r,s)\in A_z} \min\{\lambda(y), \lambda(s), 0.5\}.\end{aligned}$$

Since $a \in y \circ z \subseteq y \circ (r \circ s)$ and λ is an $(\in, \in \vee q)$-fuzzy bi-hyperideal of H, we have

$$\lambda(a) \geq \min\{\lambda(y), \lambda(s), 0.5\}.$$

Thus

$$\bigvee_{(y,z)\in A_a} \bigvee_{(r,s)\in A_z} \min\{\lambda(y), \lambda(s), 0.5\} \leq \bigvee_{(y,z)\in A_a} \bigvee_{(r,s)\in A_z} \lambda(a) = \lambda(a),$$

and consequently, $(\lambda \circ_{0.5} 1 \circ_{0.5} \lambda)(a) \leq \lambda(a)$. □

4.3 Regular and intra-regular semihypergroups

In this section, we characterize regular and intra-regular semihypergroups in terms of $(\in, \in \vee q)$-fuzzy hyperideals.

4.3.1 Theorem

A semihypergroup H is regular if and only if for every $(\in, \in \vee q)$-fuzzy bi-hyperideal λ of H we have

$$\lambda \circ_{0.5} 1 \circ_{0.5} \lambda = \lambda$$

Proof. $(\Longrightarrow)$ Let H be a regular semihypergroup and let $a \in H$. Since H is regular so there exists $x \in H$ such that $a \in a \circ x \circ a \subseteq a \circ x \circ (a \circ x \circ a) = a \circ (x \circ a \circ x) \circ a$. Then for every $s \in x \circ a \circ x$ and $r \in s \circ a$, we have $a \in a \circ r$ then

$$\begin{aligned}(\lambda \circ_{0.5} 1 \circ_{0.5} \lambda)(a) &= \bigvee_{a \in a \circ r} \min\{\lambda(a), (1 \circ_{0.5} \lambda)(r), 0.5\} \\ &\geq \min\{\lambda(a), (1 \circ_{0.5} \lambda)(r), 0.5\} \\ &= \min\{\lambda(a), \bigvee_{r \in s \circ a} \min\{1(t), \lambda(a), 0.5\}, 0.5\} \\ &\geq \min\{\lambda(a), \min\{1, \lambda(a), 0.5\}, 0.5\} \\ &= \min\{\lambda(a), 0.5\} = \lambda(a).\end{aligned}$$

Hence $\lambda(a) \leq (\lambda \circ_{0.5} 1 \circ_{0.5} \lambda)(a)$. On the other hand, by Proposition 4.2.36, we have $(\lambda \circ_{0.5} 1 \circ_{0.5} \lambda)(a) \leq \lambda(a)$. Therefore $(\lambda \circ_{0.5} 1 \circ_{0.5} \lambda)(a) = \lambda(a)$.

$(\Longleftarrow)$ Let λ be an $(\in, \in \vee q)$-fuzzy bi-hyperideal of H such that expression $\lambda \circ_{0.5} 1 \circ_{0.5} \lambda = \lambda$, is satisfied. To prove that H is regular, we will prove that $B \circ H \circ B = B$ for all bi-hyperideals B of H. Let $b \in B$, then by Corollary 4.2.14, λ_B is an $(\in, \in \vee q)$-fuzzy bi-hyperideal of H. By hypothesis

$$(\lambda_B \circ_{0.5} 1 \circ_{0.5} \lambda_B)(b) = \lambda_B(b),$$

Since $b \in B$, then $\lambda_B(b) = 1$ and we have $(\lambda_B \circ_{0.5} 1 \circ_{0.5} \lambda_B)(b) = 1$. By Proposition 1.3.7, we have $\lambda_B \circ_{0.5} 1 \circ_{0.5} \lambda_B = \lambda_{B \circ H \circ B}$ and hence $\lambda_{B \circ H \circ B}(b) = 1 \Longrightarrow b \in B \circ H \circ B$. Thus $B \subseteq B \circ H \circ B$. Since B is a bi-hyperideal of H, we $B \circ H \circ B \subseteq B$. Therefore $B \circ H \circ B = B$. □

4.3.2 Lemma

Let λ and μ be $(\in, \in \vee q)$-fuzzy bi-hyperideals of H. Then $\lambda \circ_{0.5} \mu$ is also an $(\in, \in \vee q)$-fuzzy bi-hyperideal of H.

Proof. Let λ and μ be $(\in, \in \vee q)$-fuzzy bi-hyperideals of H. Let $a, y, z \in H$. If $a \notin y \circ z$, then

$$((\lambda \circ_{0.5} \mu) \circ_{0.5} (\lambda \circ_{0.5} \mu))(a) = 0 \leq (\lambda \circ_{0.5} \mu)(a).$$

If $a \in y \circ z$, then

$$\begin{aligned}
((\lambda \circ_{0.5} \mu) \circ_{0.5} (\lambda \circ_{0.5} \mu))(a) &= \bigvee_{a \in y \circ z} \{(\lambda \circ_{0.5} \mu)(y) \wedge (\lambda \circ_{0.5} \mu)(z) \wedge 0.5\} \\
&= \bigvee_{a \in y \circ z} \left[\begin{array}{c} \bigvee_{y \in p_1 \circ q_1} \{\lambda(p_1) \wedge \mu(q_1) \wedge 0.5\} \\ \wedge \bigvee_{z \in p_2 \circ q_2} \{\lambda(p_2) \wedge \mu(q_2) \wedge 0.5\} \end{array} \right] \\
&= \bigvee_{a \in y \circ z} \bigvee_{y \in p_1 \circ q_1} \bigvee_{z \in p_2 \circ q_2} \left[\begin{array}{c} \{\lambda(p_1) \wedge \mu(q_1) \wedge 0.5\} \\ \wedge \{\lambda(p_2) \wedge \mu(q_2) \wedge 0.5\} \end{array} \right] \\
&= \bigvee_{a \in y \circ z} \bigvee_{y \in p_1 \circ q_1} \bigvee_{z \in p_2 \circ q_2} \left[\begin{array}{c} \{\lambda(p_1) \wedge \mu(p_2) \\ \wedge g(q_1) \wedge \mu(q_2) \wedge 0.5\} \end{array} \right] \\
&\leq \bigvee_{a \in y \circ z} \bigvee_{y \in p_1 \circ q_1} \bigvee_{z \in p_2 \circ q_2} \left[\begin{array}{c} \{\lambda(p_1) \wedge \lambda(p_2) \\ \wedge g(q_2) \wedge 0.5\} \end{array} \right].
\end{aligned}$$

Since $a \in y \circ z, y \in p_1 \circ q_1$ and $z \in p_2 \circ q_2$. Then $a \in y \circ z \subseteq (p_1 \circ q_1) \circ (p_2 \circ q_2) = (p_1 \circ q_1 \circ p_2) \circ q_2$. So for each $p \in (p_1 \circ q_1 \circ p_2), a \in p \circ q_2$. Then

$$\begin{aligned}
& \bigvee_{a \in y \circ z} \bigvee_{y \in p_1 \circ q_1} \bigvee_{z \in p_2 \circ q_2} [\{\lambda(p_1) \wedge \lambda(p_2) \wedge \mu(q_2) \wedge 0.5\}] \\
\leq & \bigvee_{a \in p \circ q_2} [\{\lambda(p_1) \wedge \lambda(p_2) \wedge \mu(q_2) \wedge 0.5\}].
\end{aligned}$$

Since λ is an $(\in, \in \vee \mathrm{q})$-fuzzy bi-hyperideal of H we have

$$\inf_{p \in p_1 \circ q_1 \circ p_2} \{\lambda(p)\} \geq \{\lambda(p_1) \wedge \lambda(p_2) \wedge 0.5\}.$$

Then

$$\begin{aligned}
& \bigvee_{a \in p \circ q_2} [\{\lambda(p_1) \wedge \lambda(p_2) \wedge \mu(q_2) \wedge 0.5\}] \\
\leq & \bigvee_{a \in p \circ q_2} [\{\lambda(p) \wedge \mu(q_2) \wedge 0.5\}] \\
\leq & \bigvee_{a \in p \circ q_2} [\{\lambda(p) \wedge \mu(q_2) \wedge 0.5\}] = (\lambda \circ_{0.5} \mu)(a).
\end{aligned}$$

Therefore $((\lambda \circ_{0.5} \mu) \circ_{0.5} (\lambda \circ_{0.5} \mu))(a) \leq (\lambda \circ_{0.5} \mu)(a)$, and $\lambda \circ_{0.5} \mu$ is an $(\in, \in \vee \mathrm{q})$-fuzzy subsemihypergroup of H. Let $x, y, z \in H$. Then

$$\begin{aligned}(\lambda \circ_{0.5} \mu)(x) \wedge (\lambda \circ_{0.5} \mu)(z) &= \left[\bigvee_{x \in p \circ q} \{\lambda(p) \wedge \mu(q) \wedge 0.5\}\right] \\ &\quad \wedge \left[\bigvee_{z \in r \circ s} \{\lambda(r) \wedge \mu(s) \wedge 0.5\}\right] \\ &= \bigvee_{x \in p \circ q} \bigvee_{z \in r \circ s} \left[\begin{array}{c}\{\lambda(p) \wedge \mu(q) \wedge 0.5\} \\ \wedge \{\lambda(r) \wedge \mu(s) \wedge 0.5\}\end{array}\right] \\ &= \bigvee_{x \in p \circ q} \bigvee_{z \in r \circ s} \left[\begin{array}{c}\{\lambda(p) \wedge \lambda(r) \wedge \mu(q)\} \\ \wedge \mu(s) \wedge 0.5\end{array}\right] \\ &\leq \bigvee_{x \in p \circ q} \bigvee_{z \in r \circ s} \left[\{\lambda(p) \wedge \lambda(r) \wedge \mu(s) \wedge 0.5\}\right].\end{aligned}$$

Since $x \in p \circ q$, and $z \in r \circ s$. Then for each $w \in x \circ y \circ z \subseteq (p \circ q) \circ y \circ (r \circ s) = (p \circ (q \circ y) \circ r) \circ s$ and we have $w \in (p \circ (q \circ y) \circ r) \circ s$. Thus

$$\begin{aligned}&\bigvee_{x \in p \circ q} \bigvee_{z \in r \circ s} \left[\{\lambda(p) \wedge \lambda(r) \wedge \mu(s) \wedge 0.5\}\right] \\ \leq &\bigvee_{w \in (p \circ (q \circ y) \circ r) \circ s} \left[\{\lambda(p) \wedge \lambda(r) \wedge \mu(s) \wedge 0.5\}\right].\end{aligned}$$

Since λ is an $(\in, \in \vee \mathrm{q})$-fuzzy bi-hyperideal of H, so for every $\alpha \in q \circ y$ and for every $\beta \in p \circ \alpha \circ r$, we have

$$\inf_{\beta \in p \circ \alpha \circ r} \lambda(\beta) \geq \{\lambda(p) \wedge \lambda(r) \wedge 0.5\}.$$

Hence, for each $w \in \beta \circ s \subseteq (p \circ (q \circ y) \circ r) \circ s$ and so

$$\begin{aligned}&\bigvee_{w \in \beta \circ s} \left[\{\lambda(p) \wedge \lambda(r) \wedge \mu(s) \wedge 0.5\}\right] \\ \leq &\bigvee_{w \in \beta \circ s} \left[\{\lambda(\beta) \wedge \mu(s) \wedge 0.5\}\right] \\ = &(\lambda \circ_{0.5} \mu)(w).\end{aligned}$$

Thus $(\lambda \circ_{0.5} \mu)(w) \geq (\lambda \circ_{0.5} \mu)(x) \wedge (\lambda \circ_{0.5} \mu)(z)$.
Hence $\lambda \circ_{0.5} \mu$ is an $(\in, \in \vee \mathrm{q})$-fuzzy bi-hyperideal of H. □

4.3.3 Theorem

A semihypergroup H is intra-regular if and only if $L \cap R \subseteq L \circ R$ for every left hyperideal L and for every right hyperideal R of H.

Proof. Let H be an intra-regular semihypergroup and L, R are left and right hyperideals of H respectively. Let $a \in L \cap R$ then $a \in L$ and $a \in R$. Since H is intra-regular so there exist $x, y \in H$ such that $a \in x \circ a \circ a \circ y = (x \circ a) \circ (a \circ y) \subseteq L \circ R$. Thus $L \cap R \subseteq L \circ R$.

Conversely, assume that $a \in L \cap R \subseteq L \circ R$. This implies that $a \in L \circ R = \bigcup_{l \circ r} \{l \circ r : l \in L, r \in R\}$. Since L is a left hyperideal so for some $x \in H$ such that $l = x \circ a$. Also R is a right hyperideal so for some $y \in H$ such that $r = a \circ y$. Thus

$$a \in l \circ r = (x \circ a) \circ (a \circ y) = x \circ a \circ a \circ y.$$

Hence H is intra-regular. □

4.3.4 Theorem

For a semihypergroup H, the following conditions are equivalent:

(1) H is intra-regular.

(2) $(\lambda \wedge_{0.5} \mu) \leq (\lambda \circ_{0.5} \mu)$ for every $(\in, \in \vee q)$-fuzzy left hyperideal λ and every $(\in, \in \vee q)$-fuzzy right hyperideal μ of H.

Proof. (1) $\Rightarrow$ (2) Let λ be an $(\in, \in \vee q)$-fuzzy left hyperideal and μ be an $(\in, \in \vee q)$-fuzzy right hyperideal of H. For $a \in H$, there exist $x, y \in H$ such that $a \in x \circ a \circ a \circ y = (x \circ a) \circ (a \circ y)$. Thus there exists $\beta \in x \circ a$ and $\gamma \in a \circ y$, such that $a \in \beta \circ \gamma$.

Thus

$$\begin{aligned}
(\lambda \circ_{0.5} \mu)(a) &= \bigvee_{a \in c \circ d} \{\lambda(c) \wedge \mu(d)\} \wedge 0.5 \\
&\geq \{\lambda(\beta) \wedge \mu(\gamma)\} \wedge 0.5 \\
(\text{Since } \beta &\in x \circ a \text{ and } \lambda \text{ is an } (\in, \in \vee q)\text{-fuzzy left hyperideal of } H, \\
\text{so } \inf_{z \in x \circ a} \{\lambda(z)\} &\geq \lambda(a) \wedge 0.5. \text{ Thus } \lambda(\beta) \geq \lambda(a) \wedge 0.5. \\
\text{Also since } \mu \text{ is an } (&\in , \in \vee q)\text{-fuzzy right hyperideal of } H, \\
\text{so } \inf_{z \in a \circ x} \{\mu(z)\} &\geq \mu(a) \wedge 0.5. \text{ Thus } \mu(\gamma) \geq \mu(a) \wedge 0.5) \\
&\geq \{(\lambda(a) \wedge 0.5) \wedge (\mu(a) \wedge 0.5)\} \wedge 0.5 \\
&= \lambda(a) \wedge \mu(a) \wedge 0.5 \\
&= (\lambda \wedge_{0.5} \mu)(a).
\end{aligned}$$

(2) $\Rightarrow$ (1) Let R and L be right and left hyperideals of H. Then by Corollary 4.1.12, λ_R and λ_L are $(\in, \in \vee q)$-fuzzy right and $(\in, \in \vee q)$-fuzzy left hyperideals of H,

respectively. Thus by hypothesis we have

$$\begin{aligned}\lambda_{L\circ R} &= \lambda_L \circ \lambda_R \\ &\geq \lambda_L \wedge \lambda_R \\ &= \lambda_{L\cap R}.\end{aligned}$$

Thus $L \cap R \subseteq L \circ R$. Hence it follows from Theorem 4.3.3, that H is intra-regular. □

4.3.5 Theorem

Let H be a semihypergroup. The following are equivalent:

(i) *H is both regular and intra-regular.*

(ii) *$\lambda \circ_{0.5} \lambda = \lambda$ for every $(\in, \in \vee q)$-fuzzy bi-hyperideal λ of H.*

(iii) *$\lambda \cap_{0.5} \mu = \lambda \circ_{0.5} \mu \cap_{0.5} \lambda \circ_{0.5} \mu$ for all $(\in, \in \vee q)$-fuzzy bi-hyperideals λ and μ of H.*

Proof. (i)⟶(ii). Let λ be an $(\in, \in \vee$q)-fuzzy bi-hyperideal of H and $a \in H$. Since H is regular and intra-regular there exist $x, y, z \in H$ such that $a \in a \circ x \circ a \subseteq a \circ x \circ a \circ x \circ a$, and $a \in y \circ a \circ a \circ z$. Then $a \in a \circ x \circ a \circ x \circ a \subseteq a \circ x \circ (y \circ a \circ a \circ z) \circ x \circ a = (a \circ (x \circ y) \circ a) \circ (a \circ (z \circ x) \circ a)$. Then for every $r \in x \circ y, s \in z \circ x$, $p \in a \circ r \circ a$ and $q \in a \circ s \circ a$. Then we have $a \in p \circ q$,

$$\begin{aligned}(\lambda \circ_{0.5} \lambda)(a) &= \bigvee_{a \in p \circ q} \{\lambda(p) \wedge \lambda(q) \wedge 0.5\} \\ &\geq \{\lambda(p) \wedge \lambda(q) \wedge 0.5\} \\ &\geq \left\{ \begin{array}{c} \{\lambda(a) \wedge \lambda(a) \wedge 0.5\} \\ \wedge\{\lambda(a) \wedge \lambda(a) \wedge 0.5\} \wedge 0.5 \end{array} \right\} \\ &= \{\lambda(a) \wedge 0.5\} = \lambda(a).\end{aligned}$$

On the other hand, by Proposition 4.2.34, we have $(\lambda \circ_{0.5} \lambda)(a) \leq \lambda(a)$. Thus $\lambda \circ_{0.5} \lambda = \lambda$.

(ii)⟶(iii). Let λ and g be $(\in, \in \vee$q)-fuzzy bi-hyperideals of H. Then $\lambda \cap_{0.5} \mu$ is an $(\in, \in \vee$q)-fuzzy bi-hyperideal of H. By (ii)

$$\begin{aligned}\lambda \cap_{0.5} \mu &= (\lambda \cap_{0.5} \mu) \circ_{0.5} (\lambda \cap_{0.5} \mu) \\ &\subseteq \lambda \cap_{0.5} \mu.\end{aligned}$$

Similarly, $\lambda \cap_{0.5} \mu \leq \mu \circ_{0.5} \lambda$. Thus

$$\lambda \cap_{0.5} \mu \leq \lambda \circ_{0.5} \mu \cap_{0.5} \mu \circ_{0.5} \lambda.$$

On the other hand, $\lambda \circ_{0.5} \mu$ and $\mu \circ_{0.5} \lambda$ are $(\in, \in \vee q)$-fuzzy bi-hyperideals of H by Lemma 4.3.2. Hence $\lambda \circ_{0.5} \mu \sqcap_{0.5} \mu \circ_{0.5} \lambda$ is an $(\in, \in \vee q)$-fuzzy bi-hyperideal of H. By (ii)

$$\begin{aligned}
& \lambda \circ_{0.5} \mu \sqcap_{0.5} \mu \circ_{0.5} \lambda \\
= \; & (\lambda \circ_{0.5} \mu \sqcap_{0.5} \mu \circ_{0.5} \lambda) \circ_{0.5} (\lambda \circ_{0.5} \mu \sqcap_{0.5} \mu \circ_{0.5} \lambda) \\
\subseteq \; & (\lambda \circ_{0.5} \mu) \circ_{0.5} (\mu \circ_{0.5} \lambda) = \lambda \circ_{0.5} (\mu \circ_{0.5} \mu) \circ_{0.5} \lambda \\
= \; & \lambda \circ_{0.5} \mu \circ_{0.5} \lambda \text{ (as } \mu \circ_{0.5} \mu = \mu \text{ by (i) above)} \\
\subseteq \; & \lambda \circ_{0.5} 1 \circ_{0.5} \lambda \\
= \; & \lambda \text{ (as } \lambda \circ_{0.5} 1 \circ_{0.5} \lambda = \lambda \text{ by Theorem 4.3.1).}
\end{aligned}$$

By a similar way we can prove that $\lambda \circ_{0.5} \mu \sqcap_{0.5} \mu \circ_{0.5} \lambda \subseteq \mu$. Consequently,

$$\lambda \circ_{0.5} \mu \sqcap_{0.5} \mu \circ_{0.5} \lambda \subseteq \lambda \sqcap_{0.5} \mu.$$

Therefore $\lambda \sqcap_{0.5} \mu = \lambda \circ_{0.5} \mu \sqcap_{0.5} \mu \circ_{0.5} \lambda$.

(iii)$\longrightarrow$(i). To prove that S is regular we prove that $A \cap B = A \circ B \cap B \circ A$ for every bi-hyperideal A, and B of H. Let $x \in A \cap B$. By Corollary 4.2.14, λ_A and λ_B are $(\in, \in \vee q)$-fuzzy bi-hyperideals of H. By (iii) $(\lambda_A \sqcap_{0.5} \lambda_B)(x) = (\lambda_A \circ_{0.5} \lambda_B \sqcap_{0.5} \lambda_B \circ_{0.5} \lambda_A)(x)$. Since $x \in A$ and $x \in B$, then $\lambda_A(x) = 1$ and $\lambda_B(x) = 1$. Then $(\lambda_A \sqcap_{0.5} \lambda_B)(x) = \min\{\lambda_A(x), \lambda_B(x), 0.5\} = 0.5$. Hence $(\lambda_A \circ_{0.5} \lambda_B \sqcap_{0.5} \lambda_B \circ_{0.5} \lambda_A)(x) = 0.5$. By Propositions 1.3.6 and 1.3.7, we have $\lambda_A \circ_{0.5} \lambda_B \sqcap_{0.5} \lambda_B \circ_{0.5} \lambda_A = \lambda_{A \circ B \cap B \circ A}$ and hence $\lambda_{A \circ B \cap B \circ A}(x) = 0.5$ implies that $x \in A \circ B \cap B \circ A$. On the other hand, if $x \in A \circ B \cap B \circ A$, then

$$\begin{aligned}
1 &= \lambda_{A \circ B \cap B \circ A}(x) \\
&= (\lambda_{A \circ B} \sqcap_{0.5} \lambda_{B \circ A})(x) \\
&= (\lambda_A \circ_{0.5} \lambda_B \sqcap_{0.5} \lambda_B \circ_{0.5} \lambda_A)(x) \\
&= (\lambda_A \sqcap_{0.5} \lambda_B)(x) \text{ (by (iii))} \\
&= \lambda_{A \cap B}(x).
\end{aligned}$$

Hence $x \in A \cap B$. Therefore $A \cap B = A \circ B \cap B \circ A$, consequently, H is both regular and intra-regular. This completes the proof. □

Chapter 5

SEMIHYPERGROUPS CHARACTERIZED BY $(\in, \in \vee q_k)$-FUZZY HYPERIDEALS

In this chapter we generalize the concepts of $(\in, \in \vee q)$-fuzzy subsemihypergroup, $(\in, \in \vee q)$-fuzzy hyperideal, $(\in, \in \vee q)$-fuzzy interior hyperideal, $(\in, \in \vee q)$-fuzzy bi-hyperideal, and $(\in, \in \vee q)$-fuzzy quasi hyperideal and define $(\in, \in \vee q_k)$-fuzzy subsemihypergroup, $(\in, \in \vee q_k)$-fuzzy hyperideal, $(\in, \in \vee q_k)$-fuzzy interior hyperideal, $(\in, \in \vee q_k)$-fuzzy bi-hyperideal, and $(\in, \in \vee q_k)$-fuzzy quasi hyperideal of a semihypergroup H and study some basic properties. Also we characterize regular and intra-regular semihypergroups using these notions.

In what follows, let H denote a semihypergroup and k an arbitrary element of $[0, 1)$ unless otherwise specified.

Generalizing the concept of $x_t q \lambda$, Jun [30, 31] defined $x_t q_k \lambda$, if $\lambda(x) + t + k > 1$ and $x_t \in \vee q_k \lambda$ if $x_t \in \lambda$ or $x_t q_k \lambda$.

5.1 $(\in, \in \vee q_k)$-fuzzy subsemihypergroups

5.1.1 Definition

A fuzzy subset λ of H is called an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup of H if for all $x, y \in H$ and $t, r \in (0, 1]$ the following condition holds

$$x_t \in \lambda, y_r \in \lambda \longrightarrow (z)_{\min\{t,r\}} \in \vee q_k \lambda, \text{ for each } z \in x \circ y.$$

5.1.2 Theorem

Let A be a non-empty subset of H and λ a fuzzy subset in H defined by

$$\lambda(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) *If A is a subsemihypergroup of H then λ is a $(q, \in \vee q_k)$-fuzzy subsemihypergroup of H.*

(2) *A is a subsemihypergroup of H if and only if λ is an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup of H.*

Proof. (1) Let $x, y \in H$ and $t, r \in (0, 1]$ be such that $x_t, y_r q \lambda$. Then $x, y \in A$, $\lambda(x) + t > 1$ and $\lambda(x) + r > 1$. Since A is a subsemihypergroup of H, we have $x \circ y \subseteq A$.

Thus for every $z \in x \circ y, \lambda(z) \geq \frac{1-k}{2}$. If $\min\{t,r\} \leq \frac{1-k}{2}$, then $\lambda(z) \geq \min\{t,r\}$ and so $(z)_{\min\{t,r\}} \in \lambda$. If $\min\{t,r\} > \frac{1-k}{2}$, then $\lambda(z) + \min\{t,r\} + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $(z)_{\min\{t,r\}} q_k \lambda$. Therefore $(z)_{\min\{t,r\}} \in \vee q_k \lambda$.

(2) Let $x, y \in H$ and $t, r \in (0,1]$ be such that $x_t, y_r \in \lambda$. Then $\lambda(x) \geq t > 0$ and $\lambda(y) \geq r > 0$. Thus $\lambda(x) \geq \frac{1-k}{2}$ and $\lambda(y) \geq \frac{1-k}{2}$, this implies $x, y \in A$. Since A is a subsemihypergroup of H, we have $x \circ y \subseteq A$. Thus for every $z \in x \circ y, \lambda(z) \geq \frac{1-k}{2}$. If $\min\{t,r\} \leq \frac{1-k}{2}$, then $\lambda(z) \geq \min\{t,r\}$ and so $(z)_{\min\{t,r\}} \in \lambda$. If $\min\{t,r\} > \frac{1-k}{2}$, then $\lambda(z) + \min\{t,r\} + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$ and so $(z)_{\min\{t,r\}} q_k \lambda$. Therefore $(z)_{\min\{t,r\}} \in \vee q_k \lambda$.

Conversely, assume that λ is a $(\in, \in \vee q_k)$-fuzzy subsemihypergroup of H and $x, y \in A$. Then $\lambda(x) \geq \frac{1-k}{2}, \lambda(y) \geq \frac{1-k}{2}$ that is $x_{\frac{1-k}{2}} \in \lambda$ and $y_{\frac{1-k}{2}} \in \lambda$. Now by hypothesis, $z_{\frac{1-k}{2}} \in \vee q \lambda$ for every $z \in x \circ y$. If $z_{\frac{1-k}{2}} \in \lambda$ then $\lambda(z) \geq \frac{1-k}{2}$ and so $z \in A$. If $z_{\frac{1-k}{2}} q_k \lambda$ then $\lambda(z) + \frac{1-k}{2} + k > 1$ implies $\lambda(z) > \frac{1-k}{2}$. Thus $z \in A$. Hence $x \circ y \subseteq A$, that is, A is a subsemihypergroup of H. □

5.1.3 Corollary

(1) *If a non-empty subset A of H is a subsemihypergroup of H, then the characteristic function of A is a $(q, \in \vee q_k)$-fuzzy subsemihypergroup of H.*

(2) *A non-empty subset A of H is a subsemihypergroup of H if and only if λ_A is an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup of H.*

If we take $k = 0$ in Theorem 5.1.2, then we get the Theorem 4.1.2.

5.1.4 Theorem

Let λ be a fuzzy subset of H. Then λ is an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup of H if and only if $\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$.

Proof. Let λ be an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup of H. On the contrary, assume that there exist $x, y \in H$ such that $\inf_{z \in x \circ y} \{\lambda(z)\} < \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$. Then there exists $z \in x \circ y$ such that $\lambda(z) < \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$. Choose $t \in (0,1]$ such that $\lambda(z) < t \leq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$. Then $x_t \in \lambda$ and $y_t \in \lambda$ but $\lambda(z) < t$ and $\lambda(z) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $z_t \overline{\in \vee q_k} \lambda$, which is a contradiction. Hence $\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$.

Conversely, assume that $\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$. Let $x_t \in \lambda$ and $y_r \in \lambda$ for $t, r \in (0,1]$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq r$. Now

$$\inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\} \geq \min\{t, r, \frac{1-k}{2}\}.$$

If $t \wedge r > \frac{1-k}{2}$, then $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq \frac{1-k}{2}$. So for every $z \in x \circ y$,

$$\lambda(z) + t \wedge r + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1,$$

which implies that $(z)_{\min\{t,r\}} q_k \lambda$. If $t \wedge r \leq \frac{1-k}{2}$, then $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq t \wedge r$. So $(z)_{\min\{t,t\}} \in \lambda$. Thus $(z)_{\min\{t,r\}} \in \vee q_k \lambda$. Therefore λ is an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup of H. □

If we take $k = 0$ in Theorem 5.1.4, then we get the Theorem 4.1.3.

5.1.5 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup of H if and only if $U(\lambda; t)(\neq \emptyset)$ is a subsemihypergroup of H for all $t \in (0, \frac{1-k}{2}]$.

Proof. Let λ be an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup of H and $x, y \in U(\lambda; t)$ for some $t \in (0, \frac{1-k}{2}]$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq t$. It follows from Theorem 5.1.4 that $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\} = t$. Thus for every $z \in x \circ y$, $\lambda(z) \geq t$ and so $z \in U(\lambda; t)$, that is $x \circ y \subseteq U(\lambda; t)$. Hence $U(\lambda; t)$ is a subsemihypergroup of H.

Conversely, assume that $U(\lambda; t)(\neq \emptyset)$ is a subsemihypergroup of H for all $t \in (0, \frac{1-k}{2}]$. Suppose that there exist $x, y \in H$ such that

$$\inf_{z \in x \circ y} \{\lambda(z)\} < \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}.$$

Thus there exists $z \in x \circ y$ such that $\lambda(z) < \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$. Choose $t \in (0, \frac{1-k}{2}]$ such that $\lambda(z) < t < \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$. Then $x, y \in U(\lambda; t)$ but $z \notin U(\lambda; t)$ i.e. $x \circ y \not\subseteq U(\lambda; t)$, which contradicts our hypothesis.

Hence $\inf\limits_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$ and so λ is an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup of H. □

If we take $k = 0$ in Theorem 5.1.5, then we get the Theorem 4.1.4.

5.2 $(\in, \in \vee q_k)$-fuzzy hyperideals

5.2.1 Definition

A fuzzy subset λ of H is an $(\in, \in \vee q_k)$-fuzzy left (right) hyperideal of H if for all $x, y \in H$ and $t \in (0, 1]$ the following condition holds;

$y_t \in \lambda \longrightarrow z_t \in \vee q_k \lambda$ for every $z \in x \circ y$ ($y_t \in \lambda \longrightarrow z_t \in \vee q_k \lambda$ for every $z \in y \circ x$).

5.2.2 Theorem

Let L be a subset of H and λ a fuzzy subset in H defined by

$$\lambda(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in L \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) If L is a left (resp. right) hyperideal of H then λ is a $(q, \in \vee q_k)$-fuzzy left (resp. right) hyperideal of H.

(2) L is a left (resp. right) hyperideal of H if and only if λ is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) hyperideal of H.

Proof. The proof is similar to the proof of Theorem 5.1.2. □

5.2.3 Corollary

(1) If a non-empty subset L of H is a left (resp. right) hyperideal of H, then the characteristic function of L is a $(q, \in \vee q_k)$-fuzzy left (resp. right) hyperideal of H.

(2) A non-empty subset L of H is a left (resp. right) hyperideal of H if and only if λ_L is an $(\in, \in \vee q_k)$-fuzzy sleft (resp. right) hyperideal of H.

5.2.4 Theorem

A fuzzy subset λ of H is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) hyperideal of H if and only if $\inf\limits_{z \in x \circ y}\{\lambda(z)\} \geq \min\{\lambda(y), \frac{1-k}{2}\}$ (resp. $\inf\limits_{z \in x \circ y}\{\lambda(z)\} \geq \min\{\lambda(x), \frac{1-k}{2}\}$).

Proof. Let λ be an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H. On the contrary, assume that there exist $x, y \in H$ such that $\inf\limits_{z \in x \circ y}\{\lambda(z)\} < \min\{\lambda(y), \frac{1-k}{2}\}$. Then there exists $z \in x \circ y$ such that $\lambda(z) < \min\{\lambda(y), \frac{1-k}{2}\}$. Choose $t \in (0, 1]$ such that $\lambda(z) < t \leq \min\{\lambda(y), \frac{1-k}{2}\}$. Then $y_t \in \lambda$ but $\lambda(z) < t$ and $\lambda(z) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $z_t \overline{\in \vee q_k} \lambda$, which is a contradiction. Hence $\inf\limits_{z \in x \circ y}\{\lambda(z)\} \geq \min\{\lambda(y), \frac{1-k}{2}\}$.

Conversely, assume that $\inf\limits_{z \in x \circ y}\{\lambda(z)\} \geq \min\{\lambda(y), \frac{1-k}{2}\}$. Let $x, y \in H$ and $t \in (0, 1]$ be such that $y_t \in \lambda$. Then $\lambda(y) \geq t$. Thus $\inf\limits_{z \in x \circ y}\{\lambda(z)\} \geq \min\{\lambda(y), \frac{1-k}{2}\} \geq \min\{t, \frac{1-k}{2}\}$. If $t > \frac{1-k}{2}$, then $\inf\limits_{z \in x \circ y}\{\lambda(z)\} \geq \frac{1-k}{2}$. So for every $z \in x \circ y, \lambda(z) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, which imlpies that $z_t q_k \lambda$. If $t \leq \frac{1-k}{2}$, then $\inf\limits_{z \in x \circ y}\{\lambda(z)\} \geq t$. So for every $z \in x \circ y, z_t \in \lambda$. Thus $z_t \in \vee q_k \lambda$ for every $z \in x \circ y$. Therefore λ is an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H. □

5.2.5 Corollary

A fuzzy subset λ of H is an $(\in, \in \vee q_k)$-fuzzy hyperideal of H if and only if $\inf\limits_{z \in x \circ y}\{\lambda(z)\} \geq \min\{\lambda(y), \frac{1-k}{2}\}$ and $\inf\limits_{z \in x \circ y}\{\lambda(z)\} \geq \min\{\lambda(x), \frac{1-k}{2}\}$.

5.2.6 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) hyperideal of H if and only if $U(\lambda; t)(\neq \emptyset)$ is a left (resp. right) hyperideal of H for all $t \in (0, \frac{1-k}{2}]$.

Proof. The proof is similar to the proof of Theorem 5.1.5. □

5.2.7 Theorem

If λ *is an $(\in, \in \vee q_k)$-fuzzy left hyperideal and* μ is an *$(\in, \in \vee q_k)$-fuzzy right hyperideal of H* then $\lambda \circ \mu$ is an *$(\in, \in \vee q_k)$-fuzzy two-sided hyperideal of H.*

Proof. Let $x, y \in H$. Then

$$\begin{aligned}(\lambda \circ \mu)(y) \wedge \frac{1-k}{2} &= \left(\bigvee_{y \in p \circ q} \{\lambda(p) \wedge \mu(q)\}\right) \wedge \frac{1-k}{2} \\ &= \bigvee_{y \in p \circ q} \left\{\lambda(p) \wedge \mu(q) \wedge \frac{1-k}{2}\right\} \\ &= \bigvee_{y \in p \circ q} \left\{\lambda(p) \wedge \frac{1-k}{2} \wedge \mu(q)\right\}.\end{aligned}$$

(If $y \in p \circ q$, then $x \circ y \subseteq x \circ (p \circ q) = (x \circ p) \circ q$. Now for each $z \in x \circ y$, there exists $a \in x \circ p$ such that $z \in a \circ q$. Since λ is an $(\in, \in \vee q_k)$-fuzzy left hyperideal, therefore by Theorem 5.2.4, we have $\inf_{a \in x \circ p} \{\lambda(a)\} \geq \min\{\lambda(p), \frac{1-k}{2}\}$ that is $\lambda(a) \geq \min\{\lambda(p), \frac{1-k}{2}\}$.)

Thus

$$\begin{aligned}(\lambda \circ \mu)(y) \wedge \frac{1-k}{2} &= \bigvee_{y \in p \circ q} \left\{\lambda(p) \wedge \frac{1-k}{2} \wedge \mu(q)\right\} \\ &\leq \bigvee_{z \in a \circ q} \{\lambda(a) \wedge \mu(q)\} \text{ because } \lambda(a) \geq \min\{\lambda(p), \frac{1-k}{2}\} \\ &= \bigvee_{z \in c \circ d} \{\lambda(c) \wedge \mu(d)\} \\ &= (\lambda \circ \mu)(z), \text{ for every } z \in x \circ y \subseteq a \circ q.\end{aligned}$$

So

$$\min\left\{(\lambda \circ \mu)(y), \frac{1-k}{2}\right\} \leq \inf_{z \in x \circ y} \{(\lambda \circ \mu)(z)\}.$$

Similarly we can show that $\inf_{z \in x \circ y} \{(\lambda \circ \mu)(z)\} \geq \min\left\{(\lambda \circ \mu)(x), \frac{1-k}{2}\right\}$. Thus $\lambda \circ \mu$ is an $(\in, \in \vee q_k)$-fuzzy two-sided hyperideal of H. □

Next we show that if λ and μ are $(\in, \in \vee q_k)$-fuzzy hyperideals of a semihypergroup H, then $\lambda \circ \mu \nleq \lambda \wedge \mu$.

5.2.8 Example

Consider the semihypergroup $H = \{a, b, c, d\}$ with the following table:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	$\{a, b\}$	a
d	a	a	$\{a, b\}$	$\{a, b\}$

One can easily check that $\{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}$ and $\{a, b, c, d\}$ are all hyperideals of H.

Define fuzzy sets λ, μ of H by

$$\begin{aligned} \lambda(a) &= 0.7, \quad \lambda(b) = 0.3, \quad \lambda(c) = 0.4, \quad \lambda(d) = 0, \\ \mu(a) &= 0.8, \quad \mu(b) = 0.3, \quad \mu(c) = 0.4, \quad \mu(d) = 0.2. \end{aligned}$$

Then we have

$$U(\lambda; t) = \begin{cases} \{a, b, c\} & \text{if } 0 < t \leq 0.3, \\ \{a, c\} & \text{if } 0.3 < t \leq 0.4, \\ \{a\} & \text{if } 0.4 < t \leq 0.7, \\ \emptyset & \text{if } 0.7 < t \leq 1. \end{cases}$$

$$U(\mu; t) = \begin{cases} \{a, b, c, d\} & \text{if } 0 < t \leq 0.2, \\ \{a, b, c\} & \text{if } 0.2 < t \leq 0.3, \\ \{a, c\} & \text{if } 0.3 < t \leq 0.4, \\ \{a\} & \text{if } 0.4 < t \leq 0.8, \\ \emptyset & \text{if } 0.8 < t \leq 1. \end{cases}$$

Thus by Theorem 5.2.6, λ, μ are $(\in, \in \vee q_k)$-fuzzy hyperideals of H with $k = 0.4$. Now

$$\begin{aligned} (\lambda \circ \mu)(b) &= \bigvee_{b \in x \circ y} \{\lambda(x) \wedge \mu(y)\} \\ &= \bigvee \{0.4, 0, 0\} \\ &= 0.4 \nleq (\lambda \wedge \mu)(b) = 0.3. \end{aligned}$$

Hence $\lambda \circ \mu \nleq \lambda \wedge \mu$ in general.

5.2.9 Lemma

The intersection of any family of $(\in, \in \vee q_k)$-fuzzy left (resp. right) hyperideals of H is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) hyperideal of H.

Proof. Let $\{\lambda_i\}_{i\in I}$ be a family of $(\in, \in \vee q_k)$-fuzzy left hyperideals of H and $x, y \in H$. Then $\inf\limits_{z\in x\circ y}\{(\bigwedge\limits_{i\in I}\lambda_i)(z)\} = \bigwedge\limits_{i\in I}\{\inf\limits_{z\in x\circ y}\{\lambda_i\}(z)\}$.

(Since each λ_i is an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H, so $\inf\limits_{z\in x\circ y}\{\lambda_i\}(z) \geq \min\left\{\lambda_i(y), \frac{1-k}{2}\right\}$ for all $i \in I$.)

Thus

$$\begin{aligned}
\inf_{z\in x\circ y}\{\left(\bigwedge_{i\in I}\lambda_i\right)(z)\} &= \bigwedge_{i\in I}\{\inf_{z\in x\circ y}\{\lambda_i\}(z)\} \\
&\geq \bigwedge_{i\in I}\left\{\lambda_i(y)\wedge\frac{1-k}{2}\right\} \\
&= \left(\bigwedge_{i\in I}\lambda_i(y)\right)\wedge\frac{1-k}{2} \\
&= \left(\bigwedge_{i\in I}\lambda_i\right)(y)\wedge\frac{1-k}{2}.
\end{aligned}$$

Hence $\bigwedge\limits_{i\in I}\lambda_i$ is an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H. □

5.2.10 Lemma

The union of any family of $(\in, \in \vee q_k)$-fuzzy left (resp. right) hyperideals of H is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) hyperideal of H.

Proof. Let $\{\lambda_i\}_{i\in I}$ be a family of $(\in, \in \vee q_k)$-fuzzy left hyperideals of H and $x, y \in H$. Then $\inf\limits_{z\in x\circ y}\{(\bigvee\limits_{i\in I}\lambda_i)(z)\} = \bigvee\limits_{i\in I}\{\inf\limits_{z\in x\circ y}\{\lambda_i\}(z)\}$.

(Since each λ_i is an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H, so $\inf\limits_{z\in x\circ y}\{\lambda_i\}(z) \geq \min\left\{\lambda_i(y), \frac{1-k}{2}\right\}$ for all $i \in I$.)

Thus

$$\begin{aligned}
\inf_{z\in x\circ y}\left\{\left(\bigvee_{i\in I}\lambda_i\right)\right\}(z) &= \bigvee_{i\in I}\{\inf_{z\in x\circ y}\{\lambda_i\}(z)\} \\
&\geq \bigvee_{i\in I}\left\{\lambda_i(y)\wedge\frac{1-k}{2}\right\} \\
&= \left(\bigvee_{i\in I}\lambda_i(y)\right)\wedge\frac{1-k}{2} \\
&= \left(\bigvee_{i\in I}\lambda_i\right)(y)\wedge\frac{1-k}{2}.
\end{aligned}$$

Hence $\bigvee_{i\in I}\lambda_i$ is an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H. □

5.3 $(\in, \in \vee q_k)$-fuzzy bi-hyperideals

5.3.1 Definition

An $(\in, \in \vee q_k)$-fuzzy subsemihypergroup λ of H is called an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H if for all $x, y, z \in H$ and $t, r \in (0, 1]$ the following condition holds;

$$x_t \in \lambda \text{ and } z_r \in \lambda \longrightarrow (w)_{\min\{t,r\}} \in \vee q_k \lambda \text{ for every } w \in x \circ y \circ z.$$

5.3.2 Theorem

Let B be a non-empty subset of H and λ be a fuzzy subset in H defined by

$$\lambda(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) *If B is a bi-hyperideal of H then λ is a $(q, \in \vee q_k)$-fuzzy bi-hyperideal of H.*

(2) *B is a bi-hyperideal of H if and only if λ is an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H.*

Proof. The proof is similar to the proof of Theorem 5.1.2. □

5.3.3 Corollary

(1) *If a non-empty subset B of a semihypergroup H is a bi-hyperideal of H then the characteristic function of B is a $(q, \in \vee q_k)$-fuzzy bi-hyperideal of H.*

(2) *A non-empty subset B of a semihypergroup H is a bi-hyperideal of H if and only if λ_B is an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H.*

5.3.4 Theorem

A fuzzy subset λ of a semihypergroup H is *an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H if and only if it satisfies the following conditions,*

(1) $\inf_{z \in x \circ y}\{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$ *for all $x, y \in H$.*

(2) $\inf_{w \in x \circ y \circ z}\{\lambda(w)\} \geq \min\{\lambda(x), \lambda(z), \frac{1-k}{2}\}$ *for all $x, y, z \in H$.*

Proof. The proof is similar to the proof of Theorem 5.1.4. □

If we take $k = 0$ in Thoerem 5.3.4, then we get the Theorem 4.2.15.

5.3.5 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H if and only if $U(\lambda; t)(\neq \emptyset)$ is a bi-hyperideal of H for all $t \in (0, \frac{1-k}{2}]$.

Proof. The proof is similar to the proof of Theorem 5.1.5. □

If we take $k = 0$ in Thoerem 5.3.5, then we get the Theorem 4.2.16.

5.4 $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideals

5.4.1 Definition

A fuzzy subset λ of a semihypergroup H is called an $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal of H if for all $x, y, z \in H$ and $t, r \in (0, 1]$ the following condition holds;

$$x_t \in \lambda \text{ and } z_t \in \lambda \longrightarrow (w)_{\min\{t,r\}} \in \vee q_k \lambda \text{ for every } w \in x \circ y \circ z.$$

5.4.2 Theorem

Let B be a non-empty subset of H and λ be a fuzzy subset in H such that

$$\lambda(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) If B is a generalized bi-hyperideal of H then λ is a $(q, \in \vee q_k)$-fuzzy generalized bi-hyperideal of H.

(2) B is a generalized bi-hyperideal of H if and only if λ is a $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal of H.

Proof. The proof is similar to the proof of Theorem 5.1.2. □

5.4.3 Corollary

(1) If a non-empty subset B of a semihypergroup H is a generalized bi-hyperideal of H then the characteristic function of B is a $(q, \in \vee q)$-fuzzy generalized bi-hyperideal of H.

(2) A non-empty subset B of a semihypergroup H is a generalized bi-hyperideal of H if and only if λ_B is a $(\in, \in \vee q)$-fuzzy generalized bi-hyperideal of H.

5.4.4 Theorem

A fuzzy subset λ of H is *an $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal of H if and only if it satisfies the following condition,*

$$\inf_{w \in x \circ y \circ z} \{\lambda(w)\} \geq \min\{\lambda(x), \lambda(z), \frac{1-k}{2}\} \text{ for all } x, y, z \in H.$$

Proof. The proof is similar to the proof of Theorem 5.1.4. □

5.4.5 Theorem

A fuzzy subset λ of H is an $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal of H if and only if $U(\lambda;t)(\neq \emptyset)$ is a generalized bi-hyperideal of H for all $t \in (0, \frac{1-k}{2}]$.

Proof. The proof is similar to the proof of Theorem 5.1.5. □

It is clear that every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of a semihypergroup H is an $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal of H. The next example shows that the $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal of H is not necessarily an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H.

5.4.6 Example

Consider the semihypergroup $H = \{a, b, c, d\}$ with the following table:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	$\{a, b\}$	a
d	a	a	$\{a, b\}$	$\{a, b\}$

One can easily check that $\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}$ and $\{a, b, c, d\}$ are all generalized bi-hyperideals of H and $\{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}$ and $\{a, b, c, d\}$ are all bi-hyperideals of H.

Define a fuzzy subset λ of H by

$$\lambda(a) = 0.8, \quad \lambda(b) = 0, \quad \lambda(c) = 0.4, \quad \lambda(d) = 0.$$

Then, we have

$$U(\lambda;t) = \begin{cases} \{a, c\} & \text{if } 0 < t \leq 0.4, \\ \{a\} & \text{if } 0.4 < t \leq 0.8, \\ \emptyset & \text{if } 0.8 < t \leq 1. \end{cases}$$

Thus by Theorem 5.4.5, λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal of H for every $k \in [0, 1)$ but λ is not an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H, because $U(\lambda; 0.4) = \{a, c\}$ is a generalized bi-hyperideal of H but not a bi-hyperideal of H.

5.4.7 Lemma

Every $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal of a regular semihypergroup H is an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H.

Proof. Let λ be an $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal of H and $a, b \in H$. Then there exists $x \in H$ such that $b \in b \circ x \circ b$. Thus we have

$$a \circ b \subseteq a \circ (b \circ x \circ b) = a \circ (b \circ x) \circ b.$$

Thus

$$\inf_{z \in a \circ b} \{\lambda(z)\} \geq \inf_{z \in a \circ y \circ b} \{\lambda(z)\} \geq \min\{\lambda(a), \lambda(b), \frac{1-k}{2}\}.$$

This shows that λ is an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup of H and so λ is an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H. □

5.5 $(\in, \in \vee q_k)$-fuzzy interior hyperideals

5.5.1 Definition

A fuzzy subset λ of a semihypergroup H is called an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of H if for all $x, y, a \in H$ and $t, r \in (0, 1]$ the following conditions hold;

(1) $x_t \in \lambda$ and $y_r \in \lambda \longrightarrow (z)_{\min\{t,r\}} \in \vee q_k \lambda$ for every $z \in x \circ y$,

(2) $a_t \in \lambda \longrightarrow w_t \in \vee q_k \lambda$ for every $w \in x \circ a \circ y$.

5.5.2 Theorem

Let A be a non-empty subset of H and λ be a fuzzy subset in H such that

$$\lambda(x) = \begin{cases} \geq \frac{1-k}{2} & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Then

(1) *If A is an interior hyperideal of H then λ is a $(q, \in \vee q_k)$-fuzzy interoir hyperideal of H.*

(2) *A is an interior hyperideal of H if and only if λ is an $(\in, \in \vee q_k)$-fuzzy interoir hyperideal of H.*

Proof. The proof is similar to the proof of Theorem 5.1.2. □

5.5.3 Corollary

(1) *If a non-empty subset A of a semihypergroup H is an interior hyperideal of H then the characteristic function of A is a $(q, \in \vee q)$-fuzzy interior hyperideal of H.*

(2) *A non-empty subset A of a semihypergroup H is an interior hyperideal of H if and only if λ_A is a $(\in, \in \vee q)$-fuzzy interior hyperideal of H.*

If we take $k = 0$ in Thoerem 5.5.2, then we get the Theorem 4.1.19

5.5.4 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of H if and only if it satisfies the following conditions,

(1) $\inf\limits_{z\in x\circ y}\{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$ *for all* $x, y \in H$.

(2) $\inf\limits_{w\in x\circ a\circ y}\{\lambda(w)\} \geq \min\{\lambda(a), \frac{1-k}{2}\}$ *for all* $a, x, y \in H$.

Proof. The proof is similar to the proof of Theorem 5.1.4. □

5.5.5 Lemma

The intersection of any family of $(\in, \in \vee q_k)$-fuzzy interior hyperideals of a semihypergroup H is an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of H.

Proof. Straightfoward. □

5.5.6 Theorem

A fuzzy subset λ of a semihypergroup H is an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of H if and only if $U(\lambda; t)(\neq \emptyset)$ is an interior hyperideal of H for all $t \in (0, \frac{1-k}{2}]$.

Proof. The proof is similar to the proof of Theorem 5.1.5. □

5.5.7 Lemma

Every $(\in, \in \vee q_k)$-fuzzy hyperideal of a semihypergroup H is an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of H.

Proof. Let λ be an $(\in, \in \vee q_k)$-fuzzy hyperideal of H. Then

$$\inf_{\alpha\in x\circ y}\{\lambda(\alpha)\} \geq \min\{\lambda(x), \frac{1-k}{2}\} \geq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}.$$

Thus λ is an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup.

Let $x, a, y \in H$. Then for $w \in x \circ a \circ y = x \circ (a \circ y)$, so there exists $z \in a \circ y$, such that $w \in x \circ z$.

Thus

$$\inf_{w\in x\circ z}\{\lambda(w)\} \geq \min\{\lambda(z), \frac{1-k}{2}\}.$$

$$\Rightarrow \lambda(w) \geq \min\{\lambda(z), \frac{1-k}{2}\}. \qquad (*)$$

As $z \in a \circ y$, so

$$\inf_{\gamma\in a\circ y}\{\lambda(\gamma)\} \geq \min\{\lambda(a), \frac{1-k}{2}\}.$$

$$\Rightarrow \lambda(z) \geq \min\{\lambda(a), \frac{1-k}{2}\}.$$

$$\Rightarrow \min\{\lambda(z), \frac{1-k}{2}\} \geq \min\{\lambda(a), \frac{1-k}{2}\}.$$

Thus from (*) we have

$$\lambda(w) \geq \min\{\lambda(a), \frac{1-k}{2}\}.$$

$\Rightarrow \inf_{w \in x \circ a \circ y} \{\lambda(w)\} \geq \min\{\lambda(a), \frac{1-k}{2}\}$. Hence λ is an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of H. □

The following example shows that an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of H need not be an $(\in, \in \vee q_k)$-fuzzy hyperideal of H. Also union of $(\in, \in \vee q_k)$-fuzzy interior hyperideals of H need not be an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of H.

5.5.8 Example

Let $H = \{a, b, c, d\}$ be a semihypergroup with the following multiplication table:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	$\{a, d\}$	a
c	a	a	a	a
d	a	a	a	a

Then the interior hyperideals of H are $\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, d\}, \{a, c, d\}$ and H but $\{a, b\}$ is not a hyperideal of H. Define fuzzy subsets λ, μ of H by

$$\begin{aligned} \lambda(a) &= 0.8 = \lambda(b), \quad \lambda(c) = 0 = \lambda(d); \\ \mu(a) &= 0.8 = \mu(c), \quad \mu(b) = 0 = \mu(d). \end{aligned}$$

Then we have

$$U(\lambda; t) = \begin{cases} \{a, b\} & \text{if } 0 < t \leq 0.8, \\ \emptyset & \text{if } 0.8 < t \leq 1. \end{cases}$$

$$U(\mu; t) = \begin{cases} \{a, c\} & \text{if } 0 < t \leq 0.8, \\ \emptyset & \text{if } 0.8 < t \leq 1. \end{cases}$$

Thus by Theorem 5.5.6, λ, μ are $(\in, \in \vee q_k)$-fuzzy interior hyperideals of H for every $k \in [0, 1)$. But $U(\lambda \vee \mu; t) = \{a, b, c\}$ if $t \in (0, \frac{1-k}{2}]$ for $k = 0.4$, which is not an interior hyperideal of H, so $\lambda \vee \mu$ is not an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of H.

Also λ is not an $(\in, \in \vee q_k)$-fuzzy hyperideal of H because $\{a, b\}$ is not a hyperideal of H.

5.6 $(\in, \in \vee q_k)$-fuzzy quasi-hyperideals

5.6.1 Definition

A fuzzy subset λ of a semihypergroup H is called an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H, if it satisfies,

$$\lambda(x) \geq \min\{(1 \circ \lambda)(x), (\lambda \circ 1)(x), \frac{1-k}{2}\}.$$

5.6.2 Theorem

Let λ be an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H. Then the set

$$\lambda_0 = \{x \in H \mid \lambda(x) > 0\}$$

is a quasi-hyperideal of H.

Proof. In order to show that λ_0 is a quasi-hyperideal of H, we have to show that $H \circ \lambda_0 \cap \lambda_0 \circ H \subseteq \lambda_0$. Let $a \in H \circ \lambda_0 \cap \lambda_0 \circ H$. This means $a \in H \circ \lambda_0$ and $a \in \lambda_0 \circ H$. So $a \in s \circ x$ and $a \in y \circ t$ for some $s, t \in H$ and $x, y \in \lambda_0$. Thus $\lambda(x) > 0$ and $\lambda(y) > 0$.
Since

$$\begin{aligned}
(1 \circ \lambda)(a) &= \bigvee_{a \in s \circ x} \{1(s) \wedge \lambda(x)\} \\
&\geq \{1(s) \wedge \lambda(x)\} \\
&= \{1 \wedge \lambda(x)\} \\
&= \lambda(x).
\end{aligned}$$

Similarly $(\lambda \circ 1)(a) \geq \lambda(y)$.
Thus

$$\begin{aligned}
\lambda(a) &\geq \min\{(1 \circ \lambda)(a), (\lambda \circ 1)(a), \frac{1-k}{2}\} \\
&\geq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\} \\
&> 0 \text{ because } \lambda(x) > 0 \text{ and } \lambda(y) > 0.
\end{aligned}$$

Thus $a \in \lambda_0$. Hence λ_0 is a quasi-hyperideal of H. □

5.6.3 Lemma

A non-empty subset Q of a semihypergroup H is a quasi-hyperideal of H if and only if the characteristic function λ_Q of Q is an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H.

Proof. Suppose Q is a quasi-hyperideal of H and λ_Q is the characteristic function of Q. Let $x \in H$. If $x \notin Q$ then $x \notin H \circ Q$ or $x \notin Q \circ H$. Thus $(1 \circ \lambda_Q)(x) = 0$ or $(\lambda_Q \circ 1)(x) = 0$ and so $\min\{(1 \circ \lambda_Q)(x), (\lambda_Q \circ 1)(x), \frac{1-k}{2}\} = 0 = \lambda_Q(x)$. If $x \in Q$ then $\lambda_Q(x) = 1 \geq \min\{(1 \circ \lambda_Q)(x), (\lambda_Q \circ 1)(x), \frac{1-k}{2}\}$. Hence λ_Q is an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H.

Conversely, assume that λ_Q is an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H. Then Q is a quasi-hyperideal of H, by Theorem 5.6.2 □

5.6.4 Theorem

Every $(\in, \in \vee q_k)$-fuzzy left (right) hyperideal λ of H is an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H.

Proof. Let λ be an $(\in, \in \vee q_k)$-fuzzy left hyperideal λ of H and $x \in H$. Then

$$(1 \circ \lambda)(x) = \bigvee_{x \in y \circ z} \{1(y) \wedge \lambda(z)\} = \bigvee_{x \in y \circ z} \lambda(z).$$

This implies that

$$\begin{aligned}
(1 \circ \lambda)(x) \wedge \frac{1-k}{2} &= \left(\bigvee_{x \in y \circ z} \lambda(z)\right) \wedge \frac{1-k}{2} \\
&= \bigvee_{x \in y \circ z} \left\{\lambda(z) \wedge \frac{1-k}{2}\right\} \\
&\leq \bigvee_{x \in y \circ z} \lambda(x) \\
&\quad \left(\begin{array}{c} \text{Since } \lambda \text{ is an } (\in, \in \vee q_k)\text{-fuzzy left hyperideal of } H, \\ \text{so } \inf\limits_{x \in y \circ z} \lambda(x) \geq \lambda(z) \wedge \frac{1-k}{2}. \end{array}\right) \\
\text{So } (1 \circ \lambda)(x) \wedge \frac{1-k}{2} &\leq \bigvee_{x \in y \circ z} \lambda(x) = \lambda(x).
\end{aligned}$$

Hence $\lambda(x) \geq (1 \circ \lambda)(x) \wedge \frac{1-k}{2} \geq \min\{(1 \circ \lambda)(x), (\lambda \circ 1)(x), \frac{1-k}{2}\}$. Thus λ is an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H. □

5.6.5 Lemma

Every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of a semihypergroup H is an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H.

Proof. Suppose λ is an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of semihypergroup H. Let $x, y \in H$. Now for every $\alpha \in x \circ y$, we have

$$\begin{aligned}
\lambda(\alpha) &\geq \min\{(1 \circ \lambda)(\alpha), (\lambda \circ 1)(\alpha), \frac{1-k}{2}\} \\
&= \left[\bigvee_{\alpha \in x \circ y} \{1(x) \wedge \lambda(y)\}\right] \wedge \left[\bigvee_{\alpha \in x \circ y} \{\lambda(x) \wedge 1(y)\}\right] \wedge \frac{1-k}{2} \\
&\geq \{1(x) \wedge \lambda(y)\} \wedge \{\lambda(x) \wedge 1(y)\} \wedge \frac{1-k}{2} \\
&\geq \{1 \wedge \lambda(y)\} \wedge \{\lambda(x) \wedge 1\} \wedge \frac{1-k}{2} \\
&= \lambda(y) \wedge \lambda(x) \wedge \frac{1-k}{2}.
\end{aligned}$$

So $\inf\limits_{\alpha \in x \circ y} \{\lambda(\alpha)\} \geq \min\{\lambda(x), \lambda(y), \frac{1-k}{2}\}$ for all $x, y \in H$.

Also for all $x, y, z \in H$ and for every $w \in x \circ y \circ z$, there exists $a \in x \circ y$ such that $w \in a \circ z$. Also there exists $b \in y \circ z$ such that $w \in x \circ b$. Thus

$$\begin{aligned}
\lambda(w) &\geq \min\{(1 \circ \lambda)(w), (\lambda \circ 1)(w), \frac{1-k}{2}\} \\
&= \left[\bigvee_{w \in a \circ z} \{1(a) \wedge \lambda(z)\}\right] \wedge \left[\bigvee_{w \in x \circ b} \{\lambda(x) \wedge 1(b)\}\right] \wedge \frac{1-k}{2} \\
&\geq \{1(a) \wedge \lambda(z)\} \wedge \{\lambda(x) \wedge 1(b)\} \wedge \frac{1-k}{2} \\
&\geq \{1 \wedge \lambda(z)\} \wedge \{\lambda(x) \wedge 1\} \wedge \frac{1-k}{2} \\
&= \lambda(z) \wedge \lambda(x) \wedge \frac{1-k}{2}.
\end{aligned}$$

So $\inf\limits_{w \in x \circ y \circ z} \{\lambda(w)\} \geq \min\{\lambda(x), \lambda(z), \frac{1-k}{2}\}$ for all $x, y, z \in H$. Thus λ is an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H. □

The following example shows that the converse of the above Theorem is not true.

5.6.6 Example

Consider the semihypergroup $H = \{a, b, c, d\}$ with the following table:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	$\{a, b\}$
d	a	a	$\{a, b\}$	$\{a, b, c\}$

One can easily check that $\{a, c\}$ is a bi-hyperideal of H but not a quasi-hyperideal of H.

Define a fuzzy subset λ of H by

$$\lambda(a) = 0.8 = \lambda(c), \qquad \lambda(b) = 0 = \lambda(d).$$

Then λ is an $(\in, \in \vee q_k)$-fuzzy bi-hyperideal of H for every $k \in [0, 1)$ but not an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H. Because $(1 \circ \lambda)(b) = 0.8 = (\lambda \circ 1)(b)$ but $\lambda(b) = 0 \ngeq \min\{(1 \circ \lambda)(b), (\lambda \circ 1)(b), \frac{1-k}{2}\}$.

5.7 Regular semihypergroups

In this section we characterize regular semihypergroups by the properties of their $(\in, \in \vee q_k)$-fuzzy hyperideals, $(\in, \in \vee q_k)$-fuzzy quasi- hyperideals and $(\in, \in \vee q_k)$-fuzzy bi-hyperideals.

5.7.1 Definition

Let λ, μ be fuzzy subsets of H. We define the fuzzy subsets $\lambda_k, \lambda \wedge_k \mu, \lambda \vee_k \mu$ and $\lambda \circ_k \mu$ of H as follows;

$$\begin{aligned} \lambda_k(x) &= \lambda(x) \wedge \frac{1-k}{2} \\ (\lambda \wedge_k \mu)(x) &= (\lambda \wedge \mu)(x) \wedge \frac{1-k}{2} \\ (\lambda \vee_k \mu)(x) &= (\lambda \vee \mu)(x) \wedge \frac{1-k}{2} \\ (\lambda \circ_k \mu)(x) &= (\lambda \circ \mu)(x) \wedge \frac{1-k}{2} \qquad \text{for all } x \in H. \end{aligned}$$

5.7.2 Lemma

Let λ, μ be fuzzy subsets of H. Then the following hold.

(1) $(\lambda \wedge_k \mu) = (\lambda_k \wedge \mu_k)$

(2) $(\lambda \vee_k \mu) = (\lambda_k \vee \mu_k)$

(3) $(\lambda \circ_k \mu) = (\lambda_k \circ \mu_k)$.

Proof. Let $x \in H$.

(1)

$$\begin{aligned}(\lambda \wedge_k \mu)(x) &= (\lambda \wedge \mu)(x) \wedge \frac{1-k}{2} \\ &= \lambda(x) \wedge \mu(x) \wedge \frac{1-k}{2} \\ &= \left(\lambda(x) \wedge \frac{1-k}{2}\right) \wedge \left(\mu(x) \wedge \frac{1-k}{2}\right) \\ &= \lambda_k(x) \wedge \mu_k(x) \\ &= (\lambda_k \wedge \mu_k)(x).\end{aligned}$$

(2)

$$\begin{aligned}(\lambda \vee_k \mu)(x) &= (\lambda \vee \mu)(x) \wedge \frac{1-k}{2} \\ &= (\lambda(x) \vee \mu(x)) \wedge \frac{1-k}{2} \\ &= \left(\lambda(x) \wedge \frac{1-k}{2}\right) \vee \left(\mu(x) \wedge \frac{1-k}{2}\right) \\ &= \lambda_k(x) \vee \mu_k(x) \\ &= (\lambda_k \vee \mu_k)(x).\end{aligned}$$

(3) If $x \notin y \circ z$ for all $y, z \in H$, then $(\lambda \circ \mu)(x) = 0$. Thus

$$(\lambda \circ_k \mu)(x) = (\lambda \circ \mu)(x) \wedge \frac{1-k}{2} = 0.$$

If $x \in y \circ z$ for some $y, z \in H$, then

$$\begin{aligned}(\lambda \circ_k \mu)(x) &= (\lambda \circ \mu)(x) \wedge \frac{1-k}{2} \\ &= \left[\bigvee_{x \in y \circ z} \{\lambda(y) \wedge \mu(z)\}\right] \wedge \frac{1-k}{2} \\ &= \bigvee_{x \in y \circ z} \left(\{\lambda(y) \wedge \mu(z)\} \wedge \frac{1-k}{2}\right) \\ &= \bigvee_{x \in y \circ z} \left\{\lambda(y) \wedge \mu(z) \wedge \frac{1-k}{2}\right\} \\ &= \bigvee_{x \in y \circ z} \left\{\left(\lambda(y) \wedge \frac{1-k}{2}\right) \wedge \left(\mu(z) \wedge \frac{1-k}{2}\right)\right\} \\ &= \bigvee_{x \in y \circ z} \{\lambda_k(y) \wedge \mu_k(z)\} \\ &= (\lambda_k \circ \mu_k)(x).\end{aligned}$$

□

5.7.3 Lemma

Let A and B be nonempty subsets of a semihypergroup H. Then the following hold.

(1) $(\lambda_A \wedge_k \lambda_B) = (\lambda_{A\cap B})_k$

(2) $(\lambda_A \vee_k \lambda_B) = (\lambda_{A\cup B})_k$

(3) $(\lambda_A \circ_k \lambda_B) = (\lambda_{A\circ B})_k$.

Proof. Straightforward. □

Next we show that if λ is an $(\in, \in \vee q_k)$-fuzzy left (right) hyperideal of H then λ_k is a fuzzy left (right) hyperideal of H.

5.7.4 Lemma

Let λ be an $(\in, \in \vee q_k)$-fuzzy left (right) hyperideal of H. Then λ_k is a fuzzy left (right) hyperideal of H.

Proof. Let λ be an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H. Then for all $x, y \in H$, we have $\inf\limits_{z\in x\circ y}\{\lambda(z)\} \geq \lambda(y) \wedge \frac{1-k}{2}$. This implies that

$$\inf_{z\in x\circ y}\{\lambda(z)\} \wedge \frac{1-k}{2} \geq \lambda(y) \wedge \frac{1-k}{2}.$$

So $\inf\limits_{z\in x\circ y}\{\lambda_k(z)\} \geq \lambda_k(y)$. Thus λ_k is a fuzzy left hyperideal of H. □

5.7.5 Lemma

A nonempty subset L of H is a left (right) hyperideal of H if and and only if $(\lambda_L)_k$ is an $(\in, \in \vee q_k)$-fuzzy left (right) hyperideal of H.

Proof. The proof follows from Theorem 5.2.2. □

5.7.6 Lemma

A non-empty subset Q of a semihypergroup H is a quasi-hyperideal of H if $(\lambda_Q)_k$ is an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H.

Proof. The proof follows from Theorem 5.2.2. □

5.7.7 Proposition

Let λ be a fuzzy subsemihypergroup H. Then $\lambda \circ_k \lambda \leq \lambda_k$.

Proof. Let λ be a fuzzy subsemihypergroup H. If $A_x = \emptyset$, then

$$(\lambda \circ \lambda)(x) = 0 \leq \lambda(x).$$

If $A_x \neq \emptyset$, then$(\lambda \circ_k \lambda)(x) = (\lambda \circ \mu)(x) \wedge \frac{1-k}{2}$. As λ is a fuzzy subsemihypergroup of H, so for each $x \in y \circ z$,

$$\lambda(x) \geq \min\{\lambda(y), \lambda(z)\}, \text{ for all } y, z \in H.$$

This implies that

$$\lambda_k(x) = \lambda(x) \wedge \frac{1-k}{2} \geq \min\left\{\lambda(y), \lambda(z), \frac{1-k}{2}\right\}, \text{ for all } y, z \in H.$$

Hence $(\lambda \circ_k \lambda)(x) = (\lambda \circ \mu)(x) \wedge \frac{1-k}{2} = \bigvee_{x \in y \circ z} \min\{\lambda(y), \lambda(z), \frac{1-k}{2}\} \leq \lambda_k(x)$. Thus, $\lambda \circ_k \lambda \leq \lambda_k$. □

Next we characterize regular semihypergroups by the properties of $(\in, \in \vee q_k)$-fuzzy hyperideals, quasi-hyperideals, bi-hyperideals and generalized bi-hyperideals.

5.7.8 Lemma

Let λ be an $(\in, \in \vee q_k)$-fuzzy right hyperideal and μ be an $(\in, \in \vee q_k)$-fuzzy left hyperideal of a semihypergroup H. Then $\lambda \circ_k \mu \leq \lambda \wedge_k \mu$.

Proof. Let λ be an $(\in, \in \vee q_k)$-fuzzy right hyperideal and μ be an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H.

Let $y, z \in H$. Then for every $a \in y \circ z$ we have

$$\begin{aligned}
(\lambda \circ_k \mu)(a) &= (\lambda \circ \mu)(a) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a \in y \circ z} \{\lambda(y) \wedge \mu(z)\}\right) \wedge \frac{1-k}{2} \\
&= \bigvee_{a \in y \circ z} \left(\{\lambda(y) \wedge \mu(z)\} \wedge \frac{1-k}{2}\right) \\
&= \bigvee_{a \in y \circ z} \left\{\left(\lambda(y) \wedge \frac{1-k}{2}\right) \wedge \left(\mu(z) \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2}\right\} \\
&\leq \bigvee_{a \in y \circ z} \left\{\lambda(a) \wedge \mu(a) \wedge \frac{1-k}{2}\right\}.
\end{aligned}$$

Thus

$$\begin{aligned}(\lambda \circ_k \mu)(a) &\leq \bigvee_{a\in y\circ z}\left\{\lambda(a)\wedge\mu(a)\wedge\frac{1-k}{2}\right\}\\ &= \lambda(a)\wedge\mu(a)\wedge\frac{1-k}{2}\\ &= (\lambda\wedge_k\mu)(a).\end{aligned}$$

So $(\lambda\circ_k\mu)\leq(\lambda\wedge_k\mu)$. □

5.7.9 Theorem

The following statements for a semihypergroup H are equivalent:

(1) *H is regular.*

(2) *$\lambda\wedge_k\mu=\lambda\circ_k\mu$ for every $(\in,\in\vee q_k)$-fuzzy right hyperideal λ and every $(\in,\in\vee q_k)$-fuzzy left hyperideal μ of H.*

Proof. (1) ⇒ (2) Since H is regular, so there exists $x\in H$ such that $a\in a\circ x\circ a=(a\circ x)\circ a$. Thus there exists some $\beta\in a\circ x$ such that $a\in\beta\circ a$. So

$$\begin{aligned}(\lambda \circ_k \mu)(a) &= (\lambda\circ\mu)(a)\wedge\frac{1-k}{2}\\ &= \left(\bigvee_{a\in c\circ d}\{\lambda(c)\wedge\mu(d)\}\right)\wedge\frac{1-k}{2}\\ &\geq \{\lambda(\beta)\wedge\mu(a)\}\wedge\frac{1-k}{2}\end{aligned}$$

$$\left(\begin{array}{c}\text{Since } \beta\in a\circ x \text{ and } \lambda \text{ is an } (\in,\in\vee q_k)\text{-fuzzy right}\\ \text{hyperideal of } H, \text{ so } \inf\limits_{\gamma\in\theta\circ\delta}\{\lambda(z)\}\geq\lambda(\theta)\wedge\frac{1-k}{2}\\ \text{therefore } \lambda(\beta)\geq\lambda(\theta)\wedge\frac{1-k}{2}.\end{array}\right)$$

$$\begin{aligned}\text{Therefore } (\lambda \circ_k \mu)(a) &\geq \left\{\lambda(a)\wedge\frac{1-k}{2}\wedge\mu(a)\right\}\wedge\frac{1-k}{2}\\ &= \{\lambda(a)\wedge\mu(a)\}\wedge\frac{1-k}{2}\\ &= (\lambda\wedge_k\mu)(a).\end{aligned}$$

So $\lambda\circ_k\mu\geq\lambda\wedge_k\mu$. But by Lemma 5.7.8, $\lambda\circ_k\mu\leq\lambda\wedge_k\mu$. Hence $\lambda\wedge_k\mu=\lambda\circ_k\mu$.

(2) ⇒ (1) Let R and L be right and left hyperideals of H. Then by Lemma 5.7.5, $(\lambda_R)_k$ and $(\lambda_L)_k$ are $(\in,\in\vee q_k)$-fuzzy right and $(\in,\in\vee q_k)$-fuzzy left hyperideals of H, respectively. Thus by hypothesis

$$\begin{aligned}(\lambda_{R\circ L})_k &= (\lambda_R\circ_k\lambda_L)\\ &= (\lambda_R\wedge_k\lambda_L)\ \text{ by (2)}\\ &= (\lambda_{R\cap L})_k.\end{aligned}$$

This implies $R \cap L = R \circ L$. Hence it follows from Proposition 3.2.5 that H is regular. □

5.7.10 Theorem

For a semihypergroup H, the following conditions are equivalent:

(1) *H is regular.*

(2) $(\lambda \wedge_k \mu \wedge_k \nu) \leq (\lambda \circ_k \mu \circ_k \nu)$ *for every $(\in, \in \vee q_k)$-fuzzy right hyperideal λ, for every $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal μ and for every $(\in, \in \vee q_k)$-fuzzy left hyperideal ν of H.*

(3) $(\lambda \wedge_k \mu \wedge_k \nu) \leq (\lambda \circ_k \mu \circ_k \nu)$ *for every $(\in, \in \vee q_k)$-fuzzy right hyperideal λ, for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal μ and for every $(\in, \in \vee q_k)$-fuzzy left hyperideal ν of H.*

(4) $(\lambda \wedge_k \mu \wedge_k \nu) \leq (\lambda \circ_k \mu \circ_k \nu)$ *for every $(\in, \in \vee q_k)$-fuzzy right hyperideal λ, for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal μ and for every $(\in, \in \vee q_k)$-fuzzy left hyperideal ν of H.*

Proof. (1) ⇒ (2) Let λ, μ and ν be any $(\in, \in \vee q_k)$-fuzzy right hyperideal, $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal and $(\in, \in \vee q_k)$-fuzzy left hyperideal of H, respectively. Let $a \in H$. Since H is regular, so there exists $x \in H$ such that $a \in a \circ x \circ a = (a \circ x) \circ a$. Thus there exists $\beta \in a \circ x$ such that $a \in \beta \circ a$. So

$$\begin{aligned}
(\lambda \circ_k \mu \circ_k \nu)(a) &= (\lambda \circ \mu \circ \nu)(a) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a \in \beta \circ a} \{\lambda(\beta) \wedge (\mu \circ \nu)(a)\} \right) \wedge \frac{1-k}{2} \\
&\geq \{\lambda(\beta) \wedge (\mu \circ \nu)(a)\} \wedge \frac{1-k}{2} \\
&\qquad \left(\text{since } \inf_{\beta \in a \circ x} \{\lambda(\beta)\} \geq \lambda(a) \wedge \frac{1-k}{2}. \right) \\
\text{so, } (\lambda \circ_k \mu \circ_k \nu)(a) &\geq \left(\lambda(a) \wedge \frac{1-k}{2} \right) \wedge (\mu \circ \nu)(a) \wedge \frac{1-k}{2} \qquad \text{(i)}
\end{aligned}$$

Since $a \in a \circ x \circ a = a \circ (x \circ a)$, so there exists $\gamma \in x \circ a$, such that $a \in a \circ \gamma$. Thus

$$\begin{aligned}
(\mu \circ \nu)(a) &= \bigvee_{a \in a \circ \gamma} \{\mu(a) \wedge \nu(\gamma)\} \\
&\geq \{\mu(a) \wedge \nu(\gamma)\} \\
&\qquad \left(\text{because } \inf_{\gamma \in x \circ a} \{\nu(\gamma)\} \geq \nu(a) \wedge \frac{1-k}{2}. \right) \\
&\geq \left(\mu(a) \wedge \nu(a) \wedge \frac{1-k}{2} \right)
\end{aligned}$$

Thus substitiuting value of $(\mu \circ \nu)(a)$ in (i), we have

$$\begin{aligned}(\lambda \circ_k \mu \circ_k \nu)(a) &\geq \left(\lambda(a) \wedge \frac{1-k}{2}\right) \wedge \left(\mu(a) \wedge \nu(a) \wedge \frac{1-k}{2}\right) \\ &= (\lambda(a) \wedge \mu(a) \wedge \nu(a)) \wedge \frac{1-k}{2} \\ &= (\lambda \wedge_k \mu \wedge_k \nu)(a).\end{aligned}$$

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) Straightforward.

(4) $\Rightarrow$ (1) Let λ and ν be any $(\in, \in \vee q_k)$-fuzzy right and $(\in, \in \vee q_k)$-fuzzy left hyperideals of H, respectively. Since "1" is an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H, so by hypothesis, we have

$$\begin{aligned}(\lambda \wedge_k \nu)(a) &= (\lambda \wedge \nu)(a) \wedge \frac{1-k}{2} \\ &= (\lambda \wedge 1 \wedge \nu)(a) \wedge \frac{1-k}{2} \\ &= (\lambda \wedge_k 1 \wedge_k \nu)(a) \\ &\leq (\lambda \circ_k 1 \circ_k \nu)(a) \\ &= (\lambda \circ 1 \circ \nu)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in b \circ c} \{(\lambda \circ 1)(b) \wedge \nu(c)\}\right) \wedge \frac{1-k}{2} \\ &\quad \left(\text{As } (\lambda \circ 1)(b) = \bigvee_{b \in p \circ q} \{\lambda(p) \wedge 1(q)\}\right).\end{aligned}$$

Thus

$$\begin{aligned}(\lambda \wedge_k \nu)(a) &= \left(\bigvee_{a\in b\circ c}\left\{\left(\bigvee_{b\in p\circ q}\{\lambda(p)\wedge 1(q)\}\right)\wedge \nu(c)\right\}\right)\wedge \frac{1-k}{2}\\ &= \left(\bigvee_{a\in b\circ c}\left\{\left(\bigvee_{b\in p\circ q}\{\lambda(p)\wedge 1\}\right)\wedge \nu(c)\right\}\right)\wedge \frac{1-k}{2}\\ &= \left(\bigvee_{a\in b\circ c}\left\{\left(\bigvee_{b\in p\circ q}\lambda(p)\right)\wedge \nu(c)\right\}\right)\wedge \frac{1-k}{2}\\ &= \left(\bigvee_{a\in b\circ c}\left\{\left(\bigvee_{b\in p\circ q}\lambda(p)\right)\wedge \nu(c)\right\}\wedge \frac{1-k}{2}\right)\wedge \frac{1-k}{2}\\ &= \left(\bigvee_{a\in b\circ c}\left\{\left(\bigvee_{b\in p\circ q}\left\{\lambda(p)\wedge \frac{1-k}{2}\right\}\right)\wedge \nu(c)\right\}\right)\wedge \frac{1-k}{2}\end{aligned}$$

$$\left(\begin{array}{c}\text{Since } \lambda \text{ is an } (\in,\in\vee q_k)\text{-fuzzy right hyperidealof } H, \text{ so}\\ \inf_{b\in p\circ q}\{\lambda(b)\}\geq \lambda(p)\wedge \frac{1-k}{2} \text{ that is}\\ \lambda(b)\geq \lambda(p)\wedge \frac{1-k}{2} \text{ for every } b\in p\circ q.\end{array}\right)$$

Hence

$$\begin{aligned}(\lambda \wedge_k \nu)(a) &\leq \left(\bigvee_{a\in b\circ c}\{\lambda(b)\wedge \nu(c)\}\right)\wedge \frac{1-k}{2}\\ &= \left(\bigvee_{a\in b\circ c}\lambda(b)\wedge \nu(c)\right)\wedge \frac{1-k}{2}\\ &= (\lambda \circ_k \nu)(a).\end{aligned}$$

Thus it follows that $\lambda \wedge_k \nu \leq \lambda \circ_k \nu$ for every $(\in,\in\vee q_k)$-fuzzy right hyperideal λ of H and for every $(\in,\in\vee q_k)$-fuzzy left hyperideal ν of H. But by Lemma 5.7.8, $\lambda \wedge_k \nu \geq \lambda \circ_k \nu$. So $\lambda \wedge_k \nu = \lambda \circ_k \nu$. Hence by Theorem 5.7.9, H is regular. □

5.7.11 Theorem

For a semihypergroup H, the following conditions are equivalent:

(1) *H is regular.*

(2) *$\lambda_k = (\lambda \circ_k 1 \circ_k \lambda)$ for every $(\in,\in\vee q_k)$-fuzzy generalized bi-hyperideal λ of H.*

(3) *$\lambda_k = (\lambda \circ_k 1 \circ_k \lambda)$ for every $(\in,\in\vee q_k)$-fuzzy bi-hyperideal λ of H.*

(4) *$\lambda_k = (\lambda \circ_k 1 \circ_k \lambda)$ for every $(\in,\in\vee q_k)$-fuzzy quasi-hyperideal λ of H.*

Proof. (1) ⇒ (2) Let λ be an $(\in,\in\vee q_k)$-fuzzy generalized bi-hyperideal of H and $a\in H$. Since H is regular, so there exists $x\in H$ such that $a\in a\circ x\circ a=(a\circ x)\circ a$.

Thus there exists $\beta \in a \circ x$ such that $a \in \beta \circ a$. So

$$\begin{aligned}
(\lambda \circ_k 1 \circ_k \lambda)(a) &= (\lambda \circ 1 \circ \lambda)(a) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a \in \beta \circ a} \{(\lambda \circ 1)(\beta) \wedge \lambda(a)\} \right) \wedge \frac{1-k}{2} \\
&\geq (\lambda \circ 1)(\beta) \wedge \lambda(a) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{\beta \in a \circ x} \{\lambda(a) \wedge 1(x)\} \right) \wedge \lambda(a) \wedge \frac{1-k}{2} \\
&\geq \{\lambda(a) \wedge 1(x)\} \wedge \lambda(a) \wedge \frac{1-k}{2} \\
&= \{\lambda(a) \wedge 1\} \wedge \lambda(a) \wedge \frac{1-k}{2} \\
&= \lambda_k(a).
\end{aligned}$$

Thus $(\lambda \circ_k 1 \circ_k \lambda) \geq \lambda_k$.
Since λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal of H. So we have

$$\begin{aligned}
(\lambda \circ_k 1 \circ_k \lambda)(a) &= (\lambda \circ 1 \circ \lambda)(a) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a \in x \circ y} \{(\lambda \circ 1)(x) \wedge \lambda(y)\} \right) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a \in x \circ y} \left\{ \left(\bigvee_{x \in p \circ q} \{\lambda(p) \wedge 1(q)\} \right) \wedge \lambda(y) \right\} \right) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a \in x \circ y} \left\{ \left(\bigvee_{x \in p \circ q} \{\lambda(p) \wedge 1\} \right) \wedge \lambda(y) \right\} \right) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a \in x \circ y} \left\{ \bigvee_{x \in p \circ q} \{\lambda(p) \wedge \lambda(y)\} \right\} \right) \wedge \frac{1-k}{2} \\
&= \bigvee_{a \in x \circ y} \left\{ \bigvee_{x \in p \circ q} \{\lambda(p) \wedge \lambda(y)\} \wedge \frac{1-k}{2} \right\}
\end{aligned}$$

$$\left(\begin{array}{c} \text{Since } a \in p \circ q \circ y \text{ and } \lambda \text{ is an } (\in, \in \vee q_k)\text{-fuzzy} \\ \text{generalized bi-hyperideal of } H \text{ so} \\ \inf\limits_{a \in p \circ q \circ y} \{\lambda(a)\} \geq \{\lambda(p) \wedge \lambda(y)\} \wedge \frac{1-k}{2}. \end{array} \right)$$

Thus $(\lambda \circ_k 1 \circ_k \lambda)(a) \leq \lambda(a) \wedge \frac{1-k}{2} = \lambda_k(a)$. This implies $(\lambda \circ_k 1 \circ_k \lambda) \leq \lambda_k$. Hence $\lambda_k = (\lambda \circ_k 1 \circ_k \lambda)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1) Let A be any quasi-hyperideal of H. Then λ_A is an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H. Hence, by hypothesis,

$$(\lambda_A)_k = (\lambda_A \circ_k 1 \circ_k \lambda_A) = (\lambda_A \circ_k \lambda_H \circ_k \lambda_A) = (\lambda_{A \circ H \circ A})_k.$$

This implies $A = A \circ H \circ A$. Hence it follows from Proposition 3.2.4, that H is regular. □

5.7.12 Theorem

For a semihypergroup H, the following conditions are equivalent:

(1) *H is regular.*

(2) $(\lambda \wedge_k \mu) = (\lambda \circ_k \mu \circ_k \lambda)$ *for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy hyperideal μ of H.*

(3) $(\lambda \wedge_k \mu) = (\lambda \circ_k \mu \circ_k \lambda)$ *for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy interior hyperideal μ of H.*

(4) $(\lambda \wedge_k \mu) = (\lambda \circ_k \mu \circ_k \lambda)$ *for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy hyperideal μ of H.*

(5) $(\lambda \wedge_k \mu) = (\lambda \circ_k \mu \circ_k \lambda)$ *for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy interior hyperideal μ of H.*

(6) $(\lambda \wedge_k \mu) = (\lambda \circ_k \mu \circ_k \lambda)$ *for every $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy hyperideal μ of H.*

(7) $(\lambda \wedge_k \mu) = (\lambda \circ_k \mu \circ_k \lambda)$ *for every $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy interior hyperideal μ of H.*

Proof. (1) $\Rightarrow$ (7) Let λ and μ be any $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal and

$(\in, \in \vee q_k)$-fuzzy interior hyperideal of H, respectively. Then

$$\begin{aligned}
(\lambda \circ_k \mu \circ_k \lambda)(a) &= (\lambda \circ \mu \circ \lambda)(a) \wedge \frac{1-k}{2} \\
&\leq (\lambda \circ 1 \circ \lambda)(a) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a \in x \circ y} \{(\lambda \circ 1)(x) \wedge \lambda(y)\} \right) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a \in x \circ y} \left\{ \left(\bigvee_{x \in p \circ q} \{\lambda(p) \wedge 1(q)\} \right) \wedge \lambda(y) \right\} \right) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a \in x \circ y} \left\{ \left(\bigvee_{x \in p \circ q} \{\lambda(p) \wedge 1\} \right) \wedge \lambda(y) \right\} \right) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a \in x \circ y} \left\{ \bigvee_{x \in p \circ q} \{\lambda(p) \wedge \lambda(y)\} \right\} \right) \wedge \frac{1-k}{2} \\
&= \bigvee_{a \in x \circ y} \left\{ \bigvee_{x \in p \circ q} \{\lambda(p) \wedge \lambda(y)\} \wedge \frac{1-k}{2} \right\}.
\end{aligned}$$

Since $a \in p \circ q \circ y$ and λ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal of H so

$$\inf_{a \in p \circ q \circ y} \{\lambda(a)\} \geq \{\lambda(p) \wedge \lambda(y)\} \wedge \frac{1-k}{2}.$$

Thus

$$\begin{aligned}
(\lambda \circ_k \mu \circ_k \lambda)(a) &\leq \bigvee_{a \in p \circ q \circ y} \left\{ \lambda(a) \wedge \frac{1-k}{2} \right\} \\
&\leq \lambda(a) \wedge \frac{1-k}{2} \\
&= \lambda_k(a).
\end{aligned}$$

Thus $(\lambda \circ_k \mu \circ_k \lambda)(a) \leq \lambda_k(a)$. Also

$$\begin{aligned}
(\lambda \circ_k \mu \circ_k \lambda)(a) &\leq (1 \circ_k \mu \circ_k 1)(a) \\
&= (1 \circ \mu \circ 1)(a) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a \in x \circ y} \{(1 \circ \mu)(x) \wedge 1(y)\} \right) \wedge \frac{1-k}{2} \\
&\quad \left(\begin{array}{c} \text{Since } a \in x \circ y \text{ but for } x \in p \circ q, \text{ we have} \\ (1 \circ \mu)(x) = \bigvee_{x \in p \circ q} \{1(p) \wedge \mu(q)\} \end{array} \right) \\
&= \left(\bigvee_{a \in x \circ y} \left\{ \left(\bigvee_{x \in p \circ q} \{1(p) \wedge \mu(q)\} \right) \wedge 1(y) \right\} \right) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a \in x \circ y} \left\{ \left(\bigvee_{x \in p \circ q} \{1 \wedge \mu(q)\} \right) \wedge 1 \right\} \right) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a \in x \circ y} \left\{ \bigvee_{x \in p \circ q} \mu(q) \right\} \right) \wedge \frac{1-k}{2} \\
&= \bigvee_{a \in x \circ y \subseteq p \circ q \circ y} \left\{ \mu(q) \wedge \frac{1-k}{2} \right\}
\end{aligned}$$

Since μ is an $(\in, \in \vee q_k)$-fuzzy interior hyperideal of H, so there exist $r, s, \theta \in H$ such that $\inf_{z \in r \circ \theta \circ s} \{\mu(z)\} \geq \mu(\theta) \wedge \frac{1-k}{2}$. Thus $\mu(a) \geq \mu(q) \wedge \frac{1-k}{2}$. But $a \in x \circ y \subseteq (p \circ q) \circ y$, because $x \in p \circ q$, therefore,

$$\begin{aligned}
(\lambda \circ_k \mu \circ_k \lambda)(a) &\leq \bigvee_{a \in p \circ q \circ y} \left\{ \mu(a) \wedge \frac{1-k}{2} \right\} \\
&= \mu(a) \wedge \frac{1-k}{2} = \mu_k(a).
\end{aligned}$$

Hence $(\lambda \circ_k \mu \circ_k \lambda) \leq (\lambda_k \wedge \mu_k) = (\lambda \wedge_k \mu)$.
Now let $a \in H$. Since H is regular, so there exists $x \in H$ such that $a \in a \circ x \circ a = a \circ (x \circ a \circ x \circ a)$. Thus there exists $\gamma \in x \circ a \circ x$, and $\beta \in \gamma \circ a$

such that $a \in a \circ \beta$. So

$$\begin{aligned}
(\lambda \circ_k \mu \circ_k \lambda)(a) &= (\lambda \circ \mu \circ \lambda)(a) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a \in a \circ \beta} \{\lambda(a) \wedge (\mu \circ \lambda)(\beta)\} \right) \wedge \frac{1-k}{2} \\
&\geq \{\lambda(a) \wedge (\mu \circ \lambda)(\beta)\} \wedge \frac{1-k}{2} \\
&= \lambda(a) \wedge \left(\bigvee_{\beta \in \gamma \circ a} \{\mu(\gamma) \wedge \lambda(a)\} \right) \wedge \frac{1-k}{2} \\
&\quad \left(\begin{array}{c} \text{Since every } (\in, \in \vee q_k)\text{-fuzzy hyperideal of } H \text{ is} \\ \text{an } (\in, \in \vee q_k)\text{-fuzzy hyperideal of } H \\ \text{so } \inf\limits_{\gamma \in x \circ a \circ x} \{\mu(\gamma)\} \geq \mu(a) \wedge \frac{1-k}{2}. \end{array} \right) \\
&\geq \lambda(a) \wedge \left(\mu(a) \wedge \frac{1-k}{2} \wedge \lambda(a) \right) \wedge \frac{1-k}{2} \\
&= \lambda(a) \wedge \mu(a) \wedge \frac{1-k}{2} \\
&= (\lambda \wedge_k \mu)(a).
\end{aligned}$$

So $(\lambda \circ_k \mu \circ_k \lambda) \geq (\lambda \wedge_k \mu)$. Hence $(\lambda \circ_k \mu \circ_k \lambda) = (\lambda \wedge_k \mu)$.

(7) $\Rightarrow$ (5) $\Rightarrow$ (3) $\Rightarrow$ (2) and (7) $\Rightarrow$ (6) $\Rightarrow$ (4) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (1) Let λ be any $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H. Then, since "1" is an $(\in, \in \vee q_k)$-fuzzy two-sided hyperideal of H, we have

$$\begin{aligned}
\lambda_k(a) &= \lambda(a) \wedge \frac{1-k}{2} \\
&= (\lambda \wedge 1)(a) \wedge \frac{1-k}{2} \\
&= (\lambda \wedge_k 1)(a) \\
&= (\lambda \circ_k 1 \circ_k \lambda)(a).
\end{aligned}$$

Thus it follows from Theorem 5.7.11 that H is regular. □

5.7.13 Theorem

For a semihypergroup H, the following conditions are equivalent:

(1) *H is regular.*

(2) $(\lambda \wedge_k \mu) \leq (\lambda \circ_k \mu)$ *for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy left hyperideal μ of H.*

(3) $(\lambda \wedge_k \mu) \leq (\lambda \circ_k \mu)$ *for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal λ and every $(\in, \in \vee q_k)$-fuzzy left hyperideal μ of H.*

(4) $(\lambda \wedge_k \mu) \leq (\lambda \circ_k \mu)$ *for every* $(\in, \in \vee q_k)$*-fuzzy generalized bi-hyperideal* λ *and every* $(\in, \in \vee q_k)$*-fuzzy left hyperideal* μ *of* H.

Proof. (1) $\Rightarrow$ (4) Let λ and μ be an $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal and any $(\in, \in \vee q_k)$-fuzzy left hyperideal of H, respectively. Let $a \in H$. Since H is regular, so there exists $x \in H$ such that $a \in a \circ x \circ a = a \circ (x \circ a)$. Then there exists $\beta \in x \circ a$ such that $a \in a \circ \beta$. Thus we have

$$\begin{aligned}
(\lambda \circ_k \mu)(a) &= (\lambda \circ \mu)(a) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a \in a \circ \beta} \{\lambda(a) \wedge \mu(\beta)\} \right) \wedge \frac{1-k}{2} \\
&\geq \lambda(a) \wedge \mu(\beta) \wedge \frac{1-k}{2} \\
&\geq \lambda(a) \wedge \mu(a) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2} \quad \text{(because } \beta \in x \circ a.\text{)} \\
&= \lambda(a) \wedge \mu(a) \wedge \frac{1-k}{2} \\
&= (\lambda \wedge_k \mu)(a).
\end{aligned}$$

So $(\lambda \circ_k \mu) \geq (\lambda \wedge_k \mu)$.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) are clear.

(2) $\Rightarrow$ (1) Let λ and μ be an $(\in, \in \vee q_k)$-fuzzy right hyperideal and any $(\in, \in \vee q_k)$-fuzzy left hyperideal of H, respectively. Since every $(\in, \in \vee q_k)$-fuzzy right hyperideal of H is an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H. So $(\lambda \circ_k \mu) \geq (\lambda \wedge_k \mu)$. By Lemma 5.7.8, $\lambda \circ_k \mu \leq \lambda \wedge_k \mu$. Thus $(\lambda \circ_k \mu) = (\lambda \wedge_k \mu)$ for every $(\in, \in \vee q_k)$-fuzzy right hyperideal and for every $(\in, \in \vee q_k)$-fuzzy left hyperideal of H. Hence by Theorem 5.7.9 H is regular. □

5.8 Intra-regular semihypergroups

Recall that a semihypergroup H is intra-regular if for each $a \in H$, there exist $x, y \in H$ such that $a \in x \circ a \circ a \circ y$. In general neither intra-regular semihypergroups are regular nor regular semihypergroups are intra-regular semihypergroups. However, in commutative semihypergroups both the concepts coincide.

5.8.1 Theorem

For a semihypergroup H*, the following conditions are equivalent:*

(1) H *is intra-regular.*

(2) $(\lambda \wedge_k \mu) \leq (\lambda \circ_k \mu)$ *for every* $(\in, \in \vee q_k)$*-fuzzy left hyperideal* λ *and every* $(\in, \in \vee q_k)$*-fuzzy right hyperideal* μ *of* H.

Proof. (1) $\Rightarrow$ (2) Let λ be an $(\in, \in \vee q_k)$-fuzzy left hyperideal and μ be an $(\in, \in \vee q_k)$-fuzzy right hyperideal of H. For $a \in H$, there exist $x, y \in H$ such that $a \in x \circ a \circ a \circ y = (x \circ a) \circ (a \circ y)$. Thus there exists $\beta \in x \circ a$ and $\gamma \in a \circ y$, such that $a \in \beta \circ \gamma$.

Thus

$$
\begin{aligned}
(\lambda \circ_k \mu)(a) &= (\lambda \circ \mu)(a) \wedge \frac{1-k}{2} \\
&= \left(\bigvee_{a \in c \circ d} \{\lambda(c) \wedge \mu(d)\} \right) \wedge \frac{1-k}{2} \\
&\geq \{\lambda(\beta) \wedge \mu(\gamma)\} \wedge \frac{1-k}{2} \\
&\quad \left(\begin{array}{c} \text{Since } \beta \in x \circ a \text{ and } \lambda \text{ is an } (\in, \in \vee q_k)\text{-fuzzy left hyperideal of } H, \\ \text{so } \inf\limits_{z \in x \circ a} \{\lambda(z)\} \geq \lambda(a) \wedge \frac{1-k}{2}. \text{ Thus } \lambda(\beta) \geq \lambda(a) \wedge \frac{1-k}{2}. \\ \text{Also since } \mu \text{ is an } (\in, \in \vee q_k)\text{-fuzzy right hyperideal of } H, \\ \text{so } \inf\limits_{z \in a \circ x} \{\mu(z)\} \geq \mu(a) \wedge \frac{1-k}{2}. \text{ Thus } \mu(\gamma) \geq \mu(a) \wedge \frac{1-k}{2} \end{array} \right) \\
&\geq \left\{ \left(\lambda(a) \wedge \frac{1-k}{2} \right) \wedge \left(\mu(a) \wedge \frac{1-k}{2} \right) \right\} \wedge \frac{1-k}{2} \\
&= \lambda(a) \wedge \mu(a) \wedge \frac{1-k}{2} \\
&= (\lambda \wedge_k \mu)(a).
\end{aligned}
$$

$(\lambda \circ_k \mu) \geq (\lambda \wedge_k \mu)$.

(2) $\Rightarrow$ (1) Let R and L be right and left hyperideals of H. Then by Lemma 5.7.5, $(\lambda_R)_k$ and $(\lambda_L)_k$ are $(\in, \in \vee q_k)$-fuzzy right and $(\in, \in \vee q_k)$-fuzzy left hyperideals of H, respectively. Thus by hypothesis we have

$$
\begin{aligned}
(\lambda_{L \circ R})_k &= (\lambda_L \circ_k \lambda_R) \\
&\geq (\lambda_L \wedge_k \lambda_R) \\
&= (\lambda_{L \cap R})_k.
\end{aligned}
$$

Thus $L \cap R \subseteq L \circ R$. Hence it follows from Theorem 4.3.3, that H is intra-regular. □

5.8.2 Theorem

The following conditions are equivalent for a semihypergroup H.

(1) *H is both regular and intra-regular.*

(2) *$\lambda \circ_k \lambda = \lambda_k$ for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal λ of H.*

(3) *$\lambda \circ_k \lambda = \lambda_k$ for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal λ of H.*

(4) *$\lambda \circ_k \mu \geq \lambda \wedge_k \mu$ for all $(\in, \in \vee q_k)$-fuzzy quasi-hyperideals λ, μ of H.*

(5) $\lambda \circ_k \mu \geq \lambda \wedge_k \mu$ *for every* $(\in, \in \vee q_k)$*-fuzzy quasi-hyperideal* λ *and for every* $(\in, \in \vee q_k)$*-fuzzy bi-hyperideal* μ *of* H.

(6) $\lambda \circ_k \mu \geq \lambda \wedge_k \mu$ *for all* $(\in, \in \vee q_k)$*-fuzzy bi-hyperideals* λ, μ *of* H.

Proof. (1) ⇒ (6) Let λ, μ be $(\in, \in \vee q_k)$-fuzzy bi-hyperideals of H and $a \in H$. Then there exist $x, y, z \in H$ such that $a \in a \circ x \circ a$ and $a \in y \circ a \circ a \circ z$. So

$$\begin{aligned} a &\in a \circ x \circ a \\ &\subseteq a \circ x \circ a \circ x \circ a \\ &= (a \circ x) \circ a \circ (x \circ a) \\ &\subseteq a \circ x \circ (y \circ a \circ a \circ z) \circ x \circ a \\ &= (a \circ x \circ y \circ a) \circ (a \circ z \circ x \circ a). \end{aligned}$$

Thus there exist $p \in x \circ y, q \in z \circ x, b \in a \circ p \circ a$ and $c \in a \circ q \circ a$ such that $a \in b \circ c$.
Therefore

$$\begin{aligned} (\lambda \circ_k \mu)(a) &= (\lambda \circ \mu)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in d \circ e} \{\lambda(d) \wedge \mu(e)\} \right) \wedge \frac{1-k}{2} \\ &\geq \{\lambda(b) \wedge \mu(c)\} \wedge \frac{1-k}{2} \\ &\quad \left(\begin{array}{c} \text{Since } b \in a \circ p \circ a \text{ and } \lambda \text{ is an } (\in, \in \vee q_k)\text{-fuzzy bi-hyperideal of } H, \\ \text{we have } \inf\limits_{\alpha \in a \circ p \circ a} \{\lambda(\alpha)\} \geq \min\{\lambda(a), \lambda(a), \frac{1-k}{2}\}. \\ \text{Thus } \lambda(b) \geq \min\{\lambda(a), \frac{1-k}{2}\}. \\ \text{Similarly } \mu(c) \geq \min\{\mu(a), \frac{1-k}{2}\}. \end{array} \right) \\ &\geq \left[\left(\lambda(a) \wedge \frac{1-k}{2} \right) \wedge \left(\mu(a) \wedge \frac{1-k}{2} \right) \right] \wedge \frac{1-k}{2} \\ &= [\lambda(a) \wedge \mu(a)] \wedge \frac{1-k}{2} = (\lambda \wedge_k \mu)(a). \end{aligned}$$

Thus $\lambda \circ_k \mu \geq \lambda \wedge_k \mu$ for all $(\in, \in \vee q_k)$-fuzzy bi-hyperideals λ, μ of H.

(6) ⇒ (5) ⇒ (4) are obvious.

(4) ⇒ (2) Take $\lambda = \mu$ in (4), we get $\lambda \circ_k \lambda \geq \lambda_k$. Since every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal is an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup, so $\lambda \circ_k \lambda \leq \lambda_k$. Hence $\lambda \circ_k \lambda = \lambda_k$.

(6) ⇒ (3) Take $\lambda = \mu$ in (6), we get $\lambda \circ_k \lambda \geq \lambda_k$. Since every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal is an $(\in, \in \vee q_k)$-fuzzy subsemihypergroup, so $\lambda \circ_k \lambda \leq \lambda_k$. Hence $\lambda \circ_k \lambda = \lambda_k$.

(3) ⇒ (2) Obvious.

(2) $\Rightarrow$ (1) Let Q be a quasi-hyperideal of H. Then by Lemma 5.6.3, λ_Q is an $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal of H. Hence, by hypothesis, $\lambda_Q \circ_k \lambda_Q = (\lambda_Q)_k$. Thus $(\lambda_{Q \circ Q})_k = \lambda_Q \circ_k \lambda_Q = (\lambda_Q)_k$ implies $Q \circ Q = Q$. So by Theorem 3.2.8, S is both regular and intra-regular. □

5.8.3 Theorem

The following conditions are equivalent for a semihypergroup H.

(1) *H is both regular and intra-regular.*

(2) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every $(\in, \in \vee q_k)$-fuzzy right hyperideal λ and for every $(\in, \in \vee q_k)$-fuzzy left hyperideal μ of H.*

(3) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every $(\in, \in \vee q_k)$-fuzzy right hyperideal λ and for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal μ of H.*

(4) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every $(\in, \in \vee q_k)$-fuzzy right hyperideal λ and for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal μ of H.*

(5) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every $(\in, \in \vee q_k)$-fuzzy right hyperideal λ and for every $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal μ of H.*

(6) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every $(\in, \in \vee q_k)$-fuzzy left hyperideal λ and for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal μ of H.*

(7) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every $(\in, \in \vee q_k)$-fuzzy left hyperideal λ and for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal μ of H.*

(8) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every $(\in, \in \vee q_k)$-fuzzy left hyperideal λ and for every $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal μ of H.*

(9) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideals λ and μ of H.*

(10) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal λ and for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal μ of H.*

(11) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every $(\in, \in \vee q_k)$-fuzzy quasi-hyperideal λ and for every $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal μ of H.*

(12) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideals λ and μ of H.*

(13) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every $(\in, \in \vee q_k)$-fuzzy bi-hyperideal λ and for every $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideal μ of H.*

(14) $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$ *for every $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideals λ and μ of H.*

Proof. (1) $\Rightarrow$ (14) Let λ, μ be $(\in, \in \vee q_k)$-fuzzy generalized bi-hyperideals of H and

$a \in H$. Then there exist $x, y, z \in H$ such that $a \in a \circ x \circ a$ and $a \in y \circ a \circ a \circ z$. So

$$\begin{aligned} a &\in a \circ x \circ a \\ &\subseteq a \circ x \circ a \circ x \circ a \\ &= (a \circ x) \circ a \circ (x \circ a) \\ &\subseteq a \circ x \circ (y \circ a \circ a \circ z) \circ x \circ a \\ &= (a \circ x \circ y \circ a) \circ (a \circ z \circ x \circ a). \end{aligned}$$

Thus there exist $p \in x \circ y, q \in z \circ x, b \in a \circ p \circ a$ and $c \in a \circ q \circ a$ such that $a \in b \circ c$. Therefore

$$\begin{aligned} (\lambda \circ_k \mu)(a) &= (\lambda \circ \mu)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in d \circ e} \{\lambda(d) \wedge \mu(e)\} \right) \wedge \frac{1-k}{2} \\ &\geq \{\lambda(b) \wedge \mu(c)\} \wedge \frac{1-k}{2} \end{aligned}$$

$$\left(\begin{array}{c} \text{Since } b \in a \circ p \circ a \text{ and } c \in a \circ q \circ a \text{ and } \lambda \text{ and } \mu \text{ are} \\ (\in, \in \vee q_k)\text{-fuzzy generalized bi-hyperideals of } H, \\ \text{so } \inf_{w \in x \circ y \circ z} \{\lambda(w)\} \geq \min\{\lambda(x), \lambda(z), \tfrac{1-k}{2}\} \text{ and} \\ \inf_{w \in x \circ y \circ z} \{\mu(w)\} \geq \min\{\mu(x), \mu(z), \tfrac{1-k}{2}\} \\ \text{Thus } \lambda(b) \geq \lambda(a) \wedge \tfrac{1-k}{2} \text{ so and } \mu(c) \geq \mu(a) \wedge \tfrac{1-k}{2}. \end{array} \right)$$

$$\begin{aligned} \text{Therefore } (\lambda \circ_k \mu)(a) &\geq \left[\left(\lambda(a) \wedge \frac{1-k}{2} \right) \wedge \left(\mu(a) \wedge \frac{1-k}{2} \right) \right] \wedge \frac{1-k}{2} \\ &= [\lambda(a) \wedge \mu(a)] \wedge \frac{1-k}{2} = (\lambda \wedge_k \mu)(a). \end{aligned}$$

Similarly we can prove that $(\mu \circ_k \lambda) \geq (\lambda \wedge_k \mu)$. Hence $(\lambda \circ_k \mu) \wedge (\mu \circ_k \lambda) \geq \lambda \wedge_k \mu$.
(14) $\Rightarrow$ (13) $\Rightarrow$ (12) $\Rightarrow$ (10) $\Rightarrow$ (9) $\Rightarrow$ (3) $\Rightarrow$ (2),
(14) $\Rightarrow$ (11) $\Rightarrow$ (10),
(14) $\Rightarrow$ (8) $\Rightarrow$ (7) $\Rightarrow$ (6) $\Rightarrow$ (2) and
(14) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (1) Let λ be an $(\in, \in \vee q_k)$-fuzzy right hyperideal and μ be an $(\in, \in \vee q_k)$-fuzzy left hyperideal of H.

For $a \in H$, we have

$$\begin{aligned}(\lambda \circ_k \mu)(a) &= (\lambda \circ \mu)(a) \wedge \frac{1-k}{2} \\ &= \left(\bigvee_{a \in y \circ z} \{\lambda(y) \wedge \mu(z)\}\right) \wedge \frac{1-k}{2} \\ &= \bigvee_{a \in y \circ z} \left\{\lambda(y) \wedge \mu(z) \wedge \frac{1-k}{2}\right\} \\ &= \bigvee_{a \in y \circ z} \left\{\left(\lambda(y) \wedge \frac{1-k}{2}\right) \wedge \left(\mu(z) \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2}\right\}\end{aligned}$$

$\left(\text{Because } \inf_{a \in y \circ z}\{\lambda(a)\} \geq \lambda(y) \wedge \frac{1-k}{2} \text{ and } \inf_{a \in y \circ z}\{\mu(a)\} \geq \mu(z) \wedge \frac{1-k}{2}\right)$.
Thus

$$\begin{aligned}(\lambda \circ_k \mu)(a) &\leq \bigvee_{a \in y \circ z} \left\{\lambda(a) \wedge \mu(a) \wedge \frac{1-k}{2}\right\} \\ &= \lambda(a) \wedge \mu(a) \wedge \frac{1-k}{2} \\ &= (\lambda \wedge_k \mu)(a).\end{aligned}$$

So $(\lambda \circ_k \mu) \leq (\lambda \wedge_k \mu)$. By hypothesis $(\lambda \circ_k \mu) \geq (\lambda \wedge_k \mu)$. Thus $(\lambda \circ_k \mu) = (\lambda \wedge_k \mu)$. Hence by Theorem 5.7.9, H is regular. Also by hypothesis $(\lambda \circ_k \mu) \geq (\lambda \wedge_k \mu)$, so by Theorem 5.8.1, H is intra-regular. □

Chapter 6

SEMIHYPERGROUPS CHARACTERIZED BY $(\in_\gamma, \in_\gamma \vee q_\delta)$-FUZZY HYPERIDEALS

In this chapter we generalize the concept of $(\in, \in \vee q_k)$-fuzzy hyperideals of a semihypergroup H and we define $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) hyperideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior hyperideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal of a semihypergroup H and study some basic properties. Also we characterize regular and intra-regular semihypergroups using these notions.

Recall that a fuzzy point x_t "belongs to" (resp. "quasi coincident with") a fuzzy set λ, written as $x_t \in \lambda$ (resp. $x_t q \lambda$) if $\lambda(x) \geq t$ (resp. $\lambda(x) + t > 1$). If $x_t \in \lambda$ or $x_t q\lambda$ then we write $x_t \in \vee q\lambda$. If $x_t \in \lambda$ and $x_t q\lambda$ then we write $x_t \in \wedge q\lambda$. If $\lambda(x) < t$ (resp. $\lambda(x) + t \leq 1$), then we say that $x_t \overline{\in} \lambda$ (resp. $x_t \overline{q} \lambda$). Similarly $\overline{\in \vee q}$ (resp. $\overline{\in \wedge q}$) means that $\in \vee q$ (resp. $\in \wedge q$) does not hold.

6.1 (α, β)-fuzzy hyperideals

Throughout this chapter $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$ and $\alpha, \beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$. For a fuzzy point x_t and a fuzzy subset λ of H, we say

(1) $x_t \in_\gamma \lambda$ if $\lambda(x) \geq t > \gamma$.

(2) $x_t q_\delta \lambda$ if $\lambda(x) + t > 2\delta$.

(3) $x_t \in_\gamma \vee q_\delta \lambda$ if $x_t \in_\gamma \lambda$ or $x_t q_\delta \lambda$.

(4) $x_t \in_\gamma \wedge q_\delta \lambda$ if $x_t \in_\gamma \lambda$ and $x_t q_\delta \lambda$.

(5) $x_t \overline{\alpha} \lambda$ if $x_t \alpha \lambda$ does not hold for $\alpha \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$.

Let λ be a fuzzy subset of a semhypergroup H such that $\lambda(x) \leq \delta$. Let $x \in H$ and $t \in (0, 1]$ be such that $x_t \in_\gamma \wedge q_\delta \lambda$. Then $\lambda(x) \geq t > \gamma$ and $\lambda(x) + t > 2\delta$. It follows that $2\delta < \lambda(x) + t \leq \lambda(x) + \lambda(x) = 2\lambda(x)$, so that $\lambda(x) > \delta$. This means that $\{x_t : x_t \in_\gamma \wedge q_\delta \lambda\} = \emptyset$. Therefore we do not take $\alpha = \in_\gamma \wedge q_\delta$.

6.1.1 Definition

A fuzzy subset λ of a semihypergroup H is called an (α, β)-fuzzy subsemihypergroup of H if for all $x, y \in H$ and $t, r \in (\gamma, 1]$ the following condition holds

(1) $x_t \alpha \lambda, y_r \alpha \lambda \longrightarrow (z)_{\min\{t,r\}} \beta \lambda$, for each $z \in x \circ y$.

6.1.2 Definition

A fuzzy subset λ of a semihypergroup H is called an (α,β)-fuzzy left (resp. right) hyperideal of H if for all $t\in(\gamma,1]$ and for all $x,y\in H$, we have

(2) $y_t\alpha\lambda \longrightarrow z_t\beta\lambda$ (resp. $z_t\beta\lambda$) for each $z\in x\circ y$ (resp. for each $z\in y\circ x$).

A fuzzy subset λ of a semihypergroup H is called an (α,β)-fuzzy hyperideal of H if it is both (α,β)-fuzzy left hyperideal and (α,β)-fuzzy right hyperideal of H.

6.1.3 Definition

A fuzzy subset λ of a semihypergroup H is called an (α,β)-fuzzy interior hyperideal of H, if for all $x,y,z\in H$ and for all $t\in(\gamma,1]$ it satisfies the following condition:

(3) $x_t\alpha\lambda \longrightarrow w_t\beta\lambda$ for every $w\in y\circ x\circ z$.

6.1.4 Definition

A fuzzy subset λ of a semihypergroup H is called an (α,β)-fuzzy generalized bi-hyperideal of H, if for all $x,y,z\in H$ and for all $t,r\in(\gamma,1]$ it satisfies the following condition:

(4) $x_t\alpha\lambda, y_r\alpha\lambda \longrightarrow (w)_{\min\{t,r\}}\beta\lambda$ for every $w\in x\circ z\circ y$.

6.1.5 Definition

A fuzzy subset λ of a semihypergroup H is called an (α,β)-fuzzy bi-hyperideal of H, if it satisfies the condition (1) and (4).

6.1.6 Definition

A fuzzy subset λ of a semihypergroup H is called an (α,β)-fuzzy quasi-hyperideal of H, if for all $x,y\in H$ and for all $t\in(\gamma,1]$ it satisfies the following condition:

(5) $x_t\alpha\lambda \longrightarrow z_t\beta\lambda$ for every $z\in y\circ x$ and $z\in x\circ y$.

6.1.7 Theorem

Let $2\delta=1+\gamma$ and λ be an (α,β)-fuzzy subsemihypergroup of H. Then

$$\lambda_\gamma=\{x\in H:\lambda(x)>\gamma\}$$

is a subsemihypergroup of H.

Proof. Let $x,y\in\lambda_\gamma$. Then $\lambda(x)>\gamma$ and $\lambda(y)>\gamma$. Suppose that there exists $z\in x\circ y$ such that $\lambda(z)\le\gamma$. If $\alpha\in\{\in_\gamma,\in_\gamma\vee q_\delta\}$, then $x_{\lambda(x)}\alpha\lambda$ and $y_{\lambda(y)}\alpha\lambda$ but $(z)_{\min\{\lambda(x),\lambda(y)\}}\overline{\beta}\lambda$ for $z\in x\circ y$ and for each $\beta\in\{\in_\gamma,q_\delta,\in_\gamma\vee q_\delta,\in_\gamma\wedge q_\delta\}$, a contradiction (because $\lambda(z)\le\gamma<\min\{\lambda(x),\lambda(y)\}$ so $(z)_{\min\{\lambda(x),\lambda(y)\}}\overline{\in_\gamma}\lambda$ and $\lambda(z)+\min\{\lambda(x),\lambda(y)\}\le\gamma+\min\{\lambda(x),\lambda(y)\}\le\gamma+1=2\delta$, so $(z)_{\min\{\lambda(x),\lambda(y)\}}\overline{q_\delta}\lambda$).

Hence $\lambda(z) > \gamma$, that is $z \in \lambda_\gamma$ for every $z \in x \circ y$. If $\alpha = q_\delta$. Then $x_1 q_\delta \lambda$ and $y_1 q_\delta \lambda$ (because $\lambda(x)+1 > \gamma+1 = 2\delta$ and $\lambda(y)+1 > \gamma+1 = 2\delta$). But for $z \in x \circ y$, $z_1 \overline{\beta} \lambda$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$ (because $\lambda(z) \leq \gamma$, so $z \overline{\in_\gamma} \lambda$ and $\lambda(z) + 1 \leq \gamma + 1 = 2\delta$, so $z_1 \overline{q_\delta} \lambda$). Hence $\lambda(z) > \gamma$, that is $z \in \lambda_\gamma$ for every $z \in x \circ y$. This shows that λ_γ is a subsemihypergroup of H. □

6.1.8 Theorem

Let $2\delta = 1 + \gamma$ and λ be an (α, β)-fuzzy left (right) hyperideal of H. Then

$$\lambda_\gamma = \{x \in H : \lambda(x) > \gamma\}$$

is a left (right) hyperideal of H.

Proof. Let λ be an (α, β)-fuzzy left hyperideal of H and $x \in \lambda_\gamma$. Suppose there exists $y \in H$ such that $\lambda(z) \leq \gamma$ for some $z \in y \circ x$. If $\alpha \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$, then $x_{\lambda(x)} \alpha \lambda$. But $z \in y \circ x$ such that $\lambda(z) \leq \gamma < \lambda(x)$, so $z_{\lambda(x)} \overline{\in_\gamma} \lambda$. Also $\lambda(z) + \lambda(x) \leq \gamma + \lambda(x) \leq \gamma + 1 = 2\delta$, so $z_{\lambda(x)} \overline{q_\delta} \lambda$. This shows that for $z \in y \circ x$, $z_{\lambda(x)} \overline{\beta} \lambda$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, which is a contradiction. Hence $\lambda(z) > \gamma$, that is $z \in \lambda_\gamma$ for every $z \in y \circ x$. If $\alpha = q_\delta$. Then $x_1 q_\delta \lambda$. But there exists $z \in y \circ x$ such that $\lambda(z) \leq \gamma$, so $z_1 \overline{\in_\gamma} \lambda$ and $\lambda(z) + 1 \leq \gamma + 1 = 2\delta$, so $z_1 \overline{q_\delta} \lambda$. Thus for $z \in y \circ x$, $z_1 \overline{\beta} \lambda$ for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, which is a contradiction. Hence $\lambda(z) > \gamma$, that is $z \in \lambda_\gamma$ for every $z \in y \circ x$. This shows that λ_γ is a left hyperideal of H. □

6.1.9 Theorem

(1) *Let $2\delta = 1 + \gamma$ and λ be an (α, β)-fuzzy generalized bi-hyperideal of H. Then λ_γ is a generalized bi-hyperideal of H.*

(2) *Let $2\delta = 1 + \gamma$ and λ be an (α, β)-fuzzy bi-hyperideal of H. Then λ_γ is a bi-hyperideal of H.*

(3) *Let $2\delta = 1 + \gamma$ and λ be an (α, β)-fuzzy interior hyperideal of H. Then λ_γ is an interior hyperideal of H.*

(4) *Let $2\delta = 1 + \gamma$ and λ be an (α, β)-fuzzy quasi-hyperideal of H. Then λ_γ is a quasi-hyperideal of H.*

Proof. (1) Proof is similar to the proof of Theorem 6.1.8.

(2) Proof follows from Theorem 6.1.7 and part (1).

(3) Proof is similar to the proof of part (2).

(4) Proof is similar to the proof of Theorem 6.1.8. □

6.1.10 Theorem

Let $2\delta = 1 + \gamma$ and A be a non-empty subset of H. Then A is a subsemihypergroup of H if and only if the fuzzy subset λ of H defined by

$$\lambda(x) = \begin{cases} \geq \delta & \text{if } x \in A, \\ \leq \gamma & \text{otherwise} \end{cases}$$

is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H.

Proof. Let A be a subsemihypergroup of H.

(1) Let $x, y \in H$ and $t, r \in (\gamma, 1]$ be such that $x_t \in_\gamma \lambda, y_r \in_\gamma \lambda$. Then $\lambda(x) \geq t > \gamma$ and $\lambda(y) \geq r > \gamma$. Thus $x, y \in A$ and so $z \in A$ for every $z \in x \circ y$, that is $\lambda(z) > \delta$. If $\min\{t, r\} \leq \delta$, then $\lambda(z) \geq \delta \geq \min\{t, r\} > \gamma$. This implies $z_{\min\{t,r\}} \in_\gamma \lambda$. If $\min\{t, r\} > \delta$, then $\lambda(z) + \min\{t, r\} > \delta + \delta = 2\delta$. This implies $z_{\min\{t,r\}} q_\delta \lambda$. Hence λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H.

(2) Let $x, y \in H$ and $t, r \in (\gamma, 1]$ be such that $x_t q_\delta \lambda, y_r q_\delta \lambda$. Then $\lambda(x) + t > 2\delta$ and $\lambda(y) + t > 2\delta$. This implies $\lambda(x) > 2\delta - t \geq 2\delta - 1 = \gamma$ and $\lambda(y) > 2\delta - t \geq 2\delta - 1 = \gamma$, that is $x, y \in A$. Thus $z \in A$ for every $z \in x \circ y$ and so $\lambda(z) \geq \gamma$. If $\min\{t, r\} \leq \delta$, then $\lambda(z) \geq \delta \geq \min\{t, r\} > \gamma$ and so $z_{\min\{t,r\}} \in_\gamma \lambda$. If $\min\{t, r\} > \delta$, then $\lambda(z) + \min\{t, r\} > \delta + \delta = 2\delta$. Thus $z_{\min\{t,r\}} q_\delta \lambda$. Hence λ is a $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H.

(3) Let $x, y \in H$ and $t, r \in (\gamma, 1]$ be such that $x_t \in_\gamma \lambda$ and $y_r q_\delta \lambda$ ($x_t q_\delta \lambda$ and $y_r \in_\gamma \lambda$). Then $\lambda(x) \geq t > \gamma$ and $\lambda(y) + t > 2\delta$. $\lambda(y) + t > 2\delta$ implies $\lambda(y) > 2\delta - t \geq 2\delta - 1 = \gamma$. Thus $x, y \in A$ and so $z \in A$ for every $z \in x \circ y$. Analogous to (1) and (2) we obtain $z_{\min\{t,r\}} \in_\gamma \vee q_\delta \lambda$, that is λ is an $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H.

Conversely, assume that λ is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H. Then $A = \lambda_\gamma$. It follows from Theorem 6.1.7 that A is a subsemihypergroup of H. □

6.1.11 Corollary

Let $2\delta = 1 + \gamma$ and A be a non-empty subset of H. Then A is a subsemihypergroup of H if and only if λ_A, the characteristic function of A is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H.

Similarly we can prove the following Theorem.

6.1.12 Theorem

Let $2\delta = 1 + \gamma$ and A be a non-empty subset of H. Define a fuzzy subset λ of H as

$$\lambda(x) = \begin{cases} \geq \delta & \text{if } x \in A, \\ \leq \gamma & \text{otherwise} \end{cases}$$

Then

(1) λ *is an* $(\alpha, \in_\gamma \vee q_\delta)$*-fuzzy left (right) hyperideal of* H *if and only if* A *is a left (right) hyperideal of* H.

(2) λ *is an* $(\alpha, \in_\gamma \vee q_\delta)$*-fuzzy generalized bi-hyperideal (bi-hyperideal) of* H *if and only if* A *is a generalized bi-hyperideal (bi-hyperideal) of* H.

(3) λ *is an* $(\alpha, \in_\gamma \vee q_\delta)$*-fuzzy interior hyperideal of* H *if and only if* A *is an interior hyperideal of* H.

(4) λ *is an* $(\alpha, \in_\gamma \vee q_\delta)$*-fuzzy quasi-hyperideal of* H *if and only if* A *is an quasi-hyperideal of* H.

6.1.13 Corollary

(1) *Let* $2\delta = 1 + \gamma$ *and* A *be a non-empty subset of* H. *Then* A *is a left (right) hyperideal of* H *if and only if* λ_A, *the characteristic function of* A *is an* $(\alpha, \in_\gamma \vee q_\delta)$*-fuzzy left (right) hyperideal of* H.

(2) *Let* $2\delta = 1 + \gamma$ *and* A *be a non-empty subset of* H. *Then* A *is a generalized bi-hyperideal (bi-hyperideal) of* H *if and only if* λ_A, *the characteristic function of* A *is an* $(\alpha, \in_\gamma \vee q_\delta)$*-fuzzy generalized bi-hyperideal (bi-hyperideal) of* H.

(3) *Let* $2\delta = 1 + \gamma$ *and* A *be a non-empty subset of* H. *Then* A *is an interior hyperideal of* H *if and only if* λ_A, *the characteristic function of* A *is an* $(\alpha, \in_\gamma \vee q_\delta)$*-fuzzy interior hyperideal of* H.

(4) *Let* $2\delta = 1+\gamma$ *and* A *be a non-empty subset of* H. *Then* A *is a quasi-hyperideal of* H *if and only if* λ_A, *the characteristic function of* A *is an* $(\alpha, \in_\gamma \vee q_\delta)$*-fuzzy quasi-hyperideal of* H.

It is easy to see that each (α, β)-fuzzy subsemihypergroup (left hyperideal, right hyperideal, generalized bi-hyperideal, bi-hyperideal, interior hyperideal, quasi hyperideal) of a semihypergroup H is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup (left hyperideal, right hyperideal, generalized bi-hyperideal, bi-hyperideal, interior hyperideal, quasi-hyperideal) of H.

6.1.14 Theorem

(1) *Every* $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$*-fuzzy subsemihypergroup of* H *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy subsemihypergroup of* H.

(2) *Every* $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$*-fuzzy left (right) hyperideal of* H *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left (right) hyperideal of* H.

(3) *Every* $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$*-fuzzy generalized bi-hyperideal (bi-hyperideal) of* H *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy generalized bi-hyperideal (bi-hyperideal) of* H.

(4) *Every* $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$*-fuzzy interior hyperideal of* H *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy interior hyperideal of* H.

(5) *Every* $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$*-fuzzy quasi-hyperideal of* H *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy quasi-hyperideal of* H.

Proof. The proof follows from the fact that if $x_t \in_\gamma \lambda$ then $x_t \in_\gamma \vee q_\delta \lambda$. □

6.1.15 Theorem

(1) *Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H.*

(2) *Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy left (right) hyperideal of H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) hyperideal of H.*

(3) *Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal (bi-hyperideal) of H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal (bi-hyperideal) of H.*

(4) *Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy interior hyperideal of H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior hyperideal of H.*

(5) *Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H.*

Proof. We prove only (1). Proofs of (2), (3), (4) and (5) are similar to the proof of (1).

Let λ be a $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H. Let $x, y \in H$ and $t, r \in (\gamma, 1]$ be such that $x_t \in_\gamma \lambda, y_r \in_\gamma \lambda$. Then $\lambda(x) \geq t > \gamma$ and $\lambda(y) \geq r > \gamma$. Suppose there exists $z \in x \circ y$ such that $z_{\min\{t,r\}}\overline{\in_\gamma \vee q_\delta}\lambda$. Then $\lambda(z) < \min\{t, r\}$ and $\lambda(z) + \min\{t, r\} \leq 2\delta \Rightarrow \lambda(z) < \delta$. Now $\max\{\lambda(z), \gamma\} < \min\{\lambda(x), \lambda(y), \delta\}$. Then choose an $s \in (\gamma, 1]$ such that

$$2\delta - \max\{\lambda(z), \gamma\} > s \geq 2\delta - \min\{\lambda(x), \lambda(y), \delta\}.$$

$$\Rightarrow 2\delta - \lambda(z) \geq 2\delta - \max\{\lambda(z), \gamma\} > s \geq \max\{2\delta - \lambda(x), 2\delta - \lambda(y), \delta\}.$$

This implies $\lambda(x)+s \geq 2\delta$, $\lambda(y)+s \geq 2\delta$ and $\lambda(z)+s < 2\delta$ and $\lambda(z) < \delta < s$. Hence $x_s q_\delta \lambda$ and $y_s q_\delta \lambda$ but $z_s\overline{\in_\gamma \vee q_\delta}\lambda$. This is a contradiction. Hence $z_{\min\{t,r\}} \in_\gamma \vee q_\delta \lambda$, that is λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H. □

The above discussion shows that every (α, β)-fuzzy subsemihypergroup (left hyperideal, right hyperideal, generalized bi-hyperideal, bi-hyperideal, interior hyperideal, quasi-hyperideal) of a semihypergroup H is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup (left hyperideal, right hyperideal, generalized bi-hyperideal, bi-hyperideal, interior hyperideal, quasi-hyperideal) of H. Also every $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup (left hyperideal, right hyperideal, generalized bi-hyperideal, bi-hyperideal, interior hyperideal, quasi-hyperideal) of H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup (left hyperideal, right hyperideal, generalized bi-hyperideal, bi-hyperideal, interior hyperideal, quasi-hyperideal) of H. Thus in the theory of (α, β)-fuzzy subsemihypergroup (left hyperideal, right hyperideal, generalized bi-hyperideal, bi-hyperideal, interior hyperideal, quasi-hyperideal), $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup (left hyperideal, right hyperideal, generalized bi-hyperideal, bi-hyperideal, interior hyperideal, quasi-hyperideal) plays central role.

6.2 $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideals

We start this section with the following Theorem:

6.2.1 Theorem

For any fuzzy subset λ of a semihypergroup H and for all $x, y, z \in H$ and $t, r \in (\gamma, 1]$, $(1a)$ is equivalent to $(1b)$, $(2a)$ is equivalent to $(2b)$ and $(3a)$ is equivalent to $(3b)$ and $(4a)$ is equivalent to $(4b)$. Where

$(1a)$ $x_t, y_r \in_\gamma \lambda \Rightarrow z_{\min\{t,r\}} \in_\gamma \vee q_\delta \lambda$ for every $z \in x \circ y$.

$(1b)$ $\max\{\inf\limits_{z \in x \circ y}\{\lambda(z)\}, \gamma\} \geq \min\{\lambda(x), \lambda(y), \delta\}$.

$(2a)$ $x_t \in_\gamma \lambda \Rightarrow z_t \in_\gamma \vee q_\delta \lambda$ for every $z \in y \circ x$ ($z_t \in_\gamma \vee q_\delta \lambda$ for every $z \in x \circ y$).

$(2b)$ $\max\{\inf\limits_{z \in y \circ x}\{\lambda(z)\}, \gamma\} \geq \min\{\lambda(x), \delta\} \left(\max\{\inf\limits_{z \in x \circ y}\{\lambda(z)\}, \gamma\} \geq \min\{\lambda(x), \delta\}\right)$.

$(3a)$ $x_t, y_r \in_\gamma \lambda \Rightarrow w_{\min\{t,r\}} \in_\gamma \vee q_\delta \lambda$ for every $w \in x \circ z \circ y$.

$(3b)$ $\max\{\inf\limits_{w \in x \circ z \circ y}\{\lambda(w)\}, \gamma\} \geq \min\{\lambda(x), \lambda(y), \delta\}$.

$(4a)$ $x_t \in_\gamma \lambda \Rightarrow w_t \in_\gamma \vee q_\delta \lambda$ for every $w \in y \circ x \circ z$.

$(4b)$ $\max\{\inf\limits_{w \in y \circ x \circ z}\{\lambda(w)\}, \gamma\} \geq \min\{\lambda(x), \delta\}$.

$(5a)$ $x_t \in_\gamma \lambda \Rightarrow z_t \in_\gamma \vee q_\delta \lambda$ for every $z \in y \circ x$ and $z \in x \circ y$.

$(5b)$ $\max\{\{\lambda(z)\}, \gamma\} \geq \min\{(1 \circ \lambda)(z), (\lambda \circ 1)(z), \delta\}$.

Proof. We prove only $(1a) \Leftrightarrow (1b)$. Proofs of the remaining parts are similar to this.

$(1a) \Rightarrow (1b)$ Suppose λ is a fuzzy subset of H which satisfies $(1a)$. Let $x, y \in H$ be such that

$$\max\{\inf_{z \in x \circ y}\{\lambda(z)\}, \gamma\} < \min\{\lambda(x), \lambda(y), \delta\}.$$

Select $t \in (\gamma, 1]$ such that

$$\max\{\inf_{z \in x \circ y}\{\lambda(z)\}, \gamma\} < t \leq \min\{\lambda(x), \lambda(y), \delta\}.$$

Then $\lambda(x) \geq t > \gamma$, $\lambda(y) \geq t > \gamma$, and there exists $z \in x \circ y$ such that $\lambda(z) < t \leq \delta$ and $\lambda(z) + t < \delta + \delta = 2\delta$, that is $x_t \in_\gamma \lambda, y_r \in_\gamma \lambda$ but $z_t \overline{\in_\gamma \vee q_\delta} \lambda$ which is a contradiction. Hence

$$\max\{\inf_{z \in x \circ y}\{\lambda(z)\}, \gamma\} \geq \min\{\lambda(x), \lambda(y), \delta\}.$$

$(1b) \Rightarrow (1a)$ Suppose λ is a fuzzy subset of H which satisfies $(1b)$. Let $x, y \in H$ and $t, r \in (\gamma, 1]$ be such that $x_t \in_\gamma \lambda, y_r \in_\gamma \lambda$ but $z_{\min\{t,r\}} \overline{\in_\gamma \vee q_\delta} \lambda$ for some $z \in x \circ y$. Then

$$\begin{aligned} \lambda(x) &\geq t > \gamma && (1) \\ \lambda(y) &\geq r > \gamma && (2) \\ \lambda(z) &< \min\{t, r\} && (3) \\ \text{and } \lambda(z) + \min\{t, r\} &\leq 2\delta. && (4) \end{aligned}$$

It follows from (3) and (4) that $\lambda(z) < \delta$ for some $z \in x \circ y$.
Now $\max\{\lambda(z), \gamma\} < \delta$ and $\max\{\lambda(z), \gamma\} < \min\{\lambda(x), \lambda(y)\}$. Thus

$$\max\{\lambda(z), \gamma\} < \min\{\lambda(x), \lambda(y), \delta\}$$

which is a contradiction. Hence $z_{\min\{t,r\}} \in_\gamma \vee q_\delta \lambda$ for every $z \in x \circ y$. □

From the above theorem we deduce that

6.2.2 Definition

A fuzzy subset λ of a semihypergroup H is called an
(i) $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H if it satisfies $(1b)$.
(ii) $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) hyperideal of H if it satisfies $(2b)$.
(iii) $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal of H if it satisfies $(3b)$.
(iv) $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal of H if it satisfies $(1b)$ and $(3b)$.
(v) $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior hyperideal of H if it satisfies $(4b)$.
(vi) $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H if it satisfies $(5b)$.

6.2.3 Definition

Let λ be a fuzzy subset of a semihypergroup H and $r \in (\gamma, 1]$, we define

$$\begin{aligned}
\lambda_r &= \{x \in H : x_r \in_\gamma \lambda\} = \{x \in H : \lambda(x) \geq r > \gamma\} = U(\lambda; r) \\
\lambda_r^\delta &= \{x \in H : x_r q_\delta \lambda\} = \{x \in H : \lambda(x) + r > 2\delta\} \\
[\lambda]_r^\delta &= \{x \in H : x_r \in_\gamma \vee q_\delta \lambda\} = \lambda_r \cup \lambda_r^\delta.
\end{aligned}$$

6.2.4 Theorem

Let λ be a fuzzy subset of a semihypergroup H. Then

(1) *λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H if and only if $U(\lambda; t)(\neq \emptyset)$ is a subsemihypergroup of H for all $t \in (\gamma, \delta]$.*

(2) *λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) hyperideal of H if and only if $U(\lambda; t)(\neq \emptyset)$ is a left (right) hyperideal of H for all $t \in (\gamma, \delta]$.*

(3) *λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal (bi-hyperideal) of H if and only if $U(\lambda; t)(\neq \emptyset)$ is a generalized bi-hyperideal (bi-hyperideal) of H for all $t \in (\gamma, \delta]$.*

(4) *λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior hyperideal of H if and only if $U(\lambda; t)(\neq \emptyset)$ is an interior hyperideal of H for all $t \in (\gamma, \delta]$.*

(5) *λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H if and only if $U(\lambda; t)(\neq \emptyset)$ is a quasi-hyperideal of H for all $t \in (\gamma, \delta]$.*

Proof. Let λ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H and $x, y \in U(\lambda; t)$ for some $t \in (\gamma, \delta]$. Then $\lambda(x) \geq t$ and $\lambda(y) \geq t$. By hypothesis

$$\max\{\inf_{z \in x \circ y}\{\lambda(z)\}, \gamma\} \geq \min\{\lambda(x), \lambda(y), \delta\} \geq \min\{t, \delta\} = t$$

$\Rightarrow \inf_{z \in x \circ y}\{\lambda(z)\} \geq t$, because $t > \gamma$. Hence $z \in U(\lambda; t)$ for every $z \in x \circ y$, that is $U(\lambda; t)$ is a subsemihypergroup of H.

Conversely, assume that $U(\lambda; t)(\neq \emptyset)$ is a subsemihypergroup of H for all $t \in (\gamma, \delta]$. Suppose that there exist $x, y \in H$ such that

$$\max\{\inf_{z \in x \circ y}\{\lambda(z)\}, \gamma\} < \min\{\lambda(x), \lambda(y), \delta\}.$$

Choose $t \in (\gamma, \delta]$ such that

$$\max\{\inf_{z \in x \circ y}\{\lambda(z)\}, \gamma\} < t \leq \min\{\lambda(x), \lambda(y), \delta\}.$$

This implies $\lambda(x) \geq t, \lambda(y) \geq t$ and $\lambda(z) < t$ for some $z \in x \circ y$, that is $x, y \in U(\lambda; t)$ but $z \notin U(\lambda; t)$ for some $z \in x \circ y$. Which is a contradiction. Hence $\max\{\inf_{z \in x \circ y}\{\lambda(z)\}, \gamma\} \geq \min\{\lambda(x), \lambda(y), \delta\}$, that is λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H.

Similarly we can prove (2), (3), (4) and (5). □

From the above Theorem it follows that

(1) Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal of a semihypergroup H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior hyperideal of H.

(2) Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) hyperideal of a semihypergroup H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal of H.

(3) Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal of a semihypergroup H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal of H.

(4) Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) hyperideal of a semihypergroup H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H.

(5) Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of a semihypergroup H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal of H.

6.2.5 Theorem

Let λ be a fuzzy subset of a semihypergroup H and $2\delta = 1 + \gamma$. Then

(1) λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H if and only if $\lambda_r^\delta(\neq \emptyset)$ is a subsemihypergroup of H for all $r \in (\delta, 1]$.

(2) λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) hyperideal of H if and only if $\lambda_r^\delta(\neq \emptyset)$ is a left (right) hyperideal of H for all $r \in (\delta, 1]$.

(3) λ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy generalized bi-hyperideal (bi-hyperideal) of* H *if and only if* $\lambda_r^\delta(\neq \emptyset)$ *is a generalized bi-hyperideal (bi-hyperideal) of* H *for all* $r \in (\delta, 1]$.

(4) λ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy interior hyperideal of* H *if and only if* $\lambda_r^\delta(\neq \emptyset)$ *is an interior hyperideal of* H *for all* $r \in (\delta, 1]$.

(5) λ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy quasi-hyperideal of* H *if and only if* $\lambda_r^\delta(\neq \emptyset)$ *is a quasi-hyperideal of* H *for all* $r \in (\delta, 1]$.

Proof. Suppose λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H and $x, y \in \lambda_r^\delta$. Then $x_r, y_r q_\delta \lambda$, That is $\lambda(x) + r > 2\delta$ and $\lambda(y) + r > 2\delta$. This implies $\lambda(x) > 2\delta - r \geq 2\delta - 1 = \gamma$ and similarly $\lambda(y) > \gamma$. By hypothesis

$$\max\{\inf_{z \in x \circ y}\{\lambda(z)\}, \gamma\} \geq \min\{\lambda(x), \lambda(y), \delta\}$$

$\Rightarrow \lambda(z) \geq \min\{\lambda(x), \lambda(y), \delta\} > \min\{2\delta - r, 2\delta - r, \delta\}$.

Since $r \in (\delta, 1]$ so $\delta < r \leq 1 \Rightarrow 2\delta - r < \delta$. Thus

$$\lambda(z) > 2\delta - r \Rightarrow \lambda(z) + r > 2\delta.$$

This implies $z \in \lambda_r^\delta$ for every $z \in x \circ y$. Hence λ_r^δ is a subsemihypergroup of H.

Conversely, assume that $\lambda_r^\delta(\neq \emptyset)$ is a subsemihypergroup of H for all $r \in (\delta, 1]$. Let $x, y \in H$ be such that for some $z \in x \circ y$

$$\max\{\lambda(z), \gamma\} < \min\{\lambda(x), \lambda(y), \delta\}$$

$\Rightarrow 2\delta - \min\{\lambda(x), \lambda(y), \delta\} < 2\delta - \max\{\lambda(z), \gamma\}$

$\Rightarrow \max\{2\delta - \lambda(x), 2\delta - \lambda(y), \delta\} < \min\{2\delta - \lambda(z), 2\delta - \gamma\}$. Take $r \in (\delta, 1]$ such that

$$\max\{2\delta - \lambda(x), 2\delta - \lambda(y), \delta\} < r \leq \min\{2\delta - \lambda(z), 2\delta - \gamma\}.$$

Then $2\delta - \lambda(x) < r, 2\delta - \lambda(y) < r$ and $r \leq 2\delta - \lambda(z)$.

$\Rightarrow \lambda(x) + r > 2\delta, \lambda(y) + r > 2\delta$ but $\lambda(z) + r \leq 2\delta$, that is $x_r q_\delta \lambda, y_r q_\delta \lambda$ but $z_r \overline{q_\delta} \lambda$ for some $z \in x \circ y$, which is a contradiction. Hence

$$\max\{\inf_{z \in x \circ y}\{\lambda(z)\}, \gamma\} \geq \min\{\lambda(x), \lambda(y), \delta\}.$$

Similarly we can prove (2), (3) and (4). □

6.2.6 Theorem

Let λ *be a fuzzy subset of a semihypergroup* H *and* $2\delta = 1 + \gamma$. *Then*

(1) λ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy subsemihypergroup of* H *if and only if* $[\lambda]_r^\delta(\neq \emptyset)$ *is a subsemihypergroup of* H *for all* $r \in (\gamma, 1]$.

(2) λ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left (right) hyperideal of* H *if and only if* $[\lambda]_r^\delta(\neq \emptyset)$ *is a left (right) hyperideal of* H *for all* $r \in (\gamma, 1]$

(3) *λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal (bi-hyperideal) of H if and only if $[\lambda]_r^\delta (\neq \emptyset)$ is a generalized bi-hyperideal (bi-hyperideal) of H for all $r \in (\gamma, 1]$.*

(4) *λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior hyperideal of H if and only if $[\lambda]_r^\delta (\neq \emptyset)$ is an interior hyperideal of H for all $r \in (\gamma, 1]$.*

(5) *λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H if and only if $[\lambda]_r^\delta (\neq \emptyset)$ is a quasi-hyperideal of H for all $r \in (\gamma, 1]$.*

Proof. Suppose λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H and $x, y \in [\lambda]_r^\delta$. Then $x_r \in_\gamma \vee q_\delta \lambda$ and $y_r \in_\gamma \vee q_\delta \lambda$, that is $\lambda(x) \geq r > \gamma$ or $\lambda(x) + r > 2\delta$ and $\lambda(y) \geq r > \gamma$ or $\lambda(y) + r > 2\delta$. Thus

$$\left.\begin{array}{r}\lambda(x) \geq r > \gamma \text{ or } \lambda(x) > 2\delta - r > 2\delta - 1 = \gamma \\ \text{and } \lambda(y) \geq r > \gamma \text{ or } \lambda(y) > 2\delta - r > 2\delta - 1 = \gamma.\end{array}\right] \quad (*)$$

If $r \in (\gamma, \delta]$, then $\gamma < r \leq \delta$. This implies $2\delta - r \geq \delta \geq r$. Then it follows from (*) that $\lambda(x) \geq r$ and $\lambda(y) \geq r$. By hypothesis

$$\max\{\inf_{z \in x \circ y} \{\lambda(z)\}, \gamma\} \geq \min\{\lambda(x), \lambda(y), \delta\}$$

$$\Rightarrow \inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), \delta\}$$

$$\Rightarrow \lambda(z) \geq \min\{\lambda(x), \lambda(y), \delta\}$$

$$\Rightarrow \lambda(z) \geq \min\{r, r, r\} = r \text{ for all } z \in x \circ y.$$

and so $z_r \in_\gamma \lambda$. Thus $z \in [\lambda]_r^\delta$ for every $z \in x \circ y$.

If $r \in (\delta, 1]$, then $\delta < r \leq 1$. This implies $2\delta - r < \delta < r$. Then it follows from (*) that $\lambda(x) > 2\delta - r$ and $\lambda(y) > 2\delta - r$.

Now by hypothesis

$$\max\{\inf_{z \in x \circ y} \{\lambda(z)\}, \gamma\} \geq \min\{\lambda(x), \lambda(y), \delta\}$$

$$\Rightarrow \inf_{z \in x \circ y} \{\lambda(z)\} \geq \min\{\lambda(x), \lambda(y), \delta\}$$

$$\Rightarrow \lambda(z) \geq \min\{\lambda(x), \lambda(y), \delta\}$$

$$\Rightarrow \lambda(z) > \min\{2\delta - r, 2\delta - r, 2\delta - r\} = 2\delta - r \text{ for all } z \in x \circ y.$$

This implies $\lambda(z) + r > 2\delta$, that is $z_r q_\delta \lambda$. Thus $z \in [\lambda]_r^\delta$ for every $z \in x \circ y$. Hence $[\lambda]_r^\delta$ is a subsemihypergroup of H.

Conversely, assume that $[\lambda]_r^\delta$ is a subsemihypergroup of H for all $r \in (\gamma, 1]$. Let $x, y \in H$ be such that for some $z \in x \circ y$

$$\max\{\lambda(z), \gamma\} < \min\{\lambda(x), \lambda(y), \delta\}.$$

Select $r \in (\gamma, 1]$ such that

$$\max\{\lambda(z), \gamma\} < r \leq \min\{\lambda(x), \lambda(y), \delta\}.$$

Then $x_r \in_\gamma \lambda$ and $y_r \in_\gamma \lambda$ but $z_r \overline{\in_\gamma \vee q_\delta} \lambda$ which contradicts our hypothesis. Hence

$$\max\{\inf_{z \in x o y}\{\lambda(z)\}, \gamma\} \geq \min\{\lambda(x), \lambda(y), \delta\},$$

that is λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H.
Similary, we can prove the parts (2), (3), (4) and (5). □

6.2.7 Theorem

(1) *The intersection of any family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroups of a semihypergroup H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H.*

(2) *The intersection of any family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) hyperideals of a semihypergroup H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) hyperideal of H.*

(3) *The intersection of any family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideals (bi-hyperideals) of a semihypergroup H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal (bi-hyperideal) of H.*

(4) *The intersection of any family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior hyperideals of a semihypergroup H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior hyperideal of H.*

(5) *The intersection of any family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideals of a semihypergroup H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H.*

Proof. (1) Let $\{\lambda_i\}_{i \in I}$ be a family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H and $x, y \in H$. Then

$$\begin{aligned}
\left(\inf_{z \in x o y}\left(\bigwedge_{i \in I}\lambda_i\right)(z)\right) \vee \gamma &= \left(\bigwedge_{i \in I}\left(\inf_{z \in x o y}(\lambda_i)(z)\right)\right) \vee \gamma \\
&= \left(\bigwedge_{i \in I}\left(\inf_{z \in x o y}(\lambda_i)(z)\right) \vee \gamma\right) \\
&\geq \left(\bigwedge_{i \in I}(\min\{\lambda_i(x), \lambda_i(y), \delta\})\right) \\
&= \left(\bigwedge_{i \in I}\lambda_i(x)\right) \wedge \left(\bigwedge_{i \in I}\lambda_i(y)\right) \wedge \delta \\
&= \left(\left(\bigwedge_{i \in I}\lambda_i\right)(x)\right) \wedge \left(\left(\bigwedge_{i \in I}\lambda_i\right)(y)\right) \wedge \delta
\end{aligned}$$

Thus $\bigwedge_{i\in I}\lambda_i$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H.

Similarly, we can prove (2), (3), (4) and (5). □

6.2.8 Theorem

The union of any family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) hyperideals of a semihypergroup H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) hyperideal of H.

Proof. Let $\{\lambda_i\}_{i\in I}$ be a family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideals of H and $x, y \in H$. Then

$$\begin{aligned}
\left(\inf_{z\in y\circ x}\left(\bigvee_{i\in I}\lambda_i\right)(z)\right)\vee\gamma &= \left(\bigvee_{i\in I}\left(\inf_{z\in y\circ x}(\lambda_i)(z)\right)\right)\vee\gamma \\
&= \left(\bigvee_{i\in I}\left(\inf_{z\in y\circ x}(\lambda_i)(z)\right)\vee\gamma\right) \\
&\geq \bigvee_{i\in I}((\lambda_i)(x)\wedge\delta) \\
&= \left(\bigvee_{i\in I}\lambda_i(x)\right)\wedge\delta \\
&= \left(\bigvee_{i\in I}\lambda_i\right)(x)\wedge\delta
\end{aligned}$$

This shows that $\bigvee_{i\in I}\lambda_i$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H. □

6.2.9 Proposition

Let λ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal and μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal of a semihypergroup H. Then $\lambda\circ\mu$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal of H.

Proof. Let λ, μ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left and right hyperideals of H, respectively. Let $x, y \in H$. Then

$$\begin{aligned}
(\lambda\circ\mu)(y)\wedge\delta &= \left(\bigvee_{y\in p\circ q}\{\lambda(p)\wedge\mu(q)\}\right)\wedge\delta \\
&= \bigvee_{y\in p\circ q}\{\lambda(p)\wedge\mu(q)\wedge\delta\} \\
&= \bigvee_{y\in p\circ q}\{\lambda(p)\wedge\delta\wedge\mu(q)\}.
\end{aligned}$$

(If $y \in p \circ q$ for some $p, q \in H$, then $x \circ y \subseteq x \circ (p \circ q) = (x \circ p) \circ q$. Now for each $z \in x \circ y$, there exists $a \in x \circ p$ such that $z \in a \circ q$. Since λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal, therefore by Theorem 6.2.1, we have $\{\inf_{a \in x \circ p} \{\lambda(a)\} \vee \gamma\} \geq \{\lambda(p) \wedge \delta\}$ that is $\lambda(a) \vee \gamma \geq \{\lambda(p) \wedge \delta\}$.)

Thus

$$\begin{aligned}
(\lambda \circ \mu)(y) \wedge \delta &= \bigvee_{y \in p \circ q} \{\lambda(p) \wedge \delta \wedge \mu(q)\} \\
&\leq \bigvee_{z \in a \circ q} \{(\lambda(a) \vee \gamma) \wedge \mu(q)\} \text{ because } \lambda(a) \vee \gamma \geq \{\lambda(p) \wedge \delta\} \\
&\leq \bigvee_{z \in c \circ d} \{\lambda(c) \wedge \mu(d)\} \vee \gamma \\
&= (\lambda \circ \mu)(z) \vee \gamma, \text{ for every } z \in x \circ y \subseteq a \circ q.
\end{aligned}$$

So

$$\min \{(\lambda \circ \mu)(y), \delta\} \leq \max\{\inf_{z \in x \circ y} \{(\lambda \circ \mu)(z)\}, \gamma\}.$$

Similarly we can show that $\max\{\inf_{z \in x \circ y} \{(\lambda \circ \mu)(z)\}, \gamma\} \geq \min \{(\lambda \circ \mu)(x), \delta\}$. Thus $\lambda \circ \mu$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal of H. □

Next we show that if λ and μ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideals of a semihypergroup H, then $\lambda \circ \mu \nleq \lambda \wedge \mu$.

6.2.10 Example

Consider the semihypergroup $H = \{a, b, c, d\}$ with the following table:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	$\{a, b\}$	a
d	a	a	$\{a, b\}$	$\{a, b\}$

Define fuzzy sets λ, μ of H by

$$\begin{aligned}
\lambda(a) &= 0.6, \quad \lambda(b) = 0.3, \quad \lambda(c) = 0.4, \quad \lambda(d) = 0.1, \\
\mu(a) &= 0.65, \quad \mu(b) = 0.3, \quad \mu(c) = 0.4, \quad \mu(d) = 0.2.
\end{aligned}$$

Then we have

$$U(\lambda; t) = \begin{cases} \{a, b, c, d\} & \text{if } 0 < t \leq 0.1, \\ \{a, b, c\} & \text{if } 0.1 < t \leq 0.3, \\ \{a, c\} & \text{if } 0.3 < t \leq 0.4, \\ \{a\} & \text{if } 0.4 < t \leq 0.6, \\ \emptyset & \text{if } 0.6 < t. \end{cases}$$

$$U(\mu;t)=\begin{cases}\{a,b,c,d\} & \text{if } 0<t\leq 0.2,\\ \{a,b,c\} & \text{if } 0.2<t\leq 0.3,\\ \{a,c\} & \text{if } 0.3<t\leq 0.4,\\ \{a\} & \text{if } 0.4<t\leq 0.65,\\ \emptyset & \text{if } 0.65<t.\end{cases}$$

Thus by Theorem 6.2.4, λ,μ are $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy hyperideals of H for $\gamma=0$ and $\delta=0.3$.

But

$$\begin{aligned}(\lambda\circ\mu)(b) &= \bigvee_{b\in x\circ y}\{\lambda(x)\wedge\mu(y)\}\\ &= \bigvee\{0.4,0.1,0.1\}\\ &= 0.4\nleq(\lambda\wedge\mu)(b)=0.3.\end{aligned}$$

Hence $\lambda\circ\mu\nleq\lambda\wedge\mu$, in general.

6.2.11 Definition

Let λ,μ be fuzzy subsets of a semihypergroup H. We define the fuzzy subsets $\dot{\lambda},\lambda\,\dot{\wedge}\,\mu,\lambda\,\dot{\vee}\,\mu$, and $\lambda*\mu$ of H as follows;

$$\begin{aligned}\dot{\lambda}(x) &= (\lambda(x)\vee\gamma)\wedge\delta\\ (\lambda\,\dot{\wedge}\,\mu)(x) &= ((\lambda\wedge\mu)(x)\vee\gamma)\wedge\delta\\ (\lambda\,\dot{\vee}\,\mu)(x) &= ((\lambda\vee\mu)(x)\vee\gamma)\wedge\delta\\ (\lambda*\mu)(x) &= ((\lambda\circ\mu)(x)\vee\gamma)\wedge\delta\end{aligned}$$

for all $x\in H$.

6.2.12 Lemma

Let λ,μ be fuzzy subsets of a semihypergroup H. Then the following conditions hold:

(1) $\lambda\,\dot{\wedge}\,\mu=\dot{\lambda}\wedge\dot{\mu}$.

(2) $\lambda\,\dot{\vee}\,\mu=\dot{\lambda}\vee\dot{\mu}$.

(3) $\lambda*\mu\geq\dot{\lambda}\circ\dot{\mu}$.

Proof. Let $x\in H$.

(1)

$$\begin{aligned}(\lambda \mathbin{\dot\wedge} \mu)(x) &= ((\lambda \wedge \mu)(x) \vee \gamma) \wedge \delta \\ &= ((\lambda(x) \wedge \mu(x)) \vee \gamma) \wedge \delta \\ &= ((\lambda(x) \vee \gamma) \wedge (\mu(x) \vee \gamma)) \wedge \delta \\ &= ((\lambda(x) \vee \gamma) \wedge \delta) \wedge ((\mu(x) \vee \gamma) \wedge \delta) \\ &= \dot{\lambda}(x) \wedge \dot{\mu}(x) \\ &= (\dot{\lambda} \wedge \dot{\mu})(x).\end{aligned}$$

(2) Proof is similar to the proof of (1).
(3) If there does not exist $y, z \in H$ such that $x \in y \circ z$, then

$$\begin{aligned}(\lambda * \mu)(x) &= ((\lambda \circ \mu)(x) \vee \gamma) \wedge \delta \\ &= (0 \vee \gamma) \wedge \delta = \gamma \wedge \delta \\ &\geq 0 = (\dot{\lambda} \circ \dot{\mu})(x).\end{aligned}$$

Otherwise

$$\begin{aligned}(\lambda * \mu)(x) &= ((\lambda \circ \mu)(x)) \vee \gamma) \wedge \delta \\ &= \left(\left(\bigvee_{x \in y \circ z} \{\lambda(y) \wedge \mu(z)\}\right) \vee \gamma\right) \wedge \delta \\ &= \left(\left(\bigvee_{x \in y \circ z} \{(\lambda(y) \vee \gamma) \wedge (\mu(z) \vee \gamma)\}\right)\right) \wedge \delta \\ &= \bigvee_{x \in y \circ z} (((\lambda(y) \vee \gamma) \wedge \delta) \wedge ((\mu(z) \vee \gamma) \wedge \delta)) \\ &= \bigvee_{x \in y \circ z} \left(\dot{\lambda}(y) \wedge \dot{\mu}(z)\right) \\ &= \left(\dot{\lambda} \circ \dot{\mu}\right)(x).\end{aligned}$$

□

6.2.13 Lemma

Let A, B be non-empty subsets of a semihypergroup H. Then the following conditions hold:

(1) $\lambda_A \mathbin{\dot\wedge} \lambda_B = \dot{\lambda}_{A \cap B}$.

(2) $\lambda_A \mathbin{\dot\vee} \lambda_B = \dot{\lambda}_{A \cup B}$.

(3) $\lambda_A * \lambda_B = \dot{\lambda}_{A \circ B}$.

Proof. Straightforward. □

6.2.14 Theorem

Let λ be a fuzzy subset of a semihypergroup H.

(1) *If λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H, then $\dot{\lambda}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H.*

(2) *If λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) hyperideal of H, then $\dot{\lambda}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) hyperideal of H.*

(3) *If λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal (bi-hyperideal) of H, then $\dot{\lambda}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal (bi-hyperideal) of H.*

(4) *If λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior hyperideal of H, then $\dot{\lambda}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior hyperideal of H.*

(5) *If λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H, then $\dot{\lambda}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H.*

Proof. Let λ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H and $x, y \in H$. Then for each $z \in x \circ y$, we have

$$\begin{aligned}
\max\{\inf_{z\in x\circ y}\{\dot{\lambda}(z)\}, \gamma\} &= \left(\inf_{z\in x\circ y}(\lambda(z)\vee\gamma)\wedge\delta\right)\vee\gamma \\
&\geq ((\lambda(x)\wedge\lambda(y)\wedge\delta)\wedge\delta)\vee\gamma \\
&= (((\lambda(x)\wedge\delta)\wedge(\lambda(y)\wedge\delta))\wedge\delta)\vee\gamma \\
&= (((\lambda(x)\vee\gamma\wedge\delta)\wedge(\lambda(y)\vee\gamma\wedge\delta))\wedge\delta \\
&= \dot{\lambda}(x)\wedge\dot{\lambda}(y)\wedge\delta \\
&= \min\{\dot{\lambda}(x), \dot{\lambda}(y), \delta\}.
\end{aligned}$$

Thus $\dot{\lambda}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H.
Similarly, we can prove the part (2), (3), (4) and (5). □

6.2.15 Corollary

(1) *A non-empty subset A of a semihypergroup H is a subsemihypergroup of H if and only if $\dot{\lambda}_A$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H.*

(2) *A non-empty subset A of a semihypergroup H is a left (right) hyperideal of H if and only if $\dot{\lambda}_A$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) hyperideal of H.*

(3) *A non-empty subset A of a semihypergroup H is a generalized bi-hyperideal (bi-hyperideal) of H if and only if $\dot{\lambda}_A$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal (bi-hyperideal) of H.*

(4) *A non-empty subset A of semihypergroup H is an interior hyperideal of H if and only if* $\dot{\lambda}_A$ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy interior hyperideal of H.*

(5) *A non-empty subset A of semihypergroup H is a quasi-hyperideal of H if and only if* $\dot{\lambda}_A$ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy quasi-hyperideal of H.*

6.2.16 Theorem

(1) *A non-empty subset* λ *of a semihypergroup H is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy subsemihypergroup of H if and only if* $\lambda * \lambda \leq \dot{\lambda}$.

(2) *A non-empty subset* λ *of a semihypergroup H is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left (right) hyperideal of H if and only if* $1 * \lambda \leq \dot{\lambda}(\lambda * 1 \leq \dot{\lambda})$.

(3) *A non-empty subset* λ *of a semihypergroup H is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy generalized bi-hyperideal of H if and only if* $\lambda * 1 * \lambda \leq \dot{\lambda}$.

(4) *A non-empty subset* λ *of a semihypergroup H is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy bi-hyperideal of H if and only if* $\lambda * 1 * \lambda \leq \dot{\lambda}$ and $\lambda * \lambda \leq \dot{\lambda}$.

(5) *A non-empty subset* λ *of a semihypergroup H is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy interior hyperideal of H if and only if* $1 * \lambda * 1 \leq \dot{\lambda}$.

(6) *A non-empty subset* λ *of a semihypergroup H is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy quasi-hyperideal of H if and only if* $1 * \lambda \wedge \lambda * 1 \leq \dot{\lambda}$.

Proof. (1) Let λ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H and $x \in H$. If $(\lambda * \lambda)(x) = \gamma \wedge \delta$, then

$$(\lambda * \lambda)(x) = \gamma \wedge \delta \leq \lambda(x) \vee (\gamma \wedge \delta) = (\lambda(x) \vee \gamma) \wedge \delta = \dot{\lambda}(x).$$

Otherwise

$$\begin{aligned}
(\lambda * \lambda)(x) &= ((\lambda \circ \lambda)(x)) \vee \gamma) \wedge \delta \\
&= \left(\left(\bigvee_{x \in a \circ b} \{\lambda(a) \wedge \lambda(b)\} \right) \vee \gamma \right) \wedge \delta \\
&= \left(\bigvee_{x \in a \circ b} \{\lambda(a) \wedge \lambda(b) \wedge \delta\} \vee \gamma \right) \wedge \delta \\
&\leq \left(\bigvee_{x \in a \circ b} \{\lambda(x) \vee \gamma\} \vee \gamma \right) \wedge \delta \\
&= (\lambda(x) \vee \gamma) \wedge \delta \\
&= \dot{\lambda}(x).
\end{aligned}$$

Thus $\lambda * \lambda \leq \dot{\lambda}$.

Conversely, assume that $\lambda * \lambda \leq \dot{\lambda}$ and $x, y \in H$. Then for each $z \in x \circ y$,

$$\begin{aligned} (\lambda(z) \vee \gamma) \wedge \delta &= \dot{\lambda}(z) \geq (\lambda * \lambda)(z) \\ &= \left(\left(\bigvee_{z \in a \circ b} \{\lambda(a) \wedge \lambda(b)\} \right) \vee \gamma \right) \wedge \delta \\ &\geq (\{\lambda(x) \wedge \lambda(y)\} \vee \gamma) \wedge \delta \\ &\geq \lambda(x) \wedge \lambda(y) \wedge \delta = \min\{\lambda(x), \lambda(y), \delta\}. \end{aligned}$$

Thus $\max\{\inf_{z \in x \circ y} \{\lambda(z)\}, \gamma\} \geq \min\{\lambda(x), \lambda(y), \delta\}$.

Hence λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup of H.

(3) Let λ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal of H and $x \in H$. If $(\lambda * 1 * \lambda)(x) = \gamma \wedge \delta$, then

$$(\lambda * 1 * \lambda)(x) = \gamma \wedge \delta \leq \lambda(x) \vee (\gamma \wedge \delta) = (\lambda(x) \vee \gamma) \wedge \delta = \dot{\lambda}(x).$$

Otherwise

$$\begin{aligned} (\lambda * 1 * \lambda)(x) &= ((\lambda \circ (1 * \lambda)(x)) \vee \gamma) \wedge \delta \\ &= \left(\left(\bigvee_{x \in a \circ b} \{\lambda(a) \wedge (1 * \lambda)(b)\} \right) \vee \gamma \right) \wedge \delta \\ &= \left(\bigvee_{x \in a \circ b} \left\{ \lambda(a) \wedge \left[\left(\left(\bigvee_{b \in c \circ d} \{1(c) \wedge \lambda(d)\} \right) \vee \gamma \right) \wedge \delta \right] \right\} \vee \gamma \right) \wedge \delta \\ &= \left(\bigvee_{x \in a \circ b} \left\{ \lambda(a) \wedge \left[\left(\left(\bigvee_{b \in c \circ d} \lambda(d) \right) \vee \gamma \right) \wedge \delta \right] \right\} \vee \gamma \right) \wedge \delta \\ &= \left(\left(\bigvee_{x \in a \circ b} \left\{ \bigvee_{b \in c \circ d} [\lambda(a) \wedge \lambda(d)] \vee \gamma \right\} \wedge \delta \right) \vee \gamma \right) \wedge \delta \\ &= \left(\left(\bigvee_{x \in a \circ b} \left\{ \bigvee_{b \in c \circ d} [\lambda(a) \wedge \lambda(d) \wedge \delta] \vee \gamma \right\} \right) \vee \gamma \right) \wedge \delta \\ &= \left(\bigvee_{x \in a \circ b} \left\{ \bigvee_{b \in c \circ d} [\lambda(a) \wedge \lambda(d) \wedge \delta] \vee \gamma \right\} \right) \\ &\leq \left(\bigvee_{x \in a \circ b \subseteq a \circ c \circ d} \{\lambda(x) \vee \gamma\} \vee \gamma \right) \wedge \delta \\ &= \lambda(x) \vee \gamma \wedge \delta \\ &= \dot{\lambda}(x). \end{aligned}$$

Thus $\lambda * 1 * \lambda \leq \overset{\bullet}{\lambda}$.

Conversely, assume that $\overset{\bullet}{\lambda} \geq \lambda * 1 * \lambda$ and $x, y, z \in H$. Then for every $w \in x \circ y \circ z$,

$$\begin{aligned}
\{\lambda(w)\} \vee \gamma\} \wedge \delta &= \overset{\bullet}{\lambda}(w) \\
&\geq (\lambda * 1 * \lambda)(w) \\
&= \left(\left(\bigvee_{w \in x \circ p} \{\lambda(x) \wedge (1 * \lambda)(p)\}\right) \vee \gamma\right) \wedge \delta \\
&\quad \text{(because there exist } p \in y \circ z \text{ such that } w \in x \circ p) \\
&\geq ((\lambda(x) \wedge (1 * \lambda)(p)) \vee \gamma) \wedge \delta \\
&\geq \left(\left(\lambda(x) \wedge \left[\left(\bigvee_{p \in y \circ z} \{1(y) \wedge \lambda(z)\} \vee \gamma\right)\right] \wedge \delta\right) \vee \gamma\right) \wedge \delta \\
&\geq ((\lambda(x) \wedge ([\lambda(z) \vee \gamma] \wedge \delta)) \vee \gamma) \wedge \delta \\
&\geq (((\lambda(x) \wedge \lambda(z)) \wedge \delta) \vee \gamma) \wedge \delta \\
&\geq ((\lambda(x) \wedge \lambda(z)) \wedge \delta) \wedge \delta \\
&= \lambda(x) \wedge \lambda(z) \wedge \delta.
\end{aligned}$$

Thus $\max\{\inf_{w \in x \circ y \circ z} \{\lambda(w)\}, \gamma\} \geq \lambda(x) \wedge \lambda(z) \wedge \delta$.

Thus λ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal of H.

(2) The proof is similar to the proof of (3).

(4) The proof follows from (1) and (3).

(5) Like the proof of (3), we can prove this part also.

(6) The proof follows from (2). □

6.2.17 Lemma

Let λ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal and μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of a semihypergroup H. Then $\lambda * \mu \leq \overset{\bullet}{\lambda} \wedge \overset{\bullet}{\mu}$.

Proof. Let λ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal and μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H. Let $y, z \in H$. Then for every $a \in y \circ z$ we have

$$
\begin{aligned}
(\lambda * \mu)(a) &= ((\lambda \circ \mu)(a) \vee \gamma) \wedge \delta \\
&= \left(\bigvee_{a \in y \circ z} \{\lambda(y) \wedge \mu(z)\} \vee \gamma \right) \wedge \delta \\
&= \bigvee_{a \in y \circ z} (\{(\lambda(y) \vee \gamma) \wedge (\mu(z) \vee \gamma)\} \wedge \delta) \\
&= \bigvee_{a \in y \circ z} \{((\lambda(y) \vee \gamma) \wedge \delta) \wedge ((\mu(z) \vee \gamma) \wedge \delta)\} \\
&\leq \bigvee_{a \in y \circ z} \{(\lambda(a) \vee \gamma) \wedge (\mu(a) \vee \gamma) \wedge \delta\} \\
&= ((\lambda(a) \vee \gamma) \wedge \delta) \wedge ((\mu(a) \vee \gamma) \wedge \delta) \\
&= (\dot{\lambda} \wedge \dot{\mu})(a).
\end{aligned}
$$

So $(\lambda * \mu) \leq (\dot{\lambda} \wedge \dot{\mu})$. □

6.3 Regular semihypergroups

In this section we characterize regular semihypergroups by the properties of their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi- hyperideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideals.and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideals.

6.3.1 Theorem

The following statements for a semihypergroup H are equivalent:

(1) *H is regular.*

(2) *$\dot{\lambda} \wedge \dot{\mu} = \lambda * \mu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal λ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal μ of H.*

Proof. (1) $\Rightarrow$ (2) Let λ, μ be $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy right and left hyperideals of* H, respectively and $a \in H$ Since H is regular, so there exists $x \in H$ such that $a \in$

$a \circ x \circ a = (a \circ x) \circ a$. Thus there exists some $\beta \in a \circ x$ such that $a \in \beta \circ a$. So

$$\begin{aligned}
(\lambda * \mu)(a) &= ((\lambda \circ \mu)(a) \vee \gamma) \wedge \delta \\
&= \left(\bigvee_{a \in c \circ d} \{\lambda(c) \wedge \mu(d)\} \vee \gamma \right) \wedge \delta \\
&\geq \{(\lambda(\beta) \vee \gamma) \wedge (\mu(a) \vee \gamma)\} \wedge \delta \\
&\quad \left(\begin{array}{c} \text{Since } \beta \in a \circ x \text{ and } \lambda \text{ is an } (\in_\gamma, \in_\gamma \vee q_\delta)\text{-fuzzy right} \\ \text{hyperideal of } H, \text{ so } \inf_{z \in \theta \circ \delta} \{\lambda(z)\} \vee \gamma \geq \lambda(\theta) \wedge \delta \\ \text{therefore } \lambda(\beta) \vee \gamma \geq \lambda(a) \wedge \delta. \end{array} \right) \\
\text{Therefore } (\lambda * \mu)(a) &\geq ((\lambda(a) \vee \gamma) \wedge \delta) \wedge ((\mu(a) \vee \gamma) \wedge \delta) \\
&= \left(\overset{*}{\lambda}(a) \wedge \overset{*}{\mu}(a) \right) \\
&= (\overset{*}{\lambda} \wedge \overset{*}{\mu})(a).
\end{aligned}$$

By Lemma 6.2.17 $\lambda * \mu \leq \overset{*}{\lambda}$. Thus $\lambda * \mu \leq \overset{*}{\lambda} \wedge \overset{*}{\mu}$.

(2) $\Rightarrow$ (1) Let R and L be right and left hyperideals of H. Then by Lemma 6.2.15, $\overset{*}{\lambda}_R$ and $\overset{*}{\lambda}_L$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideals of H, respectively. Thus by hypothesis

$$\begin{aligned}
\overset{*}{\lambda}_{R \circ L} &= \lambda_R * \lambda_L \\
&= \overset{*}{\lambda}_R \wedge \overset{*}{\lambda}_L \quad \text{by Lemma 6.2.12 and by (2)} \\
&= \lambda_R \overset{*}{\wedge} \lambda_L \quad \text{by Lemma 6.2.12} \\
&= \overset{*}{\lambda}_{R \cap L} \quad \text{by Lemma 6.2.13.}
\end{aligned}$$

This implies $R \cap L = R \circ L$. Hence it follows from Proposition 3.2.4 that H is regular. □

6.3.2 Theorem

For a semihypergroup H, the following conditions are equivalent:

(1) *H is regular.*

(2) $\left(\overset{*}{\lambda} \wedge \overset{*}{\mu} \wedge \overset{*}{\nu}\right) \leq (\lambda * \mu * \nu)$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal λ, for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal μ and for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal ν of H.*

(3) $\left(\overset{*}{\lambda} \wedge \overset{*}{\mu} \wedge \overset{*}{\nu}\right) \leq (\lambda * \mu * \nu)$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal λ, for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal μ and for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal ν of H.*

(4) $\left(\dot{\lambda} \wedge \dot{\mu} \wedge \dot{\nu}\right) \leq (\lambda * \mu * \nu)$ *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy right hyperideal* λ, *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy quasi-hyperideal* μ *and for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left hyperideal* ν *of* H.

Proof. (1) $\Rightarrow$ (2) Let λ, μ and ν be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H, respectively. Let $a \in H$. Since H is regular, so there exists $x \in H$ such that $a \in a \circ x \circ a = (a \circ x) \circ a$. Thus there exists $\beta \in a \circ x$ such that $a \in \beta \circ a$. So

$$\begin{aligned}
(\lambda * \mu * \nu)(a) &= ((\lambda \circ \mu \circ \nu)(a) \vee \gamma) \wedge \delta \\
&= \left(\bigvee_{a \in c \circ d} \{\lambda(c) \wedge (\mu \circ \nu)(d)\} \vee \gamma\right) \wedge \delta \\
&\geq (\{\lambda(\beta) \wedge (\mu \circ \nu)(a)\} \vee \gamma) \wedge \delta \\
&\quad \left(\text{since } \inf_{\beta \in a \circ x} \{\lambda(\beta)\} \vee \gamma \geq \lambda(a) \wedge \delta.\right) \\
\text{so, } (\lambda * \mu * \nu)(a) &\geq ((\lambda(a) \vee \gamma) \wedge \delta) \wedge ((\mu \circ \nu)(a) \vee \gamma) \wedge \delta \qquad \text{(i)}
\end{aligned}$$

Since $a \in a \circ x \circ a = a \circ (x \circ a)$, so there exists $r \in x \circ a$, such that $a \in a \circ r$. Thus

$$\begin{aligned}
(\mu \circ \nu)(a) \vee \gamma &= \bigvee_{a \in e \circ f} \{\mu(e) \wedge \nu(f)\} \vee \gamma \\
&\geq \{\mu(a) \wedge \nu(r)\} \vee \gamma \\
&\quad \left(\text{because } \inf_{\gamma \in x \circ a} \{\nu(r)\} \vee \gamma \geq \nu(a) \wedge \delta.\right) \\
&\geq (\mu(a) \vee \gamma) \wedge (\nu(a) \vee \gamma)) \wedge \delta.
\end{aligned}$$

Thus substitiuting value of $(\mu \circ \nu)(a)$ in (i), we have

$$\begin{aligned}
(\lambda * \mu * \nu)(a) &\geq ((\lambda(a) \vee \gamma) \wedge \delta) \wedge ((\mu \circ \nu)(a) \vee \gamma) \wedge \delta \\
&= ((\lambda(a) \vee \gamma) \wedge \delta) \wedge (\mu(a) \vee \gamma) \wedge (\nu(a) \vee \gamma)) \wedge \delta \\
&= ((\lambda(a) \vee \gamma) \wedge \delta) \wedge ((\mu(a) \vee \gamma) \wedge \delta) \wedge ((\nu(a) \vee \gamma) \wedge \delta) \\
&= \left(\dot{\lambda}(a) \wedge \dot{\mu}(a) \wedge \dot{\nu}(a)\right) \\
&= \left(\dot{\lambda} \wedge \dot{\mu} \wedge \dot{\nu}\right)(a)
\end{aligned}$$

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) Straightforward.

(4) $\Rightarrow$ (1) Let λ and ν be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideals of H, respectively. Then $\dot{\lambda}$ and $\dot{\nu}$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right and

$(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideals of H, respectively. Since "1" is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H, so by hypothesis, we have

$$\begin{aligned}
\left(\dot{\lambda} \wedge \dot{\nu}\right)(a) &= ((\lambda \wedge \nu)(a) \vee \gamma) \wedge \delta \\
&= ((\lambda \wedge 1 \wedge \nu)(a) \vee \gamma) \wedge \delta \\
&= \left(\dot{\lambda} \wedge 1 \wedge \dot{\nu}\right)(a) \\
&\leq (\lambda * 1 * \nu)(a) \\
&= ((\lambda \circ 1 \circ \nu)(a) \vee \gamma) \wedge \delta \\
&= \left(\bigvee_{a \in b \circ c} \{(\lambda \circ 1)(b) \wedge \nu(c)\} \vee \gamma\right) \wedge \delta \\
&\quad \left(\text{As } (\lambda \circ 1)(b) = \bigvee_{b \in p \circ q} \{\lambda(p) \wedge 1(q)\}\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\left(\dot{\lambda} \wedge \dot{\nu}\right)(a) &\leq \left(\bigvee_{a \in b \circ c} \left\{\left(\bigvee_{b \in p \circ q} \{\lambda(p) \wedge 1(q)\}\right) \wedge \nu(c)\right\} \vee \gamma\right) \wedge \delta \\
&= \left(\bigvee_{a \in b \circ c} \left\{\left(\bigvee_{b \in p \circ q} \{\lambda(p) \wedge 1\}\right) \wedge \nu(c)\right\} \vee \gamma\right) \wedge \delta \\
&= \left(\bigvee_{a \in b \circ c} \left\{\left(\bigvee_{b \in p \circ q} \lambda(p)\right) \wedge \nu(c)\right\} \vee \gamma\right) \wedge \delta \\
&= \left(\bigvee_{a \in b \circ c} \left\{\left(\bigvee_{b \in p \circ q} \lambda(p) \vee \gamma\right) \wedge (\nu(c) \vee \gamma)\right\} \wedge \delta\right) \wedge \delta \\
&= \left(\bigvee_{a \in b \circ c} \left\{\left(\bigvee_{b \in p \circ q} \{(\lambda(p) \vee \gamma) \wedge \delta\}\right) \wedge (\nu(c) \vee \gamma)\right\}\right) \wedge \delta \\
&\quad \left(\begin{array}{c} \text{Since } \lambda \text{ is an } (\in_\gamma, \in_\gamma \vee q_\delta)\text{-fuzzy right hyperidealof } H, \text{ so} \\ \inf_{b \in p \circ q} \{\lambda(b)\} \vee \gamma \geq \lambda(p) \wedge \delta \text{ that is} \\ \lambda(b) \vee \gamma \geq \lambda(p) \wedge \delta \text{ for every } b \in p \circ q. \end{array}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(\overset{\bullet}{\lambda}\wedge\overset{\bullet}{\nu}\right)(a) &\leq \left(\bigvee_{a\in b\circ c}\{(\lambda(b)\vee\gamma)\wedge(\nu(c)\vee\gamma)\}\right)\wedge\delta \\
&= \left(\left(\bigvee_{a\in b\circ c}\lambda(b)\wedge\nu(c)\right)\vee\gamma\right)\wedge\delta \\
&= ((\lambda\circ\nu)(a)\vee\gamma)\wedge\delta \\
&= (\lambda*\nu)(a).
\end{aligned}$$

Hence, it follows that $\overset{\bullet}{\lambda}\wedge\overset{\bullet}{\nu}\leq\lambda*\nu$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy right hyperideal λ of H and for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left hyperideal ν of H. But by Lemma 6.2.17, $\overset{\bullet}{\lambda}\wedge\overset{\bullet}{\nu}\geq\lambda*\nu$. So $\overset{\bullet}{\lambda}\wedge\overset{\bullet}{\nu}=\lambda*\nu$. Hence by Theorem 6.3.1, H is regular. □

6.3.3 Theorem

For a semihypergroup H, the following conditions are equivalent:

(1) *H is regular.*

(2) *$\overset{\bullet}{\lambda}=(\lambda*1*\lambda)$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy generalized bi-hyperideal λ of H.*

(3) *$\overset{\bullet}{\lambda}=(\lambda*1*\lambda)$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-hyperideal λ of H.*

(4) *$\overset{\bullet}{\lambda}=(\lambda*1*\lambda)$ for every $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy quasi-hyperideal λ of H.*

Proof. (1) ⇒ (2) Let λ be an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy generalized bi-hyperideal of H and $a\in H$. Since H is regular, so there exists $x\in H$ such that $a\in a\circ x\circ a=(a\circ x)\circ a$. Thus there exists $\beta\in a\circ x$ such that $a\in\beta\circ a$. So

$$\begin{aligned}
(\lambda*1*\lambda)(a) &= ((\lambda\circ1\circ\lambda)(a)\vee\gamma)\wedge\delta \\
&= \left(\bigvee_{a\in\beta\circ a}\{(\lambda\circ1)(\beta)\wedge\lambda(a)\}\vee\gamma\right)\wedge\delta \\
&\geq (((\lambda\circ1)(\beta)\vee\gamma)\wedge(\lambda(a)\vee\gamma))\wedge\delta \\
&= \left(\left(\bigvee_{\beta\in a\circ x}\{\lambda(a)\wedge1(x)\}\vee\gamma\right)\wedge(\lambda(a)\vee\gamma)\right)\wedge\delta \\
&\geq ((\lambda(a)\wedge1(x))\vee\gamma)\wedge(\lambda(a)\vee\gamma))\wedge\delta \\
&= (\lambda(a)\wedge1\wedge\lambda(a)\vee\gamma)\wedge\delta \\
&= (\lambda(a)\vee\gamma)\wedge\delta \\
&= \overset{\bullet}{\lambda}(a).
\end{aligned}$$

Thus $(\lambda*1*\lambda)\geq\overset{\bullet}{\lambda}$.

But from part (3) of Theorem 6.2.16, it follows that $(\lambda * 1 * \lambda) \leq \dot{\lambda}$.

Thus $\dot{\lambda} = (\lambda * 1 * \lambda)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (1) Let A be any quasi-hyperideal of H. Then by Corollary 6.2.15, $\dot{\lambda}_A$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H. Hence, by hypothesis,

$$\dot{\lambda}_A = (\lambda_A * 1 * \lambda_A) = (\lambda_A * \lambda_H * \lambda_A) = \dot{\lambda}_{A \circ H \circ A}.$$

This implies $A = A \circ H \circ A$. Hence it follows from Theorem 3.2.6, that H is regular. □

6.3.4 Theorem

For a semihypergroup H, the following conditions are equivalent:

(1) *H is regular.*

(2) $\left(\dot{\lambda} \wedge \dot{\mu}\right) = (\lambda * \mu * \lambda)$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal λ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal μ of H.*

(3) $\left(\dot{\lambda} \wedge \dot{\mu}\right) = (\lambda * \mu * \lambda)$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal λ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior hyperideal μ of H.*

(4) $\left(\dot{\lambda} \wedge \dot{\mu}\right) = (\lambda * \mu * \lambda)$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal λ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal μ of H.*

(5) $\left(\dot{\lambda} \wedge \dot{\mu}\right) = (\lambda * \mu * \lambda)$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal λ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior hyperideal μ of H.*

(6) $\left(\dot{\lambda} \wedge \dot{\mu}\right) = (\lambda * \mu * \lambda)$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal λ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideal μ of H.*

(7) $\left(\dot{\lambda} \wedge \dot{\mu}\right) = (\lambda * \mu * \lambda)$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal λ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior hyperideal μ of H.*

Proof. (1) $\Rightarrow$ (7) Let λ and μ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior hyperideal of H, respectively. Then

$$\begin{aligned} (\lambda * \mu * \lambda)(a) &\leq (\lambda * 1 * \lambda)(a) \\ &\leq (\lambda)(a) \text{ by Theorem 6.2.16.} \end{aligned}$$

Also

$$\begin{aligned} (\lambda * \mu * \lambda)(a) &\leq (1 * \mu * 1)(a) \\ &\leq \mu(a) \text{ by Theorem 6.2.16.} \end{aligned}$$

Hence $(\lambda * \mu * \lambda) \leq \left(\overset{\bullet}{\lambda} \wedge \overset{\bullet}{\mu}\right)$.

Now let $a \in H$. Since H is regular, so there exists $x \in H$ such that $a \in a \circ x \circ a = a \circ (x \circ a \circ x \circ a)$. Thus there exists $r \in x \circ a \circ x$, and $\beta \in r \circ a$ such that $a \in a \circ \beta$. So

$$\begin{aligned}
(\lambda * \mu * \lambda)(a) &= ((\lambda \circ \mu \circ \lambda)(a) \vee \gamma) \wedge \delta \\
&= \left(\bigvee_{a \in a \circ \beta} \{\lambda(a) \wedge (\mu \circ \lambda)(\beta)\} \vee \gamma\right) \wedge \delta \\
&\geq (\{\lambda(a) \wedge (\mu \circ \lambda)(\beta)\} \vee \gamma) \wedge \delta \\
&= ((\lambda(a) \vee \gamma) \wedge \delta) \wedge \left(\bigvee_{\beta \in r \circ a} \{\mu(r) \wedge \lambda(a)\} \vee \gamma\right) \wedge \delta \\
&\quad \left(\begin{array}{c} \text{Since every } (\in_\gamma, \in_\gamma \vee q_\delta)\text{-fuzzy hyperideal of } H \text{ is} \\ \text{an } (\in_\gamma, \in_\gamma \vee q_\delta)\text{-fuzzy hyperideal of } H \\ \text{so } \inf_{r \in x \circ a \circ x} \{\mu(r)\} \vee \gamma \geq \mu(a) \wedge \delta. \end{array}\right) \\
&\geq ((\lambda(a) \vee \gamma) \wedge \delta) \wedge (\mu(a) \vee \gamma) \wedge \delta) \wedge (\lambda(a) \vee \gamma) \wedge \delta \\
&= ((\lambda(a) \vee \gamma) \wedge \delta) \wedge (\mu(a) \vee \gamma) \wedge \delta)) \\
&= \left(\overset{\bullet}{\lambda}(a) \wedge \overset{\bullet}{\mu}(a)\right) \\
&= \left(\overset{\bullet}{\lambda} \wedge \overset{\bullet}{\mu}\right)(a).
\end{aligned}$$

So $(\lambda * \mu * \lambda) \geq \left(\overset{\bullet}{\lambda} \wedge \overset{\bullet}{\mu}\right)$. Hence $(\lambda * \mu * \lambda) = \left(\overset{\bullet}{\lambda} \wedge \overset{\bullet}{\mu}\right)$.

$(7) \Rightarrow (5) \Rightarrow (3) \Rightarrow (2)$ and $(7) \Rightarrow (6) \Rightarrow (4) \Rightarrow (2)$ are obvious.

$(2) \Rightarrow (1)$ Let λ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H. Then $\overset{\bullet}{\lambda}$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H. Since "1" is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two-sided hyperideal of H, we have

$$\begin{aligned}
\overset{\bullet}{\lambda}(a) &= (\lambda(a) \vee \gamma) \wedge \delta \\
&= ((\lambda \wedge 1)(a) \vee \gamma) \wedge \delta \\
&= \left(\overset{\bullet}{\lambda} \wedge \overset{\bullet}{1}\right)(a) \\
&= (\lambda * 1 * \lambda)(a).
\end{aligned}$$

Thus it follows from Theorem 6.3.3 that H is regular. □

6.3.5 Theorem

For a semihypergroup H, the following conditions are equivalent:

(1) *H is regular.*

(2) $\left(\overset{\bullet}{\lambda} \wedge \overset{\bullet}{\mu}\right) \leq (\lambda * \mu)$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal λ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal μ of H.*

(3) $\left(\overset{\bullet}{\lambda} \wedge \overset{\bullet}{\mu}\right) \leq (\lambda * \mu)$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal λ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal μ of H.*

(4) $\left(\overset{\bullet}{\lambda} \wedge \overset{\bullet}{\mu}\right) \leq (\lambda * \mu)$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal λ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal μ of H.*

Proof. (1) $\Rightarrow$ (4) Let λ and μ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideal and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H, respectively. Let $a \in H$. Since H is regular, so there exists $x \in H$ such that $a \in a \circ x \circ a = a \circ (x \circ a)$. Then there exists $\beta \in x \circ a$ such that $a \in a \circ \beta$. Thus we have

$$\begin{aligned}
(\lambda * \mu)(a) &= ((\lambda \circ \mu)(a) \vee \gamma) \wedge \delta \\
&= \left(\bigvee_{a \in c \circ d} \{\lambda(c) \wedge \mu(d)\} \vee \gamma\right) \wedge \delta \\
&\geq ((\lambda(a) \wedge \mu(\beta)) \vee \gamma) \wedge \delta \\
&\geq ((\lambda(a) \vee \gamma) \wedge (\mu(a) \vee \gamma \wedge \delta)) \wedge \delta \quad \text{(because } \beta \in x \circ a.) \\
&= ((\lambda(a) \vee \gamma) \wedge \delta) \wedge ((\mu(a) \vee \gamma) \wedge \delta) \\
&= \left(\overset{\bullet}{\lambda}(a) \wedge \overset{\bullet}{\mu}(a)\right) \\
&= \left(\overset{\bullet}{\lambda} \wedge \overset{\bullet}{\mu}\right)(a).
\end{aligned}$$

So $(\lambda * \mu) \geq \left(\overset{\bullet}{\lambda} \wedge \overset{\bullet}{\mu}\right)$.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) are clear.

(2) $\Rightarrow$ (1) Let λ and μ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H, respectively. Since every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal of H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H, we have $(\lambda * \mu) \geq \left(\overset{\bullet}{\lambda} \wedge \overset{\bullet}{\mu}\right)$. By Lemma 6.2.17, $\lambda * \mu \leq \left(\overset{\bullet}{\lambda} \wedge \overset{\bullet}{\mu}\right)$. Thus $(\lambda * \mu) = \left(\overset{\bullet}{\lambda} \wedge \overset{\bullet}{\mu}\right)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal λ and for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal μ of H. Hence by Theorem 6.3.1 H is regular. □

Recall that a semihypergroup H is intra-regular if for each $a \in H$, there exist $x, y \in H$ such that $a \in x \circ a \circ a \circ y$. In general neither intra-regular semihypergroups are regular nor regular semihypergroups are intra-regular semihypergroups. However, in commutative semihypergroups both the concepts coincide.

6.4 Intra-regular semihypergroups

6.4.1 Theorem

For a semihypergroup H, the following conditions are equivalent:

(1) *H is intra-regular.*

(2) $\left(\dot{\lambda} \wedge \dot{\mu}\right) \leq (\lambda * \mu)$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal λ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal μ of H.*

Proof. (1) $\Rightarrow$ (2) Let λ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal and μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal of H. For $a \in H$, there exist $x, y \in H$ such that $a \in x \circ a \circ a \circ y = (x \circ a) \circ (a \circ y)$. Thus there exists $r \in x \circ a$ and $s \in a \circ y$, such that $a \in r \circ s$.

Thus

$$\begin{aligned}
(\lambda * \mu)(a) &= ((\lambda \circ \mu)(a) \vee \gamma) \wedge \delta \\
&= \left(\bigvee_{a \in c \circ d} \{\lambda(c) \wedge \mu(d)\} \vee \gamma \right) \wedge \delta \\
&\geq (\{\lambda(r) \wedge \mu(s)\} \vee \gamma) \wedge \delta \\
&\quad \left(\begin{array}{c} \text{Since } r \in x \circ a \text{ and } \lambda \text{ is an } (\in_\gamma, \in_\gamma \vee q_\delta)\text{-fuzzy left hyperideal of } H, \\ \text{so } \inf_{z \in x \circ a} \{\lambda(z)\} \vee \gamma \geq \lambda(a) \wedge \delta. \text{ Thus } \lambda(r) \vee \gamma \geq \lambda(a) \vee \gamma \wedge \delta. \\ \text{Also since } \mu \text{ is an } (\in_\gamma, \in_\gamma \vee q_\delta)\text{-fuzzy right hyperideal of } H, \\ \text{so } \inf_{z \in a \circ x} \{\mu(z)\} \vee \gamma \geq \mu(a) \wedge \delta. \text{ Thus } \mu(s) \vee \gamma \geq \mu(a) \vee \gamma \wedge \delta \end{array} \right) \\
&\geq \{((\lambda(a) \vee \gamma) \wedge \delta) \wedge ((\mu(a) \vee \gamma) \wedge \delta)\} \wedge \delta \\
&= (\lambda(a) \vee \gamma \wedge \delta) \wedge (\mu(a) \vee \gamma \wedge \delta) \\
&= \left(\dot{\lambda}(a) \wedge \dot{\mu}(a)\right) \\
&= \left(\dot{\lambda} \wedge \dot{\mu}\right)(a).
\end{aligned}$$

Thus $(\lambda * \mu) \geq (\dot{\lambda} \wedge \dot{\mu})$.

(2) $\Rightarrow$ (1) Let R and L be right and left hyperideals of H. Then by Corollary 6.2.15, $\dot{\lambda}_R$ and $\dot{\lambda}_L$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideals

of H, respectively. Thus by hypothesis we have

$$\begin{aligned}\dot{\lambda}_{L\circ R} &= \lambda_L * \lambda_R \\ &\geq \dot{\lambda}_L \wedge \dot{\lambda}_R \quad \text{by Lemma 6.2.12 and by (2)} \\ &= \lambda_L \dot{\wedge} \lambda_R \quad \text{by Lemma 6.2.12} \\ &= \dot{\lambda}_{L\cap R} \quad \text{by Lemma 6.2.13.}\end{aligned}$$

Thus $L \cap R \subseteq L \circ R$. Hence it follows from Theorem 4.3.3, that H is intra-regular. □

6.4.2 Theorem

The following conditions are equivalent for a semihypergroup H.

(1) *H is both regular and intra-regular.*

(2) $\lambda * \lambda = \dot{\lambda}$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal λ of H.*

(3) $\lambda * \lambda = \dot{\lambda}$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal λ of H.*

(4) $\lambda * \mu \geq \dot{\lambda} \wedge \dot{\mu}$ *for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideals λ, μ of H.*

(5) $\lambda * \mu \geq \dot{\lambda} \wedge \dot{\mu}$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal λ and for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal μ of H.*

(6) $\lambda * \mu \geq \dot{\lambda} \wedge \dot{\mu}$ *for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideals λ, μ of H.*

Proof. (1) ⇒ (6) Let λ, μ be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideals of H and $a \in H$. Then there exist $x, y, z \in H$ such that $a \in a \circ x \circ a$ and $a \in y \circ a \circ a \circ z$. So

$$\begin{aligned} a &\in a \circ x \circ a \\ &\subseteq a \circ x \circ a \circ x \circ a \\ &= (a \circ x) \circ a \circ (x \circ a) \\ &\subseteq a \circ x \circ (y \circ a \circ a \circ z) \circ x \circ a \\ &= (a \circ x \circ y \circ a) \circ (a \circ z \circ x \circ a).\end{aligned}$$

Thus there exist $p \in x \circ y, q \in z \circ x, b \in a \circ p \circ a$ and $c \in a \circ q \circ a$ such that $a \in b \circ c$.

Therefore

$$\begin{aligned}
(\lambda * \mu)(a) &= ((\lambda \circ \mu)(a) \vee \gamma) \wedge \delta \\
&= \left(\bigvee_{a \in d \circ e} \{\lambda(d) \wedge \mu(e)\} \vee \gamma \right) \wedge \delta \\
&\geq (\{\lambda(b) \wedge \mu(c)\} \vee \gamma) \wedge \delta \\
&\left(\begin{array}{c} \text{Since } b \in a \circ p \circ a \text{ and } \lambda \text{ is an } (\in_\gamma, \in_\gamma \vee q_\delta)\text{-fuzzy bi-hyperideal of } H, \\ \text{we have } \inf_{\alpha \in a \circ p \circ a} \{\lambda(\alpha)\} \vee \gamma \geq \min\{\lambda(a), \lambda(a), \delta\}. \\ \text{Thus } \lambda(b) \vee \gamma \geq \min\{\lambda(a) \vee \gamma, \delta\}. \\ \text{Similarly } \mu(c) \vee \gamma \geq \min\{\mu(a) \vee \gamma, \delta\}. \end{array} \right) \\
&\geq [((\lambda(a) \vee \gamma) \wedge \delta) \wedge ((\mu(a) \vee \gamma) \wedge \delta)] \wedge \delta \\
&= ((\lambda(a) \vee \gamma) \wedge \delta) \wedge ((\mu(a) \vee \gamma) \wedge \delta) \\
&= \left(\dot{\lambda}(a) \wedge \dot{\mu}(a) \right) \\
&= \left(\dot{\lambda} \wedge \dot{\mu} \right)(a).
\end{aligned}$$

Thus $\lambda * \mu \geq \dot{\lambda} \wedge \dot{\mu}$ for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideals λ, μ of H.

(6) $\Rightarrow$ (5) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (2) Take $\lambda = \mu$ in (4), we get $\lambda * \lambda \geq \dot{\lambda}$. Since every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup, so $\lambda * \lambda \leq \dot{\lambda}$. Hence $\lambda * \lambda = \dot{\lambda}$.

(6) $\Rightarrow$ (3) Take $\lambda = \mu$ in (6), we get $\lambda * \lambda \geq \dot{\lambda}$. Since every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-hyperideal is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subsemihypergroup, so $\lambda * \lambda \leq \dot{\lambda}$. Hence $\lambda * \lambda = \dot{\lambda}$.

(3) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (1) Let Q be a quasi-hyperideal of H. Then by Corollary 6.2.15, $\dot{\lambda}_Q$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H. Hence, by hypothesis, $\lambda_Q * \lambda_Q = \dot{\lambda}_Q$. Thus $\dot{\lambda}_{Q \circ Q} = \lambda_Q * \lambda_Q = \dot{\lambda}_Q$ implies $Q \circ Q = Q$. So by Theorem 3.2.8, S is both regular and intra-regular. □

6.4.3 Theorem

The following conditions are equivalent for a semihypergroup H.

(1) *H is both regular and intra-regular.*

(2) $(\lambda * \mu) \wedge (\mu * \lambda) \geq \left(\dot{\lambda} \wedge \dot{\mu} \right)$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal λ and for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal μ of H.*

(3) $(\lambda * \mu) \wedge (\mu * \lambda) \geq \left(\dot{\lambda} \wedge \dot{\mu}\right)$ *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy right hyperideal* λ *and for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy quasi-hyperideal* μ *of* H.

(4) $(\lambda * \mu) \wedge (\mu * \lambda) \geq \left(\dot{\lambda} \wedge \dot{\mu}\right)$ *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy right hyperideal* λ *and for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy bi-hyperideal* μ *of* H.

(5) $(\lambda * \mu) \wedge (\mu * \lambda) \geq \left(\dot{\lambda} \wedge \dot{\mu}\right)$ *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy right hyperideal* λ *and for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy generalized bi-hyperideal* μ *of* H.

(6) $(\lambda * \mu) \wedge (\mu * \lambda) \geq \left(\dot{\lambda} \wedge \dot{\mu}\right)$ *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left hyperideal* λ *and for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy quasi-hyperideal* μ *of* H.

(7) $(\lambda * \mu) \wedge (\mu * \lambda) \geq \left(\dot{\lambda} \wedge \dot{\mu}\right)$ *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left hyperideal* λ *and for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy bi-hyperideal* μ *of* H.

(8) $(\lambda * \mu) \wedge (\mu * \lambda) \geq \left(\dot{\lambda} \wedge \dot{\mu}\right)$ *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left hyperideal* λ *and for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy generalized bi-hyperideal* μ *of* H.

(9) $(\lambda * \mu) \wedge (\mu * \lambda) \geq \left(\dot{\lambda} \wedge \dot{\mu}\right)$ *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy quasi-hyperideals* λ *and* μ *of* H.

(10) $(\lambda * \mu) \wedge (\mu * \lambda) \geq \left(\dot{\lambda} \wedge \dot{\mu}\right)$ *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy quasi-hyperideal* λ *and for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy bi-hyperideal* μ *of* H.

(11) $(\lambda * \mu) \wedge (\mu * \lambda) \geq \left(\dot{\lambda} \wedge \dot{\mu}\right)$ *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy quasi-hyperideal* λ *and for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy generalized bi-hyperideal* μ *of* H.

(12) $(\lambda * \mu) \wedge (\mu * \lambda) \geq \left(\dot{\lambda} \wedge \dot{\mu}\right)$ *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy bi-hyperideals* λ *and* μ *of* H.

(13) $(\lambda * \mu) \wedge (\mu * \lambda) \geq \left(\dot{\lambda} \wedge \dot{\mu}\right)$ *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy bi-hyperideal* λ *and for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy generalized bi-hyperideal* μ *of* H.

(14) $(\lambda * \mu) \wedge (\mu * \lambda) \geq \left(\dot{\lambda} \wedge \dot{\mu}\right)$ *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy generalized bi-hyperideals* λ *and* μ *of* H.

Proof. (1) $\Rightarrow$ (14) Let λ, μ be $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-hyperideals of H and

$a \in H$. Then there exist $x, y, z \in H$ such that $a \in a \circ x \circ a$ and $a \in y \circ a \circ a \circ z$. So

$$\begin{aligned} a &\in a \circ x \circ a \\ &\subseteq a \circ x \circ a \circ x \circ a \\ &= (a \circ x) \circ a \circ (x \circ a) \\ &\subseteq a \circ x \circ (y \circ a \circ a \circ z) \circ x \circ a \\ &= (a \circ x \circ y \circ a) \circ (a \circ z \circ x \circ a). \end{aligned}$$

Thus there exist $p \in x \circ y, q \in z \circ x, b \in a \circ p \circ a$ and $c \in a \circ q \circ a$ such that $a \in b \circ c$.
Therefore

$$\begin{aligned} (\lambda * \mu)(a) &= ((\lambda \circ \mu)(a) \vee \gamma) \wedge \delta \\ &= \left(\bigvee_{a \in d \circ e} \{\lambda(d) \wedge \mu(e)\} \vee \gamma \right) \wedge \delta \\ &\geq (\{\lambda(b) \wedge \mu(c)\} \vee \gamma) \wedge \delta \end{aligned}$$

$$\left(\begin{array}{c} \text{Since } b \in a \circ p \circ a \text{ and } c \in a \circ q \circ a \text{ and } \lambda \text{ and } \mu \text{ are} \\ (\in_\gamma, \in_\gamma \vee q_\delta)\text{-fuzzy generalized bi-hyperideals of } H, \\ \text{so } \inf\limits_{w \in x \circ y \circ z} \{\lambda(w)\} \vee \gamma \geq \min\{\lambda(x), \lambda(z), \delta\} \text{ and} \\ \inf\limits_{w \in x \circ y \circ z} \{\mu(w)\} \vee \gamma \geq \min\{\mu(x), \mu(z), \delta\} \\ \text{Thus } \lambda(b) \vee \gamma \geq \lambda(a) \wedge \delta \text{ so and } \mu(c) \vee \gamma \geq \mu(a) \wedge \delta. \end{array} \right)$$

$$\begin{aligned} \text{Therefore } (\lambda * \mu)(a) &\geq (\lambda(a) \vee \gamma) \wedge \delta) \wedge ((\mu(a) \vee \gamma) \wedge \delta) \\ &= \left(\dot{\lambda}(a) \wedge \dot{\mu}(a) \right) \\ &= \left(\dot{\lambda} \wedge \dot{\mu} \right)(a). \end{aligned}$$

Similarly we can prove that $(\mu * \lambda) \geq \left(\dot{\lambda} \wedge \dot{\mu} \right)$. Hence $(\lambda * \mu) \wedge (\mu * \lambda) \geq \left(\dot{\lambda} \wedge \dot{\mu} \right)$.

$(14) \Rightarrow (13) \Rightarrow (12) \Rightarrow (10) \Rightarrow (9) \Rightarrow (3) \Rightarrow (2)$,
$(14) \Rightarrow (11) \Rightarrow (10)$,
$(14) \Rightarrow (8) \Rightarrow (7) \Rightarrow (6) \Rightarrow (2)$ and
$(14) \Rightarrow (5) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2)$ are obvious.

$(2) \Rightarrow (1)$ Let λ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right hyperideal and μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H. As every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left hyperideal of H is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-hyperideal of H, we have

$$(\lambda * \mu) \wedge (\mu * \lambda) \geq \left(\dot{\lambda} \wedge \dot{\mu} \right).$$

As $\dot{\lambda} \wedge \dot{\mu} \leq \lambda * \mu$ always true, so we have $\lambda * \mu = \dot{\lambda} \wedge \dot{\mu}$ and $\mu * \lambda \geq \dot{\lambda} \wedge \dot{\mu}$.

Hence by Theorem 6.3.1, H is regular and by Theorem 6.4.1, H is intra-regular. □

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