A Study on Fuzzy Riesz Spaces

By

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2020

On Ordered Mordell Elliptic Curves and Their Applications in Cryptography

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A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

Supervised by

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2020

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A Study on Fuzzy Riesz Spaces

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Abstract

Fuzzy Riesz Space is an attempt to study vector spaces with a fuzzy order for more complicated scenarios. In this dissertation, we study the fuzzy order convergence, fuzzy Riesz homomorphisms, and fuzzy order continuous positive operator, which help to prove the existence of fuzzy Dedekind completion of Archimedean fuzzy Riesz spaces. Theory of the fuzzy Riesz space of all fuzzy order bounded linear operators are investigated. We define and study unbounded fuzzy order convergence and some of its applications. Furthermore, study the fuzzy norms compatible with fuzzy ordering (fuzzy normed Riesz space) and discuss the relationship between the fuzzy order dual and topological dual of a locally convex solid fuzzy Riesz space. Besides, we throw light on the unbounded fuzzy norm convergence and its applications in fuzzy Banach lattice, which is topological.

Contents

Chapter 1

Introduction

One can marvel at the complexities of real-life that cannot be modeled with perfection. In an attempt to handle vague and imprecise scenarios, Zadeh gave the notion of fuzzy set [65]. Since then, the fuzzy set theory has been developed enormously and with many applications. Many classical structures like topological spaces, metric spaces, and norm spaces have been designed in the fuzzy framework. Chang and Michalek [16, 45] defined and characterized the basic theory of fuzzy topological spaces. Warren [58] formalized many concepts in fuzzy topological spaces like neighbourhood, continuity, and bases. Lawn [41, 42] defined and studied the fuzzy compactness and connectedness in fuzzy topological spaces. Seenivasan and Kamala [53] proposed and investigated the theory of fuzzy e-continuity and fuzzy e-open sets. Vadivela and Vijayalakshmib [55] studied and characterized the fuzzy $\wedge_e$ sets and continuity in fuzzy topological spaces. Authors in [50] investigated the theory of a new pairwise fuzzy topology through a fuzzy ideal.

In a series of papers, Katsaras and Liu [36, 37, 38] defined fuzzy vector spaces, fuzzy topological vector spaces and proved many related results. Amudhambigaia and Madhuria [6] defined and studied fuzzy irresolute topological vector spaces. The notion of fuzzy metric space was first proposed in [40]. Then Gregori et al. [24, 25] studied the convergence criteria in fuzzy metric spaces. George and Veeramani [23] gave several theocratical concepts in fuzzy metric spaces. Kaleva and Seikkala [34] investigated the fuzzy metric spaces in detail. Katsaras [38] defined a fuzzy norm in fuzzy vector spaces. After that, Felbin [21] further developed the structure of fuzzy normed space. Saadati and Vaezpour [54] investigated the theory of fuzzy Banach spaces. Cheng and Mordeson [17] developed the theory of a fuzzy linear operator. The completeness of the fuzzy norm space of linear operators is discussed by Xaio and Zhu [61]. Then, Bag and Samanta [7, 8] defined and studied the notion of a fuzzy bounded linear operator and observed a type of convergence of sequence in fuzzy normed linear spaces. Moradi [47] characterized the complete fuzzy normed spaces. Rano and Bag [51] redefined fuzzy normed linear space. Since then, many authors have done more work on these structures.

The ordering of objects is much more important in many problems, to handle the situations where it is not easy to determine the relative position of an object, fuzzy ordering, and fuzzy ordered sets are defined by Venugopalan [56] and Zadeh [64]. Chon [18] further developed the notions of fuzzy partial order relations and fuzzy lattices. Xie et al. [62] introduced the theory of the Dedekind–MacNeille completions for fuzzy posets. Yuan and Wu [60] proposed and studied the notion of fuzzy ideals and sublattices in fuzzy lattices. Ajmal and Thomas [2] explored fuzzy lattices with an algebraic point of view and gave a characterization of fuzzy sublattices. Bodenhofer [15] defined similarity-based fuzzy ordering and worked on its representation. Amroune et al. [5] investigated the notions of fuzzy t-filters, fuzzy prime t-filters, and defined fuzzy lattice isomorphism. Davvaz and Kazanci [19, 35] defined a new kind of fuzzy-sublattices (ideal, filter) and characterized lattices in many classes by using their properties. Kadji et al. [33] investigated the theory of L-fuzzy filter, L-fuzzy prime filter for giving residuated lattice M and lattice L. Konecny and Krupka [39] studied the complete fuzzy lattices, residuated lattice valued ordered sets and proved many essential results. Yang [59] discussed fuzzy weak regular, strong, and preassociative filters in residuated lattices. In [48] Panneerselvam and Jayadevi investigated the notion of anti monotonic P-fuzzy G– distributive lattices. Micić et al. [46] devised practical ways to compute the greatest right invariant fuzzy quasi-order, greatest right, and left-invariant fuzzy equivalences for fuzzy automaton.

Beg and Islam [9] initiated to study vector spaces with fuzzy ordering and come up with the notion of fuzzy Riesz space [10, 11]. In continuation, Beg [12, 13, 14] defined and characterized the notion of the fuzzy positive operator and discussed the properties of fuzzy order relations. Hong [26] defined and studied fuzzy Riesz subspaces in detail. Park et al. [49] gave the concept of Riesz fuzzy normed spaces. Recently, Kadhim [32] discussed the concept of fuzzy statistical (O)-convergence in fuzzy Riesz space. The precise details of the fuzzy set theory and classical Riesz spaces can be found in [52, 66] and [1, 3, 4, 20, 22, 27, 44, 57], respectively.

In this dissertation, we study fuzzy Riesz space in many directions. We define several theoretical concepts and prove many essential results, thus significantly contributing to the existing theory of fuzzy Riesz space. The details of our findings are given below.

- In Chapter 3 we prove the existence of fuzzy Dedekind completion of an Archimedean fuzzy Riesz space. Consider (K, ϕ) is a fuzzy Riesz space with K a vector space and ϕ a fuzzy partial order on K. Our idea is to construct a vector space K^δ with fuzzy partial order ψ based on (K, ϕ) in a way that (K^δ, ψ) is fuzzy Dedekind complete. Then, a one to one positive linear map $P : K \to K^\delta$ is constructed, which preserves the lattice and algebraic structures of K that coincides with subspace lattice structure $P(K)$ inherits from K^δ. Furthermore, $P(K)$ is somehow dense in K^δ. However, the existing theory was not adequate to implement this plan, as its many parts do not make sense in the existing literature. Therefore, many new concepts are needed to be defined. For fuzzy Riesz subspace, the notions of fuzzy order dense, fuzzy majorizing, and fuzzy regularness are defined and studied in collaboration with fuzzy order convergence. The fuzzy order convergence of a net in a fuzzy Riesz space is another important concept closely related to fuzzy Dedekind completeness. Still, there is a certain defect in its definition. Therefore, it is redefined and studied in depth. For fuzzy positive operator between two fuzzy Riesz spaces, the notions of fuzzy order continuity, fuzzy boundedness, fuzzy Riesz homomorphisms, and isomorphisms are proposed and characterized nicely for practical uses.

- In Chapter 4 we study the space of all fuzzy order bounded positive linear operators denoted by $L_b(K, H)$. We show that it is fuzzy Dedekind complete when (H, μ) is fuzzy Dedekind complete by defining suitable fuzzy lattice operations. The set of all fuzzy order (σ-order) continuous bounded linear operators denoted $L_n(K, H)(L_c(K, H))$ are showed to be fuzzy bands of $L_b(K, H)$ when H is fuzzy Dedekind complete and $L_n(K, H) \subseteq L_c(K, H)$. As a special case, we also study other related concepts like separation properties, fuzzy order continuous dual, and $\sigma-$fuzzy order continuous dual on K.

- In Chapter 5, to handle the imprecise and vague scenarios more effectively, we propose a novel concept unbounded fuzzy order convergence to deal with unbounded fuzzy ordered nets. Then, an in-depth theoretical investigation is done to study its various properties. However, it is much harder to determine the unbounded fuzzy order convergence of a given net in several practical scenarios. To resolve this issue, the notion of the fuzzy weak order unit is proposed to reduce the labour of checking unbounded fuzzy order convergence. Thus, unbounded fuzzy order convergence is nicely characterized in fuzzy Dedekind complete Riesz spaces. To further develop the theory for practical use, the completeness of a fuzzy Riesz space is also explored with respect to unbounded fuzzy order convergence. For this purpose, we study the fuzzy ideals and fuzzy bands connected with fuzzy order convergence, and some results are given in the end as applications.

- In Chapter 6, we study the fuzzy normed Riesz space, which is an experiment to investigate vector space with fuzzy ordering and a compatible fuzzy norm. We prove several results and investigate fuzzy Banach lattices (complete fuzzy normed Riesz space). Also, we study the connections between the topological structure and fuzzy lattice structure of a fuzzy Riesz space, when the induced topology of the fuzzy norm is locally convex-solid. We show that topological dual is a fuzzy ideal in the fuzzy order dual. Furthermore, to deal with unbounded nets, we define a novel notion unbounded fuzzy norm convergence in a fuzzy Banach lattice and investigate its various properties. Unbounded fuzzy order convergence implies unbounded fuzzy norm convergence when a fuzzy norm is order continuous and for fuzzy norm bounded nets both notions unbounded fuzzy norm convergence and fuzzy norm convergence coincides. We define the notion of a fuzzy quasi interior point to ease the practical checking of unbounded fuzzy norm convergence. Besides, every disjoint sequence is unbounded fuzzy order convergent to zero, but this fact is not valid for unbounded fuzzy norm convergence. However, any sequence which is unbounded norm convergent to zero has an asymptotically disjoint subsequence. Lastly, we work with the topological aspect of unbounded fuzzy order convergence. We know unbounded fuzzy order convergence is not topological, but the same is not true for unbounded fuzzy order convergence and we can construct its compatible topology.

The results of Chapter 3 and Chapter 5 are reported in [29] and [28], respectively. Whereas, the remaining work is currently submitted to well-reputed international journals, see the preprints [30, 31].

Chapter 2

Basics concepts

We present here some basic concepts about fuzzy Riesz spaces, which is a vector space compatible with fuzzy ordering. Since these spaces have several structures on them, and we provide a brief overview of the background material. We recall some basic concepts and advise reader to consult them for further explanation if needed [9, 13, 16, 21, 26, 28, 37, 38, 41, 43, 54, 57]. We start by reviewing the theory of a fuzzy ordered set.

2.1 Fuzzy order set

Definition 2.1.1. *A fuzzy subset C of K is characterized through a membership function $\phi_C : K \to [0,1]$, which corresponds with every point $k \in K$ its grade or degree of membership $\phi_C(k) \in [0,1]$. If $\phi_C(k)$ closer to one higher the degree of membership of k in C.*

Definition 2.1.2. *A fuzzy order ϕ on a set K is a fuzzy set on $K \times K$ with the understanding that k precedes g iff $\phi(k,g) > 1/2$ for $k, g \in K$ and the following conditions are also satisfied:*

(i) $\forall\ k \in K\ \phi(k,k) = 1$ (reflexivity);

(ii) for $k, g \in K\ \phi(k,g) + \phi(g,k) > 1$ *implies* $k = g$ (antisymmetric);

(iii) for $k, h \in K\ \phi(k,h) \geq \bigvee_{g \in K}[\phi(k,g) \wedge \phi(g,h)]$(transitivity).

The space (K,ϕ) *is called* fuzzy ordered set (FOS).

Let (K,ϕ) be an FOS, for $k \in K$ two related fuzzy sets $\uparrow k$ and $\downarrow k$ are known as $(\uparrow k)(g) = \phi(k,g)$ and $(\downarrow k)(g) = \phi(g,k)$ for $g \in K$, respectively.

Definition 2.1.3. *Let* (K,ϕ) *be an FOS. For* $C \subseteq K$, *then two fuzzy sets* $U(C)$ *and* $L(C)$ *are defined as follows.*

$$U(C)(g) = \begin{cases} 0 & \textit{if } (\uparrow k)(g) \leq 1/2 \; for \textit{ some } k \in C; \\ (\cap_{k\in C} \uparrow k)(g) & \textit{otherwise.} \end{cases}$$

$$L(C)(g) = \begin{cases} 0 & \textit{if } (\downarrow k)(g) \leq 1/2 \; for \textit{ some } k \in C; \\ (\cap_{k\in C} \downarrow k)(g) & \textit{otherwise.} \end{cases}$$

Let $(C)^u$ *denotes the set of all upper bounds of* C *and* $k \in (C)^u$ *if* $U(C)(k) > 0$. *Analogously,* $(C)^l$ *denotes the set of all lower bounds and* $k \in (C)^l$ *if* $L(C)(k) > 0$. *Also,* $d \in K$ *is known as the* supremum *of* C *in* K *if (i)* $d \in (C)^u$ *(ii)* $g \in (C)^u$ *implies* $g \in (d)^u$. *The infimum is defined analogously. A subset* C *is said to be* fuzzy order bounded *if* $(C)^u$ *and* $(C)^l$ *are non-empty. For* $C \subseteq K$, $(C)^u$ *denotes* $(suppC)^u$, *where* $C = \{k \in K; \phi_C(k) > 0\}$ *is said to be the support of* C. *Analogously,* $(C)^l$ *denotes* $(suppC)^l$.

Proposition 2.1.4. *If* C *is a subset of FOS* K *then* $\inf C(\sup C)$, *if it exists is unique.*

The notations $k \vee g = \sup\{k,g\}$ and $k \wedge g = \inf\{k,g\}$.

Proposition 2.1.5. *If* K *is a FOS then the following identities hold,*

(i) $g \wedge g = g$ *and* $g \vee g = g$ *(idempotent).*

(ii) $k \wedge g = g \wedge k$ *and* $k \vee g = g \vee k$ *(commutative).*

(iii) $k \wedge (k \vee g) = k \vee (k \wedge g)$ *(absorption).*

(iv) $\phi(k,g) > 1/2$ *iff* $k \wedge k = k$ *and* $k \vee k = g$

2.2 Fuzzy order linear space

Definition 2.2.1. *A real vector space K with fuzzy order ϕ is known as* fuzzy ordered vector space (FOVS) *if ϕ satisfies:*

(i) for $k, g \in K$ if $\phi(k,g) > 1/2$ then $\phi(k,g) \leq \phi(k+h, g+h)$ for all $h \in K$;

(ii) for $k, g \in K$ if $\phi(k,g) > 1/2$ then $\phi(k,g) \leq \phi(\lambda k, \lambda g)$ for all $0 \leq \lambda \in \mathbb{R}$;

For $k \in K$ is known as *positive* if $\phi(0,k) > 1/2$, and *negative* if $\phi(k,0) > 1/2$. Also K^+ as set of all positive elements in K, i.e. $K^+ = \{k \in K : \phi(0,k) > 1/2\}$ is referred to as the *fuzzy positive cone*. $C \subseteq K$ is called *directed upwards* if for each finite subset D of C we have $C \cap (D)^u \neq \emptyset$. The *directed downwards* set is defined analogously. Furthermore, for a net $(k_\lambda)_{\lambda \in \Lambda}$ $k_\lambda \uparrow k$ reads as the net (k_λ) is directed upwards to k i.e. for $\lambda_0 \leq \lambda$ we have $\phi(k_{\lambda_0}, k_\lambda) > 1/2$ and $\sup\{k_\lambda\} = k$. $k_\lambda \downarrow k$ is defined analogously.

It is observed that from Definition 2.2.1, if $\phi(k_1,k_2) > 1/2$ and $\phi(k_3,k_4) > 1/2$ then $\phi(k_1+k_3, k_2+k_4) > 1/2$. Now some identities are presented in the following proposition.

Proposition 2.2.2. *Let K be an FOVS, $k, k_1, k_2 \in K$ and λ, γ real numbers then:*

(i) if k_1 and k_2 are positive then their sum $k_1 + k_2$ is also positive.

(ii) if k is positive and $\lambda \geq 0$ then λk is too;

(iii) $\phi(k_1,k_2) > 1/2$ and $\lambda \leq 0$ then $\phi(\lambda k_2, \lambda k_1) > 1/2$;

(iv) $\phi(k_1,k_2) > 1/2$ and $\lambda \leq \gamma$ then $\phi(\lambda k_1, \gamma k_2) > 1/2$.

Proposition 2.2.3. *Let $\{k_\lambda\}_{\lambda \in \Lambda}$ be a system of elements in a FOVS. If the element $\bigvee_{\lambda \in \Lambda} k_\lambda$ exists then $\bigwedge_{\lambda \in \Lambda}(-k_\lambda)$ too, and the following equality holds*

$$\bigvee_{\lambda \in \Lambda} k_\lambda = -\bigwedge_{\lambda \in \Lambda}(-k_\lambda).$$

Proposition 2.2.4. *Let* $\{k_\lambda\}_{\lambda\in\Lambda}$ *and* $\{g_\lambda\}_{\gamma\in\Gamma}$ *be two systems of elements in a FOVS. If* $\bigvee_{\lambda\in\Lambda} k_\lambda$ *and* $\bigvee_{\gamma\in\Gamma} g_\gamma$ *exist then* $\bigvee_{\lambda\in\Lambda,\gamma\in\Gamma}(k_\lambda + g_\gamma)$ *exists too, and the following equality holds*

$$\bigvee_{\lambda\in\Lambda,\gamma\in\Gamma}(k_\lambda + g_\gamma) = \bigvee_{\lambda\in\Lambda} k_\lambda + \bigvee_{\gamma\in\Gamma} g_\gamma.$$

Definition 2.2.5. *An FOVS* (K, ϕ) *is said to be* Archimedean *if* $\phi(nk, g) > 1/2$ *for all* $n \in \mathbb{N}$ *implies that* $\phi(k, 0) > 1/2$ *for all* $k, g \in K$*. Therefore,* $\{\frac{k}{n}\} \downarrow 0$ *and* $\{nk\}$ *is unbounded from above for all* $0 \neq k \in K^+$*.*

2.3 Fuzzy Riesz space

Definition 2.3.1. *An FOVS* (K, ϕ) *is said to be* fuzzy Riesz space (FRS) *if* $k \vee g$ *and* $k \wedge g$ *exist in* K *for all* $k, g \in K$*.*

Definition 2.3.2. *An FRS* (K, ϕ) *is called:*

(i) fuzzy order complete *if each non-empty subset of* K *has a supremum and infimum in* K*;*

(ii) fuzzy $\sigma-$ order complete *if each nonempty countable subset of* K *has a supremum and infimum in* K*;*

(iii) fuzzy Dedekind complete *if each non-empty subset of* K *which is bounded from above has a supremum in* K*;*

(iv) fuzzy $\sigma-$ Dedekind complete *if each nonempty countable subset of* K *which is bounded from above has a supremum in* K*.*

Remark 2.3.3. *The notion of order completeness and Dedekind completeness is considered to be the same by some authors, see [3, 4]. However, we follow the approach of Zaanen [66] and differentiate between these terms as defined above. We write Dedekind complete FRS.*

Lemma 2.3.4. *If (K, ϕ) is a Dedekind complete FRS then K is fuzzy Archimedean.*

For $k \in K$, $k^+ = k \vee 0$ and $k^- = (-k) \vee 0$ are defined to be the positive and negative part of k, respectively, whereas the absolute value of k is known as $|k| = (-k) \vee k$. Furthermore, for $C \subseteq K$ one can easily prove that:

i. $k + \sup C = \sup(k + C)$;

ii. $k + \inf C = \inf(k + C)$.

Proposition 2.3.5. *If k and g are elements of an FRS (K, ϕ), then*

(i) $k = k^+ - k^-$;

(ii) $k^+ \wedge k^- = 0$;

(iii) $|k| = k^+ + k^-$;

(iv) $|k| = 0 \Leftrightarrow k = 0$;

(v) $k + g = k \vee g + k \wedge g$.

Theorem 2.3.6. *If $k, h \in K$ and real number $\lambda \geq 0$ then following hold:*

(i) $\phi((k + h)^-, k^- + h^-) > 1/2$;

(ii) $\phi((k + h)^+, k^+ + h^+) > 1/2$;

(iii) $(\lambda k)^+ = \lambda k^+$;

(iv) $(\lambda k)^- = \lambda k^-$.

Theorem 2.3.7. *Let (K, ϕ) be an FRS. Then following properties are satisfied:*

(i) $\phi(|k + g|, |k| + |g|) > 1/2$;

(ii) $\phi(|k| - |g|, |k - g|) > 1/2$;

(iii) $|\lambda k| = |\lambda||k|$;

(iv) $|k - g| = k \vee g - k \wedge g$.

The decomposition property of an FRS can be seen in the upcoming theorem.

Theorem 2.3.8. *If $\phi(|k|, |g_1 + ... + g_n|) > 1/2$ in an FRS (K, ϕ) then there exist $k_1, ..., k_n$ satisfying $k = k_1, ..., k_n$ and $\phi(k_i, g_i) > 1/2$ for each $i = 1, ..., n$. Furthermore, for positive k, k_i are also positive.*

Let $k_1, k_2 \in K$ are called *orthogonal or disjoint* if $k_1 \wedge k_2 = 0$ and written as $k_1 \perp k_2$. Also, for $C_1, C_2 \subset K$ are called disjoint and denoted by $C_1 \perp C_2$ if $k_1 \perp k_2 = 0$ for each $k_1 \in C_1$ and $k_2 \in C_2$. Moreover, if $\emptyset \neq C \subseteq K$, then its *disjoint complement* is defined as $C^d = \{k \in K : k \perp g \ for \ each \ g \in C\}$. Notation C^{dd} represents the disjoint complement of C^d.

Theorem 2.3.9. *If $k \perp g$ then:*

(i) $(k + g)^+ = k^+ + g^+$;

(ii) $(k + g)^- = k^- + g^-$;

(iii) $|k + g| = |k| + |g|$.

Let (K, ϕ) be an FRS and $k, g \in K$ with $\phi(k, g) > 1/2$. Then the *fuzzy order interval* $[k, g] \subseteq K$ is given by

$$[k, g] = \{h \in K : \phi(k, h) > 1/2 \ and \ \phi(h, g) > 1/2\}.$$

A fuzzy positive operator, P between two FRSs, is a linear map $P : K \to H$ such that $P(k) \in H^+$ for all $k \in K^+$.

Lemma 2.3.10. *If P is an additive fuzzy positive operator between an FRS (K, ϕ) and Archimedean FRS (H, μ) then P is homogeneous.*

Lemma 2.3.11. *If $P : K^+ \to H$ is a fuzzy positive operator with H is a vector space, satisfying $P(k+g) = P(k) + P(g)$ for each $k, g \in K^+$ then P extends uniquely to an additive positive operator $V : K \to H$. Furthermore,*

$$V(k) = P(k^+) - P(k^-)$$

for $k \in K$.

2.3.1 Fuzzy Riesz subspaces

A vector subspace L of an FRS (K, ϕ) is known as *fuzzy Riesz subspace* if L is closed under the fuzzy Riesz operations $\vee$ and $\wedge$.

Definition 2.3.12. *A net $(k_\lambda)_{\lambda\in\Lambda}$ in an FRS (K, ϕ) is known as* fuzzy order convergent *to $k \in K$ denoted $k_\lambda \xrightarrow{fo} k$ if there exists another net $(g_\lambda)_{\lambda\in\Lambda}$ in K^+ directed downwards to zero such that $\phi(|k_\lambda - k|, g_\lambda) > 1/2$ for each $\lambda \in \Lambda$.*

Definition 2.3.13. *Let (K, ϕ) be an FRS.*

(i) A subset C of K is said to be fuzzy order closed (fo-closed for short)*, if for any net $(k_\lambda)_{\lambda\in\Lambda} \subset C$ and $k \in K$ with $k_\lambda \xrightarrow{fo} k$ in K implies $k \in C$.*

(ii) A subset C of K is said to be σ-fuzzy order closed *, if for any net $(k_n)_{n\in N} \subset C$ and $k \in K$ with $k_\lambda \xrightarrow{fo} k$ in K implies $k \in C$.*

(iii) A subset S of K is called fuzzy solid *if $\phi(|k|, |g|) > 1/2$ and $g \in S$ implies $k \in S$.*

(iv) A fuzzy solid vector subspace is called a fuzzy ideal *of K.*

(v) A fuzzy order closed ideal in K is said to be a fuzzy band.

Suppose C is a subset of an FRS (K, ϕ). I_C is the smallest fuzzy ideal generated by C. If a vector $k \in K$ generates a fuzzy ideal denoted I_k is called the principal fuzzy ideal.

Theorem 2.3.14. *Let (K, ϕ) be an FRS and $C \subseteq K$. Then there exists a unique I_C that can be described as*

$$I_C = \{k \in K \exists k_1, ..., k_m \text{ and } \alpha \geq 0 \text{ such that } \phi(|k|, \alpha \sum_{j=1}^{m} |k_j|) > 1/2\}.$$

Corollary 2.3.15. *If (K, ϕ) is an FRS and $k \in K$ then fuzzy ideal generated by k is described as*

$$I_k = \{g \in K \text{ and } \alpha \geq 0 \text{ such that } \phi(|g|, \alpha|k|) > 1/2\}.$$

Theorem 2.3.16. *Let (K, ϕ) be an FRS and $C \subseteq K$. Then following statements are true:*

(i) $C \subset C^{dd}$;

(ii) $C^d = C^{ddd}$;

(iii) $C^d \cap C^{dd} = \{0\}$;

(iv) *If* $C^d = \{0\}$ *then* $C^{dd} = K$;

(v) C^d *is a fuzzy ideal of* K.

2.4 Fuzzy normed spaces

Definition 2.4.1. *Consider a vector space K over a field F and $\star$ a continuous t-norm on $[0, 1]$. A fuzzy norm on K is a mapping $N : K \times (0, \infty) \to [0, 1]$ if for each $k, g \in K$ and $\lambda \in F$:*

(i) *for each* $t \in (0, \infty)$ *with* $t > 0$,$N(k, t) = 1 \Leftrightarrow k = 0$;

(ii) *for each* $t \in (0, \infty)$ *with* $t > 0$, $N(\lambda k, t) = N(k, \frac{t}{\lambda})$;

(iii) *for each* $t, s \in (0, \infty)$ *with* $t \geq 0$, $N(k, t) \star N(g, s) \leq N(k + g, t + s)$;

(iv) $N(k,.):(0,\infty)\to[0,1]$ *is continuous;*

(v) $\lim_{t\to\infty} N(k,t)=1$*;*

for each $k,g\in K$ *and* $t,s>0$*. The triple* $(K,N,\star)$ *is called fuzzy norm space.*

Definition 2.4.2. *A sequence* $(k_n)_{n\in\mathbb{N}}$ *in a fuzzy norm space is said to be:*

(i) convergent *to* $k\in K$ *denoted* $k_n \xrightarrow{fn} k$ *if for each* $\alpha\in(0,1)$ *and* $s>0$*,* $\exists\ n_0\in\mathbb{N}$ *such that*

$$N(k_n-k,s)>1-\alpha$$

for each $n>n_0$*.*

(ii) fuzzy cauchy *if for each* $\alpha\in(0,1)$ *and* $s>0$*,* $\exists\ n_0\in\mathbb{N}$ *such that*

$$N(k_n-k_m,s)>1-\alpha$$

for each $n,m>n_0$*.*

2.4.1 Fuzzy topological spaces

Definition 2.4.3. *A family* τ *of fuzzy sets of* K *is known as fuzzy topology if*

(i) τ *contain all constant fuzzy sets in* K*;*

(ii) for $\{C_i\}_{i\in\Delta}\in\tau$ *we have* $\sup_{i\in\Delta} C_i\in\tau$*;*

(iii) if $C,D\in\tau$ *then* $C\wedge D\in\tau$*.*

The pair (K,τ) *is said to be* fuzzy topological space (FTS)*.*

A fuzzy set C in (K,τ) is a *neighborhood* of a point $k\in K$ iff $N\leq C$ and $N(k)=C(k)>0$ for $N\in\tau$. A map U from a FTS K to a FTS H is called *continuous* at some $k\in K$ if $U^{-1}(N)$ is a neighborhood of k in K for each neighborhood N of $U(k)$ in H. A net $(k_\lambda)_{\lambda\in\Lambda}$ in a FTS

(K, τ) converges to a point k denoted $k_\lambda \xrightarrow{f\tau} k$ if given a neighborhood N of k, there exists a $\lambda_0 \in \Lambda$ such that $k_\lambda \in N$ whenever $\lambda \geq \lambda_0$.

A fuzzy set $C \subseteq K$ is called *convex* if $\alpha C + (1-\alpha)C \leq C$ for each $\alpha \in [0, 1]$, *balanced* if $\alpha C \leq C$ for each scalar α with $|\alpha| \leq 1$ and *absorbing* if $\sup_{\alpha>0} \alpha C = 1$.

Definition 2.4.4. *A fuzzy topology τ on a vector space K over field $\mathbb{R}$ is a fuzzy linear topology if the two mappings*

(i) $+ : K \times K \to K,\ (k, g) \mapsto k + g;$

(ii) $\cdot : \mathbb{R} \times K \to K,\ (k, g) \mapsto \lambda k.$

are continuous when $\mathbb{R} \times K$, $K \times K$ the corresponding product fuzzy topologies and $\mathbb{R}$ has the usual fuzzy topology. The pair (K, τ) is called fuzzy topological vector space.

Chapter 3

The existence of fuzzy Dedekind completion of Archimedean fuzzy Riesz space

Completeness plays an essential role in fuzzy metric space, fuzzy norm space, and fuzzy inner product space. Therefore, it is essential to question whether or not the fuzzy Dedekind completion of incomplete fuzzy Riesz space exists? To address this issue, we prove the existence of fuzzy Dedekind completion of Archimedean fuzzy Riesz spaces.

3.1 Fuzzy order convergence

Although the notion of fuzzy order convergence is a central tool in studying fuzzy Riesz spaces, the Definition 2.3.12 given in [9, 26] has some limitations that cannot truly fulfill the concept of convergence. Intuitively if we add some terms at the start of the net then the convergence should not change. The following example illustrates our point.

Example 3.1.1. *Let (K, μ) be an Archimedean FRS. Then for $k \in K^+$ the net $\{\frac{k}{n}\} \downarrow 0$. Therefore, $\frac{k}{n} \xrightarrow{fo} 0$ according to Definition 5.0.1. On the other hand, negative integers are* *added and placed between* 1 *and* 2 *in the index set. Thus the new index set is denoted as*

$$\Lambda = \{1, -1, -2, -3, ..., 2, 3, 4, ...\}.$$

The extended net (g_n) *is defined as*

$$g_n = \begin{cases} k, & \text{if } n = 1 \\ |n|k, & \text{if } n \in -\mathbb{N}; \\ \frac{k}{n}, & \text{otherwise.} \end{cases}$$

Clearly, (g_n) *is not fuzzy order convergent to zero according to the Definition 5.0.1.*

This deficiency is pointed out in [1] for the classical order convergence. Therefore, a new definition is proposed in [28] to overcome this issue, given as follows.

Definition 3.1.2. *A net* $(k_\lambda)_{\lambda\in\Lambda}$ *in an FRS* (K, μ) *is known as* fuzzy order convergent *to* $k \in K$ *denoted* $k_\lambda \xrightarrow{fo} k$ *if there exists another net* $(g_\gamma)_{\gamma\in\Gamma}$ *in* K^+ *directed downwards to zero and for each* $\gamma \in \Gamma$ *there exist* $\lambda_0 \in \Lambda$ *such that* $\mu(|k_\lambda - k|, g_\gamma) > 1/2$ *whenever* $\lambda \geq \lambda_0$.

One can check that the extended net in Example 3.1.1 is fuzzy order convergent to zero according to Definition 3.1.2. Clearly, Definition 2.3.12 implies Definition 3.1.2 and many proven results for Definition 2.3.12 also hold for Definition 3.1.2 with analogous proofs given in [9, 12, 26]. The next theorem has the same proof as in [10, 26] according to definition 3.1.2.

Theorem 3.1.3. *If* $(k_\lambda)_{\lambda\in\Lambda}$ *and* $(g_\gamma)_{\gamma\in\Gamma}$ *are nets in an FRS* (K, ϕ) *then the following statements are true:*

(i) the fuzzy order limit is unique;

(ii) every fuzzy order convergent net is fuzzy order bounded;

(iii) if $k_\lambda \uparrow$ *(*$k_\lambda \downarrow$*) then* $k_\lambda \xrightarrow{fo} k$ *iff* $k_\lambda \uparrow k$ *(*$k_\lambda \downarrow k$*);*

(iv) *if* $k_\lambda \xrightarrow{fo} \mathrm{k}$ *then* $k_\lambda^+ \xrightarrow{fo} k$, $k_\lambda^- \xrightarrow{fo} k$ *and* $|k_\lambda| \xrightarrow{fo} |k|$;

(v) *if* $k_\lambda \xrightarrow{fo} k$ *then any subnet of* k_λ *is fuzzy order convergent to* k;

(vi) *if* $k_\lambda \xrightarrow{fo} k$ *and* $g_\gamma \xrightarrow{fo} g$ *then* $k_\lambda \vee g_\gamma \xrightarrow{fo} k \vee g$ *and* $k_\lambda \wedge g_\gamma \xrightarrow{fo} k \wedge g$;

(vii) *if* $k_\lambda \xrightarrow{fo} k$ *and* $g_\gamma \xrightarrow{fo} g$ *then* $ak_\lambda + bg_\gamma \xrightarrow{fo} ak + bg$ *for each* $a, b \in \mathbb{R}$.

Now we characterized fuzzy Dedekind complete FRS in terms of fuzzy order convergence.

Proposition 3.1.4. *An FRS* (K, ϕ) *is fuzzy Dedekind complete if and only if for every increasing net* (k_λ) *in* K^+, $(k_\lambda)^u \neq \emptyset$ *there exists some* $h \in K$ *such that* $k_\lambda \uparrow h$ *i.e.* $k_\lambda \xrightarrow{fo} h$.

Proof. The forward implication is obvious. Conversely, take $\emptyset \neq C \subseteq K$ which is bounded above. Without loss, we assume that C admits supremum of its finite elements. Take $k \in C$, consider the set $C_1 = \{k \vee g : g \in C\}$, then $(C_1)^u = (C)^u \neq \emptyset$ and C_1 is directed upward. Therefore, it is sufficient to show that supremum of C_1 exists. Define $C_2 = \{j - k : j \in C_1\}$ then $C_2 \subseteq K^+$ and also it is directed upward so we can consider it as a net $(j-k)_{(j-k)\in C_2}$, which is increasing and bounded above. Thus, by hypothesis there is some h with $(j-k) \uparrow h$ i.e. $\sup(j-k) = h$. Whereas, $h = \sup(C_2) = \sup(C_1) - k$, hence $\sup(C) = h + k$ exits. □

Remark 3.1.5. *Let* (K, ϕ) *be a Dedekind complete FRS with* $(k_\lambda)_{\lambda\in\Lambda}$ *a fuzzy order bounded net. Then* $k_\lambda \xrightarrow{fo} k$ *iff* $k = \limsup_\lambda(k_\lambda) = \liminf_\lambda(k_\lambda)$.

The rest of this section is dedicated to develop the basic notions and prove detailed results that will be applied in Section 3.4 to define fuzzy Dedekind completion and prove its existence. In this regard, many important notions for fuzzy Riesz subspaces are defined and studied in detail.

Definition 3.1.6. *A fuzzy Riesz subspace* L *of an FRS* (K, ϕ) *is said to be:*

(i) fuzzy order dense *if* $(k)^l \cap L^+ - \{0\} \neq \emptyset$ *for* $0 \neq k \in K^+$;

(ii) fuzzy majorizing *if* $(k)^u \cap L^+ \neq \emptyset$ *for* $0 \neq k \in K^+$;

(iii) fuzzy dense with respect to the fuzzy order convergence *if every vector of K is a fuzzy order limit of a net in L;*

(iv) fuzzy regular *for $A \subseteq L$,* $\inf A$ *is same in K and L whenever* $\inf A$ *exists in L.*

One can easily prove the following simple proposition.

Proposition 3.1.7. *Let L be a fuzzy Riesz subspace of an FRS (K, ϕ). Then following statements are equivalent:*

(i) L is a fuzzy regular Riesz subspace of K;

(ii) if $k_\lambda \downarrow 0$ in L, then $k_\lambda \downarrow 0$ in K;

(iii) if $k_\lambda \xrightarrow{fo} k$ in L, then $k_\lambda \xrightarrow{fo} k$ in K.

The relationship between fuzzy regular and fuzzy order dense Riesz subspace is discussed in the following result.

Proposition 3.1.8. *Every fuzzy order dense Riesz subspace L of an FRS (K, ϕ) is fuzzy regular.*

Proof. Take a net $g_\lambda \downarrow 0$ in L. Suppose on the contrary that $(g_\lambda)_{\lambda \in \Lambda}$ is not directed downwards to zero in K then there exists some $0 \neq k \in K^+$ with $\phi(k, g_\lambda) > 1/2$ for all λ. Since L is fuzzy order dense, there exists $0 \neq g \in L^+$ such that $\phi(g, k) > 1/2$. Thus $\phi(g, g_\lambda) > 1/2$ in L for all λ, a contradiction. Hence by the Proposition 3.1.7 L is fuzzy regular. □

The following result characterizes fuzzy order denseness and majorizingness for the Archimedean FRS (k, ϕ).

Theorem 3.1.9. *Let L be a fuzzy Riesz subspace of an Archimedean FRS (K, ϕ). Then following statements are true:*

(i) L is fuzzy order dense iff $k = \sup\{g \in L^+ : \phi(g, k) > 1/2\}$ holds for each $k \in K^+$.

(ii) L is fuzzy majorizing iff $k = \inf\{g \in L^+ : \phi(k,g) > 1/2\}$ holds for each $k \in K^+$.

Proof. (i): If $k = \sup\{g \in L^+ : \phi(g,k) > 1/2\}$ holds for all $k \in K^+$, then clearly L is fuzzy order dense in K.

Conversely, let L is fuzzy order dense, so the set $\{g \in L^+ : \phi(g,k) > 1/2\} \neq \emptyset$ for each k in K^+. Assume that there is some $k \in K^+$ such that $k \neq \sup\{g \in L^+ : \phi(g,k) > 1/2\}$. Therefore, there exists $u \in K^+$ with $u \neq k$, $\phi(u,k) > 1/2$ and whenever $\phi(g,k) > 1/2$ for $g \in L^+$ we have $\phi(g,u) > 1/2$. Since L is fuzzy order dense in K, so there exists $0 \neq d \in L^+$ with $\phi(d, k-u) > 1/2$, by transitivity $\phi(d,k) > 1/2$ and hence $\phi(d,u) > 1/2$. Therefore, $\phi(2d = d+d, k-u+u) > 1/2$, by induction $\phi(nd,k) > 1/2$ for each $n \in \mathbb{N}$, a contradiction to the fuzzy Archimedean property.

(ii): If $k = \inf\{g \in L^+ : \phi(k,g) > 1/2\}$ for each $k \in K^+$, then clearly L is fuzzy majorizing.

Conversely, let L be fuzzy majorizing. Then, the set $C := \{g \in L^+ : \phi(k,g) > 1/2\} \neq \emptyset$, so there exists $u \in L^+$ with $\phi(k,u) > 1/2$ and $[k,u] \cap L^+ \subseteq C$. Thus,

$$\begin{aligned}
\inf\left\{[k,u] \cap L^+\right\} &= u - \sup\left\{[0, u-k] \cap L^+\right\}, \\
&= u - (u-k) = k.
\end{aligned}$$

Hence, $\inf C = k$. □

Fuzzy order convergence in FRS (K,ϕ) does not imply fuzzy order convergence in its fuzzy Riesz subspaces unless the fuzzy Riesz subspace is fuzzy order dense and majorizing.

Proposition 3.1.10. *Let $(k_\lambda)_{\lambda\in\Lambda}$ be a net in the fuzzy order dense and majorizing Riesz subspace L. Then $k_\lambda \xrightarrow{fo} 0$ in L iff $k_\lambda \xrightarrow{fo} 0$ in FRS (K,ϕ).*

Proof. Since L is fuzzy order dense by the Proposition 3.1.8 it is fuzzy regular. Therefore the forward implication is obvious due to the Proposition 3.1.7.

Conversely, in K we let $k_\lambda \xrightarrow{fo} 0$, such that there exists a net $g_\gamma \downarrow 0$ in K^+ with, for all

$\gamma \in \Gamma$ there exists λ_0 such that for all $\lambda \geq \lambda_0$ we have $\phi(|k_\lambda|, g_\gamma) > 1/2$. Put

$$C = \{h \in L^+ : \phi(g_\gamma, h) > 1/2 \text{ for some } \gamma \in \Gamma\}.$$

Then, $C \neq \emptyset$ as L is fuzzy majorizing. Let $k = \inf C$ in K then $\phi(k, g_\gamma) > 1/2$ for all $\gamma \in \Gamma$. Therefore, $k = 0$ as $g_\gamma \downarrow 0$, but L is fuzzy regular so $\inf C = 0$ as well in L. Thus, there exists a net $h_\gamma \downarrow 0$ in L^+ with $\phi(g_\gamma, h_\gamma) > 1/2$. Hence, $k_\lambda \xrightarrow{fo} 0$ in L. □

3.2 Fuzzy positive linear operators

Fuzzy positive operators play a vital role in studying the fuzzy Riesz spaces, especially to find the fuzzy Dedekind completion. Therefore, they are studied in detail first.

Definition 3.2.1. *A fuzzy positive operator P between the two FRSs (K, ϕ) and (H, μ) is said to be:*

(i) fuzzy order bounded *if $P(C) \subseteq H$ is fuzzy order bounded whenever $C \subseteq K$ is fuzzy order bounded;*

(ii) fuzzy σ-order continuous *if $(k_n)_{n\in\mathbb{N}} \xrightarrow{fo} 0$ in K implies $P(k_n)_{n\in\mathbb{N}} \xrightarrow{fo} 0$ in H;*

(iii) fuzzy order continuous *if $k_\lambda \xrightarrow{fo} 0$ in K implies $P(k_\lambda) \xrightarrow{fo} 0$ in H.*

The fuzzy order continuity of fuzzy positive operator implies the fuzzy order boundedness.

Proposition 3.2.2. *Every fuzzy order continuous positive operator P is fuzzy order bounded.*

Proof. Since P is a fuzzy order continuous positive operator between the two FRSs (K, ϕ) and (H, μ). For $g \in K^+$, consider the net $c_k := g - k$ for $k \in [0, g]$. Then $c_k \downarrow 0$, therefore, $P(c_k) \xrightarrow{fo} 0$. Thus, the net

$$(P(c_k) : k \in [0, g]) = (P(g - k) : k \in [0, g]) = P([0, g])$$

is fuzzy order bounded in H. □

The notion of fuzzy order continuous operators have nice characterizing conditions.

Theorem 3.2.3. *If P is a fuzzy order continuous operator between two FRSs (K, ϕ) and (H, μ) with H fuzzy Dedekind complete then underlying statements are equivalent:*

(i) P is fuzzy order continuous ;

(ii) if $k_\lambda \downarrow 0$ in K then $P(k_\lambda) \downarrow 0$ in H;

(iii) if $k_\lambda \downarrow 0$ in K then $\inf(P(k_\lambda)) = 0$ in H;

(iv) P^+ and P^- are both fuzzy order continuous;

(v) $|P|$ is fuzzy order continuous.

Proof. $(i) \to (ii)$ and $(ii) \to (iii)$ are obvious.

$(iii) \to (iv)$ Let $k_\lambda \downarrow 0$ in K and $P(k_\lambda) \downarrow h$ in H for $h \in H^+$. We have to show that $h = 0$. Fixed some γ and put $k = k_\gamma$. Take $k, g \in K^+$ such that $\phi(g, k) > 1/2$ and for each $\lambda \succeq \gamma$ we have

$$g - g \wedge k_\lambda = g \wedge g - g \wedge k_\lambda \text{ and } \phi(g - g \wedge k_\lambda, k - k_\lambda) > 1/2.$$

Therefore,

$$P(g) - P(g \wedge k_\lambda) = P(g - g \wedge k_\lambda),\ \ \mu(P(g - g \wedge k_\lambda), P^+(k - k_\lambda)) > 1/2$$

and

$$P^+(k - k_\lambda) = P^+(k) - P^+(k_\lambda) \text{ implies } \mu(P(g - g \wedge k_\lambda), P^+(k) - P^+(k_\lambda)) > 1/2.$$

It follows that

$$\mu(h, P^+(k_\lambda)) > 1/2 \text{ and } \mu(P^+(k_\lambda), P^+(k) + P(g \wedge k_\lambda) - P(g)) > 1/2. \tag{3.2.1}$$

Since$\phi(g,k) > 1/2$ we have $g \wedge k_\lambda \downarrow 0$ for each $\lambda \succeq \gamma$. It follows from (iii) that $\inf(P(g \wedge k_\lambda)) = 0$. Therefore, from Equation 3.2.1 $\mu(h, P^+(k) - P(g)) > 1/2$ for each $\phi(g,k) > 1/2$. Thus, $P^+(k) = \sup\{P(g) : \phi(g,k) > 1/2\}$ implies that $h = 0$.

$(iv) \to (v)$ is straightforward.

$(v) \to (i)$ immediately follows from $\mu(|P(k)|, |P|(|k|)) > 1/2$.

□

For $k \in K$ we have $k = k^+ - k^-$, therefore, $K = K^+ - K^+$. Now the question is if a fuzzy positive operator is defined from K^+ to FRS (H, μ) then can it be extended to the entire FRS (K, ϕ) Beg [13, Lemma 2.4] gave a positive answer. Our work is related to the extension of fuzzy order continuous positive operator P from the fuzzy order dense and majorizing Riesz subspace L to the FRS (K, ϕ).

Theorem 3.2.4. *Let L be a fuzzy order dense and majorizing Riesz subspace of an FRS (K, ϕ) and (H, μ) is a Dedekind complete FRS. If P is a fuzzy order continuous positive operator from L into H then for $k \in K^+$*

$$P(k) := \sup\{P(g) : g \in L^+ \textit{ and } \phi(g,k) > 1/2\},$$

P extend uniquely as a fuzzy order continuous linear operator to all of K.

Proof. For $k \in K^+$

$$V(k) = \sup\{P(g) : g \in L^+ \text{ and } \phi(g,k) > 1/2\},$$

exists in H because it is fuzzy Dedekind complete and L is fuzzy order dense in K. Take $k_\lambda \uparrow k$ in L^+. If $g \in L^+$ satisfies $\phi(g,k) > 1/2$ then $k_\lambda \wedge g \uparrow g$ in L^+. Since P is fuzzy order continuous, therefore

$$P(g) = \sup\{P(k_\lambda \wedge g)\}.$$

So,

$$\mu(\sup\{P(k_\lambda \wedge g)\}, \sup\{P(k_\lambda)\}) > 1/2 \text{ and } \mu(\sup\{P(k_\lambda)\}, V(k)) > 1/2.$$

Hence, $P(k_\lambda) \uparrow V(k)$ in H.

Let $k, h \in K^+$, then by Theorem 3.1.9(i) there exist two nets $k_\lambda \uparrow k$ and $h_\gamma \uparrow h$ in L^+. So, $k_\lambda + h_\gamma \uparrow k+h$. Hence, $P(k_\lambda) \uparrow V(k)$, $P(g_\gamma) \uparrow V(h)$ and $P(k_\lambda)+P(h_\gamma) = P(k_\lambda+h_\gamma) \uparrow V(k+h)$. Therefore,

$$V(k + h) = V(k) + V(h).$$

Now as $V : K^+ \to H^+$ is an additive map, by [13, Lemma 2.4] it is the unique extension of P from K to H.

Lastly, V is proved to be fuzzy order continuous. Let $k_\lambda \downarrow 0$ be a net in K^+ then there exists $h_\lambda \downarrow 0$ in L^+ with $\phi(k_\lambda, h_\lambda) > 1/2$ for all $\lambda \in \Lambda$, due to the fact that L is fuzzy majorizing.

Moreover, $V(h_\lambda) = P(h_\lambda) \downarrow 0$ because P is fuzzy order continuous. Therefore, V is deduced that $\mu(V(k_\lambda), V(h_\lambda)) > 1/2$ for all $\lambda \in \Lambda$. Hence, finally $V(k_\lambda) \downarrow 0$.

□

3.3 Fuzzy Riesz homomorphism

Here we will define and study the particular class of positive operators known as fuzzy Riesz homomorphism that preserves the fuzzy lattice structure.

Definition 3.3.1. *A linear operator P between the two FRSs (K, ϕ) and (H, μ) is called* fuzzy Riesz homomorphism *if $P(k \vee g) = P(k) \vee P(g)$ holds for all $k, g \in K$. In addition, if P is bijective then it is said to be* fuzzy Riesz isomorphism.

Note that a fuzzy Riesz homomorphism is indeed a fuzzy positive operator. Indeed, for $k \in K^+$ $P(k) = P(k^+) = P(k \vee 0) = P(k) \vee P(0) = [P(k)]^+$. The notion of fuzzy Riesz homomorphism has several nice characterizing conditions.

Proposition 3.3.2. *If P is a fuzzy positive operator between the two FRSs (K, ϕ) and (H, μ) then the following statements are equivalent:*

(i) *P is a fuzzy Riesz homomorphism;*

(ii) *$P(k^+) = [P(k)]^+$ for each $k \in K$;*

(iii) *$P(k \wedge h) = P(k) \wedge P(h)$ for each $k, h \in K$;*

(iv) *if $k \wedge h = 0$ in K, then $P(k) \wedge P(h) = 0$ holds in H;*

(v) *$P(|k|) = |Pk|$ for each $k \in K$.*

Proof. It is obvious that $(i) \Rightarrow (ii)$.
$(ii) \Rightarrow (iii)$

$$\begin{aligned} P(h \wedge k) &= P(k - (k-h)^+) = P(k) - P(k-h)^+ \\ &= P(k) - (Pk - Ph)^+ = Pk \wedge Ph. \end{aligned}$$

$(iii) \Rightarrow (iv)$ If $k \wedge h = 0$ then $Ph \wedge Pk = P(h \wedge k) = P(0) = 0$.
$(iv) \Rightarrow (v)$ In view if $k^+ \wedge k^- = 0$ we have

$$\begin{aligned} |Pk| &= |Pk^+ - Pk^-| = Pk^+ \vee Pk^- - Pk^+ \wedge Pk^- \\ &= Pk^+ \vee Pk^- = Pk^+ + Pk^- = P(k^+ + k^-) \\ &= P(|k|). \end{aligned}$$

$(v) \Rightarrow (i)$ observe that

$$\begin{aligned} P(k \vee h) &= P(1/2[k + h + |k-h|]) = 1/2[Pk + Ph + P(|k-h|) \\ &= 1/2[Pk + Ph + |Pk - Ph|] = Pk \vee Ph. \end{aligned}$$

□

The fuzzy Riesz isomorphisms between the surjective fuzzy positive operators are characterized as follows.

Proposition 3.3.3. *Let P be a bijective linear operator between the two FRSs (K, ϕ) and (H, μ). Then P is a fuzzy Riesz isomorphism iff both P and P^{-1} are fuzzy positive operators.*

Proof. Since P is a fuzzy Riesz isomorphism, therefore, both P and P^{-1} are fuzzy positive operators.

Conversely, let $k, g \in K$. Then $\phi(k, k \vee g) > 1/2$ and $\phi(g, k \vee g) > 1/2$, thus

$$\mu(Pk, P(g \vee k)) > 1/2 \text{ and } \mu(Pg, P(g \vee k)) > 1/2.$$

Therefore,

$$\mu(Pg \vee Pk, P(g \vee k)) > 1/2. \tag{3.3.1}$$

Also, P^{-1} is fuzzy positive operator, so by similar arguments as above.

$$\phi(P^{-1}a \vee P^{-1}b, P^{-1}(a \vee b)) > 1/2,$$

for each $a, b \in H$. Particularly, for $a = Pk$ and $b = Pg$ we have

$$\phi(k \vee g, P^{-1}(Pk \vee Pg)) > 1/2,$$

which implies

$$\mu(P(k \vee g), Pk \vee Pg) > 1/2.$$

Hence, the above combined with the Equation 3.3.1, we have

$$P(k \vee g) = Pk \vee Pg.$$

□

3.4 Fuzzy Dedekind completion

This section aims to prove the existence of fuzzy Dedekind completion for an incomplete FRS. In this regard, we know that a Dedekind complete FRS is fuzzy Archimedean [26, Lemma 5.9]. Thus, one can hope to find the fuzzy Dedekind completion only for an Archimedean FRS.

Definition 3.4.1. *Let (K, ϕ) be an FRS. A Dedekind complete FRS (H, μ) is said to be the* fuzzy Dedekind completion *of K if K is fuzzy Riesz isomorphic to a fuzzy majorizing and order dense Riesz subspace of H. Furthermore, (H, μ) is unique up to fuzzy Riesz isomorphisms.*

The fundamental result in this section will be that any Archimedean FRS has a fuzzy Dedekind completion indeed. To achieve our goal, we adopt a somewhat more general approach and start our work on fuzzy ordered sets.

Two FOSs (K, ϕ) and (H, μ) are called fuzzy order isomorphic, if their exists a bijective map P between them that preserves the fuzzy ordering i.e. $\mu(P(k), P(g)) > 1/2$ iff $\phi(k, g) > 1/2$ for all $k, g \in K$. Furthermore, a mapping P from an FOS (K, ϕ) into FOS (H, μ) preserves the suprema and infima, if $u = \sup C$ and $l = \inf C$ in K then $P(u) = \sup P(C)$ and $P(l) = \inf P(C)$ in H for any subset C of K.

Consider the following conditions for an injective mapping P between the two FOSs (K, ϕ) and (H, μ):

(a) K and its image $P(K)$ are fuzzy order isomorphic;

(b) P preserves suprema and infima.

The above conditions do not imply each other in general.

Proposition 3.4.2. *Let P be an injective mapping between the two FOSs (K, ϕ) and (H, μ). Then following hold:*

(i) if K is a fuzzy lattice then (b) implies (a);

(ii) if K and H are two fuzzy lattices and also for $g \in H$ the following holds

$$g = \sup\{P(k) : \mu(P(k), g) > 1/2\} = \inf\{P(k) : \mu(g, P(k)) > 1/2\},$$

then (a) implies (b).

Proof. (i): Let $\mu(P(k_1), P(k_2)) > 1/2$ for k_1, k_2 in K. As P preserves the suprema, therefore

$$P(k_1 \vee k_2) = P(k_1) \vee P(k_2) = P(k_2).$$

Thus, $k_1 \vee k_2 = k_2$, hence $\phi(k_1, k_2) > 1/2$. Analogously, $\phi(k_1, k_2) > 1/2$ implies $\mu(P(k_1), P(k_2)) > 1/2$.

(ii): Take two fuzzy lattices K and H, satisfying the given condition. Let $k_0 = \sup\{C\}$ for $C \subset K$. Suppose on the contrary that $P(k_0) \neq \sup\{P(C)\}$, then there exists $g_1 \in (P(C))^u$ such that $g_1 \neq P(k_0)$ and $\mu(g_1, P(k_0)) > 1/2$. Therefore, $g_1 = \inf\{P(k) : \mu(g_1, P(k)) > 1/2\}$, so there is some k_1 in K with $\mu(g_1, P(k_1)) > 1/2$, $P(k_1) \neq P(k_0)$ and $\mu(P(k_1), P(k_0)) > 1/2$. But, on the other hand $\phi(z, k_1) > 1/2$ for each $z \in C$ which implies $\phi(k_0, k_1) > 1/2$, hence $\mu(P(k_0), P(k_1)) > 1/2$, a contradiction. □

Definition 3.4.3. *Consider (K, ϕ) is an FOS. Then a subset C of K is called* fuzzy cut *if $(C^u)^l = C$.*

From here on $(C^u)^l$ will be written as C^{ul}. Clearly, $\emptyset$ and K are the fuzzy cuts. Therefore, the set of all fuzzy cuts is non-empty. Also, one can easily prove that C^{ul} is a fuzzy cut. It is the smallest fuzzy cut containing C. Let K^δ be the set of all fuzzy cuts except $\emptyset$ and K. The fuzzy order ψ on K^δ is defined as follows.

$$\psi(C, D) = \begin{cases} 1 & \text{if } C = D; \\ 2/3 & \text{if } C \subset D; \\ 0 & \text{otherwise,} \end{cases} \tag{3.4.1}$$

for $C, D \in K^\delta$.

Consider an FOS (K^δ, ψ). If K has no largest or smallest element, then K must contain at least two elements. Moreover, for $k_0 \neq k_1$, we have $\{k_0\}^{ul} \neq \{k_1\}^{ul}$.

Theorem 3.4.4. *Let (K, ϕ) be an FOS containing no smallest or largest element. Then (K^δ, ψ) is a fuzzy order complete lattice. Moreover, the map $P(k) := \{k\}^{ul}$ from K to K^δ satisfies condition (ii). In addition, every fuzzy cut C satisfies the following*

$$C = \sup\{\{k\}^{ul} : \psi(\{k\}^{ul}, C) > 1/2\} = \inf\{\{k\}^{ul} : \psi(C, \{k\}^{ul}) > 1/2\}.$$

Proof. Let (K, ϕ) be an FOS containing no smallest or largest element. Take $F \subset K^\delta$, then $D = (\cup_{C \in F} C)^{ul}$ is an upper bound of F in K^δ. Let D_1 be another upper bound, so $\cup_{C \in F} C \subseteq D_1$, therefore, $\psi(D, D_1) > 1/2$. Hence, $D = \sup(F)$. Analogously, $E = (\cap_{C \in F} C)^{ul}$ is the infima of F in K^δ. Thus K^δ is a fuzzy order complete lattice.

Take $C \subset K$ with $k_0 = \sup(C)$. Then $D = (\cup_{k \in C} \{k\}^{ul})^{ul}$, is the supremum of $P(C)$ in K^δ. As $C^u = (\cup_{k \in C} \{k\}^{ul})^u$ and k_0 is the smallest element of C^u, therefore $D = (\cup_{k \in C} \{k\}^{ul})^{ul} = \{k : \phi(k, k_0) > 1/2\} = \{k_0\}^{ul}$. Hence P preserves the suprema. Analogously, it can be shown that P preserves the infima.

Finally, take $C \in K^\delta$, we have $k \in C$ iff $\psi(\{k\}^{ul}, C) > 1/2$. Thus

$$C = \bigcup_{\psi(\{k\}^{ul}, C) > 1/2} \{k\}^{ul}.$$

Hence $C = \sup\{\{k\}^{ul} : \psi(\{k\}^{ul}, C) > 1/2\}$. Analogously, $C = \inf\{\{k\}^{ul} : \psi(C, \{k\}^{ul}) > 1/2\}$. □

We are coming closer to our goal, indeed (K^δ, ψ) is the completion of an unbounded FOS (K, ϕ). As an Archimedean FRS (K, ϕ) has no smallest and largest element; therefore, the Theorem 3.4.4 holds for it. The next step is to define the algebraic operations on K^δ so that it can become an FRS for an Archimedean FRS (K, ϕ).

Definition 3.4.5. *Given an Archimedean FRS* (K, ϕ)*, we define addition and scalar multiplication on* K^δ *as follows*

$$C \oplus D = (C + D)^{ul}, \tag{3.4.2}$$

$$\lambda \circ C = \begin{cases} \lambda C & \text{if } \lambda > 0; \\ \{0\}^{ul} & \text{if } \lambda = 0; \\ \lambda C^u & \text{if } \lambda < 0, \end{cases} \tag{3.4.3}$$

where $C, D \in K^\delta$ *and* $\lambda \in \mathbb{R}$.

Proposition 3.4.6. *The FOS* (K^δ, ψ) *is an FRS with the operations* $\oplus$ *and* $\circ$ *given in the Definition 3.4.5.*

Proof. Take $A, B \in K^\delta$. Then $A \oplus B = (A + B)^{ul} \neq K$ because $A \neq K$ and $B \neq K$. Hence $\oplus$ is well defined on K^δ. The other axioms are verified as follows.

Associative property: Take $A, B, C \in K^\delta$. Let $Y = \{a + b + c : a \in A, b \in B, c \in C\}$, clearly, $Y \subseteq (A \oplus B) \oplus C$, therefore, Y^{ul} also contained in $(A \oplus B) \oplus C$. Conversely, let $y \in Y^u$, then $\phi(a+b+c, y) > 1/2$ holds for $a \in A$, $b \in B$ and $c \in C$. So $\phi(a+b, y-c) > 1/2$, therefore, $y - c \in (A + B)^{ul}$. Thus $\phi(z, y - c) > 1/2$ holds for each $z \in A + B$. However, $\phi(z+c, y) > 1/2$ holds and implies $y \in \{(A+B)+C\}^u$. Hence $(A \oplus B) \oplus C = Y^{ul}$. Analogously, $A \oplus (B \oplus C) = Y^{ul}$. Therefore,

$$(A \oplus B) \oplus C = A \oplus (B \oplus C).$$

Additive identity: The element $C_0 = \{0\}^{ul} = -K^+$ plays the role of identity for operation $\oplus$. Take $C \in K^\delta$, now for $a \in C$ and $m \in C_0$ we have $\phi(c + m, c) > 1/2$ with equality holds if $m = 0$. Therefore, $M = \{c + m : c \in C, m \in C_0\}$ satisfies $M^u = C^u$, then $M^{ul} = C^{ul} = C$. Hence

$$C \oplus C_0 = C_0 \oplus C = C.$$

Also the additive identity is unique, indeed $B_0 \in K^\delta$ satisfies $C \oplus B_0 = B_0 \oplus C = C$ for all $C \in K^\delta$, then $C_0 = B_0$.

Additive inverse: For $C \in K^\delta$, consider $C' = -C^u$. First, we show that C' belongs to K^δ and secondly, it is the inverse of C for operation $\oplus$. Indeed, we have

$$k \in C'^u \Leftrightarrow \phi(d,k) > 1/2, \forall\, d \in C' \Leftrightarrow \phi(-k,-d) > 1/2, \forall -d \in C^u$$
$$\Leftrightarrow \phi(-k,g) > 1/2,\ \forall\, g \in C^u \Leftrightarrow -k \in C^{ul} = C.$$

Take $m \in C'^{ul}$ then $\phi(m,k) > 1/2$ for each $k \in C'^u$. But $k \in C'^u$ iff $k = -c$ for some $c \in C$. Therefore, $\phi(m,-c) > 1/2$ for all $c \in C$. Thus $-m \in C^u$ and eventually $m \in C'$. Hence, $C'^{ul} = C'$.

Next, consider $V = \{c + d : c \in C, d \in C'\}$, then

$$V = \{c - k : c \in C, k \in C^u\}.$$

Hence, $\phi(c-k,0) > 1/2$ so $\psi(V, C_0) > 1/2$, therefore, $K^+ \subseteq V^u$. Let $s \in V^u$, then $\phi(r,0) > 1/2$ and $\phi(r,s) > 1/2$ for each $r \in V$. Also, $\phi(r, s \wedge 0) > 1/2$ i.e. $\phi(c-k, -s^-) > 1/2$ for each $c \in C$ and $k \in C^u$. It follows that $\phi(c + s^-, k) > 1/2$, then $c + s^- \in C^{ul} = C$. By induction, for fix $c \in C$ and $k \in C^u$, we have $\phi(ns^-, k - c) > 1/2$ for $n = 1, 2....$ Since K is an Archimedean FOS, this is possible iff $s^- = 0$. Hence $V^u \subset K^+$, so $V^u = K^+$. Therefore,

$$C \oplus C' = C' \oplus C = C_0.$$

Moreover, the inverse is unique.

It is not hard to verify the distributional properties of operations $\oplus$ and $\circ$ and compatibility between algebraic and order structures in K^δ. Hence, $(K^\delta, \oplus, \circ, \psi)$ is an FRS. $\square$

Finally, the existence of fuzzy Dedekind completion is proved as follows.

Theorem 3.4.7. *If (K, ϕ) is an Archimedean FRS then the FRS $(K^\delta, \oplus, \circ, \psi)$ is its fuzzy Dedekind completion.*

Proof. The Archimedean FRS (K, ϕ) has no smallest and largest element, thus by the Theorem 3.4.4 $P(K)$ is fuzzy order isomorphic to K under the map $P : K \to K^\delta$ with the definition $P(k) = \{k\}^{ul}$. Now we show that P preserves the algebraic structure as well.

Indeed, for $\{w\}^{ul}, \{z\}^{ul} \in P(K)$, $\{w\}^{ul} \oplus \{z\}^{ul} \subseteq \{w + z\}^{ul}$ is clear. Take $s \in \{w + z\}^{ul}$, then $\phi(s, w + z) > 1/2$. Let $s = w + (s - w)$ with $w \in \{w\}^{ul}$ and $s - w \in \{z\}^{ul}$. Thus,

$$\{w + z\}^{ul} = \{w\}^{ul} \oplus \{z\}^{ul}.$$

Therefore, $P(w + z) = P(w) \oplus P(z)$ for all $w, z \in K$. Also, it is not hard to prove that $P(\lambda w) = \lambda \circ P(w)$ for $\lambda \in \mathbb{R}$ and $w \in K$. Hence, P is a fuzzy Riesz isomorphism between K and $P(K)$.

Lastly, $P(K)$ is also fuzzy order dense and majorizing Riesz subspace of K^δ due to the Theorem 3.1.9. □

Proposition 3.1.8 implies that (K, ϕ) is a fuzzy regular Riesz subspace of (K^δ, ψ). In the end, as an application of Theorem 3.4.7 some results are proved for a fuzzy regular Riesz subspaces of an FRS (K, ϕ).

Theorem 3.4.8. *If L is a fuzzy regular Riesz subspace of an Archimedean FRS (K, ϕ) then L^δ is a fuzzy regular Riesz subspace of (K^δ, ψ).*

Proof. Suppose K is fuzzy regular in K^δ, we have L fuzzy regular in K^δ. Then without loss, assume that $K = K^\delta$. Let $V : L \to K$ be the inclusion map. Therefore, V is a fuzzy order continuous by the fuzzy regularity of L.

Theorem 3.2.4 yields that V can be extended to a fuzzy order continuous positive operator $P : L^\delta \to K$. It will be shown that P is a fuzzy Riesz isomorphism from L^δ into K.

Let $g \in L^\delta$. Take two nets (a_λ) and (b_λ) in L^+ such that $a_\lambda \uparrow g^+$ and $b_\lambda \uparrow g^-$ in L^δ. Then clearly

$$a_\lambda = Pa_\lambda \xrightarrow{fo} Pg^+,$$

in K. Moreover, $a_\lambda - b_\lambda \xrightarrow{fo} g$ in L^δ, so $a_\lambda - b_\lambda = P(a_\lambda - b_\lambda) \xrightarrow{fo} Pg$ in K.

Also, $g^+ \wedge g^- = 0$ that implies $a_\lambda \wedge b_\lambda = 0$ in L^δ for any $\lambda \in \Lambda$ and hence

$$a_\lambda = (a_\lambda - b_\lambda)^+ = P(a_\lambda - b_\lambda)^+ \xrightarrow{fo} (Pg)^+,$$

in K. Therefore, $(Pg)^+ = P(g^+)$ for any $g \in L^\delta$. By Theorem 5.1.5 P is a fuzzy Riesz homomorphism.

Now let $Pg = 0$ for some $g \in L^\delta$, we can assume without loss of generality that $g \in L^+$, because P is a fuzzy Riesz homomorphism. Take a net (a_λ) in L^+ with $a_\lambda \uparrow g$ in L^δ. Then $a_\lambda = P(a_\lambda)$ and $\phi(Pa_\lambda, 0) > 1/2$ implies that $a_\lambda = 0$ for all λ, hence $g = 0$ and P is proved to be injective.

The fuzzy regularity of L^δ in K follows from the fuzzy order continuity of P. □

Proposition 3.4.9. *Let L be a fuzzy regular Dedekind complete Riesz subspace of an Archimedean FRS (K, ϕ). Take a net (g_λ) in L and $k \in K$. If $g_\lambda \xrightarrow{fo} k$ in K then $k \in L$ and $g_\lambda \xrightarrow{fo} k$ in L.*

Proof. Without loss, assume that $K = K^\delta$. By Remark 3.1.5 $k = \limsup_\lambda(g_\lambda) = \liminf_\lambda(g_\lambda)$, as L is a fuzzy Dedekind complete, so the $\limsup_\lambda(g_\lambda)$ and $\liminf_\lambda(g_\lambda)$ exist in L. Also, it is fuzzy regular therefore, the limits are same as in K. Hence k also belongs to L and $g_\lambda \xrightarrow{fo} k$ in L. □

A fuzzy order dense Riesz subspace L of an FRS (K, ϕ) is fuzzy dense with respect to the fuzzy order convergence, but the backward implication is not valid in general. The next result yields the equivalence between fuzzy order denseness and fuzzy denseness concerning to fuzzy order convergence.

Theorem 3.4.10. *If a fuzzy regular Riesz subspace L is dense with respect to the fuzzy order convergence in an Archimedean FRS (K, ϕ) then L is fuzzy order dense in FRS (K, ϕ). In addition, If L is a fuzzy Dedekind complete then L is a fuzzy ideal of an FRS (K, ϕ).*

Proof. By Theorem 3.4.8, L^δ is a fuzzy regular Riesz subspace of K^δ. First we show that L^δ is a fuzzy ideal of K^δ.

Take $u \in (K^\delta)^+$ and $v \in (L^\delta)^+$ such that $\psi(u, v) > 1/2$. Then $K \cap \sup([0, u]) = u$ in K^δ by Theorem 3.1.9. For $k \in [0, u] \cap K$ there is a net (g_λ) in L with $g_\lambda \xrightarrow{fo} k$ due to the fact that L is dense with respect to the fuzzy order convergence. Let $w_\lambda = g_\lambda \wedge v$ for $\lambda \in \Lambda$. Then (w_λ) is a net in L^δ and $w_\lambda \xrightarrow{fo} k$ in K^δ as well. Therefore, by the Proposition 5.1.3 k belongs to L^δ.

Thus $\sup\big([0, u] \cap L^\delta\big) = u$ in K^δ but L^δ is fuzzy regular so the supremum is same in L^δ. Hence, $u \in L^\delta$ and this proves that L^δ is indeed a fuzzy ideal of K^δ.

Now, take $0 \neq k \in K^+$ as L is dense with respect to the fuzzy order convergence in K, so there exists a net (g_λ) in L such that $g_\lambda \xrightarrow{fo} k$ in K. Then $|g_\lambda| \wedge k \xrightarrow{fo} k$ in K too.

Put $w = |g_{\lambda_0}| \wedge k$ for some $\lambda_0 \in \Lambda$. Then $\phi(w, |g_{\lambda_0}|) > 1/2$, as L^δ is fuzzy ideal and g_{λ_0} is in L, therefore w is in $(L^\delta)^+$. Furthermore, L is fuzzy order dense in L^δ so there is some u in L^+ with $\phi(u, w) > 1/2$. Hence, by transitivity $\phi(u, k) > 1/2$ and this leads to fuzzy order denseness of L in K.

Lastly, if L is a fuzzy Dedekind complete in its own right then $L = L^\delta$ and hence it is a fuzzy ideal of K. □

Chapter 4

Fuzzy order bounded linear operators in fuzzy Riesz spaces

In this chapter, we study the space of all fuzzy order bounded positive linear operators $L_b(K,H)$. In order to investigate $L_b(K,H)$ we start our work in the set of all linear operators $L(K,H)$ between (K,ϕ) and (H,μ). Of course, $L(K,H)$ is a vector space with pointwise operations. But the natural pointwise ordering i.e. $P \leq V$ if $\mu(P(k),V(k)) > 1/2$ for each $k \in K$, does not induce lattice structure on $L(K,H)$. Thus, to define proper fuzzy lattice operations on $L_b(K,H)$ first, we work on the modulus of a fuzzy positive linear operator.

Definition 4.0.1. *A fuzzy positive operator P between two FRSs (K,ϕ) and (H,μ) possesses a* modulus *if $|P| = P \vee (-P)$. The modulus of P means the supremum of the set $\{-P,P\}$ in $L(K,H)$.*

The next proposition gives the existence of a modulus of a fuzzy positive operator.

Proposition 4.0.2. *If P is a fuzzy positive operator between two FRSs (K,ϕ) and (H,μ) such that $\sup\{|Pg| : \phi(|g|,k) > 1/2\}$ exists in H for all $k \in K^+$ then modulus of P exists and*

$$|P|(k) = \sup\{|Pg| : \phi(|g|,k) > 1/2\}.$$

Proof. Suppose $V : K^+ \to H^+$ is defined by $V(k) = \sup\{|Pg| : \phi(|g|, k) > 1/2\}$ for $k \in K^+$. Since $\phi(|g|, k) > 1/2$ implies $|\pm g| = |g|$, $\phi(|\pm g|, k) > 1/2$. Now we show that V is additive.

Let $h, l \in K^+$. If $\phi(|g|, h) > 1/2$ and $\phi(|r|, l) > 1/2$ then

$$\phi(|g + r|, |g| + |r|) > 1/2 \text{ and } \phi(|g| + |r|, h + l) > 1/2.$$

Thus,

$$P(g) + P(r) = P(g + r) \text{ and } \mu(P(g + r), V(h + l)) > 1/2.$$

Therefore, $\mu(V(h) + V(l), V(h + l)) > 1/2$. Conversely, if $\phi(|g|, h + l) > 1/2$, then by [10, Theorem 4.12] there exist g_1 and g_2 with $\phi(|g_1|, h) > 1/2$, $\phi(|g_2|, l) > 1/2$ and $g = g_1 + g_2$. Therefore,

$$P(g) = P(g_1) + P(g_2) \text{ and } \mu(P(g_1) + P(g_2), V(h) + V(l)) > 1/2,$$

so $\mu(V(h + l), V(h) + V(l)) > 1/2$. Hence, V is additive. By [13, Lemma 2.4] V from K to H extends uniquely to a fuzzy positive operator.

It is left to show that V is a supremum of $\{-P, P\}$. Observe that $P \leq V$ and $-P \leq V$. Assume that $\pm P \leq R$. Thus, R is a fuzzy positive operator. Fix $k \in K^+$. If $\phi(|g|, k) > 1/2$ then

$$Pg = Pg^+ - Pg^- \text{ and } \mu(Pg^+ - Pg^-, Rg^+ + Rg^-) > 1/2.$$

Therefore,

$$Rg^+ + Rg^- = R|g| \text{ and } \mu(R|g|, Rk) > 1/2.$$

Then, $\mu(V(k), R(k)) > 1/2$ for $k \in K^+$. Hence $V = P \vee (-P)$ in $L(K, H)$. □

Remark 4.0.3. *Now we know that* $\sup\{|Pg| : \phi(|g|, k) > 1/2\}$ *exists when* H *is a fuzzy Dedekind complete. Thus, with the help of Proposition 4.0.2 we define the fuzzy lattice operations in* $L_b(K, H)$ *for fuzzy Dedekind complete* (H, μ).

Theorem 4.0.4. *If (K, ϕ) and (H, μ) are FRSs with H fuzzy Dedekind complete then the fuzzy order vector space $L_b(K, H)$ is a Dedekind complete FRS with the following fuzzy lattice operations,*

$$|P|(h) = \sup\{|Pg| : \phi(|g|, h) > 1/2\},$$

$$[V \vee P](h) = \sup\{V(g) + P(k) : g, k \in K^+ \text{ and } g + k = h\},$$

$$[V \wedge P](h) = \inf\{V(g) + P(k) : g, k \in K^+ \text{ and } g + k = h\}$$

for each $V, P \in L_b(K, H)$ and $k \in K^+$. In addition, $P_\lambda \downarrow 0$ in $L_b(K, H)$ iff $P_\lambda(h) \downarrow 0$ in H for all $h \in K^+$.

Proof. Since P is fuzzy order bounded,

$$\sup\{|Pg| : \phi(|g|, h) > 1/2\} = \sup\{Pg : \phi(|g|, h) > 1/2\} = \sup P[-h, h]$$

exists in H for $h \in K^+$. By Proposition 4.0.2 the modulus of P exists and also

$$|P|(h) = \sup\{Pg : \phi(|g|, h) > 1/2\}.$$

Now we show that $L_b(K, H)$ is an FRS. Let $V, P \in L_b(K, H)$ and $h \in K^+$ satisfying $g + k = h$ iff there exits some $l \in K$ such that $\phi(|l|, h) > 1/2$ with $g = 1/2(h + l)$ and $k = 1/2(h - l)$ for $g, k \in K^+$. It follows from [10, Theorem 4.11] that

$$\begin{aligned}
[V \vee P](h) &= 1/2[V(h) + P(h) + |V - P|(h)] \\
&= 1/2[V(h) + P(h) + \sup\{(V - P)(l) : \phi(|l|, h) > 1/2\}] \\
&= 1/2 \sup\{V(h) + V(l) + P(h) - P(l) : \phi(|l|, h) > 1/2\} \\
&= \sup\{V(1/2(h + l)) + P(1/2(h - l)) : \phi(|l|, h) > 1/2\} \\
&= \sup\{V(g) + P(k) : g, k \in K^+ \text{ and } g + k = h\}.
\end{aligned}$$

$[V \wedge P]$ can be proven analogously.

Now we have to show that $L_b(K, H)$ is fuzzy Dedekind complete. Let $P_\lambda \uparrow P$ in $L_b(K, H)$. Assume that $V(h) = \sup\{P_\lambda(h)\}$ implies that $P_\lambda(h) \uparrow V(h)$ for each $h \in K^+$. As $P_\lambda(h+g) = P_\lambda(h) + P_\lambda(g)$, it follows that $V : K^+ \to H^+$ is additive. Then V from K to H defines a fuzzy positive operator. Clearly, $P_\lambda \uparrow V$ in $L_b(K, H)$. □

Remark 4.0.5. *Theorem 4.0.4 yields that if (K, ϕ) and (H, μ) are FRSs with H fuzzy Dedekind complete then every fuzzy order bounded operator $P : K \to H$ satisfies*

$$P^+(k) = \sup\{Pg : \phi(g, k) > 1/2\}$$

$$P^-(k) = \sup\{-Pg : \phi(g, k) > 1/2\}$$

for each $k \in K^+$ and we have $P = P^+ - P^-$. To derive some formulas for fuzzy positive operators, we first prove the approximation properties of fuzzy positive operators, which are discussed as follows.

Lemma 4.0.6. *If P is a fuzzy positive operator between two FRSs (K, ϕ) and (H, μ) with H $\sigma-$fuzzy Dedekind complete then there exists a fuzzy positive operator $V : K \to H$ for each $k \in K^+$ such that:*

(i) $V \leq P$;

(ii) $V(k) = P(k)$;

(iii) $V(g) = 0$ *for each* $g \perp k$.

Proof. The proof is basically the same as for the Proposition 4.0.2 with the use of [13, Lemma 2.4]. □

The next result is proved by using Lemma 4.0.6 .

Theorem 4.0.7. *If P is a fuzzy positive operator between two FRSs (K, ϕ) and (H, μ) with H σ-fuzzy Dedekind complete then for each $k \in K$ we have:*

(i) $P(k^+) = \max\{V(k) : V \in L(K,H) \text{ and } V \leq P\}$;

(ii) $P(k^-) = \max\{-V(k) : V \in L(K,H) \text{ and } V \leq P\}$;

(iii) $P(|k|) = \max\{V(k) : V \in L(K,H) \text{ and } -P \leq V \leq P\}$.

Proof. (i) Fix $k \in K$. By Lemma 4.0.6 there exists a fuzzy positive operator $V : K \to H$ such that $V \leq P$, $V(k^+) = P(k^+)$ and $V(k^-) = 0$. If $R \in L(K,H)$ with $R \leq P$ then

$$\mu(R(k), R(k^+)) > 1/2 \text{ and } \mu(R(k^+), P(k^+)) > 1/2.$$

(ii) The proof of this part can be obtained from (i) by using identity $k^- = (-k)^+$.

(iii) Suppose that operator $R : K \to H$ satisfies $-P \leq R \leq P$, then $R(k) = R(k^+) - R(k^-)$ such that

$$\mu(R(k^+) - R(k^-), P(k^+) + P(k^-)) > 1/2 \text{ and } P(k^+) + P(k^-) = P(|k|).$$

Hence $R(k) = P(|k|)$.

On the other hand, by Lemma 4.0.6 there exist two fuzzy positive operators $V_1, V_2 : K \to H$ such that

$$V_1(k^+) = P(k^+) \text{ and } V_1(k^-) = 0.$$

And

$$V_2(k^-) = P(k^-) \text{ and } V_2(k^+) = 0.$$

Therefore, $R = V_1 - V_2$ satisfies $-P \leq R \leq P$ and $R(k) = P(|k|)$.

□

Remark 4.0.8. *The set of all fuzzy order continuous operator of $L_b(K,H)$ are denoted by*

$L_n(K,H)$ i.e.

$$L_n(K,H) = \{P \in L_b(K,H) : P \text{ is fuzzy order continuous}\}.$$

Analogously, $L_c(K,H)$ denotes the set of all σ-fuzzy order continuous operator i.e.

$$L_c(K,H) = \{P \in L_b(K,H) : P \text{ is } \sigma\text{−fuzzy order continuous}\}.$$

Both $L_c(K,H)$ and $L_n(K,H)$ are vector subspaces of $L_b(K,H)$. Furthermore, $L_n(K,H) \subseteq L_c(K,H)$. The following proposition shows $L_n(K,H)$ and $L_c(K,H)$ are fuzzy bands of $L_b(K,H)$.

Proposition 4.0.9. *If (K,ϕ) and (H,μ) are FRSs with H fuzzy Dedekind complete then both $L_n(K,H)$ and $L_c(K,H)$ are fuzzy bands of $L_b(K,H)$.*

Proof. If $|V| \leq |P|$ in $L_b(K,H)$ with $P \in L_b(K,H)$ then by Theorem 3.2.3 $V \in L_b(K,H)$. Thus, $L_n(K,H)$ are fuzzy ideal of $L_b(K,H)$.

Now we show that $L_n(K,H)$ is a fuzzy band. Let $(P_\gamma)_{\gamma\in\Gamma} \in L_n(K,H)$ and $P_\gamma \uparrow P$ in $L_b(K,H)$. Let $k_\lambda \uparrow k$ in K^+. For fixed γ, we have

$$\mu(P(k-k_\lambda), (P-P_\gamma)(k) + P_\gamma(k-k_\lambda)) > 1/2,$$

and $k - k_\lambda \downarrow 0$. As $P_\gamma \in L_n(K,H)$ implies that

$$\mu(\inf(P(k-k_\lambda)), (P-P_\gamma)(k)) > 1/2$$

for each γ. Thus, $(P-P_\gamma) \downarrow 0$. Therefore, $\inf(P(k-k_\lambda) = 0$ and so $P(k_\lambda) \uparrow P(k)$. Hence $P \in L_n(K,H)$.

$L_c(K,H)$ can be proved analogously. □

4.1 Fuzzy order dual

A fuzzy positive linear functional u between an FRS (K, ϕ) and $\mathbb{R}$ is a linear map $u : K \to \mathbb{R}$ such that $u(k) \in \mathbb{R}^+$ for all $k \in K^+$. $K^\frown = L_b(K, \mathbb{R})$, the *fuzzy order dual* of K is a vector space of all fuzzy order bounded linear functionals on K. Also $(K^\frown)^+$ is the set of all fuzzy order bounded positive linear functionals. By Theorem 4.0.4, $K^\frown$ is a fuzzy Dedekind complete Riesz space. Also, according to Theorem 4.0.4, the following fuzzy lattice operations hold for $K^\frown$.

Proposition 4.1.1. *If $K^\frown$ is a fuzzy order dual of an FRS (K, ϕ) then $u, v \in K^\frown$ and $h \in K^+$ the following statements are true:*

(i) $u^+(h) = \sup\{u(g) : g \in K^+ \text{ and } \phi(g, h) > 1/2\}$;

(ii) $u^-(h) = \sup\{-u(g) : g \in K^+ \text{ and } \phi(g, h) > 1/2\}$;

(iii) $|u|(h) = \sup\{|u(g)| : \phi(|g|, h) > 1/2\}$;

(iv) $[u \vee v](h) = \sup\{u(g) + v(\mathrm{k}) : g, \mathrm{k} \in K^+ \text{ and } g + \mathrm{k} = h\}$;

(v) $[u \wedge v](h) = \inf\{u(g) + v(\mathrm{k}) : g, \mathrm{k} \in K^+ \text{ and } g + \mathrm{k} = h\}$.

Now we discussed the FRSs whose fuzzy order dual separates the points of the spaces.

Definition 4.1.2. *The fuzzy order dual $K^\frown$ of an FRS (K, ϕ)* separates the points *of K if for all $0 \neq k \in K^+$ there exists $0 \neq u \in (K^\frown)^+$ with $u(k) \neq 0$.*

Proposition 4.1.3. *If $K^\frown$ separates the points of an FRS (K, ϕ) then $k \in K^+$ iff $u(k) \geq 0$ holds for all $u \in (K^\frown)^+$.*

Proof. The forward implication is obvious.

Conversely, let $k \in K$ satisfies $u(k) \geq 0$ for each $u \in (K^\frown)^+$. If $u \in (K^\frown)^+$ is fixed then Theorem 4.0.7 yields that there exists some $v \in (K^\frown)^+$ such that $u(k^-) = -v(k)$. As $v(k) \geq 0$ implies that $-v(k) \leq 0$. Thus $u(k^-) = 0$. Therefore, $K^\frown$ separates the points of K , we have $k^- = 0$. Hence $k = k^+ - k^- = k^+$ i.e. $\phi(0, k^+) > 1/2$. □

Remark 4.1.4. *In addition to fuzzy order dual of an FRS, one can assume the fuzzy bands of fuzzy order (σ-fuzzy order) continuous linear functionals. Let $L_n(K,\mathbb{R})$ be the set of all fuzzy order continuous linear functionals denoted by $K_n^{\frown}$ i.e. $K_n^{\frown} := L_n(K,\mathbb{R})$. Analogously, $\sigma-$fuzzy order continuous linear functionals is denoted by $K_c^{\frown} := L_c(K,\mathbb{R})$. Where $K_n^{\frown}$ and $K_c^{\frown}$ are called* fuzzy order continuous dual *and* $\sigma-$fuzzy order continuous dual *of K, respectively. By Proposition 4.0.9 both $K_n^{\frown}$ and $K_c^{\frown}$ are fuzzy bands of $K^{\frown}$.*

Definition 4.1.5. *If $u \in K^{\frown}$ then:*

(i) the null fuzzy ideal *of u denoted N_u and defined as $N_u := \{k \in K : |u|(|k|) = 0\}$;*

(ii) the disjoint complement of null fuzzy ideal denoted $C_u = N_u^d$ is said to be fuzzy carrier *of u and is defined as $C_u := \{k \in K : k \perp N_u\}$. Note that a fuzzy carrier is indeed a fuzzy band.*

One can easily prove that the null fuzzy ideal is a fuzzy band if fuzzy order bounded linear functional is fuzzy order continuous. The following proposition shows that the two fuzzy linear functionals are disjoint iff their fuzzy carriers are disjoint.

Proposition 4.1.6. *If (K,ϕ) is an Archimedean FRS then $u,v \in K_n^{\frown}$ then following statements are equivalent:*

(i) $u \perp v$;

(ii) $C_u \subseteq N_v$

(iii) $C_v \subseteq N_u$

(iv) $C_u \perp C_v$.

Proof. Without loss, assume that positive $u,v \in K_n^{\frown}$.

(i) $\to$ *(ii)* Let positive $c \in C_u = N_u^d$ and $\epsilon \in (0,1)$. Since $u \wedge v = 0$, there exists a sequence (k_n) in K^+ satisfying $k_n \uparrow k$ and $u(k_n) + v(k - k_n) < \epsilon$ for each n.

Take $g_n = \bigwedge_{i=1}^{n} k_i$ and $g_n \downarrow 0$ in K^+. Indeed, if $\phi(g, g_n) > 1/2$ then $\mu(u(g), u(g_n)) > 1/2$ and $u(g_n) < \epsilon$ implies $u(g) = 0$. Therefore, $g \in C_n \cap N_u = \{0\}$ implies $g = 0$.

Since $v \in (K_n^{\smile})^+$, we have $v(k - g_n) \uparrow v(k)$. But

$$v(k - g_n) = v(\bigvee_{i=1}^{n}(k - k_i)),$$

$$\mu(v(\bigvee_{i=1}^{n}(k - k_i)), \sum_{i=1}^{n} v(k - k_i)) > 1/2 \textit{ and } \sum_{i=1}^{n} v(k - k_i) < \epsilon.$$

Consequently, positive $v(k) \leq \epsilon$ for each $\epsilon \in (0, 1)$. Therefore, $v(k) = 0$ and hence $C_u \subseteq N_v$.

$(ii) \to (iii)$ Since N_u is a fuzzy band, so $C_u = N_u^d$. By [26, Theorem 5.8] $C_v = N_v^d \subseteq N_v^{dd} = N_v$.

$(iii) \to (iv)$ $C_v \subseteq N_u$ and $N_u \perp C_u$, we have $C_v \subseteq N_u \perp C_u$ implies $C_v \perp C_u$.

$(iv) \to (i)$ Suppose $C_v \perp C_u$. If $k = g + h \in N_v \oplus C_v$ then

$$[u \wedge v](k) = [u \wedge v](g) + [u \wedge v](h),$$

$$\mu([u \wedge v](g) + [u \wedge v](h), v(g) + u(h)) > 1/2 \textit{ and } v(g) + u(h) = 0.$$

Therefore, by [26, Theorem 4.7(ii)] $u \wedge v = 0$ holds for fuzzy order dense ideal $N_v \oplus C_v$. Hence, $u \perp v$. $\square$

Chapter 5

Unbounded fuzzy order convergence

Here we study the generalization of fuzzy order convergence in fuzzy Riesz spaces known as unbounded fuzzy order convergence. With the help of a fuzzy weak order unit, the unbounded fuzzy order convergence is nicely characterized in FRSs. Also, we discuss the fuzzy order closeness with respect to this convergence. Furthermore, in $\sigma-$fuzzy Dedekind complete FRS, the disjoint sequences are fuzzy order convergent to zero.

Definition 5.0.1. *A net $(k_\lambda)_{\lambda\in\Lambda}$ in an FRS (K,ϕ) is said to be* unbounded fuzzy order convergent (ufo-convergent for short) *to $k \in K$ denoted $k_\lambda \xrightarrow{ufo} k$ if $|k_\lambda - k| \wedge g \xrightarrow{fo} 0$ for each $g \in K^+$.*

Note that fo-convergence implies ufo-convergence. The ufo-convergence has many nice characterizing conditions.

Proposition 5.0.2. *Let $(k_\lambda)_{\lambda\in\Lambda}$ and $(g_\gamma)_{\gamma\in\Gamma}$ be nets in an FRS (K,ϕ). Then the following statements are true:*

(i) $k_\lambda \xrightarrow{ufo} k$ iff $(k_\lambda - k) \xrightarrow{ufo} 0$;

(ii) if $k_\lambda \xrightarrow{ufo} k$ and $g_\gamma \xrightarrow{ufo} g$ then $ak_\lambda + bg_\gamma \xrightarrow{ufo} ak + bg$ for each $a, b \in \mathbb{R}$;

(iii) if $k_\lambda \xrightarrow{ufo} k$ and $k_\lambda \xrightarrow{ufo} g$ then $k = g$;

(iv) if $k_\lambda \xrightarrow{ufo} k$ *then*

(a) $(k_\lambda)^+ \xrightarrow{ufo} k^+$*;*

(b) $(k_\lambda)^- \xrightarrow{ufo} k^-$.

Furthermore, (a) and (b) imply that

$$|k_\lambda| \xrightarrow{ufo} |k|.$$

(v) If a positive net $k_\lambda \xrightarrow{ufo} k$ *and* $\phi(k_\lambda, g_\gamma) > 1/2$, $g_\gamma \xrightarrow{ufo} g$ *then* $\phi(k, g) > 1/2$.

Proof. (i) Suppose $k_\lambda \xrightarrow{ufo} k$. Then $|(k_\lambda - k) - 0| \wedge g = |k_\lambda - k| \wedge g \xrightarrow{fo} 0$ for each $g \in K^+$, hence $(k_\lambda - k) \xrightarrow{ufo} 0$. The converse can be proved analogously.

(ii) Suppose $k_\lambda \xrightarrow{ufo} k$ and $g_\gamma \xrightarrow{ufo} g$, we have

$$\phi(|(k_\lambda + g_\gamma) - (k + g)| \wedge h, (|k_\lambda - k| + |g_\gamma - g|) \wedge h) > 1/2,$$

and

$$\phi((|k_\lambda - k| + |g_\gamma - g|) \wedge h, |k_\lambda - k| \wedge h + |g_\gamma - g| \wedge h) > 1/2$$

for each λ, γ and $h \in K^+$. it follows that $k_\lambda + g_\gamma \xrightarrow{fo} k + g$. Fix $a \in \mathbb{R}$ and let $h \in K^+$. Check that $|ak_\lambda - ak| \wedge h = |a||k_\lambda - k| \wedge h$. If $|a| \leq 1$, then $\phi(|a||k_\lambda - k| \wedge h, |k_\lambda - k| \wedge h) > 1/2$ and $|k_\lambda - k| \wedge h \xrightarrow{fo} 0$. If $|a| > 1$ then $|h| \leq |a|h$ and $\phi(|a||k_\lambda - k| \wedge h, |a||k_\lambda - k| \wedge |a|h) > 1/2$ and $|a|(|k_\lambda - k| \wedge h) \xrightarrow{fo} 0$. In each case $ak_\lambda \xrightarrow{ufo} ak$.

(iii) Let $\phi(|k - g|, |k - k_\lambda| + |g - k_\lambda|) > 1/2$ for each λ. Let $h = |k - g|$. Observe that $|k - g| = |k - g| \wedge h$. Also

$$\phi(|k - g| \wedge h, |k - k_\lambda| \wedge h + |g - k_\lambda| \wedge h) > 1/2.$$

Hence, $k = g$.

(iv) Suppose $|k_\lambda - k| \xrightarrow{ufo} 0$. As $\phi(|(k_\lambda)^+ - k^+|, |k_\lambda - k|) > 1/2$ for each λ. So $|(k_\lambda)^+ - k^+| \xrightarrow{ufo} 0$. Hence, $(k_\lambda)^+ \xrightarrow{ufo} k^+$. Thus $-k_\lambda \xrightarrow{ufo} -k$ this gives that $(k_\lambda)^- \xrightarrow{ufo} k^-$. The final statement follows from $\phi(||k_\lambda| - |k||, |k_\lambda - k|) > 1/2$.

(v) By (iv) $k_\lambda = |k_\lambda| \xrightarrow{ufo} |k|$. Since $k = |k|$ by uniqueness of fuzzy order limit. As $\phi(0, g_\gamma - k_\lambda) > 1/2$, then $g_\gamma - k_\lambda \xrightarrow{ufo} g - k$, we have $\phi(k, g) > 1/2$.

□

Remark 5.0.3. *Let (K, ϕ) be a Dedekind complete FRS and $(k_\lambda)_{\lambda \in \Lambda}$ be a fuzzy order bounded net in K. Then $k_\lambda \xrightarrow{fo} k$ iff $k = \limsup_\lambda (k_\lambda) = \liminf_\lambda (k_\lambda)$. Moreover, two sequences (k_n) and (k_m) are called* disjoint *if $|k_n| \wedge |k_m| = 0$ or ($k_n \perp k_m$) holds for $m \neq n$. The ufo-convergence for disjoints sequences in σ-Dedekind complete FRS are discussed in the following proposition.*

Proposition 5.0.4.

(i) Suppose $(k_n)_{n \in N}$ is a disjoint sequence in σ-Dedekind complete FRS (K, ϕ). Then $k_n \xrightarrow{ufo} 0$ in K.

(ii) Suppose $(k_n)_{n \in N}$ is a sequence in an FRS (K, ϕ). If $k_n \xrightarrow{ufo} 0$ then $\inf_m (k_{n_m}) = 0$ for each increasing sequence (n_m) of natural numbers.

Proof.

(i) Fix $k \in K^+$. We will show that $\limsup_n (|k_n| \wedge k) = 0$. Indeed, let $g \in K^+$ such that $\phi(g, \sup_n (|k_n| \wedge k)) > 1/2$. Therefore,

$$\phi(g \wedge |k_n|, (\sup_{n+1}(|k_{n+1}| \wedge k) \wedge |k_n|) > 1/2 \text{ and } \sup_{n+1}(|k_{n+1}| \wedge |k_n| \wedge k) = 0.$$

Thus, $g \wedge |k_n| = 0$ for each $n \in \mathbb{N}$. It follows that

$$g = g \wedge \sup_{n \geq 1}(k \wedge |k_n|) = \sup_{n \geq 1}(g \wedge k \wedge |k_n|) = 0.$$

Hence, $|k_n| \wedge k \xrightarrow{fo} 0$.

(ii) Suppose $k_n \xrightarrow{ufo} 0$. Take (n_m) as an increasing sequence of $\mathbb{N}$. Clearly, $k_{n_m} \xrightarrow{ufo} 0$. Let $\phi(k, k_{n_m}) > 1/2$ for each $m \in \mathbb{N}$ and $k \in K^+$. Therefore, $k = k_{n_m} \wedge k \xrightarrow{fo} 0$, implies that $k = 0$. Hence, $\inf_m(k_{n_m}) = 0$.

□

5.1 Fuzzy weak order unit

Our next goal is to reduce the task of checking ufo-convergence at every positive vector to a single special vector known as a fuzzy weak order unit that allows us to characterize ufo-convergence nicely. For $k \in K$ the fuzzy band generated by k is known as *principal fuzzy band* and defined as $B_k = \{g \in K : |g| \wedge n|k| \uparrow |g|\}$ by Corollary [26, 5.4]. The fuzzy band generated by a non-zero positive element is discussed as follows.

Definition 5.1.1. *Let (K, ϕ) be an FRS and $0 \neq w \in K^+$ is called* fuzzy weak order unit *if w generates fuzzy band satisfying either $k \wedge nw \uparrow k : n \in \mathbb{N}$ for each $k \in K^+$ or $B_w = K$.*

Proposition 5.1.2. *Let (K, ϕ) be an Archimedean FRS. $0 \neq w \in K^+$ is a fuzzy weak order unit iff $k \perp w$ implies $k = 0$ for each $k \in K^+$.*

Proof. It follows from the definition of fuzzy weak order unit, [26, Theorems 4.7 and 5.8]. □

Now we use the Proposition 5.1.2 to prove the underlying result.

Proposition 5.1.3. *Let (K, ϕ) be a Dedekind complete FRS with a fuzzy weak order unit w. Then $k_\lambda \xrightarrow{ufo} 0$ iff $|k_\lambda| \wedge w \xrightarrow{fo} 0$.*

Proof. Suppose $k_\lambda \xrightarrow{ufo} 0$. Take any $g \in K^+$. As K is fuzzy Dedekind complete, then

$$(\limsup_\lambda(|k_\lambda| \wedge g)) \wedge w = (\limsup_\lambda(|k_\lambda| \wedge w)) \wedge g = 0 \wedge g = 0.$$

Thus, w being a fuzzy weak order unit implies that $\limsup_\lambda(|k_\lambda| \wedge g) = 0$. Hence, $|k_\lambda| \wedge g \xrightarrow{fo} 0$.

The converse follows from the Definition 5.0.1. $\square$

Now we defined and studied the properties of the fuzzy component in which ufo-convergence is nicely characterized.

Definition 5.1.4. *Let (K, ϕ) be an FRS. A vector $k \in K^+$ is said to be* fuzzy component *of w whenever $k \wedge (w - k) = 0$ for $w \in K^+$.*

Remark 5.1.5. *For $k \in K$, w_k denotes the fuzzy component of w in the fuzzy band generated by k. So for each $\alpha \in \mathbb{R}$, $w_{(k-\alpha w)^+}$ is the fuzzy component of w in the fuzzy band generated by $(k - \alpha w)^+$ and we set $e(\alpha) = w_{(k-\alpha w)^+}$. In a Dedekind complete FRS (K, ϕ) with $k \in K^+$ and let $e = w_{k^+}$ then $k_e = k^+$.*

Now, many lemmas are proved to characterize the ufo-convergence with the help of fuzzy components.

Lemma 5.1.6. *If (K, ϕ) be a Dedekind complete FRS for $k \in K^+$ then*

$$\phi(e(\alpha), \frac{1}{\alpha}k) > 1/2$$

for $\alpha > 0$.

Proof. Remark 5.1.5 yields that $(k - \alpha w)_{e(\alpha)} = (k - \alpha w)^+$ and $\phi(0, (k - \alpha w)^+) > 1/2$. Therefore, $(k - \alpha w)_{e(\alpha)} = k_{e(\alpha)} - \alpha w_{e(\alpha)}$ and $\phi(0, k - \alpha e(\alpha)) > 1/2$ implies $\phi(e(\alpha), \frac{1}{\alpha}k) > 1/2$. $\square$

Lemma 5.1.7. *Let (K, ϕ) be a Dedekind complete FRS and (k_λ) a net in K^+. Then $\wedge_\lambda w_{k_\lambda} = 0$ implies $\wedge_\lambda k_\lambda = 0$. But converse is not true.*

Proof. For each λ, $\phi(k_\lambda \wedge w, w_{k_\lambda}) > 1/2$, so $(\wedge_\lambda k_\lambda) \wedge w = \wedge_\lambda(k_\lambda \wedge w)$ and $\phi(\wedge_\lambda(k_\lambda \wedge w), \wedge_\lambda w_{k_\lambda}) > 1/2$. Thus $(\wedge_\lambda k_\lambda) \wedge w = 0$. Hence $\wedge_\lambda k_\lambda = 0$. To see that the converse is false, take a set $k_n = \frac{1}{n}w$ for all $n \in \mathbb{N}$. $\square$

Lemma 5.1.8. *Let (K, ϕ) be a Dedekind complete FRS and (k_λ) a net in K^+. Then $\wedge_\lambda w_{(k_\lambda - \alpha w)^+} = 0$ for each $\alpha > 0$ iff $\wedge_\lambda k_\lambda = 0$.*

Proof. For the forward implication we show that $\phi(\wedge_\lambda k_\lambda, \alpha w) > 1/2$ for each $\alpha > 0$. Fix α. By Lemma 5.1.7 $\wedge_\lambda (k_\lambda - \alpha w)^+ = 0$. So $\phi(\wedge_\lambda (k_\lambda - \alpha w), 0) > 1/2$ that implies $\phi(\wedge_\lambda k_\lambda, \alpha w) > 1/2$.

The converse follows from the Lemma 5.1.6. □

Lemma 5.1.9. *Let (K, ϕ) be a Dedekind complete FRS and (k_λ) a net in K^+. Then $w_{k_\lambda} \xrightarrow{fo} 0$ implies $k_\lambda \xrightarrow{ufo} 0$. The converse is not true.*

Proof. The proof is essentially the same as for Lemma 5.1.7, using Lemma 5.1.3.

□

Now the characterization of ufo-convergence is established in the next theorem.

Theorem 5.1.10. *Suppose (K, ϕ) is a Dedekind complete FRS and (k_λ) a net in K^+. Then $w_{(k_\lambda - \alpha w)} \xrightarrow{fo} 0$ for each $\alpha > 0$ iff $k_\lambda \xrightarrow{ufo} 0$.*

Proof. For the forward implication, suppose the net (k_λ) is fuzzy order bounded. We show that

$$\phi(\limsup_\lambda k_\lambda, \alpha w) > 1/2 \ \forall \ \alpha > 0.$$

Fix α. By Lemma 5.1.9 $(k_\lambda - \alpha w)^+ \xrightarrow{fo} 0$. In particular, $\limsup_\lambda (k_\lambda - \alpha w)^+ = 0$, thus $\phi(\limsup_\lambda (k_\lambda - \alpha w), 0) > 1/2$ such that $\phi(\limsup_\lambda k_\lambda, \alpha w) > 1/2$.

Now drop the supposition that (k_λ) is fuzzy order bounded. For every $\alpha > 0$,

$$\phi(w_{(k_\lambda \wedge w - \alpha w)^+}, w_{(k_\lambda - \alpha w)^+}) > 1/2 \text{ and } w_{(k_\lambda - \alpha w)^+} \xrightarrow{fo} 0.$$

Since $k_\lambda \wedge w$ is fuzzy order bounded, then $k_\lambda \wedge w \xrightarrow{fo} 0$.

The backward implication is followed by the Lemma 5.1.6. □

5.1.1 Fuzzy ideals and completeness with respect to ufo-convergence

The fuzzy ideal is a useful structure with important properties that can help to study ufo-convergence. To work in the fuzzy ideal is much easier than to work in the whole space. Indeed it is shown that ufo-convergence in the fuzzy ideal is equivalent to ufo-convergence in the entire space.

Remark 5.1.11. *Let (K,ϕ) be an FRS, I a fuzzy ideal of K and $(k_\lambda) \subset I$. If $k_\lambda \xrightarrow{fo} 0$ in I, then $k_\lambda \xrightarrow{fo} 0$ in K. Conversely, If (k_λ) is fuzzy order bounded in I and $k_\lambda \xrightarrow{fo} 0$ in K, then $k_\lambda \xrightarrow{fo} 0$ in I.*

Proposition 5.1.12. *Let (K,ϕ) be a Dedekind complete FRS and (k_λ) a net in fuzzy ideal I of K. Then $k_\lambda \xrightarrow{ufo} 0$ in I iff $k_\lambda \xrightarrow{ufo} 0$ in K.*

Proof. Suppose $k_\lambda \xrightarrow{ufo} 0$ in K. Then for any $g \in I^+$ such that $|k_\lambda| \wedge g \xrightarrow{fo} 0$ in K, Remark 5.1.11 yields $|k_\lambda| \wedge g \xrightarrow{fo} 0$ in I. Hence $k_\lambda \xrightarrow{ufo} 0$ in I. Conversely, take any $g \in I^+$, then $|k_\lambda| \wedge g \xrightarrow{fo} 0$ in I, again by Remark 5.1.11 $|k_\lambda| \wedge g \xrightarrow{fo} 0$ in K. It follows that, for any $g \in I^+$ and positive $h \in I^d$ such that $|k_\lambda| \wedge (g+h) \xrightarrow{fo} 0 = |k_\lambda| \wedge g \xrightarrow{fo} 0$ in K.

For any $u \in K^+$ and $z \in (I \oplus I^d)^+$, we have $u \wedge z \in (I \oplus I^d)^+$. Therefore, by Remark 5.1.11 $|k_\lambda| \wedge (u+z) \xrightarrow{fo} 0$ in K, or equivalently,

$$\lim_\lambda \sup(|k_\lambda| \wedge u) \wedge z = \lim_\lambda \sup(|k_\lambda| \wedge (u \wedge z)) = 0.$$

[26, Theorem 4.7 (i)] yields that $(I \oplus I^d)^d = \{0\}$. Thus,

$$\lim_\lambda \sup(|k_\lambda| \wedge u) = 0.$$

Hence, $|k_\lambda| \wedge u \xrightarrow{fo} 0$ in K. □

The closeness of ufo-convergence is defined and discussed as follows.

Definition 5.1.13. *Let (K, ϕ) be an FRS. For $C \subset K$ is said to be* unbounded fuzzy order closed (ufo-closed for short)*, if for any net $(k_\lambda) \subset C$ and $k \in K$ with $k_\lambda \xrightarrow{ufo} k$ in K implies $k \in C$.*

Proposition 5.1.14. *Let L be a fuzzy Riesz subspace of FRS (K, ϕ). Then L is ufo-closed in K iff L is fo-closed in K.*

Proof. The forward implication is straightforward.

Conversely, suppose L is fo-closed in K. Let $(g_\lambda) \subseteq L$ and $k \in K$ such that $g_\lambda \xrightarrow{ufo} k$ in K. By Lemma 5.0.2(iv) $|g_\lambda| \xrightarrow{ufo} |k|$ in K. Therefore, without loss generality, consider $(g_\lambda) \subseteq L^+$ and $k \in K^+$. Observe that for each $w \in K^+$, then

$$\phi(|g_\lambda \wedge w - \mathrm{k} \wedge w|, |g_\lambda - k| \wedge \mathrm{w}) > 1/2 \textit{ and } |g_\lambda - k| \wedge \mathrm{w} \xrightarrow{fo} 0 \text{ in } K. \tag{5.1.1}$$

Consequently, for any $g \in L^+$, $g_\lambda \wedge g \xrightarrow{fo} k \wedge g$ in K. As L is fo-closed, then $k \wedge g \in L$. On the other hand, for any $w \in (L^d)^+$, then $g_\lambda \wedge w = 0$ for each λ, so that by ((i)) $k \wedge w = 0$. Thus, $k \in L^{dd}$, which is fuzzy band generated by L in K.

It follows that there is a net (w_γ) in the fuzzy ideal generated by L^+ such that $w_\gamma \uparrow k$ in K. Moreover, for each γ there exists $z_\gamma \in L$ such that $\phi(w_\gamma, z_\gamma) > 1/2$. So

$$\phi(w_\gamma \wedge k, z_\gamma \wedge k) > 1/2 \textit{ and } \phi(z_\gamma \wedge k, k) > 1/2$$

implies that $w_\gamma \uparrow k$ in K. Therefore, $z_\gamma \wedge k \xrightarrow{fo} k$ in K. Hence, $z_\gamma \wedge k \in L$ and L is fo-closed then $k \in L$. □

Chapter 6

Fuzzy normed Riesz space

In the current chapter, we study the fuzzy norm in view of fuzzy ordering and defined fuzzy normed Riesz space. Later on, we investigate the theory of locally convex-solid fuzzy Riesz space. Toward the end of this chapter, we define unbounded fuzzy norm convergence in fuzzy Banach lattices.

Definition 6.0.1. *A fuzzy norm N on an FRS (K, ϕ) is called* fuzzy Riesz norm *if $\phi(|k|, |g|) > 1/2$ implies $N(k,t) \geq N(g,t)$ for each $k, g \in K$ and $0 < t \in \mathbb{R}$. If N is a fuzzy Riesz norm on K then (K, N, ϕ) is said to be* fuzzy normed Riesz space (FNRS)*. A norm complete fuzzy normed Riesz space is said to be* fuzzy Banach lattice*.*

The following result shows that a closed unit ball in a fuzzy Riesz norm is fuzzy solid.

Proposition 6.0.2. *Let a fuzzy norm on an FRS (K, ϕ) is a fuzzy Riesz norm iff its closed unit ball with radius r*

$$B_N = \{k \in K,\ N(k,t) \geq 1 - r\ \forall\ 0 < t \in \mathbb{R}\}$$

is a fuzzy solid subset.

Proof. If N is an FNRS then clearly B_N is a fuzzy solid subset.

Conversely, if B_N is a fuzzy solid and $\phi(|k|,|g|) > 1/2$ in K then

$$\phi(|\frac{1}{N(k,t)+\epsilon}k|,|\frac{1}{N(k,t)+\epsilon}g|) > 1/2$$

for all $\epsilon \in (0,1)$. As $\frac{1}{N(k,t)+\epsilon}g \in B_N$ implies $\frac{1}{N(k,t)+\epsilon}k \in B_N$. Therefore, $N(g,t) \leq N(k,t)+\epsilon$ for all ϵ. Hence $N(g,t) \leq N(k,t)$. □

It is obvious in an FNRS (K,N,ϕ), $N(|k|,t) = N(k,t)$ for each $k \in K$ and $0 < t \in \mathbb{R}$. A few properties of FNRS are discussed in the underlying proposition.

Proposition 6.0.3. *If (K,N,ϕ) is an FNRS then following statements are true:*

(i) K is an Archimedean FRS;

(ii) the fuzzy lattice operations $(k,g) \mapsto k \wedge g$, $(k,g) \mapsto k \vee g$, $k \mapsto k^-$, $k \mapsto k^+$, and $k \mapsto |k|$ are fuzzy continuous from $K \times K$ (or from K, resp.) into K;

(iii) the positive cone is fuzzy norm closed;

(iv) the closure of a fuzzy ideal is a fuzzy ideal;

(v) the closure of a fuzzy Riesz subspace is a fuzzy Riesz subspace;

(vi) every fuzzy band is closed.

Proof. (i) For $k,g \in K$ and $\phi(nk,g) > 1/2$ for each $n \in \mathbb{N}$, it follows that $\phi(nk^+,g^+) > 1/2$ and $N(g^+,t) \leq N(nk^+,t)$ for each $0 < t \in \mathbb{R}$. Thus, $k^+ = 0$. Hence, $\phi(k,0) > 1/2$.

(ii) Suppose $k_\lambda \xrightarrow{fn} k$ in K. Then for each $0 < t \in \mathbb{R}$

$$N(|k_\lambda| - |k|,t) \geq N(|k_\lambda - k|,t) = N(k_\lambda - k,t) = 1.$$

So, $|k_\lambda| \xrightarrow{fn} |k|$. Hence the modulus operation is fuzzy continuous. Since the modulus can express all fuzzy lattice operations, so these lattice operations are fuzzy continuous as well.

(iii) Let $K^+ = \{k, k^- = 0\}$ be the fuzzy positive cone. Therefore, it is the inverse image of fuzzy closed set $\{0\}$ with respect to fuzzy continuous map $k \mapsto k^-$.

(iv) Suppose C is a fuzzy ideal of K. Then $\phi(|k|, |g|) > 1/2$ we have $g \in \overline{C}$. Take a sequence $(g_n)_{n\in\mathbb{N}} \subseteq C$ with $g_n \xrightarrow{fn} g$. Define $k_n^+ := k^+ \wedge |g_n|$ and $k_n^- := k^- \wedge |g_n|$ for each n, we have $k_n = k_n^+ - k_n^- \in C$. Clearly, $\phi(k_n, g_n) > 1/2$ for each n. Since C is a fuzzy ideal, we have $k_n \xrightarrow{fn} k$. By (ii), hence $k \in \overline{C}$.

(v) It is the immediate consequences of (ii) and (iv).

(vi) Let $k_n \uparrow k$ in K i.e. $k = \sup_n k_n$. Suppose C is a σ-fuzzy ideal in K and $(h_n) \subseteq C$ such that $h_n \xrightarrow{fn} h$ in K. Let $g_n := |h_n| \wedge |h|$ we have $(g_n) \subseteq C$. By (ii) (g_n) fn-converges to $|h|$. Defining $k_n := \sup_n g_n$ in C and satisfying $\phi(g_n, k_n) > 1/2$ and $\phi(k_n, |h|) > 1/2$. Thus, we have for each $0 < t \in \mathbb{R}$

$$N(k_n - |h|, t) \geq N(g_n - |h|, t)$$

which shows that (k_n) fn-convergent to $|h|$ in K. But $k_n \uparrow |h|$. Since C is a σ-fuzzy ideal, we have $|h| \in C$ implies that $h \in C$. Hence C is closed.

□

Lemma 6.0.4. *Let (K, N, ϕ) be an FNRS. If a net $k_\lambda \uparrow k$ and $\lim_\lambda k_\lambda = k$ then $k = \sup_\lambda k_\lambda$.*

Proof. For fixed γ, take $\gamma \leq \lambda$, we have $\phi(k_\gamma, k_\lambda) > 1/2$. Since K^+ is fuzzy closed, it follows that $\phi(k_\gamma, k) > 1/2$. Thus k is an upper bound of (k_λ). If $g \in K$ is another upper bound, then $\phi(k_\gamma, g) > 1/2$ implies $\phi(k, g) > 1/2$. Hence $k = \sup_\lambda k_\lambda$. □

An important result about continuity of fuzzy positive operators between fuzzy Banach lattice is given as follow.

Proposition 6.0.5. *If P is a fuzzy positive operator between fuzzy Banach lattice (K, N_1, ϕ) to FNRS (H, N_2, μ) then it is fuzzy continuous.*

Proof. Let $P : K \to H$ is a fuzzy positive operator. Suppose on contrary that, P is not fuzzy continuous then it must be fuzzy unbounded. Therefore, there exists a sequence $(k_n)_{n\in\mathbb{N}}$ in K such that $k_n \downarrow 0$ satisfying $N_1(k_n, t) = 1$ and $N_2(P(k_n), t) \leq M$ for each $0 < t \in \mathbb{R}$ and $M \in (0, 1)$. Since K is fuzzy norm complete then $g := \sum_n k_n$ exists in K. Clearly, $\phi(k_n, g) > 1/2$ for each n. Thus, $\mu(P(k_n), P(g)) > 1/2$ and so

$$N_2(P(g), t) \leq N_2(P(k_n), t) \leq M$$

for each n, a contradiction. □

6.0.1 Fuzzy order continuous Banach lattice

A fuzzy Riesz norm on an FRS (K, ϕ) is known as *fuzzy order continuous* if $k_\lambda \downarrow 0$ implies $k_\lambda \xrightarrow{fn} 0$ and *σ-fuzzy order continuous* if $(k_n)_{n\in\mathbb{N}} \downarrow 0$ implies $(k_n)_{n\in\mathbb{N}} \xrightarrow{fn} 0$. Next lemma is conducive to characterized the fuzzy order continuous Banach lattice (K, N, ϕ).

Lemma 6.0.6. *Let (K, ϕ) be an Archimedean FRS. If a net $k_\lambda \uparrow k$ in K, then the set $C = \{g \in K, \phi(k_\lambda, g) > 1/2 \ for\ each\ \lambda\}$ is directed downward and $g - k_\lambda \downarrow 0$.*

Proof. Clearly, C is directed downward. Let $\phi(z, g - k_\lambda) > 1/2$ holds for each λ and $g \in C$. Then $\phi(k_\lambda, g - z) > 1/2$ for all λ. Therefore, $g - z \in C$. By induction, $g - nz \in C$ for each n. In particular, $\phi(nz, k) > 1/2$ for each n. Since K is Archimedean FRS then $z = 0$. Hence $g - k_\lambda \downarrow 0$. □

A characterization of fuzzy Banach lattice (K, N, ϕ) with fuzzy order continuous norms are given in the following results.

Theorem 6.0.7. *If (K, N, ϕ) is a fuzzy Banach lattice then underlying statements are equivalent:*

(i) K has fuzzy order continuous norm;

(ii) if $k_n \uparrow k$ in K^+ then (k_n) is a fuzzy norm cauchy sequence;

(iii) K is a σ-fuzzy Dedekind complete and $k_n \downarrow 0$ in K implies $N(k_n, t) \downarrow 0$ for $t > 0$.

Proof. (i) $\Rightarrow$ (ii) Let $k_\lambda \uparrow k$ in K^+. Take $\alpha \in (0,1)$ and by continuity of $\star$, we can find $r \in (0,1)$ such that $(1-r) \star (1-r) > 1-\alpha$. By Lemma 6.0.6 there exists a net $(g_\gamma) \subseteq K$ with $g_\gamma - k_\lambda \downarrow 0$. Therefore, there exist λ_0 and γ_0 such that $N(g_\gamma - k_\lambda, t) > 1-\alpha$ for each $\lambda \geq \lambda_0$ and $\gamma \geq \gamma_0$. The inequality

$$N(k_\beta - k_\lambda, t+s) \geq N(k_\beta - g_{\lambda_0}, t) \star N(k_\lambda - g_{\lambda_0}, s)$$
$$> (1-r) \star (1-r) > 1-\alpha$$

for each $\lambda, \beta \geq \lambda_0$ and $t, s > 0$. Hence (k_λ) is fuzzy norm cauchy net.

(ii) $\Rightarrow$ (i) Suppose (k_n) is a decreasing sequence in K^+. Let (k_n) be a fuzzy norm cauchy sequence. Take $\alpha \in (0,1)$ and by continuity of $\star$, we can discover $r \in (0,1)$ such that $(1-r) \star (1-r) \star (1-r) > 1-\alpha$. Let some $g_n \in K^+$ with $N(k_n - g_n, t) > 1 - r/2^n$ for $t > 0$. Put $u_n = \bigwedge_{i=1}^n g_i$ and note (u_n) is a decreasing sequence in K^+. By our assumption, there exists some n_0 with $N(u_n - u_m, t) > 1-r$ for each $n, m \geq n_0$. So,

$$k_n - u_n = \bigvee_{i=1}^n (k_n - g_i)$$

such that

$$\phi(\bigvee_{i=1}^n (k_n - g_i), \bigvee_{i=1}^n (k_i - g_i)) > 1/2 \text{ and } \phi(\bigvee_{i=1}^n (k_i - g_i), \sum_{i=1}^n |k_i - g_i|) > 1/2.$$

Now

$$-(k_n - u_n) = \bigwedge_{i=1}^n (g_i - k_n)$$

such that

$$\phi(\bigwedge_{i=1}^{n}(g_i - k_n), g_n - k_n) > 1/2 \text{ and } \phi(g_n - k_n, \sum_{i=1}^{n}|k_i - g_i|) > 1/2.$$

We have $\phi(|k_n - u_n|, \sum_{i=1}^{n}|k_i - g_i|) > 1/2$. Thus,

$$N(k_n - u_n, t) \geq \sum_{i=1}^{n} N(k_i - g_i, s) > 1 - r/2^i > 1 - r \text{ for each } n.$$

In particular, for $m, n \geq n_0$ and $t, s, p > 0$ we have,

$$\begin{aligned} N(k_n - k_m, t + s + p) &\geq N(k_n - u_n, t) \star N(u_m - u_n, s) \star N(u_m - k_m, p) \\ &> (1 - \mathrm{r}) \star (1 - \mathrm{r}) \star (1 - \mathrm{r}) > 1 - \alpha. \end{aligned}$$

(ii) $\Rightarrow$ (iii) is straight forward.

(iii) $\Rightarrow$ (i) Let $k_\lambda \downarrow 0$. Suppose the net (k_λ) is not fuzzy norm cauchy, therefore, there exists some $\alpha \in (0, 1)$ and an increasing sequence (λ_n) such that $N(k_{\lambda_n} - k_{\lambda_{n+1}}, t) < 1 - \alpha$ for each n. As K is σ-fuzzy Dedekind complete, there exists some $k \in K$ with $k_{\lambda_n} \downarrow k$. So, k_{λ_n} is fuzzy norm cauchy sequence. A contradiction, thus, $N(k_\lambda, t) \downarrow 0$. □

6.1 Locally convex-solid fuzzy Riesz space

With the induced topology τ of fuzzy norms (K, τ) is a topological vector space. Furthermore, a topological vector space is said to be *locally convex* if it has a local base at zero consisting of convex sets. We aim to study the relationship between the fuzzy lattice structure of K under the fuzzy order ϕ and topological structure of K.

In order to construct a relation between topological dual denoted $K^{'}$ (set of all continuous linear functionals with respect to topology τ) and fuzzy order dual, we adopt a general approach by considering a locally convex topology τ on FRS (K, ϕ) generated by a family of complete fuzzy Riesz norms on K. We call the triple (K, ϕ, τ) *locally convex-solid fuzzy Riesz space(LCSFRS)*. Every locally convex-solid topology on a fuzzy Riesz space makes fuzzy lattice operations continuous functions.

Proposition 6.1.1. *If (K, ϕ, τ) is a LCSFRS then following statements are true:*

(i) $k \mapsto k^+$ *from K to K is continuous;*

(ii) $k \mapsto k^-$ *from K to K is continuous;*

(iii) $k \mapsto |k|$ *from K to K is continuous;*

(iv) $(k, g) \mapsto k \vee g$ *from $K \times K$ to K is continuous;*

(v) $(k, g) \mapsto k \wedge g$ *from $K \times K$ to K is continuous.*

Proof. It is an immediate consequence of [10, Theorem 4.11]. □

The next result shows some essential characterization of LCSFRSs.

Proposition 6.1.2. *If (K, ϕ, τ) is a LCSFRS then following statements are true:*

(i) the FRS K is a fuzzy Archimedean;

(ii) the fuzzy positive cone K^+ is a $\tau-$ closed;

(iii) $\tau-$ closure of fuzzy solid subset C of K is also a fuzzy solid;

(iv) $\tau-$ closure of fuzzy Riesz subspace of an FRS is a fuzzy Riesz subspace;

(v) every fuzzy band is a $\tau-$ closed;

(vi) if $k_\lambda \xrightarrow{\tau} k$ in K then $k_\lambda \uparrow k$ in K;

(vii) if two nets (k_λ) and (g_λ) satisfied $\phi(k_\lambda, g_\lambda) > 1/2$ and $k_\lambda - g_\lambda \xrightarrow{\tau} 0$ then $k_\lambda \downarrow k$ iff $g_\lambda \downarrow k$.

Proof. (i) Let $\phi(nk, g) > 1/2$ for each $n \in \mathbb{N}$ and $k, g \in K^+$. As $\phi(k, \frac{1}{n}g) > 1/2$ and $\frac{1}{n}g \xrightarrow{\tau} 0$ implies that $k = 0$.

(ii) By Proposition 6.1.1 $k \mapsto k^-$ is continuous. Therefore, $K^+ = \{k \in K : k^- = 0\}$. Hence K^+ is $\tau - closed$.

(iii) Let C be a fuzzy solid subset of K. Then $\phi(|k|, |g|) > 1/2$ we have $g \in \overline{C}$. Take a net $(g_\lambda) \subseteq C$ with $g_\lambda \xrightarrow{\tau} g$. Define $k_\lambda := (k \wedge g_\lambda) \vee (| - g_\lambda|)$ for each λ. Clearly, $\phi(k_\lambda, g_\lambda) > 1/2$ for each λ. Therefore, $(k_\lambda) \subseteq C$ such that $k_\lambda \xrightarrow{\tau} k$. Hence, $k \in \overline{C}$.

(iv) Suppose L is a fuzzy Riesz subspace of K. Clearly, $\overline{L}$ is a vector subspace of K. If $g \in \overline{G}$ therefore, there exists a net (g_λ) in L such that $g_\lambda \xrightarrow{\tau} g$. By Proposition 6.1.1 $(g_\lambda)^+ \subseteq L$ such that $g_\lambda^+ \xrightarrow{\tau} g^+$. Hence $g^+ \in \overline{L}$.

(v) Take a $\emptyset \neq C \subseteq K$ with disjoint complement $C^d = \{k \in K, |k| \wedge |g| = 0, \forall\ g \in C\}$ is $\tau - closed$. Indeed, if $k \in \overline{C^d}$ then $k_\lambda \xrightarrow{\tau} k$ for some $(k_\lambda) \subseteq C$. Thus, $0 = |g| \wedge |k_\lambda| \xrightarrow{\tau} |g| \wedge |k|$ for each $g \in C$. Since $|k| \wedge |g| = 0$ implies that $k \in C^d$. Hence, C^d is $\tau - closed$.

Since K fuzzy Archimedean then by [26, Theorem 5.8] every fuzzy band satisfies $C = C^{dd}$. Hence, C is $\tau - closed$.

(vi) For fixed λ, $\gamma \succeq \lambda$ we have $\phi(k \vee k_\lambda - k, k \vee k_\gamma - k) > 1/2$ and $\phi(k_\gamma - k, |k_\gamma - k|) > 1/2$. Thus, $|k_\gamma - k| \xrightarrow{\tau} 0$. Therefore, $k \vee k_\lambda - k = 0$ i.e. $\phi(k_\lambda, k) > 1/2$ for each λ.

Suppose there exists some $g \in K$ such that $\phi(k_\lambda, g) > 1/2$ for each λ. Thus, $g - k_\lambda \xrightarrow{\tau} g - k$ implies that $g - k \in K^+$ i.e. $\phi(k, g) > 1/2$. Hence, $k_\lambda \uparrow k$ in K.

(vii) Suppose $k_\lambda \downarrow k$ and $\phi(k, g) > 1/2$, $\phi(g, g_\lambda) > 1/2$ for each λ. Then $\phi((g - k_\lambda)^+, g_\lambda - k_\lambda) > 1/2$ for each λ. Thus, $(g - k_\lambda)^+ \xrightarrow{\tau} 0$. Therefore, $(g - k_\lambda)^+ \uparrow (g - k)^+ = g - k$. By (vi) we have $g - k = 0$ implies that $g_\lambda \downarrow k$.

Conversely, suppose that $g_\lambda \downarrow k$ in K. Then $\phi(k - k \wedge k_\lambda, g_\lambda - k_\lambda) > 1/2$, therefore, $k - k \wedge k_\lambda \xrightarrow{\tau} 0$. So $k - k \wedge k_\lambda \uparrow$ by (vi) $k - k \wedge k_\lambda \uparrow 0$ implies that $k - k \wedge k_\lambda = 0$ for each λ, thus $\phi(k, k_\lambda) > 1/2$. Hence, $k_\lambda \downarrow k$ in K.

□

Theorem 6.1.3. *If (K, ϕ, τ) is an LCSFRS, then the topological dual $K^{'}$ (is fuzzy Dedekind complete in its own right) is a fuzzy ideal of the fuzzy order dual $K^{\backsim}$. Moreover, if for each $u, v \in K^{'}$ and $h \in K^+$ then*

$$[u \vee v](h) = \sup\{u(k) + v(g) : k, g \in K^+ \text{ and } k + g = h\}$$

and

$$[u \wedge v](h) = \inf\{u(k) + v(g) : k, g \in K^+ \text{ and } k + g = h\}.$$

Proof. We show that $K^{'}$ is a vector subspace of $K^{\backsim}$. Suppose on contrary that some $u \in K^{'}$ and u does not exist in $K^{\backsim}$ there exist some $k \in K^+$ and a sequence $(k_n)_{n\in\mathbb{N}} \subseteq [0, k]$ satisfying $u(k_n) \geq n$ for each n. Now $\phi(\frac{1}{n}k_n, \frac{1}{n}k) > 1/2$ and $\frac{1}{n}k \xrightarrow{f\tau} 0$. Therefore, $\lim u(\frac{1}{n}k_n) = 0$, a contradiction.

Now we show that $K^{'}$ is a fuzzy ideal of $K^{\backsim}$. Suppose $\psi(|u|, |v|) > 1/2$ in $K^{\backsim}$ with $v \in K^{'}$. Let $k_\lambda \xrightarrow{f\tau} 0$ and $\epsilon \in (0, 1)$. The Theorem 4.0.4 yields that there exists a net (g_λ) with $\phi(g_\lambda, k_\lambda) > 1/2$ and $|v|(|k_\lambda|) \leq |v(g_\lambda)| + \epsilon$ for each λ. Clearly, $g_\lambda \xrightarrow{f\tau} 0$. Thus

$$\psi(|u(k_\lambda)|, |u|(|k_\lambda|)) > 1/2 \text{ and } |u|(|k_\lambda|) \leq |v|(|k_\lambda|).$$

So $|u(k_\lambda)| \leq |v(g_\lambda)| + \epsilon$ then $\limsup |u(k_\lambda)| \leq \epsilon$ for each $\epsilon \in (0, 1)$. Therefore, $\limsup |u(k_\lambda)| = 0$ and $\lim u(k_\lambda) = 0$. Hence $u \in K^{'}$. The fuzzy lattice operations are acquired from Theorem 4.0.4. □

6.2 Unbounded fuzzy norm convergence

This section aims to define and study unbounded fuzzy norm convergence in fuzzy Banach lattice, which is closely related to unbounded fuzzy order convergence. Some other theoretical concepts like fuzzy quasi interior-point and disjoint sequences are studied in relation to unbounded fuzzy norm convergence. Moreover, we define a topology that is compatible with unbounded fuzzy norm convergence.

Definition 6.2.1. *A net $(k_\lambda)_{\lambda\in\Lambda}$ in a fuzzy Banach lattice (K, N, ϕ) is known as* unbounded fuzzy norm convergent (ufn-convergent for short) *to $k \in K$ denoted $k_\lambda \xrightarrow{ufn} k$ if $|k_\lambda - k| \wedge g \xrightarrow{fn} 0$ for each $g \in K^+$ and read as k_λ ufn-convergent to k.*

Note that fn-convergence implies ufn-convergence. The following result provides some basic properties of ufn-convergence.

Lemma 6.2.2. *Let $(k_\lambda)_{\lambda\in\Lambda}$ and $(g_\gamma)_{\gamma\in\Gamma}$ be nets in a fuzzy Banach lattice (K, N, ϕ). Then following statements are true:*

(i) $k_\lambda \xrightarrow{ufn} k$ iff $(k_\lambda - k) \xrightarrow{ufn} 0$;

(ii) if $k_\lambda \xrightarrow{ufn} k$ and $g_\gamma \xrightarrow{ufn} g$ then $ak_\lambda + bg_\gamma \xrightarrow{ufn} ak + bg$ for each $a, b \in \mathbb{R}$;

(iii) if $k_\lambda \xrightarrow{ufn} k$, then $|k_\lambda| \xrightarrow{ufn} |k|$;

(iv) if $k_\lambda \xrightarrow{ufn} k$ and $k_\lambda \xrightarrow{ufn} g$, then $k = g$.

(v) if $k_n \xrightarrow{ufn} k$ and $g_n \xrightarrow{ufn} g$, then $k_n \vee g_n \xrightarrow{ufn} k \vee g$ and $k_n \wedge g_n \xrightarrow{ufn} k \wedge g$.

(vi) if $k_n \xrightarrow{ufn} k$, then $k_{n_p} \xrightarrow{ufn} k$ for any subsequence (k_{n_p}) of (k_n).

Proof. The proof of (i) and (ii) are obvious.

(iii) It follows from $\phi(||k_\lambda| - |k||, |k_\lambda - k|) > 1/2$.

(iv) Let $\phi(|k-g|, |k-k_\lambda|+|g-k_\lambda|) > 1/2$ for each λ. Let $h = |k-g|$. Thus, $|k-g| = |k-g| \wedge h$. Therefore,

$$\phi(|k - g| \wedge h, |k - k_\lambda| \wedge h + |g - k_\lambda| \wedge h) > 1/2.$$

Hence, $k = g$.

(v) It follows from

$$\phi(|k_n \vee g_n - k \vee g|, |k \vee g_n - k \vee g| + |k_n \vee g_n - k \vee g_n|) > 1/2$$

and

$$\phi(|k \vee g_n - k \vee g| + |k_n \vee g_n - k \vee g_n|, |k_n - k| + |g_n - g|) > 1/2.$$

Analogously, $k_n \wedge g_n \xrightarrow{ufn} k \wedge g$.

(vi) Assume that $k_n \xrightarrow{ufn} k$. Let (k_{n_p}) be subsequence of (k_n). For each $\alpha \in (0,1)$ and $z \in K^+$ there is an $n_0 \in \mathbb{N}$ such that $N((k_n - k) \wedge z, t) > 1 - \alpha$ for $t > 0$ whenever $n \geq n_0$. Then $n_0 \leq n_p \leq p$ we have $N((k_{n_p} - k) \wedge z, t) > 1 - \alpha$. Hence $k_{n_p} \xrightarrow{ufn} k$.

□

Note that if a net is fuzzy order bounded in K then ufn-convergence implies fn-convergence.

Lemma 6.2.3. *Let (k_λ) be a fuzzy order bounded net in fuzzy Banach lattice (K, N, ϕ). Then $k_\lambda \xrightarrow{fn} 0$ iff $k_\lambda \xrightarrow{ufn} 0$.*

Proof. Without loss, suppose the net (k_λ) in K^+. Then for each $w \in K^+$, $\phi(k_\lambda \wedge w, k_\lambda) > 1/2$ and $k_\lambda \xrightarrow{ufn} 0$. Conversely, suppose $k_\lambda \xrightarrow{ufn} 0$ there exists some $z \in K^+$ such that $\phi(k_\lambda, z) > 1/2$ for each λ. Thus $k_\lambda = k_\lambda \wedge z \xrightarrow{fn} 0$. □

The following result shows that ufo-convergence implies ufn-convergence when fuzzy Banach lattice is fuzzy order continuous.

Lemma 6.2.4. *If (K, N, ϕ) is a fuzzy order continuous Banach lattice then ufo-convergence implies ufn-convergence.*

Proof. fo-convergence implies fn-convergence in fuzzy order continuous spaces. So if $k_\lambda \xrightarrow{ufo} 0$ then we have $k_\lambda \wedge z \xrightarrow{fo} 0$ for each $z \in K^+$. Hence fuzzy order continuity gives $k_\lambda \wedge z \xrightarrow{fn} 0$. □

To reduce the checking work of ufn-convergence to each positive vector to a single special vector is known as a fuzzy quasi-interior point as in [28, Proposition 5], which is discussed as follows.

Definition 6.2.5. *A non-zero positive q in a fuzzy Banach lattice (K, N, ϕ) is known as* fuzzy quasi-interior point *if $k \wedge nq \xrightarrow{fn} k$ for each $k \in K^+$.*

Theorem 6.2.6. *Let (K, N, ϕ) be a fuzzy Banach lattice with fuzzy quasi-interior point q. Then $k_\lambda \xrightarrow{ufn} 0$ if and only if $|k_\lambda| \wedge q \xrightarrow{fn} 0$.*

Proof. As $k_\lambda \wedge q \xrightarrow{fn} k$ implies $k_\lambda \xrightarrow{ufn} 0$. Conversely, let $z \in K^+$, $t > 0$. Take $\alpha \in (0,1)$ and by continuity of $\star$, we can find $r \in (0,1)$ such that $(1-r) \star (1-r) > 1-\alpha$. Then

$$\phi(|k_\lambda| \wedge z, |k_\lambda| \wedge (z - z \wedge nq) + |k_\lambda| \wedge (z \wedge nq)) > 1/2$$

and

$$\phi(|k_\lambda| \wedge (z - z \wedge nq) + (|k_\lambda| \wedge (z \wedge nq), (z - z \wedge nq) + n(|k_\lambda| \wedge q)) > 1/2.$$

Thus,

$$N(|k_\lambda| \wedge z, t+s) \geq N(z - z \wedge nq, t) \star N(n(|k_\lambda| \wedge q), s)$$

for each λ and $n \in \mathbb{N}$. As q is fuzzy quasi-interior point we may find n such that $N(z - z \wedge nq, t) > 1-r$ for each n and $t > 0$. Moreover, it follows from $k_\lambda \wedge q \xrightarrow{fn} k$ there exists λ_0 such that $N(k_\lambda \wedge q, s) > (1-r)/n$ whenever $\lambda \geq \lambda_0$ and $s > 0$. Therefore, $N(|k_\lambda| \wedge z, t) > (1-r) \star n(1-\mathrm{r})/n > 1-\alpha$. Hence $|k_\lambda| \wedge z \xrightarrow{fn} 0$. □

6.2.1 Disjoint Sequences

Two sequences (k_n) and (k_m) are called disjoint if $|k_n| \wedge |k_m| = 0$ for $m \neq n$ and read as $k_n \perp k_m$. Disjoint sequences in an FRS is ufo-convergent to zero. But this fact is not true for ufn-convergence. The ufn-convergent sequences are almost disjoint. The next result is useful to prove our key result.

Lemma 6.2.7. *Let (K,ϕ) be an FRS. If $|k| = g + h$ for some $k \in K$ and some $g, h \in K^+$ then there exist a and b such that $k = a + b$, $|a| = g$ and $|b| = h$.*

Proof. Suppose $k^+ + k^- = g + h$, Theorem 2.3.8 yields the four positive vector w, x, y and z such that $g = w + x$, $h = y + z$, $k^+ = w + y$ and $k^- = x + z$. Put $a = w - x$ and $b = y - z$. Thus $a + b = k^+ - k^- = k$. Now as $\phi(w, k^+) > 1/2$ and $\phi(x, k^-) > 1/2$ that $w \perp x$. Hence $|a| = |w - x| = |w + x| = w + x = g$. Analogously, $y \perp z$. Hence $|b| = h$. □

Theorem 6.2.8. *If a net $k_\lambda \xrightarrow{ufn} 0$ in a fuzzy Banach lattice (K, N, ϕ) then there exists a disjoint sequence (δ_n) and an increasing sequence of indices (λ_n) such that $k_{\lambda_n} - \delta_n \xrightarrow{fn} 0$.*

Proof. Suppose that $(k_\lambda) \subseteq K^+$. Take any λ_1 and construct $\lambda_1, ..., \lambda_{n-1}$. Observe that $k_\lambda \wedge k_{\lambda_j} \xrightarrow{fn} 0$, $1 \leq j \leq n-1$. Take $\lambda_n > \lambda_{n-1}$ such that $N(k_\lambda \wedge k_{\lambda_j}, t) \geq 1 - \frac{1}{2^{n+j}}$ for each j and $t > 0$. Therefore, we get an increasing sequence of indices (λ_n). Let $g_{nj} = k_\lambda \wedge k_{\lambda_j}$ such that $N(g_{nj}, t) \geq 1 - \frac{1}{2^{n+j}}$, $1 \leq j \leq n-1$.

For each n, put $h_n = \sum_{j=1}^{n-1} g_{jn} + \sum_{i=n}^{\infty} g_{ni}$. Clearly, h_n is convergent and $N(h_n, t_n) > 1 - \frac{1}{2^{n+j}}$ for $\sum_{n=1}^{\infty} t_n < \infty$. Put $\delta_n = (k_{\lambda_n} - h_n)^+$. Clearly, $\phi(k_{\lambda_n} - \delta_n, h_n) > 1/2$ such that $N(k_{\lambda_n} - \delta_n, t) > 1 - \frac{1}{2^{n+j}}$ for each n and $t > 0$. Thus, $k_{\lambda_n} - \delta_n \xrightarrow{fn} 0$.

Now we show that (δ_n) is disjoint sequence. Let $n < p$.

$$\delta_n = \phi((k_{\lambda_n} - h_n)^+, (k_{\lambda_n} - h_{np})^+) > 1/2 \text{ and } (k_{\lambda_n} - h_{np})^+ = k_{\lambda_n} - k_{\lambda_n} \wedge k_{\lambda_p}$$

and

$$\delta_p = \phi((k_{\lambda_p} - h_p)^+, (k_{\lambda_n} - h_{np})^+) > 1/2 \text{ and } (k_{\lambda_p} - h_{np})^+ = k_{\lambda_p} - k_{\lambda_n} \wedge k_{\lambda_p}.$$

Clearly, $\delta_n \perp \delta_p = 0$.

For the general case, we first take $|k_\lambda|$ and increasing sequence of indices. Also we take a positive disjoint sequence (a_n) and $b_n \xrightarrow{fn} 0$, we have $|k_{\lambda_n}| = a_n + b_n$. Lemma 6.2.7 yields that there exists two sequence (δ_n) and (g_n) in K with $|\delta_n| = a_n$, $|g_n| = b_n$ and $k_{\lambda_n} = \delta_n + g_n$. It follows that (δ_n) is disjoint and $g_n \xrightarrow{fn} 0$. Hence $k_{\lambda_n} - \delta_n \xrightarrow{fn} 0$.

□

Remark 6.2.9. *Let $(k_\lambda)_{\lambda\in\Lambda}$ be a net in fuzzy norm space, if $k_\lambda \xrightarrow{fn} k$ then there exists an increasing sequence of indices (λ_n) such that $k_{\lambda_n} \xrightarrow{fn} k$. One can reduce nets to sequences while studying the fuzzy norm convergence.*

Proposition 6.2.10. *If a sequence $k_n \xrightarrow{fn} k$ in a fuzzy Banach lattice (K, N, ϕ) then there exists a subsequence (k_{n_p}) such that $k_{n_p} \xrightarrow{fo} k$ in K.*

Proof. Let $k_n \xrightarrow{fn} k$. Take a subsequence $(k_{n_p})(p = 1, 2, ...)$ of (k_n) such that $N(k_{n_p}, t) > 1 - p^{-3}$ for each p and $t > 0$. There exists an element $g \in K^+$ such that $\phi(p|k_{n_p} - k|, g) > 1/2$ implies $\phi(|k_{n_p} - k|, p^{-1}g) > 1/2$ for each p. □

For fuzzy order continuous Banach lattice, ufn-convergence is sequential, so one can always use ufn-convergent sequences instead of nets.

Proposition 6.2.11. *If a net $k_\lambda \xrightarrow{ufn} 0$ in a fuzzy order continuous Banach lattice (K, N, ϕ) then there exists an increasing sequence of indices (λ_n) such that $k_{\lambda_n} \xrightarrow{ufo} 0$ and $k_{\lambda_n} \xrightarrow{ufn} 0$.*

Proof. Let (λ_n) be an increasing sequence of indices and (δ_n) disjoint sequence as in Theorem 6.2.8. Since $\delta_n \xrightarrow{ufo} 0$, by Lemma 6.2.4 $\delta_n \xrightarrow{ufn} 0$. It follows from $k_{\lambda_n} - \delta_n \xrightarrow{fn} 0$ implies $k_{\lambda_n} - \delta_n \xrightarrow{ufn} 0$. Thus, $k_{\lambda_n} \xrightarrow{ufn} 0$. Moreover, since $k_{\lambda_n} - \delta_n \xrightarrow{fn} 0$ implies $k_{\lambda_n} - \delta_n \xrightarrow{fo} 0$. Therefore, $k_{\lambda_n} - \delta_n \xrightarrow{ufo} 0$. Hence $k_{\lambda_n} \xrightarrow{ufo} 0$. □

It is of interest to note that Lemma 6.2.4 is an extension of the forward direction of this result to general fuzzy Banach lattice. The following results show the characterization of ufn-convergence in terms of ufo-convergence in fuzzy order continuous Banach lattices.

Proposition 6.2.12. *If a sequence $k_n \xrightarrow{ufn} 0$ in a fuzzy Banach lattice (K, N, ϕ) then there exists a subsequence (k_{n_p}) such that $k_{n_p} \xrightarrow{ufo} 0$ in K.*

Proof. Let B_e be the fuzzy band generated by $0 \neq e \in K^+$. As $k_n \xrightarrow{ufn} 0$ i.e. $k_n \wedge e \xrightarrow{fn} 0$ in K and therefore, in B_e. By Proposition 6.2.10 there exists a subsequence (k_{n_p}) of (k_n) such that $k_{n_p} \wedge e \xrightarrow{fo} 0$ in B_e. Since e is a fuzzy weak order unit in B_e by [28, Proposition 3.4] $k_n \xrightarrow{ufo} 0$ in K. □

Proposition 6.2.13. *Let (k_n) be a sequence in fuzzy order continuous Banach lattice (K, N, ϕ). Then $k_n \xrightarrow{ufn} 0$ if and only if each subsequence (k_{n_p}) has 1further subsequence $(k_{n_{p_i}})$ such that $k_{n_{p_i}} \xrightarrow{ufo} 0$.*

Proof. The forward implication is same as Proposition 6.2.12. Conversely, suppose on contrary that $k_n \overset{ufn}{\not\longrightarrow} 0$. Then there exists $\alpha \in (0,1)$ and $g \in K^+$ and a subsequence (k_{n_p}) of (k_n) such that $N(k_{n_p}, t) < 1 - \alpha$ for all p. By assumption there is a subsequence $(k_{n_{p_i}})$ of (k_{n_p}) such that $k_{n_{p_i}} \xrightarrow{ufo} 0$. By Lemma 6.2.4 $k_{n_{p_i}} \xrightarrow{ufn} 0$, a contradiction. □

6.2.2 ufn-convergence is topological

This section is dedicated to define topology that is compatible with ufn- convergence. We work through by defining a local base of zero. Given $\alpha \in (0,1)$ and $0 \neq z \in K^+$.

$$C_{z,\alpha} = \{k \in K : N(|k| \wedge z, t) > 1 - \alpha \text{ for } t > 0\}.$$

Where $C_{z,\alpha}$ is a neighborhood of zero. The set of all neighborhood of zero is represented as

$$N_0 = \{C_{z,\alpha} : \alpha \in (0,1), 0 \neq z \in K^+\}.$$

The next result shows that N_0 is a base of neighborhoods of zero for topology.

Proposition 6.2.14. *The family N_0 is a base of neighborhoods of zero for topological vector space.*

Proof. Clearly, every set in N_0 contains zero.

Now, we show that intersection of any two sets in N_0 contains another set in N_0. Let $C_{z_1,\alpha_1}, C_{z_2,\alpha_2} \in N_0$. Take $\alpha = \min\{\alpha_1, \alpha_2\}$ and $z = \max\{z_1, z_2\}$. We claim that there exists a

$C_{z,\alpha}$ such that $C_{z,\alpha} \subseteq C_{z_1,\alpha_1} \cap C_{z_2,\alpha_2}$. Let $c \in C_{z,\alpha}$. Therefore, $N(|c| \wedge z, t) > 1 - \alpha$. It follows from $\phi(|c| \wedge z_1, |c| \wedge z) > 1/2$ i.e.

$$N(|c| \wedge z_1, t) \geq N(|c| \wedge z, t) > 1 - \alpha \geq 1 - \alpha_1 \text{ for } t > 0.$$

Thus $c \in C_{z_1,\alpha_1}$. Analogously, $c \in C_{z_2,\alpha_2}$.

Clearly, $C_{z,\alpha} + C_{z,\alpha} \subseteq C_{z,2\alpha}$. It is easy to see that for every $|\lambda| \leq 1$, we have $\lambda C_{z,\alpha} \subseteq C_{z,\alpha}$.

Finally we have to show that for every $c \in C_{z,\alpha}$ there exists $C_{g,\beta} \in N_0$ such that

$$c + C_{g,\beta} \subseteq C_{z,\alpha}$$

for some $\alpha \in (0, 1)$ and $0 \neq z \in K^+$. We need to find $\beta \in (0, 1)$ and $0 \neq g \in K^+$. Put $g := z$. It follows from $c \in C_{z,\alpha}$ that $N(|c| \wedge z, t) > 1 - \alpha$ for $t > 0$. Take $\beta := N(|c| \wedge z, s) + \alpha - 1$. It is enough to show that $c + d \in C_{z,\alpha}$. Indeed,

$$\phi(|c + d| \wedge z, |c| \wedge z + |d| \wedge z) > 1/2.$$

Thus,

$$N(|c + d| \wedge z, t + s) \geq N(|c| \wedge z, t) \wedge N(|d| \wedge z, s) \geq N(|c| \wedge z, t) - \beta \geq 1 - \alpha.$$

□

The next proposition shows that ufn-convergence is topological.

Proposition 6.2.15. *ufn-convergence in a fuzzy Banach lattice is the same as the convergence in topology whose base neighbourhood of zero is given by* N_0.

Proof. If $k_\lambda \xrightarrow{ufn} 0$ then for every $\alpha \in (0, 1)$ and $0 \neq z \in K^+$ there is some λ_0 such that $N(|k_\lambda| \wedge z, t) > 1 - \alpha$ whenever $\lambda \geq \lambda_0$. For every $C_{z,\alpha} \in N_0$ there is λ_0 such that $k_\lambda \in$

$C_{z,\alpha}$ whenever $\lambda \geq \lambda_0$. Therefore, the natural convergence in this topology is exactly ufn-convergence. □

Concluding Remarks:

Now, in the end, we summarize the novel contributions and present some future research lines.

- The existence of fuzzy Dedekind completion of a incomplete fuzzy Riesz space is proved, whereas to achieve this goal other related concepts like fuzzy order convergence, fuzzy positive operators and their related results are also explored to enrich the theory of fuzzy Riesz spaces.

- Fuzzy lattice operations are defined on the space of all fuzzy order bounded linear operators between two fuzzy Riesz spaces to make it fuzzy Riesz space when the range is fuzzy Dedekind complete. As a special case separation property of fuzzy order dual spaces are discussed.

- Fuzzy order convergence is generalized as unbounded fuzzy order convergence. Many other concepts like fuzzy weak order unit and fuzzy component are studied and many related results are proved.

- Fuzzy norms with respect to fuzzy ordering, to develop fuzzy Riesz norm, which leads to the fuzzy Banach lattices are defined and studied. Moreover, we proved that the topological dual is a fuzzy ideal of its fuzzy order dual in locally convex-solid fuzzy Riesz spaces.

- Unbounded fuzzy norm convergence is defined and studied in fuzzy Banach lattices. Many other concepts like fuzzy quasi interior point and disjoint sequences are investigated. Moreover, a topology is defined in which convergence is same as ufn-convergence.

For future research lines, one can define and explore the notion of fuzzy Riesz orthomorphism,unbounded fuzzy norm topology, unbounded fuzzy order convergence in fuzzy order dual spaces and unbounded absolute fuzzy weak convergence in fuzzy Banach lattices. Fuzzy order convergence and fuzzy positive operators can be applied to other spaces like soft sets, intuitionistic fuzzy sets and Rough set.

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