PARTON DISTRIBUTIONS AND THEIR SYMMETRIES

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Certificate

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To My Parents

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Abstract

We develop the field theoretical tools necessary to understand the complicated structure of nucleonic parton distributions and wave functions. In order to parametrize these non-perturbative objects in QCD symmetries imposed by parity, time-reversal, and hermitian conjugation are used. After some general discussions about the classification of all leading -twist distribution functions a special kind of distribution function, which is odd under time reversal, is considered. This T-odd distribution function, which was recently been shown to be non zero, was recalculated and the results in the literature were confirmed.

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Chapter 1

Introduction

It has been a dream of physicists to find a single set of laws which will describe all phenomena from the largest galaxies to the smallest atoms and beyond. At present physics of the very large scale is governed by Einstein's General Relativity. This theory describes accurately the orbits of the planets in our solar system, the bending of light by our Sun, and the behavior of radiation emitted by distant galaxies. On the other hand physics of very small scale is thought to be described by the Standard Model. This model is well-known for its extraordinarily accurate predictions of the properties of subatomic physics. This model consists of three parts. These are electromagnetic, weak and strong interactions which are described by their corresponding quantum field theories. In this work, we will be mainly concerned with the strong interaction part which is described by quantum chromodynamics (QCD).

The strong interactions are responsible for the structure of nuclei and the existence of certain decay modes. In order to obtain the experimental information about the structure of nuclei we need much higher energies and larger momentum transfers to obtain the higher resolution. The first series of such experiment to probe the structure of proton was initiated in the 1960's at SLAC (Stanford Linear Accelerator Centre) and the process was called electron-proton deep inelastic scattering (DIS). In 1969 Bjorken reported that the scaling property of structure functions in the DIS was expected in the deep inelastic region where the momentum transfer squared q^2 and energy transfer ν of electrons are very large with the ratio q^2/ν kept fixed. Immediately after Bjorken's proposal, experimental confirmation was found for it. At this time R. P. Feynman presented his parton model to explain Bjorken scaling. In his model he assumed that the projectile electrons scattered off almost-free point like constituents inside the nucleon which were called *partons*. For deep inelastic electron-nucleon scatterings, the momentum transfer squared q^2 is high so that the spatial resolution for observing the target nucleon by the projectile electron is high. Thus Bjorken scaling implies that the constituents of the nucleon look almost free and point-like when observed with high spatial resolution. Hence, if one accepts the parton idea, the dynamics governing the parton system should have the property that the interaction between partons becomes weaker at shorter distances. The partons were later identified with quarks of Gell-Mann's quark model since experimentally it was suggested that their quantum numbers such as charges and spins were practically the same as those of quark.

Searches for quark dynamics were initiated right after the foundation of the parton model. All known quantum field theories at that time were surveyed as possible candidates for quark dynamics and were shown not to enjoy the above-mentioned property that the interaction between quarks gets weaker at shorter distances. Soon after it was found that the non-Abelian gauge theories satisfied the desired property which is now called asymptotic freedom. This was the main reason that quantum chromodynamics (QCD) gained full attention of physicists. The term "chromo" refers to the color symmetry associated with this theory. Just like the photon which is an Abelian guage field mediating electromagnetic interactions between charged particles in quantum electrodynamics (QED), the non-abelian gauge field in QCD mediates color interactions between quarks. This non-abelian gauge field in QCD is called the gluon as it is responsible for binding the quarks together.

In the quark model with color symmetries hadrons appear as colorless states while quarks carry color quantum numbers. It is assumed that only colorless states are physically realized and hence quarks cannot be observed in isolated states. There is a possibility of explaining this assumption as a dynamical effect in QCD. In fact serious infrared divergences due to massless gluons may be responsible for confining quarks at long distances. Thus QCD has a desirable property that it enjoys the asymptotic freedom at short distances while it has a possibility of quark confinement at long distances.

The theoretical foundations and extensive experimental tests of the standard model in general and QCD in particular are so compelling that the focus is not on testing QCD but rather on understanding QCD. According to the property of asymptotic freedom of QCD, one may safely use perturbation theory to discuss short-distance physics. But for the long distance physics we cannot use it. In the past, there were no quantitative tools to calculate non-perturbative QCD. Now, however, the combination of theoretical tools and experimental probes presently available offers an unprecedented opportunity to make decisive progress in understanding how QCD works. In operator product expansion (OPE) we separate the incalculable non-perturbative physics from the calculable perturbative physics. The non-perturbative physics is then grouped into matrix elements which can be measured in the experiments. Once fixed, these matrix elements can be used to predict the outcome of other experiments. Thus these matrix elements can be used to understand the nucleon structure in terms of its constituents.

Understanding the structure of the nucleon in terms of the quark and gluon constituents of QCD is one of the outstanding fundamental problems in physics. Deep-inelastic scattering with charged beams has been the key tool for probing the structure of the nucleon. In the naive parton model, the unpolarized structure function is expressed in terms of a probability density $f(x)$ to find a parton of a specific flavor with a certain fraction x of the parent hadron momentum. The underlying probabilistic picture for the scattering process relies on the fact that the constituents in a hadron boosted to the infinite momentum frame behave as collections of noninteracting quanta due to time dilation. With polarized beams and targets the spin structure of the nucleon becomes accessible. We can relate the spin structure function of the nucleon in terms of different parton distributions from which we can extract the information about the spin of the nucleon in terms of its constituents. These parton model results arise as a lowest order term in the expansion in the coupling constant and inverse power of the hard momentum transfer of QCD factorization formulas. Now I shall describe briefly the contents of this dissertation.

In chapter 2, I used the deep inelastic scattering (DIS) process to describe the factorization of perturbative and non-perturbative physics, parton distribution functions and operator product expansion (OPE). DIS process is an excellent choice for developing the field theoratical background necessary to understand the ways by which we can extract information about the structure of the nucleon in QCD.

Chapter 3 is devoted to the construction of the wave function. Since the noninteracting massless QCD lagrangian respects the conformal symmetry, we can realize it as an approximate symmetry. Therefore, using conformal expansion, we construct the ρ-meson wave function at the leading twist. This will be the main result of chapter 3.

In Chapter 4, I will construct the leading twist parton distribution fuctions using the parity and time reversal invariance. Particularly, I shall concentrate on Sivers distribution function which is odd under time reversal. As time reversal is a symmetry of QCD this T-odd distribution function is apparently zero in light cone gauge where the gauge link operator between the quark fields becomes unity. But this result changes if we consider a gauge link in some other gauge. So I recalculated the Sivers distribution function in the Feynman gauge and found it to be non-zero, a fact which was discussed recently in the literature.

Chapter 2

Deep Inelastic Scattering

It is convenient to start from deep-inelastic scattering (DIS) to illustrate the field-theoretical definitions of quark (and antiquark) distribution functions. We will describe the parton model, which was the first candidate to describe the Bjorken scaling observed in DIS. We then present the formal field theoretical apparatus required to describe the short distance behaviour. This is operator product expansion(OPE).

2.1 Kinematics

In DIS a lepton is scattered off a hadron at very high energy. The final state of this process is a scattered lepton and a complicated hadronic state. We will consider the inclusive lepton-nucleon scattering(see Fig. 2.1)[1]

$$l(\ell) + N(P) \rightarrow l'(\ell') + X(P_X), \tag{2.1.1}$$

where X is some undetected hadronic final state with total four-momentum P_X. We will use the following notation:

$M(m_l) \equiv$ nucleon (lepton) mass.

[1] the dominance of one photon is assumed.

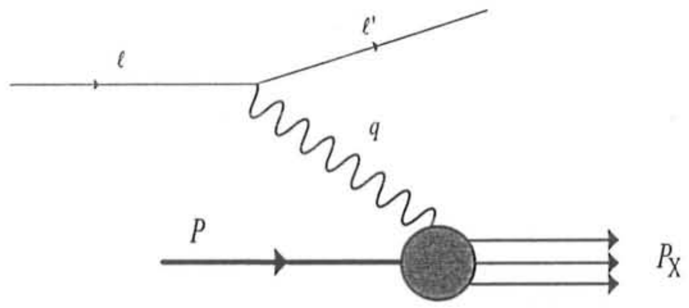

Figure 2-1: Deeply-inelastic scattering

$S(s_l) \equiv$ spin four-vector of the nucleon (lepton).

$\ell(E, \vec{\ell}\,)\equiv$ lepton four momentum.

For a given incident energy E, two kinematic variables are needed to describe the reaction (2.1.1). They can be chosen among the following in-variants (unless otherwise stated, we neglect lepton masses):

$$\begin{aligned} q^2 &= (\ell - \ell')^2 = -4EE' \sin^2(\theta/2) \leq 0, \\ Q^2 &= -q^2 \geq 0, \\ \nu &= \frac{P \cdot q}{M}, \\ x_B &= \frac{Q^2}{2M\nu}, \\ y &= \frac{P \cdot q}{P \cdot \ell}, \end{aligned}$$

where θ is the scattering angle. Since the nucleon is the lightest state with baryon number 1, therefore baryon number conservation implies that the invariant mass of the final state $M_X = \sqrt{(P+q)^2}$ must be larger than M. This gives the range of the Bjorken variable x_B

$$(P+q)^2 > M^2 \Longrightarrow 0 < x_B < 1.$$

Also using the target rest frame, we obtain the range of the lepton energy loss $y =$

$(E-E')/E$ as

$$0 < E' < E \Rightarrow 0 < y < 1.$$

At leading order in electromagnetic coupling, the S-matrix elements for the reaction(2.1.1) are given as

$$S_{fi} = \left\langle \ell' s_{\ell'}, X \left| T\left(-i\int d^4x J^\mu_\ell(x) A_\mu(x)(-i)\int d^4y J^\nu_h(y) A_\nu(y)\right)\right| \ell s_\ell, PS\right\rangle. \quad (2.1.2)$$

The currents J^μ_ℓ and J^ν_h are the usual leptonic and hadronic currents, respectively. Recognizing that there are no photons in either external state and that the two currents are fundamentally different allows us to seperate the matrix elements as

$$S_{fi} = (-i)^2 \int d^4x d^4y \left\langle \ell' s_{\ell'} \left| J^\mu_\ell(x) \right| \ell s_\ell \right\rangle \left\langle 0 \left| T\left(A_\mu(x) A_\nu(y)\right) \right| 0 \right\rangle \left\langle X \left| J^\nu_h(y) \right| PS \right\rangle \quad (2.1.3)$$

with

$$\langle 0 | T(A_\mu(x) A_\nu(y)) | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{(-ig_{\mu\nu})}{k^2 + i\varepsilon} e^{-ik.(x-y)}. \quad (2.1.4)$$

Now using the operator identity

$$\mathcal{O}(x) = e^{i\hat{P}\cdot x}\mathcal{O}(0)\mathrm{e}^{-i\hat{P}\cdot x}, \quad (2.1.5)$$

alongwith (2.1.4) and doing integration over x, y and k, we get

$$S_{fi} = i(2\pi)^4 \delta^4(P + q - P_X)\left[\left\langle \ell' s_{\ell'} \left| J^\mu_\ell(x) \right| \ell s_\ell \right\rangle \frac{g_{\mu\nu}}{q^2} \left\langle X \left| J^\nu_h(0) \right| PS \right\rangle\right]. \quad (2.1.6)$$

Using (2.1.6), inclusive DIS cross-section can be written as[2]

$$d\sigma = \frac{1}{4\ell \cdot P}\left[4\pi \frac{e^4}{Q^4} L_{\mu\nu} W^{\mu\nu}\right] \frac{d^3\ell'}{(2\pi)^3 2E'}, \quad (2.1.7)$$

[2]see the cross-section formula in Appendix A.

where the leptonic tensor $L_{\mu\nu}$ is defined as (lepton masses are retained here),

$$\begin{aligned} L_{\mu\nu} &= \sum_{s_{\ell'}} \left[\bar{u}_{\ell'}(\ell', s_{\ell'})\gamma_\mu u_\ell(\ell, s_\ell)\right]^* \left[\bar{u}_{\ell'}(\ell', s_{\ell'})\gamma_\nu u_\ell(\ell, s_\ell)\right] \\ &= Tr\left[(\not{\ell} + m_\ell)\frac{1}{2}(1+\gamma_5 \not{s})\gamma_\mu(\not{\ell}' + m_{\ell'})\gamma_\nu\right], \end{aligned} \tag{2.1.8}$$

and the hadronic tensor $W^{\mu\nu}$ is

$$W^{\mu\nu} = \frac{1}{4\pi}\sum_X \int \frac{d^3P_X}{(2\pi)^3 2E_X} (2\pi)^4 \delta^4(P+q-P_X) \langle PS|J^\mu(0)|X\rangle \langle X|J^\nu(0)|PS\rangle . \tag{2.1.9}$$

Writing the delta function as

$$\delta^4(P+q-P_X) = \int \frac{d^4\xi}{(2\pi)^4} e^{i\xi\cdot(P+q-P_X)}, \tag{2.1.10}$$

and using the identity (2.1.5), Eqn.(2.1.9) becomes

$$W^{\mu\nu} = \frac{1}{4\pi}\int d^4\xi e^{iq\cdot\xi} \langle PS|J^\mu(\xi)J^\nu(0)|PS\rangle . \tag{2.1.11}$$

Note that in (2.1.8) and (2.1.9) we summed over the final lepton spin $s_{\ell'}$ but did not average over the initial lepton spin s_ℓ, nor over the nucleon spin S. Thus we are describing, in general, the scattering of polarized leptons on a polarized target, with no measurement of the outgoing lepton polarization. In the target rest frame, (2.1.7) reads

$$\frac{d\sigma}{dE' d\Omega} = \frac{\alpha_{em}^2}{MQ^4}\frac{E'}{E} L_{\mu\nu}W^{\mu\nu}, \tag{2.1.12}$$

where $d\Omega = d\cos\theta d\varphi$.

The leptonic tensor $L_{\mu\nu}$ can be decomposed into a symmetric and antisymmetric part under $\mu - \nu$ interchange

$$L_{\mu\nu} = L^{(S)}_{\mu\nu}(\ell, \ell') + iL^{(A)}_{\mu\nu}(\ell, s_\ell; \ell') \tag{2.1.13}$$

and, computing the trace in (2.1.8), we obtain

$$L_{\mu\nu}^{(S)} = 2\left[\ell_\mu \ell'_\nu + \ell_\nu \ell'_\mu - g_{\mu\nu}(\ell \cdot \ell')\right], \tag{2.1.14a}$$

$$L_{\mu\nu}^{(A)} = 2m_\ell \varepsilon_{\mu\nu\rho\sigma} s_\ell^\rho (\ell - \ell')^\sigma. \tag{2.1.14b}$$

If the incoming lepton is longitudinally polarised, its spin vector is

$$s_\ell^\mu = \frac{\lambda_\ell}{m_\ell} \ell^\mu, \ \lambda_\ell = \pm 1, \tag{2.1.15}$$

and (2.1.14b) becomes

$$L_{\mu\nu}^{(A)} = 2\lambda_\ell \varepsilon_{\mu\nu\rho\sigma} \ell^\rho q^\sigma. \tag{2.1.16}$$

Note that the lepton mass m_ℓ appearing in (2.1.14b) has been cancelled by the denominator of (3.1.15). In contrast, if the lepton is transversely polarized, i.e $s_\ell^\mu = s_{\ell\perp}^\mu$, no such cancellation occurs and the process is suppressed by a factor m_ℓ / E.

We split the hadronic tensor $W^{\mu\nu}$ as

$$W_{\mu\nu} = W_{\mu\nu}^{(S)}(q, P) + i W_{\mu\nu}^{(A)}(q; P, S). \tag{2.1.17}$$

Hermiticity, parity invariance[3] and current conservation imply the following constraints on the form of the hadronic tensor $W^{\mu\nu}$:

$$[\text{Hermiticity}] \quad \Rightarrow \quad (W_{\mu\nu})^* = W_{\nu\mu}, \tag{2.1.18}$$

$$[\text{Parity invariance}] \quad \Rightarrow \quad W_{\mu\nu}(q; P, S) = W^{\mu\nu}\left(\tilde{q}; \tilde{P}, -\tilde{S}\right), \tag{2.1.19}$$

$$[\text{Current conservation}] \quad \Rightarrow \quad q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0, \tag{2.1.20}$$

where the tilde four-vectors are defined as $\tilde{q}^\mu = (q^0, -\mathbf{q})$. These relations allow us to

[3] see Appendix C.

deduce the tensor structure of $W_{\mu\nu}$ as

$$\begin{aligned} W_{\mu\nu}^{(S)} &= \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) F_1\left(x, Q^2\right) \\ &+ \left(P_\mu - \frac{P\cdot q}{q^2} q_\mu\right)\left(P_\nu - \frac{P\cdot q}{q^2} q_\nu\right)\frac{1}{P\cdot q} F_2\left(x, Q^2\right), \end{aligned} \tag{2.1.21a}$$

$$W_{\mu\nu}^{(A)} = -\frac{M}{P\cdot q}\epsilon_{\mu\nu\rho\sigma} q^\rho \left\{ S^\sigma G_1\left(x, Q^2\right) + \left(S^\sigma - \frac{S\cdot q}{P\cdot q} P^\sigma\right) G_2\left(x, Q^2\right)\right\}, \tag{2.1.21b}$$

where F_1, F_2 and G_1, G_2 are real and dimensionless structure functions of the nucleon. If nucleon mass is not important, F's and G's being dimensionless cannot depend on dimensionful variable Q^2 and one might expect that scale invariance holds in asymptotic (Bjorken) limit $\nu, Q^2 \to \infty$ with $x_B = \frac{Q^2}{2M\nu}$ fixed. Using (2.1.13) and (2.1.17), the cross-section (2.1.12) can be written as

$$\frac{d\sigma}{dE' d\Omega} = \frac{\alpha_{em}^2}{MQ^4}\frac{E'}{E}\left[L_{\mu\nu}^{(S)} W^{\mu\nu(S)} - L_{\mu\nu}^{(A)} W^{\mu\nu(A)}\right]. \tag{2.1.22}$$

The unpolarized cross-section is then obtained by averaging over the spins of the incoming lepton (s_ℓ) and of the nucleon (S) and reads,

$$\frac{d\sigma^{unp}}{dE' d\Omega} = \frac{1}{2}\sum_{s_\ell}\frac{1}{2}\sum_{S}\frac{d\sigma}{dE' d\Omega} = \frac{\alpha_{e.m}^2}{MQ^4}\frac{E'}{E} L_{\mu\nu}^{(S)} W^{\mu\nu(S)}. \tag{2.1.23}$$

Taking the direction of the incoming lepton to be along z-axis we have,

$$\begin{aligned} \ell^\mu &= E(1,0,0,1), \\ \ell'^\mu &= E'(1,\sin\theta,0,\cos\theta). \end{aligned} \tag{2.1.24}$$

Then using equations (2.1.14a) and (2.1.21a), we obtain the unpolarised cross-section (as a function of x and y),

$$\frac{d\sigma^{unp}}{dE' d\Omega} = \frac{4\pi\alpha_{em}^2 s}{Q^4}\left\{ xy^2 F_1\left(x, Q^2\right) + \left(1 - y - \frac{xyM^2}{s}\right) F_2\left(x, Q^2\right)\right\}, \tag{2.1.25}$$

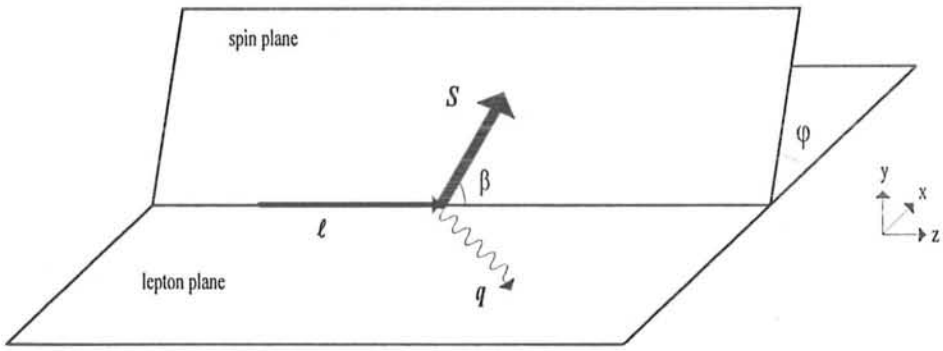

Figure 2-2: Lepton and Spin planes. The lepton plane is taken here to coincide with the xz plane.

where $s = (\ell + P)^2$.

Differences of cross-sections with opposite target spin probe the antisymmetric part of the leptonic and hadronic tensors,

$$\frac{d\sigma(+S)}{dE'd\Omega} - \frac{d\sigma(-S)}{dE'd\Omega} = -\frac{\alpha_{em}^2}{MQ^4}\frac{E'}{E}2L_{\mu\nu}^{(A)}W^{\mu\nu(A)}. \tag{2.1.26}$$

In the target rest frame the spin of the nucleon can be parameterized as (assuming $|\mathrm{S}| = 1$),

$$S^\mu = (0, \mathrm{S}) = (0, \sin\beta\cos\varphi, \sin\beta\sin\varphi, \cos\beta)\,, \tag{2.1.27}$$

where β is the angle between S and ℓ (see Fig.2.2.). Inserting (2.1.16) and (2.1.21b) in Eqn. (2.1.26) with the above parameterization for the spin, we get the cross-section asymmetry,

$$\begin{aligned}\frac{d\sigma(+S)}{dxdyd\varphi} - \frac{d\sigma(-S)}{dxdyd\varphi} &= \lambda_\ell\frac{4\pi\alpha_{em}^2}{Q^4}\left[yG_1\left(x,Q^2\right)\cos\beta\right.\\ &\left.+\frac{2Mx}{Q}\sqrt{1-y}\left\{yG_1\left(x,Q^2\right) + 2G_2\left(x,Q^2\right)\sin\beta\cos\varphi\right\}\right].\end{aligned} \tag{2.1.28}$$

Note that the term containing G_2 is suppressed by one power of Q. This makes the measurement of G_2 quite a difficult task. For a longitudinally polarized nucleon (i.e

$\beta = 0$), DIS depends only on G_1. If it is transversely polarized (i.e $\beta = \frac{\pi}{2}$) the sum of $G_1$and G_2 is measured.

2.2 The Parton Model

In the parton model the photon is assumed to scatter incoherently off the constituents of the nucleon (quarks and antiquarks). Currents are treated in free field theory and any interaction between the struck quark and target remnant is ignored. The hadronic tensor $W^{\mu\nu}$ is, then represented by the handbag diagram shown in Fig.2.3 and by inserting the current $J^\mu = \bar{\psi}\gamma^\mu\psi$, one obtains in the tree approximation

$$\begin{aligned} W^{\mu\nu} &= \frac{1}{4\pi}\sum_q e_q^2 \sum_X \int \frac{d^3\mathbf{P}_X}{(2\pi)^3 2E_X} \int \frac{d^3\kappa}{(2\pi)^3 2E_\kappa} \int \frac{d^4k}{(2\pi)^4} \\ &\times \left[\left\langle PS \left| \overline{\psi}_j(0) \right| X \right\rangle (\gamma^\mu)_{jk} \left\langle 0 \left| \psi_k(0) \right| \kappa \right\rangle \left\langle \kappa \left| \overline{\psi}_l(0) \right| 0 \right\rangle (\gamma^\nu)_{li} \left\langle X \left| \psi_i(0) \right| PS \right\rangle \right. \\ &+ \left. \left\langle 0 \left| \overline{\psi}_k(0) \right| \kappa \right\rangle (\gamma^\mu)_{ki} \left\langle PS \left| \psi_i(0) \right| X \right\rangle \left\langle X \left| \overline{\psi}_j(0) \right| PS \right\rangle (\gamma^\nu)_{jl} \left\langle X \left| \psi_l(0) \right| 0 \right\rangle \right] \\ &\times (2\pi)^4 \delta^4 (P + k - P_X) (2\pi)^4 \delta^4 (k + q - \kappa), \end{aligned} \quad (2.2.1)$$

where $\sum_X$ includes the summation over the number of particles populated the final states as well as their quantum numbers, $\sum_q$ is a sum over flavours and e_q is the quark charge in the units of e. We define the quark-quark correlation matrix $\Phi_{ij}(k, P, S)$ as

$$\Phi_{ij}(k, P, S) = \sum_X \int \frac{d^3\mathbf{P}_X}{(2\pi)^3 2E_X} \left\langle PS \left| \overline{\psi}_j(0) \right| X \right\rangle \left\langle X \left| \psi_i(0) \right| PS \right\rangle. \quad (2.2.2)$$

Using (2.1.5, 2.1.10) and the completeness of the $\mid X \rangle$ states this matrix can be re-expressed as

$$\Phi_{ij}(k, P, S) = \int d^4\xi e^{ik\cdot\xi} \left\langle PS \left| \overline{\psi}_j(0)\psi_i(\xi) \right| PS \right\rangle. \quad (2.2.3a)$$

In the same way for antiquarks we have,

$$\bar{\Phi}_{ij}(k, P, S) = \int d^4\xi e^{ik\cdot\xi} \left\langle PS \left| \psi_i(0)\overline{\psi}_j(\xi) \right| PS \right\rangle = -\Phi_{ij}(-k, P, S). \quad (2.2.3b)$$

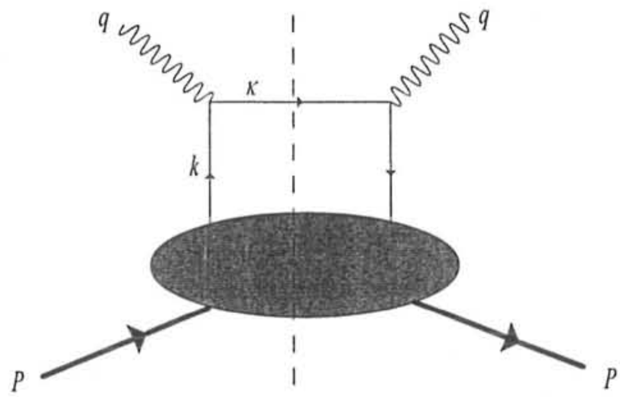

Figure 2-3: The so-called hand bag diagram. Also the diagram with opposite fermion flow has to be added.

Using the above definitions, the hadronic tensor becomes,

$$W^{\mu\nu} = \frac{1}{2}\sum_q e_q^2 \int \frac{d^4k}{(2\pi)^4}\delta\left((k+q)^2\right) \times Tr\left[\Phi\gamma^\mu(\not{k}+\not{q})\gamma^\nu\right] + \left\{ \begin{array}{c} q \leftrightarrow -q \\ \mu \leftrightarrow \nu \end{array} \right\}. \tag{2.2.4}$$

To simplify the presentation for the moment we consider only quarks, the extension to anti-quarks being rather straightforward.

In order to calculate $W^{\mu\nu}$, it is convenient to use a Sudakov parametrization[4] of the four-momenta at hand. We introduce the null vectors p^μ and n^μ satisfying,

$$p^2 = n^2 = 0, \quad p\cdot n = 1, \quad n^+ = p^- = 0 \tag{2.2.5}$$

and work in a frame where the virtual photon and the proton are collinear. The proton is taken to be directed along the positive z direction (see Fig.2.4). In terms of p^μ and n^μ the proton momentum can be parameterized as

$$P^\mu = p^\mu + \frac{M^2}{2}n^\mu \simeq p^\mu. \tag{2.2.6}$$

[4]see Appendix A.

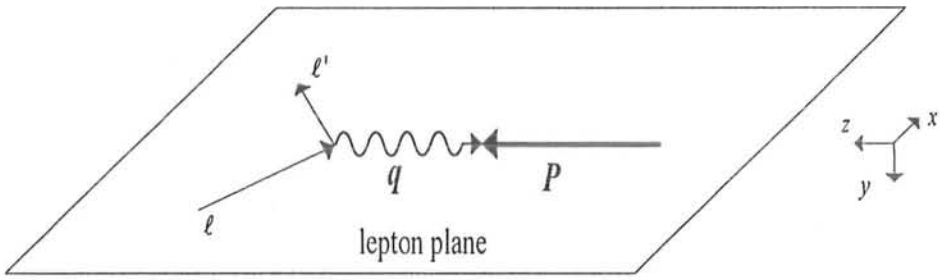

Figure 2-4: The $\gamma^* N$ collinear frame. Note our convention for the axes.

The momentum of the virtual photon can be written as

$$q^\mu \simeq (P \cdot q) n^\mu - x_B p^\mu, \tag{2.2.7}$$

where we are implicitly ignoring terms $\mathcal{O}(M^2/Q^2)$. Finally the Sudakov decomposition of the quark momentum is,

$$k^\mu = x p^\mu + \frac{(k^2 + k_\perp^2)}{2x} n^\mu + k_\perp^\mu. \tag{2.2.8}$$

In the parton model one assumes that the handbag diagram contributes to the hadronic tensor is dominated by small values of k^2 and $k_\perp^2$. This means that we can write k^μ approximately as

$$k^\mu \simeq x p^\mu. \tag{2.2.9}$$

The on-shell condition of the outgoing quark then implies

$$\delta\left((k+q)^2\right) \simeq \delta\left(-Q^2 + 2xP \cdot q\right) = \frac{1}{2P \cdot q}\delta(x - x_B), \tag{2.2.10}$$

that is $k^\mu \simeq x_B p^\mu$. Thus the Bjorken variable $x_B = Q^2/2M\nu$ is the fraction of the longitudinal momentum of the nucleon carried by the struck quark: $x_B = k^+/P^+$.

Returning to the hadronic tensor (2.2.4) the identity

$$\gamma^\mu\gamma^\rho\gamma^\nu = \left(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma} + i\varepsilon^{\mu\rho\nu\sigma}\gamma^5\right)\gamma_\sigma, \tag{2.2.11}$$

allows us to split $W^{\mu\nu}$ into symmetric (S) and anti-symmetric (A) parts under $\mu \leftrightarrow \nu$ interchange. Let us first consider $W^{(S)}_{\mu\nu}$ (i.e., unpolarized DIS):

$$\begin{aligned} W^{(S)}_{\mu\nu} &= \frac{1}{2}\frac{1}{2P\cdot q}\sum_q e_q^2 \int \frac{d^4k}{(2\pi)^4}\delta\left(x_B - \frac{k^+}{P^+}\right) \\ &\times \left[(k_\mu + q_\mu)Tr[\Phi\gamma_\nu] + (k_\nu + q_\nu)Tr[\Phi\gamma_\mu] - g_{\mu\nu}(k^\rho + q^\rho)Tr[\Phi\gamma_\rho]\right] \end{aligned} \tag{2.2.12}$$

From (2.2.7) and (2.2.8) we have $k_\mu + q_\mu \simeq (P\cdot q)n^\mu$ and (2.2.12) becomes

$$\begin{aligned} W^{(S)}_{\mu\nu} &= \frac{1}{4}\sum_q e_q^2 \int \frac{d^4k}{(2\pi)^4}\delta\left(x_B - \frac{k^+}{P^+}\right) \\ &\times \left[n_\mu Tr[\Phi\gamma_\nu] + n_\nu Tr[\Phi\gamma_\mu] - g_{\mu\nu}n^\rho Tr[\Phi\gamma_\rho]\right]. \end{aligned} \tag{2.2.13}$$

Introducing the notation[5]

$$\begin{aligned} \langle\Gamma\rangle &= \int \frac{d^4k}{(2\pi)^4}\delta\left(x_B - \frac{k^+}{P^+}\right) Tr[\Gamma\Phi] \\ &= \int \frac{d\lambda}{2\pi} e^{i\lambda x}\left\langle PS\left|\overline{\psi}(0)\Gamma\psi(\lambda n)\right|PS\right\rangle, \end{aligned} \tag{2.2.14}$$

where Γ is a Dirac matrix, $W^{(S)}_{\mu\nu}$ is written as

$$W^{(S)}_{\mu\nu} = \frac{1}{4}\sum_q e_q^2\left[n_\mu\langle\gamma_\nu\rangle + n_\nu\langle\gamma_\mu\rangle - g_{\mu\nu}n^\rho\langle\gamma_\rho\rangle\right]. \tag{2.2.15}$$

We now have to parametrize $\langle\gamma_\mu\rangle$, which is a vector quantity containing information on the quark dynamics. At leading twist, i.e., considering contibutions $O(P^+)$ in the infinite momentum frame, the only vector at our disposal is $p^\mu \simeq P^\mu$ (recall that $n^\mu = O(1/P^+)$

[5] where $\lambda n = (0, \xi^-, 0_\perp)$.

and $k^\mu \simeq xP^\mu$). Thus we can write

$$\langle\gamma^\mu\rangle = \int \frac{d\lambda}{2\pi} e^{i\lambda x} \left\langle PS \left| \bar{\psi}(0)\gamma^\mu\psi(\lambda n)\right| PS\right\rangle = 2f(x)P^\mu, \tag{2.2.16}$$

where $f(x)$ is the quark number density. This will become clear later on (see Sec.2.4). From (2.2.16) we obtain the following expression for $f(x)$,

$$f(x) = \frac{1}{2P^+}\int \frac{d\lambda}{2\pi} e^{i\lambda x} \left\langle PS \left| \bar{\psi}(0)\gamma^+\psi(\lambda n)\right| PS\right\rangle. \tag{2.2.17}$$

Inserting (2.2.16) into (2.2.15) yields

$$W^{(S)}_{\mu\nu} = \frac{1}{2}\sum_q e_q^2 \left[n_\mu P_\nu + n_\nu P_\mu - g_{\mu\nu}\right] f_q(x). \tag{2.2.18}$$

Structure functions F_1 and F_2 can be extracted from (2.1.21a) as

$$-\frac{1}{2} g_\perp^{\mu\nu} W^{(S)}_{\mu\nu} = F_1\left(x, Q^2\right) \tag{2.2.19a}$$

$$W^{(S)}_{++} = \frac{(q_+)^2}{q^2}\left[F_1\left(x, Q^2\right) - \frac{1}{2x}F_2\left(x, Q^2\right)\right]. \tag{2.2.19b}$$

Comparing (2.2.19a, 2.2.19b) and (2.2.18) we obtain the Callan-Gross relation[1] along with the scaling (i,e., F's are independent of Q^2)

$$F_1(x) = \frac{F_2(x)}{2x} = \frac{1}{2}\sum_q e_q^2 f_q(x), \tag{2.2.20}$$

which is the well-known parton model expression for the unpolarized structure functions, restricted to quarks. In order to complete the discussion, we introduce the antiquark distribution function[6] using (2.2.3b) as

$$\bar{f}(x) = \frac{1}{2P^+}\int \frac{d\lambda}{2\pi} e^{i\lambda x} \left\langle PS \left| Tr[\gamma^+\psi(0)\overline{\psi}(\lambda n)]\right| PS\right\rangle. \tag{2.2.21}$$

[6]these were not considered in the above discussion.

Including $\bar{f}_q(x)$, the structure functions become

$$F_1(x) = \frac{F_2(x)}{2x} = \frac{1}{2}\sum_q e_q^2\left[f_q(x) + \bar{f}_q(x)\right]. \qquad (2.2.22)$$

2.2.1 Polarized DIS in the Parton Model

Let us turn now to polarized DIS. The parton-model expression of the antisymmetric part of the hadronic tensor is

$$W_{\mu\nu}^{(A)} = -\frac{1}{2}\frac{1}{2P\cdot q}\sum_q e_q^2\int\frac{d^4k}{(2\pi)^4}\delta\left(x_B - \frac{k^+}{P^+}\right)\varepsilon_{\mu\nu\rho\sigma}(k+q)^\rho Tr\left[\gamma^\sigma\gamma^5\Phi\right], \qquad (2.2.23)$$

with $k^\mu \simeq x_B P^\mu$ this becomes, using the notation (2.2.14)

$$W_{\mu\nu}^{(A)} = -\varepsilon_{\mu\nu\rho\sigma}n^\rho\sum_q\frac{e_q^2}{4}\left\langle\gamma^\sigma\gamma^5\right\rangle. \qquad (2.2.24)$$

At leading twist the only pseudovector at hand is $S^\sigma = (\lambda_N/M)P^\sigma$ and $\langle\gamma^\sigma\gamma^5\rangle$ is parametrized as

$$\left\langle\gamma^\sigma\gamma^5\right\rangle = 2Mg(x_B)S^\sigma = 2\lambda_N g(x_B)P^\sigma. \qquad (2.2.25)$$

Here $g(x)$, given explicitly by,

$$g(x) = \frac{1}{2P^+}\int\frac{d\lambda}{2\pi}e^{i\lambda x}\left\langle PS\left|\overline{\psi}(0)\gamma^+\gamma_5\psi(\lambda n)\right|PS\right\rangle, \qquad (2.2.26)$$

is the helicity distribution of the quarks. Inserting (2.2.25) in (2.2.24), we find

$$W_{\mu\nu}^{(A)} = -2\lambda_N\varepsilon_{\mu\nu\rho\sigma}n^\rho p^\sigma\sum_q\frac{e_q^2}{4}g_q(x). \qquad (2.2.27)$$

Comparing with the longitudinal part of the hadronic tensor (2.1.21b), which can be written as

$$W_{\mu\nu,long}^{(A)} = -\lambda_N\varepsilon_{\mu\nu\rho\sigma}n^\rho p^\sigma G_1\left(x, Q^2\right). \qquad (2.2.28)$$

We obtain the usual parton model expression for the polarized structure function G_1

$$G_1(x) = \frac{1}{2}\sum_q e_q^2 g_q(x). \tag{2.2.29}$$

Again, antiquark distributions $g_{\bar{q}}$ should be added to (2.2.29) to obtain the full parton model expression for G_1

$$G_1(x) = \frac{1}{2}\sum_q e_q^2 \left[g_q(x) + g_{\bar{q}}(x)\right]. \tag{2.2.30}$$

We have defined the parton distributions in the naive parton model. In QCD, in order to make these distributions guage invariant, a path ordered link operator,

$$[\xi^-;0] \equiv P\exp\left(-ig\int_0^{\xi^-} d\eta^- A^+(\eta^-,0)\right), \tag{2.2.31}$$

where P denotes path-ordering, must be inserted between the quark fields. How this link is introduced, will be discussed in the chapter 4. In the light cone gauge $A^+ = 0$, by choosing an appropriate path, this gauge link can be reduced to unity. The important lesson we learned in this section is that, at leading twist, only longitudinal polarization contributes to DIS.

2.2.2 Probabilistic Interpretation of Distribution Functions

Distribution functions are essentially the probability densities for finding partons with a given momentum fraction and a given polarization inside a hadron. We shall now see how this interpretation comes about from the field-theoretical definition of quark (and antiquark) distribution functions presented above.

Let us first decompose the quark fields into "good" and "bad" components:

$$\psi = \psi_+ + \psi_-, \tag{2.2.32}$$

where

$$\psi_{\pm} = \frac{1}{2}\gamma^{\mp}\gamma^{\pm}\psi. \tag{2.2.33}$$

The usefulness of this procedure lies in the fact that "bad" components are not dynamically independent. Using the equation of motion $(i\not{D} - m)\psi = 0$ and light cone guage $A^{+} = 0$ this can be shown as:

$$(i\not{D} - m)\psi_{-} = -(i\not{D} - m)\psi_{+},$$

using $(\gamma^{+})^2 = (\gamma^{-})^2 = 0$ the above equation becomes

$$\left(i\gamma^{-}\partial^{+} - i\mathbf{D}\cdot\boldsymbol{\gamma} - m\right)\psi_{-} = -\left(i\gamma^{+}D^{-} - i\mathbf{D}\cdot\boldsymbol{\gamma} - m\right)\psi_{+},$$

multiplying by γ^{+} from the left, we get

$$\psi_{-} = \frac{1}{2\partial^{+}}\left(\gamma^{+}\mathbf{D}\cdot\boldsymbol{\gamma} - im\gamma^{+}\right)\psi_{+}. \tag{2.2.34}$$

Thus "bad" components can be eliminated in favour of "good" components and terms containing quark masses and gluon fields. Since in the $P^{+} \to \infty$ limit ψ_{+} dominates over ψ_{-}, the presence of "bad" components in a parton distribution function signals higher twists. Thus using the relations

$$\overline{\psi}\gamma^{+}\psi = \sqrt{2}\psi_{+}^{\dagger}\psi_{+}, \tag{2.2.35a}$$

$$\overline{\psi}\gamma^{+}\gamma_5\psi = \sqrt{2}\psi_{+}^{\dagger}\gamma_5\psi_{+}, \tag{2.2.35b}$$

the distribution functions can be expressed as,

$$f(x) = \int \frac{d\xi^{-}}{2\sqrt{2}\pi} e^{ixP^{+}\xi^{-}} \left\langle PS \left| \psi_{+}^{\dagger}(0)\psi_{+}(0,\xi^{-},0) \right| PS \right\rangle, \tag{2.2.36a}$$

$$g(x) = \int \frac{d\xi^{-}}{2\sqrt{2}\pi} e^{ixP^{+}\xi^{-}} \left\langle PS \left| \psi_{+}^{\dagger}(0)\gamma_5\psi_{+}(0,\xi^{-},0) \right| PS \right\rangle. \tag{2.2.36b}$$

Inserting a complete set of intermediate states $\{| n\rangle\}$, $f(x)$ can be written as

$$f(x) = \frac{1}{\sqrt{2}} \sum_n \left|\langle n \left| \psi_+(0) \right| PS\rangle\right|^2 \delta\left(P_n^+ - (1-x)P^+\right). \qquad (2.2.37)$$

Here, $\sum_n$ denotes the sum over all the quantum numbers of the state $| n \rangle$ and integration over $d^3\mathbf{P}_n$. Eqn. (2.2.37) clearly gives the probability of finding a quark of longitudinal momentum $x = k^+/P^+$ inside the nucleon irrespective of its polarization.

For the polarized distribution $g(x)$, we use the projector[7] $P^{\pm} = \frac{1}{2}(1 \pm \gamma^5)$ and obtain

$$\begin{aligned} g(x) &= \frac{1}{\sqrt{2}} \sum_n \delta\left(P_n^+ - (1-x)P^+\right) \\ &\quad \times \left\{\left|\langle n \left| P^+\psi_+(0) \right| PS\rangle\right|^2 - \left|\langle n \left| P^-\psi_+(0) \right| PS\rangle\right|^2\right\}. \end{aligned} \qquad (2.2.38)$$

Hence, $g(x)$ represents the number density of quarks with helicity + minus the number density of the quarks with helicity − (assuming the parent nucleon to have helicity +).

2.3 Operator Product Expansion

In previous sections, we have implicitly used the hadronic tensor as

$$W^{\mu\nu} = \sum_a f_{a/T} w_a^{\mu\nu}, \qquad (2.3.1)$$

where $f_{a/T}$ is the probability of finding a parton of type "a" in our nucleon target T and $w_a^{\mu\nu}$ is the scattering tensor for a partonic 'target' of type "a". The sum extends over all parton species, spin and momentum.

Equation (2.3.1) is a mathematical expression of the physical separation of scales in our process. It states that the full amplitude can be factorized into a product of two

[7] see Appendix B.

parts. The functions $f_{a/T}$ describing the probability of finding various partons within our target, called *parton distribution functions*, are nonperturbative in nature. These objects are related to soft, low-energy physics and do not depend on the specific scattering process. On the other hand, the partonic scattering tensors $w_a^{\mu\nu}$ are expressed entirely in terms of the fundamental degrees of freedom in QCD and can in principle be calculated in perturbative theory. These are associated with the hard, high energy physics of the scattering and donot depend on the target we consider. This separation between hard and soft physics is called *factorization*. Furthermore, this separation is independent of the target. Since the target dependence of the amplitude,

$$W^{\mu\nu} = \frac{1}{4\pi} \int d^4 z e^{-iqz} \langle PS \,| J^{\mu}(0) J^{\nu}(z) |\, PS \rangle \,, \tag{2.3.2}$$

is contained entirely within the external states, it is natural to ask whether or not we can strip these states away and express factorization as an operator relation. This idea was first introduced by Wilson in 1969 [2] and has been used extensively in deep inelastic scattering and other perturbative QCD processes.

The fundamental idea of OPE is that for large Q^2 the integral in (2.3.2) has support only in the region of small z. This means that our amplitude receives contributions only from regions in which the operator product $J^{\mu} J^{\nu}$ is nearly local, which suggests a kind of Taylor expansion of the operator $J^{\mu}(z)$ about $z = 0$. In this way, we express the product of operators as an infinite series known as an Operator Product Expansion (OPE).

For the product of two currents separated near the light cone, the expansion is three-fold. Primarily it is a twist expansion in which twist-2 contributions are leading whereas the higher twist terms are suppressed by powers of $1/Q^2$. Each term in the twist expansion contains an infinite number of local operators of the relevant twist. Finally the coefficients of local operators (Wilson coefficients) are themselves expansions in the strong coupling constant.

It is convenient to consider the amplitude involving time-ordered products rather than products

$$T^{\mu\nu} = i \int d^4 z e^{-iqz} \langle PS \left| T J^\mu(0) J^\nu(z) \right| PS \rangle . \tag{2.3.3}$$

The relation between (2.3.2) and (2.3.3) can be establish as follows. Consider the tensor

$$\int d^4 z e^{-iqz} \langle PS \left| J^\nu(z) J^\mu(0) \right| PS \rangle . \tag{2.3.4}$$

Introducing a complete set of intermediate states and using the identity (2.1.5), we see that it is equal to

$$\sum_X \langle PS \left| J^\nu(0) \right| X \rangle \langle X \left| J^\mu(0) \right| PS \rangle (2\pi)^4 \delta^4(q + P_X - P).$$

The sum extends over all physical states with energy-momentum conservation $P_X = P - q$. In the target rest frame $q_0 = \nu > 0$, there is no intermediate state $| \ X\rangle$ with energy $E_X = M - \nu \leq M$ which can contribute; thus the above term vanishes. Hence, we can write $W^{\mu\nu}$ as

$$W^{\mu\nu} \sim \frac{1}{4\pi} \int d^4 z e^{-iqz} \langle PS \left| [J^\mu(0), J^\nu(z)] \right| PS \rangle , \tag{2.3.5}$$

where the '$\sim$' is meant to remind that this identification is only valid in the physical region of scattering, i.e. $q_0 > 0$. Using the same trick on the tensor $T^{\mu\nu}$, we obtain

$$T^{\mu\nu} \sim i \int d^4 z \Theta(-z^0) e^{-iqz} \langle PS \left| [J^\mu(0), J^\nu(z)] \right| PS \rangle \tag{2.3.6}$$

and its complex conjugate

$$\begin{aligned} (T^{\mu\nu})^* &\sim -i \int d^4 z \Theta(-z^0) e^{iqz} \left\langle PS \left| [J^\mu(0), J^\nu(z)]^\dagger \right| PS \right\rangle \\ &= -i \int d^4 z \Theta(-z^0) e^{iqz} \langle PS \left| [J^\nu(0), J^\mu(-z)] \right| PS \rangle \\ &= -i \int d^4 z \Theta(z^0) e^{-iqz} \langle PS \left| [J^\nu(0), J^\mu(z)] \right| PS \rangle , \end{aligned} \tag{2.3.7}$$

which differs from $T^{\nu\mu}$ only in the argument of Θ-function. Writing

$$\begin{aligned} \Theta(z^0) &= \frac{1}{2}\left(1+\varepsilon(z^0)\right), \\ \text{with} \quad \varepsilon(z^0) &= 1 \quad for \quad z^0 > 0, \\ \varepsilon(z^0) &= -1 \quad for \quad z^0 < 0, \end{aligned} \tag{2.3.8}$$

it becomes obvious that

$$\mathrm{Im}\, T^{(\mu\nu)} \sim 2\pi W^{(\mu\nu)}, \tag{2.3.9a}$$

for the symmetric parts and

$$\mathrm{Re}\, T^{[\mu\nu]} \sim 2\pi i W^{[\mu\nu]}, \tag{2.3.9b}$$

for the antisymmetric parts. Decomposing $T^{\mu\nu}$ as

$$\begin{aligned} T^{\mu\nu} &= \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) T_1\left(\nu, q^2\right) + \left(P^\mu - \frac{P\cdot q}{q^2} q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2} q^\nu\right) \frac{1}{P\cdot q} T_2\left(x, Q^2\right) \\ &\quad - \frac{M}{P\cdot q} \epsilon^{\mu\nu\rho\sigma} q_\rho \left\{ S_\sigma T_1\left(x, Q^2\right) + \left(S_\sigma - \frac{S\cdot q}{P\cdot q} P_\sigma\right) T_2\left(x, Q^2\right)\right\}, \end{aligned} \tag{2.3.10}$$

we see that the structure functions associated with W are just $\frac{1}{2\pi}$ times the imaginary part of those associated with T.

Now using Wick's theorem we can expand the time-ordered product. By inspection, one can see that singular behaviour occurs only when we contract two quark fields from the different currents together to form a propagator. Thus stripping external states from (2.3.3) we have[8],

$$\begin{aligned} & i\int d^4 z e^{-iqz} T\left[J^\mu(0) J^\nu(z)\right] \\ = \; & i\int d^4 z \frac{d^4 k}{(2\pi)^4} \Theta(-z^0) e^{i(k-q)\cdot z} e^2 \overline{\psi}(0) \gamma^\mu \frac{i}{\not{k}} \gamma^\nu \psi(z) + \left\{ \begin{array}{c} \mu \leftrightarrow \nu \\ q \leftrightarrow -q \end{array} \right\} \end{aligned}$$

[8]The replacements ($\mu \leftrightarrow \nu$ & $q \leftrightarrow -q$) for second term is true only for diagonal matrix elements.

$$= i\int d^4z \frac{d^4k}{(2\pi)^4}\Theta(-z^0)e^2\overline{\psi}(0)\gamma^\mu \frac{i}{\not{q} - i\not{\partial}} e^{i(k-q)\cdot z}\gamma^\nu\psi(z) + \left\{ \begin{array}{c} \mu \leftrightarrow \nu \\ q \leftrightarrow -q \end{array} \right\} . \quad (2.3.11)$$

Here we have dropped the summation over q and will retain it at the end. Using

$$\frac{i}{\not{q} - i\not{\partial}} = \frac{\not{q} - i\not{\partial}}{q^2}\sum_{n=0}^{\infty}\left(\frac{\partial^2 + 2q\cdot i\partial}{q^2}\right)^n, \quad (2.3.12)$$

and integrating by parts (2.3.11) becomes

$$\begin{aligned} & i\int d^4z e^{-iqz} T\left[J^\mu(0)J^\nu(z)\right] \\ = & \int d^4z \frac{d^4k}{(2\pi)^4}\Theta(-z^0)e^{i(k-q)\cdot z} \\ & \overline{\psi}(0)\gamma^\mu\left(\frac{\not{q} + i\not{\partial}}{Q^2}\right)\sum_{n=0}^{\infty}\left(\frac{2q\cdot i\partial - \partial^2}{Q^2}\right)^n\gamma^\nu\psi(z) + \left\{ \begin{array}{c} \mu \leftrightarrow \nu \\ q \leftrightarrow -q \end{array} \right\}. \quad (2.3.13) \end{aligned}$$

Integrating over k and z we obtain the expansion

$$\begin{aligned} i\int d^4z e^{-iqz} T\left[J^\mu(0)J^\nu(z)\right] = & \overline{\psi}(0)\left\{\sum_{n=0}^{\infty}\gamma^\mu\left(\frac{i\not{\partial} + \not{q}}{Q^2}\right)\gamma^\nu\left(\frac{2q\cdot i\partial - \partial^2}{Q^2}\right)^n\right. \\ & \left. + \gamma^\nu\left(\frac{i\not{\partial} - \not{q}}{Q^2}\right)\gamma^\mu\left(\frac{-2q\cdot i\partial - \partial^2}{Q^2}\right)^n\right\}\psi(z), \quad (2.3.14) \end{aligned}$$

involving only local operators. Since the operator ∂^2 cannot compete with Q^2 in the Bjorken limit, so we can safely neglect this term. Using (2.2.11) and rearranging, we arrive at

$$\begin{aligned} & i\int d^4z e^{-iq\cdot z} T\left[J^\mu(0)J^\nu(z)\right] \\ = & \frac{2}{Q^2}\left(g^{\mu\alpha} + \frac{q^\mu q^\alpha}{Q^2}\right)\left(g^{\nu\beta} + \frac{q^\nu q^\beta}{Q^2}\right){\sum_{n=0}^{\infty}}'\bar{\psi}(i\partial_\alpha\gamma_\beta + i\partial_\beta\gamma_\alpha)\left(\frac{2q\cdot i\partial}{Q^2}\right)^n\psi \quad (2.3.15) \\ & -\frac{2}{Q^2}\left(g^{\mu\nu} + \frac{q^\mu q^\nu}{Q^2}\right){\sum_{n=1}^{\infty}}'\bar{\psi}\not{q}\left(\frac{2q\cdot i\partial}{Q^2}\right)^n\psi \quad (2.3.16) \end{aligned}$$

$$-i\varepsilon^{\mu\nu\alpha\beta}\frac{2}{Q^2}\sum_{n=0}^{\infty}{}' \,\bar{\psi}\left(q_\alpha+\frac{2q\cdot i\partial}{Q^2}i\partial_\alpha\right)\gamma_\beta\gamma_5\left(\frac{2q\cdot i\partial}{Q^2}\right)^n\psi \tag{2.3.17}$$

$$+\frac{2}{Q^2}\left(\frac{q^\mu q^\alpha}{Q^2}g^{\mu\beta}-\frac{q^\nu q^\beta}{Q^2}g^{\mu\alpha}\right)\sum_{n=0}^{\infty}{}' \,\bar{\psi}(i\partial_\alpha\gamma_\beta-i\partial_\beta\gamma_\alpha)\left(\frac{2q\cdot i\partial}{Q^2}\right)^n\psi \tag{2.3.18}$$

$$-\frac{2}{Q^2}g^{\mu\nu}\sum_{n=0}^{\infty}\bar{\psi}i\not{\partial}\left(\frac{2q\cdot i\partial}{Q^2}\right)^n\psi. \tag{2.3.19}$$

Here, a primed summation indicates that only even or odd numbers are summed over, depending on whether the sum starts with 0 or 1. Contracting with q_μ, we immediately see that (2.3.15), (2.3.16) and first term in (2.3.17) is automatically gauge invariant. However, (2.3.18), (2.3.19) and the second term in (2.3.17) does not satisfy this condition. To see why, let us imagine taking diagonal hadronic matrix elements of our expression. Since the operators on right-hand-side do not have any intrinsic dependence on q^μ, Lorentz invariance requires all their vector indices to be carried by the hadronic momentum P^μ and all their axial indices to be carried by spin S^μ. This implies that (2.3.18) will become zero[9]. After taking the matrix element the nonvanishing gauge dependent piece in (2.3.17) will have following form

$$\varepsilon^{\mu\nu\alpha\beta}P_\alpha S_\beta. \tag{2.3.20}$$

In the Bjorken limit, this tensor must vanish since S_β is parallel to P_β. The last term vanishes by the equation of motion[10].

Now that we have taken care of electromagnetic gauge invariance, it is natural to consider chromodynamic gauge invariance. As it stands, none of the operators on the right-hand-side of our expansion are gauge-invariant. The reason for this is very simple- we are working only to leading order in QCD. To maintain explicit gauge invariance, we can introduce a gauge link in-between the two remaining fields. The effect of this procedure is a simple replacement of all partial derivatives with covariant derivatives.

[9] because (2.5.18) is antisymmetric in vector indices.

[10] which to the order at which we work, is simply $i\not{\partial}=0$.

Stripping away the factors of q^μ, we see that our expansion involves local operators of the form

$$\bar{\psi}\gamma^{\mu_1} iD^{\mu_2} \cdots iD^{\mu_n}\psi. \tag{2.3.21}$$

This n-index tensor can be separated into several disjoint pieces. For our purpose leading contibution comes from that piece which consist of a tensor with highest spin. This can be achieved by symmetrizing and removing the traces of (2.3.21). With this view, we define

$$\mathcal{O}_n^{\mu_1\mu_2\cdots\mu_n} \equiv \bar{\psi}\gamma^{(\mu_1} iD^{\mu_2} \cdots iD^{\mu_n)}\psi, \tag{2.3.22}$$

$$\widetilde{\mathcal{O}}_n^{\mu_1\mu_2\cdots\mu_n} \equiv \bar{\psi}\gamma^{(\mu_1} iD^{\mu_2} \cdots iD^{\mu_n)}\gamma_5\psi. \tag{2.3.23}$$

In terms of these operators, we can write our expansion as

$$\begin{aligned} & i\int d^4z e^{-iq\cdot z} T\left[J^\mu(0)J^\nu(z)\right] \\ = & \frac{2}{Q^2}\sum_{n=0}^{\infty}{}' \frac{(2q)_{\mu_1}(2q)_{\mu_2}\cdots(2q)_{\mu_n}}{(Q^2)^n} \times \left[\left\{2\left(\delta^\mu_\alpha + \frac{q^\mu q_\alpha}{Q^2}\right)\left(\delta^\nu_\beta + \frac{q^\nu q_\beta}{Q^2}\right)\right.\right. \\ & \left.\left. -2\frac{q_\alpha q_\beta}{Q^2}\left(g^{\mu\nu} + \frac{q^\mu q^\nu}{Q^2}\right)\right\}\mathcal{O}_{n+2}^{\alpha\beta\mu_1\cdots\mu_n} - i\varepsilon^{\mu\nu\alpha\beta} q_\alpha g_{\beta\lambda}\widetilde{\mathcal{O}}_{n+1}^{\lambda\mu_1\cdots\mu_n}\right]. \end{aligned} \tag{2.3.24}$$

This is the desired expansion of the product of two electromagnetic currents in terms of an infinite series of local operators. Removing trivial kinematical factors, a general operator of spin s and mass-dimension m appears in our expansion appears as

$$\frac{(2q)_{\mu_1}(2q)_{\mu_2}\cdots(2q)_{\mu_n}}{(Q^2)^n}\frac{1}{Q^{m-s-2}}O^{\mu_1\mu_2\cdots\mu_n}. \tag{2.3.25}$$

Taking the matrix element with momentum P such that $\frac{2P\cdot q}{Q^2} = \frac{1}{x_B}$ we see that the contribution of this operator is suppressed by Q^{m-s-2}. Since it governs the suppression of these operators, the twist

$$t \equiv m - s \tag{2.3.26}$$

is of great importance. A look at equations (2.3.22) and (2.3.23) tells us that the twist of the operators in our leading expansion is $(n-1)+3-n=2$. Hence these operators are not suppressed in the Bjorken limit.

To see the relation between our expansion (2.3.24) and the structure functions of (2.3.10), we must take forward matrix elements of $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ in terms of pure numbers:

$$\langle P | O_n^{\mu_1\mu_2\cdots\mu_n} | P\rangle \equiv 2a_n P^{(\mu_1}\ldots P^{\mu_n)}, \tag{2.3.27}$$

$$\left\langle P \left| \widetilde{O}_n^{\mu_1\mu_2\cdots\mu_n} \right| P\right\rangle \equiv 4\tilde{a}_n M S^{(\mu_1}\ldots P^{\mu_n)}. \tag{2.3.28}$$

In this way, we arrive at

$$T_1\left(\nu,Q^2\right) = 2\sum_{n=2}^{\infty}{}'\left(a_n\right)\left(\frac{1}{x_B}\right)^n, \tag{2.3.29}$$

$$T_2\left(\nu,Q^2\right) = 4\sum_{n=1}^{\infty}{}'\left(a_{n+1}\right)\left(\frac{1}{x_B}\right)^n, \tag{2.3.30}$$

$$S_1\left(\nu,Q^2\right) = 2\sum_{n=1}^{\infty}{}'\left(\tilde{a}_n\right)\left(\frac{1}{x_B}\right)^n. \tag{2.3.31}$$

Relating scalar matrix elements with quark distributions as

$$\begin{aligned}
&\int dx x^{n-1} f(x) \\
=\ & \frac{1}{2}\int dx\frac{d\lambda}{2\pi}e^{i\lambda x}x^{n-1}\left\langle PS\left|\overline{\psi}(0)\left[0;\lambda n\right]\not{n}\psi(\lambda n)\right|PS\right\rangle \\
=\ & \frac{1}{2}\int dx\frac{d\lambda}{2\pi}\left(\left(\frac{1}{i}\frac{d}{d\lambda}\right)^{n-1}e^{i\lambda x}\right)\left\langle PS\left|\overline{\psi}(0)\left[0;\lambda n\right]\not{n}\psi(\lambda n)\right|PS\right\rangle \\
=\ & \frac{1}{2}\int dx\frac{d\lambda}{2\pi}e^{i\lambda x}\left\langle PS\left|\overline{\psi}(0)\left[0;\lambda n\right]\not{n}\left(\left(n\cdot i\mathcal{D}\right)^{n-1}\right)\psi(\lambda n)\right|PS\right\rangle \\
=\ & \frac{1}{2}\left\langle PS\left|\overline{\psi}(0)\not{n}(n\cdot i\mathcal{D})^{n-1}\psi(0)\right|PS\right\rangle,
\end{aligned} \tag{2.3.32}$$

where we have ignored a surface term. Performing the same manipulations for the polarized distribution and comparing with (2.3.27) and (2.3.28), we see that

$$\int dx x^{n-1} \, f(x) = a_n, \tag{2.3.33}$$

$$\int dx x^{n-1} \, g(x) = \tilde{a}_n. \tag{2.3.34}$$

Substituting these expressions into (2.3.29), (2.3.30), and (2.3.31), switching the order of summation and integration, and re-summing, we arrive at

$$T_1(\nu, Q^2) = \int \frac{dx}{x} \frac{x^2}{x_B} \frac{1}{x_B - x} f(x) + (x_B \to -x_B), \tag{2.3.35}$$

$$T_2(\nu, Q^2) = \int \frac{dx}{x} 2x^2 \frac{1}{x_B - x} f(x) - (x_B \to -x_B), \tag{2.3.36}$$

$$S_1(\nu, Q^2) = \int \frac{dx}{x} x \frac{1}{x_B - x} g(x) - (x_B \to -x_B). \tag{2.3.37}$$

At this point we are free to take imaginary parts[11] and obtain the physical structure functions

$$F_1(x) = \frac{F_2(x)}{2x_B} = \frac{1}{2} \sum_q e_q^2 \left[f_q(x) + \bar{f}_q(x) \right], \tag{2.3.38}$$

$$G_1(x) = \frac{1}{2} \sum_q e_q^2 \left[g_q(x) + g_{\bar{q}}(x) \right]. \tag{2.3.39}$$

Note that here we have introduced the antiquarks. Their inclusion does not significantly alter the derivation above.

[11] use the identity

$$\frac{1}{x_B - x \pm i\varepsilon} = P\left(\frac{1}{x_B - x}\right) \mp i\pi\delta\left(x_B - x\right)$$

where P represent the principal part of the integral.

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Chapter 3

Conformal Invariance in QCD

One of the main goals in QCD is to understand the non-perturbative aspects of this theory. In this chapter I describe a way of parametrizing these non-perturbative effects in a covariant and gauge-invariant way: the "meson wave functions" or "light cone distribution amplitudes of meson". By definition, "wavefunction" is the matrix element of gluon and quark operators on the light cone between the vacuum and a light meson.

To compute a whole non-perturbative function seems an impossible task. Therefore the "conformal expansion" is used (it is like a partial-wave expansion, but referred to the conformal invariance of QCD). Within this expansion the renormalization-scale dependence is also controlled. To understand the conformal expansion we will review the conformal group and its transformation of fields.

3.1 Structure of the Conformal Group

3.1.1 Conformal Group

The conformal group is defined by the co-ordinate transformation $x \to x'$, for which the metric tensor remains invariant up to a scale factor,

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x). \tag{3.1.1}$$

The set of these transformations manifestly forms a group, and it has the Poincare group as a sub-group, since the latter correspond to the special case $\Lambda(x) = 1$. Conformal transformations preserve the angle between any two vectors.

Now, we investigate the consequences of the definition (3.1.1) on an infinitesimal transformation

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu. \tag{3.1.2}$$

The metric, at first order in ε, changes as follows

$$\begin{aligned} g_{\mu\nu} \quad &\to \quad g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}\, g_{\alpha\beta}(x) \\ &= \quad g_{\mu\nu}(x) - (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu) \; + \; O(\varepsilon^2). \end{aligned} \tag{3.1.3}$$

The requirement that the transformation be conformal implies that

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \; g_{\mu\nu}(x)\; h(x), \tag{3.1.4}$$

where $h(x) = 1 - \Lambda(x)$. Multiplying the above equation by $g^{\mu\nu}(x)$, we get

$$2\, \partial.\varepsilon = \; d\, h(x) \Rightarrow \; h(x) = \frac{2}{d} \partial.\varepsilon, \tag{3.1.5}$$

where d is the dimension of the space. By applying ∂_ρ on Eq.(3.1.4), permuting the indices and taking a linear combination, we arrive at

$$2\partial_\mu\partial_\nu\varepsilon_\rho = g_{\mu\rho}\,\partial_\nu\, h(x) + g_{\nu\rho}\,\partial_\mu\, h(x) - g_{\mu\nu}\,\partial_\rho\, h(x). \tag{3.1.6}$$

Upon contracting with $g^{\mu\nu}$ this becomes

$$2\partial^2\varepsilon_\mu = (2-d)\,\partial_\mu\, h(x). \tag{3.1.7}$$

Applying ∂_ν on this expression and ∂^2 on Eq. (3.1.4) we find

$$(2-d)\,\partial_\nu\partial_\mu\, h(x) = g_{\mu\nu}\partial^2 h(x). \tag{3.1.8}$$

Finally contracting with $g^{\mu\nu}$, we end up with

$$(d-1)\partial^2\, h(x) = 0. \tag{3.1.9}$$

Using Eqs.(3.1.4) $-$ (3.1.9) we can derive the explicit form of conformal transformations in d dimensions.

First, if $d = 1$, the above equations do not impose any constraint on the function $h(x)$, so any smooth transformation is conformal in one dimension. The notion of angle does not exist in this situation . The case $d = 2$ is relevant only for statistical mechanics and string theories, so we will not discuss it.

Now consider the case $d \geq 3$. Eqns (3.1.9) and (3.1.8) imply that $\partial_\mu\partial_\nu\, h(x) = 0$ (i.e., the function $h(x)$ is at most linear in co-ordinates):

$$h(x) = A + B_\mu x^\mu, \tag{3.1.10}$$

where A and B_μ are constants. If we substitute this expression into Eq.(3.1.6), we see that $\partial_\mu\partial_\nu\varepsilon_\rho$ is constant, which means that ε_μ is at most quadratic in the co-ordinates.

We therefore write the general expression

$$\varepsilon_\mu = a_\mu + b_{\mu\nu}\, x^\nu + c_{\mu\nu\rho}\, x^\nu x^\rho, \tag{3.1.11}$$

with $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. Now we can treat each power of co-ordinate separately, as constraints (3.1.4) to (3.1.6) holds for all x.

Constant Term (a^μ):

This term is free of constraint and correspond to an infinitesimal translation i.e.,

$$x'^\mu = x^\mu + a^\mu. \tag{3.1.12}$$

Linear Term ($b_{\mu\nu}x^\nu$):

Putting this term ($\varepsilon_\mu = b_{\mu\sigma}x^\sigma$) in Eq.(3.1.4), we get

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} g_{\mu\nu} b^\rho_\rho. \tag{3.1.13}$$

Writing $b_{\mu\nu}$ as a sum of symmetric and antisymmetric parts we get

$$b_{\mu\nu} = \alpha g_{\mu\nu} + m_{\mu\nu}, \tag{3.1.14}$$

where $\alpha = \frac{1}{d} b^\rho_\rho$ and $m_{\mu\nu} = -m_{\nu\mu}$. The first term represents an infinitesimal scale transformation, whereas the 2nd term is an infinitesimal rigid rotation, i.e

$$x'^\mu = \alpha\, x^\mu \qquad \text{(dilation)}, \tag{3.1.15}$$

$$x'^\mu = M^\mu_\nu x^\nu \qquad \text{(rigid rotation)}. \tag{3.1.16}$$

Quadratic Term ($c_{\mu\nu\rho}x^\nu x^\rho$):

Putting this term ($\varepsilon_\mu = c_{\mu\nu\rho}x^\nu x^\rho$) in Eq.(3.1.6), we get

$$\partial_\mu \partial_\nu \varepsilon_\rho = \frac{1}{d}[\ g_{\mu\rho}\partial_\nu \partial_\sigma \varepsilon^\sigma + g_{\nu\rho}\partial_\mu \partial_\sigma \varepsilon^\sigma - g_{\mu\nu}\partial_\rho \partial_\sigma \varepsilon^\sigma], \tag{3.1.17}$$

applying $\partial_\mu \partial_\nu$ on the quadratic term, we get

$$\partial_\mu \partial_\nu \varepsilon_\rho = c_{\rho\nu\mu} + c_{\rho\mu\nu} = 2\ c_{\rho\mu\nu}. \tag{3.1.18}$$

Putting this in Eq.(3.1.17) and arranging indices we obtain

$$c_{\rho\mu\nu} = \eta_{\mu\nu}b_\rho + \eta_{\mu\rho}b_\nu - \eta_{\nu\rho}b_\mu,$$

where $b_\mu \equiv \frac{1}{d}c^\sigma_{\sigma\mu}$ and the corresponding infinitesimal transformation becomes

$$x'^\mu = x^\mu + 2(x.b)x^\mu - b^\mu x^2, \tag{3.1.19}$$

which bears the name of special conformal transformation (SCT).

The finite transformations corresponding to above infinitesimal transformations are given as

$$\begin{aligned} x'^\mu &= x^\mu + a^\mu && \text{(translation)}, \\ x'^\mu &= e^\alpha x^\mu && \text{(dilation)}, \\ x'^\mu &= M^\mu_\nu x^\nu && \text{(rigid rotation)}, \\ x'^\mu &= \frac{x^\mu - b^\mu x^2}{1 - 2b.x + b^2 x^2} && \text{(SCT)}. \end{aligned} \tag{3.1.20}$$

The first three equations in (3.1.20) have a simple interpretation, whereas the last one has not. To understand it, we can write it as

$$\frac{x'^{\mu}}{x'^2} = \frac{x^{\mu}}{x^2} - b^{\mu}. \tag{3.1.21}$$

Thus, the SCT is nothing but a translation, preceded and followed by an inversion $\left(x^{\mu} \rightarrow \frac{x^{\mu}}{x^2}\right)$ i.e.,

$$x^{\mu} \xrightarrow{\text{(inversion)}} \frac{x^{\mu}}{x^2} \xrightarrow{\text{(translation)}} \frac{(x^{\mu} - b^{\mu})}{x^2 - 2b \cdot x + b^2}$$

$$\xrightarrow{\text{(inversion)}} \frac{(\frac{x^{\mu}}{x^2} - b^{\mu})}{1/x^2 - 2b^{\mu}(x_{\mu}/x^2) + b^2} = \frac{x^{\mu} - b^{\mu}x^2}{1 - 2b \cdot x + b^2x^2} \text{ (STC)}.$$

Now we will see how the fields transform under conformal transformations.

3.1.2 Behavior of Local Fields under Conformal Transformations.

Consider an arbitrary function of the space-time point P. In a given coordinate system, where P is located at x^{μ}, this function will be denoted by $f(x^{\mu})$; in another where P is at x'^{μ} it will be written as $f'(x'^{\mu})$. For infinitesimal transformation, the change in the function will be

$$\begin{aligned}
\delta f &= f'(x') - f(x) \\
\delta f &= f'(x + \delta x) - f(x) \\
\delta f &= f'(x) - f(x) + \delta x^{\mu}\partial_{\mu} f' + O(\delta x^2)
\end{aligned}$$

To $O(\delta x^0)$, we can replace $\partial_{\mu} f'$ by $\partial_{\mu} f$. Then

$$\delta f = \delta_0 f + \delta x^{\mu}\partial_{\mu} f, \tag{3.1.22}$$

where we have introduced the functional change at the same x

$$\delta_0 f \equiv f^{'}(x) - f(x).$$

In operator form, Eq.(3.1.22) becomes

$$\delta_0 = \delta - \delta x^\mu \partial_\mu = \delta + \delta_d. \qquad (3.1.23)$$

Differential Representation:

We define the generator T of an infinitesimal transformation as

$$\delta_0 \Phi = i\omega T \Phi(x), \qquad (3.1.24)$$

where ω is the parameter of the group. To obtain differential representation T_d, we suppose that fields are unaffected by the transformation i.e., $\delta = 0$ and therefore

$$\delta_0 \Phi = \delta_d \Phi = i\omega T_d \Phi(x). \qquad (3.1.25)$$

Using infinitesimal transformations we can write the differential representation of the conformal generators as follows:

$$\begin{aligned} P_\mu &= i\partial_\mu && \text{(translation)}, \\ D &= ix^\mu \partial_\mu && \text{(dilatation)}, \\ L_{\mu\nu} &= i(x_\mu \partial_\nu - x_\nu \partial_\mu) && \text{(rotation)}, \\ K_\mu &= i[2x_\mu\, x.\partial - x^2 \partial_\mu] && \text{(STC)}. \end{aligned} \qquad (3.1.26)$$

These generators obey the following commutation rules which in fact define the conformal algebra:

$$\begin{aligned}
[D, P_\mu] &= -iP_\mu \\
[D, L_{\mu\nu}] &= 0 \\
[D, K_\mu] &= iK_\mu \\
[K_\mu, P_\nu] &= -2i(g_{\mu\nu}D + L_{\mu\nu}) \\
[K_\rho, L_{\mu\nu}] &= i(g_{\rho\mu}K_\nu - g_{\rho\nu}K_\mu) \\
[P_\rho, L_{\mu\nu}] &= i(g_{\rho\mu}P_\nu - g_{\rho\nu}P_\mu) \\
[L_{\mu\nu}, L_{\rho\sigma}] &= i(g_{\nu\rho}L_{\mu\sigma} + g_{\mu\sigma}L_{\nu\rho} - g_{\mu\rho}L_{\nu\sigma} - g_{\nu\sigma}L_{\mu\rho}).
\end{aligned} \tag{3.1.27}$$

Representation of the Full Generators of Conformal Symmetry:

Now fields are also affected by the transformation i.e.,

$$\delta\Phi = i\omega T_t \Phi(x). \tag{3.1.28}$$

The generator T_t must be added to the differential part T_d [Eqn.(3.1.23)] to obtain the full generator T [Eqn.(3.1.24)] of the conformal symmetry i.e.,

$$T = T_d + T_t. \tag{3.1.29}$$

In order to find out the allowed form of these generators, we shall consider the subgroup of the conformal group that leaves the point $x = 0$ invariant. It consist of conformal group excluding space-time translation P_μ. Let $\Sigma_{\mu\nu}, \Delta$ and κ_μ be the respective values of the generators $L_{\mu\nu}$, D and K_μ at $x = 0$ i.e.,

$$\begin{aligned}
L_{\mu\nu}\Phi(0) &= \Sigma_{\mu\nu}\Phi(0) \\
D\Phi(0) &= \Delta\Phi(0)
\end{aligned} \tag{3.1.30}$$

$$K_\mu \Phi(0) = \kappa_\mu \Phi(0).$$

It follows for every element X of the conformal algebra

$$\begin{aligned} X(x, \partial)\Phi(x) &= e^{(-ix^\rho P_\rho)}\left(X(0,\partial)\Phi(0)\right) \\ &= X'\Phi(x), \end{aligned} \tag{3.1.31}$$

where

$$\begin{aligned} X' &= e^{(-ix^\rho P_\rho)} X(0,\partial) e^{(ix^\rho P_\rho)} \\ &= X - ix^\rho [P_\rho, X] + \frac{i^2 x^\rho x^\sigma}{2!}[P_\rho, [P_\sigma, X]] - \cdots. \end{aligned} \tag{3.1.32}$$

Using commutation relations (3.1.27), we can translate $L_{\mu\nu}, D$ and K_μ to nonzero value of x,

$$\begin{aligned} e^{-ix^\rho p_\rho} L_{\mu\nu} e^{ix^\rho p_\rho} &= \Sigma_{\mu\nu} + x_\mu P_\nu - x_\nu P_\mu \\ e^{-ix^\rho p_\rho} D e^{ix^\rho p_\rho} &= \Delta + x^\nu P_\nu \\ e^{-ix^\rho p_\rho} K_\mu e^{ix^\rho p_\rho} &= \kappa_\mu + 2x^\nu(g_{\mu\nu}\Delta + \Sigma_{\mu\nu}) + 2x_\mu(x^\nu P_\nu) - x^2 P_\mu. \end{aligned} \tag{3.1.33}$$

Using these values alongwith the Eqns. (3.1.31) and (3.1.32), we get

$$\begin{aligned} L_{\mu\nu}\Phi(x) &= \left(i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu}\right)\Phi(x) \\ D\Phi(x) &= (\Delta + ix^\nu \partial_\nu)\Phi(x) \\ K_\mu \Phi(x) &= \left(\kappa_\mu + 2x^\nu(g_{\mu\nu}\Delta + \Sigma_{\mu\nu}) + 2ix_\mu(x\cdot\partial) - ix^2\partial_\mu\right)\Phi(x). \end{aligned} \tag{3.1.34}$$

Generators of the little group form a matrix representation of the reduced algebra

$$\begin{aligned} [\Delta, \Sigma_{\mu\nu}] &= 0 \\ [\Delta, \kappa_\mu] &= i\kappa_\mu \end{aligned}$$

$$
\begin{aligned}
[\kappa_\rho, \Sigma_{\mu\nu}] &= i(\eta_{\rho\mu}\kappa_\nu - \eta_{\rho\nu}\kappa_\mu) \\
[\Sigma_{\mu\nu}, \Sigma_{\mu\nu}] &= i(\eta_{\nu\rho}\Sigma_{\mu\sigma} + \eta_{\mu\sigma}\Sigma_{\nu\rho} - \eta_{\mu\rho}\Sigma_{\nu\sigma} - \eta_{\nu\sigma}\Sigma_{\mu\rho}).
\end{aligned}
\qquad (3.1.35)
$$

If we demand that the field $\Phi(x)$ belong to an irreducible representation of the Lorentz group, then by Schur's lemma[1] $\Delta = il_\Phi I$, where l_Φ is the scaling dimension of the field $\Phi(x)$ and the algebra (3.1.35) forces all the matrices $\kappa_\mu$to vanish. Therefore we have

$$
\begin{aligned}
P_\mu\Phi(x) &= i\partial_\mu\Phi(x) \\
L_{\mu\nu}\Phi(x) &= \left(i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu}\right)\Phi(x) \\
D\Phi(x) &= i\left(x^\nu\partial_\nu + l_\Phi\right)\Phi(x) \\
K_\mu\Phi(x) &= i\left(2x_\mu(x^\nu\partial_\nu) - x^2\partial_\mu + 2x_\mu l_\Phi - 2ix^\nu\Sigma_{\mu\nu}\right)\Phi(x).
\end{aligned}
\qquad (3.1.36)
$$

These are the transformation rules for the classical fields. In quantum field theory, we have field operators acting on Hilbert space. Therefore transformation rules for quantum field operators are[2]

$$
\begin{aligned}
[P_\mu, \Phi(x)] &= -i\partial_\mu\Phi(x) \\
[L_{\mu\nu}, \Phi(x)] &= \left(i(x_\nu\partial_\mu - x_\mu\partial_\nu) - \Sigma_{\mu\nu}\right)\Phi(x) \\
[D, \Phi(x)] &= -i(x^\nu\partial_\nu + l_\Phi)\Phi(x) \\
[K_\mu, \Phi(x)] &= -i\left(2x_\mu(x^\nu\partial_\nu) - x^2\partial_\mu + 2x_\mu l_\Phi - 2ix^\nu\Sigma_{\mu\nu}\right)\Phi(x).
\end{aligned}
\qquad (3.1.37)
$$

[1]it states that if D is an irreducible representation of the group, and if

$$[D, A] = 0,$$

then A is multiple of the Identity matrix.

[2]using the correspondence between the transformaion of classical fields and the transformaion of the matix elements of the quantum field operators.

3.2 Broken Conformal Invariance

The unrenormalized QCD-action (with massless quark) does not contain dimensionful constants, therefore it should be scale invariant. This can be seen as follows: A field operator $\Phi(x)$ transforms under scale transformation ($x \to x' = e^{\alpha}x$) as

$$\Phi(x) \to \Phi'(x) = U^{\dagger}(\alpha)\Phi(x)U(\alpha) = \exp(-\alpha l_{\Phi})\Phi(e^{-\alpha}x) \tag{3.2.1}$$

with $U(\alpha) = \exp(i\alpha D)$. For free fields we have the canonical equal time commutation relations

$$\begin{aligned} \left[\varphi(\mathbf{x},t), \dot{\varphi}(\mathbf{y},t)\right] &= i\delta^3(\mathbf{x}-\mathbf{y}) \\ \{\psi(\mathbf{x},t), \bar{\psi}(\mathbf{y},t)\} &= i\delta^3(\mathbf{x}-\mathbf{y}) \end{aligned} \tag{3.2.2}$$

and the scale dimension l_{Φ} is then defined so that the commutation relations (3.2.2) remains invariant under the scale transformation. Transforming the fields in relations (3.2.2) according to (3.2.1) we obtain the so-called canonical scale dimensions, $l_{\Phi} = 1$ for scalar fields and $l_{\Phi} = \frac{3}{2}$ for spinor fields. We see that the canonical scale dimensions of fields coincide with their ordinary dimensions defined on purely dimensional grounds. So QCD lagrangian will be transformed as

$$\begin{aligned} \delta\mathcal{L}(x) &= \mathcal{L}(\Phi'(x)) - \mathcal{L}(\Phi(x)) \\ &= -\alpha\,(4 + x\cdot\partial)\,\mathcal{L} \end{aligned} \tag{3.2.3}$$

and the corresponding action integral is scale invariant i.e.,

$$\delta I = \int d^4x\delta\mathcal{L}(x) = 0. \tag{3.2.4}$$

Since conformal invariance includes scale invariance and Poincare invariance, and the unrenormalized QCD-action is conformally invariant. But this symmetry is broken by two effects:

1) Quark-mass terms,

2) Renormalization.

We will now consider these effects one by one.

3.2.1 Quark Mass Effect on the Conformal Invariance

Scale invariance cannot be an exact symmetry of the real world. If it were all particles would have to be massless or their mass spectra continuous. Indeed, it follows from commutation relation

$$[D, P_\mu] = -iP_\mu,$$

that

$$\left[D, P^2\right] = -2iP^2,$$

or, exponentiating,

$$\exp(i\alpha D)P^2 \exp(-i\alpha D) = \exp(2\alpha)P^2. \tag{3.2.5}$$

Acting on a single particle state$\mid p\rangle$ with four-momentum p ($P^2 \mid p\rangle = p^2 \mid p\rangle$) we get

$$P^2 \exp(-i\alpha D) \mid p\rangle = \exp(2\alpha)p^2 \exp(-i\alpha D) \mid p\rangle, \tag{3.2.6}$$

i.e. the state $\exp(-i\alpha D) \mid p\rangle$ is an eigenstate of P^2 with eigenvalue $\exp(2\alpha)p^2$. If we assume in addition that the vacuum is unique under scale transformations (that is if scale invariance is not spontaneously broken)

$$\exp(-i\alpha D) \mid 0\rangle =\mid 0\rangle,$$

then we conclude that

$$\exp(-i\alpha D) \mid p\rangle = \exp(-i\alpha D)b^\dagger(p) \mid 0\rangle = \exp(-d_b)b^\dagger(e^{-\alpha}p) \mid 0\rangle, \tag{3.2.7}$$

where $b^\dagger(p)$ is the creation operator for the cosidered particle with momentum p and dimension d_b. Result (3.2.7) means that the state $\exp(-i\alpha D) \mid p\rangle$ is a quantum of the same field as the state $\mid p\rangle$ but with rescaled momentum and therefore by (3.2.6), all particles must be massless or the mass spectrum must be continuous.

3.2.2 Renormalization Effect on the Conformal Invariance

It is important to realize that the conformal invariance at the quantum level generally does not follow from conformal invariance at the classical level. Scale invariance requires that there be no dimensionful parameters whereas the regularization of the a quantum field theory is effected by introducing a dimensionful cut-off or dimensionful coupling constants in the dimension regularization procedure. Equivalently, an unavoidable renormalization procedure necessarily introduces a scale at which the theory is renormalized and this breaks scale invariance.

Consider an operator in massless QCD, which renormalizes multiplicatively, so its matrix elements Γ in $4-\varepsilon$ dimension satisfy the equation

$$\Gamma_R(p_i, g_R, \mu, \varepsilon) = Z(g_R, \varepsilon)\Gamma_B(p_i, g_B, \varepsilon), \tag{3.2.8}$$

where $\Gamma_R(\Gamma_B)$ is the renormalized(bare) Green's function, depending on the renormalized coupling constant $g_R(g_B)$ and the renormalization scale μ, and Z is the renormalization constant. The bare Green's function Γ_B is obviously renormalization scale μ independent. Renormalized coupling constant g_R depends on g_B, μ and ε. The finite limit for $\varepsilon \to 0$ exists since the theory is renormalizable. Differentiating (3.2.8) and using $\frac{d\Gamma_B}{d\mu} = 0$, we obtain

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g_R, \varepsilon)\frac{\partial}{\partial g}\right)\Gamma_R(p_i, g_R, \mu, \varepsilon) = \gamma_\Gamma(g_R, \varepsilon)\Gamma_R(p_i, g_R, \mu, \varepsilon), \tag{3.2.9}$$

where β and γ_Γ are defined as

$$\begin{aligned} \beta(g_R,\varepsilon) &= \mu\frac{\partial g_R}{\partial\mu} \underset{\varepsilon\to 0}{\to} \beta(g_R), & (3.2.10) \\ \gamma_\Gamma(g_R,\varepsilon) &= \mu\frac{\partial \ln Z}{\partial\mu} \underset{\varepsilon\to 0}{\to} \gamma_\Gamma(g_R). & (3.2.11) \end{aligned}$$

Taking the limit $\varepsilon \to 0$ in (3.2.9) and dropping the subscript R, we obtain the Callan-Symanzik equation:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right)\Gamma(p_i, g, \mu) = \gamma_\Gamma(g)\Gamma(p_i, g, \mu). \tag{3.2.12}$$

As this matrix element belongs to massless theory, it is an homogeneous function of p and μ. So with the rescaled momenta $p_i \to \rho p_i$ Euler's theorem implies that

$$\left(\mu\frac{\partial}{\partial\mu} + \rho\frac{\partial}{\partial\rho}\right)\Gamma(\rho p_i, g, \mu) = l_\Gamma\Gamma(\rho p_i, g, \mu), \tag{3.2.13}$$

where l_Γ is the canonical dimension of the operator considered. Thus equation (3.2.12) can be written as

$$\left(\rho\frac{\partial}{\partial\rho} - \beta(g)\frac{\partial}{\partial g}\right)\Gamma(\rho p_i, g, \mu) = (l_\Gamma - \gamma_\Gamma(g))\,\Gamma(\rho p_i, g, \mu). \tag{3.2.14}$$

If we consider the theory at one loop order, $\beta(g)$ can be neglected because it starts with g^3 and we get

$$\rho\frac{\partial}{\partial\rho}\Gamma(\rho p_i, g, \mu) = (l_\Gamma - \gamma_\Gamma(g))\,\Gamma(\rho p_i, g, \mu), \tag{3.2.15}$$

with the solution

$$\Gamma(\rho p_i, g, \mu) = \rho^{(l_\Gamma - \gamma_\Gamma(g))}\Gamma(p_i, g, \mu). \tag{3.2.16}$$

This equation shows that the theory is scale invariant with canonical dimension l_Γ replaced by $l_\Gamma - \gamma_\Gamma(g)$ i.e., the operator has gained an extra dimension $\gamma_\Gamma(g)$ which is called the anomalous dimension of the operator. So if the interaction is switched off

($i.e., \gamma_\Gamma(g) = 0$), we get the matrix element which transforms in itself under a dilatation: it is the matrix element of a conformal operator.

So if an operator is renormalized multiplicatively at one loop order, it is conformal if the interaction is switched off. Is the converse true?

Suppose we have a set of operators. A basis which diagonalizes the anomalous dimension matrix at one loop order is built. From the preceding arguments, we know that this basis is made of conformal operators. If we directly build such a basis of conformal operators (like in the next subsection), almost the same basis of multiplicatively renormalized operator is obtained, except if the eigenvalues of the anomalous matrix are degenerate.

Although conformal operators renormalize almost multiplicatively at one loop order, they can mix with the ones which have the same conformal representation. This property of conformal operators works also in the leading logarithmic approximation, otherwise it would contradict this property at one loop order . The coefficients of the conformal expansion of the wave function are matrix elements of the local conformal operators. So they are renormalized mutiplicatively, that is why conformal expansion of the wave function make sense.

3.3 Wave Function

The two point wave function for ρ-meson is defined as

$$\left\langle 0 \left| \bar{u}(x) \left[x; -x\right] \Gamma d(-x) \right| \rho^-(P, s) \right\rangle . \tag{3.3.1}$$

Where x is almost on the light cone, Γ any kind of product of γ^μ matrices and $[x; y]$ a path-ordered gauge link connecting x and y along the straight line :

$$[x; y] = P \exp\left[ig(x-y)_\mu \int_0^1 dt A^\mu \left(tx + (1-t)y\right)\right] \tag{3.3.2}$$

This wave function depends on three vectors:

P^μ: momentum of the ρ-meson,

$\varepsilon^\mu_{(s)}$: polarisation vector of the ρ-meson,

x^μ,

with the relations:

$$P^2 = m_\rho^2, \tag{3.3.3}$$

$$\varepsilon_{(s)} \cdot \varepsilon_{(s)} = -1, \tag{3.3.4}$$

$$P \cdot \varepsilon_{(s)} = 0. \tag{3.3.5}$$

The parametrization of the ρ meson wave function is based on the operator product expansion on the light cone. So we need light-like vectors p_μ and n_μ,

$$p^2 = n^2 = 0, \tag{3.3.6}$$

such that $P^\mu \to p^\mu$ in the limit $m_\rho^2 \to 0$ and $x^\mu \to \xi^\mu = \lambda n^\mu$ for $x^2 \to 0$:

$$P^\mu = p^\mu + \frac{m_\rho^2}{2} n^\mu \tag{3.3.7}$$

$$x^\mu = \lambda n^\mu + \frac{1}{m_\rho^2}\left[x \cdot P - \sqrt{(x \cdot P)^2 - x^2 m_\rho^2}\right] P^\mu \tag{3.3.8}$$

The polarization vector $\varepsilon^\mu_{(s)}$ can be decomposed in projections onto the two light-like vectors and the orthogonal plane:

$$\varepsilon^\mu_{(s)} = \varepsilon_{(s)} \cdot n \left(p^\mu - \frac{m_\rho^2}{2} n^\mu\right) + \varepsilon^\mu_{\perp(s)}. \tag{3.3.9}$$

In the remaining part of this chapter we will use the following notation for any four-vector A^μ

$$A^+ \equiv A \cdot \lambda n$$

$$A^- \equiv A \cdot p.$$

Using these decompositions we parametrize the two-point wave function of the vector ρ-meson as

$$\begin{aligned}
&\langle 0\, |\bar{u}(\xi)\gamma^\mu\, [\xi; -\xi]\, d(-\xi)|\, \rho^-(P,s)\rangle \\
= \;& f_\rho m_\rho \left[\left(\varepsilon^{(s)} \cdot n \int_0^1 dz e^{iz\lambda} \phi_\parallel(z,\mu^2) \right) p^\mu + \left(\int_0^1 dz e^{iz\lambda} g_\perp(z,\mu^2) \right) \varepsilon^\mu_{\perp(s)} \right. \\
& \left. - \left(\frac{m_\rho^2}{2} \varepsilon^{(s)} \cdot n \int_0^1 dz e^{iz\lambda} g_3(z,\mu^2) \right) n^\mu \right], \qquad (3.3.10)
\end{aligned}$$

where μ is the renormalization scale and f_ρ is defined by the following marix element:

$$\langle 0\, |\bar{u}(0)\gamma^\mu d(0)|\, \rho^-(P,s)\rangle = f_\rho m_\rho e^\mu_{(s)}. \qquad (3.3.11)$$

All the three functions $\phi_\parallel$, $g_\perp$, g_3 are normalized as,

$$\int_0^1 dz \phi_\parallel(z) = 1. \qquad (3.3.12)$$

Here $\phi_\parallel$ is the twist-2 contribution, $g_\perp$ is twist -3 and g_3 is twist-4. We will discuss only the twist-2 contribution $\phi_\parallel$. A physical meaning for this wave function can be given: it describes the probability amplitude to find the ρ-meson in a state with a quark and an antiquark which carry momentum fractions z for the quark and $1 - z$ for the antiquark respectively and a small transverse separation of the order $\frac{1}{\mu}$.

Various functions can extracted from non-local matrix elements like (3.3.1) and classified. Then, one needs to compute them. The method used to compute these functions is the "conformal expansion". The basic idea is to expand the wave function in a series of polynomials whose coefficients are renormalized (almost) multiplicatively. In the next subsection we will see how to calculate $\phi_\parallel$.

3.3.1 Local operators on the Light-Cone

Using the above mentioned parametrization it can be shown that only the components P^+, D, M^{-+} and K^- of the conformal algebra act non-trivially on the field $\Phi(\lambda n)$

$$\begin{aligned}
\left[P^+, \Phi(\lambda n)\right] &= -i\partial^+\Phi(\lambda n) && (3.3.13a)\\
\left[M^{-+}, \Phi(\lambda n)\right] &= -\left(i\partial^+ + \Sigma^{-+}\right)\Phi(\lambda n) && (3.3.13b)\\
\left[D, \Phi(\lambda n)\right] &= -i(\partial^+ + l)\Phi(\lambda n) && (3.3.13c)\\
\left[K^-, \Phi(\lambda n)\right] &= -2i\left(\partial^+ + l - i\Sigma^{-+}\right)\Phi(\lambda n). && (3.3.13d)
\end{aligned}$$

These operators satisfy the subalgebra

$$\begin{aligned}
\left[D, P^+\right] &= -iP^+ && (3.3.14a)\\
\left[D, M^{-+}\right] &= 0 && (3.3.14b)\\
\left[K^-, D\right] &= -iK^- && (3.3.14c)\\
\left[K^-, P^+\right] &= -2i(D + M^{-+}), && (3.3.14d)
\end{aligned}$$

of the subgroup, called the collinear conformal subgroup $SO(2,1) \cong SU(1,1)$. In order to bring this algebra into a more standard form, we consider the following linear combinations:

$$\begin{aligned}
J_+ &= J_1 + J_2 = \frac{i}{\sqrt{2}}P^+, && (3.3.15a)\\
J_- &= J_1 - J_2 = \frac{i}{\sqrt{2}}K^-, && (3.3.15b)\\
J_3 &= \frac{i}{2}(D + M^{-+}), && (3.3.15c)\\
E &= \frac{i}{2}(D - M^{-+}), && (3.3.15d)\\
J^2 &= J_3^2 - J_1^2 - J_2^2 = J_3^2 - J_3 - J_+J_-. && (3.3.15e)
\end{aligned}$$

Then commutators (3.3.14a-d) become

$$\begin{aligned}
[J_+, J_-] &= -2J_3 \\
[J_3, J_\pm] &= \pm 2J_\pm \\
[E, J_i] &= 0 \\
\left[J^2, J_i\right] &= 0 \quad for \quad i = +, -, 3.
\end{aligned} \tag{3.3.16}$$

In order to build a representation of this subgroup, all the generators P^+, D, M^{-+} and K^- must reduce to differential operators. Equations (3.3.13a-d) imply that fields must be eigenvectors of the operator Σ^{-+}, that is fields having fixed projections (s) of the Lorentz spin on to the line n^μ:

$$\Sigma^{-+}\Phi(\lambda n) = is\Phi(\lambda n). \tag{3.3.17}$$

A spinor field ψ has two components with the projection $s = \pm\frac{1}{2}$, namely $\gamma^-\psi$ and $\gamma^+\psi$. One can see this using $\Sigma_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu}$ for Dirac particles,

$$\begin{aligned}
\Sigma^{-+}\gamma^+\psi &= \frac{1}{2}\sigma^{-+}\gamma^+\psi = +\frac{i}{2}\gamma^+\psi, \\
\Sigma^{-+}\gamma^-\psi &= \frac{1}{2}\sigma^{-+}\gamma^-\psi = -\frac{i}{2}\gamma^-\psi.
\end{aligned} \tag{3.3.18}$$

If one has a field $\Phi(\alpha\xi)$ (α is a real number) which has a fixed projection s of the Lorenz spin on to the line λn^μ, the generators of the collinear conformal subgroup act in the following way:

$$[J_+, \Phi(\alpha\xi)] = \frac{1}{\sqrt{2}}\frac{d}{d\alpha}\Phi(\alpha\xi) \tag{3.3.19a}$$

$$[J_3, \Phi(\alpha\xi)] = \frac{1}{2}\left(l + s + 2\alpha\frac{d}{d\alpha}\right)\Phi(\alpha\xi) \tag{3.3.19b}$$

$$[J_-, \Phi(\alpha\xi)] = \sqrt{2}\left(\alpha(l + s) + \alpha^2\frac{d}{d\alpha}\right)\Phi(\alpha\xi) \tag{3.3.19c}$$

$$[E, \Phi(\alpha\xi)] = \frac{1}{2}(l - s)\Phi(\alpha\xi). \tag{3.3.19d}$$

From this relation the irreducible representarions of the collinear conformal subgroup SO(2,1) which contains the non-trivial conformal trnsformation on operators on the light cone can be constructed. These representations are classified by the eigenvalues of the Casimir operator J^2. Equations (3.3.19a-d) describe the construction of operators $\Phi(\alpha\xi)$ which are eigenvectors of J^2:

$$[J^2, \Phi(\alpha\xi)] = j(j-1)\Phi(\alpha\xi), \tag{3.3.20}$$

where $j = \frac{1}{2}(l + s)$ is called "conformal spin". For $\Phi(\alpha\xi)$, the algebra is reduced to diffrential operators, so one can work with functions of one real variable instead of quantum fields. Having in mind the application to the computation of the wave functions, one can make the Fourier transform of the quantum field Φ :

$$\Phi(\alpha\lambda) = \int dz e^{-i\alpha z\lambda}\phi(z). \tag{3.3.21}$$

In this formalism, the irreducible representations of the collinear conformal subgroup can be built, classified by the eigenvalues of the J^2 and $-J_3$. They are the set of the following functions:

$$|j, n\rangle = \frac{1}{\Gamma(j+n)}(\frac{v}{i\sqrt{2}})^{j+n-1}. \tag{3.3.22}$$

Equations (3.3.19a-c) give the action of the collinear conformal subgroup on the states $|j, n\rangle$:

$$J_+ |j, n\rangle = (j+n) |j, n+1\rangle , \tag{3.3.23a}$$

$$J_- |j, n\rangle = (n-j) |j, n-1\rangle , \tag{3.3.23b}$$

$$J_3 |j, n\rangle = -n |j, n\rangle . \tag{3.3.23c}$$

One can see that J_+ transforms a state one step upper and J_- one step lower. The lowest value of the n is j.

3.3.2 Polylocal Operators on the Light-Cone

In this section we will find the irreducible representation of the product of n local operators $\Phi_1(\alpha_1\xi), ..., \Phi_n(\alpha_n\xi)$ which have a fixed conformal spin. To construct such irreducible representation of the colinear conformal subgroup, the corresponding "Clebsch-Gordon coefficents" of a tensor product of different irreducible representation are needed:

$$|j,n\rangle = \sum_{n_1+...+n_k=n} C^{j,n}_{j_1,n_1,...,j_k,n_k} |j_1,n_1\rangle ... |j_k,n_k\rangle \,. \tag{3.3.24}$$

This seems a very complicated task, but one has to recall what is the conformal collinear subgroup: it is the Lorenz group in $2+1$ dimensions (SO(2,1)). So the irreducible representations $|j,n\rangle$ can be interpreted as relativistic particles in an abstract or internal space of $2+1$ dimensions. With this point of view, the state $|j,n\rangle$ is a particle with a mass j and energy n (that is why $n \geq j$).

Consider a system of k-particles. The lowest invariant mass of this system is the sum of masses of all the particles, $j_{\rm min} = j_1 + \cdots + j_k$. If the state of the lowest invariant mass has also the lowest energy, all the particles must be in their lowest energy level, $n_i = j_i$. This state is non-degenerate in the language of irreducible representations of $SO(2,1)$ and reads

$$\begin{aligned} \left| j_{\rm min} = \sum_{i=1}^{k} j_k, \; n_{\rm min} = j_{\rm min} \right\rangle \;&\sim\; |j_1, j_1\rangle \cdots |j_k, j_k\rangle \\ &\sim\; v_1^{2j_1-1} \cdots v_k^{2j_k-1}. \end{aligned} \tag{3.3.25}$$

If $n > j_{\rm min}$, the operator $J_+ = \sum_{i=1}^{k} J_+^{(i)}$ can be used to "raise" the "energy" of the state.

Since $J_+\phi(z) = \frac{1}{\sqrt{2i}} z\phi(z)$, one gets:

$$\left| j_{\min} = \sum_{i=1}^{k} j_i,\ n \right\rangle \sim (z_1 + z_2)^{n-j} \, |j_1, j_1\rangle \cdots |j_k, j_k\rangle$$
$$\sim (z_1 + z_2)^{n-j} \, z_1^{2j_1 - 1} \cdots z_k^{2j_k - 1}. \qquad (3.3.26)$$

To have the whole sum for the state $|j, n\rangle$ (equation 3.3.24), not only are states at rest needed but also the higher ones ($|j_i, n_i\rangle$ with $n_i > j_i$). In that case, there are different possibilities and the Clebsch-Gordon coefficients contain binomial coeficients (see 2). For a bilocal operator, one gets

$$|j, n\rangle = (z_1 + z_2) \sum_{n_1 + n_2 = j - j_1 - j_2} \begin{pmatrix} n_1 + n_2 \\ n_1 \end{pmatrix} |j_1, j_1 + n_1\rangle \, |j_2, j_2 + n_2\rangle \,. \qquad (3.3.27)$$

The summation gives:

$$|j, n\rangle \sim (z_1 + z_2)^{n+j+1} \, (1 + \zeta)^{2j_1 - 1} (1 - \zeta)^{2j_2 - 1} P_{j - j_1 - j_2}^{(2j_1 - 1, 2j_2 - 1)}(\zeta), \qquad (3.3.28)$$

with

$$\zeta = \frac{(z_1 - z_2)}{(z_1 + z_2)}, \qquad (3.3.29)$$

where $P_n^{(\nu_1, \nu_2)}(\zeta)$ are the Jacobi polynomials. The important thing to remark about equation (3.3.28) is that the only dependence in n is in the factor $(z_1 + z_2)^{n+j+1}$, the other parts of this equation depends only on the conformal spins j_1, j_2, and j.

An important property will be needed here. For any kind of irreducible representation of the collinear conformal subgroup $|j, n\rangle$ which is constructed from the tensor product of k-irreducible representations, we have the following homogeneity property:

$$|j, n\rangle \equiv \phi_{j,n}(z_1, \cdots, z_k) = (z_1 + \cdots + z_k)^{n+j-1} \phi_{j,n}(\hat{z}_1, \cdots, \hat{z}_k), \qquad (3.3.30)$$

where $\hat{z}_i = \frac{z_i}{z_1+\cdots+z_k}$. This property comes almost directly from the constructions of the Clebsch-Gorden coefficients (equations (3.3.26)and (3.3.27)). Hence irreducible representations of the collinear conformal subgroup on the space of k variables induce the representations on the functions defined on the simplex $z_1 + \cdots + z_k = 1$.

Now consider the product of local operators $\Phi(\alpha_1\lambda) \cdots \Phi(\alpha_k\lambda)$ on the light cone between the vacuum and a massless h-meson state of momentum p. This matrix element can be parametrized as

$$\begin{aligned} &\langle 0 \,|\Phi_1(\alpha_1\lambda) \cdots \Phi_k(\alpha_k\lambda)|\, h(p)\rangle \\ &= \int dz_1 \cdots dz_k \delta\left(\sum_{i=1}^{k} z_i - 1\right) e^{-i\lambda(\alpha_1 z_1+\cdots+\alpha_k z_k)}\phi(z_1,\cdots,z_k), \end{aligned} \tag{3.3.31}$$

According to the property of equation (3.3.30), the wave function $\phi(z_1, \cdots, z_k)$ can be expanded in different parts which have a fixed conformal spin j. The minimum value of the conformal spin is the sum of the conformal spins of every conformal operators $\Phi_i(\alpha_i\lambda)$. The different parts of this conformal expansion are mutually orthogonal polynomials (Jacobi polynomials if there are only two operators) and form the complete set of the function on the simplex $z_1 + \cdots + z_k = 1$.

In front of each of these polynomials there is a coefficient which is renormalized multiplicatively; the terms with lowest conformal spin is the only one which survives in the formal limit $Q^2 \to \infty$. This conformal expansion is justified by the construction of irreducible representations of the colinear conformal subgroup.

First term of such an expansion is called "asymptotic" wave function and can be easily computed (equation 3.3.26) :

$$\phi_{as}(z_1, \cdots, z_k) = \frac{\Gamma(2j_1 + \cdots + 2j_k)}{\Gamma(2j_1) + \cdots + \Gamma(2j_k)} z_1^{2j_1-1} \cdots z_k^{2j_k-1}, \tag{3.3.32}$$

with

$$\int dz_1 \cdots dz_k \delta \left(\sum_{i=1}^{k} z_i - 1 \right) \phi_{as}(z_1, \cdots, z_k) = 1. \tag{3.3.33}$$

Now we can understand better the conformal expansion of the wave function $\phi_{\parallel}$ given as,

$$\phi_{\parallel}(z) = 6z(1-z) \sum_{n=0}^{\infty} a_n^{\parallel} C_n^{3/2}(\zeta). \tag{3.3.34}$$

This is the contribution of the matrix element $\langle 0 \left| \bar{u}(\xi) \gamma^{\mu} \left[\xi; -\xi \right] d(-\xi) \right| \rho^{-}(P, s) \rangle$ where each quark field has a positive spin projection $s = +\frac{1}{2}$ (see equation 3.3.18). Knowing that a quark field has canonical dimension $l = \frac{3}{2}$, each quark field has a conformal spin $j_q = \frac{(l+s)}{2} = 1$. So the asymptotic distribution amplitude (3.3.32) equals $\phi_{as}(z_q, z_{\bar{q}}) = 6z_q z_{\bar{q}}$ and has the conformal spin $j = 2$. The higher terms in (3.3.34) correspond to higher values of j and n in equation.(3.3.28). Denoting $z = z_q$ with $z_q + z_{\bar{q}} = 1$, we get the expansion (3.3.34) (the Gegenbauer polynomials are $C_n^{3/2}$ and are proportional to the Jacobi ones $P_n^{(1,1)}$ which appear when $j_1 = j_2 = 1$).

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Chapter 4

Transverse Momentum Distribution Functions

4.1 Systematics of Quark Distribution Functions

In this section we present in detail the systematics of quark and antiquark distribution functions. Our focus will be on leading-twist distributions.

4.1.1 The Quark-Quark Correlation Matrix:

Let us consider the quark-quark correlation matrix introduced in Sec. 2.2 and represented in Fig.(4.1),

$$\Phi_{ij}(k,P,S) = \int d^4\xi e^{-ik\cdot\xi} \langle PS | \bar{\psi}_j(\xi)\psi_i(0) | PS \rangle . \tag{4.1.1}$$

Here, we recall, i and j are Dirac indices and a summation over colour is implicit. The quark distribution functions are essentially integrals over k of traces of the form

$$Tr\left(\Gamma\Phi\right) = \int d^4\xi e^{-ik\cdot\xi} \langle PS | \bar{\psi}(\xi)\Gamma\psi(0) | PS \rangle , \tag{4.1.2}$$

where Γ is a Dirac matrix structure.

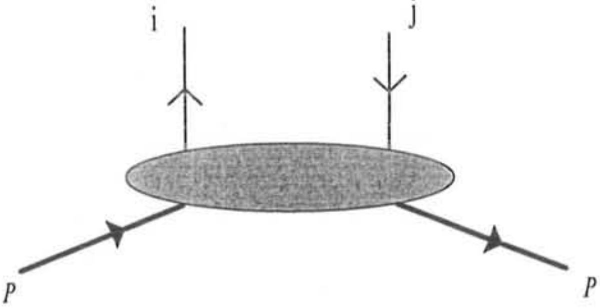

Figure 4-1: The quark-quark correlation matrix Φ.

The Φ matrix satisfies certain relations arising from hermiticity, parity invariance and time-reversal invariance[1]:

$$[\text{Hermiticity}] \quad \Rightarrow \quad \Phi^{\dagger}(k,P,S) = \gamma^0 \Phi(k,P,S)\gamma^0 \tag{4.1.3a}$$

$$[\text{Parity invariance}] \quad \Rightarrow \quad \Phi(k,P,S) = \gamma^0 \Phi(\tilde{k},\tilde{P},-\tilde{S})\gamma^0 \tag{4.1.3b}$$

$$[\text{Time reversal invariance}] \quad \Rightarrow \quad \Phi^{*}(k,P,S) = T_0 \Phi(\tilde{k},\tilde{P},\tilde{S})T_0 \tag{4.1.3c}$$

where $T_0 = i\gamma^1\gamma^3$. As we shall see, the time-reversal condition (4.1.3c) plays an important role in the phenomenology of transverse polarisation distribution functions.

The most general decomposition of Φ in a basis of Dirac matrices,

$$\Gamma = \left\{ \mathbb{1}, \gamma^\mu, \gamma^\mu\gamma^5, i\gamma^5, i\sigma^{\mu\nu}\gamma^5 \right\}, \tag{4.1.4}$$

is (we introduce a factor of $\frac{1}{2}$ for later convenience)

$$\Phi(k,P,S) = \frac{1}{2}\left\{ \mathcal{S}\mathbb{1} + \mathcal{V}_\mu\gamma^\mu + \mathcal{A}_\mu\gamma^5\gamma^\mu + i\mathcal{P}_5\gamma^5 + \frac{1}{2}i\mathcal{T}_{\mu\nu}\sigma^{\mu\nu}\gamma^5 \right\}. \tag{4.1.5}$$

The quantities $\mathcal{S}$, $\mathcal{V}^\mu$, $\mathcal{A}^\mu$, $\mathcal{P}_5$ and $\mathcal{T}^{\mu\nu}$are constructed with the vectors k^μ, P^μ and the

[1]see Appendix C

pseudovector S^μ. Imposing the constraints (4.1.3a-b) we have, in general,

$$\mathcal{S} = \frac{1}{2}Tr\left(\Phi\right) = C_1, \tag{4.1.6a}$$
$$\mathcal{V}^\mu = \frac{1}{2}Tr\left(\gamma^\mu \Phi\right) = C_2 P^\mu + C_3 k^\mu + C_4 \varepsilon^{\mu\nu\rho\sigma} P_\nu k_\rho S_\sigma, \tag{4.1.6b}$$
$$\mathcal{A}^\mu = \frac{1}{2}Tr\left(\gamma^\mu \gamma_5 \Phi\right) = C_5 S^\mu + C_6 \left(k \cdot S\right) P^\mu + C_7 \left(k \cdot S\right) k^\mu, \tag{4.1.6c}$$
$$\mathcal{P}_5 = \frac{1}{2}Tr\left(\gamma_5 \Phi\right) = C_8 \left(k \cdot S\right), \tag{4.1.6d}$$
$$\mathcal{T}^{\mu\nu} = \frac{1}{2i}Tr\left(\sigma^{\mu\nu}\gamma_5 \Phi\right) = C_9 P^{[\mu} S^{\nu]} + C_{10} S^{[\mu} k^{\nu]} + C_{11}\left(k \cdot S\right) P^{[\mu} k^{\nu]} + C_{12}\varepsilon^{\mu\nu\rho\sigma} P_\rho k_\sigma, \tag{4.1.6e}$$

where the coefficients $C_i = C_i(k^2, k.P)$ are real functions, owing to hermiticity. The amplitudes C_4 C_8 and C_{12} vanish when time reversal invariance applies but this result changes if we consider a gauge link operator between the quark fields and these amplitudes will not be zero in general as will be shown below.

4.1.2 Leading-Twist Distribution Functions:

We are mainly interested in the leading-twist contributions, that is the terms in equations (4.1.6a-e) which are of order $\mathcal{O}(P^+)$ in the infinite momentum frame. The vectors at our disposal are P^μ, $k^\mu \simeq xP^\mu + k_\perp^\mu$ and $S^\mu \simeq \lambda_N \frac{P^\mu}{M} + S_\perp^\mu$, where the approximate equality signs indicate that we are neglecting terms suppressed by $(P^+)^{-2}$. Remember that the transverse spin vector $S_\perp^\mu$ and quark momentum are of order $(P^+)^0$. At leading twist, only the vector, axial and tensor terms in (4.1.5) appear and equations. (4.1.6 b, c, e) become[2]

$$\mathcal{V}^\mu = A_1 P^\mu + A_2 \frac{\varepsilon^{\mu\nu\rho\sigma} p_\nu k_{\perp\rho} S_{\perp\sigma}}{M}, \tag{4.1.6b$'$}$$
$$\mathcal{A}^\mu = \left(\lambda_N A_3 + \frac{(k_\perp \cdot S_\perp)}{M} A_4\right) P^\mu, \tag{4.1.6c$'$}$$

[2] using Sadakov decomposition (see Appendix A).

$$T^{\mu\nu} = A_5 p^{[\mu} S_\perp^{\nu]} + \left(\lambda_N A_6 + \frac{(k_\perp \cdot S_\perp)}{M} A_7\right) \frac{p^{[\mu} k_\perp^{\nu]}}{M} + A_8 \frac{\varepsilon^{\mu\nu\rho\sigma} p_\rho k_{\perp\sigma}}{M}, \quad (4.1.6e')$$

where we have introduced new real functions $A_i(x, k_\perp^2)$ and the powers of M, so that all coefficients have the same dimension. The leading-twist quark correlation matrix (4.1.5) is then

$$\begin{aligned} \Phi(k,P,S) &= \frac{1}{2}\left\{ A_1 \not{P} + A_2 \frac{\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu P^\nu k_\perp^\rho S_\perp^\sigma}{M} \right. \\ &+ \left(\lambda_N A_3 + \frac{k_\perp \cdot S_\perp}{M} A_4\right)\gamma_5 \not{P} \\ &+ A_5 i\sigma_{\mu\nu}\gamma_5 P^\mu S_\perp^\nu + \left(\lambda_N A_6 + \frac{(k_\perp \cdot S_\perp)}{M} A_7\right)\frac{i\sigma_{\mu\nu}\gamma^5 P^\mu k_\perp^\nu}{M} \\ &\left. + A_8 \frac{\sigma_{\mu\nu} k_\perp^\mu P^\nu}{M} \right\}. \end{aligned} \quad (4.1.7)$$

Integrating (4.1.1) over k^+ and k^-, with the constraint $x = \frac{k^+}{P^+}$, we get

$$\begin{aligned} \Phi_{ij}(x, k_\perp, S_\perp) &= \int \frac{dk^+ dk^-}{(2\pi)^4} \Phi_{ij}(k,P,S)\, \delta\left(x - \frac{k^+}{P^+}\right) \\ &= P^+ \int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{(2\pi)^3} e^{-i\left(xP^+\xi^- - \mathbf{k}_\perp \cdot \boldsymbol{\xi}_\perp\right)} \left\langle PS \left| \bar{\psi}_j(\xi^-, \boldsymbol{\xi}_\perp)\psi_i(0) \right| PS \right\rangle \end{aligned} \quad (4.1.8)$$

with

$$\Phi^{[\Gamma]} = \int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{2(2\pi)^3} e^{-i\left(xP^+\xi^- - \mathbf{k}_\perp \cdot \boldsymbol{\xi}_\perp\right)} \left\langle PS \left| \bar{\psi}(\xi^-, \boldsymbol{\xi}_\perp)\Gamma\psi(0) \right| PS \right\rangle. \quad (4.1.9)$$

Therefore (4.1.7) becomes

$$\begin{aligned} \Phi(x, k_\perp) &= \frac{1}{2}\left\{ f_1 \not{p} + f_{1T}^\perp \frac{\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu S_\perp^\nu p^\rho k_\perp^\sigma}{M} + g_{1s}\gamma_5 \not{p} \right. \\ &+ h_{1T} i\sigma_{\mu\nu}\gamma_5 p^\mu S_\perp^\nu + h_{1s}^\perp \frac{i\sigma_{\mu\nu}\gamma_5 p^\mu k_\perp^\nu}{M} \\ &\left. + h_1^\perp \frac{\sigma_{\mu\nu} k_\perp^\mu p^\nu}{M} \right\}, \end{aligned} \quad (4.1.10)$$

with arguments $f_1 = f_1(x, \mathbf{k}_\perp^2)$ etc. The quantity g_{1s} (and similarly $h_{1s}^\perp$) is shorthand for

$$g_{1s}(x, k_\perp) = \lambda_N g_{1L}(x, \mathbf{k}_\perp^2) + \frac{k_\perp \cdot S_\perp}{M} g_{1T}(x, \mathbf{k}_\perp^2). \tag{4.1.11}$$

Here a subscript 1 labels the leading-twist quantities, subscripts L and T indicate that the parent hadron is longitudinally or transversely polarized and a superscript $\perp$ signals the presence of transverse momenta with uncontracted Lorentz indices. Therefore at leading twist, we have eight distribution functions (see [1]). Integrating over $\mathbf{k}_\perp$ we are left with the three distribution functions[3]

$$\begin{aligned} f_1(x) &= \int d^2\mathbf{k}_\perp f_1(x, \mathbf{k}_\perp^2), \\ g_1(x) &= \int d^2\mathbf{k}_\perp g_{1L}(x, \mathbf{k}_\perp^2), \\ h_1(x) &= \int d^2\mathbf{k}_\perp \left(h_{1T}(x, \mathbf{k}_\perp^2) + \frac{\mathbf{k}_\perp^2}{2M^2} h_{1T}^\perp(x, \mathbf{k}_\perp^2) \right), \end{aligned}$$

and therefore,

$$\Phi(x) = \frac{1}{2}\left\{ f_1(x)\not{p} + \lambda_N g_1(x)\gamma_5\not{p} + h_1(x)\not{p}\gamma_5\not{S}_\perp \right\}. \tag{4.1.12}$$

We are already familiar with the first two functions in subsection 2.2.2, but here the third function $h_1(x)$ is new. It is given by the equation (4.1.9) as

$$h(x) = \frac{1}{2P^+}\int \frac{d\lambda}{2\pi} e^{i\lambda x} \left\langle PS \left| \bar{\psi}(0)\gamma^+\gamma^1\gamma_5\psi(\lambda n) \right| PS \right\rangle. \tag{4.1.13}$$

---

[3]the terms, which are odd in $k_\perp$ will become zero in integration. Also consider the term

$$\begin{aligned} & i\sigma_{\mu\nu}\gamma_5 p^\mu \left(S_\perp^\nu h_{1T} + \frac{k_\perp^\nu (k_\perp \cdot S_\perp)}{M^2} h_{1T}^\perp \right) \\ = \; & \not{p}\gamma_5\gamma_i \left(S_\perp^i h_{1T} + \frac{1}{M^2} (\mathbf{k}_\perp \cdot \mathbf{S}_\perp) k_\perp^i h_{1T}^\perp \right) \\ = \; & \not{p}\gamma_5\gamma_i \left\{ S_\perp^i \left(h_{1T} + \frac{\mathbf{k}_\perp^2}{2M^2} h_{1T}^\perp \right) - \frac{1}{M^2} \left(k_\perp^i k_\perp^j + \frac{1}{2}\mathbf{k}_\perp^2 g_\perp^{ij} \right) S_{\perp j} h_{1T}^\perp \right\} \end{aligned}$$

it is easy to see that the second term will be zero after integration.

Using

$$\bar{\psi}\gamma^1\gamma_5\psi = \sqrt{2}\psi_+^\dagger\gamma^1\gamma_5\psi_+, \tag{4.1.14}$$

and the transverse polarization projector $\mathcal{P}_{\uparrow\downarrow} = \frac{1}{2}(1 \pm \gamma^1\gamma_5)$ we can write $h(x)$ in the form

$$\begin{aligned} h(x) &= \frac{1}{\sqrt{2}}\sum_n \delta(P_n^+ - (1-x)P^+) \\ &\times \left\{ \left| \langle n \left| \mathcal{P}_\uparrow \psi_+(0) \right| PS \rangle \right|^2 - \left| \langle n \left| \mathcal{P}_\downarrow \psi_+(0) \right| PS \rangle \right|^2 \right\}. \end{aligned} \tag{4.1.15}$$

This expression exhibits the probabilistic content of $h(x)$: it is the number density of quarks with transverse polarization $\uparrow$ minus the number density of quarks with transverse polarization $\downarrow$ (assuming the parent nucleon to have transverse polarization $\uparrow$).

Let us see the partonic interpretation of the $k_\perp$-dependent distribution functions. Considering only T-even distributions, we have from equation (4.1.9)

$$\Phi^{[\gamma^+]} = \mathcal{P}_{q/N}(x, k_\perp) = f_1(x, \mathbf{k}_\perp^2), \tag{4.1.16a}$$

$$\Phi^{[\gamma^+\gamma_5]} = \mathcal{P}_{q/N}(x, k_\perp)\lambda(x, k_\perp) = \lambda_N g_{1L}(x, \mathbf{k}_\perp^2) + \frac{\mathbf{k}_\perp \cdot \mathbf{S}_\perp}{M} g_{1T}(x, \mathbf{k}_\perp^2), \tag{4.1.16b}$$

$$\begin{aligned} \Phi^{[i\sigma^{i+}\gamma_5]} &= \mathcal{P}_{q/N}(x, k_\perp) s_\perp^i(x, k_\perp) \\ &= S_\perp^i h_{1T}(x, \mathbf{k}_\perp^2) + \frac{\lambda_N}{M} k_\perp^i h_{1L}^\perp(x, \mathbf{k}_\perp^2) + \frac{k_\perp \cdot S_\perp}{M^2} k_\perp^i h_{1T}^\perp(x, \mathbf{k}_\perp^2), \end{aligned} \tag{4.1.16c}$$

where $\mathcal{P}_{q/N}(x, k_\perp)$ is the probability of the finding quark of flavor q with longitudinal momentum fraction x and transverse momentum $k_\perp$ in the nucleon N, and $\lambda(x, k_\perp)$, $\mathbf{s}_\perp(x, k_\perp)$ are the quark helicity and transverse spin densities, respectively. If the target nucleon is unpolarized, the only measurable quantity is $f_1(x, \mathbf{k}_\perp^2)$. If the target nucleon is transversely polarized, there is some probability of finding the quarks transversely polarized along the same direction as the nucleon, along a different direction, or longitudinally polarized. This variety of situations is allowed by the presence of $\mathbf{k}_\perp$. Integrating over $\mathbf{k}_\perp$, the transverse polarization asymmetry of quarks along a different direction with re-

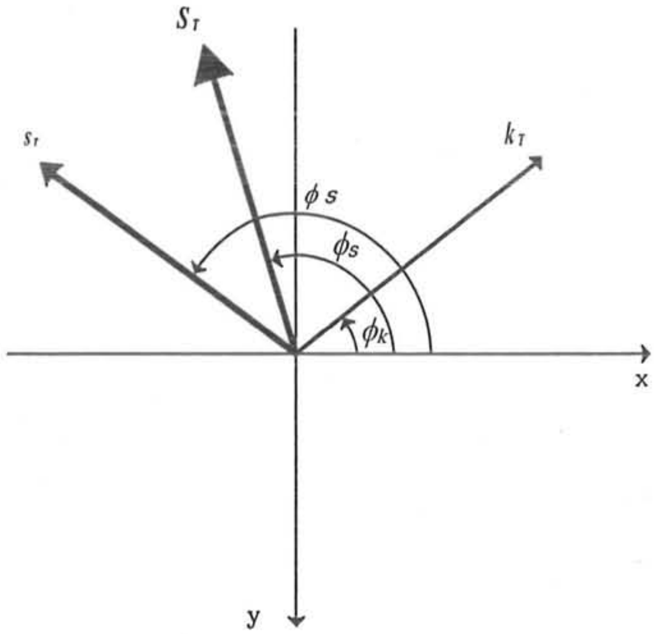

Figure 4-2: Definition of the azimuthal angles in the plane orthogonal to the $\gamma^* N$ axis. The proton momentum is directed along the positive z axis and points inwards.

spect to the nucleon polarization, and the longitudinal polarization asymmetry of quarks in a transversely polarized nucleon disappear: only the case $\mathbf{s}_\perp \parallel \mathbf{S}_\perp$ survives. Referring to Fig. (4-2) for the geometry in the azimuthal plane and using the following parametrizations for the vectors at hand (we assume full polarization of the nucleon):

$$\begin{aligned} k_\perp &= |k_\perp| (\cos\phi_k, \text{-}\sin\phi_k), \\ S_\perp &= (\cos\phi_S, \text{-}\sin\phi_S), \\ s_\perp &= |s_\perp| (\cos\phi_s, \text{-}\sin\phi_s), \end{aligned}$$

we find the following interpretation of distribution functions:

$$\mathcal{P}_{q+/N+}(x, k_\perp) - \mathcal{P}_{q-/N+}(x, k_\perp) = 2g_{1L}(x, \mathbf{k}_\perp^2), \tag{4.1.17a}$$

$$\mathcal{P}_{q+/N\uparrow}(x, k_\perp) - \mathcal{P}_{q-/\uparrow N}(x, k_\perp) = \frac{|\mathbf{k}_\perp|}{M} \cos(\phi_S - \phi_k) g_{1T}(x, k_\perp^2), \tag{4.1.17b}$$

$$\mathcal{P}_{q\uparrow/N\uparrow}(x,k_\perp)-\mathcal{P}_{q\downarrow/N\uparrow}(x,k_\perp) = \cos(\phi_S-\phi_s)h_{1T}(x,\mathbf{k}_T^2) + \frac{\mathbf{k}_\perp^2}{2M^2}\cos(2\phi_k-\phi_S-\phi_s)h_{1T}^\perp(x,\mathbf{k}_\perp^2), \tag{4.1.17c}$$

$$\mathcal{P}_{q\uparrow/N+}(x,k_\perp)-\mathcal{P}_{q\downarrow/N+}(x,k_\perp) = \frac{|\mathbf{k}_\perp|}{M}\cos(\phi_s-\phi_k)h_{1L}^\perp(x,k_\perp^2), \tag{4.1.17d}$$

i.e., $g_{1L}(x,\mathbf{k}_T^2)$ is the longitudinal quark asymmetry in the longitudinally polarized nucleon, etc. For T-odd distributions, we have from equation (4.1.9)

$$\Phi^{[\gamma^+]} = -\frac{\varepsilon_\perp^{ij}k_{i\perp}S_{j\perp}}{M}f_{1T}^\perp(x,\mathbf{k}_\perp^2), \tag{4.1.18a}$$

$$\Phi^{[i\sigma^{i+}\gamma_5]} = -\frac{\varepsilon_\perp^{ij}k_{i\perp}}{M}h_1^\perp(x,k_\perp^2), \tag{4.1.18b}$$

and

$$\mathcal{P}_{q/N\uparrow}(x,k_\perp)-\mathcal{P}_{q/N\downarrow}(x,k_\perp) = -2\frac{|\mathbf{k}_\perp|}{M}\sin(\phi_k-\phi_S)f_{1T}^\perp(x,k_\perp^2), \tag{4.1.19a}$$

$$\mathcal{P}_{q\uparrow/N}(x,k_\perp)-\mathcal{P}_{q\downarrow/N}(x,k_\perp) = -\frac{|\mathbf{k}_\perp|}{M}\sin(\phi_k-\phi_s)h_1^\perp(x,k_\perp^2). \tag{4.1.19b}$$

Therefore $f_{1T}^\perp(x,k_\perp^2)$ is related to the transversely polarized asymmetry of target nucleon with unpolarized quarks and vice-versa for $h_1^\perp(x,k_\perp^2)$. The "Sivers function" $f_{1T}^\perp(x,k_\perp^2)$ is needed to describe the asymmetrical production of pions by highly virtual photons from a target proton. In the $A^+=0$ gauge the link operator is apparently unity, which had led to the erroneous conclusion that $f_{1T}^\perp(x,k_\perp^2)$ (and, by similar analysis, $h_1^\perp$) vanishes upon demanding time reversal invariance. Recently, however, Brodsky and collaborators[5] used a simple di-quark model for the nucleon to show that this conclusion is unwarranted. They showed that final-state interactions in deep-inelastic scattering, and initial state interactions in Drell-Yan processes, permit single spin asymmetries. Subsequently, Collins[6] showed that the gauge-link in an arbitrary gauge allowed for T-odd distributions. The singular nature of the $A^+=0$ gauge, which had earlier led to misleading conclusions, was tackeled by Ji and Yuan[7], and then in full detail by Belitsky et al[8], who showed that transverse components of gauge field gave an additional scaling contribution at light cone infinity, and a transverse gauge link. Now we will see how these contributions arise.

4.2 QCD Gauge Invariant Parton Distribution

In this section we demonstrate how a gauge invariant parton density arise in QCD description of the structure functions at leading twist. In particularly, we consider transverse momentum dependent distributions. In QCD, the parton distributions are defined as hadronic matrix elements of quark bilocal operators. Since the parton fields enter at distinct space-time points, a gauge link is needed to make the operators gauge invariant. This link is generated in a hard scattering process by the final state interactions between the struck parton and the target remnants. Since the struck parton moves with a high energy, its interaction can be approximated through an eikonal phase. We show that the conventional light-cone link is not the only contribution.

The following definition of the transverse-momentum dependent quark distribution has been unequivocally accepted in the literature

$$\begin{aligned}\Phi^{[\gamma^+]}(x,\mathbf{k}) &= \int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{2(2\pi)^3} e^{-i(xp^+\xi^- - \mathbf{k}_\perp\cdot\boldsymbol{\xi}_\perp)} \\ &\quad \left\langle PS\left|\bar{q}(\xi^-,\boldsymbol{\xi}_\perp)\left[\infty,\boldsymbol{\xi}_\perp;\xi^-,\boldsymbol{\xi}_\perp\right]^\dagger\gamma^+[\infty,\mathbf{0}_\perp;0,\mathbf{0}_\perp]\,q(0,\mathbf{0}_\perp)\right|PS\right\rangle \quad (4.2.1)\end{aligned}$$

where the path-ordered gauge link extends along the light-cone

$$[\infty,\boldsymbol{\xi}_\perp;\xi^-,\boldsymbol{\xi}_\perp] \equiv P\exp\left(-ig\int_{\xi^-}^{\infty} \mathcal{A}^+(\eta^-,\boldsymbol{\xi}_\perp)d\eta^-\right). \quad (4.2.2)$$

Since the coordinate $\xi^+ = 0$, we do not display it, for brevity, here and the following presentation. As we will see, Eq (4.2.1) is true only in a class of gauges where the gluon potential vanishes at $\xi^- = \infty$.

For the photon scattering on a single parton with momentum ℓ, we have for the hadronic tensor

$$
\begin{aligned}
W^{\mu\nu} &= \frac{1}{4\pi}\sum_X \int \frac{d^4 p_J}{(2\pi)^4}(2\pi)\,\delta_+\left(p_J^2\right) \\
&\quad (2\pi)^4\,\delta^4\left(P+q-p_J-P_X\right)\left\langle PS\left|J^\mu(0)\right|p_J,\ X\right\rangle\left\langle p_J,\ X\left|J^\nu(0)\right|PS\right\rangle \qquad (4.2.3)
\end{aligned}
$$

where $\delta_+\left(p_J^2\right)=\Theta(p_{J0})\delta\left(p_J^2\right)$ imposes the on-mass-shell condition for the observed final quark (jet) of momentum p_J after multiple rescattering with spectators X in the target fragments, and the summation over X involves summation over the number of particles populating the final states as well as their quantum numbers. The tree scattering amplitude corresponding to Fig. (4-3a) reads

$$
\left\langle p_J, X\left|J^\nu(0)\right|P\right\rangle_{(0)} = \bar{u}(p_J)\gamma^\nu\left\langle X\left|q(0)\right|P\right\rangle, \qquad (4.2.4)
$$

where $\bar{u}$ is the Dirac spinor of the scattered quark. Substituting Eq. (4.2.4) into (4.2.3) and taking the component W^{11}, we get the structure function $F_1\left(x_{\mathrm{B}},Q^2\right)$ in the parton model

$$
F_1\left(x_{\mathrm{B}},Q^2\right)=\int d^2\mathbf{k}_\perp f(x,\mathbf{k}_\perp^2),
$$

in terms of the parton distribution

$$
f(x,\mathbf{k}_\perp^2)=\int\frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{2(2\pi)^3}e^{-i(xp^+\xi^- - \mathbf{k}_\perp\cdot\boldsymbol{\xi}_\perp)}\times\left\langle P\left|\bar{q}(\xi^-,\boldsymbol{\xi}_\perp)\gamma^+ q(0,\mathbf{0}_\perp)\right|P\right\rangle. \qquad (4.2.5)
$$

As it stands the above correlation is not gauge invariant.

4.2.1 Light-Cone Gauge Link

Consider now the contribution from the diagram in Fig. (4-3 b). The one-gluon amplitude reads

$$
\left\langle p_J, X\left|J^\nu(0)\right|P\right\rangle_{(1)} = g\bar{u}(p_J)\int\frac{d^4k_1}{(2\pi)^4}\left\langle X\left|\not{A}(k_1)S_F(p_J-k_1)\gamma^\nu q(0,\mathbf{0}_\perp)\right|P\right\rangle, \qquad (4.2.6)
$$

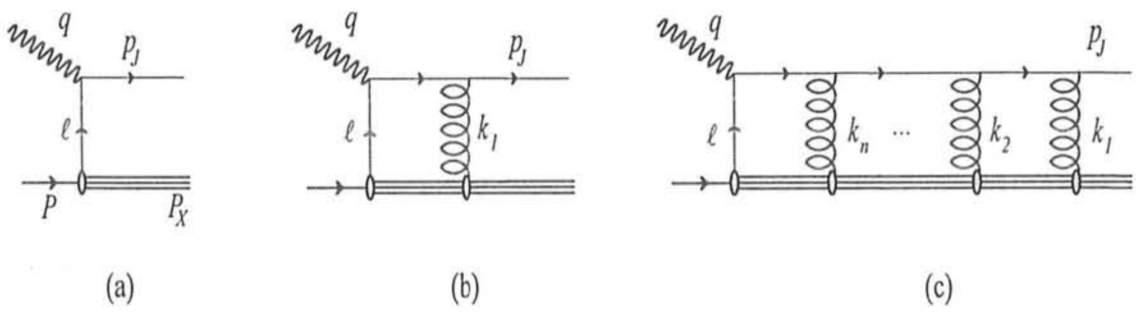

Figure 4-3: Multi-gluon attachments to the struck quark in deeply inelastic scattering which factorize to form the path-ordered exponential.

where the free quark propagator is

$$S_F(p_J - k_1) = \frac{\not{p}_J - \not{k}_1}{(p_J - k_1)^2 + i\varepsilon}. \tag{4.2.7}$$

Momentum conservation gives

$$p_J = q + \ell + k_1,$$

so that the on-mass-shell condition $p_J^2 = 0$ results into

$$\ell^+ + k_1^+ = x_B p^+,$$

in the Bjorken limit $q^- \to \infty$. By expanding all vectors in Sudakov components, one can easily keep track of leading contributions in the scaling limit. For instance, the struck quark propagator contains a scaling term,

$$S_F(p_J - k_1) \approx \frac{q^-\gamma^+}{2q^-(\ell^+ - x_B p^+) + i\varepsilon} = -\frac{1}{2}\frac{\gamma^+}{k_1^+ - i\varepsilon}, \tag{4.2.8}$$

with the remainder naively suppressed by extra powers of $\frac{1}{q^-}$.

Making the light-cone decomposition of the gluon field

$$A^{\mu} = p^{\mu} A \cdot n + n^{\mu} A \cdot p + A_{\perp}^{\mu}, \tag{4.2.9}$$

one notices that the leading twist contribution comes from the first term on the right-hand side only, since the second one vanishes due to nilpotence of $(\gamma^-)^2 = 0$, while the transverse component of the gauge field is of twist-three. Since the leading contribution in the quark density matrix $u \otimes \bar{u}$ comes from the large term $q^- \gamma^+$, we are allowed to replace $\bar{u}\gamma^-\gamma^+$ by $2\bar{u}$ on the right-hand side of Eq. (4.2.6). Finally, integrating with respect to k_1 we get

$$\int \frac{d^4 k_1}{(2\pi)^4} \frac{1}{k_1^+ - i\varepsilon} A(k_1) = i \int_{-\infty}^{\infty} d\eta^- \Theta(\xi^-) A^+(\eta^-, 0, \mathbf{0}_\perp). \tag{4.2.10}$$

Using these results, we find that the amplitude in the one-gluon approximation differs from the tree-level result (4.2.4) only by an extra factor, namely,

$$\langle p_J, X \,| J^\nu(0) |\, P \rangle_{(1)} = \bar{u}(p_J)\gamma^\nu \left\langle X \left| (-ig) \int_0^\infty d\eta^- A^+(\eta^-, 0, \mathbf{0}_\perp) q(0, \mathbf{0}_\perp) \right| P \right\rangle. \tag{4.2.11}$$

These considerations are generalizable to an arbitrary order. Namely, for n-gluon exchange, see Fig. (4-3 c), we have

$$\langle p_J, X \,| J^\nu(0) |\, P \rangle_{(n)} = g^n \bar{u}(p_J) \int \prod_{i=1}^{n} \frac{d^4 k_i}{(2\pi)^4} \langle X \mid \not{A}(k_1) S_F(p_J - k_1) \not{A}(k_2) S_F(p_J - k_1 - k_2) \cdots$$

$$\times \not{A}(k_n) S_F \left(p_J - \sum_{j=1}^{n} k_j \right) \gamma^\nu q(0, \mathbf{0}_\perp) \mid P \rangle. \tag{4.2.12}$$

Making repeatedly the same set of approximations as above, one simplifies the Dirac structure to

$$\langle p_J, X \,| J^\nu(0) |\, P \rangle_{(n)} = (-g)^n \int \prod_{i=1}^{n} \frac{d^4 k_i}{(2\pi)^4} \frac{1}{k_1^+ - i\varepsilon} \frac{1}{k_1^+ + k_2^+ - i\varepsilon} \cdots \frac{1}{\sum_{j=1}^{n} k_j^+ - i\varepsilon}$$

$$\times\bar{u}(p_J)\gamma^\nu \left\langle X \left| A^+(k_1)A^+(k_2)\cdots A^+(k_n)q(0,0_\perp)\right| P\right\rangle, \quad (4.2.13)$$

where we have used the momentum conservation $\ell^+ + \sum_{i=1}^n k_i^+ = x_B p^+$. The calculation of the momentum integrals is trivial and gives, similarly to Eq. (4.2.10),

$$\int \prod_{i=1}^n \frac{d^4k_i}{(2\pi)^4} \frac{1}{k_1^+ - i\varepsilon} \frac{1}{k_1^+ + k_2^+ - i\varepsilon} \cdots \frac{1}{\sum_{j=1}^n k_j^+ - i\varepsilon} A^+(k_1)A^+(k_2)\cdots A^+(k_n)$$

$$= i^n \int_{-\infty}^{\infty} \prod_{i=1}^n d\xi_i^- \Theta\left(\xi_i^- - \xi_{(i+1)}^-\right) A^+(\xi_1^-)A^+(\xi_2^-)\cdots A^+(\xi_n^-), \quad (4.2.14)$$

where $\xi_{(n+1)}^- = 0$. Thus,

$$\begin{aligned} \langle p_J, X\,|J^\nu(0)|\,P\rangle_{(n)} &= \bar{u}(p_J)\gamma^\nu \langle X \mid (-ig)^n \int_0^\infty d\xi_1^- \int_0^{\xi_1^-} d\xi_2^- \cdots \\ &\int_0^{\xi_{(n-1)}^-} d\xi_n^- A^+(\xi_1^-)A^+(\xi_2^-)\cdots A^+(\xi_n^-)q(0)\,|P\rangle \,. \quad (4.2.15) \end{aligned}$$

Here, one immediately recognizes the nth term in the expansion of the path-ordered exponential. Therefore, for the amplitude resummed to all orders, one gets

$$\begin{aligned} \langle p_J, X\,|J^\nu(0)|\,P\rangle &= \sum_{n=0}^{\infty} \langle p_J, X\,|J^\nu(0)|\,P\rangle_{(n)} \\ &= \bar{u}(p_J)\gamma^\nu \left\langle X \middle| P\exp\left(-ig\int_0^\infty d\eta^- \mathcal{A}^+(\eta^-,0,0_\perp)\right) q(0) \middle| P\right\rangle (4.2.16) \end{aligned}$$

Substitution of this result into the hadronic tensor yields, indeed, the conventional quark distribution (4.2.1).

4.2.2 Transverse Gauge Link

In the previous subsection we were not quite accurate and actually omitted contributions which scale in the Bjorken region. These terms survive only at a point of the momentum space and are, normally, assumed to be vanishing. However,this is not the case for all gauge potentials. And a counter-example is provided by the light-cone gauge where the potential is singular at the very same point.

In the present circumstances it is more convenient to work in a frame in which the four-momentum of the current jet p_J is light-like, $p_J^2 = 0$, and can be chosen as one of the light-like vectors with the other one still fixed by the hadron momentum $P^\mu = p^\mu$. Thus, we define

$$\tilde{n}^\mu \equiv \frac{p_J^\mu}{p_J \cdot p}. \tag{4.2.17}$$

Obviously, $\tilde{n}^2 = p^2 = 0$ and $p \cdot \tilde{n} = 1$. We decompose all Lorentz tensors via light-cone and transverse components defined in this basis

$$V^\mu = \tilde{n}^\mu V^- + p^\mu V^+ + \mathbf{V}_\perp^\mu, \tag{4.2.18}$$

and keep the same + index for contractions with $\tilde{n}$, $V^+ \equiv \tilde{n} \cdot V$. In the Bjorken limit $q^- \to \infty$, the differences between p_J^- and q^- is negligible and, therefore, both frames, used before and here, coincide.

Let us discuss first one-gluon exchange contribution in Eq. (4.2.6). By looking at the denominator of the quark propagator one immediately notices that the scaling contribution in the Bjorken limit $p_J^- \to \infty$

$$\frac{1}{(p_J - k_1)^2 + i\varepsilon} \approx \frac{1}{2p_J^- k_1^+ + \mathbf{k}_{\perp 1}^2 - i\varepsilon}, \tag{4.2.19}$$

arises not only when one extracts the large p_J^- component from the numerator but also when $k_1^+ \sim 1/p_J^- \to 0$ and keeps finite contributions in the numerator. A simple algebra gives

$$\begin{aligned} &\bar{u}(p_J)\gamma_\mu(\not{p}_J - \not{k}_1)\gamma^\nu \langle X\,|A^\mu(k_1)q(0)|\,P\rangle \\ &\approx 2p_J^-\bar{u}(p_J)\gamma^\nu \langle X\,|A^+(k_1)q(0)|\,P\rangle - \bar{u}(p_J)\gamma_{\perp\alpha}(\not{k}_{\perp 1})\gamma^\nu \langle X\,|\mathbf{A}_\perp^\alpha(k_1)q(0)|P\rangle \end{aligned} \tag{4.2.20}$$

where we have used the fact that, when keeping the transverse part of the gluon field, $\not{p}_J$

can now be pushed through $\not{A}_\perp$, giving zero when acting on the on-shell spinor due to the equation of motion. Thus, we find

$$
\begin{aligned}
&\bar{u}(p_J) \langle X | \not{A}(k_1) S_F(p_J - k_1)\gamma^\nu q(0) | P\rangle \\
\approx\ & \frac{1}{k_1^+ - i\varepsilon} \bar{u}(p_J)\gamma^\nu \left\langle X \left| A^+(k_1) q(0) \right| P\right\rangle \\
&+\bar{u}(p_J)\frac{\gamma_{\perp\alpha}(\not{k}_{\perp 1})\gamma^\nu}{2p_J^- k_1^+ + \mathbf{k}_{\perp 1}^2 - i\varepsilon} \langle X | \mathbf{A}_\perp^\alpha(k_1) q(0) | P\rangle
\end{aligned}
\tag{4.2.21}
$$

The first term on the right-hand side is a contribution to the conventional light-cone link. When summed with contribution of multi-gluon exchanges it results as before into Eq. (4.2.16). We will drop, therefore, these terms completely and concentrate on effects of the second kind from transverse components of the gluon field.

As we already emphasized before, in the scaling limit of $p_J^- \to \infty$, a finite contribution comes from $k_1^+ = 0$ i.e., when the exchanged gluon carries no longitudinal momentum. To see this, we exponentiate the denominator via the Chisholm representation

$$
\frac{1}{2p_J^- k_1^+ + \mathbf{k}_{\perp 1}^2 - i\varepsilon} = i\int_0^\infty d\lambda e^{-i\lambda(2p_J^- k_1^+ + \mathbf{k}_{\perp 1}^2 - i\varepsilon)}. \tag{4.2.22}
$$

Substituting it into (4.2.21) and then into Eq. (4.2.6), we can perform the integrations with respect to k_1^+ and k_1^- which merely yield the Fourier transform of the gauge potential in these momentum components $A^\mu(\xi^- = 2\lambda p_J^-, \xi^+, \mathbf{k}_{\perp 1})$. Thus, in the scaling limit, the argument of $A(\xi)$ is set to $\xi^- = \infty$. The integration over λ can now be trivially performed giving the propagator in the transverse space,

$$
\langle p_J, X | J^\nu(0) | P\rangle_{(1)} = g\bar{u}(p_J)\int \frac{d^2\mathbf{k}_{\perp 1}}{(2\pi)^2}\gamma_\alpha \frac{\not{k}_1}{\mathbf{k}_1^2 - i\varepsilon}\gamma^\nu \left\langle X \left| \mathbf{A}^\alpha(\xi^- = \infty, \xi^+ = 0, \mathbf{k}_{\perp 1}) q(0) \right| P\right\rangle . \tag{4.2.23}
$$

To proceed further, we note that $\mathbf{A}^\alpha(\xi^- = \infty, \xi^+ = 0, \boldsymbol{\xi})$ must be a pure gauge

$$
\mathbf{A}^\alpha(\xi^- = \infty, \xi^+ = 0, \boldsymbol{\xi}) = \nabla^\alpha \phi(\boldsymbol{\xi}), \tag{4.2.24}
$$

since the field strength vanishes. The fourier transform of this potential to the mixed, light-cone coordinate-transverse momentum representation, is $\mathbf{A}^\alpha(\xi^- = \infty, \xi^+ = 0, \mathbf{k}) = i\mathbf{k}^\alpha \tilde{\phi}(\mathbf{k})$, with $\tilde{\phi}$ being the Fourier transform of $\phi(\boldsymbol{\xi})$. Substituting it into Eq. (4.2.23) we cancel the denominator $\mathbf{k}_1^2$, making use of $\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = -2\delta^{\alpha\beta}$, and get

$$\langle p_J, X \left| J^\nu(0) \right| P\rangle_{(1)} = -ig\bar{u}(p_J)\gamma^\nu \left\langle X \left| \phi(\mathbf{0})q(0) \right| P \right\rangle . \tag{4.2.25}$$

It is easy to see that $\phi(\mathbf{0})$ can be represented as a line integral

$$\phi(\mathbf{0}) = -\int_0^\infty d\boldsymbol{\xi} \cdot \mathbf{A}^\alpha(\infty, \boldsymbol{\xi}). \tag{4.2.26}$$

The above expression is the first term in the expansion of an additional eikonal phase to the convention one (4.2.11), which has been neglected in the literature. Therefore, even in the light-cone gauge $A^+ = 0$, Fig. (4-3b) does generate a non-zero contribution to the parton distribution.

The momentum space procedure just outlined cannot be easily extended beyond single-gluon exchange. Therefore, the strategy will be to transform all factors in the integrand of Eq. (4.2.23) into the transverse coordinate space and do all manipolations there. To this end, we use

$$\frac{\mathbf{k}^\alpha}{\mathbf{k}^2} = -\frac{i}{2\pi}\int d^2\boldsymbol{\xi} e^{i\mathbf{k}\cdot\boldsymbol{\xi}} \frac{\boldsymbol{\xi}^\alpha}{\boldsymbol{\xi}^2} = -\frac{i}{2\pi}\int d^2\boldsymbol{\xi} e^{i\mathbf{k}\cdot\boldsymbol{\xi}} \nabla^\alpha \ln|\boldsymbol{\xi}| , \tag{4.2.27}$$

where we do not display a mass parameter which makes the argument of the logarithm dimensionless. Thus, the right-hand side of Eq. (4.2.23) can be equivalently written as

$$-ig\bar{u}(p_J)\gamma_\alpha\gamma_\beta\gamma^\nu \int \frac{d^2\boldsymbol{\xi}}{2\pi} \nabla^\beta \ln|\boldsymbol{\xi}| \left\langle X \left| \nabla^\alpha \phi(\boldsymbol{\xi}) q(0) \right| P \right\rangle . \tag{4.2.28}$$

Next, one integrates by parts so that both derivatives act on the logarithm, $\nabla^\alpha\nabla^\beta \ln|\boldsymbol{\xi}|$. This is a symmetric tensor in its two-dimensional Lorentz indices so that the contracted

Dirac matrices can be also symmetrized and reduced to the Kronecker symbol by means of the Cliford algebra relation. Finally, since $\ln|\boldsymbol{\xi}|$ is a two-dimensional Green function, one has $\nabla^2 \ln|\boldsymbol{\xi}| = 2\pi\delta^{(2)}(\boldsymbol{\xi})$. These manipolations lead to the result in Eq. (4.2.25).

For n-gluon exchanges, we have the following contribution from the transverse components of the gluon field, similarly to Eq. (4.2.23),

$$\langle p_J, X\,|J^\nu(0)|\,P\rangle_{(n)} = g^n \bar{u}(p_J)\langle X\,|\int \frac{d^2\mathbf{k}_1}{(2\pi)^2} \not{A}(\infty, k_1)\frac{\mathbf{k}_1/}{\mathbf{k}_1^2 - i\varepsilon}\int \frac{d^2\mathbf{k}_2}{(2\pi)^2}\not{A}(\infty, k_2)$$

$$\times\frac{\not{k}_1 + \not{k}_2}{(\not{k}_1 + \not{k}_2)^2 - i\varepsilon}\cdots\int \frac{d^2\mathbf{k}_n}{(2\pi)^2}\not{A}(\infty, k_n)\frac{\sum_{i=1}^n \not{k}_i}{(\sum_{i=1}^n \not{k}_i)^2 - i\varepsilon}\gamma^\nu q(0)\mid P\rangle. \tag{4.2.29}$$

The Fourier transformation gives

$$\langle p_J, X\,|J^\nu(0)|\,P\rangle_{(n)} = (-ig)^n \bar{u}(p_J)\langle X\,|\int \prod_{i=1}^{n} \frac{d^2\boldsymbol{\xi}_i}{(2\pi)^2}\not{A}(\infty, \boldsymbol{\xi}_1)\nabla_1 \ln|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|$$

$$\not{A}(\infty, \boldsymbol{\xi}_2)\nabla_2 \ln|\boldsymbol{\xi}_2 - \boldsymbol{\xi}_3|\cdots\not{A}(\infty, \boldsymbol{\xi}_{n-1})\nabla_{n-1}\ln|\boldsymbol{\xi}_{n-1} - \boldsymbol{\xi}_n|\,\not{A}(\infty, \boldsymbol{\xi}_n)\nabla_n \ln|\boldsymbol{\xi}_n|\,\gamma^\nu q(0)\mid P\rangle. \tag{4.2.30}$$

Now, exploiting Eq. (4.2.24), we integrate by parts starting with ξ_1 and using

$$\nabla_1^2 \ln|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| = -\nabla_2^2 \ln|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| = -2\pi\delta^{(2)}(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2),$$

then with respect to $\boldsymbol{\xi}_2$, first noting that

$$\phi(\boldsymbol{\xi})\not{A}^\alpha(\boldsymbol{\xi}) = \frac{1}{2}\nabla^\alpha\phi(\boldsymbol{\xi}), \tag{4.2.31}$$

and then performing the same partial integrations as for $\boldsymbol{\xi}_1$. In this way one can integrate out all $\boldsymbol{\xi}$'s. At the end, one gets

$$\langle p_J, X\,|J^\nu(0)|\,P\rangle_{(n)} = (-ig)^n \bar{u}(p_J)\gamma^\nu\left\langle X\left|\frac{1}{n!}\phi^n(\mathbf{0})q(0)\right|P\right\rangle. \tag{4.2.32}$$

As a final step, one simply notices that

$$\frac{1}{n!}\phi^n(\mathbf{0}) = \int_0^\infty d\boldsymbol{\xi}_1 \cdot \mathbf{A}(\infty, \boldsymbol{\xi}_1) \int_0^{\boldsymbol{\xi}_1} d\boldsymbol{\xi}_2 \cdot \mathbf{A}(\infty, \boldsymbol{\xi}_2) \cdots \int_0^{\boldsymbol{\xi}_{n-1}} d\boldsymbol{\xi}_n \cdot \mathbf{A}(\infty, \boldsymbol{\xi}_n), \quad (4.2.33)$$

so that it forms a gauge link once resummed to all orders

$$[\infty, \vec{\infty}; \infty, \mathbf{0}] = P \exp\left(ig \int_0^\infty d\boldsymbol{\xi} \cdot \mathbf{A}(\infty, \boldsymbol{\xi}) \right). \quad (4.2.34)$$

Therefore, restoring the light-cone gauge link, one finds the complete result for the amplitude

$$\langle p_J, X \left| J^\nu(\xi) \right| P \rangle = \bar{u}(p_J)\gamma^\nu \left\langle X \left| \left[\infty, \vec{\infty}; \xi^-, \boldsymbol{\xi}\right]_C q(\xi^-, \boldsymbol{\xi}) \right| P \right\rangle, \quad (4.2.35)$$

where

$$\left[\infty, \vec{\infty}; \xi^-, \boldsymbol{\xi}\right]_C \equiv \left[\infty, \vec{\infty}; \infty, \boldsymbol{\xi}\right] \left[\infty, \boldsymbol{\xi}; \xi^-, \boldsymbol{\xi}\right]. \quad (4.2.36)$$

Multiplying (4.2.35) by its complex conjugate, we deduce the gauge invariant transverse momentum-dependent parton distribution

$$q(x, \mathbf{k}) = \int \frac{d\xi^- d^2\boldsymbol{\xi}}{2(2\pi)^3} e^{-i(x\xi^- - \mathbf{k}\cdot\boldsymbol{\xi})} \left\langle P \left| \bar{\psi}(\xi^-, \boldsymbol{\xi}) \left[\infty, \infty; \xi^-, \boldsymbol{\xi}\right]_C^\dagger \gamma^+ \left[\infty, \infty; 0, \mathbf{0}\right]_C q(0, \mathbf{0}) \right| P \right\rangle. \quad (4.2.37)$$

The unitarity implies a partial cancellation of links at light-cone infinity

$$\left[\infty, \vec{\infty}; \xi^-, \boldsymbol{\xi}\right]_C^\dagger \left[\infty, \vec{\infty}; 0, \mathbf{0}\right]_C = \left[\infty, \boldsymbol{\xi}; \xi^-, \boldsymbol{\xi}\right]^\dagger \left[\infty, \boldsymbol{\xi}; \infty, \mathbf{0}\right] \left[\infty, \mathbf{0}; 0, \mathbf{0}\right], \quad (4.2.38)$$

so that the definition (4.2.1) of the parton distribution, accepted in the literature, acquires an additional transverse link.

4.3 Model Calculation of Time-Reversal Odd Quark Distribution Function

We shall calculate the single-spin asymmetry in semi-inclusive electroproduction $\gamma^* p \to HX$ induced by final-state interactions in a simple diquark model (see[5],[7]). In this model a proton of mass M is directly coupled to a charged quark and scalar diquark of masses m and λ respectively. Introducing the interaction between the diquark and gluons, and the nucleon and quark-diquark,

$$\mathcal{L}_{int} = -g\bar{q}N\phi^* - ie_2\phi^*(\overleftrightarrow{\partial^\mu})\phi A_\mu + h \cdot c \tag{4.3.1}$$

where q represents the quark field, N the nucleon, ϕ the charged scalar diquark with charge e_2. A complementary view of the gauge links is to regard the physical quark Q as a free quark field q with an attached gauge link extending from the quark position to positive infinity. This physical quark field is gauge invariant and contains all the final state interaction effects. The parton distributions are then the densities of these physical quarks in a bound state.Thus the conventional transverse momentum quark distribution in this context is,

$$\begin{aligned} \Phi^{[\gamma^+]}(x, k_\perp) &= \int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{2(2\pi)^3} e^{-i(\xi^- k^+ - \boldsymbol{\xi}_\perp \cdot \mathbf{k}_\perp)} \times \langle PS \left| \overline{Q}(\xi^-, \boldsymbol{\xi}_\perp)\gamma^+ Q(0, \mathbf{0}_\perp) \right| PS \rangle \\ &= f_1(x, k_\perp^2) + f_{1T}^\perp(x, k_\perp^2) \frac{\varepsilon^{+\alpha\beta\gamma} k_{\perp\alpha} p_\beta S_{\perp\gamma}}{Mp^+}, \end{aligned} \tag{4.3.2}$$

where $k^+ = xP^+$ and physical quark Q are defined as

$$Q(\xi^-, \boldsymbol{\xi}_\perp) = [\infty, \boldsymbol{\xi}_\perp; \xi^-, \boldsymbol{\xi}_\perp]\, q(\xi^-, \boldsymbol{\xi}_\perp) = P \exp\left(-ie_1 \int_{\xi^-}^{\infty} \mathcal{A}^+(\eta^-, \boldsymbol{\xi}_\perp) d\eta^-\right) q(\xi^-, \boldsymbol{\xi}_\perp) \tag{4.3.3}$$

and e_1 is the charge of the struck quark.

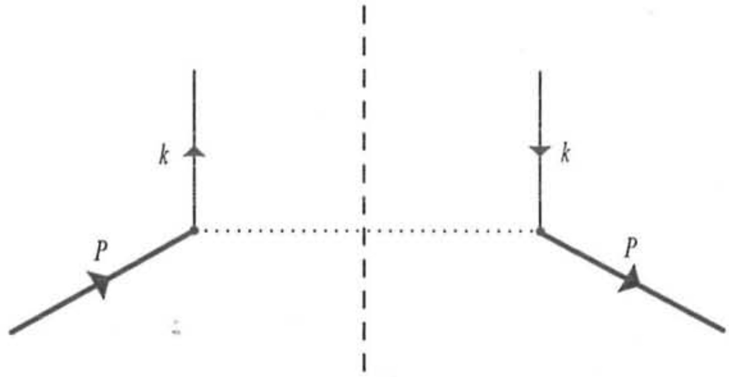

Figure 4-4: Tree contribution to the spin-independent transverse momentum distribution.

4.3.1 Unpolarized Distribution Function

The spin independent transverse-momentum distribution function $f(x, k_\perp^2)$ is given by the first term in the expansion (4.3.2)

$$f_1(x, k_\perp^2) = \int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{2(2\pi)^3} e^{-i(\xi^- k^+ - \boldsymbol{\xi}_\perp \cdot \mathbf{k}_\perp)} \left\langle P \left| \bar{q}(\xi^-, \boldsymbol{\xi}_\perp)\gamma^+ q(0) \right| P \right\rangle . \tag{4.3.4}$$

Considering the tree contribution (see Fig. 4-4)

$$\begin{aligned} f_1(x, k_\perp^2) &= \int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{2(2\pi)^3} e^{-i(\xi^- k^+ - \boldsymbol{\xi}_\perp \cdot \mathbf{k}_\perp)} \\ &\quad \times \int d^4x_1 d^4x_2 \frac{d^4\ell}{(2\pi)^4} e^{-i\ell.(x_2 - x_1)} \Theta(\ell_0)(2\pi)\delta(\ell^2 - \lambda^2) \\ &\quad \times \left\langle P \left| \overline{N}(x_2) \right| 0 \right\rangle (-ig) i S_F(x_2 - \xi)\gamma^+ i S_F(0 - x_1)(-ig) \left\langle 0 \left| N(x_1) \right| P \right\rangle , \end{aligned}$$

or,

$$\begin{aligned} f_1(x, k_\perp^2) &= \int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{2(2\pi)^3} d^4x_1 d^4x_2 \frac{d^4\ell}{(2\pi)^4} \frac{d^4\ell_1}{(2\pi)^4} \frac{d^4\ell_2}{(2\pi)^4} (2\pi)\delta(\ell^2 - \lambda^2) \\ &\quad \times e^{-ix_2 \cdot (\ell_1 - P + \ell)} e^{-ix_1 \cdot (-\ell_2 + P - \ell)} e^{-i\xi^- (k^+ - l_1^+)} e^{-i\xi_\perp \cdot (\ell_{1\perp} - \mathbf{k}_\perp)} \\ &\quad \times \overline{U}(P)(-ig) \frac{i(\not{\ell}_1 + m)}{\ell_1^2 - m^2 + i\varepsilon} \gamma^+ \frac{i(\not{\ell}_2 + m)}{\ell_2^2 - m^2 + i\varepsilon} (-ig) U(P). \end{aligned} \tag{4.3.5}$$

Integrating over x_1,x_2, ξ, ℓ_1 and ℓ_2 we get

$$
\begin{aligned}
f_1(x,k_\perp^2) &= \frac{g^2}{2(2\pi)^3}\int \frac{dk^-}{(2\pi)}(\pi)\delta\left((P-k)^2-\lambda^2\right) \\
&\times Tr\left[(\not{P}+M)(\not{k}+m)\gamma^+(\not{k}+m)\right]\frac{1}{(k^2-m^2)^2}. \qquad (4.3.6)
\end{aligned}
$$

Working in the target rest frame $P^\mu = \left(p^+, \frac{M^2}{2p^+}, \mathbf{0}_\perp\right)$ we obtain

$$
(P-k)^2-\lambda^2 = -2p^+(1-x)\left[k^- - \frac{1}{p^+}\left\{\frac{M^2}{2} - \frac{k_\perp^2}{2(1-x)} - \frac{\lambda^2}{2(1-x)}\right\}\right], \qquad (4.3.7)
$$

and the unpolarised distribution function becomes

$$
f_1(x,k_\perp^2) = \frac{g^2(1-x)\left[k_\perp^2+(xM+m)^2\right]}{2(2\pi)^3\Lambda^2(k_\perp^2)}, \qquad (4.3.8)
$$

with

$$
\Lambda(k_\perp^2) = k_\perp^2 + x(1-x)\left(-M^2+\frac{m^2}{x}+\frac{\lambda^2}{(1-x)}\right). \qquad (4.3.9)
$$

4.3.2 Polarized Distribution Function

Now consider the term, which is first order in e_1, in the expanion (4.3.2)

$$
\begin{aligned}
f(x,k_\perp,S_\perp) &= \sum_n \int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{2(2\pi)^3} e^{-i(\xi^-k^+-\boldsymbol{\xi}_\perp\cdot\mathbf{k}_\perp)}\left\langle P\left|\bar{q}(\xi^-,\boldsymbol{\xi}_\perp)\right|n\right\rangle \\
&\times\left\langle n\left|\left(-ie_1\int_0^\infty d\eta^- A^+(\eta^-,\mathbf{0})\right)\gamma^+ q(0)\right|P\right\rangle + h.c. \quad (4.3.10)
\end{aligned}
$$

At one loop order, we have the following expression from Fig. 4-5,

$$
\begin{aligned}
f(x,k_\perp,S_\perp) &= \int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{2(2\pi)^3} e^{-i(\xi^-k^+-\boldsymbol{\xi}_\perp\cdot\mathbf{k}_\perp)} \\
&\times \int d^4x_1 d^4x_2 d^4y \frac{d^4\ell}{(2\pi)^4} e^{-i\ell\cdot x_2}\Theta(\ell_0)(2\pi)\delta(\ell^2-\lambda^2) \\
&\times \left\langle PS\left|\overline{N}(x_2)\right|0\right\rangle(-ig)iS_F(x_2-\xi)\gamma^+ iS_F(0-x_1)(-ig)\left\langle 0\left|N(x_1)\right|PS\right\rangle
\end{aligned}
$$

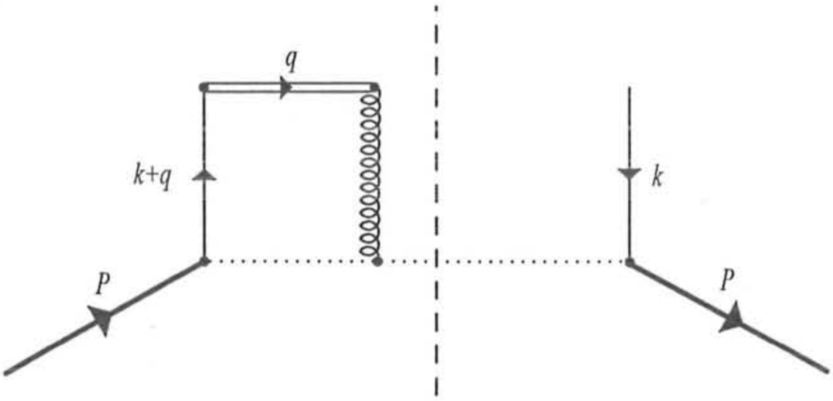

Figure 4-5: One-loop contribution to the spin-dependent transverse momentum distribution.

$$\times e_2\left(-ie_1\int_0^\infty d\eta^- iD^{+\mu}(\eta^- - y)\right)\left(e^{i\ell\cdot y}\overleftrightarrow{\partial}_{y\,\mu}\, i\Delta_F(y-x_1)\right), \tag{4.3.11}$$

or

$$\begin{aligned} f(x,k_\perp,S_\perp) &= \int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{2(2\pi)^3} d^4x_1 d^4x_2 d^4y \frac{d^4\ell}{(2\pi)^4}\frac{d^4\ell_1}{(2\pi)^4}\frac{d^4\ell_2}{(2\pi)^4}\frac{d^4\ell_3}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} \\ &\quad \times e^{-i\xi^-(k^+-l_1^+)}e^{-i\xi_\perp\cdot(\ell_{1\perp}-\mathrm{k}_\perp)}e^{-ix_2\cdot(\ell_1-P+\ell)}e^{-ix_1\cdot(-\ell_2+P-\ell_3)}e^{-iy\cdot(\ell_3-\ell-q)} \\ &\quad \times e_2\left(-ie_1\int_0^\infty d\eta^- e^{-iq^+\eta^-}\right) i\left(\ell^+ + \ell_3^+\right)(2\pi)\delta(\ell^2-\lambda^2) \\ &\quad \times \overline{U}(P,S)(-ig)(\not{\ell}_1+m)\gamma^+(\not{\ell}_2+m)(-ig)U(P,S)(ie_2) \\ &\quad \times \frac{1}{\ell_1^2-m^2+i\varepsilon}\frac{1}{\ell_2^2-m^2+i\varepsilon}\frac{1}{\ell_3^2-\lambda^2+i\varepsilon}\frac{1}{q^2+i\varepsilon} + h.c. \end{aligned} \tag{4.3.12}$$

Using

$$\int_0^\infty d\eta^- e^{\mp iq^+\eta^-} = \mathcal{L}im_{\varepsilon\to 0}\frac{i}{q^+ \pm i\varepsilon} \tag{4.3.13}$$

and integrating over x_1, x_2, y, ξ, ℓ_1, ℓ_2 and ℓ_3 we get

$$\begin{aligned} f(x,k_\perp,S_\perp) &= \left(\frac{ig^2e_1e_2}{4(2\pi)^3}\right)\int \frac{d^4q}{(2\pi)^4}\frac{dk^-}{(2\pi)}(2\pi)\delta\left((P-k)^2-\lambda^2\right)\frac{1}{(k^2-m^2)} \\ &\quad \times Tr\left[(\not{P}+M)(1+\gamma_5\not{S})(\not{k}+m)\gamma^+(\not{k}+\not{q}+m)\right]\frac{2(1-x)p^+-q^+}{q^++i\varepsilon} \end{aligned}$$

$$\times \frac{1}{(k+q)^2 - m^2 + i\varepsilon} \frac{1}{(P-k-q)^2 - \lambda^2 + i\varepsilon} \frac{1}{q^2 + i\varepsilon} + h.c. \quad (4.3.14)$$

Now consider only the spin dependent part of above equation[4]

$$\begin{aligned} f(x,k_\perp,S_\perp) &= \left(\frac{-ig^2 e_1 e_2}{8(2\pi)^3 \Lambda(k_\perp^2) p^+}\right) \int \frac{d^4q}{(2\pi)^4} \frac{2(1-x)p^+ - q^+}{q^+ + i\varepsilon} \\ &\times Tr\left[(\not P + M)(\gamma_5 \not S)(\not k + m)\gamma^+(\not k + \not q + m)\right] \\ &\times \frac{1}{(k+q)^2 - m^2 + i\varepsilon} \frac{1}{(P-k-q)^2 - \lambda^2 + i\varepsilon} \frac{1}{q^2 + i\varepsilon} + h.c \quad (4.3.15) \end{aligned}$$

Using

$$\frac{1}{q^+ \pm i\varepsilon} = P(\frac{1}{q^+}) \mp \pi\delta(q^+) \quad (4.3.16)$$

and adding the hermitian conjugating contribution[5], we obtain

$$\begin{aligned} f(x,k_\perp,S_\perp) &= \left(\frac{-ig^2 e_1 e_2}{4(2\pi)^3 \Lambda(k_\perp^2)}\right) \frac{(m + xM)}{(p^+)^2} \int \frac{d^2 q_\perp}{(2\pi)^2} \frac{dq^-}{(2\pi)} Tr\left[\gamma^+ \not q_\perp \not p \gamma_5 \not S_\perp\right] \\ &\times (-i) \frac{2(1-x)}{-q_\perp^2 + i\varepsilon} \frac{1}{2x\left[q^- - \frac{(k_\perp + q_\perp)^2}{2xp^+} + k^- + i\varepsilon\right]} \\ &\times \frac{1}{2(x-1)\left[q^- - \frac{q_\perp^2 + 2q_\perp \cdot k_\perp}{2(x-1)p^+} - i\varepsilon\right]}. \quad (4.3.17) \end{aligned}$$

---

[4]here k^- is given as

$$k^- = \frac{1}{p^+}\left\{\frac{M^2}{2} - \frac{k_\perp^2}{2(1-x)} - \frac{\lambda^2}{2(1-x)}\right\}.$$

[5]the hermitian conjugating contribution is

$$\begin{aligned} f(x,k_\perp,S_\perp) &= \left(\frac{+ig^2 e_1 e_2}{8(2\pi)^3 \Lambda(k_\perp^2)}\right) \int \frac{d^4q}{(2\pi)^4} Tr\left[(\not P + M)(\gamma_5 \not S)(\not k + m)\gamma^+(\not k + \not q + m)\right] \\ & \frac{2(1-x) - q^+}{q^+ - i\varepsilon} \frac{1}{(k+q)^2 - m^2 + i\varepsilon} \frac{1}{(P-k-q)^2 - \lambda^2 + i\varepsilon} \frac{1}{q^2 + i\varepsilon}. \end{aligned}$$

Performing contour integration over q^- and replacing $\not{q}_\perp \to \not{q}_\perp - \not{k}_\perp$, we get

$$f(x, k_\perp, S_\perp) = \left(\frac{ig^2 e_1 e_2}{4x(2\pi)^3 \Lambda(k_\perp^2)}\right) \frac{(m + xM)}{p^+} \int \frac{d^2 q_\perp}{(2\pi)^2} Tr\left[\gamma^+ (\not{q}_\perp - \not{k}_\perp) \not{p} \gamma_5 \not{S}_\perp\right] \cdot \frac{1}{q_\perp^2 + k_\perp^2 - 2q_\perp \cdot k_\perp - i\varepsilon} \frac{x(x-1)}{q_\perp^2 + \Lambda(0) - i\varepsilon}. \quad (4.3.18)$$

Using the Feynman parametrization,

$$\frac{1}{ab} = \int_0^1 \frac{dz}{(b + (a-b)z)^2} \quad (4.3.19)$$

this becomes

$$f(x, k_\perp, S_\perp) = \left(\frac{ig^2 e_1 e_2}{4x(2\pi)^3 \Lambda(k_\perp^2)}\right) \frac{(m + xM)}{p^+} \int \frac{d^2 q_\perp}{(2\pi)^2} \int_0^1 dz Tr\left[\gamma^+ (\not{q}_\perp - \not{k}_\perp) \not{p} \gamma_5 \not{S}_\perp\right] \frac{x(x-1)}{\left[q_\perp^2 - 2q_\perp \cdot k_\perp (1-z) + k_\perp^2 (1-z) + \Lambda(0) z - i\varepsilon\right]^2}. \quad (4.3.20)$$

Replacing $\not{q}_\perp \to \not{q}_\perp + \not{k}_\perp (1-z)$ and doing integration over $q_\perp$ and z we finally arrive at

$$\begin{aligned} f(x, k_\perp, S_\perp) &= f_{1T}^\perp(x, k_\perp^2) \frac{\varepsilon^{+\alpha\beta\gamma} k_{\perp\alpha} p_\beta S_{\perp\gamma}}{M p^+} \\ &= \frac{g^2 e_1 e_2}{(2\pi)^4} \frac{(1-x)(m + xM)}{4\Lambda(k_\perp^2) p^+} \varepsilon^{+\alpha\beta\gamma} k_{\perp\alpha} p_\beta S_{\perp\gamma} \frac{1}{k_\perp^2} \ln\left|\frac{\Lambda(k_\perp^2)}{\Lambda(0)}\right|. \end{aligned} \quad (4.3.21)$$

The ratio of the spin-dependent and independent distributions is

$$\frac{f(x, k_\perp, S_\perp)}{f(x, k_\perp^2)} = \frac{e_1 e_2 (m + xM)}{4\pi p^+} \frac{\Lambda(k_\perp^2)}{[k_\perp^2 + (xM + m)^2]} \varepsilon^{+\alpha\beta\gamma} k_{\perp\alpha} P_\beta S_\gamma \frac{1}{k_\perp^2} \ln\left|\frac{\Lambda(k_\perp^2)}{\Lambda(0)}\right|, \quad (4.3.22)$$

which is the same result as obtained in Ref. [5]. This shows that the conventional definition of the quark distribution in the non-singular gauge does take into account properly the effects of the final-state interactions.

Now we will show that the same result can be reproduced with the new definition of the quark distribution (4.2.37) in the light cone gauge

$$\Phi^{[\gamma^+]}(x,k_\perp) = \int \frac{d\xi^- d^2\boldsymbol{\xi}}{2(2\pi)^3} e^{-i(x\xi^- - \mathbf{k}\cdot\boldsymbol{\xi})} \left\langle P \left| \bar{q}(\xi^-,\boldsymbol{\xi}) \left[\infty,\infty;\infty,\boldsymbol{\xi}\right]^\dagger \gamma^+ \left[\infty,\infty;\infty,0\right] q(0,0) \right| P \right\rangle . \tag{4.3.23}$$

Expanding it and consider the term which is first order in e_1,

$$\begin{aligned} f(x,k_\perp,S_\perp) &= \sum_n \int \frac{d\xi^- d^2\boldsymbol{\xi}_\perp}{2(2\pi)^3} e^{-i(\xi^- k^+ - \boldsymbol{\xi}_\perp\cdot\mathbf{k}_\perp)} \left\langle P \left| \bar{q}(\xi^-,\boldsymbol{\xi}_\perp) \right| n \right\rangle \\ &\times \left\langle n \left| \left(-ie_1 \int_{\boldsymbol{\xi}_\perp}^{\infty} d\boldsymbol{\eta}_\perp \cdot A(\infty,\boldsymbol{\eta}_\perp) \right) \gamma^+ q(0,0) \right| P \right\rangle + h.c. \end{aligned}$$

Going to the momentum space, we get

$$\begin{aligned} f(x,k_\perp,S_\perp) &= \left(\frac{-ig^2 e_1 e_2}{8(2\pi)^3 \Lambda(k_\perp^2) p^+} \right) \int \frac{d^4q}{(2\pi)^4} \frac{(2(1-x)p^+ - q^+)\, e^{iq^+\infty}}{q^+} \\ &\times Tr\left[(\not{P} + M)(\gamma_5 \not{S})(\not{k} + m)\gamma^+ (\not{k} + \not{q} + m) \right] \\ &\times \frac{1}{(k+q)^2 - m^2 + i\varepsilon} \frac{1}{(P-k-q)^2 - \lambda^2 + i\varepsilon} \frac{1}{q^2 + i\varepsilon} + h.c. \end{aligned} \tag{4.3.24}$$

where $1/q^+$ comes from the light-cone propagator for the gluon[6]. Using

$$\frac{e^{iq^+\infty}}{q^+} = i\pi\delta(q^+), \tag{4.3.25}$$

which is true in the sense of distribution, we recover the result in Eq. (4.3.21).

4.4 Conclusion

To maintain QCD gauge invariance we insert a gauge link between the quark fields in the parton distribution functions. In the conventional parton distribution functions only A^+ component of the gauge potential is present in the gauge link. This results in the vanishing of the Sivers asymmetry because of the time reversal invariance in the light-cone

[6] see Appendix A.

gauge where $A^+ = 0$ and gauge link becomes unity (see [4]). We recalculated the Sivers asymmetry in the Feynman gauge, using the conventional parton distribution function and got the same result as obtained earlier by Brodsky, Hwang and Schmidt [5]. Since this result should be gauge invariant we are forced to reconsider the definition of the conventional parton distribution functions, carefully, in order to get the same result in the light cone gauge too.

As it turns out that we had been missing an additional scaling contribution which appeared in the definition of the parton distribution functions in the light-cone gauge. Our study demonstrates the existence of this extra scaling contribution from transverse components of the gauge potential at the light-cone infinity. They form a transverse gauge link which is indispensable for restoration of the gauge invariance of parton distributions in the light-cone gauge where the gauge potential does not vanish asymptotically.

In the light-cone gauge, the gauge link along the light-cone disappears, $[\infty, \boldsymbol{\xi}; \xi^-, \boldsymbol{\xi}] = 1$, and we are left with the link in the transverse direction, $[\infty, \infty; \infty, \boldsymbol{\xi}]$, which is crucial to maintain the gauge independence under residual gauge transformations. This transverse gauge link gives exactly the same result we obtained in Feynman gauge, where this transverse gauge link becomes unity. Thus the transverse link in our new definition is responsible for the final state interactions which generates the Sivers distribution function $f_{1T}^{\perp}(x, k_{\perp}^2)$: the parton transverse momentum distribution asymmetry in a transversely polarized nucleon. This distribution is fully responsible for the single transverse-spin asymmetry discovered in [5] and explained in [6]. The mechanism which led to non-vanishing asymmetry was thought to be time-reversal odd and therefore prohibited in QCD. Our analysis supports the conclusion in [6] that time-reversal does not impose any constraint on the Sivers function and hence the rich phenomenology of transverse momentum-dependent parton distributions as related to spin physics.

The transverse link at infinity cancels when integration over $\mathbf{k}$ is performed. Therefore, independent of the residual gauge fixing, ordinary Feynman distribution $f(x)$ is the correct formula for the parton distribution in any gauge. So the parton distributions will be uniquely determined by the light-cone wave functions, $f(x) \sim |\psi|^2$, and may be interepreted as the parton density in the spirit of the conventional Feynman parton model.

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Appendix A

Standard Conventions and Formulae

Now I list the conventions and some usefull formulae used in this thesis. Throughout, I use the metric tensor of signature-2,

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{A.1}$$

The totally antisymmetric tensor $\varepsilon^{\mu\nu\rho\sigma}$ is normalised so that

$$\varepsilon^{0123} = -\varepsilon_{0123} = +1. \tag{A.2}$$

A generic four-vector A^μ is written, in Cartesian contravariant components, as

$$A^\mu = (A^0, A^1, A^2, A^3) = (A^0, \mathbf{A}). \tag{A.3}$$

The light-cone components of A^μ are defined as

$$A^\pm = \frac{1}{\sqrt{2}}(A^0 \pm A^3), \tag{A.4}$$

and in these components A^μ is written as

$$A^\mu = \left[A^+, A^-, \mathbf{A}\right]. \tag{A.5}$$

Scalar product of two four-vectors A^μ and B^μ is

$$A \cdot B = A^0 B^0 - \mathbf{A} \cdot \mathbf{B} = A^+ B^- + A^- B^+ - \mathbf{A}_\perp \cdot \mathbf{B}_\perp. \tag{A.6}$$

A.1 Sudakov Decomposition of Vectors

We introduce the two light like vectors(the Sudakov vectors)

$$p^\mu = \frac{1}{\sqrt{2}}(\Lambda, 0, 0, \Lambda), \tag{A.7}$$

$$n^\mu = \frac{1}{\sqrt{2}}(\Lambda^{-1}, 0, 0, -\Lambda^{-1}), \tag{A.8}$$

where Λ is arbitrary. These vectors satisfy

$$p^2 = n^2 = 0, \; p \cdot n = 1, \; n^+ = p^- = 0. \tag{A.9}$$

In light-cone components they read

$$p^\mu = \left[\Lambda, 0^-, \mathbf{0}_\perp\right] \tag{A.10}$$

$$n^\mu = \left[0, \Lambda^{-1}, \mathbf{0}_\perp\right]. \tag{A.11}$$

A generic vector A^μ can be parametrized as (a Sudakov decomposition)

$$A^\mu = (A \cdot n)\, p^\mu + (A \cdot p)\, n^\mu + A_\perp^\mu \tag{A.12}$$

with $A_\perp^\mu = (0, \mathbf{A}_\perp, 0)$.

A.1.1 The $\gamma^* N$ Collinear Frames

In DIS processes, we call the frames where the virtual photon and the target nucleon move collinearly "$\gamma^* N$ collinear frames". If the motion takes place along the z-axis, we can represent the nucleon momemtum P and the photon momentum q in terms of the Sudakov vectors p and n as

$$P^\mu = p^\mu + \frac{1}{2}M^2 n^\mu \simeq p^\mu, \tag{A.13}$$

$$q^\mu \simeq P.q n^\mu - x p^\mu = M\nu n^\mu - x p^\mu, \tag{A.14}$$

where the approximation equality sign indicates that we are neglecting M^2 with respect to large scales such as Q^2, or $(P^+)^2$ in the infinite momentum frame. Conventionally we always take the nucleon to be directed in the positive z direction.

With the identification (A.13) the parameter Λ appearing in the definition of the Sudakov vectors coincides with P^+ and fixes the specific frame.

In particular:

- in the *target rest frame* (TRF) one has

$$P^\mu = (M, 0, 0, 0), \tag{A.15}$$

$$q^\mu = \left(\nu, 0, 0, -\sqrt{\nu^2 + Q^2}\right), \tag{A.16}$$

and $\Lambda \equiv P^+ = M/\sqrt{2}$. The Bjorken limit in this frame corresponds to $q^- = \sqrt{2}\nu \to \infty$ with $q^+ = -M/\sqrt{2}$ fixed.

- in the *infinite momentum frame* (IMF) the momenta are

$$P^\mu \simeq \frac{1}{\sqrt{2}}(P^+, 0, 0, P^+) \tag{A.17}$$

$$q^\mu \simeq \frac{1}{\sqrt{2}}\left(\frac{M\nu}{P^+} - xP^+, 0, 0, -\frac{M\nu}{P^+} - xP^+\right) \tag{A.18}$$

Here we have $P^- \to 0$ and $\Lambda \equiv P^+ \to \infty$. In this frame the vector n^μ is suppressed by a factor of $(1/P^+)^2$ with respect to p^μ.

A.2 Normalisation of States, S-Matrix and Cross-Section

S-Matix and Invariant Scattering Amplitude:

$$\begin{aligned} S &= T\exp\left[-i\int dx\mathcal{H}(x)\right] = I + iT \\ \langle f\left|T\right|i\rangle &= (2\pi)^4\delta^4(P_f - P_i)\mathcal{T}_{fi}. \end{aligned} \tag{A.19}$$

Differential Cross-Section:

Differential cross-section for the scattering from an initial state i={1,2} into a final state f={3,4,...,n} is

$$d\sigma = \frac{1}{j!}\frac{\left|\mathcal{T}_{fi}\right|^2}{FD}dLips, \tag{A.20}$$

where F is the incident particle flux

$$F = \rho_1\left|\mathbf{v}_{rel}\right| = 2E_1\left|\mathbf{v}_2 - \mathbf{v}_1\right|, \tag{A.21}$$

D is the target-particle density

$$D = \rho_2 = 2E_2 \tag{A.22}$$

and $dLips$ is the Lorentz invariant phase space factor written as

$$dLips = (2\pi)^4\delta^{(4)}(P_i - P_f)\frac{d^3p_3}{(2\pi)^3 2E_3}\cdots\frac{d^3p_n}{(2\pi)^3 2E_n}. \tag{A.23}$$

The statistical factor $\frac{1}{j!}$ is for each group of j identical particles in the final state. The product FD can be written in the Lorentz invariant form as

$$FD = 4\left[(p_1 \cdot p_2)^2 - m_1^2 m_2^2\right]^{1/2}. \tag{A.24}$$

Spin 1/2 Particle:

Spin 1/2 particle of mass m is described by a field $\psi(x)$, which in the absence of interaction, satisfies the Dirac equation

$$\left(i\overrightarrow{\not\partial} - m\right)\psi(x) = 0. \tag{A.25}$$

The adjoint of $\psi(x)$, $\bar{\psi}(x) = \psi(x)^\dagger\gamma^0$ satisfies the Dirac equation

$$\bar{\psi}(x)\left(-i\overleftarrow{\not\partial} - m\right) = 0. \tag{A.26}$$

The fourier decomposition of $\psi(x)$ is

$$\psi(x) = \sum_s \int \frac{d^3\mathrm{p}}{(2\pi)^3 2E}\left[b(p,s)u(p,s)e^{-ip\cdot x} + d^\dagger(p,s)v(p,s)e^{+ip\cdot x}\right] \tag{A.27}$$

and

$$\bar{\psi}(x) = \sum_s \int \frac{d^3\mathrm{p}}{(2\pi)^3 2E}\left[b^\dagger(p,s)\bar{u}(p,s)e^{+ip\cdot x} + d(p,s)\bar{v}(p,s)e^{-ip\cdot x}\right] \tag{A.28}$$

where u and v satisfy the equations

$$(\not p - m)u(p,s) = 0 \quad , \quad (\not p + m)v(p,s) = 0, \tag{A.29}$$

$$\bar{u}(p,s)(\not p - m) = 0 \quad , \quad \bar{v}(p,s)(\not p + m) = 0 \tag{A.30}$$

and the operators b and d satisfy the anticommutation relations

$$\left\{b(p,s), b^\dagger(p',s')\right\} = (2\pi)^3 2E\delta^3(\mathrm{p}' - \mathrm{p})\delta_{s's}, \tag{A.31}$$

$$\left\{d(p,s), d^\dagger(p',s')\right\} = (2\pi)^3 2E\delta^3(\mathbf{p}'-\mathbf{p})\delta_{s's}, \tag{A.32}$$

with the all other anticommutation relations zore. If $\mid 0\rangle$ denotes the vacuum state[1], then one particle states

$$\mid p,s\rangle = b^\dagger(p,s)\mid 0\rangle$$

are normalised as

$$\langle p',s' \mid p,s\rangle = (2\pi)^3 2E\delta^3(\mathbf{p}'-\mathbf{p})\delta_{s's}. \tag{A.33}$$

The spinors u and v satisfy the following orthogonality relation

$$\bar{u}(p,s)u(p,s') = 2m\delta_{ss'} = -\bar{v}(p,s)v(p,s'). \tag{A.34}$$

In unpolarized cross-section we use

$$\sum_s u(p,s)\otimes\bar{u}(p,s) = (\not{p}+m), \tag{A.35}$$
$$\sum_s v(p,s)\otimes\bar{v}(p,s) = (\not{p}-m), \tag{A.36}$$

while in polarized one we use

$$u(p,s)\otimes\bar{u}(p,s) = \frac{1}{2}(\not{p}+m)(1+\gamma_5\not{s}), \tag{A.37}$$
$$v(p,s)\otimes\bar{v}(p,s) = \frac{1}{2}(\not{p}-m)(1+\gamma_5\not{s}). \tag{A.38}$$

A.2.1 Quantum Chromodynamics:

$$\mathcal{L} = \bar{\psi}\,(i\not{\partial}-m)\,\psi - \frac{1}{4}(\partial_\mu\mathcal{A}^a_\nu - \partial_\nu\mathcal{A}^a_\mu)^2 + g\bar{\psi}\not{\mathcal{A}}\psi$$

---

[1] i.e.,

$$a\mid 0\rangle = 0\ ,\ b\mid 0\rangle = 0$$

$$-gf^{abc}(\partial_\mu \mathcal{A}_\nu^a)\mathcal{A}^{\mu b}\mathcal{A}^{\nu c} - g^2(f^{eab}\mathcal{A}_\mu^a\mathcal{A}_\nu^b)(f^{ecd}\mathcal{A}^{\mu c}\mathcal{A}^{\nu d}). \tag{A.39}$$

Dirac Propagator:

$$\langle 0 \left| T\psi(x)\bar{\psi}(y) \right| 0 \rangle = iS_F(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i(\not{k}+m)}{k^2 - m + i\varepsilon}. \tag{A.40}$$

Gluon Propagator in Feynman Gauge:

$$\langle 0 \left| T\mathcal{A}^\mu(x)\mathcal{A}^\nu(y) \right| 0 \rangle = iD^{\mu\nu}(x-y) = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(x-y)} \frac{-ig^{\mu\nu}}{q^2 + i\varepsilon}. \tag{A.41}$$

Gluon Propagator in Light Cone Gauge:

$$iD^{\mu\nu}(x-y) = \int \frac{d^4q}{(2\pi)^4} \frac{ie^{-iq\cdot(x-y)}}{q^2 + i\varepsilon} \left(-g^{\mu\nu} + \frac{n^\mu q^\nu + n^\nu q^\mu}{n\cdot q} - \frac{q^2}{(n\cdot q)^2} n^\mu n^\nu \right). \tag{A.42}$$

A.2.2 Decomposition of Tensor Products

In particle physics we are mostly concerned with unitary groups. A unitary group $U(N)$ can be defined in terms of unitary transformations

$$\xi^\alpha \to \xi'^\alpha = U^\alpha_\beta \xi^\beta \tag{A.43}$$

where ξ^α is a contravariant vector. Its complex conjugate transform as

$$(\xi^\alpha)^* \to (\xi'^\alpha)^* = (\xi^\beta)^* (U^\dagger)^\alpha_\beta. \tag{A.44}$$

Since this transformation rule is identical to a corresponding covariant vector η_α, we have

$$\eta_\alpha = (\xi^\alpha)^* \tag{A.45}$$

and it is easy to see that the product $\eta\cdot\xi = \eta_\alpha \xi^\alpha$ is invariant under group transformations. Now we want to discuss the behavior of different products of these vectors. In general these products will be reducible. To understand this we consider a simple example of product of two vectors ξ^α and η^β

$$\zeta^{\alpha\beta} = \xi^\alpha \eta^\beta.$$

This can be decomposed as

$$\zeta^{\alpha\beta} = \left[\frac{1}{2}(\xi^\alpha\eta^\beta + \xi^\beta\eta^\alpha) - \frac{1}{N}g^{\alpha\beta}\xi\cdot\eta\right] + \frac{1}{N}g^{\alpha\beta}\xi\cdot\eta + \frac{1}{2}(\xi^\alpha\eta^\beta - \xi^\beta\eta^\alpha) \tag{A.46}$$

into a symmetric and traceless tensor, antisymmetric tensor and an invariant tensor. These tensors transform into itself and constitute the irreducible decomposition of the given tensor. In general for a mixed tensor by doing symmetrization and removal of traces we obtain a tensor which transform into itself and describes the spin of the given tensor in the Lorentz group O(3,1).

Appendix B

Longitudinal and Transverse Polarization

The representation of the Poincare group are labelled by the eigenvalues of the two Casmir operators, P^2 and W^2. P^μ is the energy-momentun operator and W^μ is the Pauli-Lubanski operator, constructed from P^μ and the angular-momentum operator $J^{\mu\nu}$ as

$$W^\mu = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma. \tag{B.1}$$

The eigenvalues of P^2 and W^2 are m^2 and $-m^2 s(s+1)$ respectively, where m is the mass of the particle and s its spin. The states of a Dirac particle ($s = \frac{1}{2}$) are eigenvectors of P^μ and of the polarisation operator $\Pi \equiv -\frac{W \cdot s}{m}$

$$P^\mu \mid p, s\rangle = p^\mu \mid p, s\rangle \tag{B.2}$$

$$-\frac{W \cdot s}{m} \mid p, s\rangle = \pm\frac{1}{2} \mid p, s\rangle \tag{B.3}$$

where s^μ is the spin (or polarisation) vector of the particle, with the properties

$$s^2 = -1, \qquad s \cdot p = 0. \tag{B.4}$$

Using

$$J_{\nu\rho} = (x_\nu P_\rho - x_\rho P_\nu) + \frac{1}{2}\sigma_{\nu\rho} \tag{B.5}$$

for the Dirac particle, the polarisation operator Π can be re-expressed as

$$\Pi = \pm\frac{1}{2}\gamma_5 \not{s}. \tag{B.6}$$

where above (below) sign is for Dirac particle (antiparticle). Note that the polarisation operator in the above form is also well defined for massless particles. Thus the eigenvalue equations for the polarisation operator read ($\alpha = 1,2$)

$$\Pi u_{(\alpha)} = +\frac{1}{2}\gamma_5 \not{s} u_{(\alpha)} = \pm\frac{1}{2}u_{(\alpha)} \tag{B.7}$$

$$\Pi v_{(\alpha)} = -\frac{1}{2}\gamma_5 \not{s} v_{(\alpha)} = \pm\frac{1}{2}v_{(\alpha)}. \tag{B.8}$$

B.1 Longitudinal Polarization

In particle rest frame $s^\mu = (0, \mathbf{n})$ where $\mathbf{n}$ is a unit vector identifying the spin-quantization direction. For a longitudinally polarised particle ($\mathbf{n} = \mathbf{p}/|\mathbf{p}|$), we can boost the spin vector as

$$s^\mu = \left(\frac{|\mathbf{p}|}{m}, \frac{p^0}{m}\frac{\mathbf{p}}{|\mathbf{p}|}\right) \tag{B.9}$$

and polarisation vector becomes

$$\Pi = \left(\frac{1}{2}\Sigma\right) \cdot \frac{\mathbf{p}}{|\mathbf{p}|}. \tag{B.10}$$

Denoting $u(p,\pm)$, $v(p,\pm)$ by the helicity eigenstates, we have

$$\begin{aligned} \Sigma \cdot \frac{\mathbf{p}}{|\mathbf{p}|} u(p,\pm) &= \pm u(p,\pm) \\ \Sigma \cdot \frac{\mathbf{p}}{|\mathbf{p}|} v(p,\pm) &= \mp v(p,\pm). \end{aligned} \tag{B.11}$$

In case of massless particles one has

$$\Pi = \frac{1}{2}\gamma_5 \tag{B.12}$$

and Eq.(B.11) becomes

$$\begin{aligned} \gamma_5 u(p,\pm) &= \pm u(p,\pm) \\ \gamma_5 v(p,\pm) &= \mp v(p,\pm). \end{aligned} \tag{B.13}$$

Thus helicity coincides with chirality for particle states, while it is opposite to chirality for the antiparticle states. The helicity projector for the massless particles are then

$$\begin{aligned} \mathcal{P} &= \frac{1}{2}(1 \pm \gamma_5) \qquad \text{for particle states} \\ &= \frac{1}{2}(1 \mp \gamma_5) \qquad \text{for antiparticle states.} \end{aligned} \tag{B.14}$$

B.2 Transverse Polarization

For transversely polarised particles $\mathbf{n} \cdot \mathbf{p} = 0$. Assuming that the particle moves along the z direction, the spin vector can be written as

$$s^\mu = s^\mu_\perp = (0, \mathbf{n}_\perp, 0) \tag{B.15}$$

where $\mathbf{n}_\perp$ is a transverse two-vector. The polarisation vector takes the form

$$\Pi = \pm\frac{1}{2}\gamma_5 \not{s}_\perp = \mp\frac{1}{2}\gamma_5 \gamma_\perp \cdot \mathbf{n}_\perp \tag{B.16}$$

and its eigenvalue equations are

$$\frac{1}{2}\gamma_5 \not{s}_\perp u(p, \uparrow\downarrow) = \pm u(p, \uparrow\downarrow)$$

$$\frac{1}{2}\gamma_5 \not{s}_\perp u(p,\uparrow\downarrow) = \mp u(p,\uparrow\downarrow).$$

The transverse polarisation projectors along the directions x and y are

$$\begin{aligned} \mathcal{P}^{(x)}_{\uparrow\downarrow} &= \frac{1}{2}(1 \pm \gamma^1\gamma_5) \\ \mathcal{P}^{(y)}_{\uparrow\downarrow} &= \frac{1}{2}(1 \pm \gamma^2\gamma_5) \end{aligned} \tag{B.17}$$

for particle states, and

$$\begin{aligned} \mathcal{P}^{(x)}_{\uparrow\downarrow} &= \frac{1}{2}(1 \mp \gamma^1\gamma_5) \\ \mathcal{P}^{(y)}_{\uparrow\downarrow} &= \frac{1}{2}(1 \mp \gamma^2\gamma_5) \end{aligned} \tag{B.18}$$

for antiparticle states. In the high-energy limit the polarisation vector can be written as

$$s^\mu = \lambda \frac{p^\mu}{m} + s^\mu_\perp, \tag{B.19}$$

where λ is (twice) the helicity of the particle.

Appendix C

Discrete Symmertries

Three type of discrete symmetries will be discussed here, which are of fundamental importance: parity $\mathcal{P}$, charge conjugation $\mathcal{C}$, and time reversal $\mathcal{T}$. All noninteracting field theories are invariant under these transformation but this will change if interactions are present that break the symmery. Transformation properties of spinor field operators under $\mathcal{P}$, $\mathcal{C}$ and $\mathcal{T}$ will be considered here.

C.1 Parity

The transformation law for spinor field operators is

$$\mathcal{P}\psi(x)\mathcal{P}^{-1} = \gamma_0\psi(\tilde{x}) \tag{C.1}$$

and

$$\mathcal{P}\bar{\psi}(x)\mathcal{P}^{-1} = \bar{\psi}(\tilde{x})\gamma_0 \tag{C.2}$$

with $\tilde{x}^\mu = (t, -\mathbf{x})$. The different forward matrix elements ,which we are encountered in chapter 4 will be transformed as

$$\mathcal{S}(k, P, S) \;=\; \frac{1}{2}Tr\,(\Phi) = \frac{1}{2}\int d^4\xi e^{-ik\cdot\xi}\left\langle P, S\left|\bar{\psi}(\xi)\psi(0)\right| P, S\right\rangle$$

$$
\begin{aligned}
&= \frac{1}{2}\int d^4\xi e^{-ik\cdot\xi} \left\langle P,S\left|\mathcal{P}^{-1}\left(\mathcal{P}\bar{\psi}(\xi)\mathcal{P}^{-1}\right)\left(\mathcal{P}\psi(0)\mathcal{P}^{-1}\right)\mathcal{P}\right| P,S\right\rangle \\
&= \frac{1}{2}\int d^4\xi e^{-ik\cdot\xi} \left\langle \tilde{P},-\tilde{S}\left|\left(\bar{\psi}(\tilde{\xi})\gamma_0\right)\left(\gamma_0\psi(0)\right)\right| \tilde{P},-\tilde{S}\right\rangle \\
&= \frac{1}{2}\int d^4\xi e^{-i\tilde{k}\cdot\xi} \left\langle \tilde{P},-\tilde{S}\left|\bar{\psi}(\xi)\psi(0)\right| \tilde{P},-\tilde{S}\right\rangle = \mathcal{S}(\tilde{k},\tilde{P},-\tilde{S}). \qquad \text{(C.3)}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathcal{V}^{\mu}(k,P,S) &= \frac{1}{2}Tr\left(\gamma^{\mu}\Phi\right) = \mathcal{V}_{\mu}(\tilde{k},\tilde{P},-\tilde{S}), & \text{(C.4)}\\
\mathcal{A}^{\mu}(k,P,S) &= \frac{1}{2}Tr\left(\gamma^{\mu}\gamma_5\Phi\right) = -\mathcal{A}_{\mu}(\tilde{k},\tilde{P},-\tilde{S}), & \text{(C.5)}\\
\mathcal{P}_5(k,P,S) &= \frac{1}{2i}Tr\left(\gamma_5\Phi\right) = -\mathcal{P}_5(\tilde{k},\tilde{P},-\tilde{S}), & \text{(C.6)}\\
\mathcal{T}^{\mu\nu}(k,P,S) &= \frac{1}{2i}Tr\left(\sigma^{\mu\nu}\gamma_5\Phi\right) = -\mathcal{T}_{\mu\nu}(\tilde{k},\tilde{P},-\tilde{S}). & \text{(C.7)}
\end{aligned}
$$

C.2 Charge Conjugation

Charge conjugation exchanges the roles of particle and antiparticle spinors. Therefore field operators are transformed as

$$
\mathcal{C}\psi(x)\mathcal{C}^{-1} = C\bar{\psi}^{T}(x) \qquad \text{(C.8)}
$$

and

$$
\mathcal{C}\bar{\psi}(x)\mathcal{C}^{-1} = -\psi^{T}(x)C^{\dagger}. \qquad \text{(C.9)}
$$

Here C is the charge cojugation matrix and has the following properties:

$$
C\gamma^{\mu}C^{-1} = -\gamma^{\mu T} \qquad \text{(C.10)}
$$

and

$$
C^{-1} = C^{\dagger} = C^{T} = -C. \qquad \text{(C.11)}
$$

Within the standard Dirac matrices it is given by

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}. \tag{C.12}$$

C.3 Time Reversal

It is better to understand antiunitary operators to discuss the time reversal symmetry.

Let $\mathcal{T}$ be an antilinear operator in Hilbert space which satisfies

$$\mathcal{T}\left(c_1 \mid \Phi_1\rangle + c_2 \mid \Phi_2\rangle\right) = c_1^* \mathcal{T} \mid \Phi_1\rangle + c_2^* \mathcal{T} \mid \Phi_2\rangle. \tag{C.13}$$

The hermitean cojugate operator $\mathcal{O}$ is definerd through

$$\left\langle \Phi_1 \left| \mathcal{T}^\dagger \right| \Phi_2 \right\rangle = \left\langle \Phi_2 \left| \mathcal{T} \right| \Phi_1 \right\rangle \tag{C.14}$$

and is antilinear as well. The operator is antiunitary if it satisfies

$$\mathcal{T}\mathcal{T}^\dagger = \mathcal{T}^\dagger \mathcal{T} = 1. \tag{C.15}$$

An antiunitary transformation in Hilbert space,

$$\mid \Phi_1^{'}\rangle = \mathcal{T} \mid \Phi_1\rangle \quad , \quad \mid \Phi_2^{'}\rangle = \mathcal{T} \mid \Phi_2\rangle, \tag{C.16}$$

preserves the norm of a state in the same way as a unitary transformation does. However, when applied to scalar products it interchanges the "bra" and "ket" vectors since

$$\begin{aligned} \left\langle \Phi_2^{'} \mid \Phi_1^{'} \right\rangle &= \left\langle \Phi_2^{'} \left| \mathcal{T} \right| \Phi_1 \right\rangle = \left\langle \Phi_1 \left| \mathcal{T}^\dagger \right| \Phi_2^{'} \right\rangle \\ &= \left\langle \Phi_1 \left| \mathcal{T}^\dagger \mathcal{T} \right| \Phi_2 \right\rangle = \left\langle \Phi_1 \mid \Phi_2 \right\rangle. \end{aligned} \tag{C.17}$$

The matrix elements of a linear operator $\mathcal{O}$ can be rewritten as

$$\begin{aligned}\langle \Phi_2 | \mathcal{O} | \Phi_1 \rangle &= \langle \Phi_2 | \mathcal{O} \mathcal{T}^\dagger \mathcal{T} | \Phi_1 \rangle = \left\langle \mathcal{O}^\dagger \Phi_2 \left| \mathcal{T}^\dagger \right| \Phi_1' \right\rangle \\ &= \left\langle \Phi_1' \left| \mathcal{T} \right| \mathcal{O}^\dagger \Phi_2 \right\rangle = \left\langle \Phi_1' \left| \mathcal{T} \mathcal{O}^\dagger \right| \Phi_2 \right\rangle \\ &= \left\langle \Phi_1' \left| \mathcal{T} \mathcal{O}^\dagger \mathcal{T}^\dagger \mathcal{T} \right| \Phi_2 \right\rangle = \left\langle \Phi_1' \left| \mathcal{T} \mathcal{O}^\dagger \mathcal{T}^\dagger \right| \Phi_2' \right\rangle . \end{aligned} \tag{C.18}$$

Now the transformation law of spinor field operator under the time reversal is

$$\mathcal{T} \psi(x) \mathcal{T}^{-1} = T_0 \psi(-\tilde{x}) \tag{C.19}$$

and

$$\mathcal{T} \bar{\psi}(x) \mathcal{T}^{-1} = \bar{\psi}(-\tilde{x}) T_0^\dagger \tag{C.20}$$

where the time reversal matrix T_0 is related to C as

$$T_0 = -i\gamma^5 C = i\gamma^1\gamma^3. \tag{C.21}$$

The forward matrix elements of chapter 4 will be transformed as

$$\begin{aligned}\mathcal{S}(k,P,S) &= \frac{1}{2} Tr\,(\Phi) = \frac{1}{2}\int d^4\xi e^{-ik\cdot\xi} \langle P,S | \bar{\psi}(\xi)\psi(0) | P,S \rangle \\ &= \frac{1}{2}\int d^4\xi e^{-ik\cdot\xi} \langle P,S | \mathcal{T}^{-1} \left(\mathcal{T}\bar{\psi}(\xi)\mathcal{T}^{-1}\right) \left(\mathcal{T}\psi(0)\mathcal{T}^{-1}\right) \mathcal{T} | P,S \rangle \\ &= \frac{1}{2}\int d^4\xi e^{-ik\cdot\xi} \left\langle P,S \left| \mathcal{T}^{-1} \left(\bar{\psi}(-\tilde{\xi}) T_0^\dagger\right) \left(T_0\psi(0)\right) \mathcal{T} \right| P,S \right\rangle . \end{aligned}$$

Now using the identity (C.18), we get

$$\begin{aligned}\mathcal{S}(k,P,S) &= \frac{1}{2}\int d^4\xi e^{-ik\cdot\xi} \left\langle \tilde{P},\tilde{S} \left| \left[\bar{\psi}(-\tilde{\xi})\psi(0)\right]^\dagger \right| \tilde{P},\tilde{S} \right\rangle \\ &= \frac{1}{2}\int d^4\xi e^{-ik\cdot\xi} \left\langle \tilde{P},\tilde{S} \left| \bar{\psi}(0)\psi(-\tilde{\xi}) \right| \tilde{P},\tilde{S} \right\rangle . \end{aligned}$$