Best Proximity Points of Contraction Type Operators in Metric and Metric-like Spaces

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Best Proximity Points of Contraction Type Operators in Metric and Metric-like Spaces

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Dedicated to

My Father

&

My Husband

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Abstract

The origin of approximation theory is dated to the middle of twentieth century [49] and rests in the consideration of researchers to setup existence of approximate solutions for the operator equations of the type $m = Om$. It is significant that the best proximity point theory took off based on work of Eldred and Veeramani [47]. The best proximity points are generally employed to discover approximate solution of the operator equation $Om = m$, which is optimal, when some contraction O fails to have fixed point.

The motivation behind the dissertation is to explore best proximity points of various proximal contraction operators in metric and metric-like spaces. We prove best proximity point theorems for some new generalized multivalued proximal contractions. We study nonself Presic-type operator and the presence of optimal approximate solution for them. Also we give several examples to explain our results. We get some fascinating fixed point outcomes for Presic operator as consequence of our results. We demonstrate best proximity point results in few generalizations of metric space for example; modular metric space and gauge space, for operators satisfying new type of contraction inequality. We introduce fuzzy multiplicative metric space and prove best proximity points for Feng-Liu type multivalued proximal contraction.

Preface

Analysis is the field of Mathematics in which we analyze classes of functions and equations having general properties. The field of analysis has been developed into various distinct but related fields such as Fourier analysis, complex analysis, real analysis, numerical analysis and functional analysis etc. Fixed point theory is perhaps the most dynamic territories of functional analysis. The theory originated in response to needs of non-linear analysis with the method of successive approximations that was used to establish existence and uniqueness of solutions of differential equations. Many authors including Charles Emile in 1890 and Joseph Liouville in 1837 contributed for the same.

Metric fixed point theory started with an interesting and valuable result, known as Banach contraction principle, given by Stephen Banach in 1922 [19]. The principle of Banach contraction is important because it not only gives the presence and uniqueness of solution but also provides the sequence of successive approximations that converges to the solution of the problem that can be modeled in the form $x = f(x)$. Metric fixed point theory is now expanding its domain due to its diversified applications in Mathematics like existence of solutions of differential equations, integral equations [97, 117] just as in various fields like mathematical economics [95], game theory [28], computer science [68], engineering, physics, telecommunication [9], and many others [70, 116, 127]. A critical observation of literature depicts that the principle of Banach contraction has been extended in following manner;

(i) Generalizing the operator or contraction conditions of the operator(See for example [32, 39, 66, 78, 115]).

(ii) Generalizing the metric space(See for example [41, 69, 83, 75]).

(iii) Development of metric-like spaces(See for example [24, 92, 87])

Best proximity point theory is generalization of fixed point theory as best proximity point theorem bring down to fixed point theorem if the nonself map reduces to self map. The theory evolved with the work of K. Fan in 1969 [49]. Some extensions of the theorem are then given by Prolla [112],Seghal and Singh [122] and Vetrivel [132]. In first chapter, we have given brief introduction of best proximity point theory, some generalizations of metric space, some metric-like spaces and of contraction operator that we have to use in upcoming chapters.

The theory of best proximity points now become field of attention for the researchers working in the field of analysis. In 2012, S. Basha [23] introduced the proximal contractions of nonself maps and established best proximity point theorems. Several authors then used the concept and shows the presence of best proximity points of different generalizations of proximal contractions [12, 74, 101, 107]. A useful generalization of Banach contraction Principle is fixed point theorem for multivalued contraction which is given by S. B. Nadler [106]. In first section of chapter two, we used a class of auxiliary functions, written as $\mathcal{F}$, introduced by Wardowski [137]. We proved best proximity point theorems for a new class of multivalued generalized proximal contractions. With the help of example, we also have shown that our results generalize some existing results.In second section, we have proved best proximity point theorems for Presic type proximal contractions in the framework of metric space endowed with graph. As consequences of our results, we also have shown presence of fixed points of Presic type operators in the metric space furnished with graph.

Since we mentioned earlier that generalization of metric space is a manner to extend principle of Banach contraction. Modular metric space is the generalization of metric space as modular metric reduces to metric if λ is taken as 1. Fixed point theorems in the framework of modular metric space with applications are provided by several authors [2, 30, 38, 43]. Our chapter three contains best proximity point theorems of proximal contractions of first kind and second kind defined using the class of functions $\mathcal{F}$, in the framework of modular metric space. Pseudo-metric is a metric in which the distance between two distinct elements can be zero. A remarkable observation is that every metric space is a pseudo metric space. Although pseudo-metrics are rare than metrics but pseudo-metrics have their own importance as they emerge in a characteristic manner in the theory of hyperbolic complex manifolds and in functional analysis [88]. Gauge spaces are thoroughly discussed in [46] that are generated by the collection of balls of family of pseudo-metrics. In 2000, Frigon [55] proved fixed point theorems in the framework of complete gauge space. Jleli et al. [76] in 2015 proved some fixed point theorems and showed their applications in gauge spaces. In second part of chapter three, we have shown existence of best proximity points in gauge spaces of proximal contractions defined by using a class of auxiliary functions.

In metric spaces, if a space is exceptional instance of another space then the latter is termed generalization of first one. For instance metric space is exceptional instance of modular metric space. There are some spaces that are not generalizations of metric space but are analogous to metric space, for instance, fuzzy metric space, multiplicative metric space etc.Fuzzy metric space was presented by Kramosil and Michalek [92]. Fixed point theorems and best proximity point theorems of single valued and multivalued contractions in fuzzy metric space were presented by many authors [60, 119, 33, 123, 133, 134, 42, 50, 52, 57, 58, 61, 62, 86, 100]. In chapter four we introduced fuzzy multiplicative metric space and some related terminologies. We proved best proximity point theorems for some single valued and multivalued proximal contractions in the newly introduced space.

Contents

Chapter 1

Preliminaries and basic concepts

This chapter introduces basic concepts regarding some generalizations of metric space and contraction operators. The definitions and some essential outcomes from literature are incorporated that will be helpful in the entire dissertation.

1.1 Best Proximity Point Theory

The theory of fixed points is concerned with determining adequate conditions for the presence and uniqueness of the functions that satisfy non-linear equations denoted by $Om = m$, when O is a function mapping a subset of metric space or some relevant framework to itself. The equations may not have solutions for some nonlinear operator O. This one, for example, has no solution if $O : G \to H$ and $G \cap H = \phi$. In this case, we may identify a point $m \in G$ which is close to Om, that is, the distance between Om and m is shortest among the G elements. Such a point $m \in G$ is termed best proximity point of O. A point $m \in G$, where G and H are subsets of a metric space (M, d), is termed *best proximity point* of $O : G \to H$ if $d(m, Om) = d(G, H)$. Fan [49] was one who first proposed the concept of best proximity point. The definitions and findings for the study of best proximity point theory are listed below;

Definition 1.1.1. [22] Allow (M, d) to be a metric space. For $G, H \subseteq M$;

$$d(G, H) = \inf\{d(g, h) : g \in G, h \in H\}$$

$$G_0 = \{g \in G : d(G, H) = d(g, h) \text{ for some } h \in H\}$$
$$H_0 = \{h \in H : d(G, H) = d(g, h) \text{ for some } g \in G\}$$

Definition 1.1.2. [22] The set H is termed approximatively compact concerning the set G, if each $\{v_a\}$ in H with $d(m, v_a) \to d(m, H)$ for some $m \in G$ has a convergent subsequence.

The theory of best proximity points for different versions of contractions has been studied [5, 10, 20, 21, 47]. The accompanying best proximity point theorem demonstrated by Basha and Shahzad [23] for generalized proximal contraction:

Theorem 1.1.1. *Allow G and H to be closed non-empty subsets of a complete metric space (M, d). Let G_0 is not empty and $O : G \to H$ is a mapping such that for each $m_1, m_2, u_1, u_2 \in G$ with $d(u_1, Om_1) = d(G, H) = d(u_2, Om_2)$, we have*

$$d(u_1, u_2) \leq \varsigma_1 d(m_1, m_2) + \varsigma_2 d(m_1, u_1) + \varsigma_3 d(m_2, u_2) + \varsigma_4 [d(m_1, u_2) + d(m_2, u_1)] \quad (1.1.1)$$

where $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4 \geq 0$ satisfying $\varsigma_1 + \varsigma_2 + \varsigma_3 + 2\varsigma_4 < 1$. Further, consider the accompanying conditions:

(i) H is approximatively compact with respect to G,

(ii) $O(G_0)$ is contained in H_0.

Then O possesses best proximity point.

1.2 Gauge Spaces

Pseudo metric is the generalized concept of metric satisfies the property that two distinct points may not be separated. The topological space generated by the family of pseudo metrics is called gauge space. Following are some basic definitions in the perspective of gauge space;

Definition 1.2.1. [34] Consider a nonempty set M, a function $d : M \times M \to [0, \infty)$ such that for each $m, n, p \in M$;

(i) $d(m, m) = 0$ for each $m \in M$,

(ii) $d(m,n) = d(n,m)$,

(iii) $d(m,p) \leq d(m,n) + d(n,p)$.

The function d then termed as pseudometric in M.

Definition 1.2.2. [34] Consider (M,d) be a pseudo metric. The set

$$B(m,d,\epsilon) = \{n \in M : d(m,n) < \epsilon\}.$$

is termed d-ball having radius $\epsilon > 0$ and center $m \in M$.

Definition 1.2.3. [34] Consider a family $\mathfrak{P} = \{d_b | b \in \mathfrak{V}\}$ of pseudo metrics. If for each pair (m,n) with $m \neq n$, there exists $d_b \in \mathfrak{P}$ with $d_b(m,n) \neq 0$, then the family $\mathfrak{P}$ is termed separating.

Definition 1.2.4. [34] Let $\mathfrak{P} = \{d_b | b \in \mathfrak{V}\}$ be a family of pseudo metrics on a nonempty set M. Let the family of balls

$$\mathfrak{B}(\mathfrak{P}) = \{B(m,d_b,\epsilon) : m \in M, d_b \in \mathfrak{P} \text{ and } \epsilon > 0\}$$

being subbases generates the topology $\mathfrak{T}(\mathfrak{P})$, termed topology induced by $\mathfrak{P}$. The set M with topology $\mathfrak{T}(\mathfrak{P})$ is termed gauge space. It is noticed that the gauge space $(M, \mathfrak{T}(\mathfrak{P}))$ is Hausdorff if the family $\mathfrak{P}$ is separating.

Definition 1.2.5. [34] Consider a family $\mathfrak{P} = \{d_b | b \in \mathfrak{V}\}$ of pseudometrics on a set $M \neq \phi$ which induces a gauge space $(M, \mathfrak{T}(\mathfrak{P}))$. Let $\{m_a\}$ be a sequence in M and $m \in M$;

(i) if for each $\epsilon > 0$ and $b \in \mathfrak{V}$, there is some $a_1 \in \mathbb{N}$ such that $d_b(m_a, m) < \epsilon$ for each $a \geq a_1$ then the sequence $\{m_a\}$ is termed convergent to m . Symbolically we write it as $m_a \to^{\mathfrak{F}} m$,

(ii) if for each $\epsilon > 0$ and $b \in \mathfrak{V}$, there is some $a_1 \in \mathbb{N}$ such that $d_b(m_a, m_{\acute{a}}) < \epsilon$ for each $a, \acute{a} \geq a_1$ then the sequence $\{m_a\}$ is termed Cauchy sequence,

(iii) $(M, \mathfrak{T}(\mathfrak{P}))$ is complete if a sequence in $(M, \mathfrak{T}(\mathfrak{P}))$ is Cauchy then it must converge in M,

(iv) A subset C of M is termed closed if each sequence of elements of C converges in C.

1.3 Modular Metric Spaces

Chistyakov[37] proposed and built up the theory of modular metric space in 2008. Modular metric space is the generalization of metric space. Roughly we can say that the quantity $\omega(\mu, m, n)$ is the absolute value of velocity between m and n in time $\mu > 0$.

Here we define some terminologies in modular metric space that will be useful in third chapter;

Definition 1.3.1. [37, 2] A function $\omega : (0, \infty) \times M \times M \to [0, \infty]$ termed modular metric on a non-empty set M, in the event that it fulfills the accompanying conditions, for all $m, n, p \in M$:

(i) $m = n$ if and only if $\omega(\mu, m, n) = 0$ $\forall \mu > 0$;

(ii) $\omega(\mu, m, n) = \omega(\mu, n, m), \forall \mu > 0$;

(iii) $\omega(\mu + \nu, m, n) \leq \omega(\mu, m, p) + \omega(\nu, p, n)$ for all $\mu, \nu > 0$.

If the following condition, as replacement of (i), is satisfied:

$$(\acute{i}) : \omega(\mu, m, m) = 0, \forall \mu > 0, m \in M$$

then ω is termed pseudomodular metric. A regular modular metric is weaker form of modular metric where condition (i) is weakened as follows;

$$m = n \text{ if and only if } \omega(\mu, m, n) = 0 \text{ for some } \mu > 0.$$

Example 1.3.1. [37, 2] Let $M = \mathbb{R}$ and ω is defined by $\omega(\mu, m, n) = \infty$ if $\mu < 1$, and $\omega(\mu, m, n) = \frac{|m-n|}{\mu}$ if $\mu \geq 1$, it is simple to verify that ω is regular however not modular metric on M.

Definition 1.3.2. [2] Consider a pseudomodular ω on a non-empty set M. For fixed $m_0 \in M$, the set

$$M_\omega = \{m \in M : \omega(\lambda, m, m_0) \to 0 \text{ as } \lambda \to \infty\}$$

is termed modular space.

Definition 1.3.3. [2] Consider a modular metric space (M, ω).

(i) The sequence $\{m_a\}$ in M_ω is termed ω-convergent to $m \in M_\omega$ if and only if $\omega(1, m_a, m) \to 0$, as $a \to \infty$.

(ii) The sequence $\{m_a\}$ in M_ω is termed ω-Cauchy if $\omega(1, m_a, m_b) \to 0$, as $a, b \to \infty$.

(iii) If any ω-Cauchy sequence in a subset W of M_ω is ω-convergent in W then W is termed ω-complete.

(iv) If each sequence of elements of W is ω-convergent in W then the subset W of M_ω is termed ω-closed .

(v) A subset W of M_ω is termed ω-bounded if we have

$$\delta_\omega(W) = sup\,\{\omega(1, m, n) : m, n \in W\} < \infty.$$

(vi) A subset W of M_ω is termed ω-compact if there is some subsequence $\{m_{a_k}\}$ and $m \in W$ for any sequence $\{m_a\}$ in W with $\omega(1, m_{a_k}, m) \to 0$ as $k \to \infty$.

Definition 1.3.4. [110] Let (M, ω) be a modular metric space and $\{m_a\}$ be a sequence in M_ω. We state that ω fulfills the Δ_M-condition if $\lim_{a,b\to\infty} \omega(a-b, m_b, m_a) = 0$ for $(a, b \in \mathbb{N}, a > b)$ implies $\lim_{a,b\to\infty} \omega(\mu, m_b, m_a) = 0$ for all $\mu > 0$.

Definition 1.3.5. [72] If for any two sequences $\{m_a\}$ and $\{n_a\}$ ω-convergent to m and n in a modular metric space (M, ω), the accompanying condition hold true

$$\omega(1, m, n) \leq \liminf_{a\to\infty} \omega(1, m_a, n_a)$$

then ω is termed to possess Fatou property .

1.4 Multiplicative Metric Space

In [24] Bashirov et al. brought up the attention of researchers to multiplicative calculus which was remained unimportant from 1972, when first book on multiplicative calculus was published by Grossman and Katz [65]. Bashirov in [25] encouraged the researchers to investigate the materiality of multiplicative calculus in variuos fields by presenting some valuable applications in finance, economics and social sciences. Applications in different fields are being studied by [53, 54, 104, 98].

By using the concept of multiplicative distance, Ozavsar and Cevikal [108] developed the theory of multiplicative metric space. Here we give some basic terminologies.

Definition 1.4.1. [108] A non-empty set M with a mapping $d : M \times M \to \mathbb{R}$ is termed multiplicative metric space if d fulfills the accompanying axioms;

(i) $d(m, n) \geq 1$ for all $m, n \in M$ and $d(m, n) = 1$ if and only if $m = n$,

(ii) $d(m, n) = d(n, m)$ for all $m, n \in M$,

(iii) $d(m, p) \leq d(m, n).d(n, p)$ for all $m, n, p \in M$

Example 1.4.1. Let $d^* : (\mathbb{R}^+)^a \times (\mathbb{R}^+)^a \to \mathbb{R}$ be described as

$$d^*(m, n) = |\tfrac{m_1}{n_1}|^*.|\tfrac{m_2}{n_2}|^*...|\tfrac{m_a}{n_a}|^*$$

where $m = (m_1, m_2, m_3, ..., m_a)$ and $n = (n_1, n_2, n_3, ..., n_a) \in (\mathbb{R}^+)^a$ and $|.|^* : \mathbb{R}^+ \to \mathbb{R}^+$ is characterized as follows;

$$|r|^* = \begin{cases} r \text{ if } r \geq 1 \\ \frac{1}{r} \text{ if } r < 1 \end{cases}$$

Then d^* satisfies all axioms of multiplicative metric and hence $(\mathbb{R}^+)^a$ with d^* is a multiplicative metric space.

Definition 1.4.2. [108] For a multiplicative metric space (M, d), $m \in M$ and $\epsilon > 1$, the following set

$$B_\epsilon = \{n \in M : d(m, n) < \epsilon\}$$

is termed multiplicative open ball, where radius is ϵ and center is m.
The following set

$$\overline{B}_\epsilon = \{n \in M : d(m, n) \leq \epsilon\}$$

is termed multiplicative closed ball.

Definition 1.4.3. [108] Consider a sequence $\{m_a\}$ in a multiplicative metric space (M, d) and $m \in M$. The sequence $\{m_a\}$ is termed multiplicative convergent to m, if there is some $a_1 \in \mathbb{N}$ with $m_a \in B_\epsilon(m)$ for all $a \geq a_1$ and for any multiplicative open ball $B_\epsilon(m)$. Symbolically it is written as $m_a \to_* m$ as $a \to \infty$.

Lemma 1.4.1. *[108] Consider a multiplicative metric space (M, d), a sequence $\{m_a\}$ in M and $m \in M$. Then $m_a \to_* m$ as $a \to \infty$ if and only if $d(m_a, m) \to_* 1$ as $a \to \infty$.*

Lemma 1.4.2. *[108] The multiplicative limit point of a multiplicative convergent sequence $\{m_a\}$ in multiplicative metric space (M, d) is unique.*

Definition 1.4.4. [108] A sequence $\{m_a\}$ in a multiplicative metric space (M, d) is termed multiplicative Cauchy sequence if, for all $\epsilon > 1$, there exist $a_1 \in \mathbb{N}$ such that $d(m_a, m_b) < \epsilon$ for all $a, b \geq a_1$.

Lemma 1.4.3. *[108] Consider a sequence $\{m_a\}$ in a multiplicative metric space (M, d). The sequence $\{m_a\}$ is multiplicative Cauchy if and only if $d(m_a, m_b) \to_* 1$ as $a, b \to \infty$.*

Theorem 1.4.4. *[108] Consider two sequences $\{m_a\}$ and $\{n_b\}$ in a multiplicative metric space (M, d) such that $m_a \to_* m$ and $n_b \to_* n$ as $a, b \to \infty$. Then $(d(m_a, n_b)) \to_* d(m, n)$ as $a, b \to \infty$.*

Definition 1.4.5. [108] Consider a multiplicative metric space (M, d) and $G \subseteq M$. Then a point $m \in G$ is termed a multiplicative interior point of G if there exist an $\epsilon > 1$ such that $B_\epsilon(m) \subseteq G$. The collection of all multiplicative interior points of G, symbolically written $int(G)$, is termed multiplicative interior of G.

Definition 1.4.6. [108] Consider a multiplicative metric space (M, d) and $G \subseteq M$. If $G = int(G)$ i.e. every point of G is multiplicative interior point of G, then G is termed multiplicative open set.

Definition 1.4.7. [108] Consider a multiplicative metric space (M, d). If all limit points of a subset $G \subseteq M$ are contained in it then G is termed multiplicative closed in (M, d).

Theorem 1.4.5. *[108] Consider a subset G in a multiplicative metric space (M, d). Then the complement of G in M is multiplicative open if and only if G is multiplicative closed.*

Theorem 1.4.6. *[108] Consider a subset G in a multiplicative metric space (M, d). Then (G, d) is complete if and only if G is multiplicative closed.*

Definition 1.4.8. [103] Consider a subset G of a multiplicative metric space (M, d). If any sequence $\{m_a\}$ in G with $d(n, m_a) \to_* d(n, G)$ as $a \to \infty$ for some $n \in H$ possesses a subsequence which is convergent in G then G termed multiplicative approximatively compact concerning H.

1.5 Fuzzy Metric Space

Definition 1.5.1. [121] A continuous t-norm is a binary operation $\star : [0,1]^2 \to [0,1]$ satisfying;

(i) $\star$ is continuous,

(ii) $\star$ is associative and commutative,

(iii) If $w \le y$ and $x \le z$ then $w \star x \le y \star z$ for each $w, x, y, z \in [0, 1]$,

(iv) $x \star 1 = x$ for all $x \in [0, 1]$

Some typical examples of a continuous t-norm are $x \star_1 y = min\{x, y\}, x \star_2 y = \frac{xy}{max\{x,y,\lambda\}}$ for $0 < \lambda < 1, x \star_3 y = max\{x + y - 1, 0\}, x \star_4 y = xy$.

Definition 1.5.2. [58] Consider M an arbitrary set. A 3-tuple $(M, F_M, \star)$ is termed fuzzy metric space if $\star$ is continuous $t-$norm and F_M is a fuzzy set on $M \times M \times (0, \infty)$ fulfilling; for all $m, n, p \in M$ and $t, s > 0$,

FM1: $F_M(m, n, t) > 0$,

FM2: $F_M(m, n, t) = 1$ if and only if $m = n$,

FM3: $F_M(m,n,t) = F_M(n,m,t)$,

FM4: $F_M(m,p,t+s) > F_M(m,n,t) \star F_M(n,p,s)$,

FM5: $F_M(m,n,.) : (0,\infty) \to [0,1]$ is continuous.

The idea of fuzzy set provides the interpretation of $F_M(m,n,t)$ as amount of closeness of m and n regarding t, since we stated that F_M is a fuzzy set on $M^2 \times [0,\infty)$.
It is notable that $F_M(m,n,.)$ is a nondecreasing mapping on $(0,\infty)$ for each $m,n \in M$.

Definition 1.5.3. [58] Consider a fuzzy metric space $(M, F_M, \star)$. For $t > 0$, the set

$$B(m,\epsilon,t) = \{n \in M : F_M(m,n,t) > 1-\epsilon\}$$

is termed an open ball $B(m,\epsilon,t)$ with center $m \in M$ and radius $\epsilon, 0 < \epsilon < 1$.

Definition 1.5.4. [58] Consider a fuzzy metric space $(M, F_M, \star)$. We state that a sequence $\{m_a\}$ in M is convergent which converges to m if and only if there exist $a_1 \in \mathbb{N}$ with $F_M(m_a, m, t) > 1-\epsilon$ for all $a \geq a_1$ and for each $\epsilon > 0, t > 0$.

Theorem 1.5.1. *[58] Consider a sequence $\{m_a\}$ in a fuzzy metric space $(M, F_M, \star)$, $m \in M$. Then $\{m_a\}$ is convergent to m if and only if $F_M(m_a, m, t) \to 1$ as $a \to \infty$ for each $t > 0$.*

Definition 1.5.5. [58] A sequence $\{m_a\}$ in a fuzzy metric space $(M, F_M, \star)$ is termed Cauchy sequence if and only if there exist $a_1 \in \mathbb{N}$ such that $F_M(m_a, m_b, t) > 1-\epsilon$ for all $a, b \geq a_1$ and for each $\epsilon > 0, t > 0$.

Theorem 1.5.2. *[58] Allow $(M, F_M, \star)$ to be a fuzzy metric space and $\{m_a\}$ a sequence in M. Then $\{m_a\}$ is Cauchy if and only if $F_M(m_a, m_b, t) \to 1$ as $a, b \to \infty$ for each $t > 0$.*

Definition 1.5.6. [58] A subset G of a fuzzy metric space $(M, F_M, \star)$ is termed closed if $m \in G$, for each convergent sequence $\{m_a\}$ in G with $m_a \to m$.

Lemma 1.5.3. *[86] Allow $(M, F_M, \star)$ to be a fuzzy metric space such that for $m, n \in M, t > 0$ and $h > 1$*

$$\lim_{a\to\infty} \star_{i=a}^{\infty} F_M(m, n, th^i) = 1.$$

Suppose $\{m_a\}$ is a sequence in M such that for all $a \in \mathbb{N}$

$$F_M(m_a, m_{a+1}, \alpha t) \geq F_M(m_{a-1}, m_a, t)$$

where $0 < \alpha < 1$. Then $\{m_a\}$ is a Cauchy sequence.

Definition 1.5.7. [133] Let $G, H \neq \phi$ be two subsets of a fuzzy metric space $(M, F_M, \star)$. For $t > 0$;

$$G_0(t) = \{m \in G : F_M(m, n, t) = F_M(G, H, t) \text{ for some } n \in H\},$$
$$H_0(t) = \{n \in H : F_M(m, n, t) = F_M(G, H, t) \text{ for some } m \in G\}$$

where,

$$F_M(G, H, t) = sup\{F_M(m, n, t) : m \in G, n \in H\}$$

Definition 1.5.8. [118] Allow $(M, F_M, \star)$ to be a fuzzy metric space and G, H are two subsets of M which are not empty. A set H is said to be fuzzy approximatively compact concerning G if $F_M(m, n_a, t) \to F_M(m, H, t)$ implies that $m \in G_0(t)$ for every sequence $\{n_a\}$ in H and for some $m \in G$.

1.6 Contractions via Auxiliary Functions

There are many contractions in the literature that have been generalized using different auxiliary functions. In this section, we define some auxiliary functions and contraction type operators that have been generalized using the auxiliary functions.

Following class of functions was introduced by M.U. Ali [11]. The author used the class of functions and proved implicit type fixed point theorems:

Definition 1.6.1. Let $\psi : [0, \infty) \to [0, \infty)$ be a non-decreasing function satisfying; $\psi(x) < x, \forall x > 0$ and for all $x \geq 0$, $\sum_{a=1} \psi^a(x) < \infty$. A class of continuous mappings $\phi : (\mathbb{R}^+)^4 \to \mathbb{R}^+$ holding the accompanying statements;

(i) In each coordinate, ϕ is nondecreasing,

(ii) Let $m_1, m_2 \in \mathbb{R}^+$ such that if $m_1 < m_2$ and $m_1 \leq \phi(m_2, m_2, m_1, m_2)$, then $m_1 \leq \psi(m_2)$. If $m_1 \geq m_2$ and $m_1 \leq \phi(m_1, m_2, m_1, m_1)$, then $m_1 = 0$,

(iii) If $m \in \mathbb{R}^+$ with $m \leq \phi(0, 0, m, \frac{1}{2}m)$, then $m = 0$,

is represented by Φ_ψ.

Example 1.6.1. Following are some examples of $\phi \in \Phi_\psi$ that are all taken from [11]:

(i) Let $\phi_1(m_1, m_2, m_3, m_4) = \alpha \max(m_1, m_2, m_3, m_4)$ with $\psi(x) = \alpha x$, where $\alpha \in [0, \infty)$.

(ii) Let $\phi_2(m_1, m_2, m_3, m_4) = \alpha m_4$ with $\psi(x) = \alpha x$, where $\alpha \in [0, \infty)$.

(iii) Let $\phi_3(m_1, m_2, m_3, m_4) = \alpha \max(m_1, m_2, m_3)$ with $\psi(x) = \alpha x$, where $\alpha \in [0, \infty)$.

(iv) Let $\phi_4(m_1, m_2, m_3, m_4) = \alpha \max(m_2, m_3)$ with $\psi(x) = \alpha x$, where $\alpha \in [0, \infty)$.

(v) Let $\phi_5(m_1, m_2, m_3, m_4) = \alpha m_1$ with $\psi(x) = \alpha x$, where $\alpha \in [0, \infty)$.

(vi) Let $\phi_6(m_1, m_2, m_3, m_4) = \frac{\alpha}{2}(m_2 + m_3)$ with $\psi(x) = \frac{\alpha}{2}x$, where $\alpha \in [0, \infty)$.

(vii) Let $\phi_7(m_1, m_2, m_3, m_4) = \alpha \max(m_1, \frac{1}{2}(m_2 + m_3), m_4)$ with $\psi(x) = \alpha x$, where $\alpha \in [0, \infty)$.

(viii) Let $\phi_8(m_1, m_2, m_3, m_4) = \varsigma_1 m_1 + \varsigma_2(m_2 + m_3) + \varsigma_3 m_4$ with $\psi(x) = (\varsigma_1 + \varsigma_2 + \varsigma_3)x$, where $\varsigma_1, \varsigma_2, \varsigma_3$ are non-negative real numbers such that $\varsigma_1 + \varsigma_2 + \varsigma_3 \in [0, \infty)$.

(ix) Let $\phi_9(m_1, m_2, m_3, m_4) = \varsigma_1 m_2 + \varsigma_2 m_3 + \varsigma_3 m_1$ with $\psi(x) = (\varsigma_1 + \varsigma_2 + \varsigma_3)x$, where ς_1, ς_2 and ς_3 are non-negative real numbers such that $\varsigma_1 + \varsigma_2 + \varsigma_3 \in [0, \infty)$.

Wardowski [137] generalized the principle of Banach contraction by defining the notion of F-contraction. For this purpose he first introduced the specific type of functions denoted by F. These functions are defined as following:

Definition 1.6.2. Let a function $F : (0, \infty) \rightarrow \mathbb{R}$ hold the accompanying statements:

(F_1) every sequence $\{m_a\}$ in $(0, \infty)$ satisfy, $\lim_{a \to \infty} m_a = 0$ if and only if $\lim_{a \to \infty} F(m_a) = -\infty$,

(F_2) there exists a number $k \in (0, 1)$ such that $\lim_{m \to 0^+} m^k F(m) = 0$,

(F_3) F is strictly increasing on $(0, \infty)$.

The notation $\mathfrak{F}$ is used for the family of all such functions.

Example 1.6.2. Following functions belong the family $\mathfrak{F}$

(i)$F(m) = \ln m$

(ii) $F(m) = \ln m + m$.

1.7 Prešić Type Operators

Let O be a continuous operator from $J^k \subset \mathbb{R}^k$ into $J \subset \mathbb{R}$ then the equation

$$m_{a+k} = O(m_a, m_{a+1}, m_{a+2}, ..., m_{a+k-1}) \tag{1.7.1}$$

is called the nonlinear difference equation of order k and a point $m \in J$ is termed as equilibrium point of equation (1.7.1), if $m = O(m, m, m, ..., m)$. Different iteration methods like homotopy perturbation method and the method of variational iteration [17, 138] are used to find equilibrium point of nonlinear difference equations. Prešić [111] proved that the sequence mentioned in (1.7.1) converges that inevitably guarantees the presence of equilibrium point of nonlinear difference equation.

Prešić in [111] presented an operator defined on product spaces:

Definition 1.7.1. Allow (M, d) to be a metric space and k, a positive integer then a mapping $O : M^k \to M$ satisfying;

$$d(O(m_1, m_2, ..., m_k), O(m_2, m_3, ..., m_{k+1})) \leq \sum_{i=1}^{k} a_i d(m_i, m_{i+1}) \tag{1.7.2}$$

for every $m_1, m_2, ..., m_{k+1} \in M$ where $a_1, a_2, ..., a_k$ are non-negative scalars with $\sum_{i=1}^{k} a_i < 1$ is called Prešić type operator.

Prešić also proved a result which states that;

Theorem 1.7.1. *Allow (M,d) to be a metric space which is complete, k be a positive integer and $O : M^k \to M$ be Prešić type operator. Then there exist a unique point $m \in M$ such that $O(m,m,...,m) = m$. Moreover, for $m_1, m_2, ..., m_k \in M$ and each $a \in \mathbb{N}$, we have;*

$$m_{a+k} = O(m_a, m_{a+1}, ..., m_{a+k-1}) \tag{1.7.3}$$

then the sequence $\{m_a\}$ converges and $\lim m_a = O(\lim m_a, \lim m_a, ..., \lim m_a)$.

This theorem is reduced to the notion of Banach contraction if the value of k is taken as 1. So, it is the generalization of the principle of Banach contraction. An extension of this result was given by Ćirić and Prešić[40] as:

Theorem 1.7.2. *Allow (M,d) to be a complete metric space, k be a positive integer and $O : M^k \to M$ be a mapping such that:*

$$d(O(m_1, m_2, ..., m_k), O(m_2, m_3, ..., m_{k+1})) \leq \lambda \max\{d(m_i, m_{i+1}) : 1 \leq i \leq k\}$$

for every $m_1, m_2, ..., m_{k+1} \in M$, where $\lambda \in (0,1)$. Then there exist a point $m \in M$ with $O(m,m,...,m) = m$. Moreover, if $m_1, m_2, ..., m_k$ are arbitrary elements in M and

$$m_{a+k} = O(m_a, m_{a+1}, ..., m_{a+k-1})$$

for each $a \in \mathbb{N}$, then the sequence $\{m_a\}$ converges and $\lim m_a = O(\lim m_a, \lim m_a, ..., \lim m_a)$.

Chapter 2

Best proximity points of some generalized proximal contractions on metric space

The approximation theory evolved with the result of K. Fan [49]. But solutions produced by best approximation theorems may not be optimal solutions. The need to guarantee the presence of optimal approximations innovate best proximity point results that give adequate conditions to fulfil the need. To demonstrate best proximity point theorems, the notion of a very useful property was introduced by S. Raj [113] called P-property. By employing the notion of P-property, best proximity point results for various contractive mappings were studied [18, 29, 91]. The modification of notion of P-property, was introduced by Sadiq Basha [22], named as proximal contraction of first kind and proximal contraction of second kind. Several authors then introduced generalizations of proximal contractions [4, 101, 107]. In this chapter we introduced some new generalizations of proximal contractions and showed the presence of best proximity points.

2.1 Some generalizations of multivalued proximal contractions

An extensive powerful generalization of Banach contraction principle is the multivalued version of contraction map due to S. B. Nadler [106]. Another very useful generalization is due to Ciric [40], who generalized the contraction condition of a self map. Hardy and Rogers also generalized the contraction condition [66]. Motivation and inspiration of these researches compelled us to introduce some new generalizations of multivalued contractions. We introduced several new F type proximal contractions in this section and for such contractions, we demonstrated some best proximity theorems. We also used examples to demonstrate our findings. Our findings are generalisation of several previous best proximity results. Theorem 1.1.1, in particular, becomes a specific instance of one of our results(Theorem 2.1.1). The results in this section have been published in [82].

Definition 2.1.1. Allow (M, d) to be a metric space and $G, H \neq \phi$ be subsets of M. A *Hardy Rogers type α_F-proximal contraction* is a mapping $O : G \to CB(H)$ if there exist a constant $\tau > 0$ and two functions $F \in \mathfrak{F}$, $\alpha : G \times G \to [0, \infty)$ such that for each $r_1, r_2, u_1, u_2 \in G$ and $v_1 \in Or_1, v_2 \in Or_2$ with $\alpha(r_1, r_2) \geq 1$ and $d(u_1, v_1) = d(G, H) = d(u_2, v_2)$, we have;

$$\alpha(u_1, u_2) \geq 1 \text{ and } \tau + F(d(u_1, u_2)) \leq F(N(r_1, r_2)) \tag{2.1.1}$$

whenever $min\{d(u_1, u_2), N(r_1, r_2)\} > 0$, where

$$N(r_1, r_2) = \varsigma_1 d(r_1, r_2) + \varsigma_2 d(r_1, u_1) + \varsigma_3 d(r_2, u_2) + \varsigma_4 [d(r_1, u_2) + d(r_2, u_1)]$$

with $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4 \geq 0$ satisfying $\varsigma_1 + \varsigma_2 + \varsigma_3 + 2\varsigma_4 = 1$ and $\varsigma_3 \neq 1$.

Here, we prove a result which guarantees presence of best proximity point of proximal contraction defined above.

Theorem 2.1.1. *Consider a metric space (M, d) which is complete and $G, H \neq \phi$ be subsets of M. Assume G_0 to be not empty and $O : G \to CB(H)$ be a Hardy Rogers type α_F-proximal contraction fulfilling the below mentioned assumptions:*

(i) for each $r \in G_0$, $Or \subseteq H_0$,

(ii) there exist $r_1, r_2 \in G_0$ and $v_1 \in Or_1$ such that $\alpha(r_1, r_2) \geq 1$ and $d(r_2, v_1) = d(G, H)$,

(iii) H is approximatively compact concerning G,

(iv) any sequence $\{r_a\} \subseteq G$ converging to r such that $\alpha(r_a, r_{a+1}) \geq 1$, $\forall a \in \mathbb{N}$, satisfies $\alpha(r_a, r) \geq 1$, $\forall a \in \mathbb{N}$, or,

O is continuous.

Then O possess best proximity point.

Proof. Hypothesis (ii) yields, $r_1, r_2 \in G_0$ and $v_1 \in Or_1$ for which

$$\alpha(r_1, r_2) \geq 1 \text{ and } d(r_2, v_1) = d(G, H).$$

As $v_2 \in Or_2 \subseteq H_0$, there is $r_3 \in G_0$ satisfying

$$d(r_3, v_2) = d(G, H).$$

From 2.1.1, we get $\alpha(r_2, r_3) \geq 1$ and

$$\begin{aligned} \tau + F(d(r_2, r_3)) &\leq F(\varsigma_1 d(r_1, r_2) + \varsigma_2 d(r_1, r_2) + \varsigma_3 d(r_2, r_3) + \varsigma_4 [d(r_1, r_3) + d(r_2, r_2)]) \\ &\leq F(\varsigma_1 d(r_1, r_2) + \varsigma_2 d(r_1, r_2) + \varsigma_3 d(r_2, r_3) + \varsigma_4 [d(r_1, r_2) + d(r_2, r_3)]) \\ &= F((\varsigma_1 + \varsigma_2 + \varsigma_4) d(r_1, r_2) + (\varsigma_3 + \varsigma_4) d(r_2, r_3)). \qquad (2.1.2) \end{aligned}$$

As F is strictly increasing, using inequality 2.1.2, we get

$$d(r_2, r_3) < (\varsigma_1 + \varsigma_2 + \varsigma_4) d(r_1, r_2) + (\varsigma_3 + \varsigma_4) d(r_2, r_3).$$

That is,

$$(1 - \varsigma_3 - \varsigma_4) d(r_2, r_3) < (\varsigma_1 + \varsigma_2 + \varsigma_4) d(r_1, r_2).$$

As $\varsigma_1 + \varsigma_2 + \varsigma_3 + 2\varsigma_4 = 1$ and $\varsigma_3 \neq 1$, the above inequality implies that

$$d(r_2, r_3) < d(r_1, r_2)$$

Thus 2.1.2 yields,

$$\tau + F(d(r_2, r_3)) \leq F(d(r_1, r_2)). \tag{2.1.3}$$

From above we have $r_2, r_3 \in G_0$ and $v_2 \in Or_2$ satisfying

$$\alpha(r_2, r_3) \geq 1 \text{ and } d(r_3, v_2) = d(G, H).$$

As $v_3 \in Or_3 \subseteq H_0$, there is $r_4 \in G_0$ satisfying

$$d(r_4, v_3) = d(G, H).$$

From 2.1.1, we get $\alpha(r_3, r_4) \geq 1$ and

$$\begin{aligned} \tau + F(d(r_3, r_4)) &\leq F(\varsigma_1 d(r_2, r_3) + \varsigma_2 d(r_2, r_3) + \varsigma_3 d(r_3, r_4) + \varsigma_4 [d(r_2, r_4) + d(r_3, r_3)]) \\ &\leq F(\varsigma_1 d(r_2, r_3) + \varsigma_2 d(r_2, r_3) + \varsigma_3 d(r_3, r_4) + \varsigma_4 [d(r_2, r_3) + d(r_3, r_4)]) \\ &= F((\varsigma_1 + \varsigma_2 + \varsigma_4) d(r_2, r_3) + (\varsigma_3 + \varsigma_4) d(r_3, r_4)). \end{aligned}$$

After simplification we get

$$\tau + F(d(r_3, r_4)) \leq F(d(r_2, r_3)). \tag{2.1.4}$$

From 2.1.4 and 2.1.3, we obtain

$$F(d(r_3, r_4)) \leq F(d(r_1, r_2)) - 2\tau.$$

We get sequences $\{r_a\}$ in G_0 and $\{v_a\}$ in H_0 by repeating the same process such that $v_a \in Or_a, \alpha(r_a, r_{a+1}) \geq 1, d(r_{a+1}, v_a) = d(G, H)$ and

$$F(d(r_a, r_{a+1})) \leq F(d(r_1, r_2)) - a\tau \text{ for each } a \in \mathbb{N} \setminus \{1\}. \tag{2.1.5}$$

Letting $a \to \infty$ in 2.1.5, we get $\lim_{a\to\infty} F(d(r_a, r_{a+1})) = -\infty$. The property (F_1) then implies that $\lim_{a\to\infty} d(r_a, r_{a+1}) = 0$. Let $d_a = d(r_a, r_{a+1})$ for each $a \in \mathbb{N}$. Then using (F_1), there is some $k \in (0, 1)$ such that

$$\lim_{a\to\infty} d_a^k F(d_a) = 0.$$

From 2.1.5 we have

$$d_a^k F(d_a) - d_a^k F(d_1) \leq -d_a^k a\tau \leq 0 \tag{2.1.6}$$

Letting $a \to \infty$ in 2.1.6, we get

$$\lim_{a\to\infty} ad_a^k = 0.$$

This implies that there is some $a_1 \in \mathbb{N}$ such that $ad_a^k \leq 1$ for each $a \geq a_1$. Thus, we have

$$d_a \leq \frac{1}{a^{1/k}}, \text{ for each } a \geq a_1. \tag{2.1.7}$$

We now claim that $\{r_a\}$ is a Cauchy sequence in G. Let $a, b \in \mathbb{N}$ with $b > a > a_1$. Firstly using the triangular inequality and then using 2.1.7, we get;

$$\begin{aligned} d(r_a, r_b) &\leq d(r_a, r_{a+1}) + d(r_{a+1}, r_{a+2}) + ... + d(r_{b-1}, r_b) \\ &= \sum_{i=a}^{b-1} d_i \leq \sum_{i=a}^{\infty} d_i \leq \sum_{i=a}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Hence, $\{r_a\}$ is a Cauchy sequence in G because of convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$. So, there is some r^* in G such that $r_a \to r^*$ as $a \to \infty$ because G is closed subset of a complete metric space. As $d(r_{a+1}, v_a) = d(G, H)$, we have $\lim_{a\to\infty} d(r^*, v_a) = d(G, H)$. As H is approximatively compact concerning G, we get a subsequence $\{v_{a_k}\}$ of $\{v_a\}$ with $v_{a_k} \in Or_{a_k}$ that converges to v^*. As a result,

$$d(r^*, v^*) = \lim_{a\to\infty} d(r_{a_k}, v_{a_k}) = d(G, H).$$

Assumption (iv), O is continuous, yields $v^* \in Or^*$. Hence, $d(G, H) \leq d(r^*, Or^*) \leq d(r^*, v^*) = d(G, H)$. This implies that $d(G, H) = d(r^*, Or^*)$.

We now show the theorem for other part of assumption (iv), that is $\alpha(r_a, r^*) \geq 1$ for each $a \in \mathbb{N}$. Since $r^* \in G_0$, then $Or^* \subseteq H_0$. This suggests that for $z^* \in Or^*$, there is $w^* \in G_0$ satisfying $d(w^*, z^*) = d(G, H)$. Also, keep in mind that $d(r_{a+1}, v_a) = d(G, H)$.

We assert that $d(r^*, w^*) = 0$.

Contrarily assume that $d(r^*, w^*) \neq 0$. Then 2.1.1, yields

$$d(r_{a+1}, w^*) < \varsigma_1 d(r_a, r^*) + \varsigma_2 d(r_a, r_{a+1}) + \varsigma_3 d(r^*, w^*) + \varsigma_4[d(r_a, w^*) + d(r^*, r_{a+1})].$$

Letting $a \to \infty$, we get

$$d(r^*, w^*) \leq (\varsigma_3 + \varsigma_4) d(r^*, w^*),$$

which could be possible only when $d(r^*, w^*) = 0$. As a result,

$$d(G, H) \leq d(r^*, Or^*) \leq d(r^*, z^*) = d(G, H),$$

and the proof is completed. $\square$

Remark 2.1.1. Theorem 1.1.1 is unique instance of Theorem 2.1.1.

Proof. Let $G, H \neq \phi \subseteq M$ of a complete metric space (M, d). Assuming G_0 is not empty. Also consider $\alpha(r_1, r_2) = 1, F(r) = \ln r$ for each $r \in (0, \infty)$ and $CB(H) = H$. Then the contraction operator $O : G \to CB(H)$ in definition 2.1.1 reduces to

$$\tau + \ln(d(u_1, u_2)) \leq \ln(\varsigma_1 d(r_1, r_2) + \varsigma_2 d(r_1, u_1) + \varsigma_3 d(r_2, u_2) + \varsigma_4[d(r_1, u_2) + d(r_2, u_1)])$$

for each $r_1, r_2, u_1, u_2 \in G$ and $d(u_1, Or_1) = d(G, H) = d(u_2, Or_2)$
and hence

$$(d(u_1, u_2)) \leq (\varsigma_1 d(r_1, r_2) + \varsigma_2 d(r_1, u_1) + \varsigma_3 d(r_2, u_2) + \varsigma_4[d(r_1, u_2) + d(r_2, u_1)])$$

Now the proof can be completed by performing similar steps as of theorem 2.1.1. $\square$

Example 2.1.1. Let $M = \mathbb{R} \times \mathbb{R}$ and a metric on M be defined as $d((r_1, r_2), (s_1, s_2)) = |r_1 - s_1| + |r_2 - s_2|$ for each $r, s \in M$. Take $G = \{(0, r) : -1 \leq r \leq 1\}$ and $H = \{(1, r) : -1 \leq r \leq 1\}$. Define $O : G \to CB(H)$ as;

$$O(0, r) = \begin{cases} \{(1, \frac{r+1}{2})\}, & \text{if } r \geq 0 \\ \{(1, r), (1, r^2)\}, & \text{otherwise,} \end{cases}$$

and $\alpha : G \times G \to [0, \infty)$ as;

$$\alpha((0, r), (0, s)) = \begin{cases} 1, & \text{if } r, s \in [0, 1] \\ 0, & \text{otherwise,} \end{cases}$$

Proof. $d(G, H) = 1$.
For each $(0, m) \in G$ there exist $(1, m) \in H$ such that

$$d((0, m), (1, m)) = 1 = d(G, H).$$

Take $\tau = \frac{1}{2}$ and $F(m) = \ln m$ for each $m \in (0, \infty)$.
By taking $\varsigma_1 = 1$ and $\varsigma_2 = \varsigma_3 = \varsigma_4 = 0$, we now check that O is α_F-proximal contraction of Hardy Rogers type.
Case:I Let $(0, m_1), (0, m_2) \in G$ for $m_1, m_2 \geq 0$

Then $O((0,m_1)) = \{(1, \frac{m_1+1}{2})\}$ and $O((0,m_2)) = \{(1, \frac{m_2+1}{2})\}$
So $v_1 = (1, \frac{m_1+1}{2})$ and $v_2 = (1, \frac{m_2+1}{2})$
We need u_1 and u_2 such that $d(u_1, v_1) = d(G, H) = d(u_2, v_2)$ for each $v_1 \in Om_1, v_2 \in Om_2$.

$$\begin{aligned}
d(u_1, v_1) = d((0, u_1), (1, \frac{m_1+1}{2})) &= 1 \\
\Rightarrow |0-1| + |u_1 - \frac{m_1+1}{2}| &= 1 \\
\Rightarrow 1 + |u_1 - \frac{m_1+1}{2}| &= 1 \\
\Rightarrow |u_1 - \frac{m_1+1}{2}| &= 0 \\
\Rightarrow u_1 &= \frac{m_1+1}{2}
\end{aligned}$$

Similarly,

$$u_2 = \tfrac{m_2+1}{2}$$

Now,
For $\alpha((0,m_1),(0,m_2)) = 1$ since both $m_1, m_2 \geq 0$
We have $\alpha((0,u_1),(0,u_2)) = 1$ since both $u_1, u_2 \geq 0$
Also

$$\begin{aligned}
\tau + F(d(0,u_1),(0,u_2)) &= \frac{1}{2} + F(|u_1 - u_2|) \\
&= \frac{1}{2} + F(|\frac{m_1+1}{2} - \frac{m_2+1}{2}|) \\
&= \frac{1}{2} + F(|\frac{m_1 - m_2}{2}|) \\
&= \frac{1}{2} + \ln(|\frac{m_1 - m_2}{2}|) \\
&\leq \ln(|m_1 - m_2|) \\
&= F(d(0,m_1),(0,m_2))
\end{aligned}$$

Case:II Let $(0,m_1),(0,m_2) \in G$ for $m_1 < 0, m_2 \geq 0$
Then $O((0,m_1)) = \{(1,m_1),(1,m_1^2)\}$ and $O((0,m_2)) = \{(1, \frac{m_2+1}{2})\}$

Take $v_1 = (1, m_1)$ and $v_2 = (1, \frac{m_2+1}{2})$
We need u_1 and u_2 such that $d(u_1, v_1) = d(G, H) = d(u_2, v_2)$ for each $v_1 \in Om_1, v_2 \in Om_2$

$$\begin{aligned} d(u_1, v_1) = d((0, u_1), (1, m_1) &= 1 \\ \Rightarrow |0-1| + |u_1 - m_1| &= 1 \\ \Rightarrow 1 + |u_1 - m_1| &= 1 \\ \Rightarrow |u_1 - m_1| &= 0 \\ \Rightarrow u_1 &= m_1 \end{aligned}$$

Similarly,

$$u_2 = \tfrac{m_2+1}{2}$$

Now,
For $\alpha((0, m_1), (0, m_2)) = 0$ since $m_1 < 0, m_2 \geq 0$
Also

$$\begin{aligned} \tau + F(d(0, u_1), (0, u_2)) &= \frac{1}{2} + F(|u_1 - u_2|) \\ &= \frac{1}{2} + F(|m_1 - \frac{m_2+1}{2}|) \\ &= \frac{1}{2} + F(\frac{2m_1 - m_2 + 1}{2}) \\ &= \frac{1}{2} + \ln(\frac{2m_1 - m_2 + 1}{2}) \\ &\leq \ln(m_1 - m_2) \\ &= F(d(0, m_1), (0, m_2)) \end{aligned}$$

Take $v_1 = (1, m_1^2)$ and $v_2 = \frac{m_2+1}{2}$. We need u_1 and u_2 such that $d(u_1, v_1) = d(G, H) = d(u_2, v_2)$ for each $v_1 \in Om_1, v_2 \in Om_2$

$$
\begin{aligned}
d(u_1, v_1) = d((0, u_1), (1, m_1^2) &= 1 \\
\Rightarrow |0-1| + |u_1 - m_1^2| &= 1 \\
\Rightarrow 1 + |u_1 - m_1^2| &= 1 \\
\Rightarrow |u_1 - m_1^2| &= 0 \\
\Rightarrow u_1 &= m_1^2
\end{aligned}
$$

Similarly,

$$u_2 = \tfrac{m_2+1}{2}$$

Now,
For $\alpha((0, m_1), (0, m_2)) = 0$ since $m_1 < 0, m_2 \geq 0$
Also

$$
\begin{aligned}
\tau + F(d(0, u_1), (0, u_2)) &= \frac{1}{2} + F(|u_1 - u_2|) \\
&= \frac{1}{2} + F(|m_1^2 - \frac{m_2+1}{2}|) \\
&= \frac{1}{2} + F(\frac{2m_1^2 - m_2 + 1}{2}) \\
&= \frac{1}{2} + \ln(\frac{2m_1^2 - m_2 + 1}{2}) \\
&\leq \ln(m_1 - m_2) \\
&= F(d(0, m_1), (0, m_2))
\end{aligned}
$$

For each $m \in G_0$, we have $Om \subseteq H_0$. Also for $m_1 = (0, \frac{1}{2}) \in G_0$ and $v_1 = (1, \frac{3}{4}) \in Om_1$, we have $m_2 = (0, \frac{3}{4})$ such that $\alpha(m_1, m_2) = 1$ and $d(m_2, v_1) = d(G, H)$. Moreover, for any sequence $\{m_a\} \subseteq G$ such that $m_a \to m$ as $a \to \infty$ and $\alpha(m_a, m_{a+1}) = 1$ for each $a \in \mathbb{N}$, we have $\alpha(m_a, m) = 1$ for each $a \in \mathbb{N}$. Further note that H is approximatively compact concerning G, therefore, O possess best proximity point, by Theorem 2.1.1. $\square$

Remark 2.1.2. It is noticed that the above example is not valid for the Theorem 1.1.1. Therefore, our theorem properly generalizes Theorem2.1.1

Definition 2.1.2. Allow (M, d) to be a metric space and $G, H \neq \phi$ be subsets of M. A *Ćirić type α_F-proximal contraction* is an operator $O : G \to CB(H)$ if there exist a constant $\tau > 0$ and two functions $F \in \mathfrak{F}$, $\alpha : G \times G \to [0, \infty)$ such that for each $r_1, r_2, u_1, u_2 \in G$ and $v_1 \in Or_1, v_2 \in Or_2$ with $\alpha(r_1, r_2) \geq 1$ and $d(u_1, v_1) = d(G, H) = d(u_2, v_2)$, we have

$$\alpha(u_1, u_2) \geq 1 \text{ and } \tau + F(d(u_1, u_2)) \leq F(M(r_1, r_2)) \tag{2.1.8}$$

whenever $min\{d(u_1, u_2), M(r_1, r_2)\} > 0$, where

$$M(r_1, r_2) = \max\{d(r_1, r_2), d(r_1, u_1), d(r_2, u_2), \tfrac{d(r_1,u_2)+d(r_2,u_1)}{2}\}.$$

Theorem 2.1.2. *Allow (M, d) to be a metric space which is complete and $G, H \neq \phi \subseteq M$. Assume that G_0 is non-empty and $O : G \to CB(H)$ is a Ćirić type α_F-proximal contraction satisfying the below mentioned conditions:*

(i) for each $r \in G_0$, $Or \subseteq H_0$,

(ii) there exist $r_1, r_2 \in G_0$ and $v_1 \in Or_1$ such that $\alpha(r_1, r_2) \geq 1$ and $d(r_2, v_1) = d(G, H)$,

(iii) H is approximatively compact concerning G,

(iv) O is continuous, or,

any sequence $\{r_a\} \subseteq G$ converging to r such that $\alpha(r_a, r_{a+1}) \geq 1$, $\forall a \in \mathbb{N}$, satisfies $\alpha(r_a, r) \geq 1$, $\forall a \in \mathbb{N}$.

Then O possess best proximity point.

Proof. Hypothesis (ii) yields, $r_1, r_2 \in G_0$ and $v_1 \in Or_1$ for which

$$\alpha(r_1, r_2) \geq 1 \text{ and } d(r_2, v_1) = d(G, H)$$

As $v_2 \in Or_2 \subseteq H_0$, there is $r_3 \in G_0$ satisfying

$$d(r_3, v_2) = d(G, H).$$

From 2.1.8, we get $\alpha(r_2, r_3) \geq 1$ and

$$\begin{aligned}\tau + F(d(r_2, r_3)) &\leq F(\max\{d(r_1, r_2), d(r_1, r_2), d(r_2, r_3), \frac{d(r_1, r_3) + d(r_2, r_2)}{2}\}) \\ &= F(\max\{d(r_1, r_2), d(r_2, r_3)\}) \\ &= F(d(r_1, r_2)), \qquad (2.1.9)\end{aligned}$$

otherwise we have a contradiction. From above, we have $r_2, r_3 \in G_0$ and $v_2 \in Or_2$ satisfying;

$$\alpha(r_2, r_3) \geq 1 \text{ and } d(r_3, v_2) = d(G, H).$$

As $v_3 \in Or_3 \subseteq H_0$, there is $r_4 \in G_0$ satisfying;

$$d(r_4, v_3) = d(G, H).$$

From 2.1.8, we get $\alpha(r_3, r_4) \geq 1$ and

$$\begin{aligned}\tau + F(d(r_3, r_4)) &\leq F(\max\{d(r_2, r_3), d(r_2, r_3), d(r_3, r_4), \frac{d(r_2, r_4) + d(r_3, r_3)}{2}\}) \\ &= F(\max\{d(r_2, r_3), d(r_3, r_4)\}) \\ &= F(d(r_2, r_3)), \qquad (2.1.10)\end{aligned}$$

otherwise, we have a contradiction. From inequalities 2.1.9 and 2.1.10, we have

$$F(d(r_3, r_4)) \leq F(d(r_1, r_2)) - 2\tau.$$

Proceeding with a similar procedure we get sequences $\{r_a\}$ in G_0 and $\{v_a\}$ in H_0 such that $v_a \in Or_a, \alpha(r_a, r_{a+1}) \geq 1, d(r_{a+1}, v_a) = d(G, H)$ and

$$F(d(r_a, r_{a+1})) \leq F(d(r_1, r_2)) - a\tau \text{ for each } a \in \mathbb{N} \setminus \{1\}.$$

Sine the above inequality looks same as 2.1.5 so, by following the steps of Theorem 2.1.1 proof, it tends to be demonstrated that $\{r_a\}$ is a Cauchy sequence in G. As a result, there is some r^* in G such that $r_a \to r^*$ as $a \to \infty$ because, G is closed subset of a complete metric space. As $d(r_{a+1}, v_a) = d(G, H)$, we have $\lim_{a\to\infty} d(r^*, v_a) = d(G, H)$. As H is approximatively compact concerning G, we get a subsequence $\{v_{a_k}\}$ of $\{v_a\}$ with $v_{a_k} \in Or_{a_k}$ that converges to v^*. As a result,

$$d(r^*, v^*) = \lim_{a\to\infty} d(r_{a_k}, v_{a_k}) = d(G, H).$$

Assumption (iv), O is continuous, yields $v^* \in Or^*$. Hence, $d(G,H) \leq d(r^*, Or^*) \leq d(r^*, v^*) = d(G,H)$. Now, assume that $\alpha(r_a, r^*) \geq 1$ for each $a \in \mathbb{N}$. Since $r^* \in G_0$, then $Or^* \subseteq H_0$. This suggests that for $z^* \in Or^*$, we have $w^* \in G_0$ satisfying $d(w^*, z^*) = d(G,H)$. Also keep in mind that $d(r_{a+1}, v_a) = d(G,H)$.

We assert that $d(r^*, w^*) = 0$.

Contrarily, we make assumption that $d(r^*, w^*) \neq 0$. Then 2.1.8 yields,

$$\tau + F(d(r_{a+1}, w^*)) < F(\max\{d(r_a, r^*), d(r_a, r_{a+1}), d(r^*, w^*), \tfrac{d(r_a, w^*) + d(r^*, r_{a+1})}{2}\}).$$

Letting $a \to \infty$, we get

$$\tau + F(d(r^*, w^*)) \leq F(d(r^*, w^*)),$$

which is only possible when $d(r^*, w^*) = 0$. As a result,

$$d(G,H) \leq d(r^*, Or^*) \leq d(r^*, z^*) = d(G,H),$$

and the proof is accomplished. □

Example 2.1.2. Let $M = \mathbb{R} \times \mathbb{R}$ and a metric defined on M as $d((r_1, r_2), (s_1, s_2)) = |r_1 - s_1| + |r_2 - s_2|$ for each $r, s \in M$. Take $G = \{(0, r) : r \in [-1, 1]\}$ and $H = \{(1, r) : r \in [-1, 1]\}$. Define $O : G \to CB(H)$ as;

$$O(0,r) = \begin{cases} \{(1, \frac{r}{3}), (1, \frac{r}{2})\}, & \text{if } m \geq 0 \\ \{(1, r), (1, r^2)\}, & \text{otherwise,} \end{cases}$$

and $\alpha : G \times G \to [0, \infty)$ as;

$$\alpha((0,r),(0,s)) = \begin{cases} 1, & \text{if } r, s \in [0,1] \\ 0, & \text{otherwise,} \end{cases}$$

Define $F(r) = \ln r$ for each $r \in (0, \infty)$ and $\tau = \frac{1}{2}$. It is definitely not hard to see that O is Ćirić type α_F-proximal contraction. For each $r \in G_0$, we have $Or \subseteq H_0$. Also for $r_1 = (0, \frac{1}{2}) \in G_0$ and $v_1 = (1, \frac{1}{6}) \in Or_1$, we have $r_2 = (0, \frac{1}{6})$ such that $\alpha(r_1, r_2) = 1$ and $d(r_2, v_1) = d(G,H)$. Further, note that H is approximatively compact concerning G. Moreover, for each $a \in \mathbb{N}$, $\alpha(r_a, r_{a+1}) = 1$ and for any sequence $\{r_a\} \subseteq G$ such that $r_a \to r$ as $a \to \infty$, we have $\alpha(r_a, r) = 1$ for each $a \in \mathbb{N}$. Therefore, O possess best proximity point by Theorem 2.1.2.

2.1.1 Consequences

Our results are immediately followed by the following two theorems if $\alpha(r,s) = 1$ for each $r, s \in G$.

Theorem 2.1.3. *Allow G and H to be subsets of a complete metric space (M,d) which are not empty. Make assumptions that $G_0 \neq \phi$ and $O : G \to CB(H)$ is a mapping for which there is some constant $\tau > 0$ and a continuous function $F \in \mathfrak{F}$ such that for each $r_1, r_2, u_1, u_2 \in G$ and $v_1 \in Or_1, v_2 \in Or_2$ and $d(u_1, v_1) = d(G,H) = d(u_2, v_2)$, we have*

$$\tau + F(d(u_1, u_2)) \leq F(N(r_1, r_2))$$

whenever min$\{d(u_1, u_2), N(r_1, r_2)\} > 0$, where

$$N(r_1, r_2) = \varsigma_1 d(r_1, r_2) + \varsigma_2 d(r_1, u_1) + \varsigma_3 d(r_2, u_2) + \varsigma_4 [d(r_1, u_2) + d(r_2, u_1)]$$

with $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4 \geq 0$ satisfying $\varsigma_1 + \varsigma_2 + \varsigma_3 + 2\varsigma_4 = 1$ and $\varsigma_3 \neq 1$. Further consider that the accompanying assumptions hold:

(i) for each $r \in G_0$, we have $Or \subseteq H_0$;

(ii) H is approximatively compact concerning G.

Then O possesses best proximity point.

Theorem 2.1.4. *Allow G and H to be subsets of a complete metric space (M,d) which are not empty. Make assumptions that G_0 is non-empty and $O : G \to CB(H)$ is a mapping for which there exist a constant $\tau > 0$ as well as a continuous function $F \in \mathfrak{F}$ such that for each $r_1, r_2, u_1, u_2 \in G$ and $v_1 \in Or_1, v_2 \in Or_2$ with $\alpha(r_1, r_2) \geq 1$ and $d(u_1, v_1) = d(G,H) = d(u_2, v_2)$, we have*

$$\tau + F(d(u_1, u_2)) \leq F(M(r_1, r_2))$$

whenever min$\{d(u_1, u_2), M(r_1, r_2)\} > 0$, where

$$M(r_1, r_2) = \max\{d(r_1, r_2), d(r_1, u_1), d(r_2, u_2), \tfrac{d(r_1, u_2) + d(r_2, u_1)}{2}\}.$$

Further consider that the accompanying assumptions hold:

(i) H is approximatively compact concerning G,

(ii) for each $r \in G_0$, we have $Or \subseteq H_0$.

Then O possesses best proximity point.

If $G = H = M$, then the accompanying fixed point theorems are gotten from our outcomes.

Theorem 2.1.5. *Allow (M, d) to be a complete metric space. Make assumption that $O : M \to CB(M)$ is a mapping for which there is a constant $\tau > 0$ as well as two functions $\alpha : M \times M \to [0, \infty)$ and $F \in \mathfrak{F}$ such that for each $r_1, r_2 \in M$ and $u_1 \in Or_1, u_2 \in Or_2$ with $\alpha(r_1, r_2) \geq 1$, we have*

$$\alpha(u_1, u_2) \geq 1 \text{ and } \tau + F(d(u_1, u_2)) \leq F(N(r_1, r_2))$$

whenever $min\{d(u_1, u_2), N(r_1, r_2)\} > 0$, where

$$N(r_1, r_2) = \varsigma_1 d(r_1, r_2) + \varsigma_2 d(r_1, u_1) + \varsigma_3 d(r_2, u_2) + \varsigma_4 [d(r_1, u_2) + d(r_2, u_1)]$$

with $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4 \geq 0$ satisfying $\varsigma_1 + \varsigma_2 + \varsigma_3 + 2\varsigma_4 = 1$ and $\varsigma_3 \neq 1$. Further assume that O is continuous, any sequence $\{r_a\} \subseteq G$ converging to r such that $\alpha(r_a, r_{a+1}) \geq 1$, $\forall a \in \mathbb{N}$, satisfies $\alpha(r_a, r) \geq 1$, $\forall a \in \mathbb{N}$. Then O possesses a fixed point.

Theorem 2.1.6. *Allow (M, d) to be a complete metric space. Make assumption that $O : M \to CB(M)$ is a mapping for which there is a constant $\tau > 0$, $\alpha : M \times M \to [0, \infty)$ and a continuous function $F \in \mathfrak{F}$ such that for each $r_1, r_2 \in M$ and $u_1 \in Or_1, u_2 \in Or_2$ with $\alpha(r_1, r_2) \geq 1$, we have*

$$\alpha(u_1, u_2) \geq 1 \text{ and } \tau + F(d(u_1, u_2)) \leq F(M(r_1, r_2))$$

whenever $min\{d(u_1, u_2), M(r_1, r_2)\} > 0$, where

$$M(r_1, r_2) = \max\{d(r_1, r_2), d(r_1, u_1), d(r_2, u_2), \tfrac{d(r_1, u_2) + d(r_2, u_1)}{2}\}.$$

Further, assume that O is continuous, any sequence $\{r_a\} \subseteq G$ converging to r such that $\alpha(r_a, r_{a+1}) \geq 1$, $\forall a \in \mathbb{N}$, satisfies $\alpha(r_a, r) \geq 1$, $\forall a \in \mathbb{N}$. Then O possesses a fixed point.

2.2 Best proximity points of Prešić Type proximal contractions

As we have stated Prešić type operator $O : M^k \to M$ satisfying equation 1.7.2 and representation of k^{th} order nonlinear difference equation 1.7.1 in Chapter 1. It can be observed that the equilibrium points of difference equation 1.7.1 are same as the fixed points of O. Therefore, the study of fixed points of Presic type operators became as important as the equilibrium points of nonlinear difference equations. The importance of study of nonlinear difference equations can be envisioned in modeling various problems appearing in Probability theory, Biology, Economics, Psychology, and others (See for example [48],[89], [90],[128],[129]). Some generalizations of Prešić's results are proved by some renowned authors [27],[84],[109],[124, 125]. Shukla in [124] also showed applications of Presic type operators to second order difference equation.

This part of the chapter showed the presence of an approximate solution of the equation $r = O(r, r, ..., r)$, where $O : G^k \to H$. The solution of this equation only possible if $G \cap H \neq \phi$, otherwise it has no solution; hence, if $G \cap H = \phi$ then the approximate solution is only possible debate.

The approximate solution of the equation $r = O(r, r, ..., r)$ with the error $d(G, H)$ is termed *best proximity point of* $O : G^k \to H$.

All the results in this section are published in [15].

All through this section, we will use the notation $\mathfrak{G} = (V, E)$ for a directed graph characterized on a metric space (M, d), where V and E is the set of vertices and edges respectively, with $V = M$ and E comprises of all loops excluding parallel edges.

Definition 2.2.1. Allow G and H to be non-empty subsets of metric space (M, d) endowed with the graph $\mathfrak{G}$. A mapping $O : G \times G \to H$ is termed as *path admissible* if;

$$\begin{cases} d(u_1, O(g_1, g_2)) = d(G, H) \\ d(u_2, O(g_1, g_2)) = d(G, H) \quad \Rightarrow (u_1, u_2) \in E \\ g_1 P g_3 \end{cases} \tag{2.2.1}$$

where $g_1, g_2, g_3, u_1, u_2 \in G$. Here $g_1 P g_3$ means, for the above mentioned $g_1, g_2, g_3 \in V$ we have $(g_1, g_2) \in E$ and $(g_2, g_3) \in E$.

Theorem 2.2.1. *Consider a complete metric space (M, d) furnished with graph $\mathfrak{G}$ and let G, H be nonempty closed subsets of M. Let $O : G \times G \to H$ be a mapping such that for each $g_1, g_2, g_3, u_1, u_2 \in G$ with $g_1 P g_3$, that is $(g_1, g_2) \in E$ and $(g_2, g_3) \in E$, and $d(u_1, O(g_1, g_2)) = d(G, H) = d(u_2, O(g_2, g_3))$, we have:*

$$d(u_1, u_2) \leq \gamma \max\{d(g_1, g_2), d(g_2, g_3)\} \tag{2.2.2}$$

where $\gamma \in [0, 1)$. Also, consider the below mentioned assumptions:

(i) G_0 is non-empty,

(ii) there exist $g_0, g_1, g_2 \in G$ satisfying $d(g_2, O(g_0, g_1)) = d(G, H)$ and $g_0 P g_2$,

(iii) H is approximatively compact concerning G,

(iv) $O(G \times G_0) \subseteq H_0$,

(v) O is path admissible,

(vi) let a sequence $\{m_a\}$ in M satisfying $m_a P m_{a+2}$ for each $a \in \mathbb{N}$ and $m_a \to m$ as $a \to \infty$, then $(m_a, m) \in E$ for all $a \in \mathbb{N}$ and $(m, m) \in E$.

Then O possess a best proximity point, that is $d(g^*, O(g^*, g^*)) = d(G, H)$ for some $g^* \in G$.

Proof. From hypothesis (ii) we have $g_0, g_1, g_2 \in G$ satisfying $d(g_2, O(g_0, g_1)) = d(G, H)$ and $g_0 P g_2$, that is $(g_0, g_1), (g_1, g_2) \in E$. Hypothesis (iv) implies that $O(g_1, g_2) \in H_0$, and by definition of H_0,as defined in Definition 1.1.1, we have $g_3 \in G_0$ satisfying $d(g_3, O(g_1, g_2)) = d(G, H)$. Since O is path admissible, so by Definition 2.2.1, we have $(g_2, g_3) \in E$. Thus $g_1 P g_3$. By proceeding with a similar procedure, we obtain a sequence $\{g_a\}_{a \geq 2} \in G_0$ satisfying;

$$d(g_{a+1}, O(g_{a-1}, g_a)) = d(G, H) \text{ for each } a \in \mathbb{N}$$

and:

$$g_{a-1}Pg_{a+1}, \text{ that is } (g_{a-1}, g_a), (g_a, g_{a+1}) \in E \text{ for each } a \in \mathbb{N}.$$

The inequality 2.2.2 yields,

$$d(g_a, g_{a+1}) \leq \gamma \max\{d(g_{a-2}, g_{a-1}), d(g_{a-1}, g_a)\} \text{ for each } a = 2, 3, 4, ... \tag{2.2.3}$$

For convenience, we take $d_a = d(g_a, g_{a+1})$ for each $a \in \mathbb{N} \cup \{0\}$. By using induction, we can get:

$$d_{a-1} \leq Z\psi^a \text{ for each } a \in \mathbb{N} \tag{2.2.4}$$

where $\psi = \gamma^{1/2}$ and $Z = \max\{d_0/\psi, d_1/\psi^2\}$. Clearly, $d_0 \leq Z\psi$ and $d_1 \leq Z\psi^2$. We obtain:

$$\begin{aligned}
d_2 &\leq \gamma \max\{d_0, d_1\} \leq \gamma \max\{Z\psi, Z\psi^2\} \leq \gamma Z\psi = Z\psi^3. \\
&. \\
&. \\
&. \\
d_m &\leq \gamma \max\{d_{b-1}, d_{b-2}\} \leq \gamma \max\{Z\psi^b, Z\psi^{b-1}\} \leq \gamma Z\psi^{b-1} = Z\psi^{b+1}.
\end{aligned}$$

Thus, $d_{a-1} \leq Z\psi^a$ for each $a \in \mathbb{N}$. With the help of triangular inequality, for each $b, q \in \mathbb{N}$, we get:

$$\begin{aligned}
d(g_b, g_{b+q}) &\leq d(g_b, g_{b+1}) + d(g_{b+1}, g_{b+2}) + ... + d(g_{b+q-1}, g_{b+q}) \\
&\leq Z\psi^{b+1} + Z\psi^{b+2} + Z\psi^{b+q} \\
&\leq \frac{\psi^{b+1}}{1-\psi} Z
\end{aligned}$$

Note that $\psi = \gamma^{1/2} < 1$. Therefore, $\{g_a\}$ in G is a Cauchy sequence. Since M is complete and $G \subseteq M$ is a closed. So, there exist a point $g^* \in G$ such that $g_a \to g^*$. Furthermore,

$$\begin{aligned}
d(g^*, H) &\leq d(g^*, O(g_{a-1}, g_a)) && (2.2.5) \\
&\leq d(g^*, g_{a+1}) + d(g_{a+1}, O(g_{a-1}, g_a)) && (2.2.6) \\
&= d(g^*, g_{a+1}) + d(G, H) \\
&\leq d(g^*, g_{a+1}) + d(g^*, H).
\end{aligned}$$

Therefore, $d(g^*, g_{a+1}) \to d(g^*, H)$ as $a \to \infty$. Since H is approximatively compact concerning G, then by Definition 1.1.2, the sequence $\{O(g_{a-1}, g_a)\}$ has a subsequence $\{O(g_{a_k-1}, g_{a_k})\}$ convergent to a point $h^* \in H$. This implies that:

$$d(g^*, h^*) = lim_{k\to\infty} d(g_{a_{k+1}}, O(g_{a_k-1}, g_{a_k})) = d(G, H).$$

Here $g^* \in G_0$. As we know $O(g_a, g^*) \in H_0$, there exist $u \in G$ such that $d(u, O(g_a, g^*)) = d(G, H)$. Hypothesis (vi) implies that $(g_a, g^*) \in E$ for each $a \in \mathbb{N}$. Thus, we have $g_{a-1}Pg^*$, that is $(g_{a-1}, g_a), (g_a, g^*) \in E$, for all $a \in \mathbb{N}$. Hence, inequality 2.2.2 implies:

$$d(g_{a+1}, u) \leq \gamma \max\{d(g_{a-1}, g_a), d(g_a, g^*)\} \text{ for each } a \in \mathbb{N}.$$

Now by letting a approaches to infinity, we come by $d(g^*, u) = 0$, that is $u = g^*$. Moreover, notice that $O(g^*, g^*) \in H_0$, hypothesis (iv) facilitates that for some $s \in G$, $d(s, O(g^*, g^*)) = d(G, H)$. Using hypothesis (vi), we have $(g^*, g^*) \in E$. Hence, $d(g^*, O(g_a, g^*)) = d(G, H), d(s, O(g^*, g^*)) = d(G, H)$, and $g_a P g^*$, that is $(g_a, g^*) \in E$ and $(g^*, g^*) \in E$. Thus, from 2.2.2 we come by the accompanying inequality:

$$d(g^*, s) \leq \gamma \max\{d(g_a, g^*), d(g^*, g^*)\} \text{ for each } a \in \mathbb{N}.$$

We get $d(g^*, s) = 0$ by applying limit a tends to infinity, and hence, $s = g^*$. Thus, we have $d(g^*, O(g^*, g^*)) = d(G, H)$. □

Example 2.2.1. Let $M = \mathbb{R} \times \mathbb{R}$ and metric $d((m_1, m_2), (n_1, n_2)) = |m_1 - n_1| + |m_2 - n_2|$ for each $m, n \in M$ endowed with graph $\mathfrak{G}$ be defined as $V = M$ and $E = \{((m_1, m_2), (n_1, n_2)) : m_1, m_2, n_1, n_2 \in [0, 1]\} \cup \{(m, m) : m \in M\}$. Take $G = \{(0, m) : m \in [-2, 2]\}$ and $H = \{(1, m) : m \in [-2, 2]\}$. Define $O : G \times G \to H$ as:

$$O((0, m), (0, n)) = \begin{cases} (1, \frac{m+n+2}{4}), & \text{if } m, n \geq 0 \\ (1, |m + n| - 2), & \text{otherwise.} \end{cases}$$

Then, for each $\overline{g_1} = (0, g_1), \overline{g_2} = (0, g_2), \overline{g_3} = (0, g_3), \overline{u_1} = (0, u_1) = (0, \frac{g_1+g_2+2}{4}), \overline{u_2} = (0, u_2) = (0, \frac{g_2+g+3+2}{4}) \in G$ with $\overline{g_1} P \overline{g_3}$ and $d(\overline{u_1}, O(\overline{g_1}, \overline{g_2})) = d(G, H) = d(\overline{u_2}, O(\overline{g_2}, \overline{g_3}))$, we have:

$$d(\overline{u_1}, \overline{u_2}) = \frac{1}{4}|g_1 - g_3| = \gamma \max\{d(\overline{g_1}, \overline{g_2}), d(\overline{g_2}, \overline{g_3})\}$$

where $\gamma = \frac{1}{2}$. Consider $\overline{g_1} = (0, g_1), \overline{g_2} = (0, g_2), \overline{g_3} = (0, g_3) \in G$ such that $\overline{g_1} P \overline{g_3}$ and $d((0, u_1), O((0, g_1), (0, g_2))) = d(G, H) = d((0, u_2), O((0, g_2), (0, g_3)))$, then $(0, u_1), (0, u_2) \in E$. Since $(0, u_1) = (0, \frac{g_1+g_2+2}{4})$ and $(0, u_2) = (0, \frac{g_2+g+3+2}{4})$. Thus, O is path admissible. We also have $\overline{g_1} = (0, 0)$ and $\overline{g_2} = (0, 1/2)$, and $\overline{g_3} = (0, \frac{5}{8})$ such that $d((0, 5/8), O((0, 0), (0, 1/2))) = d(G, H)$ and $\overline{g_1} P \overline{g_3}$. Moreover, H is approximatively compact concerning G and each sequence $\{g_a\}$ in M satisfies $g_a P g_{a+2}$ for each $a \in \mathbb{N}$ and $g_a \to m$ as $a \to \infty$, then $(g_a, m) \in E$ for each $a \in \mathbb{N}$ and $(m, m) \in E$. Hence, O possess best proximity point as all suppositions of the Theorem 2.2.1 are fulfilled.

Theorem 2.2.2. *Allow $G, H \neq \phi$ to be closed subsets of a complete metric space (M, d) furnished with graph $\mathfrak{G}$. Let $O : G \times G \to H$ be a mapping such that for each $g_1, g_2, g_3, u_1, u_2 \in G$ with $g_1 P g_3$, that is $(g_1, g_2) \in E$ and $(g_2, g_3) \in E$, and $d(u_1, O(g_1, g_2)) = d(G, H) = d(u_2, O(g_2, g_3))$, we have:*

$$d(g_3, u_2) \leq \gamma \max\{d(g_1, g_2), d(g_2, u_1)\} \tag{2.2.7}$$

where $\gamma \in [0, 1)$. Moreover, consider the below assumptions:

(i) G_0 is non-empty,

(ii) there exist $g_0, g_1, g_2 \in G$ satisfying $d(g_2, O(g_0, g_1)) = d(G, H)$ and $g_0 P g_2$,

(iii) H is approximatively compact with respect to G,

(iv) $O(G \times G_0) \subseteq H_0$,

(v) O is path admissible,

(vi) if $\{m_a\}$ is a sequence in M satisfying $m_a P m_{a+2}$ for each $a \in \mathbb{N}$ and $m_a \to m$ as $a \to \infty$, then $(m_a, m) \in E$ for each $a \in \mathbb{N}$ and $(m, m) \in E$.

Then, there exist $g^* \in G$ satisfying $d(g^*, O(g^*, g^*)) = d(G, H)$ that is, O possess best proximity point.

Proof. By proceeding on steps similar to that of theorem 2.2.1, a sequence $\{g_a : a \in \mathbb{N} \setminus \{1\}\}$ can be constructed in G_0 satisfying:

$$d(g_{a+1}, O(g_{a-1}, g_a)) = d(G, H) \text{ for each } a \in \mathbb{N}$$

and:

$$g_{a-1}Pg_{a+1}, \text{ that is } (g_{a-1}, g_a), (g_a, g_{a+1}) \in E \text{ for each } a \in \mathbb{N}.$$

From 2.2.7, we have:

$$d(g_a, g_{a+1}) \leq \gamma \max\{d(g_{a-2}, g_{a-1}), d(g_{a-1}, g_a)\} \text{ for each } a = 2, 3, 4, ... \tag{2.2.8}$$

Since, inequality 2.2.8 looks same as 2.2.3, therefore using the same arguments as of the proof of theorem 2.2.1, we reach at the conclusion that $\{g_a\}$ is a Cauchy sequence in G and $g_a \to g^*$ where $g^* \in G_0$. As $O(g_a, g^*) \in H_0$, from hypothesis (iv) there is some $u \in G$ such that $d(u, O(g_a, g^*)) = d(G, H)$. By hypothesis (vi), $(g_a, g^*) \in E$ for all $a \in \mathbb{N}$. Thus, we have $g_{a-1}Pg^*$, that is $(g_{a-1}, g_a), (g_a, g^*) \in E$, for all $a \in \mathbb{N}$. Hence, from 2.2.7:

$$\begin{aligned} d(g^*, u) &\leq \gamma \max\{d(g_{a-1}, g_a), d(g_a, g_{a+1})\} \text{ for all } a \in \mathbb{N}. \\ \lim_{a\to\infty} d(g^*, u) &\leq \gamma \max \lim_{a\to\infty} \{d(g_{a-1}, g_a), d(g_a, g_{a+1})\} \text{ for all } a \in \mathbb{N} \\ d(g^*, u) &\leq \gamma \max\{d(g^*, g^*), d(g^*, g^*)\} \text{ for all } a \in \mathbb{N}. \end{aligned}$$

Hence, $d(g^*, u) = 0$, implies that $u = g^*$. Further, note that $O(g^*, g^*) \in H_0$, so there is some $s \in G$ such that $d(s, O(g^*, g^*)) = d(G, H)$. Hypothesis (vi) implies that $(g^*, g^*) \in E$. Hence, $d(g^*, O(g_a, g^*)) = d(G, H), d(s, O(g^*, g^*)) = d(G, H)$, and g_aPg^*, that is $(g_a, g^*) \in E$ and $(g^*, g^*) \in E$. Thus from 2.2.7 we come by:

$$\begin{aligned} d(g^*, s) &\leq \gamma \max\{d(g_a, g^*), d(g^*, g^*)\} \text{for each} a \in \mathbb{N}. \\ \lim_{a\to\infty} d(g^*, s) &\leq \gamma \max \lim_{a\to\infty} \{d(g_a, g^*), d(g^*, g^*)\} \text{for each} a \in \mathbb{N}. \\ d(g^*, s) &\leq \gamma \max\{d(g^*, g^*), d(g^*, g^*)\} \text{for each} a \in \mathbb{N}. \end{aligned}$$

Hence, $d(g^*, s) = 0$, implies that, $s = g^*$. Thus $d(g^*, O(g^*, g^*) = d(G, H)$ □

Example 2.2.2. Let $M = \mathbb{R} \times \mathbb{R}$ be furnished with a metric $d((m_1, m_2), (n_1, n_2)) = |m_1 - n_1| + |m_2 - n_2|$ for each $m, n \in M$ and a graph $\mathfrak{G}$ be defined as $V = M$ and $E = \{((m_1, m_2), (n_1, n_2)) : m_1, m_2, n_1, n_2 \in [0, 1]\} \cup \{(m, m) : m \in M\}$. Take $G = \{(0, m) : m \in [-2, 2]\}$ and $H = \{(1, m) : m \in [-2, 2]\}$. Define $O : G \times G \to H$ as:

$$O((0,m),(0,n)) = (1,n) \text{ for each } (0,m),(0,n) \in G.$$

Then, for each $\overline{g_1} = (0,g_1), \overline{g_2} = (0,g_2), \overline{g_3} = (0,g_3), \overline{u_1} = (0,u_1) = (0,g_2), \overline{u_2} = (0,u_2) = (0,g_3) \in G$ with $\overline{g_1}P\overline{g_3}$ and $d(\overline{u_1}, O(\overline{g_1}, \overline{g_2})) = d(G,H) = d(\overline{u_2}, O(\overline{g_2}, \overline{g_3}))$, we have:

$$d(\overline{g_3}, \overline{u_2}) = 0 \leq \gamma \max\{d(\overline{g_1}, \overline{g_2}), d(\overline{g_2}, \overline{u_1})\}$$

where $\gamma = \frac{1}{2}$. All other suppositions of Theorem 2.2.2 are obvious. Thus O possess a best proximity point.

Remark 2.2.1. By using $\overline{g_1} = (0, \frac{5}{8}), \overline{g_2} = (0, \frac{1}{2})$ and $\overline{g_3} = (0,0)$ in inequality 2.2.2, it can be verified that the above example does not ensures the presence of best point under the hypotheses of theorem 2.2.1

Theorem 2.2.3. *Allow G, H to be nonempty closed subsets of a complete metric space (M,d) endowed with graph $\mathfrak{G}$. Let $O : G \times G \to H$ be a mapping such that for each $g_1, g_2, g_3, u_1, u_2 \in G$ with $g_1 P g_3$, that is $(g_1, g_2) \in E$ and $(g_2, g_3) \in E$, and $d(u_1, O(g_1, g_2)) = d(G,H) = d(u_2, O(g_2, g_3))$, we have:*

$$d(O(g_2, u_1), O(g_3, u_2)) \leq \gamma d(O(g_1, g_2), O(g_2, g_3)) \tag{2.2.9}$$

where $\gamma \in [0,1)$. Moreover, consider the below assumptions:

(i) G_0 is non-empty,

(ii) there exist $g_0, g_1, g_2 \in G$ satisfying $d(g_2, O(g_0, g_1)) = d(G,H)$ and $g_0 P g_2$,

(iii) G is approximatively compact concerning H,

(iv) $O(G \times G_0) \subseteq H_0$,

(v) O is path admissible,

(vi) if $\{g_a\}$ and $\{\overline{g_a}\}$ are sequences in M with $g_a \to g$ and $\overline{g_a} \to \overline{g}$, then $O(g_a, \overline{g_a}) \to O(g, \overline{g})$.

Then, there exist $g^* \in G$ satisfying $d(g^*, O(g^*, g^*)) = d(G,H)$ that is O possess best proximity point.

Proof. By proceeding on steps similar to that of theorem 2.2.1, a sequence $\{g_a : a \in \mathbb{N} \setminus \{1\}\}$ can be constructed in G_0 satisfying:

$$d(g_{a+1}, O(g_{a-1}, g_a)) = d(G, H) \text{ for each } a \in \mathbb{N}$$

and:

$$g_{a-1} P g_{a+1}, \text{ that is } (g_{a-1}, g_a), (g_a, g_{a+1}) \in E \text{ for each } a \in \mathbb{N}.$$

From 2.2.9, we have:

$$d(O(g_{a-1}, g_a), O(g_a, g_{a+1})) \leq \gamma d(O(g_{a-2}, g_{a-1}), O(g_{a-1}, g_a)) \text{ for each } a = 2, 3, 4, ...$$

Inductively, we get:

$$d(O(g_{a-1}, g_a), O(g_a, g_{a+1})) \leq \gamma^{a-1} d(O(g_0, g_1), O(g_1, g_2)) \text{ for each } a = 2, 3, 4, ... \tag{2.2.10}$$

By triangular inequality, for each $b, c \in \mathbb{N}$, we have:

$$d(O(g_b, g_{b+1}), O(g_{b+c}, g_{b+c+1})) \leq \sum_{i=b}^{b+c-1} d(O(g_i, g_{i+1}), O(g_{i+1}, g_{i+2}))$$

Using inequality 2.2.10 in the above inequality;

$$d(O(g_b, g_{b+1}), O(g_{b+c}, g_{b+c+1})) \leq d(O(g_0, g_1), O(g_1, g_2)) \sum_{i=b}^{b+c-1} \gamma^i$$

which shows that $\{O(g_{a-1}, g_a)\}$ is a Cauchy sequence in H. Hence, $O(g_{a-1}, g_a) \to h^*$ for some $h^* \in H$ because H is closed subset of complete metric space M. Further, we have:

$$\begin{aligned} d(h^*, G) &\leq d(h^*, g_{a+1}) && (2.2.11) \\ &\leq d(h^*, O(g_{a-1}, g_a)) + d(O(g_{a-1}, g_a), g_{a+1}) && (2.2.12) \\ &= d(h^*, O(g_{a-1}, g_a)) + d(G, H) \\ &\leq d(h^*, O(g_{a-1}, g_a)) + d(h^*, G) \end{aligned}$$

Therefore, $d(h^*, g_{a+1}) \to d(h^*, G)$ as $a \to \infty$. Since G is approximatively compact concerning H, a subsequence $\{g_{a_k}\}$ of the sequence $\{g_a\}$ converges to a point g^* in G. Hence,

$$d(g^*, O(g^*, g^*)) = \lim_{k \to \infty} d(g_{a_{k+1}}, O(g_{a_k - 1}, g_{a_k})) = d(G, H),$$

which completes the proof. $\square$

Example 2.2.3. Let $M = \mathbb{R} \times \mathbb{R}$ be furnished with a metric $d((m_1, m_2), (n_1, n_2)) = |m_1 - n_1| + |m_2 - n_2|$ for each $m, n \in M$ and a graph $\mathfrak{G}$ be defined as $V = M$ and $E = \{((m_1, m_2), (n_1, n_2)) : m_1, m_2, n_1, n_2 \in [0, 1]\} \cup \{(m, m) : m \in M\}$. Take $G = \{(0, m) : m \in [-2, 2]\}$ and $H = \{(1, m) : m \in [-2, 2]\}$. Define $O : G \times G \to H$ as:

$$O((0, m), (0, n)) = (1, \tfrac{n}{2}) \text{ for each } (0, m), (0, n) \in G.$$

Then, for each $\overline{g_1} = (0, g_1), \overline{g_2} = (0, g_2), \overline{g_3} = (0, g_3), \overline{u_1} = (0, u_1) = (0, \frac{g_2}{2}), \overline{u_2} = (0, u_2) = (0, \frac{g_3}{2}) \in G$ with $d(\overline{u_1}, O(\overline{g_1}, \overline{g_2})) = d(G, H) = d(\overline{u_2}, O(\overline{g_2}, \overline{g_3}))$, we have:

$$\begin{aligned} d(O(\overline{g_2}, \overline{u_1}), O(\overline{g_3}, \overline{u_2})) &= d((1, \frac{g_2}{4}), (1, \frac{g_3}{4})) \\ &= \frac{1}{4}|g_2 - g_3| \\ &= \frac{1}{2} d((1, \frac{g_2}{2}), (1, \frac{g_3}{2})) \\ &= \gamma d(O(g_1, g_2), O(g_2, g_3)) \end{aligned}$$

where $\gamma = \frac{1}{2}$. All other suppositions of Theorem 2.2.3 can easily be verified. Thus O possess a best proximity point.

Theorem 2.2.4. *Allow G, H to be nonempty closed subsets of a complete metric space (M, d) furnished with graph $\mathfrak{G}$. Let $O : G \times G \to H$ be a mapping such that for each $g_1, g_2, g_3, u_1, u_2 \in G$ with $g_1 P g_3$, that is $(g_1, g_2) \in E$ and $(g_2, g_3) \in E$, and $d(u_1, O(g_1, g_2)) = d(G, H) = d(u_2, O(g_2, g_3))$, we have:*

$$d(O(g_2, u_1), O(g_3, u_2)) \leq \gamma \max\{d(O(g_1, g_2), O(g_2, g_3)), d(O(g_2, g_3), O(u_1, u_2))\} \tag{2.2.13}$$

where $\gamma \in [0, 1)$. Also, consider the below assumptions:

(i) G_0 is non-empty,

(ii) there exist $g_0, g_1, g_2 \in G$ satisfying $d(g_2, O(g_0, g_1)) = d(G, H)$ and $g_0 P g_2$,

(iii) G is approximatively compact concerning H,

(iv) $O(G \times G_0) \subseteq H_0$,

(v) O is path admissible,

(vi) if $\{g_a\}$ and $\{\overline{g_a}\}$ are sequences in M with $g_a \to g$ and $\overline{g_a} \to \overline{g}$, then $O(g_a, \overline{g_a}) \to O(g, \overline{g})$.

Then, there exist $g^* \in G$ satisfying $d(g^*, O(g^*, g^*)) = d(G, H)$ that is O possess best proximity point.

Proof. It is explained in theorem 2.2.1 that a sequence $\{g_a : a \in \mathbb{N} \setminus \{1\}\}$ in G_0 can be constructed which satisfies:

$$d(g_{a+1}, O(g_{a-1}, g_a)) = d(G, H) \text{ for each } a \in \mathbb{N}$$

and:

$$g_{a-1} P g_{a+1}, \text{ that is } (g_{a-1}, g_a), (g_a, g_{a+1}) \in E \text{ for each } a \in \mathbb{N}.$$

From 2.2.13, we have:

$$\begin{aligned} d(O(g_{a-1}, g_a), O(g_a, g_{a+1})) &\leq \gamma \max\{d(O(g_{a-1}, g_{a-1}), O(g_{a-1}, g_a)) \\ &\quad\ d(O(g_{a-1}, g_a), O(g_a, g_{a+1}))\} \\ &= \gamma d(O(g_{a-1}, g_{a-1}), O(g_{a-1}, g_a)) \text{ for each } a = 2, 3, 4, ... \end{aligned}$$

else we have an inconsistency. Iteratively, we get:

$$d(O(g_{a-1}, g_a), O(g_a, g_{a+1})) \leq \gamma^{a-1} d(O(g_0, g_1), O(g_1, g_2)) \text{ for each } a = 2, 3, 4, ...$$

The proof can be completed by following the same process as of theorem 2.2.3. $\square$

Theorem 2.2.5. *Allow G, H to be nonempty closed subsets of a complete metric space (M, d) endowed with graph $\mathfrak{G}$. Let $O : G \times G \to H$ be a mapping such that for each $g_1, g_2, g_3, u_1, u_2 \in G$ with $g_1 P g_3$, that is $(g_1, g_2) \in E$ and $(g_2, g_3) \in E$, and $d(u_1, O(g_1, g_2)) = d(G, H) = d(u_2, O(g_2, g_3))$, we have:*

$$d(O(g_2, g_3), O(u_1, u_2)) \leq \gamma \max\{d(O(g_1, g_2), O(g_2, g_3)), d(O(g_2, u_1), O(g_3, u_2))\} \tag{2.2.14}$$

where $\gamma \in [0, 1)$. Also, consider the below assumptions:

(i) O is path admissible,

(ii) there exist $g_0, g_1, g_2 \in G$ satisfying $d(g_2, O(g_0, g_1)) = d(G, H)$ and $g_0 P g_2$,

(iii) G is approximatively compact concerning H,

(iv) $O(G \times G_0) \subseteq H_0$,

(v) G_0 is non-empty,

(vi) if $\{g_a\}$ and $\{\overline{g_a}\}$ are sequences in M with $g_a \to g$ and $\overline{g_a} \to \overline{g}$, then $O(g_a, \overline{g_a}) \to O(g, \overline{g})$.

Then, there exist $g^* \in G$ satisfying $d(g^*, O(g^*, g^*)) = d(G, H)$ that is O possess best proximity point.

Proof. This theorem can be demonstrated likewise to the proof of Theorem 2.2.4. □

2.2.1 Best Proximity Point Theorems of extended Pre$\check{s}$i$\acute{c}$ Type proximal contractions

This section contains the augmentations, for the operators from G^k into H, where $k \in \mathbb{N}$, of previously mentioned theorems .

Theorem 2.2.6. *Allow G, H to be nonempty closed subsets of a complete metric space (M, d) endowed with graph $\mathfrak{G}$. Let $O : G^k \to H$ be a mapping such that for each $g_1, g_2, g_3, ..., g_k, g_{k+1}, u_1, u_2 \in G$ with $g_1 P g_{k+1}$, that is $(g_1, g_2), (g_2, g_3), ..., (g_k, g_{k+1}) \in E$, and $d(u_1, O(g_1, g_2, ..., g_k)) = d(G, H) = d(u_2, O(g_2, g_3, ..., g_{k+1}))$, satisfies one of the below mentioned inequalities:*

$$d(u_1, u_2) \leq \gamma \max\{d(g_i, g_{i+1}) : 1 \leq i \leq k\}$$

$$d(g_{k+1}, u_2) \leq \gamma \max\{d(g_i, g_{i+1}) : 1 \leq i \leq k-1, d(g_k, u_1)\}$$

where $\gamma \in [0, 1)$. Also, consider the below assumptions:

(i) G_0 is non-empty,

(ii) there exist $g_0, g_1, g_2, ..., g_k \in G$ satisfying $d(g_k, O(g_0, g_1, ..., g_{k-1})) = d(G, H)$ and $g_0 P g_k$,

(iii) H is approximatively compact with respect to G,

(iv) $O(G^{k-1} \times G_0) \subseteq H_0$,

(v) O is path admissible,

(vi) if $\{g_a\}$ is a sequence in M such that $g_a P g_{a+k}$ for each $a \in \mathbb{N}$ and $g_a \to g$ as $a \to \infty$, then $(g_a, g) \in E$ for each $a \in \mathbb{N}$ and $(g, g) \in E$.

Then there exist $g^* \in G$ satisfying $d(g^*, O(g^*, g^*, g^*, ..., g^*)) = d(G, H)$ that is O possess best proximity point.

Proof. Following the same procedure as of theorems 2.2.1 and 2.2.2, we can demonstrate this theorem. □

Theorem 2.2.7. *Allow G, H to be nonempty closed subsets of a complete metric space (M, d) endowed with graph $\mathfrak{G}$. Let $O : G^k \to H$ be a mapping such that for each $g_1, g_2, g_3, ..., g_k, g_{k+1}, u_1, u_2 \in G$ with $g_1 P g_{k+1}$, that is $(g_1, g_2), (g_2, g_3), ..., (g_k, g_{k+1}) \in E$, and $d(u_1, O(g_1, g_2, ..., g_k)) = d(G, H) = d(u_2, O(g_2, g_3, ..., g_{k+1}))$ satisfies one of the below mentioned inequalities:*

$$d(O(g_2, ..., g_k, u_1), O(g_3, ...g_{k+1}, u_2)) \leq \gamma d(O(g_1, g_2, ..., g_k), O(g_2, g_3, ..., g_{k+1}));$$

$$\begin{aligned} d(O(g_2, ..., g_k, u_1), O(g_3, ..., g_{k+1}, u_2)) \leq{}& \gamma \max\{d(O(g_1, g_2, ..., g_k), O(g_2, g_3, ..., g_{k+1})), \\ & d(O(g_2, g_3, ..., g_{k+1}), O(g_4, g_5, ..., g_{k+1}, u_1, u_2))\}; \end{aligned}$$

$$\begin{aligned} d(O(g_2, g_3, ..., g_{k+1}), O(g_4, g_5, ..., g_{k+1}, u_1, u_2)) \leq{}& \gamma \max\{d(O(g_1, g_2, ..., g_k), O(g_2, g_3, ..., g_{k+1})), \\ & d(O(g_2, ..., g_k, u_1), O(g_3, ..., g_{k+1}, u_2))\}, \end{aligned}$$

where $\gamma \in [0, 1)$. Also, consider the below assumptions:

(i) G_0 is non-empty,

(ii) there exist $g_0, g_1, g_2 \in G$ satisfying $d(g_2, O(g_0, g_1)) = d(G, H)$ and $g_0 P g_2$,

(iii) G is approximatively compact concerning H,

(iv) $O(G^{k-1} \times G_0) \subseteq H_0$,

(v) O *is continuous with respect to each coordinate,*

(vi) O *is path admissible.*

Then, there exist $g^* \in G$ *satisfying* $d(g^*, O(g^*, g^*)) = d(G, H)$ *that is* O *possess best proximity point.*

Proof. By proceeding on similar steps as of theorems 2.2.3 and 2.2.4, we can demonstrate this theorem. $\square$

Remark 2.2.2. The map $O : G^k \to H$ is path admissible if for each $g_1, g_2, g_3, ..., g_k$, $g_{k+1}, u_1, u_2 \in G$ with $g_1 P g_{k+1}$ that is $(g_1, g_2), (g_2, g_3), ..., (g_k, g_{k+1}) \in E$ and $d(u_1, O(g_1, g_2, ..., g_k)) = d(G, H) = d(u_2, O(g_2, g_3, ..., g_{k+1}))$, we have $(u_1, u_2) \in E$.

2.2.2 Consequences

Considering $G = H = M$ in Theorems 2.2.6 and 2.2.7, following fixed point theorems are obtained for the operator $O : M^k \to M$.

Theorem 2.2.8. *Consider a complete metric space* (M, d) *furnished with graph* $\mathfrak{G}$*. Let* $O : M^k \to M$ *be a mapping such that for each* $r_1, r_2, r_3, ..., r_k, r_{k+1} \in M$ *with* $r_1 P r_{k+1}$*, that is* $(r_1, r_2), (r_2, r_3), ..., (r_k, r_{k+1}) \in E$*,satisfies one of the below mentioned inequalities:*

$$d(O(r_1, r_2, ..., r_k), O(r_2, r_3, ..., r_{k+1})) \leq \gamma \max\{d(r_i, r_{i+1}) : 1 \leq i \leq k\}$$

$$d(r_{k+1}, O(r_2, r_3, ..., r_{k+1})) \leq \gamma \max\{d(r_i, r_{i+1}) : 1 \leq i \leq k-1, d(r_k, O(r_1, r_2, ..., r_k))\}$$

where $\gamma \in [0, 1)$*. Also, consider the below conditions:*

(i) $r_1 P r_{k+1}$ *that is* $(r_1, r_2), (r_2, r_3), ..., (r_k, r_{k+1}) \in E$*, then we have;*

$$O(r_1, r_2, ..., r_k), O(r_{2,3}, ..., r_{k+1}) \in E,$$

(ii) there exist $r_0, r_1, r_2, ..., r_k \in M$ *with* $r_k = O(r_1, r_2, ..., r_{k-1})$*and* $r_0 P r_k$*,*

(iii) if $\{r_a\}$ is a sequence in M such that $r_a P r_{a+k}$ for each $a \in \mathbb{N}$ and $r_a \to r$ as $a \to \infty$, then $(r_a, r) \in E$ for each $a \in \mathbb{N}$ and $(r, r) \in E$.

Then O possess a fixed point in M, that is $r^* = O(r^*, r^*, r^*, ..., r^*)$ for some $r^* \in M$.

Theorem 2.2.9. *Consider (M, d) be a complete metric space furnished with graph $\mathfrak{G}$. Let $O : M^k \to M$ be an operator such that for each $\mu_1, \mu_2, \mu_3, ..., \mu_k, \mu_{k+1} \in M$ with $\mu_1 P \mu_{k+1}$, that is $(\mu_1, \mu_2), (\mu_2, \mu_3), ..., (\mu_k, \mu_{k+1}) \in E$, satisfies one of the below inequalities:*

$$d(O(\mu_2, ..., \mu_k, O(\mu_1, \mu_2, ..., \mu_k)), O(\mu_3, ..., \mu_{k+1}, O(\mu_2, \mu_3, ..., \mu_{k+1}))) \leq \gamma d(O(\mu_1, \mu_2, ..., \mu_k), O(\mu_2, \mu_3, ..., \mu_{k+1}));$$

$$\begin{aligned} d(O(\mu_2, ..., \mu_k, O(\mu_1, \mu_2, ..., \mu_k)), O(\mu_3, ..., \mu_{k+1}, O(\mu_2, \mu_3, ..., \mu_{k+1}))) \leq \\ \gamma \max\{d(O(\mu_1, \mu_2, ..., \mu_k), O(\mu_2, \mu_3, ..., \mu_{k+1})), d(O(\mu_2, \mu_3, ..., \mu_{k+1}), \\ O(\mu_4, \mu_5, ..., \mu_{k+1}, O(\mu_1, \mu_2, ..., \mu_k), O(\mu_2, \mu_3, ..., \mu_{k+1})))\}; \end{aligned}$$

$$\begin{aligned} d(O(\mu_2, \mu_3, ..., \mu_{k+1}), O(\mu_4, \mu_5, ..., \mu_{k+1}, O(\mu_1, \mu_2, ..., \mu_k), O(\mu_2, \mu_3, ..., \mu_{k+1}))) \leq \\ \gamma \max\{d(O(\mu_1, \mu_2, ..., \mu_k), O(\mu_2, \mu_3, ..., \mu_{k+1})), d(O(\mu_2, ..., \mu_k, O(\mu_1, \mu_2, ..., \mu_k)), \\ O(\mu_3, ..., \mu_{k+1}, O(\mu_2, \mu_3, ..., \mu_{k+1})))\}, \end{aligned}$$

where $\gamma \in [0, 1)$. Also, consider following conditions:

(i) $\mu_1 P \mu_{k+1}$ that is $(\mu_1, \mu_2), (\mu_2, \mu_3), ..., (\mu_k, \mu_{k+1}) \in E$, then we have;

$$O(\mu_1, \mu_2, ..., \mu_k), O(\mu_2, \mu_3, ..., \mu_{k+1}) \in E,$$

(ii) there exist $\mu_0, \mu_1, \mu_2, ..., \mu_k \in M$ with $\mu_k = O(\mu_1, \mu_2, ..., \mu_{k-1})$and $\mu_0 P \mu_k$,

(iii) O is continuous in each coordinate.

Then, there exist $\mu^* \in M$ satisfying $\mu^* = O(\mu^*, \mu^*, \mu^*, ..., \mu^*)$ that is, O possess a fixed point in M.

Remark 2.2.3. If $O : M^k \to M$is an operator which satisfies the Theorem 2.2.8 or Theorem 2.2.9 and $\{\mu_a\}$ is a sequence in M such that $\mu_a P \mu_b$ for each $b > a \in \mathbb{N}$ and $\mu_{a+k+1} = O(\mu_{1+a}, \mu_{2+a}, ..., \mu_{k+a})$ for each $a \in \mathbb{N}$, then $\{\mu_a\}$ converges and hence, O possess fixed point.

Let the graph $\mathfrak{G} = (V, E)$ be characterized as $V = M$ and $E = M \times M$, then Theorem 2.2.6 and Theorem2.2.7 boils down to the below corollaries, respectively.

Corollary 2.2.10. *Let $O : M^k \to M$ be an operator, where (M, d) is a complete metric space and for each $\mu_1, \mu_2, \mu_3, ..., \mu_k, \mu_{k+1} \in M$, one of the below mentioned inequalities is satisfied:*

$$d(O(\mu_1, \mu_2, ..., \mu_k), O(\mu_2, \mu_3, ..., \mu_{k+1})) \leq \gamma \max\{d(\mu_i, \mu_{i+1}) : 1 \leq i \leq k\}$$

$$d(\mu_{k+1}, O(\mu_2, \mu_3, ..., \mu_{k+1})) \leq \gamma \max\{d(g_i, g_{i+1}) : 1 \leq i \leq k-1, d(\mu_k, O(\mu_1, \mu_2, ..., \mu_k))\}$$

where $\gamma \in [0, 1)$. Then O possess a fixed point in M, that is there exist $\mu^* \in M$ satisfying $\mu^* = O(\mu^*, \mu^*, \mu^*, ..., \mu^*)$.

Corollary 2.2.11. *Let $O : M^k \to M$ be an operator which is continuous in each coordinate, where (M, d) is a complete metric space and for each $\mu_1, \mu_2, \mu_3, ..., \mu_k, \mu_{k+1} \in M$ one of the below inequalities is satisfied:*

$$d(O(\mu_2, ..., \mu_k, O(\mu_1, \mu_2, ..., \mu_k)), O(\mu_3, ...\mu_{k+1}, O(\mu_2, \mu_3, ..., \mu_{k+1}))) \leq \gamma d(O(\mu_1, \mu_2, ..., \mu_k), O(\mu_2, \mu_3, ..., \mu_{k+1}));$$

$$\begin{aligned} d(O(\mu_2, ..., \mu_k, O(\mu_1, \mu_2, ..., \mu_k)), O(\mu_3, ..., \mu_{k+1}, O(\mu_2, \mu_3, ..., \mu_{k+1}))) \\ \leq \gamma \max\{d(O(\mu_1, \mu_2, ..., \mu_k), O(\mu_2, \mu_3, ..., \mu_{k+1})), \\ d(O(\mu_2, \mu_3, ..., \mu_{k+1}), O(\mu_4, \mu_5, ..., \mu_{k+1}, O(\mu_1, \mu_2, ..., \mu_k), O(\mu_2, \mu_3, ..., \mu_{k+1})))\}; \end{aligned}$$

$$\begin{aligned} d(O(\mu_2, \mu_3, ..., \mu_{k+1}), O(\mu_4, \mu_5, ..., \mu_{k+1}, O(\mu_1, \mu_2, ..., \mu_k), O(\mu_2, \mu_3, ..., \mu_{k+1}))) \\ \leq \gamma \max\{d(O(\mu_1, \mu_2, ..., \mu_k), O(\mu_2, \mu_3, ..., \mu_{k+1})), \\ d(O(\mu_2, ..., \mu_k, O(\mu_1, \mu_2, ..., \mu_k)), O(\mu_3, ..., \mu_{k+1}, O(\mu_2, \mu_3, ..., \mu_{k+1})))\}, \end{aligned}$$

where $\gamma \in [0, 1)$. Then O possess a fixed point.

Chapter 3

Best Proximity Points in some Generalized Metric spaces

The generalisation of metric space is a novel technique to studying fixed point theory and best proximity point theory. There are many generalizations of metric space e.g. modular metric space, b-metric space etc.

Researchers addressed the presence of fixed points for mappings satisfying the proximal contraction conditions including various auxiliary functions, as research in the field of fixed point theory progressed. The theory of contractions via auxiliary functions was developed by Lim [96] when he characterized Mier-Keeler contraction [99] by a mapping using a class of functions(L-functions). Many researchers then proved fixed point theorems for contractions via auxiliary functions ([67],[71],[120], [130], [93], [59]).

This chapter comprises of two sections. We developed best proximity point theorems for generalised F-proximal contractions in modular metric spaces in the first section. In the second part, we used the class of auxiliary functions described in Definition 1.6.1 to propose non-self proximal contraction requirements and proved best proximity point theorems for contractions in gauge space setting. We also used examples to demonstrate our findings and looked into the implications of our findings for self-mappings.

3.1 Best Proximity Point Theorems in Modular Metric Spaces

Mongkolkeha [102] proved fixed point theorems for contraction mapping in modular metric space. The fixed point property in modular metric space has been characterized and examined by numerous researchers(See for example [3],[8],[31]).

Firstly we introduce some notions that we need in our results.

A *strongly regular* modular metric ω on M is weaker form of modular metric satisfying;

$$m = n \text{ if and only if } \omega(1, m, n) = 0.$$

instead of (i) of Definition 1.3.1. Let $G, H \neq \phi$ subsets of a modular metric space (M, ω) then

$$\begin{aligned} \omega(1, g, H) &= \inf \{\omega(1, g, h) : h \in H\} \\ dist(G, H) &= \inf \{\omega(1, g, h) : g \in G, h \in H\} \\ G_0 &= \{m \in G : \omega(1, m, n) = dist(G, H), \text{ for some } n \in H\} \\ H_0 &= \{n \in H : \omega(1, m, n) = dist(G, H), \text{ for some } m \in G\}. \end{aligned}$$

Definition 3.1.1. Allow G and H to be subsets of modular metric space (M, ω) which are not empty. Then H is termed *approximatively ω-compact* concerning G if each $\{v_a\}$ in H with $\omega(1, m, v_a) \to \omega(1, m, H)$ for some m in G, has a ω-convergent subsequence.

Presently we present another contraction termed generalized F-proximal contraction of type I.

Definition 3.1.2. Let $G, H \neq \phi$ subsets of a modular metric space (M, ω). A mapping $O : G \to H$ is a *generalized F-proximal contraction of type I* if there is a constant $\tau > 0$ and a function $F \in \mathfrak{F}$ satisfying for each $m_1, m_2, u_1, u_2 \in G$ $\omega(1, u_1, Om_1) = dist(G, H) = \omega(1, u_2, Om_2)$, implies

$$\tau + F(\omega(1, u_1, u_2)) \leq F(W(m_1, m_2)) \tag{3.1.1}$$

whenever $\min\{\omega(1,u_1,u_2), W(m_1,m_2)\} > 0$, where

$$W(m_1,m_2) = \varsigma_1\omega(1,m_1,m_2) + \varsigma_2\omega(1,m_1,u_1) + \varsigma_3\omega(1,m_2,u_2) + \varsigma_4[\omega(2,m_1,u_2) + \omega(1,m_2,u_1)]$$

with $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4 \geq 0$ satisfying $\varsigma_1 + \varsigma_2 + \varsigma_3 + 2\varsigma_4 = 1$ and $\varsigma_3 \neq 1$.

Theorem 3.1.1. *Let M be a set which is not empty furnished with ω as a strongly regular modular metric that fulfills the Δ_M-condition [Definition1.3.4] as well as the Fatou property [Definition1.3.5]. Allow W to be ω-complete as well as ω-bounded subset [Definition1.3.3] of M_ω. Further, G and H are nonempty ω-closed subsets of W. Also assume that G_0 is not empty and $O : G \to H$ is a generalized F-proximal contraction of type I satisfies;*

(i) H is approximatively ω-compact concerning G,

(ii) $O(G_0)$ is contained in H_0.

Then O posses a best proximity point.

Proof. Let $m_0 \in G_0$. Hypothesis (ii) yields, $Om_0 \in H_0$, thus we have $m_1 \in G_0$ such that $\omega(1, m_1, Om_0) = dist(G,H)$. Similarly for $m_1 \in G_0$ we have $Om_1 \in H_0$, thus we have $m_2 \in G_0$ such that $\omega(1, m_2, Om_1) = dist(G,H)$. Continuing this process we have $m_a, m_{a+1} \in G_0$ such that

$$\omega(1, m_{a+1}, Om_a) = dist(G,H) \text{ for each } a \in \mathbb{N}.$$

Thus, from inequality (3.1.1), we have;

$$\begin{aligned} \tau + F(\omega(1,m_a,m_{a+1})) &\leq F(\varsigma_1\omega(1,m_{a-1},m_a) + \varsigma_2\omega(1,m_{a-1},m_a) + \varsigma_3\omega(1,m_a,m_{a+1}) \\ &\quad +\varsigma_4[\omega(2,m_{a-1},m_{a+1}) + \omega(1,m_a,m_a)]) \\ &\leq F((\varsigma_1+\varsigma_2+\varsigma_4)\omega(1,m_{a-1},m_a) + (\varsigma_3+\varsigma_4)\omega(1,m_a,m_{a+1})) \\ &\quad \text{for each } a \in \mathbb{N}. \end{aligned} \tag{3.1.2}$$

By using strictly increasing property of F and above inequality, we have

$$\omega(1,m_a,m_{a+1}) < (\varsigma_1+\varsigma_2+\varsigma_4)\omega(1,m_{a-1},m_a) + (\varsigma_3+\varsigma_4)\omega(1,m_a,m_{a+1}) \text{ for each } a \in \mathbb{N}.$$

That is,

$$(1-\varsigma_3-\varsigma_4)\omega(1,m_a,m_{a+1}) < (\varsigma_1+\varsigma_2+\varsigma_4)\omega(1,m_{a-1},m_a) \text{ for each } a\in\mathbb{N}.$$

Since $\varsigma_1+\varsigma_2+\varsigma_3+2\varsigma_4=1$ and $\varsigma\neq 1$, the above inequality implies that

$$\omega(1,m_a,m_{a+1}) < \omega(1,m_{a-1},m_a) \text{ for each } a\in\mathbb{N}.$$

Thus, from inequality (3.1.2), we have;

$$\tau + F(\omega(1,m_a,m_{a+1})) \leq F(\omega(1,m_{a-1},m_a)) \text{ for each } a\in\mathbb{N}.$$

Iteratively, we get

$$F(\omega(1,m_a,m_{a+1})) \leq F(\omega(1,m_0,m_1)) - a\tau \text{ for each } a\in\mathbb{N}. \tag{3.1.3}$$

Letting $a\to\infty$ in the above inequality, we get $\lim_{a\to\infty} F(\omega(1,m_a,m_{a+1})) = -\infty$. Thus, by property (F_1), we have $\lim_{a\to\infty}\omega(1,m_a,m_{a+1}) = 0$. Let for each $a\in\mathbb{N}$ $\omega_a = \omega(1,m_a,m_{a+1})$, then (F_2) yields that there is some $k\in(0,1)$ such that

$$\lim_{a\to\infty}\omega_a^k F(\omega_a) = 0.$$

The inequality (3.1.3) yields

$$\omega_a^k F(\omega_a) - \omega_a^k F(\omega_1) \leq -\omega_a^k a\tau \leq 0 \text{ for each } a\in\mathbb{N}. \tag{3.1.4}$$

Letting $a\to\infty$ in (3.1.4), we get

$$\lim_{a\to\infty} a\omega_a^k = 0.$$

That is, there is some $a_1\in\mathbb{N}$ such that $a\omega_a^k\leq 1$ for each $a\geq a_1$. Thus, we have;

$$\omega_a \leq \frac{1}{a^{1/k}}, \quad \text{for each } a\geq a_1. \tag{3.1.5}$$

Take arbitrary $a,b\in\mathbb{N}$ with $b>a>a_1$. By using the triangular inequality and (3.1.5), we have

$$\begin{aligned}\omega(b-a,m_a,m_b) &\leq \omega(1,m_a,m_{a+1}) + \omega(1,m_{a+1},m_{a+2}) + \cdots + \omega(1,m_{b-1},m_b)\\ &= \sum_{i=a}^{b-1}\omega_i \leq \sum_{i=a}^{\infty}\omega_i \leq \sum_{i=a}^{\infty}\frac{1}{i^{1/k}}.\end{aligned}$$

Since $\sum_{i=1}^{\infty}\frac{1}{i^{1/k}}$ is convergent series. Thus, $\lim_{a,b\to\infty}\omega(b-a,m_a,m_b)=0$. Because of Δ_M-condition, this implies that $\lim_{a,b\to\infty}\omega(1,m_a,m_b)=0$. Hence $\{m_a\}$ is ω-Cauchy sequence in G. Since W is ω-complete and G is ω-closed in W, there exists m^* in G such that $\{m_a\}$ is ω-convergent to m^*. That is, $\lim_{a\to\infty}\omega(1,m_a,m^*)=0$. Also, we have

$$\begin{aligned}
\omega(1,m^*,H) &\leq \omega(1,m^*,Om_a)\\
&\leq \omega(\frac{1}{2},m^*,m_{a+1})+\omega(\frac{1}{2},m_{a+1},Om_a)\\
&= \omega(\frac{1}{2},m^*,m_{a+1})+dist(G,H)\\
&\leq \omega(\frac{1}{2},m^*,m_{a+1})+\omega(1,m^*,H)
\end{aligned}$$

In the aforementioned inequality if we set $a\to\infty$, we get $\omega(1,m^*,Om_a)\to\omega(1,m^*,H)$. As H is approximatively ω-compact concerning G, we have a subsequence $\{Om_{a_k}\}$ of $\{Om_a\}$ which ω-converges to v^*. This implies that

$$\omega(1,m^*,v^*)\leq\lim_{k\to\infty}\omega(1,m_{a_k+1},Om_{a_k})=dist(G,H).$$

Thus we have $\omega(1,m^*,v^*)=dist(G,H)$. Since $m^*\in G_0$, we have $Om^*\in H_0$, this infers that there is $w^*\in G_0$ with $\omega(1,w^*,Om^*)=dist(G,H)$. Also, we have $\omega(1,m_{a+1},Om_a)=dist(G,H)$. We claim that $\omega(1,m^*,w^*)=0$. Contrarily suppose $\omega(1,m^*,w^*)\neq 0$. Then, (3.1.1) yields

$$\begin{aligned}
\omega(1,m_{a+1},w^*) &< \varsigma_1\omega(1,m_a,m^*)+\varsigma_2\omega(1,m_a,m_{a+1})+\varsigma_3\omega(1,m^*,w^*)\\
&\quad+\varsigma_4[\omega(2,m_a,w^*)+\omega(1,m^*,m_{a+1})]\\
&\leq \varsigma_1\omega(1,m_a,m^*)+\varsigma_2\omega(1,m_a,m_{a+1})+\varsigma_3\omega(1,m^*,w^*)\\
&\quad+\varsigma_4[\omega(1,m_a,m^*)+\omega(1,m^*,w^*)+\omega(1,m^*,m_{a+1})].
\end{aligned}$$

Letting $a\to\infty$ in the above expression, we come by;

$$\omega(1,m^*,w^*)\leq(\varsigma_3+\varsigma_4)\omega(1,m^*,w^*)<\omega(1,m^*,w^*).$$

which contradicts our assumption. Hence, $\omega(1,m^*,w^*)=0$. That is $m^*=w^*$. As a result, we have $\omega(1,m^*,Om^*)=dist(G,H)$. □

Example 3.1.1. Let $M = \mathbb{R} \times \mathbb{R}$ furnished with a strongly regular modular metric $\omega(\lambda, m, n) = \frac{1}{\lambda}(|n_1 - m_1| + |n_2 - m_2|)$ for all $m = (m_1, m_2)$ and $n = (n_1, n_2) \in M$. Very simple calculation shows that $M_\omega = M$, so, δ_M-condition and Fatou property are satisfied. Consider $W = [0, 4] \times [0, 4] \subset M_\omega$. Then W is ω-closed and ω-bounded. Let $G = \{(0, m) : 0 \leq m \leq 1\}$ and $H = \{(1, m) : 0 \leq m \leq 1\}$. Then, we have $dist(G, H) = 1$ and $G_0 = G$, $H_0 = H$. Clearly H is approximatively compact concerning G and G and H are ω-closed subsets of W. Define $O : G \to H$ as

$$O(0, m) = \begin{cases} (1, \frac{m}{4}) \text{ if } 0 \leq m < 1 \\ (1, 0) \text{ if } m = 1. \end{cases}$$

It is simple to verify that, with respect to $F(m) = \ln m$, O is generalized F-proximal contraction of type I with $\tau = \frac{1}{2}$, $\varsigma_1 = \varsigma_2 = \varsigma_3 = \frac{1}{3}$ and $\varsigma_4 = 0$. Also $O(G_0) \subseteq H_0$. Thus, Theorem 3.1.1 is satisfied. Hence, O has best proximity point.

Presently, we present generalized F-proximal contraction of type II.

Definition 3.1.3. Allow G and H to be nonempty subsets of a modular metric space (M, ω). If there is a constant $\tau > 0$ and a function $F \in \mathfrak{F}$, then a mapping $O : G \to H$ is a *generalized F-proximal contraction of type II* if it satisfies for each $m_1, m_2, u_1, u_2 \in G$ with $\omega(1, u_1, Om_1) = dist(G, H) = \omega(1, u_2, Om_2)$, implies

$$\tau + F(\omega(1, Ou_1, Ou_2)) \leq F(W(Om_1, Om_2)) \tag{3.1.6}$$

whenever $\min\{\omega(1, Ou_1, Ou_2), W(Om_1, Om_2)\} > 0$, where

$$\begin{aligned} W(Om_1, Om_2) &= \varsigma_1\omega(1, Om_1, Om_2) + \varsigma_2\omega(1, Om_1, Ou_1) + \varsigma_3\omega(1, Om_2, Ou_2) \\ &\quad + \varsigma_4[\omega(2, Om_1, Ou_2) + \omega(1, Om_2, Ou_1)] \end{aligned}$$

with $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4 \geq 0$ satisfying $\varsigma_1 + \varsigma_2 + \varsigma_3 + 2\varsigma_4 = 1$ and $\varsigma_3 \neq 1$.

Theorem 3.1.2. *Allow M to be a set which is not empty furnished with ω as a strongly regular modular metric that fulfills Fatou property as well as Δ_M-condition. Allow W to be ω-bounded as well as ω-complete subset of M_ω. Further, G and H are nonempty ω-closed subsets of W. Also suppose that G_0 is not empty and $O : G \to H$ is a generalized F-proximal contraction of type II satisfies;*

(i) G is approximatively ω-compact concerning H,

(ii) O is continuous,

(iii) $O(G_0)$ is contained in H_0.

Then O posses a best proximity point.

Proof. Let $m_0 \in G_0$. Hypothesis (iii) allows us to construct a sequence $\{m_a\}$ in G_0 such that

$$\omega(1, m_{a+1}, Om_a) = dist(G, H) \text{ for each } a \in \mathbb{N}.$$

Thus, from (3.1.6), for each $a \in \mathbb{N}$ we have

$$\begin{aligned} \tau + F(\omega(1, Om_a, Om_{a+1})) &\leq F(\varsigma_1 \omega(1, Om_{a-1}, Om_a) + \varsigma_2 \omega(1, Om_{a-1}, Om_a) \\ &\quad + \varsigma_3 \omega(1, Om_a, Om_{a+1}) \\ &\quad + \varsigma_4 [\omega(2, Om_{a-1}, Om_{a+1}) + \omega(1, Om_a, Om_a)]) \\ &\leq F((\varsigma_1 + \varsigma_2 + \varsigma_4)\omega(1, Om_{a-1}, Om_a) \qquad (3.1.7) \\ &\quad + (\varsigma_3 + \varsigma_4)\omega(1, Om_a, Om_{na+1})). \end{aligned} \tag{3.1.8}$$

From above inequality and (F_3) of definition 1.6.2, we have

$$\omega(1, Om_a, Om_{a+1}) < (\varsigma_1 + \varsigma_2 + \varsigma_4)\omega(1, Om_{a-1}, Om_a) + (\varsigma_3 + \varsigma_4)\omega(1, Om_a, Om_{a+1}) \text{ for each } a \in \mathbb{N}.$$

That is,

$$(1 - \varsigma_3 - \varsigma_4)\omega(1, Om_a, Om_{a+1}) < (\varsigma_1 + \varsigma_2 + \varsigma_4)\omega(1, Om_{a-1}, Om_a) \text{ for each } a \in \mathbb{N}.$$

Since $\varsigma_1 + \varsigma_2 + \varsigma_3 + 2\varsigma_4 = 1$ and $\varsigma_3 \neq 1$, the above inequality implies that

$$\omega(1, Om_a, Om_{a+1}) < \omega(1, Om_{a-1}, Om_a) \text{ for each } a \in \mathbb{N}.$$

Thus, from (3.1.7), we have

$$\tau + F(\omega(1, Om_a, Om_{a+1})) \leq F(\omega(1, Om_{a-1}, Om_a)) \text{ for each } a \in \mathbb{N}.$$

Iteratively, we get

$$F(\omega(1, Om_a, Om_{a+1})) \le F(\omega(1, Om_0, Om_1)) - a\tau \text{ for each } a \in \mathbb{N}. \tag{3.1.9}$$

Letting $a \to \infty$ in the above expression, we come by $\lim_{a\to\infty} F(\omega(1, Om_a, Om_{a+1})) = -\infty$. Thus, by property (F_1), we have $\lim_{a\to\infty} \omega(1, Om_a, Om_{a+1}) = 0$. Let, for each $a \in \mathbb{N}$, $\omega_a = \omega(1, Om_a, Om_{a+1})$. Then (F_2) yields that there is $k \in (0, 1)$ satisfying

$$\lim_{a\to\infty} \omega_a^k F(\omega_a) = 0.$$

The inequality (3.1.9) yields,

$$\omega_a^k F(\omega_a) - \omega_a^k F(\omega_1) \le -\omega_a^k a\tau \le 0 \text{ for each } a \in \mathbb{N}. \tag{3.1.10}$$

Letting $a \to \infty$ in (3.1.10), we get

$$\lim_{a\to\infty} a\omega_a^k = 0.$$

That is, there is some $a_1 \in \mathbb{N}$ such that $a\omega_a^k \le 1$ for each $a \ge a_1$. Thus, we have

$$\omega_a \le \frac{1}{a^{1/k}}, \quad \text{for each } a \ge a_1. \tag{3.1.11}$$

Take arbitrary $a, b \in \mathbb{N}$ with $b > a > a_1$. By using the triangular inequality and (3.1.11), we have

$$\begin{aligned} \omega(b-a, Om_a, Om_a) &\le \omega(1, Om_a, Om_{a+1}) + \omega(1, Om_{a+1}, Om_{a+2}) + \cdots + \omega(1, Om_{b-1}, Om_b) \\ &= \sum_{i=a}^{b-1} \omega_i \le \sum_{i=a}^{\infty} \omega_i \le \sum_{i=a}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series. Thus, $\lim_{a,b\to\infty} \omega(b-a, Om_a, Om_b) = 0$. Due to Δ_M-condition, this implies that $\lim_{a,b\to\infty} \omega(1, Om_a, Om_b) = 0$. Hence $\{Om_a\}$ is ω-Cauchy sequence in H. Since W is ω-complete and H is ω-closed in W, there exists n^* in H with $\{Om_a\}$ ω-convergent to n^*. That is, $\lim_{a\to\infty} \omega(1, Om_a, n^*) = 0$. Also, we have

$$\begin{aligned} \omega(1, n^*, G) &\le \omega(1, n^*, m_{a+1}) \\ &\le \omega(\frac{1}{2}, n^*, Om_a) + \omega(\frac{1}{2}, Om_a, m_{a+1}) \\ &= \omega(\frac{1}{2}, n^*, Om_a) + dist(G, H) \\ &\le \omega(\frac{1}{2}, n^*, Om_a) + \omega(1, n^*, G). \end{aligned}$$

Letting $a \to \infty$ in the above expression, we come by $\omega(1, n^*, m_a) \to \omega(1, n^*, G)$. As G is approximatively ω-compact concerning H, we have a subsequence $\{m_{a_k}\}$ of $\{m_a\}$ which ω-converges to m^*. This implies that

$$\omega(1, m^*, n^*) \leq \lim_{k\to\infty} \omega(1, m_{a_k+1}, Om_{a_k}) = dist(G, H).$$

Thus we have $\omega(1, m^*, n^*) = dist(G, H)$. As $m_{a_k} \to m^*$ and O is continuous, then we have $Om_{a_k} \to Om^*$. Since the limit point is unique so we have $n^* = Om^*$. As a result, $\omega(1, m^*, Om^*) = dist(G, H)$

□

Example 3.1.2. Let $M = \mathbb{R} \times \mathbb{R}$ furnished with a strongly regular modular metric $\omega(\lambda, m, n) = \frac{1}{\lambda}(|n_1 - m_1| + |n_2 - m_2|)$ for all $m = (m_1, m_2)$ and $n = (n_1, n_2) \in M$. Very simple calculation shows that $M_\omega = M$, so, δ_M-condition and Fatou property are satisfied. Consider $W = [0, 3] \times [0, 3] \subset X_\omega$. Then W is ω-closed and ω-bounded. Let $G = \{(0, m) : 0 \leq m \leq 1\}$ and $H = \{(t, m) : 1 \leq t \leq 2 \text{ and } 0 \leq m \leq 1\}$. Then, we have $dist(G, H) = 1$ and $G_0 = G$, $H_0 = \{(1, m) : 0 \leq m \leq 1\}$. Clearly G and H are ω-closed subsets of W and G is approximatively ω-compact concerning H. Define $O : G \to H$ as

$$O(0, m) = (1, \tfrac{m}{4}).$$

It can be easily checked that O is generalized F-proximal contraction of type II concerning $F(m) = \ln m$, $\tau = \frac{1}{2}$, $\varsigma_1 = 1$ and $\varsigma_2 = \varsigma_3 = \varsigma_4 = 0$. Also $O(G_0) \subseteq H_0$ and O is continuous. Thus, Theorem 3.1.2 is satisfied. Hence O has best proximity point.

3.2 Best proximity point theorems in gauge spaces

Fixed point results for generalised contractions on gauge spaces were proved by Frigon in [55, 56]. Others [13, 35, 36, 73, 94] also proved fixed point theorems on gauge spaces.

$(M, \mathfrak{T}(\mathfrak{P}))$ is a gauge space concerning the family $\mathfrak{P} = \{d_b | b \in \mathfrak{V}\}$ of pseudo metrics on M in this section. The notations that follow have the same meanings.

Definition 3.2.1. Given that $G, H \neq \phi$ subsets of a metric space M. Then

$$\begin{aligned} d_b(G,H) &= \inf\{d_b(g,h) : g \in G, h \in H\} \\ G_0 &= \{g \in G : d_b(g,h) = d_b(G,H) \text{ for each } b \in \mathfrak{V}, \text{ for some } h \in H\} \\ H_0 &= \{h \in H : d_b(g,h) = d_b(G,H) \text{ for each } b \in \mathfrak{V}, \text{ for some } g \in G\} \end{aligned}$$

The definition that follows is an expanded version of Basha and Shahzad's[22] definition 1.1.1.

Definition 3.2.2. Let $G, H \neq \phi$ subsets of M. Then H is termed *approximatively compact* concerning G if each $\{v_a\}$ in H with $d_b(m, v_a) \to d_b(m, H)$ for all $b \in \mathfrak{V}$ for some $m \in G$, has a convergent subsequence.

Following that, we give implicit generalised proximal contraction mappings of the first and second kinds, and show the best proximity point theorem in gauge space for the mappings. This section contains the results published in [16].

Definition 3.2.3. Allow G and H to be non-empty subsets of M. An *implicit type generalized proximal contraction of first kind* is a mapping $O : G \to H$ such that for each $g_1, g_2, u_1, u_2 \in G$, there exist $\phi \in \Phi_\psi$ satisfying $d_b(u_1, Og_1) = d_b(G,H) = d_b(u_2, Og_2)$ implies

$$\begin{aligned} d_b(u_1,u_2) \;\; \leq \;\; & \phi(d_b(g_1,g_2), d_b(g_1,u_1), d_b(g_2,u_2), \\ & 1/2(d_b(g_2,u_1) + d_b(g_1,u_2))) \end{aligned} \tag{3.2.1}$$

for each $b \in \mathfrak{V}$

Theorem 3.2.1. *Consider $\mathfrak{P} = \{d_b | b \in \mathfrak{V}\}$ a family of pseudometrics which is separating and $(M, \mathfrak{T}(\mathfrak{P}))$ a complete gauge space induced by $\mathfrak{P}$. Let G and H be non-empty closed subsets of M such that $G_0 \neq \phi$ and H is approximatively compact concerning G. Let $O : G \to H$ be implicit type generalized proximal contraction of first kind and $O(G_0) \subseteq H_0$. Then O possesses a best proximity point, that is there exist $m \in G$ such that $d_b(m, Om) = d_b(G,H) \forall b \in \mathfrak{V}$.*

Proof. Let $m_0 \in G_0$. Since $O(G_0) \subseteq H_0$, so $Om_0 \in H_0$, thus we have $m_1 \in G_0$ such that $d_b(m_1, Om_0) = d_b(G,H), \forall b \in \mathfrak{V}$. Similarly, for $m_1 \in G_0$ we have $Om_1 \in H_0$, thus we get $m_2 \in G_0$ such that $d_b(m_2, Om_1) = d_b(G,H), \forall b \in \mathfrak{V}$. We have, for each $a \in \mathbb{N}$, by continuing this process, $m_a, m_{a+1} \in G_0$ such that

$$d_b(m_{a+1}, Om_a) = d_b(G, H), \forall b \in \mathfrak{V}$$

Assume that $m_a \neq m_{a+1}$, otherwise m_a is a best proximity point. As a result, (3.2.1) yields

$$\begin{aligned} d_b(m_a, m_{a+1}) &\leq \phi(d_b(m_{a-1}, m_a), d_b(m_{a-1}, m_a), \\ &\quad d_b(m_a, m_{a+1}), 1/2(d_b(m_{a-1}, m_{a+1}) + d_b(m_a, m_a))) \\ &= \phi(d_b(m_{a-1}, m_a), d_b(m_{a-1}, m_a), \\ &\quad d_b(m_a, m_{a+1}), 1/2(d_b(m_{a-1}, m_{a+1}))) \\ &\leq \phi[d_b(m_{a-1}, m_a), d_b(m_{a-1}, m_a), \\ &\quad d_b(m_a, m_{a+1}), 1/2(d_b(m_{a-1}, m_a) + d_b(m_a, m_{a+1}))] \end{aligned} \tag{3.2.2}$$

We claim that $d_b(m_a, m_{a+1}) < d_b(m_{a-1}, m_a) \forall b \in \mathfrak{V}$ for each $a \in \mathbb{N}$. Suppose on contrary that $d_b(m_a, m_{a+1}) \geq d_b(m_{a-1}, m_a) \forall b \in \mathfrak{V}$ for some a. We can use non-decreasing of ϕ in (3.2.2),

$$d_b(m_a, m_{a+1}) \leq \phi(d_b(m_a, m_{a+1}), d_b(m_{a-1}, m_a), d_b(m_a, m_{a+1}), (d_b(m_a, m_{a+1}))) \tag{3.2.3}$$

for all $b \in \mathfrak{V}$. We get in (3.2.3) by using property (ii) of Φ_ψ,

$$d_b(m_a, m_{a+1}) = 0 \forall b \in \mathfrak{V}$$

which contradicts our assumption, since $m_{a+1} \neq m_a$ for each $a \in \mathbb{N} \cup \{0\}$.
Thus $d_b(m_a, m_{a+1}) < d_b(m_{a-1}, m_a) \forall b \in \mathfrak{V}$ for each $a \in \mathbb{N}$.Therefore (3.2.2) becomes

$$d_b(m_a, m_{a+1}) \leq \phi[d_b(m_{a-1}, m_a), d_b(m_{a-1}, m_a), d_b(m_a, m_{a+1}), (d_b(m_{a-1}, m_a))] \tag{3.2.4}$$

By using (3.2.4) and property (ii) of Φ_ψ from definition 1.6.1 for each $a \in \mathbb{N}$ we have

$$d_b(m_a, m_{a+1}) \leq \psi[d_b(m_{a-1}, m_a)] \forall b \in \mathfrak{V}$$

Consequently, for each $a \in \mathbb{N}$ we get

$$d_b(m_a, m_{a+1}) \leq \psi^{a+1}[d_b(m_0, m_1)] \forall b \in \mathfrak{V}.$$

Let $a > a'$, we have

$$\begin{aligned} d_b(m_{a'}, m_a) &\leq d_b(m_{a'}, m_{a'+1}) + d_b(m_{a'+1}, m_{a'+2}) + ... + d_b(m_{a-1}, m_a) \\ &\leq \psi^{a'}(d_b(m_0, m_1)) + \psi^{a'+1}(d_b(m_0, m_1)) + ... + \psi^{a-1}(d_b(m_0, m_1)) \\ &= (\sum_{i=a'})^{a-1} \psi^i(d_b(m_0, m_1)) < \infty \forall b \in \mathfrak{V}. \end{aligned} \tag{3.2.5}$$

As a result, in $(M, \mathfrak{T}(\mathfrak{P}))$, $\{m_a\}$ is Cauchy sequence. Because G is closed in M and M is complete. So, there is a point m^* in G such that $m_a \to m^*$. Moreover,

$$\begin{aligned} d_b(m^*, H) &\leq d_b(m^*, Om_a) \\ &\leq d_b(m^*, m_{a+1}) + d_b(m_{a+1}, Om_a) \\ &= d_b(m^*, m_{a+1}) + d_b(G, H) \\ &\leq d_b(m^*, m_{a+1}) + d_b(m^*, H) \end{aligned}$$

Therefore, $d_b(m^*, Om_a) \to d_b(m^*, H) \forall b \in \mathfrak{V}$ as $a \to \infty$. Since, H is approximatively compact concerning G, there is a subsequence $\{Om_{a_k}\}$ of the sequence $\{Om_a\}$ which converges to some point n^* in H. Hence

$$d_b(m^*, n^*) = \lim_{a \to \infty} d_b(m_{a_{k+1}}, Om_{a_k}) = d_b(G, H).$$

Since, for $m^* \in G_0$, we have $Om^* \in H_0$, thus we have $u \in G$ such that $d_b(u, Ox^*) = d_b(G, H) \forall b \in \mathfrak{V}$. Thus, from 3.2.1, we have

$$d_b(m_{a+1}, u) \leq \phi[d_b(m_a, m^*), d_b(m_a, m_{a+1}), d_b(m^*, u), 1/2(d_b(m_a, u) + d_b(m^*, m_{a+1}))],$$

for all $b \in \mathfrak{V}$. In the inequality above, applying $a \to \infty$ yields,

$$d_b(m^*, u) \leq \phi(0, 0, d_b(m^*, u), 1/2 d_b(m^*, u)) \forall b \in \mathfrak{V}.$$

Axiom (iii) of ϕ gives, $d_b(m^*, u) = 0 \forall b \in \mathfrak{V}$. We can deduce that $m^* = u$ because M is separating gauge space. Therefore

$$d_b(m^*, Om^*) = d_b(u, Om^*) = d_b(G, H) \forall b \in \mathfrak{V}.$$

□

Example 3.2.1. Consider the space encompassing all pairs of bounded and continuous real functions defined on the interval $[0,10]$ denoted as $M = C([0,10],\mathbb{R}) \times C([0,10],\mathbb{R})$ which is endowed with pseudo metrics $d_b(m(t),n(t)) = \max_{t\in[0,b]}\{|m_1(t)-n_1(t)| + |m_2(t)-n_2(t)|\}$ for all $m(t) = (m_1(t),m_2(t)), n(t) = (n_1(t),n_2(t)) \in M$ and $b \in \{1,2,3,...,10\}$. Define $G = \{(0,m(t)) : t \in [0,10]\}$ and $H = \{(10,m(t)) : t \in [0,10]\}$. Let $O : G \to H$ by

$$O(0,m(t)) = (10, \tfrac{m(t)}{2}) \text{ for each } t \in [0,10].$$

Consider $\phi(u_1,u_2,u_3,u_4) = \frac{u_1}{2}$.Then all the conditions of the Theorem3.2.1 holds Thus O possesses best proximity point.

Corollary 3.2.2. *Consider $\mathfrak{P} = \{d_b | b \in \mathfrak{V}\}$ a family of pseudometrics which is separating and $(M, \mathfrak{T}(\mathfrak{P}))$ a complete gauge space induced by $\mathfrak{P}$. Let G and H be closed subsets of M which are not empty such that $G_0 \neq \phi$ and H is approximatively compact concerning G. Moreover, suppose that a mapping $O : G \to H$ meets the following requirements*

(a) For all u_1,u_2,m_1,m_2 in G, there exists a non-negative real number $\alpha < 1$ such that

$$d_b(u_1,Om_1) = d_b(G,H) = d_b(u_2,Om_2) \Rightarrow d_b(u_1,u_2) \leq \alpha d_b(m_1,m_2),$$

(b) $O(G_0) \subseteq H_0$

Then O possesses a best proximity point, that is, $d_b(m,Om) = d_b(G,H) \forall b \in \mathfrak{V}$ for some element m in G.

Proof. Take $\phi(u_1,u_2,u_3,u_4) = \alpha u_1$ with $\psi(t) = \alpha t$, where $\alpha \in [0,1)$. From (3.2.1), we have $d_b(u_1,Om_1) = d_b(G,H) = d_b(u_2,Om_2) \Rightarrow d_b(u_1,u_2) \leq \alpha d_b(m_1,m_2) \forall b \in \mathfrak{V}$ for all $u_1,u_2,m_1,m_2 \in G$. As a result of Theorem 3.2.1, O possesses a best proximity point $m \in G$ that is $d_b(m,Om) = d_b(G,H) \forall b \in \mathfrak{V}$. □

Definition 3.2.4. Allow G and H to be non-empty subsets of M. An *implicit type generalized proximal contraction of second kind* is a mapping $O : G \to H$ such that for each $m_1, m_2, u_1, u_2 \in G$, there exist $\phi \in \Phi_\psi$ satisfying $d_b(u_1, Om_1) = d_b(G, H) = d_b(u_2, Om_2)$ implies

$$\begin{aligned} d_b(Ou_1, Ou_2) &\leq \phi(d_b(Om_1, Om_2), d_b(Om_1, Ou_1), d_b(Om_2, Ou_2), \\ &\quad 1/2(d_b(Om_2, Ou_1) + d_b(Om_1, Ou_2))) \end{aligned} \tag{3.2.6}$$

for each $b \in \mathfrak{V}$.

Theorem 3.2.3. *Consider $\mathfrak{P} = \{d_b | b \in \mathfrak{V}\}$ a family of pseudometrics which is separating and $(M, \mathfrak{T}(\mathfrak{P}))$ a complete gauge space induced by $\mathfrak{P}$. Let G and H be non-empty closed subsets of M such that $G_0 \neq \phi$ and H is approximatively compact concerning G. Consider implicit type generalized proximal contraction of second kind $O : G \to H$ which is continuous such that $O(G_0) \subseteq H_0$. Then O possesses a best proximity point, that is there exist $m \in G$ such that $d_b(m, Om) = d_b(G, H) \forall b \in \mathfrak{V}$.*

Proof. Let $m_0 \in G_0$. Then we can find a sequence m_a in G_0 by following the same steps as in Theorem3.2.1 such that for all $a \in \mathbb{N} \cup \{0\}$

$$d_b(m_{a+1}, Om_a) = d_b(G, H) \forall b \in \mathfrak{V}.$$

Assume that $Om_a \neq Om_{a+1}$ for each $a \in \mathbb{N} \cup \{0\}$, otherwise m_{a+1} is a best proximity point. As a result, (3.2.6) yields,

$$\begin{aligned} d_b(Om_a, Om_{a+1}) &\leq \phi[d_b(Om_{a-1}, Om_a), d_b(Om_{a-1}, Om_a), \\ &\quad d_b(Om_a, Om_{a+1}), 1/2(d_b(Om_{a-1}, Om_{a+1}) + d_b(Om_a, Om_a))] \\ &\leq \phi(d_b(Om_{a-1}, Om_a), d_b(Om_{a-1}, Om_a), \\ &\quad d_b(Om_a, Om_{a+1}), 1/2(d_b(Om_{a-1}, Om_{a+1}))) \\ &\leq \phi[d_b(Om_{a-1}, Om_a), d_b(Om_{a-1}, Om_a), \\ &\quad d_b(Om_a, Om_{a+1}), 1/2(d_b(Om_{a-1}, Om_a) + d_b(Om_a, Om_{a+1}))] \end{aligned} \tag{3.2.7}$$

We claim that $d_b(Om_a, Om_{a+1}) < d_b(Om_{a-1}, Om_a) \forall b \in \mathfrak{V}$ for each $a \in \mathbb{N} \cup \{0\}$.

Suppose on contrary that $d_b(Om_a, Om_{a+1}) \geq d_b(Om_{a-1}, Om_a) \forall b \in \mathfrak{V}$ and some a. Using non-decreasing of ϕ in (3.2.7) yields,

$$d_b(Om_a, Om_{a+1}) \leq \phi(d_b(Om_a, Om_{a+1}), d_b(Om_{a-1}, Om_a), d_b(Om_a, Om_{a+1}), (d_b(Om_a, Om_{a+1}))) \tag{3.2.8}$$

Using property (ii) of Φ_ψ in (3.2.8), we have

$$d_b(Om_a, Om_{a+1}) = 0 \forall b \in \mathfrak{V}$$

which contradicts our assumption, i.e. $Om_{a+1} \neq Om_a$ for each $a \in \mathbb{N} \cup \{0\}$. Thus $d_b(Om_a, Om_{a+1}) < d_b(Om_{a-1}, Om_a) \forall b \in \mathfrak{V}$ for all a.Therefore (3.2.7) becomes

$$d_b(Om_a, Om_{a+1}) \leq \phi[d_b(Om_{a-1}, Om_a), d_b(Om_{a-1}, Om_a), d_b(Om_a, Om_{a+1}), (d_b(Om_{a-1}, Om_a))] \tag{3.2.9}$$

By using (3.2.9) and property (ii) of Φ_ψ, we have

$$d_b(Om_a, Om_{a+1}) \leq \psi[d_b(Om_{a-1}, Om_a)] \forall b \in \mathfrak{V} \text{ for all } a \in \mathbb{N}.$$

Consequently, we get

$$d_b(Om_a, Om_{a+1}) \leq \psi^a[d_b(Om_0, Om_1)] \forall b \in \mathfrak{V} \text{ for each } a \in \mathbb{N} \cup \{0\}$$

Let $a > a'$, we have

$$\begin{aligned} d_b(Om_{a'}, Om_a) &\leq d_b(Om_{a'}, Om_{a'+1}) + d_b(Om_{a'+1}, Om_{a'+2}) + ... + d_b(Om_{a-1}, Om_a) \\ &\leq \psi^{a'}(d_b(Om_0, Om_1)) + \psi^{a'+1}(d_b(Om_0, Om_1)) + ... + \psi^{a-1}(d_b(Om_0, Om_1)) \\ &= \sum_{i=a'}^{a-1} \psi^i(d_b(Om_0, Om_1)) < \infty \forall b \in \mathfrak{V}. \end{aligned}$$

Hence $\{Om_a\}$ is Cauchy sequence in H. Because $(M, \mathfrak{T}(\mathfrak{P}))$ is complete gauge space and H is closed in M. So, $\{Om_a\}$ converges to n^* in H. We get the following by utilising triangular inequality,

$$\begin{aligned} d_b(n^*, G) &\leq d_b(n^*, m_a) \\ &\leq d_b(n^*, Om_{a-1}) + d_b(Om_{a-1}, m_a) \\ &= d_b(n^*, Om_{a-1}) + d_b(G, H) \\ &\leq d_b(n^*, Om_{a-1}) + d_b(n^*, G) \end{aligned}$$

Therefore, $d_b(n^*, m_a) \to d_b(n^*, G) \forall b \in \mathfrak{V}$. Since G is approximatively compact concerning H, so there is a subsequence $\{m_{a_k}\}$ of the sequence $\{m_a\}$ which converges to some point m^* in G. By using continuity of O, we get the following

$$d_b(m^*, Om^*) = \lim_{a\to\infty} d_b(m_{a+1}, Om_a) = d_b(G, H) \forall b \in \mathfrak{V}.$$

$\square$

Corollary 3.2.4. *Consider $\mathfrak{P} = \{d_b | b \in \mathfrak{V}\}$ a family of pseudometrics which is separating and $(M, \mathfrak{T}(\mathfrak{P}))$ a complete gauge space induced by $\mathfrak{P}$. Let G and H be non-empty closed subsets of M such that $G_0 \neq \phi$ and H is approximatively compact concerning G. Further assume that the mapping $O : G \to H$ fulfills the accompanying assumptions:*

(a) For all u_1, u_2, m_1, m_2 in G, there is some nonnegative real number $\alpha < 1$ such that,

$$d_b(u_1, Om_1) = d_b(G, H) = d_b(u_2, Om_2) \Rightarrow d_b(Ou_1, Ou_2) \leq \alpha d_b(Om_1, Om_2)$$

for all $b \in \mathfrak{V}$,

(b) $O(G_0) \subseteq H_0$,

(c) O is continuous.

Then there is some element m in G with $d_b(m, Om) = d_b(G, H)$.

3.2.1 Consequences

Suppose a complete metric space (M, d). A gauge space can be generated from the family $\mathfrak{P} = \{d_b = d : b \in \mathfrak{V}\}$ which is complete as well as separating. Accordingly, from Theorem 3.2.1 and Theorem 3.2.3 respectively, we get the accompanying outcomes.

Theorem 3.2.5. *Allow (M, d) to be complete metric space and $G, H \neq \phi \subseteq M$ be closed with G as approximatively compact concerning H and G_0 is non-empty. Suppose that the mapping $O : G \to H$ satisfies the accompanying assumptions:*

(i) There exist $\phi \in \Phi_\psi$ and $d(u_1, Om_1) = d(G, H) = d(u_2, Om_2)$ implies

$$d(u_1, u_2) \leq \phi(d(m_1, m_2), d(m_1, u_1), d(m_2, u_2), 1/2(d(m_1, u_2) + d(m_2, u_1)));$$

for each $m_1, m_2, u_1, u_2 \in G$

(ii) $O(G_0) \subseteq H_0$.

Then a best proximity point is possessed by O.

Theorem 3.2.6. *Allow* (M, d) *to be complete metric space and* $G, H \neq \phi \subseteq M$ *be closed such that* G_0 *is non-empty and* H *is approximatively compact concerning* G. *Suppose that the mapping* $O : G \to H$ *satisfies the accompanying assumptions:*

(i) There exist $\phi \in \Phi_\psi$ *and* $d(u_1, Om_1) = d(G, H) = d(u_2, Om_2)$ *implies*

$$d(Ou_1, Ou_2) \leq$$
$$\phi(d(Om_1, Om_2), d(Om_1, Ou_1), d(Om_2, Ou_2), 1/2(d(Om_1, Ou_2) +$$
$$d(Om_2, Ou_1)))$$

for each $m_1, m_2, u_1, u_2 \in G$,

(ii) O *is continuous,*

(iii) $O(G_0) \subseteq H_0$.

Then O *possesses a best proximity point.*

Chapter 4

Best Proximity Point Theorems in Metric-like Spaces

Multiplicative calculus introduced by Grossman and Kartz during 1967 and 1970 [65], but unfortunately it remained unpopular for many years. In 2008, Bashirov et al. [24] brought up the researcher's attention to the multiplicative calculus by demonstrating its usefulness in the branch of analysis and presented multiplicative metric. He also presented few examples of multiplicative metric space in his article. Ozavsar and Cevikel [108] then investigated the topological properties of multiplicative metric space and emphasized its importance by showing that $\mathbb{R}^+$ is complete multiplicative metric space while it is not complete in the sense of usual metric. In the same article they demonstrated few fixed point theorems of multiplicative contraction mappings. In multiplicative metric space, fixed point theorems of different contractions are explored [1]. Some interesting surveys on multiplicative metric space are written [6, 7, 44, 45, 126]. The theorems to demonstrate presence of best proximity points for multiplicative proximal contractions are given by [103].

Then again fuzzy metric, presented by Kramosil and modified by George and Veeramnai [57], are of great importance because of its usefulness in a variety of applications such as color image filtering [77, 105]. As of late, Gregori et al. [64] indicated some intriguing applications of fuzzy metric in engineering methods. Fixed point theory studied by many researchers [100, 60, 134, 62].

In this chapter we established fuzzy multiplicative metric space with few of its topological aspects. We also given best proximity point theorems for the proximal contraction and multivalued contraction of Feng-Liu type. This chapter is published as research article [51].

4.1 Fuzzy multiplicative metric spaces

Definition 4.1.1. A fuzzy multiplicative metric space is a 3-tuple $(M, F_{MM}, \star)$ if $\star$ is continuous $t-$ norm, M is arbitrary set and F_{MM} is fuzzy set on $M \times M \times (1, \infty)$ fulfilling the accompanying conditions for all $m, n, p \in M, t, s > 1$

FMM1: $F_{MM}(m, n, t) > 0$

FMM2: $m = n$ if and only if $F_{MM}(m, n, t) = 1$

FMM3: $F_{MM}(m, n, t) = F_{MM}(n, m, t)$

FMM4: $F_{MM}(m, p, t.s) > F_{MM}(m, n, t) \star F_{MM}(n, p, s)$

FMM5: $F_{MM}(m, n, .) : (1, \infty) \to [0, 1]$ is continuous.

Here we have an example of fuzzy multiplicative metric which can not be fuzzy metric.

Example 4.1.1. Let $M = \mathbb{R}^+$ and $F_{MM}(m, n, t) = \frac{t+1}{t+|\frac{m}{n}|^*}$ and consider a continuous $t-$norm $\star : [0, 1] \times [0, 1] \to [0, 1]$ as $x \star y = xy$. Then M is fuzzy multiplicative metric space.

Remark 4.1.1. **1.** Allow $(M, F_{MM}, \star)$ to be a fuzzy multiplicative metric space. Whenever $F_{MM}(m, n, t) > 1 - \epsilon$ for $m, n \in M$ and $t > 1$, $0 < \epsilon < 1$, we can find a t_0, $1 < t_0 < t$ such that $F_{MM}(m, n, t_0) > 1 - \epsilon$.

2. Let $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5 \in (0, 1)$. For any $\epsilon_1 > \epsilon_2$, we are able to locate an ϵ_3 such that $\epsilon_1 \star \epsilon_3 \geq \epsilon_2$ and for any ϵ_4 we can find an ϵ_5 such that $\epsilon_5 \star \epsilon_5 \geq \epsilon_4$.

Here we discuss fuzzy multiplicative metric space with its some topological properties.

Definition 4.1.2. Allow $(M, F_{MM}, \star)$ to be a fuzzy multiplicative metric space and $0 < \epsilon < 1$ then an open ball of centre m and radius ϵ is defined as

$$B(m, \epsilon, t) = \{n \in M : F_{MM}(m, n, t) > 1 - \epsilon\}.$$

Definition 4.1.3. Allow $(M, F_{MM}, \star)$ to b fuzzy multiplicative metric space and $G \subseteq M$. Then G is called open set if and only if for every $m \in G$, there exist an open ball $B(m, \epsilon, t)$ for $t > 1$ and $0 < \epsilon < 1$ such that $B(m, \epsilon, t) \subseteq G$.

Proposition 4.1.1. *Every open ball in fuzzy multiplicative metric space is an open set.*

Proof. Allow $B(m, \epsilon, t)$ to be an open ball and let $n \in B(m, \epsilon, t)$. This implies that $F_{MM}(m, n, t) > 1 - \epsilon$. Since $F_{MM}(m, n, t) > 1 - \epsilon$, using Remark 4.1.1, we can find a t_0, $1 < t_0 < t$, such that $F_{MM}(m, n, t_0) > 1 - \epsilon$. Let $\epsilon_0 = F_{MM}(m, n, t_0) > 1 - \epsilon$. Since $\epsilon_0 > 1 - \epsilon$, therefore by using Remark 4.1.1, we are able to locate an ϵ_1, $0 < \epsilon_1 < 1$, such that $\epsilon_0 > 1 - \epsilon_1 > 1 - \epsilon$. Now for a given ϵ_0 and ϵ_1 satisfying $\epsilon_0 > 1 - \epsilon_1$, we are able to locate $\epsilon_2, 0 < \epsilon_2 < 1$ such that $\epsilon_0 \star \epsilon_2 \geq 1 - \epsilon_1$. Now, think about the ball $B(n, 1 - \epsilon_2, \frac{t}{t_0})$. We claim that

$$B(n, 1 - \epsilon_2, \tfrac{t}{t_0}) \subset B(m, \epsilon, t).$$

Now, $p \in B(n, 1 - \epsilon_2, \frac{t}{t_0})$ implies that $F_{MM}(n, p, \frac{t}{t_0}) > \epsilon_2$. Therefore,

$$\begin{aligned} F_{MM}(m, p, t) &\geq F_{MM}(m, n, t_0) \star F_{MM}(n, p, \frac{t}{t_0}) \\ &\geq \epsilon_0 \star \epsilon_2 \\ &\geq 1 - \epsilon_1 \\ &> 1 - \epsilon. \end{aligned}$$

Therefore, $p \in B(m, \epsilon, t)$ and hence,

$$B(n, 1 - \epsilon_2, \tfrac{t}{t_0}) \subset B(m, \epsilon, t).$$

□

Proposition 4.1.2. *Allow $(M, F_{MM}, \star)$ to be a fuzzy multiplicative metric space. Define*

$$\tau = \{G \subset M : m \in G \text{ if and only if } B(m, \epsilon, t) \subset G \text{ for } t > 1 \text{ and } 0 < \epsilon < 1\}.$$

The set τ is then a topology on M.

Theorem 4.1.3. *Hausdorff axioms are fulfilled by every fuzzy multiplicative metric space .*

Proof. Assume that $(M, F_{MM}, \star)$ is a given fuzzy multiplicative metric space. Allow m, n to be distinct points of M, then $0 < F_{MM}(m, n, t) < 1$. Let $F_{MM}(m, n, t) = \epsilon$, $0 < \epsilon < 1$. For each ϵ_0, $\epsilon < \epsilon_0 < 1$, using Remark 4.1.1, we can find an ϵ_1 such that $\epsilon_1 \star \epsilon_1 \geq \epsilon_0$. Now think about the open balls $B(m, 1 - \epsilon_1, t^{\frac{1}{2}})$ and $B(n, 1 - \epsilon_1, t^{\frac{1}{2}})$. Obviously,

$$B(m, 1 - \epsilon_1, t^{\frac{1}{2}}) \cap B(n, 1 - \epsilon_1, t^{\frac{1}{2}}) = \phi.$$

Because, if there is

$$p \in B(m, 1 - \epsilon_1, t^{\frac{1}{2}}) \cap B(n, 1 - \epsilon_1, t^{\frac{1}{2}})$$

Then

$$\begin{aligned} \epsilon &= F_{MM}(m, n, t) \\ &\geq F_{MM}(m, p, t^{\frac{1}{2}}) \star F_{MM}(p, n, t^{\frac{1}{2}}) \\ &\geq \epsilon_1 \star \epsilon_1 \\ &\geq \epsilon_0 \\ &> \epsilon, \end{aligned}$$

which is a contradiction. Therefore, $(M, F_{MM}, \star)$ is Hausdorff. □

Definition 4.1.4. In a fuzzy multiplicative metric space $(M, F_{MM}, \star)$, a sequence $\{m_a\}$ is a convergent sequence which converges to m if and only if there is some $a_1 \in \mathbb{N}$ with $M(m_a, m, t) > 1 - \epsilon$ for all $a \geq a_1$ and for each $\epsilon > 0, t > 1$.

Theorem 4.1.4. *Allow $(M, F_{MM}, \star)$ to be a fuzzy multiplicative metric space, $m \in M$ and $\{m_a\}$ be a sequence in M. Then $\{m_a\}$ converges to m if and only if $F_{MM}(m_a, m, t) \to 1$ as $a \to \infty$ for each $t > 1$.*

Proof. Suppose that $m_a \to m$. Then for each $t > 1$ and $\epsilon \in (0,1)$, there exist a natural number a_1 such that $F_{MM}(m_a, m, t) > 1 - \epsilon$ for all $a \geq a_1$. We have $1 - F_{MM}(m_a, m, t) < r$. Hence, $F_{MM}(m_a, m, t) \to 1$ as $a \to \infty$.

Conversely, suppose that $F_{MM}(m_a, m, t) \to 1$ as $a \to \infty$. Then for each $t > 1$ and $\epsilon \in (0,1)$, there exist a natural number a_1 such that $1 - F_{MM}(m_a, m, t) < \epsilon$ for all $a \geq a_1$. In that case, $F_{MM}(m_a, m, t) > 1 - \epsilon$. Hence, $m_a \to m$ as $a \to \infty$. □

Definition 4.1.5. Let a sequence $\{m_a\}$ in a fuzzy multiplicative metric space $(M, F_{MM}, \star)$. If for each $\epsilon > 0, t > 1$, there exist $a_1 \in \mathbb{N}$ such that $F_{MM}(m_a, m_b, t) > 1 - \epsilon$ for all $a, b \geq a_1$ then $\{m_a\}$ is termed Cauchy sequence in M.

Theorem 4.1.5. *Let $(M, F_{MM}, \star)$ be a fuzzy multiplicative metric space, $m \in M$ and $\{m_a\}$ be a sequence in M. Then $\{m_a\}$ is Cauchy if and only if $F_{MM}(m_a, m_b, t) \to 1$ as $a, b \to \infty$ for each $t > 1$.*

Proof. Suppose that m_a is a Cauchy sequence in M. Then for each $t > 1$ and $\epsilon \in (0,1)$, there exist a natural number a_1 such that $F_{MM}(m_a, m_b, t) > 1 - \epsilon$ for all $a, b \geq a_1$. We have $1 - F_{MM}(m_a, m_b, t) < \epsilon$. Hence, $F_{MM}(m_a, m_b, t) \to 1$ as $a, b \to \infty$.

Conversely, suppose that $F_{MM}(m_a, m_b, t) \to 1$ as $a, b \to \infty$. Then for each $t > 1$ and $\epsilon \in (0,1)$, there exist a natural number a_1 such that $1 - F_{MM}(m_a, m_b, t) < \epsilon$ for all $a, b \geq a_1$. In that case, $F_{MM}(m_a, m_b, t) > 1 - \epsilon$. Hence, m_a is a Cauchy sequence. □

Proposition 4.1.6. *In a fuzzy multiplicative metric space $(M, F_{MM}, \star)$, if a sequence $\{m_a\}$ converges in M, then $\{m_a\}$ is Cauchy.*

Proof. Let ϵ and t be real numbers with $\epsilon \in (0,1), t > 1$. Since $\epsilon \in (0,1)$, there is some $\epsilon_0 \in (0,1)$ such that $(1 - \epsilon_0) \star (1 - \epsilon_0) > 1 - \epsilon$. Also suppose that $\{m_a\}$ converges in M, say it converges to $m \in M$. Then there exists $a_0 \in \mathbb{N}$ such that for each $a \geq a_0$;

$$F_{MM}(m_a, m, t^{\frac{1}{2}}) > 1 - \epsilon_0$$

Thus for $a, b \geq a_0$ we have

$$\begin{aligned} F_{MM}(m_a, m_b, t) &\geq F_{MM}(m_a, m, t^{\frac{1}{2}}) \star F_{MM}(m_b, m, t^{\frac{1}{2}}) \\ &> (1-\epsilon_0) \star (1-\epsilon_0) \\ &> 1-\epsilon \end{aligned} \tag{4.1.1}$$

That is $\{m_a\}$ is a Cauchy sequence. □

Definition 4.1.6. A fuzzy multiplicative metric space $(M, F_{MM}, \star)$ is termed complete if and only if every sequence in M which is Cauchy must converge in M.

Definition 4.1.7. Allow $(M, F_{MM}, \star)$ to be a fuzzy multiplicative metric space. A subset G of M is closed if for each sequence $\{m_a\}$ in G which is convergent with $m_a \to m$, we have $m \in G$.

Allow $(M, F_{MM}, \star)$ to be a complete fuzzy multiplicative metric space. A subset G of M is closed if and only if $(G, F_{MM}, \star)$ is complete.

Lemma 4.1.7. *Allow $(M, F_{MM}, \star)$ to be a fuzzy metric space such that for $m, n \in M, t > 1$ and $h > 1$*

$$\lim_{a\to\infty} \star_{i=a}^{\infty} F_{MM}(m, n, t^{h^i}) = 1.$$

A sequence $\{m_a\}$ in M is Cauchy if for all $a \in \mathbb{N}$ and $0 < \alpha < 1$

$$F_{MM}(m_a, m_{a+1}, t^{\alpha}) \geq F_{MM}(m_{a-1}, m_a, t).$$

Proof. Each $a \in \mathbb{N}$ and $t > 1$ yields

$$F_{MM}(m_a, m_{a+1}, t) \geq F_{MM}(m_{a-1}, m_a, t^{\frac{1}{\alpha}}) \geq F_{MM}(m_{a-2}, m_{a-1}, t^{\frac{1}{\alpha^2}}) \geq \ldots \geq F_{MM}(m_0, m_1, t^{\frac{1}{\alpha^{a-1}}})$$

Thus for each $a \in \mathbb{N}$ we get

$$F_{MM}(m_a, m_{a+1}, t) \geq F_{MM}(m_0, m_1, t^{\frac{1}{\alpha^{a-1}}}) \tag{4.1.2}$$

Settle the numbers $h > 1$ and $l \in \mathbb{N}$ such that $h\alpha < 1$ and $\sum_{i=l}^{\infty} \frac{1}{h^i} = \frac{\frac{1}{h^l}}{1-\frac{1}{h}} < 1$ Hence for $b \geq a$

$$\begin{aligned} F_{MM}(m_a, m_b, t) &\geq F_{MM}(m_a, m_b, t^{(\frac{1}{h^l}+\frac{1}{h^{l+1}}+\ldots+\frac{1}{h^{l+b}})}) \\ &\geq F_{MM}(m_a, m_{a+1}, t^{\frac{1}{h^l}}) \star F_{MM}(m_{a+1}, m_{a+2}, t^{\frac{1}{h^{l+1}}}) \star \ldots \\ &\quad \star F_{MM}(m_{b-1}, m_b, t^{\frac{1}{h^{l+b}}}) \end{aligned} \tag{4.1.3}$$

Using 4.1.2 in above inequality, we come by;

$$F_{MM}(m_a, m_b, t) \geq$$
$$F_{MM}(m_0, m_1, t^{\frac{1}{\alpha^{a-1}h^l}}) \star F_{MM}(m_0, m_1, t^{\frac{1}{\alpha^a h^{l+1}}}) \star ... \star F_{MM}(m_0, m_1, t^{\frac{1}{\alpha^{b-2}h^{l+b-a-2}}})$$

That is;

$$F_{MM}(m_a, m_b, t) \geq$$
$$F_{MM}(m_0, m_1, t^{\frac{1}{(\alpha h)^{a-1}}}) \star F_{MM}(m_0, m_1, t^{\frac{1}{(\alpha h)^a}}) \star ... \star F_{MM}(m_0, m_1, t^{\frac{1}{(\alpha h)^{b-2}}})$$

The above expression can be simplified as;

$$F_{MM}(m_a, m_b, t) \geq \star_{i=a}^{\infty} F_{MM}(m_0, m_1, t^{\frac{1}{(\alpha h)^{i-1}}})$$

Then from the above, we have

$$\lim_{a,b\to\infty} F_{MM}(m_a, m_b, t) \geq \lim_{a\to\infty} \star_{i=a}^{\infty} F_{MM}(m_0, m_1, t^{\frac{1}{(\alpha h)^{i-1}}}) = 1$$

for each $t > 1$. Hence for each $t > 1$

$$\lim_{a,b\to\infty} F_{MM}(m_a, m_b, t) = 1$$

which shows that $\{m_a\}$ is a Cauchy sequence. □

Definition 4.1.8. Consider a fuzzy multiplicative metric space $(M, F_{MM}, \star)$ and $G, H \subset M$ then for all $t > 1$;

$$G_0 = \{m \in G : F_{MM}(m, n, t) = F_{MM}(G, H, t), \text{ for some } n \in H\}$$
$$H_0 = \{n \in H : F_{MM}(m, n, t) = F_{MM}(G, H, t), \text{ for some } m \in G\}$$

where

$$F_{MM}(G, H, t) = Sup\{F_{MM}(m, n, t), m \in G, n \in H\} \text{ for all } t > 1.$$

Definition 4.1.9. Allow $(M, F_{MM}, \star)$ to be a fuzzy multiplicative metric space and $G, H \subset M$. If every sequence $\{m_a\}$ of G fulfilling the condition that $F_{MM}(n, m_a, t) \to F_{MM}(n, m, t)$ for some n in H and for all $t > 1$ has a convergent subsequence then G is termed approximatively compact concerning H

4.2 Best proximity point theorems in fuzzy multiplicative metric spaces

Definition 4.2.1. Allow $(M, F_{MM}, \star)$ to be a fuzzy multiplicative metric space and $G, H \subset M$. A mapping $O : G \to H$ is named as multiplicative contraction of first kind if there exists $\alpha \in [0, 1)$ such that for all $u, v, m, n \in G$

$$F_{MM}(u, Om, t) = F_{MM}(G, H, t) \text{ and } F_{MM}(v, On, t) = F_{MM}(G, H, t) \Rightarrow \\ F_{MM}(u, v, t^{\alpha}) \geq F_{MM}(m, n, t)$$

Theorem 4.2.1. *Allow $(M, F_{MM}, \star)$ to be a complete fuzzy multiplicative metric space and $G, H \subset M$ such that H is approximatively compact concerning G. Assume that $\lim_{t\to\infty} F_{MM}(m, n, t) = 1$. Let $O : G \to H$ be multiplicative contraction of first kind and $O(G_0) \subset H_0$. Then O possesses best proximity point.*

Proof. Let $m_0 \in G_0$ then for $Om_0 \in OG_0 \subset H_0$ there exist $m_1 \in G_0$ such that

$$F_{MM}(m_1, Om_0, t) = F_{MM}(G, H, t)$$

Further, since $Om_1 \in OG_0 \subset H_0$ there exist $m_2 \in G_0$ such that

$$F_{MM}(m_2, Om_1, t) = F_{MM}(G, H, t)$$

Similarly for $Om_2 \in OG_0 \subset H_0$ there exist $m_3 \in G_0$ such that

$$F_{MM}(m_3, Om_2, t) = F_{MM}(G, H, t)$$

By continuing the similar steps we get;

$$F_{MM}(m_{a+1}, Om_a, t) = F_{MM}(G, H, t) \text{ for all } a \in \mathbb{N} \tag{4.2.1}$$

By successive application of fuzzy multiplicative contraction we have for all $a \in$

$\mathbb{N} \cup \{0\}$

$$\begin{aligned} F_{MM}(m_a, m_{a+1}, t^{\alpha}) &\geq F_{MM}(m_{a-1}, m_a, t) \\ &\geq F_{MM}(m_{a-2}, m_{a-1}, t^{1/\alpha}) \\ &\geq F_{MM}(m_{a-3}, m_{a-2}, t^{1/\alpha^2}) \\ &. \\ &. \\ &. \\ &\geq F_{MM}(m_0, m_1, t^{1/\alpha^{a-1}}) \end{aligned} \quad (4.2.2)$$

For any $q \in \mathbb{N}$;

$$F_{MM}(m_a, m_{a+q}, t) \geq F_{MM}(m_a, m_{a+1}, t^{1/q}) \star F_{MM}(m_{a+1}, m_{a+2}, t^{1/q}) \star ... \star F_{MM}(m_{a+q-1}, m_{a+q}, t^{1/q})$$

Using 4.2.2 in above inequality

$$F_{MM}(m_a, m_{a+q}, t) \geq F_{MM}(m_0, m_1, t^{1/q\alpha^a}) \star F_{MM}(m_0, m_1, t^{1/q\alpha^{a+1}}) \star ... \star F_{MM}(m_0, m_1, t^{1/q\alpha^{a+q-1}})$$

By assumption $\lim_{t\to\infty} F_{MM}(m, n, t) = 1$

$$\lim_{a\to\infty} F_{MM}(m_a, m_{a+q}, t) = 1 \star 1 \star ... \star 1 = 1$$

As a result, $\{m_a\}$ is a Cauchy sequence. The completeness of fuzzy multiplicative metric space $(M, F_{MM}, \star)$ implies that $\{m_a\}$ converges to $m^* \in G$ that is

$$\lim_{a\to\infty} F_{MM}(m_a, m^*, t) = 1 \text{ for all } t > 1$$

Take a note that

$$\begin{aligned} F_{MM}(m, H, t) &\geq F_{MM}(m, Om_a, t) \\ &\geq F_{MM}(m, m_{a+1}, t^{1/2}) \star F_{MM}(m_{a+1}, Om_a, t^{1/2}) \\ &= F_{MM}(x, x_{a+1}, t^{1/2}) \star F_{MM}(G, H, t) \\ &\geq F_{MM}(m, m_{a+1}, t^{1/2}) \star F_{MM}(m, H, t). \end{aligned}$$

Therefore $F_{MM}(m, Om_a, t) \to F_{MM}(m, H, t)$ as $a \to \infty$. Since H is approximatively compact concerning G, so $\{Om_a\}$ has a convergent sequence $\{Om_{a_k}\}$ converging to some $p \in H$. Further for each $k \in \mathbb{N}$ we have

$$\begin{aligned} F_{MM}(G,H,t) &\geq F_{MM}(m,p,t) \\ &\geq F_{MM}(m,m_{a_{k+1}},t^{1/3}) \star F_{MM}(m_{a_{k+1}},Om_{a_k},t^{1/3}) \star F_{MM}(Om_{a_k},p,t^{1/3}) \\ &= F_{MM}(m,m_{a_{k+1}},t^{1/3}) \star F_{MM}(G,H,t^{1/3}) \star F_{MM}(Om_{a_k},p,t^{1/3}) \end{aligned}$$

Letting $k \to \infty$, we get $F_{MM}(m,p,t) = F_{MM}(G,H,t)$ which implies that $m \in G_0$ and $O(G_0) \subseteq H_0$ implies that $Om \in H_0$, there exist $m^* \in G$ such that $F_{MM}(m^*,Om,t) = F_{MM}(G,H,t)$. This equation and equation 4.2.1 implies that

$$F_{MM}(m_{a+1},m^*,t) \geq F_{MM}(m_a,m,t^{1/\alpha})$$

Applying limit $a \to \infty$ to above inequality gives $F_{MM}(m,m^*,t) = 1$ which implies that $m = m^*$. Hence $F_{MM}(m,Om,t) = F_{MM}(G,H,t)$ which demonstrates that O possesses best proximity point m. $\square$

Example 4.2.1. Let $M = \mathbb{R}^+ \times \mathbb{R}^+$ and $F_{MM}(m,n,t) = \frac{t+1}{t+d(m,n)}$ where $d(m,n) = |\frac{m_1}{n_1}|^*.|\frac{m_2}{n_2}|^*$ for $m = (m_1,m_2)$ and $n = (n_1,n_2)$. Then $(M,F_{MM},\star)$ is complete fuzzy multiplicative metric space with $\star : [0,1]^2 \to [0,1]$ defined as $a \star b = ab$. Let $G = \{(1,m) : m \in \mathbb{R}^+\}$ and $H = \{(2,n) : n \in \mathbb{R}^+\}$ then G and H are closed subsets of M and $F_{MM}(G,H,t) = \frac{t+1}{t+2}$, $G_0 = G, H_0 = H$. Define $O : G \to H$ as

$$O(1,m) = (2, \tfrac{m^2}{2})$$

Let $m = (1,m), n = (1,n) \in G$ then $u = (1,\frac{m^2}{2})$ and $v = (1,\frac{n^2}{2}) \in G$ such that $F_{MM}(u,Om,t) = F_{MM}(G,H,t) = F_{MM}(v,Om,t)$. It can be easily checked that O is proximal contraction in fuzzy multiplicative metric space M with $\alpha = \frac{2}{3}$. Also the condition $\lim_{t\to\infty} F_{MM}(m,n,t) = 1$ holds.
Since all statements of theorem 4.3.1 hold so, O possesses best proximity points.

Theorem 4.2.2. *Allow $(M,F_{MM},\star)$ to be complete fuzzy multiplicative metric space. $G,H \subseteq M$ be two non-empty closed subsets of M having P-property and $G_0 \neq \phi$. Let $O : G \to C(H)$ be a mapping such that $O(G_0) \subseteq H_0$ and for all $m \in G_0$ and $n \in Om$ there exist $p \in G_0$ satisfying*

$$F_{MM}(n,p,t) = F_{MM}(G,H,t) and F_{MM}(n,Op,t^c) \geq F_{MM}(m,p,t) \tag{4.2.3}$$

for some $c \in (0,1)$ and $t > 1$. Suppose $(M, F_{MM}, \star)$ satisfy

$$\lim_{a\to\infty} \star_{i=a}^{\infty} F_{MM}(m, n, t^{h^i}) = 1 \tag{4.2.4}$$

for every $m, n \in M, t > 1$ and $h > 1$. Then O has best proximity point in G provided that $f(m,n) = F_{MM}(n, Om, t)$ is upper semi-continuous.

Proof. Allow $m_0 \in G_0$ to be an arbitrary point. Choose $n_0 \in Om_0$. Then by assumption there exist $m_1 \in G_0$ such that

$$F_{MM}(n_0, m_1, t) = F_{MM}(G, H, t) \text{ and } F_{MM}(n_0, Om_1, t^c) \geq F_{MM}(m_0, m_1, t)$$

Presently let $b \in (c, 1)$, then we can discover $n_1 \in Om_1$ such that

$$F_{MM}(n_0, n_1, t) \geq F_{MM}(n_0, Om_1, t^b)$$

Again by assumption there exist $m_2 \in G_0$ such that

$$F_{MM}(n_1, m_2, t) = F_{MM}(G, H, t) \text{ and } F_{MM}(n_1, Om_2, t^c) \geq F_{MM}(m_1, m_2, t)$$

Also we can find $n_2 \in Om_2$ such that

$$F_{MM}(n_1, n_2, t) \geq F_{MM}(n_1, Om_2, t^b)$$

Proceeding in similar manner we develop two sequences $\{m_a\}$ and $\{n_a\}$ in G and H respectively, with $m_a \in G_0$, $n_a \in Om_a$ and

$$F_{MM}(n_a, m_{a+1}, t) = F_{MM}(G, H, t) \tag{4.2.5}$$

$$F_{MM}(n_a, Om_{a+1}, t^c) \geq F_{MM}(m_a, m_{a+1}, t) \tag{4.2.6}$$

$$F_{MM}(n_a, n_{a+1}, t) \geq F_{MM}(n_a, Om_{a+1}, t) \tag{4.2.7}$$

for all $a \in \mathbb{N}$ and $t > 1$. Then again, since G and H have P-property so from equation 4.2.6 we get

$$F_{MM}(m_a, m_{a+1}, t) = F_{MM}(n_{a-1}, n_a, t)$$

Therefore from inequality 4.2.7 we have

$$\begin{aligned} F_{MM}(m_a, m_{a+1}, t) &= F_{MM}(n_{a-1}, n_a, t) \\ &\geq F_{MM}(n_{a-1}, Om_a, t^b) \end{aligned} \tag{4.2.8}$$

From inequality 4.2.7 we have

$$F_{MM}(n_{a-1}, Om_a, t) \geq F_{MM}(m_{a-1}, m_a, t^{\frac{1}{c}}) \tag{4.2.9}$$

Combining inequality 4.2.8 and 4.2.9 we get

$$F_{MM}(m_a, m_{a+1}, t) \geq F_{MM}(m_{a-1}, m_a, t^{\frac{b}{c}}) \tag{4.2.10}$$

for all $a \geq 1$ and $t > 1$.
Let $k = \frac{c}{b}$ then $0 < k < 1$. The inequality 4.2.10 gives

$$F_{MM}(m_a, m_{a+1}, t^k) \geq F_{MM}(m_{a-1}, m_a, t)$$

for $0 < k < 1$ and $t > 1$ By our assumption 4.2.4 and lemma 4.1.7 $\{m_a\}$ is Cauchy sequence.
Now from inequality 4.2.7 and 4.2.8 we have

$$\begin{aligned} F_{MM}(n_a, Om_{a+1}, t^c) &\geq F_{MM}(m_a, m_{a+1}, t) \\ &\geq F_{MM}(n_{a-1}, Om_a, t^b) \\ \Rightarrow F_{MM}(n_a, Om_{a+1}, t) &\geq F_{MM}(n_{a-1}, Om_a, t^{\frac{b}{c}}) \end{aligned} \tag{4.2.11}$$

Also from inequality 4.2.7 and 4.2.11 we have

$$\begin{aligned} F_{MM}(n_a, n_{a+1}, t^{\frac{1}{b}}) &\geq F_{MM}(n_a, Om_{a+1}, t) \\ &\geq F_{MM}(n_{a-1}, Om_a, t^{\frac{b}{c}}) \\ \Rightarrow F_{MM}(n_a, n_{a+1}, t^c) &\geq F_{MM}(n_{a-1}, Om_a, t) \end{aligned}$$

for $0 < c < 1$ and $t > 1$. Hence $\{n_a\}$ is Cauchy sequence.
Since G, H are closed subsets of complete fuzzy metric space so, $\{m_a\}, \{n_a\}$ are convergent sequences in G and H respectively. Thus, there exist $m^* \in G$ and $n^* \in H$ such that $m_a \to m^*$ and $n_a \to n^*$ as $a \to \infty$.
Letting $a \to \infty$ in equation 4.2.6 we have

$$F_{MM}(m^*, n^*, t) = F_{MM}(G, H, t)$$

for $t > 1$. The inequality 4.2.11 shows that the sequence $f(m_a, n_a) = F_{MM}(n_a, Om_a, t)$ is increasing and it converges to 1. Since $f(m, n)$ is upper semi-continuous so,

$$1 = \lim\sup_{a\to\infty} f(m_a, n_a) \leq f(m^*, n^*) \leq 1$$

implies to the fact that $f(m^*, n^*) = 1$ that is $F_{MM}(n^*, Om^*, t) = 1$ and hence $n^* \in Om^*$.
Therefore,

$$\begin{aligned} F_{MM}(G, H, t) &\geq F_{MM}(m^*, Om^*, t) \\ &\geq F_{MM}(m^*, n^*, t) \\ &= F_{MM}(G, H, t) \end{aligned}$$

that is $F_{MM}(m^*, Om^*, t) = F_{MM}(G, H, t)$. This shows that O possesses best proximity point m^*. $\square$

4.3 Best proximity point theorems of Feng-Liu type mappings in fuzzy metric space

Theorem 4.3.1. *Allow $(M, F_M, \star)$ to be complete fuzzy metric space. $G, H \neq \phi \subseteq M$ be closed having P-property and $G_0 \neq \phi$. Let $O : G \to C(H)$ be a mapping such that $O(G_0) \subseteq H_0$ and for all $m \in G_0$ and $n \in Om$ there exist $p \in G_0$ satisfying*

$$F_M(n, p, t) = F_M(G, H, t) \text{ and } F_M(n, Op, ct) \geq F_M(m, p, t) \tag{4.3.1}$$

for some $c \in (0, 1)$ and $t > 0$. Suppose $(M, F_M, \star)$ satisfy

$$\lim_{a\to\infty} \star_{i=a}^{\infty} F_M(m, n, th^i) = 1 \tag{4.3.2}$$

for every $t > 0, h > 1$ and $m, n \in M$. Then O possesses best proximity point in G provided that $f(m, n) = F_M(n, Om, t)$ is upper semi-continuous.

Proof. Allow $m_0 \in G_0$ to be arbitrary point. Choose $n_0 \in Om_0$. Then by assumption there exist $m_1 \in G_0$ such that

$$F_M(n_0, m_1, t) = F_M(G, H, t) \text{ and } F_M(n_0, Om_1, ct) \geq F_M(m_0, m_1, t)$$

Presently let $b \in (c, 1)$, then we can discover $n_1 \in Om_1$ such that

$$F_M(n_0, n_1, t) \geq F_M(n_0, Om_1, bt)$$

Again by assumption there exist $m_2 \in G_0$ such that

$$F_M(n_1, m_2, t) = F_M(G, H, t) \text{ and } F_M(n_1, Om_2, ct) \geq F_M(m_1, m_2, t)$$

Also we can find $n_2 \in Om_2$ such that

$$F_M(n_1, n_2, t) \geq F_M(n_1, Om_2, bt)$$

Proceeding in similar manner we develop two sequences $\{m_a\}$ and $\{n_a\}$ in G and H respectively, with $m_a \in G_0$, $n_a \in Om_a$ and

$$F_M(n_a, m_{a+1}, t) = F_M(G, H, t) \tag{4.3.3}$$

$$F_M(n_a, Om_{a+1}, ct) \geq F_M(m_a, m_{a+1}, t) \tag{4.3.4}$$

$$F_M(n_a, n_{a+1}, t) \geq F_M(n_a, Om_{a+1}, t) \tag{4.3.5}$$

for all $a \in \mathbb{N}$ and $t > 0$. Then again, since G and H have P-property so from equation 4.3.4 we get

$$F_M(m_a, m_{a+1}, t) = F_M(n_{a-1}, n_a, t)$$

Therefore from inequality 4.3.5 we have

$$\begin{aligned} F_M(m_a, m_{a+1}, t) &= F_M(n_{a-1}, n_a, t) \\ &\geq F_M(n_{a-1}, Om_a, bt) \end{aligned} \tag{4.3.6}$$

From inequality 4.3.5 we have

$$F_M(n_{a-1}, Om_a, t) \geq F_M(m_{a-1}, m_a, \frac{1}{c}t) \tag{4.3.7}$$

Combining inequality 4.3.6 and 4.3.7 we get

$$F_M(m_a, m_{a+1}, t) \geq F_M(m_{a-1}, m_a, \frac{b}{c}t) \tag{4.3.8}$$

for all $a \geq 1$ and $t > 0$.
Let $k = \frac{c}{b}$ then $0 < k < 1$. The inequality 4.3.8 gives

$$F_M(m_a, m_{a+1}, kt) \geq F_M(m_{a-1}, m_a, t)$$

for $0 < k < 1$ and $t > 0$ By our assumption 4.3.2 and lemma 4.1.7 $\{m_a\}$ is Cauchy sequence.

Now from inequality 4.3.5 and 4.3.6 we have

$$\begin{aligned} F_M(n_a, Om_{a+1}, ct) &\geq F_M(m_a, m_{a+1}, t) \\ &\geq F_M(n_{a-1}, Om_a, bt) \\ \Rightarrow F_M(n_a, Om_{a+1}, t) &\geq F_M(n_{a-1}, Om_a, \frac{b}{c}t) \end{aligned} \tag{4.3.9}$$

Also from inequality 4.3.5 and 4.3.9 we have

$$\begin{aligned} F_M(n_a, n_{a+1}, \frac{1}{b}t) &\geq F_M(n_a, Om_{a+1}, t) \\ &\geq F_M(n_{a-1}, Om_a, \frac{b}{c}t) \\ \Rightarrow F_M(n_a, n_{a+1}, ct) &\geq F_M(n_{a-1}, Om_a, t) \end{aligned}$$

for $0 < c < 1$ and $t > 0$. Hence $\{n_a\}$ is Cauchy sequence.

Since G, H are closed subsets of complete fuzzy metric space so, $\{m_a\}, \{n_a\}$ are convergent sequences in G and H respectively. Thus, there is some $m^* \in G$ and $n^* \in H$ such that $m_a \to m^*$ and $n_a \to n^*$ as $a \to \infty$.

Letting $a \to \infty$ in equation 4.3.4 we have

$$F_M(m^*, n^*, t) = F_M(G, H, t)$$

for $t > 0$. The inequality 4.3.9 shows that the sequence $f(m_a, n_a) = F_M(n_a, Om_a, t)$ is increasing sequence, so it converges to 1. Since $f(m, n)$ is upper semi-continuous so,

$$1 = \limsup_{a\to\infty} f(m_a, n_a) \leq f(m^*, n^*) \leq 1$$

implies to the fact that $f(m^*, n^*) = 1$ that is $F_M(n^*, Om^*, t) = 1$ and hence $n^* \in Om^*$.

Therefore,

$$\begin{aligned} F_M(G, H, t) &\geq F_M(m^*, Om^*, t) \\ &\geq F_M(m^*, n^*, t) \\ &= F_M(G, H, t) \end{aligned}$$

that is $F_M(m^*, Om^*, t) = F_M(G, H, t)$. This shows that O possesses best proximity point m^*. $\square$

Example 4.3.1. Let $J = \{0, 1\} \cup \{\frac{1}{2^n} : n \in \mathbb{N}\}$ and $M = J \times J$,

$$F_M(m, n, t) = \tfrac{t}{t+d(m,n)} \text{ and } d(m, n) = |m_1 - n_1| + |m_2 - n_2|$$

for $m = (m_1, m_2)$ and $n = (n_1, n_2) \in M$ Then $(M, F_M, \star)$ is complete fuzzy metric space where $\star : [0, 1]^2 \to [0, 1]$ defined by $a \star b = ab$. Let $G = \{(0, \frac{1}{2^a}) : a \in \mathbb{N}\} \cup \{(0, 0), (0, 1)\}$ and $H = \{(1, \frac{1}{2^a}) : a \in \mathbb{N}\} \cup \{(1, 0), (1, 1)\}$. Then $G_0 = G$, $H_0 = H$ and $F_M(G, H, t) = \frac{t}{t+1}$. Define $O : G \to C(H)$ as;

$$O(1, m) = \begin{cases} \{(0, \frac{1}{2^{a+1}}), (0, 1)\} \text{ if } m = \frac{1}{2^a}, a = 0, 1, 2, ... \\ \{(0, 0), (0, \frac{1}{2})\} \text{ if } m = 0. \end{cases}$$

For all $m, n \in M$, $\lim_{a\to\infty} \star_{i=a}^{\infty} F_M(m, n, th^i) = 1$ which implies that M satisfies 4.3.2 . Let $m = (1, \frac{1}{2^a}) \in G_0$ and $n = (0, \frac{1}{2^{a_1}}) \in Om = O(1, \frac{1}{2^a})$ then for $p = (1, \frac{1}{2^{a_1}}) \in G_0$ we have

$$F_M(n, p, t) = F_M(G, H, t) \text{ and } F_M(n, Op, t) = 1 \geq F_M(m, n, t)$$

Also

$$f(m, n) = F_M(n, Om, t) = \tfrac{t}{t+d(n,Om)} = \begin{cases} \frac{t}{t+\frac{1}{2^{a+1}}} \text{ for } m = (1, \frac{1}{2^a}) \\ 1 \text{ for } m = (1, 0), (1, 1) \end{cases}$$

is continuous. Because the theorem's 4.3.1 requirements are all met, so, best proximity points for O exists. Furthermore, for $u = (1, \frac{1}{2^a}), v = (1, 0) \in G_0$

$$H_{F_M}(O(1, \tfrac{1}{2^a}), O(1, 0), ct) = \tfrac{ct}{ct+\frac{1}{2}} \text{ and } F_M((1, \tfrac{1}{2^a}), (1, 0), t) = \tfrac{t}{t+\frac{1}{2^a}}$$

Assume that for $c \in (0, 1)$, $H_{F_M}(O(1, \frac{1}{2^a}), O(1, 0), ct) \geq F_M((1, \frac{1}{2^a}), (1, 0), t)$ That is

$$\tfrac{ct}{ct+\frac{1}{2}} \geq \tfrac{t}{t+\frac{1}{2^a}}$$

which implies that $c \geq 2^{a-1}$ for $a \in \mathbb{N}$ which is a contradiction. This shows that O does not satisfies contraction condition of Nadler's multivalued mapping.

As corollary of Theorem 4.1, we obtain a result which was proved in [14]. We get the corollary by taking $A = B = M$.

Corollary 4.3.2. *Allow $(M, F_M, \star)$ to be complete fuzzy metric space. Let $O : M \to C(M)$ be a mapping, for all $m \in M$ and $n \in I_b^m$ (where $I_b^m = \{n \in Om | F_M(m, n, t) \geq F_M(m, Om, bt)\} \subset M$ for some $b \in (0, 1)$) satisfying*

$$F_M(n, On, ct) \geq F_M(m, n, t)$$

for some $c \in (0, 1)$ and $t > 1$. Then O possesses fixed point provided that $c < b$ and $f(m) = F_M(m, Om, t)$ is upper semi-continuous.

4.4 Conclusion

Zadeh [139] introduced the notion of fuzzy logic to cope with the problem of uncertainty, that occurs essentially while studying real life problem. Many researcher found easiness to study the phenomenon of different fields that were too complex to be analyzed by conventional techniques. Fuzzy metric introduced by Kaleva [81] measures the imprecision of distance between elements. Fuzzy metric has been applied in variety of applications like color image filtering [105] and in engineering methods [64]. Multiplicative calculus has its great applications in various fields, few of which are in biomedical image analysis [53], contour detection in images [104]. We introduced fuzzy multiplicative metric space in this chapter and demonstrated some best proximity point and fixed point results in this new framework. The above discussion shows the possible applications in this framework in future.

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