*On The Continuity Of Some Integral Operators On Function Spaces*

By

*Muhammad Asim*

**Department of Mathemtics**
**Quaid-i-Azam University**
**Islamabad, Pakistan**

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT

OF THE REQUIREMENT FOR THE DEGREE OF

**DOCTOR OF PHILOSOPHY**

IN

***MATHEMATICS***


Supervised By

*Dr. Amjad Hussain*

# Department of Mathemtics
# Quaid-i-Azam University
# Islamabad, Pakistan


**2022**

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Student Name: **Muhammad Asim** Signature:________

External committee:

a) **External Examiner 1**: Signature:________
Name: **Dr. Muhammad Mushtaq**
Designation: Associate Professor
Office Address: Department of Mathematics, COMSATS University, Park Road Chak Shahzad, Islamabad.

b) **External Examiner 2**: Signature:________
Name: **Dr. Matloob Anwar**
Designation: Professor
Office Address: School of Natural Sciences (SNS), National University of Sciences and Technology (NUST) , Islamabad.

c) **Internal Examiner** Signature:________
Name: **Dr. Amjad Hussain**
Designation: Associate Professor
Office Address: Department of Mathematics, QAU Islamabad.

**Supervisor Name:** Signature:________
**Dr. Amjad Hussain**

**Name of Dean/ HOD** Signature:________

**Prof. Dr. Tariq Shah**

# On The Continuity Of Some Integral Operators on Function Spaces

**By**

## Muhammad Asim

CERTIFICATE

A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE

**DOCTOR OF PHILOSOPHY IN MATHEMATICS**

**We accept this thesis as conforming to the required standard**

1. ______________________
**Prof. Dr. Tariq Shah**
(Chairman)

2. ______________________
**Dr. Amjad Hussain**
(Supervisor)

3. ______________________
**Dr. Muhammad Mushtaq**
(External Examiner)

Department of Mathematics, COMSATS University, Park Road Chak Shahzad, Islamabad.

4. ______________________
**Prof. Dr. Matloob Anwar**
(External Examiner)

School of Natural Sciences (SNS), National University of Sciences and Technology (NUST) , Islamabad.

**Department of Mathematics**
**Quaid-I-Azam University**
**Islamabad, Pakistan**
**2022**

# Abstract

This thesis aims to study the continuity criteria for the Hardy type operators on the variable exponent function spaces. More specifically, in this thesis, we consider the boundedness of the fractional Hardy and rough fractional Hardy operators on Morrey and Herz-type spaces with variable exponents. Similar results for the commutators generated by these operators and variable $\lambda$-central bounded mean oscillation (BMO) functions are likewise obtained. Also, the continuity of Hardy-type operators and their commutators on variable exponent function spaces of weighted type took less attention by the research community worldwide. The same is with weighted Morrey and Herz-type space with variable exponents. The present thesis also aims to fill this gap by proving the boundedness of the fractional Hardy-type operators, along with their commutators with weighted variable $\lambda$-central bounded BMO functions, on these spaces.

# Contents

# Preface

This thesis aims to study Hardy type operators on some function spaces. We thus include the results discussing boundedness of the fractional Hardy operator and rough Hardy operator along with the commutators of these operators on variable exponent function spaces.

In Chapter 2, we come up with the boundedness of fractional Hardy operator on central Morrey space, we also obtained the inequalities for commutator generated by fractional Hardy operator and the $\lambda$-central bounded mean oscillation (BMO) functions on variable exponent central Morrey space. These results published in the Journal of mathematics [1].

In Chapter 3, we investigate the boundedness of fractional rough Hardy operator on central Morrey space, we also obtained the inequalities for commutators of the rough fractional Hardy operators on central Morrey space with variable exponent. In the end, we study the boundedness of rough fractional Hardy operator on Herz type spaces. The content of this chapter are drafted in the form of a manuscript which will be submitted soon.

In Chapter 4, we prove the boundedness for fractional type Hardy operators on weighted variable exponents Herz-Morrey spaces. The contents of this chapter have been published in The Journal of Inequalities and Applications [3].

In Chapter 5, we established the boundedness of commutators of fractional Hardy operators on a class of function spaces called weighted Herz-Morrey space with variable exponents. The contents of this chapter have been published in the Journal of Function spaces [2].

In Chapter 6, some results are added to demonstrate the boundedness of Hardy operators and related commutators on weighted variable exponent Morrey spaces.

Muhammad Asim
Islamabad, Pakistan
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# Chapter 1
# Introduction

The boundedness of operators on function spaces is one of the core issues in harmonic analysis. It is mainly because many problems in the theory of partial differential equations, in their simplified form, are reduced to the boundedness of operators on function spaces. It stimulates the research community to embark on such problems in this field. The same is the case of the Hardy operator, which is considered a significant averaging operator in mathematical analysis. Hardy operators and related commutators play an indispensable role in the theory of partial differential equations [4, 5] and the characterization of function spaces [6–8].

Having in hand results discussing the boundedness of Hardy type operators on function spaces with constant exponents, this thesis aims to obtain similar boundedness results on variable exponent function spaces. Nonetheless, boundedness results on the latter type of function spaces are far from perfect than the same results on the earlier ones. In addition, this thesis also studies the weighted boundedness of the Hardy-tye operators and related commutators on generalized function spaces. However, before starting our main results, we need to introduce the reader to the Hardy type operators and some basic definitions and preliminary results regarding variable exponent function spaces. The purpose of the coming few sections is to fulfill this objective.

## 1.1 One Dimensional Hardy Operator

In mathematical analysis, the Hardy operator is considered a significant averaging operator and has been exercised a lot during the recent past. In [9], Hardy defined the classical Hardy operator as:

$$Hg(z) = \frac{1}{z}\int_0^z g(t)dt, \quad z > 0, \tag{1.1.1}$$

and established the inequality:

$$\|Hg\|_{L^p(\mathbb{R}^+)} \leq \frac{p}{p-1}\|g\|_{L^p(\mathbb{R}^+)}, \quad 1 < p < \infty \tag{1.1.2}$$

where $\frac{p}{p-1}$ was shown to be the best possible constant.

It is customary to denote the adjoint operator of $H$ by $H^*$ :

$$H^*g(x) = \int_z^\infty \frac{g(t)}{|t|}dt, \quad t \in \mathbb{R}^n \setminus \{0\}.$$

In [10] Hardy found inequality for adjoint Hardy operator

$$\int_0^\infty (H^*g(x))^p dx \leq p^p \int_0^\infty (xf(x))^p dx, \quad f(x) \geq 0, p > 1.$$

Let $b \in L_{loc}(\mathbb{R}^+)$, the commutators of One dimensional Hardy operator $H$ and its adjoint operator are defined as follow

$$[b, H]g(x) = b(x)Hg(x) - H(bg)(x),$$

$$[b, H^*]g(x) = b(x)H^*g(x) - H^*(bg)(x).$$

Wong and Long [11] obtained the inequalities for commutator generated by one dimensional Hardy operator and the adjoint operator of $H$. Later on, Xue and Zhang [12] developed two weights inequalities for its commutator.

## 1.2 Multi-Dimensional Hardy Operator

Faris [13] gave an $n$-dimensional extension of (1.1.1) of which the equivalent form is given by

$$Hg(y) = |B(0,|y|)|^{-1} \int_{B(0,|y|)} g(z)dz, \tag{1.2.1}$$

where $|B(0,|y|)|$ is the Lebesgue measure of the ball $B(0,|y|)$ in $n$-dimensional Euclidean space $\mathbb{R}^n$. Recently, in [14], it was shown that $H$ satisfies:

$$\|Hg\|_{L^p(\mathbb{R}^n)} \leq p'\|g\|_{L^p(\mathbb{R}^n)}, \qquad 1 < p \leq \infty, \tag{1.2.2}$$

where the same constant, as given in the inequality (1.1.2), was declared sharp. Inequalities (1.1.2) and (1.2.2) were recently extended to power weighted Lebesgue spaces in [15] and [16] where sharp constants, depending upon the weight indices, were again fixed.

Inequalities (1.1.2) and (1.2.2) are known as strong-type $(p,p)$ Hardy inequalities because in these inequalities Hardy operator maps $L^p$ to $L^p$. The authors in [17] have established the weak-type $(p,p)$ Hardy inequalities in which Hardy operator maps $L^p$ to $L^{p,\infty}$. However, it was shown that the optimal constant for weak-type Hardy inequalities is 1 which is obviously less than $p/(p-1)$. Subsequently, the sharp constants for weak-type Hardy inequalities on Morrey type spaces were obtained in [18] and [19].

## 1.3 Multi-Dimensional Fractional Hardy Operator

Subsequently, in [20], the authors defined the fractional Hardy operator and its adjoint operator as:

$$Hg(z) = \frac{1}{|z|^{n-\beta}} \int_{|t|\leq|z|} g(t)dt, \quad H^*g(z) = \int_{|t|>|z|} \frac{g(t)}{|t|^{n-\beta}}dt, \quad z \in \mathbb{R}^n \backslash \{0\}, \tag{1.3.1}$$

where $|z| = \sqrt{\sum_{i=1}^{n} z_i^2}$.

Likewise the sharp constant for high dimensional fractional Hardy operator on Lebesgue spaces were not fixed until 2015. Zhao and Lu, in [22], solved this problem by extending the Bliss results (see [21]) for one dimensional fractional Hardy operator. It has been shown in [22] that for $1 < p < q < \infty,\ 0 < \beta < n$, and $\beta/n + 1/q = 1/p$, the operator $H_\beta$ is of strong type $(p, q)$ that is it satisfies:

$$\|H_\beta g\|_{L^q(\mathbb{R}^n)} \leq A\|g\|_{L^p(\mathbb{R}^n)}, \tag{1.3.2}$$

where the constant:

$$A = \left(\frac{p'}{q}\right)^{1/q} \left(\frac{n}{q\beta} \cdot B\left(\frac{n}{q\beta}, \frac{n}{q'\beta}\right)\right)^{-\beta/n},$$

is not only sharp but also, unlike the Hardy inequality (1.2.2), depends on the dimension $n$ of the Euclidean space $\mathbb{R}^n$.

Hardy inequalities have been a main focus of interest in various monographs [23, 24]. The optimal bounds for Hardy type inequalities are established only in a few cases and the research in this area is an active part of modern analysis. Some recent publication in this area include [25–30]. Besides, Sharp constants for Hardy type inequalities on product of some function spaces have also been reported in the literature [31–34].

## 1.4 Multi-Dimensional Rough Hardy Operator

Recently, Fu and Lu [35] gave the definition of $n$-dimensional rough Hardy operator and and it's commutator as follow:

$$H_\Omega g(z) = \frac{1}{|z|^{n-\beta}} \int_{|\tau|\leq|z|} \Omega(z-\tau) g(\tau) d\tau, \quad H_\Omega^* g(z) = \int_{|\tau|>|z|} \Omega(z-\tau) \frac{g(\tau)}{|\tau|^{n-\beta}} d\tau,$$

where $\Omega \in L^s(S^{n-1})$, $1 < s \leq \infty$, is homogeneous of degree zero. The commutators generated by a locally integrable function $b$ and the rough

Hardy operators are defined as below:

$$H_{\Omega}^{b}g(z) = \frac{1}{|z|^{n-\beta}} \int_{|\tau|\le|z|} (b(z) - b(\tau))\Omega(z-\tau)g(\tau)d\tau$$

$$H_{\Omega}^{*,b}g(z) = \int_{|\tau|>|z|} (b(z) - b(\tau))\Omega(z-\tau)\frac{g(\tau)}{|\tau|^{n-\beta}}d\tau.$$

In [36], Ren and Tao obtained the weighted boundedness for the commutator of $n$-dimensional rough Hardy operator and central $BMO$ functions on the weighted Lebesgue spaces, the weighted Morrey-Herz spaces and the weighted Herz spaces.

## 1.5 Variable Exponent Function Spaces

Variable exponent function spaces are a general form of the classical function spaces in which the constant exponents are replaced with variable exponents. Many aspects of the resultant Banach function spaces are comparable to those of classical spaces, but they differ from each other surprisingly and subtly. As a result, variable function spaces are fascinating, but they are also essential in the context of partial differential equations(PDE's) and variational integrals with non-standard growth conditions. The study of these and similar spaces has grown dramatically during the last 20 years, particularly in the past few decades.

Now, we give an outline of the historical backdrop of variable Lebesgue spaces. Describing this set of experiences up to the mid-1990s is generally direct since somewhat a couple of mathematicians worked around here. From that time the field has blossomed, and we just note a couple of features. It is for the most part acknowledged that the splitting line between the early and present day time spans in the investigation of variable Lebesgue spaces is the basic paper of Kováčik and Rákosník [37] from 1991. However, the beginning of the variable Lebesgue spaces originates before their work by 60 years, since they were first concentrated by Orlicz [38] in 1931. For a variable exponent $p(x) \in (1, \infty)$, he exposed that if

$$\int_0^1 |g(x)|^{p(x)} dx < \infty,$$

then for function $f$ the necessary and sufficient condition so that

$$\int_0^1 g(x) f(x) dx < \infty,$$

is that for some $\sigma > 0$,

$$\int_0^1 \left( \frac{|f(x)|}{\sigma} \right)^{p'(x)} dx < \infty,$$

Nonetheless, this piece of information is basically the main commitment of Orlicz to the investigation of the variable Lebesgue spaces. All things considered, Orlicz directed his concentration toward the investigation of the spaces presently called Orlicz spaces, which he likewise presented in 1931 in a collaborative article with Birnbaum [39]. The following stage in the improvement of the Lebesgue spaces along with variable exponent came twenty years after the fact in crafted by Nakano [40] who fostered the hypothesis of modular spaces, in some cases alluded to as Nakano spaces. Modular space is a topological vector space outfitted with a modular: the speculation of a standard. A significant illustration of a modular space is the function space comprising of all functions $f$ such that for some $\lambda > 0$,

$$\int_E \Phi \left( x, \frac{|g(x)|}{\sigma} \right) dx < \infty,$$

Where $\Phi : E \times [0, \infty) \to [0, \infty)$ is a function for $x \in E$, $\Phi(x, .)$ acts like a Young function. These spaces are alluded to as Musielak- Orlicz spaces, given in [41]. If $\Phi = \Phi(t)$ is only function of $t$ then they are called Orlicz spaces. If $\Phi(x, t)$ is equal to $t^{p(x)}$, they are grow into variable Lebesgue spaces. If $\Phi(x,t)$ is equal to $w(x)t^{p(x)}$, they are nominated weighted Lebesgue spaces.

From this point onward we give a brief introduction to the notations and definitions related to the variable exponent Lebesgue, Morrey and Herz type spaces. From the start to finish of this work, we represent by $|B|$ and $\chi_B$ the Lebesgue measure and characteristic function of a measurable set $B \subset \mathbb{R}^n$, respectively. Also, $B_j = B(0,2^j) = \{x \in \mathbb{R}^n : |x| \le 2^j\}$ with $A_j = \{x \in \mathbb{R}^n : 2^{j-1} < |x| \le 2^j\}$ and $\chi_j = \chi_{A_j}$ for $j \in \mathbb{Z}$. The notation $g \approx f$ implies that there exist two constants $C_1 > 0$ and $C_2 > 0$ such that $C_1 f \le g \le C_2 f$. Furthermore, $E \subseteq \mathbb{R}^n$ represents an open set and $p(\cdot) : E \to [1,\infty)$ is a measurable function, $p'(\cdot)$ denotes the conjugate exponent of $p(\cdot)$ which satisfies:

$$\frac{1}{p'(\cdot)} = -\frac{1}{p(\cdot)} + 1.$$

The set *P(E)* consist of all $p(\cdot)$ and $p'(\cdot)$ such that:

$$1 < p^- = \operatorname{essinf}\{p(x) : x \in E\} < p^+ = \operatorname{esssup}\{p(x) : x \in E\} < \infty.$$

**Definition 1.5.1** *Suppose $p(\cdot)$ is a real valued function on $\mathbb{R}^n$. We say that*
*(i) $\mathcal{C}^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ is the collection of all local log-Hölder continuous function $p(\cdot)$ contented*

$$|p(x) - p(y)| \lesssim \frac{-C}{\log(|x-y|)}, \quad |x-y| < \frac{1}{2}, \quad x, y \in \mathbb{R}^n.$$

*(ii) $\mathcal{C}^{\log}_{0}(\mathbb{R}^n)$ is the collection of all local log-Hölder continuous function $p(\cdot)$ fulfill at origin*

$$|p(x) - p(0)| \lesssim \frac{C}{\log(|e + \frac{1}{|x|}|)}, \quad x \in \mathbb{R}^n.$$

*(iii)* $\mathcal{C}_\infty^{\log}(\mathbb{R}^n)$ *is a coolection of all log-Hölder continuous function satisfy the following condition at infinity*

$$|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}, \quad x \in \mathbb{R}^n.$$

*(iv)* $\mathcal{C}^{\log}(\mathbb{R}^n) = \mathcal{C}_\infty^{log} \bigcap \mathcal{C}_{\mathrm{loc}}^{log}$ *denotes the set of all global log-Hölder continuous function* $p(\cdot)$*. It was proved in [42] that if* $p(\cdot) \in P(\mathbb{R}^n) \cap \mathcal{C}^{\log}(\mathbb{R}^n)$ *then Hardy-Littlewood maximal operator* $M$ *is bounded on* $L^{p(\cdot)}(\mathbb{R}^n)$*.*

### 1.5.1 Variable Exponent Lebesgue Space

The variable exponent Lebesgue spaces $L^{p(\cdot)}$ were firstly introduced by Kováčik and Rákosník in [37]. After that, the development of variable Lebesgue spaces was started along with the investigation of boundedness of several operators including the maximal operator on $L^{p(\cdot)}$ [43,46]. The space $L^{p(\cdot)}$ is a set of all measurable function $f$ on the open set $E$, in such a away that for positive $\eta$,

$$\int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} dx < \infty,$$

which becomes a Banach function space when equipped with the Luxemburg-norm

$$||f||_{L^{p(\cdot)}(E)} = \inf\left\{\eta > 0 : \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} dx \leq 1\right\}.$$

Local version of variable exponent Lebesgue space is denoted by $L_{\mathrm{loc}}^{p(\cdot)}(E)$ and is defined by

$$L_{\mathrm{loc}}^{p(\cdot)}(E) = \left\{f : f \in L^{p(\cdot)}(B) \ \forall \text{ compact subset B} \subset \text{E}\right\}.$$

For existing findings in variable exponent Lebesgue spaces, we direct the researcher to the recent monographs [44] and [45].

Recently, the theory of variable exponent analysis is modeled in terms of the boundedness of the Hardy Littlewood maximal operator $M$ [43, 46, 47]:

$$Mf = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f| dy,$$

where $B_r = \{y \in \mathbb{R}^n : |x - y| < r\}$, on the Lebesgue spaces. We use $\mathfrak{B}(\mathbb{R}^n)$ to denote a set containing $p(\cdot) \in P(\mathbb{R}^n)$ satisfying the condition that the $M$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

The theory of Lebesgue spaces showed deep concern in many fields of mathematical analysis like for example in the field of image processing [48], in the analysis of electrorheological fluids models [49] and in the theory of partial differential equations with nonstandard growth conditions [50].

### 1.5.2 Variable Exponent Morrey Space

The Morrey space $L^{p,\lambda}$, which was proposed in [51] concerning the study of partial differential equations, have been widely discussed in the literature (e.g, [51, 52]). We also refer to a recent review study [53], which contains multiple variants of Morrey type spaces as well as their generalizations. During the last few decades, many classical operators such as singular, potential and maximum operators have been studied in Morrey type spaces. Researchers in the field of variable exponent analysis were also drawn to the Morrey spaces. First time, variable exponent Morrey space $L^{p(\cdot),\lambda(\cdot)}$ over an open set $E \subset \mathbb{R}^n$ was introduce by Almeida and Hasanov in [54].

**Definition 1.5.2** *For $0 < \lambda(\cdot) < n$ and $p(\cdot) \in P(\mathbb{R}^n)$, variable exponent Morrey space $L^{p(\cdot),\lambda(\cdot)}$ is defined as:*

$$\|f\|_{L^{p(\cdot),\lambda(\cdot)}(E)} = \inf \left\{ V : I^{p(\cdot),\lambda(\cdot)}\left(\frac{f}{V}\right) \leq 1 \right\},$$

*where*

$$I^{p(\cdot),\lambda(\cdot)}(f) = \sup_{r>0, x\in E} \frac{1}{r(x)^{\lambda(\cdot)}} \int_{\bar{B}(x,r)} |f(y)|^{p(x)} dy,$$

and $\bar{B}(x,r) = B(x,r) \cap E$. When $p(\cdot) = p$ and $\lambda(\cdot) = \lambda$, we get classical Morrey space $L^{p,\lambda}$.

### 1.5.3 Variable Exponent Central Morrey Space

The $\lambda-$central Morrey space, the central bounded mean oscillation (BMO) space and associated function spaces have attractive applications by exploring estimates for operators along with their commutators [55–62]. Initially, for constant exponent $\lambda-$central Morrey space was defined in [55]. Mizuta et al. defined the variable exponent nonhomogeneous $\lambda-$central Morrey space in [63].

**Definition 1.5.3** *[64] Let $p(\cdot) \in P(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$. Then the variable exponent central Morrey space $\dot{B}^{p(\cdot),\lambda}(\mathbb{R}^n)$ is defined as:*

$$\dot{B}^{p(\cdot),\lambda}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{Loc}(\mathbb{R}^n) : \|f\|_{\dot{B}^{p(\cdot),\lambda}(\mathbb{R}^n)} < \infty \right\},$$

*where*

$$\|f\|_{\dot{B}^{p(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{R>0} \frac{\|f\chi_{B(0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

Replace variable exponent by constant exponent we get classical $\lambda-$central Morrey.

### 1.5.4 Variable Exponent Herz Space

Herz spaces $\dot{K}^{\alpha,p}_{q}$ are a type of function space presented by Herz in [65] as part of his research on absolutely convergent Fourier transformations. More lately, there has been a lot of focus on the research of the Herz spaces since there have been numerous notable works that have helped to advance the study of the Herz spaces. Initially , Izuku introduced Herz spaces with variable exponent $\dot{K}_q^{\alpha,p(\cdot)}$ in [66,67]. Later on, Almeida and Drihemn [68] gave a new definition of Herz spaces by taking the exponent alpha as a variable. They also proved Hardy-littlewood-sobolev theorems for fractional integrals on Herz Variable exponent Spaces in the meanwhile. However, Herz space having all the exponents as variables was defined and studied in [69]. Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $A_k = B_k \setminus B_{k-1}$, and $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$.

**Definition 1.5.4** *Let $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. and $p(\cdot), q(\cdot) \in P(\mathbb{R}^n)$. The space $\dot{K}^{\alpha(\cdot)}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$ is the set of all measurable functions $f$ given by*

$$\dot{K}^{\alpha(\cdot)}_{q(\cdot),p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha(\cdot)}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

*where*

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)} = \|\{2^{k\alpha(\cdot)}\|f\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\}_{k=-\infty}^{\infty}\|_{\ell^{q(\cdot)}(\mathbb{Z})}.$$

If $\alpha(\cdot) = \alpha$, $q(\cdot) = q$ and $p(\cdot) = p$ then we have the classical Herz space $\dot{K}_q^{\alpha,p}$.

### 1.5.5 Variable Exponent Morrey-Herz Space

Herz-Morrey spaces $M\dot{K}^{\alpha,\lambda}_{q,p}$ are the extended form of Herz spaces and Morrey spaces . The mapping properties of the singular integral operators on the Herz-Morrey spaces is one of the pioneering research on the Herz-Morrey spaces by Lu and Xu [70]. When $\lambda = 0$ then $M\dot{K}^{\alpha,\lambda}_{q,p} = \dot{K}^{\alpha,p}_q$. Herz-Morrey spaces with variable exponent $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}$ are first appeared in [71]. The ensuing paper [72] made some generalization in the definition of Herz-Morrey spaces given in [71] by replacing the exponent $\alpha$ with $\alpha(\cdot)$. Few important considerations in this regard can be found in [73–76].

**Definition 1.5.5** *Let* $0 < q < \infty$, $\lambda \in [0, \infty)$, $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ *with* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. *and* $p(\cdot) \in P(\mathbb{R}^n)$. *The space* $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ *is the set of all measurable functions* $f$ *given by*

$$M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

*where*

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} = \sup_{k_0 \in Z} 2^{-k_0\lambda} \left( \sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)} f\chi_k\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q}.$$

Obviously, $M\dot{K}^{\alpha(\cdot),0}_{q,p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha(\cdot)}_{q,p(\cdot)}(\mathbb{R}^n)$ is the Herz space with variable exponent (see [68]). If $\alpha(\cdot)$ is constant then we have $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = \dot{M}K^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ [73]. When $\alpha(\cdot)$ and $q(\cdot)$ both are constant and $\lambda = 0$, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha}_{q,p}(\mathbb{R}^n)$ are the classical Herz space.

### 1.5.6 Bounded Mean Oscillation

The absolute value of a function subtracted from its average over a set is the oscillation of that function over the set. So, the average of the oscillation over a set is called mean oscillation. A function is said to be bounded mean oscillation (BMO) if the boundedness of the mean oscillation over the set is satisfied. Let $b \in L^1_{loc}(\mathbb{R}^n)$ and measurable set $B$ in $\mathbb{R}^n$, represented by

$$\underset{B}{Avg}\, b = b_B = \frac{1}{|B|} \int_B b(x)dx$$

the average or mean of function $b$ over $B$. Then the oscillation of $b$ over $B$ is the function $|b - b_B|$ and the mean oscillation of $b$ over $B$ is

$$\frac{1}{|B|} \int_B |b(x) - b_B| dx.$$

**Definition 1.5.6** *[77] let $b \in L^1_{loc}(\mathbb{R}^n)$, set*

$$\|b\|_{BMO(\mathbb{R}^n)} = \sup_B \frac{1}{B} \int_B |b(x) - b_B| dx,$$

where supremum is taken all over the ball $B \subset \mathbb{R}^n$ and $b_B = |B|^{-1} \int_B b(y)dy$. The function $b$ is known as bounded mean oscillation if $\|b\|_{BMO(\mathbb{R}^n)} < \infty$ and $BMO(\mathbb{R}^n)$ consist of all $b \in L^1_{loc}(\mathbb{R}^n)$ with $BMO(\mathbb{R}^n) < \infty$. We can say that $BMO(\mathbb{R}^n)$ is a linear space $i.e$ if $g, f \in BMO(\mathbb{R}^n)$ and $\lambda \in C$ then also $f + g, \lambda f \in BMO(\mathbb{R}^n)$ and

$$\|g + f\|_{BMO} \leq \|g\|_{BMO} + \|f\|_{BMO}$$

$$\|\lambda g\|_{BMO} = |\lambda| \|g\|_{BMO}.$$

But here is problem arise if $\|g\|_{BMO} = 0$, which is not implies $g = 0$ gives $g$ is a constant. So, we conclude that $\|\|_{BMO}$ is not a norm.

### 1.5.7 Variable Exponent Central BMO Space

Central BMO space along with variable exponent $CBMO^{p(\cdot)}$ was introduce by D. Wang and Z. Liu see [78]. By keeping $p(\cdot)$ constant we get the classical central BMO space $CBMO^p$ which was defined by Lu and Yang [79]. Variable exponent $CBMO^{p(\cdot)}$ defined as follow

**Definition 1.5.7** *Let $p(\cdot) \in P(\mathbb{R}^n)$ . Then the variable exponent central BMO space $CBMO^{p(\cdot)}(\mathbb{R}^n)$ is defined as:*

$$\|f\|_{CBMO^{p(\cdot)}(\mathbb{R}^n)} = \sup_{R>0} \frac{\|(f - f_{B(0,R)})\chi_{B(0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B(0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

### 1.5.8 Variable Exponent $\lambda$-Central BMO Space

Meanwhile, the authors in [64] gave the definition of $\lambda$-central BMO space along with some important results regarding the estimation of some operators.

**Definition 1.5.8** *Let $p(\cdot) \in P(\mathbb{R}^n)$ and $\lambda < \frac{1}{n}$. Then the variable exponent $\lambda$- central BMO space $CBMO^{p(\cdot),\lambda}(\mathbb{R}^n)$ is defined as:*

$$CBMO^{p(\cdot),\lambda}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{Loc}(\mathbb{R}^n) : \|f\|_{CBMO^{p(\cdot),\lambda}(\mathbb{R}^n)} < \infty \right\},$$

*where*

$$\|f\|_{CBMO^{p(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{R>0} \frac{\|(f - f_{B(0,R)})\chi_{B(0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}.$$

If $p(\cdot) = p$ then we get the classical $\lambda$- central BMO defined in [55].

## 1.6 Variable Exponent Weighted Function Spaces

**Definition 1.6.1** *Consider a weight $w$ on $\mathbb{R}^n$ and $q(\cdot) : \mathbb{R}^n \to [1, \infty)$ with $q_+ < \infty$. 1. Variable exponent weighted Lebesgue space $L^{q(\cdot)}(w)$ contain all measurable function $f$ hold the inequality given below*

$$||f||_{L^{q(\cdot)}(w)} = \inf \left\{ \eta > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\eta} \right)^{q(x)} w(x)dx \leq 1 \right\}.$$

*is finite. $L^{q(\cdot)}(w) = L^{q(\cdot)}$ and $\|f\|_{L^{q(\cdot)}} = \|f\|_{L^{q(\cdot)}(w)}$ when $w = 1$.*

*2. Let $q(\cdot) \in P(\mathbb{R}^n)$. The set $L^{q(\cdot)}_{loc}(w)$ contain all measurable function $f$ so that $f\chi_i \in L^{q(\cdot)}(w)$ for each any every compact set $i \subset \mathbb{R}^n$.*

### 1.6.1 The $A_p$ Condition

In this thesis we represent weight by $w$ which is non-negative locally integrable function on $\mathbb{R}^n$, that acquire value between $(0, \infty)$ almost everywhere. Because of this, weights are permitted to be infinite or zero only on a set of Lebesgue measure zero. Consequently, when $\frac{1}{w}$ is locally integrable and $w$ is weight then $\frac{1}{w}$ will also weight. For measurable set $B$ we will used notation for a given weight $w$

$$w(B) = \int_B w(x)dx$$

to express the $w-$ measure of the set $B$. Generally, weighted $L^p$ spaces denoted by $L^p(w)$.

**Definition 1.6.2** *A function $w(x) \geq 0$ is known as an $A_1$ weight if*

$$M(w)(x) \leq Cw(x),$$

*$C$ is a constant. If $w$ is considered an $A_1$ weights, then the equation given below*

$$[w]_{A_1} = \sup_{Q \text{ cubes in } \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q w(t)dt \right) \| \frac{1}{w} \|_{L^\infty(Q)}$$

*is an $A_1$ Muckenhoupt characteristic constant.*

**Definition 1.6.3** *Let $p \in (0, \infty)$. A weight $w$ is known to be of class $A_p$ if the following condition hold*

$$\sup_{Q \text{ cubes in } \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q w(t)^{-(p-1)^{-1}} dt \right)^{p-1} \left( \frac{1}{|Q|} \int_Q w(t)dt \right) < \infty$$

*The above expression is said to Muckenhoupt characteristic constant of $w$ and express by $[w]_{A_p}$.*

Suppose $w(x)$ is a weight function on $\mathbb{R}^n$, which is nonnegative and locally integrable function on $\mathbb{R}^n$. Let $L^{p(\cdot)}(w)$ be the space of all complex-valued functions $f$ on $\mathbb{R}^n$ such that: $fw^{\frac{1}{p(\cdot)}} \in L^{p(\cdot)}(\mathbb{R}^n)$. The space $L^{p(\cdot)}(w)$ is a Banach function space equipped with the norm:

$$\|f\|_{L^{p(\cdot)}(w)} = \|fw^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}}.$$

Benjamin Muckenhoupt introduced the theory of $A_p$ $(1 < p < \infty)$ weights on $\mathbb{R}^n$ in [80]. Recently, in [81, 82] Izuki and Noi generalized the Muckenhoupt $A_p$ class by taking $p$ as variable.

**Definition 1.6.4** *Let $p(\cdot) \in P(\mathbb{R}^n)$. A weight $w$ is an $A_{p(\cdot)}$ weight if*

$$\sup_B \frac{1}{|B|} \|w^{1/p(\cdot)}\chi_B\|_{L^{p(\cdot)}} \|w^{-1/p(\cdot)}\chi_B\|_{L^{p'(\cdot)}} < \infty.$$

*In [83], authors proved that $w \in A_{p(\cdot)}$ if and only if $M$ is bounded on the space $L^{p(\cdot)}$.*

**Remark**

( [81]) Suppose $p(\cdot), q(\cdot) \in P(\mathbb{R}^n) \cap \mathcal{C}^{log}(\mathbb{R}^n)$ and $p(\cdot) \leq q(\cdot)$, then we have

$$A_1 \subset A_{p(\cdot)} \subset A_{q(\cdot)}.$$

**Definition 1.6.5** *Suppose $p_1(\cdot), p_2(\cdot) \in P(\mathbb{R}^n)$ and $\beta \in (0, n)$ such that $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\beta}{n}$. A weight $w$ is said to be $A(p_1(\cdot), p_2(\cdot))$ weight if*

$$\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \|\chi_B\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \leq C|B|^{1-\frac{\beta}{n}}.$$

**Definition 1.6.6** *[81] Suppose $p_1(\cdot), p_2(\cdot) \in P(\mathbb{R}^n)$ and $\beta \in (0, n)$ such that $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\beta}{n}$. Then $w \in A_{(p_1(\cdot), p_2(\cdot))}$ if and only if $w^{p_2(\cdot)} \in A_{1+p_2(\cdot)/p_1'(\cdot)}$*

### 1.6.2 Variable Exponent Weighted $\lambda-$ Central Morrey Space

In this thesis we introduce weighted central Morrey space along with variable exponent.

**Definition 1.6.7** *If $p(\cdot) \in P$ and $\lambda \in \mathbb{R}$. The weighted central Morrey space along with variable exponent $\dot{B}^{p(\cdot),\lambda}(w)$ is defined as*

$$\dot{B}^{p(\cdot),\lambda}(w) = \{f \in L^{p(\cdot)}_{loc}(w) : \ \|f\|_{\dot{B}^{p(\cdot),\lambda}(w)} < \infty\},$$

*where*

$$\|f\|_{\dot{B}^{p(\cdot),\lambda}(w)} = \sup_{R>0} \frac{\|f\chi_{B(0,R)}\|_{L^{p(\cdot)}(w)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{p(\cdot)}(w)}}$$

### 1.6.3 Weighted $\lambda-$ Central BMO Space With Variable Exponent

Here we introduce weighted variable exponent $\lambda-$ central BMO space.

**Definition 1.6.8** *If $p(\cdot) \in P$ and $\lambda \in \frac{1}{n}$. The weighted $\lambda-$ BMO space along with variable exponent $CBMO^{p(\cdot),\lambda}(w)$ is defined as*

$$CBMO^{p(\cdot),\lambda}(w) = \{f \in L^{p(\cdot)}_{loc}(w) : \ \|f\|_{CBMO^{p(\cdot),\lambda}(w)} < \infty\},$$

*where*

$$\|f\|_{CBMO^{p(\cdot),\lambda}(w)} = \sup_{R>0} \frac{\|(f - f_{B(0,R)})\chi_{B(0,R)}\|_{L^{p(\cdot)}(w)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{p(\cdot)}(w)}}$$

### 1.6.4 Variable Exponent Weighted Herz Space

Recent advancement in the field of variable exponent function spaces include the development of its weighted theory based on the Muckenhoupt weights [80]. In [83, 84], Cruz-Uribe with different co-authors gave the continuity criteria for Hardy-Littlewood maximal operator $M$ on weighted $L^{p(\cdot)}(w)$ spaces. Equivalence between the boundedness of $M$ on $L^{p(\cdot)}(w)$ and the Muckenhoupt condition was proved by Diening and Hasto in [85]. For constant exponent Lu and Yung [86] introduce the weighted Herz Space. Izuki and Noi defined the weighted Herz spaces with variable exponents in [81].

**Definition 1.6.9** *Let $w$ is a weight on $\mathbb{R}^n$, $q \in (0, \infty)$, $p(\cdot) \in P(\mathbb{R}^n)$ $\alpha \in \mathbb{R}$. The space $\dot{K}^{\alpha}_{q,p(\cdot)}(w)$ is the set of all measurable functions $f$ given by*

$$\dot{K}^{\alpha}_{q,p(\cdot)}(w) = \left\{f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\}, w) : \|f\|_{\dot{K}^{\alpha}_{q,p(\cdot)}(w)} < \infty\right\},$$

*where*

$$\|f\|_{\dot{K}^{\alpha}_{q,p(\cdot)}(w)} = \sup_{k_0 \in Z} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha q}\|f\chi_k\|_{L^{p(\cdot)}(w)}\right)^{1/q}.$$

For $p(\cdot) = p$, we obtained $\dot{K}^{\alpha}_{q,p(\cdot)}(w) = \dot{K}^{\alpha}_{q,p}(w)$.

### 1.6.5 Weighted Morrey-Herz Space With Variable Exponent

However, weighted Herz-Morrey spaces with variable exponents are defined and studied in [87]. Now, we define variable exponent weighted Morrey-Herz space $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w)$.

**Definition 1.6.10** *Let $w$ be a weight on $\mathbb{R}^n$, $\lambda \in [0,\infty)$, $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $q \in (0,\infty)$, and $p(\cdot) \in P(\mathbb{R}^n)$. The space $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w)$ is the set of all measurable functions $f$ given by*

$$M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w) = \left\{ f \in L^{p(\cdot)}_{loc}(\mathbb{R}^n \setminus \{0\}, w) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w)} < \infty \right\},$$

*where*

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w)} = \sup_{k_0 \in Z} 2^{-k_0\lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot)q} \|f\chi_k\|^q_{L^{p(\cdot)}(w)} \right)^{1/q}.$$

Obviously, $M\dot{K}^{\alpha(\cdot),0}_{q,p(\cdot)}(w)=\dot{K}^{\alpha(\cdot)}_{q,p(\cdot)}(w)$ is the weighted Herz space with variable exponent (see [68]), and $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w)=M\dot{K}^{\alpha,\lambda}_{q,p}(w)$ is the classical weighted Herz-Morrey spaces (see [88]).

## 1.7 Our Contribution

We contribute to the theory of Hardy operator in many ways. Firstly, we obtained the boundedness of fractional Hardy operator and fractional rough Hardy operator on $\lambda-$ central Morrey space. We also found the estimate of commutator led by by fractional Hardy operator and fractional rough Hardy operator on central Morrey space. Secondly, we demonstrated the estimates for fractional Hardy operator and their commutator on weighted $\lambda-$Central Morrey space and on weighted Herz

Morrey space with variable exponent. At last, we extend the results of fractional Hardy operator on Herz Morrey space for fractional rough Hardy operator.

### 1.7.1 References of Contribution

We have published three papers [1–3] and the remaining results submitted in reputed journals.

# Chapter 2
# Variable $\lambda$-central Morrey Space Estimates for the Fractional Hardy Operators and Commutators

## 2.1 Introduction

One of the central issues in harmonic analysis is the boundedness of operators on function spaces. It is mainly because many problems in the theory of partial differential equations, in their simplified form, are reduced to the boundedness of operators on function spaces. It stimulates the research community to embark on such problems in this field. In this chapter, we mainly obtain the boundedness of fractional Hardy operators on variable exponent central Morrey spaces. In addition, commutators of these operators:

$$[b, H]g = bHg - H(bg), \qquad [b, H^*]g = bH^*g - H^*(bg). \quad (2.1.1)$$

with symbol functions $b$ in variable $\lambda$-central BMO spaces are shown bounded on central Morrey spaces with variable exponent. However, before stating our main results, we need to introduce the reader to some basic definitions and preliminary results regarding variable exponent function spaces.

Notably, the function spaces with variable exponents have considerable importance in Harmonic analysis as well. Back in 1931, Orlicz [38] started the theory of variable exponent Lebesgue space. Musielak Orlicz spaces were defined and studied in [40]. The study of Sobolev and Lebesgue spaces with variable exponents in [37, 44, 45, 89] further stimulated the subject. The central BMO space was first appeared in [78]. Meanwhile, the authors in [64] gave the definition of variable exponent central Morrey and $\lambda$-central BMO space along with some important results regarding the estimation of some operators. Recently, some publications [90–92] discussing the continuity of multi-linear integral operators on these function spaces have added substantially to the existing literature on this topic.

The aim of this chapter is to prove that on central Morrey space with variable exponent, the fractional Hardy operator and its adjoint operator are bounded. When the symbol functions belong to $\lambda$-central BMO space with variable exponent, similar conclusions for their commutators are achieved.

Let's describe the framework of this Chapter . We will remind some lemmas and propositions related to variable exponent function spaces. In the second section of this chapter, we will demonstrate the boundedness for Hardy operators and their commutators on central Morrey space with variable exponent. In Section 2.3, we shall investigate the similar estimates for the adjoint fractional Hardy operator and its commutators.

**Proposition 2.1.1** *[42, 93] Let $E$ denotes an open set and $p(\cdot) \in P(E)$ fulfill the following inequalities:*

$$|p(x) - p(y)| \leq \frac{-C}{log(|x-y|)}, \quad \frac{1}{2} \geq |x-y|, \tag{2.1.2}$$

$$|p(x) - p(y)| \leq \frac{C}{log(|x|+e)}, \quad |x| \leq |y|, \tag{2.1.3}$$

then $p(\cdot) \in \mathfrak{B}(E)$, where $C$ is a positive constant independent of $x$ and $y$.

**Lemma 2.1.2** *[37] ( Generalized Hölder inequality) Let* $p(\cdot), p_1(\cdot), p_2(\cdot) \in P(E)$.

(a) If $g \in L^{p(\cdot)}(E)$ and $f \in L^{p'(\cdot)}(E)$, then we have

$$\int_E |g(x)f(x)| \leq r_p \|g\|_{L^{p(\cdot)}(E)} \|f\|_{L^{p'(\cdot)}(E)},$$

where $r_p = 1 + \frac{1}{p_-} - \frac{1}{p_+}$.

(b) If $g \in L^{p_1(\cdot)}(E),\ f \in L^{p_2(\cdot)}(E)$ and $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)}$, then we have

$$\|gf|_{L^{p(\cdot)}(E)} \leq r_{p,p_1} \|g\|_{L^{p_1(\cdot)}(E)} \|f\|_{L^{p_2(\cdot)}(E)},$$

where $r_{p,p_1} = \left(1 + \frac{1}{(p_1)_-} - \frac{1}{(p_1)_+}\right)^{1/p_-}$.

**Lemma 2.1.3** *[94] If* $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, *then there exist constants* $0 < \delta < 1$ *and a positive constant* $C$ *such that for all balls* $B$ *in* $\mathbb{R}^n$ *and all measurable subsets* $S \subset B$,

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta}.$$

**Remarks** Let $p(\cdot) \in P(\mathbb{R}^n)$ and meet conditions (2.1.2) and (2.1.3) in Proposition 2.1.1, then so does $p'(\cdot)$. This implies that $p(\cdot), p'(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$. Therefore, using Lemma 2.1.3, we have a constant $\delta_1 \in (0, \frac{1}{(p_2)_+})$ such that the inequality:

$$\frac{\|\chi_S\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \tag{2.1.4}$$

is satisfied for all balls $B \subset \mathbb{R}^n$ and for $S \subset B$. Similarly, if $p_1(\cdot), p_1'(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, then by Lemma 2.1.3, we have constants $\delta_2 \in (0, \frac{1}{(p_1)_+})$, $\delta_3 \in (0, \frac{1}{(p_1')_+})$ such that

$$\frac{\|\chi_S\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_2}, \qquad \frac{\|\chi_S\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_3}, \tag{2.1.5}$$

for each and every balls $B \subset \mathbb{R}^n$ and for $S \subset B$.

**Lemma 2.1.4** *[94] Assuming that $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, the for all balls $B \subset \mathbb{R}^n$ and for a positive constant $C$, the following inequality holds:*

$$C^{-1} \leq \frac{1}{|B|}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C.$$

**Lemma 2.1.5** *[95] Let $q(\cdot) \in P(\mathbb{R}^n)$, then for all $b \in BMO$ and all $l, i \in \mathbb{Z}$ with $l > i$ we have*

$$C^{-1}\|b\|_{BMO} \leq \sup_{B:Ball} \frac{1}{\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}}\|(b-b_B)\chi_B\|_{L^{q(\cdot)}} \leq C\|b\|_{BMO}, \quad (2.1.6)$$

$$\|(b-b_{B_i})\chi_{B_l}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C(l-i)\|b\|_{BMO}\|\chi_{B_l}\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \quad (2.1.7)$$

While proving our results we control the boundedness of the fractional Hardy operator using the boundedness of the fractional integral operator $I_\beta$ :

$$I_\beta(g)(z) = \int_{\mathbb{R}^n} \frac{g(t)}{|z-t|^{n-\beta}}dt,$$

on variable Lebesgue space. In this regard, we need the following Proposition.

**Proposition 2.1.6** *[96] Let $p_1(\cdot) \in P(\mathbb{R}^n)$, $0 < \beta < \frac{n}{(p_1)_+}$ and define $p_2(\cdot)$ by*

$$\frac{1}{p_2(\cdot)} = \frac{1}{p_1(\cdot)} - \frac{\beta}{n}.$$

*Then*

$$\|I_\beta f\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.$$

Proposition 2.1.6 is useful in establishing the following Lemma (see [97]).

**Lemma 2.1.7** *Suppose $\beta$, $p_1(\cdot)$ $p_2(\cdot)$ be as defined in Proposition 2.1.6, then*

$$\|\chi_{B_j}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C2^{-j\beta}\|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)},$$

*for each and every balls $B_j = \{x \in \mathbb{R}^n : |x| \leq 2^j\}$ with $j \in \mathbb{Z}$.*

## 2.2 Boundedness for Fractional Hardy Operator and Commutator

In this section, we present theorems on the boundedness of the fractional Hardy operator and commutators on central Morrey space with their proofs.

**Theorem 2.2.1** *Let $p_1(\cdot) \in P(\mathbb{R}^n)$ and satisfying the condition (2.1.2) and (2.1.3) in Proposition 2.1.1. Define the variable exponent $p_2(\cdot)$ by*

$$\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\beta}{n}.$$

*If $\lambda_2 = \lambda_1 + \frac{\beta}{n}$ and $\lambda_2 > -(\delta_1 + \delta_3)$, where $\delta_1$ and $\delta_3$ are the same constants as appeared in inequalities (2.1.4) and (2.1.5), then*

$$\|H_\beta f\|_{\dot{B}^{p_2(\cdot),\lambda_2}} \le C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}}.$$

**proof**

By definition of the fractional Hardy operator and Lemma 2.1.2, it's easy to see that

$$\begin{aligned}|H_\beta f(x)\chi_k(x)| &\le \frac{1}{|x|^{n-\beta}} \int_{B_k} |f(t)|dt\chi_k(x)\\ &\le C2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \chi_k(x).\end{aligned}$$

Taking the $L^{p_2(\cdot)}(\mathbb{R}^n)$ norm on both sides, we have

$$\begin{aligned}\|H_\beta f\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\le C2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\\ &\le C2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Through the use of Lemma 2.1.4 and the inequality (2.1.5), it is easy to see that

$$
\begin{aligned}
\|H_\beta f\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\le C2^{k\beta}\sum_{j=-\infty}^{k}\|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|^{-1}_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{k\beta}\sum_{j=-\infty}^{k}\|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}}\\
&\quad \|\chi_{B_k}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|^{-1}_{L^{p_2'(\cdot)}(\mathbb{R}^n)}\\
&\le C2^{k\beta}\sum_{j=-\infty}^{k}2^{n\delta_3(j-k)}\|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|^{-1}_{L^{p_2'(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
\tag{2.2.1}
$$

In view of the condition $1/p_1'(x) = 1/p_2'(x) - \beta/n$ and Lemma 2.1.7, the last inequality reduces to the following inequality:

$$
\|H_\beta f\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\sum_{j=-\infty}^{k}2^{n\delta_3(j-k)}|B_j|^{\lambda_1}\|\chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.
\tag{2.2.2}
$$

Since

$$
\|\chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \approx |B|^{\frac{1}{p_1(\cdot)}} \approx |B|^{\frac{1}{p_2(\cdot)}+\frac{\beta}{n}} \approx |B|^{\frac{\beta}{n}}\|\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}.
\tag{2.2.3}
$$

Therefore, from the inequality (2.2.2), we infer that

$$\|H_\beta f\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \sum_{j=-\infty}^{k} 2^{n\delta_3(j-k)}|B_j|^{\lambda_1+\frac{\beta}{n}}\|\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}$$

$$= C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \sum_{j=-\infty}^{k} 2^{n\delta_3(j-k)}|B_k|^{\lambda_2}\frac{|B_j|^{\lambda_2}}{|B_k|^{\lambda_2}}$$

$$\|\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}$$

$$\leq C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}|B_k|^{\lambda_2}\|\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \sum_{j=-\infty}^{k} 2^{n\delta_3(j-k)}$$

$$\frac{|B_j|^{\lambda_2}}{|B_k|^{\lambda_2}}\frac{\|\chi_j\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}.$$

Finally, inequality (2.1.4) helps us to have

$$\|H_\beta f\|_{\dot{B}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \leq C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)} \sum_{j=-\infty}^{k} 2^{n(j-k)(\delta_3+\delta_1+\lambda_2)}.$$

Since $\delta_3+\delta_1+\lambda_2>0$, so we get

$$\|H_\beta f\|_{\dot{B}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \leq C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

**Theorem 2.2.2** *Let $0<\beta<n$ and let $p(\cdot),q(\cdot),r(\cdot)\in P(\mathbb{R}^n)$ and satisfying the condition (2.1.2) and (2.1.3) in Proposition 2.1.1 with $p(\cdot)<n/\beta, p'(\cdot)<r(\cdot)$ and*

$$\frac{1}{q(\cdot)}+\frac{\beta}{n}=\frac{1}{p(\cdot)}+\frac{1}{r(\cdot)}.$$

*Let $0<\nu<1/n$ and $-1/q_+<\mu$. If $\mu=\nu+\lambda+\frac{\beta}{n}$, with $\max\{-(\nu+1),-(\delta_1+\delta_3+\beta/n)\}<\lambda$, where $\delta_1,\delta_3$ are the same constant as appeared in inequalities (2.1.4) and (2.1.5), and $b\in\|b\|_{CBMO^{r(\cdot),\nu}}$ then*

$$\|[b,H_\beta]f\|_{\dot{B}^{q(\cdot),\mu}} \leq C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}}$$

**proof**

We decompose the integral appearing in the commutator operator as:

$$\begin{aligned}
|[b, H_\beta]f(x) \cdot \chi_B(x)| &\leq \left| \frac{1}{|x|^{n-\beta}} \int_{|y|\leq|x|} |(b(x) - b(y))f(y)|dy \cdot \chi_B(x) \right| \\
&\leq \left| \frac{1}{|x|^{n-\beta}} \int_{|y|\leq|x|} |(b(x) - b_B)f(y)|dy \cdot \chi_B(x) \right| \\
&+ \left| \frac{1}{|x|^{n-\beta}} \int_{|y|\leq|x|} |(b(y) - b_B)f(y)|dy \cdot \chi_B(x) \right| \\
&= A_1 + A_2.
\end{aligned}$$

Let us first estimate $A_1$. By taking the variable Lebesgue space norm on both sides, we get

$$\|A_1\|_{L^{q(\cdot)}(\mathbb{R}^n)} = \|(b(\cdot) - b_B)H_\beta f(\cdot)\chi_B(\cdot)\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

Taking into consideration the condition $\frac{1}{q(\cdot)} = \frac{1}{s(\cdot)} + \frac{1}{r(\cdot)}$, $\left(\frac{1}{s(\cdot)} = \frac{1}{p(\cdot)} - \frac{\beta}{n}\right)$, the generalized Hölder inequality gives us the following estimation of $A_1$ :

$$\begin{aligned}
\|A_1\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leq \|(b - b_B)\chi_B\|_{L^{r(\cdot)}(\mathbb{R}^n)}\|(H_\beta f)\chi_B\|_{L^{s(\cdot)}(\mathbb{R}^n)} \\
&= C\|b\|_{CBMO^{r(\cdot),\nu}}|B|^\nu\|\chi_B\|_{L^{r(\cdot)}(\mathbb{R}^n)}\|H_\beta f\|_{\dot{B}^{s(\cdot),\sigma}}|B|^\sigma\|\chi_B\|_{L^{s(\cdot)}(\mathbb{R}^n)},
\end{aligned}$$

where $\sigma = \lambda + \frac{\beta}{n}$. Using the result of Theorem 2.2.1, we obtain

$$\|A_1\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_{CBMO^{r(\cdot),\nu}}|B|^{\nu+\sigma}\|\chi_B\|_{L^{r(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{s(\cdot)}(\mathbb{R}^n)}\|f\|_{\dot{B}^{p(\cdot),\lambda}}.$$

Here it is easy to see that

$$\|\chi_B\|_{L^{r(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{s(\cdot)}(\mathbb{R}^n)} \approx |B|^{\frac{1}{r(\cdot)}}|B|^{\frac{1}{s(\cdot)}} = |B|^{\frac{1}{q(\cdot)}} \approx \|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}. \tag{2.2.4}$$

Therefore, on account of the condition $\mu = \nu + \sigma$, for $A_1$ we have

$$\|A_1\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C|B|^\mu\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}}.$$

Next, we consider $A_2$ for approximation:

$$A_2 = \frac{1}{|x|^{n-\beta}} \int_{|y|\le|x|} |(b(y) - b_B)f(y)|dy \cdot \chi_B(x),$$

which can be decomposed further as:

$$\begin{aligned}
A_2 &= \sum_{k=-\infty}^{0} \frac{1}{|x|^{n-\beta}} \sum_{j=-\infty}^{k} \int_{2^jB\backslash 2^{j-1}B} |(b(y) - b_B)f(y)|dy \cdot \chi_{2^kB\backslash 2^{k-1}B}(x) \\
&\le \sum_{k=-\infty}^{0} \frac{1}{|x|^{n-\beta}} \sum_{j=-\infty}^{k} \int_{2^jB\backslash 2^{j-1}B} |(b(y) - b_{2^jB})f(y)|dy \cdot \chi_{2^kB\backslash 2^{k-1}B}(x) \\
&+ \sum_{k=-\infty}^{0} \frac{1}{|x|^{n-\beta}} \sum_{j=-\infty}^{k} \int_{2^jB\backslash 2^{j-1}B} |(b_B - b_{2^jB})f(y)|dy \cdot \chi_{2^kB\backslash 2^{k-1}B}(x) \\
&= A_{21} + A_{22},
\end{aligned}$$

where

$$A_{21} = \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta}{n}-1} \sum_{j=-\infty}^{k} \int_{2^jB\backslash 2^{j-1}B} |(b(y) - b_{2^jB})f(y)|dy \cdot \chi_{2^kB\backslash 2^{k-1}B}(x).$$

We define a new variable $t(\cdot)$ such that $\frac{1}{t(\cdot)} = \frac{1}{p'(\cdot)} - \frac{1}{r(\cdot)}$, then by the generalized Hölder inequality, we have

$$A_{21}$$
$$\leq C \sum_{k=-\infty}^{0} |2^k B|^{\frac{\beta}{n}-1} \chi_{2^kB\backslash 2^{k-1}B}(x)$$
$$\sum_{j=-\infty}^{k} \|(b(y) - b_{2^jB})\chi_{2^jB}\|_{L^{r(\cdot)}} \|f\chi_{2^jB}\|_{L^{p(\cdot)}} \|\chi_{2^jB}\|_{L^{t(\cdot)}}$$
$$= C \sum_{k=-\infty}^{0} |2^k B|^{\frac{\beta}{n}-1} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=-\infty}^{k} \|b\|_{CBMO^{r(\cdot),\nu}} |2^jB|^{\nu} \|\chi_{2^jB}\|_{L^{r(\cdot)}} \|f\|_{\dot{B}^{p(\cdot),\lambda}}$$
$$|2^jB|^{\lambda} \|\chi_{2^jB}\|_{L^{p(\cdot)}} \|\chi_{2^jB}\|_{L^{t(\cdot)}}$$
$$= C \sum_{k=-\infty}^{0} |2^k B|^{\frac{\beta}{n}-1} \chi_{2^kB\backslash 2^{k-1}B}(x) \|b\|_{CBMO^{r(\cdot),\nu}} \|f\|_{\dot{B}^{p(\cdot),\lambda}}$$
$$\sum_{j=-\infty}^{k} |2^jB|^{\nu+\lambda} |2^jB|^{\frac{1}{t(\cdot)}+\frac{1}{p(\cdot)}+\frac{1}{r(\cdot)}}$$
$$= C\|b\|_{CBMO^{r(\cdot),\nu}} \|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} |2^kB|^{\mu} \chi_{2^kB\backslash 2^{k-1}B}(x).$$

With the Lebesgue space with variable exponent norm on both sides, above inequality takes the following form:

$$\begin{aligned}
\|A_{21}\|_{L^{q(\cdot)}} &\leq C\|b\|_{CBMO^{r(\cdot),\nu}} \|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} |2^kB|^{\mu} \|\chi_{2^kB}\|_{L^{q(\cdot)}} \\
&= C\|b\|_{CBMO^{r(\cdot),\nu}} \|f\|_{\dot{B}^{p(\cdot),\lambda}} |B|^{\frac{1}{q(\cdot)}} |B|^{\mu} \sum_{k=-\infty}^{0} 2^{k(\mu+\frac{1}{q(\cdot)})} \\
&= C\|b\|_{CBMO^{r(\cdot),\nu}} \|f\|_{\dot{B}^{p(\cdot),\lambda}} \|\chi_B\|_{L^{q(\cdot)}} |B|^{\mu} \sum_{k=-\infty}^{0} 2^{k(\mu+\frac{1}{q_+})}.
\end{aligned}$$

Hence

$$\|A_{21}\|_{L^{q(\cdot)}} \leq C\|b\|_{CBMO^{r(\cdot),\nu}} \|f\|_{\dot{B}^{p(\cdot),\lambda}} \|\chi_B\|_{L^{q(\cdot)}} |B|^{\mu}.$$

Finally, consider

$$A_{22} = \sum_{k=-\infty}^{0} |2^k B|^{\frac{\beta}{n}-1} \sum_{j=-\infty}^{k} \int_{2^j B \setminus 2^{j-1} B} |(b_B - b_{2^j B}) f(y)| dy \cdot \chi_{2^k B \setminus 2^{k-1} B}(x).$$

The factor $(b_B - b_{2^j B})$ in the above inequality needs to be dealt with first. So,

$$\begin{aligned} |b_B - b_{2^j B}| &= \sum_{i=j}^{-1} |b_{2^{i+1} B} - b_{2^i B}| \\ &= \sum_{i=j}^{-1} \frac{1}{|2^i B|} \int_{2^i B} |b(y) - b_{2^{i+1} B}| dy \\ &\leq C \sum_{i=j}^{-1} \frac{1}{|2^i B|} \|(b - b_{2^{i+1} B}) \chi_{2^{i+1} B}\|_{L^{r(\cdot)}} \|\chi_{2^{i+1} B}\|_{L^{r'(\cdot)}}. \end{aligned}$$

Next, Lemma 2.1.4 helps us to write

$$\begin{aligned} |b_B - b_{2^j B}| &\leq C \sum_{i=j}^{-1} \frac{1}{|2^i B|} \|(b - b_{2^{i+1} b}) \chi_{2^{i+1} B}\|_{L^{r(\cdot)}} \frac{|2^{i+1} B|}{\|\chi_{2^{i+1} B}\|_{L^{r(\cdot)}}} \\ &\leq C \sum_{i=j}^{-1} \|b\|_{CBMO^{r(\cdot),\nu}} |2^{i+1} B|^{\nu} \\ &\leq C \|b\|_{CBMO^{r(\cdot),\nu}} \sum_{i=j}^{-1} |2^{i+1} B|^{\nu} \\ &\leq C \|b\|_{CBMO^{r(\cdot),\nu}} |2^{j+1} B|^{\nu} |j| \qquad (2.2.5) \end{aligned}$$

In turn, $A_{22}$ satisfies the below inequality:

$$A_{22} \le C \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1} \sum_{j=-\infty}^{k} \|b\|_{CBMO^{r(\cdot),\nu}}|2^{j+1}B|^{\nu}|j|$$

$$\|f\chi_{2^jB}\|_{L^{p(\cdot)}}\|\chi_{2^jB}\|_{L^{p'(\cdot)}}$$

$$\le C\|b\|_{CBMO^{r(\cdot),\nu}} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1} \sum_{j=-\infty}^{k} |2^{j+1}B|^{\nu}|j|$$

$$\|f\chi_{2^jB}\|_{L^{p(\cdot)}}\frac{|2^jB|}{\|\chi_{2^jB}\|_{L^{p(\cdot)}}}$$

$$\le C\|b\|_{CBMO^{r(\cdot),\nu}} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1} \sum_{j=-\infty}^{k} |2^jB|^{\nu+1}|j|\|f\|_{\dot{B}^{p(\cdot),\lambda}}|2^jB|^{\lambda}$$

$$\le C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1} \sum_{j=-\infty}^{k} |2^jB|^{\nu+\lambda+1}|j|$$

$$\le C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} |k||2^kB|^{\mu}\chi_{2^kB\backslash 2^{k-1}B}(x).$$

Ultimately, our last step would be applying the norm on both sides to get

$$\begin{aligned}
\|A_{22}\|_{L^{q(\cdot)}} &\le C\|b\|_{CBMO^{r(\cdot),\mu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} |k||2^kB|^{\mu}\|\chi_{2^kB}\|_{L^{q(\cdot)}} \\
&\le C\|b\|_{CBMO^{r(\cdot),\mu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} |k||2^kB|^{\mu}|2^kB|^{\frac{1}{q(\cdot)}} \\
&\le C\|b\|_{CBMO^{r(\cdot),\mu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} |k||2^k|^{\mu+\frac{1}{q(\cdot)}}|B|^{\mu+\frac{1}{q(\cdot)}} \\
&\le C\|b\|_{CBMO^{r(\cdot),\mu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}}|B|^{\mu}\|\chi_B\|_{L^{q(\cdot)}}.
\end{aligned}$$

Combine all the approximations of $A_1$, $A_2$, $A_{21}$, $A_{22}$, we obtained the required result

$$\|[b, H_\beta]f\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}}|B|^{\mu}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}.$$

## 2.3 Estimates for Adjoint Fractional Hardy Operator and Commutator

In this last section, we first establish the boundedness of adjoint fractional Hardy operator and then use it to prove the boundedness of commutator generated by this operator and $\lambda$-central BMO function $b$. The first result is as under.

**Theorem 2.3.1** *Let $p_1(\cdot) \in P(\mathbb{R}^n)$ and satisfying the condition (2.1.2) and (2.1.3) in Proposition 2.1.1. Define $p_2(\cdot)$ by*

$$\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\beta}{n}.$$

*If $\lambda_2 = \lambda_1 + \frac{\beta}{n}$ and $\lambda_1 < (\delta_2 - 1) - \beta/n$, where $\delta_2$ is the same constant as appeared in inequality (2.1.5), then*

$$\|H_\beta^* f\|_{\dot{B}^{p_2(\cdot),\lambda_2}} \leq C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}}.$$

**proof**

Since

$$\begin{aligned} |H_\beta^* f(x)\chi_k(x)| &\leq \int_{\mathbb{R}^n \setminus B_k} |f(t)||t|^{\beta-n} dt \chi_k(x) \\ &\leq C \sum_{j=k+1}^{\infty} 2^{j(\beta-n)} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \chi_k(x), \end{aligned}$$

from which we infer that

$$\|H_\beta^* f(x)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C \sum_{j=k+1}^{\infty} 2^{j(\beta-n)} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_j\|_{L^{p_1'(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}.$$

Lemma 2.1.4 guide us to have the following inequality:

$$\|H_\beta^* f(x)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C \sum_{j=k+1}^{\infty} 2^{j\beta} \|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_j\|^{-1}_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)},$$

which by Lemma 2.1.7 reduces to the following one

$$\begin{aligned}\|H_\beta^* f(x)\chi_k\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} &\le C\sum_{j=k+1}^{\infty} 2^{j\beta}\|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_j\|^{-1}_{L^{p_1(\cdot)}(\mathbb{R}^n)}2^{-k\beta}\|\chi_k\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\\ &\le C\sum_{j=k+1}^{\infty} 2^{\beta(j-k)}\|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_k\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}\\ &\le C\sum_{j=k+1}^{\infty} 2^{(j-k)(\beta-n\delta_2)}\|f_j\|_{L^{p_1(\cdot)}(\mathbb{R}^n)},\end{aligned}$$

where we made use of inequality (2.1.5) in the last step of the above result. Hence, we obtain

$$\|H_\beta^* f\|_{\dot{B}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \le C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\sum_{j=k+1}^{\infty} 2^{(j-k)(\beta-n\delta_2)}\frac{|B_j|^{\lambda_1}}{|B_k|^{\lambda_2}}\frac{\|\chi_{B_j}\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_{B_k}\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}}.$$

Utilizing the condition $\lambda_2 = \lambda_1 + \beta/n$, a result similar to (2.2.3) and the Lemma 2.1.3, we obtain

$$\|H_\beta^* f\|_{\dot{B}^{p_2(\cdot),\lambda_2}(\mathbb{R}^n)} \le C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(\mathbb{R}^n)}\sum_{j=k+1}^{\infty} 2^{n(j-k)(\beta/n+\lambda_1-\delta_2+1)}.$$

Finally, the series is convergent due to the fact that $\lambda_1 < (\delta_2 - 1) - \beta/n$, and hence the result.

Keeping in view the analysis made in the previous section, we only outline the proof of the following theorem without going into many details.

**Theorem 2.3.2** *Let $p(\cdot), q(\cdot), r(\cdot) \in P(\mathbb{R}^n)$ and satisfying the condition (2.1.2) and (2.1.3) in Proposition 2.1.1 with $p(\cdot) < n/\beta, p'(\cdot) < r(\cdot)$ and*

$$\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{r(\cdot)} - \frac{\beta}{n}.$$

*Let $0 < \nu < 1/n$ and $-1/q_+ < \mu < 0$. If $\mu = \nu + \lambda + \frac{\beta}{n}$, with $\lambda < \min\{(\delta_2 - 1) - \beta/n, -(\nu + \beta/n)\}$, where $\delta_2$ is the same constant as* *appeared in inequality (2.1.4) and* $b \in \|b\|_{CBMO^{r(\cdot),\nu}}$ *then*

$$\|[b, H^*_\beta]f\|_{\dot{B}^{q(\cdot),\mu}} \leq C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}}$$

**proof:**

As in the previous section, we start from decomposing the integral:

$$\begin{aligned}
|[b, H^*_\beta]f(x)\cdot\chi_B(x)| &\leq \left|\int_{|\tau|>|x|}\frac{|(b(x)-b(\tau))f(\tau)|}{|\tau|^{n-\beta}}d\tau\cdot\chi_B(x)\right| \\
&\leq \left|\int_{|\tau|>|x|}\frac{|(b(x)-b_B)f(\tau)|}{|\tau|^{n-\beta}}d\tau\cdot\chi_B(x)\right| \\
&+ \left|\int_{|\tau|>|x|}\frac{|(b(\tau)-b_B)f(\tau)|}{|\tau|^{n-\beta}}d\tau\cdot\chi_B(x)\right| \\
&= D_1 + D_2.
\end{aligned}$$

Following the steps taken to approximate $A_1$ in Theorem 2.2.2, we directly estimate $D_1$ as below

$$\begin{aligned}
\|D_1\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leq C\|(b-b_B)\chi_B\|_{L^{r(\cdot)}(\mathbb{R}^n)}\|(H^*_\beta f)\chi_B\|_{L^{s(\cdot)}(\mathbb{R}^n)} \\
&= C\|b\|_{CBMO^{r(\cdot),\nu}}|B|^\nu\|\chi_B\|_{L^{r(\cdot)}(\mathbb{R}^n)}|B|^\sigma\|\chi_B\|_{L^{s(\cdot)}(\mathbb{R}^n)}\|H^*_\beta f\|_{\dot{B}^{s(\cdot),\sigma}},
\end{aligned}$$

where $\sigma = \lambda + \frac{\beta}{n}$. Now, the result (2.2.4) and the Theorem 2.3.1 helps us to write:

$$\|D_1\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_{CBMO^{r(\cdot),\nu}}|B|^\mu\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|f\|_{\dot{B}^{p(\cdot),\lambda}},$$

Next comparing $D_2$ with $A_2$ of the Theorem 2.2.2, we arrive at:

$$\begin{aligned}
D_2 &\leq \sum_{k=-\infty}^{0}\chi_{2^kB\setminus 2^{k-1}B}(x)\sum_{j=k+1}^{\infty}|2^jB|^{\frac{\beta}{n}-1}\int_{2^jB\setminus 2^{j-1}B}|(b(\tau)-b_{2^jB})f(\tau)|d\tau \\
&+ \sum_{k=-\infty}^{0}\chi_{2^kB\setminus 2^{k-1}B}(x)\sum_{j=k+1}^{\infty}|2^jB|^{\frac{\beta}{n}-1}\int_{2^jB\setminus 2^{j-1}B}|(b_B-b_{2^jB})f(\tau)|d\tau \\
&= D_{21} + D_{22}
\end{aligned}$$

And for the approximation of $D_{21}$, we follow a procedure similar to one followed in the estimation of $A_{21}$. Hence, we get

$$D_{21} \leq C \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1} \|(b(\tau)-b_{2^jB})\chi_{2^jB}\|_{L^{r(\cdot)}}$$
$$\|f\chi_{2^jB}\|_{L^{p(\cdot)}}\|\chi_{2^jB}\|_{L^{t(\cdot)}}$$
$$= C \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1} \|b\|_{CBMO^{r(\cdot),\nu}}|2^jB|^{\nu}\|\chi_{2^jB}\|_{L^{r(\cdot)}}$$
$$\|f\|_{\dot{B}^{p(\cdot),\lambda}}|2^jB|^{\lambda}\|\chi_{2^jB}\|_{L^{p(\cdot)}}\|\chi_{2^jB}\|_{L^{t(\cdot)}}$$
$$= C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1}$$
$$|2^jB|^{\nu+\lambda}|2^jB|^{\frac{1}{t(\cdot)}+\frac{1}{p(\cdot)}+\frac{1}{r(\cdot)}}$$
$$= C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\mu}$$
$$= C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} |2^{(k+1)}B|^{\mu}\chi_{2^kB\backslash 2^{k-1}B}(x).$$

Eventually, it is easy to see that

$$\|D_{21}\|_{L^{q(\cdot)}} \leq C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} |2^{(k+1)}B|^{\mu}\|\chi_{2^kB}\|_{L^{q(\cdot)}}$$
$$\leq C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} |2^{(k+1)}B|^{\mu}|2^kB|^{\frac{1}{q(\cdot)}}$$
$$\leq C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}}|B|^{\mu+\frac{1}{q(\cdot)}} \sum_{k=-\infty}^{0} 2^{(k+1)(\mu+\frac{1}{q(\cdot)})}$$
$$\leq C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}}|B|^{\mu}\|\chi_B\|_{L^{q(\cdot)}}.$$

Next, by virtue of inequality (2.2.5), $D_{22}$ satisfies:

$$
\begin{aligned}
D_{22} \le C \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1}\|b\|_{CBMO^{r(\cdot),\nu}}|2^{j+1}B|^{\nu}|j| \\
\|f\chi_{2^jB}\|_{L^{p(\cdot)}}\|\chi_{2^jB}\|_{L^{p'(\cdot)}} \\
\le C\|b\|_{CBMO^{r(\cdot),\nu}} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1}|2^{j+1}B|^{\nu}|j| \\
\|f\chi_{2^jB}\|_{L^{p(\cdot)}}\frac{|2^jB|}{\|\chi_{2^jB}\|_{L^{p(\cdot)}}} \\
\le C\|b\|_{CBMO^{r(\cdot),\nu}} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1}|2^jB|^{\nu+1}|j|\|f\|_{\dot{B}^{p(\cdot),\lambda}}|2^jB|^{\lambda} \\
\le C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\nu+\lambda+\frac{\beta}{n}}|j| \\
\le C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^{k+1}B|^{\mu}|k+1| \\
\le C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} |k+1||2^{k+1}B|^{\mu}\chi_{2^kB\backslash 2^{k-1}B}(x).
\end{aligned}
$$

To finish the estimation, we take norm on both sides of the above inequality to obtain:

$$
\begin{aligned}
\|D_{22}\|_{L^{q(\cdot)}} &\le C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} |k+1||2^{k+1}B|^{\mu}\|\chi_{2^kB}\|_{L^{q(\cdot)}} \\
&\le C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} |k+1||2^{k+1}B|^{\mu}|2^kB|^{\frac{1}{q(\cdot)}} \\
&\le C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}} \sum_{k=-\infty}^{0} |k+1||2^{k+1}|^{\mu+\frac{1}{q(\cdot)}}|B|^{\mu+\frac{1}{q(\cdot)}} \\
&\le C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}}|B|^{\mu}\|\chi_B\|_{L^{q(\cdot)}}.
\end{aligned}
$$

In the end, combining all the estimates of $D_1$, $D_2$, $D_{21}$, $D_{22}$ we arrive at the following conclusive inequality:

$$\|[b, H_\beta^*]f\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_{CBMO^{r(\cdot),\nu}}\|f\|_{\dot{B}^{p(\cdot),\lambda}}|B|^\mu\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)},$$

which is as desired.

## 2.4 Conclusion

By utilizing, some conditions and lemma we have bounded the Hardy operator on Central Morrey space and for their commutator also.

# Chapter 3

# Boundedness for the Fractional rough Hardy Operators and Commutators on Variable $\lambda$-central Morrey Space

## 3.1 Introduction

In this chapter we are going to examine the boundedness for Rough fractional Hardy operators as well as boundedness of commutators on variable exponent $\lambda-$ central Morrey space.

Let's clarify the framework of this chapter. In section 3.2 of this chapter, we will demonstrate the boundedness for rough Hardy operators in central Morrey space about variable exponent. In section 3.3, we shall be able to investigate the estimates of commutator led by rough Hardy operator and $\lambda-$ central BMO function on central Morrey space along with variable exponent.

## 3.2 Boundedness of Fractional Hardy Operators

**Theorem 3.2.1** *Assume that* $\Omega \in L^s(S^{n-1})$, $\frac{n}{n-1} < s$. *Let* $q_1(\cdot), p(\cdot), q_2(\cdot) \in P(\mathbb{R}^n)$ *hold inequalities (2.1.2) and (2.1.3) in proposition 2.1.1 and de-*

*fine the variable exponent* $p(\cdot)$ *by*

$$\frac{1}{q_1(\cdot)} = \frac{1}{p(\cdot)} + \frac{\beta}{n}.$$

*Let* $\lambda_1$ *satisfy the following condition:*

*When* $\frac{1}{q_2(\cdot)} = \frac{1}{q_1'(\cdot)} - \frac{1}{s}$, *there is* $-\lambda_1 > \frac{\beta}{n}$ *and* $\lambda_2 = \lambda_1 + \frac{\beta}{n}$. *If* $\delta_3 - \frac{1}{s} + \lambda_2 + \delta_1 > 0$, *then the fractional rough Hardy operator is bounded from* $\dot{B}^{q_1(\cdot),\lambda_1}$ *to* $\dot{B}^{p(\cdot),\lambda_2}$ *and the following inequality satisfy*

$$||H_{\beta,\Omega} f \cdot \chi_k||_{\dot{B}^{p(\cdot),\lambda_2}} \leq C||f||_{\dot{B}^{q_1(\cdot),\lambda_1}}.$$

**proof**

$$|H_{\beta,\Omega} f(x) \cdot \chi_k(x)| \leq \frac{1}{|x|^{n-\beta}} \int_{B_k} |f(t)||\Omega(x-t)|dt \cdot \chi_k(x)$$

$$\leq C2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\Omega(x-t)\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \chi_k(x).$$

By using of $\frac{1}{q_2(\cdot)} + \frac{1}{s} = \frac{1}{q_1'(\cdot)}$

$$|H_{\beta,\Omega} f(x) \cdot \chi_k(x)|$$
$$\leq C2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\Omega(x-t)\chi_j\|_{L^s(\mathbb{R}^n)} \|\chi_j\|_{L^{q_2(\cdot)}} \cdot \chi_k(x).$$

$$\|H_{\beta,\Omega} f \cdot \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\leq C2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\Omega(x-t)\chi_j\|_{L^s(\mathbb{R}^n)} \|\chi_j\|_{L^{q_2(\cdot)}} \cdot \|\chi_k\|_{L^{p(\cdot)}}.$$

Since we have

$$\|\chi_k\|_{L^{q_2 \cdot}} \approx |B_k|^{\frac{1}{q_2(\cdot)}} \approx |B_k|^{\frac{1}{q_1'(\cdot)} - \frac{1}{s}} \approx |B_k|^{-\frac{1}{s}} \|\chi_k\|_{L^{q_1'(\cdot)}}$$

$$\|H_{\beta,\Omega} f \cdot \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\leq C2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\Omega(x-t)\chi_j\|_{L^s(\mathbb{R}^n)} |B_j|^{-\frac{1}{s}} \|\chi_j\|_{L^{q_1'(\cdot)}} \|\chi_k\|_{L^{p(\cdot)}}. \tag{3.2.1}$$

Based on preposition 2.1.6, we have

$$I_\beta(\chi_{B_k})(x) \geq C2^{k\beta}\chi_{B_k}(x)$$

$$\chi_{B_k}(x) \leq C2^{-k\beta}I_\beta(\chi_{B_k})(x)$$

$$\begin{aligned} \|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq C2^{-k\beta}\|I_\beta\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leq C2^{-k\beta}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\ &\leq C2^{k(n-\beta)}\|\chi_{B_k}\|^{-1}_{L^{q_1'(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.2.2}$$

Using inequality (3.2.2) in (3.2.1)

$$\begin{aligned} &\|H_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\leq C\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\chi_j\|_{L^s(\mathbb{R}^n)}|B_j|^{-\frac{1}{s}}\|\chi_j\|_{L^{q_1'(\cdot)}}\|\chi_k\|^{-1}_{L^{q_1'(\cdot)}}. \end{aligned} \tag{3.2.3}$$

Using condition (2.1.5)

$$\|H_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\sum_{j=-\infty}^{k}2^{(j-k)n\delta_3}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\chi_j\|_{L^s(\mathbb{R}^n)}|B_j|^{-\frac{1}{s}}. \tag{3.2.4}$$

For $t \in C_j$ and $x \in C_k$ and $j \leq k$, we have $0 \leq |x-t| \leq |x| + 2^j \leq 2.2^k$, and

$$\int_{C_j}|\Omega(x-t)|^s dt \leq \int_0^{2^{k+1}}\int_{s^{n-1}}|\Omega(x^{'})|^s d\sigma(x^{'})r^{n-1}dr \leq C2^{kn}$$

$$\begin{aligned} \|H_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq C\sum_{j=-\infty}^{k}2^{(j-k)(n\delta_3-\frac{n}{s})}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\ &\leq C\sum_{j=-\infty}^{k}2^{(j-k)(n\delta_3-\frac{n}{s})}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)}|B_j|^{\lambda_1}||\chi_j||_{L^{q_1(\cdot)}}. \end{aligned} \tag{3.2.5}$$

$$\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \approx |B|^{\frac{1}{q_1(\cdot)}} \approx |B|^{\frac{1}{p(\cdot)}+\frac{\beta}{n}} \approx |B|^{\frac{\beta}{n}}\|\chi_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\|H_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\sum_{j=-\infty}^{k}2^{(j-k)(n\delta_3-\frac{n}{s})}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)}|B_j|^{\lambda_1+\frac{\beta}{n}}||\chi_j||_{L^{p(\cdot)}}$$

$$\le C\sum_{j=-\infty}^{k}2^{(j-k)(n\delta_3-\frac{n}{s})}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)}\frac{|B_j|^{\lambda_2}}{|B_k|^{\lambda_2}}|B_k|^{\lambda_2}\frac{||\chi_j||_{L^{p(\cdot)}}}{||\chi_k||_{L^{p(\cdot)}}}||\chi_k||_{L^{p(\cdot)}}. \tag{3.2.6}$$

Using inequality (2.1.4)

$$\|H_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C\sum_{j=-\infty}^{k}2^{n(j-k)(\delta_3-\frac{1}{s}+\lambda_2+\delta_1)}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)}|B_k|^{\lambda_2}||\chi_k||_{L^{p(\cdot)}}. \tag{3.2.7}$$

Since $\delta_3-\frac{1}{s}+\lambda_2+\delta_1>0$, we get

$$\|H_{\beta,\Omega}f\cdot\chi_k\|_{\dot{B}^{p(\cdot),\lambda_2}(\mathbb{R}^n)}\le C\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)}. \tag{3.2.8}$$

**Theorem 3.2.2** *Let $p(\cdot), q_1(\cdot)$, $q_2(\cdot)$ and $\beta$ are defined same as in theorem 3.2.1 and $\Omega\in L^s(S^{n-1})$. If $\lambda_2=\lambda_1+\frac{\beta}{n}$ and $\lambda_2<\frac{1}{s}-\frac{\beta}{n}-1$, then*

$$\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{\dot{B}^{p(\cdot),\lambda_2}}\le C\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}.$$

**proof**

$$|H^*_{\beta,\Omega}f(x)\cdot\chi_k|\le\int_{\mathbb{R}^n\setminus B_k}|f(t)\Omega(x-t)||t|^{\beta-n}dt\cdot\chi_k(x)$$

$$\le C\sum_{j=k+1}^{\infty}2^{j(\beta-n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\cdot\chi_k(x).$$

By using $\frac{1}{q_2(\cdot)}+\frac{1}{s}=\frac{1}{q_1'(\cdot)}$

$$\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\le C\sum_{j=k+1}^{\infty}2^{j(\beta-n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\|_{L^s}\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Using inequality (3.2.2)

$$\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\sum_{j=k+1}^{\infty}2^{(j-k)(\beta-n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\|_{L^s}\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|\chi_k\|^{-1}_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.$$

As we know

$$\|\chi_j\|_{L^{q_2(\cdot)}}\approx|B_j|^{\frac{1}{q_2(\cdot)}}\approx|B_j|^{\frac{1}{q_1'(\cdot)}-\frac{1}{s}}\approx|B_j|^{-\frac{1}{s}}\|\chi_j\|_{L^{q_1'(\cdot)}}$$

$$\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\sum_{j=k+1}^{\infty}2^{(j-k)(\beta-n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\|_{L^s}|B_j|^{-\frac{1}{s}}\|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|^{-1}_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.$$

Now by using lemma (2.1.3)

$$\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\sum_{j=k+1}^{\infty}2^{(j-k)(\beta-n+n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\|_{L^s}|B_j|^{-\frac{1}{s}}.$$

For further calculation follows theorem 3.2.1, we get

$$\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\sum_{j=k+1}^{\infty}2^{(j-k)(\beta-\frac{n}{s})}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.$$

Hence we have

$$\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{\dot{B}^{p(\cdot),\lambda_2}(\mathbb{R}^n)}\leq C\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)}\sum_{j=k+1}^{\infty}2^{(j-k)(\beta-\frac{n}{s}+n\lambda_2+n)}.$$

By using $\lambda_2<\frac{1}{s}-\frac{\beta}{n}-1$ we get the required result

$$\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{\dot{B}^{p(\cdot),\lambda_2}(\mathbb{R}^n)}\leq C\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

## 3.3 Boundedness of Commutators for Fractional Hardy Operators

**Theorem 3.3.1** *Let* $0 < \beta < n, \Omega \in L^s(S^{n-1}),\ \frac{n}{n-1} < s$. *Let* $q_1(\cdot), p(\cdot), q(\cdot) \in P(\mathbb{R}^n)$ *hold conditions (2.1.2) and (2.1.3) in proposition 2.1.1 and variable exponent* $q_2(\cdot)$ *define by*

$$\frac{1}{q_1(\cdot)} = \frac{1}{q_2(\cdot)} - \frac{1}{q(\cdot)} + \frac{\beta}{n}.$$

*Let* $\lambda_1$ *satisfy the following condition:*

*When* $\frac{1}{q(\cdot)} = \frac{1}{q_1'(\cdot)} - \frac{1}{s}$, *there is* $\lambda_1 > -\lambda - \frac{\beta}{n}$ *and* $\lambda_2 = \lambda_1 + \lambda + \frac{\beta}{n}$. *If* $b \in \|b\|_{CBMO^{q(\cdot),\lambda}}$ *and* $\delta_3 - \frac{1}{s} + \lambda_2 + \delta_1 > 0$, *then the following inequality satisfy*

$$||[b, H_{\beta,\Omega}]f \cdot \chi_k||_{\dot{B}^{q_2(\cdot),\lambda_2}} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}}||f||_{\dot{B}^{q_1(\cdot),\lambda_1}}.$$

**proof**

$$\begin{aligned}
|[b, H_\beta]f(x) \cdot \chi_B(x)| &\leq \frac{1}{|x|^{n-\beta}} \int_{B(0,|x|)} |f(\tau)(b(x) - b(\tau))\Omega(x - \tau)|d\tau \cdot \chi_B(x) \\
&\leq |x|^{-n+\beta} \int_{B(0,|x|)} |f(\tau)(b(x) - b_B)\Omega(x - \tau)|d\tau \cdot \chi_B(x) \\
&+ |x|^{-n+\beta} \int_{B(0,|x|)} |f(\tau)(b(\tau) - b_B)\Omega(x - \tau)|d\tau \cdot \chi_B(x) \\
&= A_1 + A_2.
\end{aligned}$$

First we estimate $A_1$. Denote $\frac{1}{p(x)} = \frac{1}{q_1(x)} - \frac{\beta}{n}$, then $\frac{1}{q_2(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}$.

$$\begin{aligned}
A_1 &= |x|^{-n+\beta} \int_{B(0,|x|)} |f(\tau)(b(x) - b_B)\Omega(x - \tau)|d\tau \cdot \chi_B(x) \\
&= |(b(x) - b_B)\chi_B(x)||H_{\beta,\Omega}f(x)|,
\end{aligned}$$

$$\|A_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} = \|(b(x) - b_B)\chi_B(x)H_{\beta,\Omega}f\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}.$$

By using Holder inequality $(\frac{1}{q_2(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)})$

$$\begin{aligned}\|A_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} &\leq C\|(b(x)-b_B)\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|H_{\beta,\Omega}f\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &= C\|b\|_{CBMO^{q(\cdot),\lambda}}|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}|B|^{\mu}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|H_{\beta,\Omega}f\|_{\dot{B}^{\mu,p(\cdot)}},\end{aligned}$$

$$\|A_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}}|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}|B|^{\mu}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|H_{\beta,\Omega}f\|_{\dot{B}^{\mu,p(\cdot)}}.$$

Given that $\mu = \lambda_1 + \frac{\beta}{n}$, using the result of theorem 3.2.1

$$\|A_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}}|B|^{\lambda_2}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}},$$

$$\begin{aligned}A_2 &= \sum_{k=-\infty}^{0}\frac{1}{|x|^{n-\beta}}\sum_{j=-\infty}^{k}\int_{2^jB\setminus 2^{j-1}B}|f(\tau)(b(\tau)-b_B)\Omega(x-\tau)|d\tau\cdot\chi_{2^kB\setminus 2^{k-1}B}(x) \\ &\leq \sum_{k=-\infty}^{0}\frac{1}{|x|^{n-\beta}}\sum_{j=-\infty}^{k}\int_{2^jB\setminus 2^{j-1}B}|f(\tau)(b(\tau)-b_{2^jB})\Omega(x-\tau)|d\tau\cdot\chi_{2^kB\setminus 2^{k-1}B}(x) \\ &+ \sum_{k=-\infty}^{0}\frac{1}{|x|^{n-\beta}}\sum_{j=-\infty}^{k}\int_{2^jB\setminus 2^{j-1}B}|f(\tau)(b_B-b_{2^jB})\Omega(x-\tau)|d\tau\cdot\chi_{2^kB\setminus 2^{k-1}B}(x) \\ &= A_{21}+A_{22}\end{aligned}$$

$$A_{21} = \sum_{k=-\infty}^{0}|2^kB|^{\frac{\beta}{n}-1}\sum_{j=-\infty}^{k}\int_{2^jB\setminus 2^{j-1}B}|f(\tau)(b(\tau)-b_{2^jB})\Omega(x-\tau)|d\tau\cdot\chi_{2^kB\setminus 2^{k-1}B}(x)$$

Using Holder inequality $(\frac{1}{q(\cdot)} + \frac{1}{q_1(\cdot)} + \frac{1}{s} = 1)$.

$$A_{21} \leq C \sum_{k=-\infty}^{0} |2^k B|^{\frac{\beta}{n}-1} \chi_{2^k B \setminus 2^{k-1}B}(x) \sum_{j=-\infty}^{k} \|(b(\tau) - b_{2^j B})\chi_{2^j B}\|_{L^{q(\cdot)}}$$

$$\|f\chi_{2^j B}\|_{L^{q_1(\cdot)}} \|\Omega(x-\tau)\chi_{2^j B}\|_{L^s}$$

$$= C \sum_{k=-\infty}^{0} |2^k B|^{\frac{\beta}{n}-1} \chi_{2^k B \setminus 2^{k-1}B}(x) \sum_{j=-\infty}^{k} \|b\|_{CBMO^{q(\cdot),\lambda}} |2^j B|^{\lambda} \|\chi_{2^j B}\|_{L^{q(\cdot)}}$$

$$\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} |2^j B|^{\lambda_1} \|\chi_{2^j B}\|_{L^{q_1(\cdot)}} \|\Omega(x-\tau)\chi_{2^j B}\|_{L^s}$$

$$= C \sum_{k=-\infty}^{0} |2^k B|^{\frac{\beta}{n}-1} \chi_{2^k B \setminus 2^{k-1}B}(x) \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}$$

$$\sum_{j=-\infty}^{k} |2^j B|^{\lambda+\lambda_1} |2^j B|^{\frac{1}{s}+\frac{1}{q_1(\cdot)}+\frac{1}{q(\cdot)}}$$

$$= C \sum_{k=-\infty}^{0} |2^k B|^{\frac{\beta}{n}-1} \chi_{2^k B \setminus 2^{k-1}B}(x) \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}$$

$$\sum_{j=-\infty}^{k} |2^j|^{\lambda+\lambda_1+1} |B|^{\lambda+\lambda_1+1}$$

$$= C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |2^k|^{\frac{\beta}{n}+\lambda+\lambda_1} \chi_{2^k B \setminus 2^{k-1}B}(x) |B|^{\lambda+\lambda_1+\frac{\beta}{n}}$$

$$\|A_{21}\|_{L^{q_2(\cdot)}} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |2^k|^{\frac{\beta}{n}+\lambda+\lambda_1} \|\chi_{2^k B}\|_{L^{q_2(\cdot)}} |B|^{\lambda+\lambda_1+\frac{\beta}{n}}$$

$$= C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |2^k|^{\frac{\beta}{n}+\lambda+\lambda_1} |2^k B|^{\frac{1}{q_2(\cdot)}} |B|^{\lambda+\lambda_1+\frac{\beta}{n}}$$

$$= C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} |B|^{\frac{1}{q_2(\cdot)}} |B|^{\lambda_2} \sum_{k=-\infty}^{0} |2|^{k(\lambda_2+\frac{1}{q_2(\cdot)})}$$

$$\|A_{21}\|_{L^{q_2(\cdot)}} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \|\chi_B\|_{L^{q_2(\cdot)}} |B|^{\lambda_2}$$

$$A_{22} = \sum_{k=-\infty}^{0} |2^k B|^{\frac{\beta}{n}-1} \sum_{j=-\infty}^{k} \int_{2^j B \setminus 2^{j-1}B} |(b_B - b_{2^j B}) f(\tau)\Omega(x-\tau)| d\tau \cdot \chi_{2^k B \setminus 2^{k-1}B}(x)$$

$$\begin{aligned}
|(b_B - b_{2^j B})| &= \sum_{i=j}^{-1} |(b_{2^{i+1}B} - b_{2^i B})\\
&= \sum_{i=j}^{-1} \frac{1}{|2^i B|} \int_{2^i B} |b(\tau) - b_{2^{i+1}B}| dy\\
&\leq C \sum_{i=j}^{-1} \frac{1}{|2^i B|} \|(b - b_{2^{i+1}b})\chi_{2^{i+1}B}\|_{L^{q(\cdot)}} \|\chi_{2^{i+1}B}\|_{L^{q'(\cdot)}}
\end{aligned}$$

By virtue of lemma 2.1.4, we have

$$\begin{aligned}
|(b_B - b_{2^j B})| &\leq C \sum_{i=j}^{-1} \frac{1}{|2^i B|} \|(b - b_{2^{i+1}b})\chi_{2^{i+1}B}\|_{L^{q(\cdot)}} \frac{|2^{i+1}B|}{\|\chi_{2^{i+1}B}\|_{L^{q(\cdot)}}}\\
&\leq C \sum_{i=j}^{-1} \|b\|_{CBMO^{q(\cdot),\lambda}} |2^{i+1}B|^{\lambda}\\
&\leq C \|b\|_{CBMO^{q(\cdot),\lambda}} \sum_{i=j}^{-1} |2^{i+1}B|^{\lambda}\\
&\leq C \|b\|_{CBMO^{q(\cdot),\lambda}} |2^{j+1}B|^{\lambda} |j| \qquad (3.3.1)
\end{aligned}$$

$$A_{22} \leq C \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1} \sum_{j=-\infty}^{k} \|b\|_{CBMO^{q(\cdot),\lambda}}|2^{j+1}B|^{\lambda}|j|$$

$$\|f\chi_{2^jB}\|_{L^{q_1(\cdot)}}\|\Omega(x-\tau)\chi_{2^jB}\|_{L^s}\|\chi_{2^jB}\|_{L^{q(\cdot)}}$$

$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1} \sum_{j=-\infty}^{k} |2^jB|^{\lambda+\lambda_1}|j|\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}$$

$$\|\chi_{2^jB}\|_{L^{q_1(\cdot)}}\|\Omega(x-\tau)\chi_{2^jB}\|_{L^s}\|\chi_{2^jB}\|_{L^{q(\cdot)}}$$

$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1} \sum_{j=-\infty}^{k} |2^jB|^{\lambda+\lambda_1}|j|$$

$$\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}|2^jB|^{\frac{1}{q_1(\cdot)}+\frac{1}{s}+\frac{1}{q(\cdot)}}$$

$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1} \sum_{j=-\infty}^{k} |2^jB|^{\lambda+\lambda_1+1}|j|$$

$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1}|2^kB|^{\lambda+\lambda_1+1}|k|$$

$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |k||2^kB|^{\lambda+\lambda_1+\frac{\beta}{n}}\chi_{2^kB\backslash 2^{k-1}B}(x)$$

$$\|A_{22}\|_{L^{q_2(\cdot)}} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |k||2^kB|^{\lambda+\lambda_1+\frac{\beta}{n}}\|\chi_{2^kB}\|_{L^{q_2(\cdot)}}$$

$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |k||2^kB|^{\lambda+\lambda_1+\frac{\beta}{n}}|2^kB|^{\frac{1}{q_2(\cdot)}}$$

$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |k||2^k|^{\lambda_2+\frac{1}{q_2(\cdot)}}|B|^{\lambda_2+\frac{1}{q_2(\cdot)}}$$

$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}|B|^{\lambda_2}\|\chi_B\|_{L^{q_2(\cdot)}}$$

Combine all results of $A_1$, $A_2$, $A_{21}$, $A_{22}$, we obtained the required result

$$\|[b,H_{\beta,\Omega}]f\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}|B|^{\lambda_2}\|\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}$$

$$\|[b,H_{\beta,\Omega}]f\chi_B\|_{\dot{B}^{q_2(\cdot),\lambda_2}} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}$$

**Theorem 3.3.2** *Let $p(\cdot), q_1(\cdot)$, $q_2(\cdot)$ and $\beta$ are defined same as in theorem 3.2.1 and $\Omega \in L^s(S^{n-1})$. If $b \in \|b\|_{CBMO^{q_1(\cdot),v}}$ , $\lambda_2 = \lambda + \lambda_1 + \frac{\beta}{n}$ and $\beta < n(1 - \delta_3 - \delta_1 - \lambda_2 + \frac{1}{s})$, then*

$$\|[b, H^*_{\beta,\Omega}]f\|_{\dot{B}^{q_2(\cdot),\lambda_2}} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}$$

**proof**

$$\begin{aligned}
|[b, H^*_{\beta,\Omega}]f(x) \cdot \chi_B(x)| &\leq \int_{B(0,|x|)^c} \frac{|f(\tau)(b(x) - b(\tau))\Omega(x-\tau)|}{|\tau|^{n-\beta}} d\tau \cdot \chi_B(x) \\
&\leq \int_{B(0,|x|)^c} \frac{|f(\tau)(b(x) - b_B)\Omega(x-\tau)|}{|\tau|^{n-\beta}} d\tau \cdot \chi_B(x) \\
&+ \int_{B(0,|x|)^c} \frac{|f(\tau)(b(\tau) - b_B)\Omega(x-\tau)|}{|\tau|^{n-\beta}} d\tau \cdot \chi_B(x) \\
&= D_1 + D_2.
\end{aligned}$$

$$\begin{aligned}
D_1 &= \int_{B(0,|x|)^c} \frac{|f(\tau)(b(x) - b_B)\Omega(x-\tau)|}{|\tau|^{n-\beta}} d\tau \cdot \chi_B(x) \\
&= |(b(x) - b_B)\chi_B(x)||H^*_{\beta,\Omega}f(x)|,
\end{aligned}$$

$$\|D_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} = \|(b(x) - b_B)\chi_B(x)H^*_{\beta,\Omega}f(x)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}.$$

By using Holder inequality ($\frac{1}{q_2(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}$)

$$\begin{aligned}
\|D_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} &\leq C\|(b(x) - b_B)\chi_B(x)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|H^*_{\beta,\Omega}f(x)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&= C\|b\|_{CBMO^{q(\cdot),\lambda}}|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}|B|^{\mu}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|H^*_{\beta,\Omega}f\|_{\dot{B}^{\mu,p(\cdot)}},
\end{aligned}$$

Given that $\mu = \lambda_1 + \frac{\beta}{n}$, using the result of theorem 3.2.2

$$\|D_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}}|B|^{\lambda_2}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}},$$

$$D_2 = \int_{B(0,|x|)^c} \frac{|f(\tau)(b(\tau) - b_B)\Omega(x-\tau)|}{|\tau|^{n-\beta}} dt \cdot \chi_B(x).$$

$$D_2 = \sum_{k=-\infty}^{0} \int_{2^jB\setminus 2^{j-1}B} \frac{|f(\tau)(b(\tau)-b_B)\Omega(x-\tau)|}{|\tau|^{n-\beta}} d\tau \cdot \chi_{2^kB\setminus 2^{k-1}B}(x)$$

$$\leq \sum_{k=-\infty}^{0} \chi_{2^kB\setminus 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1} \int_{2^jB\setminus 2^{j-1}B} |f(\tau)(b(\tau)-b_{2^jB})\Omega(x-\tau)|dt$$

$$+ \sum_{k=-\infty}^{0} \chi_{2^kB\setminus 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1} \int_{2^jB\setminus 2^{j-1}B} |f(\tau)(b_B-b_{2^jB})\Omega(x-\tau)|d\tau$$

$$= D_{21} + D_{22}$$

$$D_{21} = \sum_{k=-\infty}^{0} \chi_{2^kB\setminus 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1} \int_{2^jB\setminus 2^{j-1}B} |f(\tau)(b(\tau)-b_{2^jB})\Omega(x-\tau)|d\tau$$

Using Holder inequality $(\frac{1}{q(\cdot)} + \frac{1}{q_1(\cdot)} + \frac{1}{s} = 1)$.

$$D_{21} \leq C \sum_{k=-\infty}^{0} \chi_{2^kB\setminus 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1} \|(b(\tau)-b_{2^jB})\chi_{2^jB}\|_{L^{q(\cdot)}}$$

$$\|f\chi_{2^jB}\|_{L^{q_1(\cdot)}} \|\Omega(x-\tau)\chi_{2^jB}\|_{L^s}$$

$$= C \sum_{k=-\infty}^{0} \chi_{2^kB\setminus 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1} \|b\|_{CBMO^{q(\cdot),\lambda}} |2^jB|^{\lambda} \|\chi_{2^jB}\|_{L^{q(\cdot)}}$$

$$\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} |2^jB|^{\lambda_1} \|\chi_{2^jB}\|_{L^{q_1(\cdot)}} \|\Omega(x-\tau)\chi_{2^jB}\|_{L^s}$$

$$= C \sum_{k=-\infty}^{0} \chi_{2^kB\setminus 2^{k-1}B}(x) \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}$$

$$\sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1} |2^jB|^{\lambda+\lambda_1} |2^jB|^{\frac{1}{s}+\frac{1}{q_1(\cdot)}+\frac{1}{q(\cdot)}}$$

$$= C \sum_{k=-\infty}^{0} \chi_{2^kB\setminus 2^{k-1}B}(x) \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{j=k+1}^{\infty} |2^jB|^{\lambda_2}$$

$$= C \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |2^{(k+1)}B|^{\lambda_2} \chi_{2^kB\setminus 2^{k-1}B}(x)$$

$$\|D_{21}\|_{L^{q_2(\cdot)}} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |2^{(k+1)}B|^{\lambda_2}\|\chi_{2^kB}\|_{L^{q_2(\cdot)}}$$

$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |2^{(k+1)}B|^{\lambda_2}|2^kB|^{\frac{1}{q_2(\cdot)}}$$

$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}|B|^{\lambda_2+\frac{1}{q_2(\cdot)}} \sum_{k=-\infty}^{0} |2^{(k+1)}|^{\lambda_2+\frac{1}{q_2(\cdot)}}$$

$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}|B|^{\lambda_2+\frac{1}{q_2(\cdot)}}$$

$$D_{22} = \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1} \int_{2^jB\backslash 2^{j-1}B} |(b_B - b_{2^jB})f(\tau)|d\tau$$

Here we use inequality (2.2.5)

$$D_{22} \leq C \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1}\|b\|_{CBMO^{q(\cdot),\lambda}}|2^{j+1}B|^{\lambda}|j|\|f\chi_{2^jB}\|_{L^{q_1(\cdot)}}$$

$$\|\Omega(x-\tau)\chi_{2^jB}\|_{L^s}\|\chi_{2^jB}\|_{L^{q(\cdot)}}$$

$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1}|2^jB|^{\lambda+\lambda_1}|j|\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}$$

$$\|\Omega(x-\tau)\chi_{2^jB}\|_{L^s}\|\chi_{2^jB}\|_{L^{q(\cdot)}}\|\chi_{2^jB}\|_{L^{q_1(\cdot)}}$$

$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\lambda+\lambda_1+\frac{\beta}{n}}|j|$$

$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^{k+1}B|^{\lambda_2}|k+1|$$

$$\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |k+1||2^{k+1}B|^{\lambda_2}\chi_{2^kB\backslash 2^{k-1}B}(x)$$

$$
\begin{aligned}
\|D_{22}\|_{L^{q_2(\cdot)}} &\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\sum_{k=-\infty}^{0}|k+1||2^{k+1}B|^{\lambda_2}\|\chi_{2^kB}\|_{L^{q_2(\cdot)}} \\
&\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\sum_{k=-\infty}^{0}|k+1||2^{k+1}B|^{\lambda_2}|2^kB|^{\frac{1}{q_2(\cdot)}} \\
&\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\sum_{k=-\infty}^{0}|k+1||2^{k+1}|^{\lambda_2+\frac{1}{q_2(\cdot)}}|B|^{\lambda_2+\frac{1}{q_2(\cdot)}} \\
&\leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}|B|^{\lambda_2}\|\chi_B\|_{L^{q_2(\cdot)}}
\end{aligned}
$$

Combine all results of $D_1$, $D_2$, $D_{21}$, $D_{22}$, we obtained the required result

$$
\|[b,H^*_{\beta,\Omega}]f\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}|B|^{\lambda_2}\|\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}
$$

$$
\|[b,H^*_{\beta,\Omega}]f\chi_B\|_{\dot{B}^{q_2(\cdot),\lambda_2}} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}.
$$

## 3.4 Conclusion

From the above four results we come by the boundedness of Rough Hardy operator on central Morrey space. Also, the boundedness of commutator generated by rough Hardy operator on central Morrey space.

# Chapter 4
# Weighted Variable Morrey-Herz Estimates for Fractional Hardy Operators

## 4.1 Introduction

The aim of this chapter is to study the continuity criteria for fractional type Hardy operators on weighted variable exponents Herz-Morrey spaces. It is worth mentioning here that our idea is based on Muckenhoupt theory and on Banach function spaces. The variable Lebegue spaces boundedness property of Riesz potential is reported in [96]. On the weighted Herz spaces, the boundedness of fractional Integral operator is obtained by Izuki and Noi [81].

The presentation of this chapter include two sections. In Section 4.2, we furnish key lemmas which are helpful in proving our main results in Section 4.3.

## 4.2 Key Lemmas

This section begins with some useful lemmas that will aid in the proof of our results.

**Lemma 4.2.1** *[98] If $X$ is Banach function space. Then* *(i) the associated space $X'$ is also Banach function space.*

*(ii) $\|\cdot\|_{(X')'}$ and $\|\cdot\|_X$ are equivalent.*

*(iii) If $g \in X$ and $f \in X'$, then*

$$\int_{\mathbb{R}^n} |f(x)g(x)| \leq \|g\|_X \|f\|_{X'}$$

*is the generalized Hölder inequality.*

**Lemma 4.2.2** *Suppose $X$ is a Banach function space, we have that for all balls $B$,*

$$1 \leq \frac{1}{|B|}\|\chi_B\|_X \|\chi_B\|_{X'}.$$

**Lemma 4.2.3** *[99] Consider Banach function space $X$. Let $M$ be a Hardy Littlewood maximal operator is bounded weakly on $X$, i.e*

$$\|\chi_{\{Mf>\sigma\}}\|_X \lesssim \sigma^{-1}\|f\|_X$$

*is true for $\sigma > 0$ and every $f \in X$. Next we have*

$$\sup_{B:ball} \frac{1}{|B|}\|\chi_B\|_X \|\chi_B\|_{X'} < \infty.$$

**Lemma 4.2.4** *[96] (1) $X(\mathbb{R}^n, W)$ is Banach function space equipped with the norm*

$$\|f\|_{X(\mathbb{R}^n,W)} = \|fw\|_X,$$

*where*

$$X(\mathbb{R}^n, W) = \{f \in \mathbb{M} : fW \in X\}.$$

*(2) The associate space $X'(\mathbb{R}^n, W^{-1})$ is also Banach function space.*

**Lemma 4.2.5** *[81] Let $X$ be a Banach function space and $M$ be bounded on $X'$, then there exists a constant $\delta \in (0,1)$ for all $B \subset \mathbb{R}^n$ and $E \subset B$,*

$$\frac{\|\chi_E\|_X}{\|\chi_B\|_X} \lesssim \left(\frac{|E|}{|B|}\right)^{\delta}.$$

The paper [37] shows that $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space and the associated space $L^{p'(\cdot)}(R^n)$ with equivalent norm.

**Remark:**

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and by compering the Lebesgue space $L^{p(\cdot)}(w^{p(\cdot)})$ and $L^{p'(\cdot)}(w^{-p'(\cdot)})$ with the definition of $X(\mathbb{R}^n, W)$, we have

1: If we take $W = w$ and $X = L^{p(\cdot)}(\mathbb{R}^n)$, further we get $L^{p(\cdot)}(\mathbb{R}^n, w) = L^{p(\cdot)}(w^{p(\cdot)})$.

2: If we consider $W = w^{-1}$ and $X = L^{p'(\cdot)}(\mathbb{R}^n)$, then we have $L^{p'(\cdot)}(w^{-p'(\cdot)}) = L^{p'(\cdot)}(\mathbb{R}^n, w^{-1})$.

By virtue of lemma 4.2.4, we get

$$(L^{p(\cdot)}(\mathbb{R}^n, w))' = (L^{p(\cdot)}(w^{p(\cdot)}))' = L^{p'(\cdot)}(w^{-p'(\cdot)}) = L^{p'(\cdot)}(\mathbb{R}^n, w^{-1}).$$

**Lemma 4.2.6** *[100] Let $p(\cdot) \in P(\mathbb{R}^n) \cap \mathcal{C}^{log}(\mathbb{R}^n)$ be a Log Hölder continuous function both at infinity and at origin, if $w^{p_2(\cdot)} \in A_{p_2(\cdot)}$ implies $w^{-p_2'(\cdot)} \in A_{p_2'(\cdot)}$. Thus the Hardy Littlewood operator is bounded on $L^{p_2'(\cdot)}(w^{-p_2'(\cdot)})$ and there exist constants $\delta_1, \delta_2 \in (0,1)$ such that*

$$\frac{\|\chi_E\|_{L^{P_2(\cdot)}(w^{p_2(\cdot)})}}{\|\chi_B\|_{L^{P_2(\cdot)}(w^{p_2(\cdot)})}} = \frac{\|\chi_E\|_{(L^{P_2'(\cdot)}w^{-p_2'(\cdot)})'}}{\|\chi_B\|_{(L^{P_2'(\cdot)}w^{-p_2'(\cdot)})'}} \lesssim \left(\frac{|E|}{|B|}\right)^{\delta_1}. \tag{4.2.1}$$

*and*

$$\frac{\|\chi_E\|_{(L^{P_2(\cdot)}w^{p_2(\cdot)})'}}{\|\chi_B\|_{(L^{P_2(\cdot)}w^{p_2(\cdot)})'}} \lesssim \left(\frac{|E|}{|B|}\right)^{\delta_2}, \tag{4.2.2}$$

*for each and every $B$ and for all measurable sets $E \subset B$.*

**Lemma 4.2.7** *[83] Let Hardy Littlewood maximal operator is bounded on Banach function space $X$, then for measurable sets $E \subset B$ we have the following result*

$$\frac{\|\chi_B\|_X}{\|\chi_E\|_X} \lesssim \frac{|B|}{|E|}$$

**Lemma 4.2.8** *[100]Let consider a weight $w$ on $\mathbb{R}^n$. There exist $p \in [1, \infty)$ such that $w \in A_p$, then for any measurable sets $E$ subset of $B$, we have*

$$\frac{w(B)}{w(E)} \leq C \left( \frac{|B|}{|E|} \right)^p$$
$$\frac{w(E)}{w(B)} \leq C \left( \frac{|E|}{|B|} \right)^\delta,$$

*where $0 < \delta < 1$ represent a constant independent of $E$ and $B$.*

**Lemma 4.2.9** *[81] Let $p_1(\cdot) \in P(\mathbb{R}^n) \cap \mathcal{C}^{log}(\mathbb{R}^n)$ and $0 < \beta < \frac{n}{p_{1+}}$, and $\frac{1}{p_2(\cdot)} = \frac{1}{p_1(\cdot)} - \frac{\beta}{n}$. If $w \in A(p_1(\cdot), p_2(\cdot))$, then $I^\beta$ is bounded from $L^{p_1(\cdot)}(w^{P_1(\cdot)})$ to $L^{p_2(\cdot)}(w^{P_2(\cdot)})$.*

## 4.3 Main Results and Proofs

Following Proposition was proved in [101].

**Proposition 4.3.1** *Let $p(\cdot) \in P(\mathbb{R}^n)$, $0 < q < \infty$ and $0 \leq \lambda < \infty$. If $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{C}^{log}(\mathbb{R}^n)$, then*

$$\begin{aligned}
&\|f\|^q_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w^{p(\cdot)})} \\
&= \sup_{k_0 \in Z} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot) q} \|f\chi_k\|^q_{L^{p(\cdot)}(w^{p(\cdot)})} \\
&\approx \max \Bigg\{ \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0 \lambda q} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0) q} \|f\chi_k\|^q_{L^{p(\cdot)}(w^{p(\cdot)})}, \\
&\quad \sup_{\substack{k_0 \in Z \\ k_0 \geq 0}} 2^{-k_0 \lambda q} \Bigg( \sum_{k=-\infty}^{-1} 2^{k\alpha(0) q} \|f\chi_k\|^q_{L^{p(\cdot)}(w^{p(\cdot)})} \\
&+ \sum_{k=0}^{k_0} 2^{k\alpha(\infty) q} \|f\chi_k\|^q_{L^{p(\cdot)}(w^{p(\cdot)})} \Bigg) \Bigg\}.
\end{aligned}$$

One of the main result of this study is as below:

**Theorem 4.3.2** *Let* $0 < q_1 \leq q_2 < \infty$, $p_2(\cdot) \in P(\mathbb{R}^n) \cap \mathcal{C}^{log}(\mathbb{R}^n)$ *and* $p_1(\cdot)$ *be such that* $\frac{1}{p_2(\cdot)} = \frac{1}{p_1(\cdot)} - \frac{\beta}{n}$. *Also, let* $w^{p_2(\cdot)} \in A_1$, $\lambda > 0$ *and* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{C}^{log}(\mathbb{R}^n)$ *be log Hölder continuous at the origin, with* $\alpha(0) \leq \alpha(\infty) < \lambda + n\delta_2 - \beta$, *where* $\delta_2 \in (0,1)$ *is the constant come into view in (4.2.2), then*

$$\|H_\beta f(x)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})} \leq C\|f(x)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}.$$

**proof**

For any $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})$, if we represent $f_j = f \cdot \chi_{A_j} = f \cdot \chi_j$, for every $j \in \mathbb{Z}$. After that, we write

$$f(x) = \sum_{j=-\infty}^{\infty} f_j(x) = \sum_{j=-\infty}^{\infty} f(x) \cdot \chi_j(x).$$

The generalized Hölder inequality yields

$$\begin{aligned} |H_\beta f(x) \cdot \chi_k(x)| &\leq \frac{1}{|x|^{n-\beta}} \int_{B_k} |f(t)| dt \cdot \chi_k(x) \\ &\leq C 2^{-kn} \sum_{j=-\infty}^{k} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} 2^{k\beta} \chi_k(x). \end{aligned} \tag{4.3.1}$$

Making use of Lemmas 4.2.3 and 4.2.6, respectively, we obtain

$$\begin{aligned}
&\|H_\beta f(x)\cdot\chi_k(x)\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\le C2^{k\beta}\sum_{j=-\infty}^{k}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}2^{-kn}\|\chi_k(x)\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\le C2^{k\beta}\sum_{j=-\infty}^{k}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_k(x)\|^{-1}_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\\
&\le C2^{k\beta}\sum_{j=-\infty}^{k}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_j(x)\|^{-1}_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\\
&\frac{\|\chi_j(x)\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}{\|\chi_k(x)\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}\\
&\le C2^{k\beta}\sum_{j=-\infty}^{k}2^{n\delta_2(j-k)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_j(x)\|^{-1}_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}.
\end{aligned}\tag{4.3.2}$$

To proceed further, we take $f=\chi_{B_j}$ in the definition of $I_\beta$ to get

$$I_\beta(\chi_{B_j})(x)\ge C2^{j\beta}\chi_{B_j}(x),$$

which implies that

$$\chi_{B_j}(x)\le C2^{-j\beta}I_\beta(\chi_{B_j})(x).$$

Taking the norm on both sides and using Lemmas 4.2.9 and 4.2.3, respectively, we get

$$\begin{aligned}
\|\chi_{B_j}(x)\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}&\le C2^{-j\beta}||I_\beta(\chi_{B_j})(x)||_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\le C2^{-j\beta}||(\chi_{B_j})(x)||_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\le 2^{j(n-\beta)}||(\chi_{B_j})(x)||^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}
\end{aligned}\tag{4.3.3}$$

Inserting (4.3.3) into (4.3.2), we are down to

$$\begin{aligned}
&\|H_\beta f(x)\cdot\chi_k(x)\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\le C2^{k\beta}\sum_{j=-\infty}^{k}2^{n\delta_2(j-k)}2^{j(n-\beta)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_j\|^{-1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_j(x)\|^{-1}_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\\
&=C\sum_{j=-\infty}^{k}2^{(\beta-n\delta_2)(k-j)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\left(2^{-jn}\|\chi_j\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_j(x)\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\right)^{-1}\\
&\le C\sum_{j=-\infty}^{k}2^{(\beta-n\delta_2)(k-j)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}\tag{4.3.4}$$

In the rest of the proof, in order to estimate $\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}$, we consider two cases as below.

Case 1: We take $j<0$, and start estimating as

$$\begin{aligned}
\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}&=2^{-j\alpha(0)}\left(2^{j\alpha(0)q_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{\frac{1}{q_1}}\\
&\le 2^{-j\alpha(0)}\left(\sum_{i=-\infty}^{j}2^{i\alpha(0)q_1}\|f_i\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{\frac{1}{q_1}}\\
&\le 2^{j(\lambda-\alpha(0))}2^{-j\lambda}\left(\sum_{i=-\infty}^{j}2^{i\alpha(\cdot)q_1}\|f_i\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{\frac{1}{q_1}}\\
&\le C2^{j(\lambda-\alpha(0))}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}\tag{4.3.5}$$

Case 2: For $j \geq 0$, we get

$$
\begin{aligned}
\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} &= 2^{-j\alpha(\infty)}\left(2^{j\alpha(\infty)q_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{\frac{1}{q_1}} \\
&\leq 2^{-j\alpha(\infty)}\left(\sum_{i=0}^{j}2^{i\alpha(\infty)q_1}\|f_i\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{\frac{1}{q_1}} \\
&\leq 2^{j(\lambda-\alpha(\infty))}2^{-j\lambda}\left(\sum_{i=-\infty}^{j}2^{i\alpha(\cdot)q_1}\|f_i\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{\frac{1}{q_1}} \\
&\leq C2^{j(\lambda-\alpha(\infty))}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned} \tag{4.3.6}
$$

By definition of variable exponent Herz-Morrey space along with the use of Proposition (4.3.1) we arrive at the following inequality:

$$
\begin{aligned}
&\|H_\beta f(x)\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})} = \sup_{k_0\in Z}2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}2^{k\alpha(\cdot)q_1}\|H_\beta f(x)\cdot\chi_k(x)\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\leq \max\Bigg\{\sup_{\substack{k_0\in Z\\k_0<0}}2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}2^{k\alpha(0)q_1}\|H_\beta f(x)\cdot\chi_k(x)\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}, \\
&\sup_{\substack{k_0\in Z\\k_0\geq 0}}2^{-k_0\lambda q_1}\Bigg(\sum_{k=-\infty}^{-1}2^{k\alpha(0)q_1}\|H_\beta f(x)\cdot\chi_k(x)\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\
&+\sum_{k=0}^{k_0}2^{k\alpha(\infty)q_1}\|H_\beta f(x)\cdot\chi_k(x)\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Bigg)\Bigg\} \\
&= \max\{Y_1, Y_2+Y_3\},
\end{aligned} \tag{4.3.7}
$$

where

$$
Y_1 = \sup_{\substack{k_0\in Z\\k_0<0}}2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}2^{k\alpha(0)q_1}\|H_\beta f(x)\cdot\chi_k(x)\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})},
$$

$$Y_2 = \sup_{\substack{k_0 \in Z \\ k_0 \geq 0}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{-1} 2^{k\alpha(0) q_1} \| H_\beta f(x) \cdot \chi_k(x) \|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})},$$

$$Y_3 = \sup_{\substack{k_0 \in Z \\ k_0 \geq 0}} 2^{-k_0 \lambda q_1} \sum_{k=0}^{k_0} 2^{k\alpha(\infty) q_1} \| H_\beta f(x) \cdot \chi_k(x) \|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.$$

First, we approximate $Y_1$. Since $\alpha(0) \leq \alpha(\infty) < n\delta_2 + \lambda - \beta$,

$$\begin{aligned}
Y_1 &\leq C \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0) q_1} \left( \sum_{j=-\infty}^{k} 2^{(\beta - n\delta_2)(k-j)} \|f\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\
&\leq C \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0) q_1} \left( \sum_{j=-\infty}^{k} 2^{(\beta - n\delta_2)(k-j)} 2^{j(\lambda - \alpha(0))} \|f\|_{M\dot{k}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\
&\leq C \|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})} \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0) q_1} \left( \sum_{j=-\infty}^{k} 2^{(\beta - n\delta_2)(k-j)} 2^{j(\lambda - \alpha(0))} \right)^{q_1} \\
&\leq C \|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})} \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0 \lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\lambda q_1} \left( \sum_{j=-\infty}^{k} 2^{(j-k)(-\beta + n\delta_2 - \alpha(0) + \lambda)} \right)^{q_1} \\
&\leq C \|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}$$

The estimate of $Y_2$ is similar to that of $Y_1$. Lastly, we estimate $Y_3$

$$\begin{aligned}
Y_3 &\leq C \sup_{\substack{k_0\in Z\\ K_0\geq 0}} 2^{-k_0\lambda q_1} \sum_{k=0}^{K_0} 2^{k\alpha(\infty)q_1} \Big( \sum_{j=-\infty}^{k} 2^{(\beta-n\delta_2)(k-j)} \|f\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \Big)^{q_1} \\
&\leq C \sup_{\substack{k_0\in Z\\ k_0\geq 0}} 2^{-k_0\lambda q_1} \sum_{k=0}^{K_0} 2^{k\alpha(\infty)q_1} \Big( \sum_{j=-\infty}^{k} 2^{(\beta-n\delta_2)(k-j)} 2^{j(\lambda-\alpha(\infty))} \|f\|_{M\dot{k}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})} \Big)^{q_1} \\
&\leq C \|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})} \sup_{\substack{k_0\in Z\\ k_0\geq 0}} 2^{-k_0\lambda q_1} \sum_{k=0}^{k_0} 2^{k\alpha(\infty)q_1} \Big( \sum_{j=-\infty}^{k} 2^{(\beta-n\delta_2)(k-j)} 2^{j(\lambda-\alpha(\infty))} \Big)^{q_1} \\
&\leq C \|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})} \sup_{\substack{k_0\in Z\\ k_0\geq 0}} 2^{-k_0\lambda q_1} \sum_{k=0}^{k_0} 2^{k\lambda q_1} \Big( \sum_{j=-\infty}^{k} 2^{(j-k)(-\beta+n\delta_2-\alpha(\infty)+\lambda)} \Big)^{q_1} \\
&\leq C \|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}$$

The desired result is obtained by inserting the approximations of $Y_1, Y_2$ and $Y_3$ into (4.3.7).

**Theorem 4.3.3** *Let $q_1, q_2, p_1(\cdot), p_2(\cdot), \beta, \alpha(\cdot)$ and $w$ be as in Theorem 4.3.2. In addition, if $-n\delta_1 + \lambda < \alpha(0) \leq \alpha(\infty)$, where $\delta_1 \in (0,1)$ is the constant appearing in (4.2.1), then*

$$\|H^*_\beta f(x)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})} \leq C\|f(x)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}.$$

**proof**

An application of the Holder inequality gives

$$\begin{aligned}
|H^*_\beta f(x) \cdot \chi_k(x)| &\leq \int_{R^n\setminus B_k} |f(t)||x|^{\beta-n} dt \cdot \chi_k(x) \\
&\leq C \sum_{j=k+1}^{\infty} 2^{j(\beta-n)} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \chi_k(x).
\end{aligned}$$

Now, using Lemma 4.2.3, we have

$$\begin{aligned}\|H^*_\beta f(x)\cdot\chi_k(x)\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\le C\sum_{j=k+1}^{\infty}2^{j(\beta-n)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\ &\quad \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_k(x)\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\ &\le C\sum_{j=k+1}^{\infty}2^{j\beta}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\ &\quad \|\chi_j\|^{-1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_k(x)\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}. \end{aligned} \tag{4.3.8}$$

In view of inequality (4.3.3), we obtain

$$\begin{aligned}\|H^*_\beta f(x)\cdot\chi_k(x)\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\le C\sum_{j=k+1}^{\infty}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\frac{\|\chi_k(x)\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}}{\|\chi_j(x)\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}}\\ &\le C\sum_{j=k+1}^{\infty}2^{n\delta_1(k-j)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}, \end{aligned} \tag{4.3.9}$$

where we used Lemma 4.2.6 in the last step.

In the remaining proof of this theorem, we follow the procedure as followed in Theorem 4.3.2, to have

$$\|H^*_\beta f(x)\cdot\chi_k(x)\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} = \max\{Z_1, Z_2+Z_3\}, \tag{4.3.10}$$

where

$$Z_1 = \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}2^{k\alpha(0)q_1}\|H^*_\beta f(x)\cdot\chi_k(x)\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.$$

$$Z_2 = \sup_{\substack{k_0\in Z\\ k_0\ge 0}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{-1}2^{k\alpha(0)q_1}\|H^*_\beta f(x)\cdot\chi_k(x)\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.$$

$$Z_3 = \sup_{\substack{k_0\in Z\\ k_0\ge 0}} 2^{-k_0\lambda q_1}\sum_{k=0}^{k_0}2^{k\alpha(\infty)q_1}\|H^*_\beta f(x)\cdot\chi_k(x)\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.$$

The estimates of $Z_i$ $(i=1,2,3)$ are similar to that of $Y_i$ $(i=1,2,3)$ of Theorem 4.3.2. Here we conclude our result.

## 4.4 Conclusion

In chapter we discussed boundedness for the fractional Hardy operators with variable exponent weighted Morrey-Herz spaces $(M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w))$.

# Chapter 5
# Commutators of fractional Hardy operator on weighted variable Herz-Morrey spaces

## 5.1 Introduction

In this peace of work, our main focus is on establishing the boundedness of commutators of fractional Hardy operators on a class of function spaces called weighted Herz-Morrey space with variable exponents. We seek to fined the boundedness of these commutators with symbol function in $BMO$ spaces.

## 5.2 Main Results and Their Proofs

**Lemma 5.2.1** *Let $q(\cdot) \in P(\mathbb{R}^n)$ and $w$ be an $A_{q(\cdot)}$ weight. Then for all $b \in BMO$ and all $l, i \in \mathbb{Z}$ with $l > i$ we have*

$$\|b\|_{BMO} \sim \sup_{B:Ball} \frac{1}{\|\chi_B\|_{L^{q(\cdot)}(w^{q(\cdot)})}} \|(b - b_B)\chi_B\|_{L^{q(\cdot)}(w^{q(\cdot)})} \tag{5.2.1}$$

$$\|(b - b_{B_i})\chi_{B_l}\|_{L^{q(\cdot)}(w^{q(\cdot)})} \leq C(l - i)\|b\|_{BMO}\|\chi_{B_l}\|_{L^{q(\cdot)}(w^{q(\cdot)})} \tag{5.2.2}$$

**proof**

First part of this lemma is a consequence of theorem 4 in [100]. Next, we will prove (5.2.2), for all $l, i \in \mathbb{Z}$ with $l > i$

$$\begin{aligned} &\|(b - b_{B_i})\chi_{B_l}\|_{L^{q(\cdot)}(w^{q(\cdot)})} \\ &\leq C\|(|b - b_{B_l}| + |b_{B_l} - b_{B_i}|)\chi_{B_l}\|_{L^{q(\cdot)}(w^{q(\cdot)})} \\ &\leq C\Big\{\|(b - b_{B_l})\chi_{B_l}\|_{L^{q(\cdot)}(w^{q(\cdot)})} + \|(b_{B_l} - b_{B_i})\chi_{B_l}\|_{L^{q(\cdot)}(w^{q(\cdot)})}\Big\}. \end{aligned} \tag{5.2.3}$$

In the view of (5.2.1) we have

$$\|(b - b_{B_l})\chi_{B_l}\|_{L^{q(\cdot)}(w^{q(\cdot)})} \leq C\|b\|_{BMO}\|\chi_{B_l}\|_{L^{q(\cdot)}(w^{q(\cdot)})} \tag{5.2.4}$$

Also, it is easy to see that

$$\begin{aligned} |b_{B_l} - b_{B_i}| &\leq \sum_{n=i}^{l-1} |b_{n+1} - b_n| \\ &\leq \sum_{n=i}^{l-1} \frac{1}{|B_n|}\int_{B_n} |b_{n+1} - b(x)|dx \\ &\leq C\sum_{n=i}^{l-1} \frac{1}{|B_{n+1}|}\int_{B_{n+1}} |b_{n+1} - b(x)|dx \\ &= C(l - i)\|b\|_{BMO(\mathbb{R}^n)} \end{aligned} \tag{5.2.5}$$

Combining (5.2.3), (5.2.4) and (5.2.5) we get (5.2.2).

**Theorem 5.2.2** *Let $0 < p_1 \leq p_2 < \infty$, $q_2(\cdot) \in P(\mathbb{R}^n) \cap \mathcal{C}^{log}(\mathbb{R}^n)$ and $q_1(\cdot)$ be such that $\frac{1}{q_1(\cdot)} = \frac{1}{q_2(\cdot)} - \frac{\beta}{n}$. Also, let $w^{q_2(\cdot)} \in A_1$, $b \in BMO(\mathbb{R}^n)$, $\lambda > 0$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{C}^{log}(\mathbb{R}^n)$ be log Hölder continuous at the origin, with $\alpha(0) \leq \alpha(\infty) < \lambda + n\delta_2 - \beta$, where $0 < \delta_2 < 1$, then*

$$\|[b, H_\beta]f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(w^{q_2(\cdot)})} \leq C\|b\|_{BMO}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_1(\cdot)})}$$

**proof**

For any $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_1(\cdot)})$, if we represent $f_l = f \cdot \chi_l = f \cdot \chi_{A_l}$, for each $l \in \mathbb{Z}$, then we denote

$$f(x) = \sum_{l=-\infty}^{\infty} f(x) \cdot \chi_l(x) = \sum_{l=-\infty}^{\infty} f_l(x)$$

$$
\begin{aligned}
|[b,H_\beta]f(x)\chi_j(x)| &\le \frac{1}{|x|^{n-\beta}}\int_{B_j}|(b(x)-b(y))f(y)|dy\cdot\chi_j(x)\\
&\le 2^{-j(n-\beta)}\sum_{l=-\infty}^{j}\int_{B_l}|(b(x)-b(y))f(y)|dy\cdot\chi_j(x)\\
&\le 2^{-j(n-\beta)}\sum_{l=-\infty}^{j}\int_{B_l}|(b(x)-b_{B_l})f(y)|dy\cdot\chi_j(x)\\
&+2^{-j(n-\beta)}\sum_{l=-\infty}^{j}\int_{B_l}|(b(y)-b_{B_l})f(y)|dy\cdot\chi_j(x)\\
&=E_1+E_2
\end{aligned}
\tag{5.2.6}
$$

The generalized Hölder inequality yield the following inequality for $E_1$ :

$$
\begin{aligned}
E_1 &= 2^{-j(n-\beta)}\sum_{l=-\infty}^{j}\int_{B_l}|(b(x)-b_{B_l})f(y)|dy\cdot\chi_j(x)\\
&\le 2^{-j(n-\beta)}\sum_{l=-\infty}^{j}|b(x)-b_{B_l}|\cdot\chi_j(x)\|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_{B_l}\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}
\end{aligned}
\tag{5.2.7}
$$

Applying norm on both sides and using Lemma 5.2.1, we get

$$
\begin{aligned}
&\|E_1\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\le 2^{-j(n-\beta)}\sum_{l=-\infty}^{j}\|(b(x)-b_{B_l})\cdot\chi_{B_j}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_{B_l}\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\\
&\le 2^{-j(n-\beta)}\sum_{l=-\infty}^{j}(j-l)\|b\|_{BMO}\|\chi_{B_j}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_{B_l}\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}
\end{aligned}
\tag{5.2.8}
$$

Now, we turn to estimate $E_2$. For this, we have

$$\begin{aligned} E_2 &\le 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \|(b(y)-b_{B_l})\cdot \chi_l\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \cdot \chi_j(x) \\ &\le 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \|(b(y)-b_{B_l})\cdot \chi_{B_l}\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \cdot \chi_j(x) \end{aligned} \tag{5.2.9}$$

Similar to the estimation for $E_1$, we take norm on both sides of above inequality and use Lemma 5.2.1 to obtain

$$\begin{aligned} &\|E_2\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \\ &\le 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \|(b(y)-b_{B_l})\cdot \chi_{B_l}\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \cdot \|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \\ &\le 2^{-j(n-\beta)} \sum_{l=-\infty}^{j} \|b\|_{BMO} \|\chi_{B_l}\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \cdot \|\chi_{B_j}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}. \end{aligned} \tag{5.2.10}$$

Hence from inequalities (5.2.8) and (5.2.10), one has

$$\begin{aligned} &\|[b,H_\beta]f(x)\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \\ &\le 2^{-j(n-\beta)} \|b\|_{BMO} \sum_{l=-\infty}^{j} (j-l) \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\chi_{B_j}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \|\chi_{B_l}\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}. \end{aligned} \tag{5.2.11}$$

By using Lemma 4.2.3

$$\begin{aligned} &\|[b,H_\beta]f(x)\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \\ &\le 2^{j\beta} \|b\|_{BMO} \sum_{l=-\infty}^{j} (j-l) \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\chi_{B_l}\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \|\chi_{B_j}\|^{-1}_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'} \end{aligned} \tag{5.2.12}$$

Now using Lemma 4.2.6

$$\begin{aligned}
&\|[b,H_\beta]f(x)\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\le 2^{j\beta}\|b\|_{BMO}\sum_{l=-\infty}^{j}(j-l)\|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\frac{\|\chi_{B_l}\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}}{\|\chi_{B_l}\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}}\frac{\|\chi_{B_l}\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}}{\|\chi_{B_j}\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}}\\
&\le 2^{j\beta}\|b\|_{BMO}\sum_{l=-\infty}^{j}(j-l)2^{(l-j)n\delta_2}\|b\|_{BMO}\|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\frac{\|\chi_{B_l}\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}}{\|\chi_{B_l}\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}}
\end{aligned} \tag{5.2.13}$$

In the definition of fraction integral $I_\beta$, we replace $f$ by $\chi_{B_l}$ to obtain

$$I_\beta(\chi_{B_l})(x)\ge C2^{l\beta}\chi_{B_l}(x),$$

from which we infer that

$$\chi_{B_l}(x)\le C2^{-l\beta}I_\beta(\chi_{B_l})(x).$$

Taking the norm on both sides and using Lemmas 4.2.9 and 4.2.3, respectively, we get

$$\begin{aligned}
\|\chi_{B_l}\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} &\le C2^{-l\beta}\|I_\beta(\chi_{B_l})\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\le C2^{-l\beta}\|\chi_{B_l}\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\\
&\le C2^{l(n-\beta)}\|\chi_{B_l}\|^{-1}_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}.
\end{aligned} \tag{5.2.14}$$

The use of (5.2.14) into (5.2.13) results in the following inequality:

$$\|[b,H_\beta]f(x)\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}$$
$$\leq C\|b\|_{BMO}\sum_{l=-\infty}^{j}2^{l(n-\beta)}2^{j\beta}(j-l)2^{(l-j)n\delta_2}\|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}$$
$$\left(\|\chi_l\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\|\chi_l\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\right)^{-1}$$
$$\leq C\|b\|_{BMO}\sum_{l=-\infty}^{j}2^{(j-l)(\beta-n\delta_2)}(j-l)\|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}$$
$$\left(2^{-ln}\|\chi_l\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\|\chi_l\|_{(L^{q_2(\cdot)}(w^{q_2(\cdot)}))'}\right)^{-1}$$
$$\leq C\|b\|_{BMO}\sum_{l=-\infty}^{j}2^{(j-l)(\beta-n\delta_2)}(j-l)\|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \tag{5.2.15}$$

In view of the condition $p_1\leq p_2$ and the Proposition 4.3.1, we have

$$\|[b,H_\beta]f(x)\chi_j\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(w^{q_2(\cdot)})}$$
$$\leq max\Bigg\{\sup_{\substack{k_0\in Z\\k_0<0}}2^{-k_0\lambda p_1}\left(\sum_{j=-\infty}^{k_0}2^{j\alpha(0)p_1}\|[b,H_\beta]f(x)\cdot\chi_j\|^{p_1}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\right),$$
$$\sup_{\substack{k_0\in Z\\k_0\geq 0}}\Bigg(2^{-k_0\lambda p_1}\left(\sum_{j=-\infty}^{-1}2^{j\alpha(0)p_1}\|[b,H_\beta]f(x)\cdot\chi_j\|^{p_1}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\right)$$
$$+2^{-k_0\lambda p_1}\left(\sum_{j=0}^{k_0}2^{j\alpha(\infty)p_1}\|[b,H_\beta]f(x)\cdot\chi_j\|^{p_1}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\right)\Bigg)\Bigg\}$$
$$=max\{X_1,X_2,X_3\}, \tag{5.2.16}$$

where

$$X_1=\sup_{\substack{k_0\in Z\\k_0<0}}2^{-k_0\lambda p_1}\left(\sum_{j=-\infty}^{k_0}2^{j\alpha(0)p_1}\|[b,H_\beta]f(x)\cdot\chi_j\|^{p_1}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\right)$$

$$X_2 = \sup_{\substack{k_0 \in Z \\ k_0 \geq 0}} 2^{-k_0 \lambda p_1} \left( \sum_{j=-\infty}^{-1} 2^{j\alpha(0)p_1} \|[b, H_\beta] f(x) \cdot \chi_j\|^{p_1}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \right)$$

$$X_3 = \sup_{\substack{k_0 \in Z \\ k_0 \geq 0}} 2^{-k_0 \lambda p_1} \left( \sum_{j=0}^{k_0} 2^{j\alpha(\infty)p_1} \|[b, H_\beta] f(x) \cdot \chi_j\|^{p_1}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \right)$$

To estimate $X_1,\ X_2,\ X_3$, we make use of the conditions on $\alpha(\cdot)$, such that for $l < 0$, we have

$$\begin{aligned}
\|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} &= 2^{-l\alpha(0)} \left( 2^{j\alpha(0)p_1} \|f_l\|^{p_1}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \right)^{\frac{1}{p_1}} \\
&\leq 2^{-l\alpha(0)} \left( \sum_{i=-\infty}^{l} 2^{i\alpha(0)p_1} \|f_i\|^{p_1}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \right)^{\frac{1}{p_1}} \\
&\leq 2^{l(\lambda-\alpha(0))} 2^{-l\lambda} \left( \sum_{i=-\infty}^{l} 2^{i\alpha(\cdot)p_1} \|f_i\|^{p_1}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \right)^{\frac{1}{p_1}} \\
&\leq C 2^{l(\lambda-\alpha(0))} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_1(\cdot)})},
\end{aligned} \tag{5.2.17}$$

and for $l \geq 0$, we obtain

$$\begin{aligned}
\|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} &= 2^{-l\alpha(\infty)} \left( 2^{l\alpha(\infty)p_1} \|f_l\|^{p_1}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \right)^{\frac{1}{p_1}} \\
&\leq 2^{-l\alpha(\infty)} \left( \sum_{i=0}^{l} 2^{i\alpha(\infty)p_1} \|f_i\|^{p_1}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \right)^{\frac{1}{p_1}} \\
&\leq 2^{l(\lambda-\alpha(\infty))} 2^{-l\lambda} \left( \sum_{i=-\infty}^{l} 2^{i\alpha(\cdot)p_1} \|f_i\|^{p_1}_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \right)^{\frac{1}{p_1}} \\
&\leq C 2^{l(\lambda-\alpha(\infty))} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_1(\cdot)})}.
\end{aligned} \tag{5.2.18}$$

In order to calculate $X_1$, we need to use $\alpha(0) \leq \alpha(\infty) < n\delta_2 + \lambda - \beta$.

$$X_1 \leq \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1} \left( \sum_{l=-\infty}^{j} (j-l) 2^{(j-l)(\beta - n\delta_2)} \|b\|_{BMO} \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \right)^{p_1}$$

$$\leq C \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1} \left( \sum_{l=-\infty}^{j} (j-l) 2^{(j-l)(\beta - n\delta_2)} 2^{l(\lambda - \alpha(0))} \right.$$

$$\left. \|b\|_{BMO} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_1(\cdot)})} \right)^{p_1}$$

$$\leq C \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1} \left( \sum_{l=-\infty}^{j} (j-l) 2^{(j-l)(\beta - n\delta_2)} 2^{l(\lambda - \alpha(0))} \right)^{p_1}$$

$$\|b\|^{p_1}_{BMO} \|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_1(\cdot)})}$$

$$\leq C \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\lambda p_1} \left( \sum_{l=-\infty}^{j} (j-l) 2^{(l-j)(-\beta + n\delta_2 - \alpha(0) + \lambda)} \right)^{p_1}$$

$$\|b\|^{p_1}_{BMO} \|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_1(\cdot)})}$$

$$\leq C \|b\|^{p_1}_{BMO} \|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_1(\cdot)})} \tag{5.2.19}$$

The result of $X_2$ is similar to that of $X_1$. Next, we estimate of $X_3$ as below

$$
\begin{aligned}
X_3 &\leq \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(\infty)p_1} \left( \sum_{l=-\infty}^{j} (j-l) 2^{(j-l)(\beta-n\delta_2)} \|b\|_{BMO} \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \right)^{p_1} \\
&\leq C \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(\infty)p_1} \Bigg( \sum_{l=-\infty}^{j} (j-l) 2^{(j-l)(\beta-n\delta_2)} 2^{l(\lambda-\alpha(\infty))} \\
&\quad \|b\|_{BMO} \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_1(\cdot)})} \Bigg)^{p_1} \\
&\leq C \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\alpha(\infty)p_1} \left( \sum_{l=-\infty}^{j} (j-l) 2^{(j-l)(\beta-n\delta_2)} 2^{l(\lambda-\alpha(\infty))} \right)^{p_1} \\
&\quad \|b\|^{p_1}_{BMO} \|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_1(\cdot)})} \\
&\leq C \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1} \sum_{j=-\infty}^{k_0} 2^{j\lambda p_1} \left( \sum_{l=-\infty}^{j} (j-l) 2^{(l-j)(-\beta+n\delta_2-\alpha(\infty)+\lambda} \right)^{p_1} \\
&\quad \|b\|^{p_1}_{BMO} \|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_1(\cdot)})} \\
&\leq C \|b\|^{p_1}_{BMO} \|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_1(\cdot)})}.
\end{aligned}
\tag{5.2.20}
$$

Finally, we combine the estimates for $X_i$ $(i = 1, 2, 3)$, to have the desired result.

**Theorem 5.2.3** *Let $p_1, p_2, q_1(\cdot), q_2(\cdot), \beta, \alpha(\cdot)$ and $w$ be as in Theorem 5.2.2. In addition if $\lambda - n\delta_1 < \alpha(0) \leq \alpha(\infty)$, where $1 < \delta_1 < 0$, then*

$$\|[b, H^*_\beta]f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(w^{q_2(\cdot)})} \leq C\|b\|_{BMO}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(w^{q_1(\cdot)})}$$

**proof**

We write

$$
\begin{aligned}
&[b, H^*_\beta]f(x)\chi_j(x)\\
&\le \int_{\mathbb{R}^n\setminus B_j} |y|^{\beta-n}|(b(x)-b(y))f(y)|dy\cdot\chi_j(x)\\
&\le \sum_{l=j+1}^{\infty}\int_{B_l} |y|^{\beta-n}|(b(x)-b(y))f(y)|dy\cdot\chi_j(x)\\
&\le \sum_{l=j+1}^{\infty}\int_{B_l} |y|^{\beta-n}|b(x)-b_{B_l}|f(y)dy\cdot\chi_j(x)\\
&+ \sum_{l=j+1}^{\infty}\int_{B_l} |y|^{\beta-n}|(b(y)-b_{B_l})f(y)|dy\cdot\chi_j(x)\\
&= F_1+F_2 \qquad (5.2.21)
\end{aligned}
$$

Estimating $F_1$ and $F_2$ separately. A use of generalized inequality results in the following:

$$
\begin{aligned}
F_1 &\le C\sum_{l=j+1}^{\infty} 2^{-l(n-\beta)}\int_{B_l}|(b(x)-b_{B_l})f(y)|dy\cdot\chi_j(x)\\
&\le C\sum_{l=j+1}^{\infty} 2^{-l(n-\beta)}\|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_{B_l}\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}|b(x)-b_{B_l}|\cdot\chi_j
\end{aligned}
\qquad (5.2.22)
$$

Applying weighted Lebesgue space norm on both sides and using Lemma 5.2.1, we obtain

$$
\begin{aligned}
\|F_1\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} &\le C\sum_{l=j+1}^{\infty} 2^{-l(n-\beta)}\|(b(x)-b_{B_j})\cdot\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\quad \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_{B_l}\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}\\
&\le C\sum_{l=j+1}^{\infty} 2^{-l(n-\beta)}\|b\|_{BMO}\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\\
&\quad \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}\|\chi_{B_l}\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \qquad (5.2.23)
\end{aligned}
$$

Similarly

$$F_2 \leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} \int_{B_l} |(b(y) - b_{B_l})f(y)|dy \cdot \chi_j(x)$$
$$\leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} \|(b(y) - b_{B_l}) \cdot \chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \cdot \chi_j(x)(x) \tag{5.2.24}$$

In view of weighted Lebesgue norm and Lemma 5.2.1, we get

$$\|F_2\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} \|(b(y) - b_{B_l}) \cdot \chi_j\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}$$
$$\|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \cdot \|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}$$
$$\leq C \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} (l-j) \|b\|_{BMO} \|\chi_l\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}$$
$$\|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \cdot \|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \tag{5.2.25}$$

Hence, from (5.2.21), (5.2.23) and (5.2.25), we obtain

$$\|[b, H_\beta^*] f(x)\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}$$
$$\leq \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} (l-j) \|b\|_{BMO} \|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\chi_{B_l}\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'}$$
$$\leq \sum_{l=j+1}^{\infty} 2^{-l(n-\beta)} (l-j) \|b\|_{BMO} \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \|\chi_{B_l}\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \|\chi_l\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}$$
$$\frac{\|\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}{\|\chi_l\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}}$$
$$\leq \sum_{l=j+1}^{\infty} 2^{n\delta_1(j-l)} 2^{-l(n-\beta)} (l-j) \|b\|_{BMO} \|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}$$
$$\|\chi_{B_l}\|_{(L^{q_1(\cdot)}(w^{q_1(\cdot)}))'} \|\chi_l\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \tag{5.2.26}$$

Using the condition of $A(q_1(\cdot), q_2(\cdot))$ weights given in the Definition 1.6.5, above inequality reduces to

$$\|[b, H^*_\beta]f(x)\chi_j\|_{L^{q_2(\cdot)}(w^{q_2(\cdot)})} \leq \sum_{l=j+1}^{\infty} 2^{n\delta_1(j-l)}(l-j)\|b\|_{BMO}\|f_l\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})} \tag{5.2.27}$$

Next, the condition $p_1 < p_2$ and the Proposition 4.3.1 help us to write

$$\|[b, H^*_\beta]f\chi_j\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_2,q_2(\cdot)}(w^{q_2(\cdot)})} = max\{Y_1, Y_2, Y_3\}, \tag{5.2.28}$$

where

$$Y_1 = \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0\lambda p_1}\left(\sum_{j=-\infty}^{k_0} 2^{j\alpha(0)p_1}\|[b, H^*_\beta]f \cdot \chi_j\|^{p_1}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\right),$$

$$Y_2 = \sup_{\substack{k_0 \in Z \\ k_0 \geq 0}} 2^{-k_0\lambda p_1}\left(\sum_{j=-\infty}^{-1} 2^{j\alpha(0)p_1}\|[b, H^*_\beta]f \cdot \chi_j\|^{p_1}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\right),$$

and

$$Y_3 = \sup_{\substack{k_0 \in Z \\ k_0 \geq 0}} 2^{-k_0\lambda p_1}\left(\sum_{j=0}^{k_0} 2^{j\alpha(\infty)p_1}\|[b, H^*_\beta]f \cdot \chi_j\|^{p_1}_{L^{q_2(\cdot)}(w^{q_2(\cdot)})}\right).$$

Lastly, in view of the condition $-n\delta_1+\lambda < \alpha(0) \leq \alpha(\infty)$, we estimate $Y_i,\ i = 1, 2, 3$, as we estimated $X_i,\ i = 1, 2, 3$, in Theorem 5.2.2. Hence, we finish the proof.

## 5.3 Conclusion

In the present chapter we achieved our aim is to established the boundedness of commutators of fractional Hardy operator and its adjoint operator on weighted Herz-Morrey spaces with variable exponents $M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(w)$.

# Chapter 6
# Variable Weighted $\lambda$-central Morrey Space Boundedness for the Fractional Hardy Operators and Commutators

## 6.1 Introduction

In this chapter, our main focus is on the weighted $\lambda-$ central Morrey space, which has importance in the theory of Harmonic analysis. For first time, the idea of $\lambda-$ central Morrey space for variable exponent and $\lambda-$ central BMO space was discussed in [64]. Central Morrey space, central BMO and the related function spaces have delightful application by investigating the results for operators along with singular integral operators, cleared in [61, 63]. Reisz type potential operator is used for the boundedness of Hardy operator.

Let's clarify the framework of this chapter. In section 6.2 of the analysis of this chapter, we will analyze the boundedness for Hardy operators in weighted central Morrey space about variable exponent. In section 6.3, we will be able to investigate the estimates of commutator led by Hardy operator and weighted $\lambda-$ central BMO function on central Morrey space along with variable exponent.

## 6.2 Boundedness of Fractional Hardy Operators

**Theorem 6.2.1** *Let* $p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap \mathcal{C}^{log}(\mathbb{R}^n)$. *Define the variable exponent* $p_2(\cdot)$ *by*

$$\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\alpha}{n}.$$

*If* $w^{P_1(\cdot)} \in A_1,\ \lambda_2 = \lambda_1 + \frac{\alpha}{n}$ *and* $\delta_2 + \delta\lambda_2 + \delta_1 > 0,$ *then*

$$\|H_\alpha f\|_{\dot{B}^{p_2(\cdot),\lambda_2}(w^{p_2(\cdot)})} \leq C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}$$

**proof**

Using generalized Holder inequality given in Lemma 4.2.1.

$$|H_\alpha f(x) \cdot \chi_k(x)| \leq \frac{1}{|x|^{n-\alpha}} \int_{B_k} |f(t)|dt \cdot \chi_k(x)$$
$$\leq C2^{-k(n-\alpha)} \sum_{j=-\infty}^{k} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \cdot \chi_k(x).$$

$$\|H_\alpha f(x) \cdot \chi_k\|_{L^{P_2(\cdot)}(w^{p_2(\cdot)})}$$
$$\leq C2^{-k(n-\alpha)} \sum_{j=-\infty}^{k} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \|\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.$$

By mean of lemma 4.2.3 we have

$$\|H_\alpha f(x) \cdot \chi_k\|_{L^{P_2(\cdot)}(w^{p_2(\cdot)})}$$
$$\leq C2^{k\alpha} \sum_{j=-\infty}^{k} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \|\chi_k\|^{-1}_{(L^{p_2(\cdot))}(w^{p_2(\cdot)}))'}$$
$$\leq C2^{k\alpha} \sum_{j=-\infty}^{k} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \frac{\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}{\|\chi_k\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}$$
$$\|\chi_k\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \|\chi_k\|^{-1}_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}.$$

As a results of lemma 4.2.6 and condition (4.2.2) leads to

$$\|H_\alpha f(x)\cdot\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \le C2^{k\alpha}\sum_{j=-\infty}^{k} 2^{n\delta_2(j-k)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_k\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_k\|^{-1}_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}. \quad (6.2.1)$$

Based on lemmas 4.2.9 and 4.2.3, we have

$$I_\alpha(\chi_{B_k})(x) \ge C2^{k\alpha}\chi_{B_k}(x)$$

$$\begin{aligned}\chi_{B_k}(x) &\le C2^{-k\alpha}I_\alpha(\chi_{B_k})(x)\\ \|\chi_{B_k}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\le C2^{-k\alpha}\|I_\alpha(\chi_{B_k})\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\ &\le C2^{-k\alpha}\|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \qquad (6.2.2)\\ &\le C2^{k(n-\alpha)}\|\chi_{B_k}\|^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}.\end{aligned}$$

Using inequality (6.2.2) in (6.2.1) and by the results of lemma 4.2.3

$$\begin{aligned}\|H_\alpha f(x)\cdot\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\le C2^{k\alpha}\sum_{j=-\infty}^{k} 2^{n\delta_2(j-k)}2^{k(n-\alpha)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\ &\quad \|\chi_k\|^{-1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_k\|^{-1}_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\\ &\le C\sum_{j=-\infty}^{k} 2^{n\delta_2(j-k)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\ &\quad \left(2^{-kn}\|\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_k\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\right)^{-1}\\ &\le C\sum_{j=-\infty}^{k} 2^{n\delta_2(j-k)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\ &\le C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\sum_{j=-\infty}^{k} 2^{n\delta_2(j-k)}|B_j|^{\lambda_1}\|\chi_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\ &\le C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\sum_{j=-\infty}^{k} 2^{n\delta_2(j-k)}w(B_j)^{\lambda_1}\|\chi_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.\end{aligned}$$

$$\|\chi_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \approx w(B)^{\frac{1}{p_1(\cdot)}} \approx w(B)^{\frac{1}{p_2(\cdot)}+\frac{\alpha}{n}} \approx w(B)^{\frac{\alpha}{n}}\|\chi_j\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.$$

$$\|H_\alpha f(x)\cdot\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \le C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\sum_{j=-\infty}^{k}2^{n\delta_2(j-k)}$$
$$w(B_j)^{\lambda_1+\frac{\alpha}{n}}\|\chi_j\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}$$
$$= C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\sum_{j=-\infty}^{k}2^{n\delta_2(j-k)}$$
$$w(B_k)^{\lambda_2}\frac{w(B_j)^{\lambda_2}}{w(B_k)^{\lambda_2}}\|\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\frac{\|\chi_j\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}}{\|\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}}.$$

Applying Lemmas 4.2.6 and 4.2.8

$$\|H_\alpha f(x)\cdot\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}$$
$$\le C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\sum_{j=-\infty}^{k}2^{n\delta_2(j-k)}$$
$$|B_k|^{\lambda_2}\left(\frac{|B_j|}{|B_k|}\right)^{\delta\lambda_2}\|\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\frac{\|\chi_j\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}}{\|\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}}$$
$$\le C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|B_k|^{\lambda_2}\|\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\sum_{j=-\infty}^{k}2^{(j-k)(n\delta_3+n\delta_1+n\delta\lambda_2)},$$

$$\|H_\alpha f(x)\cdot\chi_k\|_{\dot{B}^{p_2(\cdot),\lambda_2}(w^{p_2(\cdot)})} \le C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}$$
$$\sum_{j=-\infty}^{k}2^{n(j-k)(\delta_2+\delta_1+\delta\lambda_2)}.$$

Since, it is given that $\delta_2+\delta_1+\delta\lambda_2>0$, which give the required result

$$\|H_\alpha f(x)\cdot\chi_k\|_{\dot{B}^{p_2(\cdot),\lambda_2}(w^{p_2(\cdot)})} \le C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}.$$

**Theorem 6.2.2** *Let $p_1(\cdot)$, $p_2(\cdot)$ and $\alpha$ be same as in theorem 6.2.1 . If $\lambda_2=\lambda_1+\frac{\alpha}{n}$, $w^{p_1(\cdot)}\in A_1$ and $\alpha<-n(1+\lambda_2)$, then*

$$\|H^*_\alpha f\|_{\dot{B}^{p_2(\cdot),\lambda_2}(w^{p_2(\cdot)})} \le C\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}.$$

**proof**

$$
\begin{aligned}
|H_\alpha^* f(x)\cdot \chi_k(x)| &\le \int_{\mathbb{R}^n\setminus B_k} |f(t)||t|^{\alpha-n}dt\cdot \chi_k(x)\\
&\le C\sum_{j=k+1}^{\infty} 2^{j(\alpha-n)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\chi_k(x).
\end{aligned}
$$

$$
\begin{aligned}
&\|H_\alpha^* f(x)\cdot \chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\le C\sum_{j=k+1}^{\infty} 2^{j(\alpha-n)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.\\
&\le C\sum_{j=k+1}^{\infty} 2^{j(\alpha-n)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\frac{\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}{\|\chi_k\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}\|\chi_k\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.
\end{aligned}
$$

By virtue of lemmas 4.2.6, 4.2.7, and by definition of $A(p_2(\cdot),p_1(\cdot))$ we get

$$
\begin{aligned}
&\|H_\alpha^* f(x)\cdot \chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\le C\sum_{j=k+1}^{\infty} 2^{j(\alpha-n)}2^{n(j-k)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_k\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.\\
&\le C\sum_{j=k+1}^{\infty} 2^{(j-k)(\alpha-n)}2^{n(j-k)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}
$$

Hence we have

$$
\|H_\alpha^* f(x)\cdot \chi_k\|_{\dot B^{p_2(\cdot),\lambda_2}(w^{p_2(\cdot)})} \le C\|f\|_{\dot B^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\sum_{j=k+1}^{\infty} 2^{(j-k)(\alpha+n+n\lambda_2)}.
$$

By using $\alpha < -n(1+\lambda_2)$, we get the final result

$$
\|H_\alpha^* f\|_{\dot B^{p_2(\cdot),\lambda_2}(w^{p_2(\cdot)})} \le C\|f\|_{\dot B^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}.
$$

## 6.3 Estimates of Commutators for Fractional Hardy Operators

**Theorem 6.3.1** *Let* $\alpha \in (0,n)$ *and* $p_1(\cdot), p(\cdot) \in P(\mathbb{R}^n) \cap \mathcal{C}^{log}(\mathbb{R}^n)$ . *Define the variable exponent* $p_2(\cdot)$ *by*

$$\frac{1}{p_2(x)} = \frac{1}{p(x)} + \frac{1}{p_1(x)} - \frac{\alpha}{n}.$$

*If* $w^{P_1(\cdot)} \in A_1$, $b \in \|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}$ , $\mu = \lambda_1 + \frac{\alpha}{n}$ *and* $\lambda_2 = \lambda + \lambda_1 + \frac{\alpha}{n}$, *then*

$$\|[b, H_\alpha]f \cdot \chi_j\|_{\cdot B^{p_2(\cdot),\lambda_2}(w^{p_2(\cdot)})} \leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}.$$

**proof**

$$\begin{aligned}
|[b, H_\alpha]f(x) \cdot \chi_B(x)| &\leq \frac{1}{|x|^{n-\alpha}} \int_{B(0,|x|)} |(b(x) - b(y))f(y)|dy \cdot \chi_B(x) \\
&\leq \frac{1}{|x|^{n-\alpha}} \int_{B(0,|x|)} |(b(x) - b_B)f(y)|dy \cdot \chi_B(x) \\
&+ \frac{1}{|x|^{n-\alpha}} \int_{B(0,|x|)} |(b(y) - b_B)f(y)|dy \cdot \chi_B(x) \\
&= A_1 + A_2.
\end{aligned}$$

First we estimate $A_1$. Denote $\frac{1}{S(x)} = \frac{1}{p_1(x)} - \frac{\alpha}{n}$, then $\frac{1}{p_2(x)} = \frac{1}{S(x)} + \frac{1}{p(x)}$.

$$\begin{aligned}
A_1 &= |x|^{-n+\alpha} \int_{B(0,|x|)} |(b(x) - b_B)f(y)|dy \cdot \chi_B(x) \\
&= |(b(x) - b_B)\chi_B(x)||H_\alpha f(x)|,
\end{aligned}$$

$$\|A_1\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} = \|(b(x) - b_B)\chi_B(x)H_\alpha f(x)\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.$$

By using Holder inequality $(\frac{1}{p_2(\cdot)} = \frac{1}{S(\cdot)} + \frac{1}{p(\cdot)})$

$$\begin{aligned}\|A_1\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\leq \|(b(x)-b_B)\chi_B(x)\|_{L^{p(\cdot)}(w^{p(\cdot)})}\|H_\alpha f(x)\chi_B\|_{L^{S(\cdot)}(w^{S(\cdot)})} \\ &= C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(w^{p(\cdot)})} \\ &\quad |B|^{\mu}\|\chi_B\|_{L^{S(\cdot)}(w^{S(\cdot)})}\|H_\alpha f\|_{\dot{B}^{\mu,S(\cdot)}(w^{S(\cdot)})},\end{aligned}$$

$$\begin{aligned}\|A_1\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|B|^{\lambda}\|\chi_B\|_{L^{p(\cdot)}(w^{p(\cdot)})} \\ |B|^{\mu}\|\chi_B\|_{L^{S(\cdot)}(w^{S(\cdot)})}\|H_\alpha f\|_{\dot{B}^{\mu,S(\cdot)}(w^{S(\cdot)})}.\end{aligned}$$

$$\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \approx w(B)^{\frac{1}{p_2(\cdot)}} \approx w(B)^{\frac{1}{S(\cdot)}+\frac{1}{p(\cdot)}} \approx \|\chi_B\|_{L^{p(\cdot)}(w^{p(\cdot)})}\|\chi_B\|_{L^{S(\cdot)}(w^{S(\cdot)})}$$

Given that $\mu = \lambda_1 + \frac{\alpha}{n}$, using the result of theorem 6.2.1

$$\|A_1\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|B|^{\lambda_2}\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})},$$

$$A_2 = \frac{1}{|x|^{n-\alpha}}\int_{B(0,|x|)}|f(y)(b(y)-b_B)|dy \cdot \chi_B(x).$$

$$\begin{aligned}A_2 &= \sum_{k=-\infty}^{0}\frac{1}{|x|^{n-\alpha}}\sum_{j=-\infty}^{k}\int_{2^jB\backslash 2^{j-1}B}|(b(y)-b_B)f(y)|dy \cdot \chi_{2^kB\backslash 2^{k-1}B}(x) \\ &\leq \sum_{k=-\infty}^{0}\frac{1}{|x|^{n-\alpha}}\sum_{j=-\infty}^{k}\int_{2^jB\backslash 2^{j-1}B}|(b(y)-b_{2^jB})f(y)|dy \cdot \chi_{2^kB\backslash 2^{k-1}B}(x) \\ &+ \sum_{k=-\infty}^{0}\frac{1}{|x|^{n-\alpha}}\sum_{j=-\infty}^{k}\int_{2^jB\backslash 2^{j-1}B}|(b_B-b_{2^jB})f(y)|dy \cdot \chi_{2^kB\backslash 2^{k-1}B}(x) \\ &= A_{21} + A_{22}\end{aligned}$$

$$A_{21} = \sum_{k=-\infty}^{0}|2^kB|^{\frac{\alpha}{n}-1}\sum_{j=-\infty}^{k}\int_{2^jB\backslash 2^{j-1}B}|(b(y)-b_{2^jB})f(y)|dy \cdot \chi_{2^kB\backslash 2^{k-1}B}(x)$$

Using Holder inequality ($\frac{1}{p_1(\cdot)} + \frac{1}{t(\cdot)} + \frac{1}{p(\cdot)} = 1$).

$$A_{21} \leq C \sum_{k=-\infty}^{0} |2^k B|^{\frac{\alpha}{n}-1} \chi_{2^k B \setminus 2^{k-1}B}(x) \sum_{j=-\infty}^{k} \|(b(y) - b_{2^j B})\chi_{2^j B}\|_{L^{p(\cdot)}(w^{p(\cdot)})}$$

$$\|f\chi_{2^j B}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_{2^j B}\|_{L^{t(\cdot)}(w^{t(\cdot)})}$$

$$= C \sum_{k=-\infty}^{0} |2^k B|^{\frac{\alpha}{n}-1} \chi_{2^k B \setminus 2^{k-1}B}(x) \sum_{j=-\infty}^{k} \|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}$$

$$|2^j B|^{\lambda} \|\chi_{2^j B}\|_{L^{p(\cdot)}(w^{p(\cdot)})}$$

$$\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} |2^j B|^{\lambda_1} \|\chi_{2^j B}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_{2^j B}\|_{L^{t(\cdot)}(w^{t(\cdot)})}$$

$$\|\chi_{2^j B}\|_{(L^{p_1'(\cdot)}(w^{p_1(\cdot)}))'} \approx w(2^j B)^{\frac{1}{p_1'(\cdot)}} \approx w(2^j B)^{\frac{1}{p(\cdot)} + \frac{1}{t(\cdot)}} \approx \|\chi_{2^j B}\|_{L^{p(\cdot)}(w^{p(\cdot)})} \|\chi_{2^j B}\|_{L^{t(\cdot)}(w^{t(\cdot)})}$$

$$A_{21} = C \sum_{k=-\infty}^{0} |2^k B|^{\frac{\alpha}{n}-1} \chi_{2^k B \setminus 2^{k-1}B}(x) \|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}$$

$$\sum_{j=-\infty}^{k} |2^j B|^{\lambda+\lambda_1} \|\chi_{2^j B}\|_{(L^{p_1'(\cdot)}(w^{p_1(\cdot)}))'} \|\chi_{2^j B}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}$$

By using the result of lemma 4.2.3, we come to have

$$A_{21} = C \sum_{k=-\infty}^{0} |2^k B|^{\frac{\alpha}{n}-1} \chi_{2^k B \setminus 2^{k-1}B}(x) \|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}$$

$$\sum_{j=-\infty}^{k} |2^j|^{\lambda+\lambda_1+1} |B|^{\lambda+\lambda_1+1}$$

$$= C \|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} |2^k|^{\frac{\alpha}{n}+\lambda+\lambda_1} \chi_{2^k B \setminus 2^{k-1}B}(x) |B|^{\lambda+\lambda_1+\frac{\alpha}{n}}$$

$$\|A_{21}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}$$

$$\sum_{k=-\infty}^{0}|2^k|^{\frac{\alpha}{n}+\lambda+\lambda_1}\|\chi_{2^kB}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}|B|^{\lambda+\lambda_1+\frac{\alpha}{n}}$$

$$= C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}$$

$$\sum_{k=-\infty}^{0}|2^k|^{\frac{\alpha}{n}+\lambda+\lambda_1}w(2^kB)^{\frac{1}{p_2(\cdot)}}|B|^{\lambda+\lambda_1+\frac{\alpha}{n}}$$

$$= C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}w(B)^{\frac{1}{p_2(\cdot)}}|B|^{\lambda_2}$$

$$\sum_{k=-\infty}^{0}|2|^{k(\lambda_2+\frac{1}{p_2(\cdot)})}$$

$$\|A_{21}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}|B|^{\lambda_2}$$

$$A_{22} = \sum_{k=-\infty}^{0}|2^kB|^{\frac{\alpha}{n}-1}\sum_{j=-\infty}^{k}\int_{2^jB\setminus 2^{j-1}B}|(b_B - b_{2^jB})f(y)|dy \cdot \chi_{2^kB\setminus 2^{k-1}B}(x)$$

$$|(b_B - b_{2^jB})| = \sum_{i=j}^{-1}|(b_{2^{i+1}B} - b_{2^iB})$$

$$= \sum_{i=j}^{-1}\frac{1}{|2^iB|}\int_{2^iB}|b(y) - b_{2^{i+1}B}|dy$$

$$\leq C\sum_{i=j}^{-1}\frac{1}{|2^iB|}\|(b - b_{2^{i+1}b})\chi_{2^{i+1}B}\|_{L^{p(\cdot)}(w^{p(\cdot)})}\|\chi_{2^{i+1}B}\|_{(L^{p(\cdot)}(w^{p_1(\cdot)}))'}$$

By virtue of lemma 4.2.3, we have

$$
\begin{aligned}
|(b_B - b_{2^j B})| &\leq C \sum_{i=j}^{-1} \frac{1}{|2^i B|} \|(b - b_{2^{i+1}b})\chi_{2^{i+1}B}\|_{L^{p(\cdot)}(w^{p(\cdot)})} \frac{|2^{i+1}B|}{\|\chi_{2^{i+1}B}\|_{L^{p(\cdot)}(w^{p(\cdot)})}} \\
&\leq C \sum_{i=j}^{-1} \|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} |2^{i+1}B|^{\lambda} \\
&\leq C \|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \sum_{i=j}^{-1} |2^{i+1}B|^{\lambda} \\
&\leq C \|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} |2^{j+1}B|^{\lambda} |j|
\end{aligned}
\tag{6.3.1}
$$

$$A_{22} \leq C \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\alpha}{n}-1} \sum_{j=-\infty}^{k} \|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}$$

$$|2^{j+1}B|^{\lambda}|j|\|f\chi_{2^jB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{2^jB}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}$$

$$\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\alpha}{n}-1}$$

$$\sum_{j=-\infty}^{k} |2^{j+1}B|^{\lambda}|j|\|f\chi_{2^jB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\frac{|2^jB|}{\|\chi_{2^jB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}}$$

$$\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\alpha}{n}-1}$$

$$\sum_{j=-\infty}^{k} |2^jB|^{\lambda+1}|j|\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|2^jB|^{\lambda_1}$$

$$\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\alpha}{n}-1}$$

$$\sum_{j=-\infty}^{k} |2^jB|^{\lambda+\lambda_1+1}|j|$$

$$\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}$$

$$\sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)|2^kB|^{\frac{\alpha}{n}-1}|2^kB|^{\lambda+\lambda_1+1}|k|$$

$$\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} |k||2^kB|^{\lambda+\lambda_1+\frac{\alpha}{n}}\chi_{2^kB\backslash 2^{k-1}B}(x)$$

$$\begin{aligned}\|A_{22}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \\ &\sum_{k=-\infty}^{0}|k||2^kB|^{\lambda+\lambda_1+\frac{\alpha}{n}}\|\chi_{2^kB}\|_{L^{p_2(\cdot)}} \\ &\le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \\ &\sum_{k=-\infty}^{0}|k||2^kB|^{\lambda+\lambda_1+\frac{\alpha}{n}}w(2^kB)^{\frac{1}{p_2(\cdot)}} \\ &\le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \\ &\sum_{k=-\infty}^{0}|k||2^k|^{\lambda_2+\frac{1}{p_2(\cdot)}}|B|^{\lambda_2}w(B)^{\frac{1}{p_2(\cdot)}} \\ &\le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|B|^{\lambda_2}\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\end{aligned}$$

Combine all results of $A_1$, $A_2$, $A_{21}$, $A_{22}$, we obtained the required result

$$\|[b,H_\alpha]f\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|B|^{\lambda_2}\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}$$

$$\|[b,H_\alpha]f\|_{\dot{B}^{p_2(\cdot),\lambda_2}(w^{p_2(\cdot)})} \le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}$$

**Theorem 6.3.2** *Let $p_1(\cdot)$, $p_2(\cdot)$, $p(\cdot)$ and $\alpha$ is define same as in theorem 6.3.1.*

*If $b \in \|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}$ , $\mu = \lambda_1 + \frac{\alpha}{n}$ and $\lambda_2 = \lambda + \lambda_1 + \frac{\alpha}{n}$, then*

$$\|[b,H^*_\alpha]f\|_{\dot{B}^{p_2(\cdot),\lambda_2}(w^{p_2(\cdot)})} \le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}.$$

**proof**

$$\begin{aligned}|[b,H^*_\alpha]f(x)\cdot\chi_B(x)| &\le \int_{B(0,|x|)^c}\frac{|(b(x)-b(y))f(y)|}{|y|^{n-\alpha}}dy\cdot\chi_B(x) \\ &\le \int_{B(0,|x|)^c}\frac{|(b(x)-b_B)f(y)|}{|y|^{n-\alpha}}dy\cdot\chi_B(x) \\ &+ \int_{B(0,|x|)^c}\frac{|(b(y)-b_B)f(y)|}{|y|^{n-\alpha}}dy\cdot\chi_B(x) \\ &= D_1 + D_2.\end{aligned}$$

$$D_1 = \int_{B(0,|x|)^c} \frac{|(b(x)-b_B)f(y)|}{|y|^{n-\alpha}} dy \cdot \chi_B(x)$$
$$= |(b(x)-b_B)\chi_B(x)||H^*_\alpha f(x)|,$$
$$\|D_1\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} = \|(b(x)-b_B)\chi_B H^*_\alpha f\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.$$

By using Holder inequality ($\frac{1}{p_2(\cdot)} = \frac{1}{S(\cdot)} + \frac{1}{p(\cdot)}$)

$$\begin{aligned}\|D_1\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\le C\|(b(x)-b_B)\chi_B\|_{L^{p(\cdot)}(w^{p(\cdot)})}\|H^*_\alpha f\chi_B\|_{L^{S(\cdot)}(w^{S(\cdot)})} \\ &= C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|B|^\lambda\|\chi_B\|_{L^{p(\cdot)}(w^{p(\cdot)})}|B|^\mu\|\chi_B\|_{L^{S(\cdot)}(w^{S(\cdot)})} \\ &\|H^*_\alpha f\|_{\dot{B}^{\mu,S(\cdot)}(w^{S(\cdot)})},\end{aligned}$$

Given that $\mu = \lambda_1 + \frac{\alpha}{n}$, using the result of theorem 6.2.2

$$\begin{aligned}\|D_1\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|B|^{\lambda_2}\|\chi_B\|_{L^{p(\cdot)}(w^{p(\cdot)})}\|\chi_B\|_{L^{S(\cdot)}(w^{S(\cdot)})} \\ &\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}, \\ &\le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|B|^{\lambda_2}\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})},\end{aligned}$$

$$D_2 = \int_{B(0,|x|)^c} \frac{|f(y)(b(y)-b_B)|}{|y|^{n-\alpha}} dy \cdot \chi_B(x).$$

$$\begin{aligned}D_2 &= \sum_{k=-\infty}^{0} \int_{2^jB\setminus 2^{j-1}B} \frac{|(b(y)-b_B)f(y)|}{|y|^{n-\alpha}} dy \cdot \chi_{2^kB\setminus 2^{k-1}B}(x) \\ &\le \sum_{k=-\infty}^{0} \chi_{2^kB\setminus 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\alpha}{n}-1} \int_{2^jB\setminus 2^{j-1}B} |(b(y)-b_{2^jB})f(y)|dy \\ &+ \sum_{k=-\infty}^{0} \chi_{2^kB\setminus 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\alpha}{n}-1} \int_{2^jB\setminus 2^{j-1}B} |(b_B-b_{2^jB})f(y)|dy \\ &= D_{21} + D_{22}\end{aligned}$$

$$D_{21} = \sum_{k=-\infty}^{0} \chi_{2^kB\setminus 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\alpha}{n}-1} \int_{2^jB\setminus 2^{j-1}B} |(b(y)-b_{2^jB})f(y)|dy$$

Using Holder inequality $(\frac{1}{t(\cdot)}+\frac{1}{p_1(\cdot)}+\frac{1}{p(\cdot)}=1)$.

$$D_{21}\leq C\sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{j=k+1}^{\infty}|2^jB|^{\frac{\alpha}{n}-1}\|(b(y)-b_{2^jB})\chi_{2^jB}\|_{L^{p(\cdot)}(w^{p(\cdot)})}$$

$$\|f\chi_{2^jB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{2^jB}\|_{L^{t(\cdot)}(w^{t(\cdot)})}$$

$$=C\sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{j=k+1}^{\infty}|2^jB|^{\frac{\alpha}{n}-1}\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|2^jB|^{\lambda}\|\chi_{2^jB}\|_{L^{p(\cdot)}(w^{p(\cdot)})}$$

$$\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|2^jB|^{\lambda_1}\|\chi_{2^jB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{2^jB}\|_{L^{t(\cdot)}(w^{t(\cdot)})}$$

$$\leq C\sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{j=k+1}^{\infty}|2^jB|^{\frac{\alpha}{n}-1}\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}|2^jB|^{\lambda}$$

$$\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|2^jB|^{\lambda_1}\|\chi_{2^jB}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{2^jB}\|_{(L^{p(\cdot)}(w^{p(\cdot)}))'}$$

$$=C\sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}$$

$$\sum_{j=k+1}^{\infty}|2^jB|^{\frac{\alpha}{n}-1}|2^jB|^{\lambda+\lambda_1+1}$$

$$=C\sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\sum_{j=k+1}^{\infty}|2^jB|^{\lambda_2}$$

$$=C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}\sum_{k=-\infty}^{0}|2^{(k+1)}B|^{\lambda_2}\chi_{2^kB\backslash 2^{k-1}B}(x)$$

$$\begin{aligned}
\|D_{21}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \\
&\sum_{k=-\infty}^{0}|2^{(k+1)}B|^{\lambda_2}\|\chi_{2^kB}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \\
&\sum_{k=-\infty}^{0}|2^{(k+1)}B|^{\lambda_2}w(2^kB)^{\frac{1}{p_2(\cdot)}} \\
&\le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|B|^{\lambda_2}w(B)^{\frac{1}{p_2(\cdot)}} \\
&\sum_{k=-\infty}^{0}|2^{(k+1)}|^{\lambda_2+\frac{1}{p_2(\cdot)}} \\
&\le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|B|^{\lambda_2}\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}
\end{aligned}$$

$$D_{22} = \sum_{k=-\infty}^{0}\chi_{2^kB\setminus 2^{k-1}B}(x)\sum_{j=-\infty}^{k}|2^jB|^{\frac{\alpha}{n}-1}\int_{2^jB\setminus 2^{j-1}B}|(b_B - b_{2^jB})f(y)|dy$$

Here we use inequality (6.3.1)

$$
\begin{aligned}
D_{22} \le C \sum_{k=-\infty}^{0} \chi_{2^k B\setminus 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^j B|^{\frac{\alpha}{n}-1} \|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} |2^{j+1}B|^{\lambda} |j| \\
\|f\chi_{2^j B}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_{2^j B}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \\
\le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \sum_{k=-\infty}^{0} \chi_{2^k B\setminus 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^j B|^{\frac{\alpha}{n}-1} |2^{j+1}B|^{\lambda} |j| \\
\|f\chi_{2^j B}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \frac{|2^j B|}{\|\chi_{2^j B}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}} \\
\le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \sum_{k=-\infty}^{0} \chi_{2^k B\setminus 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^j B|^{\frac{\alpha}{n}-1} |2^j B|^{\lambda+1} |j| \\
\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} |2^j B|^{\lambda_1} \\
\le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} \chi_{2^k B\setminus 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^j B|^{\lambda+\lambda_1+\frac{\alpha}{n}} |j| \\
\le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} \chi_{2^k B\setminus 2^{k-1}B}(x) |2^{k+1}B|^{\lambda_2} |k+1| \\
\le C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})} \|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{0} |k+1| |2^{k+1}B|^{\lambda_2} \chi_{2^k B\setminus 2^{k-1}B}(x)
\end{aligned}
$$

$$\begin{aligned}
\|D_{22}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \\
&\sum_{k=-\infty}^{0} |k+1||2^{k+1}B|^{\lambda_2}\|\chi_{2^kB}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \\
&\sum_{k=-\infty}^{0} |k+1||2^{k+1}B|^{\lambda_2}w(2^kB)^{\frac{1}{p_2(\cdot)}} \\
&\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})} \\
&\sum_{k=-\infty}^{0} |k+1||2^{k+1}|^{\lambda_2+\frac{1}{p_2(\cdot)}}|B|^{\lambda_2}w(B)^{\frac{1}{p_2(\cdot)}} \\
&\leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|B|^{\lambda_2}\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}
\end{aligned}$$

Combine all results of $D_1$, $D_2$, $D_{21}$, $D_{22}$, we obtained the required result

$$\|[b, H_\alpha^*]f\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}|B|^{\lambda_2}\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}$$

$$\|[b, H_\alpha^*]f\|_{\dot{B}^{p_2(\cdot),\lambda_2}(w^{p_2(\cdot)})} \leq C\|b\|_{CBMO^{p(\cdot),\lambda}(w^{p(\cdot)})}\|f\|_{\dot{B}^{p_1(\cdot),\lambda_1}(w^{p_1(\cdot)})}$$

## 6.4 Conclusion

We concluded that the fractional Hardy operator and its adjoint operator can be bounded on weighted central Morrey space with variable exponent. Similar results for their commutators are obtained when the symbol functions belong to weighted $\lambda$-central BMO space with variable exponent.

# Chapter 7
# Variable Morrey-Herz Estimates for Fractional Rough Hardy Operators

## 7.1 Introduction

In this chapter our main focus on the fractional rough hardy operators. Here we study continuity criteria for fractional rough hardy on Herz Morrey space along with variable exponent.

## 7.2 Boundedness of Fractional Hardy Operators

**Proposition 7.2.1** *[96] Let $p_1(\cdot) \in P(\mathbb{R}^n)$, $0 < \beta < \frac{n}{(p_1)_+}$ and define $p_2(\cdot)$ by*

$$\frac{1}{p_2(\cdot)} = \frac{1}{p_1(\cdot)} - \frac{\beta}{n}.$$

*Then*

$$\|I_\beta f\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.$$

**Theorem 7.2.2** *Let $0 < p_1 \leq q < \infty$, $\Omega \in L^s(S^{n-1})$, $\frac{n}{n-1} < s$. Let $q_1(\cdot), p(\cdot), q_2(\cdot) \in P(\mathbb{R}^n)$ hold inequalities (2.1.2) and (2.1.3) in propo-*

*sition 2.1.1 and define the variable exponent* $p(\cdot)$ *by*

$$\frac{1}{q_1(\cdot)} = \frac{1}{p(\cdot)} + \frac{\beta}{n}.$$

*Let* $\lambda$ *satisfy the following condition:*

*When* $\frac{1}{q_2(\cdot)} = \frac{1}{q_1'(\cdot)} - \frac{1}{s}$, *there is* $-\lambda > \frac{\beta}{n}$ *and* $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{C}^{log}(\mathbb{R}^n)$ *be log Hölder continuous at the origin, with* $\alpha(0) \leq \alpha(\infty) < \lambda + n\delta_3 - \frac{n}{s}$ *then*

$$\|H_{\beta,\Omega} f(x)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}} \leq C\|f(x)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}}.$$

**proof**

$$|H_{\beta,\Omega} f(x) \cdot \chi_k(x)| \leq \frac{1}{|x|^{n-\beta}} \int_{B_k} |f(t)||\Omega(x-t)|dt \cdot \chi_k(x)$$

$$\leq C2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\Omega(x-t)\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \chi_k(x).$$

By using of $\frac{1}{q_2(\cdot)} + \frac{1}{s} = \frac{1}{q_1'(\cdot)}$

$$|H_{\beta,\Omega} f(x) \cdot \chi_k(x)|$$

$$\leq C2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\Omega(x-t)\chi_j\|_{L^s(\mathbb{R}^n)} \|\chi_j\|_{L^{q_2(\cdot)}} \cdot \chi_k(x).$$

$$\|H_{\beta,\Omega} f \cdot \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\leq C2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\Omega(x-t)\chi_j\|_{L^s(\mathbb{R}^n)} \|\chi_j\|_{L^{q_2(\cdot)}} \cdot \|\chi_k\|_{L^{p(\cdot)}}.$$

Since we have

$$\|\chi_k\|_{L^{q_2\cdot}} \approx |B_k|^{\frac{1}{q_2(\cdot)}} \approx |B_k|^{\frac{1}{q_1'(\cdot)} - \frac{1}{s}} \approx |B_k|^{-\frac{1}{s}} \|\chi_k\|_{L^{q_1'(\cdot)}}$$

$$\|H_{\beta,\Omega} f \cdot \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\leq C2^{-k(n-\beta)} \sum_{j=-\infty}^{k} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\Omega(x-t)\chi_j\|_{L^s(\mathbb{R}^n)} |B_j|^{-\frac{1}{s}} \|\chi_j\|_{L^{q_1'(\cdot)}} \|\chi_k\|_{L^{p(\cdot)}}. \tag{7.2.1}$$

Using inequality (3.2.2) in (7.2.1)

$$\|H_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\chi_j\|_{L^s(\mathbb{R}^n)}|B_j|^{-\frac{1}{s}}\|\chi_j\|_{L^{q_1'(\cdot)}}\|\chi_k\|^{-1}_{L^{q_1'(\cdot)}}. \tag{7.2.2}$$

Using condition (2.1.5)

$$\|H_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\sum_{j=-\infty}^{k}2^{(j-k)n\delta_3}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\chi_j\|_{L^s(\mathbb{R}^n)}|B_j|^{-\frac{1}{s}}. \tag{7.2.3}$$

For $t\in C_j$ and $x\in C_k$ and $j\leq k$, we have $0\leq|x-t|\leq|x|+2^j\leq 2.2^k$, and

$$\int_{C_j}|\Omega(x-t)|^s dt \leq \int_0^{2^{k+1}}\int_{S^{n-1}}|\Omega(x^{'})|^s d\sigma(x^{'})r^{n-1}dr \leq C2^{kn}$$

$$\|H_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\sum_{j=-\infty}^{k}2^{(j-k)(n\delta_3-\frac{n}{s})}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}. \tag{7.2.4}$$

In the rest of the proof, in order to estimate $\|f_j\|_{L^{q_1(\cdot)}(w^{q_1(\cdot)})}$, we consider two cases as below.

Case 1: We take $j<0$, and start estimating as

$$\begin{aligned}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} &= 2^{-j\alpha(0)}\left(2^{j\alpha(0)p_1}\|f_j\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{p_1}}\\ &\leq 2^{-j\alpha(0)}\left(\sum_{i=-\infty}^{j}2^{i\alpha(0)p_1}\|f_i\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{p_1}}\\ &\leq 2^{j(\lambda-\alpha(0))}2^{-j\lambda}\left(\sum_{i=-\infty}^{j}2^{i\alpha(\cdot)p_1}\|f_i\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{p_1}}\\ &\leq C2^{j(\lambda-\alpha(0))}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}. \end{aligned}\tag{7.2.5}$$

Case 2: For $j \geq 0$, we get

$$\begin{aligned}
\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} &= 2^{-j\alpha(\infty)}\left(2^{j\alpha(\infty)p_1}\|f_j\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{p_1}} \\
&\leq 2^{-j\alpha(\infty)}\left(\sum_{i=-\infty}^{j} 2^{i\alpha(\infty)p_1}\|f_i\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{p_1}} \\
&\leq 2^{j(\lambda-\alpha(\infty))}2^{-j\lambda}\left(\sum_{i=-\infty}^{j} 2^{i\alpha(\cdot)p_1}\|f_i\|^{p_1}_{L^{q_1(\cdot)}(\mathbb{R}^n)}\right)^{\frac{1}{p_1}} \\
&\leq C2^{j(\lambda-\alpha(\infty))}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}. \qquad (7.2.6)
\end{aligned}$$

By definition of variable exponent Herz-Morrey space along with the use of Proposition (4.3.1) we arrive at the following inequality:

$$\begin{aligned}
\|H_{\beta,\Omega}f(x)\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} &= \sup_{k_0\in Z} 2^{-k_0\lambda p_1}\sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot)p_1}\|H_{\beta,\Omega}f(x)\cdot\chi_k(x)\|^{p_1}_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leq \max\left\{\sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\|H_{\beta,\Omega}f(x)\cdot\chi_k(x)\|^{p_1}_{L^{p(\cdot)}(\mathbb{R}^n)},\right. \\
&\quad \sup_{\substack{k_0\in Z\\ k_0\geq 0}} 2^{-k_0\lambda p_1}\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p_1}\|H_{\beta,\Omega}f(x)\cdot\chi_k(x)\|^{p_1}_{L^{p(\cdot)}(\mathbb{R}^n)}\right. \\
&\quad \left.\left.+\sum_{k=0}^{k_0} 2^{k\alpha(\infty)p_1}\|H_{\beta,\Omega}f(x)\cdot\chi_k(x)\|^{p_1}_{L^{p(\cdot)}(\mathbb{R}^n)}\right)\right\} \\
&= \max\{A_1, A_2+A_3\}, \qquad (7.2.7)
\end{aligned}$$

where

$$A_1 = \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\|H_{\beta,\Omega}f(x)\cdot\chi_k(x)\|^{p_1}_{L^{p(\cdot)}(\mathbb{R}^n)},$$

$$A_2 = \sup_{\substack{k_0 \in Z \\ k_0 \geq 0}} 2^{-k_0 \lambda p_1} \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p_1} \|H_{\beta,\Omega} f(x) \cdot \chi_k(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_1},$$

$$A_3 = \sup_{\substack{k_0 \in Z \\ k_0 \geq 0}} 2^{-k_0 \lambda p_1} \sum_{k=0}^{k_0} 2^{k\alpha(\infty)p_1} \|H_{\beta,\Omega} f(x) \cdot \chi_k(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_1}.$$

First, we approximate $A_1$. Since $\alpha(0) \leq \alpha(\infty) < n\delta_2 + \lambda - \beta$,

$$\begin{aligned}
Y_1 &\leq C \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0 \lambda p_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1} \left( \sum_{j=-\infty}^{k} 2^{(n\delta_3 - \frac{n}{s})(j-k)} \|f\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\
&\leq C \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0 \lambda p_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1} \left( \sum_{j=-\infty}^{k} 2^{(n\delta_3 - \frac{n}{s})(j-k)} 2^{j(\lambda - \alpha(0))} \|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)} \right)^{p_1} \\
&\leq C \|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{p_1} \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0 \lambda p_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1} \left( \sum_{j=-\infty}^{k} 2^{(\frac{n}{s} - n\delta_3)(k-j)} 2^{j(\lambda - \alpha(0))} \right)^{p_1} \\
&\leq C \|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{p_1} \sup_{\substack{k_0 \in Z \\ k_0 < 0}} 2^{-k_0 \lambda p_1} \sum_{k=-\infty}^{k_0} 2^{k\lambda p_1} \left( \sum_{j=-\infty}^{k} 2^{(j-k)(-\frac{n}{s} + n\delta_3 - \alpha(0) + \lambda)} \right)^{p_1} \\
&\leq C \|f\|_{M\dot{K}_{p_1,q_1(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)}^{p_1}.
\end{aligned}$$

The estimate of $A_2$ is similar to that of $A_1$. Lastly, we estimate $A_3$

$$\begin{aligned}
A_3 &\leq C \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(\infty)p_1} \left( \sum_{j=-\infty}^{k} 2^{(n\delta_3-\frac{n}{s})(j-k)} \|f\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\
&\leq C \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(\infty)p_1} \left( \sum_{j=-\infty}^{k} 2^{(n\delta_3-\frac{n}{s})(j-k)} 2^{j(\lambda-\alpha(\infty))} \|f\|_{M\dot{k}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)} \right)^{p_1} \\
&\leq C \|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)} \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(\infty)p_1} \left( \sum_{j=-\infty}^{k} 2^{(\frac{n}{s}-n\delta_3)(k-j)} 2^{j(\lambda-\alpha(\infty))} \right)^{p_1} \\
&\leq C \|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)} \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1} \sum_{k=-\infty}^{k_0} 2^{k\lambda p_1} \left( \sum_{j=-\infty}^{k} 2^{(j-k)(-\frac{n}{s}+n\delta_3-\alpha(\infty)+\lambda)} \right)^{p_1} \\
&\leq C \|f\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

The desired result is obtained by inserting the approximations of $A_1, A_2$ and $A_3$ into (7.2.7).

**Theorem 7.2.3** *Let $p_1, q, q_1(\cdot), q_2(\cdot), p(\cdot), \beta$ and $\alpha(\cdot)$ be as in Theorem 7.2.2. In addition, if $\beta - \frac{n}{s} + \lambda < \alpha(0) \leq \alpha(\infty)$, then*

$$\|H^*_{\beta,\Omega} f(x)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}} \leq C \|f(x)\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{p_1,q_1(\cdot)}}.$$

**proof**

$$\begin{aligned}
|H^*_{\beta,\Omega} f(x)\cdot\chi_k| &\leq \int_{\mathbb{R}^n\setminus B_k} |f(t)\Omega(x-t)||t|^{\beta-n} dt \cdot \chi_k(x) \\
&\leq C \sum_{j=k+1}^{\infty} 2^{j(\beta-n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\Omega(x-t)\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \cdot \chi_k(x).
\end{aligned}$$

By using $\frac{1}{q_2(\cdot)} + \frac{1}{s} = \frac{1}{q_1'(\cdot)}$

$$\begin{aligned}
&\|H^*_{\beta,\Omega} f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leq C \sum_{j=k+1}^{\infty} 2^{j(\beta-n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\Omega(x-t)\|_{L^s} \|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Using inequality (3.2.2)

$$\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\le C\sum_{j=k+1}^{\infty}2^{(j-k)(\beta-n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\|_{L^s}\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|\chi_k\|^{-1}_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.$$

As we know

$$\|\chi_j\|_{L^{q_2(\cdot)}}\approx|B_j|^{\frac{1}{q_2(\cdot)}}\approx|B_j|^{\frac{1}{q_1'(\cdot)}-\frac{1}{s}}\approx|B_j|^{-\frac{1}{s}}\|\chi_j\|_{L^{q_1'(\cdot)}}$$

$$\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\le C\sum_{j=k+1}^{\infty}2^{(j-k)(\beta-n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\|_{L^s}|B_j|^{-\frac{1}{s}}\|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|^{-1}_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.$$

Now by using Lemma 2.1.3

$$\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\le C\sum_{j=k+1}^{\infty}2^{(j-k)(\beta-n+n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\|_{L^s}|B_j|^{-\frac{1}{S}}.$$

For further calculation follows theorem 7.2.2, we get

$$\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\le C\sum_{j=k+1}^{\infty}2^{(j-k)(\beta-\frac{n}{s})}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.$$

Hence we have

$$\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{\dot{B}^{p(\cdot),\lambda_2}(\mathbb{R}^n)}\le C\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)}\sum_{j=k+1}^{\infty}2^{(j-k)(\beta-\frac{n}{s}+n\lambda_2+n)}.$$

$$\begin{aligned}
\|H^*_{\beta,\Omega}f(x)\|^{p_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)} &= \sup_{k_0\in Z} 2^{-k_0\lambda p_1}\sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot)p_1}\|H^*_{\beta,\Omega}f(x)\cdot\chi_k(x)\|^{p_1}_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\le \max\Bigg\{\sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\|H^*_{\beta,\Omega}f(x)\cdot\chi_k(x)\|^{p_1}_{L^{p(\cdot)}(\mathbb{R}^n)},\\
&\quad \sup_{\substack{k_0\in Z\\ k_0\ge 0}} 2^{-k_0\lambda p_1}\Bigg(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p_1}\|H^*_{\beta,\Omega}f(x)\cdot\chi_k(x)\|^{p_1}_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\quad + \sum_{k=0}^{k_0} 2^{k\alpha(\infty)p_1}\|H^*_{\beta,\Omega}f(x)\cdot\chi_k(x)\|^{p_1}_{L^{p(\cdot)}(\mathbb{R}^n)}\Bigg)\Bigg\}\\
&= \max\{B_1, B_2+B_3\},
\end{aligned}$$

where

$$B_1 = \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda p_1}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)p_1}\|H^*_{\beta,\Omega}f(x)\cdot\chi_k(x)\|^{p_1}_{L^{p(\cdot)}(\mathbb{R}^n)},$$

$$B_2 = \sup_{\substack{k_0\in Z\\ k_0\ge 0}} 2^{-k_0\lambda p_1}\sum_{k=-\infty}^{-1} 2^{k\alpha(0)p_1}\|H^*_{\beta,\Omega}f(x)\cdot\chi_k(x)\|^{p_1}_{L^{p(\cdot)}(\mathbb{R}^n)},$$

$$B_3 = \sup_{\substack{k_0\in Z\\ k_0\ge 0}} 2^{-k_0\lambda p_1}\sum_{k=0}^{k_0} 2^{k\alpha(\infty)p_1}\|H^*_{\beta,\Omega}f(x)\cdot\chi_k(x)\|^{p_1}_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

## 7.3 Conclusion

In chapter we discussed the boundedness criteria for the fractional Hardy operators on weighted variable exponent Morrey-Herz spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w)$.

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