A study on multi-criteria group decision making techniques based on fuzzy and quantitative information

QUAID-I-AZAM UNIVERSITY
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This dissertation is submitted for the degree of
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Abstract

Decision making activities are prevalent in the human race. Till date, numerous multi-criteria group decision making (MCGDM) techniques have been put forward under various fuzzy contexts to address scenarios of vague nature. However, there are some imperfections in these developed studies. The present work mainly aims to remove the existing limitations and to construct some robust MCGDM techniques in order to tackle the complex problems more accurately. For this, we revise the basic operational laws, comparison method and a series of aggregation operators for the probabilistic uncertain linguistic term set (PULTS) and uncertain probabilistic linguistic term set (UPLTS). In addition, some innovative fuzzy tools, namely probabilistic hesitant intuitionistic linguistic term set (PHILTS) and weighted interval-valued dual hesitant fuzzy set (WIVDHFS), and their related theories, are introduced. Then, based on the proposed theory, some well-known MCGDM techniques, viz. technique of order preference similarity to the ideal solution (TOPSIS), gained and lost dominance score (GLDS), weighted aggregated sum product assessment (WASPAS) and aggregation-based method are extended to more generalized fuzzy frameworks. Meanwhile, several criteria weight determination models are built according to different fuzzy environments. To validate the practicality of the proposed work, we address some real world problems, including investment problem, supplier selection and teaching quality assessment. Further, to testify the potentiality and weaknesses of the provided techniques, comparative study and sensitivity analysis are also delineated.

Contents

List of Abbreviations

MCGDM	Multi-criteria group decision making
DMs	Decision makers
FS	Fuzzy set
IFS	Intuitionistic fuzzy set
IVHFS	Interval-valued hesitant fuzzy set
IVIFS	Interval-valued intuitionistic fuzzy set
IVDHFS	Interval-valued dual hesitant fuzzy set
LTS	Linguistic term set
HFLTS	Hesitant fuzzy linguistic term set
HIFLTS	Hesitant intuitionistic fuzzy linguistic term set
TOPSIS	Technique of order preference similarity to the ideal solution
PLTS	Probabilistic linguistic term set
PHILTS	Probabilistic hesitant intuitionistic linguistic term set
PULTS	Probabilistic uncertain linguistic term set
UPLTS	Uncertain probabilistic linguistic term set

IVDHFS	Interval-valued dual hesitant fuzzy set
GLDS	Gained and lost dominance score
WASPAS	Weighted aggregated sum product assessment
WIVDHFS	Weighted interval-valued dual hesitant fuzzy set
PIS	Positive ideal solution
NIS	Negative ideal solution
PHILE	Probabilistic hesitant intuitionistic linguistic element
PHILA	Probabilistic hesitant intuitionistic linguistic averaging
PHILG	Probabilistic hesitant intuitionistic linguistic geometric
PHILWA	Probabilistic hesitant intuitionistic linguistic weighted averaging
PHILWG	Probabilistic hesitant intuitionistic linguistic weighted geometric
PULE	Probabilistic uncertain linguistic element
PULWA	Probabilistic uncertain linguistic weighted averaging
PULWG	Probabilistic uncertain linguistic weighted geometric
FIC	Flagship investment company
WIVDHFE	Weighted interval-valued dual hesitant fuzzy element
GWIVDHFWA	Generalized weighted interval-valued dual hesitant fuzzy weighted averaging
GWIVDHFWG	Generalized weighted interval-valued dual hesitant fuzzy weighted geometric
WIVDHFWA	Weighted interval-valued dual hesitant fuzzy weighted averaging
WIVDHFWG	Weighted interval-valued dual hesitant fuzzy weighted geometric
WIVDHFHWA	Weighted interval-valued dual hesitant fuzzy Hammer weighted averaging

WIVDHFHWG	Weighted interval-valued dual hesitant fuzzy Hammer weighted geometric
WIVDHFEWA	Weighted interval-valued dual hesitant fuzzy Einstein weighted averaging
WIVDHFEWG	Weighted interval-valued dual hesitant fuzzy Einstein weighted geometric
WIVDHFFWA	Weighted interval-valued dual hesitant fuzzy Frank weighted averaging
WIVDHFFWG	Weighted interval-valued dual hesitant fuzzy Frank weighted geometric
UPLWA	Uncertain probabilistic linguistic weighted averaging
UPLWG	Uncertain probabilistic linguistic weighted geometric
UPLSWG	Uncertain probabilistic linguistic simple weighted geometry
WSM	Weighted sum measure
WPM	Weighted product measure

Research paper related to the thesis

It is the requirement of Higher Education Commission for the award of Ph.D degree that the student must have at least one paper in an ISI listed impact factor journal. The content of this thesis are reported in the following journals.

(1) J. Ali, Z. Bashir and T. Rashid, "WASPAS-based decision making methodology with unknown weight information under uncertain evaluations", Expert Systems with Applications, vol. 168, p. 114143, 2021.

(2) J. Ali, Z. Bashir and T. Rashid, "Weighted interval valued dual hesitant fuzzy set and its application to teaching quality assessment", Soft Computing, vol. 25, no. 5, pp. 3503-3530, 2021.

(3) Z. Bashir, J. Ali and T. Rashid, "Consensus-based robust decision making methods under a novel study of probabilistic uncertain linguistic information and their application in Forex investment", Artificial Intelligence Review, vol. 54 no. 3, pp. 2091-2132, 2021.

(4) M. G. Malik, Z. Bashir, T. Rashid and J. Ali, "Probabilistic hesitant intuitionistic linguistic term sets in multi-attribute group decision making", Symmetry, vol. 10, no. 9, p. 392, 2018.

Chapter 1

Introduction

This chapter mainly focuses on presenting limitations of existing theory and to point out the novel contributions.

1.1 Background of research

Due to growing competitions, it is noticed that decision making has become one of the fastest emerging research areas concerned with real world problems. Multi-criteria group decision making (MCGDM) is one of the key components of the decision making process has become most popular, see [1, 2, 3]. Its primary goal is to allow a group of decision makers (DMs) to rate alternatives according to a specific set of criteria and then choose the best one. Since, the criteria conflict with each other, it may not have a unique solution that simultaneously meets each criterion. Years ago, several MCGDM techniques were developed in classical mathematics as comfortable and stable frameworks to make rational decisions. However, there is a wide range of vague phenomena in the current global competitive environment, imprecise concepts and unknown quantities that fall beyond the scope of classical mathematics. To cope with such vague phenomena, a concept of 'Fuzzy' has been coined in mathematics. Fuzzy sets (FSs) originally initiated by Zadeh [4] in 1965, has emerged as one of the fruitful decision aid tool having the capability to tackle vagueness and hesitancy. Numerous MCGDM techniques have been built in a fuzzy environment over the years with extensive theoretical and practical backgrounds [5, 6, 7, 8, 9].

Up to date, several extensions have been done of FSs to enable a better depiction of the real world's imprecision. Atanassov [10] presented the base of intuitionistic fuzzy sets (IFSs) marked by the membership and non-membership functions ranges from 0 to 1. Numerous decision ranking methodologies and related theory have been described in the framework of IFSs [11, 12, 13]. Atanassov and Gargov [14] commenced the doctrine of interval-valued intuitionistic fuzzy sets (IVIFSs) based on IFSs. Following their pioneering work, the theory of IVIFSs was further developed. In [15, 16, 17, 18], many concepts like correlation coefficient, topological properties, the correlation and decomposition theorem, related algorithms and the relationship between other FSs were discussed.

Ju et al. [19] enhanced the concept of interval-valued hesitant fuzzy set (IVHFS) [20] to the interval-valued dual hesitant fuzzy sets (IVDHFSs), in which the membership and non-membership grades are represented by two sets of several interval values. After the introduction of this extraordinary representation, several scholars engaged in this direction. For instance, Peng et al. [21] designed several interval-valued dual hesitant fuzzy aggregation operators based on Archimedean t-norm and t-conorm. Zang et al. [22] investigated a set of interval-valued dual hesitant fuzzy Heronian mean operators and then according to these operators, proposed the procedure for addressing the group decision making challenges. IVDHFS is an important extension of FS and has drawn the attention of many experts. However, in the theme of IVDHFS, each membership and non-membership value possesses the same weightage. To illustrate our point, consider a person who has to purchase a commodity C, and he is certain that his degree of agreement towards purchasing the commodity is 60% towards $[0.2, 0.3]$ and 40% towards $[0.3, 0.5]$ and analogously, for the non-membership case, he is 70% favouring to the $[0.3, 0.4]$ rejection level and 30% favouring the $[0.2, 0.3]$ rejection level. Thus, under IVDHFS context, this information is captured as:

$$\langle \{[0.2, 0.3], [0.3, 0.5]\}, \{[0.3, 0.4], [0.2, 0.3]\} \rangle .$$

Here the weightage value is ignored. So, situations like this, in which the hesitation has some preference over another hesitant value arise new challenges to IVDHFS.

In certain instances, the aforementioned fuzzy sets are incapable of modelling real world decision problems due to the qualitative nature of many attributes. Indeed DMs may feel suitable to state their views by linguistic terms rather than quantitative form. Though the linguistic terms are less precise than numerical values, they are more close to human cognitive processes. Zadeh [23] mounted the root for linguistic decision-making which further achieved great concern from the work of Herrera et al. [24, 25]. After that, Rodriguez et al. [26] got inspiration from the strength of linguistic term sets (LTSs) [27], and HFSs [28], and investigated the idea of hesitant fuzzy linguistic term sets (HFLTSs) which made a breakthrough in terms of adjusting vagueness and uncertainty. Beg and Rashid [29] further generalized the notion of HFLTSs to hesitant intuitionistic fuzzy linguistic term sets (HIFLTSs) and studied the distance measure and technique of order preference similarity to the ideal solution (TOPSIS) methodology in the proposed setting along with the application. Though HFLTSs and HIFLTSs are efficient tools, they assign equal weight to each term and cannot reflect the original assessments of DMs in specific problems.

To overcome the limitations of HFLTSs, Pang et al. [30] put forward the notion of probabilistic linguistic term sets (PLTSs), by adjoining each term with its corresponding probability. Here, the probability meanings can be the weight, the importance value or the possible degree. Many researchers have explored various properties of PLTSs in-depth and on its viewpoint [31, 32, 33]. For instance, the basic and improved operations [34], distance measure [35], aggregation operators [36] and some decision ranking methods have been presented [32, 37, 38]. To provide more adaptability in DMs choices, Lin et al. [32] designed the probabilistic uncertain linguistic term set (PULTS). They studied basic operational laws, aggregation operators for PULTSs. They also put forward an extended TOPSIS technique to cope with MCGDM problems in the PULTS framework. Even though PULTS makes it achievable to present uncertain linguistic terms in a more realistic manner, but still a challenge emerges since PULTS is completely founded on a qualitative scale mapped to a sequence of uniformly distributed integers. Such a representative model is doubtable, and DMs must legitimize the choice of uniformly distributed integers.

In the theory of PLTSs, owing to factors such as time pressure, inadequate experience, nature of objects, some DMs may not provide their assessment information. In such a situation, normalization is required before using PLTSs. Thus, the ignorance information is proportionally allotted to each linguistic term in PLTS, which may cause the loss of information and may result in unreasonable results. To circumvent this issue, Jin et al. [33] stated the concept of uncertain probabilistic linguistic term set (UPLTS) as an exciting expansion of PLTS. They presented some basic operations, score function and comparison method for UPLTSs. However, their proposed operational law faces several challenges; in particular, it lacks basic operational properties such as the distributive property of multiplication over addition, which is a significant drawback. Also, the results obtained after operations are no longer UPLTSs.

Further, in some instances, the score function provided by [33] may be impractical (see Example 6.1.5). Another major issue regarding UPLTSs is that of aggregating the opinions of DMs. Though Jin et al. [33] studied some basic aggregation operators and their weighted form under uncertain probabilistic linguistic environment, their proposed operators are based on such operations whose influence on the final solution may be unrealistic. Besides this, these cannot handle the situation in which DM's weights are taken into account. For fruitful decision making, selecting the appropriate operational laws during the aggregation phase is crucial. Therefore, some novel theories under uncertain probabilistic linguistic environment are required.

1.2 Motivations

Based on the above literature analysis, the motivations of the present work are pointed out.

- ➤ HIFLTS and IVDHFS are powerful tools for handling complex problems with linguistic or quantitative variables. However, sometimes it becomes cumbersome to model the scenarios of more vague nature by utilizing these notions. Thus, there is a need to introduce novel fuzzy tools and their related mathematical studies to cope with this predicament.

- ➤ The basic operational laws given by [32] and [33] for UPLTSs and PULTSs, respectively, are irrational:

  (a) The probabilistic uncertain linguistic operations [32, 33] are directly based on indices of linguistic terms and are valid only if the linguistic scale function is balanced. Nevertheless, there does not exist any study about the unbalanced scenarios.

  (b) Likewise, probabilistic uncertain linguistic operations, the existing operational laws of UPLTSs are also based on the doctrine to operate the indices of linguistic terms with their associated probabilities. This way is unreasonable as these two are different dimensions. Besides this, the results obtained by these operational laws lose the probability information and are no longer UPLTSs.

  The detail discussion about the limitations (a) and (b) can be seen in Chapters 4 and 6, respectively.

- ➤ The present distance measures between two PULTSs need to insert artificial linguistic terms to the smaller one in order to equalize their length, which is rude and would lead to unfruitful decision ranking results.

- ➤ The available comparison method fails to differentiate UPLTSs in some cases (see Example 6.1.5).

- The aggregated results obtained by the aggregation operators proposed by [32] and [33] generate some specific terms which are unstated in the given LTS, and thus calls a challenge to design some novel aggregation operators.

- As claimed by Gou et al. [39] that selection of an appropriate alternative from the set of available alternatives under the complex environment is an interesting challenge. Thereby some popular MCGDM techniques such as TOPSIS, gained and lost dominance score (GLDS), weighted aggregated sum product assessment (WASPAS) and aggregation based methods must be effectively expanded to probabilistic hesitant intuitionistic linguistic term set (PHILTS), PULTS and UPLTS, respectively, under unknown weight information.

1.3 Contributions

To circumvent the challenges mentioned above, the key contributions of this study are listed in a nutshell below:

- To assess the DMs information accurately, the present study originate some novel notions viz., PHILTS, weighted interval-valued dual hesitant fuzzy set (WIVDHFS) and their extended concepts.

- By analyzing the weaknesses of the existing operational laws of PULTSs and UPLTSs, we redefine these laws and examine their relevant properties.

- As mentioned above, to overcome the shortcomings of existing probabilistic uncertain linguistic distance measures, some new distance measures are devised.

- We have also formulated a novel uncertain probabilistic linguistic comparative method based on the revised score function and deviation degree to cover the imperfections of the existing ones.

- The present work also focuses on aggregation operators. Certain existing aggregation operators are revised to aggregate the rating of DMs with precise aim. Besides this, several innovative aggregation operators, namely uncertain probabilistic linguistic simple weighted geometry (UPLSWG) and weighted interval-valued dual hesitant fuzzy operators in terms of t-norm and t-conorm are delineated along with their silent features.

- We build certain robust decision making techniques including TOPSIS, GLDS, WASPAS and aggregation-based method for some advance fuzzy tools in order to cope with uncertainty problems more accurately.

To manifest the practicality and advantages of the designed techniques, several real world problems are addressed, and detail comparison is given with existing literature.

1.4 Arrangement of the thesis

There are seven chapters in the thesis, including this one. The details of the remaining chapters are given below:

Chapter 2 recalls some basic definitions and concepts of PLTS, PULTS, UPLTS, IVDHFS, and related concepts, which will be helpful in later chapters.

Chapter 3 introduces a novel notion, namely PHILTs and its related concepts like basic operations, aggregation operators, normalization process, and distance measure. Also, an in-depth theoretical investigation is done to extend the maximizing deviation method for criteria weight determination. Two practical decision-making models: aggregation based method and the TOPSIS method for the proposed notion are designed along with their application.

Chapter 4 mainly focuses on improving the existing theory of PULTS, including operational rules, score function deviation degree, distance measure and aggregation operators. Further, to suit the needs of different semantics, two robust decision making methods named as consensus-based PUL-gained and lost dominance score method and consensus-based PUL-aggregation method are proposed. A practical case concerning the decision of the best commodity for investment in Forex is presented to illustrate the application of the provided methods.

Chapter 5 deals with an innovative fuzzy set, namely WIVDHFS and its related mathematical study. In line with Archimedean t-norm and t-conorm, two types of aggregation operators, namely, the generalized weighted interval-valued dual hesitant fuzzy weighted averaging (GWIVDHFWA) operator and the generalized weighted interval-valued dual hesitant fuzzy weighted geometric (GWIVDHFWG) operator are designed along with their relevant properties. Meanwhile, some of their particular cases and relationships are also explored. A novel MCGDM approach under a weighted interval-valued dual hesitant fuzzy environment is then constructed based on particular cases. Finally, an application case about teaching quality assessment is provided, and some analysis and comparisons are presented to testify the constructed approach.

Chapter 6, we first pointed out some weaknesses of the existing operational laws and score function of UPLTSs through some critical examples and then redefined them to overcome existing flaws to acquire more accurate results in practical decision making problems. Also, we establish various properties of the revised operational laws along with proofs. To design a novel comparison method, the concept of deviation degree is put forward in order to accommodate the situation in which two different UPLTSs have the same score values. Based on the proposed operational laws, several existing aggregation operators are modified, and a novel aggregation operator, namely uncertain probabilistic linguistic simple weighted geometry (UPLSWG) operator is designed. Meanwhile, some interesting properties of these proposed operators are carefully analyzed. Furthermore, an entropy technique under uncertain probabilistic linguistic information is structured for computing the completely unknown weights of criteria. Following this, a new extension of WASPAS method called uncertain probabilistic linguistic-WASPAS (UPL-WASPAS) methodology based on the proposed aggregation operators is studied under the UPLTS context for ranking objects in MCGDM problems. To demonstrate the applicability and potentiality of the originated method, an example of supplier selection is addressed, and a detailed performance comparison analysis is conducted.

Lastly, concluding remarks and various proposals for possible future developments of this doctoral work are given.

Chapter 2

Some fundamental concepts

This chapter presents some essential concepts related to FSs, particularly, HIFLTS, PLTS, PULTS, IVDHFS and UPLTS that are required to understand the work proposed in succeeding chapters. For a detailed study, we encourage the readers to see [19, 29, 30, 32, 33, 40, 41, 42].

2.1 Linguistic scale function and some extensions of FS

To facilitate the presentation let us first review linguistic scale function, some extensions of FS and their relevant concepts.

Definition 2.1.1. *[40] Let S be an LTS and $\theta_\beta \in [0,1]$ then linguistic scale function ℓ is a monotonically increasing function which can be mathematically characterised as:*

$$\ell : \pounds_\beta \to \theta_\beta \quad ; \ell^{-1} : \theta_\beta \to \pounds_\beta \; \forall \; \pounds_\beta \in S.$$

Actually, the function value θ_β reflects the semantics of linguistic term $\pounds_\beta$.

On the basis of different types of LTSs, balanced and unbalanced, the linguistic scale function is categorized into the following three types.

i. If the semantics are uniformly distributed, then

$$\begin{cases} \ell(\pounds_\beta) = \dfrac{\beta}{2\tau} \\ \ell^{-1}(\pounds_\beta) = \pounds_{2\tau\theta_\beta} \end{cases} , \text{if } \beta \in [0, 2\tau] \tag{2.1.1}$$

$$\begin{cases} \ell(\pounds_\beta) = \dfrac{\beta + \tau}{2\tau} \\ \ell^{-1}(\theta_\beta) = \pounds_{2\tau\theta_\beta - \tau} \end{cases} , \text{if } \beta \in [-\tau, \tau] \tag{2.1.2}$$

ii. If the semantics are unequally distributed included that the semantic deviations between the adjacent linguistic terms are increasing with enlargement from medium, then

$$\begin{cases} \ell(\pounds_\beta) = \dfrac{\mu_1^\tau - \mu_1^{\tau-\beta}}{2\mu_1^\tau - 2} \times 1_{\{\beta\in[0,\tau]\}} + \dfrac{\mu_2^\tau - \mu_2^{\beta-\tau} - 2}{2\mu_2^\tau - 2} \times 1_{\{\beta\in[\tau,2\tau]\}} \\ \ell^{-1}(\theta_\beta) = \pounds_{(\tau - \log_{\mu_1}(\mu_1^\tau - (2\mu_1^\tau - 2)\theta_\beta)) \times 1_{\{\beta\in[0,\tau]\}} + (\tau + \log_{\mu_2}((2\mu_2^\tau - 2)\theta_\beta - (\mu_2^\tau - 2))) \times 1_{\{\beta\in[\tau,2\tau]\}}} \end{cases} , \text{if } \beta \in [0, 2\tau] \tag{2.1.3}$$

$$\begin{cases} \ell(\pounds_\beta) = \dfrac{\mu_1^\tau - \mu_1^{-\beta}}{2\mu_1^\tau - 2} \times 1_{\{\beta\in[-\tau,0]\}} + \dfrac{\mu_2^\tau - \mu_2^{\beta} - 2}{2\mu_2^\tau - 2} \times 1_{\{\beta\in[0,\tau]\}} \\ \ell^{-1}(\theta_\beta) = \pounds_{(-\log_{\mu_1}(\mu_1^\tau - (2\mu_1^\tau - 2)\theta_\beta)) \times 1_{\{\beta\in[-\tau,0]\}} + (\tau + \log_{\mu_2}((2\mu_2^\tau - 2)\theta_\beta - (\mu_2^\tau - 2))) \times 1_{\{\beta\in[0,\tau]\}}} \end{cases} , \text{if } \beta \in [-\tau, \tau] \tag{2.1.4}$$

where $1_{\{\beta\in[0,\tau]\}} = \begin{cases} 1, & \text{if } \beta \in [0, \tau]; \\ 0, & \text{otherwise.} \end{cases}$ is a significative function. The other significative functions have the same meanings. μ_1 and μ_2 are parameters and must be larger than 1.

iii. In case that semantics are unequally distributed included that the semantic deviations between the adjacent linguistic terms are decreasing with the enlargement from medium, then

$$\begin{cases} \ell(\pounds_\beta) = \dfrac{\tau^{\psi_1} - (\tau-\beta)^{\psi_1}}{2\tau^{\psi_1}} \times 1_{\{\beta\in[0,\tau]\}} + \dfrac{\tau^{\psi_2} + (\beta-\tau)^{\psi_2}}{2\tau^{\psi_2}} \times 1_{\{\beta\in[\tau,2\tau]\}} \\ \ell^{-1}(\theta_\beta) = \pounds_{(\tau-\tau\ \sqrt[\psi_1]{1-2\theta_\beta})\times 1_{\{\beta\in[0,\tau]\}} + (\tau+\tau\ \sqrt[\psi_2]{2\theta_\beta-1})\times 1_{\{\beta\in[\tau,2\tau]\}}} \end{cases} \text{, if } \beta \in [0, 2\tau] \tag{2.1.5}$$

$$\begin{cases} \ell(\pounds_\beta) = \dfrac{\tau^{\psi_1} - (-\beta)^{\psi_1}}{2\tau^{\psi_1}} \times 1_{\{\beta\in[-\tau,0]\}} + \dfrac{\tau^{\psi_2} + (\beta)^{\psi_2}}{2\tau^{\psi_2}} \times 1_{\{\beta\in[0,\tau]\}} \\ \ell^{-1}(\theta_\beta) = \pounds_{(-\tau\ \sqrt[\psi_1]{1-2\theta_\beta})\times 1_{\{\beta\in[-\tau,0]\}} + (\tau\ \sqrt[\psi_2]{2\theta_\beta-1})\times 1_{\{\beta\in[0,\tau]\}}} \end{cases} \text{, if } \beta \in [-\tau, \tau] \tag{2.1.6}$$

where ψ_1 and ψ_2 are parameters belongs to $(0, 1]$.

Definition 2.1.2. *[26] Let $S = \{\pounds_\alpha; \alpha = 0, 1, 2, \ldots, \tau\}$ be a linguistic term set; then, HFLTS, H_S, is a finite and ordered subset of the consecutive linguistic terms of S.*

Example 2.1.3. *Let* $S = \left\{ \begin{array}{c} \pounds_0 = \textit{extremely poor}, \pounds_1 = \textit{very poor}, \pounds_2 = \textit{poor}, \pounds_3 = \textit{medium}, \\ \pounds_4 = \textit{good}, \pounds_5 = \textit{ very good }, \pounds_6 = \textit{ extremely good} \end{array} \right\}$, *be a linguistic term set. Then, two different HFLTSs may be defined as:*

$H_S(x) = \{\pounds_1 = \textit{very poor}, \pounds_2 = \textit{poor}, \pounds_3 = \textit{medium}, \pounds_4 = \textit{good}\}$ *and* $H_S(y) = \{\pounds_3 = \textit{medium}, \pounds_4 = \textit{good}, \pounds_5 = \textit{very good}\}$.

Definition 2.1.4. *[26] Let $S = \{\pounds_\alpha; \alpha = 0, 1, 2, \ldots, \tau\}$ be an ordered finite set of linguistic terms and E be an ordered finite subset of the consecutive linguistic terms of S. Then, the operators "max" and "min" on E are given as follows:*

(i) $\max(E) = \max(\pounds_\alpha) = \pounds_m$; $\pounds_\alpha \in E$ *and* $\pounds_\alpha \leq \pounds_m\ \forall\alpha$;

(ii) $\min(E) = \min(\pounds_\alpha) = \pounds_n$; $\pounds_\alpha \in E$ *and* $\pounds_\alpha \geq \pounds_n$ $\forall\alpha$.

Definition 2.1.5. *[29] Let Z be a universal set, and $S = \{\pounds_\alpha; \alpha = 0, 1, 2, \ldots, \tau\}$ be a linguistic term set, then HIFLTS on Z are two functions h and h' that when applied to an element of Z return finite and ordered subsets of consecutive linguistic terms of S, this can be expressed mathematically as:*

$$\mathscr{A} = \left\{\left\langle z, h(z), h'(z)\right\rangle | z \in Z\right\},$$

where $h(z)$ and $h'(z)$ denote the possible membership and non-membership degree in terms of consecutive linguistic terms of the element $z \in Z$ to the set $\mathscr{A}$ such that the following conditions are satisfied:

(i) $\max(h(z)) + \min\left(h'(z)\right) \leq \pounds_\tau$;

(ii) $\min(h(z)) + \max\left(h'(z)\right) \leq \pounds_\tau$.

Definition 2.1.6. *[30] Let $S = \{\pounds_\alpha; \alpha = 0, 1, 2, \ldots, \tau\}$ be a linguistic term set, then a PLTS can be presented as:*

$$\pounds(p) = \left\{\pounds^{(i)}\left(p^{(i)}\right) \mid \pounds^{(i)} \in S,\ p^{(i)} \geq 0\ i = 1, 2, \ldots, \#\pounds(p),\ \sum_{i=1}^{\#\pounds(p)} p^{(i)} \leq 1\right\}, \qquad (2.1.7)$$

where $\pounds^{(i)}\left(p^{(i)}\right)$ is the ith linguistic term $\pounds^{(i)}$ associated with the probability $p^{(i)}$, and $\#\pounds(p)$ denotes the number of linguistic terms in $\pounds(p)$.

Definition 2.1.7. *[30] Let $\pounds(p) = \left\{\pounds^{(i)}\left(p^{(i)}\right)\ ;\ i = 1, 2, \ldots, \#\pounds(p)\right\}$, $r^{(i)}$ be the lower index of linguistic term $\pounds^{(i)}$, $\pounds(p)$ is called an ordered PLTS, if all the elements $\pounds^{(i)}\left(p^{(i)}\right)$ in $\pounds(p)$ are ranked according to the values of $r^{(i)} \times p^{(i)}$ in descending order.*

However, in a PLTS, it is possible for two or more linguistic terms with equal values of $r^{(i)} \times p^{(i)}$. Taking a PLTS $\pounds(p) = \{\pounds_1(0.4), \pounds_2(0.2), \pounds_3(0.4)\}$, here $r^{(1)} \times p^{(1)} = r^{(2)} \times p^{(2)} = 0.4$.

According to the above rule, these two values cannot be arranged. To handle such type of problem, Zhang et al. [43] defined the following ranking rule.

Definition 2.1.8. *Let* $\pounds(P) = \left\{ \pounds^{(i)} \left(p^{(i)}\right) \ ; \ i = 1, 2, \ldots, \#\pounds(p) \right\}$, $r^{(i)}$ *be the lower index of linguistic term* $\pounds^{(i)}$.

(1) *If the values of* $r^{(i)}\left(p^{(i)}\right)$ *are different for all elements in PLTS, then arrange all the elements according to the values of* $r^{(i)}\left(p^{(i)}\right)$ *directly.*

(2) *If all the values of* $r^{(i)}\left(p^{(i)}\right)$ *become equal for two or more elements, then*

(a) *When the lower indices* $r^{(i)}\,(i = 1, 2, \ldots, \#\pounds(p))$ *are unequal, rank* $r^{(i)}\left(p^{(i)}\right)(i = 1, 2, \ldots, \#\pounds(p))$ *accordant with the values of* $r^{(i)}\,(i = 1, 2, \ldots, \#\pounds(p))$ *in descending order.*

(b) *When the lower indices* $r^{(i)}\,(i = 1, 2, \ldots, \#\pounds(p))$ *are incomparable, rank* $r^{(i)}\left(p^{(i)}\right)$ $(i = 1, 2, \ldots, \#\pounds(p))$ *accordant with the values of* $p^{(i)}\,(i = 1, 2, \ldots, \#\pounds(p))$ *in descending order.*

Definition 2.1.9. *[30] Let* $\pounds(p)$ *be a PLTS such that* $\sum\limits_{i=1}^{\#\pounds(p)} p^{(i)} < 1$, *then the associated PLTS is denoted and defined as*

$$\pounds^{\cdot}(p) = \left\{ \pounds^{(i)} \left(p^{\cdot(i)} \right) \ ; \ i = 1, 2, \ldots, \#\pounds(p) \right\}, \tag{2.1.8}$$

where $p^{\cdot(i)} = \frac{p^{(i)}}{\sum\limits_{i=1}^{\#\pounds(p)} p^{(i)}}, \forall i = 1, 2, \ldots, \#\pounds(p)$.

Definition 2.1.10. *[30] Let* $\pounds_1(p) = \left\{ \pounds_1^{(i)} \left(p_1^{(i)} \right) \ ; \ i = 1, 2, \ldots, \#\pounds_1(p) \right\}$ *and* $\pounds_2(p) = \left\{ \pounds_2^{(i)} \left(p_2^{(i)} \right) \ ; i = 1, 2, \ldots, \#\pounds_2(p) \right\}$ *be two PLTSs, where* $\#\pounds_1(p)$ *and* $\#\pounds_2(p)$ *denote the number of linguistic terms in* $\pounds_1(p)$ *and* $\pounds_2(p)$, *respectively. If* $\#\pounds_1(p) > \#\pounds_2(p)$, *then* $\#\pounds_1(p) - \#\pounds_2(p)$ *linguistic terms will be added to* $\pounds_2(p)$ *so that the number of elements in* $\pounds_1(p)$ *and* $\pounds_2(p)$ *becomes equal. The inserted linguistic terms are the smallest one's in* $\pounds_2(p)$ *with zero probabilities.*

Let $\pounds_1(p) = \left\{\pounds_1^{(i)}\left(p_1^{(i)}\right)\ ;\ i = 1, 2, \ldots, \#\pounds_1(p)\right\}$ and $\pounds_2(p) = \left\{\pounds_2^{(i)}\left(p_2^{(i)}\right)\ ;\ i = 1, 2, \ldots, \#\pounds_2(p)\right\}$, then the normalized PLTSs denoted by $\widetilde{\pounds_1}(p) = \left\{\widetilde{\pounds_1^{(i)}}\left(p_1^{(i)}\right)\ ;\ i = 1, 2, \ldots, \#\pounds_1(p)\right\}$ and $\widetilde{\pounds_2}(p) = \left\{\widetilde{\pounds_2^{(i)}}\left(p_2^{(i)}\right)\ ;\ i = 1, 2, \ldots, \#\pounds_2(p)\right\}$ can be obtained according to the following two steps:

(1) If $\sum_{i=1}^{\#\pounds_k(p)} p_k^{(i)} < 1$, then $\pounds^{\bullet}_k(p), k = 1, 2$ is calculated according to Definition 2.1.9.

(2) If $\#\pounds_1(p) \neq \#\pounds_2(p)$, then according to Definition 2.1.10, add some linguistic terms to the one with less number of elements.

Definition 2.1.11. *[32] Let $S = \{\pounds_\alpha | \alpha = -\tau, ... -1, 0, 1, ...\tau\}$ be an LTS and $z_i \in Z$ be fixed. A PULTS on S is $U_s(P) = \{\langle z_i, u_s^i(p)\rangle | z_i \in Z\}$ with*

$$u_s^i(p) = \left\{\left\langle [\pounds^{i(\jmath)}, U^{i(\jmath)}], p^{i(\jmath)}\right\rangle | \pounds^{i(\jmath)}, U^{i(\jmath)} \in S, p^{i(\jmath)} \geq 0, \jmath = 1, 2, ..., \pounds, \sum_{\jmath=1}^{\pounds} p^{i(\jmath)} \leq 1\right\}, \quad (2.1.9)$$

where $\left\langle [\pounds^{i(\jmath)}, U^{i(\jmath)}], p^{i(\jmath)}\right\rangle$ is the $\jmath$th uncertain linguistic variable $[\pounds^{i(\jmath)}, U^{i(\jmath)}]$ associated with probability $p^{i(\jmath)}$, $\pounds^{i(\jmath)}$ and $U^{i(\jmath)}$ are the linguistic terms, provided that $\pounds^{i(\jmath)} \leq U^{i(\jmath)}$ and $\pounds$ denote the cardinality of $u_s^i(p)$.

For the sake of convenience, $u_s^i(p)$ is named as the probabilistic uncertain linguistic element (PULE).

Example 2.1.12. *Suppose 'Z' represents the "set of candidates". Further assume that three experts are called to evaluate the candidates represented by set Z on the basis of attribute/criteria 'ability'. The invited experts are provided with the linguistic term set $S = \Big\{\pounds_{-3} =$ very bad, $\pounds_{-2} =$ bad, $\pounds_{-1} =$ a little bad, $\pounds_0 =$ medium, $\pounds_1 =$ a little good, $\pounds_2 =$ good, $\pounds_3 =$ very good$\Big\}$. One expert may deem that he is 80% sure that the candidate $z \in Z$ lies between medium and a little good and 20% sure that he lies between a little good and* *good. Expert 2 may judge that he is* 100% *sure that z lies between medium and good. Expert 3 may say that he is* 40% *sure that z lies between medium and a little good and* 50 *sure that z lies between good and very good. The opinions of these three experts can be expressed in terms of PULEs as* $u_s^1(p) = \{\langle[\pounds_0, \pounds_1], 0.8\rangle, \langle[\pounds_1, \pounds_2], 0.2\rangle\}$, $u_s^2(p) = \{\langle[\pounds_0, \pounds_2], 1\rangle\}$, $u_s^3(p) = \{\langle[\pounds_0, \pounds_1], 0.4\rangle, \langle[\pounds_2, \pounds_3], 0.5\rangle\}$.

Definition 2.1.13. *[32] Given a PULE* $u_s(p) = \left\{\langle[\pounds^\jmath, U^\jmath], p^\jmath\rangle \mid \sum_{\jmath=1}^{\pounds} p^{(\jmath)} \leq 1\right\}$, *if* $\sum_{\jmath=1}^{\pounds} p^{(\jmath)} < 1$, *then the associated PULE is denoted and defined as*

$$\widetilde{u}_s(p) = \left\{\left\langle[\pounds^\jmath, U^\jmath], \widetilde{p}^{(\jmath)}\right\rangle \mid \sum_{\jmath=1}^{\pounds} p^{(\jmath)} = 1\right\}, \tag{2.1.10}$$

where $\widetilde{p}^{(\jmath)} = \frac{p^{(\jmath)}}{\sum_{\jmath=1}^{\pounds} p^{(l)}} \forall \jmath = 1, 2, ..., \pounds$.

Definition 2.1.14. *[32] Let* $u_s^1(p) = \left\{\left\langle[\pounds^{1(\jmath)}, U^{1(\jmath)}], p^{1(\jmath)}\right\rangle | \jmath = 1, 2, ..., \pounds_1\right\}$ *and* $u_s^2(p) = \left\{\left\langle[\pounds^{1(\jmath)}, U^{1(\jmath)}], p^{2(\jmath)}\right\rangle | \jmath = 1, 2, ..., \pounds_2\right\}$ *be two PULEs, and let* $\pounds_1 \neq \pounds_2$. *If* $\pounds_1 > \pounds_2$, *then* $\pounds_1 - \pounds_2$ *uncertain linguistic terms with probability* 0 *are added to* $u_s^2(p)$ *so that the numbers of uncertain linguistic terms in* $u_s^1(p)$ *and* $u_s^2(p)$ *becomes identical. The inserted uncertain linguistic terms are the smallest one in* $u_s^2(p)$.

Let $u_s^1(p) = \left\{\left\langle[\pounds^{1(\jmath)}, U^{1(\jmath)}], p^{1(\jmath)}\right\rangle | \jmath = 1, 2, ..., \pounds_1\right\}$ and $u_s^2(p) = \left\{\left\langle[\pounds^{1(\jmath)}, U^{1(\jmath)}], p^{2(\jmath)}\right\rangle | \jmath = 1, 2, ..., \pounds_2\right\}$ be any two PULEs, then the normalization process can be carried out by below steps:

i. If $\left\{\sum_{\jmath=1}^{\pounds} p^{i(\jmath)} \leq 1\right\}$, then according to the designed Formula 2.1.10, compute $\widetilde{u}_s^i(p), i = 1, 2$.

ii. If $\pounds_1 \neq \pounds_2$, then according to Definition 2.1.14, adjoin some uncertain linguistic terms to the one with the less number of uncertain linguistic terms. The resultant PULEs are known normalized PULEs. For the sake of simplicity, normalized PULEs are symbolized by $u_s^1(p)$ and $u_s^2(p)$ as well.

If the lower and upper linguistic terms are not consecutive then PULE should be preprocessed by splitting its inconsecutive uncertain linguistic terms into a fixed number of consecutive uncertain linguistic terms and they distribute the probabilities uniformly. In Example 2.1.3, $\{\langle[\pounds_0, \pounds_2], 1\rangle\}$ the inconsecutive uncertain linguistic term $[\pounds_0, \pounds_2]$ is split into two consecutive uncertain linguistic terms $[\pounds_0, \pounds_1]$ and $[\pounds_1, \pounds_2]$, and their probabilities are 0.5. Therefore, the preprocessed PULE is $\{\langle[\pounds_0, \pounds_1], 0.5\rangle, \langle[\pounds_1, \pounds_2], 0.5\rangle\}$.

Definition 2.1.15. *[32] Let $u_s(p) = \{\langle[\pounds^{(\jmath)}, U^{(\jmath)}], p^{\jmath}\rangle \,|\jmath = 1, 2, ..., \pounds\}$ be a preprocessed PULE then, it is said to be ordered PULE if all the uncertain linguistic terms are arranged on the basis of ascending order of linguistic terms $\pounds^{(\jmath)}$ or $U^{(\jmath)}, \jmath = 1, 2, ..., \pounds$.*

Remark 2.1.16. *All the PULEs throughout this thesis are considered to be ordered PULEs.*

Definition 2.1.17. *[33] Let $\pounds = \{\pounds_\alpha | \alpha = 0, 1, ..., 2\tau\}$ be an LTS, then UPLTS is characterized as follows:*

$$U(P) = \left\{ \left(\pounds^k, [p^k, q^k]\right) |q^k \geq p^k \geq 0, k = 1, 2, ..., \pounds, \sum_{k=1}^{\pounds} p^k \leq 1, \sum_{k=1}^{\pounds} q^k \geq 1 \right\}, \qquad (2.1.11)$$

where $\left(\pounds^k, [p^k, q^k]\right)$ is the kth linguistic term $\pounds^k$ associated with uncertain probability $[p^k, q^k]$, $q^k \geq p^k$ and $\pounds$ denote the cardinality of $U(P)$.

In an UPLTS, positions of elements can be swapped arbitrarily. To make sure the operational results are straightforwardly ascertained, Jin et al. [33] proposed ordered UPLTS.

Definition 2.1.18. *Given an UPLTS $U(P) = \{\left(\pounds^k, [p^k, q^k]\right) |k = 1, 2, ..., \pounds\}$, if all the linguistic terms $\left(\pounds^k, [p^k, q^k]\right)$ are arranged according to the values of $[p^k \pounds^k, q^k \pounds^k]$ in descending order, then it is called an ordered UPLTS.*

Definition 2.1.19. *[19] Let Z be a universal set, an IVDHFS on Z is defined in terms of two functions $h_F(z)$ and $g_F(z)$ as follows:*

$$F = \{\langle z, h_F(z), g_F(z)\rangle \,|z \in Z\}, \qquad (2.1.12)$$

where $h_F(z) = \bigcup_{[\gamma^l,\gamma^u]\in h_F(z)} \left\{[\gamma^l,\gamma^u]\right\}$ *and* $g_F(z) = \bigcup_{[\eta^l,\eta^u]\in g_F(z)} \left\{[\eta^l,\eta^u]\right\}$ *are two sets of some interval values in* $[0,1]$, *representing the possible membership degree and non-membership degree of the element* $z \in Z$ *to the set* F, *respectively, with the condition*

$$[\gamma^l,\gamma^u],[\eta^l,\eta^u] \subset [0,1],\ 0 \leq (\gamma^u)^+ + (\eta^u)^+ \leq 1,$$

where $(\gamma^u)^+ = \max \gamma^u$, *and* $(\eta^u)^+ = \max \eta^u\ \forall\ z \in Z$.

For convenience, Ju et al. [19] called the pair $f(z) = (h_f(z), g_f(z))$ as an IVDHFE. For the sake of simplification, it is symbolized by $f = (h_f, g_f)$ where F is the set of all IVDHFEs.

Definition 2.1.20. *[41]* $T : [0,1] \times [0,1] \longrightarrow [0,1]$ *is described as T-norm if it meets the below axioms.*

i. $T(z_1, 1) = z_1$, *for all* $z_1 \in [0,1]$.

ii. $T(z_1, z_2) = T(z_2, z_1)$, *for all* z_1 *and* z_2.

iii. $T(z_1, T(z_2, z_3)) = T(T(z_1, z_2), z_3)$, *for all* z_1, z_2 *and* z_3.

iv. If $z_1 \leq z_1^{'}$ *and* $z_2 \leq z_2^{'}$, *then* $T(z_1, z_2) \leq T\left(z_1^{'}, z_2^{'}\right)$.

Definition 2.1.21. *[41]* $S : [0,1] \times [0,1] \longrightarrow [0,1]$ *is described as T-conorm if it meets the following axioms.*

i. $S(z_1, 0) = z_1$, *for all* $z_1 \in [0,1]$.

ii. $S(z_1, z_2) = S(z_2, z_1)$, *for all* z_1 *and* z_2.

iii. $S(z_1, S(z_2, z_3)) = S(S(z_1, z_2), z_3)$, *for all* z_1, z_2 *and* z_3.

iv. If $z_1 \leq z_1^{'}$ *and* $z_2 \leq z_2^{'}$, *then* $S(z_1, z_2) \leq S\left(z_1^{'}, z_2^{'}\right)$.

Definition 2.1.22. *A t-norm function* $T(z_1, z_2)$ *is known as Archimedean t-norm if it is continuous and* $T(z_1, z_1) < z_1\ \forall\ z_1 \in (0,1)$. *An Archimedean t-norm is called strictly Archimedean t-norm if it is strictly increasing in each variable for* $z_1, z_2 \in (0,1)$.

Definition 2.1.23. *A t-conorm function $S(z_1, z_2)$ is known as Archimedean t-conorm if it is continuous and $S(z_1, z_1) > z_1 \ \forall \ z_1 \in (0, 1)$. An Archimedean t-conorm is called strictly Archimedean t-conorm if it is strictly increasing in each variable for $z_1, z_2 \in (0, 1)$.*

Related study [42] shows that the strict Archimedean t-norm can be obtained by a strictly increasing additive function $g : [0, 1] \longrightarrow [0, \infty]$ is expressed as $T(z_1, z_2) = g^{-1} (g(z_1) + g(z_2))$, where $g(1) = 0$ and $g(0) = 1$, and similarly, dual Archimedean t-conorm also known as the strict Archimedean s-norm can be expressed as $S(z_1, z_2) = f^{-1} (f(z_1) + f(z_2))$, where $f(t) = g(1 - t)$. Thus, $f(t)$ is strictly increasing function, and $f(0) = 0$, $f(1) = 1$.

Lemma 2.1.24. *[44] Let $\alpha_i > 0$, $\omega_i > 0$, $i = 1, 2, ..., n$, and $\sum_{i=1}^{n} \omega_i = 1$; then $\prod_{i=1}^{n} (\alpha_i)^{\omega_i} \leq \sum_{i=1}^{n} \alpha_i \omega_i$, and equality holds if and only if $\alpha_1 = \alpha_2 = \cdots = \alpha_n$.*

Lemma 2.1.25. *[41] Let a_j, b_j, c_j, d_j $(j = 1, 2, ..., n)$ be four collection of nonnegative numbers; if $a_j - b_j - c_j - d_j \geq 0$, $j = 1, 2, ..., n$, then*

$$\prod_{j=1}^{n} a_j - \prod_{j=1}^{n} b_j - \prod_{j=1}^{n} c_j - \prod_{j=1}^{n} d_j \geq 0. \tag{2.1.13}$$

2.2 Comparative methods

In what follows, we give comparison methods given by Ju et al. [19] and Jin et al. [33] for comparing IVDHFEs and UPLTSs, respectively.

Definition 2.2.1. *Score function of IVDHFE $f = (h_f, g_f)$ is defined as follows:*

$$S(f) = \frac{1}{2} \left(\frac{1}{\#h_f} \sum_{[\gamma^l, \gamma^u] \in h_f} \left(\gamma^l + \gamma^u\right) - \frac{1}{\#g_f} \sum_{[\eta^l, \eta^u] \in g_f} \left(\eta^l + \eta^u\right) \right), \tag{2.2.1}$$

where $\#h_f$ and $\#g_f$ are the number of interval values in h_f and g_f, respectively. The larger the score $S(f)$, the greater the IVDHFE f.

Definition 2.2.2. *Accuracy function of IVDHFE $f = (h_f, g_f)$ is defined as follows:*

$$A(f) = \frac{1}{2}\left(\frac{1}{\#h_f}\sum_{[\gamma^l,\gamma^u]\in h_f}\left(\gamma^l+\gamma^u\right)+\frac{1}{\#g_f}\sum_{[\eta^l,\eta^u]\in g_f}\left(\eta^l+\eta^u\right)\right), \tag{2.2.2}$$

where $\#h_f$ and $\#g_f$ are the number of interval values in h_f and g_f, respectively. The larger accuracy $A(f)$, the greater the IVDHFE f.

Definition 2.2.3. *Let $f_1 = (h_{1f}, g_{1f})$ and $f_2 = (h_{2f}, g_{2f})$ be two IVDHFEs, then*

i. If $S(f_1) > S(f_2)$, then $f_1 > f_2$;

ii. If $S(f_1) = S(f_2)$, then

(a) *If $A(f_1) > A(f_2)$, then $f_1 > f_2$;*

(b) *If $A(f_1) = A(f_2)$, then $f_1 = f_2$.*

Definition 2.2.4. *Let $U(P) = \left\{\left(\pounds^k, [p^k, q^k]\right)|k = 1, 2, ..., \pounds\right\}$ be an UPLTS, then the score of $U(P)$ is given by*

$$S\left(U(P)\right) = \left[\pounds_{\overline{\alpha}}, \pounds_{\overline{\beta}}\right], \tag{2.2.3}$$

where $\overline{\alpha} = \frac{\sum_{k=1}^{\pounds} r^k p^k}{\sum_{k=1}^{\pounds} p^k}$, $\overline{\beta} = \frac{\sum_{k=1}^{\pounds} r^k q^k}{\sum_{k=1}^{\pounds} q^k}$ and r^k is the subscript of the linguistic term $\pounds^k$.

Definition 2.2.5. *Let $U_1(p)$ and $U_2(P)$ be two UPLTSs with score values $S\left(U_1(p)\right) = \left[\pounds_{\overline{\alpha_1}}, \pounds_{\overline{\beta_1}}\right]$ and $S\left(U_2(p)\right) = \left[\pounds_{\overline{\alpha_2}}, \pounds_{\overline{\beta_2}}\right]$, respectively. Then, degree of possibility of $U_1(p) \geq U_2(p)$ is given by the following formula*

$$\mathcal{P}\left(U_1(p) \geq U_2(p)\right) = \begin{cases} 1, & \text{if } \overline{\alpha_1} \geq \overline{\beta_2} \\ \frac{\overline{\beta_1}-\overline{\alpha_2}}{\overline{\beta_1}-\overline{\alpha_1}+\overline{\beta_2}-\overline{\alpha_2}}, & \text{if } \overline{\beta_1} > \overline{\alpha_2} \text{ and } \overline{\alpha_1} < \overline{\beta_2} \\ 0, & \overline{\beta_1} \leq \overline{\alpha_2}. \end{cases}$$

Using the possibility degree formula, a comparison rule is defined as follows:

i. If $\mathcal{P}\left(U_1(p) \geq U_2(p)\right) > 0.5$, then $U_1(p) > U_2(p)$.

ii. If $\mathcal{P}\left(U_1(p) \geq U_2(p)\right) = 0.5$, then $U_1(p) = U_2(p)$.

iii. If $\mathcal{P}\left(U_1(p) \geq U_2(p)\right) < 0.5$, then $U_1(p) < U_2(p)$.

2.3 Distance measure and aggregation operators

In the following, we present the formula for computing the distance measure between PLTSs. Beyond this, we state several aggregation operators under uncertain probabilistic linguistic context.

Definition 2.3.1. *[30] Let $\pounds_1(p) = \left\{\pounds_1^{(i)}\left(p_1^{(i)}\right)\ ;\ i = 1, 2, \ldots, \#\pounds_1(p)\right\}$ and $\pounds_2(p) = \left\{\pounds_2^{(i)}\left(p_2^{(i)}\right)\ ; i = 1, 2, \ldots, \#\pounds_2(p)\right\}$ be two PLTSs, where $\#\pounds_1(p)$ and $\#\pounds_2(p)$ denote the number of linguistic terms in $\pounds_1(p)$ and $\pounds_2(p)$, respectively, with $\#\pounds_1(p) = \#\pounds_2(p)$. Then, the distance between these two PLTSs can be defined as*

$$d\left(\pounds_1(p), \pounds_2(p)\right) = \sqrt{\frac{1}{\#\pounds_1(p)} \sum_{i=1}^{\pounds_1(p)} \left(p_1^{(i)} r_1^{(i)} - p_2^{(i)} r_2^{(i)}\right)^2}, \tag{2.3.1}$$

where $r_1^{(i)}$ and $r_2^{(i)}$ denote the lower indices of linguistic terms $\pounds_1^{(i)}$ and $\pounds_2^{(i)}$, respectively.

Definition 2.3.2. *[33] Let $U_i(p) = \left\{\left(\pounds_i^k, [p_i^k, q_i^k]\right) | k = 1, 2, ..., \pounds_i\right\}$ $(i = 1, 2, ..., n)$ be a collection of UPLTSs. Then, the uncertain probabilistic linguistic averaging (UPLA) operator is defined as*

$$UPLA\left(U_1(p), U_2(p), ..., U_n(p)\right) = \oplus_{i=1}^{n} \frac{1}{n} U_i(p). \tag{2.3.2}$$

Definition 2.3.3. *[33] Let $U_i(p) = \left\{\left(\pounds_i^k, [p_i^k, q_i^k]\right) | k = 1, 2, ..., \pounds_i\right\}$ $(i = 1, 2, ..., n)$ be a collection of UPLTSs, $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ denotes the weighting vector of $U_i(p)$ and $\omega_i \in [0, 1]$.*

Then, the uncertain probabilistic linguistic weighted averaging (UPLWA) operator has the following form

$$UPLWA\left(U_1(p), U_2(p), ..., U_n(p)\right) = \oplus_{i=1}^{n} \omega_i U_i(p). \tag{2.3.3}$$

Definition 2.3.4. *[33] Let* $U_i(p) = \left\{\left(\pounds_i^k, [p_i^k, q_i^k]\right) | k = 1, 2, ..., \pounds_i\right\}$ $(i = 1, 2, ..., n)$ *be a collection of UPLTSs. Then, the uncertain probabilistic linguistic geometric (UPLG) operator is defined as*

$$UPLG\left(U_1(p), U_2(p), ..., U_n(p)\right) = \otimes_{i=1}^{n} \left(U_i(p)\right)^{\frac{1}{n}}. \tag{2.3.4}$$

Definition 2.3.5. *[33] Let* $U_i(p) = \left\{\left(\pounds_i^k, [p_i^k, q_i^k]\right) | k = 1, 2, ..., \pounds_i\right\}$ $(i = 1, 2, ..., n)$ *be a collection of UPLTSs,* $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ *denotes the weighting vector of* $U_i(p)$ *and* $\omega_i \in [0, 1]$. *Then, the uncertain probabilistic linguistic weighted geometric (UPLWG) operator has the following form*

$$UPLWG\left(U_1(p), U_2(p), ..., U_n(p)\right) = \otimes_{i=1}^{n} \left(U_i(p)\right)^{\omega_i}. \tag{2.3.5}$$

2.4 Basic operational laws

The present section concentrates on the existing operational laws of PULEs and UPLTSs, which assist in the aggregation of assessment information.

Definition 2.4.1. *[32] Let* $u_s^1(p) = \left\{\left\langle [\pounds^{1(\jmath)}, U^{1(\jmath)}], p^{1(\jmath)} \right\rangle | \jmath = 1, 2, ..., \pounds_1\right\}$, $u_s^2(p) = \left\{\left\langle [\pounds^{2(\jmath)}, U^{2(\jmath)}], p^{2(\jmath)} \right\rangle | \jmath = 1, 2, ..., \pounds_2\right\}$ *be two normalized PULEs and* $\gamma \geq 0$. *Then*

i. $u_s^1(p) \oplus u_s^2(p) = \bigcup_{\left\langle [\pounds^{1(\jmath)}, U^{1(\jmath)}], p^{1(\jmath)} \right\rangle \in u_s^1(p), \left\langle [\pounds^{2(\jmath)}, U^{2(\jmath)}], p^{2(\jmath)} \right\rangle \in u_s^2(p)} \left\{ p^{1(\jmath)} [\pounds^{1(\jmath)}, U^{1(\jmath)}] \oplus p^{2(\jmath)} [\pounds^{2(\jmath)}, U^{2(\jmath)}] \right\}$;

ii. $u_s^1(p) \otimes u_s^2(p) = \bigcup_{\left\langle [\pounds^{1(\jmath)}, U^{1(\jmath)}], p^{1(\jmath)} \right\rangle \in u_s^1(p), \left\langle [\pounds^{2(\jmath)}, U^{2(\jmath)}], p^{2(\jmath)} \right\rangle \in u_s^2(p)} \left\{ [\pounds^{1(\jmath)}, U^{1(\jmath)}]^{p^{1(\jmath)}} \otimes [\pounds^{2(\jmath)}, U^{2(\jmath)}]^{p^{2(\jmath)}} \right\}$;

iii. $\gamma\left(u_s^1(p)\right) = \bigcup_{\left\langle[\pounds^{1(\jmath)}, U^{1(l)}], p^{1(\jmath)}\right\rangle \in u_s^1(p)} \left\{\gamma p^{1(\jmath)}[\pounds^{1(\jmath)}, U^{1(\jmath)}]\right\}$;

iv. $\left(u_s^1(p)\right)^{\gamma} = \bigcup_{\left\langle[\pounds^{1(\jmath)}, U^{1(\jmath)}], p^{1(\jmath)}\right\rangle \in u_s^1(p)} \left\{[\pounds^{1(\jmath)}, U^{1(\jmath)}]^{\gamma p^{1(\jmath)}}\right\}$.

Definition 2.4.2. *[33] Let* $U(p) = \left\{\left(\pounds^k, [p^k, q^k]\right) | k = 1, 2, ..., \pounds\right\}$ *and* $U_1(P) = \left\{\left(\pounds_1^k, [p_1^k, q_1^k]\right) | k = 1, 2, ..., \pounds_1\right\}$ *be two ordered UPLTSs, and* λ *be a positive real number, then*

i. $U(p) \oplus U_1(p) = \bigcup_{\left(\pounds^k, [p^k, q^k]\right) \in U(p), \left(\pounds_1^k, [p_1^k, q_1^k]\right) \in U_1(P)} \left\{\left(\frac{p^k + q^k}{2}\right)\pounds^k \oplus \left(\frac{p_1^k + q_1^k}{2}\right)\pounds_1^k\right\}$;

ii. $U(p) \otimes U_1(p) = \bigcup_{\left(\pounds^k, [p^k, q^k]\right) \in U(p), \left(\pounds_1^k, [p_1^k, q_1^k]\right) \in U_1(P)} \left\{\left(\pounds^k\right)^{\left(\frac{p^k + q^k}{2}\right)} \otimes \left(\pounds_1^k\right)^{\left(\frac{p_1^k + q_1^k}{2}\right)}\right\}$;

iii. $\lambda U(p) = \bigcup_{\left(\pounds^k, [p^k, q^k]\right) \in U(P)} \left\{\lambda\left(\frac{p^k + q^k}{2}\right)\pounds^k\right\}$;

iv. $(U(p))^{\lambda} = \bigcup_{\left(\pounds^k, [p^k, q^k]\right) \in U(P)} \left\{\left(\pounds^k\right)^{\lambda\left(\frac{p^k + q^k}{2}\right)}\right\}$.

Chapter 3

Probabilistic hesitant intuitionistic linguistic term sets in multi-attribute group decision making

In the present chapter, we originate a new fuzzy tool known as PHILTS. Meanwhile, some related concepts, distance measure and primary aggregation operators are also explored. Afterwards, two practical decision making techniques, i.e., aggregation based method and TOPSIS method with unknown weight information are put forward under the proposed notion. To validate the practicality and effectiveness of the designed set and methods a practical problem about selecting the best alternative is solved. The research work of this chapter is published in [45].

3.1 Probabilistic hesitant intuitionistic linguistic term set

Although HIFLTS allow the DM to state his assessments by using several linguistic terms, it cannot reflect the probabilities of the assessments of DM.

In the present section, the concept of PHILTS, based on the concept of HIFLTS and PLTS is proposed. Furthermore, some basic operations for PHILTS are also designed.

Definition 3.1.1. *Let Z be a universal set, and $S = \{£_\alpha; \alpha = 0, 1, 2, \ldots, \tau\}$ be a linguistic term set, then a* PHILTS *on Z are two functions $£$ and $£'$ that when applied to an element of Z return finite and ordered subsets of the consecutive linguistic terms of S along with their occurrence probabilities, which can be mathematically expressed as*

$$A(p) = \left\{ \begin{array}{c} \left\langle z, £(z)(p(z)) = \left\{£^{(i)}(z)\left(p^{(i)}(z)\right)\right\}, £'(z)\left(p'(z)\right) = \left\{£'^{(j)}(z)\left(p'^{(j)}(z)\right)\right\}\right\rangle \\ |p^{(i)}(z) \geq 0, i = 1, 2, \ldots, \#£(z)(p(z)), \sum\limits_{i=1}^{\#£(z)(p(z))} p^{i}(z) \leq 1 \ \& \\ p'^{(j)}(z) \geq 0, j = 1, 2, \ldots, \#£'(z)\left(p'(z)\right), \sum\limits_{j=1}^{\#£'(z)\left(p'(z)\right)} p'^{(j)}(z) \leq 1 \end{array} \right\}, \tag{3.1.1}$$

where $£(z)(p(z))$ and $£'(z)p'(z)$ are the PLTSs, presenting the membership and non-membership degree of the element $z \in Z$ to the set $A(p)$ such that the following two conditions are satisfied:

(i) $\max(£(z)) + \min\left(£'(z)\right) \leq £_\tau$;

(ii) $\min(£(z)) + \max\left(£'(z)\right) \leq £_\tau$.

For the sake of simplicity and convenience, we call the pair $A(z)(p(z)) = \langle £(z)(p(z)), £'(z)\left(p'(z)\right)\rangle$ as the intuitionistic probabilistic linguistic term element (PHILTE), denoted by $A(p) = \left\langle £(p), £'\left(p'\right)\right\rangle$ for short.

Remark 3.1.2. *Particularly, if the probabilities of all linguistic terms in membership part and non-membership part become equal, then PHILTE reduces to HIFLTE.*

Example 3.1.3. *Let* $S = \left\{ \begin{array}{c} \pounds_0 = \textit{extremely poor}, \pounds_1 = \textit{very poor}, \pounds_2 = \textit{poor}, \pounds_3 = \textit{medium}, \\ \pounds_4 = \textit{good}, \pounds_5 = \textit{ very good }, \pounds_6 = \textit{ extremely good} \end{array} \right\}$ *be a linguistic term set. A PHILTS is* $A(p) = \left\{ \begin{array}{c} \langle z_1, \{\pounds_1(0.4), \pounds_2(0.1), \pounds_3(0.35)\}, \{\pounds_3(0.3), \pounds_4(0.4) \\ \}\rangle, \langle z_2, \{\pounds_4(0.33), \pounds_5(0.5)\}, \{\pounds_1(0.2), \pounds_2(0.45)\}\rangle \end{array} \right\}$.

One can easily check the conditions of PHILTS for $A(p)$.

To illustrate the PHILTS more straightforwardly, in the following, a practical life example is given to depicting the difference between the PHILTS and HIFLTS:

Example 3.1.4. *Take the evaluation of a vehicle on the comfortable degree attribute/criteria as an example. Let S be a linguistic term set used in the above example. An expert provides an HIFLTE $\langle \{\pounds_1, \pounds_2, \pounds_3\}, \{\pounds_3, \pounds_4\}\rangle$ on the comfortable degree due to his/her hesitation for this evaluation. However, he/she is more confident in the linguistic term $\pounds_2$ for the membership degree set and the linguistic term $\pounds_4$ for the non-membership degree set. The HIFLTS fails to express his/her confidence. Therefore, we utilize the PHILTS to present his/her evaluations. In this case, his/her evaluations can be expressed as* $A(p) = \langle \{\pounds_1(0.2), \pounds_2(0.6), \pounds_3(0.2)\}, \{\pounds_3(0.2), \pounds_4(0.8)\}\rangle$.

In the following, the ordered PHILTE is defined to make sure that the operational results among PHILTEs can be investigated easily.

Definition 3.1.5. *A PHILTE* $A(p) = \left\langle \pounds(p), \pounds'\left(p'\right) \right\rangle$ *is known to be an ordered PHILTE, if* $l(p)$ *and* $\pounds'\left(p'\right)$ *are ordered PLTSs.*

Example 3.1.6. *Consider a PHILTE* $A(p) = \langle \{\pounds_1(0.4), \pounds_2(0.1), \pounds_3(0.35)\}, \{\pounds_3(0.3), \pounds_4(0.4) \}\rangle$ *used in the Example 3.1.3. Then, according to Definition 3.1.5 the ordered PHILTE is* $A(p) = \langle \{\pounds_3(0.35), \pounds_1(0.4), \pounds_2(0.1)\}, \{\pounds_4(0.4), \pounds_3(0.3)\}\rangle$.

3.1.1 The normalization of PHILTEs

Ideally, the sum of the probabilities is one, but in PHILTE if either of the membership probabilities or non-membership probabilities have sum less than one than this issue is resolved as follows.

Definition 3.1.7. *Consider a PHILTE* $A(p) = \left\langle £(p), £^{'}\left(p^{'}\right)\right\rangle$*, the associated PHILTE* $A^{\boldsymbol{\cdot}}(p) = \left\langle £(p^{\boldsymbol{\cdot}}), £^{'}\left(p^{'\boldsymbol{\cdot}}\right)\right\rangle$ *is defined, where*

$$£(p^{\boldsymbol{\cdot}}) = \left\{£^{(i)}\left(p^{\boldsymbol{\cdot}(i)}\right) | i = 1, 2, \ldots, \#£(p)\right\}; p^{\boldsymbol{\cdot}(i)} = \frac{p^{(i)}}{\sum\limits_{i=1}^{\#£(p)} p^{(i)}}, \forall i = 1, 2, \ldots, \#£(p), \quad (3.1.2)$$

and

$$£^{'}\left(p^{'\boldsymbol{\cdot}}\right) = \left\{£^{'(j)}\left(\left(p^{'\boldsymbol{\cdot}(j)}\right)\right) | j = 1, 2, \ldots, £^{'}\left(p^{'}\right)\right\}; p^{'\boldsymbol{\cdot}(j)} = \frac{p^{'(j)}}{\sum\limits_{j=1}^{£^{'}(p^{'})} p^{'(j)}}, \forall j = 1, 2, \ldots, £^{'}\left(p^{'}\right). \quad (3.1.3)$$

Example 3.1.8. *Consider a PHILTE* $A(p) = \langle\{£_1(0.4), £_2(0.1), £_3(0.35)\}, \{£_3(0.3), £_4(0.4)\}\rangle$*. Here, we see that* $\sum\limits_{i=1}^{\#£(p)} p^{(i)} = 0.85 < 1$ *also* $\sum\limits_{j=1}^{\#£^{'}(p^{'})} p^{'(j)} = 0.7 < 1$ *so the associated PHILTE* $A^{\boldsymbol{\cdot}}(p) = \left\langle £(p^{\boldsymbol{\cdot}}), £^{'}\left(p^{'\boldsymbol{\cdot}}\right)\right\rangle = \left\langle\left\{£_1\left(\frac{0.4}{0.85}\right), £_2\left(\frac{0.1}{0.85}\right), £_3\left(\frac{0.35}{0.85}\right)\right\}, \left\{£_3\left(\frac{0.3}{0.7}\right), £_4\left(\frac{0.4}{0.7}\right)\right\}\right\rangle$.

In decision making process, experts usually face such problems in which the length of PHILTEs is different. Let $A(p) = \left\langle £(p), £^{'}\left(p^{'}\right)\right\rangle$ and $A_1(p_1) = \left\langle £_1(p_1), £_1^{'}\left(p_1^{'}\right)\right\rangle$ be two PHILTEs of different lengths. Then, the following three cases are possible

$(I)\ \#£(p) \neq \#£_1(p_1), (II)\ \#£^{'}\left(p^{'}\right) \neq \#£_1^{'}\left(p_1^{'}\right), (III)\ \#£(p) \neq \#£_1(p_1)$ and $\#£^{'}\left(p^{'}\right) \neq \#£_1^{'}\left(p_1^{'}\right)$. In such situation, they need to equalize their lengths by increasing the number of probabilistic linguistic terms in that PLTS in which the number of probabilistic linguistic terms are relatively small because PHILTEs of different lengths create great problems in operations, aggregation operators and finding the deviation degree between two PHILTEs.

Definition 3.1.9. *Given any two PHILTEs* $A(p) = \left\langle \pounds(p), \pounds^{'}\left(p^{'}\right)\right\rangle$ *and* $A_1(p_1) = \langle \pounds_1(p_1),$ $\pounds_1^{'}\left(p_1^{'}\right)\rangle$ *if* $\#\pounds(p) > \#\pounds_1(p_1)$ *then* $\#\pounds(p) - \#\pounds_1(p_1)$ *linguistic terms should be added to* $\pounds_1(p_1)$ *to make their cardinalities identical. The inserted linguistic terms are the smallest one(s) in* $\pounds_1(p_1)$*, and the probabilities of all the linguistic terms are zero.*

The remaining cases are analogous to Case (I).

Let $A_1(p_1) = \left\langle \pounds_1(p_1), \pounds_1^{'}\left(p_1^{'}\right)\right\rangle$ and $A_2(p_2) = \left\langle \pounds_2(p_2), \pounds_2^{'}\left(p_2^{'}\right)\right\rangle$ be two PHILTEs. Then, the following two simple steps are involved in normalization process.

Step 1: If $\sum\limits_{i=1}^{\#\pounds_j(p_j)} p_j^{(i)} < 1$ or $\sum\limits_{i=1}^{\#\pounds_j^{'}\left(p_j^{'}\right)} p_j^{'(i)} < 1$; $j = 1, 2$, then we calculate $\pounds_j\left(p_j^{\cdot}\right)$, $\pounds_j^{'}\left(p_j^{'\cdot}\right)$; $j = 1, 2$ using Equations (3.1.2) and (3.1.3).

Step 2: If $\#\pounds_1(p_1) \neq \#\pounds_2(p_2)$ or $\#\pounds_1^{'}\left(p_1^{'}\right) \neq \#\pounds_2^{'}\left(p_2^{'}\right)$, then we add some elements according to Definition 3.1.9 to the one with small number of elements.

The resultant PHILTEs are called the normalized PHILTEs which are denoted as $\widetilde{A}(p)$ and $\widetilde{A_1}(p_1)$.

Note: For the convenience of presentation, we denote the normalized PHILTEs by $A(p)$ and $A_1(p_1)$ as well.

3.1.2 The comparison between PHILTEs

In this section, the comparison between two PHILTEs is presented. For this purpose, the score function and the deviation degree of the PHILTE are defined.

Definition 3.1.10. *Let* $A(p) = \left\langle \pounds(p), \pounds^{'}\left(p^{'}\right)\right\rangle = \left\langle \pounds^{(i)}\left(p^{(i)}\right), \pounds^{'(j)}\left(p^{'(j)}\right)\right\rangle; i = 1, 2, \ldots, \#\pounds(p),$ $j = 1, 2, \ldots, \pounds^{'}\left(p^{'}\right)$ *be a PHILTE with a linguistic term set* $S = \{\pounds_\alpha; \alpha = 0, 1, 2, \ldots, \tau\}$ *such that* $r^{(i)}$ *and* $r^{'(j)}$ *denote, respectively, the lower indices of linguistic terms* $\pounds^{(i)}$ *and* $\pounds^{'(j)}$*, then* *the score of $A(p)$ is denoted and defined as follows:*

$$E(A(p)) = \pounds_{\overline{\gamma}}, \tag{3.1.4}$$

where $\overline{\gamma} = \frac{\tau+\varpi-\beta}{2}$ *;* $\varpi = \frac{\sum_{i=1}^{\#\pounds(p)} r^{(i)}p^{(i)}}{\sum_{i=1}^{\#\pounds(p)} p^{(i)}}$ *and* $\beta = \frac{\sum_{j=1}^{\#\pounds'\left(p'\right)} r'^{(j)} p'^{(j)}}{\sum_{j=1}^{\#\pounds'\left(p'\right)} p'^{(j)}}$.

It is easy to see that $0 \leq \frac{\tau+\varpi-\beta}{2} \leq \tau$ *which means* $\pounds_{\overline{\gamma}} \in \overline{S} = \{\pounds_\alpha | \alpha \in [0,\tau]\}$.

Apparently, the score function represents the averaging linguistic term of PHILTE.

For two PHILTEs $A(p)$ and $A_1(p_1)$, if $E(A(p)) > E(A_1(p_1))$, then $A(p)$ is superior to $A_1(p_1)$, denoted as $A(p) > A_1(p_1)$; if $E(A(p)) < E(A_1(p_1))$, then $E(A(p))$ is inferior to $A_1(p_1)$, denoted as $A(p) < A_1(p_1)$; and, if $E(A(p)) = E(A_1(p_1))$, then we cannot distinguish between them. Thus, in this case, we define another indicator, named as the deviation degree as follows:

Definition 3.1.11. *Let* $A(p) = \left\langle \pounds(p), \pounds'\left(p'\right)\right\rangle = \left\langle \pounds^{(i)}\left(p^{(i)}\right), \pounds'^{(j)}\left(p'^{(j)}\right)\right\rangle; i = 1,2,\ldots \#\pounds(p)$, $j = 1,2,\ldots,\pounds'\left(p'\right)$ *be a PHILTE such that* $r^{(i)}$ *and* $r'^{(j)}$ *denote, respectively, the lower indices of linguistic terms* $\pounds^{(i)}$ *and* $\pounds'^{(j)}$*, then the deviation degree of* $A(p)$ *is denoted and defined as follows:*

$$\sigma(A(p)) = \left(\frac{\sum_{i=1}^{\#\pounds(p)} \left(p^{(i)}\left(r^{(i)} - \overline{\gamma}\right)\right)^2}{\sum_{i=1}^{\#\pounds(p)} p^{(i)}} + \frac{\sum_{j=1}^{\#\pounds'\left(p'\right)} \left(p'^{(j)}\left(r'^{(j)} - \overline{\gamma}\right)\right)^2}{\sum_{j=1}^{\#\pounds'\left(p'\right)} p'^{(j)}} \right)^{\frac{1}{2}}. \tag{3.1.5}$$

The deviation degree shows the distance from the average value in the PHILTE. The greater value of σ implies lower consistency, while the lesser value of σ indicates higher consistency.

Thus, $A(p)$ and $A_1(p_1)$ can be ranked by the following procedure:

(1) if $E(A(p)) > E(A_1(p_1))$, then $A(p) > A_1(p_1)$;

(2) if $E(A(p)) = E(A_1(p_1))$ and

(a) $\sigma\ (A(p)) > \sigma(A_1(p_1))$, then $A(p) < A_1(p_1)$;

(b) $\sigma\ (A(p)) < \sigma(A_1(p_1))$, then $A(p) > A_1(p_1)$;

(c) $\sigma\ (A(p)) = \sigma(A_1(p_1))$, then $A(p)$ is indifferent to $A_1(p_1)$ and is denoted as $A(p) \sim A_1(p_1)$.

In the following, we present a theorem which shows that the association does not affect the score and deviation degree of PHILTE.

Theorem 3.1.12. *Let* $A(p) = \left\langle \pounds(p), \pounds'\left(p'\right)\right\rangle$ *be a PHILTE and* $A^{\boldsymbol{\cdot}}(p) = \left\langle \pounds(p^{\boldsymbol{\cdot}}), \pounds'\left(p'^{\boldsymbol{\cdot}}\right)\right\rangle$ *be the associated PHILTE then* $E(A(p)) = E(A^{\boldsymbol{\cdot}}(p))$ *and* $\sigma(A(p)) = \sigma(A^{\boldsymbol{\cdot}}(p))$.

Proof. $E(A^{\boldsymbol{\cdot}}(p)) = \pounds_{\dot{\overline{\gamma}}}$ where $\dot{\overline{\gamma}} = \frac{g+\dot{\varpi}-\dot{\beta}}{2}$ and $\dot{\alpha} = \frac{\sum_{i=1}^{\#\pounds(p^{\boldsymbol{\cdot}})} r^{(i)} p^{\boldsymbol{\cdot}(i)}}{\sum_{i=1}^{\#\pounds(p^{\boldsymbol{\cdot}})} p^{\boldsymbol{\cdot}(i)}}$. Since $\sum_{i=1}^{\#\pounds(p^{\boldsymbol{\cdot}})} p^{\boldsymbol{\cdot}(i)} = 1$ and $p^{\boldsymbol{\cdot}(i)} = \frac{p^{(i)}}{\sum_{i=1}^{\#\pounds(p)} p^{(i)}}$, which implies that $\dot{\alpha} = \frac{\sum_{i=1}^{\#\pounds(p)} r^{(i)} p^{(i)}}{\sum_{i=1}^{\#\pounds(p)} p^{(i)}} = \alpha$ and $\dot{\beta} = \frac{\sum_{j=1}^{\#\pounds'\left(p'^{\boldsymbol{\cdot}}\right)} r^{(j)} p'^{\boldsymbol{\cdot}(j)}}{\sum_{j=1}^{\#\pounds'\left(p'^{\boldsymbol{\cdot}}\right)} p'^{\boldsymbol{\cdot}(j)}}$. Since $\sum_{j=1}^{\#\pounds'\left(p'^{\boldsymbol{\cdot}}\right)} p'^{\boldsymbol{\cdot}(j)} = 1$ and $p'^{\cdot(j)} = \frac{p'^{(j)}}{\sum_{j=1}^{\#\pounds'\left(p'\right)} p'^{(j)}}$ which further implies that $\dot{\beta} = \frac{\sum_{j=1}^{\#\pounds'\left(p'\right)} r^{(i)} p'^{(i)}}{\sum_{i=1}^{\#\pounds'\left(p'\right)} p'^{(i)}} = \beta$. Hence, $E(A^{\boldsymbol{\cdot}}(p)) = E(A(p))$.

Next, $\sigma\ (A^{\boldsymbol{\cdot}}(p)) = \left(\frac{\sum_{i=1}^{\#\pounds(p^{\boldsymbol{\cdot}})} \left(p^{\cdot(i)}\left(r^{(i)} - \dot{\overline{\gamma}}\right)\right)^2}{\sum_{i=1}^{\#\pounds(p^{\cdot})} p^{\cdot(i)}} + \frac{\sum_{j=1}^{\#\pounds'\left(p'^{\cdot}\right)} \left(p'^{\boldsymbol{\cdot}(j)}\left(r'^{(j)} - \dot{\overline{\gamma}}\right)\right)^2}{\sum_{j=1}^{\#\pounds'\left(p'^{\boldsymbol{\cdot}}\right)} p'^{\boldsymbol{\cdot}(j)}} \right)^{\frac{1}{2}}$

Since $\sum_{i=1}^{\#\pounds(p^{\boldsymbol{\cdot}})} p^{\boldsymbol{\cdot}(i)} = 1$, $p^{\boldsymbol{\cdot}(i)} = \frac{p^{(i)}}{\sum_{i=1}^{\#\pounds(p)} p^{(i)}}$, $\sum_{j=1}^{\#\pounds'\left(p'^{\boldsymbol{\cdot}}\right)} p'^{\boldsymbol{\cdot}(j)} = 1$, $p'^{\boldsymbol{\cdot}(j)} = \frac{p'^{(j)}}{\sum_{j=1}^{\#\pounds'\left(p'\right)} p'^{(j)}}$ and $\dot{\overline{\gamma}} = \overline{\gamma}$.

It yields that $\sigma\left(A^{\centerdot}(p)\right)=\left(\frac{\sum_{i=1}^{\#\pounds(p)}\left(p^{(i)}\left(r^{(i)}-\overline{\gamma}\right)\right)^{2}}{\sum_{i=1}^{\#\pounds(p)}p^{(i)}}+\frac{\sum_{j=1}^{\#\pounds'\left(p'\right)}\left(p'^{(j)}\left(r'^{(j)}-\overline{\gamma}\right)\right)^{2}}{\sum_{j=1}^{\#\pounds'\left(p'\right)}p'^{(j)}}\right)^{\frac{1}{2}}=\sigma\left(A\left(p\right)\right)$. □

The following theorem shows that order of comparison between two PHILTEs remains unaltered after normalization.

Theorem 3.1.13. *Let* $A(p)=\left\langle \pounds(p),\pounds'\left(p'\right)\right\rangle$ *and* $A_1(p_1)=\left\langle \pounds_1(p_1),\pounds'_1\left(p'_1\right)\right\rangle$ *be any two PHILTEs,* $\widetilde{A}(p)=\left\langle \widetilde{\pounds}(p),\widetilde{\pounds'}\left(p'\right)\right\rangle$ *and* $\widetilde{A}_1(p_1)=\left\langle \widetilde{\pounds}_1(p_1),\widetilde{\pounds'}_1\left(p'_1\right)\right\rangle$ *be the corresponding normalized PHILTEs, respectively, then* $A(p)<A_1(p_1)\iff\widetilde{A}(p)<\widetilde{A}_1(p_1)$.

Proof. The proof is quite clear because, according to Theorem 3.1.12, $E\left(A\left(p\right)\right)=E\left(A^{\centerdot}(p)\right)$ and $\sigma\left(A\left(p\right)\right)=\sigma\left(A^{\centerdot}(p)\right)$, so order of comparison in Step (1) of normalization process is preserved and so for Step (2) is concerned in that step we add some elements to PHILTEs though it does not change the order as we attach zero probabilities with the corresponding added elements so this means $E\left(\widetilde{A}(p)\right)=E\left(\widetilde{A}_1(p_1)\right)$ and $\sigma\left(\widetilde{A}(p)\right)=\sigma\left(\widetilde{A}_1(p_1)\right)$. Hence, the result holds true. □

In the following definition, we summarize the fact that comparison of any two PHILTEs can be done by their corresponding normalized PHILTEs.

Definition 3.1.14. *Let* $A(p)=\left\langle \pounds(p),\pounds'\left(p'\right)\right\rangle$ *and* $A_1(p_1)=\left\langle \pounds_1(p_1),\pounds'_1\left(p'_1\right)\right\rangle$ *be any two PHILTEs,* $\widetilde{A}(p)=\left\langle \widetilde{\pounds}(p),\widetilde{\pounds'}\left(p'\right)\right\rangle$ *and* $\widetilde{A}_1(p_1)=\left\langle \widetilde{\pounds}_1(p_1),\widetilde{\pounds'}_1\left(p'_1\right)\right\rangle$ *be the corresponding normalized PHILTEs, respectively, then*

(I) If $E\left(\widetilde{A}(p)\right)>E\left(\widetilde{A_1}(p_1)\right)$ then $A(p)>A_1(p_1)$.

(II) If $E\left(\widetilde{A}(p)\right)<E\left(\widetilde{A_1}(p_1)\right)$ then $A(p)<A_1(p_1)$.

(III) If $E\left(\widetilde{A}(p)\right)=E\left(\widetilde{A_1}(p_1)\right)$, then in this case, we are unable to decide which one is superior. Thus, in this case, we do the comparison of PHILTEs on the bases of the deviation degree of normalized PHILTEs as follows.

(1) If $\delta\left(\widetilde{A}(p)\right) > \delta\left(\widetilde{A_1}(p_1)\right)$ then $A(p) < A_1(p_1)$.

(2) If $\delta\left(\widetilde{A}(p)\right) < \delta\left(\widetilde{A_1}(p_1)\right)$ then $A(p) > A_1(p_1)$.

(3) If $\delta\left(\widetilde{A}(p)\right) = \delta\left(\widetilde{A_1}(p_1)\right)$ in such case we say that $A(p)$ is indifferent to $A_1(p_1)$ and is denoted by $A(p) \sim A_1(p_1)$.

3.1.3 Basic operations of PHILTEs

Inspired by the operational laws of PLTSs [30], in what follows, we present some basic operational framework of PHILTEs and investigate their properties in preparation for applications to the practical real life problems. Hereafter, it is assumed that all PHILTEs are normalized.

Definition 3.1.15. *Let* $A(p) = \langle £(p), £'(p') \rangle = \left\langle £^{(i)}\left(p^{(i)}\right), £^{'(j)}\left(p^{'(j)}\right)\right\rangle ; i = 1, 2, \ldots, \#£(p)$, $j = 1, 2, \ldots, \#£'(p')$ *and* $A_1(p_1) = \langle £_1(p_1), £_1'(p_1') \rangle = \left\langle £_1^{(i)}\left(p_1^{(i)}\right), £_1^{'(j)}\left(p_1^{'(j)}\right)\right\rangle ; i = 1, 2, \ldots, \#£_1(p_1)$, $j = 1, 2, \ldots, \#£_1'(p_1')$ *be two normalized and ordered PHILTEs, then*

i. $A(p) \oplus A_1(p_1) = \langle £(p) \oplus £_1(p_1), £'(p') \oplus £_1'(p_1') \rangle$

$$= \left\langle \begin{array}{c} \cup_{£^{(i)} \in £(p), £_1^{(i)} \in £_1(p_1)} \left\{ p^{(i)} £^{(i)} \oplus p_1^{(i)} £_1^{(i)} \right\}, \\ \cup_{£^{'(j)} \in £'(p'), £_1^{'(j)} \in £_1^{'(j)}\left(p_1^{'(j)}\right)} \left\{ p^{'(j)} £^{'(j)} \oplus p_1^{'(j)} £_1^{'(j)} \right\} \end{array} \right\rangle;$$

ii. $A(p) \otimes A_1(p_1) = \langle £(p) \otimes £_1(p_1), £'(p') \otimes £_1'(p_1') \rangle$

$$= \left\langle \begin{array}{c} \cup_{£^{(i)} \in £(p), £_1^{(i)} \in £_1(p_1)} \left\{ \left(£^{(i)}\right)^{p^{(i)}} \otimes \left(£_1^{(i)}\right)^{p_1^{(i)}} \right\}, \\ \cup_{£^{'(j)} \in £'(p'), £_1^{'(j)} \in £_1'(p_1)} \left\{ \left(£^{'(j)}\right)^{p^{'(j)}} \otimes \left(£_1^{'(j)}\right)^{p_1^{'(j)}} \right\} \end{array} \right\rangle;$$

iii. $\gamma(A(p)) = \langle \gamma £(p), \gamma £'(p') \rangle = \left\langle \cup_{£^{(i)} \in £(p)} \gamma p^{(i)} £^{(i)}, \cup_{£^{'(j)} \in £'(p')} \gamma p^{'(j)} £^{'(j)} \right\rangle;$

iv. $(A(p))^{\gamma} = \left\langle (\pounds(p))^{\gamma}, \left(\pounds'\left(p'\right)^{\gamma}\right)\right\rangle = \left\langle \cup_{\pounds^{(i)}\in\pounds(p)} \left(\pounds^{(i)}\right)^{\gamma p^{(i)}}, \cup_{\pounds'^{(j)}\in\pounds'\left(p'\right)} \left(\pounds'^{(j)}\right)^{\gamma p'^{(j)}} \right\rangle;$

where $\pounds^{(i)}$ *and* $\pounds_1^{(i)}$ *are the ith linguistic terms in* $\pounds(p)$ *and* $\pounds_1(p_1)$, *respectively;* $\pounds'^{(j)}$ *and* $\pounds_1'^{(j)}$ *are the jth linguistic terms in* $\pounds'\left(p'\right)$ *and* $\pounds_1'\left(p_1'\right)$, *respectively;* $p^{(i)}$ *and* $p_1^{(i)}$ *are the probabilities of the ith linguistic terms in* $\pounds(p)$ *and* $\pounds_1(p_1)$, *respectively;* $p'^{(j)}$ *and* $p_1'^{(j)}$ *are the probabilities of the jth linguistic terms in* $\pounds'\left(p'\right)$ *and* $\pounds_1'\left(p_1'\right)$, *respectively; and* γ *denote a nonnegative scalar.*

Theorem 3.1.16. *Let* $A(p) = \left\langle \pounds(p), \pounds'\left(p'\right)\right\rangle$, $A_1(p_1) = \left\langle \pounds_1(p_1), \pounds_1'\left(p_1'\right)\right\rangle$, $A_2(p_2) = \langle \pounds_2(p_2),$ $\pounds_2'\left(p_2'\right)\rangle$ *be any three ordered and normalized PHILTEs,* $\gamma_1,\gamma_2,\gamma_3 \geq 0$, *then*

(1) $A(p) \oplus A_1(p_1) = A_1(p_1) \oplus A(p);$

(2) $A(p) \oplus (A_1(p_1) \oplus A_2(p_2)) = (A(p) \oplus A_1(p_1)) \oplus A_2(p_2);$

(3) $\gamma(A(p) \oplus A_1(p_1)) = \gamma A(p) \oplus \gamma A_1(p_1);$

(4) $(\gamma_1 + \gamma_2) A(p) = \gamma_1 A(p) \oplus \gamma_2 A(p);$

(5) $A(p) \otimes A_1(p_1) = A_1(p_1) \otimes A(p);$

(6) $A(p) \otimes (A_1(p_1) \otimes A_2(p_2)) = (A(p) \otimes A_1(p_1)) \otimes A_2(p_2);$

(7) $(A(p) \otimes A_1(p_1))^{\gamma} = (A(p))^{\gamma} \otimes (A_1(p_1))^{\gamma};$

(8) $(A(p))^{\gamma_1+\gamma_2} = (A(p))^{\gamma_1} \otimes (A(p))^{\gamma_2}.$

3.2 Aggregation operators and criteria weights

This section is dedicated to discussion on some basic aggregation operators of PHILTS. Deviation degree between two PHILTEs is also defined in this section. Finally, we calculate the criteria weights in the light of PHILTEs.

3.2.1 The aggregation operators for PHILTEs

The aggregation operators are powerful tools to deal with linguistic information. To make a better usage of PHILTEs in real world problems, in the following, aggregation operators for PHILTEs have been developed.

Definition 3.2.1. *Let $A_k(p_k) = \left\langle \pounds_k(p_k), \pounds_k'\left(p_k'\right)\right\rangle$ $(k = 1, 2, \ldots, n)$ be n ordered and normalized PHILTEs. Then*

$$\begin{aligned}&PHILA\left(A_1(p_1), A_2(p_2), \ldots, A_n(p_n)\right)\\&= \tfrac{1}{n}\left(\left\langle \pounds_1(p_1), \pounds_1'\left(p_1'\right)\right\rangle \oplus \left\langle \pounds_2(p_2), \pounds_2'\left(p_2'\right)\right\rangle \oplus \ldots \oplus \left\langle \pounds_n(p_n), \pounds_n'\left(p_n'\right)\right\rangle\right)\\&= \tfrac{1}{n}\left\langle \pounds_1(p_1) \oplus \pounds_2(p_2) \oplus \ldots \oplus \pounds_n(p_n), \pounds_1'\left(p_1'\right) \oplus \pounds_2'\left(p_2'\right) \oplus \ldots \oplus \pounds_n'\left(p_n'\right)\right\rangle\end{aligned}$$

$$= \frac{1}{n}\left\langle \begin{array}{c} \cup_{\pounds_1^{(i)} \in \pounds_1(p_1), \pounds_2^{(i)} \in \pounds_2(p_2), \ldots, \pounds_n^{(i)} \in \pounds_n(p_n)} \left\{p_1^{(i)}\pounds_1^{(i)} \oplus p_2^{(i)}\pounds_2^{(i)} \oplus \ldots \oplus p_n^{(i)}\pounds_n^{(i)}\right\}, \\ \cup_{\pounds_1'^{(j)} \in \pounds_1'\left(p_1'\right), \pounds_2'^{(j)} \in \pounds_2'\left(p_2'\right), \ldots, \pounds_n'^{(j)} \in \pounds_n'\left(p_n'\right)} \left\{p_1'^{(j)}\pounds_1'^{(j)} \oplus p_2'^{(j)}\pounds_2'^{(j)} \oplus \ldots \oplus p_n'^{(j)}\pounds_n'^{(j)}\right\} \end{array}\right\rangle, \tag{3.2.1}$$

is called the probabilistic hesitant intuitionistic linguistic averaging (PHILA) operator.

Definition 3.2.2. *Let $A_k(p_k) = \left\langle \pounds_k(p_k), \pounds_k'\left(p_k'\right)\right\rangle$ $(k = 1, 2, \ldots, n)$ be n ordered and normalized PHILTEs. Then*

$$\begin{aligned}&PHILWA\left(A_1(p_1), A_2(p_2), \ldots, A_n(p_n)\right)\\&= w_1\left\langle \pounds_1(p_1), \pounds_1'\left(p_1'\right)\right\rangle \oplus w_2\left\langle \pounds_2(p_2), \pounds_2'\left(p_2'\right)\right\rangle \oplus \ldots \oplus w_n\left\langle \pounds_n(p_n), \pounds_n'\left(p_n'\right)\right\rangle\\&= \left\langle w_1\pounds_1(p_1) \oplus w_2\pounds_2(p_2) \oplus \ldots \oplus w_n\pounds_n(p_n), w_1\pounds_1'\left(p_1'\right) \oplus w_2\pounds_2'\left(p_2'\right) \oplus \ldots \oplus w_n\pounds_n'\left(p_n'\right)\right\rangle\end{aligned}$$

$$= \left\langle \begin{array}{c} \cup_{\pounds_1^{(i)} \in \pounds_1(p_1)} \left\{w_1p_1^{(i)}\pounds_1^{(i)}\right\} \oplus \cup_{\pounds_2^{(i)} \in \pounds_2(p_2)} \left\{w_2p_2^{(i)}\pounds_2^{(i)}\right\} \oplus \ldots \oplus \cup_{\pounds_n^{(i)} \in \pounds_n(p_n)} \left\{w_np_n^{(i)}\pounds_n^{(i)}\right\}, \\ \cup_{\pounds_1'^{(j)} \in \pounds_1'\left(p_1'\right)} \left\{w_1p_1'^{(j)}\pounds_1'^{(j)}\right\} \oplus \cup_{\pounds_2'^{(j)} \in \pounds_2'\left(p_2'\right)} \left\{w_2p_2'^{(j)}\pounds_2'^{(j)}\right\} \oplus \ldots \oplus \cup_{\pounds_n'^{(j)} \in \pounds_n'\left(p_n'\right)} \left\{w_np_n'^{(j)}\pounds_n'^{(j)}\right\} \end{array}\right\rangle, \tag{3.2.2}$$

is called the probabilistic hesitant intuitionistic linguistic weighted averaging (PHILWA) operator, where $w = (w_1, w_2, \dots, w_n)^t$ *is the weight vector of* $A_k(p_k)$ $(k = 1, 2, \dots, n)$*,* $w_k \geq 0$*,* $k = 1, 2, \dots, n$*, and* $\sum_{k=1}^{n} w_k = 1$.

Particularly, if we take $w = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^t$, then the PHILWA operator reduces to the PHILA operator.

Definition 3.2.3. *Let* $A_k(p_k) = \left\langle \pounds_k(p_k), \pounds_k^{'}\left(p_k^{'}\right)\right\rangle$ $(k = 1, 2, \dots, n)$ *be* n *ordered and normalized PHILTEs. Then,*

$$PHILG\left(A_1(p_1), A_2(p_2), \dots, A_n(p_n)\right)$$
$$= \left(\left\langle \pounds_1(p_1), \pounds_1^{'}\left(p_1^{'}\right)\right\rangle \otimes \left\langle \pounds_2(p_2), \pounds_2^{'}\left(p_2^{'}\right)\right\rangle \otimes \dots \otimes \left\langle \pounds_n(p_n), \pounds_n^{'}\left(p_n^{'}\right)\right\rangle\right)^{\frac{1}{n}}$$
$$= \left(\left\langle \pounds_1(p_1) \otimes \pounds_2(p_2) \otimes \dots \otimes \pounds_n(p_n), \pounds_1^{'}\left(p_1^{'}\right) \otimes \pounds_2^{'}\left(p_2^{'}\right) \otimes \dots \otimes \pounds_n^{'}\left(p_n^{'}\right)\right\rangle\right)^{\frac{1}{n}}$$

$$= \left(\left\langle \begin{array}{l} \cup_{\pounds_1^{(i)} \in \pounds_1(p_1), \pounds_2^{(i)} \in \pounds_2(p_2), \dots, \pounds_n^{(i)} \in \pounds_n(p_n)} \left\{ \left(\pounds_1^{(i)}\right)^{p_1^{(i)}} \otimes \left(\pounds_2^{(i)}\right)^{p_2^{(i)}} \otimes \dots \otimes \left(\pounds_n^{(i)}\right)^{p_n^{(i)}} \right\}, \\ \cup_{\pounds_1^{'(j)} \in \pounds_1^{'}\left(p_1^{'}\right), \pounds_2^{'(j)} \in \pounds_2^{'}\left(p_2^{'}\right), \dots, \pounds_n^{'(j)} \in \pounds_n^{'}\left(p_n^{'}\right)} \left\{ \left(\pounds_1^{'(j)}\right)^{p_1^{'(j)}} \otimes \left(\pounds_2^{'(j)}\right)^{p_2^{'(j)}} \otimes \dots \otimes \left(\pounds_n^{'(j)}\right)^{p_n^{'(j)}} \right\} \end{array} \right\rangle \right)^{\frac{1}{n}}, \tag{3.2.3}$$

is called the probabilistic hesitant intuitionistic linguistic geometric (PHILG) operator.

Definition 3.2.4. *Let* $A_k(p_k) = \left\langle \pounds_k(p_k), \pounds_k^{'}\left(p_k^{'}\right)\right\rangle$ $(k = 1, 2, \dots, n)$ *be* n *ordered and normalized PHILTEs. Then*

$$PHILWG\left(A_1(p_1), A_2(p_2), \dots, A_n(p_n)\right)$$
$$= \left\langle \pounds_1(p_1), \pounds_1^{'}\left(p_1^{'}\right)\right\rangle^{w_1} \otimes \left\langle \pounds_2(p_2), \pounds_2^{'}\left(p_2^{'}\right)\right\rangle^{w_2} \otimes \dots \otimes \left\langle \pounds_n(p_n), \pounds_n^{'}\left(p_n^{'}\right)\right\rangle^{w_n}$$
$$= \left\langle \left(\pounds_1(p_1)\right)^{w_1} \otimes \left(\pounds_2(p_2)\right)^{w_2} \otimes \dots \otimes \left(\pounds_n(p_n)\right)^{w_n}, \left(\pounds_1^{'}\left(p_1^{'}\right)\right)^{w_1} \otimes \left(\pounds_2^{'}\left(p_2^{'}\right)\right)^{w_2} \otimes \dots \otimes \left(\pounds_n^{'}\left(p_n^{'}\right)\right)^{w_n} \right\rangle$$

$$= \left\langle \begin{array}{c} \cup_{\pounds_1^{(i)} \in \pounds_1(p_1)} \left\{ \left(\pounds_1^{(i)}\right)^{w_1 p_1^{(i)}} \right\} \otimes \cup_{\pounds_2^{(i)} \in \pounds_2(p_2)} \left\{ \left(\pounds_2^{(i)}\right)^{w_2 p_2^{(i)}} \right\} \otimes \ldots \otimes \cup_{\pounds_n^{(i)} \in \pounds_n(p_n)} \left\{ \left(\pounds_n^{(i)}\right)^{w_n p_n^{(i)}} \right\}, \\ \cup_{\pounds_1^{\prime(j)} \in \pounds_1^{\prime}(p_1^{\prime})} \left\{ \left(\pounds_1^{\prime(j)}\right)^{w_1 p_1^{\prime(j)}} \right\} \otimes \cup_{\pounds_2^{\prime(j)} \in \pounds_2^{\prime}(p_2^{\prime})} \left\{ \left(\pounds_2^{\prime(j)}\right)^{w_2 p_2^{\prime(j)}} \right\} \otimes \ldots \otimes \cup_{\pounds_n^{\prime(j)} \in \pounds_n^{\prime}(p_n^{\prime})} \left\{ \left(\pounds_n^{\prime(j)}\right)^{w_n p_n^{\prime(j)}} \right\} \end{array} \right\rangle, \tag{3.2.4}$$

is called the probabilistic hesitant intuitionistic linguistic weighted geometric (PHILWG) operator, where $w = (w_1, w_2, \ldots, w_n)^t$ *is the weight vector of* $A_k(p_k)$ $(k = 1, 2, \ldots, n)$*,* $w_k \geq 0$*,* $k = 1, 2, \ldots, n$*, and* $\sum_{k=1}^{n} w_k = 1$*.*

Particularly, if we take $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^t$, then the PHILWG operator reduces to the PHILG operator.

3.2.2 Maximizing deviation method for calculating the criteria weights

The selection of weights directly affects the performance of MCGDM approach. Thereby, in this part, the maximizing deviation method is adopted to determine weight vector in MCGDM when weights are not known or partly known. Based on Definition 2.3.1, the deviation degree measure between two PHILTEs is defined as follows:

Definition 3.2.5. *Let* $A(p)$ *and* $A_1(p_1)$ *be any two PHILTEs of equal length. Then, the deviation degree* D *between* $A(p)$ *and* $A_1(p_1)$ *is given by*

$$D\left(A(p), A_1(p_1)\right) = d\left(\pounds(p), \pounds_1(p_1)\right) + d\left(\pounds^{\prime}\left(p^{\prime}\right), \pounds_1^{\prime}\left(p_1^{\prime}\right)\right), \tag{3.2.5}$$

where

$$d\left(\pounds(p), \pounds_1(p_1)\right) = \sqrt{\frac{\sum_{i=1}^{\#\pounds(p)} \left(p^{(i)} r^{(i)} - p_1^{(i)} r_1^{(i)}\right)}{\#\pounds(p)}}, \tag{3.2.6}$$

$$d\left(\pounds^{'}\left(p^{'}\right),\pounds_{1}^{'}\left(p_{1}^{'}\right)\right)=\sqrt{\frac{\sum\limits_{j=1}^{\#\pounds^{'}\left(p^{'}\right)}\left(p^{'(j)}r^{'(j)}-p_{1}^{'(j)}r_{1}^{'(j)}\right)}{\#\pounds^{'}\left(p^{'}\right)}}, \tag{3.2.7}$$

$r^{(i)}$ denote the lower index of the ith linguistic term of $\pounds\left(p\right)$ and $r^{'(j)}$ denote the lower index of the jth linguistic term of $\pounds^{'}\left(p^{'}\right)$.

Based on the above definition, in what follows, we obtain criteria weight vector because working on the probabilistic linguistic data to deal with the MCGDM problems, in which the weight information of criteria values is completely unknown or partly known, we must find the criteria weights in advance.

Given the set of alternatives $z=\{z_1,z_2,\ldots,z_m\}$ and the set of "n" attributes $c=\{c_1,c_2,\ldots,c_n\}$, respectively, then, by using Equation (3.2.5), the deviation measure between the alternative "z_i" and all other alternatives with respect to the criteria "c_j" can be given as:

$$D_{ij}\left(w\right)=\sum_{q=1,q\neq i}w_{j}D\left(h_{ij},h_{qj}\right),i=1,2,\ldots,m,j=1,2,\ldots,n. \tag{3.2.8}$$

Accordant with the theme of the maximizing deviation method, if the deviation degree among alternatives is smaller for an criteria, then the criteria should give a smaller weight. This one reveals that the alternatives are homologous to the criteria. Contradictorily, it should give a larger weight. Let

$$\begin{aligned}D_{j}\left(w\right)&=\sum_{i=1}^{m}D_{ij}\left(w\right)=\sum_{i=1}^{m}\sum_{q\neq i}^{m}w_{j}D\left(h_{ij},h_{qj}\right)\\&=\sum_{i=1}^{m}\sum_{q\neq i}^{m}w_{j}\left(d\left(\pounds_{ij}\left(p_{ij}\right),\pounds_{qj}\left(p_{qj}\right)\right)+d\left(\pounds_{ij}^{'}\left(p_{ij}^{'}\right),\pounds_{qj}^{'}\left(p_{qj}^{'}\right)\right)\right),\end{aligned} \tag{3.2.9}$$

show the deviation degree of one alternative and others with respect to the criteria "c_j" and let

$$
\begin{aligned}
D(w) &= \sum_{j=1}^{n} D_j(w) = \sum_{j=1}^{n}\sum_{i=1}^{m} D_{ij}(w) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{q\neq i}^{m} w_j D(h_{ij}, h_{qj}) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{q\neq i}^{m} w_j \left(d\left(\pounds_{ij}(p_{ij}), \pounds_{qj}(p_{qj})\right) + d\left(\pounds'_{ij}\left(p'_{ij}\right), \pounds'_{qj}\left(p'_{qj}\right)\right)\right) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{q\neq i}^{m} w_j \left(\begin{array}{c} \sqrt{\frac{1}{\#\pounds_{ij}(p_{ij})} \sum\limits_{k_1=1}^{\#\pounds_{ij}(p_{ij})} \left(p_{ij}^{(k_1)} r_{ij}^{(k_1)} - p_{qj}^{(k_1)} r_{qj}^{(k_1)}\right)^2} + \\ \sqrt{\frac{1}{\#\pounds'_{ij}(p'_{ij})} \sum\limits_{k_2=1}^{\#\pounds'_{ij}(p'_{ij})} \left(p_{ij}^{\prime(k_2)} r_{ij}^{\prime(k_2)} - p_{qj}^{\prime(k_2)} r_{qj}^{\prime(k_2)}\right)^2} \end{array} \right),
\end{aligned}
\tag{3.2.10}
$$

express the sum of the deviation degrees among all attributes.

To attain the criteria weights vector $w = (w_1, w_2, \ldots, w_n)^t$, we build the following single objective optimization model (named as M_1) to drive the deviation degree $d(w)$ as large as possible.

$$
M_1 = \begin{cases} \max D(w) = \sum\limits_{j=1}^{n}\sum\limits_{i=1}^{m}\sum\limits_{q\neq i}^{m} w_j D(h_{ij}, h_{qj}) \\ w_j \geq 0, j = 1, 2, \ldots, n, \sum\limits_{j=1}^{n} w_j^2 = 1. \end{cases}
$$

To solve the above model M_1, we use the Lagrange multiplier function:

$$
L(w, \eta) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{q\neq i}^{m} w_j D(h_{ij}, h_{qj}) + \frac{\eta}{2}\left(\sum_{j=1}^{n} w_j^2 - 1\right), \tag{3.2.11}
$$

where η is the Lagrange parameter.

Then, we compute the partial derivatives of Lagrange function with respect to w_j and η and let them be zero:

$$\begin{cases} \frac{\partial L(w,\eta)}{\partial w_j} = \sum_{i=1}^{m}\sum_{q\neq i}^{m} w_j D\left(h_{ij}, h_{qj}\right) + \eta w_j = 0, j = 1, 2, \ldots, n. \\ \frac{\delta L(w,\eta)}{\partial \eta} = \sum_{j=1}^{n} w_j^2 - 1 = 0. \end{cases} \tag{3.2.12}$$

By solving Equation (3.2.12), one can get the optimal weight $w = (w_1, w_2, \ldots, w_n)^t$.

$$w_j = \frac{\sum_{i=1}^{m}\sum_{q\neq i}^{m} D(h_{ij},h_{qj})}{\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{q\neq i} D(h_{ij},h_{qj})\right)^2}} = \frac{\sum_{i=1}^{m}\sum_{q\neq i}^{m}\left(d(\pounds_{ij}(p_{ij}),\pounds_{qj}(p_{qj}))+d\left(\pounds'_{ij}\left(p'_{ij}\right),\pounds'_{qj}\left(p'_{qj}\right)\right)\right)}{\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{q\neq i}\left(d(\pounds_{ij}(p_{ij}),\pounds_{qj}(p_{qj}))+d\left(\pounds'_{ij}\left(p'_{ij}\right),\pounds'_{qj}\left(p'_{qj}\right)\right)\right)\right)^2}}$$

$$w_j = \frac{\sum_{i=1}^{m}\sum_{q\neq i}^{m}\left(\sqrt{\frac{1}{\#\pounds_{ij}(p_{ij})}\sum_{k_1=1}^{\#\pounds_{ij}(p_{ij})}\left(p_{ij}^{(k_1)}r_{ij}^{(k_1)} - p_{qj}^{(k_1)}r_{qj}^{(k_1)}\right)^2 + \sqrt{\frac{1}{\#\pounds'_{ij}\left(p'_{ij}\right)}\sum_{k_2=1}^{\#\pounds'_{ij}\left(p'_{ij}\right)}\left(p_{ij}^{\prime(k_2)}r_{ij}^{\prime(k_2)} - p_{qj}^{\prime(k_2)}r_{qj}^{\prime(k_2)}\right)^2}}\right)}{\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{q\neq i}\left(\sqrt{\frac{1}{\#\pounds_{ij}(p_{ij})}\sum_{k_1=1}^{\#\pounds_{ij}(p_{ij})}\left(p_{ij}^{(k_1)}r_{ij}^{(k_1)} - p_{qj}^{(k_1)}r_{qj}^{(k_1)}\right)^2 + \sqrt{\frac{1}{\#\pounds'_{ij}\left(p'_{ij}\right)}\sum_{k_2=1}^{\#\pounds'_{ij}\left(p'_{ij}\right)}\left(p_{ij}^{\prime(k_2)}r_{ij}^{\prime(k_2)} - p_{qj}^{\prime(k_2)}r_{qj}^{\prime(k_2)}\right)^2}}\right)\right)^2}}, \tag{3.2.13}$$

where $j = 1, 2, \ldots, n$.

Clearly, $w_j \geq 0 \ \forall\ j$. By normalizing Equation (3.2.13), we get

$$w_j = \frac{\sum_{i=1}^{m}\sum_{q\neq i}^{m} D\left(h_{ij}, h_{qj}\right)}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{q\neq i} D\left(h_{ij}, h_{qj}\right)}$$

$$= \frac{\sum_{i=1}^{m}\sum_{q\neq i}^{m}\left(\begin{array}{c}\sqrt{\frac{1}{\#\pounds_{ij}(p_{ij})}\sum_{k_1=1}^{\#\pounds_{ij}(p_{ij})}\left(p_{ij}^{(k_1)}r_{ij}^{(k_1)} - p_{qj}^{(k_1)}r_{qj}^{(k_1)}\right)^2} + \\ \sqrt{\frac{1}{\#\pounds'_{ij}\left(p'_{ij}\right)}\sum_{k_2=1}^{\#\pounds'_{ij}\left(p'_{ij}\right)}\left(p_{ij}^{\prime(k_2)}r_{ij}^{\prime(k_2)} - p_{qj}^{\prime(k_2)}r_{qj}^{\prime(k_2)}\right)^2}\end{array}\right)}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{q\neq i}\left(\begin{array}{c}\sqrt{\frac{1}{\#\pounds_{ij}(p_{ij})}\sum_{k_1=1}^{\#\pounds_{ij}(p_{ij})}\left(p_{ij}^{(k_1)}r_{ij}^{(k_1)} - p_{qj}^{(k_1)}r_{qj}^{(k_1)}\right)^2} + \\ \sqrt{\frac{1}{\#\pounds'_{ij}\left(p'_{ij}\right)}\sum_{k_2=1}^{\#\pounds'_{ij}\left(p'_{ij}\right)}\left(p_{ij}^{\prime(k_2)}r_{ij}^{\prime(k_2)} - p_{qj}^{\prime(k_2)}r_{qj}^{\prime(k_2)}\right)^2}\end{array}\right)}, \tag{3.2.14}$$

where $j = 1, 2, \ldots, n$.

The above end result can be applied to the situations where the information of criteria weights is completely unknown. However, in real life decision making problems, the weight information is most often partly known. In such cases, let H be a set of the known weight information, which can be given in the following ways.

Rank 1. A weak ranking: $\{w_i \geq w_j\}$ $(i \neq j)$.

Rank 2. A strict ranking: $\{w_i - w_j \geq \beta_i\}$ $(i \neq j)$.

Rank 3. A ranking of differences: $\{w_i - w_j \geq w_k - w_l\}$ $(j \neq k \neq l)$.

Rank 4. A ranking with multiples: $\{w_i \geq \beta_i w_j\}$ $(i \neq j)$.

Rank 5. An interval form: $\{\beta_i \leq w_j \leq \beta_i + \epsilon_i\}$ $(i \neq j)$.

β_i and ϵ_i denote the non-negative numbers.

With the set H, we can build the below model:

$$M_2 = \begin{cases} \max D(w) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{q\neq i}^{m} w_j D\left(h_{ij}, h_{qj}\right) \\ w_j \in H, w_j \geq 0, j = 1, 2, \ldots, n, \sum_{j=1}^{n} w_j^2 = 1 \end{cases}$$

from which the optimal weight vector $w = (w_1, w_2, \ldots, w_n)^t$ attained.

3.3 MCGDM with probabilistic hesitant intuitionistic linguistic information

In this section, two practical methods, i.e., an extended TOPSIS method and an aggregation based method, for MCGDM problems are proposed, where the opinions of DMs take the form of PHILTSs.

3.3.1 Extended TOPSIS method for MCGDM with probabilistic hesitant intuitionistic linguistic information

Of the numerous MCGDM methods, TOPSIS is one of the effective methods for ranking and selecting a number of possible alternatives by measuring Euclidean distances. It has been successfully applied to solve evaluation problems with a finite number of alternatives and criteria [29, 30, 46] because it is easy to understand and implement, and can measure the relative performance for each alternative.

In what follows, we discuss the complete construction of extended TOPSIS method in PHILTS regard. This methodology involves the following steps.

Step 1: Analyze the given MCGDM problem; since the problem is group decision making, so let there be "t" decision makers or experts $D = \{d_1, d_2, \ldots, d_t\}$ involved in the given problem. The set of alternatives is $z = \{z_1, z_2, \ldots, z_m\}$ and the set of attributes is $c = \{c_1, c_2, \ldots, c_n\}$. The experts provide their linguistic evaluation values for membership and non-membership by using linguistic term set $S = \{\pounds_0, \pounds_1, \ldots, \pounds_\tau\}$ over the alternative $z_i\,(i = 1, 2, \ldots, m)$ with respect to the criteria $c_j\,(j = 1, 2, \ldots, n)$.

The DM $d_k\,(k = 1, 2, \ldots, t)$ states his membership and non-membership linguistic evaluation values keeping in mind all the alternatives and attributes in the form of PHILTEs.

Thus, intuitionistic probabilistic linguistic decision matrix $H^k = \left[\left\langle \pounds_{ij}^k\left(p_{ij}\right), \pounds_{ij}^{'(k)}\left(p_{ij}^{'}\right)\right\rangle\right]_{m\times n}$ is constructed. It should be noted that preference of alternative "z_i" with respect to decision maker "m_k" and criteria "c_j" is denoted as PHILTE $A_{ij}^k\left(p_{ij}\right)$ in a group decision making problem with "l" experts.

Step 2: Calculate the one probabilistic hesitant intuitionistic linguistic decision matrix H by aggregating the opinions of DMs$\left(H^{(1)}, H^{(2)}, \ldots, H^{(l)}\right)$; $H = [h_{ij}]$, where

$h_{ij} = \left\langle \left\{s_{m_{ij}}\left(p_{ij}\right), s_{n_{ij}}\left(q_{ij}\right)\right\}, \left\{s_{m_{ij}}^{'}\left(p_{ij}^{'}\right), s_{n_{ij}}^{'}\left(q_{ij}^{'}\right)\right\}\right\rangle$ where

$$s_{m_{ij}}\left(p_{ij}\right) = \min\left\{\min_{k=1}^{\pounds}\left(\max \pounds_{ij}^k\left(p_{ij}\right)\right), \max_{k=1}^{\pounds}\left(\min \pounds_{ij}^k\left(p_{ij}\right)\right)\right\},$$

$$s_{n_{ij}}\left(q_{ij}\right) = \max\left\{\min_{k=1}^{\pounds}\left(\max \pounds_{ij}^k\left(q_{ij}\right)\right), \max_{k=1}^{\pounds}\left(\min \pounds_{ij}^k\left(q_{ij}\right)\right)\right\},$$

$$s_{m_{ij}^{'}}\left(p_{ij}^{'}\right) = \min\left\{\min_{k=1}^{\pounds}\left(\max \pounds_{ij}^{'k}\left(p_{ij}^{'}\right)\right), \max_{k=1}^{\pounds}\left(\min \pounds_{ij}^{'k}\left(p_{ij}^{'}\right)\right)\right\},$$

$$s_{n_{ij}^{'}}\left(q_{ij}^{'}\right) = \max\left\{\min_{k=1}^{\pounds}\left(\max \pounds_{ij}^{'k}\left(q_{ij}^{'}\right)\right), \max_{k=1}^{\pounds}\left(\min \pounds_{ij}^{'k}\left(q_{ij}^{'}\right)\right)\right\}.$$

Here, $\max \pounds_{ij}^k\left(p_{ij}\right)$ and $\min \pounds_{ij}^k\left(p_{ij}\right)$ are taken according to the maximum and minimum value of $p_{ij} \times r_{ij}^l$, $l = 1, 2, \ldots, \#\pounds_{ij}^k\left(p_{ij}\right)$, respectively, where r_{ij}^l denotes the lower index of the lth linguistic term and p_{ij} is its corresponding probability.

In this aggregated matrix H, the preference of alternative a_i with respect to criteria c_j is denoted as h_{ij}.

Each term of the aggregated matrix H, i.e., h_{ij} is also an PHILTE; for this, we have to prove that

$s_{m_{ij}}\left(p_{ij}\right)+s_{n_{ij}}^{'}\left(q_{ij}^{'}\right) \leq \pounds_\tau$ and $s_{n_{ij}}\left(q_{ij}\right)+s_{m_{ij}}^{'}\left(p_{ij}^{'}\right) \leq \pounds_\tau$. Since we know that $\left[\pounds_{ij}^k\left(p_{ij}\right), \pounds_{ij}^{'}\left(p_{ij}^{'}\right)\right]$ is a PHILTS for every k^{th} expert, i^{th} alternative and j^{th} criteria, a PHILTS it must satisfy the conditions:

$\min\left(\pounds_{ij}^{(k)}\right) + \max\left(\pounds_{ij}^{'(k)}\right) \leq \pounds_\tau$, $\max\left(\pounds_{ij}^{(k)}\right) + \min\left(\pounds_{ij}^{'(k)}\right) \leq \pounds_\tau$.

Thus, the above simple construction of $s_{m_{ij}}(p_{ij})$, $s_{n_{ij}}(q_{ij})$, $s'_{m_{ij}}\left(p'_{ij}\right)$, and $s_{n'_{ij}}\left(q'_{ij}\right)$ guarantees that the h_{ij} is a PHILTE.

Step 3: Normalize the probabilistic hesitant intuitionistic linguistic decision matrix $H = [h_{ij}]$ according to the method in Section 3.1.1.

Step 4: Obtain the weight vector $w = (w_1, w_2, \ldots, w_n)^t$ of the attributes $c_j \, (j = 1, 2, \ldots, n)$.
$w_j = \frac{\sum_{i=1}^{m}\sum_{q\neq i} D(h_{ij},h_{qj})}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{q\neq i} D(h_{ij},h_{qj})} = \frac{\sum_{i=1}^{m}\sum_{q\neq i} d(\pounds_{ij}(p_{ij}),\pounds_{qj}(p_{qj}))+d\left(\pounds'_{ij}\left(p'_{ij}\right),\pounds'_{qj}\left(p'_{qj}\right)\right)}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{q\neq i} d(\pounds_{ij}(p_{ij}),\pounds_{qj}(p_{qj}))+d\left(\pounds'_{ij}\left(p'_{ij}\right),\pounds'_{qj}\left(p'_{qj}\right)\right)}$, $j = 1, 2, \ldots, n$.

Step 5: The PHILTS positive ideal solution (PHILTS-PIS) of alternatives, denoted by $A^+ = \left\langle \pounds^+(p), \pounds'^+(p) \right\rangle$, is formulated as follows:

$$A^+ = \left\langle \pounds^+(p) = \left(\pounds_1^+(p), \pounds_2^+(p), \ldots, \pounds_n^+(p)\right), \pounds'^+(p) = \left(\pounds_1'^+(p), \pounds_2'^+(p), \ldots, \pounds_n'^+(p)\right) \right\rangle, \tag{3.3.1}$$

where $\pounds_j^+(p) = \left\{\left(\pounds_j^{(k_1)}\right)^+ | k_1 = 1, 2, \ldots, \#\pounds_{ij}(p)\right\}$ and $\left(\pounds_j^{(k_1)}\right)^+ = s_{\max_i}\left\{p_{ij}^{(k_1)} r_{ij}^{(k_1)}\right\}, k_1 = 1, 2, \ldots, \#\pounds_{ij}(p), j = 1, 2, \ldots, n$ and $r_{ij}^{(k_1)}$ is lower index of the linguistic term $\pounds_{ij}^{(k_1)}$ while $\pounds_j'^+(p) = \left\{\left(\pounds_j'^{(k_2)}\right)^+ | k_2 = 1, 2, \ldots, \#\pounds'_{ij}(p)\right\}$ and $\left(\pounds_j'^{(k_2)}\right)^+ = s_{\min_i}\left\{p_{ij}'^{(k_2)} r_{ij}'^{(k_2)}\right\}, k_2 = 1, 2, \ldots, \#\pounds'_{ij}(p), j = 1, 2, \ldots, n$ and $r_{ij}'^{(k_2)}$ is lower index of the linguistic term $\pounds_{ij}'^{(k_2)}$. Similarly, the PHILTS negative ideal solution (PHILTS-NIS) of alternatives, denoted by $A^- = \left\langle \pounds^-(p), \pounds'^-(p) \right\rangle$, is formulated as follows:

$$A^- = \left\langle \pounds^-(p) = \left(\pounds_1^-(p), \pounds_2^-(p), \ldots, \pounds_n^-(p)\right), \pounds'^-(p) = \left(\pounds_1'^-(p), \pounds_2'^-(p), \ldots, \pounds_n'^-(p)\right) \right\rangle, \tag{3.3.2}$$

where $\pounds_j^-(p) = \left\{\left(\pounds_j^{(k_1)}\right)^- | k_1 = 1, 2, \ldots, \#\pounds_{ij}(p)\right\}$ and $\left(\pounds_j^{(k_1)}\right)^- = s_{\min_i}\left\{p_{ij}^{(k_1)} r_{ij}^{(k_1)}\right\}$, $k_1 = 1, 2, \ldots, \#\pounds_{ij}(p), j = 1, 2, \ldots, n$ and $r_{ij}^{(k_1)}$ is lower index of the linguistic term $\pounds_{ij}^{(k_1)}$ while $\pounds_j'^-(p) = \left\{\left(\pounds_j'^{(k_2)}\right)^- | k_2 = 1, 2, \ldots, \#\pounds'_{ij}(p)\right\}$ and $\left(\pounds_j'^{(k_2)}\right)^+ = s_{\max_i}\left\{p_{ij}'^{(k_2)} r_{ij}'^{(k_2)}\right\}$,

$k_2 = 1,2,\ldots,\ \#\pounds'_{ij}(p); j = 1,2,\ldots,n$ and $r'^{(k_2)}_{ij}$ is lower index of the linguistic term $\pounds'^{(k_2)}_{ij}$.

Step 6: Compute the deviation degree between each alternative z_i PHILTS-PIS A^+ as follows:
$D(z_i, A^+) = \sum_{j=1}^{n} w_j D(h_{ij}, A^+) = \sum_{j=1}^{n} w_j \left(d\left(\pounds_{ij}(p), \pounds_j^+(p)\right) + d\left(\pounds'_{ij}(p), \pounds'^+_j(p)\right)\right)$

$$= \sum_{j=1}^{n} w_j \left(\begin{array}{c} \sqrt{\frac{1}{\#\pounds_{ij}(p)} \sum_{k_1=1}^{\#\pounds_{ij}(p)} \left(p_{ij}^{(k_1)} r_{ij}^{(k_1)} - \left(p_j^{(k_1)} r_j^{(k_1)} \right)^+ \right)^2} + \\ \sqrt{\frac{1}{\#\pounds'_{ij}(p)} \sum_{k_2=1}^{\#\pounds'_{ij}(p)} \left(p'^{(k_1)}_{ij} r'^{(k_2)}_{ij} - \left(p'^{(k_2)}_j r'^{(k_2)}_j \right)^+ \right)^2} \end{array} \right). \tag{3.3.3}$$

The smaller is the deviation degree $D(z_i, A^+)$, the better is alternative z_i. Similarly, compute the deviation degree between each alternative z_i PHILTS-NIS A^- as follows:
$D(z_i, A^-) = \sum_{j=1}^{n} w_j D(h_{ij}, A^-) = \sum_{j=1}^{n} w_j \left(d\left(\pounds_{ij}(p), \pounds_j^-(p)\right) + d\left(\pounds'_{ij}(p), \pounds'^-_j(p)\right)\right)$

$$= \sum_{j=1}^{n} w_j \left(\begin{array}{c} \sqrt{\frac{1}{\#\pounds_{ij}(p)} \sum_{k_1=1}^{\#\pounds_{ij}(p)} \left(p_{ij}^{(k_1)} r_{ij}^{(k_1)} - \left(p_j^{(k_1)} r_j^{(k_1)} \right)^- \right)^2} + \\ \sqrt{\frac{1}{\#\pounds'_{ij}(p)} \sum_{k_2=1}^{\#\pounds'_{ij}(p)} \left(p'^{(k_1)}_{ij} r'^{(k_2)}_{ij} - \left(p'^{(k_2)}_j r'^{(k_2)}_j \right)^- \right)^2} \end{array} \right). \tag{3.3.4}$$

The larger is the deviation degree $D(z_i, A^-)$, the better is alternative z_i.

Step 7: Determine $D_{\min}(z_i, A^+)$ and $D_{\max}(z_i, A^-)$, where

$$D_{\min}\left(z_i, A^+\right) = \min_{1 \le i \le m} D\left(z_i, A^+\right), \tag{3.3.5}$$

and

$$D_{\max}\left(z_i, A^-\right) = \max_{1\leq i\leq m} D\left(z_i, A^-\right). \tag{3.3.6}$$

Step 8: Determine the closeness coefficient Cl of each alternative z_i to rank the alternatives.

$$Cl\left(z_i\right) = \frac{D\left(z_i, A^-\right)}{D_{\max}\left(z_i, A^-\right)} - \frac{D\left(z_i, A^+\right)}{D_{\min}\left(z_i, A^+\right)}. \tag{3.3.7}$$

Step 9: Pick the best alternative z_i on the basis of the closeness coefficient Cl, where the larger is the closeness coefficient $Cl\left(z_i\right)$, the better is alternative z_i. Thus, the best alternative

$$z^b = \left\{ z_i | \max_{1\leq i\leq m} Cl\left(z_i\right) \right\}. \tag{3.3.8}$$

3.3.2 The aggregation-based method for MCGDM with probabilistic hesitant intuitionistic linguistic information

In this subsection, the aggregation-based method for MCGDM is presented, where the preference opinions of DMs are represented by PHILTS. In Section 3.2, we have developed some aggregation operators, i.e., PHILA, PHILWA, PHILG and PHILWG. In this algorithm, we use PHILWA operator to aggregate the criteria values of each alternative z_i, into the overall criteria values. The following steps are involved in this algorithm. The first four steps are similar to the extended TOPSIS method. Therefore, we go to Step 5.

Step 5: Determine the overall criteria values $\widetilde{Z}_i\left(w\right)\left(i = 1, 2, \ldots, m\right)$, where $w = \left(w_1, w_2, \ldots, w_n\right)^T$ is the weight vector of attributes, using PHILWA operator, this can be expressed as follows:

$$\widetilde{Z}_i(w) = w_1\left\langle \pounds_{i1}(p), \pounds'_{i1}(p')\right\rangle \oplus w_2\left\langle \pounds_{i2}(p), \pounds'_{i2}(p')\right\rangle \oplus \ldots \oplus w_n\left\langle \pounds_{in}(p), \pounds'_{in}(p')\right\rangle$$

$$= \left\langle w_1\pounds_{i1}(p) \oplus w_2\pounds_{i2}(p) \oplus \ldots \oplus w_n\pounds_{in}(p), w_1\pounds'_{i1}(p') \oplus w_2\pounds'_{i2}(p') \oplus \ldots \oplus w_n\pounds'_{in}(p')\right\rangle$$

$$= \left\langle \begin{array}{c} \cup_{\pounds_{i1}^{(k_1)} \in \pounds_{i1}(p)}\left\{w_1 p_{i1}^{(k_1)} \pounds_{i1}^{(k_1)}\right\} \oplus \cup_{\pounds_{i2}^{(k_1)} \in \pounds_{i2}(p)}\left\{w_2 p_{i2}^{(k_1)} \pounds_{i2}^{(k_1)}\right\} \oplus \ldots \oplus \cup_{\pounds_{in}^{(k_1)} \in \pounds_{in}(p)}\left\{w_n p_{in}^{(k_1)} \pounds_{in}^{(k_1)}\right\}, \\ \cup_{\pounds_{i1}^{\prime(k_2)} \in \pounds'_{i1}(p')}\left\{w_1 p_{i1}^{\prime(k_2)} \pounds_{i1}^{\prime(k_2)}\right\} \oplus \cup_{\pounds_{i2}^{\prime(k_2)} \in \pounds'_{i2}(p')}\left\{w_2 p_{i2}^{\prime(k_2)} \pounds_{i2}^{\prime(k_2)}\right\} \oplus \ldots \oplus \cup_{\pounds_{in}^{\prime(k_2)} \in \pounds'_{in}(p')}\left\{w_n p_{in}^{\prime(k_2)} \pounds_{in}^{\prime(k_2)}\right\} \end{array} \right\rangle, \tag{3.3.9}$$

where $i = 1, 2, \ldots, m$.

Step 6: Compare the overall criteria values $\widetilde{Z}_i(w)\,(i = 1, 2, \ldots, m)$ mutually, based on their score function and deviation degree whose detail is given in Section 3.2.

Step 7: Rank the alternatives $z_i\,(i = 1, 2, \ldots, m)$ according to the order of $\widetilde{Z}_i(w)\,(i = 1, 2, \ldots, m)$ and pick the best alternative.

The flow chart of the proposed models is presented in Figure 3.1.

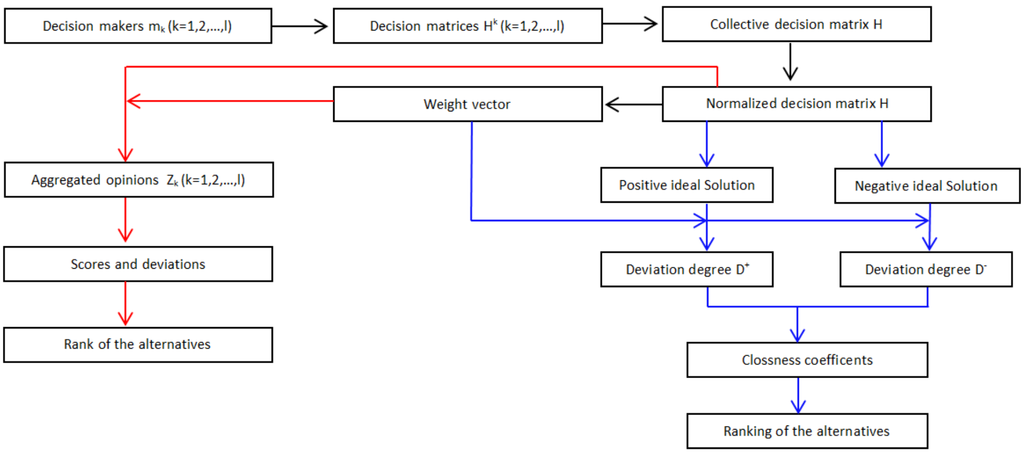

Figure 3.1: Extended TOPSIS and Aggregation-based models.

3.4 A case study

To validate the proposed theory and decision making models, in this section, a practical example taken from [29] is solved. A group of seven peoples d_t $(t = 1, 2, 3, \ldots, 7)$ need to invest their savings in a most profitable way. They considered five possibilities: z_1 is real estate, z_2 is stock market, z_3 is T-bills, z_4 is national saving scheme, and z_5 is insurance company. To determine best option, the following attributes are taken into account: c_1 is the risk factor, c_2 is the growth, c_3 is quick refund, and c_4 is complicated documents requirement. Base upon their knowledge and experience, they provide their opinion in terms of following HIFLTSs.

3.4.1 The extended TOPSIS method for the considered case

We handle the above problem by applying the extended TOPSIS method.

Step 1: The probabilistic hesitant intuitionistic linguistic decision matrices derived from Tables 3.1–3.3 are shown in Tables 3.4–3.6, respectively.

Table 3.1: Decision matrix provided by the DMs 1, 2, 3 (d_1, d_2, d_3)

	c_1	c_2	c_3	c_4
z_1	$\langle\{\pounds_3, \pounds_4, \pounds_5\}, \{\pounds_1, \pounds_2\}\rangle$	$\langle\{\pounds_4, \pounds_5\}, \{\pounds_0, \pounds_1\}\rangle$	$\langle\{\pounds_1, \pounds_2\}, \{\pounds_3, \pounds_4\}\rangle$	$\langle\{\pounds_1, \pounds_2\}, \{\pounds_3, \pounds_4\}\rangle$
z_2	$\langle\{\pounds_1, \pounds_2\}, \{\pounds_3, \pounds_4\}\rangle$	$\langle\{\pounds_3, \pounds_4, \pounds_5\}, \{\pounds_1, \pounds_2\}\rangle$	$\langle\{\pounds_3, \pounds_4\}, \{\pounds_0, \pounds_1\}\rangle$	$\langle\{\pounds_4, \pounds_5\}, \{\pounds_1, \pounds_2\}\rangle$
z_3	$\langle\{\pounds_4, \pounds_5)\}, \{\pounds_0, \pounds_1, \pounds_2\}\rangle$	$\langle\{\pounds_3, \pounds_4\}, \{\pounds_1, \pounds_2\}\rangle$	$\langle\{\pounds_5, \pounds_6\}, \{\pounds_0\}\rangle$	$\langle\{\pounds_1, \pounds_2\}, \{\pounds_2, \pounds_3, \pounds_4\}\rangle$
z_4	$\langle\{\pounds_5, \pounds_6\}, \{\pounds_0, \pounds_1\}\rangle$	$\langle\{\pounds_1, \pounds_2\}, \{\pounds_3, \pounds_4\}\rangle$	$\langle\{\pounds_1, \pounds_2\}, \{\pounds_3, \pounds_4\}\rangle$	$\langle\{\pounds_3, \pounds_4, \pounds_5\}, \{\pounds_1, \pounds_2\}\rangle$
z_5	$\langle\{\pounds_6\}, \{\pounds_0\}\rangle$	$\langle\{\pounds_1, \pounds_2\}, \{\pounds_3, \pounds_4, \pounds_5\}\rangle$	$\langle\{\pounds_0, \pounds_1\}, \{\pounds_2, \pounds_3\}\rangle$	$\langle\{\pounds_4, \pounds_5\}, \{\pounds_1, \pounds_2\}\rangle$

Table 3.2: Decision matrix provided by the DMs 4, 5 (d_4, d_5)

	c_1	c_2	c_3	c_4
z_1	$\langle\{\pounds_1, \pounds_2\}, \{\pounds_3, \pounds_4\}\rangle$	$\langle\{\pounds_5, \pounds_6\}, \{\pounds_0, \pounds_1\}\rangle$	$\langle\{\pounds_0, \pounds_1\}, \{\pounds_3, \pounds_4\}\rangle$	$\langle\{\pounds_3, \pounds_4\}, \{\pounds_1, \pounds_2\}\rangle$
z_2	$\langle\{\pounds_0, \pounds_1\}, \{\pounds_2, \pounds_3\}\rangle$	$\langle\{\pounds_1, \pounds_2\}, \{\pounds_2, \pounds_3, \pounds_4\}\rangle$	$\langle\{\pounds_4, \pounds_5\}, \{\pounds_0, \pounds_1\}\rangle$	$\langle\{\pounds_5, \pounds_6\}, \{\pounds_0\}\rangle$
z_3	$\langle\{\pounds_3, \pounds_4\}, \{\pounds_0, \pounds_1\}\rangle$	$\langle\{\pounds_1, \pounds_2\}, \{\pounds_3, \pounds_4\}\rangle$	$\langle\{\pounds_4, \pounds_5\}, \{\pounds_1, \pounds_2)\}\rangle$	$\langle\{\pounds_0, \pounds_1\}, \{\pounds_2, \pounds_3\}\rangle$
z_4	$\langle\{\pounds_5, \pounds_6\}, \{\pounds_0\}\rangle$	$\langle\{\pounds_3, \pounds_4\}, \{\pounds_0, \pounds_1, \pounds_2\}\rangle$	$\langle\{\pounds_1, \pounds_2\}, \{\pounds_2, \pounds_3, \pounds_4\}\rangle$	$\langle\{\pounds_4, \pounds_5\}, \{\pounds_0\}\rangle$
z_5	$\langle\{\pounds_4, \pounds_5\}, \{\pounds_1, \pounds_2\}\rangle$	$\langle\{\pounds_3, \pounds_4\}, \{\pounds_1, \pounds_2, \pounds_3\}\rangle$	$\langle\{\pounds_1, \pounds_2\}, \{\pounds_3, \pounds_4\}\rangle$	$\langle\{\pounds_5, \pounds_6\}, \{\pounds_0\}\rangle$

Table 3.3: Decision matrix provided by the DMs 6, 7 (d_6, d_7)

	c_1	c_2	c_3	c_4
z_1	$\langle\{\pounds_4, \pounds_5\}, \{\pounds_0, \pounds_1\}\rangle$	$\langle\{\pounds_5, \pounds_6\}, \{\pounds_0\}\rangle$	$\langle\{\pounds_3, \pounds_4\}, \{\pounds_1, \pounds_2\}\rangle$	$\langle\{\pounds_0, \pounds_1\}, \{\pounds_3, \pounds_4\}\rangle$
z_2	$\langle\{\pounds_3, \pounds_4\}, \{\pounds_1, \pounds_2, \pounds_3\}\rangle$	$\langle\{\pounds_1, \pounds_2\}, \{\pounds_3, \pounds_4\}\rangle$	$\langle\{\pounds_5, \pounds_6\}, \{\pounds_0\}\rangle$	$\langle\{\pounds_3, \pounds_4\}, \{\pounds_1, \pounds_2\}\rangle$
z_3	$\langle\{\pounds_1, \pounds_2\}, \{\pounds_2, \pounds_3, \pounds_4\}\rangle$	$\langle\{\pounds_5, \pounds_6\}, \{\pounds_0\}\rangle$	$\langle\{\pounds_4, \pounds_5\}, \{\pounds_0, \pounds_1\}\rangle$	$\langle\{\pounds_0, \pounds_1\}, \{\pounds_3, \pounds_4\}\rangle$
z_4	$\langle\{\pounds_4, \pounds_5\}, \{\pounds_1, \pounds_2\}\rangle$	$\langle\{\pounds_4, \pounds_5\}, \{\pounds_0, \pounds_1\}\rangle$	$\langle\{\pounds_0, \pounds_1, \pounds_2\}, \{\pounds_2, \pounds_3\}\rangle$	$\langle\{\pounds_3, \pounds_4, \pounds_5\}, \{\pounds_1, \pounds_2\}\rangle$
z_5	$\langle\{\pounds_3, \pounds_4\}, \{\pounds_0, \pounds_1, \pounds_2\}\rangle$	$\langle\{\pounds_1, \pounds_2\}, \{\pounds_2, \pounds_3, \pounds_4\}\rangle$	$\langle\{\pounds_2, \pounds_3\}, \{\pounds_3, \pounds_4\}\rangle$	$\langle\{\pounds_6\}, \{\pounds_0\}\rangle$

Table 3.4: Probabilistic hesitant intuitionistic linguistic decision matrix H_1 with respect to DMs 1, 2, 3 (d_1, d_2, d_3)

	c_1	c_2
z_1	$\langle\{(\pounds_3(0.14), \pounds_4(0.28), \pounds_5(0.28))\}, \{\pounds_1(0.28), \pounds_2(0.14)\}\rangle$	$\langle\{\pounds_4(0.14), \pounds_5(0.42)\}, \{\pounds_0(0.42), \pounds_1(0.28)\}\rangle$
z_2	$\langle\{\pounds_1(0.28), \pounds_2(0.14)\}, \{\pounds_3(0.42), \pounds_4(0.14)\}\rangle$	$\langle\{\pounds_3(0.14), \pounds_4(.14), \pounds_5(0.14)\}, \{\pounds_1(0.14), \pounds_2(0.28)\}\rangle$
z_3	$\langle\{\pounds_4(0.28), \pounds_5(0.14)\}, \{\pounds_0(0.28), \pounds_1(0.28), \pounds_2(0.28)\}\rangle$	$\langle\{\pounds_3(0.14), \pounds_4(0.28)\}, \{\pounds_1(0.14), \pounds_2(0.14)\}\rangle$
z_4	$\langle\{\pounds_5(0.42), \pounds_6(0.28)\}, \{\pounds_0(0.28), \pounds_1(0.28)\}\rangle$	$\langle\{\pounds_1(0.14), \pounds_2(0.14)\}, \{\pounds_3(0.14), \pounds_4(0.14)\}\rangle$
z_5	$\langle\{\pounds_6(0.14)\}, \{\pounds_0(0.28)\}\rangle$	$\langle\{\pounds_1(0.28), \pounds_2(0.28)\}, \{\pounds_3(0.42), \pounds_4(0.28), \pounds_5(0.14)\}\rangle$
	c_3	c_4
z_1	$\langle\{\pounds_1(0.28), \pounds_2(0.14)\}, \{\pounds_3(0.28), \pounds_4(0.28)\}\rangle$	$\langle\{\pounds_1(0.28), \pounds_2(0.14)\}, \{\pounds_3(0.28), \pounds_4(0.28)\}\rangle$
z_2	$\langle\{\pounds_3(0.14), \pounds_4(0.28)\}, \{\pounds_0(0.42), \pounds_1(0.28)\}\rangle$	$\langle\{\pounds_4(0.14), \pounds_5(0.28)\}, \{\pounds_1(0.28), \pounds_2(0.28)\}\rangle$
z_3	$\langle\{\pounds_5(0.42), \pounds_6(0.14)\}, \{\pounds_0(0.28)\}\rangle$	$\langle\{\pounds_1(0.42), \pounds_2(0.14)\}, \{\pounds_2(0.28), \pounds_3(0.42), \pounds_4(0.28)\}\rangle$
z_4	$\langle\{\pounds_1(0.42), \pounds_2(.42)\}, \{\pounds_3(0.42), \pounds_4(0.28)\}\rangle$	$\langle\{\pounds_3(0.28), \pounds_4(0.42), \pounds_5(0.42)\}, \{\pounds_1(0.28), \pounds_2(0.28)\}\rangle$
z_5	$\langle\{\pounds_0(0.14), \pounds_1(0.28)\}, \{\pounds_2(0.28), \pounds_3(0.42)\}\rangle$	$\langle\{\pounds_4(0.14), \pounds_5(0.28)\}, \{\pounds_1(0.14), \pounds_2(0.14)\}\rangle$

Table 3.5: Probabilistic hesitant intuitionistic linguistic decision matrix H_2 with respect to DMs $4,5\,(d_4,d_5)$

	c_1	c_2
z_1	$\langle\{\pounds_1(0.14), \pounds_2(0.14)\}, \{\pounds_3(0.14), \pounds_4(0.14)\}\rangle$	$\langle\{\pounds_5(0.42), \pounds_6(0.28)\}, \{\pounds_0(0.42), \pounds_1(0.28)\}\rangle$
z_2	$\langle\{\pounds_0(0.14), \pounds_1(0.28)\}, \{\pounds_2(0.28), \pounds_3(0.42)\}\rangle$	$\langle\{\pounds_1(0.28), \pounds_2(0.28)\}, \{\pounds_2(0.28), \pounds_3(0.28), \pounds_4(0.28)\}\rangle$
z_3	$\langle\{\pounds_3(0.14), \pounds_4(.28)\}, \{\pounds_0(0.28), \pounds_1(0.28)\}\rangle$	$\langle\{\pounds_1(0.14), \pounds_2(0.14)\}, \{\pounds_3(0.14), \pounds_4(0.14)\}\rangle$
z_4	$\langle\{\pounds_5(0.42), \pounds_6(0.28)\}, \{\pounds_0(0.28)\}\rangle$	$\langle\{\pounds_3(0.14), \pounds_4(0.28)\}, \{\pounds_0(0.28), \pounds_1(0.28), \pounds_2(0.14)\}\rangle$
z_5	$\langle\{\pounds_4(0.28), \pounds_5(0.14)\}, \{\pounds_1(0.28), \pounds_2(0.28)\}\rangle$	$\langle\{\pounds_3(0.14), \pounds_4(0.14)\}, \{\pounds_1(0.14), \pounds_2(0.28), \pounds_3(0.42)\}\rangle$
	c_3	c_4
z_1	$\langle\{\pounds_{0.}(0.14), \pounds_1(0.28)\}, \{\pounds_3(0.28), \pounds_4(0.28)\}\rangle$	$\langle\{\pounds_3(0.14), \pounds_4(0.14)\}, \{\pounds_1(0.14), \pounds_2(0.14)\}\rangle$
z_2	$\langle\{\pounds_4(0.28), \pounds_5(0.28)\}, \{\pounds_0(0.42), \pounds_1(0.28)\}\rangle$	$\langle\{\pounds_5(0.28), \pounds_6(0.14)\}, \{\pounds_0(0.14)\}\rangle$
z_3	$\langle\{\pounds_4(0.28), \pounds_5(0.42)\}, \{\pounds_1(0.28), \pounds_2(0.14)\}\rangle$	$\langle\{\pounds_0(0.28), \pounds_1(0.42)\}, \{\pounds_2(0.28), \pounds_3(0.42)\}\rangle$
z_4	$\langle\{\pounds_1(0.42), \pounds_2(0.42)\}, \{\pounds_2(0.28), \pounds_3(0.42), \pounds_4(0.28)\}\rangle$	$\langle\{\pounds_4(0.42), \pounds_5(0.42)\}, \{\pounds_0(0.14)\}\rangle$
z_5	$\langle\{\pounds_1(0.28), \pounds_2(0.14)\}, \{\pounds_3(0.42), \pounds_4(0.28)\}\rangle$	$\langle\{\pounds_5(0.28), \pounds_6(0.28)\}, \{\pounds_0(0.28)\}\rangle$

Table 3.6: Probabilistic hesitant intuitionistic linguistic decision matrix H_3 with respect to DMs $6, 7\,(d_6, d_7)$

	c_1	c_2
z_1	$\langle\{\pounds_4\,(0.28)\,, \pounds_5\,(0.28)\}\,, \{\pounds_0\,(0.14)\,, \pounds_1\,(0.28)\}\rangle$	$\langle\{\pounds_5\,(0.42)\,, \pounds_6\,(0.28)\}\,, \{\pounds_0\,(0.42)\}\rangle$
z_2	$\langle\{\pounds_3\,(0.14)\,, \pounds_4\,(0.14)\}\,, \{\pounds_1\,(0.14)\,, \pounds_2\,(0.28)\,, \pounds_3\,(0.42)\}\rangle$	$\langle\{\pounds_1\,(0.28)\,, \pounds_2\,(0.28)\}\,, \{\pounds_3\,(0.28)\,, \pounds_4\,(0.28)\}\rangle$
z_3	$\langle\{\pounds_1\,(0.14)\,, \pounds_2\,(0.14)\}\,, \{\pounds_2\,(0.28)\,, \pounds_3\,(0.14)\,, \pounds_4\,(0.14)\}\rangle$	$\langle\{\pounds_5\,(0.28)\,, \pounds_6\,(0.14)\}\,, \{\pounds_0\,(0.14)\}\rangle$
z_4	$\langle\{\pounds_4\,(0.14)\,, \pounds_5\,(0.42)\}\,, \{\pounds_1\,(0.28)\,, \pounds_2\,(0.14)\}\rangle$	$\langle\{\pounds_4\,(0.28)\,, \pounds_5\,(0.14)\}\,, \{\pounds_0\,(0.28)\,, \pounds_1\,(0.28)\}\rangle$
z_5	$\langle\{\pounds_3\,(0.14)\,, \pounds_4\,(0.28)\}\,, \{\pounds_0\,(0.28)\,, \pounds_1\,(0.28)\,, \pounds_2\,(0.28)\}\rangle$	$\langle\{\pounds_1\,(0.28)\,, \pounds_2\,(0.28)\}\,, \{\pounds_2\,(0.28)\,, \pounds_3\,(0.42)\,, \pounds_4\,(0.28)\}\rangle$

	c_3	c_4
z_1	$\langle\{\pounds_3\,(0.14)\,, \pounds_4\,(0.14)\}\,, \{\pounds_1\,(0.14)\,, \pounds_2\,(0.14)\}\rangle$	$\langle\{\pounds_0\,(0.14)\,, \pounds_1\,(0.28)\}\,, \{\pounds_3\,(0.28)\,, \pounds_4\,(0.28)\}\rangle$
z_2	$\langle\{\pounds_5\,(0.28)\,, \pounds_6\,(0.14)\}\,, \{\pounds_0\,(0.42)\}\rangle$	$\langle\{\pounds_3\,(0.14)\,, \pounds_4\,(0.28)\}\,, \{\pounds_1\,(0.28)\,, \pounds_2\,(0.28)\}\rangle$
z_3	$\langle\{\pounds_4\,(0.28)\,, \pounds_5\,(0.42)\}\,, \{\pounds_0\,(0.28)\,, \pounds_1\,(0.28)\}\rangle$	$\langle\{\pounds_0\,(0.28)\,, \pounds_1\,(0.42)\}\,, \{\pounds_3\,(0.42)\,, \pounds_4\,(0.28)\}\rangle$
z_4	$\langle\{\pounds_0\,(0.14)\,, \pounds_1\,(0.42)\,, \pounds_2\,(0.42)\}\,, \{\pounds_2\,(0.28)\,, \pounds_3\,(0.42)\}\rangle$	$\langle\{\pounds_3\,(0.28)\,, \pounds_4\,(0.42)\,, \pounds_5\,(0.42)\}\,, \{\pounds_1\,(0.28)\,, \pounds_2\,(0.28)\}\rangle$
z_5	$\langle\{\pounds_2\,(0.14)\,, \pounds_3\,(0.14)\}\,, \{\pounds_3\,(0.28)\,, \pounds_4\,(0.28)\}\rangle$	$\langle\{\pounds_6\,(0.28)\}\,, \{\pounds_0\,(0.28)\}\rangle$

Step 2: The decision matrix H in Table 3.7 is constructed by utilizing Tables 3.4–3.6.

Table 3.7: Decision matrix H

	c_1	c_2
z_1	$\langle\{\pounds_2(0.14), \pounds_4(0.28)\}, \{\pounds_1(0.28), \pounds_3(0.14)\}\rangle$	$\langle\{\pounds_6(0.28), \pounds_5(0.42)\}, \{\pounds_0(0.42), \pounds_0(0.42)\}\rangle$
z_2	$\langle\{\pounds_1(0.28), \pounds_3(0.14)\}, \{\pounds_4(0.14), \pounds_3(0.42)\}\rangle$	$\langle\{\pounds_2(0.28), \pounds_3(0.14)\}, \{\pounds_2(0.28), \pounds_3(0.28)\}\rangle$
z_3	$\langle\{\pounds_2(0.14), \pounds_0(0.14)\}, \{\pounds_1(0.28), \pounds_3(0.14)\}\rangle$	$\langle\{\pounds_2(0.14), \pounds_6(0.14)\}, \{\pounds_0(0.14), \pounds_3(0.14)\}\rangle$
z_4	$\langle\{\pounds_6(0.28), \pounds_5(0.42)\}, \{\pounds_0(0.28), \pounds_1(0.28)\}\rangle$	$\langle\{\pounds_2(0.14), \pounds_5(0.14)\}, \{\pounds_1(0.28), \pounds_3(0.14)\}\rangle$
z_5	$\langle\{\pounds_6(0.14), \pounds_6(0.14)\}, \{\pounds_0(0.28), \pounds_1(0.28)\}\rangle$	$\langle\{\pounds_3(0.14), \pounds_2(0.28)\}, \{\pounds_5(0.14), \pounds_3(0.42)\}\rangle$

	c_3	c_4
z_1	$\langle\{\pounds_1(0.28), \pounds_3(0.14)\}, \{\pounds_2(0.14), \pounds_3(0.28)\}\rangle$	$\langle\{\pounds_1(0.28), \pounds_3(0.14)\}, \{\pounds_2(0.14), \pounds_3(0.28)\}\rangle$
z_2	$\langle\{\pounds_4(0.28), \pounds_4(0.14)\}, \{\pounds_0(0.42), \pounds_0(0.42)\}\rangle$	$\langle\{\pounds_1(0.28), \pounds_3(0.14)\}, \{\pounds_0(0.14), \pounds_3(0.28)\}\rangle$
z_3	$\langle\{\pounds_4(0.28), \pounds_5(0.42)\}, \{\pounds_0(0.28), \pounds_1(0.28)\}\rangle$	$\langle\{\pounds_1(0.14), \pounds_2(0.42)\}, \{\pounds_4(0.28), \pounds_3(0.42)\}\rangle$
z_4	$\langle\{\pounds_1(0.42), \pounds_2(0.42)\}, \{\pounds_{4.}(0.28), \pounds_3(0.42)\}\rangle$	$\langle\{\pounds_4(0.42), \pounds_5(0.42)\}, \{\pounds_0(0.14), \pounds_2(0.28)\}\rangle$
z_5	$\langle\{\pounds_1(0.28), \pounds_2(0.14)\}, \{\pounds_{4.}(0.28), \pounds_3(0.42)\}\rangle$	$\langle\{\pounds_5(0.28), \pounds_6(0.28)\}, \{\pounds_0(0.28), \pounds_1(0.14)\}\rangle$

Step 3: The normalized probabilistic hesitant intuitionistic linguistic decision matrix of the group is shown in Table 3.8.

Table 3.8: The normalized probabilistic hesitant intuitionistic linguistic decision matrix

	c_1	c_2
z_1	$\langle\{\pounds_4(0.6667), \pounds_2(0.3333)\}, \{\pounds_3(0.3333), \pounds_1(0.6667)\}\rangle$	$\langle\{\pounds_5(0.6), \pounds_6(0.4)\}, \{\pounds_0(0.5), \pounds_0(0.5)\}\rangle$
z_2	$\langle\{\pounds_3(0.3333), \pounds_1(0.6667)\}, \{\pounds_3(0.75), \pounds_4(0.25)\}\rangle$	$\langle\{\pounds_3(0.3333), \pounds_2(0.6667)\}, \{\pounds_3(0.5), \pounds_2(0.5)\}\rangle$
z_3	$\langle\{\pounds_0(0.5), \pounds_2(0.5)\}, \{\pounds_3(0.3333), \pounds_1(0.6667)\}\rangle$	$\langle\{\pounds_6(0.5), \pounds_2(0.5)\}, \{\pounds_3(0.5), \pounds_0(0.5)\}\rangle$
z_4	$\langle\{\pounds_5(0.6), \pounds_6(0.4)\}, \{\pounds_1(0.5), \pounds_0(0.5)\}\rangle$	$\langle\{\pounds_5(0.5), \pounds_2(0.5)\}, \{\pounds_3(0.3333), \pounds_1(0.6667)\}\rangle$
z_5	$\langle\{\pounds_6(0.5), \pounds_6(0.5)\}, \{\pounds_0(0.5), \pounds_1(0.5)\}\rangle$	$\langle\{\pounds_2(0.6667), \pounds_3(0.3333)\}, \{\pounds_3(0.75), \pounds_5(0.25)\}\rangle$

	c_3	c_4
z_1	$\langle\{\pounds_3(0.3333), \pounds_1(0.6667)\}, \{\pounds_3(0.6667), \pounds_2(0.3333)\}\rangle$	$\langle\{\pounds_3(0.3333), \pounds_1(0.6667)\}, \{\pounds_3(0.6667), \pounds_2(0.3333)\}\rangle$
z_2	$\langle\{\pounds_4(0.6667), \pounds_4(0.3333)\}, \{\pounds_0(0.5), \pounds_0(0.5)\}\rangle$	$\langle\{\pounds_3(0.3333), \pounds_1(0.6667)\}, \{\pounds_3(0.6667), \pounds_0(0.3333)\}\rangle$
z_3	$\langle\{\pounds_5(0.6), \pounds_4(0.4)\}, \{\pounds_5(0.6), \pounds_4(0.4)\}\rangle$	$\langle\{\pounds_2(0.75), \pounds_1(0.25)\}, \{\pounds_3(0.6), \pounds_4(0.4)\}\rangle$
z_4	$\langle\{\pounds_1(0.5), \pounds_2(0.5)\}, \{\pounds_3(0.6), \pounds_{4.}(0.4)\}\rangle$	$\langle\{\pounds_5(0.5), \pounds_4(0.5)\}, \{\pounds_0(0.3333), \pounds_2(0.6667)\}\rangle$
z_5	$\langle\{\pounds_1(0.6667), \pounds_2(0.3333)\}, \{\pounds_3(0.6), \pounds_4(0.4)\}\rangle$	$\langle\{\pounds_6(0.5), \pounds_5(0.5)\}, \{\pounds_1(0.3333), \pounds_0(0.6667)\}\rangle$

Step 4: The weight vector is derived from Equation (3.2.14) as follows:

$w = (0.2715, 0.2219, 0.2445, 0.2621)^t$.

Step 5: The PHILTS-PIS "A^+" and the PHILTS-NIS "A^-" of each alternative are derived using Equations (3.3.1) and (3.3.2) as follows:

$A^+ = (\langle\{3,3\}, \{0,0\}\rangle, \langle\{3,2.4\}, \{0,0\}\rangle, \langle\{3,1.6\}, \{0,0\}\rangle, \langle\{3,2.5\}, \{0,0\}\rangle)$.

$A^- = (\langle\{0,0.661\}, \{2.25,1\}\rangle, \langle\{1,1\}, \{2.25,1.25\}\rangle, \langle\{.5,0.66\}, \{2,1.6\}\rangle, \langle\{1,0.2\}, \{2,1.6\}\rangle)$.

$D(z_1, A^+) = 2.1211$, $D(z_2, A^+) = 2.5516$, $D(z_3, A^+) = 2.9129$, $D(z_4, A^+) = 1.7999$, $D(z_5, A^+) = 1.6494$.

$D(z_1, A^-) = 2.0142$, $D(z_2, A^-) = 1.5861$, $D(z_3, A^-) = 1.6204$, $D(z_4, A^-) = 2.4056$, $D(z_5, A^-) = 2.2812$.

Step 7: Calculate $D_{\min}(z_i, A^+)$ and $D_{\max}(z_i, A^-)$ by Equations (3.3.5) and (3.3.6 as:

$D_{\min}(z_i, A^+) = 1.6494$, $D_{\max}(z_i, A^-) = 2.4050$.

Step 8: Determine the closeness coefficient of each alternative z_i by Equation (3.3.7) as:

$Cl(z_1) = -0.4486$, $Cl(z_2) = -0.8876$, $Cl(z_3) = -1.0924$, $Cl(z_4) = -0.0912$, $Cl(z_5) = -0.0519$.

Step 9: Sort the alternatives according to the ranking of $Cl(z_i)$ $(i = 1, 2, \ldots, 5)$: $z_5 > z_4 > z_1 > z_2 > z_3$, and thus, z_5(insurance company) is the best alternative.

3.4.2 The aggregation-based method for the considered case

We can also apply the aggregation-based method to attain the ranking of alternatives for the case study.

Step 1: Construct the probabilistic hesitant intuitionistic fuzzy decision matrices of the group as listed in Tables 3.4–3.6, and then aggregated and normalized as shown in Tables 3.7 and 3.8.

Step 2: Utilize Equation (3.2.14) to obtain the weight vector

$w = (0.2715, 0.2219, 0.2445, 0.2621)^t$.

Step 3: Derive the overall criteria value of each alternative z_i $(i = 1, 2, 3, 4, 5)$ by using Equation (3.3.9) as:

$\widetilde{Z_1}(w) = \langle \{\pounds_{1.8962}, \pounds_{0.5187}\}, \{\pounds_{1.2847}, \pounds_{0.5187}\} \rangle$,

$\widetilde{Z_2}(w) = \langle \{\pounds_{1.4074}, \pounds_{0.9776}\}, \{\pounds_{1.4679}, \pounds_{0.4934}\} \rangle$,

$\widetilde{Z_3}(w) = \{\pounds_{1.7923}, \pounds_{1.1256}\}, \{\pounds_{1.8096}, \pounds_{0.9915}\}$,

$\widetilde{Z_4}(w) = \langle \{\pounds_{2.1467}, \pounds_{1.642}\}, \{\pounds_{0.7977}, \pounds_{0.8886}\} \rangle$,

$\widetilde{Z_5}(w) = \langle \{\pounds_{2.0596}, \pounds_{1.8546}\}, \{\pounds_{1.0267}, \pounds_{0.8043}\} \rangle$.

Step 4: Compute the score of each criteria value $\widetilde{Z_i}(w)$ by Definition 6.2.1 as:

$E\left(\widetilde{Z_1}(w)\right) = \pounds_{3.1528}$, $E\left(\widetilde{Z_2}(w)\right) = \pounds_{3.1059}$, $E\left(\widetilde{Z_3}(w)\right) = \pounds_{3.0584}$, $E\left(\widetilde{Z_4}(w)\right) = \pounds_{4.0512}$, $E\left(\widetilde{Z_5}(w)\right) = \pounds_{5.8726}$.

Step 5: Compare the overall criteria values of alternatives according to the values of the score function. It is obvious, that $z_5 > z_4 > z_1 > z_2 > z_3$. Thus, again, we get the best alternative z_5.

3.5 Discussions and comparison

For the purpose of comparison, in this subsection, the case study is again solved by applying the TOPSIS method with traditional HIFLTSs.

Step 1: The decision matrix X in Table 3.9 is constructed by utilizing Tables 3.1–3.3 as follows:

Table 3.9: Decision matrix X

	c_1	c_2	c_3	c_4
z_1	$([\pounds_2,\pounds_4],[\pounds_1,\pounds_3])$	$([\pounds_5,\pounds_5],[\pounds_0,\pounds_0])$	$([\pounds_1,\pounds_3],[\pounds_2,\pounds_3])$	$([\pounds_1,\pounds_3],[\pounds_2,\pounds_3])$
z_2	$([\pounds_1,\pounds_3],[\pounds_3,\pounds_3])$	$([\pounds_2,\pounds_3],[\pounds_2,\pounds_3])$	$([\pounds_4,\pounds_5],[\pounds_0,\pounds_0])$	$([\pounds_4,\pounds_5],[\pounds_0,\pounds_1])$
z_3	$([\pounds_2,\pounds_4],[\pounds_1,\pounds_2])$	$([\pounds_3,\pounds_5],[\pounds_0,\pounds_3])$	$([\pounds_5,\pounds_5],[\pounds_0,\pounds_1])$	$([\pounds_1,\pounds_1],[\pounds_3,\pounds_3])$
z_4	$([\pounds_5,\pounds_5],[\pounds_0,\pounds_1])$	$([\pounds_2,\pounds_4],[\pounds_1,\pounds_3])$	$([\pounds_1,\pounds_2],[\pounds_3,\pounds_3])$	$([\pounds_4,\pounds_5],[\pounds_1,\pounds_2])$
z_5	$([\pounds_4,\pounds_6],[\pounds_0,\pounds_1])$	$([\pounds_2,\pounds_3],[\pounds_3,\pounds_3])$	$([\pounds_1,\pounds_2],[\pounds_3,\pounds_3])$	$([\pounds_5,\pounds_6],[\pounds_0,\pounds_1])$

Step 2: Determine the HIFLTS-PIS "P^+" and the HIFLTS-NIS "P^-" for cost criteria c_1,c_4 and benefit criteria c_2, c_3 as follows:

$P^+ = [([\pounds_0,\pounds_1],[\pounds_3,\pounds_4]),([\pounds_5,\pounds_6],[\pounds_0,\pounds_0]),([\pounds_5,\pounds_6],[\pounds_0,\pounds_0]),([\pounds_0,\pounds_1],[\pounds_3,\pounds_4])]$,

$P^- = [([\pounds_6,\pounds_6],[\pounds_0,\pounds_0]),([\pounds_1,\pounds_2],[\pounds_3,\pounds_5]),([\pounds_0,\pounds_1],[\pounds_3,\pounds_4]),([\pounds_6,\pounds_6],[\pounds_0,\pounds_0])]$.

Note: One can see the detail of HIFLTS-PIS "P^+" and the HIFLTS-NIS "P^-" in [29].

Step 3: Calculate the positive ideal matrix D^+ and the negative ideal matrix D^- as follows:

$$D^+ = \begin{bmatrix} 8+1+12+5 \\ 4+11+2+14 \\ 9+7+2+2 \\ 15+9+14+12 \\ 15+12+14+16 \end{bmatrix} = \begin{bmatrix} 26 \\ 31 \\ 20 \\ 50 \\ 57 \end{bmatrix}$$

$D^+_{11} = d\left(z_{11}, v_1^+\right) + d\left(z_{12}, v_2^+\right) + d\left(z_{13}, v_3^+\right) + d\left(z_{14}, v_4^+\right)$, in which $d\left(z_{11}, v_1^+\right) = d\left(([\pounds_2,\pounds_4],[\pounds_1,\pounds_3])\right.$

$, ([\pounds_0, \pounds_1], [\pounds_3, \pounds_4])) = |2-0| + |4-1| + |1-3| + |3-4| = 8.$

Other entries can be found by similar calculation.

$$D^- = \begin{bmatrix} 10+15+5+13 \\ 14+5+15+4 \\ 9+9+15+16 \\ 3+7+3+6 \\ 3+4+3+2 \end{bmatrix} = \begin{bmatrix} 43 \\ 38 \\ 49 \\ 19 \\ 12 \end{bmatrix}.$$

Step 4: The relative closeness (RC) of each alternative to the ideal solution can be obtained as follows:

$RC(z_1) = 43/(26+43) = 0.6232$, $RC(z_2) = 38/(31+38) = 0.5507$.

The RC of other alternatives can be find by similar calculations.

$RC(z_3) = 0.7101$, $RC(z_4) = 0.2754$, $RC(z_5) = 0.1739$.

Step 5: The ranking of alternatives of alternatives $z_i\,(i = 1, 2, \ldots, 5)$ according to the closeness coefficient $RC(z_i)$ is $z_3 > z_1 > z_2 > z_4 > z_5$.

- In Table 3.9, the disadvantages of HIFLTS are apparent because in HIFLTS the probabilities of the linguistic terms is not considered which means that all possible linguistic terms in HIFLTS have same occurrence possibility which is unrealistic, whereas the inspection of Table 3.7 shows that PHILTS not only contains the linguistic terms, but also considers the probabilities of linguistic terms, and, thus, PHILTS constitutes an extension of HIFLTS.

- The inspection of Table 3.10 reveals that the extended TOPSIS method and the aggregation-based method give the same best alternative z_5. The TOPSIS method with the traditional HIFLTSs gives z_3 as the best alternative.

- This difference of best alternative in Table 3.10 is due to the effect of probabilities of membership and non-membership linguistic terms, which highlight the critical role of probabilities. Thus, our methods are more rational to get the ranking of alternatives and further to find the best alternative.

- Extended TOPSIS method and aggregation-based method for MCGDM with PLTS information explained in [30] are more promising and better than extended TOPSIS method and aggregation-based method for MCGDM with HFLTS information. However, a clear superiority of PHILTS is that it assigns to each element the degree of belongingness and also the degree of non-belongingness along with probability. PLTS only assigns to each element a belongingness degree along with probability. Using PLTSs, various frameworks have been developed by DMs [30, 43] but they are still intolerant, since there is no mean of attributing reliability or confidence information to the degree of belongingness.

Table 3.10: Comparison of Results

TOPSIS [29]	$z_3 > z_1 > z_2 > z_4 > z_5$
Proposed extend TOPSIS	$z_5 > z_4 > z_1 > z_2 > z_3$
Proposed aggregation model	$z_5 > z_4 > z_1 > z_2 > z_3$

The comparisons and other aspects are summarized in Table 3.11.

Table 3.11: The advantages and limitations of the proposed methods

Advantages	**Limitations**
1. PHILTS generalize the existing PLTS models since PHILTS take more information from the DMs into account. 2. PHILTS is not affected by partial vagueness. 3. PHILTS is more in line with people's language, leading to much more fruitful decisions. 4. The criteria weights are calculated with objectivity (without favor).	1. It is essential to take membership as well as non-membership probabilistic data. 2. Its computational index is high.

Chapter 4

Consensus-based robust decision making methods under a novel study of probabilistic uncertain linguistic information and their application in Forex investment

The current chapter focuses on the generalization of aggregation formula and the derivation of adjusting rule of probability to adjust the probability distribution of two or more than two PULEs. Novel operations, comparative method, distance measure and aggregation operators are studied for PULTSs based on linguistic scale function. To suit the needs of different semantics, two robust decision making methods, such as consensus-based PUL-gained and lost dominance score method and consensus-based PUL aggregation method are presented along with the application. The research work of this chapter is published in [47].

4.1 Aggregation formula and adjusting rule of probability

This section presents aggregation formula and adjusting rule of probability for PULTSs.

4.1.1 Aggregation formula

It is demanding to aggregate the PULTS given by DMs to get an overall assessment of a group. Motivated by [48] in the following an aggregation formula is developed to integrate DMs opinions which are expressed in terms of PULEs to group assessment, taking into account both the distribution of uncertain linguistic terms and weights of DMs.

Definition 4.1.1. *Let $D = \{d_t | t = 1, 2, ..., T\}$ be a set of DMs whose weight vector is $\pi = (\pi^{(1)}, \pi^{(2)}, ..., \pi^{(T)})^t$ such that $\sum_{t=1}^{T} \pi^{(t)} = 1$ and S be an LTS. Assume that $u_s^t(p) = \left\{ \left\langle [\pounds^{t(\jmath)}, U^{t(\jmath)}], p^{t(\jmath)} \right\rangle | \jmath = 1, 2, ..., \pounds \right\}$ $(t = 1, 2, ..., Q)$ are PULEs on S stated by 'Q' DMs (there are $(T - Q)$ DMs who do not give any opinion). Then the overall assessment of DMs group is given by a PULE as follows:*

$$u_s(p) = \left\{ \left\langle [\pounds^{(\jmath)}, U^{(\jmath)}], p^{(\jmath)} \right\rangle, p^{(\jmath)} = \sum_{t=1}^{Q} w^{t(\jmath)} \pi^t, \jmath = 1, 2, ..., \pounds_1 \right\}, \tag{4.1.1}$$

where $w^{t(\jmath)}$ denotes the probability of $[\pounds^{t(\jmath)}, U^{t(\jmath)}]$ in $u_s^t(p)$ and

$$w^{t(\jmath)} = \begin{cases} p^{t(\jmath)}, & \text{if } [\pounds^{t(\jmath)}, U^{t(\jmath)}] \in u_s^t(p) \\ 0, & \text{otherwise.} \end{cases} \tag{4.1.2}$$

Remark 4.1.2. *With Eq. (4.1.1) opinions provided by DMs in uncertain linguistic terms can be aggregated into the PULEs. If the DMs opinions are expressed in PULEs then with Eq. (4.1.1) one can also obtain the overall PULE, in this case, actual probabilities are utilized to replace the value of $w^{t(\jmath)}$. If weights of DMs are equal, we can assume that $\pi^t = \frac{1}{T}$; $t =$*

$1, 2, ..., T$.

Example 4.1.3. *All experts opinions in Example 2.1.12 are integrated with Eq. (4.1.1) as* $u_s(p) = \{\langle[\pounds_0, \pounds_1], 0.6\rangle, \langle[\pounds_1, \pounds_2], 0.233\rangle, \langle[\pounds_2, \pounds_3], 0.167\rangle\}$.

It represents the original group judgments in full extent.

4.1.2 Adjusting rule of probability for 'n' PULEs

Uncertain linguistic terms and associated probabilities of two or more PULEs are always different, which create problems in operational laws, distance measure and correlation measure. Multiplying the indices of the uncertain linguistic terms along with their associated probability like Lin et al. [32] or taking the average of probabilities [49] are irrational and biased results are obtained. To get satisfactory results, a novel rule is introduced to adjust the probability distribution and equalize the length of two or more than two PULEs into the same probability distribution.

Let $S = \{\pounds_\alpha | \alpha = -\tau, ... -1, 0, 1, ..., \tau\}$ be an LTS, $\widetilde{u}_s^1(p) = \left\{ \left\langle [\pounds^{1(\jmath)}, U^{1(\jmath)}], \widetilde{p}^{1(\jmath)} \right\rangle | \jmath = 1, 2, ..., \pounds_1 \right\}, \widetilde{u}_s^2(p) = \left\{ \left\langle [\pounds^{2(\jmath)}, U^{2(\jmath)}], \widetilde{p}^{2(\jmath)} \right\rangle | \jmath = 1, 2, ..., \pounds_2 \right\}, \widetilde{u}_s^3(p) = \left\{ \left\langle [\pounds^{3(\jmath)}, U^{3(\jmath)}], \widetilde{p}^{3(\jmath)} \right\rangle | \jmath = 1, 2, ..., \pounds_3 \right\}, ..., \widetilde{u}_s^n(p) = \left\{ \left\langle [\pounds^{n(\jmath)}, U^{n(\jmath)}], \widetilde{p}^{n(\jmath)} \right\rangle | \jmath = 1, 2, ..., \pounds_n \right\}$ be n associated PULEs. Assume that re-arranged probability distribution set of $\widetilde{u}_s^1(p), \widetilde{u}_s^2(p), ..., \widetilde{u}_s^n(p)$ is $p = \left\{p^{*(1)}, p^{*(2)}, ..., p^{*(E)}\right\}^t$. Then the adjusted PULEs are

$u_s^{*1}(p) = \left\{ \left\langle [\pounds^{1(e)}, U^{1(e)}], p^{*(e)} \right\rangle | e = 1, 2, ..., E \right\}$,
$u_s^{*2}(p) = \left\{ \left\langle [\pounds^{2(e)}, U^{2(e)}], p^{*(e)} \right\rangle | e = 1, 2, ..., E \right\}$,
$\vdots$
$u_s^{*n}(p) = \left\{ \left\langle [\pounds^{n(e)}, U^{n(e)}], p^{*(e)} \right\rangle | e = 1, 2, ..., E \right\}$,

- $p^{*(1)} = \min\{\widetilde{p}^{11}, \widetilde{p}^{21}, \widetilde{p}^{31}, ..., \widetilde{p}^{n1}\}$
  if $p^{*(1)} = \widetilde{p}^{11}$,
  $p^{*(2)} = \min\{\widetilde{p}^{12}, \widetilde{p}^{21} - p^{*(1)}, \widetilde{p}^{31} - p^{*(1)}, ..., \widetilde{p}^{n1} - p^{*(1)}\}$

if $p^{*(1)} = \widetilde{p}^{21}$,

$p^{*(2)} = \min\{\widetilde{p}^{11} - p^{*(1)}, \widetilde{p}^{22}, \widetilde{p}^{31} - p^{*(1)}, ..., \widetilde{p}^{n1} - p^{*(1)}\}$

$\vdots$

if $p^{*(1)} = \widetilde{p}^{n1}$,

$p^{*(2)} = \min\{\widetilde{p}^{11} - p^{*(1)}, \widetilde{p}^{21} - p^{*(1)}, \widetilde{p}^{31} - p^{*(1)}, ..., \widetilde{p}^{(n-1)1} - p^{*(1)}, \widetilde{p}^{n2}\}$.

➤ If $p^{*(1)} = \widetilde{p}^{11}$ and $p^{*(2)} = \widetilde{p}^{12}$

$p^{*(3)} = \min\{\widetilde{p}^{13}, \widetilde{p}^{21} - p^{*(1)} - p^{*(2)}, \widetilde{p}^{31} - p^{*(1)} - p^{*(2)}, ..., \widetilde{p}^{n1} - p^{*(1)} - p^{*(2)}\}$

if $p^{*(1)} = \widetilde{p}^{11}$ and $p^{*(2)} = \widetilde{p}^{21} - p^{*(1)}$,

$p^{*(3)} = \min\{\widetilde{p}^{12} - p^{*(2)}, \widetilde{p}^{22}, \widetilde{p}^{31} - p^{*(1)} - p^{*(2)}, ..., \widetilde{p}^{n1} - p^{*(1)} - p^{*(2)}\}$

if $p^{*(1)} = \widetilde{p}^{11}$ and $p^{*(2)} = \widetilde{p}^{31} - p^{*(1)}$,

$p^{*(3)} = \min\{\widetilde{p}^{12} - p^{*(2)}, \widetilde{p}^{21} - p^{*(1)} - p^{*(2)}, \widetilde{p}^{32}, \widetilde{p}^{41} - p^{*(1)} - p^{*(2)} ..., \widetilde{p}^{n1} - p^{*(1)} - p^{*(2)}\}$

$\vdots$

if $p^{*(1)} = \widetilde{p}^{11}$ and $p^{*(2)} = \widetilde{p}^{n1} - p^{*(1)}$,

$p^{*(3)} = \min\{\widetilde{p}^{12} - p^{*(2)}, \widetilde{p}^{21} - p^{*(1)} - p^{*(2)}, \widetilde{p}^{31} - p^{*(1)} - p^{*(2)}, ..., \widetilde{p}^{(n-1)1} - p^{*(1)} - p^{*(2)}, \widetilde{p}^{n2}\}$.

➤ If $p^{*(1)} = \widetilde{p}^{21}$ and $p^{*(2)} = \widetilde{p}^{22}$

$p^{*(3)} = \min\{\widetilde{p}^{11} - p^{*(1)} - p^{*(2)}, \widetilde{p}^{23}, \widetilde{p}^{31} - p^{*(1)} - p^{*(2)}, ..., \widetilde{p}^{n1} - p^{*(1)} - p^{*(2)}\}$

if $p^{*(1)} = \widetilde{p}^{11}$ and $p^{*(2)} = \widetilde{p}^{21} - p^{*(1)}$,

$p^{*(3)} = \min\{\widetilde{p}^{12} - p^{*(2)}, \widetilde{p}^{22}, \widetilde{p}^{31} - p^{*(1)} - p^{*(2)}, ..., \widetilde{p}^{n1} - p^{*(1)} - p^{*(2)}\}$

if $p^{*(1)} = \widetilde{p}^{21}$ and $p^{*(2)} = \widetilde{p}^{11} - p^{*(1)}$,

$p^{*(3)} = \min\{\widetilde{p}^{12}, \widetilde{p}^{22} - p^{*(2)}, \widetilde{p}^{31} - p^{*(1)} - p^{*(2)}, ..., \widetilde{p}^{n1} - p^{*(1)} - p^{*(2)}\}$

if $p^{*(1)} = \widetilde{p}^{21}$ and $p^{*(2)} = \widetilde{p}^{31} - p^{*(1)}$,

$p^{*(3)} = \min\{\widetilde{p}^{11} - p^{*(1)} - p^{*(2)}, \widetilde{p}^{22} - p^{*(2)}, \widetilde{p}^{32}, \widetilde{p}^{41} - p^{*(1)} - p^{*(2)} ..., \widetilde{p}^{n1} - p^{*(1)} - p^{*(2)}\}$

$\vdots$

if $p^{*(1)} = \widetilde{p}^{21}$ and $p^{*(2)} = \widetilde{p}^{n1} - p^{*(1)}$,

$p^{*(3)} = \min\{\widetilde{p}^{11} - p^{*(1)} - p^{*(2)}, \widetilde{p}^{22} - p^{*(2)}, \widetilde{p}^{31} - p^{*(1)} - p^{*(2)}, ..., \widetilde{p}^{(n-1)1} - p^{*(1)} - p^{*(2)}, \widetilde{p}^{n2}\}$.

➤ If $p^{*(1)} = \widetilde{p}^{31}$ and $p^{*(2)} = \widetilde{p}^{32}$

$p^{*(3)} = \min\{\widetilde{p}^{11} - p^{*(1)} - p^{*(2)}, \widetilde{p}^{21} - p^{*(1)} - p^{*(2)}, \widetilde{p}^{33}, \widetilde{p}^{41} - p^{*(1)} - p^{*(2)}, ..., \widetilde{p}^{n1} - p^{*(1)} - p^{*(2)}\}$
if $p^{*(1)} = \widetilde{p}^{31}$ and $p^{*(2)} = \widetilde{p}^{11} - p^{*(1)}$,
$p^{*(3)} = \min\{\widetilde{p}^{12}, \widetilde{p}^{21} - p^{*(1)} - p^{*(2)}, \widetilde{p}^{32} - p^{*(2)}, \widetilde{p}^{41} - p^{*(1)} - p^{*(2)} ..., \widetilde{p}^{n1} - p^{*(1)} - p^{*(2)}\}$
if $p^{*(1)} = \widetilde{p}^{31}$ and $p^{*(2)} = \widetilde{p}^{21} - p^{*(1)}$,
$p^{*(3)} = \min\{\widetilde{p}^{11} - p^{*(1)} - p^{*(2)}, \widetilde{p}^{22}, \widetilde{p}^{32} - p^{*(2)}, \widetilde{p}^{41} - p^{*(1)} - p^{*(2)}, ..., \widetilde{p}^{n1} - p^{*(1)} - p^{*(2)}\}$
if $p^{*(1)} = \widetilde{p}^{31}$ and $p^{*(2)} = \widetilde{p}^{n1} - p^{*(1)}$,
$p^{*(3)} = \min\{\widetilde{p}^{11} - p^{*(1)} - p^{*(2)}, \widetilde{p}^{21} - p^{*(1)} - p^{*(2)}, \widetilde{p}^{32} - p^{*(2)}, \widetilde{p}^{41} - p^{*(1)} - p^{*(2)} ..., \widetilde{p}^{n-1} - p^{*(1)} - p^{*(2)}, \widetilde{p}^{n2}\}$
$\vdots$
$p^{*(E)} = \min\{\widetilde{p}^{1\pounds_1}, \widetilde{p}^{*(2\pounds_2)}, ..., \widetilde{p}^{*(n\pounds_n)}\}$.

Following this pattern one can get adjusted probability distribution $p = \{p^{*(1)}, p^{*(2)}, ..., p^{*(E)}\}$. Furthermore, they must satisfy $\widetilde{p}^{1\jmath} = p^{*(e)} + p^{*(e+1)} + ... + p^{*(e+r)}$ too, where $\widetilde{p}^{1(\jmath)}$ is the probability of $[\pounds^{1(\jmath)}, U^{1(\jmath)}]$ in $\widetilde{u}_s^1(p)$ and $p^{*(e)}, p^{*(e+1)}, ..., p^{*(e+r)}$ are the probabilities of $[\pounds^{1(\jmath)}, U^{1(\jmath)}]$ in $u_s^{*1}(p)$. r denotes the number of linguistic terms in $u_s^{*1}(p)$. Similarly, the elements in $u_s^{*2}(p), u_s^{*3}(p), ..., u_s^{*n}(p)$ also meet this condition. The uncertain linguistic terms and sum of probability remain unaltered in the adjusted PULEs.

Note: A PULTS is said to be adjusted PULTS if all of its PULEs are adjusted.

Next example is provided for a better understanding.

Example 4.1.4. *Consider the following four associated PULEs.*
$\widetilde{u}_s^1(p) = \big\{\langle[\pounds_0, \pounds_1], 0.3\rangle, \langle[\pounds_1, \pounds_2], 0.2\rangle, \langle[\pounds_2, \pounds_3], 0.5\rangle\big\}$, $\widetilde{u}_s^2(p) = \big\{\langle[\pounds_{-2}, \pounds_{-1}], 0.4\rangle, \langle[\pounds_1, \pounds_2], 0.6\rangle\big\}$, $\widetilde{u}_s^3(p) = \big\{\langle[\pounds_2, \pounds_3], 1\rangle\big\}$, $\widetilde{u}_s^4(p) = \big\{\langle[\pounds_{-1}, \pounds_0], 0.4\rangle, \langle[\pounds_1, \pounds_2], 0.5\rangle, \langle[\pounds_3, \pounds_4], 0.1\rangle\big\}$. *Take*
$p^{*1} = \min\big(0.3, 0.4, 1, 0.4\big) = 0.3$, $p^{*2} = \big(0.2, 0.1, 0.7, 0.1\big) = 0.1$, $p^{*3} = \big(0.1, 0.6, 0.6, 0.5\big) = 0.1$, $p^{*4} = \big(0.5, 0.5, 0.5, 0.4\big) = 0.4$, $p^{*5} = \big(0.1, 0.1, 0.1, 0.1\big) = 0.1$.
The adjusted PULEs are $\widetilde{u}_s^{*1}(p) = \big\{\langle[\pounds_0, \pounds_1], 0.3\rangle, \langle[\pounds_1, \pounds_2], 0.1\rangle, \langle[\pounds_1, \pounds_2], 0.1\rangle, \langle[\pounds_2, \pounds_4], 0.4\rangle, \langle[\pounds_2, \pounds_3], 0.1\rangle\big\}$, $\widetilde{u}_s^{*2}(p) = \big\{\langle[\pounds_{-2}, \pounds_{-1}], 0.3\rangle, \langle[\pounds_{-2}, \pounds_{-1}], 0.1\rangle, \langle[\pounds_1, \pounds_2], 0.1\rangle, \langle[\pounds_1, \pounds_2], 0.4\rangle, \langle[\pounds_1, \pounds_2], 0.1\rangle\big\}$, $\widetilde{u}_s^{*3}(p) = \big\{\langle[\pounds_2, \pounds_3], 0.3\rangle, \langle[\pounds_2, \pounds_3], 0.1\rangle, \langle[\pounds_2, \pounds_3], 0.1\rangle, \langle[\pounds_2, \pounds_3], 0.4\rangle, \langle[\pounds_2, \pounds_3], 0.1\rangle\big\}$, $\widetilde{u}_s^{*4}(p) = \big\{\langle[\pounds_{-1}, \pounds_0], 0.3\rangle, \langle[\pounds_{-1}, \pounds_0], 0.1\rangle, \langle[\pounds_1, \pounds_2], 0.1\rangle, \langle[\pounds_1, \pounds_2], 0.4\rangle, \langle[\pounds_3, \pounds_4], 0.1\rangle\big\}$.

4.2 Novel operations and comparison

This section concentrates on designing some novel operations, studying their properties and comparison of PULEs based upon adjusting rule of probability and linguistic scale function.

4.2.1 Novel operations

Operational laws cannot be defined effectively when integrating the indices of uncertain linguistic terms directly along with corresponding probabilities. It is bothersome to handle PULEs in comparison with other linguistic expression models as there is one extra dimension, i.e., probabilities along with uncertain linguistic terms. Most of the scholars [32, 49] aggregate the indices of the uncertain linguistic terms with associated probabilities directly. However, these operational laws have some shortcomings. Here we only take the additive operation between two ordered and normalized PULEs given in Definition 2.4.2 and provide an example to demonstrate its limitations.

Example 4.2.1. *Let* $S = \{\pounds_\alpha | \alpha = -3, ..., -1, 0, 1, ..., 3\}$ *be an LTS, and let* $u_s^1(p) = \big\{\langle[\pounds_1, \pounds_2], 0.5\rangle, \langle[\pounds_2, \pounds_3], 0.5\rangle\big\}$, $u_s^2(p) = \big\{\langle[\pounds_2, \pounds_3], 0.8\rangle, \langle[\pounds_{-1}, \pounds_0], 0.2\rangle\big\}$ *according to additive operation of [32], we get*

$$u_s^1(p) \oplus u_s^2(p) = \big\{[\pounds_{2.1}, \pounds_{3.4}], [\pounds_{0.8}, \pounds_{1.5}]\big\}. \tag{4.2.1}$$

Clearly, the result of Eq. (4.2.1) not only loses the probability information but also the linguistic terms $\pounds_{3.4}$ cross the boundary of the boundary $[\pounds_{-3}, \pounds_3]$. The major flaws of existing operations are summarized as follows.

i. It is not very meaningful to operate the indices of linguistic terms with their corresponding probabilities because these dimensions have absolutely different meanings.

ii. The results obtained after operations cannot reflect the probabilities of uncertain linguistic terms.

iii. The resultant linguistic terms may cross the boundary of provided LTS while directly operating indices of the linguistic terms.

iv. The existing operations can be operated only on normalized PULEs. In the normalization process, we need to add an uncertain linguistic term to the smallest one. The added artificial uncertain linguistic terms would lead to inaccurate calculation and different techniques may obtain different results for adding uncertain linguistic terms.

v. The existing operations fail to handle the case of unbalanced linguistic terms.

In order to stop losing of probability, recently, [49] designed some interesting operations of PULEs by taking an average of the corresponding probabilities.

With this innovation flaw 2 is resolved but the other flaws are still unresolved. In order to avoid all these flaws, novel operations are defined below.

Definition 4.2.2. *Let* $u_s^{*1}(p) = \big\{\big\langle[\pounds^{1(e)}, U^{1(e)}], p^{*(e)}\big\rangle\,|e = 1, 2, ..., E\big\}$ *and* $u_s^{*2}(p) = \big\{\,\big\langle[\pounds^{2(e)}, U^{2(e)}], p^{*(e)}\big\rangle\,|e = 1, 2, ..., E\big\}$ *be two adjusted PULEs and* ℓ, ℓ^{-1} *be the equivalent linguistic scale functions and* $\lambda \geq 0$*; Then:*

i. $u_s^1(p) \oplus u_s^2(p) = u_s^{*1}(p) \oplus u_s^{*2}(p) = \big\{\big\langle[\ell^{-1}(\ell(\pounds^{1(e)}) + \ell(\pounds^{2(e)})), \ell^{-1}(\ell(U^{1(e)}) + \ell(U^{2(e)}))], p^{*(e)}\big\rangle|e = 1, 2, ..., E\big\}$;

ii. $u_s^1(p) \otimes u_s^2(p) = u_s^{*1}(p) \otimes u_s^{*2}(p) = \big\{\big\langle[\ell^{-1}(\ell(\pounds^{1(e)})\ell(\pounds^{2(e)})) \ell^{-1}(\ell(U^{1(e)})\ell(U^{2(e)}))], p^{*(e)}\big\rangle|e = 1, 2, ..., E\big\}$;

iii. $\lambda u_s^1(p) = \big\{\big\langle[\ell^{-1}(\lambda(\ell(\pounds^{1(e)}))), \ell^{-1}(\lambda(\ell(U^{1(e)})))], p^{*(e)}\big\rangle\,|e = 1, 2, ..., E\big\}$;

iv. $\big(u_s^1(p)\big)^\lambda = \big\{\big\langle[\ell^{-1}((\ell(\pounds^{1(e)}))^\lambda), \ell^{-1}((\ell(U^{1(e)}))^\lambda)], p^{*(e)}\big\rangle\,|e = 1, 2, ..., E\big\}$.

In what follows, some interesting properties of the novel operations are presented.

Theorem 4.2.3. *Let* $S = \{\pounds_\alpha|\alpha = -\tau, ..., -1, 0, 1, ..., -\tau\}$ *be an LTS,* $u_s^*(p), u_s^{*1}(p), u_s^{*2}(p)$ *be any three adjusted PULEs and* $\lambda, \lambda_1, \lambda_2$ *be three positive real numbers. Then:*

i. $u_s^{*1}(p) \oplus u_s^{*2}(p) = u_s^{*2}(p) \oplus u_s^{*1}(p)$;

ii. $\big(u_s^{*}(p) \oplus u_s^{*1}(p)\big) \oplus u_s^{*2}(p) = u_s^{*}(p) \oplus \big(u_s^{*1}(p) \oplus u_s^{*2}(p)\big)$;

iii. $\lambda\big(u_s^{*1}(p) \oplus u_s^{*2}(p)\big) = \lambda u_s^{*1}(p) \oplus \lambda u_s^{*2}(p)$;

iv. $\big(\lambda_1 + \lambda_2\big)u_s^{*}(p) = \lambda_1 u_s^{*}(p) \oplus \lambda_2 u_s^{*}(p)$;

v. $u_s^{*1}(p) \otimes u_s^{*2}(p) = u_s^{*2}(p) \otimes u_s^{*1}(p)$;

vi. $\big(u_s^{*}(p) \otimes u_s^{*1}(p)\big) \otimes u_s^{*2}(p) = u_s^{*}(p) \otimes \big(u_s^{*1}(p) \otimes u_s^{*2}(p)\big)$;

vii. $\big(u_s^{*1}(p) \otimes u_s^{*2}(p)\big)^{\lambda} = \big(u_s^{*1}(p)\big)^{\lambda} \otimes \big(u_s^{*2}(p)\big)^{\lambda}$;

viii. $\big(u_s^{*}(p)\big)^{\lambda_1+\lambda_2} = \big(u_s^{*}(p)\big)^{\lambda_1} \oplus \big(u_s^{*}(p)\big)^{\lambda_2}$.

Proof. Listed items i and v can be proven easily. Therefore, their proofs are skipped. Selecting the linguistic scale function described in Eq. (2.1.2), we have

$$
\begin{aligned}
2.\ \big(u_s^*(p) \oplus u_s^{*1}(p)\big) \oplus u_s^{*2}(p) &= \big\{\big\langle[\ell^{-1}(\ell(\pounds^{1(e)}) + \ell(\pounds^{2(e)})), \ell^{-1}(\ell(U^{1(e)}) + \ell(U^{2(e)}))],\\
&p^{*(e)}\big\rangle|e = 1, 2, ..., E\big\} \oplus u_s^{*2}(p)\\
&= \big\{\big\langle[\ell^{-1}(\frac{\alpha^{(e)}+\tau}{2\tau} + \frac{\alpha^{(1e)}+\tau}{2\tau}), \ell^{-1}(\frac{\beta^{(e)}+\tau}{2\tau} + \frac{\beta^{(1e)}+\tau}{2\tau})],\\
&p^{*(e)}\big\rangle|e = 1, 2, ..., E\big\} \oplus u_s^{*2}(p)\\
&= \big\{\big\langle[\pounds_{\alpha^{(e)}+\alpha^{(1e)}+2\tau}, \pounds_{\beta^{(e)}+\beta^{(1e)}+2\tau}], p^{*(e)}\big\rangle\,|e = 1, 2, ..., E\big\}\\
&\oplus u_s^{*2}(p)\\
&= \Big\{\big\langle[\ell^{-1}\big(\ell(l_{\alpha^{(e)}+\alpha^{1(e)}+2\tau}) + \ell(\pounds^{2(e)})\big), \ell^{-1}\big(\ell(\pounds_{\beta^{(e)}+\beta^{1(e)}+2\tau})\\
&+ \ell(U^{2(e)})\big)], p^{*(e)}\big\rangle \mid e = 1, 2, ..., E\Big\} \oplus u_s^{*2}(p)\\
&= \Big\{\big\langle[\ell^{-1}(\frac{\alpha^{(e)} + \alpha^{1(e)} + \alpha^{2(e)} + 3\tau}{2\tau}), \ell^{-1}(\frac{\beta^{(e)+\beta^{1(e)}+\beta^{2(e)}} + 3\tau}{2\tau})],\\
&p^{*(e)}\big\rangle|e = 1, 2, ..., E\Big\} \oplus u_s^{*2}(p)\\
&= \big\{\big\langle[\pounds_{\alpha^{(e)}+\alpha^{1(e)}+\alpha^{2(e)}+2\tau}, \pounds_{\beta^{(e)}+\beta^{1(e)}+\beta^{2(e)}+2\tau}], p^{*(e)}\big\rangle\,|e = 1, 2, ..., E\big\}\,.
\end{aligned} \tag{4.2.2}
$$

Now

$$
\begin{aligned}
u_s^*(p) \oplus \big(u_s^{*1}(p) \oplus u_s^{*2}(p)\big) &= u_s^*(p) \oplus \big\{\big\langle[\ell^{-1}(\ell(\pounds^{1(e)}) + \ell(\pounds^{2(e)})), \ell^{-1}(\ell(U^{1(e)}) + \ell(U^{2(e)}))],\\
&p^{*(e)}\big\rangle|e = 1, 2, ..., E\big\}\\
&= u_s^*(p) \oplus \big\{\big\langle[\ell^{-1}(\frac{\alpha^{1(e)} + \tau}{2\tau} + \frac{\alpha^{(2e)} + \tau}{2\tau}), \ell^{-1}(\frac{\beta^{1(e)} + \tau}{2\tau} + \frac{\beta^{2(e)} + \tau}{2\tau})],\\
&p^{*(e)}\big\rangle|e = 1, 2, ..., E\big\}\\
&= u_s^*(p) \oplus \big\{\ \big\langle[\pounds_{\alpha^{1(e)}+\alpha^{2(e)}+2\tau}, \pounds_{\beta^{(e)}+\beta^{(1e)}+2\tau}], p^{*(e)}\big\rangle\\
&|e = 1, 2, ..., E\big\} \oplus u_s^{*2}(p)\\
&= \big\{\big\langle[\ell^{-1}\big(\ell(\pounds^{(e)}) + \ell(\pounds_{\alpha^{1(e)}+\alpha^{2(e)}+2\tau})\big), \ell^{-1}\big(\ell(\pounds^{(e)})+\\
&\ell(\pounds_{\beta^{1(e)}+\beta^{2(e)}+2\tau})\big)], p^{*(e)}\big\rangle|e = 1, 2, ..., E\big\} \oplus u_s^{*2}(p)\\
&= \big\{\big\langle[\ell^{-1}(\frac{\alpha^{(e)} + \alpha^{1(e)} + \alpha^{2(e)} + 3\tau}{2\tau}), \ell^{-1}(\frac{\beta^{(e)+\beta^{1(e)}+\beta^{2(e)}} + 3\tau}{2\tau})],\\
&p^{*(e)}\big\rangle|e = 1, 2, ..., E\big\} \oplus u_s^{*2}(p)\\
&= \big\{\big\langle[\pounds_{\alpha^{(e)}+\alpha^{1(e)}+\alpha^{2(e)}+2\tau}, \pounds_{\beta^{(e)}+\beta^{1(e)}+\beta^{2(e)}+2\tau}], p^{*(e)}\big\rangle\ |e = 1, 2, ..., E\big\}\ .
\end{aligned}
\tag{4.2.3}
$$

From (4.2.2) and (4.2.3) the desired proof is obtained.

3. $\lambda u_s^{*1}(p) \oplus \lambda u_s^{*2}(p) = \big\{\big\langle[\ell^{-1}(\lambda(\ell(\pounds^{1(e)}))), \ell^{-1}(\lambda(\ell(U^{1(e)})))], p^{*(e)}\big\rangle \,|e = 1, 2, ..., E\big\}$

$$\oplus \left\{\left\langle[\ell^{-1}(\lambda(\ell(\pounds^{2(e)}))), \ell^{-1}(\lambda(\ell(U^{2(e)})))], p^{*(e)}\right\rangle|e = 1, 2, ..., E\right\}$$

$$= \big\{\big\langle[\ell^{-1}(\lambda(\ell(\pounds^{1(e)}) + \ell(\pounds^{2(e)}))), \ell^{-1}(\lambda(\ell(U^{1(e)}) + \ell(U^{2(e)})))],$$

$$p^{*(e)}\big\rangle|e = 1, 2, ..., E\big\}$$

$$= \big\{\big\langle[\ell^{-1}(\lambda(\ell(\ell^{-1}(\ell(\pounds^{1(e)}) + \ell(\pounds^{2(e)}))))), \ell^{-1}(\lambda(\ell(\ell^{-1}(\ell(U^{1(e)})+$$

$$\ell(U^{2(e)})))))], p^{*(e)}\big\rangle|e = 1, 2, ..., E\big\}$$

$$= \lambda\big(\big\{\,\big\langle[\ell^{-1}(\ell(\pounds^{1(e)}) + \ell(\pounds^{2(e)})), \ell^{-1}(\ell(U^{1(e)}) + \ell(U^{2(e)}))], p^{*(e)}\big\rangle$$

$$|e = 1, 2, ..., E\big\}\big)$$

$$= \lambda\big(u_s^{*1}(p) \oplus u_s^{*2}(p)\big).$$

4. $\lambda_1 u_s^{*}(p) \oplus \lambda_2 u_s^{*}(p) = \big\{\big\langle[\ell^{-1}(\lambda_1(\ell(\pounds^{(e)}))), \ell^{-1}(\lambda_1(\ell(U^{(e)})))], p^{*(e)}\big\rangle \,|e = 1, 2, ..., E\big\} \oplus$

$$\left\{\left\langle[\ell^{-1}(\lambda_2(\ell(\pounds^{(e)}))), \ell^{-1}(\lambda_2(\ell(U^{(e)})))], p^{*(e)}\right\rangle|e = 1, 2, ..., E\right\}$$

$$= \left\{\left\langle[\ell^{-1}(\ell(\ell^{-1}(\lambda_1(\ell(\pounds^{(e)}))))) + \ell^{-1}(\ell(\ell^{-1}(\lambda_2(\ell(\pounds^{(e)}))))),\right.\right.$$

$$\left.\left.\ell^{-1}(\ell(\ell^{-1}(\lambda_1(\ell(U^{(e)}))))) + \ell^{-1}(\ell(\ell^{-1}(\lambda_2(\ell(U^{(e)})))))], p^{*(e)}\right\rangle|e = 1, 2, ..., E\right\}$$

$$= \big\{\big\langle[\ell^{-1}(\lambda_1(\ell(\pounds^{(e)}))) + \ell^{-1}(\lambda_2(\ell(\pounds^{(e)}))), \ell^{-1}(\lambda_1(\ell(U^{(e)})))+$$

$$\ell^{-1}(\lambda_2(\ell(U^{(e)})))], p^{*(e)}\big\rangle|e = 1, 2, ..., E\big\}$$

$$= \big\{\big\langle[\ell^{-1}((\lambda_1 + \lambda_2)(\ell(\pounds^{(e)}))), \ell^{-1}((\lambda_1 + \lambda_2)(\ell(U^{(e)})))], p^{*(e)}\big\rangle \,|e = 1, 2, ..., E\big\}$$

$$= (\lambda_1 + \lambda_2)u_s^{*}(p).$$

6. $\big(u_s^*(p) \otimes u_s^{*1}(p)\big) \otimes u_s^{*2}(p) = \Big\{ \Big\langle [\ell^{-1}(\ell(\pounds^{1(e)})\ell(\pounds^{2(e)})), \ell^{-1}(\ell(U^{1(e)})\ell(U^{2(e)}))], p^{*(e)} \Big\rangle$

$$
\begin{aligned}
&|e = 1, 2, ..., E\Big\} \otimes u_s^{*2}(p) \\
&= \Big\{\Big\langle [\ell^{-1}(\frac{\alpha^{(e)} + \tau}{2\tau} + \frac{\alpha^{(1e)} + \tau}{2\tau}), \ell^{-1}(\frac{\beta^{(e)} + \tau}{2\tau} + \frac{\beta^{(1e)} + \tau}{2\tau})], \\
&p^{*(e)}\Big\rangle|e = 1, 2, ..., E\Big\} \otimes u_s^{*2}(p) \\
&= \Big\{\Big\langle [\pounds_{\alpha^{(e)}+\alpha^{(1e)}+2\tau}, \pounds_{\beta^{(e)}+\beta^{(1e)}+2\tau}], p^{*(e)}\Big\rangle|e = 1, 2, ..., E\Big\} \otimes u_s^{*2}(p) \\
&= \Big\{\Big\langle [\ell^{-1}\big(\ell(\pounds_{\alpha^{(e)}+\alpha^{1(e)}+2\tau})\ell(\pounds^{2(e)})\big), \ell^{-1}\big(\ell(\pounds_{\beta^{(e)}+\beta^{1(e)}+2\tau})\ell(U^{2(e)})\big)], \\
&p^{*(e)}\Big\rangle|e = 1, 2, ..., E\Big\} \oplus u_s^{*2}(p) \\
&= \Big\{\Big\langle [\ell^{-1}(\frac{\alpha^{(e)} + \alpha^{1(e)} + \alpha^{2(e)} + 3\tau}{2\tau}), \ell^{-1}(\frac{\beta^{(e)+\beta^{1(e)}+\beta^{2(e)}} + 3\tau}{2\tau})], \\
&p^{*(e)}\Big\rangle|e = 1, 2, ..., E\Big\} \otimes u_s^{*2}(p) \\
&= \Big\{\Big\langle [\pounds_{\alpha^{(e)}+\alpha^{1(e)}+\alpha^{2(e)}+2\tau}, \pounds_{\beta^{(e)}+\beta^{1(e)}+\beta^{2(e)}+2\tau}], p^{*(e)}\Big\rangle|e = 1, 2, ..., E\Big\}.
\end{aligned} \tag{4.2.4}
$$

Now

$$
\begin{aligned}
u_s^*(p) \otimes \left(u_s^{*1}(p) \otimes u_s^{*2}(p)\right) &= u_s^*(p) \otimes \left\{ \left\langle [\ell^{-1}(\ell(\pounds^{1(e)})\ell(\pounds^{2(e)})), \ell^{-1}(\ell(U^{1(e)})\ell(U^{2(e)}))], p^{*(e)} \right\rangle \right.\\
&\left. |e = 1, 2, ..., E \right\} \\
&= u_s^*(p) \otimes \left\{ \left\langle [\ell^{-1}(\frac{\alpha^{1(e)} + \tau}{2\tau} + \frac{\alpha^{(2e)} + \tau}{2\tau}), \ell^{-1}(\frac{\beta^{1(e)} + \tau}{2\tau} + \frac{\beta^{2(e)} + \tau}{2\tau})] \right.\right.\\
&\left.\left. , p^{*(e)} \right\rangle |e = 1, 2, ..., E \right\} \\
&= u_s^*(p) \otimes \left\{ \left\langle [\pounds_{\alpha^{1(e)}+\alpha^{2(e)}+2\tau}, \pounds_{\beta^{(e)}+\beta^{(1e)}+2\tau}], p^{*(e)} \right\rangle |e = 1, 2, ..., E \right\} \\
&\otimes u_s^{*2}(p) \\
&= \left\{ \left\langle [\ell^{-1}\left(\ell(\pounds^{(e)}) + \ell(\pounds_{\alpha^{1(e)}+\alpha^{2(e)}+2\tau})\right), \ell^{-1}\left(\ell(\pounds^{(e)}) + \right.\right.\right.\\
&\left.\left.\left. \ell(\pounds_{\beta^{1(e)}+\beta^{2(e)}+2\tau})\right)], p^{*(e)} \right\rangle |e = 1, 2, ..., E \right\} \otimes u_s^{*2}(p) \\
&= \left\{ \left\langle [\ell^{-1}(\frac{\alpha^{(e)} + \alpha^{1(e)} + \alpha^{2(e)} + 3\tau}{2\tau}), \ell^{-1}(\frac{\beta^{(e)+\beta^{1(e)}+\beta^{2(e)}} + 3\tau}{2\tau})] \right.\right.\\
&\left.\left. , p^{*(e)} \right\rangle |e = 1, 2, ..., E \right\} \otimes u_s^{*2}(p) \\
&= \left\{ \left\langle [\pounds_{\alpha^{(e)}+\alpha^{1(e)}+\alpha^{2(e)}+2\tau}, \pounds_{\beta^{(e)}+\beta^{1(e)}+\beta^{2(e)}+2\tau}], p^{*(e)} \right\rangle |e = 1, 2, ..., E \right\}.
\end{aligned} \tag{4.2.5}
$$

From (4.2.4) and (4.2.5) we get the desired proof.

7. $\left(u_s^{*1}(p)\right)^{\lambda} \otimes \left(u_s^{*1}(p)\right)^{\lambda} = \left\{\left\langle[\ell^{-1}(\ell(\pounds^{1(e)}))^{\lambda}, \ell^{-1}(\ell(U^{1(e)}))^{\lambda}], p^{*(e)}\right\rangle |e=1,2,...,E\right\}$

$$\otimes \left\{\left\langle[\ell^{-1}(\ell(\pounds^{2(e)}))^{\lambda}, \ell^{-1}(\ell(U^{2(e)}))^{\lambda}], p^{*(e)}\right\rangle |e=1,2,...,E\right\}$$

$$= \left\{\left\langle[\ell^{-1}((\ell(\pounds^{1(e)}))^{\lambda}(\ell(\pounds^{2(e)}))^{\lambda}), \ell^{-1}((\ell(U^{1(e)}))^{\lambda}(\ell(U^{2(e)}))^{\lambda})],\right.\right.$$

$$\left.\left. p^{*(e)}\right\rangle |e=1,2,...,E\right\}$$

$$= \left\{\left\langle[\ell^{-1}(\ell(\pounds^{1(e)})\ell(\pounds^{2(e)}))^{\lambda}, \ell^{-1}(\ell(U^{1(e)})(\ell(U^{2(e)})))^{\lambda}], p^{*(e)}\right\rangle\right.$$

$$\left. |e=1,2,...,E\right\}$$

$$= \left\{\left\langle[\ell^{-1}((\ell(\ell^{-1}(\ell(\pounds^{1(e)})\ell(\pounds^{2(e)})))))^{\lambda}, \ell^{-1}(\ell(\ell^{-1}(\ell(U^{1(e)})(\ell(U^{2(e)})))))^{\lambda}],\right.\right.$$

$$\left.\left. p^{*(e)}\right\rangle |e=1,2,...,E\right\}$$

$$= \left\{\left\langle[\ell^{-1}((\ell(\pounds^{1(e)})\ell(\pounds^{2(e)})))^{\lambda}, \ell^{-1}((\ell(U^{1(e)})(\ell(U^{2(e)}))))^{\lambda}], p^{*(e)}\right\rangle\right.$$

$$\left. |e=1,2,...,E\right\}$$

$$= \left(u_s^{*1}(p) \otimes u_s^{*2}(p)\right)^{\lambda}.$$

8. $\left(u_s^{*}(p)\right)^{\lambda_1} \otimes \left(u_s^{*}(p)\right)^{\lambda_2} = \left\{\left\langle[\ell^{-1}(\ell(\pounds^{(e)}))^{\lambda_1}, \ell^{-1}(\ell(U^{(e)}))^{\lambda_1}], p^{*(e)}\right\rangle |e=1,2,...,E\right\}$

$$\otimes \left\{\left\langle[\ell^{-1}(\ell(\pounds^{(e)}))^{\lambda_2}, \ell^{-1}(\ell(U^{(e)}))^{\lambda_2}], p^{*(e)}\right\rangle |e=1,2,...,E\right\}$$

$$= \left\{\left\langle[\ell^{-1}(\ell(\ell^{-1}(\ell(\pounds^{(e)})))^{\lambda_1}\ell(\ell^{-1}(\ell(U^{(e)})))^{\lambda_2})], p^{*(e)}\right\rangle |e=1,2,...,E\right\},$$

$$\left\{\left\langle[\ell^{-1}(\ell(\ell^{-1}(\ell(\pounds^{(e)})))^{\lambda_1}\ell(\ell^{-1}(\ell(U^{(e)})))^{\lambda_2})], p^{*(e)}\right\rangle |e=1,2,...,E\right\}$$

$$= \left\{\left\langle[\ell^{-1}(\ell(\ell^{-1}(\ell(\pounds^{(e)})))^{\lambda_1+\lambda_2})], p^{*(e)}\right\rangle |e=1,2,...,E\right\},$$

$$\left\{\left\langle[\ell^{-1}(\ell(\ell^{-1}(\ell(\pounds^{(e)})))^{\lambda_1+\lambda_2})], p^{*(e)}\right\rangle |e=1,2,...,E\right\}$$

$$= \left(u_s^{*}(p)\right)^{\lambda_1+\lambda_2}.$$

$\square$

4.2.2 Comparison

In view of various linguistic scale functions for the semantics of linguistic terms, a new score function and deviation degree of a PULE are put forwarded to compare PULEs with one another.

Definition 4.2.4. *Let* $u_s(p) = \left\{\left\langle[\pounds^{(\jmath)}, U^{(\jmath)}], p^{*(\jmath)}\right\rangle |\jmath = 1, 2, ..., \pounds\right\}$ *be a PULE on* Z *and* ℓ *be a linguistic scale function, then score function of* $u_s(p)$ *is mathematically defined as:*

$$F(u_s(p)) = \pounds_{\overline{\alpha}} \ , \tag{4.2.6}$$

where $\overline{\alpha} = \sum_{\jmath=1}^{\pounds}\left(\frac{\ell(\pounds^{(\jmath)})+\ell(U^{(\jmath)})}{2} \cdot p^{(\jmath)}\right)$*, and the deviation degree of* $u_s(p)$ *is*

$$\sigma(u_s(p)) = \pounds_{\overline{\beta}} \ , \tag{4.2.7}$$

where $\overline{\beta} = \left(\sum_{\jmath=1}^{\pounds}\left(\left(\frac{\ell(\pounds^{(\jmath)})+\ell(U^{(\jmath)})}{2} - F(u_s(p))\right)^2 \cdot p^{(\jmath)}\right)\right)^{\frac{1}{2}}$.

For any two PULEs $u_s^1(p)$ and $u_s^2(p)$, comparison method between them is constructed as follows:

$$\begin{cases} \text{If } F(u_s^1(p)) > F(u_s^2(p)) \text{ then} & u_s^1(p) > u_s^2(p); \\ \text{If } F(u_s^1(p)) < F(u_s^2(p)) \text{ then} & u_s^1(p) < u_s^2(p); \\ \text{If } F(u_s^1(p)) = F(u_s^2(p)) \text{ then} & \begin{cases} \text{If } \sigma(u_s^1(p)) > \sigma(u_s^2(p)), & u_s^1(p) < u_s^2(p) \\ \text{If } \sigma(u_s^1(p)) < \sigma(u_s^2(p)), & u_s^1(p) > u_s^2(p) \\ \text{If } \sigma(u_s^1(p)) = \sigma(u_s^2(p)), & u_s^1(p) \sim u_s^2(p). \end{cases} \end{cases}$$

4.3 Aggregation operators, distance and correlation measure

To aggregate preferences of DMs more rationally, the present section first improves existing probabilistic uncertain linguistic aggregation operators and then by gaining motivations from [32, 50], construct the distance measures and introduce the concept of correlation measure in the PULTS context. We also seek out the relationship between score function and distance measure of PULTSs.

4.3.1 Aggregation operators

In order to aggregate DMs opinions, Lin et al. [32] proposed some probabilistic uncertain aggregation operators. Xie et al. [49] improved these operators by introducing new operations of PULEs but they are still tasteless because the operations they used are irrational as mentioned earlier. Therefore, these operators should be redefined. Here we present only probabilistic uncertain linguistic weighted averaging (PULWA) operator and probabilistic uncertain linguistic weighted geometric (PULWG) operator.

Definition 4.3.1. *Given n adjusted PULEs $u_s^{*i}(p) = \left\{\left\langle[\pounds^{i(e)}, U^{i(e)}], p^{*i(e)}\right\rangle|e = 1, 2, ..., E\right\}$. Then*

$$\begin{aligned} PULWA\left(u_s^{*1}(p), u_s^{*2}(p), ..., u_s^{*n}(p)\right) &= \pi_1 u_s^{*1}(p) \oplus \pi_2 u_s^{*2}(p) \oplus ... \oplus \pi_n u_s^{*n}(p) \\ &= \Bigg\{\Bigg\langle[\ell^{-1}\left(\pi_1\ell(\pounds^{1(e)}) + \pi_2\ell(\pounds^{2(e)}) + ... + \pi_n\ell(\pounds^{n(e)})\right), \\ &\ell^{-1}\Bigg(\pi_1\ell(U^{1(e)}) + \pi_2\ell(U^{2(e)}) + ... + \pi_n\ell(U^{n(e)})\Bigg)], p^{*(e)}\Bigg\rangle \\ &|e = 1, 2, ..., E\Bigg\}, \end{aligned} \tag{4.3.1}$$

is called the PULWA operator, $\pi = [\pi_1, \pi_2, ..., \pi_n]^t$ is the weight vector of these PULEs, $\pi_i \in [0, 1], \sum_{i=1}^n \pi_i = 1$ and ℓ is the linguistic scale function.

Definition 4.3.2. *Given n adjusted PULEs $u_s^{*i}(p) = \left\{\left\langle[\pounds^{i(e)}, U^{i(e)}], p^{*i(e)}\right\rangle |e = 1, 2, ..., E\right\}$. Then*

$$
\begin{aligned}
PULWG\left(u_s^{*1}(p), u_s^{*2}(p), ..., u_s^{*n}(p)\right) &= \left(u_s^{*1}(p)\right)^{\pi_1} \otimes \left(u_s^{*2}(p)\right)^{\pi_2} \otimes ... \otimes \left(u_s^{*n}(p)\right)^{\pi_n} \\
&= \Bigg\{\Bigg\langle[\ell^{-1}\left(\left(\ell(\pounds^{1(e)})\right)^{\pi_1}\left(\ell(\pounds^{2(e)})\right)^{\pi_2} ... \left(\ell(\pounds^{n(e)})\right)^{\pi_n}\right), \\
&\ell^{-1}\left(\left(\ell(U^{1(e)})\right)^{\pi_1}\left(\ell(U^{2(e)})\right)^{\pi_2} ... \left(\ell(U^{n(e)})\right)^{\pi_n}\right)], p^{*(e)}\Bigg\rangle \\
&|e = 1, 2, ..., E\Bigg\},
\end{aligned}
\tag{4.3.2}
$$

is called the PULWG operator, $\pi = [\pi_1, \pi_2, ..., \pi_n]^t$ is the weight vector of these PULEs, $\pi_i \in [0, 1], \sum_{i=1}^n \pi_i = 1$ and ℓ is the linguistic scale function.

It can be easily observed that the aggregated values obtained by Eqs. (4.3.1) and (4.3.2) are also a PULEs.

4.3.2 Distance measure

Some well-known distance measures for PULTSs based on the linguistic scale function including Hamming distance, Euclidean distance and generalized distance are proposed as follows:

Definition 4.3.3. *For two associated PULEs $\widetilde{u}_s^1(p)$ and $\widetilde{u}_s^1(p)$ with adjusted forms $u_s^{*1}(p) = \left\{\left\langle[\pounds^{1(e)}, U^{1(e)}], p^{*(e)}\right\rangle |e = 1, 2, ..., E\right\}$ and $u_s^{*2}(p) = \left\{\left\langle[\pounds^{2(e)}, U^{2(e)}], p^{*(e)}\right\rangle |e = 1, 2, ..., E\right\}$. Then the Hamming distance between $\widetilde{u}_s^1(p)$ and $\widetilde{u}_s^2(p)$ can be defined as:*

$$
d_{hd}\left(\widetilde{u}_s^1(p), \widetilde{u}_s^2(p)\right) = \sum_{e=1}^{E} p^{*(e)}\left(\left|\ell(\pounds^{1(e)} - \ell(\pounds^{2(e)})\right| + \left|\ell(U^{1(e)} - \ell(U^{2(e)})\right|\right). \tag{4.3.3}
$$

Euclidean distance can be defined as:

$$d_{ed}\left(\widetilde{u}_s^1(p), \widetilde{u}_s^1(p)\right) = \sqrt{\sum_{e=1}^{E} p^{*(e)} \left(\left|\ell(\pounds^{1(e)} - \ell(\pounds^{2(e)})\right|^2 + \left|\ell(U^{1(e)} - \ell(U^{2(e)})\right|^2\right)}. \quad (4.3.4)$$

The generalized distance can be defined as:

$$d_{gd}\left(\widetilde{u}_s^1(p), \widetilde{u}_s^1(p)\right) = \left(\sum_{e=1}^{E} p^{*(e)} \left(\left|\ell(\pounds^{1(e)} - \ell(\pounds^{2(e)})\right|^\zeta + \left|\ell(U^{1(e)} - \ell(U^{2(e)})\right|^\lambda\right)\right)^{1/\zeta}. \quad (4.3.5)$$

The function ℓ is a linguistic scale function and can be chosen from the aforementioned three types of linguistic scale functions under different semantics. $\zeta > 0$ is a parameter.

The major advantages of proposed distance measure over existing ones are outlined below.

i. Multiplying the probabilities with indices of lower and upper limits of the corresponding uncertain linguistic terms of a PULE like Ref. [32] or taking the average of probabilities of PULEs [49] are irrational. The proposed distances are based on the adjusting rule of probability which leave off the probability calculations.

ii. Unlike existing distance formulas, there is no requirement to insert artificial linguistic terms to the smaller one with our adjusting rule of probability given in Section 4.1.

iii. The existing distance measures fail to compute the distance between unbalanced PULEs while proposed distances are based on the linguistic scale function and DMs can choose an appropriate linguistic scale function under different circumstances.

Proposition 4.3.4. *Let $u_s^1(p)$, $u_s^2(p)$, $u_s^3(p)$ and $u_s^4(p)$ be four PULEs, if $|F\left(u_s^1(p)\right) - F\left(u_s^2(p)\right)| \geq |F\left(u_s^3(p)\right) - F\left(u_s^4(p)\right)|$, then $d\left(u_s^1(p), u_s^2(p)\right) \geq d\left(u_s^3(p), u_s^4(p)\right)$.*

Proof. Suppose that $u_s^{*1}(p)$, $u_s^{*2}(p)$, $u_s^{*3}(p)$ and $u_s^{*4}(p)$ are the adjusted forms of the given

PULEs with rearranged probability set $p = \left\{p^{*(1)}, p^{*(2)}, ..., p^{*(E)}\right\}^t$. From Eq. (4.2.6) we got

$$F\left(u_s(p)\right) = \sum_{e=1}^{E} \Big(\frac{\ell(\pounds^{(e)}) + \ell(U^{(e)})}{2} \cdot p^{(e)}\Big) = \sum_{e=1}^{E} \Big(\frac{\ell(\pounds^{(e)}) + \ell(U^{(e)})}{2} \cdot p^{*(e)}\Big)$$

$$\implies F\left(u_s^1(p)\right) - F\left(u_s^2(p)\right) = \sum_{e=1}^{E} \Big(\frac{\ell(\pounds^{1(e)}) + \ell(U^{1(e)})}{2} \cdot p^{*(e)}\Big) - \sum_{e=1}^{E} \Big(\frac{\ell(\pounds^{2(e)}) + \ell(U^{2(e)})}{2} \cdot p^{*(e)}\Big)$$

$$= \sum_{e=1}^{E} \Big(\frac{\ell(\pounds^{1(e)}) - \ell(\pounds^{2(e)}) + \ell(U^{1(e)} - \ell(U^{2(e)})}{2} \cdot p^{*(e)}\Big).$$

Since $\quad |F\left(u_s^1(p)\right) - F\left(u_s^2(p)\right)| \geq |F\left(u_s^1(p)\right) - F\left(u_s^2(p)\right)|$
therefore,

$$\sum_{e=1}^{E} \left|\frac{\ell(\pounds^{1(e)}) - \ell(\pounds^{2(e)}) + \ell(U^{1(e)} - \ell(U^{2(e)})}{2}\right| \cdot p^{*(e)} \geq \sum_{e=1}^{E} \left|\frac{\ell(\pounds^{3(e)}) - \ell(\pounds^{4(e)}) + \ell(U^{3(e)} - \ell(U^{4(e)})}{2}\right| \cdot p^{*(e)},$$

since $\zeta > 0$ so we can write

$$\sum_{e=1}^{E} \left|\left(\ell(\pounds^{1(e)}) - \ell(\pounds^{2(e)})\right) + \left(\ell(U^{1(e)}) - \ell(U^{2(e)})\right)\right|^{\zeta} \cdot p^{*(e)} \geq \sum_{e=1}^{E} \left|\left(\ell(\pounds^{3(e)}) - \ell(\pounds^{4(e)})\right) + \left(\ell(U^{3(e)} - \ell(U^{4(e)}))\right)\right|^{\zeta} \cdot p^{*(e)}.$$

Thus, $d\left(u_s^1(p), u_s^2(p)\right) \geq d\left(u_s^3(p), u_s^4(p)\right)$. $\square$

4.3.3 Correlation coefficient

Based on the above distance measure, we propose PUL-correlation coefficient for PULTSs, which is defined as follows:

Definition 4.3.5. *Let $U_s^{*1}(p) = \{u_s^{*11}(p), u_s^{*12}(p), u_s^{*13}(p), ..., u_s^{*1m}(p)\}$ and $U_s^{*2}(p) = \{u_s^{*21}(p), u_s^{*22}(p), u_s^{*23}(p), ..., u_s^{*2m}(p)\}$ be two adjusted PULTS then the probabilistic uncertain linguistic correlation coefficient (PUL-correlation coefficient) between $U_s^{*1}(p)$ and $U_s^{*2}(p)$ can be mathematically* *defined as:*

$$R\left(U_s^{*1}(p),U_s^{*2}(p)\right)$$
$$=\frac{\sum_{i=1}^{m}\left(\left(\frac{d\left(u_s^{*1+}(p),u_s^{*1i}(p)\right)}{d\left(u_s^{*1+}(p),u_s^{*1-}(p)\right)}-\frac{1}{m}\sum_{i=1}^{m}\frac{d\left(u_s^{*1+}(p),u_s^{*1i}(p)\right)}{d\left(u_s^{*1+}(p),u_s^{*1-}(p)\right)}\right)\times\left(\frac{d\left(u_s^{*2+}(p),u_s^{*2i}(p)\right)}{d\left(u_s^{*2+}(p),u_s^{*2-}(p)\right)}-\frac{1}{m}\sum_{i=1}^{m}\frac{d\left(u_s^{*2+}(p),u_s^{*1i}(p)\right)}{d\left(u_s^{*2+}(p),u_s^{*2-}(p)\right)}\right)\right)}{\sqrt{\sum_{i=1}^{m}\left(\frac{d\left(u_s^{*1+}(p),u_s^{*1i}(p)\right)}{d\left(u_s^{*1+}(p),u_s^{*1-}(p)\right)}-\frac{1}{m}\sum_{i=1}^{m}\frac{d\left(u_s^{*1+}(p),u_s^{*1i}(p)\right)}{d\left(u_s^{*1+}(p),u_s^{*1-}(p)\right)}\right)^2}\times\sqrt{\sum_{i=1}^{m}\left(\frac{d\left(u_s^{*2+}(p),u_s^{*2i}(p)\right)}{d\left(u_s^{*2+}(p),u_s^{*2-}(p)\right)}-\frac{1}{m}\sum_{i=1}^{m}\frac{d\left(u_s^{*2+}(p),u_s^{*1i}(p)\right)}{d\left(u_s^{*2+}(p),u_s^{*2-}(p)\right)}\right)^2}},\tag{4.3.6}$$

where m is the number of PULEs in each PULTS, $u_s^{*q+}(p)$ and $u_s^{*q-}(p)$ are the best and worst PULEs in $U_s^{*q}(p)$, respectively, $q=1,2$. $u_s^{*q+}(p)$ and $u_s^{*q-}(p)$ can be determined by Eqs. (4.2.6) and (4.2.7), $d\left(u_s^{*1+}(p),u_s^{*1i}(p)\right)$ denotes the distance between $u_s^{*1+}(p)$ and $u_s^{*1i}(p)$.

Proposition 4.3.6. *The PUL-correlation coefficient satisfy the characteristic of the Pearson correlation coefficient i.e., $R\left(U_s^{*1}(p),U_s^{*2}(p)\right)\in[-1,1]$.*

Proof. From

$$u_s^{*q+}(p)=\begin{cases}\max_i u_s^{*qi}(p) & \text{for the benefit criteria}\\ \min_i u_s^{*qi}(p) & \text{for the cost criteria}\end{cases}$$

and

$$u_s^{*q-}(p)=\begin{cases}\min_i u_s^{*qi}(p) & \text{for the benefit criteria}\\ \max_i u_s^{*qi}(p) & \text{for the cost criteria}\end{cases}$$

we have $\left|F\left(u_s^{*q+}(p)\right)-F\left(u_s^{*q-}(p)\right)\right|\geq\left|F\left(u_s^{*q+}(p)\right)-F\left(u_s^{*qi}(p)\right)\right|$.

Proposition 4.3.4, presents us $d\left(u_s^{*q+}(p),u_s^{*q-}(p)\right)\geq d\left(u_s^{*q+}(p),u_s^{*qi}(p)\right),\ i=1,2,...,m$. Therefore, $\frac{d\left(u_s^{*q+}(p),u_s^{*qi}(p)\right)}{d\left(u_s^{*q+}(p),u_s^{*q-}(p)\right)}\leq 1$. Let $\frac{d\left(u_s^{*q+}(p),u_s^{*qi}(p)\right)}{d\left(u_s^{*q+}(p),u_s^{*q-}(p)\right)}-\frac{1}{m}\sum_{i=1}^{m}\left(\frac{d\left(u_s^{*q+}(p),u_s^{*qi}(p)\right)}{d\left(u_s^{*q+}(p),u_s^{*q-}(p)\right)}\right)=z^{qi}, q=1,2$. Then $z^{qi}\in[-1,1]$. The denominator in Eq. (4.3.6) can be written as $\sqrt{\sum_{i=1}^{m}(z^{1i})^2}\times\sqrt{\sum_{i=1}^{m}(z^{2i})^2}\geq\sum_{i=1}^{m}\mid z^{1i}\times z^{2i}\mid$. The numerator can be written as $\sum_{i=1}^{m}\left(z^{1i}\times z^{2i}\right)$. Therefore, $R\left(U_s^{*1}(p),U_s^{*2}(p)\right)\in[-1,1]$.

□

Correlation coefficient formula is used to find how strong a relationship is between data. As proved above, the formulae return a value between -1 and 1, where:

- 1 indicates a strong positive relationship,
- -1 indicates a strong negative relationship,
- A result of zero indicates no relationship at all.

The absolute value of the correlation coefficient gives us the relationship strength. The larger the number, the stronger the relationship, $|-0.75| = 0.75$ has a stronger relationship than 0.65. Generally, the correlation strength of two variables $U_s^1(p)$ and $U_s^2(p)$ can be ranked by the following ranking rule [51].

➤ $R\left(U_s^{*1}(p), U_s^{*2}(p)\right) \in (0.8, 1] \in$ extremely strong correlation.

➤ $R\left(U_s^{*1}(p), U_s^{*2}(p)\right) \in (0.6, 0.8] \in$ strong correlation.

➤ $R\left(U_s^{*1}(p), U_s^{*2}(p)\right) \in (0.4, 0.6] \in$ moderate strong correlation.

➤ $R\left(U_s^{*1}(p), U_s^{*2}(p)\right) \in (0.2, 0.4] \in$ weak correlation.

➤ $R\left(U_s^{*1}(p), U_s^{*2}(p)\right) \in (0, 0.2] \in$ extremely weak correlation.

4.4 Consensus-based robust decision making methods for MCGDM problems

This section is devoted to present a consensus-based probabilistic uncertain linguistic (PUL)-GLDS method and consensus-based PUL-aggregation method to handle the MCGDM problems with the decision information in PULEs form. A general MCGDM problem contains a set of alternatives $Z = \{z_1, z_2, z_3, ..., z_m\}$ $(m \geq 2)$, a set of attributes $C = \{c_1, c_2, c_3, ..., c_n\}$ $(n \geq 2)$ with the weight vector $\pi = \{\pi_1, \pi_2, \pi_3, ..., \pi_n\}^t$, and a group of DMs $D = \{d_1, d_2, d_3, ..., d_T\}$

$(T \geq 2)$. The decision information of DM d_T on z_i regarding to A_j are transformed to PULEs $u_s^{ij(t)}(p)(i = 1, 2, ..., m; j = 1, 2, ..., n; t = 1, 2, ..., T)$. Thus, a group of individual probabilistic uncertain linguistic decision matrices are attained as $M^t = \left(u_s^{ij(t)}(p)\right)_{m\times n}$ $(t = 1, 2, ..., T)$. By Eqs. (4.1.1) and (4.1.2), we can attain the overall probabilistic uncertain linguistic opinion matrix $\widehat{M} = (\widehat{u}_s^{ij}(p))_{m\times n}$.

4.4.1 The PUL-consensus reaching approach

The PUL-consensus reaching approach can be conducted into two steps: PUL-consensus checking approach and PUL-consensus improving approach.

i. PUL-consensus checking approach

Definition 4.4.1. *Let $M^t = \left(u_s^{ij(t)}(p)\right)_{m\times n}$ be the individual probabilistic uncertain linguistic opinion matrix of DM d_t and $\widehat{M} = (\widehat{u}_s^{ij}(p))_{m\times n}$ be the overall probabilistic uncertain linguistic opinion matrix. The PUL-consensus level of DM d_t can be defined as follows:*

$$\sigma^{(t)} = \frac{1}{n}\sum_{j=1}^{n}\theta_j^{(t)}, \tag{4.4.1}$$

where

$$\theta_j^{(t)} = \frac{\sum_{i=1}^{m}\left[\left(\frac{d_{ij}^{(t)}}{d_j^{(t)}} - \frac{1}{m}\frac{\sum_{i=1}^{m} d_{ij}^{(t)}}{d_j^{(t)}}\right) \times \left(\frac{d_{ij}}{d_j} - \frac{1}{m}\frac{\sum_{i=1}^{m} d_{ij}}{d_j}\right)\right]}{\sqrt{\sum_{i=1}^{m}\left(\frac{d_{ij}^{(t)}}{d_j^{(t)}} - \frac{1}{m}\frac{\sum_{i=1}^{m} d_{ij}^{(t)}}{d_j^{(t)}}\right)^2} \times \sqrt{\sum_{i=1}^{m}\left(\frac{d_{ij}}{d_j} - \frac{1}{m}\frac{\sum_{i=1}^{m} d_{ij}}{d_j}\right)^2}}$$

where $d_{ij}^{(t)} = d\left(u_s^{j+(t)}(p), u_s^{ij(t)}(p)\right)$, $d_j^{(t)} = d\left(u_s^{j+(t)}(p), u_s^{j-(t)}(p)\right)$, $d_{ij} = d\left(\widehat{u}_s^{j+}(p), \widehat{u}_s^{ij}(p)\right)$ and $d_j = d\left(\widehat{u}_s^{j+}(p), \widehat{u}_s^{j}(p)\right)$

$$u_s^{j+}(p) = \begin{cases} \max_i u_s^{ij(t)}(p), & \textit{for the benefit criteria} \\ \min_i u_s^{ij(t)}(p), & \textit{for the cost criteria}. \end{cases}$$

$$\widehat{u}_s^{j+}(p) = \begin{cases} \max_i \widehat{u}_s^{ij}(p), & \textit{for the benefit criteria} \\ \min_i \widehat{u}_s^{ij}(p), & \textit{for the cost criteria .} \end{cases}$$

$$u_s^{j-}(p) = \begin{cases} \min_i u_s^{ij(t)}(p), & \textit{for the benefit criteria} \\ \max_i u_s^{ij(t)}(p), & \textit{for the cost criteria .} \end{cases}$$

$$\widehat{u}_s^{j-}(p) = \begin{cases} \min_i \widehat{u}_s^{ij}(p), & \textit{for the benefit criteria} \\ \max_i \widehat{u}_s^{ij}(p), & \textit{for the cost criteria .} \end{cases}$$

According to Proposition 4.3.6, we can attain that $\sigma^t \in [-1, 1]$. The larger the value of σ^t is, the more powerful the PUL-correlation consensus level of DM d_t to the group would be regarding to each criteria. If $\sigma^t > 0$, there is consensus with various strength; if $\sigma^t \leq 0$, the DM d_t posses no consensus with other DMs. It is rational to remove the biased opinions of DMs d_t if σ^t and $\theta_j^{(t)}$ are extremely poor. Based upon the ranking of correlation strength, the ranking of consensus level is given below:

- $\sigma^{(t)} \in (0.8, 1] \in$ extremely strong correlation;
- $\sigma^{(t)} \in (0.6, 0.8] \in$ strong correlation;
- $\sigma^{(t)} \in (0.4, 0.6] \in$ moderate strong correlation;
- $\sigma^{(t)} \in (0.2, 0.4] \in$ weak correlation;
- $\sigma^{(t)} \in (0, 0.2] \in$ extremely weak correlation.

ii. PUL-consensus improving approach:
As mentioned earlier, perfect consistency is not expected in a hesitant environment. When the consistency level of a DM is not satisfied, then DM should be asked to reconsider their decisions and provide new preferences. This subsection provides an algorithm to guide this consistency improvement approach.

Let the minimal level of DMs group be $\sigma^* = \sigma^{(\mu)} = \min_t \sigma^{(t)}(t = 1, 2, ..., T)$, θ and σ be stated correlation and consensus threshold, respectively. If $\sigma^* = \sigma^{(\mu)} < \sigma$ then the corresponding individual probabilistic uncertain linguistic opinion matrix $M^{(\mu)}$ with minimal consensus level $\sigma^{(\mu)}$ need to be repaired as:

$$\acute{M}^{(\mu)} = \left(\acute{u}_s^{ij(\mu)}(p)\right)_{m \times n}, \tag{4.4.2}$$

where

$$\acute{u}^{ij(\mu)} = \begin{cases} \frac{1}{2}\left(u_s^{ij(\mu)}(p) \oplus u_s^{ij}(p)\right), & \text{if } \theta_j^{\mu} < \theta \\ u_s^{ij(\mu)}(p), & \text{otherwise.} \end{cases}$$

If the consensus level of all opinion matrices are greater than or equal to the consensus threshold σ, then, integrate the individual opinion matrices into final opinion matrix $M = (u_s^{ij}(p))_{m \times n}$. We can choose the consensus threshold as $\sigma, \theta \in (0.4, 0.8]$. If the demand for consensus level is hard then DMs can choose a high value of threshold. If the demand for consensus level is not hard, then a low value of threshold can be imposed.

4.4.2 The PUL-GLDS method

The GLDS method was originally designed by Wu and Liao [48] to cope with complex MCGDM problems with probabilistic linguistic assessments. It is a novel outranking method based upon both the gained and lost dominance score between alternatives. Based on the algorithm of PL-GLDS method, we generalize this method to probabilistic uncertain linguistic scenario. To do so, in the following, we are going to define some concepts.

Definition 4.4.2. *Let the DMs opinions of alternatives z_i and z_ς under criteria c_j be two PULEs $u_s^{ij}(p)$ and $u_s^{\varsigma j}(p)$, respectively. The adjusted PULEs are $u_s^{*ij}(p) = \{\langle [\pounds^{ij(e)}, U^{ij(e)}], p^{*(e)}\rangle$ $|e = 1, 2, ..., E\}$ and $u_s^{*\varsigma j}(p) = \{\langle [\pounds^{\varsigma j(e)}, U^{\varsigma j(e)}],$*

$p^{*(e)}\rangle | e = 1, 2, ..., E\}$. *Let*

$$h_{i\varsigma}^{(e)} = \begin{cases} \max\left\{mid[\ell(\pounds^{ij(e)}), \ell(U^{ij(e)})] - mid[\ell(\pounds^{\varsigma j(e)}), \ell(U^{\varsigma j(e)})], 0\right\}, & \textit{for benefit criteria} \\ \max\left\{mid[\ell(\pounds^{\varsigma j(e)}), \ell(U^{\varsigma j(e)})] - mid[\ell(\pounds^{ij(e)}), \ell(U^{ij(e)})], 0\right\}, & \textit{for cost criteria} \end{cases}$$

; $i, \varsigma = 1, 2, ..., m$ and 'ℓ' is a linguistic scale function. The PUL-dominance flow of z_i over z_ς under criteria c_j can be characterised as:

$$df_j\left(z_i, z_\varsigma\right) = \sum_{e=1}^{E}\left(h_{i\varsigma}^{(e)} \cdot p^{*(e)}\right). \tag{4.4.3}$$

Since $mid[\ell(\pounds), \ell(U)] \in [0, 1]$, this implies $h_{i\varsigma}^{(e)} \in [0, 1]$. Also $\sum_{e=1}^{E} p^{*(e)} = 1$ and $p^{*(e)} > 0 \ \forall\ e = 1, 2, ..., E$, this results $\sum_{e=1}^{E}\left(h_{i\varsigma \cdot p^{*(e)}}^{(e)} \leq \sum_{e=1}^{E} p^{*(e)} = 1\right)$. Therefore, $df_j\left(z_i, z_\varsigma\right) \in [0, 1]$. Normalize the PUL-dominance flow by using the following formula:

$$df_j^N\left(z_i, z_\varsigma\right) = \frac{df_j\left(z_i, z_\varsigma\right)}{\sqrt{\sum_{\varsigma=1}^{m}\sum_{i=1}^{m}\left(df_j\left(z_i, z_\varsigma\right)\right)^2}}. \tag{4.4.4}$$

Since $df_j\left(z_i, z_\varsigma\right) \in [0, 1]$, it is clear that $df_j^N\left(z_i, z_\varsigma\right) \in [0, 1]$.

Definition 4.4.3. *The probabilistic uncertain linguistic gained dominance score (PUL-gained dominance score) of alternative z_i under criteria c_j can be mathematically described as:*

$$gd_j(z_i) = \sum_{\varsigma=1}^{m} df_j^N\left(z_i, z_\varsigma\right). \tag{4.4.5}$$

Definition 4.4.4. *The probabilistic uncertain linguistic lost dominance score (PUL-lost dominance score) of alternative z_i under criteria c_j is defined as:*

$$ld_j(z_i) = \max_{\varsigma} df_j^{\varsigma}\left(z_\varsigma, z_i\right). \tag{4.4.6}$$

Definition 4.4.5. *The probabilistic uncertain linguist overall gained dominance score (PUL-overall gained dominance score) of alternative z_i under criteria c_j is denoted and described as:*

$$OGD(z_i) = \sum_{j=1}^{n} \pi_j gd_j(z_i), \tag{4.4.7}$$

where π_j is the weight of the criteria c_j.

PUL-overall gained dominance score represents the "group utility" value of each alternative. The alternatives are positioned in decreasing order of $OGD(z_i)(i = 1, 2, ..., m)$ and thus, a rank set $S_1 = \{r_1(z_1), r_1(z_2), ..., r_1(z_m)\}$ is achieved.

Definition 4.4.6. *The probabilistic uncertain linguist overall lost dominance score (PUL-overall lost dominance score) of alternative z_i under criteria c_j is denoted and defined as follows:*

$$OLD(z_i) = \max_i \pi_j ld_j(z_i), \tag{4.4.8}$$

where π_j is the weight of the criteria c_j.

The PUL-overall dominance score presents the maximum "individual regret" value of each alternative. The alternatives are positioned in increasing order of $OLD(z_i)(i = 1, 2, ..., m)$ and this results a second ranked set $S_2 = \{r_2(z_1), r_2(z_2), ..., r_2(z_m)\}$.
Normalize $OGD(z_i)$ and $OLD(z_i)$ by normalization formulas as Eqs. (4.4.9) and (4.4.10).

$$OGD^N(z_i) = \frac{OGD(z_i)}{\sqrt{\sum_{i=1}^{m} (OGD(z_i))^2}}. \tag{4.4.9}$$

$$OLD^N(z_i) = \frac{OLD(z_i)}{\sqrt{\sum_{i=1}^{m} (OLD(z_i))^2}}. \tag{4.4.10}$$

The overall score is obtained by the following aggregation operator.

$$OS_i = OGD^N(z_i) \cdot \frac{m - r_1(z_i) + 1}{m(m+1)/2} - OLD^N(z_i) \cdot \frac{m - r_2(z_i) + 1}{m(m+1)/2}. \tag{4.4.11}$$

The alternatives are positioned in descending order of $OS_i (i = 1, 2, ..., m)$. Thus, a final rank set $S = \{r(z_1), r(z_2), ..., r(z_m)\}$ is obtained.

4.4.3 Consensus-based PUL-aggregation method

This subsection aims to present the consensus-based PUL-aggregation method for MCGDM problems, where assessment information of DMs are presented by PULEs. In this method, we collect and transform the values of cost criteria in global opinion matrix to be greatest. Let the assessment information of alternative z_i under (cost)criteria A_j be the PULE $u_s^{*ij}(p) = \left\{\left\langle [\pounds^{ij(e)}, U^{ij(e)}], p^{*i(e)} \right\rangle |e = 1, 2, ..., E\right\}$ with $\alpha^{ij(e)}$ and $\beta^{ij(e)}$ being the indices of the linguistic terms $\pounds^{ij(e)}$ and $U^{ij(e)}$, respectively. Then, transform it to

$$\bar{u}_s^{*ij}(p) = \left\{\left\langle [\bar{\pounds}^{ij(e)}, \bar{U}^{ij(e)}], p^{*i(e)} \right\rangle |e = 1, 2, ..., E\right\}, \tag{4.4.12}$$

with $\bar{\alpha}^{ij(e)} = -\beta^{ij(e)}$ and $\bar{\beta}^{ij(e)} = -\alpha^{ij(e)}$ being the indices of the linguistic terms $\bar{\pounds}^{ij(e)}$ and $\bar{U}^{ij(e)}$, respectively. Then, the aggregation operators defined in Section 4.3 are applied to aggregate the criteria values of alternatives into overall criteria value.

Definition 4.4.7. *Let $\bar{D} = [U_s^{*ij}(p)]_{m \times n}$ be a probabilistic uncertain linguistic global opinion matrix after maximization of the values of cost criteria. Then, the overall criteria value of alternative z_i can be computed as:*

case 1: *If PULWA operator is used:*

$$u_s^i(p) = \pi_1 u_s^{*i1}(p) \oplus \pi_2 u_s^{*i2}(p) \oplus ... \oplus \pi_n u_s^{*in}(p)$$
$$= \big\{\big\langle[\ell^{-1}\big(\pi_1\ell(\pounds^{i1}) + \pi_2\ell(\pounds^{i2}) + ... + \pi_n\ell(\pounds^{in})\big), \ell^{-1}\big(\pi_1\ell(U^{i1}) +$$
$$\pi_2\ell(U^{i2}) + ... + \pi_n\ell(U^{in})\big)], p^{*(e)}\big\rangle | e = 1, 2, ..., E\big\}, \tag{4.4.13}$$

where $i = 1, 2, ..., m$.

case 2: *If PULWG operator is used:*

$$u_s^i(p) = \big(u_s^{*i1}(p)\big)^{\pi_1} \otimes \big(u_s^{*i2}(p)\big)^{\pi_2} \otimes ... \otimes \big(u_s^{*in}(p)\big)^{\pi_n}$$
$$= \big\{\big\langle[\ell^{-1}\left(\big(\ell(\pounds^{i1})\big)^{\pi_1}\big(\ell(\pounds^{i2})\big)^{\pi_2} ... \big(\ell(\pounds^{in})\big)^{\pi_n}\right), g^{-1}\big(\big(\ell(U^{i1})\big)^{\pi_1}\big(\ell(U^{i2})\big)^{\pi_2}$$
$$... \big(\ell(U^{in})\big)^{\pi_n}\big)], p^{*(e)}\big\rangle | e = 1, 2, ..., E\big\}, \tag{4.4.14}$$

where $i = 1, 2, ..., m$.

After this, employ the comparison method presented in Section 4.2, to compare $u_s^i(p)(i = 1, 2, ..., m)$ mutually. As a result, the ranking of alternatives is collected.

4.4.4 The Decision making procedure

Based on the above analysis, we come up with two MCGDM methods named as consensus-based PUL-GLDS method and consensus-based PUL-aggregation method. This subsection mainly concerns to summarize their stepwise procedure as Algorithm 1 and Algorithm 2.

Algorithm 1

Step 1: Construct the individual opinion matrices:

Obtain the linguistic expressions from DMs and then construct the individual probabilistic uncertain linguistic opinion matrices $M^{(t)} = [u_s^{ij(t)}(p)]_{m\times n}(t = 1, 2, ..., T)$.

Step 2: Determine the overall opinion matrix:

Based on Eqs. (4.1.1) and (4.1.2) calculate the overall opinions of DMs and establish the overall probabilistic uncertain linguistic opinion matrix $\widehat{M}^{(t)} = [\widehat{u}_s^{ij}(p)]_{m\times n}$. Let $I = 1$.

Step 3. Determine the alternatives which need repairing:

Determine the correlation threshold θ and consensus threshold σ, which are generally within the bounds of 0.4 and 0.8. Then obtain the correlation coefficient and the consensus level σ^t in the light of Eq. (4.4.1). If $\sigma^t \geq \sigma(t = 1, 2, ..., T)$ then proceed to Step 5; otherwise proceed to the coming next step.

Step 3: Derive repair opinion matrices:

Find the opinion matrix $M^{(t)}$ with $\sigma^{(t)} = \min_\mu \sigma^\mu$. Build its improved opinion matrix $\acute{M}^{(t)}$ according to Eq. (4.4.2). Let $I = I + 1$, go to Step 2.

Step 4: Compute the global opinion matrix:

Combine the individual opinion matrices into a global opinion matrix $M = [u_s^{ij}(p)]_{m\times n}$ by using Eqs. (4.1.1) and (4.1.2).

Step 5: Calculate the PUL-dominance flows:

Based on Eq. (4.4.3) calculate the PUL-dominance flows and normalize them according to Eq. (4.4.4). Derive the PUL-gained dominance score by using Eq. (4.4.5) and the PUL-lost dominance score by using Eq. (4.4.6).

Step 6: Derive the overall dominance scores:

Derive the overall PUL-gained dominance score by Eq. (4.4.7) and the overall PUL-lost dominance score by Eq. (4.4.8).

Step 7: Determine the accessory ranks of alternatives:

Normalize the overall PUL-gained dominance score and the overall PUL-lost dominance score according to Eqs. (4.4.9) and (4.4.10) and thus got the accessory rank sets.

Step 8: Obtain the final ranking of alternatives:

Integrate the accessory ranking sets by using Eq. (4.4.11) and obtain the final ranking set. End.

Algorithm 2

Since, the first four steps are same as Steps 1-4 of Algorithm 1. Therefore, we go to Step $5'$.

Step $5'$: Maximize the values of cost attributes: Transform the values of cost attributes in global opinion matrix according to Eq. (4.4.12).

Step $6'$: Aggregate the attributes of alternatives: Based on Eq. (4.4.13) or (4.4.14) compute the overall criteria values of each alternative. Then follow next step.

Step $7'$: Obtain the ranking of alternatives: Get the ranking of alternatives according to Eqs. (4.2.6) and (4.2.7) and pick the best alternative. End.

For the facilitation of understanding, the flowchart of the developed methods is presented in Fig. 4.1.

4.5 A case study

This section presents a real world problem (adapted from [52]) concerning the selection of a commodity for investment in Forex. The consensus-based PUL-GLDS method and consensus-based PUL-aggregation method are utilized to handle this problem. In addition, some comparative analyses are made with the existing method to highlight the feasibility and efficiency of the proposed methods.

Case description: The Flagship Investment Company (FIC) provides its service for different kinds of investment plans. An investor is interested in Forex trading and wants to choose the best profitable commodity among gold, wheat, and oil denoted, as z_1, z_2 and z_3

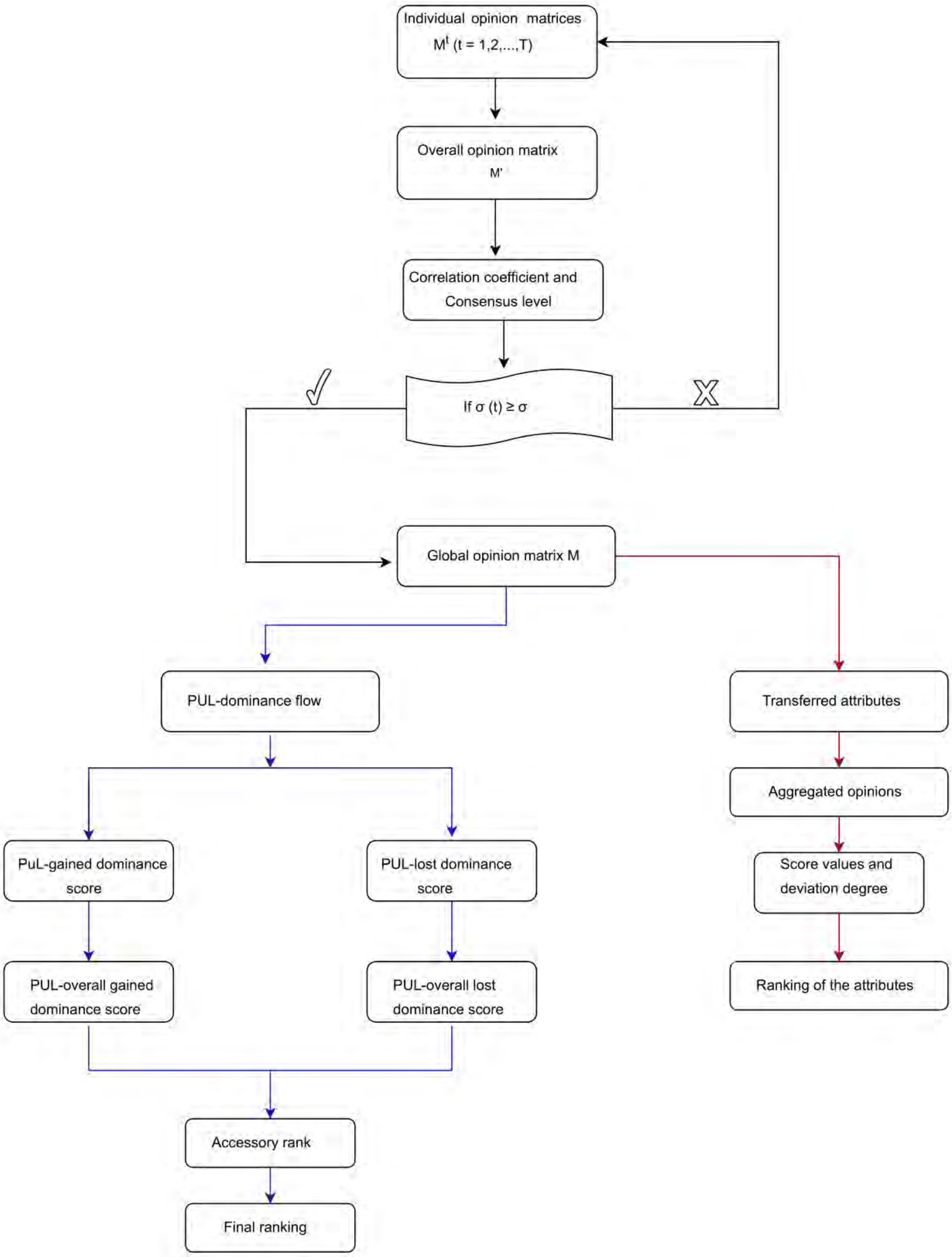

Figure 4.1: Consensus-based PUL-GLDS and Consensus-based PUL-aggregation methods

respectively. The profit of a commodity depends upon many factors; among them, four major factors are:

c_1 : Price stability(benefit), c_2 : Market demand(benefit), c_3 : Supply(benefit), c_4 : Trading cost(cost). Regard these factors $c_j(j = 1, 2, 3, 4)$ as four attributes and the weight information is $\pi_1 = 0.3$, $\pi_2 = 0.25$, $\pi_3 = 0.2$, $\pi_4 = 0.25$. The investor requested the FIC to suggest him/her the best commodity to invest in based on these attributes. The FIC formed a committee of four economic experts (DMs) d_1, d_2, d_3 and d_4 with the same importance to carry out the assessment on the basis of the LTS $S = \Big\{\pounds_{-3} = \text{very poor}, \pounds_{-2} = \text{poor}, \pounds_{-1} = \text{somewhat poor}, \pounds_0 = \text{medium}, \pounds_1 = \text{somewhat good}, \pounds_2 = \text{good}, \pounds_3 = \text{very good}\Big\}$. The four DMs have evaluated various commodities based on four criteria. Their opinions are displayed in based on the information in Table 4.1.

Table 4.1: Decision matrix

DMs	Factors	Criteria			
		c_1	c_2	c_3	c_4
d_1	z_1	$\{\langle[\pounds_{-1},\pounds_0],0.6\rangle,\langle[\pounds_0,\pounds_1],0.4\rangle\}$	$\{\langle[\pounds_1,\pounds_3],1\rangle\}$	$\{\langle[\pounds_{-2},\pounds_{-1}],1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],1\rangle\}$
	z_2	$\{\langle[\pounds_{-1},\pounds_0],1\rangle\}$	$\{\langle[\pounds_1,\pounds_2],0.8\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],0.7\rangle,\langle[\pounds_0,\pounds_1],0.3\rangle\}$	$\{\langle[\pounds_1,\pounds_2],0.8\rangle\}$
	z_3	$\{\langle[\pounds_{-1},\pounds_1],1\rangle\}$	$\{\langle[\pounds_0,\pounds_1],0.6\rangle,\langle[\pounds_1,\pounds_2],0.4\rangle\}$	$\{\langle[\pounds_1,\pounds_2],1\rangle\}$	$\{\langle[\pounds_0,\pounds_1],0.6\rangle,\langle[\pounds_1,\pounds_2],0.4\rangle\}$
d_2	z_1	$\{\langle[\pounds_0,\pounds_1],0.4\rangle,\langle[\pounds_1,\pounds_2],0.6\rangle\}$	$\{\langle[\pounds_1,\pounds_2],0.9\rangle,\langle[\pounds_2,\pounds_3],0.1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],1\rangle\}$	$\{\langle[\pounds_0,\pounds_1],0.6\rangle,\langle[\pounds_1,\pounds_2],0.4\rangle\}$
	z_2	$\{\langle[\pounds_0,\pounds_1],0.4\rangle,\langle[\pounds_1,\pounds_2],0.6\rangle\}$	$\{\langle[\pounds_{-2},\pounds_{-1}],1\rangle\}$	$\{\langle[\pounds_{-2},\pounds_0],1\rangle\}$	$\{\langle[\pounds_2,\pounds_3],1\rangle\}$
	z_3	$\{\langle[\pounds_1,\pounds_2],1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],0.4\rangle,\langle[\pounds_0,\pounds_1],0.6\rangle\}$	$\{\langle[\pounds_0,\pounds_1],0.8\rangle\}$	$\{\langle[\pounds_0,\pounds_1],1\rangle\}$
d_3	z_1	$\{\langle[\pounds_1,\pounds_2],1\rangle\}$	$\{\langle[\pounds_0,\pounds_1],1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],0.4\rangle,\langle[\pounds_0,\pounds_1],0.6\rangle\}$
	z_2	$\{\langle[\pounds_0,\pounds_1],0.4\rangle,\langle[\pounds_1,\pounds_2],0.6\rangle\}$	$\{\langle[\pounds_1,\pounds_2],0.75\rangle\}$	$\{\langle[\pounds_{-2},\pounds_{-1}],0.4\rangle,\langle[\pounds_{-1},\pounds_0],0.4\rangle\}$	$\{\langle[\pounds_1,\pounds_2],1\rangle\}$
	z_3	$\{\langle[\pounds_0,\pounds_1],0.9\rangle\}$	$\{\langle[\pounds_0,\pounds_1],0.6\rangle,\langle[\pounds_1,\pounds_2],0.4\rangle\}$	$\{\langle[\pounds_0,\pounds_1],1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],1\rangle\}$
d_4	z_1	$\{\langle[\pounds_{-2},\pounds_1],0.8\rangle,\langle[\pounds_{-1},\pounds_0],0.2\rangle\}$	$\{\langle[\pounds_1,\pounds_2],0.75\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],1\rangle\}$	$\{\langle[\pounds_{-3},\pounds_{-2}],0.9\rangle\}$
	z_2	$\{\langle[\pounds_1,\pounds_2],1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_1],1\rangle\}$	$\{\langle[\pounds_2,\pounds_3],1\rangle\}$
	z_3	$\{\langle[\pounds_{-1},\pounds_0],0.7\rangle,\langle[\pounds_0,\pounds_1],0.3\rangle\}$	$\{\langle[\pounds_1,\pounds_2],0.4\rangle,\langle[\pounds_0,\pounds_1],0.6\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],0.6\rangle,\langle[\pounds_0,\pounds_1],0.4\rangle\}$	$\{\langle[\pounds_{-3},\pounds_{-2}],1\rangle\}$

4.5.1 Handling the case by PUL-GLDS based method

Below the proposed consensus-based PUL-GLDS method is used to solve the case study.

Step 1: The ordered and associated decision matrix derived from the provided data is depicted in Table 4.2.

Table 4.2: Ordered and associated decision matrix

DMs	Factors	Criteria			
		c_1	c_2	c_3	c_4
d_1	z_1	$\{\langle[\pounds_{-1},\pounds_0],0.6\rangle,\langle[\pounds_0,\pounds_1],0.4\rangle\}$	$\{\langle[\pounds_1,\pounds_2],0.5\rangle,\langle[\pounds_2,\pounds_3],0.5\rangle\}$	$\{\langle[\pounds_{-2},\pounds_{-1}],1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],1\rangle\}$
	z_2	$\{\langle[\pounds_{-1},\pounds_0],1\rangle\}$	$\{\langle[\pounds_1,\pounds_2],1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],0.7\rangle,\langle[\pounds_0,\pounds_1],0.3\rangle\}$	$\{\langle[\pounds_1,\pounds_2],1\rangle\}$
	z_3	$\{\langle[\pounds_{-1},\pounds_0],0.5\rangle,\langle[\pounds_0,\pounds_1],0.5\rangle\}$	$\{\langle[\pounds_0,\pounds_1],0.6\rangle,\langle[\pounds_1,\pounds_2],0.4\rangle\}$	$\{\langle[\pounds_1,\pounds_2],1\rangle\}$	$\{\langle[\pounds_0,\pounds_1],0.6\rangle,\langle[\pounds_1,\pounds_2],0.4\rangle\}$
d_2	z_1	$\{\langle[\pounds_0,\pounds_1],0.4\rangle,\langle[\pounds_1,\pounds_2],0.6\rangle\}$	$\{\langle[\pounds_1,\pounds_2],0.9\rangle,\langle[\pounds_2,\pounds_3],0.1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],1\rangle\}$	$\{\langle[\pounds_0,\pounds_1],0.6\rangle,\langle[\pounds_1,\pounds_2],0.4\rangle\}$
	z_2	$\{\langle[\pounds_0,\pounds_1],0.4\rangle,\langle[\pounds_1,\pounds_2],0.6\rangle\}$	$\{\langle[\pounds_{-2},\pounds_{-1}],1\rangle\}$	$\{\langle[\pounds_{-2},\pounds_{-1}],0.5\rangle,\langle[\pounds_{-1},\pounds_0],0.5\rangle\}$	$\{\langle[\pounds_2,\pounds_3],1\rangle\}$
	z_3	$\{\langle[\pounds_1,\pounds_2],1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],0.4\rangle,\langle[\pounds_0,\pounds_1],0.6\rangle\}$	$\{\langle[\pounds_0,\pounds_1],1\rangle\}$	$\{\langle[\pounds_0,\pounds_1],1\rangle\}$
d_3	z_1	$\{\langle[\pounds_1,\pounds_2],1\rangle\}$	$\{\langle[\pounds_0,\pounds_1],1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],0.4\rangle,\langle[\pounds_0,\pounds_1],0.6\rangle\}$
	z_2	$\{\langle[\pounds_0,\pounds_1],0.4\rangle,\langle[\pounds_1,\pounds_2],0.6\rangle\}$	$\{\langle[\pounds_1,\pounds_2],1\rangle\}$	$\{\langle[\pounds_{-2},\pounds_{-1}],0.5\rangle,\langle[\pounds_{-1},\pounds_0],0.5\rangle\}$	$\{\langle[\pounds_1,\pounds_2],1\rangle\}$
	z_3	$\{\langle[\pounds_0,\pounds_1],1\rangle\}$	$\{\langle[\pounds_0,\pounds_1],0.6\rangle,\langle[\pounds_1,\pounds_2],0.4\rangle\}$	$\{\langle[\pounds_0,\pounds_1],1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],1\rangle\}$
d_4	z_1	$\{\langle[\pounds_{-2},\pounds_{-1}],0.8\rangle,\langle[\pounds_{-1},\pounds_0],0.2\rangle\}$	$\{\langle[\pounds_1,\pounds_2],1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],1\rangle\}$	$\{\langle[\pounds_{-3},\pounds_{-2}],1\rangle\}$
	z_2	$\{\langle[\pounds_1,\pounds_2],1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],1\rangle,\}$	$\{\langle[\pounds_{-1},\pounds_0],0.5\rangle\langle[\pounds_0,\pounds_1],0.5\rangle\}$	$\{\langle[\pounds_2,\pounds_3],1\rangle\}$
	z_3	$\{\langle[\pounds_{-1},\pounds_0],0.7\rangle,\langle[\pounds_0,\pounds_1],0.3\rangle\}$	$\{\langle[\pounds_0,\pounds_1],0.6\rangle,\langle[\pounds_1,\pounds_2],0.4\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],0.6\rangle,\langle[\pounds_0,\pounds_1],0.4\rangle\}$	$\{\langle[\pounds_{-3},\pounds_{-2}],1\rangle\}$

Step 2: Since it is assumed that the four DMs have equal importance. Following this, we aggregate the four individual opinion matrices into the overall opinion matrix by using Eq. (4.1.1). Thus, the overall opinion matrix is built as:

Table 4.3: Overall Opinion matrix

	c_1	c_2	c_3	c_4
z_1	$\{\langle[\mathcal{L}_{-2},\mathcal{L}_{-1}],0.2\rangle,\langle[\mathcal{L}_{-1},\mathcal{L}_{0}],0.2\rangle,\langle[\mathcal{L}_{0},\mathcal{L}_{1}],0.2\rangle,\langle[\mathcal{L}_{1},\mathcal{L}_{2}],0.4\rangle\}$	$\{\langle[\mathcal{L}_{0},\mathcal{L}_{1}],0.25\rangle,\langle[\mathcal{L}_{1},\mathcal{L}_{2}],0.6\rangle,\langle[\mathcal{L}_{2},\mathcal{L}_{3}],0.15\rangle\}$	$\{\langle[\mathcal{L}_{-2},\mathcal{L}_{-1}],0.25\rangle,\langle[\mathcal{L}_{-1},\mathcal{L}_{0}],0.75\rangle\}$	$\{\langle[\mathcal{L}_{-3},\mathcal{L}_{-2}],0.25\rangle,\langle[\mathcal{L}_{-1},\mathcal{L}_{0}],0.35\rangle,\langle[\mathcal{L}_{0},\mathcal{L}_{1}],0.3\rangle,\langle[\mathcal{L}_{1},\mathcal{L}_{2}],0.1\rangle\}$
z_2	$\{\langle[\mathcal{L}_{-1},\mathcal{L}_{0}],0.25\rangle,\langle[\mathcal{L}_{0},\mathcal{L}_{1}],0.2\rangle,\langle[\mathcal{L}_{1},\mathcal{L}_{2}],0.55\rangle\}$	$\{\langle[\mathcal{L}_{-2},\mathcal{L}_{-1}],0.25\rangle,\langle[\mathcal{L}_{-1},\mathcal{L}_{0}],0.25\rangle,\langle[\mathcal{L}_{1},\mathcal{L}_{2}],0.5\rangle\}$	$\{\langle[\mathcal{L}_{-2},\mathcal{L}_{-1}],0.25\rangle,\langle[\mathcal{L}_{-1},\mathcal{L}_{0}],0.55\rangle,\langle[\mathcal{L}_{0},\mathcal{L}_{1}],0.2\rangle\}$	$\{\langle[\mathcal{L}_{1},\mathcal{L}_{2}],0.5\rangle,\langle[\mathcal{L}_{2},\mathcal{L}_{3}],0.5\rangle\}$
z_3	$\{\langle[\mathcal{L}_{-1},\mathcal{L}_{0}],0.3\rangle,\langle[\mathcal{L}_{0},\mathcal{L}_{1}],0.45\rangle,\langle[\mathcal{L}_{1},\mathcal{L}_{2}],0.25\rangle\}$	$\{\langle[\mathcal{L}_{-1},\mathcal{L}_{0}],0.1\rangle,\langle[\mathcal{L}_{0},\mathcal{L}_{1}],0.6\rangle,\langle[\mathcal{L}_{1},\mathcal{L}_{2}],0.3\rangle\}$	$\{\langle[\mathcal{L}_{-1},\mathcal{L}_{0}],0.15\rangle,\langle[\mathcal{L}_{0},\mathcal{L}_{1}],0.6\rangle,\langle[\mathcal{L}_{1},\mathcal{L}_{2}],0.25\rangle\}$	$\{\langle[\mathcal{L}_{-3},\mathcal{L}_{-2}],0.25\rangle,\langle[\mathcal{L}_{-1},\mathcal{L}_{0}],0.25\rangle,\langle[\mathcal{L}_{0},\mathcal{L}_{1}],0.4\rangle,\langle[\mathcal{L}_{1},\mathcal{L}_{2}],0.1\rangle\}$

Step 3: Adjust the opinion matrices Tables 4.1, 4.3 according to the adjusting rule of probability studied in Section 4.1.2, and then choose the linguistic scale function given in Eq. (2.1.2). Taking the correlation threshold $\theta = 0.4$ and the consensus threshold $\sigma = 0.6$. Based on Eqs. (4.3.6) and (4.4.1) the computed values of correlation coefficients and consensus degrees of the DMs are depicted in Table 4.4.

Table 4.4: Correlation coefficient and consensus level of DMs

DMs	Correlation coefficient				Consensus level
d_1	-0.0931	0.9776	0.9721	0.7402	0.6492
d_2	0.2078	0.9844	0.9538	-0.4030	0.4357
d_3	0.9479	0.9783	0.9547	-0.4136	0.6168
d_4	0.9897	0.9264	0.1651	0.9998	0.7703

Step 4: From the shown results in Table 4.4, it is clear that the PUL-consensus level of d_2 is less than the consensus threshold, in addition, the correlation coefficient with respect

to attributes c_1 and c_4 is also less than the correlation threshold. Therefore, based on Eq. (4.4.2), repair the opinions of d_2 with respect to attributes c_1 and c_4.

Table 4.5: Repair opinion matrix R_2' provided by d_2

	c_1	c_2	c_3	c_4
z_1	$\{\langle[\pounds_{-1},\pounds_0],0.2\rangle,\langle[\pounds_{-0.5},\pounds_{0.5}],0.2\rangle,\langle[\pounds_{0.5},\pounds_{1.5}],0.2\rangle,\langle[\pounds_1,\pounds_2],0.4\rangle\}$	$\{\langle[\pounds_1,\pounds_2],0.9\rangle,\langle[\pounds_2,\pounds_3],0.1\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],1\rangle\}$	$\{\langle[\pounds_{-1.5},\pounds_{-0.5}],0.25\rangle,\langle[\pounds_{-0.5},\pounds_{0.5}],0.35\rangle,\langle[\pounds_{0.5},\pounds_{1.5}],0.3\rangle,\langle[\pounds_1,\pounds_2],0.1\rangle\}$
z_2	$\{\langle[\pounds_{-0.5},\pounds_{0.5}],0.25\rangle,\langle[\pounds_0,\pounds_1],0.15\rangle,\langle[\pounds_{0.5},\pounds_{1.5}],0.05\rangle,\langle[\pounds_1,\pounds_2],0.55\rangle\}$	$\{\langle[\pounds_{-2},\pounds_{-1}],1\rangle\}$	$\{\langle[\pounds_{-2},\pounds_0],1\rangle\}$	$\{\langle[\pounds_{1.5},\pounds_{2.5}],0.5\rangle,\langle[\pounds_2,\pounds_3],0.5\rangle\}$
z_3	$\{\langle[\pounds_0,\pounds_1],0.3\rangle,\langle[\pounds_{0.5},\pounds_{1.5}],0.45\rangle,\langle[\pounds_1,\pounds_2],0.25\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],0.4\rangle,\langle[\pounds_0,\pounds_1],0.6\rangle\}$	$\{\langle[\pounds_0,\pounds_1],0.8\rangle\}$	$\{\langle[\pounds_{-1.5},\pounds_{-0.5}],0.25\rangle,\langle[\pounds_{-0.5},\pounds_{0.5}],0.25\rangle,\langle[\pounds_0,\pounds_1],0.4\rangle,\langle[\pounds_{0.5},\pounds_{1.5}],0.1\rangle\}$

Step 5: By Eq. (4.1.1), the repair overall opinion matrix D is established as shown in Table 4.6:

Table 4.6: Global opinion matrix

	c_1	c_2	c_3	c_4
z_1	$\{\langle[\pounds_{-2},\pounds_{-1}],0.2\rangle,\langle[\pounds_{-1},\pounds_0],0.25\rangle,\langle[\pounds_{-0.5},\pounds_{0.5}],0.05\rangle,\langle[\pounds_0,\pounds_1],0.1\rangle,\langle[\pounds_{0.5},\pounds_{1.5}],0.05\rangle,\langle[\pounds_1,\pounds_2],0.35\rangle\}$	$\{\langle[\pounds_0,\pounds_1],0.25\rangle,\langle[\pounds_1,\pounds_2],0.6\rangle,\langle[\pounds_2,\pounds_3],0.15\rangle\}$	$\{\langle[\pounds_{-2},\pounds_{-1}],0.25\rangle,\langle[\pounds_{-1},\pounds_0],0.75\rangle\}$	$\{\langle[\pounds_{-3},\pounds_{-2}],0.25\rangle,\langle[\pounds_{-1.5},\pounds_{-0.5}],0.0625\rangle,\langle[\pounds_{-1},\pounds_0],0.35\rangle,\langle[\pounds_{-0.5},\pounds_{0.5}],0.0875\rangle,\langle[\pounds_0,\pounds_1],0.15\rangle,\langle[\pounds_{0.5},\pounds_{1.5}],0.075\rangle,\langle[\pounds_1,\pounds_2],0.025\rangle\}$
z_2	$\{\langle[\pounds_{-1},\pounds_0],0.25\rangle,\langle[\pounds_{-0.5},\pounds_{0.5}],0.0625\rangle,\langle[\pounds_0,\pounds_1],0.1375\rangle,\langle[\pounds_{0.5},\pounds_{1.5}],0.0125\rangle,\langle[\pounds_1,\pounds_2],0.5375\rangle\}$	$\{\langle[\pounds_{-2},\pounds_{-1}],0.25\rangle,\langle[\pounds_{-1},\pounds_0],0.25\rangle,\langle[\pounds_1,\pounds_2],0.5\rangle\}$	$\{\langle[\pounds_{-2},\pounds_{-1}],0.25\rangle,\langle[\pounds_{-1},\pounds_0],0.55\rangle,\langle[\pounds_0,\pounds_1],0.2\rangle\}$	$\{\langle[\pounds_1,\pounds_2],0.5\rangle,\langle[\pounds_{1.5},\pounds_{2.5}],0.125\rangle,\langle[\pounds_2,\pounds_3],0.375\rangle\}$
z_3	$\{\langle[\pounds_{-1},\pounds_0],0.3\rangle,\langle[\pounds_0,\pounds_1],0.525\rangle,\langle[\pounds_{0.5},\pounds_{1.5}],0.1125\rangle\langle[\pounds_1,\pounds_2],0.0625\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],0.1\rangle,\langle[\pounds_0,\pounds_1],0.6\rangle,\langle[\pounds_1,\pounds_2],0.3\rangle\}$	$\{\langle[\pounds_{-1},\pounds_0],0.15\rangle,\langle[\pounds_0,\pounds_1],0.6\rangle,\langle[\pounds_1,\pounds_2],0.25\rangle\}$	$\{\langle[\pounds_{-3},\pounds_{-2}],0.0625\rangle,\langle[\pounds_{-1.5},\pounds_{-0.5}],0.0625\rangle,\langle[\pounds_{-1},\pounds_0],0.3375\rangle,\langle[\pounds_{-0.5},\pounds_{0.5}],0.0625\rangle,\langle[\pounds_0,\pounds_1],0.325\rangle,\langle[\pounds_{0.5},\pounds_{1.5}],0.025\rangle,\langle[\pounds_1,\pounds_2],0.125\rangle\}$

Derive the PUL-consensus level by using Eq. (4.4.1), one can determine that the group assessments rise to the consensus threshold. Therefore, we stop the repairing process.

Step 6: Compute the normalized dominance flow according to Eqs. (4.4.3) and (4.4.4), which are placed in Table 4.7.

Table 4.7: Normalized Dominance flow

	c_1	c_2	c_3	c_4
$z_1 \to z_2$	0.0000	0.0367	0.0000	0.0000
$z_1 \to z_3$	0.0018	0.0136	0.0000	0.0000
$z_2 \to z_1$	0.0153	0.0000	0.0011	0.1850
$z_2 \to z_3$	0.0056	0.0011	0.0000	0.1056
$z_3 \to z_1$	0.0045	0.0000	0.0506	0.0109
$z_3 \to z_2$	0.000001	0.0108	0.0367	0.0000

Now based on Eqs. (4.4.5) and (4.4.6), calculate the PUL-gained dominance score and the PUL-lost dominance score, which are documented in Table 4.8 and 4.9, respectively.

Table 4.8: The PUL-gained dominance score of each alternative

Alternatives	Gained dominance score				Net gained dominance score
	c_1	c_2	c_3	c_4	
z_1	0.2589	1.2365	0.0000	0.0000	0.3868
z_2	1.2041	0.1335	0.1120	1.3754	0.7608
z_3	0.4107	0.4342	1.4016	0.1908	0.5598

Table 4.9: The PUL-lost dominance score of each alternative

Alternatives	Lost dominance score				Net lost dominance score
	c_1	c_2	c_3	c_4	
z_1	0.2589	0.6804	0.0000	0.7492	0.1921
z_2	0.2248	0.0333	0.0224	0.1958	0.0224
z_3	0.1213	0.1085	0.1513	0.6556	0.1514

Step 7: Determine the normalized net gained dominance score and the normalized net lost dominance score of each alternative in the light of Eqs. (4.4.9) and (4.4.10), respectively, which are written below.

The normalized net gained dominance score: $DS_1^N(z_1) = 0.3789$, $DS_1^N(z_2) = 0.7454$, $DS_1^N(z_3) = 0.5484$.

The normalized net lost dominance score: $DS_2^N(z_1) = 0.5784$, $DS_2^N(z_2) = 0.6766$, $DS_2^N(z_3) = 0.4557$.

Step 8: By Eqs. (4.4.9), and (4.4.10), calculate the accessory rank sets of alternatives as: $r_1(z_1) = 0.3868, r_1(z_2) = 0.7608, r_1(z_3) = 0.5598$ and $r_2(z_1) = 0.1921, r_2(z_2) = 0.2248, r_2(z_3) = 0.1514$.

Step 9: Lastly, on the basis of Eq. (4.4.11) we get the score of all alternatives as: $OS_1 = -0.1389$, $OS_2 = -0.0233$, $OS_3 = 0.0221$.

Thus z_3 is at the top of the range.

4.5.2 Handling the case by PUL-aggregation based method

Now, we deal with the same problem by the consensus-based PUL-aggregation method introduced in Section 4.4.3 to determine the optimal alternative. The decision-making steps are presented as follows:

Since the first four steps are similar to that of Algorithm 1. Therefore, we go to Step $5'$.
Step $5'$: Since the criteria c_4 is cost type, on that account, transform the values along c_4 according to Eq. (4.4.12). The obtained matrix is shown in Table 4.10.

Table 4.10: Transformed opinion matrix

	c_1	c_2	c_3	c_4
z_1	$\{\langle[\pounds_{-2}, \pounds_{-1}], 0.2\rangle, \langle[\pounds_{-1}, \pounds_{0}], 0.25\rangle, \langle[\pounds_{-0.5}, \pounds_{0.5}], 0.05\rangle, \langle[\pounds_{0}, \pounds_{1}], 0.1\rangle, \langle[\pounds_{0.5}, \pounds_{1.5}], 0.05\rangle, \langle[\pounds_{1}, \pounds_{2}], 0.35\rangle\}$	$\{\langle[\pounds_{0}, \pounds_{1}], 0.25\rangle, \langle[\pounds_{1}, \pounds_{2}], 0.6\rangle, \langle[\pounds_{2}, \pounds_{3}], 0.15\rangle\}$	$\{\langle[\pounds_{-2}, \pounds_{-1}], 0.25\rangle, \langle[\pounds_{-1}, \pounds_{0}], 0.75\rangle\}$	$\{\langle[\pounds_{-2}, \pounds_{-1}], 0.025\rangle, \langle[\pounds_{-1.5}, \pounds_{-0.5}], 0.075\rangle, \langle[\pounds_{-1}, \pounds_{0}], 0.15\rangle, \langle[\pounds_{-0.5}, \pounds_{0.5}], 0.0875\rangle, \langle[\pounds_{0.5}, \pounds_{1.5}], 0.0625\rangle, \langle[\pounds_{0}, \pounds_{1}], 0.35\rangle, \langle[\pounds_{2}, \pounds_{3}], 0.25\rangle\}$
z_2	$\{\langle[\pounds_{-1}, \pounds_{0}], 0.25\rangle, \langle[\pounds_{-0.5}, \pounds_{0.5}], 0.0625\rangle, \langle[\pounds_{0}, \pounds_{1}], 0.1375\rangle, \langle[\pounds_{0.5}, \pounds_{1.5}], 0.0125\rangle, \langle[\pounds_{1}, \pounds_{2}], 0.5375\rangle\}$	$\{\langle[\pounds_{-2}, \pounds_{-1}], 0.25\rangle, \langle[\pounds_{-1}, \pounds_{0}], 0.25\rangle, \langle[\pounds_{1}, \pounds_{2}], 0.5\rangle\}$	$\{\langle[\pounds_{-2}, \pounds_{-1}], 0.25\rangle, \langle[\pounds_{-1}, \pounds_{0}], 0.55\rangle, \langle[\pounds_{0}, \pounds_{1}], 0.2\rangle\}$	$\{\langle[\pounds_{-3}, \pounds_{-2}], 0.375\rangle, \langle[\pounds_{-2.5}, \pounds_{-1.5}], 0.125\rangle, \langle[\pounds_{-2}, \pounds_{-1}], 0.5\rangle\}$
z_3	$\{\langle[\pounds_{-1}, \pounds_{0}], 0.3\rangle, \langle[\pounds_{0}, \pounds_{1}], 0.525\rangle, \langle[\pounds_{0.5}, \pounds_{1.5}], 0.1125\rangle\langle[\pounds_{1}, \pounds_{2}], 0.0625\rangle\}$	$\{\langle[\pounds_{-1}, \pounds_{0}], 0.1\rangle, \langle[\pounds_{0}, \pounds_{1}], 0.6\rangle, \langle[\pounds_{1}, \pounds_{2}], 0.3\rangle\}$	$\{\langle[\pounds_{-1}, \pounds_{0}], 0.15\rangle, \langle[\pounds_{0}, \pounds_{1}], 0.6\rangle, \langle[\pounds_{1}, \pounds_{2}], 0.25\rangle\}$	$\{\langle[\pounds_{-2}, \pounds_{-1}], 0.125\rangle, \langle[\pounds_{-1.5}, \pounds_{-0.5}], 0.025\rangle, \langle[\pounds_{-1}, \pounds_{0}], 0.325\rangle, \langle[\pounds_{-0.5}, \pounds_{0.5}], 0.0625\rangle, \langle[\pounds_{0}, \pounds_{1}], 0.3375\rangle, \langle[\pounds_{0.5}, \pounds_{1.5}], 0.0625\rangle, \langle[\pounds_{2}, \pounds_{3}], 0.0625\rangle\}$

Step $6'$: Utilize the PULWA operator to aggregate individual decision opinions into a collective one, as shown below. The calculation process of the PULWA operator can be found as Eq. (4.4.13).

$$
\begin{aligned}
u_s^1(p) =\big\{ & \langle[\pounds_{-1.5}, \pounds_{-0.4998}], 0.025\rangle, \langle[\pounds_{-1.3752}, \pounds_{-0.375}], 0.075\rangle, \langle[\pounds_{-1.2528}, \pounds_{-0.2502}], 0.1\rangle, \\
& \langle[\pounds_{-0.9498}, \pounds_{0.0498}], 0.05\rangle, \langle[\pounds_{-0.6252}, \pounds_{0.375}], 0.0875\rangle, \langle[\pounds_{-0.2502}, \pounds_{0.75}], 0.1125\rangle, \\
& \langle[\pounds_{-0.1002}, \pounds_{0.9}], 0.05\rangle, \langle[\pounds_{0.0498}, \pounds_{1.05}], 0.1\rangle\langle[\pounds_{0.1998}, \pounds_{1.2}], 0.05\rangle, \\
& \langle[\pounds_{0.3498}, \pounds_{1.35}], 0.0375\rangle, \langle[\pounds_{0.4746}, \pounds_{1.4748}], 0.0625\rangle, \langle[\pounds_{0.8502}, \pounds_{1.8498}], 0.1\rangle, \\
& \langle[\pounds_{1.0998}, \pounds_{2.1}], 0.15\rangle\big\},
\end{aligned}
$$

$$
\begin{aligned}
u_s^2(p) =\big\{ & \langle[\pounds_{-1.95}, \pounds_{-0.9504}], 0.25\rangle, \langle[\pounds_{-1.35}, \pounds_{-0.3498}], 0.0625\rangle, \langle[\pounds_{-1.6002}, \pounds_{-0.1998}], 0.0625\rangle, \\
& \langle[\pounds_{-1.0752}, \pounds_{-0.075}], 0.075\rangle, \langle[\pounds_{-0.9252}, \pounds_{0.075}], 0.0125\rangle, \langle[\pounds_{-0.7752}, \pounds_{0.225}], 0.0375\rangle, \\
& \langle[\pounds_{-0.15}, \pounds_{0.8496}], 0.3\rangle, \langle[\pounds_{0.0498}, \pounds_{1.05}], 0.2\rangle\big\},
\end{aligned}
$$

$$
\begin{aligned}
u_s^3(p) =& \big\{\langle[\pounds_{-1.2498}, \pounds_{-0.2502}], 0.1\rangle, \langle[\pounds_{-1.0002}, \pounds_{0}], 0.025\rangle, \langle[\pounds_{-0.8748}, \pounds_{0.1248}], 0.025\rangle,\\
&\langle[\pounds_{-0.5502}, \pounds_{0.45}], 0.15\rangle, \langle[\pounds_{-0.2502}, \pounds_{0.75}], 0.175\rangle, \langle[\pounds_{-0.1254}, \pounds_{0.8748}], 0.0625\rangle,\\
&\langle[\pounds_{0}, \pounds_{1.002}], 0.1625\rangle, \langle[\pounds_{0.2502}, \pounds_{1.2498}], 0.05\rangle, \langle[\pounds_{0.45}, \pounds_{1.4502}], 0.075\rangle,\\
&\langle[\pounds_{0.6}, \pounds_{1.6002}], 0.05\rangle, \langle[\pounds_{0.7248}, \pounds_{1.725}], 0.0625\rangle, \langle[\pounds_{1.2498}, \pounds_{2.25}], 0.0625\rangle\big\}.
\end{aligned}
$$

Step $7'$: Calculate the score values $F(z_i)(i = 1, 2, 3)$ of the overall assessment values in the light of Eq. (4.2.6). Hence, the following results are achieved:

$F(z_1) = 0.4473,\ F(z_2) = 0.5456,\ F(z_3) = 0.5667.$

According to the derived score values $F(z_i)(i = 1, 2, 3)$, it is clear that commodity z_3 is the best alternative.

4.5.3 Comparative analysis and discussion

Next, to illustrate the strength of the proposed methods, the existing aggregation-based method is utilized given by Lin et al. [32] to solve the considered problem. The following steps of the Lin et al. [32] approach have been executed as follows:

Step $1''$: Firstly, normalize overall opinion matrix listed in Table 4.3 according to Definition 2.2.2.

Step $2''$: Since, the weight vector of attributes is given in advance, so there is no need to compute it.

Step $3''$: Taking $W = (0.3, 0.25, 0.2, 0.25)$ and use PULWA operator given in Eq. (5) of [32] to get the fused value of each alternative z_i.

$$
\begin{aligned}
u_s^1(p) &= \{\langle[\pounds_{-0.4075}, \pounds_{-0.1725}]\rangle, \langle[\pounds_{-0.1475}, \pounds_{0.3}]\rangle, \langle[\pounds_{0.06}, \pounds_{0.2475}]\rangle, \langle[\pounds_{0.145}, \pounds_{0.29}]\rangle\},\\
u_s^2(p) &= \{\langle[\pounds_{-0.25}, \pounds_{0.1375}]\rangle, \langle[\pounds_{0.0775}, \pounds_{0.435}]\rangle, \langle[\pounds_{0.29}, \pounds_{0.62}]\rangle\rangle\},\\
u_s^3(p) &= \{\langle[\pounds_{-0.3325}, \pounds_{-0.125}]\rangle, \langle[\pounds_{-0.0625}, \pounds_{0.405}]\rangle, \langle[\pounds_{0.2}, \pounds_{0.5}]\rangle, \langle[\pounds_{0.025}, \pounds_{0.05}]\rangle\}.
\end{aligned}
$$

Step $4''$: By Definition 9 of Ref. [32], we get $F(z_1) = 0.69125,\ F(z_2) = 0.655,\ F(z_3) = 0.33.$

Step $5''$: Since $F(z_1) > F(z_2) > F(z_3)$ and thus ranking order of their corresponding alternatives is $z_1 > z_2 > z_3$.

Contrary to PULWA operator, if we utilize PULWG operator then the following steps are executed as:

Step $1'''$: Similar to above Step $1''$.

Step $2'''$: Similar to above Step $2''$.

Step $3'''$: Utilize PULWG operator given in Eq. (7) of [32] to get the fused value of each alternative z_i.

$$\begin{aligned}
u_s^1(p) &= \{\langle[\pounds_{-3.1488}, \pounds_{-2.0443}]\rangle, \langle[\pounds_{-2}, \pounds_{1.1096}]\rangle, \langle[\pounds_{2.0263}, \pounds_{4.042}]\rangle, \langle[\pounds_{3}, \pounds_{4.0104}]\rangle\}, \\
u_s^2(p) &= \{\langle[\pounds_{-2.0795}, \pounds_{-0.9095}]\rangle, \langle[\pounds_{-0.9095}, \pounds_{2.1472}]\rangle, \langle[\pounds_{3}, \pounds_{4.2117}]\rangle\rangle\}, \\
u_s^3(p) &= \{\langle[\pounds_{-4.071}, \pounds_{-1.0443}]\rangle, \langle[\pounds_{-1}, \pounds_{3}]\rangle, \langle[\pounds_{3}, \pounds_{4.1419}]\rangle, \langle[\pounds_{-2}, \pounds_{1.0175}]\rangle\}.
\end{aligned}$$

Step $4'''$: The score values by using Definition 9 of Ref. [32], are $F(z_1) = 3.5445,\ F(z_2) = 2.1452,\ F(z_3) = 1.0308$.

Step $5'''$: Since $F(z_1) > F(z_2) > F(z_3)$ and thus ranking order of their corresponding alternatives is $z_1 > z_2 > z_3$.

From Table 4.11, it is evident that according to the proposed methods, z_3 is the best alternative and z_1 is the worst one. However, the ranking results of [32] approach are quite different. Here are the reasons for this:

i. Improved operations:

   The main reason behind the variation in the sequence of alternatives is the redefined operational laws and aggregation operators. The utilized aggregation operators in [32]

Table 4.11: Results obtained by three methods

Methods	Alternative ranking
Consensus-based PUL-GLDS method	$z_3 > z_2 > z_1$
Consensus-based PUL-aggregation method	$z_3 > z_2 > z_1$
Aggregation-based method (using PULWA operator) [32]	$z_1 > z_2 > z_3$
Aggregation-based method (utilizing PULWG operator) [32]	$z_1 > z_2 > z_3$

approach are based on such operations in which we need to add artificial terms in order to equalize the length of the data. These added artificial terms render the evaluation rough and one-sided.

ii. Consensus reaching approach:

The second major reason behind this difference is that the designed methods make use of consensus reaching approach. Due to the limited ability of human thinking and experiences, one expert may be familiar with some attributes but unfamiliar with other attributes. It is rational to remove biased assessments of DM if the correlation coefficient and consensus level are extremely low. Therefore, we repair the assessments of DM under those attributes that have less consensus with other DMs. The results displayed in Table 4.4 shows that the PUL-consensus level of d_2 is less than consensus threshold. Also, the correlation coefficient with respect to attributes c_1 and c_4 of d_2 is less than the correlation threshold. Therefore, we have improved the assessments of d_2 regarding c_1 and c_4 as displayed in Table 4.5.

iii. Attributes classification:

Tables 4.8 and 4.9 indicates that regarding different attributes (benefit and cost type), normalization technique is conducted before integrating the dominance flow and overall dominance score while the existing method [32] does not categorize the attributes into benefit and cost type. Though criteria c_4 is of cost type, but still it is treated as other attributes which cause a strong effect on the final decision.

iv. "Group utility rates" and " individual regret rates":

In the PUL-GLDS method, we are provided with the overall PUL-gained dominance score (group utility rate) of all alternative based on the weighted arithmetic operator. The results derived by the consensus-based PUL-GLDS method listed in the Table 4.8 show that z_1 performs badly under attributes c_3 and c_4. Therefore, its overall score is less. Anyhow, we are with difficulty to choose an alternative which behaves poorly with respect to some attributes although its net score is high. In that sense, we should determine the poorest value of each alternative under all attributes by the weighted maximum operator. The overall lost dominance score (individual regret rates) displayed in Table 4.9, indicate that the overall score of z_1 is high which indicates that its performance is bad. Therefore, it's kept at the bottom of the range. The existing method claims that the utility value of z_1 is higher than z_2 and z_3 which is contrary to the results obtained by our methods. The reason may be that the existing method does not take into account the "Group utility rates" and "individual regret rates".

Some major advantages of the proposed methods over existing ones are outlined below:

- Unbalanced linguistic environment:

  The unbalanced situation is a common case in this complex world and it is necessary to handle this, but existing approaches with respect to PULTSs [32, 49] do not work in an unbalanced scenario. On the other hand, the approaches proposed in this article are capable of working under different semantics which is the major beauty of our work. The score function $F(u_s(p))$, deviation degree $\sigma(u_s(p))$ and distance measure $d_{gd}\left(u_s^1(p), u_s^1(p)\right)$ used in the proposed approaches are based on linguistic scale function g. According to different decision making environments, the DMs can select a different linguistic scale function g described in Eqs. (2.2.3-2.1.6) on the basis of their preferences.

- Weights of DMs:

  The weights of DMs are not considered in the methods proposed by [32, 49]. This means if we assign different weights to the DMs the existing models will fail to solve the MCGDM problems. From Tables 4.3 and 4.6 it is clear that the developed schemes give importance to the weights of DMs, which shows the validity of the constructed work.

- Keep the originality of data:

  The PUL-GLDS method and the PUL-aggregation method guarantee the completeness of the original data while addressing the MCGDM problems. Unlike, the aggregation-based methodology [32] the results shown in Tables 4.6 and 4.10 after operating the uncertain linguistic terms reflect the probabilities of the uncertain linguistic terms. Additionally, the PULWA and PULWG operators proposed by Lin et al. [32] multiply the probabilities with the indices of the linguistic terms of the corresponding uncertain linguistic terms which are unrealistic. The aggregated values obtained by these operators are not PULEs. Whereas the values obtained in Step $6'$ by the improved operators after aggregation are still PULEs.

Some limitations of the proposed methodologies are listed in a nutshell below:

- Arithmetic complexity:

  Since the proposed methods are based on adjusted PULEs. It can be noticed from Example 4.1.4, that after adjusting the probability distribution, the length of the PULEs becomes large and then, it becomes cumbersome to handle these PULEs.

- Weight vector:

The developed methods have a superiority of DMs weight determination, but on the other hand, these methods cannot work under the circumstances where the weight information of attributes is incompletely known, the weight information should be completely known for the purpose of implementation.

- Complication:

  Though the proposal is effective in reflecting hesitation and gains high attraction under theoretic context, but contents are too lengthy and complicated, which causes puzzling to understanding and thus practical sense of the proposal is tough for DMs to adopt.

Chapter 5

Weighted interval-valued dual hesitant fuzzy sets and its application in teaching quality assessment

This chapter gives the generalized form of IVDHFS, namely WIVDHFS and its related mathematically study. Based on the Archimedean t-norm and t-conorm, some primary aggregation operators, their relevant properties, special cases relationships are explored in weighted interval-valued dual hesitant fuzzy environment. In addition, an aggregation based method with weighted interval-valued dual hesitant fuzzy information is framed. Later on, an example of the teaching quality assessment is solved by the proposed and related approaches to show the applicability and efficiency of the provided method. The research work of this chapter is published in [53].

5.1 Weighted interval-valued dual hesitant fuzzy set

As we know, the membership grips with epistemic certainty while the non-membership grips with epistemic uncertainty, and thus can reflect the original information given by DMs as much as possible, similar to IVHFSs, the uncertainty on the possible values should be considered. Based on this idea, Ju et al. [19] defined the notion of IVDHFS in terms of two functions that return two sets of membership and non-membership values, respectively, for each element in the domain. In real decision-making problems, IVDHFS fails to describe the importance of membership and non-membership degree of an element to a given set. To resolve this issue, the concept, operational laws and comparison laws of WIVDHFS are put forward.

5.1.1 The concept of weighted interval-valued dual hesitant fuzzy set

This section mainly concentrates on the introduction of a novel fuzzy set, namely WIVDHS. Further, for better understanding of the proposed notion, a practical example regarding PhD thesis evaluation is also provided in this part.

Definition 5.1.1. *Let Z be a fixed set, a WIVDHFS on Z is described as*

$$D = \{\langle z, h_D(z), g_D(z)\rangle \,|z \in Z\}\,, \tag{5.1.1}$$

where $\mathrm{h}_D(z) = \bigcup_{(\gamma,\mathrm{w}_\gamma)\in h_D(z)} \{(\gamma, \mathrm{w}_\gamma)\}$ *and* $\mathrm{g}_D(z) = \bigcup_{(\eta,\mathrm{w}_\eta)\in\, g_D(z)} \{(\eta, \mathrm{w}_\eta)\}$ *in which* $\gamma = [\gamma^l, \gamma^u]$ *and* $\eta = [\eta^l, \eta^u]$ *are two sets of some possible interval values in* $[0, 1]$*, denoting the possible membership and non-membership degrees of the element* $z \in Z$ *to the set* D*, respectively,* w_γ *and* w_η *are the corresponding weight for these two types of degrees.*

Also there is $[\gamma^l, \gamma^u],\ [\eta^l, \eta^u] \subset [0, 1],\ 0 \le \gamma^{u+} + \eta^{u+} \le 1$ *and* $\mathrm{w}_\gamma \in [0, 1],\ \mathrm{w}_\eta \in [0, 1],\ \sum_{(\gamma,\mathrm{w}_\gamma)\in h_D(z)}^{\#h_D(z)} \mathrm{w}_\gamma \le 1,\ \sum_{(\eta,\mathrm{w}_\eta)\in\, g_D(z)}^{\#g_D(z)} \mathrm{w}_\eta \le 1,$ *where* $\gamma^{u+} = \bigcup_{[\gamma^l,\gamma^u]\in h_D(z)} \max\{\gamma^u\}$, $\eta^{u+} = \bigcup_{[\eta^l,\eta^u]\in g_D(z)} \max\{\eta^u\}$*. The symbols* $\#\mathrm{h}_D(z)$ *and* $\#\mathrm{g}_D(z)$ *denote the crisp scalar cardinality of the components* $\mathrm{h}_D(z)$ *and* $\mathrm{g}_D(z)$*, respectively.*

Remark 5.1.2. *Especially, if all the weight values in membership part and non-membership part become equal, then WIVDHFS reduces to IVDHFS and further, if intervals are single valued, then D reduces to DHFS.*

For sake of simplicity, we call the pair $d(z) = \langle h_d(z), g_d(z)\rangle$ as the weighted interval-valued dual hesitant fuzzy element (WIVDHFE), marked by $d = \langle h_d, g_d\rangle$ and D is the set of all WIVDHFEs. To illustrate the WIVDHFS more straightforwardly, we describe a practical example to depict the difference between the WIVDHFS and IVDHFS.

Example 5.1.3. *Take the evaluation of PhD thesis as an example. Here, in Pakistan, the review of PhD thesis is always taken by three experts. Due to the complexity of PhD thesis and little time availability, it is cumbersome for an expert to provide exact evaluating values. The first expert believes that the chance that the PhD thesis meets the requirement is* $[0.4, 0.5]$ *and that of it not being up to the standard is* $[0.3, 0.4]$. *The second one thinks that the possibility of the PhD thesis meeting the requirement is* $[0.5, 0.6]$ *while the contrary is* $[0.2, 0.3]$. *The third expert regards the compliance to be* $[0.4, 0.5]$ *and the non-compliance to be* $[0.2, 0.3]$. *In these situations, the degree to which PhD thesis meets the requirements can be expressed as a WIVDHFE* $\left\langle \left\{([0.4, 0.5], \frac{2}{3}), ([0.5, 0.6], \frac{1}{3})\right\}, \left\{([0.2, 0.3], \frac{2}{3}), ([0.3, 0.4], \frac{1}{3})\right\}\right\rangle$. *If we utilize IVHFE to represent this evaluation, the result is* $\langle\{[0.4, 0.5], [0.5, 0.6]\}, \{[0.2, 0.3], [0.3, 0.4]\}\rangle$. *The IVDHFE fails to reflect the importance value/weight for the membership and non-membership degrees to which the PhD thesis meets the requirement. Therefore, it is far better to represent the situation by utilizing WIVDHFE than an IVDHFE.*

Due to the complex nature of the world or limited knowledge about the scenarios, some DMs cannot provide complete information. In such situation $\sum_{(\gamma, \mathrm{w}_\gamma)\in h_d(z)}^{\# h_d(z)} \mathrm{w}_\gamma < 1$, $\sum_{(\eta, \mathrm{w}_\eta)\in\, g_d(z)}^{\# g_d(z)} \mathrm{w}_\eta < 1$. A feasible way to accomplish it is introduced below:

Definition 5.1.4. *If a WIVDHFE* $d = \langle h_d, g_d\rangle$ *is given by* $\sum_{(\gamma, \mathrm{w}_\gamma)\in h_d}^{\# h_d} \mathrm{w}_\gamma < 1$, $\sum_{(\eta, \mathrm{w}_\eta)\in\, g_d}^{\# g_d} \mathrm{w}_\eta < 1$. *Then, its connected WIVDHFE* $\hat{d}$ *is defined as* $\hat{d} = \langle h_{\hat{d}}, g_{\hat{d}}\rangle = \left\langle \bigcup_{(\gamma, \hat{\mathrm{w}}_\gamma)\in h_{\hat{d}}} \{(\gamma, \hat{\mathrm{w}}_\gamma)\}, \bigcup_{(\eta, \hat{\mathrm{w}}_\eta)\in g_{\hat{d}}} \{(\eta, \hat{\mathrm{w}}_\eta)\} \right\rangle$ *; where* $\hat{\mathrm{w}}_\gamma = \mathrm{w}_\gamma / \sum_{(\gamma, \mathrm{w}_\gamma)\in h_d}^{\# h_d} \mathrm{w}_\gamma$, $\hat{\mathrm{w}}_\eta = \mathrm{w}_\eta / \sum_{(\eta, \mathrm{w}_\eta)\in h_d}^{\# h_d} \mathrm{w}_\eta$.

Definition 5.1.4 is an effective mean to estimate the ignorance of weightage information. However, the method may result in the loss of a certain amount of information. There may be more complicated ways to resolve this problem [33]. In the current paper, we do not talk more about it, and some further researches on it will be done in our future work.

5.1.2 The comparison of WIVDHFEs

The comparison of WIVDHFEs is essential if we tend to apply this theory to decision making problems. Hence, we define the score function and accuracy function of the WIVDHFE, making it possible to rank WIVDHFEs.

Definition 5.1.5. *Let $d = \langle \mathrm{h}_d, \mathrm{g}_d \rangle$ be WIVDHFE, then the score function is defined as follows:*

$$S(d) = \frac{1}{2}\left(\frac{1}{\#\mathrm{h}_d}\sum_{[\gamma^l,\gamma^u]\in \mathrm{h}_d}\left(\gamma^l+\gamma^u\right)\cdot \mathrm{w}_\gamma - \frac{1}{\#\mathrm{g}_d}\sum_{[\eta^l,\eta^u]\in \mathrm{g}_d}\left(\eta^l+\eta^u\right)\cdot \mathrm{w}_\eta\right), \tag{5.1.2}$$

where $\#\mathrm{h}_d$ and $\#\mathrm{g}_d$ is the cardinality of h_d and g_d, respectively.

Definition 5.1.6. *Let $d = \langle \mathrm{h}_d, \mathrm{g}_d \rangle$ be WIVDHFE, then the accuracy function of d is given by formula*

$$A(d) = \frac{1}{2}\left(\frac{1}{\#\mathrm{h}_d}\sum_{[\gamma^l,\gamma^u]\in \mathrm{h}_d}\left(\gamma^l+\gamma^u\right)\cdot \mathrm{w}_\gamma + \frac{1}{\#\mathrm{g}_d}\sum_{[\eta^l,\eta^u]\in \mathrm{g}_d}\left(\eta^l+\eta^u\right)\cdot \mathrm{w}_\eta\right), \tag{5.1.3}$$

where $\#\mathrm{h}_d$ and $\#\mathrm{g}_d$ is the cardinality of h_d and g_d, respectively.

Based on the score function and the accuracy function of WIVDHFE, we define the following method for comparing two PDHFEs.

Definition 5.1.7. *Let $d_l (l = 1, 2)$ be two WIVDHFEs, $S(d_l)(l = 1, 2)$ and $A(d_l)(l = 1, 2)$ are the score function and accuracy function, respectively. Then*

i. If $S(d_1) > S(d_2)$, then $d_1 > d_2$; on the contrary, there is $d_1 < d_2$.

ii. If $S(d_1) = S(d_2)$, then

(a) *If* $A(d_1) > A(d_2)$*, then* $d_1 > d_2$*;*

(b) *If* $A(d_1) < A(d_2)$*, then* $d_1 < d_2$*;*

(c) *If* $A(d_1) = A(d_2)$*, then* $d_1 = d_2$*.*

5.1.3 The basic operations of WIVDHFEs

Based on the Archimedean t-norm and Archimedean t-conorm, we propose some basic operation rules of WIVDHFEs and explore their characteristics in preparation for applications to the practical problems.

Definition 5.1.8. *The complement of a given WIVDHFE* $d = \langle \mathrm{h}_d, \mathrm{g}_d \rangle = \bigcup_{\gamma \in \mathrm{h}_d, \eta \in \mathrm{g}_d} \langle \{(\gamma, \mathrm{w}_\gamma)\}, \{(\eta, \mathrm{w}_\eta)\} \rangle$ *is defined as follows:*

$$d^c = \begin{cases} \bigcup_{\gamma \in \mathrm{h}_d, \eta \in \mathrm{g}_d} \langle \{(\eta, \mathrm{w}_\eta)\}, \{(\gamma, \mathrm{w}_\gamma)\} \rangle, & \text{if } \mathrm{h}_d \neq \varnothing \text{ and } \mathrm{g}_d \neq \varnothing \\ \bigcup_{\gamma \in \mathrm{h}_d} \left\langle \left\{ ([1-\gamma^l, 1-\gamma^u], \mathrm{w}_\gamma) \right\}, \varnothing \right\rangle, & \text{if } \mathrm{h}_d \neq \varnothing \text{ and } \mathrm{g}_d = \varnothing \\ \bigcup_{\eta \in \mathrm{g}_d} \left\langle \varnothing, \left\{ ([1-\eta^l, 1-\eta^u], \mathrm{w}_\eta) \right\} \right\rangle, & \text{if } \mathrm{h}_d = \varnothing \text{ and } \mathrm{g}_d \neq \varnothing. \end{cases} \tag{5.1.4}$$

Definition 5.1.9. *Let* Z *be a fixed set,* $d = \langle \mathrm{h}_d, \mathrm{g}_d \rangle$ *and* $d_1 = \langle \mathrm{h}_{d_1}, \mathrm{g}_{d_1} \rangle$ *be two WIVDHFEs, then:*

$$i.\ \ d \oplus d_1 = \bigcup_{\gamma_d \in \mathrm{h}_d, \eta_d \in \mathrm{g}_d, \gamma_{d1} \in \mathrm{h}_{d_1}, \eta_{d1} \in \mathrm{g}_{d_1}} \left\langle \begin{array}{c} \left\{ \left(S\left([\gamma_d^l, \gamma_d^u], [\gamma_{d1}^l, \gamma_{d1}^u] \right), \mathrm{w}_{\gamma_d} \mathrm{w}_{\gamma_{d1}} \right) \right\}, \\ \left\{ \left(T\left([\eta_d^l, \eta_d^u], [\eta_{d1}^l, \eta_{d1}^u] \right), \mathrm{w}_{\eta_d} \mathrm{w}_{\eta_{d1}} \right) \right\} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} \bigcup_{\gamma_d \in \mathrm{h}_d, \gamma_{d1} \in \mathrm{h}_{d_1}} \left\{ \left(\left[f^{-1}\left(f\left(\gamma_d^l\right) + f\left(\gamma_{d1}^l\right) \right), f^{-1}\left(f\left(\gamma_d^u\right) + f\left(\gamma_{d1}^u\right) \right) \right], \mathrm{w}_{\gamma_d} \mathrm{w}_{\gamma_{d1}} \right) \right\}, \\ \bigcup_{\eta_d \in \mathrm{g}_d, \eta_{d1} \in \mathrm{g}_{d_1}} \left\{ \left(\left[g^{-1}\left(g\left(\eta_d^l\right) + g\left(\eta_{d1}^l\right) \right), g^{-1}\left(g\left(\eta_d^u\right) + g\left(\eta_{d1}^u\right) \right) \right], \mathrm{w}_{\eta_d} \mathrm{w}_{\eta_{d1}} \right) \right\} \end{array} \right\rangle;$$

$$ii.\ \ d \otimes d_1 = \bigcup_{\gamma_d \in \mathrm{h}_d, \eta_d \in \mathrm{g}_d, \gamma_{d1} \in \mathrm{h}_{d_1}, \eta_{d1} \in \mathrm{g}_{d_1}} \left\langle \begin{array}{c} \left\{ \left(T\left([\gamma_d^l, \gamma_d^u], [\gamma_{d1}^l, \gamma_{d1}^u] \right), \mathrm{w}_{\gamma_d} \mathrm{w}_{\gamma_{d1}} \right) \right\}, \\ \left\{ \left(S\left([\eta_d^l, \eta_d^u], [\eta_{d1}^l, \eta_{d1}^u] \right), \mathrm{w}_{\eta_d} \mathrm{w}_{\eta_{d1}} \right) \right\} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} \bigcup_{\gamma_d \in h_d, \gamma_{d1} \in h_{d_1}} \left\{ \left(\left[g^{-1} \left(g \left(\gamma_d^l \right) + g \left(\gamma_{d1}^l \right) \right), g^{-1} \left(g \left(\gamma_d^u \right) + g \left(\gamma_{d1}^u \right) \right) \right], \mathrm{w}_{\gamma_d} \mathrm{w}_{\gamma_{d1}} \right) \right\}, \\ \bigcup_{\eta_d \in g_d, \eta_{d1} \in g_{d_1}} \left\{ \left(\left[f^{-1} \left(f \left(\eta_d^l \right) + f \left(\eta_{d1}^l \right) \right), f^{-1} \left(f \left(\eta_d^u \right) + f \left(\eta_{d1}^u \right) \right) \right], \mathrm{w}_{\eta_d} \mathrm{w}_{\eta_{d1}} \right) \right\} \end{array} \right\rangle;$$

iii. $$\alpha d = \bigcup_{\gamma_d \in h_d, \eta_d \in g_d} \left\langle \begin{array}{c} \left\{ \left(\left[f^{-1} \left(\alpha f \left(\gamma_d^l \right) \right), f^{-1} \left(\alpha f \left(\gamma_d^u \right) \right) \right], \mathrm{w}_{\gamma_d} \right) \right\}, \\ \left\{ \left(\left[g^{-1} \left(\alpha g \left(\eta_d^l \right) \right), g^{-1} \left(\alpha g \left(\eta_d^u \right) \right) \right], \mathrm{w}_{\eta_d} \right) \right\} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} \bigcup_{\gamma_d \in h_d} \left\{ \left(\left[f^{-1} \left(\alpha f \left(\gamma_d^l \right) \right), f^{-1} \left(\alpha f \left(\gamma_d^u \right) \right) \right], \mathrm{w}_{\gamma_d} \right) \right\}, \\ \bigcup_{\eta_d \in g_d} \left\{ \left(\left[g^{-1} \left(\alpha g \left(\eta_d^l \right) \right), g^{-1} \left(\alpha g \left(\eta_d^u \right) \right) \right], \mathrm{w}_{\gamma_d} \right) \right\} \end{array} \right\rangle;$$

iv. $$d^\alpha = \bigcup_{\gamma_d \in h_d, \eta_d \in g_d} \left\langle \begin{array}{c} \left\{ \left(\left[g^{-1} \left(\alpha g \left(\gamma_d^l \right) \right), g^{-1} \left(\alpha g \left(\gamma_d^u \right) \right) \right], \mathrm{w}_{\gamma_d} \right) \right\}, \\ \left\{ \left(\left[f^{-1} \left(\alpha f \left(\eta_d^l \right) \right), f^{-1} \left(\alpha f \left(\eta_d^u \right) \right) \right], \mathrm{w}_{\eta_d} \right) \right\} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} \bigcup_{\gamma_d \in h_d} \left\{ \left(\left[g^{-1} \left(\alpha g \left(\gamma_d^l \right) \right), g^{-1} \left(\alpha g \left(\gamma_d^u \right) \right) \right], \mathrm{w}_{\gamma_d} \right) \right\}, \\ \bigcup_{\eta_d \in g_d} \left\{ \left(\left[f^{-1} \left(\alpha f \left(\eta_d^l \right) \right), f^{-1} \left(\alpha f \left(\eta_d^u \right) \right) \right], \mathrm{w}_{\eta_d} \right) \right\} \end{array} \right\rangle.$$

Theorem 5.1.10. *Let d, d_1 and d_2 be three WIVDHFEs, and $\alpha, \alpha_1 \geq 0$; then:*

i. $d \oplus d_1 = d_1 \oplus d$;

ii. $d \oplus (d_1 \oplus d_2) = (d \oplus d_1) \oplus d_2$;

iii. $\alpha (d \oplus d_1) = \alpha d \oplus \alpha d_1$;

iv. $\alpha d \oplus \alpha_1 d = (\alpha + \alpha_1) d$;

v. $d \otimes d_1 = d_1 \otimes d$;

vi. $d \otimes (d_1 \otimes d_2) = (d \otimes d_1) \otimes d_2$;

vii. $(d \otimes d_1)^\alpha = d^\alpha \otimes d_1^\alpha$;

viii. $d^\alpha \otimes d^{\alpha_1} = d^{\alpha + \alpha_1}$.

5.2 Development of generalized weighted interval-valued dual hesitant fuzzy information operators

When we apply the WIVDHFSs to decision making, a significant problem is how to fuse or analyze the information provided by the experts or the DMs. The information aggregation operators will be the solution to this problem. In what follows, the GWIVDHFWA operator and the GWIVDHFWG operator are studied, and then the related properties are established.

5.2.1 The GWIVDHFWA operator

In this part, we give the definition of GWIVDHFWA operator and its relevant properties along with proof.

Definition 5.2.1. *Let* $d_j = \left\langle \mathrm{h}_{dj}, \mathrm{g}_{dj} \right\rangle (j = 1, 2, 3, ..., n)$ *be a collection of WIVDHFEs, and let* $\omega = (\omega_1, \omega_2, ..., \omega_n)$ *with* $\omega_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \omega_j = 1$*. Then, the GWIVDHFWA operator is a mapping* $D^n \longrightarrow D$*, such that*

$$GWIVDHFWA(d_1, d_2, ..., d_n) = \oplus_{j=1}^{n} (\omega_j d_j) . \tag{5.2.1}$$

Theorem 5.2.2. *Let* $d_j = \left\langle \mathrm{h}_{dj}, \mathrm{g}_{dj} \right\rangle (j = 1, 2, 3, ..., n)$ *be a collection of WIVDHFEs. Then, the aggregated value by using GWIVDHFWA operator is also a WIVDHFE, and*

$$GWIVDHFWA(d_1, d_2, ..., d_n) = \oplus_{j=1}^{n} (\omega_j d_j) = \left\langle \bigcup_{\substack{(\gamma_j, \omega_j) \in h_{dj} \\ (j=1,2,...,n)}} \left\{ \left(\left[f^{-1} \left(\sum_{j=1}^{n} \omega_j f(\gamma_j^l) \right), f^{-1} \left(\sum_{j=1}^{n} \omega_j f(\gamma_j^u) \right) \right], \prod_{j=1}^{n} \mathrm{w}_{\gamma_{dj}} \right\}, \right.$$
$$\left. \bigcup_{\substack{(\eta_j, \omega_j) \in g_{dj} \\ (j=1,2,...,n)}} \left\{ \left(\left[g^{-1} \left(\sum_{j=1}^{n} \omega_j g(\eta_j^l) \right), g^{-1} \left(\sum_{j=1}^{n} \omega_j g(\eta_j^u) \right) \right], \prod_{j=1}^{n} \mathrm{w}_{\eta_{dj}} \right\} \right\rangle .$$

Proof. Using mathematical induction on n:

For $n = 2$, we have

$$\begin{aligned}
GWIVHFWA(d_1, d_2) &= \omega_1 d_1 \oplus \omega_2 d_2 \\
&= \Bigg\langle \bigcup_{(\gamma_1,\omega_1)\in h_{d1}} \left\{\left(\left[f^{-1}\left(\omega_1 f\left(\gamma_1^l\right)\right), f^{-1}\left(\omega_1 f\left(\gamma_1^u\right)\right)\right], p_2\right)\right\}, \\
&\quad \bigcup_{(\eta_1,\omega_1)\in g_{d1}} \left\{\left(\left[g^{-1}\left(\omega_1 g\left(\eta_1^l\right)\right), g^{-1}\left(\omega_1 g\left(\eta_1^u\right)\right)\right], q_2\right)\right\} \Bigg\rangle \\
&\quad \oplus \Bigg\langle \bigcup_{(\gamma_2,\omega_2)\in h_{d2}} \left\{\left(\left[f^{-1}\left(\omega_2 f\left(\gamma_2^l\right)\right), f^{-1}\left(\omega_2 f\left(\gamma_2^u\right)\right)\right], p_2\right)\right\}, \\
&\quad \bigcup_{(\eta_2,\omega_2)\in g_{d2}} \left\{\left(\left[g^{-1}\left(\omega_2 g\left(\eta_2^l\right)\right), g^{-1}\left(\omega_2 g\left(\eta_2^u\right)\right)\right], q_2\right)\right\} \Bigg\rangle \\
&= \Bigg\langle \bigcup_{\substack{(\gamma_1,\omega_1)\in h_{d1},\\ (\gamma_2,\omega_2)\in h_{d2}}} \left\{\left(\left[f^{-1}\left(f\left(f^{-1}\left(\omega_1 f\left(\gamma_1^l\right)\right)\right) + f\left(f^{-1}\left(\omega_2 f\left(\gamma_2^l\right)\right)\right)\right),\right.\right.\right. \\
&\qquad \left.\left.\left. f^{-1}\left(f\left(f^{-1}\left(\omega_1 f\left(\gamma_1^u\right)\right)\right) + f\left(f^{-1}\left(\omega_2 f\left(\gamma_2^u\right)\right)\right)\right)\right], p_2\right)\right\}, \\
&\quad \bigcup_{\substack{(\eta_1,\omega_1)\in g_{d1},\\ (\eta_2,\omega_2)\in g_{d2}}} \left\{\left(\left[g^{-1}\left(g\left(g^{-1}\left(\omega_1 g\left(\eta_1^l\right)\right)\right) + g\left(g^{-1}\left(\omega_2 g\left(\eta_2^l\right)\right)\right)\right),\right.\right.\right. \\
&\qquad \left.\left.\left. g^{-1}\left(g\left(g^{-1}\left(\omega_1 g\left(\eta_1^u\right)\right)\right) + g\left(g^{-1}\left(\omega_2 g\left(\eta_2^u\right)\right)\right)\right)\right], q_2\right)\right\} \Bigg\rangle \\
&= \Bigg\langle \bigcup_{\substack{(\gamma_1,\omega_1)\in h_{d1},\\ (\gamma_2,\omega_2)\in h_{d2}}} \left\{\left(\left[f^{-1}\left(\omega_1 f\left(\gamma_1^l\right) + \omega_2 f\left(\gamma_2^l\right)\right), f^{-1}\left(\omega_1 f\left(\gamma_1^u\right) + \omega_2 f\left(\gamma_2^u\right)\right)\right]\right.\right. \\
&\quad \left.\left., p_2\right)\right\}, \bigcup_{\substack{(\eta_1,\omega_1)\in g_{d1},\\ (\eta_2,\omega_2)\in g_{d2}}} \left\{\left(\left[g^{-1}\left(\omega_1 g\left(\eta_1^l\right) + \omega_2 g\left(\eta_2^l\right)\right), g^{-1}\left(\omega_1 g\left(\eta_1^u\right) + \right.\right.\right.\right. \\
&\quad \left.\left.\left.\left. \omega_2 g\left(\eta_2^u\right)\right)\right], q_2\right)\right\} \Bigg\rangle
\end{aligned}$$

where $p_2 = \prod_{j=1}^{2} w_{\gamma_{dj}}$ and $q_2 = \prod_{j=1}^{2} w_{\eta_{dj}}$. Suppose the above equality holds for $n = t$, that is

$$
\begin{aligned}
GWIVDHFWA(d_1, d_2, ..., d_t) &= \oplus_{j=1}^{t}\left(\omega_j d_j\right) \\
&= \left\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{d_j} \\ (j=1,2,...,n)}} \left\{\left(\left[f^{-1}\left(\sum_{j=1}^{t}\omega_j f(\gamma_j^l)\right), f^{-1}\left(\sum_{j=1}^{t}\omega_j f(\gamma_j^u)\right)\right], p_t\right)\right\}, \right. \\
&\quad \left. \bigcup_{\substack{(\eta_j,\omega_j)\in g_{d_j} \\ (j=1,2,...,n)}} \left\{\left(\left[g^{-1}\left(\sum_{j=1}^{t}\omega_j g(\eta_j^l)\right), g^{-1}\left(\sum_{j=1}^{t}\omega_j g(\eta_j^u)\right)\right], q_t\right)\right\} \right\rangle
\end{aligned}
$$

where $p_t = \prod_{j=1}^{t} w_{\gamma_{dj}}$ and $q_t = \prod_{j=1}^{t} w_{\eta_{dj}}$. Then, when $n = t + 1$, we have

$$
\begin{aligned}
GWIVDHFWA(d_1, d_2, ..., d_{t+1}) &= \oplus_{j=1}^{t+1} (\omega_j d_j) = \oplus_{j=1}^{t} (\omega_j d_j) \oplus (\omega_{t+1} d_{t+1}) \\
&= \Bigg\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{dj} \\ (j=1,2,...,t)}} \left\{ \left(\left[f^{-1}\left(\sum_{j=1}^{t} \omega_j f(\gamma_j^l)\right), f^{-1}\left(\sum_{j=1}^{t} \omega_j f(\gamma_j^u)\right) \right], p_{t+1} \right) \right\}, \\
&\bigcup_{\substack{(\eta_j,\omega_j)\in h_{dj} \\ (j=1,2,...,n)}} \left\{ \left(\left[g^{-1}\left(\sum_{j=1}^{t} \omega_j g(\eta_j^l)\right), g^{-1}\left(\sum_{j=1}^{t} \omega_j g(\eta_j^u)\right) \right], q_{t+1} \right) \right\} \Bigg\rangle \oplus \\
&\Bigg\langle \bigcup_{(\gamma_{t+1},\omega_{t+1})\in h_{dt+1}} \left\{ \left(\left[f^{-1}\left(\omega_{t+1} f(\gamma_{t+1}^l)\right), f^{-1}\left(\omega_{t+1} f(\gamma_{t+1}^u)\right) \right], p_{t+1} \right) \right\}, \\
&\bigcup_{(\eta_{t+1},\omega_{t+1})\in g_{dt+1}} \left\{ \left(\left[g^{-1}\left(\omega_{t+1} g(\eta_{t+1}^l)\right), g^{-1}\left(\omega_{t+1} g(\eta_{t+1}^u)\right) \right], q_{t+1} \right) \right\} \Bigg\rangle \\
&= \Bigg\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{dj}, \\ (\gamma_{t+1},\omega_{t+1})\in h_{dt+1}}} \Bigg\{ \Bigg(\Bigg[f^{-1}\left(\sum_{j=1}^{t} f\left(f^{-1}\left(\omega_j f\left(\gamma_j^l\right)\right)\right) + f\left(f^{-1}\left(\omega_{t+1} f\left(\gamma_{t+1}^l\right)\right)\right) \right), \\
&f^{-1}\left(\sum_{j=1}^{t} f\left(f^{-1}\left(\omega_j f\left(\gamma_j^u\right)\right)\right) + f\left(f^{-1}\left(\omega_{t+1} f\left(\gamma_{t+1}^u\right)\right)\right) \right) \Bigg], p_{t+1} \Bigg) \Bigg\}, \\
&\bigcup_{\substack{(\eta_j,\omega_j)\in g_{dj}, \\ (\eta_{t+1},\omega_{t+1})\in g_{dt+1}}} \Bigg\{ \Bigg(\Bigg[g^{-1}\left(\sum_{j=1}^{t} g\left(g^{-1}\left(\omega_j g\left(\eta_j^l\right)\right)\right) + g\left(g^{-1}\left(\omega_{t+1} g\left(\eta_{t+1}^l\right)\right)\right) \right), \\
&g^{-1}\left(\sum_{j=1}^{t} g\left(g^{-1}\left(\omega_j g\left(\eta_j^u\right)\right)\right) + g\left(g^{-1}\left(\omega_{t+1} g\left(\eta_{t+1}^u\right)\right)\right) \right) \Bigg], q_{t+1} \Bigg) \Bigg\} \Bigg\rangle \\
&= \Bigg\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{dj} \\ (j=1,2,...,t+1)}} \left\{ \left(\left[f^{-1}\left(\sum_{j=1}^{t+1} \omega_j f\left(\gamma_j^l\right)\right), f^{-1}\left(\sum_{j=1}^{t+1} \omega_j f\left(\gamma_j^u\right)\right) \right], p_{t+1} \right) \right\}, \\
&\bigcup_{\substack{(\eta_j,\omega_j)\in h_{dj} \\ (j=1,2,...,t+1)}} \left\{ \left(\left[g^{-1}\left(\sum_{j=1}^{t+1} \omega_j g\left(\eta_j^l\right)\right), g^{-1}\left(\sum_{j=1}^{t+1} \omega_j g\left(\eta_j^u\right)\right) \right], q_{t+1} \right) \right\} \Bigg\rangle
\end{aligned}
$$

where $p_{t+1} = \prod_{j=1}^{t+1} w_{\gamma_{dj}}$ and $q_{t+1} = \prod_{j=1}^{t+1} w_{\eta_{dj}}$. □

5.2.2 The GWIVDHFWG operator

Here, we give the definition of GWIVDHFWG operator and its relevant properties along with detail proof.

Definition 5.2.3. *Let $d_j = \left\langle \mathrm{h}_{dj}, \mathrm{g}_{dj} \right\rangle (j = 1, 2, 3, ..., n)$ be a collection of WIVDHFEs, and let $\omega = (\omega_1, \omega_2, ..., \omega_n)$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$. Then, the GWIVDHFWG operator is a mapping $D^n \longrightarrow D$, such that*

$$GWIVDHFWG(d_1, d_2, ..., d_n) = \otimes_{j=1}^{n} (\omega_j d_j)\,. \tag{5.2.2}$$

Theorem 5.2.4. *Let $d_j = \left\langle \mathrm{h}_{dj}, \mathrm{g}_{dj} \right\rangle (j = 1, 2, 3, ..., n)$ be a collection of WIVDHFEs. Then, the aggregated value by using GWIVDHFWG operator is also a WIVDHFE, and*

$$GWIVDHFWG(d_1, d_2, ..., d_n) = \otimes_{j=1}^{n} (\omega_j d_j) = \left\langle \bigcup_{\substack{(\gamma_j, \omega_j) \in h_{dj} \\ (j=1,2,...,n)}} \left\{ \left(\left[g^{-1} \left(\sum_{j=1}^{n} \omega_j g(\gamma_j^l) \right), g^{-1} \left(\sum_{j=1}^{n} \omega_j g(\gamma_j^u) \right) \right], \prod_{j=1}^{n} \mathrm{w}_{\gamma_{dj}} \right\}, \right.$$
$$\left. \bigcup_{\substack{(\eta_j, \omega_j) \in h_{dj} \\ (j=1,2,...,n)}} \left\{ \left(\left[f^{-1} \left(\sum_{j=1}^{n} \omega_j g(\eta_j^l) \right), f^{-1} \left(\sum_{j=1}^{n} \omega_j f(\eta_j^u) \right) \right], \prod_{j=1}^{n} \mathrm{w}_{\eta_{dj}} \right\} \right\rangle.$$

Proof. Based on the lines of Theorem 5.2.2, one can easily prove Theorem 5.2.4. □

5.2.3 The properties of the WIVDHFWA operator and the WIVDHFWG operator

The proposed aggregation operators enjoy some interesting properties including idempotency, monotonicity, boundedness and symmetry which are presented below:

Property 1: Idempotency

Let $d_j = \left\langle \mathrm{h}_{dj}, \mathrm{g}_{dj} \right\rangle (j = 1, 2, 3, ..., n)$ be a collection of WIVDHFEs. If all d_j are equal, i.e., $d_j = d \; \forall \; j$, then:

$$GWIVDHFWA\,(d_1, d_2, ..., d_n) = d, \tag{5.2.3}$$

$$GWIVDHFWG\,(d_1, d_2, ..., d_n) = d, \tag{5.2.4}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is the weight vector of d_j, with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Proof. It is given that $d_j = d \; \forall \; j$, therefore

$$\begin{aligned}
GWIVDHFWA\,(d_1, d_2, ..., d_n) =& GWIVDHFWA\,(d, d, ..., d) \\
=& \left\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{d_j} \\ (j=1,2,...,n)}} \left\{ \left(\left[f^{-1}\left(\sum_{j=1}^{n} \omega_j f(\gamma^l) \right), f^{-1}\left(\sum_{j=1}^{n} \omega_j f(\gamma^u) \right) \right], p_n \right) \right\}, \right. \\
& \left. \bigcup_{\substack{(\eta_j,\omega_j)\in h_{d_j} \\ (j=1,2,...,n)}} \left\{ \left(\left[g^{-1}\left(\sum_{j=1}^{n} \omega_j g(\eta^l) \right), g^{-1}\left(\sum_{j=1}^{n} \omega_j g(\eta^u) \right) \right], q_n \right) \right\} \right\rangle \\
=& \left\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{d_j} \\ (j=1,2,...,n)}} \left\{ \left(\left[f^{-1}\left(f(\gamma^l) \sum_{j=1}^{n} \omega_j \right), f^{-1}\left(f(\gamma^u) \sum_{j=1}^{n} \omega_j \right) \right], p_n \right) \right\}, \right. \\
& \left. \bigcup_{\substack{(\eta_j,\omega_j)\in h_{d_j} \\ (j=1,2,...,n)}} \left\{ \left(\left[g^{-1}\left(g(\eta^l) \sum_{j=1}^{n} \omega_j \right), g^{-1}\left(g(\eta^u) \sum_{j=1}^{n} \omega_j \right) \right], q_n \right) \right\} \right\rangle \\
=& \left\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{d_j} \\ (j=1,2,...,n)}} \left\{ \left(\left[\gamma^l, \gamma^u \right], p_n \right) \right\}, \bigcup_{\substack{(\eta_j,\omega_j)\in h_{d_j} \\ (j=1,2,...,n)}} \left\{ \left(\left[\eta^l, \eta^u \right], q_n \right) \right\} \right\rangle \\
=& d
\end{aligned}$$

where $p_n = \prod_{j=1}^{n} w_{\gamma_{dj}}$ and $q_n = \prod_{j=1}^{n} w_{\eta_{dj}}$. Hence the proof. □

Property 2: Monotonicity

Let

$$d_j = \left\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{d_j} \\ (j=1,2,...,n)}} \left\{ \left(\left[f^{-1}\left(\sum_{j=1}^{n} \omega_j f(\gamma_j^l) \right), f^{-1}\left(\sum_{j=1}^{n} \omega_j f(\gamma_j^u) \right) \right], w_{\gamma_{dj}} \right) \right\}, \bigcup_{\substack{(\eta_j,\omega_j)\in h_{d_j} \\ (j=1,2,...,n)}} \left\{ \left(\left[g^{-1}\left(\sum_{j=1}^{n} \omega_j g(\eta_j^l) \right), g^{-1}\left(\sum_{j=1}^{n} \omega_j g(\eta_j^u) \right) \right], w_{\eta_{dj}} \right) \right\} \right\rangle$$

and

$$d_j^* = \left\langle \bigcup_{\substack{(\gamma_j^*, \omega_j^*) \in h_{d_j}^* \\ (j=1,2,...,n)}} \left\{ \left(\left[f^{-1}\left(\sum_{j=1}^{n} \omega_j^* f(\gamma_j^{*l}) \right), f^{-1}\left(\sum_{j=1}^{n} \omega_j^* f(\gamma_j^{*u}) \right) \right], \mathrm{w}_{\gamma^* d_j} \right) \right\}, \bigcup_{\substack{(\eta_j^*, \omega_j^*) \in g_{d_j}^* \\ (j=1,2,...,n)}} \left\{ \left(\left[g^{-1}\left(\sum_{j=1}^{n} \omega_j^* g(\eta_j^{*l}) \right), g^{-1}\left(\sum_{j=1}^{n} \omega_j^* g(\eta_j^{*u}) \right) \right], \mathrm{w}_{\eta^* d_j} \right) \right\} \right\rangle$$

;$j = 1, 2, ..., n$ be two collections of WIVDHFEs. If $\gamma_j^l \le \gamma_j^{*l}$, $\gamma_j^u \le \gamma_j^{*u}$, $\eta_j^l \ge \eta_j^{*l}$, $\eta_j^u \ge \eta_j^{*u}$, $\mathrm{w}_{\gamma d_j} \le \mathrm{w}_{\gamma^* d_j}$, $\mathrm{w}_{\eta d_j} \ge \mathrm{w}_{\eta^* d_j}$ $\forall\, j = 1, 2, ..., n$. Then:

$$GWIVDHFWA\left(d_1, d_2, ..., d_n\right) \le GWIVDHFWA\left(d_1^*, d_2^*, ..., d_n^*\right), \tag{5.2.5}$$

$$GWIVDHFWG\left(d_1, d_2, ..., d_n\right) \le GWIVDHFWG\left(d_1^*, d_2^*, ..., d_n^*\right), \tag{5.2.6}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is the weight vector of d_j, with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Proof. As given that, $\gamma_j^l \le \gamma_j^{*l}$ and $\gamma_j^u \le \gamma_j^{*u}$ $\forall\, j = 1, 2, ..., n$. Since, f is monotonic increasing function. Therefore,

$$f\left(\gamma_j^l\right) \le f\left(\gamma_j^{*l}\right) \text{ and } f\left(\gamma_j^u\right) \le f\left(\gamma_j^{*u}\right)$$

$$\Rightarrow f^{-1}\left(\sum_{j=1}^{n} \omega_j f(\gamma_j^l) \right) \le f^{-1}\left(\sum_{j=1}^{n} \omega_j f(\gamma_j^{*l}) \right) \text{ and } f^{-1}\left(\sum_{j=1}^{n} \omega_j f(\gamma_j^u) \right) \le f^{-1}\left(\sum_{j=1}^{n} \omega_j f(\gamma_j^{*u}) \right)$$

$$\Rightarrow \left[f^{-1}\left(\sum_{j=1}^{n} \omega_j f(\gamma_j^l) \right), f^{-1}\left(\sum_{j=1}^{n} \omega_j f(\gamma_j^u) \right) \right] \le \left[f^{-1}\left(\sum_{j=1}^{n} \omega_j^* f(\gamma_j^{*l}) \right), f^{-1}\left(\sum_{j=1}^{n} \omega_j^* f(\gamma_j^{*u}) \right) \right]$$

$$\Rightarrow \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{dj}\\(j=1,2,...,n)}} \left\{\left[f^{-1}\left(\sum_{j=1}^{n}\omega_j f(\gamma_j^l)\right), f^{-1}\left(\sum_{j=1}^{n}\omega_j f(\gamma_j^u)\right)\right]\right\} \leq \bigcup_{\substack{(\gamma_j^*,\omega_j^*)\in h_{dj}^*\\(j=1,2,...,n)}} \left\{\left[f^{-1}\left(\sum_{j=1}^{n}\omega_j^* f(\gamma_j^{*l})\right), f^{-1}\left(\sum_{j=1}^{n}\omega_j^* f(\gamma_j^{*u})\right)\right]\right\}. \tag{5.2.7}$$

Hence, we obtain the result for membership part.

Also, we have $\eta_j^l \geq \eta_j^{*l}$ and $\eta_j^u \geq \eta_j^{*u}$ $\forall$ $j = 1, 2, ..., n$.

Since, g is monotonic decreasing function. Therefore,

$$g\left(\eta_j^l\right) \leq g\left(\eta_j^{*l}\right) \text{ and } g\left(\eta_j^u\right) \leq g\left(\eta_j^{*u}\right)$$

$$\Rightarrow g^{-1}\left(\sum_{j=1}^{n}\omega_j g(\eta_j^l)\right) \leq g^{-1}\left(\sum_{j=1}^{n}\omega_j g(\eta_j^{*l})\right) \text{ and } g^{-1}\left(\sum_{j=1}^{n}\omega_j g(\eta_j^u)\right) \leq g^{-1}\left(\sum_{j=1}^{n}\omega_j g(\eta_j^{*u})\right)$$

$$\Rightarrow \left[g^{-1}\left(\sum_{j=1}^{n}\omega_j g(\eta_j^l)\right), g^{-1}\left(\sum_{j=1}^{n}\omega_j g(\eta_j^u)\right)\right] \leq \left[g^{-1}\left(\sum_{j=1}^{n}\omega_j^* g(\eta_j^{*l})\right), g^{-1}\left(\sum_{j=1}^{n}\omega_j^* g(\gamma_j^{*u})\right)\right]$$

$$\Rightarrow \bigcup_{\substack{(\eta_j,\omega_j)\in h_{dj}\\(j=1,2,...,n)}} \left\{\left[g^{-1}\left(\sum_{j=1}^{n}\omega_j g(\eta_j^l)\right), g^{-1}\left(\sum_{j=1}^{n}\omega_j g(\eta_j^u)\right)\right]\right\} \leq \bigcup_{\substack{(\eta_j^*,\omega_j^*)\in g_{dj}^*\\(j=1,2,...,n)}} \left\{\left[g^{-1}\left(\sum_{j=1}^{n}\omega_j^* g(\eta_j^{*l})\right), g^{-1}\left(\sum_{j=1}^{n}\omega_j^* f(\eta_j^{*u})\right)\right]\right\}. \tag{5.2.8}$$

Hence, the required result for non-membership part is worked out.

Now, for importance degrees, since $w_{\gamma_{dj}} \leq w_{\gamma^*{dj}}$ and $w_{\eta_{dj}} \geq w_{\eta^*{dj}}$ $\forall$ $j = 1, 2, ..., n$, which implies that $\prod_{j=1}^{n} w_{\gamma_{dj}} \leq \prod_{j=1}^{n} w_{\gamma^*{dj}}$ and $\prod_{j=1}^{n} w_{\eta_{dj}} \geq \prod_{j=1}^{n} w_{\eta^*{dj}}$.

Now, according to the score function as defined in Definition 5.1.5, we ca write from Eq. (5.2.7) and Eq. (5.2.8),

$$\left\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{d_j}\\(j=1,2,...,n)}} \left\{ \left(\left[f^{-1}\left(\sum_{j=1}^{n} \omega_j f(\gamma_j^l) \right), f^{-1}\left(\sum_{j=1}^{n} \omega_j f(\gamma_j^u) \right) \right], \prod_{j=1}^{n} \mathrm{w}_{\gamma_{dj}} \right) \right\}, \bigcup_{\substack{(\eta_j,\omega_j)\in h_{d_j}\\(j=1,2,...,n)}} \left\{ \left(\left[g^{-1}\left(\sum_{j=1}^{n} \omega_j g(\eta_j^l) \right), g^{-1}\left(\sum_{j=1}^{n} \omega_j g(\eta_j^u) \right) \right], \prod_{j=1}^{n} \mathrm{w}_{\eta_{dj}} \right) \right\} \right\rangle$$

$$\leq \left\langle \bigcup_{\substack{(\gamma_j^*,\omega_j^*)\in h_{d_j}^*\\(j=1,2,...,n)}} \left\{ \left(\left[f^{-1}\left(\sum_{j=1}^{n} \omega_j^* f(\gamma_j^{*l}) \right), f^{-1}\left(\sum_{j=1}^{n} \omega_j^* f(\gamma_j^{*u}) \right) \right], \prod_{j=1}^{n} \mathrm{w}_{\gamma^*{dj}} \right) \right\}, \bigcup_{\substack{(\eta_j^*,\omega_j^*)\in g_{d_j}^*\\(j=1,2,...,n)}} \left\{ \left(\left[g^{-1}\left(\sum_{j=1}^{n} \omega_j^* g(\eta_j^{*l}) \right), g^{-1}\left(\sum_{j=1}^{n} \omega_j^* g(\eta_j^{*u}) \right) \right], \prod_{j=1}^{n} \mathrm{w}_{\eta^*{dj}} \right) \right\} \right\rangle$$

$$\Rightarrow GWIVDHFWA\left(d_1, d_2, ..., d_n\right) \leq GWIVDHFWA\left(d_1^*, d_2^*, ..., d_n^*\right).$$

□

Property 3: Boundedness

If $d_j = \bigcup_{\substack{\gamma_j\in h_{d_j},\eta_j\in g_{d_j}\\(j=1,2,...,n)}} \left\langle \left\{ (\gamma_j, \mathrm{w}_{\gamma_j}) \right\}, \left\{ (\eta_j, \mathrm{w}_{\eta_j}) \right\} \right\rangle$, $j = 1, 2, ..., n,$ be a set of WIVDHFEs, then

$$d^- \leq GWIVDHFWA\left(d_1, d_2, ..., d_n\right) \leq d^+, \tag{5.2.9}$$

$$d^- \leq GWIVDHFWG\left(d_1, d_2, ..., d_n\right) \leq d^+, \tag{5.2.10}$$

where $d^- = \langle h_d^-, g_d^+ \rangle = \left\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{dj}\\(j=1,2,...,n)}} \left\{\left(\left[\gamma^{l-}, \gamma^{u-}\right], w_{\gamma_j^-}\right)\right\}, \bigcup_{\substack{(\eta_j,\omega_j)\in h_{dj}\\(j=1,2,...,n)}} \left\{\left(\left[\eta^{l+}, \eta^{u+}\right], w_{\eta_j^+}\right)\right\} \right\rangle =$

$$\bigcup_{\substack{\left(\gamma_j{}^l,\gamma_j{}^u\right)\in h_{dj}\\(j=1,2,...,n)}} \left\langle \left\{(\left[\min \gamma_j{}^l, \min \gamma_j{}^u\right], \min w_{\gamma_j})\right\}, \bigcup_{\substack{(\eta_j,\omega_j)\in g_{dj}\\(j=1,2,...,n)}} \left\{(\left[\max \eta_j{}^l, \max \eta_j{}^u\right], \max w_{\eta_j})\right\} \right\rangle,$$

$$d^+ = \langle h_d^+, g_d^- \rangle = \left\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{dj}\\(j=1,2,...,n)}} \left\{\left(\left[\gamma^{l+}, \gamma^{u+}\right], w_{\gamma_j^+}\right)\right\}, \bigcup_{\substack{(\eta_j,\omega_j)\in h_{dj}\\(j=1,2,...,n)}} \left\{\left(\left[\eta^{l-}, \eta^{u-}\right], w_{\eta_j^-}\right)\right\} \right\rangle =$$

$$\bigcup_{\substack{\left(\gamma_j{}^l,\gamma_j{}^u\right)\in h_{dj},\left(\eta_j{}^l,\eta_j{}^u\right)\in g_{dj}\\(j=1,2,3,...,n)}} \left\langle \left\{(\left[\max \gamma_j{}^l, \max \gamma_j{}^u\right], \max w_{\gamma_j})\right\}, \left\{(\left[\min \eta_j{}^l, \min \eta_j{}^u\right], \min w_{\eta_j})\right\} \right\rangle$$

and $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is the weight vector of d_j, with $\omega_j \in [0,1]$ and $\sum_{j=1}^n \omega_j = 1$.

Proof. Since for all $j = 1, 2, ..., n$, $\gamma^{l-} \le \gamma_j{}^l \le \gamma^{l+}$, $\gamma^{u-} \le \gamma_j{}^u \le \gamma^{u+}$
therefore,

$$f\left(\gamma^{l-}\right) \le f\left(\gamma_j{}^l\right) \le f\left(\gamma^{l+}\right),\ f\left(\gamma^{u-}\right) \le f\left(\gamma_j{}^u\right) \le f\left(\gamma^{u+}\right)$$

$$\Rightarrow f^{-1}\left(\sum_{j=1}^n \omega_j f\left(\gamma^{l-}\right)\right) \le f^{-1}\left(\sum_{j=1}^n \omega_j f\left(\gamma_j{}^l\right)\right) \le f^{-1}\left(\sum_{j=1}^n \omega_j f\left(\gamma^{l+}\right)\right), f^{-1}\left(\sum_{j=1}^n \omega_j f\left(\gamma^{u-}\right)\right) \le f^{-1}\left(\sum_{j=1}^n \omega_j f\left(\gamma_j{}^u\right)\right) \le f^{-1}\left(\sum_{j=1}^n \omega_j f\left(\gamma^{u+}\right)\right)$$

$$\Rightarrow \sum_{j=1}^n \omega_j f^{-1}\left(f\left(\gamma^{l-}\right)\right) \le f^{-1}\left(\sum_{j=1}^n \omega_j f\left(\gamma_j{}^l\right)\right) \le \sum_{j=1}^n \omega_j f^{-1}\left(f\left(\gamma^{l+}\right)\right), \sum_{j=1}^n \omega_j f^{-1}\left(f\left(\gamma^{u-}\right)\right) \le f^{-1}\left(\sum_{j=1}^n \omega_j f\left(\gamma_j{}^u\right)\right) \le \sum_{j=1}^n \omega_j f^{-1}\left(f\left(\gamma^{u+}\right)\right)$$

$$\Rightarrow \gamma^{l-} \le f^{-1}\left(\sum_{j=1}^n \omega_j f\left(\gamma_j{}^l\right)\right) \le \gamma^{l+},\ \gamma^{u-} \le f^{-1}\left(\sum_{j=1}^n \omega_j f\left(\gamma_j{}^u\right)\right) \le \gamma^{u+}$$

$$\Rightarrow \left[\gamma^{l-}, \gamma^{u-}\right] \leq \left[f^{-1}\left(\sum_{j=1}^{n} \omega_j f\left({\gamma_j}^{l}\right)\right),\ f^{-1}\left(\sum_{j=1}^{n} \omega_j f\left({\gamma_j}^{u}\right)\right)\right] \leq \left[\gamma^{l+}, \gamma^{u+}\right]$$

$$\Rightarrow \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{d_j}\\(j=1,2,\ldots,n)}} \left\{\left[\gamma^{l-}, \gamma^{u-}\right]\right\} \leq \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{d_j}\\(j=1,2,\ldots,n)}} \left\{\left[f^{-1}\left(\sum_{j=1}^{n} \omega_j f\left({\gamma_j}^{l}\right)\right), f^{-1}\left(\sum_{j=1}^{n} \omega_j f\left({\gamma_j}^{u}\right)\right)\right]\right\} \leq \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{d_j}\\(j=1,2,\ldots,n)}} \left\{\left[\gamma^{l+}, \gamma^{u+}\right]\right\}. \tag{5.2.11}$$

Hence, we get the result for membership part.

Since, for all $j = 1, 2, ..., n$, $\eta^{l-} \leq {\eta_j}^{l} \leq \eta^{l+}$, $\eta^{u-} \leq {\eta_j}^{u} \leq \eta^{u+}$.

Therefore,

$$g\left(\eta^{l+}\right) \leq g\left({\eta_j}^{l}\right) \leq g\left(\eta^{l-}\right), g\left(\eta^{u+}\right) \leq g\left({\eta_j}^{u}\right) \leq g\left(\eta^{u-}\right)$$

$$\Rightarrow g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left(\eta^{l+}\right)\right) \leq g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left({\eta_j}^{l}\right)\right) \leq g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left(\eta^{l+}\right)\right), g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left(\eta^{u-}\right)\right) \leq g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left({\eta_j}^{u}\right)\right) \leq g^{-1}\left(\sum_{j=1}^{n} \omega_j f\left(\eta^{u-}\right)\right)$$

$$\Rightarrow \sum_{j=1}^{n} \omega_j g^{-1}\left(g\left(\eta^{l+}\right)\right) \leq g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left({\eta_j}^{l}\right)\right) \leq \sum_{j=1}^{n} \omega_j g^{-1}\left(g\left(\eta^{l-}\right)\right),\ \sum_{j=1}^{n} \omega_j g^{-1}\left(g\left(\eta^{u-}\right)\right) \leq g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left({\eta_j}^{u}\right)\right) \leq \sum_{j=1}^{n} \omega_j g^{-1}\left(g\left(\eta^{u-}\right)\right)$$

$$\Rightarrow \eta^{l+} \leq g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left({\eta_j}^{l}\right)\right) \leq \eta^{l-}, \eta^{u+} \leq g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left({\eta_j}^{u}\right)\right) \leq \eta^{u-}$$

$$\Rightarrow \left[\eta^{l+}, \eta^{u+}\right] \leq \left[g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left(\gamma_j{}^{l}\right)\right), g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left(\gamma_j{}^{u}\right)\right)\right] \leq \left[\eta^{l-}, \eta^{u-}\right]$$

$$\Rightarrow \bigcup_{\substack{(\eta_j,\omega_j)\in h_{d_j}\\(j=1,2,\ldots,n)}} \left\{\left[\eta^{l+}, \eta^{u+}\right]\right\} \leq \bigcup_{\substack{(\eta_j,\omega_j)\in h_{d_j}\\(j=1,2,\ldots,n)}} \left\{\left[g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left(\eta_j{}^{l}\right)\right), g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left(\eta_j{}^{u}\right)\right)\right]\right\} \leq \bigcup_{\substack{(\eta_j,\omega_j)\in h_{d_j}\\(j=1,2,\ldots,n)}} \left\{\left[\eta^{l-}, \eta^{u-}\right]\right\}. \tag{5.2.12}$$

Hence, the required result for non-membership part is attained.

Next, for importance degrees, since $\min \mathrm{w}_{\gamma_{dj}} \leq \mathrm{w}_{\gamma_{dj}} \leq \max \mathrm{w}_{\gamma_{dj}}$ and $\min \mathrm{w}_{\eta_{dj}} \leq \mathrm{w}_{\eta_{dj}} \leq \max \mathrm{w}_{\eta_{dj}}$ which implies that $\prod_{j=1}^{n} \min \mathrm{w}_{\gamma_{dj}} \leq \prod_{j=1}^{n} \mathrm{w}_{\gamma_{dj}} \leq \prod_{j=1}^{n} \max \mathrm{w}_{\gamma_{dj}}$.

According to score function as defined in Definition 5.1.5 we can write up from Eqs. (5.2.9) and (5.2.10),

$$\left\langle \left(\left[\gamma^{l-}, \gamma^{u-}\right], \prod_{j=1}^{n} \min \mathrm{w}_{\gamma_{dj}}\right), \left(\left[\eta^{l+}, \eta^{u+}\right], \prod_{j=1}^{n} \max \mathrm{w}_{\gamma_{dj}}\right)\right\rangle \leq \left\langle \left(\left[f^{-1}\left(\sum_{j=1}^{n} \omega_j f\left(\gamma_j{}^{l}\right)\right), f^{-1}\left(\sum_{j=1}^{n} \omega_j f\left(\gamma_j{}^{u}\right)\right)\right], \mathrm{w}_{\gamma_{dj}}\right),\right.$$

$$\left.\left(\left[g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left(\eta_j{}^{l}\right)\right), g^{-1}\left(\sum_{j=1}^{n} \omega_j g\left(\eta_j{}^{u}\right)\right)\right], \mathrm{w}_{\eta_{dj}}\right)\right\rangle$$

$$\leq \left\langle \left(\left[\gamma^{l+}, \gamma^{u+}\right], \prod_{j=1}^{n} \max \mathrm{w}_{\gamma_{dj}}\right), \left(\left[\eta^{l-}, \eta^{u-}\right], \prod_{j=1}^{n} \min \mathrm{w}_{\gamma_{dj}}\right)\right\rangle$$

$$\leq \left\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{d_j}, (\eta_j,\omega_j)\in g_{d_j}\\(j=1,2,\ldots,n)}} \left\{\left(\left[f^{-1}\left(\sum_{j=1}^{t} \omega_j f(\gamma_j^{l})\right), f^{-1}\left(\sum_{j=1}^{t} \omega_j f(\gamma_j^{u})\right)\right], \mathrm{w}_{\gamma_j}\right)\right\},\right.$$

$$\left.\left\{\left(\left[g^{-1}\left(\sum_{j=1}^{t} \omega_j g(\eta_j^{l})\right), g^{-1}\left(\sum_{j=1}^{t} \omega_j g(\eta_j^{u})\right)\right], \mathrm{w}_{\eta_j}\right)\right\}\right\rangle$$

$$\leq \left\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{d_j},(\eta_j,\omega_j)\in g_{d_j} \\ (j=1,2,...,n)}} \left\{ \left(\left[\gamma^{l+}, \gamma^{u+} \right], w_{\gamma_j}^{+} \right), \left(\left[\eta^{l-}, \eta^{u-} \right], w_{\eta_j}^{-} \right) \right\} \right\rangle .$$

This implies that

$$d^{-} \leq GWIVDHFWA\left(d_1, d_2, ..., d_n\right) \leq d^{+}.$$

□

Property 4: Symmetry

Let $d_j = \left\langle h_{dj}, g_{dj} \right\rangle$ $(j = 1, 2, 3, ..., n)$ be a collection of WIVDHFEs. Then, if $d_j' = \left\langle h_{dj}', g_{dj}' \right\rangle$ $(j = 1, 2, 3, ..., n)$ be any permutation of $d_j = \left\langle h_{dj}, g_{dj} \right\rangle$, then we have:

$$GWIVDHFWA\left(d_1, d_2, ..., d_n\right) = GWIVDHFWA\left(d_1', d_2', ..., d_n'\right), \tag{5.2.13}$$

$$GWIVDHFWG\left(d_1, d_2, ..., d_n\right) = GWIVDHFWG\left(d_1', d_2', ..., d_n'\right), \tag{5.2.14}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)$ is the weight vector of d_j, with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

Proof. The proof is obvious, therefore, it is omitted. □

5.3 Relationship among the weighted interval-valued dual hesitant fuzzy information aggregation operators

In this part, we investigate some special cases of the proposed aggregation operators by using different values of f and g. After that, the relationships among these operators are constructed.

Case 5.3.1. *If $g(t) = -\ln(t)$, then the GWIVDHFWA operator and the GWIVDHFG operator reduce to weighted interval-valued dual hesitant fuzzy weighted averaging (WIVDHFWA) operator and weighted interval-valued dual hesitant fuzzy weighted geometric (WIVDHFWG) operator respectively, as follows:*

$$WIVDHFWA(d_1, d_2, ..., d_n) = \left\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{dj} \\ (j=1,2,...,n)}} \left\{ \left(\left[1 - \prod_{j=1}^{n} (1-\gamma_j^l)^{\omega_j}, 1 - \prod_{j=1}^{n} (1-\gamma_j^u)^{\omega_j} \right], \prod_{j=1}^{n} \mathrm{w}_{\gamma_{dj}} \right) \right\}, \bigcup_{\substack{(\eta_j,\omega_j)\in h_{dj} \\ (j=1,2,...,n)}} \left\{ \left(\left[\prod_{j=1}^{n} (\eta_j^l)^{\omega_j}, \prod_{j=1}^{n} (\eta_j^u)^{\omega_j} \right], \prod_{j=1}^{n} \mathrm{w}_{\eta_{dj}} \right) \right\} \right\rangle, \tag{5.3.1}$$

$$WIVDHFWG(d_1, d_2, ..., d_n) = \left\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{dj} \\ (j=1,2,...,n)}} \left\{ \left(\left[\prod_{j=1}^{n} (\gamma_j^l)^{\omega_j}, \prod_{j=1}^{n} (\gamma_j^u)^{\omega_j} \right], \prod_{j=1}^{n} \mathrm{w}_{\gamma_{dj}} \right) \right\}, \bigcup_{\substack{(\eta_j,\omega_j)\in h_{dj} \\ (j=1,2,...,n)}} \left\{ \left(\left[1 - \prod_{j=1}^{n} (1-\eta_j^l)^{\omega_j}, 1 - \prod_{j=1}^{n} (1-\eta_j^u)^{\omega_j} \right], \prod_{j=1}^{n} \mathrm{w}_{\eta_{dj}} \right) \right\} \right\rangle. \tag{5.3.2}$$

Case 5.3.2. *If $g(t) = -\ln(2 - t/t)$, then the GWIVDHFWA operator and the GWIVDHFG operator reduce to weighted interval-valued dual hesitant fuzzy Einstein weighted averaging (WIVDHFEWA) operator and weighted interval-valued dual hesitant fuzzy Einstein weighted geometric (WIVDHFEWG) operator respectively, as follows:*

$$\begin{aligned} WIVDHFEWA(d_1, d_2, ..., d_n) = \Bigg\langle & \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{dj} \\ (j=1,2,...,n)}} \left\{ \left(\left[\frac{\prod_{j=1}^{n}(1+\gamma_j^l)^{\omega_j} - \prod_{j=1}^{n}(1-\gamma_j^l)^{\omega_j}}{\prod_{j=1}^{n}(1+\gamma_j^l)^{\omega_j} + \prod_{j=1}^{n}(1-\gamma_j^l)^{\omega_j}}, \frac{\prod_{j=1}^{n}(1+\gamma_j^u)^{\omega_j} - \prod_{j=1}^{n}(1-\gamma_j^u)^{\omega_j}}{\prod_{j=1}^{n}(1+\gamma_j^u)^{\omega_j} + \prod_{j=1}^{n}(1-\gamma_j^u)^{\omega_j}} \right], \prod_{j=1}^{n} \mathrm{w}_{\gamma_{dj}} \right) \right\}, \\ & \bigcup_{\substack{(\eta_j,\omega_j)\in h_{dj} \\ (j=1,2,...,n)}} \left\{ \left(\left[\frac{2\prod_{j=1}^{n}(\eta_j^l)^{\omega_j}}{\prod_{j=1}^{n}(2-\eta_j^l)^{\omega_j} + \prod_{j=1}^{n}(\eta_j^l)^{\omega_j}}, \frac{2\prod_{j=1}^{n}(\eta_j^u)^{\omega_j}}{\prod_{j=1}^{n}(2-\eta_j^u)^{\omega_j} + \prod_{j=1}^{n}(\eta_j^u)^{\omega_j}} \right], \prod_{j=1}^{n} \mathrm{w}_{\eta_{dj}} \right) \right\} \Bigg\rangle, \end{aligned} \tag{5.3.3}$$

$$WIVDHFEWG(d_1,d_2,...,d_n)=\left\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{d_j}\\(j=1,2,...,n)}} \left\{\left(\left[\frac{2\prod_{j=1}^n(\gamma_j^l)^{\omega_j}}{\prod_{j=1}^n(2-\gamma_j^l)^{\omega_j}+\prod_{j=1}^n(\gamma_j^l)^{\omega_j}}, \frac{2\prod_{j=1}^n(\gamma_j^u)^{\omega_j}}{\prod_{j=1}^n(2-\gamma_j^u)^{\omega_j}+\prod_{j=1}^n(\gamma_j^u)^{\omega_j}}\right], \prod_{j=1}^n \mathrm{w}_{\gamma_{dj}}\right)\right\},\right.$$
$$\left. \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{d_j}\\(j=1,2,...,n)}} \left\{\left(\left[\frac{\prod_{j=1}^n(1+\eta_j^l)^{\omega_j}-\prod_{j=1}^n(1-\eta_j^l)^{\omega_j}}{\prod_{j=1}^n(1+\eta_j^l)^{\omega_j}+\prod_{j=1}^n(1-\eta_j^l)^{\omega_j}}, \frac{\prod_{j=1}^n(1+\eta_j^u)^{\omega_j}-\prod_{j=1}^n(1-\eta_j^u)^{\omega_j}}{\prod_{j=1}^n(1+\eta_j^u)^{\omega_j}+\prod_{j=1}^n(1-\eta_j^u)^{\omega_j}}\right], \prod_{j=1}^n \mathrm{w}_{\eta_{dj}}\right)\right\}\right\rangle. \tag{5.3.4}$$

Case 5.3.3. *If* $g(t)=-\ln(\tau+(1-\tau)t/t), \tau>0$ *then the GWIVDHFWA operator and the GWIVDHFG operator reduce to weighted interval-valued dual hesitant fuzzy Hammer weighted averaging (WIVDHFHWA) operator and weighted interval-valued dual hesitant fuzzy Hammer weighted geometric (WIVDHFHWG) operator respectively, as follows:*

$$WIVDHFHWA(d_1,d_2,...,d_n)=\left\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{d_j}\\(j=1,2,...,n)}} \left\{\left(\left[\frac{\prod_{j=1}^n(1-(1-\tau)\gamma_j^l)^{\omega_j}-\prod_{j=1}^n(1-\gamma_j^l)^{\omega_j}}{\prod_{j=1}^n(1-(1-\tau)\gamma_j^l)^{\omega_j}-(1-\tau)\prod_{j=1}^n(1-\gamma_j^l)^{\omega_j}}, \frac{\prod_{j=1}^n(1-(1-\tau)\gamma_j^u)^{\omega_j}-\prod_{j=1}^n(1-\gamma_j^u)^{\omega_j}}{\prod_{j=1}^n(1-(1-\tau)\gamma_j^u)^{\omega_j}-(1-\tau)\prod_{j=1}^n(1-\gamma_j^u)^{\omega_j}}\right], \prod_{j=1}^n \mathrm{w}_{\gamma_{dj}}\right)\right\},\right.$$
$$\left. \bigcup_{\substack{(\eta_j,\omega_j)\in h_{d_j}\\(j=1,2,...,n)}} \left\{\left(\left[\frac{\tau\prod_{j=1}^n(\eta_j^l)^{\omega_j}}{\prod_{j=1}^n(1-(1-\tau)(1-\eta_j^l))^{\omega_j}-(1-\tau)\prod_{j=1}^n(\eta_j^l)^{\omega_j}}, \frac{\tau\prod_{j=1}^n(\eta_j^u)^{\omega_j}}{\prod_{j=1}^n(1-(1-\tau)(1-\eta_j^u))^{\omega_j}-(1-\tau)\prod_{j=1}^n(\eta_j^u)^{\omega_j}}\right], \prod_{j=1}^n \mathrm{w}_{\eta_{dj}}\right)\right\}\right\rangle, \tag{5.3.5}$$

$$WIVDHFHWG(d_1,d_2,...,d_n)=\left\langle \bigcup_{\substack{(\gamma_j,\omega_j)\in h_{d_j}\\(j=1,2,...,n)}} \left\{\left(\left[\frac{\tau\prod_{j=1}^n(\gamma_j^l)^{\omega_j}}{\prod_{j=1}^n(1-(1-\tau)(1-\gamma_j^l))^{\omega_j}-(1-\tau)\prod_{j=1}^n(\gamma_j^l)^{\omega_j}}, \frac{\tau\prod_{j=1}^n(\gamma_j^u)^{\omega_j}}{\prod_{j=1}^n(1-(1-\tau)(1-\gamma_j^u))^{\omega_j}-(1-\tau)\prod_{j=1}^n(\gamma_j^u)^{\omega_j}}\right], \prod_{j=1}^n \mathrm{w}_{\gamma_{dj}}\right)\right\},\right.$$
$$\left. \bigcup_{\substack{(\eta_j,\omega_j)\in h_{d_j}\\(j=1,2,...,n)}} \left\{\left(\left[\frac{\prod_{j=1}^n(1-(1-\tau)\eta_j^l)^{\omega_j}-\prod_{j=1}^n(1-\eta_j^l)^{\omega_j}}{\prod_{j=1}^n(1-(1-\tau)\eta_j^l)^{\omega_j}-(1-\tau)\prod_{j=1}^n(1-\eta_j^l)^{\omega_j}}, \frac{\prod_{j=1}^n(1-(1-\tau)\eta_j^u)^{\omega_j}-\prod_{j=1}^n(1-\eta_j^u)^{\omega_j}}{\prod_{j=1}^n(1-(1-\tau)\eta_j^u)^{\omega_j}-(1-\tau)\prod_{j=1}^n(1-\eta_j^u)^{\omega_j}}\right], \prod_{j=1}^n \mathrm{w}_{\eta_{dj}}\right)\right\}\right\rangle. \tag{5.3.6}$$

Especially, if $\tau=1$*, then the WIVDHFHWA and WIVDHFHWG operator reduces to the WIVDHFWA and WIVDHFWG operator and, if* $\tau=2$*, then the WIVDHFHWA and WIVDHFHWG operator reduce to the WIVDHFEWA and WIVDHFEWG operator.*

Case 5.3.4. *If* $g(t)=-\ln((\tau-1)t/(\tau^t-1)), \tau>1$ *then the GWIVDHFWA operator and the GWIVDHFG operator reduce to weighted interval-valued dual hesitant fuzzy Frank weighted averaging (WIVDHFFWA) operator and weighted interval-valued dual hesitant fuzzy Frank* *weighted geometric (WIVDHFFWG) operator respectively, as follows:*

$$WIVDHFWA(d_1, d_2, ..., d_n) = \Bigg\langle \bigcup_{\substack{(\gamma_j, \omega_j) \in h_{d_j} \\ (j=1,2,...,n)}} \left\{ \left(\left[1 - \ln_\tau \left(1 + \frac{\prod_{j=1}^n (\tau^{1-\gamma_j^l} - 1)^{\omega_j}}{\tau - 1} \right), 1 - \ln_\tau \left(1 + \frac{\prod_{j=1}^n (\tau^{1-\gamma_j^u} - 1)^{\omega_j}}{\tau - 1} \right) \right], \prod_{j=1}^n \mathrm{w}_{\gamma_{dj}} \right) \right\},$$
$$\bigcup_{\substack{(\eta_j, \omega_j) \in h_{d_j} \\ (j=1,2,...,n)}} \left\{ \left(\left[\ln_\tau \left(1 + \frac{\prod_{j=1}^n (\tau^{\eta_j^l} - 1)^{\omega_j}}{\tau - 1} \right), \ln_\tau \left(1 + \frac{\prod_{j=1}^n (\tau^{\eta_j^u} - 1)^{\omega_j}}{\tau - 1} \right) \right], \prod_{j=1}^n \mathrm{w}_{\eta_{dj}} \right) \right\} \Bigg\rangle, \tag{5.3.7}$$

$$WIVDHFWG(d_1, d_2, ..., d_n) = \Bigg\langle \bigcup_{\substack{(\gamma_j, \omega_j) \in h_{d_j} \\ (j=1,2,...,n)}} \left\{ \left(\left[\ln_\tau \left(1 + \frac{\prod_{j=1}^n (\tau^{\gamma_j^l} - 1)^{\omega_j}}{\tau - 1} \right), \ln_\tau \left(1 + \frac{\prod_{j=1}^n (\tau^{\gamma_j^u} - 1)^{\omega_j}}{\tau - 1} \right) \right], \prod_{j=1}^n \mathrm{w}_{\gamma_{dj}} \right) \right\},$$
$$\bigcup_{\substack{(\eta_j, \omega_j) \in h_{d_j} \\ (j=1,2,...,n)}} \left\{ \left(\left[1 - \ln_\tau \left(1 + \frac{\prod_{j=1}^n (\tau^{1-\eta_j^l} - 1)^{\omega_j}}{\tau - 1} \right), 1 - \ln_\tau \left(1 + \frac{\prod_{j=1}^n (\tau^{1-\eta_j^u} - 1)^{\omega_j}}{\tau - 1} \right) \right], \prod_{j=1}^n \mathrm{w}_{\eta_{dj}} \right) \right\} \Bigg\rangle. \tag{5.3.8}$$

Especially, if $\tau \to 1$, then the WIVDHFFWA and WIVDHFFWG operator reduces to the WIVDHFWA and WIVDHFWG operator, respectively.

Theorem 5.3.5. *Let $d_j = \langle h_{dj}, g_{dj} \rangle$ $(j = 1, 2, 3, ..., n)$ be a collection of WIVDHFEs and let $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the associated weight vector with $\omega_j \in [0, 1]$ and $\sum_{j=1}^n \omega_j = 1$; then*

$$WIVDHFEWA(d_1, d_2, ..., d_n) \leq WIVDHFWA(d_1, d_2, ..., d_n). \tag{5.3.9}$$

If $d_1 = d_2 = \cdots = d_n$, the equality is established.

Proof. Since $\sum_{j=1}^n \omega_j = 1$ and $\omega_j \geq 0$, $j = 1, 2, ..., n$, and according to Lemma 2.1.24, for any $\gamma_1 \in h_{d1}$,$\gamma_2 \in h_{d2}, ..., \gamma_n \in h_{dn}$, we have

$$\prod_{j=1}^n \left(1 + \gamma_j^l\right)^{\omega_j} + \prod_{j=1}^n \left(1 - \gamma_j^l\right)^{\omega_j} \leq \sum_{j=1}^n \omega_j \left(1 + \gamma_j^l\right) + \sum_{j=1}^n \omega_j \left(1 - \gamma_j^l\right) = \sum_{j=1}^n \omega_j \left(1 + \gamma_j^l + 1 - \gamma_j^l\right) = 2. \tag{5.3.10}$$

Then

$$\frac{\prod_{j=1}^{n}\left(1+\gamma_j^l\right)^{\omega_j}-\prod_{j=1}^{n}\left(1-\gamma_j^l\right)^{\omega_j}}{\prod_{j=1}^{n}\left(1+\gamma_j^l\right)^{\omega_j}+\prod_{j=1}^{n}\left(1-\gamma_j^l\right)^{\omega_j}}=1-\frac{2\prod_{j=1}^{n}\left(1-\gamma_j^l\right)^{\omega_j}}{\prod_{j=1}^{n}\left(1+\gamma_j^l\right)^{\omega_j}+\prod_{j=1}^{n}\left(1-\gamma_j^l\right)^{\omega_j}}\leq 1-\frac{2\prod_{j=1}^{n}\left(1-\gamma_j^l\right)^{\omega_j}}{2}=1-\prod_{j=1}^{n}\left(1-\gamma_j^l\right)^{\omega_j}. \tag{5.3.11}$$

where the equality holds if and only if $\gamma_1^l=\gamma_2^l=\cdots=\gamma_n^l$.

Similarly, we also get that

$$\frac{\prod_{j=1}^{n}\left(1+\gamma_j^u\right)^{\omega_j}-\prod_{j=1}^{n}\left(1-\gamma_j^u\right)^{\omega_j}}{\prod_{j=1}^{n}\left(1+\gamma_j^u\right)^{\omega_j}+\prod_{j=1}^{n}\left(1-\gamma_j^u\right)^{\omega_j}}\leq 1-\prod_{j=1}^{n}\left(1-\gamma_j^u\right)^{\omega_j}. \tag{5.3.12}$$

where the equality holds if and only if $\gamma_1^u=\gamma_2^u=\cdots=\gamma_n^u$.

In the same fashion, one can get the result for non-membership part.

Since, the weights corresponding to $\left[1-\prod_{j=1}^{n}(1-\gamma_j^l)^{\omega_j}, 1-\prod_{j=1}^{n}(1-\gamma_j^u)^{\omega_j}\right]$ and $\left[\frac{\prod_{j=1}^{n}(1+\gamma_j^l)^{\omega_j}-\prod_{j=1}^{n}(1-\gamma_j^l)^{\omega_j}}{\prod_{j=1}^{n}(1+\gamma_j^l)^{\omega_j}+\prod_{j=1}^{n}(1-\gamma_j^l)^{\omega_j}}, \frac{\prod_{j=1}^{n}(1+\gamma_j^u)^{\omega_j}-\prod_{j=1}^{n}(1-\gamma_j^u)^{\omega_j}}{\prod_{j=1}^{n}(1+\gamma_j^u)^{\omega_j}+\prod_{j=1}^{n}(1-\gamma_j^u)^{\omega_j}}\right]$ are the same as $w_{\gamma_{d1}}, w_{\gamma_{d2}}, ..., w_{\gamma_{dn}}$, also the weights corresponding to $\left[\prod_{j=1}^{n}(\eta_j^l)^{\omega_j}, \prod_{j=1}^{n}(\eta_j^u)^{\omega_j}\right]$ and $\left[\frac{2\prod_{j=1}^{n}(\eta_j^l)^{\omega_j}}{\prod_{j=1}^{n}(2-\eta_j^l)^{\omega_j}+\prod_{j=1}^{n}(\eta_j^l)^{\omega_j}}, \frac{2\prod_{j=1}^{n}(\eta_j^u)^{\omega_j}}{\prod_{j=1}^{n}(2-\eta_j^u)^{\omega_j}+\prod_{j=1}^{n}(\eta_j^u)^{\omega_j}}\right]$ are the same as $w_{\eta_{d1}}, w_{\eta_{d2}}, ..., w_{\eta_{dn}}$. It follows that

$$WIVDHFEWA(d_1,d_2,...,d_n)\leq WIVDHFWA(d_1,d_2,...,d_n).$$

If $d_1=d_2=\cdots=d_n$, the equality is established, which completes the proof. □

Theorem 5.3.6. *Let $d_j=\langle h_{dj}, g_{dj}\rangle$ $(j=1,2,3,...,n)$ be a collection of WIVDHFEs and let $\omega=(\omega_1,\omega_2,...,\omega_n)^T$ be the associated weight vector with $\omega_j\in[0,1]$ and $\sum_{j=1}^{n}\omega_j=1$; then*

$$WIVDHFWG(d_1,d_2,...,d_n)\leq WIVDHFEWG(d_1,d_2,...,d_n). \tag{5.3.13}$$

If $d_1=d_2=\cdots=d_n$, the equality is established.

Proof. Since $\sum_{j=1}^{n}\omega_j=1$ and $\omega_j\geq 0$, $j=1,2,...,n$, and according to Lemma 2.1.25, for any

$\gamma_1 \in h_{d1}$,$\gamma_2 \in h_{d2}, ..., \gamma_n \in h_{dn}$, we have

$$\prod_{j=1}^{n}\left(2-\gamma_j^l\right)^{\omega_j}+\prod_{j=1}^{n}\left(\gamma_j^l\right)^{\omega_j} \leq \sum_{j=1}^{n}\omega_j\left(2-\gamma_j^l\right)+\sum_{j=1}^{n}\omega_j\left(\gamma_j^l\right)=\sum_{j=1}^{n}\omega_j\left(2-\gamma_j^l+\gamma_j^l\right)=2. \tag{5.3.14}$$

Then

$$\frac{2\prod_{j=1}^{n}\left(\gamma_j^l\right)^{\omega_j}}{\prod_{j=1}^{n}\left(2-\gamma_j^l\right)^{\omega_j}+\prod_{j=1}^{n}\left(\gamma_j^l\right)^{\omega_j}} \geq \frac{2\prod_{j=1}^{n}\left(\gamma_j^l\right)^{\omega_j}}{2}=\prod_{j=1}^{n}\left(\gamma_j^l\right)^{\omega_j}, \tag{5.3.15}$$

where the equality holds if and only if $\gamma_1^l=\gamma_2^l=\cdots=\gamma_n^l$.

Similarly, we also get that

$$\frac{2\prod_{j=1}^{n}\left(\gamma_j^u\right)^{\omega_j}}{\prod_{j=1}^{n}\left(2-\gamma_j^u\right)^{\omega_j}+\prod_{j=1}^{n}\left(\gamma_j^u\right)^{\omega_j}} \geq \prod_{j=1}^{n}\left(\gamma_j^u\right)^{\omega_j}, \tag{5.3.16}$$

where the equality holds if and only if $\gamma_1^u=\gamma_2^u=\cdots=\gamma_n^u$.

In the same fashion, we can get the result for non-membership part.

Since, the weights corresponding to $\left[\prod_{j=1}^{n}(\gamma_j^l)^{\omega_j},\prod_{j=1}^{n}(\gamma_j^u)^{\omega_j}\right]$ and $\left[\frac{2\prod_{j=1}^{n}(\gamma_j^l)^{\omega_j}}{\prod_{j=1}^{n}(2-\gamma_j^l)^{\omega_j}+\prod_{j=1}^{n}(\gamma_j^l)^{\omega_j}},\right.$ $\left.\frac{2\prod_{j=1}^{n}(\gamma_j^u)^{\omega_j}}{\prod_{j=1}^{n}(2-\gamma_j^u)^{\omega_j}+\prod_{j=1}^{n}(\gamma_j^u)^{\omega_j}}\right]$ are the same as $w_{\gamma_{d1}}, w_{\gamma_{d2}}, \cdots, w_{\gamma_{dn}}$, also the weights corresponding to $\left[1-\prod_{j=1}^{n}(1-\eta_j^l)^{\omega_j},1-\prod_{j=1}^{n}(1-\eta_j^u)^{\omega_j}\right]$ and $\left[\frac{\prod_{j=1}^{n}(1+\eta_j^l)^{\omega_j}-\prod_{j=1}^{n}(1-\eta_j^l)^{\omega_j}}{\prod_{j=1}^{n}(1+\eta_j^l)^{\omega_j}+\prod_{j=1}^{n}(1-\eta_j^l)^{\omega_j}},\frac{\prod_{j=1}^{n}(1+\eta_j^u)^{\omega_j}-\prod_{j=1}^{n}(1-\eta_j^u)^{\omega_j}}{\prod_{j=1}^{n}(1+\eta_j^u)^{\omega_j}+\prod_{j=1}^{n}(1-\eta_j^u)^{\omega_j}}\right]$ are the same as $w_{\eta_{d1}}, w_{\eta_{d2}}, \cdots, w_{\eta_{dn}}$. It follows that

$$WIVDHFWG(d_1,d_2,...,d_n) \leq WIVDHFEWG(d_1,d_2,...,d_n).$$

If $d_1=d_2=\cdots=d_n$, the equality is established, which completes the proof. □

Theorem 5.3.7. *Let* $d_j=\left\langle h_{dj},g_{dj}\right\rangle (j=1,2,3,...,n)$ *be a collection of WIVDHFEs and let*

$\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ be the associated weight vector with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$; then

$$WIVDHFEWG(d_1, d_2, ..., d_n) \leq WIVDHFEWA(d_1, d_2, ..., d_n). \tag{5.3.17}$$

If $d_1 = d_2 = \cdots = d_n$, the equality is established.

Proof. Since $\sum_{j=1}^{n} \omega_j = 1$ and $\omega_j \geq 0$, $j = 1, 2, ..., n$, and according to Lemma 2.1.25, for any $\gamma_1 \in h_{d1}, \gamma_2 \in h_{d2}, ..., \gamma_n \in h_{dn}$, we have

$$\begin{aligned}
&\frac{\prod_{j=1}^{n}\left(1+\gamma_j^l\right)^{\omega_j} - \prod_{j=1}^{n}\left(1-\gamma_j^l\right)^{\omega_j}}{\prod_{j=1}^{n}\left(1+\gamma_j^l\right)^{\omega_j} + \prod_{j=1}^{n}\left(1-\gamma_j^l\right)^{\omega_j}} - \frac{2\prod_{j=1}^{n}\left(\gamma_j^l\right)^{\omega_j}}{\prod_{j=1}^{n}\left(2-\gamma_j^l\right)^{\omega_j} + \prod_{j=1}^{n}\left(\gamma_j^l\right)^{\omega_j}} \\
&= \frac{\prod_{j=1}^{n}\left(2+\gamma_j^l-\left(\gamma_j^l\right)^2\right)^{\omega_j} - \prod_{j=1}^{n}\left(2-3\gamma_j^l+\left(\gamma_j^l\right)^2\right)^{\omega_j} - \prod_{j=1}^{n}\left(\gamma_j^l+\left(\gamma_j^l\right)^2\right)^{\omega_j} - 3\prod_{j=1}^{n}\left(\gamma_j^l-\left(\gamma_j^l\right)^2\right)^{\omega_j}}{\left(\prod_{j=1}^{n}\left(1+\gamma_j^l\right)^{\omega_j} + \prod_{j=1}^{n}\left(1-\gamma_j^l\right)^{\omega_j}\right) \times \left(\prod_{j=1}^{n}\left(2-\gamma_j^l\right)^{\omega_j} + \prod_{j=1}^{n}\left(\gamma_j^l\right)^{\omega_j}\right)} \\
&= \frac{\prod_{j=1}^{n}\left(2+\gamma_j^l-\left(\gamma_j^l\right)^2\right)^{\omega_j} - \prod_{j=1}^{n}\left(2-3\gamma_j^l+\left(\gamma_j^l\right)^2\right)^{\omega_j} - \prod_{j=1}^{n}\left(\gamma_j^l+\left(\gamma_j^l\right)^2\right)^{\omega_j} - \prod_{j=1}^{n}\left(\sqrt[n]{3}\gamma_j^l-\sqrt[n]{3}\left(\gamma_j^l\right)^2\right)^{\omega_j}}{\left(\prod_{j=1}^{n}\left(1+\gamma_j^l\right)^{\omega_j} + \prod_{j=1}^{n}\left(1-\gamma_j^l\right)^{\omega_j}\right) \times \left(\prod_{j=1}^{n}\left(2-\gamma_j^l\right)^{\omega_j} + \prod_{j=1}^{n}\left(\gamma_j^l\right)^{\omega_j}\right)}.
\end{aligned} \tag{5.3.18}$$

As $0 \leq \gamma_j^l \leq \gamma_j^l \leq 1$, $\forall j = 1, 2, ..., n$, then

$$2+\gamma_j^l-\left(\gamma_j^l\right)^2 \geq 0$$

$$2-3\gamma_j^l+\left(\gamma_j^l\right)^2 \geq 0$$

$$\gamma_j^l+\left(\gamma_j^l\right)^2 \geq 0$$

$$\sqrt[n]{3}\gamma_j^l-\sqrt[n]{3}\left(\gamma_j^l\right)^2 \geq 0 \; \forall j = 1, 2, ..., n.$$

Note that

$$\begin{aligned}
\left(2+\gamma_j^l-\left(\gamma_j^l\right)^2\right) - \left(2-3\gamma_j^l+\left(\gamma_j^l\right)^2\right) - \left(\gamma_j^l+\left(\gamma_j^l\right)^2\right) - \left(\sqrt[n]{3}\gamma_j^l-\sqrt[n]{3}\left(\gamma_j^l\right)^2\right) \\
= \gamma_j^l\left(3-\sqrt[n]{3}\right)\left(1-\gamma_j^l\right) \geq 0
\end{aligned}$$

Using Lemma 2.1.25, we have

$$\prod_{j=1}^{n}\left(2+\gamma_j^l-\left(\gamma_j^l\right)^2\right)^{\omega_j}-\prod_{j=1}^{n}\left(2-3\gamma_j^l+\left(\gamma_j^l\right)^2\right)^{\omega_j}-\prod_{j=1}^{n}\left(\gamma_j^l+\left(\gamma_j^l\right)^2\right)^{\omega_j}-\prod_{j=1}^{n}\left(\sqrt[n]{3}\gamma_j^l-\sqrt[n]{3}\left(\gamma_j^l\right)^2\right)^{\omega_j}\geq 0. \tag{5.3.19}$$

It follows that

$$\frac{\prod_{j=1}^{n}\left(1+\gamma_j^l\right)^{\omega_j}-\prod_{j=1}^{n}\left(1-\gamma_j^l\right)^{\omega_j}}{\prod_{j=1}^{n}\left(1+\gamma_j^l\right)^{\omega_j}+\prod_{j=1}^{n}\left(1-\gamma_j^l\right)^{\omega_j}}-\frac{2\prod_{j=1}^{n}\left(\gamma_j^l\right)^{\omega_j}}{\prod_{j=1}^{n}\left(2-\gamma_j^l\right)^{\omega_j}+\prod_{j=1}^{n}\left(\gamma_j^l\right)^{\omega_j}}\geq 0.$$

Similarly, we also get that

$$\frac{\prod_{j=1}^{n}\left(1+\gamma_j^u\right)^{\omega_j}-\prod_{j=1}^{n}\left(1-\gamma_j^u\right)^{\omega_j}}{\prod_{j=1}^{n}\left(1+\gamma_j^u\right)^{\omega_j}+\prod_{j=1}^{n}\left(1-\gamma_j^u\right)^{\omega_j}}-\frac{2\prod_{j=1}^{n}\left(\gamma_j^u\right)^{\omega_j}}{\prod_{j=1}^{n}\left(2-\gamma_j^u\right)^{\omega_j}+\prod_{j=1}^{n}\left(\gamma_j^u\right)^{\omega_j}}\geq 0.$$

In the same fashion, for non-membership part.
Since, the weights corresponding to $\left[\frac{2\prod_{j=1}^{n}(\gamma_j^l)^{\omega_j}}{\prod_{j=1}^{n}(2-\gamma_j^l)^{\omega_j}+\prod_{j=1}^{n}(\gamma_j^l)^{\omega_j}},\frac{2\prod_{j=1}^{n}(\gamma_j^u)^{\omega_j}}{\prod_{j=1}^{n}(2-\gamma_j^u)^{\omega_j}+\prod_{j=1}^{n}(\gamma_j^u)^{\omega_j}}\right]$ and $\left[\frac{\prod_{j=1}^{n}(1+\gamma_j^l)^{\omega_j}-\prod_{j=1}^{n}(1-\gamma_j^l)^{\omega_j}}{\prod_{j=1}^{n}(1+\gamma_j^l)^{\omega_j}+\prod_{j=1}^{n}(1-\gamma_j^l)^{\omega_j}},\frac{\prod_{j=1}^{n}(1+\gamma_j^u)^{\omega_j}-\prod_{j=1}^{n}(1-\gamma_j^u)^{\omega_j}}{\prod_{j=1}^{n}(1+\gamma_j^u)^{\omega_j}+\prod_{j=1}^{n}(1-\gamma_j^u)^{\omega_j}}\right]$ are the same as $w_{\gamma_{d1}}, w_{\gamma_{d2}},\cdots,w_{\gamma_{dn}}$, also the weights corresponding to $\left[\frac{\prod_{j=1}^{n}(1+\eta_j^l)^{\omega_j}-\prod_{j=1}^{n}(1-\eta_j^l)^{\omega_j}}{\prod_{j=1}^{n}(1+\eta_j^l)^{\omega_j}+\prod_{j=1}^{n}(1-\eta_j^l)^{\omega_j}},\frac{\prod_{j=1}^{n}(1+\eta_j^u)^{\omega_j}-\prod_{j=1}^{n}(1-\eta_j^u)^{\omega_j}}{\prod_{j=1}^{n}(1+\eta_j^u)^{\omega_j}+\prod_{j=1}^{n}(1-\eta_j^u)^{\omega_j}}\right]$ and $\left[\frac{2\prod_{j=1}^{n}(\eta_j^l)^{\omega_j}}{\prod_{j=1}^{n}(2-\eta_j^l)^{\omega_j}+\prod_{j=1}^{n}(\eta_j^l)^{\omega_j}},\frac{2\prod_{j=1}^{n}(\eta_j^u)^{\omega_j}}{\prod_{j=1}^{n}(2-\eta_j^u)^{\omega_j}+\prod_{j=1}^{n}(\eta_j^u)^{\omega_j}}\right]$ are the same as $w_{\eta_{d1}}, w_{\eta_{d2}},\cdots,w_{\eta_{dn}}$. It follows that

$$WIVDHFEWG(d_1,d_2,...,d_n)\leq WIVDHFEWA(d_1,d_2,...,d_n).$$

So, we complete the proof of Theorem 5.3.7. □

Theorem 5.3.8. *Let* $d_j=\left\langle h_{dj},g_{dj}\right\rangle(j=1,2,3,...,n)$ *be a collection of WIVDHFEs and let*

$\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ *be the associated weight vector with* $\omega_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \omega_j = 1$; *then*

$$WIVDHFWG(d_1, d_2, ..., d_n) \leq WIVDHFEWG(d_1, d_2, ..., d_n) \leq WIVDHFEWA(d_1, d_2, ..., d_n) \leq WIVDHFWA(d_1, d_2, ..., d_n). \tag{5.3.20}$$

If $d_1 = d_2 = \cdots = d_n$, *the equality is established.*

5.4 Proposed decision framework

In the following, we use the proposed weighted interval-valued dual hesitant fuzzy aggregation operators to build an approach to multiple criteria group decision-making with interval-valued dual hesitant fuzzy information. First, an MCGDM with interval-valued dual hesitant fuzzy information can be outlined as follows:

Let $Z = \{z_1, z_2, ..., z_m\}$ be a set of m alternatives, $C = \{c_1, c_2, ..., c_n\}$ a collection of n criteria, whose weight vector is $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$, and let $E = \{e_1, e_2, ..., e_l\}$ be a set of l DMs, whose weight vector is $W = (w_1, w_2, ..., w_l)^T$ with $w_k \in [0, 1]$ and $\sum_{k=1}^{l} w_k = 1$. Let $R^{(k)} = (r_{ij}^{(k)})_{m \times n}$ be an interval-valued dual hesitant fuzzy decision matrix, where $r_{ij}^{(k)} = \left\{ \nu_{ij}^{(k)} \mid \nu_{ij}^{(k)} \in r_{ij}^{(k)} \right\} = \left\{ \left([\gamma^l_{\nu_{ij}^{(k)}}, \gamma^u_{\nu_{ij}^{(k)}}], [\eta^l_{\nu_{ij}^{(k)}}, \eta^u_{\nu_{ij}^{(k)}}] \right) \mid \nu_{ij}^{(k)} \in r_{ij}^{(k)} \right\} \in D$ is an IVDHFE stated by the DM $e_k \in E$, where $[\gamma^l_{\nu_{ij}^{(k)}}, \gamma^u_{\nu_{ij}^{(k)}}]$ indicates the possible degree range that the alternative $z_i \in Z$ satisfies the criteria $c_j \in C$, while $[\eta^l_{\nu_{ij}^{(k)}}, \eta^u_{\nu_{ij}^{(k)}}]$ indicates the possible degree range that the alternative $z_i \in Z$ does not satisfy the criteria $c_j \in C$.

In general, there are benefit attributes (i.e., the bigger the criteria values, the better) and cost attributes (i.e., the smaller the criteria values, the better) in a MCGDM problem. In such cases, we transform the criteria values of cost type into the criteria values of benefit type; that is, normalize the interval-valued dual hesitant fuzzy decision matrix $R^{(k)} = (r_{ij}^{(k)})_{m \times n}$ by the following method:

$$r_{ij}^{(k)} = \begin{cases} r_{ij}^{(k)} \ , & \text{if } c_j \text{ is benefit type criteria ,} \\ \left(r_{ij}^{(k)}\right)^c, & \text{if } c_j \text{ is cost type criteria ,} \end{cases} \tag{5.4.1}$$

$i = 1, 2, ..., m, \ j = 1, 2, ..., n, \ k = 1, 2, ..., l.$
where $\left(r_{ij}^{(k)}\right)^c$ is the complement of $r_{ij}^{(k)}$.

The developed weighted interval-valued dual hesitant fuzzy information aggregation operators are utilized to construct an approach for solving the above MCGDM problems. The proposed methodology is described through the following steps:

Step 1: Establish the individual decision matrices:

Analyze the decision-making problem and determine the set of alternatives $Z = \{z_1, z_2, ..., z_m\}$, the set of criteria $C = \{c_1, c_2, ..., c_n\}$ and the weight vector $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$. Then, invite a group of DMs to make evaluations on alternatives with respect to each criteria in WIVDHFEs form and build the individual decision matrices $R^{(k)} = (r_{ij}^{(k)})_{m\times n}$. Go to Step 2.

Step 2: Normalize the decision matrix:

Transform the interval-valued dual hesitant fuzzy decision matrix $R^{(k)} = (r_{ij}^{(k)})_{m\times n}$ into the normalized interval-valued dual hesitant fuzzy decision matrix $D^{(k)} = (d_{ij}^{(k)})_{m\times n}$ according to Eq. (5.4.1).

Step 3: Derive the comprehensive decision matrix.

Compute the comprehensive decision matrix M as

$$M = [d_{ij}]_{m\times n} = [h_{dij}, g_{dij}]_{m\times n}, \tag{5.4.2}$$

where $h_{dij} = \left\{\bigcup_{k=1}^{l} \left(\gamma_{ij}^k, (w_\gamma)_{ij}\right)\right\}$ and $g_{dij} = \left\{\bigcup_{k=1}^{l} \left(\eta_{ij}^k, (w_\eta)_{ij}\right)\right\}$.

Furthermore, $w_{\gamma_{ij}}$ and $w_{\eta_{ij}}$ represents the importance values of the members of h_{dij} and g_{dij}, respectively.

Step 4: Obtain the fuse values:

Choose the aggregation operator WIVDHFWA/WIVDHFWG/WIVDHFEWA/ WIVDHFEWG/WIVDHFHWA/WIVDHFHWG/WIVDHFFWA or WIVDHFFWG and take the weight ω_j of each criteria c_j into consideration to derive the fused values of each alternative.

Step 5: Compute the score values:

According to Definition 5.1.5, determine the score values $S(z_i)(i = 1, 2, ..., m)$ of each alternative z_i.

Step 6: Rank all the alternatives:

Get the ranking of the alternatives $z_i(i = 1, 2, ..., m)$ in order to choice the best one(s) in accordance with the Definition 5.1.7.

The aforementioned stepwise procedure is shown diagrammatically in Fig. 5.1.

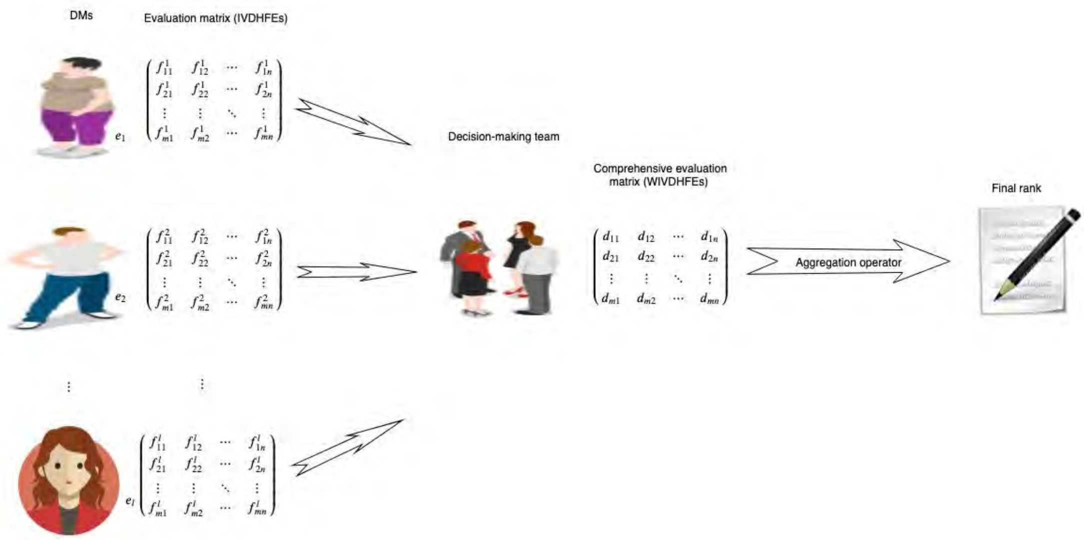

Figure 5.1: Weighted interval-valued dual hesitant fuzzy MCGDM.

5.5 The application of the developed approach in group decision-making problems

In the following, the above-described decision making approach has been manifested with a real life example of the teaching quality assessment.

Higher education teaching quality assessment is a crucial job. Its primary assessment target is the teacher and their teaching activities. It has been one of the main focus of universities for a long time. Several scholars have done work on how to improve the teaching quality and how to evaluate it adequately. Gao and his colleagues [54], applied factor analysis evaluation technique for teaching quality. Yingying et al. [55] developed a teaching quality assessment model based on the analytic hierarchy and neural networks. Han built a teaching quality assessment scale [56]. Li [57] used an uncertain linguistic weighted averaging operator to fuse the data corresponding to each alternative for evaluating the computer web-based multimedia college English reaching with uncertain linguistic information. Chen et al. [58] developed a new framework to assess teaching performance by combining the analytic hierarchy process (AHP) and comprehensive evaluation method. Chang and Wang [59] presented a cloud model for evaluating teachers in higher education. Recently, Peng and Dai [60] researched on the assessment of classroom teaching quality with q-rung orthopair fuzzy information based on multiparametric similarity measure and combinative distance-based assessment. In the current study, efforts are made to address such MCGDM problem using proposed weighted interval-valued dual hesitant fuzzy aggregation approach.

Kohat University of Science and Technology (KUST) one of the well-known universities in Khyber Pakhtunkhwa (KP), Pakistan, has placed great importance on teaching quality assessment. Currently, there are three teachers, namely Abrar (z_1), Nisar (z_2) and Israr (z_3) in the Institute of Numerical Sciences (INS) need to be evaluated, and the evaluation is processed from the following aspects, namely teaching method (c_1), number of publications (c_2) and student feedback (c_3). The weight vector of the three parameters is supposed as $(0.3, 0.4, 0.3)^T$. The evaluation information given by DMs for the three teachers are shown in Tables 5.1, 5.2 and 5.3.

Table 5.1: The decision matrix M_1 provided by D_1

	c_1	c_2	c_3
z_1	$\langle[0.4, 0.6], [0.1, 0.3]\rangle$	$\langle[0.3, 0.5], [0.3, 0.4]\rangle$	$\langle[0.6, 0.7], [0.1, 0.3]\rangle$
z_2	$\langle[0.3, 0.4], [0.5, 0.6]\rangle$	$\langle[0.3, 0.4], [0.3, 0.5]\rangle$	$\langle[0.2, 0.4], [0.3, 0.4]\rangle$
z_3	$\langle[0.4, 0.5], [0.3, 0.5]\rangle$	$\langle[0.6, 0.7], [0.1, 0.3]\rangle$	$\langle[0.4, 0.5], [0.4, 0.5]\rangle$

Table 5.2: The decision matrix M_2 provided by D_2

	c_1	c_2	c_3
z_1	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$	$\langle[0.2, 0.4], [0.2, 0.4]\rangle$	$\langle[0.5, 0.6], [0.2, 0.3]\rangle$
z_2	$\langle[0.3, 0.4], [0.4, 0.5]\rangle$	$\langle[0.4, 0.5], [0.1, 0.5]\rangle$	$\langle[0.3, 0.6], [0.2, 0.3]\rangle$
z_3	$\langle[0.1, 0.3], [0.3, 0.5]\rangle$	$\langle[0.5, 0.6], [0.1, 0.3]\rangle$	$\langle[0.4, 0.5], [0.1, 0.3]\rangle$

Table 5.3: The decision matrix M_3 provided by D_3

	c_1	c_2	c_3
z_1	$\langle[0.6, 0.7], [0.2, 0.3]\rangle$	$\langle[0.2, 0.4], [0.3, 0.5]\rangle$	$\langle[0.5, 0.6], [0.1, 0.3]\rangle$
z_2	$\langle[0.3, 0.4], [0.4, 0.6]\rangle$	$\langle[0.4, 0.5], [0.3, 0.5]\rangle$	$\langle[0.2, 0.4], [0.3, 0.4]\rangle$
z_3	$\langle[0.4, 0.5], [0.4, 0.5]\rangle$	$\langle[0.6, 0.7], [0.2, 0.3]\rangle$	$\langle[0.3, 0.5], [0.4, 0.5]\rangle$

Step 2: Since all the criteria are benefit type, so there is no need to transform the DMs assessment information to normalize form.

Step 3: Collect the individual decision matrices M_1, M_2 and M_3 into the collective decision matrix M, which is presented in Table 5.4.

Table 5.4: The Collective decision matrix M

	c_1
z_1	$\langle\{([0.4,0.6],1/3),([0.1,0.3],1/3),([0.6,0.7],1/3)\},\{([0.1,0.3],1/3),([0.2,0.3],2/3)\}\rangle$
z_2	$\langle\{([0.3,0.4],1)\},\{([0.5,0.6],1/3),([0.4,0.5],1/3),([0.4,0.6],1/3)\}\rangle$
z_3	$\langle\{([0.4,0.5],2/3),([0.1,0.3],1/3)\},\{([0.3,0.5],2/3),([0.4,0.5],1/3)\}\rangle$
	c_2
z_1	$\langle\{([0.3,0.5],1/3),([0.2,0.4],2/3)\},\{([0.3,0.4],1/3),([0.2,0.4],1/3),([0.3,0.5],1/3)\}\rangle$
z_2	$\langle\{([0.3,0.4],1/3),([0.4,0.5],2/3)\},\{([0.3,0.5],2/3),([0.1,0.5],1/3)\}\rangle$
z_3	$\langle\{([0.6,0.7],2/3),([0.5,0.6],1/3)\},\{([0.1,0.3],2/3),([0.2,0.3],1/3)\}\rangle$
	c_3
z_1	$\langle\{([0.6,0.7],1/3),([0.5,0.6],2/3)\},\{([0.1,0.3],2/3),([0.2,0.3],1/3)\}\rangle$
z_2	$\langle\{([0.2,0.4],2/3),([0.3,0.6],1/3)\},\{([0.3,0.4],2/3),([0.2,0.3],1/3)\}\rangle$
z_3	$\langle\{([0.4,0.5],2/3),([0.3,0.5],1/3)\},\{([0.4,0.5],2/3),([0.1,0.3],1/3)\}\rangle$

5.5.1 Applying the WIVDHFWA operator

In the following paragraph, we will employ WIVDHFWA operator to handle the above considered MCGDM problem.

Step 4: The fused values are

$$
\begin{aligned}
z_1 = \langle\{&([0.4349,0.5988],1/27),([0.3958,0.5627],2/27),([0.4039,0.5685],2/27),([0.3627,0.5296],4/27),([0.3618,0.5255],1/27),\\
&([0.3177,0.4827],2/27),([0.3268,0.4896],2/27),([0.2802,0.4436],4/27),([0.4996,0.6319],1/27),([0.4650,0.5988],2/27),\\
&([0.4722,0.6041],2/27),([0.4357,0.5685],4/27)\},\{([0.1552,0.3366],2/27),([0.1910,0.3366],1/27),([0.1319,0.3366],2/27),\\
&([0.1625,0.3366],1/27),([0.1552,0.3680],2/27),([0.1910,0.3680],1/27),([0.1910,0.3366],4/27),([0.2352,0.3366],2/27),\\
&([0.1625,0.3366],4/27),([0.2,0.3366],2/27),([0.1910,0.3680],4/27),([0.2352,0.3680],2/27)\}\rangle,
\end{aligned}
$$

$$z_2 = \langle \{([0.2714, 0.4], 2/9), ([0.3, 0.4687], 1/9), ([0.3149, 0.4422], 4/9), ([0.3419, 0.5060], 2/9)\}, \{([0.3497, 0.4939], 4/27), ([0.3096, 0.4530], 2/27), ([0.2253, 0.4939], 2/27), ([0.1995, 0.4530], 1/27), ([0.3270, 0.4676], 4/27), ([0.2896, 0.4289], 2/27), ([0.2107, 0.4676], 2/27), ([0.1866, 0.4289], 1/27), ([0.3270, 0.4939], 4/27), ([0.2896, 0.4530], 2/27), ([0.2107, 0.4939], 2/27), ([0.1866, 0.4530], 1/27)\} \rangle,$$

$$z_3 = \langle \{([0.4898, 0.5924], 8/27), ([0.4657, 0.5924], 4/27), ([0.4422, 0.5427], 4/27), ([0.4158, 0.5427], 2/27), ([0.4238, 0.5491], 4/27), ([0.3966, 0.5491], 2/27), ([0.3700, 0.4941], 2/27), ([0.3402, 0.4941], 1/27)\}, \{([0.2107, 0.4076], 8/27), ([0.1390, 0.3497], 4/27), ([0.2780, 0.4076], 4/27), ([0.1835, 0.3497], 2/27), ([0.2297, 0.4076], 4/27), ([0.1516, 0.3497], 2/27), ([0.3031, 0.4076], 2/27), ([0.2, 0.3497], 1/27)\} \rangle.$$

Step 5: According to the Definition 5.1.5, the score value of each alternative $z_i (i = 1, 2, 3)$ is tabulated in Table 5.5.

Table 5.5: Score values while using WIVDHFWA operator

	z_1	z_2	z_3
$S(z_i)$	0.0163	0.0663	0.0251

5.5.2 Applying the WIVDHFWG operator

In what follows, the WIVDHFWG operator is applied to the considered problem to get the ranking of each alternative.

Step 4: The fused values are

$$z_1 = \langle \{([0.4026, 0.5842], 1/27), ([0.3812, 0.5578], 2/27), ([0.3424, 0.5343], 2/27), ([0.3241, 0.5102], 4/27), ([0.2656, 0.4745], 1/27), ([0.2515, 0.4530], 2/27), ([0.2259, 0.4340], 2/27), ([0.2138, 0.4144], 4/27), ([0.4547, 0.6119], 1/27), ([0.4305, 0.5842], 2/27), ([0.3866, 0.5596], 2/27), ([0.3660, 0.5343], 4/27)\}, \{([0.1860, 0.3419], 2/27), ([0.2143, 0.3419], 1/27), ([0.1414, 0.3419], 2/27), ([0.1712, 0.3419], 1/27), ([0.1860, 0.3881], 2/27), ([0.2143, 0.3881], 1/27), ([0.2143, 0.3419], 4/27), ([0.2416, 0.3419], 2/27), ([0.1712, 0.3419], 4/27), ([0.2, 0.3419], 2/27), ([0.2143, 0.3881], 4/27), ([0.2416, 0.3881], 2/27)\} \rangle,$$

$$z_2 = \langle \{([0.2656, 0.4], 2/9), ([0.3, 0.4517], 1/9), ([0.2980, 0.4373], 4/9), ([0.3366, 0.4939], 2/9)\}, \{([0.3894, 0.5060], 4/27), ([0.3413, 0.4827], 2/27), ([0.3003, 0.5060], 2/27), ([0.2717, 0.4827], 1/27), ([0.3316, 0.4719], 4/27), ([0.3043, 0.4469], 2/27), ([0.2609, 0.4719], 2/27), ([0.2307, 0.4469], 1/27), ([0.3316, 0.5060], 4/27), ([0.3043, 0.4827], 2/27), ([0.2609, 0.5060], 2/27), ([0.2307, 0.4827], 1/27)\} \rangle,$$

$$z_3 = \langle \{([0.4704, 0.5720], 8/27), ([0.4315, 0.5720], 4/27), ([0.4373, 0.5378], 4/27), ([0.4012, 0.5378], 2/27), ([0.3104, 0.4908], 4/27), ([0.2847, 0.4908], 2/27), ([0.2885, 0.4614], 2/27), ([0.2647, 0.4614], 1/27)\}, \{([0.2609, 0.4279], 8/27), ([0.1654, 0.3672], 4/27), ([0.2949, 0.4279], 4/27), ([0.2038, 0.3672], 2/27), ([0.2944, 0.4279], 4/27), ([0.2030, 0.3672], 2/27), ([0.3268, 0.4279], 2/27), ([0.2398, 0.3672], 1/27)\} \rangle.$$

Step 5: According to the Definition 5.1.5, the score value of each alternative $z_i (i = 1, 2, 3)$ is displayed in Table 5.6.

Table 5.6: Score values while adopting WIVDHFWG operator

	z_1	z_2	z_3
$S(z_i)$	0.0115	0.0553	0.0169

5.5.3 Applying the WIVDHFEWA operator

To rank the alternatives, the WIVDHFEWA operator is chosen for information aggregation.

Step 4: The fused values are

$$z_1 = \langle \{([0.4291, 0.5966], 1/27), ([0.3933, 0.5619], 2/27), ([0.3936, 0.5633], 2/27), ([0.3566, 0.5266], 4/27), ([0.3468, 0.5174], 1/27), ([0.3080, 0.47808], 2/27), ([0.3087, 0.4796], 2/27), ([0.2692, 0.4384], 4/27), ([0.4927, 0.6291], 1/27), ([0.4594, 0.5966], 2/27), ([0.4597, 0.5978], 2/27), ([0.4250, 0.5633], 4/27)\}, \{([0.1574, 0.3373], 2/27), ([0.1930, 0.3373], 1/27), ([0.1325, 0.3507], 2/27), ([0.1636, 0.3373], 1/27), ([0.1574, 0.3708], 2/27), ([0.1929, 0.3708], 1/27), ([0.1930, 0.3373], 4/27), ([0.2358, 0.3373], 2/27), ([0.1632, 0.3507], 4/27), ([0.2, 0.3373], 2/27), ([0.1930, 0.3708], 4/27), ([0.2358, 0.3708], 2/27)\} \rangle,$$

$$z_2 = \langle \{([0.2706, 0.4], 2/9), ([0.3, 0.4656], 1/9), ([0.3123, 0.4414], 4/9), ([0.3409, 0.5039], 2/9)\}, \{([0.3676, 0.4959], 4/27), ([0.3135, 0.4579], 2/27), ([0.2326, 0.4959], 2/27), ([0.2056, 0.4579], 1/27), ([0.3326, 0.4683], 4/27), ([0.2914, 0.4319], 2/27), ([0.2154, 0.4683], 2/27), ([0.1902, 0.4319], 1/27), ([0.3276, 0.4959], 4/27), ([0.2918, 0.4579], 2/27), ([0.2154, 0.4959], 2/27), ([0.1902, 0.4579], 1/27)\} \rangle,$$

$$z_3 = \langle \{([0.4865, 0.5892], 8/27), ([0.4599, 0.5892], 4/27), ([0.4414, 0.5449], 4/27), ([0.4134, 0.5489], 2/27), ([0.4089, 0.5403], 4/27), ([0.3801, 0.5403], 2/27), ([0.3599, 0.4890], 2/27), ([0.3298, 0.4890], 1/27)\}, \{([0.2269, 0.4246], 8/27), ([0.1406, 0.3519], 4/27), ([0.2799, 0.4108], 4/27), ([0.1852, 0.3676], 2/27), ([0.2364, 0.4108], 4/27), ([0.1550, 0.3519], 2/27), ([0.3061, 0.4078], 2/27), ([0.2035, 0.3519], 1/27)\} \rangle.$$

Step 5: The score value of each alternative $z_i (i = 1, 2, 3)$ is shown in Table 5.7.

Table 5.7: Score values while employing WIVDHFEWA operator

	z_1	z_2	z_3
$S(z_i)$	0.0155	0.0628	0.0236

5.5.4 Applying the WIVDHFEWG operator

In the following, we utilize the WIVDHFEWG operator in the developed approach for solving the MCGDM problem.

Step 4: The fused values are

$$z_1 = \langle \{([0.4072, 0.5866], 1/27), ([0.3833, 0.5586], 2/27), ([0.3505, 0.5400], 2/27), ([0.3292, 0.5135], 4/27), ([0.2761, 0.4826], 1/27), ([0.2587, 0.4579], 2/27), ([0.2250, 0.4418], 2/27), ([0.2198, 0.4186], 4/27), ([0.4503, 0.6154], 1/27), ([0.4359, 0.5866], 2/27), ([0.3997, 0.5675], 2/27), ([0.3760, 0.5400], 4/27)\}, \{([0.1819, 0.3409], 2/27), ([0.2115, 0.3409], 1/27), ([0.1403, 0.3409], 2/27), ([0.1703, 0.3409], 1/27), ([0.1819, 0.3846], 2/27), ([0.2115, 0.3846], 1/27), ([0.2115, 0.3409], 4/27), ([0.2406, 0.3409], 2/27), ([0.1703, 0.3409], 4/27), ([0.2, 0.3409], 2/27), ([0.2115, 0.3846], 4/27), ([0.2406, 0.3846], 2/27)\}\rangle,$$

$$z_2 = \langle \{([0.2663, 0.4], 2/9), ([0.3, 0.4542], 1/9), ([0.2678, 0.4380], 4/9), ([0.3373, 0.4959], 2/9)\}, \{([0.3793, 0.5039], 4/27), ([0.3358, 0.4780], 2/27), ([0.2893, 0.5039], 2/27), ([0.2597, 0.4780], 1/27), ([0.3308, 0.4712], 4/27), ([0.3019, 0.4441], 2/27), ([0.2543, 0.4712], 2/27), ([0.2242, 0.4441], 1/27), ([0.3308, 0.5039], 4/27), ([0.3019, 0.4780], 2/27), ([0.2543, 0.5039], 2/27), ([0.2241, 0.4777], 1/27)\}\rangle,$$

$$z_3 = \langle \{([0.4734, 0.5753], 8/27), ([0.4367, 0.5753], 4/27), ([0.4380, 0.5547], 4/27), ([0.4034, 0.5386], 2/27), ([0.3255, 0.5004], 4/27), ([0.2979, 0.5004], 2/27), ([0.2989, 0.4669], 2/27), ([0.2734, 0.4669], 1/27)\}, \{([0.2307, 0.4246], 8/27), ([0.1617, 0.3639], 4/27), ([0.2922, 0.4246], 4/27), ([0.2013, 0.3639], 2/27), ([0.2862, 0.4246], 4/27), ([0.1947, 0.3639], 2/27), ([0.3232, 0.4246], 2/27), ([0.2339, 0.3639], 1/27)\}\rangle.$$

Step 5: According to the Definition 5.1.5, the score value of each alternative $z_i (i = 1, 2, 3)$ is listed in Table 5.8.

Table 5.8: Score values while applying WIVDHFEWG operator

	z_1	z_2	z_3
$S(z_i)$	0.0122	0.0583	0.0187

5.6 Comparison analysis

To examine the rationality and effectiveness of the proposed aggregation operators, a comparative analysis is carried out with some existing aggregation operators of Ref. [19] on the considered problem.

Step 3: We firstly convert the WIVDHFEs presented in Table 5.4 into IVDHFEs by excluding the importance values. The results obtained in this step are shown in Table 5.9.

Table 5.9: Interval-valued dual hesitant fuzzy collective decision matrix

	c_1
z_1	$\langle \{[0.4, 0.6], [0.1, 0.3], [0.6, 0.7]\}, \{[0.1, 0.3], [0.2, 0.3]\} \rangle$
z_2	$\langle \{[0.3, 0.4]\}, \{[0.5, 0.6], [0.4, 0.5], [0.4, 0.6], \} \rangle$
z_3	$\langle \{[0.4, 0.5], [0.1, 0.3]\}, \{[0.3, 0.5], [0.4, 0.5]\} \rangle$
	c_2
z_1	$\langle \{[0.3, 0.5], [0.2, 0.4]\}, \{[0.3, 0.4], [0.2, 0.4], [0.3, 0.5]\} \rangle$
z_2	$\langle \{[0.3, 0.4], [0.4, 0.5]\}, \{[0.3, 0.5], [0.1, 0.5]\} \rangle$
z_3	$\langle \{[0.6, 0.7], [0.5, 0.6]\}, \{[0.1, 0.3], [0.2, 0.3]\} \rangle$
	c_3
z_1	$\langle \{[0.6, 0.7], [0.5, 0.6]\}, \{[0.1, 0.3], [0.2, 0.3]\} \rangle$
z_2	$\langle \{[0.2, 0.4], [0.3, 0.6]\}, \{[0.3, 0.4], [0.2, 0.3]\} \rangle$
z_3	$\langle \{[0.4, 0.5], [0.3, 0.5]\}, \{[0.4, 0.5], [0.1, 0.3]\} \rangle$

Now we are able to apply the approach given by [19], on the considered example. Under it, we aggregate the preference values by utilizing the operators and hence obtain the score value of each alternative.

5.6.1 Applying the IVDHFWA operator

To derive the ranking of the alternatives, here we employ the IVDHFWA operator [19] to aggregate the interval-valued dual hesitant fuzzy information of the alternatives $z_i (i = 1, 2, 3)$ on all criteria $c_j (j = 1, 2, 3)$.

Step 4: The fused values are

$$\begin{aligned} z_1 = \langle \{ & [0.4349, 0.5988], [0.3958, 0.5627], [0.4039, 0.5685], [0.3627, 0.5296], [0.3618, 0.5255], [0.3177, 0.4827], \\ & [0.3268, 0.4896], [0.2802, 0.4436], [0.4996, 0.6319], [0.4650, 0.5988], [0.4722, 0.6041], [0.4357, 0.5685] \}, \\ \{ & [0.1552, 0.3366], [0.1910, 0.3366], [0.1319, 0.3366], [0.1625, 0.3366], [0.1552, 0.3680], [0.1910, 0.3680], \\ & [0.1910, 0.3366], [0.2352, 0.3366], [0.1625, 0.3366], [0.2, 0.3366], [0.1910, 0.3680], [0.2352, 0.3680] \} \rangle, \end{aligned}$$

$$\begin{aligned} z_2 = \langle \{ & [0.2714, 0.4], [0.3, 0.4687], [0.3149, 0.4422], [0.3419, 0.5060] \}, \{ [0.3497, 0.4939], [0.3096, 0.4530], \\ & [0.2253, 0.4939], [0.1995, 0.4530], [0.3270, 0.4676], [0.2896, 0.4289], [0.2107, 0.4676], [0.1866, 0.4289], \\ & [0.3270, 0.4939], [0.2896, 0.4530], [0.2107, 0.4939], [0.1866, 0.4530] \} \rangle, \end{aligned}$$

$$\begin{aligned} z_3 = \langle \{ & [0.4898, 0.5924], [0.4657, 0.5924], [0.4422, 0.5427], [0.4158, 0.5427], [0.4238, 0.5491], [0.3966, 0.5491], \\ & [0.3700, 0.4941], [0.3402, 0.4941] \}, \{ [0.2107, 0.4076], [0.1390, 0.3497], [0.2780, 0.4076], [0.1835, 0.3497], \\ & [0.2297, 0.4076], [0.1516, 0.3497], [0.3031, 0.4076], [0.2, 0.3497] \} \rangle. \end{aligned}$$

Step 5: According to the Definition 2.2.1, the score value of each alternative $z_i (i = 1, 2, 3)$ is noted in Table 5.10.

Table 5.10: Score values while using IVDHFWA operator

	z_1	z_2	z_3
$S(z_i)$	0.2080	0.0185	0.1859

5.6.2 Applying the IVDHFWG operator

For comparative study, in this part, the teaching quality assessment problem is again solved by the IVDHFWG operator of Ref. [19].

Step 4: The fused values are

$$\begin{aligned} z_1 = \langle &\{[0.4026, 0.5842], [0.3812, 0.5578], [0.3424, 0.5343], [0.3241, 0.5102], [0.2656, 0.4745], [0.2515, 0.4530], \\ &[0.2259, 0.4340], [0.2138, 0.4144], [0.4547, 0.6119], [0.4305, 0.5842], [0.3866, 0.5596], [0.3660, 0.5343]\}, \\ &\{[0.1860, 0.3419], [0.2143, 0.3419], [0.1414, 0.3419], [0.1712, 0.3419], [0.1860, 0.3881], [0.2143, 0.3881], \\ &[0.2143, 0.3419], [0.2416, 0.3419], [0.1712, 0.3419], [0.2, 0.3419], [0.2143, 0.3881], [0.2416, 0.3881]\}\rangle, \end{aligned}$$

$$\begin{aligned} z_2 = \langle &\{[0.2656, 0.4], [0.3, 0.4517], [0.2980, 0.4373], [0.3366, 0.4939]\}, \{[0.3894, 0.5060], [0.3413, 0.4827], \\ &[0.3003, 0.5060], [0.2717, 0.4827], [0.3316, 0.4719], [0.3043, 0.4469], [0.2609, 0.4719], [0.2307, 0.4469], \\ &[0.3316, 0.5060], [0.3043, 0.4827], [0.2609, 0.5060], [0.2307, 0.4827]\}\rangle, \end{aligned}$$

$$\begin{aligned} z_3 = \langle &\{[0.4704, 0.5720], [0.4315, 0.5720], [0.4373, 0.5378], [0.4012, 0.5378], [0.3104, 0.4908], [0.2847, 0.4908], \\ &[0.2885, 0.4614], [0.2647, 0.4614]\}, \{[0.2609, 0.4279], [0.1654, 0.3672], [0.2949, 0.4279], [0.2038, 0.3672], \\ &[0.2944, 0.4279], [0.2030, 0.3672], [0.3268, 0.4279], [0.2398, 0.3672]\}\rangle. \end{aligned}$$

Step 5: According to the Definition 2.2.1, the score value of each alternative $z_i(i = 1, 2, 3)$ is listed in Table 5.11.

Table 5.11: Score values while applying IVDHFWG operator

	z_1	z_2	z_3
$S(z_i)$	0.1505	−0.0167	0.1152

The score values of alternatives are graphically represented in Fig. 5.2.

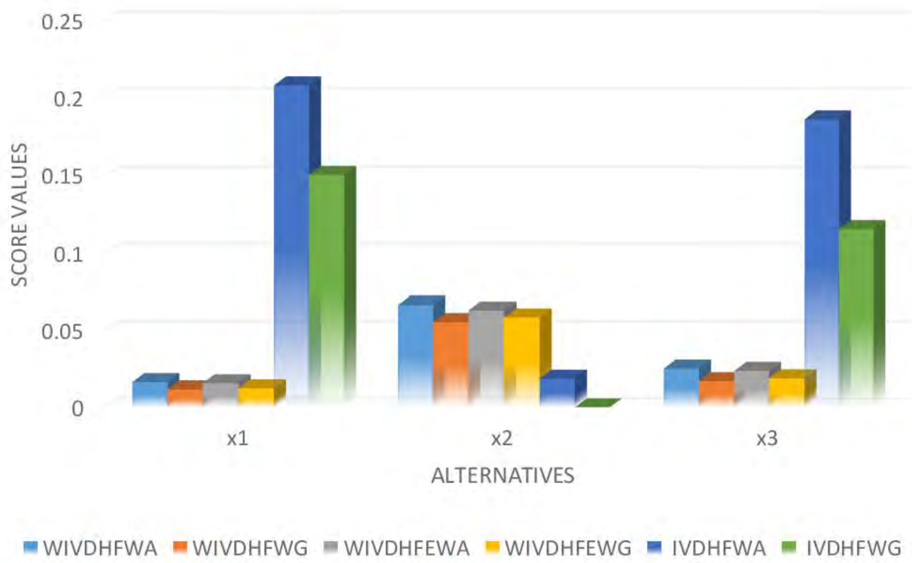

Figure 5.2: The score values of alternatives

The raking results of alternatives are documented in Table 5.12.

Table 5.12: The ranking order of alternatives

Operator	z_1	z_2	z_3	Order
WIVDHFWA	3	1	2	$z_2 > z_3 > z_1$
WIVDHFWG	3	1	2	$z_2 > z_3 > z_1$
WIVDHFEWA	3	1	2	$z_2 > z_3 > z_1$
WIVDHFEWG	3	1	2	$z_2 > z_3 > z_1$
IVDHFWA [19]	1	3	2	$z_1 > z_3 > z_2$
IVDHFWG [19]	1	3	2	$z_1 > z_3 > z_2$

Table 5.12 depicts the ranking order from different aggregation operators, and we can see from this table that the ranking order obtained by the proposed operators is the same. Further, from Tables 5.5, 5.6, 5.7 and 5.8, it is observed that the score values of the proposed operators are also approximately same, which shows that the generalized weighted interval-valued dual hesitant fuzzy information operators proposed in this study have inherent consistency.

The presented Tables 5.10 and 5.11, clearly demonstrate that the score values of the IVDHFWA operator and IVDHFWG operator which uses IVDHFEs as preference information vary from each other. So the Score values obtained by these aggregation operators are biased. Secondly, It can be noticed from Table 5.12 that the ranking obtained by these operators are different from the ones through WIVDHFWA, WIVDHFWG, WIVDHFEWA and WIVDHFEWG with the first and third alternatives swapped. This fluctuation in results is due to the ability of the stated aggregation operators to capture the evaluation data along with their importance degrees whereas the approach outlined by Ju et al. [19] by utilizing IVD-

HFWA and IVDHFWG operators eliminate the values of importance. This negligence of the importance values causes a severe loss of information, leading to erroneous decision results.

Chapter 6

WASPAS-based decision making methodology with unknown weight information under uncertain evaluations

In this chapter, to enrich the existing theory of UPLTSs, we redefine the operational laws, comparison method, and some fundamental aggregation operators. Furthermore, a novel aggregation operator UPLSWG is also designed. Following this, a novel extension of WASPAS method called uncertain probabilistic linguistic (UPL)-WASPAS is extended to UPLTS context with altogether unknown weight information for ranking objects. In the end, an example of supplier selection is addressed, and detail analysis of results is performed. The research work of this chapter is published in [61].

6.1 Novel operations and comparison method of UPLTSs

Based on the limitations of the existing study, in this section, we first improve the basic operational laws and then put forward a novel comparison method to accommodate the situations where the input arguments are UPLTSs.

6.1.1 Novel operational law of UPLTSs

The proposed operational law for UPLTSs, which assists in the aggregation of criteria values, is not reasonable. To observe the flaws of existing operational law better, consider the following example.

Example 6.1.1. *Suppose that $S = \{\pounds_\alpha | \alpha = 0, 1, ..., 6\}$ be an LTS, and let $U_1(p) = \{(\pounds_3, [0.5, 0.7]), (\pounds_2, [0.3, 0.5])\}$ and $U_2(P) = \{(\pounds_2, [0.3, 0.5]), (\pounds_1, [0.4, 0.8])\}$ be two UPLTSs, then by the add operation in Definition 2.2.3, it can be obtained that $U_1(p) \oplus U_2(p) = \left\{\left(\frac{0.5+0.7}{2}\right)\pounds_3 \oplus \left(\frac{0.3+0.5}{2}\right)\pounds_2, \left(\frac{0.3+0.5}{2}\right)\pounds_2 \oplus \left(\frac{0.4+0.8}{2}\right)\pounds_1\right\} = \{\pounds_{2.6}, \pounds_{1.4}\}$.*

Clearly, the generated result after operation lost the probability information and can no longer be considered a UPLTS. Another weak point to be elucidated is that the existing operations operate the subscripts of linguistics terms with their corresponding interval probability which is irrational because these two are totally different dimensions. To avoid these defects, in the following, we will define some new operations for UPLTSs.

Definition 6.1.2. *Let $U(p) = \left\{\left(\pounds^k, [p^k, q^k]\right) | k = 1, 2, ..., \pounds\right\}$ and $U_1(p) = \left\{\left(\pounds_1^k, [p_1^k, q_1^k]\right) | k = 1, 2, ..., \pounds_1\right\}$ be two ordered and scaled UPLTSs, and λ be a positive real number; then*

i. $U(p) \oplus U_1(p) = \bigcup_{\left(\pounds^k,[p^k,q^k]\right)\in U(P),\left(\pounds_1^k,[p_1^k,q_1^k]\right)\in U_1(P)} \left\{\left(\pounds^k \oplus \pounds_1^k, \left[\frac{p^k p_1^k}{\sum_{k=1}^{\pounds} p^k \sum_{k=1}^{\pounds_1} p_1^k}, \frac{q^k + q_1^k}{\sum_{k=1}^{\pounds} q^k \sum_{k=1}^{\pounds_1} q_1^k}\right]\right)\right\}$;

ii. $U(p) \otimes U_1(p) = \bigcup_{\left(\pounds^k,[p^k,q^k]\right)\in U(P),\left(\pounds_1^k,[p_1^k,q_1^k]\right)\in U_1(P)} \left\{\left(\pounds^k \otimes \pounds_1^k, \left[\frac{p^k p_1^k}{\sum_{k=1}^{\pounds} p^k \sum_{k=1}^{\pounds_1} p_1^k}, \frac{q^k + q_1^k}{\sum_{k=1}^{\pounds} q^k \sum_{k=1}^{\pounds_1} q_1^k}\right]\right)\right\}$;

iii. $\lambda U(p)=\bigcup_{\left(\pounds^k,[p^k,q^k]\right)\in U(P)}\left\{\left(\lambda\pounds^k,[p^k,q^k]\right)\right\}$;

iv. $(U(p))^\lambda=\bigcup_{\left(\pounds^k,[p^k,q^k]\right)\in U(P)}\left\{\left((\pounds^k)^\lambda,[p^k,q^k]\right)\right\}$.

Remark 6.1.3. *In the above defined operational law, one should take notice of an important fact: If there are n number of UPLTSs (say) $U_1(P)$,$U_2(P)$,...,$U_n(P)$, and n is very large number, then the $p_1^k p_2^k ... p_n^k$ is nearly equal to zero that is the lower bound of the interval probability is completely ignored. On the other hand, the upper bound outcome, i.e., $q_1^k+q_2^k+...+q_n^k$ is the negation of the lower bound. It does not confirm the actual situation. A feasible way to accomplish this issue is the normalization technique. Therefore, in the operation we divide them by $\sum_{k=1}^{\pounds_1}p_1^k\sum_{k=1}^{\pounds_2}p_2^k...\sum_{k=1}^{\pounds_n}p_n^k$ and $\sum_{k=1}^{\pounds_1}q_1^k\sum_{k=1}^{\pounds_2}q_2^k...\sum_{k=1}^{\pounds_n}q_n^k$, respectively.*

Theorem 6.1.4. *Given any three ordered and scaled UPLTSs $U(p)=\left\{\left(\pounds^k,[p^k,q^k]\right)|k=1,2,..,\pounds\right\}$, $U_1(P)=\left\{\left(\pounds_1^k,[p_1^k,q_1^k]\right)|k=1,2,...,\pounds_1\right\}$ and $U_2(P)=\left\{\left(\pounds_2^k,[p_2^k,q_2^k]\right)|k=1,2,....,\pounds_2\right\}$, $\lambda,\lambda_1\geq 0$, then*

i. $U(p)\oplus U_1(p)=U_1(p)\oplus U(P)$;

ii. $(U(p)\oplus U_1(p))\oplus U_2(p)=U(p)\oplus(U_1(p)\oplus U_2(p))$;

iii. $\lambda(U(p)\oplus U_1(p))=\lambda U(p)\oplus\lambda U_1(p)$;

iv. $U(p)\otimes U_1(p)=U_1(p)\otimes U(p)$;

v. $(U(p)\otimes U_1(p))\otimes U_2(p)=U(p)\otimes(U_1(p)\otimes U_2(p))$;

vi. $(U(p)\otimes U_1(p))^\lambda=(U(p))^\lambda\otimes(U_1(p))^\lambda$.

Proof. i.

$$\begin{aligned}U(p)\oplus U_1(p)&=\bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in E(p),\left(\pounds_1^k,\left[p_1^k,q_1^k\right]\right)\in E_1(p)}\left\{\left(\pounds^k\oplus\pounds_1^k,\left[\frac{p^kp_1^k}{\sum_{k=1}^{\pounds}p^k\sum_{k=1}^{\pounds_1}p_1^k},\frac{q^k+q_1^k}{\sum_{k=1}^{\pounds}q^k\sum_{k=1}^{\pounds_1}q_1^k}\right]\right)\right\}\\&=\bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in E(p),\left(\pounds_1^k,\left[p_1^k,q_1^k\right]\right)\in E_1(p)}\left\{\left(\pounds_1^k\oplus\pounds^k,\left[\frac{p_1^kp^k}{\sum_{k=1}^{\pounds_1}p_1^k\sum_{k=1}^{\pounds}p^k},\frac{q^k+q_1^k}{\sum_{k=1}^{\pounds_1}q_1^k\sum_{k=1}^{\pounds}q^k}\right]\right)\right\}\\&=U_1(p)\oplus U(p).\end{aligned}$$

ii.

$$
\begin{aligned}
(U(p) \oplus U_1(p)) \oplus U_2(p) =& \bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in E(P),\left(\pounds_1^k,\left[p_1^k,q_1^k\right]\right)\in E_1(p)} \left\{\left(\pounds^k \oplus \pounds_1^k, \left[\frac{p^k p_1^k}{\sum_{k=1}^{\pounds} p^k \sum_{k=1}^{\pounds_1} p_1^k}, \frac{q^k q_1^k}{\sum_{k=1}^{\pounds} q^k \sum_{k=1}^{\pounds_1} q_1^k}\right]\right)\right\} \oplus U_2(p) \\
=& \bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in E(p),\left(\pounds_1^k,\left[p_1^k,q_1^k\right]\right)\in E_1(p),\left(\pounds_2^k,\left[p_2^k,q_2^k\right]\right)\in E_2(p)} \left\{\left(\left(\pounds^k \oplus \pounds_1^k\right) \oplus \pounds_2^k, \left[\frac{\left(p^k p_1^k\right) p_2^k}{\left(\sum_{k=1}^{\pounds} p^k \sum_{k=1}^{\pounds_1} p_1^k\right) \sum_{k=1}^{\pounds_2} p_2^k}, \frac{\left(q^k + q_1^k\right) + q_2^k}{\left(\sum_{k=1}^{\pounds} q^k \sum_{k=1}^{\pounds_1} q_1^k\right) \sum_{k=1}^{\pounds_2} q_2^k}\right]\right)\right\} \\
=& \bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in E(p),\left(\pounds_1^k,\left[p_1^k,q_1^k\right]\right)\in E_1(p),\left(\pounds_2^k,\left[p_2^k,q_2^k\right]\right)\in E_2(p)} \left\{\left(\pounds^k \oplus \left(\pounds_1^k \oplus \pounds_2^k\right), \left[\frac{p^k \left(p_1^k p_2^k\right)}{\sum_{k=1}^{\pounds} p^k \left(\sum_{k=1}^{\pounds_1} p_1^k \sum_{k=1}^{\pounds_2} p_2^k\right)}, \frac{q^k + \left(q_1^k + q_2^k\right)}{\sum_{k=1}^{\pounds} q^k \left(\sum_{k=1}^{\pounds_1} q_1^k \sum_{k=1}^{\pounds_2} q_2^k\right)}\right]\right)\right\} \\
=& U(p) \oplus \left(U_1(p) \oplus U_2(p)\right).
\end{aligned}
$$

iii.

$$
\begin{aligned}
\lambda\left(U(p) \oplus U_1(p)\right) =& \lambda\left(\bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in E(p),\left(\pounds_1^k,\left[p_1^k,q_1^k\right]\right)\in E_1(p)} \left\{\left(\pounds^k \oplus \pounds_1^k, \left[\frac{p^k p_1^k}{\sum_{k=1}^{\pounds} p^k \sum_{k=1}^{\pounds_1} p_1^k}, \frac{q^k q_1^k}{\sum_{k=1}^{\pounds} q^k \sum_{k=1}^{\pounds_1} q_1^k}\right]\right)\right\}\right) \\
=& \bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in E(p),\left(\pounds_1^k,\left[p_1^k,q_1^k\right]\right)\in E_1(p)} \left\{\left(\lambda\left(\pounds^k \oplus \pounds_1^k\right), \left[\frac{p^k p_1^k}{\sum_{k=1}^{\pounds} p^k \sum_{k=1}^{\pounds_1} p_1^k}, \frac{q^k q_1^k}{\sum_{k=1}^{\pounds} q^k \sum_{k=1}^{\pounds_1} q_1^k}\right]\right)\right\} \\
=& \bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in E(P),\left(\pounds_1^k,\left[p_1^k,q_1^k\right]\right)\in E_1(p)} \left\{\left(\lambda\pounds^k \oplus \lambda\pounds_1^k, \left[\frac{p^k p_1^k}{\sum_{k=1}^{\pounds} p^k \sum_{k=1}^{\pounds_1} p_1^k}, \frac{q^k q_1^k}{\sum_{k=1}^{\pounds} q^k \sum_{k=1}^{\pounds_1} q_1^k}\right]\right)\right\} \\
=& \lambda E(p) \oplus \lambda E_1(p).
\end{aligned}
$$

iv.

$$
\begin{aligned}
U(p) \otimes U_1(p) =& \bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in E(P),\left(\pounds_1^k,\left[p_1^k,q_1^k\right]\right)\in E_1(p)} \left\{\left(\pounds^k \otimes \pounds_1^k, \left[\frac{p^k p_1^k}{\sum_{k=1}^{\pounds} p^k \sum_{k=1}^{\pounds_1} p_1^k}, \frac{q^k + q_1^k}{\sum_{k=1}^{\pounds} q^k \sum_{k=1}^{\pounds_1} q_1^k}\right]\right)\right\} \\
=& \bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in E(p),\left(\pounds_1^k,\left[p_1^k,q_1^k\right]\right)\in E_1(p)} \left\{\left(\pounds_1^k \otimes \pounds^k, \left[\frac{p_1^k p^k}{\sum_{k=1}^{\pounds_1} p_1^k \sum_{k=1}^{\pounds} p^k}, \frac{q^k + q_1^k}{\sum_{k=1}^{\pounds_1} q_1^k \sum_{k=1}^{\pounds} q^k}\right]\right)\right\} \\
=& U_1(p) \otimes U(p).
\end{aligned}
$$

v.

$$\begin{aligned}
(U(p)\otimes U_1(p))\otimes U_2(p) =& \bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in E(p),\left(\pounds_1^k,\left[p_1^k,q_1^k\right]\right)\in E_1(p)} \left\{\left(\pounds^k\otimes \pounds_1^k, \left[\frac{p^kp_1^k}{\sum_{k=1}^{\pounds} p^k \sum_{k=1}^{\pounds_1} p_1^k}, \frac{q^kq_1^k}{\sum_{k=1}^{\pounds} q^k \sum_{k=1}^{\pounds_1} q_1^k}\right]\right)\right\} \otimes U_2(p)\\
=& \bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in E(P),\left(\pounds_1^k,\left[p_1^k,q_1^k\right]\right)\in E_1(p),\left(\pounds_2^k,\left[p_2^k,q_2^k\right]\right)\in E_2(p)} \left\{\left(\left(\pounds^k\otimes \pounds_1^k\right)\otimes \pounds_2^k, \left[\frac{\left(p^kp_1^k\right)p_2^k}{\left(\sum_{k=1}^{\pounds} p^k \sum_{k=1}^{\pounds_1} p_1^k\right)\sum_{k=1}^{\pounds_2} p_2^k}, \frac{\left(q^k+q_1^k\right)+q_2^k}{\left(\sum_{k=1}^{\pounds} q^k \sum_{k=1}^{\pounds_1} q_1^k\right)\sum_{k=1}^{\pounds_2} q_2^k}\right]\right)\right\}\\
=& \bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in E(p),\left(\pounds_1^k,\left[p_1^k,q_1^k\right]\right)\in E_1(p),\left(\pounds_2^k,\left[p_2^k,q_2^k\right]\right)\in E_2(p)} \left\{\left(\pounds^k\otimes \left(\pounds_1^k\otimes \pounds_2^k\right), \left[\frac{p^k\left(p_1^kp_2^k\right)}{\sum_{k=1}^{\pounds} p^k \left(\sum_{k=1}^{\pounds_1} p_1^k\sum_{k=1}^{\pounds_2} p_2^k\right)}, \frac{q^k+\left(q_1^k+q_2^k\right)}{\sum_{k=1}^{\pounds} q^k \left(\sum_{k=1}^{\pounds_1} q_1^k\sum_{k=1}^{\pounds_2} q_2^k\right)}\right]\right)\right\}\\
=& U(p)\otimes\left(U_1(p)\otimes U_2(p)\right).
\end{aligned}$$

vi.

$$\begin{aligned}
(U(p)\otimes U_1(p))^{\lambda} =& \left(\bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in E(p),\left(\pounds_1^k,\left[p_1^k,q_1^k\right]\right)\in E_1(p)} \left\{\left(\pounds^k\otimes \pounds_1^k, \left[\frac{p^kp_1^k}{\sum_{k=1}^{\pounds} p^k \sum_{k=1}^{\pounds_1} p_1^k}, \frac{q^kq_1^k}{\sum_{k=1}^{\pounds} q^k \sum_{k=1}^{\pounds_1} q_1^k}\right]\right)\right\}\right)^{\lambda}\\
=& \bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in E(p),\left(\pounds_1^k,\left[p_1^k,q_1^k\right]\right)\in E_1(p)} \left\{\left(\left(\pounds^k\otimes \pounds_1^k\right)^{\lambda}, \left[\frac{p^kp_1^k}{\sum_{k=1}^{\pounds} p^k \sum_{k=1}^{\pounds_1} p_1^k}, \frac{q^kq_1^k}{\sum_{k=1}^{\pounds} q^k \sum_{k=1}^{\pounds_1} q_1^k}\right]\right)\right\}\\
=& \bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in E(p),\left(\pounds_1^k,\left[p_1^k,q_1^k\right]\right)\in E_1(p)} \left\{\left((\pounds^k)^{\lambda}\otimes (\pounds_1^k)^{\lambda}, \left[\frac{p^kp_1^k}{\sum_{k=1}^{\pounds} p^k \sum_{k=1}^{\pounds_1} p_1^k}, \frac{q^kq_1^k}{\sum_{k=1}^{\pounds} q^k \sum_{k=1}^{\pounds_1} q_1^k}\right]\right)\right\}\\
=& (E(p))^{\lambda}\otimes (E_1(p))^{\lambda}.
\end{aligned}$$

□

6.1.2 The ranking of the UPLTSs

The ranking result derived in Definition 2.2.4, may fail in certain situations. To clarify this drawback, consider the following example.

Example 6.1.5. *Let* $S = \{\pounds_{\alpha}|\alpha = 0, 1, ..., 6\}$ *be an LTS,* $U_1(p)$ *and* $U_2(p)$ *be two different UPLTSs based on* S*. Suppose that* $U_1(p) = \{(\pounds_3, [0.5, 0.6]), (\pounds_2, [0.4, 0.5])\}$ *and* $U_2(p) = \{(\pounds_0, [0.4, 0.5]), (\pounds_1, [0.3, 0.4]), (\pounds_2, [0.2, 0.3])\}$.

Then, by Eq. (2.2.3), we get for $U_1(P)$

$$\left[\overline{\alpha}, \overline{\beta}\right] = [2.55, 2.54],$$

which does not make sense because the upper bound of the interval is smaller than the lower bound. It is a shortcoming that will result in an inaccurate decision conclusion. What is more, sometimes we face the situation that two different UPLTSs have the same score values. At that time, we need another ranking parameter, deviation degree, which has not been defined for UPLTSs context yet.

To overcome the current flaws of the existing score function, it becomes very necessary to seek a new comparison method which can be capable of comparing UPLTSs more effectively. Inspired by the score function of PLTSs [37], in the following, we first revise the definition of score function and then define the deviation degree of UPLTSs.

Definition 6.1.6. *Let $S = \{£_\alpha; \alpha = 0, 1, ..., 2\tau\}$ be an LTS, ℓ be a linguistic scale function and $U(p) = \left\{\left(£^k, \left[p^k, q^k\right]\right) |k = 1, 2, ..., £\right\}$ be a UPLTS on S_2. Then, the score function of $U(p)$ is defined as*

$$S\left(U(p)\right) = \sum_{k=1}^{£} \left(\frac{p^k + q^k}{2}\right) \ell(£^k) / \sum_{k=1}^{£} q^k, \tag{6.1.1}$$

the deviation degree of $U(p)$ is

$$F\left(U(p)\right) = \left(\sum_{k=1}^{£} \left(\left(\frac{p^k + q^k}{2}\right) \left(\ell(£^k) - S\left(U(P)\right)\right)^2\right) / \sum_{k=1}^{£} q^k\right)^{1/2}. \tag{6.1.2}$$

In summary, the comparison rules for two arbitrary UPLTSs $U_l(p) (l = 1, 2)$ are shown in the following.

1. If $S\left(U_1(p)\right) > S\left(U_2(p)\right)$, then $U_1(p) > U_2(p)$;
2. If $S\left(U_1(p)\right) < S\left(U_2(p)\right)$, then $U_1(p) < U_2(p)$;
3. If $S\left(U_1(p)\right) = S\left(U_2(p)\right)$, then:

3.1. If $\digamma(U_1(p)) > \digamma(U_2(p))$, then $U_1(p) < U_2(p)$;

3.2. If $\digamma(U_1(p)) < \digamma(U_2(p))$, then $U_1(p) > U_2(p)$;

3.3. If $\digamma(U_1(p)) = \digamma(U_2(p))$, then $U_1(p) = U_2(p)$.

Example 6.1.7. *(Continued to Example 6.1.5) The score function of $U_1(p)$ and $U_1(p)$ are computed with Eq. (6.1.1) as $S(U_1(p)) = 0.7583$ and $S(U_2(p)) = 0.4236$. Thus, $S(U_1(p)) > S(U_2(p))$, and hence, $U_1(p) > U_2(p)$.*

6.2 The aggregation operators

Inspired by the aggregation operator developed by Krishankumar et al. [62], this section is dedicated to focusing on the improvement of some existing aggregation operators. Besides this, the development of a robust aggregation operator, UPLSWG is also a part of this section.

6.2.1 Improved aggregation operators

Here, we first improve some existing aggregation operators according to the novel operational law and then based on these operators, some impressive results are derived.

Definition 6.2.1. *Let $U_i(p) = \{(\pounds_i^k, [p_i^k, q_i^k]) \,|k = 1, 2, ..., \pounds_i\}$ $(i = 1, 2, ..., n)$ be n UPLTSs, where $\pounds_i^k$ and p_i^k are the kth linguistic term and its probability interval, respectively in $U_i(p)$. Then*

$$\begin{aligned} UPLA(U_1(p), U_2(p), ..., U_n(p)) &= \frac{1}{n}(U_1(p) \oplus U_2(p) \oplus ... \oplus U_n(p)) \\ &= \bigcup_{\substack{(\pounds_i^k, [p_i^k, q_i^k]) \in U_i(p) \\ (i=1,2,...,n)}} \left\{ \left(\frac{1}{n} \oplus_{i=1}^n \pounds_i^k, \left[\frac{\prod_{i=1}^n p_i^k}{\prod_{i=1}^n \left(\sum_{k=1}^{\pounds_i} p_i^k\right)}, \frac{\bigoplus_{i=1}^n q_i^k}{\prod_{i=1}^n \left(\sum_{k=1}^{\pounds_i} q_i^k\right)} \right] \right) \right\}, \end{aligned} \tag{6.2.1}$$

is called the uncertain probabilistic linguistic averaging (UPLA) operator.

Definition 6.2.2. *Let* $U_i(p) = \left\{ \left(\pounds_i^k, \left[p_i^k, q_i^k \right] \right) | k = 1, 2, ..., \pounds_i \right\} (i = 1, 2, ..., n)$ *be* n *UPLTSs, where* $\pounds_i^k$ *and* p_i^k *are the* k*th linguistic term and its probability interval, respectively in* $U_i(p)$*. Then*

$$UPLWA\left(U_1(p), U_2(p), ..., U_n(p)\right) = \frac{1}{n}\left(\omega_1 U_1(p) \oplus \omega_2 U_2(p) \oplus ... \oplus \omega_n U_n(p)\right)$$
$$= \bigcup_{\substack{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) \in U_i(p) \\ (i=1,2,...,n)}} \left\{ \left(\oplus_{i=1}^n \omega_i \pounds_i^k, \left[\frac{\prod_{i=1}^n p_i^k}{\prod_{i=1}^n \left(\sum_{k=1}^{\pounds_i} p_i^k \right)}, \frac{\oplus_{i=1}^n q_i^k}{\prod_{i=1}^n \left(\sum_{k=1}^{\pounds_i} q_i^k \right)} \right] \right) \right\}, \tag{6.2.2}$$

is called the uncertain probabilistic linguistic weighted averaging (UPLWA) operator, where $w = (\omega_1, \omega_2, ..., \omega_n)^T$ *is the weight vector of* $\pounds_i(p)(i = 1, 2, ..., n)$*, and* $\sum_{i=1}^n \omega_i = 1$*. Particularly, if* $w = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$*, then the UPLWA operator will be the same as the UPLA operator.*

Definition 6.2.3. *Let* $U_i(p) = \left\{ \left(\pounds_i^k, \left[p_i^k, q_i^k \right] \right) | k = 1, 2, ..., \pounds_i \right\} (i = 1, 2, ..., n)$ *be* n *UPLTSs, where* $\pounds_i^k$ *and* p_i^k *are the* k*th linguistic term and its probability interval, respectively in* $U_i(p)$*. Then*

$$UPLG\left(U_1(p), U_2(p), ..., U_n(p)\right) = \left(U_1(p) \otimes U_2(p) \otimes ... \otimes U_n(p)\right)^{\frac{1}{n}}$$
$$= \bigcup_{\substack{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) \in U_i(p) \\ (i=1,2,...,n)}} \left\{ \left(\left(\otimes_{i=1}^n \pounds_i^k\right)^{\frac{1}{n}}, \left[\frac{\prod_{i=1}^n p_i^k}{\prod_{i=1}^n \left(\sum_{k=1}^{\pounds_i} p_i^k \right)}, \frac{\oplus_{i=1}^n q_i^k}{\prod_{i=1}^n \left(\sum_{k=1}^{\pounds_i} q_i^k \right)} \right] \right) \right\}, \tag{6.2.3}$$

is called the uncertain probabilistic linguistic geometric (UPLG) operator.

Definition 6.2.4. *Let* $U_i(p) = \left\{ \left(\pounds_i^k, \left[p_i^k, q_i^k \right] \right) | k = 1, 2, ..., \pounds_i \right\} (i = 1, 2, ..., n)$ *be* n *UPLTSs, where* $\pounds_i^k$ *and* p_i^k *are the* k*th linguistic term and its probability interval, respectively in* $U_i(p)$*.*

Then

$$UPLWG\left(U_1(p), U_2(p), ..., U_n(p)\right) = \left(U_1(p)^{\omega_1} \otimes U_2(p)^{\omega_2} \otimes ... \otimes U_n(p)^{\omega_n}\right)^{\frac{1}{n}}$$

$$= \bigcup_{\substack{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) \in U_i(p) \\ (i=1,2,...,n)}} \left\{\left(\otimes_{i=1}^n \pounds_i^{k\omega_i}, \left[\frac{\prod_{i=1}^n p_i^k}{\prod_{i=1}^n \left(\sum_{k=1}^{\pounds_i} p_i^k\right)}, \frac{\oplus_{i=1}^n q_i^k}{\prod_{i=1}^n \left(\sum_{k=1}^{\pounds_i} q_i^k\right)}\right]\right)\right\}, \tag{6.2.4}$$

is called the uncertain probabilistic linguistic weighted geometric (UPLWG) operator, where $w = (\omega_1, \omega_2, ..., \omega_n)^T$ *is the weight vector of* $U_i(p) (i = 1, 2, ..., n)$*, and* $\sum_{i=1}^n \omega_i = 1$*. Especially, if* $w = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$*, then the UPLWG degenerate into the UPLG operator.*

Remark 6.2.5. *In the aforementioned definitions, we demand that* $\sum_{i=1}^n \omega_i = 1$*. It conforms our habits and makes the aggregation operators fit for implementation. But in practical decision-making process, sometimes, we may face the situation that* $\sum_{i=1}^n \omega_i < 1$ *which is also reasonable. This issue must be resolved, but, luckily, it is not a big issue. One should do a normalization to the weight vector, and then newly weight vector meets the property that* $\sum_{i=1}^n \omega_i = 1$.

Now, let us look at all sorts of excellent properties of the UPLWA operator.

Theorem 6.2.6. *If all UPLTSs* $U_i(p) (i = 1, 2, ..., n)$ *satisfy*

$$U_i(p) = U(p) = \bigcup_{\left(\pounds^k, \left[p^k, q^k\right]\right) \in U(P)} \left\{\left(\pounds^k, \left[p^k, q^k\right]\right) | k = 1, 2, ..., \pounds\right\}, \forall\, i,$$

then

$$UPLWA\left(U_1(p), U_2(p), ..., U_n(p)\right) = UPLWA\left(U(p), U(p), ..., U(p)\right) = U(p).$$

Proof. By definition of UPLWA operator, we have

$$UPLWA\left(U_1(p), U_2(p), ..., U_n(p)\right) = \bigcup_{\substack{\left(\pounds_i^k,\left[p_i^k,q_i^k\right]\right)\in U_i(p)\\(i=1,2,...,n)}} \left\{\left(\oplus_{i=1}^n \omega_i \pounds_i^k, \left[\frac{\prod_{i=1}^n p_i^k}{\prod_{i=1}^n\left(\sum_{k=1}^{\pounds_i} p_i^k\right)}, \frac{\oplus_{i=1}^n q_i^k}{\prod_{i=1}^n\left(\sum_{k=1}^{\pounds_i} q_i^k\right)}\right]\right)\right\}$$

$$= \bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in U(p)} \left\{\left(\oplus_{i=1}^n \omega_i \pounds^k, \left[\frac{\prod_{i=1}^n p^k}{\prod_{i=1}^n\left(\sum_{k=1}^{\pounds} p^k\right)}, \frac{\oplus_{i=1}^n q^k}{\prod_{i=1}^n\left(\sum_{k=1}^{\pounds} q^k\right)}\right]\right)\right\}$$

$$\bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in U(p)} \left\{\left(\sum_{i=1}^n \omega_i \pounds^k, \left[\frac{\prod_{i=1}^n p^k}{\prod_{i=1}^n\left(\sum_{k=1}^{\pounds} p^k\right)}, \frac{\oplus_{i=1}^n q^k}{\prod_{i=1}^n\left(\sum_{k=1}^{\pounds} q^k\right)}\right]\right)\right\}$$

$$= U(p).$$

□

Theorem 6.2.7. *Suppose* $U_i(p) = \bigcup_{\left(\pounds_i^k,\left[p_i^k,q_i^k\right]\right)\in U_i(p)} \left\{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) | k = 1, 2, ..., \pounds_i\right\}$ *be a collection of UPLTSs. If* $U(p) = \bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in U(P)} \left\{\left(\pounds^k, \left[p^k, q^k\right]\right) | k = 1, 2, ..., \pounds\right\}$ *is a UPLTS, then*

$$UPLWA\left(U_1(p) \oplus U(p), U_2(p) \oplus U(p), ..., U_n(p) \oplus U(p)\right) = UPLWA\left(U_1(p), U_2(p), ..., U_n(p)\right) \oplus U(p).$$

Proof. Since for any i

$$U_i(p) \oplus U(p) = \bigcup_{\substack{\left(\pounds_i^k,\left[p_i^k,q_i^k\right]\right)\in U_i(p),\\ \left(\pounds^k,\left[p^k,q^k\right]\right)\in U(p)}} \left(\left\{\left(\omega_i(\pounds_i^k \oplus \pounds^k), \left[\frac{p_i^k p^k}{\sum_{k=1}^{\pounds_i} p_i^k \sum_{k=1}^{\pounds} p^k}, \frac{q_i^k q^k}{\sum_{k=1}^{\pounds_i} q_i^k \sum_{k=1}^{\pounds} q^k}\right]\right)\right\}\right)$$

$$UPLWA\left(U_1(p) \oplus U(p), U_2(p) \oplus U(p), ..., U_n(p) \oplus U(p)\right)$$

$$= \bigcup_{\substack{\left(\pounds_i^k,\left[p_i^k,q_i^k\right]\right)\in U_i(p),\\ \left(\pounds^k,\left[p^k,q^k\right]\right)\in U(P)}} \left(\left\{\left(\oplus_{i=1}^n \omega_i(\pounds_i^k \oplus \pounds^k), \left[\frac{\prod_{i=1}^n p_i^k p^k}{\prod_{i=1}^n \sum_{k=1}^{\pounds_i} p_i^k \sum_{k=1}^{\pounds} p^k}, \frac{\oplus_{i=1}^n q_i^k q^k}{\prod_{i=1}^n \sum_{k=1}^{\pounds_i} q_i^k \sum_{k=1}^{\pounds} q^k}\right]\right)\right\}\right)$$

$$= \bigcup_{\left(\pounds_i^k,\left[p_i^k,q_i^k\right]\right)\in U_i(p)} \left(\left\{\left(\oplus_{i=1}^n \omega_i \pounds_i^k, \left[\frac{\prod_{i=1}^n p_i^k}{\prod_{i=1}^n \sum_{k=1}^{\pounds_i} p_i^k}, \frac{\oplus_{i=1}^n q_i^k}{\prod_{i=1}^n \sum_{k=1}^{\pounds_i} q_i^k}\right]\right)\right\}\right) \oplus \bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in U(P)} \left\{\left(\pounds^k, \left[p^k, q^k\right]\right) | k = 1, 2, ..., \pounds\right\}$$

$$= UPLWA\left(U_1(p), U_2(p), ..., U_n(p)\right) \oplus U(p).$$

□

Theorem 6.2.8. *Let* $U_i(p) = \bigcup_{\left(\pounds_i^k,\left[p_i^k,q_i^k\right]\right)\in U_i(p)} \left\{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) | k = 1, 2, ..., \pounds_i\right\} (i = 1, 2, ..., n)$ *be a collection of* n *UPLTSs. If* $d > 0$*, then*

$$UPLWA\left(dU_1(p), dU_2(p), ..., dU_n(p)\right) = dUPLWA\left(U_1(p), U_2(p), ..., U_n(p)\right).$$

Proof. According to Definition 6.1.2, we have

$$dU_i(p) = \bigcup_{\left(\pounds_i^k,\left[p_i^k,q_i^k\right]\right)\in U_i(p)} \left\{\left(d\pounds_i^k, \left[p_i^k, q_i^k\right]\right) | k = 1, 2, ..., \pounds_i\right\}.$$

By definition of UPLWA operator, we have

$$\begin{aligned} & UPLWA\left(dU_1(p), dU_2(p), ..., dU_n(p)\right) \\ &= \bigcup_{\left(\pounds_i^k,\left[p_i^k,q_i^k\right]\right)\in U_i(p)} \left(\left\{\left(\oplus_{i=1}^n \omega_i d\pounds_i^k, \left[\frac{\prod_{i=1}^n p_i^k}{\prod_{i=1}^n \sum_{k=1}^{\pounds_i} p_i^k}, \frac{\oplus_{i=1}^n q_i^k}{\prod_{i=1}^n \sum_{k=1}^{\pounds_i} q_i^k}\right]\right)\right\}\right) \\ &= d \bigcup_{\left(\pounds_i^k,\left[p_i^k,q_i^k\right]\right)\in U_i(p)} \left(\left\{\left(\oplus_{i=1}^n \omega_i \pounds_i^k, \left[\frac{\prod_{i=1}^n p_i^k}{\prod_{i=1}^n \sum_{k=1}^{\pounds_i} p_i^k}, \frac{\oplus_{i=1}^n q_i^k}{\prod_{i=1}^n \sum_{k=1}^{\pounds_i} q_i^k}\right]\right)\right\}\right) \\ &= dUPLWA\left(U_1(p), U_2(p), ..., U_n(p)\right). \end{aligned}$$

Thus completing the proof of Theorem. □

According to Theorem 6.2.7 and 6.2.8, we can get Theorem 6.2.9 easily.

Theorem 6.2.9. *Suppose* $U_i(p) = \bigcup_{\left(\pounds_i^k,\left[p_i^k,q_i^k\right]\right)\in U_i(p)} \left\{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) | k = 1, 2, ..., \pounds_i\right\}$ *be a collection of UPLTSs. If* $U(p) = \bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in U(P)} \left\{\left(\pounds^k, \left[p^k, q^k\right]\right) | k = 1, 2, ..., \pounds\right\}$ *is a UPLTS, then*

$$UPLWA\left(dU_1(p) \oplus U(p), dU_2(p) \oplus U(p), ..., dU_n(p) \oplus U(p)\right) = dUPLWA\left(U_1(p), U_2(p), ..., U_n(p)\right) \oplus U(p).$$

Theorem 6.2.10. *Let* $U_i(p) = \left\{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) | k = 1, 2, ..., \pounds_i\right\} (i = 1, 2, ..., n)$ *and* $U_i^{'}(p) = \left\{\left(\pounds_i^{'k}, \left[p_i^{'k}, q_i^{'k}\right]\right) | k = 1, 2, ..., \pounds_i\right\} (i = 1, 2, ..., n)$ *be two collections of UPLTSs, then*

$$UPLWA\left(U_1(p) \oplus U_1^{'}(p), U_2(p) \oplus U_2^{'}(p), ..., U_n(p) \oplus U_n^{'}(p)\right) = UPLWA\left(U_1(p), U_2(p), ..., U_n(p)\right) \oplus UPLWA\left(U_1^{'}(p), U_2^{'}(p), ..., U_n^{'}(p)\right).$$

Proof. According to Definition 6.1.2, we have

$$U_i(p) \oplus U_i^{'}(p) = \bigcup_{\substack{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) \in U_i(p), \\ \left(\pounds_i^{'k}, \left[p_i^{'k}, q_i^{'k}\right]\right) \in U_i^{'}(P)}} \left\{\left(\pounds_i^k \oplus \pounds_i^{'k}, \left[\frac{p_i^k p_i^{'k}}{\sum_{k=1}^{\pounds_i} p_i^k \sum_{k=1}^{\pounds_i} p_i^{'}}, \frac{q_i^k q_i^{'k}}{\sum_{k=1}^{\pounds_i} q_i^k \sum_{k=1}^{\pounds_i} q_i^{'k}}\right]\right)\right\}.$$

By definition of UPLWA operator, we have

$$\begin{aligned} UPLWA\left(U_1(p) \oplus U_1^{'}(p), U_2(p) \oplus U_2^{'}(p), ..., U_n(p) \oplus U_n^{'}(p)\right) &= \bigcup_{\substack{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) \in U_i(p), \\ \left(\pounds_i^{'k}, \left[p_i^{'k}, q_i^{'k}\right]\right) \in U_i^{'}(p)}} \left(\left\{\left(\oplus_{i=1}^n \omega_i \left(\pounds_i^k \oplus \pounds_i^{'k}\right), \left[\frac{\prod_{i=1}^n \left(p_i^k p_i^{'k}\right)}{\prod_{i=1}^n \left(\sum_{k=1}^{\pounds_i} p_i^k \sum_{k=1}^{\pounds_i^{'}} p_i^{'k}\right)}, \frac{\oplus_{i=1}^n \left(q_i^k q_i^{'k}\right)}{\prod_{i=1}^n \left(\sum_{k=1}^{\pounds_i} q_i^k \sum_{k=1}^{\pounds_i^{'}} q_i^{'k}\right)}\right]\right)\right\}\right) \\ &= \bigcup_{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) \in U_i(p)} \left(\left\{\left(\oplus_{i=1}^n \omega_i \pounds_i^k, \left[\frac{\prod_{i=1}^n \left(p_i^k\right)}{\prod_{i=1}^n \left(\sum_{k=1}^{\pounds_i} p_i^k\right)}, \frac{\oplus_{i=1}^n \left(q_i^k\right)}{\prod_{i=1}^n \left(\sum_{k=1}^{\pounds_i} q_i^k\right)}\right]\right)\right\}\right) \\ &\oplus \bigcup_{\left(\pounds_i^{'k}, \left[p_i^{'k}, q_i^{'k}\right]\right) \in U_i^{'}(p)} \left(\left\{\left(\oplus_{i=1}^n \omega_i \pounds_i^{'k}, \left[\frac{\prod_{i=1}^n \left(p_i^{'k}\right)}{\prod_{i=1}^n \left(\sum_{k=1}^{\pounds_i^{'}} p_i^{'k}\right)}, \frac{\oplus_{i=1}^n \left(q_i^k q_i^{'k}\right)}{\prod_{i=1}^n \left(\sum_{k=1}^{\pounds_i^{'}} q_i^{'k}\right)}\right]\right)\right\}\right) \\ &= UPLWA\left(U_1(p), U_2(p), ..., U_n(p)\right) \oplus UPLWA\left(U_1^{'}(p), U_2^{'}(p), ..., U_n^{'}(p)\right). \end{aligned}$$

□

Theorem 6.2.11. *Let* $U_i^{\star}(p)$ *be any permutation of* $U_i(p) (i = 1, 2, ..., n)$*, then*

$$UPLWA\left(U_1(p), U_2(p), ..., U_n(p)\right) = UPLWA\left(U_1^{\star}(p), U_2^{\star}(p), ..., U_n^{\star}(p)\right).$$

Proof. Because $(U_1^{\star}(p), U_2^{\star}(p), ..., U_n^{\star}(p))$ is any permutation of $(U_1(p), U_2(p), ..., U_n(p))$, by definition of UPLWA and the operational law (1) of Definition 6.1.2, we can conclude that

$$UPLWA\left(U_1(p), U_2(p), ..., U_n(p)\right) = UPLWA\left(U_1^{\star}(p), U_2^{\star}(p), ..., U_n^{\star}(p)\right).$$

Therefore, we complete the proof of Theorem 6.2.11. □

Theorem 6.2.12. *Let* $U_i(p) = \left\{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) | k = 1, 2, ..., \pounds_i\right\}$ *be UPLTSs, then the fused value of UPLTSs, computed by applying UPLWA operator, is still UPLTS.*

Proof. According to the definition of UPLWA operator, we have

$$UPLWA\left(U_1(p), U_2(p), ..., U_n(p)\right) = \bigcup_{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) \in U_i(p)} \left(\left\{\left(\oplus_{i=1}^n \omega_i \pounds_i^k, \left[\frac{\prod_{i=1}^n p_i^k}{\prod_{i=1}^n \sum_{k=1}^{\pounds_i} p_i^k}, \frac{\oplus_{i=1}^n q_i^k}{\prod_{i=1}^n \sum_{k=1}^{\pounds_i} q_i^k}\right]\right)\right\}\right).$$

According to the operational laws of (Xu, 2004), $\oplus_{i=1}^n \omega_i \pounds_i^k$ is an LTS, further from definition of UPLTS, we have $\sum_{k=1}^{\pounds_i} p_i^k \leq 1, \sum_{k=1}^{\pounds_i} q_i^k \geq 1 \; \forall \; i$, from this we can set forth $\frac{\prod_{i=1}^n p_i^k}{\prod_{i=1}^n \sum_{k=1}^{\pounds_i} p_i^k} \leq 1$ $\frac{\oplus_{i=1}^n q_i^k}{\prod_{i=1}^n \sum_{k=1}^{\pounds_i} q_i^k} \geq 1.$ □

Based on Definition 6.1.2, some relevant theorems can be obtained and described as follows:

Theorem 6.2.13. *Suppose* $U_i(p) = \bigcup_{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) \in U_i(p)} \left\{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) | k = 1, 2, ..., \pounds_i\right\}$ *be a collection of UPLTSs. If* $U(p) = \bigcup_{\left(\pounds^k, \left[p^k, q^k\right]\right) \in U(p)} \left\{\left(\pounds^k, \left[p^k, q^k\right]\right) | k = 1, 2, ..., \pounds\right\}$ *is a UPLTS, then*

$$UPLWG\left(U_1(p) \otimes U(p), U_2(p) \otimes U(p), ..., U_n(p) \otimes U(p)\right) = UPLWG\left(U_1(p), U_2(p), ..., U_n(p)\right) \otimes U(p).$$

Proof. The proof of Theorem 6.2.13, is similar to that of Theorem 6.2.7. □

Theorem 6.2.14. *If all UPLTSs* $U_i(p)(i = 1, 2, ..., n)$ *satisfy*

$$U_i(p) = U(p) = \bigcup_{\left(\pounds^k, \left[p^k, q^k\right]\right) \in U(P)} \left\{\left(\pounds^k, \left[p^k, q^k\right]\right) | k = 1, 2, ..., \pounds\right\}, \forall \; i,$$

then

$$UPLWG\left(U_1(p),U_2(p),...,U_n(p)\right)=UPLWG\left(U(p),U(p),...,U(p)\right)=U(p).$$

Theorem 6.2.15. *Let* $U_i(p)=\bigcup_{\left(\pounds_i^k,\left[p_i^k,q_i^k\right]\right)\in U_i(p)}\left\{\left(\pounds_i^k,\left[p_i^k,q_i^k\right]\right)|k=1,2,...,\pounds_i\right\}$ $(i=1,2,...,n)$ *be a collection of* n *UPLTSs. If* $d>0$, *then*

$$UPLWG\left(dU_1(p),dU_2(p),...,dU_n(p)\right)=\left(UPLWG\left(U_1(p),U_2(p),...,U_n(p)\right)\right)^d.$$

Proof. The proof of Theorem 6.2.15 is similar to that of Theorem 6.2.8. □

Using Theorems 6.2.13 and 6.2.15, we can get Theorem 6.2.16 easily.

Theorem 6.2.16. *Suppose* $U_i(p)=\bigcup_{\left(\pounds_i^k,\left[p_i^k,q_i^k\right]\right)\in U_i(p)}\left\{\left(\pounds_i^k,\left[p_i^k,q_i^k\right]\right)|k=1,2,...,\pounds_i\right\}$ *be a collection of UPLTSs. If* $U(p)=\bigcup_{\left(\pounds^k,\left[p^k,q^k\right]\right)\in U(p)}\left\{\left(\pounds^k,\left[p^k,q^k\right]\right)|k=1,2,...,\pounds\right\}$ *is a UPLTS, then*

$$UPLWA\left((U_1(p))^d\otimes U(p),(U_2(p))^d\otimes U(p),...,(U_n(p))^d\otimes U(p)\right)=\left(UPLWA\left(U_1(p),U_2(p),...\right.\right.$$
$$\left.\left.,U_n(p)\right)\right)^d\otimes U(p).$$

The proof of Theorem 6.2.17 is similar to that of Theorem 6.2.13, and thus it is not provided here.

Theorem 6.2.17. *Let* $U_i(p)=\left\{\left(\pounds_i^k,\left[p_i^k,q_i^k\right]\right)|k=1,2,...,\pounds_i\right\}$ $(i=1,2,...,n)$ *and* $U_i^{'}(p)=\left\{\left(\pounds_i^{'k},\left[p_i^{'k},q_i^{'k}\right]\right)|k=1,2,...,\pounds_i\right\}$ $(i=1,2,...,n)$ *be two collections of UPLTSs, then*

$$UPLWG\left(U_1(p)\otimes U_1^{'}(p),U_2(p)\otimes U_2^{'}(p),...,U_n(p)\otimes U_n^{'}(p)\right)=UPLWG\left(U_1(p),U_2(p),...,U_n(p)\right)$$
$$\otimes\, UPLWG\left(U_1^{'}(p),U_2^{'}(p),...,U_n^{'}(p)\right).$$

Theorem 6.2.18. *Let $U_i^\star(p)$ be any permutation of $U_i(p)(i = 1, 2, ..., n)$, then*

$$UPLWA\left(U_1(p), U_2(p), ..., U_n(p)\right) = UPLWA\left(U_1^\star(p), U_2^\star(p), ..., U_n^\star(p)\right).$$

Proof. The proof is similar to Theorem 6.2.11, which is discussed already. □

Theorem 6.2.19. *Let $U_i(p) = \left\{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) | k = 1, 2, ..., \pounds_i\right\}$ be UPLTSs, then the fused value of UPLTSs, computed by applying UPLWG operator, is still UPLTS.*

Proof. Similarly, as proof of Theorem 6.2.12, it is omitted here. □

6.2.2 The UPLSWG operator

This part concentrates on the construction of UPLSWG operator, which makes full use of all the input arguments and considers DMs weight among multi-input arguments.

Definition 6.2.20. *Let $U_i(p) = \bigcup_{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) \in U_i(p)} \left\{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) | k = 1, 2, ..., \pounds_i\right\}$ be a collection of UPLTSs, then the aggregation of UPLTS information is a mapping $U^n \longrightarrow U$ such that,*

$$UPLSWG_\pi\left(U_1(p), U_2(p), ..., U_n(p)\right) = U^*(p) = \bigcup_{\left(\pounds^{*k}, \left[p^{*k}, q^{*k}\right]\right) \in U^*(P)} \left\{\left(\pounds^{*k}, \left[p^{*k}, q^{*k}\right]\right) | k = 1, 2, ..., \pounds^*\right\}, \tag{6.2.5}$$

here

$$\pounds^{*k} = \begin{cases} Scheme1, & \text{if all the linguistic terms are unique} \\ Scheme2, & \text{otherwise.} \end{cases} \tag{6.2.6}$$

$$\left[p^{*k}, q^{*k}\right] = \left[\prod_{l=1}^{m} p_i^{k^{\pi_l}}, \prod_{l=1}^{m} q_i^{k^{\pi_l}}\right], \tag{6.2.7}$$

where π_l is the relative importance of lth DM with $0 \leq \pi_l \leq 1$ and $\sum_{l=1}^{m} \pi_l = 1$, m is the total number of DMs.

Scheme1= The average of the subscripts is computed and rounding off principal is applied to get a non-artificial linguistic term.

Scheme2= According to Eq. (6.2.7), the interval probability of each linguistic term is computed, and the term with maximum interval probability is selected as the aggregated value.

Following are a few properties of UPLSWG operator that immediately follow from its definition.

Property 1. (Idempotency) If all UPLTSs $U_i(p)(i = 1, 2, ..., n)$ satisfy

$$U_i(p) = U(p) = \bigcup_{\left(\pounds^k, \left[p^k, q^k\right]\right) \in U(P)} \left\{ \left(\pounds^k, \left[p^k, q^k\right]\right) | k = 1, 2, ..., \pounds \right\}, \forall\, i,$$

then

$$UPLSWG_\pi \left(U_1(p), U_2(p), ..., U_n(p)\right) = UPLSWG_\pi \left(U(p), U(p), ..., U(p)\right) = U(p).$$

Property 2. (Boundedness) Let $U_i(p) = \bigcup_{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) \in U_i(p)} \left\{ \left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) | k = 1, 2, ..., \pounds_i \right\}$ be a set of UPLTSs, then

$$U(p)^- \leq UPLSWG_\pi \left(U_1(p), U_2(p), ..., U_n(p)\right) \leq U(p)^+$$

where $U(p)^- = \min\left(U_i(p)\right)$ and $U(p)^+ = \max\left(U_i(p)\right)\ \forall\, i = 1, 2, ..., n$.

Property 3. (Monotonicity) Let $U_i(p) = \bigcup_{\left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) \in U_i(p)} \left\{ \left(\pounds_i^k, \left[p_i^k, q_i^k\right]\right) | k = 1, 2, ..., \pounds_i \right\}$ and $U_i'(p) = \bigcup_{\left({\pounds'}_i^k, \left[{p'}_i^k, {q'}_i^k\right]\right) \in {U'}_i(p)} \left\{ \left({\pounds'}_i^k, \left[{p'}_i^k, {q'}_i^k\right]\right) | k = 1, 2, ..., {\pounds'}_i \right\}$ be two sets of UPLTSs, if $\pounds_i^k \leq {\pounds'}_i^k$ and $\left[p_i^k, q_i^k\right] \leq \left[{p'}_i^k, {q'}_i^k\right]\ \forall\, i = 1, 2, ..., n$, then

$$UPLSWG_\pi \left(U_1(p), U_2(p), ..., U_n(p)\right) \leq UPLSWG_\pi \left(U_1'(p), U_2'(p), ..., U_n'(p)\right).$$

Property 4. (Commutativity) Let $U_i^{\star}(p)$ be any permutation of $U_i(p)(i = 1, 2, ..., n)$, then

$$UPLSWG_{\pi}\left(U_1(p), U_2(p), ..., U_n(p)\right) = UPLSWG_{\pi}\left(U_1^{\star}(p), U_2^{\star}(p), ..., U_n^{\star}(p)\right).$$

Theorem 6.2.21. *The aggregated value of UPLTSs by using UPLSWG operator is also an UPLTS.*

Proof. From the formulation presented in Eq. (6.2.6), it is evident that the aggregation of linguistic terms also form a linguistic term which is within the defined LTS. Next using Lemma 2.1.24, we have the following inequalities $\forall\ i = 1, 2, ..., n$

$$0 \leq \prod_{l=1}^{m} \left(q_i^k\right)^{\pi_l} \leq \sum_{l=1}^{m} \pi_l \left(q_i^k\right) \leq \sum_{l=1}^{m} \pi_l = 1. \tag{6.2.8}$$

Also,

$$0 \leq \prod_{l=1}^{m} \left(1 - p_i^k\right)^{\pi_l} \leq \sum_{l=1}^{m} \pi_l \left(1 - p_i^k\right) \leq \sum_{l=1}^{m} \pi_l = 1$$

$$1 \geq 1 - \prod_{l=1}^{m} \left(1 - p_i^k\right)^{\pi_l} \geq 1 - \sum_{l=1}^{m} \pi_l \left(1 - p_i^k\right) \geq 0. \tag{6.2.9}$$

From these two inequalities, it is clear that both the bounds of interval probability are within the range of $[0, 1]$. Further, $\sum_{k=1}^{\pounds_i} p_i^k \leq 1$, $\sum_{k=1}^{\pounds_i} q_i^k \geq 1$ from this we can write $\sum_{k=1}^{\pounds_i^*} 1 - \prod_{l=1}^{m} \left((1 - p_i)^k\right) \leq 1$, $\sum_{k=1}^{\pounds_i^*} \prod_{l=1}^{m} \left((q_i)^k\right) \geq 1$ (being weighted geometric means).

Thus, we can say that the aggregated linguistic term and its associated interval probability follow the constraints given in Definition 2.2.1. □

An example can be set up for clarification.

Example 6.2.22. *Let $S = \{\pounds_{\alpha} | \alpha = 0, 1, ..., 6\}$ be an LTS. Suppose that the comments of two DMs with their weight values $(0.6, 0.4)$, for a particular scenario are $U_1(p) = \{(\pounds_5, [0.5, 0.6]),$*

$(\pounds_4, [0.4, 0.5])\}$ *and* $U_2(p) = \{(\pounds_1, [0.4, 0.5]), (\pounds_0, [0.5, 0.6])\}$*, respectively. Now, if we employed the UPLSWG operator on the provided information, then the aggregated result is given by* $\{(\pounds_3, [0.4573, 0.5578]), (\pounds_2, [0.4373, 0.5378])\}$*. But if we take the rating* $U_1(p)$ *as:* $U_1(p) = \{(\pounds_5, [0.5, 0.6]), (\pounds_6, [0.4, 0.5])\}$*, then the aggregated result generated by UPLSWG operator is* $\{(\pounds_3, [0.4573, 0.5578]), (\pounds_3, [0.4373, 0.5378])\} = \{(\pounds_3, [0.4573, 0.5578])\}$.

Some major advantages of the proposed UPLSWG operator are outlined below.

i. It can be noticed from the formulation given in Eq. (6.2.6) that the aggregation of UPLTSs utilizing UPLSWG operator produces no artificial term. Which confirms that the aggregation process of UPLTSs is stable and acceptable.

ii. Unlike UPLWA and UPLWG operators, the aggregated linguistic terms do not exceed the bounds of LTS, which can be realized from Example 6.2.22.

iii. The relative importance of each DM is considered in the aggregation process of associated interval probabilities of linguistic terms. Therefore, we obtain better results, especially in a situation where DMs are of different rank.

6.3 MCGDM with uncertain probabilistic linguistic information

In the following, we study a decision-making methodology in which the information related to the weight vector of criteria is entirely unknown, and the assessment information takes the form of UPLTSs.

6.3.1 Entropy method to determine the criteria weights

Deriving criteria weight is an important step that arises in most MCGDM models. In this section, efforts are made to generalize the entropy method to UPLTS context. To achieve this task, the scheme of the proposal is presented below.

Firstly, it is assumed that there is a set of m feasible alternatives $z_i(i = 1, 2, ..., m)$ and n criteria $c_j(j = 1, 2, ..., n)$.

Step 1: The uncertain probabilistic linguistic decision matrix U which shows the performance of different alternatives with respect to various criteria is constructed.

$$U = [u_{ij}] = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mn} \end{pmatrix} (i = 1, 2, ..., m,\ j = 1, 2, ..., n), \tag{6.3.1}$$

here, each u_{ij} is an UPLTS representing the performance value of ith alternative with respect to jth criteria.

Step 2: Firstly, utilize one of the proposed aggregation operators to obtain the aggregated decision matrix U and then normalize all the entries of U by the following formula

$$f_{ij} = \frac{\sum_{k=1}^{u_{ij}} \left(\left(\frac{p_{ij}^k + q_{ij}^k}{2} \right) \ell(\pounds_{ij}^k) \right)}{\sum_{i=1}^{m} \sum_{k=1}^{u_{ij}} \left(\left(\frac{p_{ij}^k + q_{ij}^k}{2} \right) \ell(\pounds_{ij}^k) \right)},\ j = 1, 2, ..., n. \tag{6.3.2}$$

Step 3: The entropy values (e_j) are derived for each criteria by

$$e_j = \frac{-\sum_{i=1}^{m} f_{ij} \ln f_{ij}}{\ln m}\ ; i = 1, 2, ..., m,\ j = 1, 2, ..., n, \tag{6.3.3}$$

here $0 < e_j < 1$. If f_{ij} are all the same, then the entropy values of each criteria is the maximum $(e_j = 1)$. If $f_{ij} = 0$, then $f_{ij} \ln f_{ij} = 0$ [63].

Step 4: Entropy weights are calculated as

$$\omega_j = \frac{1 - e_j}{n - \sum_{j=1}^{n} e_j} \; ; \; \sum_{j=1}^{n} \omega_j = 1. \tag{6.3.4}$$

In the above equation, $1 - e_j$ represents the degree of divergence of each criterion's inherent information. The larger the value of the entropy, the smaller the entropy-based weight and vice-versa.

6.3.2 WASPAS method for MCGDM under UPLTS context

This section presents an extension of the classical WASPAS method to UPLTS setting for choosing an appropriate alternative from the range of available alternatives.

WASPAS methodology is based on a combination of two models WSM (weighted sum model) and WPM (weighted product model) [64]. The stated collection improves its capability of accuracy compared to other MCGDM techniques. It is one of the advance multi-index decision-making model which has been copped and employed in several domains [62, 65, 66].

Let $Z = \{z_1, z_2, ..., z_m\}$ be a set of m alternatives and $C = \{c_1, c_2, ..., c_n\}$ be a set of n criteria whose weight vector is unknown. Further, assume that there is a set of l DMs $D = \{d_1, d_2, ..., d_l\}$ whose weight vector is $\pi = \{\pi_1, \pi_2, ..., \pi_l\}$ where $\pi_k > 0$ and $\sum_{k=1}^{l} \pi_k = 1$ such that each DM has provided his assessment values in the form of UPLTSs $U_{ij}^k(P)$. Then, the proposed methodology has been summarized into the following steps to tackle MCGDM problems under UPLTS environment.

Step 1: Build the aggregated uncertain probabilistic linguistic decision matrix:

To aggregate all the individual decision matrices and create a single group decision matrix, utilize one of the proposed aggregation operator to obtain the aggregated decision matrix.

Step 2: Determine the criteria weights:

Compute the criteria weights according to the entropy approach shown in Section 6.3.1.

Step 3: Weighted sum measure of alternatives:

Calculate the weighted sum measure (WSM) of each alternative $z_i(i = 1, 2, ..., m)$ by using the following formula

$$WSM = Q_i^1 = \sum_{j=1}^{n} \omega_j \left(\sum_{k=1}^{u_{ij}} \left(\frac{p_{ij}^k + q_{ij}^k}{2} \right) + \ell \left(\pounds_{ij}^k \right) \right). \tag{6.3.5}$$

Step 4: Weighted product measure of alternatives:

Derive the weighted product measure (WPM) of each alternative $z_i(i = 1, 2, ..., n)$ by using the following formula

$$WPM = Q_i^2 = \prod_{j=1}^{n} \left(\sum_{k=1}^{u_{ij}} \left(\frac{p_{ij}^k + q_{ij}^k}{2} \right) \ell \left(\pounds_{ij}^k \right) \right)^{\omega_j}. \tag{6.3.6}$$

Step 5: Aggregated measure of alternatives:

Compute the aggregated measure of each alternative $z_i(i = 1, 2, ..., n)$, as follows:

$$Q_i = \lambda \left(\frac{Q_i^1 - Q_i^{1^-}}{Q_i^{1^+} - Q_i^{1^-}} \right) + (1 - \lambda) \left(\frac{Q_i^2 - Q_i^{2^-}}{Q_i^{2^+} - Q_i^{2^-}} \right), \tag{6.3.7}$$

where $Q_i^{1^-} = \min Q_i^1$, $Q_i^{1^+} = \max Q_i^1$, $Q_i^{2^-} = \min Q_i^2$, $Q_i^{2^+} = \max Q_i^2$ and λ is the aggregating coefficient of decision precision. It is developed to estimate the accuracy of WASPAS based on initial criteria exactness and $\lambda \in [0, 1]$(when $\lambda = 0$, and $\lambda = 1$, WASPAS is transformed to the WPM and the WSM, respectively). It has been proven that the accuracy of the aggregating methods is higher than the accuracy of single ones.

Step 6: Sort the alternatives:

For comparison, arrange the alternatives $z_i (i = 1, 2, ..., n)$ according to decreasing values of Q_i.

Step 7: End.

6.4 Implementation of WASPAS method

This section addresses a case study of selecting the best supplier from the bunch of suppliers to validate the feasibility and effectiveness of the developed model. In order to deal in the present competitive environment, a company must supply top quality products and services at a reasonable rate and shorter lead time. It is not practicable to score such goals except for the right input products from the right suppliers. On account of this, choosing appropriate suppliers becomes one of the primary decision-making problem faced by a company.

The supplier selection is complex and uncertain since it involves different selection criteria and several DMs with different prospectives. Therefore, many authors utilised MCGDM techniques [67, 68, 69] in supplier selection problem, but these studies are unable to model the situations in which DMs assessments take the form of UPLTSs. In this segment, we apply the constructed algorithm to a supplier selection problem in a food company by considering three supplier alternatives z_1, z_2 and z_3, which we want to select the best one among them using four criteria c_1=delivery, c_2=salary package, c_3=service and c_4=quality. Three DMs d_1, d_2 and d_3 whose weight vector is $\pi = \{0.3, 0.4, 0.3\}$ assess the alternatives with respect to the four independent criteria by using the linguistic term set $S = \{\pounds_0, \pounds_1, \pounds_2, \pounds_3, \pounds_4, \pounds_5, \pounds_6\} = \{\text{very low}, \text{low}, \text{medium}, \text{high}, \text{very high}, \text{perfect}\}$. The DMs assessments information (after the transformation from PLTSs to UPLTSs framework, utilizing the technique mentioned in Ref. [33] are listed in Table 6.1.

Table 6.1: The uncertain probabilistic linguistic decision matrix

DMs	Suppliers	Evaluation criteria			
		c_1	c_2	c_3	c_4
d_1	z_1	$\{(\pounds_2, [0.3, 0.5]), (\pounds_3, [0.5, 0.7])\}$	$\{(\pounds_3, [0.4, 0.6]), (\pounds_4, [0.4, 0.6])\}$	$\{(\pounds_3, 0.6), (\pounds_2, 0.4)\}$	$\{(\pounds_0, 0.4), (\pounds_1, 0.6)\}$
	z_2	$\{(\pounds_2, 0.4), (\pounds_3, 0.2), (\pounds_1, 0.4)\}$	$\{(\pounds_2, [0.5, 0.6]), (\pounds_3, [0.4, 0.5])\}$	$\{(\pounds_2, 0.4), (\pounds_0, 0.3), (\pounds_1, 0.3)\}$	$\{(\pounds_1, [0.2, 0.4]), (\pounds_2, [0.6, 0.8])\}$
	z_3	$\{(\pounds_0, 0.2), (\pounds_1, 0.2), (\pounds_2, 0.6)\}$	$\{(\pounds_3, 0.6), (\pounds_2, 0.2), (\pounds_1, 0.2)\}$	$\{(\pounds_2, [0.3, 0.5]), (\pounds_1, [0.5, 0.7])\}$	$\{(\pounds_3, 0.4), (\pounds_2, 0.3), (\pounds_1, 0.3)\}$
d_2	z_1	$\{(\pounds_3, 0.6), (\pounds_2, 0.4)\}$	$\{(\pounds_0, 0.5), (\pounds_1, 0.3), (\pounds_2, 0.2)\}$	$\{(\pounds_2, 0.6), (\pounds_1, 0.4)\}$	$\{(\pounds_4, [0.4, 0.5]), (\pounds_5, [0.5, 0.6])\}$
	z_2	$\{(\pounds_4, [0.3, 0.5]), (\pounds_3, [0.5, 0.7])\}$	$\{(\pounds_0, 0.7), (\pounds_1, 0.3)\}$	$\{(\pounds_2, 0.5), (\pounds_3, 0.5), \}$	$\{(\pounds_2, [0.6, 0.8]), (\pounds_1, [0.2, 0.4])\}$
	z_3	$\{(\pounds_4, 0.4), (\pounds_5, 0.3), (\pounds_3, 0.3)\}$	$\{(\pounds_1, [0.5, 0.6]), (\pounds_0, [0.4, 0.5])\}$	$\{(\pounds_2, 0.7), (\pounds_3, 0.3)\}$	$\{(\pounds_2, 0.6), (\pounds_1, 0.4)\}$
d_3	z_1	$\{(\pounds_0, [0.5, 0.7]), (\pounds_1, [0.3, 0.5])\}$	$\{(\pounds_3, 0.3), (\pounds_1, 0.5), (\pounds_2, 0.2)\}$	$\{(\pounds_1, 0.5), (\pounds_2, 0.5)\}$	$\{(\pounds_1, [0.4, 0.5]), (\pounds_0, [0.5, 0.6])\}$
	z_2	$\{(\pounds_3, [0.3, 0.4]), (\pounds_2, [0.6, 0.7]), (\pounds_2, 0)\}$	$\{(\pounds_5, 0.2), (\pounds_4, 0.2), (\pounds_3, 0.6)\}$	$\{(\pounds_2, 0.25), (\pounds_1, 0.75)\}$	$\{(\pounds_2, [0.2, 0.4]), (\pounds_1, [0.6, 0.8])\}$
	z_3	$\{(\pounds_3, 0.4), (\pounds_2, 0.3), (\pounds_1, 0.3)\}$	$\{(\pounds_1, 0.7), (\pounds_2, 0.3)\}$	$\{(\pounds_3, [0.5, 0.6]), (\pounds_2, [0.4, 0.5])\}$	$\{(\pounds_1, 0.5), (\pounds_0, 0.5)\}$

We first scaled and ordered the UPLTSs in the presented in Table 6.1, and then proceed to aggregation step.

The scaled and ordered UPLTSs are depicted in Table 6.2.

Table 6.2: The scaled and ordered uncertain probabilistic linguistic decision matrix

DMs	Suppliers	Evaluation criteria			
		c_1	c_2	c_3	c_4
d_1	z_1	$\{(\pounds_3, [0.5, 0.7]), (\pounds_2, [0.3, 0.5]), (\pounds_2, 0)\}$	$\{(\pounds_4, [0.4, 0.6]), (\pounds_3, [0.4, 0.6]), (\pounds_3, 0)\}$	$\{(\pounds_3, 0.6), (\pounds_2, 0.4), (\pounds_2, 0)\}$	$\{(\pounds_1, 0.6), (\pounds_0, 0.4), (\pounds_0, 0)\}$
	z_2	$\{(\pounds_2, 0.4), (\pounds_3, 0.2), (\pounds_1, 0.4)\}$	$\{(\pounds_3, [0.4, 0.5]), (\pounds_2, [0.5, 0.6]), (\pounds_2, 0)\}$	$\{(\pounds_2, 0.4), (\pounds_1, 0.3), (\pounds_0, 0.3)\}$	$\{(\pounds_2, [0.6, 0.8]), (\pounds_1, [0.2, 0.4]), (\pounds_1, 0)\}$
	z_3	$\{(\pounds_2, 0.6), (\pounds_1, 0.2), (\pounds_0, 0.2)\}$	$\{(\pounds_3, 0.6), (\pounds_2, 0.2), (\pounds_1, 0.2)\}$	$\{(\pounds_2, [0.3, 0.5]), (\pounds_1, [0.5, 0.7]), (\pounds_1, 0)\}$	$\{(\pounds_3, 0.4), (\pounds_2, 0.3), (\pounds_1, 0.3)\}$
d_2	z_1	$\{(\pounds_3, 0.6), (\pounds_2, 0.4), (\pounds_2, 0)\}$	$\{(\pounds_2, 0.2), (\pounds_1, 0.3), (\pounds_0, 0.5)\}$	$\{(\pounds_2, 0.6), (\pounds_1, 0.4), (\pounds_1, 0)\}$	$\{(\pounds_5, [0.5, 0.6]), (\pounds_4, [0.4, 0.5]), (\pounds_4, 0)\}$
	z_2	$\{(\pounds_3, [0.5, 0.7]), (\pounds_4, [0.3, 0.5]), (\pounds_3, 0)\}$	$\{(\pounds_1, 0.3), (\pounds_0, 0.7), (\pounds_0, 0)\}$	$\{(\pounds_3, 0.5), (\pounds_2, 0.5), (\pounds_2, 0)\}$	$\{(\pounds_2, [0.6, 0.8]), (\pounds_1, [0.2, 0.4], (\pounds_1, 0))\}$
	z_3	$\{(\pounds_4, 0.4), (\pounds_5, 0.3), (\pounds_3, 0.3)\}$	$\{(\pounds_1, [0.5, 0.6]), (\pounds_0, [0.4, 0.5]), (\pounds_0, 0)\}$	$\{(\pounds_2, 0.7), (\pounds_3, 0.3), (\pounds_2, 0)\}$	$\{(\pounds_2, 0.6), (\pounds_1, 0.4), (\pounds_1, 0)\}$
d_3	z_1	$\{(\pounds_1, [0.3, 0.5]), (\pounds_0, [0.5, 0.7]), (\pounds_0, 0)\}$	$\{(\pounds_3, 0.3), (\pounds_1, 0.5), (\pounds_2, 0.2)\}$	$\{(\pounds_2, 0.5), (\pounds_1, 0.5), (\pounds_1, 0)\}$	$\{(\pounds_1, [0.4, 0.5]), (\pounds_0, [0.5, 0.6], (\pounds_0, 0))\}$
	z_2	$\{(\pounds_2, [0.6, 0.7]), (\pounds_3, [0.3, 0.4]), (\pounds_2, 0)\}$	$\{(\pounds_3, 0.6), (\pounds_5, 0.2), (\pounds_4, 0.2)\}$	$\{(\pounds_1, 0.75), (\pounds_2, 0.25), (\pounds_1, 0)\}$	$\{(\pounds_1, [0.6, 0.8]), (\pounds_2, [0.2, 0.4]), (\pounds_1, 0)\}$
	z_3	$\{(\pounds_3, 0.4), (\pounds_2, 0.3), (\pounds_1, 0.3)\}$	$\{(\pounds_1, 0.7), (\pounds_2, 0.3), (\pounds_1, 0)\}$	$\{(\pounds_3, [0.5, 0.6]), (\pounds_2, [0.4, 0.5]), (\pounds_2, 0)\}$	$\{(\pounds_1, 0.5), (\pounds_0, 0.5), (\pounds_0, 0)\}$

The results obtained after aggregation are presented in Table 6.3.

Table 6.3: The uncertain probabilistic linguistic decision matrix after aggregation

	c_1	c_2	c_3	c_4
z_1	$\{(\pounds_2, [0.4614, 0.5949]), (\pounds_1, [0.3923, 0.5059])\}$	$\{(\pounds_3, [0.2780, .3140]), (\pounds_2, [0.3812, 0.4305])\}$	$\{(\pounds_2, 0.5680), (\pounds_1, 0.4277)\}$	$\{(\pounds_2, [0.4939, 0.5680]), (\pounds_1, [0.4277, 0.4939])\}$
z_2	$\{(\pounds_2, [0.4939, 0.5918]), (\pounds_3, [0.2656, 0.3552])\}$	$\{(\pounds_2, [0.4345, 0.4589])\}$	$\{(\pounds_2, 0.5281)\}$	$\{(\pounds_2, [0.6, 0.8]), (\pounds_1, [0.2, 0.4])\}$
z_3	$\{(\pounds_3, 0.4517), (\pounds_2, 0.2656), (\pounds_1, 0.2656)\}$	$\{(\pounds_2, [0.5842, 0.6284]), (\pounds_1, [0.2980, 0.3259])\}$	$\{(\pounds_2, [0.4908, 0.6042])\}$	$\{(\pounds_2, 0.5030), (\pounds_1, 0.3923)\}$

The entropy values are computed for each criteria and entropy weights are determined by means of Eq. (6.3.3) and Eq. (6.3.4).

Table 6.4: Entropy values and entropy weights (UPL-WASPAS)

	c_1	c_2	c_3	c_4
Entropy values (e_j)	0.9898	0.9690	0.9848	0.9967
Entropy weights (ω_j)	0.1709	0.5192	0.2546	0.0553

According to the Table 6.4, the c_2 (salary package) is the most important criteria with the highest entropy weight. c_1 (delivery), c_4 (quality) and c_3 (service) follow this criteria, respectively.

Table 6.5: Sensitivity analysis (UPL-WASPAS)

λ values	Suppliers	UPL-WASPAS score	Ranking order
0.1	z_1	1	$z_1 > z_3 > z_2$
	z_2	0	
	z_3	0.7253	
0.2	z_1	1	$z_1 > z_3 > z_2$
	z_2	0	
	z_3	0.7266	
0.3	z_1	1	$z_1 > z_3 > z_2$
	z_2	0	
	z_3	0.7279	
0.4	z_1	1	$z_1 > z_3 > z_2$
	z_2	0	
	z_3	0.7292	
0.5	z_1	1	$z_1 > z_3 > z_2$
	z_2	0	
	z_3	0.7305	
0.6	z_1	1	$z_1 > z_3 > z_2$
	z_2	0	
	z_3	0.7317	
0.7	z_1	1	$z_1 > z_3 > z_2$
	z_2	0	
	z_3	0.7330	
0.8	z_1	1	$z_1 > z_3 > z_2$
	z_2	0	
	z_3	0.7343	
0.9	z_1	1	$z_1 > z_3 > z_2$
	z_2	0	
	z_3	0.7356	

6.5 Comparative study and discussion

In this section, the developed method's outcomes are investigated based on a comparison and a sensitivity analysis. Besides, managerial implications are also documented in this section.

6.5.1 Ranking of alternatives applying classical approach

For comparison, in this section, we utilise the existing aggregation operator to carry out the aggregation process in the proposed method. By doing so, the aggregated assessment information takes the form of HFLTSs as a substitute for UPLTSs.

Here, we aggregate the provided decision matrix by utilizing the UPLA operator of Ref. [33].

Table 6.6: The uncertain probabilistic linguistic decision matrix after aggregation

	c_1	c_2	c_3	c_4
z_1	$\{\pounds_{1.3333}, \pounds_{0.5334}, \pounds_0\}$	$\{\pounds_{1.1}, \pounds_{0.7767}, \pounds_{0.1333}\}$	$\{\pounds_{1.3333}, \pounds_{0.5667}, \pounds_0\}$	$\{\pounds_{1.2667}, \pounds_{0.6}, \pounds_0\}$
z_2	$\{\pounds_{1.3}, \pounds_{1.0833}, \pounds_{0.1333}\}$	$\{\pounds_{1.15}, \pounds_{0.7}, \pounds_{0.2667}\}$	$\{\pounds_{1.0167}, \pounds_{1.4}, \pounds_0\}$	$\{\pounds_{2.1103}, \pounds_{1.2}, \pounds_0\}$
z_3	$\{\pounds_{1.3433}, \pounds_{0.7667}, \pounds_{0.6}\}$	$\{\pounds_{1.0166}, \pounds_{0.3333}, \pounds_{0.0667}\}$	$\{\pounds_{1.2834}, \pounds_{0.8}, \pounds_0\}$	$\{\pounds_{0.9667}, \pounds_{0.3333}, \pounds_{0.1}\}$

Table 6.6 depicts that the aggregated data is no more in the form of UPLTSs. It has lost the probability information and reduce to hesitant fuzzy linguistic context. Therefore, we employ the hesitant fuzzy linguistic (HFL)-WASPAS approach. The steps involved in the mechanism of this approach are almost similar to that of UPL-WASPAS method. Anyway, the Formulas (6.3.2), (6.3.5), and (6.3.6) reduce to

$$f_{ij} = \frac{\sum_{k=1}^{u_{ij}} \left(\ell(\pounds_{ij}^{k})\right)}{\sum_{i=1}^{m} \sum_{k=1}^{u_{ij}} \left(\ell(\pounds_{ij}^{k})\right)},\ j = 1, 2, ..., n, \tag{6.5.1}$$

$$WSM = Q_i^1 = \sum_{j=1}^{n} \omega_j \left(\sum_{k=1}^{u_{ij}} \ell \left(\pounds_{ij}^k \right) \right), \tag{6.5.2}$$

and

$$WPM = Q_i^2 = \prod_{j=1}^{n} \left(\sum_{k=1}^{u_{ij}} \ell \left(\pounds_{ij}^k \right) \right)^{\omega_j}, \tag{6.5.3}$$

respectively.

Table 6.7: Entropy values and entropy weights (HFL-WASPAS)

	c_1	c_2	c_3	c_4
Entropy values (e_j)	0.9890	0.9868	0.9955	0.9396
Entropy weights (ω_j)	0.1235	0.1481	0.0505	0.6779

The evaluation parameters for UPL-WASPAS and HFL-WASPAS are depicted in Table 6.8.

Table 6.8: Evaluation parameters for WASPAS method

Suppliers	UPL-WASPAS		HFL-WASPAS	
	WSM	WPM	WSM	WPM
z_1	1.5128	0.2699	0.3148	0.3149
z_2	0.9955	0.1851	0.4983	0.4809
z_3	1.3767	0.2465	0.2664	0.2587

Table 6.9: Sensitivity analysis (HFL-WASPAS)

λ values	Suppliers	HFL-WASPAS score	Ranking order
0.1	z_1	0.2485	$z_2 > z_1 > z_3$
	z_2	1	
	z_3	0	
0.2	z_1	0.2440	$z_2 > z_1 > z_3$
	z_2	1	
	z_3	0	
0.3	z_1	0.2396	$z_2 > z_1 > z_3$
	z_2	1	
	z_3	0	
0.4	z_1	0.2352	$z_2 > z_1 > z_3$
	z_2	1	
	z_3	0	
0.5	z_1	0.2308	$z_2 > z_1 > z_3$
	z_2	1	
	z_3	0	
0.6	z_1	0.2264	$z_2 > z_1 > z_3$
	z_2	1	
	z_3	0	
0.7	z_1	0.2219	$z_2 > z_1 > z_3$
	z_2	1	
	z_3	0	
0.8	z_1	0.2175	$z_2 > z_1 > z_3$
	z_2	1	
	z_3	0	
0.9	z_1	0.2131	$z_2 > z_1 > z_3$
	z_2	1	
	z_3	0	

To present a more realistic view of the comparison results, we sketch the ranking results yielded by UPL-WASPAS and HFL-WASPAS methods into Fig. 6.1. From Fig. 6.1, it is clear that the ranking order of suppliers obtained by these two approaches is quite different. In the following, we make explanations and analysis for this difference.

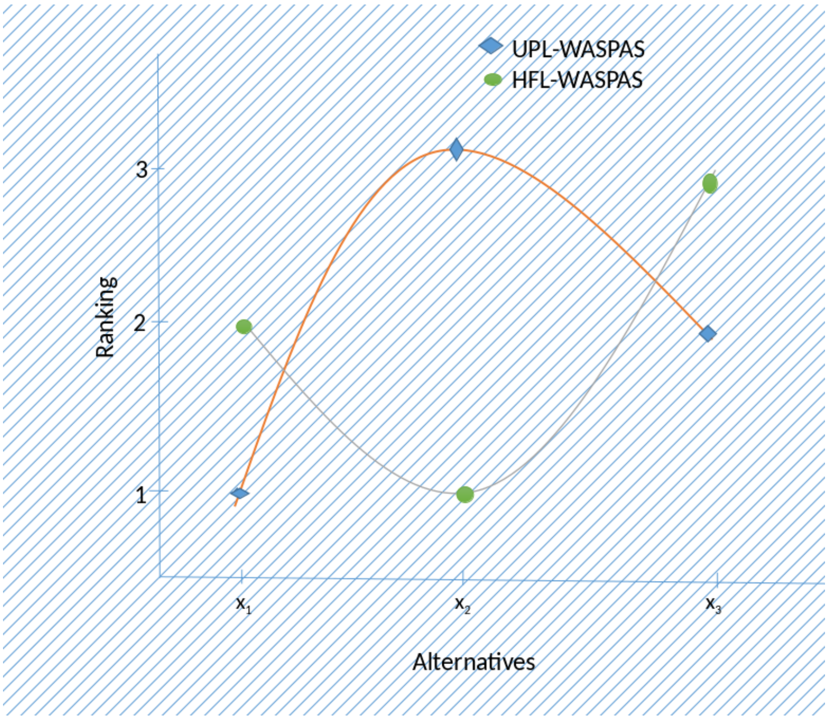

Figure 6.1: The representation of the UPL-WASPAS and HFL-WASPAS methods ranking

Firstly, the aggregation operator utilised in HFL-WASPAS method is based on existing operational law. By dint of the defects of that operational law, as discussed in Example 6.1.1, the ranking result obtained by this model is not acceptable. Secondly, the linguistic terms generated after aggregation based on UPLA operator are not the possible linguistic terms of the provided linguistic term set and may cross the boundary of a linguistic term set in some cases. It is not reliable. Our proposed operator makes sure that the aggregated linguistic terms are the possible linguistic terms in UPLTS within the boundary of the linguistic term set. Thirdly, from Tables 6.4 and 6.7 it is evident that the weight vector obtained by the proposed aggregation based method and the existing aggregation operator based method is not the same. According to the weight vector of Table 6.4, the criteria c_2 is the most important criteria while the weight vector obtained in Table 6.7 insist upon criteria c_4. The difference in evaluation is due to the information loss that occurs in the existing aggregation operator based method. Actually, the Eq. (6.5.1), and used in the mechanism of this approach is based on HFLTSs which have lost their probability information, which is an obvious limitation. Therefore, the weight vector derived by this method is not acceptable. It also causes a difference in the ranking results obtained by the two methods. Last but not least, the reason behind the difference of ranking result is that the proposed operator utilised in UPL-WASPAS method considers the DMs weight information during the aggregation process. In contrast, the existing aggregation operators pay no attention to the DMs weight, which may result in an irrational ranking of alternatives.

One of the measures that can be used to evaluate various approaches in the field of MCGDM is the ability to differentiate between alternatives. One way to determine this measure is by using the coefficient of variation. The coefficient of variation is formulated by dividing the standard deviation by average. After computing this index for the developed method, it is proved that this method has the useful ability to distinguish between alternatives. In the UPL-WASPAS method, the coefficient of variation is 0.8970, while for the HFL-WASPAS method, it is equal to 1.2761.

MCGDM methods have been widely used in supplier selection problems. In this research, we have considered the process of multi criteria group evaluation of suppliers in the context of UPLTSs for the first time. UPLTSs can reduce incomplete information under uncertainty. The WASPAS approach is one of the advanced MCGDM methods and has been applied to many practical problems [70, 71, 72]. In the present study, we have developed a novel WASPAS method based on the proposed UPLSWG operator to handle the MCGDM problems under the background of UPLTSs. In the construction of this approach, some novel concepts, scaled UPLTSs, UPLSWG operator have been used, and some alterations have been executed in the existing theory. The alterations are related to basic operations, aggregation operators, WSM and WPM measures. The classical WASPAS method utilises the existing aggregation operators [33], which fail to handle the situation where DMs are of different rank because these operators pay no attention to DMs' weight. Further, in the aggregation process, the product of two different dimensions, i.e., subscripts of linguistic terms and their associated probabilities are taken. Their strategy is simple, but it weakens the effect of the probabilities in UPLTSs. However, in the constructed algorithm, the probability information is separately considered, and the linguistic information is handled more effectively than the existing ones. In the proposed method, a novel framework based on the proposed entropy measure has been designed to obtain the unknown criteria weights objectively with information characterised by UPLTSs.

From the aforestated analysis, it can be concluded that the main merits of our constructed approach are not only due to its ability to overcome the flaws of the existing studies efficiently but also due to its capability to manage the assessment information that is expressed by UPLTSs. Thus, we avoid losing and distorting of the original information, which makes the ranking results of MCGDM problems more reliable.

In addition to the above discussion, we give some characteristic comparison of our constructed method and classical method in Table 6.10.

Table 6.10: Characteristic comparison of HFL-WASPAS and UPL-WASPAS

Characteristic	HFL-WASPAS	UPL-WASPAS
Aggregation operator	Existing UPLA operator [33]	Proposed UPLSWG operator
DMS' weight	Ignore DMS' weight	Consider DMS' weight
Data context after aggregation	HFLTSs (discard the probability information)	UPLTSs (having the ability to reflect probability)
Flexibility according to DMs preferences	Yes	Yes
Computational complexity	Less	Comparatively high

6.5.2 Sensitivity analysis

In this subsection, results of the designed model are analyzed based on sensitivity. The objective of the sensitivity analysis is to investigate the impact of various setting of the WASPAS parameter λ.

Varying the value of λ from 0.1 to 0.9 can helps us to assess the sensitivity of the proposed method. The impact of the change in the value of λ can be examined from Tables 6.5 and 6.9. Also, the results are plotted graphically in Figs. 6.2 and 6.3. The analysis reveals that the score of each alternative obtained by the UPL-WASPAS and the HFL-WASPAS methods differ according to each case of the parameter λ. From Table 6.5, it can be seen that the score value of the alternative z_3 increases with the increase in the value of λ. While from Table 6.9, it can be observed that the score value of the alternative z_1 decreases with increases in the value of λ. However, the overall ranking of alternatives retains the same for each value of λ. Therefore, it can be stated that the designed method has good stability with various values of λ. Moreover, the parameter λ found in the proposed method can carefully reflect the DMs risk preferences. Thus, the presented method gives the DMs more choices as they can choose the value of λ according to their preferences.

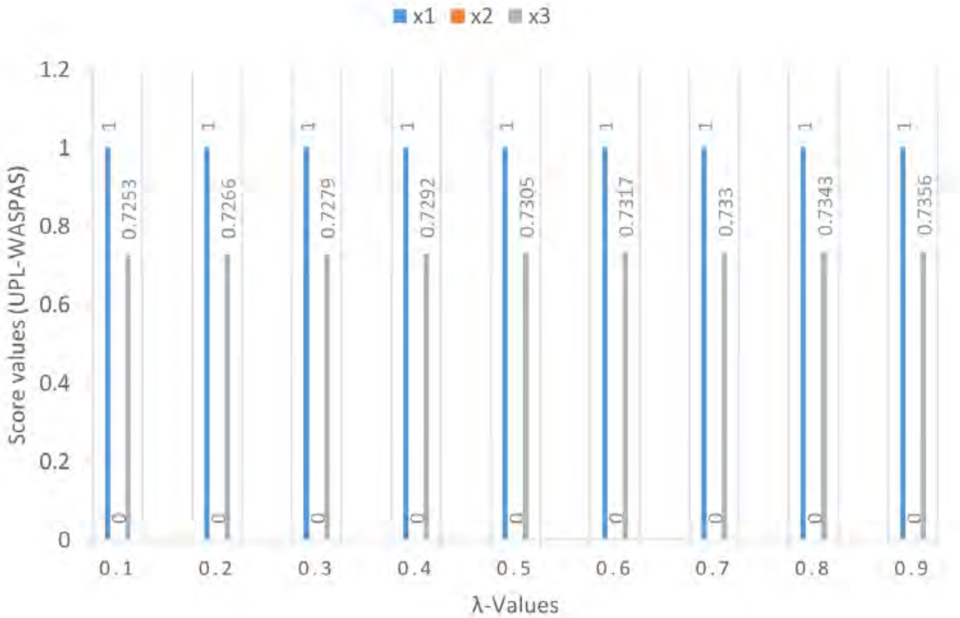

Figure 6.2: Rank acceptability indices of alternatives with respect to UPL-WASPAS decision mechanism coefficient

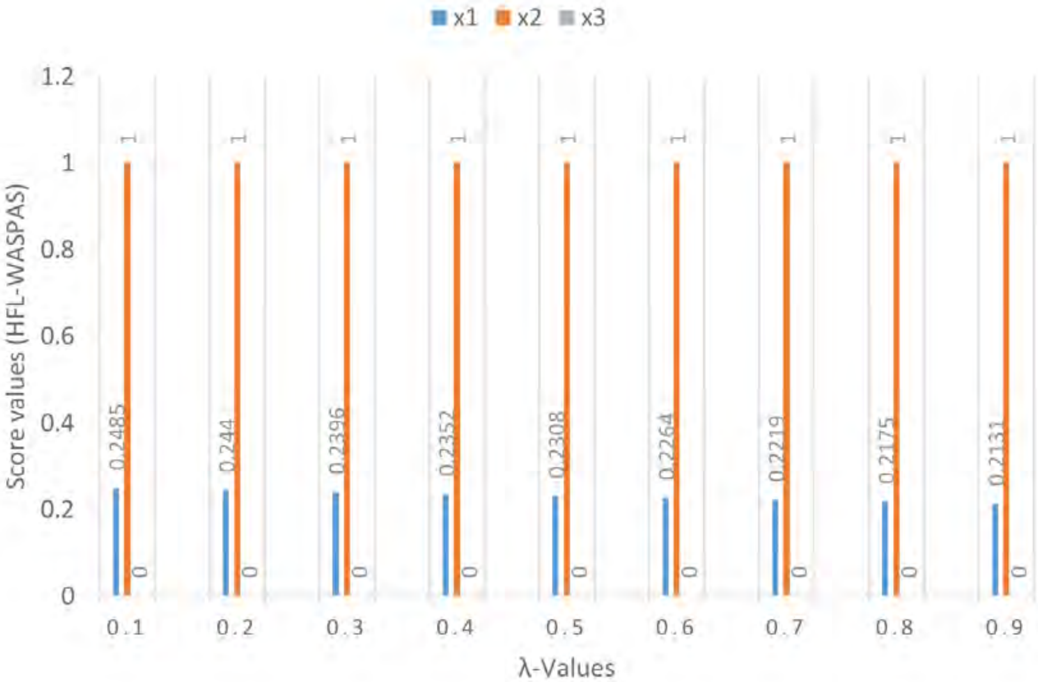

Figure 6.3: Rank acceptability indices of alternatives with respect to HFL-WASPAS decision mechanism coefficient

To compare the results of the proposed approach with the HFL-WASPAS, we use Spearman's ranking correlation test [73]. According to this test, the value of correlation coefficient is -0.5. As we can see the results of the UPL-WASPAS method are entirely different from the ones obtained from HFL-WASPAS method. Which is due to the loss of information during the aggregation process in HFL-WASPAS.

Chapter 7

Concluding remarks and outlooks

The present chapter is devoted to summarising this doctoral thesis and pointing out some suggestions about future work.

7.1 Conclusions

In what follows, we disclose the conclusions of the provided study.

MCGDM is one of the widely known subjects of making decisions. Fuzzy logic stipulates an effective way to tackle MCGDM problems. Very often in these problems, DMs face hurdles to make decisions, owing to uncertain and fuzzy data. To handle such cases, numerous MCGDM techniques have been framed until now. However, there are still needs of some advance fuzzy techniques because the existing ones are not perfect. In this thesis, some MCGDM techniques are developed based on the proposed theory to more accurately model the complex scenarios. At the beginning of this thesis, we introduced a novel fuzzy set, namely PHILTS, to extend the existing HIFLTS and PLTS. To facilitate the calculation of the PHILTSs, a normalization process, essential operations and aggregation operators for PHILTSs are also designed. An extended TOPSIS method and aggregation-based method are constructed to solve the group's decision ranking problems with the multiple conflict criteria in PHILTS. The proposed models are compared with the existing model of TOPSIS. The PLTS and HIFLTS are special cases of PHILTS; it grants the freedom to DMs to express their opinions in a more dynamic way. Furthermore, the occurrence probabilities of membership and non-membership linguistic term sets greatly affects the decision making, validating the importance of designed theory and models in this manuscript. The probability is one of the best tools to handle future uncertainties; thus, our proposed models are more suitable for decision making related to the possible future scenarios. However, its arithmetic complexity is high.

In the upcoming chapter, efforts were made to construct robust techniques for correctly solving MCGDM problems under the probabilistic uncertain linguistic context. Firstly, an aggregation formula is studied in PULTS context to compute probabilities of uncertain linguistic terms in group assessment, which can upgrade the adaptability of PULTSs on a practical level. Further, to facilitate calculations of PULEs the adjusting rule of probability is put forward to adjust probabilities of any finite number of PULEs. Meanwhile, based on this adjusting rule of probability and linguistic scale function, novel operations of PULEs are proposed. These novel operations can avoid the operation values crossing the boundary of LTSs and works under different semantics of linguistic terms. Aggregation operators and distance measure were also redefined in terms of linguistic scale function which overcome the drawbacks of existing ones. Correlation measure was also discussed in detail. Furthermore, to remove the biased assessments in MCGDM problems, the PUL-consensus reaching approach was proposed in which first the consensus level of each expert is calculated and then adjusted opinion matrices are constructed before aggregating. Finally, two types of multiple-criteria decision making methodologies were proposed to handle MCGDM problems in balanced as well as in unbalanced scenarios. Subsequently, an application of commodity selection in Forex investment is provided to demonstrate the practicality and feasibility of the proposed methods, which are further compared to the existing one.

The fifth chapter focused on the notion of WIVDHFS, which is the extension of one of the broadly applicable fuzzy set IVDHFS. The proposed set's prominent characteristic is that the membership degree and non-membership degree of an element to a given set is signified by two sets of possible different interval values along with their importance values, respectively. Based on the Archimedean norms, we have defined some basic operational laws for WIVDHFEs and studied their properties and comparison method. Based on these operational laws, some generalized aggregation operators have been developed to aggregate weighted interval-valued dual hesitant fuzzy information, including the GWIVDHFWA and the GWIVDHFWG operator. Some related properties and special cases of the developed operators have been investigated. The proposed operators can cover a wide variety of existing aggregation operators and have the capability to capture the uncertainty efficiently in MCGDM problems. Based on presented operators, an approach for solving MCGDM problems with weighted interval-valued dual hesitant fuzzy information has been constructed. The paper has also provided an illustrative example of teaching quality assessment to verify the proposed framework and demonstrate its applicability. Further, the superiority of the stated method has been illustrated by comparing the past study.

To improve relevant theoretical research regarding UPLTS, the basic operational laws and score function are redefined in the second last chapter to avoid the drawbacks of existing ones, as discussed in the previous section. The revised formulas' effectiveness is justified by employing examples, which revealed that the obtained results are credible. Meanwhile, the concept of deviation degree is introduced in order to accommodate the situation in which two different UPLTSs have the same score values. Afterwards, some existing aggregation operators are redefined in terms of the revised operational laws. In addition, a newly operator, namely, UPLSWG is designed, and some of its properties such as idempotency, boundedness, monotonicity and commutativity, are studied to open a novel perspective. A vital issue of decision making with incomplete weight information is to find the proper way to determine the criteria weights. Thus, the entropy-based method is proposed, which determines the weighThusiteria objectively without considering the DMs' preferences. Next, the UPL-WASPAS model is put forward with the help of proposed theoretical notions. The proposed model can handle the qualitative data even when judgments are made based on partly unknown data. To this extent, it is easy for DMs to provide their preferences while respecting subjectivity and the lack of information on specific criteria.

Further, in the presented method, the coefficient of variation of score values, which is the basis for the sorting of alternatives, is high. The developed methodology is used for the supplier selection problem to establish its effectiveness and application. Finally, some comparative analysis and detailed sensitivity analysis are made to show the provided method's appropriateness and stability, respectively. Dealing with calculation, we took different values for λ and found the ranking results' robustness. Also, it is noticed that the score values of alternatives have the monotonic behaviour concerning the parameter λ and can influence the risk preference of the DMs.

7.2 Future research directions

This part focuses on future developments with regard to the themes analysed in this study.

The following future research lines are marked up as a guide for scholars working in this research area.

i. The developed fuzzy notions, namely PHILTS and WIVDHFS can be considered as the cornerstones for providing other MCGDM techniques, which have not been opened yet for these contexts such as TODIM, WASPAS, GLDS, best and worst method and multi-objective optimization based on ratio analysis plus full multiplicative form (MULTIMOORA).

ii. Though, we have improved the operational laws of UPLTSs, but some limitations still exist, they can be further enriched by using Archimedean t-norm and t-conorm.

iii. In the presented models, DMs are considered as perfectly rational people. However, in practice, DMs do not behave in an entirely rational manner. So the research about the irrational characteristic of DMs [74] will be studied further.

iv. It is predictable that forthcoming studies may also extend other aggregation operators such as power aggregation operators, prioritized operators and induced generalized aggregation operators to PHILTS, PULTS and WIDHFS setting.

v. As future work, the presented methods can be employed in other fields, such as risk evaluation, hotel location, project selection and other domains under ambiguous environments.

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