Some Contribution To Finite State Machine on The Basis of Soft Set

By

ASIM HUSSAIN

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan

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Supervised By

Prof. MUHAMMAD SHABIR

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan

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Dedication

To my parents

The reason of what I become today. Thanks for your great support and continuous care. You will always be in my heart.

Acknowledgements

All praises to almighty Allah, the most beneficent and most merciful, who created this universe and gave us the idea to discover. I am highly grateful to Almighty Allah for His blessings, guidance and help in each and every step of my life. He blessed us with the **Holy Prophet Muhammad** (S.A.W), who is forever source of guidance and knowledge for humanity.

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May Almighty Allah shower His choicest blessings and prosperity on all those who assisted me in any way during completion of my thesis.

ASIM HUSSAIN
(2017)

Introduction

A finite state machine (FSM) or finite state automaton or a simply state machine is a mathematical model of computation used, to design both computer programs and sequential logic circuits. It is conceived an abstract machine that can be in one of a finite number of states. The machine is only one state at a time; the state it is in at any given time is called the current state. It can change from one state to another when initiated by a triggering event or condition; this is called a transition. A particular FSM is defined by a list of its states, and the triggering condition for each transition.
The behavior of state machines can be observed in many devices in modern society which perform a predetermined sequence of actions depending on a sequence of events with which they are presented. Simple examples are vending machines which dispense products when the proper combination of coins is deposited, elevators which drop riders off at upper floors before going down, traffic lights which change sequence when cars are waiting, and combination locks which require the input of combination numbers in the proper order.

In 1965 Zadeh introduced the concept of fuzzy set in his definitive paper [56]. Many authors used this concept to generalize basic notions of algebra. Concept of fuzzy automata was introduced by Wee in [55]. The theory of fuzzy automata has been developed by many researchers like, Adamek and Wechler [1], Arbib and Manes [12, 13], Brunner and Wechler [16], Mizumoto et al. [41], Peeva and Nedyalkova. [45], Peeva [46, 47] and Sentos and Wee [50]. Malik et al. [36, 37, 38, 39, 40] used algebraic techniques to study the fuzzy finite state machine, and introduced the notion of submachine of a finite state machine, retrievable, separated and connected fuzzy finite state machines and discussed their basic properties. Latter different approaches for the fuzzification of finite state machine presented in detail in [31, 44].

Today there are many practical problems which in different fields, such as engineering, medical sciences, economics and social sciences involve data that contain uncertainties. Due to various types of uncertainties, we cannot use traditional methods. Because, most of traditional methods for formal modeling, reasoning and computing are crisp, deterministic and precise in character. With the passage of time, different theories have been developed which we can consider as mathematical tools for dealing with ambiguity and manifold types of uncertainties, such as probability theory, theory of fuzzy sets, theory of intuitionistic fuzzy sets, theory of vague sets, theory of interval mathematics and theory of rough sets. All of them have their advantages as well as inherent limitations in dealing with ambiguity and uncertainties. For example, theory of probabilities can deal only with stochastically stable phenomena. Consequently, Molodtsov proposed a completely new approach for modeling ambiguity, vagueness and uncertainty [43]. This so called soft set theory is free from the difficulties affecting existing methods. Algebraic operations on soft set can be found in [9].

During recent years soft set theory has gained popularity among the researchers due to its applications in various areas. Number of publications related to soft sets has risen exponentially. Basic aim of this theory is to introduce a mathematical model with enough parameters to handle uncertainty. Prior to soft set theory, probability theory, fuzzy set theory, rough set theory and interval mathematics were common tools to discuss uncertainty. But unfortunately difficulties were attached with these theories, for details see [35, 43]. As mentioned above soft set theory has enough number of parameters, so it is free from difficulties associated with other theories. Soft set theory has been applied to various fields very successfully. Molodtsov in [43] point out several directions for the application of soft set theory, such as smoothness of function, operation research, game theory, probability theory of measurement and so on.

Chapter wise Study

The present work in this thesis is written in the theoretical background of finite state machines and soft sets. It contains the necessary part of automata theory and shows to formulate in an elegant way various concepts and facts about the soft finite state machine. Prerequisites are minimal and the work is self-contained.

In this thesis, we have six chapters. Chapter 1 provides some basic material needed for an understanding of finite state machine and algebra.

In second chapter, using the notion of soft sets, we introduced the concept of soft finite state machines (SFSM) as a generalization of fuzzy finite state machines. Study of soft finite state machines is interesting and worthwhile because there are results which hold for fuzzy finite state machine but do not hold for soft finite machine. For example, [20], in fuzzy finite state machines if p is successor of q and r is a successor of p, then r is a successor of q. In general this result holds no longer in soft finite machines. In this chapter, we also introduce a congruence relation which can be naturally established in such a way that each associates a semigroup with a soft finite state machine (SFSM). We introduce a (strong) homomorphism of SFSM and then we investigate the related properties. Using a SFSM we make three finite semigroups with identity and show that they are isomorphic. We defined soft admissible relation and establish a relation between SFSM and the quotient structure of another soft finite state machine. Finally soft transformation semigroups are defined and related properties investigated.

Concept of covering, cascade product, and wreath product play an important role in the study of automata and their associated semigroups. In 3rd chapter, we examine these concepts for soft finite state machines.

In chapter 4, we continue our study of a soft finite state machine utilizing algebraic techniques. We defined the concept of soft submachine, separability, connectivity and decomposition of soft finite state machine. With the help of these concepts, we will prove Decomposition Theorem for soft finite state machine.

In 5th chapter concepts of soft subsystem, strong soft subsystem, switching, commutative and soft finite switchboard state machine are introduced and some of its properties are discussed.

In last chapter we study a new product of two soft finite machines M_1 and M_2, written $M_1 \cdot M_2$ and called the Cartesian composition of M_1 and M_2. We also define the concept of soft admissible partitions and construct the quotient structure of SFSM with the help of soft admissible partitions. Finally we discuss the associativity of wreath product, sum and cascade products of soft finite state machines.

Contents

The University of New South Wales

School of Electrical Engineering and
Computer Science and Engineering

ASIM HUSSAIN

DOCTORATE OF PHILOSOPHY

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Supervisor: Prof. Dr. Muhammad Shabir
Assessor:

Chapter 1

Preliminaries

The theory of machines has had a major impact on the development of computer systems and their associated languages and softwares. It has also found applications in such areas of science as biology, biochemistry, psychology and others [49].

This chapter briefly summarizes the known results about connections between finite state machine and algebra.

1.1 Finite State Machines

A finite state machine (FSM) or simply state machine is a mathematical model of computation used, to design both computer programs and sequential logic circuits. It is conceived an abstract machine that can be in one of a finite number of states. The machine is only one state at a time, the state it is in at any given time is called the current state. It can change from one state to another when initiated by a triggering event or condition, this is called a transition. A particular FSM is defined by a list of its states, and the triggering condition for each transition.

The behavior of state machines can be observed in many devices in modern society which perform a predetermined sequence of actions depending on a sequence of events with which they are presented. Simple examples are vending machines which dispense products when the proper combination of coins is deposited, elevators which drop riders off at upper floors before going down, traffic lights which change sequence when cars are waiting, and combination locks which require the input of combination numbers in the proper order.

Finite state machines can model a large number of problems, among which are electronic design automation, communication protocol design, language parsing and other engineering applications. In biology and artificial intelligence research, state machines have been used to describe neurological systems.

In this section we review some basic results of finite state machines. We are indebted [23, 44, 49].

1.1.1 Definition

A six tuple $\Upsilon = (Q, X, Y, f, g, s)$ is called a finite state machine, where Q, X, and Y are finite non-empty sets, $f : Q \times X \to Q$, $g : Q \times X \longrightarrow Y$, and $s \in Q$.

The members of Q are called states. The members of X and Y are called input and output symbols, respectively. The functions f and g are called the state transition and output functions, respectively. The state s is called the initial state.

1.1.2 Example

Let $Q = \{q_1, q_2\}$, $X = \{a, b\}$ and $Y = \{0, 1\}$. Define the functions $f : Q \times X \longrightarrow Q$ and $g : Q \times X \longrightarrow Y$ as described in following table

	f		g	
$Q \setminus X$	a	b	a	b
q_1	q_1	q_2	0	1
q_2	q_2	q_2	1	0

The interpretation of the above table is as follows:

$$\begin{array}{ll} f(q_1, a) = q_1 & g(q_1, a) = 0 \\ f(q_1, b) = q_2 & g(q_1, b) = 1 \\ f(q_2, a) = q_2 & g(q_2, a) = 1 \\ f(q_2, b) = q_2 & g(q_2, b) = 0 \end{array}$$

Then $\Upsilon = (Q, X, Y, f, g, q_1)$ is a finite state machine.

The next state and output functions can also be defined by a transition diagram.

1.1.3 Example

We draw the transition diagram for the finite state machine of Example 1.1.2, see Figure 1.

The transition diagram is known as digraph in graph theory. The vertices are the states. The initial state is indicated by an arrow as shown. If the finite state machine is in state q_0 and inputting x causes output y and a move to state q_1, a directed edge is drawn from the vertex q_0 to q_1 and labelled x/y. The transition diagram of Figure

1 is

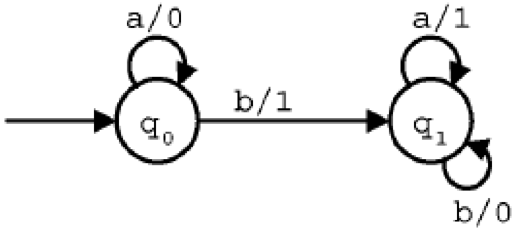


Figure 1

A word or string (over X) is a finite sequence of elements of some nonempty set X. If $x = x_1x_2...x_m$ and $y = y_1y_2...y_n$ are strings over X, where $x_1, x_2, ..., x_m, y_1, y_2, ..., y_n \in X$, then $xy = x_1x_2...x_my_1y_2...y_n$ is also a string over X. Let X^* denote the set of all words of element of X of finite length, λ denote the empty word in X^* and $|x|$ denote the length of x for every $x \in X^*$.

1.1.4 Definition

Let $\Upsilon = (Q, X, Y, f, g, q_0)$ be a finite state machine. An input string for Υ is a string over X and output string for Υ is a string over Y. The string

$$v_1v_2v_3....v_n$$

is the output string for Υ corresponding to the input string

$$u_1u_2u_3....u_n$$

if there exist states $q_0, q_1, q_2, ..., q_n \in Q$, such that

$$\begin{aligned} f(q_{i-1}, u_i) &= q_i \text{ for } i = 1, 2, 3....n \\ g(q_{i-1}, u_i) &= v_i \text{ for } i = 1, 2, 3....n. \end{aligned}$$

1.1.5 Definition

A finite state automaton $A = (Q, X, Y, f, g, s)$ is a finite-state machine such that the set of output symbols Y is $\{0, 1\}$ and where the current state determines the last output. Those states for which the last output was 1 are called accepting states.

1.1.6 Example

The transition diagram of the finite-state machine $A = (Q, X, Y, f, g, s)$, where $Q = \{q_0, q_1, q_2\}$, $X = \{a, b\}$ and $Y = \{0, 1\}$, is defined by the table below. The initial state is q_0.

	f		g	
$Q \setminus X$	a	b	a	b
q_0	q_0	q_1	0	1
q_1	q_0	q_2	0	1
q_2	q_0	q_2	0	1

The transition diagram is shown in Figure 2. If the finite state machine is in state q_0, the last output was 0. If the machine is in either state q_1 or q_2, the last output was 1. Consequently, A is a finite-state automaton and the accepting states are q_1 and q_2.

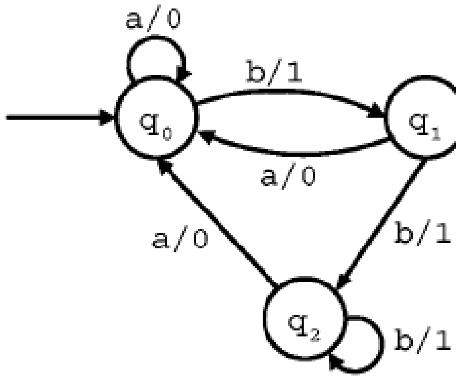


Figure 2

We see from Example 1.1.6, that the finite-state machine defined by a transition diagram is a finite state automaton if the set of output symbols is $\{0, 1\}$ and if, for each state q, all incoming edges to q have the same output label.

The transition diagram of a finite-state automaton is often drawn with the accepting states in double circles and with the output symbols omitted. If we draw the transition diagram of Figure 2 in this way, we obtain the transition diagram of Figure 3.

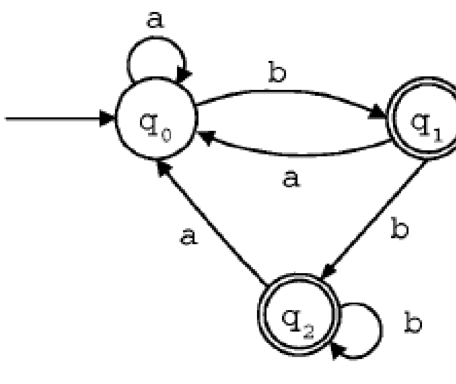


Figure 3

In Figure 3, accepting states are in double circles and output symbols are omitted.

1.1.7 Example

We now draw the transition diagram of the following finite-state automaton of Figure 4 as a transition diagram of a finite-state machine.

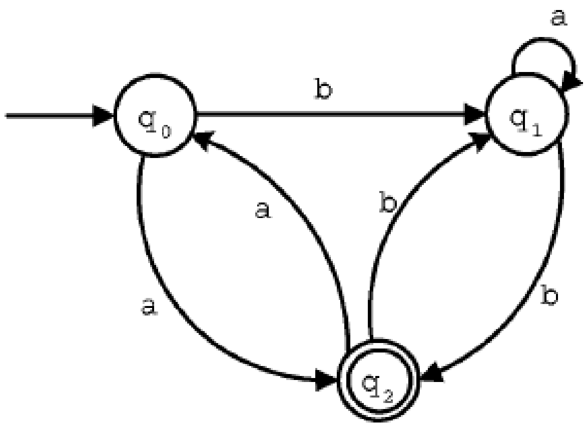


Figure 4

Since q_2 is an accepting state, we label all its incoming edges with output 1 as in Figure 5. Since the states q_0 and q_1 are not accepting, all their incoming edges are labelled with output 0 as in Figure 5. Then the transition diagram of FSM of Figure 5 is obtained.

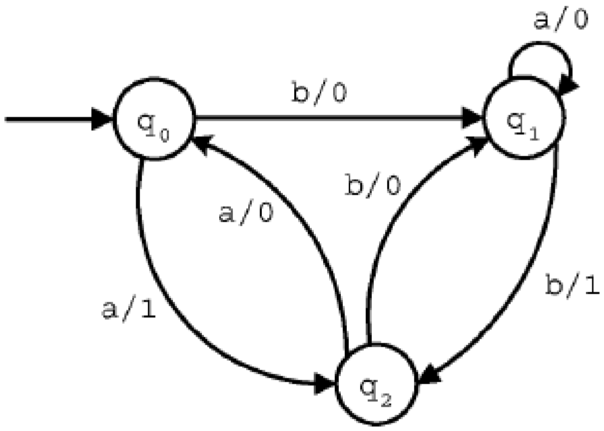


Figure 5

1.1.8 Definition

An alternative to Definition 1.1.5 can be obtained by regarding a finite state automaton A as consisting of

(1) a finite set Q of states,

(2) a finite set X of input symbols,

(3) a next-state function f from $Q \times X$ into Q,

(4) a subset $\mathcal{A}$ of Q of accepting states,

(5) an initial state $q \in Q$.

In this case, we write $A = (Q, X, f, \mathcal{A}, q)$.

1.1.9 Example

The transition diagram of the finite-state automaton $A = (Q, X, f, \mathcal{A}, g)$, where $Q = \{q_0, q_1, q_2\}$, $X = \{a, b\}$, $\mathcal{A} = \{q_2\}$, $q = qo$, and f is given by the following table,

	f	
$Q \setminus X$	a	b
q_0	q_1	q_0
q_1	q_2	q_0
q_2	q_2	q_0

is shown in Figure 6

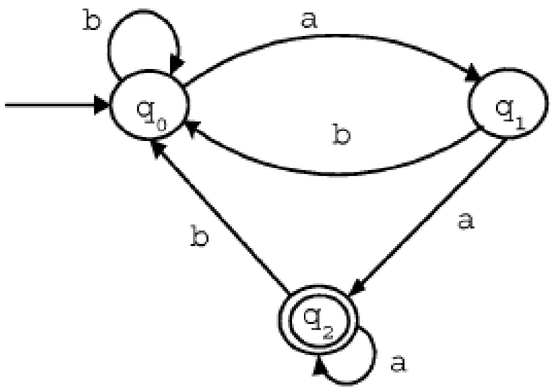


Figure 6

1.1.10 Definition

Let $A = (Q, X, f, \mathcal{A}, q)$ be a finite-state automaton. Let $x = x_1 x_2 ... x_n$ be a string over X. If there exists states $q_0, q_1, ... q_n$ satisfying

(1) $q_0 = q$

(2) $f(q_{i-1}, x_i) = q_i$ for $i = 1, 2, ..., n$.

(3) $q_n \in \mathcal{A}$.

Then x is said to be accepted by A. The empty string is accepted if and only if $q_0 = q \in \mathcal{A}$. Let $Ac(A)$ denote the set of strings accepted by A.

Let $x = x_1 x_2 ... x_n$ be a string over X. Define states $q_0, q_1, ..., q_n$ by conditions (1) and (2) above. Then the (directed) path $(q_0, q_1, ..., q_n)$ is called the path representing x in A.

It follows from Definition 1.1.10 that if the path P represents the string x in a finite-state automaton A, then A accepts x if and only if P ends at an accepting state.

The next two examples illustrate design problems.

1.1.11 Example

We design a finite-state automaton that accepts precisely those strings over $\{a, b\}$ that contain no a's.

We use two states :

A : There exists an a.

B : There does not exist an a.

The state B is the initial state and the only accepting state. The finite state automaton in Figure 7 correctly accepts the empty string.

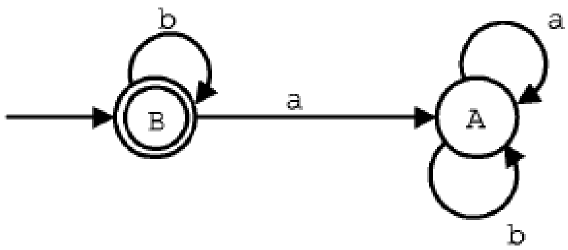


Figure 7

1.1.12 Example

We design a finite-state automaton that accepts precisely those strings over $\{a, b\}$ that contain an odd number of a's.

We use two states :

E : An even number of a's are in the string.

O : An odd number of $a's$ are in the string.

The initial state is E and the accepting state is O. We obtain the transition diagram shown in Figure 8.

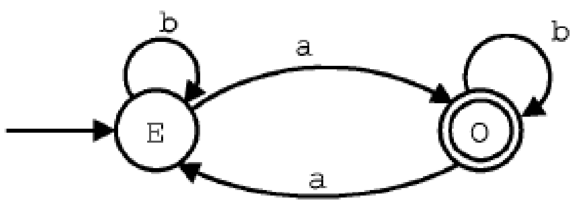


Figure 8

A finite-state automaton is essentially an algorithm to decide whether or not a given string is accepted.

1.1.13 Definition

Finite-state automata A and A' are called equivalent if $Ac(A) = Ac(A')$.

1.1.14 Example

It can be shown that the finite-state automata of Figures 7 and 9 are equivalent.

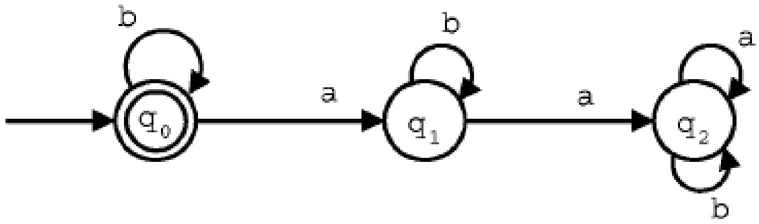


Figure 9

1.2 Semigroups and Monoids

During the past few decades connection in the theory of semigroups and the theory of machines, became of increasing importance, both theories enriching each other. In association with the study of machines and automata, other areas of applications such as formal languages and the software use the language of modern algebra in terms of Boolean algebra, semigroups and others. But also parts of other areas, such as biology, psychology, biochemistry and sociology make use of semigroups.

The algebraic theory of automata, which uses algebraic concepts to formalize and study certain types of finite-state machines. One of the main algebraic tools used to do this is the theory of semigroups. Automaton is an abstract model of computing device. Using this models different types of problems can be solved.

In this section, we review some basic results of semigroups and monoids.

1.2.1 Definition

A nonempty set S with a binary operation $*$ is called a semigroup if it satisfies the associative law: $(x * (y * z)) = ((x * y) * z)$ for all $x, y, z \in S$. A semigroup is a monoid if it has an element e for which $s * e = e * s = s$, for all $s \in S$.

If S is finite and it has n elements, then we also say that S is a finite semigroup of order n. S is called commutative or abelian if $x * y = y * x$ holds for every $x, y \in S$. Otherwise, we say that S is non commutative or non abelian. One writes instead of $x * y$ simply xy. Associativity guarantees that products written without parentheses have well-defined values in S.

1.2.2 Example

(1) The set $\mathbb{N}\backslash\{1\} = \{2, 3, 4, ...\}$ of all natural numbers, excluding 1, forms a semigroup under usual multiplication of natural numbers.

(2) The set of non-negative integers forms a semigroup with identity (monoid) under usual addition of integers.

Assume $(S, *)$ is a semigroup, $a \in S$, and for any pair $H_1, H_2 \subseteq S$, we write

$$\begin{aligned} a * H_1 &= \{ah : h \in H_1\} \\ H_1 * a &= \{ha : a \in H_1\} \\ H_1 H_2 &= \{h_1 h_2 \mid h_1 \in H_1, h_2 \in H_2\}. \end{aligned}$$

In particular, if $H_1 = H_2 = H$, then we put $H^2 = HH$ and, in general, let $H^1 = H$, $H^k = H^{k-1}H$ for every positive integer $k > 1$. Associativity of S guarantees that $(H_1 (H_2 H_3)) = ((H_1 H_2) H_3)$ for every choice of subsets H_1, H_2 and $H_3 \subseteq S$, so $H_1 H_2 H_3$, written without any parentheses, is a well-defined subset of S. Therefore, the set of all subsets of S is itself a semigroup under this multiplication. A subsemigroup of $(S, *)$ is a non empty subset $S^{'}$ of S which is closed under the binary operation $*$ of S, that is it satisfies $S^{'} * S^{'} \subseteq S^{'}$. The subsemigroup of S generated by a subset $H \subseteq S$ is the smallest subsemigroup of S containing H. It is denoted by $\langle H \rangle$. In the case of singleton $H = \{h\}$ or finite $H = \{h_1, h_2, ...h_n\}$, we write $\langle h \rangle$ for $\langle \{h\} \rangle$ and may write $\langle h_1, h_2, ...h_n \rangle$ for $\langle \{h_1, h_2, ...h_n\} \rangle$, respectively. A semigroup is called monogenic if it is generated by a single element. H is a generating system for S if $\langle H \rangle = S$. A generating system H is minimal if for every $h \in H$, $H\backslash\{h\}$ is not a generating system. A minimal generating system is also called a basis. In addition, a semigroup S is finitely generated if it has a generating system with finitely many elements. If H is a finite generating system for S, then for an appropriate $K \subseteq H$, K is a basis of S.

1.2.3 Definition

Let S and T be semigroups having a mapping $\phi : S \longrightarrow T$ such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in S$. Then, we say that ϕ is a homomorphism from S to T. If ϕ is surjective, we also say that S can be mapped homomorphically onto T and that T is a homomorphic image of S. If ϕ is bijective, then we say that S is isomorphic to T (or S and T are isomorphic) and ϕ is called an isomorphism. An automorphism is an isomorphism from a semigroup S to itself.

If the semigroup S_1 is a homomorphic (isomorphic) image of the semigroup S_2 and S_2 is a homomorphic (isomorphic) image of the semigroup S_3, then S_1 is homomorphic (isomorphic) image of S_3.

1.2.4 Definition

An element $r \in S$ is called a right-zero (left-zero) element of S if $sr = r$ $(rs = r)$ for all $s \in S$. In addition, $0 \in S$ is the zero element if $0s = s0 = 0$, $s \in S$.

1.2.5 Example

Let S be a nonempty set. There is a simple semigroup structure on S with the multiplication given by $xy = x$, for all $x, y \in S$, is called the left-zero semigroup over S.

If the semigroup has both a left-zero element l and a right-zero element r, then it has an unambiguously determined zero element $0 = lr = l = r$. It follows that the zero element of a semigroup S is uniquely determined if it exists. Just as for a zero element, the element e of a semigroup is uniquely determined if it exists and is called the identity element of S. If $\phi : S \longrightarrow T$ is a homomorphism, and S and T are both monoids with respective identity elements 1_S and 1_T, it need not be the case that $\phi(1_S) = 1_T$. The monoid with two right-zero elements or flip-flop monoid is $F = \{e, l, r\}$ with multiplication is defined as

	e	l	r
e	e	l	r
l	l	l	r
r	r	l	r

Then, for example, the constant function from F to F taking every element of F to l is a homomorphism but does not take the identity element e to itself. A homomorphism between monoids that does take the identity to the identity is called a monoid homomorphism.

1.2.6 Definition

Let X be any nonempty set. A word or string (over X) is a finite sequence of elements of some nonempty set X. Let X^* denote the set of all words of element of X of finite length, λ denote the empty word in X^* and $|x|$ denote the length of x for every $x \in X^*$. X^* is the free monoid generated by X with operation, concatenation. Note that $X^+ = X^* - \{\lambda\}$ is the free semigroup over X.

1.2.7 Theorem

Let X^+ be a free semigroup over X, S is an arbitrary semigroup, and $\alpha : X \longrightarrow S$ is an arbitrary map from X into S. Then there is a unique semigroup homomorphism $\overline{\alpha} : X^+ \longrightarrow S$ extending α.

1.2.8 Remark

Let $X = \{x\}$. The free semigroup X^* over X is isomorphic $(W, +)$, where $W = \{0, 1, 2, ...\}$. Mapping $n \in W$ to x^n, we obtain an isomorphism from W to X^*.

1.2.9 Definition

Let S be a set. An equivalence relation R on S is a reflexive, symmetric and transitive binary relation, that is, for all $x, y, z \in S$,

(1) xRx

(2) xRy imply yRx and

(3) xRy and yRz imply xRz.

1.2.10 Definition

A congruence relation ρ on a semigroup S is an equivalence relation such that $x\rho y$ and $u\rho v$ implies $ux\rho vy$.

If $x\rho y$ for some $x, y \in S$, then sometimes we write $x \cong y \pmod{\rho}$ or, in short, $x \cong y$. (Then, we say that x is congruent to y modulo ρ or, in short, x is congruent to y.) Let S and T be semigroups having a homomorphism $f : S \longrightarrow T$. Then, determines a congruence relation ρ with $x \cong y \pmod{\rho}$ if and only if $f(x) = f(y)\,(x, y \in S)$. A partition S/ρ of the semigroup S is called compatible if ρ is a congruence relation. In this case, multiplication in S induces a semigroup structure on S/ρ. Letting $[x]$ denote the ρ equivalence class x/ρ of $x \in S$, compatibility means that the multiplication $[x]\,[y] = [xy]$ is well defined for all $x, y \in S$ and is associative. Thus, we have the following result.

1.2.11 Theorem

Let ρ be congruence relation on a semigroup (monoid) S. Then $S/\rho = \{[x] : x \in S\}$ is also a semigroup (monoid). Moreover, there exists a homomorphism from S onto S/ρ.

Chapter 2

Soft Finite State Machine

Soft set theory [43] was firstly proposed by a Russian researcher Molodtsov, in 1999. This theory is a general mathematical tool for dealing with uncertain, fuzzy, not clearly defined objects. Soft sets have received much attention in the last era because of their application in decision making problems. Molodtsov presented the concept of soft sets to deal with uncertainty. Prior to soft set theory, probability theory, fuzzy set theory, rough set theory and interval mathematics were common tools to discuss uncertainty. But unfortunately difficulties were attached with these theories, for details see [35, 43]. As mentioned above soft set theory has enough number of parameters, so it is free from difficulties associated with other theories. Soft set theory has been applied to various fields very successfully.

Now we recall the basic definitions of soft set. Throughout U refers to an initial universe set, $P(U)$ the power set of U and E the set of all possible parameters under consideration with respect to U.

A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \longrightarrow P(U)$ and A is a subset of E. Therefore, a soft set over U gives a parametrized family of subsets of the universe U. For $a \in A$, $F(a)$ may be considered as the set of a-approximate elements of U by the soft set (F, A).

Soft set theory is a generalization of fuzzy set theory see [7]. In this chapter, using the notion of soft sets, we introduced the concept of soft finite state machines $(SFSM)$ as a generalization of fuzzy finite state machines. Study of soft finite state machines is interesting and worthwhile because there are results which hold for fuzzy finite state machine but do not hold for soft finite machine. For example, [20], in fuzzy finite state machines if p is successor of q and r is a successor of p, then r is a successor of q. In general this result holds no longer in soft finite machines. In this chapter, we also introduce a congruence relation which can be naturally established in such a way that each associates a semigroup with a soft finite state machine (SFSM).

We introduce a (strong) homomorphism of SFSM and then we investigate the related properties. Using a SFSM we make three finite semigroups with identity and show that they are isomorphic. We defined soft admissible relation and establish a relation between SFSM and the quotient structure of another soft finite state machine. Finally, soft transformation semigroups are defined and related properties are investigated.

2.0.12 Definition

A soft finite state machine (SFSM) is a quadruple (Q, X, F, U) where Q and X are finite non-empty sets called the soft states and the set of soft input symbols, respectively, and $(F, Q \times X \times Q)$ is a soft set over U.

Let X^* denote the set of all words of element of X of finite length, λ denote the empty word in X^* and $|x|$ denote the length of x for every $x \in X^*$.

2.0.13 Definition

Let $\Upsilon = (Q, X, F, U)$ be SFSM. Define soft set $(F^*, Q \times X^* \times Q)$ by

$$\begin{aligned} F^*(q, \lambda, p) &= \begin{cases} U & \text{if } q = p, \\ \emptyset & \text{otherwise.} \end{cases} \\ F^*(q, a, p) &= F(q, a, p) \end{aligned}$$

and

$$F^*(q, xa, p) = \bigcup_{r \in Q} \{F^*(q, x, r) \cap F(r, a, p)\},$$

for all $x \in X^*$, $p, q \in Q$ and $a \in X$.

Let $X^+ = X^* - \{\lambda\}$. Then, X^+ is a semigroup. For F^* given in Definition 2.0.13, we let $F^+ = F^*$ restricted to $Q \times X^+ \times Q$.

2.0.14 Example

Let $Q = \{a, b\}$ be the set of soft states and $X = \{x, y\}$ be the set of soft inputs. Consider the universal set $U = \{1, 2, 3, 4\}$. Then $\Upsilon = (Q, X, F, U)$ is described as

	a	b
(a, x)	$\emptyset$	$\{1, 3\}$
(a, y)	$\{1, 2, 3\}$	$\emptyset$
(b, x)	$\{2, 3\}$	$\{1, 2\}$
(b, y)	$\emptyset$	$\{2, 4\}$

The above table can be read as, the image of (a, y, a) under F is $\{1, 2, 3\}$. The image of (a, xy, a) under F^* is $\emptyset$. Since

$$\begin{aligned} F^*(a, xy, a) &= \bigcup_{r \in Q} [F^*(a, x, r) \cap F(r, y, a)] \\ &= [F^*(a, x, a) \cap F(a, y, a)] \cup [F^*(a, x, b) \cap F(b, y, a)] \\ &= [\emptyset \cap \{1, 2, 3\}] \cup [\{1, 3\} \cap \emptyset] \\ &= \emptyset. \end{aligned}$$

2.0.15 Lemma

Let $\Upsilon = (Q, X, F, U)$ be a SFSM. Then,

$$F^*(q, xy, p) = \bigcup_{r \in Q} \{F^*(q, x, r) \cap F^*(r, y, p)\},$$

for all $p, q \in Q$ and $x, y \in X^*$.

Proof. Let $p, q \in Q$ and $x, y \in X^*$. We prove the result by induction on $|y| = n$ if $n = 0$, then $y = \lambda$ and so $xy = x\lambda = x$, hence

$$\begin{aligned} \bigcup_{r \in Q} [F^*(q, x, r) \cap F^*(r, y, p)] &= \bigcup_{r \in Q} [F^*(q, x, r) \cap F^*(r, \lambda, p)] \\ &= F^*(q, x, p). \end{aligned}$$

Thus the result is true for $n = 0$.

Suppose that the result is valid for all $u \in X^*$ such that $|u| = n - 1$. Let $y = ua$ where $u \in X^*$, $a \in X$ and $|u| = n - 1$. Then,

$$\begin{aligned} F^*(q, xy, p) &= F^*(q, xua, p) = \bigcup_{r \in Q} [F^*(q, xu, r) \cap F(r, a, p)] \\ &= \bigcup_{r \in Q} \left[\bigcup_{s \in Q} [F^*(q, x, s) \cap F^*(s, u, r)] \cap F(r, a, p) \right] \\ &= \bigcup_{r, s \in Q} [F^*(q, x, s) \cap F^*(s, u, r) \cap F(r, a, p)] \\ &= \bigcup_{s \in Q} \left[F^*(q, x, s) \cap \left\{ \bigcup_{r \in Q} [F^*(s, u, r) \cap F(r, a, p)] \right\} \right] \\ &= \bigcup_{s \in Q} [F^*(q, x, s) \cap F^*(s, ua, p)] \\ &= \bigcup_{s \in Q} [F^*(q, x, s) \cap F^*(s, y, p)]. \end{aligned}$$

Hence the result is valid for $|y| = n$. This completes the proof. ■

Define a relation $\cong$ on X^* by $x \cong y$ if and only if $F^*(q,x,p) = F^*(q,y,p)$ for all $p,q \in Q$. Clearly $\cong$ is an equivalence relation on X^*. Let $z \in X^*$ and $x \cong y$. Then,

$$\begin{aligned} F^*(q,zx,p) &= \bigcup_{r\in Q} \{F^*(q,z,r) \cap F^*(r,x,p)\} \\ &= \bigcup_{r\in Q} \{F^*(q,z,r) \cap F^*(r,y,p)\} \\ &= F^*(q,zy,p). \end{aligned}$$

Thus $zx \cong zy$. Similarly $xz \cong yz$. Thus $\cong$ is a congruence relation on the semigroup X^*. Thus we have the following result.

2.0.16 Theorem

Let $\Upsilon = (Q,X,F,U)$ be a SFSM. Define a relation $\cong$ on X^* by $x \cong y$ if and only if $F^*(q,x,p) = F^*(q,y,p)$ for all $p,q \in Q$. Then $\cong$ is a congruence relation on X^*.

Let $x \in X^*$, we denote $[x] = \{y \in X^* \mid x \cong y\}$ and $S(\Upsilon) = \{[x] \mid x \in X^*\}$. Define a binary operation $\circledast$ on $S(\Upsilon)$ by $[x] \circledast [y] = [xy]$ for all $[x],[y] \in S(\Upsilon)$. Obviously $\circledast$ is well defined and associative. It can be easily checked that $[\lambda]$ is the identity of $S(\Upsilon)$ under $\circledast$. Hence, we have the following theorem.

2.0.17 Theorem

Let $\Upsilon = (Q,X,F,U)$ be a SFSM. Define a binary operation $\circledast$ on $S(\Upsilon)$ by $[x] \circledast [y] = [xy]$ for all $[x],[y] \in S(\Upsilon)$. Then $(S(\Upsilon),\circledast)$ is a finite semigroup with identity $[\lambda]$.

Now we define another type of congruence relation on X^*. Let $x,y \in X^*$, define $x \equiv y$ iff $F^*(q,x,p) = U \iff F^*(q,y,p) = U$ for all $q,p \in Q$.

Clearly $\equiv$ is an equivalence relation. Assume that $x \equiv y$, for every $p,q \in Q$ and for any $z \in X^*$, we have $F^*(q,zx,p) = U \iff \bigcup_{r\in Q} [F^*(q,z,r) \cap F^*(r,x,p)] = U$

$\iff \bigcup_{r\in Q} [F^*(q,z,r) \cap F^*(r,y,p)] = U$

$\iff F^*(q,zy,p) = U.$

Hence $zx \equiv zy$. Similarly $xz \equiv yz$. Thus $\equiv$ is a congruence relation on X^*. For any $x \in X^*$, we denote $\widetilde{x} = \{y \in X^* : x \equiv y\}$ and $\widetilde{S}(\Upsilon) = \{\widetilde{x} : x \in X^*\}$.

Define a binary operation $\odot$ on $\widetilde{S}(\Upsilon)$ by $\widetilde{x} \odot \widetilde{y} = \widetilde{xy}$ for all $\widetilde{x},\widetilde{y} \in \widetilde{S}(\Upsilon)$. Clearly $\odot$ is well defined and associative, For any $\widetilde{y} \in \widetilde{S}(\Upsilon)$, we have $\widetilde{\lambda} \odot \widetilde{y} = \widetilde{\lambda y} = \widetilde{y} = \widetilde{y\lambda} = \widetilde{y} \odot \widetilde{\lambda}$. This means $\widetilde{\lambda}$ is the identity of $\left(\widetilde{S}(\Upsilon),\odot\right)$.

Let $f : S(\Upsilon) \to \widetilde{S}(\Upsilon)$ by $f([x]) = \widetilde{x}$ for all $[x] \in S(\Upsilon)$. Let $[x],[y] \in S(\Upsilon)$ such that $[x] = [y] \implies F^*(q,x,p) = F^*(q,y,p)$ for all $p,q \in Q$.

Thus $F^*(q,x,p) = U \iff F^*(q,y,p) = U$ for all $p,q \in Q$

$\implies \widetilde{x} = \widetilde{y} \implies f$ is well defined. Obviously f is onto and one-one. For every $[x],[y] \in S(\Upsilon)$, we have

$f([x] \circledast [y]) = f([xy]) = \widetilde{xy} = \widetilde{x} \odot \widetilde{y} = f([x]) \odot f([x])$.

Thus, we have the following result.

2.0.18 Theorem

Let $\Upsilon = (Q,X,F,U)$ be a SFSM. Define a binary operation $\odot$ on $\widetilde{S}(\Upsilon)$ by $\widetilde{x} \odot \widetilde{y} = \widetilde{xy}$ for all $\widetilde{x},\widetilde{y} \in \widetilde{S}(\Upsilon)$. Then $\left(\widetilde{S}(\Upsilon), \odot\right)$ is a finite semigroup with identity and $[x] \longrightarrow \widetilde{x}$ is a bijective homomorphism of $S(\Upsilon)$ to $\widetilde{S}(\Upsilon)$.

Let $\Upsilon = (Q,X,F,U)$ be a SFSM. For all $x \in X^*$, define a soft set F_x in $Q \times Q$ by $F_x((p,q)) = F^*(p,x,q)$ for all $(p,q) \in Q \times Q$.

Let P, R and T be non-empty subsets of Q. Let F_1 and F_2 be soft sets in $P \times R$ and $R \times T$. Define $F_1 \circ F_2$ in $P \times T$ by $F_1 \circ F_2(p,t) = \bigcup_{r \in Q} [F_1(p,r) \cap F_2(r,t)]$.

2.0.19 Theorem

Let $\Upsilon = (Q,X,F,U)$ be a SFSM and $S_\Upsilon = \{F_x : x \in X^*\}$. Then

(i) $F_x \circ F_y = F_{xy}$ for all $x,y \in X^*$.

(ii) $(S_\Upsilon, \circ)$ is a finite semigroup with identity, and it is isomorphic to $(S(\Upsilon), \circledast)$.

Proof. (i) Let $p,q \in Q$. Then

$$\begin{aligned} F_{xy}(q,p) &= F^*(q,xy,p) = \bigcup_{r \in Q} \{F^*(q,x,r) \cap F^*(r,y,p)\} \\ &= \bigcup_{r \in Q} \{F_x(q,r) \cap F_y(r,p)\} \\ &= F_x \circ F_y(q,p). \end{aligned}$$

So $F_x \circ F_y = F_{xy}$.

(ii) Obviously $(S_\Upsilon, \circ)$ is a finite semigroup with identity F_λ. Note that S_Υ is finite because Q and the image of F are finite. Define $f : S_\Upsilon \longrightarrow S(\Upsilon)$ by $f(F_x) = [x]$ for all $F_x \in S_\Upsilon$. Let $F_x, F_y \in S_\Upsilon$. Then, $F_x = F_y$

$$\begin{aligned} &\iff F_x(q,p) = F_y(q,p) \text{ for all } (q,p) \in Q \times Q \\ &\iff F^*(q,x,p) = F^*(q,y,p) \\ &\iff [x] = [y]. \end{aligned}$$

Thus, f is well defined. Obviously, f is one-one and onto.

Now, $f(F_x \circ F_y) = f(F_{xy}) = [xy] = [x] \circledast [y] = f(F_x) \circledast f(F_y)$. So f is a homomorphism. This completes the proof. ∎

2.1 Homomorphisms

Homomorphism of machines play a central role in problems of machine decomposition and in different hardware and software implementations. In particular, automorphism are applied in automata decomposition, characterization of different automata types, and investigations of structure of the characteristic semigroup of an automaton.

2.1.1 Definition

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$, over the same universal set U. A pair (f, g) of mappings $f : Q_1 \longrightarrow Q_2$ and $g : X_1 \longrightarrow X_2$ is called a homomorphism if

$$F_1(q, x, p) \subseteq F_2(f(q), g(x), f(p)), \text{ for all } p, q \in Q_1 \text{ and } x \in X_1.$$

It is written as $(f, g) : \Upsilon_1 \longrightarrow \Upsilon_2$.

The pair (f, g) is called a strong homomorphism if it satisfies

$$F_2(f(q), g(x), f(p)) = \bigcup \{F_1(q, x, r) \mid r \in Q_1,\ f(r) = f(p)\},$$

for all $p, q \in Q_1$ and $x \in X_1$.

A homomorphism (strong homomorphism) $(f, g) : \Upsilon_1 \longrightarrow \Upsilon_2$ is called an isomorphism (strong isomorphism) if f and g are both one-one and onto. It is written as $\Upsilon_1 \cong \Upsilon_2$.

If $X_1 = X_2$ and g is the identity map, then we simply write $f : \Upsilon_1 \longrightarrow \Upsilon_2$ and say that f is a homomorphism or strong homomorphism accordingly. If (f, g) is a strong homomorphism with f is one-one, then

$$F_1(q, x, p) = F_2(f(q), g(x), f(p)) \text{ for all } p, q \in Q_1 \text{ and } x \in X_1.$$

2.1.2 Lemma

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$, over the same universal set U. Let $(f, g) : \Upsilon_1 \longrightarrow \Upsilon_2$, be a strong homomorphism. If $F_2(f(q), g(x), f(r)) \neq \emptyset$ for all $q, r \in Q_1$ and $x \in X_1$, then there exists $t \in Q_1$ such that $F_1(q, x, t) \neq \emptyset$ and $f(t) = f(r)$.

Furthermore, if $f(q) = f(p)$ for all $p \in Q$, then

$$F_1(p, x, r) \subseteq \bigcup \{F_1(q, x, t) : t \in Q,\ f(t) = f(r)\}.$$

It can be written as $F_1(p, x, r) \subseteq \bigcup_{i \in I} F_1(q, x, t_i)$ such that $t_i \in Q$, and $f(t_i) = f(r)$ for all $i \in I$.

Proof. Let $p, q, r \in Q_1$, $x \in X_1$ and $F_2(f(q), g(x), f(r)) \neq \emptyset$.

Then, $\bigcup \{F_1(q,x,s) \mid s \in Q_1,\ f(r) = f(s)\} \neq \emptyset$. Since Q_1 is finite so there exist $t \in Q$, such that $f(t) = f(r)$ and $F_1(q,x,t) \neq \emptyset$. Suppose $f(q) = f(p)$ for all $p \in Q$. Then,

$$\begin{aligned} F_1(p,x,r) &\subseteq \bigcup \{F_1(p,x,s) \mid s \in Q_1,\ f(s) = f(r)\} \\ &= F_2(f(p), g(x), f(r)) \\ &= F_2(f(q), g(x), f(r)) \\ &= \bigcup \{F_1(q,x,t) \mid t \in Q_1,\ f(t) = f(r)\} \end{aligned}$$

$\Longrightarrow F_1(p,x,r) \subseteq \bigcup \{F_1(q,x,t) \mid t \in Q_1,\ f(t) = f(r)\}$. ■

2.1.3 Definition

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$, over the same universal set U. Let $(f,g) : \Upsilon_1 \longrightarrow \Upsilon_2$, be a homomorphism. A mapping $g : X_1 \longrightarrow X_2$ can be extended to $g : X_1^* \longrightarrow X_2^*$ by $g^*(x) = g(x_1)\, g(x_2) .....g(x_n)$, where $x = x_1 x_2 ....x_n$ and $x_j \in X_1$, $j = 1, 2, ..., n$, and $g(\lambda) = \lambda$.

2.1.4 Lemma

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$. Let $(f,g) : \Upsilon_1 \longrightarrow \Upsilon_2$, be a homomorphism. Then, $g^*(xy) = g^*(x)\, g^*(y)$ for all $x, y \in X_1^*$.

Proof. Let $x, y \in X_1^*$ and $|y| = n$. If $n = 0$, Then, $y = \lambda$ and hence $g^*(xy) = g^*(x) = g^*(x)\, g^*(y)$.

Suppose now the result is true for all $y \in X_1^*$ such that $|y| = n-1$, $n > 0$. Let $u = ya$ where $y \in X_1^*$, $a \in X_1$ and $|y| = n - 1$. Then, $g^*(xu) = g^*(xya) = g^*(xy)\, g(a) = g^*(x)\, g^*(y)\, g(a) = g^*(x)\, g^*(ya) = g^*(x)\, g^*(u)$. The result now follows by induction. ■

2.1.5 Proposition

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$, over the same universal set U. Let $(f,g) : \Upsilon_1 \longrightarrow \Upsilon_2$, be a homomorphism. Then,

$$F_1^*(q,x,p) \subseteq F_2^*(f(q), g^*(x), f(p)) \text{ for all } p, q \in Q_1 \text{ and } x \in X_1^*.$$

Proof. Let $p, q \in Q_1$ and $x \in X_1^*$. We prove the result by induction on $|x| = n$. If $n = 0$, then $x = \lambda$ so $g^*(\lambda) = \lambda$.

If $p = q$, then $F_1^*(q,x,p) = F_1^*(q,\lambda,p) = U = F_2^*(f(q), \lambda, f(p)) = F_2^*(f(q), g^*(x), f(p))$.

If $p \neq q$, then $F_1^*(q,x,p) = F_1^*(q,\lambda,p) = \emptyset \subseteq F_2^*(f(q), g^*(x), f(p))$.

Hence the result is true for $n = 0$. Suppose that the result is true for all $y \in X_1^*$ such that $|y| = n-1$, $n > 0$. Let $x = ya$, where $y \in X_1^*$, $a \in X_1$ and $|y| = n-1$. Then,

$$\begin{aligned}
F_1^*(q, x, p) &= F_1^*(q, ya, p) \\
&= \bigcup_{r \in Q_1} \{F_1^*(q, y, r) \cap F_1^*(r, a, p)\} \\
&\subseteq \bigcup_{r \in Q_1} \{F_2^*(f(q), g^*(y), f(r)) \cap F_2^*(f(r), g(a), f(p))\} \\
&\subseteq \bigcup_{r' \in Q_2} \left\{F_2^*(f(q), g^*(y), r') \cap F_2^*(r', g(a), f(p))\right\} \\
&= F_2^*(f(q), g^*(y) g(a), f(p)) \\
&= F_2^*(f(q), g^*(ya), f(p)) \\
&= F_2^*(f(q), g^*(x), f(p)).
\end{aligned}$$

This completes the proof. ■

2.1.6 Theorem

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$, over the same universal set U. Let $(f, g) : \Upsilon_1 \longrightarrow \Upsilon_2$, be a strong homomorphism. Then, f is one-one, if and only if

$$F_1^*(q, x, p) = F_2^*(f(q), g^*(x), f(p)) \text{ for all } p, q \in Q_1 \text{ and } x \in X_1^*.$$

Proof. Suppose f is one-one. Let $p, q \in Q_1$ and $x \in X_1^*$. We prove the result by induction on $|x| = n$. If $n = 0$, then $x = \lambda$ so $g^*(\lambda) = \lambda$.

Since $p = q$ iff $f(p) = f(q)$, then $F_1^*(q, x, p) = U = F_2^*(f(q), g^*(x), f(p))$.

Suppose that the result is true for all $y \in X_1^*$ such that $|y| = n-1$, $n > 0$. Let $x = ya$ where $y \in X_1^*$, $a \in X_1$ and $|y| = n-1$. Then,

$$\begin{aligned}
F_2^*(f(q), g^*(x), f(p)) &= F_2^*(f(q), g^*(ya), f(p)) \\
&= F_2^*(f(q), g^*(y) g(a), f(p)) \\
&= \bigcup_{r \in Q_1} \{F_2^*(f(q), g^*(y), f(r)) \cap F_2(f(r), g(a), f(p))\} \\
&= \bigcup_{r \in Q_1} \{F_1^*(q, y, r) \cap F_1(r, a, p)\} \\
&= F_1^*(q, ya, p) \\
&= F_1^*(q, x, p).
\end{aligned}$$

Conversely, let $p, q \in Q_1$ and let $f(p) = f(q)$. Then, $F_2^*(f(q), \lambda, f(p)) = F_1^*(q, \lambda, p) = U$. Hence, $q = p$, i.e., f is one-one. This completes the proof. ■

2.2 Soft Admissible Relations

In this section, we define soft admissible relation for soft finite state machines. Soft admissible relation naturally lead to a quotient structure of soft finite state machines. These admissible partitions appeared in the literature under several names, out-licit equivalences, quotient, covering [2, 3], simulation [4], bisimulation [5] and block-stochastic matrices [15].

2.2.1 Definition

Let $\Upsilon = (Q, X, F, U)$ be a SFSM and "$\sim$" be an equivalence relation on Q. Then, "$\sim$" is called a soft admissible relation if it satisfies,

$$\begin{aligned} p &\sim q \text{ and } F(p,a,r) \neq \emptyset \text{ then there exist } t_i \in Q \text{ for } i = 1,2,...n \text{ such that} \\ t_i &\sim r \text{ for each } i \text{ and } \bigcup_{i=1}^{n} F(q,a,t_i) \supseteq F(p,a,r) \text{ for all } p,q,r \in Q \text{ and } a \in X. \end{aligned}$$

2.2.2 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a SFSM and let "$\sim$" be an equivalence relation on Q. Then ,"$\sim$" is a soft admissible relation if and only if it satisfies,

$$\begin{aligned} p &\sim q \text{ and } F^*(p,x,r) \neq \emptyset, \text{ then there exist } t_i \in Q \text{ for } i = 1,2,...n \text{ such that} \\ t_i &\sim r \text{ for each } i \text{ and } \bigcup_{i=1}^{n} F^*(q,x,t_i) \supseteq F^*(p,x,r) \text{ for all } p,q,r \in Q \text{ and } x \in X^*. \end{aligned}$$

Proof. Suppose "$\sim$" is a soft admissible relation on Q. Let $p, q, r \in Q$ and $x \in X^*$ be such that $p \sim q$ and $F^*(p,x,r) \neq \emptyset$.

Suppose $|x| = n$, if $n = 0$, then $x = \lambda$. Thus, $F^*(p,\lambda,r) = F^*(p,x,r) = U$

$\Longrightarrow$ $p = r$ and $F^*(p,x,r) = U$. If we take $t = q$, then $t \sim r$ and $F^*(q,x,t) = F^*(q,\lambda,q) = U$

$\Longrightarrow$ $F^*(q,x,t) \supseteq F^*(p,x,r)$ and $t \sim r$.

Thus, the result is valid for $n = 0$

Suppose the result is true for all $y \in X^*$ with $|y| = n-1$, $n > 0$. Let $x = ya$, where $a \in X$. Then,

$$F^*(p,x,r) = F^*(p,ya,r) = \bigcup_{q_1 \in Q} [F^*(p,x,q_1) \cap F(q_1,a,r)] \neq \emptyset.$$

Let $s \in Q$ be such that $F^*(p,x,s) \cap F(s,a,r) \neq \emptyset$. It follows from induction hypothesis that there exist $t_{s_i} \in Q$ such that $t_{s_i} \sim s$ for each i and $\bigcup_{i=1}^{n} F^*(q,y,t_{s_i}) \supseteq F^*(p,y,s)$.

Now, $F^*(s,a,r) \neq \emptyset$ and $t_{s_i} \sim s$. Since "$\sim$" is soft admissible relation on Q, then there exist $t_{j_i} \in Q$ such that $t_{j_i} \sim r$ for each j_i and $\bigcup_{j_i=1}^{m_i} F(t_{s_i}, a, t_{j_i}) \supseteq F(s,a,r)$ for all $i = 1, 2, ...n$.

Thus,

$$\begin{aligned} F^*(p,x,r) &= \bigcup_{s\in Q} \left[F^*(p,y,s) \cap F(s,a,r)\right] \\ &\subseteq \bigcup_{s\in Q} \left[\left[\bigcup_{i=1}^{n} F^*(q,y,t_{s_i})\right] \cap \left[\bigcup_{j_i=1}^{m_i} F(t_{s_i},a,t_{j_i})\right]\right] \\ &\subseteq \bigcup_{\substack{1\leq i\leq n \\ 1\leq j_i\leq m_i}} \bigcup_{t_{s_i}\in Q} \left[F^*(q,y,t_{s_i}) \cap F(t_{s_i},a,t_{j_i})\right] \\ &= \bigcup_{\substack{1\leq i\leq n \\ 1\leq j_i\leq m_i}} F^*(q,x,t_{j_i}). \end{aligned}$$

So $t_{j_i} \sim r$ for each t_{j_i} and $\bigcup_{\substack{1\leq i\leq n \\ 1\leq j_i\leq m_i}} F^*(q,x,t_{j_i}) \supseteq F^*(p,x,r)$. Therefore, the result follows by induction.

The converse is trivial. ■

Let $\Upsilon = (Q, X, F, U)$ be a SFSM and let "$\sim$" be a soft admissible relation on Q. For $q \in Q$, let $[q]$ denote the equivalence class of q. Let $\widetilde{Q} = \{[q] : q \in Q\}$ and define soft set $\widetilde{F}$ in $\widetilde{Q} \times X \times \widetilde{Q}$ by

$$\widetilde{F}([q], x, [p]) = \bigcup_{t\in[p]} F(q,x,t) \text{ for all } p, q \in Q \text{ and } a \in X.$$

Suppose that $[q] = [q']$, $x = y$ and $[p] = [p']$. Then, $q \sim q'$. Now, $\widetilde{F}([q], x, [p]) = \bigcup_{t\in[p]} F(q,x,t)$. Let $r \in [p]$ be such that $F^*(q,x,r) \neq \emptyset$ and "$\sim$" is soft admissible.
Then, there exist $t_i \in Q$ such that $t_i \sim r$ and $\bigcup_{i\in I} F(q',x,t_i) \supseteq F(q,x,r)$.

Now, $t_i \sim r \in [p] \implies t_i \in [p] = [p']$, similarly if $F(q',x,t) \neq \emptyset$, then there exist $r_j \in Q$ such that $r_j \sim t$ and $\bigcup_{j\in J} F(q,x,r_j) \supseteq F(q',x,t)$

$\implies \widetilde{F}([q], x, [p]) = \widetilde{F}([q'], x, [p'])$. This shows that $\widetilde{F}$ is single valued. Therefore, $\left(\widetilde{Q}, X, \widetilde{F}, U\right)$ is SFSM. Now define $\underline{f}$: $Q \longrightarrow \widetilde{Q}$ by $\underline{f}(q) = [q]$ for all $q \in Q$. Clearly, $\underline{f}$ is well defined onto map. Let $g : X \longrightarrow X$ be the identity map. Let $q, t \in Q$ and $x \in X$. Then,

$$\widetilde{F}\left(f(q), x, \underline{f}(t)\right) = \widetilde{F}([q], x, [t]) = \bigcup_{r\in[t]} F(q,x,r) \supseteq F(q,x,t).$$

Hence, $\left(\underline{f}, g\right)$ is a homomorphism.

2.2.3 Definition

Let $\Upsilon_i = (Q_i, X, F_i, U)$ be SFSMs, $i = 1, 2$ and let $f : \Upsilon_1 \longrightarrow \Upsilon_2$ be a strong homomorphism. Then, the kernel of f, denoted Kerf, is defined to be the set

$$Kerf = \{(p, q) \mid f(p) = f(q)\}.$$

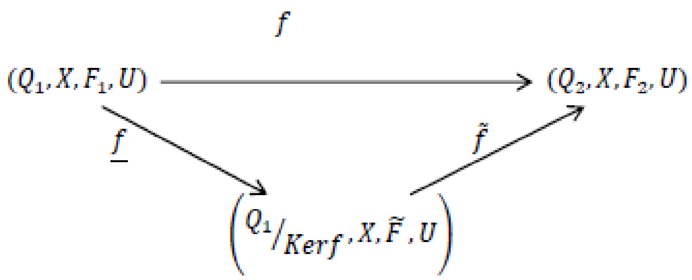
2.2.4 Lemma

Let $\Upsilon_i = (Q_i, X, F_i, U)$ be SFSMs, $i = 1, 2$ and let $f : \Upsilon_1 \longrightarrow \Upsilon_2$ be a strong homomorphism. Then, the $Kerf$, is a soft admissible relation on Q_1.

Proof. Clearly $Kerf$ is an equivalence relation on Q_1. Let $p, q \in Q_1$ and $(p, q) \in Kerf$. Then, $f(p) = f(q)$. Let $a \in X, r \in Q_1$, and $F_1(p, a, r) \neq \emptyset$. Then, $F_2(f(q), a, f(r)) = F_2(f(p), a, f(r)) \supseteq F_1(p, a, r) \neq \emptyset$. By Lemma 2.1.2, there exist $t_i \in Q_1$, $i = 1, 2, ..., n$ such that $F_1(p, x, r) \subseteq \bigcup_{i=1}^{n} F_1(q, x, t_i)$ and $f(t_i) = f(r)$. Since $f(t_i) = f(r)$, $(t_i, r) \in Kerf$ for all $i = 1, 2..., n$. Thus, $Kerf$ is admissible. ■

2.2.5 Theorem

Let $\Upsilon_i = (Q_i, X, F_i, U)$ be SFSMs, $i = 1, 2$ and let $f : \Upsilon_1 \longrightarrow \Upsilon_2$ be an onto strong homomorphism. Then, there exists an isomorphism $\widetilde{f} : \left(Q_1/Kerf, X, \widetilde{F}, U\right) \longrightarrow (Q_2, X, F_2, U)$ such that the following diagram commutes.



Proof. Define $\widetilde{f} : Q_1/Kerf \longrightarrow Q_2$ by $\widetilde{f}([q]) = f(q)$. Let $p, q \in Q_1$ be such that $[p] = [q]$. Then, $(p, q) \in Kerf$, and hence, $f(p) = f(q)$ or $\widetilde{f}([p]) = \widetilde{f}([q])$. Thus, $\widetilde{f}$ is well defined. Now, let $q, p \in Q_1$ and $x \in X$. Then,

$$\begin{aligned} \widetilde{F}([q], x, [p]) &= \bigcup_{t \in [p]} F_1(q, x, t) \\ &= \bigcup \{F_1(q, x, t) \mid f(t) = f(p),\ t \in Q_1\} \\ &= F_2(f(q), x, f(p)) \\ &= F_2\left(\widetilde{f}([q]), x, \widetilde{f}([p])\right). \end{aligned}$$

Thus, $\widetilde{f}$ is a homomorphism. Obviously, $\widetilde{f}$ is one-to-one and onto, and $\widetilde{f} \circ \underline{f} = f$. This completes the proof. ■

2.3 Soft Transformation Semigroups

The concept of transformation semigroups has played an important role in the theory of finite automata [54]. In this section, we define the concepts of soft transformation semigroups, faithful soft transformation semigroups and related properties discussed.

2.3.1 Definition

A soft transformation semigroup (STS) is a quadruple (Q, S, F_s, U) in which Q is finite non-empty set, S is a finite semigroup and F_s is a soft set of $Q \times S \times Q$ over U such that

(i) $F_s(q, xy, p) = \bigcup_{r \in Q} \{F_s(q, x, r) \cap F_s(r, y, p)\}$

(ii) If S is a semigroup with identity e, then

$$F_s(q, e, p) = \begin{cases} U & \text{if } q = p \\ \emptyset & \text{otherwise} \end{cases}$$

If, in addition, the following property holds, then (Q, S, F_s, U) is called faithful.

(iii) Let $x, y \in S$, if $F_s(q, x, p) = F_s(q, y, p)$ for all $q, p \in Q$, then $x = y$

Let $\Upsilon = (Q, S, F_s, U)$ be a soft transformation semigroup. This, STS may not be faithful. Define a relation R on S, xRy if and only if $F_s(q, x, p) = F_s(q, y, p)$ for all $q, p \in Q$ and for all $x, y \in S$. Clearly, R is an equivalence relation on S. Suppose that $x, y, z \in S$ and xRy. Then,

$$\begin{aligned} F_s(q, xz, p) &= \bigcup_{r \in Q} \{F_s(q, x, r) \cap F_s(r, z, p)\} \\ &= \bigcup_{r \in Q} \{F_s(q, y, r) \cap F_s(r, z, p)\} \\ &= F_s(q, yz, p) \end{aligned}$$

for all $q, p \in Q$. Similarly, we can show that, $F_s(q, zx, p) = F_s(q, zy, p)$ for all $q, p \in Q$. Hence, R is an congruence relation on S. Let $[x]$ denote the equivalence class of R. Let $S/R = \{[x] \mid x \in S\}$. Define $\overline{F_s} : Q \times S/R \times Q \longrightarrow P(U)$ by $\overline{F_s}(q, [x], p) = F_s(q, x, p)$ for all $q, p \in Q$ and $[x] \in S/R$. Clearly, $\overline{F_s}$ is single valued. Now,

$$\begin{aligned} \overline{F_s}(q, [x][y], p) &= \overline{F_s}(q, [xy], p) \\ &= F_s(q, xy, p) \\ &= \bigcup_{r \in Q} \{F_s(q, x, r) \cap F_s(r, y, p)\} \\ &= \bigcup_{r \in Q} \{\overline{F_s}(q, [x], r) \cap \overline{F_s}(r, [y], p)\} \end{aligned}$$

also

$$\overline{F_s}(q,[e],p)=\begin{cases} U & \text{if } q=p \\ \emptyset & \text{otherwise.}\end{cases}$$

Suppose that $\overline{F_s}(q,[x],p)=\overline{F_s}(q,[y],p)$ for all $q,p\in Q$. Then, $F_s(q,x,p)=F_s(q,y,p)$ for all $q,p\in Q$. Hence xRy and so $[x]=[y]$. Thus, $\left(Q,S/R,\overline{F_s},U\right)$ is a faithful STS. We call $\left(Q,S/R,\overline{F_s},U\right)$ the faithful soft transformation semigroup represented by the quadruple (Q,S,F,U).

2.3.2 Theorem

Let $\Upsilon=(Q,X,F,U)$ be a SFSM. Then, $\left(Q,S(\Upsilon),F_{S(\Upsilon)},U\right)$ is a faithful soft transformation semigroup, where

$$F_{S(\Upsilon)}(p,[x],q)=F^*(p,x,q) \text{ for all } p,q\in Q \text{ and } x\in X^*.$$

Proof. By Theorem 2.0.17, $(S(\Upsilon),\circledast)$ is a finite semigroup with identity $[\lambda]$. Clearly, $F_{S(\Upsilon)}$ is single valued. Let $[x],[y]\in S(\Upsilon)$ and $q,p\in Q$. Then,

$$\begin{aligned} F_{S(\Upsilon)}(p,[x]\circledast[y],q) &= F_{S(\Upsilon)}(p,[xy],q) \\ &= F^*(p,xy,q) \\ &= \bigcup_{r\in Q}\{F^*(p,x,r)\cap F^*(r,y,q)\} \\ &= \bigcup_{r\in Q}\left\{F_{S(\Upsilon)}(p,[x],r)\cap F_{S(\Upsilon)}(r,[y],q)\right\}. \end{aligned}$$

Now,

$$F_{S(\Upsilon)}(p,[\lambda],q)=\begin{cases} F^*(p,\lambda,q)=U & \text{if } q=p \\ F^*(p,\lambda,q)=\emptyset & \text{otherwise.}\end{cases}$$

Suppose $F_{S(\Upsilon)}(p,[x],p)=F_{S(\Upsilon)}(q,[y],p)$ for all $q,p\in Q$. Then, $F^*(p,x,q)=F^*(p,y,q)$ for all $q,p\in Q$. Thus, $x\cong y$ or $[x]=[y]$. Hence, $\left(Q,S(\Upsilon),F_{S(\Upsilon)},U\right)$ is a faithful soft transformation semigroup. ■

Let $\Upsilon=(Q,X,F,U)$ be a SFSM. Then, by Theorem 2.3.2, $\left(Q,S(\Upsilon),F_{S(\Upsilon)},U\right)$ is a soft transformation semigroup and it is denoted by STS(Υ), we call STS(Υ) the soft transformation semigroup associated with Υ.

Let $\Upsilon=(Q,X,F,U)$ be a SFSM. Define the relation "$\cong^+$" on X^+ by $x\cong^+ y$ if and only if $F^+(q,x,p)=F^+(q,y,p)$ for all $p,q\in Q$. Then, "$\cong^+$" is a restriction of $\cong$ to $X^+\times X^+$. Let $S(\Upsilon)^+$ denote the set of all equivalence classes induced by $\cong^+$. Then, $S(\Upsilon)^+=S(\Upsilon)-\{[\lambda]\}$ and $S(\Upsilon)^+$ is a subsemigroup of $S(\Upsilon)$. So, we can define $\left(Q,S(\Upsilon)^+,F_{S(\Upsilon)^+},U\right)$ as a soft transformation semigroup associated with Υ.

2.3.3 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a SFSM and STS(Υ)=$\left(Q, S(\Upsilon), F_{S(\Upsilon)}, U\right)$ associated with Υ. If $f : Q \to Q$ is the identity map and $g : X \to$S(Υ) by $g(x) = [x]$ for all $x \in X$. Then, (f, g) is a strong homomorphism.

Proof. Let for any $p, q \in Q$ and $x \in X$. Then,
$F(q, x, p) = F^*(q, x, p) = F_{S(\Upsilon)}(q, [x], p)$
$\implies F(q, x, p) = F_{S(\Upsilon)}(f(q), [x], f(p))$. ■

2.3.4 Definition

Let (Q, S, F_s, U) be a STS. Let "$\sim$" be an equivalence relation on Q. Then, "$\sim$" is called a soft admissible relation if and only if,

$$\begin{aligned} p &\sim q \text{ and } F_s(p, a, r) \neq \emptyset \text{ then there exist } t_i \in Q \text{ for } i = 1, 2, ...n \text{ such that} \\ t_i &\sim r \text{ for each } i \text{ and } \bigcup_{i=1}^{n} F_s(q, a, t_i) \supseteq F_s(p, a, r) \text{ for all } p, q, r \in Q \text{ and } a \in S. \end{aligned}$$

2.3.5 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a SFSM and $\sim$ be an equivalence relation on Q. Then, "$\sim$" is soft admissible relation for Υ if and only if $\sim$ is soft admissible relation for the soft transformation semigroup, STS(Υ) $= \left(Q, S(\Upsilon), F_{S(\Upsilon)}, U\right)$.

Proof. Suppose "$\sim$" is a soft admissible relation for Υ. Let $p, q \in Q$ be such that $p \sim q$ and $[x] \in S(\Upsilon)$. Let $F_{S(\Upsilon)}(p, [x], r) \neq \emptyset$ for some $r \in Q$. Then, $F^*(p, x, r) \neq \emptyset$. Hence, by Theorem 2.2.2, there exist $t_i \in Q$ for $i = 1, 2, ...n$ such that

$$t_i \sim r \text{ for each } i \text{ and } \bigcup_{i=1}^{n} F^*(q, x, t_i) \supseteq F^*(p, x, r) \text{ for all } p, q, r \in Q \text{ and } x \in X^*.$$

Thus,

$$\bigcup_{i=1}^{n} F_{S(\Upsilon)}(q, [x], t_i) = \bigcup_{i=1}^{n} F^*(q, x, t_i) \supseteq F^*(p, x, r) = F_{S(\Upsilon)}(p, [x], r).$$

Hence, $\sim$ is soft admissible relation for STS(Υ) $= \left(Q, S(\Upsilon), F_{S(\Upsilon)}, U\right)$.

Conversely, let $p, q \in Q$ be such that $p \sim q$ and $a \in X$. Let $F^*(p, a, r) \neq \emptyset$ for some $r \in Q$. Then, $F_{S(\Upsilon)}(p, [x], r) \neq \emptyset$. Then, there exist $t_i \in Q$ for $i = 1, 2, ...n$ such that

$$t_i \sim r \text{ for each } i \text{ and } \bigcup_{i=1}^{n} F_{S(\Upsilon)}(q, [a], t_i) \supseteq F_{S(\Upsilon)}(p, [a], r)$$

Now,

$$\bigcup_{i=1}^{n} F^* (q, a, t_i) = \bigcup_{i=1}^{n} F_{S(\Upsilon)} (q, [a], t_i) \supseteq F_{S(\Upsilon)} (p, [a], r) = F^* (p, a, r) \text{ and } t_i \sim r.$$

Hence, "$\sim$" is a soft admissible for Υ. ∎

2.3.6 Definition

Let (Q_1, S_1, F_{S_1}, U) and (Q_2, S_2, F_{S_2}, U) be two soft transformation semigroups. A pair (f, g) of mappings $f : Q_1 \longrightarrow Q_2$ and $g : S_1 \longrightarrow S_2$ is called a homomorphism from (Q_1, S_1, F_{S_1}, U) to (Q_2, S_2, F_{S_2}, U) if

(1) $g (xy) = g (x) g (y)$

(2) If e_1 is the identity of S_1 and e_2 is the identity of S_2, then $g (e_1) = g (e_2)$.

(3) $F_{S_1} (q, x, p) \subseteq F_{S_2} (f (q), g (x), f (p))$ for all $q, p \in Q_1$, $x \in S_1$.

(f, g) is called a strong homomorphism if it satisfies (1), (2) and

$$F_{S_2} (f (q), g (x), f (p)) = \bigcup \{F_{S_1} (q, x, t) \mid t \in Q_1,\ f (t) = f (p)\},$$

for all $q, p \in Q_1$, $x \in S_1$.

A homomorphism (strong homomorphism) $(f, g) : (Q_1, S_1, F_{S_1}, U) \longrightarrow (Q_2, S_2, F_{S_2}, U)$ is called an isomorphism if f and g are both one-one and onto.

2.3.7 Theorem

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$. Let $(f, g) : \Upsilon_1 \longrightarrow \Upsilon_2$, be a strong homomorphism with f is one-one and onto. Then, there exists a strong homomorphism (α_f, β_g) from STS(Υ_1) to STS(Υ_2).

Proof. Define $\alpha_f : Q_1 \longrightarrow Q_2$ by $\alpha_f (q) = f (q)$ for all $q \in Q_1$ and let $\beta_g : S (\Upsilon_1) \longrightarrow S (\Upsilon_2)$ by $\beta_g ([x]) = [g^* (x)]$ for all $[x] \in S (\Upsilon_1)$. Let $[x], [y] \in S (\Upsilon_1)$ and $[x] = [y]$. $F_1^* (q, x, p) = F_1^* (q, y, p)$ for all $q, p \in Q_1$. Now,

$$\begin{aligned} F_2^* (f (q), g^* (x), f (p)) &= F_1^* (q, x, p) \\ &= F_1^* (q, y, p) \\ &= F_2^* (f (q), g^* (y), f (p)), \end{aligned}$$

for all $p, q \in Q_1$. Thus, $[g^* (x)] = [g^* (y)]$. Hence β_g is well defined. Now, $\beta_g ([x] \circledast [y]) = \beta_g ([xy]) = [g^* (xy)] = [g^* (x) g^* (y)] = [g^* (x)] \circledast [g^* (y)] = \beta_g ([x]) \circledast \beta_g ([y])$ and $\beta_g ([\lambda]) = [g^* (\lambda)] = [\lambda]$. Also,

$$\begin{aligned} F_{S(\Upsilon_1)} (q, [x], p) &= F_1^* (q, x, p) \\ &= F_2^* (f (q), g^* (x), f (p)) \\ &= F_{S(\Upsilon_2)} (\alpha_f (q), \beta_g ([x]), \alpha_f (p)). \end{aligned}$$

Hence, by definition $\left(\alpha_f, \beta_g\right)$ is a strong homomorphism. ■

Chapter 3

Coverings and Products of Soft Finite State Machines

The concept of, covering, cascade product, and wreath product play an important role in the study of automata and their associated semigroups [49]. In this chapter, we examine these concepts for soft finite state machines.

3.1 Covering and General Product Of Soft Finite State Machines

We have seen in Theorem 2.3.2 that "each soft finite state machine $\Upsilon = (Q, X, F, U)$ has an associated soft transformation semigroup STS(Υ) $= \left(Q, S(\Upsilon), F_{S(\Upsilon)}, U\right)$". In this section, we defined the concept of covering, general direct product, full and restricted direct product of soft finite state machines. Using these concepts, we will show that full (restricted) direct product of soft transformation semigroups associated with machines cover by soft transformation semigroup associated with the full (restricted) direct product of soft machines.

3.1.1 Definition

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$. Let η be a function from Q_2 onto Q_1 and let ζ be a function from X_1 into X_2. Extend ζ to a function ζ^* of X_1^* into X_2^* by $\zeta^*(x) = \zeta(x_1)\zeta(x_2)\zeta(x_3)....\zeta(x_n)$ where, $x = x_1x_2x_3...x_n$ and $x_i \in X_1$, $i = 1, 2, ..., n$ and $\zeta^*(\lambda) = \lambda$. Then (η, ζ) is called a covering of Υ_1 by Υ_2, written $\Upsilon_1 \leq \Upsilon_2$, if and only if $\forall q_2 \in Q_2$, $q_1 \in Q_1$ and $x \in X_1^*$,

$$F_1^*(\eta(q_2), x, q_1) = \bigcup \{F_2^*(q_2, \zeta^*(x), r_2) \mid \eta(r_2) = q_1,\ r_2 \in Q_2\}.$$

Clearly, (η, ζ) is a covering of Υ_1 by Υ_2 if, and only if, for all $q_2 \in Q_2$, $q_1 \in Q_1$, and $x \in X_1^*$, $F_1^*(\eta(q_2), x, q_1) \supseteq F_2^*(q_2, \zeta^*(x), r_2)$ such that $\eta(r_2) = q_1$ and there exist $r_i \in Q_2$ for $i = 1, 2, .., n$ such that $\eta(r_i) = q_1$ for all i and $F_1^*(\eta(q_2), x, q_1) = \bigcup_{i=1}^{n} F_2^*(q_2, \zeta^*(x), r_i)$.

3.1.2 Example

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$, such that $Q_1 = \{q_1, q_2\}$, $X_1 = \{a, b\}$, $Q_2 = \{q_1', q_2', q_3'\}$, $X_2 = \{a, b\}$ and $U = \{1, 2, 3, 4\}$. Then $\Upsilon_1 = (Q_1, X_1, F_1, U)$ and $\Upsilon_2 = (Q_2, X_2, F_2, U)$ are described as, respectively

	q_1	q_2
(q_1, a)	$\{1, 2\}$	$\{1, 4\}$
(q_1, b)	$\emptyset$	$\{1, 2, 3\}$
(q_2, a)	$\emptyset$	$\{1, 2\}$
(q_2, b)	$\{1, 2, 3\}$	$\{3, 4\}$

The above table can be read as, the image of (q_1, b, q_2) under F_1 is $\{1, 2, 3\}$.

	q_1'	q_2'	q_3'
(q_1', a)	$\emptyset$	$\{1, 4\}$	$\{1, 2\}$
(q_1', b)	$\emptyset$	$\{1, 2, 3\}$	$\emptyset$
(q_2', a)	$\emptyset$	$\{1, 2\}$	$\emptyset$
(q_2', b)	$\emptyset$	$\{3, 4\}$	$\{1, 2, 3\}$
(q_3', a)	$\{1, 2\}$	$\emptyset$	$\emptyset$
(q_3', b)	$\emptyset$	$\{2\}$	$\emptyset$

The above table can be read as, the image of (q_1', a, q_2') under F_2 is $\{1, 4\}$. Define $\eta : Q_2 \longrightarrow Q_1$ by $\eta(q_1') = \eta(q_3') = q_1$ and $\eta(q_2') = q_2$. Let ζ be the identity map on $X_1 \longrightarrow X_2$. Now, for all $x \in X_1^*$

$$F_1^*\left(\eta\left(q_1'\right), x, q_1\right) = \bigcup \left\{F_2^*\left(q_1', x, r_2\right) \mid \eta(r_2) = q_1,\ r_2 \in Q_2\right\}$$

$$\Longrightarrow F_1^*(q_1, x, q_1) = F_2^*\left(q_1', x, q_1'\right) \cup F_2^*\left(q_1', x, q_3'\right).$$

Similarly, we can write

$$\begin{aligned} F_1^*(q_1, x, q_2) &= F_2^*\left(q_1', x, q_2'\right) \\ F_1^*(q_2, x, q_1) &= F_2^*\left(q_2', x, q_1'\right) \cup F_2^*\left(q_2', x, q_3'\right) \\ F_1^*(q_2, x, q_2) &= F_2^*\left(q_2', x, q_2'\right). \end{aligned}$$

Thus, (η, ζ) is a covering of Υ_1 by Υ_2.

3.1.3 Definition

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$. Let $\overline{X}$ be a finite set and f a function from $\overline{X}$ into $X_1 \times X_2$. Let π_i be the projection map of $X_1 \times X_2$ onto X_i, $i = 1, 2$, respectively.

Then, $\Upsilon_1 * \Upsilon_2 = \left(Q_1 \times Q_2, \overline{X}, F_f, U\right)$ is called the general direct product of Υ_1 and Υ_2. Where $F_f : (Q_1 \times Q_2) \times \overline{X} \times (Q_1 \times Q_2) \longrightarrow P(U)$ is defined as follows: for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$ and $a \in \overline{X}$,

$$F_f((q_1, q_2), a, (p_1, p_2)) = F_1(q_1, \pi_1(f(a)), p_1) \cap F_2(q_2, \pi_2(f(a)), p_2).$$

If $\overline{X} = X_1 \times X_2$ and f is the identity map, then $\Upsilon_1 * \Upsilon_2$ is called the full direct product of Υ_1 and Υ_2 and we write $\Upsilon_1 \times \Upsilon_2 = (Q_1 \times Q_2, X_1 \times X_2, F_1 \times F_2, U)$ for $\Upsilon_1 * \Upsilon_2$. Where $F_1 \times F_2 : (Q_1 \times Q_2) \times (X_1 \times X_2) \times (Q_1 \times Q_2) \longrightarrow P(U)$ is defined as follows: for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$ and $(a_1, a_2) \in X_1 \times X_2$,

$$F_1 \times F_2((q_1, q_2), (a_1, a_2), (p_1, p_2)) = F_1(q_1, a_1, p_1) \cap F_2(q_2, a_2, p_2).$$

If $X_1 = X_2$, $\overline{X} = \{(a, a) \mid a \in X_1\}$ and f is the identity map from $\overline{X}$ into X, where X is the set of all diagonal elements of $X_1 \times X_1$. Then, $\Upsilon_1 * \Upsilon_2$ is called the restricted direct product of Υ_1 and Υ_2 and we write $\Upsilon_1 \times_R \Upsilon_2 = \left(Q_1 \times Q_2, \overline{X}, F_1 \times_R F_2, U\right)$ for $\Upsilon_1 * \Upsilon_2$, where

$$F_1 \times_R F_2((q_1, q_2), (a_1, a_1), (p_1, p_2)) = F_1(q_1, a_1, p_1) \cap F_2(q_2, a_2, p_2).$$

(To obtain the restricted direct product we could also let $\overline{X} = X_1 = X_2$ and $f : \overline{X} \longrightarrow \{(x_1, x_2) \mid x_i \in X_i,\ i = 1, 2,\ x_1 = x_2\}$ where $f(x) = (x, x)$, and we write $\Upsilon_1 \times_R \Upsilon_2 = (Q_1 \times Q_2, X, F_1 \times_R F_2, U)$, where

$$F_1 \times_R F_2((q_1, q_2), a, (p_1, p_2)) = F_1(q_1, a, p_1) \cap F_2(q_2, a, p_2).)$$

For every result concerning $\Upsilon_1 * \Upsilon_2$, there is a corresponding result for $\Upsilon_1 \times_R \Upsilon_2$. We see this by making the identifications $(x, x) \longrightarrow x$ for all $x \in X_1 = X_2$ and $(x_1, x_1) ... (x_n, x_n) \longrightarrow x_1...x_n$ for $x_i \in X_1$, $i = 1, 2..., n$.

3.1.4 Example

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be a SFSMs, $i = 1, 2$, where $Q_1 = \{p, q\}$, $X_1 = \{a\}$, $Q_2 = \{p', q'\}$, $X_2 = \{a\}$, $U = \{1, 2, 3\}$, and F_1 and F_2 are defined as follows

	p	q
(p, a)	$\emptyset$	$\{1, 3\}$
(q, a)	$\{1, 3\}$	$\emptyset$

	p'	q'
(p', a)	$\emptyset$	$\{2, 3\}$
(q', a)	$\emptyset$	$\{2, 3\}$

The above table can be read as the image of (p, a, q) under F_1 is $\{1, 3\}$ and image of (p', a, q') under F_2 is $\{2, 3\}$. Then, $\Upsilon_1 \times \Upsilon_2 = (Q_1 \times Q_2, X_1 \times X_2, F_1 \times F_2, U)$, where $F_1 \times F_2$ is defined as

$$\begin{aligned} F_1 \times F_2 \left(\left(p, p'\right), (a, a), \left(q, q'\right) \right) &= F_1 (p, a, q) \cap F_2 \left(p', a, q'\right) = \{3\} \\ F_1 \times F_2 \left(\left(q, p'\right), (a, a), \left(p, p'\right) \right) &= F_1 (q, a, p) \cap F_2 \left(p', a, p'\right) = \emptyset. \end{aligned}$$

Thus,

$\Upsilon_1 \times \Upsilon_2$

	(p, p')	(p, q')	(q, p')	(q, q')
$((p, p'), (a, a))$	$\emptyset$	$\emptyset$	$\emptyset$	$\{3\}$
$((p, q'), (a, a))$	$\emptyset$	$\emptyset$	$\emptyset$	$\{3\}$
$((q, p'), (a, a))$	$\emptyset$	$\{3\}$	$\emptyset$	$\emptyset$
$((q, q'), (a, a))$	$\emptyset$	$\{3\}$	$\emptyset$	$\emptyset$

The above table can be read as the image of $((p, p'), (a, a), (q, q'))$ under $F_1 \times F_2$ is $\{3\}$.

Now, $\cong_1$ is the congruence relation on Υ_1 defined in Theorem 2.0.16.

Note that $F_1^* (q, a, p) = \{1, 3\}$ and

$$\begin{aligned} F_1^* (q, aaa, p) &= F_1^* (q, aaa, p) = \bigcup_{r \in Q_1} \{F_1^* (q, aa, r) \cap F_1^* (r, a, p)\} \\ &= \{F_1^* (q, aa, q) \cap F_1^* (q, a, p)\} \cup \{F_1^* (q, aa, p) \cap F_1^* (p, a, p)\} \\ &= \{F_1^* (q, aa, q) \cap \{1, 3\}\} \cup \{F_1^* (q, aa, p) \cap \emptyset\} \\ &= F_1^* (q, aa, q) \cap \{1, 3\} \\ &= \bigcup_{r \in Q_1} \{F_1^* (q, a, r) \cap F_1^* (r, a, q)\} \cap \{1, 3\} \\ &= \left\{ \begin{array}{c} \{F_1^* (q, a, q) \cap F_1^* (q, a, q)\} \cup \\ \{F_1^* (q, a, p) \cap F_1^* (p, a, q)\} \end{array} \right\} \cap \{1, 3\} \\ &= \{\emptyset \cup \{1, 3\}\} \cap \{1, 3\} \\ &= \{1, 3\}. \end{aligned}$$

Thus, $F_1^* (q, a, p) = F_1^* (q, aaa, p)$ for all $q, p \in Q_1 \Longrightarrow a \cong_1 aaa$. Similarly, $aa \cong_1 aaaa$.

Hence, obviously, $x \cong_1 y$ if and only if length of x and y are both even or both odd, other then zero, for all $x, y \in X^*$ and $\lambda \cong_1 \lambda$. Hence, $S (\Upsilon_1) = \left\{ [\lambda], [a], \left[a^2\right] \right\}$.

For Υ_2, $\cong_2$ is the congruence relation on Υ_2 defined in Theorem 2.0.16.

Obviously, $x \cong_2 y$ for all $x, y \in X^*$ such that $|x|$, $|y| > 0$ and $\lambda \cong_2 \lambda$. Thus, $S (\Upsilon_2) = \{[\lambda], [a]\}$.

For $\Upsilon_1 \times \Upsilon_2$, $\cong$ is the congruence relation on $\Upsilon_1 \times \Upsilon_2$ defined in Theorem 2.0.16. In this case, $S (\Upsilon_1 \times \Upsilon_2) = \left\{ [\lambda], [(a, a)], \left[(a, a)^2\right] \right\}$.

3.1.5 Example

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$, where $Q_1 = \{p, q\}$, $X_1 = \{a\}$, $Q_2 = \{p', q'\}$, $X_2 = \{a, b\}$, $U = \{1, 2, 3\}$, and F_1 and F_2 are defined as follows

	p	q
(p, a)	$\emptyset$	$\{1, 2\}$
(q, a)	$\{1, 2\}$	$\emptyset$

	p'	q'
(p', a)	$\emptyset$	$\{2, 3\}$
(q', a)	$\emptyset$	$\{2, 3\}$
(p', b)	$\{2, 3\}$	$\emptyset$
(q', b)	$\{2, 3\}$	$\emptyset$

The above table can be read as the image of (p, a, q) under F_1 is $\{1, 2\}$ and image of (p', b, p') under F_2 is $\{2, 3\}$. Then,

$$\begin{aligned}
F_1 \times F_2 \left(\left(p, p'\right), (a, b), \left(q, q'\right)\right) &= F_1 (p, a, q) \cap F_2 \left(p', b, q'\right) = \{2\} \\
F_1 \times F_2 \left(\left(q, p'\right), (a, b), \left(p, p'\right)\right) &= F_1 (q, a, p) \cap F_2 \left(p', b, p'\right) = \{2\} \\
F_1 \times F_2 \left(\left(p, p'\right), (a, a), \left(q, q'\right)\right) &= F_1 (p, a, q) \cap F_2 \left(p', a, q'\right) = \{2\} \\
F_1 \times F_2 \left(\left(q, q'\right), (a, b), \left(p, p'\right)\right) &= F_1 (q, a, p) \cap F_2 \left(q', b, p'\right) = \{2\} \\
F_1 \times F_2 \left(\left(q, p'\right), (a, a), \left(p, q'\right)\right) &= F_1 (p, a, q) \cap F_2 \left(p', a, q'\right) = \{2\} \\
F_1 \times F_2 \left(\left(p, q'\right), (a, b), \left(q, p'\right)\right) &= F_1 (p, a, q) \cap F_2 \left(q', b, p'\right) = \{2\} \\
F_1 \times F_2 \left(\left(p, q'\right), (a, a), \left(q, q'\right)\right) &= F_1 (p, a, q) \cap F_2 \left(q', a, q'\right) = \{2\} \\
F_1 \times F_2 \left(\left(q, q'\right), (a, a), \left(p, q'\right)\right) &= F_1 (q, a, q) \cap F_2 \left(q', a, q'\right) = \{2\}.
\end{aligned}$$

Image of $F_1 \times F_2$ is empty set elsewhere. Then, $S(\Upsilon_1) = \left\{[\lambda], [a], \left[a^2\right]\right\}$ and $S(\Upsilon_1)^+ = \left\{[a], \left[a^2\right]\right\}$ is a group with identity $[\lambda]$ and $\left[a^2\right]$, respectively. $S(\Upsilon_2) = \{[\lambda], [a], [b]\}$, where $[a] = \left[a^2\right]$, $[b] = \left[b^2\right]$, $[ab] = [b]$ and $[ba] = [a]$. Thus, $S(\Upsilon_1)^+ = \left\{[a], \left[a^2\right]\right\}$ and $S(\Upsilon_2)^+ = \{[a], [b]\}$.

$S(\Upsilon_1)^+ \times S(\Upsilon_2)^+ = \left\{([a], [a]),\ ([a], [b]),\ \left(\left[a^2\right], [a]\right),\ \left(\left[a^2\right], [b]\right)\right\}$. It follows that $S(\Upsilon_1 \times \Upsilon_2)^+ = \left\{[(a, a)],\ \left[(a, a)^2\right],\ [(a, b)],\ \left[(a, b)^2\right]\right\}$. The operation tables of $S(\Upsilon_1)^+ \times S(\Upsilon_2)^+$ and $S(\Upsilon_1 \times \Upsilon_2)^+$ are given below.

$S(\Upsilon_1)^+ \times S(\Upsilon_2)^+$

	$([a],[a])$	$([a^2],[a])$	$([a],[b])$	$([a^2],[b])$
$([a],[a])$	$([a^2],[a])$	$([a],[a])$	$([a^2],[b])$	$([a],[b])$
$([a^2],[a])$	$([a],[a])$	$([a^2],[a])$	$([a],[b])$	$([a^2],[b])$
$([a],[b])$	$([a^2],[a])$	$([a],[a])$	$([a^2],[b])$	$([a],[b])$
$([a^2],[b])$	$([a],[a])$	$([a^2],[a])$	$([a],[b])$	$([a^2],[b])$

$S(\Upsilon_1 \times \Upsilon_2)^+$

	$[(a,a)]$	$[(a,a)^2]$	$[(a,b)]$	$[(a,b)^2]$
$[(a,a)]$	$[(a,a)^2]$	$[(a,a)]$	$[(a,b)^2]$	$[(a,b)]$
$[(a,a)^2]$	$[(a,a)]$	$[(a,a)^2]$	$[(a,b)]$	$[(a,b)^2]$
$[(a,b)]$	$[(a,a)^2]$	$[(a,a)]$	$[(a,b)^2]$	$[(a,b)]$
$[(a,b)^2]$	$[(a,a)]$	$[(a,a)^2]$	$[(a,b)]$	$[(a,b)^2]$

We see that $S(\Upsilon_1)^+ \times S(\Upsilon_2)^+ \cong S(\Upsilon_1 \times \Upsilon_2)^+$ under

$$\begin{aligned} ([a],[a]) &\longrightarrow [(a,a)], \ \ ([a^2],[a]) \longrightarrow [(a,a)^2], \\ ([a],[b]) &\longrightarrow [(a,b)], \ \ ([a^2],[b]) \longrightarrow [(a,b)^2]. \end{aligned}$$

3.1.6 Lemma

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$. Consider the general direct product $\Upsilon_1 * \Upsilon_2$. Then,

(*i*) $F_f^*((q_1,q_2), \lambda, (p_1,p_2)) = F_1^*(q_1, \lambda, p_1) \cap F_2^*(q_2, \lambda, p_2)$ for all $q_1, p_1 \in Q_1, q_2, p_2 \in Q_2$.

$$(ii)\ \ F_f^*((q_1,q_2), \overline{a_1}...\overline{a_n}, (p_1,p_2)) = \begin{cases} F_1^*(q_1, \pi_1(f(\overline{a_1}))...\pi_1(f(\overline{a_n})), p_1) \cap \\ F_2^*(q_2, \pi_2(f(\overline{a_1}))...\pi_2(f(\overline{a_n})), p_2), \end{cases}$$

where $\overline{a_i} \in \overline{X}, i = 1, 2, ...n$, for all $q_1, p_1 \in Q_1, q_2, p_2 \in Q_2$.

Proof. (i) The proof is straightforward.

(ii)

$$F_f^* \left((q_1, q_2), \overline{a_1} ... \overline{a_n}, (p_1, p_2) \right) = \bigcup_{\substack{\left(r_1^{(i)}, r_2^{(i)}\right) \in Q_1 \times Q_2 \\ i=1,2,...n-1}} \left\{ \begin{array}{c} F_f \left((q_1, q_2), \overline{a_1}, \left(r_1^{(1)}, r_2^{(1)}\right) \right) \cap \\ F_f \left(\left(r_1^{(1)}, r_2^{(1)}\right), \overline{a_2}, \left(r_1^{(2)}, r_2^{(2)}\right) \right) \cap ....\cap \\ F_f \left(\left(r_1^{(n-1)}, r_2^{(n-1)}\right), \overline{a_n}, (p_1, p_2) \right) \end{array} \right\}$$

$$= \bigcup_{\substack{\left(r_1^{(i)}, r_2^{(i)}\right) \in Q_1 \times Q_2 \\ i=1,2,...n-1}} \left\{ \begin{array}{c} F_1 \times F_2 \left((q_1, q_2), \left(\pi_1 \left(f\left(\overline{a_1}\right)\right), \pi_2 \left(f\left(\overline{a_1}\right)\right)\right), \left(r_1^{(1)}, r_2^{(1)}\right) \right) \cap \\ F_1 \times F_2 \left(\left(r_1^{(1)}, r_2^{(1)}\right), \left(\pi_1 \left(f\left(\overline{a_2}\right)\right), \pi_2 \left(f\left(\overline{a_2}\right)\right)\right), \left(r_1^{(2)}, r_2^{(2)}\right) \right) \cap ....\cap \\ F_1 \times F_2 \left(\left(r_1^{(n-1)}, r_2^{(n-1)}\right), \left(\pi_1 \left(f\left(\overline{a_n}\right)\right), \pi_2 \left(f\left(\overline{a_n}\right)\right)\right), (p_1, p_2) \right) \end{array} \right\}$$

$$= \bigcup_{\substack{r_1^{(i)} \in Q_1,\ r_2^{(i)} \in Q_2 \\ i=1,2,...n-1}} \left\{ \begin{array}{c} F_1 \left(q_1, \pi_1 \left(f\left(\overline{a_1}\right)\right), r_1^{(1)} \right) \cap F_2 \left(q_2, \pi_2 \left(f\left(\overline{a_1}\right)\right), r_2^{(1)} \right) \cap \\ F_1 \left(r_1^{(1)}, \pi_1 \left(f\left(\overline{a_2}\right)\right), r_1^{(2)} \right) \cap F_2 \left(r_2^{(1)}, \pi_2 \left(f\left(\overline{a_2}\right)\right), r_2^{(2)} \right) \cap ....\cap \\ F_1 \left(r_1^{(n-1)}, \pi_1 \left(f\left(\overline{a_n}\right)\right), p_1 \right) \cap F_2 \left(r_2^{(n-1)}, \pi_2 \left(f\left(\overline{a_n}\right)\right), p_2 \right) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \left\{ \bigcup_{\substack{r_1^{(i)} \in Q_1 \\ i=1,2,...n-1}} \left\{ \begin{array}{c} F_1 \left(q_1, \pi_1 \left(f\left(\overline{a_1}\right)\right), r_1^{(1)} \right) \cap F_1 \left(r_1^{(1)}, \pi_1 \left(f\left(\overline{a_2}\right)\right), r_1^{(2)} \right) \cap ....\cap \\ F_1 \left(r_1^{(n-1)}, \pi_1 \left(f\left(\overline{a_n}\right)\right), p_1 \right) \end{array} \right\} \right\} \bigcap \\ \left\{ \bigcup_{\substack{r_2^{(i)} \in Q_2 \\ i=1,2,...n-1}} \left\{ \begin{array}{c} F_2 \left(q_2, \pi_2 \left(f\left(\overline{a_1}\right)\right), r_2^{(1)} \right) \cap F_2 \left(r_2^{(1)}, \pi_2 \left(f\left(\overline{a_2}\right)\right), r_2^{(2)} \right) \cap ....\cap \\ F_2 \left(r_2^{(n-1)}, \pi_2 \left(f\left(\overline{a_n}\right)\right), p_2 \right) \end{array} \right\} \right\} \end{array} \right.$$

$$= \left\{ \begin{array}{l} F_1^* \left(q_1, \pi_1 \left(f\left(\overline{a_1}\right)\right) ... \pi_1 \left(f\left(\overline{a_n}\right)\right), p_1 \right) \cap \\ F_2^* \left(q_2, \pi_2 \left(f\left(\overline{a_1}\right)\right) ... \pi_2 \left(f\left(\overline{a_n}\right)\right), p_2 \right). \end{array} \right.$$

■

3.1.7 Corollary

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$. Then,

(i) $(F_1 \times F_2)^* = F_1^* \times F_2^*$.

(ii) $(F_1 \times_R F_2)^* = F_1^* \times_R F_2^*$.

3.1.8 Proposition

Let $\Upsilon_1 = (Q_1, X, F_1, U)$ and $\Upsilon_2 = (Q_2, X, F_2, U)$ be a soft finite state machines. Define the relation $\cong_{12}$ on X^* by $x \cong_{12} y$ if, and only if $x \cong_1 y$ and $x \cong_2 y$, where $\cong_1$ and

$\cong_2$ is the congruence relation on Υ_1 and Υ_2 defined in Theorem 2.0.16. Then, $\cong_{12}$ is a congruence relation on X^*.

Proof. Straightforward. ■

3.1.9 Definition

Let $\Upsilon_i = (Q_i, X, F_i, U)$ be SFSMs, $i = 1, 2$. Let $\langle x \rangle = \{y \in X^* : x \cong_{12} y\}$ for all $x \in X^*$ and $T = \{\langle x \rangle : x \in X^*\}$. Then,

$$\text{STS}(\Upsilon_1) \times_R \text{STS}(\Upsilon_2) = \left(Q_1 \times Q_2, T, F_{S(\Upsilon_1)} \times_R F_{S(\Upsilon_2)}, U\right),$$

where

$$F_{S(\Upsilon_1)} \times_R F_{S(\Upsilon_2)} \left((q_1, q_2), \langle x \rangle, (p_1, p_2)\right) = F_{S(\Upsilon_1)} \left(q_1, [x]_1, p_1\right) \cap F_{S(\Upsilon_2)} \left(q_2, [x]_2, p_2\right).$$

3.1.10 Theorem

Let $\Upsilon_i = (Q_i, X, F_i, U)$ be SFSMs, $i = 1, 2$. Then, the following assertion holds.

$$\text{STS}(\Upsilon_1 \times_R \Upsilon_2) \geq \text{STS}(\Upsilon_1) \times_R \text{STS}(\Upsilon_2).$$

Proof. $(r_1, r_2), (s_1, s_2) \in Q_1 \times Q_2$ and $x \in X^*$. Then,
$F_{S(\Upsilon_1) \times_R S(\Upsilon_2)} \left((r_1, r_2), [x], (s_1, s_2)\right)$

$$\begin{aligned} &= (F_1 \times_R F_2)^* \left((r_1, r_2), x, (s_1, s_2)\right) \\ &= F_1^* (r_1, x, s_1) \cap F_2^* (r_2, x, s_2) \\ &= F_{S(\Upsilon_1)} \left(r_1, [x]_1, s_1\right) \cap F_{S(\Upsilon_2)} \left(r_2, [x]_2, s_2\right) \\ &= F_{S(\Upsilon_1)} \times_R F_{S(\Upsilon_2)} \left((r_1, r_2), \langle x \rangle, (s_1, s_2)\right) \end{aligned}$$

Let $T = \{\langle x \rangle : x \in X^*\}$. Define $f : T \longrightarrow \text{S}(\Upsilon_1 \times_R \Upsilon_2)$ by $f(\langle x \rangle) = [x]$ for all $x \in X^*$.

Let $x, y \in X^*$ and $\langle x \rangle = \langle y \rangle \iff x \cong_{12} y \iff x \cong_1 y$ and $x \cong_2 y \iff$

$$\begin{aligned} F_1^* (q_1, x, p_1) &= F_1^* (q_1, y, p_1) \text{ and} \\ F_2^* (q_2, x, p_2) &= F_2^* (q_2, y, p_2) \text{ for all } q_1, p_1 \in Q_1 \text{ and } q_2, p_2 \in Q_2 \end{aligned}$$

$\implies F_1^* (q_1, x, p_1) \cap F_2^* (q_2, x, p_2) = F_1^* (q_1, x, p_1) \cap F_2^* (q_2, x, p_2)$ for all $q_1, p_1 \in Q_1$ and $q_2, p_2 \in Q_2$

$\implies (F_1 \times_R F_2)^* \left((q_1, q_2), x, (p_1, p_2)\right) = (F_1 \times_R F_2)^* \left((q_1, q_2), y, (p_1, p_2)\right)$
for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$

$\implies x \cong y$ for $\Upsilon_1 \times_R \Upsilon_2 \implies [x] = [y] \implies f(\langle x \rangle) = f(\langle y \rangle)$.

Thus, f is well defined and if we take g to be the identity map of $Q_1 \times Q_2$, then we have (g, f) is covering of $\text{STS}(\Upsilon_1) \times_R \text{STS}(\Upsilon_2)$ by $\text{STS}(\Upsilon_1 \times_R \Upsilon_2)$. This completes the proof. ■

3.1.11 Theorem

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$. Then, the following assertion holds.

$$\mathrm{STS}\left(\Upsilon_1 \times \Upsilon_2\right) \geq \mathrm{STS}\left(\Upsilon_1\right) \times \mathrm{STS}\left(\Upsilon_2\right).$$

Proof. Let $a_i \in X_1$ and $b_i \in X_2$, $i = 1, 2, ...n$. Then, $(a_1, b_1) . (a_2, b_2) ... (a_n, b_n) = (a_1 a_2 ... a_n, b_1 b_2 ... b_n) = (x_1, x_2)$, where $x_1 = a_1 a_2 ... a_n$ and $x_2 = b_1 b_2 ... b_n$.

Now $\Upsilon_1 \times \Upsilon_2 = (Q_1 \times Q_2, X_1 \times X_2, F_1 \times F_2, U)$,

STS$(\Upsilon_1 \times \Upsilon_2) = \left(Q_1 \times Q_2, S\left(\Upsilon_1 \times \Upsilon_2\right), F_{S(\Upsilon_1 \times \Upsilon_2)}, U\right)$ and

STS(Υ_1) $\times$STS(Υ_2) $= \left(Q_1 \times Q_2, S\left(\Upsilon_1\right) \times S\left(\Upsilon_2\right), F_{S(\Upsilon_1)} \times F_{S(\Upsilon_2)}, U\right)$. Let $x_1 \in X_1^*$ and $x_2 \in X_2^*$, then

$F_{S(\Upsilon_1 \times \Upsilon_2)}\left((q_1, q_2), [(x_1, x_2)], (p_1, p_2)\right)$

$$\begin{aligned}
&= (F_1 \times F_2)^*\left((q_1, q_2), (x_1, x_2), (p_1, p_2)\right) \\
&= F_1^*\left(q_1, x, p_1\right) \cap F_2^*\left(q_2, x, p_2\right) \\
&= F_{S(\Upsilon_1)}\left(q_1, [x_1], p_1\right) \cap F_{S(\Upsilon_2)}\left(q_2, [x_2], p_2\right) \\
&= F_{S(\Upsilon_1)} \times F_{S(\Upsilon_2)}\left((q_1, q_2), ([x_1], [x_2]), (p_1, p_2)\right).
\end{aligned}$$

Now, define $f : S\left(\Upsilon_1\right) \times S\left(\Upsilon_2\right) \longrightarrow S\left(\Upsilon_1 \times \Upsilon_2\right)$ by

$f\left(([x_1], [x_2])\right) = [(x_1, x_2)]$ for all $x_1 \in X_1^*$ and $x_2 \in X_2^*$.

Let $([x_1], [x_2]), ([y_1], [y_2]) \in S\left(\Upsilon_1\right) \times S\left(\Upsilon_2\right)$ be such that $([x_1], [x_2]) = ([y_1], [y_2]) \implies [x_1] = [y_1]$ and $[x_2] = [y_2] \implies$

$$\begin{aligned}
F_1^*\left(q_1, x_1, p_1\right) &= F_1^*\left(q_1, y_1, p_1\right) \text{ and} \\
F_2^*\left(q_2, x_2, p_2\right) &= F_2^*\left(q_2, y_2, p_2\right) \text{ for all } q_1, p_1 \in Q_1 \text{ and } q_2, p_2 \in Q_2
\end{aligned}$$

$\implies F_1^*\left(q_1, x_1, p_1\right) \cap F_2^*\left(q_2, x_2, p_2\right) = F_1^*\left(q_1, y_1, p_1\right) \cap F_2^*\left(q_2, y_2, p_2\right)$

for all $q_1, p_1 \in Q_1$ and $q_2, p_2 \in Q_2$

$\implies (F_1 \times F_2)^*\left((q_1, q_2), (x_1, x_2), (p_1, p_2)\right) = (F_1 \times F_2)^*\left((q_1, q_2), (y_1, y_2), (p_1, p_2)\right)$

for all $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2 \implies [(x_1, x_2)] = [(y_1, y_2)]$. Thus, f is well defined and g is the identity map of $Q_1 \times Q_2$, then (g, f) is the covering of STS(Υ_1) $\times$STS(Υ_2) by STS$(\Upsilon_1 \times \Upsilon_2)$. ■

3.2 Cascade and Wreath Product of Soft Finite State Machines

Cascade and Wreath Product has various applications in the theory of automata [22, 23, 48, 49]. In this section, we define the concepts of cascade product, weak covering and wreath product of soft finite state machines. Using these concepts, we will show that there exists a weak covering of soft transformation semigroup associated with wreath product of soft finite state machines by a wreath product of soft transformation semigroup associated with machines. In general, soft transformation semigroup associated with cascade product of soft finite state machines weakly cover by a wreath product of soft transformation semigroup associated with machines.

3.2.1 Definition

Let $\Upsilon_1 = (Q_1, X_1, F_1, U)$ and $\Upsilon_2 = (Q_2, X_2, F_2, U)$ be soft finite state machines. Let θ be a function from $Q_2 \times X_2$ into X_1. Let $Q = Q_1 \times Q_2$, define $F^\theta : Q \times X_2 \times Q \longrightarrow P(U)$ as follows

$$F^\theta((q_1, q_2), x, (p_1, p_2)) = F_1(q_1, \theta(q_2, x), p_1) \cap F_2(q_2, x, p_2),$$

for all $((q_1, q_2), x, (p_1, p_2)) \in Q \times X_2 \times Q$. Then $\Upsilon = \left(Q, X_2, F^\theta, U\right)$ is a $SFSM$. Υ is called cascade product of Υ_1 and Υ_2 and we write $\Upsilon = \Upsilon_1 \theta \Upsilon_2$. F^θ is called separable if for all $(q_1, q_2), (p_1, p_2) \in Q$ and for all $x = x_1 x_2 x_3 ... x_n \in X_2^*$

$$F^{\theta*}((q_1, q_2), x, (p_1, p_2)) = F_1^*\left(q_1, \theta(q_2, x_1)\,\theta\left(q_2^{(1)}, x_2\right) ... \theta\left(q_2^{(n-1)}, x_n\right), p_1\right) \cap F_2^*(q_2, x, p_2),$$

for some $q_2^{(i)} \in Q_2$, $i = 1, 2, 3 ... n-1$.

3.2.2 Example

Let $\Upsilon_1 = (Q_1, X_1, F_1, U)$ and $\Upsilon_2 = (Q_2, X_2, F_2, U)$ be soft finite state machines, where $Q_1 = \{p_1, q_1\}$, $X_1 = \{a, b\}$, $Q_2 = \{p_2, q_2\}$, $X_2 = \{a\}$, $U = \{1, 2, 3, 4\}$ and F_1 and F_2 are defined as follows

Υ_1

	p_1	q_1
(p_1, a)	$\emptyset$	$\{1, 2\}$
(p_1, b)	$\{2, 3, 4\}$	$\emptyset$
(q_1, a)	$\emptyset$	$\{1, 2, 4\}$
(q_1, b)	$\{1, 2, 3\}$	$\emptyset$

Υ_2

	p_2	q_2
(p_2, a)	$\emptyset$	$\{2, 4\}$
(q_2, a)	$\{2, 3\}$	$\emptyset$

Let $\theta : Q_2 \times X_2 \longrightarrow X_1$ be defined as follows:

$$\theta(p_2, a) = a \quad \text{and} \quad \theta(q_2, a) = b.$$

Let $Q = Q_1 \times Q_2$. Then, $F^\theta : Q \times X_2 \times Q \longrightarrow P(U)$ is such that

$$\begin{aligned}
F^\theta((p_1, p_2), a, (q_1, q_2)) &= F_1(p_1, \theta(p_2, a), q_1) \cap F_2(p_2, a, q_2) = \{1, 2\} \cap \{2, 4\} \\
F^\theta((q_1, q_2), a, (p_1, p_2)) &= F_1(q_1, \theta(q_2, a), p_1) \cap F_2(q_2, a, p_2) = \{1, 2, 3\} \cap \{2, 3\} \\
F^\theta((p_1, q_2), a, (p_1, p_2)) &= F_1(p_1, \theta(q_2, a), p_1) \cap F_2(q_2, a, p_2) = \{2, 3, 4\} \cap \{2, 3\} \\
F^\theta((q_1, p_2), a, (q_1, q_2)) &= F_1(q_1, \theta(p_2, a), q_1) \cap F_2(p_2, a, q_2) = \{1, 2, 4\} \cap \{2, 4\}
\end{aligned}$$

and F^{θ} is $\emptyset$ elsewhere.

Note that

$$\begin{aligned} F^{\theta *}\left(\left(p_1,q_2\right),aa,\left(q_1,q_2\right)\right) &= F_1^{*}\left(p_1,\theta\left(q_2,a\right)\theta\left(p_2,a\right),q_1\right)\cap F_2^{*}\left(q_2,aa,q_2\right) \\ &= F_1^{*}\left(p_1,ba,q_1\right)\cap F_2^{*}\left(q_2,aa,q_2\right) \\ &= F_1\left(p_1,b,p_1\right)\cap F_1\left(p_1,a,q_1\right)\cap F_2^{*}\left(q_2,aa,q_2\right) \\ &= \{2,3,4\}\cap\{1,2\}\cap\{2,3\}\cap\{2,4\}. \end{aligned}$$

It follows that F^{θ} is separable.

3.2.3 Proposition

Let $\Upsilon_1 = (Q_1, X_1, F_1, U)$ and $\Upsilon_2 = (Q_2, X_2, F_2, U)$ be soft finite state machines. Let $\Upsilon = \Upsilon_1\theta\Upsilon_2$ for some θ. Then, for all $(q_1,q_2),(p_1,p_2)\in Q$ and for all $y = y_1y_2....y_n \in X_2^*$

$F^{\theta *}\left(\left(q_1,q_2\right),y,\left(p_1,p_2\right)\right) =$

$$\bigcup_{q_2^{(i)}\in Q_2}\left\{\begin{array}{c} F_1^{*}\left(q_1,\theta\left(q_2,y_1\right)\theta\left(q_2^{(1)},y_2\right)....\theta\left(q_2^{(n-1)},y_n\right),p_1\right)\cap \\ F_2\left(q_2,y_1,q_2^{(1)}\right)\cap F_2\left(q_2^{(1)},y_2,q_2^{(2)}\right)\cap....\cap F_2\left(q_2^{(n-1)},y_n,p_2\right) \end{array}\right\}.$$

Proof. $F^{\theta *}\left(\left(q_1,q_2\right),y,\left(p_1,p_2\right)\right)$

$$= \bigcup_{\left(q_1^{(i)},q_2^{(i)}\right)\in Q}\left\{\begin{array}{c} F^{\theta}\left(\left(q_1,q_2\right),y_1,\left(q_1^{(1)},q_2^{(1)}\right)\right)\cap F^{\theta}\left(\left(q_1^{(1)},q_2^{(1)}\right),y_2,\left(q_1^{(2)},q_2^{(2)}\right)\right)\cap.... \\ \cap F^{\theta}\left(\left(q_1^{(n-1)},q_2^{(n-1)}\right),y_n,\left(p_1,p_2\right)\right) \end{array}\right\}$$

$$= \bigcup_{\left(q_1^{(i)},q_2^{(i)}\right)\in Q}\left\{\begin{array}{c} F_1\left(q_1,\theta\left(q_2,y_1\right),q_1^{(1)}\right)\cap F_2\left(q_2,y_1,q_2^{(1)}\right)\cap \\ F_1\left(q_1^{(1)},\theta\left(q_2^{(1)},y_2\right),q_1^{(2)}\right)\cap F_2\left(q_2^{(1)},y_2,q_2^{(2)}\right)\cap.... \\ \cap F_1\left(q_1^{(n-1)},\theta\left(q_2^{(n-1)},y_n\right),p_1\right)\cap F_2\left(q_2^{(n-1)},y_n,p_2,\right) \end{array}\right\}$$

$$= \bigcup_{\left(q_1^{(i)},q_2^{(i)}\right)\in Q}\left\{\begin{array}{c} F_1\left(q_1,\theta\left(q_2,y_1\right),q_1^{(1)}\right)\cap F_1\left(q_1^{(1)},\theta\left(q_2^{(1)},y_2\right),q_1^{(2)}\right)\cap... \\ \cap F_1\left(q_1^{(n-1)},\theta\left(q_2^{(n-1)},y_n\right),p_1\right)\cap \\ F_2\left(q_2,y_1,q_2^{(1)}\right)\cap F_2\left(q_2^{(1)},y_2,q_2^{(2)}\right)\cap....\cap F_2\left(q_2^{(n-1)},y_n,p_2,\right) \end{array}\right\}$$

$$\bigcup_{q_2^{(i)}\in Q_2}\left\{\begin{array}{c} F_1^{*}\left(q_1,\theta\left(q_2,y_1\right)\theta\left(q_2^{(1)},y_2\right)....\theta\left(q_2^{(n-1)},y_n\right),p_1\right)\cap \\ F_2\left(q_2,y_1,q_2^{(1)}\right)\cap F_2\left(q_2^{(1)},y_2,q_2^{(2)}\right)\cap....\cap F_2\left(q_2^{(n-1)},y_n,p_2\right) \end{array}\right\}.$$

■

3.2.4 Definition

Let $\Upsilon_1 = (Q_1, X_1, F_1, U)$ and $\Upsilon_2 = (Q_2, X_2, F_2, U)$ be soft finite state machines. Let f be a function from Q_2 into X_1. Let $Q = Q_1 \times Q_2$, define $F^{\circ} : Q \times \left(X_1^{Q_2} \times X_2\right) \times Q \longrightarrow P(U)$ as follows, for all $((q_1, q_2), (f, x), (p_1, p_2)) \in Q \times \left(X_1^{Q_2} \times X_2\right) \times Q$,

$$F^{\circ}((q_1, q_2), (f, x), (p_1, p_2)) = F_1(q_1, f(q_2), p_1) \cap F_2(q_2, x, p_2).$$

Then, $\Upsilon = \left(Q, X_1^{Q_2} \times X_2, F^{\circ}, U\right)$ is a SFSM. $\Upsilon = \Upsilon_1 \circ \Upsilon_2$ is called the wreath product of Υ_1 and Υ_2. F° is called separable if for all $(q_1, q_2), (p_1, p_2) \in Q$ and for all $(f_1, b_1)(f_2, b_2) .... (f_n, b_n) \in \left(X_1^{Q_2} \times X_2\right)^*$,

$F^{\circ *}((q_1, q_2), (f_1, b_1)(f_2, b_2) .... (f_n, b_n), (p_1, p_2)) =$

$$F_1^*\left(q_1, f_1(q_1) f_2\left(q_1^{(1)}\right) ... f_n\left(q_1^{(n-1)}\right), p_1\right) \cap F_2^*(q_2, b_1 b_2 ... b_n, p_2),$$

for some $q_2^{(i)} \in Q_2$, $i = 1, 2, ...n-1$.

3.2.5 Example

Let $\Upsilon_1 = (Q_1, X_1, F_1, U)$ and $\Upsilon_2 = (Q_2, X_2, F_2, U)$ be soft finite state machines, where $Q_1 = \{q_1, q_2\}$, $X_1 = \{a, b\}$, $Q_2 = \{q_1', q_2\prime\}$, $X_2 = \{a\}$, U is any universal set and F_1 and F_2 are defined as follows,

Υ_1

	q_1	q_2
(q_1, a)	$\emptyset$	A
(q_1, b)	B	$\emptyset$
(q_2, a)	$\emptyset$	C
(q_2, b)	D	$\emptyset$

Υ_2

	q_1'	$q_2\prime$
(q_1', a)	$\emptyset$	E
$(q_2\prime, a)$	F	$\emptyset$

where A, B, C, D, E and F are subsets of U. Let $Q = Q_1 \times Q_2$, and $X_1^{Q_2} = \{f_1, f_2, f_3, f_4\}$, where

$$f_1(q_1') = a, \quad f_1(q_2') = a, \quad f_2(q_1') = a, \quad f_2(q_2') = b,$$
$$f_3(q_1') = b, \quad f_3(q_2') = a, \quad f_4(q_1') = b, \quad f_4(q_2') = b.$$

Define $F^{\circ} : Q \times \left(X_1^{Q_2} \times X_2\right) \times Q \longrightarrow P(U)$ as follows

$F^{\circ}((q_1, q_1'), (f_1, a), (q_2, q_2')) = F_1(q_1, f_1(q_1'), q_2) \cap F_2(q_1', a, q_2') = A \cap E$

$F^{\circ}((q_2, q_1'), (f_1, a), (q_2, q_2')) = F_1(q_2, f_1(q_1'), q_2) \cap F_2(q_1', a, q_2') = C \cap E$

$F^{\circ}((q_1, q_2'), (f_1, a), (q_2, q_1')) = F_1(q_1, f_1(q_2'), q_2) \cap F_2(q_2', a, q_1') = A \cap F$

$F^{\circ}((q_2, q_2'), (f_1, a), (q_2, q_1')) = F_1(q_2, f_1(q_2'), q_2) \cap F_2(q_1', a, q_1') = C \cap F$

$F^{\circ}((q_1, q_1'), (f_2, a), (q_2, q_2')) = F_1(q_1, f_2(q_1'), q_2) \cap F_2(q_1', a, q_2') = A \cap E$

$$
\begin{aligned}
&F^{\circ}\left(\left(q_2,q_1'\right),\left(f_2,a\right),\left(q_2,q_2'\right)\right)=F_1\left(q_2,f_2\left(q_1'\right),q_2\right)\cap F_2\left(q_1',a,q_2'\right)=C\cap E\\
&F^{\circ}\left(\left(q_1,q_2'\right),\left(f_2,a\right),\left(q_1,q_1'\right)\right)=F_1\left(q_1,f_2\left(q_2'\right),q_1\right)\cap F_2\left(q_2',a,q_1'\right)=B\cap F\\
&F^{\circ}\left(\left(q_2,q_2'\right),\left(f_2,a\right),\left(q_1,q_1'\right)\right)=F_1\left(q_2,f_2\left(q_2'\right),q_1\right)\cap F_2\left(q_2',a,q_1'\right)=D\cap F\\
&F^{\circ}\left(\left(q_1,q_1'\right),\left(f_3,a\right),\left(q_1,q_2'\right)\right)=F_1\left(q_1,f_3\left(q_1'\right),q_1\right)\cap F_2\left(q_1',a,q_2'\right)=B\cap E\\
&F^{\circ}\left(\left(q_2,q_1'\right),\left(f_3,a\right),\left(q_1,q_2'\right)\right)=F_1\left(q_2,f_3\left(q_1'\right),q_1\right)\cap F_2\left(q_1',a,q_2'\right)=D\cap E\\
&F^{\circ}\left(\left(q_1,q_2'\right),\left(f_3,a\right),\left(q_2,q_1'\right)\right)=F_1\left(q_1,f_3\left(q_2'\right),q_2\right)\cap F_2\left(q_2',a,q_1'\right)=A\cap F\\
&F^{\circ}\left(\left(q_2,q_2'\right),\left(f_3,a\right),\left(q_2,q_1'\right)\right)=F_1\left(q_2,f_3\left(q_2'\right),q_2\right)\cap F_2\left(q_1',a,q_1'\right)=C\cap F\\
&F^{\circ}\left(\left(q_1,q_1'\right),\left(f_4,a\right),\left(q_1,q_2'\right)\right)=F_1\left(q_1,f_4\left(q_1'\right),q_1\right)\cap F_2\left(q_1',a,q_2'\right)=B\cap E\\
&F^{\circ}\left(\left(q_2,q_1'\right),\left(f_4,a\right),\left(q_1,q_2'\right)\right)=F_1\left(q_2,f_4\left(q_1'\right),q_1\right)\cap F_2\left(q_1',a,q_2'\right)=D\cap E\\
&F^{\circ}\left(\left(q_1,q_2'\right),\left(f_4,a\right),\left(q_1,q_1'\right)\right)=F_1\left(q_1,f_4\left(q_2'\right),q_1\right)\cap F_2\left(q_2',a,q_1'\right)=B\cap F\\
&F^{\circ}\left(\left(q_2,q_2'\right),\left(f_4,a\right),\left(q_1,q_1'\right)\right)=F_1\left(q_2,f_4\left(q_2'\right),q_1\right)\cap F_2\left(q_2',a,q_1'\right)=D\cap F.
\end{aligned}
$$

Then, $\Upsilon=\left(Q,X_1^{Q_2}\times X_2,F^{\circ},U\right)$ is a SFSM.

3.2.6 Proposition

For all $(q_1,q_2),(p_1,p_2)\in Q_1\times Q_2$ and for all $(f_1,b_1)(f_2,b_2)\ldots(f_n,b_n)\in\left(X_1^{Q_2}\times X_2\right)^*$,

$$
F^{\circ *}\left(\left(q_1,q_2\right),\left(f_1,b_1\right)\left(f_2,b_2\right),\ldots\left(f_n,b_n\right),\left(p_1,p_2\right)\right)=
$$

$$
\bigcup_{q_2^{(i)}\in Q_2}\left\{\begin{array}{c}F_1^*\left(q_1,f_1\left(q_2\right)f_2\left(q_2^{(1)}\right)\ldots f_n\left(q_2^{(n-1)}\right),p_1\right)\cap\\ F_2\left(q_2,b_1,q_2^{(1)}\right)\cap F_2\left(q_2^{(1)},b_2,q_2^{(2)}\right)\cap\ldots\cap F_2\left(q_2^{(n-1)},b_n,p_2\right)\end{array}\right\}.
$$

Proof. $F^{\circ *}\left(\left(q_1,q_2\right),\left(f_1,b_1\right)\left(f_2,b_2\right),\ldots\left(f_n,b_n\right),\left(p_1,p_2\right)\right)$

$$
=\bigcup_{\substack{q_1^{(i)}\in Q_1\\ q_2^{(i)}\in Q_2}}\left\{\begin{array}{c}F_1\left(q_1,f_1\left(q_2\right),q_1^{(1)}\right)\cap F_2\left(q_2,b_1,q_2^{(1)}\right)\cap\\ F_1\left(q_1^{(1)},f_2\left(q_2^{(1)}\right),q_1^{(2)}\right)\cap F_2\left(q_2^{(1)},b_2,q_2^{(2)}\right)\cap\ldots\\ \cap F_1\left(q_1^{(n-1)},f_n\left(q_2^{(n-1)}\right),p_1\right)\cap F_2\left(q_2^{(n-1)},b_n,p_2\right)\end{array}\right\}
$$

$$
=\bigcup_{\substack{q_1^{(i)}\in Q_1\\ q_2^{(i)}\in Q_2}}\left\{\begin{array}{c}F_1\left(q_1,f_1\left(q_2\right),q_1^{(1)}\right)\cap F_1\left(q_1^{(1)},f_2\left(q_2^{(1)}\right),q_1^{(2)}\right)\cap\ldots\\ \cap F_1\left(q_1^{(n-1)},f_n\left(q_2^{(n-1)}\right),p_1\right)\cap\\ F_2\left(q_2,b_1,q_2^{(1)}\right)\cap F_2\left(q_2^{(1)},b_2,q_2^{(2)}\right)\cap\ldots\\ \cap F_2\left(q_2^{(n-1)},b_n,p_2\right)\end{array}\right\}
$$

$$
=\bigcup_{q_2^{(i)}\in Q_2}\left\{\begin{array}{c}F_1^*\left(q_1,f_1\left(q_2\right)f_2\left(q_2^{(1)}\right)\ldots f_n\left(q_2^{(n-1)}\right),p_1\right)\cap\\ F_2\left(q_2,b_1,q_2^{(1)}\right)\cap F_2\left(q_2^{(1)},b_2,q_2^{(2)}\right)\cap\ldots\cap F_2\left(q_2^{(n-1)},b_n,p_2\right)\end{array}\right\}.
$$

Which completes the proof. ■

With the help of cascade and wreath product of soft finite state machines we derive covering results of soft transformation semigroup. For this we have the following definition.

3.2.7 Definition

Let $\Upsilon = (Q, X, F, U)$ and $\Upsilon_i = (Q_i, X_i, F_i, U)$, $i = 1, 2$, be soft finite state machines. Let η be function of $Q_1 \times Q_2$ onto Q and ζ a function of X^* into $S(\Upsilon_1)^{Q_2} \times S(\Upsilon_2)$. Then, (η, ζ) is said to be weak covering of STS(Υ) by STS(Υ_1) $\circ\, STS(\Upsilon_2)$, written STS(Υ) $\leq_w$STS(Υ_1) $\circ\, STS(\Upsilon_2)$, if

$F_{S(\Upsilon)}(\eta(q_1, q_2), [x], \eta(p_1, p_2)) =$

$$\bigcup \left\{ \widehat{F}^{\circ}((q_1, q_2), \zeta(x), (r_1, r_2)) \mid \eta(r_1, r_2) = \eta(p_1, p_2), (r_1, r_2) \in Q_1 \times Q_2 \right\},$$

for all $x \in X^*$ and $(q_1, q_2), (p_1, p_2) \in Q_1 \times Q_2$.

3.2.8 Theorem

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$. If F° is separable, then

$$STS(\Upsilon_1 \circ \Upsilon_2) \leq_w STS(\Upsilon_1) \circ STS(\Upsilon_2).$$

Proof. Now, $\Upsilon_1 \circ \Upsilon_2 = \left(Q_1 \times Q_2, X_1^{Q_2} \times X_2, F^{\circ}, U\right)$, where $F^{\circ}((q_1, q_2), (g, b), (p_1, p_2)) = F_1(q_1, g(q_2), p_1) \cap F_2(q_2, b, p_2)$ and STS($\Upsilon_1 \circ \Upsilon_2$) $= \left(Q_1 \times Q_2, S(\Upsilon_1 \circ \Upsilon_2), F^{\circ}_{S(\Upsilon_1 \circ \Upsilon_2)}, U\right)$, where $F^{\circ}_{S(\Upsilon_1 \circ \Upsilon_2)}((q_1, q_2), [(g_1, b_1)(g_2, b_2) ... (g_n, b_n)], (p_1, p_2)) =$

$$F^{\circ *}((q_1, q_2), (g_1, b_1)(g_2, b_2) ... (g_n, b_n), (p_1, p_2)).$$ Also,

$$STS(\Upsilon_1) \circ STS(\Upsilon_2) = \left(Q_1 \times Q_2, S(\Upsilon_1)^{Q_2} \times S(\Upsilon_2), \widehat{F}^{\circ}, U\right),$$ where

$$\begin{aligned} \widehat{F}^{\circ}((q_1, q_2), (f, [b_1 b_2 ... b_n]), (p_1, p_2)) &= F_{S(\Upsilon_1)}(q_1, f(q_2), p_1) \cap F_{S(\Upsilon_2)}(q_2, [b_1 b_2 ... b_n], p_2) \\ &= F_1^*(q_1, x_1, p_1) \cap F_2^*(q_2, b_1 b_2 ... b_n, p_2), \end{aligned}$$

where $f(q_2) = [x_1]$ and $x_1 \in X_1^*$ is selected below.

Let η be the identity map of $Q_1 \times Q_2$. Define $\zeta : \left(X_1^{Q_2} \times X_2\right)^* \longrightarrow S(\Upsilon_1)^{Q_2} \times S(\Upsilon_2)$ as follows:

$$\zeta((g_1, b_1)(g_2, b_2) ... (g_n, b_n)) = (f, [b_1 b_2 ... b_n])$$

Now, $(g_1, b_1)(g_2, b_2) ... (g_n, b_n) = (h_1, a_1)(h_2, a_2) ... (h_m, a_m)$ if, and only if $n = m$ and $(g_i, b_i) = (h_i, a_i)$ $i = 1, 2, ...n$ if, and only if $n = m$ and $g_i = h_i$, $b_i = a_i$, $i = 1, 2, ..., n$. Thus, ζ is single valued.

$$\begin{aligned}
& F^{\circ}_{S(\Upsilon_1\circ\Upsilon_2)}\left((q_1,q_2),[(g_1,b_1)(g_2,b_2)...(g_n,b_n)],(p_1,p_2)\right)\\
= & F^{\circ *}\left((q_1,q_2),(g_1,b_1)(g_2,b_2)...(g_n,b_n),(p_1,p_2)\right)\\
= & \bigcup_{q_2^{(i)}\in Q_2}\left\{\begin{array}{c} F_1^*\left(q_1,g_1(q_2)\,g_2\left(q_2^{(1)}\right)...g_n\left(q_2^{(n-1)}\right),p_1\right)\cap \\ F_2\left(q_2,b_1,q_2^{(1)}\right)\cap F_2\left(q_2^{(1)},b_2,q_2^{(2)}\right)\cap ....\cap F_2\left(q_2^{(n-1)},b_n,p_2\right)\end{array}\right\}\\
= & F_1^*(q_1,x_1,p_1)\cap F_2^*(q_2,x_2,p_2) \text{ where } x_1=g_1(q_2)\,g_2\left(q_2^{(1)}\right)...g_n\left(q_2^{(n-1)}\right)\\
= & F_{S(\Upsilon_1)}(q_1,f(q_2),p_1)\cap F_{S(\Upsilon_2)}(q_2,[x_2],p_2)\\
= & \widehat{F}^{\circ}\left((q_1,q_2),(f,[x_2]),(p_1,p_2)\right)\\
= & \bigcup\left\{\widehat{F}^{\circ}\left((q_1,q_2),\zeta\left((g_1,b_1)(g_2,b_2)...(g_n,b_n)\right),(r_1,r_2)\right)\mid \eta(r_1,r_2)=(p_1,p_2)\right\}
\end{aligned}$$

Thus, (η,ζ) is a weak covering of $STS(\Upsilon_1\circ\Upsilon_2)$ by $STS(\Upsilon_1)\circ STS(\Upsilon_2)$. ■

3.2.9 Theorem

Let $\Upsilon_i=(Q_i,X_i,F_i,U)$ be SFSMs, $i=1,2$. If F^{θ} is separable, then

$$STS(\Upsilon_1\theta\Upsilon_2)\leq_w STS(\Upsilon_1)\circ STS(\Upsilon_2).$$

Proof. Now, $\Upsilon_1\theta\Upsilon_2=\left(Q_1\times Q_2,X_2,F^{\theta},U\right)$, where $\theta:Q_2\times X_2\longrightarrow X_1$ and $F^{\theta}((q_1,q_2),b,(p_1,p_2))=F_1(q_1,\theta(q_2,b),p_1)\cap F_2(q_2,b,p_2)$ for all $(q_1,q_2),(p_1,p_2)\in Q_1\times Q_2$, $b\in X_2^*$ and $STS(\Upsilon_1\theta\Upsilon_2)=\left(Q_1\times Q_2,S(\Upsilon_1\theta\Upsilon_2),F_{S(\Upsilon_1\theta\Upsilon_2)},U\right)$, where $F_{S(\Upsilon_1\theta\Upsilon_2)}((q_1,q_2),[x_2],(p_1,p_2))=F^{\theta *}((q_1,q_2),x_2,(p_1,p_2))$ for all $(q_1,q_2),(p_1,p_2)\in Q_1\times Q_2$, $b\in X_2^*$. Also,

STS$(\Upsilon_1)\circ STS(\Upsilon_2)=\left(Q_1\times Q_2,S(\Upsilon_1)^{Q_2}\times S(\Upsilon_2),\widehat{F}^{\circ},U\right)$, where

$$\begin{aligned}
\widehat{F}^{\circ}\left((q_1,q_2),(f,[x_2]),(p_1,p_2)\right) &= F_{S(\Upsilon_1)}(q_1,f(q_2),p_1)\cap F_{S(\Upsilon_2)}(q_2,[x_2],p_2)\\
&= F_1^*(q_1,x_1,p_1)\cap F_2^*(q_2,x_2,p_2),
\end{aligned}$$

for all $(q_1,q_2),(p_1,p_2)\in Q_1\times Q_2$, $x_2\in X_2^*$. Let η be the identity map of $Q_1\times Q_2$. Since F^{θ} is separable, so we can define ζ in the following manner. Define

$$\zeta:X_2^*\longrightarrow S(\Upsilon_1)^{Q_2}\times S(\Upsilon_2)$$

by

$$\zeta(x_2)=(f_{x_2},[x_2]) \text{ for all } x_2\in X_2^*,$$

where $f_{x_2}(q_2)=[x_1]$ for $x_2=b_1b_2...b_n$ and $x_1=a_1a_2...a_n$ and $\theta(q_1,b_1)=a_1$ and $\theta\left(q_1^{(i-1)},b_i\right)=a_i$, $i=1,2,...n$ for those $q_1^{(i-1)}$ so that

$$\begin{aligned}
& \bigcup_{q_2^{(i)}\in Q_2}\left\{\begin{array}{c} F_1^*\left(q_1,\theta(q_2,b_1)\,\theta\left(q_2^{(1)},b_2\right)....\theta\left(q_2^{(n-1)},b_n\right),p_1\right)\cap \\ F_2\left(q_2,b_1,q_2^{(1)}\right)\cap F_2\left(q_2^{(1)},b_2,q_2^{(2)}\right)\cap ....\cap F_2\left(q_2^{(n-1)},b_n,p_2\right)\end{array}\right\}\\
= & F_1^*(q_1,a_1a_2...a_n,p_1)\cap F_2^*(q_2,x_2,p_2).
\end{aligned}$$

Now ζ is single valued. For this $y_2 = x_2$ if, and only if $d_1 d_2 ... d_j = b_1 b_2 ... b_n$ if and only if $j = n$ and $d_i = b_i$ for $i = 1, 2, ...n$.

Then, $F_{S(\Upsilon_1 \theta \Upsilon_2)}((q_1, q_2), [x_2], (p_1, p_2)) = F^{\theta *}((q_1, q_2), x_2, (p_1, p_2))$

$$\begin{aligned}
&= \bigcup_{q_2^{(i)} \in Q_2} \left\{ \begin{array}{c} F_1^* \left(q_1, \theta(q_2, b_1) \theta \left(q_2^{(1)}, b_2 \right) ....\theta \left(q_2^{(n-1)}, b_n \right), p_1 \right) \cap \\ F_2 \left(q_2, b_1, q_2^{(1)} \right) \cap F_2 \left(q_2^{(1)}, b_2, q_2^{(2)} \right) \cap .... \cap F_2 \left(q_2^{(n-1)}, b_n, p_2 \right) \end{array} \right\} \\
&= F_1^* (q_1, a_1 a_2 ... a_n, p_1) \cap F_2^* (q_2, x_2, p_2) \\
&= F_{S(\Upsilon_1)} (q_1, f(q_2), p_1) \cap F_{S(\Upsilon_2)} (q_2, [x_2], p_2) \\
&= \widehat{F}^{\circ} ((q_1, q_2), (f, [x_2]), (p_1, p_2)) \\
&= \widehat{F}^{\circ} ((q_1, q_2), \zeta(x_2), (p_1, p_2)).
\end{aligned}$$

Hence, (η, ζ) is a weak covering of $STS(\Upsilon_1 \theta \Upsilon_2)$ by $STS(\Upsilon_1) \circ STS(\Upsilon_2)$. ■

Chapter 4

Decomposition of Soft Finite State Machine

In this chapter, we continue our study of a soft finite state machine utilizing algebraic techniques. We defined the concept of soft submachine, separability, connectivity and decomposition of soft finite state machine. With the help of these concepts, we will prove Decomposition Theorem for soft finite state machine.

4.1 Soft Submachines

In this section, we define the concept of soft immediate successor, soft successor, strongly soft connected, weakly soft connected. In this section, we will also give examples regarding to some results, which holds in fuzzy finite state machines, but may not hold in soft finite state machines.

First, we review some basic results of soft finite state machine. We are indebted [20].

4.1.1 Definition

Let $\Upsilon = (Q, X, F, U)$ be a SFSM and let $p, q \in Q$. Then, p is called a soft immediate successor of q if there exists $a \in X$ such that $F(q, a, p) \neq \emptyset$. We denote by $SIS(q)$ the set of all soft immediate successors of q.

We say that p is a soft successor of q if there exists $x \in X^*$ such that $F^*(q, x, p) \neq \emptyset$. We denote by $SS(q)$ the set of all soft successors of q. For any subset T of Q, the set of all soft successors of T denoted by $SS(T)$ is defined to be the set

$$SS(T) = \cup \{SS(q) : \ q \in T\}.$$

4.1.2 Proposition

Let $\Upsilon = (Q, X, F, U)$ be a SFSM. Then, $q \in SS(q)$ for every $q \in Q$.

Proof. It is obvious. Since

$$F^*(q, \lambda, q) = U,$$

so we have $q \in SS(q)$. ■

4.1.3 Remark

Let $\Upsilon = (Q, X, F, U)$ be a SFSM. For any $p, q, r \in Q$.

(i) $SIS(p) \subseteq SS(p)$ for all $p \in Q$.

(ii) If $p \in SS(q)$ and $r \in SS(p)$, then r need not be in $SS(q)$. This, is the interesting point which holds in fuzzy frame, but may not hold in soft frame. For this see the following example.

4.1.4 Example

Let $Q = \{a, b, c\}$ be the set of soft states and $X = \{x, y, z\}$ be the set of soft inputs. Consider the universal set $U = \{1, 2, 3, 4\}$. Then, $\Upsilon = (Q, X, F, U)$ be described as

	a	b	c
(a, x)	$\{1, 2\}$	$\emptyset$	$\emptyset$
(a, y)	$\emptyset$	$\{1, 3\}$	$\emptyset$
(a, z)	$\emptyset$	$\emptyset$	$\emptyset$
(b, x)	$\emptyset$	$\emptyset$	$\emptyset$
(b, y)	$\emptyset$	$\{2, 4\}$	$\emptyset$
(b, z)	$\emptyset$	$\emptyset$	$\{4\}$
(c, x)	$\{2, 3, 4\}$	$\emptyset$	$\emptyset$
(c, z)	$\emptyset$	$\emptyset$	$\{1, 3\}$
(c, y)	$\emptyset$	$\{1\}$	$\emptyset$

The above table can be read as, the image of (a, x, a) under F is $\{1, 2\}$.

Obviously, $SIS(q) \subseteq SS(q)$ for all $q \in Q \implies SIS(a) \subseteq SS(a)$ and clearly in the above table $SIS(a) = \{a, b\}$. We want to show that $SIS(a) = SS(a)$, it is enough to prove that $c \notin SS(a)$. For this, we show $F^*(a, w, c) = \emptyset$ for all $w \in X^*$. Here, we use mathematical induction on $|w| = n$.

If $n = 0$, then $w = \lambda$, hence, $F^*(a, w, c) = F^*(a, \lambda, c) = \emptyset$. This result is true for $n = 0$.

If $n = 1$, then $w \in X$, clearly $F^*(a, w, c) = \emptyset$ for all $w \in X$. This result is also true for $n = 1$.

Suppose that $F^*(a,w,c)=\emptyset$ for all $w\in X^*$ such that $|w|=n-1$. Let $u=wt$ where $w\in X^*$, $t\in X$ and $|w|=n-1$. Then,

$$\begin{aligned} F^*(a,u,c) &= F^*(a,wt,c)=\bigcup_{r\in Q}[F^*(a,w,r)\cap F(r,t,c)] \\ &= \left\{\begin{array}{c}[F^*(a,w,a)\cap F(a,t,c)]\cup[F^*(a,w,b)\cap F(b,t,c)]\cup \\ {[F^*(a,w,c)\cap F(c,t,c)]}\end{array}\right\} \end{aligned}$$

Since $t\in X$, so if $t=x$ or $t=y$, we have $F^*(a,u,c)=\emptyset$. In particular if $t=z$, then we have

$$F^*(a,u,c)=F^*(a,w,b)\cap F(b,z,c)=F^*(a,w,b)\cap\{4\}\text{ .... }(1)$$

It can easily be prove that $4\notin F^*(a,w,b)$ for all $w\in X^*$ by using mathematical induction on $|w|=n$. Hence, from (1) we have $F^*(a,u,c)=\emptyset$ for all $u\in X^*$ such that $|u|=n$.

Since $F^*(a,w,c)=\emptyset$ for all $w\in X^*$ $\Longrightarrow$ $c\notin SS(a)$. Thus $SS(a)=\{a,b\}$. Similarly, we can calculate $SS(b)=\{a,b,c\}$ and $SS(c)=\{a,b,c\}$.

Note that $b\in SS(a)$ and $c\in SS(b)$ but $c\notin SS(a)$.

4.1.5 Proposition

Let $\Upsilon=(Q,X,F,U)$ be a SFSM. For any subsets A and B of Q, the following assertions hold.

(i) If $A\subseteq B$, then $SS(A)\subseteq SS(B)$.

(ii) $A\subseteq SS(A)$.

(iii) $SS(A)\subseteq SS(SS(A))$.

(iv) $SS(A\cup B)=SS(A)\cup SS(B)$.

(v) $SS(A\cap B)\subseteq SS(A)\cap SS(B)$.

Proof. (i) Let $q\in SS(A)=\cup\{SS(a):\ a\in A\}$ $\Longrightarrow$ $q\in SS(a)$ for some $a\in A$. Then, there exists $x\in X^*$ such that $F^*(a,x,q)\neq\emptyset$.

Note that $a\in A\subseteq B$ $\Longrightarrow$ $q\in SS(B)$. Thus, $SS(A)\subseteq SS(B)$.

(ii) It is obvious.

(iii) Since $A\subseteq SS(A)$ $\Longrightarrow$ $SS(A)\subseteq SS(SS(A))$ using (i).

(iv) $A\subseteq A\cup B$ and $B\subseteq A\cup B$

$$\begin{aligned} &\Longrightarrow\quad SS(A)\subseteq SS(A\cup B)\text{ and }SS(B)\subseteq SS(A\cup B) \\ &\Longrightarrow\quad SS(A)\cup SS(B)\subseteq SS(A\cup B). \end{aligned}$$

Conversely, let $q \in SS(A \cup B) = \cup \{SS(z) : z \in A \cup B\} \implies q \in SS(z)$ for some $z \in A \cup B$. Then there exists some $x \in X^*$ such that $F^*(z, x, q) \neq \emptyset$

$$\implies q \in SS(A) \text{ or } q \in SS(B)$$
$$\implies q \in SS(A) \cup SS(B)$$

Thus, $SS(A \cup B) \subseteq SS(A) \cup SS(B)$. Hence $SS(A \cup B) = SS(A) \cup SS(B)$.

(v) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$

$$\implies SS(A \cap B) \subseteq SS(A) \text{ and } SS(A \cap B) \subseteq SS(B)$$

$$\implies SS(A \cap B) \subseteq SS(A) \cap SS(B). \blacksquare$$

Note that equality in (v) may not hold in general. For this, in Example 4.1.4, let $A = \{a, b\}$ and $B = \{a, c\}$. Then $A \cap B = \{a\}$.

Clearly, $SS(A \cap B) \neq SS(A) \cap SS(B)$, because $SS(A \cap B) = \{a, b\}$, $SS(A) = \{a, b, c\}$ and $SS(B) = \{a, b, c\}$.

4.1.6 Definition

Let $\Upsilon = (Q, X, F, U)$ be a SFSM. We say that Υ satisfies the soft exchange property if for all $p, q \in Q$ and $T \subseteq Q$, whenever

$$p \in SS(T \cup \{q\}) \text{ and } p \notin SS(T), \text{ then } q \in SS(T \cup \{p\}).$$

4.1.7 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a SFSM. Then, the following assertions are equivalent.

(i) Υ satisfies the soft exchange property.

(ii) (for all $p, q \in Q$) $p \in SS(q) \iff q \in SS(p)$.

Proof. Assume that Υ satisfies the soft exchange property.

Let $p, q \in Q$ be such that $p \in SS(q) = SS(\emptyset \cup \{q\})$. Note that $p \notin SS(\emptyset)$ and so $q \in SS(\emptyset \cup \{p\}) = SS(p)$.

Similarly, if $q \in SS(p)$ then $p \in SS(q)$.

Conversely, suppose that (ii) is valid.

Let $p, q \in Q$ and $T \subseteq Q$. If $p \in SS(T \cup \{q\})$ and $p \notin SS(T)$, then $p \in SS(q)$. It follows from (ii) that $q \in SS(p) \subseteq SS(T \cup \{p\})$. Hence, Υ satisfies the soft exchange property. $\blacksquare$

4.1.8 Definition

Let $\Upsilon = (Q, X, F, U)$ be a SFSM and $T \subseteq Q$. Let $(f, T \times X \times T)$ be a soft set and let $\Im = (T, X, f, U)$ be a SFSM. Then, $\Im$ is called a soft submachine of Υ if

(i) $F\mid_{T\times X\times T}=f$,

(ii) $SS(T)\subseteq T$.

We assume that $\phi=(\emptyset,X,f,U)$ is a soft submachine of Υ. Obviously, if $\Im'$ is a soft submachine of $\Im$, and $\Im$ is soft submachine of Υ, then $\Im'$ is a soft submachine of Υ. A soft submachine $\Im=(T,X,f,U)$ of a SFSM $\Upsilon=(Q,X,F,U)$ is said to be proper if $T\neq\emptyset$ and $T\neq Q$.

4.1.9 Theorem

Let $\Upsilon=(Q,X,F,U)$ be a SFSM and let $\Im_i=(T_i,X,F_i,U)$ $i\in I$ be a family of soft submachines of Υ. Then, we have

(i) $\bigcap_{i\in I}\Im_i=(\cap_{i\in I}T_i,X,\Cap_{i\in I}F_i,U)$ is a soft submachine of Υ.

(ii) $\bigcup_{i\in I}\Im_i=(\cup_{i\in I}T_i,X,F',U)$ is a soft submachine of Υ where $F'=F\mid_{\cup_{i\in I}T\times X\times\cup_{i\in I}T}$

Proof. Let $(q,x,p)\in\cap_{i\in I}T_i\times X\times\cap_{i\in I}T_i$. Then,

$$\begin{aligned}(\Cap_{i\in I}F_i)(q,x,p) &= \Cap_{i\in I}F_i(q,x,p)=\Cap_{i\in I}F(q,x,p)\\ &= F(q,x,p).\end{aligned}$$

Hence, $F\mid_{\cap_{i\in I}T_i\times X\times\cap_{i\in I}T_i}=\Cap_{i\in I}F_i$

Now,

$$SS(\cap_{i\in I}T_i)\subseteq\cap_{i\in I}SS(T_i)\subseteq\cap_{i\in I}T_i$$

Hence, $\bigcap_{i\in I}\Im_i$ is a soft submachine of Υ.

(ii) Since

$$\begin{aligned}SS(\cup_{i\in I}T_i) &= \cup_{i\in I}SS(T_i)\\ &\subseteq \cup_{i\in I}T_i\end{aligned}$$

Hence, $\bigcup_{i\in I}\Im_i$ is a soft submachine of Υ. ■

4.1.10 Definition

A SFSM $\Upsilon=(Q,X,F,U)$ is said to be strongly soft connected if $p\in SS(q)$ and weakly soft connected if $p\in SS(SS(q))$, for every $p,q\in Q$.

4.1.11 Remark

Every strongly soft connected SFSM $\Upsilon=(Q,X,F,U)$ is weakly soft connected, but converse may not be true in general.

4.1.12 Example

Consider the SFSM, which is described in Example 4.1.4. Clearly SFSM is weakly soft connected but not strongly soft connected, because $c \notin SS(a)$.

4.1.13 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine which is weakly soft connected. Then, Υ has no proper soft submachine.

Proof. Suppose that $\Upsilon = (Q, X, F, U)$ is weakly soft connected. Let $\Im = (T, X, F', U)$ be a soft submachine of Υ such that $T \neq \emptyset$. Then, there exists $q \in T$ such that if $p \in Q$, then $p \in SS(SS(q))$, because Υ is weakly soft connected. It follows that $p \in SS(SS(q)) \subseteq SS(T) \subseteq T$ so that $T = Q$. Hence, $\Upsilon = \Im$, that is Υ has no proper soft submachine. ■

4.1.14 Proposition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and $T \subseteq Q$. Then, $\Im = (SS(T), X, F_T, U)$ is a soft submachine of Υ if and only if $SS(T) = SS(SS(T))$, where $F_T = F\mid_{SS(T)\times X\times SS(T)}$.

Proof. Suppose $\Im = (SS(T), X, F', U)$ is a soft submachine of Υ. Then $SS(SS(T)) \subseteq SS(T)$. Also from Proposition 4.1.5, $SS(T) \subseteq SS(SS(T))$ for any $T \subseteq Q$. Hence, $SS(T) = SS(SS(T))$. Converse is trivial. ■

4.1.15 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $T \subseteq Q$ and $\{\Im_i : i \in I\}$ be the collection of all soft submachines of Υ whose state set contains T and $\Im_i = (Q_i, X, F_i, U)$, $i \in I$. Define $\langle T\rangle = \bigcap_{i\in I} \Im_i$. Then, $\langle T\rangle$ is called the soft submachine generated by T, written as $\langle T\rangle = (Q_T, X, F_T, U)$, where Q_T denotes the intersection of all soft submachines states set which contains T, that is $Q_T = \bigcap_{i\in I} Q_i$.

It is clear that $\langle T\rangle$ is the smallest soft submachine of Υ whose state set contains T.

4.1.16 Proposition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and let $T \subseteq Q$. If $(SS(T), X, F_T, U)$ is a soft submachine of Υ, then $\langle T\rangle = (SS(T), X, F_T, U)$.

Proof. Now $\langle T\rangle = \left(\bigcap_{i\in I} Q_i, X, \Cap_{i\in I} F_i, U\right)$, where $\{N_i \mid i \in I\}$ is the collection of all submachines of Υ, whose state set contains T and $N_i = (Q_i, X, F_i, U)$, $i \in I$. It is enough to prove that $SS(T) = \bigcap_{i\in I} Q_i = Q_T$. Since $(SS(T), X, F_T, U)$ is a soft submachine of Υ such that $T \subseteq SS(T)$, we have that $\bigcap_{i\in I} Q_i \subseteq SS(T)$. Let $p \in SS(T)$. Then, there exist $t \in T$ and $x \in X^*$ such that $F^*(t, x, p) \neq \emptyset$. Now, $t \in \bigcap_{i\in I} Q_i$ and since $\langle T\rangle$ is a soft submachine of Υ so $p \in \bigcap_{i\in I} Q_i$. Thus, $SS(T) \subseteq \bigcap_{i\in I} Q_i$. Hence $SS(T) = \bigcap_{i\in I} Q_i$. ■

4.1.17 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Then, Υ is called singly generated if there exist $q \in Q$ such that $\Upsilon = \langle\{q\}\rangle$. In this case q is called a generator of Υ and we say that Υ is generated by q.

4.1.18 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and let $T \subseteq Q$. T is called free if for all $t \in T$, $t \notin SS(T \setminus \{t\})$.

4.1.19 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and let $T \subseteq Q$. If T is free and $\Upsilon = \langle T\rangle$, then T is called a basis of Υ.

4.1.20 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $\Upsilon_i = (Q_i, X, F_i, U)$, $i = 1, 2$, be a soft finite submachines of Υ. If $\Upsilon = \langle Q_1 \cup Q_2\rangle$, then we say that Υ is the union of Υ_1 and Υ_2 and we write $\Upsilon = \Upsilon_1 \cup \Upsilon_2$.

If $\Upsilon = \Upsilon_1 \cup \Upsilon_2$ and $Q_1 \cap Q_2 = \emptyset$, then we say that Υ is the internal direct union of Υ_1 and Υ_2.

4.1.21 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and $A_1, A_2 \subseteq Q$ such that $SS(A_i) = SS(SS(A_i))$ for $i = 1, 2$. Then, the following assertions hold.

(i) $\langle A_1 \cup A_2\rangle = \langle A_1\rangle \cup \langle A_2\rangle$

(*ii*) $\langle A_1 \cap A_2 \rangle \subseteq \langle A_1 \rangle \cap \langle A_2 \rangle$, whenever $SS(A_1 \cap A_2) = SS(SS(A_1 \cap A_2))$.

Proof. By Proposition 4.1.14, $\left(SS(A_1), X, F_{SS(A_1)}, U\right)$, $\left(SS(A_2), X, F_{SS(A_2)}, U\right)$ and $\left(SS(A_1 \cup A_2), X, F_{SS(A_1 \cup A_2)}, U\right)$ are soft submachines of Υ. Thus,

$\langle A_1 \rangle = \left(SS(A_1), X, F_{SS(A_1)}, U\right)$, $\langle A_2 \rangle = \left(SS(A_2), X, F_{SS(A_2)}, U\right)$ and $\langle A_1 \cup A_2 \rangle = \left(SS(A_1 \cup A_2), X, F_{SS(A_1 \cup A_2)}, U\right)$ by Proposition 4.1.16.

(*i*) Since $SS(A_1 \cup A_2) = SS(A_1) \cup SS(A_2)$, by Proposition 4.1.5. Now,

$$\begin{aligned} F_{SS(A_1) \cup SS(A_2)} &= F \mid_{(SS(A_1) \cup SS(A_2)) \times X \times (SS(A_1) \cup SS(A_2))} \\ &= F \mid_{SS(A_1 \cup A_2) \times X \times SS(A_1 \cup A_2)} \\ &= F_{SS(A_1 \cup A_2)}. \end{aligned}$$

Hence, $\langle A_1 \cup A_2 \rangle = \langle A_1 \rangle \cup \langle A_2 \rangle$.

(*ii*) By above argument, $\langle A_1 \cap A_2 \rangle = \left(SS(A_1 \cap A_2), X, F_{SS(A_1 \cap A_2)}, U\right)$ and by Proposition 4.1.5, $SS(A_1 \cap A_2) \subseteq SS(A_1) \cap SS(A_2)$. Now,

$$F_{SS(A_1 \cap A_2)} = F \mid_{SS(A_1 \cap A_2) \times X \times SS(A_1 \cap A_2)}$$

and

$$F_{SS(A_1) \cap SS(A_2)} = F \mid_{(SS(A_1) \cap SS(A_2)) \times X \times (SS(A_1) \cap SS(A_2))}.$$

Thus,

$$F_{SS(A_1 \cap A_2)} = F_{SS(A_1) \cap SS(A_2)} \mid_{SS(A_1 \cap A_2) \times X \times SS(A_1 \cap A_2)}$$

Hence $\langle A_1 \cap A_2 \rangle \subseteq \langle A_1 \rangle \cap \langle A_2 \rangle$. ■

4.2 Separability and Connectivity

In this section, we define the concept of soft connected and separated submachine of soft finite state machine. Using these concepts, some characterizations of soft finite state machines are given.

4.2.1 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $\Upsilon_1 = (Q_1, X, F_1, U)$ be a soft submachine of Υ. Then, Υ_1 is said to be separated if

$$SS(Q \setminus Q_1) \cap Q_1 = \emptyset.$$

4.2.2 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $\Upsilon_1 = (Q_1, X, F_1, U)$ be a soft submachine of Υ. Then, Υ_1 is separated if, and only if, $SS(Q \setminus Q_1) = Q \setminus Q_1$.

Proof. Suppose Υ_1 is separated. Obviously, $Q \setminus Q_1 \subseteq SS(Q \setminus Q_1)$. For reverse inclusion, let $q \in SS(Q \setminus Q_1)$. Since Υ_1 is separated so $SS(Q \setminus Q_1) \cap Q_1 = \emptyset$. Hence $q \notin Q_1$. Thus $q \in Q \setminus Q_1$. Hence, $SS(Q \setminus Q_1) \subseteq Q \setminus Q_1$. Thus, $SS(Q \setminus Q_1) = Q \setminus Q_1$.

Conversely, suppose that $SS(Q \setminus Q_1) = Q \setminus Q_1$. Clearly, $SS(Q \setminus Q_1) \cap Q_1 = \emptyset$. Thus, Υ_1 is separated. ■

4.2.3 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $\Upsilon_1 = (Q_1, X, F_1, U) \neq \emptyset$ be a soft submachine of Υ. If Υ_1 is separated, then $\Upsilon_2 = (Q \setminus Q_1, X, F_2, U)$ is also a separated soft submachine of Υ where $F_2 = F |_{(Q \setminus Q_1) \times X \times (Q \setminus Q_1)}$.

Proof. Given that $\emptyset \neq Q_1 \neq Q$, so $Q \setminus Q_1 \neq \emptyset$. By Theorem 4.2.2, $SS(Q \setminus Q_1) = Q \setminus Q_1$. Hence, Υ_2 is a soft submachine of Υ. Now, $SS(Q \setminus (Q \setminus Q_1)) = SS(Q_1) = Q_1$. Thus $SS(Q \setminus (Q \setminus Q_1)) \cap (Q \setminus Q_1) = Q_1 \cap (Q \setminus Q_1) = \emptyset$. Hence, Υ_2 is separated. ■

4.2.4 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Then, Υ is said to be soft connected if Υ has no separated proper soft submachine.

4.2.5 Lemma

Every weakly soft connected finite state machine $\Upsilon = (Q, X, F, U)$ is soft connected.

Proof. Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine, which is weakly soft connected. Hence, Υ has no proper soft submachines, by Theorem 4.1.13. Thus Υ has no proper separated soft submachines. Hence, Υ is connected. ■

4.2.6 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Then, Υ is soft connected if, and only if, for all proper submachines $\Im = (Q_1, X, F, U)$ there exist $q \in Q \setminus Q_1$ and $q_1 \in Q_1$ such that $SS(q) \cap SS(q_1) \neq \emptyset$.

Proof. Suppose Υ is soft connected. Let $\Im = (Q_1, X, F, U)$ be a proper soft submachine of Υ. Then, $SS(Q \setminus Q_1) \cap Q_1 \neq \emptyset$ since Υ has no separated proper soft submachine. Thus, there exist $t \in SS(Q \setminus Q_1) \cap Q_1$. Now, $SS(Q_1) = Q_1$. Hence, $t \in SS(Q_1)$ and $t \in SS(Q \setminus Q_1)$. Thus, there exists $q_1 \in Q_1$ and $q \in Q$ such that $t \in SS(q) \cap SS(q_1)$. Hence, $SS(q) \cap SS(q_1) \neq \emptyset$.

Conversely, let $\Im = (Q_1, X, F, U)$ be a proper soft submachine of Υ. Then there exist $q \in Q \setminus Q_1$ and $q_1 \in Q_1$ such that $SS(q) \cap SS(q_1) \neq \emptyset$. Hence

$$\emptyset \neq SS(q) \cap SS(q_1) \subseteq SS(Q \setminus Q_1) \cap SS(Q_1) \subseteq SS(Q \setminus Q_1) \cap Q_1.$$

Thus, $\Im = (Q_1, X, F, U)$ is not separated soft submachine. Thus, Υ is connected. ■

4.2.7 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $p, q \in Q$. Then, p, q are called soft connected, if either $p = q$ or there exist $q_0, q_1, ...q_l \in Q$ such that $q_0 = q$ and $q_l = p$ and there exist $a_1, a_2, ...a_l \in X$ such that for all $i = 1, 2, ...l$ either $F(q_{i-1}, a_i, q_i) \neq \emptyset$ or $F(q_i, a_i, q_{i-1}) \neq \emptyset$.

From the above definition it is clearly that if q and p are soft connected and p and r are soft connected, then q and r are soft connected.

4.2.8 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. For all $q \in Q$, the collection of all soft connected to q is denoted and defined as

$$SC(q) = \{p \in Q \mid p \text{ and } q \text{ are soft connected}\}.$$

Note that, for all $q, p \in Q$ if $q \in SC(p)$, then $SC(q) = SC(p)$. Let Q_1 be any subset of Q. Then

$$SC(Q_1) = \bigcup_{q \in Q_1} SC(q).$$

4.2.9 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and let $N, T \subseteq Q$. Then, the following assertions holds.

(1) If $N \subseteq T$, then $SC(N) \subseteq SC(T)$.

(2) $T \subseteq SC(T)$.

(3) $SC(SC(T)) = SC(T)$.

(4) $SC(N \cup T) = SC(N) \cup SC(T)$.

(5) $SC(N \cap T) \subseteq SC(N) \cap SC(T)$.

(6) Let $q, p \in Q$. If $q \in SC(T \cup \{p\})$ and $q \notin SC(T)$, then $p \in SC(T \cup \{q\})$.

(7) $SS(T) \subseteq SC(T)$.

(8) $SS(SC(T)) = SC(T)$.

Proof. The proofs of (1) and (2) are straightforward.

(3) Since $SC(T) \subseteq SC(SC(T))$ by (2). For the reverse inclusion, let $t \in SC(SC(T))$. Then, $t \in SC(t_1)$ for some $t_1 \in SC(T)$. Thus, there exist $t_2 \in T$ such that $t_1 \in SC(t_2)$. Hence, $t \in SC(t_2)$ for some $t_2 \in T$. This implies $t \in SC(t_2) \subseteq SC(T)$. This completes the proof.

(4) Now $N \subseteq N \cup T$, and $T \subseteq N \cup T$, thus $SC(N) \subseteq SC(N \cup T)$, and $SC(T) \subseteq SC(N \cup T)$. Thus $SC(N) \cup SC(T) \subseteq SC(N \cup T)$. Conversely, let $q \in SC(N \cup T)$. Then, $q \in SC(p)$ for some $p \in N \cup T$. Thus,

$$\begin{aligned} q &\in SC(p) \text{ for some } p \in N \text{ or} \\ q &\in SC(p) \text{ for some } p \in T. \end{aligned}$$

Hence, $q \in SC(N)$ or $q \in SC(T)$. That is $q \in SC(N) \cup SC(T)$.

(5) is straightforward.

(6) Suppose that $q \in SC(T \cup \{p\}) = SC(T) \cup SC(p)$ and $q \notin SC(T)$. Then, $q \in SC(p)$. Hence, $p \in SC(q) \subseteq SC(T \cup \{q\})$.

(7) is straightforward.

Consider (8). By proposition 4.1.5, $SC(T) \subseteq SS(SC(T))$. Conversely, let $q \in SS(SC(T))$. Then, $q \in SS(t)$ for some $t \in SC(T)$. Now, $t \in SC(t_1)$ for some $t_1 \in T$. Thus, $q \in SS(t) \subseteq SC(t)$ by (7), and $t \in SC(t_1)$. Hence, $q \in SC(t_1) \subseteq SC(T)$. Thus $SS(SC(T)) \subseteq SC(T)$. This completes the proof. ■

4.2.10 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and let $Q_1 \subseteq Q$. Then, Q_1 is called soft connected component of Q, if for all q_1, $q_2 \in Q_1$, q_1 and q_2 are soft connected. Q_1 is called maximal soft connected component of Q, when for all $q \in Q$

$$\text{If } q \in SC(Q_1), \text{ then } q \in Q_1.$$

4.2.11 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and let $p \in Q$. Then, $SC(p)$ is a maximal soft connected component of Q.

Proof. Straightforward. ■

4.2.12 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and let $p \in Q$. Let $\Im = (SC(p), X, F', U)$. Then, $\Im$ is a soft submachine of Υ, where $F' = F|_{SC(p) \times X \times SC(p)}$.

Proof. Straightforward. ■

4.2.13 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Then, Υ is soft connected if, and only if, for all $p \in Q$, $SC(p) = Q$.

Proof. Suppose Υ is soft connected. On the contrary suppose that $SC(p) \neq Q$. Then, there exists $q \in Q$ such that $q \notin SC(p)$. By Theorem 4.2.12, $\Im = (SC(p), X, F', U)$ is a proper soft submachine of Υ. Hence, by Theorem 4.2.6, there exist $q \in Q \setminus SC(p)$ and $p \in SC(p)$ such that $SS(q) \cap SS(p) \neq \emptyset$. Let $t \in SS(q) \cap SS(p)$. Then, t and q are soft connected, and t and p are soft connected. Hence, q and p are soft connected. Thus, $q \in SC(p)$ which is a contradiction. So our supposition is wrong. Hence, $SC(p) = Q$.

Conversely, suppose that for all $p \in Q$, $SC(p) = Q$. Let $\Im = (Q_1, X, F_1, U)$ be a proper soft submachine of Υ. Suppose $\Im$ is separated. Then, $SS(Q \setminus Q_1) \cap Q_1 = \emptyset$ and $SS(Q \setminus Q_1) = Q \setminus Q_1$. Let $q \in Q \setminus Q_1$ and $q' \in Q$. Then, $SC(q) = Q = SC(q')$. Hence, q' and q are connected. Thus, there exist $q_0, q_1, ...q_l \in Q$ such that $q_0 = q$ and $q_l = q'$ and there exist $a_1, a_2, ...a_l \in X$ such that for all $i = 1, 2, ...l$ either $F(q_{i-1}, a_i, q_i) \neq \emptyset$ or $F(q_i, a_i, q_{i-1}) \neq \emptyset$. Now, there exists i such that $q_{i-1} \in Q \setminus Q_1$ and $q_i \in Q_1$. Hence, either $q_i \in SS(q_{i-1}) \subseteq SS(Q \setminus Q_1)$ or $q_{i-1} \in SS(q_i) \subseteq T$, which gives the contradiction. Thus our supposition is wrong, that is $\Im$ is not separated. Hence, Υ is soft connected. ■

4.2.14 Corollary

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Then, the following assertions are equivalent.

(1) Υ is soft connected finite state machine.

(2) $SC(q) = SC(q')$ for all $q, q' \in Q$.

4.2.15 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Then, the following assertions are equivalent.

(i) Υ is the internal direct union of its submachines.

(ii) Υ has a separated soft submachine.

Proof. $(i) \implies (ii)$ Let $\Upsilon_i = (Q_i, X, F_i, U)$, $i = 1, 2$, be soft finite submachines of Υ, such that $\Upsilon = \langle Q_1 \cup Q_2 \rangle$, and $Q_1 \cap Q_2 = \emptyset$. To prove Υ_1 is separated, it is enough to prove that $SS(Q \setminus Q_1) \cap Q_1 = \emptyset$. Contrary suppose that there exist $q \in SS(Q \setminus Q_1) \cap Q_1 \neq \emptyset$ such that $q \in SS(Q \setminus Q_1)$ and $q \in Q_1$. Note that $SS(Q \setminus Q_1) \subseteq Q_2$. Thus, $q \in Q_1 \cap Q_2 \neq \emptyset$, which contradicts the given fact. So our supposition is wrong. Hence, $SS(Q \setminus Q_1) \cap Q_1 = \emptyset$. Thus, Υ has separated soft submachine.

$(ii) \implies (i)$ Let $\Upsilon_1 = (Q_1, X, F_1, U) \neq \emptyset$ be a separated soft submachine of Υ. Then, $\Upsilon_2 = (Q \setminus Q_1, X, F_2, U)$ is also a separated soft submachine of Υ by Theorem 4.2.3. Clearly, $Q_1 \cap (Q \setminus Q_1) = \emptyset$ and $\langle Q_1 \rangle \cup \langle Q \setminus Q_1 \rangle = \langle Q \rangle = \Upsilon$. This completes the proof. ■

4.3 Decomposition of Soft Finite State Machines

In this section, we define the concept of primary soft submachine. To use this concept we prove the decomposition theorem for soft finite state machine. In this section, we will also give some results of separability and connectivity for primary soft submachine.

4.3.1 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $\Im$ be a soft submachine of Υ. Then, $\Im$ is called primary soft submachine of Υ if

(i) There exist $t \in Q$ such that $\Im = \langle t \rangle$,

(ii) For all $q \in Q$ If $\Im \subseteq \langle q \rangle$, then $\Im = \langle q \rangle$.

4.3.2 Theorem (Decomposition Theorem)

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $\{\Im_1, \Im_2, \Im_3, ..., \Im_n\}$ be the set of all distinct primary soft submachines of Υ. Then,

(1) $\Upsilon = \bigcup_{i=1}^{n} \Im_i$

(2) $\Upsilon \neq \bigcup_{\substack{i=1 \\ i \neq j}}^{n} \Im_i$ for any $j \in \{1, 2, 3, ..., n\}$.

Proof. (1) It is enough to prove $\Upsilon \subseteq \bigcup_{i=1}^{n} \Im_i$. Note that, for all $p_i \in Q$, there exist only two cases.

(a) $\langle p_i \rangle \in \{\Im_1, \Im_2, \Im_3, ..., \Im_n\}$ or

(b) there exist $p_{i+1} \in Q \setminus Q_{p_i}$ such that $\langle p_i \rangle \subset \langle p_{i+1} \rangle$.

Let $p_0 \in Q$, since Q is finite, either $\langle p_0 \rangle \in \{\Im_1, \Im_2, \Im_3, ..., \Im_n\}$ or there exists positive integer m such that $\langle p_0 \rangle \subset \langle p_m \rangle \in \{\Im_1, \Im_2, \Im_3, ..., \Im_n\}$. Thus, $Q = \bigcup_{i=1}^{n} Q_{t_i}$, where $\Im_i = \langle t_i \rangle$, for $i = 1, 2, 3, ..., n$. Hence, $\Upsilon = \bigcup_{i=1}^{n} \Im_i$.

(2) Let $N = \bigcup_{\substack{i=1 \\ i \neq j}}^{n} \Im_i$ for any $j \in \{1, 2, 3, ..., n\}$ and let $\Im_j = \langle t_j \rangle$. If $t_j \in \bigcup_{\substack{i=1 \\ i \neq j}}^{n} Q_{t_i}$, then

$t_j \in Q_{t_i}$ for some $i \neq j$. Hence, $\Im_j = \langle t_j \rangle \subset \Im_i$. This is a contradiction, because $\Im_j$ is maximal. So our supposition is wrong. Thus, $t_j \notin \bigcup\limits_{\substack{i=1 \\ i \neq j}}^{n} SS(t_i)$. Hence, $\Upsilon \neq N$. ■

4.3.3 Corollary

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Then, every singly generated soft submachine of $\Upsilon \neq \emptyset$ is a soft submachine of a primary soft submachine of Υ.

Proof. Let $\{\Im_1, \Im_2, \Im_3, ..., \Im_n\}$ be the set of all distinct primary soft submachines of Υ. Let $p_0 \in Q$, since Q is finite, either $\langle p_0 \rangle \in \{\Im_1, \Im_2, \Im_3, ..., \Im_n\}$ or there exists $p_1 \in Q \setminus Q_{p_0}$ such that $\langle p_0 \rangle \subset \langle p_1 \rangle$. Continue this process until, there exists positive integer m such that $\langle p_0 \rangle \subset \langle p_m \rangle \in \{\Im_1, \Im_2, \Im_3, ..., \Im_n\}$. Thus, every singly generated soft submachine of $\Upsilon \neq \emptyset$ is a soft submachine of a primary soft submachine of Υ. ■

4.3.4 Corollary

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Then, $\Omega = \{\Im_1, \Im_2, \Im_3, ..., \Im_n\}$ in Theorem 4.3.2 is unique.

Proof. Let $\Gamma = \{\xi_1, \xi_2, ..., \xi_m\}$ be another collection of primary soft submachines of Υ. Let $t_i \in \Im_i \in \Omega$ for any $i = 1, 2, ..., n$. Since $t_i \in \Upsilon = \bigcup\limits_{j=1}^{m} \xi_j$, so there exist j such that $\langle t_i \rangle \subseteq \xi_j$, by using the corollary 4.3.3. Thus, $\Im_i \subseteq \xi_j$. But $\Im_i$ is maximal, so $\Im_i = \xi_j$. Similarly, we can prove the converse. Thus, Ω and Γ are the similar families, which is appearing under the different name.

Hence, $\{\Im_1, \Im_2, \Im_3, ..., \Im_n\}$ in Theorem 4.3.2 is unique. ■

4.3.5 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Then, rank of Υ, rank(Υ), is the number of distinct primary soft submachines of Υ.

4.3.6 Proposition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Then, Υ is singly generated if, and only if, rank$(\Upsilon) = 1$

Proof. Straightforward. ■

4.3.7 Lemma

Every soft finite state machine $\Upsilon = (Q, X, F, U)$ has a weakly soft connected submachine.

Proof. We prove the result by induction on $|Q| = n$. If $n = 1$, then the result is obvious. Suppose the result is true for all soft finite state machines $\Im = (T, X, F, U)$ such that $|T| \leq n - 1$, $n > 0$. Let $q \in Q$. Then, $\langle q \rangle = (Q_q, X, F_q, U)$ is a submachine of Υ. If $\langle q \rangle$ is weakly soft connected, then the result follows. Suppose that $\langle q \rangle$ is not weakly soft connected. Then, there exist $p \in Q_q$ such that $q \notin SS\,(SS\,(p))$ and hence $\langle p \rangle \subset \langle q \rangle$. Now $|Q_p| \leq n - 1$. Hence, by the induction hypothesis the soft finite state machine $\langle p \rangle = (Q_p, X, F_p, U)$ has a weakly soft connected submachine. Sine $\langle p \rangle$ is a soft submachine of Υ, so Υ has a weakly soft connected submachine. ■

4.3.8 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Then, the following assertions are equivalent.

(i) Every singly generated soft submachine of Υ is primary.

(ii) Every non-empty connected submachine of Υ is primary.

Proof. $(i) \implies (ii)$ Let $\Im = (T, X, F_T, U)$ be a non-empty connected submachine of Υ. Let $q \in T$. Suppose $\langle q \rangle \neq \Im$, that is $Q_q \neq T$. Since $\Im$ is connected, $SS\,(T \setminus Q_q) \cap Q_q \neq \emptyset$. Let $p \in SS\,(T \setminus Q_q) \cap Q_q$. Then, $p \in SS\,(t)$ for some $t \in T \setminus Q_q$ and $p \in Q_q$. Now $\langle p \rangle \subseteq \langle t \rangle$ and $\langle p \rangle \subseteq \langle q \rangle$. Since $\langle p \rangle$ is primary, $\langle p \rangle = \langle t \rangle = \langle q \rangle$. Hence, $t \in Q_q$,which is a contradiction. Hence, $\langle q \rangle = \Im$ and so $\Im$ is primary.

$(ii) \implies (i)$ Let $\Im = \langle t \rangle$ be a singly generated soft submachine. By Lemma 4.3.7, $\Im$ has a weakly connected submachine, say, $N = \langle p \rangle$. Then, N is connected and hence primary. Thus, $\langle p \rangle = \langle t \rangle = \Im$. Hence, $\Im$ is primary. ■

4.3.9 Lemma

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and let $\Im = (T, X, F_T, U)$ be a separated soft submachine of Υ. Then, every primary soft submachine of $\Im$ is also a primary soft submachine of Υ.

Proof. Let $\langle t \rangle$ be a primary soft submachine of $\Im$. Suppose $\langle t \rangle$ is not a primary soft submachine of Υ. Then there exists $q \in Q \setminus Q_t$ such that $\langle t \rangle \subseteq \langle q \rangle$. Clearly $q \notin T$. Thus $q \in Q \setminus T$. Since $t \in Q_q \subseteq Q \setminus T$. Note that $Q \setminus T = SS\,(Q \setminus T)$, by Theorem 4.2.3. Thus $t \in SS\,(Q \setminus T) \cap T$, which contradicts the fact that $\Im$ is separated. Hence, $\langle t \rangle$ is a primary soft submachine of Υ. ■

4.3.10 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and let $\Im_i = (Q_i, X, F_i, U)$, $i = 1, 2, 3, ...n$, be the primary soft submachines of Υ. Then, a proper soft submachine

$N = (T, X, F_T, U)$ of Υ is separated if, and only if, for some $J \subseteq \{1, 2, 3, ..., m\}$, $J \neq \emptyset$, $Q \setminus T = \bigcup_{j \in J} Q_j$.

Proof. Let $N = (T, X, F_T, U)$ be a proper separated soft submachine of Υ. Then, by Theorem 4.2.3, $SS(Q \setminus T) = Q \setminus T$. Note that $\langle Q \setminus T \rangle$ is non-empty because N is proper. Thus by decomposition Theorem 4.3.2, $\langle Q \setminus T \rangle$ is the union of all its primary soft submachines. Since $\langle Q \setminus T \rangle$ is separated soft submachine, so every primary soft submachine of $\langle Q \setminus T \rangle$ is a primary soft submachine of Υ, by Theorem 4.3.9. Thus, $SS(Q \setminus T) = \bigcup_{j \in J} Q_j$ for some non-empty $J \subseteq \{1, 2, 3, ..., m\}$. Since $Q \setminus T = SS(Q \setminus T) = \bigcup_{j \in J} Q_j$ for some non-empty $J \subseteq \{1, 2, 3, ..., m\}$.

Conversely, let $N = (T, X, F_T, U)$ be a proper soft submachine of Υ such that $Q \setminus T = \bigcup_{j \in J} Q_j$ for some $J \subseteq \{1, 2, 3, ..., m\}$. Then

$$SS(Q \setminus T) = SS\left(\bigcup_{j \in J} Q_j\right) = \bigcup_{j \in J} SS(Q_j) = \bigcup_{j \in J} Q_j = Q \setminus T.$$

Hence by Theorem 4.2.2, $N = (T, X, F_T, U)$ is separated. ■

4.3.11 Corollary

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Then, Υ is soft connected if, and only if, Υ has no proper soft submachine $\Im = (Q_1, X, F_1, U)$ such that $Q \setminus Q_1$ is the union of the sets of state of all primary soft submachines of Υ.

Proof. Suppose that Υ is soft connected. On the contrary, Υ has proper soft submachine $\Im = (Q_1, X, F_1, U)$ such that $Q \setminus Q_1 = \bigcup_{i \in I} Q_i$, where $\Im_i = (Q_i, X, F_i, U)$, $i = 1, 2, 3, ...n$, be the primary soft submachines of Υ. Then, by Theorem 4.3.10, $\Im$ is connected which contradicts the given assumption. So our supposition is wrong, hence Υ has no proper soft submachine $\Im = (Q_1, X, F_1, U)$ such that $Q \setminus Q_1$ is the union of the sets of state of all primary soft submachines of Υ. Converse is similar. ■

4.3.12 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and let $\Im = (T, X, F_1, U)$ be a soft submachine of Υ. A subset $R \subseteq Q$ is called a generating set of $\Im$ if $\Im = \langle R \rangle$.

4.3.13 Lemma

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and let $\Im_i = (Q_i, X, F_i, U)$, $i = 1, 2, 3, ...n$, be the primary soft submachines of Υ. Let $R \subseteq Q$. Then, R generates Υ

if, and only if, for all i, $1 \leq i \leq n$, there exist $r_i \in R$ such that $\Im_i = \langle r_i \rangle$.

Proof. Suppose that R generates Υ. Then, $\Upsilon = \langle R \rangle = \bigcup_{r \in R} \langle r \rangle$. Let $q_i \in Q_i$ be such that $\Im_i = \langle q_i \rangle$, for all i, $1 \leq i \leq n$. Then, $q_i \in \bigcup_{r \in R} \langle r \rangle$ and so $q_i \in \langle r \rangle$, for some $r \in R$. Thus $\langle q_i \rangle \subseteq \langle r \rangle$. Since $\langle q_i \rangle$ is primary, we have $\langle q_i \rangle = \langle r \rangle$.

Converse is trivial. ■

4.3.14 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and let $R \subseteq Q$ be a generating set of Υ. Then, R is minimal generating set of Υ if

(i) $\Upsilon = \langle R \rangle$ and

(ii) For all $r \in R$, $\langle R \setminus \{r\} \rangle \neq \Upsilon$.

4.3.15 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and $\Upsilon = \langle R \rangle$ for any $R \subseteq Q$. Then R is minimal generating set of Υ if, and only if, $|R|$ =rank(Υ) .

Proof. Let rank$(\Upsilon) = n$ and let $\Im_i = (Q_i, X, F_i, U)$, $i = 1, 2, 3, ...n$, be the primary soft submachines of Υ. By Lemma 4.3.13, since R generates Υ, there exist $r_i \in R$ such that $\Im_i = \langle r_i \rangle$ for all i, $1 \leq i \leq n$. Since $\Im_i$ are distinct, then r_i are distinct. Thus, $|R|$ $\geq$rank(Υ) . Now, assume that R is a minimal generating set. Suppose $|R|$ $>$rank(Υ) . Then, there exist $r \in R$ such that $r \notin \{r_1, r_2, ..., r_n\}$. Thus $\langle R \setminus \{r\} \rangle = \Upsilon$ Hence, R is not minimal, which is a contradiction. Thus, $|R|$ =rank(Υ) .

Conversely, suppose that $|R|$ =rank(Υ) . Then, $R = \{r_1, r_2, ..., r_n\}$. Hence, $\langle R \setminus \{r_i\} \rangle = \bigcup_{j \neq i} \Im_j \neq \Upsilon$. Thus, R is minimal. ■

Chapter 5

Soft Subsystems and Finite Switch Board State Machine

In this chapter, the concepts of soft subsysytem, strong soft subsystem, switching, commutative and soft finite switchboard state machine are introduced and some of its properties are discussed.

5.1 Soft Subsystems of Soft Finite State Machines

In this section, the concept of soft subsysytem and strong soft subsystem are defined. After defining these concepts, we consider the relation between these concepts, that is, every strong soft subsystem is a soft subsystem. But, converse is not true in general. In this section, we will also prove that, homomorphic image of soft subsystem (strong soft subsystem) is a soft subsystem (strong soft subsystem), if the homomorphism between soft finite state machine is onto.

5.1.1 Definition

Let $\Upsilon = (Q, X, F, U)$ be a SFSM. Let $\left(\widetilde{F}, Q\right)$ be a soft set. Then, $\left(Q, \widetilde{F}, X, F, U\right)$ is called a soft subsystem of Υ if

$$\widetilde{F}(p) \cap F(p, a, q) \subseteq \widetilde{F}(q) \quad \text{for all } p, q \in Q \text{ and } a \in X.$$

If $\left(Q, \widetilde{F}, X, F, U\right)$ is a soft subsystem of Υ, then we simply write $\widetilde{\Upsilon}$ for $\left(Q, \widetilde{F}, X, F, U\right)$.

5.1.2 Example

Let $Q = \{a, b, c\}$ be the set of soft states and $X = \{x, y, z\}$ be the set of soft inputs. Consider the universal set $U = \{1, 2, 3, 4\}$. Then, $\Upsilon = (Q, X, F, U)$ is described as

	a	b	c
(a, x)	$\{1, 2\}$	$\emptyset$	$\emptyset$
(a, y)	$\emptyset$	$\{1, 3\}$	$\emptyset$
(a, z)	$\emptyset$	$\emptyset$	$\emptyset$
(b, x)	$\emptyset$	$\emptyset$	$\emptyset$
(b, y)	$\emptyset$	$\{2, 4\}$	$\emptyset$
(b, z)	$\emptyset$	$\emptyset$	$\{4\}$
(c, x)	$\{2, 3, 4\}$	$\emptyset$	$\emptyset$
(c, z)	$\emptyset$	$\emptyset$	$\{1, 3\}$
(c, y)	$\emptyset$	$\{1\}$	$\emptyset$

The above table can be read as, the image of (a, x, a) under F is $\{1, 2\}$.

Let $\left(\widetilde{F}, Q\right)$ be a soft set over U, defined as

$$\widetilde{F}(a) = \{1, 4\}, \ \widetilde{F}(b) = \{1, 2, 4\} \text{ and } \widetilde{F}(c) = \{1, 4\}.$$

Then, clearly $\widetilde{\Upsilon}$ is a soft subsystem of Υ.

5.1.3 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a SFSM and $\left(\widetilde{F}, Q\right)$ be a soft set over U. Then, $\widetilde{\Upsilon}$ is a soft subsystem of Υ if and only if

$$F^*(p, x, q) \cap \widetilde{F}(p) \subseteq \widetilde{F}(q) \text{ for all } p, q \in Q \text{ and } x \in X^*.$$

Proof. Suppose that $\widetilde{\Upsilon}$ is a soft subsystem of Υ. Let $p, q \in Q$ and $x \in X^*$. The proof is by induction on $|x| = n$

If $n = 0$, then $x = \lambda$. Now, if $p = q$, then

$$F^*(q, \lambda, q) \cap \widetilde{F}(q) = \widetilde{F}(q)$$

If $q \neq p$, then

$$F^*(p, \lambda, q) \cap \widetilde{F}(p) = \emptyset \subseteq \widetilde{F}(q)$$

Thus, the result is true for $n = 0$.

Suppose that result is true for all $y \in X^*$, with $|y| = n - 1$, $n > 0$

Let $x = ya$ where $a \in X$. Then,

$$\begin{aligned}
\widetilde{F}(p) \cap F^*(p,x,q) &= \widetilde{F}(p) \cap F^*(p,ya,q) \\
&= \widetilde{F}(p) \cap \left[\bigcup_{r\in Q} \{F^*(p,y,r) \cap F(r,a,q)\} \right] \\
&= \bigcup_{r\in Q} \left[\widetilde{F}(p) \cap F^*(p,y,r) \cap F(r,a,q) \right] \\
&\subseteq \bigcup_{r\in Q} \left[\widetilde{F}(r) \cap F(r,a,q) \right] \\
&\subseteq \widetilde{F}(q)
\end{aligned}$$

$\Longrightarrow \widetilde{F}(p) \cap F^*(p,x,q) \subseteq \widetilde{F}(q)$.

Converse is trivial. ■

5.1.4 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $\widetilde{\Upsilon}_1$ and $\widetilde{\Upsilon}_2$ be soft subsystems of Υ. Then, the following assertions hold.

(1) $\widetilde{\Upsilon}_1 \cup \widetilde{\Upsilon}_2$ is a soft subsystem of Υ.

(2) $\widetilde{\Upsilon}_1 \cap \widetilde{\Upsilon}_2$ is a soft subsystem of Υ.

Proof. Straightforward. ■

5.1.5 Definition

Let $\Upsilon = (Q, X, F, U)$ be a SFSM and $\left(\widetilde{F}, Q\right)$ be a soft set over U. For all $x \in X^*$, define $\left(\widetilde{F_x}, Q\right)$ as follows

$$\widetilde{F_x}(q) = \bigcup_{p\in Q} \left\{ \widetilde{F}(p) \cap F^*(p,x,q) \right\} \text{ for all } q \in Q.$$

5.1.6 Proposition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Then, for all soft sets $\left(\widetilde{F}, Q\right)$ and for all $x, y \in X^*$

$$\left(\widetilde{F_x}\right)_y = \widetilde{F_{xy}}.$$

Proof. Let $\left(\widetilde{F}, Q\right)$ be a soft set and $x, y \in X^*$. We prove the result by induction on $|y| = n$. If $n = 0$, then $y = \lambda$. Let $q \in Q$. Now,

$$\begin{aligned}
\left(\widetilde{F_x}\right)_\lambda (q) &= \bigcup_{p\in Q} \left\{ \widetilde{F_x}(p) \cap F^*(p,\lambda,q) \right\} \\
&= \widetilde{F_x}(q).
\end{aligned}$$

Hence $\left(\widetilde{F_x}\right)_\lambda = \widetilde{F_x} = \widetilde{F_{x\lambda}}$.

Suppose the result is true for all $u \in X^*$, such that $|u| = n-1, n > 0$. Let $y = ua$, where $a \in X$, $u \in X^*$ and $|u| = n-1$. Let $q \in Q$. Then,

$$\begin{aligned}
\widetilde{F_{xy}}(q) &= \widetilde{F_{xua}}(q) \\
&= \left(\widetilde{F_{xu}}\right)_a (q) \\
&= \bigcup_{p\in Q} \left\{\widetilde{F_{xu}}(p) \cap F^*(p,a,q)\right\} \\
&= \bigcup_{p\in Q} \left\{\left\{\bigcup_{r\in Q} \left\{\widetilde{F_x}(r) \cap F^*(r,u,p)\right\}\right\} \cap F^*(p,a,q)\right\} \\
&= \bigcup_{r\in Q} \left\{\widetilde{F_x}(r) \cap \left\{\bigcup_{p\in Q} \{F^*(r,u,p) \cap F^*(p,a,q)\}\right\}\right\} \\
&= \bigcup_{r\in Q} \left\{\widetilde{F_x}(r) \cap F^*(r,ua,q)\right\} \\
&= \left(\widetilde{F_x}\right)_{ua}(q) \\
&= \left(\widetilde{F_x}\right)_y (q).
\end{aligned}$$

This completes the proof. ■

5.1.7 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a SFSM. Let $\left(\widetilde{F}, Q\right)$ be a soft set. Then, $\widetilde{\Upsilon} = \left(Q, \widetilde{F}, X, F, U\right)$ is a soft subsystem of Υ if, and only if, $\widetilde{F_x} \subseteq \widetilde{F}$ for all $x \in X^*$.

Proof. Let $\widetilde{\Upsilon} = \left(Q, \widetilde{F}, X, F, U\right)$ be a soft subsystem of Υ. Let $x \in X^*$ and $q \in Q$. Then,

$$\widetilde{F_x}(q) = \bigcup_{p\in Q} \left\{\widetilde{F}(p) \cap F^*(p,x,q)\right\} \subseteq \widetilde{F}(q),$$

by Theorem 5.1.3. Hence, $\widetilde{F_x} \subseteq \widetilde{F}$.

Conversely, suppose $\widetilde{F_x} \subseteq \widetilde{F}$ for all $x \in X^*$. Let $q \in Q$ and $x \in X^*$. Now,

$$\widetilde{F}(p) \cap F^*(p,x,q) \subseteq \bigcup_{p\in Q} \left\{\widetilde{F}(p) \cap F^*(p,x,q)\right\} = \widetilde{F_x}(q) \subseteq \widetilde{F}(q),$$

for all $p \in Q$. Hence, $\widetilde{\Upsilon} = \left(Q, \widetilde{F}, X, F, U\right)$ is a soft subsystem of Υ. ■

5.1.8 Definition

Let $\Upsilon_i = (Q_i, X, F_i, U)$, $i = 1, 2$, be soft finite machines. Let $(f, g) : \Upsilon_1 \longrightarrow \Upsilon_2$ be a homomorphism and $\left(\widetilde{F}, Q_1\right)$ be a soft set. Define soft set $\left(f_{\widetilde{F}}, Q_2\right)$ by

$$f_{\widetilde{F}}\left(q'\right) = \begin{cases} \bigcup\left\{\widetilde{F}(q) \mid q \in Q_1,\ f(q) = q'\right\} & \text{if } f^{-1}(q) \neq \emptyset \\ \emptyset & \text{otherwise,} \end{cases}$$

for all $q' \in Q_2$.

5.1.9 Theorem

Let $\Upsilon_i = (Q_i, X, F_i, U)$, $i = 1, 2$, be soft finite machines. Let $(f, g) : \Upsilon_1 \longrightarrow \Upsilon_2$ be an onto strong homomorphism. If $\left(Q_1, \widetilde{F}, X, F_1, U\right)$ is a soft subsystem of Υ_1, then $\left(Q_2, f_{\widetilde{F}}, X, F_2, U\right)$ is a soft subsystem of Υ_2.

Proof. Let $p', q' \in Q_2$ and $x \in X$. Then,

$$\begin{aligned} f_{\widetilde{F}}\left(p'\right) \cap F_2\left(p', a, q'\right) &= \bigcup\left\{\widetilde{F}(p) \mid p \in Q_1,\ f(p) = p'\right\} \cap F_2\left(p', a, q'\right) \\ &= \bigcup\left\{\widetilde{F}(p) \cap F_2\left(p', a, q'\right) \mid p \in Q_1,\ f(p) = p'\right\}. \end{aligned}$$

Let $p, q \in Q_1$ be such that $f(p) = p'$and $f(q) = q'$. Then,

$$\begin{aligned} \widetilde{F}(p) \cap F_2\left(p', a, q'\right) &= \widetilde{F}(p) \cap F_2(f(p), a, f(q)) \\ &= \widetilde{F}(p) \cap \left(\bigcup\left\{F_1(p, a, r) \mid r \in Q_1,\ f(r) = f(q) = q'\right\}\right) \\ &= \bigcup\left\{\widetilde{F}(p) \cap F_1(p, a, r) \mid r \in Q_1,\ f(r) = f(q) = q'\right\} \\ &\subseteq \bigcup\left\{\widetilde{F}(r) \mid r \in Q_1,\ f(r) = q'\right\} \\ &= f_{\widetilde{F}}\left(q'\right). \end{aligned}$$

Hence,

$$\begin{aligned} f_{\widetilde{F}}\left(p'\right) \cap F_2\left(p', a, q'\right) &\subseteq \bigcup\left\{f_{\widetilde{F}}\left(q'\right) \mid p \in Q_1,\ f(p) = p'\right\} \\ &= f_{\widetilde{F}}\left(q'\right). \end{aligned}$$

Thus, $\left(Q_2, f_{\widetilde{F}}, X, F_2, U\right)$ is a soft subsystem of Υ_2. ■

The next example shows that the above result need not be true if f is not onto.

5.1.10 Example

Let $Q_1 = Q_2 = Q = \{p, q\}$, $X = \{x\}$ and $U = \{a, b, c, d\}$. Then, $\Upsilon_1 = \Upsilon_2 = (Q, X, F, U)$ is a soft finite state machine which is defined as

	p	q
(p, x)	U	$\{a, b, c\}$
(q, x)	$\{b, d\}$	$\{a, b\}$

The above table can be read as the image of (p, x, q) under F is $\{a, b, c\}$.

Let $f : Q \longrightarrow Q$ be a mapping such that $f(p) = f(q) = p$. Clearly, f is a strong homomorphism which is not onto. Let $\left(\widetilde{F}, Q\right)$ defined as $\widetilde{F}(p) = \widetilde{F}(q) = \{b\}$. Then,

$$\widetilde{F}(t) = \{b\} = \widetilde{F}(s) \cap F(s, x, t) \text{ for all } s, t \in Q.$$

Thus, $\left(Q, \widetilde{F}, X, F, U\right)$ is a soft subsystem of Υ_1. Now,

$$f_{\widetilde{F}}(p) \cap F_2(p, x, q) = \{b\}$$

and $f_{\widetilde{F}}(q) = \emptyset$. Thus

$$f_{\widetilde{F}}(p) \cap F_2(p, x, q) \nsubseteq f_{\widetilde{F}}(q).$$

Hence, $\left(Q_2, f_{\widetilde{F}}, X, F, U\right)$ is not a soft subsystem of Υ_2.

5.1.11 Definition (Strong Soft Subsystems)

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $\left(\widetilde{F}, Q\right)$ be a soft set over U. Then, $\left(Q, \widetilde{F}, X, F, U\right)$ is a strong soft subsystem of Υ if and only if, there exists $a \in X$ such that $F(p, a, q) \neq \emptyset$ (for all $p, q \in Q$), then $\widetilde{F}(p) \subseteq \widetilde{F}(q)$.

If $\left(Q, \widetilde{F}, X, F, U\right)$ is a strong soft subsystem of Υ, then we simply write $\widetilde{\Upsilon}$ for $\left(Q, \widetilde{F}, X, F, U\right)$.

5.1.12 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $\left(\widetilde{F}, Q\right)$ be a soft set over U. Then, $\widetilde{\Upsilon}$ is a strong soft subsystem of Υ if and only if, if there exist $x \in X^*$ such that $F^*(p, x, q) \neq \emptyset$ (for all $p, q \in Q$), then $\widetilde{F}(q) \supseteq \widetilde{F}(p)$.

Proof. Suppose $\widetilde{\Upsilon}$ is a strong soft subsystem of Υ. We prove the result by induction on $|x| = n$. If $n = 0$, then $x = \lambda$. Now, if $p = q$, then $F^*(p, x, q) = U$ and $\widetilde{F}(p) = \widetilde{F}(q)$. If $p \neq q$, then $F^*(p, x, q) = \emptyset$. Thus, the result is true if $n = 0$. Suppose the result is true for all $u \in X^*$ such that $|u| = n-1$, $n > 0$. Let $x = ua$, $|u| = n-1$, $u \in X^*$, $a \in X$. Suppose that $F^*(p,x,q) \neq \emptyset$. Then,

$$\begin{aligned} \bigcup_{r \in Q} \{F^*(p,u,r) \cap F(r,a,q)\} &= F^*(p,ua,q) \\ &= F^*(p,x,r) \\ &\neq \emptyset. \end{aligned}$$

Thus there exist $r \in Q$ such that $F^*(p,u,r) \cap F(r,a,q) \neq \emptyset$. Clearly, $F^*(p,u,r) \neq \emptyset$ and $F(r,a,q) \neq \emptyset$. Hence, by hypothesis $\widetilde{F}(q) \supseteq \widetilde{F}(r)$ and $\widetilde{F}(r) \supseteq \widetilde{F}(p)$. Thus, $\widetilde{F}(q) \supseteq \widetilde{F}(p)$. The converse is trivial. ■

5.1.13 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and let $\left(\widetilde{F}, Q\right)$ be a soft set over U. If $\widetilde{\Upsilon}$ is a strong soft subsystem of Υ then, $\widetilde{\Upsilon}$ is a soft subsystem of Υ.

Proof. Let $\widetilde{\Upsilon} = \left(Q, \widetilde{F}, X, F, U\right)$ be a strong soft subsystem of Υ. If $F^*(p,x,q) = \emptyset$ for all $p, q \in Q$, then

$$\widetilde{F}(p) \cap F^*(p,x,q) \subseteq \widetilde{F}(q).$$

If $F^*(p,x,q) \neq \emptyset$, then $\widetilde{F}(p) \subseteq \widetilde{F}(q)$, because $\widetilde{\Upsilon}$ is a strong soft subsystem of Υ. Hence, $\widetilde{F}(p) \cap F^*(p,x,q) \subseteq \widetilde{F}(q)$. ■

5.1.14 Remark

Every soft subsystem of the soft finite state machine Υ need not be a strong soft subsystem of Υ.

5.1.15 Example

Let $Q = \{a, b\}$, $X = \{x\}$ and $U = \{1, 2, 3, 4\}$. Then, $\Upsilon = (Q, X, F, U)$ is described as

	a	b
(a,x)	$\{1,2\}$	$\{1,4\}$
(b,x)	$\{2,4\}$	$\{1,2,4\}$

The above table can be read as, the image of (a, y, b) under F is $\{1, 4\}$.

Let $\left(\widetilde{F}, Q\right)$ be a soft set over U, defined as

$$\widetilde{F}(a) = \{1,2\}, \text{ and } \widetilde{F}(b) = \{1,2,3\}.$$

Then, $\widetilde{\Upsilon} = \left(Q, \widetilde{F}, X, F, U\right)$ is a soft subsystem of Υ. Note that $F(b,x,a) \neq \emptyset$, but $\widetilde{F}(b) \nsubseteq \widetilde{F}(a)$. Thus, $\widetilde{\Upsilon} = \left(Q, \widetilde{F}, X, F, U\right)$ is not a strong soft subsystem of Υ.

5.1.16 Theorem

Let $\Upsilon_i = (Q_i, X, F_i, U)$, $i = 1, 2$, be a soft finite machines. Let $f : \Upsilon_1 \longrightarrow \Upsilon_2$ be an onto strong homomorphism. If $\left(Q_1, \widetilde{F}, X, F_1, U\right)$ is a strong soft subsystem of Υ_1, then $\left(Q_2, f_{\widetilde{F}}, X, F_2, U\right)$ is a strong soft subsystem of Υ_2.

Proof. Let $q, p \in Q_1$ and $x \in X$ be such that $F_2(f(p), x, f(q)) \neq \emptyset$.

Now,

$$f_{\widetilde{F}}(f(q)) = \bigcup \left\{\widetilde{F}(r) \mid r \in Q_1,\ f(r) = f(q)\right\}$$

and

$$f_{\widetilde{F}}(f(p)) = \bigcup \left\{\widetilde{F}(s) \mid s \in Q_1,\ f(s) = f(p)\right\}.$$

Let $s \in Q_1$ be such that $\widetilde{F}(s) \neq \emptyset$ and $f(s) = f(p)$. Then,

$$F_2(f(s), x, f(q)) = F_2(f(p), x, f(q)) \neq \emptyset.$$

Hence,

$$\bigcup \{F_1(s, a, r) \mid r \in Q_1,\ f(r) = f(q)\} \neq \emptyset.$$

Thus, there exist $r \in Q_1$ such that $F_1(s, a, r) \neq \emptyset$ and $f(r) = f(q)$. Since $\left(Q_1, \widetilde{F}, X, F_1, U\right)$ is a strong soft subsystem of Υ_1, $\widetilde{F}(r) \supseteq \widetilde{F}(s)$.Hence, $f_{\widetilde{F}}(f(q)) \supseteq f_{\widetilde{F}}(f(p))$. Thus, $\left(Q_2, f_{\widetilde{F}}, X, F_2, U\right)$ is a strong soft subsystem of Υ_2. ■

The following example shows that the above result need not be true if f is not onto.

5.1.17 Example

Let $Q_1 = Q_2 = Q = \{p, q\}$, $X = \{x\}$ and $U = \{a, b, c, d\}$. Then, $\Upsilon_1 = \Upsilon_2 = (Q, X, F, U)$ is a soft finite state machine which is defined as

	p	q
(p, x)	U	$\{a, b, c\}$
(q, x)	$\{b, d\}$	$\{a, b\}$

The above table can be read as the image of (p, x, q) under F is $\{a, b, c\}$.

Let $f : Q \longrightarrow Q$ be a mapping such that $f(p) = f(q) = p$. Clearly f is a strong homomorphism which is not onto. Let $\left(\widetilde{F}, Q\right)$ be a soft set over U which is defined as $\widetilde{F}(p) = \widetilde{F}(q) = \{b\}$. Then,

$$\text{If } F(s, x, t) \neq \emptyset \text{ (for all } s, t \in Q) \text{ then } \widetilde{F}(t) \supseteq \widetilde{F}(s).$$

Thus, $\left(Q, \widetilde{F}, X, F, U\right)$ is a strong soft subsystem of Υ_1. Now,

$$F_2(p, x, q) \neq \emptyset,$$

$f_{\widetilde{F}}(p) = \{b\}$ and $f_{\widetilde{F}}(q) = \emptyset$. Thus,

$$f_{\widetilde{F}}(p) \nsubseteq f_{\widetilde{F}}(q).$$

Hence, $\left(Q_2, f_{\widetilde{F}}, X, F, U\right)$ is not a strong soft subsystem of Υ_2.

5.2 Soft Finite Switch Board State Machine

In this section, we define switching soft finite state machine and prove that every switching soft finite state machine satisfies the soft exchange property. We will also define commutative soft finite state machine and prove that homomorphic image of commutative soft finite state machine is a commutative soft finite state machine, if the strong homomorphism between soft finite state machine is onto.

5.2.1 Definition

A SFSM $\Upsilon = (Q, X, F, U)$ is said to be switching if

$$F(q, x, p) = F(p, x, q), \text{ for all } p, q \in Q \text{ and } x \in X.$$

5.2.2 Definition

A SFSM $\Upsilon = (Q, X, F, U)$ is said to be commutative if

$$F^*(q, xy, p) = F^*(q, yx, p), \text{ for all } p, q \in Q \text{ and } x, y \in X.$$

5.2.3 Definition

A SFSM $\Upsilon = (Q, X, F, U)$ is a soft finite switchboard state machine (SFSSM, for short) if it is both switching and commutative.

5.2.4 Proposition

If $\Upsilon = (Q, X, F, U)$ is a commutative SFSM, then

$$F^*(q, xa, p) = F^*(q, ax, p) \text{ for all } p, q \in Q,\ a \in X \text{ and } x \in X^*.$$

Proof. Let $p, q \in Q$, $a \in X$ and $x \in X^*$. We prove the result by induction on $|x| = n$.

If $n = 0$, then $x = \lambda$ and so $xa = \lambda a = a = a\lambda = ax$. Hence, $F^*(q, xa, p) = F^*(q, ax, p)$. Thus, the result is true for $n = 0$.

Suppose that the result is valid for all $u \in X^*$ such that $|u| = n - 1$. Let $y = ub$, where $u \in X^*$, $b \in X$ and $|u| = n - 1$. Then,

$$\begin{aligned} F^*(q, ya, p) &= F^*(q, uba, p) = \bigcup_{r \in Q} \{F^*(q, u, r) \cap F^*(r, ba, p)\} \\ &= \bigcup_{r \in Q} \{F^*(q, u, r) \cap F^*(r, ab, p)\} \\ &= F^*(q, uab, p) = \bigcup_{r \in Q} \{F^*(q, ua, r) \cap F^*(r, b, p)\} \\ &= \bigcup_{r \in Q} \{F^*(q, au, r) \cap F^*(r, b, p)\} \\ &= F^*(q, aub, p) = F^*(q, ay, p). \end{aligned}$$

This completes the proof. ■

5.2.5 Proposition

Let $\Upsilon = (Q, X, F, U)$ be a SFSSM. Then, $F^*(q, x, p) = F^*(p, x, q)$ for all $p, q \in Q$ and $x \in X^*$.

Proof. Let $p, q \in Q$ and $x \in X^*$. Suppose $|x| = n$. Since $x = \lambda$ whenever $n = 0$, we have

$$F^*(q, x, p) = F^*(q, \lambda, p) = F^*(p, \lambda, q) = F^*(p, x, q).$$

Assume that the result is true for all $u \in X^*$ such that $|u| = n - 1$, $n > 0$. Let $y = ua$ where $u \in X^*$, $a \in X$ and $|u| = n - 1$. Then

$$\begin{aligned} F^*(q, y, p) &= F^*(q, ua, p) = \bigcup_{r \in Q} \{F^*(q, u, r) \cap F^*(r, a, p)\} \\ &= \bigcup_{r \in Q} \{F^*(r, u, q) \cap F^*(p, a, r)\} \\ &= \bigcup_{r \in Q} \{F^*(p, a, r) \cap F^*(r, u, q)\} \\ &= F^*(p, au, q) = F^*(p, ua, q) = F^*(p, y, q). \end{aligned}$$

This shows that the result is true for $|x| = n$. This completes the proof. ■

5.2.6 Proposition

Let $\Upsilon = (Q, X, F, U)$ be a SFSSM. Then, $F^*(q, xy, p) = F^*(q, yx, p)$ for all $p, q \in Q$ and $x, y \in X^*$.

Proof. Let $p, q \in Q$ and $x \in X^*$. Suppose $|y| = n$. Since $y = \lambda$, whenever $n = 0$, we have

$$F^*(q, xy, p) = F^*(q, x\lambda, p) = F^*(q, x, p) = F^*(q, \lambda x, p) = F^*(q, yx, p).$$

Assume that $F^*(q, xy, p) = F^*(q, yx, p)$ for all $y \in X^*$ such that $|y| = n-1$, $n > 0$. Let $z = ua$ where $u \in X^*$, $a \in X$ and $|u| = n-1$. Then,

$$\begin{aligned}
F^*(q, xz, p) &= F^*(q, xua, p) = \bigcup_{r \in Q} \{F^*(q, xu, r) \cap F(r, a, p)\} \\
&= \bigcup_{r \in Q} \{F^*(q, ux, r) \cap F(r, a, p)\} \\
&= \bigcup_{r \in Q} \{F^*(r, ux, q) \cap F(p, a, r)\} \\
&= \bigcup_{r \in Q} \{F(p, a, r) \cap F^*(r, ux, q)\} \\
&= F^*(p, aux, q) = \bigcup_{r \in Q} \{F^*(p, au, r) \cap F^*(r, x, q)\} \\
&= \bigcup_{r \in Q} \{F^*(p, ua, r) \cap F^*(r, x, q)\} \\
&= F^*(p, uax, q) = F^*(q, uax, p) = F^*(q, zx, p).
\end{aligned}$$

This completes the proof. ■

5.2.7 Theorem

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$, $i = 1, 2$, be SFSM. Let $(f, g) : \Upsilon_1 \longrightarrow \Upsilon_2$, be an onto strong homomorphism. If Υ_1 is commutative, then so is Υ_2.

Proof. Let $p_2, q_2 \in Q_2$. Then, there exist $p_1, q_1 \in Q_1$ such that $f(p_1) = p_2$ and $f(q_2) = q_2$. Let $a_2, b_2 \in X_2$. Then, there are $a_1, b_1 \in X_1$ such that $g(a_1) = a_2$, and $g(b_1) = b_2$. Since Υ_1 is commutative, we have

$$\begin{aligned}
F_2^*(q_2, a_2b_2, p_2) &= F_2^*(f(q_1), g(a_1)g(b_1), f(p_1)) \\
&= F_2^*(f(q_1), g^*(a_1b_1), f(p_1)) \\
&= \bigcup \{F_1^*(q_1, a_1b_1, r_1) \mid r_1 \in Q_1,\ f(r_1) = f(p_1)\} \\
&= \bigcup \{F_1^*(q_1, b_1a_1, r_1) \mid r_1 \in Q_1,\ f(r_1) = f(p_1)\} \\
&= F_2^*(f(q_1), g^*(b_1a_1), f(p_1)) \\
&= F_2^*(f(q_1), g(b_1)g(a_1), f(p_1)) \\
&= F_2^*(q_2, b_2a_2, p_2).
\end{aligned}$$

Hence, Υ_2 is commutative SFSM. This completes the proof. ■

5.2.8 Theorem

Every switching soft finite state machine $\Upsilon = (Q, X, F, U)$ satisfies the soft exchange property.

Proof. Let $\Upsilon = (Q, X, F, U)$ be a switching SFSM, that is

$$F(q, x, p) = F(p, x, q) \text{ for all } p, q \in Q \text{ and } x \in X.$$

From Theorem 4.1.7, it is enough to prove (for all $p, q \in Q$) $p \in SS(q) \iff q \in SS(p)$.

Let $p \in SS(q)$, then there exist $x \in X^*$ such that $F^*(q, x, p) \neq \emptyset$, where $x = x_1 x_2 ... x_n$ and $x_1, x_2, ... x_n \in X$. Since $F^*(q, x, p) \neq \emptyset$, so there exist $r_i, s \in Q$ such that

$$F(q, x_1, r_1) \cap F(r_1, x_2, r_2) \cap ... \cap F(r_{n-1}, x_n, p) \neq \emptyset.$$

Since Υ is switching, so it can be written as

$$F(p, x_n, r_{n-1}) \cap F(r_{n-1}, x_{n-1}, r_{n-2}) \cap ... \cap F(r_1, x_1, q) \neq \emptyset.$$

$$\Longrightarrow \bigcup_{r_i \in Q} F(p, x_n, r_{n-1}) \cap F(r_{n-1}, x_{n-1}, r_{n-2}) \cap ... \cap F(r_1, x_1, q) \neq \emptyset$$

$$\Longrightarrow F^*(p, y, q) \neq \emptyset.$$

Hence, $q \in SS(p)$. Similarly, if $q \in SS(p)$ we can prove $p \in SS(q)$. This completes the proof. ■

Chapter 6

Cartesian Composition of Soft Finite State Machines

In this chapter, we present a new product of two soft finite machines Υ_1 and Υ_2, written as $\Upsilon_1 \cdot \Upsilon_2$ and called the cartesian composition of Υ_1 and Υ_2. We also define the concept of soft admissible partitions and construct the quotient structure of SFSM with the help of soft admissible partitions. Finally, we discuss the associativity of wreath product, sum and cascade products of soft finite state machines.

6.1 Cartesian Composition of Soft Finite State Machines

In this section, we study the concept of the cartesian composition of soft finite state machines. We show that soft finite state machines and their cartesian composition share many structural properties. Some of these properties are connectedness, strong connectedness, commutativity and perfection.

6.1.1 Definition

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$, $i = 1, 2$, be soft finite machines and let $X_1 \cap X_2 = \emptyset$. Define

$$\Upsilon_1 \cdot \Upsilon_2 = (Q_1 \times Q_2, X_1 \cup X_2, F_1 \cdot F_2, U),$$

where

$$F_1 \cdot F_2((p_1, p_2), a, (q_1, q_2)) = \begin{cases} F_1(p_1, a, q_1) & \text{if } a \in X_1 \text{ and } p_2 = q_2 \\ F_2(p_2, a, q_2) & \text{if } a \in X_2 \text{ and } p_1 = q_1 \\ \emptyset & \text{otherwise,} \end{cases}$$

for all $(p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$, $a \in X_1 \cup X_2$. Then, $\Upsilon_1 \cdot \Upsilon_2$ is a soft finite state machine and is called the cartesian composition of Υ_1 and Υ_2.

6.1.2 Theorem

Let $\Upsilon_1 \cdot \Upsilon_2 = (Q_1 \times Q_2, X_1 \cup X_2, F_1 \cdot F_2, U)$ be the cartesian composition of Υ_1 and Υ_2, where $\Upsilon_i = (Q_i, X_i, F_i, U)$, $i = 1, 2$, are soft finite machines such that $X_1 \cap X_2 = \emptyset$. Then, for all $x \in X_1^* \cup X_2^*$, such that $x \neq \lambda$,

$$(F_1 \cdot F_2)^* ((p_1, p_2), x, (q_1, q_2)) = \begin{cases} F_1^* (p_1, x, q_1) & \text{if } x \in X_1^* \text{ and } p_2 = q_2 \\ F_2^* (p_2, x, q_2) & \text{if } x \in X_2^* \text{ and } p_1 = q_1 \\ \emptyset & \text{otherwise,} \end{cases}$$

for all (p_1, p_2) and $(q_1, q_2) \in Q_1 \times Q_2$.

Proof. Let $x \in X_1^* \cup X_2^*$, such that $x \neq \lambda$ and $|x| = n$. Suppose $n = 1$, then clearly it is true. Suppose the result is true for all $u \in X_1^*$, such that $|u| = n-1$, where $n > 1$. Let $x = ua$ where $u \in X_1^*$ and $a \in X_1$. Now,

$$\begin{aligned}
(F_1 \cdot F_2)^* ((p_1, p_2), x, (q_1, q_2)) &= (F_1 \cdot F_2)^* ((p_1 \times p_2), ua, (q_1, q_2)) \\
&= \bigcup_{(r_1, r_2) \in Q_1 \times Q_2} \left\{ \begin{array}{c} (F_1 \cdot F_2)^* ((p_1, p_2), u, (r_1, r_2)) \cap \\ (F_1 \cdot F_2) ((r_1, r_2), a, (q_1, q_2)) \end{array} \right\} \\
&= \bigcup_{r_1 \in Q_1} \left\{ \begin{array}{c} (F_1 \cdot F_2)^* ((p_1, p_2), u, (r_1, q_2)) \cap \\ F_1 (r_1, a, q_1) \end{array} \right\} \\
&= \begin{cases} \bigcup_{r_1 \in Q_1} \{F_1^* (p_1, u, r_1) \cap F_1 (r_1, a, q_1)\} & \text{if } p_2 = q_2 \\ \emptyset \text{ otherwise} \end{cases} \\
&= \begin{cases} F_1^* (p_1, ua, q_1) & \text{if } p_2 = q_2 \\ \emptyset \text{ otherwise.} \end{cases}
\end{aligned}$$

Hence, the result is valid for $|x| = n$. Similarly, we can prove if $x \in X_2^*$. This completes the proof. ■

6.1.3 Theorem

Let $\Upsilon_1 \cdot \Upsilon_2 = (Q_1 \times Q_2, X_1 \cup X_2, F_1 \cdot F_2, U)$ be the cartesian composition of Υ_1 and Υ_2, where $\Upsilon_i = (Q_i, X_i, F_i, U)$, $i = 1, 2$, are soft finite machines such that $X_1 \cap X_2 = \emptyset$. Then, for all $x \in X_1^*$, $y \in X_2^*$

$$\begin{aligned}
(F_1 \cdot F_2)^* ((p_1, p_2), xy, (q_1, q_2)) &= F_1^* (p_1, x, q_1) \cap F_2^* (p_2, y, q_2) \\
&= (F_1 \times F_2)^* ((p_1, p_2), yx, (q_1, q_2)),
\end{aligned}$$

for all (p_1, p_2) and $(q_1, q_2) \in Q_1 \times Q_2$.

Proof. Let (p_1, p_2) and $(q_1, q_2) \in Q_1 \times Q_2$ and $x \in X_1^*$, $y \in X_2^*$. If $x = y = \lambda$, then $xy = \lambda$. Suppose $p_1 = q_1$ and $q_2 = p_2$, then $(p_1, p_2) = (q_1, q_2)$. Hence,

$$\begin{aligned}(F_1 \cdot F_2)^* ((p_1, p_2), xy, (q_1, q_2)) &= U \\ &= F_1^* (p_1, x, q_1) \cap F_2^* (p_2, y, q_2).\end{aligned}$$

If $(p_1, p_2) \neq (q_1, q_2)$, then either $p_1 \neq q_1$ or $p_2 \neq q_2$. Thus, $F_1^* (p_1, x, q_1) \cap F_2^* (p_2, y, q_2) = \emptyset$. Hence,

$$\begin{aligned}F_1^* (p_1, x, q_1) \cap F_2^* (p_2, y, q_2) &= \emptyset \\ &= (F_1 \cdot F_2)^* ((p_1, p_2), xy, (q_1, q_2)).\end{aligned}$$

If $x \neq \lambda$ and $y = \lambda$ or $x = \lambda$ and $y \neq \lambda$. Then, by Theorem 6.1.2, it can easily verify that

$$(F_1 \cdot F_2)^* ((p_1, p_2), xy, (q_1, q_2)) = F_1^* (p_1, x, q_1) \cap F_2^* (p_2, y, q_2).$$

If $x \neq \lambda$ and $y \neq \lambda$, then

$$\begin{aligned}(F_1 \cdot F_2)^* ((p_1, p_2), xy, (q_1, q_2)) &= \bigcup_{(r_1, r_2) \in Q_1 \times Q_2} \left\{ \begin{array}{c} (F_1 \cdot F_2)^* ((p_1, p_2), x, (r_1, r_2)) \cap \\ (F_1 \cdot F_2)^* ((r_1, r_2), y, (q_1, q_2)) \end{array} \right\} \\ &= \bigcup_{r_1 \in Q_1} \left\{ \bigcup_{r_2 \in Q_2} \left\{ \begin{array}{c} (F_1 \cdot F_2)^* ((p_1, p_2), x, (r_1, r_2)) \cap \\ (F_1 \cdot F_2)^* ((r_1, r_2), y, (q_1, q_2)) \end{array} \right\} \right\} \\ &= \bigcup_{r_1 \in Q_1} \left\{ \begin{array}{c} (F_1 \cdot F_2)^* ((p_1, p_2), x, (r_1, p_2)) \cap \\ (F_1 \cdot F_2)^* ((r_1, p_2), y, (q_1, q_2)) \end{array} \right\} \\ &= \left\{ \begin{array}{c} (F_1 \cdot F_2)^* ((p_1, p_2), x, (q_1, p_2)) \cap \\ (F_1 \cdot F_2)^* ((q_1, p_2), y, (q_1, q_2)) \end{array} \right\} \\ &= F_1^* (p_1, x, q_1) \cap F_2^* (p_2, y, q_2).\end{aligned}$$

Similarly, we can prove

$$(F_1 \cdot F_2)^* ((p_1, p_2), yx, (q_1, q_2)) = F_1^* (p_1, x, q_1) \cap F_2^* (p_2, y, q_2).$$

This completes the proof. ■

6.1.4 Theorem

Let $\Upsilon_1 \cdot \Upsilon_2 = (Q_1 \times Q_2, X_1 \cup X_2, F_1 \times F_2, U)$ be the cartesian composition of Υ_1 and Υ_2, where $\Upsilon_i = (Q_i, X_i, F_i, U)$, $i = 1, 2$, are soft finite machines such that $X_1 \cap X_2 = \emptyset$. Then, for all $x \in (X_1 \cup X_2)^*$, there exist $u \in X_1^*$ and $v \in X_2^*$ such that

$$\begin{aligned}(F_1 \cdot F_2)^* ((p_1, p_2), x, (q_1, q_2)) &= (F_1 \cdot F_2)^* ((p_1, p_2), uv, (q_1, q_2)) \\ &= F_1^* (p_1, u, q_1) \cap F_2^* (p_2, v, q_2),\end{aligned}$$

for all (p_1, p_2) and $(q_1, q_2) \in Q_1 \times Q_2$.

Proof. Let (p_1, p_2) and $(q_1, q_2) \in Q_1 \times Q_2$ and $x \in (X_1 \cup X_2)^*$. If $x = \lambda$, then we can choose $u = v = \lambda$. So, the result is true in this case. Suppose $x \neq \lambda$, if $x \in X_1^*$ or $x \in X_2^*$, again in this case result is trivially true. Suppose $x \notin X_1^*$ and $x \notin X_2^*$, then we have the following possibilities

Case 1: If $x = uv$, $u \in X_1^+$, $v \in X_2^+$, then the result follows by Theorem 6.1.3.

Case 2: Suppose $x = u_1 v_1 u_2$, $u_1, u_2 \in X_1^*$ and $v_1 \in X_2^*$, $u_i \neq \lambda$ and $v_1 \neq \lambda$ for $i = 1, 2$. Let $u = u_1 u_2 \in X_1^*$ and $v = v_1 \in X_2^*$. Then, by Theorem 6.1.3,

$$(F_1 \cdot F_2)^* \left((p_1, p_2), v_1 u_2, (q_1, q_2) \right) = (F_1 \times F_2)^* \left((p_1, p_2), u_2 v_1, (q_1, q_2) \right),$$

for all (p_1, p_2) and $(q_1, q_2) \in Q_1 \times Q_2$. Thus,

$$\begin{aligned} (F_1 \cdot F_2)^* \left((p_1, p_2), x, (q_1, q_2) \right) &= (F_1 \cdot F_2)^* \left((p_1, p_2), u_1 v_1 u_2, (q_1, q_2) \right) \\ &= \bigcup_{(r_1, r_2) \in Q_1 \times Q_2} \left\{ \begin{array}{c} (F_1 \cdot F_2)^* \left((p_1, p_2), u_1, (r_1, r_2) \right) \cap \\ (F_1 \cdot F_2)^* \left((r_1, r_2), v_1 u_2, (q_1, q_2) \right) \end{array} \right\} \\ &= \bigcup_{(r_1, r_2) \in Q_1 \times Q_2} \left\{ \begin{array}{c} (F_1 \cdot F_2)^* \left((p_1, p_2), u_1, (r_1, r_2) \right) \cap \\ (F_1 \cdot F_2)^* \left((r_1, r_2), u_2 v_1, (q_1, q_2) \right) \end{array} \right\} \\ &= (F_1 \cdot F_2)^* \left((p_1, p_2), u_1 u_2 v_1, (q_1, q_2) \right) \\ &= (F_1 \cdot F_2)^* \left((p_1, p_2), uv, (q_1, q_2) \right). \end{aligned}$$

Case 3: Suppose $x = v_1 u_1 v_2$, $u_1 \in X_1^*$ and $v_1, v_2 \in X_2^*$ $u_1 \neq \lambda$ and $v_i \neq \lambda$, for $i = 1, 2$. Let $v = v_1 v_2 \in X_2^*$ and $u = u_1 \in X_1^*$. Then, by Theorem 6.1.3,

$$(F_1 \cdot F_2)^* \left((p_1, p_2), v_1 u_1, (q_1, q_2) \right) = (F_1 \cdot F_2)^* \left((p_1, p_2), u_1 v_1, (q_1, q_2) \right),$$

for all (p_1, p_2) and $(q_1, q_2) \in Q_1 \times Q_2$. Thus,

$$\begin{aligned} (F_1 \cdot F_2)^* \left((p_1, p_2), x, (q_1, q_2) \right) &= (F_1 \cdot F_2)^* \left((p_1, p_2), v_1 u_1 v_2, (q_1, q_2) \right) \\ &= \bigcup_{(r_1, r_2) \in Q_1 \times Q_2} \left\{ \begin{array}{c} (F_1 \cdot F_2)^* \left((p_1, p_2), v_1 u_1, (r_1, r_2) \right) \cap \\ (F_1 \cdot F_2)^* \left((r_1, r_2), v_2, (q_1, q_2) \right) \end{array} \right\} \\ &= \bigcup_{(r_1, r_2) \in Q_1 \times Q_2} \left\{ \begin{array}{c} (F_1 \cdot F_2)^* \left((p_1, p_2), u_1 v_1, (r_1, r_2) \right) \cap \\ (F_1 \cdot F_2)^* \left((r_1, r_2), v_2, (q_1, q_2) \right) \end{array} \right\} \\ &= (F_1 \cdot F_2)^* \left((p_1, p_2), u_1 v_1 v_2, (q_1, q_2) \right) \\ &= (F_1 \cdot F_2)^* \left((p_1, p_2), uv, (q_1, q_2) \right). \end{aligned}$$

Case 4: Suppose $x = u_1 v_1 u_2 v_2$, $u_1, u_2 \in X_1^*$ and $v_1, v_2 \in X_2^*$, $u_i \neq \lambda$ and $v_i \neq \lambda$, for $i = 1, 2$. Let $u = u_1u_2 \in X_1^*$ and $v = v_1v_2 \in X_2^*$. Thus,

$$\begin{aligned}
(F_1 \cdot F_2)^* ((p_1, p_2), x, (q_1, q_2)) &= (F_1 \cdot F_2)^* ((p_1, p_2), u_1v_1u_2v_2, (q_1, q_2)) \\
&= \bigcup_{(r_1,r_2)\in Q_1\times Q_2} \left\{ \begin{array}{c} (F_1 \cdot F_2)^* ((p_1, p_2), u_1, (r_1, r_2)) \cap \\ (F_1 \cdot F_2)^* ((r_1, r_2), v_1u_2v_2, (q_1, q_2)) \end{array} \right\} \\
&= \bigcup_{(r_1,r_2)\in Q_1\times Q_2} \left\{ \begin{array}{c} (F_1 \cdot F_2)^* ((p_1, p_2), u_1, (r_1, r_2)) \cap \\ (F_1 \cdot F_2)^* ((r_1, r_2), u_2v_1v_2, (q_1, q_2)) \end{array} \right\} \\
&= (F_1 \cdot F_2)^* ((p_1, p_2), u_1u_2v_1v_2, (q_1, q_2)) \\
&= (F_1 \cdot F_2)^* ((p_1, p_2), uv, (q_1, q_2)).
\end{aligned}$$

Case 5: Let $x = v_1u_1v_2u_2$, $u_1, u_2 \in X_1^*$ and $v_1, v_2 \in X_2^*$, $u_i \neq \lambda$ and $v_i \neq \lambda$ for $i = 1, 2$. Let $u = u_1u_2 \in X_1^*$ and $v = v_1v_2 \in X_2^*$. In this case, the proof is similar to case 4.

Case 6: Suppose the result is true for all $x = u_1v_1u_2v_2....u_{n-1}v_{n-1} \in (X_1 \cup X_2)^*$ for $n > 1$. Let $u' = u_1u_2....u_{n-1}$ and $v' = v_1v_2....v_{n-1}$. Now, $u = u'u_n$ and $v = v'v_n$, then

$$\begin{aligned}
& (F_1 \cdot F_2)^* ((p_1, p_2), u_1v_1u_2v_2....u_nv_n, (q_1, q_2)) \\
= & \bigcup_{(r_1,r_2)\in Q_1\times Q_2} \left\{ \begin{array}{c} (F_1 \cdot F_2)^* ((p_1, p_2), u_1v_1u_2v_2....u_{n-1}v_{n-1}, (r_1, r_2)) \cap \\ (F_1 \cdot F_2)^* ((r_1, r_2), u_nv_n, (q_1, q_2)) \end{array} \right\} \\
= & \bigcup_{(r_1,r_2)\in Q_1\times Q_2} \left\{ \begin{array}{c} (F_1 \cdot F_2)^* ((p_1, p_2), u'v', (r_1, r_2)) \cap \\ (F_1 \cdot F_2)^* ((r_1, r_2), u_nv_n, (q_1, q_2)) \end{array} \right\} \\
= & (F_1 \cdot F_2)^* \left((p_1, p_2), u'v'u_nv_n, (q_1, q_2)\right) \\
= & (F_1 \cdot F_2)^* ((p_1, p_2), uv, (q_1, q_2)).
\end{aligned}$$

The result follows by induction. Similarly, we can prove when $x = v_1u_1v_2u_2...v_nu_n \in (X_1 \cup X_2)^*$. This completes the proof. ■

6.1.5 Theorem

Let $\Upsilon_1 \cdot \Upsilon_2 = (Q_1 \times Q_2, X_1 \cup X_2, F_1 \cdot F_2, U)$ be the cartesian composition of Υ_1 and Υ_2, where $\Upsilon_i = (Q_i, X_i, F_i, U)$, $i = 1, 2$, are soft finite machines such that $X_1 \cap X_2 = \emptyset$. Then, $\Upsilon_1 \cdot \Upsilon_2$ is soft connected if, and only if, Υ_1 and Υ_2 are soft connected.

Proof. Suppose Υ_1 and Υ_2 are soft connected. To prove $\Upsilon_1 \cdot \Upsilon_2$ is soft connected, for this it is enough to prove $SC((p, p')) = Q_1 \times Q_2$, by Theorem 4.2.13. Let $(q, q') \in Q_1 \times Q_2$, then $q, p \in Q_1$ and $q', p' \in Q_2$. As Υ_1 is soft connected, so there exist $q_0, q_1, ...q_k \in Q_1$, $q = q_0$, $p = q_k$, and there exist $x_1, x_2, ..., x_n \in X_1$ such that for all $i = 1, 2, ..., k$ either $F_1(q_i, x_i, q_{i-1}) \neq \emptyset$ or $F_1(q_{i-1}, x_i, q_i) \neq \emptyset$. Similarly, for Υ_2, there exist $q'_0, q'_1, ...q'_l \in Q_1$, $q' = q'_0$, $p' = q'_l$, and there exist $y_1, y_2, ..., y_l \in X_2$ such that for all $i = 1, 2, ..., l$ either $F_2\left(q'_i, y_i, q'_{i-1}\right) \neq \emptyset$ or $F_2\left(q'_{i-1}, y_i, q'_i\right) \neq \emptyset$. Now, consider the sequences $(q, q') = (q_0, q'_0), (q_1, q'_0), (q_2, q'_0), ..., (q_k, q'_0), (q_k, q'_1), (q_k, q'_2), ..., (q_k, q'_l) = (p, p') \in Q_1 \times Q_2$ and the sequence $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in X_1 \cup X_2$. Then, it can easily check that for all $i = 1, 2, ..., k$ either

$$\begin{aligned} (F_1 \cdot F_2)\left(\left(q_i, q'_0\right), x_i, \left(q_{i-1}, q'_0\right)\right) &\neq \emptyset \text{ or} \\ (F_1 \cdot F_2)\left(\left(q_{i-1}, q'_0\right), x_i, \left(q_i, q'_0\right)\right) &\neq \emptyset \end{aligned}$$

and for all $i = 1, 2, ..., l$ either

$$\begin{aligned} (F_1 \cdot F_2)\left(\left(q_k, q'_i\right), y_i, \left(q_k, q'_{i-1}\right)\right) &\neq \emptyset \text{ or} \\ (F_1 \cdot F_2)\left(\left(q_k, q'_{i-1}\right), y_i, \left(q_k, q'_i\right)\right) &\neq \emptyset. \end{aligned}$$

Hence. (q, q') and (p, p') soft connected. This implies $(q, q') \in SC((p, p'))$. Hence, $SC((p, p')) = Q_1 \times Q_2$.

Conversely, suppose $\Upsilon_1 \cdot \Upsilon_2$ is soft connected. Let $q, p \in Q_1$ and $q' \in Q_2$. If $q = p$, then q and p are soft connected. If $q \neq p$, then there exist $(q, q') = (q_0, q'_0), (q_1, q'_1), (q_2, q'_2), ..., (q_m, q'_m) = (p, q') \in Q_1 \times Q_2$ and $x_1, x_2, ..., x_m \in X_1 \cup X_2$ such that for all $i = 1, 2, ..., m$ either

$$\begin{aligned} (F_1 \cdot F_2)\left(\left(q_{i-1}, q'_{i-1}\right), x_i, \left(q_i, q'_i\right)\right) &\neq \emptyset \text{ or} \\ (F_1 \cdot F_2)\left(\left(q_i, q'_i\right), x_i, \left(q_{i-1}, q'_{i-1}\right)\right) &\neq \emptyset. \end{aligned}$$

Clearly, if $q_{i-1} \neq q_i$, then $q'_{i-1} = q'_i$ and if $q'_{i-1} \neq q'_i$, then $q_{i-1} = q_i$ for all $i = 1, 2, 3, ..., m$. Let $\{q = q_{i_1}, q_{i_2}, ..., q_{i_n} = p\}$ be the set of all distinct $q_i \in \{q_0, q_1, ..., q_m\}$ and let $x_{i_1}, x_{i_2}, ..., x_{i_n}$ be the corresponding x_i's. Then $x_{i_1}, x_{i_2}, ..., x_{i_n} \in X_1$ and for all $j = 1, 2, ..., n$ either

$$\begin{aligned} F_1\left(q_{i_{j-1}}, x_{i_j}, q_{i_j}\right) &\neq \emptyset \text{ or} \\ F_1\left(q_{i_j}, x_{i_j}, q_{i_{j-1}}\right) &\neq \emptyset. \end{aligned}$$

Thus, q and p are soft connected. Hence, Υ_1 is soft connected. Similarly, we can prove Υ_2 is soft connected. ■

6.1.6 Theorem

Let $\Upsilon_1 \cdot \Upsilon_2 = (Q_1 \times Q_2, X_1 \cup X_2, F_1 \cdot F_2, U)$ be the cartesian composition of Υ_1 and Υ_2, where $\Upsilon_i = (Q_i, X_i, F_i, U)$, $i = 1, 2$, are soft finite machines such that $X_1 \cap X_2 = \emptyset$. Then, Υ_1 and Υ_2 are weakly soft connected if $\Upsilon_1 \cdot \Upsilon_2$ is weakly soft connected.

Proof. Suppose $\Upsilon_1 \cdot \Upsilon_2$ is weakly soft connected. Then, $SS(SS((q,p))) = Q_1 \times Q_2$ for all $(q,p) \in Q_1 \times Q_2$. First we prove Υ_1 is weakly soft connected. Let $q \in Q_1$, then $SS(SS(q)) \subseteq Q_1$. For any $q' \in Q_1$ and $p \in Q_2$, then $(q',p) \in Q_1 \times Q_2 = SS(SS((q,p)))$. Then, there exists $(q_1,p_1) \in SS((q,p))$ such that $(q',p) \in SS((q_1,p_1))$. Since $(q',p) \in SS((q_1,p_1))$ so there exists $x \in (X_1 \cup X_2)^*$ such that

$$(F_1 \cdot F_2)^* \left((q_1,p_1), x, \left(q',p\right)\right) \neq \emptyset.$$

Then, by Theorem 6.1.4, there exist $u \in X_1^*$ and $v \in X_2^*$ such that

$$F_1\left(q_1,u,q'\right) \cap F_2(p_1,v,p) \neq \emptyset,$$

that is,

$$F_1\left(q_1,u,q'\right) \neq \emptyset \text{ and } F_2(p_1,v,p) \neq \emptyset,$$

thus $q' \in SS(q_1)$ and $p \in SS(p_1)$ .... (1)

Also, $(q_1,p_1) \in SS((q,p))$ so there exist $y \in (X_1 \cup X_2)^*$ such that

$$(F_1 \cdot F_2)^* ((q,p), y, (q_1,p_1)) \neq \emptyset.$$

Then, by Theorem 6.1.4, there exist $w \in X_1^*$ and $z \in X_2^*$ such that

$$F_1(q,w,q_1) \cap F_2(p,z,p_1) \neq \emptyset,$$

that is,

$$F_1(q,w,q_1) \neq \emptyset \text{ and } F_2(p,z,p_1) \neq \emptyset,$$

thus $q_1 \in SS(q)$ and $p_1 \in SS(p)$ .... (2)

Thus, from (1) and (2), $q' \in SS(q_1) \subseteq SS(SS(q))$and $p \in SS(p_1) \subseteq SS(SS(p))$.

Hence, $Q_1 \subseteq SS(SS(q))$ and $Q_2 \subseteq SS(SS(p))$. This completes the proof. ■

6.1.7 Theorem

Let $\Upsilon_1 \cdot \Upsilon_2 = (Q_1 \times Q_2, X_1 \cup X_2, F_1 \cdot F_2, U)$ be the cartesian composition of Υ_1 and Υ_2, where $\Upsilon_i = (Q_i, X_i, F_i, U)$, $i = 1,2$, are soft finite machines such that $X_1 \cap X_2 = \emptyset$. Then, $\Upsilon_1 \cdot \Upsilon_2$ is commutative if, and only if, Υ_1 and Υ_2 are commutative.

Proof. Suppose $\Upsilon_1 \cdot \Upsilon_2$ is commutative. Let $p,q \in Q_1$ and $x,y \in X_1^*$ and $p' \in Q_2$. Then, $\left(p,p'\right), \left(q,p'\right) \in Q_1 \times Q_2$ such that

$$(F_1 \cdot F_2)^* \left(\left(p,p'\right), xy, \left(q,p'\right)\right) = (F_1 \cdot F_2)^* \left(\left(p,p'\right), yx, \left(q,p'\right)\right) \qquad (1)$$

because $\Upsilon_1 \cdot \Upsilon_2$ is commutative. Then, equation (1) becomes $F_1^*(p,xy,q) = F_1^*(p,yx,q)$. Similarly, we can prove Υ_2 is commutative. Converse is trivial. ■

6.1.8 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. If Υ is weakly soft connected and commutative, then Υ is said to be weakly soft perfect.

6.1.9 Theorem

Let $\Upsilon_1 \cdot \Upsilon_2 = (Q_1 \times Q_2, X_1 \cup X_2, F_1 \cdot F_2, U)$ be the cartesian composition of Υ_1 and Υ_2, where $\Upsilon_i = (Q_i, X_i, F_i, U)$, $i = 1, 2$, are soft finite machines such that $X_1 \cap X_2 = \emptyset$. Then, Υ_1 and Υ_2 are weakly soft perfect if $\Upsilon_1 \cdot \Upsilon_2$ is weakly soft perfect.

Proof. Straightforward. ■

6.1.10 Theorem

Let $\Upsilon_i = (Q_i, X, F_i, U)$, $i = 1, 2$, be a soft finite machines such that $X_1 \cap X_2 = \emptyset$. Let $\Im_i = \left(Q_i', X, F_{Q_i}, U\right)$ be a soft submachine of Υ_i for $i = 1, 2$. Then $\Im_1 \cdot \Im_2$ is a submachine of $\Upsilon_1 \cdot \Upsilon_2$. Conversely, if $\Im = \left(Q_1' \times Q_2', X_1 \cup X_2, F, U\right)$ is a soft submachine of $\Upsilon_1 \cdot \Upsilon_2$, then there exist soft submachines $\Im_1$ of Υ_1 and $\Im_2$ of Υ_2 such that $\Im = \Im_1 \cdot \Im_2$

Proof. Let $\Im_i = \left(Q_i', X, F_{Q_i}, U\right)$ be a soft submachine of Υ_i for $i = 1, 2$. Now,

$$\Im_1 \cdot \Im_2 = \left(Q_1' \times Q_2', X_1 \cup X_2, F_{Q_1'} \cdot F_{Q_2'}, U\right).$$

Let $(p_1, q_1) \in SS\left(Q_1' \times Q_2'\right)$. Then, there exist $x \in (X_1 \cup X_2)^*$ and $(p, q) \in Q_1' \times Q_2'$ such that

$$(F_1 \cdot F_2)^* \left((p, q), x, (p_1, q_1)\right) \neq \emptyset.$$

Let $x = uv$ be the standard form of x, where $u \in X_1^*$ and $v \in X_2^*$. Now, by Theorem 6.1.4,

$$\begin{aligned} F_1^* (p, u, p_1) \cap F_2^* (q, v, q_1) &= (F_1 \cdot F_2)^* \left((p, q), uv, (p_1, q_1)\right) \\ &= (F_1 \cdot F_2)^* \left((p, q), x, (p_1, q_1)\right) \neq \emptyset. \end{aligned}$$

Thus, $F_1^* (p, u, p_1) \neq \emptyset$ and $F_2^* (q, v, q_1) \neq \emptyset$. So $p_1 \in SS(p) \subseteq SS\left(Q_1'\right) = Q_1'$ and $q_1 \in SS(q) \subseteq SS\left(Q_2'\right) = Q_2'$. Thus, $(p_1, q_1) \in Q_1' \times Q_2'$. Thus, $SS\left(Q_1' \times Q_2'\right) \subseteq Q_1' \times Q_2'$. Now, we want to prove

$$F_1 \cdot F_2 \mid_{\left(Q_1' \times Q_2'\right) \times (X_1 \cup X_2) \times \left(Q_1' \times Q_2'\right)} = F_{Q_1'} \cdot F_{Q_2'}.$$

For this, let $(p,q),(p_1,q_1) \in Q_1' \times Q_2'$ and $x \in X_1 \cup X_2$. Then,

$$\begin{aligned}
\left(F_{Q_1'} \cdot F_{Q_2'}\right)((p,q),x,(p_1,q_1)) &= \begin{cases} F_{Q_1'}(p,x,p_1) & \text{if } x \in X_1 \text{ and } q = q_1 \\ F_{Q_2'}(q,x,q_1) & \text{if } x \in X_2 \text{ and } p = p_1 \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} F_1(p,x,p_1) & \text{if } x \in X_1 \text{ and } q = q_1 \\ F_2(q,x,q_1) & \text{if } x \in X_2 \text{ and } p = p_1 \\ \emptyset & \text{otherwise} \end{cases} \\
&= (F_1 \cdot F_2)((p,q),x,(p_1,q_1)).
\end{aligned}$$

Thus, $\Im_1 \cdot \Im_2 = \left(Q_1' \times Q_2', X_1 \cup X_2, F_{Q_1'} \cdot F_{Q_2'}, U\right)$ is a soft submachine of $\Upsilon_1 \cdot \Upsilon_2$.

Conversely, suppose that $\Im = \left(Q_1' \times Q_2', X_1 \cup X_2, F, U\right)$ is a soft submachine of $\Upsilon_1 \cdot \Upsilon_2$. Let $F_{Q_1'} = F_1 \mid_{Q_1' \times X_1 \times Q_1'}$, $F_{Q_2'} = F_2 \mid_{Q_2' \times X_2 \times Q_2'}$, $\Im_1 = \left(Q_1', X, F_{Q_1'}, U\right)$, $\Im_2 = \left(Q_2', X, F_{Q_2'}, U\right)$. Let $r \in SS\left(Q_1'\right)$. Then, there exist $p \in Q_1'$ and $x \in X_1^*$ such that $F_1(p,x,r) \neq \emptyset$. Let $s \in Q_2'$. Then, $(F_1 \cdot F_2)^*((p,s),x,(r,t)) = F_1^*(p,x,r) \neq \emptyset$. Thus, $(r,t) \in SS\left(Q_1' \times Q_2'\right) = Q_1' \times Q_2'$. Thus, $r \in Q_1'$. Hence, $SS\left(Q_1'\right) \subseteq Q_1'$. Thus, $\Im_1$ is a soft submachine of Υ_1. Similarly, we can prove $\Im_2$ is a soft submachine of Υ_2.

Let $(p,q),(r,s) \in Q_1' \times Q_2'$, $x \in X_1 \cup X_2$. Now

$$\begin{aligned}
F((p,q),x,(r,s)) &= (F_1 \cdot F_2)((p,q),x,(r,s)) \\
&= \begin{cases} F_1(p,x,r) & \text{if } x \in X_1 \text{ and } q = s \\ F_2(q,x,s) & \text{if } x \in X_2 \text{ and } p = r \\ \emptyset & \text{otherwise} \end{cases} \\
&= \begin{cases} F_{Q_1'}(p,x,r) & \text{if } x \in X_1 \text{ and } q = s \\ F_{Q_2'}(q,x,s) & \text{if } x \in X_2 \text{ and } p = r \\ \emptyset & \text{otherwise} \end{cases} \\
&= \left(F_{Q_1'} \cdot F_{Q_2'}\right)((p,q),x,(r,s)).
\end{aligned}$$

Thus, $\Im = \Im_1 \cdot \Im_2$ ■

6.1.11 Theorem

Let $\Upsilon_i = (Q_i, X, F_i, U)$, $i = 1,2$, be soft finite machines such that $X_1 \cap X_2 = \emptyset$. Let $\sim_i$ be an admissible relation on Υ_i. Then, $\sim_1 \cdot \sim_2$ is an admissible relation on $\Upsilon_1 \cdot \Upsilon_2$.

Proof. Let $\sim_i$ be admissible relation on $\Upsilon_i = (Q_i, X, F_i, U)$. Define $\sim_1 \cdot \sim_2$ on $\Upsilon_1 \cdot \Upsilon_2$ by

$$(p_1,q_1) \sim_1 \cdot \sim_2 \left(p_1', q_1'\right) \text{ if, and only if, } p_1 \sim_1 p_1' \text{ and } q_1 \sim_2 q_1',$$

for all $(p_1, q_1), \left(p_1^{'}, q_1^{'}\right) \in Q_1 \times Q_2$.

It can easily be prove that $\sim_1 \cdot \sim_2$ is an equivalence relation on $Q_1 \times Q_2$. Let $(p_1, q_1), \left(p_1^{'}, q_1^{'}\right) \in Q_1 \times Q_2$ be such that $(p_1, q_1) \sim_1 \cdot \sim_2 \left(p_1^{'}, q_1^{'}\right)$. Let $a \in X_1 \cup X_2$, and for some $(r_1, s_1) \in Q_1 \times Q_2$ such that

$$(F_1 \cdot F_2)\left((p_1, q_1), a, (r_1, s_1)\right) \neq \emptyset.$$

Since $a \in X_1 \cup X_2$, then either $a \in X_1$ or $a \in X_2$. If $a \in X_1$, then

$$F_1(p_1, a, r_1) = (F_1 \cdot F_2)\left((p_1, q_1), a, (r_1, s_1)\right) \neq \emptyset.$$

Thus, $q_1 = s_1$. Now $p_1 \sim_1 p_1^{'}$, $F_1(p_1, a, r_1) \neq \emptyset$, and $\sim_1$ is an admissible relation on Q_1, then there exist $t_i \in Q_1$, for $i = 1, 2, ..., n$ such that

$$t_i \sim r_1 \text{ for each } i \text{ and } \bigcup_{i=1}^{n} F\left(p_1^{'}, a, t_i\right) \supseteq F(p_1, a, r_1).$$

Hence,

$$\begin{aligned} \bigcup_{i=1}^{n} (F_1 \cdot F_2)\left(\left(p_1^{'}, q_1^{'}\right), a, \left(t_i, q_1^{'}\right)\right) &= \bigcup_{i=1}^{n} F\left(p_1^{'}, a, t_i\right) \\ &\supseteq F(p_1, a, r_1) \\ &= (F_1 \cdot F_2)\left((p_1, q_1), a, (r_1, q_1)\right). \end{aligned}$$

Note that $t_i \sim r_1$ for each $i = 1, 2, ..., n$ and $q_1^{'} \sim_2 q_1$. Hence, $\left(t_i, q_1^{'}\right) \sim_1 \cdot \sim_2 (r, q_1)$ for each $i = 1, 2, ..., n$. Thus, $\sim_1 \cdot \sim_2$ is an admissible relation on $\Upsilon_1 \cdot \Upsilon_2$. This completes the proof. ■

6.2 Soft Admissible Partitions

Here we study the concept of soft admissible partition of the Q, where Q is the state set of the soft finite state machine. Using this concept, we construct a quotient structure of soft finite state machine and soft irreducible finite state machine. In this section, we will also prove that soft admissible partition is maximal if, and only if, corresponding quotient soft finite state machine is irreducible.

6.2.1 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and $\wp = \{Q_1, Q_2, ..., Q_l\}$ be a partition of Q. Then, $\wp$ is called soft admissible partition of Q, if the following holds.

Let $a \in X$, then for all i, there exist j, where $i, j \in \{1, 2, ...l\}$, such that for all $q_1, q_2 \in Q_i$, if $F(q_1, a, r) \neq \emptyset$, for some $r \in Q$, then there exist $t_k \in Q$, for $k = 1, 2, ..n$ such that

$$\bigcup_{i=1}^{n} F(q_2, a, t_k) \supseteq F(q_1, a, r) \text{ and } t_k, r \in Q_j, \text{ for all } k = 1, 2, .., n.$$

6.2.2 Proposition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine.

(1) $I_Q = \{\{q\} : q \in Q\}$. Then, I_Q is a soft admissible partition of Q.

(2) $\{Q\}$ is a soft admissible partition of Q.

Proof. Straightforward. ■

6.2.3 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and $\wp = \{Q_1, Q_2, ..., Q_l\}$ be a partition of Q. Then, the following assertions are equivalent.

(1) $\wp$ is a soft admissible partition of Q.

(2) Let $x \in X^*$. Then, for all i, there exist j, where $i, j \in \{1, 2, ...l\}$, such that for all $q_1, q_2 \in Q_i$, if $F^*(q_1, x, r) \neq \emptyset$, for some $r \in Q$, then there exist $t_k \in Q$, for $k = 1, 2, ..n$, such that

$$\bigcup_{k=1}^{n} F^*(q_2, x, t_k) \supseteq F^*(q_1, x, r) \text{ and } t_k, r \in Q_j, \text{ for all } k = 1, 2, ..., n.$$

Proof. $(1) \Rightarrow (2)$. Let $x \in X^*$, we prove this result by induction on $|x| = n$. Let $q_1, q_2 \in Q_i$ and $F^*(q_1, x, r) \neq \emptyset$ for some $r \in Q$. If $n = 0$, then $x = \lambda$ and $F^*(q_1, x, r) \neq \emptyset$ means that $F^*(q_1, x, r) = U$, that is $q_1 = r$. In this case, $i = j$, hence the result is true for $n = 0$.

Suppose that the result is true for all $u \in X^*$, such that $|u| = n - 1$ where $n > 0$. Let $x = ua$ where $a \in X$. Now,

$$F^*(q_1, x, r) = F^*(q_1, ua, r) = \bigcup_{s \in Q} \{F^*(q_1, u, s) \cap F^*(s, a, r)\} \neq \emptyset.$$

Thus, there exist finite number of $s \in Q$ such that $F^*(q_1, u, s) \cap F^*(s, a, r) \neq \emptyset$, implies that $F^*(q_1, u, s) \neq \emptyset$ and $F^*(s, a, r) \neq \emptyset$. By the induction hypothesis, there exist $t_k \in Q$, for $k = 1, 2, ...m$, such that

$$\bigcup_{k=1}^{m} F^*(q_2, u, t_k) \supseteq F^*(q_1, u, s) \text{ and } t_k, s \in Q_j, \text{ for all } k = 1, 2, ..., m.$$

Now, $t_k, s \in Q_j$ for all $k = 1, 2, ..., m$ and $F(s, a, r) = F^*(s, a, r) \neq \emptyset$. By given (1), for each $k = 1, 2, ..., m$ there exist $t'_{k'} \in Q$, for $k' = 1, 2, ..., n$ such that

$$\bigcup_{k'=1}^{n} F^*\left(t_k, a, t'_{k'}\right) \supseteq F(s, a, r) \text{ and } t'_{k'}, r \in Q_{j'}, \text{ for all } k' = 1, 2, ..., n.$$

Now,

$$\begin{aligned} F^*(q_1, x, r) &= F^*(q_1, ua, r) = \bigcup_{s \in Q} \{F^*(q_1, u, s) \cap F^*(s, a, r)\} \\ &\subseteq \bigcup_{s \in Q} \left\{ \left(\bigcup_{k=1}^{m} F^*(q_2, u, t_k) \right) \cap \left(\bigcup_{k'=1}^{n} F^*\left(t_k, a, t'_{k'}\right) \right) \right\} \\ &= \bigcup_{k=1}^{m} \bigcup_{k'=1}^{n} \left\{ F^*(q_2, u, t_k) \cap F^*\left(t_k, a, t'_{k'}\right) \right\} \\ &\subseteq \bigcup_{k'=1}^{n} F^*\left(q_2, ua, t'_{k'}\right) \\ &= \bigcup_{k'=1}^{n} F^*\left(q_2, x, t'_{k'}\right) \end{aligned}$$

and $t'_{k'}, r \in Q_l$, for all $k' = 1, 2, ..., n$. Thus,

$$\bigcup_{k'=1}^{n} F^*\left(q_2, x, t'_{k'}\right) \supseteq F^*(q_1, x, r) \text{ and } t'_{k'}, r \in Q_{j'}, \text{ for all } k' = 1, 2, ..., n.$$

Hence, the result is valid for $|x| = n$. This completes the proof.

$(2) \Longrightarrow (1)$. Converse is obvious. ■

6.2.4 Corollary

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Then, every soft admissible partition $\wp$ of Q induces a soft admissible relation $\sim$ on Q such that the set of all equivalence classes of $\sim$ is $\wp$. Conversely, the set of all equivalence classes of a soft admissible relation on Q is an admissible partition $\wp$ of Q.

6.2.5 Lemma

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and $\wp = \{Q_1, Q_2, ..., Q_l\}$ be a soft admissible partition of Q. For $i, j \in \{1, 2, ...l\}$, for all $p, p' \in Q_i$ and $a \in X$, then

$$\bigcup_{r \in Q_j} F(p, a, r) = \bigcup_{r \in Q_j} F\left(p', a, r\right).$$

Proof. Let $p, p^{'} \in Q_i$, $a \in X$. Let $H_1 = \{F(p,a,r) \mid r \in Q_j\}$ and $H_2 = \left\{F\left(p^{'},a,r\right) \mid r \in Q_j\right\}$. Now, suppose that $r \in Q_j$ such that $F(p,a,r) \neq \emptyset$. Since $\wp$ is soft admissible partition of Q, there exist $t_k \in Q$, for $k = 1,2,...,n$, such that

$$\bigcup_{k=1}^{n} F\left(p^{'},a,t_k\right) \supseteq F(p,a,r) \text{ and } t_k, r \in Q_j, \text{ for all } k = 1,2,...,n. \quad (1)$$

Similarly, if $F\left(p^{'},a,r^{'}\right) \neq \emptyset$ for some $r^{'} \in Q_j$, there exist $t^{'}_{k^{'}} \in Q$, for $k^{'} = 1,2,...,m$, such that

$$\bigcup_{k^{'}=1}^{m} F\left(p,a,t^{'}_{k^{'}}\right) \supseteq F\left(p^{'},a,r^{'}\right) \text{ and } t^{'}_{k^{'}}, r^{'} \in Q_j, \text{ for all } k^{'} = 1,2,...,m. \quad (2)$$

Note that $F(p,a,r) = \emptyset$, for all $r \in Q_j$ if, and only if, $F\left(p^{'},a,r\right) = \emptyset$, for all $r \in Q_j$, because $\wp$ is soft admissible partition of Q. Hence from (1) and (2)

$$\bigcup_{r \in Q_j} F(p,a,r) = \bigcup_{r \in Q_j} F\left(p^{'},a,r\right).$$

This completes the proof. ∎

6.2.6 Theorem

Let $\Upsilon = (Q,X,F,U)$ be a soft finite state machine and $\wp = \{Q_1,Q_2,...,Q_l\}$ be a soft admissible partition of Q. Define

$$F^{\wp} : \wp \times X \times \wp \longrightarrow P(U)$$

by

$$F^{\wp}(Q_i,a,Q_j) = \bigcup_{r \in Q_j} F(p,a,r),$$

for all $Q_i, Q_j \in \wp$ and $a \in X$, where $p \in Q_i$. Then, $\Upsilon/\wp$ is a soft finite state machine. It is called the quotient soft finite state machine with respect to $\wp$.

Proof. In Lemma 6.2.5, it is proved, $F^{\wp}$ is well defined. ∎

6.2.7 Proposition

Let $\Upsilon = (Q,X,F,U)$ be a soft finite state machine and $\wp = \{Q_1,Q_2,...,Q_l\}$ be a soft admissible partition of Q. Then, for $p \in Q_i$

$$F^{\wp *}(Q_i,x,Q_j) \subseteq \bigcup_{r \in Q_j} F^{*}(p,x,r),$$

for all $Q_i, Q_j \in \wp$ and $x \in X^{*}$.

Proof. Let $Q_i, Q_j \in \wp$ and $x \in X^*$. To prove this result, we use mathematical induction on $|x| = n$. If $n = 0$, then $x = \lambda$. If $Q_i = Q_j$, then $F^{\wp *}(Q_i, x, Q_j) = U$ and

$$\bigcup_{r \in Q_j} F^*(p, x, r) = \bigcup_{r \in Q_i} F^*(p, x, r) = F^*(p, x, p) = U,$$

where $p \in Q_i$.

If $Q_i \neq Q_j$, then $F^{\wp *}(Q_i, x, Q_j) \neq \emptyset$ and $Q_i \cap Q_j = \emptyset$. Since $Q_i \cap Q_j = \emptyset$ and $p \in Q_i$, then $\bigcup_{r \in Q_j} F^*(p, x, r) = \emptyset$. Hence, $F^{\wp *}(Q_i, x, Q_j) = \bigcup_{r \in Q_j} F^*(\ p, x, r)$. Suppose that the result is valid for all $x \in X^*$ such that $|x| = n - 1$, where $n > 0$. Let $n > 0$ and $y = xa$, where $x \in X^*$ $a \in X$, and $|x| = n - 1$. Now, for $p \in Q_i$ and $r_1 \in Q_k$,

$$\begin{aligned}
F^{\wp *}(Q_i, y, Q_j) &= F^{\wp *}(Q_i, xa, Q_j) \\
&= \bigcup_{Q_k \in \wp} \{F^{\wp *}(Q_i, x, Q_k) \cap F^{\wp *}(Q_k, a, Q_j)\} \\
&= \bigcup_{Q_k \in \wp} \left\{ \left\{ \bigcup_{r \in Q_k} F^*(p, x, r) \right\} \cap \left\{ \bigcup_{q \in Q_j} F^*(r_1, a, q) \right\} \right\} \\
&= \bigcup_{Q_k \in \wp} \left\{ \bigcup_{r \in Q_k} \bigcup_{q \in Q_j} \{F^*(p, x, r) \cap F^*(r_1, a, q)\} \right\} \\
&\subseteq \bigcup_{Q_k \in \wp} \left\{ \bigcup_{r \in Q_k} \bigcup_{q \in Q_j} \{F^*(p, x, r) \cap F^*(r, a, q)\} \right\} \\
&= \bigcup_{q \in Q_j} \left\{ \bigcup_{Q_k \in \wp} \bigcup_{r \in Q_k} \{F^*(p, x, r) \cap F^*(r, a, q)\} \right\} \\
&= \bigcup_{q \in Q_j} \left\{ \bigcup_{r \in Q} \{F^*(p, x, r) \cap F^*(r, a, q)\} \right\} \\
&= \bigcup_{q \in Q_j} F^*(p, xa, q) \\
&= \bigcup_{q \in Q_j} F^*(p, y, q).
\end{aligned}$$

This completes the proof. ■

6.2.8 Proposition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and $\wp = \{Q_1, Q_2, ..., Q_l\}$ be a soft admissible partition of Q. Then, for all $x \in X^*$

$$F^*(p, x, q) \subseteq F^{\wp *}(Q_i, x, Q_j),$$

where $p \in Q_i, q \in Q_j$.

Proof. Let $p \in Q_i$ and $q \in Q_j$. To prove this result, we use mathematical induction on $|x| = n$, where $x \in X^*$. If $n = 0$, then $x = \lambda$. If $p = q$, then $Q_i = Q_j$ and $F^{\wp *}(Q_i, x, Q_j) = U = F^*(p, x, q)$.

If $p \neq q$, then $F^*(p, x, q) = \emptyset \subseteq F^{\wp *}(Q_i, x, Q_j)$. If $n = 1$, then $x = a \in X$ and

$$\begin{aligned} F^*(p, a, q) &= F(p, a, q) \\ &\subseteq \bigcup_{r \in Q_j} \{F(p, a, r)\} \\ &= F^{\wp}(Q_i, a, Q_j) \\ &= F^{\wp *}(Q_i, a, Q_j). \end{aligned}$$

Hence, the result is true for $n = 0$ and $n = 1$. Suppose that the result is valid for all $x \in X^*$ such that $|x| = n - 1$, where $n > 0$. Let $n > 0$ and $y = xa$, where $x \in X^*$ $a \in X$, and $|x| = n - 1$. Then,

$$\begin{aligned} F^*(p, y, q) &= F^*(p, xa, q) \\ &= \bigcup_{r \in Q} \{F^*(p, x, r) \cap F^*(r, a, q)\} \\ &= \bigcup_{Q_k \in \wp} \left\{ \bigcup_{r \in Q_k} \{F^*(p, x, r) \cap F^*(r, a, q)\} \right\} \\ &\subseteq \bigcup_{Q_k \in \wp} \left\{ \bigcup_{r \in Q_k} \{F^{\wp *}(Q_i, x, Q_k) \cap F^{\wp}(Q_k, a, Q_j)\} \right\} \\ &= \bigcup_{Q_k \in \wp} \{F^{\wp *}(Q_i, x, Q_k) \cap F^{\wp}(Q_k, a, Q_j)\} \\ &= F^{\wp *}(Q_i, xa, Q_j) \\ &= F^{\wp *}(Q_i, y, Q_j). \end{aligned}$$

This completes the proof. ∎

6.2.9 Corollary

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and $\wp = \{Q_1, Q_2, ..., Q_l\}$ be a soft admissible partition of Q. Then, for all $x \in X^*$ and $Q_i, Q_j \in \wp$

$$\bigcup_{r \in Q_j} F^*(p, x, r) \subseteq F^{\wp *}(Q_i, x, Q_j),$$

where $p \in Q_i$.

From the Proposition 6.2.7, and Corollary 6.2.9, we conclude the following theorem.

6.2.10 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and $\wp = \{Q_1, Q_2, ..., Q_l\}$ be a soft admissible partition of Q. Then, for $p \in Q_i$

$$F^{\wp *}(Q_i, x, Q_j) = \bigcup_{r \in Q_j} F^*(p, x, r),$$

for all $Q_i, Q_j \in \wp$ and $x \in X^*$.

Proof. Straightforward. ■

6.2.11 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $\wp$ and Ω be soft admissible partitions of Q. Then, $\wp$ and Ω are F-orthogonal if

(1) $\wp \cap \Omega = I_Q = \{\{q\} : q \in Q\}$

(2) For all $Q_i, Q_u \in \wp$, $H_j, H_v \in \Omega$ and $a \in X$, if $Q_i \cap H_j = \{q\}$ and $Q_u \cap H_v = \{p\}$, then

$$F(q, a, p) = \bigcup_{\substack{r \in Q_u \\ r' \in H_v}} \left\{F(q, a, r) \cap F\left(q, a, r'\right)\right\}.$$

6.2.12 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $\wp$ and Ω be a soft admissible partitions of Q. Then, $\wp$ and Ω are F-orthogonal if

(1) $\wp \cap \Omega = I_Q = \{\{q\} : q \in Q\}$

(2) For all $Q_i, Q_u \in \wp$, $H_j, H_v \in \Omega$ and $x \in X^*$, if $Q_i \cap H_j = \{q\}$ and $Q_u \cap H_v = \{p\}$, then

$$F^*(q, x, p) = \bigcup_{\substack{r \in Q_u \\ r' \in H_v}} \left\{F^*(q, x, r) \cap F^*\left(q, x, r'\right)\right\}.$$

Proof. Suppose $\wp$ and Ω are F-orthogonal, then obviously (1) holds.

(2) Let $Q_i, Q_u \in \wp$, $H_j, H_v \in \Omega$, $x \in X^*$, $Q_i \cap H_j = \{q\}$ and $Q_u \cap H_v = \{p\}$. To prove this result, we use mathematical induction on $|x| = n$, where $x \in X^*$. If $n = 0$, then $x = \lambda$. If $q \neq p$ then $F^*(q, x, p) = \emptyset$ and if $q = p$ then $F^*(q, x, p) = U$. Suppose, $q \neq p$. Then either $Q_i \cap Q_u = \emptyset$ or $H_j \cap H_v = \emptyset$, say, $Q_i \cap Q_u = \emptyset$. Thus, $q \notin Q_u$ and $q \notin H_v$. Hence,

$$\bigcup_{\substack{r \in Q_u \\ r' \in H_v}} \left\{F^*(q, x, r) \cap F^*\left(q, x, r'\right)\right\} = \bigcup_{\substack{r \in Q_u \\ r' \in H_v}} \left\{\emptyset \cap F^*\left(q, x, r'\right)\right\} = \emptyset.$$

Suppose $q = p$. Then, $Q_i = Q_u$ and $H_j = H_v$. Thus,

$$\bigcup_{\substack{r \in Q_u \\ r' \in H_v}} \left\{ F^* (q, x, r) \cap F^* \left(q, x, r'\right) \right\} = F^* (q, x, p) \cap F^* (q, x, p) = U.$$

Hence, if $n = 0$, then

$$F^* (q, x, p) = \bigcup_{\substack{r \in Q_u \\ r' \in H_v}} \left\{ F^* (q, x, r) \cap F^* \left(q, x, r'\right) \right\}.$$

Suppose that the result is true for all $x \in X^*$ such that $|x| = n - 1$, $n > 0$. Let $n > 0$ and $y = xa$, where $x \in X^*$, $a \in X$, where $|x| = n - 1$. Then,

$$\begin{aligned}
F^* (q, y, p) &= F^* (q, xa, p) \\
&= \bigcup_{s \in Q} \left\{ F^* (q, x, s) \cap F^* (s, a, p) \right\} \\
&= \bigcup_{s \in Q} \left\{ F^* (q, x, s) \cap \left\{ \bigcup_{\substack{r \in Q_u \\ r' \in H_v}} \left\{ F^* (s, a, r) \cap F^* \left(s, a, r'\right) \right\} \right\} \right\} \\
&= \bigcup_{s \in Q} \left\{ \left\{ F^* (q, x, s) \cap F^* (s, a, r) \right\} \cap \left\{ \bigcup_{\substack{r \in Q_u \\ r' \in H_v}} \left\{ F^* (q, x, s) \cap F^* \left(s, a, r'\right) \right\} \right\} \right\} \\
&= \left\{ \bigcup_{s \in Q} \left\{ F^* (q, x, s) \cap F^* (s, a, r) \right\} \right\} \cap \left\{ \bigcup_{\substack{r \in Q_u \\ r' \in H_v}} \left\{ \bigcup_{s \in Q} \left\{ F^* (q, x, s) \cap F^* \left(s, a, r'\right) \right\} \right\} \right\} \\
&= \bigcup_{\substack{r \in Q_u \\ r' \in H_v}} \left\{ F^* (q, xa, r) \cap F^* \left(q, xa, r'\right) \right\} \\
&= \bigcup_{\substack{r \in Q_u \\ r' \in H_v}} \left\{ F^* (q, y, r) \cap F^* \left(q, y, r'\right) \right\}.
\end{aligned}$$

The result now follows by induction . The converse is trivial. ■

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $\wp$ and Ω be soft admissible partitions of Q. Consider the soft finite state machine $\Upsilon/\wp = (\wp, X, F^{\wp}, U)$ and $\Upsilon/\Omega = \left(\Omega, X, F^{\Omega}, U\right)$. Define

$$F^{\Diamond} : (\wp \times \Omega) \times X \times (\wp \times \Omega) \longrightarrow P(U) \text{ by}$$

$$F^{\Diamond} \left((Q_i, H_j), a, (Q_u, H_v) \right) = F^{\wp} (Q_i, a, Q_u) \cap F^{\Omega} (H_j, a, H_v)$$

For all $Q_i, Q_u \in \wp$, $H_j, H_v \in \Omega$ and $a \in X$. Then, $\Upsilon/\wp \times_R \Upsilon/\Omega = \left(\wp \times \Omega, X, F^{\Diamond}, U\right)$ is a soft finite state machine. Note that

$$F^{\Diamond *}\left(\left(Q_i, H_j\right), x, \left(Q_u, H_v\right)\right) = F^{\wp *}\left(Q_i, x, Q_u\right) \cap F^{\Omega *}\left(H_j, x, H_v\right)$$

For all $Q_i, Q_u \in \wp$, $H_j, H_v \in \Omega$ and $x \in X^*$.

6.2.13 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $\wp$ and Ω be soft admissible partitions of Q such that $\wp$ and Ω are F-orthogonal. Then, $\Upsilon \leq \Upsilon/\wp \times_R \Upsilon/\Omega$.

Proof. Define $\eta : \wp \times \Omega \longrightarrow Q$ by $\eta\left(\left(Q_i, H_j\right)\right) = q$, where $Q_i \cap H_j = \{q\}$, because $\wp$ and Ω are F-orthogonal. Note that η is one-to-one. Let ξ be the identity map on X. Let $Q_i, Q_u \in \wp$, $H_j, H_v \in \Omega$ and $x \in X^*$. Suppose $Q_i \cap H_j = \{q\}$ and $Q_u \cap H_v = \{p\}$, then

$$F^*\left(\eta\left(Q_i, H_j\right), x, p\right) = F^*\left(q, x, p\right).$$

Also,

$$\begin{aligned} F^{\Diamond *}\left(\left(Q_i, H_j\right), x, \left(Q_u, H_v\right)\right) &= F^{\wp *}\left(Q_i, x, Q_u\right) \cap F^{\Omega *}\left(H_j, x, H_v\right) \\ &= \left\{\bigcup_{r \in Q_u} F^*\left(q, x, r\right)\right\} \cap \left\{\bigcup_{r' \in H_v} F^*\left(q, x, r'\right)\right\} \\ &= \bigcup_{\substack{r \in Q_u \\ r' \in H_v}} \left\{F^*\left(q, x, r\right) \cap F^*\left(q, x, r'\right)\right\} \\ &= F^*\left(q, x, p\right). \end{aligned}$$

Since $\wp$ and Ω are F-orthogonal, so by using Theorem 6.2.12, the last inequality holds. Thus,

$$F^{\Diamond *}\left(\left(Q_i, H_j\right), x, \left(Q_u, H_v\right)\right) = F^*\left(\eta\left(Q_i, H_j\right), x, p\right).$$

Now,

$$\begin{aligned} F^*\left(\eta\left(Q_i, H_j\right), x, p\right) &= F^{\Diamond *}\left(\left(Q_i, H_j\right), x, \left(Q_u, H_v\right)\right) \\ &= \bigcup \left\{ \begin{array}{c} F^{\Diamond *}\left(\left(Q_i, H_j\right), x, \left(Q_r, H_s\right)\right) \mid \\ \eta\left(\left(Q_r, H_s\right)\right) = \eta\left(Q_u, H_v\right), \left(Q_r, H_s\right) \in \wp \times \Omega \end{array} \right\}. \end{aligned}$$

Since η is one-to-one. Hence,

$$F^*\left(\eta\left(Q_i, H_j\right), x, p\right) = \bigcup \left\{ \begin{array}{c} F^{\Diamond *}\left(\left(Q_i, H_j\right), x, \left(Q_r, H_s\right)\right) \mid \\ \eta\left(\left(Q_r, H_s\right)\right) = \eta\left(Q_u, H_v\right), \left(Q_r, H_s\right) \in \wp \times \Omega \end{array} \right\}.$$

Consequently, $\Upsilon \leq \Upsilon/\wp \times_R \Upsilon/\Omega$. ■

6.2.14 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $\wp$ and Ω be soft admissible partitions of Q such that $\wp$ and Ω are F-orthogonal. Then, $\Upsilon \leq N\theta\Upsilon/\wp$.

Proof. Let $\wp = \{Q_i : i \in I\}$ and $\Omega = \{H_j : j \in J\}$ be F-orthogonal partitions of Q. Let $N = (\Omega, \wp \times X, F_N, U)$, where

$$F_N(H_j, (Q_i, a), H_v) = \bigcup_{r \in H_v} \{F(q, a, r)\},$$

where $Q_i \cap H_j = \{q\}$. F_N is well defined because Ω is a soft admissible partitions of Q. Define $\theta : \wp \times X \longrightarrow \wp \times X$ to be the identity map. Define $\eta : \Omega \times \wp \longrightarrow Q$ by $\eta((H_j, Q_i)) = q$, where $Q_i \cap H_j = \{q\}$. Then, it can easily prove that η is one to one and onto. Let ζ be the identity map on X. Now, we want to prove

$$F^*(\eta(H_j, Q_i), x, \eta(H_v, Q_u)) = F^{\theta *}((H_j, Q_i), x, (H_v, Q_u))$$

where $(H_j, Q_i), (H_v, Q_u) \in \Omega \times \wp$ and $x \in X^*$. We use mathematical induction on $|x| = n$, where $x \in X^*$. For $n = 1$, $x = a \in X$, then for $Q_u \cap H_v = \{p\}$,

$$\begin{aligned}
F(\eta(H_j, Q_i), a, \eta(H_v, Q_u)) &= F(q, a, p) \\
&= \bigcup_{(r', r) \in H_v \times Q_u} \{F(q, x, r') \cap F(q, x, r)\} \\
&\quad \text{because } \wp \text{ and } \Omega \text{ are } F\text{-orthogonal.} \\
&= \left\{ \bigcup_{r' \in H_v} \{F(q, x, r')\} \right\} \cap \left\{ \bigcup_{r \in Q_u} \{F(q, x, r)\} \right\} \\
&= F_N(H_j, (Q_i, a), H_v) \cap F^{\wp}(Q_i, a, Q_u) \\
&= F_N(H_j, \theta(Q_i, a), H_v) \cap F^{\wp}(Q_i, a, Q_u) \\
&= F^{\theta}((H_j, Q_i), a, (H_v, Q_u)).
\end{aligned}$$

Suppose that it is true for all $x \in X^*$ such that $|x| = n - 1$, where $n > 0$. Let $y = xa$, where $x \in X^*$ and $a \in X$. Now,

$$\begin{aligned}
F^*(\eta(H_j, Q_i), y, \eta(H_v, Q_u)) &= F^*(\eta(H_j, Q_i), xa, \eta(H_v, Q_u)) \\
&= \bigcup_{\eta(H_s, Q_r) \in \eta(\Omega \times \wp)} \left\{ \begin{array}{c} F^*(\eta((H_j, Q_i)), x, \eta((H_s, Q_r))) \cap \\ F(\eta(H_s, Q_r), a, \eta(H_v, Q_u)) \end{array} \right\} \\
&\quad \text{Since } \eta \text{ is onto} \\
&= \bigcup_{(H_s, Q_r) \in (\Omega \times \wp)} \left\{ \begin{array}{c} F^{\theta *}((H_j, Q_i), x, (H_s, Q_r)) \cap \\ F^{\theta}((H_s, Q_r), a, (H_v, Q_u)) \end{array} \right\} \\
&= F^{\theta *}((H_j, Q_i), xa, (H_v, Q_u)) \\
&= F^{\theta *}((H_j, Q_i), y, (H_v, Q_u)).
\end{aligned}$$

Also note that

$$F^{*}\left(\eta\left(H_j,Q_i\right),\lambda,\eta\left(H_v,Q_u\right)\right)=U$$

if, and only if, $\eta\left(\left(H_j,Q_i\right)\right)=\eta\left(\left(H_v,Q_u\right)\right)$ if, and only if, $\left(H_j,Q_i\right)=\left(H_v,Q_u\right)$, since η is one to one, if, and only if,

$$F^{\theta*}\left(\left(H_j,Q_i\right),\lambda,\left(H_v,Q_u\right)\right)=U.$$

Thus,

$$F^{*}\left(\eta\left(H_j,Q_i\right),\lambda,\eta\left(H_v,Q_u\right)\right)=F^{\theta*}\left(\left(H_j,Q_i\right),\lambda,\left(H_v,Q_u\right)\right)=U.$$

Hence, for all $x\in X^{*}$,

$$F^{*}\left(\eta\left(H_j,Q_i\right),x,\eta\left(H_v,Q_u\right)\right)=F^{\theta*}\left(\left(H_j,Q_i\right),x,\left(H_v,Q_u\right)\right).$$

Thus (η,ζ), is a covering of Υ by $N\theta\Upsilon/\wp$. ■

6.2.15 Definition

Let $\Upsilon=(Q,X,F,U)$ be a soft finite state machine. Let $\wp$ and Ω be partitions of Q. Then, $\wp\subseteq\Omega$ if for all $A\in\wp$, there exists $B\in\Omega$ such that $A\subseteq B$.

6.2.16 Lemma

Let $\Upsilon=(Q,X,F,U)$ be a soft finite state machine. Let $\wp=\{Q_1,Q_2,...,Q_n\}$ and $\Omega=\{H_1,H_2,...,H_m\}$ be partitions of Q such that $\wp\subseteq\Omega$ and $m\leq n$. Then, for all $H_j\in\Omega$, there exist some $Q_{j_1},Q_{j_2},...,Q_{j_r}\in\wp$ such that

$$H_j=Q_{j_1}\cup Q_{j_2}\cup...\cup Q_{j_r}.$$

If $m=n$, then $\wp=\Omega$.

Proof. Let $\Upsilon=(Q,X,F,U)$ be a soft finite state machine. Let $\wp=\{Q_1,Q_2,...,Q_n\}$ and $\Omega=\{H_1,H_2,...,H_m\}$ be a partitions of Q such that $\wp\subseteq\Omega$ and $m\leq n$. Then $Q=Q_1\cup Q_2\cup...\cup Q_n$ and $Q=H_1\cup H_2\cup...\cup H_m$. Let $H_j\in\Omega$, and $q_1\in H_j\subseteq Q=\bigcup_{i=1}^{n}Q_i$, there exist $Q_{j_1}\in\wp$ such that $Q_{j_1}\subseteq H_j$. Let $q_2\in H_j\setminus Q_{j_1}\subseteq Q=\bigcup_{i=1}^{n}Q_i$, there exist $Q_{j_2}\in\wp$ such that $Q_{j_2}\subseteq H_j$. Continue in this way, there exist some $Q_{j_1},Q_{j_2},...,Q_{j_r}\in\wp$ such that

$$H_j=Q_{j_1}\cup Q_{j_2}\cup...\cup Q_{j_r}.$$

Obviously, $\wp=\Omega$, if $m=n$. ■

6.2.17 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $\wp$ be a soft admissible partition of Q. Then, $\wp$ is called maximal soft admissible partition of Q if

(1) $\wp$ is nontrivial.

(2) If Ω is any soft admissible partition of Q such that $\wp \subseteq \Omega \subseteq \{Q\}$, then either $\wp = \Omega$ or $\Omega = \{Q\}$.

6.2.18 Definition

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Then Υ is called soft irreducible finite state machine if I_Q and $\{Q\}$ are the only soft admissible partition of Q, where $|Q| \geq 2$.

6.2.19 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine. Let $\wp = \{Q_1, Q_2, ..., Q_n\}$ be a soft admissible partition of Q. Then, $\wp$ is maximal soft admissible partition of Q if and only if $\Upsilon/\wp = (\wp, X, F^{\wp}, U)$ is soft irreducible finite state machine.

Proof. Suppose $\wp$ is maximal soft admissible partition of Q. Consider $\Upsilon/\wp = (\wp, X, F^{\wp}, U)$, where

$$F^{\wp}(Q_i, a, Q_j) = \bigcup_{r \in Q_j} F(p, a, r),$$

for all $Q_i, Q_j \in \wp$ and $a \in X$ where $p \in Q_i$. Since $\wp$ is maximal, so $\wp \neq \{Q\}$. Thus, $|\wp| > 1$. Let $\overline{\wp}$ be a soft admissible partition of $\wp$. Suppose $\overline{\wp} \neq I_{\wp}$. Then, there exist $\Omega \subseteq \wp$ such that $\Omega \in \overline{\wp}$ and $|\Omega| > 1$. Suppose that $\Omega \neq \wp$. Without loss of generality, we may assume that $\Omega = \{Q_1, Q_2, ..., Q_m\}$, where $1 < m < n$. Let

$$\wp' = \{Q_1 \cup Q_2 \cup ... \cup Q_m, Q_{m+1} \cup Q_{m+2} \cup ... \cup Q_n\}.$$

Then, $\wp \subseteq \wp'$ and $\wp'$ is a soft partition of Q. Next we show that $\wp'$ is soft admissible partition of Q.

Consider $p, q \in Q_1 \cup Q_2 \cup ... \cup Q_m$ with $p \in Q_1$ and $q \in Q_2$. Suppose $F(q, a, s) \neq \emptyset$ where $s \in Q_i \in \wp$. Then,

$$F^{\wp}(Q_2, a, Q_i) = \bigcup_{s \in Q_i} F(q, a, s) \neq \emptyset.$$

Since $\overline{\wp}$ is a soft admissible partition of $\wp$, so there exist $Q_j' \in \wp$, for $j = 1, 2, ..., r$, such that

$$\bigcup_{j=1}^{r} F^{\wp}\left(Q_1, a, Q_j'\right) \supseteq F^{\wp}(Q_2, a, Q_i)$$

and, $Q_j^{'}$ and Q_i are belong to the same element of $\overline{\wp}$ for $j = 1, 2, ..., r$. Hence

$$\bigcup_{j=1}^{r}\left(\bigcup_{t^{'}\in Q_j^{'}} F\left(p, a, t^{'}\right)\right) \supseteq \bigcup_{s\in Q_i} F(q, a, s)$$

this implies that there exists $t^{'} \in Q_j^{'}$ for $j = 1, 2, ..., r$ such that

$$\bigcup_{j=1}^{r}\left(\bigcup_{t^{'}\in Q_j^{'}} F\left(p, a, t^{'}\right)\right) \supseteq F(q, a, s).$$

Now, if $Q_i \in \Omega$ if, and only if, $Q_j^{'} \in \Omega$, for $j = 1, 2, ..., r$, because $Q_j^{'}$ and Q_i are belong to the same element of $\overline{\wp}$. If Q_i,$Q_j^{'} \in \Omega$, then $s, t^{'} \in Q_1 \cup Q_2 \cup ... \cup Q_m$ for all $t^{'} \in Q_j^{'}$ $j = 1, 2, ..., r$. If Q_i,$Q_j^{'} \notin \Omega$, then $s, t^{'} \in Q_{m+1} \cup Q_{m+2} \cup ... \cup Q_n$ for all $t^{'} \in Q_j^{'}$ $j = 1, 2, ..., r$. That is, for all $t^{'} \in Q_j^{'}$ such that $j = 1, 2, ..., r$ and s belongs to the same element of $\wp^{'}$. Hence, $\wp^{'}$ is soft admissible. Since $\wp$ is maximal soft admissible partition of Q, it follows that $\wp^{'} = \{Q\}$ and $\Omega = \{Q\}$. This implies that $\overline{\wp} = \{\wp\}$. Thus, $\Upsilon/\wp = (\wp, X, F^{\wp}, U)$ is soft irreducible finite state machine.

Conversely, assume that $\Upsilon/\wp = (\wp, X, F^{\wp}, U)$ is soft irreducible finite state machine. Let Ω be a soft admissible partition of Q such that $\wp \subseteq \Omega \subseteq \{Q\}$. Suppose that $\wp \neq \Omega$. By Lemma 6.2.16, without any loss of generality, we may assume that

$$\Omega = \{Q_1 \cup Q_2 \cup ... \cup Q_m,\ Q_{m+1}, Q_{m+2}, ..., Q_n\},$$

where $1 < m \leq n$. As $\wp \neq \Omega$,

$$\overline{\Omega} = \{\{Q_1, Q_2, ..., Q_m\}, \{Q_{m+1}\}, \{Q_{m+2}\}, ... \{Q_n\}\} \neq I_{\wp}.$$

We now show that $\overline{\Omega}$ is soft admissible partition of $\wp$.

For this consider, Q_1, Q_2 and $F^{\wp}(Q_1, a, Q_i) \neq \emptyset$, for some $Q_i \in \wp$. Then,

$$F^{\wp}(Q_1, a, Q_i) = \bigcup_{s\in Q_i} F(q, a, s) \neq \emptyset \text{ where } s \in Q_i.$$

Since Q is finite, so Q_i is also finite. Thus, we can find all those $s_{ij} \in Q_i$ such that $F(q, a, s_{ij}) \neq \emptyset$ and

$$F^{\wp}(Q_1, a, Q_i) = \bigcup_{s_{ij}\in Q_i} F(q, a, s_{ij}) = \bigcup_{j=1}^{r} F(q, a, s_{ij}).$$

Since $F(q, a, s_{ij}) \neq \emptyset$, for all $j = 1, 2, 3, ..., r$ and Ω is soft admissible partition of Q, for all $p \in Q_2$, there exist $t_{ijk} \in Q$, such that $\bigcup_{k=1}^{r^{'}} F(p, a, t_{ijk}) \supseteq F(q, a, s_{ij})$ and

t_{ijk}, s_{ij} are in the same element of Ω for all $k = 1, 2, ..., r'$ and $j = 1, 2, ..., r$. Now, if $Q_i \notin \{Q_1 \cup Q_2 \cup ... \cup Q_m\}$, then $t_{ijk}, s_{ij} \in Q_i$, for all j,k and for all $p \in Q_2$, and

$$\begin{aligned} F^{\wp}(Q_2, a, Q_i) &= \bigcup_{t \in Q_i} F(p, a, t_i) \\ &\supseteq \bigcup_{j=1}^{r} \left\{ \bigcup_{k=1}^{r'} F(p, a, t_{ijk}) \right\} \\ &\supseteq \bigcup_{j=1}^{r} F(q, a, s_{ij}) \\ &= F^{\wp}(Q_1, a, Q_i). \end{aligned}$$

Suppose $Q_i \in \{Q_1 \cup Q_2 \cup ... \cup Q_m\}$, then $t_{ijk}, s_{ij} \in \{Q_1 \cup Q_2 \cup ... \cup Q_m\}$, for all j,k and for all $p \in Q_2$. Since $t_{ijk} \in \{Q_1 \cup Q_2 \cup ... \cup Q_m\} = \bigcup_{l=1}^{m} Q_l$, for all j and k, hence

$$\begin{aligned} \bigcup_{l=1}^{m} F^{\wp}(Q_2, a, Q_l) &= \bigcup_{l=1}^{m} \left\{ \bigcup_{t_l \in Q_l} F(q, a, t_l) \right\} \\ &\supseteq \bigcup_{j=1}^{r} \left\{ \bigcup_{k=1}^{r'} F(p, a, t_{ijk}) \right\} \\ &\supseteq \bigcup_{j=1}^{r} F(q, a, s_{ij}) \\ &= F^{\wp}(Q_1, a, Q_i) \end{aligned}$$

and $Q_l, Q_i \in \{Q_1, Q_2, ..., Q_m\}$. Consequently, $\overline{\Omega}$ is soft admissible partition of $\wp$. Since $\Upsilon/\wp$ is soft irreducible finite state machine, so $\overline{\Omega} = \{\wp\}$ and $\Omega = \{Q\}$. Hence, $\wp$ is maximal. ■

6.2.20 Theorem

Let $\Upsilon = (Q, X, F, U)$ be a soft finite state machine and $|Q| = n > 1$. Then,

$$\Upsilon \leq \Upsilon_1 \theta_1 \Upsilon_2 \theta_2 .... \theta_{p-1} \Upsilon_p,$$

where $\Upsilon_1, \Upsilon_2, ... \Upsilon_p$ are soft irreducible finite state machines and the state sets Q_i of Υ_i such that $|Q_i| < n$.

Proof. Given $|Q| = n > 1$, so we can choose a maximal soft admissible partition $\wp$ of Q. Obviously, $|\wp| < |Q|$. From Theorem 6.2.14, there exist a soft finite state machine N_1 such that $\Upsilon \leq N_1 \theta_1 \Upsilon/\wp$ for some suitable θ_1. We have proved that in Theorem 6.2.19, if $\wp$ is maximal soft admissible partition of Q, then $\Upsilon/\wp$ is soft irreducible.

Note that $\wp$ is the state set for $\Upsilon/\wp$ and $|\wp| < |Q|$. If R_1 is the state set of N_1, then $|R_1| < |Q|$ by the construction of N_1. Again, we can use Theorem 6.2.14, there exist a soft finite state machine N_2 such that $N_1 \leq N_2\theta_2 N_1/\wp_1$ and $N_1/\wp_1$ is soft irreducible. Note that the number of state in N_2 is less then the number of state in N_1. Thus, we have

$$\Upsilon \leq N_1\theta_1\Upsilon/\wp \leq N_2\theta_2 N_1/\wp_1\theta_1\Upsilon/\wp,$$

and we can write

$$\Upsilon \leq N_2\theta_2 N_1/\wp_1\theta_1\Upsilon/\wp.$$

Continue in this way and apply Theorem 6.2.14, to N_2. This process must terminate after a finite number of steps, because $|Q|$ is finite. Hence,

$$\Upsilon \leq \Upsilon_1\theta_1\Upsilon_2\theta_2....\theta_{p-1}\Upsilon_p,$$

where $\Upsilon_1, \Upsilon_2, ...\Upsilon_p$ are soft irreducible finite state machines and the state sets Q_i of Υ_i such that $|Q_i| < n$. ■

6.3 Associative Properties of Products

In this section, we discuss the associativity of wreath product, sum and cascade products of soft finite state machines.

Recall the Definition 2.1.1, of homomorphism, which is given as.

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$, over the same universal set U. A pair (f, g) of mappings $f : Q_1 \longrightarrow Q_2$ and $g : X_1 \longrightarrow X_2$ is called a homomorphism if

$$F_1(q, x, p) \subseteq F_2(f(q), g(x), f(p)), \text{ for all } p, q \in Q_1 \text{ and } x \in X_1.$$

It is written as $(f, g) : \Upsilon_1 \longrightarrow \Upsilon_2$.

A homomorphism $(f, g) : \Upsilon_1 \longrightarrow \Upsilon_2$ is called an isomorphism, if f and g are both one-one and onto. It is written as $\Upsilon_1 \cong \Upsilon_2$.

6.3.1 Theorem

Let Υ, Υ_1 and Υ_2 be soft finite state machines. Then the following properties hold.

(1) $(\Upsilon \circ \Upsilon_1) \circ \Upsilon_2 \cong \Upsilon \circ (\Upsilon_1 \circ \Upsilon_2)$

(2) $(\Upsilon\theta_1\Upsilon_1)\theta_2\Upsilon_2 \cong \Upsilon\theta_3(\Upsilon_1\theta_4\Upsilon_2)$

Proof. Let $\Upsilon = (Q, X, F, U), \Upsilon_1 = (Q_1, X_1, F_1, U)$, and $\Upsilon_2 = (Q_2, X_2, F_2, U)$. Then,

$$(\Upsilon \circ \Upsilon_1) \circ \Upsilon_2 = \Big((Q \times Q_1) \times Q_2, \left(X^{Q_1} \times X_1\right)^{Q_2} \times X_2, (F \circ F_1) \circ F_2, U\Big)$$

and

$$\Upsilon \circ (\Upsilon_1 \circ \Upsilon_2) = \left(Q \times (Q_1 \times Q_2), X^{Q_1 \times Q_2} \times \left(X_1^{Q_1} \times X_2\right), F \circ (F_1 \circ F_2), U\right).$$

Let $f : (Q \times Q_1) \times Q_2 \longrightarrow Q \times (Q_1 \times Q_2)$ be the natural map. Let γ_1 and γ_2 be the natural projection mappings such that $\gamma_1 : X^{Q_1} \times X_1 \longrightarrow X^{Q_1}$ and $\gamma_2 : X^{Q_1} \times X_1 \longrightarrow X_1$.

Suppose a function $\alpha : Q_2 \longrightarrow X^{Q_1} \times X_1$, $\alpha_1 = \gamma_1 \circ \alpha$ and let $\alpha_2 = \gamma_2 \circ \alpha$. Define $\overline{\alpha_1} : Q_1 \times Q_2 \longrightarrow X$ by $\overline{\alpha_1}((q_1, q_2)) = \alpha_1(q_2)(q_1)$. Define $g : \left(X^{Q_1} \times X_1\right)^{Q_2} \times X_2 \longrightarrow X^{Q_1 \times Q_2} \times \left(X_1^{Q_1} \times X_2\right)$ by

$$g((\alpha, x_2)) = (\overline{\alpha_1}, (\alpha_2, x_2)).$$

It can easily be prove that g is well defined. Now we prove g is one-one. For this, let

$$\begin{aligned} & g((\alpha, x_2)) = g\left(\left(\alpha', x_2'\right)\right) \\ \Longrightarrow\ & (\overline{\alpha_1}, (\alpha_2, x_2)) = \left(\overline{\alpha_1}', \left(\alpha_2', x_2'\right)\right) \\ \Longrightarrow\ & \overline{\alpha_1} = \overline{\alpha_1}', \alpha_2 = \alpha_2', \text{ and } x_2 = x_2' \end{aligned}$$

Thus, $\alpha_1(q_2)(q_1) = \alpha_1'(q_2)(q_1), \alpha_2 = \alpha_2'$, and $x_2 = x_2'$. Implies that $\alpha_1 = \alpha_1', \alpha_2 = \alpha_2'$, and $x_2 = x_2'$. Thus $\gamma_1 \circ \alpha = \gamma_1 \circ \alpha', \gamma_2 \circ \alpha = \gamma_2 \circ \alpha'$, and $x_2 = x_2'$. Therefore, $\alpha = \alpha'$, and $x_2 = x_2'$, and so $(\alpha, x_2) = \left(\alpha', x_2'\right)$. Thus, β is one-one. Now, we prove β is surjective. For this, let $\left(\alpha', (\varphi, x_2)\right) \in X^{Q_1 \times Q_2} \times \left(X_1^{Q_1} \times X_2\right)$. Define $\alpha : Q_2 \longrightarrow X^{Q_1} \times X_1$ by $\alpha(q_2) = \left(\alpha_{q_2}', \varphi(q_2)\right)$, where $\alpha_{q_2}'(q_1) = \alpha'(q_1, q_2)$. Then, $g((\alpha, x_2)) = \left(\alpha', (\varphi, x_2)\right)$. Thus, g is onto.

Now, consider

$$\begin{aligned} & F \circ (F_1 \circ F_2)(f((q, q_1), q_2), g((\alpha, x_2)), f((p, p_1), p_2)) \\ = \ & F \circ (F_1 \circ F_2)((q, (q_1, q_2)), (\overline{\alpha_1}, (\alpha_2, x_2)), (p, (p_1, p_2))) \\ = \ & F(q, \overline{\alpha_1}((p_1, p_2)), p) \cap \{F_1 \circ F_2((q_1, q_2), (\alpha_2, x_2), (p_1, p_2))\} \\ = \ & F(q, \alpha_1(p_2)(p_1), p) \cap \{F_1(q_1, \alpha_2(p_2), p_1) \cap F_2(q_2, x_2, p_2)\} \\ = \ & F(q, \gamma_1 \circ \alpha(p_2)(p_1), p) \cap \{F_1(q_1, \gamma_2 \circ \alpha(p_2), p_1) \cap F_2(q_2, x_2, p_2)\} \\ = \ & F(q, \gamma_1(\delta_1, y_1)(p_1), p) \cap \{F_1(q_1, \gamma_2(\delta_1, y_1), p_1) \cap F_2(q_2, x_2, p_2) \text{ where } \alpha(p_2) = (\delta_1, y_1)\} \\ = \ & \{F(q, \delta_1(p_1), p) \cap F_1(q_1, y_1, p_1)\} \cap F_2(q_2, x_2, p_2) \\ = \ & (F \circ F_1)((q, q_1), (\delta_1, y_1), (p, p_1)) \cap F_2(q_2, x_2, p_2) \\ = \ & (F \circ F_1) \circ F_2(((q, q_1), q_2), (\alpha, x_2), ((p, p_1), p_2)). \end{aligned}$$

Thus,

$$\begin{aligned} & F \circ (F_1 \circ F_2)(f((q, q_1), q_2), g((\alpha, x_2)), f((p, p_1), p_2)) \\ = \ & (F \circ F_1) \circ F_2((q, q_1), q_2, (\alpha, x_2), (p, p_1), p_2). \end{aligned}$$

It follows that (f, g) is required homomorphism.

Thus, we can write $(\Upsilon \circ \Upsilon_1) \circ \Upsilon_2 \cong \Upsilon \circ (\Upsilon_1 \circ \Upsilon_2)$.

(2) Let $\Upsilon = (Q, X, F, U)$ and $\Upsilon_1 = (Q_1, X_1, F_1, U)$ be soft finite state machines. Let θ_1 be a function of $Q_1 \times X_1$ into X. Define $F\theta_1 F_1 : (Q \times Q_1) \times X_1 \times (Q \times Q_1) \longrightarrow P(U)$ as follows

$$F\theta_1 F_1 ((q, q_1), x_1, (p, p_1)) = F(q, \theta_1(q_1, x_1), p) \cap F_1(q_1, x_1, p_1),$$

for all $((q, q_1), x_1, (p, p_1)) \in Q \times X_1 \times Q$. Then, $(Q, X_1, F\theta_1 F_1, U)$ is a $SFSM$ and it is called cascade product of Υ and Υ_1 and we write $\Upsilon\theta_1\Upsilon_1$.

Consider $(\Upsilon\theta_1\Upsilon_1)\theta_2\Upsilon_2 = ((Q \times Q_1) \times Q_2, X_2, (F\theta_1 F_1)\theta_2 F_2, U)$ and $\theta_2 : Q_2 \times X_2 \longrightarrow X_1$ and

$(F\theta_1 F_1)\theta_2 F_2 (((q, q_1), q_2), x_2, ((p, p_1), p_2))$

$$\begin{aligned} &= F\theta_1 F_1 ((q, q_1), \theta_2(q_2, x_2), (p, p_1)) \cap F_2(q_2, x_2, p_2) \\ &= \{F(q, \theta_1(q_1, \theta_2(q_2, x_2)), p) \cap F_1(q_1, \theta_2(q_2, x_2), p_1)\} \cap F_2(q_2, x_2, p_2). \end{aligned}$$

Now, define $\theta_3 : (Q_1 \times Q_2) \times X_2 \longrightarrow X$ by $\theta_3((q_1, q_2), x_2) = \theta_1(q_1, \theta_2(q_2, x_2))$ and take $\theta_4 = \theta_2$. Then, we can write $\Upsilon_1\theta_4\Upsilon_2 = (Q_1 \times Q_2, X_2, F_1\theta_4 F_2, U)$ and

$$F_1\theta_4 F_2 ((q_1, q_2), x_2, (p_1, p_2)) = F_1(q_1, \theta_4(q_2, x_2), p_1) \cap F_2(q_2, x_2, p_2).$$

Thus, $\Upsilon\theta_3(\Upsilon_1\theta_4\Upsilon_2) = (Q \times (Q_1 \times Q_2), X_2, F\theta_3(F_1\theta_4 F_2), U)$ and

$F\theta_3(F_1\theta_4 F_2)(q \times (q_1 \times q_2), x_2, p \times (p_1 \times p_2))$

$$\begin{aligned} &= F(q, \theta_3((q_1, q_2), x_2), p) \cap F_1\theta_4 F_2 ((q_1, q_2), x_2, (p_1, p_2)) \\ &= F(q, \theta_3((q_1, q_2), x_2), p) \cap F_1(q_1, \theta_4(q_2, x_2), p_1) \cap F_2(q_2, x_2, p_2). \end{aligned}$$

Let $f : (Q \times Q_1) \times Q_2 \longrightarrow Q \times (Q_1 \times Q_2)$ be the natural mapping and g is the identity map on X_2. Then,

$(F\theta_1 F_1)\theta_2 F_2 (((q, q_1), q_2), x_2, ((p, p_1), p_2))$

$$\begin{aligned} &= F\theta_1 F_1 ((q, q_1), \theta_2(q_2, x_2), (p, p_1)) \cap F_2(q_2, x_2, p_2) \\ &= \{F(q, \theta_1(q_1, \theta_2(q_2, x_2)), p) \cap F_1(q_1, \theta_2(q_2, x_2), p_1)\} \cap F_2(q_2, x_2, p_2) \\ &= F(q, \theta_3((q_1, q_2), x_2), p) \cap \{F_1(q_1, \theta_4(q_2, x_2), p_1) \cap F_2(q_2, x_2, p_2)\} \\ &= F(q, \theta_3((q_1, q_2), x_2), p) \cap F_1\theta_4 F_2 ((q_1, q_2), x_2, (p_1, p_2)) \\ &= F\theta_3(F_1\theta_4 F_2)((q, (q_1, q_2)), x_2, (p, (p_1, p_2))). \\ &= F\theta_3(F_1\theta_4 F_2)(f((q, q_1), q_2), g(x_2), f((p, p_1), p_2)). \end{aligned}$$

Thus, we can write $(\Upsilon\theta_1\Upsilon_1)\theta_2\Upsilon_2 \cong \Upsilon\theta_3(\Upsilon_1\theta_4\Upsilon_2)$. ■

6.3.2 Theorem

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$. Then,
$\Upsilon_1 \theta \Upsilon_2 \leq \Upsilon_1 \circ \Upsilon_2$.

Proof. Let $\Upsilon_1 = (Q_1, X_1, F_1, U)$ and $\Upsilon_2 = (Q_2, X_2, F_2, U)$. Then,

$$\Upsilon_1 \circ \Upsilon_2 = \left(Q_1 \times Q_2, X_1^{Q_2} \times X_2, F_1 \circ F_2, U\right),$$

where

$$F_1 \circ F_2 \left(\left(q_1, q_2\right), \left(f, x_2\right), \left(p_1, p_2\right)\right) = F_1 \left(q_1, f\left(q_2\right), p_1\right) \cap F_2 \left(q_2, x_2, p_2\right),$$

and

$$\Upsilon_1 \theta \Upsilon_2 = \left(Q_1 \times Q_2, X_2, F_1 \theta F_2, U\right),$$

where $\theta : Q_2 \times X_2 \longrightarrow X_1$ and

$$F_1 \theta F_2 \left(\left(q_1, q_2\right), x_2, \left(p_1, p_2\right)\right) = F_1 \left(q_1, \theta\left(q_2, x_2\right), p_1\right) \cap F_2 \left(q_2, x_2, p_2\right).$$

Let for any $x_2 \in X_2$, we can define $\theta_{x_2} : Q_2 \longrightarrow X_1$ by $\theta_{x_2}\left(q_2\right) = \theta\left(q_2, x_2\right)$ for all $q_2 \in Q_2$ and $x_2 \in X_2$. It can easily prove that θ_{x_2} is well defined. Now, define $\zeta : X_2 \longrightarrow X_1^{Q_2} \times X_2$ by $\zeta\left(x'\right) = \left(\theta_{x'}, x'\right)$ and η be the identity map on $Q_1 \times Q_2$. Then,

$$\begin{aligned}
F_1 \theta F_2 \left(\eta\left(q_1, q_2\right), x_2, \left(p_1, p_2\right)\right) &= F_1 \theta F_2 \left(\left(q_1, q_2\right), x_2, \left(p_1, p_2\right)\right) \\
&= F_1 \left(q_1, \theta\left(q_2, x_2\right), p_1\right) \cap F_2 \left(q_2, x_2, p_2\right) \\
&= F_1 \left(q_1, \theta_{x_2}\left(q_2\right), p_1\right) \cap F_2 \left(q_2, x_2, p_2\right) \\
&= F_1 \circ F_2 \left(\left(q_1, q_2\right), \left(\theta_{x_2}, x_2\right), \left(p_1, p_2\right)\right) \\
&= F_1 \circ F_2 \left(\left(q_1, q_2\right), \zeta\left(x_2\right), \left(p_1, p_2\right)\right).
\end{aligned}$$

This implies

$$F_1 \theta F_2 \left(\eta\left(q_1, q_2\right), x_2, \left(p_1, p_2\right)\right) = F_1 \circ F_2 \left(\left(q_1, q_2\right), \zeta\left(x_2\right), \left(p_1, p_2\right)\right).$$

Thus, (η, ζ) is a covering of $\Upsilon_1 \theta \Upsilon_2$ by $\Upsilon_1 \circ \Upsilon_2$, that is $\Upsilon_1 \theta \Upsilon_2 \leq \Upsilon_1 \circ \Upsilon_2$. ■

6.3.3 Definition

Let $\Upsilon_i = (Q_i, X_i, F_i, U)$ be SFSMs, $i = 1, 2$. Let η be a partial function of Q_2 onto Q_1 and let ζ be a function of X_1 into X_2. Extended ζ to a function ζ^+ of X_1^+ into X_2^+ by $\zeta^+(x) = \zeta\left(x_1\right)\zeta\left(x_2\right)\zeta\left(x_3\right)....\zeta\left(x_n\right)$, where $x = x_1 x_2 x_3 ... x_n$ and $x_i \in X_1$. Then, the pair (η, ζ) is called a strong covering of Υ_1 by Υ_2, written $\Upsilon_1 \leq_s \Upsilon_2$ if, and only if,

$$F_1^+ \left(\eta\left(q_1\right), x, \eta\left(p_1\right)\right) \subseteq F_2^+ \left(q_1, \zeta^+(x), p_1\right),$$

for all $x \in X_1^+$ and q_1, p_1 belong to the domain of η.

6.3.4 Theorem

Let $\Upsilon = (Q, X, F, U), \Upsilon_1 = (Q_1, X_1, F_1, U)$ and $\Upsilon_2 = (Q_2, X_2, F_2, U)$ be soft finite state machines such that $\Upsilon \leq_s \Upsilon_1$. Then the following assertions hold:

(1) $\Upsilon \times \Upsilon_2 \leq_s \Upsilon_1 \times \Upsilon_2$

(2) $\Upsilon_2 \times \Upsilon \leq_s \Upsilon_2 \times \Upsilon_1$

(3) For any $\theta_1 : Q_2 \times X_2 \longrightarrow X$, then there exist $\theta_2 : Q_2 \times X_2 \longrightarrow X_1$ such that $\Upsilon\theta_1\Upsilon_2 \leq_s \Upsilon_1\theta_2\Upsilon_2$.

(4) If (η, ζ) is a covering of Υ by Υ_1 such that ζ is onto, then for any $\theta_1 : Q \times X \longrightarrow X_2$, there exist $\theta_2 : Q_1 \times X_1 \longrightarrow X_2$ such that $\Upsilon_2\theta_1\Upsilon \leq_s \Upsilon_2\theta_2\Upsilon_1$.

(5) $\Upsilon \circ \Upsilon_2 \leq_s \Upsilon_1 \circ \Upsilon_2$.

Proof. Since $\Upsilon \leq_s \Upsilon_1$, so there exist pair (η, ζ) where η is a partial function of Q_1 onto Q and ζ be a function of X into X_1 such that

$$F^+(\eta(q_1), x, \eta(p_1)) \subseteq F_1^+(q_1, \zeta(x), p_1),$$

for all $x \in X_1^+$.

(1) : Define $\eta_1 : Q_1 \times Q_2 \longrightarrow Q \times Q_2$ by $\eta_1((p_1, p_2)) = (\eta(p_1), p_2)$ and $\zeta_1 : X \times X_2 \longrightarrow X_1 \times X_2$ by $\zeta_1((x, x_2)) = (\zeta(x), x_2)$. Clearly, η_1 is the partial function $Q_1 \times Q_2$ onto $Q \times Q_2$. Now,

$$\begin{aligned} F^+ \times F_2^+(\eta_1((q_1, q_2)), (x, x_2), \eta_1((p_1, p_2))) &= F^+ \times F_2^+((\eta(q_1), q_2), (x, x_2), (\eta(p_1), p_2)) \\ &= F^+ \times F_2^+((\eta(q_1), q_2), (x, x_2), (\eta(p_1), p_2)) \\ &= F^+(\eta(q_1), x, \eta(p_1)) \cap F_2^+(q_2, x_2, p_2) \\ &\subseteq F_1^+(q_1, \zeta(x), p_1) \cap F_2^+(q_2, x_2, p_2) \\ &= F_1^+ \times F_2^+((q_1, q_2), (\zeta(x), x_2), (p_1, p_2)) \\ &= F_1^+ \times F_2^+((q_1, q_2), \zeta_1((x, x_2)), (p_1, p_2)). \end{aligned}$$

Thus

$$F^+ \times F_2^+(\eta_1((q_1, q_2)), (x, x_2), \eta_1((p_1, p_2))) \subseteq F_1^+ \times F_2^+((q_1, q_2), \zeta_1((x, x_2)), (p_1, p_2)).$$

So (η_1, ζ_1) is the required covering of $\Upsilon \times \Upsilon_2$ by $\Upsilon_1 \times \Upsilon_2$.

(2) : Define $\eta_2 : Q_2 \times Q_1 \longrightarrow Q_2 \times Q$ by $\eta_2((p_2, p_1)) = (p_2, \eta(p_1))$ and $\zeta_2 : X_2 \times X \longrightarrow X_2 \times X_1$ by $\zeta_2((x_2, x)) = (x_2, \zeta(x))$. Clearly, η_2 is the partial function

$Q_2 \times Q_1$ onto $Q_2 \times Q$. Now,

$$\begin{aligned} F_2^+ \times F^+ (\eta_2 ((q_2, q_1)), (x_2, x), \eta_2 ((p_2, p_1))) &= F_2^+ \times F^+ ((q_2, \eta (q_1)), (x_2, x), (p_2, \eta (p_1))) \\ &= F_2^+ (q_2, x_2, p_2) \cap F^+ (\eta (q_1), x, \eta (p_1)) \\ &\subseteq F_2^+ (q_2, x_2, p_2) \cap F_1^+ (q_1, \zeta (x), p_1) \\ &= F_2^+ \times F_1^+ ((q_2, q_1), (x_2, \zeta (x)), (p_2, p_1)) \\ &= F_2^+ \times F_1^+ ((q_2, q_1), \zeta_2 ((x_2, x)), (p_2, p_1)) . \end{aligned}$$

Thus,

$$F_2^+ \times F^+ (\eta_2 ((q_2, q_1)), (x_2, x), \eta_2 ((p_2, p_1))) \subseteq F_2^+ \times F_1^+ ((q_2, q_1), \zeta_2 ((x_2, x)), (p_2, p_1)) .$$

So, (η_2, ζ_2) is the required covering of $\Upsilon_2 \times \Upsilon$ by $\Upsilon_2 \times \Upsilon_1$.

(3) : Since $\Upsilon \theta_1 \Upsilon_2 = (Q \times Q_2, X_2, F\theta_1 F_2, U)$, where $\theta_1 : Q_2 \times X_2 \longrightarrow X$ and

$$F\theta_1 F_2 ((q, q_1), x_2, (p, p_2)) = F (q, \theta_1 (q_2, x_2), p) \cap F_2 (q_2, x_2, p_2) .$$

Define $\theta_2 : Q_2 \times X_2 \longrightarrow X_1$ by $\theta_2 ((q_2, x_2)) = \zeta \circ \theta_1 ((q_2, x_2))$. Then $\Upsilon_1 \theta_2 \Upsilon_2 = (Q_1 \times Q_2, X_2, F_1 \theta_2 F_2, U)$ and

$$F_1 \theta_2 F_2 ((q_1, q_2), x_2, (p_1, p_2)) = F_1 (q_1, \theta_2 (q_2, x_2), p_1) \cap F_2 (q_2, x_2, p_2) .$$

Let $\eta' : Q_1 \times Q_2 \longrightarrow Q \times Q_2$ defined by $\eta' ((q_1, q_2)) = (\eta (q_1), q_2)$. Clearly, η' is partial surjective map and take ζ' be the identity map on X_2. Then,

$$\begin{aligned} F^+ \theta_1 F_2^+ \left(\eta' ((q_1, q_2)), x_2, \eta' ((p_1, p_2))\right) &= F^+ \theta_1 F_2^+ ((\eta (q_1), q_2), x_2, (\eta (p_1), p_2)) \\ &= F^+ (\eta (q_1), \theta_1 (q_2, x_2), \eta (p_1)) \cap F_2^+ (q_2, x_2, p_2) \\ &\subseteq F_1^+ (q_1, \zeta (\theta_1 (q_2, x_2)), p_1) \cap F_2^+ (q_2, x_2, p_2) \\ &= F_1^+ (q_1, \zeta \circ \theta_1 (q_2, x_2), p_1) \cap F_2^+ (q_2, x_2, p_2) \\ &= F_1^+ (q_1, \theta_2 ((q_2, x_2)), p_1) \cap F_2^+ (q_2, x_2, p_2) \\ &= F_1^+ \theta_2 F_2^+ ((q_1, q_2), x_2, (p_1, p_2)) . \end{aligned}$$

Thus,

$$F^+ \theta_1 F_2^+ \left(\eta' ((q_1, q_2)), x_2, \eta' ((p_1, p_2))\right) \subseteq F_1^+ \theta_2 F_2^+ ((q_1, q_2), x_2, (p_1, p_2)) .$$

So, $\left(\eta', \zeta'\right)$ is the required covering of $\Upsilon_2 \theta_1 \Upsilon$ by $\Upsilon_2 \theta_2 \Upsilon_1$.

(4) : Since $\Upsilon_2 \theta_1 \Upsilon = (Q_2 \times Q, X, F_2 \theta_1 F, U)$, where $\theta_1 : Q \times X \longrightarrow X_2$ and

$$F_2 \theta_1 F ((q_2, q), x, (p_2, p)) = F_2 (q_2, \theta_1 (q, x), p_2) \cap F (q, x, p) .$$

Define $\theta_2 : Q_1 \times X_1 \longrightarrow X_2$ by $\theta_2 (q_1, x_1) = \theta_2 ((q_1, \zeta (x))) = \theta_1 ((\eta (q_1), x))$, then $\Upsilon_2\theta_2\Upsilon_1 = (Q_2 \times Q_1, X_1, F_2\theta_2F_1, U)$ and

$$F_2\theta_2F_1 ((q_2, q_1), x_1, (p_2, p_1)) = F_2 (q_2, \theta_2 (q_1, x_1), p_2) \cap F_1 (q_1, x_1, p_1).$$

Let $\eta^{'} : Q_2 \times Q_1 \longrightarrow Q_2 \times Q$ defined by $\eta^{'} ((q_2, q_1)) = (q_2, \eta (q_1))$ and take $\zeta^{'} = \zeta$. Then $\left(\eta^{'}, \zeta^{'}\right)$ is the required covering.

(5) : Since

$$\Upsilon \circ \Upsilon_2 = \left(Q \times Q_2, X^{Q_2} \times X_2, F \circ F_2, U\right),$$

where

$$F \circ F_2 ((q, q_2), (f, x_2), (p, p_2)) = F (q, f (q_2), p) \cap F_2 (q_2, x_2, p_2),$$

and

$$\Upsilon_1 \circ \Upsilon_2 = \left(Q_1 \times Q_2, X_1^{Q_2} \times X_2, F_1 \circ F_2, U\right),$$

where

$$F_1 \circ F_2 ((q_1, q_2), (g, x_2), (p_1, p_2)) = F_1 (q_1, g (q_2), p_1) \cap F_2 (q_2, x_2, p_2).$$

Define $\eta^{'} : Q_1 \times Q_2 \longrightarrow Q \times Q_2$ by $\eta^{'} ((q_1, q_2)) = (\eta (q_1), q_2)$ and $\zeta^{'} : X^{Q_2} \times X_2 \longrightarrow X_1^{Q_2} \times X_2$ by $\zeta^{'} ((f, x_2)) = (\zeta \circ f, x_2)$. Then, $\left(\eta^{'}, \zeta^{'}\right)$ is the required covering. ■

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