Study of Electrostatic Modes in Self-Gravitating Dusty Plasma

MASTER OF PHILOSOPHY

IN

PHYSICS

BY

NOOR UL AIN

DEPARTMENT of PHYSICS

QUAID-I-AZAM UNIVERSITY

ISLAMABAD, PAKISTAN

Study of Electrostatic Modes in Self-Gravitating Dusty Plasma

A dissertation submitted to the department of Physics, Quaid-i-Azam university Islamabad, in the partial fulfillment of the requirement for the degree of

Master of Philosophy

in

Physics

by

NOOR UL AIN

Department of Physics
Quaid-i-Azam University
Islamabad, Pakistan.


بسم الله الرحمن الرحيم

CERTIFICATE

This is to certify that ***Miss. NOOR UL AIN*** has carried out the work contained in this dissertation under my supervision and is accepted by the Department of Physics, Quaid-i-Azam University, Islamabad as satisfying the dissertation requirement for the degree of Master of Philosophy in Physics.

Prof. Dr. Arshad M. Mirza

Supervisor

Department of Physics

Quaid-i-Azam University

Islamabad, Pakistan.

Submitted Through

Prof. Dr. Muhammad Zaka-ullah

Chairman

Department of Physics

Quaid-i-Azam University

Islamabad, Pakistan.

Dedicated

To

My Loving Parents,

Family

And

My honorable Supervisor

Prof.Dr.Arshad M. Mirza

Acknowledgements

In the name of Allah the most Gracious, the most Merciful. All praise is due to Allah; we praise Him, seek His help and ask for His forgiveness. I am thankful to **Allah**, Who has bestowed me with the strength, the guidance and the love to complete this dissertation. Also, I cannot forget the ideal man of the world and more respectable personality, **Prophet Mohammad (S.A.W)**. May the blessings and peace of Allah be upon Muhammad (S.A.W), His family and companions.

I am truly indebted and grateful to my supervisor **Prof. Dr. Arshad M. Mirza** for his invaluable advice and supervision from finding an appropriate title of research to the process of writing the dissertation. His vast knowledge, calm nature and positive criticism motivated me to complete this work.

I feel great pleasure and gratitude to acknowledge **Prof. Dr. Muhammad Zaka-ullah,** chairman department of Physics, for providing me the opportunities to improve learning capacities and to dig out things in depth.

I express my sincerest appreciation for Sidra Ali, for her assistance in any way that I may have asked. I feel great pleasure in expressing my thanks to my senior research fellows Hina, Qurat ul ain and my co-fellows Tahira, Tashfeen and Kashifa. I convey my thanks to my caring and loving friends Tayyaba, and Amina for their cheerful and refreshing company during my research work.

I owe my deepest gratitude to my maternal uncle Javed Afzal and his wife, Rehana Shaheen for their affectionate and supportive behavior in every crucial time during my research work. I am grateful to Ejaz Saeed Aslam and Naheed Ejaz for their help whenever I needed. My ineffable thanks to Miss. Farida Asghar, Sir Muqadas and Mr. Mubarak for their co-operation and encouragement during my research work.

My sincere and profound gratitude are due to my mother and father. Their prayer and moral support will always boost my progress. To them, I am eternally grateful. I am particularly grateful for the assistance given by my sister Kiran Aqsa. My heartiest well wishes for my brothers Afra Siab, Zulqarnain Turabi and Saqlain Ali. Deep and heartfelt thanks again to my brother Zulqarnain, who proved astonishingly supportive towards my success. I thank to each and everyone who helped me at any point and anytime either mentioned or not in this acknowledgement.

Finally, I pay thanks to my husband Rashid for his cooperation

NOOR UL AIN

Abstract

In this dissertation, a review of some electrostatic modes in self-gravitating dusty plasma is carried out. First of all the basic normal modes (dust-acoustic wave (DAW) and dust-ion-acoustic wave (DIAW)), in unmagnetized and collisionless dusty plasma are discussed.

The electrostatic waves with fluctuating dust charge and self-gravitational effects are considered in a dense dusty plasma. The self-gravitational attraction arises due to massive dust particles and leads to Jeans type instability. This Jeans instability is discussed for tenuous (low density dust), dilute (medium density dust) and dense (high density) regimes that corresponds to dust-acoustic wave (DAW), dust charge density wave (DCDW) and dust coulomb wave (DCW), respectively. A review of general dispersion relation for electrostatic dust waves for multi-ion system is carried out. The linear instability of dusty plasma is discussed for electrostatic case with limit $Gm_d^2/q_d^2 \approx 0$. The dispersion relation of self-gravitating, collisional (dust-ion collision only) dusty plasma is also reviewed.

A new dispersion relation is re-derived considering the attractive forces among the dust particles and concluded that the attraction among the charge particles could be the cause of collapse of astrophysical bodies and hence leads to the formation of stars and planets in dusty plasma.

Contents

Chapter 1

Introduction

Dusty plasmas have been, extensively investigated at the end of nineteenth century. The charged dust particles and plasma are the two components of the Universe that are present everywhere. The dust particles in the plasma give rise to new characteristic features for the research interest. The properties of the dusty plasma are entirely different from ordinary plasma with multi-ionic species. An interesting feature of the dusty plasma is formation of plasma crystals, i.e., it can form crystals, when particles arranged in an ordered form [1, 2]. The dusty plasma is present in our solar system such as in circumsolar dust rings, in asteroids (a space object made up of dust and plasma that orbits the Sun in the space between Mars and Jupiter), in comets (a huge ball of frozen gases and dust orbiting around the Sun), in space between the planets, interstellar clouds, and in circumstellar disks, in noctilucent clouds, in light coming from clouds to the ground including air polluted with smoke during thunderstorm, and in the flame of candle [3, 4]. The space between stars is called interstellar space. This space is occupied completely with dust particles and frozen gases. The large molecular clouds collide with each other due to gravitational forces and cause the formation of new stars. The charged dust grains are also present in interplanetary medium. The space between planets is filled with dust, whose presence was indicated from the zodiacal light. The dust particles in the inner solar system are dominantly due to the asteroid belt (a wide band of asteroid orbiting the Sun between the Mars and Jupiter).

1.1 Characteristics and Parameters of Dusty Plasma

The neutral dust particles when covered completely in an ordinary plasma, they get electrically charged [3, 5]. In dusty plasma, the plasma species (electrons and ions) can recombine. While sometimes electrons are emitted by dust particles through thermionic or secondary electron emission. This influences the ionization balance of plasma [6]. The dust grains size ranges from nano- to -micronmetre and the dust charging could be a thousand electron. The grains are assumed to have equal radius with same charges. The basic processes of dust grains charging mainly depend on the environment around the dust grains.

The dust grains are charged by the electrons and ions flowing onto their surfaces. The dust particles absorbed in a gaseous plasma are generally negatively charged. The energetic electrons or ions are either reflected by the dust particulates or they pass through the dust material when they interact with dust particle surface. In this interaction they may lose their energy partly or fully. This energy can excite other electrons that may cause them to leave the material and are named as Secondary-Electrons. The emission of secondary electrons from the dust particles, make the particle surface positive. The photons incident onto the dust grain surface give rise to the Photo-Emission of electrons from the dust particle surface. Due to emission of photoelectrons, the dust particles gain positive charge. The emitted electrons interact with other dust particles and are absorbed by a number of dust particles, which may become negatively charged [7]. Other charging processes are field emission, ultra-violet ray irradiation, radioactivity, impact ionization etc. (Feuerbacher *et al.*, 1973; Fechting *et al.*, 1979; Whipple *et al.*, 1985; Havnesetal., 1987) [8].

The charge neutrality condition (when there are no external perturbations), is given as

$$\alpha z_d n_{d0} + n_{e0} = n_{i0} \tag{1.1}$$

here $\alpha = 1$ or -1, for negative or positive charged dust particle respectively. In Eq. (1) z_d is total number of charges accumulated on the dust particle surface.

The effective Debye length in dusty plasma is given as

$$\lambda_D = \frac{\lambda_{De}\lambda_{Di}}{\sqrt{\lambda_{De}^2 + \lambda_{Di}^2}} \tag{1.2}$$

where λ_{De} and λ_{Di} are the Debye lengths of the electron and ions respectively.

The dust plasma frequency is defined as

$$\omega_{pd} = \sqrt{\frac{4\pi q_{d0}^2 n_{d0}}{m_d}} \tag{1.3}$$

where q_{d0}, n_{d0} and m_d are the equilibrium dust charge, number density and mass respectively.

The length scale parameters for the dusty plasma are the radius of the dust particle, the average distance between the dust particles and the effective Debye length, denoted by "R", "d" and "λ_D" respectively. Here the dust particle size is smaller than the other two scale lengths. Now if the dust particles are arranged so that $R \ll \lambda_D \ll d$ " then each particle is surrounded by the electrons and form a Debye shielding. In other case when "$R \ll d \ll \lambda_D$", the distance between dust particles is less than the effective Debye length. In this region heavy charged dust particles play role in the collective behavior [7].

1.2 Wave Instabilities in Dusty Plasma

Since the wave is defined as periodic motion, so the change in distribution of kinetic and potential energy in any medium can also be termed as wave.

In dusty plasma, the dust particles introduces new kind of instabilities. The dust particles bears many forces like gravitational force, electrostatic and electromagnetic forces, and pressure gradients etc. These forces are the sources of free energy to cause different sort of instabilities in space dusty plasma. Instabilities in dusty plasma varies with respect to space or time. Thus, there is a strong relationship between wave frequency ω and wave vector k, showing that perturbation belong to free energy (kinetic energy of the particle transferred to the perturbation fields) of the system. Any perturbation in the system changes the quasi-neutrality condition and the space plasmas does not remain at thermodynamic equilibrium. This variation decreases the free energy.

The frequency and wave vector can be written as in term of real and imaginary parts, i.e., $\omega = \omega_r + i\omega_i$, and $k = k_r + ik_i$, respectively. The imaginary part of the frequency ω_i represents the damping or growth rate of instabilities. The increase in free energy increases the value of ω_i form negative to a limit when $\omega_i = 0$, is reached. This is the limit of threshold for instability. This shows that the further increase in free energy at particular wave vector makes the imaginary part of frequency ω_i positive and hence the wave grows. The dispersion relation changes to the cold plasma case when the free energy is increased to a range where maximum growth rate reaches. This regime is taken as a fluid instability.

1.3 Self-Gravitation

In dusty plasma with heavy dust particles, the gravitational interaction among dust particles come into play on the same footing as the electrostatic forces behave. This gravitational interaction among dust particles is termed as Self-Gravitation.

1.4 Effects of Self-Gravitation in Dusty Plasma

The gravitational instability of an infinite homogenous medium was first considered by Sir James Jeans [9, 10]. It is a facts that the formation of stars is due to the gravitational collapse of dust particles and the clouds of interstellar gas. The interstellar radiations and other processes ionizes the gas clouds. Thus, due to interstellar gaseous cloud-cloud collisions, stars or stellar clusters are formed. In these collisions a gaseous slab is shaped with shock fronts propagating away from the collision boundary. These gaseous slab increase in mass and become unstable against gravitational instability, which results in fragmentation of these slabs. Thus, these fragments collapse further and results in stars or stellar clumps [11]. Mainly the Jeans Instability of self-gravitating interstellar gaseous cloud is supposed to be the cause of formation of stars.

1.5 Review of Research Work on Dusty Plasma

Here only that research work is summarized that is related to the rest analysis of this dissertation.

In 1985 Bliokh *et al.* [12], had investigated the electrostatic waves in Saturn's rings. In 1988 Havnes [13], discovered the relative drift between the solar wind and dust particles. In 1990 Rao *et al.* [14], had discussed the low frequency "dust acoustic wave" in unmagnetized, collisionless dusty plasma with Boltzmann distributed electrons and ions. In 1992 Shukla [15] had investigated the linear electrostatic and electromagnetic waves in low frequency regime. In 1992 Shukla *et al.* [16], investigated another normal dust mode i.e., "dust ion acoustic wave" arising due to depletion of electrons and immobile dust particles with high charge.

In 1993 Melando *et al.* [17], had first time properly discussed the effect of fluctuation of dust particle's charge on collective behavior in dusty plasma. In 1994 Pandey *et al.* [18], discussed the linear and nonlinear effects of Jeans instability in dusty plasma. In 1995 Barken *et al.* [19], had presented the laboratory results of the instabilities of dust acoustic modes. In 1996 Pandey *et al.* [20], has investigated the self-gravitational instabilities by considering the dynamics of ionic specie as well. In 1997 Verheest *et al.* [21], had pointed out that work of Pandey *et. al* [20], is erroneous, because there is no need to consider the ion dynamics in comparison to heavy dust particles. They also have found the dispersion relation for dust acoustic and dust ion acoustic waves in self-gravitating dusty plasma with fluctuating dust charge. In 1998 Mammun *et al.* [22], had described the ultra-low frequency waves of electrostatic dust modes in inhomogeneous, magnetized dusty plasma including self-gravitation.

In 2000 Rao *et al.* [23], had investigated the effects of dust charge fluctuations and self-gravitation on electrostatic dust waves in dense dusty plasma. They had also discussed the Jeans instability for different density regimes. In 2002 Jacobi *et al.* [24], had represented the effects of self-gravitation and collisions (among dust particles and ions) on low frequency, electrostatic waves in dusty plasma. In 2004 Misra *et al.* [25], had discussed the linear electrostatic dust modes in collisional, self-gravitating dusty plasma with different charge impurities. In 2005 Delzanno *et al.* [26], had considered the Lennard Jones potential in a model of collisionless dusty plasma. In 2006 Shukla *et al.* [27] have investigated the Jeans type instability due to an attractive forces among charged particles in self-gravitating dusty plasma.

1.6 Motivation

Dusty plasma is the rapidly growing field of research and it provides a rapid development in plasma technology. In the beginning, the main focus of research was on the study of characteristics of isolated dust particle e.g., the charging processes, creation of dust particles, shielding of charges and drag forces etc. were investigated. These processes alongwith some other studies provide the information about the field of gravito-electrodynamics.

The recent research interests are the study of collective processes in dusty plasma. Among these processes, the waves propagating under different conditions and corresponding instabilities are much focused. Different types of new dust modes are investigated for collective dynamics of electrons, ions and the charged as well as neutral dust particles. Self-gravitational forces appearing due to attraction among heavy dust particles, is the basic source of formation of stars, planets etc. Thus, the processes of formation astrophysical bodies is the rapidly developing research field.

The growing research interest in studying dusty plasma provides great motivation to study different processes in dusty plasma.

1.7 Layout of Dissertation

The dissertation is arranged in the following fashion: The first Chapter contains a brief introduction to dusty plasma, its parameters and characteristics, self-gravitation and the Jeans instability and review of research work on instabilities in dusty plasma.

In Chapter 2, some waves in dusty plasma namely the "dust-acoustic wave" and "dust-ion acoustic wave" are discussed. In last section the introduction to self-gravitation and Jeans instability is presented.

In Chapter 3, the dense dusty plasma is considered. Firstly a fluid model is discussed, which carries constant dust charge. Then in next section, new dust wave mode, i.e., 'dust-coulomb wave' is discussed, which arise due to self-gravitation and fluctuation of dust particle charge in dense dusty plasma. The Jeans type instability and critical wavelengths for different density regime are also discussed.

The last Chapter 4, deals with the dispersion relations, derived for different cases. A general formalism is described for self-gravitating dusty plasma with more than one ionic specie. Then in next section, linear Jeans instability is described for heavy dust particles obeying the condition $Gm_d^2/q_d^2 \approx 0$, and is then discussed for various limits. In the last section, takes into account the electrostatic energy in addition to self-gravitation and dust charge fluctuation.

Chapter 2

Wave Phenonmena in Dusty Plasma

2.1 Introduction

In this chapter we shall discuss the physics of "waves in dusty plasma" and derive linear dispersion relations. The introduction to the phenomena of self-gravitation in dusty plasma is also discussed. We consider here dusty plasma consisting of electrons, ions and negatively charged dust grains.

2.2 Waves in Dusty Plasma

In dusty plasma, the dust particles are charged, it sustains wave modes in the same way as an electron-ion plasma supports ion-acoustic waves [28]. There are two kind of waves in dusty plasma. These wave mechanisms arising due to dust dynamics, have been widely studied theoretically as well as experimentally. First type of electrostatic waves called the "Dust–Acoustic Wave" was theoretically discussed by Rao *et al*, [14]. The second type of wave was discussed by Shukla *et al.*, [16] and named it the "Dust-Ion–Acoustic Wave". Both the dust-acoustic and dust-ion-acoustic waves have been detected in laboratory as well [29].

2.2.1 Dust Acoustic Waves

In dust-acoustic wave, the dust particles are mobile and these are compressional waves in which disturbance propagates through the dust layers [30, 28]. For this mode, we consider collisionless, unmagnetized dusty plasma [7, 29].

The Dust-Acoustic wave is very low frequency wave with frequencies of the order of the dust plasma frequency $\omega_{pd}\left(=\sqrt{4\pi n_{d_0}q_{d0}/m_d}\right)$ [30]. This frequency is much less than the electron (ω_{pe}) and ion (ω_{pi}) plasma frequency due to high dust mass [30]. The phase velocity of the dust-acoustic waves is smaller than both the electron and ion thermal velocities [7]. Thus, in Dust-Acoustic waves, the dust grains provide the inertia which is balanced by the restoring force of the pressures of Boltzmannen-distributed electrons and ions.

Fluid Equations

The electron and ion have small mass as compared to the mass of dust particle, thus we call these species as inertialess. These inertialess species maintain equilibrium in the dust-acoustic wave potential "ϕ". When we ignore the inertia of electrons and ions, then electric force is balanced by the pressure gradient and hence the number densities of electron and ion are given by Boltzmann type distribution, [14] as,

$$n_e = n_{e0}\exp\left(\frac{e\phi}{k_BT_e}\right) \tag{2.1}$$

$$n_i = n_{i0}\exp\left(\frac{-e\phi}{k_BT_i}\right) \tag{2.2}$$

respectively. Here $n_{e0}\,(n_{i0})$, $T_e\,(T_i)$ are the equilibrium number densities and temperature of electrons (ions), ϕ is the electrostatic potential, and e is the electronic charge. The dust fluid equations consist of continuity equation,

$$\frac{\partial}{\partial t}(n_d) + \boldsymbol{\nabla}\cdot(\mathbf{n}_d\mathbf{v}_d) = 0 \tag{2.3}$$

and momentum equation given by

$$m_dn_d\left(\frac{\partial}{\partial t} + \mathbf{v}_d\cdot\boldsymbol{\nabla}\right)\mathbf{v}_d = q_dn_d\mathbf{E} - \boldsymbol{\nabla}p_d \tag{2.4}$$

where m_d, n_d, $\mathbf{v}_d$, $q_d\,(= -ez_d)$, are mass, number density, velocity, and charge of dust particles respectively. While $\mathbf{E}$ and $\boldsymbol{\nabla}p_d$ are the electric field and dust pressure gradient, respectively. Here the electrostatic waves propagate along the $x-$axis. We replace "$\mathbf{E}$" by $(-\partial_x\,\phi)$ and "p_d" by $(n_dk_BT_d)$, thus after simplification, we get

$$\left(\frac{\partial}{\partial t} + v_d\frac{\partial^2}{\partial x^2}\right)v_d = -\frac{q_d}{m_d}\frac{\partial}{\partial x}\ \phi - \frac{k_BT_d}{m_dn_d}\frac{\partial}{\partial x}n_{d1} \tag{2.5}$$

To close the set of fluid equations, the Poisson's equation is given by,

$$\frac{\partial^2}{\partial x^2}\phi = \mathbf{4}\pi\left(en_e - en_i - q_d n_d\right) \tag{2.6}$$

Now we first separate the dependant variables into equilibrium and perturbed parts denoted by subscripts "0" and "1" respectively,

$$n_e = n_{e0} + n_{e1}, \quad n_i = n_{i0} + n_{i1}, \quad n_d = n_{d0} + n_{d1}\ , v_d = v_{d0} + v_{d1},$$

here equilibrium parts represent the plasma parameters without oscillations. Then, after linearization, we get the simplified equations. By linearization we mean that the amplitude of oscillations is so small that the terms with higher power of amplitude can be ignored. Hence, we get the set of equations: For $\phi \ll k_B T_{e,i}/e$, the exponential factor "$\exp\left(\pm e\phi/k_B T_{e,i}\right)$" in the electron and ion number densities are,

$$n_{e,i} = n_{e0,i0}\left[1 \pm \frac{e\phi_1}{k_B T_{e,i}} + \left(\frac{e\phi_2}{k_B T_{e,i}}\right)^2 + ...\right]$$

Therefore, the first order perturbed electron and ion number densities are,

$$n_{e1} = n_{e0}\frac{e\phi_1}{k_B T_e} \tag{2.7}$$

$$n_{i1} = -n_{i0}\frac{e\phi_1}{k_B T_i} \tag{2.8}$$

The linearized dust continuity equation can be written as,

$$\frac{\partial}{\partial t}n_{d1} + n_{d0}\frac{\partial}{\partial x}v_{d1} = 0 \tag{2.9}$$

and dust momentum equation

$$\frac{\partial}{\partial t}v_{d1} = -\frac{q_d}{m_d}\boldsymbol{\phi} - \frac{3k_B T_d}{m_d n_{d0}}\frac{\partial}{\partial x}n_{d1}$$

For one dimensional case, $m_d V_{td}^2/2 = k_B T_d/2$, where T_d is the dust temperature

$$\frac{\partial}{\partial t}v_{d1} = -\frac{q_d}{m_d}\frac{\partial}{\partial x}\ \phi - \frac{3V_{td}^2}{n_{d0}}\frac{\partial}{\partial x}n_{d1} \tag{2.10}$$

Assuming all the perturbed quantities are proportional to $\exp\left[-i\left(\omega t-kx\right)\right]$, the Eqs. (2.9) and (2.10) yields the following result,

$$-i\omega n_{d1}+n_{d0}ik\cdot v_{d1}=0 \tag{2.11}$$

$$-i\omega v_{d1}=-ik\frac{q_d}{m_d}\phi{-}ik\frac{3V_{Td}^2}{n_{d0}}n_{d1} \tag{2.12}$$

Solving Eqs. (2.11) and (2.12), we obtain

$$n_{d1}=\frac{k^2n_{d0}q_{d0}}{m_{d\left(\omega^2-3k^2V_{td}^2\right)}}\phi \tag{2.13}$$

Now writing perturbed form of Poisson's equation as,

$$\frac{\partial^2}{\partial x^2}\phi_1{=}\,\mathbf{4}\pi\left(en_{e1}-en_{i1}-q_{d0}n_{d1}\right) \tag{2.14}$$

Dispersion Relation

Substituting the values of n_{e1}, n_{i1} from equations (2.7) and (2.8), we get

$$\frac{\partial^2}{\partial x^2}\phi_1=\mathbf{4}\pi\left(e\left(n_{e0}\frac{e\phi}{k_BT_e}\right)-e\left(-n_{i0}\frac{e\phi}{k_BT_i}\right)-q_dn_{d1}\right)$$

After rearranging the terms, we get

$$\frac{\partial^2}{\partial x^2}\phi_1=\mathbf{4}\pi\left(n_{e0}\frac{e^2}{k_BT_e}+n_{i0}\frac{e^2}{k_BT_i}\right)\phi-\mathbf{4}\pi q_dn_{d1} \tag{2.15}$$

Now defining Debye lengths of electron and ions as, $\lambda_{De}=\sqrt{\mathbf{4}\pi n_{e0}e^2/k_BT_e}$ and $\lambda_{Di}=\sqrt{\mathbf{4}\pi n_{i0}e^2/k_BT_i}$, we get

$$\mathbf{4}\pi n_{e0}\frac{e^2}{k_BT_e}+4\pi n_{i0}\frac{e^2}{k_BT_i}=\frac{1}{\lambda_{De}^2}+\frac{1}{\lambda_{Di}^2}=\frac{1}{\lambda_D^2}$$

Where λ_D is the effective Debye length. Now replacing the $\lambda_D^{-2}=k_D^2$, the Eq. (2.15) becomes

$$\frac{\partial^2}{\partial x^2}\phi_1=k_D^2\phi-\mathbf{4}\pi q_dn_{d1}$$

Now taking Fourier transform, we get

$$-k^2\phi_1 = k_D^2\phi_1 - 4\pi q_d n_{d1}$$

or

$$\left(k^2 + k_D^2\right)\phi_1 = 4\pi q_d n_{d1}$$

Substituting n_{d1} from Eq. (2.13), in the above equation and replacing $k_D^2 = \lambda_D^{-2}$ and then multiplying with λ_D^2, we get the dispersion relation for Dust-Acoustic wave as:

$$\omega^2 = 3k^2V_{td}^2 + \frac{k^2\omega_{pd}^2\lambda_D^2}{1 + k^2\lambda_D^2}$$

we can write $\omega_{pd}\lambda_D = C_{DA}$, representing speed of Dust-Acoustic wave. Finally, we obtain the following dispersion relation for the dust-acoustic wave,

$$\omega^2 = 3k^2V_{td}^2 + \frac{k^2C_{DA}^2}{1 + k^2\lambda_D^2} \tag{2.16}$$

Limiting Cases

For cold plasma i.e., $\omega/k \gg V_{Td}$ and in long-wavelength limit i.e., $k^2\lambda_D^2 \ll 1$, Eq. (2.16) takes the following result,

$$\omega = k\omega_{pd}\lambda_D$$

Substituting the values of λ_D and ω_{pd}, we get

$$\omega = k\sqrt{\frac{4\pi z_{d0}^2 e^2 n_{d0}}{m_d}}\left[\frac{4\pi e^2}{k_B}\left\{\frac{n_{i0}}{T_i} + \frac{n_{e0}}{T_e}\right\}\right]^{-\frac{1}{2}}$$

or

$$\omega = kz_{d0}\sqrt{\frac{n_{d0}}{m_d}}\left[\frac{n_{i0}}{k_BT_i}\left\{1 + \frac{T_in_{e0}}{T_en_{i0}}\right\}\right]^{-\frac{1}{2}}$$

Using charge neutrality condition, we can write "$n_{e0}/n_{i0} = 1 - z_{d0}n_{d0}/n_{i0}$" for negatively charged dust grains. Thus, substituting this in above dispersion relation, we get

$$\omega = k\sqrt{\frac{n_{d0}}{n_{i0}}}\sqrt{\frac{k_BT_i}{m_d}}\left[1 + \frac{T_i}{T_e}\left(1 - \frac{z_{d0}n_{d0}}{n_{i0}}\right)\right]^{-\frac{1}{2}} \tag{2.17}$$

Phase Velocity of Dust-Acoustic waves

From this equation we can found the phase velocity of the Dust-Acoustic waves if we know the plasma and dust parameters T_e, T_i, n_{i0}, and n_{d0}, z_{d0},

$$V_p = \frac{\omega}{k} = \sqrt{\frac{n_{d0}}{n_{i0}}}\sqrt{\frac{k_B T_i}{m_d}}\left[1 + \frac{T_i}{T_e}\left(1 - \frac{z_{d0} n_{d0}}{n_{i0}}\right)\right]^{-\frac{1}{2}}$$

2.2.2 Dust Ion Acoustic Waves

Shukla *et al.* *[16]* have discussed dust-ion acoustic waves in dusty plasma. In the Dust-Ion acoustic waves, the dust particles are considered to be stationary, but the dust grain influences the wave propagation. The influence of the dust is that it reduces the electron number density because a fraction of electrons, sticks to the surface of dust particles. The wave frequency is of the order of the ion plasma frequency ω_{pi} which is greater than dust plasma frequency [30]. In this case, the phase velocity of the Dust-Ion acoustic waves is much greater than the thermal velocity of the ions and dust particles i.e., $\omega/k \gg V_{Th,i}$, V_{td}. Whereas, the electrons thermal velocity is much greater than the phase velocity of Dust-Ion-acoustic wave, [7]. The "ion-acoustic wave" can still exists even when the electron and ion temperatures are equal.

Fluid Model for Immobile Dust Particles

The electrons number density here obey Boltzmann distribution as given by Eq. (2.7). The set of one dimensional equations for dust ion acoustic waves with immobile dust particles consist of the ion continuity equation,

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i v_i) = 0$$

the ion momentum equation,

$$m_i n_i \frac{\partial v_i}{\partial t} = e n_{i0} E - \frac{\partial}{\partial x} p_i$$

and the Poisson's equation,

$$\frac{\partial^2}{\partial x^2}\phi = \mathbf{4}\pi\left(e n_e - e n_i\right)$$

These equations differs from that described for dust acoustic waves in this aspect that the kinetic pressure of ions and perturbed dust number density, both are neglected [30]. The dust particles contributions is described only by quasineutrality equation,

$$n_{i0} = n_{e0} + z_d n_{d0}$$

Dispersion Relation

From above fluid equations the perturbed ion number density, after Fourier transformation, is evaluated to be,

$$n_{i1} = -\frac{k^2}{\omega^2}\left(\frac{en_{i0}}{m_i}\phi_1\right)$$

After taking Fourier transform of Poisson's equation and substituting the perturbed electron and ion number densities, the dispersion relation comes out to be

$$\omega^2 = \frac{\omega_{pi}^2 k^2 \lambda_{De}^2}{1 + k^2\lambda_{De}^2} = \frac{n_{i0}}{n_{e0}}\left(\frac{k^2}{1 + k^2\lambda_{De}^2}\right)\frac{k_B T_e}{m_i}$$

where $\omega_{pi} = \sqrt{4\pi e^2 n_{i0}/m_i}$, and $\lambda_{De} = \sqrt{k_B T_e/4\pi e^2 n_{e0}}$, are the ion plasma frequency and electrons Debye length, respectively. This dispersion relation is same as that for ion acoustic waves with additional term $n_{i0}/n_{e0} > 1$.

Fluid Model for Mobile Dust Particle

The electrons number density here obey Boltzmann distribution as given by Eq. (2.7). The ion number density is found from perturbed continuity equation [7],

$$\frac{\partial n_{i1}}{\partial t} + n_{i0}\frac{\partial}{\partial x}v_{i1} = 0 \tag{2.18}$$

and momentum equation for ions

$$m_i n_{i0}\frac{\partial v_{i1}}{\partial t} = en_{i0}E_1 - \frac{\partial}{\partial x}p_{i1} \tag{2.19}$$

Substituting $E_1 = -\partial_x\phi_1$, and $p_i = n_{i1}k_B T_i$, we get

$$\frac{\partial v_{i1}}{\partial t} = -\frac{e}{m_i}\frac{\partial}{\partial x}\phi_1 - \frac{k_B T_i}{m_i n_{i0}}\frac{\partial}{\partial x}n_{i1} \tag{2.20}$$

The Poisson's equation is given by Eq. (2.14). Now for finding perturbed ion number density, we take Fourier transform of Eqs. (2.18) and (2.20). Let us, simply replace "∂_t" by "$-i\omega$" and "∂_x" by "ik", we get,

$$-i\omega n_{i1} + n_{i0} ikv_{i1} = 0 \tag{2.21}$$

$$-i\omega v_{i1} = -\frac{e}{m_i} ik\phi_1 - 3V_{Ti}^2 ikn_{i1} \tag{2.22}$$

and solving both equations simultaneously, we get

$$n_{i1} = \left(\frac{\omega^2}{k^2} - 3V_{Td}^2 n_{i0}\right)^{-1} \left(-\frac{en_{i0}}{m_i}\phi_1\right) \tag{2.23}$$

The dust number density perturbations remains same as that for Dust-Acoustic waves, given by Eq. (2.13). Whereas for stationary dust particles, the dust charge perturbations are zero, i.e., $n_{d1} = 0$, and the dust ion acoustic wave comes out for a very short time scale as compared to the dust plasma period ($t = 2\pi/\omega_{pd}$).

Dispersion Relation

Combining equations (2.1), (2.13), (2.14) and (2.23). Also here we assume that $\omega \gg kV_{Ti}$, kV_{Td}, for obtaining the dispersion relation.

$$-k^2 = \frac{4\pi e^2 n_{e0}}{k_B T_e} - \frac{k^2}{\omega^2}\left(\frac{4\pi e^2 n_{i0}}{m_i} + \frac{4\pi q_{d0}^2 n_{d0}}{m_d}\right)$$

Thus, the dispersion relation for the Dust-Ion acoustic waves is (Shukla and Silin 1992) [7, 31]

$$1 + \frac{k_{De}^2}{k^2} - \frac{{\omega_{pi}}^2 + \omega_{pd}^2}{\omega^2} = 0 \tag{2.24}$$

where $\lambda_{De} = 1/k_{De} = \sqrt{k_B T_e/4\pi e^2 n_{e0}}$, $\omega_{pi} = \sqrt{4\pi e^2 n_{i0}/m_i}$, and $\omega_{pd} = \sqrt{4\pi q_{d0}^2 n_{d0}/m_d}$, are the electron's Debye length, ion plasma frequency and dust plasma frequency, respectively.

Limiting cases

Since $m_d \gg m_i$, thus ion plasma frequency is much greater than the dust plasma frequency i.e., $\omega_{pi} \gg \omega_{pd}$, thus Eq. (2.24), yields

$$\omega^2 = \frac{k^2\omega_{pi}^2\lambda_{De}^2}{1 + k^2\lambda_{De}^2} \tag{2.25}$$

or

$$\omega^2 = \frac{k^2 C_s^2}{1 + k^2 \lambda_{De}^2} \tag{2.26}$$

Here, we have substituted

$$C_s = \omega_{pi} \lambda_{De} = \sqrt{\frac{4\pi e^2 n_{i0}}{m_i} \frac{k_B T_e}{4\pi e^2 n_{e0}}} = \left(\sqrt{\frac{n_{i0}}{n_{e0}}} \right) c_s \tag{2.27}$$

where C_s is the speed of sound of dust-ion-acoustic waves and $c_s = \sqrt{k_B T_e / m_i}$is the speed of sound of ion-acoustic waves. For the long wavelength limit i.e., $k^2 \lambda_{De}^2 \ll 1$, equation (2.30) is simplified as

$$\omega = k \left(\sqrt{\frac{n_{i0}}{n_{e0}}} \right) c_s$$

Phase Velocity of Dust-Ion-Acoustic wave

The phase velocity of the dust-ion acoustic waves is same as that of pure ion-acoustic waves with an additional factor of $\sqrt{n_{i0}/n_{e0}} > 1$ [30] i.e.,

$$\frac{\omega}{k} = \left(\sqrt{\frac{n_{i0}}{n_{e0}}} \right) c_s$$

This frequency is typically for laboratory plasma are tons of kHz and the propagating velocity is few cm/s [28]. The speed of sound of dust- ion-acoustic wave is larger than that of ion-acoustic wave by a factor $\sqrt{n_{i0}/n_{e0}}$ in Eq. (2.27).

Since, we know that a fraction of free electron number density is attached to the dust particles [30], thus by increasing dust charged density, the free electrons number density can be reduced. In this way due to reduction of free electron number density, the speed of sound of dust-ion-acoustic waves would increase in comparison with Ion-acoustic waves [30].

2.3 Self-Gravitation in Dusty Plasma

2.3.1 Introduction

The previous discussed normal waves show the stable excitation in dusty plasma. Whereas the space and laboratories plasmas are not at thermodynamic equilibrium. The self-gravitating force introduces instability in dusty plasmas. The instability in plasma is determined by the influence of exponentially growing unstable collective modes [7]. The gravitational instability was first studied by Jeans [10]. We assume that the plasma particle's density is much smaller than the dust density, i.e., $m_e n_e \ll m_i n_i \ll m_d n_d$ and hence, the gravitational potential is only determined by the dust grain density [7]. The charged plasma species are subjected to the electrostatic and gravitational forces. Here, we study the mathematical formalism of Jeans-Instability.

2.3.2 Fluid Model

The dynamics of the interstellar gas clouds, consist of continuity equations,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (v\rho) = 0 \tag{2.28}$$

and momentum equation

$$\rho \left[\frac{\partial}{\partial t} + (\bar{v} \cdot \boldsymbol{\nabla}) \right] \bar{v} = -\rho \nabla \psi - \nabla P \tag{2.29}$$

respectively. Here ρ is the mass density, v is the fluid velocity, ψ is the self gravitational potential and P is the gas pressure. Where the gas pressure P is derived from speed of sound, i.e., "$c_s = \sqrt{\gamma k_B T/M}$" given as

$$\nabla P = c_s^2 \nabla \rho \tag{2.30}$$

These set of equation is closed with the help of gravitational Poisson's equation,

$$\nabla^2 \psi = 4\pi \rho G \tag{2.31}$$

where G is the gravitational constant.

2.3.3 Dispersion Relation

Separating now the dependant variables into equilibrium and perturbed parts with subscript "0" and "1" as $v = v_0 + v_1$, $\rho = \rho_0 + \rho_1$. Here we assume that equilibrium variables are spatially invariant and are stationary as well. Now we write the linearized set of equation

$$\frac{\partial \rho_1}{\partial t} + \boldsymbol{\nabla} \cdot (\bar{v}_0 \rho_1 + \bar{v}_1 \rho_0) = 0$$

$$\frac{\partial \rho_1}{\partial t} = -\rho_0 \boldsymbol{\nabla} \cdot \bar{v}_1 \tag{2.32}$$

Since $v_0 = 0$. The perturbed momentum equation is

$$\frac{\partial \bar{v}_1}{\partial t} = -\nabla \psi_1 - c_s^2 \frac{\nabla \rho_1}{\rho_0} \tag{2.33}$$

The gravitational Poisson's equation now becomes

$$\nabla^2 \psi_1 = 4\pi G \rho_1 \tag{2.34}$$

Assuming that all the perturbed variables vary as $\exp\left[-i\left(\omega t + \bar{k}.\bar{r}\right)\right]$, thus

$$-i\omega \rho_1 = -\rho_0 i\bar{k} \cdot \bar{v}_1 \tag{2.35}$$

for "$\bar{v}_0 = 0$" and momentum equation transforms as

$$-i\omega v_1 = -ik\psi_1 - c_s^2 \frac{ik\rho_1}{\rho_0} \tag{2.36}$$

$$\psi_1 = -\frac{4\pi G \rho_1}{k^2} \tag{2.37}$$

Solving these equations simultaneously, we get the dispersion relation

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0 \tag{2.38}$$

Or we can write it as

$$\omega^2 = c_s^2 \left(k^2 - k_J^2\right) \tag{2.39}$$

where

$$k_J^2 = \frac{4\pi G \rho_0}{c_s^2} = \frac{4\pi G \rho_0 m_p \mu}{k_B T} \tag{2.40}$$

In this equation m_p is the mass of proton, μ is the mean molecular weight, k_B, and T, are the Boltzmann constant and gas temperature respectively [32].

2.3.4 Jeans Instability

For $k \ll k_J$, the ω^2 in the Eq. (2.39) becomes negative

$$\omega^2 = -c_s^2\left(k_J^2 - k^2\right) \tag{2.41}$$

which shows that the perturbation is growing exponentially and causes Jeans instability. k_J defines a minimum mass scale

$$M_J = \left(\frac{2\pi}{k_J}\right)^3 \rho_0 = \left[\frac{\pi k_B T}{G m_p \mu}\right]^{\frac{3}{2}} \frac{1}{\sqrt{\rho_0}} \tag{2.42}$$

This is named as Jeans Mass. Oscillation in gas cloud larger than this size will grow and become Self-Gravitating and collapse, because due to heavy dust particles they become densely packed which reduces their electrical charging. At that time the electrostatic pressure cannot balance the gravitational force, this leads to the formation of stars and planets, *etc.*., like in astrophysical bodies.

2.4 Summary

In this chapter we have reviewed the work of Rao *et al.* [14] and Shukla *et al.* [16]. Rao et al. [14] have discovered that the dust-acoustic wave is low-frequency wave in which restoring force is provided by the pressure of inertialess electrons and ions, while dust inertia helps in the propagation of the wave. Shukla *et al.* [16] have described the dust-ion-acoustic wave in dusty plasma. They concluded that the large phase velocity of the dust-ion-acoustic wave is due to the negative potential, which is set up due to the depletion of electrons in the background plasma. We have also studied the effects of self-gravitational force in dusty plasma. Self-gravitation in dusty plasma leads to Jeans type instability.

Chapter 3

Dust Wave Modes in Self-Gravitating Dense Dusty Plasma

3.1 Introduction

In this chapter, we shall study the effects of self-gravitational force and Jeans type instability for different density regimes of dust particles. The dust particle in dusty plasma gets electrically charged due to electrons, ions and other surrounding effects. These charged particles play important role than being just additional specie. The charging of dust grains and its fluctuating due to electron and ion currents reaching the dust particle surfaces makes dusty plasma entirely different from normal electron-ion plasma. Due to dust charged fluctuations, the waves propagating through dusty plasma, are weakly damped. These waves are named as "dust coulomb wave". We shall study about dust-acoustic, dust-charge-density and dust-coulomb waves existing in tenuous, dilute and dense dusty plasma regimes respectively. The critical wavelengths for different for different density regimes are also investigated.

3.2 Self-Gravitating Dusty Plasma

First, we study the ultra-low frequency wave, in self-gravitating dusty plasma without dust charge fluctuations and named as "Dust-Acoustic wave". The number densities of Boltzmann distributed electrons and ions, are given as

$$n_e = n_{e0} \exp\left(\frac{e\phi}{k_B T_e}\right), \tag{3.1}$$

$$n_i = n_{i0} \exp\left(-\frac{e\phi}{k_B T_i}\right), \tag{3.2}$$

where $n_e\,(n_i)$ and $T_e\,(T_i)$ are the number densities and temperatures of electrons and ions, $n_{e0}\,(n_{i0})$ are the equilibrium number densities, ϕ is the electrostatic potential and k_B is the Boltzmann constant [23]. The fluid equations in one-dimension consist of dust continuity equation

$$\frac{\partial n_d}{\partial t} + \frac{\partial}{\partial z}(n_d v_d) = 0, \tag{3.3}$$

and the dust momentum equation

$$\frac{\partial v_d}{\partial t} + v_d \frac{\partial v_d}{\partial z} = -\frac{q_d}{m_d}\frac{\partial \phi}{\partial z} - \frac{k_B T_d}{n_d m_d}\frac{\partial n_d}{\partial z} - \frac{\partial \psi}{\partial z}, \tag{3.4}$$

where ϕ and ψ are the electrostatic and the gravitational potential respectively, and q_d is the grain charge with equilibrium value Q_d [23]. The electrostatic Poisson's equation is

$$\frac{\partial^2 \phi}{\partial z^2} + 4\pi\left(en_i + q_d n_d - en_e\right) = 0, \tag{3.5}$$

while the gravitational Poisson's equation is

$$\frac{\partial^2 \psi}{\partial z^2} = 4\pi G n_d m_d \tag{3.6}$$

here, $V_{td} = \sqrt{k_B T_d / m_d}$, is the dust thermal velocity. Expanding the dependent variables as

$$n_d = n_{d0} + n_{d1}, \quad v_d = v_{d0} + v_{d1}, \quad \phi = \phi_0 + \phi_1, \quad \psi = \psi_0 + \psi_1$$

Here, equilibrium quantities are spatially invariant and stationary as well. Thus, writing the linearized set of equations as

$$n_{e1} = n_{e0}\frac{e\phi_1}{k_B T_e} \tag{3.7}$$

$$n_{i1} = -n_{i0}\frac{e\phi_1}{k_B T_i} \tag{3.8}$$

Obtained for limit $\phi \ll T_{e,i}/e$, in Eq. (3.1) and (3.2). The continuity equation becomes

$$\frac{\partial n_{d1}}{\partial t} + n_{d0}\frac{\partial v_{d1}}{\partial z} = 0 \tag{3.9}$$

and the momentum equation is

$$\frac{\partial v_{d1}}{\partial t} = -\frac{q_d}{m_d}\frac{\partial \phi_1}{\partial z} - \frac{V_{td}^2}{n_{d0}}\frac{\partial n_{d1}}{\partial z} - \frac{\partial \psi}{\partial z} \tag{3.10}$$

The electrostatic Poisson's equation is

$$\frac{\partial^2 \phi_1}{\partial z^2} + 4\pi\left(en_{i1} + q_d n_{d1} - en_{e1}\right) = 0, \tag{3.11}$$

The gravitational Poisson's equation is

$$\frac{\partial^2 \psi_1}{\partial z^2} = 4\pi G m_d n_{d1}, \tag{3.12}$$

Taking the time derivative of continuity equation,

$$\frac{\partial^2 n_{d1}}{\partial t^2} + n_{d0}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial z}v_{d1}\right) = 0 \tag{3.13}$$

Now, taking space derivative of momentum equation and using gravitational Poisson's equation, we get

$$\frac{\partial}{\partial z}\left(\frac{\partial v_{d1}}{\partial t}\right) = -\frac{q_d}{m_d}\frac{\partial^2 \phi_1}{\partial z^2} - \frac{V_{td}^2}{n_{d0}}\frac{\partial^2 n_{d1}}{\partial z^2} - \frac{\partial^2 \psi}{\partial z^2} \tag{3.14}$$

Solving above perturbed equations simultaneously, we get

$$\left(\frac{\partial^2}{\partial t^2} - 4\pi G n_{d0} m_d - V_{td}^2\frac{\partial^2}{\partial z^2}\right) n_{d1} = -\frac{q_d n_{d0}}{m_d}\frac{\partial^2 \phi_1}{\partial z^2} \tag{3.15}$$

Assuming all the perturbed quantities are oscillating and varies sinusoidally as $\exp\left[-i\left(\omega t - kz\right)\right]$. Thus, Eq. (3.14) transforms into the following result

$$\left(\omega^2 + 4\pi G n_{d0} m_d - V_{td}^2 k^2\right) n_{d1} = -\frac{q_d n_{d0}}{m_d}k^2\phi_1 \tag{3.16}$$

Let,

$$A = \omega^2 + \omega_{Jd}^2 - V_{td}^2 k^2 \tag{3.17}$$

where ω_{Jd} is the dust Jeans frequency given by

$$\omega_{Jd} = \sqrt{4\pi G n_{d0} m_d} \tag{3.18}$$

Thus, the perturbed dust number density from Eq. (3.15) can be written as,

$$n_{d1} = -\frac{q_d n_{d0}}{A m_d} k^2 \phi_1 \tag{3.19}$$

Now substituting the perturbed electron and ion number densities in perturbed electrostatic Poisson's equation

$$\frac{\partial^2 \phi_1}{\partial z^2} - \frac{4\pi e^2}{k_B}\left[\frac{1}{T_e} + \frac{1}{T_i}\right]\phi_1 + 4\pi q_d n_{d1} = 0, \tag{3.20}$$

Taking Fourier transform of above equations

$$-k^2\phi_1 - \left[\frac{1}{\lambda_{De}^2} + \frac{1}{\lambda_{Di}^2}\right]\phi_1 + 4\pi q_d n_{d1} = 0, \tag{3.21}$$

Here λ_{De} and λ_{Di} are the electron and ion Debye lengths respectively. Now substituting the value of n_{d1} in above equation we get,

$$A\left(\frac{1 + k^2\lambda_D^2}{\lambda_D^2}\right) - \frac{4\pi N_d q_d^2 k^2}{m_d} = 0, \tag{3.22}$$

where $\lambda_D = \sqrt{\lambda_{De}^{-2} + \lambda_{Di}^{-2}}$, is the effective Debye length. After substituting $\omega_{pd} = \sqrt{4\pi n_{d0} q_d^2 / m_d}$, and with re-substitution of value of A from Eq. (3.16), we get the following result

$$\frac{\omega^2}{k^2} = \frac{\omega_{pd}^2 \lambda_D^2}{1 + k^2\lambda_D^2} + V_{td}^2 - \frac{\omega_{Jd}^2}{k^2}, \tag{3.23}$$

which is the desire dispersion relation of the "Dust-Acoustic wave" in Self-Gravitating dusty plasma.

3.3 Electrostatic Modes in Dense Dusty Plasma

In dusty plasma for ultra-low frequency case, the electron and ion number densities are supposed to obey Boltzmann type distribution [23], as written in Eqs. (3.1)and (3.2). Now assuming that the equilibrium parts are spatially invariant and also stationary. Thus, we get the perturbed fluid equations consisting of number densities of electron and ion respectively,

$$n_{e1} = n_{eo}\left(\frac{e\phi_1}{k_B T_e}\right) \tag{3.24}$$

$$n_{i1} = -n_{i0}\left(\frac{e\phi}{k_B T_i}\right) \tag{3.25}$$

The one dimensional dust continuity equation,

$$\frac{\partial n_{d1}}{\partial t} + n_{d0}\frac{\partial}{\partial z}v_{d1} = 0 \tag{3.26}$$

and the equation of motion,

$$\frac{\partial v_{d1}}{\partial t} = -\frac{q_{d0}}{m_d}\frac{\partial \phi_1}{\partial z} - \frac{V_{td}^2}{n_{d0}}\frac{\partial n_{d1}}{\partial z} - \frac{\partial \psi_1}{\partial z} \tag{3.27}$$

where $V_{td} = \sqrt{k_B T_d/m_d}$, is the dust thermal velocity. Here ϕ_1 is the perturbed electrostatic and ψ_1 is the gravitational potential. The electrostatic Poisson's equation is given as

$$\frac{\partial^2 \phi}{\partial z^2} + 4\pi\left(en_{i1} + q_{d0}n_{d1} + q_{d1}n_{d0} - en_e\right) = 0 \tag{3.28}$$

While the gravitational Poisson's equation is

$$\frac{\partial^2 \psi_1}{\partial z^2} = 4\pi G m_d n_{d1} \tag{3.29}$$

The charge fluctuation is determined from current balance equation [23]

$$\frac{\partial q_d}{\partial t} + v_d\frac{\partial q_q}{\partial z} = I_e + I_i \tag{3.30}$$

where I_e, I_i are the electron and ion currents respectively. These are given as

$$I_e = -\pi R^2 e n_e(\phi)\sqrt{\frac{8k_B T_e}{\pi m_e}}\exp\left[\frac{eV}{k_B T_e}\right] \tag{3.31}$$

$$I_i = \pi R^2 e n_i(\phi)\sqrt{\frac{8k_B T_i}{\pi m_i}}\left[1 - \frac{eV}{k_B T_i}\right] \tag{3.32}$$

where R and V are the dust grain radius and surface potential, respectively.

3.3.1 Perturbed Dust Number Density

Now for finding the perturbed dust number density, we follow these steps: After taking temporal derivative of continuity equation and spatial derivative of momentum equation, we get the following results

$$\frac{\partial^2 n_{d1}}{\partial t^2} + n_{d0}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial z}v_{d1}\right) = 0 \tag{3.33}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial v_{d1}}{\partial t}\right) = -\frac{q_{d0}}{m_d}\frac{\partial^2 \phi_1}{\partial z^2} - \frac{V_{td}^2}{n_{d0}}\frac{\partial^2 n_{d1}}{\partial z^2} - 4\pi G m_d n_{d1} \tag{3.34}$$

Considering that v_{d1} is continuous, i.e., $\partial_z\left(\partial_t v_{d1}\right) = \partial_t\left(\partial_z v_{d1}\right)$, we get the following result

$$\left(\frac{\partial^2}{\partial t^2} - V_{td}^2\frac{\partial^2}{\partial z^2} - 4\pi G m_d n_{d0}\right) n_{d1} = -\frac{n_{d0}q_{d0}}{m_d}\frac{\partial^2 \phi_1}{\partial z^2} \tag{3.35}$$

Assuming all the perturbed variables are proportional to $\exp\left[-i\left(\omega t - kz\right)\right]$, Eq. (3.34) gives the following result,

$$n_{d1} = \frac{n_{d0}q_{d0}}{A m_d}\phi_1 \tag{3.36}$$

where

$$A = \frac{\omega^2}{k^2} - V_{td}^2 + \frac{\omega_{Jd}^2}{k^2} \tag{3.37}$$

3.3.2 Dust Charge Fluctuation

The perturbed current balance equation is

$$\frac{\partial\left(q_{d0} + q_{d1}\right)}{\partial t} + v_{d0}\frac{\partial q_{d1}}{\partial z} = I_{e0} + I_{i0} + I_{e1} + I_{i1} \tag{3.38}$$

Here v_{d0}, $I_{e0}(I_{i0})$ are the equilibrium velocity and the electron(ion) current respectively. At equilibrium these quantities are equal to zero, thus Eq. (3.37) is simplified as,

$$\frac{\partial q_{d1}}{\partial t} = I_{e1} + I_{i1} \tag{3.39}$$

After taking Fourier transform of Eq. (3.38), we get

$$-i\omega q_{d1} = I_{e1} + I_{i1} \tag{3.40}$$

Now after simplify the right hand side of Eq. (3.38), by substituting the perturbed values of I_e, I_i, we get the following equation,

$$\begin{aligned} -i\omega q_{d1} &= -\pi R^2 e\left(n_{e0}+n_{e1}\right)\sqrt{\frac{8k_BT_e}{\pi m_e}}\exp\left[\frac{e\left(V_0+V_1\right)}{k_BT_e}\right]+ \\ &\quad \pi R^2 e\left(n_{i0}+n_{i1}\right)\sqrt{\frac{8k_BT_i}{\pi m_i}}\left[1-\frac{e\left(V_0+V_1\right)}{k_BT_i}\right] \end{aligned}$$

or

$$\begin{aligned} -i\omega q_{d1} = -\pi R^2 e\sqrt{\frac{8k_BT_e}{\pi m_e}}\left[\left(n_{e0}+n_{e1}\right)\exp\left[\frac{eV_0}{k_BT_e}\right]\exp\left[\frac{eV_1}{k_BT_e}\right]\right]+ \\ \pi R^2 e\sqrt{\frac{8k_BT_i}{\pi m_i}}\left[\left(n_{i0}+n_{i1}\right)\left[1-\frac{eV_0}{k_BT_i}-\frac{eV_1}{k_BT_i}\right]\right] \end{aligned} \tag{3.41}$$

Now for $V_1 \ll T_e/e$, we can expand $\exp\left(eV_1/k_BT_e\right)=1+eV_1/k_BT_e+...$, thus

$$-i\omega q_{d1} = -\pi R^2 e\sqrt{\frac{8k_BT_e}{\pi m_e}}\left[\left(n_{e0}+n_{e1}\right)\exp\left(\frac{eV_0}{k_BT_e}\right)\left[1+\frac{eV_1}{k_BT_e}\right]\right]+ \tag{3.42}$$

$$\pi R^2 e\sqrt{\frac{8k_BT_i}{\pi m_i}}\left[\left(n_{i0}+n_{i1}\right)\left[1-\frac{eV_0}{k_BT_i}-\frac{eV_1}{k_BT_i}\right]\right]$$

After linearization, we get

$$-i\omega q_{d1} = -\pi R^2 e\sqrt{\frac{8k_BT_e}{\pi m_e}}\left[n_{e0}\exp\left(\frac{eV_0}{k_BT_e}\right)\left[\frac{eV_1}{k_BT_e}\right]+n_{e1}\exp\left(\frac{eV_0}{k_BT_e}\right)\right]+ \tag{3.43}$$

$$\pi R^2 e\sqrt{\frac{8k_BT_i}{\pi m_i}}\left[n_{i1}\left(1-\frac{eV_0}{k_BT_i}\right)-n_{i0}\left(\frac{eV_1}{k_BT_i}\right)\right]$$

as $V_1 = q_{d1}/4\pi\epsilon_0 R$, and $n_{e1,i1} = n_{e0,i0}\left(\pm e\phi_1/k_BT_{e,i}\right)$, thus

$$-i\omega q_{d1} = -\pi R^2\sqrt{\frac{8e^2k_BT_e}{\pi m_e}}\left[n_{e0}\exp\left(\frac{eV_0}{k_BT_e}\right)\left[\frac{eq_{d1}}{4\pi\epsilon_0 Rk_BT_e}\right]+\frac{n_{e0}e\phi}{k_BT_e}\exp\left(\frac{eV_0}{k_BT_e}\right)\right]+$$

$$\pi R^2\sqrt{\frac{8e^2k_BT_i}{\pi m_i}}\left[n_{i0}\left(-\frac{e\phi_1}{k_BT_i}\right)\left(1-\frac{eV_0}{k_BT_i}\right)-n_{i0}\left(\frac{eq_{d1}}{4\pi\epsilon_0 Rk_BT_i}\right)\right]$$

or

$$\begin{aligned} -i\omega q_{d1} = -R\sqrt{\frac{8\pi n_{e0}^2e^4}{k_BT_em_e}}\exp\left(\frac{eV_0}{k_BT_e}\right)q_{d1}-R^2\sqrt{\frac{8\pi n_{e0}^2e^2}{k_BT_em_e}}\exp\left(\frac{eV_0}{k_BT_e}\right)\phi_1- \\ R^2\sqrt{\frac{8\pi n_{i0}^2e^4}{k_BT_im_i}}\left(1-\frac{eV_0}{k_BT_i}\right)\phi_1-R\sqrt{\frac{8\pi n_{i0}^2e^4}{k_BT_im_i}}q_{d1} \end{aligned} \tag{3.44}$$

Re-arranging the terms, we get

$$-i\omega q_{d1} = -\frac{R}{\sqrt{2\pi}}\left[\sqrt{\frac{4\pi n_{e0}e^2}{k_BT_e}\frac{4\pi e^2 n_{e0}}{m_e}}\exp\left(\frac{eV_0}{k_BT_e}\right)+\sqrt{\frac{4\pi n_{i0}e^2}{k_BT_i}\frac{4\pi e^2 n_{i0}}{m_i}}\right]q_{d1}$$

$$-\frac{R^2}{\sqrt{2\pi}}\left[\sqrt{\frac{4\pi n_{e0}e^2}{k_BT_e}\frac{4\pi e^2 n_{e0}}{m_e}}\exp\left(\frac{eV_0}{k_BT_e}\right)+\sqrt{\frac{4\pi n_{i0}e^2}{k_BT_i}\frac{4\pi e^2 n_{i0}}{m_i}}\left(1-\frac{eV_0}{k_BT_i}\right)\right]\phi_1 \tag{3.45}$$

or

$$-i\omega q_{d1} = -\frac{R}{\sqrt{2\pi}}\left[\frac{\omega_{pe}}{\lambda_{De}}\exp\left(\frac{eV_0}{k_BT_e}\right)+\frac{\omega_{pi}}{\lambda_{Di}}\right]q_{d1}-$$
$$\frac{R^2}{\sqrt{2\pi}}\left[\frac{\omega_{pe}}{\lambda_{De}}\exp\left(\frac{eV_0}{k_BT_e}\right)+\frac{\omega_{pi}}{\lambda_{Di}}\left(1-\frac{eV_0}{k_BT_i}\right)\right]\phi_1 \tag{3.46}$$

where $\omega_{p\alpha}$ and $\lambda_{D\alpha}$ are the plasma frequency and Debye length of the $\alpha\,(=e,\,i)$ specie. We suppose that characteristic dust grain charging frequencies are given by,

$$\omega_1 = \frac{R}{\sqrt{2\pi}}\left[\frac{\omega_{pe}}{\lambda_{De}}\exp\left(\frac{eV_0}{k_BT_e}\right)+\frac{\omega_{pi}}{\lambda_{Di}}\right] \tag{3.47}$$

$$\omega_2 = \frac{R}{\sqrt{2\pi}}\left[\frac{\omega_{pe}}{\lambda_{De}}\exp\left(\frac{eV_0}{k_BT_e}\right)+\frac{\omega_{pi}}{\lambda_{Di}}\left(1-\frac{eV_0}{k_BT_i}\right)\right] \tag{3.48}$$

Thus, we get the following result,

$$-i\omega q_{d1} = -\omega_1 q_{d1} - \omega_2 R\phi_1 \tag{3.49}$$

or

$$q_{d1} = -R\frac{\omega_2}{\omega_1 - i\omega}\phi_1 \tag{3.50}$$

Let $\Delta = \omega_2/\omega_1 - i\omega$, thus the dust charge fluctuation is given by the following equation:

$$q_{d1} = -R\Delta\phi_1 \tag{3.51}$$

3.3.3 Dispersion Relation

Now substituting perturbed electron and ion number densities from Eqs. (3.1) and (3.2) in electrostatic Poisson's equation given by Eq. (3.27), thus

$$\frac{\partial^2\phi_1}{\partial z^2} - \frac{4\pi e^2}{k_B}\left[\frac{N_e}{T_e}+\frac{N_i}{T_i}\right]\phi_1 + 4\pi\left[q_{d1}n_{d0} + q_{do}n_{d1}\right] = 0$$

or

$$\frac{\partial^2 \phi_1}{\partial z^2} - \frac{1}{\lambda_D^2}\phi_1 + 4\pi\left[q_{d1}n_{d0} + q_{do}n_{d1}\right] = 0$$

where $\lambda_D = \left(\lambda_{Di}^{-2} + \lambda_{De}^{-2}\right)^{-\frac{1}{2}}$ is the effective Debye length and $\lambda_{Di,De} = \sqrt{k_B T_{i,e}/4\pi e^2 n_{i,e}}$ are the ion and electron Debye lengths, respectively. After taking Fourier transform of above equation and substituting the values of q_{d1}, n_{d1} and then simplifying, we get the following equation

$$\frac{1 + k^2\lambda_D^2 + f\Delta}{k^2} + \frac{4\pi N_d q_{d0}^2 \lambda_D^2}{k^2 A m_d} = 0 \tag{3.52}$$

where $f = 4\pi R N_d \lambda_D^2$, is the fugacity parameter, which is a measure of grain packing. Separating A from Eq. (3.53), we get,

$$A = \frac{\omega_{pd}^2 \lambda_D^2}{1 + k^2\lambda_D^2 + f\Delta} \tag{3.53}$$

Substituting back the value of "A" from Eq. (3.36), into Eq. (3.51), we get

$$\frac{\omega^2}{k^2} - V_{td}^2 + \frac{\omega_{Jd}^2}{k^2} = \frac{C_{DA}^2}{1 + \lambda_D^2 k^2 + f\Delta}$$

where $C_{DA} = \omega_{pd}\lambda_D$, is the dust-acoustic velocity. Thus, we get the dispersion relation of self gravitating dusty plasma with fluctuating charge of dust particles as,

$$\frac{\omega^2}{k^2} = \frac{C_{DA}^2}{1 + \lambda_D^2 k^2 + f\Delta} + V_{td}^2 - \frac{\omega_{Jd}^2}{k^2} \tag{3.54}$$

3.3.4 Discussion on Jeans Type Instability

For the limit $\omega \ll \omega_1$, when the dust charge fluctuation is small and dust particles gets equilibrium charge more quickly, so $\Delta \to \delta = \omega_2/\omega_1$, then Eq. (3.53) changes as follow,

$$\frac{\omega^2}{k^2} = \frac{C_{DA}^2}{1 + \lambda_D^2 k^2 + f\delta} + V_{td}^2 - \frac{\omega_{Jd}^2}{k^2} \tag{3.55}$$

This relation is analyzed in different fugacity limits as $f\delta \ll 1$, $f\delta \gg 1$, for tenuous and dense dusty plasma respectively.

Tenuous Dusty Plasma

When dilute dusty plasma is considered, i.e., $f\delta \ll 1$, the dispersion relation becomes

$$\frac{\omega^2}{k^2} = \frac{C_{DA}^2}{1 + k^2\lambda_D^2} + V_{td}^2 - \frac{\omega_{Jd}^2}{k^2} \tag{3.56}$$

which leads to usual Jeans instability of dust-acoustic waves [23]. This relation can further be simplifies, when we consider the cold dusty plasma $V_{td} \ll C_{DA}$ and the large wavelength limit, $k^2\lambda_D^2 \ll 1$ the dispersion relation becomes

$$\frac{\omega^2}{k^2} = C_{DA}^2 - \frac{\omega_{Jd}^2}{k^2} \tag{3.57}$$

To find the instability rate we re-arrange the above equation as follows,

$$\frac{\omega^2}{\omega_{Jd}^2} = \frac{C_{DA}^2 k^2}{\omega_{Jd}^2} - 1$$

$$\frac{\omega^2}{\omega_{Jd}^2} = k^2\lambda_{JDA}^2 - 1 \tag{3.58}$$

where $\lambda_{JDA} = C_{DA}/\omega_{Jd}$, is the Jeans length [23]. In the limit $k^2\lambda_{JDA}^2 \ll 1$, i.e., Jeans instability arises due to the smaller Jeans length than system scale length. Thus, from Eq. (3.57), we get the following roots

$$\omega = \pm i\omega_{Jd} \tag{3.59}$$

The instability is growing for $\omega = i\omega_{Jd}$, while it is damped for frequency, $\omega = -i\omega_{Jd}$

Dense Dusty Plasma

We have dense dusty plasma when $f\delta \gg 1$, showing that the dust grain packing (fugacity) is large. The dispersion relation in Eq. (3.54) becomes,

$$\frac{\omega^2}{k^2} = \frac{C_{DA}^2}{f\delta\left(1 + \frac{\lambda_D^2 k^2}{f\delta}\right)} + V_{td}^2 - \frac{\omega_{Jd}^2}{k^2} \tag{3.60}$$

$$\frac{\omega^2}{k^2} = \frac{C_{DC}^2}{\delta\left(1 + \lambda_R^2 k^2\right)} + V_{td}^2 - \frac{\omega_{Jd}^2}{k^2} \tag{3.61}$$

where $C_{DC}^2 = C_{DA}^2/f$ and $\lambda_R^2 = \lambda_D^2/f\delta$.

$$\frac{\omega^2}{\omega_{Jd}^2} = \frac{k^2 C_{DC}^2}{\omega_{Jd}^2\delta} - 1, \tag{3.62}$$

let the critical Jeans length is

$$\lambda_{JDC} = \frac{C_{DC}}{\omega_{Jd}\delta},$$

Thus,

$$\frac{\omega^2}{\omega_{Jd}^2} = k^2\lambda_{JDC}^2 - 1, \tag{3.63}$$

As, due to large fugacity $C_{DC}^2 \ll C_{DA}^2$, this implies that $\lambda_{JDC} \ll \lambda_{JDA}$, which shows that the system will be unstable for much smaller scale size. The Jeans instability occur when the system scale length is greater than Jeans length i.e., $k^2\lambda_{JDC}^2 \ll 1$ thus,

$$\omega = \pm i\omega_{Jd} \tag{3.64}$$

General Case

The dispersion relation in Eq. (3.55) in the limits $k^2\lambda_D^2$, $k^2\lambda_R^2 \ll 1$, become

$$\frac{\omega^2}{k^2} = \frac{C_{DA}^2 C_{DC}^2}{C_{DC}^2 + C_{DA}^2\delta} + V_{td}^2 - \frac{\omega_{Jd}^2}{k^2} \tag{3.65}$$

Critical Jeans Wavenumber

For cold dusty plasma ($V_{td} = 0$), the critical Jeans wavelength is found from Eq. (3.64), and is given as

$$k_J^2 = \frac{\omega_{Jd}^2}{C_{DA}^2} + \frac{\omega_{Jd}^2\delta}{C_{DC}^2} = \frac{\omega_{Jd}^2\,(1 + f\delta)}{C_{DA}^2} \tag{3.66}$$

which gives different cases of dust-acoustic and dust-coulomb waves for specific fugacity limits. From Eq. (3.65), we can see that for a given values of ω_{Jd} and C_{DA}, the critical Jeans length "k_J" varies with "$1 + f\delta$ " only. Thus, for high fugacity dusty plasmas, the system becomes unstable at shorter scale length. We can note that Jeans length λ_{JDA} is greater than critical length λ_{JDC} i.e.,

$$\frac{\lambda_{JDA}^2}{\lambda_{JDC}^2} = \frac{\lambda_D^2}{\lambda_R^2} = f\delta \tag{3.67}$$

which is true for all fugacity limits. The nature of the dust modes and corresponding Jeans instability is determined by the shorter scale lengths.

3.3.5 Critical Wavenumber " k_c "

Since we know that Jeans instability is the growing instability for the perturbed system with wavenumber $k < k_c$, where k_c is the critical wavenumber. The expression for critical wavelength can be determined by substituting "$-i\omega = \Omega$" in Eq.(3.54) and rearranging the terms we get the following result

$$\frac{\Omega^2}{k^2} + \frac{C_{DA}^2}{1 + k^2\lambda_D^2 + f\frac{\omega_2}{\omega_1+\Omega}} + V_{td}^2 - \frac{\omega_{Jd}^2}{k^2} = 0 \tag{3.68}$$

Now factorizing the above equation and after simplification, we get

$$\frac{C_{DA}^2 k^2 (\Omega + \omega_1) + \left(\left(1 + k^2\lambda_D^2\right)(\Omega + \omega_1) + f\omega_2\right)\left(k^2 V_{td}^2 + \Omega^2 - \omega_{Jd}^2\right)}{k^2\left(\left(1 + k^2\lambda_D^2\right)(\Omega + \omega_1) + f\omega_2\right)} = 0 \tag{3.69}$$

Collecting terms with "Ω"

$$\Omega^3\left(1 + k^2\lambda_D^2\right) + \Omega^2\omega_1\left(1 + k^2\lambda_D^2 + f\frac{\omega_2}{\omega_1}\right) + \Omega\left[k^2C_{DA}^2 + \left(k^2V_{td}^2 - \omega_{Jd}^2\right)\left(1 + k^2\lambda_D^2\right)\right] \tag{3.70}$$

$$+\omega_1\left[k^2C_{DA}^2 + \left(k^2V_{td}^2 - \omega_{Jd}^2\right)\left(1 + k^2\lambda_D^2 + f\right)\right] = 0$$

After rearranging the above equation, we get

$$(\Omega + \omega_1)\left(k^2C_{DA}^2 + \left(1 + k^2\lambda_D^2\right)\left(k^2V_{td}^2 + \Omega^2 - \omega_{Jd}^2\right)\right) + f\omega_1\delta\left(k^2V_{td}^2 + \Omega^2 - \omega_{Jd}^2\right) = 0 \tag{3.71}$$

Separating the terms with ω_1

$$\omega_1\left[\left(1 + k^2\lambda_D^2\right)\left(k^2V_{td}^2 + \Omega^2 - \omega_{Jd}^2\right) + f\delta\left(k^2V_{td}^2 + \Omega^2 - \omega_{Jd}^2\right)\right]$$

$$+\Omega\left[C_{DA}^2k^2 + \left(1 + k^2\lambda_D^2\right)\left(k^2V_{td}^2 + \Omega^2 - \omega_{Jd}^2\right)\right] = -C_{DA}^2k^2\omega_1$$

Simplifying and comparing coefficients of ω_1, we get

$$\left(1 + k^2\lambda_D^2 + f\delta\right)\left(k^2V_{td}^2 + \Omega^2 - \omega_{Jd}^2\right) = -C_{DA}^2k^2 \tag{3.72}$$

Rearranging the terms, we obtain

$$\Omega^2 + k^2V_{td}^2 - \omega_{Jd}^2 = \frac{-C_{DA}^2k^2}{1 + k^2\lambda_D^2 + f\delta} \tag{3.73}$$

or

$$\Omega = \sqrt{-\left(\frac{C_{DA}^2 k^2}{1 + k^2\lambda_D^2 + f\delta} + k^2 V_{td}^2 - \omega_{Jd}^2\right)}$$

Thus, it has only one real positive root when the wavenumber k satisfies the following condition

$$\frac{k^2 C_{DA}^2}{1 + k^2\lambda_D^2 + f\delta} + k^2 V_{td}^2 < \omega_{Jd}^2 \tag{3.74}$$

The critical wavenumber is determined by finding the roots of biquadratic equation derived from above condition

$$V_{td}^2\lambda_D^2 \left(k_c^2\right)^2 + \left[C_{DA}^2 + V_{td}^2\left(1 + f\delta\right) - \omega_{Jd}^2\lambda_D^2\right] k_c^2 - \omega_{Jd}^2\left(1 + f\delta\right) = 0 \tag{3.75}$$

It has only one positive root. For limit $k^2\lambda_D^2 \ll (1 + f\delta)$ in Eq. (3.74), the critical wavenumber k_c, turns out to be

$$k_c = \omega_{Jd}\left[\frac{C_{DA}^2}{1 + f\delta} + V_{td}^2\right]^{-1/2} \tag{3.76}$$

This illustrates that the system becomes unstable at shorter scale length, when the critical wavenumber increases for increasing the dust fugacity. It also shows that the thermal pressure of the dust particle increases the critical wavelength, thus for Jeans instability, long-wavelength perturbations are necessary. Physically, we can say that the thermal pressure of the dust particles causes an opposing force to avoid instability, thus prior to the establishment of gravitational, the system scale size should be larger. For neutral dust particle Eq. (3.75) give the following classical result,

$$k_c = \frac{\omega_{Jd}}{V_{td}} \tag{3.77}$$

For "dust-acoustic waves" in tenuous dusty plasma ($f\delta \ll 1$), the critical wavenumber is

$$k_c^{DAW} = \omega_{Jd}\left[C_{DA}^2 + {V_{td}}^2\right]^{-1/2} \tag{3.78}$$

Whereas, the critical wavenumber for "dust-coulomb waves" in dense dusty ($f\delta \gg 1$), plasma is given as

$$k_c^{DCW} = \omega_{Jd}\left[\frac{C_{DA}}{\delta} + V_{td}^2\right]^{-1/2} \tag{3.79}$$

Generally the critical wavelength, for any fugacity regime including dilute dust plasma ($f\delta \sim 1$) corresponding to "dust-charged-density waves" can be written as

$$k_c = \omega_{Jd}\left[\frac{C_{DA}^2 C_{DC}^2}{C_{DC}^2 + C_{DC}^2 f\delta} + V_{td}^2\right]^{-1/2} \tag{3.80}$$

which is obtained by multiplying first term on right side with "C_{DC}^2/C_{DC}^2". Replacing "$C_{DA}^2 = C_{DC}^2 f$ " thus

$$k_c = \omega_{Jd}\left[\frac{C_{DA}^2 C_{DC}^2}{C_{DC}^2 + C_{DA}^2 \delta} + V_{td}^2\right]^{-1/2} \tag{3.81}$$

This equation gives the same result as that derived above for dust-acoustic and dust-coulomb waves in respective limits.

3.4 Summary

In this chapter, we have reviewed the research work of Rao *et al.* [23], in which they a systematic investigation on the effects of self-gravitational forces in dusty plasma for different density regimes has been carried out. They found that for dust-acoustic waves, the Jeans instability sets in, for tenuous dusty plasma, i.e., "$f\delta \ll 1$" whereas for dust-coulomb wave, it occurs for dense dusty plasma, i.e., "$f\delta \gg 1$", which propagate due to dust charge fluctuation. These dust modes exist when the dust charging frequency is much greater than the wave frequency [33]. The dust-acoustic wave also exist when the dust charging frequency is much less than wave frequency for dilute dusty plasma regime [33]. They also concluded that the critical scale length for Jeans instability of dust-coulomb wave is smaller than that for dust-acoustic waves.

Chapter 4

Electrostatic Dust Modes in Dusty Plasma

In this chapter, we shall derive the different dispersion relations of Jeans instability. In unmagnetized dusty plasma two dust modes of electrostatic waves have been discovered in last decade of 19^{th} century. These modes are the dust-acoustic wave [14] and dust-ion acoustic wave [16]. It has been discovered that the electron and ion currents reaching the dust particle's surface are oscillatory in the presence of above said modes. Thus, the dust charge fluctuation become new dynamic variable, which causes a damping mode in a dusty plasma. It also causes the damping of dust-acoustic and dust-ion-acoustic waves, which is different from Landau and collisional damping. In this chapter we shall discuss the following models :

4.1 Self-Gravitational Effect in Multi-Ion System

Consider a self-gravitating dusty plasma composed of electrons, ions and charged dust particles, where dust grains are supposed to be point charges having interparticle distance much smaller than the effective Debye length of dusty plasma. We study the electrostatic oscillation only.

4.1.1 Model Equations

We consider the dusty plasma composed of electrons, ions and charged dust particles. The average distance among dust particles is less than the effective Debye length of the dusty plasma. We does not consider the ion gravitational force because, for most plasma conditions, the ion mass is much smaller than the dust mass per charge. The number density of electrons obey Boltzmann distribution, because we can neglect the electrons inertia due to smaller phase velocity of the dust-acoustic and dust-ion-acoustic waves, thus we have electron number density given as

$$n_e = n_{e0} \exp\left(\frac{e\phi}{k_B T_e}\right),$$

The dynamics of the plasma is controlled by continuity equation,

$$\frac{\partial n_j}{\partial t} + \frac{\partial}{\partial z}(n_j v_j) = 0,$$

and the equation of motion,

$$\frac{\partial v_j}{\partial t} = \frac{-q_j}{m_j}\frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial z},$$

and the electrostatic and gravitational Poisson's equations, given by

$$\epsilon_0 \frac{\partial^2 \phi}{\partial z^2} - e n_e + \sum_j n_j q_j = 0, \tag{4.1}$$

$$\frac{\partial^2 \psi}{\partial z^2} = 4\pi G \sum_j n_j m_j, \tag{4.2}$$

where n_j, v_j, q_j and m_j is the number density, velocity, charge and mass of the j^{th} ionic-specie respectively. And ϕ and ψ are the electrostatic and gravitational potential, where G is the gravitational constant. We can define the charge neutrality condition as,

$$e n_{e0} = \sum_j q_{j0} n_{j0},$$

Here n_{e0} and q_{j0} are the equilibrium number density and charge of j^{th} specie respectively.

$$n_{e1} = n_{e0}\left(\frac{e\phi}{k_B T_e}\right), \tag{4.3}$$

The perturbed continuity equation,

$$\frac{\partial n_{j1}}{\partial t} + \frac{\partial}{\partial z}(n_{j0} v_{j1}) = 0, \tag{4.4}$$

the equation of motion is,

$$\frac{\partial v_{j1}}{\partial t} = \frac{-q_{j0}}{m_j}\frac{\partial \phi_1}{\partial z} - \frac{\partial \psi_1}{\partial z}, \tag{4.5}$$

and the electrostatic and gravitational Poisson's equation are

$$\epsilon_0 \frac{\partial^2 \phi_1}{\partial z^2} - en_{e1} + \sum_j n_{j1} q_j = 0, \tag{4.6}$$

$$\frac{\partial^2 \psi_1}{\partial z^2} = 4\pi G \sum_j n_{j1} m_j, \tag{4.7}$$

respectively, where n_{j1}, v_{j1}, q_j, and m_j are the number density, the velocity, the charge, and the mass of the ionic species. Also ψ and G are the gravitational potential and the gravitation constant. Taking Fourier transform of perturbed continuity and the momentum equation as:

$$n_{j1} = \frac{k}{\omega} n_{j0} v_{j1}, \tag{4.8}$$

$$v_{j1} = \frac{k}{\omega}\left(\frac{q_{j0}}{m_j}\phi_1 + \psi_1\right), \tag{4.9}$$

Solving above equations simultaneously, we get

$$n_{j1} = \frac{k^2}{\omega^2} n_{j0}\left(\frac{q_{j0}}{m_j}\phi_1 + \psi_1\right), \tag{4.10}$$

Taking Fourier transform of gravitational Poisson's equation

$$\psi_1 = -\frac{4\pi G}{k^2} \sum_j m_j n_{j1}, \tag{4.11}$$

and substituting the perturbed ion number density from Eq. (4.10) into Eq. (4.11), we get,

$$\psi_1 = -\frac{4\pi G}{\omega^2} \sum_j m_j n_{j0}\left(\frac{q_{j0}}{m_j}\phi_1 + \psi_1\right), \tag{4.12}$$

or

$$\psi_1 + \frac{\sum_j 4\pi G m_j n_{j0}}{\omega^2}\psi_1 = -\frac{4\pi G}{\omega^2} \sum_j q_{j0} n_{j0} \phi_1,$$

The perturbed gravitational potential is thus given as,

$$\psi_1 = -\frac{\frac{4\pi G}{\omega^2}\sum\limits_j q_{j0}n_{j0}\phi_1}{1+\frac{\sum\limits_j \omega_{Jj}^2}{\omega^2}}, \tag{4.13}$$

where $\omega_{Jj} = \sqrt{\sum_j 4\pi G m_j n_{j0}}$, is the Jeans frequency of j^{th} specie.

4.1.2 Dispersion Relation

Now substituting the values of n_{e1}, n_{j1} and ψ_1 from Eq. (4.3), Eq. (4.8) and Eq. (4.13) in electrostatic Poisson's equation given by Eq. (4.6), after taking its Fourier transform as follows,

$$\epsilon_0 (ik)^2 \phi_1 - \frac{n_{e0}e^2}{k_B T_e}\phi_1 + \sum_j q_j \frac{k^2}{\omega^2} n_{j0}\left(\frac{q_{j0}}{m_j}\phi_1 - \frac{\frac{4\pi G}{\omega^2}\sum\limits_j q_{j0}n_{j0}\phi_1}{1+\frac{\sum\limits_j \omega_{Jj}^2}{\omega^2}}\right) = 0,$$

or

$$\left(\epsilon_0 k^2 + \frac{n_{e0}e^2}{k_B T_e} - \sum_j q_j \frac{k^2}{\omega^2}\frac{n_{j0}}{m_j}\right)\left(1+\frac{\sum\limits_j \omega_{Jj}^2}{\omega^2}\right) + \sum_j \frac{4\pi G k^2}{\omega^4} q_{j0}^2 n_{j0}^2 = 0,$$

After rearranging the above terms, we obtain a dispersion relation,

$$\left(1+\frac{1}{k^2\lambda_{De}^2} - \sum_j \frac{\omega_{pj}^2}{\omega^2}\right)\left(1+\frac{\sum\limits_j \omega_{Jj}^2}{\omega^2}\right) + \frac{4\pi G}{\epsilon_0}\left(\sum_j \frac{q_{j0}n_{j0}}{\omega^2}\right)^2 = 0, \tag{4.14}$$

where $\lambda_{De} = \sqrt{n_{e0}e^2/\epsilon_0 k_B T_e}$ and $\omega_{pj} = \sqrt{q_{j0}^2 n_{j0}^2/\epsilon_0 m_j}$ are the electron's Debye length and plasma frequency of the j^{th} ionic specie. Since due to charge neutrality condition $en_{e0} = \sum\limits_j q_{j0}n_{j0}$, thus we can write,

$$\frac{4\pi G}{\epsilon_0}\left(\sum_j \frac{q_{j0}n_{j0}}{\omega^2}\right)^2 = \frac{4\pi G}{\epsilon_0}(en_{e0})^2 = \frac{n_{e0}e^2}{\epsilon_0 me} \times 4\pi G m_e n_{e0} = \omega_{pe}^2\omega_{Je}^2$$

We can introduce the global frequency for plasma oscillation and Jeans frequency as "$\omega_p^2 = \sum_j \omega_{pj}^2$" and "$\omega_J^2 = \sum_j \omega_{Jj}^2$", thus the above dispersion relation becomes,

$$\left(1 + \frac{1}{k^2\lambda_{De}^2} - \frac{\omega_p^2}{\omega^2}\right)\left(1 + \frac{\omega_J^2}{\omega^2}\right) + \frac{4\pi G}{\epsilon_0}\left(\sum_j \frac{q_{j0}n_{j0}}{\omega^2}\right)^2 = 0, \tag{4.15}$$

Let

$$A^{-1} = 1 + \frac{1}{k^2\lambda_{De}^2}, \tag{4.16}$$

Eq. (4.15) becomes

$$\left(\omega^4 - A\omega_p^2\omega\right)\left(1 + \frac{\omega^2}{\omega_J^2}\right) + \omega_{pe}^2\omega_{Je}^2 = 0,$$

Simplifying, the above dispersion relation, we get the following result

$$\omega^4 + \left(\omega_J^2 - A\omega_p^2\right)\omega^2 - A\left(\omega_p^2\omega_J^2 - \omega_{pe}^2\omega_{Je}^2\right) = 0, \tag{4.17}$$

The constant term is greater than or equal to zero, because

$$\omega_p^2\omega_J^2 - \omega_{pe}^2\omega_{Je}^2 = \frac{2\pi G}{\epsilon_0}\sum_j\sum_l \frac{n_{j0}n_{l0}}{m_j m_l}\left(q_{j0}m_l - q_{l0}m_j\right)^2 \geq 0 \tag{4.18}$$

The double summation states that "$j \neq l$". Since this term is equal to, or greater than zero, thus dispersion relation in Eq. (4.17), always have a constant term equal to zero or with negative sign. For such plasmas with one ion, like hydrogen-ion, we have the constant term equal to zero. Thus, the dispersion relation reduces to

$$\omega^4 + \left(\omega_{Ji}^2 - A\omega_{pi}^2\right)\omega^2 = 0 \tag{4.19}$$

It has two roots one is "$\omega = 0$", and other comes from

$$\omega^2 = \omega_{ss}^2 - \omega_{Ji}^2 \tag{4.20}$$

where "$\omega_{ss}^2 = A\omega_{pi}^2$". For plasma consisting of more than one type of ionic specie, we have roots

$$\omega^2 = \frac{-\left(\omega_J^2 - A\omega_p^2\right) \pm \sqrt{\left(\omega_J^2 - A\omega_p^2\right)^2 + 4A\left(\omega_p^2\omega_J^2 - \omega_{pe}^2\omega_{Je}^2\right)}}{2}$$

$$\omega^2 = \frac{\left(A\omega_p^2 - \omega_J^2\right) \pm \sqrt{\left(A\omega_p^2 + \omega_J^2\right)^2 - 4A\omega_{pe}^2\omega_{Je}^2}}{2} \tag{4.21}$$

The instability grows for the negative root of the equation.

4.1.3 Self-gravitational Effects in Dusty Plasma

In this section, we consider the self-gravitation in dusty plasma for different ionic specie. We consider here protons and massive dust particles with negative charge, labeled with "i and d" respectively. Substituting "$j = i$ and $l = e$", thus Eq. (4.18) can be written as,

$$\omega_p^2\omega_J^2 - \omega_{pe}^2\omega_{Je}^2 = \frac{4\pi G n_{i0} n_{d0}}{\epsilon_0 m_i m_d}\left(q_{i0}m_d - q_{d0}m_i\right)^2 \tag{4.22}$$

The dust particle charge in equilibrium is $q_{d0} = z_{d0}e$ and $q_{i0} = e$, thus we can write

$$\omega_p^2\omega_J^2 - \omega_{pe}^2\omega_{Je}^2 = \frac{4\pi G n_{i0} n_{d0}}{\epsilon_0 m_i m_d}\left(e m_d + e z_{d0} m_i\right)^2$$

or

$$\omega_p^2\omega_J^2 - \omega_{pe}^2\omega_{Je}^2 = 4\pi G m_d n_{d0} \times \frac{e^2 n_{i0}}{\epsilon_0 m_i}\left(1 + z_{d0}\frac{m_i}{m_d}\right)^2 \tag{4.23}$$

$$\omega_p^2\omega_J^2 - \omega_{pe}^2\omega_{Je}^2 = \omega_{Jd}^2\omega_{pi}^2\left(1 + z_{d0}\frac{m_i}{m_d}\right)^2 \tag{4.24}$$

where ω_{Jd} and ω_{pi} are the dust Jeans frequency and ion plasma frequency. In Eq. (4.24) we can use the approximation that "$z_{d0}m_i \ll m_d$", thus we get,

$$A\left(\omega_p^2\omega_J^2 - \omega_{pe}^2\omega_{Je}^2\right) = A\omega_{pi}^2\omega_{Jd}^2 = \omega_{ss}^2\omega_{Jd}^2 \tag{4.25}$$

This result is valid when self-gravitational effects due to ions are neglected in comparison to that of heavy dust particles. If we consider the self-gravitational effects due to the ions then we consider the result of Eq. (4.24). For bi-ion plasma instability arises due to some modified Jeans instabilities. In dusty plasma heavy dust particles are the cause of self-gravitational effects leading to Jeans type instabilities, but in usual bi-ion plasma, the instabilities have weak effects.

4.2 Linear Instability of Dusty Plasma

As stated earlier, the dust grain size is very large usually in the range $1\mu m$ to $1cm$. The flux of these micrometer sized grains observed Helios probes is 10^{-2}-10^{-9}g in the range of 1 $A.U.$ from sun which is much greater than the mass of both electron and ion. The charge on dust grains is also 10^4 times large as compared to the charge on electron. As the dynamics of electrons and ions is controlled by electromagnetic forces, while the dynamics of astrophysical bodies like stars, planets and satellites is governed by gravitational forces. For micro or sub-micron sized dust particles these two forces become comparable. For only electrostatic forces to be considered we should have $Gm_d^2/q_d^2 \approx 0$. Since the dust grains are very massive thus we assume that the dust particles are cold i.e., $T_d \ll T_e = T_i = T$ showing that electrons and ions are thermalized [**?**].

4.2.1 Fluid Model

Here we consider spatially uniform density of electrons, ions and dust particles. The charge of these species is denoted by $q_e = q_i$ and q_d. The continuity equation is

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha v_\alpha) = 0 \tag{4.26}$$

where α represents electrons, ions and dust particles. The momentum equation for electrons and ions is given as

$$m_\alpha n_\alpha \frac{dv_\alpha}{dt} = -n_\alpha q_\alpha \nabla \phi - n_\alpha m_\alpha \nabla \psi - T \nabla n_\alpha \tag{4.27}$$

here $\alpha = e, i$. Whereas the momentum equation for cold dust particles is $(T_d \ll T_e = T_i = T)$

$$m_d n_d \frac{dv_d}{dt} = n_d q_d \nabla \phi - m_d n_d \nabla \psi \tag{4.28}$$

Where v_d is the fluid velocity. Since

$$\nabla \cdot E = \frac{\rho}{\varepsilon_\circ} = \frac{n_\alpha q_\alpha}{\varepsilon_\circ}$$

The electrostatic Poisson's equation is

$$\nabla^2 \phi = -4\pi \left[q_i n_i - q_e n_e - q_d n_d\right] \tag{4.29}$$

while gravitational Poisson's equation is

$$\nabla^2\psi = 4\pi G m_d n_d \tag{4.30}$$

The quasineutral equilibrium is $q_d n_d = q(n_i - n_e)$. For linear gravitational stability of homogeneous dusty plasma, we assume that : $n_{e0} = constant$, $n_{i0} = constant$, $n_{d0} = constant$, where $n_{e0}, n_{i0} \gg n_{d0}$, $T_e = T_i = T = constant$, $T_d = 0$, $\psi_0 = 0$, $\phi_0 = 0$, $v_{d0} = v_{e0} = v_{i0} = 0$. The linearized set of fluid equations are the continuity equation for α specie,

$$\frac{\partial \delta n_\alpha}{\partial t} + n_{\alpha 0}\nabla\delta v_\alpha = 0 \tag{4.31}$$

and the momentum equation for electron, ions and dust particles are

$$m_d n_{d0}\frac{d\delta v_d}{dt} = n_{d0} q_d \nabla\delta\phi - m_d n_{d0}\nabla\delta\psi \tag{4.32}$$

$$m_e n_{e0}\frac{d\delta v_e}{dt} = n_{e0} q \nabla\delta\phi - m_e n_{e0}\nabla\delta\psi - T\nabla\delta n_e \tag{4.33}$$

$$m_i n_{i0}\frac{d\delta v_i}{dt} = -n_{i0} q \nabla\delta\phi - m_i n_{i0}\nabla\delta\psi - T\nabla\delta n_i \tag{4.34}$$

respectively. The gravitational Poisson's equation,

$$\nabla^2\delta\psi = 4\pi G m_d \delta n_d \tag{4.35}$$

and the electrostatic Poisson's equation

$$\nabla^2\delta\phi = -4\pi\left[q\left(\delta n_i - \delta n_e\right) - q_d\delta n_d\right] \tag{4.36}$$

For homogeneous equilibrium, the perturbations varies with $\exp i(kx - \omega t)$. Now we take the Fourier transform of linearized equations and write them in term of δn_d. Eq. (4.30), gives the following result,

$$-i\omega\delta n_d + n_{d0}(ik)\delta v_d = 0 \tag{4.37}$$

$$(-i\omega)\delta n_e + n_{e0}(ik)\delta v_e = 0,$$

$$(-i\omega)\delta n_i + n_{i0}(ik)\delta v_i = 0$$

The perturbed dust particles velocity is

$$\delta v_d = \frac{\omega \delta n_d}{k n_{do}} \tag{4.38}$$

and the perturbed electron and ion velocities are

$$\delta v_e = \frac{\omega \delta n_e}{k n_{e0}} \tag{4.39}$$

$$\delta v_i = \frac{\omega \delta n_i}{k n_{i0}} \tag{4.40}$$

respectively. The gravitational potential is found from Fourier transform of Eq. (4.34) as,

$$-k^2 \delta\psi = 4\pi G m_d \delta n_d$$

$$\delta\psi = \frac{-4\pi G m_d \delta n_d}{k^2} \tag{4.41}$$

4.2.2 Dispersion Relation

Taking Fourier transform of momentum equations of electrons and ions, then substituting perturbed electron and ion velocities and gravitational potential from Eq. (4.39), (4.40) and (4.41), respectively in transformed electron and ion momentum equations we get the following results

$$m_e n_{e0}(-i\omega)\frac{\omega \delta n_e}{k n_{e0}} = n_{e0} q (ik)\delta\phi - m_e n_{e0}(ik)(\frac{-4\pi G m_d \delta n_d}{k^2}) - T(ik)\delta n_e$$

$$m_i n_{i0}(-i\omega)\frac{\omega \delta n_i}{k n_{i0}} = -n_{i0} q (ik)\delta\phi - m_i n_{i0}(ik)(\frac{-4\pi G m_d \delta n_d}{k^2}) - T(ik)\delta n_i$$

Dividing above equations with "ikT " we obtain

$$\left[1 - \frac{\omega^2}{k^2}\left(\frac{m_e}{T}\right)\right]\delta n_e = \frac{n_{e0} q \delta\phi}{T} + \left(\frac{m_e}{T}\right)\left(\frac{4\pi G m_d n_{d0}}{n_{d0} k^2}\right)\delta n_d$$

$$\left[1 - \frac{\omega^2}{k^2}\left(\frac{m_i}{T}\right)\right]\delta n_i = -\frac{n_{i0} q \delta\phi}{T} + \left(\frac{m_i}{T}\right)\left(\frac{4\pi G m_d n_{d0}}{n_{d0} k^2}\right)\delta n_d$$

Substituting "$m_\alpha/T = 1/v_{th_\alpha}$" where ($\alpha$ stands for electrons and ions) v_{th} is the thermal velocity and "$\omega_{Jd} = \sqrt{4\pi G m_d n_{d0}}$" is the Jeans frequency for dust particles. Now after rearranging we can write δn_e and δn_i in term of δn_d and $\delta\phi$, as

$$\delta n_e = \left[\frac{n_{e0}q\delta\phi}{T} + \frac{\omega_{Jd}^2}{k^2 v_{th_e}^2}\frac{n_{e0}}{n_{d0}}\delta n_d\right]\left[1 - \frac{\omega^2}{k^2 v_{th_e}^2}\right]^{-1} \tag{4.42}$$

$$\delta n_i = \left[-\frac{n_{i0}q\delta\phi}{T} + \frac{\omega_{Jd}^2}{k^2 v_{th_i}^2}\frac{n_{i0}}{n_{d0}}\delta n_d\right]\left[1 - \frac{\omega^2}{k^2 v_{th_i}^2}\right]^{-1} \tag{4.43}$$

respectively. Now we find perturbed dust number density by substituting perturbed dust velocity δv_d from Eq. (4.38) and perturbed gravitational potential $\delta\psi$ from Eq. (4.41) in perturbed dust momentum Eq. (4.33), after taking its Fourier transform, we get,

$$m_d n_{d0}\left(-i\omega\right)\left(\frac{\omega\delta n_d}{kn_{d0}}\right) = n_{d0}q_d\left(ik\right)\delta\phi - m_d n_{d0}\left(ik\right)\left(-\frac{4\pi G m_d \delta n_d}{n_{d0}k^2}\right) \tag{4.44}$$

Dividing above equation with "ikm_d" and rearranging again, we obtain

$$\left(\frac{\omega^2}{k^2} + \frac{4\pi G m_d n_{d0}}{k^2}\right)\delta n_d = -\frac{n_{d0}q_d}{m_d}\delta\phi$$

or

$$\delta n_d = -\frac{n_{d0}q_d}{m_d}\left(\frac{\omega^2}{k^2} + \frac{\omega_{Jd}^2}{k^2}\right)^{-1}\delta\phi \tag{4.45}$$

here $\omega_{Jd} = \sqrt{4\pi G m_d n_{d0}}$, is Jeans frequency of dust particles. Now using δn_e and δn_i from Eq. (4.42) and Eq. (4.43) in Poisson's Eq. (4.36) after taking its Fourier transform, we get the following result,

$$k^2\delta\phi = 4\pi q\left[-\frac{n_{i0}q\delta\phi}{T} + \frac{\omega_{Jd}^2}{k^2 v_{th_i}^2}\frac{n_{i0}}{n_{d0}}\delta n_d\right]\left[1 - \frac{\omega^2}{k^2 v_{th_i}^2}\right]^{-1} -$$

$$4\pi q\left[\frac{n_{e0}q\delta\phi}{T} + \frac{\omega_{Jd}^2}{k^2 v_{th_e}^2}\frac{n_{e0}}{n_{d0}}\delta n_d\right]\left[1 - \frac{\omega^2}{k^2 v_{th_e}^2}\right]^{-1} - 4\pi q_d \delta n_d \tag{4.46}$$

Substituting"$A_\alpha = 1 - \omega^2/k^2 v_{th_\alpha}^2$" where α stands for electrons and ions. For thermal $v_{th_e}^2 \gg v_{th_i}^2$ and $n_{io} \approx n_{e0} \approx n_0$

$$k^2\delta\phi = -\frac{4\pi n_0 q^2}{T}\left[\frac{1}{A_i} + \frac{1}{A_e}\right]\delta\phi - 4\pi q_d\left[1 - \frac{\omega_{Jd}^2}{k^2 v_{th_i}^2}\frac{n_{i0}}{n_{d0}}\frac{1}{A_i k^2}\right]\delta n_d$$

Here "$\lambda_D = \sqrt{T/4\pi n_0 q^2}$" is the effective Debye length thus

$$\left[k^2\lambda_D^2 + A_i^{-1} + A_e^{-1}\right]\delta\phi = -4\pi q_d \lambda_D^2 \left[1 - \frac{q}{q_d}\frac{\omega_{Jd}^2}{v_{th_i}^2}\frac{n_{i0}}{n_{d0}}\frac{1}{A_i}\right]\delta n_d \tag{4.47}$$

or

$$\delta\phi = -4\pi q_d \left[1 - \frac{q}{q_d}\frac{\omega_{Jd}^2}{v_{th_i}^2}\frac{n_{i0}}{n_{d0}}\frac{1}{A_i}\right]\delta n_d \left[k^2\lambda_D^2 + A_i^{-1} + A_e^{-1}\right]^{-1} \tag{4.48}$$

Substituting this value of $\delta\phi$ in perturbed dust number density Eq. (4.45), we obtain

$$\left(\frac{\omega^2}{k^2} + \frac{\omega_{Jd}^2}{k^2}\right) = -\frac{4\pi n_{d0} q_d^2 \lambda_D^2}{\left[k^2\lambda_D^2 + A_i^{-1} + A_e^{-1}\right] m_d}\left[1 - \frac{q}{q_d}\frac{\omega_{Jd}^2}{v_{th_i}^2}\frac{n_{i0}}{n_{d0}}\frac{1}{A_i}\right] \tag{4.49}$$

Simplifying the Eq. (4.49), we obtain a dispersion relation for Jeans instability of dusty plasma with heavy dust particles such that $Gm_d^2/q_d^2 \simeq 1$, as

$$\omega^2 = \frac{-k^2\lambda_D^2\omega_{pd}^2}{\left[k^2\lambda_D^2 + A_i^{-1} + A_e^{-1}\right]}\left[1 - \frac{q}{q_d}\frac{\omega_{Jd}^2}{v_{th_i}^2}\frac{n_{i0}}{n_{d0}}\frac{1}{A_i}\right] - \omega_{Jd}^2 \tag{4.50}$$

Where "$\omega_{pd} = \sqrt{4\pi n_{d0} q_d^2/m_d}$" is the dust plasma frequency.

Limiting Cases

If there is no charge on dust particles then dust plasma frequency $\omega_{pd} = 0$ and gives the Jeans instability of gas. Electrostatic oscillation of the dusty plasma can be obtained if $\omega_{Jd}^2 = 0$. For the ultra low frequency case one has the limits $\omega^2 \approx \omega_{Jd}^2 \approx \omega_{pd}^2 \ll \omega_{pe}^2, \omega_{pi}^2$. In this limit we have $\omega^2/\left(k^2 v_{th_i}^2\right) \sim \left(\omega_{pd}^2/\omega_{pi}^2\right)\left(1/k^2\lambda_D^2\right)$ and $\omega_{Jd}^2/\omega_{pi}^2 \sim (m_i/m_d)\left(q_d^2/q_i^2\right)(n_{d0}/n_0)$. Now we substitute the typical values as $(m_i/m_d \approx 10^{-12})$, $(q_d^2/q_i^2 \approx 10^6)$ and $(n_{d0}/n_0 \approx 10^{-3})$, we conclude that $\omega^2/\left(k^2 v_{th_i}^2\right) \sim \omega_{Jd}^2\left(k^2 v_{th_i}^2\right) \ll 1$, $A_i \approx 1$, $A_e \sim 1$. Thus, we can neglect the second term in above dispersion relation. The electron and ions both follow Boltzmann's relation because the gravitational and inertial effects are weaker on both species. Thus, the simplified dispersion relation is

$$\omega^2 = \frac{k^2\lambda_D^2\omega_{pd}^2}{2 + k^2\lambda_D^2} - \omega_{Jd}^2 \tag{4.51}$$

For the scale size of perturbation of order same as the Debye length, i.e., $k^2\lambda_D^2 \geqslant 1$, the gravitational condensation is affected by unshielded electric field (which is due to charge separation). Hence we have

$$\omega^2 = \omega_{pd}^2 - \omega_{Jd}^2 \tag{4.52}$$

While in this limit the electrons and ions are supposed to be fixed in background thus "$\delta n_e = \delta n_i = 0$". The gravitational condensation of dust particles in the limit $Gm_d^2/q_d^2 < 1$ is maintained by space charge electric field.

4.3 Electrostatic Mode in Self-Gravitating, Collisional Dusty Plasma

We consider in this model the dust-ion collisions in dusty plasma comprising of electrons, ion and dust particles. We consider the electrostatic waves only that propagates along x-axis with wave period quite different from dust particle's fluctuating charge time to ensure that dust fluctuations are zero.

4.3.1 Fluid Model

The number density of the electrons is Boltzmann distributed given as

$$n_e = n_{e0} \exp\left(\frac{e\phi}{k_B T_e}\right), \tag{4.53}$$

The one dimensional (along $x-$axis) continuity equations and equations of motion for ion and dust grains are

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i v_i) = 0 \tag{4.54}$$

$$\frac{\partial n_d}{\partial t} + \frac{\partial}{\partial x}(n_d v_d) = 0 \tag{4.55}$$

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial x} + \frac{q_i}{m_i}\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial x} + \frac{v_{Ti}^2}{n_i}\frac{\partial n_i}{\partial x} + \nu_{id}(v_i - v_d) = 0 \tag{4.56}$$

$$\frac{\partial v_d}{\partial t} + v_d \frac{\partial v_d}{\partial x} + \frac{q_d}{m_d}\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial x} + \frac{v_{Td}^2}{n_d}\frac{\partial n_d}{\partial x} + \nu_{di}(v_d - v_i) = 0 \tag{4.57}$$

Here

$$\nu_{di} = \frac{m_i n_{i0}}{m_d n_{d0}} \nu_{id}$$

where n_α, v_α, q_α, and m_α, are the number density, velocity, charge and mass of the α specie respectively, whereas ν_{id}, ν_{di} are the collisional frequency The electrostatic and gravitational Poisson's equations are

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{\epsilon_0}\left(n_e e - n_i q_i - n_d q_d\right). \tag{4.58}$$

$$\frac{\partial \psi}{\partial x^2} = 4\pi G\left(m_i n_i + m_d n_d\right) \tag{4.59}$$

respectively. We write here the perturbed as well as linearized set of equations: The linearized electron number density is

$$n_{e1} = n_{e0}\left(\frac{e\phi_1}{k_B T_e}\right). \tag{4.60}$$

the perturbed ion and dust continuity equation are

$$\frac{\partial n_{i1}}{\partial t} + n_{i0}\frac{\partial}{\partial x}v_{i1} = 0 \tag{4.61}$$

$$\frac{\partial n_{d1}}{\partial t} + n_{d0}\frac{\partial}{\partial x}v_{d1} = 0. \tag{4.62}$$

respectively. While perturbed ion's and dust grain's equation of motion are

$$\frac{\partial v_{i1}}{\partial t} + \frac{q_i}{m_i}\frac{\partial \phi_1}{\partial x} + \frac{\partial \psi_1}{\partial x} + \frac{v_{Ti}^2}{n_i}\frac{\partial n_{i1}}{\partial x} + \nu_{id}\left(v_{i1} - v_{d1}\right) = 0. \tag{4.63}$$

$$\frac{\partial v_{d1}}{\partial t} + \frac{q_d}{m_d}\frac{\partial \phi_1}{\partial x} + \frac{\partial \psi_1}{\partial x} + \frac{v_{Td}^2}{n_{d0}}\frac{\partial n_{d1}}{\partial x} + \nu_{di}\left(v_{d1} - v_{i1}\right) = 0 \tag{4.64}$$

respectively. The perturbed Poisson's equations are,

$$\frac{\partial^2 \phi_1}{\partial x^2} = \frac{1}{\epsilon_0}\left(n_{e1} e - n_{i1} q_i - n_d q_d\right). \tag{4.65}$$

$$\frac{\partial \psi_1}{\partial x^2} = 4\pi G\left(m_i n_{i1} + m_d n_{d1}\right) \tag{4.66}$$

All the perturbed quantities are assumed to be proportional to $\exp\left[i\left(kx - \omega t\right)\right]$. Thus, the ion continuity Eq. (4.61) and dust continuity Eq.(4.62), transforms as

$$-i\omega n_{i1} + ikn_{i0}v_{i1} = 0$$

$$-i\omega n_{d1} + ikn_{d0}v_{d1} = 0$$

or

$$n_{i1} = \frac{kn_{i0}}{\omega} \tag{4.67}$$

$$n_{d1} = \frac{kn_{d0}}{\omega} \tag{4.68}$$

respectively. The momentum equation of ions and dust particles in Eq. (4.63) and (4.64), respectively transforms to

$$-i\omega v_{i1} + ik\frac{q_i}{m_i}\phi_1 + ik\psi_1 + ik\frac{v_{Ti}^2}{n_{io}}n_{i1} + \nu_{id}\left(v_{i1} - v_{d1}\right) = 0 \tag{4.69}$$

$$-i\omega v_{d1} + ik\frac{q_d}{m_d}\phi_1 + ik\psi_1 + ik\frac{v_{Td}^2}{n_{d0}}n_{d1} + \nu_{di}\left(v_{d1} - v_{i1}\right) = 0 \tag{4.70}$$

respectively. The electrostatic Poisson's equation becomes,

$$-k^2\phi_1 = \frac{1}{\epsilon_0}\left(\frac{e^2 n_{e0}}{k_B T} + n_{i1}e + n_{d1}q_d\right)$$

$$\phi_1 = \frac{1}{k^2\epsilon_0}\frac{e^2 n_{e0}}{k_B T} + \frac{1}{k^2\epsilon_0}\left(n_{i1}e + n_{d1}q\right)$$

$$\left(1 + \frac{1}{k^2\lambda_{De}^2}\right)\phi = \frac{1}{k^2\epsilon_0}\left(n_{i1}e + n_{d1}q_d\right) \tag{4.71}$$

where $\lambda_{De} = \sqrt{e^2 n_{e0}/k_B T}$, is the electron Debye length. Let us suppose that

$$A = \left(1 + \frac{1}{k^2\lambda_{De}^2}\right)^{-1} \tag{4.72}$$

Substituting the n_{i1} and n_{d1} from Eq. (4.67) and (4.68) in Eq. (4.71), we get the following equation

$$\phi_1 = \frac{A}{k^2\epsilon_o}\frac{k}{\omega}\left(n_{io}v_{i1}q_{i0} + n_{i0}v_{d1}q_d\right) \tag{4.73}$$

The gravitational Poisson's Eq. (4.66) transforms as,

$$-k^2\psi_1 = 4\pi G\frac{k}{\omega}\left(m_{i1}n_{i0}v_{i1} + m_d n_{d0}v_{d1}\right) \tag{4.74}$$

Substituting the perturbed electrostatic and gravitational potential from Eq. (4.73) and (4.74) into the ion's equation of motion, Eq. (4.56), we get

$$-i\omega v_{i1} + \nu_{id}v_{i1} + \frac{ikq_{io}}{m_i}\left[\frac{A}{\epsilon_o k\omega}\left(n_{i0}q_{i0}v_{i1}\right)\right] + \frac{ikq_{io}}{m_i}\left[\frac{A}{\epsilon_0 k\omega}\left(n_{d0}q_{d0}v_{d1}\right)\right] +$$

$$ik\left[-\frac{4\pi G}{k\omega}\left(m_i n_{i0}v_{i1}\right)\right] + ik\left[-\frac{4\pi G}{k\omega}\left(m_d n_{d0}v_{d1}\right)\right] + \frac{ikv_{Ti}^2}{n_{d0}}\frac{k}{\omega}n_{e0}v_{i1} - \nu_{id}v_{d1} = 0 \tag{4.75}$$

Multiplying with $(i\omega)$ and simplifying the Eq. (4.75), we get the following result,

$$[\omega\left(\omega + i\nu_{id}\right) - A\frac{q_{i0}^2 n_{i0}}{\epsilon_o m_i} + 4\pi G m_i n_{i0} - k^2 v_{Ti}^2]v_{i1} = -\left[-A\frac{q_{i0}n_{d0}}{m_i\epsilon_0}q_d + 4\pi G m_d n_{d0} - i\omega\nu_{id}\right]v_{d1}$$

here we define plasma and Jeans frequency as: $\sqrt{q_{i0}^2 n_{i0}/\epsilon_0 m_i} = \omega_{pi}$ and $\sqrt{4\pi G m_i n_{i0}} = \omega_{Ji}$ respectively. Thus,

$$\left[\omega(\omega + i\nu_{id}) - A\omega_{pi}^2 + \omega_{Ji}^2 - k^2 v_{Ti}^2\right]v_{i1} = \left[A\frac{q_{i0}q_d}{\epsilon_o m_i} - \omega_{Jd}^2 + i\omega\nu_{id}\right]v_{d1} \tag{4.76}$$

Now substituting Eq. (4.73) and (4.74) into the dust equation of motion, i.e., Eq. (4.57),

$$(\omega + i\nu_{di}) - k\frac{q_d}{m_d}\left[\frac{A}{\epsilon_o k\omega}n_{d0}q_d v_{d1}\right] - k\frac{q_d}{m_d}\left[\frac{An_{i0}q_{i0}}{\epsilon_o k\omega}\right]v_{i1} - k\left[-\frac{4\pi G}{k\omega}m_d n_{d0}v_{d1}\right] -$$

$$k\left[-\frac{4\pi G}{k\omega}m_i n_{i0}v_{i1}\right] - \frac{k^2}{\omega}\frac{v_{Td}^2}{n_{d0}}n_{d0}v_{d1} - iv_{d1}v_{i1} = 0 \tag{4.77}$$

Multiplying Eq. (4.77) with "ω"

$$\left[\omega\left(\omega + i\nu_{di}\right) - A\frac{q_d^2 n_{d0}}{\epsilon_o m_d} + 4\pi G m_d n_{d0} - k^2 v_{Td}^2\right]v_{d1} = -\left[-A\frac{q_d q_{io}n_{io}}{m_d\epsilon_o} + 4\pi G m_i n_{io} - i\omega v_{d1}\right]v_{i1}$$

where,

$$\nu_{di} = \frac{m_i n_{io}}{m_d n_{do}}\nu_{id}$$

It implies that,

$$\left[\omega\left(\omega + i\nu_{id}\frac{4\pi G m_i n_{io}}{4\pi G m_d n_{do}}\right) - A\omega_{pd}^2 + \omega_{Jd}^2 - k^2 v_{Td}^2\right]v_{d1} = \left[A\frac{v_{do}q_{io}n_{io}}{\epsilon_o m_d} - \omega_{Ji}^2 + i\omega\nu_{id}\frac{\omega_{Ji}^2}{\omega_{Jd}^2}\right]v_{i1}$$

or

$$\left[\omega\left(\omega+i\omega\nu_{id}\frac{\omega_{ji}^2}{\omega_{Jd}^2}\right)-k^2v_{Td}^2+\omega_{Jd}^2-A\omega_{pd}^2\right]v_{d1}=\left[A\omega_{pi}^2\frac{q_d}{q_{i0}}\frac{m_i}{m_d}-\omega_{Ji}^2+i\omega\nu_{id}\frac{\omega_{Ji}}{\omega_{Jd}^2}\right]v_{i1} \tag{4.78}$$

Comparing Eq (4.76) and Eq. (4.78), we get

$$\left[\omega\left(\omega+iv_{id}\frac{\omega_{Ji}^2}{\omega_{Jd}^2}\right)-k^2v_{Td}^2+\omega_{Jd}^2-A\omega_{pd}^2\right]\times\left[\omega\left(\omega+i\nu_{id}\right)-k^2v_{Ti}^2+\omega_{Ji}^2-A\omega_{pd}^2\right]=$$

$$\left[A\frac{q_{i0}q_dn_{d0}}{\epsilon_om_i}-\omega_{Jd}^2+i\omega\nu_{id}\right]\times\left[A\omega_{pi}^2\frac{q_d}{q_{i0}}\frac{m_i}{m_d}-\omega_{Jd}^2+i\omega\nu_{id}\frac{\omega_{Ji}^2}{\omega_{Jd}^2}\right] \tag{4.79}$$

Simplifying the right hand side of Eq. (4.79) as follow :

$$A^2\omega_{pi}^2\frac{q_{do}^2n_{do}}{\epsilon_om_d}+\omega_{Ji}^2\omega_{Jd}^2+(i\omega\nu_{id})^2\left(\frac{\omega_{Ji}}{\omega_{Jd}}\right)^2-A\omega_{Ji}^2\frac{q_{io}q_{do}n_{do}}{\epsilon_om_i}-A\omega_{pi}^2\omega_{Jd}^2\frac{q_{do}}{q_{io}}\frac{m_i}{m_d}+$$

$$A\frac{q_{i0}q_dn_{d0}}{\epsilon_om_i}\left(i\omega\nu_{id}\right)\frac{\omega_{Ji}^2}{\omega_{Jd}^2}+A\omega_{pi}^2\frac{q_{do}m_i}{q_{io}m_d}\left(i\omega\nu_{id}\right)-i\omega\nu_{id}\omega_{Ji}^2-i\omega\nu_{id}\omega_{Ji}^2$$

It implies that,

$$A^2\omega_{pi}^2\omega_{pd}^2+\omega_{Jd}^2\omega_{Ji}^2+\left(\omega i\nu_{id}\frac{\omega_{Ji}}{\omega_{Jd}}\right)^2-2i\omega\nu_{id}\omega_{Ji}^2-A\omega_{Ji}\sqrt{4\pi Gn_{d0}m_d\frac{q_d^2n_{d0}}{\epsilon_om_d}\frac{q_{i0}^2n_{i0}}{\epsilon_om_i}}-$$

$$A\omega_{pi}\omega_{Jd}\sqrt{\frac{4\pi q_{i0}^2n_{i0}}{\epsilon_om_i}Gm_dn_{d0}\frac{q_{d0}^2m_i^2}{q_{i0}^2m_d^2}}+A\frac{\omega_{Ji}}{\omega_{Jd}}i\omega\nu_{id}\sqrt{\frac{4\pi Gm_in_{i0}}{4\pi Gm_dn_{d0}}\times\frac{q_{i0}^2q_d^2n_{d0}^2}{\epsilon_om_i^2}}+$$

$$A\omega_{pi}i\omega\nu_{id}\sqrt{\frac{q_{d0}^2m_i^2}{q_{i0}^2m_d^2}\times\frac{4\pi q_{i0}^2n_{i0}}{m_i}},$$

or

$$A^2\omega_{pi}^2\omega_{pd}^2+\omega_{Jd}^2\omega_{Ji}^2+\left(\omega i\nu_{id}\frac{\omega_{Ji}}{\omega_{Jd}}\right)^2-2i\omega\nu_{id}\omega_{Ji}^2-A\left(\omega_{Ji}\omega_{Jd}\omega_{pi}\omega_{pd}+\omega_{Ji}\omega_{Jd}\omega_{pi}\omega_{pd}\right)+$$

$$A\left(i\omega\nu_{id}\omega_{pi}\omega_{pd}\frac{\omega_{Ji}}{\omega_{Jd}}+i\omega\nu_{id}\omega_{pi}\omega_{pd}\frac{\omega_{Ji}}{\omega_{Jd}}\right),$$

Thus

$$(A\omega_{pi}\omega_{pd})^2 + (\omega_{Ji}\omega_{Jd})^2 + \left(\omega i\nu_{id}\frac{\omega_{Ji}}{\omega_{Jd}}\right)^2 - 2i\omega v_{id}\omega_{Ji}^2 - 2A\omega_{Ji}\omega_{Ji}\omega_{pi}\omega_{pd} + 2Ai\omega\nu_{id}\omega_{pi}\omega_{pd}\frac{\omega_{Ji}}{\omega_{Jd}}$$

$$= \left(A\omega_{pi}\omega_{pd} - \omega_{Ji}\omega_{Jd} + i\omega\nu_{id}\frac{\omega_{Ji}}{\omega_{Jd}}\right)^2, \tag{4.80}$$

So, we have the desired dispersion relation for self-gravitating dusty plasma with dust-ion collisions, as

$$\left[\omega\left(\omega + i\nu_{id}\frac{\omega_{Ji}^2}{\omega_{Jd}^2}\right) - k^2v_{Td}^2 + \omega_{Jd}^2 - A\omega_{pd}^2\right] \times \left[\omega\left(\omega + i\nu_{id}\right) - k^2v_{Ti}^2 + \omega_{Ji}^2 - A\omega_{pd}^2\right] =$$

$$\left(A\omega_{pi}\omega_{pd} - \omega_{Ji}\omega_{Jd} + i\omega\nu_{id}\frac{\omega_{Ji}}{\omega_{Jd}}\right)^2,$$

4.4 Multi-fluid Theory for Jeans Type Instability

The collapse of interstellar gas clouds and subsequent formation of stars and planetesimals is caused by the "Jeans instability" in the interstellar space [34]. In astrophysical plasmas, Jeans instability is produced due to the effects of self-gravitating force [27]. In dusty plasma consisting of plasma species (electrons and ions) and heavy dust particles, the mass density is mainly due to the dust particles. The total pressure is written as the sum of pressures of both plasma specie. And the dust-acoustic waves establish a potential for Jeans instability same as sound waves in usual electron-ion plasma. In the present model, we consider high charging of dust particles, which is fluctuating due to electron-ion currents reaching the dust particle surface in dusty plasma. We also consider that there is an attractive force between overlapping Deby spheres [35] of the two separate dust charges of the same polarity due to electrostatic energy. The equilibrium is established, when this electrostatic energy balances the self-gravitational forces. The equilibrium gravitational potential is given as,

$$\phi_{go} = \frac{z_{d0}^2e^2}{m_d d}\exp\left(-\frac{d}{\lambda_D}\right)\left(1 - \frac{d}{2\lambda_D}\right)$$

here z_{d0}, e, m_d, and d are the dust charge number at equilibrium, electronic charge, mass of dust particle and separation distance between dust particles.

4.4.1 Fluid Model

The set of fluid equations governing the dynamics of Dust-acoustic perturbation are given as follow : The perturbed number densities of Boltzmann distributed electron and ions are given as

$$n_{e1} = n_{e0}\left(\frac{e\phi}{T_e}\right), \tag{4.81}$$

$$n_{i1} = n_{i0}\left(\frac{-e\phi}{T_i}\right), \tag{4.82}$$

respectively, which are derived from

$$n_{e,i} = n_{e0,i0}\exp\left(\pm\frac{e\phi}{T_{e,i}}\right),$$

where $n_{e0}\,(n_{i0})$ is the equilibrium electron (ion) number density, $T_e\,(T_i)$ is the electron (ion) temperature, ϕ is the electrostatic potential of the dust-acoustic perturbations. The linearized continuity equation for dust particles is

$$\frac{\partial n_{d1}}{\partial t} + n_{d0}\nabla.v_d = 0, \tag{4.83}$$

and the dust momentum equation is

$$\left(\frac{\partial}{\partial t} + \nu_d\right)v_d = \frac{Z_{d0}e}{m_d}\nabla\phi - \frac{3V_{Td}^2}{n_{d0}}\nabla n_{d1} - \nabla\psi_1 + \frac{\nabla U}{m_d}, \tag{4.84}$$

where $n_{\alpha1}\,(\ll n_{\alpha0})$ is a small density perturbation (α stands for electrons "e" and ions "i" and dust particles "d"), ν_d is the dust-neutral collision frequency, v_d is the dust fluid velocity, V_{Td} is the dust thermal speed, ψ is the gravitational potential. The gravitational Poisson's equation is given as

$$\nabla^2\psi_1 = 4\pi G m_d n_{d1}, \tag{4.85}$$

where G is the gravitational constant and m_d is the dust mass. The perturbed interaction potential energy between two dust grains is given as

$$U \approx \frac{2Z_{d0}Z_{d1}e^2}{d}\exp\left(-\frac{d}{\lambda_D}\right)\left(1-\frac{d}{2\lambda_D}\right), \tag{4.86}$$

where Z_{d1} is a small perturbation in the equilibrium dust charge, $\lambda_D = \lambda_{Di}\lambda_{De}/\sqrt{\lambda_{Di}^2 + \lambda_{De}^2}$ is the effective Debye length of the dusty plasma and λ_{Di}, λ_{De} are the ion and electron Debye lengths respectively. The quasi-neutrality condition is

$$n_{e1} - n_{i1} + Z_{do}n_{d1} + Z_{d1}n_{d0} = 0, \tag{4.87}$$

here substituting n_{e1}, n_{i1} and Z_{d1}, we get

$$\frac{1}{4\pi}\left[\frac{4\pi n_{e0}e^2}{T_e} + \frac{4\pi n_{i0}e^2}{T_i}\right]\phi + n_{d0}r_d\frac{v_2}{v_1}\phi = -Z_{d0}en_{d1},$$

or

$$\left[1 + 4\pi n_{d0}r_d\lambda_D^2\frac{v_2}{v_1}\right]\phi = -4\pi Z_{d0}e\lambda_D^2 n_{d1},$$

or

$$\phi = -4\pi Z_{d0}e\lambda_D^2\left(1+f\right)^{-1}n_{d1}, \tag{4.88}$$

where $f = 4\pi n_{d0}r_d\lambda_D^2\nu_2/\nu_1$ is related with the dust charge perturbation. The perturbed dust charge is given as,

$$Z_{d1} = \left(r_d\nu_2/e\nu_1\right)\phi, \tag{4.89}$$

The Eq. (4.86) now can be written as

$$U \approx \frac{2Z_{d0}e^2}{d}\exp\left(-\frac{d}{\lambda_D}\right)\left(1 - \frac{d}{2\lambda_D}\right)\left(\frac{r_d\nu_2}{e\nu_1}\right)\phi \tag{4.90}$$

This relation has been derived by using current balance equation

$$\frac{\partial q_d}{\partial t} + v_d\frac{\partial q_d}{\partial z} = I_e + I_i, \tag{4.91}$$

where $I_e\,(I_i)$ is the electron (ion) current onto the grain surface

$$I_e = -\pi e r_d^2 n_e\left(\phi\right)\sqrt{\frac{8k_BT_e}{\pi m_e}}\exp\left[\frac{eV}{k_BT_e}\right],$$

$$I_i = \pi e r_d^2 n_i\left(\phi\right)\sqrt{\frac{8k_BT_i}{\pi m_i}}\exp\left[1 - \frac{eV}{k_BT_i}\right], \tag{4.92}$$

here $V = q_d/4\pi\epsilon_0 r_d$ is the grain surface potential. The perturbed dust charge is found in the same way as derived in section [3.3.2]. The right hand side of dust momentum equation shows that various forces are acting on the dust fluid. The first term corresponds to the electrostatic forces, the second term represents the dust pressure gradient, while third term arises due to self-gravitational force and the last term due to electrostatic energy. The attractive force is due to the sum of the electrostatic energies of the electron's and ion's Debye spheres and the overlapping dust Debye sphere.

4.4.2 Dispersion Relation

Now we take Fourier transform of set of dust fluid equations and simplify these equations as follow: Fourier transform of continuity equation of dust particles is,

$$(-i\omega)\, n_{d1} + n_{d0}\,(ik)\, v_d = 0, \tag{4.93}$$

or

$$\omega n_{d1} = n_{d0} k v_d, \tag{4.94}$$

Fourier transform of dust momentum and gravitational Poisson's equation is :

$$((-i\omega) + \nu_d)\, v_d = \frac{Z_{d0}e}{m_d}(ik)\,\phi - \frac{3V_{Td}^2}{n_{d0}}(ik)\, n_{d1} - (ik)\,\psi_1 + \frac{(ik)\,U}{m_d}, \tag{4.95}$$

$$\psi_1 = \frac{-4\pi G m_d n_{d1}}{k^2}, \tag{4.96}$$

respectively. Substituting the interaction energy and dust charge perturbation from Eqs. (4.90) and (4.96) into Eq. (4.95), we get

$$(\omega + i\nu_d)\, v_d = \frac{k}{(1+f)}\left(\frac{4\pi Z_{d0}^2 e^2 \lambda_D^2}{m_d}\right) n_{d1} + \frac{3V_{Td}^2 k}{n_{d0}} n_{d1} - k\left(\frac{4\pi G m_d n_{d1}}{k^2}\right) +$$

$$\frac{k}{m_d}\left(\frac{2Z_{d0}e^2}{d}\exp\left(\frac{-d}{\lambda_D}\right)\left(1 - \frac{d}{2\lambda_D}\right)\frac{r_d\nu_2}{\nu_1}\left(\frac{4\pi Z_{d0} e \lambda_D^2}{1+f}\right)\right) n_{d1}, \tag{4.97}$$

Since $v_d = \omega n_{d1}/n_{d0}k$, after simplifying the Eq. (4.97), we get

$$\omega\,(\omega + i\nu_d) = \frac{k^2\omega_{pd}^2\lambda_D^2}{(1+f)} + 3V_{Td}^2 k^2 n - \Omega_J^2 + \left(\frac{2k^2\omega_{pd}^2\lambda_D^2}{1+f}\right)$$

$$\times \exp\left(\frac{-d}{\lambda_D}\right)\left(1 - \frac{d}{2\lambda_D}\right)\frac{r_d\nu_2}{\nu_1 d}, \tag{4.98}$$

where $\omega_{pd} = \sqrt{4\pi Z_{d0}^2 e^2 n_{d0}/m_d}$, is the dust plasma frequency and $\Omega_J = \sqrt{4\pi G m_d n_{d0}}$ is the Jeans frequency. Let $C_{DP} = \omega_{pd}\lambda_D (1+f)^{-1}$, thus the final dispersion relation is

$$\omega(\omega + i\nu_d) = k^2 C_{DP}^2 + 3V_{Td}^2 k^2 - \Omega_J + \left(2k^2 C_{DP}^2 \frac{r_d}{d}\frac{\nu_2}{\nu_1}\right.$$

$$\left.\exp\left(\frac{-d}{\lambda_D}\right)\left(1 - \frac{d}{2\lambda_D}\right)\right), \tag{4.99}$$

where ω and k are the frequency and the wavenumbers. Rearranging the above dispersion relation and writing in quadratic form as follow

$$\omega^2 + i\nu_d\omega + \Omega_J - k^2 C_{DP}^2 \left[1 + \frac{2r_d}{d}\frac{\nu_2}{\nu_1}\exp\left(\frac{-d}{\lambda_D}\right)\right] - 3k^2 V_{td}^2 +$$

$$k^2 C_{DP}^2 \frac{r_d}{\lambda_D}\frac{\nu_2}{\nu_1}\exp\left(\frac{-d}{\lambda_D}\right) = 0,$$

Let $V = [C_{DP}[1 + (2r_d\nu_2/\nu_1)\exp(-d/\lambda_D)] + 3kV_{Td}^2]^{1/2}$ and $S = r_d\nu_2/\nu_1\lambda_D \exp(-d/\lambda_D)$, thus

$$\omega^2 + i\nu_d\omega + \Omega_J - V^2k^2 + k^2 C_{DP}^2 S = 0, \tag{4.100}$$

The solution of Eq. (4.100) is

$$\omega = -i\frac{\nu_d}{2} \pm \frac{i}{2}\sqrt{4\Omega^2 + 4SC_{DP}^2 k^2 - 4k^2V^2 + v_d^2}, \tag{4.101}$$

The Jeans instability occurs if

$$4\Omega^2 + 4SC_{DP}^2 k^2 > 4k^2V^2 - v_d^2, \tag{4.102}$$

The maximum growth rate is

$$\gamma = \sqrt{\Omega^2 + SC_{DP}^2 k^2}, \tag{4.103}$$

shows that the Jeans instability increases with dust grains attraction. From above equations it is clear that the induced Jeans instability due the grain attraction is dominant for the limit $Sk^2C_{Dp}^2 > \Omega_J^2$.

4.5 Summary

In this chapter, we have reviewed the research work of Pandey *et. al* [18], Verheest *et al.* [21], Jacobs *et. al* [24] and Shukla *et al.* [27]. In [18], we have reviewed the linear instability analysis only and found the dispersion relation for Jeans instability for limit $Gm_d^2/q_d^2 \approx 0$, showing that dust grains are much heavier. Also discussed the linear dispersion relation for low-frequency case, and considered the electron and ions as Boltzmannian. In reference [21], we have re-derived the linear dispersion relation, for cold multi-ionic species in the presence of mutual effects of self-gravitational and electrostatic forces in dusty plasma. Dispersion relation for the hydrogen plasma is derived for single ion plasma. The general dispersion relation for collisional (dust-ion collisions only), low frequency dusty plasma for electrostatic case has been rederived [24]. New dispersion relation for dusty plasma by taking into account the electrostatic energy and self-gravitational effect has been rederived. They found that the electrostatic force assists the Jeans instability and makes the system further unstable. They also found that in the presence of electrostatic force, the system becomes unstable even if there is no self-gravitational effects in dusty plasma.

References

[1] J. H. Chu and Lin. I, Phys. Rev. Lett. **72**, 4009 (1994).

[2] H. Thomas, G. E. Morfill, V. Demmel, J. Goree, B.Feuerbacher and D. Mohlmann, Phys. Rev. Lett., **73**, 652 (1994).

[3] P. K. Shukla, Phys. Plasmas, **8**, 1791 (2001).

[4] M. Rosenberg, Astrophys. and Space Science, **277**, 125, (2001).

[5] P. K. Shukla, Dust Plasma Interaction in Space, Nova Sci. Publisher (2002).

[6] V E Fortov, A G Khrapak, S A Khrapak, V I Molotkov, and O F Petrov, Phys–USP, **47(5)**, 447, (2004).

[7] P K Shukla, A. A. Mamun, Introduction to Dusty Plasma Physics, (**IOP** 2002).

[8] N. N. Rao, P. K. Shukla and M. Y. YU, Planet. Space Sci., **38**, 543, (1990).

[9] S. Shaikh , A. Khan, and P. K. Bhatia and Z. Naturforsch. **61a**, 275, (2006).

[10] J. H. Jeans and Philos. Trans, R. Soc., London, **199**, 1, (1902).

[11] D. Tsiklauri, The Astrophysical Journal, **507**, 226, (1998).

[12] P. V. Bliokh and V.V Yaroshenko, Soviet Aston, **29**, 330, (1985).

[13] O. Havens, Astron. Astrophys., **193**, 309, (1988).

[14] N. N. Rao, P. K. Shukla, and M. Yu, Planet. Space Sci., **38**, 541 (1990).

[15] P. K. Shukla, Phys. Scripta, **45**, 504, (1992).

[16] P. K. Shukla, and V. P. Silin, Phys. Scr., **45**, 508, (1992).

[17] F. Melandso, T. K. Aslaksen, and O. Havnes, Planetry Space Sci., **41**, 321, (1993a).

[18] B. P. Pandey, K. Avinash and C. B. Dwivedi, Physical Rev. E, **49**, 5599, (1994).

[19] A. Barkan, R. L. Merlino and N. dl'Angelo, Phys.Plasmas, **2**, 3563, (1995).

[20] B. P. Pandey and C. B. Dwivedi, J. Plasma Phys., **65**, 395, (1996).

[21] F. Verheest, P. K. Shukla, N. N. Rao and P. Meuris, J. Plasma Physics, **58**, 163, (1997).

[22] A. A. Mamun, M. Salahuddin, and M. Salimullah, **47**, 79, (1998).

[23] N. N. Rao and F. Verheest, Phys. Lett. A, **268** , 390, (2000).

[24] G. Jacobs, V. V. Yaroshenko and F. Verheest, Physical Rev. E, **66**, 26407 (2002).

[25] A. P. Misra, A. R. Chowdhurya and K. R. Chowdhury, Physics Lett. A , **323**, 110, (2004).

[26] G. L. Delzanno and G. Lapenta, Phys. Rev. Lett., **94**, 175005, (2005).

[27] P.K Shukla and L Stenflo, Proc. R. Soc. A, **462**, 403, (2006).

[28] R. L. Merlino, Plasma Phys Applied, **73**, 110, (2006).

[29] N. N. Rao, Phys. Plasmas, **6**, 4414, (1999).

[30] Andr e Melzer, Introduction to Colloidal (Dusty Plasma), Lecture Notes, (2012).

[31] Shukla P K and Silin V P, Phys. Scripta, **45**, 508 (1992).

[32] D. Bhattacharya, Fundamentals of Astrophysics, PH217, Aug-Dec (2003).

[33] N. N. Rao, Phys. Plasmas **7**, 795 (2000).

[34] Eddington, A. S, Cambridge, UK: Cambridge University Press, (1988).

[35] Resendes, D. P., Mendonça, J. T. & Shukla, P. K. Phys. Lett. A, **239**, 181 (1998).