Fixed point theorems for single-valued and multi-valued dynamical systems in metric type spaces

By

Wahid Ullah

Department of Mathematics
Quaid-I-Azam University
Islamabad, Pakistan
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A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF

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IN

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Supervised by

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Department of Mathematics
Quaid-I-Azam University
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We accept this thesis as conforming to the required standard

1. ______________________
Prof. Dr. Tariq Shah
(Chairman)

2. ______________________
Dr. Maria Samreen
(Supervisor)

3. ______________________
Dr. Mujeeb Ur Rehman
(External Examiner)

School of Natural Sciences, National University of Science and Technology (NUST), Islamabad.

4. ______________________
Dr. Tahir Mahmood
(External Examiner)

Department of Mathematics & Statistics, International Islamic University, Islamabad.

Department of Mathematics
Quaid-I-Azam University
Islamabad, Pakistan
2022

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This is to certify that the research work presented in this thesis entitled **Fixed point theorems for single-valued and multi-valued dynamical systems in metric type spaces** was conducted by **Mr. Wahid Ullah** under the kind supervision of **Dr. Maria Samreen**. No part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the Department of Mathematics, Quaid-i-Azam University, Islamabad in partial fulfillment of the requirements for the degree of Doctor of Philosophy in field of Mathematics from Department of Mathematics, Quaid-i-Azam University Islamabad, Pakistan.

Student Name: **Wahid Ullah** Signature:

External committee:

a) **External Examiner 1**: Signature:
Name: **Dr. Mujeeb Ur Rehman**
Designation: Associate Professor
Office Address: School of Natural Sciences, National University of Sciences and Technology (NUST), Islamabad.

b) **External Examiner 2**: Signature:
Name**: Dr. Tahir Mahmood**
Designation: Assistant Professor
Office Address: Department of Mathematics & Statistics, International Islamic University, Islamabad.

c) **Internal Examiner:** Signature:
Name: **Dr. Maria Samreen**
Designation: Assistant Professor
Office Address: Department of Mathematics, Quaid-I-Azam University, Islamabad.

Supervisor Name: Signature:
Dr. Maria Samreen

Name of Dean/ HOD Signature:

Prof. Dr. Tariq Shah

This thesis is dedicated

to

My Family

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Preface

The theory of fixed point is a growing field of research with several applications in various fields. It is concerned with the results which state that a single-valued dynamical systems $\digamma : \mathfrak{Q} \to \mathfrak{Q}$ or a multi-valued dynamical system $\digamma : \mathfrak{Q} \to P(\mathfrak{Q})$ admits one or more fixed points under particular circumstances. The necessity to prove theorems about the existence of solutions to differential and integral equations drove the further growth in this theory. There are three major topics of theory of fixed points: Metric, Topological and Discrete fixed point theory. Some of the most well-known and significant results in these fields are: Banach, Brouwer and Tarski fixed point theorems respectively. In 1922, Banach was working on integral equations and proved a theorem known as the Banach contraction principle, which guarantee to exists a unique fixed point in a complete metric space. The Banach contraction principle is a very useful tool in nonlinear analysis with many applications to operator equations, fractal theory, optimization theory and other topics.

After Banach, many researchers introduced new type of contractions in metric spaces. It has been observed that a Banach contraction $\digamma$ is always a continuous map. This brings up the question whether some contraction conditions exist which guarantee to exists of unique fixed point of discontinuous mappings. In 1968, Kannan and Cheatterja gave positive answer to this question for complete metric spaces. Another important contraction in this perspective which generalizes both Banach and Kannan contractions for a complete metric spaces was proved by Reich in 1971. Due to the wide range of applications of Banach contraction principle, many authors have refined the contraction condition or changed the metric space to different abstract spaces to generalize/extend this elegant result.

Nadler extended first time the Banach contraction for the multi-valued dynamical systems *i.e.*, $\digamma : \mathfrak{Q} \to CB(\mathfrak{Q})$, where $CB(\mathfrak{Q}) = \{\mathcal{M} \subseteq \mathfrak{Q} \,:\, \mathcal{M} \text{ is bounded and closed}\}$. He proved for a complete metric space $(\mathfrak{Q}, \rho)$ that if a map $\digamma : \mathfrak{Q} \to CB(\mathfrak{Q})$ satisfies the following condition:

$$H(\digamma p, \digamma q) \leq \kappa \rho(p, q)$$

for each $p, q \in \mathfrak{Q}$ where H is a Hausdorff metric and $k \in [0, 1)$, then there is a point $u_0 \in \mathfrak{Q}$ which is a $\mathcal{F} \cdot \mathcal{P}$ of the map $\digamma$. After Nadler, a number of authors worked in this direction. Some of the refinements of Nadler fixed point theorem are by Reich, where he used $H(\mathfrak{Q})$ the collection of all compact subsets of a metric space $\mathfrak{Q}$ and by Kamran who used $Cl(\mathfrak{Q})$, the collection of all non-void closed subsets of $\mathfrak{Q}$ instead of $CB(\mathfrak{Q})$.

Due to the importance of fixed point theory in diverse fields, some researchers have extended the idea of metric space in various ways. In 1993, Czerwik introduced the notion of a b-metric space by replacing the triangular property of a metric space with $\rho(p, t) \leq b[\rho(p, q) + \rho(q, t)]$, where $b \geq 1$. Later on, in 2017 Kamran *et al.* further extended the concept of b-metric space by introducing extended b-metric spaces. They introduced a function $\theta : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty)$ instead of b in triangular inequality condition. In 2018, Mlaiki *et al.* gave the idea of controlled metric type spaces. They used $\theta : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty)$ instead of b in triangular inequality condition of metric spaces from a different approach from Kamran *et al.*

In 2007, Huang and Zhang initiated the concept of cone metric space over a Banach space as the generalization of metric spaces. They used ordered Banach space E instead of $\mathbb{R}$ as the range set of metric ρ, *i.e.* they used $\rho : \mathfrak{Q} \times \mathfrak{Q} \to E$. They also discussed Banach type contraction and proved some fixed point results. After that, many researcher concentrated to investigate such spaces and proved a number of fixed point theorems. According to rough statistics, by using cone metric spaces, more than six hundred articles have been published. But recently some scholars obtained the equivalent results of usual metric space $(\mathfrak{Q}, d^*)$ and that of cone metric space $(\mathfrak{Q}, \rho)$. They defined the real valued metric function d^* as the non-linear scalarization function ξ. However, Liu and Xu in 2013 introduced cone metric space by using a real Banach algebra instead of Banach space and defined generalized Lipschitz mapping. They presented an example which established that results of fixed point in metric spaces are not equivalent to that of results in cone metric spaces over Banach algebras.

The concept of distances in uniform spaces and metric spaces was first time presented by Valyi in 1985. We call it a Valyi-distances. After Valyi, some other researchers introduced different type of distances in metric spaces and in uniform spaces. Some well known distances are Tataru-distances by Tataru in 1992, ω-distance by Kada in 1996 and τ-distance by Suzuki in 2001. Recently in 2010 Wlodarczyk gave an idea of distances which provide a handy research tool to obtain more general results with weaker assumptions in uniform space known as generalized pseudo-distances. He also introduced generalized Hausdorff distances, gauge spaces, quasi-gauge spaces, triangular spaces, quasi-triangular spaces.

The main objective of this thesis is to prove some fixed point theorems and proximity fixed point theorems for single-valued and multi-valued dynamical systems in metric type spaces. This thesis has been organized into six chapters.

In Chapter 1, We have recollect some fundamental notions, some well-known contractions, abstract spaces and results in such spaces. Also, we present some basic concepts of comparison functions, introduction and basic theory of fractals in metric type spaces. At the end, we gave the theory of proximity fixed point in metric type spaces and generalized distances.

In Chapter 2, we introduced a new geometrical structure which is the hybrid of cone metric space over Banach algebra and extended b-metric space. We prove analogues of

Banach, Kannan and Reich type fixed point theorems in our predefined space. We also furnish with various concrete examples to establish the validity of our results. At the end, we have added some consequences and applications of our results. Recently, this work has been published in the journal of Filomat.

Chapter 3, is concerned with the study of a new type of metric type space which we call a controlled cone metric type space over Banach algebra. By using such spaces we proved some fixed point theorems for generalized R-type contraction and generalized lipschitz mappings. We add an example to show the validity of our results. Work of this chapter has been published in the Journal of Inequalities and Applications.

The aim of chapter 4 is two fold. Firstly, we produced several results concern with fixed point for the family of multi-valued contractions by using comparison functions in extended b-metric spaces. Then, we constructed some new multi-valued fractals based on a fixed point approach in the framework of extended b-metric spaces. Later on, using the idea of well-posed problem of fixed point is studied. Some of the results of this chapter has been published in the Journal of function spaces.

Chapter 5 is intended to the study of theory of proximity points in controlled metric type spaces. We introduced generalized distances in controlled metric type spaces. We proved some global maximality results by using the defined generalized distances.

Chapter 6 is the last chapter of this thesis, where we have introduced a new type of space which we named controlled quasi-triangular space. We introduced left(right) families generated by controlled quasi-triangular space. We proved Banach type theorem by using such families in controlled quasi-triangular space. At the end, we gave some concrete examples to validate our definitions and results.

Contents

Chapter 1

Introduction and Preliminaries

The aim of this chapter is to recollect some definitions, results and their origin which are needed in the sequel. Throughout in this thesis, by $N(\mathfrak{Q})$, $CB(\mathfrak{Q})$ and $H(\mathfrak{Q})$, we mean the collection of all the nonempty subsets of $\mathfrak{Q}$, the collection of all the closed and bounded subsets of $\mathfrak{Q}$ and the collection of all compact subsets of $\mathfrak{Q}$ respectively. By $\mathcal{F}\cdot\mathcal{P}$ we mean fixed point, by $\mathcal{B}\cdot\mathcal{P}\cdot\mathcal{P}$ we mean best proximity point and by an $\mathcal{M}\cdot\mathcal{S}$ we mean a metric space. We denote the set of non-negative real numbers by $\mathbb{R}_+$.

1.1 Fixed points of dynamical systems

By a single-valued dynamical system, we mean a pair $(\mathfrak{Q}, F)$, where $\mathfrak{Q}$ is a phase space and $F : \mathfrak{Q} \to \mathfrak{Q}$ is a single-valued map. We think of $\mathfrak{Q}$ as a phase space of possible states of the system and the map F as a "law of evolution" of the system. The set of $\mathcal{F}\cdot\mathcal{P}$ of a single-valued dynamical system F is defined as $\text{Fix}(F) = \{u \in \mathfrak{Q} : Fu = u\}$. Similarly, by a multi-valued (set-valued) dynamical system we mean the pair $(\mathfrak{Q}, F)$, where $\mathfrak{Q}$ is a phase space and $F : \mathfrak{Q} \to 2^{\mathfrak{Q}}$ is a multi-valued map: here $2^{\mathfrak{Q}} = \{\mathcal{M} \subseteq \mathfrak{Q} : \mathcal{M} \neq \emptyset\}$. The collection of all $\mathcal{F}\cdot\mathcal{P}$ of F is defined as $\text{Fix}(F) = \{u \in \mathfrak{Q} : u \in Fu\}$. Given $\varpi_0 \in \mathfrak{Q}$ and a single-valued dynamical system $(\mathfrak{Q}, F)$, the orbit of ϖ_0 is defined as:

$$O(\varpi_0) = \{\varpi_n = F\varpi_{n-1} = F^n(\varpi_0) \; : \; n \in \mathbb{N} \cup \{0\}\}.$$

For multi-valued dynamical system the orbit is defined as follows:

$$O(\varpi_0) = \{\varpi_n : \varpi_n \in \digamma \varpi_{n-1} : n \in \mathbb{N} \cup \{0\}\}.$$

The main goal of the theory of dynamical system is to describe and classify the possible structure which arise from the iteration of single-valued and multi-valued maps.

1.2 Some well-known contractions in metric spaces

In 1932 Banach [6] introduced a principle, the "Banach contraction" which is pioneer of the theory of metric fixed point.

1.2.1 Banach contraction

Let $(\mathfrak{Q}, \rho)$ be a $\mathcal{M} \cdot \mathcal{S}$ and $\digamma : \mathfrak{Q} \to \mathfrak{Q}$ be a dynamical system. We say that $\digamma$ is a Banach contraction if

$$\{\exists \lambda \in [0,1) \text{ such that } \forall\, v, t \in \mathfrak{Q},\ \rho(\digamma v, \digamma t) \leq \lambda \rho(v, t)\}. \tag{1.2.1}$$

Banach proved that if $\mathfrak{Q}$ is complete and $\digamma : \mathfrak{Q} \to \mathfrak{Q}$ satisfy the contraction (1.2.1), then there exists a unique u_0 in $\mathfrak{Q}$ which is a $\mathcal{F} \cdot \mathcal{P}$ of the map $\digamma$.

After Banach, many researchers introduce new contractions for $\mathcal{M} \cdot \mathcal{S}$s. One well known contraction for complete $\mathcal{M} \cdot \mathcal{S}$ is presented by Kannan [28].

1.2.2 Kannan contraction

Let $(\mathfrak{Q}, \rho)$ be a $\mathcal{M} \cdot \mathcal{S}$ and $\digamma : \mathfrak{Q} \to \mathfrak{Q}$ be a dynamical system. We say that $\digamma$ is a Kannan contraction if there is a $\lambda \in [0, \frac{1}{2})$ such that

$$\rho(\digamma u, \digamma v) \leq \lambda \cdot [\rho(u, \digamma u) + \rho(v, \digamma v)] \ \ \forall u, v \in \mathfrak{Q}. \tag{1.2.2}$$

Kannan proved that if $\mathfrak{Q}$ is complete and $\digamma : \mathfrak{Q} \to \mathfrak{Q}$ satisfy the contraction (1.2.2), then there exists a unique point u_0 in $\mathfrak{Q}$ which is a $\mathcal{F} \cdot \mathcal{P}$ of the map $\digamma$.

One another important result in this perspective which generalizes both Banach and Kannan contractions for a complete $\mathcal{M} \cdot \mathcal{S}$ $(\mathfrak{Q}, \rho)$ was proved by Reich [41] in 1971.

1.2.3 Reich contraction

Let $(\mathfrak{Q}, \rho)$ be a $\mathcal{M} \cdot \mathcal{S}$ and $F : \mathfrak{Q} \to \mathfrak{Q}$ be a dynamical system. F is said to be a Reich contraction if for all $u, v \in \mathfrak{Q}$, there exist three non-negative real numbers α, β, γ such that $\alpha + \beta + \gamma < 1$ and the following inequality holds:

$$\rho(Fu, Fv) \leq \alpha\rho(u, Fu) + \beta\rho(v, Fv) + \gamma\rho(u, v). \tag{1.2.3}$$

Reich proved that if $\mathfrak{Q}$ is complete and $F : \mathfrak{Q} \to \mathfrak{Q}$ satisfy the contraction (1.2.3), then there exists a unique point in $\mathfrak{Q}$ which is a $\mathcal{F} \cdot \mathcal{P}$ of the map F.
If we put $\alpha = \beta = 0$ in (1.2.3), then we get the Banach contraction (1.2.1) and for $\alpha = \beta,\ \gamma = 0$ we get Kannan contraction (1.2.2).

1.2.4 Nadler contraction

Nadler [37] extended first time the Banach contraction for the multi-valued mapping $F : \mathfrak{Q} \to CB(\mathfrak{Q})$. He used Hausdorff metric on a $\mathcal{M} \cdot \mathcal{S}$ $(\mathfrak{Q}, \rho)$ to establish the result of $\mathcal{F} \cdot \mathcal{P}$ for multi-valued dynamical systems. The Hausdorff metric H on $CB(\mathfrak{Q})$ is denoted and defined as

$$\{\forall\, U, V \in CB(\mathfrak{Q}),\ H(U, V) = \max\{D(U, V), D(V, U)\}\},$$

where $D(U, V) = \sup_{a \in U} \rho(a, V)$ and $\rho(a, V) = \inf_{b \in V} \rho(a, b)$.

Definition 1.2.1. Let $F : \mathfrak{Q} \to CB(\mathfrak{Q})$ be a multi-valued dynamical system with a $\mathcal{M} \cdot \mathcal{S}$ $(\mathfrak{Q}, \rho)$. F is said to be a Nadler's contraction if for all $p, q \in \mathfrak{Q}$, the following inequality holds:

$$H(Fp, Fq) \leq \kappa\rho(p, q), \tag{1.2.4}$$

where H is a Hausdorff metric and $\kappa \in [0, 1)$.

Nadler proved that if $(\mathfrak{Q}, \rho)$ is complete and $F : \mathfrak{Q} \to CB(\mathfrak{Q})$ satisfy (1.2.4), then there is a point $u \in \mathfrak{Q}$ which is fixed under F.

After Nadler, a number of authors worked in this direction. Some of the refinement of Nadler $\mathcal{F} \cdot \mathcal{P}$ theorem are by Reich [42], where he used $H(\mathfrak{Q})$ the collection of all compact subsets of a $\mathcal{M} \cdot \mathcal{S}$ $\mathfrak{Q}$ and by Kamran [18] who used $Cl(\mathfrak{Q})$, the collection of all nonempty closed subsets of $\mathfrak{Q}$ instead of $CB(\mathfrak{Q})$.

1.3 Some abstract spaces

This section consists of some well-known generalizations of $\mathcal{M} \cdot \mathcal{S}$ which we will use in the upcoming chapters.

1.3.1 b-metric spaces

By transforming the condition of triangle inequality of $\mathcal{M} \cdot \mathcal{S}$s, Czerwik [14] introduced first time the idea of b-$\mathcal{M} \cdot \mathcal{S}$.

Definition 1.3.1. For any non-empty set $\mathfrak{Q}$, a b-metric on $\mathfrak{Q}$ is a function $d_b : \mathfrak{Q} \times \mathfrak{Q} \to \mathbb{R}_+$ satisfying the following axioms:

$B_1 : d_b(p, v) = 0$ iff $p = v \quad : \quad \forall\, p, v \in \mathfrak{Q};$

$B_2 : d_b(p, v) = d_b(v, p) \quad : \quad \forall\, p, v \in \mathfrak{Q};$

$B_3 : \exists\, b \geq 1$ such that $d_b(p, u) \leq b[d_b(p, v) + d_b(v, u)] \quad : \quad \forall\, p, v, u \in \mathfrak{Q}.$

The pair $(\mathfrak{Q}, d_b)$ is then termed as b-metric space with coefficient b.

Example 1.3.2. Let $\mathfrak{Q} = \mathbb{R}$ and $d_b : \mathfrak{Q} \times \mathfrak{Q} \to \mathbb{R}$ defined as

$$d(\eta, \xi) = (\eta - \xi)^2.$$

Then d_b is a b-metric with $b = 2$, and d_b is not a metric on $\mathfrak{Q}$.

Every metric is clearly a b-metric with $b = 1$. Apparently, one can say that the class of b-$\mathcal{M} \cdot \mathcal{S}$s is super-class of the class of $\mathcal{M} \cdot \mathcal{S}$s.

1.3.2 Extended b-metric spaces

In 2017, Kamran *et al.*[29] more generalized the idea of a b-$\mathcal{M}\cdot\mathcal{S}$s by introducing a map $s : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty)$ instead of $b \geq 1$ in b-$\mathcal{M}\cdot\mathcal{S}$s. They called this space, an extended b-$\mathcal{M}\cdot\mathcal{S}$.

Definition 1.3.3. Let $\mathfrak{Q}$ be a non empty set and $s : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty)$. A function $d_s : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ is called an extended b-metric (in short $Eb-M$) if for all $p, v, t \in \mathfrak{Q}$ it satisfies:

(i) $d_s(p, v) = 0$ iff $p = v$;

(ii) $d_s(p, v) = d_s(v, p)$;

(iii) $d_s(p, t) \leq s(p, t)[d_s(p, v) + d_s(v, t)]$.

The pair $(\mathfrak{Q}, d_s)$ is then called an $Eb - M$ space (extended b-metric space).

If $\forall p_1, p_2 \in \mathfrak{Q}, \quad s(p_1, p_2) = b$ for some $b \geq 1$, then the Definition 1.3.3 becomes equivalent to Definition 1.3.1 with coefficient b.

Example 1.3.4. [47] Let $\mathfrak{Q} = \mathbb{N}$ and $d_s : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ be defined by by $d_s(\sigma, \varpi) = (\sigma - \varpi)^4$. Define $s : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty)$ as follow:

$$s(\sigma, \varpi) = \begin{cases} |\sigma - \varpi|^3 & \text{if } \sigma \neq \varpi \text{ ;} \\ 1 & \text{if } \sigma = \varpi. \end{cases}$$

Then $(\mathfrak{Q}, d_s)$ is an $Eb - M$ space. space.

Definition 1.3.5. [29] Consider an $Eb - M$ space $(\mathfrak{Q}, d_s)$. A sequence $\{\sigma_r\}$ in $\mathfrak{Q}$ is said to be:

(i) convergent which converges to some $\sigma \in \mathfrak{Q}$ iff $d_s(\sigma_r, \sigma) \to 0$ as $r \to \infty$, we write $\lim_{r\to\infty} \sigma_r = \sigma$;

(ii) Cauchy sequence if $d_s(\sigma_r, \sigma_k) \to 0$ as $r, k \to \infty$.

If every Cauchy sequence in $\mathfrak{Q}$ converges in $\mathfrak{Q}$ with respect to d_s, then we say that the $Eb-M$ space $(\mathfrak{Q}, d_s)$ is complete. It has been noted that the $Eb-M$ d_s is not always continuous and every convergent sequence has a unique limit in $\mathfrak{Q}$. Following is the main theorem in [29] related to an $Eb-M$ spaces.

Theorem 1.3.6. [29] Let $(\mathfrak{Q}, d_s)$ be a complete $Eb-M$ space with d_s continuous. Let F be a self-map on $\mathfrak{Q}$ which satisfy

$$d_s(F\eta, F\xi) \leq \kappa d_s(\eta, \xi) \quad \text{for all } \eta, \xi \in \mathfrak{Q}, \tag{1.3.1}$$

where $\kappa \in [0, 1)$ be such that for each $t_0 \in \mathfrak{Q}$, $\lim_{j,i\to\infty} s(t_{j+1}, t_i) < \frac{1}{\kappa}$, here $t_j = F^j t_0$, $j = 1, 2, \cdots$. Then F has precisely one $\mathcal{F} \cdot \mathcal{P}$ ϱ. Moreover for each $y \in \mathfrak{Q}$, the iterative sequence $F^j y$ converges to ϱ.

1.3.3 Controlled metric type spaces

After Kamran *et al.*, in 2018, Mlaiki *et al.* [36] introduced controlled metric type spaces by different approach to that of Kamran.

Definition 1.3.7. [36] Let $\mathfrak{Q}$ be a non empty set and $s : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty)$. A controlled metric type (CMT) is a function $d_s : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ such that for all $\sigma, \varrho, \varpi \in \mathfrak{Q}$ it satisfies the following:

(i) $d_s(\sigma, \varrho) = 0$ iff $\sigma = \varrho$;

(ii) $d_s(\sigma, \varrho) = d_s(\varrho, \sigma)$;

(iii) $d_s(\sigma, \varpi) \leq s(\sigma, \varrho) d_s(\sigma, \varrho) + s(\varrho, \varpi) d_s(\varrho, \varpi)$.

The pair $(\mathfrak{Q}, d_s)$ is then called a CMT-space (controlled metric type space).

Remark 1.3.8.
1. If $\forall\, p, q \in \mathfrak{Q}, s(p, q) = b$ for some $b \geq 1$, then the Definition 1.3.7 coincides with the Definition 1.3.1.
2. Mlaiki *et al.* gave an example which provides that CMT and $Eb-M$ are two different notions.

In the present work, throughout we assume that the CMT d_s is continuous on $\mathfrak{Q}\times\mathfrak{Q}$.

Definition 1.3.9. [36] Let $(\mathfrak{Q}, d_s)$ be a CMT space. We say that a sequence σ_n is a:

(i) convergent sequence and converges to σ if and only if for every $\epsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ such that $d_s(\sigma_n, \sigma) < \epsilon$ for all $n \geq n_0$. We write $\lim_{n\to\infty} \sigma_n = \sigma$;

(ii) Cauchy sequence if $d_s(\sigma_r, \sigma_k) \to 0$ as $r, k \to \infty$.

If every Cauchy sequence in $\mathfrak{Q}$ converges in $\mathfrak{Q}$, then the CMT space $(\mathfrak{Q}, d_s)$ is said to be complete. The main result of Mlaiki *et al*. [36] is given below.

Theorem 1.3.10. [36] Let $(\mathfrak{Q}, d_s)$ be a complete CMT space with d_s continuous. Let $\digamma : \mathfrak{Q} \to \mathfrak{Q}$ satisfy

$$d_s(\digamma\varrho, \digamma\varpi) \leq \kappa d_s(\varrho, \varpi) \quad \text{for all } \varrho, \varpi \in \mathfrak{Q}, \tag{1.3.2}$$

where $\kappa \in [0, 1)$ be such that for each $\sigma_0 \in \mathfrak{Q}$,

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{s(\sigma_{i+1}, \sigma_{i+2})}{s(\sigma_i, \sigma_{i+1})} s(\sigma_{i+1}, \sigma_m) < \frac{1}{\kappa}, \tag{1.3.3}$$

here $\sigma_i = \digamma^i\sigma_0$. Furthermore, assume that for every $q \in \mathfrak{Q}$, we have

$$\lim_{n\to\infty} s(\sigma_n, q) \text{ and } \lim_{n\to\infty} s(q, \sigma_n), \tag{1.3.4}$$

exist and finite. Then $\digamma$ has only one $\mathcal{F}\cdot\mathcal{P}$ σ.

1.3.4 Cone b-metric space over Banach algebra

Before defining cone b-$\mathcal{M}\cdot\mathcal{S}$ (in short $CbMS$)over Banach algebra, we recall some basic definitions and notions from the theory of Banach algebras [43].
Let $\mathcal{A}$ be a real Banach algebra with zero element ϑ. A cone $\mathfrak{P}$ in $\mathcal{A}$ is a nonempty closed subset of $\mathcal{A}$ such that $\mathfrak{P} \cap (-\mathfrak{P}) = \vartheta$, $\mathfrak{P} + \mathfrak{P} \subseteq \mathfrak{P}$, $\mathfrak{P}\cdot\mathfrak{P} \subseteq \mathfrak{P}$ and $\mu\mathfrak{P} \subseteq \mathfrak{P}$ for all $\mu \geq 0$. If the interior of $\mathfrak{P}$ denoted by int$\mathfrak{P}$ is nonempty, then the cone $\mathfrak{P}$ is called a solid cone. If we define a relation $\preceq$ on $\mathcal{A}$ by $\varsigma \preceq \varpi$ iff $\varpi - \varsigma \in \mathfrak{P}$, then $\preceq$ is a partial order on $\mathcal{A}$. We write $\varsigma \preceq c\varpi$ iff $\varpi - \varsigma \in \mathfrak{P}$ and $\varsigma \neq \varpi$. Define another

partial order $\ll$ on $\mathcal{A}$ by $\varsigma \ll \varpi$ iff $\varpi - \varsigma \in \text{int}\mathfrak{P}$. A cone $\mathfrak{P}$ in $\mathcal{A}$ is said to be a normal cone if for all $\varsigma, \varpi \in \mathcal{A}$ with $\vartheta \preceq \varsigma \preceq \varpi$, there exists a real number $M > 0$ such that $||\varsigma|| \leq M||\varpi||$. The normal constant of $\mathfrak{P}$ is the least positive constant M for which the above inequality holds.

Consider a unital Banach algebra $\mathcal{A}$ with identity element e. An element ς in $\mathcal{A}$ is said to be invertible if there exists ϖ in $\mathcal{A}$ such that $\varsigma\varpi = \varpi\varsigma = e$. A complex number $\mu \in \mathbb{C}$ is said to be spectral value of $\varpi \in \mathcal{A}$ if $\varpi - \mu e$ is non-invertible in $\mathcal{A}$. The set of all spectral values of $\varpi \in \mathcal{A}$ denoted by $\sigma(\varpi)$ is called the spectrum of ϖ. The number $r_\sigma(\varpi)$ (or $r(\varpi)$) defined by $r_\sigma(\varpi) = \sup\{|\mu| \ : \ \mu \in \sigma(\varpi)\}$ is called the spectral radius of $\varpi \in \mathcal{A}$.

Lemma 1.3.11. ([43]) Let $\mathcal{A}$ be a Banach algebra with identity e. Then the spectral radius $r(\varrho)$ of $\varrho \in \mathcal{A}$ satisfies:

$$r(\varrho) = \lim_{n\to\infty} ||\varrho^n||^{1/n}. \tag{1.3.5}$$

Furthermore, if $r(\varpi) < |\mu|$ for some $\varpi \in \mathcal{A}$, then $(\mu e - \varpi)$ is invertible,

$$(\mu e - \varpi)^{-1} = \sum_{i=0}^{\infty} \frac{\varpi^i}{\mu^{i+1}} \quad \text{and} \quad r[(\mu e - \varpi)^{-1}] \leq \frac{1}{|\mu| - r(\varpi)}.$$

Lemma 1.3.12. [43] Let $\mathcal{A}$ be a Banach algebra and $\varpi, \varrho \in \mathcal{A}$ be such that ϖ and ϱ commute. Then we have

$$r(\varpi + \varrho) \leq r(\varpi) + r(\varrho) \qquad r(\varpi\varrho) \leq r(\varpi)r(\varrho).$$

Definition 1.3.13. ([27]) Let $\mathcal{A}$ be a Banach algebra with solid cone $\mathfrak{P}$. A c-sequence is a sequence $\{\varpi_i\}$ in $\mathfrak{P}$ such that for every $c \in \mathcal{A}$ with $c \gg \vartheta$, there exists $k \in \mathbb{N}$ such that

$$\varpi_i \ll c \qquad \forall i \geq k.$$

Lemma 1.3.14. ([23]) Let α, $\beta \in \mathfrak{P}$ be any two arbitrary vectors and $\{u_n\}$, $\{q_n\}$ be two c-sequences in a solid cone $\mathfrak{P}$ of a Banach algebra $\mathcal{A}$. Then $\{\alpha u_n + \beta q_n\}$ is a c-sequence.

Lemma 1.3.15. ([61]) Let $\mathfrak{P}$ be a cone in a Banach algebra $\mathcal{A}$ (not necessary a normal cone). Then the following assertions hold:

(u_1) If for each c with $c \gg \vartheta$ and $\vartheta \preceq \varpi \ll c$, implies that $\varpi = \vartheta$.

(u_2) If $\varpi \in \mathfrak{P}$ is such that $r(\varpi) < 1$, then $||\varpi^j|| \to 0$ as $j \to \infty$.

(u_3) Let $c \in int\mathfrak{P}$ and $\varpi_j \to \vartheta$ in $\mathcal{A}$ as $j \to \infty$. Then $\exists\ M \in \mathbb{N}$ such that $\forall j \geq M$, $\varpi_j \ll c$.

(u_4) If $\varpi \preceq \varpi k$, where $\varpi, k \in \mathfrak{P}$ and $r(k) < 1$, then $\varpi = \vartheta$.

Definition 1.3.16. [23] For a nonempty set $\mathfrak{Q}$ and a constant $b \geq 1$. A mapping $d_b : \mathfrak{Q} \times \mathfrak{Q} \to \mathcal{A}$ is called a $CbMS$ over a Banach algebra $\mathcal{A}$ if the following axioms hold:

$B_1 : \quad \forall\ \eta, \xi \in \mathfrak{Q}, \quad d_b(\eta, \xi) \succeq \vartheta$ and $d_b(\eta, \xi) = \vartheta$ iff $\eta = \xi$;

$B_2 : \quad \forall\ \eta, \xi \in \mathfrak{Q}, \quad d_b(\eta, \xi) = d_b(\xi, \eta)$;

$B_3 : \quad \forall\ \eta, \xi, \zeta \in \mathfrak{Q},\ d_b(\eta, \zeta) \preceq b[d_b(\eta, \xi) + d_b(\xi, \zeta)]$.

The pair $(\mathfrak{Q}, d_b)$ is called a $CbMS$ over a Banach algebra $\mathcal{A}$ (in short $CbMS$ over $\mathcal{A}$).

Remark 1.3.17. If $b = 1$, then we say that d_1 is a cone metric over a Banach algebra $\mathcal{A}$. So we can say that cone b-metric is the generalization of a cone metric.

Example 1.3.18. Consider the Banach algebra $\mathcal{A} = C([0, 1])$ with unit element $e(t) = 1$ and supremum norm where multiplication is defined point wise. Let $\mathfrak{Q} = \mathbb{R}$ and $\mathfrak{P} = \{f \in \mathcal{A} : f(h) \geq 0\ ; \forall h \in [0, 1]\}$. Define $d_b : \mathfrak{Q} \times \mathfrak{Q} \to \mathcal{A}$ by

$$d_b(\eta, \xi)(\varpi) = |\eta - \xi|^a e^{\varpi} \quad \forall \eta, \xi \in \mathfrak{Q}\ \&\ a > 1.$$

Then d_b is a $CbMS$ over $\mathcal{A}$ with $b = 2^{a-1}$ but it is not a cone metric on $\mathfrak{Q}$.

Definition 1.3.19. ([23]) Let $\{\sigma_k\}$ be a sequence in $\mathfrak{Q}$ where $(\mathfrak{Q}, d_b)$ is a $CbMS$ over $\mathcal{A}$. We say that $\{\sigma_k\}$ is:

(i) a convergent sequence which converges to $\sigma \in \mathfrak{Q}$ if for every $c \in \text{int}\mathfrak{P}$ $(i.e. \vartheta \ll c)$, $\exists\ N \in \mathbb{N}$ such that $d_b(\sigma_k, \sigma) \ll c$ for all $k \geq N$;

(ii) a Cauchy sequence if for every $c \in \text{int}\mathfrak{P}$ $(i.e. \vartheta \ll c)$, there exists a natural number N such that $d_b(\sigma_k, \sigma_i) \ll c$ for all $k,\ i \geq N$.

If every Cauchy sequence in $\mathfrak{Q}$ is convergent in $\mathfrak{Q}$, then the space $(\mathfrak{Q}, d_b)$ is called a complete $CbMS$ over $\mathcal{A}$.

Remark 1.3.20. [23, 61] 1. If $\{\sigma_n\}$ converges to σ in $\mathfrak{Q}$, then $\{d_b(\sigma_k, \sigma)\}$ and $\{d_b(\sigma_k, \sigma_{k+i})\}$ are c-sequences for any $i \in \mathbb{N}$.
2. If $||\sigma_k|| \to 0$ as $k \to \infty$, then for any $c \gg \vartheta$, there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $\sigma_k \ll c$.

Theorem 1.3.21. [23] Let $(\mathfrak{Q}, d)$ be a complete $CbMS$ over $\mathcal{A}$ with coefficient $b \geq 1$ and $\mathfrak{P}$ be the associated solid cone (not necessary normal) in $\mathcal{A}$. Suppose that a mapping $F : \mathfrak{Q} \to \mathfrak{Q}$ satisfies one of the following generalized Lipschitz conditions for all $\sigma,\ \varrho \in \mathfrak{Q}$:

(i) $d(F\sigma, F\varrho) \preceq \kappa d(\sigma, \varrho)$ where $\kappa \in \mathfrak{P}$ be such that $r(\kappa) < \dfrac{1}{b}$.

(ii) $d(F\sigma, F\varrho) \preceq \kappa(d(F\sigma, \sigma) + d(F\varrho, \varrho))$ where $\kappa \in \mathfrak{P}$ be such that $r(\kappa) < \dfrac{1}{1+b}$.

Then there exists a unique point $\varpi \in \mathfrak{Q}$ which is a $\mathcal{F} \cdot \mathcal{P}$ of the map F.

Now we want to recall the definition of generalized α-admissible, α-regular and generalized R-type mapping in the setting of cone b-$\mathcal{M} \cdot \mathcal{S}$s over Banach algebras.

Definition 1.3.22. [56] Let $(\mathfrak{Q}, d_b)$ be an cone b-$\mathcal{M} \cdot \mathcal{S}$ over a Banach algebra $\mathcal{A}$ with $\mathfrak{P}$ an underlying solid cone. Let $\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ and $F : \mathfrak{Q} \to \mathfrak{Q}$ be mappings. Then:

(i) F is said to be a generalized α-admissible mapping if for $p, q \in \mathfrak{Q}, \alpha(p, q) \geq b$ implies that $\alpha(Fp, Fq) \geq b$;

(ii) $(\mathfrak{Q}, d_b)$ is said to be α-regular if any sequence $\{u_k\} \in \mathfrak{Q}$ with $\alpha(u_k, u_{k+1}) \geq b$ for all $k \in \mathbb{N}$ and $u_k \to p$ implies that $\alpha(u_k, p) \geq b$.

Definition 1.3.23. [56] Let $(\mathfrak{Q}, d_b)$ be a cone b-$\mathcal{M} \cdot \mathcal{S}$ over a Banach algebra $\mathcal{A}$ with coefficient b, $\mathfrak{P}$ an underlying solid cone and $\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ be mapping. Then the mapping $F : \mathfrak{Q} \to \mathfrak{Q}$ is called a generalized Reich type contraction if there exists $v_1, v_2, v_3 \in \mathfrak{P}$ such that for all $p, q \in \mathfrak{Q}$ with $\alpha(p, q) \geq b$:

(i) $2br(v_1) + (b + 1)r(v_2 + v_3) < 2$;

(ii) $d(Fp, Fq) \preceq v_1 d(p, q) + v_2 d(p, Fp) + v_3 d(q, Fq)$.

1.4 Comparison functions in metric type spaces

An increasing self-map ϕ on $[0,\infty)$ is termed as a comparison function if for all $p \in [0,\infty)$, $\lim_{r\to\infty} \phi^r(p) = 0$, see [35].
An increasing self-map ϕ on $[0,\infty)$ is said to be a c-comparison function if for every $p > 0$, the following series converges

$$\sum_{r=1}^{\infty} \phi^r(p).$$

It is evident from the above definitions that every c-comparison function is itself a comparison function but the converse is not be true in general, see example in [44].
Now consider an increasing self-map ϕ on $[0,\infty)$ and a b-$\mathcal{M}\cdot\mathcal{S}$ $(\mathfrak{Q}, d_b)$. The map ϕ is called a b-comparison function if for all $\varrho \in [0,\infty)$, the following series converges ([8, 44])

$$\sum_{r=0}^{\infty} b^r \phi^r(\varrho).$$

Let $(\mathfrak{Q}, d_b)$ be a b-$\mathcal{M}\cdot\mathcal{S}$ with $b \geq 1$ and let $0 < p < \dfrac{1}{b}$. Then the function $\phi(\eta) = p\eta$ is a b-comparison function.
We noted that for $b = 1$, the defined b-comparison function becomes equivalent to the definition of a comparison function.
Next, in $Eb - M$ spaces, we define the idea of F-orbital lower semi-continuity (lsc), which we will utilise in the next chapters.

Definition 1.4.1. [21] Let $F : D \subset \mathfrak{Q} \to \mathfrak{Q}$, $\varpi_0 \in D$ and the orbit of $\varpi_0 \in D$, $\mathcal{O}(\varpi_0) = \{\varpi_0, F(\varpi_0), F^2\varpi_0, \cdots\} \subset D$. A function $G : D \to \mathbb{R}$ is called F-orbitally lsc at $v \in D$ if $\varpi_r \to v$ and $(\varpi_r) \subset \mathcal{O}(\varpi_0)$ implies $G(v) \leq \lim_{r\to\infty} \inf G(\varpi_r)$.

1.5 Fractals and multi-fractals in metric spaces

Fractals and multi-fractals play an important role in a variety of applications including digital photography, fluid mechanics, soil mechanics, dynamical systems, computer graphics, signal and image compression, and computer graphics etc. We can obtain most of these fractals (multi-fractals) by using the approach of iterated function (multi-function) systems IFS (IMFS). Hutchinson [25] in 1981 defined first time the iterated function systems (IFS) and Barnsley [7] developed further the iterated function systems theory. This theory is called the Hutchinson–Barnsley (HB) theory. The collection of finite number of contractive self mappings is said to by IFS by Hutchinson. He also introduced the notion of HB operator which is defined on the hyper space of nonempty compact sets. He defined a fractal (attractor) to be a unique fixed point of the HB operator. The theory of $\mathcal{F}\cdot\mathcal{P}$s plays a prominent role in order to construct new fractals. For years, IFS has been used by the researchers to develop different new techniques and generate new fractal objects. To construct fractals and self-similar sets, numerous development, new results and extensions of IFS are made, see for example ([9, 12, 13]).

Let $(\mathfrak{Q}, d)$ be a $\mathcal{M}\cdot\mathcal{S}$ and $\mathscr{P}_{cp}(\mathfrak{Q})$ be the collection of all non-empty and compact subsets of $\mathfrak{Q}$. Consider a collection of multi-valued operators $\mathcal{G}_j : \mathfrak{Q} \to \mathscr{P}_{cp}(\mathfrak{Q})$ (for $j = 1, 2, \cdots, m$), where each $\mathcal{G}_j$ is upper semicontinuous. An operator denoted and defined as:

$$\left\{\text{for each } \mathscr{Y} \in \mathscr{P}_{cp}(\mathfrak{Q}),\ \ F_{\mathcal{G}}(\mathscr{Y}) = \bigcup_{j=1}^{m} \mathcal{G}_j(\mathscr{Y})\right\},$$

is called a multi-fractal operator which is produced by the iterated multi-functions system (IMFS) $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_m)$. By using the condition of upper semicontinuity on $\mathcal{G}_j$, we can say that the operator $F_{\mathcal{G}}$ maps from $\mathscr{P}_{cp}(\mathfrak{Q})$ to $\mathscr{P}_{cp}(\mathfrak{Q})$. An element of $\mathscr{P}_{cp}(\mathfrak{Q})$ (say $\mathscr{A}^*$) is called a multi-valued fractal generated by the IMFS $\mathcal{G} = (\mathcal{G}_1, \cdots, \mathcal{G}_m)$ if and only if it is fixed under the associated multi-fractal operator $F_{\mathcal{G}}$.

If we take g_j, the single-valued continuous operators instead of $\mathcal{G}_j$, then a fractal (sometime we call it a self-similar set) is a point which is fixed under the fractal operator $F_g : \mathscr{P}_{cp}(\mathfrak{Q}) \to \mathscr{P}_{cp}(\mathfrak{Q})$ generated by the IFS $g = (g_1, g_2, \cdots, g_m)$, where F_g is defined as follows:

$$\left\{\text{for each } \mathscr{Y} \in \mathscr{P}_{cp}(\mathfrak{Q}),\ \ F_g(\mathscr{Y}) = \bigcup_{j=1}^{m} g_j(\mathscr{Y}),\ \right\}.$$

1.6 Best proximity point in metric type spaces

Let $\mathcal{U}, \mathcal{V}$ be the subsets of a $\mathcal{M} \cdot \mathcal{S}$ $(\mathfrak{Q}, d)$ and $\digamma : \mathcal{U} \to \mathcal{V}$ be a non-self mapping. Then it is not necessary that there will exist u in $\mathcal{U}$ such that $d(u, \digamma u) = 0$. Thus it is contemplated to find some u in $\mathcal{U}$ such that the *error* $d(u, \digamma u)$ is minimum which will be consider as the highest closeness between the element u and its image $\digamma u$ under $\digamma$. Since $d(u, \digamma u) \geq Dist(\mathcal{U}, \mathcal{V})$ for each $u \in \mathcal{U}$. The optimal solution for minimizing the problem of *error* $d(u, \digamma u)$ will be the one for which the value $Dist(\mathcal{U}, \mathcal{V})$ is attained. The best approximation theory has been derived from this idea. In view of this idea Kay Fan[17] presented the following theorem.

Theorem 1.6.1. [17] Let $\mathfrak{Q}$ be a normed space and $\digamma : \mathcal{U} \to \mathfrak{Q}$ be a continuous mapping where $\mathcal{U}$ is a compact and convex subset of $\mathfrak{Q}$. Then $\exists\, u \in \mathcal{U}$ such that

$$||u - \digamma u|| = \inf\{||v - \digamma u|| : \ v \in \mathcal{U}\}.$$

Definition 1.6.2. Let $\mathcal{U}, \mathcal{V}$ be the subsets of $(\mathfrak{Q}, d)$. A point $p \in \mathcal{U}$ is called a $\mathcal{B} \cdot \mathcal{P} \cdot \mathcal{P}$ of the non-self mapping $\digamma : \mathcal{U} \to \mathcal{V}$ if $d(p, \digamma p) = Dist(\mathcal{U}, \mathcal{V})$.

Similarly a point $p \in \mathcal{U}$ is called a $\mathcal{B} \cdot \mathcal{P} \cdot \mathcal{P}$ of a multi-valued mapping $\digamma : \mathcal{U} \to 2^{\mathcal{V}}$ if $D(p, \digamma p) = dist(\mathcal{U}, \mathcal{V})$.

Remark 1.6.3. If $\mathcal{U} = \mathcal{V} = \mathfrak{Q}$, then $Dist(\mathcal{U}, \mathcal{V}) = 0$ and p becomes a $\mathcal{F} \cdot \mathcal{P}$ of $\digamma$.

1.6.1 WP-property and P-property

In 2014 Gabeleh [19] generalized the $\mathcal{F} \cdot \mathcal{P}$ theorem in [51] by using an appropriate geometric property and established an interesting $\mathcal{B} \cdot \mathcal{P} \cdot \mathcal{P}$s theorem. We start by recalling some definitions and notations.

Definition 1.6.4. [18] Let $\mathcal{U}, \mathcal{V}$ be the subsets of $(\mathfrak{Q}, d)$. Define:

$$\mathcal{U}_0 = \{e \in \mathcal{U} : d(e, f) = Dist(\mathcal{U}, \mathcal{V}) \text{ for some } f \in \mathcal{V}\},$$

$$\mathcal{V}_0 = \{f \in \mathcal{V} : d(e, f) = Dist(\mathcal{U}, \mathcal{V}) \text{ for some } e \in \mathcal{U}\}.$$

Example 1.6.5. Let $\mathfrak{Q} = \mathbb{R}$ with the usual metric d and let $\mathcal{U} = [0, 1]$, $\mathcal{V} = \{-2, -1, 2\}$. Then $Dist(\mathcal{U}, \mathcal{V}) = 1$, $\mathcal{U}_0 = \{0, 1\}$, and $\mathcal{V}_0 = \{-1, 2\}$.

Definition 1.6.6. [18] Let $\mathcal{U}, \mathcal{V}$ be the subsets of $(\mathfrak{Q}, d)$ such that $\mathcal{U}_0 \neq \emptyset$. We say that the pair $(\mathcal{U}, \mathcal{V})$ has the WP-property if and only if

$$d(u_1, v_1) = Dist(\mathcal{U}, \mathcal{V}),$$

$$d(u_2, v_2) = Dist(\mathcal{U}, \mathcal{V})$$

implies

$$d(u_1, u_2) \leq d(v_1, v_2),$$

where $u_1, u_2 \in \mathcal{U}_0$, $v_1, v_2 \in \mathcal{V}_0$.

Example 1.6.7. Let $\mathfrak{Q} = \mathbb{R}$ with usual metric d and let $\mathcal{U} = [9, 10]$, $\mathcal{V} = [1, 4] \cup [15, 19]$. Then $Dist(\mathcal{U}, \mathcal{V}) = 5$, $\mathcal{U}_0 = \{9, 10\}$, $\mathcal{V}_0 = \{4, 15\}$. Let $u_1 = 9, u_2 = 10, v_1 = 4, v_2 = 15$. Then $d(u_1, v_1) = d(9, 4) = d(10, 15) = d(u_2, v_2) = 5 = Dist(\mathcal{U}, \mathcal{V})$ and $d(u_1, u_2) = d(9, 10) = 1 < d(v_1, v_2) = d(4, 15) = 11$. Thus $(\mathcal{U}, \mathcal{V})$ has the WP-property.

Example 1.6.8. Let $\mathfrak{Q} = \mathbb{R}$ with usual metric d and let $\mathcal{U} = \{-1, 0, 3\}$, $\mathcal{V} = [1, 2]$. Then $Dist(\mathcal{U}, \mathcal{V}) = 1$, $\mathcal{U}_0 = \{0, 3\}$, $\mathcal{V}_0 = \{1, 2\}$. Let $u_1 = 0, u_2 = 3, v_1 = 1, v_2 = 2$. Then $d(u_1, v_1) = d(0, 1) = d(3, 2) = d(u_2, v_2) = 1 = Dist(\mathcal{U}, \mathcal{V})$ while $d(u_1, u_2) = d(0, 3) = 3 > 1 = d(1, 2) = d(v_1, v_2)$. Thus $(\mathcal{U}, \mathcal{V})$ does not have the WP-property.

Definition 1.6.9. [2] Let $\mathcal{U}, \mathcal{V}$ be the subsets of $(\mathfrak{Q}, d)$ such that $\mathcal{U}_0 \neq \emptyset$. We say that the pair $(\mathcal{U}, \mathcal{V})$ has the P-property if and only if

$$d(u_1, v_1) = Dist(\mathcal{U}, \mathcal{V}),$$

$$d(u_2, v_2) = Dist(\mathcal{U}, \mathcal{V})$$

implies

$$d(u_1, u_2) = d(v_1, v_2),$$

where $u_1, u_2 \in \mathcal{U}_0$ and $v_1, v_2 \in \mathcal{V}_0$.

Remark 1.6.10. It is obvious that if a pair $(\mathcal{U}, \mathcal{V})$ has the P-property, then it has the WP-property but in general its converse may not true.

Remark 1.6.11. Note that the the definitions of $\mathcal{B}\cdot\mathcal{P}\cdot\mathcal{P}$, P-property and WP-property in b-$\mathcal{M}\cdot\mathcal{S}$s is similar to the definitions of these notions defined in $\mathcal{M}\cdot\mathcal{S}$s.

Example 1.6.12. Let $\mathcal{U}$ be the subset of $(\mathfrak{Q}, d)$. Then the pair $(\mathcal{U},\mathcal{U})$ always has the P-property.

Example 1.6.13. Let $\mathfrak{Q} = \mathbb{R}$ with usual metric d and let
$\mathcal{U} = \{1, 2, 3...\}$, $\mathcal{V} = \{... - 2, -1, 0, 1, 2...\}$. Then $Dist(\mathcal{U}, \mathcal{V}) = 0$, $\mathcal{U}_0 = \mathcal{U}$, $\mathcal{V}_0 = \mathcal{V}$ and if $u_1, u_2 \in \mathcal{U}_0$, $v_1, v_2 \in \mathcal{V}_0$ such that
$d(u_1, v_1) = 0 = Dist(\mathcal{U}, \mathcal{V})$ and $d(u_2, v_2) = 0 = Dist(\mathcal{U}, \mathcal{V})$. Then it implies that $u_1 = v_1$ and $u_2 = v_2$, and so $d(u_1, u_2) = d(v_1, v_2)$. Thus the pair $(\mathcal{U}, \mathcal{V})$ has the P-property.

Chapter 2

Fixed points of single-valued dynamical systems on extended cone b-metric space over Banach algebra

In this chapter, we have introduced a new geometrical structure which is the hybrid of CMS over Banach algebra and $Eb-M$ space. We prove analogues of Banach, Kannan and Reich type $\mathcal{F}\cdot\mathcal{P}$ theorems in our introduced space. We also established various concrete examples to validate our results. The main results due to Vujakovic *et al.*, Hussain *et al.*, Huang, Radenovic, Xu become special cases of our results. At the end, we have added some consequences of our results and application in the existence of solution of integral equations. The work of this chapter has been published in an esteemed international journal Filomat [53]. Throughout this chapter, we will consider only real Banach algebras.

2.1 Extended cone b-metric space over Banach algebras

We start this section by the definition of an $ECbMS$ over Banach algebra.

Definition 2.1.1. Let $\mathcal{A}$ be a real Banach algebra with cone $\mathfrak{P}$, $\mathfrak{Q}$ be a non empty set and $s : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty)$ be a mapping. An extended cone b-metric (in short $ECbM$) on $\mathfrak{Q}$ over $\mathcal{A}$ is a function $d_s : \mathfrak{Q} \times \mathfrak{Q} \to \mathcal{A}$ such that:

(E_1) $d_s(\eta, \xi) \succeq \vartheta$ and $d_s(\eta, \xi) = \vartheta$ iff $\eta = \xi$ for all $\eta, \xi \in \mathfrak{Q}$;

(E_2) $d_s(\eta, \xi) = d_s(\xi, \eta)$ for all $\eta, \xi \in \mathfrak{Q}$;

(E_3) $d_s(\eta, \zeta) \preceq s(\eta, \zeta)[d_s(\eta, \xi) + d_s(\xi, \zeta)]$ for all $\eta, \xi, \zeta \in \mathfrak{Q}$.

The pair$(\mathfrak{Q}, d_s)$ is then called an extended cone b-$\mathcal{M} \cdot \mathcal{S}$ over a Banach algebra $\mathcal{A}$ (in short $ECbMS$ over $\mathcal{A}$).

Remark 2.1.2. It is clear that the class of $ECbMS$ over Banach algebras is larger than the classes of CbM spaces and cone metric spaces over Banach algebras.
The definitions of Cauchy sequence, convergent sequence and completeness for $ECbM$ space over $\mathcal{A}$ are similar to that of CbM spaces over Banach algebra defined in the Definition 1.3.19.
In general d_s is not necessarily a continuous function but in this chapter, d_s will always mean a continuous function $d_s : \mathfrak{Q} \times \mathfrak{Q} \to \mathcal{A}$.

Example 2.1.3. Let $s : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty)$ be defined as $s(p, q) = 1 + p + q$ for $\mathfrak{Q} = \{1, 2, 3\}$. Consider the real Banach algebra $\mathcal{A} = \mathbb{R}^2$ with solid cone $\mathfrak{P} = \{(a, b) \in \mathbb{R}^2 \ : \ a, \ b \geq 0\}$. If we define $d_s : \mathfrak{Q} \times \mathfrak{Q} \to \mathcal{A}$ by:

$$\begin{aligned}
&d_s(1, 2) = d_s(2, 1) = (80, 80);\\
&d_s(1, 3) = d_s(3, 1) = (1000, 1000);\\
&d_s(3, 2) = d_s(2, 3) = (600, 600);\\
&d_s(1, 1) = d_s(2, 2) = d_s(3, 3) = (0, 0) = \vartheta.
\end{aligned}$$

Clearly the first and second conditions of an $ECbMS$ over $\mathcal{A}$ are satisfied. For the third condition we have:

$$s(1, 2)[d_s(1, 3) + d_s(3, 2)] - d_s(1, 2) = 4[(1000, 1000) + (600, 600)] - (80, 80) = (6320, 6320) \in \mathfrak{P};$$

$s(1,3)[d_s(1,2)+d_s2,3)]-d_s(1,3) = 5[(80,80)+(600,600)]-(1000,1000) = (2400,2400) \in \mathfrak{P}$;

$s(2,3)[d_s(2,1)+d_s1,3)]-d_s(2,3) = 6[(80,80)+(1000,1000)]-(600,600) = (5880,5880) \in \mathfrak{P}$.

Hence for all $\eta, \xi, \zeta \in \mathfrak{Q}$,

$$d_s(\eta, \xi) \preceq s(\eta, \xi)[d_s(\eta, \zeta) + d_s(\zeta, \xi)].$$

Thus $(\mathfrak{Q}, d_s)$ is an $ECbMS$ over $\mathcal{A} = \mathbb{R}^2$.

Remark 2.1.4. Let $(\mathfrak{Q}, d_s)$ be an $ECbMS$ over $\mathcal{A}$ with $s : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty)$. If $\mathcal{A} = \mathbb{R}$ and $\mathfrak{P} = [0, \infty)$, then $(\mathfrak{Q}, d_s)$ is an $Eb - M$ space.

We now define generalized α-admissible mapping and α-regular space in term of $ECbMS$ over Banach algebra.

Definition 2.1.5. Consider $(\mathfrak{Q}, d_s)$ an $ECbMS$ over $\mathcal{A}$ with $\mathfrak{P}$ an underlying solid cone in $\mathcal{A}$ and a self-map $\digamma$ on $\mathfrak{Q}$. Let $\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$. Then:

- (i) $\digamma$ is said to be a generalized α-admissible mapping if for $\eta, \xi \in \mathfrak{Q}, \alpha(\eta, \xi) \geq s(\eta, \xi)$ implies that $\alpha(\digamma\eta, \digamma\xi) \geq s(\digamma\eta, \digamma\xi)$;
- (ii) $(\mathfrak{Q}, d_s)$ is said to be α-regular if any sequence $\{\varpi_k\} \in \mathfrak{Q}$ with $\alpha(\varpi_k, \varpi_{k+1}) \geq s(\varpi_k, \varpi_{k+1})$ for all $k \in \mathbb{N}$ and $\varpi_k \to \varpi$ implies that $\alpha(\varpi_k, \varpi) \geq s(\varpi_k, \varpi)$.

2.2 Generalized Reich type contraction in $ECbMS$ over Banach algebra

In this section, we have introduced generalized Reich type mapping in the setting of $ECbMS$ over $\mathcal{A}$. Then we proved a couple of theorems and established an example to prove the validity of the result.

Definition 2.2.1. Let $(\mathfrak{Q}, d_s)$ be an $ECbMS$ over $\mathcal{A}$ with $\mathfrak{P}$ an underlying solid cone and $\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ be a mapping. Then a self-map $\digamma$ on $\mathfrak{Q}$ is called a generalized R-type (Reich type) contraction if there exists three vectors $\varpi_1, \varpi_2, \varpi_3$ in $\mathfrak{P}$ such that for all $\eta, \xi \in \mathfrak{Q}$ with $\alpha(\eta, \xi) \geq s(\eta, \xi)$:

(*i*) $2s(\eta,\xi)r(\varpi_1)+(s(\eta,\xi)+1)r(\varpi_2+\varpi_3)<2$ and for each $\varrho_0\in\mathfrak{Q}$ with $\varrho_j=F^j\varrho_0$,

$$\lim_{k,i\to\infty} s(\varrho_{j+1},\varrho_i)<\frac{1}{||\kappa||} \text{ where } \kappa=(2e-\varpi)^{-1}(2\varpi_1+\varpi) \text{ for } \varpi=\varpi_2+\varpi_3;$$

(*ii*) $d_s(F\eta,F\xi)\preceq\varpi_1 d_s(\eta,\xi)+\varpi_2 d_s(\eta,F\eta)+\varpi_3 d_s(\xi,F\xi)$.

One of the main results of this chapter is given as follows:

Theorem 2.2.2. Let $(\mathfrak{Q},d_s)$ be a complete *ECbMS* over $\mathcal{A}$ with $\alpha:\mathfrak{Q}\times\mathfrak{Q}\to[0,\infty)$ be a mapping and $\mathfrak{P}$ an underlying solid cone. Suppose that the self-map F on $\mathfrak{Q}$ is a generalized R-type contraction with vectors $v_1,v_2,v_3\in\mathfrak{P}$ such that:

1. F is a generalized α-admissible;
2. there exists an element $u_0\in\mathfrak{Q}$ such that $\alpha(u_0,Fu_0)\geq s(u_0,Fu_0)$;
3. $(\mathfrak{Q},d_s)$ is regular or F is continuous.

Then there exists a point ϱ in $\mathfrak{Q}$ which is a $\mathcal{F}\cdot\mathcal{P}$ of the map F.

Proof. Let u_0 be a point in $\mathfrak{Q}$ such that $\alpha(u_0,Fu_0)\geq s(u_0,Fu_0)$. For $u_0\in\mathfrak{Q}$, if we define $u_1=Fu_0,\ u_2=Fu_1=F(Fu_0)=T^2u_0,\cdots,\ u_{n+1}=Fu_n=F^{n+1}u_0$, then

$$\alpha(u_0,u_1)\geq s(u_0,u_1).$$

But F is generalized α-admissible, so

$$\alpha(Fu_0,Fu_1)=\alpha(u_1,u_2)\geq s(u_1,u_2),$$

and so by induction we get

$$\alpha(u_n,u_{n+1})\geq s(u_n,u_{n+1}).$$

By using Definition 2.2.1, we have

$$\begin{aligned} d_s(u_n,u_{n+1}) &= d_s(Fu_{n-1},Fu_n)\\ &\preceq v_1 d_s(u_{n-1},u_n)+v_2 d_s(u_{n-1},Fu_{n-1})+v_3 d_s(u_n,Fu_n), \text{i.e.} \end{aligned}$$

$$(e - v_3)d_s(u_n, u_{n+1}) \preceq (v_1 + v_2)d_s(u_{n-1}, u_n). \tag{2.2.1}$$

Similarly

$$\begin{aligned} d_s(u_{n+1}, u_n) &= d_s(\digamma u_n, \digamma u_{n-1}) \\ &\preceq v_1 d_s(u_n, u_{n-1}) + v_2 d_s(u_n, \digamma u_n) + v_3 d_s(u_{n-1}, \digamma u_{n-1}), \text{i.e.} \end{aligned}$$

$$(e - v_2)d_s(u_{n+1}, u_n) \preceq (v_1 + v_3)d_s(u_{n-1}, u_n). \tag{2.2.2}$$

Adding (2.2.1) and (2.2.2), we obtain

$$(2e - v_2 - v_3)d_s(u_n, u_{n+1}) \preceq (2v_1 + v_2 + v_3)d_s(u_{n-1}, u_n).$$

If we take $v = v_2 + v_3$, then we obtain

$$(2e - v)d_s(u_{n+1}, u_n) \preceq (2v_1 + v)d_s(u_{n-1}, u_n). \tag{2.2.3}$$

Note that

$$2r(v) \leq (s(u_n, u_{n+1}) + 1)r(v) \leq 2r(v_1) + (s(u_n, u_{n+1}) + 1)r(v) < 2.$$

Hence $r(v) < 1 < 2 \implies r(v) < 2$. Thus by using Lemma 1.3.11, we obtain that the element $2e - v$ is invertible and $(2e - v)^{-1} = \sum_{n=0}^{\infty} \frac{v^n}{2^{n+1}}$, $r((2e - v)^{-1}) < \frac{1}{2 - r(v)}$. Hence (2.2.3) becomes

$$d_s(u_n, u_{n+1}) \preceq \kappa d_s(u_{n-1}, u_n), \tag{2.2.4}$$

where $\kappa = (2e - v)^{-1}(2v_1 + v)$. The inequality (2.2.4) then implies that for all $n \in \mathbb{N}$

$$\begin{aligned} d_s(u_n, u_{n+1}) &\preceq \kappa d_s(u_{n-1}, u_n) \\ &\preceq \kappa^2 d_s(u_{n-1}, u_n) \\ &\vdots \\ &\preceq \kappa^n d_s(u_0, u_1). \end{aligned} \tag{2.2.5}$$

Now if we take $m > n$, then by using (2.2.5) and Definition 2.1.1, (iii) we have

$$
\begin{aligned}
d_s(u_n, u_m) &\preceq s(u_n, u_{n+1})d_s(u_n, u_{n+1}) + s(u_n, u_{n+1})s(u_{n+1}, u_{n+2})d_s(u_{n+1}, u_{n+2}) + \cdots + \\
&s(u_n, u_{n+1})s(u_{n+1}, u_{n+2}) \ldots s(u_{m-1}, u_m)(d_s(u_{m-1}, u_m)) \\
&\preceq s(u_n, u_m)\kappa^n d_s(u_0, u_1) + s(u_n, u_m)s(u_{n+1}, u_m)\kappa^{n+1} d_s(u_0, u_1) + \cdots + \\
&s(u_n, u_m)s(u_{n+1}, u_m)s(u_{n+2}, u_m) \ldots s(u_{m-2}, u_m)s(u_{m-1}, u_m)\kappa^{m-1} d_s(u_0, u_1) \\
&\preceq d_s(u_0, u_1)\Big[s(u_1, u_m)s(u_2, u_m) \ldots s(u_{n-1}, u_m)s(u_n, u_m)\kappa^n + \\
&s(u_1, u_m)s(u_2, u_m) \ldots s(u_n, u_m)s(u_{n+1}, u_m)\kappa^{n+1} + \cdots + \\
&\{s(u_1, u_m)s(u_2, u_m) \ldots s(u_n, u_m)s(u_{n+1}, u_m) \ldots s(u_{m-2}, u_m)s(u_{m-1}, u_m)\}\kappa^{m-1}\Big] \\
&= d_s(u_0, u_1)\Big[\kappa^n \prod_{j=1}^{n} s(u_j, u_m) + \kappa^{n+1} \prod_{j=1}^{n+1} s(u_j, u_m) + \cdots + \kappa^{m-1} \prod_{j=1}^{m-1} s(u_j, u_m)\Big].
\end{aligned}
$$

Let $a_n = \kappa^n \prod_{j=1}^{n} s(u_j, u_m)$ and $S = \sum_{n=1}^{\infty} a_n$.
Since by Definition 2.2.1, $||\kappa|| \lim_{n,m\to\infty} s(u_{n+1}, u_m) < 1$, so the series S converges absolutely. Because by using ratio test we have

$$
\lim_{n\to\infty} \frac{||a_{n+1}||}{||a_n||} \leq \lim_{n\to\infty} \frac{||\kappa||||\kappa^n||s(u_{n+1}, u_m)}{||\kappa^n||} = ||\kappa|| \lim_{n,m\to\infty} s(u_{n+1}, u_m) < 1.
$$

But $\mathcal{A}$ is a Banach algebra and the series S is absolutely convergent, so it converges in $\mathcal{A}$. Thus $S_{m-1} - S_n = \Big[\kappa^n \prod_{j=1}^{n} s(u_j, u_m) + \cdots + \kappa^{m-1} \prod_{j=1}^{m-1} s(u_j, u_m)\Big] \to \vartheta$ as $n, m \to \infty$ and so is $d_s(u_0, u_1)(S_{m-1} - S_n)$. By Lemma 1.3.15, for every $c \gg \vartheta$, there exists a natural number n_0 such that for all $n \geq n_0$, $d_s(u_n, u_m) \ll c$. Thus by Definition 1.3.19 $\{u_n\}$ is a Cauchy sequence in $\mathfrak{Q}$. But $\mathfrak{Q}$ is complete so there exists $\varrho \in \mathfrak{Q}$ such that $u_n \to \varrho$ as $n \to \infty$. We show that ϱ is a $\mathcal{F} \cdot \mathcal{P}$ of the map F.
Suppose that F is continuous. It follows that $u_{n+1} = F u_n \to F \varrho$ as $n \to \infty$. But limit of a sequence is unique, so we must have $F \varrho = \varrho$. Hence ϱ is a $\mathcal{F} \cdot \mathcal{P}$ of the map F in this case.
However, if $(\mathfrak{Q}, d_s)$ is α-regular, then by Definition 2.1.5 we have

$$
\alpha(u_n, \varrho) \geq s(u_n, \varrho), \qquad \text{for all } n \in \mathbb{N}.
$$

$$
\begin{aligned}
d_s(\varrho, F\varrho) &\preceq s(\varrho, F\varrho)\big[d_s(\varrho, Fu_n) + d_s(Fu_n, F\varrho)\big] \\
&\preceq s(\varrho, F\varrho)d_s(\varrho, Fu_n) + s(\varrho, F\varrho)\big[v_1 d_s(u_n, \varrho) + v_2 d_s(u_n, Fu_n) + v_3 d_s(\varrho, F\varrho)\big] \\
&\preceq s(\varrho, F\varrho)d_s(\varrho, Fu_n) + s(\varrho, F\varrho)v_1 d_s(u_n, \varrho) + s(\varrho, F\varrho)v_3 d_s(\varrho, F\varrho) \\
&\quad + s(\varrho, F\varrho)s(u_n, u_{n+1})v_2\big[d_s(u_n, \varrho) + d_s(\varrho, u_{n+1})\big] \\
&= s(\varrho, F\varrho)(e + s(u_n, u_{n+1})v_2)d_s(\varrho, u_{n+1}) + s(\varrho, F\varrho)v_3 d_s(\varrho, F\varrho) \\
&\quad + s(\varrho, F\varrho)(v_1 + s(u_n, u_{n+1})v_2)d_s(u_n, \varrho),
\end{aligned}
$$

which further implies that

$$
(e - s(\varrho, F\varrho)v_3)d_s(\varrho, F\varrho) \preceq s(\varrho, F\varrho)(e + s(u_n, u_{n+1})v_2)d_s(u_{n+1}, \varrho) + s(\varrho, F\varrho)(v_1 + s(u_n, u_{n+1})v_2)d_s(u_n, \varrho) \tag{2.2.6}
$$

Similarly,

$$
\begin{aligned}
d_s(\varrho, F\varrho) &\preceq s(\varrho, F\varrho)\big[d_s(\varrho, Fu_n) + d_s(Fu_n, F\varrho)\big] \\
&= s(\varrho, F\varrho)d_s(\varrho, Fu_n) + s(\varrho, F\varrho)d_s(F\varrho, Fu_n) \\
&\preceq s(\varrho, F\varrho)d_s(\varrho, Fu_n) + s(\varrho, F\varrho)\big[v_1 d_s(\varrho, u_n) + v_2 d_s(\varrho, F\varrho) + v_3 d_s(u_n, Fu_n)\big] \\
&\preceq s(\varrho, F\varrho)d_s(\varrho, Fu_n) + s(\varrho, F\varrho)v_1 d_s(\varrho, u_n) + s(\varrho, F\varrho)v_2 d_s(\varrho, F\varrho) \\
&\quad + s(\varrho, F\varrho)s(u_n, u_{n+1})v_3\big[d_s(u_n, \varrho) + d_s(\varrho, u_{n+1})\big] \\
&= s(\varrho, F\varrho)(e + s(u_n, u_{n+1})v_3)d_s(\varrho, u_{n+1}) + s(\varrho, F\varrho)v_2 d_s(\varrho, F\varrho) \\
&\quad + s(\varrho, F\varrho)(v_1 + s(u_n, u_{n+1})v_3)d_s(u_n, \varrho),
\end{aligned}
$$

which further implies that

$$
(e - s(\varrho, F\varrho)v_2)d_s(\varrho, F\varrho) \preceq s(\varrho, F\varrho)(e + s(u_n, u_{n+1})v_3)d_s(u_{n+1}, \varrho) + s(\varrho, F\varrho)(v_1 + s(u_n, u_{n+1})v_3)d_s(u_n, \varrho). \tag{2.2.7}
$$

Therefore, by combining (2.2.6) and (2.2.7), we get

$$
\begin{aligned}
(2e - s(\varrho, F\varrho)v_2 - s(\varrho, F\varrho)v_3)d_s(\varrho, F\varrho) &\preceq s(\varrho, F\varrho)(2e + s(\varrho, F\varrho)v_2 + s(\varrho, F\varrho)v_3)d_s(u_{n+1}, \varrho) \\
&\quad + s(\varrho, F\varrho)(2v_1 + s(\varrho, F\varrho)v_2 + s(\varrho, F\varrho)v_3)d_s(u_n, \varrho), \textit{ i.e.}
\end{aligned}
$$

$$
\begin{aligned}
(2e - s(\varrho, F\varrho)v)d_s(\varrho, F\varrho) &\preceq s(\varrho, F\varrho)(2e + s(\varrho, F\varrho)v)d_s(u_{n+1}, \varrho) \\
&\quad + s(\varrho, F\varrho)(2v_1 + s(\varrho, F\varrho)v)d_s(u_n, \varrho).
\end{aligned} \tag{2.2.8}
$$

We also note that

$$r(s(\varrho,\digamma\varrho)v) = s(\varrho,\digamma\varrho)r(v) \leq 2s(\varrho,\digamma\varrho)r(v_1) + (s(\varrho,\digamma\varrho)+1)r(v) < 2.$$

Thus by Lemma 1.3.11, $2e - s(\varrho,\digamma\varrho)v$ is invertible and so (2.2.8) implies that

$$\begin{aligned} d_s(\varrho,\digamma\varrho) \preceq (2e - s(\varrho,\digamma\varrho)v)^{-1}\big[&s(\varrho,\digamma\varrho)(2e + s(\varrho,\digamma\varrho)v)d_s(u_{n+1},\varrho) \\ &+ s(\varrho,\digamma\varrho)(2v_1 + s(\varrho,\digamma\varrho)v)d_s(u_n,\varrho)\big]. \end{aligned} \tag{2.2.9}$$

By using Remark 1.3.20 the sequences $\{d_s(u_{n+1},\varrho)\}$ and $\{d_s(u_n,\varrho)\}$ are c-sequences. Hence by Lemma 1.3.14, the sequence $\{\tau_1 d_s(u_{n+1},\varrho)+\tau_2 d_s(u_n,\varrho)\}$ is a c-sequence (where $\tau_1 = (2e-s(\varrho,\digamma\varrho)v)^{-1}s(\varrho,\digamma\varrho)(2e+s(\varrho,\digamma\varrho)v)$ and $\tau_2 = (2e-s(\varrho,\digamma\varrho)v)^{-1}s(\varrho,\digamma\varrho)(2v_1+s(\varrho,\digamma\varrho)v))$. Therefore, for any $c \in int(\mathfrak{P}) \exists\, n_0 \in \mathbb{N}$ such that

$$d_s(\varrho,\digamma\varrho) \preceq \tau_1 d_s(u_{n+1},\varrho) + \tau_2 d_s(u_n,\varrho) \ll c.$$

Which further implies by using Lemma 1.3.15 that $d_s(\varrho,\digamma\varrho) = \vartheta$. Therefore, $\digamma\varrho = \varrho$ and this complete the proof. □

Example 2.2.3. Let $\mathcal{A} = C^1_{\mathbb{R}}[0,1]$ and $||f|| = ||f||_\infty + ||f'||_\infty$. If we define point wise multiplication of functions on $\mathcal{A}$, then $\mathcal{A}$ becomes a real Banach algebra with identity $e(t) = 1$. If we take $\mathfrak{P} = \{T \in \mathcal{A} \ : \ T(p) \geq 0,\ p \in [0,1]\}$, then it can be seen that $\mathfrak{P}$ is a non-normal cone (see [26]). Let $\mathfrak{Q} = [0,\infty)$ and $s : \mathfrak{Q} \times \mathfrak{Q} \to [1,\infty)$ be defined as

$$s(\varrho,\varpi) = \begin{cases} \varrho + \varpi + 2 & \text{if } \varrho,\varpi \in\ [0,1]; \\ 2 & \text{elsewhere.} \end{cases}$$

Define $d_s : \mathfrak{Q} \times \mathfrak{Q} \to \mathcal{A}$ by

$$d_s(\varrho,\varpi)(t) = (\varrho - \varpi)^2 e^t.$$

Then d_s is an $ECbMS$ over $\mathcal{A}$. Also note that $\mathfrak{Q}$ is complete with respect to d_s. Define two maps $\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0,\infty)$ and $\digamma : \mathfrak{Q} \to \mathfrak{Q}$ by:

$$\alpha(x,y) = \begin{cases} s(p,q) & \text{if } p,q \in\ [0,1]; \\ 0 & \text{elsewhere.} \end{cases}$$

$$\digamma(p) = \begin{cases} \frac{\sqrt{5}}{3}p & \text{if } p \in \ [0,1]; \\ p-1 & \text{if } p > 1. \end{cases}$$

Note that for every $p \in [0,1]$, $\digamma p \in [0,1]$. By choosing $v_1(t) = \frac{1}{36} + \frac{1}{36}t$, $v_2(t) = \frac{1}{18} + \frac{1}{18}t$ and $v_3(t) = \frac{1}{24} + \frac{1}{24}t$ we obtain that $r(v_1) = \frac{2}{9}$, $r(v) = r(v_2+v_3) = \frac{7}{36}$. Simple calculations show that $2(2+2)r(v_1) + (2+2+1)r(v) = \frac{51}{36} < 2$ and so $\digamma$ is a generalized R-type contraction as;

$$2s(x,y)r(v_1) + (s(x,y)+1)r(v) \leq 2(2+2)r(v_1) + (2+2+1)r(v) = \frac{51}{36} < 2.$$

Also for each $u_0 \in \mathfrak{Q}$, the limit $\lim_{n,m\to\infty} s(u_{n+1}, u_m) = 2$ and $||\kappa|| = ||(2e-v)^{-1}(2v_1 + v)|| \leq \left(\frac{72}{130}\right)\left(\frac{46}{72}\right) = \frac{23}{65} < \frac{1}{2} = \frac{1}{\lim_{n,m\to\infty} s(u_{n+1},u_m)}$. Similarly by easily calculation one can show that

$$d_s(\digamma p, \digamma q) \preceq v_1 d_s(p,q) + v_2 d_s(p, \digamma p) + v_3 d_s(q, \digamma q).$$

Next we show that there is a point u_0 in $\mathfrak{Q}$ such that $\alpha(u_0, \digamma u_0) \geq s(u_0, \digamma u_0)$. Indeed, for $u_0 = 1$, we have

$$\alpha(1, \digamma 1) = \alpha(1, \frac{\sqrt{5}}{3}) \geq s(1, \frac{\sqrt{5}}{3}) = s(1, \digamma 1).$$

Next we show that $\digamma$ is a generalized α-admissible mapping. In fact, if $p, q \in \mathfrak{Q}$ are such that $\alpha(p,q) \geq s(p,q)$, then by definition of α, the points p, q is in $[0,1]$. Therefore, $\digamma p, \digamma q \in [0,1]$ and so

$$\alpha(\digamma p, \digamma q) \geq s(\digamma p, \digamma q).$$

Finally we show that $(\mathfrak{Q}, d_s)$ is α-regular. If we assume a sequence $\{\sigma_n\}$ in $\mathfrak{Q}$ such that $\alpha(\sigma_n, \sigma_{n+1}) \geq s(\sigma_n, \sigma_{n+1})$ for all $n \in \mathbb{N}$ and $\sigma_n \to q \in \mathfrak{Q}$, then $\{\sigma_n\} \subseteq [0,1]$. But $[0,1]$ is closed, so $q \in [0,1]$. This implies that $\alpha(v_n, q) \geq s(\sigma_n, q)$ for all $n \in \mathbb{N}$. Hence all the axioms of Theorem 2.2.2 satisfied, and so there is a point $\varrho = 0$ (say) which is a $\mathcal{F} \cdot \mathcal{P}$ of the map $\digamma$.

Theorem 2.2.4. Let $\mathcal{A}$ be a Banach algebra with solid cone $\mathfrak{P}$. Let $(\mathfrak{Q}, d_s)$ be a complete $ECbMS$ over $\mathcal{A}$ with $\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ a mapping. Suppose that the self-map $\digamma$ on $\mathfrak{Q}$ is a generalized R-type contraction with vectors v_1, v_2, v_3 in $\mathfrak{P}$ such that v_1 commutes with $v_2 + v_3$ and:

1. $\digamma$ is a generalized α-admissible;

2. there exists $u_0 \in \mathfrak{Q}$ such that $\alpha(u_0, \digamma u_0) \geq s(u_0, \digamma u_0)$;

3. $\digamma$ is continuous or $(\mathfrak{Q}, d_s)$ is regular;

4. for any two fixed points ϖ, ζ of $\digamma$, there exists z in $\mathfrak{Q}$ such that $\alpha(\varpi, z) \geq s(\varpi, z)$ and $\alpha(\zeta, z) \geq s(\zeta, z)$.

Then there exists a unique point ϱ in $\mathfrak{Q}$ which is a $\mathcal{F} \cdot \mathcal{P}$ of the map $\digamma$.

Proof. From the hypothesis and the first three conditions, in Theorem 2.2.2, it has been proved that exists a point $\varrho \in \mathfrak{Q}$ which is a $\mathcal{F} \cdot \mathcal{P}$ of the map $\digamma$. We show that this point is unique and for this let $\zeta \in Fix(\digamma)$ such that $\varrho \neq \zeta$. Then by using Condition 4, there exists $z \in \mathfrak{Q}$ with

$$\alpha(\varrho, z) \geq s(\varrho, z) \quad \text{and} \quad \alpha(\zeta, z) \geq s(\zeta, z). \tag{2.2.10}$$

Since $\digamma$ is a generalized α-admissible mapping and $\varrho, \zeta \in Fix(\digamma)$ so by (2.2.10) we get

$$\alpha(\varrho, \digamma^i z) \geq s(\varrho, \digamma^i z) \quad \text{and} \quad \alpha(\zeta, \digamma^i z) \geq s(\zeta, \digamma^i z)\ , \quad \text{for all } i \in \mathbb{N}. \tag{2.2.11}$$

By using Definition 2.2.1 and (2.2.11) we obtain

$$\begin{aligned} d_s(\varrho, \digamma^i z) &= d_s(\digamma \varrho, \digamma(\digamma^{i-1} z)) \\ &\preceq v_1 d_s(\varrho, \digamma^{i-1} z) + v_2 d_s(\varrho, \digamma \varrho) + v_3 d_s(\digamma^{i-1} z, \digamma^i z) \\ &\preceq v_1 d_s(\varrho, \digamma^{i-1} z) + v_3 s(\digamma^{i-1} z, \digamma^i z)[d_s(\digamma^{i-1} z, \varrho) + d_s(\varrho, \digamma^i z)], \end{aligned}$$

which further implies that

$$(e - s(\digamma^{i-1} z, \digamma^i z) v_3) d_s(\varrho, \digamma^i z) \preceq (v_1 + s(\digamma^{i-1} z, \digamma^i z)) v_3) d_s(\varrho, \digamma^{i-1} z). \tag{2.2.12}$$

Similarly,

$$\begin{aligned} d_s(\digamma^i z, \varrho) &= d_s(\digamma(\digamma^{i-1} z), \digamma \varrho) \\ &\preceq v_1 d_s(\digamma^{i-1} z, \varrho) + v_2 d_s(\digamma^{i-1} z), \digamma^i z)) + v_3 d_s(\varrho, \digamma \varrho) \\ &\preceq v_1 d_s(\digamma^{i-1} z, \varrho) + v_2 s(\digamma^{i-1} z, \digamma^i z)[d_s(\digamma^{i-1} z, \varrho) + d_s(\varrho, \digamma^i z)], \end{aligned}$$

which further implies that

$$(e - s(F^{i-1}z, F^{i}z)v_2)d_s(F^{i}z, \varrho) \preceq (v_1 + s(F^{i-1}z, F^{i}z))v_2)d_s(F^{i-1}z, \varrho). \qquad (2.2.13)$$

Adding (2.2.12) and (2.2.13) we have

$$(2e - s(F^{i-1}z, F^{i}z)v_2 - s(F^{i-1}z, F^{i}z)v_3)d_s(\varrho, F^{i}z) \preceq (2v_1 + s(F^{i-1}z, F^{i}z)v_2 + s(F^{i-1}z, F^{i}z)v_3)d_s(\varrho, F^{i-1}z)$$

$$(2e - s(F^{i-1}z, F^{i}z)v)d_s(\varrho, F^{i}z) \preceq (2v_1 + s(F^{i-1}z, F^{i}z)v)d_s(\varrho, F^{i-1}z).$$

Note that $2r(s(F^{i-1}z, F^{i}z)v) \leq (s(u_n, u_{n+1})+1)r(s(F^{i-1}z, F^{i}z)v) \leq 2r(v_1)+(s(u_n, u_{n+1})+1)r(s(F^{i-1}z, F^{i}z)v) < 2$. Which implies that $r(s(F^{i-1}z, F^{i}z)v) < 1 < 2$. Thus by Lemma 1.3.11, we can say that $2e-s(F^{i-1}z, F^{i}z)v$ is invertible and $(2e-s(F^{i-1}z, F^{i}z)v)^{-1} = \sum_{n=0}^{\infty} \frac{(s(F^{i-1}z,F^{i}z)v)^n}{2^{n+1}}$,
$r((2e - s(F^{i-1}z, F^{i}z)v)^{-1}) < \frac{1}{2-r(s(F^{i-1}z,F^{i}z)v)}$. Thus we have

$$d_s(\varrho, F^{i}z) \preceq (2e - s(F^{i-1}z, F^{i}z)v)^{-1}(2v_1 + s(F^{i-1}z, F^{i}z)v)d_s(\varrho, F^{i-1}z) \text{ , } i.e.$$

$$d_s(\varrho, F^{i}z) \preceq \tau d_s(\varrho, F^{i-1}z) \qquad (2.2.14)$$

where $\tau = (2e - s(F^{i-1}z, F^{i}z)v)^{-1}(2v_1 + s(F^{i-1}z, F^{i}z)v)$. Therefore, we have

$$\begin{aligned} d_s(\varrho, F^{i}z) &\preceq \tau d_s(\varrho, F^{i-1}z) \\ &\preceq \tau^2 d_s(\varrho, F^{i-2}z) \\ &\vdots \\ &\preceq \tau^i d_s(\varrho, z) \text{ for all } i \in \mathbb{N}. \end{aligned}$$

Since v_1 commutes with $v_2 + v_3 = v$, so

$$\begin{aligned} (2e - s(F^{i-1}z, F^{i}z)v)^{-1}(2v_1 + s(F^{i-1}z, F^{i}z)v) &= \Big(\sum_{n=0}^{\infty} \frac{(s(F^{i-1}z, F^{i}z)v)^n}{2^{n+1}}\Big)(2v_1 + s(F^{i-1}z, F^{i}z)v) \\ &= 2v_1\Big(\sum_{n=0}^{\infty} \frac{(s(F^{i-1}z, F^{i}z)v)^n}{2^{n+1}}\Big) + s(F^{i-1}z, F^{i}z)v\Big(\sum_{n=0}^{\infty} \frac{(s(F^{i-1}z, F^{i}z)v)^n}{2^{n+1}}\Big) \\ &= (2v_1 + s(F^{i-1}z, F^{i}z)v)(2e - s(F^{i-1}z, F^{i}z)v)^{-1}. \end{aligned}$$

Which shows that $(2e - s(\digamma^{i-1}z, \digamma^i z)v)^{-1}$ commutes with $(2v_1 + s(\digamma^{i-1}z, \digamma^i z)v)$. Hence by applying Lemma 1.3.11 and Lemma 1.3.12 we obtain that;

$$\begin{aligned} r(\tau) &= r((2e - s(\digamma^{i-1}z, \digamma^i z)v)^{-1}(2v_1 + s(\digamma^{i-1}z, \digamma^i z)v)) \\ &\leq r((2e - s(\digamma^{i-1}z, \digamma^i z)v)^{-1}) \cdot r((2v_1 + s(\digamma^{i-1}z, \digamma^i z)v)) \\ &\leq \frac{1}{2 - r(s(\digamma^{i-1}z, \digamma^i z)v)}(2r(v_1) + r(s(\digamma^{i-1}z, \digamma^i z)v)) \\ &< \frac{1}{s(u_n, u_{n+1})} < 1 \end{aligned}$$

By Lemma 1.3.15 it follows that $||\tau^i|| \to 0$ as $i \to \infty$ and so

$$||\tau^i d_s(\varrho, z)|| \leq ||\tau^i|| ||d_s(\varrho, z)|| \to 0 \quad (i \to \infty).$$

By Remark 1.3.20 we conclude that for any $c \gg \vartheta$, $\exists\, M \in \mathbb{N}$ such that

$$d_s(\varrho, \digamma^i z) \preceq \tau^i d_s(\varrho, z) \preceq c \quad \forall\, i \geq M.$$

Thus by Lemma 1.3.15 $\digamma^i z \to \varrho$ as $i \to \infty$. Similarly we obtain that $\digamma^i z \to \zeta$ as $i \to \infty$. Now by uniqueness of limit, we conclude that $\varrho = \zeta$. □

2.3 Generalized Lipschitz contractions in $ECbMS$ over Banach algebras

In this section, we have discussed the theory of $\mathcal{F} \cdot \mathcal{P}$s of generalized Lipschitz mappings in $ECbMS$ over $\mathcal{A}$.

Theorem 2.3.1. Let $(\mathfrak{Q}, d_s)$ be a complete $ECbMS$ over $\mathcal{A}$ with $\mathfrak{P}$ an associated cone in $\mathcal{A}$. Let $\digamma$ be a self-map on $\mathfrak{Q}$ such that for all $p, q \in \mathfrak{Q}$;

$$d_s(\digamma p, \digamma q) \preceq \kappa d_s(p, q), \tag{2.3.1}$$

where $\kappa \in \mathfrak{P}$ be such that $r(\kappa) < 1$ and for each $u_0 \in \mathfrak{Q}$, $\lim_{n,m\to\infty} s(u_{n+1}, u_m) < \frac{1}{||\kappa||}$. Then there exists a unique point $\varrho \in \mathfrak{Q}$ which is a $\mathcal{F} \cdot \mathcal{P}$ of the map $\digamma$. Furthermore for each $u_0 \in \mathfrak{Q}$, the iterative sequence $u_n = \digamma(u_{n-1}) = \digamma^n u_0$ converges to ϱ.

Proof. If we take $v_1 = \kappa$, $v_2 = v_3 = \vartheta$ and $\alpha(p,q) = s(p,q)$, then all the conditions of Theorem 2.2.2 are satisfied, i.e. $\digamma$ satisfies the condition of Definition 2.2.1, $\digamma$ is generalized α-admissible, $(\mathfrak{Q}, d_s)$ is regular and for every $u_0 \in \mathfrak{Q}$ $\alpha(u_0, \digamma u_0) \succeq s(u_0, \digamma u_0)$. Hence there exists ϱ in $\mathfrak{Q}$ which is a $\mathcal{F}\cdot\mathcal{P}$ of the map $\digamma$. Now it remains only to show that this $\mathcal{F}\cdot\mathcal{P}$ is unique. For this, let there is ζ in $\mathfrak{Q}$ such that $\digamma\zeta = \zeta$. Then we have

$$d_s(\varrho,\zeta) = d_s(\digamma\varrho, \digamma\zeta) \preceq \kappa d_s(\varrho,\zeta).$$

But $r(\kappa) < 1$, so by Lemma 1.3.11, $e-\kappa$ is invertible. Thus by Lemma 1.3.15 $d_s(\varrho,\zeta) = \vartheta$. □

Theorem 2.3.2. Let $(\mathfrak{Q}, d_s)$ be a complete $ECbMS$ over $\mathcal{A}$ and $\mathfrak{P}$ be the associated cone in $\mathcal{A}$. Let $\digamma$ be a self-map on $\mathfrak{Q}$ satisfies the generalized Lipschitz condition, i.e. for all $p, q \in \mathfrak{Q}$;

$$d_s(\digamma p, \digamma q) \preceq \kappa[d_s(\digamma p, p) + d_s(\digamma q, q)], \tag{2.3.2}$$

where $\kappa \in \mathfrak{P}$ be such that $r(\kappa) < \dfrac{1}{s(p,q)+1}$ and for each $u_0 \in \mathfrak{Q}$, $\lim_{n,m\to\infty} s(u_{n+1}, u_m) < \dfrac{1}{||\tau||}$ with $\tau = (e-\kappa)^{-1}\kappa$. Then there exists a unique point $\varrho \in \mathfrak{Q}$ which is $\mathcal{F}\cdot\mathcal{P}$ of the map $\digamma$.

Proof. If we take $v_1 = \vartheta$, $v_2 = v_3 = \kappa$ and $\alpha(p,q) = s(p,q)$, then all the condition of Theorem 2.2.2 are satisfied. Hence there exists ϱ in $\mathfrak{Q}$ which is a $\mathcal{F}\cdot\mathcal{P}$ of the map $\digamma$. Finally we show that ϱ is a unique $\mathcal{F}\cdot\mathcal{P}$ of the map $\digamma$. For this if ζ is another $\mathcal{F}\cdot\mathcal{P}$ of $\digamma$, then

$$d_s(\varrho,\zeta) = d_s(\digamma\varrho, \digamma\zeta) \preceq \kappa[d_s(\varrho, \digamma\varrho) + d_s(\zeta, \digamma\zeta) = \vartheta.$$

Therefore, $\varrho = \zeta$. □

The result Theorem 1.3.21 for generalized Lipschitz mappings on CbM space over Banach algebra [23] can be directly proved by using our results, Theorem 2.3.1 and Theorem 2.3.2 when we define $s(\eta,\xi) = b$ for some $b \geq 1$.

Corollary 2.3.3. Let $\mathfrak{P}$ be the associated cone in a Banach algebra $\mathcal{A}$ and $(\mathfrak{Q}, d_s)$ be a complete CM space over $\mathcal{A}$. Let $\digamma$ be a self-map on $\mathfrak{Q}$ such that for all $p, q \in \mathfrak{Q}$;

$$d_s(\digamma p, \digamma q) \preceq \kappa d_s(p,q), \tag{2.3.3}$$

where $\kappa \in \mathfrak{P}$ be such that $r(\kappa) < 1$. Then for every $\sigma_0 \in \mathfrak{Q}$, the iterative sequence $\sigma_n = \digamma(\sigma_{n-1}) = \digamma^n \sigma_0$ converges to ϖ which is a unique $\mathcal{F} \cdot \mathcal{P}$ of $\digamma$.

Proof. Take $s(\eta, \xi) = 1$ for all $\eta, \xi \in \mathfrak{Q}$ in Theorem 2.3.1, we get the required result. □

Remark 2.3.4. 1. If we define $s(p, q) = b$ for some $b \geq 1$ in Theorem 2.3.1 and in Theorem 2.3.2, we get the main results of [23] for $CbMS$ over Banach algebras.

2. By using Remark 2.1.4, we obtain Theorem 1.3.6 as a corollary of our Theorem 2.3.1.

3. If we take $s(x, y) = b$ for some $b \geq 1$ in Theorem 2.2.2 and in Theorem 2.2.4, we get the main results of [56] for $CbMS$ over Banach algebra.

2.4 Consequences and applications

This section is devoted to some important consequences of our results which generalizes the results of Hussain *et* al. [24], Xu and Radenovic [61], Malhotra *et* al. [33], Malhotra *et* al. [34] and the results of Liu and Xu [32]. We also have added the applications of our proved results in existence of solution of integral equations.

Definition 2.4.1. Let $\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ be a function for a non-empty set $\mathfrak{Q}$. A mapping $\digamma : \mathfrak{Q} \to \mathfrak{Q}$ is said to be an α-admissible mapping if $\alpha(\eta, \xi) \geq 1 \implies \alpha(\digamma\eta, \digamma\xi) \geq 1$.

Definition 2.4.2. Let $(\mathfrak{Q}, d_s)$ be a complete $ECbMS$ over $\mathcal{A}$ and $\mathfrak{P}$ be the underlying solid cone in $\mathcal{A}$. A self-map $\digamma$ on $\mathfrak{Q}$ is said to be generalized α-Lipschitz contraction if for all $\eta, \xi \in \mathfrak{Q}$ with $\alpha(\eta, \xi) \geq 1$ satisfies the following:

$$d_s(\digamma\eta, \digamma\xi) \preceq \kappa d_s(\eta, \xi),$$

where $\kappa \in \mathfrak{P}$ is such that $r(\kappa) < \frac{1}{s(\eta,\xi)}$ and for each $\varpi_0 \in \mathfrak{Q}$, $\lim_{n,m\to\infty} s(\varpi_{n+1}, \varpi_m) < \dfrac{1}{||\kappa||}$.

The following theorem becomes special case of Theorem 2.2.2 if we define $\alpha : \mathfrak{Q}\times\mathfrak{Q} \to [0,\infty)$ by $\alpha(\eta,\xi) = s(\eta,\xi) \geq 1$ for all $\eta,\xi \in \mathfrak{Q}$ and take $\kappa = \varpi_1$, $\varpi_2 = \varpi_3 = \vartheta$.

Theorem 2.4.3. Let $(\mathfrak{Q}, d_s)$ be a complete *ECbMS* over $\mathcal{A}$ and $\mathfrak{P}$ be the associated solid cone. Let $F : \mathfrak{Q} \to \mathfrak{Q}$ satisfies the generalized α-Lipschitz contraction with Lipschitz constant κ such that:

1. F is α-admissible;
2. there exists $\varpi_0 \in \mathfrak{Q}$ such that $\alpha(\varpi_0, F\varpi_0) \geq 1$;
3. F is continuous or if a sequence $\{\varpi_n\} \in \mathfrak{Q}$ with $\alpha(\varpi_n, \varpi_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\varpi_n \to \varpi$ implies that for every $n \in \mathbb{N}$, $\alpha(\varpi_n, \varpi) \geq 1$.

Then there is a point ϱ in $\mathfrak{Q}$ which is a $\mathcal{F}\cdot\mathcal{P}$ of the map F.

For uniqueness of this point, we use the following extra condition:

$$\forall \varrho, \zeta \in Fix(F), \text{ there exists } \eta \in \mathfrak{Q} \text{ such that } \alpha(\varrho,\eta) \geq 1 \text{ and } \alpha(\zeta,\eta) \geq 1. \qquad (2.4.1)$$

Theorem 2.4.4. If we add the condition (2.4.1) in the assumption of Theorem 2.4.3, then the $\mathcal{F}\cdot\mathcal{P}$ is unique.

Proof. The assertion follows simply by using Theorem 2.4.3 and Theorem 2.2.4. □

Remark 2.4.5.
1. If we take $s(\eta,\xi) = b$ for some $b \geq 1$, then we obtain the main results due to Hussain *et al*. [24, Theorems 3.1 and 3.2].
2. Results due to in Malhotra *et al*. [33, Theorems 3.1, 3.2 and 3.5] become special cases of Theorems 2.4.3 and 2.4.4 for $s(\eta,\xi) = 1$, $\varpi_1 = 1$ and $\varpi_2 = \varpi_3 = \vartheta$.
3. Results due to Malhotra *et al*. [34, Theorems 3.1, 3.2 and 3.3] become special cases of Theorems 2.4.3 and 2.4.4 for $s(\eta,\xi) = 1$, $\varpi_1 = \vartheta$ and $\varpi_2 = \varpi_3$.

If the given *ECbMS* over $\mathcal{A}$ is a partially ordered, then we have the following theorem.

Theorem 2.4.6. Let $s : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty)$ be a function, where $(\mathfrak{Q}, \trianglerighteq)$ is a partially ordered set. Let $(\mathfrak{Q}, d_s)$ be a complete $ECbMS$ over $\mathcal{A}$ with underlying solid cone $\mathfrak{P}$. Assume a self-map F on $\mathfrak{Q}$ is non-decreasing with respect to $\trianglerighteq$ and satisfies the following conditions:

(1) there exists vectors $\varpi_1, \varpi_2, \varpi_3 \in \mathfrak{P}$ such that $2s(\eta, \xi)r(\varpi_1) + (s(\eta, \xi) + 1)r(\varpi_2 + \varpi_3) < 2$, $d_s(F\eta, F\xi) \preceq \varpi_1 d_s(\eta, \xi) + \varpi_2 d_s(\eta, F\eta) + \varpi_3 d_s(\xi, F\xi)$ for all $\eta, \xi \in \mathfrak{Q}$ with $\eta \trianglerighteq \xi$ and for each $u_0 \in \mathfrak{Q}$ with $u_n = F^n u_0$,

$$\lim_{n,m\to\infty} s(u_{n+1}, u_m) < \frac{1}{||\kappa||} \text{ where } \kappa = (2e - \varpi)^{-1}(2\varpi_1 + \varpi) \text{ for } \varpi = \varpi_2 + \varpi_3;$$

(2) $\exists\, \varpi_0 \in \mathfrak{Q}$ such that $\varpi_0 \trianglerighteq F\varpi_0$;

(3) F is continuous or if $\{\varpi_n\}$ is a non-decreasing sequence in $\mathfrak{Q}$ with respect to $\trianglerighteq$ such that $\varpi_n \to \varpi \in \mathfrak{Q}$ as $(n \to \infty)$, then $\varpi_n \trianglerighteq \varpi$ for all $n \in \mathbb{N}$.

Then there exists a point ϱ in $\mathfrak{Q}$ which is a $\mathcal{F} \cdot \mathcal{P}$ of the map F.

Proof. Define a function $\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ by

$$\alpha(\eta, \xi) = \begin{cases} s(\eta, \xi) & \text{if } \eta \trianglerighteq \xi; \\ 0 & \text{elsewhere.} \end{cases}$$

By condition (1), we can say that F is a generalized R-type contraction. Now since F is non-decreasing, so F is a generalized α-admissible mapping. Definition of α and condition (2) implies that there exists $\varpi_0 \in \mathfrak{Q}$ such that $\alpha(\varpi_0, F\varpi_0) = s(\varpi_0, F\varpi_0)$. By condition (3) we can see that either F is continuous or $(\mathfrak{Q}, d_s)$ is regular. It follows that all the necessary conditions of Theorem 2.2.2 are satisfied, so we conclude that there exists a point in $\mathfrak{Q}$ which is a $\mathcal{F} \cdot \mathcal{P}$ of the map F. □

Corollary 2.4.7. Let $s : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty)$ be a function, where $(\mathfrak{Q}, \trianglerighteq)$ is a partially ordered set. Let $(\mathfrak{Q}, d_s)$ be a complete $ECbMS$ over $\mathcal{A}$ with underlying solid cone $\mathfrak{P}$. Let F be a self-map on $\mathfrak{Q}$ which is non-decreasing with respect to $\trianglerighteq$ and the following assumptions hold:

(1) there exists vectors $\kappa \in \mathfrak{P}$ such that $r(\kappa) < \frac{1}{s(\eta,\xi)}$, $d_s(\digamma\eta, \digamma\xi) \preceq \kappa d_s(\eta, \xi)$ for all $\eta, \xi \in \mathfrak{Q}$ with $\eta \trianglerighteq \xi$ and for each $u_0 \in \mathfrak{Q}$ with $u_n = \digamma^n u_0$,

$$\lim_{n,m\to\infty} s(u_{n+1}, u_m) < \frac{1}{||\kappa||};$$

(2) $\exists\ \varrho_0 \in \mathfrak{Q}$ with $\varrho_0 \trianglerighteq \digamma\varrho_0$;

(3) $\digamma$ is continuous or if $\{\varrho_n\}$ is a non-decreasing sequence in $\mathfrak{Q}$ with respect to $\trianglerighteq$ such that $\varrho_n \to \varrho \in \mathfrak{Q}$, then $\varrho_n \trianglerighteq \varrho$ for all $n \in \mathbb{N}$.

Then there exists a unique point ϱ in $\mathfrak{Q}$ which is a $\mathcal{F} \cdot \mathcal{P}$ of the map $\digamma$.

Proof. The assertion follows directly if we take $\varpi_1 = \kappa$ and $\varpi_2 = \varpi_3 = \vartheta$ in Theorem 2.4.6. □

Remark 2.4.8. 1. Theorem 2.4.6 reduces the main result due to Vujakovic [56, Theorem 3.6] for $s(p, q) = b$ and $b \geq 1$.

2. Corollary 2.4.7 reduces to the main results due to Hussain *et al.* [24, Theorems 4.2 and 4.3] for $s(p, q) = b$ and $b \geq 1$.

3. Corollary 2.4.7 reduces to the results due to Nieto and Rodreguez-Lopez [38, Theorems 2.1 and 2.2] for $s(p, q) = 1$ and $\mathcal{A} = \mathbb{R}$.

Following is given a lemma which is proved for cone b-$\mathcal{M} \cdot \mathcal{S}$s in [62] and the proof in *ECbM* spaces over Banach algebras are same.

Lemma 2.4.9. Let Ψ be a Lebesgue measurable function defined on $[0, 1]$ with $k \geq 1$. Then we have

$$\left|\int_0^1 \Psi(s)ds\right|^k \leq \int_0^1 |\Psi(s)|^k ds.$$

Example 2.4.10. Let $\mathcal{A} = \mathfrak{Q} = C^1_{\mathbb{R}}[0, 1]$ be the space of all real valued differentiable functions with continuous derivative defined on $[0, 1]$. If we take $\mathfrak{P} = \{h \in \mathcal{A} \ : \ h(a) \geq 0 \ : \ \forall a \in [0, 1]\}$, then $\mathfrak{P}$ is a cone in $\mathcal{A}$. Define a map $d_s : \mathfrak{Q} \times \mathfrak{Q} \to \mathcal{A}$ by:

$$d_s(\eta, \xi)(t) = ||\eta - \xi||^p_\infty e^t.$$

Then d_s is an extended cone b-metric over $\mathcal{A}$ with $s : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty)$ defined as $s(\eta, \xi)(t) = \max |\eta(t)| + \max |\xi(t)| + 2^p$.

Consider the following nonlinear integral equation

$$f(t) = \int_0^1 \digamma(t, f(\eta))ds, \tag{2.4.2}$$

where $\digamma$ satisfies the following:

(a) $\digamma : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous;

(b) there exists a constant $M \in [0, \frac{1}{2})$ such that for each $f_0 \in \mathfrak{Q}$ we have that: $M^p < \frac{1}{\lim_{n,m\to\infty} s(f_{n+1}, f_m)}$ and for all $t \in [0, 1]$ and $\eta, \xi \in \mathbb{R}$, $|\digamma(t, \eta) - \digamma(t, \xi)| \leq M|\eta - \xi|$.

Theorem 2.4.11. The equation (2.4.2) has a unique solution in $\mathfrak{Q} = C^1_{\mathbb{R}}$.

Proof. To show that (2.4.2) has a unique solution, define $\mathcal{G} : \mathfrak{Q} \to \mathfrak{Q}$ by

$$\mathcal{G}(g)(p) = \int_0^1 \digamma(p, g(s))ds.$$

By using Lemma 2.4.9 we have

$$\begin{aligned}
d_s(\mathcal{G}(f), \mathcal{G}(g))(t) &= e^t ||\mathcal{G}(f) - \mathcal{G}(g)||^p_\infty \\
&= e^t \max_{0\leq x\leq 1} |\mathcal{G}(f)(x) - \mathcal{G}(g)(x)|^p \\
&= e^t \max_{0\leq x\leq 1} \left| \int_0^1 \digamma(x, f(s))ds - \int_0^1 \digamma(x, g(s))ds \right|^p \\
&= e^t \max_{0\leq x\leq 1} \left| \int_0^1 (\digamma(x, f(s)) - \digamma(x, g(s)))\, ds \right|^p \\
&\leq e^t \max_{0\leq x\leq 1} \int_0^1 |\digamma(x, f(s)) - \digamma(x, g(s))|^p\, ds \\
&\leq e^t \int_0^1 (M\, |f(s) - g(s)|)^p ds \\
&= e^t M^p \int_0^1 |f(s) - g(s)|^p\, ds \\
&\leq e^t M^p \max_{0\leq s\leq 1} |f(s) - g(s)|^p\, ds \\
&= M^p d_s(f, g).
\end{aligned}$$

If we take $\kappa = M^p e$, then $r(\kappa) \leq ||M^p e|| = M^p < \frac{1}{\lim_{n,m\to\infty} s(f_{n+1}, f_m)}$. So all the conditions of Theorem 2.3.1 and thus there is a unique point in $\mathfrak{Q}$ which is a $\mathcal{F} \cdot \mathcal{P}$ of the map $\mathcal{G}$. Equivalently, 2.4.2 has a unique solution in $\mathfrak{Q} = C^1_{\mathbb{R}}$. $\square$

Chapter 3

Fixed points of single-valued dynamical systems on controlled cone metric type space over real Banach algebra

In this chapter, we introduce a new type of $\mathcal{M} \cdot \mathcal{S}$ over a real Banach algebra which we call a controlled cone metric type space over Banach algebra. By using such spaces we proved some $\mathcal{F} \cdot \mathcal{P}$ theorems for generalized R-type contraction and generalized lipschitz mapping. Our results extends/generalizes some previous well known results in the literature. The work of this chapter has been published in the Journal of Inequalities and Applications [54].

3.1 Controlled cone metric type spaces over real Banach algebras

We start this section by the definition of a $CCMT$ space over Banach algebra.

Definition 3.1.1. Let $\mathcal{A}$ be a real Banach algebra with cone $\mathfrak{P}$, $\mathfrak{Q}$ be a non empty set and $s : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty)$ be a mapping. A controlled cone metric type (in short $CCMT$) on $\mathfrak{Q}$ over a Banach algebra $\mathcal{A}$ is a function $d_s : \mathfrak{Q} \times \mathfrak{Q} \to \mathcal{A}$ such that:

(E_1) $d_s(p, q) \succeq \theta$ and $d_s(p, q) = \theta$ iff $p = q$ for all $p, q \in \mathfrak{Q}$;

(E_2) $d_s(p, q) = d_s(q, p)$ for all $p, q \in \mathfrak{Q}$;

(E_3) $d_s(p, v) \preceq s(p, q)d_s(p, q) + s(q, v)d_s(q, v)$ for all $p, q, v \in \mathfrak{Q}$.

The pair$(\mathfrak{Q}, d_s)$ is then called a controlled cone metric type space over a Banach algebra $\mathcal{A}$ (in short $CCMTS$ over $\mathcal{A}$).

Remark 3.1.2. It is clear that the class of $CCMTS$ over $\mathcal{A}$ is larger than the classes of CbM spaces and cone $\mathcal{M} \cdot \mathcal{S}$s over Banach algebras.
The definition of Cauchy sequences, convergent sequences and completeness for $CCMTS$ over $\mathcal{A}$ are same as cone b-$\mathcal{M} \cdot \mathcal{S}$s over a Banach algebra defined in 1.3.19.
In general d_s is not necessarily a continuous function but in this paper, d_s will always mean a continuous function $d_s : \mathfrak{Q} \times \mathfrak{Q} \to \mathcal{A}$.

Example 3.1.3. Let $\mathfrak{Q} = \{1, 2, 3\}$ and $s : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty)$ be defined as $s(p, q) = 1 + p + q$. Consider the real Banach algebra $\mathcal{A} = \mathbb{R}^2$ together with a solid cone
$\mathfrak{P} = \{(a, b) \in \mathbb{R}^2 : a,\, b \geq 0\}$. If we define $d_s : \mathfrak{Q} \times \mathfrak{Q} \to \mathcal{A}$ by:

$$\begin{aligned}
&d_s(1, 2) = d_s(2, 1) = (100, 100);\\
&d_s(1, 3) = d_s(3, 1) = (1200, 1200);\\
&d_s(3, 2) = d_s(2, 3) = (800, 800);\\
&d_s(1, 1) = d_s(2, 2) = d_s(3, 3) = (0, 0) = \theta.
\end{aligned}$$

Clearly the first and second conditions of a $CCMTS$ over $\mathcal{A}$ are satisfied. For the third condition we have:

$s(1, 3)d_s(1, 3)+s(3, 2)d_s(3, 2)-d_s(1, 2) = 5(1200, 1200)+6(800, 800)-(100, 100) = (10700, 10700) \in \mathfrak{P}$;

$s(1, 2)d_s(1, 2)+s(2, 3)d_s2, 3)-d_s(1, 3) = 4(100, 100)+6(800, 800)-(1200, 1200) = (4000, 4000) \in \mathfrak{P}$;

$s(2, 1)d_s(2, 1)+s(1, 3)d_s1, 3)-d_s(2, 3) = 4(100, 100)+5(1200, 1200)]-(600, 600) = (5800, 5800) \in \mathfrak{P}$.

Hence for all $p, q, v \in \mathfrak{Q}$,

$$d_s(p, v) \preceq s(p, q)d_s(p, q) + s(q, v)d_s(q, v).$$

Thus $(\mathfrak{Q}, d_s)$ is a $CCMT$ space over $\mathcal{A} = \mathbb{R}^2$.

Remark 3.1.4. Let $(\mathfrak{Q}, d_s)$ be a $CCMTS$ over $\mathcal{A}$ with $s : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty)$. If $\mathcal{A} = \mathbb{R}$ and $\mathfrak{P} = [0, \infty)$, then $(\mathfrak{Q}, d_s)$ is a CMT space.

We now define generalized α-admissible mapping and α-regular space in term of controlled cone metric type spaces over Banach algebras.

Definition 3.1.5. Consider $(\mathfrak{Q}, d_s)$, a $CCMTS$ over $\mathcal{A}$ and $\mathfrak{P}$ an underlying solid cone in $\mathcal{A}$. Let $\mathcal{G} : \mathfrak{Q} \to \mathfrak{Q}$ and $\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ be mappings. Then:

(i) F is said to be a generalized α-admissible mapping if for $p, q \in \mathfrak{Q}, \alpha(p, q) \geq s(p, q)$ implies that $\alpha(Fp, Fq) \geq s(Fp, Fq)$;

(ii) $(\mathfrak{Q}, d_s)$ is said to be α-regular if any sequence $\{u_k\} \in \mathfrak{Q}$ with $\alpha(u_k, u_{k+1}) \geq s(u_k, u_{k+1})$ for all $k \in \mathbb{N}$ and $u_k \to q$ implies that $\alpha(u_k, q) \geq s(u_k, q)$.

3.2 Generalized Reich type contraction in controlled cone metric type spaces

In this section, we have introduced generalized R-type mapping in the setting of $CCMT$ space over Banach algebra. Later on, we proved some results and gave an example to prove the validity of the results.

Definition 3.2.1. Let $(\mathfrak{Q}, d_s)$ be a $CCMTS$ over $\mathcal{A}$ with $\mathfrak{P}$ an underlying solid cone and $\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ be a mapping. Then the mapping $F : \mathfrak{Q} \to \mathfrak{Q}$ is called the generalized R-type contraction if there exist three vectors v_1, v_2, v_3 in $\mathfrak{P}$ such that for every $p, q \in \mathfrak{Q}$ with $\alpha(p, q) \geq s(p, q)$ we have:

(i) $2s(p, q)r(v_1) + (s(p, q) + 1)r(v_2 + v_3) < 2$ and for each $u_0 \in \mathfrak{Q}$ with $u_m = F^m u_0$,

$$\lim_{m,i\to\infty} \frac{s(p_{i+1}, p_{i+2})}{s(p_i, p_{i+1})} s(p_{i+1}, p_m) < \frac{1}{||\kappa||} \text{ where } \kappa = (2e - v)^{-1}(2v_1 + v) \text{ for } v = v_2 + v_3;$$

(ii) $d_s(\digamma p, \digamma q) \preceq v_1 d_s(p,q) + v_2 d_s(p, \digamma p) + v_3 d_s(q, \digamma q)$.

One of the main results of this chapter is given as follows:

Theorem 3.2.2. Let $(\mathfrak{Q}, d_s)$ be a complete $CCMTS$ over $\mathcal{A}$ with $\mathfrak{P}$ an underlying solid cone and $\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ a mapping. Suppose that the mapping $\digamma : \mathfrak{Q} \to \mathfrak{Q}$ is a generalized R-type contraction with vectors $v_1, v_2, v_3 \in \mathfrak{P}$ such that:

1. $\digamma$ is a generalized α-admissible mapping;
2. there exists $u_0 \in \mathfrak{Q}$ such that $\alpha(u_0, \digamma u_0) \geq s(u_0, \digamma u_0)$;
3. $(\mathfrak{Q}, d_s)$ is regular or $\digamma$ is continuous.

Then there exists a point ϱ in $\mathfrak{Q}$ which is a $\mathcal{F} \cdot \mathcal{P}$ of the map $\digamma$.

Proof. Let u_0 be a point in $\mathfrak{Q}$ such that $\alpha(u_0, \digamma u_0) \geq s(u_0, \digamma u_0)$. For $u_0 \in \mathfrak{Q}$, if we define $u_1 = \digamma u_0,\ u_2 = \digamma u_1 = \digamma(\digamma u_0) = T^2 u_0, \cdots,\ u_{n+1} = \digamma u_n = \digamma^{n+1} u_0$, then

$$\alpha(u_0, u_1) \geq s(u_0, u_1).$$

But $\digamma$ is generalized α-admissible, so

$$\alpha(\digamma u_0, \digamma u_1) = \alpha(u_1, u_2) \geq s(u_1, u_2),$$

and so by induction we get

$$\alpha(u_n, u_{n+1}) \geq s(u_n, u_{n+1}).$$

By using Definition 3.2.1, we have

$$\begin{aligned} d_s(u_n, u_{n+1}) &= d_s(\digamma u_{n-1}, \digamma u_n) \\ &\preceq v_1 d_s(u_{n-1}, u_n) + v_2 d_s(u_{n-1}, \digamma u_{n-1}) + v_3 d_s(u_n, \digamma u_n), \text{i.e.} \\ &(e - v_3) d_s(u_n, u_{n+1}) \preceq (v_1 + v_2) d_s(u_{n-1}, u_n). \qquad (3.2.1) \end{aligned}$$

Similarly

$$
\begin{aligned}
d_s(u_{n+1}, u_n) &= d_s(Fu_n, Fu_{n-1}) \\
&\preceq v_1 d_s(u_n, u_{n-1}) + v_2 d_s(u_n, Fu_n) + v_3 d_s(u_{n-1}, Fu_{n-1}), \text{i.e.}
\end{aligned}
$$

$$(e - v_2)d_s(u_{n+1}, u_n) \preceq (v_1 + v_3)d_s(u_{n-1}, u_n). \tag{3.2.2}$$

Adding (3.2.1) and (3.2.2), we obtain

$$(2e - v_2 - v_3)d_s(u_n, u_{n+1}) \preceq (2v_1 + v_2 + v_3)d_s(u_{n-1}, u_n).$$

If we take $v = v_2 + v_3$, then we obtain

$$(2e - v)d_s(u_{n+1}, u_n) \preceq (2v_1 + v)d_s(u_{n-1}, u_n). \tag{3.2.3}$$

Note that

$$2r(v) \leq (s(u_n, u_{n+1}) + 1)r(v) \leq 2r(v_1) + (s(u_n, u_{n+1}) + 1)r(v) < 2.$$

Hence $r(v) < 1 < 2 \implies r(v) < 2$. Thus by Lemma 1.3.11, we obtain that the element $2e - v$ is invertible and $(2e - v)^{-1} = \sum_{n=0}^{\infty} \frac{v^n}{2^{n+1}}$, $r((2e - v)^{-1}) < \frac{1}{2-r(v)}$.
Thus (3.2.3) becomes

$$d_s(u_n, u_{n+1}) \preceq \kappa d_s(u_{n-1}, u_n), \tag{3.2.4}$$

where $\kappa = (2e - v)^{-1}(2v_1 + v)$. The inequality (3.2.4) then implies that for all $n \in \mathbb{N}$

$$d_s(u_n, u_{n+1}) \preceq \kappa d_s(u_{n-1}, u_n) \preceq \kappa^2 d_s(u_{n-1}, u_n) \preceq \cdots \preceq \kappa^n d_s(u_0, u_1). \tag{3.2.5}$$

Now if we take $m > n$, then by using (3.2.5) and Definition 3.1.1, (iii) we have

$$
\begin{aligned}
d_s(u_n,u_m) &\preceq s(u_n,u_{n+1})d_s(u_n,u_{n+1}) + s(u_{n+1},u_m)d_s(u_{n+1},u_m)\\
&\preceq s(u_n,u_{n+1})d_s(u_n,u_{n+1}) + s(u_{n+1},u_m)s(u_{n+1},u_{n+2})d_s(u_{n+1},u_{n+2})\\
&+ s(u_{n+1},u_m)s(u_{n+2},u_m)d_s(u_{n+2},u_m)\\
&\preceq s(u_n,u_{n+1})d_s(u_n,u_{n+1}) + s(u_{n+1},u_m)s(u_{n+1},u_{n+2})d_s(u_{n+1},u_{n+2})\\
&+ s(u_{n+1},u_m)s(u_{n+2},u_m)s(u_{n+2},u_{n+3})d_s(u_{n+2},u_{n+3})\\
&+ s(u_{n+1},u_m)s(u_{n+2},u_m)s(u_{n+3},u_m)d_s(u_{n+3},u_m)\\
&\preceq\\
&\vdots\\
&\preceq s(u_n,u_{n+1})d_s(u_n,u_{n+1}) + \sum_{i=n+1}^{m-2} s(u_i,u_{i+1})d_s(u_i,u_{i+1})\left(\prod_{j=n+1}^{i} s(u_j,u_m)\right)\\
&+ d_s(u_{m-1},u_m)\left(\prod_{k=n+1}^{m-1} s(u_k,u_m)\right)\\
&\preceq s(u_n,u_{n+1})\kappa^n d_s(u_0,u_1) + \sum_{i=n+1}^{m-2} s(u_i,u_{i+1})\kappa^i d_s(u_0,u_1)\left(\prod_{j=n+1}^{i} s(u_j,u_m)\right)\\
&+ \kappa^{m-1}d_s(u_0,u_1)\left(\prod_{k=n+1}^{m-1} s(u_k,u_m)\right)\\
&\preceq s(u_n,u_{n+1})\kappa^n d_s(u_0,u_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i} s(u_j,u_m)\right) s(u_i,u_{i+1})\kappa^i d_s(u_0,u_1)\\
&+ \left(\prod_{k=n+1}^{m-1} s(u_k,u_m)\right) s(u_{m-1},u_m)\kappa^{m-1}d_s(u_0,u_1)\\
&\preceq s(u_n,u_{n+1})\kappa^n d_s(u_0,u_1) + \sum_{i=n+1}^{m-1} s(u_i,u_{i+1})\kappa^i d_s(u_0,u_1)\left(\prod_{j=n+1}^{i} s(u_j,u_m)\right)\\
&\preceq \kappa^n d_s(u_0,u_1)\left(\prod_{j=0}^{n} s(u_n,u_{n+1})\right) + \sum_{i=n+1}^{m-1} s(u_i,u_{i+1})\kappa^i d_s(u_0,u_1)\left(\prod_{j=0}^{i} s(u_j,u_m)\right)\\
&= d_s(u_0,u_1)\sum_{i=n}^{m-1}\left(\prod_{j=0}^{i} s(u_j,u_m)\right) s(u_i,u_{i+1})\kappa^i.
\end{aligned}
$$

In the above steps we use the fact that $s(p,q) \geq 1$ and thus $x \preceq s(p,q)x$ for any $x \in \mathcal{A}$.
Let

$$a_n = \left(\prod_{j=0}^{n} s(u_j, u_m)\right) \kappa^n s(u_n, u_{n+1}) \ \text{ and } \ S = \sum_{n=1}^{\infty} a_n.$$

Since by Definition 3.2.1, $||\kappa|| \lim_{m,i\to\infty} \frac{s(u_{i+1},u_{i+2})}{s(u_i,u_{i+1})} s(u_{i+1}, u_m) < 1$, so the series S converges absolutely. Because by using ratio test we have

$$\begin{aligned} \lim_{n\to\infty} \frac{||a_{n+1}||}{||a_n||} &\leq \lim_{n\to\infty} \frac{||\kappa||||\kappa^n|| \left(\prod_{j=1}^{n+1} s(u_j, u_m)\right) s(u_{n+1}, u_{n+2})}{||\kappa^n|| \left(\prod_{j=1}^{n} s(u_j, u_m)\right) s(u_n, u_{n+1})} \\ &= ||\kappa|| \lim_{n,m\to\infty} \frac{s(u_{n+1}, u_{n+2})}{s(u_{n+1}, u_{n+1})} s(u_{n+1}, u_m) < 1. \end{aligned}$$

But $\mathcal{A}$ is a Banach algebra and the series S is absolutely convergent, so it converges in $\mathcal{A}$. Thus $S_{m-1} - S_n = \left[\sum_{i=n}^{m-1} \left(\prod_{j=0}^{i} s(u_j, u_m)\right) s(u_i, u_{i+1})\kappa^i\right] \to \theta$ as $n, m \to \infty$ and so is $d_s(u_0, u_1)(S_{m-1} - S_n)$. Hence by Lemma 1.3.15, for every $\theta \ll \delta$, there exists a natural number N such that for all $n \geq N$, we have $d_s(u_n, u_m) \ll \delta$. Thus by Definition 1.3.19 the sequence $\{u_n\}$ is a Cauchy sequence in $\mathfrak{Q}$. But $\mathfrak{Q}$ is complete so there exists $\varrho \in \mathfrak{Q}$ such that $u_n \to \varrho$ as $n \to \infty$. We show that ϱ is a $\mathcal{F} \cdot \mathcal{P}$ of the map F.
Suppose that F is continuous. It follows that $u_{n+1} = F u_n \to F\varrho$ as $n \to \infty$. But limit of a sequence is unique, so we must have $F\varrho = \varrho$. Hence ϱ is a $\mathcal{F} \cdot \mathcal{P}$ of the map F in this case.
However, if $(\mathfrak{Q}, d_s)$ is α-regular, then by Definition 3.1.5 we have

$$\alpha(u_n, \varrho) \geq s(u_n, \varrho), \qquad \text{for all } n \in \mathbb{N},$$

and

$$\begin{aligned} d_s(\varrho, F\varrho) &\preceq s(\varrho, Fu_n)d_s(\varrho, Fu_n) + s(Fu_n, F\varrho)d_s(Fu_n, F\varrho) \\ &\preceq s(\varrho, Fu_n)d_s(\varrho, Fu_n) + s(Fu_n, F\varrho)\big[v_1 d_s(u_n, \varrho) + v_2 d_s(u_n, Fu_n) + v_3 d_s(\varrho, F\varrho)\big] \\ &\preceq s(\varrho, Fu_n)d_s(\varrho, Fu_n) + s(Fu_n, F\varrho)v_1 d_s(u_n, \varrho) + s(Fu_n, F\varrho)v_3 d_s(\varrho, F\varrho) \\ &\quad + s(Fu_n, F\varrho)v_2\big[s(u_n, \varrho)d_s(u_n, \varrho) + s(\varrho, u_{n+1})d_s(\varrho, u_{n+1})\big] \\ &= s(\varrho, FFu_n)(e + s(u_{n+1}, F\varrho)v_2)d_s(\varrho, u_{n+1}) + s(u_{n+1}, F\varrho)v_3 d_s(\varrho, F\varrho) \\ &\quad + s(u_{n+1}, F\varrho)(v_1 + s(u_n, \varrho)v_2)d_s(u_n, \varrho), \end{aligned}$$

which further implies that

$$(e - s(u_{n+1}, F\varrho)v_3)d_s(\varrho, F\varrho) \preceq s(\varrho, Fu_n)(e + s(u_{n+1}, F\varrho)v_2)d_s(u_{n+1}, \varrho) + s(u_{n+1}, F\varrho)(v_1 + s(u_n, \varrho)v_2)d_s(u_n, \varrho). \tag{3.2.6}$$

Similarly,

$$\begin{aligned} d_s(\varrho, F\varrho) &\preceq s(\varrho, Fu_n)d_s(\varrho, Fu_n) + s(Fu_n, F\varrho)d_s(Fu_n, F\varrho) \\ &= s(\varrho, Fu_n)d_s(\varrho, Fu_n) + s(Fu_n, F\varrho)d_s(F\varrho, Fu_n) \\ &\preceq s(\varrho, Fu_n)d_s(\varrho, Fu_n) + s(Fu_n, F\varrho)\big[v_1 d_s(\varrho, u_n) + v_2 d_s(\varrho, F\varrho) + v_3 d_s(u_n, Fu_n)\big] \\ &\preceq s(\varrho, Fu_n)d_s(\varrho, Fu_n) + s(Fu_n, F\varrho)v_1 d_s(u_n, \varrho) + s(Fu_n, FF\varrho)v_2 d_s(\varrho, F\varrho) \\ &\quad + s(Fu_n, F\varrho)v_3\big[s(u_n, \varrho)d_s(u_n, \varrho) + s(\varrho, u_{n+1})d_s(\varrho, u_{n+1})\big] \\ &= s(\varrho, Fu_n)(e + s(u_{n+1}, F\varrho)v_3)d_s(\varrho, u_{n+1}) + s(u_{n+1}, F\varrho)v_2 d_s(\varrho, F\varrho) \\ &\quad + s(u_{n+1}, F\varrho)(v_1 + s(u_n, \varrho)v_2)d_s(u_n, \varrho), \end{aligned}$$

which further implies that

$$(e - s(u_{n+1}, F\varrho)v_2)d_s(\varrho, F\varrho) \preceq s(\varrho, Fu_n)(e + s(u_{n+1}, F\varrho)v_3)d_s(u_{n+1}, \varrho) + s(u_{n+1}, F\varrho)(v_1 + s(u_n, \varrho)v_3)d_s(u_n, \varrho). \tag{3.2.7}$$

Therefore, by adding (3.2.6) and (3.2.7) we get

$$\begin{aligned} (2e - s(u_{n+1}, F\varrho)v_2 - s(u_{n+1}, F\varrho)v_3)d_s(\varrho, F\varrho) &\preceq s(\varrho, u_{n+1})(2e + s(u_{n+1}, F\varrho)v_2 \\ &\quad + s(u_{n+1}, F\varrho)v_3)d_s(u_{n+1}, \varrho) \\ &\quad + s(\varrho, F\varrho)(2v_1 + s(\varrho, F\varrho)v_2 \\ &\quad + s(\varrho, F\varrho)v_3)d_s(u_n, \varrho), \ i.e. \end{aligned}$$

$$\begin{aligned} (2e - s(u_{n+1}, F\varrho)v)d_s(\varrho, F\varrho) &\preceq s(\varrho, u_{n+1})(2e + s(u_{n+1}, F\varrho)v)d_s(u_{n+1}, \varrho) \\ &\quad + s(\varrho, F\varrho)(2v_1 + s(\varrho, F\varrho)v)d_s(u_n, \varrho). \end{aligned} \tag{3.2.8}$$

We also notice from Definition 3.2.1 that

$$2r(s(u_{n+1}, F\varrho)v) = 2s(u_{n+1}, F\varrho)r(v) \leq 2s(u_{n+1}, F\varrho)r(v_1) + (s(u_{n+1}, F\varrho) + 1)r(v) < 2,$$

i.e. $r(s(u_{n+1}, \digamma\varrho)v) < 1 < 2$. Thus by Lemma 1.3.11, $2e - s(u_{n+1}, \digamma\varrho)v$ is invertible and so (3.2.8) implies that

$$d_s(\varrho, \digamma\varrho) \preceq (2e - s(u_{n+1}, \digamma\varrho)v)^{-1}\big[s(\varrho, u_{n+1})(2e + s(u_{n+1}, \digamma\varrho)v)d_s(u_{n+1}, \varrho) + s(\varrho, \digamma\varrho)(2v_1 + s(\varrho, \digamma\varrho)v)d_s(u_n, \varrho)\big]. \qquad (3.2.9)$$

By using Remark 1.3.20 the sequences $\{d_s(u_{n+1}, \varrho)\}$ and $\{d_s(u_n, \varrho)\}$ are c-sequences. Hence by Lemma 1.3.14, the sequence $\{\tau_1 d_s(u_{n+1}, \varrho)+\tau_2 d_s(u_n, \varrho)\}$ is a c-sequence (where $\tau_1 = (2e-s(u_{n+1}, \digamma\varrho)v)^{-1}s(\varrho, u_{n+1})(2e+s(u_{n+1}, \digamma\varrho)v)$ and $\tau_2 = (2e-s(u_{n+1}, \digamma\varrho)v)^{-1}s(\varrho, \digamma\varrho)(2v_1+ s(\varrho, \digamma\varrho)v)$). Therefore, for any $c \in \mathcal{A}$ with $c \gg \theta$, there exists $n_0 \in \mathbb{N}$ such that

$$d_s(\varrho, \digamma\varrho) \preceq \tau_1 d_s(u_{n+1}, \varrho) + \tau_2 d_s(u_n, \varrho) \ll c.$$

Which further implies by using Lemma 1.3.15 that $d_s(\varrho, \digamma\varrho) = \theta$. Therefore, $\digamma\varrho = \varrho$ and this complete the proof. □

Example 3.2.3. Let $\mathcal{A} = C^1_{\mathbb{R}}[0, 1]$ and $||f|| = ||f||_\infty + ||f^{'}||_\infty$. If we define point wise multiplication of functions on $\mathcal{A}$, then $\mathcal{A}$ becomes a real Banach algebra with identity $e(t) = 1$. If we take $\mathfrak{P} = \{f \in \mathcal{A} \; : \; f(t) \geq 0, \, t \in [0, 1]\}$, then $\mathfrak{P}$ is a non-normal cone (see [26]). Let $\mathfrak{Q} = [0, \infty)$ and $s : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty)$ be defined as $s(p, q) = 2 + p + q$. Define $d_s : \mathfrak{Q} \times \mathfrak{Q} \to \mathcal{A}$ by

$$d_s(p, q)(t) = (p - q)^2 e^t.$$

Then d_s is a controlled type cone metric over $\mathcal{A}$. Also note that $\mathfrak{Q}$ is complete with respect to d_s. If we define $\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ and $\digamma : \mathfrak{Q} \to \mathfrak{Q}$ by:

$$\alpha(p, q) = \begin{cases} s(p, q) & \text{if } p, q \in \; [0, 1]; \\ 0 & \text{elsewhere.} \end{cases}$$

$$\digamma(p) = \begin{cases} \frac{\sqrt{5}}{3}p & \text{if } p \in \; [0, 1]; \\ p + 1 & \text{if } p > 1. \end{cases}$$

Note that for every $p \in [0, 1]$, $\digamma p \in [0, 1]$. By choosing $v_1(t) = \frac{1}{9} + \frac{1}{9}t$, $v_2(t) = \frac{1}{18} + \frac{1}{18}t$ and $v_3(t) = \frac{1}{24} + \frac{1}{24}t$ we obtain that $r(v_1) = \frac{2}{9}$, $r(v) = r(v_2+v_3) = \frac{7}{36}$. Simple calculations show that $2(2)r(v_1)+(2+1)r(v)=\frac{53}{36}$ and so $\digamma$ is a generalized R-type contraction as;

$$\frac{1}{2(p+q+2)r(v_1)+((p+q+2)+1)r(v)}\leq\frac{1}{2(2)r(v_1)+(2+1)r(v)}=\frac{36}{53}.$$

Which further implies that $2s(p,q)r(v_1)+(s(p,q)+1)r(v)\leq\frac{53}{36}<2$. Also we have $\lim_{m,i\to\infty}\frac{s(p_{i+1},p_{i+2})}{s(p_i,p_{i+1})}s(p_{i+1},p_m)=2$ and $||\kappa||=||(2e-v)^{-1}(2v_1+v)||\leq\left(\frac{72}{130}\right)\left(\frac{46}{72}\right)=\frac{23}{65}<\frac{1}{2}=\lim_{m,i\to\infty}\frac{s(p_i,p_{i+1})}{s(p_{i+1},p_{i+2})}s(p_{i+1},p_m)$. Similarly by easily calculation one can show that

$$d_s(\digamma p,\digamma q)\preceq v_1d_s(p,q)+v_2d_s(p,\digamma p)+v_3d_s(q,\digamma q).$$

Next we show that there is a point u_0 in $\mathfrak{Q}$ such that $\alpha(u_0,\digamma u_0)\geq s(u_0,\digamma u_0)$. Indeed, for $u_0=1$, we have

$$\alpha(1,\digamma 1)=\alpha(1,\frac{\sqrt{5}}{3})\geq s(1,\frac{\sqrt{5}}{3})=s(1,\digamma 1).$$

Next we show that $\digamma$ is a generalized α-admissible mapping. In fact, if $p,q\in\mathfrak{Q}$ are such that $\alpha(p,q)\geq s(p,q)$, then by definition of α, $p,q\in[0,1]$. Therefore, $\digamma p,\digamma q\in[0,1]$ and so

$$\alpha(\digamma p,\digamma q)\geq s(\digamma p,\digamma q).$$

Finally we show that $(\mathfrak{Q},d_s)$ is α-regular. If we assume a sequence $\{p_n\}$ in $\mathfrak{Q}$ such that $\alpha(p_n,p_{n+1})\geq s(p_n,p_{n+1})$ for all $n\in\mathbb{N}$ and $p_n\to q\in\mathfrak{Q}$ (as $n\to\infty$), then $\{p_n\}\subseteq[0,1]$. But $[0,1]$ is closed, so $q\in[0,1]$. This implies that $\alpha(p_n,q)\geq s(p_n,q)$ for all $n\in\mathbb{N}$. Hence all the conditions of Theorem 3.2.2 are satisfied, so $\digamma$ has a $\mathcal{F}\cdot\mathcal{P}$ $\varrho=0$ (say).

Theorem 3.2.4. Let $\mathcal{A}$ be a Banach algebra and $\mathfrak{P}$ be a solid cone in $\mathcal{A}$. Let $(\mathfrak{Q},d_s)$ be a complete $CCMTS$ over $\mathcal{A}$ and $\alpha:\mathfrak{Q}\times\mathfrak{Q}\to[0,\infty)$ be a mapping. Suppose that the mapping $\digamma:\mathfrak{Q}\to\mathfrak{Q}$ is a generalized R-type contraction with vectors v_1,v_2,v_3 in $\mathfrak{P}$ such that v_1 commutes with v_2+v_3 and:

1. $\digamma$ is a generalized α-admissible;
2. $\exists\, u_0\in\mathfrak{Q}$ such that $\alpha(u_0,\digamma u_0)\geq s(u_0,\digamma u_0)$;
3. $\digamma$ is continuous or $(\mathfrak{Q},d_s)$ is regular;

4. for any two $\mathcal{F}\cdot\mathcal{P}$s ϖ, ζ of $\digamma$, there exists $z \in \mathfrak{Q}$ such that $\alpha(\varpi, z) \geq s(\varpi, z)$ and $\alpha(\zeta, z) \geq s(\zeta, z)$.

Then there exists a unique $\mathcal{F}\cdot\mathcal{P}$ of the map $\digamma$.

Proof. Using Theorem 3.2.2 and the first three given condition we can say that there exists a point $\varrho \in \mathfrak{Q}$ which is a $\mathcal{F}\cdot\mathcal{P}$ of the map $\digamma$. We show that this point is unique and for this let $\zeta \in Fix(\digamma)$ such that $\varrho \neq \zeta$. Then by using Condition 4, there exists $z \in \mathfrak{Q}$ such that

$$\alpha(\varrho, z) \geq s(\varrho, z) \quad \text{and} \quad \alpha(\zeta, z) \geq s(\zeta, z). \tag{3.2.10}$$

Since $\digamma$ is a generalized α-admissible mapping and $\varrho, \zeta \in Fix(\digamma)$ so by (3.2.10) we get

$$\alpha(\varrho, \digamma^i z) \geq s(\varrho, \digamma^i z) \quad \text{and} \quad \alpha(\zeta, \digamma^i z) \geq s(\zeta, \digamma^i z)\ , \quad \text{for all } i \in \mathbb{N}. \tag{3.2.11}$$

By using Definition 3.2.1 and (3.2.11) we obtain

$$\begin{aligned} d_s(\varrho, \digamma^i z) &= d_s(\digamma\varrho, \digamma(\digamma^{i-1} z)) \\ &\preceq v_1 d_s(\varrho, \digamma^{i-1} z) + v_2 d_s(\varrho, \digamma\varrho) + v_3 d_s(\digamma^{i-1} z, \digamma^i z) \\ &\preceq v_1 d_s(\varrho, \digamma^{i-1} z) + v_3 s(\digamma^{i-1} z, \varrho) d_s(\digamma^{i-1} z, \varrho) + v_3 s(\varrho, \digamma^i z) d_s(\varrho, \digamma^i z)], \end{aligned}$$

which further implies that

$$(e - (\varrho, \digamma^i z) v_3) d_s(\varrho, \digamma^i z) \preceq (v_1 + s(\digamma^{i-1} z, \varrho) v_3) d_s(\varrho, \digamma^{i-1} z). \tag{3.2.12}$$

Similarly,

$$\begin{aligned} d_s(\digamma^i z, \varrho) &= d_s(\digamma(\digamma^{i-1} z), \digamma\varrho) \\ &\preceq v_1 d_s(\digamma^{i-1} z, \varrho) + v_2 d_s(\digamma^{i-1} z), \digamma^i z)) + v_3 d_s(\varrho, \digamma\varrho) \\ &\preceq v_1 d_s(\digamma^{i-1} z, \varrho) + v_2 s(\digamma^{i-1} z, \varrho) d_s(\digamma^{i-1} z, \varrho) + (\varrho, \digamma^i z) v_2 d_s(\varrho, \digamma^i z)], \end{aligned}$$

which further implies that

$$(e - (\varrho, \digamma^i z) v_2) d_s(\digamma^i z, \varrho) \preceq (v_1 + s(\digamma^{i-1} z, \varrho)) v_2) d_s(\digamma^{i-1} z, \varrho). \tag{3.2.13}$$

Adding (3.2.12) and (3.2.13) we have

$(2e-s(\varrho, F^i z)v_2-s(\varrho, F^i z)v_3)d_s(\varrho, F^i z) \preceq (2v_1+s(F^{i-1}z, \varrho)v_2+s(F^{i-1}z, \varrho)v_3)d_s(\varrho, F^{i-1}z),$ *i.e.*

$$(2e - s(\varrho, F^i z)v)d_s(\varrho, F^i z) \preceq (2v_1 + s(F^{i-1}z, \varrho)v)d_s(\varrho, F^{i-1}z). \quad (3.2.14)$$

Note that $2r(s(\varrho, F^i z)v) \leq (s(\varrho, F^i z)+1)r(v) \leq 2s(\varrho, F^i z)r(v_1)+(s(\varrho, F^i z)+1)r(v) < 2$. So that $r(s(\varrho, F^i z)v) < 1 < 2$ and by Lemma 1.3.11, we can say that $2e - s(\varrho, F^i z)v$ is invertible and $(2e - s(\varrho, F^i z)v)^{-1} = \sum_{n=0}^{\infty} \frac{(s(\varrho, F^i z)v)^n}{2^{n+1}}$,
$r((2e - s(\varrho, F^i z)v)^{-1}) < \frac{1}{2-r(s(\varrho, F^i z)v)}$. Thus by (3.2.14) we have

$$d_s(\varrho, F^i z) \preceq (2e - s(\varrho, F^i z)v)^{-1}(2v_1 + s(F^{i-1}z, \varrho)v)d_s(\varrho, F^{i-1}z) \text{ , i.e.}$$

$$d_s(\varrho, F^i z) \preceq \tau d_s(\varrho, F^{i-1}z), \quad (3.2.15)$$

where $\tau = (2e - s(\varrho, F^i z)v)^{-1}(2v_1 + s(F^{i-1}z, \varrho)v)$. Therefore, we have

$$\begin{aligned} d_s(\varrho, F^i z) &\preceq \tau d_s(\varrho, F^{i-1}z) \\ &\preceq \tau^2 d_s(\varrho, F^{i-2}z) \\ &\vdots \\ &\preceq \tau^i d_s(\varrho, z) \text{ for all } i \in \mathbb{N}. \end{aligned}$$

Since v_1 commutes with $v_2 + v_3 = v$, so

$$\begin{aligned} (2e - s(\varrho, F^i z)v)^{-1}(2v_1 + s(F^{i-1}z, \varrho)v) &= \left(\sum_{n=0}^{\infty} \frac{(s(\varrho, F^i z)v)^n}{2^{n+1}}\right)(2v_1 + s(F^{i-1}z, \varrho)v) \\ &= 2v_1 \left(\sum_{n=0}^{\infty} \frac{(s(\varrho, F^i z)v)^n}{2^{n+1}}\right) + s(F^{i-1}z, \varrho)v \left(\sum_{n=0}^{\infty} \frac{(s(\varrho, F^i z)v)^n}{2^{n+1}}\right) \\ &= (2v_1 + s(F^{i-1}z, \varrho)v)(2e - s(\varrho, F^i z)v)^{-1}. \end{aligned}$$

Which shows that $(2e - s(\varrho, F^i z)v)^{-1}$ commutes with $(2v_1 + s(F^{i-1}z, \varrho)v)$. Hence by using Lemma 1.3.11 and Lemma 1.3.12 we have

$$\begin{aligned} r(\tau) &= r((2e - s(\varrho, F^i z)v)^{-1}(2v_1 + s(F^{i-1}z, \varrho)v)) \\ &\leq r((2e - s(\varrho, F^i z)v)^{-1}) \cdot r((2v_1 + s(F^{i-1}z, \varrho)v)) \\ &\leq \frac{1}{2 - s(F^{i-1}z, \varrho)r(v)}(2r(v_1) + s(F^{i-1}z, \varrho)r(v)) \\ &< 1. \end{aligned}$$

Because,

$$2\left(r(v_1)+s(\digamma^{i-1}z,\varrho)r(v)\right)\leq 2s(\digamma^{i-1}z,\varrho)r(v_1)+\left(s(\digamma^{i-1}z,\varrho)+1\right)r(v)<2,$$

implies that

$$2r(v_1)+s(\digamma^{i-1}z,\varrho)r(v)<2-s(\digamma^{i-1}z,\varrho)r(v).$$

By Lemma 1.3.15 it follows that $||\tau^i||\to 0$ as $i\to\infty$ and so

$$||\tau^i d_s(\varrho,z)||\leq ||\tau^i||||d_s(\varrho,z)||\to 0 \quad (i\to\infty).$$

By Remark 1.3.20 we conclude that for any $c\in\mathcal{A}$ with $c\gg\theta$, there exists $N\in\mathbb{N}$ such that

$$d_s(\varrho,\digamma^i z)\preceq \tau^i d_s(\varrho,z)\preceq c \quad \forall\, i\geq N.$$

Thus by Lemma 1.3.15 $\digamma^i z\to\varrho$ as $i\to\infty$. Similarly we obtain that $\digamma^i z\to\zeta$ as $i\to\infty$. Now by uniqueness of limit, we conclude that $\varrho=\zeta$, which completes the proof. □

3.3 Generalized Lipschitz mappings in controlled cone metric type spaces

This section is concerned with the discussion of the theory of $\mathcal{F}\cdot\mathcal{P}$s of generalized Lipschitz mappings in $CCMTS$ over Banach algebra.

Theorem 3.3.1. Let $(\mathfrak{Q},d_s)$ be a complete $CCMTS$ over $\mathcal{A}$ with underlying solid cone $\mathfrak{P}$. Let $\digamma:\mathfrak{Q}\to\mathfrak{Q}$ be such that for all $p,q\in\mathfrak{Q}$;

$$d_s(\digamma p,\digamma q)\preceq \kappa d_s(p,q), \tag{3.3.1}$$

where $\kappa\in\mathfrak{P}$ be such that $r(\kappa)<1$ and for each $p_0\in\mathfrak{Q}$, $\lim_{m,i\to\infty}\frac{s(p_{i+1},p_{i+2})}{s(p_i,p_{i+1})}s(p_{i+1},p_m)<\dfrac{1}{||\kappa||}$. Then there exists a unique point $\varrho\in\mathfrak{Q}$ which is a $\mathcal{F}\cdot\mathcal{P}$ of the map $\digamma$. Furthermore for each $u_0\in\mathfrak{Q}$, the iterative sequence $u_n=\digamma(u_{n-1})=\digamma^n u_0$ converges to ϱ.

Proof. If we take $v_1=\kappa$, $v_2=v_3=\theta$ and $\alpha(p,q)=s(p,q)$, then $\digamma$ satisfied all the conditions of Theorem 3.2.2, i.e. $\digamma$ satisfies the condition of Definition 3.2.1, $\digamma$ is generalized α-admissible, $(\mathfrak{Q},d_s)$ is regular and for every $u_0\in\mathfrak{Q}$ $\alpha(u_0,\digamma u_0)\succeq s(u_0,\digamma u_0)$.

Hence there exists ϱ in $\mathfrak{Q}$ which is a $\mathcal{F}\cdot\mathcal{P}$ of the map $\digamma$. Now it remains only to show that this $\mathcal{F}\cdot\mathcal{P}$ is unique. Suppose that there is $\zeta \in \mathfrak{Q}$ such that $\digamma\zeta = \zeta$. Then we have

$$d_s(\varrho,\zeta) = d_s(\digamma\varrho,\digamma\zeta) \preceq \kappa d_s(\varrho,\zeta).$$

But $r(\kappa) < 1$, so by Lemma 1.3.11, $e-\kappa$ is invertible. Thus by Lemma 1.3.15 $d_s(\varrho,\zeta) = \theta$. □

Theorem 3.3.2. Let $(\mathfrak{Q}, d_s)$ be a complete $CCMTS$ over $\mathcal{A}$ with underlying solid cone $\mathfrak{P}$. Let $\digamma : \mathfrak{Q} \to \mathfrak{Q}$ satisfies the following generalized Lipschitz condition, i.e. for all $p,q \in \mathfrak{Q}$;

$$d_s(\digamma p,\digamma q) \preceq \kappa[d_s(\digamma p,p) + d_s(\digamma q,q)], \tag{3.3.2}$$

where $\kappa \in \mathfrak{P}$ be such that $r(\kappa) < \dfrac{1}{s(p,q)+1}$ and for each $p_0 \in \mathfrak{Q}$, we have

$$\lim_{m,i\to\infty} \frac{s(p_{i+1},p_{i+2})}{s(p_i,p_{i+1})} s(p_{i+1},p_m) < \frac{1}{||\tau||},$$

with $\tau = (e-\kappa)^{-1}\kappa$. Then there exists a unique point $\varrho \in \mathfrak{Q}$ which is a $\mathcal{F}\cdot\mathcal{P}$ of the map $\digamma$.

Proof. If we take $v_1 = \theta$, $v_2 = v_3 = \kappa$ and $\alpha(p,q) = s(p,q)$, then all the condition of Theorem 3.2.2 are satisfied. Hence there exists ϱ in $\mathfrak{Q}$ which is a $\mathcal{F}\cdot\mathcal{P}$ of the map $\digamma$. Finally we show that ϱ is a unique $\mathcal{F}\cdot\mathcal{P}$ of the map $\digamma$. For this if we have another $\mathcal{F}\cdot\mathcal{P}$ of the map $\digamma$ say ζ, then

$$d_s(\varrho,\zeta) = d_s(\digamma\varrho,\digamma\zeta) \preceq \kappa[d_s(\varrho,\digamma\varrho) + d_s(\zeta,\digamma\zeta)] = \theta.$$

Therefore, $\varrho = \zeta$. □

The main result of 1.3.21 about generalized Lipschitz mappings on CbM spaces over a Banach algebras [23] becomes a special case of our results Theorem 3.3.1 and Theorem 3.3.2 when we define $s(p,q) = b$ for some $b \geq 1$.

Corollary 3.3.3. Let $(\mathfrak{Q}, d_s)$ be a complete CMS over $\mathcal{A}$ and $\mathfrak{P}$ be the associated cone in $\mathcal{A}$. Let $\digamma : \mathfrak{Q} \to \mathfrak{Q}$ be such that for all $p,q \in \mathfrak{Q}$,

$$d_s(\digamma p,\digamma q) \preceq \kappa d_s(p,q), \tag{3.3.3}$$

where $\kappa \in \mathfrak{P}$ be such that $r(\kappa) < 1$. Then for each $u_0 \in \mathfrak{Q}$, the iterative sequence $u_n = \digamma(u_{n-1}) = \digamma^n u_0$ converges to a unique $\mathcal{F} \cdot \mathcal{P}$ of $\digamma$.

Proof. Take $b = 1$ in Theorem 1.3.21, we get the required result. □

Remark 3.3.4. 1. If we take $s(x, y) = b$ for some $b \geq 1$ in Theorem 3.3.1 and in Theorem 3.3.2, we get the main results of [23] for cone b-$\mathcal{M} \cdot \mathcal{S}$s over Banach algebra.

2. By using Remark 3.1.4, we obtain Theorem 1.3.10 as a corollary of our Theorem 3.3.1.

3. If we take $s(x, y) = b$ for some $b \geq 1$ in Theorem 3.2.2 and in Theorem 3.2.4, we get the main results of [56] for cone b-$\mathcal{M} \cdot \mathcal{S}$s over Banach algebra.

3.4 Consequences and applications

In this section, we have listed some important consequences and applications of our results which generalizes some results of Hussain *et al*. [24], Xu and Radenovic [61], Malhotra *et al*. [33, 34] and Liu and Xu [32].

Definition 3.4.1. Let $\mathfrak{Q}$ be a non-empty set and $\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ be a function. A mapping $\digamma : \mathfrak{Q} \to \mathfrak{Q}$ is said to be an α-admissible mapping if $\alpha(p, q) \geq 1$ implies that $\alpha(\digamma p, \digamma q) \geq 1$.

Definition 3.4.2. Let $(\mathfrak{Q}, d_s)$ be a complete $CCMTS$ over $\mathcal{A}$ and $\mathfrak{P}$ be the underlying solid cone. A mapping $\digamma : \mathfrak{Q} \to \mathfrak{Q}$ is said to be generalized α-Lipschitz contraction if for all $p, q \in \mathfrak{Q}$ with $\alpha(p, q) \geq 1$ satisfies the following:

$$d_s(\digamma p, \digamma q) \preceq \kappa d_s(p, q),$$

where $\kappa \in \mathfrak{P}$ is such that $r(\kappa) < \frac{1}{s(p,q)}$ and for each $p_0 \in \mathfrak{Q}$, we have

$$\lim_{m,i \to \infty} \frac{s(p_{i+1}, p_{i+2})}{s(p_i, p_{i+1})} s(p_{i+1}, p_m) < \frac{1}{||\kappa||}.$$

The following theorem becomes special case of Theorem 3.2.2 if we define $\alpha : \mathfrak{Q}\times\mathfrak{Q} \to [0, \infty)$ by $\alpha(p, q) = s(p, q) \geq 1$ for all $p, q \in \mathfrak{Q}$ and take $\kappa = v_1$, $v_2 = v_3 = \theta$.

Theorem 3.4.3. Let $(\mathfrak{Q}, d_s)$ be a complete $CCMTS$ over $\mathcal{A}$ and $\mathfrak{P}$ be the associated solid cone. Let $F : \mathfrak{Q} \to \mathfrak{Q}$ satisfies the generalized α-Lipschitz contraction with Lipschitz constant κ such that:

1. F is α-admissible;

2. there exists $u_0 \in \mathfrak{Q}$ such that $\alpha(u_0, Fu_0) \geq 1$;

3. F is continuous or if a sequence $\{u_n\} \in \mathfrak{Q}$ with $\alpha(u_n, u_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $u_n \to q$ implies that $\alpha(u_n, q) \geq 1$ for all $n \in \mathbb{N}$.

Then there is a point ϱ in $\mathfrak{Q}$ which is fixed under F.

For uniqueness of $\mathcal{F} \cdot \mathcal{P}$, we use the following extra condition:

$$\forall \varrho, \zeta \in Fix(F), \text{ there exists } z \in \mathfrak{Q} \text{ such that } \alpha(\varrho, z) \geq 1 \text{ and } \alpha(\zeta, z) \geq 1. \quad (3.4.1)$$

Theorem 3.4.4. If we add the condition (3.4.1) in the assumption of Theorem 3.4.3, then the $\mathcal{F} \cdot \mathcal{P}$ is unique.

Proof. The assertion follows simply by using Theorem 3.4.3 and Theorem 3.2.4. □

Remark 3.4.5. 1. If we take $s(p, q) = b$ for some $b \geq 1$, then we obtain the main results of Hussain *et al*. [24], Theorems 3.1 and 3.2.

2. Theorems 3.1, 3.2 and 3.5 in Malhotra *et al*. [33] become special cases of our Theorem 3.4.3 and 3.4.4 respectively with $s(x, y) = 1$, $v_1 = 1$ and $v_2 = v_3 = \theta$.

3. If we define $s(p, q) = 1$, $v_1 = \theta$ and $v_2 = v_3$, then Theorems 3.1, 3.2 and 3.3 in Malhotra *et al*. [34] become special cases of our Theorem 3.4.3 and 3.4.4 respectively.

Chapter 4

Fixed points of multi-valued dynamical systems using comparison functions and multi-fractals in extended b-metric spaces

In the first section of this chapter, we proved multiple results of $\mathcal{F}\cdot\mathcal{P}$s for the class of multi-valued φ-contractions in the setting of $Eb-M$ spaces. Then, constructed some new multi-valued fractals based on a $\mathcal{F}\cdot\mathcal{P}$ approach in the framework of $Eb-M$ spaces. Later on, the idea of well-posed problem of $\mathcal{F}\cdot\mathcal{P}$s is studied. Our results generalized some famous recent results in the theory of iterated function system. For application point of view, we discussed the Collage theorems. Some of the results in this chapter is published in the Journal of function spaces [48].

4.1 Multi-valued φ-contractions in extended b-metric spaces

The aim of this section is to produce several results of $\mathcal{F}\cdot\mathcal{P}$s for the class of multi-valued φ-contractions in $Eb-M$ spaces.

4.1.1 Extended b-comparison functions

Samreen *et al.*, [47] presented for some technical reasons a new class of comparison functions in the framework of $Eb-M$ spaces as follow.

Definition 4.1.1. Let $(\mathfrak{Q}, d_s)$ be an $Eb-M$ space. A self-map φ on $[0,\infty)$ is called an extended b-comparison function (in short EbC function) if it is increasing and there exists a function $\digamma : D \subset \mathfrak{Q} \to \mathfrak{Q}$ such that for some $\eta_0 \in D$, the orbit $\mathcal{O}(\eta_0) \subset D$ and for all $p \in [0,\infty)$, for every $k \in \mathbb{N}$, the following series converges

$$\sum_{r=0}^{\infty} \left(\prod_{i=1}^{r} s(\eta_i, \eta_k) \right) \varphi^r(p).$$

Here $\eta_r = \digamma^r \eta_0$ for $r \in \mathbb{N}$. We call a map φ to be an EbC function for $\digamma$ at η_0.

Remark 4.1.2. For an arbitrary self-map $\digamma$ on $\mathfrak{Q}$, if we take $s(p_1, p_2) = b \geq 1$ (a constant), then the Definition 4.1.1 becomes the definition of a b-comparison function. Furthermore for some $b \geq 1$, every EbC is also a b-comparison, i.e. if $s(p_1, p_2) \geq 1$ for every $p_1, p_2 \in \mathfrak{Q}$, then by setting $b = \inf_{p_1, p_2 \in \mathfrak{Q}} s(p_1, p_2)$ we have

$$\sum_{r=0}^{\infty} b^r \varphi^r(l) \leq \sum_{r=0}^{\infty} \left(\prod_{i=1}^{r} s(\omega_i, \omega_k) \right) \varphi^r(l).$$

Example 4.1.3. [47] Let $\digamma$ be a self-map on $\mathfrak{Q}$, where $(\mathfrak{Q}, d_s)$ is an $Eb-M$ space. Let $\lim_{r,k\to\infty} s(\omega_r, \omega_k)$ exists for every $\omega_0 \in \mathfrak{Q}$, and with $\omega_r = \digamma^r \omega_0$. Define $\varphi : [0,\infty) \to [0,\infty)$ as $\varphi(p) = tp$ such that

$$\lim_{r,k\to\infty} s(\omega_r, \omega_k) < \frac{1}{t}.$$

By using ratio test, the series $\sum_{r=1}^{\infty} \left(\prod_{i=1}^{r} s(\omega_i, \omega_k) \right) \varphi^r(t)$ converges. Thus φ is an EbC function for $\digamma$ for every ω_0.

Lemma 4.1.4. Let a self-map $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a comparison function. Then:

(1) each iteration ϕ^k is a comparison function for $k \geq 1$;

(2) ϕ is continuous at zero;

(3) $\phi(\eta) < \eta$ for any $\eta > 0$.

Lemma 4.1.5. Every *EbC* function is *b*-comparison and hence a comparison function

Definition 4.1.6. [49] Let $(\mathfrak{Q}, d_s)$ be an $Eb-M$ space. A function $\mathcal{H}_s : H(\mathfrak{Q}) \times H(\mathfrak{Q}) \to \mathbb{R}_+$ induced by $Eb - M$ d_s called an extended Pompeiu-Hausdorff metric is defined as follows:

$$\left\{\forall\, \mathcal{W}, \mathcal{Z} \in H(\mathfrak{Q}),\;\; \mathcal{H}_s(\mathcal{W}, \mathcal{Z}) = \max\left\{ \sup_{w\in\mathcal{W}} d_s(w, \mathcal{Z}), \sup_{z\in\mathcal{Z}} d_s(\mathcal{W}, z)\right\}\right\},$$

where $d_s(\sigma, \mathcal{Z}) = \inf\{d_s(\sigma, z)\ :\ z \in \mathcal{Z}\}$ and $s(\mathcal{W}, \mathcal{Z}) = \sup\{s(\sigma, \varrho)\ :\ \sigma \in \mathcal{W},\ \varrho \in \mathcal{Z}\}$.

Theorem 4.1.7. [49] The space $(H(\mathfrak{Q}), \mathcal{H}_s($ is complete $Eb-M$ space whenever $(\mathfrak{Q}, d_s)$ is a complete $Eb - M$ space.

4.1.2 Generalized φ-contractions in extended b-metric space

We start be the following lemma whose proof is trivial.

Lemma 4.1.8. Let $(\mathfrak{Q}, d_s)$ be an $Eb - M$ space and $\mathcal{W}, \mathcal{Z} \in H(\mathfrak{Q})$. Then for every $z \in \mathcal{Z}$ and for any $\gamma > 0$, there exist $\sigma \in \mathcal{W}$ such that

$$d_s(\sigma, z) \leq \mathcal{H}_s(\mathcal{W}, \mathcal{Z}) + \gamma.$$

Following is the main result of this section.

Theorem 4.1.9. Let $(\mathfrak{Q}, d_s)$ be an $Eb - M$ space with d_s a continuous functional on $\mathfrak{Q}$. Let $D \subseteq \mathfrak{Q}$ be a closed set and $F : D \to H(\mathfrak{Q})$ be such that $\mathcal{O}(\sigma_0)$ is subset of D. Suppose that for all $\varrho \in \mathcal{O}(\sigma_0)$ and $\varpi \in F(\varrho)$;

$$\mathcal{H}_s(F(\varrho), F(\varpi)) \leq \varphi(d_s(\varrho, \varpi)). \tag{4.1.1}$$

Moreover, the inequality (4.1.1) holds strictly if and only if $\varrho \neq \varpi$ and φ is an *EbC* function for F at $\sigma_0 \in D$. Then there is a point σ in $\mathfrak{Q}$ such that the iterative sequence σ_k converges to σ, where $\sigma_k \in F(\sigma_{k-1})$. Furthermore σ is a $\mathcal{F} \cdot \mathcal{P}$ under the map F iff the map $G(p) = d_s(p, F(p))$ is F-orbitally *lsc* at σ.

Proof. Let $\sigma_0 \in D$ and $\sigma_1 \in \digamma(\sigma_0)$. Then $\sigma_0 \neq \sigma_1$ because if it is not true, then σ_0 is a point, fixed under the $\digamma$. By using (4.1.1) for $\digamma(\sigma_0),\, \digamma(\sigma_1) \in H(\mathfrak{Q})$, we obtain

$$\mathcal{H}_s(\digamma(\sigma_0),\, \digamma(\sigma_1)) < \varphi(d_s(\sigma_0,\, \sigma_1)).$$

Choose $\epsilon_1 > 0$ such that

$$\mathcal{H}_s(\digamma(\sigma_0),\, \digamma(\sigma_1)) + \epsilon_1 \leq \varphi(d_s(\sigma_0,\, \sigma_1)). \tag{4.1.2}$$

Now $\sigma_1 \in \digamma(\sigma_0)$ and $\epsilon_1 > 0$, then by Lemma 4.1.8 there exists $\sigma_2 \in \digamma(\sigma_1)$ such that

$$\begin{aligned} d_s(\sigma_1,\, \sigma_2) &\leq \mathcal{H}_s(\digamma(\sigma_0),\, \digamma(\sigma_1)) + \epsilon_1 \\ &\leq \varphi(d_s(\sigma_0,\, \sigma_1)). \end{aligned} \tag{4.1.3}$$

Again, $\sigma_1 \neq \sigma_2$, otherwise σ_1 becomes fixed under $\digamma$. By using (4.1.1), we obtain

$$\mathcal{H}_s(\digamma(\sigma_1),\, \digamma(\sigma_2)) < \varphi(d_s(\sigma_1,\, \sigma_2)).$$

Choose $\epsilon_2 > 0$ such that

$$\begin{aligned} \mathcal{H}_s(\digamma(\sigma_1),\, \digamma(\sigma_2)) + \epsilon_2 &\leq \varphi(d_s(\sigma_1,\, \sigma_2)) \\ &\leq \varphi(\varphi(d_s(\sigma_0,\, \sigma_1))) \\ &= \varphi^2(d_s(\sigma_0,\, \sigma_1)), \end{aligned} \tag{4.1.4}$$

here the second inequality obtained by using (4.1.3). By Lemma 4.1.8, for $\sigma_2 \in \digamma(\sigma_1)$ and $\epsilon_2 > 0$ $\exists$ $\sigma_3 \in \digamma(\sigma_2)$ such that

$$\begin{aligned} d_s(\sigma_2,\, \sigma_3) &\leq \mathcal{H}_s(\digamma(\sigma_1),\, \digamma(\sigma_2)) + \epsilon_2 \\ &\leq \varphi^2(d_s(\sigma_0,\, \sigma_1)). \end{aligned}$$

Continuing in the same way we get

$$d_s(\sigma_r,\, \sigma_{r+1}) \leq \varphi^r(d_s(\sigma_0,\, \sigma_1)). \tag{4.1.5}$$

By utilizing the triangle inequality like condition of $Eb - M$ and (4.1.5) if $r < k$, then we have,

$$\begin{aligned} d_s(\sigma_r,\, \sigma_k) &\leq s(\sigma_r,\, \sigma_k) d_s(\sigma_r,\, \sigma_{r+1}) + s(\sigma_r,\, \sigma_k) s(\sigma_{r+1},\, \sigma_k) d_s(\sigma_{r+1}, \sigma_{r+2}) + \\ &\cdots + s(\sigma_r,\, \sigma_k) s(\sigma_{r+1},\, \sigma_k) \ldots s(\sigma_{k-1},\, \sigma_k) d_s(\sigma_{k-1}, \sigma_k) \\ &\leq d_s(\sigma_r, \sigma_{r+1}) \prod_{i=1}^{r} s(\sigma_i, \sigma_k) + d_s(\sigma_{r+1}, \sigma_{r+2}) \prod_{i=1}^{r+1} s(\sigma_i, \sigma_k) + \\ &\cdots + d_s(\sigma_{k-1}, \sigma_k) \prod_{i=1}^{k-1} s(\sigma_i, \sigma_k) \\ &\leq \varphi^r(d_s(\sigma_0, \sigma_1) \prod_{i=1}^{r} s(\sigma_i, \sigma_k) + \varphi^{r+1}(d_s(\sigma_0, \sigma_1) \prod_{i=1}^{r+1} s(\sigma_i, \sigma_k) + \\ &\cdots + \varphi^{k-1}(d_s(\sigma_0, \sigma_1) \prod_{i=1}^{k-1} s(\sigma_i, \sigma_k). \end{aligned} \tag{4.1.6}$$

Now φ is given to be an EbC function, so the series $S = \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} s(\sigma_i, \sigma_k) \right) \varphi^j(d_s(\sigma_0, \sigma_1))$ converges. By setting $S_n = \sum_{j=1}^{n} \left(\prod_{i=1}^{j} s(\sigma_i, \sigma_k) \right) \varphi^j(d_s(\sigma_0, \sigma_1))$, from inequality (4.1.6) we obtain that

$$d_s(\sigma_r, \sigma_k) \leq (S_{k-1} - S_{r-1}).$$

Which further implies that $\lim_{r,k \to \infty} d_s(\sigma_r, \sigma_k) \to 0$. Hence $\{\sigma_r\}$ becomes a Cauchy sequence in D. But $D \subseteq \mathfrak{Q}$ is closed set, so there must exists a point $\sigma \in D$ such that the iterative sequence σ_r converges to σ.

Using the definition $\mathcal{H}_s$ and (4.1.1), we obtain

$$\begin{aligned} d_s(\sigma_r, \sigma_{r+1}) &\leq \mathcal{H}_s(\digamma(\sigma_{r-1}), \digamma(\sigma_r)) \\ &\leq \varphi(d_s(\sigma_{r-1}, \sigma_r)) \\ &< d_s(\sigma_{r-1}, \sigma_r). \end{aligned}$$

But $\sigma_r \to \sigma$ as $r \to \infty$ which infers that $\lim_{n \to \infty} d_s(\sigma_r, \digamma(\sigma_r)) = 0$.

Assume that $G(\sigma) = d_s(\sigma, \digamma\sigma)$ is $\digamma$-orbitally lsc at σ. Then

$$d_s(\sigma, \digamma(\sigma)) = G(\sigma) \leq \liminf_{r \to \infty} G(\sigma_r) = \liminf_{r \to \infty} d_s(\sigma_r, \digamma(\sigma_r)) = 0.$$

Hence $\sigma \in \digamma(\sigma)$. But $\digamma(\sigma)$ is closed, so $\sigma \in \digamma(\sigma)$ and thus σ is a $\mathcal{F}\cdot\mathcal{P}$ of the map $\digamma$. Conversely if σ is a point $\mathcal{F}\cdot\mathcal{P}$ of the map $\digamma$, then $G(\sigma) = 0 \leq \liminf_{r\to\infty} G(\sigma_r)$. □

Example 4.1.10. Let $\mathfrak{Q} = [0, \frac{1}{4}]$ and $d_s : \mathfrak{Q} \times \mathfrak{Q} \to \mathbb{R}$ be defined as $d_s(l, q) = (l - q)^2$. Then $(\mathfrak{Q}, d_s)$ is an $Eb - M$ space with $s(l, q) = l + q + 2$. Define $\digamma : \mathfrak{Q} \to H(\mathfrak{Q})$ by $\digamma(l) = [0, l^2]$, then for each $\sigma_0 \in \mathfrak{Q}$ and $\sigma_r \in \digamma(\sigma_{r-1})$, we have $\lim_{r,k\to\infty} s(\sigma_r, \sigma_k) = \lim_{r,k\to\infty}(\sigma_r + \sigma_k + 2) = 2 < 4$. Now for every $l \in \mathfrak{Q}$ and $q \in T(l)$, we have

$$\begin{aligned}\mathcal{H}_s(\digamma l, \digamma q) &= \mathcal{H}_s([0, l^2], [0, q^2]) = (l^2 - q^2)^2 \\ &= (l + q)^2(l - q)^2 \\ &\leq \frac{1}{4}(l - q)^2.\end{aligned}$$

If we define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(j) = \frac{j}{4}$, then $\digamma$ satisfied all the conditions present in Theorem 4.1.9. So there exists σ in $\mathfrak{Q}$ such that $\sigma \in \digamma(\sigma)$, as we can see here that $\sigma = 0 \in \digamma 0$.

4.2 Multi-valued fractals and well-posedness in extended b-metric spaces

This section is based on the construction of some new multi-valued fractals using a $\mathcal{F}\cdot\mathcal{P}$ approach in the framework of $Eb - M$ spaces. Later on, the idea of well-posed problems of $\mathcal{F}\cdot\mathcal{P}$ is discussed.

4.2.1 Generalized functionals in extended b-metric spaces

We introduced the notion of generalized functionals in the framework of $Eb - M$ spaces $(\mathfrak{Q}, d_s)$ in this section as follow.

The gap functional:

$$D_s : \mathscr{P}(\mathfrak{Q}) \times \mathscr{P}(\mathfrak{Q}) \to [0, \infty) \cup \{+\infty\},$$

$$D_s(\mathscr{U},\mathscr{V}) = \begin{cases} 0, & \text{if } \mathscr{U} = \phi = \mathscr{V} \\ \inf\{d_s(p,q) \ : \ p \in \mathscr{U}, q \in \mathscr{V}\}, & \text{if } \mathscr{U} \neq \phi \neq \mathscr{V} \\ +\infty, & \text{otherwise.} \end{cases}$$

If $a \in \mathfrak{Q}$ is an arbitrary element, then $D_s(a,\mathscr{B}) = D_s(\{a\},\mathscr{B})$.

The excess generalized functional:

$$\rho_s : \mathscr{P}(\mathfrak{Q}) \times \mathscr{P}(\mathfrak{Q}) \to [0,\infty) \cup \{+\infty\},$$

$$\rho_s(\mathscr{U},\mathscr{V}) = \begin{cases} \sup\{D_s(a,\mathscr{V}) \ : \ a \in \mathscr{U}\}, & \text{if } \mathscr{U} \neq \phi \neq \mathscr{V} \\ 0, & \text{if } \mathscr{U} = \phi \\ +\infty, & \mathscr{V} = \phi \neq \mathscr{U}. \end{cases}$$

Pompeiu-Hausdorff generalized functional:

$$H_s : \mathscr{P}(\mathfrak{Q}) \times \mathscr{P}(\mathfrak{Q}) \to [0,\infty) \cup \{+\infty\},$$

$$H_s(\mathscr{U},\mathscr{V}) = \begin{cases} \max\{\rho_s(\mathscr{U},\mathscr{V}), \rho_s(\mathscr{V},\mathscr{U})\}, & \text{if } \mathscr{U} \neq \phi \neq \mathscr{V} \\ 0, & \text{if } \mathscr{U} = \phi = \mathscr{V} \\ +\infty, & \text{otherwise.} \end{cases}$$

Lemma 4.2.1. Let $\mathscr{U},\mathscr{V} \in \mathscr{P}(\mathfrak{Q})$, where $(\mathfrak{Q}, d_s)$ an $Eb-M$ space. Let there is $\mathfrak{r} > 0$ satisfying the following:

(i) for every element $u \in \mathscr{U}$ there is an element $v \in \mathscr{V}$ such that $d_s(u,v) < \mathfrak{r}$;

(ii) for every element $v \in \mathscr{V}$ there is an element $u \in \mathscr{U}$ such that $d_s(u,v) < \mathfrak{r}$.

Then $H_s(\mathscr{U},\mathscr{V}) < \mathfrak{r}$.

Lemma 4.2.2. Suppose $(\mathfrak{Q}, d_s)$ is an $Eb-M$ space and $s(\mathscr{U},\mathscr{V}) = \sup\{s(\eta,\xi) \ : \ \eta \in \mathscr{U}, \ \xi \in \mathscr{V}\}$. Then the following hold:

$$D_s(p,\mathscr{U}) \leq s(p,\mathscr{U})[D_s(p,\mathscr{V}) + H_s(\mathscr{V},\mathscr{U})], \quad \forall\, p \in \mathfrak{Q} \ \text{ and } \ \forall\, \mathscr{U},\mathscr{V} \in \mathscr{P}(\mathfrak{Q}).$$

Lemma 4.2.3. Let $(\mathscr{X}, d_s)$ be an $Eb - M$ space. Then the following hold:

$$H_s(\mathscr{U}, \mathscr{W}) \leq s(\mathscr{U}, \mathscr{W})[H_s(\mathscr{U}, \mathscr{V}) + H_s(\mathscr{V}, \mathscr{W})], \quad \forall\, \mathscr{U}, \mathscr{V}, \mathscr{W} \in \mathcal{P}(\mathfrak{Q}).$$

Lemma 4.2.4. (1) Let $\mathscr{U}, \mathscr{V}$ be a compact subsets of an $Eb - M$ space $(\mathfrak{Q}, d_s)$. Then for every $\eta \in \mathscr{U}$ there is a point $\xi \in \mathscr{V}$ such that:

$$d_s(\eta, \xi) \leq s(\eta, \xi) H_s(\mathscr{U}, \mathscr{V}).$$

(2) Let $\mathscr{U}, \mathscr{V}$ be elements of $\mathscr{P}_{cp}(\mathfrak{Q})$, where $(\mathfrak{Q}, d_s)$ is an $Eb - M$ space. Let the map d_s be a continuous functional. Then for each element η of $\mathscr{U}$, there is an element ξ in $\mathscr{V}$ such that

$$d_s(\eta, \xi) \leq H_s(\mathscr{U}, \mathscr{V}).$$

4.2.2 Picard operators in extended b-metric spaces

We start by the definition of a Picard operator and then proved a result of Picard operators by using $Eb - M$ spaces.

Definition 4.2.5. Let $(\mathfrak{Q}, d_s)$ be an $Eb - M$ space. A Picard operator is a map $\mathscr{F} : \mathfrak{Q} \to \mathfrak{Q}$ which satisfies the following conditions:

(i) $Fix(\mathscr{F}) = \{\sigma\}$;

(ii) for every point w_0 in $\mathfrak{Q}$, the sequence $\mathscr{F}^n(w_0)$ converges to σ as $n \to \infty$.

Following is the main result of this section.

Theorem 4.2.6. Let $(\mathfrak{Q}, d_s)$ be a complete $Eb - M$ space, where d_s is a continuous functional. Suppose that the self-map ϕ on $[0, \infty$ is an EbC function for $\mathscr{F}$ at some point x_0, where $\mathscr{F}$ is a self ϕ-contraction. Then the following hold.

(i) The map $\mathscr{F}$ is a Picard operator, i.e. there exists a $\mathcal{F} \cdot \mathcal{P}$ σ of the map $\mathscr{F}$ such that $\mathscr{F}^n(\omega) \to \sigma$ for all $\omega \in \mathfrak{Q}$.

(ii) (1) For all $\omega \in \mathfrak{Q}$ and with $\omega_n = F^n\omega$, $d_s(\omega_n, \sigma) \le s(\omega_n, \sigma)S_s\left(\phi^n(d_s(\omega, F\omega))\right)$.

(2) For all $\omega \in \mathfrak{Q}$ and with $\omega_n = F^n\omega$, $d_s(\omega_n, \sigma) \le s(\omega_n, \sigma)S_s\left((d_s(\omega_n, \omega_{n+1}))\right)$, where

$$S_s(t) = \sum_{k=0}^{\infty} \left(\prod_{i=1}^{k} s(\omega_{n+i}, \omega_m) \right) \phi^k(t).$$

(iii) $d_s(\omega, \sigma) \le s(\omega, \sigma)S_s(d_s(\omega, F\omega))$ for all $\omega \in \mathfrak{Q}$.

Proof. Let $\varrho_0 \in \mathfrak{Q}$ and $\varrho_n = F^n\varrho_0 = F(\varrho_{n-1})$ for $n \ge 1$. Since F is ϕ-contraction so we have

$$d_s(\varrho_n, \varrho_{n+1}) = d_s(F(\varrho_{n-1}, F(\varrho_n))) \le \phi(d_s(\varrho_{n-1}, \varrho_n)),$$

which by induction yields

$$d_s(\varrho_n, \varrho_{n+1}) \le \phi^n(d_s(\varrho_0, \varrho_1)). \tag{4.2.1}$$

But d_s is an $Eb-M$, so by triangular inequality like condition of d_s and by using (4.2.1), we have for $m > n$ that:

$$\begin{aligned} d_s(\varrho_n, \varrho_m) &\le s(\varrho_n, \varrho_{n+1})d_s(\varrho_n, \varrho_{n+1}) + s(\varrho_n, \varrho_{n+1})s(\varrho_{n+1}, \varrho_{n+2})d_s(\varrho_{n+1}, \varrho_{n+2}) + \cdots \\ &+ s(\varrho_n, \varrho_{n+1})s(\varrho_{n+1}, \varrho_{n+2}) \cdots s(\varrho_{m-1}, \varrho_m)(d_s(\varrho_{m-1}, \varrho_m)) \\ &\le s(\varrho_n, \varrho_m)\phi^n(d_s(\varrho_0, \varrho_1)) + s(\varrho_n, \varrho_m)s(\varrho_{n+1}, \varrho_m)\phi^{n+1}d_s((\varrho_0, \varrho_1)) + \cdots \\ &+ s(\varrho_n, \varrho_m)s(\varrho_{n+1}, \varrho_m)s(\varrho_{n+2}, \varrho_m)...s(\varrho_{m-2}, \varrho_m)s(\varrho_{m-1}, \varrho_m)\phi^{m-1}(d_s(\varrho_0, \varrho_1)) \\ &\le \left(\prod_{i=1}^{n} s(\varrho_i, \varrho_m) \right) \phi^n(d_s(\varrho_0, \varrho_1)) + \left(\prod_{i=1}^{n+1} s(\varrho_i, \varrho_m) \right) \phi^{n+1}(d_s(\varrho_0, \varrho_1)) + \\ &\cdots + \left(\prod_{i=1}^{m-1} s(\varrho_i, \varrho_m) \right) \phi^{m-1}(d_s(\varrho_0, \varrho_1)). \end{aligned} \tag{4.2.2}$$

Since ϕ is an *EbC* function, the series $\sum_{k=1}^{\infty} \left(\prod_{i=1}^{k} s(\varrho_i, \varrho_m) \right) \phi^k(d_s(\varrho_0, \varrho_1))$ converges. Thus if we take $S_n = \sum_{k=1}^{n} \left(\prod_{i=1}^{k} s(\varrho_i, \varrho_m) \right) \phi^k(d_s(\varrho_0, \varrho_1))$, then by (4.2.2) we have

$$d_s(\varrho_n, \varrho_m) \le (S_{m-1} - S_{n-1}) \to 0, \text{ as } n, m \to \infty.$$

That is $\{\varrho_n\}$ is a Cauchy sequence in $\mathfrak{Q}$. By completeness of $\mathfrak{Q}$, there exists a point $\sigma \in \mathfrak{Q}$ such that $\varrho_n \to \sigma$.

Next we show that σ is a $\mathcal{F}\cdot\mathcal{P}$ of the map $\digamma$ and for this we have

$$d_s(\varrho_{n+1}, \digamma(\sigma)) = d_s(\digamma(\varrho_n), \digamma(\sigma)) \leq \phi(d_s(\varrho_n, \sigma)). \tag{4.2.3}$$

But by Lemma 4.1.5 and Lemma 4.1.4, ϕ is continuous at zero and given that d_s is continuous, so if we take limit as $n \to \infty$, then from (4.2.3) we can say that $d_s(\sigma, \digamma(\sigma))$=0, which shows that σ is a point fixed under $\digamma$. It remains only to show that this point σ is unique. Suppose that there is a point $\varpi \in \mathfrak{Q}$, and $\digamma(\varpi) = \varpi$. Then we have

$$\begin{aligned} d_s(\sigma, \varpi) &= d_s(\digamma(\sigma), \digamma(\varpi)) \\ &\leq \phi(d_s(\sigma, \varpi)) \\ &< d_s(\sigma, \varpi), \end{aligned}$$

which is possible only when $d_s(\sigma, \varpi) = 0$ implies that $\sigma = \varpi$. Hence $\digamma$ is a Picard operator.

(ii) Let $u \in \mathfrak{Q}$ be such that $u_n = \digamma^n(u)$. From (4.2.2) we have

$$\begin{aligned} d_s(u_n, u_m) \leq\ & s(u_n, u_m)\phi^n(d_s(u, u_1)) + s(u_n, u_m)s(u_{n+1}, u_m)\phi^{n+1}d_s((u, u_1)) + \cdots + \\ & s(u_n, u_m)s(u_{n+1}, u_m)s(u_{n+2}, u_m)\ldots s(u_{m-2}, u_m)s(u_{m-1}, u_m)\phi^{m-1}(d_s(u, u_1)) \\ & \leq s(u_n, u_m)[\phi^0(\phi^n(d_s(u, u_1))) + s(u_{n+1}, u_m)\phi^1(\phi^n d_s((u, u_1))) + \ldots \\ & + s(u_{n+1}, u_m)s(u_{n+2}, u_m)\cdots s(u_{m-1}, u_m)\phi^{m-n-1}(\phi^n(d_s(u, u_1)))], \end{aligned} \tag{4.2.4}$$

where $n \geq 0$ and $m > n$. Letting $m \to \infty$ in (4.2.4) we get a priori estimate

$$d_s(u_n, \sigma) \leq s(u_n, \sigma)S_s\left(\phi^n(d_s(u, \digamma u))\right) \quad \forall n \geq 0.$$

On the other side, for $n \geq 1$ and $p \geq 0$ such that $n + p = m$ we have

$$d_s(u_{n+p}, u_{n+p+1}) = d_s(\digamma(u_{n+p-1}), \digamma(u_{n+p})) \leq \phi(d_s(u_{n+p-1}, u_{n+p})).$$

By using induction we get

$$d_s(u_{n+p}, u_{n+p+1}) \leq \phi^p(d_s(u_n, u_{n+1})). \tag{4.2.5}$$

By (4.2.5) and triangular inequality like condition of $Eb-M$, we have

$$\begin{aligned}d_s(u_n,u_m) = d_s(u_n,u_{n+p}) &\leq s(u_n,u_m)[d_s(u_n,u_{n+1}) + s(u_{n+1},u_m)\phi(d_s((u_n,u_{n+1})) + \cdots \\ &+ s(u_{n+1},u_m)s(u_{n+2},u_m)s(u_{n+3},u_m)\ldots s(u_{n+p-1},u_m)\phi^{p-1}(d_s(u_n,u_{n+1}))] \\ &= s(u_n,u_m)[(d_s(u_n,u_{n+1})) + s(u_{n+1},u_m)\phi(d_s((u_n,u_{n+1})) + \cdots \\ &+ \left(\prod_{j=1}^{p-1} s(u_{n+j},u_m)\right)\phi^{p-1}(d_s(u_n,u_{n+1}))]. \qquad (4.2.6)\end{aligned}$$

If we take the limit $p \to \infty$ in (4.2.5) we obtain a posteriori estimate

$$d_s(u_n,\sigma) \leq s(u_n,\sigma)S_s(d_s(u_n,u_{n+1})) \ \ \forall n \geq 0. \qquad (4.2.7)$$

(iii) Let $u_n = u$ for an arbitrary $u \in \mathfrak{Q}$ in (4.2.7). Then

$$d_s(u,\sigma) \leq s(u,\sigma)S_s(d_s(u,\digamma(u))).$$

□

4.2.3 Multi-fractal operators in extended b-metric spaces

We start this section by a lemma which under some conditions guaranties that image of a compact set under a multi-valued contractive operator is compact.

Lemma 4.2.7. Let $\mathcal{G}$ from $\mathfrak{Q}$ to $\mathscr{P}_{cp}(\mathfrak{Q})$ be a multi-valued contractive operator, where $(\mathfrak{Q}, d_s)$ is an $Eb-M$ space. i.e. $\forall\ \eta,\xi \in \mathfrak{Q}$ with $\eta \neq \xi$,

$$H_s(\mathcal{G}(\eta),\mathcal{G}(\xi)) < d_s(\eta,\xi).$$

Furthermore, suppose that $\forall x \in \mathfrak{Q}$ and for every compact set $\mathscr{Y}$, $\lim_{n\to\infty} s(\xi_n,x)$ exists and finite for all $\xi_n \in \mathscr{Y}$. Then $\mathcal{G}(\mathscr{Y})$ is compact, i.e. $\mathcal{G}(\mathscr{Y})$ lies in $\mathscr{P}_{cp}(\mathfrak{Q})$.

Proof. If we choose $\xi_n \in \mathcal{G}(\mathscr{Y})$, then there exists $\eta_n \in \mathscr{Y}$ such that $\xi_n \in \mathcal{G}(\eta_n)$ for all $n \in \mathbb{N}$. But $\mathscr{Y}$ is compact, so there exists a subsequence η_{n_k} of η_n such that η_{n_k} converges to some p (say) in $\mathscr{Y}$. Then by Lemma 4.2.4 (1) for $\xi_{n_k} \in \mathcal{G}(\eta_{n_k})$, $\exists\ u_{n_k} \in \mathcal{G}(p)$ such that

$$\begin{aligned}d_s(\xi_{n_k},u_{n_k}) &\leq s(\xi_{n_k},u_{n_k})H_s(\mathcal{G}(\eta_{n_k}),\mathcal{G}(p)) \\ &< s(\xi_{n_k},u_{n_k})d_s(\eta_{n_k},p) \to 0, \quad \text{when } n \to \infty,\end{aligned}$$

since $\lim_{k\to\infty} s(\xi_{n_k}, u_{n_k})$ exists and finite. Now $\mathcal{G}(x)$ is compact and $u_{n_k} \in \mathcal{G}(x)$, so there is a convergent subsequence of u_{n_k} which converges to some $q \in \mathcal{G}(p)$. Let us denote this subsequence by u_{n_k} too. Then we have

$$\begin{aligned} d_s(\xi_{n_k}, q) &\leq s(\xi_{n_k}, q)[d_s(\xi_{n_k}, u_{n_k}) + d_s(u_{n_k}, q)] \\ &\leq M[d_s(\xi_{n_k}, u_{n_k}) + d_s(u_{n_k}, q)] \\ &\to 0, \end{aligned}$$

as $n \to \infty$. Implies that $\xi_{n_k} \to q \in \mathcal{G}(x) \subseteq \mathcal{G}(\mathscr{Y})$. Hence $\mathcal{G}(\mathscr{Y})$ is compact and so $\mathcal{G}(\mathscr{Y}) \in \mathscr{P}_{cp}(\mathfrak{Q})$. □

Following is the main result of the present section.

Theorem 4.2.8. Let $(\mathfrak{Q}, d_s)$ be a complete $Eb - M$ space, where d_s be a continuous functional. Suppose that for each $(i \in \{1, 2, \cdots, n\})$, the map $\mathcal{G}_j : \mathfrak{Q} \to \mathscr{P}_{cp}(\mathfrak{Q})$ is a multi-valued ϕ-contraction, where the self-map ϕ on $[0, \infty)$ is an EbC function for $\digamma_{\mathcal{G}}$ at some point, where $\digamma_{\mathcal{G}}$ is a multi-valued fractal operator generated by the IMFS $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_n)$. Then:

(a) the map $\digamma_{\mathcal{G}}$ maps from $\mathscr{P}_{cp}(\mathfrak{Q})$ to $\mathscr{P}_{cp}(\mathfrak{Q})$;

(b) the map $\digamma_{\mathcal{G}}$ is a ϕ-contraction;

(c) $\digamma_{\mathcal{G}}$ is a Picard operator, i.e. there is a unique point $\mathscr{A}^*_{\mathcal{G}}$ in $\mathscr{P}_{cp}(\mathfrak{Q})$ which is a $\mathcal{F} \cdot \mathcal{P}$ of the map $\digamma_{\mathcal{G}}$;

(d) $H_s(\digamma^n_{\mathcal{G}}(\mathscr{A}), \mathscr{A}^*_{\mathcal{G}}) \leq s(\mathscr{A}, \mathscr{A}^*_{\mathcal{G}}) P_b(\phi^n(H_s(\mathscr{A}, \digamma_{\mathcal{G}}(\mathscr{A}))))$;

(e) for each $\mathscr{A}$ in $\mathscr{P}_{cp}(\mathfrak{Q})$, $H_s(\mathscr{A}, \mathscr{A}^*_{\mathcal{G}}) \leq s(\mathscr{A}, \mathscr{A}^*_{\mathcal{G}}) P_b(H_s(\mathscr{A}, \digamma_{\mathcal{G}}(\mathscr{A})))$.

Proof. (a) As given that ϕ is an EbC function, so it is also a b-comparison and hence a comparison function. Also for each $t > 0$, $\phi(t) < t$, so for each $j \in \{1, 2, \cdots n\}$, $\mathcal{G}_j$ is contractive. Therefore, by using Lemma 4.2.7, we can say that the map $\digamma_{\mathcal{G}}$ maps from $\mathscr{P}_{cp}(\mathfrak{Q})$ to $\mathscr{P}_{cp}(\mathfrak{Q})$.

(b) We prove that the map $\mathcal{F}_{\mathcal{G}}$ is a ϕ-contraction and for this we need to show that $\forall \mathscr{U}, \mathscr{V} \in \mathscr{P}_{cp}(\mathfrak{Q})$,

$$H_s(\mathcal{F}_{\mathcal{G}}(\mathscr{U}), \mathcal{F}_{\mathcal{G}}(\mathscr{V})) \leq \phi(H_s(\mathscr{U}, \mathscr{V})).$$

Let $\mathscr{U}, \mathscr{V}$ be elements of $\mathscr{P}_{cp}(\mathfrak{Q})$ and let $u \in \mathcal{F}_{\mathcal{G}}(\mathscr{U})$. Then $u \in \mathcal{G}_j(\mathscr{U})$ for some $j \in \{1, 2, \cdots n\}$, which implies $u \in \mathcal{G}_j(p)$ for some $p \in \mathscr{U}$. Since $\mathscr{U}, \mathscr{V}$ are compact and $p \in \mathscr{U}$, so there exists an element q in $\mathscr{V}$ by Lemma 4.2.4 (2) such that

$$d_s(p, q) \leq H_s(\mathscr{U}, \mathscr{V}). \tag{4.2.8}$$

Thus by Lemma 4.2.4 (2), for $u \in \mathcal{G}_j(p)$, there exists an element v in $\mathcal{G}_j(q)$ for which

$$d_s(u, v) \leq H_s(\mathcal{G}_j(p), \mathcal{G}_j(q)). \tag{4.2.9}$$

By combining the inequalities (4.2.8) and (4.2.9), we can say that for each u in $\mathcal{F}_{\mathcal{G}}(\mathscr{U})$, there is an element v in $\mathcal{F}_{\mathcal{G}}(\mathscr{V})$ for which

$$\begin{aligned} d_s(u, v) &\leq H_s(\mathcal{G}_j(p), \mathcal{G}_j(q)) \\ &\leq \phi(d_s(p, q)) \\ &\leq \phi(H_s(\mathscr{U}, \mathscr{V})). \end{aligned} \tag{4.2.10}$$

By a similar process, we obtain that for every element $v \in \mathcal{F}_{\mathcal{G}}(\mathscr{V})$ there is an element $u \in \mathcal{F}_{\mathcal{G}}(\mathscr{U})$ for which

$$d_s(u, v) \leq \phi(H_s(\mathscr{U}, \mathscr{V})). \tag{4.2.11}$$

Lemma 4.2.1 together with (4.2.10) and (4.2.11) implies that

$$H_s(\mathcal{F}_{\mathcal{G}}(\mathscr{U}), \mathcal{F}_{\mathcal{G}}(\mathscr{V})) \leq \phi(H_s(\mathscr{U}, \mathscr{V})).$$

Hence the self-map $\mathcal{F}_{\mathcal{G}}$ is a ϕ-contraction which is defined on a complete $Eb - M$ space $(\mathscr{P}_{cp}(\mathfrak{Q}), H_s)$.
By using Theorem 4.2.6, $(c) - (e)$ follows immediately. □

Remark 4.2.9. If we take $s(p, q) = b$ for some $b \geq 1$ in Theorem 4.2.8, then we obtain the main results of [9].

It is very convenient to show that any Meir-Keeler type multi-valued operator defined on an $Eb-M$ space is contractive. Thus by using Lemma 4.2.7, it is evident that $\mathcal{G}(\mathscr{Y})$ lies in $\mathscr{P}_{cp}(\mathfrak{Q})$ for every $\mathscr{Y} \in \mathscr{P}_{cp}(\mathfrak{Q})$.
So if $\mathcal{G}_j : \mathfrak{Q} \to \mathscr{P}_{cp}(\mathfrak{Q})$ is a family of finite numbers of Meir-Keeler type multi-valued operator on an $Eb - M$ space, then we can derive easily the existence and uniqueness results for the multi-valued fractal generated by the IMFS $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m)$.

4.2.4 Well-posedness in extended b-metric spaces

In this section we present the concept of well-posedness in the setting of $Eb-M$ spaces.

Definition 4.2.10. Let $\digamma$ be a self-map on $\mathfrak{Q}$, where $(\mathfrak{Q}, d_s)$ an $Eb - M$ space. Then the problem of $\mathcal{F} \cdot \mathcal{P}$ for the map $\digamma$ is said to be well-posed w.r.t d_s if and only if the following axioms hold:

(*i*) $Fix(\digamma) = \{\sigma\}$;

(*ii*) for any sequence $\{\eta_n\}$ in $\mathfrak{Q}$ satisfying $d_s(\eta_n, \digamma(\eta_n)) \to 0$ as $n \to \infty$, we have that $d_s(\eta_n, \sigma) \to 0$ as $n \to \infty$.

Definition 4.2.11. Let $\mathcal{G} : \mathfrak{Q} \to \mathscr{P}(\mathfrak{Q})$ be a multi-valued map, where $(\mathfrak{Q}, d_s)$ is an $Eb-M$ space. Then we say that the map $\mathcal{G}$ has the property of well-posedness of $\mathcal{F} \cdot \mathcal{P}$ problem with respect to:

(*i*) the generalized functional D_s if and only if there exits a unique point σ which is a $\mathcal{F} \cdot \mathcal{P}$ of the map $\mathcal{G}$ and for any sequence $\{\eta_n\}$ in $\mathfrak{Q}$ satisfying $D_s(\eta_n, \mathcal{G}(\eta_n)) \to 0$ as $n \to \infty$, implies that $d_s(\eta_n, \sigma) \to 0$ as $n \to \infty$.

(*ii*) the generalized functional H_s if and only if $SFix(\mathcal{G}) = \{\sigma\}$ and for any sequence $\{\eta_n\}$ in $\mathfrak{Q}$ satisfying $H_s(\eta_n, \mathcal{G}(\eta_n)) \to 0$ as $n \to \infty$, implies that $d_s(\eta_n, \sigma) \to 0$ as $n \to \infty$, where $SFix(\mathcal{G}) = \{u \in \mathfrak{Q} \, : \, \{u\} = \mathcal{G}(u)\}$.

Theorem 4.2.12. Let $(\mathfrak{Q}, d_s)$ be a complete $Eb - M$ space, where d_s is a continuous functional. Let $\mathcal{G} : \mathfrak{Q} \to \mathscr{P}_{cp}(\mathfrak{Q})$ be a multi-valued ϕ-contraction such that ϕ is an EbC function. Suppose that $SFix(\mathcal{G})$ is non-empty and for each $\eta \in \mathfrak{Q}$, the map

$\psi : [0,\infty) \to [0,\infty)$ defined by $\psi(t) = t - s(\eta,\sigma)\phi(t)$ is onto and strictly increasing, where σ is a $\mathcal{F}\cdot\mathcal{P}$ of $\mathcal{G}$. Then the problem of $\mathcal{F}\cdot\mathcal{P}$ is well-posed for the map $\mathcal{G}$ with respect to both D_s and H_s.

Proof. Let σ be an arbitrary element of $SFix(\mathcal{G})$. We first prove that $Fix(\mathcal{G}) = SFix(\mathcal{G}) = \{\sigma\}$. For this, if ϱ is an element of $Fix(\mathcal{G})$, then by using the fact that $\mathcal{G}$ is ϕ-contraction, we have

$$\begin{aligned} d_s(\sigma,\varrho) &= D_s(\mathcal{G}(\sigma),\mathcal{G}(\varrho)) \\ &\leq H_s(\mathcal{G}(\sigma),\mathcal{G}(\varrho)) \\ &\leq \phi(d_s(\sigma,\varrho)). \end{aligned}$$

But ϕ is an EbC function, so is comparison and thus for each $p > 0$, $\phi(p) < p$. Hence $d_s(\sigma,\varrho) = 0$ which implies that $\sigma = \varrho$.
Let $\{\eta_n\}$ be a sequence in $\mathfrak{Q}$ satisfying $D_s(\eta_n,\mathcal{G}(\eta_n)) \to 0$ as $n \to \infty$. We need to show that $d_s(\eta_n,\sigma) \to 0$ as $n \to \infty$. For this we have by Lemma 4.2.2 that

$$\begin{aligned} d_s(\eta_n,\sigma) = D_s(\eta_n,\mathcal{G}(\sigma)) &\leq s(\eta_n,\mathcal{G}(\sigma))[D_s(\eta_n,\mathcal{G}(\eta_n)) + H(\mathcal{G}(\eta_n),\mathcal{G}(\sigma))] \\ &\leq s(\eta_n,\sigma)[D_s(\eta_n,\mathcal{G}(\eta_n)) + \phi(d_s(\eta_n,\sigma))]. \end{aligned}$$

Therefore we obtain for each $n \in \mathbb{N}$ that $\psi(d_s(\eta_n,\sigma)) \leq s(\eta_n,\sigma)D_s(x_n,\mathcal{G}(x_n))$. Hence

$$d_s(\eta_n,\sigma) \leq \psi^{-1}(s(\eta_n,\sigma)D_s(\eta_n,\mathcal{G}(\eta_n)) \to \psi^{-1}(0) \to 0 \text{ as } n \to \infty.$$

Thus, the problem of $\mathcal{F}\cdot\mathcal{P}$ is well-posed for the map $\mathcal{G}$ with respect to D_s. Notice that $H_s(\eta_n,\mathcal{G}(\eta_n)) \to 0$ implies that $D_s(\eta_n,\mathcal{G}(\eta_n)) \to 0$ as $n \to \infty$. So the problem of $\mathcal{F}\cdot\mathcal{P}$ is well-posed for the map $\mathcal{G}$ with respect to H_s too. □

Theorem 4.2.13. Let $\mathcal{G} : \mathfrak{Q} \to \mathfrak{Q}$ be a multi-valued ϕ-contraction such that $\phi : [0,\infty) \to [0,\infty)$ is an EbC function, where $\mathfrak{Q}$ is a complete $Eb - M$ space with d_s a continuous functional. Suppose that $SFix(\mathcal{G})$ is non-empty and for each $\eta \in \mathfrak{Q}$, the map $\psi : [0,\infty) \to [0,\infty)$ defined by $\psi(t) = t - s(\eta,\sigma)\phi(t)$ is onto and strictly increasing, where σ is a $\mathcal{F}\cdot\mathcal{P}$ of $\mathcal{G}$. Then the problem of $\mathcal{F}\cdot\mathcal{P}$ is well-posed for the map $\mathcal{G}$ with respect to d_s.

Proof. Clearly $Fix(\mathcal{G}) = \{\sigma\}$ by Theorem 4.2.6. Now let $\{\eta_n\}$ be a sequence in $\mathfrak{Q}$ satisfying $d_s(\eta_n, \mathcal{G}(\eta_n)) \to 0$ when $n \to \infty$. We have to prove that $\eta_n \to \sigma$ or equivalently, $d_s(\eta_n, \sigma) \to 0$ when $n \to \infty$. Note that the function ψ is a bijection and $\phi^{-1}(\xi) \to 0$ as $\xi \to 0$. Thus by triangular inequality condition of d_s, we have

$$\begin{aligned} d_s(\eta_n, \sigma) &\leq s(\eta_n, \sigma)[d_s(\eta_n, \mathcal{G}(\eta_n)) + d_s(\mathcal{G}(\eta_n), \mathcal{G}(\sigma))] \\ &\leq s(\eta_n, \sigma)[d_s(\eta_n, \mathcal{G}(\eta_n)) + \phi(d_s(\eta_n, \sigma))]. \end{aligned}$$

Hence

$$d_s(\eta_n, \sigma) \leq \phi^{-1}(s(\eta_n, \sigma) d_s(\eta_n, \mathcal{G}(\eta_n))) \to 0,$$

since $s(\eta_n, \sigma) d_s(\eta_n, \mathcal{G}(\eta_n)) \to 0$ as $n \to \infty$. Therefore, the $\mathcal{F} \cdot \mathcal{P}$ problem is well-posed for $\mathcal{G}$ w.r.t d_s. □

By combining Theorem 4.2.8 and Theorem 4.2.13 we obtain the following result.

Corollary 4.2.14. Let $(\mathfrak{Q}, d_s)$ be a complete $Eb - M$ space, where d_s a continuous functional. Let for each $(j \in \{1, 2, \cdots, n\})$, the map $\mathcal{G}_j$ from $\mathfrak{Q}$ to $\mathscr{P}_{cp}(\mathfrak{Q})$ be a multi-valued ϕ-contractions such that $\phi : [0, \infty) \to [0, \infty)$ is an EbC function for $F_{\mathcal{G}}$ at some point. Suppose that for each $\eta \in \mathfrak{Q}$, the self-map ψ on $[0, \infty)$ defined by $\psi(t) = t - s(\eta, \mathcal{A}^*)\phi(t)$ is onto and strictly increasing, where $\mathcal{A}^*$ is a $\mathcal{F} \cdot \mathcal{P}$ of $F_{\mathcal{G}}$. Then the problem of $\mathcal{F} \cdot \mathcal{P}$ is well-posed for the multi-fractal operator defined by $F_{\mathcal{G}}(\mathscr{Y}) = \cup_{j=1}^{n} \mathcal{G}_j(\mathscr{Y})$.

Remark 4.2.15. By taking $s(p, q) = b$ for some $b \geq 1$ in Theorems 4.2.12 and 4.2.13, we obtain the same results in the setting of b-$\mathcal{M} \cdot \mathcal{S}$s [9].

4.3 Consequences and applications

This section consists of some important consequences of Theorem 4.1.9 which involves $\beta_* - \varphi$ multi-valued contractions on $Eb - M$ spaces. The results we have obtained generalizes/extends some results by Bota *et al.* [10] and Hasanzadde *et al.* [5]. For application point of view, we proved Collage theorems, which can be used to construct more general fractals and to find solution of inverse problems in extended Hausdorff

b-$\mathcal{M}\cdot\mathcal{S}$s. Following is one of the main results in [47] which is direct consequence of Theorem 4.1.9.

Corollary 4.3.1. Let $(\mathfrak{Q}, d_s)$ be a complete $Eb - M$ space with d_s a continuous functional. Let $F : D \subset \mathfrak{Q} \to \mathfrak{Q}$ be a map and there is some σ_0 such that $\mathcal{O}(\sigma_0) \subseteq D$. Suppose that $\forall\, q \in \mathcal{O}(\sigma_0)$

$$d_s(Fq, F^2(q)) \leq \varphi\big(d_s(q, F(q))\big),$$

where φ is an EbC function for F at σ_0. Then $\exists\, \sigma$ in $\mathfrak{Q}$ such that $F^r\sigma_0 \to \sigma$. Furthermore, the point σ is a $\mathcal{F}\cdot\mathcal{P}$ of the map F iff the map $G(t) = d_s(t, Ft)$ is F-orbitally lsc at σ.

Proof. The assertion simply follows by using Theorem 4.1.9 for a self-map F. □

Theorem 4.3.2. Let $(\mathfrak{Q}, d_s)$ be a complete $Eb - M$ space with d_s a continuous functional. Let $F : D \subset \mathfrak{Q} \to \mathfrak{Q}$ be a map and σ_0 be such $\mathcal{O}(\sigma_0) \subset D$. Assume that the limit $\lim_{r,k\to\infty} s(\sigma_r, \sigma_k)$ exists and λ is a constant such that for all $\sigma_r, \sigma_k \in \mathcal{O}(\sigma_0)$,

$$\lim_{r,k\to\infty} s(\sigma_r, \sigma_k) < \frac{1}{\lambda}.$$

Assume further that

$$d_s(F(p), F^2p) \leq \lambda\big(d_s(p, F(p))\big)$$

for every $p \in \mathcal{O}(\sigma_0)$. Then the iterative sequence $F^r\sigma_0$ converges to $\sigma \in \mathfrak{Q}$. Additionally the point of convergence σ is a $\mathcal{F}\cdot\mathcal{P}$ of the map F if and only if the map $G(t) = d_s(t, F(t))$ is F-orbitally lsc at σ.

Proof. Define a map $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\varphi(t) = \lambda t$. By taking F a self-map, by using Example 4.1.3, we obtain that φ is an EbC function for F at every point $\sigma_0 \in \mathfrak{Q}$. Hence the assertion implies by Theorem 4.1.9. □

Definition 4.3.3. Let $(\mathfrak{Q}, d_s)$ be an $Eb-M$ space. A mapping $F : \mathfrak{Q} \to P(\mathfrak{Q})$ is called a β_*-admissible map if there exists a mapping $\beta : \mathfrak{Q} \times \mathfrak{Q} \to \mathbb{R}_+$ such that

$$\beta(p, q) \geq 1 \implies \beta_*(F(p), F(q)) \geq 1 \qquad \forall p, q \in \mathfrak{Q}$$

Note that $\beta_* : P(\mathfrak{Q}) \times P(\mathfrak{Q}) \to \mathbb{R}_+$ is defined by

$$\beta_*(W, Z) = \inf\{\beta(p, q) \ : \ p \in W, \, q \in Z\}.$$

Definition 4.3.4. [10] Let $(\mathfrak{Q}, d_s)$ be an $Eb - M$ space. A mapping $F : \mathfrak{Q} \to P(\mathfrak{Q})$ is said to be an $\beta_* - \varphi$-contractive multi-valued operator of type (Eb) if there exists two functions $\varphi \in \breve{\Phi}_{Eb}$ and $\beta : \mathfrak{Q} \times \mathfrak{Q} \to \mathbb{R}_+$ such that

$$\varphi(d_s(\varrho, \sigma)) \geq \beta_*(F(\varrho), \, F(\sigma))\mathcal{H}_s(F(\varrho), \, F(\sigma)), \tag{4.3.1}$$

for all $\varrho, \sigma \in \mathfrak{Q}$. Here by $\breve{\Phi}_{Eb}$, we mean the collection of all EbC functions.

Theorem 4.3.5. Let $(\mathfrak{Q}, d_s)$ be a complete $Eb - M$ space such that d_s is continuous. Suppose that the map $F : \mathfrak{Q} \to H(\mathfrak{Q})$ is a $\beta_* - \varphi$ contractive multi-valued operator of type (Eb) which satisfies the following:

(i) F is β_*-admissible;

(ii) there exists points $\sigma_0 \in \mathfrak{Q}$ and $\sigma_1 \in F(\sigma_0)$ such that $\beta(\sigma_0, \sigma_1) \geq 1$.

Then there exists a point $\sigma \in \mathfrak{Q}$ such that the iterative sequence σ_r converges to σ, where $\sigma_r \in F(\sigma_{r-1})$. Additionally, the point σ is a $\mathcal{F} \cdot \mathcal{P}$ of F if and only if the map $G(t) = d_s(t, Ft)$ is F-orbitally lsc at σ.

Proof. Given that F is β_* admissible and $\beta(\sigma_0, \sigma_1) \geq 1$ for $\sigma_1 \in F(\sigma_0)$, so that $\beta_*(F(\sigma_0), F(\sigma_1)) \geq 1$. By using infimum property, for $\sigma_1 \in F(\sigma_0)$ and $\sigma_2 \in F(\sigma_1)$

$$\beta(\sigma_1, \sigma_2) \geq \beta_*(F(\sigma_0), F(\sigma_1)).$$

Thus $\beta(\sigma_1, \sigma_2) \geq 1$ which further implies that $\beta_*(F(\sigma_1), F(\sigma_2)) \geq 1$. Again by using the same property, for $\sigma_2 \in F(\sigma_1)$ and $\sigma_3 \in T\sigma_2$ $\beta(\sigma_2, \sigma_3) \geq \beta_*(F(\sigma_1), F(\sigma_2)) \geq 1$. Continuing in the similar way, to obtain

$$\beta_*(F(\sigma_r), \, F\sigma_{r+1}) \geq 1, \qquad r = 1, 2, 3, \cdots$$

The contractive condition 4.3.1 then implies that

$$\begin{aligned}\mathcal{H}_s(F(\sigma_r), \, F(\sigma_{r+1})) &\leq \beta_*(F(\sigma_r), \, F(\sigma_{r+1}))\mathcal{H}_s(F(\sigma_r), F(\sigma_{r+1})) \\ &\leq \varphi(d_s(F^{r-1}(\sigma_0), F^r(\sigma_0))).\end{aligned}$$

Which becomes equivalent to the following condition

$$\mathcal{H}_s(Fp_1, Fp_2) \leq \varphi(d_s(p_1, p_2)), \tag{4.3.2}$$

for every $p_1 \in \mathcal{O}(\sigma_0)$ and $p_2 \in Fp_1$. This shows that all the conditions of Theorem 4.1.9 are fulfilled and hence the assertion proved. □

By using some additional conditions on Theorem 4.2.6, we obtain the following.

Theorem 4.3.6. Let $(\mathfrak{Q}, d_s)$ be a complete $Eb - M$ space, where d_s is a continuous functional. Suppose that the self-map ϕ on $[0, \infty)$ is an EbC function for F at some $x_0 \in \mathfrak{Q}$ and $F : \mathfrak{Q} \to \mathfrak{Q}$ a ϕ-contraction. (By given hypothesis, it is clear that F admits a unique $\mathcal{F} \cdot \mathcal{P}$ say σ by Theorem 4.2.6 Then:

(1)**(Abstract Collage Theorem)** If for all $\varrho \in \mathfrak{Q}$, the function $\psi : [0, \infty) \to [0, \infty)$ defined by $\psi(t) = t - s(\varrho, \sigma)\phi(t)$ is onto and strictly increasing, then

$$d_s(\varrho, \sigma) \leq \psi^{-1}\left(s(\varrho, \sigma)d_s(\varrho, F(\varrho))\right);$$

(2)**(Abstract Anti-Collage Theorem)** Suppose that the mapping $g : [0, \infty) \to [0, \infty)$ defined by $g(\sigma) = \sigma + \phi(\sigma)$ is onto. Then we have $\forall\ \varrho \in \mathfrak{Q}$ that

$$d_s(\varrho, \sigma) \geq g^{-1}\left(\frac{1}{s(\varrho, F(\varrho))} d_s(\varrho, F(\varrho))\right).$$

Proof. (1) For an arbitrary $\varpi \in \mathfrak{Q}$ we have

$$\begin{aligned} d_s(\varpi, \sigma) &\leq s(\varpi, \sigma)[d_s(\varpi, F(\varpi)) + d_s(F(\varpi), \sigma)] \\ &\leq s(\varpi, \sigma)[d_s(\varpi, F(\varpi)) + \phi(d_s(\varpi, \sigma))], \end{aligned}$$

which implies that $d_s(\varpi, \sigma) - s(\varpi, \sigma)\phi(d_s(\varpi, \sigma)) \leq s(\varpi, \sigma)d_s(\varpi, F(\varpi))$. Hence

$$\psi(d_s(\varpi, \sigma)) \leq s(\varpi, \sigma)d_s(\varpi, F(\varpi)).$$

But ψ is increasing and onto, so is an increasing bijection and thus for every $\varpi \in \mathfrak{Q}$, we obtain that

$$d_s(\varpi, \sigma) \leq \psi^{-1}(s(\varpi, \sigma)d_s(\varpi, F(\varpi))).$$

(2) For an arbitrary element ϱ of $\mathfrak{Q}$, we have by triangular inequality like condition of d_s that:

$$
\begin{aligned}
d_s(\varrho, F(\varrho)) &\leq s(\varrho, F(\varrho))[d_s(\varrho, \sigma) + d_s(\sigma, F(\varrho))] \\
&\leq s(\varrho, F(\varrho))[d_s(\varrho, \sigma) + \phi(d_s(\varrho, \sigma))].
\end{aligned}
$$

Hence $g(d_s(\varrho, \sigma)) \geq \frac{1}{s(\varrho, F(\varrho))}(d_s(\varrho, F(\varrho)))$. Since ϕ is increasing and g is onto, so g is strictly increasing and bijective. Thus we get that:

$$
d_s(\varrho, \varpi) \geq g^{-1}\left(\frac{1}{s(\varrho, F(\varrho))} d_s(\varrho, F(\varrho))\right), \quad \forall \varrho \in \mathfrak{Q}.
$$

□

Remark 4.3.7. If we take $s(p, q) = b$ for some $b \geq 1$ in Theorem 4.3.6, then we obtain one of main results of [9].

Chapter 5

Best proximity points of multi-valued dynamical systems on controlled metric type spaces

In this chapter, we introduced a new type of generalized distances on CMT space $(\mathfrak{Q}, d_s)$ which we call controlled type generalized pseudo-distance (CTG pseudo-distance) . With the help of this generalized distance, we define $J_s(u, F)$, $J^*(u, F)$, H^{J_s} distance of Hausdorff type where $E, F \in CB(\mathfrak{Q}), u \in E$ and WP^{J_s}-property of a pair of nonempty subsets of $\mathfrak{Q}$. More precisely our newly defined mappings are more general then that of corresponding notions defined by Gabeleh and Plebaniak.

5.1 Global optimality results for multi-valued maps in b-metric spaces

In 2014 Plebaniak [39] introduced the notion of a b-generalized pseudo-distance (in short bG pseudo-distance) on a b-$\mathcal{M} \cdot \mathcal{S}$ $\mathfrak{Q}$ as below.

Definition 5.1.1. [39] Let $(\mathfrak{Q}, d_b)$ be a b-$\mathcal{M} \cdot \mathcal{S}$ (with constant $b \geq 1$). A map J_b : $\mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ is said to be a bG pseudo-distance on $\mathfrak{Q}$, if the following are satisfied:

(J_b1) $J_b(\varpi, \eta) \leq b[J_b(\varpi, \xi) + J_b(\xi, \eta)]$ for any $\varpi, \xi, \eta \in \mathfrak{Q}$;

(J_b2) For any sequence (η_m) and (ξ_m) in $\mathfrak{Q}$ such that

$$\lim_{n\to\infty} \sup_{m>n} J_b(\eta_n, \eta_m) = 0$$

and

$$\lim_{m\to\infty} J_b(\eta_m, \xi_m) = 0,$$

we have

$$\lim_{m\to\infty} d_b(\eta_m, \xi_m) = 0.$$

In 2018 Gabeleh [20] extends the main theorem of [19] by constructing the following definitions and notations.
Let $(\mathcal{U}, \mathcal{V})$ be a pair in a b-$\mathcal{M}\cdot\mathcal{S}$ $\mathfrak{Q}$ of nonempty sets. We denote and define the following:

$$J_b(u, \mathcal{V}) = \inf_{v\in\mathcal{V}} J_b(u, v),$$

$$J_b^*(u, \mathcal{V}) = \frac{1}{s} J_b(u, \mathcal{V}) - dist(\mathcal{U}, \mathcal{V}) \text{ where } u \in \mathfrak{Q} \text{ and } dist(\mathcal{U}, \mathcal{V}) = \inf_{u\in\mathcal{U}, v\in\mathcal{V}} d_b(u, v),$$

$$H_b^J(\mathcal{V}, \mathcal{V}) = \max\{\sup_{u\in\mathcal{U}} J_b(v, \mathcal{U}), \sup_{v\in\mathcal{V}}(v, \mathcal{U})\} \ \forall \mathcal{U}, \mathcal{V} \in CB(\mathfrak{Q}),$$

$$\mathcal{U}_0 = \{u \in \mathcal{U} : J_b(u, v) = dist(\mathcal{U}, \mathcal{V}) \text{ for some } v \in \mathcal{V}\},$$

$$\mathcal{V}_0 = \{v \in \mathcal{V} : J_b(u, v) = dist(\mathcal{U}, \mathcal{V}) \text{ for some } u \in \mathcal{U}\}.$$

Definition 5.1.2. [20] Let $(\mathfrak{Q}, d_b)$ be a b-$\mathcal{M}\cdot\mathcal{S}$ (with constant $b \geq 1$) and $(\mathcal{U}, \mathcal{V})$ be a pair of nonempty subsets of $\mathfrak{Q}$ with $\mathcal{U}_0 \neq \emptyset$.
(1) The pair $(\mathcal{U}, \mathcal{V})$ have the WP^{J_b}-property if and only if the conditions

$$J_b(u_1, v_1) = dist(\mathcal{U}, \mathcal{V}),$$

$$J_b(u_2, v_2) = dist(\mathcal{U}, \mathcal{V})$$

implies

$$J_b(u_1, u_2) \leq J_b(v_1, v_2)$$

where $u_1, u_2 \in \mathcal{U}_0$ and $v_1, v_2 \in \mathcal{V}_0$.

(2) A bG pseudo-distance J_b is said to be associated with the pair $(\mathcal{U}, \mathcal{V})$ if for any sequences (η_n) and (ξ_n) in $\mathfrak{Q}$ with

$$\lim_{n\to\infty} \eta_n = \eta, \quad \lim_{m\to\infty} \xi_m = \xi;$$

and

$$J_b(\eta_m, \xi_{m-1}) = dist(\mathcal{U}, \mathcal{V}) \; \forall \; m \in \mathbb{N},$$

we have $d_b(\eta, \xi) = dist(\mathcal{U}, \mathcal{V})$.

Definition 5.1.3. [20] let $(\mathfrak{Q}, \tau)$ be a topological space and $\mathcal{U}, \mathcal{V}$ be nonempty subsets of $\mathfrak{Q}$. A mapping $\digamma : \mathcal{U} \to 2^{\mathcal{V}}$ is said to be closed whenever (η_m) is a sequence in $\mathcal{U}$ and (ξ_m) is a sequence in $\mathcal{V}$ such that $\xi_m \in \digamma(\eta_m) \; \forall \; m \in \mathbb{N}$, $\eta_m \to \eta \in \mathcal{U}$, and $\xi_m \to \xi \in \mathcal{V}$ implies that $\xi \in \digamma(\eta)$.

Definition 5.1.4. [20] Let $\eta : [0,1) \to (1/2, 1]$ by $\eta(r) = \frac{1}{1+r}$. Let $\mathfrak{Q}$ be a $b \; \mathcal{M} \cdot \mathcal{S}$ (with $b \geq 1$) and the mapping $J_b : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ be a bG pseudo-distance on $\mathfrak{Q}$. Let $(\mathcal{U}, \mathcal{V})$ be a pair of nonempty subsets of $\mathfrak{Q}$. A multi-valued non-self mapping $\digamma : \mathcal{U} \to 2^{\mathcal{V}}$ is said to be a contraction of Suzuki type with respect to (in short S-type w.r.t) bG pseudo-distances if there exists $r \in [0, 1)$ such that

$$\frac{\eta(r)}{b} J_b^*(u, \digamma u) \leq J_b(u, w) \quad \text{implies} \quad bH^{J_b}(\digamma u, \digamma w) \leq \; r d_b(u, w), \; \forall \; u, w \in \mathcal{U}.$$

5.2 Global optimality results for multi-valued mappings in controlled metric type spaces

Inspired by the ideas of Mlaiki *et al*. [36] of CMT space, we define a new class of multi-valued contraction of S-type w.r.t CTG pseudo-distances. To begin with our main results, first we define the following.

Definition 5.2.1. Let $(\mathfrak{Q}, d_s)$ be a CMT space. A map $J_s : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ is said to be a CTG pseudo-distance if the following two conditions satisfy:

(J_s1) $J_s(\zeta,\gamma) \leq s(\zeta,\xi)J_s(\zeta,\xi) + s(\xi,\gamma)J_s(\xi,\gamma)$ for all $\zeta,\xi,\gamma \in \mathfrak{Q}$;

(J_s2) For any sequences (ζ_m) and (ξ_m) in $\mathfrak{Q}$ with

$$\lim_{n\to\infty} \sup_{m>n} J_s(\zeta_n,\zeta_m) = 0 \quad \text{and} \lim_{m\to\infty} J_s(\zeta_m,\xi_m) = 0,$$

we have

$$\lim_{m\to\infty} d_s(\zeta_m,\xi_m) = 0.$$

Remark 5.2.2. Every controlled metric $d_s : \mathfrak{Q} \times \mathfrak{Q} \to [1,\infty)$ on $\mathfrak{Q}$ is a CTG pseudo-distance on $\mathfrak{Q}$ but the converse is false in general.

Example 5.2.3. Let $(\mathfrak{Q}, d_s)$ be a CMT space and E be a closed subset of $\mathfrak{Q}$ such that it contain at least two different points. Let $r > 0$ be such that $r > \delta(E)$ where $\delta(E) = \sup\{d_s(\sigma,\varrho) : \sigma,\varrho \in E\}$. Define $J_s : \mathfrak{Q} \times \mathfrak{Q} \to [1,\infty)$ by

$$J_s(\varrho,\varpi) = \begin{cases} d_s(\varrho,\varpi) & \text{if} \quad \{\varrho,\varpi\} \subseteq E \\ r & \text{if} \quad \{\varrho,\varpi\} \nsubseteq E. \end{cases}$$

Then J_s is a CTG pseudo-distance.

Proof. (J_s1) Let $\sigma_0, \varrho_0, \varpi_0 \in \mathfrak{Q}$ be such that

$$J_s(\sigma_0,\varpi_0) > s(\sigma_0,\varrho_0)J_s(\sigma_0,\varrho_0) + s(\varrho_0,\varpi_0)J_s(\varrho_0,\varpi_0). \tag{5.2.1}$$

Then we have $\{\sigma_0,\varrho_0,\varpi_0\} \nsubseteq E$. Because if it is subset of E, then
$J_s(\sigma_0,\varrho_0) = d_s(\sigma_0,\varrho_0), J_s(\sigma_0,\varpi_0) = d_s(\sigma_0,\varpi_0), J_s(\varrho_0,\varpi_0) = d_s(\varrho_0,\varpi_0)$, and with this, the Inequality (5.2.1) will become $d_s(\sigma_0,\varpi_0) > s(\sigma_0,\varrho_0)d_s(\sigma_0,\varrho_0) + s(\varrho_0,\varpi_0)d_s(\varrho_0,\varpi_0)$, which is a contradiction to the fact that d_s is a controlled metric type. Thus there will exists some $u \in \{\sigma_0,\varrho_0,\varpi_0\}$ such that $u \notin E$. If $u = \sigma_0$, then $J_s(\sigma_0,\varpi_0) = r$ and $J_s(\sigma_0,\varrho_0) = r$ and so (5.2.1) becomes $r > s(\sigma_0,\varrho_0)r + s(\varrho_0,\varpi_0)J_s(\varrho_0,\varpi_0)$ which is a contradiction.
Similarly if we take $u = \varrho_0$ or $u = \varpi_0$, then we get the same contradiction. Hence the condition (J_s1) of Definition 5.2.1 is fulfilled, i.e.

$J_s(\zeta,\gamma) \leq s(\zeta,\xi)J_s(\zeta,\xi) + s(\xi,\gamma)J_s(\xi,\gamma)$ for all $\zeta,\xi,\gamma \in \mathfrak{Q}$.

(J_s2) Let $\{x_m\}_{m\in\mathbb{N}}$ and $\{y_m\}_{m\in\mathbb{N}}$ be any two sequences in $\mathfrak{Q}$ such that $\lim_{n\to\infty} \sup_{m>n} J_s(x_n, x_m) = 0$ and $\lim_{m\to\infty} J_s(x_m, y_m) = 0$. We need to show that

$$\lim_{m\to\infty} d_s(x_m, y_m) = 0.$$

As $\lim_{m\to\infty} J_s(x_m, y_m) = 0$, so we have $\lim_{m\to\infty} z_m = 0$ where $z_m = J_s(x_m, y_m) \in \mathbb{R}_+$. Which implies that for every $\epsilon > 0$ (hence for $0 < \epsilon < r$), there exists a natural number k such that $d(z_m, 0) = |z_m - 0| < \epsilon$ for all $m \geq k$. Thus $|z_m - 0| < \epsilon < r$ for all $m \geq k$, and so $z_m < \epsilon < r$ for all $m \geq k$, because $z_m \geq 0$ $\forall n \in \mathbb{N}$. So that $z_m = J_s(x_m, y_m) = d_s(x_m, y_m)$ for all $m \geq k$. Thus, $J_s(x_m, y_m) = d_s(x_m, y_m) < \epsilon < r$ for all $m \geq k$. Hence $\lim_{m\to\infty} d_s(x_m, y_m) = 0$. □

Let $(\mathfrak{Q}, d_s)$ be a CMT space and $J_s : \mathfrak{Q} \times \mathfrak{Q} \to [0,\infty)$ be a CTG pseudo-distance on $\mathfrak{Q}$. Let $(\mathcal{U},\mathcal{V})$ be a nonempty pair of subsets of $\mathfrak{Q}$. We define the following notions:

$$J_s(\varrho,\mathcal{V}) = \inf_{v\in\mathcal{V}} J_s(\varrho, v)$$

$$J_s^*(\varrho,\mathcal{V}) = \frac{1}{s(\varrho,\varrho)} J_s(\varrho,\mathcal{V}) - dist(\mathcal{U},\mathcal{V}), \qquad \forall \varrho \in \mathcal{U},$$

$$\mathcal{U}_0 = \{\varrho \in \mathcal{U} \ : \ s(\varrho, v)J_s(\varrho, v) = dist(\mathcal{U},\mathcal{V}) \text{ for some } v \in \mathcal{V}\},$$

$$\mathcal{V}_0 = \{v \in \mathcal{V} \ : \ s(\varrho, v)J_s(\varrho, v) = dist(\mathcal{U},\mathcal{V}) \text{ for some } \varrho \in \mathcal{U}\}.$$

We define $H^{J_s} : CB(\mathfrak{Q}) \times CB(\mathfrak{Q}) \to [0,\infty)$ by

$$H^{J_s}(\mathcal{U},\mathcal{V}) = \max\left\{\sup_{\varrho\in\mathcal{U}} J_s(\varrho,\mathcal{U}), \sup_{v\in\mathcal{V}} J_s(\mathcal{U}, v)\right\}, \forall\, \mathcal{U},\mathcal{V} \in CB(\mathfrak{Q}).$$

Definition 5.2.4. Let $(\mathfrak{Q}, d_s)$ be a CMT space and $J_s : \mathfrak{Q} \times \mathfrak{Q} \to [0,\infty)$ be a CTG pseudo-distance on $\mathfrak{Q}$. Let $\mathcal{U},\mathcal{V}$ be non-void subsets of $\mathfrak{Q}$ with $\mathcal{U}_0 \neq \emptyset$. Then:

1. The pair $(\mathcal{U},\mathcal{V})$ is said to have the WP^{J_s}-property if and only if the conditions

$$s(u_1, v_1)J_s(u_1, v_1) = dist(\mathcal{U},\mathcal{V}),$$

$$s(u_2, v_2)J_s(u_2, v_2) = dist(\mathcal{U}, \mathcal{V})$$

implies that

$$J_s(u_1, u_2) \leq J_s(v_1, v_2),$$

where $u_1, u_2 \in \mathcal{U}_0$ and $v_1, v_2 \in \mathcal{V}_0$.

2. A *CTG* pseudo-distance J_s is said to be associated with the pair $(\mathcal{U}, \mathcal{V})$ if for any sequences (η_m) and (ξ_m) in $\mathfrak{Q}$ such that $\lim_{m\to\infty} \eta_m = \eta$, $\lim_{m\to\infty} \xi_m = \xi$ and for all $m \in \mathbb{N}$, $s(\eta_m, \xi_{m-1})J_s(\eta_m, \xi_{m-1}) = dist(\mathcal{U}, \mathcal{V})$, we have

$$d_s(\eta, \xi) = dist(\mathcal{U}, \mathcal{V}).$$

Lemma 5.2.5. Let $(\mathfrak{Q}, d_s)$ be a complete controlled metric type space and $J_s : \mathfrak{Q}\times\mathfrak{Q} \to [0, \infty)$ be a controlled type generalized pseudo-distance on $\mathfrak{Q}$. Let a sequence (σ_m) in $\mathfrak{Q}$ be such that the limits

$$\lim_{n\to\infty} s(\sigma_{i+n}, \sigma_n) \text{ and } \lim_{n\to\infty} s(\sigma_n, \sigma_{j+n}),$$

are finite for every $i, j \in \mathbb{N}$ and satisfies

$$\lim_{n\to\infty} \sup_{m>n} J_s(\sigma_n, \sigma_m) = 0. \tag{5.2.2}$$

Then (σ_m) is a Cauchy sequence in $\mathfrak{Q}$, for $m \in \{0\} \cup \mathbb{N}$.

Proof. From (5.2.2) we can say that for all $\epsilon > 0$ there exists $n_1 = n_1(\epsilon) \in \mathbb{N}$ such that $\forall\, n > n_1$,

$$\sup\{J_s(\sigma_n, \sigma_m) : m > n\} < \epsilon.$$

In particular, $\forall\, \epsilon > 0\ \exists\, n_1 = n_1(\epsilon) \in \mathbb{N}$ such that $\forall\, n > n_1, \forall\, t \in \mathbb{N}$ we have

$$J_s(\sigma_n, \sigma_{t+n}) < \epsilon. \tag{5.2.3}$$

Let $i_0, j_0 \in \mathbb{N}, i_0 > j_0$, be fixed and arbitrary. Define the sequences

$$z_n = \sigma_{i_0+n} \; and \; u_n = \sigma_{j_0+n} \; for \; n \in \mathbb{N}. \tag{5.2.4}$$

Then (5.2.3) gives

$$\lim_{n\to\infty} J_s(\sigma_n, z_n) = \lim_{n\to\infty} J_s(\sigma_n, y_n) = 0. \tag{5.2.5}$$

Therefore by using (5.2.2),(5.2.4) and (J_s2) we have

$$\lim_{n\to\infty} d_s(\sigma_n, z_n) = \lim_{n\to\infty} d_s(\sigma_n, y_n) = 0. \tag{5.2.6}$$

By using (5.2.6) and (5.2.4), we have

$$\lim_{n\to\infty} d_s(\sigma_{i_0+n}, \sigma_n) = \lim_{n\to\infty} d_s(\sigma_n, \sigma_{j_0+n}) = 0. \tag{5.2.7}$$

Let $k, l \in \mathbb{N}$ be such that $k > l > n_0$. Then for some $i_0, j_0 \in \mathbb{N}$, $k = i_0 + n_0$ and $l = j_0 + n_0$, and that $i_0 > j_0$. Now by using (5.2.7) and the triangular inequality like condition of controlled type metric d_s, we have

$$\begin{aligned} d_s(\sigma_k, \sigma_l) &= d_s(\sigma_{i_0+n_0}, \sigma_{j_0+n_0}) \\ &\leq s(\sigma_{i_0+n_0}, \sigma_{n_0}) d_s(\sigma_{i_0+n_0}, \sigma_{n_0}) + s(\sigma_{n_0}, \sigma_{j_0+n_0}) d_s(\sigma_{n_0}, \sigma_{j_0+n_0}) \\ &\to 0 + 0 \quad \text{as } \ n_0 \to \infty, \end{aligned}$$

since the limits $\lim_{n\to\infty} s(\sigma_{i_0+n_0}, \sigma_{n_0})$ and $\lim_{n\to\infty} s(\sigma_{n_0}, \sigma_{j_0+n_0})$ are finite. Hence, we conclude that $\lim_{k,l\to\infty} d_s(\sigma_k, \sigma_l) = 0$ and so the sequence $(\sigma_m : m \in \mathbb{N})$ is a Cauchy sequence. □

Definition 5.2.6. Let $\eta : [0, 1) \to (1/2, 1]$ be defined by $\eta(r) = \frac{1}{1+r}$. Let $(\mathfrak{Q}, d_s)$ be a CMT space and the map $J_s : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ be a CTG pseudo-distance on $\mathfrak{Q}$. Let $(\mathcal{U}, \mathcal{V})$ be a pair of nonempty subsets of $\mathfrak{Q}$. A multi-valued non-self mapping $F : \mathcal{U} \to 2^{\mathcal{V}}$ is said to be a S-type w.r.t a CTG pseudo-distances if there exists $r \in [0, 1)$ such that for all $x, y \in \mathcal{U}$,

$$\frac{\eta(r)}{s(x,y)} J_s^*(x, Fx) \leq J_s(x, y) \quad \text{implies} \quad s(x, y) H^{J_s}(Fx, Fy) \leq \ r J_s(x, y) \tag{5.2.8}$$

Clearly the class of multi-valued non-self mappings which are contraction of Suzuki type with respect to CTG pseudo-distances contains the class of multi-valued non-self mappings which are S-type w.r.t b-generalized pseudo-distances. As if we take $s(x, y) = b$, then $J_s = J_b \ \forall \, x, y \in \mathfrak{Q}$.

Theorem 5.2.7. Let $\mathfrak{Q}$ be a complete CMT space and let the map $J_s : \mathfrak{Q}\times\mathfrak{Q} \to [0,\infty)$ be a CTG pseudo-distance on $\mathfrak{Q}$. Let $(\mathcal{U},\mathcal{V})$ be a nonempty closed pair of subsets of $\mathfrak{Q}$ with $\mathcal{U}_0 \neq \emptyset$ and such that $(\mathcal{U},\mathcal{V})$ has the WP^{J_s}-property and J_s is associated with $(\mathcal{U},\mathcal{V})$. Let $\digamma : \mathcal{U} \to 2^{\mathcal{U}}$ be a closed S-type w.r.t CTG pseudo-distance J_s and $r \in [0,1)$ be such that

$$\lim_{n,m\to\infty} \frac{s(\varrho_{n+1},\varrho_{n+2})}{s(\varrho_n,\varrho_{n+1})} s(\varrho_{n+1},\varrho_m) < \frac{1}{r}\ ,\ \lim_{n,m\to\infty} \frac{s(\sigma_{n+1},\sigma_{n+2})}{s(\sigma_n,\sigma_{n+1})} s(\sigma_{n+1},\sigma_m) < \frac{1}{r},$$

for every $\varrho_n \in \mathcal{U}_0$ and $\sigma_n \in \digamma\varrho_n, n = 0,1,2\ldots.$
If $\digamma(x) \in CB(\mathfrak{Q})\ \forall\ x \in \mathcal{U}$, and $\digamma(x) \subset \mathcal{V}_0$ for each $x \in\ \mathcal{U}_0$, then $\digamma$ has a $\mathcal{B}\cdot\mathcal{P}\cdot\mathcal{P}$ in $\mathcal{U}$.

Proof. Since $\mathcal{U}_0 = \emptyset$, so let $\varrho_0 \in \mathcal{U}_0\ , \sigma_0 \in \digamma\varrho_0 \subseteq \mathcal{V}_0$. Then by definition of $\mathcal{V}_0$, there exits $\varrho_1 \in \mathcal{U}$ such that

$$s(\varrho_1,\sigma_0)J_s(\varrho_1,\sigma_0) = dist(\mathcal{U},\mathcal{V}). \tag{5.2.9}$$

But since $\mathcal{V}_0 \subseteq \mathcal{V}$, so $\sigma_0 \in \mathcal{V}$ and thus from the above we conclude that $\varrho_1 \in \mathcal{U}_0$. Now we have

$$\begin{aligned} J_s(\varrho_0,\digamma\varrho_0) &= \inf_{y\in\digamma\varrho_0} J_s(\varrho_0,y) \\ &\leq J_s(\varrho_0,\sigma_0) \\ &\leq s(\varrho_0,\varrho_1)J_s(\varrho_0,\varrho_1) + s(\varrho_1,\sigma_0)J_s(\varrho_1,\sigma_0) \\ &\leq s(\varrho_0,\varrho_0)[s(\varrho_0,\varrho_1)J_s(\varrho_0,\varrho_1) + s(\varrho_1,\sigma_0)J_s(\varrho_1,\sigma_0)]. \end{aligned}$$

Thus we have

$$\begin{aligned} J_s^*(\varrho_0,\digamma\varrho_0) &= \frac{1}{s(\varrho_0,\varrho_0)} J_s(\varrho_0,\digamma\varrho_0) - dist(\mathcal{U},\mathcal{V}) \\ &\leq \frac{1}{s(\varrho_0,\varrho_0)} s(\varrho_0,\varrho_0)[s(\varrho_0,\varrho_1)J_s(\varrho_0,\varrho_1) + s(\varrho_1,\sigma_0)J_s(\varrho_1,\sigma_0)] - dist(\mathcal{U},\mathcal{V}) \\ &= s(\varrho_0,\varrho_1)J_s(\varrho_0,\varrho_1) + s(\varrho_1,\sigma_0)J_s(\varrho_1,\sigma_0) - dist(\mathcal{U},\mathcal{V}) \\ &= s(\varrho_0,\varrho_1)J_s(\varrho_0,\varrho_1), \end{aligned}$$

since by (5.2.9). Which further implies that

$$\frac{1}{s(\varrho_0,\varrho_1)} J_s^*(\varrho_0,\digamma\varrho_0) \leq \frac{1}{s(\varrho_0,\varrho_1)} s(\varrho_0,\varrho_1)J_s(\varrho_0,\digamma\varrho_0) = J_s(\varrho_0,\varrho_1).$$

Also since $\eta(r) \leq 1$, we obtain

$$\frac{\eta(r)}{s(\varrho_0, \varrho_1)} J_s^*(\varrho_0, \digamma \varrho_0) \leq J_s(\varrho_0, \varrho_1).$$

Thus by (5.2.8) we have

$$s(\varrho_0, \varrho_1) H^{J_s}(\digamma \varrho_0, \digamma \varrho_1) \leq r J_s(\varrho_0, \varrho_1)$$

$$\implies H^{J_s}(\digamma \varrho_0, \digamma \varrho_1) \leq \frac{r}{s(\varrho_0, \varrho_1)} J_s(\varrho_0, \varrho_1).$$

Since $J_s(\sigma_0, \digamma \varrho_1) \leq H^{J_s}(\digamma \varrho_0, \digamma \varrho_1) \leq \frac{r}{s(\varrho_0,\varrho_1)} J_s(\varrho_0, \varrho_1)$, so there will exists $\sigma_1 \in \digamma \varrho_1$ such that

$$J_s(\sigma_0, \sigma_1) \leq \frac{r}{s(\varrho_0, \varrho_1)} J_s(\varrho_0, \varrho_1). \tag{5.2.10}$$

Again as $\varrho_1 \in \mathcal{U}_0, \digamma \varrho_1 \subseteq \mathcal{V}_0,\ \sigma_1 \in \digamma \varrho_1$, so there exists $\varrho_2 \in \mathcal{U}_0$ such that

$$s(\varrho_2, \sigma_1) J_s(\varrho_2, \sigma_1) = dist(\mathcal{U}, \mathcal{V}). \tag{5.2.11}$$

Now we have

$$\begin{aligned} J_s(\varrho_1, \digamma \varrho_1) &= \inf_{y \in \digamma \varrho_1} J_s(\varrho_1, y) \\ &\leq J_s(\varrho_1, \sigma_1) \\ &\leq s(\varrho_1, \varrho_2) J_s(\varrho_1, \varrho_2) + s(\varrho_1, \sigma_1) J_s(\varrho_2, \sigma_1) \\ &\leq s(\varrho_1, \varrho_1)[s(\varrho_1, \varrho_2) J_s(\varrho_1, \varrho_2) + s(\varrho_2, \sigma_1) J_s(\varrho_2, \sigma_1)]. \end{aligned}$$

Thus we have

$$\begin{aligned} J_s^*(\varrho_1, \digamma \varrho_1) &= \frac{1}{s(\varrho_1, \varrho_1)} J_s(\varrho_1, \digamma \varrho_1) - dist(\mathcal{U}, \mathcal{V}) \\ &\leq \frac{1}{s(\varrho_1, \varrho_1)} s(\varrho_1, \varrho_1)[s(\varrho_1, \varrho_2) J_s(\varrho_1, \varrho_2) + s(\varrho_2, \sigma_1) J_s(\varrho_2, \sigma_1)] - dist(\mathcal{U}, \mathcal{V}) \\ &= s(\varrho_1, \varrho_2) J_s(\varrho_1, \varrho_2) + s(\varrho_2, \sigma_1) J_s(\varrho_2, \sigma_1) - dist(\mathcal{U}, \mathcal{V}) \\ &= s(\varrho_1, \varrho_2) J_s(\varrho_1, \varrho_2), \end{aligned}$$

since by (5.2.11). Which further implies that

$$\frac{1}{s(\varrho_1, \varrho_2)} J_s^*(\varrho_1, \digamma \varrho_1) \leq \frac{1}{s(\varrho_1, \varrho_2)} s(\varrho_1, \varrho_2) J_s(\varrho_1, \digamma \varrho_1) = J_s(\varrho_1, \varrho_2).$$

Also since $\eta(r) \leq 1$, we obtain

$$\frac{\eta(r)}{s(\varrho_1, \varrho_2)} J_s^*(\varrho_1, \digamma\varrho_1) \leq J_s(\varrho_1, \varrho_2).$$

Thus by (5.2.8) we have

$$s(\varrho_1, \varrho_2) H^{J_s}(\digamma\varrho_1, \digamma\varrho_2) \leq r J_s(\varrho_1, \varrho_2)$$

$$\implies H^{J_s}(\digamma\varrho_1, \digamma\varrho_2) \leq \frac{r}{s(\varrho_1, \varrho_2)} J_s(\varrho_1, \varrho_2).$$

Since $J_s(\sigma_1, \digamma\varrho_2) \leq H^{J_s}(\digamma\varrho_1, \digamma\varrho_2) \leq \frac{r}{s(\varrho_1,\varrho_2)} J_s(\varrho_1, \varrho_2)$, so there will exists $\sigma_2 \in \digamma\varrho_2$ such that

$$J_s(\sigma_1, \sigma_2) \leq \frac{r}{s(\varrho_1, \varrho_2)} J_s(\varrho_1, \varrho_2). \tag{5.2.12}$$

Continuing this process, we can find two sequences (ϱ_n) and (σ_n) for $n \in \{0\} \cup N$ such that

(1) $\varrho_n \in \mathcal{U}_0, \sigma_n \in \mathcal{V}_0 \; \forall \; n \in \mathbb{N}$.

(2) $\sigma_n \in \digamma\varrho_n \; \forall \; n \in \{0\} \cup \mathbb{N}$.

(3) $s(\varrho_n, \sigma_{n-1}) J_s(\varrho_n, \sigma_{n-1}) = dist(\mathcal{U}, \mathcal{V}) \; \forall \; n \in \mathbb{N}$.

(4) $J_s(\sigma_{n-1}, \sigma_n) \leq \frac{r}{s(\varrho_{n-1},\varrho_n)} J_s(\varrho_{n-1}, \varrho_n) \; \forall \; n \in \mathbb{N}$.

Now for any $n \in \mathbb{N}$ we have $s(\varrho_n, \sigma_{n-1}) J_s(\varrho_n, \sigma_{n-1}) = dist(\mathcal{U}, \mathcal{V})$ and $s(\varrho_{n+1}, \sigma_n) J_s(\varrho_{n+1}, \sigma_n) = dist(\mathcal{U}, \mathcal{V})$. But $(\mathcal{U}, \mathcal{V})$ satisfy the WP^{J_s}-property, so we conclude that

$$J_s(\varrho_n, \varrho_{n+1}) \leq J_s(\sigma_{n-1}, \sigma_n) \; \forall \; n \in \mathbb{N}.$$

Thereby,

$$\begin{aligned}
J_s(\varrho_n, \varrho_{n+1}) &\leq J_s(\sigma_{n-1}, \sigma_n) \\
&\leq \frac{r}{s(\varrho_{n-1}, \varrho_n)} J_s(\varrho_{n-1}, \varrho_n) \\
&\leq \frac{r}{s(\varrho_{n-1}, \varrho_n)} J_s(\sigma_{n-2}, \sigma_{n-1}) \\
&\leq \frac{r^2}{s(\varrho_{n-1}, \varrho_n) s(\varrho_{n-2}, \varrho_{n-1})} J_s(\varrho_{n-2}, \varrho_{n-1}) \\
&\leq \frac{r^2}{s(\varrho_{n-1}, \varrho_n) s(\varrho_{n-2}, \varrho_{n-1})} J_s(\sigma_{n-3}, \sigma_{n-2}) \\
&\vdots \\
&\leq \frac{r^{n-1}}{\prod_{i=2}^{n} s(\varrho_{i-1}, xi)} J_s(\sigma_0, \sigma_1) \\
&\leq \frac{r^{n}}{\prod_{i=1}^{n} s(\varrho_{i-1}, xi)} J_s(\varrho_0, \varrho_1) \qquad (5.2.13)
\end{aligned}$$

Now for each $m > n$, we have

$$
\begin{aligned}
J_s(\varrho_n,\varrho_m) &\le s(\varrho_n,\varrho_{n+1})J_s(\varrho_n,\varrho_{n+1}) + s(\varrho_{n+1},\varrho_m)J_s(\varrho_{n+1},\varrho_m)\\
&\le s(\varrho_n,\varrho_{n+1})J_s(\varrho_n,\varrho_{n+1}) + s(\varrho_{n+1},\varrho_m)s(\varrho_{n+1},\varrho_{n+2})\\
&\quad J_s(\varrho_{n+1},\varrho_{n+2}) + s(\varrho_{n+1},\varrho_m)s(\varrho_{n+2},\varrho_m)J_s(\varrho_{n+2},\varrho_m)\\
&\le s(\varrho_n,\varrho_{n+1})J_s(\varrho_n,\varrho_{n+1}) + s(\varrho_{n+1},\varrho_m)s(\varrho_{n+1},\varrho_{n+2})\\
&\quad J_s(\varrho_{n+1},\varrho_{n+2}) + s(\varrho_{n+1},\varrho_m)s(\varrho_{n+2},\varrho_m)s(\varrho_{n+2},\varrho_{n+3})J_s(\varrho_{n+2},\varrho_{n+3})\\
&\quad + s(\varrho_{n+1},\varrho_m)s(\varrho_{n+2},\varrho_m)s(\varrho_{n+3},\varrho_m)J_s(\varrho_{n+3},\varrho_m)\\
&\quad \vdots\\
&\le s(\varrho_n,\varrho_{n+1})J_s(\varrho_n,\varrho_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n}^{i} s(\varrho_j,\varrho_m)\right)s(\varrho_i,\varrho_{i+1})J_s(\varrho_i,\varrho_{i+1})\\
&+ \left(\prod_{k=n}^{m-1} s(\varrho_k,\varrho_m)\right)J_s(\varrho_{m-1},\varrho_m)\\
&\le s(\varrho_n,\varrho_{n+1})J_s(\varrho_n,\varrho_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n}^{i} s(\varrho_j,\varrho_m)\right)s(\varrho_i,\varrho_{i+1})J_s(\varrho_i,\varrho_{i+1})\\
&+ \left(\prod_{k=n}^{m-1} s(\varrho_k,\varrho_m)\right)s(\varrho_{m-1},\varrho_m)J_s(\varrho_{m-1},\varrho_m)\\
&= s(\varrho_n,\varrho_{n+1})J_s(\varrho_n,\varrho_{n+1}) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n}^{i} s(\varrho_j,\varrho_m)\right)s(\varrho_i,\varrho_{i+1})J_s(\varrho_i,\varrho_{i+1})\\
&\le \left(\prod_{j=0}^{n} s(\varrho_j,\varrho_m)\right)s(\varrho_n,\varrho_{n+1})J_s(\varrho_n,\varrho_{n+1})\\
&+ \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i} s(\varrho_j,\varrho_m)\right)s(\varrho_i,\varrho_{i+1})J_s(\varrho_i,\varrho_{i+1})\\
&= \sum_{i=n}^{m-1}\left(\prod_{j=0}^{i} s(\varrho_j,\varrho_m)\right)s(\varrho_i,\varrho_{i+1})J_s(\varrho_i,\varrho_{i+1})\\
&\le \sum_{i=n}^{m-1}\left(\prod_{j=0}^{i} s(\varrho_j,\varrho_m)\right)s(\varrho_i,\varrho_{i+1})\frac{r^i}{\prod_{k=1}^{i} s(\varrho_{k-1},xk)}J_s(\varrho_0,\varrho_1)\\
&\le \sum_{i=n}^{m-1}\left(\prod_{j=0}^{i} s(\varrho_j,\varrho_m)\right)s(\varrho_i,\varrho_{i+1})r^iJ_s(\varrho_0,\varrho_1)\\
&= J_s(\varrho_0,\varrho_1)\sum_{i=n}^{m-1}\left(\prod_{j=0}^{i} s(\varrho_j,\varrho_m)\right)s(\varrho_i,\varrho_{i+1})r^i.
\end{aligned}
$$

In the above steps we use the fact that $s(p,q) \geq 1$ and thus $x \leq s(p,q)x$ for any $x \in [0,\infty)$.
Let

$$a_n = \left(\prod_{j=0}^{n} s(\varrho_j,\varrho_m)\right) s(\varrho_n,\varrho_{n+1})r^n \text{ and } S = \sum_{n=1}^{\infty} a_n.$$

Since by hypothesis, $r \lim_{m,i\to\infty} \frac{s(\varrho_{i+1},\varrho_{i+2})}{s(\varrho_i,\varrho_{i+1})} s(\varrho_{i+1},\varrho_m) < 1$, so the series S converges, because by using ratio test we have

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &\leq \lim_{n\to\infty}\frac{r^{n+1}\left(\prod_{j=0}^{n+1} s(\varrho_j,\varrho_m)\right) s(\varrho_{n+1},\varrho_{n+2})}{r^n\left(\prod_{j=0}^{n} s(\varrho_j,\varrho_m)\right) s(\varrho_n,\varrho_{n+1})}\\ &= r\lim_{n,m\to\infty}\frac{s(\varrho_{n+1},\varrho_{n+2})}{s(\varrho_n,\varrho_{n+1})} s(\varrho_{n+1},\varrho_m) < 1.\end{aligned}$$

Thus $S_{m-1} - S_n = \left[\sum_{i=n}^{m-1}\left(\prod_{j=0}^{i} s(u_j,u_m)\right) s(u_i,u_{i+1})r^i\right] \to 0$ as $n,m\to\infty$ and so is $d_s(u_0,u_1)(S_{m-1}-S_n)$. Hence we deduce that

$$\lim_{n\to\infty}\sup_{m>n} J_s(\varrho_n,\varrho_m) = 0.$$

Similar calculation implies that

$$\lim_{n\to\infty}\sup_{m>n} J_s(\sigma_n,\sigma_m) = 0.$$

Therefore, by Lemma 5.2.5 we can say that the sequences (ϱ_n) and (σ_n) are Cauchy sequences in $\mathcal{U}$ and $\mathcal{V}$ respectively. Since $(\mathcal{U},\mathcal{V})$ is a closed pair of subsets of the complete CMT space $\mathfrak{Q}$, so there will exists $p \in \mathcal{U}$ and $q \in \mathcal{V}$ such that $\varrho_n \to p$ and $\sigma_n \to q$. Also since $\sigma_n \in \digamma\varrho_n \forall\, m \in \{0\}\cup\mathbb{N}$, so by closeness of $\digamma$ we obtain that $q \in \digamma p$.
On the other hand, since $s(\varrho_n,\sigma_{n-1})J_s(\varrho_n,\sigma_{n-1}) = dist(\mathcal{U},\mathcal{V})$ and J_s is associated with $(\mathcal{U},\mathcal{V})$, so we conclude that $d_s(p,q) = dist(\mathcal{U},\mathcal{V})$. We now have

$$dist(\mathcal{U},\mathcal{V}) \leq D(p,\mathcal{V}) \leq D(p,\digamma p) \leq d_s(p,q) = dist(\mathcal{U},\mathcal{V}).$$

i.e $D(p,\digamma p) = dist(\mathcal{U},\mathcal{V})$ and so $p \in \mathcal{U}$ is a $\mathcal{B}\cdot\mathcal{P}\cdot\mathcal{P}$ of the non-self mapping $\digamma$. □

5.3 Consequences and applications

This section consists of some important consequences of Theorem 5.2.7. The results we have obtained generalizes/extends some results by Gabeleh and Plebaniak.

Corollary 5.3.1. Let $\mathfrak{Q}$ be a complete $\mathcal{M}\cdot\mathcal{S}$ and let the mapping $J : \mathfrak{Q}\times\mathfrak{Q} \to [0,\infty)$ be a generalized pseudo-distance on $\mathfrak{Q}$. Let $(\mathcal{U},\mathcal{V})$ be a nonempty closed pair of subsets of $\mathfrak{Q}$ with $\mathcal{U}_0 \neq \emptyset$ and such that $(\mathcal{U},\mathcal{V})$ has the WP-property and J is associated with $(\mathcal{U},\mathcal{V})$. Let $F : \mathcal{U} \to 2^{\mathcal{V}}$ be a closed contraction of S-type. If $F(u) \in CB(\mathfrak{Q})\ \forall\ u \in \mathcal{U}$, and $F(u) \subset \mathcal{V}_0$ for each $u \in \mathcal{U}_0$, then F has a $\mathcal{B}\cdot\mathcal{P}\cdot\mathcal{P}$ in $\mathcal{U}$.

Proof. The assertion holds if we define $s(u,v) = 1$ for all $u,v \in \mathfrak{Q}$ in Theorem 5.2.7. □

Following is the main result in [19] which is direct consequence of the result 5.2.7.

Theorem 5.3.2. [19] Let $\eta : [0,1) \to (1/2,1]$ by $\eta(k) = \frac{1}{1+k}$. Let $\mathcal{U},\mathcal{V}$ be the closed subsets of the complete space $(\mathfrak{Q},d)$ such that $(\mathcal{U},\mathcal{V})$ has the P-property. Let $F : \mathcal{U} \to 2^{\mathcal{V}}$ be a multi-valued mapping such that

$$\eta(k)D^*(u,Fu) \leq d(u,v) \quad \text{implies} \quad H^d(Fu,Fv) \leq\ kd(u,v) \text{ for each } u,v \in \mathcal{U},$$

where $0 \leq k < 1, D^*(u,Fu) = D(u,Fu) - Dist(\mathcal{U},\mathcal{V})$. Let $Fu \in CB(\mathfrak{Q})$ for each $u \in \mathcal{U}$, $Fu \subset \mathcal{V}_0$ for each $u \in \mathcal{U}_0$. Then there exists some p in $\mathcal{U}$ such that $D(p,Fp) = Dist(\mathcal{U},\mathcal{V})$.

The main result of [20] is proved easily by Theorem 5.2.7 when we define $s(\sigma,\varrho) = b \geq 1$.

Theorem 5.3.3. Let $\mathfrak{Q}$ be a complete b-$\mathcal{M}\cdot\mathcal{S}$ (with $s \geq 1$) and let the mapping $J_b : \mathfrak{Q}\times\mathfrak{Q} \to [0,\infty)$ be a bG pseudo-distance on $\mathfrak{Q}$. Let $(\mathcal{U},\mathcal{V})$ be a nonempty closed pair of subsets of $\mathfrak{Q}$ with $\mathcal{U}_0 \neq \emptyset$ and such that $(\mathcal{U},\mathcal{V})$ has the WP^{J_b}-property and J_b is associated with $(\mathcal{U},\mathcal{V})$. Let $F : \mathcal{U} \to 2^{\mathcal{V}}$ be a closed contraction of S-type. If $F(\varrho) \in CB(\mathfrak{Q})\ \forall\ \varrho \in \mathcal{U}$, and $F(\varrho) \subset \mathcal{V}_0$ for each $\varrho \in \mathcal{U}_0$, then F has a $\mathcal{B}\cdot\mathcal{P}\cdot\mathcal{P}$ in $\mathcal{U}$.

Chapter 6

Fixed point theorems of single-valued dynamical systems in controlled quasi-triangular spaces

In this chapter, we have extended the idea of Wlodarczyk [59] and introduced a new space, which we call the controlled quasi-triangular space (in short CQT space). We introduced the left (right) families generated by such spaces and proved Banach type theorem in such spaces. Our results generalizes results proved in triangular space, QT space, CMT space, b-metric, quasi-metric, quasi b-metric and $\mathcal{M} \cdot \mathcal{S}$. Throughout this chapter by L(R), we will always mean left (right).

6.1 Controlled quasi-triangular space

We start this section by the definition of CQT family and CQT space.

Definition 6.1.1. Let $\mathfrak{Q}$ be a non-empty set and $\mathcal{I}$ be an index set. Let $C = \{S_\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty) \ : \ \alpha \in \mathcal{I}\}$.

1. We say that a family $\mathscr{P}_{C;\mathcal{I}} = \{p_\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty) \ : \ \alpha \in \mathcal{I}\}$ of distances is a CQT (in short CQT) family on $\mathfrak{Q}$ if

$$p_\alpha(\zeta, \gamma) \leq S_\alpha(\zeta, \xi)p_\alpha(\zeta, \xi) + S_\alpha(\xi, \gamma)p_\alpha(\xi, \gamma) \text{ for all } \zeta, \xi, \gamma \in \mathfrak{Q}. \tag{6.1.1}$$

A CQT space $(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}})$ is a set $\mathfrak{Q}$ together with the CQT family $\mathscr{P}_{C;\mathcal{I}}$ on $\mathfrak{Q}$.
2. Let $(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}})$ be a CQT space. We say that the family $\mathscr{P}_{C;\mathcal{I}}$ is separating if for all $\zeta, \gamma \in \mathfrak{Q}$ with $\zeta \neq \gamma$, there exists $\alpha \in \mathcal{I}$ such that

$$p_\alpha(\zeta, \gamma) > 0 \text{ or } p_\alpha(\gamma, \zeta) > 0.$$

3. If $(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}})$ is a CQT space and for all $\alpha \in \mathcal{I}$, $p_\alpha^{-1}(\zeta, \gamma) = p_\alpha(\gamma, \zeta)$ for all $\zeta, \gamma \in \mathfrak{Q}$, then $\forall \alpha \in \mathcal{I}$, and for all $\zeta, \xi, \gamma \in \mathfrak{Q}$,

$$p_\alpha^{-1}(\zeta, \gamma) \leq S_\alpha(\zeta, \xi) p_\alpha^{-1}(\zeta, \xi) + S_\alpha(\xi, \gamma) p_\alpha^{-1}(\xi, \gamma).$$

We say that the CQT space $(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}}^{-1})$ is the conjugation of $(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}})$ where $\mathscr{P}_{C;\mathcal{I}}^{-1} = \{p_\alpha^{-1} : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty) \ : \ \alpha \in \mathcal{I}\}$.

Remark 6.1.2. In general, in the space $(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}}^{-1})$ the distances $p_\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)$ for $\alpha \in \mathcal{I}$ do not vanish on the diagonal, they are asymmetric and do not satisfy triangle inequality, (i.e. the properties $p_\alpha(u, u) = 0 \ \forall u \in \mathfrak{Q}$ or $p_\alpha(u, w) = p_\alpha(w, u) \ \ \forall u, w \in \mathfrak{Q}$ or $p_\alpha(u, w) \leq p_\alpha(u, v) + p_\alpha(v, w) \ \ \forall u, v, w \in \mathfrak{Q}$ do nor hold necessary)

Definition 6.1.3. Let $\mathfrak{Q}$ be a non-void set and let $C = \{S_\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [1, \infty) \ : \ \alpha \in \mathcal{I}\}$.
1. We say that a family $\mathscr{Q}_{C;\mathcal{I}} = \{q_\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty) \ : \ \alpha \in \mathcal{I}\}$ of distances is an ultra CQT family on $\mathfrak{Q}$ if

$$q_\alpha(\varrho, \varpi) \leq \max\{S_\alpha(\varrho, \sigma) q_\alpha(\varrho, \sigma), S_\alpha(\sigma, \varpi) q_\alpha(v, \varpi)\} \quad : \forall \ \alpha \in \mathcal{I}, \ \ \forall \ \varrho, \sigma, \varpi \in \mathfrak{Q}. \tag{6.1.2}$$

An ultra CQT space $(\mathfrak{Q}, \mathscr{Q}_{C;\mathcal{I}})$ is a set $\mathfrak{Q}$ together with the CQT family $\mathscr{Q}_{C;\mathcal{I}}$ on $\mathfrak{Q}$.
2.We say that a family $\mathscr{S}_{C;\mathcal{I}} = \{p_\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty) \ : \ \alpha \in \mathcal{I}\}$ of distances is a controlled partial QT family on $\mathfrak{Q}$ if

$$p_\alpha(\zeta, \gamma) \leq S_\alpha(\zeta, \xi) p_\alpha(\zeta, \xi) + S_\alpha(\xi, \gamma) p_\alpha(\xi, \gamma) - p_\alpha(\xi, \xi) \text{ for all } \zeta, \xi, \gamma \in \mathfrak{Q}. \tag{6.1.3}$$

A controlled partial QT space $(\mathfrak{Q}, \mathscr{S}_{C;\mathcal{I}})$ is a set $\mathfrak{Q}$ together with the controlled partial QT family $\mathscr{S}_{C;\mathcal{I}}$ on $\mathfrak{Q}$.

Remark 6.1.4. 1. If we define $S_\alpha(u,w) = \mathcal{C}_\alpha \in [1,\infty)$ for each $\alpha \in \mathcal{I}$ and for each $u,w \in \mathfrak{Q}$ in (6.1.1), then we get a quasi-triangular space [59, 60].

2. It is noticing that CQT space generalize ultra CQT and controlled partial QT space.

Example 6.1.5. Let $\mathfrak{Q} = [0:6]$, and $p_1 : \mathfrak{Q}^2 \to [0,\infty)$, $S_1 : \mathfrak{Q}^2 \to [1,\infty)$ be defined by

$$p_1(\varrho,\varpi) = \begin{cases} 0 \quad \text{if} \quad \varrho \geq \varpi \\ (\varpi - \varrho)^3 \quad \text{if} \quad \varrho < \varpi. \end{cases} \tag{6.1.4}$$

$$S_1(\varrho,\varpi) = \varrho + \varpi + 4 \tag{6.1.5}$$

(1) The space $(\mathfrak{Q}, \mathscr{P}_{C;\{1\}})$, $\mathscr{P}_{C;\{1\}} = \{p_1\}$ is a CQT space. In fact for all $\varrho, \sigma, \varpi \in \mathfrak{Q}$

$$p_1(\varrho,\varpi) \leq (\varrho + \sigma + 4)p_1(\varrho,\sigma) + (\sigma + \varpi + 4)p_1(\sigma,\varpi)$$

holds. This can be prove from the following cases.

Case 1. If $\sigma \leq \varrho < \varpi$ then $p_1(\varrho,\sigma) = 0$, $\varpi - \varrho \leq \varpi - \sigma$. Consequently,

$$\begin{aligned} p_s(\varrho,\sigma) &= (\varpi - \varrho)^3 \leq (\varrho - \sigma)^3 \\ &< (\varrho + \sigma + 4)p(\varrho,\sigma)^3 \\ &= (\varrho + \sigma + 4)p_1(\varrho,\sigma) \\ &= (\varrho + \sigma + 4)p_1(\varrho,\sigma) + (\sigma + \varpi + 4)p_1(\sigma,\varpi). \end{aligned}$$

Case 2. If $\varrho < \varpi$ and $\varrho \leq \sigma \leq \varpi$ then $p_1(\varrho,\varpi) = (\varpi - \varrho)^3$ and $f(\sigma_0) = \min_{\varrho \leq \sigma \leq \varpi} f(\sigma) = (\varpi - \varrho)^3$ where $\sigma_0 = \frac{(\varrho + \varpi)}{2}$ is a minimum element of the map

$$f(\sigma) = (\varrho + \sigma + 4)p_1(\varrho,\sigma) + (\sigma + \varpi + 4)p_1(\sigma,\varpi)$$

Case 3. If $\varrho < \varpi \leq \sigma$, then $p_1(\sigma,\varpi) = 0$ and consequently

$$\begin{aligned} p_1(\varrho,\varpi) &= (\varpi - \varrho)^3 \leq (\sigma - \varrho)^3 < (\varrho + \sigma + 4)(\varrho,\sigma)^3 \\ &= (\varrho + \sigma + 4)p_1(\varrho,\sigma) \\ &= (\varrho + \sigma + 4)p_1(\varrho,\sigma) + (\sigma + \varpi + 4)p_1(\sigma,\varpi). \end{aligned}$$

(2) $\mathscr{P}_{C;\{1\}} = \{p_1\}$ is asymmetric. Indeed, we have that $0 = p_1(6,0) \neq p_1(0,6) = 216$. Therefore condition $p_1(\varrho, \varpi) = p_1(\varpi, \varrho)$ does not hold for all $\varrho, \varpi \in \mathfrak{Q}$.

(3) $\mathscr{P}_{C;\{1\}} = \{p_1\}$ vanishes on the diagonal. By (6.1.4), it is clear that $\forall \varrho \in \mathfrak{Q},\ p_1(\varrho, \varrho) = 0$.

Example 6.1.6. Let $\mathfrak{Q} = [0, \infty)$, and $p_1 : \mathfrak{Q}^2 \to [0, \infty)$, $S_1 : \mathfrak{Q}^2 \to [1, \infty)$ be defined by

$$p_1(\zeta, \eta) = \begin{cases} 0 \quad \text{if} \quad \zeta = \eta = 0 \\ \frac{\eta}{1+\eta} \quad \text{if} \quad \zeta = 0 \quad \text{and} \quad \eta \neq 0 \\ \frac{\zeta}{1+\zeta} \quad \text{if} \quad \zeta \neq 0 \quad \text{and} \quad \eta = 0 \\ \zeta + \eta \quad \text{if} \quad \zeta \neq 0 \neq \eta, \end{cases} \tag{6.1.6}$$

$$S_1(\zeta, \eta) = 2\zeta + 2\eta + 2.$$

1. The space $(\mathfrak{Q}, \mathscr{P}_{C;\{1\}})$ with $\mathscr{P}_{C;\{1\}} = \{p_1\}$ is a CQT space and for this we need to show that $\forall \zeta, \xi, \eta \in \mathfrak{Q}$ the following inequality holds:

$$p_1(\zeta, \eta) \leq S_1(\zeta, \xi) p_1(\zeta, \xi) + S_1(\xi, \eta) p_1(\xi, \eta). \tag{6.1.7}$$

For this we have the following cases.

Case 1. If $\zeta = 0 = \eta$, then (6.1.7) holds trivially.

Case 2. If $\zeta = 0$, $\eta \neq 0$ and $\xi = 0$, then we have

$$\begin{aligned} p_1(\zeta, \eta) &= \frac{\eta}{1+\eta} \\ &\leq (2)(0) + (2 + 2\eta)\frac{\eta}{1+\eta} \\ &\leq S_1(\zeta, \xi) p_1(\zeta, \xi) + S_1(\xi, \eta) p_1(\xi, \eta). \end{aligned}$$

Case 3. If $\zeta = 0$, $\eta \neq 0$ and $\xi \neq 0$, then we have

$$\begin{aligned}
p_1(\zeta,\eta) &= \frac{\eta}{1+\eta} \\
&\le \eta \le (2+2\xi+2\eta)(\xi+\eta) \\
&\le (2+2\xi)\frac{\xi}{1+\xi} + (2+2\xi+2\eta)(\xi+\eta) \\
&\le S_1(\zeta,\xi)p_1(\zeta,\xi) + S_1(\xi,\eta)p_1(\xi,\eta).
\end{aligned}$$

Case 4. If $\zeta \neq 0$ and $\eta = 0$, then by similar process as Cases 3 and 4, we obtain that (6.1.7) holds.

Case 5. If $\zeta \neq 0 \neq \eta$ and $\xi = 0$, then we have:

$$\begin{aligned}
p_1(\zeta,\eta) &= \zeta+\eta \\
&\le (2+2\zeta)\frac{\zeta}{1+\zeta} + (2+2\eta)\frac{\eta}{1+\eta} \\
&\le S_1(\zeta,\xi)p_1(\zeta,\xi) + S_1(\xi,\eta)p_1(\xi,\eta).
\end{aligned}$$

Case 6. If $\zeta \neq 0 \neq \eta$ and $\xi \neq 0$, then we have:

$$\begin{aligned}
p_1(\zeta,\eta) &= \zeta+\eta \\
&< \zeta+\xi+\xi+\eta \\
&\le (2+2\zeta+2\xi)(\zeta+\xi) + (2+2\xi+2\eta)(\xi+\eta) \\
&\le S_1(\zeta,\xi)p_1(\zeta,\xi) + S_1(\xi,\eta)p_1(\xi,\eta).
\end{aligned}$$

2. The space $(\mathfrak{Q}, \mathscr{P}_{C;\{1\}})$ with $\mathscr{P}_{C;\{1\}} = \{p_1\}$ is not a quasi-triangular space. We show this by contradiction, i.e. if it is quasi-triangular space, then there exists $S_\alpha \in [1,\infty)$ such that

$$p_1(\zeta,\eta) \le S_\alpha[p_1(\zeta,\xi) + p_1(\xi,\eta)] \quad \forall \zeta,\xi,\eta \in \mathfrak{Q}.$$

Now for any $\xi > 0$, we have

$$p_1(\xi,\xi+1) \le S_\alpha[p_1(\xi,0) + p_1(0,\xi+1)].$$

Which implies that

$$2\xi + 1 \leq S_\alpha \left[\frac{\xi}{\xi+1} + \frac{\xi+1}{\xi+2} \right]. \tag{6.1.8}$$

But when we take limit as $y \to \infty$ in (6.1.8), we get that $+\infty \leq 2S_\alpha$ which is not possible and hence $(\mathfrak{Q}, p_1)$ is not a quasi-triangular space.

3. $(\mathfrak{Q}, \mathscr{P}_{C;\{1\}})$ with $\mathscr{P}_{C;\{1\}} = \{p_1\}$ does not vanish on the diagonal, i,e, if $\eta \neq 0$, then $p_1(\eta, \eta) = 2\eta \neq 0$.

6.2 Left(Right) Families Generated by CQT families

Let $\mathscr{P}_{C;\mathcal{I}}$ be a CQT family on $\mathfrak{Q}$. It is natural to define families generated by $\mathscr{P}_{C;\mathcal{I}}$ which will provides a new structure on $\mathfrak{Q}$.

Definition 6.2.1. Let $(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}})$ be a CQT space.

(a) The family of distances $\mathscr{J}_{C;\mathcal{I}} = \{J_\alpha : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty)\ \alpha \in \mathcal{I}\}$ is said to be a left (right) (in short L(R)) family generated by $\mathscr{P}_{C;\mathcal{I}}$ if it satisfies the following:

$\mathscr{J}_1$ For all $\alpha \in \mathcal{I}$ and for all $\zeta, \xi, \gamma \in \mathfrak{Q}$,

$$J_\alpha(\zeta, \gamma) \leq S_\alpha(\zeta, \xi) J_\alpha(\zeta, \xi) + S_\alpha(\xi, \gamma) J_\alpha(\xi, \gamma); \tag{6.2.1}$$

$\mathscr{J}_2$ If the sequences $\{s_m\}$ and $\{t_m\}$ in $\mathfrak{Q}$ satisfying the following

$$\lim_{m\to\infty} \sup_{n>m} J_\alpha(s_m, s_n) = 0\ \forall\ \alpha \in \mathcal{I}, \tag{6.2.2}$$

$$\left(\lim_{m\to\infty} \sup_{n>m} J_\alpha(s_n, s_m) = 0\ \forall\ \alpha \in \mathcal{I} \right), \tag{6.2.3}$$

$$\lim_{m\to\infty} J_\alpha(t_m, s_m) = 0\ \forall\ \alpha \in \mathcal{I}, \tag{6.2.4}$$

$$\left(\lim_{m\to\infty} J_\alpha(s_m, t_m) = 0\ \forall\ \alpha \in \mathcal{I} \right), \tag{6.2.5}$$

then the following hold:

$$\lim_{m\to\infty} p_\alpha(t_m, s_m) = 0\ \forall\ \alpha \in \mathcal{I}, \tag{6.2.6}$$

$$\left(\lim_{m\to\infty} p_\alpha(s_m, t_m) = 0\ \forall\ \alpha \in \mathcal{I} \right). \tag{6.2.7}$$

(*b*) $\mathbb{J}^L_{(\mathfrak{Q},\mathscr{P}_{C;\mathcal{I}})}$ ($\mathbb{J}^R_{(\mathfrak{Q},\mathscr{P}_{C;\mathcal{I}})}$) is the collection of all $\mathscr{J}_{C;\mathcal{I}}$ on $\mathfrak{Q}$ generated by $\mathscr{P}_{C;\mathcal{I}}$.

Remark 6.2.2. (a) It can be directly seen from the Definition 6.2.1 that $\mathscr{P}_{C;\mathcal{I}}$ lies in both $\mathbb{J}^L_{(\mathfrak{Q},\mathscr{P}_{C;\mathcal{I}})}$ and $\mathbb{J}^R_{(\mathfrak{Q},\mathscr{P}_{C;\mathcal{I}})}$.

(b) The structure on $\mathfrak{Q}$ determined by L(R) families $\mathscr{J}_{C;\mathcal{I}}$ generated by $\mathscr{P}_{C;\mathcal{I}}$ are more general than the structure on $\mathfrak{Q}$ determined by $\mathscr{P}_{C;\mathcal{I}}$.

(c) If $\mathscr{J}_{C;\mathcal{I}} \in \mathbb{J}^L_{(\mathfrak{Q},\mathscr{P}_{C;\mathcal{I}})} \cap \mathbb{J}^R_{(\mathfrak{Q},\mathscr{P}_{C;\mathcal{I}})}$, then by (6.2.1), we can say that $(\mathfrak{Q}, \mathscr{J}_{C;\mathcal{I}})$ is a CQT space.

Motivated from [59, Theorem 14], we present the following result which shows the validity of Definition 6.2.1 and that $\mathbb{J}^L_{(\mathfrak{Q},\mathscr{P}_{C;\mathcal{I}})} - \mathscr{P}_{C;\mathcal{I}} \neq \emptyset$ and $\mathbb{J}^R_{(\mathfrak{Q},\mathscr{P}_{C;\mathcal{I}})}\mathscr{P}_{C;\mathcal{I}} \neq \emptyset$.

Proposition 6.2.3. Let $(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}})$ be a CQT space. Let $\mathscr{E} \subseteq \mathfrak{Q}$ be a set containing at least two different points and for every $\alpha \in \mathcal{I}$, $\mu_\alpha \in (0, \infty)$ be such that

$$\mu_\alpha \geq \frac{\delta_\alpha(\mathscr{E})}{2}, \tag{6.2.8}$$

where for all $\alpha \in \mathcal{I}$, $\delta_\alpha(\mathscr{E}) = \sup\{p_\alpha(u, w) \ : \ u, w \in \mathscr{E}\}$.
If $\mathscr{J}_{C;\mathcal{I}} = \{J_\alpha \ : \ \alpha \in \mathcal{I}\}$, where for each $\alpha \in \mathcal{I}$, the distance $J_\alpha : \mathfrak{Q}^2 \to [0, \infty)$ is defined by

$$J_\alpha(u, w) = \begin{cases} p_\alpha(u, w) & \text{if} \quad \{u, w\} \subseteq \mathscr{E} \\ \mu_\alpha & \text{if} \quad \{u, w\} \nsubseteq \mathscr{E}, \end{cases} \tag{6.2.9}$$

then $\mathscr{J}_{C;\mathcal{I}}$ is L(R) family generated by $\mathscr{P}_{C;\mathcal{I}}$.

Proof. Suppose on contrary that $\mathscr{J}_1$ does not holds. Then there exist some $\alpha_0 \in \mathcal{I}$ and $\sigma_0, \varrho_0, \varpi_0 \in \mathfrak{Q}$ such that

$$J_{\alpha_0}(\sigma_0, \varpi_0) > C_{\alpha_0}(\sigma_0, \varrho_0) J_{\alpha_0}(\sigma_0, \varrho_0) + C_{\alpha_0}(\varrho_0, \varpi_0) J_{\alpha_0}(\varrho_0, \varpi_0). \tag{6.2.10}$$

Then we have $\{\sigma_0, \varrho_0, \varpi_0\} \nsubseteq \mathscr{E}$. Because if it is subset of $\mathscr{E}$, then $J_{\alpha_0}(\sigma_0, \varrho_0) = p_{\alpha_0}(\sigma_0, \varrho_0)$, $J_{\alpha_0}(\sigma_0, \varpi_0) = p_{\alpha_0}(\sigma_0, \varpi_0)$, $J_{\alpha_0}(\varrho_0, \varpi_0) = p_{\alpha_0}(\varrho_0, \varpi_0)$, and with this, the Inequality (6.2.10) will become $p_{\alpha_0}(\sigma_0, \varpi_0) > C_{\alpha_0}(\sigma_0, \varrho_0) p_{\alpha_0}(\sigma_0, \varrho_0) + C_{\alpha_0}(\varrho_0, \varpi_0) p_{\alpha_0}(\varrho_0, \varpi_0)$, which is a contradiction to the fact that $(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}})$ is a CQT space. Thus we have the following four cases.

Case 1. If $\{\sigma_0, \varpi_0\} \subseteq \mathscr{E}$, then $\varrho_0 \notin \mathscr{E}$, then by using (6.2.9) we get that $J_{\alpha_0}(\sigma_0, \varpi_0) = p_{\alpha_0}(\sigma_0, \varpi_0)$, $J_{\alpha_0}(\sigma_0, \varrho_0) = \mu_{\alpha_0}$ and $J_{\alpha_0}(\varrho_0, \varpi_0) = \mu_{\alpha_0}$. Thus (6.2.10) and (6.2.8) implies that

$$\begin{aligned} p_{\alpha_0}(\sigma_0, \varpi_0) &> C_{\alpha_0}(\sigma_0, \varrho_0)\mu_{\alpha_0} + C_{\alpha_0}(\varrho_0, \varpi_0)\mu_{\alpha_0} \\ &\geq \mu_{\alpha_0} + \mu_{\alpha_0} \\ &\geq \frac{\delta_0(\mathscr{E})}{2} + \frac{\delta_0(\mathscr{E})}{2} \\ &= \delta_0(\mathscr{E}), \end{aligned}$$

which is impossible because $\sigma_0, \varpi_0 \in \mathscr{E}$.

Case 2. If $\sigma_0 \in \mathscr{E}$ and $\varpi_0 \notin \mathscr{E}$, then by using (6.2.9) we get that $J_{\alpha_0}(\sigma_0, \varpi_0) = \mu_{\alpha_0}$ and $J_{\alpha_0}(\varrho_0, \varpi_0) = \mu_{\alpha_0}$. Thus (6.2.10) and (6.2.8) implies that

$$\mu_{\alpha_0} > C_{\alpha_0}(\sigma_0, \varrho_0)p_{\alpha_0}(\sigma_0, \varrho_0) + C_{\alpha_0}(\varrho_0, \varpi_0)\mu_{\alpha_0},$$

which is not possible for every $\varrho_0 \in \mathfrak{Q}$.

Case 3. If $\sigma_0 \notin \mathscr{E}$ and $\varpi_0 \in \mathscr{E}$, then by using (6.2.9) we get that $J_{\alpha_0}(\sigma_0, \varpi_0) = \mu_{\alpha_0}$ and $J_{\alpha_0}(\sigma_0, \varrho_0) = \mu_{\alpha_0}$. Thus (6.2.10) and (6.2.8) implies that

$$\mu_{\alpha_0} > C_{\alpha_0}(\sigma_0, \varrho_0)\mu_{\alpha_0} + C_{\alpha_0}(\varrho_0, \varpi_0)\mu_{\alpha_0},$$

which is not possible for every $\varrho_0 \in \mathfrak{Q}$.

Case 4. If $\sigma_0 \notin \mathscr{E}$ and $\varpi_0 \notin \mathscr{E}$, then by using (6.2.9) we get that $J_{\alpha_0}(\sigma_0, \varpi_0) = \mu_{\alpha_0}$, $J_{\alpha_0}(\sigma_0, \varrho_0) = \mu_{\alpha_0}$ and $J_{\alpha_0}(\varrho_0, \varpi_0) = \mu_{\alpha_0}$. Thus (6.2.10) and (6.2.8) implies that

$$\mu_{\alpha_0} > C_{\alpha_0}(\sigma_0, \varrho_0)\mu_{\alpha_0} + C_{\alpha_0}(\varrho_0, \varpi_0)\mu_{\alpha_0},$$

which is not possible for every $\varrho_0 \in \mathfrak{Q}$.

Thus our supposition was wrong and hence for all $\alpha \in \mathcal{I}$ we have

$$J_\alpha(\zeta, \gamma) \leq S_\alpha(\zeta, \xi)J_\alpha(\zeta, \xi) + S_\alpha(\xi, \gamma)J_\alpha(\xi, \gamma) \text{ for all } \zeta, \xi, \gamma \in \mathfrak{Q}.$$

Assume that the sequence (s_m) and (t_n) in $\mathfrak{Q}$ satisfy (6.2.2) and (6.2.4). We need to show that (6.2.6) holds. Indeed, (6.2.4) implies that $\forall\ \alpha \in \mathcal{I}$, $\forall\ 0 < \epsilon < \mu_\alpha$ there exists $m_0 = m_0(\alpha) \in \mathbb{N}$ such that $\forall\ m \geq N$, we have

$$J_\alpha(s_m, t_m) < \epsilon. \tag{6.2.11}$$

Denote $m' = \min\{m_0(\alpha)\ :\ \alpha \in \mathcal{I}\}$, we can see by (6.2.11) and (6.2.9), that $\forall\ m \geq m'$, $\mathscr{E} \cap \{s_m, t_m\} = \{s_m, t_m\}$. Thus in view of Definition 6.2.1(a), (6.2.9) and (6.2.11), this implies that $\forall\ \alpha \in \mathcal{I}$, $\forall\ 0 < \epsilon < \mu_\alpha$ there exists $m' \in \mathbb{N}$ such that $\forall\ m \geq m'$, we have

$$p_\alpha(s_m, t_m) = J_\alpha(s_m, t_m) < \epsilon.$$

This show that (6.2.6) holds. Thus $\mathscr{J}_{C;\mathcal{I}}$ is a left family generated by $\mathscr{P}_{C;\mathcal{I}}$.
In the similar way, we can show that $\mathscr{J}_{C;\mathcal{I}}$ is a right family generated by $\mathscr{P}_{C;\mathcal{I}}$. We have proved that $\mathscr{J}_{C;\mathcal{I}} \in \mathbb{J}^L_{(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}})} \cap \mathbb{J}^R_{(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}})}$ holds. □

Definition 6.2.4. Let $(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}})$ be a CQT space and $\mathscr{J}_{C;\mathcal{I}}$ be a L(R) family generated by $\mathscr{P}_{C;\mathcal{I}}$. Let (η_n) be a sequence in $\mathfrak{Q}$.

1. We say that (η_n) is L(R) $\mathscr{J}_{C;\mathcal{I}}$-Cauchy sequence if $\forall \alpha \in \mathcal{I}$ we have

$$\lim_{m\to\infty} \sup_{n>m} J_\alpha(\eta_m, \eta_n) = 0 \qquad \left(\lim_{m\to\infty} \sup_{n>m} J_\alpha(\eta_n, \eta_m) = 0\right).$$

2. We say that (η_n) is L(R) $\mathscr{J}_{C;\mathcal{I}}$-convergent to $\eta \in \mathfrak{Q}$ if

$$\eta \in \mathrm{LIM}^{L-\mathscr{J}_{C;\mathcal{I}}}_{(\eta_m)} \qquad \left(\eta \in \mathrm{LIM}^{R-\mathscr{J}_{C;\mathcal{I}}}_{(\eta_m)}\right),$$

   where

$$\mathrm{LIM}^{L-\mathscr{J}_{C;\mathcal{I}}}_{(\eta_m)} = \left\{u \in \mathfrak{Q}\ :\ \lim_{m\to\infty} J_\alpha(u, \eta_m) = 0\ \ \forall\ \alpha \in \mathcal{I}\right\}$$
$$\left(\mathrm{LIM}^{R-\mathscr{J}_{C;\mathcal{I}}}_{(\eta_m)} = \left\{u \in \mathfrak{Q}\ :\ \lim_{m\to\infty} J_\alpha(\eta_m, u) = 0\ \ \forall\ \alpha \in \mathcal{I}\right\}\right).$$

3. If every L(R) $\mathscr{J}_{C;\mathcal{I}}$-Cauchy sequence (η_m) is L(R) $\mathscr{J}_{C;\mathcal{I}}$-convergent in $\mathfrak{Q}$ (i.e., $\mathrm{LIM}^{L-\mathscr{J}_{C;\mathcal{I}}}_{(\eta_m)} \neq \emptyset \left(\mathrm{LIM}^{R-\mathscr{J}_{C;\mathcal{I}}}_{(\eta_m)} \neq \emptyset\right)\right)$, then the space $(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}})$ is said to be L(R) $\mathscr{J}_{C;\mathcal{I}}$-sequential complete.

The proof of the following result is similar to the case of quasi-triangular space [59].

Theorem 6.2.5. Let $(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}})$ be a CQT space, and let $\mathscr{J}_{C;\mathcal{I}}$ be the L(R) family generated by $\mathscr{P}_{C;\mathcal{I}}$. If $\mathscr{P}_{C;\mathcal{I}}$ is separating on $\mathfrak{Q}$, i.e. if for all $\eta, \xi \in \mathfrak{Q}$ with $\eta \neq \xi$, there exists $\alpha \in \mathcal{I}$ such that

$$p_\alpha(\eta, \xi > 0 \vee p_\alpha(\xi, \eta) > 0, \tag{6.2.12}$$

then $\mathscr{J}_{C;\mathcal{I}}$ is separating on $\mathfrak{Q}$, i.e. for all $\eta, \xi \in \mathfrak{Q}$ with $\eta \neq \xi$, there exists $\alpha \in \mathcal{I}$ such that

$$J_\alpha(\eta, \xi > 0 \vee J_\alpha(\xi, \eta) > 0. \tag{6.2.13}$$

6.3 Banach type theorem in controlled quasi-triangular spaces

In this section, we discussed a Banach type theorem in the setting of CQT spaces.

Definition 6.3.1. Let $(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}})$ be a CQT space and $\mathscr{J}_{C;\mathcal{I}}$ be the L(R) family generated by $\mathscr{P}_{C;\mathcal{I}}$. Let $F : \mathfrak{Q} \to \mathfrak{Q}$ be a map, $\lambda = \{\lambda_\alpha : \alpha \in \mathcal{I}, \lambda_\alpha \in [0;1)\}$, and $\eta = 1 \,\text{or}\, 2$.

(A) If $\mathscr{J}_{C;\mathcal{I}} \in \mathbb{J}^L_{(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}})}$, then the left quasi-distance $\mathscr{D}^{L-\mathscr{J}_{C;\mathcal{I}}}_{\mathfrak{Q},\eta}$ on $\mathfrak{Q}$ is defined by

$$\mathscr{D}^{L-\mathscr{J}_{C;\mathcal{I}}}_{\mathfrak{Q},\eta} = \{D^{L-\mathscr{J}_{C;\mathcal{I}}}_{\eta;\mathfrak{Q};\alpha} : \mathfrak{Q} \times \mathfrak{Q} \to [0, \infty), \alpha \in \mathcal{I}\},$$

where $\forall\ \alpha \in \mathcal{I}\ \forall \sigma, \varrho \in \mathfrak{Q}$

$$D^{L-\mathscr{J}_{C;\mathcal{I}}}_{1;\mathfrak{Q};\alpha}(\sigma, \varrho) = \max\{J_\alpha(\sigma, \varrho), J_\alpha(\varrho, \sigma)\}, \tag{6.3.1}$$

$$D^{L-\mathscr{J}_{C;\mathcal{I}}}_{2;\mathfrak{Q};\alpha}(\sigma, \varrho) = J_\alpha(\sigma, \varrho).$$

We say that F is a left $(\mathscr{D}^{L-\mathscr{J}_{C;\mathcal{I}}}_{\mathfrak{Q},\eta}, \lambda)$-controlled quasi–contraction on $\mathfrak{Q}$ if $\forall_{\alpha \in \mathcal{I}}\ \forall_{\sigma,\varrho \in \mathfrak{Q}}$

$$S_\lambda(\sigma, \varrho) D^{L-\mathscr{J}_{C;\mathcal{I}}}_{\eta;\mathfrak{Q};\alpha}(T(\sigma), T(\varrho)) \leq \lambda_\alpha J_\alpha(\sigma, \varrho), \tag{6.3.2}$$

and for each $\sigma_0 \in \mathfrak{Q}$ with $\sigma_{n+1} = T(\sigma_n)$ we have

$$\lim_{m,i \to \infty} \frac{S_\alpha(\sigma_{i+1}, \sigma_{i+2})}{S_\alpha(\sigma_i, \sigma_{i+1})} S_\alpha(\sigma_{i+1}, \sigma_m) < \frac{1}{\lambda_\alpha}.$$

(B) If $\mathscr{J}_{C;\mathcal{I}} \in \mathbb{J}^{R}_{(\mathfrak{Q},\mathscr{P}_{C;\mathcal{I}})}$, then we define the right $\mathscr{D}^{R-\mathscr{J}_{C;\mathcal{I}}}_{\mathfrak{Q},\eta}$ quasi-distance on $\mathfrak{Q}$ by $\mathscr{D}^{R-\mathscr{J}_{C;\mathcal{I}}}_{\mathfrak{Q},\eta} = \{D^{R-\mathscr{J}_{C;\mathcal{I}}}_{\eta;\mathfrak{Q};\alpha} : \mathfrak{Q} \times \mathfrak{Q} \to [0,\infty), \alpha \in \mathcal{I}\}$ where $\forall_{\alpha\in\mathcal{I}}\ \forall_{\sigma,\varrho\in\mathfrak{Q}}$

$$D^{R-\mathscr{J}_{C;\mathcal{I}}}_{2;\mathfrak{Q};\alpha}(\sigma,\varrho) = \max\{J_\alpha(\sigma,\varrho), J_\alpha(\varrho,\sigma)\}, \tag{6.3.3}$$

$$D^{R-\mathscr{J}_{C;\mathcal{I}}}_{2;\mathfrak{Q};\alpha}(\sigma,\varrho) = J_\alpha(\sigma,\varrho).$$

We say that F is right $(\mathscr{D}^{R-\mathscr{J}_{C;\mathcal{I}}}_{\mathfrak{Q},\eta}, \lambda)$-controlled quasi–contraction on $\mathfrak{Q}$ if $\forall_{\alpha\in\mathcal{I}}\ \forall_{\sigma,\varrho\in\mathfrak{Q}}$ we have

$$S_\lambda(\sigma,\varrho) D^{R-\mathscr{J}_{C;\mathcal{I}}}_{\eta;\mathfrak{Q};\alpha}(T(\sigma),T(\varrho)) \leq \lambda_\alpha J_\alpha(\sigma,\varrho), \tag{6.3.4}$$

and for each $\sigma_0 \in \mathfrak{Q}$ with $\sigma_{n+1} = T(\sigma_n)$ we have

$$\lim_{m,i\to\infty} \frac{S_\alpha(\sigma_{i+1},\sigma_{i+2})}{S_\alpha(\sigma_i,\sigma_{i+1})} S_\alpha(\sigma_{i+1},\sigma_m) < \frac{1}{\lambda_\alpha}.$$

Definition 6.3.2. Let $(\mathfrak{Q},\mathscr{P}_{C;\mathcal{I}})$ be a CQT space and $\mathscr{J}_{C;\mathcal{I}}$ be the L(R) family generated by $\mathscr{P}_{C;\mathcal{I}}$. Let $F : \mathfrak{Q} \to \mathfrak{Q}$ be a map.

(A) Given $\varrho_0 \in \mathfrak{Q}$, we say that F is L(R) $\mathscr{J}_{C;\mathcal{I}}$-admissible in ϱ_0 if for the sequence $(\varrho_m = T^{[m]}(\varrho_0))$, we have $\mathrm{LIM}^{L-\mathscr{J}_{C;\mathcal{I}}}_{(\varrho_m} \neq \emptyset \left(\mathrm{LIM}^{R-\mathscr{J}_{C;\mathcal{I}}}_{(\varrho_m} \neq \emptyset\right)$ whenever

$$\lim_{m\to\infty} \sup_{n>m} J_\alpha(\varrho_m,\varrho_n) = 0 \left(\lim_{m\to\infty} \sup_{n>m} J_\alpha(\varrho_n,\varrho_m) = 0\right), \quad \forall\, \alpha \in \mathcal{I}. \tag{6.3.5}$$

(B) we say that F is L(R) $\mathscr{J}_{C;\mathcal{I}}$-admissible on $\mathfrak{Q}$, if F is L(R) $\mathscr{J}_{C;\mathcal{I}}$-admissible in each point $\varrho_0 \in \mathfrak{Q}$.

Following is the generalization of continuity.

Definition 6.3.3. Let $(\mathfrak{Q},\mathscr{P}_{C;\mathcal{I}})$ be a CQT space. Let $F : \mathfrak{Q} \to \mathfrak{Q}$ be a map and $k \in \mathbb{N}$. The single-valued dynamical system $(\mathfrak{Q}, F^{[k]})$ is called a L(R) $\mathscr{P}_{C;\mathcal{I}}$-closed on $\mathfrak{Q}$ if for each sequence (x_m) in $F^{[k]}(\mathfrak{Q})$, L(R) $\mathscr{P}_{C;\mathcal{I}}$-converging in $\mathfrak{Q}$ and having subsequences (v_m) and (u_m) satisfying that for all $m \in \mathbb{N}, v_m = F^{[k]}(u_m)$, the following property holds:

$$\exists\, \eta \in \mathrm{LIM}^{L-\mathscr{J}_{C;\mathcal{I}}}_{(\varrho_m)} \ (LIM^{R-\mathscr{J}_{C;\mathcal{I}}}_{(\varrho_m)}) \quad \text{such that } \eta = F^{[k]}(\eta).$$

Now we present the main result of this chapter.

Theorem 6.3.4. Let $(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}})$ be a CQT space, $\digamma : \mathfrak{Q} \to \mathfrak{Q}$ a map, $\zeta \in \{0,1\}$ and $\lambda = \{\lambda_\alpha \in [0,1) \ : \ \alpha \in \mathcal{I}\}$. Suppose that there is a L(R) family $\mathscr{J}_{C;\mathcal{I}}$ generated by $\mathscr{P}_{C;\mathcal{I}}$ and a point $\sigma_0 \in \mathfrak{Q}$ which satisfy the following properties.

(a_1) $\digamma$ is left $(\mathscr{D}^{L-\mathscr{J}_{C;\mathcal{I}}}_{\mathfrak{Q},\zeta}, \lambda)$ controlled quasi–contraction (right $(\mathscr{D}^{R-\mathscr{J}_{C;\mathcal{I}}}_{\mathfrak{Q},\zeta}, \lambda)$ controlled quasi–contraction) on $\mathfrak{Q}$.

(a_2) $\digamma$ is L(R) $\mathscr{J}_{C;\mathcal{I}}$-admissible in a point $\sigma_0 \in \mathfrak{Q}$.

Then the following hold.

(b_1) There exists a point $\varpi \in \mathfrak{Q}$ such that the sequence $\sigma_n = \digamma^n \sigma_0$ is L(R)-$\mathscr{P}_{C;\mathcal{I}}$ convergent to ϖ.

(b_2) If the dynamical system $\digamma^k$ is L(R) $\mathscr{P}_{C;\mathcal{I}}$-closed on $\mathfrak{Q}$ for some $k \in \mathbb{N}$, then $Fix(\digamma^k) \neq \emptyset$, there exists a point $\varpi \in Fix(\digamma^k)$ such that the sequence $\sigma_n = \digamma^n(\sigma_0)$ is L(R) $\mathscr{P}_{C;\mathcal{I}}$-convergent to ϖ, and for all $\alpha \in \mathcal{I}$, $\forall\, \zeta \in Fix(\digamma^k)$ we have

$$J_\alpha(\zeta, \digamma(\zeta)) = J_\alpha(\digamma(\zeta), \zeta) = 0. \tag{6.3.6}$$

(b_3) If the family $\mathscr{P}_{C;\mathcal{I}}$ is separating on $\mathfrak{Q}$ and if the map $\digamma^k$ is L(R) $\mathscr{P}_{C;\mathcal{I}}$-closed on $\mathfrak{Q}$ for some $k \in \mathbb{N}$, then there exists a point $\varpi \in \mathfrak{Q}$ such that

$$Fix(\digamma^k) = Fix(\digamma) = \{\varpi\}, \tag{6.3.7}$$

and the sequence $\sigma_n = \digamma^n(\sigma_0)$ is L(R) $\mathscr{P}_{C;\mathcal{I}}$-convergent to ϖ, and for all $\alpha \in \mathcal{I}$,

$$J_\alpha(\varpi, \varpi) = 0. \tag{6.3.8}$$

Proof. We only prove the theorem for the case of left. The proof for right is based on analogous technique.

(b_1) For $\sigma_0 \in \mathfrak{Q}$, let $\sigma_n = F^{n-1}(\sigma_{n-1}) = F^n(\sigma_0)$. By using the Definition 6.3.1 we have,

$$\begin{aligned}
J_\alpha(\sigma_n, \sigma_{n+1}) &\leq D^{L-\mathscr{J}_{C;\mathcal{I}}}_{\eta;\mathfrak{Q};\alpha}(\sigma_n, \sigma_{n+1}) \\
&= D^{L-\mathscr{J}_{C;\mathcal{I}}}_{\eta;\mathfrak{Q};\alpha}(F(\sigma_{n-1}), F(\sigma_n)) \\
&\leq \frac{\lambda_\alpha}{S_\alpha(\sigma_{n-1}, \sigma_n)} J_\alpha(\sigma_{n-1}, \sigma_n) \\
&\leq \lambda_\alpha J_\alpha(\sigma_{n-1}, \sigma_n) \\
&\leq \lambda_\alpha D^{L-\mathscr{J}_{C;\mathcal{I}}}_{\eta;\mathfrak{Q};\alpha}(\sigma_{n-1}, \sigma_n) \\
&= \lambda_\alpha D^{L-\mathscr{J}_{C;\mathcal{I}}}_{\eta;\mathfrak{Q};\alpha}(F(\sigma_{n-2}), F(\sigma_{n-1})) \\
&\leq \lambda_\alpha \frac{\lambda_\alpha}{S_\alpha(\sigma_{n-2}, \sigma_{n-1})} J_\alpha(\sigma_{n-2}, \sigma_{n-1}) \\
&\leq \lambda_\alpha^2 J_\alpha(\sigma_{n-2}, \sigma_{n-1}) \\
&\quad\vdots \\
&\leq \lambda_\alpha^n J_\alpha(\sigma_0, \sigma_1). \qquad (6.3.9)
\end{aligned}$$

Now if we take $m > n$, then by using (6.3.9) and (6.1.1), we have

$$
\begin{aligned}
J_\alpha(\sigma_n,\sigma_m) &\le S_\alpha(\sigma_n,\sigma_{n+1})J_\alpha(\sigma_n,\sigma_{n+1}) + S_\alpha(\sigma_{n+1},\sigma_m)J_\alpha(\sigma_{n+1},\sigma_m)\\
&\le S_\alpha(\sigma_n,\sigma_{n+1})J_\alpha(\sigma_n,\sigma_{n+1}) + S_\alpha(\sigma_{n+1},\sigma_m)S_\alpha(\sigma_{n+1},\sigma_{n+2})J_\alpha(\sigma_{n+1},\sigma_{n+2})\\
&+ S_\alpha(\sigma_{n+1},\sigma_m)S_\alpha(\sigma_{n+2},\sigma_m)J_\alpha(\sigma_{n+2},\sigma_m)\\
&\le S_\alpha(\sigma_n,\sigma_{n+1})J_\alpha(\sigma_n,\sigma_{n+1}) + S_\alpha(\sigma_{n+1},\sigma_m)S_\alpha(\sigma_{n+1},\sigma_{n+2})J_\alpha(\sigma_{n+1},\sigma_{n+2})\\
&+ S_\alpha(\sigma_{n+1},\sigma_m)S_\alpha(\sigma_{n+2},\sigma_m)S_\alpha(\sigma_{n+2},\sigma_{n+3})J_\alpha(\sigma_{n+2},\sigma_{n+3})\\
&+ S_\alpha(\sigma_{n+1},\sigma_m)S_\alpha(\sigma_{n+2},\sigma_m)S_\alpha(\sigma_{n+3},\sigma_m)J_\alpha(\sigma_{n+3},\sigma_m)\\
&\le \dots\\
&\le S_\alpha(\sigma_n,\sigma_{n+1})J_\alpha(\sigma_n,\sigma_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i} S_\alpha(\sigma_j,\sigma_m)\right) S_\alpha(\sigma_i,\sigma_{i+1})J_\alpha(\sigma_i,\sigma_{i+1})\\
&+ \left(\prod_{k=n+1}^{m-1} S_\alpha(\sigma_k,\sigma_m)\right) J_\alpha(\sigma_{m-1},\sigma_m)\\
&\le S_\alpha(\sigma_n,\sigma_{n+1})\lambda_\alpha^n J_\alpha(\sigma_0,\sigma_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i} S_\alpha(\sigma_j,\sigma_m)\right) S_\alpha(\sigma_i,\sigma_{i+1})\lambda_\alpha^i J_\alpha(\sigma_0,\sigma_1)\\
&+ \left(\prod_{k=n+1}^{m-1} S_\alpha(\sigma_k,\sigma_m)\right) \lambda_\alpha^{m-1} J_\alpha(\sigma_0,\sigma_1)\\
&\le S_\alpha(\sigma_n,\sigma_{n+1})\lambda_\alpha^n J_\alpha(\sigma_0,\sigma_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i} S_\alpha(\sigma_j,\sigma_m)\right) S_\alpha(\sigma_i,\sigma_{i+1})\lambda_\alpha^i J_\alpha(\sigma_0,\sigma_1)\\
&+ \left(\prod_{k=n+1}^{m-1} S_\alpha(\sigma_k,\sigma_m)\right) S_\alpha(\sigma_{m-1},\sigma_m)\lambda_\alpha^{m-1} J_\alpha(\sigma_0,\sigma_1)\\
&\le S_\alpha(\sigma_n,\sigma_{n+1})\lambda_\alpha^n J_\alpha(\sigma_0,\sigma_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i} S_\alpha(\sigma_j,\sigma_m)\right) S_\alpha(\sigma_i,\sigma_{i+1})\lambda_\alpha^i J_\alpha(\sigma_0,\sigma_1)\\
&\le \left(\prod_{j=0}^{n} S_\alpha(\sigma_n,\sigma_{n+1})\right) \lambda_\alpha^n J_\alpha(\sigma_0,\sigma_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i} S_\alpha(\sigma_j,\sigma_m)\right) S_\alpha(\sigma_i,\sigma_{i+1})\lambda_\alpha^i J_\alpha(\sigma_0,\sigma_1)\\
&= J_\alpha(\sigma_0,\sigma_1)\sum_{i=n}^{m-1}\left(\prod_{j=0}^{i} S_\alpha(\sigma_j,\sigma_m)\right) S_\alpha(\sigma_i,\sigma_{i+1})\lambda_\alpha^i.
\end{aligned}
$$

Let

$$a_n = \left(\prod_{j=0}^{n} S_\alpha(\sigma_j, \sigma_m)\right) \lambda_\alpha^n S_\alpha(\sigma_n, \sigma_{n+1}) \text{ and } S = \sum_{n=1}^{\infty} a_n.$$

Since by Definition 6.3.1, $\lambda_\alpha \lim_{m,i\to\infty} \frac{S_\alpha(\sigma_{i+1},\sigma_{i+2})}{S_\alpha(\sigma_i,\sigma_{i+1})} S_\alpha(\sigma_{i+1}, \sigma_m) < 1$, so the series S converges because by using ratio test we have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \lim_{n\to\infty} \frac{\lambda_\alpha \lambda_\alpha^n \left(\prod_{j=1}^{n+1} S_\alpha(\sigma_j, \sigma_m)\right) S_\alpha(\sigma_{n+1}, \sigma_{n+2})}{\lambda_\alpha^n \left(\prod_{j=1}^{n} S_\alpha(\sigma_j, \sigma_m)\right) S_\alpha(\sigma_n, \sigma_{n+1})}$$
$$= \lambda_\alpha \lim_{n,m\to\infty} \frac{S_\alpha(\sigma_{n+1}, \sigma_{n+2})}{S_\alpha(\sigma_{n+1}, \sigma_{n+1})} S_\alpha(\sigma_{n+1}, \sigma_m) < 1.$$

Thus $Sum_{m-1} - Sum_n = \left[\sum_{i=n}^{m-1} \left(\prod_{j=0}^{i} S_\alpha(\sigma_j, \sigma_m)\right) S_\alpha(\sigma_i, \sigma_{i+1}) \lambda_\alpha^i\right] \to 0$ as $n, m \to \infty$ and so is $J_\alpha(\sigma_0, \sigma_1)(Sum_{m-1} - Sum_n)$, where $Sum_m = \sum_{i=1}^{m} a_i$. This shows that for all $\alpha \in \mathcal{I}$ we have

$$\lim_{n\to\infty} \sup_{m>n} J_\alpha(\sigma_n, \sigma_m) = 0. \tag{6.3.10}$$

Now, since $(\mathfrak{Q}, \digamma)$ is left $\mathscr{J}_{C;\mathcal{I}}$-admissible in $\sigma_0 \in \mathfrak{Q}$, so by Definition 6.3.2 there exists a point $\varpi \in \mathfrak{Q}$ such that for all $\alpha \in \mathcal{I}$ we have

$$\lim_{n\to\infty} J_\alpha(\varpi, \sigma_n) = 0. \tag{6.3.11}$$

By defining $s_n = \sigma_n$ and $t_n = \varpi$ for all $n \in \{0\} \cup \mathbb{N}$, then by (6.3.10) and (6.3.11), we can see that the conditions (6.2.2) and (6.2.4) hold for the sequences $s_n = \sigma_n$ and $t_n = \varpi$ in $\mathfrak{Q}$. Thus by Definition 6.2.1, (6.2.6) holds, i,e, for all $\alpha \in \mathcal{I}$ we have

$$\lim_{n\to\infty} p_\alpha(\varpi, \sigma_n) = 0, \tag{6.3.12}$$

and so in particular $\varpi \in \mathrm{LIM}_{\sigma_n}^{L-\mathscr{P}_{C;\mathcal{I}}}$.

(b_2) We only prove that (6.3.6) holds. For this, suppose on contrary that $\exists\, \beta \in \mathcal{I}$ and there exists $\eta \in Fix(\digamma^k)$ such that $J_\beta(\eta, \digamma(\eta)) > 0$. But then we have $\eta = \digamma^k(\eta) = \digamma^{2k}(\eta)$, $\digamma(\eta) = \digamma^{2k}(T(\eta))$ and for $\zeta \in \{1, 2\}$, by Definition 6.3.1 we have

$$
\begin{aligned}
0 < J_\beta(\eta, \digamma(\eta)) &= J_\beta\left(\digamma^{2k}(\eta), \digamma^{2k}(\digamma(\eta))\right) \\
&\leq D^{L-\mathscr{J}_{C;\mathcal{I}}}_{\zeta;\mathfrak{Q};\beta}\left(\digamma^{2k}(\eta), \digamma^{2k}(\digamma(\eta))\right) \\
&\leq \left(\frac{\lambda_\beta}{S_\beta(\digamma^{2k-2}(\eta), \digamma^{2k-1}(\eta))}\right) J_\beta\left(\digamma^{2k-1}(\eta), \digamma^{2k-1}(\digamma(\eta))\right) \\
&\leq \left(\frac{\lambda_\beta}{S_\beta(\digamma^{2k-2}(\eta), \digamma^{2k-1}(\eta))}\right) D^{L-\mathscr{J}_{C;\mathcal{I}}}_{\zeta;\mathfrak{Q};\beta}\left(\digamma^{2k-1}(\eta), \digamma^{2k-1}(\digamma(\eta))\right) \\
&\leq \left(\frac{\lambda_\beta^2}{S_\beta(\digamma^{2k-2}(\eta), \digamma^{2k-1}(\eta)) \cdot S_\beta(\digamma^{2k-3}(\eta), \digamma^{2k-2}(\eta))}\right) J_\beta\left(\digamma^{2k-2}(\eta), \digamma^{2k-2}(\digamma(\eta))\right) \\
&\leq \\
&\vdots \\
&\leq \left(\frac{\lambda_\beta^{2k}}{\prod_{j=1}^{2k} S_\beta(\digamma^{j-2}(\eta), \digamma^{j-1}(\eta))}\right) J_\beta\left(\eta, \digamma(\eta)\right) \\
&< J_\beta\left(\eta, \digamma(\eta)\right)
\end{aligned}
$$

which is not possible. Thus for all $\alpha \in \mathcal{I}$ and for all $\eta \in Fix(\digamma^k)$ we have

$$J_\alpha(\eta, \digamma(\eta)) = 0. \tag{6.3.13}$$

Now it is easy to show that for all $\alpha \in \mathcal{I}$ and for all $\eta \in Fix(\digamma^k)$, $J_\alpha(\digamma(\eta), \eta) = 0$ by using (6.3.13) and the fact that $\eta = \digamma^k(\eta) = \digamma^{2k}(\eta)$. Thus (6.3.6) holds true.

(b_3) Next we show that properties (6.3.7) and (6.3.8) hold. For this, suppose that there exists $\eta \in Fix(\digamma^k)$ such that $\digamma(\eta) \neq \eta$. Then since the family $\mathscr{P}_{C;\mathcal{I}}$ is separating on $\mathfrak{Q}$, so there exists $\alpha \in \mathcal{I}$ such that

$$p_\alpha(\digamma(\eta), \eta) > 0 \ \vee \ p_\alpha(\eta, \digamma(\eta)) > 0.$$

Thus in view of (6.2.5), we get that there exists $\alpha \in \mathcal{I}$ such that $J_\alpha(\digamma(\eta), \eta) > 0 \vee J_\alpha(\eta, \digamma(\eta)) > 0$, which is not possible by property (6.3.6). Hence $\digamma(\eta) = \eta$ and thus $Fix(\digamma^k) = Fix(\digamma)$, which is (6.3.7).

Now we prove the property (6.3.8). By property (6.3.6), we conclude that for all

$\alpha \in \mathcal{I}$ and for all $\eta \in Fix(\digamma^k)$,

$$J_\alpha(\eta, \eta) \leq S_\alpha(\eta, \digamma(\eta)) J_\alpha(\eta, \digamma(\eta)) + S_\alpha(\digamma(\eta), \eta) J_\alpha(\digamma(\eta), \eta) = 0 + 0 = 0.$$

Finally we prove that $Fix(\digamma)$ is singleton set. For this, let $\varpi_1, \varpi_2 \in Fix(\digamma)$ and $\varpi_1 \neq \varpi_2$. Then, since the family $\mathscr{P}_{C;\mathcal{I}}$ is separating on $\mathfrak{Q}$, so there exits $\beta \in \mathcal{I}$ such that $\{p_\beta(\varpi_1, \varpi_2) > 0 \vee p_\beta(\varpi_2, \varpi_1) > 0\}$. By (6.2.5), we obtain that there exits $\beta \in \mathcal{I}$ such that $\{J_\beta(\varpi_1, \varpi_2) > 0 \vee J_\beta(\varpi_2, \varpi_1) > 0\}$. Consequently, for $\zeta \in \{1, 2\}$, by Definition 6.3.1, we conclude that there exists $\beta \in \mathcal{I}$ such that either

$$\begin{aligned}
J_\beta(\varpi_1, \varpi_2) &= J_\beta(\digamma(\varpi_1), \digamma(\varpi_2)) \\
&\leq D^{L-\mathscr{J}_{C;\mathcal{I}}}_{\zeta;\mathfrak{Q};\beta}(\digamma(\varpi_1), \digamma(\varpi_2)) \\
&\leq \left(\frac{\lambda_\beta}{S_\beta(\varpi_1, \varpi_2)}\right) J_\beta(\varpi_1, \varpi_2) \\
&< J_\beta(\varpi_1, \varpi_2),
\end{aligned}$$

or,

$$\begin{aligned}
J_\beta(\varpi_2, \varpi_1) &= J_\beta(\digamma(\varpi_2), \digamma(\varpi_1)) \\
&\leq D^{L-\mathscr{J}_{C;\mathcal{I}}}_{\zeta;\mathfrak{Q};\beta}(\digamma(\varpi_2), \digamma(\varpi_1)) \\
&\leq \left(\frac{\lambda_\beta}{S_\beta(\varpi_2, \varpi_1)}\right) J_\beta(\varpi_2, \varpi_1) \\
&< J_\beta(\varpi_2, \varpi_1),
\end{aligned}$$

which is not possible. Thus $Fix(\digamma)$ is a singleton set and hence (6.3.7) and (6.3.8) hold true.

□

6.4 Consequences and applications

In this section, we have discussed some consequences of the Theorem 6.3.4 and an example which fulfill the assumption of Theorem 6.3.4.

Example 6.4.1. Let $\mathfrak{Q} = (0, 6)$, $\gamma > 3$ and $A = A_1 \cup A_2$ where $A_1 = (0, 3]$, $A_2 = [5, 6)$. Let $p : \mathfrak{Q}^2 \to [0, \infty)$ be defined by

$$p(\zeta, \eta) = \begin{cases} 0 & \text{if} \quad A \cap \{\zeta, \eta\} = \{\zeta, \eta\} \\ \gamma & \text{if} \quad A \cap \{\zeta, \eta\} \neq \{\zeta, \eta\}, \end{cases} \tag{6.4.1}$$

and let $\mathscr{J}_{\{S\},\{1\}} = \mathscr{P}_{\{S\},\{1\}} = \{p\}$ with $S(\zeta, \eta) = 1 \quad \forall\ \zeta, \eta \in \mathfrak{Q}$. Define a map $\digamma : \mathfrak{Q} \to \mathfrak{Q}$ by

$$\digamma(\eta) = \begin{cases} 5 & \text{if} \quad \eta \in (0, 4) \\ 3 & \text{if} \quad \eta \in [4, 6). \end{cases} \tag{6.4.2}$$

1. $(\mathfrak{Q}, \mathscr{P}_{\{S\},\{1\}})$ is a CQT space. Indeed (6.4.1) implies that for all $\zeta, \xi, \eta \in \mathfrak{Q}$, $p(\zeta, \eta) \leq p(\zeta, \xi) + p(\xi, \eta)$. Because if it is not true and there exist $\zeta_0, \xi_0, \eta_0 \in \mathfrak{Q}$ such that $p(\zeta_0, \eta_0) > p(\zeta_0, \xi_0) + p(\xi_0, \eta_0)$. Then clearly $p(\zeta_0, \eta_0) = \gamma$a and $p(\zeta_0, \xi_0) = p(\xi_0, \eta_0) = 0$, implies that $A \cap \{\zeta_0, \eta_0\} \neq \{\zeta_0, \eta_0\}$, $A \cap \{\zeta_0, \xi_0\} = \{\zeta_0, \xi_0\}$ and $A \cap \{\xi_0, \eta_0\} = \{\xi_0, \eta_0\}$, which is not possible.

2. For $\lambda \in [0, 1)$, the dynamic system $(\mathfrak{Q}, \digamma)$ is $(\mathscr{D}_{1;\mathfrak{Q}}^{L-\mathscr{P}_{\{1\}:\{1\}}}, \lambda)$ controlled quasi-contraction on $\mathfrak{Q}$. Indeed, for all $\zeta, \eta \in \mathfrak{Q}$ implies that $\digamma(\zeta), \digamma(\eta) \in A$ and so we have
$$D_{1;\mathfrak{Q}}^{L-\mathscr{P}_{\{1\}:\{1\}}}(\digamma(\zeta), \digamma(\eta)) = 0 \leq \lambda p(\zeta, \eta).$$

3. The dynamical system $(\mathfrak{Q}, \digamma)$ is left and right $\mathscr{P}_{\{1\};\{1\}}$-admissible in $\mathfrak{Q}$. Clearly for any $\eta_0 \in \mathfrak{Q}$, The sequence η_n with $\eta_{n+1} = \digamma(\eta_n)$, satisfies $\lim_{n\to\infty} \sup_{m>n} p(\eta_n, \eta_m) = 0$. Thus by using (6.4.1) and (6.4.2) we have $\eta_n \in A$. This gives us that $\mathrm{LIM}_{\eta_n}^{L-\mathscr{P}_{\{1\}:\{1\}}} = \mathrm{LIM}_{\eta_n}^{R-\mathscr{P}_{\{1\}:\{1\}}} = A$.

4. The single-valued dynamical system $(\mathfrak{Q}, \digamma^2)$ is left and right $\mathscr{P}_{\{1\}:\{1\}}$-closed in $\mathfrak{Q}$. Indeed, if $(\eta_n) \subseteq \digamma^2(\mathfrak{Q}) = \{3, 5\}$ is a left $\mathscr{P}_{\{1\}:\{1\}}$-convergent sequence in $\mathfrak{Q}$ and having subsequence (u_n), (v_n) such that $\forall\ n \in \mathbb{N}, \quad v_n \in \digamma(u_n)$. Then by using (6.4.1) and (6.4.2) we have $\mathrm{LIM}_{\eta_n}^{L-\mathscr{P}_{\{1\}:\{1\}}} = \mathrm{LIM}_{\eta_n}^{R-\mathscr{P}_{\{1\}:\{1\}}} = A$. In particular, $3 = \digamma^2(3) \in \mathrm{LIM}_{\eta_n}^{L-\mathscr{P}_{\{1\}:\{1\}}}$ and $5 = \digamma^2(5) \in \mathrm{LIM}_{\eta_n}^{L-\mathscr{P}_{\{1\}:\{1\}}}$.

5. The family $\mathscr{P}_{\{1\}:\{1\}} = \{p\}$ is not separating on $\mathfrak{Q}$. Since if $\zeta, \eta \in \mathfrak{Q}$ such that $\zeta, \eta \notin A$, then $p(\zeta, \eta) = p(\eta, \zeta) = \gamma \neq 0$.

Considering the cases 1-5, we can see that (b_1) and (b_2) of Theorem 6.3.4 hold. But b_3 does not holds, because the family is not separating.

Now we have discussed some consequences of the Theorem 6.3.4. Following is one of the main results in [59] which is directly proved if we define $S_\alpha(\eta, \xi) = \mathcal{C}_\alpha$ for all $\eta, \xi \in \mathfrak{Q}$ in Theorem 6.3.4.

Theorem 6.4.2. [59] Let $(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}})$ be a quasi-triangular space and let $F : \mathfrak{Q} \to \mathfrak{Q}$ be a single-valued dynamical system. Let $\zeta \in \{0, 1\}$ and $\lambda = \{\lambda_\alpha \in [0, 1) \ : \ \alpha \in \mathcal{I}\}$. Suppose that there is a left (right) family $\mathscr{J}_{C;\mathcal{I}}$ generated by $\mathscr{P}_{C;\mathcal{I}}$ and a point $\sigma_0 \in \mathfrak{Q}$ which satisfy the following properties.

(a_1) F is $L - (\mathscr{D}^{L-\mathscr{J}_{C;\mathcal{I}}}_{\mathfrak{Q},\zeta}, \lambda)$ quasi-contraction $(R - (\mathscr{D}^{R-\mathscr{J}_{C;\mathcal{I}}}_{\mathfrak{Q},\zeta}, \lambda)$ quasi-contraction) on $\mathfrak{Q}$.

(a_2) F is left (right) $\mathscr{J}_{C;\mathcal{I}}$-admissible in a point $\sigma_0 \in \mathfrak{Q}$.

Then we have the following.

(b_1) There exists a point $\varpi \in \mathfrak{Q}$ such that the sequence $\sigma_n = F^n \sigma_0$ is left (right)-$\mathscr{P}_{C;\mathcal{I}}$ convergent to ϖ.

(b_2) If the map F^k is left (right) $\mathscr{P}_{C;\mathcal{I}}$-closed on $\mathfrak{Q}$ for some $k \in \mathbb{N}$, then $Fix(F^k) \neq \emptyset$, and thus there exists a point $\varpi \in Fix(F^k)$ such that the sequence $\sigma_n = F^n(\sigma_0)$ is left (right) $\mathscr{P}_{C;\mathcal{I}}$-convergent to ϖ, and for all $\alpha \in \mathcal{I}$, $\forall\ \zeta \in Fix(F^k)$ we have

$$J_\alpha(\zeta, F(\zeta)) = J_\alpha(F(\zeta), \zeta) = 0.$$

(b_3) If the family $\mathscr{P}_{C;\mathcal{I}}$ is separating on $\mathfrak{Q}$ and if the mapping F^k is left (right) $\mathscr{P}_{C;\mathcal{I}}$-closed on $\mathfrak{Q}$ for some $k \in \mathbb{N}$, then $\exists\ \eta \in \mathfrak{Q}$ such that

$$Fix(F^k) = Fix(F) = \{\eta\},$$

and the sequence $\sigma_n = \digamma^n(\sigma_0)$ is left (right) $\mathscr{P}_{C;\mathcal{I}}$-convergent to η, and for all $\alpha \in \mathcal{I}$,

$$J_\alpha(\eta, \eta) = 0.$$

The main results of Banach type for triangular spaces is direct consequence of our result 6.3.4 when we define $S_\alpha(\eta, \xi) = 1$ for all $\eta, \xi \in \mathfrak{Q}$.

Corollary 6.4.3. Let $(\mathfrak{Q}, \mathscr{P}_{C;\mathcal{I}})$ be a triangular space and $\digamma : \mathfrak{Q} \to \mathfrak{Q}$ be a map. Let $\zeta \in \{0, 1\}$ and $\lambda = \{\lambda_\alpha \in [0, 1) \;:\; \alpha \in \mathcal{I}\}$. Suppose that there is a left (right) family $\mathscr{J}_{C;\mathcal{I}}$ generated by $\mathscr{P}_{C;\mathcal{I}}$ and a point $\sigma_0 \in \mathfrak{Q}$ such that the following axioms satisfied.

(a_1) $\digamma$ is $L - (\mathscr{D}^{L-\mathscr{J}_{C;\mathcal{I}}}_{\mathfrak{Q},\zeta}, \lambda)$ contraction $(R - (\mathscr{D}^{R-\mathscr{J}_{C;\mathcal{I}}}_{\mathfrak{Q},\zeta}, \lambda)$ contraction) on $\mathfrak{Q}$.

(a_2) $\digamma$ is left (right) $\mathscr{J}_{C;\mathcal{I}}$-admissible in a point $\sigma_0 \in \mathfrak{Q}$.

Then we have.

(b_1) If the map $\digamma^k$ is left (right) $\mathscr{P}_{C;\mathcal{I}}$-closed on $\mathfrak{Q}$ for some $k \in \mathbb{N}$, then $Fix(\digamma^k) \neq \emptyset$, and thus there exists a point $\varpi \in Fix(\digamma^k)$ such that the sequence $\sigma_n = \digamma^n(\sigma_0)$ is left (right) $\mathscr{P}_{C;\mathcal{I}}$-convergent to ϖ, and for all $\alpha \in \mathcal{I}$, $\forall\, \zeta \in Fix(\digamma^k)$ we have

$$J_\alpha(\zeta, \digamma(\zeta)) = J_\alpha(\digamma(\zeta), \zeta) = 0.$$

(b_2) If the family $\mathscr{P}_{C;\mathcal{I}}$ is separating on $\mathfrak{Q}$ and if the mapping $\digamma^k$ is left (right) $\mathscr{P}_{C;\mathcal{I}}$-closed on $\mathfrak{Q}$ for some $k \in \mathbb{N}$, then $\exists \eta \in \mathfrak{Q}$ such that

$$Fix(\digamma^k) = Fix(\digamma) = \{\eta\},$$

and the sequence $\sigma_n = \digamma^n(\sigma_0)$ is left (right) $\mathscr{P}_{C;\mathcal{I}}$-convergent to η, and for all $\alpha \in \mathcal{I}$,

$$J_\alpha(\eta, \eta) = 0.$$

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