Periodic Points for Various Mappings in Generalized Gauge Type Metric Spaces

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A Thesis Submitted to the Department of Mathematics,
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by

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By

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CERTIFICATE

A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE

DOCTOR OF PHILOSOPHY IN MATHEMATICS

We accept this thesis as conforming to the required standard

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This is to certify that the research work presented in this thesis entitled **Periodic Points for Various Mappings in Generalized Gauge Type Metric Spaces** was conducted by **Ms. Nosheen** under the kind supervision of **Prof. Dr. Tayyab Kamran**. No part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the Department of Mathematics, Quaid-i-Azam University, Islamabad in partial fulfillment of the requirements for the degree of Doctor of Philosophy in field of Mathematics from Department of Mathematics, Quaid-i-Azam University Islamabad, Pakistan.

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Dedicated

To my Family

Preface

The clarity and effectiveness of fixed point theory has motivated several researchers to explore it not only in single valued but also in multi-valued mappings. Banach contraction principle has attained its fame in case of single valued mapping and attracted various authors for many years. The principle assures the uniqueness and existence of fixed point of specific self-maps on complete metric spaces and gives a powerful tool to estimate the fixed point. The panoptic and comprehensive aspect of Banach fixed point theorem has led to a number of generalizations of the result. Banach contraction principle is expanded by Nadler to multi valued mappings using the idea of Hausdorff metric spaces. The frequent appearance of fixed point theory in modern scientific fields has forced researchers to analyze this field from more general point of view. From various aspects this field has been explored such as by generalization of metric spaces and the contraction conditions.

The study of gauge spaces was initiated by Dugundji which generalize metric spaces. Gauge spaces have the property that even the distance between two different points of the space may be zero. This simple characterization has been the center of interest for many researchers world wide. Reilly designed quasi-gauge spaces in 1973 and showed that it is generalization of quasi uniform spaces, quasi metric spaces and topological spaces. The quasi-gauge space generates asymmetric structure which has applications in theoretical computer science.

Given a quasi-gauge space, Wlodarczyk and Plebaniak have introduced the notion of left(right) families of generalized quasi-pseudo distances in quasi-gauge space. These left(right) families generalize the quasi-gauge and provide significant and useful tools to obtain more general results with weaker assumptions in fixed point theory.

The object of this dissertation is to study the results of periodic and fixed points for single and multi-valued mappings in generalized gauge type metric spaces and to add some more widely applicable results to the literature.

This dissertation mainly contains 5 chapters. Chapter 1, contains some basic definitions, significant results and some generalizations of contraction mapping relevant to our work. Also, some generalizations of metric space along with their properties that make these spaces remarkable then metric spaces are discussed.

Chapter 2, comprises of three sections. In the first section, we introduce $\mathcal{J}_{s;\Omega}$-families of generalized pseudo-b-distances in b-gauge spaces $(U, Q_{s;\Omega})$. Moreover, by using these $\mathcal{J}_{s;\Omega}$-families on U, we define the $\mathcal{J}_{s;\Omega}$-sequential completeness which generalizes the usual $Q_{s;\Omega}$-sequential completeness. In the second section, we develop novel results for periodic and fixed point of F-type contractions in the setting of b-gauge space using $\mathcal{J}_{s;\Omega}$-family on U, which generalize and improve the existing results in the corresponding literature. An example validating our result is given at the end of the section. In the third section we derive some fixed point results for mappings in b-gauge space equipped with the graph as a consequences of our results obtained in second section. At the end of the section, the validity and importance of our theorems are shown through an application via existence theorem for integral equations.

Chapter 3 consists of four main sections. In the first section, we establish the concept of quasi b-gauge space $(U, Q_{s;\Omega})$. In the second section, we introduce the notion of left (right) $\mathcal{J}_{s;\Omega}$-families of generalized quasi-pseudo-b-distances generated by $Q_{s;\Omega}$. In the third section, we prove novel periodic and fixed point theorems in quasi b-gauge space, which generalize the existing results due to Nadler and Banach in the corresponding literature. The last section comprises of some important consequences of the obtained results.

Chapter 4 consists of five sections. In the first section, we introduce the notion of extended b-gauge spaces $(U, Q_{\varphi;\Omega})$. In the second section we establish the notion of extended $\mathcal{J}_{\varphi;\Omega}$-families of generalized extended pseudo-b-distances. In the third section, we investigate novel results for periodic and fixed points of multi valued mappings in extended b-gauge space equipped with a graph. In the fourth section, in extended b-gauge spaces the periodic points for Caristi type G-contractions are discussed. The last section contains important consequences of the results obtained.

Chapter 5 consists of four main sections. In the first section, we initiate the idea of extended quasi b-gauge space $(U, Q_{\varphi;\Omega})$. In the second section, we introduce the notion of extended left (right) $\mathcal{J}_{\varphi;\Omega}$-families of generalized extended quasi-pseudo-b-distances generated by $Q_{\varphi;\Omega}$. In the third section, we investigate novel periodic and fixed point theorems in the locale of extended quasi b-gauge space, which generalize and improve the existing results due to Banach and Rus in fixed point theory. The last section consists of some important consequences of the results obtained. Each section of this chapter includes some examples to illustrate the corresponding results.

Contents

Chapter 1

Introduction and Preliminaries

In order to make this dissertation self content and pedagogical, this chapter aims to recollect some elementary definitions, notions and results which are relevant to our work.

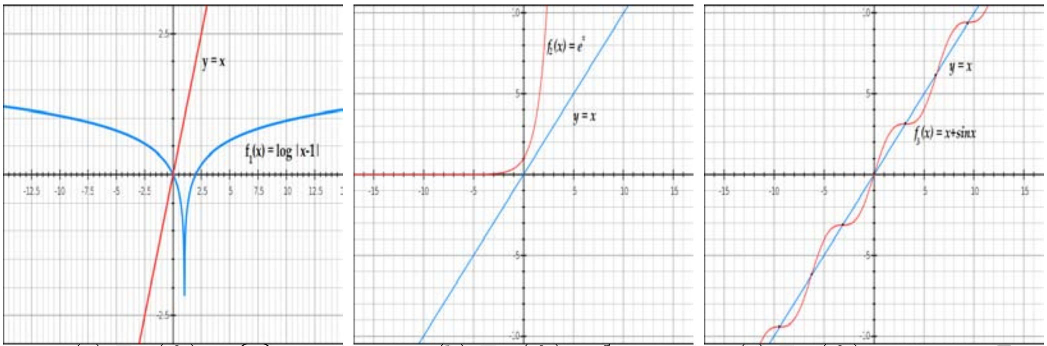
The first section defines fixed and periodic points for single and multi-valued mappings and states the renowned fixed point theorems due to Banach and Nadler. Moreover, the definition of semicontinuity of a function is discussed. The second section comprises of some generalizations of contraction mapping which we will use in our results. The third section defines some well known distance spaces and contains a detailed diagram showing hierarchy of these spaces.

1.1 Fixed and Periodic Points

In this section we discuss fixed and periodic points for single and multi-valued mappings.

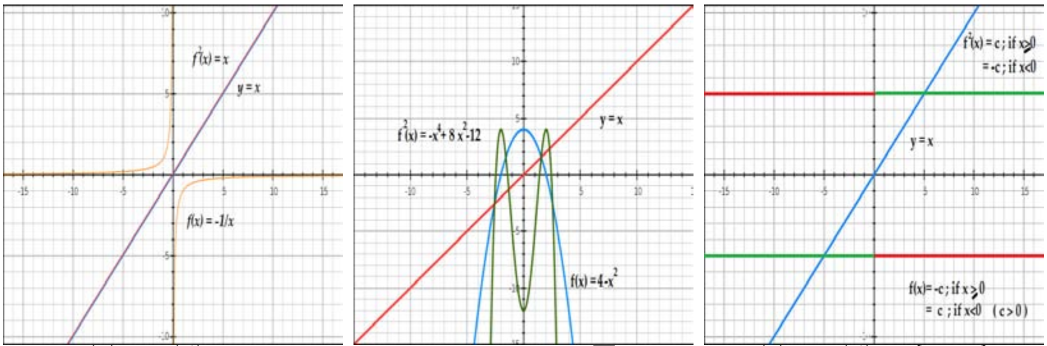
1.1.1 Single-valued Mappings

Let X be a non empty set. A point $x \in X$ is said to be a fixed point of $f : X \to X$ if $f(x) = x$. In case $X = \mathbb{R}$ we observe that fixed points are precisely the points where the graphs $y = f(x)$ and $y = x$ intersect. We denote by $\mathrm{Fix}(f)$ the set of all fixed points of f, i.e., $\mathrm{Fix}(f) := \{x \in X : x = f(x)\}$. A function may or may not have a fixed point. Further, the set of fixed points of a function may or may not be finite. It may happens that a mapping f does not has a fixed point but some of its iterate f^k, where $k \in \mathbb{N}$ has a fixed point. Such a point is called the periodic point of f with period k. we denote by $\mathrm{Per}(f)$ the set of all periodic points of f, i.e., $\mathrm{Per}(f) := \{x \in X : x = f^{[k]}(x) \text{ for some } k \text{ in } \mathbb{N}\}$.



(a) $\text{Fix}(f_1) = \{0\}$ (b) $\text{Fix}(f_2) = \emptyset$ (c) $\text{Fix}(f_3) = n\pi,\ n \in \mathbb{Z}$

Figure 1.1: Fixed points of mappings.



(a) $\text{Per}(f) = \mathbb{R}$ (b) $\text{Per}(f) = \{\frac{1}{2} \pm \frac{\sqrt{13}}{2}, -\frac{1}{2} \pm \frac{\sqrt{17}}{2}\}$ (c) $\text{Per}(f) = \{c, -c\}$

Figure 1.2: Periodic points of mappings.

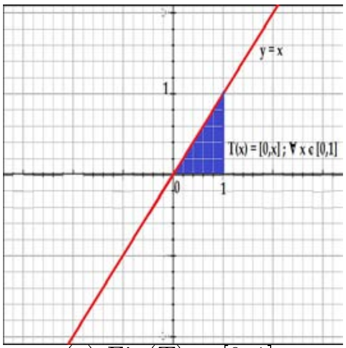
1.1.2 Multi-valued Mappings

Let X and Y be two non void sets. A multi-valued map or set-valued map is a mapping $T : X \to Y$ relating a subset $T(x)$ of a set Y with every $x \in X$. For the case when $T(x)$ has exactly one element for each $x \in X$, the map $T : X \to Y$ becomes a single-valued map. Thus single valued maps are special case of set-valued maps. Let $T : X \to 2^X$ is a multi-valued map then a point $x \in X$ is said to be the fixed point of T [55], if $x \in T(x)$. Let $\text{Fix}(T)$ indicates the set of all fixed points of of T, then $\text{Fix}(T) = \{x \in X : x \in T(x)\}$. A point $x \in X$ is called to be periodic point [52] of $T : X \to 2^X$ if $x \in T^{[k]}(x)$, for some $k \in \mathbb{N}$, where $T^{[k]}(x) = T(T^{[k-1]}(x)) = \cup_{y \in T^{[k-1]}(x)} Ty$. Let $\text{Per}(T)$ symbolizes the set of all periodic points of T then $\text{Per}(T) = \{x \in X : x \in T^{[k]}(x) \text{ for some } k \text{ in } \mathbb{N}\}$.

Example 1.1.1. Let $X = [0, 1]$ and let map $T : X \to 2^X$ is defined by

$$T(x) = [0, x], \quad \text{for all } x \in X.$$

Then $\text{Fix}(T) = [0, 1]$.



(a) $\text{Fix}(T) = [0, 1]$

Figure 1.3: Fixed points of multi-valued mapping.

1.1.3 Banach Fixed Point Theorem

Banach realized the importance of Picard method of successive approximation in finding solution of integral and differential equations and puts these ideas in to firm abstract settings [1]. The following theorems appeared first time in Banach Ph.D. thesis [1].

Theorem 1.1.2. (**Banach contraction principle** [1]) Let (X, d) be a complete metric space and $f : X \to X$, if there exists $\mu \in [0, 1)$ such that

$$d(fx, fy) \leq \mu d(x, y) \tag{1.1.1}$$

for all $x, y \in X$. Then for each $x_0 \in X$, the sequence $(x_m = f^{[m]}(x_0) : m \in \mathbb{N})$ converges to the unique fixed point of f.

Banach contraction principle is a very useful tool in finding zeroes of polynomials, solving system of algebraic equations, to obtain solution of differential and integral equations. It has also been used in computational mathematics for proving the convergence of algorithms.

Here we briefly discuss its use in image processing. Noise is an unwanted factor which attempts to fail many image processing algorithms [2]. The Banach contraction principle is effectively used in image denoising [3]. In this case we generally consider the Banach space $X = \mathbb{R}^{N\times M}$ with suitable norm such as $||x|| = \max_{i,j} |x(i, j)|$. Further, the median filter $M_W : X \to X$ for a given window W (a window simply means that it is a sliding submatrix in a given image matrix) is defined by $M_W(x)(i,j) =$ median of $\{x(i+a, j+b) : (a,b) \in W\}$. For a noisy image $V \in X$ is given and we seek to the solution of the equation

$$U = (1-\alpha)V + \alpha M_W(U) \quad \text{for } \alpha \in [0,1) \tag{1.1.2}$$

as a denoised image of V. The model defined in (1.1.2) suggest that an image is unchanged, $V = U$ if and only if $M_W(V) = V$. The recovered image using Banach contraction principle can be seen in figure 1.4.

(a) Noisy image (b) Denoised image

Figure 1.4: Image denoising using Banach contraction principle.

1.1.4 Nadler's Fixed Point Theorem

Nadler [55] expanded Banach fixed point theorem to multi-valued mappings. We begin with the definition of Hausdorff metric. Let (X,d) be a metric space and $CB(X)$ be the class of all closed and bounded subsets of X. For all $A, B \in CB(X)$ we define

$$H^d(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\},$$

where for all $x \in X$ and for all $B \in CB(X)$

$$d(x,B) = \inf_{y\in B} d(x,y).$$

The function $H^d : CB(X) \times CB(X) \to \mathbb{R}$ is said to be Hausdorff metric on $CB(X)$. It is well known that completeness of (X,d) implies the completeness of $(CB(X), H^d)$.

Theorem 1.1.3. [Nadler fixed point theorem [55]] Let (X, d) be a complete metric space and let $T : X \to CB(X)$ satisfies (H^d, μ)-contraction, i.e., there exist $\mu \in [0, 1)$ such that

$$H^d\left(T(x), T(y)\right) \leq \mu d\left(x, y\right)$$

for all $x, y \in X$. Then there exists $z \in X$ such that $z \in T(z)$, thus, $\mathrm{Fix}(T) \neq \emptyset$.

Example 1.1.4. Assume $X = [0, 1]$. Define map $T : X \to 2^X$ by

$$T(x) = [0, x^2], \quad \text{for all } x \in X.$$

Then $\mathrm{Fix}(T) = \{0, 1\}$.

1.1.5 Semicontinuity

In this subsection we define semicontinuity of functions. To define semicontinuity we define the notions of limit supremum and limit infimum of a sequence.

We know that every convergent sequence is bounded, but every bounded sequence may not be convergent. For instance, $((-1)^m : m \in \mathbb{N})$ is a bounded sequence which is not converent. For convergence in bounded sequence we define weaker condition of convergence which is known as limit supremum and limit infimum.

For this, let $(u_m : m \in \mathbb{N})$ is a bounded sequence. Let $v_N = \sup\{u_m : m \geq N\}$, thus

$$\begin{aligned} v_1 &= \sup\{u_1, u_2, u_3, ...\}, \\ v_2 &= \sup\{u_2, u_3, u_4, ...\}, \\ v_3 &= \sup\{u_3, u_4, u_5, ...\}, ... \end{aligned}$$

then $v_1 \geq v_2 \geq v_3 \geq v_4 \geq ....$
Thus (v_N) is a decreasing sequence and obviously bounded. Hence it conveges to its infimum. i.e.,

$$\lim_{N\to\infty} v_N = \inf\{v_1, v_2, v_3, ...\}.$$

This implies

$$\begin{aligned} \limsup_{m\to\infty} u_m &= \lim_{N\to\infty} v_N \\ &= \limsup_{N\to\infty} \{u_m : m \geq N\}, \end{aligned}$$

where $\limsup_{m\to\infty} u_m$ is denoting limit supremum of the sequence (u_m).
The following example illustrates the concept perceptibly.

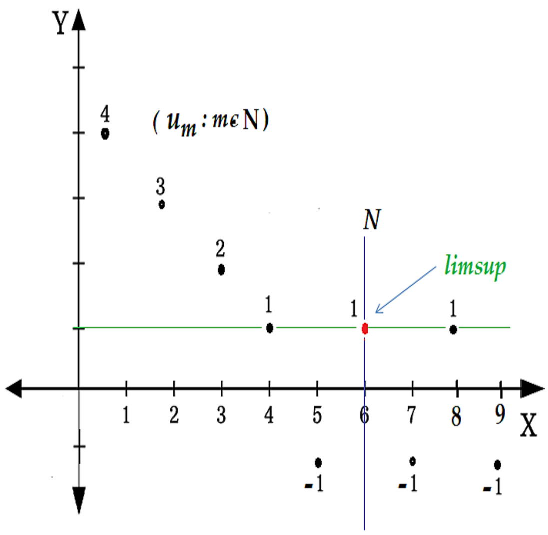
Suppose $(u_m : m \in \mathbb{N}) = 4, 3, 2, 1, -1, 1, -1, 1....$ is a bounded sequence. Although this sequence doesnot converge, but we can find its limit supremum which always exists. To do this we construct a helper seqence $v_N = \sup\{u_m : m \geq N\}$, thus

$$\begin{aligned} v_1 &= \sup\{u_1, u_2, u_3, ...\} = 4, \\ v_2 &= \sup\{u_2, u_3, u_4, ...\} = 3, \\ v_3 &= \sup\{u_3, u_4, u_5, ...\} = 2, \\ v_4 &= \sup\{u_4, u_5, u_6, ...\} = 1, \\ v_5 &= \sup\{u_5, u_6, u_7, ...\} = 1, ... \end{aligned}$$

then $v_1 \geq v_2 \geq v_3 \geq v_4 \geq ....$

Thus (v_N) is a decreasing sequence and obviously bounded. Hence it conveges to its infimum which is 1.

Thus we can write



$$\limsup_{m\to\infty} u_m = \lim_{N\to\infty} \sup\{u_m : m \geq N\} = 1.$$

From the above definition and example of limit supremum we obseve the following important points.

Remark 1.1.5. (i) We observe that even though the $\lim_{m\to\infty} u_m$ doesnot exists, but $\limsup_{m\to\infty} u_m$ always exist. Also note that if the sequence (u_m) is convergent to u, then limit supremum must be equal to u.

(ii) limit supremum is not the same as supremum, because supremum is the biggest value of the sequence which is 4 but limit supremum is 1.

(iii) If $(u_m : m \in \mathbb{N})$ is not bounded above then $\limsup_{m\to\infty} u_m = \infty$.

Analogously, we can define limit infimum of a sequence (u_m). Thus we have the following definition.

$$\liminf_{m\to\infty} u_m = \liminf_{N\to\infty}\{u_m : m \geq N\},$$

where $\liminf_{m\to\infty} u_m$ is denoting limit infimum of the sequence (u_m).

Definition 1.1.6. An extended real-valued function $f : X \to \mathbb{R} \cup \{-\infty, \infty\}$ is said to be upper (lower) semi-continuous at a point $\bar{x} \in X$, if and only if

$$\limsup_{x\to\bar{x}} f(x) \leq f(\bar{x}),$$

$$\Big(\liminf_{x\to\bar{x}} f(x) \geq f(\bar{x})\Big),$$

where limsup (liminf) is denoting limit supremum (limit infimum) of the function at $\bar{x}$. A function is upper (lower) semi-continuous function if it is upper (lower) semi-continuous function at every point of X. Also a function is continuous if and only if it is both upper and lower semi-continuous.

Following examples and diagrams illustrate the concept perceptibly.

Example 1.1.7. The real function f defined by

$$f(x) = \begin{cases} x & \text{if } x < 1, \\ x+1 & \text{if } x \geq 1. \end{cases}$$

is upper semi-continuous at $x = 1$.

Example 1.1.8. The real function f defined by

$$f(x) = \begin{cases} x & \text{if } x \leq 1, \\ x+1 & \text{if } x > 1. \end{cases}$$

is lower semi-continuous at $x = 1$.

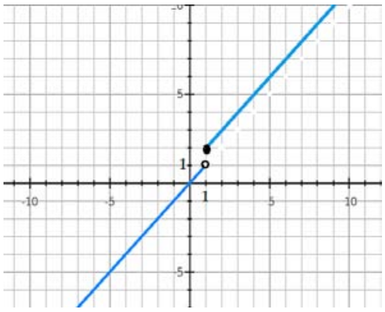


Figure 1.5: Upper semi-continuity at $x = 1$.

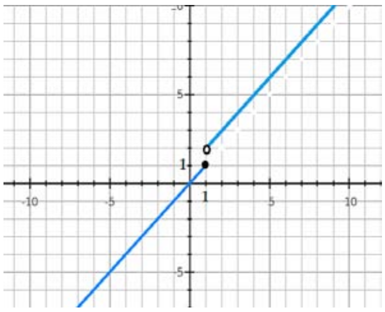


Figure 1.6: Lower semi-continuity at $x = 1$.

1.2 Generalized Contractions

A mapping $f : X \to X$ fulfilling condition (1.1.1) is said to be a contraction. This contraction condition plays pivotal rule in proving that the iterative sequence of the function is Cauchy. Banach proved that every contraction on a complete metric space has a unique fixed point. Now a question arises what happen if the given function is not a contraction or if the contractive condition does not hold for all pair of points in $X \times X$. So, attempts were made to obtain fixed point of mapping which are not contraction in the sense of (1.1.1).

In this section we will discuss some contractive conditions which are more general than the contraction condition (1.1.1). Indeed there are numerous generalization of the contraction mapping (1.1.1) but we will present only those generalization of (1.1.1) which we will use in our results. All over, this section, X denotes a nonempty set equipped with a metric d, unless otherwise stated. Further, we will call a mapping f satisfying (1.1.1) as Banach contraction. Observe that a Banach contraction is continuous.

1.2.1 Rus contractions

Rus [4] proved that a mapping f on a complete metric space satisfying contractive condition (1.1.1) for all pair of points in $X \times f(X)$ instead of $X \times X$ still has a fixed point.

Definition 1.2.1. [Rus contraction [4]] A mapping $f : X \to X$ is called Rus contraction, if there exists $\mu \in [0, 1)$ such that

$$d\left(fx, f^2x\right) \leq \mu d\left(x, fx\right) \quad \text{for all } \; x \in X. \tag{1.2.1}$$

By taking $y = f(x)$ in (1.1.1), we see that every Banach contraction is a Rus contraction. The converse is not true as exhibited in the following example.

Example 1.2.2. Let $X = [0, 1]$ is endowed with usual metric d and $f : X \to X$ is defined by

$$f(x) = \begin{cases} \frac{x}{4} & \text{for } 0 \leq x < \frac{1}{2}, \\ \frac{x}{5} & \text{for } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Note that f is not continuous at $x = \frac{1}{2}$, and thus it is not a Banach contraction. By taking $\mu = \frac{4}{5}$, we see that f is Rus contraction.

1.2.2 Caristi mapping

By introducing a lower semi continuous function from a metric space into the set of non-negative real numbers, Caristi [5] generalized Rus contraction in the following manner.

Definition 1.2.3. [Caristi mapping [5]] A self map f on X is called Caristi mapping if there is a lower semi continuous function $\phi : X \to [0, \infty)$ such that

$$d\left(x, fx\right) \leq \phi(x) - \phi(fx) \quad \text{for all } \; x \in X. \tag{1.2.2}$$

Taking $\phi(x) = \frac{d(x,f(x))}{1-\mu}$, where $\mu \in [0, 1)$, we see that every Rus contraction is a Caristi mapping. Following example substantiate that a Caristi mapping need not to be a Rus contraction.

Example 1.2.4. Let $X = \{0, 1, 2\}$, define the metric $d : X \times X \to [0, \infty)$ by

$$d(0, 1) = 1, d(2, 0) = 1, d(1, 2) = \frac{3}{2}, d(a, a) = 0, \forall a \in X \text{ and } d(a, b) = d(b, a) \forall a, b \in X.$$

Define $f : X \to X$ by

$$f(0) = 0, \quad f(1) = 2, \quad f(2) = 0$$

and $\phi : X \to [0,\infty)$ by

$$\phi(0) = 0, \quad \phi(1) = 4, \quad \phi(2) = 2.$$

Clearly f is a Caristi mapping. Since $d(f(0), f^2(0)) = d(0, f(0))$, f does not satisfy Rus contraction condition. Also f does not satisfy Banach contraction since $d(f(1), f(0)) = d(1, 0)$.

1.2.3 Banach G-contraction

Jachymaski [6] generalized Banach contraction by considering a graph $G = (V, E)$ in $X \times X$, where the vertex set $V = X$ and the edge set E contains the diagonal but includes no parallel edge. He showed that the conclusion of Banach theorem remained valid if the condition (1.1.1) holds for those ordered pairs which form edges in the graph.

Definition 1.2.5. [Banach G-contraction [6]] A mapping $f : X \to X$ is called Banach G-contraction, if there is $\mu \in [0, 1)$ such that:

(a) $(x, y) \in E \Rightarrow (fx, fy) \in E$

(b) $d(fx, fy) \leq \mu d(x, y)$

for all $(x, y) \in E$.

Every Banach contraction is Banach G-contraction, where the graph G is defined by $E = X \times X$. The converse is not true as exhibited in the following example.

Example 1.2.6. Let $X = \{0, 1, 2, 3\}$, define $d : X \times X \to [0,\infty)$ by

$$d(x, y) = |x - y|, \quad \text{for all } x, y \in X.$$

Then (X, d) is a metric space. Define $f : X \to X$ by

$$f(0) = 0, \quad f(1) = 0, \quad f(2) = 1, \quad f(3) = 1.$$

Also define $E(G) = \{(0, 1), (0, 2), (2, 3), (0, 0), (1, 1), (2, 2), (3, 3)\}$.
It is easy to see that f is a Banach G-contraction. To see that f is not a Banach contraction, observe that

$$d(f(1), f(2)) = d(0, 1) = 1 = d(1, 2).$$

It is important to note here that just as Jachymaski [6] generalized Banach contraction to Banach G-contraction by considering graph, others generalized Rus contraction to G-graphic contraction [7] and Caristi mapping to Caristi G-mapping [8] by considering graphs.

1.2.4 $\alpha - \psi$-contractive type mapping

By introducing two auxiliary functions Samet et. al., [9] generalized Banach contraction as follows. They established the concepts of α-admissible and $\alpha - \psi$-contractive type mappings. A mapping $f : X \to X$ is called α-admissible, where $\alpha : X \times X \to [0, \infty)$, if $\alpha(x, y) \geq 1$ implies $\alpha(fx, fy) \geq 1$, $x, y \in X$. Let Ψ indicates the family of non-decreasing mappings $\psi : [0, \infty) \to [0, \infty)$ fulfilling the properties: (i) $\psi(0) = 0$; (ii) $\psi(\eta t) = \eta\psi(t) < \eta t$, for every $\eta, t > 0$; and (iii) $\sum_{j=1}^{\infty} \psi^j(t) < \infty$.

Definition 1.2.7. [**$\alpha - \psi$-contractive type mapping [9]**] A self map f on X is said to be $\alpha - \psi$-contractive type mapping, if we have

$$\alpha(x, y)d(fx, fy) \leq \psi(d(x, y)) \quad \text{for all } x, y \in X. \tag{1.2.3}$$

If we take in (1.2.3) $\alpha(x, y) = 1$, for all $x, y \in X$ and $\psi(t) = \mu t$, for each $t > 0$ and $\mu \in [0, 1)$, we obtain Banach contraction. Every $\alpha - \psi$-contractive type mapping need not to be a Banach contraction as shown in the following example.

Example 1.2.8. Let $X = [0, 1]$ is endowed with usual metric. Define $f : X \to X$ by

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x < \frac{1}{2}, \\ \frac{1}{2} & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Since f is not continuous at $x = \frac{1}{2}$, so that f is not a Banach contraction. But f is $\alpha - \psi$-contractive type mapping with $\psi(t) = \frac{t}{2}$ and

$$\alpha(x, y) = \begin{cases} \frac{1}{4} & \text{if } 0 \leq x, y < \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

1.2.5 F-contraction

In 2012, Wardowski generalized Banach contraction in another direction, which he named as F-contraction. Wardowsiki [10] introduced the family $\mathfrak{F}$ of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the properties given below:

(F_1) for any $x, y \in (0, \infty)$ with $x < y$ we have $F(x) < F(y)$;

(F_2) for any sequence $(x_m : m \in \mathbb{N})$ of positive numbers, we have $\lim_{m\to\infty} x_m = 0$ iff $\lim_{m\to\infty} F(x_m) = -\infty$;

(F_3) there exist $p \in (0, 1)$ such that $\lim_{x\to 0^+} x^p F(x) = 0$.

Some examples of the functions $F : (0, \infty) \to \mathbb{R}$ are given below:

(i) $F_x = x + \ln x$ for any $x \in (0, \infty)$.

(ii) $F_y = \ln y$ for any $y \in (0, \infty)$.

Definition 1.2.9. [*F*-**contraction [10]**] A mapping $f : X \to X$ is said to be F-contraction if there exists $\tau > 0$ such that

$$d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leq F(d(x, y)) \quad \text{for all } x, y \in X. \tag{1.2.4}$$

When we examine in (1.2.4) the various kinds of the mappings F then we get the different types of contractions, which are already known to us. For instance, for $F_x = \ln x$, (1.2.4) is Banach contraction. Following example substantiate that F-contraction need not to be a Banach contraction.

Example 1.2.10. [**10**] Define the set X by

$$X := \{t_m : m \in \mathbb{N}\} \quad \text{where } t_m = \frac{m(m+1)}{2}, \text{ for all } m \in \mathbb{N}.$$

We consider the usual metric d on X. Define the map $f : X \to X$ by

$$f(t_1) = t_1, \quad f(t_m) = t_{m-1} \quad \text{for all } m > 1$$

since

$$\lim_{m\to\infty} \frac{d(f(t_m), f(t_1))}{d(t_m, t_1)} = 1.$$

Therefore, f is not a Banach contraction. Now let us define $F : (0, \infty) \to \mathbb{R}$ by

$$F(x) = x + \ln x \quad \text{for all } x > 0,$$

then the mapping f is a F-contraction, with $\tau = 1$.

Following Figure 1.7 is a hierarchy diagram showing a complete picture of the relationship between the above mentioned contraction type conditions.

1.3 Distance Spaces

In the previous section we have discussed some generalizations of the Banach contraction. This section will exhibit some generalizations of metric spaces which are relevant to our work. The idea of metric spaces was originated by the French mathematician Maurice Frechet [11] in order to generalize the notion of usual distance function on the real line to more general settings.

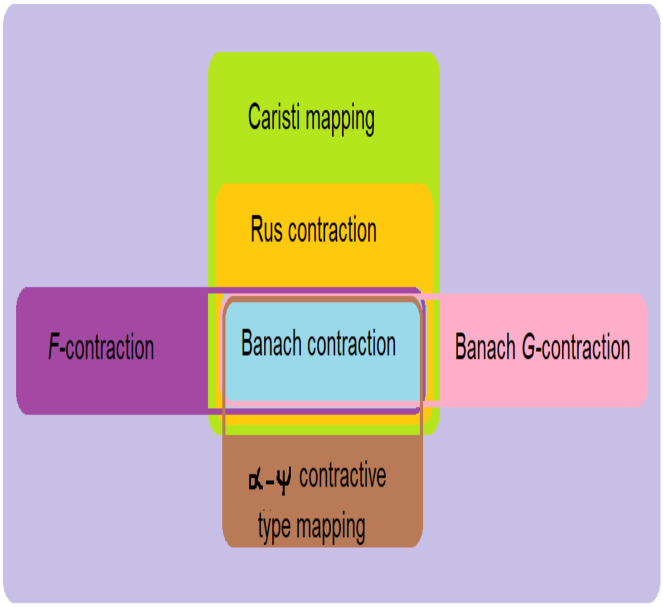


Figure 1.7: Hierarchy of contraction type mappings.

Throughout, this section, X is a nonempty set and Ω is an index set. Further, we recall that a metric space is a set X along with a function $d : X \times X \to [0, \infty)$ fulfilling for all $x, y, z \in X$ the properties: (a) $d(x, y) \geq 0$; (b) $d(x, y) = 0$ if and only if $x = y$; (c) $d(x, y) = d(y, x)$; and (d) $d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequality).

1.3.1 b-metric spaces

The b-metric space is a fascinating generalization of metric space, initiated by Bakhtin [12] in 1989. Later on, Czerwik [13] formally defined b-metric space by giving an axiom which was weaker than the triangular inequality.

Definition 1.3.1. A map $d : X \times X \to [0, \infty)$ is b-metric, if there is $s \geq 1$ fulfilling the following properties for all $x, y, z \in X$:

(a) $d(x, y) = 0$ iff $x = y$;

(b) $d(x, y) = d(y, x)$; and

(c) $d(x, z) \leq s\{d(x, y) + d(y, z)\}$.

For prescribed b-metric d on X, the pair (X, d) is called to be b-metric space.

When $s = 1$, we get the definition of a metric space. However, every b-metric space need not to be a metric space if $s > 1$. Thus b-metric space generalizes metric space. For evidence we give the following example.

Example 1.3.2. Suppose $X = [0,1]$. Describe $d : X \times X \to [0,\infty)$ for all $x, y \in X$ as:

$$d(x,y) = (x-y)^2.$$

Then d is a b-metric on X, where $s = 2$.

We observe that d is not a metric on X, since for $x = 0,\ \ y = \frac{1}{2}$ and $z = 1$, the triangular inequality does not hold.

It is important to note that b-metric is not continuous, in general. Further detail on the topic of b-metric spaces can be seen in references [14, 15, 16, 17, 18, 19, 20, 21, 22].

1.3.2 Extended b-metric spaces

In 2017, Kamran et. al., [23] enriched the notion of b-metric space by amending the triangular inequality and introduced the following definition of extended b-metric space in view of generalizing b-metric space.

Definition 1.3.3. A map $d : X \times X \to [0,\infty)$ is called to be an extended b-metric, if there exists $\varphi : X \times X \to [1,\infty)$ satisfying the following properties for all $x, y, z \in X$:

(a) $d(x,y) = 0$ iff $x = y$;

(b) $d(x,y) = d(y,x)$; and

(c) $d(x,z) \leq \varphi(x,z)\{d(x,y) + d(y,z)\}$.

For prescribed extended b-metric d, (X,d) is called extended b-metric space.

We notice from the definition that when $1 \leq \varphi(x,y) = s$(a finite constant), for all $x, y \in X$, both the definitions of extended b-metric space and b-metric space coincide. However, every extended b-metric space need not to be a b-metric space. Thus the class of extended b-metric spaces is bigger than the class of b-metric spaces.
Following are examples of extended b-metric spaces.

Example 1.3.4. [23] Let $X = C[a,b]$. Define $d : X \times X \to [0,\infty)$ and $\varphi : X \times X \to [1,\infty)$ for all $x, y \in X$ as follows:

$$d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|^2,$$

and
$$\varphi(x,y) = |x(t)| + |y(t)| + 2.$$
Then (X,d) is an extended b-metric space.

Example 1.3.5. **[24]** Let $X = [0,\infty)$. Define $d : X \times X \to [0,\infty)$ and $\varphi : X \times X \to [1,\infty)$ for all $x,y \in X$ by:
$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ x + y & \text{if } x \neq y \end{cases}$$
and $\varphi(x,y) = x + y + 1$.
Then d is an extended b-metric on X but d is not a b-metric on X.

For more examples and recent results see [24, 25, 26, 27, 28, 29, 30].

1.3.3 Gauge spaces

In 1966, Dugundji [31] initiated the idea of gauge spaces which generalizes metric spaces (or more generally pseudo-metric spaces). Gauge spaces have the characteristic that even the distance between two different points of the space may be zero. This simple characterization has been the center of interest for many researchers world wide.

Here, we discuss the topology induced by gauge spaces and the condition in which these spaces are Hausdorff.

Definition 1.3.6. A map $d : X \times X \to [0,\infty)$ is a pseudo metric, if for all $x,y,z \in X$ it satisfies:

(a) $d(x,x) = 0$;

(b) $d(x,y) = d(y,x)$; and

(c) $d(x,z) \leq d(x,y) + d(y,z)$.

The pair (X,d) is said to be pseudo metric space.

Example 1.3.7. Suppose $X = \mathbb{R}^2$ and define $d : X \times X \to [0,\infty)$ by
$d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| \quad \text{for all } (x_1,y_1),(x_2,y_2) \in X.$
Then (X,d) is a pseudo metric space. However, since $d((3,4),(3,6)) = |3-3| = 0$, that is, two distinct points have distance 0, therefore, (X,d) is not a metric space.

Definition 1.3.8. Each family $D = \{d_\beta : \beta \in \Omega\}$ of pseudo metrics $d_\beta : X \times X \to [0,\infty)$ for $\beta \in \Omega$, is said to be gauge on X.

Definition 1.3.9. The family $D = \{d_\beta : \beta \in \Omega\}$ is called to be separating if for each pair (x, y) where $x \neq y$, there is $d_\beta \in D$ such that $d_\beta(x, y) > 0$.

Definition 1.3.10. Let $D = \{d_\beta : \beta \in \Omega\}$ is a family of pseudo metrics on X. The topology $\mathcal{T}(D)$ on X whose subbase is defined by the family $\mathcal{B}(D) = \{B(x, \epsilon_\beta) : x \in X, \epsilon_\beta > 0, \beta \in \Omega\}$ of all balls $B(x, \epsilon_\beta) = \{y \in X : d_\beta(x, y) < \epsilon_\beta\}$, is called the induced topology.

Definition 1.3.11. A topological space $(X, \mathcal{T})$ is called a gauge space, if there exists gauge D on X with $\mathcal{T} = \mathcal{T}(D)$. The pair $(X, \mathcal{T}(D))$ denotes gauge space and is Hausdorff if D is separating.

Example 1.3.12. Let $X = \mathbb{R}^2$ and let $d_1, d_2 : X \times X \to [0, \infty)$ are defined for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ by

$$d_1((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| \text{ and } d_2((x_1, y_1), (x_2, y_2)) = |y_2 - y_1|.$$

Then d_1 and d_2 are pseudo metrics on X.
Note that $d_1((2, 3), (2, 5)) = |2-2| = 0$, but $(2, 3)$ and $(2, 5)$ are distinct points. Also $d_2((3, 6), (5, 6)) = |6 - 6| = 0$, but $(3, 6)$ and $(5, 6)$ are distinct points. Therefore, d_1 and d_2 are not metrics on X.
Let the family $D = \{d_1, d_2\}$ is a gauge on X. We now look for the topology $\mathcal{T}(D)$ induced by gauge D in the following manner.
First finding balls $B(x, \epsilon_1)$ for d_1, where $x = (x_1, y_1) \in X$ and $\epsilon_1 > 0$.

$$\begin{aligned} B((x_1, y_1), \epsilon_1) &= \{(x_2, y_2) \in X : d_1((x_1, y_1), (x_2, y_2)) < \epsilon_1\} \\ &= \{(x_2, y_2) \in X : x_2 \in (-\epsilon_1 + x_1, \epsilon_1 + x_1)\}. \end{aligned}$$

Thus $B((x_1, y_1), \epsilon_1)$ contains all verticle strips in the plane.
Similarly

$$\begin{aligned} B((x_1, y_1), \epsilon_2) &= \{(x_2, y_2) \in X : d_2((x_1, y_1), (x_2, y_2)) < \epsilon_2\} \\ &= \{(x_2, y_2) \in X : y_2 \in (-\epsilon_2 + y_1, \epsilon_2 + y_1)\}. \end{aligned}$$

Thus $B((x_1, y_1), \epsilon_2)$ contains all horizontal strips in the plane.
The subbase $\mathcal{B}(D)$ for induced topology $\mathcal{T}(D)$ is the collection all vertical and horizontal infinite open strips. Their intersectsion are open rectangles which form the base of induced topology shown in the Figure 1.8. The induced topology is thus the usual topology on $\mathbb{R}^2$. Therefore, (X, D) is a gauge space.

Example 1.3.13. [31] Every metric space is a gauge space, but the converse of the statement is not true, since for $d(x, y) = 0$, $x = y$ may not hold.

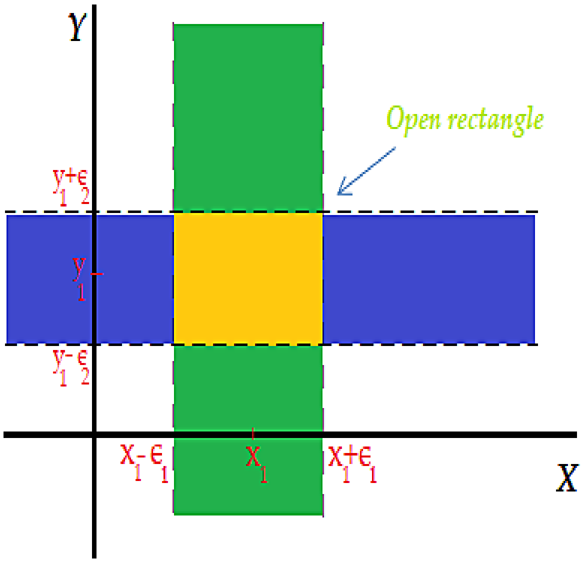


Figure 1.8: Base for usual topology in $\mathbb{R}^2$

Dugundji [31] has proved an important result which set the relationship between gauge space and topological space. In this regard, we first note the following definition of a completely regular space.
A topological space $(X, \mathcal{J})$ is completely regular iff for every closed set B in X and $y \in X$ such that $y \notin B$, there is a continuous function $f : X \to [0, 1]$ such that $f(y) = 0$ and $f(B) = 1$.
We now state the result:

Theorem 1.3.14. **[31]** A space X is a gauge space if and only if it is completely regular.

For further facts on gauge spaces see Agarwal et. al., [34], Frigon [32], Chis and Precup [33], Chifu and Petrusel [35], Lazara and Petrusel [38], Cherichi et. al., [36, 37] and Jleli et. al., [39].

1.3.4 Quasi-gauge spaces

In 1973, Reilly [40] initiated the idea of quasi-gauge spaces in order to generalize quasi-pseudo metric spaces by replacing single quasi-pseudo metric space by the family of such spaces on the set. In this way, he was also able to show that quasi-gauge spaces generalize gauge spaces.
We record the following definitions of his work.

Definition 1.3.15. The map $d : X \times X \to [0, \infty)$ is called a quasi-pseudo metric, if for all $x, y, z \in X$ the following properties hold:

(a) $d(x, x) = 0$; and

(b) $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called quasi-pseudo metric space.

Definition 1.3.16. Each family $D = \{d_\beta : \beta \in \Omega\}$ of quasi-pseudo metrics $d_\beta : X \times X \to [0, \infty)$ for $\beta \in \Omega$, is said to be quasi-gauge on X.

Definition 1.3.17. The family $D = \{d_\beta : \beta \in \Omega\}$ is separating if for every pair (x, y) where $x \neq y$, there is $d_\beta \in D$ such that either $d_\beta(x, y) > 0$ or $d_\beta(y, x) > 0$.

Definition 1.3.18. Let $D = \{d_\beta : \beta \in \Omega\}$ be the family of quasi-pseudo metrics on X. The topology $\mathcal{T}(D)$ on X whose subbase is defined by the family $\mathcal{B}(D) = \{B(x, \epsilon_\beta) : x \in X, \epsilon_\beta > 0, \beta \in \Omega\}$ of all balls $B(x, \epsilon_\beta) = \{y \in X : d_\beta(x, y) < \epsilon_\beta\}$, is called the induced topology.

Definition 1.3.19. Let $(X, \mathcal{T})$ is a topological space. If there exists a quasi-gauge D on X with $\mathcal{T} = \mathcal{T}(D)$, then the topological space $(X, \mathcal{T})$ is called to be a quasi-gauge space. It is denoted by the pair (X, D) and is Hausdorff if D is separating.

According to Reilly given a topological space $(X, \mathcal{T})$ such that $\mathcal{O} \in \mathcal{T}$, we can always define quasi-pseudo metric $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x \notin \mathcal{O}, \\ 0 & \text{if } x \in \mathcal{O},\ y \in \mathcal{O}, \\ 1 & \text{if } x \in \mathcal{O},\ y \notin \mathcal{O}, \end{cases} \tag{1.3.1}$$

Thus $D = \{d : \mathcal{O} \in \mathcal{T}\}$ is a quasi-gauge on X. Moreover,

$$B(x, \varepsilon) = \begin{cases} \mathcal{O} & \text{if } x \in \mathcal{O} \text{ and } \varepsilon \leq 1, \\ X & \text{otherwise.} \end{cases}$$

Thus $\mathcal{B}(D) = \{\mathcal{O} : \mathcal{O} \in \mathcal{T}\}$ and $\mathcal{T}(D) = \mathcal{T}$. This is illustrated through the following simple example.
Let $X = \{1, 2, 3\}$ and $\mathcal{J} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, X\}$, then $(X, \mathcal{J})$ is a topological space.
Let $\mathcal{O} = \{1, 2\} \in \mathcal{J}$. For all $x \in X$ and for $\epsilon = 1$, we obtain the balls $B(x, \epsilon)$ in the following manner.

$B(1,1) = \{y \in X : d(1,y) < 1\}$.
Using (1.3.1), we see that $d(1,1) = 0$, $d(1,2) = 0$ and $d(1,3) = 1$.
This gives $B(1,1) = \{1,2\} = \mathcal{O}$.
Similarly we have $B(2,1) = \mathcal{O}$ and $B(3,1) = X$.
Similar results are obtain when $\epsilon < 1$ and other $\mathcal{O} \in \mathcal{J}$ are used. This gives $\mathcal{B}(D) = \{\mathcal{O} : \mathcal{O} \in \mathcal{T}\}$ and $\mathcal{T}(D) = \mathcal{T}$, which implies the following important result by Reilly.

Theorem 1.3.20. [40] Every topological space is a quasi-gauge space.

Since every pseudo metric space is a quasi-pseudo metric space, it follows that every gauge space is a quasi-gauge space, but every quasi-gauge space need not to be a gauge space and hence metric space (or more generally pseudo metric space). For evidence we present the following example.

Example 1.3.21. [44] Let $X = [0,1]$ and $B = \{\frac{1}{2^m} : m \in \mathbb{N}\}$. Define $d : X \times X \to [0,\infty)$ for all $x, y \in X$ by

$$d(x,y) = \begin{cases} |x-y| & \text{if } x \in B \text{ or } y \notin B, \\ |x-y|+1 & \text{if } x \notin B \text{ and } \ y \in B. \end{cases}$$

Then d is a quasi-pseudo metric on X. Let $D = \{d\}$, then (X, D) is a quasi-gauge space.

Since the symmetric property does not hold (i.e., $d(0, \frac{1}{4}) \neq d(\frac{1}{4}, 0)$), therefore (X, D) is not a gauge space.

1.3.5 b-gauge spaces

Recently, Ali et. al., [45] introduced the notion of b-gauge spaces, thus extended the idea of gauge spaces in the locale of b-metric spaces. We note down the following definitions of their work.

Definition 1.3.22. A map $d : X \times X \to [0,\infty)$ is a b-pseudo metric, if there is $s \geq 1$ satisfying for all $x, y, z \in X$ the following conditions:

(a) $d(x,x) = 0$;

(b) $d(x,y) = d(y,x)$; and

(c) $d(x,z) \leq s\{d(x,y) + d(y,z)\}$.

For prescribed b-pseudo metric d, (X, d) is called b-pseudo metric space.

Definition 1.3.23. Each family $D = \{d_\beta : \beta \in \Omega\}$ of b-pseudo metrics $d_\beta : X \times X \to [0, \infty)$, is called b-gauge on X.

Definition 1.3.24. The family $D = \{d_\beta : \beta \in \Omega\}$ is separating if for each pair (x, y) where $x \neq y$, there is $d_\beta \in D$ such that $d_\beta(x, y) > 0$.

Definition 1.3.25. Let $D = \{d_\beta : \beta \in \Omega\}$ be the family of b-pseudo metrics on X. The topology $\mathcal{T}(D)$ on X whose subbase is defined by the family $\mathcal{B}(D) = \{B(x, \epsilon_\beta) : x \in X, \epsilon_\beta > 0, \beta \in \Omega\}$, where $B(x, \epsilon_\beta) = \{y \in X : d_\beta(x, y) < \epsilon_\beta\}$, is called the topology induced by D. The pair $(X, \mathcal{T}(D))$ is called to be a b-gauge space and is Hausdorff if D is separating.

Ali et. al., [45] presented the following example to show that b-pseudo metric space (in fact, b-gauge space) is the generalization of metric space, pseudo metric space (in fact, gauge space) and b-metric space.

Example 1.3.26. [45] Suppose $X = C([0, \infty), \mathbb{R})$ and describe $d : X \times X \to [0, \infty)$ by

$$d(u(t), v(t)) = \max_{t \in [0,1]} (u(t) - v(t))^2.$$

Then d is a b-pseudo metric, but not a metric, pseudo metric or b-metric.

The Figure 1.9 represents the relationship between the above mentioned distance spaces and topological space.

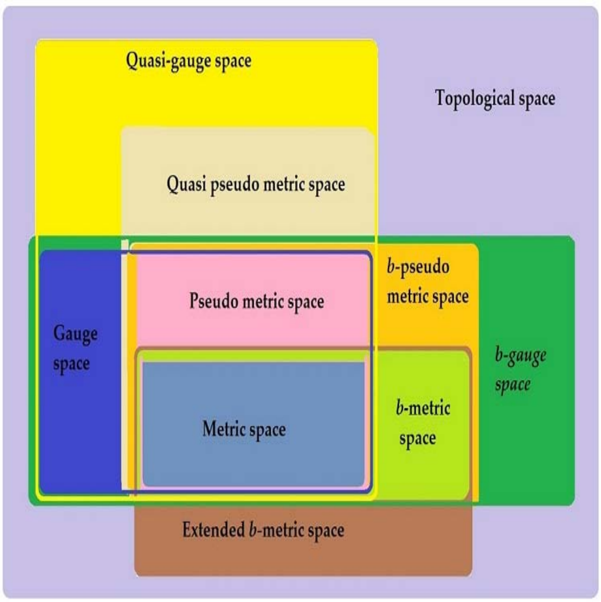


Figure 1.9: Hierarchy of distance spaces and topological space.

1.3.6 Left (right) $\mathcal{J}$-families of generalized quasi-pseudo distances

Given a quasi-gauge space (X, D), Wlodarczyk and Plebaniak [44] have introduced the notion of left(right) $\mathcal{J}$-families of generalized quasi-pseudodistances on X. These $\mathcal{J}$-families generated by quasi-gauge D, determine a structure on X which is more general than the structure on X determined by D and provide useful tools to obtain more general results with weaker assumptions which can be seen in [46, 47, 48, 49, 50, 51]. In this direction, in case of metric spaces the work done by Kada et. al., [41], Suzuki [42], and Lin and Du [43] are also appreciable.

Definition 1.3.27. [**44**] Let (X, D) is a quasi-gauge space. The family $\mathcal{J} = \{J_\beta : \beta \in \Omega\}$ where $J_\beta : X \times X \to [0, \infty)$, $\beta \in \Omega$ is called the left(right) $\mathcal{J}$-family of generalized quasi-pseudodistances on X if for all $x, y, z \in X$ and for all $\beta \in \Omega$ the following properties are fulfilled:

($\mathcal{J}1$) $J_\beta(x, z) \leq J_\beta(x, y) + J_\beta(y, z)$; and

($\mathcal{J}2$) for each sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in N)$ in X fulfilling

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(u_m, u_n) = 0, \tag{1.3.2}$$

$$\Big(\lim_{m\to\infty} \sup_{n>m} J_\beta(u_n, u_m) = 0\Big),$$

and

$$\lim_{m\to\infty} J_\beta(v_m, u_m) = 0, \tag{1.3.3}$$

$$\Big(\lim_{m\to\infty} J_\beta(u_m, v_m) = 0\Big),$$

the following holds:

$$\lim_{m\to\infty} d_\beta(v_m, u_m) = 0, \tag{1.3.4}$$

$$\Big(\lim_{m\to\infty} d_\beta(u_m, v_m) = 0\Big).$$

Note that $\mathbb{J}^L_{(X,D)}$ $(\mathbb{J}^R_{(X,D)})$ denotes the set of all left(right) $\mathcal{J}$-families on X.

Example 1.3.28. **[52]** Let (X, D) be a quasi-gauge space, where X has at least two distinct points and $D = \{d_\beta : \beta \in \Omega\}$ is the family of quasi-pseudo metrics $d_\beta : X \times X \to [0, \infty)$, $\beta \in \Omega$.

Let the set $F \subset X$ also contains at least two distinct, arbitrary and fixed points and let $c_\beta \in (0, \infty)$, $\beta \in \Omega$ satisfies $\delta_\beta(F) < c_\beta$ for all $\beta \in \Omega$, where $\delta_\beta(F) = \sup\{d_\beta(a, b) : a, b \in F\}$. Define $J_\beta : X \times X \to [0, \infty)$, $\beta \in \Omega$ for all $x, y \in X$ and for all $\beta \in \Omega$ as:

$$J_\beta(x, y) = \begin{cases} d_\beta(x, y) & \text{if } F \cap \{x, y\} = \{x, y\}, \\ c_\beta & \text{if } F \cap \{x, y\} \neq \{x, y\}. \end{cases} \tag{1.3.5}$$

Then $\mathcal{J} = \{J_\beta : \beta \in \Omega\} \in \mathbb{J}^L_{(X,Q)} \cap \mathbb{J}^R_{(X,Q)}$.
Here $J_\beta(x, z) \leq J_\beta(x, y) + J_\beta(y, z)$ for all $\beta \in \Omega$ and for all $x, y, z \in X$. Hence condition $(\mathcal{J}_1)$ is satisfied. Certainly, condition $(\mathcal{J}_1)$ does not hold only if there exists $x, y, z \in X$ and some $\beta \in \Omega$ such that $J_\beta(x, z) = c_\beta$, $J_\beta(x, y) = d_\beta(x, y)$, $J_\beta(y, z) = d_\beta(y, z)$ and $c_\beta \geq d_\beta(x, y) + d_\beta(y, z)$. However, then this implies the existence of $w \in \{x, z\}$ such that $w \notin F$ and $x, y, z \in F$, which is unfeasible.
Now let (1.3.2) and (1.3.3) hold for the sequences (u_m) and (v_m) in X. Then (1.3.3) implies that for all $\beta \in \Omega$ and for all $0 < \epsilon < c_\beta$, there exists $m_1 = m_1(\beta) \in \mathbb{N}$ such that

$$J_\beta(v_m, u_m) < \epsilon, \text{ for all } m \geq m_1. \tag{1.3.6}$$

By (1.3.6) and (1.3.5), denoting $m_2 = \min\{m_1(\beta) : \beta \in \Omega\}$, we have

$$\{v_m, u_m\} \cap F = \{v_m, u_m\}, \text{ for all } m \geq m_2$$

and

$$d_\beta(v_m, u_m) = J_\beta(v_m, u_m) < \epsilon.$$

Hence (1.3.4) holds. Thus, $\mathcal{J}$ is a left $\mathcal{J}$-family on X. Similar method follows in order to show that $\mathcal{J}$ is a right $\mathcal{J}$-family on X.

We note the following consequences from the above definition and example.

Remark 1.3.29. (i) $D \in \mathbb{J}^L_{(X,D)} \cap \mathbb{J}^R_{(X,D)}$.

(ii) There exists example of $\mathcal{J}$-family on X which shows that J_β, $\beta \in \Omega$ are not a quasi-pseudo metrics.

(iii) Let $\mathcal{J} = \{J_\beta : \beta \in \Omega\}$. If $J_\beta(v, v) = 0$, for all $v \in X$ and for all $\beta \in \Omega$ then J_β for each $\beta \in \Omega$, is a quasi-pseudo metric.

Chapter 2

Periodic and Fixed Points for Single-valued Mappings in b-Gauge Spaces

Throughout this chapter $(U, Q_{s;\Omega})$ is representing a b-gauge space, where U is the underlying nonempty set and $Q_{s;\Omega}$ is a b-gauge with s as coefficient of b-metric and Ω is an index set.

The motivation behind this chapter is to develop novel results based on periodic and fixed points of F-type contractions in b-gauge space $(U, Q_{s;\Omega})$ using $\mathcal{J}_{s;\Omega}$-family of generalized pseudo-b-distances on U.

For the said purpose this chapter is divided into three sections. The first section, introduces $\mathcal{J}_{s;\Omega}$-families of generalized pseudo-b-distances in b-gauge space $(U, Q_{s;\Omega})$. Moreover, by using these $\mathcal{J}_{s;\Omega}$-families on U, we define the $\mathcal{J}_{s;\Omega}$-sequential completeness which generalizes the usual $Q_{s;\Omega}$-sequential completeness. In the second section, we develop novel periodic and fixed point results for F-type contractions in the setting of b-gauge space using $\mathcal{J}_{s;\Omega}$-family on U, which generalize and improve all the results in [64] and some of the results of [65]. An example validating our result is given at the end of the section. In the third section, as a consequences of our results we derive some fixed point results for mappings in b-gauge space with the graph. At the end of the section, the validity and importance of our theorems are shown through an application via existence theorem for integral equations.

2.1 $\mathcal{J}_{s;\Omega}$-families of generalized pseudo-b-distances

In this section, we introduce $\mathcal{J}_{s;\Omega}$-families of generalized pseudo-b-distances in b-gauge space $(U, Q_{s;\Omega})$. The new structure determine by these families of distances are generalization of b-gauges and give valuable and important tools for inquiring periodic points and fixed points of maps in b-gauge spaces. Moreover, by using these $\mathcal{J}_{s;\Omega}$-families on U, we define the $\mathcal{J}_{s;\Omega}$-sequential completeness which generalizes the usual $Q_{s;\Omega}$-sequential completeness.

Definition 2.1.1. Let $(U, Q_{s;\Omega})$ be a b-gauge space. The family $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ where $J_\beta : U \times U \to [0, \infty)$, $\beta \in \Omega$, is said to be the $\mathcal{J}_{s;\Omega}$-family of generalized pseudo-b-distances on U (for short, $\mathcal{J}_{s;\Omega}$-family on U) if for all $\beta \in \Omega$ and for all $u, v, w \in U$ the following statements hold:

($\mathcal{J}$1) $J_\beta(u, w) \leq s_\beta\{J_\beta(u, v) + J_\beta(v, w)\}$;

($\mathcal{J}$2) for each sequences (u_m) and (v_m) in U fulfilling

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(u_m, u_n) = 0, \tag{2.1.1}$$

and

$$\lim_{m\to\infty} J_\beta(v_m, u_m) = 0, \tag{2.1.2}$$

the following holds:

$$\lim_{m\to\infty} q_\beta(v_m, u_m) = 0. \tag{2.1.3}$$

We denote
$\mathbb{J}_{(U,Q_{s;\Omega})} = \{\mathcal{J}_{s;\Omega} : \mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}\}$.
Also, we denote
$U^0_{\mathcal{J}_{s;\Omega}} = \{u \in U : \forall_{\beta\in\Omega}\{J_\beta(u, u) = 0\}\}$ and
$U^+_{\mathcal{J}_{s;\Omega}} = \{u \in U : \forall_{\beta\in\Omega}\{J_\beta(u, u) > 0\}\}$.
Then, of course $U = U^0_{\mathcal{J}_{s;\Omega}} \cup U^+_{\mathcal{J}_{s;\Omega}}$.

Example 2.1.2. Let $(U, Q_{s;\Omega})$ be a b-gauge space where U contains at least two distinct elements and suppose $Q_{s;\Omega} = \{q_\beta : \beta \in \Omega\}$ the family of b-pseudo metrics, is a b-gauge on U.

Let the set $F \subset U$ contains at least two distinct elements but arbitrary and fixed. Let $a_\beta \in (0, \infty)$ satisfies $\delta_\beta(F) < a_\beta$, where $\delta_\beta(F) = \sup\{q_\beta(e, f) : e, f \in F\}$, for all $\beta \in \Omega$. Let $J_\beta : U \times U \to [0, \infty)$ for all $e, f \in U$ and for all $\beta \in \Omega$ be defined as:

$$J_\beta(e, f) = \begin{cases} q_\beta(e, f) & \text{if } F \cap \{e, f\} = \{e, f\}, \\ a_\beta & \text{if } F \cap \{e, f\} \neq \{e, f\}. \end{cases} \tag{2.1.4}$$

Then $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\} \in \mathbb{J}_{(U,Q)}$.
We observe that $J_\beta(e, g) \leq s_\beta\{J_\beta(e, f) + J_\beta(f, g)\}$, for all $e, f, g \in U$, thus

condition ($\mathcal{J}_1$) holds. Indeed, condition ($\mathcal{J}_1$) will not hold in case if there is some $e, f, g \in U$ with $J_\beta(e, g) = a_\beta$, $J_\beta(e, f) = q_\beta(e, f)$, $J_\beta(f, g) = q_\beta(f, g)$ and $s_\beta\{q_\beta(e, f)+q_\beta(f, g)\} \leq a_\beta$. However, then this implies that there is $h \in \{e, g\}$ such that $h \notin F$ and on other hand, $e, f, g \in F$, which is impossible.

Now suppose that (2.1.1) and (2.1.2) are satisfied by the sequences (u_m : $m \in \mathbb{N}$) and (v_m : $m \in \mathbb{N}$) in U. Then (2.1.2) yields that for all $0 < \epsilon < a_\beta$, there exists $m_1 = m_1(\beta) \in \mathbb{N}$ such that

$$J_\beta(v_m, u_m) < \epsilon \text{ for all } m \geq m_1, \text{ for all } \beta \in \Omega. \tag{2.1.5}$$

By (2.1.5) and (2.1.4), denoting $m_2 = \min\{m_1(\beta) : \beta \in \Omega\}$, we have

$$F \cap \{v_m, u_m\} = \{v_m, u_m\}, \text{ for all } m \geq m_2$$

and

$$q_\beta(v_m, u_m) = J_\beta(v_m, u_m) < \epsilon.$$

Thus (2.1.3) is satisfied. Therefore, $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ is a $\mathcal{J}_{s;\Omega}$-family of generalized pseudo-b-distances on U.

We now mention some trivial properties of $\mathcal{J}_{s;\Omega}$-families.

Remark 2.1.3. (a) $Q_{s;\Omega} \in \mathbb{J}_{(U,Q_{s;\Omega})}$.

(b) Let $\mathcal{J}_{s;\Omega} \in \mathbb{J}_{(U,Q_{s;\Omega})}$. If $J_\beta(v, v) = 0$ and $J_\beta(u, v) = J_\beta(v, u)$ for all $\beta \in \Omega$ and for all $u, v \in U$ then for each $\beta \in \Omega$, J_β is b-pseudo metric.

(c) There exists examples of $\mathcal{J}_{s;\Omega} \in \mathbb{J}_{(U,Q_{s;\Omega})}$ which shows that the maps J_β, $\beta \in \Omega$ are not b-pseudo metrics.

Proposition 2.1.4. Let $(U, Q_{s;\Omega})$ is a Hausdorff b-gauge space and the family $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the $\mathcal{J}_{s;\Omega}$-family of generalized pseudo-b-distances on U. Then for each $e, f \in U$, there exists $\beta \in \Omega$ such that

$$e \neq f \Rightarrow J_\beta(e, f) > 0 \vee J_\beta(f, e) > 0.$$

Proof. Let there are $e, f \in U$ where $e \neq f$ such that $J_\beta(e, f) = 0 = J_\beta(f, e)$ for all $\beta \in \Omega$. Then by using property ($\mathcal{J}1$) we have $J_\beta(e, e) = 0$, for all $\beta \in \Omega$.

Defining sequences (u_m) and (v_m) in U by $u_m = f$ and $v_m = e$, we see that conditions (2.1.1) and (2.1.2) of property ($\mathcal{J}2$) are satisfied and therefore condition (2.1.3) holds, which implies that $q_\beta(e, f) = 0$, for all $\beta \in \Omega$. But, this denies the fact that $(U, Q_{s;\Omega})$ is a Hausdorff b-gauge space. Therefore, our supposition is wrong and there exists $\beta \in \Omega$ such that

$$e \neq f \Rightarrow J_\beta(e, f) > 0 \vee J_\beta(f, e) > 0$$

for all $e, f \in U$. □

Now, using $\mathcal{J}_{s;\Omega}$-families on U, we establish the following concept of $\mathcal{J}_{s;\Omega}$-completeness in the b-gauge space $(U, Q_{s;\Omega})$ which generalizes the usual $Q_{s;\Omega}$-sequential completeness.

Definition 2.1.5. Let $(U, Q_{s;\Omega})$ be a b-gauge space. Let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the $\mathcal{J}_{s;\Omega}$-family on U. A sequence $(v_m : m \in \mathbb{N})$ is $\mathcal{J}_{s;\Omega}$-Cauchy sequence in U if

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(v_m, v_n) = 0, \quad \text{for all } \beta \in \Omega.$$

Definition 2.1.6. Let $(U, Q_{s;\Omega})$ be a b-gauge space. Let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the $\mathcal{J}_{s;\Omega}$-family on U. The sequence $(v_m : m \in \mathbb{N})$ is called to be $\mathcal{J}_{s;\Omega}$-convergent to $v \in U$ if $\lim_{m\to\infty}^{\mathcal{J}_{s;\Omega}} v_m = v$, where
$\lim_{m\to\infty}^{\mathcal{J}_{s;\Omega}} v_m = v \Leftrightarrow \lim_{m\to\infty} J_\beta(v, v_m) = 0 = \lim_{m\to\infty} J_\beta(v_m, v)$, for all $\beta \in \Omega$.

Definition 2.1.7. Let $(U, Q_{s;\Omega})$ be a b-gauge space. Let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the $\mathcal{J}_{s;\Omega}$-family on U. If $S^{\mathcal{J}_{s;\Omega}}_{(v_m:m\in\mathbb{N})} \neq \emptyset$, where
$S^{\mathcal{J}_{s;\Omega}}_{(v_m:m\in\mathbb{N})} = \{v \in U : \lim_{m\to\infty}^{\mathcal{J}_{s;\Omega}} v_m = v\}$.
Then $(v_m : m \in \mathbb{N})$ in U is $\mathcal{J}_{s;\Omega}$-convergent sequence in U.

Definition 2.1.8. Let $(U, Q_{s;\Omega})$ be a b-gauge space. Let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the $\mathcal{J}_{s;\Omega}$-family on U. The space $(U, Q_{s;\Omega})$ is called $\mathcal{J}_{s;\Omega}$-sequentially complete, if every $\mathcal{J}_{s;\Omega}$-Cauchy in U is $\mathcal{J}_{s;\Omega}$-convergent in U.

Example 2.1.9. Let $U = [0, 4]$ and let $Q_{s;\Omega} = \{q\}$, where $q : U \times U \to [0, \infty)$ is pseudo-b-metric on U defined by

$$q(x, y) = |x - y|^2 \quad \text{for all } x, y \in U. \tag{2.1.6}$$

Then $(U, Q_{s;\Omega})$ is a b-gauge space.
Let the set $F = [\frac{1}{8}, 1] \subset U$ and let $J : U \times U \to [0, \infty)$ for all $x, y \in U$ be defined as:

$$J(x, y) = \begin{cases} q(x, y) & \text{if } F \cap \{x, y\} = \{x, y\}, \\ 4 & \text{if } F \cap \{x, y\} \neq \{x, y\}. \end{cases} \tag{2.1.7}$$

Then $\mathcal{J}_{s;\Omega} = \{J\}$ is the $\mathcal{J}_{s;\Omega}$-family on U (see Example 2.1.2).
We show that $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequential complete.
For this, let $\{v_m : m \in \mathbb{N}\}$ is $\mathcal{J}_{s;\Omega}$-Cauchy sequence. We may suppose, without loosing generality that for all $0 < \epsilon_1 < \frac{1}{64}$ and for all $n, m \in \mathbb{N}$, there exist $k_0 \in \mathbb{N}$ such that

$$J(v_m, v_n) < \epsilon_1 < \frac{1}{64}, \text{ for all } n \geq m \geq k_0. \tag{2.1.8}$$

Then by using (2.1.7), (2.1.6) and (2.1.8), we get

$$J(v_m, v_n) = q(v_m, v_n) =| v_m - v_n |^2 < \epsilon_1 < \frac{1}{64}. \tag{2.1.9}$$

$$v_m \in F = [\frac{1}{8}, 1] \text{ for all } m \geq k_0. \tag{2.1.10}$$

Rewriting (2.1.9) for all $0 < \epsilon < \frac{1}{8}$ and for $n, m, k_0 \in \mathbb{N}$ as

$$| v_m - v_n |< \epsilon < \frac{1}{8}, \text{ for all } n \geq m \geq k_0,$$

where $\epsilon = \sqrt{\epsilon_1}$.
Now since $(\mathbb{R}, | \ . \ |)$ is complete, $F = \left[\frac{1}{8}, 1\right]$ is closed in $\mathbb{R}$, also, by (2.1.10) $v_m \in F = [\frac{1}{8}, 1]\}$ for all $m \in \mathbb{N}$ such that $m \geq k_0$ and $\{v_m : m \in \mathbb{N}\}$ is Cauchy with respect to $| \ . \ |$, so there is $v \in F$ such that for all $0 < \epsilon < \frac{1}{8}$ and for all $n, m \in \mathbb{N}$, there exist $k_0 \in \mathbb{N}$ such that

$$| v - v_m |< \epsilon, \text{ for all } n \geq m \geq k_0.$$

Hence, $\{v_m : m \in \mathbb{N}\}$ is $\mathcal{J}_{s;\Omega}$-convergent to v.
This implies $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequential complete.

Remark 2.1.10. There exist examples of b-gauge spaces $(U, Q_{s;\Omega})$ and $\mathcal{J}_{s;\Omega}$-family on U with $\mathcal{J}_{s;\Omega} \neq Q_{s;\Omega}$ such that $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequential complete but not $Q_{s;\Omega}$-sequential complete (see Example 2.1.11 below).

Example 2.1.11. Let $U = [0, 1]$ and $B = \{\frac{1}{2^m} : m \in \mathbb{N}\}$.
Let $Q_{s;\Omega} = \{q\}$, where $q : U \times U \to [0, \infty)$ is b-pseudo metric on U defined for all $e, f \in U$ by

$$q(e, f) = \begin{cases} |e - f|^2 & \text{if } e = f \text{ or } \{e, f\} \cap B = \{e, f\}, \\ |e - f|^2 + 1 & \text{if } e \neq f \text{ and } \{e, f\} \cap B \neq \{e, f\}. \end{cases} \tag{2.1.11}$$

Then $(U, Q_{s;\Omega})$ is a b-gauge space.
Let the set $F = [\frac{1}{8}, 1] \subset U$ and let $J : U \times U \to [0, \infty)$ for all $e, f \in U$ be defined as:

$$J(e, f) = \begin{cases} q(e, f) & \text{if } F \cap \{e, f\} = \{e, f\}, \\ 4 & \text{if } F \cap \{e, f\} \neq \{e, f\}. \end{cases} \tag{2.1.12}$$

Then $\mathcal{J}_{s;\Omega} = \{J\}$ is the $\mathcal{J}_{s;\Omega}$-family on U (see Example 2.1.2).
First we show that $(U, Q_{s;\Omega})$ is not $Q_{s;\Omega}$-sequential complete.

For this let $\{v_m\} = \{\frac{1}{2^m} : m \in \mathbb{N}\}$, then by (2.1.11) for all $\epsilon > 0$ and for all $n, m \in \mathbb{N}$, there exist $k_0 \in \mathbb{N}$ such that

$$q(v_m, v_n) = \left|\frac{1}{2^m} - \frac{1}{2^n}\right|^2 < \epsilon, \text{ for all } n \geq m \geq k_0.$$

Thus $\{v_m : m \in \mathbb{N}\}$ is a $Q_{s;\Omega}$-Cauchy sequence. However, this sequence is not $Q_{s;\Omega}$-convergent in U. Otherwise, suppose that $\lim_{m\to\infty} v_m = v$, for some $v \in U$. We may suppose without loosing generality that for all $0 < \epsilon < 1$, there exist $k_0 \in \mathbb{N}$ such that

$$q(v, v_m) < \epsilon < 1, \text{ for all } m \geq k_0. \tag{2.1.13}$$

We have the following two cases:

(i) if $v \notin B$, then using (2.1.11) we can write

$$q(v, v_m) = |v - v_m|^2 + 1 < \epsilon < 1, \text{ for all } m \geq k_0.$$

Which is not possible.

(ii) If $v \in B$, then let $v = \frac{1}{2^{m_1}}$, for some $m_1 \in \mathbb{N}$ and using (2.1.11), we can write

$$q(v, v_m) = |v - v_m|^2 = \left|\frac{1}{2^{m_1}} - \frac{1}{2^m}\right|^2,$$

taking limit interior as $m \to \infty$, we get

$$\lim_{m\to\infty} q(v, v_m) = \frac{1}{2^{m_2}}, \text{ where } m_2 = 2m_1.$$

Which by (2.1.13) is impossible.

Thus we conclude that $(U, Q_{s;\Omega})$ is not $Q_{s;\Omega}$-sequential complete.
Next, we show that $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequential complete.
Let $\{v_m : m \in \mathbb{N}\}$ be a $\mathcal{J}_{s;\Omega}$-Cauchy sequence; without loosing generality we may let that for $0 < \epsilon < 1$, there exist $k_0 \in \mathbb{N}$ such that

$$J(v_m, v_n) < \epsilon < 1, \text{ for all } n \geq m \geq k_0. \tag{2.1.14}$$

Then by (2.1.12), (2.1.11) and (2.1.14), we obtain

$$J(v_m, v_n) = q(v_m, v_n) = |v_m - v_n|^2 < \epsilon < 1, \text{ for all } n \geq m \geq k_0, \tag{2.1.15}$$

$$v_m \in F = \left[\frac{1}{8}, 1\right] \text{ for all } m \geq k_0 \tag{2.1.16}$$

$$\text{and } v_m = v_{m_0} \text{ for all } m_0 \geq k_0 \text{ or } v_m \in B \text{ for all } m \geq k_0. \tag{2.1.17}$$

From (2.1.17), we have two cases:

(i) if $v_m = v_{m_0}$ for all $m_0 \geq k_0$. Then $\{v_m : m \in \mathbb{N}\}$ represent a constant sequence and by (2.1.16), (2.1.12), (2.1.11) and (2.1.17) the sequence $\{v_m : m \in \mathbb{N}\}$ is $\mathcal{J}_{s;\Omega}$-convergent to v_{m_0}.

(ii) If $v_m \in B$, for all $m_0 \geq k_0$.
Let $v_{k_0+s} \in B$ for all $s \in \mathbb{N}$. This together with (2.1.15), (2.1.16) and (2.1.17) implies that $v_{k_0+s} = \frac{1}{2}$ for all $s \in \mathbb{N}$ or $v_{k_0+s} = \frac{1}{4}$ for all $s \in \mathbb{N}$ or $v_{k_0+s} = \frac{1}{8}$ for all $s \in \mathbb{N}$. Therefore, $\{v_m : m \in \mathbb{N}\}$ is $\mathcal{J}_{s;\Omega}$-convergent to the point $\frac{1}{2}$ or $\frac{1}{4}$ or $\frac{1}{8}$ respectively.

Thus we conclude that $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequential complete.

Definition 2.1.12. Let $(U, Q_{s;\Omega})$ be a b-gauge space. The map $T^{[k]} : U \to U$, where $k \in \mathbb{N}$ is called to be a $Q_{s;\Omega}$-closed map on U if for each sequence (w_m) in $T^{[k]}(U)$, which is $Q_{s;\Omega}$-converging in U, i.e., $S^{Q_{s;\Omega}}_{(w_m:m\in\mathbb{N})} \neq \emptyset$ and for all $m \in \mathbb{N}$, $x_m = T^{[k]}(y_m)$ holds for its subsequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$

has the property that there exists $w \in S^{Q_{s;\Omega}}_{(w_m:m\in\mathbb{N})}$ such that $w = T^{[k]}(w)$.

2.2 Periodic and fixed point theorems for F-type contractions in b-gauge spaces

Now we present novel periodic and fixed point results for F-type contractions (utilizing idea of α-admissibility [9] and Hardy-Rogers contraction [66]) in the setting of b-gauge space using $\mathcal{J}_{s;\Omega}$-families on U. In this context we recall that Cosentino et, al., [67] modify the $\mathfrak{F}$-family introduced by Wardowski [10], in the setting of b-metric spaces as follows:
Let $s \geq 1$ be a real number. Denote by $\mathfrak{F}_{\mathfrak{s}}$ the family of functions $F : (0; \infty) \to \mathbb{R}$ fulfilling the following properties:

(F_1) for any $a, b \in (0, \infty)$ with $a < b$ we have $F(a) < F(b)$;

(F_2) for any sequence $(b_m : m \in \mathbb{N})$ of positive numbers, we have $\lim_{m\to\infty} b_m = 0$ iff $\lim_{m\to\infty} F(b_m) = -\infty$;

(F_3) for any sequence $(b_m : m \in \mathbb{N})$ in $\mathbb{R}_+$, we have $\lim_{m\to\infty} b_m = 0$, there is $p \in (0, 1)$ such that $\lim_{m\to\infty} b_m^p F(b_m) = 0$;

(F_4) if sequence $(b_m : m \in \mathbb{N})$ in $\mathbb{R}_+$ satisfying $\tau + F(sb_m) \leq F(b_{m-1})\ \forall\, m \in \mathbb{N}$ and some $\tau > 0$, then $\tau + F(s^m b_m) \leq F(s^{m-1} b_{m-1})$.

Some examples of the functions belonging to $\mathfrak{F}_{\mathfrak{s}}$ are given below:

(i) $F_x = x + \ln x$ for any $x \in (0, \infty)$.

(ii) $F_y = \ln y$ for any $y \in (0, \infty)$.

Our main results are now given below, where $\alpha : U \times U \to [0, \infty)$.

Theorem 2.2.1. Let $(U, Q_{s;\Omega})$ be a b-gauge space. Let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \to [0, \infty)$, be the $\mathcal{J}_{s;\Omega}$-family of distances generated by $Q_{s;\Omega}$ such that $U^0_{\mathcal{J}_{s;\Omega}} \neq \emptyset$ and $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequentially complete. Let map $T : U \to U$ be such that $T(U) \subset U^0_{\mathcal{J}_{s;\Omega}}$ and for which we have $F \in \mathfrak{F}_{\mathfrak{s}}$ and $\tau > 0$ such that :

$$\alpha(u, v) \geq 1 \Rightarrow \tau + F(s_\beta J_\beta(Tu, Tv)) \leq F\Big(a_\beta J_\beta(u, v) + b_\beta J_\beta(u, Tu) + c_\beta J_\beta(v, Tv) + e_\beta J_\beta(u, Tv) + L_\beta J_\beta(v, Tu)\Big) \quad (2.2.1)$$

for all $\beta \in \Omega$ and for any $u, v \in U$ whenever $J_\beta(Tu, Tv) \neq 0$.
Further, $a_\beta, b_\beta, c_\beta, e_\beta, L_\beta \geq 0$ be such that $a_\beta + b_\beta + c_\beta + (s_\beta + 1)e_\beta < 1$ for each $\beta \in \Omega$.
Assume, moreover that, the following statements hold:

(i) there exist $z^0 \in U$ such that $\alpha(z^0, z^1) \geq 1$;

(ii) if $\alpha(x, y) \geq 1$, then $\alpha(Tx, Ty) \geq 1$;

(iii) if a sequence $(z^m : m \in \mathbb{N})$ in U is such that $\alpha(z^m, z^{m+1}) \geq 1$ and $\lim^{\mathcal{J}_{s;\Omega}}_{m\to\infty} z^m = z$, then $\alpha(z^m, z) \geq 1$ and $\alpha(z, z^m) \geq 1$.

Then the following statements hold:

(I) For each $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is $Q_{s;\Omega}$-convergent sequence in U; thus, $S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) Furthermore, assume that $T^{[k]}$ for some $k \in \mathbb{N}$, is $Q_{s;\Omega}$-closed map on U and $s_\beta\{c_\beta + e_\beta s_\beta\} < 1$, for each $\beta \in \Omega$. Then

(a$_1$) $\mathrm{Fix}(T^{[k]}) \neq \emptyset$;

(a$_2$) there exists $z \in \mathrm{Fix}(T^{[k]})$ such that $z \in S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and

(a$_3$) for all $z \in \mathrm{Fix}(T^{[k]})$, $J_\beta(z, T(z)) = J_\beta(T(z), z) = 0$, for all $\beta \in \Omega$.

(III) Furthermore, let $\mathrm{Fix}(T^{[k]}) \neq \emptyset$ for some $k \in \mathbb{N}$ and $(U, Q_{s;\Omega})$ is a Hausdorff space. Then

(b$_1$) $\mathrm{Fix}(T^{[k]})$=$\mathrm{Fix}(T)$;

(b$_2$) there exists $z \in \text{Fix}(T)$ such that $z \in S^{L-Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and

(b$_3$) for all $z \in \text{Fix}(T)$, $J_\beta(z,z) = 0$, for all $\beta \in \Omega$.

Proof. (I) We first prove that the sequence $(z^m : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{J}_{s;\Omega}$-Cauchy sequence in U.
Using assumption (i) there is $z^0 \in U$ such that $\alpha(z^0, z^1) \geq 1$. Now for each $\beta \in \Omega$, using (2.2.1) we can write

$$\begin{aligned}
\tau + F(s_\beta J_\beta(z^1, z^2)) &= \tau + F(s_\beta J_\beta(Tz^0, Tz^1)) \\
&\leq F\Big(a_\beta J_\beta(z^0, z^1) + b_\beta J_\beta(z^0, Tz^0) + c_\beta J_\beta(z^1, Tz^1) \\
&\quad + e_\beta J_\beta(z^0, Tz^1) + L_\beta J_\beta(z^1, Tz^0)\Big) \\
&\leq F\Big(a_\beta J_\beta(z^0, z^1) + b_\beta J_\beta(z^0, z^1) + c_\beta J_\beta(z^1, z^2) \\
&\quad + e_\beta J_\beta(z^0, z^2) + L_\beta .0\Big) \\
&\leq F\Big(a_\beta J_\beta(z^0, z^1) + b_\beta J_\beta(z^0, z^1) + c_\beta J_\beta(z^1, z^2) \\
&\quad + e_\beta s_\beta (J_\beta(z^0, z^1) + J_\beta(z^1, z^2))\Big) \\
&= F\Big((a_\beta + b_\beta + e_\beta s_\beta) J_\beta(z^0, z^1) + (c_\beta + e_\beta s_\beta) J_\beta(z^1, z^2)\Big).
\end{aligned}$$

This gives

$$\tau + F(s_\beta J_\beta(z^1, z^2)) \leq F\Big((a_\beta + b_\beta + e_\beta s_\beta) J_\beta(z^0, z^1) + (c_\beta + e_\beta s_\beta) J_\beta(z^1, z^2)\Big). \tag{2.2.2}$$

As F is strictly increasing, we can write from above that

$$s_\beta J_\beta(z^1, z^2) < (a_\beta + b_\beta + e_\beta s_\beta) J_\beta(z^0, z^1) + (c_\beta + e_\beta s_\beta) J_\beta(z^1, z^2) \quad \text{for all } \beta \in \Omega.$$

We can also write it as

$$(s_\beta - c_\beta - e_\beta s_\beta) J_\beta(z^1, z^2) < (a_\beta + b_\beta + e_\beta s_\beta) J_\beta(z^0, z^1) \quad \text{for all } \beta \in \Omega.$$

$$(1 - \frac{c_\beta}{s_\beta} - e_\beta) s_\beta J_\beta(z^1, z^2) < (a_\beta + b_\beta + e_\beta s_\beta) J_\beta(z^0, z^1) \quad \text{for all } \beta \in \Omega.$$

Since $a_\beta + b_\beta + c_\beta + (s_\beta + 1) e_\beta < 1$, we get

$$1 - \frac{c_\beta}{s_\beta} - e_\beta \geq 1 - c_\beta - e_\beta > a_\beta + b_\beta + s_\beta e_\beta \geq 0.$$

Thus

$$s_\beta J_\beta(z^1, z^2) < J_\beta(z^0, z^1) \quad \text{for all } \beta \in \Omega.$$

Now using (2.2.2), we can write

$$\tau + F(s_\beta J_\beta(z^1, z^2)) < F(J_\beta(z^0, z^1)) \quad \text{for all } \beta \in \Omega.$$

Using assumption (ii), we have $\alpha(Tz^0, Tz^1) = \alpha(z^1, z^2) \geq 1$. Now for each $\beta \in \Omega$, using (2.2.1) we can write

$$\begin{aligned}
\tau + F(s_\beta J_\beta(z^2, z^3)) &= \tau + F(s_\beta J_\beta(Tz^1, Tz^2)) \\
&\leq F\Big(a_\beta J_\beta(z^1, z^2) + b_\beta J_\beta(z^1, Tz^1) + c_\beta J_\beta(z^2, Tz^2) \\
&\quad + e_\beta J_\beta(z^1, Tz^2) + L_\beta J_\beta(z^2, Tz^1)\Big) \\
&\leq F\Big(a_\beta J_\beta(z^1, z^2) + b_\beta J_\beta(z^1, z^2) + c_\beta J_\beta(z^2, z^3) \\
&\quad + e_\beta J_\beta(z^1, z^3) + L_\beta .0\Big) \\
&\leq F\Big(a_\beta J_\beta(z^1, z^2) + b_\beta J_\beta(z^1, z^2) + c_\beta J_\beta(z^2, z^3) \\
&\quad + e_\beta s_\beta (J_\beta(z^1, z^2) + J_\beta(z^2, z^3))\Big) \\
&= F\Big((a_\beta + b_\beta + e_\beta s_\beta) J_\beta(z^1, z^2) + (c_\beta + e_\beta s_\beta) J_\beta(z^2, z^3)\Big).
\end{aligned}$$

This gives

$$\tau + F(s_\beta J_\beta(z^2, z^3)) \leq F\Big((a_\beta + b_\beta + e_\beta s_\beta) J_\beta(z^1, z^2) + (c_\beta + e_\beta s_\beta) J_\beta(z^2, z^3)\Big). \tag{2.2.3}$$

As F is strictly increasing, we can write from above that

$$s_\beta J_\beta(z^2, z^3) < (a_\beta + b_\beta + e_\beta s_\beta) J_\beta(z^1, z^2) + (c_\beta + e_\beta s_\beta) J_\beta(z^2, z^3) \quad \text{for all } \beta \in \Omega.$$

We can also write it as

$$(s_\beta - c_\beta - e_\beta s_\beta)) J_\beta(z^2, z^3) < (a_\beta + b_\beta + e_\beta s_\beta) J_\beta(z^1, z^2) \quad \text{for all } \beta \in \Omega.$$

Since $a_\beta + b_\beta + c_\beta + (s_\beta + 1)e_\beta < 1$, we get

$$1 - \frac{c_\beta}{s_\beta} - e_\beta \geq 1 - c_\beta - e_\beta > a_\beta + b_\beta + s_\beta e_\beta \geq 0.$$

Thus

$$s_\beta J_\beta(z^2, z^3) < J_\beta(z^1, z^2) \quad \text{for all } \beta \in \Omega.$$

Now using (2.2.3), we can write

$$\tau + F(s_\beta J_\beta(z^2, z^3)) < F(J_\beta(z^1, z^2)) \quad \text{for all } \beta \in \Omega.$$

Progressing in the above manner, we get a sequence $(z^m : m \in \{0\} \cup \mathbb{N}) \subset U$ such that
$z^m = Tz^{m-1}, z^{m-1} \neq z^m$ and $\alpha(z^{m-1}, z^m) \geq 1$, for each $m \in \mathbb{N}$. Furthermore,

$$\tau + F(s_\beta J_\beta(z^m, z^{m+1})) < F(J_\beta(z^{m-1}, z^m)) \quad \text{for all } \beta \in \Omega.$$

Now using property (F_4), for all $m \in \mathbb{N}$, we can write

$$\tau + F(s_\beta^m J_\beta(z^m, z^{m+1})) < F(s_\beta^{m-1} J_\beta(z^{m-1}, z^m)) \quad \text{for all } \beta \in \Omega.$$

Furthermore,

$$F(s_\beta^m J_\beta(z^m, z^{m+1})) < F(J_\beta(z^0, z^1)) - m\tau \quad \text{for all } \beta \in \Omega \text{ and } m \in \mathbb{N}. \tag{2.2.4}$$

Now, letting $m \to \infty$, from (2.2.4) we get $\lim_{m\to\infty} F(s_\beta^m J_\beta(z^m, z^{m+1})) = -\infty$ for all $\beta \in \Omega$. Hence using property (F_2) we get $\lim_{m\to\infty} s_\beta^m J_\beta(z^m, z^{m+1}) = 0$. Let $(J_\beta)_m = J_\beta(z^m, z^{m+1})$ for all $\beta \in \Omega$ and $m \in \mathbb{N}$. From (F_3), there is $p \in (0,1)$ such that

$$\lim_{m\to\infty} (s_\beta^m (J_\beta)_m)^p F(s_\beta^m (J_\beta)_m) = 0 \quad \text{for all } \beta \in \Omega.$$

From (2.2.4), we can write for all $\beta \in \Omega$ and $m \in \mathbb{N}$

$$(s_\beta^m (J_\beta)_m)^p F((s_\beta^m J_\beta)_m) - (s_\beta^m (J_\beta)_m)^p F((J_\beta)_0) \leq -(s_\beta^m (J_\beta)_m)^p m\tau \leq 0. \tag{2.2.5}$$

Applying $m \to \infty$, we have

$$\lim_{m\to\infty} m(s_\beta^m (J_\beta)_m)^p = 0 \quad \text{for all } \beta \in \Omega. \tag{2.2.6}$$

This suggests that there is $m_1 = m_1(\beta) \in \mathbb{N}$ such that $m(s_\beta^m (J_\beta)_m)^p \leq 1$ for each $m \geq m_1$ and for all $\beta \in \Omega$. Hence, we can write

$$s_\beta^m (J_\beta)_m \leq \frac{1}{m^{\frac{1}{p}}} \quad \text{for all } m \geq m_1 \text{ and } \beta \in \Omega\ . \tag{2.2.7}$$

Now by repeated use of $(\mathcal{J}1)$ and (2.2.7) for all $m, n \in \mathbb{N}$ such that $n > m > m_1$ and for all $\beta \in \Omega$, we get

$$J_\beta(z^m, z^n) \leq \sum_{j=m}^{n-1} s_\beta^j (J_\beta)_j \leq \sum_{j=m}^{\infty} s_\beta^j (J_\beta)_j \leq \sum_{j=m}^{\infty} \frac{1}{j^{\frac{1}{p}}}.$$

Since $\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{p}}}$ is convergent series, thus we can write

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(z^m, z^n) = 0 \quad \text{for all } \beta \in \Omega. \tag{2.2.8}$$

Now, since $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequentially complete b-gauge space, so $(z^m : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{J}_{s;\Omega}$-convergent in U, thus for all $z \in S^{\mathcal{J}_{s;\Omega}}_{(z^m : m\in\{0\}\cup\mathbb{N})}$, we can write

$$\lim_{m\to\infty} J_\beta(z, z^m) = 0 \quad \text{for all} \ \beta \in \Omega. \tag{2.2.9}$$

Thus from (2.2.8) and (2.2.9), fixing $z \in S^{\mathcal{J}_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$, defining $(u_m = z^m : m \in \{0\} \cup \mathbb{N})$ and $(v_m = z : m \in \{0\} \cup \mathbb{N})$ and applying $(\mathcal{J}2)$ to these sequences, we get

$$\lim_{m\to\infty} q_\beta(z, z^m) = 0 \quad \text{for all} \ \beta \in \Omega.$$

This implies $S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) To prove (a_1), let $z^0 \in U$ be arbitrary and fixed. Since $S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$ and we have

$$z^{(m+1)k} = T^{[k]}(z^{mk}), \quad \text{for } m \in \{0\} \cup \mathbb{N}$$

thus defining $(z_m = z^{m-1+k} : m \in \mathbb{N})$, we can write

$$(z_m : m \in \mathbb{N}) \subset T^{[k]}(U),$$

$$S^{Q_{s;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset,$$

also, its subsequences

$$(y_m = z^{(m+1)k}) \subset T^{[k]}(U)$$

and

$$(x_m = z^{mk}) \subset T^{[k]}(U)$$

satisfy

$$y_m = T^{[k]}(x_m), \ \text{for all} \ m \in \mathbb{N}$$

and are $Q_{s;\Omega}$-convergent to each point $z \in S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$. Now, using the fact below

$$S^{Q_{s;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{Q_{s;\Omega}}_{(y_m:m\in\mathbb{N})} \qquad \text{and} \qquad S^{Q_{s;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{Q_{s;\Omega}}_{(x_m:m\in\mathbb{N})}$$

and the supposition that $T^{[k]}$ for some $k \in \mathbb{N}$, is a $Q_{s;\Omega}$-closed map on U, we have there exists $z \in S^{Q_{s;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$ such that $z \in T^{[k]}(z)$. Thus, (a_1) holds.

The assertion (a_2) follows from (a_1) and the fact that $S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

To prove (a_3), on contrary suppose that $J_\beta(z, Tz) > 0$ for some $\beta \in \Omega$, there exists $m_0 \in \mathbb{N}$ such that $J_\beta(z^m, Tz) > 0$ for each $m \geq m_0$. Hence for each $m \geq m_0$, use triangular inequality and inequality (2.2.1), we obtain

$$\begin{aligned} J_\beta(z, Tz) &\leq s_\beta\{J_\beta(z, z^{m+1}) + J_\beta(z^{m+1}, Tz)\} \\ &= s_\beta\{J_\beta(z, z^{m+1}) + J_\beta(Tz^m, Tz)\} \\ &\leq s_\beta\{J_\beta(z, z^{m+1}) + a_\beta J_\beta(z^m, z) + b_\beta J_\beta(z^m, Tz^m) + c_\beta J_\beta(z, Tz) \\ &\quad + e_\beta J_\beta(z^m, Tz) + L_\beta J_\beta(z, Tz^m)\} \\ &\leq s_\beta\{J_\beta(z, z^{m+1}) + a_\beta J_\beta(z^m, z) + b_\beta J_\beta(z^m, z^{m+1}) + c_\beta J_\beta(z, Tz) \\ &\quad + e_\beta s_\beta\{J_\beta(z^m, z) + J_\beta(z, Tz)\} + L_\beta J_\beta(z, z^{m+1})\}. \end{aligned}$$

Letting $m \to \infty$, we have

$$J_\beta(z, Tz) \leq s_\beta\{c_\beta + e_\beta s_\beta\} J_\beta(z, Tz) \qquad \forall \;\; \beta \in \Omega.$$

Now since we have assumed that $s_\beta\{c_\beta + e_\beta s_\beta\} < 1$, we get

$$J_\beta(z, Tz) \leq s_\beta\{c_\beta + e_\beta s_\beta\} J_\beta(z, Tz) < J_\beta(z, Tz) \qquad \forall \;\; \beta \in \Omega.$$

Which is absurd, thus $J_\beta(z, Tz) = 0$ for all $\beta \in \Omega$.
Next, we prove that $J_\beta(Tz, z) = 0$ for all $\beta \in \Omega$. On contrary suppose that $J_\beta(Tz, z) > 0$ for some $\beta \in \Omega$, there exists $m_0 \in \mathbb{N}$ such that $J_\beta(Tz, z^m) > 0$ for each $m \geq m_0$. Hence for each $m \geq m_0$, use triangular inequality and inequality (2.2.1), we obtain

$$\begin{aligned} J_\beta(Tz, z) &\leq s_\beta\{J_\beta(Tz, z^{m+1}) + J_\beta(z^{m+1}, z)\} \\ &= s_\beta\{J_\beta(Tz, Tz^m) + J_\beta(z^{m+1}, z)\} \\ &\leq s_\beta\{a_\beta J_\beta(z, z^m) + b_\beta J_\beta(z, Tz) + c_\beta J_\beta(z^m, Tz^m) + e_\beta J_\beta(z, Tz^m) \\ &\quad + L_\beta J_\beta(z^m, Tz) + J_\beta(z^{m+1}, z)\} \\ &\leq s_\beta\{a_\beta J_\beta(z, z^m) + b_\beta J_\beta(z, Tz) + c_\beta J_\beta(z^m, z^{m+1}) + e_\beta J_\beta(z, z^{m+1}) \\ &\quad + L_\beta s_\beta\{J_\beta(z^m, z) + J_\beta(z, Tz)\} + J_\beta(z^{m+1}, z)\}. \end{aligned}$$

Letting $m \to \infty$, we have

$$J_\beta(Tz, z) \leq s_\beta\{b_\beta + L_\beta s_\beta\} J_\beta(z, Tz) \qquad \forall \;\; \beta \in \Omega.$$

Now as we have proved that $J_\beta(z, Tz) = 0$ for all $\beta \in \Omega$, thus we obtain $J_\beta(Tz, z) = 0$ for all $\beta \in \Omega$. Hence assertion (a_3) holds.
(III) Since $(U, Q_{s;\Omega})$ is a Hausdorff space, using Proposition (2.1.4), assertion (a_3) suggests that for $z \in \mathrm{Fix}(T^{[k]})$, we have $z = T(z)$. This gives $z \in \mathrm{Fix}(T)$. Hence (b_1) is true.
Assertions (a_2) and (b_1) imply (b_2) .

To prove assertion (b_3), consider $(\mathcal{J}1)$ and use (a_3) and (b_1), we have for all $z \in \text{Fix}(T^{[k]}) = \text{Fix}(T)$,
$J_\beta(z,z) \leq s_\beta\{J_\beta(z,T(z)) + J_\beta(T(z),z)\} = 0$ for all $\beta \in \Omega$.

□

Theorem 2.2.2. Let $(U, Q_{s;\Omega})$ be a b-gauge space and suppose $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \to [0,\infty)$, be the $\mathcal{J}_{s;\Omega}$-family of distances generated by $Q_{s;\Omega}$ such that $U^0_{\mathcal{J}_{s;\Omega}} \neq \emptyset$ and $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequentially complete. Let map $T : U \to U$ be such that $T(U) \subset U^0_{\mathcal{J}_{s;\Omega}}$ and for which we have $F \in \mathfrak{F}_{\mathfrak{s}}$ and $\tau > 0$ such that :

$$\alpha(u,v) \geq 1 \Rightarrow \tau + F(s_\beta J_\beta(Tu,Tv)) \leq F\Big(\max\Big\{J_\beta(u,v), J_\beta(u,Tu), J_\beta(v,Tv), \frac{J_\beta(u,Tv)+J_\beta(v,Tu)}{2s_\beta}\Big\} + L_\beta J_\beta(v,Tu)\Big) \tag{2.2.10}$$

for all $\beta \in \Omega$ and for any $u, v \in U$ whenever $J_\beta(Tu,Tv) \neq 0$. Also $L_\beta \geq 0$. Assume, moreover that:

(i) there exist $z^0 \in U$ such that $\alpha(z^0, z^1) \geq 1$; and

(ii) if $\alpha(u,v) \geq 1$, then $\alpha(Tu,Tv) \geq 1$.

Then the following statements hold:

(I) For any $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is $Q_{s;\Omega}$-convergent sequence in U, thus $S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) Furthermore, let that $T^{[k]}$ for some $k \in \mathbb{N}$, is a $Q_{s;\Omega}$-closed map on U. Then

- (a$_1$) $\text{Fix}(T^{[k]}) \neq \emptyset$; and
- (a$_2$) there exists $z \in \text{Fix}(T^{[k]})$ such that $z \in S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$.

(III) Furthermore, let $\text{Fix}(T^{[k]}) \neq \emptyset$ for some $k \in \mathbb{N}$ and T is continuous. Then

- (b$_1$) $\text{Fix}(T^{[k]})$=$\text{Fix}(T)$;
- (b$_2$) there exists $z \in \text{Fix}(T)$ such that $z \in S^{L-Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and
- (b$_3$) for all $z \in \text{Fix}(T)$, $J_\beta(z,z) = 0$, for all $\beta \in \Omega$.

Proof. (I) We first prove that the sequence $(z^m : m \in \{0\} \cup \mathbb{N})$ is a $\mathcal{J}_{s;\Omega}$-Cauchy sequence in U.
Using assumption (i) there is $z^0 \in U$ such that $\alpha(z^0, z^1) \geq 1$. Now for each $\beta \in \Omega$, using (2.2.10) we can write

$$\begin{aligned}\tau + F(s_\beta J_\beta(z^1, z^2)) &= \tau + F(s_\beta J_\beta(Tz^0, Tz^1)) \\ &\leq F\Big(\max\Big\{J_\beta(z^0, z^1), J_\beta(z^0, Tz^0), J_\beta(z^1, Tz^1), \\ &\qquad \frac{J_\beta(z^0, Tz^1) + J_\beta(z^1, Tz^0)}{2s_\beta}\Big\} + L_\beta J_\beta(z^1, Tz^0)\Big) \\ &= F\Big(\max\{J_\beta(z^0, z^1), J_\beta(z^1, z^2)\}\Big).\end{aligned}$$

We observe a contradiction if we choose $\max\{J_\beta(z^0, z^1), J_\beta(z^1, z^2)\} = J_\beta(z^1, z^2)$. Hence choosing $\max\{J_\beta(z^0, z^1), J_\beta(z^1, z^2)\} = J_\beta(z^0, z^1)$ for all $\beta \in \Omega$, we get

$$\tau + F(s_\beta J_\beta(z^1, z^2)) < F(J_\beta(z^0, z^1)) \quad \text{for all } \beta \in \Omega.$$

Using assumption (ii), we have $\alpha(Tz^0, Tz^1) = \alpha(z^1, z^2) \geq 1$. Now for each $\beta \in \Omega$, using (2.2.10) we can write

$$\begin{aligned}\tau + F(s_\beta J_\beta(z^2, z^3)) &= \tau + F(s_\beta J_\beta(Tz^1, Tz^2)) \\ &\leq F\Big(\max\Big\{J_\beta(z^1, z^2), J_\beta(z^1, Tz^1), J_\beta(z^2, Tz^2), \\ &\qquad \frac{J_\beta(z^1, Tz^2) + J_\beta(z^2, Tz^1)}{2s_\beta}\Big\} + L_\beta J_\beta(z^2, Tz^1)\Big) \\ &= F\Big(\max\{J_\beta(z^1, z^2), J_\beta(z^2, z^3)\}\Big).\end{aligned}$$

We observe a contradiction if we choose $\max\{J_\beta(z^1, z^2), J_\beta(z^2, z^3)\} = J_\beta(z^2, z^3)$. Hence choosing $\max\{J_\beta(z^1, z^2), J_\beta(z^2, z^3)\} = J_\beta(z^1, z^2)$ for all $\beta \in \Omega$, we get

$$\tau + F(s_\beta J_\beta(z^2, z^3)) < F(J_\beta(z^1, z^2)) \quad \text{for all } \beta \in \Omega.$$

Progressing in the above manner, we get a sequence $(z^m : m \in \{0\} \cup \mathbb{N}) \subset U$ such that
$z^m = Tz^{m-1}, z^{m-1} \neq z^m$ and $\alpha(z^{m-1}, z^m) \geq 1$, for each $m \in \mathbb{N}$. Furthermore,

$$\tau + F(s_\beta J_\beta(z^m, z^{m+1})) < F(J_\beta(z^{m-1}, z^m)) \quad \text{for all } \beta \in \Omega.$$

Now using property (F_4), for all $m \in \mathbb{N}$, we can write

$$\tau + F(s_\beta^m J_\beta(z^m, z^{m+1})) < F(s_\beta^{m-1} J_\beta(z^{m-1}, z^m)) \quad \text{for all } \beta \in \Omega.$$

Furthermore,

$$F(s_\beta^m J_\beta(z^m, z^{m+1})) < F(J_\beta(z^0, z^1)) - m\tau \quad \text{for all } \beta \in \Omega \text{ and } m \in \mathbb{N}. \tag{2.2.11}$$

Now, letting $m \to \infty$, from (2.2.11) we get $\lim_{m\to\infty} F(s_\beta^m J_\beta(z^m, z^{m+1})) = -\infty$ for all $\beta \in \Omega$. Hence using property (F_2) we get $\lim_{m\to\infty} s_\beta^m J_\beta(z^m, z^{m+1}) = 0$. Let $(J_\beta)_m = J_\beta(z^m, z^{m+1})$ for all $\beta \in \Omega$ and $m \in \mathbb{N}$. From (F_3), there is $p \in (0,1)$ such that

$$\lim_{m\to\infty} (s_\beta^m (J_\beta)_m)^p F(s_\beta^m (J_\beta)_m) = 0 \quad \text{for all } \beta \in \Omega.$$

From (2.2.11), we can write for all $\beta \in \Omega$ and $m \in \mathbb{N}$

$$(s_\beta^m (J_\beta)_m)^p F((s_\beta^m J_\beta)_m) - (s_\beta^m (J_\beta)_m)^p F((J_\beta)_0) \leq -(s_\beta^m (J_\beta)_m)^p m\tau \leq 0. \tag{2.2.12}$$

Applying $m \to \infty$, we have

$$\lim_{m\to\infty} m(s_\beta^m (J_\beta)_m)^p = 0 \quad \text{for all } \beta \in \Omega. \tag{2.2.13}$$

This suggests that there is $m_1 = m_1(\beta) \in \mathbb{N}$ such that $m(s_\beta^m (J_\beta)_m)^p \leq 1$ for each $m \geq m_1$ and for all $\beta \in \Omega$. Hence, we can write

$$s_\beta^m (J_\beta)_m \leq \frac{1}{m^{\frac{1}{p}}} \quad \text{for all } m \geq m_1 \text{ and } \beta \in \Omega\ . \tag{2.2.14}$$

Now by repeated use of $(\mathcal{J}1)$ and (2.2.14) for all $m, n \in \mathbb{N}$, $n > m > m_1$ and for all $\beta \in \Omega$, we get

$$J_\beta(z^m, z^n) \leq \sum_{j=m}^{n-1} s_\beta^j (J_\beta)_j \leq \sum_{j=m}^{\infty} s_\beta^j (J_\beta)_j \leq \sum_{j=m}^{\infty} \frac{1}{j^{\frac{1}{p}}}.$$

Since $\sum_{j=1}^{\infty} \frac{1}{j^{\frac{1}{p}}}$ is convergent series, therefore we have

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(z^m, z^n) = 0 \ \text{ for all } \ \beta \in \Omega. \tag{2.2.15}$$

Now, since $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequentially complete b-gauge space, so $(z^m : m \in \{0\} \cup \mathbb{N})$ is $\mathcal{J}_{s;\Omega}$-convergent in U, thus for all $z \in S^{\mathcal{J}_{s;\Omega}}_{(z^m : m \in \{0\}\cup\mathbb{N})}$, we can write

$$\lim_{m\to\infty} J_\beta(z, z^m) = 0 \ \text{ for all } \ \beta \in \Omega. \tag{2.2.16}$$

Thus from (2.2.15) and (2.2.16), fixing $z \in S^{\mathcal{J}_{s;\Omega}}_{(z^m : m \in \{0\}\cup\mathbb{N})}$, defining $(u_m = z^m : m \in \{0\} \cup \mathbb{N})$ and $(v_m = z : m \in \{0\} \cup \mathbb{N})$ and applying $(\mathcal{J}2)$ to these sequences, we get

$$\lim_{m\to\infty} q_\beta(z, z^m) = 0 \ \text{ for all } \ \beta \in \Omega.$$

This implies $S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) To prove (a_1), let $z^0 \in U$ be arbitrary and fixed. Since $S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$ and we have

$$z^{(m+1)k} = T^{[k]}(z^{mk}), \quad \text{for } m \in \{0\} \cup \mathbb{N}$$

thus defining $(z_m = z^{m-1+k} : m \in \mathbb{N})$, we can write

$$(z_m : m \in \mathbb{N}) \subset T^{[k]}(U),$$

$$S^{Q_{s;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset,$$

also, its subsequences

$$(y_m = z^{(m+1)k}) \subset T^{[k]}(U)$$

and

$$(x_m = z^{mk}) \subset T^{[k]}(U)$$

satisfy

$$y_m = T^{[k]}(x_m) \text{ for all } m \in \mathbb{N}$$

and are $Q_{s;\Omega}$-convergent to each point $z \in S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$. Now, using the fact below

$$S^{Q_{s;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{Q_{s;\Omega}}_{(y_m:m\in\mathbb{N})} \quad \text{and} \quad S^{Q_{s;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{Q_{s;\Omega}}_{(x_m:m\in\mathbb{N})}$$

and the supposition that $T^{[k]}$ for some $k \in \mathbb{N}$, is $Q_{s;\Omega}$-closed map on U, we have there exists $z \in S^{Q_{s;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$ such that $z \in T^{[k]}(z)$.
Thus, (a_1) holds.
The assertion (a_2) follows from (a_1) and the fact that $S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.
(III) Since by (a_2), there is $z \in \mathrm{Fix}(T^{[k]})$ such that $z \in S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$, we have $\lim_{m\to\infty} z^m = z$.
Now, if T is continuous, then $z = \lim_{m\to\infty} z^{m+1} = \lim_{m\to\infty} Tz^m = T(\lim_{m\to\infty} z^m) = T(z)$. This gives $z \in \mathrm{Fix}(T)$. Hence (b_1) is true.
Assertions (a_2) and (b_1) imply (b_2) .
To prove assertion (b_3), since $T(U) \subset U^0_{\mathcal{J}_{s;\Omega}}$, this implies that $z = T(z) \in U^0_{\mathcal{J}_{s;\Omega}}$.
Therefore, $J_\beta(z, z) = 0$, for all $\beta \in \Omega$.

□

Example 2.2.3. Let $U = [0, 1]$ and $B = \{\frac{1}{2^m} : m \in \mathbb{N}\}$.
Let $Q_{s;\Omega} = \{q\}$, where $q : U \times U \to [0, \infty)$ is b-pseudo metric on U defined for all $u, v \in U$ by

$$q(u, v) = \begin{cases} |u - v|^2 & \text{if } u = v \text{ or } \{u, v\} \cap B = \{u, v\}, \\ |u - v|^2 + 1 & \text{if } u \neq v \text{ and } \{u, v\} \cap B \neq \{u, v\}. \end{cases} \tag{2.2.17}$$

Let the set $F = [\frac{1}{8}, 1] \subset U$ and let $J : U \times U \to [0, \infty)$ for all $u, v \in U$ be defined as:

$$J(u,v) = \begin{cases} q(u,v) & \text{if } F \cap \{u,v\} = \{u,v\}, \\ 4 & \text{if } F \cap \{u,v\} \neq \{u,v\}. \end{cases} \quad (2.2.18)$$

Define $\alpha : U \times U \to [0, \infty)$

$$\alpha(u,v) = \begin{cases} 0 & \text{if } u = v, \\ 5 & \text{if } u \neq v. \end{cases}$$

The single-valued map T is defined by

$$T(u) = \frac{u+1}{5}, \quad \text{for all } u \in U. \quad (2.2.19)$$

Note that $T(U) = [\frac{1}{5}, \frac{2}{5}] \subset U^0_{\mathcal{J}_{s;\Omega}} = [\frac{1}{8}, 1]$.
Also, take $F(u) = \ln(u)$, then $F \in \mathfrak{F}_{\mathfrak{s}}$.

(I.1) $(U, \mathcal{Q}_{s;\Omega})$ is a b-gauge space which is also Hausdorff.

(I.2) The family $\mathcal{J}_{s;\Omega} = \{J\}$ is $\mathcal{J}_{s;\Omega}$-family on U (see Example 2.1.2).

(I.3) $(U, \mathcal{Q}_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequential complete (follows from Example 2.1.11).

(I.4) Next, Applying $F(u) = \ln(u)$ to condition (2.2.1), we show that T satisfies the following condition.

$$\alpha(u,v) \geq 1 \Rightarrow J(Tu,Tv) \leq \quad aJ(u,v) + bJ(u,Tu) + cJ(v,Tv) \\ +eJ(u,Tv) + LJ(v,Tu)$$

for any $u, v \in U$ whenever $J(Tu, Tv) \neq 0$.
It is obvious that above condition holds for $a = b = c = \frac{1}{5}$ and $e = L = 0$.

(I.5) Assumption (i), (ii) and (iii) of Theorem 2.2.1 holds.
For $z_0 = 0$ and $z_1 = Tz_0 = \frac{1}{5}$, we have $\alpha(z_0, Tz_0) > 1$. Also $\alpha(Tu, Tv) > 1$, if $\alpha(u, v) > 1$. Finally, if a sequence $(z_m : m \in \mathbb{N})$ in U is such that $\alpha(z_m, z_{m+1}) \geq 1$ and $\lim^{\mathcal{J}_{s;\Omega}}_{m\to\infty} z_m = z$, then $\alpha(z_m, z) \geq 1$ and $\alpha(z, z_m) \geq 1$.

(I.6) Finally, we show that T is $\mathcal{Q}_{s;\Omega}$-closed map on U.
For this let $(z_m : m \in \mathbb{N})$ be a sequence in $T(U) = [\frac{1}{5}, \frac{2}{5}]$ which is $\mathcal{Q}_{s;\Omega}$-convergent to each point of $S^{\mathcal{Q}_{s;\Omega}}_{(z_m : m \in \{0\} \cup \mathbb{N})} \neq \emptyset$. Let the subsequences (v_m) and (u_m) satisfy $v_m = T(u_m)$, for all $m \in \mathbb{N}$.

Let $z \in S^{Q_{s;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})}$, then without loosing generality we may assume that for all $0 < \epsilon_1 < 1$ there exists $k \in \mathbb{N}$ such that

$$q(z, z_m) = |z - z_m|^2 < \epsilon_1 < 1, \text{ for all } m \geq k.$$

As a result, for $\epsilon = \sqrt{\epsilon_1}$, we can also write for all $0 < \epsilon < 1$ there exists $k \in \mathbb{N}$ such that

$$\begin{aligned}[| z - z_m |< \epsilon] \wedge [| z - u_m |< \epsilon] \wedge [| z - v_m |< \epsilon] \\ \wedge [v_m = T(u_m)], \quad \text{for all } m \geq k.\end{aligned}$$

This in particular implies that

$$\begin{aligned}| z - u_m | &=| z - 5v_m + 1 |=| 5z - 4z - 5v_m + 1 | \\ &=| 4(\tfrac{1}{4} - z) - 5(v_m - z) |< \epsilon\end{aligned}$$

and we get

$$4 \left| \frac{1}{4} - z \right|< \epsilon + 5 \mid v_m - z \mid \text{ for all } m \geq k.$$

Now since $| z - v_m |\to 0$, when $m \to \infty$, we get $| \frac{1}{4} - z |< \epsilon_2$ where $\epsilon_2 = \frac{\epsilon}{4} < \frac{1}{4}$. This gives
$S^{Q_{s;\Omega}}_{(z_m:m\in\mathbb{N})} = \{\frac{1}{4}\}$ and there exists $z = \frac{1}{4} \in S^{Q_{s;\Omega}}_{(z_m:m\in\mathbb{N})}$ such that $\frac{1}{4} = T(\frac{1}{4})$.
Hence, T is $Q_{s;\Omega}$-closed map on U.

(I.7) As all the assumptions of Theorem 2.2.1 holds, we have

$$\text{Fix}(T) = \left\{\frac{1}{4}\right\},$$

$$\overset{Q_{s;\Omega}}{\lim_{m\to\infty}} z^m = \frac{1}{4},$$

and

$$J(\frac{1}{4}, \frac{1}{4}) = 0.$$

2.3 Consequences and applications

This section exhibits some important consequences and applications.
Let $(U, Q_{s;\Omega})$ be a b-gauge space and $G = (V, E)$ be a directed graph such that set of vertices V is equal to U and set of edges E includes $\{(u, u) : u \in U\}$, but G includes no parallel edges. We obtain the following corollaries from our theorems by defining $\alpha : U \times U \to [0, \infty)$ for some $\kappa \geq 1$ in the following way.

$$\alpha(u, v) = \begin{cases} \kappa & \text{if } (u, v) \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.1}$$

Corollary 2.3.1. Let $(U, Q_{s;\Omega})$ be a b-gauge space. Let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \to [0, \infty)$, be the $\mathcal{J}_{s;\Omega}$-family of distances generated by $Q_{s;\Omega}$ such that $U^0_{\mathcal{J}_{s;\Omega}} \neq \emptyset$ and $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequentially complete. Let map $T : U \to U$ be such that $T(U) \subset U^0_{\mathcal{J}_{s;\Omega}}$ and for which we have $F \in \mathfrak{F}_\mathfrak{s}$ and $\tau > 0$ such that :

$$\begin{aligned}(u, v) \in E \Rightarrow \tau + F(s_\beta J_\beta(Tu, Tv)) \leq \; & F(a_\beta J_\beta(u, v) + b_\beta J_\beta(u, Tu) + c_\beta J_\beta(v, Tv) \\ & + e_\beta J_\beta(u, Tv) + L_\beta J_\beta(v, Tu))\end{aligned}$$

for all $\beta \in \Omega$ and for any $u, v \in U$ whenever $J_\beta(Tu, Tv) \neq 0$.
Further, $a_\beta, b_\beta, c_\beta, e_\beta, L_\beta \geq 0$ be such that $a_\beta + b_\beta + c_\beta + (s_\beta + 1)e_\beta < 1$ for each $\beta \in \Omega$.
Assume, moreover that:

(i) there exist $z^0 \in U$ such that $(z^0, z^1) \in E$;

(ii) if $(u, v) \in E$, then $(Tu, Tv) \in E$;

(iii) if a sequence $(z^m : m \in \mathbb{N})$ in U is such that $(z^m, z^{m+1}) \in E$ and $\lim^{\mathcal{J}_{s;\Omega}}_{m\to\infty} z^m = z$, then $(z^m, z) \in E$ and $(z, z^m) \in E$.

Then the following statements hold:

(I) For each $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is $Q_{s;\Omega}$-convergent sequence in U; thus, $S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) Furthermore, assume that $T^{[k]}$ for some $k \in \mathbb{N}$, is $Q_{s;\Omega}$-closed map on U and $s_\beta\{c_\beta + e_\beta s_\beta\} < 1$, for each $\beta \in \Omega$. Then

(a$_1$) $\text{Fix}(T^{[k]}) \neq \emptyset$;

(a$_2$) there exists $z \in \text{Fix}(T^{[k]})$ such that $z \in S^{Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and

(a$_3$) for all $z \in \text{Fix}(T^{[k]})$, $J_\beta(z, T(z)) = J_\beta(T(z), z) = 0$, for all $\beta \in \Omega$.

(III) Furthermore, let $\text{Fix}(T^{[k]}) \neq \emptyset$ for some $k \in \mathbb{N}$ and $(U, Q_{s;\Omega})$ is a Hausdorff space. Then

(b$_1$) $\text{Fix}(T^{[k]})=\text{Fix}(T)$;

(b$_2$) there exists $z \in \text{Fix}(T)$ such that $z \in S^{L-Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and

(b$_3$) for all $z \in \text{Fix}(T)$, $J_\beta(z, z) = 0$, for all $\beta \in \Omega$.

Corollary 2.3.2. Let $(U, Q_{s;\Omega})$ be a b-gauge space. Let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \to [0, \infty)$, be the $\mathcal{J}_{s;\Omega}$-family of distances generated by $Q_{s;\Omega}$ such that $U^0_{\mathcal{J}_{s;\Omega}} \neq \emptyset$ and $(U, Q_{s;\Omega})$ is $\mathcal{J}_{s;\Omega}$-sequentially complete. Let map $T : U \to U$ be such that $T(U) \subset U^0_{\mathcal{J}_{s;\Omega}}$ and for which we have $F \in \mathfrak{F}_5$ and $\tau > 0$ such that :

$$(u,v) \in E \Rightarrow \tau + F(s_\beta J_\beta(Tu,Tv)) \leq \; F\Big(\max\Big\{J_\beta(u,v), J_\beta(u,Tu), J_\beta(v,Tv), \frac{J_\beta(u,Tv)+J_\beta(v,Tu)}{2s_\beta}\Big\} + L_\beta J_\beta(v,Tu)\Big)$$

for all $\beta \in \Omega$ and for any $u, v \in U$, whenever $J_\beta(Tu,Tv) \neq 0$. Also, $L_\beta \geq 0$. Assume, moreover that:

(i) there exist $z^0 \in U$ such that $(z^0, z^1) \in E$;

(ii) if $(u,v) \in E$, then $(Tu,Tv) \in E$.

Then the following statements hold:

(I) For any $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is $Q_{s;\Omega}$-convergent sequence in U; thus, $S^{Q_{s;\Omega}}_{(z^m : m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) Furthermore, assume that $T^{[k]}$ for some $k \in \mathbb{N}$, is a $Q_{s;\Omega}$-closed map on U. Then

- (a_1) $\mathrm{Fix}(T^{[k]}) \neq \emptyset$;
- (a_2) there exists $z \in \mathrm{Fix}(T^{[k]})$ such that $z \in S^{Q_{s;\Omega}}_{(z^m : m\in\{0\}\cup\mathbb{N})}$.

(III) Furthermore, let $\mathrm{Fix}(T^{[k]}) \neq \emptyset$ for some $k \in \mathbb{N}$ and T is continuous. Then

- (b_1) $\mathrm{Fix}(T^{[k]})$=$\mathrm{Fix}(T)$;
- (b_2) there exists $z \in \mathrm{Fix}(T)$ such that $z \in S^{L-Q_{s;\Omega}}_{(z^m : m\in\{0\}\cup\mathbb{N})}$; and
- (b_3) for all $z \in \mathrm{Fix}(T)$, $J_\beta(z,z) = 0$, for all $\beta \in \Omega$.

Remark 2.3.3. (a) The fixed point results concerning F-type-contractions in gauge space in [64] require the completeness of the space (U, d). Therefore, our theorems and corollaries for F-type-contractions in the b-gauge space are new generalization of the results in [64] in which assumption are weaker and assertions are stronger.

(b) Our results for F-type-contractions in b-gauge space tells about periodic points as well. Hence improve the results in [64].

(c) The above theorems, Theorem 2.2.1 and Theorem 2.2.2 generalize Theorem 4.2 and Theorem 5.2 respectively in [65].

Now, we consider possible application of our results for the solution an integral equation.
The volterra integral equation

$$u(t) = f(t) + \int_0^{g(t)} K(t,s)u(s)ds \quad t,s \in [0,\infty) \tag{2.3.2}$$

is the integral equation located in the space $C[0,\infty)$ of all continuous functions defined on the interval $[0,\infty)$. Where $K(t,s) : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and $f, g : [0,\infty) \to \mathbb{R}$ are continuous functions and $g(t) \geq 0$ for all $t \in [0,\infty)$. Let $U = (C[0,\infty), \mathbb{R})$. Define the family of b-pseudo metrics by

$$q_m(u,v) = \max_{t\in[0,m]} \{|u(t) - v(t)|^2 e^{-|\tau t|}\}.$$

Obviously, $Q_{s;\Omega} = \{q_m : m \in \mathbb{N}\}$ defines a complete Hausdorff b-gauge structure on U. Here in particular we consider the case when $Q_{s;\Omega} = \mathcal{J}_{s;\Omega} = \{q_m : m \in \mathbb{N}\}$. Define the map $\alpha : U \times U \to [0,\infty)$ for some $\kappa \geq 1$ in the following way.

$$\alpha(u,v) = \begin{cases} \kappa & \text{if } u \neq v \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.3.4. Define the operator $T : C[0,\infty) \to C[0,\infty)$ as follows:

$$Tu(t) = f(t) + \int_0^{g(t)} K(t,s)u(s)ds \quad t,s \in [0,\infty) \tag{2.3.3}$$

where $K(t,s) : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and $f, g : [0,\infty) \to \mathbb{R}$ are continuous functions and $g(t) \geq 0$ for all $t \in [0,\infty)$.
Assume, moreover there exist $\gamma : U \to (0,\infty)$ and $\alpha : U \times U \to (0,\infty)$ such that the following statements hold:

(i) there exist $\tau > 0$ such that

$$|K(t,s)u(s) - K(t,s)v(s)| \leq \sqrt{\frac{e^{-\tau}}{\gamma(u+v)} q_m(u,v)}$$

for each $t, s \in [0,\infty)$ and $u, v \in U$, moreover

$$\Big| \int_0^{g(t)} \frac{1}{\sqrt{\gamma(u(s)+v(s))}} \, ds \Big|^2 \leq e^{|\tau t|};$$

(ii) there exist $z^0 \in U$ such that $\alpha(z^0, Tz^0) \geq 1$;

(iii) for $x, y \in U$ with $\alpha(x, y) \geq 1$ we have $\alpha(Tx, Ty) \geq 1$;

(iv) if a sequence $(z^m : m \in \mathbb{N})$ in U is such that $\alpha(z^m, z^{m+1}) \geq 1$ and $\lim_{m\to\infty}^{\mathcal{J}_{s;\Omega}} z^m = z$, then $\alpha(z^m, z) \geq 1$ and $\alpha(z, z^m) \geq 1$;

(v) T is $Q_{s;\Omega}$-closed map.

Then there exist at least one solution of integral equation (2.3.2).

Proof. We first prove that T satisfies condition (2.2.1). For any $u, v \in U$ with $\alpha(u, v) \geq 1$, we have

$$\begin{aligned}
|Tu(t) - Tv(t)|^2 &= \Big|f(t) + \int_0^{g(t)} K(t,s)u(s)ds - (f(t) + \int_0^{g(t)} K(t,s)v(s)ds)\Big|^2 \\
&= \Big|\int_0^{g(t)} K(t,s)u(s)ds - \int_0^{g(t)} K(t,s)v(s)ds\Big|^2 \\
&\leq \Big(\int_0^{g(t)} \Big|K(t,s)u(s)ds - K(t,s)v(s)\Big|ds\Big)^2 \\
&\leq e^{-\tau} q_m(u,v)\Big(\int_0^{g(t)} \frac{1}{\sqrt{\gamma(u(s)+v(s))}}\, ds\Big)^2 \\
&\leq e^{|\tau t|} e^{-\tau} q_m(u,v).
\end{aligned}$$

From here we can write

$$|Tu(t) - Tv(t)|^2 e^{-|\tau t|} \leq e^{-\tau} q_m(u,v).$$

This can be written as

$$q_m(Tu - Tv) \leq e^{-\tau} q_m(u,v).$$

Obviously, natural logarithm belong to the family $\mathfrak{F}_{\mathfrak{s}}$, therefore, taking logarithm on both sides, we have

$$\ln(q_m(Tu - Tv)) \leq \ln(e^{-\tau} q_m(u,v)).$$

Simplification leads to the following

$$\tau + \ln(q_m(Tu - Tv)) \leq \ln(q_m(u,v)).$$

This implies that (2.2.1) holds for $a_m = 1$ and $b_m = c_m = e_m = L_m = 0$, for all $m \in \mathbb{N}$ and $F(u) = \ln u$. Hence, Theorem (2.2.1), ensure a fixed point of the operator T, thus, there is at least one solution of the integral equation (2.3.2). □

Chapter 3

Periodic and Fixed Points in Quasi-b-Gauge Spaces

The panoptic and comprehensive aspect of Banach fixed point theorem has led to a number of generalizations of the result. In case of multi valued mapping it is extended to Nadler fixed point theorem which has its own significance. Keeping in view the importance of these theorems this chapter is designed to present novel periodic and fixed point results in the setting of quasi-b-gauge space, which generalize and improve the existing results due to Nadler and Banach in fixed point theory.

Throughout the following sections $(U, Q_{s;\Omega})$ is representing a quasi-b-gauge space (denoted by $Q-b-G$ space, for short), where U is the underlying nonempty set and $Q_{s;\Omega}$ is a quasi-b-gauge with s as coefficient of b-metric and Ω is an index set. Also, L(R) is representing the concept of left(right).

This chapter contains four sections. The first section introduces the notion of quasi-b-gauge space $(U, Q_{s;\Omega})$. In the second section we establish the concept of left(right) $\mathcal{J}_{s;\Omega}$-families of generalized quasi-pseudo-b-distances generated by $Q_{s;\Omega}$. In the third section, we investigate novel results based on periodic and fixed point in the setting of quasi-b-gauge space, which generalize and improve the existing results due to Nadler and Banach in the corresponding literature. An example is presented to support our result. The last section is devoted to some important and fascinating consequences and applications of the results obtained. This chapter is published as research article [53].

3.1 Quasi-b-gauge spaces

Reilly [40] studied quasi-gauge spaces and since then numerous researchers continued to work in this direction and obtained several important results.

In this section, we establish the concept of $Q-b-G$ spaces which generalize the existing b-gauge space and quasi-gauge space. We begin with the initiation of the concept of quasi-pseudo-b metric space.

Definition 3.1.1. Let $s \geq 1$. The map $q : U \times U \to [0, \infty)$ is said to be quasi-pseudo-b metric, if for all $e, f, g \in U$ the following properties hold:

(a) $q(e, e) = 0$; and

(b) $q(e, g) \leq s\{q(e, f) + q(f, g)\}$.

The pair (U, q) is said to be quasi-pseudo-b metric space and is called Hausdorff if it satisfies

$$e \neq f \Rightarrow q(e, f) > 0 \vee q(f, e) > 0$$

for all $e, f \in U$.

Example 3.1.2. Suppose $U = [0, 6]$. Define $q : U \times U \to [0, \infty)$ for all $x, y \in U$ as:

$$q(x, y) = \begin{cases} 0 & \text{if } x \geq y \\ t & \text{if } x < y, \end{cases}$$

where $t \geq 2$. Then q is a quasi-pseudo-b-metric on U. In fact, $q(x, x) = 0$ for all $x \in U$. Further, $q(x, z) \leq \frac{t}{2}\{q(x, y) + q(y, z)\}$ holds for all $x, y, z \in U$ and for $\frac{t}{2} = s \geq 1$. Also, (U, q) is Hausdorff.

Example 3.1.3. Let $U = l_p = \big\{\{x_n\}_{n \geq 1} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\big\}$, where $1 \leq p < \infty$. Define $q : U \times U \to [0, \infty)$ for all $x, y \in U$ by

$$q(x, y) = \begin{cases} 0 & \text{if } x \leq y, \\ (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} & \text{otherwise.} \end{cases}$$

Then q is a quasi-pseudo-b-metric on U with $s = p \geq 1$. Since symmetry property does not hold therefore q is not a pseudo-b-metric and hence not a b-metric.

Example 3.1.4. Suppose $U = [0, 6]$. Define $q : U \times U \to [0, \infty)$ for all $e, f \in U$ by

$$q(e, f) = \begin{cases} 0 & \text{if } e \geq f, \\ (e - f)^2 & \text{if } e < f. \end{cases} \tag{3.1.1}$$

Then q is a quasi-pseudo-b-metric on U. Indeed, $q(e, e) = 0$ for all $e \in U$. Further, $q(e, g) \leq 2\{q(e, f) + q(f, g)\}$ holds for all $e, f, g \in U$ and for $s = 2$. Note that, (U, q) is not quasi-pseudo-metric space.

Definition 3.1.5. Each family $Q_{s;\Omega} = \{q_\beta : \beta \in \Omega\}$ of quasi-pseudo-b metrics $q_\beta : U \times U \to [0, \infty)$ (with constant $s_\beta \geq 1$) for $\beta \in \Omega$, is said to be a quasi b-gauge on U.

Definition 3.1.6. The family $Q_{s;\Omega} = \{q_\beta : \beta \in \Omega\}$ is called to be separating if for every pair (e, f) where $e \neq f$, there exists $q_\beta \in Q_{s;\Omega}$ such that either $q_\beta(e, f) > 0$ or $q_\beta(f, e) > 0$.

Definition 3.1.7. Let the family $Q_{s;\Omega} = \{q_\beta : \beta \in \Omega\}$ is a quasi b-gauge on U. The topology $\mathcal{T}(Q_{s;\Omega})$ on U whose subbase is defined by the family $\mathcal{B}(Q_{s;\Omega}) = \{B(e, \epsilon_\beta) : e \in U, \epsilon_\beta > 0, \beta \in \Omega\}$ of all balls $B(e, \epsilon_\beta) = \{f \in U : q_\beta(e, f) < \epsilon_\beta\}$, is called the topology induced by $Q_{s;\Omega}$. The topological space $(U, \mathcal{T}(Q_{s;\Omega}))$ is a $Q - b - G$ space, denoted by $(U, Q_{s;\Omega})$. We note that $(U, Q_{s;\Omega})$ is Hausdorff if $Q_{s;\Omega}$ is separating.

Remark 3.1.8. Each quasi-gauge space is a $Q - b - G$ space (where $s_\beta = 1$, for all $\beta \in \Omega$). Also every b-gauge space is a $Q - b - G$ space. Thus, the class of $Q - b - G$ spaces is bigger than the class of quasi-gauge spaces and b-gauge spaces .

3.2 Left (right) $\mathcal{J}_{s;\Omega}$-families of generalize quasi-pseudo-b-distances

In this section, we establish the notion of L(R) $\mathcal{J}_{s;\Omega}$-families of generalize quasi-pseudo-b-distances on $Q - b - G$ spaces. Moreover, by using these L(R) $\mathcal{J}_{s;\Omega}$-families, we define the L(R) $\mathcal{J}_{s;\Omega}$-sequential completeness and establish for set-valued map $T : U \to Cl^{\mathcal{J}_{s;\Omega}}(U)$ the Nadler type contractions and for single-valued map $T : U \to U$ the Banach type contractions.

The new asymmetric structure determine by these families of distances are generalization of quasi-b-gauges and give valuable and important tools for inquiring periodic points and fixed points of maps in $Q - b - G$ spaces.

Definition 3.2.1. Let $(U, Q_{s;\Omega})$ is a $Q - b - G$ space. The family $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ where $J_\beta : U \times U \to [0, \infty)$, $\beta \in \Omega$ is called the L(R) $\mathcal{J}_{s;\Omega}$-family of generalized quasi-pseudo-b-distances on U (L(R) $\mathcal{J}_{s;\Omega}$-family on U, for short) if for all $x, y, z \in U$ and for all $\beta \in \Omega$ the following statements are fulfilled:

($\mathcal{J}1$) $J_\beta(x, z) \leq s_\beta\{J_\beta(x, y) + J_\beta(y, z)\}$;

($\mathcal{J}2$) for each sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in \mathbb{N})$ in U fulfilling

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(u_m, u_n) = 0, \tag{3.2.1}$$

$$\Big(\lim_{m\to\infty}\sup_{n>m} J_\beta(u_n,u_m)=0\Big), \tag{3.2.2}$$

and

$$\lim_{m\to\infty} J_\beta(v_m,u_m)=0, \tag{3.2.3}$$

$$\Big(\lim_{m\to\infty} J_\beta(u_m,v_m)=0\Big), \tag{3.2.4}$$

the following holds:

$$\lim_{m\to\infty} q_\beta(v_m,u_m)=0, \tag{3.2.5}$$

$$\Big(\lim_{m\to\infty} q_\beta(u_m,v_m)=0\Big). \tag{3.2.6}$$

We denote
$\mathbb{J}^L_{(U,Q_{s;\Omega})}=\{\mathcal{J}_{s;\Omega}:\mathcal{J}_{s;\Omega}=\{J_\beta:\beta\in\Omega\}$ is left $\mathcal{J}_{s;\Omega}$-family on $U\}$,
$\mathbb{J}^R_{(U,Q_{s;\Omega})}=\{\mathcal{J}_{s;\Omega}:\mathcal{J}_{s;\Omega}=\{J_\beta:\beta\in\Omega\}\}$ is right $\mathcal{J}_{s;\Omega}$-family on $U\}$.

Example 3.2.2. Let $(U,Q_{s;\Omega})$ be a $Q-b-G$ space, where U contains at least two distinct points and $Q_{s;\Omega}=\{q_\beta:\beta\in\Omega\}$ is the family of quasi-pseudo-b metrics $q_\beta:U\times U\to[0,\infty)$, $\beta\in\Omega$.

Let the set $F\subset U$ has at least two distinct, arbitrary and fixed points and let $a_\beta\in(0,\infty)$, $\beta\in\Omega$ satisfies $\delta_\beta(F)<a_\beta$ for all $\beta\in\Omega$, where $\delta_\beta(F)=\sup\{q_\beta(a,b):a,b\in F\}$. Define $J_\beta:U\times U\to[0,\infty),\beta\in\Omega$ for all $e,f\in U$ and for all $\beta\in\Omega$ as:

$$J_\beta(e,f)=\begin{cases} q_\beta(e,f) & \text{if } F\cap\{e,f\}=\{e,f\},\\ a_\beta & \text{if } F\cap\{e,f\}\neq\{e,f\}.\end{cases} \tag{3.2.7}$$

Then $\mathcal{J}_{s;\Omega}=\{J_\beta:\beta\in\Omega\}\in\mathbb{J}^L_{(U,Q_{s;\Omega})}\cap\mathbb{J}^R_{(U,Q_{s;\Omega})}$.
We notice that $J_\beta(e,g)\le s_\beta\{J_\beta(e,f)+J_\beta(f,g)\}$ for all $\beta\in\Omega$ and for all $e,f,g\in U$, where $s_\beta\ge 1$. Hence condition $(\mathcal{J}_1)$ is satisfied. Certainly, condition $(\mathcal{J}_1)$ does not hold only if there exists $e,f,g\in U$ and some $\beta\in\Omega$ such that $J_\beta(e,g)=a_\beta$, $J_\beta(e,f)=q_\beta(e,f)$, $J_\beta(f,g)=q_\beta(f,g)$ and $s_\beta\{q_\beta(e,f)+q_\beta(f,g)\}\le a_\beta$. However, then this implies that there exists $h\in\{e,g\}$ such that $h\notin F$ and on other hand $e,f,g\in F$, which is unfeasible.
Now let (3.2.1) and (3.2.3) hold for the sequences $(u_m:m\in\mathbb{N})$ and $(v_m:$

$m \in \mathbb{N}$) in U. Then in particular (3.2.3) yields that for all $\beta \in \Omega$ and for all $0 < \epsilon < a_\beta$, there exists $m_1 = m_1(\beta) \in \mathbb{N}$ such that

$$J_\beta(v_m, u_m) < \epsilon, \text{ for all } m \geq m_1 \tag{3.2.8}$$

By (3.2.8) and (3.2.7), denoting $m_2 = \min\{m_1(\beta) : \beta \in \Omega\}$, we have

$$\{v_m, u_m\} \cap F = \{v_m, u_m\}, \text{ for all } m \geq m_2 \tag{3.2.9}$$

and

$$q_\beta(v_m, u_m) = J_\beta(v_m, u_m) < \epsilon. \tag{3.2.10}$$

Hence (3.2.5) holds. Thus, $\mathcal{J}_{s;\Omega}$ is a left $\mathcal{J}_{s;\Omega}$-family on U.

Similarly, we can show that $\mathcal{J}_{s;\Omega}$ is a right $\mathcal{J}_{s;\Omega}$-family on U.

Now we state few trivial properties of L(R) $\mathcal{J}_{s;\Omega}$-families in the following remark.

Remark 3.2.3. Let $(U, Q_{s;\Omega})$ is a $Q-b-G$ space.

(a) $Q_{s;\Omega} \in \mathbb{J}^L_{(U,Q_{s;\Omega})} \cap \mathbb{J}^R_{(U,Q_{s;\Omega})}$.

(b) Let $\mathcal{J}_{s;\Omega} \in \mathbb{J}^L_{(U,Q_{s;\Omega})}$ or $\mathcal{J}_{s;\Omega} \in \mathbb{J}^R_{(U,Q_{s;\Omega})}$. If $J_\beta(v, v) = 0$ for all $\beta \in \Omega$ and for all $v \in U$, then J_β for each $\beta \in \Omega$, is a quasi-pseudo-b metric.

(c) There exists example of $\mathcal{J}_{s;\Omega} \in \mathbb{J}^L_{(U,Q_{s;\Omega})}$ and $\mathcal{J}_{s;\Omega} \in \mathbb{J}^R_{(U,Q_{s;\Omega})}$ which shows that J_β, $\beta \in \Omega$ are not a quasi-pseudo-b metrics.

Proposition 3.2.4. Let $(U, Q_{s;\Omega})$ be a Hausdorff $Q-b-G$ space and $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the L(R) $\mathcal{J}_{s;\Omega}$-family on U. Then there exists $\beta \in \Omega$ such that for all $e, f \in U$

$$e \neq f \Rightarrow J_\beta(e, f) > 0 \vee J_\beta(f, e) > 0.$$

Proof. Let $\mathcal{J}_{s;\Omega}$ is a left $\mathcal{J}_{s;\Omega}$-family on U and suppose that there are $e, f \in U$ with $e \neq f$ such that $J_\beta(e, f) = 0 = J_\beta(f, e)$ for all $\beta \in \Omega$. Then by using property $(\mathcal{J}1)$ we have $J_\beta(e, e) = 0$, for all $\beta \in \Omega$.

Defining sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in \mathbb{N})$ in U by $u_m = e$ and $v_m = f$ or $u_m = f$ and $v_m = e$, we see that conditions (3.2.1) and (3.2.3) of property $(\mathcal{J}2)$ are satisfied and therefore condition (3.2.5) holds, which implies that $q_\beta(e, f) = 0 = q_\beta(f, e)$, for all $\beta \in \Omega$. But, this denies the fact that $(U, Q_{s;\Omega})$ is a Hausdorff $Q-b-G$ space. Therefore, our supposition is wrong and there exists $\beta \in \Omega$ such that

$$e \neq f \Rightarrow J_\beta(e, f) > 0 \vee J_\beta(f, e) > 0$$

for all $e, f \in U$.

Similarly for $\mathcal{J}_{s;\Omega}$ is a right $\mathcal{J}_{s;\Omega}$-family on U, the proof is based on analogous technique. □

Definition 3.2.5. Let $(U, Q_{s;\Omega})$ is a $Q-b-G$ space and $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the L(R) $\mathcal{J}_{s;\Omega}$-family on U. Define the L(R) $\mathcal{J}_{s;\Omega}$-ball center at x^0 and radius $\epsilon = \{\epsilon_\beta\}_{\beta\in\Omega} \in (0,\infty)$ as:

$$B^{L-\mathcal{J}_{s;\Omega}}(x^0, \epsilon) = \{y \in U : \forall_{\beta\in\Omega}\{J_\beta(x^0, y) \le \epsilon_\beta\}\};$$

$$(B^{R-\mathcal{J}_{s;\Omega}}(x^0, \epsilon) = \{y \in U : \forall_{\beta\in\Omega}\{J_\beta(y, x^0) \le \epsilon_\beta\}\}).$$

Remark 3.2.6. From Example 3.2.2 it follows that there is a $Q-b-G$ space, the family $\mathcal{J}_{s;\Omega}$ on U, $x^0 \in U$ and $\epsilon = \{\epsilon_\beta\}_{\beta\in\Omega} \in (0,\infty)$ such that $x^0 \notin B^{L-\mathcal{J}_{s;\Omega}}(x^0, \epsilon)$ $(x^0 \notin B^{R-\mathcal{J}_{s;\Omega}}(x^0, \epsilon))$.

Now, using L(R) $\mathcal{J}_{s;\Omega}$-family on U, we describe L(R) $\mathcal{J}_{s;\Omega}$-completeness in the $Q-b-G$ space $(U, Q_{s;\Omega})$ which generalizes the usual $Q_{s;\Omega}$-completeness.

Definition 3.2.7. Let $(U, Q_{s;\Omega})$ is a $Q-b-G$ space and let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ is a L(R) $\mathcal{J}_{s;\Omega}$-family on U.

(A) A sequence $(v_m : m \in \mathbb{N})$ is said to be L(R) $\mathcal{J}_{s;\Omega}$-Cauchy sequence in U if for all $\beta \in \Omega$, we have

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(v_m, v_n) = 0$$

$$\Big(\lim_{m\to\infty} \sup_{n>m} J_\beta(v_n, v_m) = 0\Big).$$

(B) The sequence $(v_m : m \in \mathbb{N})$ is said to be L(R) $\mathcal{J}_{s;\Omega}$-convergent to $v \in U$ if $\lim^{L-\mathcal{J}_{s;\Omega}}_{m\to\infty} v_m = v$ $(\lim^{R-\mathcal{J}_{s;\Omega}}_{m\to\infty} v_m = v)$, where
$\lim^{L-\mathcal{J}_{s;\Omega}}_{m\to\infty} v_m = v \Leftrightarrow \lim_{m\to\infty} J_\beta(v, v_m) = 0, \text{ for all } \beta \in \Omega$
$\Big(\lim^{R-\mathcal{J}_{s;\Omega}}_{m\to\infty} v_m = v \Leftrightarrow \lim_{m\to\infty} J_\beta(v_m, v) = 0, \text{ for all } \beta \in \Omega\Big).$

(C) If $S^{L-\mathcal{J}_{s;\Omega}}_{(v_m:m\in\mathbb{N})} \neq \emptyset$ $(S^{R-\mathcal{J}_{s;\Omega}}_{(v_m:m\in\mathbb{N})} \neq \emptyset)$, where
$S^{L-\mathcal{J}_{s;\Omega}}_{(v_m:m\in\mathbb{N})} = \{v \in U : \lim^{L-\mathcal{J}_{s;\Omega}}_{m\to\infty} v_m = v\}$
$\Big(S^{R-\mathcal{J}_{s;\Omega}}_{(v_m:m\in\mathbb{N})} = \{v \in U : \lim^{R-\mathcal{J}_{s;\Omega}}_{m\to\infty} v_m = v\}\Big).$
Then the sequence $(v_m : m \in \mathbb{N})$ in U is L(R) $\mathcal{J}_{s;\Omega}$-convergent in U.

(D) The space $(U, Q_{s;\Omega})$ is L(R) $\mathcal{J}_{s;\Omega}$-sequentially complete $Q-b-G$ space, if each L(R) $\mathcal{J}_{s;\Omega}$-Cauchy sequence in U is L(R) $\mathcal{J}_{s;\Omega}$-convergent in U.

Remark 3.2.8. Let $(U, Q_{s;\Omega})$ be a $Q-b-G$ space and let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the $\mathcal{J}_{s;\Omega}$-family on U.

(i) There exist examples of $Q{-}b{-}G$ spaces $(U, Q_{s;\Omega})$ and L(R) $\mathcal{J}_{s;\Omega}$-family on U with $\mathcal{J}_{s;\Omega} \neq Q_{s;\Omega}$ such that $(U, Q_{s;\Omega})$ is L(R) $\mathcal{J}_{s;\Omega}$-sequential complete but not L(R) $Q_{s;\Omega}$-sequential complete (see Example 2.1.11).

(ii) If $(v_m : m \in \mathbb{N})$ be a L(R) $\mathcal{J}_{s;\Omega}$-convergent sequence in U, then for every of its subsequence $(u_m : m \in \mathbb{N})$ we have
$S^{L-\mathcal{J}_{s;\Omega}}_{(v_m:m\in\mathbb{N})} \subset S^{L-\mathcal{J}_{s;\Omega}}_{(u_m:m\in\mathbb{N})} \left(S^{R-\mathcal{J}_{s;\Omega}}_{(v_m:m\in\mathbb{N})} \subset S^{R-\mathcal{J}_{s;\Omega}}_{(u_m:m\in\mathbb{N})}\right)$.

(iii) There exist examples of $Q - b - G$ spaces $(U, Q_{s;\Omega})$ and $\mathcal{J}_{s;\Omega}$-family on U with $\mathcal{J}_{s;\Omega} \in \mathbb{J}^L_{(U,Q_{s;\Omega})} \cap \mathbb{J}^R_{(U,Q_{s;\Omega})}$ such that $(U, Q_{s;\Omega})$ is left $\mathcal{J}_{s;\Omega}$-Cauchy sequence, but not right $\mathcal{J}_{s;\Omega}$-Cauchy sequence (see Example 3.2.9 below).

Example 3.2.9. Let $U = \mathbb{R}$ and let $Q_{s;\Omega} = \{q\}$ where q is a quasi-pseudo-b-metric on U defined for all $x, y \in U$ by

$$q(x,y) = \begin{cases} 0 & \text{if } x \geq y, \\ t & \text{if } x < y. \end{cases}$$

Where $t \geq 2$.
Let $G = \mathbb{Z}^-$ is a subset of U. Let $\mathcal{J}_{s;\Omega} = \{J\}$ where $J : U \times U \to [0, \infty)$ is defined for all $x, y \in U$ by

$$J(x,y) = \begin{cases} q(x,y) & \text{if } G \cap \{x,y\} = \{x,y\}, \\ t^2 & \text{if } G \cap \{x,y\} \neq \{x,y\}. \end{cases}$$

Then $(U, Q_{s;\Omega})$ is a $Q - b - G$ space and $\mathcal{J}_{s;\Omega} \in \mathbb{J}^L_{(U,Q_{s;\Omega})} \cap \mathbb{J}^R_{(U,Q_{s;\Omega})}$.
Take a sequence $(u_m : m \in \mathbb{N}) = (-m : m \in \mathbb{N})$, then it is left $\mathcal{J}_{s;\Omega}$-Cauchy sequence, but not right $\mathcal{J}_{s;\Omega}$-Cauchy sequence.
Also, we observe that $(-m : m \in \mathbb{N})$ is left $\mathcal{J}_{s;\Omega}$-convergent to each point in $\mathbb{Z}^-$, but not right $\mathcal{J}_{s;\Omega}$-convergent to any point in $\mathbb{R}$.

Definition 3.2.10. Suppose $(U, Q_{s;\Omega})$ is a $Q-b-G$ space and let $T : U \to 2^U$ is a multi-valued map. The map $T^{[k]}$ is called a L(R) $Q_{s;\Omega}$-quasi-closed map on U, where $k \in \mathbb{N}$, if for each sequence $(w_m : m \in \mathbb{N})$ within $T^{[k]}(U)$, which is L(R) $Q_{s;\Omega}$-convergent in U, thus $S^{L-Q_{s;\Omega}}_{(w_m:m\in\mathbb{N})} \neq \emptyset$ $(S^{R-Q_{s;\Omega}}_{(w_m:m\in\mathbb{N})} \neq \emptyset)$, and having subsequences $(y_m : m \in \mathbb{N})$ and $(z_m : m \in \mathbb{N})$ which satisfy
$y_m \in T^{[k]}(z_m)$ for all $m \in \mathbb{N}$,
has the characteristic that there exists $w \in S^{L-Q_{s;\Omega}}_{(w_m:m\in\mathbb{N})} (w \in S^{R-Q_{s;\Omega}}_{(w_m:m\in\mathbb{N})})$ such that $w \in T^{[k]}(w) (w \in T^{[k]}(w))$.

Definition 3.2.11. Let $(U, Q_{s;\Omega})$ is a $Q-b-G$ space and let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the L(R) $\mathcal{J}_{s;\Omega}$-family on U. A set $W \in 2^U$ is a L(R) $\mathcal{J}_{s;\Omega}$-closed in U if $W = cl_U^{L-\mathcal{J}_{s;\Omega}}(W)$ $(W = cl_U^{R-\mathcal{J}_{s;\Omega}}(W))$, where $cl_U^{L-\mathcal{J}_{s;\Omega}}(W)$ $(cl_U^{R-\mathcal{J}_{s;\Omega}}(W))$, is the L(R) $\mathcal{J}_{s;\Omega}$-closure in U and defined as:
$cl_U^{L-\mathcal{J}_{s;\Omega}}(W) = \{z \in U : \lim^{L-\mathcal{J}_{s;\Omega}}_{m\to\infty} z_m = z\}$
$(cl_U^{R-\mathcal{J}_{s;\Omega}}(W) = \{z \in U : \lim^{R-\mathcal{J}_{s;\Omega}}_{m\to\infty} z_m = z)$.
Define $Cl^{L-\mathcal{J}_{s;\Omega}}(U) = \{W \in 2^U : W = cl_U^{L-\mathcal{J}_{s;\Omega}}(W)\}$ $(Cl^{R-\mathcal{J}_{s;\Omega}}(U) = \{W \in 2^U : W = cl_U^{R-\mathcal{J}_{s;\Omega}}(W)\})$. Thus $Cl^{L-\mathcal{J}_{s;\Omega}}(U)$ $(Cl^{R-\mathcal{J}_{s;\Omega}}(U))$ symbolizes the class of all L(R) $\mathcal{J}_{s;\Omega}$-closed subsets of U.

Definition 3.2.12. Let $(U, Q_{s;\Omega})$ is a $Q-b-G$ space and let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ is a L(R) $\mathcal{J}_{s;\Omega}$-family on U. The map $T : U \to 2^U$ is called L(R) $\mathcal{J}_{s;\Omega}$-admissible at a point $z^0 \in U$ if for each sequence $(z^m : m \in \{0\} \cup \mathbb{N})$ which satisfies $z^{m+1} \in T(z^m)$ for all $m \in \{0\} \cup \mathbb{N}$ and for all $\beta \in \Omega$

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(z^m, z^n) = 0$$

$$\Big(\lim_{m\to\infty} \sup_{n>m} J_\beta(z^n, z^m) = 0\Big),$$

there is $z \in U$ such that
$\lim_{m\to\infty} J_\beta(z, z^m) = 0$ $\Big(\lim_{m\to\infty} J_\beta(z^m, z) = 0\Big)$, for all $\beta \in \Omega$.
The map $T : U \to 2^U$ is called L(R) $\mathcal{J}_{s;\Omega}$-admissible in U if it is L(R) $\mathcal{J}_{s;\Omega}$-admissible at every $z^0 \in U$.

Example 3.2.13. Let $U = [0, 6]$ and let $Q_{s;\Omega} = \{q\}$ where q is a quasi-pseudo-b-metric on U defined for all $x, y \in U$ by

$$q(x, y) = \begin{cases} 0 & \text{if } x \geq y, \\ (x-y)^2 & \text{if } x < y. \end{cases} \tag{3.2.11}$$

Then $(U, Q_{s;\Omega})$ is a $Q-b-G$ space.
Let $G = [0, 3) \cup (3, 6]$ be a subset of U. Let $\mathcal{J}_{s;\Omega} = \{J\}$ where $J : U \times U \to [0, \infty)$ is defined for all $x, y \in U$ by

$$J(x, y) = \begin{cases} q(x, y) & \text{if } G \cap \{x, y\} = \{x, y\} \\ 40 & \text{if } G \cap \{x, y\} \neq \{x, y\}. \end{cases} \tag{3.2.12}$$

Then $\mathcal{J}_{s;\Omega}$ is a L(R) $\mathcal{J}_{s;\Omega}$-family on U.
The set-valued map T is defined by

$$T(x) = \begin{cases} [4, 6] & \text{for } x \in [0, 3) \cup (3, 6] \\ [5, 6] & \text{for } x = 3. \end{cases} \tag{3.2.13}$$

We show that T is left $\mathcal{J}_{s;\Omega}$-admissible at U. Thus if $z^0 \in U$ and $\{z^m : m \in \{0\} \cup \mathbb{N}\}$ fulfils the properties

$$z^{m+1} \in T(z^m), \quad \text{for all } m \in \{0\} \cup \mathbb{N} \tag{3.2.14}$$

and

$$\lim_{m\to\infty} \sup_{n>m} J(z^m, z^n) = 0, \tag{3.2.15}$$

then

$$\lim_{m\to\infty} J(z, z^m) = 0 \qquad \text{where } z = 6. \tag{3.2.16}$$

In fact, we observe

$$T^{[m]}(U) = [4, 6] \subset G \qquad \text{for } m \geq 2. \tag{3.2.17}$$

We can also write (3.2.15) in the form that for all $\epsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that for all $n > m \geq m_0$, we have $J(z^m, z^n) < \epsilon$ and so, in particular in view of (3.2.17), (3.2.11) and (3.2.12), this implies that there exist $m_1 \geq m_0$ such that for all $0 < \epsilon$ and for all $n > m \geq m_1$, we have

$$J(z^m, z^n) = q(z^m, z^n) = 0 < \epsilon. \tag{3.2.18}$$

From (3.2.17), (3.2.18), (3.2.11) and (3.2.12), we conclude that $z^m \geq z^{m+1}$ for all $m \geq m_1$, and since $6 \geq z^m$ for all m and $6 \in G$, we have $\lim_{m\to\infty} q(z, z^m) = 0$ where $z = 6$ and this implies (3.2.16). Thus, (U, T) is left $\mathcal{J}_{s;\Omega}$-admissible at U.

Remark 3.2.14. Suppose $(U, Q_{s;\Omega})$ is a $Q - b - G$ space and let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ is a L(R) $\mathcal{J}_{s;\Omega}$-family on U.

(a) There exist examples of $Q - b - G$ space $(U, Q_{s;\Omega})$ and $\mathcal{J}_{s;\Omega}$-family on U with $\mathcal{J}_{s;\Omega} \neq Q_{s;\Omega}$ such that (U, T) is L(R) $\mathcal{J}_{s;\Omega}$-admissible on U but not $Q_{s;\Omega}$-sequential complete (see Example 6.4 of [44]).

(b) If (U, Q) is a L(R) $\mathcal{J}_{s;\Omega}$-sequentially complete then $T : U \to 2^U$ is L(R) $\mathcal{J}_{s;\Omega}$-admissible on U.

(c) Note that if $s_\beta = 1$, for all $\beta \in \Omega$, we obtain all the definitions of this section in $\mathcal{J}$-family of generalized quasi-pseudo distances in quasi-gauge spaces [52].

In the $Q - b - G$ space, we describe the L(R) Hausdorff type quasi-b-distances and Nadler type L(R) contractions in the following way.

Definition 3.2.15. Let $(U, Q_{s;\Omega})$ is a $Q-b-G$ space and let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ is a L(R) $\mathcal{J}_{s;\Omega}$-family on U, let $\zeta \in \{1,2,3\}$ and suppose for all $\beta \in \Omega$, for all $e \in U$ and for all $F \in 2^U$

$$\begin{aligned} J_\beta(e,F) &= \inf\{J_\beta(e,g) : g \in F\} \\ \wedge J_\beta(F,e) &= \inf\{J_\beta(g,e) : g \in F\}. \end{aligned} \tag{3.2.19}$$

(a) Define on $Cl^{L-\mathcal{J}_{s;\Omega}}(U)$ $(Cl^{R-\mathcal{J}_{s;\Omega}}(U))$ the L(R) quasi-b-distances of Hausdorff type $\mathcal{D}_\zeta^{L-\mathcal{J}_{s;\Omega}} = \{D_{\zeta;\beta}^{L-\mathcal{J}_{s;\Omega}}, \beta \in \Omega\}$ $(\mathcal{D}_\zeta^{R-\mathcal{J}_{s;\Omega}} = \{D_{\zeta;\beta}^{R-\mathcal{J}_{s;\Omega}}, \beta \in \Omega\})$, where $D_{\zeta;\beta}^{L-\mathcal{J}_{s;\Omega}} : Cl^{L-\mathcal{J}_{s;\Omega}}(U) \times Cl^{L-\mathcal{J}_{s;\Omega}}(U) \to [0,\infty]$, $\beta \in \Omega$ $(D_{\zeta;\beta}^{R-\mathcal{J}_{s;\Omega}} : Cl^{R-\mathcal{J}_{s;\Omega}}(U) \times Cl^{R-\mathcal{J}_{s;\Omega}}(U) \to [0,\infty]$, $\beta \in \Omega)$ for all $\beta \in \Omega$ and for all $E, F \in Cl^{\mathcal{J}_{s;\Omega}}(U)$ as:

(a.1) $D_{1;\beta}^{L-\mathcal{J}_{s;\Omega}}(E,F) = \max\{\sup_{e\in E} J_\beta(e,F), \sup_{f\in F} J_\beta(E,f)\}$,
$D_{2;\beta}^{L-\mathcal{J}_{s;\Omega}}(E,F) = \max\{\sup_{e\in E} J_\beta(e,F), \sup_{f\in F} J_\beta(f,E)\}$ and
$D_{3;\beta}^{L-\mathcal{J}_{s;\Omega}}(E,F) = \sup_{e\in E} J_\beta(e,F)$, if $\mathcal{J}_{s;\Omega} \in \mathbb{J}_{(U,Q)}^L$.

(a.2) $D_{1;\beta}^{R-\mathcal{J}_{s;\Omega}}(E,F) = \max\{\sup_{e\in E} J_\beta(e,F), \sup_{f\in F} J_\beta(E,f)\}$,
$D_{2;\beta}^{R-\mathcal{J}_{s;\Omega}}(E,F) = \max\{\sup_{e\in E} J_\beta(e,F), \sup_{f\in F} J_\beta(f,E)\}$ and
$D_{3;\beta}^{R-\mathcal{J}_{s;\Omega}}(E,F) = \sup_{e\in E} J_\beta(e,F)$, if $\mathcal{J}_{s;\Omega} \in \mathbb{J}_{(U,Q)}^R$.

(b) Let $\mu = \{\mu_\beta\}_{\beta\in\Omega} \in [0,1)$. The map $T : U \to Cl^{L-\mathcal{J}_{s;\Omega}}(U)$ $(T : U \to Cl^{R-\mathcal{J}_{s;\Omega}}(U))$ is L(R) $(\mathcal{D}_\zeta^{L-\mathcal{J}_{s;\Omega}}, \mu)$-contraction on U $((\mathcal{D}_\zeta^{R-\mathcal{J}_{s;\Omega}}, \mu)$-contraction on $U)$ if for all $\beta \in \Omega$ and for all $u, v \in U$:

(b.1) $D_{\zeta;\beta}^{L-\mathcal{J}_{s;\Omega}}(T(u), T(v)) \le \mu_\beta J_\beta(u,v)$, if $\mathcal{J}_{s;\Omega} \in \mathbb{J}_{(U,Q)}^L$;

(b.2) $D_{\zeta;\beta}^{R-\mathcal{J}_{s;\Omega}}(T(u), T(v)) \le \mu_\beta J_\beta(u,v)$, if $\mathcal{J}_{s;\Omega} \in \mathbb{J}_{(U,Q)}^R$.

Remark 3.2.16. Let $(U, Q_{s;\Omega})$ is a $Q-b-G$ space and let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ of maps $J_\beta : U \times U \to [0,\infty)$, $\beta \in \Omega$, be a L(R) $\mathcal{J}_{s;\Omega}$-family on U.

(a) In general $D_{\zeta;\beta}^{L-\mathcal{J}_{s;\Omega}}(D_{\zeta;\beta}^{R-\mathcal{J}_{s;\Omega}})$ are not symmetric thus, $D_{\zeta;\beta}^{L-\mathcal{J}_{s;\Omega}}(E,F) = D_{\zeta;\beta}^{L-\mathcal{J}_{s;\Omega}}(F,E)(D_{\zeta;\beta}^{R-\mathcal{J}_{s;\Omega}}(E,F) = D_{\zeta;\beta}^{R-\mathcal{J}_{s;\Omega}}(F,E))$ not necessarily hold. Also $D_{\zeta;\beta}^{L-\mathcal{J}_{s;\Omega}}(E,E) = 0(D_{\zeta;\beta}^{R-\mathcal{J}_{s;\Omega}}(E,E) = 0)$ not necessarily hold; see Remark 3.3.6 (b) and (c) for detail.

(b) Each $(\mathcal{D}_\zeta^{L-\mathcal{J}_{s;\Omega}}, \mu)$-contraction on U $((\mathcal{D}_\zeta^{R-\mathcal{J}_{s;\Omega}}, \mu)$-contraction on $U)$, $\zeta \in \{1,2,3\}$ is $(\mathcal{D}_3^{L-\mathcal{J}_{s;\Omega}}, \mu)$-contraction on U $((\mathcal{D}_3^{R-\mathcal{J}_{s;\Omega}}, \mu)$-contraction on $U)$ but, the converse not generally true.

3.3 Periodic and fixed point theorems in quasi-b-gauge spaces

Wlodarczyk and Plebaniak [52] have investigated periodic and fixed point theorems in quasi-gauge spaces using $\mathcal{J}$-family of generalized quasi-pseudo distances. Using their technique we present novel theorems related to periodic and fixed points of certain maps in $Q-b-G$ space which generalize some of their results.

Theorem 3.3.1. Let $(U, Q_{s;\Omega})$ be a $Q-b-G$ space, let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ is a L(R) $\mathcal{J}_{s;\Omega}$-family on U and let $\zeta \in \{1, 2, 3\}$. Assume, moreover, that $\mu = \{\mu_\beta\}_{\beta\in\Omega} \in [0, 1)$ and the map $T : U \to Cl^{L-\mathcal{J}_{s;\Omega}}(U)$ $(T : U \to Cl^{R-\mathcal{J}_{s;\Omega}}(U))$ satisfy:

(i) T is $(\mathcal{D}_\zeta^{L-\mathcal{J}_{s;\Omega}}, \mu)$-contraction on U $((\mathcal{D}_\zeta^{R-\mathcal{J}_{s;\Omega}}, \mu)$-contraction on U); and

(ii) for any $u \in U$ and any $\gamma = \{\gamma_\beta\}_{\beta\in\Omega} \in (0, \infty)$, there exists $v \in T(u)$ such that for all $\beta \in \Omega$

$$J_\beta(u, v) < J_\beta(u, T(u)) + \gamma_\beta \tag{3.3.1}$$

$$\Big(J_\beta(v, u) < J_\beta(T(u), u) + \gamma_\beta\Big). \tag{3.3.2}$$

(I) If (U, T) at a point $z^0 \in U$ is L(R) $\mathcal{J}_{s;\Omega}$-admissible, then there is a sequence $(z^m : m \in \{0\} \cup \mathbb{N})$ starting at $z^0 \in U$ such that $z^m \in T(z^{m-1})$ for all $m \in \mathbb{N}$, a point $z \in U$ and $r = \{r_\beta\}_{\beta\in\Omega} \in (0, \infty)$ such that $z^m \in B^{L-\mathcal{J}_{s;\Omega}}(z^0, r)$ $(z^m \in B^{R-\mathcal{J}_{s;\Omega}}(z^0, r)\})$ for all $m \in \mathbb{N}$ and $\lim^{L-\mathcal{J}_{s;\Omega}}_{m\to\infty} z_m = z$ $\big(\lim^{R-\mathcal{J}_{s;\Omega}}_{m\to\infty} z_m = z\big)$.

(II) If (U, T) at a point $z^0 \in U$ is L(R) $\mathcal{J}_{s;\Omega}$-admissible and if $T^{[k]}$, for some $k \in \mathbb{N}$, is L(R) $Q_{s;\Omega}$-quasi-closed map on U then Fix$(T^{[k]})$ is non-empty and there is a sequence $(z^m : m \in \{0\} \cup \mathbb{N})$ starting at $z^0 \in U$ such that $z^m \in T(z^{m-1})$ for all $m \in \mathbb{N}$, a point $z \in$ Fix$(T^{[k]})$ and $r = \{r_\beta\}_{\beta\in\Omega} \in (0, \infty)$ such that $z^m \in B^{L-\mathcal{J}_{s;\Omega}}(z^0, r)(z^m \in B^{R-\mathcal{J}_{s;\Omega}}(z^0, r))$ for all $m \in \mathbb{N}$ and $\lim^{L-\mathcal{J}_{s;\Omega}}_{m\to\infty} z_m = z$ $\big(\lim^{R-\mathcal{J}_{s;\Omega}}_{m\to\infty} z_m = z\big)$.

Proof. (I) Suppose that (U, T) is L(R) $\mathcal{J}_{s;\Omega}$-admissible at a point $z^0 \in U$. From using (3.2.19) and the fact that $J_\beta : U \times U \to [0, \infty)$, $\beta \in \Omega$, we choose

$$r = \{r_\beta\}_{\beta\in\Omega} \in (0, \infty) \tag{3.3.3}$$

$$s = \{s_\beta\}_{\beta\in\Omega} \in [1, \infty) \tag{3.3.4}$$

such that for all $\beta \in \Omega$

$$J_\beta(z^0, T(z^0)) < \frac{(1-\mu_\beta)r_\beta}{s_\beta}. \tag{3.3.5}$$

$$\text{Put} \quad \gamma_\beta^{(0)} = \frac{(1-\mu_\beta)r_\beta}{s_\beta} - J_\beta(z^0, T(z^0)) \quad \text{for all } \beta \in \Omega. \tag{3.3.6}$$

From (3.3.3), (3.3.4) and (3.3.5) we have $\gamma^{(0)} = \{\gamma_\beta^{(0)}\}_{\beta\in\Omega} \in (0,\infty)$ and applying (3.3.1) to get $z^1 \in T(z^{(0)})$ such that

$$J_\beta(z^0, z^1) < J_\beta(z^0, T(z^0)) + \gamma_\beta^{(0)} \quad \text{for all } \beta \in \Omega. \tag{3.3.7}$$

We see from (3.3.6) and (3.3.7) that

$$J_\beta(z^0, z^1) < \frac{(1-\mu_\beta)r_\beta}{s_\beta} \quad \text{for all } \beta \in \Omega. \tag{3.3.8}$$

Observe that (3.3.8) implies $z^1 \in B^{L-\mathcal{J}_{s;\Omega}}(z^0, r)$.
Put now

$$\gamma_\beta^{(1)} = \mu_\beta\Big[\frac{(1-\mu_\beta)r_\beta}{s_\beta^2} - J_\beta(z^0, z^1)\Big] \quad \text{for all } \beta \in \Omega. \tag{3.3.9}$$

From (3.3.8) we have, $\gamma^{(1)} = \{\gamma_\beta^{(1)}\}_{\beta\in\Omega} \in (0,\infty)$ and we apply (3.3.1) to find $z^2 \in T(z^{(1)})$ such that

$$J_\beta(z^1, z^2) < J_\beta(z^1, T(z^1)) + \gamma_\beta^{(1)} \quad \text{for all } \beta \in \Omega. \tag{3.3.10}$$

Also note that

$$J_\beta(z^1, z^2) < \frac{\mu_\beta(1-\mu_\beta)r_\beta}{s_\beta^2} \quad \text{for all } \beta \in \Omega. \tag{3.3.11}$$

Indeed, from (3.3.10), (3.2.19), Definition 3.2.15 and (3.3.9), we get for all $\beta \in \Omega$
$J_\beta(z^1, z^2) < J_\beta(z^1, T(z^1)) + \gamma_\beta^{(1)} \le \sup_{u\in T(z^0)} J_\beta(u, T(z^1)) + \gamma_\beta^{(1)}$
$\le D_{\zeta;\beta}^{L-\mathcal{J}_{s;\Omega}}(T(z^0), T(z^1)) + \gamma_\beta^{(1)} \le \mu_\beta J_\beta(z^0, z^1) + \gamma_\beta^{(1)} = \frac{\mu_\beta(1-\mu_\beta)r_\beta}{s_\beta^2}$, $\zeta \in \{1,2,3\}$.
Thus (3.3.11) holds.
Further, by $(\mathcal{J}_1)$ there exist $s = \{s_\beta\}_{\beta\in\Omega} \in [1,\infty)$ and using (3.3.8) and

(3.3.11), we have for all $\beta \in \Omega$

$$\begin{aligned} J_\beta(z^0, z^2) &\leq s_\beta\{J_\beta(z^0, z^1) + J_\beta(z^1, z^2)\} \\ &< s_\beta\Big\{\frac{(1-\mu_\beta)r_\beta}{s_\beta} + \frac{\mu_\beta(1-\mu_\beta)r_\beta}{s_\beta^2}\Big\} \\ &\leq (1-\mu_\beta)r_\beta(1+\frac{\mu_\beta}{s_\beta}) \\ &\leq (1-\mu_\beta)r_\beta(1+\mu_\beta) \\ &\leq (1-\mu_\beta)r_\beta\sum_{k=0}^{\infty}\mu_\beta^k = r_\beta. \end{aligned}$$

Thus $z^2 \in B^{L-\mathcal{J}_{s;\Omega}}(z^0, r)$.
Repeating the above process, using Definition 3.2.15 and property (3.3.1), we find a sequence $(z^m : m \in \mathbb{N})$ in U satisfies

$$z^{m+1} \in T(z^m) \quad \text{for all} \quad m \in \{0\} \cup \mathbb{N}. \tag{3.3.12}$$

Letting $\gamma^{(m)} = \{\gamma_\beta^{(m)}\}_{\beta\in\Omega}$ for all $m \in \mathbb{N}$, where

$$\gamma_\beta^{(m)} = \mu_\beta\Big[\frac{\mu_\beta^{m-1}(1-\mu_\beta)r_\beta}{s_\beta^{m+1}} - J_\beta(z^{m-1}, z^m)\Big]$$

also, we notice that $\{\gamma^{(m)} \in (0,\infty) : m \in \mathbb{N}\}$ and for all $\beta \in \Omega$ and for all $m \in \{0\} \cup \mathbb{N}$, we have

$$J_\beta(z^m, z^{m+1}) < J_\beta(z^m, T(z^m)) + \gamma^{(m)},$$

$$J_\beta(z^m, z^{m+1}) < \frac{\mu_\beta^m(1-\mu_\beta)r_\beta}{s_\beta^{m+1}}. \tag{3.3.13}$$

For all $\beta \in \Omega$ and for all $m \in \{0\} \cup \mathbb{N}$, we can write

$$\begin{aligned}
J_\beta(z^0, z^{m+1}) &\leq s_\beta J_\beta(z^0, z^1) + s_\beta^2 J_\beta(z^1, z^2) + s_\beta^3 J_\beta(z^2, z^3) \\
&\quad + ... + s_\beta^m J_\beta(z^{m-1}, z^m) + s_\beta^m J_\beta(z^m, z^{m+1}) \\
&< s_\beta \frac{(1-\mu_\beta) r_\beta}{s_\beta} + s_\beta^2 \frac{\mu_\beta (1-\mu_\beta) r_\beta}{s_\beta^2} + s_\beta^3 \frac{\mu_\beta^2 (1-\mu_\beta) r_\beta}{s_\beta^3} \\
&\quad + ... + s_\beta^m \frac{\mu_\beta^{m-1} (1-\mu_\beta) r_\beta}{s_\beta^m} + s_\beta^m \frac{\mu_\beta^m (1-\mu_\beta) r_\beta}{s_\beta^{m+1}} \\
&= (1-\mu_\beta) r_\beta \Big\{ 1 + \mu_\beta + \mu_\beta^2 .... \mu_\beta^{m-1} + \frac{\mu_\beta^m}{s_\beta} \Big\} \\
&\leq (1-\mu_\beta) r_\beta \{ 1 + \mu_\beta + \mu_\beta^2 .... \mu_\beta^{m-1} + \mu_\beta^m \} \\
&= (1-\mu_\beta) r_\beta \sum_{k=0}^{m} \mu_\beta^k \\
&< (1-\mu_\beta) r_\beta \sum_{k=0}^{\infty} \mu_\beta^k = r_\beta.
\end{aligned}$$

Hence this implies that $z^m \in B^{L-\mathcal{J}_{s;\Omega}}(z^0, r)$ for all $m \in \mathbb{N}$.
Applying ($\mathcal{J}1$) and (3.3.13), for all $m, n \in \mathbb{N}$ such that $n > m$, we have

$$\begin{aligned}
\lim_{m\to\infty} \sup_{n>m} J_\beta(z^m, z^n) &\leq \lim_{m\to\infty} \sup_{n>m} \Big\{ s_\beta J_\beta(z^m, z^{m+1}) + s_\beta^2 J_\beta(z^{m+1}, z^{m+2}) \\
&\quad + ... + s_\beta^{n-m-1} J_\beta(z^{n-2}, z^{n-1}) + s_\beta^{n-m-1} J_\beta(z^{n-1}, z^n) \Big\} \\
&\leq \lim_{m\to\infty} \sup_{n>m} \Big\{ s_\beta \frac{\mu_\beta^m (1-\mu_\beta) r_\beta}{s_\beta^{m+1}} + s_\beta^2 \frac{\mu_\beta^{m+1} (1-\mu_\beta) r_\beta}{s_\beta^{m+2}} \\
&\quad + ... + s_\beta^{n-m-1} \frac{\mu_\beta^{n-2} (1-\mu_\beta) r_\beta}{s_\beta^{n-1}} + s_\beta^{n-m-1} \frac{\mu_\beta^{n-1} (1-\mu_\beta) r_\beta}{s_\beta^{n-1+1}} \Big\} \\
&\leq \lim_{m\to\infty} \sup_{n>m} (1-\mu_\beta) r_\beta \Big\{ \frac{\mu_\beta^m}{s_\beta^m} + \frac{\mu_\beta^{m+1}}{s_\beta^m} + + ... \frac{\mu_\beta^{n-2}}{s_\beta^m} + \frac{\mu_\beta^{n-1}}{s_\beta^{m+1}} \Big\} \\
&\leq \lim_{m\to\infty} \sup_{n>m} (1-\mu_\beta) r_\beta \{ \mu_\beta^m + \mu_\beta^{m+1} + ... + \mu_\beta^{n-2} + \mu_\beta^{n-1} \} \\
&= (1-\mu_\beta) r_\beta \lim_{m\to\infty} \sup_{n>m} \sum_{j=m}^{n-1} \mu_\beta^j \\
&\leq r_\beta \lim_{m\to\infty} \mu_\beta^m.
\end{aligned}$$

This implies

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(z^m, z^n) = 0 \quad \text{for all } \beta \in \Omega. \tag{3.3.14}$$

Given that (U,T) is left $\mathcal{J}_{s;\Omega}$-admissible on U, hence using Definition 3.2.12, properties (3.3.12) and (3.3.14) we find $z \in U$ such that

$$\lim_{m\to\infty} J_\beta(z, z^m) = 0 \ \text{ for all } \beta \in \Omega. \tag{3.3.15}$$

Let us decide $v_m = z$ and $u_m = z^m$ for $m \in \mathbb{N}$ and note that conditions (3.2.1) and (3.2.3) hold for sequences (u_m) and (v_m) in U by (3.3.14) and (3.3.15). Consequently, we get (3.2.5) by $(\mathcal{J}2)$ which gives

$$\lim_{m\to\infty} q_\beta(z, z^m) = \lim_{m\to\infty} q_\beta(v_m, u_m) = 0 \ \text{ for all } \beta \in \Omega$$

and thus, we have $z \in S^{L-Q_{s;\Omega}}_{(z_m:m\in\mathbb{N})} = \{x \in U : \lim^{L-Q_{s;\Omega}}_{m\to\infty} z^m = x\}$.

(II) Let (U,T) is left $\mathcal{J}_{s;\Omega}$-admissible at a point $z^0 \in U$ and $T^{[k]}$ is left $Q_{s;\Omega}$-quasi-closed on U, for some $k \in \mathbb{N}$.
Let $z^0 \in U$ be arbitrary and fixed. Since $S^{L-Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$ and for $m \in \{0\}\cup\mathbb{N}$, we have

$$z^{(m+1)k} \in T^{[k]}(z^{mk}),$$

thus defining $(z_m = z^{m-1+k} : m \in \mathbb{N})$, we have

$$(z_m : m \in \mathbb{N}) \subset T^{[k]}(U),$$

$$S^{L-Q_{s;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{L-Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset,$$

also, its subsequences

$$(y_m = z^{(m+1)k}) \subset T^{[k]}(U)$$

and

$$(x_m = z^{mk}) \subset T^{[k]}(U)$$

satisfy

$$y_m = T^{[k]}(x_m) \ \text{ for all } m \in \mathbb{N}$$

and are left $Q_{s;\Omega}$-convergent to each point $z \in S^{L-Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$. Now, since

$$S^{L-Q_{s;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{L-Q_{s;\Omega}}_{(y_m:m\in\mathbb{N})} \qquad \text{and} \qquad S^{L-Q_{s;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{L-Q_{s;\Omega}}_{(x_m:m\in\mathbb{N})}.$$

Using above and the assumption that $T^{[k]}$ for some $k \in \mathbb{N}$, is L(R) $Q_{s;\Omega}$-quasi-closed map on U, there exists $z \in S^{L-Q_{s;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{L-Q_{s;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$, such that $z \in T^{[k]}(z)$. □

Now extending the above theorems to the Banach type single valued L(R)-contractions.

Definition 3.3.2. Let $(U, Q_{s;\Omega})$ be a $Q-b-G$ space, let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ is a L(R) $\mathcal{J}_{s;\Omega}$-family on U and let $\zeta \in \{1,2\}$.

(c) Define on U the L(R) b-distance $\mathcal{D}_\zeta^{L-\mathcal{J}_{s;\Omega}} = \{D_{\zeta;\beta}^{L-\mathcal{J}_{s;\Omega}}, \beta \in \Omega\}(\mathcal{D}_\zeta^{R-\mathcal{J}_{s;\Omega}} = \{D_{\zeta;\beta}^{R-\mathcal{J}_{s;\Omega}}, \beta \in \Omega\})$ on U, where $D_{\zeta;\beta}^{L-\mathcal{J}_{s;\Omega}} : U \times U \to [0,\infty)$, $\beta \in \Omega$ $(D_{\zeta;\beta}^{R-\mathcal{J}_{s;\Omega}} : U \times U \to [0,\infty)$, $\beta \in \Omega)$ for all $a, b \in U$ and $\beta \in \Omega$ as follows:

(c.1) $D_{1;\beta}^{L-\mathcal{J}_{s;\Omega}}(a,b) = \max\{J_\beta(a,b), J_\beta(b,a)\}$,
$D_{2;\beta}^{L-\mathcal{J}_{s;\Omega}}(a,b) = J_\beta(a,b)$, if $\mathcal{J}_{s;\Omega} \in \mathbb{J}^L_{(U,Q_{s;\Omega})}$.

(c.2) $D_{1;\beta}^{R-\mathcal{J}_{s;\Omega}}(a,b) = \max\{J_\beta(a,b), J_\beta(b,a)\}$,
$D_{2;\beta}^{R-\mathcal{J}_{s;\Omega}}(a,b) = J_\beta(a,b)$, if $\mathcal{J}_{s;\Omega} \in \mathbb{J}^R_{(U,Q_{s;\Omega})}$.

(d) Let $\mu = \{\mu_\beta\}_{\beta\in\Omega} \in [0,1)$. A map $T : U \to U$ is $(\mathcal{D}_\zeta^{L-\mathcal{J}_{s;\Omega}}, \mu)$- contraction on U $((\mathcal{D}_\zeta^{R-\mathcal{J}_{s;\Omega}}, \mu)$-contraction on $U)$ if for all $\beta \in \Omega$ and for all $u, v \in U$:

(d.1) $D_{\zeta;\beta}^{L-\mathcal{J}_{s;\Omega}}(T(u), T(v)) \leq \mu_\beta J_\beta(u,v)$, if $\mathcal{J}_{s;\Omega} \in \mathbb{J}^L_{(U,Q_{s;\Omega})}$;

(d.2) $D_{\zeta;\beta}^{R-\mathcal{J}_{s;\Omega}}(T(u), T(v)) \leq \mu_\beta J_\beta(u,v)$, if $\mathcal{J}_{s;\Omega} \in \mathbb{J}^R_{(U,Q_{s;\Omega})}$.

As a result of above Definition 3.3.2 and Theorem 3.3.1, we have now the following result.

Theorem 3.3.3. Let $(U, Q_{s;\Omega})$ is a $Q-b-G$ space, let $\mathcal{J}_{s;\Omega} = \{J_\beta : \beta \in \Omega\}$ is a L(R) $\mathcal{J}_{s;\Omega}$-family on U and let $\zeta \in \{1,2\}$. Moreover, assume that $\mu = \{\mu_\beta\}_{\beta\in\Omega} \in [0,1)$ and $T : U \to U$ be $(\mathcal{D}_\zeta^{L-\mathcal{J}_{s;\Omega}}, \mu)$-contraction on U $((\mathcal{D}_\zeta^{R-\mathcal{J}_{s;\Omega}}, \mu)$-contraction on $U)$.

(I) If (U,T) at a point $z^0 \in U$ is L(R) $\mathcal{J}_{s;\Omega}$-admissible, then there exist a sequence $(z^m : m \in \{0\} \cup \mathbb{N})$ starting at $z^0 \in U$ such that $(z^m = T^{[m]}(z^0) : m \in \{0\} \cup \mathbb{N})$, a point $z \in U$ and $r = \{r_\beta\}_{\beta\in\Omega} \in (0,\infty)$ such that $z^m \in B^{L-\mathcal{J}_{s;\Omega}}(z^0, r)(z^m \in B^{R-\mathcal{J}_{s;\Omega}}(z^0, r))$ for all $m \in \mathbb{N}$ and $\lim_{m\to\infty}^{L-\mathcal{J}_{s;\Omega}} z_m = z$ $\left(\lim_{m\to\infty}^{R-\mathcal{J}_{s;\Omega}} z_m = z\right)$.

(II) If (U,T) at a point $z^0 \in U$ is L(R) $\mathcal{J}_{s;\Omega}$-admissible and if $T^{[k]}$ is L(R) $Q_{s;\Omega}$-quasi-closed map on U, for some $k \in \mathbb{N}$, then $\text{Fix}(T^{[k]})$ is non-empty and there exist a sequence $(z^m : m \in 0 \cup \mathbb{N})$ starting at $z^0 \in U$ such that $(z^m = T^{[m]}(z^0) : m \in \{0\} \cup \mathbb{N})$, a point $z \in \text{Fix}(T^{[k]})$ and $r = \{r_\beta\}_{\beta\in\Omega} \in (0,\infty)$ such that $z^m \in B^{L-\mathcal{J}_{s;\Omega}}(z^0, r)(z^m \in B^{R-\mathcal{J}_{s;\Omega}}(z^0, r))$ for all $m \in \mathbb{N}$, $\lim_{m\to\infty}^{L-\mathcal{J}_{s;\Omega}} z_m = z$ $\left(\lim_{m\to\infty}^{R-\mathcal{J}_{s;\Omega}} z_m = z\right)$ and we have

$$J_\beta(z, T(z)) = J_\beta(T(z), z) = 0, \tag{3.3.16}$$

for all $\beta \in \Omega$ and for all $z \in \mathrm{Fix}(T^{[k]})$.

(III) If $(U, Q_{s;\Omega})$ is a Hausdorff space, if (U, T) at a point $z^0 \in U$ is L(R) $\mathcal{J}_{s;\Omega}$-admissible and if $T^{[k]}$ is L(R) $Q_{s;\Omega}$-quasi-closed map on U, for some $k \in \mathbb{N}$, then there exist a sequence $(z^m : m \in 0 \cup \mathbb{N})$ starting at $z^0 \in U$ such that $(z^m = T^{[m]}(z^0) : m \in \{0\} \cup \mathbb{N})$, a point $z \in \mathrm{Fix}(T^{[k]}) = \mathrm{Fix}(T) = \{z\}$ and $r = \{r_\beta\}_{\beta \in \Omega} \in (0, \infty)$ such that $z^m \in B^{L-\mathcal{J}_{s;\Omega}}(z^0, r)(z^m \in B^{R-\mathcal{J}_{s;\Omega}}(z^0, r))$ for all $m \in \mathbb{N}$, $\lim^{L-\mathcal{J}_{s;\Omega}}_{m\to\infty} z_m = z$ $\left(\lim^{R-\mathcal{J}_{s;\Omega}}_{m\to\infty} z_m = z\right)$ and we have

$$J_\beta(z, z) = 0 \quad \text{for all} \quad \beta \in \Omega. \tag{3.3.17}$$

Proof. We prove only (3.3.16) and (3.3.17) here.
On contrary suppose that there exists $\beta_0 \in \Omega$ and $z \in \mathrm{Fix}(T^{[k]})$ for which $J_{\beta_0}(z, T(z)) > 0$. Indeed, $z = T^{[2k]}(z)$, $T(z) = T^{[2k]}(T(z))$ and for $\zeta \in \{1, 2\}$, by Definition (3.3.2),

$$\begin{aligned}
0 < J_{\beta_0}(z, T(z)) &= J_{\beta_0}(T^{[2k]}(z), T^{[2k]}(T(z))) \\
&\leq D^{L-\mathcal{J}_{s;\Omega}}_{\zeta;\beta_0}(T^{[2k]}(z), T^{[2k]}(T(z))) \\
&\leq \mu_{\beta_0} J_{\beta_0}(T^{[2k-1]}(z), T^{[2k-1]}(T(z))) \\
&\leq \mu_{\beta_0} D^{L-\mathcal{J}_{s;\Omega}}_{\zeta;\beta_0}(T^{[2k-1]}(z), T^{[2k-1]}(T(z))) \\
&\leq \mu^2_{\beta_0} J_{\beta_0}(T^{[2k-2]}(z), T^{[2k-2]}(T(z))) \leq \ldots \\
&\leq \mu^{2k}_{\beta_0} J_{\beta_0}(z, T(z)) < J_{\beta_0}(z, T(z)),
\end{aligned}$$

which is unfeasible.

Now suppose that there exists $\beta_0 \in \Omega$ and $z \in \mathrm{Fix}(T^{[k]})$ such that $J_{\beta_0}(T(z), z) > 0$. Then, Definition (3.3.2) and the fact that $z = T^{[k]}(z) = T^{[2k]}(z)$, implies that for $\zeta \in \{1, 2\}$,

$$\begin{aligned}
0 &< J_{\beta_0}(T(z), z) = J_{\beta_0}(T^{[k+1]}(z), T^{[2k]}(z)) \\
&\leq s_{\beta_0} J_{\beta_0}(T^{[k+1]}(z), T^{[k+2]}(z)) + s^2_{\beta_0} J_{\beta_0}(T^{[k+2]}(z), T^{[k+3]}(z)) \\
&\quad + \ldots + s^{k-2}_{\beta_0} J_{\beta_0}(T^{[2k-1]}(z), T^{[2k]}(z)) \\
&\leq s_{\beta_0} D^{L-\mathcal{J}_{s;\Omega}}_{\zeta;\beta_0}(T^{[k+1]}(z), T^{[k+2]}(z)) + s^2_{\beta_0} D^{L-\mathcal{J}_{s;\Omega}}_{\zeta;\beta_0}(T^{[k+2]}(z), T^{[k+3]}(z)) \\
&\quad + \ldots + s^{k-2}_{\beta_0} D^{L-\mathcal{J}_{s;\Omega}}_{\zeta;\beta_0}(T^{[2k-1]}(z), T^{[2k]}(z)) \\
&\leq s_{\beta_0} \mu^{k+1}_{\beta_0} J_{\beta_0}(z, T(z)) + s^2_{\beta_0} \mu^{k+2}_{\beta_0} J_{\beta_0}(z, T(z)) \\
&\quad + \ldots + s^{k-2}_{\beta_0} \mu^{2k-1}_{\beta_0} J_{\beta_0}(z, T(z)) = 0.
\end{aligned}$$

which is impossible. Thus property (3.3.16) holds.
Next, we show that property (3.3.17) holds.

If the space $(U, Q_{s;\Omega})$ is Hausdorff, then Proposition (3.2.4) and property (3.3.16) suggest that $T(z) = z$, for all $z \in \mathrm{Fix}(T^{[k]})$. Also, for all $z \in \mathrm{Fix}(T^{[k]})$, we have

$$J_\beta(z, z) \leq s_\beta J_\beta(z, T(z)) + s_\beta J_\beta(T(z), z) = 0, \text{ for all } \beta \in \Omega.$$

Thus, $\mathrm{Fix}(T^{[k]}) = \mathrm{Fix}(T)$ and for all $z \in \mathrm{Fix}(T)$, we have
$J_\beta(z, z) = 0$, for all $\beta \in \Omega$.
To prove $\mathrm{Fix}(T)$ is singleton, on contrary let $y, z \in \mathrm{Fix}(T)$ and $y \neq z$. Then, Proposition (3.2.4) implies there exists $\beta_0 \in \Omega$ such that $J_{\beta_0}(y, z) > 0 \vee J_{\beta_0}(z, y) > 0$. Obviously, for $\zeta \in \{1, 2\}$, we then have
$[J_{\beta_0}(y, z) > 0 \wedge J_{\beta_0}(y, z) = J_{\beta_0}(T(y), T(z)) \leq D^{L-\mathcal{J}_{s;\Omega}}_{\zeta;\beta_0}(T(y), T(z))$
$\leq \mu_{\beta_0} J_{\beta_0}(y, z) < J_{\beta_0}(y, z)] \vee [J_{\beta_0}(z, y) > 0 \wedge J_{\beta_0}(z, y)$
$= J_{\beta_0}(T(z), T(y)) \leq D^{L-\mathcal{J}_{s;\Omega}}_{\zeta;\beta_0}(T(z), T(y)) \leq \mu_{\beta_0} J_{\beta_0}(z, y) < J_{\beta_0}(z, y)]$,
which is unfeasible. Hence, we obtain $\mathrm{Fix}(T) = \{z\}$. □

Remark 3.3.4. (i) The proof of right case in above theorems is based on same method.

(ii) The proof of fixed point theorem due to Banach [1] and Nadler [55] require the completeness of the metric spaces (U, q) and $(CB(U), H^q)$, the continuity of q and H^q and the continuity of the mappings T. Our main theorems Theorem 3.3.1 and Theorem 3.3.3 remove these assumptions and leaving the assertion more general. Hence our results are new generalization of the fixed point theorems due to Banach and Nadler.

Example 3.3.5. Let $U = [0, 6]$ and let $Q_{s;\Omega} = \{q\}$ where q is a quasi-pseudo-b-metric on U defined for all $u, v \in U$ by

$$q(u, v) = \begin{cases} 0 & \text{if } u \geq v, \\ (u - v)^2 & \text{if } u < v. \end{cases} \tag{3.3.18}$$

Let $G = [0, 3) \cup (3, 6]$ be a subset of U. Let $\mathcal{J}_{s;\Omega} = \{J\}$ where $J : U \times U \to [0, \infty)$ is defined for all $u, v \in U$ by

$$J(u, v) = \begin{cases} q(u, v) & \text{if } G \cap \{u, v\} = \{u, v\} \\ 40 & \text{if } G \cap \{u, v\} \neq \{u, v\}. \end{cases} \tag{3.3.19}$$

The set-valued map T is defined by

$$T(u) = \begin{cases} [4, 6] & \text{for } u \in [0, 3) \cup (3, 6] \\ [5, 6] & \text{for } u = 3. \end{cases} \tag{3.3.20}$$

(I.1) $\mathcal{J}_{s;\Omega}$ is not symmetric. Indeed, $J(4,0) = 0$ and $J(0,4) = 16$.

(I.2) $(U, Q_{s;\Omega})$ is a $Q-b-G$ space and $\mathcal{J}_{s;\Omega} \in \mathbb{J}^{L}_{(U,Q_{s;\Omega})} \cap \mathbb{J}^{R}_{(U,Q_{s;\Omega})}$. See Example 3.2.2.

(I.3) Using (3.3.18) and Definitions 3.2.11 and 3.2.7(C), the property $T : U \to Cl^{L-Q_{s;\Omega}}(U)$ ($T : U \to Cl^{R-Q_{s;\Omega}}(U)$) holds.

(I.4) $T : U \to Cl^{L-Q_{s;\Omega}}(U)$ is a $(\mathcal{D}_1^{L-\mathcal{J}_{s;\Omega}}, \mu = \frac{1}{10})$-contraction on U, i.e., for all $u, v \in U$
$D_1^{L-\mathcal{J}_{s;\Omega}}(T(u), T(v)) \leq \mu J(u, v)$, where for $A, B \in 2^U$
$D_1^{L-\mathcal{J}_{s;\Omega}}(A, B) = \max\{\sup_{a\in A} J(a, B), \sup_{b\in B} J(A, b)\}$.
Denoting $D_1^{L-\mathcal{J}_{s;\Omega}} = D_1$, we prove this in the following subcases:

(I.4.1) If $u, v \in [0, 3) \cup (3, 6]$, this implies $u, v \in G$, $T(u) = T(v) = [4, 6] = E \subset G$ and by (3.3.18) for all $e \in E$ we have $\inf\{J(e, f) : f \in E\} = J(e, e) = q(e, e) = 0$. Thus, $D_1(T(u), T(v)) = 0 \leq \mu J(u, v)$.

(I.4.2) If $u \in [0, 3) \cup (3, 6]$ and $v = 3$, then $u \in G$, $v \notin G$, $J(u, v) = 40$, $T(u) = [4, 6] = E \subset G$, $T(v) = [5, 6] = F \subset G$ and by (3.3.18), $e \in E$ suggests

$$\inf\{J(e, f) = q(e, f) : f \in F\} = \begin{cases} 4 & \text{whenever } e \in [4, 5] \\ 0 & \text{whenever } e \in [5, 6]. \end{cases}$$

Whereas, $f \in F$ inferred $\inf\{J(e, f) = q(e, f) : e \in E\} = 0$. Thus, $D_1(T(u), T(v)) = 4 = \mu J(u, v)$.

(I.4.3) If $u = 3$ and $v \in [0, 3) \cup (3, 6]$, then $u \notin G$, $v \in G$, $J(u, v) = 40$, $T(u) = [5, 6] = E \subset G$, $T(v) = [4, 6] = F \subset G$. As a result, by (3.3.18), $e \in E$ implies $\inf\{J(e, f) = q(e, f) : f \in F\} = 0$. Further, by (3.3.18) $f \in F$ suggests $\inf\{J(e, f) : e \in E\} = 0$. Thus, $D_1(T(u), T(v)) = 0 \leq \mu J(u, v)$.

(I.4.4) If $u = v = 3$, then $J(u, v) = 40$, $T(u) = T(v) = [5, 6] = E \subset G$ and for all $e \in E$ $\inf\{J(e, f) = q(e, f) : f \in E\} = q(e, e) = 0$. Therefore, $D_1(T(u), T(v)) = 0 < \mu J(u, v)$.

(I.5) To prove that there exists $v \in T(u)$ such that $J(u, v) < J(u, T(u)) + \gamma$, for each $u \in U$ and for all $\gamma \in (0, \infty)$, we observe the following subcases:

(I.5.1) If $u \in [0, 3) \cup (3, 4)$ and $v = 4 \in T(u) = [4, 6]$, then $J(u, v) = q(u, v) = (u - v)^2$, $J(u, T(u)) = (u - v)^2$ and $J(u, v) < J(u, T(u)) + \gamma$ for all $\gamma \in (0, \infty)$.

(I.5.2) If $u \in [4,6]$ and $v = 4 \in T(u) = [4,6]$, then $J(u,v) = q(u,v) = 0$, $J(u,T(u)) = 0$ and $J(u,v) < J(u,T(u)) + \gamma$ for all $\gamma \in (0,\infty)$.

(I.5.3) If $u = 3$ and $v \in T(u) = [5,6]$, then $J(u,v) = J(u,T(u)) = 40$ and $J(u,v) < J(u,T(u)) + \gamma$ for all $\gamma \in (0,\infty)$.

(I.6) The map T is left $\mathcal{J}_{s;\Omega}$-admissible at U (follows from Example 3.2.13).

(I.7) To prove (U,T) is a left $Q_{s;\Omega}$-quasi-closed map in U, suppose $(w_m : m \in \mathbb{N}) \subset T(U)$ is a left $Q_{s;\Omega}$-converging sequence in U. Now, as $[4,6] \subset Cl^{L-Q_{s;\Omega}}(U)$, there exists $w \in T(U) = [4,6]$ such that $\lim_{m\to\infty} q(w,w_m) = 0$.
Equivalently, there exist $w \in T(U) = [4,6]$ and m_0 such that $q(w,w_m) < \epsilon$ for all $\epsilon > 0$ and for all $m \geq m_0$ and thus, by (3.3.19) and (3.3.18), there exist $w \in T(U) = [4,6]$ and $m_1 \geq m_0$ such that $q(w,w_m) = 0 < \epsilon$, for all $0 < \epsilon$ and $m \geq m_1$ or analogously there exist $w \in T(U) = [4,6]$ and m_1 such that $w \geq w_m$ for all $m \geq m_1$. Obviously, then $[w,6] \subset S^{L-Q_{s;\Omega}}_{(w_m:m\in\mathbb{N})}$.
The consideration above implies that if (x_m) and (y_m) are fixed and arbitrary subsequences of $\{w_m : m \in \mathbb{N}\}$ fulfilling $y_m \in T(x_m)$ for all $m \in \mathbb{N}$, then there exists m_1 such that $x_m \in [4,6] \wedge y_m \in T(x_m) \wedge z \geq x_m \wedge z \geq y_m \wedge z \in T(z)$ for all $m \geq m_1$ and for all $z \in [w,6]$.

(I.8) From (I.1)-(I.7), we observe that, in the left case all the hypotheses of Theorem 3.3.1 hold.
Thus, Fix$(T) = [4,6]$, and we declare that if $z^0 \in U$, $z^1 \in T(z^0)$, $z^2 \in T(z^1)$ and $w \in [4,6]$ are fixed and arbitrary and $z^m = w$ for all $m \geq 3$, then the sequence $\{z^m : m \in \{0\} \cup \mathbb{N}\}$ beginning at z^0 and left $Q_{s;\Omega}$-converging to each point z, satisfies $z \in T(z)$.

Remark 3.3.6. Let a $Q-b-G$ space $(U, Q_{s;\Omega})$ and a family $\mathcal{J}_{s;\Omega} = \{J\}$ be as defined in Example 3.3.5.

(a) (3.3.18) implies that q is a quasi-pseudo-b-metric, where $s = 2$ and q is not a quasi-pseudo-metric. Thus $(U, Q_{s;\Omega})$ is a $Q-b-G$ space, but not a quasi-gauge space. Hence a $Q-b-G$ space becomes a more general space than a quasi-gauge space.

(b) From cases I.4.2 and I.4.3, it follows that $4 = D_1^{L-\mathcal{J}_{s;\Omega}}(E,F) \neq D_1^{L-\mathcal{J}_{s;\Omega}}(E,F) = 0$ for $F = [5,6]$ and $E = [4,6]$.

(c) We see that $D_1^{L-\mathcal{J}_{s;\Omega}}(E,E) \neq 0$ if $E = \{3\}$.

3.4 Consequences and application

This section is concerned about some important consequences and application of the results obtained. Following corollaries are some fascinating consequences of the main results.

Corollary 3.4.1. Let (U, Q) is a quasi-gauge space, let $\mathcal{J} = \{J_\beta : \beta \in \Omega\}$ is a L(R) $\mathcal{J}$-family on U and let $\zeta \in \{1, 2, 3\}$. Assume, moreover, that $\mu = \{\mu_\beta\}_{\beta\in\Omega} \in [0, 1)$ and the map $T : U \to Cl^{L-\mathcal{J}}(U)$ $(T : U \to Cl^{R-\mathcal{J}}(U))$ satisfy:

(i) T is $(\mathcal{D}_\zeta^{L-\mathcal{J}}, \mu)$-contraction on U $((\mathcal{D}_\zeta^{R-\mathcal{J}}, \mu)$-contraction on $U)$; and

(ii) for any $u \in U$ and any $\gamma = \{\gamma_\beta\}_{\beta\in\Omega} \in (0, \infty)$, there exists $v \in T(u)$ such that for all $\beta \in \Omega$

$$J_\beta(u, v) < J_\beta(u, T(u)) + \gamma_\beta$$

$$\Big(J_\beta(v, u) < J_\beta(T(u), u) + \gamma_\beta\Big).$$

(I) If (U, T) at a point $z^0 \in U$ is L(R) $\mathcal{J}$-admissible, then there is a sequence $(z^m : m \in \{0\} \cup \mathbb{N})$ starting at $z^0 \in U$ such that $z^m \in T(z^{m-1})$ for all $m \in \mathbb{N}$, a point $z \in U$ and $r = \{r_\beta\}_{\beta\in\Omega} \in (0, \infty)$ such that $z^m \in B^{L-\mathcal{J}}(z^0, r)$ $(z^m \in B^{R-\mathcal{J}}(z^0, r)\})$ for all $m \in \mathbb{N}$ and $\lim^{L-\mathcal{J}}_{m\to\infty} z_m = z$ $\big(\lim^{R-\mathcal{J}}_{m\to\infty} z_m = z\big)$.

(II) If (U, T) at a point $z^0 \in U$ is L(R) $\mathcal{J}$-admissible and if $T^{[k]}$ is L(R) Q-quasi-closed map on U, for some $k \in \mathbb{N}$, then $\mathrm{Fix}(T^{[k]})$ is non-empty and there exist a sequence $(z^m : m \in \{0\} \cup \mathbb{N})$ starting at $z^0 \in U$ such that $z^m \in T(z^{m-1})$ for all $m \in \mathbb{N}$, a point $z \in \mathrm{Fix}(T^{[k]})$ and $r = \{r_\beta\}_{\beta\in\Omega} \in (0, \infty)$ such that $z^m \in B^{L-\mathcal{J}}(z^0, r)(z^m \in B^{R-\mathcal{J}}(z^0, r))$, for all $m \in \mathbb{N}$ and $\lim^{L-\mathcal{J}}_{m\to\infty} z_m = z$ $\big(\lim^{R-\mathcal{J}}_{m\to\infty} z_m = z\big)$.

Proof. The proof follows from the proof of Theorem 3.3.1 by taking $s_\beta = 1$ for each $\beta \in \Omega$.. □

Corollary 3.4.2. Let (U, Q) is a quasi-gauge space, let $\mathcal{J} = \{J_\beta : \beta \in \Omega\}$ is a L(R) $\mathcal{J}$-family on U and let $\zeta \in \{1, 2\}$. Moreover, assume that $\mu = \{\mu_\beta\}_{\beta\in\Omega} \in [0, 1)$ and $T : U \to U$ be $(\mathcal{D}_\zeta^{L-\mathcal{J}}, \mu)$-contraction on U $((\mathcal{D}_\zeta^{R-\mathcal{J}}, \mu)$-contraction on $U)$.

(I) If (U, T) at a point $z^0 \in U$ is L(R) $\mathcal{J}$-admissible, then there is a sequence $(z^m : m \in \{0\} \cup \mathbb{N})$ starting at $z^0 \in U$ such that $(z^m = T^{[m]}(z^0) : m \in \{0\} \cup \mathbb{N})$, a point $z \in U$ and $r = \{r_\beta\}_{\beta\in\Omega} \in (0, \infty)$ such that $z^m \in B^{L-\mathcal{J}}(z^0, r)(z^m \in B^{R-\mathcal{J}}(z^0, r))$ for all $m \in \mathbb{N}$ and $\lim^{L-\mathcal{J}}_{m\to\infty} z_m = z$ $\big(\lim^{R-\mathcal{J}}_{m\to\infty} z_m = z\big)$.

(II) If (U,T) at a point $z^0 \in U$ is L(R) $\mathcal{J}$-admissible and if $T^{[k]}$ is L(R) Q-quasi-closed map on U, for some $k \in \mathbb{N}$, then $\mathrm{Fix}(T^{[k]})$ is non-empty and there exist a sequence $(z^m : m \in \{0\} \cup \mathbb{N})$ starting at $z^0 \in U$ such that $(z^m = T^{[m]}(z^0) : m \in \{0\} \cup \mathbb{N})$, a point $z \in \mathrm{Fix}(T^{[k]})$ and $r = \{r_\beta\}_{\beta\in\Omega} \in (0,\infty)$ such that $z^m \in B^{L-\mathcal{J}}(z^0, r)(z^m \in B^{R-\mathcal{J}}(z^0, r))$ for all $m \in \mathbb{N}$, $\lim^{L-\mathcal{J}}_{m\to\infty} z_m = z$ $\left(\lim^{R-\mathcal{J}}_{m\to\infty} z_m = z\right)$ and we have

$$J_\beta(z, T(z)) = J_\beta(T(z), z) = 0,$$

for all $\beta \in \Omega$ and for all $z \in \mathrm{Fix}(T^{[k]})$.

(III) If (U,Q) is a Hausdorff space, if (U,T) at a point $z^0 \in U$ is L(R) $\mathcal{J}$-admissible and if $T^{[k]}$ is L(R) Q-quasi-closed map on U, for some $k \in \mathbb{N}$, then there is a sequence $(z^m : m \in 0 \cup \mathbb{N})$ starting at $z^0 \in U$ such that $(z^m = T^{[m]}(z^0) : m \in \{0\}\cup\mathbb{N})$, a point $z \in \mathrm{Fix}(T^{[k]}) = \mathrm{Fix}(T) = \{z\}$ and $r = \{r_\beta\}_{\beta\in\Omega} \in (0,\infty)$ such that for all $m \in \mathbb{N}$ $z^m \in B^{L-\mathcal{J}}(z^0, r)(z^m \in B^{R-\mathcal{J}}(z^0, r))$, $\lim^{L-\mathcal{J}}_{m\to\infty} z_m = z$ $\left(\lim^{R-\mathcal{J}}_{m\to\infty} z_m = z\right)$ and we have

$$J_\beta(z, z) = 0 \quad \text{for all} \quad \beta \in \Omega.$$

Proof. The proof easily follows by taking $s_\beta = 1$ for each $\beta \in \Omega$, in the proof of Theorem 3.3.3. □

Remark 3.4.3. We note that Corollary 3.4.2 is Theorem 11.1 of Wlodarczyk and Plebaniak [52]. Hence our Theorem 3.3.3 is generalization of their result.

Now we present an application on the existence of solution of integral equation.
Consider

$$u(t) = f(t) + \eta \int_0^t K(t, s, u(s))ds, \quad t \in [0,\infty) \tag{3.4.1}$$

is the Volterra integral equation located in the space $C[0,\infty)$, i.e., the space of all continuous functions defined on the interval $I = [0,\infty)$, such that $K : I \times I \times \mathbb{R} \to \mathbb{R}$ and $f : I \to \mathbb{R}$ are continuous functions and $\eta \in [0,1)$.
Let $U = (C[0,\infty), \mathbb{R})$, define the quasi-pseudo-b-metric for all $u, v \in U$ by

$$q_m(u,v) = \begin{cases} ||(u-v)||_m & \text{if } u \neq v, \\ 0 & \text{if } u = v. \end{cases} \tag{3.4.2}$$

where $||u||_m = \max_{r\in[0,m]}(u(r))^2$, for all $u \in U$, where $m \in \mathbb{N}$.
Clearly $Q_{s;\mathbb{N}} = \{q_m : m \in \mathbb{N}\}$ is a $Q-b-G$ on U and thus $(U, Q_{s;\mathbb{N}})$ is a $Q-b-G$ space which is complete and Hausdorff. Here in particular we take $Q_{s;\mathbb{N}} = \mathcal{J}_{s;\mathbb{N}}$.

Theorem 3.4.4. Define $T : C[0,\infty) \to C[0,\infty)$ as follows

$$Tu(t) = f(t) + \eta \int_0^t K(t,s,u(s))ds, \quad t \in [0,\infty) \tag{3.4.3}$$

is the integral equation located in the space $C[0,\infty)$, i.e., the space of all continuous functions defined on the interval $[0,\infty)$, where $K : I \times I \times \mathbb{R} \to \mathbb{R}$ and $f : I \to \mathbb{R}$ are continuous functions and $\eta \in [0,1)$.
Suppose that the following statements hold:

(i) for each $t, s \in [0,m]$ and $u, v \in U$, there is a continuous mapping $g : I \times I \to I$ such that $|K(t,s,u(s)) - K(t,s,v(s))| \leq \sqrt{g(t,s)q_m(u,v)}$ for each $m \in \mathbb{N}$;

(ii) $\sup_{t\geq 0} \int_0^t \sqrt{g(t,s)}ds = b < 1$;

(iii) T is $Q_{s;\mathbb{N}}$-quasi-closed map on U.

Then there exist a solution of integral equation (3.4.1).

Proof. For any $u, v \in U$ and $t \in [0,m]$, consider

$$\begin{aligned}
(Tu(t) - Tv(t))^2 &= \Big(f(t) + \eta \int_0^t K(t,s,u(s))ds - (f(t) + \eta \int_0^t K(t,s,v(s))ds)\Big)^2 \\
&= \Big(\eta \int_0^t K(t,s,u(s))ds - \eta \int_0^t K(t,s,v(s))ds\Big)^2 \\
&\leq \eta^2 \Big(\int_0^t |K(t,s,u(s)) - K(t,s,v(s))|ds\Big)^2 \\
&\leq \eta^2 \Big(\int_0^t \sqrt{g(t,s)q_m(u,v)}ds\Big)^2 \\
&\leq \eta^2 \Big(\int_0^t \sqrt{g(t,s)}ds\Big)^2 q_m(u,v) \\
&= \eta^2 b^2 q_m(u,v) \\
&= \mu q_m(u,v), \quad \text{where } \mu = \eta^2 b^2 < 1.
\end{aligned}$$

Hence, for each $u, v \in U$ such that $Tu \neq Tv$ and $m \in \mathbb{N}$, we obtain

$$q_m(Tu,Tv) \leq \mu q_m(u,v) \quad \text{where } \mu < 1. \tag{3.4.4}$$

For $Tu = Tv$, we have $q_m(Tu,Tv) = 0$, so (3.4.4) holds. Hence, by Theorem (3.3.3), the operator T has a fixed point, that is, the integral equation (3.4.1) has at least one solution. □

Chapter 4

Periodic and Fixed Points for Set-valued Mappings in Extended b-Gauge Spaces

Throughout this chapter $(U, Q_{\varphi;\Omega})$ in denoting an extended b-gauge space with underlying nonempty set U enriched with a graph $G = (V, E)$ such that the vertex set $V = U$ and the edge set E contains the diagonal but includes no parallel edge.

This chapter is aimed to introduce the notions of extended b-gauge space $(U, Q_{\varphi;\Omega})$ and extended $\mathcal{J}_{\varphi;\Omega}$-families of generalized extended pseudo-b-distances on U. Moreover, by using these extended $\mathcal{J}_{\varphi;\Omega}$-families on U, we define the extended $\mathcal{J}_{\varphi;\Omega}$-sequential completeness and investigate some periodic and fixed point theorems for set-valued mappings in the novel space equipped with a graph.

This chapter includes five sections. In the first section, we introduce the notion of extended b-gauge space $(U, Q_{\varphi;\Omega})$. In the second section we establish the notion of extended $\mathcal{J}_{\varphi;\Omega}$-families of generalized extended pseudo-b-distances on U. In the third section, we investigate some periodic and fixed point theorems for set-valued mappings in extended b-gauge space equipped with a graph. In the fourth section, in extended b-gauge spaces the periodic points for Caristi type G-contractions are discussed. The last section contains important consequences of the results obtained. A part of this chapter is published as research article [54].

4.1 Extended b-gauge spaces

Recently, gauge spaces have been defined in the locale of b-pseudo metrics by Ali et. al., [45], which are called b-gauge spaces. In this section we introduce the concept of extended b-gauge space. For this we initiate by establishing the definition of a extended pseudo-b-metric space.

Definition 4.1.1. A map $q : U \times U \to [0, \infty)$ is an extended pseudo-b-metric, if for all $u, v, w \in U$, there exists $\varphi : U \times U \to [1, \infty)$ satisfying the following properties:

(a) $q(u, u) = 0$;

(b) $q(u, v) = q(v, u)$; and

(c) $q(u, w) \leq \varphi(u, w)\{q(u, v) + q(v, w)\}$.

The pair (U, q) is called extended pseudo-b-metric space.

Example 4.1.2. Let $U = [0, 1]$. Define $q : U \times U \to [0, \infty)$ and $\varphi : U \times U \to [1, \infty)$ for all $u, v \in U$ as:

$$q(u, v) = (u - v)^2$$

and

$$\varphi(u, v) = u + v + 2.$$

Then q is an extended pseudo-b-metric on U. Indeed, $q(u, u) = 0$ and $q(u, v) = q(v, u)$ for all $u, v \in U$. Further, $q(u, w) \leq \varphi(u, w)\{q(u, v) + q(v, w)\}$ holds for all $u, v, w \in U$.

Example 4.1.3. Let $U = \{u, v, w\}$ and $\varphi : U \times U \to [1, \infty)$ such that $\varphi(u, v) = |u| + |v| + 2$. Define $q : U \times U \to [0, \infty)$ for all $u, v, w \in U$ by:

$$q(u, u) = 0,$$

$$q(u, v) = q(v, u) = 1,$$

$$q(v, w) = q(w, v) = \frac{1}{2},$$

$$\text{and } q(w, u) = q(u, w) = 2,$$

Further, $q(u, w) \leq \varphi(u, w)\{q(u, v) + q(v, w)\}$ holds. Indeed, q is an extended pseudo-b-metric on U. Note that $\frac{3}{2} = q(u, v) + q(v, w) < q(u, w) = 2$; hence q is not a pseudo metric on U. The example proves that extended pseudo-b-metric is generalization of pseudo metric.

Definition 4.1.4. Each family $Q_{\varphi;\Omega} = \{q_\beta : \beta \in \Omega\}$ of extended pseudo-b-metrics $q_\beta : U \times U \to [0, \infty)$, $\beta \in \Omega$, is called an extended b-gauge on U.

Definition 4.1.5. The family $Q_{\varphi;\Omega} = \{q_\beta : \beta \in \Omega\}$ is called to be separating if for every pair (e, f) where $e \neq f$, there exists $q_\beta \in Q_{\varphi;\Omega}$ such that $q_\beta(e, f) \neq 0$.

Definition 4.1.6. Let the family $Q_{\varphi;\Omega} = \{q_\beta : \beta \in \Omega\}$ be an extended b-gauge on U. The topology $\mathcal{T}(Q_{\varphi;\Omega})$ on U whose subbase is defined by the family $\mathcal{B}(Q_{\varphi;\Omega}) = \{B(e, \epsilon_\beta) : e \in U, \epsilon_\beta > 0, \beta \in \Omega\}$ of all balls $B(e, \epsilon_\beta) = \{f \in U : q_\beta(e, f) < \epsilon_\beta\}$, is called the topology induced by $Q_{\varphi;\Omega}$. The topological space $(U, \mathcal{T}(Q_{\varphi;\Omega}))$ is an extended b-gauge space, denoted by $(U, Q_{\varphi;\Omega})$. We note that $(U, Q_{\varphi;\Omega})$ is Hausdorff if $Q_{\varphi;\Omega}$ is separating.

Remark 4.1.7. For s_β=1, for each $\beta \in \Omega$, each gauge space is b_s-gauge space and for $\varphi_\beta(u, v) = s$, for each $\beta \in \Omega$, where $s \geq 1$, each b-gauge space is an extended b-gauge space. Hence, extended b-gauge space is the largest general space.

4.2 Extended $\mathcal{J}_{\varphi;\Omega}$-families of generalized extended pseudo-b-distances

In the following, we establish the idea of extended $\mathcal{J}_{\varphi;\Omega}$-families of generalized extended pseudo-b-distances on U (which are called extended $\mathcal{J}_{\varphi;\Omega}$-families on U, for short). These extended $\mathcal{J}_{\varphi;\Omega}$-families are the generalization of extended b-gauges.

Definition 4.2.1. Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space. The family $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ where $J_\beta : U \times U \to [0, \infty)$, $\beta \in \Omega$, is said to be the extended $\mathcal{J}_{\varphi;\Omega}$-family of generalized extended pseudo-b-distances on U if the following statements hold for all $u, v, w \in U$ and for all $\beta \in \Omega$:

($\mathcal{J}1$) $J_\beta(u, w) \leq \varphi_\beta(u, w)\{J_\beta(u, v) + J_\beta(v, w)\}$;

($\mathcal{J}2$) for each sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in N)$ in U fulfilling

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(u_m, u_n) = 0, \tag{4.2.1}$$

and

$$\lim_{m\to\infty} J_\beta(v_m, u_m) = 0, \tag{4.2.2}$$

the following holds:

$$\lim_{m\to\infty} q_\beta(v_m, u_m) = 0. \tag{4.2.3}$$

We denote
$\mathbb{J}_{(U,Q_{\varphi;\Omega})} = \{\mathcal{J}_{\varphi;\Omega} : \mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}\}$.
Also, we denote

$U^0_{\mathcal{J}_{\varphi;\Omega}} = \{u \in U : \forall_{\beta\in\Omega}\{J_\beta(u,u) = 0\}\}$ and
$U^+_{\mathcal{J}_{\varphi;\Omega}} = \{u \in U : \forall_{\beta\in\Omega}\{J_\beta(u,u) > 0\}\}$.
Then, of course $U = U^0_{\mathcal{J}_{\varphi;\Omega}} \cup U^+_{\mathcal{J}_{\varphi;\Omega}}$.

Example 4.2.2. Let U contains at least two distinct elements and suppose $Q_{\varphi;\Omega} = \{q_\beta : \beta \in \Omega\}$ be the family of extended pseudo-b-metrics, is an extended b-gauge on U. Thus $(U, Q_{\varphi;\Omega})$ is an extended b-gauge space.

Let there are at least two distinct but arbitrary and fixed elements in a set $F \subset U$. Let $a_\beta \in (0,\infty)$ satisfies $\delta_\beta(F) < a_\beta$, where $\delta_\beta(F) = \sup\{q_\beta(e,f) : e,f \in F\}$, for all $\beta \in \Omega$. Let $J_\beta : U \times U \to [0,\infty)$ for all $e,f \in U$ is define as:

$$J_\beta(e,f) = \begin{cases} q_\beta(e,f) & \text{if } F \cap \{e,f\} = \{e,f\}, \\ a_\beta & \text{if } F \cap \{e,f\} \neq \{e,f\}. \end{cases} \tag{4.2.4}$$

Then $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\} \in \mathbb{J}_{(U,Q)}$.
We observe that $J_\beta(e,g) \le \varphi_\beta(e,g)\{J_\beta(e,f) + J_\beta(f,g)\}$, for all $e,f,g \in U$, thus condition $(\mathcal{J}_1)$ holds. Indeed, condition $(\mathcal{J}_1)$ will not hold in case if there is some $e,f,g \in U$ such that $J_\beta(e,g) = a_\beta$, $J_\beta(e,f) = q_\beta(e,f)$, $J_\beta(f,g) = q_\beta(f,g)$ and $\varphi_\beta(e,g)\{q_\beta(e,f) + q_\beta(f,g)\} \le a_\beta$. However, then this implies the existence of $h \in \{e,g\}$ with $h \notin F$ and on other hand, $e,f,g \in F$, which is impossible.
Now suppose that (4.2.1) and (4.2.2) are satisfied by the sequences (u_m) and (v_m) in U. Then (4.2.2) yields that for all $0 < \epsilon < a_\beta$, there exists $m_1 = m_1(\beta) \in \mathbb{N}$ such that

$$J_\beta(v_m, u_m) < \epsilon \text{ for all } m \ge m_1, \text{ for all } \beta \in \Omega. \tag{4.2.5}$$

By (4.2.5) and (4.2.4), denoting $m_2 = \min\{m_1(\beta) : \beta \in \Omega\}$, we have

$$F \cap \{v_m, u_m\} = \{v_m, u_m\}, \text{ for all } m \ge m_2$$

and

$$q_\beta(v_m, u_m) = J_\beta(v_m, u_m) < \epsilon.$$

Thus (4.2.3) is satisfied. Therefore, $\mathcal{J}_{\varphi;\Omega}$ is a $\mathcal{J}_{\varphi;\Omega}$-family on U.

We now state few trivial characteristics of extended $\mathcal{J}_{\varphi;\Omega}$-families on U.

Remark 4.2.3. Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space. Then the following hold:

(a) $Q_{\varphi;\Omega} \in \mathbb{J}_{(U,Q_{\varphi;\Omega})}$.

(b) Let $\mathcal{J}_{\varphi;\Omega} \in \mathbb{J}_{(U,Q_{\varphi;\Omega})}$. If $J_\beta(v,v) = 0$ and $J_\beta(u,v) = J_\beta(v,u)$ for all $\beta \in \Omega$ and for all $u, v \in U$ then for each $\beta \in \Omega$, J_β is an extended pseudo-b-metric.

(c) There exists examples of $\mathcal{J}_{\varphi;\Omega} \in \mathbb{J}_{(U,Q_{\varphi;\Omega})}$ which shows that the maps J_β, $\beta \in \Omega$ are not extended pseudo-b-metrics.

Proposition 4.2.4. Let $(U, Q_{\varphi;\Omega})$ is a Hausdorff extended b-gauge space and the family $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U. Then for each $e, f \in U$, there exists $\beta \in \Omega$ such that

$$e \neq f \Rightarrow J_\beta(e,f) > 0 \vee J_\beta(f,e) > 0.$$

Proof. : Let there are $e, f \in U$ where $e \neq f$ such that $J_\beta(e,f) = 0 = J_\beta(f,e)$ for all $\beta \in \Omega$. Then by using property $(\mathcal{J}1)$ we have $J_\beta(e,e) = 0$, for all $\beta \in \Omega$. Defining sequences (u_m) and (v_m) in U by $u_m = f$ and $v_m = e$, we see that conditions (4.2.1) and (4.2.2) of property $(\mathcal{J}2)$ are satisfied and therefore condition (4.2.3) holds, which implies that $q_\beta(e,f) = 0$, for all $\beta \in \Omega$. But, this denies the fact that $(U, Q_{\varphi;\Omega})$ is a Hausdorff extended b-gauge space. Therefore, our supposition is wrong and there exists $\beta \in \Omega$ such that for all $e, f \in U$

$$e \neq f \Rightarrow J_\beta(e,f) > 0 \vee J_\beta(f,e) > 0.$$

□

We now define extended $\mathcal{J}_{\varphi;\Omega}$-completeness in the extended b-gauge space $(U, Q_{\varphi;\Omega})$, using extended $\mathcal{J}_{\varphi;\Omega}$-families on U.

Definition 4.2.5. Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space and $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U.

(A) A sequence $(v_m : m \in \mathbb{N})$ is extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U if

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(v_m, v_n) = 0, \quad \text{for all } \beta \in \Omega.$$

(B) The sequence $(v_m : m \in \mathbb{N})$ is called to be extended $\mathcal{J}_{\varphi;\Omega}$-convergent to $v \in U$ if $\lim^{\mathcal{J}_{\varphi;\Omega}}_{m\to\infty} v_m = v$, where $\lim^{\mathcal{J}_{\varphi;\Omega}}_{m\to\infty} v_m = v \Leftrightarrow \lim_{m\to\infty} J_\beta(v, v_m) = 0 = \lim_{m\to\infty} J_\beta(v_m, v)$, for all $\beta \in \Omega$.

(C) If $S^{\mathcal{J}_{\varphi;\Omega}}_{(v_m:m\in\mathbb{N})} \neq \emptyset$, where $S^{\mathcal{J}_{\varphi;\Omega}}_{(v_m:m\in\mathbb{N})} = \{v \in U : \lim^{\mathcal{J}_{\varphi;\Omega}}_{m\to\infty} v_m = v\}$.
Then $(v_m : m \in \mathbb{N})$ in U is extended $\mathcal{J}_{\varphi;\Omega}$-convergent sequence in U.

(D) The space $(U, Q_{\varphi;\Omega})$ is called extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete, if every extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy in U is an extended $\mathcal{J}_{\varphi;\Omega}$-convergent in U.

Remark 4.2.6. There exist examples of extended b-gauge space $(U, Q_{\varphi;\Omega})$ and $\mathcal{J}_{\varphi;\Omega}$-family on U with $\mathcal{J}_{\varphi;\Omega} \neq Q_{\varphi;\Omega}$ such that $(U, Q_{\varphi;\Omega})$ is $\mathcal{J}_{\varphi;\Omega}$-sequential complete but not $Q_{\varphi;\Omega}$-sequential complete (see Example 2.1.11).

Definition 4.2.7. Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space and let $T : U \to 2^U$ is a multi-valued map. The map $T^{[k]}$ is said to be extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$, if for each sequence $(w_m : m \in \mathbb{N})$ in $T^{[k]}(U)$, which is extended $Q_{\varphi;\Omega}$-converging in U, thus $S^{Q_{\varphi;\Omega}}_{(w_m:m\in\mathbb{N})} \neq \emptyset$ and its subsequences (y_m) and (z_m) satisfy
$y_m \in T^{[k]}(z_m)$, for all $m \in \mathbb{N}$
has the property that there exists $w \in S^{Q_{\varphi;\Omega}}_{(w_m:m\in\mathbb{N})}$ such that $w \in T^{[k]}(w)$.

Definition 4.2.8. Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space, let $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U. A set $Y \in 2^U$ is $\mathcal{J}_{\varphi;\Omega}$-closed in U if $Y = cl^{\mathcal{J}_{\varphi;\Omega}}_U(Y)$, where $cl^{\mathcal{J}_{\varphi;\Omega}}_U(Y)$, is the $\mathcal{J}_{\varphi;\Omega}$-closure in U, indicates the set of all $u \in U$ for which there is a sequence $(u_m : m \in \mathbb{N})$ in Y such that it $\mathcal{J}_{\varphi;\Omega}$-converges to u.
Define $Cl^{\mathcal{J}_{\varphi;\Omega}}(U) = \{Y \in 2^U : Y = cl^{\mathcal{J}_{\varphi;\Omega}}_U(Y)\}$. Thus $Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ denotes the class of all $\mathcal{J}_{\varphi;\Omega}$-closed subsets of U.

Definition 4.2.9. Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space, let $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U and let for all $\beta \in \Omega$, for all $e \in U$ and for all $F \in 2^U$

$$J_\beta(e, F) = \inf\{J_\beta(e, g) : g \in F\}.$$

Define on $Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ the distance $D^{\mathcal{J}_{\varphi;\Omega}}_\beta$ of Hausdorff type for all $\beta \in \Omega$ and for all $E, F \in Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$, where $D^{\mathcal{J}_{\varphi;\Omega}}_\beta : Cl^{\mathcal{J}_{\varphi;\Omega}}(U) \times Cl^{\mathcal{J}_{\varphi;\Omega}}(U) \to [0, \infty)$, $\beta \in \Omega$ as follows:

$$D^{\mathcal{J}_{\varphi;\Omega}}_\beta(E, F) = \begin{cases} \max\{\sup_{e\in E} J_\beta(e, F), \sup_{f\in F} J_\beta(f, E)\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise.} \end{cases}$$

4.3 Periodic and fixed point theorems in extended b-gauge spaces endowed with a graph

In this section, we prove a couple of theorems in extended b-gauge spaces equipped with a graph. For each theorem and corollary of this chapter ahead

it is assumed that $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space and $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \to [0, \infty)$, be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U such that $(U, Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete.
We first proof the following lemma.

Lemma 4.3.1. Let $(U, Q_{\varphi;\Omega})$ be an extended b-gauge space and let $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \to [0, \infty)$, be the extended $\mathcal{J}_{\varphi;\Omega}$-family on U. Then

$$J_\beta(u, A) \leq \varphi_\beta(u, A)\{J_\beta(u, v) + J_\beta(v, A)\},$$

for all $\beta \in \Omega$, for all $u, v \in U$ and for all $A \subset U$, where

$$\varphi_\beta(u, A) = \inf\{\varphi_\beta(u, a) : a \in A\}.$$

Proof. From axioms of definition, we can write for all $\beta \in \Omega$

$$J_\beta(u, a) \leq \varphi_\beta(u, a)\{J_\beta(u, v) + J_\beta(v, a)\} \quad \text{for all } u, v, a \in U$$

$$J_\beta(u, a) \leq \varphi_\beta(u, a)J_\beta(u, v) + \varphi_\beta(u, a)J_\beta(v, a).$$

By taking infimum of both sides over A, we get for all $\beta \in \Omega$

$$\inf_{a\in A} J_\beta(u, a) \leq \inf_{a\in A} \varphi_\beta(u, a)J_\beta(u, v) + \inf_{a\in A} \varphi_\beta(u, a) \inf_{a\in A} J_\beta(v, a)$$

$$J_\beta(u, A) \leq \varphi_\beta(u, A)J_\beta(u, v) + \varphi_\beta(u, A)J_\beta(v, A)$$
$$J_\beta(u, A) \leq \varphi_\beta(u, A)\{J_\beta(u, v) + J_\beta(v, A)\}.$$

□

Our main results for set-valued G-contraction in the new setting of extended b-gauge space equipped with the graph are now given below.

Theorem 4.3.2. Let the set-valued map $T : U \to Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ and $\varphi_\beta : U \times U \to [1, \infty)$ for each $\beta \in \Omega$ satisfy:

$$D_\beta^{\mathcal{J}_{\varphi;\Omega}}(Tu, Tv) \leq a_\beta J_\beta(u, v)+b_\beta J_\beta(u, Tu)+c_\beta J_\beta(v, Tv)+e_\beta J_\beta(u, Tv)+L_\beta J_\beta(v, Tu) \quad (4.3.1)$$

for all $(u, v) \in E$, where $a_\beta, b_\beta, c_\beta, e_\beta, L_\beta \geq 0$ be such that $a_\beta + b_\beta + c_\beta + 2e_\beta\varphi_\beta(z^{m-1}, Tz^m) < 1$ and $\lim_{m,n\to\infty} \varphi_\beta(z^m, z^n)\mu_\beta < 1$, for some $\mu_\beta < 1$ and each $z^0 \in U$, here $z^m \in T(z^{m-1})$, for $m \in \mathbb{N}$.
Moreover, let that

(i) there exist $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$;

(ii) if $(u,v) \in E$ and $x \in Tu$ and $y \in Tv$ such that $J_\beta(x,y) \leq J_\beta(u,v)$, for all $\beta \in \Omega$ then $(x,y) \in E$;

(iii) for any $\{r_\beta : r_\beta > 1\}_{\beta \in \Omega}$ and $u \in U$ there exists $v \in Tu$ such that

$$J_\beta(u,v) \leq r_\beta J_\beta(u,Tu), \text{ for all } \beta \in \Omega.$$

Then the following statements hold:

(I) For any $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is extended $Q_{\varphi;\Omega}$-convergent sequence in U, thus $S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})} \neq \emptyset$.

(II) Moreover, suppose that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$. Then

(a_1) $\mathrm{Fix}(T^{[k]}) \neq \emptyset$; and

(a_2) there exists $z \in \mathrm{Fix}(T^{[k]})$ such that $z \in S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})}$.

Proof. (I) We first show that $(z^m : m \in \{0\} \cup \mathbb{N})$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U.
Using supposition (i) there exists $z^0, z^1 \in U$ such that $z^1 \in Tz^0$ and $(z^0, z^1) \in E$. Now for each $\beta \in \Omega$, applying (4.3.1) we have

$$\begin{aligned} D^{\mathcal{J}_{\varphi;\Omega}}_\beta(Tz^0, Tz^1) \leq\ & a_\beta J_\beta(z^0,z^1) + b_\beta J_\beta(z^0,Tz^0) + c_\beta J_\beta(z^1,Tz^1) + e_\beta J_\beta(z^0,Tz^1) \\ & + L_\beta J_\beta(z^1,Tz^0). \end{aligned} \tag{4.3.2}$$

Now as $J_\beta(z^1,Tz^1) \leq D^{\mathcal{J}_{\varphi;\Omega}}_\beta(Tz^0,Tz^1)$ and $J_\beta(z^0,Tz^1) \leq \varphi_\beta(z^0,Tz^1)\{J_\beta(z^0,z^1) + J_\beta(z^1,Tz^1)\}$, therefore (4.3.2) implies

$$J_\beta(z^1,Tz^1) \leq \frac{1}{\zeta_\beta} J_\beta(z^0,z^1), \tag{4.3.3}$$

where $\zeta_\beta = \frac{1-c_\beta - e_\beta\varphi(z^0,Tz^1)}{a_\beta + b_\beta + e_\beta\varphi(z^0,Tz^1)} > 1$. Now using assumption (iii), we have $z^2 \in Tz^1$ such that

$$J_\beta(z^1,z^2) \leq \sqrt{\zeta_\beta} J_\beta(z^1,Tz^1). \tag{4.3.4}$$

Combining (4.3.3) and (4.3.4), we can write

$$J_\beta(z^1,z^2) \leq \frac{1}{\sqrt{\zeta_\beta}} J_\beta(z^0,z^1), \quad \forall\, \beta \in \Omega. \tag{4.3.5}$$

Assumption (ii) and (4.3.5) implies that $(z^1, z^2) \in E$. Progressing in the same manner, we find a sequence $(z^m : m \in \{0\} \cup \mathbb{N})$ in U such that $(z^m, z^{m+1}) \in E$ and

$$J_\beta(z^m, z^{m+1}) \leq \big(\frac{1}{\sqrt{\zeta_\beta}}\big)^m J_\beta(z^0, z^1), \quad \forall\, \beta \in \Omega. \tag{4.3.6}$$

For convenience let $\mu_\beta = \frac{1}{\sqrt{\zeta_\beta}}$, for each $\beta \in \Omega$.

Now by repeated use of $(\mathcal{J}1)$ and (4.3.6) for all $\beta \in \Omega$ and for all $m, n \in \mathbb{N}$ such that $n > m$, we get

$$\begin{aligned} J_\beta(z^m, z^n) &\leq \varphi_\beta(z^m, z^n)\mu_\beta^m J_\beta(z^0, z^1) + \varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n)\mu_\beta^{m+1} J_\beta(z^0, z^1) \\ &\quad + \varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n)\varphi_\beta(z^{m+2}, z^n)\mu_\beta^{m+2} J_\beta(z^0, z^1) \\ &\quad + \ldots + \varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n)\ldots\varphi_\beta(z^{n-1}, z^n)\mu_\beta^{n-1} J_\beta(z^0, z^1) \\ &\leq J_\beta(z^0, z^1)[\varphi_\beta(z^1, z^n)\varphi_\beta(z^2, z^n)\ldots\varphi_\beta(z^m, z^n)\mu_\beta^m \\ &\quad + \varphi_\beta(z^1, z^n)\varphi_\beta(z^2, z^n)\ldots\varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n)\mu_\beta^{m+1} \\ &\quad + \ldots + \varphi_\beta(z^1, z^n)\varphi_\beta(z^2, z^n)\ldots\varphi_\beta(z^m, z^n)\ldots\varphi_\beta(z^{n-1}, z^n)\mu_\beta^{n-1}]. \end{aligned}$$

Since $\lim_{n,m\to\infty} \varphi_\beta(z^{m+1}, z^n)\mu_\beta < 1$, for some $\mu_\beta < 1$, so the series $\sum_{m=1}^{\infty} \mu_\beta^m \prod_{i=1}^{m} \varphi_\beta(z^i, z^n)$ converges by ratio test. Let $S = \sum_{m=1}^{\infty} \mu_\beta^m \prod_{i=1}^{m} \varphi_\beta(z^i, z^n)$ and $S_m = \sum_{j=1}^{m} \mu_\beta^j \prod_{i=1}^{j} \varphi_\beta(z^i, z^n)$. This gives

$$J_\beta(z^m, z^n) \leq J_\beta(z^0, z^1)[S_{n-1} - S_m].$$

This implies

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(z^m, z^n) = 0 \ \text{ for all } \beta \in \Omega. \tag{4.3.7}$$

Now, since $(U, Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete b-gauge space, so $(z^m : m \in \{0\} \cup \mathbb{N})$ is extended $\mathcal{J}_{\varphi;\Omega}$ convergent in U, thus for all $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup\mathbb{N})}$, we have

$$\lim_{m\to\infty} J_\beta(z, z^m) = 0 \ \text{ for all } \beta \in \Omega. \tag{4.3.8}$$

Thus from (4.3.7) and (4.3.8), fixing $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z_m : m\in\{0\}\cup\mathbb{N})}$, defining $(u_m = z^m : m \in \{0\} \cup \mathbb{N})$ and $(v_m = z : m \in \{0\} \cup \mathbb{N})$ and applying $(\mathcal{J}2)$ to these sequences, we get

$$\lim_{m\to\infty} q_\beta(z, z^m) = \lim_{m\to\infty} q_\beta(v_m, u_m) = 0 \ \text{ for all } \beta \in \Omega$$

This implies $S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) To show (a_1), let $z^0 \in U$ is fixed and arbitrary. Since $S^{Q_{\varphi;\Omega}}_{(z^m : m\in\{0\}\cup\mathbb{N})} \neq \emptyset$ and

$$z^{(m+1)k} \in T^{[k]}(z^{mk}), \quad \text{for } m \in \{0\} \cup \mathbb{N}$$

thus describing $(z_m = z^{m-1+k} : m \in \mathbb{N})$, we get

$$(z_m : m \in \mathbb{N}) \subset T^{[k]}(U),$$

$$S^{Q_{\varphi;\Omega}}_{(z_m: m\in\{0\}\cup\mathbb{N})} = S^{Q_{\varphi;\Omega}}_{(z^m: m\in\{0\}\cup\mathbb{N})} \neq \emptyset,$$

also, its subsequences

$$(y_m = z^{(m+1)k}) \subset T^{[k]}(U)$$

and

$$(x_m = z^{mk}) \subset T^{[k]}(U)$$

satisfy

$$y_m \in T^{[k]}(x_m) \text{ for all } m \in \mathbb{N}$$

and are extended $Q_{\varphi;\Omega}$-convergent to each point $z \in S^{Q_{\varphi;\Omega}}_{(z^m: m\in\{0\}\cup\mathbb{N})}$. Thus, applying the fact below

$$S^{Q_{\varphi;\Omega}}_{(z_m: m\in\mathbb{N})} \subset S^{Q_{\varphi;\Omega}}_{(y_m: m\in\mathbb{N})} \quad \text{and} \quad S^{Q_{\varphi;\Omega}}_{(z_m: m\in\mathbb{N})} \subset S^{Q_{\varphi;\Omega}}_{(x_m: m\in\mathbb{N})}$$

and the assumption that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$, we have there exists $z \in S^{Q_{\varphi;\Omega}}_{(z_m: m\in\{0\}\cup\mathbb{N})} = S^{Q_{\varphi;\Omega}}_{(z^m: m\in\{0\}\cup\mathbb{N})}$ such that $z \in T^{[k]}(z)$.
Thus, (a_1) holds.
The statement (a_2) follows from (a_1) and the certainty that $S^{Q_{\varphi;\Omega}}_{(z^m: m\in\{0\}\cup\mathbb{N})} \neq \emptyset$. □

Let us consider $T : U \to U$, we get the following result in single-valued mapping.

Theorem 4.3.3. Let the single-valued map $T : U \to U$ and $\varphi_\beta : U \times U \to [1, \infty)$ for each $\beta \in \Omega$ satisfy:

$$J_\beta(Tu, Tv) \leq a_\beta J_\beta(u, v) + b_\beta J_\beta(u, Tu) + c_\beta J_\beta(v, Tv) + e_\beta J_\beta(u, Tv) + L_\beta J_\beta(v, Tu) \tag{4.3.9}$$

for all $(u, v) \in E$, where $a_\beta, b_\beta, c_\beta, e_\beta, L_\beta \geq 0$ be such that $a_\beta + b_\beta + c_\beta + 2e_\beta\varphi_\beta(z^{m-1}, Tz^m) < 1$ and $\lim_{m,n\to\infty} \varphi_\beta(z^m, z^n)\mu_\beta < 1$, for some $\mu_\beta < 1$ and each $z^0 \in U$, here $z^m = T^{[m]}(z^0)$, where $m \in \mathbb{N}$.
Moreover, let that

(i) there exist $z^0 \in U$ such that $(z^0, Tz^0) \in E$;

(ii) for $(u, v) \in E$ we have $(Tu, Tv) \in E$, given that $J_\beta(Tu, Tv) \leq J_\beta(u, v)$ for all $\beta \in \Omega$;

(iii) if a sequence $(z^m : m \in \mathbb{N})$ in U is such that $(z^m, z^{m+1}) \in E$ and $\lim_{m\to\infty}^{\mathcal{J}_{\varphi;\Omega}} z^m = z$, then $(z^m, z) \in E$ and $(z, z^m) \in E$.

Then the following assertions hold:

(I) For each $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is extended $Q_{\varphi;\Omega}$-convergent sequence in U, thus, $S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) Moreover, suppose that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$ and $\varphi_\beta(z, Tz)\{c_\beta + e_\beta \lim_{m\to\infty} \varphi_\beta(z^m, Tz)\} < 1$. Then

(a$_1$) $\mathrm{Fix}(T^{[k]}) \neq \emptyset$;

(a$_2$) there is $z \in \mathrm{Fix}(T^{[k]})$ such that $z \in S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\}\cup\mathbb{N})}$; and

(a$_3$) for all $z \in \mathrm{Fix}(T^{[k]})$, $J_\beta(z, T(z)) = J_\beta(T(z), z) = 0$, for all $\beta \in \Omega$.

(III) Furthermore, let $\mathrm{Fix}(T^{[k]}) \neq \emptyset$, for some $k \in \mathbb{N}$ and $(U, Q_{\varphi;\Omega})$ is a Hausdorff space. Then

(b$_1$) $\mathrm{Fix}(T^{[k]})$=$\mathrm{Fix}(T)$;

(b$_2$) there is $z \in \mathrm{Fix}(T)$ such that $z \in S^{L-Q_{\varphi;\Omega}}_{(z^m : m \in \{0\}\cup\mathbb{N})}$; and

(b$_3$) for all $z \in \mathrm{Fix}(T)$, $J_\beta(z, z) = 0$, for all $\beta \in \Omega$.

Proof. In view of Theorem 4.3.2, it remains to prove assertions (a_3) and (b_1)-(b_3) of the above stated theorem.
To prove (a_3), on contrary suppose that $J_{\beta_0}(z, Tz) > 0$ for some $\beta_0 \in \Omega$. Use $\mathcal{J}1$, assumption (iii) and inequality (4.3.9), we can write

$$\begin{aligned}
J_{\beta_0}(z, Tz) &\leq \varphi_{\beta_0}(z, Tz)\{J_{\beta_0}(z, z^{m+1}) + J_{\beta_0}(z^{m+1}, Tz)\} \\
&= \varphi_{\beta_0}(z, Tz)\{J_{\beta_0}(z, z^{m+1}) + J_{\beta_0}(Tz^m, Tz)\} \\
&\leq \varphi_{\beta_0}(z, Tz)\{J_{\beta_0}(z, z^{m+1}) + a_{\beta_0} J_{\beta_0}(z^m, z) + b_{\beta_0} J_{\beta_0}(z^m, Tz^m) + c_{\beta_0} J_{\beta_0}(z, Tz) \\
&\quad + e_{\beta_0} J_{\beta_0}(z^m, Tz) + L_{\beta_0} J_{\beta_0}(z, Tz^m)\} \\
&\leq \varphi_{\beta_0}(z, Tz)\{J_{\beta_0}(z, z^{m+1}) + a_{\beta_0} J_{\beta_0}(z^m, z) + b_{\beta_0} J_{\beta_0}(z^m, z^{m+1}) + c_{\beta_0} J_{\beta_0}(z, Tz) \\
&\quad + e_{\beta_0} \varphi_{\beta_0}(z^m, Tz)\{J_{\beta_0}(z^m, z) + J_{\beta_0}(z, Tz)\} + L_{\beta_0} J_{\beta_0}(z, z^{m+1})\}.
\end{aligned}$$

Letting $m \to \infty$, since $\lim_{m,n\to\infty} \varphi_\beta(z^m, z^n)\mu_\beta < 1$, for some $\mu_\beta < 1$ and for each $z^m, z^n \in U$, $\varphi_\beta(z^m, z^n)$ is finite and thus we obtain

$$J_{\beta_0}(z, Tz) \leq \varphi_{\beta_0}(z, Tz)\{c_{\beta_0} + e_{\beta_0} \lim_{m\to\infty} \varphi_{\beta_0}(z^m, Tz)\} J_{\beta_0}(z, Tz).$$

Now since $\varphi_\beta(z, Tz)\{c_\beta + e_\beta \lim_{m\to\infty} \varphi_\beta(z^m, Tz)\} < 1$, we get

$$J_{\beta_0}(z, Tz) \leq \varphi_{\beta_0}(z, Tz)\{c_{\beta_0} + e_{\beta_0} \lim_{m\to\infty} \varphi_{\beta_0}(z^m, Tz)\} J_{\beta_0}(z, Tz) < J_{\beta_0}(z, Tz).$$

Which is absurd, thus $J_\beta(z, Tz) = 0$, for all $\beta \in \Omega$.
Next, we prove that $J_\beta(Tz, z) = 0$, for all $\beta \in \Omega$.

$$\begin{aligned} J_\beta(Tz, z) &\leq \varphi_\beta(Tz, z)\{J_\beta(Tz, z^{m+1}) + J_\beta(z^{m+1}, z)\} \\ &= \varphi_\beta(Tz, z)\{J_\beta(Tz, Tz^m) + J_\beta(z^{m+1}, z)\} \\ &\leq \varphi_\beta(Tz, z)\{a_\beta J_\beta(z, z^m) + b_\beta J_\beta(z, Tz) + c_\beta J_\beta(z^m, Tz^m) + e_\beta J_\beta(z, Tz^m) \\ &\quad + L_\beta J_\beta(z^m, Tz) + J_\beta(z^{m+1}, z)\} \\ &\leq \varphi_\beta(Tz, z)\{a_\beta J_\beta(z, z^m) + b_\beta J_\beta(z, Tz) + c_\beta J_\beta(z^m, z^{m+1}) + e_\beta J_\beta(z, z^{m+1}) \\ &\quad + L_\beta \varphi_\beta(z^m, Tz)\{J_\beta(z^m, z) + J_\beta(z, Tz)\} + J_\beta(z^{m+1}, z)\}. \end{aligned}$$

Letting $m \to \infty$, since $\lim_{m,n\to\infty} \varphi_\beta(z^m, z^n)\mu_\beta < 1$, for some $\mu_\beta < 1$ and for each $z^m, z^n \in U$, $\varphi_\beta(z^m, z^n)$ is finite and thus we have

$$J_\beta(Tz, z) \leq \varphi_\beta(Tz, z)\{b_\beta + L_\beta \lim_{m\to\infty} \varphi_\beta(z^m, Tz)\}J_\beta(z, Tz) \qquad \forall\ \beta \in \Omega.$$

Also, since we have proved that $J_\beta(z, Tz) = 0$ for all $\beta \in \Omega$, thus we obtain $J_\beta(Tz, z) = 0$ for all $\beta \in \Omega$. Hence assertion (a_3) holds.
(III) Since $(U, Q_{\varphi;\Omega})$ is a Hausdorff space, using Proposition (4.2.4), assertion (a_3) suggests that for $z \in \text{Fix}(T^{[k]})$, we have $z = T(z)$. This gives $z \in \text{Fix}(T)$. Hence (b_1) is true.
Assertions (a_2) and (b_1) imply (b_2) .
To prove assertion (b_3), consider $(\mathcal{J}1)$ and use (a_3) and (b_1), we have for all $z \in \text{Fix}(T) = \text{Fix}(T^{[k]})$ and for all $\beta \in \Omega$

$$J_\beta(z, z) \leq \varphi(z, z)\{J_\beta(z, T(z)) + J_\beta(T(z), z)\} = 0.$$

□

Before moving ahead to the next results we first define the family Ψ_φ of mappings $\psi : [0, \infty) \to [0, \infty)$ which are non-decreasing and satisfy the following conditions:

(i) $\psi(0) = 0$;

(ii) $\psi(\eta t) = \eta\psi(t) < \eta t$ for each $\eta, t > 0$,

(iii) $\sum_{i=1}^{\infty} r^i \psi^i(t) \prod_{m=1}^{i} \varphi(z^m, z^n) < \infty$
where $r \geq 1$.

Theorem 4.3.4. Let the set-valued map $T : U \to Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ and $\varphi_\beta : U\times U \to [1, \infty)$ for each $\beta \in \Omega$ and $(u, v) \in E$ satisfy:

$$D_\beta^{\mathcal{J}_{\varphi;\Omega}}(Tu, Tv) \leq \psi_\beta(J_\beta(u, v)), \tag{4.3.10}$$

where $\psi_\beta \in \Psi_\varphi$.
Moreover, let that

(i) there exist $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$;

(ii) if $(u, v) \in E$ and $x \in Tu$ and $y \in Tv$ such that $\frac{1}{r_\beta} J_\beta(x, y) < J_\beta(u, v)$, for all $\beta \in \Omega$, where $\{r_\beta : r_\beta > 1\}_{\beta \in \Omega}$, then $(x, y) \in E$;

(iii) for each $\{r_\beta : r_\beta > 1\}_{\beta \in \Omega}$ and $x \in U$ there exists $y \in Tx$ such that

$$J_\beta(x, y) \leq r_\beta J_\beta(x, Tx), \quad \text{for all } \beta \in \Omega.$$

Then the following assertions hold:

(I) For each $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is extended $Q_{\varphi;\Omega}$-convergent sequence in U, thus, $S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})} \neq \emptyset$.

(II) Moreover, suppose that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$. Then

(c_1) $\text{Fix}(T^{[k]}) \neq \emptyset$;

(c_2) there exists $z \in \text{Fix}(T^{[k]})$ such that $z \in S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})}$.

Proof. (I) We first show that $(z^m : m \in \{0\} \cup \mathbb{N})$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U.
Using supposition (i) there exists $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$. Now for each $\beta \in \Omega$, applying (4.3.10) we have

$$J_\beta(z^1, Tz^1) \leq D^{\mathcal{J}_{\varphi;\Omega}}_\beta(Tz^0, Tz^1) \leq \psi_\beta(J_\beta(z^0, z^1)). \tag{4.3.11}$$

Now using assumption (iii), for $z^1 \in U$, there exists $z^2 \in Tz^1$ such that

$$J_\beta(z^1, z^2) \leq r_\beta J_\beta(z^1, Tz^1) \leq r_\beta \psi_\beta(J_\beta(z^0, z^1)), \quad \forall \quad \beta \in \Omega. \tag{4.3.12}$$

Applying ψ_β, we get

$$\psi(J_\beta(z^1, z^2)) \leq \psi(r_\beta \psi_\beta(J_\beta(z^0, z^1)) = r_\beta \psi^2_\beta(J_\beta(z^0, z^1)), \quad \forall \quad \beta \in \Omega.$$

Using assumption (ii), from (4.3.12) it follows that $(z^1, z^2) \in E$. Now again for each $\beta \in \Omega$, using (4.3.10) we can write

$$J_\beta(z^2, Tz^2) \leq D^{\mathcal{J}_{\varphi;\Omega}}_\beta(Tz^1, Tz^2) \leq \psi_\beta(J_\beta(z^1, z^2)).$$

Using assumption (iii), for $z^2 \in U$, there exists $z^3 \in Tz^2$ such that

$$J_\beta(z^2, z^3) \leq r_\beta J_\beta(z^2, Tz^2) \leq r_\beta \psi_\beta(J_\beta(z^1, z^2)) \leq r^2_\beta \psi^2_\beta(J_\beta(z^0, z^1)), \quad \forall \quad \beta \in \Omega.$$

It is obvious that, $(z^2, z^3) \in E$. Proceeding in the similar fashion we find a sequence $(z^m : m \in \{0\} \cup \mathbb{N})$ such that $(z^m, z^{m+1}) \in E$ and

$$J_\beta(z^m, z^{m+1}) \leq r_\beta^m \psi_\beta^m(J_\beta(z^0, z^1)), \quad \forall \quad \beta \in \Omega. \tag{4.3.13}$$

Now by repeated use of $(\mathcal{J}1)$ and (4.3.13) for all $\beta \in \Omega$ and for all $m, n \in \mathbb{N}$ such that $n > m$, we get

$$\begin{aligned} J_\beta(z^m, z^n) &\leq \varphi_\beta(z^m, z^n) r_\beta^m \psi_\beta^m(J_\beta(z^0, z^1)) + \varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n) r_\beta^{m+1}\psi_\beta^{m+1}(J_\beta(z^0, z^1)) \\ &\quad + \varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n)\varphi_\beta(z^{m+2}, z^n) r_\beta^{m+2}\psi_\beta^{m+2}(J_\beta(z^0, z^1)) \\ &\quad + ... + \varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n)...\varphi_\beta(z^{n-1}, z^n) r_\beta^{n-1}\psi_\beta^{n-1}(J_\beta(z^0, z^1)) \\ &\leq \varphi_\beta(z^1, z^n)\varphi_\beta(z^2, z^n)...\varphi_\beta(z^m, z^n) r_\beta^m \psi_\beta^m(J_\beta(z^0, z^1)) \\ &\quad + \varphi_\beta(z^1, z^n)\varphi_\beta(z^2, z^n)...\varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n) r_\beta^{m+1}\psi_\beta^{m+1}(J_\beta(z^0, z^1)) \\ &\quad + ... + \varphi_\beta(z^1, z^n)\varphi_\beta(z^2, z^n)...\varphi_\beta(z^m, z^n)...\varphi_\beta(z^{n-1}, z^n) r_\beta^{n-1}\psi_\beta^{n-1}(J_\beta(z^0, z^1)). \end{aligned}$$

Let $S_m = \sum_{j=1}^m r_\beta^j \psi_\beta^j(J_\beta(z^0, z^1)) \prod_{i=1}^j \varphi_\beta(z^i, z^n)$, we can write

$$J_\beta(z^m, z^n) \leq (S_{n-1} - S_m).$$

Since $S_m < \infty$, we can write

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(z^m, z^n) = 0 \text{ for all } \beta \in \Omega. \tag{4.3.14}$$

Now, since $(U, Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete b-gauge space, so $(z^m : m \in \{0\} \cup \mathbb{N})$ is extended $\mathcal{J}_{\varphi;\Omega}$ convergent in U, thus for all $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$, we can write

$$\lim_{m\to\infty} J_\beta(z, z^m) = 0 \text{ for all } \beta \in \Omega. \tag{4.3.15}$$

Thus from (4.3.14) and (4.3.15), fixing $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$, defining $(u_m = z^m : m \in \{0\} \cup \mathbb{N})$ and $(v_m = z : m \in \{0\} \cup \mathbb{N})$ and applying $(\mathcal{J}2)$ to these sequences, we get

$$\lim_{m\to\infty} q_\beta(z, z^m) = 0 \text{ for all } \beta \in \Omega$$

This implies $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) To show (c_1), let $z^0 \in U$ is fixed and arbitrary. Since $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$ and

$$z^{(m+1)k} \in T^{[k]}(z^{mk}), \quad \text{for } m \in \{0\} \cup \mathbb{N}$$

thus defining $(z_m = z^{m-1+k} : m \in \mathbb{N})$, we have

$$(z_m : m \in \mathbb{N}) \subset T^{[k]}(U),$$

$$S^{Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset,$$

also, its subsequences

$$(y_m = z^{(m+1)k}) \subset T^{[k]}(U)$$

and

$$(x_m = z^{mk}) \subset T^{[k]}(U)$$

satisfy

$$y_m \in T^{[k]}(x_m), \text{ for all } m \in \mathbb{N}$$

and are extended $Q_{\varphi;\Omega}$-convergent to each point $z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$. Thus, applying the fact that

$$S^{Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{Q_{\varphi;\Omega}}_{(y_m:m\in\mathbb{N})} \quad \text{and} \quad S^{Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{Q_{\varphi;\Omega}}_{(x_m:m\in\mathbb{N})}$$

and the assumption that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$, we get there exists $z \in S^{Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$ such that $z \in T^{[k]}(z)$. Thus, (c_1) holds.
The statement (c_2) follows from (c_1) and the fact $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$. □

Let us consider $T : U \to U$, we get the following result in single-valued mapping.

Theorem 4.3.5. Let the map $T : U \to U$ and $\varphi_\beta : U \times U \to [1, \infty)$ for each $\beta \in \Omega$ satisfy:

$$J_\beta(Tu, Tv) \leq \psi_\beta(J_\beta(u, v)), \quad \forall \ (u, v) \in E, \tag{4.3.16}$$

where $\psi_\beta \in \Psi_\varphi$.
Moreover, let that

(i) there exist $z^0 \in U$ such that $(z^0, Tz^0) \in E$;

(ii) for $(u, v) \in E$ we have $(Tu, Tv) \in E$, provided $J_\beta(Tu, Tv) \leq J_\beta(u, v)$, for all $\beta \in \Omega$;

(iii) if a sequence $(z^m : m \in \mathbb{N})$ in U is such that $(z^m, z^{m+1}) \in E$ and $\lim^{\mathcal{J}_{\varphi;\Omega}}_{m\to\infty} z^m = z$, then $(z^m, z) \in E$ and $(z, z^m) \in E$.

Then the statements below are satisfied:

(I) For any $z^0 \in U$, $(z^m : m \in \{0\}\cup\mathbb{N})$ is extended $Q_{\varphi;\Omega}$-convergent sequence in U, thus, $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) Moreover, suppose that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$. Then

(c$_1$) $\text{Fix}(T^{[k]}) \neq \emptyset$;

(c$_2$) there exists $z \in \text{Fix}(T^{[k]})$ such that $z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and

(c$_3$) for all $z \in \text{Fix}(T^{[k]})$, $J_\beta(z, T(z)) = J_\beta(T(z), z) = 0\}$, for all $\beta \in \Omega$.

(III) Furthermore, let $\text{Fix}(T^{[k]}) \neq \emptyset$ for some $k \in \mathbb{N}$ and $(U, Q_{\varphi;\Omega})$ is a Hausdorff space. Then

(d$_1$) $\text{Fix}(T^{[k]})$=$\text{Fix}(T)$;

(d$_2$) there exists $z \in \text{Fix}(T)$ such that $z \in S^{L-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and

(d$_3$) for all $z \in \text{Fix}(T^{[k]}) = \text{Fix}(T)$, $J_\beta(z, z) = 0$, for all $\beta \in \Omega$.

Proof. Since every single-valued mapping can be viewed as multi-valued mapping, it remains to prove assertion (c_3) and assertions (d_1)-(d_3) of the above stated theorem.
To prove (c_3), use $\mathcal{J}1$, assumption (iii) and inequality (4.3.16), we obtain for all $\beta \in \Omega$

$$\begin{aligned} J_\beta(z, Tz) &\leq \varphi_\beta(z, Tz)\{J_\beta(z, z^{m+1}) + J_\beta(z^{m+1}, Tz)\} \\ &\leq \varphi_\beta(z, Tz)\{J_\beta(z, z^{m+1}) + J_\beta(Tz^m, Tz)\} \\ &\leq \varphi_\beta(z, Tz)\{J_\beta(z, z^{m+1}) + \psi_\beta(J_\beta(z^m, z))\}. \end{aligned}$$

Letting $m \to \infty$, we have

$$J_\beta(z, Tz) = 0 \qquad \forall \;\; \beta \in \Omega.$$

Similarly we can show that $J_\beta(Tz, z) = 0 \;\; \forall \;\; \beta \in \Omega$.
(III) Since $(U, Q_{\varphi;\Omega})$ is a Hausdorff space, using Proposition (4.2.4), assertion (c_3) suggests that for $z \in \text{Fix}(T^{[k]})$, we have $z \in T(z)$. This gives $z \in \text{Fix}(T)$. Hence (d_1) is true.
Assertions (c_2) and (d_1) imply (d_2) .
To prove assertion (d_3), consider $(\mathcal{J}1)$ and use (c_3) and (d_1), we have for all $z \in \text{Fix}(T^{[k]}) = \text{Fix}(T)$,
$J_\beta(z, z) \leq \varphi(z, z)\{J_\beta(z, T(z)) + J_\beta(T(z), z)\} = 0$, for all $\beta \in \Omega$. □

Remark 4.3.6. In comparison with results in [45], our results in extended b-gauge spaces are more generalized and improved, in which assumptions are weak and claims are robust.

4.4 Periodic points for Caristi type G-contractions in extended b-gauge spaces

Indeed, the Caristi fixed point theorem [5] (also known as Caristi-Kirk fixed point theorem [68, 69]) came into being as a result of looking for a different proof of the superlative Banach contraction principle. Indeed, Caristis theorem is identical to the completeness of metric [70]. For further details to this subject, we refer to [71, 72, 73].

In this section, we develop novel periodic and fixed point results for Caristi type G-contractions $T : U \to Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ in the new setting of extended b-gauge space endowed with a graph, which generalize, enhance and unite the current results in the corresponding literature.

Our main results in multi-valued mappings are given below.

Theorem 4.4.1. Let map $T : U \to Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ be edge preserving and $\phi_\beta : U \to [0, \infty)$, $\beta \in \Omega$ is a lower semi continuous function such that for each $u \in U$ and $v \in Tu$ where $(u, v) \in E$, we have

$$J_\beta(v, Tv) \leq \phi_\beta(u) - \phi_\beta(v), \quad \forall\ \beta \in \Omega. \tag{4.4.1}$$

Moreover, let that

(i) there exist $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$;

(ii) for each $\{r_\beta : r_\beta > 1\}_{\beta\in\Omega}$ and $u \in U$ there exists $v \in Tu$ such that

$$J_\beta(u, v) \leq r_\beta J_\beta(u, Tu), \quad \forall\ \beta \in \Omega.$$

Then the following assertions hold:

(I) For any $z^0 \in U$, $(z^m : m \in \{0\}\cup\mathbb{N})$ is extended $Q_{\varphi;\Omega}$-convergent sequence in U, thus $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) Moreover, suppose that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$. Then

(a_1) $\text{Fix}(T^{[k]}) \neq \emptyset$;

(a_2) there exists $z \in \text{Fix}(T^{[k]})$ such that $z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$.

Proof. (I) We first show that $(z^m : m \in \{0\} \cup \mathbb{N})$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U.

Using supposition (i) there exists $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$. Now applying (4.4.1) we have

$$J_\beta(z^1, Tz^1) \leq \phi_\beta(z^0) - \phi_\beta(z^1), \quad \forall \quad \beta \in \Omega. \tag{4.4.2}$$

Now by using assumption (ii) and (4.4.2) we have $r_\beta > 1$ for each $\beta \in \Omega$ and $z^2 \in Tz^1$ such that

$$J_\beta(z^1, z^2) \leq r_\beta J_\beta(z^1, Tz^1) \leq r_\beta\{\phi_\beta(z^0) - \phi_\beta(z^1)\}, \quad \forall \quad \beta \in \Omega.$$

We have $(z^1, z^2) \in E$, since T is edge preserving. Moving on the same lines, we have a sequence $\{z^m : m \in \{0\} \cup \mathbb{N}\}$ such that $(z^m, z^{m+1}) \in E$ and for all $\beta \in \Omega$ and for each $m \in \mathbb{N}$, we get

$$J_\beta(z^m, z^{m+1}) \leq r_\beta J_\beta(z^m, Tz^m) \leq r_\beta\{\phi_\beta(z^{m-1}) - \phi_\beta(z^m)\}.$$

From here we observe that $\{\phi_\beta(z^m)\}$ is a non-increasing sequence, hence we can find $l_\beta \geq 0$ such that $\{\phi_\beta(z^m)\} \to l_\beta$ as $m \to \infty$. For $m, p \in \mathbb{N}$ and each $\beta \in \Omega$, we write

$$\begin{aligned}
J_\beta(z^m, z^{m+p}) \leq\ & \varphi_\beta(z^m, z^{m+p}) J_\beta(z^m, z^{m+1}) + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) J_\beta(z^{m+1}, z^{m+2}) \\
& + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p})\varphi_\beta(z^{m+2}, z^{m+p}) J_\beta(z^{m+2}, z^{m+3}) \\
& + ... + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) ... \varphi_\beta(z^{m+p-1}, z^{m+p}) J_\beta(z^{m+p-1}, z^{m+p}) \\
& \leq \varphi_\beta(z^m, z^{m+p}) r_\beta\{\phi_\beta(z^{m-1}) - \phi_\beta(z^m)\} + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) \\
& r_\beta\{\phi_\beta(z^m) - \phi_\beta(z^{m+1})\} + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p})\varphi_\beta(z^{m+2}, z^{m+p}) \\
& r_\beta\{\phi_\beta(z^{m+1}) - \phi_\beta(z^{m+2})\} + ... + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) ... \\
& \varphi_\beta(z^{m+p-1}, z^{m+p}) r_\beta\{\phi_\beta(z^{m+p-2}) - \phi_\beta(z^{m+p-1})\}.
\end{aligned}$$

Letting $m \to \infty$, we have $\{\phi_\beta(z^m)\} \to l_\beta$. This implies that $(z^m : m \in \{0\} \cup \mathbb{N})$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U, thus for all $\beta \in \Omega$, for all $\epsilon > 0$ and for all $n, m \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that

$$J_\beta(z^m, z^n) < \epsilon, \text{ for all } n \geq m \geq k. \tag{4.4.3}$$

Now, since $(U, Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete b-gauge space, so $(z^m : m \in \{0\} \cup \mathbb{N})$ is extended $\mathcal{J}_{\varphi;\Omega}$-convergent in U, thus for all $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})}$, for all $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ and for all $\beta \in \Omega$, we have

$$J_\beta(z, z^m) < \epsilon, \text{ for all } m \geq k. \tag{4.4.4}$$

Thus from (4.4.3) and (4.4.4), fixing $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})}$, defining $(u_m = z^m : m \in \{0\} \cup \mathbb{N})$ and $(v_m = z : m \in \{0\} \cup \mathbb{N})$ and applying $(\mathcal{J}2)$ to these sequences, for all $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ and for all $\beta \in \Omega$, we have

$$q_\beta(z, z^m) < \epsilon, \text{ for all } m \geq k.$$

This implies $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) To show (a_1), let $z^0 \in U$ is fixed and arbitrary. Since $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$ and

$$z^{(m+1)k} \in T^{[k]}(z^{mk}), \quad \text{for } m \in \{0\} \cup \mathbb{N}$$

thus describing $(z_m = z^{m-1+k} : m \in \mathbb{N})$, we have

$$(z_m : m \in \mathbb{N}) \subset T^{[k]}(U),$$

$$S^{Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset,$$

also, its subsequences

$$(y_m = z^{(m+1)k}) \subset T^{[k]}(U)$$

and

$$(x_m = z^{mk}) \subset T^{[k]}(U)$$

satisfy

$$y_m \in T^{[k]}(x_m), \text{ for all } m \in \mathbb{N}$$

and are extended $Q_{\varphi;\Omega}$-convergent to each point $z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$. Thus, applying the fact

$$S^{Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{Q_{\varphi;\Omega}}_{(y_m:m\in\mathbb{N})} \qquad \text{and} \qquad S^{Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{Q_{\varphi;\Omega}}_{(x_m:m\in\mathbb{N})}$$

and the assumption that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$, we get there exists $z \in S^{Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$ such that $z \in T^{[k]}(z)$. Thus, (a_1) holds.

The statement (a_2) follows from (a_1) and the certainty that $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$. □

Theorem 4.4.2. Let map $T : U \to Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ be edge preserving and $\phi_\beta : U \to [0, \infty)$, $\beta \in \Omega$ be a lower semi continuous function such that for each $u \in U$ and $v \in Tu$ where $(u, v) \in E$, we have

$$J_\beta(u, v) \leq \phi_\beta(u) - \phi_\beta(v), \quad \forall \quad \beta \in \Omega. \tag{4.4.5}$$

Moreover, let that

(i) there exist $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$;

Then the assertions below hold:

(I) For any $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is extended $Q_{\varphi;\Omega}$-convergent sequence in U, thus $S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})} \neq \emptyset$.

(II) Moreover, suppose that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$. Then

(b$_1$) $\mathrm{Fix}(T^{[k]}) \neq \emptyset$;

(b$_2$) there exists $z \in \mathrm{Fix}(T^{[k]})$ such that $z \in S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})}$.

Proof. (I) We first show that $(z^m : m \in \{0\} \cup \mathbb{N})$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U.
Using supposition (i) there exists $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$. Now applying (4.4.5) we have

$$J_\beta(z^0, z^1) \leq \phi_\beta(z^0) - \phi_\beta(z^1), \text{ for all } \beta \in \Omega. \tag{4.4.6}$$

We can write $(z^1, z^2) \in E$, since T is edge preserving. Progressing in the same way, we obtain a sequence $\{z^m : m \in \{0\} \cup \mathbb{N}\}$ such that $(z^m, z^{m+1}) \in E$ and for each $m \in \mathbb{N}$ and for each $\beta \in \Omega$, we have

$$J_\beta(z^m, z^{m+1}) \leq \phi_\beta(z^m) - \phi_\beta(z^{m+1}).$$

From here we observe that $\{\phi_\beta(z^m)\}$ is a non-increasing sequence, thus we can find $l_\beta \geq 0$ such that $\{\phi_\beta(z^m)\} \to l_\beta$ as $m \to \infty$. For $m, p \in \mathbb{N}$ and each $\beta \in \Omega$, we write

$$\begin{aligned}
J_\beta(z^m, z^{m+p}) \leq\ & \varphi_\beta(z^m, z^{m+p}) J_\beta(z^m, z^{m+1}) + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) J_\beta(z^{m+1}, z^{m+2}) \\
& + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p})\varphi_\beta(z^{m+2}, z^{m+p}) J_\beta(z^{m+2}, z^{m+3}) \\
& + ... + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) ... \varphi_\beta(z^{m+p-1}, z^{m+p}) J_\beta(z^{m+p-1}, z^{m+p}) \\
& \leq \varphi_\beta(z^m, z^{m+p})\{\phi_\beta(z^m) - \phi_\beta(z^{m+1})\} + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) \\
& \{\phi_\beta(z^{m+1}) - \phi_\beta(z^{m+2})\} + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p})\varphi_\beta(z^{m+2}, z^{m+p}) \\
& \{\phi_\beta(z^{m+2}) - \phi_\beta(z^{m+3})\} + ... + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) ... \\
& \varphi_\beta(z^{m+p-1}, z^{m+p})\{\phi_\beta(z^{m+p-1}) - \phi_\beta(z^{m+p})\}.
\end{aligned}$$

Letting $m \to \infty$, we have $\{\phi_\beta(z^m)\} \to l_\beta$. This implies that $(z^m : m \in \{0\} \cup \mathbb{N})$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U, for all $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ and for all $\beta \in \Omega$, we have

$$J_\beta(z^m, z^n) < \epsilon, \text{ for all } n \geq m \geq k. \tag{4.4.7}$$

Now, since $(U, Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete b-gauge space, so $(z^m : m \in \{0\} \cup \mathbb{N})$ is extended $\mathcal{J}_{\varphi;\Omega}$-convergent in U, thus for all $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})}$, for all $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ and for all $\beta \in \Omega$, we have

$$J_\beta(z, z^m) < \epsilon, \text{ for all } m \geq k. \tag{4.4.8}$$

Thus from (4.4.7) and (4.4.8), fixing $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})}$, defining $(u_m = z^m : m \in \{0\} \cup \mathbb{N})$ and $(v_m = z : m \in \{0\} \cup \mathbb{N})$ and applying $(\mathcal{J}2)$ to these sequences, we get, for all $\beta \in \Omega$, for all $\epsilon > 0$ and for all $m \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that

$$q_\beta(z, z^m) < \epsilon, \text{ for all } m \geq k.$$

This implies $S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})} \neq \emptyset$.

(II) To show (b_1), let $z^0 \in U$ is fixed and arbitrary. Since $S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})} \neq \emptyset$ and

$$z^{(m+1)k} \in T^{[k]}(z^{mk}), \quad \text{for } m \in \{0\} \cup \mathbb{N}$$

thus describing $(z_m = z^{m-1+k} : m \in \mathbb{N})$, we have

$$(z_m : m \in \mathbb{N}) \subset T^{[k]}(U),$$

$$S^{Q_{\varphi;\Omega}}_{(z_m : m \in \{0\} \cup \mathbb{N})} = S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})} \neq \emptyset,$$

also, its subsequences

$$(y_m = z^{(m+1)k}) \subset T^{[k]}(U)$$

and

$$(x_m = z^{mk}) \subset T^{[k]}(U)$$

satisfy

$$y_m \in T^{[k]}(x_m), \text{ for all } m \in \mathbb{N}$$

and are extended $Q_{\varphi;\Omega}$-convergent to each point $z \in S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})}$. Thus, applying the fact

$$S^{Q_{\varphi;\Omega}}_{(z_m : m \in \mathbb{N})} \subset S^{Q_{\varphi;\Omega}}_{(y_m : m \in \mathbb{N})} \quad \text{and} \quad S^{Q_{\varphi;\Omega}}_{(z_m : m \in \mathbb{N})} \subset S^{Q_{\varphi;\Omega}}_{(x_m : m \in \mathbb{N})}$$

and the assumption that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$, we get there exists $z \in S^{Q_{\varphi;\Omega}}_{(z_m : m \in \{0\} \cup \mathbb{N})} = S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})}$ such that $z \in T^{[k]}(z)$.

Thus, (b_1) holds.

The statement (b_2) follows from (b_1) and the certainty that $S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})} \neq \emptyset$. □

Theorem 4.4.3. Let map $T : U \to Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ be edge preserving and $\psi_\beta : U \to [0, \infty)$, $\beta \in \Omega$ be an upper semi continuous function such that for each $u \in U$ and $v \in Tu$ where $(u, v) \in E$, we have

$$J_\beta(v, Tv) \leq \psi_\beta(u) - \psi_\beta(v), \quad \forall \quad \beta \in \Omega. \tag{4.4.9}$$

Moreover, let that

(i) there exist $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$;

(ii) for each $\{r_\beta : r_\beta > 1\}_{\beta\in\Omega}$ and $x \in U$ there exists $y \in Tx$ such that

$$J_\beta(x, y) \leq r_\beta J_\beta(x, Tx)\}, \quad \forall \quad \beta \in \Omega.$$

Then the below assertions are satisfied:

(I) For any $z^0 \in U$, $(z^m : m \in \{0\}\cup\mathbb{N})$ is extended $Q_{\varphi;\Omega}$-convergent sequence in U, thus $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) Moreover, suppose that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$. Then

- (c$_1$) Fix$(T^{[k]}) \neq \emptyset$;
- (c$_2$) there exists $z \in$ Fix$(T^{[k]})$ such that $z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$.

Proof. (I) We first show that $(z^m : m \in \{0\} \cup \mathbb{N})$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U.
Using supposition (i) there exists $z^0 \in U$ and $z^1 \in Tz^0$ such that $(z^0, z^1) \in E$. Now using (4.4.9) we can write

$$J_\beta(z^1, Tz^1) \leq \psi_\beta(z^0) - \psi_\beta(z^1), \quad \forall \quad \beta \in \Omega. \tag{4.4.10}$$

Now by using assumption (ii) and (4.4.10) we have $r_\beta > 1$ for each $\beta \in \Omega$ and $z^2 \in Tz^1$ such that

$$J_\beta(z^1, z^2) \leq r_\beta J_\beta(z^1, Tz^1) \leq r_\beta\{\psi_\beta(z^0) - \psi_\beta(z^1)\}.$$

We have $(z^1, z^2) \in E$, since T is edge preserving. Progressing in the same manner, we obtain a sequence $\{z^m : m \in \{0\} \cup \mathbb{N}\}$ such that $(z^m, z^{m+1}) \in E$ and for each $\beta \in \Omega$ and for each $m \in \mathbb{N}$, we have

$$J_\beta(z^m, z^{m+1}) \leq r_\beta J_\beta(z^m, Tz^m) \leq r_\beta\{\psi_\beta(z^{m-1}) - \psi_\beta(z^m)\}.$$

From here we see that $\{\psi_\beta(z^m)\}$ is a non-increasing sequence, thus we can find $l_\beta \geq 0$ such that $\{\psi_\beta(z^m)\} \to l_\beta$ as $m \to \infty$. For $m, p \in \mathbb{N}$ and each $\beta \in \Omega$, we write

$$\begin{aligned} J_\beta(z^m, z^{m+p}) \leq & \varphi_\beta(z^m, z^{m+p})J_\beta(z^m, z^{m+1}) + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p})J_\beta(z^{m+1}, z^{m+2}) \\ & + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p})\varphi_\beta(z^{m+2}, z^{m+p})J_\beta(z^{m+2}, z^{m+3}) \\ & + ... + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p})...\varphi_\beta(z^{m+p-1}, z^{m+p})J_\beta(z^{m+p-1}, z^{m+p}) \\ \leq & \varphi_\beta(z^m, z^{m+p})r_\beta\{\psi_\beta(z^{m-1}) - \psi_\beta(z^m)\} + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p}) \\ & r_\beta\{\psi_\beta(z^m) - \psi_\beta(z^{m+1})\} + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p})\varphi_\beta(z^{m+2}, z^{m+p}) \\ & r_\beta\{\psi_\beta(z^{m+1}) - \psi_\beta(z^{m+2})\} + ... + \varphi_\beta(z^m, z^{m+p})\varphi_\beta(z^{m+1}, z^{m+p})... \\ & \varphi_\beta(z^{m+p-1}, z^{m+p})r_\beta\{\psi_\beta(z^{m+p-2}) - \psi_\beta(z^{m+p-1})\}. \end{aligned}$$

Letting $m \to \infty$, we have $\{\psi_\beta(z^m)\} \to l_\beta$. This implies that $(z^m : m \in \{0\} \cup \mathbb{N})$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U, thus for all $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ and for all $\beta \in \Omega$, we have

$$J_\beta(z^m, z^n) < \epsilon, \ \text{ for all } \ n \geq m \geq k. \tag{4.4.11}$$

Now, since $(U, Q_{\varphi;\Omega})$ is extended $\mathcal{J}_{\varphi;\Omega}$-sequentially complete b-gauge space, so $(z^m : m \in \{0\} \cup \mathbb{N})$ is extended $\mathcal{J}_{\varphi;\Omega}$-convergent in U, thus for all $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})}$, we can write for all $\beta \in \Omega$, for all $\epsilon > 0$ and for all $m \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that

$$J_\beta(z, z^m) < \epsilon, \ \text{ for all } \ m \geq k. \tag{4.4.12}$$

Thus from (4.4.11) and (4.4.12), fixing $z \in S^{\mathcal{J}_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})}$, defining $(u_m = z^m : m \in \{0\} \cup \mathbb{N})$ and $(v_m = z : m \in \{0\} \cup \mathbb{N})$ and applying $(\mathcal{J}2)$ to these sequences, we get, for all $\beta \in \Omega$, for all $\epsilon > 0$ and for all $m \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that

$$q_\beta(z, z^m) < \epsilon, \ \text{ for all } \ m \geq k.$$

This implies $S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})} \neq \emptyset$.

(II) To show (c_1), let $z^0 \in U$ is fixed and arbitrary. Since $S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})} \neq \emptyset$ and

$$z^{(m+1)k} \in T^{[k]}(z^{mk}), \quad \text{ for } m \in \{0\} \cup \mathbb{N}$$

thus describing $(z_m = z^{m-1+k} : m \in \mathbb{N})$, we have

$$(z_m : m \in \mathbb{N}) \subset T^{[k]}(U),$$

$$S^{Q_{\varphi;\Omega}}_{(z_m : m \in \{0\} \cup \mathbb{N})} = S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})} \neq \emptyset,$$

also, its subsequences

$$(y_m = z^{(m+1)k}) \subset T^{[k]}(U)$$

and

$$(x_m = z^{mk}) \subset T^{[k]}(U)$$

satisfy

$$y_m \in T^{[k]}(x_m), \text{ for all } m \in \mathbb{N}$$

and are extended $Q_{\varphi;\Omega}$-convergent to each point $z \in S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})}$. Thus, applying the fact

$$S^{Q_{\varphi;\Omega}}_{(z_m : m \in \mathbb{N})} \subset S^{Q_{\varphi;\Omega}}_{(y_m : m \in \mathbb{N})} \quad \text{and} \quad S^{Q_{\varphi;\Omega}}_{(z_m : m \in \mathbb{N})} \subset S^{Q_{\varphi;\Omega}}_{(x_m : m \in \mathbb{N})}$$

and the assumption that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$, we get there exists $z \in S^{Q_{\varphi;\Omega}}_{(z_m : m \in \{0\} \cup \mathbb{N})} = S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})}$ such that $z \in T^{[k]}(z)$.
Thus, (c_1) holds.
The statement (c_2) follows from (c_1) and the certainty that $S^{Q_{\varphi;\Omega}}_{(z^m : m \in \{0\} \cup \mathbb{N})} \neq \emptyset$. □

Remark 4.4.4. (a) Our main results in extended b-gauge space are new generalization and improvement of the results in [74] in which assumption are weak and assertions are robust.

(b) We must note that that by taking $\varphi_\beta(u, v) = s \geq 1$, for all $\beta \in \Omega$, we obtain the results in b-gauge space.

4.5 Consequences and applications

This section consists of important and fascinating consequences of the theorems proved in the third section.

We set up some periodic and fixed point results for mappings fulfilling contraction inequalities involving function α.
Recall that U is a non-void set and the Graph $G := (V, E)$ is defined as

$$V := U \text{ and } E := \{(a, b) \in U \times U : \alpha(a, b) \geq 1\},$$

where $\alpha : U \times U \to [0, \infty)$.

Corollary 4.5.1. Let the map $T : U \to Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ and $\varphi_\beta : U \times U \to [1, \infty)$ for each $\beta \in \Omega$ satisfy:

$$D^{\mathcal{J}_{\varphi;\Omega}}_\beta(Tu, Tv) \leq a_\beta J_\beta(u, v) + b_\beta J_\beta(u, Tu) + c_\beta J_\beta(v, Tv) + e_\beta J_\beta(u, Tv) + L_\beta J_\beta(v, Tu) \tag{4.5.1}$$

for all $\alpha(u,v) \geq 1$, where $a_\beta, b_\beta, c_\beta, e_\beta, L_\beta \geq 0$ be such that $a_\beta + b_\beta + c_\beta + 2e_\beta \varphi_\beta(z^{m-1}, Tz^m) < 1$ and $\lim_{m,n\to\infty} \varphi_\beta(z^m, z^n)\mu_\beta < 1$, for some $\mu_\beta < 1$ and each $z^0 \in U$, here $z^m \in T(z^{m-1})$, where $m \in \mathbb{N}$.
Moreover, let that

(a) there exist $z^0 \in U$ and $z^1 \in Tz^0$ such that $\alpha(z^0, z^1) \geq 1$;

(b) if $\alpha(u,v) \geq 1$ and $x \in Tu$ and $y \in Tv$ such that $J_\beta(x,y) \leq J_\beta(u,v)$, for all $\beta \in \Omega$, then $\alpha(x,y) \geq 1$;

(c) for any $\{r_\beta : r_\beta > 1\}_{\beta\in\Omega}$ and $u \in U$ there exists $v \in Tu$ such that

$$J_\beta(u,v) \leq r_\beta J_\beta(u,Tu)\}, \quad \text{for all } \beta \in \Omega.$$

Then the following assertions hold:

(I) For any $z^0 \in U$, $(z^m : m \in \{0\}\cup\mathbb{N})$ is extended $Q_{\varphi;\Omega}$-convergent sequence in U, thus $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) Moreover, suppose that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$. Then

(c$_1$) $\text{Fix}(T^{[k]}) \neq \emptyset$;

(c$_2$) there exists $z \in \text{Fix}(T^{[k]})$ such that $z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$.

Proof. Consider the graph $G = (V,E)$ and define the map $\alpha : U \times U \to [0,\infty)$ for some $\rho \geq 1$ as:

$$\alpha(u,v) = \begin{cases} \rho & \text{if } (u,v) \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5.2}$$

Now the inequality (4.5.1) takes the foam

$$D^{\mathcal{J}_{\varphi;\Omega}}_\beta(Tu,Tv) \leq a_\beta J_\beta(u,v) + b_\beta J_\beta(u,Tu) + c_\beta J_\beta(v,Tv) + e_\beta J_\beta(u,Tv) + L_\beta J_\beta(v,Tu) \tag{4.5.3}$$

for all $(u,v) \in E$. This yields that T satisfies inequality (4.3.1). Also conditions (a), (b) and (c) implies conditions (i), (ii) and (iii) of Theorem 4.3.2. Hence conclusion follows from Theorem 4.3.2. □

Let us consider $T : U \to U$, we get the following result in single-valued mapping.

Corollary 4.5.2. Let the single-valued map $T : U \to U$ and $\varphi_\beta : U \times U \to [1,\infty)$ for each $\beta \in \Omega$ satisfy:

$$J_\beta(Tu,Tv) \le a_\beta J_\beta(u,v)+b_\beta J_\beta(u,Tu)+c_\beta J_\beta(v,Tv)+e_\beta J_\beta(u,Tv)+L_\beta J_\beta(v,Tu) \tag{4.5.4}$$

for all $\alpha(u,v) \ge 1$, where $a_\beta, b_\beta, c_\beta, e_\beta, L_\beta \ge 0$ be such that $a_\beta + b_\beta + c_\beta + 2e_\beta\varphi_\beta(z^{m-1}, Tz^m) < 1$ and $\lim_{m,n\to\infty} \varphi_\beta(z^m, z^n)\mu_\beta < 1$, for some $\mu_\beta < 1$ and each $z^0 \in U$, here $z^m = T^{[m]}(z^0)$, where $m \in \mathbb{N}$.
Moreover, let that

(a) there exist $z^0 \in U$ such that $\alpha(z^0, Tz^0) \ge 1$;

(b) for $\alpha(u,v) \ge 1$ we have $\alpha(Tu,Tv) \ge 1$, provided $J_\beta(Tu,Tv) \le J_\beta(u,v)$, for all $\beta \in \Omega$;

(c) if a sequence $(z^m : m \in \mathbb{N})$ in U is such that $\alpha(z^m, z^{m+1}) \ge 1$ and $\lim^{\mathcal{J}_{\varphi;\Omega}}_{m\to\infty} z^m = z$, then $\alpha(z^m, z) \ge 1$ and $\alpha(z, z^m) \ge 1$.

Then the following assertions hold:

(I) For any $z^0 \in U$, $(z^m : m \in \{0\}\cup\mathbb{N})$ is extended $Q_{\varphi;\Omega}$-convergent sequence in U, thus, $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) Moreover, suppose that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$ and $\varphi_\beta(z,Tz)\{c_\beta + e_\beta \lim_{m\to\infty} \varphi_\beta(z^m, Tz)\} < 1$. Then

 (c_1) $\mathrm{Fix}(T^{[k]}) \neq \emptyset$;

 (c_2) there exists $z \in \mathrm{Fix}(T^{[k]})$ such that $z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and

 (c_3) for all $z \in \mathrm{Fix}(T^{[k]})$, $J_\beta(z, T(z)) = J_\beta(T(z), z) = 0$, for all $\beta \in \Omega$.

(III) Furthermore, let $\mathrm{Fix}(T^{[k]}) \neq \emptyset$ for some $k \in \mathbb{N}$ and $(U, Q_{\varphi;\Omega})$ is a Hausdorff space. Then

 (d_1) $\mathrm{Fix}(T^{[k]})=\mathrm{Fix}(T)$;

 (d_2) there exists $z \in \mathrm{Fix}(T)$ such that $z \in S^{L-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and

 (d_3) for all $z \in \mathrm{Fix}(T^{[k]}) = \mathrm{Fix}(T)$, $J_\beta(z,z) = 0$ for all $\beta \in \Omega$.

Proof. Same reasons as in the proof of Theorem 4.5.1. □

Corollary 4.5.3. Let the set-valued map $T : U \to Cl^{\mathcal{J}_{\varphi;\Omega}}(U)$ and $\varphi_\beta : U \times U \to [1,\infty)$ for each $\beta \in \Omega$ and $\alpha(u,v) \ge 1$ satisfy:

$$D^{\mathcal{J}_{\varphi;\Omega}}_\beta(Tu,Tv) \le \psi_\beta(J_\beta(u,v)), \tag{4.5.5}$$

where $\psi_\beta \in \Psi_\varphi$.
Moreover, let that

(a) there exist $z^0 \in U$ and $z^1 \in Tz^0$ such that $\alpha(z^0, z^1) \geq 1$;

(b) if $\alpha(u,v) \geq 1$ and $x \in Tu$ and $y \in Tv$ such that for all $\beta \in \Omega$, $\frac{1}{r_\beta} J_\beta(x,y) < J_\beta(u,v)$, where $\{r_\beta : r_\beta > 1\}_{\beta\in\Omega}$, then $\alpha(x,y) \geq 1$;

(c) for each $\{r_\beta : r_\beta > 1\}_{\beta\in\Omega}$ and $u \in U$ there exists $v \in Tu$ such that

$$J_\beta(u,v) \leq r_\beta J_\beta(u,Tu), \quad \text{for all } \beta \in \Omega.$$

Then the following assertions hold:

(I) For each $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is extended $Q_{\varphi;\Omega}$-convergent sequence in U, thus, $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(II) Moreover, suppose that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$. Then

(e$_1$) $\mathrm{Fix}(T^{[k]}) \neq \emptyset$;

(e$_2$) there exists $z \in \mathrm{Fix}(T^{[k]})$ such that $z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$.

Proof. Consider the graph $G = (V,E)$ and define the map $\alpha : U \times U \to [0,\infty)$ for some $\rho \geq 1$ as:

$$\alpha(x,y) = \begin{cases} \rho & \text{if } (x,y) \in E \\ 0 & \text{otherwise.} \end{cases} \tag{4.5.6}$$

Now the inequality (4.5.5) takes the foam

$$D_\beta^{\mathcal{J}_{\varphi;\Omega}}(Tu,Tv) \leq \psi_\beta(J_\beta(u,v)), \tag{4.5.7}$$

for all $(u,v) \in E$. This yields that T satisfies inequality (4.3.10). Also conditions (a), (b) and (c) implies conditions (i), (ii) and (iii) of theorem 4.3.4. Hence conclusion follows from Theorem 4.3.4. □

Let us consider $T : U \to U$, we get the following result in single-valued mapping.

Corollary 4.5.4. Let the single-valued map $T : U \to U$ and $\varphi_\beta : U \times U \to [1,\infty)$ for each $\beta \in \Omega$ satisfy:

$$J_\beta(Tu,Tv) \leq \psi_\beta(J_\beta(u,v)) \;\; \forall \;\; \alpha(u,v) \geq 1, \tag{4.5.8}$$

where $\psi_\beta \in \Psi_\varphi$.
Moreover, let that

(a) there exist $z^0 \in U$ such that $\alpha(z^0, Tz^0) \geq 1$;

(b) for $\alpha(u,v) \geq 1$ we have $\alpha(Tu,Tv) \geq 1$, provided $J_\beta(Tu,Tv) \leq J_\beta(u,v)$, for all $\beta \in \Omega$;

(c) if a sequence $(z^m : m \in \mathbb{N})$ in U is such that $\alpha(z^m, z^{m+1}) \geq 1$ and $\lim_{m\to\infty}^{\mathcal{J}_{\varphi;\Omega}} z^m = z$, then $\alpha(z^m, z) \geq 1$ and $\alpha(z, z^m) \geq 1$.

Then the following assertions hold:

(D) For any $z^0 \in U$, $(z^m : m \in \{0\} \cup \mathbb{N})$ is extended $Q_{\varphi;\Omega}$-convergent sequence in U, thus, $S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.

(E) Moreover, suppose that $T^{[k]}$ is an extended $Q_{\varphi;\Omega}$-closed map on U, for some $k \in \mathbb{N}$. Then

(e$_1$) $\mathrm{Fix}(T^{[k]}) \neq \emptyset$;

(e$_2$) there exists $z \in \mathrm{Fix}(T^{[k]})$ such that $z \in S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and

(e$_3$) for all $z \in \mathrm{Fix}(T^{[k]})$, $J_\beta(z, T(z)) = J_\beta(T(z), z) = 0$, for all $\beta \in \Omega$.

(F) Furthermore, let $\mathrm{Fix}(T^{[k]}) \neq \emptyset$ for some $k \in \mathbb{N}$ and $(U, Q_{\varphi;\Omega})$ is a Hausdorff space. Then

(f$_1$) $\mathrm{Fix}(T^{[k]})$=$\mathrm{Fix}(T)$;

(f$_2$) there exists $z \in \mathrm{Fix}(T)$ such that $z \in S^{L-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$; and

(f$_3$) for all $z \in \mathrm{Fix}(T)$, $J_\beta(z,z) = 0$, for all $\beta \in \Omega$.

Proof. Same evidences as in the proof of Theorem 4.5.3. □

Remark 4.5.5. (a) The above results in extended b-gauge space are new generalization and improvement of the results in [76] in which assumption are weak and assertions are robust.

(b) Observed that in case α greater than or equal to one is used in any contraction inequality, bringing it back to the regular contractive inequality without α (see for instance inequalities (4.5.3) and (4.5.7)). Therefore it seems that α-function plays no roll in proving the existence of the fixed point of any mapping. Hence when the under lying space is endowed with the graph, the fixed point theorems can easily be reduced to the α-type analogous.

(c) Since we have stated a few theorems for contraction inequalities involving function α. Some more analogues of the above results for contraction inequalities involving function α can simply be derived from results in [77, 78, 75].

Chapter 5

Periodic and Fixed Points in Extended Quasi-b-Gauge Spaces

The famous fixed point results due to Bannach [1] and Rus [4] (see also [79]) for single-valued mapping in complete metric space have many different versions in the literature. Their analogues in more general spaces are important, fascinating and challenging for the researchers.

This chapter aims to prove the Banach contraction principle and theorem due to Rus for single-valued mapping in more general spaces with asymmetric structure using new families of distances. Hence, our results generalize and improve the existing results due to Banach and Rus in the literature.

Throughout this chapter $(U, Q_{\varphi;\Omega})$ is representing an extended quasi-b-gauge space, where U is the underlying nonempty set and $Q_{\varphi;\Omega}$ is an extended quasi-b-gauge with $\varphi : U \times U \to [1, \infty)$ and Ω is an index set.

This chapter includes four main sections. The first section introduces the notion of extended quasi-b-gauge space $(U, Q_{\varphi;\Omega})$. In the second section, we establish the notion of left (right) $\mathcal{J}_{\varphi;\Omega}$-families of generalized extended quasi-pseudo-b-distances generated by $(U, Q_{\varphi;\Omega})$. In the third section, we investigate novel periodic and fixed point results in the setting of extended quasi-b-gauge space, which generalize and improve the existing results due to Banach and Rus in fixed point theory. The last section consists of some important consequences of the results obtained. Each section consists some examples to explain the corresponding results.

5.1 Extended quasi-b-gauge spaces

In this section, we introduce the notion of extended quasi-b-gauge space. We begin with the development of the notion of extended quasi-pseudo-b metric space.

Definition 5.1.1. The map $q : U \times U \to [0, \infty)$ is called to be an extended quasi-pseudo-b metric, if for all $e, f, g \in U$, there exists $\varphi : U \times U \to [1, \infty)$ satisfying the following conditions:

(a) $q(e, e) = 0$; and

(b) $q(e, g) \leq \varphi(e, g)\{q(e, f) + q(f, g)\}$.

The pair (U, q) is called extended quasi-pseudo-b metric space. A Hausdorff extended quasi-pseudo-b metric space (U, q) satisfies

$$e \neq f \Rightarrow q(e, f) > 0 \vee q(f, e) > 0$$

for all $e, f \in U$.

Example 5.1.2. Let $U = C([0, \infty), \mathbb{R})$ be the space of all continuous real valued function defined on $[0, \infty)$. Define $q : U \times U \to [0, \infty)$ and $\varphi : U \times U \to [1, \infty)$ for all $f, g \in U$ as:

$$q(f(t), g(t)) = \max_{t \in [0,1]} (f(t) - g(t))^2.$$

and

$$\varphi(f, g) = |f(t)| + |g(t)| + 2.$$

Then q is an extended quasi-pseudo-b-metric on U.

We observe that q is not a quasi-pseudo metric on U. For this, we take $f, g, h \in U$ defined by

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1, \\ t - 1 & \text{if } t > 1, \end{cases}$$

$g(t) = 3$ for each $t \geq 0$ and $h(t) = -3$ for each $t \geq 0$. Now we note that $q(g, h) \not\leq q(g, f) + q(f, h)$.

Example 5.1.3. Suppose $U = [0, 1]$. Define $q : U \times U \to [0, \infty)$ and $\varphi : U \times U \to [1, \infty)$ for all $e, f \in U$ as:

$$q(e, f) = \begin{cases} 0 & \text{if } e \geq f, \\ (e - f)^2 & \text{if } e < f, \end{cases}$$

and

$$\varphi(e, f) = e + f + 2.$$

Then q is an extended quasi-pseudo-b metric on U. Certainly, $q(e, e) = 0$, for all $e \in U$. Further, $q(e, g) \leq \varphi(e, g)\{q(e, f) + q(f, g)\}$, for all $e, f, g \in U$ holds. Also, (U, q) is Hausdorff.

Definition 5.1.4. Each family $Q_{\varphi;\Omega} = \{q_\beta : \beta \in \Omega\}$ of extended quasi-pseudo-b metrics $q_\beta : U \times U \to [0,\infty)$ for $\beta \in \Omega$, is said to be an extended quasi b-gauge on U.

Definition 5.1.5. The family $Q_{\varphi;\Omega} = \{q_\beta : \beta \in \Omega\}$ is called to be separating if for every pair (e,f) where $e \neq f$, there exists $q_\beta \in Q_{\varphi;\Omega}$ such that either $q_\beta(e,f) > 0$ or $q_\beta(f,e) > 0$.

Definition 5.1.6. Let the family $Q_{\varphi;\Omega} = \{q_\beta : \beta \in \Omega\}$ be an extended quasi b-gauge on U. The topology $\mathcal{T}(Q_{\varphi;\Omega})$ on U whose subbase is defined by the family $\mathcal{B}(Q_{\varphi;\Omega}) = \{B(e,\epsilon_\beta) : e \in U, \epsilon_\beta > 0, \beta \in \Omega\}$ of all balls $B(e,\epsilon_\beta) = \{f \in U : q_\beta(e,f) < \epsilon_\beta\}$, is called the topology induced by $Q_{\varphi;\Omega}$. The topological space $(U, \mathcal{T}(Q_{\varphi;\Omega}))$ is called to be an extended quasi-b-gauge space, denoted by $(U, Q_{\varphi;\Omega})$. We note that $(U, Q_{\varphi;\Omega})$ is Hausdorff if $Q_{\varphi;\Omega}$ is separating.

Remark 5.1.7. (a) Every quasi-gauge space is an extended quasi-b-gauge space (where $\varphi_\beta(u,v) = 1$ for each $\beta \in \Omega$). Also every quasi-b-gauge space is an extended quasi-b-gauge space (where $\varphi_\beta(u,v) = s_\beta$ for each $\beta \in \Omega$). Therefore, in the asymmetric structure extended quasi-b-gauge space is the largest general space.

(b) Note that if $\varphi_\beta(u,v) = 1$, for each $\beta \in \Omega$, we obtain the definitions in quasi-gauge spaces.

5.2 Extended left (right) $\mathcal{J}_{\varphi;\Omega}$-families of generalize extended quasi-pseudo-b-distances

In the following, we establish the idea of extended left (right) $\mathcal{J}_{\varphi;\Omega}$-families of generalized extended quasi-pseudo-b-distances on U (which are called extended left (right) $\mathcal{J}_{\varphi;\Omega}$-family on U, for short). These extended left (right) $\mathcal{J}_{\varphi;\Omega}$-families are the generalization of extended quasi-b-gauges. Moreover, by using these extended left (right) $\mathcal{J}_{\varphi;\Omega}$-families, the extended left (right) $\mathcal{J}_{\varphi;\Omega}$-sequential completeness are defined and the Banach and Rus types contractions $T : U \to U$ are constructed, which are not necessarily continuous.

Definition 5.2.1. Let $(U, Q_{\varphi;\Omega})$ is an extended quasi-b-gauge space. The family $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ where $J_\beta : U \times U \to [0,\infty)$, $\beta \in \Omega$ is called the extended left (right) $\mathcal{J}_{\varphi;\Omega}$-family of generalized extended quasi-pseudo-b-distances on U if for all $x,y,z \in U$ and for each $\beta \in \Omega$ the following conditions hold:

($\mathcal{J}1$) $J_\beta(x,z) \leq \varphi_\beta(x,z)\{J_\beta(x,y) + J_\beta(y,z)\}$;

($\mathcal{J}2$) for each sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in N)$ in U fulfilling

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(u_m, u_n) = 0, \tag{5.2.1}$$

$$\left(\lim_{m\to\infty} \sup_{n>m} J_\beta(u_n, u_m) = 0\right), \tag{5.2.2}$$

and

$$\lim_{m\to\infty} J_\beta(v_m, u_m) = 0, \tag{5.2.3}$$

$$\left(\lim_{m\to\infty} J_\beta(u_m, v_m) = 0\right), \tag{5.2.4}$$

the following holds:

$$\lim_{m\to\infty} q_\beta(v_m, u_m) = 0, \tag{5.2.5}$$

$$\left(\lim_{m\to\infty} q_\beta(u_m, v_m) = 0\right). \tag{5.2.6}$$

We denote
$\mathbb{J}^L_{(U,Q_{\varphi;\Omega})} = \{\mathcal{J}_{\varphi;\Omega} : \mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ is left $\mathcal{J}_{\varphi;\Omega}$-family on $U\}$,
$\mathbb{J}^R_{(U,Q_{\varphi;\Omega})} = \{\mathcal{J}_{\varphi;\Omega} : \mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ is right $\mathcal{J}_{\varphi;\Omega}$-family on $U\}$.

Example 5.2.2. Let U contains at least two distinct elements and suppose $Q_{\varphi;\Omega} = \{q_\beta : \beta \in \Omega\}$ is the family of extended quasi-pseudo-b-metrics, is an extended quasi-b-gauge on U. Thus $(U, Q_{\varphi;\Omega})$ is an extended quasi-b-gauge space.

Let there are at least two distinct but arbitrary and fixed elements in a set $F \subset U$. Let $a_\beta \in (0,\infty)$ satisfies $\delta_\beta(F) < a_\beta$, where $\delta_\beta(F) = \sup\{q_\beta(e,f) : e, f \in F\}$, for all $\beta \in \Omega$. Let $J_\beta : U \times U \to [0,\infty)$ for all $e, f \in U$ and for all $\beta \in \Omega$ be defined as:

$$J_\beta(e,f) = \begin{cases} q_\beta(e,f) & \text{if } F \cap \{e,f\} = \{e,f\}, \\ a_\beta & \text{if } F \cap \{e,f\} \neq \{e,f\}. \end{cases} \tag{5.2.7}$$

Then $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\} \in \mathbb{J}^L_{(U,Q_{\varphi;\Omega})} \cap \mathbb{J}^R_{(U,Q_{\varphi;\Omega})}$.
We observe that $J_\beta(e,g) \leq \varphi_\beta(e,g)\{J_\beta(e,f) + J_\beta(f,g)\}$, for all $e, f, g \in U$, thus condition ($\mathcal{J}_1$) holds. Indeed, condition ($\mathcal{J}_1$) will not hold in case if there exists some $e, f, g \in U$ such that $J_\beta(e,g) = a_\beta$, $J_\beta(e,f) = q_\beta(e,f)$, $J_\beta(f,g) = q_\beta(f,g)$

and $\varphi_\beta(e,g)\{q_\beta(e,f)+q_\beta(f,g)\} \leq a_\beta$. However, then this implies the existence of $h \in \{e,g\}$ with $h \notin F$ and on other hand, $e,f,g \in F$, which is impossible. Now let that (5.2.1) and (5.2.3) are fulfilled by the sequences (u_m) and (v_m) in U. Then (5.2.3) implies that for all $\beta \in \Omega$ and for all $0 < \epsilon < a_\beta$, there exists $m_1 = m_1(\beta) \in \mathbb{N}$ such that

$$J_\beta(v_m,u_m) < \epsilon \text{ for all } m \geq m_1. \tag{5.2.8}$$

By (5.2.8) and (5.2.7), denoting $m_2 = \min\{m_1(\beta) : \beta \in \Omega\}$, we have

$$F \cap \{v_m,u_m\} = \{v_m,u_m\}, \text{ for all } m \geq m_2$$

and

$$q_\beta(v_m,u_m) = J_\beta(v_m,u_m) < \epsilon.$$

Thus (5.2.5) holds. Therefore, $\mathcal{J}_{\varphi;\Omega}$ is a left $\mathcal{J}_{\varphi;\Omega}$-family.

Moving on the same lines, we can prove that $\mathcal{J}_{\varphi;\Omega}$ is an extended right $\mathcal{J}_{\varphi;\Omega}$-family.

We now state few trivial properties of extended left (right) $\mathcal{J}_{\varphi;\Omega}$-families on U as follows.

Remark 5.2.3. Let $(U, Q_{\varphi;\Omega})$ be an extended quasi-b-gauge space. Then the following is true:

(a) $Q_{\varphi;\Omega} \in \mathbb{J}^L_{(U,Q_{\varphi;\Omega})} \cap \mathbb{J}^R_{(U,Q_{\varphi;\Omega})}$.

(b) Let $\mathcal{J}_{\varphi;\Omega} \in \mathbb{J}^L_{(U,Q_{\varphi;\Omega})}$ or $\mathcal{J}_{\varphi;\Omega} \in \mathbb{J}^R_{(U,Q_{\varphi;\Omega})}$. If for each $\beta \in \Omega$, $J_\beta(v,v) = 0$, for all $v \in U$, then J_β for each $\beta \in \Omega$, is an extended quasi-pseudo-b metric.

(c) There exists example of $\mathcal{J}_{\varphi;\Omega} \in \mathbb{J}^L_{(U,Q_{\varphi;\Omega})}$ and $\mathcal{J}_{\varphi;\Omega} \in \mathbb{J}^R_{(U,Q_{\varphi;\Omega})}$ which shows that the maps J_β, $\beta \in \Omega$ are not extended quasi-pseudo-b metrics.

(d) We note that the above definition reduces to the corresponding definition in quasi-gauge space, if $\varphi_\beta(u,v) = 1$, for all $\beta \in \Omega$ and for all $u,v \in U$.

Proposition 5.2.4. Let $(U, Q_{\varphi;\Omega})$ is a Hausdorff extended quasi-b-gauge space and the family $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ be the extended left (right) $\mathcal{J}_{\varphi;\Omega}$-family on U. Then we can find $\beta \in \Omega$ such that for all $e,f \in U$

$$e \neq f \Rightarrow J_\beta(e,f) > 0 \vee J_\beta(f,e) > 0.$$

Proof. : Let $\mathcal{J}_{\varphi;\Omega}$ is a left $\mathcal{J}_{\varphi;\Omega}$-family on U and suppose that there are $e, f \in U$ with $e \neq f$ such that $J_\beta(e, f) = 0 = J_\beta(f, e)$ for all $\beta \in \Omega$. Then by using property ($\mathcal{J}1$) we have $J_\beta(e, e) = 0$, for all $\beta \in \Omega$.
Defining sequences (u_m) and (v_m) in U by $u_m = e$ and $v_m = f$ or $u_m = f$ and $v_m = e$, we see that conditions (5.2.1) and (5.2.3) of property ($\mathcal{J}2$) are satisfied and therefore condition (5.2.5) holds, which implies that $q_\beta(e, f) = 0 = q_\beta(f, e)$, for all $\beta \in \Omega$. But, this denies the fact that $(U, Q_{\varphi;\Omega})$ is a Hausdorff extended quasi-b-gauge space. Therefore, our supposition is wrong and there exists $\beta \in \Omega$ such that for all $e, f \in U$

$$e \neq f \Rightarrow J_\beta(e, f) > 0 \vee J_\beta(f, e) > 0.$$

Similar proof follows for $\mathcal{J}_{\varphi;\Omega}$ is a right $\mathcal{J}_{\varphi;\Omega}$-family on U. □

Next, we define extended left (right) $\mathcal{J}_{\varphi;\Omega}$-completeness using extended left (right) $\mathcal{J}_{\varphi;\Omega}$-families on U.

Definition 5.2.5. Let $(U, Q_{\varphi;\Omega})$ is an extended quasi-b-gauge space and the family $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$ is an extended left (right) $\mathcal{J}_{\varphi;\Omega}$-family on U.

(A) A sequence $(v_m : m \in \mathbb{N})$ is said to be extended left (right) $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U if for all $\beta \in \Omega$

$$\lim_{m\to\infty} \sup_{n>m} J_\beta(v_m, v_n) = 0$$

$$\Big(\lim_{m\to\infty} \sup_{n>m} J_\beta(v_n, v_m) = 0\Big).$$

(B) The sequence $(v_m : m \in \mathbb{N})$ is said to be extended left (right) $\mathcal{J}_{\varphi;\Omega}$-convergent to $v \in U$ if $\lim^{L-\mathcal{J}_{\varphi;\Omega}}_{m\to\infty} v_m = v$ ($\lim^{R-\mathcal{J}_{\varphi;\Omega}}_{m\to\infty} v_m = v$), where $\lim^{L-\mathcal{J}_{\varphi;\Omega}}_{m\to\infty} v_m = v \Leftrightarrow \lim_{m\to\infty} J_\beta(v, v_m) = 0$, for all $\beta \in \Omega$
$\Big(\lim^{R-\mathcal{J}_{\varphi;\Omega}}_{m\to\infty} v_m = v \Leftrightarrow \lim_{m\to\infty} J_\beta(v_m, v) = 0$, for all $\beta \in \Omega\Big)$.

(C) If $S^{L-\mathcal{J}_{\varphi;\Omega}}_{(v_m:m\in\mathbb{N})} \neq \emptyset$ ($S^{R-\mathcal{J}_{\varphi;\Omega}}_{(v_m:m\in\mathbb{N})} \neq \emptyset$), where
$S^{L-\mathcal{J}_{\varphi;\Omega}}_{(v_m:m\in\mathbb{N})} = \{v \in U : \lim^{L-\mathcal{J}_{\varphi;\Omega}}_{m\to\infty} v_m = v\}$
$\Big(S^{R-\mathcal{J}_{\varphi;\Omega}}_{(v_m:m\in\mathbb{N})} = \{v \in U : \lim^{R-\mathcal{J}_{\varphi;\Omega}}_{m\to\infty} v_m = v\}\Big)$.
Then the sequence $(v_m : m \in \mathbb{N})$ in U is extended left (right) $\mathcal{J}_{\varphi;\Omega}$-convergent in U.

(D) The extended quasi-b-gauge space $(U, Q_{\varphi;\Omega})$ is said to be extended left (right) $\mathcal{J}_{\varphi;\Omega}$-sequentially complete, if every extended left (right) $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U is an extended left (right) $\mathcal{J}_{\varphi;\Omega}$-convergent sequence in U.

Remark 5.2.6. Suppose $(U, Q_{\varphi;\Omega})$ is an extended quasi-b-gauge space.

(a) There exist examples in which $(U, Q_{\varphi;\Omega})$ is left(right) $\mathcal{J}_{\varphi;\Omega}$-sequential complete but not left(right) $Q_{\varphi;\Omega}$-sequential complete (see Example 6.4 of [44]).

(b) If $(v_m : m \in \mathbb{N})$ be an extended left (right) $\mathcal{J}_{\varphi;\Omega}$-convergent sequence in U, then for every of its subsequence $(u_m : m \in \mathbb{N})$ we have $S^{L-\mathcal{J}_{\varphi;\Omega}}_{(v_m:m\in\mathbb{N})} \subset S^{L-\mathcal{J}_{\varphi;\Omega}}_{(u_m:m\in\mathbb{N})} \left(S^{R-\mathcal{J}_{\varphi;\Omega}}_{(v_m:m\in\mathbb{N})} \subset S^{R-\mathcal{J}_{\varphi;\Omega}}_{(u_m:m\in\mathbb{N})}\right)$.

(c) We observe that if $\varphi_\beta(u, v) = 1$ for all $\beta \in \Omega$, we obtain the above definitions in quasi-gauge spaces.

Definition 5.2.7. Suppose $(U, Q_{\varphi;\Omega})$ is an extended quasi-b-gauge space. The map $T^{[k]} : U \to U$ is called an extended left (right) $Q_{\varphi;\Omega}$-quasi-closed map on U, where $k \in \mathbb{N}$, if for every sequence $(z_m : m \in \mathbb{N})$ in $T^{[k]}(U)$, which is extended left (right) $Q_{\varphi;\Omega}$-convergent in U, i.e., $S^{L-Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})} \neq \emptyset$ $(S^{R-Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})} \neq \emptyset)$, and having $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ as its subsequences which satisfy $y_m = T^{[k]}(x_m)$, for all $m \in \mathbb{N}$,
has the property that there exists $z \in S^{L-Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})}(z \in S^{R-Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})})$ such that $z = T^{[k]}(z)(z = T^{[k]}(z))$.

5.3 Periodic and fixed point theorems in extended quasi-b-gauge spaces

Wlodarczyk and Plebaniak [44] have investigated periodic and fixed point theorems in quasi-gauge spaces using $\mathcal{J}$-family of generalized quasi-pseudodistances. Using their technique we present novel periodic and fixed point results in the novel setting of extended quasi-b-gauge space, which generalize and improve the existing results due to Banach and Rus in fixed point theory.

Theorem 5.3.1. Suppose $(U, Q_{\varphi;\Omega})$ is an extended quasi-b-gauge space and let $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \to [0, \infty)$ is an extended left (right) $\mathcal{J}_{\varphi;\Omega}$-family on U such that $(U, Q_{\varphi;\Omega})$ is extended left (right) $\mathcal{J}_{\varphi;\Omega}$-sequentially complete. Let for all $v \in U$ and for all $\beta \in \Omega$, there exists $\mu_\beta \in [0, 1)$ such that $T : U \to U$ and $\varphi_\beta : U \times U \to [1, \infty)$ satisfy:

$$J_\beta(T(v), T^2(v)) \leq \mu_\beta J_\beta(v, T(v)) \qquad \text{(C1)}$$

and $\lim_{n,m\to\infty} \varphi_\beta(z^m, z^n)\mu_\beta < 1$, for each $z^0 \in U$, here $(z^m = T^{[m]}(z^0) : m \in \mathbb{N})$.
Then the statements below are satisfied:

(I) The sequence $(z^m : m \in \{0\} \cup \mathbb{N})$, for any $z^0 \in U$, is extended left (right) $Q_{\varphi;\Omega}$-convergent in U; thus $S^{L-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset \Big(S^{R-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset \Big)$.

(II) Moreover, suppose that $T^{[k]}$ is an extended left (right) $Q_{\varphi;\Omega}$-quasi-closed map on U, for some $k \in \mathbb{N}$. Then

- (a_1) $\mathrm{Fix}(T^{[k]}) \neq \emptyset$;
- (a_2) there exists $z \in \mathrm{Fix}(T^{[k]})$ such that $z \in S^{L-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \Big(z \in S^{R-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \Big)$ and
- (a_3) for all $z \in \mathrm{Fix}(T^{[k]})$, $J_\beta(z, T(z)) = J_\beta(T(z), z) = 0$, for all $\beta \in \Omega$.

(III) Furthermore, let $\mathrm{Fix}(T^{[k]}) \neq \emptyset$ for some $k \in \mathbb{N}$ and $(U, Q_{\varphi;\Omega})$ is a Hausdorff space. Then

- (b_1) $\mathrm{Fix}(T^{[k]})$=$\mathrm{Fix}(T)$;
- (b_2) there exists $z \in \mathrm{Fix}(T)$ such that $z \in S^{L-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \Big(z \in S^{R-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \Big)$ and
- (b_3) for all $z \in \mathrm{Fix}(T)$, $J_\beta(z, z) = 0$, for all $\beta \in \Omega$.

Proof. (I) We first show that $(z^m : m \in \{0\} \cup \mathbb{N})$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U.
Let $\beta \in \Omega$ and $z^0 \in U$ is fixed and arbitrary. Also using (C1) and ($\mathcal{J}1$) repeatedly, for $n, m \in \mathbb{N}$ such that $n > m$, we get

$$\begin{aligned} J_\beta(z^m, z^n) &\leq \varphi_\beta(z^m, z^n)\mu_\beta^m J_\beta(z^0, z^1) + \varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n)\mu_\beta^{m+1} J_\beta(z^0, z^1) \\ &\quad + \varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n)\varphi_\beta(z^{m+2}, z^n)\mu_\beta^{m+2} J_\beta(z^0, z^1) \\ &\quad + ... + \varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n)...\varphi_\beta(z^{n-1}, z^n)\mu_\beta^{n-1} J_\beta(z^0, z^1) \\ &\leq J_\beta(z^0, z^1)[\varphi_\beta(z^1, z^n)\varphi_\beta(z^2, z^n)...\varphi_\beta(z^m, z^n)\mu_\beta^m \\ &\quad + \varphi_\beta(z^1, z^n)\varphi_\beta(z^2, z^n)...\varphi_\beta(z^m, z^n)\varphi_\beta(z^{m+1}, z^n)\mu_\beta^{m+1} \\ &\quad + ... + \varphi_\beta(z^1, z^n)\varphi_\beta(z^2, z^n)...\varphi_\beta(z^m, z^n)...\varphi_\beta(z^{n-1}, z^n)\mu_\beta^{n-1}]. \end{aligned}$$

Since, $\lim_{n,m\to\infty} \varphi_\beta(z^{m+1}, z^n)\mu_\beta < 1$, so that the series $\sum_{m=1}^\infty \mu_\beta^m \prod_{i=1}^m \varphi_\beta(z^i, z^n)$ converges by ratio test.
Let $S = \sum_{m=1}^\infty \mu_\beta^m \prod_{i=1}^m \varphi_\beta(z^i, z^n)$ and $S_m = \sum_{j=1}^m \mu_\beta^j \prod_{i=1}^j \varphi_\beta(z^i, z^n)$.
This gives

$$J_\beta(z^m, z^n) \leq J_\beta(z^0, z^1)[S_{n-1} - S_m].$$

Letting $m \to \infty$, we have $(z^m : m \in \{0\} \cup \mathbb{N})$ is an extended $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence in U, thus for all $\beta \in \Omega$, for all $\epsilon > 0$ and for all $n, m \in \mathbb{N}$, there is $l \in \mathbb{N}$ such that

$$J_\beta(z^m, z^n) < \epsilon, \text{ for all } n \geq m \geq l. \tag{5.3.1}$$

Now, since $(U, Q_{\varphi;\Omega})$ is an extended left (right) $\mathcal{J}_{\varphi;\Omega}$-sequentially complete space, for $z \in S^{L-\mathcal{J}_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$, we can write for all $\beta \in \Omega$, for all $\epsilon > 0$ and for all $m \in \mathbb{N}$, there is $l \in \mathbb{N}$ such that

$$J_\beta(z, z^m) < \epsilon, \ \text{ for all } \ m \geq l. \tag{5.3.2}$$

Thus from (5.3.1) and (5.3.2), fixing $z \in S^{L-\mathcal{J}_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N}}$, defining $(u_m = z^m : m \in \{0\} \cup \mathbb{N})$ and $(v_m = z : m \in \{0\} \cup \mathbb{N})$ and applying $(\mathcal{J}2)$ to these sequences, we get, for all $\beta \in \Omega$, for all $\epsilon > 0$ and for all $m \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that

$$q_\beta(z, z^m) < \epsilon, \ \text{ for all } \ m \geq l.$$

This implies $S^{L-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$.
(II) To prove assertion (a_1), let $z^0 \in U$ is fixed and arbitrary. Since $S^{L-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$ and for $m \in \{0\} \cup \mathbb{N}$, we write

$$z^{(m+1)k} = T^{[k]}(z^{mk}),$$

thus defining $(z_m = z^{m-1+k} : m \in \mathbb{N})$, we get

$$(z_m : m \in \mathbb{N}) \subset T^{[k]}(U),$$

$$S^{L-Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{L-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset,$$

also, its subsequences

$$(y_m = z^{(m+1)k}) \subset T^{[k]}(U)$$

and

$$(x_m = z^{mk}) \subset T^{[k]}(U)$$

satisfy

$$y_m = T^{[k]}(x_m), \ \text{ for all } \ m \in \mathbb{N}$$

and are extended left $Q_{\varphi;\Omega}$-convergent to each point $z \in S^{L-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$. Now, since

$$S^{L-Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{L-Q_{\varphi;\Omega}}_{(y_m:m\in\mathbb{N})} \quad \text{and} \quad S^{L-Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})} \subset S^{L-Q_{\varphi;\Omega}}_{(x_m:m\in\mathbb{N})}.$$

Using above and the assumption that $T^{[k]}$ for some $k \in \mathbb{N}$ is an extended left (right) $Q_{\varphi;\Omega}$-quasi-closed map on U, there exists $z \in S^{Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})} = S^{Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})}$ such that $z = T^{[k]}(z)$. Thus assertion (a_1) holds.
Now, the assertion (a_2) follows from assertion (a_1).

To prove assertion (a_3), let $z \in \text{Fix}(T^{[k]})$ is fixed and arbitrary and on contrary suppose that there exists $\beta_0 \in \Omega$ such that $J_{\beta_0}(z, T(z)) > 0$. Now, using $T^{[k]}(z) = T^{[2k]}(z) = z$ and (C1), we have

$$\begin{aligned} J_{\beta_0}(z, T(z)) &= J_{\beta_0}(T^{[2k]}(z), T^{[2k]}(T(z))) \\ &\leq \mu_{\beta_0} J_{\beta_0}(T(T^{[2k-2]}(z)), T^{[2]}(T^{[2k-2]}(z))) \\ &\leq \mu_{\beta_0}^2 J_{\beta_0}(T^{[2k-2]}(z), T(T^{[2k-2]}(z))) \\ &\leq ... \leq \mu_{\beta_0}^{2k} J_{\beta_0}(z, T(z)) < J_{\beta_0}(z, T(z)), \end{aligned}$$

which is illogical. Thus $J_\beta(z, T(z)) = 0$, for all $\beta \in \Omega$.
Now let $J_{\beta_0}(T(z), z) > 0$ for some $\beta_0 \in \Omega$. Using $z = T^{[k]}(z) = T^{[2k]}(z)$, $(\mathcal{J}1)$, (C1) and the facts that $k+1 < 2k$ and $J_{\beta_0}(z, T(z)) = 0$, we can write

$$\begin{aligned} 0 < J_{\beta_0}(T(z), z) &= J_{\beta_0}(T(T^{[k]}(z)), T^{[2k]}(z)) \\ &= J_{\beta_0}(T^{[k+1]}(z), T^{[2k]}(z)) \\ &\leq \varphi_{\beta_0}(T^{[k+1]}(z), T^{[2k]}(z)) \mu_{\beta_0}^{k+1} J_{\beta_0}(z, T(z)) \\ &+ \varphi_{\beta_0}(T^{[k+1]}(z), T^{[2k]}(z)) \varphi_{\beta_0}(T^{[k+2]}(z), T^{[2k]}(z)) \mu_{\beta_0}^{k+2} J_{\beta_0}(z, T(z)) \\ &+ ... + \varphi_{\beta_0}(T^{[k+1]}(z), T^{[2k]}(z)) \varphi_{\beta_0}(T^{[k+2]}(z), T^{[2k]}(z)) \\ &... \varphi_{\beta_0}(T^{[2k-1]}(z), T^{[2k]}(z)) \mu_{\beta_0}^{2k-1} J_{\beta_0}(z, T(z)) = 0, \end{aligned}$$

which is absurd. Thus $J_\beta(T(z), z) = 0$, for all $\beta \in \Omega$.
Hence, the assertion (a_3) holds.
(III) Since $(U, Q_{\varphi;\Omega})$ is a Hausdorff space, using Proposition (5.2.4), assertion (a_3) suggests that for $z \in \text{Fix}(T^{[k]})$, we have $z = T(z)$. This gives $z \in \text{Fix}(T)$. Hence (b_1) is true.
Assertions (a_2) and (b_1) imply (b_2) .
To prove assertion (b_3), consider $(\mathcal{J}1)$ and use (a_3) and (b_1), we have for all $z \in \text{Fix}(T^{[k]}) = \text{Fix}(T)$,
$J_\beta(z, z) \leq \varphi(z, z)\{J_\beta(z, T(z)) + J_\beta(T(z), z)\} = 0$ for all $\beta \in \Omega$. □

Theorem 5.3.2. Suppose $(U, Q_{\varphi;\Omega})$ is an extended quasi-b-gauge space and let $\mathcal{J}_{\varphi;\Omega} = \{J_\beta : \beta \in \Omega\}$, where $J_\beta : U \times U \to [0, \infty)$ is an extended left (right) $\mathcal{J}_{\varphi;\Omega}$-family on U such that $(U, Q_{\varphi;\Omega})$ is extended left (right) $\mathcal{J}_{\varphi;\Omega}$-sequentially complete. Let for all $u, v \in U$ and for all $\beta \in \Omega$, there exists $\mu_\beta \in [0, 1)$ such that $T : U \to U$ and $\varphi_\beta : U \times U \to [1, \infty)$ satisfy:

$$J_\beta(T(u), T(v)) \leq \mu_\beta J_\beta(u, v) \qquad \text{(C2)}$$

and $\lim_{n,m\to\infty} \varphi_\beta(z^m, z^n)\mu_\beta < 1$, for each $z^0 \in U$, here $(z^m = T^{[m]}(z^0) : m \in \mathbb{N})$.
Then the statements below are satisfied:

(I) The sequence $(z^m : m \in \{0\} \cup \mathbb{N})$, for any $z^0 \in U$, is extended left (right) $Q_{\varphi;\Omega}$-convergent in U; thus $S^{L-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset \Big(S^{R-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset \Big)$.

(II) Moreover, suppose that $T^{[k]}$ is an extended left (right) $Q_{\varphi;\Omega}$-quasi-closed map on U, for some $k \in \mathbb{N}$. Then

- (c_1) $\text{Fix}(T^{[k]}) \neq \emptyset$;
- (c_2) there exists $z \in \text{Fix}(T^{[k]})$ such that $z \in S^{L-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} (z \in S^{R-Q_{\varphi;\Omega}}_{(w^m:m\in\{0\}\cup\mathbb{N})})$ and
- (c_3) for all $z \in \text{Fix}(T^{[k]})$, $J_\beta(z, T(z)) = J_\beta(T(z), z) = 0$, for all $\beta \in \Omega$.

(III) Furthermore, let $\text{Fix}(T^{[k]}) \neq \emptyset$ for some $k \in \mathbb{N}$ and $(U, Q_{\varphi;\Omega})$ is a Hausdorff space. Then

- (d_1) $\text{Fix}(T^{[k]}) = \text{Fix}(T) = \{z\}$ for some $z \in U$;
- (d_2) there exists $z \in \text{Fix}(T)$ such that $z \in S^{L-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})} (z \in S^{R-Q_{\varphi;\Omega}}_{(z^m:m\in\{0\}\cup\mathbb{N})})$ and
- (d_3) $J_\beta(z, z) = 0$, for all $\beta \in \Omega$.

Proof. Since each map T that is satisfying (C2) also satisfies (C1) therefore, it is suffices to show that (d_1) holds. We observe that for $y, z \in \text{Fix}(T)$ such that $y \neq z$, (C2) gives for all $\beta \in \Omega$, there exists $\mu_\beta \in [0, 1)$ such that

$$[J_\beta(y, z) \leq \mu_\beta J_\beta(y, z)] \wedge [J_\beta(z, y) \leq \mu_\beta J_\beta(z, y)].$$

However, using Proposition (5.2.4), as $y \neq z$, we can write that there exist $\beta_0 \in \Omega$ such that

$$[J_{\beta_0}(y, z) > 0] \vee [J_{\beta_0}(z, y) > 0].$$

This implies

$$[J_{\beta_0}(y, z) > 0 \wedge J_{\beta_0}(y, z) \leq \mu_{\beta_0} J_{\beta_0}(y, z)] \vee [J_{\beta_0}(z, y) > 0 \wedge J_{\beta_0}(z, y) \leq \mu_{\beta_0} J_{\beta_0}(z, y)].$$

Which is absurd. Hence $\text{Fix}(T) = \{z\}$.
By (d_1), we notice that (d_2) and (d_3) can be obtained from (b_2) and (b_3) respectively. □

Remark 5.3.3. The proof of right case for above theorems follow on the same lines.

Example 5.3.4. Suppose $U = [0,1]$ and $Q_{\varphi;\Omega} = \{q\}$, where $q : U \times U \to [0,\infty)$ is an extended quasi-pseudo-b metric on U defined for all $e, f \in U$ by

$$q(e,f) = \begin{cases} 0 & \text{if } e \geq f, \\ (e-f)^2 & \text{if } e < f, \end{cases} \tag{5.3.3}$$

and $\varphi : U \times U \to [1,\infty)$ is defined as:

$$\varphi(e,f) = e + f + 2.$$

Let the set $F = [\frac{1}{8}, 1] \subset U$ and let $J : U \times U \to [0,\infty)$ for all $e, f \in U$ be defined as:

$$J(e,f) = \begin{cases} q(e,f) & \text{if } F \cap \{e,f\} = \{e,f\}, \\ 4 & \text{if } F \cap \{e,f\} \neq \{e,f\}, \end{cases} \tag{5.3.4}$$

and $\varphi(e,f) = e + f + 2$.
$T : U \to U$ is described as

$$T(e) = \begin{cases} \frac{e}{2} + \frac{1}{4} & \text{if } e \in [0, \frac{1}{2}), \\ \frac{1}{2} & \text{if } e \in [\frac{1}{2}, 1]. \end{cases} \tag{5.3.5}$$

(I.1) $(U, Q_{\varphi;\Omega})$ is Hausdorff extended quasi-b-gauge space (see Example 5.1.3).

(I.2) $\mathcal{J}_{\varphi;\Omega} = \{J\}$ is an extended left $\mathcal{J}_{\varphi;\Omega}$-family on U (see Example 5.2.2).

(I.3) $(U, Q_{\varphi;\Omega})$ is extended left $\mathcal{J}_{\varphi;\Omega}$-sequential complete.
For this, let $\{v_m : m \in \mathbb{N}\}$ is an extended left $\mathcal{J}_{\varphi;\Omega}$-Cauchy sequence. Without loosing generality, let for $0 < \epsilon_1 < \frac{1}{64}$ and for all $n, m \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that

$$J(v_m, v_n) < \epsilon_1 < \frac{1}{64}, \text{ for all } n \geq m \geq k_0. \tag{5.3.6}$$

Then by using (5.3.6), (5.3.4) and (5.3.3), we get

$$J(v_m, v_n) = q(v_m, v_n) =| v_m - v_n |^2 < \epsilon_1 < \frac{1}{64}, \text{ for all } n \geq m \geq k_0. \tag{5.3.7}$$

$$v_m \in F = [\frac{1}{8}, 1], \text{ for all } m \geq k_0. \tag{5.3.8}$$

Rewriting (5.3.7) as

$$| v_m - v_n | < \epsilon < \frac{1}{8}, , \text{ for all } n \geq m \geq k_0$$

where $\epsilon = \sqrt{\epsilon_1}$.
Now since $(\mathbb{R}, |\ .\ |)$ is complete, $F = [\frac{1}{8}, 1]$ is closed in $\mathbb{R}$, also, $v_m \in F = [\frac{1}{8}, 1]$ by (5.3.8) and $\{v_m : m \in \mathbb{N}\}$ is Cauchy with respect to $|\ .\ |$, hence for all $0 < \epsilon < \frac{1}{8}$ and for all $m \in \mathbb{N}$, there exists $k_1 \in \mathbb{N}$ and we can find $v \in F$ such that

$$| v - v_m | < \epsilon, \text{ for all } m \geq k_1.$$

Hence, $\{v_m : m \in \mathbb{N}\}$ is extended left $\mathcal{J}_{\varphi;\Omega}$-convergent to v.
This implies $(U, Q_{\varphi;\Omega})$ is extended left $\mathcal{J}_{\varphi;\Omega}$-sequential complete.

(I.4) Next we show that T satisfies condition (C1) for $\mu = \frac{3}{4}$.
The following cases are considered.
Case-1: Let $e = 0$, then $T(e) = \frac{1}{4} \in F$ and $T^2(e) = \frac{3}{8} \in F$, then by using (5.3.4) and (5.3.3), we have

$$\begin{aligned} J(T(e), T^2(e)) &= q(T(e), T^2(e)) \\ &= | T(e) - T^2(e) |^2 \\ &= \left| \frac{1}{4} - \frac{3}{8} \right|^2 = \frac{1}{64} < \frac{3}{4} 4 = \mu J(e, T(e)); \end{aligned}$$

Case-2: Let $e \in (0, \frac{1}{8})$, then $\frac{1}{4} < T(e) < \frac{5}{16} < \frac{1}{2}$, this implies that $T(e) \in F$. Similarly $\frac{1}{4} < \frac{3}{8} < T^2(e) < \frac{13}{32} < \frac{1}{2}$, which implies that $T^2(e) \in F$. This gives

$$\begin{aligned} J(T(e), T^2(e)) &= q(T(e), T^2(e)) \\ &= | T(e) - T^2(e) |^2 \\ &= \left| \frac{e}{2} + \frac{1}{4} - \frac{e}{4} - \frac{3}{8} \right|^2 \\ &= \left| \frac{e}{4} - \frac{1}{8} \right|^2 \leq \frac{1}{64} < \frac{3}{4} 4 = \mu J(e, T(e)); \end{aligned}$$

Case-3: Let $e \in [\frac{1}{8}, \frac{1}{2})$, then $\frac{5}{16} < T(e) < \frac{3}{8} < \frac{1}{2}$, this implies that $T(e) \in F$, similarly $\frac{13}{32} < T^2(e) < \frac{7}{16} < \frac{1}{2}$, which implies that $T^2(e) \in F$. This gives

$$\begin{aligned} J(T(e), T^2(e)) &= q(T(e), T^2(e)) \\ &= | T(e) - T^2(e) |^2 \\ &= \left| \frac{e}{2} + \frac{1}{4} - \frac{e}{4} - \frac{3}{8} \right|^2 \\ &= \left| \frac{e}{4} - \frac{1}{8} \right|^2 \leq \frac{1}{64} < \frac{3}{4} 4 = \mu J(e, T(e)); \end{aligned}$$

Case-4: Let $e \in [\frac{1}{2}, 1]$, then $\{T(e), T^2(e)\} = \frac{1}{2}$ and $\{T(e), T^2(e)\} \in F$. This gives

$$\begin{aligned} J(T(e), T^2(e)) &= q(T(e), T^2(e)) \\ &= \mid T(e) - T^2(e) \mid^2 \\ &= \left| \frac{1}{2} - \frac{1}{2} \right|^2 = 0 < \mu J(e, T(e)). \end{aligned}$$

Hence, T satisfies condition (C1) for $\mu = \frac{3}{4}$.

(I.5) Finally, we show that T is extended left $Q_{\varphi;\Omega}$-quasi-closed map on U. Let the arbitrary and fixed sequence $(z_m : m \in \mathbb{N})$ belongs to $T(U) = [\frac{1}{4}, \frac{1}{2}]$ which is extended left $Q_{\varphi;\Omega}$-convergent to $S^{L-Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})}$ and let the subsequences (y_m) and (x_m) are satisfying $y_m = T(x_m)$, for all $m \in \mathbb{N}$. Let $z \in S^{L-Q_{\varphi;\Omega}}_{(z_m:m\in\{0\}\cup\mathbb{N})}$ be fixed and arbitrary. Then this implies for all $\epsilon_1 > 0$ and for all $m \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that

$$q(z, z^m) < \epsilon_1, \text{ for all } m \geq k.$$

As a result,

$$\begin{aligned} &[\mid z - z_m \mid < \epsilon] \wedge [\mid z - x_m \mid < \epsilon] \\ &\wedge [\mid z - y_m \mid < \epsilon] \wedge [y_m = T(x_m)], \end{aligned}$$

where $\epsilon = \sqrt{\epsilon_1}$.
We see that $S^{L-Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})} = \{\frac{1}{2}\}$, otherwise
$z \in S^{L-Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})} \setminus \{\frac{1}{2}\}$ and then for all $\epsilon > 0$ and for all $m \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that

$$\begin{aligned} &[\mid z - z_m \mid < \epsilon] \wedge [\mid z - x_m \mid < \epsilon] \\ &\wedge [\mid z - y_m \mid < \epsilon] \wedge [y_m = T(x_m)], \text{ for all } m \geq k. \end{aligned}$$

However, this implies the following

$$\mid z - x_m \mid = \left| z - 2y_m + \frac{1}{2} \right| = \left| \frac{1}{2} - z + 2(z - y_m) \right| < \epsilon$$

and we get

$$\left| \frac{1}{2} - z \right| < \epsilon + 2 \mid z - y_m \mid, \text{ for all } m \geq k.$$

Now since $\mid z - y_m \mid \to 0$, when $m \to \infty$, we get $\mid \frac{1}{2} - z \mid < \epsilon$, which is a contradiction. Thus we have shown that
$S^{L-Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})} = \{\frac{1}{2}\}$ and there exists $z = \frac{1}{2} \in S^{L-Q_{\varphi;\Omega}}_{(z_m:m\in\mathbb{N})}$ such that $\frac{1}{2} = T(\frac{1}{2})$.
Hence, T is extended left $Q_{\varphi;\Omega}$-quasi-closed map on U.

(I.6) Hence by using Theorem 5.3.1, we can write

$$\text{Fix}(T) = \frac{1}{2},$$

$$\lim_{m\to\infty}^{L-Q_{\varphi;\Omega}} z^m = \frac{1}{2},$$

and

$$J(\frac{1}{2}, \frac{1}{2}) = 0.$$

Remark 5.3.5. By taking $\varphi_\beta(u, v) = s$, where $s \geq 1$, for each $\beta \in \Omega$, we attain the above results in quasi-b-gauge space.

5.4 Consequences and applications

This section consists of some fascinating consequences of obtained results.

Corollary 5.4.1. Suppose (U, Q) is a quasi-gauge space and $\mathcal{J} = \{J_\beta : \beta \in \Omega\}$ where, $J_\beta : U \times U \to [0, \infty)$ is left (right) $\mathcal{J}$-family on U. Assume that, (U, Q) is left (right) $\mathcal{J}$-sequentially complete and let for all $v \in U$ and for all $\beta \in \Omega$, there exists $\mu_\beta \in [0, 1)$ such that $T : U \to U$ satisfies:

$$J_\beta(T(v), T^2(v)) \leq \mu_\beta J_\beta(v, T(v)). \qquad (G1)$$

Then the statements below are fulfilled:

(I) The sequence $(z^m : m \in \{0\} \cup \mathbb{N})$, for any $z^0 \in U$, is left (right) Q-convergent in U; thus $S^{L-Q}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset (S^{R-Q}_{(z^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset)$.

(II) Moreover, suppose that $T^{[k]}$ is left (right) Q-quasi-closed map on U, for some $k \in \mathbb{N}$. Then

- (u_1) $\text{Fix}(T^{[k]}) \neq \emptyset$;
- (u_2) there exists $z \in \text{Fix}(T^{[k]})$ such that $z \in S^{L-Q}_{(z^m:m\in\{0\}\cup\mathbb{N})} (z \in S^{R-Q}_{(z^m:m\in\{0\}\cup\mathbb{N})})$ and
- (u_3) for all $z \in \text{Fix}(T^{[k]})$, $J_\beta(z, T(z)) = J_\beta(T(z), z) = 0$, for all $\beta \in \Omega$.

(III) Furthermore, let $\text{Fix}(T^{[k]}) \neq \emptyset$ for some $k \in \mathbb{N}$ and (U, Q) is a Hausdorff space. Then

- (v_1) $\text{Fix}(T^{[k]})$ =$\text{Fix}(T)$;
- (v_2) there exists $z \in \text{Fix}(T)$ such that $z \in S^{L-Q}_{(x^m:m\in\{0\}\cup\mathbb{N})} (z \in S^{R-Q}_{(z^m:m\in\{0\}\cup\mathbb{N})})$ and

(v$_3$) for all $z \in \mathrm{Fix}(T)$, $J_\beta(z,z) = 0$, for all $\beta \in \Omega$.

Proof. The proof follows directly from the proof of Theorem (5.3.1) if we take $\varphi_\beta(u,v) = 1$, for each $\beta \in \Omega$. □

Corollary 5.4.2. Suppose (U,Q) is a quasi-gauge space and $\mathcal{J} = \{J_\beta : \beta \in \Omega\}$ where, $J_\beta : U \times U \to [0,\infty)$ is left (right) $\mathcal{J}$-family on U. Assume that, (U,Q) is left (right) $\mathcal{J}$-sequentially complete and let for all $u, v \in U$ and for all $\beta \in \Omega$ there is $\mu_\beta \in [0,1)$ such that $T : U \to U$ satisfies:

$$J_\beta(T(u), T(v)) \leq \mu_\beta J_\beta(u,v). \qquad (G2)$$

Then the following hold:

(I) The sequence $(z^m : m \in \{0\} \cup \mathbb{N})$, for any $z^0 \in U$, is left (right) Q-convergent in U; thus $S^{L-Q}_{(z^m : m\in\{0\}\cup\mathbb{N})} \neq \emptyset \left(S^{R-Q}_{(z^m : m\in\{0\}\cup\mathbb{N})} \neq \emptyset\right)$.

(II) Moreover, suppose that $T^{[k]}$ is left (right) Q-quasi-closed map on U, for some $k \in \mathbb{N}$. Then

- (s$_1$) $\mathrm{Fix}(T^{[k]}) \neq \emptyset$;
- (s$_2$) there exists $z \in \mathrm{Fix}(T^{[k]})$ such that $z \in S^{L-Q}_{(z^m : m\in\{0\}\cup\mathbb{N})} \left(z \in S^{R-Q}_{(z^m : m\in\{0\}\cup\mathbb{N})}\right)$ and
- (s$_3$) for all $z \in \mathrm{Fix}(T^{[k]})$, $J_\beta(z, T(z)) = J_\beta(T(z), z) = 0$, for all $\beta \in \Omega$.

(III) Furthermore, let $\mathrm{Fix}(T^{[k]}) \neq \emptyset$ for some $k \in \mathbb{N}$ and (U,Q) is a Hausdorff space. Then

- (t$_1$) $\mathrm{Fix}(T^{[k]})$ =$\mathrm{Fix}(T) = \{z\}$ for some $z \in U$;
- (t$_2$) there exists $z \in \mathrm{Fix}(T)$ such that $z \in S^{L-Q}_{(z^m : m\in\{0\}\cup\mathbb{N})}\} \left(z \in S^{R-Q}_{(z^m : m\in\{0\}\cup\mathbb{N})}\right)$ and
- (t$_3$) $J_\beta(z,z) = 0$, for all $\beta \in \Omega$.

Proof. The proof follows directly from the proof of Theorem (5.3.2) if we take $\varphi_\beta(u,v) = 1$, for each $\beta \in \Omega$. □

Remark 5.4.3. (a) We note that Corollary 5.4.1 and Corollary 5.4.2 are Theorem 4.2 and Theorem 4.1 of Wlodarczyk and Plebaniak [44] respectively. Hence our theorems are generalization of their results.

(b) The proof of fixed point theorem due to Banach [1] and Rus [4] (see also [79]) require the completeness of the metric space (U, q), the continuity of q and the continuity of the mappings T. On other hand, our Theorem 5.3.1 and Theorem 5.3.2 remove these assumptions and leaving the assertions more general. Hence our results are new generalization of the fixed point theorems due to Banach and Rus.

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