ON GENERALIZED COMMUTATIVE RINGS AND RELATED STRUCTURES

By

Inayatur-Rehman

Department of Mathematics
Quaid-i-Azam University, Islamabad
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Supervised

By

Dr. Tariq Shah

Department of Mathematics
Quaid-i-Azam University, Islamabad
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A thesis submitted in the partial fulfillment of the requirements for the degree of Doctor of Philosophy

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We accept this thesis as conforming to the required standard.

1. Dr. Tariq Shah (Supervisor)

2. Professor Dr. Muhammad Ayub (Chairperson)

3. Professor Dr. Muhammad Aslam

4. Lt. Col. Dr. Muhammad Ashiq

Department of Mathematics
Quaid-i-Azam University
Islamabad
2011

2011

Dedicated to:

The memories of my late uncle (Sardar M. Aslam Khan).

Preface

The study of commutative rings originated from the theory of algebraic number and algebraic geometry [3]. In 1921, Emmy Noether gave the first axiomatic foundation of the theory of commutative rings in her monumental paper "Ideal theory in rings". The genesis of the theory of commutative rings dates back to the early 19th century while its maturity was achieved in third decade of 20th century.

For any type of abstract algebra, a generalization is a defined class of such algebra. Of course, a generalization of a concept is an extension of the concept to less specific criteria. Generalization plays a vital role to enhance a mathematical concept and to walk around the tracks which leads to achieve new goals.

Since the introduction of the concept of commutative rings, much progress has been made by many researchers in the development of this concept through generalization [28, 29, 39].

In 1972, a generalization of a commutative semigroup has been introduced by Kazim and Naseeruddin [26]. In ternary commutative law, $abc = cba$, they introduced braces on the left of this law and explored a new pseudo associative law, that is $(ab)c = (cb)a$. This they called the left invertive law. A groupoid satisfying the left invertive law is called a left almost semigroup and is abbreviated as LA-semigroup. Despite the fact that the structure is non-associative and non-commutative, however it possesses properties which usually come across in associative and commutative algebraic structures.

In 1978, Mushtaq and Yusuf [50] established some interesting and useful results in the theory of LA-semigroups. In [51], they introduced locally associative LA-semigroups.

Later in 1991, Mushtaq and Iqbal [43] did an extensive study in decomposition of locally associative LA-semigroup. One can see that Mushtaq and his associates have a remarkable contribution to strengthen and explore new tracks in the theory of LA-semigroups.

In 1994, Protic and Stevanovic [56] have used the name Abel-Grassmann's groupoids(abbreviated as AG-groupoids), instead of LA-semigroups. For more details of their spad work regarding AG*-groupoids and AG**-groupoids, we refer [57, 58, 59, 69]. Mushtaq and Khan [46, 47, 48, 49] also focused on AG*-groupoids and AG**-groupoids and have done extensive work.

In 1996, Mushtaq and Kamran [45] extended this concept to left almost group (abbreviated as LA-group). An LA-semigroup is called an LA-group if (i) there exist a left identity $e \in G$ such that $ea = a$ for all $a \in G$, and (ii) for every $a \in G$ there exists a left inverse $a' \in$ G such that $a'a = e$. For more details see [25].

The fundamental questionnaire like, Why's, What's, If's, always insist the human mind to explore an unending task, to move from known to unknown and from visible to invisible. The passionate purposefulness has always led to the opening of new outlook of expanse.

Keeping in view the fundamental questionnaire, we thought of a purpose to generalize commutative rings by an outcome of LA-semigroups and LA-groups. Actually, this offshoot is a non-associative and a non-commutative structure, known as left almost rings (abbreviated as LA-rings), introduced by Yusuf [70]. Left almost rings are in fact a generalization of commutative rings and carries attraction due to its peculiar characteristics and structural formation.

This thesis comprises 8 chapters and on the whole it is a threefold study. In first phase we discuss AG-groupoids and Γ-AG-groupoids which are in fact single (binary) operational structures. These structures have been investigated in chapters 2 and 5. In second phase, we deal with two (binary) operational structures such as LA-rings and Γ-LA-rings. These concepts have been studied in chapters 3, 4 and 6. While in the third phase, we look into the application side and investigated the fuzzy concepts of these algebraic structures.

In chapter one, a brief history of LA-semigroups, LA-groups, Γ-semigroups and Γ-rings has been discussed. Moreover, some basics of fuzzy concepts have been provided. We have also included the fundamental information about these structures which are directly related to our study.

Chapter 2 contains two sections. In section one; we have discussed the ordering of AG-groupoids. Actually the purpose of defining ordering has been based to tackle the degree questions in LA-ring of finitely non-zero functions, which have been discussed in chapter 4 (section 2). In section 2, we have included some results regarding ideals, M-systems and N-systems in ordered AG-groupoids.

The contents of chapter 2, in the form of two papers have been published (accepted) in;

(i) **Far East J. Mathematical Sciences,** 47(2010), 13-21.
(ii) **Int. Electronic J. Pure and Applied Mathematics (**to appear**).**

Chapter 3 is distributed into three sections. In section one, we have provided some elementary properties of LA-rings. In second section, we have discussed ideals, M-system, P-system, I-system and subtractive sets in LA-rings. We have proved that if R is an LA-ring with left identity e, then R is fully prime if and only if every ideal is idempotent and the set ideal(R) is totally ordered under inclusion. Further, a left ideal I of an LA-ring R with left identity e is quasi-prime if and only if R\I is an M-system. Also we have shown that every subtractive subset of an LA-ring R is semi-subtractive. In section 3 we continued the developments made in section 2, regarding elementary concepts of ideals in LA-rings. We have taken a step forward to study the direct sum in LA-rings and have established some criterion for LA-ring to be the direct sum of its ideals.

In chapter 4, our study is distributed into two sections. In section one, we generalize the structure of commutative semigroup ring (ring of semigroup S over ring R represented as R[X; S]) to a non-associative LA-ring of commutative semigroup S over LA-ring R represented as $R[X^s; s \in S]$, consisting of finitely nonzero functions. Nevertheless it also possesses associative ring structures.

The motivational source behind this study is the book: Commutative semigroup rings, the University Chicago, Press, 1984 (see [13]). During the construction of LA-ring represented as $R[X^s; s \in S]$, we have adopted the analogous way as in [13]. In this study we have obtained various generalizations parallel to corresponding parts of commutative semigroup rings. For this we mostly followed [12, 13, 20, 54]. Some results concerning homomorphisms of LA-ring $R[X^s; s \in S]$ have been included. We have also introduced the concept of LA-modules which intuitively would be the most useful tool for further developments. For example recently, Shah and Raees [68] have investigated several results corresponding to associative modules theory over the rings.

In the second section, we have taken the case in which commutative semigroup has been replaced by an LA-semigroup and accordingly we have constructed an LA- ring through which all the established results in section one stood as particular case. Here is the right place to use the concept of ordered

AG-groupoids as discussed earlier in chapter 2; to tackle the degree problem arose during the developments of this type of LA-rings.

The contents of chapter 4 have been published in;

Int. J. Contemp. Math. Sciences, 5 (2010), no. 5, 209-222.

In chapter 5, we have divided our work into two sections. In section one; we have introduced a non-commutative and a non-associative structure, which we named as Γ-AG-groupoids. Our main objective behind this study is to generalize the concept of AG-groupoids which have been studied by several researchers [41, 42, 43, 46, 47, 50, 57, 58, 69]. We have discussed the concepts of Γ-ideals and Γ-bi-ideals in this structure. In second section, we have studied regular and intra-regular Γ-AG-groupoids. And have characterized these particular types of Γ-AG-groupoids by the properties of Γ-ideals.

Also we are with a plan to use the characteristics of Γ-AG-groupoids and to make a link with Γ-LA-rings which have been discussed in chapter 6. One can observe that Γ-LA-rings are an immediate generalization of LA-rings and also it is a generalization of commutative Γ-rings [4, 34, 35, 52].

The contents of chapter 5 also have been published in;

(i) **Proc. Pakistan Acad. Sci.,** 47 (2010), 33-39.
(ii) **Int. J. Algebra,** 4 (2010), no. 6, 267-276.
(iii) **Int. J. Applied Mathematics and statistics,** (to appear).

In chapter 6, we have introduced Γ-left almost rings (abbreviated as Γ-LA-rings). We have initiated this idea by taking inspiration from an article: “On a generalization of ring theory” published in Osaka Journal of Mathematics, 1964. In this article, Nobusava [52] have introduced the concept of Γ-rings for the first time. After his research, Barnes [4] weakened slightly the conditions in the definition of the Γ-ring in the sense of Nobusava. Barnes [4], Kyuno [34, 35] and Luh [37], studied the structure of Γ-rings and obtained various generalizations analogous to the corresponding parts in ring theory.

Γ-left almost rings are a direct generalization of LA-rings discussed earlier in chapters 3 and 4. We can easily observe that Γ-left almost rings are in fact a generalization of commutative Γ-rings and intuitively commutative Γ-rings are generalization of commutative ring theory. Consequently, Γ-left almost rings become a generalization of commutative rings.

In this study we have generalized some results which were established for LA-rings in chapter 3 earlier.

Lastly, in chapter 7 we have made our thoughts to turn to the application side and for the first time we have taken a step forward to investigate the fuzzy concepts of these algebraic structures. We have established some interesting results concerning Γ-AG-groupoids and LA-rings. For these developments, we mostly followed [1, 8, 22, 23, 31, 32, 70].

Research Profile

The work contained in this thesis has been published, accepted and submitted in international professional journals in the form of following articles.

[1] T. Shah, I. Rehman and A. Ali, On ordering of AG-groupoids, Int. Electronic J. Pure and Applied Mathematics (to appear).

[2] T. Shah, I. Rehman and R. Chinram, On M-systems in ordered AG-groupoids, Far East J. Mathematical Sciences, 47 (1)(2010), 13-21.

[3] T. Shah, I. Rehman and R. Chinram, Some characterizations of regular and intra-regular Γ-AG-groupoids, Int. J. Applied Mathematics and statistics, (to appear).

[4] T. Shah and I. Rehman, On M-systems in Γ-AG-groupoids, Proc. Pakistan Acad. Sci., 47 (1)(2010), 33-39.

[5] T. Shah and I. Rehman, On Γ-ideals and Γ-bi-ideals in Γ-AG-groupoids, Int. J. Algebra, 4 (2010), no. 6, 267-276.

[6] T. Shah and I. Rehman, On LA-ring of finitely non-zero functions, Int. J. Contemp. Math. Sciences, 5 (2010), no. 5, 209-222.

[7] T. Shah, I. Rehman and R. Salim Badar, On generalization of Commutative Semigroup rings, (submitted).

[8] T. Shah and I. Rehman, On characterizations of LA-rings through some properties of their ideals, (submitted).

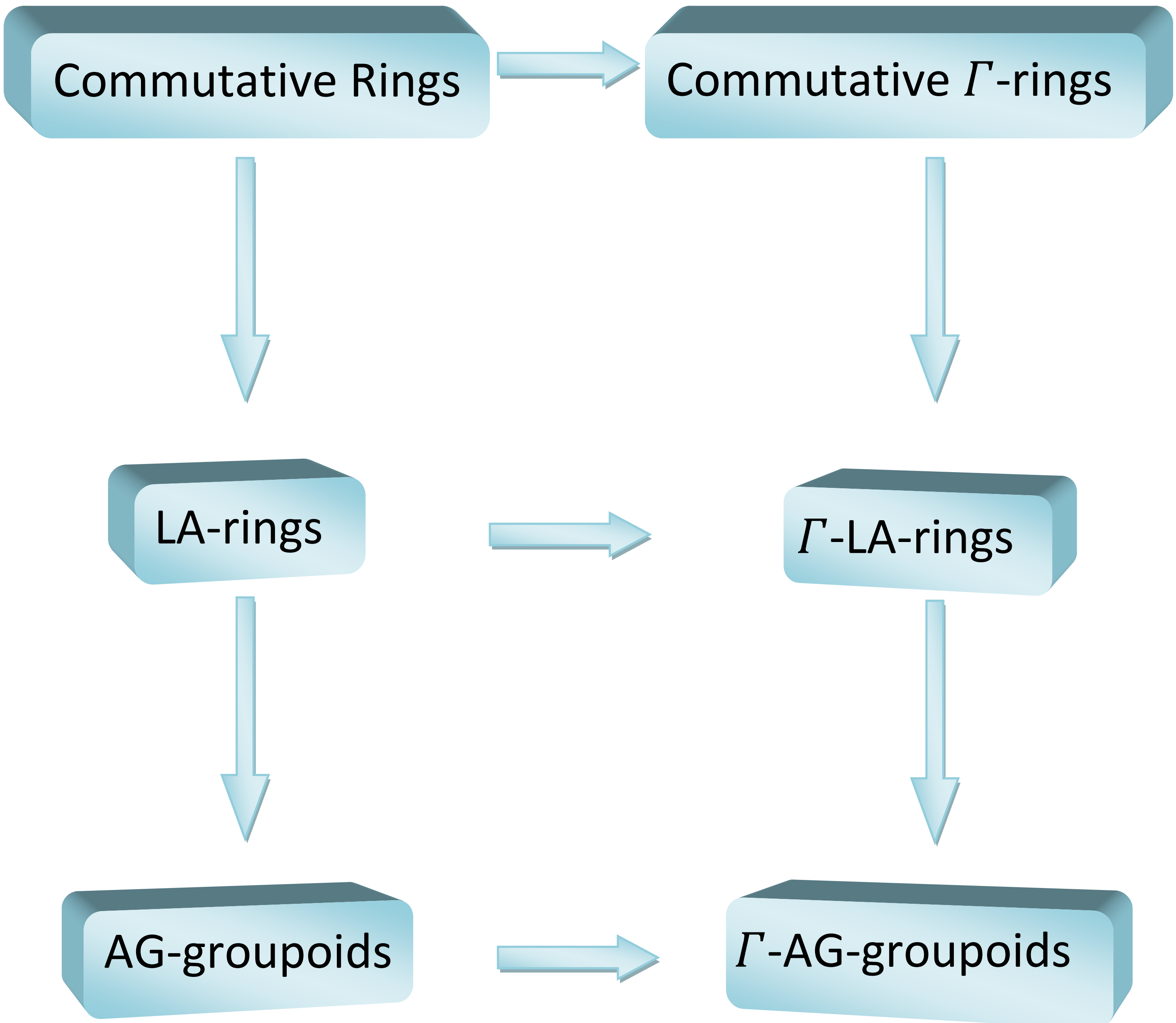
[9] T. Shah, I. Rehman and M. Raees, On direct sum in LA-rings, (submitted).

[10] T. Shah, I. Rehman and R. Chinram, On Γ-left almost rings, (submitted).

[11] T. Shah, I. Rehman and A. Khan, Fuzzy Γ-ideals in Γ-AG-groupoids, Int. J. Fuzzy Systems, (accepted).

[12] T. Shah, I. Rehman and Chinram, Regular and intra-regular Γ-AG-groupoids characterized by the properties of fuzzy Γ- ideals, (submitted).

Table of Implications



Index of main notations

Symbol	Page	Meaning
$\subseteq$	4	Set inclusion
G/H	5	Set of all right cosets of H in G
S\L	7	Complement of L in S
$C\times S/\sim$	8	Set of all equivalence classes of $C\times S$ under ~
Kerϕ	10	The kernel of ϕ
Q^+	10	The positive rational numbers
(H]	13	The set of all elements of S such that s is less than or equal to h, where $s\in S$ and $h\in H$
<H>	14	The left ideal of S generated by A
$\cap_{i\in I} J_i$	17	Intersection of the sets of quasi-prime ideals
U(R)	20	Set of all unit elements in LA-ring R
I^+ (R)	23	Set of all additively idempotent elements
V(R)	33	Set of all those elements of R having additive inverse
$\oplus_{i\in I} R_i$	35	The direct sum of LA-rings
$\Pi_{i\in I} R_i$	35	The direct product of LA-rings

$R[X^s; s \in S]$	42	The LA-ring of commutative semigroup S over LA-ring R
R[S]	45	An additive LA-group
Supp(f)	47	The support of f
$L \oplus M$	48	External direct sum of commutative semigroups L and M
$\{Y_i\}_{i=1}^{n}$	49	Finite set of indeterminates
Q_0^+	54	The set of positive rational numbers with zero adjoint
f	97	A function from a non-empty set X to the interval [0,1], called as fuzzy subset of S
$f \wedge g$	97	Minimum of f and g
$f \vee g$	97	Maximum of f and g
U(f ; t)	98	The level set of f
χ_A	98	The characteristic of A
FI(S)	104	The set of all fuzzy Γ-ideals of S
a_t	104	Fuzzy point with support t
l_{a_t}	105	Fuzzy left Γ-ideals generated by a_t

Contents

Chapter 1

Brief History and Preliminaries

Introduction

This chapter contains the brief history of LA-semigroups, LA-groups, Γ-semigroups and Γ-rings. We have provided the preliminaries of these structures and those definitions and fundamental results which are directly related to our study. We have partitioned this chapter into two sections. In first section, we deal the basic definitions, and fundamental results of LA-semigroups and LA-groups which are used successively in further discussion. In second section, we provide some preliminaries about Γ-semigroups and Γ-rings.

1.1 LA-semigroups and LA-groups

In 1972, a generalization of a commutative semigroup has been introduced by M. A. Kazim and M. Naseeruddin [26]. In ternary commutative law, abc = cba, they introduced braces on the left of this law and introduced a new pseudo associative law, that is $(ab)c = (cb)a$. They called it left invertive law. A groupoid satisfying the left invertive law is called a left almost semigroup and is abbreviated as LA-semigroup. The notion of LA-semigroups significantly contributed in the generalization of commutative semigroups. With the help of this left invertive law, they successfully manipulated subtraction and division as binary operations and proved several results. They have generalized some useful results of semigroup theory. Despite the fact that the structure is non-associative and non-commutative, however it possesses properties which usually valid for associative and commutative algebraic structures.

Later, Q. Mushtaq and his associates have investigated the structure further and added many useful results to the theory of LA-semigroups. The notion of locally associative LA-semigroups was introduced by Q. Mushtaq and S. M. Yusuf [51]. An LA-semigroup S is said to be a locally associative if $(aa)a = a(aa)$ for all $a \in S$. Later in [43], Q. Mushtaq and Q. Iqbal have done extensive study on Locally associative LA-semigroups. We genuinely acknowledge that in this field much of the spade work has been done by M. Kazim and M. Naseeruddin [26], Q. Mushtaq and his associates [40, 41, 42, 43, 44, 45, 46, 48, 49, 50, 51] and P. V. Protic and N. Stevanovic [56, 57, 58, 59, 69].

In [17], P. Holgate has called this structure as left invertive groupoid. He defined it as a groupoid S in which every a, b, c satisfy the left invertive law $(ab)c = (cb)a$. He has preferred the term left invertive groupoid instead of LA-semigroup because naturally commutative semigroup satisfy left invertive law and LA-semigroups lie between them and groupoids.

LA-semigroups are also known as Abel-Grassmann's groupoids [56] and abbreviated as AG-groupoids. In an AG-groupoid S the medial property, $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in S$, holds. A groupoid G is called a paramedial if $(ax)(yb) = (bx)(ya)$ for all $a, b, x, y \in G$ [24]. Ideals in AG-groupoids have been discussed in [69]. In an AG-groupoid S with left identity e, $S(Sa) \subseteq Sa$ and $(aS)S \subseteq aS$, where a is an idempotent in S. Also in [69], a subset Q of an AG-groupoid S is said to be a quasi-ideal if $SQ \cap QS \subseteq Q$ and since $Q^2 = Q^2 \cap Q^2 \subseteq SQ \cap QS \subseteq Q$, this implies that Q is an AG-subgroupoid of S. P. V. Protic and N. Stevanovic $[69]$ have defined AG*-groupoid. An AG-groupoid with weak associative law is called as AG*-groupoid. Also they have generalized AG-groupoid with left identity in the form of AG**-groupoid. An AG**-groupoid with a weak associative law becomes a semigroup. They have shown that a non-associative a left simple (right simple, simple) AG*-groupoid does not exist. In [59], they have also introduced the concept of AG-bands. An AG-groupoid is called an AG-band if all of its elements are idempotents.

Later in 1994, Q. Mushtaq and M. Kamran [25] extended this concept to left almost group (abbreviated as LA-group). An LA-semigroups is called an LA-group if (i) there exist a left identity such that ea = a for all, and (ii) for every $a \in G$, there exists a left inverse $a^{'} \in G$ such that $a^{'}a \in G$. By e we shall mean the left identity. It is not very hard to see that the left identity 'e' and the left inverse are unique. It is important to note that if $a'$is the left inverse of a then $aa' = (ea)a' = (a'a)e = ee = e$. This implies a' is the right inverse of a. A non-empty subset H of an LA-group G is said to be an LA-subgroup of G if H is itself an LA-group under the same binary operation as defined in G. Suppose $(G, \cdot)$ is commutative group then it is easy to see that $(G, *)$, where "$*$" is defined as $a * b = ba^{-1}$ for all $a, b \in G$ is an LA-group. If H is a non-empty subset of an LA-group G, then $H \leq G$ if and only if $ab^{-1} \in H$ for all $a, b \in H$.

Theorem 1 [45, *Theorem* 3.2] *If G is an LA-group, then*

(1) $GG = G$.

(2) $eG = Ge = G$.

Lemma 2 [45, *Lemma* 3.4] *If G is an LA-group and $H \leq G$, then*

(1) $aH = (Ha)e$.

(2) $(ab)H = H(ba)$ *for all* $a, b \in G$.

Lemma 3 [45, *Lemma* 3.5] *The relation $a \equiv b(modH)$ is an equivalence relation, where $H \leq G$.*

Theorem 4 [45, *Theorem* 3.8] *If G is an LA-group and $H \leq G$, then $G/H = \{Ha : a \in G\}$ is an LA-group.*

1.2 Γ-semigroups and Γ-rings

In this section, we give basic concept of Γ-semigroups and Γ-rings. In 1981, the notion of Γ-semigroups was introduced by M. K. Sen (see [63] and [64]). Let M and Γ be any nonempty sets. If there exists a mapping $M \times \Gamma \times M \to M$ written (a, α, c) by $a\alpha c$, M is called a Γ-semigroup if M satisfies the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Whereas the Γ-semigroups are a generalization of semigroups. Many classical notions of semigroups have been extended to Γ-semigroups. We remark that the Γ-semigroup given in [64, 65] by Sen and Saha may be called a one-sided Γ-semigroup. Later on in [10], Dutta and

Adhikari introduced a both sided Γ-semigroup in which the operation $\Gamma \times S \times \Gamma \longrightarrow \Gamma$ was also taken into consideration.

Nobusawa studies Γ-rings for the first in [52]. After his research, Barnes [4] weakened slightly the conditions in the definition of the Γ-ring in the sense of Nobusawa. Barnes [4], Kyuno [34, 35] and Luh [37] studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory. Let $(M, +)$ and $(\Gamma, +)$ be abelian groups. M is called a Γ-ring if there exists a mapping $M \times \Gamma \times M \rightarrow M$ satisfying the following conditions:

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$.

(ii) $a\alpha(b + c) = a\alpha b + a\alpha c$.

(iii) $a(\alpha + \beta)b = a\alpha b + a\beta b$.

(iv) $(a\alpha b)\beta c = a\alpha(b\beta c)$.

Chapter 2

Ordered AG-groupoids

Introduction

This chapter contains two sections. In section one we deal with ordering of AG-groupoid. The motivation behind this study is the usefulness of totally ordered semigroups. For this purpose, we have extended this concept to the AG*-groupoids with left identity e and established that: An M-torsion free and cancellative AG*-groupoid S with quotient group T, admits a total order compatible with its operation if and only if T has a total order.

In section 2, we have investigated M-systems, N-systems and I-systems in ordered AG-groupoids. We have proved that if S is an ordered AG-groupoid with left identity and L a proper left ideal of S. Then L is quasi-prime if and only if $S \setminus L$ is an M-system. Also we have shown that if N is an N-system of S and $a \in N$, then there exists an M-system M of S such that $a \in M \subseteq N$.

2.1 Ordering of AG-groupoids

The techniques we have used in this study are mainly inspired by [13]. We begin initially by the following theorem which is a generalization of [13, Theorem 1.2].

Theorem 5 *If S is an AG*-groupoid and C is a left cancellative subAG*-groupoid of S, then there exists an embedding $\phi : S \to T$, where T is an abelian monoid such that*

(1) $\phi(c)$ *has an inverse* $(\phi(c))^{-1}$ *in* T *for all* $c \in C$ *and*

(2) $T = \{(\phi(c))^{-1}\phi(s) : s \in S, c \in C\}$.

If $S = C$, then monoid T is an abelian group.

Proof. Define a relation $\sim$ on $A = C \times S$ by $(c_1, s_1) \sim (c_2, s_2)$ if and only if $c_1 s_2 = c_2 s_1$. We claim that $\sim$ is an equivalence relation. Indeed: The relation $\sim$ is reflexive, as $cs = cs$ implies $(c, s) = (c, s)$. Clearly $\sim$ is symmetric as $(c_1, s_1) \sim (c_2, s_2)$ implies $c_1 s_2 = c_2 s_1$, i.e. $c_2 s_1 = c_1 s_2$ and hence $(c_2, s_2) \sim (c_1, s_1)$. Now suppose $(c_1, s_1) \sim (c_2, s_2)$ and $(c_2, s_2) \sim (c_3, s_3)$. This implies $c_1 s_2 = c_2 s_1$ and $c_2 s_3 = c_3 s_2$. Now we have $(c_1 s_3) c_2 = (c_2 s_3) c_1 = (c_1 s_2) c_3 = (c_2 s_1) c_3 = (c_3 s_1) c_2$. This implies that $c_1 s_3 = c_3 s_1$ and hence $(c_1, s_1) \sim (c_3, s_3)$ and therefore $\sim$ is transitive. If $(c_1, s_1) \sim (c_2, s_2)$, then $c_1 s_2 = c_2 s_1$. By [50, Theorem 2.7], it implies that $s_2 c_1 = s_1 c_2$. Now $(c_3 s_4)(s_2 c_1) = (c_3 s_4)(s_1 c_2)$ implies $(c_3 s_2)(s_4 c_1) = (c_3 s_1)(s_4 c_2)$ and so $(c_3 s_2, s_4 c_2) \sim (c_3 s_1, s_4 c_1)$ or $(c_3, s_4)(c_2, s_2) \sim (c_3, s_4)(c_1, s_1)$ or $(c_3, s_4)(c_1, s_1) \sim (c_3, s_4)(c_2, s_2)$. This implies $\sim$ is left compatible. Again if $(c_1, s_1) \sim (c_2, s_2)$, then $c_1 s_2 = c_2 s_1$ and $s_2 c_1 = s_1 c_2$. Now $(s_2 c_1)(c_3 s_4) = (s_1 c_2)(c_3 s_4)$, using medial law we have $(s_2 c_3)(c_1 s_4) = (s_1 c_3)(c_2 s_4)$ and so $(c_1 s_4)(s_2 c_3) = (c_2 s_4)(s_1 c_3)$. This implies, $(c_1 s_4, s_1 c_3) \sim (c_2 s_4, s_2 c_3)$ or $(c_{1,} s_1)(c_3, s_4) \sim (c_2, s_2)(c_3, s_4)$. Hence $\sim$ is right compatible. Thus $\sim$ is compatible. Now $T = C \times S / \sim = \{[c, s] : c \in C, s \in S\}$ is the set of all equivalence classes of $C \times S$ under "$\sim$". T is a commutative monoid under the binary operation "$*$" defined by $[(c_1, s_1)] * [(c_2, s_2)] = [(c_1 c_2, s_2 s_1)] \in T$. Clearly T is closed. Now we show that $(T, *)$ is an AG-groupoid. For this consider,

$$
\begin{aligned}
([(c_1, s_1)] * [(c_2, s_2)]) * [(c_3, s_3)] &= [(c_1 c_2, s_2 s_1)] * [(c_3, s_3)] \\
&= [(c_1 c_2, s_2 s_1)] * [(c_3, s_3)] \\
&= [((c_1 c_2) c_3, s_3 (s_2 s_1))] \\
&= [((c_3 c_2) c_1, s_2 (s_3 s_1))].
\end{aligned}
$$

Now we take

$$
\begin{aligned}
([c_3, s_3] * [c_2, s_2]) * [c_1, s_1] &= [(c_3 c_2, s_2 s_3)] * [(c_1, s_1)] \\
&= [((c_3 c_2) c_1, s_1 (s_2 s_3))] \\
&= [((c_3 c_2) c_1, s_2 (s_3 s_1))]
\end{aligned}
$$

Thus $([c_1,s_1]*[c_2,s_2])*[c_3,s_3] = ([(c_3,s_3)]*[(c_2,s_2)])*[(c_1,s_1)]$. Hence $(T,*)$ is an AG-groupoid. Let $[(c_1,s_1)] \in T$, then

$$\begin{aligned}
[(c_1,s_1)]*[(c,c)] &= [(c_1c, cs_1)] = \{(c_2,s_2) \in A : (c_1c, cs_1) \sim (c_2,s_2)\} \\
&= \{(c_2,s_2) \in A : (c_1c)\, s_2 = c_2\,(cs_1)\} = \{(c_2,s_2) \in A : c\,(c_1s_2) = c\,(c_2s_1)\} \\
&= \{(c_2,s_2) \in A : c_1s_2 = c_2s_1\} \\
&= \{(c_2,s_2) \in A : (c_1,s_1) \sim (c_2,s_2)\} = [(c_1,s_1)]\,.
\end{aligned}$$

Hence $[(c,c)]$ is a right identity in T for all $c \in C$. Now since T is an AG-groupoid therefore by [50, Theorem 2.4] it becomes a commutative monoid. Now define $\phi : S \to T$ by $\phi(s) = [(c,cs)]$ for all $s \in S$. Let $s_1, s_2 \in S$ such that $s_1 = s_2$. It is easy to verify that ϕ is well-defined. Now we show that ϕ is one-one. Let $s_1, s_2 \in S$,

$$\begin{aligned}
\phi(s_1s_2) &= [(c, c\,(s_1s_2))] = [((c_2c_1), (c_2c_1)(s_1s_2))] = [((c_2c_1), (c_2s_1)(c_1s_2))] \\
&= [((c_2c_1), (c_2s_1)(c_1s_2))]\,[(e,e)] = [((c_2c_1)\,e, e((c_2s_1)(c_1s_2)))] \\
&= [((c_2c_1)\,e, e((c_2s_1)(c_1s_2)))] = [((ec_1)\,c_2), (c_2s_1)(c_1s_2)] \\
&= [(c_1, c_1s_2)][(c_2, c_2s_1)] = [(c_2, c_2s_1)][(c_1, c_1s_2)] \\
&= [(c_1s_1, c_1)]\,[(c_2s_2, c_2)] = \phi(s_1)\,\phi(s_2)\,.
\end{aligned}$$

Now

$$\begin{aligned}
\text{Ker}\phi &= \{s \in S : \phi(s) \text{ is the identity of } T\} = \{s \in S : \phi(s) = [(c,c)]\} \\
&= \{s \in S : [(c,cs)] = [(c,c)]\} = \{s \in S : (c,cs) \sim (c,c)\} \\
&= \{s \in S : cc = c(cs)\} = \{s \in S : cc = (cc)s\} \\
&= \{s \in S : e = s\} = \{s \in S : s = e\} = \{e\}.
\end{aligned}$$

Thus ϕ is one-one. Hence $\phi : S \to T$ is an embedding. Hence ϕ is one-one. Thus $\phi : S \to T$ is an embedding. Now if $c \in C$, then $\phi(c) = \left[(c, c^2)\right]$ has an inverse $(\phi(c))^{-1} = \left[(c^2, c)\right] \in T$.

Indeed, $\phi(c)(\phi(c))^{-1} = [c, c^2][(c^2, c)] = [(c.c^2, c.c^2)] = [(c_1, c_1)]$, where $c_1 = c.c^2 \in C$ and $[(c_1, c_1)]$ is an identity in T. Now an arbitrary element $[(s, c)]$ in T can be written as

$$\begin{aligned}(\phi(c))^{-1}\phi(s) &= [(c, cs)][(c^2, c)] = [(cc^2, c(cs))] \\ &= [(cc^2, (cc)s)] = [(cc^2, c^2 s)] \\ &= [(c, s)][(c^2, c^2)] = [(c, s)].\end{aligned}$$

As T is commutative, so $(\phi(c))^{-1}\phi(s) = \phi(s)(\phi(c))^{-1} = [(c, s)]$. If $S = C$, then every element of T is invertible. Consider $[(c, s)][(s, c)] = [(cs, cs)] = [(c^2, c^2)] = [(c_1.c_1)]$, which is an identity in T. Hence T is an abelian group. ■

Now in the following we extend the definition [61, Page 332] for an AG*-groupoid with left identity e.

Definition 6 *Let $(S, *)$ be an AG*-groupoid, then S is said to be M-torsion free if for all $x, y \in S$ there exist $1 \leq m \in M \subseteq \mathbb{Z}^+$ with $x^m = y^m$, then $x = y$.*

Example 7 *Take AG*-groupoid $(Q^+, *)$, in which the binary operation "$*$" is defined as $a * b = b.a^{-1}$. $(Q^+, *)$ is an O-torsion free, where O is the set of odd positive integers. In particular for $m = 3$, take $x^3 = y^3$ and by locally associative property we have $x^2 * x = y^2 * y$. Now as for all $x \in Q^+$, $x^2 = 1$, so $1 * x = 1 * y$. This implies $x = y$. Hence $(Q^+, *)$ is O-torsion free AG*-groupoid. Similarly $(\mathbb{Z}, \circ)$ is an O-torsion free defined by $a \circ b = b - a$.*

Definition 8 *Let S be a nonempty set, "$*$" a binary operation on S and $\leq$ a relation on S. $(S, *, \leq)$ is called a total ordered AG-groupoid if $(S, *)$ is a AG-groupoid, $(S, \leq)$ is a partially ordered set and for all $a, b, c \in S$, $a \leq b$ implies that $a * c \leq b * c$ and $c * a \leq c * b$.*

Lemma 9 *Let $(S, *)$ be an AG*-groupoid. If $\leq$ is total order on S compatible with $*$, then S is M-torsion free and cancellative.*

Proof. Let $a, b \in S$ and say $a < b$ (that is $a \leq b$ and $a \neq b$). If $a < b$, this implies $a * x < b * x$ for all $x \in S$. Since $\leq$ is compatible with respect to $*$, this implies S is cancellative. Now if

$a < b$, then $a * a < a * b ... (1)$ and $a * b < b * b ... (2)$. It further implies that $a * a < a * b < b * b$. From (1), we have $(a * a) * a < (a * b) * a$ and from (2), we can say $(a * b) * b < (b * b) * b$. Now for $a < b$, the compatibility of $*$ implies that $(a * b) * a < (a * b) * b$ and hence $(a * a) * a < (a * b) * b < (b * b) * b$.

Continuing this process for m-times, where m is minimal in the set M, we have $a^m < ... < b^m$. This implies $a^m < b^m$ for some $m \in M$. Hence $(S, *)$ is M-torsion free. ■

The following theorem establishes a relation between an AG*-groupoid and its quotient group.

Theorem 10 *Let T be the quotient group of a cancellative AG*-groupoid S. Then T is M-torsion free if and only if for all $x, y \in S$, $x^n = y^n$ implies $x = y$,where $n \in M \subseteq Z^+$.*

Proof. Suppose $T = C \times S/\sim$ is torsion free. This implies $[(x, x)]$ is only element of $C \times S/\sim$ of finite order. So, $[(x, x)]^n = [(x, x)]$. Assume that $x^n = y^n$, where $n \in M \subseteq Z^+$. Then $x.x^n = x.y^n$. So by power associativity of S, we have $x^{1+n} = x.y^n$ or $x^n.x = x.y^n$. This implies $(x^n, y^n) \sim (x, x)$ or $[(x, y)]^n = [(x, x)]$ and hence it implies $x = y$.

Conversely suppose that for all $x, y \in S$, $x^n = y^n$ implies $x = y$. Let $[x, y] \in C \times S/\sim$ such that $[(x, y)]^n = [(x, x)]$. This implies $(x^n, y^n) \sim (x, x)$ and therefore $x^n.x = x.y^n$. So by power associativity in S, $x^{n+1} = x.y^n$ or $x.x^n = x.y^n$. This implies $x^n = y^n$ and so $x = y$. Thus $[(x, x)]^n = [(x, x)]$ and hence $T = C \times S/\sim$ is M-torsion free. ■

Theorem 11 *Let S be a M-torsion free cancellative AG*-groupoid with quotient group T. Then S admits a total order compatible with its operation if and only if T has a total order.*

Proof. If T is totally ordered under $\leq$, then the relation $\leq$ induces a total order on S compatible with the AG*-groupoid operation. Conversely, if S is totally ordered under $\leq$, then we define a relation $\sim$ on T as follows, each element of T is expressible in the form $c^{'}s$ for some $c, s \in S$ and $c^{'}$ is inverse of c. Now for $t_1 = c_1^{'} s_1$ and $t_2 = c_3^{'} s_3$ in T, we define $t_1 \sim t_2$ by $c_1^{'} s_1 \leq c_3^{'} s_3$. It is not hard to see that $\sim$ is a well defined relation of total order on T that is consistent with the group operation on T and for the restriction of the relation $\leq$ on S, we just to verify that $\sim$ is well defined and that it agrees with the relation $\leq$ on S. Thus, if $t_1 = c_1^{'}s_1 = c_2^{'}s_2$ and $t_2 = c_3^{'}s_3 = c_4^{'}s_4$, where $c_3s_1 \leq c_1s_3$, then, $(c_3s_1)(c_2s_4) \leq (c_1s_3)(c_2s_4)$........(1)

Now for the values of c_2 and s_4, we consider $c_1^{'}s_1 = c_2^{'}s_2$, then by cancellativity we have $(c_1^{'}s_1)s_2^{'} = (c_2^{'}s_2)s_2^{'} = (s_2^{'}s_2)c_2^{'} = ec_2^{'}$. So, $(c_1^{'}s_1)s_2^{'} = c_2^{'}$ and $((c_1^{'}s_1)s_2^{'})^{'} = (c_2^{'})^{'}$ implies that $(c_1s_1^{'})s_2 = c_2$. Now for s_4, consider $c_3^{'}s_3 = c_4^{'}s_4$. Then $c_4(c_3^{'}s_3) = c_4(c_4^{'}s_4)$ or $c_4(c_3^{'}s_3) = (c_4^{'}c_4)s_4 = s_4$. Now by repeated use of definitions of AG-groupoid, AG*-groupoid and medial law in (1), it can be easily verified that if $(c_3s_1)(c_2s_4) \leq (c_1s_3)(c_2s_4)$, then $(c_4s_2) \leq (c_2s_4)$ and hence $\sim$ is well defined. Define $\phi : S \rightarrow T$ by $\phi(s) = c^{'}(cs)$, where $c^{'}$ is inverse of $c \in S$. Then ϕ is an embedding. Indeed,

$$\begin{aligned}\phi(s_1s_2) &= c^{'}(c(s_1s_2)) = (cc^{'})(s_1s_2) = e(s_1s_2) = s_1s_2 \\ &= [(cc^{'})s_1][(cc^{'})s_2] = [c^{'}(cs_1)][c^{'}(cs_2)] = \phi(s_1)\phi(s_2).\end{aligned}$$

Let $\phi(s_1) = \phi(s_2)$. This implies that $[c^{'}(cs_1)] = [c^{'}(cs_2)]$ or $(cc^{'})s_1 = (cc^{'})s_2$ and hence $s_1 = s_2$. Thus for $s, t \in S$, we have $s \sim t$ if and only if $c(cs) \leq c(ct)$ if and only if $s \leq t$. ■

2.2 M-Systems in Ordered AG-groupoids

In this section, we have studied ideals, M-systems, N-systems and I-systems of ordered AG-groupoids. We have proved that if L is a left ideal of an ordered AG-groupoid with left identity, then L is quasi-prime if and only if $S \setminus L$ is an M-system; L is quasi-semiprime if and only if $S \setminus L$ is an N-system and L is quasi-irreducible if and only if $S \setminus L$ is an I-system. Moreover, we show that every quasi-semiprime left ideal of an ordered AG-groupoid with left identity is an intersection of some quasi-prime left ideals.

Let S be a nonempty set, "$\cdot$" a binary operation on S and $\leq$ a relation on S. $(S, \cdot, \leq)$ is called an ordered AG-groupoid if $(S, \cdot)$ is a AG-groupoid, $(S, \leq)$ is a partially ordered set and for all $a, b, c \in S$, $a \leq b$ implies that $ac \leq bc$ and $ca \leq cb$. This structure is a generalization of AG-groupoids and ordered semigroups. The following theorem follows by Theorem 1 in [41] and definitions of ordered AG-groupoids and ordered semigroups.

Theorem 12 *An ordered AG-groupoid S is an ordered semigroup if and only if $a(bc) = (cb)a$ for all $a, b, c \in S$.*

For $H \subseteq S$, let $(H] = \{t \in S \mid t \leq h$ for some $h \in H\}$. This lemma is similar to the case of ordered semigroups.

Lemma 13 *Let S be an ordered AG-groupoid and A, B subsets of S. The following statements hold.*

(1) *If $A \subseteq B$, then $(A] \subseteq (B]$.*

(2) $(A](B] \subseteq (AB]$.

(3) $((A](B]] \subseteq (AB]$.

A nonempty subset A of an ordered AG-groupoid S is called a left ideal of S if $(A] \subseteq A$ and $SA \subseteq A$ and called a right ideal of S if $(A] \subseteq A$ and $AS \subseteq A$. A nonempty subset A of S is called an ideal of S if A is both left and right ideal of S.

Proposition 14 *Let S be an ordered AG-groupoid with left identity. Then every right ideal of S is a left ideal of S.*

Proof. Let R be a right ideal of S. Then $(R] \subseteq R$ and $RS \subseteq R$. We claim that $SR \subseteq R$, indeed, $SR = (eS)R = (RS)e \subseteq Re \subseteq R$. ■

For $A \subseteq S$, let $< A >$ denote the left ideal of S generated by A and for $a \in S$, $< \{a\} >$ is denoted by $< a >$.

A groupoid S is called a paramedial if $(ax)(yb) = (bx)(ya)$ for all $a, b, x, y \in S$ [24].

Lemma 15 *If S is an AG-groupoid with left identity e, then it satisfies paramedial law.*

Proof. Let $x, y, l, m \in S$, then consider

$$\begin{aligned}(xy)(lm) &= (xl)(ym), \text{ by medial law} \\ &= (e(xl))(ym) \\ &= [(ym)(xl)]e, \text{ by left invertive law} \\ &= [(yx)(ml)]e, \text{ by medial law} \\ &= [e(ml)](yx) \\ &= (ml)(yx) \\ &= (my)(lx).\end{aligned}$$

Hence S is a paramedial. ■

Lemma 16 *Let S be an ordered AG-groupoid with left identity and $A \subseteq S$. Then $S(SA) = SA$ and $S(SA] \subseteq (SA]$.*

Proof. Since S has a left identity, $S = SS$. Then by definition of AG-groupoids and paramedial law, we have $S(SA) = (SS)(SA) = (AS)(SS) = (AS)S = (SS)A = SA$. Thus $S(SA) = SA$. By Lemma 13, we have $S(SA] = (S](SA] \subseteq (S(SA)] = (SA]$. ■

Lemma 17 *Let S be an ordered AG-groupoid with left identity and $a \in S$. Then $< a >= (Sa]$.*

Proof. Since S has a left identity, $a \in (Sa]$. By Lemma 16, we have $S(Sa] \subseteq (Sa]$. So $(Sa]$ is a left ideal of S containing a. Let L be another left ideal containing a. Thus $Sa \subseteq L$, so $(Sa] \subseteq L$. ■

Let S be an ordered AG-groupoid. A nonempty subset M of S is called an M-system of S if for each $a, b \in M$, there exists $x \in S$ and $c \in M$ such that $c \leq a(xb)$. Equivalent definition: for each $a, b \in M$, there exists $c \in M$ such that $c \in (a(Sb)]$.

Remark 18 *(1) If $(S, \cdot)$ is an AG-groupoid, we endow S with the order relation $\leq$, then*

$(S,\cdot,\leq)$ is an ordered AG-groupoid. Moreover, the set M is an M-system of a AG-groupoid $(S,\cdot)$ if and only if M is an M-system of an ordered AG-groupoid $(S,\cdot,\leq)$.

(2) If an ordered AG-groupoid S is an ordered semigroup, then the set M is an M-system of an ordered AG-groupoid S if and only if M is an M-system of an ordered semigroup S.

A left ideal P of an ordered AG-groupoid S is called quasi-prime if and only if for all left ideals A, B of S, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Lemma 19 *Let L be a left ideal of an ordered AG-groupoid S with left identity e. Then L is quasi-prime if and only if for all $a, b \in S$, $a(Sb) \subseteq L$ implies $a \in L$ or $b \in L$.*

Proof. Suppose that $a(Sb) \subseteq L$. We get $S(a(Sb)) \subseteq SL \subseteq L$ and by medial law, paramedial law and the definition of AG-groupoid, we have

$$\begin{aligned} S(a(Sb)) &= (SS)(a(Sb)) = (Sa)(S(Sb)) = Sa((SS)(Sb)) \\ &= (Sa)((bS)(SS)) = (Sa)((bS)S) = (Sa)((SS)b) = (Sa)(Sb). \end{aligned}$$

Since L is a left ideal of $S, (Sa](Sb] \subseteq ((Sa)(Sb)] = (S(a(Sb))] \subseteq L$. Since $(Sa]$ and $(Sb]$ are left ideals of S and L is quasi-prime, $(Sa] \subseteq L$ or $(Sb] \subseteq L$. By Lemma 17, $a \in L$ or $b \in L$. Conversely, let A and B be left ideals of S such that $AB \subseteq L$ and $A \nsubseteq L$. Then there exists $s \in A$ and $x \notin L$. Now for all $y \in B$, we have $x(Sy) \subseteq A(SB) \subseteq AB \subseteq L$. Hence by assumption, $y \in L$ for all $y \in B$. Hence $B \subseteq L$, this implies that L is quasi-prime. ∎

Theorem 20 *Let S be an ordered AG-groupoid with left identity and L a proper left ideal of S. Then L is quasi-prime if and only if $S \setminus L$ is an M-system.*

Proof. Assume L is quasi-prime and let $a, b \in S \setminus L$. Suppose $c \notin (a(Sb)]$ for all $c \in S \setminus L$. Then $(a(Sb)] \subseteq L$, this implies $a(Sb) \subseteq L$. By Lemma 19, $a \in L$ or $b \in L$, which is impossible. Then there exist $c \in S \setminus L$ such that $c \in (aSb]$. Hence $S \setminus L$ is an M-system.

Conversely, assume that $S \setminus L$ is an M-system. Let $a, b \in S$ such that $a(Sb) \subseteq L$. Suppose that $a, b \in S \setminus L$. Since $S \setminus L$ is an M-system, there exist $c \in S \setminus L$ and $x \in S$ such that

$c \leq a(xb) \in a(Sb) \subseteq L$. Since L is a left ideal of S, we have $c \in L$, which is impossible. Hence $a \in L$ or $b \in L$. By Lemma 19, L is quasi-prime. ■

Let S be an ordered AG-groupoid. A nonempty subset N of S is called an N-system of S if for each $a \in N$, there exists $x \in S$ and $c \in N$ such that $c \leq a(xa)$. Equivalent definition: for each $a \in N$, there exists $c \in N$ such that $c \in (a(Sa)]$.

Remark 21 (1) *In [48], definition of N-systems in AG-groupoids is called a P-system. If* $(S, \cdot)$ *is a AG-groupoid, we endow* S *with the order relation* $\leq := id_S$*, then* $(S, \cdot, \leq)$ *is an ordered AG-groupoid. Moreover, the set* N *is an P-system of a AG-groupoid* $(S, \cdot)$ *if and only if* N *is an N-system of an ordered AG-groupoid* $(S, \cdot, \leq)$*.*

(2) *If an ordered AG-groupoid* S *is an ordered semigroup, then the set* M *is an M-system of an ordered AG-groupoid* S *if and only if* M *is an M-system of an ordered semigroup* S*.*

(3) *Let* S *be an ordered AG-groupoid. Each M-system of* S *is an N-system of* S*.*

A left ideal P of an ordered AG-groupoid S is called quasi-semiprime if for any left ideal A of S, $A^2 \subseteq P$ implies that $A \subseteq P$. It is obvious that a quasi-prime subset of S is a quasi-semiprime subset of S.

Lemma 22 *Let* L *be a left ideal of an ordered AG-groupoid* S *with left identity* e*. Then* L *is quasi-semiprime if and only if for all* $a \in S$*,* $a(Sa) \subseteq L$ *implies* $a \in L$*.*

Proof. Suppose that $a(Sa) \subseteq L$. We get $S(a(Sa)) \subseteq SL \subseteq L$ and by similar in the proof of Lemma 22, we have $S(a(Sa)) = (Sa)(Sa)$. Since L is a left ideal of $S, (Sa](Sa] \subseteq ((Sa)(Sa)] = (S(a(Sa))] \subseteq L$. Since $(Sa]$ is a left ideal of S and L is quasi-semiprime, $(Sa] \subseteq L$. By Lemma 17, $a \in L$. Conversely, let A be a left ideal of S such that $A^2 \subseteq L$. Now for all $x \in A$, we have $x(Sx) \subseteq A(SA) \subseteq A^2 \subseteq L$. Hence by assumption, $x \in L$ for all $x \in A$. Hence $A \subseteq L$, this implies that L is quasi-semiprime. ■

Theorem 23 *Let* S *be an ordered AG-groupoid with left identity and* L *a proper left ideal of* S*. Then* L *is quasi-semiprime if and only if* $S \setminus L$ *is an N-system.*

Proof. Assume L is quasi-semiprime and let $a \in S \setminus L$. Suppose $c \notin (a(Sa)]$ for all $c \in S \setminus L$. Thus $(a(Sa)] \subseteq L$, this implies $a(Sa) \subseteq L$. By Lemma 17, $a \in L$, which is impossible. So there exist $c \in S \setminus L$ such that $c \in (aSa]$. Hence $S \setminus L$ is an N-system. Conversely, assume that $S \setminus L$ is an N-system. Let $a \in S$ such that $a(Sa) \subseteq L$. Suppose that $a \in S \setminus L$. Since $S \setminus L$ is an N-system, there exist $c \in S \setminus L$ and $x \in S$ such that $c \leq a(xa) \in a(Sa) \subseteq L$. Then $c \in L$, which is impossible. Therefore $a \in L$. By Lemma 22, L is quasi-semiprime. ■

The intersection of quasi-prime left ideals of an ordered AG-groupoids S (if it is non empty) need not be a quasi-prime left ideals of S. The following proposition shows that it becomes quasi-semiprime.

Proposition 24 *Let J_i be any set of quasi-prime left ideals of an ordered AG-groupoid for all $i \in I$. If $P = \bigcap_{i \in I} J_i \neq \emptyset$, then P is a quasi-semiprime left ideal of S.*

Proof. Let L be a left ideal of S such that $L^2 \subseteq P$. Then $L^2 \subseteq J_i$ for all $i \in J$. This implies $L \in J_i$ for all $i \in J$. So $L \subseteq P$. Hence P is a quasi-semiprime left ideal of S. ■

Theorem 25 *Every quasi-semiprime left ideal of an ordered AG-groupoid with left identity is an intersection of some quasi-prime left ideals.*

Proof. Let L be a quasi-semiprime left ideal of S and $\{J_i \mid i \in I\}$ the set of all quasi-prime left ideal of S containing L. This set is not empty because S itself is a quasi-prime left ideal of S. Let $a \in S \setminus L$. Then $a(Sa) \nsubseteq L$, take $a_1 \in a(Sa) \subseteq (a(Sa)]$ but $a_1 \notin L$. From $a_1(Sa_1) \nsubseteq L$, we have $a_2 \in S$ such that $a_2 \in a_1(Sa_1) \subseteq (a_1(Sa_1)]$ but $a_2 \notin L$. We continue this way, take $a_i \in (a_{i-1}(Sa_{i-1})]$ but $a_i \notin L$. We put $a = a_0$ and let $A = \{a_0, a_1, a_2, \ldots\}$. So $A \cap L = \emptyset$. Next, we claim that M is an M-system. Let $a_i, a_j \in M$. Let us assume that $i \leq j$. If $i = j$, then $a_{i+1} \in (a_i(Sa_i)] = (a_j(Sa_i)]$. If $i < j$, then $a_{j+1} \in (a_j(Sa_j)] \subseteq (a_j(S(a_{j-1}(Sa_{j-1})))] \subseteq (a_j(Sa_{j-1})] \ldots \subseteq (a_j(Sa_i)]$ by Lemma 16. A similar argument takes care of the case in which $i > j$. Now we have that A is an M-system and $A \cap L = \emptyset$. Let $T = \{M \mid M$ is an M-system of S such that $a \in M$ and $M \cap L = \emptyset\}$. Then $T \neq \emptyset$ because $A \in T$. By Zorn's Lemma, there exists a maximal element, say M' in T. Again let $X = \{J \mid J$ is a left ideal of S such that $J \cap M' = \emptyset$ and $L \subseteq J\}$. Then $X \neq \emptyset$ because $L \in X$. By Zorn's Lemma, there exists a maximal element, say J' in X. Let $x, y \in S \setminus J'$. Then $(Sx \cup J'] \cap M' \neq \emptyset$ and $(Sy \cup J'] \cap M' \neq \emptyset$. So there exist $a, b \in M'$ such that $a \leq ux$ and $b \leq vy$ for some $u, v \in S$. Since M' is an M-system, there exists $m \in M$ such that $m \leq a(wb)$ for some $w \in S$. Thus

$$\begin{aligned} m &\leq (ux)(w(vy)) = ((vy)x)(wu) = (uw)(x(vy)) = (e(uw))(x(vy)) \\ &= (ex)((uw)(vy)) = x((yw)(vu)) = x(((vu)w)y). \end{aligned}$$

Then $S \setminus J'$ is an M-system. From maximality of M', $S \setminus J' = M'$ and so J' is a quasi-prime left ideal of S containing L. Since $a \notin J'$, $L = \cap\{J_i \mid i \in I\}$. ■

Theorem 26 *Let S be an ordered AG-groupoid. If N is an N-system of S and $a \in N$, then there exists an M-system M of S such that $a \in M \subseteq N$.*

Proof. Since N is an N-system and $a \in N$, there exists $c_1 \in N$ such that $c_1 \in (a(Sa)]$, So $(a(Sa)] \cap N \neq \emptyset$, take $a_1 \in (a(Sa)] \cap N$. Since N is an N-system, there exists $c_2 \in N$ such that $c_2 \in (a_1(Sa_1)]$, So $(a_1(Sa_1)] \cap N \neq \emptyset$, take $a_2 \in (a_1(Sa_1)] \cap N$. We continue this way, take $a_i \in (a_{i-1}(Sa_{i-1})] \cap N$. We put $a = a_0$ and let $M = \{a_0, a_1, a_2, \ldots\}$. We have M is an M-system and $a \in M \subseteq N$. ■

Let S be an ordered AG-groupoid and a left ideal P of S is called quasi irreducible (quasi strongly irreducible) if for any left ideals A, B of S, $A \cap B = P$ $(A \cap B \subseteq P)$ implies that $A = P$ or $B = P$ $(A \subseteq P$ or $B \subseteq P)$.

Let S be an ordered AG-groupoid with left identity. A nonempty subset I of S is called an I-system of S if for each $a, b \in I$, $(<a> \cap <b>) \cap I \neq \emptyset$.

Theorem 27 *Let S be an ordered AG-groupoid with left identity and L a proper left ideal of S. Then the following statements are equivalent.*

(1) *L is quasi-irreducible.*

(2) *For all $a, b \in S$, $<a> \cap <b> \subseteq L$ implies $a \in L$ or $b \in L$.*

(3) *$S \setminus L$ is an I-system.*

Proof. $(1) \Rightarrow (2)$ Assume L is quasi-irreducible and let $a, b \in S$ such that $< a > \cap < b > \subseteq I$. Thus $< a > \in L$ or $< b > \in L$. Then $a \in L$ or $b \in L$. $(2) \Rightarrow (3)$ Let $a, b \in S \setminus L$. Suppose $(< a > \cap < b >) \cap (S \setminus L) = \emptyset$. This implies $< a > \cap < b > \subseteq L$. So $a \in L$ or $b \in L$, it is impossible. Hence $(< a > \cap < b >) \cap (S \setminus L) \neq \emptyset$. Therefore $S \setminus L$ is an I-system. $(3) \Rightarrow (1)$ Let A, B be left ideals of S such that $A \cap B \subseteq L$. Suppose $A \nsubseteq L$ and $B \nsubseteq L$. Let $a \in A \setminus L$ and $b \in B \setminus L$. This implies that $a, b \in S \setminus L$. By hypothesis, $(< a > \cap < b >) \cap (S \setminus L) \neq \emptyset$. Then there exists $c \in < a > \cap < b >$ and $c \in S \setminus L$. It shows that $c \in < a > \cap < b > \subseteq A \cap B \subseteq L$, it is impossible. Thus $A \subseteq L$ or $B \subseteq L$. Hence L is quasi-irreducible. ■

Chapter 3

Left Almost Rings

Introduction

Actually left almost rings (abbreviated as LA-rings) [70], are an offshoot of LA-semigroups and LA-groups . LA-rings are in fact a generalization of commutative rings and carries attraction due to its peculiar characteristics and structural formation. Despite the fact that the structure is non-associative and non-commutative, however it possesses properties which usually come across in associative and commutative algebraic structures.

This chapter contains three sections. In section one, we have discussed some elementary properties of LA-rings. In second section, we deal with the concepts of ideals, M-systems, P-systems, I-systems and subtractive sets in LA-rings. In continuity to second section, in the third section, we have taken a step forward and established some results regarding the direct sum in LA-rings.

3.1 Some basics of LA-rings

Definition 28 *Let R be a set with at least two elements and two binary operations '$+$' and '$\cdot$' defined on R. Suppose $(R,+)$ is an LA-group and $(R,\cdot)$ is an LA-semigroup satisfying both left and right distributive laws: $a(b+c)=ab+ac$ and $(a+b)c=ac+bc$ for all $a,b,c\in R$. Then $(R,+,\cdot)$ is called an LA-ring.*

Since $(R,+)$ is an LA-group, it contains a left additive identity '0' such that $0+a=a$ for all $a \in R$. Now

$$
\begin{aligned}
0 \cdot a &= (0+0) \cdot a = 0 \cdot a + 0 \cdot a \\
&\Rightarrow 0 + 0 \cdot a = 0 \cdot a + 0 \cdot a
\end{aligned}
$$

As $(R,+)$ is cancellative, so $0 = 0a$ for all $a \in R$. i.e. ,0 is the left zero for $(R,\cdot)$. Similarly, it can be proved that $a0 = 0$ for all $a \in R$, so that 0 is the right zero for $(R,\cdot)$. We shall refer to 0 as the zero element of LA-ring $(R,+,\cdot)$. By the definition of an LA-group, every element a of $(R,+)$ contains a unique additive inverse $-a$. An LA-ring $(R,+,\cdot)$ is called an LA-ring with left identity e if for all $a \in R$, $e.a = a$.

Example 29 *From a commutative ring* $(R,+,\cdot)$, *we can always obtain an LA-ring* $(R,\oplus,\cdot)$ *by defining for* $a,b \in R$, $a \oplus b = b - a$, $a \cdot b$, *is same as in the ring.*

Let R be an LA-ring with left identity e. An element $a \in R$ is said to be invertible or unit in R if there exists an element $b \in R$ such that $ab = ba = e$.

We represent $U(R)$, the set of all unit elements in R. It is not hard to prove that $U(R)$ is an LA-group under the multiplication.

Theorem 30 *If* $(R,+,\cdot)$ *is an LA-ring. Then*

(1) $a0 = 0 = 0a$

(2) $a(-b) = -ab = (-a)b$

(3) $-(-a) = a$

(4) $(-a)(-b) = ab$

(5) $-(a+b) = -a-b$ *for all* $a,b \in R$.

Proof. (1)

$$
\begin{aligned}
a0 &= a(0+0) = a0 + a0 \\
&\Rightarrow 0 + a0 = a0 + a0
\end{aligned}
$$

By the cancellative law for the additive LA-group$(R,+)$, we have $0 = a0$. Likewise, $0a = (0+0)a = 0a + 0a$ implies $0a = 0$.

(2) By the definition, $-(ab)$ is the element which when added to ab gives 0. Thus in order to show $a(-b) = -(ab)$, we must show $a(-b) + ab = 0$. By the left distributive law: $a(-b) + ab = a(-b+b) = a0 = 0$. Similarly, $(-a)b + ab = (-a+a)b = 0b = 0$. (3) Since $0 = a - a$,

$$\begin{aligned} 0-(-a) &= (a-a)-(-a) \text{ adding both sided by } -(-a) \\ &= (-(-a)+(-a))+a = 0+a = a \\ &\Rightarrow \; 0-(-a) = a \\ &\Rightarrow \; -(-a) = a. \end{aligned}$$

(4) By using (2) and (3), we have $(-a)(-b) = -(a(-b)) = -(-(ab)) = ab$. (5) Since $0 = -(a+b) + (a+b)$

$$\begin{aligned} 0+(-a-b) &= (-(a+b)+(a+b))+(-a-b) \text{ adding both sides by } (-a-b) \\ (-a-b) &= ((-a-b)+(a+b)-(a+b) \text{ by left invertive law} \\ &= ((-a+a)+(-b+b))-(a+b) \\ &= (0+0)-(a+b) = 0-(a+b) = -(a+b) \\ &\Rightarrow (-a-b) = -(a+b). \end{aligned}$$

■

Lemma 31 *If an LA-ring has left identity e, then it is unique.*

Proof. Suppose there exists another left identity, say f. Then $ef = f$ and $fe = e$. Now $f = ef = (ee)f = (fe)e = ee = e$. This implies left identity is unique. ■

In an LA-ring medial law holds trivially, i.e, $(ab)(cd) = (ac)(bd)$ for all $a,b,c,d \in R$. While paramedial law: $(ab)(cd) = (db)(ca)$ holds only if R has left identity e.

Lemma 32 *Let R be an LA-ring with left identity 'e' such that $ab = cd$ for all $a, b, c, d \in R$. Then $ba = dc$.*

Proof. Let $ab = cd$ for all $a, b, c, d \in R$. Now $ba = (eb)\,a = (ab)\,e = (cd)\,e = (ed)\,c = dc$. ■

Definition 33 *Let $(R, +, \cdot)$ be an LA-ring. If S is a non-empty subset of R and S is itself an LA-ring under the binary operation induced by R, then S is called an LA-subring of $(R, +, \cdot)$.*

Lemma 34 *If S is non-empty subset of an LA-ring $(R, +, \cdot)$, then S is an LA-subring of $(R, +, \cdot)$ if and only if both $a - b$ and $ab \in S$ for all $a, b \in S$.*

Proof. Proof is straight forward. ■

Lemma 35 *The intersection of two LA-subrings of an LA-ring R is again an LA-subring.*

Proof. Straight forward. ■

Remark 36 *The intersection of any family of LA-subrings of an LA-ring R is again an LA-subring.*

Let R be an LA-ring and $0 \neq a \in R$, then a is said to be a left (right) zero divisor in R if there exists some element $0 \neq b \in R$ such that $ab = 0$ $(ba = 0)$ and is called zero divisor if it is both left zero and right zero divisor. An LA-ring R is cancellative if $ab = ac$ and $ba = ca$ (where $a \neq 0$) implies $b = c$ for all $a, b, c \in R$. Let $(R, +, \cdot)$ be an LA-ring. If $ab = 0$ for all $a, b \in R$, implies either $a = 0$ or $b = 0$, then $(R, +, \cdot)$ is called an LA-integral domain.

Lemma 37 *A left cancellative LA-ring is a cancellative LA-ring.*

Proof. Let R be a left cancellative LA-ring. Let $a, b, c \in R$ such that $ba = ca$. Consider $d \in R$ be any fixed element of R. Then $(ba)\,d = (ca)\,d$. This implies $(da)\,b = (da)\,c$. Thus $b = c$. Hence a left cancellative LA-ring is a cancellative LA-ring. ■

Remark 38 *A right cancellative LA-ring with left identity e is cancellative.*

Theorem 39 *An LA-ring R is an LA-integral domain if and only if it satisfies the cancellative laws.*

Proof. Let R be an LA-integral domain. Suppose

$$\begin{aligned} ac &= ab \text{ for all } a,b,c \in R \text{ and } a \neq 0 \\ &\Rightarrow ac - ab = 0 \\ &\Rightarrow a(c-b) = 0 \\ &\Rightarrow c - b = 0 \text{ since } a \neq 0 \\ &\Rightarrow c = b. \end{aligned}$$

Now suppose

$$\begin{aligned} ca &= ba \\ &\Rightarrow ca - ba = 0 \\ &\Rightarrow (c-b)a = 0 \\ &\Rightarrow c - b = 0 \text{ since } a \neq 0 \\ &\Rightarrow c = b. \end{aligned}$$

Hence R satisfies the cancellative laws.

Conversely, suppose R satisfies the cancellative laws. we have to show that R is an LA-integral domain. It is enough to show that R has no zero divisor. But we suppose a contradiction that R has zero divisor. Then by definition $ab = 0$ where a, b are non-zero elements of R. This implies $ab = a0$. By left cancellative law, we get $b = 0$, which a contradiction. R has no left zero divisor. Now $ba = 0 = b0$. By left cancellative law $a = 0$, which is a contradiction. This implies R has no right zero divisor. This implies R has no zero divisor. Hence R is an LA-integral domain. ■

An element a of an LA-ring R is called additively idempotent if $a + a = a$. $I^+(R)$ denotes the set of all additively idempotent elements of R. Since $I^{+}(R)$ is non-empty, because $0 \in R$ as left additive identity. If every element of R is additively idempotent then R is additively idempotent.

Lemma 40 *An LA-ring is additively idempotent if and only if* $I^{+}(R) = R$.

Proof. Suppose R is additively idempotent. This implies $a + a = a$ for all $a \in R$. As we know that $I^{+}(R) \subseteq R$. Since $a + a$ for all $a \in R$. This implies $R \subseteq I^{+}(R)$. Hence $R = I^{+}(R)$. Conversely, suppose $I^{+}(R) = R$. Let $a \in R$. This implies $a \in I^{+}(R)$ i.e., a is additively idempotent element of R. Since a was taken to arbitrary element of R. Hence R is additively idempotent. ■

Lemma 41 *If a, b, c and d are elements of an additively idempotent LA-ring R satisfying $a + c = b$ and $b + d = a$ then $a = b$.*

Proof. By additive idempotent, we have

$$\begin{aligned}
a &= a + a = a + (b + d) = b + (a + d) = (a + c) + (a + d) \\
&= (a + a) + (c + d) = a + (c + d) = (b + d) + (c + d) \\
&= (b + c) + (d + d) = (b + c) + d = ((a + c) + c) + d \\
&= ((c + c) + a) + d = (c + a) + d = (d + a) + c \\
&= (d + (b + d)) + c = (b + (d + d)) + c \\
&= (b + d) + c = a + c = b
\end{aligned}$$

Hence $b = c$. ■

An element a of an LA-ring R is called multiplicative idempotent if $a^2 = a$ (i.e., $aa = a$). $I^{\times}(R)$ denotes the set of all multiplicative idempotent elements of R. Since $I^{\times}(R)$ is non-empty, because $00 = 0 \in R$ as left additive identity. If R is an LA-ring with left identity e, then e is multiplicative idempotent element of R, since $ee = e$. If every element of R is multiplicative idempotent then R is multiplicative idempotent.

Lemma 42 *An LA-ring R is multiplicative idempotent if and only if $I^{\times}(R) = R$.*

Proof. Proof is easy. ■

3.2 Ideals in LA-rings

In this section, we discuss the concept of ideals in LA-rings. Main results are (1) Let R be an LA-ring with left identity e, then R is fully prime if and only if every ideal is idempotent and the set ideal(R) is totally ordered under inclusion. (2) A left ideal I of an LA-ring R with left identity e is quasi-prime if and only if $R \backslash I$ is an M-system. (3) Every subtractive subset of an LA-ring R is semi-subtractive. (4) Every quasi-prime ideal of an LA-ring R with left identity e is semi-subtractive.

Definition 43 *If A is an LA-subring of an LA-ring $(R, +, \cdot)$, then A is called a left ideal if $RA \subseteq A$. Right ideal and two sided ideal are defined in the usual manner.*

The following proposition provided that every right ideal in an LA-ring is a two sided ideal.

Proposition 44 *If $(R, +, \cdot)$ is an LA-ring with left identity e, then every right ideal is a left ideal.*

Proof. Let I be a right ideal of LA-ring R, this implies I is a LA-subring of R. Now let $r \in R$ and $i \in I$, then $ri = (er)\, i = (ir)\, e \in I$. Thus I is also a left ideal. ■

Now onward by ideal in LA-ring R with left identity e, we mean a right ideal. An element r of an LA-ring $(R, +, \cdot)$ is called idempotent if $r \cdot r = r$. An ideal I of an LA-ring R is called minimal if it does not contain any ideal of R other than itself.

Lemma 45 *Let R be an LA-ring with left identity e. If I is a minimal left ideal of R, then aI is a minimal left ideal of R for every idempotent a.*

Proof. Let I be a minimal left ideal of an LA-ring R and a is an idempotent element, consider $aI = \{ai : i \in I\}$. Let $ai_1, ai_2 \in aI$. Then $ai_1 - ai_2 = a\,(i_1 - i_2) = ai^{'} \in aI$, where

$i^{'} = i_1 - i_2 \in I$ and $(ai_1) \cdot (ai_2) = (a \cdot a)(i_1 \cdot i_2) = a(i_1 \cdot i_2) = ai^{''} \in aI$, by medial law. Thus aI is a subring of R. For $r \in R$, $ai \in aI$, using [51, Lemma 4], we have $r(ai) = a(ri) \in aI$. Thus aI is a left ideal of R. Next, let H be a non-empty left ideal of R which is properly contained in aI. Define $K = \{i \in I : ai \in H\}$ and let $y \in K$. Then $ay \in H$, and so we get $a(ry) = r(ay) \in RH \subseteq H$. This implies that $ry \in K$. Hence K is a left ideal properly contained in I. But this is a contradiction to the minimality of I. Thus aI is a minimal left ideal of LA-ring R. ■

Lemma 46 *If I is a right ideal of an LA-ring R with left identity e then I^2 is an ideal of R.*

Proof. Let $i \in I^2$, then we can write $i = xy$ where $x, y \in I$. Now consider $ir = (xy)r = (ry)x \in II = I^2$. This implies that I^2 is a right ideal and hence by proposition 44, I^2 is a left ideal. ■

Remark 47 *If I is a left ideal of R with left identity e, then I^2 becomes an ideal of R.*

Lemma 48 *Intersection of two left(right) ideals of an LA-ring is again a left(right) ideal.*

Proof. Let I, J be two left ideals of an LA-ring R. Let $a, b \in I \cap J$ this implies that $a, b \in I$ and $a, b \in J$. So, $a - b \in I$, $a - b \in J$ and $a \cdot b \in I$ and $a.b \in J$. This implies that $a - b \in I \cap J$ and $a \cdot b \in I \cap J$. Now let $r \in R$ and $a \in I \cap J$, so $a \in I$ and $a \in J$. This implies $ra \in I$ and $ra \in J$ and hence $ra \in I \cap J$. Thus $I \cap J$ is a left ideal. ■

Addition of ideals I and J of an LA-ring R is defined as

$$I + J = \{x + y : x \in I,\ y \in J\} \subseteq R.$$

Lemma 49 *The sum of two left ideals of an LA-ring with left identity e is again a left ideal.*

Proof. Let R be an LA-ring. Let I, J be left ideals in R. Suppose $z_1, z_2 \in I + J$. This implies $z_1 = (x_1 + y_1)$ and $z_2 = (x_2 + y_2)$. $z_1 - z_2 = (x_1 + y_1) - (x_2 + y_2) = (x_1 + y_1) +$

$(-x_2 - y_2) = (x_1 - x_2) + (y_1 - y_2) \in I + J$. Now

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + y_1) \cdot (x_2 + y_2) = x_1 (x_2 + y_2) + y_1 (x_2 + y_2) \\ &= (x_1 x_2 + x_1 y_2) + (y_1 x_2 + y_1 y_2) = (x_1 x_2 + y_1 x_2) + (x_1 y_2 + y_1 y_2) \in I + J. \end{aligned}$$

Again suppose $z \in I + J$ and $r \in R$ then $z = x + y$ for some $x \in I, y \in J$.

$rz = r(x + y) = (rx + ry) \in I + J$. Thus $I + J$ is a left ideal. ■

Consequently we obtain the following:

Corollary 50 *Let R be an LA-ring R with left identity e.Then*

(1) Sum of two right ideals of R is a right ideal of R.

(2) The sum of one left ideal and one right ideal of an LA-ring R is a left ideal of R.

(3) Addition of left ideals is not commutative as well as not associative.

Remark 51 *For an LA-ring R with left identity e, the following hold;*

(1) $I + R = R$ and $R + I \neq R$.

(2) $(0) + I = I$ and $I + (0) \neq I$.

(3) $I = -I = -x : x \in I$.

Let A, B are ideals of an LA-ring R. Then the product of A and B is defined by

$$\begin{aligned} IJ &= \{\sum_{finite} x_i y_i : x_i \in I, y_i \in J\}. \\ &= \{\sum_{finite} x_i y_i = (...(((x_1 y_1 + x_2 y_2) + x_3 y_3) + x_4 y_4) + ... + x_{n-1} y_{n-1}) + x_n y_n : x_i \in I, y_i \in J\}. \end{aligned}$$

Lemma 52 *Let R be an LA-ring with left identity e. Product of two left ideals of R is again a left ideal.*

Proof. Straight forward. ■

Lemma 53 *Let R be an LA-ring with left identity e. If I is a proper ideal of R, then $e \notin I$.*

Proof. *Assume on contrary that $e \in I$ and let $r \in R$, then consider $r = er \in IR \subseteq I$. This implies that $R \subseteq I$, but $I \subseteq R$. So, $I = R$. A contradiction. Hence $e \notin I$.* ■

Definition 54 *An LA-ring R is said to be fully idempotent if all ideals of R are idempotent. If R is an LA-ring with left identity e then the principal left ideal generated by an element a is defined as $\langle a \rangle = Ra = \{ra : r \in R\}$.*

Remark 55 *It is important to note that if I is an ideal of R, then $I = \langle I \rangle$ and also I^2 is an ideal of LA-ring R. Hence $I^2 = \langle I^2 \rangle$.*

Proposition 56 *If R is an LA-ring with left identity e and I, J are ideals of R, then the following assertions are equivalent:*

(1) R is fully idempotent,

(2) $I \cap J = \langle IJ \rangle$,

(3) the ideals of R form a semilattice $(L_S, \wedge)$, where $I \wedge J = \langle IJ \rangle$.

Proof. (1)⇒(2). Since $IJ \subseteq I \cap J$, $\langle IJ \rangle \subseteq I \cap J$. Now let $a \in I \cap J$. As $\langle a \rangle$ is principal left ideal generated by a fixed element a, so $a \in \langle a \rangle = \langle a \rangle \langle a \rangle \subseteq \langle IJ \rangle$. Hence $I \cap J = \langle IJ \rangle$. (2)⇒(3). $I \wedge J = \langle IJ \rangle = I \cap J = J \cap I = J \wedge I$ and also $I \wedge I = \langle II \rangle = I \cap I = I$. Hence $(L_S, \wedge)$ is a semilattice. (3)⇒(1). Now $I = I \wedge I = \langle II \rangle = II$. ■

Proposition 57 *Suppose R is an LA-ring with left identity e. Let I be a right ideal, then the following are equivalent;*

(1) *$I = R$.*

(2) *$e \in I$.*

(3) *I contains a unit*

(4) *I contains an element which is right invertible or left invertible.*

Proof. (1)⇒(2). Suppose $I = R$, then since $e \in R$, we have $e \in I$. Now suppose $e \in I$, then for all $x \in R$; $x = ex \in I$. Thus $R \subseteq I$ and I being an ideal in R, $I \subseteq R$. Hence

$I = R$. (2)⇒(3). If $e \in I$, I contains a unit obviously. Now suppose I contains a unit u, then $e = u \cdot u^{-1} \in I$. (3)⇒(4). Suppose I contains a unit, then obviously I contains an element which is both right and left invertible. (4)⇒(1). Now suppose I contains an element a which is right invertible or left invertible. But in an LA-ring right invertibility implies left invertibility and vice versa. Thus a is a unit. ■

An ideal P of an LA-ring R is said to be prime ideal if and only if $AB \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$,where A and B are ideals in R and it is called semi-prime if for any ideal I of R, $I^2 \subseteq P$ implies that $I \subseteq P$. An LA-ring R is said to be fully prime if every ideal of R is prime and it is fully semiprime if every ideal is semiprime.

The set of ideals of an LA-ring R is said to be a totally ordered under inclusion if for all ideals I, J of R, either $I \subseteq J$ or $J \subseteq I$ and is denoted by ideal(R).

Theorem 58 *Let R be an LA-ring with left identity e, then R is fully prime if and only if every ideal is idempotent and the set ideal(R) is totally ordered under inclusion.*

Proof. *Let R is fully prime and I be any ideal of R. By lemma 46, I^2 is an ideal of R, and so $I^2 \subseteq I$. Also $II \subseteq I$ which implies that $I \subseteq I^2$. So, $I^2 = I$ and hence I is idempotent. Now let A, B be ideals of R and $AB \subseteq A$, $AB \subseteq B$ which implies that $AB \subseteq A \cap B$. As A and B are prime ideals so $A \cap B$ is also a prime ideal of R. Then $A \subseteq A \cap B$ or $B \subseteq A \cap B$ which implies that either $A \subseteq B$ or $B \subseteq A$. Hence the set ideal(R) is totally ordered under inclusion. Conversely let every ideal of R is idempotent and ideal(R) is totally ordered under inclusion. Let L, M and N be any ideals of R with $LM \subseteq N$ such that $L \subseteq M$. Now since L is idempotent, $L = L^2 = LL \subseteq LM \subseteq N$. This implies that $L \subseteq N$ and hence R is fully prime.* ■

Definition 59 *An ideal I of an LA-ring is said to be strongly irreducible if $P \cap Q \subseteq I$ implies $P \subseteq I$ or $Q \subseteq I$.*

Theorem 60 *Let R is an LA-ring with left identity e, then an ideal I of R is prime if and only if it is semiprime and strongly irreducible.*

Proof. The proof is obvious. ■

3.2.1 M-systems, I-systems and Subtractive sets in LA-rings

In this study, we discuss M-system, P-system, I-system and subtractive sets in an LA-ring with left identity e. We prove the equivalent conditions for a left ideal to be an M-system, P-system, I-system and establish that every M-system of elements of an LA ring with left identity e is P-system. Also we prove that every subtractive subset of R is semi-subtractive. Finally, we show that every quasi-prime ideal of an LA-ring R with left identity e is semi-subtractive.

Definition 61 *A nonempty subset S of an LA-ring R is called an M-system if for $a, b \in S$ there exists r in R such that $a(rb) \in S$.*

Example 62 *Since we assume that any LA-ring R has left identity e, so any LA-semigroup or LA-monoid of $(R, \cdot)$ is an M-system.*

Definition 63 *Let I be a left ideal of an LA-ring R with left identity e. Then I is said to be a quasi-prime if $HK \subseteq I$ implies that either $H \subseteq I$ or $K \subseteq I$, where H and K are any left ideals of R. If for any left ideal H of R such that $H^2 \subseteq I$, we have $H \subseteq I$, then I is called quasi-semiprime.*

Proposition 64 *Let I be a left ideal of R with left identity e, then the following are equivalent:*

(1) *I is quasi-prime ideal.*

(2) *$HK = \langle HK \rangle \subseteq I$ implies that either $H \subseteq I$ or $K \subseteq I$, where H and K are any left ideals of R.*

(3) *If $H \nsubseteq I$ and $K \nsubseteq I$ then $HK \nsubseteq I$, where H and K are any left ideals of R.*

(4) *If h, k are elements of R such that $h \notin I$ and $k \notin I$ then $\langle h \rangle \langle k \rangle \nsubseteq I$.*

(5) *If h, k are elements of R satisfying $h(Rk) \subseteq I$, then either $h \in I$ or $k \in I$.*

Proof. (1)⇔(2). Let I is quasi-prime. Now by definition if $HK = \langle HK \rangle \subseteq I$, then obviously it implies that either $H \subseteq I$ or $K \subseteq I$ for all left ideals H and K of R. Converse is trivial.(2)⇔(3) is trivial. (1)⇒(4). Let $\langle h \rangle \langle k \rangle \subseteq I$, then either $\langle h \rangle \subseteq I$ or $\langle k \rangle \subseteq I$, which implies that either $h \in I$ or $k \in I$. (4)⇒(2). Let $HK \subseteq I$. If $h \in H$ and $k \in K$, then

$\langle h\rangle\langle k\rangle \subseteq I$ and hence by hypothesis either $h \in I$ or $k \in I$. This implies that either $H \subseteq I$ or $K \subseteq I$. (1)⇔(5). Let $h(Rk) \subseteq I$, then $R(h(Rk)) \subseteq RI \subseteq I$. Now consider

$$\begin{aligned} R(h(Rk)) &= (RR)(h(Rk)) = (Rh)(R(Rk)), \text{ by medial law} \\ &= (Rh)((RR)(Rk)) = (Rh)((kR)(RR)), \text{ by paramedial law} \\ &= (Rh)((RR)k), \text{ by left invertive law} \\ &= (Rh)(Rk) \subseteq I. \end{aligned}$$

Since Rh and Rk are left ideals for all $h \in H$ and $k \in K$, hence either $h \in I$ or $k \in I$. Conversely, let $HK \subseteq I$ where H and K are any left ideals of R. Let $H \nsubseteq I$ then there exists $l \in H$ such that $l \notin I$. For all $m \in K$, we have $l(Rm) \subseteq H(RK) \subseteq HK \subseteq I$. This implies that $K \subseteq I$ and hence I is quasi-prime ideal of R. ■

Proposition 65 *A left ideal I of an LA-ring R with left identity e is quasi-prime if and only if $R\backslash I$ is an M-system.*

Proof. Suppose I is a quasi-prime ideal. Let $a, b \in R\backslash I$ which implies that $a \notin I$ and $b \notin I$. So by Proposition 64, $a(Rb) \nsubseteq I$. This implies that there exists some $r \in R$ such that $a(rb) \notin I$ which further implies that $a(rb) \in R\backslash I$. Hence $R\backslash I$ is an M-system. Conversely, let $R\backslash I$ is an M-system. Suppose $a(Rb) \subseteq I$ and let $a \notin I$ and $b \notin I$. This implies that $a, b \in R\backslash I$. Since $R\backslash I$ is an M-system so there exists $r \in R$ such that $a(rb) \in R\backslash I$ which implies that $a(Rb) \nsubseteq I$, which is a contradiction. Hence either $a \in I$ or $b \in I$. This shows that I is a quasi-prime ideal. ■

Definition 66 *A nonempty subset Q of an LA-ring R with left identity e is called P-system if for all $a \in Q$, there exists $r \in R$ such that $a(ra) \in Q$.*

Proposition 67 *Let I be a left ideal of an LA-ring R with left identity e, then the following are equivalent:*

(1) I is quasi-semiprime.

(2) $H^2 = \langle H^2 \rangle \subseteq I \Rightarrow H \subseteq I$, where H is any left ideal of R.

(3) For any left ideal H of $R: H \nsubseteq I \Rightarrow H^2 \nsubseteq I$.

(4) If a is any element of R such that $\langle a \rangle^2 \subseteq I$, then it implies that $a \in I$.

(5) For all $a \in R: a(Ra) \subseteq I \Rightarrow a \in I$.

Proof. (1)⇔(2)⇔(3) is trivial.(1)⇒(4). Let $\langle a \rangle^2 \subseteq I$. But by hypothesis I is quasi-semiprime, so it implies that $\langle a \rangle \subseteq I$ which further implies that $a \in I$. (4)⇒(2). For all left ideals H of R, let $H^2 = \langle H^2 \rangle \subseteq I$. If $a \in H$, then by (4) $\langle a \rangle^2 \subseteq I$ implies that $a \in I$. Hence it shows that $H \subseteq I$. (1)⇔(5) is straight forward. ■

Proposition 68 *A left ideal I of an LA-ring R with left identity e is quasi-semiprime if and only if $R \backslash I$ is a P-system.*

Proof. Let I is quasi-semiprime ideal of R and let $a \in R \backslash I$. On contrary suppose that there does not exist an element $x \in R$ such that $a(xa) \in R \backslash I$. This implies that $a(xa) \in I$. Since I is quasi-semiprime, so by proposition 67, $a \in I$ which is a contradiction. Thus there exists $x \in R$ such that $a(xa) \in R \backslash I$. Hence $R \backslash I$ is a P-system. Conversely, suppose for all $a \in R \backslash I$ there exists $x \in R$ such that $a(xa) \in R \backslash I$. Let $a(Ra) \subseteq I$. This implies that there does not exist $x \in R$ such that $a(xa) \in R \backslash I$ which implies that $a \in I$. Hence by 67, I is quasi-semiprime. ■

Lemma 69 *An M-system of elements of an LA-ring R is a P-system.*

Proof. Let S be a nonempty subset of R such that S is an M-system. Then for all $a, b \in S$, there exists an element $r \in R$ such that $a(rb) \in S$. If we take $b = a$, then $a(ra) \in S$ which implies that S is a P-system. ■

Definition 70 *An ideal I of an LA-ring R with left identity e is strongly irreducible if and only if for ideal H and K of R, $H \cap K \subseteq I$ implies that $H \subseteq I$ or $K \subseteq I$ and I is said to be irreducible if for ideals H and K, $I = H \cap K$ implies that $I = H$ or $I = K$.*

Lemma 71 *Every strongly irreducible ideal of an LA-ring R with left identity e is irreducible.*

Proof. The proof is obvious. ■

Proposition 72 *An ideal I of an LA-ring R with left identity e is prime if and only if it is semiprime and strongly irreducible.*

Proof. The proof is obvious. ■

Definition 73 *A nonempty subset S of an LA-ring R with left identity e is called an I-system if for all $a, b \in S$, $(\langle a\rangle \cap \langle b\rangle) \cap S \neq \phi$.*

Proposition 74 *The following conditions on an ideal I of an LA-ring R are equivalent:*

(1) I is strongly irreducible.

(2) For all $a, b \in R : \langle a\rangle \cap \langle b\rangle \subseteq I$ implies that either $a \in I$ or $b \in I$.

(3) $R\backslash I$ is an I-system.

Proof. (1)⇒(2) is trivial. (2)⇒(3). Let $a, b \in R\backslash I$. Let $(\langle a\rangle \cap \langle b\rangle) \cap R\backslash I = \phi$. This implies that $\langle a\rangle \cap \langle b\rangle \subseteq I$ and so by hypothesis either $a \in I$ or $b \in I$ which is a contradiction. Hence $(\langle a\rangle \cap \langle b\rangle) \cap R\backslash I \neq \phi$. (3)⇒(1). Let H and K be ideal of R such that $H \cap K \subseteq I$. Suppose H and K are not contained in I, then there exist elements a, b such that $a \in H\backslash I$ and $b \in K\backslash I$. This implies that $a, b \in R\backslash I$. So by hypothesis $(\langle a\rangle \cap \langle b\rangle) \cap R\backslash I \neq \phi$ which implies that there exists an element $c \in \langle a\rangle \cap \langle b\rangle$ such that $c \in R\backslash I$. It shows that $c \in \langle a\rangle \cap \langle b\rangle \subseteq H \cap K \subseteq I$ which further implies that $H \cap K \nsubseteq I$. A contradiction. Hence either $H \subseteq I$ or $K \subseteq I$ and so I is strongly irreducible. ■

Definition 75 *A nonempty subset A of an LA-ring R with left identity e is said to be subtractive if and only if $a \in A$ and $a + b \in A$ implies that $b \in A$ and A is called semi-subtractive if and only if $a \in A \cap V(R)$ implies that $-a \in A \cap V(R)$, where $V(R)$ is a set of all those elements of R having additive inverse.*

Proposition 76 *Let A be a subtractive subset of an LA-ring R with left identity e, then*

(1) Every subtractive subset of R contains 0.

(2) Every subtractive subset of R is semi-subtractive.

Proof. (1) If $a \in A$ then $0 + a = a \in A$. Hence by definition $0 \in A$. (2) Let A be a subtractive subset of R. Let $a \in A \cap V(R)$. This implies that $a \in V$ and $a \in V(R)$. Now as A is subtractive, so $a + (-a) = 0 \in A$. This implies that $-a \in A$ and also $-a \in V(R)$. So $-a \in A \cap V(R)$. Hence A is semi-subtractive. ■

Proposition 77 *For subtractive and semi-subtractive left ideals of R, the following holds:*

(1) Intersection of subtractive left ideals of LA-ring R with left identity e is a subtractive left ideal of R.

(2) Intersection of semi-subtractive left ideals of LA-ring R with left identity e is a semi-subtractive left ideal of R.

Proof. The proof is obvious. ■

Proposition 78 *Every quasi-prime ideal of an LA-ring R with left identity e is semi-subtractive.*

Proof. Let I be a quasi-prime ideal of R and $a \in I \cap V(R)$. If $r \in R$, then $(-a)(r(-a)) + a(r(-a)) = 0$ and so $(-a)(r(-a)) = -a(r(-a))$. But on the other hand $a(ra) + a(r(-a)) = 0$ which implies that $a(ra) = -a(r(-a))$. So by uniqueness of additive inverse, we have $(-a)(r(-a)) = a(ra)$. For all $r \in R$ if $(-a)(r(-a)) = a(ra) \in I$, then by Proposition 64, $-a \in I$ and also $-a \in V(R)$, which implies that $-a \in I \cap V(R)$. Hence I is semi-subtractive.
■

Since every quasi-prime ideal is surely quasi-semiprime, so following corollary is an immediate consequence of Proposition 78.

Corollary 79 *Every quasi-semiprime ideal of an LA-ring R is semi-subtractive.*

Proof. The proof is analogous to the proof of Proposition 78. ■

3.3 Direct Sum in LA-Rings

In this section, we construct some new LA-rings from a given family $\{R_i : i \in I\}$ of LA-rings. For this purpose, we introduce complete direct sum and direct sum of this family. We discuss some criterion for LA-ring to be the direct sum of its ideals. We show that if $\{A_i : i \in I\}$ be a family of ideals of an LA-ring R, $I = \{1, 2, 3..., n\}$. Then the following conditions are equivalent. (1) $\Sigma_{i\in I} A_i$ is a direct sum. (2) $\Sigma_{i\in I} a_i = (...((a_1 + a_2) + a_3) + ... + a_{n-1}) + a_n = 0$, $a_i \in A_i, i \in I$, implies that $a_i = 0$ for all $i \in I$. (3) Each element $a \in \Sigma_{i\in I} A_i$ is uniquely expressible in the form $a = (...((a_1 + a_2) + a_3) + ... + a_{n-1}) + a_n$, where $a_i \in A_i$ and $i \in I$.

It is natural to ask is it possible to construct something new from old? In algebra, we use the notion of direct sum. Here we adopt this notion to answer this question.

3.3.1 Complete Direct Sum and Direct Sum

Let $\{R_i : i \in I\}$ be a family of LA-rings indexed by a nonempty set I. The Cartesian product $\Pi\{R_i : i \in I\}$ of the sets R_i is the set of all functions $f : I \longrightarrow U\{R_i : i \in I\}$ such that $f(i) \in R_i$ for all $i \in I$. Let $f, g \in \Pi\{R_i : i \in I\}$. Define $f + g$, fg by $f + g(i) = f(i) + g(i)$, $fg(i) = f(i)g(i)$ for all $i \in I$. Then $f + g$, $fg \in \Pi\{R_i : i \in I\}$. It can be easily verified that $\Pi\{R_i : i \in I\}$ together with the above two operations is an LA-ring. This LA-ring is called the complete direct sum of the family of LA-rings $\Pi\{R_i : i \in I\}$ and is denoted by $\Pi_{i\in I} R_i$.

Suppose that I is a finite set, say $I = \{1, 2, 3..., n\}$. In this case, the complete direct sum is denoted by $\oplus_{i\in I} R_i = (((...((R_1 \oplus R_2) \oplus R_3)...) \oplus R_{n-1}) \oplus R_n)$ and an element $\{a_i : i \in I\}$ is usually written as an n-tuple $(a_1, a_2, a_3, ..., a_n)$.

Definition 80 *The direct sum of family of LA-rings $\{R_i : i \in I\}$, denoted by $\oplus_{i\in I} R_i$ is the set $\oplus_{i\in I} R_i = \{(a_i) \in \Pi_{i\in I} R_i : a_i \neq 0$ for at most finitely many $i \in I\}$.*

Theorem 81 *Let R be an LA-ring and $\{R_i : i \in I\}$ a non empty family of LA-rings, $\Pi_{i\in I} R_i$ the direct product of LA-rings R_i, and $\oplus_{i\in I} R_i$ the direct sum the LA-rings R_i.*

(1) $\Pi_{i\in I} R_i$, is an LA-ring.

(2) $\oplus_{i\in I} R_i$ is an LA-subring of $\Pi_{i\in I} R_i$.

Proof. Straight forward. ■

Definition 82 *Let I be a nonempty set, say, $\{1, 2, 3..., n\}$, and each $\{A_i : i \in I\}$ be a family of ideals of LA-rings R. Then the sum of this family, denoted by $\Sigma_{i\in I}\ A_i$ is the set*

$\Sigma_{i\in I}$ A_i= { $(...((a_1+a_2)+a_3)+...+a_{n-1})$ +a_n: $a_i\in$ A_i}.

Theorem 83 *Let $\{A_i : i \in I\}$ be a family of ideals of an LA-ring R. Then*

(1) $\Sigma_{i\in I}\ A_i$ *is an ideal of R,*

(2) $A_i \subset \Sigma_{j\in I}\ A_j$ *for all* $i \in I$,

(3) *if A is an ideal of R such that $A_i \subset A$ for all $i \in I$, then $\Sigma_{i\in I}\ A_i \subset A$.*

Proof. Straight forward. ■

Definition 84 *Let $\{A_i : i \in I\}$ be a family of ideals of an LA-ring R, where I is finite or infinite. Then the sum of this family, denoted by $\Sigma_{i\in I}\ \ A_i$, is the set*

$$\Sigma_{i\in I}\ A_i = \{a \in R : a \in \Sigma_{i\in I}\ A_i \text{ for some finite subset } I_0 \text{ of } I\}$$

Theorem 85 *Let $\{A_i : i \in I\}$ be a family of ideals of an LA-ring R. Then $\Sigma_{i\in I}\ \ A_i$ is an ideal of R.*

Proof. It is straight forward. ■

Definition 86 *Let $\{A_i : i \in I\}$ be a family of ideals of an LA-ring R. A sum $\Sigma_{i\in I}\ A_i$ of $\{A_i : i \in I\}$ is called a direct sum, if for all $k \in I$, $A_k \cap \sum\limits_{\substack{i\in I\\ i\neq k}} A_i = \{0\}$.*

Lemma 87 *Let $\{A_i : i \in I\}$ be a family of ideals of an LA-ring R. If $\Sigma_{i\in I}\ A_i$ is a direct sum, then for all $a \in A_k$, $b \in A_l$, $k \neq l$, $ab = 0$.*

Proof. Let $a \in A_k$, $b \in A_l$, and $k \neq l$. Since A_k and A_l are ideals, $ab \in A_k$ and $ab \in A_l$. Since $A_k \subseteq \sum\limits_{\substack{i\in I\\ i\neq k}} A_i$, $ab \in \sum\limits_{\substack{i\in I\\ i\neq k}} A_i$. Therefore $ab \in A_k \cap \sum\limits_{\substack{i\in I\\ i\neq k}} A_i$. Since $\Sigma_{i\in I}\ A_i$ is a direct sum, so $A_k \cap \sum\limits_{\substack{i\in I\\ i\neq k}} A_i\ \{0\}$. Hence $ab = 0$. ■

Lemma 88 *[67] If $(S, *)$ is an LA-monoid with left identity e, then position of any element a_i in the finite product of elements of S can be changed accordingly.*

(1) *If $n-i$ is even, then*

$$\begin{aligned} & ((\ldots((a_1 * a_2) * a_3)\ldots) * a_i) * \ldots) * a_{n-1}) * a_n \\ = \; & ((\ldots((a_1 * a_2) * a_3)\ldots) * a_{i-1}) * [a_{i+1} * e]) * \ldots) * [a_{n-1} * e]) * [a_n * e]) * a_i \end{aligned}$$

(2) *If $n-i$ is odd, then*

$$\begin{aligned} & ((\ldots((a_1 * a_2) * a_3)\ldots) * a_i) * \ldots) * a_{n-1}) * a_n \\ = \; & ((\ldots((a_1 * a_2) * a_3)\ldots) * a_{i-1}) * [a_{i+1} * e]) * \ldots) * [a_{n-1} * e]) * [a_n * e]) * [a_i * e], \end{aligned}$$

where $a_i \in S$, $i = 1, 2, \ldots, n$.

Theorem 89 *Let $\{A_i : i \in I\}$ be a family of ideals of an LA-ring R, $I = \{1, 2, 3\ldots, n\}$. Then the following conditions are equivalent.*

(1) *$\Sigma_{i\in I}\, A_i$ is a direct sum.*

(2) *$\Sigma_{i\in I}\, a_i = (\ldots((a_1 + a_2) + a_3) + \ldots + a_{n-1}) + a_n = 0$, $a_i \in A_i, i \in I$, implies that $a_i = 0$ for all $i \in I$.*

(3) *Each element $a \in \Sigma_{i\in I}\, A_i$ is uniquely expressible in the form*

$$a = (\ldots((a_1 + a_2) + a_3) + \ldots + a_{n-1}) + a_n, \text{ where } a_i \in A_i \text{ and } i \in I.$$

Proof. Case I: (when $k-i$ is even);

$(1) \Rightarrow (2)$, let $(\ldots((a_1+a_2)+a_3)+\ldots+a_{n-1})+a_n=0, a_i \in A_i, i \in I$. Let $k \in I$. Now

$$\begin{aligned}
0 &= ((((\ldots((a_1+a_2)+a_3)+\ldots+a_{k-1})+(a_{k+1}+0))+\ldots+(a_{n-1}+0))+(a_n+0))+a_k \\
0-a_k &= (((((\ldots((a_1+a_2)+a_3)+\ldots+a_{k-1})+(a_{k+1}+0))+\ldots+(a_{n-1}+0))+(a_n+0))+a_k)-a_k \\
-a_k &= (a_k-a_k)+((((\ldots((a_1+a_2)+a_3)+\ldots+a_{k-1})+(a_{k+1}+0))+\ldots+(a_{n-1}+0))+(a_n+0)) \\
-a_k &= 0+((((\ldots((a_1+a_2)+a_3)+\ldots+a_{k-1})+(a_{k+1}+0))+\ldots+(a_{n-1}+0))+(a_n+0)) \\
-a_k &= ((((\ldots((a_1+a_2)+a_3)+\ldots+a_{k-1})+(a_{k+1}+0))+\ldots+(a_{n-1}+0))+(a_n+0)) \\
&\in A_k \cap \sum_{\substack{i\in I \\ i\neq k}} A_i = \{0\}\,.
\end{aligned}$$

Hence $a_k = 0$.

$(2) \Rightarrow (3)$,

$$a=(\ldots((a_1+a_2)+a_3)+\ldots+a_{n-1})+a_n=(\ldots((b_1+b_2)+b_3)+\ldots+b_{n-1})+b_n,$$

where $a_i, b_i \in A_i$, for all $i \in I$. Then

$$\begin{aligned}
0 &= ((\ldots((a_1+a_2)+a_3)+\ldots+a_{n-1})+a_n)-((\ldots((b_1+b_2)+b_3)+\ldots+b_{n-1})+b_n) \\
0 &= ((\ldots((a_1+a_2)+a_3)+\ldots+a_{n-1})-(\ldots((b_1+b_2)+b_3)+\ldots+b_{n-1}))+(a_n-b_n) \\
0 &= (((\ldots((a_1+a_2)+a_3)+\ldots+a_{n-2})-(\ldots((b_1+b_2)+b_3)+\ldots+b_{n-2}))) \\
&\quad +(a_{n-1}-b_{n-1})+(a_n-b_n)\,.
\end{aligned}$$

Continuing in this way, we get $(\ldots(((a_1-b_1)+(a_2-b_2))+(a_3-b_3))+\ldots+(a_{n-1}-b_{n-1}))+$ $(a_n-b_n)=0$. Hence by (2), $a_i-b_i=0$ for all $i \in I$. That is $a_i=b_i$ for all $i \in I$. $(3) \Rightarrow (1)$, Let $a \in A_k \cap \sum_{\substack{i\in I \\ i\neq k}} A_i$. Then there exist $a_i \in A_i$, $i=1,2,3\ldots,n$, such that

$a = a_k = (\dots((\dots((a_1 + a_2) + a_3) + \dots + a_{k-1}) + a_{k+1}) + \dots + a_{n-1}) + a_n$, this implies that

$$\begin{aligned} 0 &= ((\dots((\dots((a_1 + a_2) + a_3) + \dots + a_{k-1}) + a_{k+1}) + \dots + a_{n-1}) + a_n) - a_k \\ &= ((\dots((\dots((a_1 + a_2) + a_3) + \dots + a_{k-1}) + (-a_k) + (a_{k+1} + 0)) + \dots + (a_{n-1} + 0)) + (a_n + 0)) \\ &= 0. \end{aligned}$$

Also $0 + 0 + \dots + 0 = 0$. Therefore, by (3), $a_i = 0$ for all $i \in I$. Thus, $A_k \cap \sum_{\substack{i \in I \\ i \neq k}} A_i = \{0\}$ and so $\Sigma_{i \in I} A_i$ is a direct sum.

Case II: (when $k - i$ odd): It is trivial. ■

Definition 90 *An LA-ring R is said to be an internal direct sum of a finite family of ideals* $\{A_i : i \in I\}$ *if*

(i) $R = (\dots((A_1 + A_2) + A_3) + \dots + A_{n-1}) + A_n$.

(ii) $(\dots((A_1 + A_2) + A_3) + \dots + A_{n-1}) + A_n$ *is a direct sum.*

Theorem 91 *Let R be an LA-ring with the property that $(R, +)$ is an abelian group. If $\{A_i : i \in I\}$ is a finite family of ideals R and R is an internal direct sum of $\{A_i : i \in I\}$, then $R \simeq \oplus_{i \in I} A_i$.*

Proof. Let $I = \{1, 2, 3, \dots, n\}$. Suppose R is an internal direct sum of ideals $A_1, A_2, \dots, A_n$. Let $a \in R$. Then a is uniquely expressible in the form

$$a = (\dots((\dots((a_1 + a_2) + a_3) + \dots + a_{k-1}) + a_{k+1}) + \dots + a_{n-1}) + a_n, \text{ where } a_i \in A_i,\ i \in I.$$

Now $(a_1, a_2, a_3, \dots, a_{n-1}, a_n) \in \oplus_{i \in I} A_i$. Define

$$f : R \longrightarrow \oplus_{i \in I} A_i \text{ by } f(a) = (a_1, a_2, a_3, \dots, a_{n-1}, a_n).$$

Let $a, b \in R$. Then there exist $a_i, b_i \in A_i, i \in I$ such that

$$\begin{aligned} a &= (\ldots\ldots((a_1 + a_2) + a_3) + \ldots + a_{n-1}) + a_n \\ \text{and } b &= (\ldots\ldots((b_1 + b_2) + b_3) + \ldots + b_{n-1}) + b_n. \end{aligned}$$

$$\begin{aligned} \text{Let } a &= b \\ &\Leftrightarrow (\ldots((a_1 + a_2) + a_3) + \ldots + a_{n-1}) + a_n = (\ldots((b_1 + b_2) + b_3) + \ldots + b_{n-1}) + b_n \\ &\Leftrightarrow a_i = b_i \text{ for all } i \in I \\ &\Leftrightarrow (a_1, a_2, a_3, \ldots, a_{n-1}, a_n) = (b_1, b_2, b_3, \ldots, b_{n-1}, b_n) \\ &\Leftrightarrow f(a) = f(b) \end{aligned}$$

This shows that f is a one to one function. Let $(a_1, a_2, a_3, \ldots, a_{n-1}, a_n) \in \oplus_{i \in I} A_i$. Then $a = (\ldots\ldots((a_1 + a_2) + a_3) + \ldots + a_{n-1}) + a_n \in \Sigma_{i \in I} \ A_i = R$ and $f(a) = (a_1, a_2, a_3, \ldots, a_{n-1}, a_n)$. Hence, f is onto $\oplus_{i \in I} A_i$. Finally, we shows that f is a homomorphism. Since $a + b = (\ldots(((a_1 + b_1) + (a_2 + b_2)) + (a_3 + b_3)) + \ldots + (a_{n-1} + b_{n-1})) + (a_n + b_n)$, we have,

$$\begin{aligned} f(a + b) &= ((a_1 + b_1), (a_2 + b_2), (a_3 + b_3), \ldots, (a_n + b_n)) \\ &= (a_1, a_2, a_3, \ldots, a_n) + (b_1, b_2, b_3, \ldots, b_n) = f(a) + f(b). \end{aligned}$$

By lemma 87, $ab = (((a_1 b_1 + a_2 b_2) + a_3 b_3) + \ldots + a_{n-1} b_{n-1}) + a_n b_n = \sum_{i=1}^{n} a_i b_i$. $ab = \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i$, where, $a = \sum_{i=1}^{n} a_i, b = \sum_{i=1}^{n} b_i,\ a_i, b_i \in A_i,$ for all $i \in I.$

$$\begin{aligned} ab &= \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i = \left(\sum_{i=1}^{n-1} a_i + a_n\right)\left(\sum_{i=1}^{n-1} b_i + b_n\right) \\ &= \left(\sum_{i=1}^{n-1} a_i + a_n\right)\sum_{i=1}^{n-1} b_i + \left(\sum_{i=1}^{n-1} a_i + a_n\right) b_n \\ &= \left(\sum_{i=1}^{n-1} a_i \sum_{i=1}^{n-1} b_i + a_n \sum_{i=1}^{n-1} b_i\right) + \left(\sum_{i=1}^{n-1} a_i b_n + a_n b_n\right) \\ &= \left(\sum_{i=1}^{n-1} a_i b_i + 0\right) + (0 + a_n b_n) = 0 + \left(\sum_{i=1}^{n-1} a_i b_i + a_n b_n\right) \end{aligned}$$

by using $(a+b)+c = b+(a+c) = \left(\sum_{i=1}^{n-1} a_i b_i + a_n b_n\right) = \sum_{i=1}^{n} a_i b_i$. This implies that $\mathsf{f}(ab) = \mathsf{f}\left(\sum_{i=1}^{n} a_i b_i\right) = (a_1 b_1, a_2 b_2, ..., a_n b_n) = (a_1, a_2, ..., a_n)(b_1, b_2, ..., b_n) = \mathsf{f}(a)\,\mathsf{f}(b)$. So, f is LA-ring homomorphism. Hence $R \simeq \oplus_{i \in I} A_i$. ∎

Theorem 92 *Let R be an LA-ring with 1 such that $(R,+)$ is an abelian group and $\{A_i : i \in I\}$ be a finite family of ideals of R. Then $R = (((...((A_1 \oplus A_2) \oplus A_3)...) \oplus A_{n-1}) \oplus A_n)$ if and only if there exist idempotents $e_i \in A_i$, $i = 1, 2, ..., n$, such that*

(1) $1 = (...((e_1 + e_2) + e_3) + ... + e_{n-1}) + e_n,$

(2) $e_i R = A_i$ *for all* $i = 1, 2, 3..., n,$

(3) $e_i e_{j=} e_j e_i = 0$ *for* $i \neq j$.

Proof. Let $R = (((...((A_1 \oplus A_2) \oplus A_3)...) \oplus A_{n-1}) \oplus A_n)$. Now $1 \in R$. Thus, there exist $e_i \in A_i$, $i = 1, 2, ..., n$, such that $1 = (...((e_1 + e_2) + e_3) + ... + e_{n-1}) + e_n$. Then

$$\begin{aligned} e_i &= (((...((...((e_1 + e_2) + e_3) + ...+) e_{i-1}) + e_i) + e_{i+1} + ... + e_{n-1}) + e_n) e_i \\ &= \left(\left(\left(...((...((e_1 e_i + e_2 e_i) + e_3 e_i) + ...+) e_{i-1} e_i) + e_i^2\right) + e_{i+1} e_i + ... + e_{n-1} e_i\right) + e_n e_i\right) \\ &= \left(\left(\left(...((...((0+0)+0) + ...+) 0) + e_i^2\right) + 0 + ... + 0\right) + 0\right), \text{ by Lemma } 87. \end{aligned}$$

Hence $e_i = \left\{\begin{smallmatrix} e_i^2 & \text{if n}-\text{i=even} \\ e_i^2+0 & \text{if } n-i\text{=odd} \end{smallmatrix}\right\}$. If $e_i = e_i^2 + 0$, then $e_i = 0 + \left(e_i^2 + 0\right) = \left(e_i^2 + 0\right) + 0$, by using given property that $(a+b)+c = b+(a+c)$, so, this implies that $e_i = (0+0) + e_i^2$. So, $e_i = e_i^2$. Hence e_i is an idempotent for all $i = 1, 2, 3, ..., n$. Since $e_i \in A_i$ and A_i is an ideal, so

$e_iR \subseteq A_i$. Let $a \in A_i$. Then

$$\begin{aligned} \text{a} = 1\text{a} = & \quad ((\ldots((e_1 + e_2) + e_3) + \ldots + e_{n-1}) + e_n)\,\text{a} \\ = & \quad (\ldots((e_1a + e_2a) + e_3a) + \ldots + e_{n-1}a) + e_n\text{a} \\ = & \quad \left\{ \begin{matrix} e_ia & \text{if n-i is even} \\ e_ia + 0 & \text{if n-i is odd} \end{matrix} \right\} \end{aligned}$$

because $e_ie_j = 0$ for all $i \neq j$. If $a = e_ia + 0$, then $a = 0 + (e_ia + 0) = (e_ia + 0) + 0$, by using given property that $(a + b) + c = b + (a + c)$, so, this implies that $a = (0 + 0) + e_ia$, by left invertive law. So, $a = e_ia$. Hence $a = e_ia \in e_iR$. Therefore, $e_iR = A_i$. Conversely, assume that there exist idempotents $e_i \in A_i$, $i = 1, 2, 3..., n$, satisfying the given conditions. Let $a \in R$.

$$\begin{aligned} a &= 1a = ((\ldots((e_1 + e_2) + e_3) + \ldots + e_{n-1}) + e_n)\,a \\ &= (\ldots((e_1a + e_2a) + e_3a) + \ldots + e_{n-1}a) + e_na \\ &\in (((\ldots((e_1R_1 + e_1R_2) + e_1R_3)\ldots) + e_1R_{n-1}) + e_1R_n) \\ &\subseteq (((\ldots((A_1 + A_2) + A_3)\ldots) + A_{n-1}) + A_n) \end{aligned}$$

Hence, $R \subseteq (((\ldots((A_1 + A_2) + A_3)\ldots) + A_{n-1}) + A_n)$. But $(((\ldots((A_1 + A_2) + A_3)\ldots) + A_{n-1}) + A_n) \subseteq R$. Hence, $R = (((\ldots((A_1 + A_2) + A_3)\ldots) + A_{n-1}) + A_n)$. Now we show that this sum is direct sum. Let $a \in A_i \cap \sum_{\substack{i \in I \\ i \neq k}} A_i$. Then there exist $a_1, a_2, ..., a_n \in R$ such that, $a = (\ldots((\ldots((a_1 + a_2) + a_3) + \ldots + a_{i-1}) + a_{i+1}) + \ldots + a_{n-1}) + a_n$. This implies $e_ia = (\ldots((\ldots((e_ia_1 + e_ia_2) + e_ia_3) + \ldots + e_ia_{i-1}) + e_ia_{i+1}) + \ldots + e_ia_{n-1}) + e_ia_n$. Thus,

$$e_ia = (\ldots((\ldots((0 + 0) + 0) + \ldots + 0) + 0) + \ldots + 0) + 0, \text{ because } e_ie_j = 0 \text{ for all } i \neq j.$$

Thus $e_ia = 0$. Since $0 \neq e_i \in A_i$, therefore $e_ia = 0$ implies that $a = 0$. Hence $R = (((\ldots((A_1 \oplus A_2) \oplus A_3)\ldots) \oplus A_{n-1}) \oplus A_n)$. ■

Chapter 4

A generalization of Commutative Semigroup Rings

Introduction

On the basis of developments made in chapter 2, regarding the ordering of AG-groupoids and in chapter 3, regarding the concepts of ideals and direct sum in LA-rings, we take a step forward into broad vision and deal with an area which is comparatively hard. This chapter contains two sections and main objective of this study is to generalize commutative semigroup rings. For this in first section, we construct LA-rings of finitely non-zero functions. We adopt the analogous way as in [13] and obtain various generalizations parallel to corresponding parts of commutative semigroup rings. During construction we also introduce the concept of LA-module, which intuitively would be the most useful tool for further developments. For example recently in [68], T. Shah and M. Raees have investigated several results parallel to associative modules theory over the rings.

In second section, we generalize the results established in first section. In this study, we actually consider a case in which we replace commutative semigroups by LA-semigroups (or AG-groupoids) and construct an LA-ring of finitely non-zero functions. Here is the due place to use the concept of ordering of AG-groupoids as discussed earlier in chapter 2; to tackle the degree problem arose during the developments of this type of LA-rings.

4.1 LA-rings of finitely non-zero functions

In this study, we generalize the structure of a commutative semigroup ring (ring of functions from a commutative semigroup S to ring R represented as $R[X;S]$) to an LA-ring of commutative semigroup S over LA-ring R represented as $R[X^s; s \in S]$, which is a non-associative structure, consisting of finitely nonzero functions from a commutative semigroup S into LA-ring R. Generally, the concepts of degree and order are not defined in semigroup rings unless we consider S, a totally ordered semigroup with 0 adjoined. Analogous to commutative semigroup rings $R\left[X;S\right]$, we may define degree and order of an element of LA-ring $R[X^s; s \in S]$.

4.1.1 The Construction

Let $(R, +, \cdot)$ be an LA-ring with left identity and S be a commutative semigroup under binary operation "$*$". Let $T = \{f \mid f : S \to R$, where f are finitely nonzero$\}$. Define the binary operation "$+$" in T as $(f+g)(s) = f(s) + g(s)$.

$(T, +)$ is an LA-group. Indeed, let $f, g \in T$. Now as $f(s), g(s) \in R$ for all $s \in S$, so, $(f+g)(s) = f(s) + g(s) \in R$ and hence $f + g \in T$. Let $f, g, h \in T$. As $f(s), g(s), h(s) \in R$, so by left invertive law in $(R, +)$, we have

$$\begin{aligned} ((f+g)+h)(s) &= (f+g)(s) + h(s) = (f(s)+g(s)) + h(s) \\ &= (h(s)+g(s)) + f(s) = \left(h+g\right)(s) + f(s) \\ &= ((h+g)+f)(s). \end{aligned}$$

Hence $(f+g)+h = (h+g)+f$. Thus left invertive law holds in T. Define the map $o : S \to R$ such that $o(s) = 0$ for all $s \in S$,

$$\begin{aligned} (o+f)(s) &= o(s) + f(s) = 0 + f(s) = f(s) \\ o + f &= f. \end{aligned}$$

Thus o is left additive identity in T. For every $f \in T$ there exists a function $-f : S \to R$ defined by $(-f)(s) = -f(s)$ for all $s \in S$ and $((-f)+f)(s) = (-f(s)) + f(s) = -f(s) + f(s) = 0 = o(s)$. This implies $(-f) + f = o$. So the left inverses exist in $(T, +)$. Hence $(T, +)$ is an LA-group. We can say $f + (-f) = 0$ as $-f(s)$ is also the right inverse of $f(s)$ in R by [45]. Now we define binary operation "$\odot$" in T as follows $f \odot g\,(s) = \sum_{t*u=s} f(t) \cdot g(u)$. We claim that $(T, \odot)$ is an LA-semigroup. As for $f(t)$ and $g(u) \in R$, where $t, u \in (S, *)$ and $(R, \cdot)$ is LA-ring, $f \odot g(s) \in R$. Since f, g are finitely nonzero on S, therefore $f \odot g \in T$. For $f, g, h \in T$ and $s \in S$, consider

$$\begin{aligned}
[(f \odot g) \odot h](s) &= \sum_{t*u=s} (f \odot g)(t) \cdot h(u) = \sum_{t*u=s} \{ \sum_{t=p*q} ((f(p) \cdot g(q))\} \cdot h(u) \\
&= \sum_{(p*q)*u=s} (f(p) \cdot g(q)) \cdot h(u) = \sum_{(u*q)*p=s} (h(u) \cdot g(q)) \cdot f(p).
\end{aligned}$$

As every commutative semigroup implies an LA-semigroup, so $(p * q) * u = (u * q) * p$ for all $p, q, u \in (S, *)$. Hence

$$\begin{aligned}
[(f \odot g) \odot h](s) &= \sum_{(p*q)*u=s} (f(p) \cdot g(q)) \cdot h(u) = \sum_{(u*q)*p=s} (h(u) \cdot g(q)) \cdot f(p) \\
&= \sum_{r'*p=s} \{ \sum_{r'=u*q} (h(u) \cdot g(q))\} \cdot f(p) = \sum_{r'*p=s} (h \odot g)(r') \cdot f(p) \\
&= [(h \odot g) \odot f] (s) .
\end{aligned}$$

Thus $(T, \odot)$ is an LA-semigroup. Now we verify that the binary operation "$\odot$" is distributive over addition. Indeed as $f(t), g(u)$ and $h(u) \in R$ and multiplication is distributive over addition in R, so

$$\begin{aligned}
[f \odot (g+h)] (s) &= \sum_{t*u=s} f(t) \cdot (g+h)(u) = \sum_{t*u=s} f(t) \cdot (g(u) + h(u)) \\
&= \sum_{t*u=s} (f(t) \cdot g(u) + f(t) \cdot h(u)) = \sum_{t*u=s} f(t) \cdot g(u) + \sum_{t*u=s} f(t) \cdot h(u) \\
&= (f \odot g)(s) + (f \odot h)(s) = [f \odot g + f \odot h](s) .
\end{aligned}$$

Hence $f \odot (g+h) = f \odot g + f \odot h$. Similarly

$$\begin{aligned}
[(g+h) \odot f](s) &= \sum_{t*u=s} (g+h)(t) \cdot f(u) = \sum_{t*u=s} (g(t)+h(t)) \cdot f(u) \\
&= \sum_{t*u=s} (g(t) \cdot f(u) + h(t) \cdot f(u)) = \sum_{t*u=s} g(t) \cdot f(u) + \sum_{t*u=s} h(t) \cdot f(u) \\
&= (g \odot f)(s) + (h \odot f)(s) = [g \odot f + h \odot f](s).
\end{aligned}$$

Hence $(g+h) \odot f = g \odot f + h \odot f$. Thus $(T, +, \odot)$ is an LA- ring of commutative semigroup $(S, *)$ over LA-ring $(R, +, \cdot)$.

Remark 93 *If we take $S = \mathbb{Z}_0$ then polynomial LA-ring becomes a particular case of LA-ring $(T, +, \odot)$.*

4.1.2 Representation of elements of T

To represent the elements of ring $(T, +, \odot)$, we first define LA- modules over an LA-ring R.

Definition 94 *Let $(R, +, .)$ be an LA-ring with left identity e. An LA-group $(M, +)$ is said to be LA-module over R if $R \times M \to M$ defined as $(a, m) \mapsto am \in M$, where $a \in R$, $m \in M$ satisfies*

(i) $(a+b)m = am + bm$,

(ii) $a(m+n) = am + an$,

(iii) $a(bm) = b(am)$,

(iv) $1.m = m$,

for all $a, b \in R$, $m, n \in M$.

For instance, let $(R, +, \cdot)$ be an LA-ring with left identity and S be a commutative semigroup. It is important to note that every commutative semigroup is an LA-semigroup. Now it is easy to verify that $R[S] = \{\sum_{j=1}^{n} a_j s_j : a_j \in R, s_j \in S\}$ is an additive LA-group. We claim that $R[S]$ is an LA-module over R. Indeed, let $R \times R[S] \mapsto R[S]$ be defined as $(a, \sum_{j=1}^{n} a_j s_j) \mapsto \sum_{j=1}^{n} (aa_j) s_j$ which is obviously well-defined.

The first two and fourth properties are easy to verify. We verify the third property. Consider $a(b\sum_{j=1}^{n} a_j s_j) = a(\sum_{j=1}^{n} (ba_j)\, s_j) = (\sum_{j=1}^{n} (a\,(ba_j))\, s_j)$.

As R is an LA-ring with left identity, so by [51, Lemma], $a(bc) = b(ac)$ holds for all $a, b, c \in R$. Hence $a(b\sum_{j=1}^{n} a_j s_j) = (\sum_{j=1}^{n} (b\,(aa_j))\, s_j) = b(\sum_{j=1}^{n} (aa_j)\, s_j) = b(a\sum_{j=1}^{n} a_j s_j)$ Thus $a(b\sum_{j=1}^{n} a_j s_j) = b(a\sum_{j=1}^{n} a_j s_j)$.

Remark 95 *If $(S, \cdot)$ is a commutative semigroup, then $T = R[S]$ and elements of LA-ring T are written either in the form of $\sum_{s\in S} f(s)s$ or $\sum_{i=1}^{n} f(s_i)s_i$. Thus (1) S is a free basis for $R[S]$ as an LA-module over LA-ring R and (2) multiplication in $R[S]$ is determined by using distributivity and by setting $(r_1s_1)(r_2s_2) = (r_1r_2)(s_1s_2)$ where $r_1, r_2 \in R$ and $s_1, s_2 \in S$. Indeed, consider $f(s_1)s_1 + f(s_2)s_2 + ... + f(s_n)s_n = 0$. Since $s_1, s_2, ..., s_n \in S$ and $s_i \neq 0$ for all $i = 1, 2, ..., n$, so, $f(s_i) = 0$ for all $i = 1, 2, ..., n$. Thus $s_1, s_2, ..., s_n$ are linearly independent. Now let $f(s_1), f(s_1), ..., f(s_n) \in R$ and $s_1, s_2, ..., s_n \in S$. Then, $f(s_1)s_1 + f(s_2)s_2 + ... + f(s_n)s_n$ is a linear combination of elements of S whose coefficients are from the LA-ring R. Thus S is a free basis for ring T as an LA-module over an LA-ring R. Now let $f = \sum_{i=1}^{n} f(s_i)s_i$ and $g = \sum_{i=1}^{m} g(t_i)t_i$.*

$$\begin{aligned}
For\ n &= 1, m = 1,\ f \cdot g = (f(s_1)s_1) \cdot (g(t_1)t_1) \\
&= f(s_1)g(t_1) \cdot s_1t_1 \\
For\ n &= 2, m = 2,\ (f(s_1)s_1 + f(s_2)s_2) \cdot (g(t_1)t_1 + g(t_2)t_2) \\
&= f(s_1)s_1 \cdot (g(t_1)t_1 + g(t_2)t_2) + f(s_2)s_2 \cdot (g(t_1)t_1 + g(t_2)t_2) \\
&= f(s_1)s_1 \cdot g(t_1)t_1 + f(s_1)s_1 \cdot g(t_2)t_2 + f(s_2)s_2 \cdot g(t_1)t_1 + f(s_2)s_2 \cdot g(t_2)t_2) \\
&= f(s_1)g(t_1) \cdot s_1t_1 + f(s_1)g(t_2) \cdot s_1t_2 + f(s_2)g(t_1) \cdot s_2t_1 + f(s_2)g(t_2) \cdot s_2t_2 \\
&= \sum_{i+j=2}^{4} f(s_i)g(t_j)s_it_j.\ \text{Thus in general, } f \cdot g = \sum_{i+j=2}^{m+n} f(s_i)g(t_j)s_it_j.
\end{aligned}$$

Remark 96 *If $(S, +)$ is a commutative semigroup, then the elements of T are written either in the form $\sum_{s\in S} f(s)X^s$ or $\sum_{i=1}^{n} f(s_i)X^{s_i}$.*

From next lemma, it is obvious that the introduction of symbol X and notation X^s has the effect of transforming $(S,+)$ into $(\{X^s \mid s\in S\},\cdot)$ by means of isomorphism.

Lemma 97 *For a semigroup $(S,+)$, there exists a semigroup $(\{X^s : s\in S\},\cdot)$ which is isomorphic to S, where "$\cdot$" is usual multiplication.*

Thus by the effect of this isomorphism, the representation of an element f of T gets the form $f=\sum_{i=1}^{n} f(s_i)X^{s_i}$ or $\sum_{i=1}^{n} f_iX^{s_i}$, where $f_i=f(s_i)$. We shall represent T by $R\left[X^s; s\in S\right]$.

4.1.3 Degree and order of elements of LA-ring $R\left[X^s; s\in S\right]$

The concepts of degree and order are not generally defined in semigroup rings unless we have to consider S, a totally ordered semigroup with 0 adjoined (that is ordered monoid). The structure of LA-ring $R\left[X^s; s\in S\right]$ is also not convenient for defining degree and order of an element unless $(S,*)$ is totally ordered. Here we define support of $f=\sum_{i=1}^{n} f_iX^{s_i}$ abbreviated as Supp$(f)=\{s_i : f_i\neq 0\}$. The order and degree of f is defined as ord(f) =min(supp(f)) and deg(f) =max(supp(f)).

Lemma 98 (1) *If R is an LA-ring with left identity, then for $f,g\in R\left[X^s; s\in S\right]$,*

$$\text{deg}\,(f.g)\leq \text{deg}\,(f)+\text{deg}\,(g)\,.$$

(2) *If R is an LA-integral domain, then deg$(f.g)$ =deg(f) +deg(g).*

Proof. (1) *Let $f=(f_0+f_1X+...,f_nX^n)$ and $g=(g_0+g_1X+...,g_mX^m)$ where $f_n,g_m\neq 0$. So*

$$\begin{aligned} f\cdot g &= (f_0g_0,(f_1g_0+f_0g_1)X \ +...+f_ng_mX^{n+m}. \\ \textit{Now if } f_n\cdot g_m &\neq 0, \textit{ then, } \text{deg}(f\cdot g)=n+m=\text{deg}(f)+\text{deg}(g) \textit{ and if } f_n\cdot g_m=0, \textit{ then} \\ \text{deg}(f\cdot g) &< \text{deg}(f)+\text{deg}(g). \textit{ Thus } \text{deg}\,(f.g)\leq \text{deg}\,(f)+\text{deg}\,(g)\,. \end{aligned}$$

(2) *As R is an LA-integral domain, so for $f_n\neq 0, g_m\neq 0$, the product $f_n\cdot g_m\neq 0$. Thus* *clearly deg*$(f.g)$ =*deg*(f) +*deg*(g) . ■

4.1.4 Further Developments

A mapping φ of an LA-ring $(R,+,\cdot)$ into an LA-ring $(R',+,\cdot)$ is called a homomorphism if $\varphi(a+b) = \varphi(a)+\varphi(b)$ and $\varphi(ab) = \varphi(a)\cdot\varphi(b)$ and for the two sided ideal I of R, the mapping $\nu : R \to R/I$ defined as $\nu(a) = a+I$ is called the natural epimorphism of LA-ring R onto R/I.

Let θ be an epimorphism of an LA-ring R to an LA-ring R', then $R/Ker\theta \simeq R'$.

Theorem 99 *Let R be an LA-ring and L, M be commutative semigroups. Then*

$$R\left[X^{(l,m)};(l,m)\in L\oplus M\right] \simeq (R\left[X^{l};l\in L\right])([X^{m};m\in M]),$$

where $L\oplus M$ the is external direct sum of commutative semigroups L and M.

Proof. Here we regard the elements of LA- ring R, as finitely non-zero functions from commutative semigroup to LA-ring R.

Assume that $(R\left[X^{l};l\in L\right])([X^{m};m\in M]) = A$. Define $\phi : A \to R\left[X^{(l,m)};(l,m)\in L\oplus M\right]$ by $[\phi(f)](l,m) = [f(m)](l)$ where $f(m)\in R[X^{l};l\in L]$, $f\in A$ and $f : M\to R[X^{l};l\in L]$. Clearly ϕ is surjective because if $h\in R\left[X^{(l,m)};(l,m)\in L\oplus M\right]$, then the element $f\in A$ defined by

$$\begin{aligned}[f(m)](l) &= h(l,m) \text{ is such that} \\ [\phi(f)](l,m) &= [f(m)](l) = h(l,m).\end{aligned}$$

$$
\begin{aligned}
\text{Now Suppose } f &\neq g. \\
\text{Then } f(m) &\neq g(m) \text{ for some } m \in M. \\
[f(m)](l) &\neq [g(m)](l) \text{ for some } l \in L, m \in M \\
\phi(f)(l,m) &\neq \phi(g)(l,m) \\
\phi(f) &\neq \phi(g).
\end{aligned}
$$

Thus ϕ is one-one. It is an LA-ring homomorphism. Indeed, let $f, g \in A$ and $(l,m) \in L \oplus M$.

$$
\begin{aligned}
[\phi(f+g)](l,m) &= [(f+g)(m)](l) = [f(m)+g(m)](l) \\
&= [f(m)](l) + [g(m)](l) = [\phi(f)](l,m) + [\phi(g)](l,m) \\
&= [\phi(f) + \phi(g)](l,m)
\end{aligned}
$$

Now consider

$$
\begin{aligned}
[\phi(f \odot g)](l,m) &= [(f \odot g)(m)](l) = [\sum_{a*b=m} (f(a) \cdot g(b))](l) \\
&= \sum_{a*b=m} (\sum_{c*d=l} (f(a))(c)) \cdot ((g(b))(d)) \\
&= \sum_{(c*d,a*b)=(l,m)} (f(a))(c)) \cdot ((g(b))(d) \\
&= \sum_{(c*a)+(d*b)=(l,m)} \phi(f)(c,a) \cdot \phi(g)(d,b) \\
&= [\phi(f) \odot \phi(g)](l,m) \text{ for all } (l,m) \in L \oplus M.
\end{aligned}
$$

Thus ϕ is an isomorphism. ■

Remark 100 *From above result, for the finite set $\{Y_i\}_{i=1}^{n}$ of indeterminates, isomorphism of $R[Y_1, Y_2, ..., Y_n]$ and $R[X; \mathbb{Z}_0^n]$ follows by induction.*

The following theorem is an extended form of theorem 99.

Theorem 101 *The polynomial LA-ring $R[\{Y_\lambda\}_{\lambda \in \Lambda}]$, where R is an LA-ring and $\{Y_\lambda\}_{\lambda \in \Lambda}$ is a* *family of commuting indeterminates and* $F=\sum_{\lambda\in\Lambda}^{w} Z_\lambda$ *such that* $Z_\lambda \simeq Z_0$*. Then* $R\left[\{Y_\lambda\}_{\lambda\in\Lambda}\right]$ *is isomorphic to LA-ring* $R[X;F]$ *of free commutative semigroup* $(F,+)$ *over* R*.*

Proof. As $F=\sum_{\lambda\in\Lambda}^{w} Z_\lambda$ such that $Z_\lambda \simeq Z_0$ for each λ and $\{e_\lambda\}_{\lambda\in\Lambda}$ be the standard free basis for F that is the λ-th coordinate of e_λ is 1 and all others coordinates are 0. Each element of F is uniquely expressible in the form $a=\sum k_\lambda e_\lambda$, for some $k_\lambda \geq 0\,(k_\lambda \in Z_\lambda)$. For each $r_aX^a \in R[X;F]$, we have

$\sum r_a \Pi_{\lambda\in\Lambda} Y_\lambda^{k_\lambda} \in R\left[\{Y_\lambda\}_{\lambda\in\Lambda}\right]$. We define

$$\phi : R[X;F] \to R\left[[Y_\lambda]_{\lambda\in\Lambda}\right] \text{ by } \phi(\sum_{a\in F} r_a X^a)=\sum_{a\in F} r_a \Pi_{\lambda\in\Lambda} Y_\lambda^{k_\lambda}.$$

$$\begin{aligned}
\text{Suppose } \sum_{a\in F} r_a X^a &= \sum_{b\in F} r_b X^b. \\
\text{Here, } a &= \sum k_\lambda e_\lambda \text{ and } b=\sum k_\lambda^{'} e_\lambda, \text{ where } k_\lambda, k_\lambda^{'} \geq 0. \\
\sum_{a\in F} r_a X^{\sum k_\lambda e_\lambda} &= \sum_{b\in F} r_b X^{\sum k_\lambda^{'} e_\lambda} \\
\sum_{a\in F} r_a X^{k_1e_1+k_2e_2+...+k_\lambda e_\lambda+...} &= \sum_{b\in F} r_b X^{k_1^{'}e_1+k_2^{'}e_2+...+k_\lambda^{'} e_\lambda+...} \\
\sum_{a\in F} r_a X^{k_1e_1} X^{k_2e_2}...X^{k_\lambda e_\lambda}... &= \sum_{b\in F} r_b X^{k_1^{'}e_1} X^{k_2^{'}e_2}...X^{k_\lambda^{'} e_\lambda}... \\
\sum_{a\in F} r_a X^{(k_1,0,0,...)}...X^{(..,0,0,0,k_\lambda...)}... &= \sum_{b\in F} r_b X^{(k_1^{'},0,0,...)}...X^{(0,...,0,k_\lambda^{'},0...)}... \\
\sum_{a\in F} r_a Y_1^{k_1}.Y_2^{k_2}...Y_\lambda^{k_\lambda}... &= \sum_{b\in F} r_b Y_1^{k_1^{'}}.Y_2^{k_2^{'}}...Y_\lambda^{k_\lambda^{'}}... \\
\sum_{a\in F} r_a \Pi_{\lambda\in\Lambda} Y_\lambda^{k_\lambda} &= \sum_{b\in F} r_b \Pi_{\lambda\in\Lambda} Y_\lambda^{k_\lambda^{'}} \\
\phi(\sum_{a\in F} r_a X^a) &= \phi(\sum_{b\in F} r_b X^b).
\end{aligned}$$

Thus ϕ is well-defined. Now it is straight forward to prove that ϕ is an isomorphism. ■

The following is a generalized form of [13, Theorem 8.1].

Theorem 102 *Let* R *be an LA-ring with left identity* e *and let* $(S,*)$ *be a commutative* *semigroup. Let $R[X^s; s \in S]$ be an LA-ring of S over R. Then $R[X^s; s \in S]$ is an LA-integral domain if and only if R is an LA-integral domain and S is an torsion free and cancellative.*

Proof. Assume that R is an LA-integral domain. Now S is torsion free and cancellative if and only if S admits a total order $\leq$ compatible with its operation [13, Corollary 3.4]. Let $f, g \in R[X^s; s \in S] \setminus \{0\}$ such that $f = \sum_{i=1}^{m} f_i X^{s_i}, g = \sum_{i=1}^{n} g_i X^{t_i}$, where $s_1 \leq s_2 \leq ... \leq s_m$ and $t_1 \leq t_2 \leq ... \leq t_n$. If $f_1 \neq 0, g_1 \neq 0$, then $s_1 + t_1 \in$supp$(f \odot g)$ and $f_1 g_1 X^{s_1+t_1}$ is the corresponding term in $f \odot g$. In particular, $f \cdot g \neq 0$ hence $R[X^s; s \in S]$ is an LA-integral domain. Conversely, assume that $R[X^s; s \in S]$ is an LA-integral domain. On the contrary, suppose that R is not an LA- integral domain then for $a, b \in R \setminus \{0\}$, we have $a.b = 0$. If $s \in S$, then $aX^s \odot bX^s = 0$, where $aX^s \neq 0, bX^s \neq 0$. This implies that $R[X^s; s \in S]$ is not an LA-integral domain. Similarly, if S is not cancellative and $s, t, u \in S$ are such that $s+t = s+u$ but $t \neq u$, then for $r \in R \setminus \{0\}$, we have $rX^s \odot \left(rX^t - rX^u\right) = 0$, where rX^s and $rX^t - rX^u$ are nonzero. Hence $R[X^s; s \in S]$ is not an LA-integral domain. Finally, assume that R is an LA-integral domain and that S is cancellative but not torsion free. Let $s, t \in S$ be such that $s \neq t$ while $ns = nt$ for some $n \in \mathbb{Z}^+$ and choose $k \in \mathbb{Z}^+$ minimal so that $ks = kt$. If $0 \neq r \in R$, then $0 = (r^2 X^{ks} - r^2 X^{kt}) = (rX^s - rX^t) \odot (\sum_{i=0}^{k-1} rX^{(k-i-1)s+it})$. Since S is cancellative the choice of k implies $(k - i_1 - 1)s + i_1 t \neq (k - i_2 - 1)s + i_2 t$ for $0 \leq i_1 < i_2 \leq k - 1$.Thus $\sum_{i=0}^{k-1} rX^{(k-i-1)s+it} \neq 0$. Hence again $R[X^s; s \in S]$ is not an LA-integral domain. ■

The next result generalizes [13, Theorem 7.2] and contains some basic informations concerning homomorphisms of LA-rings of commutative semigroups over LA-rings.

Theorem 103 *Let $\mu : R \longrightarrow R_0$ be an LA-ring homomorphism. Let $A =$kerμ and $\phi : S \longrightarrow S_0$ be a semigroup homomorphism, where S, S_0 are commutative semigroups with 0 adjoined to them. Then the following statements holds;*

(1) $\mu^* : R[X^s; s \in S] \longrightarrow R_0[X^s; s \in S]$ *is defined as* $\mu^*(\sum_{i=1}^{n} r_i X^{s_i}) = \sum_{i=1}^{n} \mu(r_i) X^{s_i}$, *is LA-ring homomorphism such that ker$\mu^* = A[X^s; s \in S] =ker\mu[X^s; s \in S]$. μ^* is surjective if μ is surjective.*

(2) $\phi^* : R[X^s; s \in S] \to R[X^s; s \in S_0]$ *defined as* $\phi^*(\sum_{i=1}^{n} r_i X^{s_i}) = \sum_{i=1}^{n} r_i X^{\phi(s_i)}$, *is sur-*

jective and ker$\phi^* = I$. *The ideal of* $R[X^s; s \in S]$ *generated by* $\left\{rX^a - rX^b : \phi(a) = \phi(b), r \in R\right\}$. ϕ^* *is surjective if* ϕ *is surjective.*

(3) $\tau : R[X^s; s \in S] \to R_0[X^s; s \in S_0]$ *defined as* $\tau(\sum_{i=1}^n r_i X^{s_i}) = \sum_{i=1}^n \mu(r_i) X^{\phi(s_i)}$, *is an LA-ring homomorphism such that ker*τ = *ker*μ^*+*ker*$\phi^* = A[X^s; s \in S] + I$. *Then* τ *is surjective if* μ *and* ϕ *are surjective.*

Proof. Same as in [13, Theorem 7.2]. ■

Corollary 104 *Assume that* A *is an ideal of the LA-ring* R *and that* $\sim$ *is a congruence on commutative semigroup* $S = S \cup \{0\}$. *Let* $I = \left\{rX^a - rX^b : \phi(a) = \phi(b), r \in R\right\}$, $\phi : S \to S/\sim$ *is a canonical epimorphism. Then* $R[X^s; s \in S]/A[X^s; s \in S] \simeq \frac{R}{A}[X^s; s \in S]$ *and* $R[X^s; s \in S]/I \simeq R[X^s; s \in S/\sim]$, *where the ideal* I *is called the kernel ideal of congruence.*

Proof. As $\phi : S \to S/\sim$ is defined by $\phi(s) = [s]$. So we define $\phi^* : R[X^s; s \in S] \to R[X^s; s \in S/\sim]$ as $\phi^*(\sum_{i=1}^n r_i X^{s_i}) = \sum_{i=1}^n r_i X^{\phi(s_i)} = \sum_{i=1}^{n} r_i X^{[s_i]}$ which is an LA-ring epimorphism. Therefore $R[X^s; s \in S]/\text{ker}\phi^* \simeq R[X^s; s \in S/\sim]$. Similarly we may define a map $\mu^* : R[X^s; s \in S] \to \frac{R}{A}[X^s; s \in S]$ by $\mu^*(\sum_{i=1}^n r_i X^{s_i}) = \sum_{i=1}^n \mu(r_i) X^{s_i}$, where $\mu : R \to \frac{R}{A}$ is a surjective LA-ring homomorphism defined as $\mu(r) = r + A$. μ^* is also LA- ring epimorphism. Hence $R[X^s; s \in S]/\text{ker}\mu^* \simeq \frac{R}{A}[X^s; s \in S]$. It can be shown that $\text{ker}\mu^* = A[X^s; s \in S]$. Therefore, $R[X^s; s \in S]/A[X^s; s \in S] \simeq \frac{R}{A}[X^s; s \in S]$. ■

Definition 105 *An ideal* P *of an LA-ring* R *is called prime if and only if* $AB \subseteq P$ *implies that either* $A \subseteq P$ *or* $B \subseteq P$, *where* A *and* B *are ideals in* R.

Theorem 106 *Let* R *be an LA-ring with left identity* 1. *Then* P *is a prime ideal in* R *if and only if* R/P *is an LA-integral domain having the left identity* $P + 1$.

Proof. Same as [19, Theorem 2.16]. ■

The following is a generalized form of [13, Corollary 8.2].

Corollary 107 *Let* A *be a proper ideal of LA-ring* R, *then* $A[X^s; s \in S]$ *is prime ideal in* $R[X^s; s \in S]$ *if and only if* A *is prime ideal in* R *and* S *is cancellative and torsion free semigroup.*

Proof. Follows from theorem 102 and corollary 104. ■

4.2 Generalized LA-rings of finitely non-zero functions

In continuation to first section, in this section, we have investigated a case in which commutative semigroup S is taken as an LA-semigroup and almost all the established results in first section stand as a particular case. We construct an LA-ring of finitely non-zero functions from LA-semigroup S to LA-ring R, represented as $R\left[X^{s};s\in S\right]$ and also we discuss the concepts of degree and order of $R\left[X^{s};s\in S\right]$.

Let $(R,+,\cdot)$ be an LA-ring with left identity and S be an LA-semigroup under "$*$". We name the set $\{f:f:S\rightarrow R$, where f are finitely nonzero$\}$ as T. Similar to section one, we define the binary operations "$+$" and "$\odot$" in T as $(f+g)(s)=f(s)+g(s)$ and $f\odot g\left(s\right)=\sum_{t*u=s}f(t)\cdot g(u)$.

It is not hard to see that $(T,+,\odot)$ is an LA- ring of LA-semigroup $(S,*)$ over LA-ring $(R,+,\cdot)$.

To represent the elements of LA-ring $(T,+,\odot)$, we follow the same procedure as in first section. Ultimately the canonical form of an element of T is $\sum\limits_{i=1}^{n}f(s_i)s_i$ or $\sum\limits_{i=1}^{n}f_is_i$.

The following is a generalization of lemma 97 of first section.

Lemma 108 *For an LA-semigroup $(S,*)$, there exists an LA-semigroup $(\{X^{s}\mid s\in S\},\odot)$ which is isomorphic to S, where "$\odot$" is not a usual multiplication.*

Proof. Suppose $A=\{X^{s}\mid s\in S\}$. We claim that A is an LA-semigroup. Let $X^{s_1},X^{s_2}\in A$, so $X^{s_1}\odot X^{s_2}=X^{s_1*s_2}\in A$, as S under $*$ is an LA-semigroup. Hence A is closed under "$\odot$".Let $X^{s_1},X^{s_2},X^{s_3}\in A$ then $(X^{s_1}\odot X^{s_2})\odot X^{s_3}=X^{\left(s_1*s_2\right)}\odot X^{s_3}=X^{(s_1*s_2)*s_3}=X^{(s_3*s_2)*s_1}=X^{(s_3*s_2)}\odot X^{s_1}=(X^{s_3}\odot X^{s_2})\odot X^{s_1}$.

Hence A is an LA-semigroup under "$\odot$". Now define $\phi:S\rightarrow A$ as $\phi\left(s\right)=X^{s}$. Let for $s_1,s_2\in S,\ s_1=s_2$. This implies $X^{s_1}=X^{s_2}$ and $\phi\left(s_1\right)=\phi\left(s_2\right)$. Thus ϕ is well-defined. suppose $\phi\left(s_1\right)=\phi(s_2)$. Thus $X^{s_1}=X^{s_2}$,so $s_1=s_2$. Hence ϕ is one-one. Clearly ϕ is onto. Now ϕ is homomorphism as for $s_1,s_2\in S,\ \phi(s_1*s_2)=X^{\left(s_1*s_2\right)}=X^{s_1}\odot X^{s_2}=\phi\left(s_1\right)\odot\phi\left(s_2\right)$. Thus ϕ is an LA-semigroup isomorphism. ■

Thus by the effect of above isomorphism, the representation of an element f of T gets the form $f=\sum_{i=1}^{n} f(s_i)X^{s_i}$ or $\sum_{i=1}^{n} f_iX^{s_i}$, where $f_i=f(s_i)$. We shall represent T by $R\left[X^s; s\in S\right]$.

The concept of degree and order are not generally defined in semigroup rings unless we have to consider S, a totally ordered semigroup with 0 adjoined (that is ordered monoid). The structure of LA-ring $R\left[X^s; s\in S\right]$ is also not convenient for defining degree and order of an element unless $(S,*)$ is totally ordered. The concept of ordering of LA-semigroups (or AG-groupoids) has been discussed earlier in Chapter 2.

Here we define support of $f=\sum_{i=1}^{n} f_iX^{s_i}$ abbreviated as Supp$(f)=\{s_i : f_i\neq 0\}$. The order and degree of f is defined as $ord\left(f\right)=\min\text{Supp}(f)$ and $\deg\left(f\right)=\max Supp\left(f\right)$.

Let $(\mathbb{Q}^+,\cdot)$ denote the group of all positive rational numbers. If we take $S=\left(\mathbb{Q}_0^+\right)^{ILAS}$, where ILAS abbreviates Initial LA-semigroup, which is made an LA-semigroup by defining the binary operation $*$ as

$$\begin{aligned} a*b &= 0 \text{ if } a=0 \text{ } or \text{ } b=0, \\ &= b\cdot a^{-1} \text{ if } a\neq 0 \text{ and } b\neq 0. \end{aligned}$$

If R is an LA-ring with left identity, then for $f,g\in R[X^s; s\in(\mathbb{Q}_0^+,*)]$, the degree and order of $f\left(X\right)=a_0+a_1X^{\frac{a_1}{b_1}}+...+a_nX^{\frac{a_n}{b_n}}\in R\left[X^s; s\in S\right]$ with $\frac{a_1}{b_1}<\frac{a_2}{b_2}<...<\frac{a_n}{b_n}$, is defined as, $ord\left(f\right)=0$ if $a_0\neq 0$ and $\deg\left(f\right)=\frac{a_n}{b_n}$.

In a polynomial ring $R[X]$, for $f,g\in R\left[X\right]$, $\deg\left(f\cdot g\right)\leq\deg\left(f\right)+\deg\left(g\right)$ and $ord\left(f\cdot g\right)\geq ord\left(f\right)+ord\left(g\right)$. If R is an integral domain, then $\deg\left(f\cdot g\right)=\deg\left(f\right)+\deg\left(g\right)$ and $ord\left(f\cdot g\right)=ord\left(f\right)+ord\left(g\right)$. But the following lemma shows a deviation for LA-ring $R\left[X^s; s\in S\right]$ of LA-semigroup $(\mathbb{Q}_0^+,*)=S$ over LA-ring R.

Proposition 109 *Let R be an LA-ring with left identity and $f,g\in R\left[X^s; s\in S\right]$ such that $f\left(X\right)=f_0+f_1X^{\frac{a_1}{b_1}}+...+f_{m-1}X^{\frac{a_{m-1}}{b_{m-1}}}+f_mX^{\frac{a_m}{b_m}}$ and $g\left(X\right)=g_0+g_1X^{\frac{c_1}{d_1}}+...+g_{n-1}X^{\frac{c_{n-1}}{d_{n-1}}}+g_nX^{\frac{c_n}{d_n}}$.*

(1) *If R is an LA-integral domain, then $\deg\left(f\odot g\right)=\deg\left(g\right)\cdot\left(\deg\left(f\right)\right)^{-1}$ if $\left(\deg\left(f\right)\right)^{-1}>1$ and $\deg\left(f\odot g\right)>\deg\left(g\right)\cdot\left(\deg\left(f\right)\right)^{-1}$ if $\left(\deg\left(f\right)\right)^{-1}<1$.*

(2) *If R is not LA-integral domain, then* $\deg(f \odot g) < \deg(g) \cdot (\deg(f))^{-1}$ *if* $\frac{b_{m-1}}{a_{m-1}} > 1$ *and* $\deg(f \odot g) > \deg(g) \cdot (\deg(f))^{-1}$ *if* $\frac{b_{m-1}}{a_{m-1}} < 1$.

Proof. (1) As R is an LA-integral domain, so for $f_m \neq 0, g_n \neq 0, f_m \cdot g_n \neq 0$.

$$\begin{aligned} f(X) \odot g(X) &= f_0 g_0 + f_1 g_0 X^{\frac{a_1}{b_1}} + f_0 g_1 X^{\frac{c_1}{d_1}} + ... + f_m g_n X^{\frac{a_m}{b_m} * \frac{c_n}{d_n}}. \\ \text{As } \frac{c_t}{d_t} &< \frac{c_n}{d_n}, \text{ where t} = 1, 2, ..., \text{n} - 1. \text{ So } \frac{a_m}{b_m} * \frac{c_t}{d_t} < \frac{a_m}{b_m} * \frac{c_n}{d_n} \\ \text{if } \frac{b_m}{a_m} &> 1. \text{ Hence } \deg(f \odot g) = \frac{a_m}{b_m} * \frac{c_n}{d_n} = \frac{c_n}{d_n} \cdot \frac{b_m}{a_m} \\ &= \deg(g) \cdot (\deg(f))^{-1}. \\ \text{Now for } t &= 1, 2, ..., n-1, \ \frac{c_t}{d_t} < \frac{c_n}{d_n} \text{ implies } \frac{a_m}{b_m} * \frac{c_t}{d_t} > \frac{a_m}{b_m} * \frac{c_n}{d_n} \\ \text{if } \frac{b_m}{a_m} &< 1 \text{ so, } \deg(f \odot g) > \frac{a_m}{b_m} * \frac{c_n}{d_n} = \frac{c_n}{d_n} \cdot \frac{b_m}{a_m} \\ &= \deg(g) \cdot (\deg(f))^{-1} \text{ if } \frac{b_m}{a_m} < 1. \end{aligned}$$

(2) If R is not an LA-integral domain, then there is a possibility that for $f_m \neq 0, g_n \neq 0$, $f_m \cdot g_n = 0$.

$$\begin{aligned} \text{As } \frac{a_{m-1}}{b_{m-1}} * \frac{c_k}{d_k} &< \frac{a_{m-1}}{b_{m-1}} * \frac{c_{n-1}}{d_{n-1}}, \text{ where } k = 1, 2, ..., n-2. \\ \text{Hence } \deg(f \odot g) &= \frac{a_{m-1}}{b_{m-1}} * \frac{c_{n-1}}{d_{n-1}} < \frac{a_m}{b_m} * \frac{c_n}{d_n} \\ &= \frac{c_n}{d_n} \cdot \frac{b_m}{a_m} = \deg(g) \cdot (\deg(f))^{-1}. \\ \text{Thus } \deg(f \odot g) &< \deg(g) \cdot (\deg(f))^{-1}. \\ \text{Now } \frac{a_{m-1}}{b_{m-1}} * \frac{c_{n-1}}{d_{n-1}} &> \frac{a_m}{b_m} * \frac{c_n}{d_n} = \frac{c_n}{d_n}.\frac{b_m}{a_m} \\ \text{Thus in this case, } \deg(f \odot g) &> \deg(g) \cdot (\deg(f))^{-1} \text{ if } \frac{b_{m-1}}{a_{m-1}} < 1. \end{aligned}$$

■

Remark 110 *There could be two possibilities for* $\deg(f \odot g)$.

(i) *If R is an LA-integral domain,* $\frac{a_m}{b_m} * \frac{c_n}{d_n} < \frac{a_m}{b_m}$ *or* $\frac{a_m}{b_m} * \frac{c_n}{d_n} < \frac{c_n}{d_n}$*, then in this case,* $\deg(f \odot g) < \deg(f)$ *or* $\deg(f \odot g) < \deg(g)$.

(ii) If R is not an LA-integral domain $\frac{a_{m-1}}{b_{m-1}} * \frac{c_n}{d_n} < \frac{a_m}{b_m}$ or $\frac{c_n}{d_n}$, then in this case, $\deg(f \odot g) < \deg(f)$ or $\deg(f \odot g) < \deg(g)$.

Remark 111 *If we take an LA-semigroup other than $(\mathbb{Q}_0^+, *)$, then the results for degree and order will be different from those of $(\mathbb{Q}_0^+, *)$.*

4.2.1 Main Results

In this section we generalize the results as established in first section of this chapter. Specifically, we show the necessary and sufficient condition for an LA-ring $R[X^s; s \in S]$ to be an LA-integral domain. We also discuss the homomorphisms of LA-rings.

Following external direct sum of semigroups as in [13, Page 18], we define external direct sum of LA-semigroups $(S, *)$ and $(T, \#)$ as $S \oplus T = \{(s,t) : s \in S, t \in T\}$, whereas the binary operation in $S \oplus T$ is defined as $(s_1, t_1) \circledast (s_2, t_2) = (s_1 * s_1, t_1 \# t_2)$ for $s_1, s_2 \in S$ and $t_1, t_2 \in T$.

A mapping φ of an LA-ring $(R, +, \cdot)$ into an LA-ring $(R', +, \cdot)$ is called a homomorphism if $\varphi(a+b) = \varphi(a) + \varphi(b)$ and $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$ and for the ideal I of R, the mapping $\nu : R \to R/I$ defined as $\nu(a) = a + I$ is called the natural epimorphism of LA-ring R onto R/I. Let θ be an epimorphism of an LA-ring R to an LA-ring R', then $R/Ker\theta \simeq R'$.

The following is a generalized form of Theorem 99 of first section.

Theorem 112 *Let R be an LA-ring and S, T be LA-semigroups. Then $R\left[X^{(s,t)}; (s,t) \in S \oplus T\right] \simeq (R[X^s; s \in S])(\left[X^t; t \in T\right])$, where $S \oplus T$ the is external direct sum of LA-Semigroups S and T.*

Proof. Here we regard the elements of LA-ring of S (respectively of T) over R as finitely nonzero functions from the LA-semigroup S (respectively LA-semigroup T) into LA-ring R.

Assume that $(R[X^s; s \in S])(\left[X^t; t \in T\right]) = A$. Define $\phi : A \to R\left[X^{(s,t)}; (s,t) \in S \oplus T\right]$ as follows; if $f \in A$ and $(s,t) \in S \oplus T$, then $[\phi(f)](s,t) = [f(t)](s)$, where $f(t) \in R[X^s; s \in S]$ and $f : T \to R[X^s; s \in S]$, which is well-defined map. Clearly ϕ is surjective because if $h \in R\left[X^{(s,t)}; (s,t) \in S \oplus T\right]$, then the element $f \in A$ defined by $[f(t)](s) = h(s,t)$ is such that

$[\phi(f)](s,t) = [f(t)](s) = h(s,t)\,.$

$$\begin{aligned} \text{Now Suppose } f &\neq g. \\ \text{Then } f(t) &\neq g(t) \text{ for some } t \in T. \\ [f(t)](s) &\neq [g(t)](s) \text{ for some } s \in S, t \in T \\ \phi(f)(s,t) &\neq \phi(g)(s,t) \\ \phi(f) &\neq \phi(g). \end{aligned}$$

Thus ϕ is one-one. Let $f, g \in A$ and $(s,t) \in S \oplus T$.

$$\begin{aligned} [\phi(f+g)](s,t) &= [(f+g)(t)](s) = [f(t) + g(t)](s) \\ &= [f(t)](s) + [g(t)](s) = [\phi(f)](s,t) + [\phi(g)](s,t) \\ &= [\phi(f) + \phi(g)](s,t)\,. \end{aligned}$$

Now consider

$$\begin{aligned} [\phi(f \odot g)](s,t) &= [(f \odot g)(t)](s) = [\sum_{a*b=t} (f(a) \cdot g(b))](s) \\ &= \sum_{a*b=t} (\sum_{c*d=s} (f(a))(c)) \cdot ((g(b))(d)) \\ &= \sum_{(c*d, a*b)=(s,t)} (f(a))(c)) \cdot ((g(b))(d) \\ &= \sum_{(c*a)+(d*b)=(s,t)} \phi(f)(c,a) \cdot \phi(g)(d,b) \\ &= [\phi(f) \odot \phi(g)](s,t) \text{ for all } (s,t) \in S \oplus T. \end{aligned}$$

Thus ϕ is an isomorphism. ■

Corollary 113

$$\mathsf{R}[\mathsf{X}^{(r,s)}; (\mathsf{r},\mathsf{s}) \in \left(\mathbb{Q}_0^+\right)^{ILAS} \oplus \left(\mathbb{Q}_0^+\right)^{ILAS}] \simeq (\mathsf{R}\left[X^r; r \in \left(\mathbb{Q}_0^+\right)^{ILAS}\right])(\left[X^s; s \in \left(\mathbb{Q}_0^+\right)^{ILAS}\right]$$

By [61, Page 332], a semigroup S is said to be m-torsion free if for any $x, y \in S$ there exists $m \geq 1$ with $x^m = y^m$, then $x = y$. We extend this for LA*-semigroup with left identity e.

Definition 114 *An LA*-semigroup $(S, *)$ with left identity e is said to be an M-torsion free if for all $x, y \in S$ there exists a subset M of Z^+ such that $1 \leq m \in M$ with $x^m = y^m$ implies $x = y$.*

Example 115 *$(Q_0^+, *)$ is an LA*-semigroup with left identity 1, defined as*

$$\begin{aligned} a * b &= 0 \text{ if } a = 0 \text{ or } b = 0, \\ &= b \cdot a^{-1} \text{ if } a \neq 0 \text{ and } b \neq 0, \end{aligned}$$

is an O-torsion free where O is the set of odd positive integers. For this, consider $m = 3$, Let $x^3 = y^3$. As $(Q_0^+, *)$ is an LA-semigroup, so $x^2 * x = y^2 * y$. Now as for all $x \in Q_0^+$, $x^2 = 1$, so $1 * x = 1 * y$. This implies $x = y$. Hence $(Q_0^+, *)$ is an O-torsion free LA*-semigroup. Similarly $(\mathbb{Z}, *)$, an LA*-semigroup with left identity 0 defined as $a * b = b - a$, is an O-torsion free where O is a the set of odd positive integers.*

The following is a generalized form of Theorem 102, proved in first section.

Theorem 116 *Let R be an LA-ring with left identity and let $(S, *)$ be an LA*-semigroup. Let $R[X^s; s \in S]$ be an LA-ring of S over R. Then $R[X^s; s \in S]$ is an LA-integral domain if and only if R is an LA-integral domain and S is an M-torsion free and cancellative.*

Proof. Assume that R is an LA-integral domain. Now S is an M-torsion free and cancellative if and only if S admits a total order $\leq$ compatible with its operation (see chapter 2, section one). Let $f, g \in R[X^s; s \in S] \setminus \{0\}$ such that $f = \sum_{i=1}^{n} f_i X^{s_i}$, $g = \sum_{i=1}^{n} g_i X^{t_i}$, where $s_1 \leq s_2 \leq ... \leq s_n$ and $t_1 \leq t_2 \leq ... \leq t_n$. If $f_1 \neq 0, g_1 \neq 0$, then $s_1 * t_1 \in$ Supp$(f \odot g)$ and $f_1 g_1 X^{s_1 * t_1}$ is the corresponding term in $f \odot g$. In particular $f.g \neq 0$ hence $R[X^s; s \in S]$ is an LA-integral domain. Conversely, assume that $R[X^s; s \in S]$ is an LA-integral domain. On the contrary, suppose that R is not an LA- integral domain then for $a, b \in R\backslash\{0\}$, we have $a.b = 0$. If $s \in S$, then $aX^s \odot bX^s = 0$, where $aX^s \neq 0, bX^s \neq 0$. This implies that $R[X^s; s \in S]$ is not an LA-integral domain. Similarly, if S is not cancellative and $s, t, u \in S$ are such that $s*t = s*u$ but $t \neq u$, then for $r \in R\backslash\{0\}$, we have $rX^s \odot \left(rX^t - rX^u\right) = 0$, where rX^s and $rX^t - rX^u$ are nonzero. Hence $R[X^s; s \in S]$ is not an LA-integral domain. Finally, assume that R is an LA-integral domain and that S is cancellative but not an M-torsion free. Let $s, t \in S$ be such that $s \neq t$ while $s^m = t^m$ for some $m \in M \subseteq \mathbb{Z}^+$ and choose $k \in \mathbb{Z}^+$ minimal in M so that $s^k = t^k$. If $0 \neq r \in R$, then $0 = (r^2X^{s^k} - r^2X^{t^k}) = (rX^s - rX^t) \odot (\sum_{i=0}^{k-1} rX^{s^{(k-i-1)}*it})$. Since S is cancellative the choice k implies that $ts^{(k-i_1-1)}*i_1t \neq s^{(k-i_2-1)}*i_2t$ for $0 \leq i_1 < i_2 \leq k-1$. Thus $\sum_{i=0}^{k-1} rX^{s^{(k-i-1)}*it} \neq 0$. Hence again $R[X^s; s \in S]$ is not an LA-integral domain. ■

The next result generalizes Theorem 103 and contains some basic information concerning homomorphisms of LA-rings.

Theorem 117 *Let $\mu : R \longrightarrow R_0$ be an LA-ring homomorphism. Let $A = \ker \mu$ and $\phi : S \longrightarrow S_0$ be an LA-semigroup homomorphism, where S, S_0 are LA-semigroups with 0 adjoined to them. Then the following statements holds;*

(1) $\mu^* : R[X^s; s \in S] \longrightarrow R_0[X^s; s \in S]$ *is defined as* $\mu^*(\sum_{i=1}^{n} r_iX^{s_i}) = \sum_{i=1}^{n} \mu(r_i) X^{s_i}$, *is LA-ring homomorphism such that* $\ker \mu^* = A[X^s; s \in S] = \ker \mu [X^s; s \in S]$. μ^* *is surjective if* μ *is surjective.*

(2) $\phi^* : R[X^s; s \in S] \to R[X^s; s \in S_0]$ *defined as* $\phi^*(\sum_{i=1}^{n} r_iX^{s_i}) = \sum_{i=1}^{n} r_iX^{\phi(s_i)}$, *is surjective and* $\ker \phi^* = I$. *The ideal of* $R[X^s; s \in S]$ *generated by*

$\left\{rX^a - rX^b : \phi(a) = \phi(b), r \in R\right\}$. ϕ^* *is surjective if* ϕ *is surjective.*

(3) $\tau : R[X^s; s \in S] \to R_0[X^s; s \in S_0]$ *defined as* $\tau(\sum_{i=1}^{n} r_iX^{s_i}) = \sum_{i=1}^{n} \mu(r_i) X^{\phi(s_i)}$, *is an LA-ring homomorphism such that* $\ker \tau = \ker \mu^* + \ker \phi^* = A[X^s; s \in S] + I$. *Then* τ *is surjective if* μ *and* ϕ *are surjective.*

Proof. Same as [13, Theorem 7.2]. ■

The congruences on LA-semigroups have been discussed in [42].

The following generalizes Corollary104.

Corollary 118 *Assume that A is an ideal of the LA-ring R and that $\sim$ is a congruence on an LA-semigroup $S = S \cup \{0\}$. Let $I = \left\{rX^a - rX^b : \phi(a) = \phi(b), r \in R\right\}$, $\phi : S \to S/\sim$ is a canonical epimorphism. Then $R[X^s; s \in S]/A[X^s; s \in S] \simeq R/A[X^s; s \in S]$ and $R[X^s; s \in S]/I \simeq R[X^s; s \in S/\sim]$, where the ideal I is called the kernel ideal of congruence.*

Proof. As $\phi : S \to S/\sim$ is defined by $\phi(s) = [s]$. So we define $\phi^* : R[X^s; s \in S] \to R[X^s; s \in S/\sim]$ as $\phi^*(\sum_{i=1}^n r_i X^{s_i}) = \sum_{i=1}^n r_i X^{\phi(s_i)} = \sum_{i=1}^{n} r_i X^{[s_i]}$ which is an LA-ring epimorphism. Therefore, $R[X^s; s \in S]/\ker\phi^* \simeq R[X^s; s \in S/\sim]$.

Similarly we may define a map $\mu^* : R[X^s; s \in S] \to R/A[X^s; s \in S]$ by $\mu^*(\sum_{i=1}^n r_i X^{s_i}) = \sum_{i=1}^n \mu(r_i) X^{s_i}$, where $\mu : R \to R/A$ is a surjective LA-ring homomorphism defined as $\mu(r) = r + A$. μ^* is also LA-ring epimorphism. Hence $R[X^s; s \in S]/\ker\mu^* \simeq R/A[X^s; s \in S]$. It can be shown that $\ker\mu^* = A[X^s; s \in S]$. Therefore, $R[X^s; s \in S]/A[X^s; s \in S] \simeq R/A[X^s; s \in S]$. ■

Definition 119 *An ideal P of an LA-ring R is called prime if and only if $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$, where A and B are ideals in R.*

Corollary 120 *Let A be a proper ideal of LA-ring R, then $A[X^s; s \in S]$ is prime ideal in $R[X^s; s \in S]$ if and only if A is prime ideal in R and S is cancellative M-torsion free LA-semigroup.*

Proof. Proof is straight forward. ■

Chapter 5

Γ-AG-groupoids

Introduction

In this study, we have introduced a non-commutative and a non-associative structure, which we named as Γ-AG-groupoids. Γ-AG-groupoids are a direct generalization of AG-groupoids which have been studied by several researchers [41, 42, 43, 46, 47, 50, 57, 58, 69]. The motivation behind this study is an article ; On Γ-semigroups, by M. K. Sen [63], published in 1981. Γ-semigroups are a generalization of semigroups. Many classical notions of semigroups were extended to Γ-semigroups.

In this chapter, we present our work into two sections. In section one, we have defined Γ-AG-groupoids and also constructed some examples. Γ-AG-groupoids are in fact a generalization of AG-groupoids. We discussed some basic characteristics of this concept and established some important results which have been used for further developments in the theory of Γ-AG-groupoids. Moreover, we investigated the concept of Γ-ideals and M-systems in Γ-AG-groupoids. In second section, we characterized regular and intra-regular Γ-AG-groupoids by the properties of Γ-ideals.

5.1 The characteristics of elements of Γ-AG-groupoids

In this section, we extend some properties of AG-groupoids to Γ-AG-groupoids and correlate Γ-AG-groupoids and Γ-semigroups [63, 64].

We initiate with the following definition:

Definition 121 *Let S and Γ be nonempty sets. We call S to be a Γ-AG-groupoid if there* *exists a mapping $S \times \Gamma \times S \to S$, written (a, γ, c) by $a\gamma c$, such that S satisfies the identity $(a\gamma b)\mu c = (c\gamma b)\mu a$ for all $a, b, c \in S$ and $\gamma, \mu \in \Gamma$.*

Example 122 *An AG-groupoid is Γ-AG-groupoid. Indeed, let S be an arbitrary AG-groupoid and Γ any nonempty set. Define a mapping $S \times \Gamma \times S \to S$, by $a\gamma b = ab$ for all $a, b \in S$ and $\gamma \in \Gamma$. It is easy to see that S is a Γ-AG-groupoid. Indeed, consider*

$$\begin{aligned} (a\gamma b)\mu c &= (ab)\mu c = (ab)c = (cb)a. \\ \text{Now take } (c\gamma b)\mu a &= (cb)\mu a = (cb)a. \\ \text{Hence } (a\gamma b)\mu c &= (c\gamma b)\mu a \text{ for all } a, b, c \in S \text{ and } \gamma, \mu \in \Gamma. \end{aligned}$$

Thus every AG-groupoid is a Γ-AG-groupoid.

Example 123 *Let $\Gamma = \{1, 2, 3\}$. Define a mapping $\mathbb{Z} \times \Gamma \times \mathbb{Z} \to \mathbb{Z}$ by $a\gamma b = b - \gamma - a$ for all $a, b \in \mathbb{Z}$ and $\gamma \in \Gamma$, where "$-$" is a usual subtraction of integers. Then $\mathbb{Z}$ is a Γ-AG-groupoid. Indeed, $(a\gamma b)\mu c = (b - \gamma - a)\mu c = c - \mu - (b - \gamma - a) = c - \mu - b + \gamma + a$. and $(c\gamma b)\mu a = (b - \gamma - c)\mu a = a - \mu - (b - \gamma - c) = a - \mu - b + \gamma + c = c - \mu - b + \gamma + a$. Hence $(a\gamma b)\mu c = (c\gamma b)\mu a$ for all $a, b, c \in \mathbb{Z}$ and $\gamma, \mu \in \Gamma$.*

Example 124 *In general a Γ-AG-groupoid S is not an AG-groupoid but if we define $a * b = a\gamma b$ for all $a, b \in S$ and γ a fixed element in Γ. Then it can be easily verified that $(S, *)$ is an AG-groupoid and we denote this by S_γ.*

An element $e \in S$ is called a left identity of Γ-AG-groupoid if $e\gamma a = a$ for all $a \in S$ and $\gamma \in \Gamma$. Let G and Γ be non-empty sets. If there exists a mapping $G \times \Gamma \times G \to G$, written (x, γ, y) by $x\gamma y$, G is called a Γ-medial if it satisfies the identity $(x\alpha y)\beta(l\gamma m) = (x\alpha l)\beta(y\gamma m)$ for all $x, y, l, m \in G$ and $\alpha, \beta, \gamma \in \Gamma$, and if G satisfies the identity $(x\alpha y)\beta(l\gamma m) = (m\alpha y)\beta(l\gamma x)$, then it is called Γ-paramedial. A Γ-AG-groupoid S is said to be commutative if for any $a, b \in S$, there exists $\mu \in \Gamma$ such that $a\mu b = b\mu a$.Let G and Γ be non-empty sets.

Theorem 125 *If a Γ-AG-groupoid S has a left identity e, then it is unique.*

Proof. Suppose there exists another left identity f. Then $e\gamma f = f$ and $f\gamma e = e$. Now $f = e\gamma f = (e\gamma e)\gamma f = (f\gamma e)\gamma e = e\gamma e = e$. ■

Theorem 126 *In a Γ-AG-groupoid, a right identity becomes a left identity.*

Proof. Let S be a Γ-AG-groupoid with right identity e, then $a\gamma e = a$ for all $a \in S$ and $\gamma \in \Gamma$. Consider $e\gamma a = (e\gamma e)\gamma a = (a\gamma e)\gamma e = a\gamma e = a$. Hence this implies that e is also a left identity. ■

Theorem 127 *Every Γ-AG-groupoid is a Γ-medial.*

Proof. Let S be a Γ-AG-groupoid and for all $x, y, l, m \in S$ and $\alpha, \beta, \gamma \in \Gamma$, using definition of Γ-AG-groupoid repeatedly we have $(x\alpha y)\beta(l\gamma m) = [(l\gamma m)\alpha y]\beta x = [(y\gamma m)\alpha l]\beta x = (x\alpha l)\beta(y\gamma m)$. ■

Theorem 128 *If a Γ-AG-groupoid S has left identity e, then S is Γ-paramedial.*

Proof. Consider

$$\begin{aligned}(x\alpha y)\beta(l\gamma m) &= (x\alpha l)\beta(y\gamma m) = [e\mu(x\alpha l)]\beta(y\gamma m)\\ &= [(y\gamma m)\mu(x\alpha l)]\beta e = [(y\gamma x)\mu(m\alpha l)]\beta e\\ &= [e\mu(m\alpha l)]\beta(y\gamma x) = (m\alpha l)\beta(y\gamma x) = (m\alpha y)\beta(l\gamma x).\end{aligned}$$

Hence S is a Γ-paramedial. ■

Theorem 129 *In a Γ-AG-groupoid with left identity e, associativity and commutativity imply each other.*

Proof. Suppose S be a Γ-AG-groupoid with left identity e and let S be associative, then for all $a, b \in S$ and $\mu \in \Gamma$. Consider

$$\begin{aligned}a\mu b &= e\gamma(a\mu b) = (e\gamma a)\mu b = (b\gamma a)\mu e = b\gamma(a\mu e) = (e\alpha b)\gamma(a\mu e)\\ &= (e\alpha a)\gamma(b\mu e) = a\gamma(b\mu e) = (a\gamma b)\mu e = (e\gamma b)\mu a = b\mu a.\end{aligned}$$

Thus S is commutative. Conversely let S be commutative and let for any $a, b, c \in S$, there exists $\mu \in \Gamma$. Then consider $a\mu(b\mu c) = (b\mu c)\mu a = (c\mu b)\mu a = (a\mu b)\mu c$. Hence S is associative. ■

Theorem 130 *In a Γ-AG-groupoid S, the following statements are equivalent:*

(1) $(a\alpha b)\beta c = b\alpha(c\beta a)$

(2) $(a\alpha b)\beta c = b\alpha(a\beta c)$

Proof. (1) $\Rightarrow$ (2), $a\mu(b\mu c) = (b\mu c)\mu a = (c\mu b)\mu a = (a\mu b)\mu c$. (2) $\Rightarrow$ (1), $(a\alpha b)\beta c = (c\alpha b)\beta a = b\alpha(c\beta a)$ ■

Theorem 131 *Let S be a Γ-AG-groupoid with left identity e. If for any $a, b, c, d \in S$, there exists $\alpha \in \Gamma$, then $a\alpha b = c\alpha d$ implies $b\alpha a = d\alpha c$.*

Proof. $\mathrm{b}\alpha\mathrm{a} = (\mathrm{e}\alpha\mathrm{b})\alpha\mathrm{a} = (\mathrm{a}\alpha\mathrm{b})\alpha\mathrm{e} = (\mathrm{c}\alpha\mathrm{d})\alpha\mathrm{e} = (\mathrm{e}\alpha\mathrm{d})\alpha\mathrm{c} = \mathrm{d}\alpha\mathrm{c}$. ■

Lemma 132 *If S is a Γ-AG-groupoid with left identity e, then $(a\alpha b)^2 = (b\alpha a)^2$ for all $a, b \in S$ and $\alpha, \beta \in \Gamma$.*

Proof. $(\mathrm{a}\alpha\mathrm{b})^2 = (\mathrm{a}\alpha\mathrm{b})\beta(\mathrm{a}\alpha\mathrm{b}) = (\mathrm{a}\alpha\mathrm{a})\beta(\mathrm{b}\alpha\mathrm{b}) = (\mathrm{b}\alpha\mathrm{a})\beta(\mathrm{b}\alpha\mathrm{a}) = (\mathrm{b}\alpha\mathrm{a})^2$. ■

Definition 133 *A Γ-AG-groupoid S is said to be a left cancellative if for all $a, b, c \in S$ and any $\alpha \in \Gamma$, if $c\alpha a = c\alpha b$ implies $a = b$.*

Theorem 134 (1) *A left cancellative Γ-AG-groupoid is also a right cancellative.*

(2) *A right cancellative Γ-AG-groupoid S with left identity e is a left cancellative.*

Proof. (1) Let S be a left cancellative Γ-AG-groupoid. Now for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$, suppose $a\alpha c = b\alpha c$. Let $t \in S$ be any fixed element. Then $(a\alpha c)\beta t = (b\alpha c)\beta t$ implies $(t\alpha c)\beta a = (t\alpha c)\beta b$. So $a = b$.

(2) Let S be a right cancellative Γ-AG-groupoid S with left identity e and let $a\alpha x = a\alpha y$. Now by theorem 131, $a\alpha x = a\alpha y$ implies $x\alpha a = y\alpha a$. As S is right cancellative, therefore $x = y$ and hence a left cancellative. ■

In the following we have an interesting result which may be useful in further development in Γ-AG-groupoids.

Theorem 135 *Let S be a Γ-AG-groupoid with left identity e, then $a\alpha(b\beta c) = b\alpha(a\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.*

Proof. For all $a, b, c \in S$ and $\alpha, \beta, \gamma \in \Gamma$, consider $a\alpha(b\beta c) = (e\gamma a)\alpha(b\beta c) = (e\gamma b)\alpha(a\beta c) = b\alpha(a\beta c)$. ■

5.1.1 Relationship between Γ-AG-groupoids and Γ-semigroups

We have established some interesting results which show a relationship between Γ-AG-groupoids and Γ-semigroups.

Theorem 136 *A Γ-AG-groupoid is a Γ-semigroup if and only if $a\alpha(b\beta c) = (c\alpha b)\beta a$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.*

Proof. Let S be a Γ-AG-groupoid and let $a\alpha(b\beta c) = (c\alpha b)\beta a$ holds for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. $a\alpha(b\beta c) = (c\alpha b)\beta a = (a\alpha b)\beta c$. This implies that S is a Γ-semigroup. Now conversely let S is a Γ-semigroup, then $(a\alpha b)\beta c = a\alpha(b\beta c)$, Also $(a\alpha b)\beta c = (c\alpha b)\beta a$. Hence $a\alpha(b\beta c) = (c\alpha b)\beta a$. ■

Theorem 137 *If a Γ-AG-groupoid S has a right identity e, then S is a commutative Γ-semigroup with identity.*

Proof. Let S be a Γ-AG-groupoid with right identity e, then by theorem 126, e is also a left identity in S. Let for any $a, b \in S$, there exists $\mu \in \Gamma$. Then $a\mu b = (e\mu a)\mu b = (b\mu a)\mu e = b\mu a$. Hence S is commutative. Further consider $(a\mu b)\mu c = (c\mu b)\mu a = (b\mu c)\mu a = a\mu(b\mu c)$. Thus S is a commutative Γ-semigroup with identity. ■

Theorem 138 *A Γ-AG-groupoid S with left identity e is a commutative Γ-semigroup with identity if and only for any $a, b, c \in S$, there exists $\alpha \in \Gamma$ such that $a\alpha(b\alpha c) = (c\alpha b)\alpha a$.*

Proof. Suppose S is a Γ-AG-groupoid with left identity e such that $a\alpha(b\alpha c) = (c\alpha b)\alpha a$. Consider $a\alpha e = a\alpha(e\alpha e) = (e\alpha e)\alpha a = e\alpha a = a$. Which implies e is the right identity of S and hence by theorem 137, S is a commutative Γ-semigroup with identity. Conversely suppose S is a commutative Γ-semigroup with identity. Then for all $a, b, c \in S$ and $\alpha \in \Gamma$, we have $a\alpha(b\alpha c) = (a\alpha b)\alpha c = c\alpha(a\alpha b) = c\alpha(b\alpha a) = (c\alpha b)\alpha a$. ■

The following corollary is a consequence of theorems 129 and 136.

Corollary 139 *For a Γ-AG-groupoid S, the following are equivalent:*

(1) *Associativity*

(2) *Commutativity*

(3) $a\alpha(b\beta c) = (c\alpha b)\beta a$ *for all* $a, b, c \in S$ *and* $\alpha, \beta \in \Gamma$.

5.1.2 Γ-Ideals in Γ-AG-groupoids

In this study, we have discussed Γ-ideals and Γ-bi-ideals of Γ-AG-groupoids which are in fact a generalization of ideals and bi-ideals of AG-groupoids (for a suitable choice of Γ). we have studied some characteristics of Γ-ideals and Γ-bi-ideals of Γ-AG-groupoids. Specifically, we have proved that a Γ-AG-groupoid S with left identity e is fully Γ-prime if and only if every Γ-ideal in S is Γ-idempotent and the set of Γ-ideals of S is totally ordered under inclusion. We also proved the equivalent conditions for Γ-bi-ideals of S that is (1) every Γ-bi-ideal of S is Γ-idempotent, (2) $H \cap K = H\Gamma K$, where H and K are any Γ-bi-ideals of S and (3) the Γ-ideals of S form a semilattice $(L_S, \wedge)$, where $H \wedge K = H\Gamma K$. Also we have shown that every Γ-bi-ideal of a Γ-AG-groupoid S with left identity e is a Γ-prime if and only if it is Γ-idempotent and the set of Γ-bi-ideals of S is totally ordered under inclusion. In the end we have established some results regarding m-systems in Γ-AG-groupoids

We initiated with the following lemma:

Lemma 140 *If S is a Γ-AG-groupoid with left identity e then $S\Gamma S = S$ and $S = e\Gamma S = S\Gamma e$.*
Proof. *Let $x \in S$, then for any $\gamma \in \Gamma$, we have $x = e\gamma x \in S\Gamma S$ and so $S \subseteq S\Gamma S$. Hence $S = S\Gamma S$. Now as e is left identity in S, so for any $\gamma \in \Gamma$, it is obvious that $e\Gamma S = S$. Now*

consider $S\Gamma e = (S\Gamma S)\Gamma e = (e\Gamma S)\Gamma S = S\Gamma S = S$. *Hence* $S = e\Gamma S = S\Gamma e$. ■

Definition 141 *Let* S *be a* Γ*-AG-groupoid. A nonempty subset* M *of* S *is called a sub*Γ*-AG-groupoid of* S *if* $a\gamma b \in M$ *for all* $a, b \in M$ *and* $\gamma \in \Gamma$. *A sub*Γ*-AG-groupoid* I *of* S *is called a left(right)* Γ*-ideal of* S *if* $S\Gamma I \subseteq I$ $(I\Gamma S \subseteq I)$ *and is called an* Γ*-ideal if it is left as well as right* Γ*-ideal.*

Proposition 142 *If a* Γ*-AG-groupoid* S *has a left identity* e, *then every right* Γ*-ideal is a left* Γ*-ideal.*

Proof. *Let* I *be a right* Γ*-ideal of* S. *Then for* $i \in I$, $s \in S$ *and* $\alpha \in \Gamma$, *consider* $s\alpha i = (e\gamma s)\alpha i = (i\gamma s)\alpha e \in I$. *Hence* I *is a left* Γ*-ideal.* ■

Lemma 143 *If* I *is a left* Γ*-ideal of a* Γ*-AG-groupoid* S *with left identity* e, *and if for any* $a \in S$, *there exists* $\gamma \in \Gamma$, *then* $a\gamma I$ *is a left* Γ*-ideal of* S.

Proof. Let I is a left Γ-ideal of S,consider $s\gamma(a\gamma i) = (e\gamma s)\gamma(a\gamma i) = (e\gamma a)\gamma(s\gamma i) = a\gamma(s\gamma i) \in a\gamma I$. Hence $a\gamma I$ is a left Γ-ideal of S. ■

Lemma 144 *If* I *is a right* Γ*-ideal of a* Γ*-AG-groupoid* S *with left identity* e, *then* $I\Gamma I$ *or a* Γ*-ideal of* S.

Proof. Let $x \in I\Gamma I$, then $x = i\gamma j$ where $i, j \in I$ and $\gamma \in \Gamma$. Now consider $x\alpha s = (i\gamma j)\alpha s = (s\gamma j)\alpha i \in I\Gamma I$. This implies that $I\Gamma I$ is a right Γ-ideal and hence by proposition 142, $I\Gamma I$ is a Γ-ideal of S. ■

Corollary 145 *If* I *is a left* Γ*-ideal of* S *then* $I\Gamma I$ *becomes a* Γ*-ideal of* S.

Definition 146 *A* Γ*-ideal* I *of* S *is called minimal* Γ*-ideal, if it does not properly contain any* Γ*-ideal of* S.

Lemma 147 *A proper* Γ*-ideal* M *of a* Γ*-AG-groupoid* S *with left identity* e, *is minimal if and only if* $M = a^2\Gamma M$, *for all* $a \in S$.

Proof. Assume that M is a minimal Γ-ideal of S. Now as $M\Gamma M$ is a Γ-ideal of S so $M = M\Gamma M$. It is easy to see that $a^2\Gamma M$ is a Γ-ideal and is contained in M. But as M is minimal so $M = a^2\Gamma M$. Conversely let $M = a^2\Gamma M$, for all $a \in S$. On contrary let K be a minimal Γ-ideal of S which is properly contained in M containing a, the $M = a^2\Gamma M \subseteq K$, which is a contradiction. ■

A Γ-ideal P of Γ-AG-groupoid S is said to be Γ-prime if $A\Gamma B \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$, for all Γ-ideals A and B in S. A Γ-ideal P is called Γ-semiprime if $I\Gamma I \subseteq P$ implies that $I \subseteq P$, for any Γ-ideals I of S. If every Γ-ideal of Γ-AG-groupoid S is Γ-semiprime, then S is said to be fully Γ-semiprime and if every Γ-ideal is Γ-prime, then S is called fully Γ-prime. A Γ-ideal I of a Γ-AG-groupoid S is called a Γ-idempotent if $I\Gamma I = I$ and if every Γ-ideal of S is Γ-idempotent then S is called fully Γ-idempotent. The set of Γ-ideals of Γ-AG-groupoid S is said to be totally ordered under inclusion if for all Γ-ideals H, K, either $H \subseteq K$ or $K \subseteq H$ and we denote it by Γ-ideal(S).

Theorem 148 *A Γ-AG-groupoid S with left identity e is fully Γ-prime if and only if every Γ-ideal in S is Γ-idempotent and Γ-ideal(S) is totally ordered under inclusion.*

Proof. Let S is fully Γ-prime. Let I be a Γ-ideal in S. Then by lemma 144, $I\Gamma I$ will also be a Γ-ideal in S and hence $I\Gamma I \subseteq I$. Also $I\Gamma I \subseteq I\Gamma I$. But as S is fully Γ-prime, so it implies that $I \subseteq I\Gamma I$. Thus $I\Gamma I = I$ and hence I is Γ-idempotent. Now let H, K be Γ-ideals of S and $H\Gamma K \subseteq H$, $H\Gamma K \subseteq K$ which imply that $H\Gamma K \subseteq H \cap K$. Now as $H \cap K$ is prime, so $H \subseteq H \cap K$ or $K \subseteq H \cap K$ which further imply that $H \subseteq K$ or $K \subseteq H$. Hence Γ-ideal(S) is totally ordered under inclusion. Conversely, let every Γ-ideal is Γ-idempotent and Γ-ideal(S) is totally ordered under inclusion. Let I, J and P be Γ-ideals in S with $I\Gamma J \subseteq P$ such that $I \subseteq J$. As I is Γ-idempotent, so $I = I\Gamma I \subseteq I\Gamma J \subseteq P$ which imply that S is fully Γ-prime. ■

Definition 149 *If S is a Γ-AG-groupoid S with left identity e, then the principal left Γ-ideal generated by x is defined as $\langle x \rangle = S\Gamma x = \{s\gamma x : s \in S\}$, for all $x \in S$ and $\gamma \in \Gamma$.*

Definition 150 *Let P be a left Γ-ideal of a Γ-AG-groupoid S, then P is said to be a qausi Γ-prime if for left Γ-ideals A and B of S such that $A\Gamma B \subseteq P$, we have $A \subseteq P$ or $B \subseteq P$*

and P is called qausi Γ-semiprime if for any left Γ-ideal of S such that $I\Gamma I \subseteq P$ implies that $I \subseteq P$.

Theorem 151 *If S is a Γ-AG-groupoid S with left identity e, then a left Γ-ideal P of S is qausi Γ-prime if and only if $a\alpha(S\beta b) \subseteq P$ implies $a \in P$ or $b \in P$, for all $a, b \in S$ and any $\alpha, \beta \in \Gamma$.*

Proof. Let P be a qausi Γ-prime in Γ-AG-groupoid S with left identity e. Assume that $a\alpha(S\beta b) \subseteq P$, then $S\gamma(a\alpha(S\beta b)) \subseteq S\Gamma P \subseteq P$. So by lemma 140, Γ-medial and Γ-paramedial, we get

$$\begin{aligned} S\gamma(a\alpha(S\beta b)) &= (S\delta S)\gamma(a\alpha(S\beta b)) = (S\delta a)\gamma(S\alpha(S\beta b)) = (S\delta a)\gamma((S\delta S)\alpha(S\beta b)) \\ &= (S\delta a)\gamma((b\delta S)\alpha(S\beta S)) = (S\delta a)\gamma((b\delta S)\alpha S) = (S\delta a)\gamma((S\delta S)\alpha b) \\ &= (S\delta a)\gamma(S\alpha b). \end{aligned}$$

This implies that $\langle a\rangle\gamma\langle b\rangle \subseteq S\Gamma P \subseteq P$. But P is qausi Γ-prime, hence either $a \in P$ or $b \in P$. Conversely, assume that $A\Gamma B \subseteq P$, where A and B are left Γ-ideals of S such that $A \nsubseteq P$. Then there exists $x \in A$ such that $x \notin P$. Now $x\alpha(S\beta y) \subseteq A\Gamma(S\Gamma B) \subseteq A\Gamma B \subseteq P$, for all $y \in B$ and $\alpha, \beta \in \Gamma$. So by hypothesis, $y \in P$ for all $y \in B$ implies that $B \subseteq P$. Hence P is qausi Γ-prime. ■

Corollary 152 *If S is a Γ-AG-groupoid with left identity e, then a left Γ-ideal P of S is qausi Γ-semiprime if and only if $a\alpha(S\beta a) \subseteq P$ implies $a \in P$, for all $a \in S$ and any $\alpha, \beta \in \Gamma$.*

Lemma 153 *If I is a proper right(left) Γ-ideal of a Γ-AG-groupoid S with left identity e, then $e \notin I$.*

Proof. On contrary let $e \in I$. Then for any $\gamma \in \Gamma$, we have $S = e\gamma S \in I\Gamma S \subseteq I$ and consequently $I = S$. A contradiction arises because I is proper Γ-ideal of S. Hence $e \notin I$. ■

Definition 154 *Let S be a Γ-AG-groupoid. A sub Γ-AG-groupoid B of S is said to be Γ-bi-ideal of S if $(B\Gamma S)\Gamma B \subseteq B$.*

Example 155 *Let $S = \{1, 2, 3, 4, 5\}$. Define a binary operation "$\cdot$" in S as follows:*

$\cdot$	1	2	3	4	5
1	x	x	x	x	x
2	x	x	x	x	x
3	x	x	x	x	x
4	x	x	x	x	x
5	x	x	3	x	x

Then $(S, \cdot)$ becomes an AG-groupoid, where $x \in \{1, 2, 4\}$. Now let $\Gamma = \{1\}$ and define a mapping $S \times \Gamma \times S \to S$, by

$a\Gamma b = ab$ for all $a, b \in S$. Then it is easy to see that S is a Γ-AG-groupoid. If we take $B = \{3, x\}$, then B becomes

Γ-bi-ideal of S.

Remark 156 *Example 155 shows that Γ-bi-ideals in Γ-AG-groupoids are in fact a generalization of bi-ideals in AG-groupoids (for a suitable choice of Γ).*

Proposition 157 *Let A be a left Γ-ideal and B be a bi-Γ-ideal of a Γ-AG-groupoid S with left identity e, then $B\Gamma A$ and $(A\Gamma A)\Gamma B$ are Γ-bi-ideals of S.*

Proof. To show that $B\Gamma A$ is a Γ-bi-ideal of S, let consider $((B\Gamma A)\Gamma S)\Gamma(B\Gamma A) = ((S\Gamma A)\Gamma B)\Gamma(B\Gamma A) = ((B\Gamma A)\Gamma B)\Gamma(S\Gamma A) \subseteq ((B\Gamma S)\Gamma B)\Gamma A \subseteq B\Gamma A$. Also by Γ-medial law, it can be verified that $(B\Gamma A)\Gamma(B\Gamma A) = (B\Gamma B)\Gamma(A\Gamma A) \subseteq B\Gamma A$. Hence $B\Gamma A$ is a Γ-bi-ideal of S. Now by corollary 145, Γ-medial law and the fact that $S\Gamma S = S$, we have

$$\begin{aligned}(((A\Gamma A)\Gamma B)\Gamma S)\Gamma((A\Gamma A)\Gamma B) &= (((A\Gamma A)\Gamma S)\Gamma(B\Gamma S))\Gamma((A\Gamma A)\Gamma B) \\ &\subseteq ((A\Gamma A)\Gamma(B\Gamma S))\Gamma((A\Gamma A)\Gamma B) \\ &= ((A\Gamma A)\Gamma(A\Gamma A))\Gamma((B\Gamma S)\Gamma B) \subseteq (A\Gamma A)\Gamma B.\end{aligned}$$

Hence $(A\Gamma A)\Gamma B$ is a Γ-bi-ideal of S. ■

Proposition 158 *The product of two Γ-bi-ideals of a Γ-AG-groupoid S with left identity e is again a Γ-bi-ideal of S.*

Proof. Let H and K be two Γ-bi-ideals of S. Then using Γ-medial law and $S\Gamma S = S$, we get

$((H\Gamma K)\Gamma S)\Gamma(H\Gamma K) = ((H\Gamma K)\Gamma(S\Gamma S))\Gamma(H\Gamma K) = ((H\Gamma S)\Gamma(K\Gamma S))\Gamma(H\Gamma K) = ((H\Gamma S)\Gamma H)\Gamma((K\Gamma S)\Gamma K)$ $\subseteq H\Gamma K$. Hence $H\Gamma K$ is a Γ-bi-ideal of S. ■

Theorem 159 *Let S be a Γ-AG-groupoid and H_i a Γ-bi-ideal of S for all $i \in I$. If $\cap_{i\in I} H_i \neq \varnothing$, then $\cap_{i\in I} H_i$ is a Γ-bi-ideal of S.*

Proof. Let S be a Γ-AG-groupoid and H_i a Γ-bi-ideal of S for all $i \in I$. Assume that $\cap_{i\in I} H_i \neq \varnothing$. Let $x, y \in \cap_{i\in I} H_i$, $s \in S$ and $\alpha, \beta \in \Gamma$. Now $x, y \in H_i$ for all $i \in I$ and since for each $i \in I$, H_i is a Γ-bi-ideal of S, so $x\alpha y \in H_i$ and $(x\alpha s)\beta y \in (H_i\Gamma S)\Gamma H_i \subseteq H_i$ for all $i \in I$. Therefore $x\alpha y \in \cap_{i\in I} H_i$ and $(x\alpha s)\beta y \in \cap_{i\in I} H_i$. Hence $\cap_{i\in I} H_i$ is a Γ-bi-ideal of S for all $i \in I$. ■

Theorem 160 *If B is Γ-idempotent Γ-bi-ideal of a Γ-AG-groupoid S with left identity e, then B is a Γ-ideal of S.*

Proof. Consider $B\Gamma S = (B\Gamma B)\Gamma S = (S\Gamma B)\Gamma B = (S\Gamma(B\Gamma B))\Gamma B = ((B\Gamma B)\Gamma S)\Gamma B = (B\Gamma S)\Gamma B \subseteq B$. Which implies that B is a right Γ-ideal and so is left Γ-ideal of S. Hence B is a Γ-ideal of S. ■

Lemma 161 *If B is a proper Γ-bi-ideal of a Γ-AG-groupoid S with left identity e, then $e \notin B$.*

Proof. On contrary let $e \in B$. Now consider $s\alpha b = (e\gamma s)\alpha b \in B$. Also for any $s \in S$ and any $\gamma \in \Gamma$, we have $s = (e\gamma e)\gamma s = (s\gamma e)\gamma e \in (S\Gamma B)\Gamma B \subseteq B$ which implies that $S \subseteq B$. A contradiction to the hypothesis. Hence $e \notin B$. ■

Proposition 162 *If H and K are Γ-bi-ideals of a Γ-AG-groupoid S with left identity e, then the following assertions are equivalent:*

(1) *every Γ-bi-ideals of S is Γ-idempotent,*

(2) $H \cap K = H\Gamma K$,

(3) *the* Γ*-ideals of* S *form a semilattice* $(L_S, \wedge)$*, where* $H \wedge K = H\Gamma K$.

Proof. $(1) \Rightarrow (2)$, By lemma 160, it is obvious that $H\Gamma K \subseteq H \cap K$. For reverse inclusion, as $H \cap K \subseteq H$ and also $H \cap K \subseteq K$, so $(H \cap K)\Gamma(H \cap K) \subseteq H\Gamma K$ which implies that $H \cap K \subseteq H\Gamma K$. Hence $H \cap K = H\Gamma K$.

$(2) \Rightarrow (3)$, $H \wedge K = H\Gamma K = H \cap K = K \cap H = K \wedge H$. Also $H \wedge H = H\Gamma H = H \cap H = H$. Similarly associativity follows. Hence $(L_S, \wedge)$ is a semilattice. $(3) \Rightarrow (1)$, $H = H \wedge H = H\Gamma H$. ■

Definition 163 *A* Γ*-bi-ideal* P *of a* Γ*-AG-groupoid* S *is said to be prime* Γ*-bi-ideal if for all* Γ*-bi-ideals* A *and* B *of* S, $A\Gamma B \subseteq P$ *implies either* $A \subseteq P$ *or* $B \subseteq P$.

Definition 164 *The set of* Γ*-bi-ideals of* S *is totally ordered under inclusion if for all* Γ*-bi-ideals* I, J *either* $I \subseteq J$ *or* $J \subseteq I$.

The following theorem gives necessary and sufficient conditions for a Γ-bi-ideal to be a Γ-prime ideal.

Theorem 165 *Every* Γ*-bi-ideal of a* Γ*-AG-groupoid* S *with left identity* e *is a* Γ*-prime if and only if it is* Γ*-idempotent and the set of* Γ*-bi-ideals of* S *is totally ordered under inclusion.*

Proof. Let P be a Γ-bi-ideal of Γ-AG-groupoid S and assume that each Γ-bi-ideal of S is Γ-prime. Since $P\Gamma P$ is a Γ-ideal, so it is Γ-prime which implies that $P \subseteq P\Gamma P$, hence P is Γ-idempotent. Now let A and B be any Γ-bi-ideals of S. As $A \cap B$ is also a Γ-bi-ideal, so by hypothesis $A \cap B$ is Γ-prime. Now by lemma 160, either $A \subseteq A \cap B$ or $B \subseteq A \cap B$ which further implies that either $A \subseteq B$ or $B \subseteq A$. Hence the set of bi-Γ-ideals of S is totally ordered under inclusion. Conversely, let every Γ-bi-ideal of S is Γ-idempotent and the set of Γ-bi-ideals of S is totally ordered under inclusion. Let A, B and P are Γ-bi-ideals of S with $A\Gamma B \subseteq P$ and also assume that $A \subseteq B$. Now as A is Γ-idempotent, so $A = A\Gamma A \subseteq A\Gamma B \subseteq P$. Hence every Γ-bi-ideal of a Γ-AG-groupoid S with left identity e is a Γ-prime. ■

A nonempty subset A of a Γ-AG-groupoid S is called an M-system if for $a, b \in A$ there exist $r \in S$ and $\alpha, \beta \in \Gamma$ such that $a\alpha(r\beta b) \in A$ and A is called P-system if for all $a \in A$, there exist $r \in S$ and $\alpha, \beta \in \Gamma$ such that $a\alpha(r\beta a) \in A$. Let I be a left Γ-ideal of a Γ-AG-groupoid S. Then I is said to be a quasi-Γ-prime if $H\Gamma K \subseteq I$ implies that either $H \subseteq I$ or $K \subseteq I$, where H and K are any left Γ-ideals of S. If for any left Γ-ideal H of S such that $H\Gamma H \subseteq I$, we have $H \subseteq I$, then I is called quasi-Γ-semiprime.

Proposition 166 *Let I be a left Γ-ideal of S with left identity e, then the following are equivalent:*

(1) *I is quasi-Γ-prime ideal.*

(2) *$H\Gamma K = \langle H\Gamma K\rangle \subseteq I$ implies that either $H \subseteq I$ or $K \subseteq I$, where H and K are any left Γ-ideals of S.*

(3) *If $H \nsubseteq I$ and $K \nsubseteq I$ then $H\Gamma K \nsubseteq I$, where H and K are any left Γ-ideals of S.*

(4) *If h, k are elements of S such that $h \notin I$ and $k \notin I$, then $\langle h\rangle\Gamma\langle k\rangle \nsubseteq I$.*

(5) *If $h, k \in S$ and $\alpha, \beta \in \Gamma$ satisfying $h\alpha(S\beta k) \subseteq I$, then either $h \in I$ or $k \in I$.*

Proof. $(1) \Leftrightarrow (2)$. Let I be a quasi-Γ-prime. Now by definition if $H\Gamma K = \langle H\Gamma K\rangle \subseteq I$, then obviously it implies that either $H \subseteq I$ or $K \subseteq I$ for all left Γ-ideals H and K of S. Converse is trivial. $(2) \Leftrightarrow (3)$ is trivial. $(1) \Rightarrow (4)$. Let $\langle h\rangle\Gamma\langle k\rangle \subseteq I$. Then either $\langle h\rangle \subseteq I$ or $\langle k\rangle \subseteq I$, which implies that either $h \in I$ or $k \in I$. $(4) \Rightarrow (2)$. Let $H\Gamma K \subseteq I$. If $h \in H$ and $k \in K$, then $\langle h\rangle\Gamma\langle k\rangle \subseteq I$ and hence by hypothesis either $h \in I$ or $k \in I$. This implies that either $H \subseteq I$ or $K \subseteq I$. $(1) \Leftrightarrow (5)$. Let $h\alpha(S\beta k) \subseteq I$. Then $S\Gamma(h\alpha(S\beta k)) \subseteq S\Gamma I \subseteq I$. Now consider

$$\begin{aligned}
S\Gamma(h\alpha(S\beta k)) &= (S\Gamma S)\Gamma(h\alpha(S\beta k)) \\
&= (S\Gamma h)\Gamma(S\Gamma(S\beta k)), \text{ by } \Gamma\text{-medial law} \\
&= (S\Gamma h)\Gamma((S\Gamma S)\Gamma(S\beta k)) \\
&= (S\Gamma h)\Gamma((k\Gamma S)\Gamma(S\Gamma S)), \text{ by } \Gamma\text{-paramedial law} \\
&= (S\Gamma h)((S\Gamma S)\Gamma k), \text{ by definition of } \Gamma\text{-AG-groupoid} \\
&= (S\Gamma h)(S\Gamma k) \subseteq I.
\end{aligned}$$

Since $S\Gamma h$ and $S\Gamma k$ are left Γ-ideals for all $h \in H$ and $k \in K$, hence either $h \in I$ or $k \in I$. Conversely, let $H\Gamma K \subseteq I$ where H and K are any left Γ-ideals of S. Let $H \nsubseteq I$ then there exists $l \in H$ such that $l \notin I$. For all $m \in K$, we have $l\Gamma(S\Gamma m) \subseteq H\Gamma(S\Gamma K) \subseteq H\Gamma K \subseteq I$. This implies that $K \subseteq I$ and hence I is quasi-Γ-prime ideal of S. ■

Proposition 167 *A left Γ-ideal I of a Γ-AG-groupoid S with left identity e is quasi-Γ-prime if and only if $R\backslash I$ is an M-system.*

Proof. Suppose I is a quasi-Γ-prime ideal. Let $a, b \in S\backslash I$ which implies that $a \notin I$ and $b \notin I$. So by proposition 166 $a\Gamma(S\Gamma b) \nsubseteq I$. This implies that there exist some $r \in S$ and $\alpha, \beta \in \Gamma$ such that $a\alpha(r\beta b) \notin I$ which further implies that $a\alpha(r\beta b) \in S\backslash I$. Hence $S\backslash I$ is an M-system. Conversely, let $S\backslash I$ is an M-system. Suppose $a\Gamma(S\Gamma b) \subseteq I$ and let $a \notin I$ and $b \notin I$. This implies that $a, b \in R\backslash I$. Since $R\backslash I$ is an M-system so there exist $r \in R$ and $\alpha, \beta \in \Gamma$ such that $a\alpha(r\beta b) \in R\backslash I$ which implies that $a\Gamma(S\Gamma b) \nsubseteq I$. A contradiction. Hence either $a \in I$ or $b \in I$. This shows that I is a quasi-Γ-prime ideal. ■

Lemma 168 *An M-system of elements of Γ-AG-groupoid S is a P-system.*

Proof. Let A be a nonempty subset of S such that A is an M-system. Then for all $a, b \in A$, there exist an element $r \in S$ and $\alpha, \beta \in \Gamma$ such that $a\alpha(r\beta b) \in S$. If we take $b = a$, then $a\alpha(r\beta a) \in S$ which implies that S is a P-system. ■

5.2 Some characterizations of regular and intra-regular Γ-AG-groupoids

In this section we have investigated the characterizations of regular Γ-AG-groupoids by the properties of Γ-ideals.

Definition 169 *A Γ-AG-groupoid S is said to be a regular Γ-AG-groupoid if for each a in S there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$.*

Lemma 170 *Every right Γ-ideal of a regular Γ-AG-groupoid is a Γ-ideal.*

Proof. Let S is a regular Γ-AG-groupoid and let I be its right Γ-ideal. Now for each $s \in S$ there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $s = (s\alpha x)\beta s$. If $a \in S$ and $\gamma \in \Gamma$, then consider $s\gamma a = ((s\alpha x)\beta s)\gamma a = (a\beta s)\gamma(s\alpha x) \in I$. Which implies that I is a left Γ-ideal. Hence I is a Γ-ideal of S. ■

Lemma 171 *Every regular Γ-AG-groupoid is fully Γ-idempotent.*

Proof. Let S be a regular Γ-AG-groupoid and I be a Γ-ideal of S. It is always true that $I\Gamma I \subseteq I$. Now if $a \in I$, then as S is regular Γ-AG-groupoid, so there exists $b \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha b)\beta a \in I\Gamma I$. Thus $I \subseteq I\Gamma I$, and hence S is fully Γ-idempotent. ■

Lemma 172 *If S is a regular Γ-AG-groupoid then $H\Gamma K = H \cap K$, where H is right Γ-ideal and K is left Γ-ideal.*

Proof. Let H and K be right and left Γ-ideals of S with $H\Gamma K \subseteq H \cap K$. Now let $x \in H \cap K$, then there exist $y \in S$ and $\alpha, \beta \in \Gamma$ such that $x = (x\alpha y)\beta x \in H\Gamma K$. Hence $H\Gamma K = H \cap K$.
■

Theorem 173 *A regular Γ-AG-groupoid S is fully Γ-prime if and only if Γ-ideal(S) is totally ordered under inclusion.*

Proof. Proof follows from theorem 148 and lemma 172. ■

Definition 174 *A Γ-ideal I of a regular Γ-AG-groupoid S is said to be strongly irreducible if for Γ-ideals P and Q of S, $P \cap Q \subseteq I$ implies that either $P \subseteq I$ or $K \subseteq I$.*

Theorem 175 *Every Γ-ideal in a regular Γ-AG-groupoid S is Γ-prime if and only if it is strongly irreducible.*

Proof. Assume that P is a prime Γ-ideal of S. Then there exist Γ-ideals A and B in S such that $A\Gamma B \subseteq P$. Now by lemma 172 $A\Gamma B = A \cap B$ implies that either $A \subseteq P$ or $B \subseteq P$. Hence P is strongly irreducible. Now conversely let every Γ-ideal of a regular Γ-AG-groupoid S is strongly irreducible. Then for any Γ-ideals A and B of S, $A \cap B \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$. But by lemma 172, $A\Gamma B = A \cap B$. Hence P is a prime Γ-ideal of S. ■

Definition 176 *A Γ-AG-groupoid S is said to be an anti-rectangular Γ-AG-groupoid if $x = (y\alpha x)\beta y$, for all $x, y \in S$ and $\alpha, \beta \in \Gamma$.*

Proposition 177 *If A and B are Γ-ideals of an anti-rectangular Γ-AG-groupoid S, then there product is also a Γ-ideal.*

Proof. Using Γ-medial law and $S\Gamma S = S$, we have $(A\Gamma B)\Gamma S = (A\Gamma B)\Gamma(S\Gamma S) = (A\Gamma S)\Gamma(B\Gamma S) \subseteq AB$, and $S\Gamma(A\Gamma B) = (S\Gamma S)\Gamma(A\Gamma B) = (S\Gamma A)\Gamma(S\Gamma B) \subseteq AB$. Hence $A\Gamma B$ is a Γ-ideal. ∎

As a consequence, if I_1, I_2, $I_3, ...,$ I_n are Γ-ideals of S, then $(...((I_1\Gamma I_2)\Gamma I_3)\Gamma...\Gamma I_n)$ is also a a Γ-ideal of S.

Theorem 178 *Any subset of an anti-rectangular Γ-AG-groupoid S is left Γ-ideal if and only if it is right Γ-ideal.*

Proof. Let I be right Γ-ideal of S. Now for all $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$, consider $x\gamma i = ((y\alpha x)\beta y)\gamma i = (i\beta y)\gamma(y\alpha x) \in I$. Conversely, assume that I is a left Γ-ideal, then $i\gamma x = ((y\alpha i)\beta y)\gamma x = (x\beta y)\gamma(y\alpha i) \in I$. ∎

Lemma 179 *If I is a Γ-ideal of an anti-rectangular Γ-AG-groupoid S, then*

$$H\Gamma(a) = \{x \in S : (x\alpha a)\beta x = x, \text{ for } a \in I \text{ and } \alpha, \beta \in \Gamma\} \subseteq I.$$

Proof. Let $y \in H\Gamma(a)$, then for any $a \in I$ and $\alpha, \beta \in \Gamma$, we have $y = (y\alpha a)\beta y \in (S\Gamma I)\Gamma S \subseteq I$. ∎

Proposition 180 *If H and K are Γ-ideals of an anti-rectangular Γ-AG-groupoid S, then the following assertions are equivalent:*

(1) *S is fully Γ-idempotent,*

(2) *$H \cap K = H\Gamma K$,*

(3) *The Γ-ideals of S form a semilattice $(L_S, \wedge)$, where $H \wedge K = H\Gamma K$.*

Proof. The proof follows from proposition 162. ■

Theorem 181 *Every Γ-ideal of an anti-rectangular Γ-AG-groupoid S is Γ-prime if and only if it is Γ-idempotent and Γ-ideals(S) is totally ordered under inclusion.*

Proof. The proof follows from theorem 148. ■

5.2.1 Quasi-Γ-ideals in Γ-AG-groupoids

Now we study quasi-Γ-ideals of Γ-AG-groupoids analogous to quasi-Γ-ideals in Γ-semigroups [7].

Definition 182 *Let S be a Γ-AG-groupoid. A nonempty subset Q of S is called a quasi-Γ-ideal of S if $S\Gamma Q \cap Q\Gamma S \subseteq Q$.*

Remark 183 (1) *Each quasi-Γ-ideal Q of Γ-AG-groupoid S is a subΓ-AG-groupoid of S. In fact, $Q\Gamma Q \subseteq S\Gamma Q \cap Q\Gamma \subseteq Q$.*

(2) *Every right Γ-ideal and every left Γ-ideal of a Γ-AG-groupoid S is a quasi-Γ-ideal of S.*

The proof of the next theorem similar to the case of quasi-Γ-ideals of Γ-semigroups (see [7]).

Theorem 184 *Let S be a Γ-AG-groupoid and Q_i a quasi-Γ-ideal of S for each $i \in I$. If $\bigcap_{i\in I} Q_i$ is nonempty set, then $\bigcap_{i\in I} Q_i$ is a quasi-Γ-ideal of S.*

Remark 185 *In theorem 184, it is necessary that $\bigcap_{i\in I} Q_i$ is a nonempty set. For example, let $\Gamma = \{1,2\}$ and N be a set of positive integers. Define a mapping $N \times \Gamma \times N \to N$, by $a\alpha b = a + b + \alpha$ for all $a, b \in N$ where $+$ is the usual addition on N. Obviously N is a Γ-AG-groupoid. Take $Q_n = \{n+1, n+2, n+3, ...\}$ for $n \in N$. Then Q_n is a quasi-Γ-ideal of N for all $n \in N$ but $\bigcap_{n\in\mathbb{N}} Q_n$ is an empty set.*

Theorem 186 *The intersection of a left Γ-ideal and a right Γ-ideal of a Γ-AG-groupoid S is a quasi-Γ-ideal of S.*

Proof. Let L and R be left and right Γ-ideal of S. Consider $a \in R, b \in L$ and $\alpha \in \Gamma$, we have $a\alpha b \in L \cap R$. So $L \cap R \neq \emptyset$. Consider $S\Gamma(L \cap R) \cap (L \cap R)\Gamma S \subseteq S\Gamma L \cap R\Gamma S \subseteq L \cap R$. Hence $L \cap R$ is a quasi-Γ-ideal of S. ■

In the following we give an example of Γ-AG-groupoid with left identity e in which the condition $(x\alpha e)\beta S = x\beta S$ holds for all $x \in S$ and $\alpha, \beta \in \Gamma$.

Example 187 *Let $S = \{a, b, c, d, e\}$. Define a binary operation $\cdot$ in S as follows:*

$\cdot$	a	b	c	d	e
a	a	b	c	d	e
b	b	b	b	d	e
c	c	b	a	d	e
d	e	e	e	b	d
e	d	d	d	e	b

Then $(S, \cdot)$ becomes an AG-groupoid. Now let $\Gamma = \{1\}$ and define a mapping $S\times\Gamma\times S \to S$, by $x1y = xy$ for all $x, y \in S$. Then S is a Γ-AG-groupoid. Also we can see that a is a left identity in S and $(x\alpha a)\beta S = x\beta S$ holds for all $x \in S$ and $\alpha, \beta \in \Gamma$. However, a is not an identity of S.

Remark 188 *Let S be a Γ-AG-semigroup with left identity e. We have $S\Gamma e = S$, in deed, $S\Gamma e = (S\Gamma S)\Gamma e = (e\Gamma S)\Gamma S = S\Gamma S = S$. However, e need not be an identity.*

Theorem 189 *Let S be a Γ-AG-groupoid with left identity e such that $(x\alpha e)\beta S = x\beta S$ for all $x \in S$ and $\alpha, \beta \in \Gamma$. Every quasi-Γ-ideal of S is the intersection of a left Γ-ideal and a right Γ-ideal of S.*

Proof. Let Q be a quasi-Γ-ideal of S. Set $L = Q \cup S\Gamma Q$ and $R = Q \cup Q\Gamma S$. Then $S\Gamma L = S\Gamma(Q \cup S\Gamma Q) = S\Gamma Q \cup S\Gamma(S\Gamma Q) = S\Gamma Q \cup (S\Gamma e)\Gamma(S\Gamma Q) = S\Gamma Q \cup (S\Gamma S)\Gamma(e\Gamma Q) = S\Gamma Q \subseteq L$. So, $S\Gamma L \subseteq L$. This implies that L is a left Γ-ideal of S. Similarly, $R\Gamma S = (Q \cup Q\Gamma S)\Gamma S = Q\Gamma S \cup (Q\Gamma S)\Gamma S = Q\Gamma S \cup (Q\Gamma S)\Gamma(e\Gamma S) = Q\Gamma S \cup (Q\Gamma e)\Gamma(S\Gamma S) = Q\Gamma S \cup (Q\Gamma e)\Gamma S =$

$Q\Gamma S \subseteq R$. So R is a right Γ-ideal of S. We have $L \cap R = (Q \cup S\Gamma Q) \cap (Q \cup Q\Gamma S) = Q \cup (S\Gamma Q \cap Q\Gamma S) = Q$. Hence $Q = L \cap R$. ■

Similar to [7], bi-Γ-ideals of Γ-AG-groupoids are defined as follows:

Definition 190 *A subΓ-AG-groupoid B of Γ-AG-groupoid S is said to be a bi-Γ-ideal of S if $(B\Gamma S)\Gamma B \subseteq B$.*

Theorem 191 *Let S be a Γ-AG-groupoid with left identity e such that $(x\alpha e)\beta S = x\beta S$ for all $x \in S$ and $\alpha, \beta \in \Gamma$. Then every quasi-Γ-ideal Q of S is a bi-Γ-ideal of S.*

Proof. Since $Q\Gamma Q \subseteq Q\Gamma S$ and $Q\Gamma Q \subseteq S\Gamma Q$, $Q\Gamma Q \subseteq Q\Gamma S \cap S\Gamma Q \subseteq Q$. This implies that Q is a subΓ-AG-groupoid of S. Now $(Q\Gamma S)\Gamma Q \subseteq (S\Gamma S)\Gamma Q = S\Gamma Q$. Also $(Q\Gamma S)\Gamma Q \subseteq (Q\Gamma S)\Gamma S = (Q\Gamma S)\Gamma(e\Gamma S) = (Q\Gamma e)\Gamma(S\Gamma S) = (Q\Gamma e)\Gamma S = Q\Gamma S$. This implies that $(Q\Gamma S)\Gamma Q \subseteq S\Gamma Q \cap Q\Gamma S \subseteq Q$. Hence Q is a bi-Γ-ideal of S. ■

5.2.2 Regular and intra-regular Γ-AG-groupoids

In this study, we give necessary and sufficient conditions for regular Γ-AG-groupoids with left identity. Moreover, in last theorem, we give necessary and sufficient conditions for regular and intra-regular Γ-AG-groupoids with left identity.

Lemma 192 *Let S be a Γ-AG-groupoid with left identity e, then $a\alpha(b\beta c) = b\alpha(a\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.*

Proof. For all $a, b, c \in S$ and $\alpha, \beta, \gamma \in \Gamma$, consider $a\alpha(b\beta c) = (e\gamma a)\alpha(b\beta c) = (e\gamma b)\alpha(a\beta c) = b\alpha(a\beta c)$. ■

Lemma 193 *Let S be a Γ-AG-groupoid with left identity e such that $(x\alpha e)\beta S = x\beta S$ for all $x \in S$ and $\alpha, \beta \in \Gamma$. The following statements are true.*

(1) *$S\Gamma a$ is the smallest left Γ-ideal of S containing a.*

(2) *$a\Gamma S \cup S\Gamma a$ is the smallest right Γ-ideal of S containing a.*

Proof. (1) Since S has a left identity, $a \in S\Gamma a$. Consider $S\Gamma(S\Gamma a) = (S\Gamma e)\Gamma(S\Gamma a) = (S\Gamma S)\Gamma(e\Gamma a) = S\Gamma a$, thus $S\Gamma a$ is a left Γ-ideal of S containing a. Next, let L be any left Γ-ideal containing a. So $S\Gamma a \subseteq S\Gamma L \subseteq L$. Hence $S\Gamma a$ is the smallest left Γ-ideal of S containing a.

(2) Claim that $(a\Gamma S \cup S\Gamma a)\Gamma S \subseteq (a\Gamma S \cup S\Gamma a)$. Consider

$$\begin{aligned}
(a\Gamma S \cup S\Gamma a)\Gamma S &= (a\Gamma S)\Gamma S \cup (S\Gamma a)\Gamma S \\
&= (S\Gamma S)\Gamma a \cup (S\Gamma a)\Gamma(e\Gamma S) \\
&= S\Gamma a \cup (S\Gamma e)\Gamma(a\Gamma S) \\
&= S\Gamma a \cup S\Gamma(a\Gamma S) \\
&= S\Gamma a \cup a\Gamma(S\Gamma S) \text{ by lemma 192} \\
&= S\Gamma a \cup a\Gamma S = a\Gamma S \cup S\Gamma a.
\end{aligned}$$

Hence $a\Gamma S \cup S\Gamma a$ is a right Γ-ideal of S. Since $a \in S\Gamma a$, $a \in a\Gamma S \cup S\Gamma a$. Let R be any right Γ-ideal of S containing a. Since $a\Gamma S \in R\Gamma S \subseteq R, a\Gamma S \subseteq R$. Also $S\Gamma a = (S\Gamma S)\Gamma a = (a\Gamma S)\Gamma S \subseteq (R\Gamma S)\Gamma S \subseteq R\Gamma S \subseteq R$. It follows that $a\Gamma S \cup S\Gamma a \subseteq R$. Therefore $a\Gamma S \cup S\Gamma a$ is the smallest right Γ-ideal of S containing a. ■

Theorem 194 *Let S be a Γ-AG-groupoid with left identity e such that $(x\alpha e)\beta S = x\beta S$ for all $x \in S$ and $\alpha, \beta \in \Gamma$. Then the following statements are equivalent:*

(1) *S is regular.*

(2) *$Q = (Q\Gamma S)\Gamma Q$ for every quasi-Γ-ideal Q of S.*

(3) *$R \cap L = R\Gamma L$ for every right Γ-ideal R and every left Γ-ideal L of S.*

Proof. $(1) \Rightarrow (2)$. Let Q be a quasi-Γ-ideal of S. Let $x \in Q$ and $\alpha, \beta \in \Gamma$. Since S is regular, so there exist $a \in S$ and $\alpha, \beta \in \Gamma$ such that $x = (x\alpha a)\beta x$. Thus $x = (x\alpha a)\beta x \in (Q\Gamma S)\Gamma Q$. This implies that $Q \subseteq (Q\Gamma S)\Gamma Q$. Now $(Q\Gamma S)\Gamma Q \subseteq (Q\Gamma S)\Gamma S = (Q\Gamma S)\Gamma(e\Gamma S) = (Q\Gamma e)\Gamma(S\Gamma S) \subseteq (Q\Gamma e)\Gamma S = Q\Gamma S$. Thus $(Q\Gamma S)\Gamma Q \subseteq Q\Gamma S$. Also $(Q\Gamma S)\Gamma Q \subseteq (Q\Gamma S)\Gamma S = (S\Gamma S)\Gamma Q \subseteq S\Gamma Q$. It follows that $(Q\Gamma S)\Gamma Q \subseteq Q\Gamma S \cap S\Gamma Q \subseteq Q$ and hence $Q = (Q\Gamma S)\Gamma Q$. $(2) \Rightarrow$

(3). Let R be a right Γ-ideal and L a left Γ-ideal of S. So by theorem 186, $R\cap L$ is a quasi-Γ-ideal of S. Then by hypothesis, $R\cap L = ((R\cap L)\Gamma S)\Gamma(R\cap L) \subseteq (R\Gamma S)\Gamma L \subseteq R\Gamma L$. This implies that $R\cap L \subseteq R\Gamma L$. Obviously, $R\Gamma L \subseteq R\cap L$. Hence $R\cap L = R\Gamma L$. $(3) \Rightarrow (1)$. Let $x \in S$. By lemma 193, $S\Gamma x$ is a left Γ-ideal of S containing x and $x\Gamma S\cup S\Gamma x$ is a right Γ-ideal of S containing x. So by hypothesis $(x\Gamma S\cup S\Gamma x)\cap S\Gamma x = (x\Gamma S\cup S\Gamma x)\Gamma(S\Gamma x) = (x\Gamma S)\Gamma(S\Gamma x)\cup(S\Gamma x)\Gamma(S\Gamma x)$. Since $x \in (x\Gamma S \cup S\Gamma x) \cap S\Gamma x, x \in (x\Gamma S)\Gamma(S\Gamma x) \cup (S\Gamma x)\Gamma(S\Gamma x)$ which further implies that $x \in (x\Gamma S)\Gamma(S\Gamma x)$ or $x \in (S\Gamma x)\Gamma(S\Gamma x)$.

Case 1: $x \in (x\Gamma S)\Gamma(S\Gamma x)$. By the definition of Γ-AG-groupoids and lemma 192, we have

$$\begin{aligned} x &= (x\alpha a)\beta(b\gamma x) = ((b\gamma x)\alpha a)\beta x \\ &= (((e\mu b)\gamma x)\alpha a)\beta x = (((x\mu b)\gamma e)\alpha a)\beta x \\ &= ((a\gamma e)\alpha(x\mu b))\beta x = (x\alpha((a\gamma e)\mu b))\beta x \\ &= (x\alpha s)\beta x, \text{ where } s = (a\gamma e)\mu b \in S. \end{aligned}$$

Hence $x = (x\alpha s)\beta x$. So x is regular.

Case 2: $x \in (S\Gamma x)\Gamma(S\Gamma x)$. Since $(S\Gamma x)\Gamma(S\Gamma x) = ((e\Gamma S)\Gamma x)\Gamma(S\Gamma x) = ((x\Gamma S)\Gamma e)\Gamma(S\Gamma x) = (x\Gamma S)\Gamma(S\Gamma x)$, $x \in (x\Gamma S)\Gamma(S\Gamma x)$. By using the process of case 1, we have x is regular. In both two cases, therefore S is regular. ■

Definition 195 *A nonempty subset A of Γ-AG-groupoid S is called a generalized bi-Γ-ideal of S if $(A\Gamma S)\Gamma A \subseteq A$.*

Remark 196 (1) *Generalized bi-Γ-ideal of S need not be a subAG-groupoids of S.*

(2) *If S is a Γ-AG-groupoid with left identity e such that $(x\alpha e)\beta S = x\beta S$ for all $x \in S$ and $\alpha, \beta \in \Gamma$, then every quasi-Γ-ideal Q of S is a generalized bi-Γ-ideal of S.*

Theorem 197 *Let S be a Γ-AG-groupoid with left identity e such that $(x\alpha e)\beta S = x\beta S$ for all $x \in S$ and $\alpha, \beta \in \Gamma$. Then the following statements are equivalent:*

(1) *S is regular.*

(2) *$Q = (Q\Gamma S)\Gamma Q$ for every quasi-Γ-ideal Q of S.*

(3) $A = (A\Gamma S)\Gamma A$ *for every generalized bi-*Γ*-ideal* A *of* S.

Proof. $(1) \Rightarrow (3)$. Let A be a generalized bi-Γ-ideal of S, then $(A\Gamma S)\Gamma A \subseteq A$. Now let $a \in A$. Since S is regular, there exist x in S and α, β in Γ such that $a = (a\alpha x)\beta a$. It follows that $a = (a\alpha x)\beta a \in (A\Gamma S)\Gamma A$ which further implies that $A \subseteq (A\Gamma S)\Gamma A$. Hence $A = (A\Gamma S)\Gamma A$. $(3) \Rightarrow (2)$. Let Q be a quasi-Γ-ideal of S. Since every quasi-Γ-ideal of S is a generalized bi-Γ-ideal of S and by hypothesis, $Q = (Q\Gamma S)\Gamma Q$. $(2) \Rightarrow (1)$. By theorem 194. ■

Definition 198 *A nonempty subset* A *of* Γ*-AG-groupoid* S *is called an interior* Γ*-ideal of* S *if* $(S\Gamma A)\Gamma S \subseteq A$.

Lemma 199 *Let* S *be a* Γ*-AG-groupoid with left identity* e. *Then a non-empty subset* I *of* S *is an interior* Γ*-ideal of* S *if and only if* I *is a* Γ*-ideal of* S.

Proof. Suppose I is an interior Γ-ideal of S. Let $a \in I$ and $s \in S$. This implies $a \in S$, and so we have $a = e\gamma a$, where $\gamma \in \Gamma$. Now consider $a\alpha s = (e\gamma a)\alpha s \in (S\Gamma I)\Gamma S \subseteq I$, it follows that $a\alpha s \in I$ and so $I\Gamma S \subseteq I$. Hence I is a right Γ-ideal of S and hence I is a Γ-ideal of S. Conversely, let I be a Γ-ideal of S. Then $(S\Gamma I)\Gamma S \subseteq I\Gamma S \subseteq I$. This implies $(S\Gamma I)\Gamma S \subseteq I$. Hence I is an interior Γ-ideal of S. ■

Theorem 200 *Let* S *be a* Γ*-AG-groupoid with left identity* e *such that* $(x\alpha e)\beta S = x\beta S$ *for all* $x \in S$ *and* $\alpha, \beta \in \Gamma$. *Then the following statements are equivalent:*

(1) S *is regular.*

(2) $Q \cap J = (Q\Gamma J)\Gamma Q$ *for every quasi-*Γ*-ideal* Q *and every* Γ*-ideal* J *of* S.

(3) $B \cap J = (B\Gamma J)\Gamma B$ *for every bi-*Γ*-ideal* B *and every* Γ*-ideal* J *of* S.

(4) $G \cap J = (G\Gamma J)\Gamma G$ *for every generalized bi-*Γ*-ideal* G *and every* Γ*-ideal* J *of* S.

Proof. $(1) \Rightarrow (4)$. Let $a \in G \cap J$, where G is a generalized bi-Γ-ideal and J is a Γ-ideal of S. Since S is regular, there exist x in S and α, β in Γ such that $a = (a\alpha x)\beta a$. Then consider $a = (a\alpha x)\beta a = (((a\alpha x)\beta a)\alpha x)\beta a = ((x\beta a)\alpha(a\alpha x))\beta a = (a\alpha((x\beta a)\alpha x))\beta a$ by lemma 192. Since $(x\beta a)\alpha x \in (S\Gamma J)\Gamma S \subseteq J\Gamma S \subseteq J$, $(x\beta a)\alpha x \in J$. Thus $a = (a\alpha((x\beta a)\alpha x))\beta a \in (G\Gamma J)\Gamma G$.

It follows that $G \cap J \subseteq (G\Gamma J)\Gamma G$. On the other hand, let G be a generalized bi-Γ-ideal and J be a Γ-ideal of S. Then $(G\Gamma J)\Gamma G \subseteq (G\Gamma S)\Gamma G \subseteq ((e\Gamma G)\Gamma S)\Gamma G \subseteq G\Gamma G \subseteq G$. Also $(G\Gamma J)\Gamma G \subseteq (S\Gamma J)\Gamma S \subseteq J$. It follows that $(G\Gamma J)\Gamma G \subseteq G \cap J$. Hence $G \cap J = (G\Gamma J)\Gamma G$. Now by theorem 191 and it is fact that every bi-Γ-ideal is a generalized bi-Γ-ideal, we have $(4) \Rightarrow (3)$ and $(3) \Rightarrow (2)$. $(2) \Rightarrow (1)$. Let Q be a quasi-Γ-ideal of S. Since S itself is a Γ-ideal of S, it follows that $Q = Q \cap S = (Q\Gamma S)\Gamma Q$. Hence by theorem 194, S is regular. ■

Theorem 201 *Let S be a Γ-AG-groupoid with left identity e such that $(x\alpha e)\beta S = x\beta S$ for all $x \in S$ and $\alpha, \beta \in \Gamma$. Then the following statements are equivalent:*

(1) *S is regular.*

(2) *$Q \cap R \subseteq R\Gamma Q$ for every quasi-Γ-ideal Q and every right Γ-ideal R of S.*

(3) *$B \cap R \subseteq R\Gamma B$ for every bi-Γ-ideal B and every right Γ-ideal R of S.*

(4) *$G \cap R \subseteq R\Gamma G$ for every generalized bi-Γ-ideal G and every right Γ-ideal R of S.*

Proof. $(1) \Rightarrow (4)$. Let G be a generalized bi-Γ-ideal and R a right Γ-ideal of S. Let $a \in G \cap R$. Since S is regular, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$. Then $a = (a\alpha x)\beta a = (a\alpha x)\beta((a\alpha x)\beta a) \in R\Gamma G$. This implies that $G \cap R \subseteq R\Gamma G$. $(4) \Rightarrow (3)$ and $(3) \Rightarrow (2)$ are trivial. $(2) \Rightarrow (1)$. Suppose that L is a left Γ-ideal and R is a right Γ-ideal of S. Obviously, $R\Gamma L \subseteq L \cap R$. Since every left Γ-ideal of S is a quasi-Γ-ideal of S and by hypothesis, $L \cap R \subseteq R\Gamma L$. Hence $L \cap R = R\Gamma L$. So By theorem 194, S is regular. ■

Definition 202 *A Γ-AG-groupoid S is said to be an intra-regular Γ-AG-groupoid if for each a in S there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (x\alpha a)\beta(a\gamma y)$.*

Proposition 203 *Let S be a Γ-AG-groupoid with left identity e. Then A is a Γ-ideal of S if and only if A is an interior Γ-ideal of S.*

Proof. Let A be a Γ-ideal of S. Then $(S\Gamma A)\Gamma S \subseteq A\Gamma S \subseteq A$, this implies that A is an interior Γ-ideal of S. Conversely, suppose A is an interior Γ-ideal of S. Now let $t \in A\Gamma S$, then $t = a\alpha b$ for some $a \in A$, $\alpha \in \Gamma$ and $b \in S$. Since S is intra-regular, so there exist p, q in S and α, β, γ in Γ such that $b = (p\alpha b)\beta(b\gamma q)$. Then $t = a\alpha b = a\alpha((p\alpha b)\beta(b\gamma q)) = (p\alpha b)\alpha(a\beta(b\gamma q)) =$

$(p\alpha a)\alpha(b\beta(b\gamma q)) \in (S\Gamma A)\Gamma S \subseteq A$. It follows that $A\Gamma S \subseteq A$, which further implies that A is a right Γ-ideal of S and hence A is a Γ-ideal of S. ■

Theorem 204 *Let S be a Γ-AG-groupoid with left identity e such that $(x\alpha e)\beta S = x\beta S$ for all $x \in S$ and $\alpha, \beta \in \Gamma$. Then S is regular and intra-regular if and only if every quasi-Γ-ideal of S is Γ-idempotent.*

Proof. Assume that S is regular and intra-regular and let Q be a quasi-Γ-ideal of S. Then $Q\Gamma Q \subseteq Q$. On the other hand, let $a \in Q$. Since S is regular and intra-regular, there exist $x, y, z \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (a\alpha x)\beta a$ and $a = (y\alpha a)\beta(a\gamma z)$. Using definition of Γ-AG-groupoid repeatedly and lemma 192, we have

$$\begin{aligned}
a &= (a\alpha x)\beta a = (((y\alpha a)\beta(a\gamma z))\alpha x)\beta a \\
&= ((x\beta(a\gamma z))\alpha(y\alpha a))\beta a = ((a\beta(x\gamma z))\alpha(y\alpha a))\beta a \\
&= (((y\alpha a)\beta(x\gamma z))\alpha a)\beta a = ((((e\mu y)\alpha a)\beta(x\gamma z))\alpha a)\beta a \\
&= ((((a\mu y)\alpha e)\beta(x\gamma z))\alpha a)\beta a = ((((x\gamma z)\alpha e)\beta(a\mu y))\alpha a)\beta a \\
&= (((x\gamma z)\beta(a\mu y))\alpha a)\beta a = ((a\beta((x\gamma z)\mu y))\alpha a)\beta a \\
&= ((a\beta p)\alpha a)\beta a, \text{ where } p = (x\gamma z)\mu y \in S \\
&\in ((Q\Gamma S)\Gamma Q)\Gamma Q = Q\Gamma Q.
\end{aligned}$$

This implies that $Q \subseteq Q\Gamma Q$. Hence $Q\Gamma Q = Q$, which shows that Q is Γ-idempotent.

Conversely, assume every quasi-Γ-ideal of S is Γ-idempotent. Let $a \in S$. Then $S\Gamma a$ is a quasi-Γ-ideal of S. So,

$$\begin{aligned}
S\Gamma a &= (S\Gamma a)\Gamma(S\Gamma a) = (S\Gamma a)\Gamma((e\Gamma S)\Gamma a) = (S\Gamma a)\Gamma((a\Gamma S)\Gamma e) \\
&= (S\Gamma a)\Gamma((a\Gamma S)\Gamma(e\Gamma S)) = (S\Gamma a)\Gamma((a\Gamma e)\Gamma(S\Gamma S)) = (S\Gamma a)\Gamma(a\Gamma S).
\end{aligned}$$

Thus $a \in (S\Gamma a)\Gamma(a\Gamma S)$, which further implies that $a = (x\alpha a)\beta(a\gamma y)$ for some $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Hence a is intra-regular. So S is intra-regular. Next, we show that S is regular by using theorem 194 $(3) \Rightarrow (1)$, let R and L be right and left ideals of S, respectively. Claim that $R \cap L = R\Gamma L$. Clearly $R\Gamma L \subseteq R \cap L$. Conversely, by theorem 186, $R \cap L$ is a quasi-Γ-ideal of S. By assumption, $R \cap L = (R \cap L)\Gamma(R \cap L) \subseteq R\Gamma L$. Hence $R \cap L = R\Gamma L$. By theorem 194, S is regular. ■

Chapter 6

Γ-Left Almost Rings

Introduction

In this chapter, we introduce gamma left almost rings (Γ-LA-rings) which are in fact a generalization of left almost rings (LA-rings) introduced by S. M. Yusuf [70] and also a generalization of commutative Γ-rings [52]. We have initiated this idea by taking inspiration from an article: "On a generalization of ring theory" published in Osaka Journal of Mathematics, 1964. In this article, Nobusawa [52] have introduced the concept of Γ-rings for the first time. After his research, Barnes [4] weakened slightly the conditions in the definition of the Γ-ring in the sense of Nobusawa. Barnes [4], Kyuno [34, 35] and Luh [37], studied the structure of Γ-rings and obtained various generalizations analogous to corresponding parts in ring theory.

Γ-left almost rings are a direct generalization of LA-rings discussed earlier in chapters 3, 4. We can easily observe that Γ-left almost rings are in fact a generalization of commutative Γ-rings and intuitively commutative Γ-rings are generalization of commutative ring theory. Consequently, Γ-left almost rings become a generalization of commutative rings.

Let $(M, +)$ and $(\Gamma, +)$ be abelian groups. M is called a Γ-ring if there exists a mapping $M \times \Gamma \times M \to M$ satisfying the following conditions:

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$.

(ii) $a\alpha(b + c) = a\alpha b + a\alpha c$.

(iii) $a(\alpha + \beta)b = a\alpha b + a\beta b$.

(iv) $(a\alpha b)\beta c = a\alpha(b\beta c)$.

In this study we have generalized some results which were established for LA-rings in chapter 3 earlier.

6.1 Main Results

We first define Γ-LA-ring.

Definition 205 *Let $(M,+)$ and $(\Gamma,+)$ be additive LA-groups. If there exists a mapping $M \times \Gamma \times M \to M$ satisfying the following conditions:*

(L1) $(a+b)\alpha c = a\alpha c + b\alpha c$

(L2) $a\alpha(b+c) = a\alpha b + a\alpha c$

(L3) $a(\alpha+\beta)b = a\alpha b + a\beta b$

(L4) $(a\alpha b)\beta c = (c\alpha b)\beta a$

for all $a,b,c \in M$ and $\alpha,\beta \in \Gamma$, then M is called a gamma left almost ring (Γ-LA-ring). Throughout the matter below M denotes a Γ-LA-ring.

Definition 206 *An additive LA-subgroup A of M is said to be a left Γ-ideal if $m\alpha a \in A$ for all $a \in A, \alpha \in \Gamma$ and $m \in M$. Right Γ-ideals and two sided Γ-ideals are defined in the usual manner.*

Definition 207 *An element $e \in M$ is called a left identity of M if $e\gamma a = a$ for all $a \in M$ and $\gamma \in \Gamma$.*

Proposition 208 *If M is a Γ-LA-ring with left identity e, then every right Γ-ideal of M is a left Γ-ideal of M.*

Proof. Let I be a right Γ-ideal of M, this implies I is an additive LA-subgroup of M. Now let $m \in M$, $i \in I$ and $\gamma \in \Gamma$, then $m\gamma i = (e\alpha m)\gamma i = (i\alpha m)\gamma e \in I$. Thus I is also a left Γ-ideal of M. ■

Now onward by Γ-ideal in Γ-LA-ring M with left identity e, we mean a right Γ-ideal of M.

Lemma 209 *If I is a right Γ-ideal of M with left identity e then $I\Gamma I$ is a Γ-ideal of M.*

Proof. Let $i \in I\Gamma I$, then we can write $i = x\alpha y$ where $x,y \in I$ and $\alpha \in \Gamma$. Now consider $i\gamma m = (x\alpha y)\gamma m = (m\alpha y)\gamma x \in I\Gamma I$. This implies that $I\Gamma I$ is a right Γ-ideal and by Proposition 208, $I\Gamma I$ is a left Γ-ideal. Hence it follows that $I\Gamma I$ is a Γ-ideal of M. ■

Lemma 210 *Let M be a Γ-LA-ring with left identity e. If I is a proper right (left) Γ-ideal of M, then $e \notin I$.*

Proof. *Assume on contrary that $e \in I$ and let $m \in M$ and $\alpha \in \Gamma$, then consider $m = e\alpha m \in I\Gamma M \subseteq I$. This implies that $M \subseteq I$, but $I \subseteq M$. So $I = M$, a contradiction. Hence $e \notin I$.* ■

A Γ-LA-ring M is said to be fully Γ-idempotent if all Γ-ideals of M are Γ-idempotent. If M is a Γ-LA-ring with left identity e then the principal left Γ-ideal generated by an element m is defined as $\langle m \rangle = M\Gamma m = \{n\alpha m : n \in M$ and $\alpha \in \Gamma\}$.

It is important to note that if I is a Γ-ideal of M then $I = \langle I \rangle$, and as $I\Gamma I$ is a Γ-ideal of M, hence $I\Gamma I = \langle I\Gamma I \rangle$.

Proposition 211 *If M is a Γ-LA-ring with left identity e and I, J are Γ-ideals of M, then the following assertions are equivalent:*

(1) *M is fully Γ-idempotent,*

(2) *$I \cap J = \langle I\Gamma J \rangle$,*

(3) *the Γ-ideals of M form a semilattice $(L_M, \wedge)$, where $I \wedge J = \langle I\Gamma J \rangle$.*

Proof. $(1) \Rightarrow (2)$. Since $I\Gamma J \subseteq I \cap J$, $\langle I\Gamma J \rangle \subseteq I \cap J$. Now let $m \in I \cap J$. As $\langle m \rangle$ is a principal left Γ-ideal generated by a fixed element m, so $m \in \langle m \rangle = \langle m \rangle \Gamma \langle m \rangle \subseteq \langle I\Gamma J \rangle$. Hence $I \cap J = \langle I\Gamma J \rangle$. $(2) \Rightarrow (3)$. $I \wedge J = \langle I\Gamma J \rangle = I \cap J = J \cap I = J \wedge I$ and also $I \wedge I = \langle I\Gamma I \rangle = I \cap I = I$. Hence $(L_M, \wedge)$ is a semilattice. $(3) \Rightarrow (1)$. Now $I = I \wedge I = \langle I\Gamma I \rangle = I\Gamma I$.
■

A Γ-ideal P of a Γ-LA-ring M is said to be prime Γ-ideal of M if and only if $A\Gamma B \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$, where A and B are Γ-ideals in M and P is called semiprime if for any Γ-ideal I of M, $I\Gamma I \subseteq P$ implies that $I \subseteq P$. A Γ-LA-ring M is said to be fully prime if every Γ-ideal of M is prime and it is fully semiprime if every Γ-ideal is semiprime.

The set of Γ-ideals of M is said to be a totally ordered under inclusion if for all Γ-ideals I, J of M, either $I \subseteq J$ or $J \subseteq I$ and is denoted by Γ-ideal(M).

Theorem 212 *Let M be a Γ-LA-ring with left identity e, then M is fully prime if and only if every Γ-ideal is Γ-idempotent and the set Γ-ideal(M) is totally ordered under inclusion.*

Proof. *Let M be fully prime and I be any Γ-ideal of M. By Lemma 209, $I\Gamma I$ is a Γ-ideal of M, and so $I\Gamma I \subseteq I$. Also $I\Gamma I \subseteq I$ which implies that $I\Gamma I = I$ and hence I is Γ-idempotent. Now let A, B be Γ-ideals of M and $A\Gamma B \subseteq A$, $A\Gamma B \subseteq B$ which implies that $A\Gamma B \subseteq A \cap B$. As A and B are prime Γ-ideals so $A \cap B$ is also a prime Γ-ideal of M. Then $A \subseteq A \cap B$ or $B \subseteq A \cap B$ which implies that either $A \subseteq B$ or $B \subseteq A$. Hence the set Γ-ideal(M) is totally ordered under inclusion. Conversely, assume that every Γ-ideal of M is Γ-idempotent and Γ-ideal(M) is totally ordered under inclusion. Let Q, R and S be any Γ-ideals of M with $Q\Gamma R \subseteq S$ such that $Q \subseteq R$. Now since Q is Γ-idempotent, $Q = Q\Gamma Q \subseteq Q\Gamma R \subseteq S$. This implies that $Q \subseteq S$ and hence M is fully prime.* ■

Definition 213 *A Γ-ideal I of a Γ-LA-ring M is said to be strongly irreducible if $P \cap Q \subseteq I$ implies $P \subseteq I$ or $Q \subseteq I$.*

Theorem 214 *Let M be a Γ-LA-ring with left identity e, then a Γ-ideal I of M is prime if and only if it is semiprime and strongly irreducible.*

Proof. The proof is straight forward. ■

Definition 215 *A nonempty subset A of a Γ-LA-ring M with left identity e is called an M-system if for $a, b \in A$ there exist $m \in M$ and $\alpha, \beta \in \Gamma$ such that $a\alpha(m\beta b) \in A$.*

Let M be a Γ-LA-ring, $a, b, c, d \in M$ and $\alpha, \beta, \gamma \in \Gamma$, we have that $(a\alpha b)\beta(c\gamma d) = [(c\gamma d)\alpha b]\beta a = [(b\gamma d)\alpha c]\beta a = (a\alpha c)\beta(b\gamma d)$. So $(a\alpha b)\beta(c\gamma d) = (a\alpha c)\beta(b\gamma d)$, this is called the Γ-medial law. Next, let M be a Γ-LA-ring with left identity e, $a, b, c, d \in M$ and $\alpha, \beta, \gamma \in \Gamma$, we have that

$$\begin{aligned}(a\alpha b)\beta(c\gamma d) &= [e\beta(a\alpha b)]\beta(c\gamma d) = [(c\gamma d)\beta(a\alpha b)]\beta e \\ &= [(c\gamma a)\beta(d\alpha b)]\beta e = [e\beta(d\alpha b)]\beta(c\gamma a) \\ &= (d\alpha b)\beta(c\gamma a).\end{aligned}$$

Then $(a\alpha b)\beta(c\gamma d) = (d\alpha b)\beta(c\gamma a)$, this is called the Γ-paramedial law.

Definition 216 *Let I be a left Γ-ideal of a M with left identity e. Then I is said to be quasi-Γ-prime if $H\Gamma K \subseteq I$ implies that either $H \subseteq I$ or $K \subseteq I$, where H and K are any left Γ-ideals of M. If for any left Γ-ideal H of M such that $H\Gamma H \subseteq I$, we have $H \subseteq I$, then I is called quasi-Γ-semiprime.*

Proposition 217 *Let I be a left Γ-ideal of M with left identity e, then the following are equivalent:*

(1) *I is a quasi-Γ-prime ideal of M.*

(2) *$H\Gamma K = \langle H\Gamma K\rangle \subseteq I$ implies that either $H \subseteq I$ or $K \subseteq I$, where H and K are any left Γ-ideals of M.*

(3) *If $H \nsubseteq I$ and $K \nsubseteq I$ then $H\Gamma K \nsubseteq I$, where H and K are any left Γ-ideals of M.*

(4) *If h, k are elements of M such that $h \notin I$ and $k \notin I$ then $\langle h\rangle\Gamma\langle k\rangle \nsubseteq I$.*

(5) *If $h, k \in M$ satisfying $h\Gamma(M\Gamma k) \subseteq I$, then either $h \in I$ or $k \in I$.*

Proof. $(1) \Leftrightarrow (2)$. Let I be quasi-Γ-prime. Now by definition if $H\Gamma K = \langle H\Gamma K\rangle \subseteq I$, then obviously it implies that either $H \subseteq I$ or $K \subseteq I$ for all left Γ-ideals H and K of M. Converse is trivial. $(2) \Leftrightarrow (3)$ is trivial.

$(1) \Rightarrow (4)$. Let $\langle h\rangle\Gamma\langle k\rangle \subseteq I$, then either $\langle h\rangle \subseteq I$ or $\langle k\rangle \subseteq I$, which implies that either $h \in I$ or $k \in I$.

$(4) \Rightarrow (2)$. Let $H\Gamma K \subseteq I$. If $h \in H$ and $k \in K$, then $\langle h\rangle\Gamma\langle k\rangle \subseteq I$ and hence by hypothesis either $h \in I$ or $k \in I$. This implies that either $H \subseteq I$ or $K \subseteq I$. $(1) \Leftrightarrow (5)$. Let $h\Gamma(M\Gamma k) \subseteq I$, then $M\Gamma(h\Gamma(M\Gamma k)) \subseteq M\Gamma I \subseteq I$. Now consider

$$\begin{aligned} M\Gamma(h\Gamma(M\Gamma k)) &= (M\Gamma M)\Gamma(h\Gamma(M\Gamma k)) = (M\Gamma h)\Gamma(M\Gamma(M\Gamma k)) \\ &= (M\Gamma h)\Gamma((M\Gamma M)\Gamma(M\Gamma k)) = (M\Gamma h)\Gamma((k\Gamma M)\Gamma(M\Gamma M)) \\ &= (M\Gamma h)\Gamma((k\Gamma M)\Gamma M), = (M\Gamma h)\Gamma((M\Gamma M)\Gamma k) \\ &= (M\Gamma h)\Gamma(M\Gamma k). \end{aligned}$$

Hence $(M\Gamma h)\Gamma(M\Gamma k) \subseteq I$. Since $M\Gamma h$ and $M\Gamma k$ are left Γ-ideals for all $h \in H$ and $k \in K$, hence either $h \in I$ or $k \in I$. Conversely, let $H\Gamma K \subseteq I$ where H and K are any left Γ-ideals of M. Let $H \nsubseteq I$ then there exists $l \in H$ such that $l \notin I$. For all $m \in K$, we have $l\Gamma(M\Gamma m) \subseteq H\Gamma(M\Gamma K) \subseteq H\Gamma K \subseteq I$. This implies that $K \subseteq I$ and hence I is a quasi-Γ-prime ideal of M. ■

Proposition 218 *A left Γ-ideal I of a Γ-LA-ring M with left identity e is quasi-Γ-prime if and only if $R\backslash I$ is an M-system.*

Proof. Suppose I is a quasi-Γ-prime ideal. Let $a, b \in M\backslash I$ which implies that $a \notin I$ and $b \notin I$. So by Proposition 217, we have $a\Gamma(M\Gamma b) \nsubseteq I$. This implies that there exist some $m \in M$ and $\alpha, \beta \in \Gamma$ such that $a\alpha(m\beta b) \notin I$ which further implies that $a\alpha(m\beta b) \in M\backslash I$. Hence $M\backslash I$ is an M-system. Conversely, let $M\backslash I$ is an M-system. Suppose $a\Gamma(M\Gamma b) \subseteq I$ and let $a \notin I$ and $b \notin I$. This implies that $a, b \in M\backslash I$. Since $M\backslash I$ is an M-system so there exist $m \in M$ and $\alpha, \beta \in \Gamma$ such that $a\alpha(m\beta b) \in M\backslash I$ which implies that $a\Gamma(M\Gamma b) \nsubseteq I$, a contradiction. Hence either $a \in I$ or $b \in I$. This shows that I is a quasi-Γ-prime ideal. ■

Definition 219 *A nonempty subset Q of a Γ-LA-ring M with left identity e is called P-system if for all $a \in Q$, there exist $m \in M$ and $\alpha, \beta \in \Gamma$ such that $a\alpha(m\beta a) \in Q$.*

Proposition 220 *Let I be a left Γ-ideal of M with left identity e, then the following are equivalent:*

(1) *I is quasi-Γ-semiprime.*

(2) *$H\Gamma H = \langle H\Gamma H\rangle \subseteq I \Rightarrow H \subseteq I$, where H is any left Γ-ideal of M.*

(3) *For any left Γ-ideal H of M : $H \nsubseteq I \Rightarrow H\Gamma H \nsubseteq I$.*

(4) *If m is any element of M and $\gamma \in \Gamma$ such that $\langle m\gamma m\rangle \subseteq I$, then it implies that $m \in I$.*

(5) *For all $m \in M$ and $\alpha, \beta \in \Gamma$: $m\alpha(M\beta m) \subseteq I \Rightarrow m \in I$.*

Proof. $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ is trivial. $(1) \Rightarrow (4)$. Let $\langle m\gamma m\rangle \subseteq I$. But by hypothesis I is quasi-Γ-semiprime, so it implies that $\langle m\rangle \subseteq I$ which further implies that $m \in I$. $(4) \Rightarrow (2)$

For all left Γ-ideals H of M, let $H\Gamma H = \langle H\Gamma H\rangle \subseteq I$. If $m \in H$ and $\gamma \in \Gamma$, then by (4) $\langle m\gamma m\rangle \subseteq I$ implies that $m \in I$. Hence it shows that $H \subseteq I$. $(1) \Leftrightarrow (5)$ is straight forward. ■

Proposition 221 *A left Γ-ideal I of M with left identity e is quasi-Γ-semiprime if and only if $M\backslash I$ is a P-system.*

Proof. Let I be a quasi-Γ-semiprime ideal of M and let $m \in M\backslash I$. On contrary suppose that there do not exist an element $x \in M$ and $\alpha, \beta \in \Gamma$ such that $m\alpha(x\beta m) \in M\backslash I$. This implies that $m\alpha(x\beta m) \in I$. Since I is quasi-Γ-semiprime, so by Proposition 220, $m \in I$ which is a contradiction. Thus there exist $x \in M$ and $\alpha, \beta \in \Gamma$ such that $m\alpha(x\beta m) \in M\backslash I$. Hence $M\backslash I$ is a P-system. Conversely, suppose for all $m \in M\backslash I$ there exist $x \in M$ and $\alpha, \beta \in \Gamma$ such that $m\alpha(x\beta m) \in M\backslash I$. Let $m\Gamma(M\Gamma m) \subseteq I$. This implies that there does not exist $x \in M$ such that $m\alpha(x\beta m) \in M\backslash I$ which implies that $m \in I$. Hence by Proposition 220, I is quasi-Γ-semiprime. ■

Lemma 222 *An M-system of elements of Γ-LA-ring is a P-system.*

Proof. Let A be a nonempty subset of M such that A is an M-system. Then for all $a, b \in A$, there exist an element $m \in M$ and $\alpha, \beta \in \Gamma$ such that $a\alpha(m\beta b) \in A$. If we take $b = a$, then $a\alpha(m\beta a) \in A$ which implies that A is a P-system. ■

Definition 223 *A Γ-ideal I of a Γ-LA-ring M with left identity e is strongly irreducible if and only if for Γ -ideals H and K of M, $H \cap K \subseteq I$ implies that $H \subseteq I$ or $K \subseteq I$ and I is said to be irreducible if for Γ-ideals H and K, $I = H \cap K$ implies that $I = H$ or $I = K$.*

Lemma 224 *Every strongly irreducible Γ-ideal of M with left identity e is irreducible.*

Proof. The proof is obvious. ■

Proposition 225 *A Γ-ideal I of M with left identity e is Γ-prime if and only if it is Γ-semiprime and strongly irreducible.*

Proof. The proof is straight forward. ■

Definition 226 *A nonempty subset B of Γ-LA-ring M with left identity e is called an I-system if for all $a, b \in B$, $(\langle a\rangle \cap \langle b\rangle) \cap B \neq \phi$.*

Proposition 227 *The following conditions on Γ-ideal I of M with left identity e are equivalent:*

(1) *I is strongly irreducible.*

(2) *For all $a, b \in M$: $\langle a\rangle \cap \langle b\rangle \subseteq I$ implies that either $a \in I$ or $b \in I$.*

(3) *$M \backslash I$ is an I-system.*

Proof. $(1) \Rightarrow (2)$ is trivial. $(2) \Rightarrow (3)$ Let $a, b \in M \backslash I$. Let $(\langle a\rangle \cap \langle b\rangle) \cap M \backslash I = \phi$. This implies that $\langle a\rangle \cap \langle b\rangle \subseteq I$ and so by hypothesis either $a \in I$ or $b \in I$ which is a contradiction. Hence $(\langle a\rangle \cap \langle b\rangle) \cap M \backslash I \neq \phi$. $(3) \Rightarrow (1)$ Let H and K be Γ-ideals of M such that $H \cap K \subseteq I$. Suppose H and K are not contained in I, then there exist elements a, b such that $a \in H \backslash I$ and $b \in K \backslash I$. This implies that $a, b \in M \backslash I$. So by hypothesis $(\langle a\rangle \cap \langle b\rangle) \cap M \backslash I \neq \phi$ which implies that there exists an element $c \in \langle a\rangle \cap \langle b\rangle$ such that $c \in M \backslash I$. It shows that $c \in \langle a\rangle \cap \langle b\rangle \subseteq H \cap K \subseteq I$ which further implies that $H \cap K \nsubseteq I$. A contradiction. Hence either $H \subseteq I$ or $K \subseteq I$ and so I is strongly irreducible. ■

Definition 228 *A nonempty subset T of Γ-LA-ring M with left identity e is said to be subtractive if and only if $t \in T$ and $t + s \in T$ implies that $s \in T$ and $T$is called semi-subtractive if and only if $t \in T \cap V(M)$ implies that $-t \in T \cap V(M)$, where $V(M)$ is a set of all those elements of M having additive inverse.*

Proposition 229 *Let T be a subtractive subset of M with left identity e, then*

(1) *every subtractive subset of M contains 0 and*

(2) *every subtractive subset of M is semi-subtractive.*

Proof. (1) If $t \in T$ then $0 + t = t \in T$. Hence by definition $0 \in T$.

(2) Let T be a subtractive subset of M. Let $t \in T \cap V(M)$. This implies that $t \in T$ and $t \in V(M)$. Now as T is subtractive, so $t + (-t) = 0 \in T$. This implies that $-t \in T$ and also $-t \in V(M)$. So, $-t \in T \cap V(M)$. Hence T is semi-subtractive. ■

Proposition 230 *For subtractive and semi-subtractive left Γ-ideals of M, the following holds.*

(1) *Intersection of subtractive left Γ-ideals of M with left identity e is a subtractive left Γ-ideal of M.*

(2) *Intersection of semi-subtractive left Γ-ideals of M with left identity e is a semi-subtractive left Γ-ideal of M.*

Proof. The proof is obvious. ■

Proposition 231 *Every quasi-prime Γ-ideal of M with left identity e is semi-subtractive.*

Proof. Let I be a quasi-prime Γ-ideal of M and $a \in I \cap V(M)$. If $m \in M$ and $\alpha, \beta \in \Gamma$ then $(-a)\alpha(m\beta(-a)) + a\alpha(m\beta(-a)) = 0$ and so $(-a)\alpha(m\beta(-a)) = -a\alpha(m\beta(-a))$. But on the other hand $a\alpha(m\beta a) + a\alpha(m\beta(-a)) = 0$ which implies that $a\alpha(m\beta a) = -a\alpha(m\beta(-a))$. So by uniqueness of additive inverse, we have $(-a)\alpha(m\beta(-a)) = a\alpha(m\beta a)$. For all $m \in M$ if $(-a)\alpha(m\beta(-a)) = a\alpha(m\beta a) \in I$, then by Proposition 217, $-a \in I$ and also $-a \in V(M)$, which implies that $-a \in I \cap V(M)$. Hence I is semi-subtractive. ■

Since every quasi-prime Γ-ideal is surely quasi-semiprime, so following corollary is an immediate consequence of Proposition 231.

Corollary 232 *Every quasi-semiprime Γ-ideal of Γ-LA-ring M is semi-subtractive.*

Proof. The proof is analogous to the proof of Proposition 231 ■

Chapter 7

Fuzzy Concepts in Γ-AG-groupoids and LA-rings

Introduction

Earlier in chapter 5, we have introduced the notion of Γ-AG-groupoids and have studied this structure from different aspects. The aim of this study is to extend the results established in chapter 5, to fuzzy concepts. We have investigate fuzzy Γ-ideals and prime, semiprime fuzzy Γ-ideals of a Γ-AG-groupoid S. We have proved that, if S is a Γ-AG-groupoid with left identity e, then every fuzzy Γ-ideal of S is idempotent if and only if every fuzzy Γ-ideal of S is semiprime. We have also shown that, if S is a Γ-AG-groupoid with left identity, then every fuzzy Γ-ideal of S is prime if and only if every fuzzy Γ-ideal of S is idempotent and the set of fuzzy Γ-ideals of S is totally ordered by inclusion.

In [71], Zadeh introduced the notion of a fuzzy subset f of a set S as a function from S into $[0, 1]$. The notion of a fuzzy ideal in Γ-rings was first introduced by Jun and Lee [23]. They studied some preliminary properties of fuzzy ideals of Γ-rings. Dutta and Chanda [11], studied the structures of fuzzy ideals of a Γ-ring and characterize Γ-field, Noetherian Γ-ring, etc. with the help of fuzzy ideals via operator rings of Γ-ring. Jun [22] defined fuzzy prime ideal of a Γ-ring and obtained a number of characterizations for a fuzzy ideal to be a fuzzy prime ideal.

In 1981, the notion of Γ-semigroups was introduced by Sen (see [63, 64, 65]). Let M and Γ be any nonempty sets. If there exists a mapping $M \times \Gamma \times M \longrightarrow M$ written (a, γ, c) by $a\gamma c$, M is called a Γ-semigroup if M satisfies the identity $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. Whereas the Γ-semigroups are a generalization of semigroups. Many classical notions of semigroups have been extended to Γ-semigroups.

In this paper we define fuzzy Γ-ideals in a Γ-AG-groupoids and study some of its properties. We also define prime and semiprime fuzzy Γ-ideals in Γ-AG-groupoids and study those Γ-AG-groupoids in which each fuzzy Γ-ideal is (prime) semiprime.

7.1 Fuzzy Γ-ideals in Γ-AG-groupoids

We recall certain definitions and preliminaries from chapter 5, which are needed for our discussion.

Definition 233 *Let S and Γ be nonempty sets. We call S to be a Γ-AG-groupoid if there exists a mapping $S \times \Gamma \times S \longrightarrow S$, written as (a, γ, c) and denoted by $a\gamma c$ such that S satisfies the identity $(a\gamma b)\mu c = (c\gamma b)\mu a$ for all $a, b, c \in S$ and $\gamma, \mu \in \Gamma$.*

In a Γ-AG groupoid S the medial law holds, that is

$$(a\alpha b)\beta(c\gamma d) = (a\alpha c)\beta(b\gamma d) \text{ for all } a, b, c \in S \text{ and } \alpha, \beta, \gamma \in \Gamma.$$

An element e of a Γ-AG groupoid S is called left identity if $e\gamma a = a$ for all $a \in S$ and $\gamma \in \Gamma$.

A non-empty subset A of a Γ-AG groupoid S is called a sub Γ-AG groupoid of S if $a\gamma b \in A$ for all $a, b \in S$ and $\gamma \in \Gamma$.

An Γ-AG groupoid S whose all elements are idempotent, that is $a\gamma a = a$ for all $a \in S$ and $\gamma \in \Gamma$ is called a band.

In a Γ-AG band the following laws are true

1. $(a\alpha b)\beta a = a\alpha(b\beta a)$ for all $a, b \in S$ and $\alpha, \beta \in \Gamma$.
2. $(a\alpha b)\beta c = (a\alpha c)\beta(b\gamma c)$ for all $a, b, c \in S$ and $\alpha, \beta, \gamma \in \Gamma$.
3. $(a\alpha b)\beta b = b\alpha a$ for all $a, b \in S$ and $\alpha, \beta \in \Gamma$.

If S is Γ-AG groupoid and $A, B \subseteq S$ we denote

$$A\Gamma B := \{a\gamma b | a \in A, \gamma \in \Gamma, b \in B\}.$$

A non-empty subset I of a Γ-AG groupoid S is called a left (right) Γ-ideal of S if $S\Gamma I \subseteq I$ ($I\Gamma S \subseteq I$). A non-empty subset I of a Γ-AG groupoid S is called a Γ-ideal if it is both a left and right Γ-ideal of S. The intersection and union of a non-empty family of left (right) Γ-ideals of a Γ-AG groupoid is again a Γ-ideal of S. If S is a Γ-AG groupoid with left identity e, then every right Γ-ideal of S is a left Γ-ideal of S.

7.1.1 Fuzzy Subset of Γ-AG-groupoids

A function f from a non empty set X to the unit interval $[0, 1]$ is called a fuzzy subset of S. For fuzzy subsets f, g of X, $f \subseteq g$ means that $f(x) \leq g(x)$ for all $x \in X$. The fuzzy subsets $f \wedge g$ and $f \vee g$ of X are defined as

$$\begin{aligned} (f \wedge g)(x) &= f(x) \wedge g(x) = \min\{f(x), g(x)\}, \\ (f \vee g)(x) &= f(x) \vee g(x) = \max\{f(x), g(x)\} \text{ for all } x, y \in S. \end{aligned}$$

More generally, if $\{f_i | i \in I\}$ is a family of fuzzy subsets of X, then $\bigwedge\limits_{i \in I} f_i$ and $\bigvee\limits_{i \in I} f_i$ are defined as follows:

$$\begin{aligned} \left(\bigwedge_{i \in I} f_i\right)(x) &= \bigwedge_{i \in I} f_i(x) = \inf_{i \in I}\{f_i | i \in I\}, \\ \left(\bigvee_{i \in I} f_i\right)(x) &= \bigvee_{i \in I} f_i(x) = \sup_{i \in I}\{f_i | i \in I\} \end{aligned}$$

and will be the intersection and union of the family $\{f_i | i \in I\}$ of fuzzy subset of X.

Let f be a fuzzy subset and $t \in (0,1]$ then the set

$$\mathbb{U}(f;t) := \{x \in X | f(x) \geq t\}$$

is called the level set of f.

If S is an Γ-AG groupoid and f, g are any fuzzy subsets of S. We define the product $f\Gamma g$ of f and g as follows:

$$(f\Gamma g)(x) := \begin{cases} \bigvee\limits_{x=y\gamma z} \min\{f(y), g(z)\} \text{ if } \exists\ x, y \in S \text{ and } \gamma \in \Gamma \text{ such that } x = y\gamma z, \\ 0 \qquad \text{if } x \neq y\gamma z \end{cases}$$

7.1.2 Fuzzy Γ-ideals

Let S be a Γ-AG groupoid and $\emptyset \neq A \subseteq S$. Then the characteristic function χ_A of A is defined by:

$$\chi_A : S \longrightarrow [0,1], \longmapsto \chi_A(x) := \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A \end{cases}$$

Let S be a Γ-AG groupoid and f a fuzzy subset of S. Then f is called a fuzzy sub Γ-AG groupoid of S if $f(x\gamma y) \geq$min$\{f(x), f(y)\}$ for all $x, y \in S$ and $\gamma \in \Gamma$.

Let S be a Γ-AG groupoid and f a fuzzy subset of S. Then f is called a fuzzy left (right) Γ-ideal of S if $f(x\gamma y) \geq f(y)(f(x\gamma y) \geq f(x))$ for all $x, y \in S$ and $\gamma \in \Gamma$.

If f is both a fuzzy left Γ-ideal and a fuzzy right Γ-ideal of S. Then f is called a two-sided fuzzy Γ-ideal of S.

Note that if f is fuzzy right Γ-ideal of a Γ-AG groupoid S with left identity e. Then

$$f(x) = f(ex) \geq f(e) \text{ for all } x \in S.$$

Proposition 234 *Let S be a Γ-AG groupoid and $\emptyset \neq A \subseteq S$. Then A is a sub Γ-AG groupoid if and only if the characteristic function χ_A of A is a fuzzy sub Γ-AG groupoid.*

Proof. Suppose that A is a Γ-AG groupoid and $a, b \in S$, and $\gamma \in \Gamma$. If $a, b \in A$, then $a\gamma b \in A$ and we have $\chi_A(a\gamma b) = 1 =$min$\{1, 1\} =$min$\{\chi_A(a), \chi_A(b)\}$. If $a \notin A$ or $b \notin A$, then min$\{\chi_A(a), \chi_A(b)\} = 0 \leq \chi_A(a\gamma b)$. Hence χ_A is a fuzzy sub Γ-AG groupoid.

Conversely, assume that χ_A is a fuzzy sub Γ-AG groupoid. Let $a, b \in A$ and $\gamma \in \Gamma$, then $1 =$min$\{\chi_A(a), \chi_A(b)\} \leq \chi_A(a\gamma b)$. Thus $a\gamma b \in A$ and A is a sub Γ-AG groupoid. ■

Proposition 235 *Let S be a Γ-AG groupoid and $\emptyset \neq A \subseteq S$. Then A is a left (right) Γ-ideal of S if and only if the characteristic function χ_A of A is a fuzzy left (right) Γ-ideal of S.*

Proof. The proof is similar to Proposition 234. ■

Proposition 236 *Let S be a Γ-AG groupoid f a fuzzy subset of S. Then f if a fuzzy sub Γ-AG groupoid if and only if $U(f; t)(\neq \emptyset)$ is a sub Γ-AG groupoid for all $t \in (0, 1]$.*

Proof. Suppose that f is a fuzzy sub Γ-AG groupoid. Let $a, b \in U(f; t)(\neq \emptyset)$. Then $f(a) \geq t$ and $f(b) \geq t$. Since $f(a\gamma b) \geq$min$\{f(a), f(b)\} = t$. Hence $a\gamma b \in U(f; t)$ and $U(f; t)$ is a sub Γ-AG groupoid of S.

Conversely, assume that $U(f; t)(\neq \emptyset)$ is a sub Γ-AG groupoid of S for all $t \in (0, 1]$. Let $a, b \in S$ and $\gamma \in \Gamma$ such that $f(a\gamma b) <$min$\{f(a), f(b)\} = t$. Then $t \in (0, 1]$ and $a, b \in U(f; t)$ but $a\gamma b \notin U(f; t)$. This is a contradiction. Hence $f(a\gamma b) \geq$min$\{f(a), f(b)\}$ for all $a, b \in S$ and $\gamma \in \Gamma$. ■

Proposition 237 *Let S be a Γ-AG groupoid and f a fuzzy subset of S. Then f is a fuzzy left (right) Γ-ideal of S if and only if $U(f; t)(\neq \emptyset)$ is a left (right) Γ-ideal of S for all $t \in (0, 1]$.*

Proof. The proof is similar to Proposition 236. ■

Example 238 *Let $S = \{1, 2, 3, 4, 5\}$ and define a binary operation "$\cdot$" in S as follows:*

$\cdot$	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	5	3	4
4	1	1	4	5	3
5	1	1	3	4	5

Then $(S, \cdot)$ is an AG-groupoid. Now let $\Gamma = \{1\}$ and define mapping $S \times \Gamma \times S \longrightarrow S$ by $a1b = ab$ for all $a, b \in S$. Then S is a Γ-AG groupoid (see example 155 chapter 5) and $\{1\}, \{1,2\}, \{1,3,4,5\}$ and S are ideal of S.

Define $f : S \longrightarrow [0,1]$ by $f(1) = 0.9$, $f(2) = 0.8$, $f(3) = 0.5$, $f(4) = 0.5$, $f(5) = 0.5$.

$$\mathbb{U}(f;t) := \begin{cases} S & \text{if } t \in (0, 0.5] \\ \{1,2\} & \text{if } t \in (0.5, 0.8] \\ \{1\} & \text{if } t \in (0.8, 0.9] \\ \emptyset & \text{if } t \in (0.9, 1] \end{cases}$$

Then by Proposition 237, f is a fuzzy ideal of S.

Lemma 239 *If S is Γ-AG groupoid with left identity e. Then every fuzzy right Γ-ideal of S is a fuzzy left Γ-ideal of S.*

Proof. Let S be a Γ-AG groupoid with left identity e and f a fuzzy right Γ-ideal of S. Let $a, b \in S$ and $\alpha, \beta \in \Gamma$ then

$$\begin{aligned} f(a\alpha b) &= f((e\alpha a)\beta b) \text{ because } e \text{ is left identity} \\ &= f((b\alpha a)\beta e) \text{ by left invertive law} \\ &\geq f(b\alpha a) \text{ because } f \text{ is a fuzzy right } \Gamma\text{-ideal} \\ &\geq f(b) \text{ because } f \text{ is a fuzzy right } \Gamma\text{-ideal.} \end{aligned}$$

Thus $f(a\alpha b) \geq f(b)$ for all $a, b \in S$ and $\alpha, \beta \in \Gamma$. Hence f is a fuzzy left Γ-ideal. ■

Note that the intersection and union of any family of fuzzy Γ-ideals of a Γ-AG groupoid S is a fuzzy Γ-ideal of S.

Proposition 240 *If S is a Γ-AG groupoid and f, g, h are fuzzy subsets of S. Then*

$$(f\Gamma g)\Gamma h = (h\Gamma g)\Gamma f.$$

Proof. Let $x \in S$ and $\alpha, \beta \in \Gamma$, then

$$\begin{aligned}
((f\Gamma g)\Gamma h)(x) &= \bigvee_{x=y\alpha z} \{(f\Gamma g)(y) \wedge h(z)\} \\
&= \bigvee_{x=y\alpha z} \left\{ \left(\bigvee_{y=a\beta b} (f(a) \wedge g(b)) \right) \wedge h(z) \right\} \\
&= \bigvee_{x=(a\beta b)\alpha z} \{(f(a) \wedge g(b)) \wedge h(z)\} \\
\text{(since } x &= (a\beta b)\alpha z = (z\beta b)\alpha a \text{ by left invertive law)} \\
\text{so } ((f\Gamma g)\Gamma h)(x) &= \bigvee_{x=(z\beta b)\alpha a} [(h(z) \wedge g(b)) \wedge f(a)] \\
\text{(since } h(z) \wedge g(b) &\leq \bigvee_{z\alpha b=c\alpha d} \{h(c) \wedge g(d)\}) \\
\text{so, } ((f\Gamma g)\Gamma h)(x) &\leq \bigvee_{x=(z\beta b)\alpha a} \left[\left\{ \bigvee_{z\alpha b=c\alpha d} \{h(c) \wedge g(d)\} \right\} \wedge f(a) \right] \\
&= \bigvee_{x=(z\beta b)\alpha a} \{(h\Gamma g)(z\beta b) \wedge f(a)\} \\
&\leq \bigvee_{x=s\gamma t} \{(h\Gamma g)(s) \wedge f(t)\} \\
&= ((h\Gamma g)\Gamma f)(x).
\end{aligned}$$

Hence $(f\Gamma g)\Gamma h \leq (h\Gamma g)\Gamma f$. Similarly, $(h\Gamma g)\Gamma f \leq (f\Gamma g)\Gamma h$. Thus $(f\Gamma g)\Gamma h = (h\Gamma g)\Gamma f$.

■

Remark 241 *If S is a Γ-AG groupoid and $F(S)$ is the collection of all fuzzy subsets of S, then $(F(S), \Gamma)$ is a Γ-AG groupoid.*

Lemma 242 *Let S be an Γ-AG groupoid with left identity e and f, g, h be fuzzy subsets of S, then*

$$f\Gamma(g\Gamma h) = g\Gamma(f\Gamma h).$$

Proof. Let $x \in S$, then

$$\begin{aligned}
(f\Gamma(g\Gamma h))(x) &= \bigvee_{x=y\alpha z} [f(y) \wedge (g\Gamma h)(z)] \\
&= \bigvee_{x=y\alpha z} \left[f(y) \wedge \bigvee_{z=a\beta b} \{g(a) \wedge h(b)\} \right] \\
&= \bigvee_{x=y\alpha(a\beta b)} \{f(y) \wedge (g(a) \wedge h(b))\} \\
(\text{since } x &= y\alpha(a\beta b) \text{ implies } x = a\alpha(y\beta b) \text{ because } S \text{ has left identity}) \\
\text{So, } (f\Gamma(g\Gamma h))(x) &= \bigvee_{x=a\alpha(y\beta b)} \{g(a) \wedge (f(y) \wedge h(b))\} \\
(\text{since } f(y) \wedge h(b) &\leq \bigvee_{y\beta b=t\gamma s} (f(t) \wedge h(s))) \\
\text{So, } (f\Gamma(g\Gamma h))(x) &\leq \bigvee_{x=a\alpha(y\beta b)} \left[g(a) \wedge \left\{ \bigvee_{y\beta b=t\gamma s} (f(t) \wedge h(s)) \right\} \right] \\
&= \bigvee_{x=a\alpha(y\beta b)} [g(a) \wedge \{(f\Gamma h)(y\beta b)\}] \\
&\leq \bigvee_{x=p\alpha q} [g(p) \wedge \{(f\Gamma h)(q)\}] \\
&= (g\Gamma(f\Gamma h))(x).
\end{aligned}$$

Thus $f\Gamma(g\Gamma h) \leq g\Gamma(f\Gamma h)$.

Similarly, $g\Gamma(f\Gamma h) \leq f\Gamma(g\Gamma h)$. Therefore $f\Gamma(g\Gamma h) = g\Gamma(f\Gamma h)$. ■

Lemma 243 *Let S be a Γ-AG groupoid with left identity e and f a fuzzy right Γ-ideal of S, then $f\Gamma f$ is a fuzzy Γ-ideal of S.*

Proof. Since f is a fuzzy right Γ-ideal of S, by Proposition 235, f is a fuzzy left Γ-ideal of S.

Let $a, b \in S$. If $(f\Gamma f)(a) = 0$, then $(f\Gamma f)(ab) \geq (f\Gamma f)(a)$. Otherwise

$$\begin{aligned}
(f\Gamma f)(ab) &= \bigvee_{a=y\gamma z} \{f(y) \wedge f(z)\} \\
(\text{If } a &= y\gamma z, \text{ then } a\alpha b = (y\gamma z)\alpha b = (b\gamma z)\alpha y \text{ by left invertive law}) \\
\text{So, } (f\Gamma f)(ab) &= \bigvee_{a=y\gamma z} \{f(y) \wedge f(z)\} \\
&\leq \bigvee_{a=y\gamma z} \{f(b\gamma z) \wedge f(y)\} \text{ since } f \text{ is a fuzzy left } \Gamma\text{-ideal} \\
&\leq \bigvee_{a\alpha b=c\gamma d} \{f(c) \wedge f(d)\} = (f\Gamma f)(ab).
\end{aligned}$$

Thus $(f\Gamma f)(ab) \geq (f\Gamma f)(a)$. Hence $f\Gamma f$ is fuzzy right Γ-ideal of S and by Lemma 239, a fuzzy Γ-ideal of S. ■

Lemma 244 *If S is a Γ-AG groupoid and f, g are fuzzy Γ-ideals of S, then $f\Gamma g \subseteq f \cap g$.*

Proof. Let f and g be fuzzy Γ-ideals of S and $x \in S$. If $(f\Gamma g)(x) = 0$, then $(f\Gamma g)(x) \leq (f \cap g)(x)$, otherwise

$$\begin{aligned}
(f\Gamma g)(x) &= \bigvee_{x=y\alpha z} (f(y) \wedge g(z)) \\
&\leq \bigvee_{x=y\alpha z} (f(y) \wedge g(z))(f(yz) \wedge g(yz)) \text{ since } f \text{ and } g \text{ are fuzzy } \Gamma\text{-ideals of } S \\
&= \bigvee_{x=y\alpha z} (f(x) \wedge g(x)) \\
&= (f \cap g)(x).
\end{aligned}$$

Thus $f\Gamma g \subseteq f \cap g$. ■

Remark 245 *If S is a Γ-AG groupoid with left identity e and f and g are fuzzy right Γ-ideals of S, then $f\Gamma g \subseteq f\Gamma g$. If S is a Γ-AG groupoid and f a fuzzy Γ-ideal of S, then $f\Gamma f \subseteq f$.*

Lemma 246 *If S is a Γ-AG groupoid with left identity e and f, g are fuzzy Γ-ideals of S, then $f\Gamma g$ is a fuzzy Γ-ideal of S.*

Proof. Let f, g be fuzzy Γ-ideals of S and $a, b \in S$, and $\alpha, \beta, \gamma \in \Gamma$. If $(f\Gamma g)(a) = 0$, then $(f\Gamma g)(a) \leq (f\Gamma g)(ab)$, otherwise

$$\begin{aligned}
(f\Gamma g)(a) &= \bigvee_{a=c\gamma d} (f(c) \wedge g(d)) \\
(\text{since } a &= c\gamma d, \text{ so } a\beta b = (c\gamma d)\beta b = (c\gamma d)\beta(e\alpha b) = (c\gamma e)\beta(d\alpha b) \text{ by medial law}) \\
(f\Gamma g)(a) &\leq \bigvee_{a=c\gamma d} (f(c\gamma e) \wedge g(d\alpha b))(\text{ since } f \text{ and } g \text{ are fuzzy } \Gamma\text{-ideals}) \\
&\leq \bigvee_{a\beta b=x\gamma y} (f(x) \wedge g(y)) \\
&= (f\Gamma g)(a\beta b).
\end{aligned}$$

Thus $(f\Gamma g)(a\beta b) \geq (f\Gamma g)(a)$.

Therefore $f\Gamma g$ is a fuzzy right Γ-ideal of S and by Lemma 239, $f\Gamma g$ is a fuzzy Γ-ideal of S. ■

Remark 247 *Let S be a Γ-AG-groupoid with left identity e and let $FI(S)$ be the set of all fuzzy Γ-ideals of S, then $(FI(S), \Gamma)$ forms a Γ-AG-groupoid.*

7.1.3 Fuzzy points in Γ-AG-groupoids

Let S be a Γ-AG-groupoid and $x \in S$. Then for $a \in S$ and $t \in (0, 1]$, we define

$$a_t : S \longrightarrow [0,1], x \longmapsto a_t(x) := \begin{cases} t & \text{if } x = a \\ 0 & \text{otherwise} \end{cases}$$

Then a_t is a fuzzy subset of S and is called a fuzzy point with support t.

By $a_t \in f$, we mean $f(a) \geq t$.

Theorem 248 *Let S be a Γ-AG groupoid with left identity e and let it holds Γ-medial law . If f is a fuzzy left Γ-ideal of S, then $a_t\Gamma f$ is a fuzzy left Γ-ideal of S, where a_t is a fuzzy point of S.*

Proof. Suppose that f is a fuzzy left Γ-ideal of S. Let $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$. If $(a_t\Gamma f)(x) = 0$, then $(a_t\Gamma f)(x\gamma y) \geq (a_t\Gamma f)(x)$, otherwise

$$\begin{aligned}
(a_t\Gamma f)(x) &= \bigvee_{y=p\alpha q} \{a_t(p) \wedge f(q)\} \\
(\text{Since } y &= p\alpha q, \text{ so } x\gamma y = x\gamma(p\alpha q) \\
&= (e\beta x)\gamma(p\alpha q) = (e\beta p)\gamma(x\alpha q) = p\gamma(x\alpha q) \text{ by medial law}) \\
\text{Thus, } (a_t\Gamma f)(x) &= \bigvee_{y=p\alpha q} \{a_t(p) \wedge f(q)\} \\
&\leq \bigvee_{y=p\alpha q} \{a_t(p) \wedge f(x\alpha q)\} \text{ since } f \text{ is a fuzzy left } \Gamma\text{-ideal of } S \\
&\leq \bigvee_{x\gamma y=c\alpha d} \{a_t(c) \wedge f(d)\} \\
&= (a_t\Gamma f)(x\gamma y).
\end{aligned}$$

Thus $(a_t\Gamma f)(x\gamma y) \geq (a_t\Gamma f)(x)$. Consequently, $a_t\Gamma f$ is a fuzzy left Γ-ideal of S. ■

Let S be a Γ-AG groupoid and $a \in S$. A fuzzy left (two-sided) Γ-ideal f of S is called the fuzzy left (two-sided) Γ-ideal of S, generated by a_t for $t \in (0, 1]$ if f is the smallest fuzzy left (two-sided) Γ-ideal of S containing a_t.

Theorem 249 *Let S be a Γ-AG groupoid with left identity e and a_t a fuzzy point of S. Then the fuzzy left Γ-ideal of S, generated by a_t is l_{a_t}, defined by:*

$$l_{a_t} : S \longrightarrow [0, 1], x \longmapsto l_{a_t}(x) := \begin{cases} t \text{ if } x \in S\Gamma a \\ 0 \text{ otherwise} \end{cases}.$$

Proof. Let $x, y \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Case 1. If $y \notin S\Gamma a$, then $l_{a_t}(y) = 0 \leq l_{a_t}(x\gamma y)$.

Case 2. If $y \in S\Gamma a$, then $y = s\alpha a$ for some $a \in S$ and $\alpha \in \Gamma$. Hence

$$\begin{aligned}
x\gamma y &= x\gamma(s\alpha a) = (e\beta x)\gamma(s\alpha a) \\
&= ((s\alpha a)\beta x)\gamma e \text{ by left invertive law} \\
&= ((s\alpha a)\beta(e\delta x))\gamma e \\
&= ((s\alpha e)\beta(a\delta x))\gamma e \text{ by medial law} \\
&= (e\beta(a\delta x))\gamma(s\alpha e) \text{ by left invertive law} \\
&= (a\delta x)\gamma(s\alpha e) \\
&= ((s\alpha e)\delta x)\gamma a \in Sa.
\end{aligned}$$

Hence $l_{a_t}(y) = t = l_{a_t}(x\gamma y)$. Thus in any case $l_{a_t}(x\gamma y) \geq l_{a_t}(x)$. Consequently, l_{a_t} is a fuzzy left Γ-ideal of S.

Also by the definition of a_t, we have $a_t \leq l_{a_t}$.

Now let f be a fuzzy left Γ-ideal of S containing a_t.

Case 1. If $x \in S\Gamma a$, then $x = s\alpha a$ for some $s \in S$ and $\alpha \in \Gamma$ and so $l_{a_t}(x) = t$.

Also

$$\begin{aligned}
t &= a_t(a) \leq f(a) \\
&\Rightarrow t \leq f(a) \leq f(s\alpha a) = f(x) \\
&\Rightarrow f(x) \geq t = l_{a_t}(x).
\end{aligned}$$

Case 2. If $x \notin S\Gamma a$, then $l_{a_t}(x) = 0 \leq f(x)$.

Thus $l_{a_t} \subseteq f$ in any case. This shows that l_{a_t} is the smallest fuzzy left Γ-ideal of S containing a_t. ■

Theorem 250 *Let S be a Γ-AG-groupoid with left identity e and a_t a fuzzy point of S. Then* *the fuzzy right* Γ*-ideal* t_{a_t} *of* S*, generated by* a_t *is defined by:*

$$t_{a_t} : S \longrightarrow [0,1], x \longmapsto t_{a_t}(x) := \begin{cases} t \text{ if } x \in a\Gamma S \cup S\Gamma a, \\ 0 \qquad \text{otherwise} \end{cases}.$$

Proof. Let $x, y \in S$ and .

Case 1. If $x \notin a\Gamma S \cup S\Gamma a$, then $t_{a_t}(x) = 0 \leq t_{a_t}(x\gamma y)$.

Case 2. If $x \in a\Gamma S \cup S\Gamma a$, then $x \in a\Gamma S$ or $x \in S\Gamma a$. If $x \in a\Gamma S$, then $x = a\alpha c$ for some $c \in S$ and $\alpha \in \Gamma$. So $x\gamma y = (a\alpha c)\gamma y = (y\alpha c)\gamma a \in S\Gamma a$.

Let $x \in S\Gamma a$, then $x = s\beta a$ for some $s \in S$ and $\alpha \in \Gamma$ and so

$$\begin{aligned} x\gamma y &= (s\beta a)\gamma y = (s\beta a)\gamma(e\alpha y) \\ &= (s\beta e)\gamma(a\alpha y) \\ &= a\gamma((s\beta e)\alpha y) \in a\Gamma S \text{ because } a\alpha(b\beta c) = b\alpha(a\beta c). \end{aligned}$$

Hence $x\gamma y \in a\Gamma S \cup S\Gamma a$, and we have $t_{a_t}(x) = t = t_{a_t}(x\gamma y)$. Thus in any case $t_{a_t}(x\gamma y) \geq t_{a_t}(x)$. Therefore t_{a_t} is a fuzzy right Γ-ideal of S.

Also by definition by the definition of a_t, we have $a_t \leq t_{a_t}$.

Now let f be a fuzzy right Γ-ideal of S containing a_t. Then by Lemma 239, f is a fuzzy Γ-ideal of S. Then for any $x \in a\Gamma S \cup S\Gamma a$, we have $x = a\alpha s$ or $x = s\alpha a$ for some $s \in S$ and $\alpha \in \Gamma$. Hence $t_{a_t}(x) = t$.

If $x = a\alpha s$, then

$$t = a_t(a) \leq f(a) \leq f(a\alpha s) = f(x).$$

If $x = s\alpha a$, then

$$t = a_t(a) \leq f(a) \leq f(s\alpha a) = f(x), \text{ since } f \text{ is a fuzzy } \Gamma\text{-ideal of } S.$$

Hence $f(x) \geq t = t_{a_t}(x)$.

If $x \notin a\Gamma S \cup S\Gamma a$, then $t_{a_t}(x) = 0 \leq f(x)$. Thus, we have $t_{a_t} \subseteq f$. Consequently t_{a_t} is the smallest fuzzy Γ-ideal of S generated by a_t. ■

A Γ-AG-groupoid S is called is called regular if for every $a \in S$, there exists x in S and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$, or equivalently, $a \in (a\Gamma S)\Gamma a$.

For a regular Γ-AG-groupoid it is easy to see that $S\Gamma S = S$.

Proposition 251 *Every fuzzy right Γ-ideal of a regular Γ-AG-groupoid is a fuzzy left Γ-ideal of S.*

Proof. Let f be a fuzzy right Γ-ideal of S and $a, b \in S$ and $\gamma \in \Gamma$. Since S is regular, there exist $x \in S$, and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$. Then

$$\begin{aligned} f(a\gamma b) &= f(((a\alpha x)\beta a)\gamma b) \\ &= f((b\alpha a)\beta(a\gamma x)) \\ &\geq f(b\alpha a) \text{ since } f \text{ is a fuzzy right } \Gamma\text{-ideal of } S \\ &\geq f(b). \end{aligned}$$

Thus $f(a\gamma b) \geq f(b)$. Therefore f is a fuzzy left Γ-ideal of S. ■

Corollary 252 *In a regular Γ-AG groupoid S, every fuzzy right Γ-ideal of S is a fuzzy Γ-ideal of S.*

Lemma 253 *If f and g are fuzzy right Γ-ideals of a regular Γ-AG groupoid S, then $f\Gamma g = f \cap g$.*

Proof. Since S is regular, by Proposition 251, every fuzzy right Γ-ideal of S is a fuzzy Γ-ideal of S. Since S is regular, so for every $a \in S$ there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that

$a = (a\alpha x)\beta a$. Thus

$$\begin{aligned}(f \cap g)(a) &= f(a) \wedge g(a) \\ &\leq f(a\alpha x) \wedge g(a) \text{ since } f \text{ is a fuzzy right } \Gamma\text{-ideal} \\ &\leq \bigvee_{a=y\gamma z} (f(a) \wedge g(b)) \\ &= (f\Gamma g)(a).\end{aligned}$$

Thus $f \cap g \subseteq f\Gamma g$. On the other hand by Lemma 244, we have $f\Gamma g \subseteq f \cap g$. Therefore $f\Gamma g = f \cap g$. ■

Corollary 254 *Let f be a fuzzy right Γ-ideal of a regular Γ-AG groupoid S, then $f = f\Gamma f$.*

Remark 255 *It is clear that if S is a regular Γ-AG groupoid, then $FI(S)$ (the set of all fuzzy Γ-ideals) is a commutative semigroup.*

A fuzzy Γ-ideal f of a Γ-AG groupoid S is called prime (semiprime) if:

$g\Gamma h \subseteq f$ ($g\Gamma g \subseteq f$) implies $g \subseteq f$ or $h \subseteq f$ ($g \subseteq f$) for every fuzzy Γ-ideal g and h of S.

Note that every prime fuzzy Γ-ideal of a Γ-AG groupoid is semiprime.

A fuzzy Γ-ideal f of a Γ-AG groupoid S is called irreducible if:

$g \cap h \subseteq f$ implies $g \subseteq f$ or $h \subseteq f$ for every fuzzy Γ-ideal g and h of S.

Proposition 256 *A fuzzy Γ-ideal f of a Γ-AG groupoid S is prime if and only if it is both semiprime and irreducible.*

Proof. Suppose that f is a prime fuzzy Γ-ideal of S. Then clearly, f is semiprime. Let g and

h be fuzzy Γ-ideals of S such that $g \cap h \subseteq f$. As

$$\begin{aligned} g\Gamma h &\subseteq g \cap h \subseteq f \Rightarrow g\Gamma h \subseteq f \\ &\Rightarrow g \subseteq f \text{ or } h \subseteq f, \text{ since f is irreducible.} \end{aligned}$$

Hence f is prime fuzzy Γ-ideal of S. ■

A Γ-AG groupoid is called fully fuzzy prime if every fuzzy Γ-ideal of S is prime fuzzy Γ-ideal of S.

Theorem 257 *A regular Γ-AG groupoid S is fully fuzzy prime if and only if the set of all fuzzy Γ-ideals $FI(S)$ of S is totally ordered under inclusion.*

Proof. Suppose that S is fully fuzzy prime. Let f, g be fuzzy Γ-ideals of S. Then by Lemma 244, $f\Gamma g \subseteq f \cap g$. As $f \cap g$ is a prime fuzzy Γ-ideal of S, hence either $f \subseteq f \cap g$ or $g \subseteq f \cap g$. This implies that either $f \subseteq g$ or $g \subseteq f$. Hence $FI(S)$ is totally ordered under inclusion.

Conversely, assume that $FI(S)$ is totally ordered under inclusion. Let f, g, h be fuzzy Γ-ideals of S such that $g\Gamma h \subseteq f$. As $FI(S)$ is totally ordered under inclusion, so either $g \subseteq h$ or $h \subseteq g$. Suppose that $g \subseteq h$. Then $g = g\Gamma g \subseteq g\Gamma h \subseteq f$. It follows that $g \subseteq f$. Thus f is prime fuzzy Γ-ideal of S and hence S is fully fuzzy prime. ■

Theorem 258 *Every fuzzy Γ-ideal f in a regular Γ-AG groupoid S is prime if and only if f is irreducible.*

Proof. Let f be a prime fuzzy Γ-ideal of S. Let g, h be fuzzy Γ-ideals of S such that $g \cap h \subseteq f$. By Lemma 244,

$$\begin{aligned} g\Gamma h &\subseteq g \cap h \subseteq f \\ &\Rightarrow g \subseteq f \text{ or } h \subseteq f, \text{ since } f \text{ is prime.} \end{aligned}$$

Hence f is irreducible.

Conversely, assume that f is irreducible. Let g, h be fuzzy Γ-ideals of S such that $g\Gamma h \subseteq f$. Since S is regular, by Lemma 242, $g\Gamma h = g \cap h$. Thus $g \cap h \subseteq f$, since f is irreducible, we have $g \subseteq f$ or $h \subseteq f$. Hence f is prime. ■

A fuzzy subset f of a Γ-AG groupoid is called a Γ-AG band if all its elements are idempotent.

Lemma 259 *The concepts of fuzzy right and fuzzy left Γ-ideals in a Γ-AG band coincide.*

Proof. Let f be a fuzzy right Γ-ideal of S and $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Then

$$\begin{aligned} f(a\alpha b) &= f((a\alpha a)\beta b) \\ &= f((b\alpha a)\beta a) \text{ by left invertive law} \\ &\geq f(b\alpha a), \text{ since } f \text{ is a fuzzy right } \Gamma\text{-ideal} \\ &\geq f(b) \end{aligned}$$

Hence $f(a\alpha b) \geq f(b)$ and f is a fuzzy left Γ-ideal of S.

Conversely, assume that f is a fuzzy left Γ-ideal. Then for $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$, we have

$$\begin{aligned} f(a\alpha b) &= f((a\alpha a)\beta b) \\ &= f((b\alpha a)\beta a) \text{ by left invertive law} \\ &\geq f(a) \text{ since } f \text{ is fuzzy left } \Gamma\text{-ideal.} \end{aligned}$$

Hence f is a fuzzy right Γ-ideal of S. ■

Lemma 260 *Every fuzzy Γ-ideal of a Γ-AG-band is idempotent.*

Proof. Straightforward. ■

Theorem 261 *Every fuzzy Γ-ideal of a Γ-AG-band is prime if and only if $FI(S)$ is totally ordered under inclusion.*

Proof. Suppose that every fuzzy Γ-ideal of S is prime. Let f and g be fuzzy Γ-ideals of S, then $f \wedge g$ is a fuzzy Γ-ideal of S and hence prime. Thus

$$\begin{aligned} f\Gamma g &\leq f \wedge g \\ &\Rightarrow f \leq f \wedge g \text{ or } g \leq f \wedge g \\ &\Rightarrow f \leq g \text{ or } g \leq f. \end{aligned}$$

Hence $FI(S)$ is totally ordered under inclusion.

Conversely, suppose that $FI(S)$ is totally ordered by inclusion. Let $f, g,$ and h be fuzzy Γ-ideal of S such that $f\Gamma g \leq h$. Since $FI(S)$ is totally ordered under inclusion so either $f \leq g$ or $g \leq f$. Assume that $f \leq g$, since S is a band, so every fuzzy Γ-ideal is idempotent and hence

$$\begin{aligned} f &= f\Gamma f \leq f\Gamma g \leq h \\ &\Rightarrow f \leq h. \end{aligned}$$

Hence f is prime. ■

7.2 Regular and Intra-regular Γ-AG-groupoids characterized by the properties of fuzzy Γ-ideals.

In this section, we characterize regular and intra-regular Γ-AG-groupoids by the properties of fuzzy Γ-quasi-ideals, fuzzy Γ-interior ideals and fuzzy Γ-bi-ideals and fuzzy Γ-generalized bi-ideals.

7.2.1 Definitions and Preliminary Lemmas

Throughout this study S will denote a Γ-AG-groupoid unless otherwise stated. For definitions and lemmas we refer to chapter 5.

In the following we recall certain definitions and results from first section of this chapter.

Definition 262 *Let S be a Γ-AG groupoid and $\emptyset \neq A \subseteq S$. Then the characteristic function*

χ_A *of* A *is defined by:*

$$\chi_A : S \longrightarrow [0,1], \longmapsto \chi_A(x) := \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A \end{cases}$$

Definition 263 *A function* f *from a non empty set* X *to the unit interval* $[0,1]$ *is called a fuzzy subset of* S*. For fuzzy subsets* f, g *of* X*,* $f \subseteq g$ *means that* $f(x) \leq g(x)$ *for all* $x \in X$*. The fuzzy subsets* $f \wedge g$ *and* $f \vee g$ *of* X *are defined as*

$$\begin{aligned} (f \wedge g)(x) &= f(x) \wedge g(x) = \min\{f(x), g(x)\}, \\ (f \vee g)(x) &= f(x) \vee g(x) = \max\{f(x), g(x)\} \text{ for all } x, y \in S. \end{aligned}$$

If f, g *are any fuzzy subsets of* S*. We define the product* $f\Gamma g$ *of* f *and* g *as follows:*

$$(f\Gamma g)(x) := \begin{cases} \bigvee_{x=y\gamma z} \min\{f(y), g(z)\} \text{ if } \exists\ x, y \in S \text{ and } \gamma \in \Gamma \text{ such that } x = y\gamma z, \\ 0 \qquad\qquad \text{if } x \neq y\gamma z \end{cases}$$

Definition 264 *Let* S *be a* Γ*-AG groupoid and* f *a fuzzy subset of* S*. Then* f *is called a fuzzy sub* Γ*-AG groupoid of* S *if* $f(x\gamma y) \geq \min\{f(x), f(y)\}$ *for all* $x, y \in S$ *and* $\gamma \in \Gamma$ *and* f *is called a fuzzy left (right)* Γ*-ideal of* S *if* $f(x\gamma y) \geq f(y)(f(x\gamma y) \geq f(x))$ *for all* $x, y \in S$ *and* $\gamma \in \Gamma$*. If* f *is both a fuzzy left* Γ*-ideal and a fuzzy right* Γ*-ideal of* S*. Then* f *is called a two-sided fuzzy* Γ*-ideal of* S*.*

Lemma 265 *Let* S *be a* Γ*-*AG *groupoid with left identity* e*. Then every fuzzy right* Γ*-ideal of* S *is a fuzzy left* Γ*-ideal.*

Lemma 266 *If* S *is a* Γ*-*AG *groupoid with left identity* e *and* f, g *are fuzzy* Γ*-ideals of* S*, then* $f\Gamma g$ *is a fuzzy* Γ*-ideal of* S*.*

Lemma 267 *If* S *is a* Γ*-*AG *groupoid and* f, g *are fuzzy* Γ*-ideals of* S*, then* $f\Gamma g \subseteq f \cap g$*.*

Definition 268 *A Γ-AG-groupoid S is said to be a regular Γ-AG-groupoid if for each a in S there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$.*

7.2.2 Fuzzy ideals in regular Γ-AG-groupoids

We initiate with the following lemma:

Lemma 269 *In a regular Γ-AG-groupoid S, $f\Gamma g = f \cap g$, where f is a fuzzy right Γ-ideal and g is a fuzzy left Γ-ideal.*

Proof. Since $f\Gamma g \subseteq f \cap g$, so we only show that $f \cap g \subseteq f\Gamma g$. Let $a \in S$, then there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$. Now $(f\Gamma g)(a) = \vee_{a=y\beta z}\{f(y) \wedge g(z)\} \geq f(a\alpha x) \wedge g(a) \geq f(a) \wedge g(a) = f(a) \wedge g(a) = (f \cap g)(a)$. This implies $f \cap g \subseteq f\Gamma g$. Hence $f\Gamma g = f \cap g$. ■

Lemma 270 *Every fuzzy right Γ-ideal of a regular Γ-AG-groupoid is fuzzy Γ-idempotent.*

Proof. Let S be a regular Γ-AG-groupoid and f a fuzzy right Γ-ideal of S. Since $f\Gamma f \subseteq f$, so we only show that $f \subseteq f\Gamma f$. Let $a \in S$ then there exist an element x in S and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$. Now $(f\Gamma f)(a) = \vee_{a=y\gamma z}\{f(y) \wedge f(z)\} \geq f(a\alpha x) \wedge f(a) \geq f(a) \wedge f(a) = f(a)$. This implies $f \subseteq f\Gamma f$. ■

Definition 271 *Let A be a subset of Γ-AG-groupoid S. The characteristic function of A is denoted by C_A and defined by*

$$C_A(a) = \begin{cases} 1 \text{ if } a \in A \\ 0 \text{ otherwise} \end{cases}$$

We note that a Γ-AG-groupoid S can be considered a fuzzy Γ-subset of itself and write $S = C_S$, i.e., $S(x) = 1$ for all $x \in S$.

Proposition 272 *Let A and B be any non-empty subsets of Γ-AG-groupoids S. Then the following properties hold.*

Lemma 273 (1) *If $A \subseteq B$ then $C_A \subseteq C_B$.*

(2) $C_A \Gamma C_B = C_{A\Gamma B}$.

(3) $C_A \cup C_B = C_{A\cup B}$.

(4) $C_A \cap C_B = C_{A\cap B}$.

Proof. (1) Let a be any element of S. Suppose $a \in A$, this implies $a \in B$. Thus $C_A(a) = 1 = C_B(a)$. This implies $C_A \subseteq C_B$. If $a \notin A$, this implies $a \notin B$. This implies $C_A(a) = 0 = C_B(a)$. Thus $C_A \subseteq C_B$.

(2) Let x be any element of S. Suppose $x \in A\Gamma B$. This implies $x = a\alpha b$ for some $a \in A$, $b \in B$ and $\alpha \in \Gamma$. $(C_A\Gamma C_B)(x) = \vee_{x=y\gamma z}\{C_A(y) \wedge C_B(z)\} \geq C_A(a) \wedge C_B(b) = 1 \wedge 1 = 1 = C_{A\Gamma B}(x)$.

Now suppose $x \notin A\Gamma B$. This implies $x \neq a\alpha b$ for some $a \in A$, $b \in B$ and $\alpha \in \Gamma$. $(C_A\Gamma C_B)(x) = \vee_{x=y\gamma z}\{C_A(y) \wedge C_B(z)\} = 0 \wedge 0 = 0 = C_{A\Gamma B}(x)$. Hence $C_A\Gamma C_B = C_{A\Gamma B}$.

(3) Let a be any element of S. Suppose $a \in A \cup B$. Then there are three cases

(i) when $a \in A$ and $a \in B$. $(C_A \cup C_B)(a) = C_A(a) \vee C_B(a) = 1 \vee 1 = 1 = C_{A\cup B}(a)$.

(ii) when $a \in A$ and $a \notin B$. $(C_A \cup C_B)(a) = C_A(a) \vee C_B(a) = 1 \vee 0 = 1 = C_{A\cup B}(a)$.

(iii) when $a \notin A$ and $a \in B$. $(C_A \cup C_B)(a) = C_A(a) \vee C_B(a) = 0 \vee 1 = 1 = C_{A\cup B}(a)$.

If $a \notin A \cup B$. This implies $a \notin A$ and $a \notin B$. This implies $(C_A \cup C_B)(a) = C_{A\cup B}(a)$. Hence in all cases $C_A \cup C_B = C_{A\cup B}$.

(4) Let a be any element of S. Suppose $a \in A \cap B$. This implies $a \in A$ and $a \in B$. Now $(C_A \cap C_B)(a) = C_A(a) \wedge C_B(a) = 1 \wedge 1 = 1 = C_{A\cap B}(a)$. Suppose $a \notin A \cap B$. This implies $a \notin A$ and $a \notin B$. Now $(C_A \cap C_B)(a) = C_A(a) \wedge C_B(a) = 0 \wedge 0 = 0 = C_{A\cap B}(a)$. Hence $C_A \cap C_B = C_{A\cap B}$. ■

Definition 274 *A fuzzy Γ-subAG-groupoid f of a Γ-AG-groupoid S is called a fuzzy Γ-bi-ideal of S if $f((x\alpha y)\beta z) \geq f(x) \wedge f(z)$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.*

Lemma 275 *Every fuzzy right (two-sided) Γ-ideal of a Γ-AG-groupoid S is a fuzzy Γ-bi-ideal of S.*

Proof. Let f is a fuzzy right Γ-ideal of S. Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Now $f\left((x\alpha y)\beta z\right) = f\left(x\alpha y\right) \geq f\left(x\right)$ and $f((x\alpha y)\beta z) = f((z\alpha y)\beta x) \geq f(z\alpha y) \geq f(z)$. It follows that $f((x\alpha y)\beta z) \geq f(x) \wedge f(z)$. This implies f is a fuzzy Γ-bi-ideal of S. Similarly it is fairly easy to prove that every two sided fuzzy Γ-ideal is also a fuzzy Γ-bi-ideal of S. ■

Lemma 276 *Let S be a regular Γ-AG-groupoid. Then for every fuzzy Γ-bi-ideal f, $(f\Gamma S)\Gamma f = f$.*

Proof. Since $(f\Gamma S)\Gamma f \subseteq f$, because f is a fuzzy Γ-bi-ideal of S. Let $a \in S$. This implies there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$. Now

$$\begin{aligned} ((f\Gamma S)\Gamma f)(a) &= \vee_{a=y\gamma z}\left\{(f\Gamma S)\left(y\right) \wedge f\left(z\right)\right\} \geq (f\Gamma S)(a\alpha x) \wedge f(a) \\ &= \vee_{a\alpha x=p\delta q}\left\{(f\left(p\right) \wedge S\left(q\right)\right\} \wedge f(a) \geq f(a) \wedge S(x) \wedge f(a) = 1 \wedge f(a) = f(a) \end{aligned}$$

This implies $f \subseteq (f\Gamma S)\Gamma f$. Thus $(f\Gamma S)\Gamma f = f$. ■

Lemma 277 *Every fuzzy right Γ-ideal of a regular Γ-AG-groupoid is a fuzzy Γ-ideal of S.*

Proof. Let f is a fuzzy right Γ-ideal of S. Let $a, b \in S$. This implies there exist elements $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$. $f(a\gamma b) = f((a\alpha x)\beta a)\gamma b) = f((b\beta a)\gamma(a\alpha x)) \geq f(b\beta a) \geq f(b)$. This implies $f(a\gamma b) \geq f(b)$. This implies f is a fuzzy left Γ-ideal and hence f is a fuzzy Γ-ideal of S. ■

Definition 278 *A fuzzy subset f of Γ-AG-groupoid S is called a fuzzy Γ-interior ideal of S if $f\left((x\alpha y)\beta z\right) \geq f\left(y\right)$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.*

Lemma 279 *Let S be a regular Γ-AG-groupoid. Then any non-empty fuzzy subset f of S is a fuzzy Γ-interior ideal of S if and only if f is a fuzzy Γ-ideal of S.*

Proof. Suppose f is a fuzzy Γ-interior ideal of S. Let $a, b \in S$. This implies there exist elements $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = (a\alpha x)\beta a$. $f(a\gamma b) = f((a\alpha x)\beta a)\gamma b) = f((b\beta a)\gamma(a\alpha x)) \geq f(a)$. This implies $f(a\gamma b) \geq f(a)$. It follows that f is a fuzzy Γ-right ideal of S and hence by Lemma 277, f is a fuzzy Γ-ideal of S. Conversely, let f be a fuzzy Γ-ideal of S. Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Now $f\left((x\alpha y)\beta z\right) = f\left(x\alpha y\right) \geq f\left(y\right)$. Thus f is a fuzzy interior Γ-ideal of S. ■

Remark 280 *Fuzzy interior Γ-ideal and fuzzy Γ-ideal coincide if S is regular Γ-AG-groupoid.*

Definition 281 *A fuzzy subset f of Γ-AG-groupoid S is called a fuzzy Γ-quasi-ideal of S if $(f\Gamma S) \cap (S\Gamma f) \subseteq f$.*

Proposition 282 *In a regular Γ-AG-groupoid S with left identity e, $(f\Gamma S) \cap (S\Gamma f) = f$ for every fuzzy right Γ-ideal f of S.*

Proof. Let f be a fuzzy right Γ-ideal of S. This implies $(f\Gamma S) \cap (S\Gamma f) \subseteq f$, because every fuzzy right Γ-ideal is fuzzy Γ-quasi-ideal. Let $a \in S$. This implies there exist $x \in S$ and $\alpha, \beta \in \Gamma$, such that $a = (a\alpha x)\beta a$. $(f\Gamma S)(a) = \vee_{a=y\gamma z}\left\{f\left(y\right) \wedge S\left(z\right)\right\} \geq f(a\alpha x) \wedge S(a) = f(a\alpha x) \wedge 1 = f(a\alpha x) \geq f(a)$. This implies $f \subseteq f\Gamma S$. This implies $(S\Gamma f)(a) = \vee_{a=l\delta m}\left\{S\left(l\right) \wedge f\left(m\right)\}\right\} \geq S(a\alpha x) \wedge f(a) = 1 \wedge f(a) = f(a).f \subseteq S\Gamma f$

$$\begin{aligned}(S\Gamma f)(a) &= \vee_{a=l\delta m}\left\{S\left(l\right) \wedge f\left(m\right)\}\right\} \geq S(a\alpha x) \wedge f(a) = 1 \wedge f(a) = f(a) \\ f &\subseteq S\Gamma f \\ &\Rightarrow f \subseteq (f\Gamma S) \cap (S\Gamma f).\end{aligned}$$

Hence $(f\Gamma S) \cap (S\Gamma f) = f$. ■

Lemma 283 *Let S be a groupoid with left identity e such that Γ-medial law holds. Then every fuzzy Γ-quasi-ideal is a fuzzy Γ-bi-ideal of S.*

Proof. Let f is a fuzzy quasi-ideal of S. Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. This implies $f((x\alpha y)\beta z) \geq ((f\Gamma S) \cap (S\Gamma f))((x\alpha y)\beta z) = (f\Gamma S)((x\alpha y)\beta z) \wedge (S\Gamma f)((x\alpha y\beta)z)$, since f is a quasi-ideal of S. Now $(S\Gamma f)((x\alpha y)\beta z)=\vee_{(x\alpha y)\beta z=l\gamma m}\{S(l)\wedge f(m)\}=\vee_{(x\alpha y)\beta z=l\gamma m}\{1\wedge f(m)\}\geq f(z)$. This implies $(S\Gamma f)((x\alpha y)\beta z)\geq f(z)$. $(f\Gamma S)((x\alpha y)\beta z)=\vee_{(x\alpha y)\beta z=p\gamma q}\{f(p)\wedge S(q)\}=\vee_{(x\alpha y)\beta z=p\gamma q}\{f(p)\wedge 1\}$.

Now $(x\alpha y)\beta z=(x\alpha y)\beta(e\delta z)=(x\alpha e)\beta(y\delta z)\in(x\alpha e)\Gamma S=x\Gamma S$. This implies $(x\alpha y)\beta z\in x\Gamma S$, so $(x\alpha y)\beta z=x\eta r$ for some $r\in S$ and $\eta\in\Gamma$. This implies $(f\Gamma S)((x\alpha y)\beta z)=\vee_{(x\alpha y)\beta z=x\eta r=pq}\{f(p)\wedge S(q)\}=\vee_{x\eta r=pq}\{f(p)\wedge 1\}\geq f(x)$. This implies$(f\Gamma S)((x\alpha y)\beta z)\geq f(x)$. Thus $f((x\alpha y)\beta z)\geq(f\Gamma S)((x\alpha y)\beta z)\wedge(S\Gamma f)((x\alpha y)\beta z)\geq f(x)\wedge f(z)$. This implies $f((x\alpha y)\beta z)\geq f(x)\wedge f(z)$. This implies f is a fuzzy Γ-bi-ideal of S. ■

Theorem 284 *Let S be a Γ-AG-groupoid with left identity e. Then the following conditions are equivalent.*

(1) *S is regular.*

(2) *$f\cap g=f\Gamma g$ for every fuzzy right Γ-ideal f and every fuzzy left Γ-ideal g of S.*

(3) *$h=(h\Gamma S)\Gamma h$ for every fuzzy Γ-quasi-ideal h of S.*

Proof. $(1)\Rightarrow(2)$. By Lemma 269. $(1)\Rightarrow(3)$. Let $a\in S$. This implies there exist elements x in S and α,β in Γ such that $a=(a\alpha x)\beta a$. Now

$$\begin{aligned}((h\Gamma S)\Gamma h)(a) &= \vee_{a=y\gamma z}\{(h\Gamma S)(y)\wedge h(z)\}\geq(h\Gamma S)(a\alpha x)\wedge h(a)\\ &= (\vee_{a\alpha x=l\delta m}\{h(l)\wedge S(m)\})\wedge h(a)\geq h(a)\wedge S(x)\wedge h(a)=h(a)\\ &\Rightarrow h\subseteq(h\Gamma S)\Gamma h.\end{aligned}$$

Since every fuzzy Γ-quasi-ideal is fuzzy Γ-bi-ideal of S by lemma 283. This implies $(h\Gamma S)\Gamma h\subseteq h$. Hence it follows that $h=(h\Gamma S)\Gamma h$. $(3)\Rightarrow(2)$. Let f is a fuzzy right Γ-ideal and g is a fuzzy left Γ- ideal of S. Then obviously f and g are fuzzy Γ-quasi-ideals of S. Since intersection of any two fuzzy Γ-quasi-ideals of S is also a fuzzy Γ-quasi-ideal of S by, so $f\cap g$ is a fuzzy Γ-quasi-ideal of S. $f\cap g=((f\cap g)\Gamma S)\Gamma(f\cap g)\subseteq(f\Gamma S)\Gamma g\subseteq f\Gamma g$. This implies $f\cap g\subseteq f\Gamma g$. Since $f\Gamma g\subseteq f\cap g$. Hence $f\Gamma g=f\cap g$. $(2)\Rightarrow(1)$. Let $a\in S$. Since $S\Gamma a$ is a left Γ-ideal of S generated by a and $a\Gamma S\cup S\Gamma a$ is a right Γ-ideal of S containing

a. This implies $C_{S\Gamma a}$ and $C_{a\Gamma S\cup S\Gamma a}$ are fuzzy Γ-left and fuzzy Γ-right ideals of S respectively and by hypothesis $C_{a\Gamma S\cup S\Gamma a} \cap C_{S\Gamma a} = C_{a\Gamma S\cup S\Gamma a}\Gamma C_{S\Gamma a}$. So by Proposition 272, we have $C_{(a\Gamma S\cup S\Gamma a)\cap S\Gamma a} = C_{(a\Gamma S\cup S\Gamma a)S\Gamma a}$. Consequently $(a\Gamma S \cup S\Gamma a) \cap S\Gamma a = (a\Gamma S \cup S\Gamma a)S\Gamma a$. Now as $a \in (a\Gamma S \cup S\Gamma a) \cap S\Gamma a$, so $a \in (a\Gamma S \cup S\Gamma a)S\Gamma a$. This implies a is regular and hence S is regular. ■

Theorem 285 *Let S be a Γ-AG-groupoid with left identity e. Then the following conditions are equivalent.*

(1) *S is regular.*

$(2\ )$ *$f = (f\Gamma S)\Gamma f$ for every quasi-ideal f of S.*

(3) *$f = (f\Gamma S)\Gamma f$ for every bi-ideal f of S.*

Proof. $(1) \Rightarrow (3)$. Straight forward. $(3) \Rightarrow (2)$. Because every fuzzy Γ-quasi -ideal is fuzzy Γ-bi-ideal by Lemma 283. $(2) \Rightarrow (1)$. By Theorem 284. ■

Theorem 286 *Let S be a Γ-AG-groupoid with left identity e. Then the following conditions are equivalent.*

(1) *S is regular.*

(2) *$f \cap g = (f\Gamma g)\Gamma f$ for every fuzzy Γ-quasi-ideal f and every fuzzy Γ-ideal g of S.*

(3) *$h \cap k = (h\Gamma k)\Gamma h$ for every fuzzy Γ-bi-ideal h and every fuzzy Γ-ideal k of S.*

Proof. $(1) \Rightarrow (3)$. Let h be a fuzzy Γ-bi-ideal and k be a fuzzy Γ-ideal of S. Then $(h\Gamma k)\Gamma h \subseteq (S\Gamma k)\Gamma S \subseteq k\Gamma S \subseteq k$ and $(h\Gamma k)\Gamma h \subseteq (h\Gamma S)\Gamma h \subseteq h$. It follows that $(h\Gamma k)\Gamma h \subseteq h \cap k$.

Let $a \in S$ and $\alpha, \beta \in \Gamma$. Then $a = (a\alpha x)\beta a = (((a\alpha x)\beta a)\alpha x)\beta a = ((x\beta a)\alpha(a\alpha x))\beta a = (a\alpha((x\beta a)\alpha x))\beta a$. Now

$$\begin{aligned}
((h\Gamma k)\Gamma h)(a) &= \vee_{a=l\eta m}\left\{(h\Gamma k)(l) \wedge h(m)\right\} \\
&\geq (h\Gamma k)(a\alpha((x\beta a)\alpha x)) \wedge h(a) \\
&= (\vee_{a\alpha((x\beta a)\alpha x)=y\delta z}\left\{h(y) \wedge k(z)\right\}) \wedge h(a) \\
&\geq h(a) \wedge k((x\beta a)\alpha x) \wedge h(a) \geq h(a) \wedge k(a) \\
&= (h \cap k)(a) \\
&\Rightarrow h \cap k \subseteq (h\Gamma k)\Gamma h
\end{aligned}$$

Consequently $h \cap k = (h\Gamma k)\Gamma h.(3) \Rightarrow (2)$. Straight forward because every fuzzy Γ-quasi-ideal of S is a fuzzy Γ-bi-ideal of $S.(2) \Rightarrow (1)$. Since S is fuzzy Γ-ideal,so $f \cap S = (f\Gamma S)\Gamma f$. It follows that $f = (f\Gamma S)\Gamma f$, where f is a fuzzy Γ-quasi-ideal of S. Hence by Theorem 284, S is regular. ■

Theorem 287 *Let S be a Γ-AG-groupoid with left identity e. Then the following conditions are equivalent.*

(1) *S is regular.*

(2) *$f \cap g \subseteq g\Gamma f$ for every fuzzy Γ-quasi-ideal f and every fuzzy right Γ-ideal g of S.*

(3) *$h \cap g \subseteq g\Gamma h$ for every fuzzy Γ-bi-ideal h and every fuzzy right Γ-ideal g of S.*

(4) *$k \cap g \subseteq g\Gamma k$ for every fuzzy generalized Γ-bi-ideal k and every fuzzy right Γ-ideal g of S.*

Proof. $(1) \Rightarrow (4)$. Let $a \in S$ and $\alpha, \beta \in \Gamma$, then by hypothesis, $a = (a\alpha x)\beta a$. Now $(g\Gamma k)(a) = \vee_{a=l\gamma m}\left\{(g)(l) \wedge k(m)\right\} \geq g(a\alpha x) \wedge k(a) \geq g(a) \wedge k(a) = k(a) \wedge g(a)$. This implies $(k \wedge g)(a)$ and consequently $k \cap g \subseteq g\Gamma k$ for every fuzzy generalized Γ-bi-ideal k and every fuzzy right Γ-ideal g of S. $(4) \Rightarrow (3) \Rightarrow (2)$ are straight forward.

$(2) \Rightarrow (1)$. Let f be a fuzzy right Γ-ideal and g be a fuzzy left Γ-ideal of S.Then clearly $g\Gamma f \subseteq f \cap g$. Also since every fuzzy left Γ- ideal is fuzzy Γ-quasi-ideal of S, so by hypothesis

$f \cap g \subseteq g\Gamma f$. It follows that $f \cap g = g\Gamma f$ for every fuzzy right Γ-ideal f and g is a fuzzy left Γ-ideal of S. This implies that $g \cap f = g\Gamma f$. Hence S is regular 284. ■

7.2.3 Fuzzy ideals in intra-regular Γ-AG-groupoids

Definition 288 *A Γ-AG-groupoid S is called intra-regular if for each $a \in S$, there exist x, $y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (x\alpha a)\beta(a\gamma y)$.*

Lemma 289 *In an intra-regular Γ-AG-groupoid S, every fuzzy Γ-ideal is fuzzy Γ-idempotent.*

Proof. Let f be a any fuzzy Γ-ideal of S. Since $f\Gamma f \subseteq f$. We only show that $f \subseteq f\Gamma f$. Let $a \in S$. So by definition $a = (x\alpha a)\beta(a\gamma y)$. Now $(f\Gamma f)(a) = \vee_{a=y\delta z} \{f(y) \wedge f(z)\} \geq f(x\alpha a) \wedge f(a\gamma y) \geq f(a) \wedge f(a) = f(a)$. This implies $f \subseteq f\Gamma f$. Hence $f = f\Gamma f$. ■

Lemma 290 *Let S be an intra-regular Γ-AG-groupoid S, then $g \cap f \subseteq f\Gamma g$ for every fuzzy left Γ-ideal f and for every fuzzy right Γ-ideal g of S.*

Proof. Let $a \in S$. This implies there exist x, $y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a = (x\alpha a)\beta(a\gamma y)$. Now $(f\Gamma g)(a) = \vee_{a=y\delta z} \{f(y) \wedge g(z)\} \geq f(x\alpha a) \wedge g(a\gamma y) \geq f(a) \wedge g(a) = g(a) \wedge f(a) = (g \cap f)(a)$. This implies $g \cap f \subseteq f\Gamma g$. ■

Lemma 291 *Every fuzzy right Γ-ideal of an intra-regular Γ-AG-groupoid S is a fuzzy Γ-ideal of S.*

Proof. Let f be a fuzzy right Γ-ideal of S. Let $a, b \in S$. Then by definition, $a = (x\alpha a)\beta(a\gamma y)$. Now $f(a\delta b) = f(((x\alpha a)\beta(a\gamma y))\delta b) = f((b\beta(a\gamma y))\delta(x\alpha a)) \geq f(b\beta(a\gamma y)) \geq f(b)$.

This implies f is a fuzzy left Γ-ideal of S and consequently f is a fuzzy Γ-ideal of S. ■

Theorem 292 *Let S be an intra-regular Γ-AG-groupoid with left identity e. Then any non-empty fuzzy subset f of S is a fuzzy Γ-interior ideal of S if and only if f is a fuzzy Γ-ideal of S.*

Proof. Suppose f is a fuzzy Γ-interior ideal of S. Let $a, b \in S$. Then there exist element $x,\ y \in S$ and $\alpha, \beta, \gamma \in \Gamma$, such that $a = (x\alpha a)\beta(a\gamma y)$. $f(a\delta b) = f((a\alpha x)\beta(a\gamma y))\delta b) = f((b\beta(a\gamma y))\delta(a\alpha x)) \geq f(a\gamma y) \geq f((e\eta a)\gamma y) \geq f(a)$. This implies $f(a\delta b) \geq f(a)$. It follows that f is a fuzzy right Γ-ideal of S and hence by Lemma 291, f becomes a fuzzy Γ-ideal of S. Conversely, let f be a fuzzy Γ-ideal of S. Let $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Now $f\left((x\alpha y)\beta z\right) = f\left(x\alpha y\right) \geq f\left(y\right)$. Thus f is a fuzzy interior Γ-ideal of S. ■

Remark 293 *In an intra-regular Γ-AG-groupoid S, fuzzy Γ-interior ideal and fuzzy Γ-ideal coincide.*

Theorem 294 *Let S be a Γ-AG-groupoid with left identity e. Then the following conditions are equivalent.*

(1) *S is an intra-regular.*

(2) *$g \cap f \subseteq f\Gamma g$ for every fuzzy left Γ-ideal f and for every fuzzy right Γ-ideal g of S.*

Proof. $(1) \Rightarrow (2)$. Straight forward by lemma 290. $(2) \Rightarrow (1)$. Let $a \in S$, then $S\Gamma a$ is a left Γ-ideal of S generated by a and $a\Gamma S \cup S\Gamma a$ is the right Γ-ideal of S containing a. It follows that $C_{S\Gamma a}$ and $C_{a\Gamma S\cup S\Gamma a}$ are fuzzy Γ-left and fuzzy Γ-right ideals of S respectively. So by hypothesis $C_{a\Gamma S\cup S\Gamma a} \cap C_{S\Gamma a} \subseteq C_{S\Gamma a} \circ C_{a\Gamma S\cup S\Gamma a}$. Then by Proposition 272, we have $C_{(a\Gamma S\cup S\Gamma a)S\Gamma a} \subseteq C_{S\Gamma a(a\Gamma S\cup S\Gamma a)}$. It follows that $(a\Gamma S \cup S\Gamma a) \cap S\Gamma a \subseteq S\Gamma a(a\Gamma S \cup S\Gamma a)$. Since $a \in (a\Gamma S \cup S\Gamma a) \cap S\Gamma a$, so $a \in S\Gamma a(a\Gamma S \cup S\Gamma a)$. This implies a is intra-regular. Hence S is intra-regular. ■

Theorem 295 *Let S be a Γ-AG-groupoid with left identity e. Then the following conditions are equivalent.*

(1) *S is intra-regular.*

(2) *$f \cap g = (f\Gamma g)\Gamma f$ for every fuzzy Γ-quasi-ideal f and for every fuzzy Γ- ideal g of S.*

(3) *$h \cap k = (h\Gamma k)\Gamma h$ for every fuzzy Γ-bi-ideal h and for every fuzzy Γ-ideal k of S.*

Proof. $(1) \Rightarrow (3)$. Let $x \in S$, so there exist $s,\ t \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $x = (s\alpha x)\beta(x\gamma t)$. Consider

$$\begin{aligned}
x\gamma t &= ((s\alpha x)\beta(x\gamma t))\gamma t = (x\beta((s\alpha x)\gamma t))\gamma t = (x\beta((s\alpha x)\gamma t))\gamma(e\eta t) \\
&= (x\beta e)\gamma(((s\alpha x)\gamma t)\eta t) = (x\beta e)\gamma((t\gamma t)\eta(s\alpha x)) = (x\beta e)\gamma(((e\eta t)\gamma t)\eta(s\alpha x) \\
&= (x\beta e)\gamma(((t\eta t)\gamma e)\eta(s\alpha x)) = (x\beta e)\gamma(((t\eta t)\gamma s)\eta(e\alpha x) = (x\beta e)\gamma(((t\eta t)\gamma s)\eta x) \\
&= (x\beta((t\eta t)\gamma s))\gamma(e\eta x) = (x\beta((t\eta t)\gamma s))\gamma x
\end{aligned}$$

and

$$\begin{aligned}
s\alpha x &= s\alpha((s\alpha x)\beta(x\gamma t)) = s\alpha(x\beta((s\alpha x)\gamma t)) \\
&= (e\eta s)\alpha(x\beta((s\alpha x)\gamma t)) = (e\eta x)\beta(s\alpha((s\alpha x)\gamma t)) \\
&= x\beta(s\alpha((s\alpha x)\gamma t))
\end{aligned}$$

Now

$$\begin{aligned}
((h\Gamma k)\Gamma h)(x) &= \vee_{x=y\gamma z}\left\{(h\Gamma k)(y) \wedge h(z)\right\} \\
&\geq (h\Gamma k)(s\alpha x) \wedge h(x\gamma t) \\
&= (\vee_{s\alpha x=l\varepsilon m}\left\{h(l) \wedge k(m)\right\}) \wedge h((x\beta((t\eta t)\gamma s))\gamma x) \\
&\geq h(x) \wedge k(s\alpha((s\alpha x)\gamma t)) \wedge h(x) \text{ since } h \text{ is a fuzzy } \Gamma\text{-bi-ideal.} \\
&\geq h(x) \wedge k(x) = h \cap k(x) \text{ since } k \text{ is a fuzzy } \Gamma\text{-ideal.} \\
&\Rightarrow h \cap k \subseteq (h\Gamma k)\Gamma h
\end{aligned}$$

Also

$$\begin{aligned}(h\Gamma k)\Gamma h &\subseteq (S\Gamma k)\Gamma S \subseteq k\Gamma S \subseteq k \\ &\Rightarrow (h\Gamma k)\Gamma h \subseteq k \\ (h\Gamma k)\Gamma h &\subseteq (h\Gamma S)\Gamma h \subseteq h \\ &\Rightarrow (h\Gamma k)\Gamma h \subseteq h \\ &\Rightarrow (h\Gamma k)\Gamma h \subseteq h \cap k.\end{aligned}$$

Hence $h \cap k = (h\Gamma k)\Gamma h$. $(3) \Rightarrow (2)$. Straight forward because every fuzzy Γ-quasi-ideal of S is a fuzzy Γ-bi-ideal of S. $(2) \Rightarrow (1)$. Let E be a fuzzy right Γ-ideal and J be a fuzzy two-sided Γ-ideal of S. Since every fuzzy right Γ-ideal of S is a fuzzy Γ-quasi-ideal of S. Then by hypothesis $E \cap J = (E\Gamma J)\Gamma E \subseteq (S\Gamma J)\Gamma E \subseteq J\Gamma E$. Since E is a fuzzy right Γ-ideal and J is also a fuzzy left Γ-ideal. Hence S is intra-regular by Theorem 294. ■

Theorem 296 *Let S be a Γ-AG-groupoid with left identity e. Then the following conditions are equivalent.*

(1) *S is intra-regular.*

(2) *$f \cap g \subseteq g\Gamma f$ for every fuzzy Γ-quasi-ideal f and every fuzzy left Γ-ideal g of S.*

(3) *$h \cap g \subseteq g\Gamma h$ for every fuzzy Γ-bi-ideal h and every fuzzy left Γ-ideal g of S.*

Proof. $(1) \Rightarrow (3)$. Let h be fuzzy Γ-bi-ideal and g be a fuzzy left Γ-ideal of S. Let $x \in S$. Then by definition $x = (s\alpha x)\beta(x\gamma t)$. Now

$$\begin{aligned}x\gamma t &= ((s\alpha x)\beta(x\gamma t))\delta t = (x\beta((s\alpha x)\gamma t))\delta t \\ &= (x\beta((s\alpha x)\gamma t))\delta(e\eta t) = (x\beta e)\delta(((s\alpha x)\gamma t)\eta t),\ \Gamma\text{-medial law} \\ &= (x\beta e)\delta((t\gamma t)\eta(s\alpha x)) = (x\beta e)\delta(((e\eta t)\gamma t)\eta(s\alpha x)) \\ &= (x\beta e)\delta(((t\eta t)\gamma e)\eta(s\alpha x)) = (x\beta e)\delta(((t\eta t)\gamma s)\eta(e\alpha x)) \\ &= (x\beta e)\delta(((t\eta t)\gamma s)\eta x) = (x\beta((t\eta t)\gamma s))\delta(e\eta x) \\ &= (x\beta((t\eta t)\gamma s))\delta x\end{aligned}$$

Now

$$\begin{aligned}(g\Gamma h)(x) &= \vee_{x=a\mu b}\{g(a)\wedge h(b)\}\geq g(s\alpha x)\wedge h((x\beta((t\eta t)\gamma s))\delta x) \\ &\geq g(x)\wedge h(x)=h(x)\wedge g(x)=(h\cap g)(x) \\ &\Rightarrow h\cap g\subseteq g\Gamma h\end{aligned}$$

$(3)\Rightarrow(2)$. Obvious because every fuzzy Γ-quasi-ideal of S is fuzzy Γ-bi-ideal of S. $(2)\Rightarrow(1)$. Let R be a fuzzy right Γ-ideal of S and L be a fuzzy left Γ-ideal of S. Since every fuzzy right Γ-ideal of S is fuzzy Γ-quasi-ideal of S. Consequently R is a fuzzy Γ-quasi-ideal of S. So by hypothesis $R\cap L\subseteq L\Gamma R$. Hence S is intra-regular by 294. ■

7.2.4 Fuzzy idempotent ideals in regular and intra-regular Γ-AG-groupoids.

Theorem 297 *Let S be a Γ-AG-groupoid with left identity e. Then the following conditions are equivalent.*

(1) *S is Regular and intra-regular.*

(2) *$f\Gamma f=f$ for all fuzzy Γ-bi-ideals of S.*

(3) *$f_1\cap f_2=(f_1\Gamma f_2)\cap(f_2\Gamma f_1)$ for all fuzzy Γ-bi-ideals f_1,f_2 of S.*

Proof. $(1)\Rightarrow(2)$. Let $x\in S$. Since S is regular, so there exist elements $a\in R$ and $\alpha,\beta\in\Gamma$ such that $x=(x\alpha a)\beta x$, also S is intra-regular, so by definition $x=(s\alpha x)\beta(x\gamma t)$. Now

$x=(x\alpha a)\beta x=(x\alpha a)\beta((x\alpha a)\beta x)$.

$$\begin{aligned}
\text{Also } x\alpha a &= ((s\alpha x)\beta(x\gamma t))\alpha a, \text{ since } x=(s\alpha x)\beta(x\gamma t)\\
&= ((s\alpha x)\beta(x\gamma t))\alpha(e\eta a)=((s\alpha x)\beta e)\alpha((x\gamma t)\eta a)\\
&= ((e\alpha x)\beta s)\alpha((x\gamma t)\eta a)=(x\beta s)\alpha((x\gamma t)\eta a)\\
&= (x\beta s)\alpha((((x\alpha a)\beta x)\gamma t)\eta a)=(x\beta s)\alpha(((t\beta x)\gamma(x\alpha a))\eta a)\\
&= (x\beta s)\alpha((a\gamma(x\alpha a))\eta(t\beta x))=(x\beta s)\alpha((x\gamma(a\alpha a))\eta(t\beta x))\\
&= (x\beta s)\alpha(((e\mu x)\gamma(a\alpha a))\eta(t\beta x))=(x\beta s)\alpha((((a\alpha a)\mu x)\gamma e)\eta(t\beta x))\\
&= (x\beta s)\alpha((((a\alpha a)\mu x)\gamma t)\eta(e\beta x))=(x\beta s)\alpha((((a\alpha a)\mu x)\gamma t)\eta x)\\
&= (x\beta s)\alpha(((t\mu x)\gamma(a\alpha a))\eta x)=(x\beta s)\alpha((((e\mu t)\mu x)\gamma(a\alpha a))\eta x)\\
&= (x\beta s)\alpha((((x\mu t)\mu e)\gamma(a\alpha a))\eta x)=(x\beta s)\alpha((((a\alpha a)\mu e)\gamma(x\mu t))\eta x)\\
&= (x\beta s)\alpha((((e\alpha a)\mu a)\gamma(x\mu t))\eta x)=(x\beta s)\alpha(((a\mu a)\gamma(x\mu t))\eta x)\\
&= (x\beta s)\alpha((x\gamma((a\mu a)\mu t))\eta x)=(x\beta s)\alpha((x\gamma z)\eta x),\ \text{where } z=(a\mu a)\mu t\\
&= (((x\gamma z)\eta x)\beta s)\alpha x=((s\eta x)\beta(x\gamma z))\alpha x=(x\beta((s\eta x)\gamma z))\alpha x
\end{aligned}$$

So we have $x=(x\alpha a)\beta((x\alpha a)\beta x)=((x\beta((s\eta x)\gamma z))\alpha x)\beta((x\alpha a)\beta x)$, since $x\alpha a=(x\beta((s\eta x)\gamma z))\alpha x$.

Now consider

$$\begin{aligned}
(f\Gamma f)(x) &= \vee_{x=a\rho b}\{f(a)\wedge f(b)\}\geq f((x\beta((s\eta x)\gamma z))\alpha x)\wedge f((x\alpha a)\beta x)\\
&\geq (f(x)\wedge f(x)=f(x)\\
\text{So, } f &\subseteq f\Gamma f.
\end{aligned}$$

Also $f\Gamma f\subseteq f$. Hence it follows that $f=f\Gamma f$.

$(2)\Rightarrow(3)$. Let $f_1,\ f_2$ are fuzzy Γ-bi-ideals of S. Then obviously $f_1\cap f_2$ is also a fuzzy Γ-bi-ideal of S. So by hypothesis, we have $f_1\cap f_2=(f_1\cap f_2)\Gamma(f_1\cap f_2)\subseteq f_1\Gamma f_2$. Also $f_1\cap f_2=(f_1\cap f_2)\Gamma(f_1\cap f_2)\subseteq f_2\Gamma f_1$. It follows that $f_1\cap f_2\subseteq(f_1\Gamma f_2)\cap(f_2\Gamma f_1)$. Now we claim that

that $f_1\Gamma f_2$ is a fuzzy Γ-bi-ideal of S. For this we show that $((f_1\Gamma f_2)\Gamma S)\Gamma(f_1\Gamma f_2) \subseteq (f_1\Gamma f_2)$.

$$\begin{aligned}((f_1\Gamma f_2)\Gamma S)\Gamma(f_1\Gamma f_2) &= ((f_1\Gamma f_2)\Gamma(S\Gamma S))\Gamma(f_1\Gamma f_2)\\ &= ((f_1\Gamma S)\Gamma(f_2\Gamma S))\Gamma(f_1\Gamma f_2)\\ &= ((f_1\Gamma S)\Gamma f_1)((f_2\Gamma S)\Gamma f_2) \subseteq f_1\Gamma f_2\end{aligned}$$

Consequently $f_1\Gamma f_2$ is a fuzzy Γ-bi-ideal of S. Similarly $f_2\Gamma f_1$ is also a fuzzy Γ-bi-ideal of S. As intersection of fuzzy Γ-bi-ideals of S is also fuzzy Γ-bi-ideal of S, so $(f_1\Gamma f_2)\cap(f_2\Gamma f_1)$ is a fuzzy Γ-bi-ideal. Then by hypothesis

$$\begin{aligned}(f_1\Gamma f_2)\cap(f_2\Gamma f_1) &= ((f_1\Gamma f_2)\cap(f_2\Gamma f_1))\Gamma((f_1\Gamma f_2)\cap(f_2\Gamma f_1))\\ &\subseteq (f_1\Gamma f_2)\Gamma(f_2\Gamma f_1) \subseteq (f_1\Gamma S)\Gamma(S\Gamma f_1)\\ &= ((S\Gamma f_1)\Gamma S)\Gamma f_1 = (((Se)\Gamma f_1)\Gamma S)\Gamma f_1\\ &= (((f_1 e)\Gamma S)\Gamma S)\Gamma f_1 = ((f_1\Gamma S)\Gamma S)\Gamma f_1\\ &= ((S\Gamma S)\Gamma f_1)\Gamma f_1 = (S\Gamma f_1)\Gamma f_1\\ &= ((Se)\Gamma f_1)\Gamma f_1 = ((f_1 e)\Gamma S)\Gamma f_1 = (f_1\Gamma S)\Gamma f_1 \subseteq f_1.\end{aligned}$$

On same lines we have, $(f_1\Gamma f_2)\cap(f_2\Gamma f_1) \subseteq f_2$ and hence it follows that $(f_1\Gamma f_2)\cap(f_2\Gamma f_1) \subseteq f_1\cap f_2$. Consequently $f_1\cap f_2 = (f_1\Gamma f_2)\cap(f_2\Gamma f_1)$.

$(3)\Rightarrow(1)$. Let R be a fuzzy right Γ- ideal and L a fuzzy Γ-ideal of S. Then R and L are fuzzy Γ-bi-ideals of S. By lemma 275, every fuzzy right Γ-ideal and fuzzy two-sided Γ-ideal is a fuzzy Γ-bi-ideal of S. Then by hypothesis we have $R\cap L = (R\Gamma L)\cap(L\Gamma R)$. This implies $R\cap L \subseteq (R\Gamma L)\cap(L\Gamma R)$ and $R\cap L \subseteq L\Gamma R \Rightarrow R\cap L \subseteq R\Gamma L$. Since $R\Gamma L \subseteq R\cap L$ always true. Hence it follows that $R\cap L = R\Gamma L$ and $R\cap L \subseteq L\Gamma R$. Hence S is regular and intra-regular which completes the proof. ■

Theorem 298 *Let S be a Γ-AG-groupoid with left identity e. Then following conditions are equivalent.*

(1) S *is regular and intra-regular.*

(2) *Every fuzzy* Γ*-quasi-ideal of* S *is* Γ*-idempotent.*

Proof. $(1) \Rightarrow (2)$. Let g be a fuzzy Γ-quasi-ideal of S. Since $g\Gamma g \subseteq g$, so we only show that $g \subseteq g\Gamma g$. Let $x \in S$. Then S being a regular we have by definition $x = (x\alpha a)\beta x$, also S is intra-regular, so there exist $s,\ t \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $x = (s\alpha x)\beta(x\gamma t)$. Consider

$$\begin{aligned} x &= (x\alpha a)\beta x = (((s\alpha x)\beta(x\gamma t))\alpha a)\beta x = ((a\beta(x\gamma t))\alpha(s\alpha x))\beta x \\ &= ((x\beta(a\gamma t))\alpha(s\alpha x))\beta x = (((s\alpha x)\beta(a\gamma t))\alpha x)\beta x \\ &= ((((e\eta s)\alpha x)\beta(a\gamma t))\alpha x)\beta x = ((((x\eta s)\alpha e)\beta(a\gamma t))\alpha x)\beta x \\ &= ((((a\gamma t)\alpha e)\beta(x\eta s))\alpha x)\beta x = ((x\beta(((a\gamma t)\alpha e)\eta s)))\alpha x)\beta x \end{aligned}$$

Now

$$\begin{aligned} (g\Gamma g)(x) &= \vee_{x=y\mu z}\left\{g\left(y\right) \wedge g\left(z\right)\right\} \geq g((x\beta(((a\gamma t)\alpha e)\eta s)))\alpha x) \wedge g(x) \\ &\geq (g(x) \wedge g(x)) \wedge g(x) = g(x) \\ &\Rightarrow g \subseteq g\Gamma g \end{aligned}$$

Hence it follows that $g = g\Gamma g$.

$(2) \Rightarrow (1)$, let $a \in S$. Then $S\Gamma a$ is a right Γ-ideal of S containing a. Since every right Γ-ideal is Γ-quasi ideal of S, so $S\Gamma a$ is a Γ-quasi-ideal of S. This implies $C_{S\Gamma a}$ a fuzzy Γ-quasi- ideal of S. Then by (2), $C_{S\Gamma a} = C_{S\Gamma a}\Gamma C_{S\Gamma a} = C_{(S\Gamma a)(S\Gamma a)}$. It follows that $S\Gamma a = (S\Gamma a)(S\Gamma a)$. As $a \in S\Gamma a$, so $a \in (S\Gamma a)(S\Gamma a)$. Hence a is regular and intra-regular. Consequently S is regular and intra-regular. ■

Theorem 299 *Let* S *be a* Γ*-AG-groupoid with left identity* e*. Then following conditions are equivalent.*

(1) S *is regular and intra-regular.*

(2) *Every fuzzy* Γ*-quasi-ideal of* S *is* Γ*-idempotent.*

(3) *Every fuzzy* Γ*-bi-ideal of* S *is* Γ*-idempotent.*

Proof. $(1) \Rightarrow (3)$. Obvious by Theorem 297. $(3) \Rightarrow (2)$. Straight forward because every fuzzy Γ-quasi-ideal of S is a fuzzy Γ-bi-ideal of S by Lemma 283.

$(2) \Rightarrow (1)$. Follows from Theorem 298. ■

Theorem 300 *Let* S *be a* Γ*-AG-groupoid with left identity* e*. Then following conditions are equivalent.*

(1) S *is regular and intra-regular.*

(2) $f_1 \cap f_2 \subseteq f_1 \Gamma f_2$ *for all fuzzy* Γ*-quasi-ideals* $f_1,\ f_2$ *of* S*.*

(3) $f \cap g \subseteq f \Gamma g$ *for every fuzzy* Γ*-quasi-ideal* f *and every fuzzy* Γ*-bi-ideal* g *of* S*.*

(4) $g \cap f \subseteq g \Gamma f$ *for every fuzzy* Γ*- bi-ideal* g *and every fuzzy* Γ*-quasi-ideal* f *of* S*.*

(5) $g_1 \cap g_2 \subseteq g_1 \Gamma g_2$ *for all fuzzy* Γ*-bi-ideals* g_1, g_2 *of* S*.*

Proof. $(1) \Rightarrow (5)$. Let $g_1,\ g_2$ are fuzzy Γ-bi-ideals of S. Then $g_1 \cap g_2$ is also a fuzzy Γ-bi-ideal of S. Also by Theorem 297, every fuzzy Γ-bi-ideal in S is Γ-idempotent. Then it follows that $g_1 \cap g_2 = (g_1 \cap g_2)\Gamma(g_1 \cap g_2) \subseteq g_1 \Gamma g_2$.$(5) \Rightarrow (4) \Rightarrow (2)$ and $(5) \Rightarrow (3) \Rightarrow (2)$. Because every fuzzy Γ-quasi-ideal of S is fuzzy Γ-bi-ideal of S by Lemma 283.$(2) \Rightarrow (1)$. Let R be a fuzzy right Γ- ideal and L is a fuzzy left Γ-ideal of S. Since every fuzzy right and fuzzy left Γ-ideal of S is fuzzy Γ-quasi-ideal of S. Then by hypothesis, we have $R \cap L \subseteq R \Gamma L$. Also it is always true that $R \Gamma L \subseteq R \cap L$. So consequently $R \cap L = R \Gamma L$. Hence S is regular by Theorem 284. Again by (2), $R \cap L = L \cap R \subseteq L \Gamma R$. It follows that $R \cap L \subseteq L \Gamma R$. Hence by Theorem 294, S is intra-regular. ■

7.3 Fuzzy ideals in LA-rings

In this section, we introduce the concept of fuzzy ideals in LA-rings. Specifically we have shown that: (1) If R is an LA-ring with left identity e, then every fuzzy right ideal of R is a fuzzy ideal of R. (2) In an LA-ring R with left identity e, a non-empty fuzzy subset f of R is a fuzzy interior ideal if and only if f is a fuzzy ideal of R. (3) If R is a fully fuzzy quasi-prime LA-ring, then every fuzzy left ideal is fuzzy idempotent.

Definition 301 *Let R be an LA-ring. A fuzzy subset f of R is a function from R into the closed unit interval $[0,1]$, that is $f\colon R \rightarrow [0,1]$.*

Remark 302 *$F(R)$ denote the collection of all fuzzy subsets of R.*

Definition 303 *Let A be a subset of R. The characteristic function of A is denoted by C_A and defined by*

$$C_A(a) = \begin{cases} 1 \text{ if } a \in A \\ 0 \text{ otherwise} \end{cases}$$

We note that an LA-ring R can be considered a fuzzy subset of itself and write $R = C_R$, i.e., $R(x) = 1$ for all $x \in R$.

Definition 304 *Let f and g be two fuzzy subsets of an LA-ring R. The product $f \circ g$ is defined by $(f \circ g)(x) = \vee_{x=\sum\limits_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n}\{f(a_i) \wedge g(b_i)\}\}$,otherwise 0.*

Lemma 305 *Let A and B be any non-empty subset of an LA-ring R. Then the following properties hold.*

(1) *If $A \subseteq B$ then $C_A \subseteq C_B$.*

(2) $C_A \circ C_B = C_{AB}$.

(3) $C_A \cup C_B = C_{A\cup B}$.

(4) $C_A \cap C_B = C_{A\cap B}$.

Proof. (1) Let a be any element of R. Suppose $a \in A$, this implies $a \in B$. Thus $C_A(a) = 1 = C_B(a)$. This implies $C_A \subseteq C_B$. If $a \notin A$, this implies $a \notin B$. This implies $C_A(a) = 0 = C_B(a)$. Thus $C_A \subseteq C_B$.

(2) Let x be any element of R. Suppose $x \in AB$. This implies $x = ab$ for some $a \in A$ and $b \in B$. $(C_A \circ C_B)(x) = \vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{C_A(a_i) \wedge C_B(b_i)\}\} \geq C_A(a) \wedge C_B(b) = 1 \wedge 1 = 1 = C_{AB}(x)$.

Now suppose $x \notin AB$. This implies $x \neq ab$ for some $a \in A$ and $b \in B$. $(C_A \circ C_B)(x) = \vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{C_A(a_i) \wedge C_B(b_i)\}\} = 0 \wedge 0 = 0 = C_{AB}(x)$. Hence $C_A \circ C_B = C_{AB}$.

(3) Let a be any element of R. Suppose $a \in A \cup B$. Then there are three cases

(1) $a \in A$ and $a \in B$. Now $(C_A \cup C_B)(a) = C_A(a) \vee C_B(a) = 1 \vee 1 = 1 = C_{A \cup B}(a)$.

(2) $a \in A$ and $a \notin B$. Now $(C_A \cup C_B)(a) = C_A(a) \vee C_B(a) = 1 \vee 0 = 1 = C_{A \cup B}(a)$.

(3) $a \notin A$ and $a \in B$. Now $(C_A \cup C_B)(a) = C_A(a) \vee C_B(a) = 0 \vee 1 = 1 = C_{A \cup B}(a)$. If $a \notin A \cup B$. This implies $a \notin A$ and $a \notin B$. This implies $(C_A \cup C_B)(a) = C_{A \cup B}(a)$. Hence in all cases $C_A \cup C_B = C_{A \cup B}$.

(4) Let a be any element of R. Suppose $a \in A \cap B$. This implies $a \in A$ and $a \in B$. Now $(C_A \cap C_B)(a) = C_A(a) \wedge C_B(a) = 1 \wedge 1 = 1 = C_{A \cap B}(a)$. Suppose $a \notin A \cap B$. This implies $a \notin A$ and $a \notin B$. Now $(C_A \cap C_B)(a) = C_A(a) \wedge C_B(a) = 0 \wedge 0 = 0 = C_{A \cap B}(a)$. Hence $C_A \cap C_B = C_{A \cap B}$. ■

Proposition 306 *If f, g, h are fuzzy subsets of an LA-ring R, then*

$$(f \circ g) \circ \mathrm{h} = (h \circ g) \circ \mathrm{f}.$$

Proof. Let $x \in R$ and suppose $f \circ g = k$. Now

$$\begin{aligned}
(k \circ h)(x) &= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{k(a_i) \wedge h(b_i)\}\} \\
\texttt{Now } k(a_i) &= (f \circ g)(a_i) = \vee_{a_i=\sum_{i=1}^{n} c_i d_i} \{\wedge_{i=1}^{n} \{f(c_i) \wedge g(d_i)\}\} \\
((f \circ g) \circ h)(x) &= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \left\{(\vee_{a_i=\sum_{i=1}^{n} c_i d_i} \{\wedge_{i=1}^{n} \{f(c_i) \wedge g(d_i)\}\}) \wedge h(b_i)\right\}\} \\
&= \vee_{x=\sum_{i=1}^{n} (\sum_{i=1}^{n} c_i d_i) b_i} \{\wedge_{i=1}^{n} (\{\wedge_{i=1}^{n} \{f(c_i) \wedge g(d_i)\}\} \wedge h(b_i))\} \\
&= \vee_{x=\sum_{i=1}^{n} (c_i d_i) b_i} \{\wedge_{i=1}^{n} ((f(c_i) \wedge g(d_i)) \wedge h(b_i))\} \\
&= \vee_{x=\sum_{i=1}^{n} (b_i d_i) c_i} \{\wedge_{i=1}^{n} ((h(b_i) \wedge g(d_i)) \wedge f(c_i))\} \\
&= \vee_{x=\sum_{i=1}^{n} (\sum_{i=1}^{n} b_i d_i) c_i} \{\wedge_{i=1}^{n} (\{\wedge_{i=1}^{n} \{h(b_i) \wedge g(d_i)\}\} \wedge f(c_i))\} \\
&= \vee_{x=\sum_{i=1}^{n} m_i b_i} \{\wedge_{i=1}^{n} \left\{(\vee_{m_i=\sum_{i=1}^{n} b_i d_i} \{\wedge_{i=1}^{n} \{h(b_i) \wedge g(d_i)\}\}) \wedge f(c_i)\right\}\} \\
&= \vee_{x=\sum_{i=1}^{n} m_i b_i} \{\wedge_{i=1}^{n} \{l(m_i) \wedge f(b_i)\}\} \\
&= (l \circ f)(x) = ((h \circ g) \circ f)(x)
\end{aligned}$$

where $l(m_i) = (h \circ g)(m_i) = \vee_{m_i=\sum_{i=1}^{n} b_i d_i} \{\wedge_{i=1}^{n} \{h(b_i) \wedge g(d_i)\}\}$. Thus $(f \circ g) \circ h = (h \circ g) \circ f$. ■

Remark 307 *Let R be an LA-ring with left identity e, then it easy to verify that $f \circ (g \circ h) = g \circ (f \circ h)$, $(f \circ g) \circ (h \circ k) = (f \circ h) \circ (g \circ k)$ and $(f \circ g) \circ (h \circ k) = (k \circ h) \circ (g \circ f)$ for all fuzzy subset f, g, h and k of R.*

Definition 308 *A fuzzy subset f of an LA-ring R is called fuzzy LA-subring of R if $f(a - b) \geq f(a) \wedge f(b)$ and $f(ab) \geq f(a) \wedge f(b)$ for all $a, b \in R$.*

A fuzzy LA-subring f of an LA-ring R is called a fuzzy left ideal of R if $f(ab) \geq f(b)$ for all $a, b \in R$. Similarly, f is called a fuzzy right ideal of R if $f(ab) \geq f(a)$ for all $a, b \in R$. f is called a fuzzy ideal of R if f is a fuzzy right ideal and a fuzzy left ideal of R.

In an LA-ring if f and g are two fuzzy LA-subrings of R then obviously $f \cap g$ is also a fuzzy LA-subring of R and similarly the intersection of two fuzzy left (right, two-sided) ideals is again a fuzzy left (right, two-sided) ideal of R.

Theorem 309 *Let A be a non-empty subset of an LA-ring R. Then the following properties hold.*

(1) *A is an LA-subring of R if and only if C_A is a fuzzy LA-subring of R.*

(2) *A is a left (right, two-sided) ideal of R if and only if C_A is a fuzzy left (right, two-sided) ideal of R.*

Proof. (1) Suppose A is an LA-subring of R. Let $a, b \in R$. If $a, b \notin A$. This implies $C_A(a) = 0 = C_A(b)$, so $C_A(a - b) \geq 0 = C_A(b) \wedge C_A(b)$ and $C_A(ab) \geq 0 = C_A(b) \wedge C_A(b)$. If $a, b \in A$. This implies $C_A(a) = 1 = C_A(b)$, so $C_A(a - b) = 1 = C_A(a) \wedge C_A(b)$ and $C_A(ab) = 1 = C_A(a) \wedge C_A(b)$. This implies $C_A(a - b) \geq C_A(a) \wedge C_A(b)$ and $C_A(ab) \geq C_A(a) \wedge C_A(b)$. This implies C_A is a fuzzy LA-subring of R. Conversely, assume that C_A is a fuzzy LA-subring of R. Let $a, b \in A$. Since $C_A(a - b) \geq C_A(a) \wedge C_A(b) = 1$ and $C_A(ab) \geq C_A(a) \wedge C_A(b) = 1$. This implies $C_A(a - b) = 1$ and $C_A(ab) = 1$. Thus $a - b$ and $ab \in A$. Hence A is an LA-subring.

(2) Suppose A is a left ideal of R. Let $a, b \in R$. If $b \notin A$, then $C_A(b) = 0$, so $C_A(ab) \geq 0 = C_A(b)$. If $b \in A$, then $ab \in A$, since A is a left ideal. This implies $C_A(ab) = 1 = C_A(b)$. This implies $C_A(ab) \geq C_A(b)$. Thus C_A is fuzzy left ideal of R. Conversely, assume that C_A is a fuzzy left ideal of R. Let $r \in R$ and $a \in A$. Since $C_A(ra) \geq C_A(a) = 1$. This implies $C_A(ra) = 1$. This implies $ra \in A$. Thus A is a left ideal of R. Similarly we can prove for right and two-sided ideal. ■

Theorem 310 *Let f be a fuzzy LA-subring of an LA-ring R, then the following assertions are true.*

(1) f is a fuzzy LA-subring of R if and only if $f \circ f \subseteq f$ and $f(x-y) \geq f(x) \wedge f(y)$ for all $x, y \in R$.

(2) f is a fuzzy left ideal of R if and only if $R \circ f \subseteq f$.

(3) f is a fuzzy right ideal of R if and only if $f \circ R \subseteq f$.

Proof. (1) Suppose f is a fuzzy LA-subring of R. Let $x \in R$. If $(f \circ f)(x) = 0 \leq f(x)$. This implies $f \circ f \subseteq f$. Otherwise ,$(f \circ f)(x) = \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{f(a_i) \wedge f(b_i)\}\} \leq \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{f(a_i b_i)\}\} = f(x)$. This implies$f \circ f \subseteq f$.

Conversely, assume that $f \circ f \subseteq f$ and $f(x-y) \geq f(x) \wedge f(y)$ for all $x, y \in R$. Let $x, y \in R$, then $f(xy) \geq (f \circ f)(xy) = \vee_{xy=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{f(a_i) \wedge f(b_i)\}\} \geq f(x) \wedge f(y)$. This implies f is a fuzzy LA-subring of R.

(2) Suppose f is a fuzzy left ideal of R. Let $x \in R$. If $(R \circ f)(x) = 0 \leq f(x)$. This implies $R \circ f \subseteq f$. Otherwise,

$$\begin{aligned}(R \circ f)(x) &= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{R(a_i) \wedge f(b_i)\}\} = \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{1 \wedge f(b_i)\}\} \\ &= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{f(b_i)\}\} \leq \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{f(a_i b_i)\}\} = f(x)\end{aligned}$$

Conversely, assume that $R \circ f \subseteq f$. Let $y, z \in R$. Set $x = yz$. Now $f(yz) = f(x) \geq (R \circ f)(x) = \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{R(a_i) \wedge f(b_i)\}\} \geq R(y) \wedge f(z) = 1 \wedge f(z) = f(z)$. This implies f is a fuzzy left ideal of R. Similarly we can prove (3). ■

Remark 311 *f is a fuzzy ideal of R if and only if $R \circ f \subseteq f$ and $f \circ R \subseteq f$.*

Lemma 312 *If f, g are fuzzy LA-subrings of an LA-ring R with left identity e, then $f \circ g$ is a fuzzy LA-subring of R.*

Proof. Let f, g are fuzzy LA-subrings of R. Now we have to show that $f \circ g$ is a fuzzy LA-subring of R.

(1) $(f \circ g) \circ (f \circ g) = (f \circ f) \circ (g \circ g) \subseteq f \circ g$ by 307. This implies $(f \circ g) \circ (f \circ g) \subseteq f \circ g$.

(2) Since g is a fuzzy LA-subring. This implies $g(x - y) \geq g(x) \wedge g(y)$ for all $x, y \in R$. This implies $f(g(x - y)) \geq f(g(x) \wedge g(y))$. So $(f \circ g)(x - y) \geq (f \circ g)(x) \wedge (f \circ g)(y)$, by use part (1) of theorem 310. ■

Lemma 313 *If R is an LA-ring with left identity e, then every fuzzy right ideal of R is a fuzzy ideal of R.*

Proof. Let R be an LA-ring with left identity e and f be a fuzzy right ideal of R. Let $a, b \in R$. Now $f(ab) = f((ea)b) = f((ba)e) \geq f(ba) \geq f(b)$ Thus f is a fuzzy ideal of R. ■

Lemma 314 *If R is an LA-ring with left identity e and f, g are fuzzy ideals of R, then $f \circ g$ is a fuzzy ideal of R.*

Proof. Let $x, y \in R$. It is enough to show that $f \circ g$ is a fuzzy right ideal of R. If $(f \circ g)(x) = 0 \leq (f \circ g)(xy)$, otherwise $(f \circ g)(x) = \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{f(a_i) \wedge g(b_i)\}\}$. Since $x = \sum_{i=1}^{n} a_i b_i$, so, $xy = (\sum_{i=1}^{n} a_i b_i)y = \sum_{i=1}^{n}(a_i b_i)y = \sum_{i=1}^{n}(a_i b_i)(ey) = \sum_{i=1}^{n}(a_i e)(b_i y)$. Now

$$\begin{aligned} (f \circ g)(x) &= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{f(a_i) \wedge g(b_i)\}\} \\ &\leq \vee_{xy=\sum_{i=1}^{n} (a_i e)(b_i y)} \{\wedge_{i=1}^{n} \{f(a_i e) \wedge g(b_i y)\}\} \\ &\leq \vee_{xy=\sum_{i=1}^{n} c_i d_i} \{\wedge_{i=1}^{n} \{f(c_i) \wedge g(d_i)\}\} = (f \circ g)(xy) \\ &\Rightarrow (f \circ g)(xy) \geq (f \circ g)(x) \end{aligned}$$

This implies $f \circ g$ is a fuzzy ideal of R by 313. ■

Theorem 315 *If R is an LA-ring and f, g are fuzzy ideals of R with left identity e, then $f \circ g \subseteq f \cap g$.*

Proof. Let f, g be fuzzy ideals of R and $x \in R$. If $(f \circ g)(x) = 0 \leq (f \cap g)(x)$, then $(f \circ g)(x) \leq (f \cap g)(x)$, otherwise

$$\begin{aligned} f \circ g(x) &= \vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{f(a_i) \wedge g(b_i)\}\} \\ &\leq \vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{f(a_i b_i) \wedge g(a_i b_i)\}\} \\ &= \vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{f \cap g(a_i b_i)\}\} = (f \cap g)(x) \\ &\Rightarrow f \circ g \subseteq f \cap g. \end{aligned}$$

■

Remark 316 *If R is an LA-ring and f is a fuzzy ideal of R, then $f \circ f \subseteq f$.*

Theorem 317 *Let R be an LA-ring and f is a fuzzy right ideal of R, then $f \circ f$ is a fuzzy ideal of R.*

Proof. Since f is fuzzy right ideal of R and R is an LA-ring. Let $x, y \in R$. We show that $(f \circ f)(xy) \geq (f \circ f)(x)$. If $(f \circ f)(x) = 0 \leq (f \circ f)(xy)$, otherwise

$$\begin{aligned} (f \circ f)(x) &= \vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{f(a_i) \wedge f(b_i)\}\} \\ &= \vee_{x=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{f(b_i) \wedge f(a_i)\}\} \\ &= \vee_{xy=(\sum_{i=1}^{n} a_i b_i)y}\{\wedge_{i=1}^{n}\{f(b_i) \wedge f(a_i)\}\} \\ &= \vee_{xy=\sum_{i=1}^{n}(a_i b_i)y}\{\wedge_{i=1}^{n}\{f(b_i) \wedge f(a_i)\}\} \\ &\leq \vee_{xy=\sum_{i=1}^{n}(y b_i)a_i}\{\wedge_{i=1}^{n}\{f(y b_i) \wedge f(a_i)\}\} \\ &\leq \vee_{xy=\sum_{i=1}^{n} c_i d_i}\{\wedge_{i=1}^{n}\{f(c_i) \wedge f(d_i)\}\} = (f \circ f)(xy) \\ &\Rightarrow (f \circ f)(xy) \geq (f \circ f)(x) \end{aligned}$$

This implies $f \circ f$ is a fuzzy ideal of R. ■

Remark 318 *Let R be an LA-ring and f is a fuzzy left ideal of R with left identity, then $f \circ f$ is a fuzzy ideal of R.*

Lemma 319 *Let R be an LA-ring. Then $f \circ g \subseteq f \cap g$ for every fuzzy right ideal f and fuzzy left ideal g of R.*

Proof. Let f is a fuzzy right ideal and g is a fuzzy left ideal of R. Let $x \in R$. If $f \circ g(x) = 0 \leq f \cap g(x)$, otherwise

$$\begin{aligned}
(f \circ g)(x) &= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{f(a_i) \wedge g(b_i)\}\} \\
&\leq \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{f(a_i b_i) \wedge g(a_i b_i)\}\} \\
&= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} (f \cap g)(a_i b_i)\} = (f \cap g)(x) \\
&\Rightarrow f \circ g \subseteq f \cap g.
\end{aligned}$$

■

Lemma 320 *In an LA-ring R with left identity e, $R \circ R = R$.*

Proof. Every x in R can be written as $x = ex$, where e is the left identity in R. So $(R \circ R)(x) = \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{R(a_i) \wedge f(b_i)\}\} \geq \{R(e) \wedge R(x)\} = 1$ This implies $(R \circ R)(x) = 1 = R(x)$ for all x in R. Hence $R \circ R = R$. ■

A fuzzy LA-subring f of an LA-ring R is called a fuzzy interior ideal of R if $f((xy)z) \geq f(y)$ and f is called a fuzzy bi-ideal if $f((xy)z) \geq f(x) \wedge f(z)$ for all $x, y, z \in R$. A fuzzy LA-subring f of an LA-ring R is called a fuzzy quasi-ideal of R if $(f \circ R) \cap (R \circ f) \subseteq f$. A fuzzy ideal f of an LA-ring R is called a fuzzy idempotent if $f \circ f = f$.

As in theorem 309, it can be easily verified that " A is a bi (interior,quasi) ideal of R if and only if C_A is a fuzzy bi (interior,quasi) ideal of R".

Theorem 321 *Let f be a fuzzy LA-subring of an LA-ring R. Then f is a fuzzy interior ideal of R if and only if $(R \circ f) \circ R \subseteq f$.*

Proof. Suppose f is a fuzzy interior ideal of R. Let $x \in R$. If $((R \circ f) \circ R)(x) = 0 \leq f(x)$. This implies $(R \circ f) \circ R \subseteq f$. Otherwise there exist $a_i, b_i, c_i, d_i \in R$ such that $x = \sum_{i=1}^{n} a_i b_i$ and $a_i = \sum_{i=1}^{n} c_i d_i$. Since f is a fuzzy interior ideal of R. This implies $f((c_i d_i) b_i) \geq f(d_i)$. Now

$$\begin{aligned}
((R \circ f) \circ R)(x) &= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{(R \circ f)(a_i) \wedge R(b_i)\}\} \\
&= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \left\{(\vee_{a_i=\sum_{i=1}^{n} c_i d_i} \{\wedge_{i=1}^{n} \{R(c_i) \wedge f(d_i)\}\}) \wedge R(b_i)\right\}\} \\
&= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \left\{(\vee_{a_i=\sum_{i=1}^{n} c_i d_i} \{\wedge_{i=1}^{n} \{1 \wedge f(d_i)\}\}) \wedge 1\right\}\} \\
&= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \left\{(\vee_{a_i=\sum_{i=1}^{n} c_i d_i} \{\wedge_{i=1}^{n} f(d_i)\}) \wedge 1\right\}\} \\
&= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \left\{\vee_{a_i=\sum_{i=1}^{n} c_i d_i} \{\wedge_{i=1}^{n} f(d_i)\}\right\}\} \\
&= \vee_{x=\sum_{i=1}^{n} (c_i d_i) b_i} \{\wedge_{i=1}^{n} f(d_i)\} \\
&\leq \vee_{x=\sum_{i=1}^{n} (c_i d_i) b_i} \{\wedge_{i=1}^{n} f((c_i d_i) b_i)\}, \text{ by def of fuzzy interior ideal.} \\
&= f(x) \\
&\Rightarrow ((R \circ f) \circ R) \subseteq f.
\end{aligned}$$

Conversely, assume that $((R \circ f) \circ R) \subseteq f$. Let $x, y, z \in R$ and set $a = (xy)z$. Now

$$\begin{aligned}
f((xy)z) &= f(a) \geq ((R \circ f) \circ R)(a) \\
&= \vee_{a=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{(R \circ f)(a_i) \wedge R(b_i)\}\} \\
&\geq (R \circ f)(xy) \wedge R(z) \\
&= \vee_{xy=\sum_{i=1}^{n} c_i d_i} \{\wedge_{i=1}^{n} \{R(c_i) \wedge f(d_i)\} \wedge R(z) \\
&\geq R(x) \wedge f(y) \wedge R(z) = 1 \wedge f(y) \wedge 1 = f(y) \\
&\Rightarrow f((xy)z) \geq f(y).
\end{aligned}$$

This implies f is a fuzzy interior ideal of R. ■

Theorem 322 *Let f be a fuzzy LA-subring of an LA-ring R. Then f is a fuzzy bi-ideal of R if and only if $(f \circ R) \circ f \subseteq f$.*

Proof. Suppose f is a fuzzy bi-ideal of R. Let $x \in R$. If $((f \circ R) \circ f)(x) = 0 \leq f(x)$. So, $(f \circ R) \circ f \subseteq f$. Otherwise there exist $a_i, b_i, c_i, d_i \in R$ such that $x = \sum_{i=1}^{n} a_i b_i$ and $a_i = \sum_{i=1}^{n} c_i d_i$. Since f is a fuzzy bi- ideal of R. This implies $f((c_i d_i) b_i) \geq f(c_i) \wedge f(b_i)$. Now

$$\begin{aligned}
((f \circ R) \circ f)(x) &= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{(f \circ R)(a_i) \wedge f(b_i)\}\} \\
&= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \left\{(\vee_{a_i=\sum_{i=1}^{n} c_i d_i} \{\wedge_{i=1}^{n} \{f(c_i) \wedge R(d_i)\}\}) \wedge f(b_i)\right\}\} \\
&= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \left\{(\vee_{a_i=\sum_{i=1}^{n} c_i d_i} \{\wedge_{i=1}^{n} \{f(c_i) \wedge 1\}\}) \wedge f(b_i)\right\}\} \\
&= \vee_{x=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \left\{(\vee_{a_i=\sum_{i=1}^{n} c_i d_i} \{\wedge_{i=1}^{n} f(c_i)\}) \wedge f(b_i)\right\}\} \\
&= \vee_{x=\sum_{i=1}^{n} (c_i d_i) b_i} \{\wedge_{i=1}^{n} \{f(c_i) \wedge f(b_i)\}\} \\
&\leq \vee_{x=\sum_{i=1}^{n} (c_i d_i) b_i} \{\wedge_{i=1}^{n} f((c_i d_i) b_i)\} = f(x) \\
&\Rightarrow ((f \circ R) \circ f) \subseteq f.
\end{aligned}$$

Conversely, assume that $((f \circ R) \circ f) \subseteq f$. Let $x, y, z \in R$ and set $a = (xy)z$. Now

$$\begin{aligned}
f((xy)z) &= f(a) \geq ((f \circ R) \circ f)(a) \\
&= \vee_{a=\sum_{i=1}^{n} a_i b_i} \{\wedge_{i=1}^{n} \{(f \circ R)(a_i) \wedge f(b_i)\}\} \\
&\geq (f \circ R)(xy) \wedge f(z) \\
&= \vee_{xy=\sum_{i=1}^{n} c_i d_i} \{\wedge_{i=1}^{n} \{f(c_i) \wedge R(d_i)\} \wedge f(z) \\
&\geq f(x) \wedge R(y) \wedge f(z) = f(x) \wedge 1 \wedge f(z) = f(x) \wedge f(z) \\
&\Rightarrow f((xy)z) \geq f(x) \wedge f(z).
\end{aligned}$$

This implies f is a fuzzy bi-ideal of R. ■

Proposition 323 *In an LA-ring R, the following hold:*

(1) *Every fuzzy ideal of R is a fuzzy interior ideal of R.*

(2) *Every fuzzy left (right, two-sided) ideal of R is a fuzzy quasi-ideal of R.*

(3) *Every fuzzy right (two-sided) ideal of R is a fuzzy bi-ideal of R.*

Proof. The proof is straight forward. ■

Remark 324 *If f and g be two fuzzy interior $($bi,quasi$)$ ideals of an LA-ring R. Then it is straight forward to show that $f \cap g$ is a fuzzy interior $($bi, quasi$)$ ideal of R.*

Lemma 325 *Let R be an LA-ring with left identity e. Then any non-empty fuzzy subset f of R is a fuzzy interior ideal if and only if f is a fuzzy ideal of R.*

Proof. Suppose f is a fuzzy interior ideal of R. Let $x, y \in R$. Now $f(xy) = f((ex)y) \geq f(x)$. This implies f is a fuzzy ideal of by lemma 313. Converse is true by Proposition 297. ■

Lemma 326 *Every fuzzy quasi-ideal of an LA-ring R with left identity e, is a fuzzy bi-ideal of R.*

Proof. Let f is a fuzzy quasi-ideal of R. Let $x, y, z \in R$. This implies $f((xy)z) \geq ((f \circ R) \cap (R \circ f))((xy)z) = (f \circ R)((xy)z) \wedge (R \circ f)((xy)z)$, since f is a quasi-ideal of R. Now

$$\begin{aligned}
(R \circ f)((xy)z) &= \vee_{(xy)z=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{R(a_i) \wedge f(b_i)\}\} \\
&= \vee_{(xy)z=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{1 \wedge f(b_i)\}\} \geq f(z) \\
&\Rightarrow (R \circ f)((xy)z) \geq f(z). \\
(f \circ R)((xy)z) &= \vee_{(xy)z=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{f(a_i) \wedge R(b_i)\}\} \\
&= \vee_{(xy)z=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{f(a_i) \wedge 1\}\}
\end{aligned}$$

Now $(xy)z = (xy)(ez) = (xe)(yz) \in (xe)R = xR$. This implies $(xy)z \in xR$, so $(xy)z = xr$ for some $r \in R$. This implies

$$\begin{aligned}
(f \circ R)((xy)z) &= \vee_{(xy)z=xr=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{f(a_i) \wedge 1\}\} \\
&= \vee_{xr=\sum_{i=1}^{n} a_i b_i}\{\wedge_{i=1}^{n}\{f(a_i) \wedge 1\}\} \geq f(x) \\
&\Rightarrow (f \circ R)((xy)z) \geq f(x).
\end{aligned}$$

Thus $f((xy)z) \geq (f \circ R)((xy)z) \wedge (R \circ f)((xy)z) \geq f(x) \wedge f(z)$. This implies $f((xy)z) \geq f(x) \wedge f(z)$. This implies f is a fuzzy bi-ideal of R. ■

7.3.1 Fuzzy Quasi-Prime ideals in LA-ring

Definition 327 *A fuzzy left ideal P of an LA-ring R is called fuzzy quasi-prime ideal of R if $f \circ g \subseteq P$ implies that either $f \subseteq P$ or $g \subseteq P$, for all fuzzy left ideals f and g of R. And R is called fully fuzzy quasi-prime if every fuzzy left ideal of R is fuzzy quasi-prime.*

Lemma 328 *Let R be a fully fuzzy quasi-prime LA-ring with left identity e, then every fuzzy left ideal is fuzzy idempotent.*

Proof. Let f be a fuzzy left ideal of R. This implies f is a fuzzy quasi-prime ideal of R. Since $f \circ f$ is a fuzzy ideal of R by Remark 318. This implies $f \circ f$ is a fuzzy quasi-prime ideal of R. Since $f \circ f \subseteq f \circ f$. This implies $f \subseteq f \circ f$. Since $f \circ f \subseteq f$. Hence $f \circ f = f$. ■

Proposition 329 *If R is a fully fuzzy quasi-prime LA-ring with left identity e, then every fuzzy left ideal is a fuzzy ideal of R.*

Proof. Let f be a fuzzy left ideal of R. Now we have to show that f is a fuzzy right ideal of R. $f \circ R = (f \circ f) \circ R = (R \circ f) \circ f \subseteq f \circ f \subseteq R \circ f \subseteq f$. This implies $f \circ R \subseteq f$. This implies f is a fuzzy right ideal of R. Hence f is a fuzzy ideal of R. ■

Theorem 330 *If R is fully fuzzy quasi-prime LA-ring with left identity e. Then for every fuzzy left ideal f of R following are true.*

(1) *$R \circ f$ is an idempotent.*

(2) *Every fuzzy right ideal g of R commutes with f.*

Proof. (1) $(R \circ f)^2 = (R \circ f) \circ (R \circ f) = (R \circ R) \circ (f \circ f) = R \circ f$ by Remark 307.

(2) $f \circ g = (f \circ f) \circ g = (g \circ f) \circ f \subseteq (g \circ R) \circ f \subseteq B \circ f$. $g \circ f = g \circ (f \circ f) = f \circ (g \circ f) \subseteq f \circ (g \circ R) \subseteq f \circ g$. This implies $f \circ g = g \circ f$. ■

Proposition 331 *Let R be a fully fuzzy quasi-prime LA-ring with left identity e, then for all fuzzy left ideals f and g of R, $f \circ g = f \cap g$.*

Proof. Let f and g are fuzzy left ideals of R. $f \circ g \subseteq R \circ g \subseteq g$ and $g \circ f \subseteq R \circ f \subseteq f$. Now $f \circ g = (f \circ f) \circ (g \circ g) = (g \circ g) \circ (f \circ f) = g \circ f$. This implies $f \circ g \subseteq f \cap g$. Now $(f \cap g)^2 = (f \cap g) \circ (f \cap g) \subseteq f \circ g$. This implies $f \cap g \subseteq f \circ g$. Thus $f \cap g = f \circ g$. Since $f \cap g$ is a left ideal of R and $f \cap g$ is idempotent by Lemma 328. ■

Theorem 332 *Let R be a fully fuzzy quasi-prime LA-ring with left identity e. Then for every fuzzy left ideal f, then the following conditions are equivalent.*

(1) f *is a fuzzy ideal.*

(2) f *is a fuzzy interior ideal.*

(3) f *is fuzzy bi-ideal.*

(4) f *is fuzzy quasi-ideal.*

Proof. $(1) \Rightarrow (2)$, is obvious by Proposition 323. $(2) \Rightarrow (3)$, suppose f is a fuzzy interior ideal. $(f \circ R) \circ f = (f \circ R) \circ (f \circ f) = (f \circ f) \circ (R \circ f) \subseteq (R \circ f) \circ f \subseteq (R \circ f) \circ R \subseteq f$. This implies $(f \circ R) \circ f \subseteq f$. Thus f is fuzzy bi-ideal. $(3) \Rightarrow (4)$, suppose f is a fuzzy bi-ideal.

$$\begin{aligned}(f \circ R) \cap (R \circ f) &\subseteq f \circ R = (f \circ f) \circ R = (R \circ f) \circ f = (R \circ (f \circ f)) \circ f \\ &= (f \circ (R \circ f)) \circ f \subseteq (f \circ f) \circ f \subseteq (f \circ R) \circ f \subseteq f \\ &\Rightarrow (f \circ R) \cap (R \circ f) \subseteq f.\end{aligned}$$

Hence f is a fuzzy quasi-ideal of R. $(4) \Rightarrow (1)$, suppose f is a fuzzy quasi-ideal. Now we have to show that f is a fuzzy ideal of R. It is enough to show that $f \circ R \subseteq f$. $f \circ R = (f \circ f) \circ R = (R \circ f) \circ f \subseteq f \circ f \subseteq R \circ f$. This implies $f \circ R \subseteq R \circ f$. Thus $f \circ R \subseteq (f \circ R) \cap (R \circ f) \subseteq f$. This implies $f \circ R \subseteq f$. Hence f is a fuzzy ideal. ■

Chapter 8

Conclusions

The whole study in this thesis can be divided into three phases. In first phase we discuss AG-groupoids and Γ-AG-groupoids which are in fact single (binary) operational structures. These structures have been investigated in chapters $2, 5$. In second phase, we deal with two (binary) operational structures such as LA-rings and Γ-LA-rings. These concepts have been studied in chapters $3, 4$, and 6. While in the third phase, we look into the application aspect and investigated the fuzzy concepts of these algebraic structures.

AG-groupoids (also known as LA-semigroups) have been studied by several researchers. We genuinely acknowledge that in this field much of the fundamental work has been done by M. Kazim and M. Naseeruddin [24], Q. Mushtaq and his associates [40, 41, 42, 43, 44, 45, 47, 48, 49] and P. V. Protic and N. Stevanovic [56, 57, 58, 59, 69].

In phase one, we first time introduce the concept of ordered AG-groupoids. Also we initiate the concept of Γ-AG-groupoids which are a generalization of AG-groupoids.

In the second phase, we have discussed left almost rings (abbreviated as LA-rings), which are in fact a generalization of commutative rings. Despite the fact that this structure is non-associative and non-commutative, interestingly it possesses properties which usually are valid in associative and commutative algebraic structures. Moreover, we have studied Γ-LA-rings which also lead to the generalizations of commutative rings. The concepts focused in phase one, have significant contribution in phase 2 and phase 3. For instance, the purpose of defining the concept of ordered AG-groupoids was to deal with the 'degree questions' arose in LA-ring of finitely non- zero functions. Likewise, the characteristics of Γ-AG-groupoids can be seen in the set up of Γ-LA-rings discussed in chapter 6.

Some of the important conclusions which can be drawn from the chapters $2-7$ are as follow.

(1) It is beyond doubt that total ordering of semigroups plays a vital role to enhance the theory of semigroups. Keeping in view the importance of this concept, we first time investigate the ordering of AG-groupoids. For this study we followed mostly [13].

(2) By following $[27, 48]$, we also study ideals, M-systems, N-systems and I-systems of ordered AG-groupoids.

(3) We introduce a non-commutative and a non-associative structure, which we named as Γ-AG-groupoids. This structure is a direct generalization of AG-groupoids. The motivation behind this study is an article ; On Γ-semigroups, by M. K. Sen [63], published in 1981. Γ-semigroups are a generalization of semigroups. Many classical notions of semigroups were extended to Γ-semigroups. The supporting literature for this study which we followed is in $[7, 10, 15, 16, 63, 64, 65]$.

(4) Several authors, for example, E. Kleinfeld in $[28, 29]$ and D. C. Murdoch and O. Ore in [39] have generalized the concept of commutative rings and investigated the structural properties of these generalizations. In this study we also generalize the concept of commutative rings but however with a different mode. For this we discuss left almost rings (abbreviated as LA-rings) [70]. We investigate some elementary concepts in LA-rings. Moreover, we introduce the concepts of ideals and establish some results analogous to associative ring theory. We also discuss M-systems, P-systems and I-systems and subtractive sets in LA-rings. The main objective of this study is to generalize the results corresponding the commutative rings and to find the characteristics hidden in this non-commutative and non-associative structure. Also we discuss direct sums in LA-rings using the concepts of ideals.

(5) We discuss an area which is comparatively hard. For this we construct LA-rings of finitely nonzero functions, represented as $R\left[X^{s}; s \in S\right]$. Also we introduce the concept of LA-modules which will play a vital role to enhance the theory LA-rings. Recently in [68], T. Shah and M. Raees have investigated this concept and generalized several results corresponding to associative modules theory over the rings. For this study we followed $[12, 13, 20, 54]$.

(6) Taking motivations from Nobosawa [52], we introduce Γ-left almost rings, which are a direct generalization of LA-rings. It is not hard to see that Γ-left almost rings are also a generalization commutative Γ-rings. During the discussions on Γ-left almost rings, we followed [4, 34, 35, 37, 52].

(7) We also attempt the fuzzy concepts of algebraic structures such as LA-rings and Γ-AG-groupoids which have been discussed in earlier chapters. For this we mostly followed [1, 8, 11, 18, 23, 36, 62, 71].

8.1 Future prospects of the work

(1) In phase one, we deal with single (binary) operational structures such as AG-groupoids and Γ-AG-groupoids, which have been discussed in chapters 2 and 5.

In 1972, the concept of LA-semigroups (or AG-groupoids) has been introduced by M. Kazim and M. Naseerudin. We have investigated some characteristics of ordered AG-groupoids in chapter 2. Keeping in view the importance of ordered semigroup theory, we can see a lot of space to be filled in, by doing work in this particular area. In the same phase, we have also introduced a non-commutative and a non-associative structure, which we named as Γ-AG-groupoids (chapter 5). We have discussed its elementary characteristics and established some results regarding its ideals. In future, this structure can be investigated in many different aspects. For example, semilattice decompositions of Γ-AG-groupoids, orthodox Γ-AG-groupoids, archimedean Γ-AG-groupoids and theory of ordered Γ-AG-groupoids. This structure can further be extended to Γ-AG*-groupoids and Γ-AG**-groupoids.

(2) In second phase, we discuss LA-rings and Γ-LA-rings which are in fact two (binary) operational structures. These concepts have been studied in chapters 3, 4 and 6. Though LA-ring is a non-associative and non-commutative structure, but due to its peculiar characteristics, it possesses properties which we usually encounter in associative algebraic structures. In future we can see a lot of room to extend and study this concept in different parameters. For example on the basis of developments made in chapter 3, we have constructed LA-rings of finitely non-zero functions (chapter 4, section one), represented as $R[X^s; s \in S]$. By Remark 93, we

got liberty to extend this to integral dependence. Also we are planning to use $R[X^s; s \in S]$ in coding theory and cryptography. Several authors have investigated these concepts, for instance, Reed and Solomon [60], I. F. Blake [5] and A. A. Andrade and R. Palazzo [2]. We may replace the polynomial codes by taking bits from a finite LA-ring or from finite almost field. During the construction of $R[X^s; s \in S]$, we have also defined LA-module which intuitively would be the most useful tool for further developments. In this regard recently in [68], T. Shah and M. Raees have investigated several results parallel to associative modules theory over the rings.

(3) In the third phase, we have initiated fuzzy concepts of the structures discussed in phase one and phase two. We investigated the concept of fuzzy ideals in Γ-AG-groupoids and in LA-rings (chapter 7). In future we are planning to investigate the concepts of soft AG-groupoids, soft ordered AG-groupoids, (α, β)-fuzzy ideals, generalized fuzzy ideals, intuitionistic fuzzy ideals and rough ideals in these newly established structures (ordered AG-groupoids, Γ-AG-groupoids and LA-rings).

References

[1] J. Ahsan, Kui Yuan Li and M. Shabir, Semigroups characterized by their fuzzy bi-ideals, J. Fuzzy Math., 10(2002), 441-449.

[2] A. A. Andrade and R. Palazzo, Construction and decoding of BCH codes over finite commutative rings, Linear Algebra & its Application, 286(1999), 69-85.

[3] M. F. Atiyah, Introduction to Commutative Algebra, Addison-Wesley Publishing Company, Menlo Park, California, 1969.

[4] W. E. Barnes, On the Γ-rings of Nobusawa, Pacific J. Math., 18(1966), 411-422.

[5] I. F. Blake, Codes over integer residue rings, Information & Computation/ Information & Control, IANDG, 29(1975), 295-300.

[6] G. L. Booth and N. J. Groenwald, On uniformly strongly prime gamma rings, Bull. Austral. Math. Soc., 37(1988), 437-445.

[7] R. Chinram, On quasi-Γ-ideals in Γ-semigroups, Science Asia, 32(2006), 351-353.

[8] P. S. Das, Fuzzy groups and level subgroups, J. Math. Anal. Appl., 84(1981), 264-269.

[9] J. Denes and A. D. Keedwell, Latin squares and their applications, Akademia Kiado, Budapest, 1974.

[10] T. K. Dutta and N. C. Adhikari, On Γ-semigroups with right and left unities, Soochow J. Math., 19,4(1993), 461-474.

[11] T. K. Dutta and T. Chanda, Structures of fuzzy ideals of Γ-ring, Bull. Malaysian Math. Sci. Soc., 28(2005), 9–18.

[12] R. Gilmar, A note on semigroup rings, Amer. Math. Monthly, 75(1969), 36-37.

[13] R. Gilmer, Commutative Semigroup Rings, The Univ. Chicago Press, 1984.

[14] R. D. Giri and A. K. Wazalwar, Prime ideals and prime radicals in non-commutative semigroups, Kyungpook Math. J., 33(1993), 37-48.

[15] K. Hila, On regular semiprime and quasi-reflexive Γ-semigroup and minimal quasi-ideals, LJM, 29(2008), 141-152.

[16] K. Hila, On quasi-prime, weakly quasi-prime left ideals in ordered-Γ-semigroups, Math. Slovaca, 60(2010), 195-212.

[17] P. Holgate, Groupoids satisfying a simple invertive law, Math. Stud., 61(1992), 101-106.

[18] S. M. Hong and Y. B. Jun, A note on fuzzy ideals in Γ-ring, Bull. Honam. Math. Soc., 12(1995), 39-48.

[19] T. W. Hungerford, Algebra, Holt, Rinehart & Winston, Inc., 1974.

[20] B. D. Janeway, Zero divisor in commutative semigroup rings, Univ. Houstan Dissertation, 1981.

[21] J. Jezek and T. Kepka, Medial groupoids, Rozpravy CSAV, 93/2(1983).

[22] Y. B. Jun, On fuzzy prime ideals of Γ-rings, Soochow J. Math., 21(1)(1995), 41-48.

[23] Y. B. Jun and C. Y. Lee, Fuzzy Γ-rings, Pusan Kyongnam Math. J., 84(1981), 264-269.

[24] R. Jung Cho, Pusan, J. Jezek and T. Kepka Praha, Paramedial Groupoids, Czec. Math. J., 49(1999), 277-290.

[25] M. S. Kamran, Conditions for LA-semigroups to resemble associative structures, Ph.D Thesis, Quaid-i-Azam Univ., Islamabad, 1993.

[26] M. A. Kazim and M. Naseerudin, On almost semigroups, Alig. Bull. Math., 2(1972), 1-7.

[27] N. Kehayopulu, M-systems and N-systems in ordered semigroups, Quasigroups & Related Systems, 11 (2004), 55-58.

[28] E. Kleinfeld, A generalization of commutative and associative rings, Pacific J. Math., 38(1971), 95-101.

[29] E. Kleinfeld, A Generalization of Commutative and Alternative Rings. IV, Proc. Amer. Math. Soc., 46(1974), 21-23.

[30] M. Kondo, W. A. Dudek, On the Transfer Principle in fuzzy theory, Mathware Soft Comput., 12(2005), 41-55.

[31] N. Kuroki, Fuzzy bi-ideals in semigroups, Comment. Math. Univ. St. Pauli, 1(1979), 17-21.

[32] N. Kuroki, On fuzzy ideals and fuzzy bi-ideals in semigroups, Fuzzy Sets & Systems, 5(1981), 203-215.

[33] N. Kuroki, On fuzzy semigroups, Inform. Sci., 53(1991), 203-236.

[34] S. Kyuno, On prime Γ-rings, Pacific J. Math., 75(1978), 185-190.

[35] S. Kyuno, A Γ-ring with the right and left unities, Math. Japonica, 24(1979), 191-193.

[36] W. J. Liu, Fuzzy invariant subgroups and fuzzy ideal, Fuzzy Sets & Systems, 8(1982), 133-139.

[37] J. Luh, On the theory of simple Γ-rings, Michigan Math. J., 16(1969), 65-75.

[38] D. S. Malik, M. Mordeson and M. K Sen, Fundamentals of Abstract Algebra, McGraw-Hill Companies, Inc., 1997.

[39] D. C. Murdoch and O. Ore, On Generalized Rings, Amer. J. Math., 63(1941), 73-86.

[40] Q. Mushtaq, Abelian group defined by LA-semigroup, Studia. Sci. Math. Hangar, 18(1983), 427-428.

[41] Q. Mushtaq, Zeroids and idempoids in AG-groupoids, Quasigroups & Related Systems, 11(2004), 79-84.

[42] Q. Mushtaq and M. Iqbal, Partial ordering and congruences on LA-semigroups, Indian J. Pure Appl. Math., 22(4)(1991), 331-336.

[43] Q. Mushtaq and Q. Iqbal, Decomposition of a locally associative LA-semigroup, Semigroup Forum, 41(1991), 155-164.

[44] Q. Mushtaq and M. S. Kamran, On LA-semigroup with weak associative law, Sci. Khyber, 1(1989), 69-71.

[45] Q. Mushtaq and M. S. Kamran, Left almost group, Proc. Pak. Acad. Sciences, 33(1996), 1-2.

[46] Q. Mushtaq, and M. Khan, Ideals in AG-band and AG*-groupoid, Quasigroups & Related Systems, 14(2006), 207-215.

[47] Q. Mushtaq and M. Khan, The decomposition of a locally associative AG**-groupoid, Adv. Algebra & Analysis, 2(2006), 115-122.

[48] Q. Mushtaq and M. Khan, M-systems in LA-semigroups, Southeast Asian Bull. Math., 33(2009), 321-327.

[49] Q. Mushtaq and M. Khan, Semilattice decomposition of locally associative AG**-groupoids, Algebra Colloquium, 16, 1(2009), 17-22.

[50] Q. Mushtaq and S. M. Yusuf, On LA-semigroups, Alig. Bull. Math., 8(1978), 65-70.

[51] Q. Mushtaq and S. M. Yusuf, On locally associative LA-semigroups, J. Nat. Sci. Math., 1, 19(1979), 57-62.

[52] N. Nobusawa, On a generalization of the ring theory, Osaka J. Math., 1(1964), 81-89.

[53] Y. S. Park and J. P. Kim, Prime and semiprime ideals in semigroups, Kyungpook Math. J., 32(1992), 629-633.

[54] T. Parker, The semigroup ring, Florida State Univ. Dissertation, 1973.

[55] M. Petrich, Introduction to semigroups, Merrill Publ. Co., A Bell & Howell Comp., Columbus, 1973.

[56] P. V. Protic and N. Stevanovic, On Abel-Grassmann's groupoids, Proc. Math. Conf. Pristina, (1994), 31-38.

[57] P. V. Protic and N. Stevanovic, AG-test and some general properties of Abel-Grassmann's groupoids, PU. M. A., 4, 6(1995), 371-383.

[58] P. V. Protic and M. Bozinovic, Some congruence on an AG^{**}-groupoid, Algebra Logic & Discrete Math., Nic, 3, 9(1995), 879-886.

[59] P. V. Protic and N. Stevanovic, Abel-Grassmann's bands, Quasigroups & Related Systems, 11(2004), 95-101.

[60] I. S. Reed, G. Solomon, Polynomial codes over certain finite fields, J. Soc. Industrial & Appl. Math., 8(1960), 300-304.

[61] M. Roitman: On finite generation of powers of ideals, J. Pure & Appl. Algebra, 61(2001), 327-340.

[62] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl., 35(1971) 512-517.

[63] M. K. Sen, On Γ-semigroups, Proc. Inter. Sympos. Algebra & its Application, Decker Publication, New York, (1981), 301-308.

[64] M. K. Sen and N. K. Saha, On Γ-semigroups I, Bull. Cal. Math. Soc., 78(1986), 180-186.

[65] M. K. Sen and N. K. Saha, On orthodox Γ-semigroups, Int. J. Math. Math. Sci., 15(1992), 527-534.

[66] A. Seth, Γ-groups on regular Γ-semigroups, Int. J. Math. Math. Sci., 15(1992), 103-106.

[67] T. Shah, G. Ali and F. Rehman, Direct Sum of Ideals in a generalized LA-Ring, Int. J. Contemp. Math. Sci., 6(2011), 1095-1101.

[68] T. Shah, M. Raees and G. Ali, On LA-Modules, Int. J. Contemp. Math. Sci., 6(2011), 999-1006.

[69] N. Stevanovic and P. V. Protic, The structural theorem for AG*-groupoids, Ser. Math. Inform., 10(1995), 25-33.

[70] S. M. Yusuf, On left almost ring, 7th Int. Pure Math. Conf. Islamabad, Pakistan, 2006.

[71] L. A. Zadeh, Fuzzy Sets, Information & Control, 8(1965), 338-353.