Some Fixed Point Theorems for Multi-valued Maps

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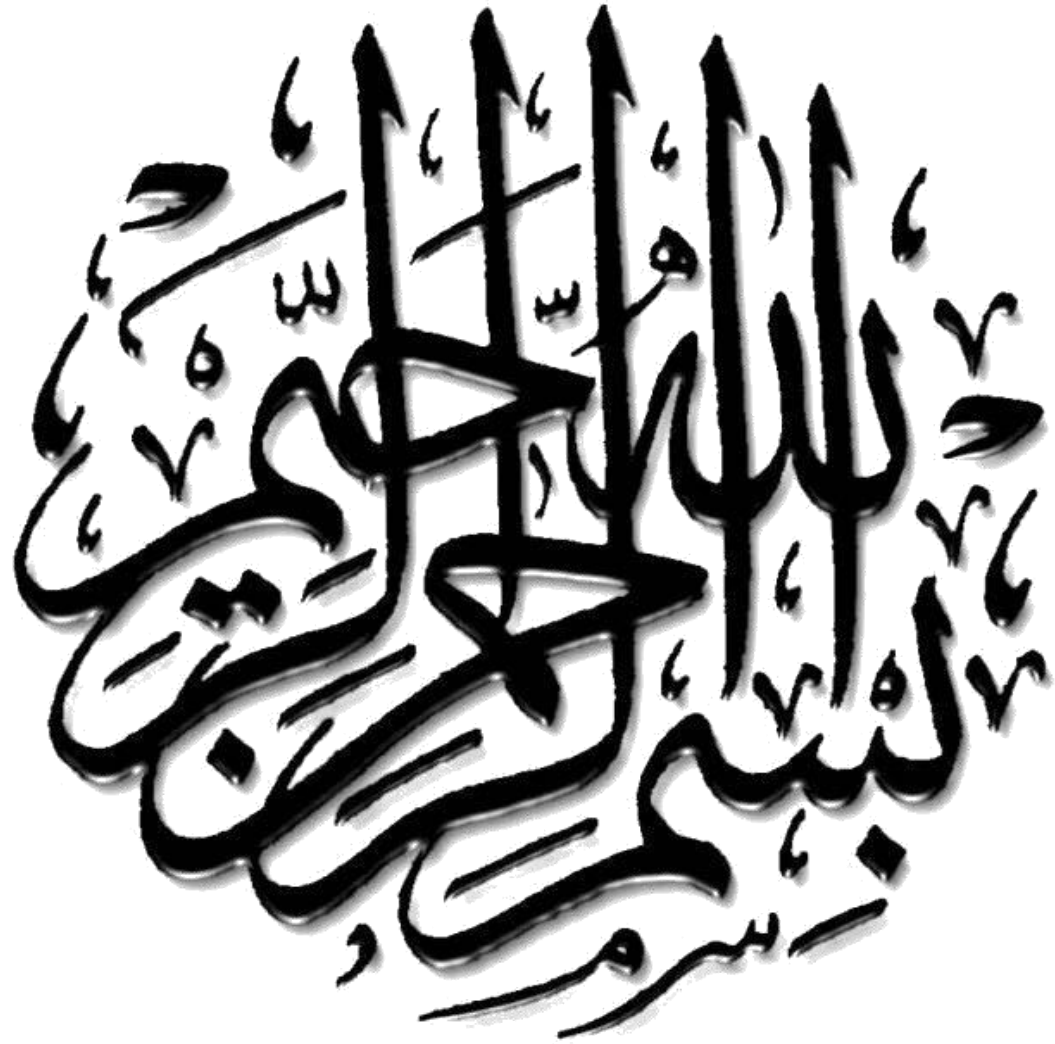
This Thesis is dedicated to

The ideal personalities of my life,
The reasons of my success,

Mother & Father

You have given me so much, thanks for your faith in me.



In the Name of Allah,
The Most Gracious, The Most Merciful.

Acknowledgement

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Sharafat Hussain

Abstract

The notion of Picard operator was introduced by Rus. The notion of weakly Picard operators was introduced and used by Rus and his collaborators. Berinde extended the notion of weakly Picard operator to multi-valued case. Kamran introduced the notion of *f*-weakly Picard operators. Popescu introduced the notion of (s,r)-contractive multi-valued operator. He presented some basic problem for fixed point and strict fixed point theory for (s,r)-contractive multi-valued operator.

In this dissertation, we extend the notion of (s,r)-contractive multi-valued operator and proved some fixed points theorems for newly defined contraction. In chapter one, we recollect some basic definitions and results, which are needed for subsequent chapters. In chapter two, we review some fixed point theorems for (s,r)-contractive multi-valued operator. We study, in detail, some results obtained by Popescu. In chapter three, we define the notion of weakly (s,r)-contractive multi-valued operator and *f*-(s,r)-contractive multi-valued operator and using these conditions we obtain some new fixed point theorems.

Contents

Chapter 1

Preliminaries

In this chapter we present basic concepts and results which will be used in subsequent chapters. Moreover, we shall fix our notions and terminologies to be used in this dissertation. Throughout, this dissertation X is a metric space endowed with a metric d unless stated otherwise.

1.1 Fixed points for single-valued maps

A point $x \in X$ is called fixed point of a mapping $f : X \to X$ if

$$f(x) = x$$

We denote the set of fixed points of f by $Fixf, i.e.,$

$$Fixf = \{x \in X : f(x) = x\}.$$

Note that a mapping need not have a fixed point. Further, if fixed point of a mapping exist then it is not always unique.

1.1.1 Examples

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 - 3x + 4$ then $Fixf = \{2\}$.

1.1.2 Example

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$ then $Fixf = \{0, 1\}$.

1.1.3 Example

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$ then $Fixf = \{0, 1, -1\}$.

1.1.4 Example

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x + 1$ then $Fixf = \Phi$.

1.1.5 Example

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x$ then $Fixf = \mathbb{R}$.

1.2 Lipschtizian Mappings

A mapping $f : X \to X$ is said to be Lipschtizian if there exist a constant $c > 0$ such that $d(fx, fy) \leq cd(x, y)$, for all $x \in X$. Note that a Lipschtizian mapping is uniformly continuous. Now, we give some subclasses for the class of Lipschtizian mapping.

1.2.1 Example

Let $X = \mathbb{R}$ be endowed with the usual metric. Define $f : X \to X$ by $fx = 2x$, then f is Lipschtizian mapping.

1.2.2 Contraction Mappings

A mapping f is said to be a contraction if there exist a constant $0 < c < 1$ such that

$$d(fx, fy) \leq cd(x, y), \forall\ x, y \in X.$$

1.2.3 Example

Let $X = \mathbb{R}$ be endowed with the usual metric. Define $f : X \rightarrow X$ by $fx = 1 + \frac{x}{4}$, then f is a contraction on X.

1.2.4 Contractive Mappings

A mapping f is said to be a contractive mappings if there exist a constant $0 < c < 1$ such that

$$d(fx, fy) < cd(x, y), \forall\ x, y \in X, x \neq y.$$

1.2.5 Example

Let $X = [1, \infty)$ be endowed with the usual metric space. Define $f : X \rightarrow X$ by $fx = x + \frac{1}{x}$, then f is a contractive mapping on X.

1.2.6 Non-expansive Mappings

A mapping f is said to be a non-expansive mappings if,

$$d(fx, fy) \leq d(x, y), \forall\ x, y \in X, x \neq y.$$

1.2.7 Example

Let $X = \mathbb{R}$ be endowed with the usual metric space. Define $f : X \to X$ by $fx = x$, then f is non-expansive on X.

Therefore, Contraction $\Rightarrow$ Contractive $\Rightarrow$ Non-expansive $\Rightarrow$ Lipschtizian

Remark 1 *Note that a contraction mapping is contractive, a contractive mapping is non-expansive and a non-expansive mapping is Lipschtizian.*

1.2.8 Banach Fixed Point Theorem (Contraction Theorem)

Banach showed that every contraction on a complete metric space has a unique fixed point. This result appeared explicitly first time in Banach's doctorial thesis and commonly known as Banach contraction principal. This principle is an existence and uniqueness theorem for fixed points of self-mappings. Banach' contraction principle extensively used to study the existence of solutions for nonlinear integral and differential equations and to prove the convergence of algorithms in computational mathematics.

1.2.9 Theorem[11]

Let (X, d) be a complete metric space and $f : X \to X$ be a contraction on X with contraction constant c. Then f has unique fixed point z. Moreover, for any $x_0 \in X$:

1. The iterative sequence $\{f^n x_0\}$ converge to z.

2. The following prior estimate hold

$$d(x_m, z) \leq \frac{c^m}{1-c} d(x_0, x_1).$$

3. The following posterior estimate hold

$$d(x_m, z) \leq \frac{c}{1-c} d(x_{m-1}, x_m).$$

Proof. Fix any arbitrary element $x_0 \in X$ and define the iterative sequence $\{x_n\}$ by $x_0,\ x_1 = fx_0,\ x_2 = fx_1 = f^2x_0, \cdots, x_n = f^n x_0$.

By triangle inequality, we have

$$d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n)$$

$$\begin{aligned}
&= d(fx_{m-1}, fx_m) + d(fx_m, fx_{m+1}) + \cdots + d(fx_n, fx_{n-1}) \\
&\leq cd(x_{m-1}, x_m) + cd(x_m, x_{m+1}) + \cdots + cd(x_n, x_{n-1}) \\
&= cd(fx_{m-2}, fx_{m-1}) + cd(fx_{m-1}, fx_m) + \cdots + cd(fx_{n-1}, fx_{n-2}) \\
&\leq c^2 d(x_{m-2}, x_{m-1}) + c^2 d(x_{m-1}, x_m) + \cdots + c^2 d(x_{n-1}, x_{n-2}).
\end{aligned}$$

Continuing this process we obtain

$$\begin{aligned}
d(x_m, x_n) &\leq (c^m + c^{m+1} + \cdots + c^{n-1}) d(x_0, x_1) \\
&= c^m \frac{1 - c^{n-m}}{1-c} d(x_0, x_1)
\end{aligned}$$

Since $0 < c < 1$, so $1 - c^{n-m} < 1$, consequently,

$$d(x_m, x_n) \leq \frac{c^m}{1-c} d(x_0, x_1) \tag{1}$$

On the right, $0 < c < 1$ and $d(x_0, x_1)$ is fixed, so that we can make the right-hand side as small as we please by taking m sufficiently large (and $n > m$). This proves that $\{x_m\}$ is a Cauchy sequence. Since X is complete, $\{x_m\}$ converges, *i.e.* there is $z \in X$ such that $x_m \to z$ as $n \to \infty$. Now

$$d(z, fz) \leq d(z, x_m) + d(x_m, fz) \leq d(z, x_m) + cd(x_m, z)$$

letting $m \to \infty$, we get

$$d(z, fz) = 0, \text{ which implies } z = fz.$$

This show that z is fixed point of f. For uniqueness suppose on contrary that y and z are two fixed points of f. Now

$$d(y, z) = d(fy, fz) \leq cd(y, z) < d(y, z), \text{since } 0 < c < 1.$$

this yields a contraction.

letting $n \to \infty$ in equation (1), we get

$$d(x_m, z) \leq \frac{c^m}{1-c} d(x_0, x_1) \tag{2}$$

Taking $m = 1$ and writing y_0 for x_0 and y_1 for x_1 in (2),we have

$$d(y_1, z) \leq \frac{c}{1-c} d(y_0, y_1).$$

setting $y_0 = x_{m-1}$, we have $y_1 = fy_0 = x_m$, we obtain

$$d(x_m, z) \leq \frac{c}{1-c} d(x_{m-1}, x_m).$$

■

1.3 Multi-valued Maps

Let X, Y be two non-empty sets. We say that T is a multi-valued mapping from X into Y if for each $x \in X$, $T(x)$ is a subset of Y. Clearly, a single-valued mappings are special case of multi-valued mappings. We denote $T : X \rightsquigarrow Y$ to represent that T is a multi-valued from X into Y. The trigonometric, hyperbolic and exponential functions are all single-valued mappings, their inverses are multi-valued mappings.

1.3.1 Example

Let $f : X \rightarrow Y$ is a continuous mapping. Then its inverse can be consider as multi-valued mappings $S : Y \rightsquigarrow X$ defined by

$$S(y) = f^{-1}(y), \text{ for } y \in Y.$$

1.3.2 Example

Let $X = [0, 1]$ and let $N(X)$ denote the family of all non-empty subsets of X. Define $T : X \rightarrow N(X)$ by $Tx = [x, 1]$ and $S : X \rightarrow N(X)$

$$Sx = \begin{cases} [0,1] \text{ if } x \neq \frac{1}{2} \\ \{a, b\} \text{ if } x = \frac{1}{2} \end{cases}$$

then T and S are multi-valued mappings.

1.4 Fixed point for multi-valued mapping

Let X and Y be two metric (or topological) spaces and $T : X \rightsquigarrow Y$ be a multi-valued mapping an element $x \in X$ is called fixed point of T if $x \in T(x)$.

1.4.1 Example

Let $X = \{1, 2, 3\}$, $CB(X)$ denotes the set of non empty closed and bounded subsets of X and $d(x, y) = |x - y|$ $\forall$ $x, y \in X$. Define $T : X \rightarrow CB(X)$ by $T1 = T2 = \{1, 2\}$, $T3 = \{3\}$. Then, every $x \in X$ is a fixed point of T.

1.4.2 Example

Let $X = [0,1]$ be endowed with the usual metric d and let $S : X \rightsquigarrow X$ be given by

$$S(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2}, 0 \leq x \leq \frac{1}{2} \\ -\frac{1}{2}x + 1, \frac{1}{2} < x \leq 1 \end{cases}.$$

Define $T : X \to CB(X)$ by $T(x) = \{0\} \cup \{S(x)\}$ for each $x \in X$, then the set of fixed points of T is $\{0, \frac{2}{3}\}$.

1.4.3 Definition

Let X and Y be two metric spaces and $T, S : X \rightsquigarrow Y$ be two multi-valued mapping. An element $x \in X$ is said to be common fixed point of T and S, if $x \in Tx$ and $x \in Sx$.

1.4.4 Example

Define $T, S : [a,b] \to [a,b]$, where $b > a$ and $a, b \in \mathbb{R}$ by

$$Tx = \begin{cases} \{a\} \text{ if } x = \{a,b\}; \\ [x,b] \text{ if } a < x < b, \end{cases}$$

and

$$Sx = [a,x] \text{ for all } x \in [a,b].$$

then each $x \in [a,b]$ is a common fixed point of T and S.

1.5 Hausdorff Metric

The key to the classical Banach fixed point theorem is that one is working in a complete metric space. To get an analogous result for multi-valued mappings,we have to equip the powerset of a metric space with a metric. One such metric on the power set of a metric space X was given by Hausdorff and it is commonly known as Hausdorff metric. Now we give some detail to explain the notion of Hausdorff metric.

Let M, N be subsets of X then,

$$D(x, N) = \inf\{d(x, y) : y \in N\}$$

$$D(M, N) = \sup\{D(x, N) : x \in M\}$$

1.5.1 Example

Let $X = [0.1]$ be endowed with the usual metric $d, A = (0, 1)$ be a non empty subset of X. Then for all $x \in X$ $D(x, A) = 0$.

1.5.2 Definition

The Hausdorff metric on the family of all non-empty closed bounded subsets of a metric space is defined by

$$H(M, N) = \max\{\ D(M, N),\ D(N, M)\}.$$

1.5.3 Example

Let $X = \mathbb{R}, A = [1, 3], B = [2, 5]$ and $d(x, y) = |x - y|$ for all $x, y \in A, B$. Then $H(A, B) = 2$.

It is well known that if (X, d) is a complete metric space, then $(CB(X), H)$ is a complete metric space, where H is Hausdorff metric induced by d[7].

1.6 Multi-valued Contraction

Nadler [6] gave a generalization of Banach's contraction principle to the case of multi-valued mappings. In this section we give the proof of Nadler theorem.

1.6.1 Definition

Let (X, d) be a metric space. A map $T : X \to CB(X)$ is called multi-valued contraction if

$$H(Tx, Ty) \leq kd(x, y) \text{ for all } x, y \in X,$$

for some $k \in [0, 1)$.

1.6.2 Example

Let $X = [0, 1]$ be endowed with the usual metric space d and let $S : X \to X$ be given by

$$S(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2}, 0 \leq x \leq \frac{1}{2} \\ -\frac{1}{2}x + 1, \frac{1}{2} < x \leq 1 \end{cases}.$$

Define $T : X \to CB(X)$ by $T(x) = \{0\} \cup \{S(x)\}$ for each $x \in X$, then T is multi-valued contraction mapping.

1.6.3 Lemma[6]

If $A, B \in CB(X)$ and $a \in A$, then for each $k > 0$, there exists $b \in B$ such that

$$d(a, b) \leq H(A, B) + k.$$

1.6.4 Nadler fixed point Theorem [6]

Let (X, d) be a complete metric space and $T : X \to CB(X)$ is a multi-valued contraction mapping. Then T has a fixed point.

Proof. Let $x_0 \in X$. Choose $x_1 \in Tx_0$. Since Tx_0, $Tx_1 \in CB(X)$ and $x_1 \in Tx_0$, there exist $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq H(Tx_0, Tx_1) + k.$$

Since $Tx_1, Tx_2 \in CB(X)$ and $x_2 \in Tx_1$, there exist $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq H(Tx_1, Tx_2) + k^2.$$

continuing in the same way, we get a sequence $\{x_n\}$ such that $x_{n+1} \in Tx_n$ and

$$d(x_n, x_{n+1}) \leq H(Tx_{n-1}, Tx_n) + k^n, \text{for all } n \in \mathbb{N}.$$

Now we have

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq H(Tx_{n-1}, Tx_n) + k^n \\
&\leq kd(x_{n-1}, x_n) + k^n \\
&\leq k[H(Tx_{n-2}, Tx_{n-1}) + k^n] + k^n \\
&\leq k^2 d(x_{n-2}, x_{n-1}) + 2k^n \\
&\vdots \\
&\leq k^n d(x_1, x_0) + nk^n, \forall n \in \mathbb{N}.
\end{aligned}$$

Hence

$$\begin{aligned} d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+m-1}, x_{n+m}) \\ &\leq k^n d(x_1, x_0) + nk^n + k^{n+1} d(x_1, x_0) + (n+1)k^{n+1} \\ &\quad + \cdots + k^{n+m-1} d(x_1, x_0) + (n+m-1)k^{n+m-1} \\ &= \sum_{i=n}^{n+m-1} (k^i) d(x_1, x_0) + \sum_{i=n}^{n+m-1} (ik^i), \forall n, m \in \mathbb{N}. \end{aligned}$$

It follows that $\{x_n\}$ is Cauchy sequence. Since (X, d) is complete, so the sequence $\{x_n\}$ converges to some $x \in X$. Therefore the sequence $\{Tx_n\}$ converges to Tx, since $x_n \in Tx_{n-1}$ for all n, it follows that $x \in Tx$. ■

1.6.5 Example

Let $X = [0, \infty)$ be endowed with the usual metric and $T : X \to CB(X)$ be a multi-valued mapping such that $Tx = \frac{1}{3}\{x, x+u\}$, for each $x \in X$. where u is some finite real number. For each $x, y \in X$, we have

$$H(Tx, Ty) = \frac{1}{3}|x - y| = \frac{1}{3} d(x, y).$$

Hence by theorem 1.6.4, T has fixed point.

Chapter 2

(s, r)-Contractive Multi-valued Operator

Rus [8] introduced the notion of a multi-valued weakly Picard operator. Popescu [4] introduced the notion of (s, r)-contractive multi-valued operators and showed that they are weakly picard operators. He also obtained fixed point and strict fixed point theorems for (s, r)-contractive multi-valued operators. This chapter is a review of the paper by Popescu [4].

2.1 Multi-valued Weakly Picard Operators

Berinde and Berinde [9] extended the notion of weak picard operator from single valued mapping to multi-valued mapping. They also introduced, the notion of multi-valued (θ, L) weak contraction[10].

A multi-valued operator T from metric space X to set of all non-empty closed and bounded subsets of X is said to be multi-valued weakly picard operator iff $\forall\ x \in X$ and $y \in Tx$, there exit a sequence $\{x_n\}$ in X such that

(i) $x_0 = x, x_1 = y$,

(ii) $x_{n+1} \in Tx_n$ for all $n \geq 0$,

(iii) $\{x_n\}$ is convergent and its limit is fixed point of T.

2.1.1 Example [6]

Let (X, d) be complete metric space and $T : X \to CB(X)$ be a multi-valued contraction. Then T is a multi-valued weakly Picard operator.

2.1.2 Definition [9]

Let $T : X \to CB(X)$ be a multi-valued operator T is said to be multi-valued weak contraction or a multi-valued (θ, L) weak contraction iff for all $x, y \in X$ there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(x, y) + L(y, Tx).$$

2.2 Multi-valued f-weak contraction

Kamran [3] extended the notion of weak contraction and presented the notions of multi-valued f-weak contraction and generalized multi-valued f-weak contraction.

2.2.1 Definition [3]

Let (X, d) be a metric space, $f : X \to X$ and $T : X \to CB(X)$ be a multi-valued operator. T is said to be an $f-$ weakly picard operator iff for all $x \in X$ and $fy \in Tx$ $(y \in X)$, there exit a sequence $\{x_n\}$ in X such that

(i) $x_0 = x, x_1 = y,$

(ii) $fx_{n+1} \in Tx_n$ for all $n \geq 0,$

(iii) $\{fx_n\}$ is converges to fp where p is the coincidence point of f and T.

2.2.2 Example [3]

Let $g : X \to X$ and $S : X \to CL(X)$ be a multi-valued operator such that $SX \subset gX$, and

$$H(Sx, Sy) \leq h[tE(x, y) + (1 - t)F(x, y)]$$

for all $x, y \in X, 0 \leq h < 1, 0 \leq t \leq 1$, where

$$E(x, y) = \max\{d(gx, gy), d(gx, Sx), d(gy, Ty), \frac{1}{2}[d(gx, Sy) + d(gy, Sx)]\},$$

$$\begin{aligned} F(x, y) &= [\max\{d^2(gx, gy), d(gx, Sx)d(gy, Ty), d(gx, Sy)d(gy, Sx), \\ &\quad \frac{1}{2}d(gx, Sx)d(gy, Tx), \frac{1}{2}d(gx, Sy)d(gy, Ty)\}]^2 \end{aligned}$$

T is an $f-$weakly Picard operator if one of SX and gX is a complete subspace of X.

2.2.3 Definition [3]

Let (X, d) be a metric space, $f : X \to X$ and $T : X \to CB(X)$ be a multi-valued operator. T is called a multi-valued f-weak contraction or a multi-valued (f, θ, L)-weak contraction iff there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(fx, fy) + Ld(fy, Tx), \text{ for all } x, y \in X.$$

2.3 (s, r) Contraction

Popescu [4] introduce the notion of a (s, r) contractive multi-valued operator as follows.

2.3.1 Definition [4]

A multi-valued operator T from metric space X to set of all non-empty closed and bounded subsets of X is said to be a (s, r)-contractive multi-valued operator if for $r \in [0, 1)$, $s \geq r$ such that,

$$D(y, Tx) \leq sd(y, x) \;\Rightarrow\; H(Tx, Ty) \leq rM_T(x, y), \forall\, x, y \in X.$$

where

$$M_T(x, y) = \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\}.$$

2.4 Fixed Point Theorems For (s, r)-Contractive Multi-valued Operator

In this section we study, in detail, some results obtained by Popescu [4] for (s, r) contractive multi-valued operator.

2.4.1 Theorem [4]

Let T be a $(s, r)-$ contractive multi-valued operator from complete metric space X to set of all non-empty closed and bounded subsets of X with $s > r$. Then T is a multi-valued weakly picard operator.

Proof. Take a real number $t < 1$ such that $0 \leq r < t < s$. Let $x_1 \in X$ and $x_2 \in Tx_1$.Then $D(x_2, Tx_1) = 0 \leq sd(x_2, x_1)$ and by hypothesis we have

$$\begin{aligned} D(x_2, Tx_2) &\leq H(Tx_1, Tx_2) \\ &\leq r \max\{d(x_1, x_2), D(x_1, Tx_1), D(x_2, Tx_2), \\ &\quad \frac{D(x_1, Tx_2) + D(x_2, Tx_1)}{2}\} \\ D(x_2, Tx_2) &\leq r \max\{d(x_1, x_2), D(x_2, Tx_2), \\ &\quad \frac{d(x_1, x_2) + D(x_2, Tx_2)}{2}\}. \end{aligned}$$

As $r < 1$, so we have $D(x_2, Tx_2) \leq rd(x_1, x_2)$. Then there exist $x_3 \in Tx_2$ such that $d(x_2, x_3) \leq rd(x_1, x_2)$. Therefore a sequence $\{x_n\}$ can be constructed in X such that $x_{n+1} \in Tx_n$ and $d(x_{n+1}, Tx_{n+2}) \leq td(x_n, x_{n+1})$ for all $n \in N$. Therefore, we have

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=1}^{\infty} t^{n-1} d(x_1, x_2) < \infty$$

which implies $\{x_n\}$ is a Cauchy sequence. Since X is complete, there is some point $z \in X$ such that $\{x_n\}$ converges to z Now,we will show that there exist a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$D(z, Tx_{n(k)}) \leq sd(z, x_{n(k)}) \ \forall\ k \in \mathbb{N}.$$

Suppose that there exist a positive integer $N \in \mathbb{N}$ such that

$$D(z, Tx_n) > sd(z, x_n) \ \ \forall\ n \geq N.$$

This implies

$$d(z, x_{n+1}) > sd(z, x_n) \ \forall\ n \geq N.$$

By induction, we have

$$d(z, x_{n+p}) > s^p d(z, x_n) \ \forall\ n \geq N, p \geq 1 \tag{1}$$

Since

$$d(x_{n+p}, x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}).$$

$$\begin{aligned} d(x_{n+p}, x_n) &\leq d(x_n, x_{n+1})(1 + t + t^2 + \cdots + t^{p-1}) \\ &= \frac{1 - t^p}{1 - t} d(x_n, x_{n+1}) \ \forall\ n \geq N, p \geq 1. \end{aligned}$$

Letting $p \to \infty$,we obtain

$$d(z, x_n) \leq \frac{1}{1-t} d(x_n, x_{n+1}) \ \forall\ n \geq 1.$$

Thus we have,

$$\begin{aligned} d(z, x_{n+p}) &\leq \frac{1}{1-t} d(x_{n+p}, x_{n+p+1}) \\ &\leq \frac{t^p}{1-t} d(x_n, x_{n+1}) \ \forall\ n \geq 1, p \geq 1 \end{aligned} \tag{2}$$

(1) and (2) implies,

$$d(z, x_n) < \frac{(\frac{t}{s})^p}{1-t} d(x_n, x_{n+1}) \ \forall \ n \geq N, p \geq 1.$$

By letting $p \to \infty$ we obtain

$$d(z, x_n) = 0 \ \forall \ n \geq N.$$

which contradicts (1). Therefore, there exist a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$D(z, Tx_{n(k)}) \leq sd(z, x_{n(k)}) \ \forall \ k \in \mathbb{N}.$$

Therefore, we have

$$\begin{aligned} H(Tz, Tx_{n(k)}) &\leq r \max\{d(z, x_{n(k)}), D(z, Tz), D(x_{n(k)}, Tx_{n(k)}), \\ &\quad \frac{D(z, Tx_{n(k)}) + D(x_{n(k)}, Tz)}{2}\}. \end{aligned}$$

Hence

$$\begin{aligned} D(x_{n(k)+1}, Tz) &\leq r \max\{d(z, x_{n(k)}), D(z, Tz), D(x_{n(k)}, Tx_{n(k)+1}), \\ &\quad \frac{D(z, Tx_{n(k)+1}) + D(x_{n(k)}, Tz)}{2}\}. \end{aligned}$$

Letting $k \to \infty$ we obtain

$$D(z, Tz) \leq r \max\{D(z, Tz), \frac{D(z, Tz)}{2}\}$$

As $r < 1$ which yields $D(z, Tz) = 0$.

Since $Tz \in CB(X)$ so $z \in Tz$. ■

2.4.2 Theorem [4]

Let (X, d) be a complete metric space and $T : X \to X$ be an $(s, r)-$contractive single-valued operator. Then T has a fixed point. Moreover, if $s \geq 1$ then T has a unique fixed point.

Proof. From Theorem 2.4.1 T has a fixed point. If $s \geq 1$ Suppose that T has two distinct fixed points x and y. Then

$$d(y, Tx) = d(y, x) \leq sd(y, x)$$

So by hypothesis, $d(Tx, Ty) \leq rM_T(x, y)$. It follows that $d(x, y) \leq rd(x, y)$. Since $r < 1$ which contradict our supposition. ■

2.4.3 Theorem [4]

Let T be a mapping from complete metric space X to $CB(X)$. Assume that there exist $r, s \in [0, 1)$ such that $r < s$ and

$$\frac{1}{1+r} D(x, Tx) \leq d(y, x) \leq \frac{1}{1-s} D(x, Tx) \;\Rightarrow\; H(Tx, Ty) \leq rM_T(x, y).$$

Where

$$M_T(x, y) = \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\}.$$

Then T is a multi-valued weakly picard operator.

Proof. Take a real number $t < 1$ such that $0 \leq r < t < s$. Let $x_1 \in X$ and $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq \frac{1-t}{1-s} D(x_1, Tx_1).$$

Then

$$\frac{1}{1+r} D(x_1, Tx_1) \leq D(x_1, Tx_1) \leq d(x_1, x_2) \leq \frac{1}{1-s} D(x_1, Tx_1).$$

and by hypothesis we have

$$\begin{aligned} D(x_2, Tx_2) &\leq H(Tx_1, Tx_2) \\ &\leq r \max\{d(x_1, x_2), D(x_1, Tx_1), D(x_2, Tx_2), \frac{D(x_1, Tx_2) + D(x_2, Tx_1)}{2}\}. \\ &\leq r \max\{d(x_1, x_2), D(x_2, Tx_2), \frac{d(x_1, x_2) + D(x_2, Tx_2)}{2}\}. \end{aligned}$$

As $r < 1$, so we have

$$D(x_2, Tx_2) \leq rd(x_1, x_2).$$

Then there exist $x_3 \in Tx_2$ such that $d(x_2, x_3) \leq rd(x_1, x_2)$. Therefore a sequence $\{x_n\}$ can be constructed in X such that $x_{n+1} \in Tx_n$ and $d(x_{n+1}, Tx_{n+2}) \leq td(x_n, x_{n+1})$ for all $n \in N$. Therefore, we have

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=1}^{\infty} t^{n-1} d(x_1, x_2) < \infty$$

which implies $\{x_n\}$ is a Cauchy sequence. Since X is complete, there is some point $z \in X$ such that $\{x_n\}$ converges to z. Since

$$d(x_{n+p}, x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}).$$

$$\begin{aligned} d(x_{n+p}, x_n) &\leq d(x_n, x_{n+1})(1 + t + t^2 + \cdots + t^{p-1}) \\ &= \frac{1 - t^p}{1 - t} d(x_n, x_{n+1}) \ \forall\ n \geq N, p \geq 1. \end{aligned}$$

Now letting $p \to \infty$ we have

$$d(z, x_n) \leq \frac{1}{1 - t} d(x_n, x_{n+1}) \ \forall\ n \geq 1.$$

Since

$$d(x_n, x_{n+1}) \leq \frac{1-t}{1-s} D(x_n, Tx_n).$$

we have

$$d(z, x_n) \leq \frac{1}{1-s} D(x_n, Tx_n) \ \forall \ n \geq 1.$$

Now suppose that there exist $N > 0$ such that

$$d(z, x_n) < \frac{1}{1+r} D(x_n, Tx_n) \ \forall \ n \geq N.$$

Thus we have,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(z, x_n) + d(z, x_{n+1}) \\ &< \frac{1}{1+r} \left[D(x_n, Tx_n) + D(x_{n+1}, Tx_{n+1}) \right] \\ &< \frac{1}{1+r} \left[D(x_n, Tx_n) + r d(x_n, x_{n+1}) \right]. \end{aligned}$$

This implies that

$$d(x_n, x_{n+1}) < D(x_n, Tx_n).$$

which is not possible. So there exist a subsequence$\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$d(z, x_{n(k)}) \geq \frac{1}{1+r} D(x_{n(k)}, Tx_{n(k)}) \ \forall \ k \geq N.$$

Since

$$d(z, x_n) \leq \frac{1}{1-s} D(x_n, Tx_n) \ \forall \ n \geq 1,$$

Thus we have

$$\begin{aligned} H(Tz, Tx_{n(k)}) \leq \ & r \max\{d(z, x_{n(k)}), D(z, Tz), D(x_{n(k)}, Tx_{n(k)}), \\ & \frac{D(z, Tx_{n(k)}) + D(x_{n(k)}, Tz)}{2}\}. \end{aligned}$$

Hence

$$D(x_{n(k)+1}, Tz) \leq r\max\{d(z, x_{n(k)}), D(z, Tz), D(x_{n(k)}, Tx_{n(k)+1}), \frac{D(z, Tx_{n(k)+1}) + D(x_{n(k)}, Tz)}{2}\}.$$

Let $k \to \infty$ we have

$$D(z, Tz) \leq r\max\{D(z, Tz), \frac{D(z, Tz)}{2}\}.$$

$$D(z, Tz) \leq r\max\{D(z, Tz), \frac{D(z, Tz)}{2}\}.$$

As $r < 1$,then we get $D(z, Tz) = 0$ and since $Tz \in CB(X), z \in Tz$. ■

2.4.4 Corollary

Let (X, d) be a complete metric space, $T : X \to X$ be a mapping . Assume that there exist $r \in [0, 1]$ such that

$$\frac{1}{1+r}D(x, Tx) \leq d(x, y) \leq \frac{1}{1-r}D(x, Tx) \;\Rightarrow\; H(Tx, Ty) \leq rM_T(x, y)$$

where

$$M_T(x, y) = \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\}.$$

The there exist $z \in X$ such that $Tz = z$.

Proof. One can easily prove that for every $x_1 \in X$ the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ be such that $d(x_{n+1}, x_{n+2}) \leq rd(x_n, x_{n+1})$ also $\{x_n\}$ is Cauchy and there is point $z \in X$ such that $\lim_{n\to\infty} x_n = z$. From above theorem we have $d(z, x_n) \leq \frac{1}{1-r}d(x_n, x_{n+1})$ $\forall$ $n \geq 1$ and there exist a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$d(z, x_{n(k)}) \geq \frac{1}{1+r} d(x_{n(k)}, x_{n(k)+1}) \ \forall \ k \geq N.$$

Therefore, we obtain

$$\begin{aligned} d(x_{n(k)+1}, Tz) \quad &\leq \quad r \max\{d(z, x_{n(k)}), d(z, Tz), d(x_{n(k)}, Tx_{n(k)+1}), \\ &\frac{d(z, Tx_{n(k)+1}) + d(x_{n(k)}, Tz)}{2}\}. \end{aligned}$$

by taking $k \to \infty$ we obtain $Tz = z$. ■

Chapter 3

Generalization of (s, r) Contractive Multi-valued Operator

3.1 Introduction

In this Chapter we use the concept of $f-$weakly picard operator given by Kamran [3] to extend the results presented by Popescu [4] for (s, r) contractive multi-valued operator.

Throughout this chapter, we denote set of all non-empty closed and bounded subsets of a metric space X by $CB(X)$ and all non-empty closed subsets of a metric space X by $CL(X)$.

3.2 Weakly (s, r)-Contractive Multi-valued Operator

In this section we introduce the notion of weakly (s, r)-contractive multi-valued operator and extend the results given by Popescu [4]. We start this section with following definition.

3.2.1 Definition

Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multi-valued operator. T is said to be weakly (s,r)-contractive multi-valued operator if $r \in [0,1), s \geq r, L \geq 0$ with

$$D(y,Tx) \leq sd(y,x) \Rightarrow H(Tx,Ty) \leq rM(x,y) \; \forall \; x,y \in X.$$

where,

$$\begin{aligned} M(x,y) &= \max\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty)+D(y,Tx)}{2}\} \\ &\quad +L\min\{(d(x,y), d(y,Tx)\}. \end{aligned}$$

Remark 2 *When $L = 0$ the above definition reduce to definition 2.3.1*

3.2.2 Theorem

Let $T : X \to CB(X)$ be weakly $(s,r)-$ contractive multi-valued operator with $s > r$ and $L \geq 0$ where (X,d) is a complete metric space. Then T is multi-valued weakly picard operator.

Proof. Take a real number $t < 1$ such that $0 \leq r < t < s$. Let $x_1 \in X$ and

$x_2 \in Tx_1$.Then $D(x_2, Tx_1) = 0 \leq sd(x_2, x_1)$ and by hypothesis we have.

$$\begin{aligned} D(x_2, Tx_2) &\leq H(Tx_1, Tx_2) \\ &\leq r \max\{d(x_1, x_2), D(x_1, Tx_1), D(x_2, Tx_2), \frac{D(x_1, Tx_2) + D(x_2, Tx_1)}{2}\} \\ &\quad +L \min\{(d(x_1, x_2), d(x_2, Tx_1). \\ D(x_2, Tx_2) &\leq r \max\{d(x_1, x_2), D(x_2, Tx_2), \frac{d(x_1, x_2) + D(x_2, Tx_2)}{2}\} + 0. \end{aligned}$$

As $r < 1$, so we have $D(x_2, Tx_2) \leq rd(x_1, x_2)$. Then there exist $x_3 \in Tx_2$ such that $d(x_2, x_3) \leq rd(x_1, x_2)$. Therefore a sequence $\{x_n\}$ can be constructed in X such that $x_{n+1} \in Tx_n$ and $d(x_{n+1}, Tx_{n+2}) \leq td(x_n, x_{n+1})$ for all $n \in N$. Therefore, we have

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=1}^{\infty} t^{n-1} d(x_1, x_2) < \infty$$

which implies $\{x_n\}$ is a Cauchy sequence. Since X is complete, there is some point $z \in X$ such that $\{x_n\}$ converges to z Now, we claim that there exist a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$D(z, Tx_{n(k)}) \leq sd(z, x_{n(k)}) \ \forall \ k \in \mathbb{N}.$$

Suppose on contrary that there exist a positive integer $N \in \mathbb{N}$ such that

$$D(z, Tx_n) > sd(z, x_n) \ \ \forall \ n \geq N.$$

This implies

$$d(z, x_{n+1}) > sd(z, x_n) \ \forall \ n \geq N.$$

By induction, we obtain

$$d(z, x_{n+p}) > s^p d(z, x_n) \ \forall \ n \geq N, p \geq 1. \quad (1)$$

Since

$$\begin{aligned} d(x_{n+p}, x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}). \\ d(x_{n+p}, x_n) &\leq d(x_n, x_{n+1})(1 + t + t^2 + \cdots + t^{p-1}) \\ &= \frac{1 - t^p}{1 - t} \ \ d(x_n, x_{n+1}) \ \forall \ n \geq N, p \geq 1. \end{aligned}$$

Letting $p \to \infty$,we obtain

$$d(z, x_n) \leq \frac{1}{1-t} d(x_n, x_{n+1}) \ \forall \ n \geq 1.$$

Thus we have,

$$\begin{aligned} d(z, x_{n+p}) &\leq \frac{1}{1-t} d(x_{n+p}, x_{n+p+1}) \\ &\leq \frac{t^p}{1-t} d(x_n, x_{n+1}) \ \forall \ n \geq 1, p \geq 1. \end{aligned} \tag{2}$$

From (1) and (2) we obtained

$$d(z, x_n) < \frac{(\frac{t}{s})^p}{1-t} d(x_n, x_{n+1}) \ \forall \ n \geq N, p \geq 1.$$

for all $n \geq N, p \geq 1$.By letting $p \to \infty$ we have $d(z, x_n) = 0$ for all $n \geq N$ which contradicts (1) therefore there exist a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$D(z, Tx_{n(k)}) \leq sd(z, x_{n(k)}) \ \forall \ k \in \mathbb{N}.$$

Thus we have

$$\begin{aligned}
D(x_{n(k)+1}, Tz) &\leq H(Tz, Tx_{n(k)}) \\
&\leq r\max\{d(z, x_{n(k)}), D(z, Tz), D(x_{n(k)}, Tx_{n(k)}), \frac{D(z, Tx_{n(k)}) + D(x_{n(k)}, Tz)}{2}\} \\
&\quad + L\min\{(d(x_{n(k)}, z), d(x_{n(k)}, Tz)\} \\
D(x_{n(k)+1}, Tz) &\leq r\max\{d(z, x_{n(k)}), D(z, Tz), D(x_{n(k)}, Tx_{n(k)+1}), \frac{D(z, Tx_{n(k)+1}) + D(x_{n(k)}, Tz)}{2}\} \\
&\quad + L\min\{(d(x_{n(k)}, z), d(x_{n(k)}, Tz)\}.
\end{aligned}$$

Letting $k \to \infty$ we have

$$D(z, Tz) \leq r\max\{D(z, Tz), \frac{D(z, Tz)}{2}\} + L\min\{(d(z, z), d(z, Tz)\}.$$

As $r < 1$,then we get $D(z, Tz) = 0$ and since $Tz \in CB(X), z \in Tz$. ■

Remark 3 *When $L = 0$ then above theorem reduces to theorem 2.4.1*

3.2.3 Example

Let $X = \{1, 2, 3\}$ and $d(x, y) = |x - y| \ \forall \ x, y \in X$. Let $T : X \to CB(X)$ be such that $T1 = T2 = \{1, 2\}, T3 = \{3\}$.Then:

(a) T is a weakly (s, r)-contractive multi-valued operator with $r = 0.3, s = 0.6$ and $L = 1$;

(b) Every $x \in X$ is a fixed point of T;

(c) T is not an $(s, r)-$contractive multi-valued operator.

Proof. (a) We have

$$H(T1, T1) = H(T1, T2) = H(T3, T3) = H(T2, T2) = 0,$$

and

$$D(3,T1) = 1 < sd(3,1) = 1.2,$$

implies

$$\begin{aligned} H(T3,T1) &= 1 \\ &< r\max\{d(3,1), D(3,T3), D(1,T1), \frac{D(3,T1)+D(1,T3)}{2}\} \\ +L\min\{d(3,1), d(1,T3)\} &= 2.6, \end{aligned}$$

$$\begin{aligned} D(1,T3) &= 2 > sd(1,3) = 1.2, \\ D(2,T3) &= 1 > sd(2,3) = 0.6, \\ D(3,T2) &= 1 > sd(3,2) = 0.6, \end{aligned}$$

so T is a weakly (s,r)-contractive multi-valued operator with $r = 0.3, s = 0.6$ and $L = 1$.

(b) It is obvious.

(c)

$$D(3,T1) = 1 < sd(3,1) = 1.2,$$

but

$$\begin{aligned} H(T3,T1) &= 1 \\ &> r\max\{d(3,1), D(3,T3), D(1,T1), \frac{D(3,T1)+D(1,T3)}{2}\} \\ &= 0.6; \end{aligned}$$

■

3.2.4 Theorem

Let (X, d) be a complete metric space and $T : X \to X$ be a weakly $(s, r)-$contractive single-valued operator. Then T has a fixed point. Moreover, if $s \geq 1$ and $L + r < 1$ then T has a unique fixed point.

Proof. From Theorem 3.2.2 T has a fixed point. If $s \geq 1$ Suppose that T has two distinct fixed points x and y.

Then

$$d(y, Tx) = d(y, x) \leq sd(y, x)$$

Thus

$$d(Tx, Ty) \leq rM(x, y).$$

It follows that $d(x, y) \leq (r + L)d(x, y)$,since $(r + L) < 1$ which is a contradiction. ∎

3.2.5 Theorem

Let T be a mapping from complete metric space X to set of all non-empty closed and bounded subsets of X. Assume that there exist $r, s \in [0, 1], r < s$ such that

$$\frac{1}{1+r}D(x, Tx) \leq d(x, y) \leq \frac{1}{1-s}D(x, Tx) \Rightarrow H(Tx, Ty) \leq rM(x, y)$$

where

$$\begin{aligned} M(x, y) &= \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2}\} \\ &\quad + L\min\{d(x, y), d(y, Tx). \end{aligned}$$

Then T is a multi-valued weakly picard operator.

Proof. Take a real number $t < 1$ such that $0 \leq r < t < s$. Let $x_1 \in X$ and $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq \frac{1-t}{1-s} D(x_1, Tx_1)$$

Then

$$\frac{1}{1+r} D(x_1, Tx_1) \leq D(x_1, Tx_1) \leq d(x_1, x_2) \leq \frac{1}{1-s} D(x_1, Tx_1)$$

and by hypothesis we have

$$\begin{aligned} D(x_2, Tx_2) &\leq H(Tx_1, Tx_2) \\ &\leq r \max\{d(x_1, x_2), D(x_1, Tx_1), D(x_2, Tx_2), \frac{D(x_1, Tx_2) + D(x_2, Tx_1)}{2}\} \\ &\quad + L \min\{(d(x_1, x_2), d(x_2, Tx_1) \\ &\leq r \max\{d(x_1, x_2), D(x_2, Tx_2), \frac{d(x_1, x_2) + D(x_2, Tx_2)}{2}\} + 0. \end{aligned}$$

As $r < 1$, so we have

$$D(x_2, Tx_2) \leq rd(x_1, x_2).$$

Then there exist $x_3 \in Tx_2$ such that $d(x_2, x_3) \leq rd(x_1, x_2)$. Therefore a sequence $\{x_n\}$ can be constructed in X such that $x_{n+1} \in Tx_n$ and $d(x_{n+1}, Tx_{n+2}) \leq td(x_n, x_{n+1})$ for all $n \in N$. Therefore, we have

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \sum_{n=1}^{\infty} t^{n-1} d(x_1, x_2) < \infty$$

which implies $\{x_n\}$ is a Cauchy sequence. Since X is complete, there is some point $z \in X$ such that $\{x_n\}$ converges to z. Since

$$d(x_{n+p}, x_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+p-1}, x_{n+p}).$$

$$\begin{aligned} d(x_{n+p}, x_n) &\leq d(x_n, x_{n+1})(1 + t + t^2 + \cdots + t^{p-1}) \\ &= \frac{1 - t^p}{1 - t} d(x_n, x_{n+1}) \ \forall \ n \geq N, p \geq 1. \end{aligned}$$

Now letting $p \to \infty$ we have

$$d(z, x_n) \leq \frac{1}{1 - t} d(x_n, x_{n+1}) \ \forall \ n \geq 1.$$

as

$$d(x_n, x_{n+1}) \leq \frac{1 - t}{1 - s} D(x_n, Tx_n).$$

we have

$$d(z, x_n) \leq \frac{1}{1 - s} D(x_n, Tx_n) \ \forall \ n \geq 1.$$

Now suppose that there exist $N > 0$ such that

$$d(z, x_n) < \frac{1}{1 + r} D(x_n, Tx_n) \ \forall \ n \geq N.$$

Therefore, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(z, x_n) + d(z, x_{n+1}) \\ &< \frac{1}{1 + r} \left[D(x_n, Tx_n) + D(x_{n+1}, Tx_{n+1}) \right] \\ &< \frac{1}{1 + r} \left[D(x_n, Tx_n) + r d(x_n, x_{n+1}) \right]. \end{aligned}$$

This implies that

$$d(x_n, x_{n+1}) < D(x_n, Tx_n).$$

which is not possible. So there exist a subsequence$\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$d(z, x_{n(k)}) \geq \frac{1}{1 + r} D(x_{n(k)}, Tx_{n(k)}) \ \forall \ k \geq N..$$

Since

$$d(z, x_n) \leq \frac{1}{1-s} D(x_n, Tx_n) \ \forall \ n \geq 1,$$

By hypothesis we have

$$\begin{aligned} H(Tz, Tx_{n(k)}) &\leq r \max\{d(z, x_{n(k)}), D(z, Tz), D(x_{n(k)}, Tx_{n(k)}), \\ &\quad \frac{D(z, Tx_{n(k)}) + D(x_{n(k)}, Tz)}{2}\} \\ &\quad + L \min\{(d(x_{n(k)}, z), d(x_{n(k)}, Tz)\} \end{aligned}$$

Hence

$$\begin{aligned} D(x_{n(k)+1}, Tz) &\leq r \max\{d(z, x_{n(k)}), D(z, Tz), D(x_{n(k)}, Tx_{n(k)+1}), \\ &\quad \frac{D(z, Tx_{n(k)+1}) + D(x_{n(k)}, Tz)}{2}\} \\ &\quad + L \min\{(d(x_{n(k)}, z), d(x_{n(k)}, Tz)\}. \end{aligned}$$

Letting $k \to \infty$ we have

$$D(z, Tz) \leq r \max\{D(z, Tz), \frac{D(z, Tz)}{2}\} + L \min\{(d(z, z), d(z, Tz)\}.$$

$$D(z, Tz) \leq r \max\{D(z, Tz), \frac{D(z, Tz)}{2}\} + 0$$

As $r < 1$, so we get $D(z, Tz) = 0$ and since $Tz \in CB(X), z \in Tz$. ■

Remark 4 *When $L = 0$ then above theorem reduces to theorem 2.4.3*

3.2.6 Corollary

Let (X, d) be a complete metric space, $T : X \to X$ be a mapping . Assume that there exist $r \in [0, 1]$ such that

$$\frac{1}{1+r}D(x,Tx) \leq d(x,y) \leq \frac{1}{1-r}D(x,Tx) \quad \Rightarrow \; H(Tx,Ty) \leq rM(x,y)$$

where

$$\begin{aligned} M(x,y) &= \max\{d(x,y), D(x,Tx), D(y,Ty), \frac{D(x,Ty)+D(y,Tx)}{2}\} \\ &\quad +L\min\{(d(x,y), d(y,Tx)\}. \end{aligned}$$

The there exist $z \in X$ such that $Tz = z$.

Proof. One can easily prove that for every $x_1 \in X$ the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ be such that $d(x_{n+1}, x_{n+2}) \leq rd(x_n, x_{n+1})$ also $\{x_n\}$ is Cauchy and there is point $z \in X$ such that $\lim_{n\to\infty} x_n = z$. From above theorem we have $d(z, x_n) \leq \frac{1}{1-r}d(x_n, x_{n+1})$ for all $n \geq 1$ and there exist a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$d(z, x_{n(k)}) \geq \frac{1}{1+r}d(x_{n(k)}, x_{n(k)+1}).$$

holds for every $k \geq N$.Therefore, we get

$$\begin{aligned} d(x_{n(k)+1}, Tz) &\leq r\max\{d(z, x_{n(k)}), d(z,Tz), d(x_{n(k)}, Tx_{n(k)+1}), \frac{d(z, Tx_{n(k)+1}) + d(x_{n(k)}, Tz)}{2}\} \\ &\quad +L\min \; (d(x_{n(k)}, z), d(x_{n(k)}, Tz)\}. \end{aligned}$$

by taking $k \to \infty$ we obtain $Tz = z$. ■

3.3 $f-(s,r)$-Contractive Multi-valued Operator

In this section we extend the results given by Popescu [4] for (s,r)-contractive multi-valued operator by using the concept of multi-valued $f-$weak contractions given by Kamran [3].

3.3.1 Definition

Let (X,d) be a metric space and $f : X \to X$,$T : X \to CB(X)$ be a multi-valued operator T is said to be an $f-(s,r)$-contractive multi-valued operator if $r \in [0,1), s \geq r$ and $u, v \in X$ with

$$D(fx,Ty) \leq sd(fx,y) \Rightarrow H(Tx,Ty) \leq rM_T(fx,fy)$$

where

$$M_T(fx,fy) = \max\{d(fx,fy), D(fx,Ty), D(fx,Ty), \frac{D(fx,Ty)+D(fx,Ty)}{2}\}$$

Remark 5 *When $f = I$ then above definition reduces to definition 2.3.1*

3.3.2 Theorem

Let (X,d) be a metric space, $f : X \to X$ and $T : X \to CB(X)$ be an $f-(s,r)-$ contractive multi-valued operator with $s > r$ such that $TX \subset fX$. Suppose fX is complete. Then T is an $f-$multi-valued weakly picard operator

Proof. Take a real number $t < 1$ such that $0 \leq r < t < s$. Let $x_1 \in X$ and

$fx_2 \in Tx_1$.Then $D(fx_2, Tx_1) = 0 \leq sd(fx_2, fx_1)$ and by hypothesis we have,

$$\begin{aligned} D(fx_2, Tx_2) &\leq H(Tx_1, Tx_2). \\ &\leq r\max\{d(fx_1, fx_2), D(fx_1, Tx_1), D(fx_2, Tx_2), \frac{D(fx_1, Tx_2) + D(fx_2, Tx_1)}{2}\}. \\ &\leq r\max\{d(fx_1, fx_2), D(fx_2, Tx_2), \frac{d(fx_1, fx_2) + D(fx_2, Tx_2)}{2}\}. \end{aligned}$$

As $r < 1$, so we have

$$D(fx_2, Tx_2) \leq rd(fx_1, fx_2).$$

Then there exist $fx_3 \in Tx_2$ such that

$$d(fx_2, fx_3) \leq rd(fx_1, fx_2).$$

Thus we can construct a sequence $\{fx_n\}$ in E such that $fx_{n+1} \in Tx_n$ and $d(fx_{n+1}, Tx_{n+2}) \leq td(fx_n, fx_{n+1})$ for all $n \in N$. Therefore, we have

$$\sum_{n=1}^{\infty} d(fx_n, fx_{n+1}) \leq \sum_{n=1}^{\infty} t^{n-1} d(fx_1, fx_2) < \infty$$

which implies $\{fx_n\}$ is a Cauchy sequence. Since fX is complete, there is some point $fz \in fX$ such that $\{fx_n\}$ converges to fz. Now,we will show that there exist a subsequence $\{fx_{n(k)}\}$ of $\{fx_n\}$ such that

$$D(fz, Tx_{n(k)}) \leq sd(fz, fx_{n(k)}) \quad \forall \;\; k \in \mathbb{N}.$$

Suppose on contrary that there exist a positive integer N such that

$$D(fz, Tx_n) > sd(fz, fx_n) \; \forall \;\; n \geq N.$$

as $fx_{n+1} \in Tx_n$ so,

$$d(fz, fx_{n+1}) > sd(fz, fx_n) \qquad \forall \;\; n \geq N.$$

By induction, we get for all $n \geq N, p \geq 1$ that

$$d(fz, fx_{n+p}) > s^p d(fz, fx_n) \qquad (1)$$

$$d(fx_{n+p}, fx_n) \leq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2}) + \cdots + d(fx_{n+p-1}, fx_{n+p}).$$

$$\begin{aligned} d(fx_{n+p}, fx_n) &\leq d(fx_n, fx_{n+1})(1 + t + t^2 + \cdots + t^{p-1}) \\ &= \frac{1-t^p}{1-t} d(fx_n, fx_{n+1}). \end{aligned}$$

for all $n \geq N, p \geq 1$.Now letting $p \to \infty$ we get,

$$d(z, fx_n) \leq \frac{1}{1-t} d(fx_n, fx_{n+1}) \ \forall \ n \geq 1.$$

Now for all $n \geq 1, p \geq 1$ we have

$$d(z, fx_{n+p}) \leq \frac{1}{1-t} d(fx_{n+p}, fx_{n+p+1}) \leq \frac{t^p}{1-t} \ d(fx_n, fx_{n+1}) \ . \qquad (2)$$

From (1) and (2) we obtained

$$d(fz, fx_n) < \frac{(\frac{t}{s})^p}{1-t} d(fx_n, fx_{n+1}), \ \forall \ n \geq N, p \geq 1.$$

By taking limit as $p \to \infty$ we have $d(fz, fx_n) = 0$ for all $n \geq N$ which contradicts (1) therefore there exist a subsequence $\{fx_{n(k)}\}$ of $\{fx_n\}$ such that

$$D(fz, Tx_{n(k)}) \leq sd(fz, fx_{n(k)}) \ \ \forall \ k \in N.$$

By hypothesis We have

$$\begin{aligned} H(Tz, Tx_{n(k)}) \leq \ & r \max\{d(fz, fx_{n(k)}), D(fz, Tz), D(fx_{n(k)}, Tx_{n(k)}), \\ & \frac{D(fz, Tx_{n(k)}) + D(fx_{n(k)}, Tz)}{2}\}. \end{aligned}$$

Hence

$$\begin{aligned} D(fx_{n(k)+1}, Tz) &\leq r\max\{d(fz, fx_{n(k)}), D(fz, Tz), D(fx_{n(k)}, Tx_{n(k)+1}), \\ &\quad \frac{D(fz, Tx_{n(k)+1}) + D(fx_{n(k)}, Tz)}{2}\}. \end{aligned}$$

by letting $k \to \infty$ we have

$$D(fz, Tz) \leq r\max\{D(fz, Tz), \frac{D(z, Tz)}{2}\}.$$

As $r < 1$,so we get $D(fz, Tz) = 0$ and since $Tz \in CB(X)$, $fz \in Tz$. ■

Remark 6 *When $f = I$ then above theorem reduces to theorem 2.4.1*

3.3.3 Example

Let $X = \{1, 2, 3\}$ and $d(x, y) = |x - y| \ \forall \ x, y \in X$. Let $f : X \to X$ and $T : X \to CB(X)$ be such that $T1 = T2 = \{1, 2\}, T3 = \{2, 3\}$ and $f1 = f2 = 1, f3 = 3$. Then:

(a) T is a $f - (s, r)$-contractive multi-valued operator with $r = 0.3, s = 0.4$;

(b) Every $e \in X$ is coincidence point of f and T.;

(c) T is not $(s, r)-$contractive multi-valued operator.

Proof. (a) We have

$$H(T1, T1) = H(T1, T2) = H(T2, T2) = H(T3, T3) = 0,$$

and

$$\begin{aligned} D(f3, T1) &= 1 > sd(f3, f1) = 0.8, \\ D(f1, T3) &= 2 > sd(f1, f3) = 0.8, \end{aligned}$$

$$D(f2, T3) = 1 > sd(f2, f3) = 0.8,$$

$$D(f3, T2) = 1 > sd(f3, f2) = 0.8,$$

so T is a $f-(s,r)$-contractive multi-valued operator with $r = 0.3, s = 0.4$.

(b) It is obvious.

(c)

$$D(2, T3) = 0 < sd(2, 3) = 0.4,$$

but

$$H(T2, T3) = 1 > r\max\{d(2,3), D(2,T2), D(3,T3), \frac{D(3,T2) + D(2,T3)}{2}\} = 0.3;$$

■

3.3.4 Theorem

Let (X, d) be a metric space, $f : X \to X$ and $T : X \to X$ be an $f-(s,r)-$contractive single-valued operator such that $TX \subset fX$. Suppose fX is complete. Then T and f has a coincidence point. Moreover, if $s \geq 1$ then T and f has a unique coincidence point.

Proof. From Theorem 3.3.2 T and f has a coincidence point. If $s \geq 1$ suppose that there exist $fx, fy \in C(f,T), fx \neq fy$. Then

$$d(fy, Tx) = d(fy, fx) \leq sd(fy, fx).$$

so by hypothesis

$$d(Tx, Ty) \leq rM_T(fx, fy).$$

It follows that

$$d(fx, fy) \leq rd(fx, fy).$$

which is a contradiction. ■

3.3.5 Example

Let $X = \{a, b, c, d\}$ and $d : X \times X \to X$ be a metric space such that $d(a, b) = d(b, d) = 4, d(b, c) = d(a, c) = d(a, d) = d(c, d) = 5$ further $f : X \to X$ and $T : X \to X$ be such that $fa = fc = c, fc = a, fd = d$ and $Ta = Tc = b, Tb = Td = d$ then

(a) X is complete metric space and T and f has a coincidence point.

(b) T is a $f-(s,r)$-contractive single valued operator with $r = 0.9, s = 1.2$;

(c) T is not $(s, r)-$contractive multi-valued operator.

Proof. (a) It is obvious.

(b) we have $d(Ta, Tb) = d(Tb, Td) = 0$, in remaining cases we have,

1) If $x = a, y = b$ or $x = b, y = a$, then $d(fx, Ty) = 5 > sd(fx, fy) = 0$.

2) $d(fa, Td) = 5 < sd(fa, fd) = 6$ and $d(fd, Ta) = 4 < sd(fa, fd) = 6$,also $d(Ta, Td) = 4$ and $M_T(fa, fd) = 5$,hence $d(Ta, Td) < rM_T(fa, fd)$.

3) If $x = b, y = c$ or $x = c, y = b$, then $d(fx, Ty) = 5 < sd(fx, fy) = 6$,also $d(Tx, Ty) = 4, M_T(fx, fy) = 5$ hence $d(Tx, Ty) < rM_T(fx, fy)$.

4) $d(fc, Td) = 5 < sd(fc, fd) = 6$ and $d(fd, Tc) = 4 < sd(fc, fd) = 6$,also $d(Tc, Td) = 4$ and $M_T(fc, fd) = 5$,hence $d(Tc, Td) < rM_T(fc, fd)$.

(c) $d(b, Ta) = 0 < sd(b, a) = 4.8$ but $d(Ta, Ta) = 4 > rM_T(a, b) = 3.6$. ■

3.3.6 Theorem

Let (X, d) be a metric space, $f : X \to X$ and $T : X \to CB(X)$ be a mapping . Assume that there exist $r, s \in [0, 1], r < s$ such that $\frac{1}{1+r}D(fx, Tx) \le d(fx, fy) \le \frac{1}{1-s}D(fx, Tx)$

$\Rightarrow H(Tx, Ty) \leq rM_T(fx, fy)$ and $TX \subset fX$ where

$$M_T(fx, fy) = \max\{d(fx, fy), D(fx, Tx), D(fy, Ty), \frac{D(fx, Ty) + D(fy, Tx)}{2}\}.$$

Suppose fX is complete. Then T is an multi-valued weakly picard operator

Proof. Take a real number $t < 1$ such that $0 \leq r < t < s$. Let $x_1 \in X$ and $fx_2 \in Tx_1$ such that

$$d(fx_1, fx_2) \leq \frac{1-t}{1-s} D(fx_1, Tx_1).$$

Then

$$\frac{1}{1+r} D(fx_1, Tx_1) \leq D(fx_1, Tx_1) \leq d(fx_1, fx_2) \leq \frac{1}{1-s} D(fx_1, Tx_1)$$

and by hypothesis we have

$$\begin{aligned} D(fx_2, Tx_2) \quad \leq \quad & H(Tx_1, Tx_2) \leq r \max\{d(fx_1, fx_2), D(fx_1, Tx_1), D(fx_2, Tx_2), \\ & \frac{D(fx_1, Tx_2) + D(fx_2, Tx_1)}{2}\} \end{aligned}$$

$$\leq r \max\{d(fx_1, fx_2), D(fx_2, Tx_2), \frac{d(fx_1, fx_2) + D(fx_2, Tx_2)}{2}\}$$

As $r < 1$, so we have

$$d(fx_2, Tx_2) \leq rd(fx_1, fx_2).$$

Then there exist $fx_3 \in Tx_2$ such that

$$d(fx_2, fx_3) \leq rd(fx_1, fx_2).$$

Thus we can construct a sequence $\{fx_n\}$ in E such that $fx_{n+1} \in Tx_n$ and $d(fx_{n+1}, Tx_{n+2}) \leq td(fx_n, fx_{n+1})$ for all $n \in N$. Therefore, we have

$$\sum_{n=1}^{\infty} d(fx_n, fx_{n+1}) \leq \sum_{n=1}^{\infty} t^{n-1} d(fx_1, fx_2) < \infty$$

which implies $\{fx_n\}$ is a Cauchy sequence. sequence. Since fX is complete, there is some point $fz \in fX$ such that $\{fx_n\}$ converges to fz. and Therefore, we have,

$$d(fx_{n+p}, fx_n) \leq d(fx_n, fx_{n+1}) + d(fx_{n+1}, fx_{n+2}) + \cdots + d(fx_{n+p-1}, fx_{n+p})$$

$$\begin{aligned} d(fx_{n+p}, fx_n) &\leq d(fx_n, fx_{n+1})(1 + t + t^2 + \cdots + t^{p-1}) \\ &= \frac{1 - t^p}{1 - t} \ d(fx_n, fx_{n+1}) \ \forall \ n \geq N, p \geq 1. \end{aligned}$$

Letting $p \to \infty$

$$d(fz, fx_n) \leq \frac{1}{1 - t} \ d(fx_n, fx_{n+1}) \ \forall \ n \geq 1.$$

as

$$d(fx_n, fx_{n+1}) \leq \frac{1 - t}{1 - s} D(fx_n, Tx_n).$$

we have

$$d(fz, fx_n) \leq \frac{1}{1 - s} D(fx_n, Tx_n) \ \forall \ \ n \geq 1.$$

Now suppose that there exist $N > 0$ such that

$$d(fz, fx_n) < \frac{1}{1 + r} D(fx_n, Tx_n) \ \forall \ \ n \geq N.$$

Then we have

$$
\begin{aligned}
d(fx_n, fx_{n+1}) &\leq d(fz, fx_n) + d(fz, fx_{n+1}) \\
&< \frac{1}{1+r}\left[D(fx_n, Tx_n) + D(fx_{n+1}, Tx_{n+1})\right] \\
&< \frac{1}{1+r}\left[D(fx_n, Tx_n) + rd(fx_n, fx_{n+1})\right].
\end{aligned}
$$

This implies that

$$d(fx_n, fx_{n+1}) < D(fx_n, Tx_n).$$

which is not possible. So there exist a subsequence$\{fx_{n(k)}\}$ of $\{fx_n\}$ such that

$$d(fz, fx_{n(k)}) \geq \frac{1}{1+r} D(fx_{n(k)}, Tx_{n(k)}).$$

holds for every $k \geq N$. Since

$$d(fz, fx_n) \leq \frac{1}{1-s} D(fx_n, Tx_n).$$

for all $n \geq 1$,By hypothesis we have

$$
\begin{aligned}
H(Tz, Tx_{n(k)}) \leq\ & r \max\{d(fz, fx_{n(k)}), D(fz, Tz), D(fx_{n(k)}, Tx_{n(k)}), \\
& \frac{D(fz, Tx_{n(k)}) + D(fx_{n(k)}, Tz)}{2}\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
D(fx_{n(k)+1}, Tz) \leq\ & r \max\{d(fz, fx_{n(k)}), D(fz, Tz), D(fx_{n(k)}, Tx_{n(k)+1}), \\
& \frac{D(fz, Tx_{n(k)+1}) + D(fx_{n(k)}, Tz)}{2}\}.
\end{aligned}
$$

by letting $k \to \infty$ we have

$$D(fz, Tz) \leq r \max\{D(fz, Tz), \frac{D(fz, Tz)}{2}\}.$$

As $r < 1$,then we get $D(fz, Tz) = 0$ and since $Tz \in CB(X), fz \in Tz$. ■

Remark 7 *When $L = 0$ then above theorem reduces to theorem 2.4.3*

3.3.7 Corollary

Let (X, d) be a metric space, $f : X \to X$ and $T : X \to X$ be a mapping. Assume that there exist $r \in [0, 1]$, such that

$$\frac{1}{1+r}d(fx, Tx) \leq d(fx, fy) \leq \frac{1}{1-r}d(fx, Tx) \;\Rightarrow\; H(Tx, Ty) \leq rM_T(fx, fy)$$

and $TX \subset fX$ where

$$M_T(fx, fy) = \max\{d(fx, fy), D(fx, Tx), D(fy, Ty), \frac{D(fx, Ty) + D(fy, Tx)}{2}\}$$

.Suppose fX is complete. Then there exist $fz \in X$ such that $fz = Tz$.

Proof. One can easily prove that for every $fx_1 \in X$ the sequence $\{fx_n\}$ defined by $fx_{n+1} = Tx_n$ be such that $d(fx_{n+1}, fx_{n+2}) \leq rd(fx_n, fx_{n+1})$ also $\{fx_n\}$ is Cauchy and there is point $fz \in X$ such that $\lim_{n\to\infty} fx_n = fz$. From above theorem we have $d(fz, fx_n) \leq \frac{1}{1-r}d(fx_n, fx_{n+1})$ for all $n \geq 1$ and there exist a subsequence $\{fx_{n(k)}\}$ of $\{fx_n\}$ such that

$$d(fz, fx_{n(k)}) \geq \frac{1}{1+r}d(fx_{n(k)}, fx_{n(k)+1}).$$

holds for every $k \geq N$.Therefore, we get

$$\begin{aligned} d(fx_{n(k)+1}, Tz) \;&\leq\; r\max\{d(fz, fx_{n(k)}), d(fz, Tz), d(fx_{n(k)}, Tx_{n(k)+1}), \\ &\quad \frac{d(fz, Tx_{n(k)+1}) + d(fx_{n(k)}, Tz)}{2}\}. \end{aligned}$$

by taking $k \to \infty$ we obtain $Tzb = fz$. ■

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