Mixed Convection Flows of Nonlinear Fluids

By

Muhammad Bilal Ashraf

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
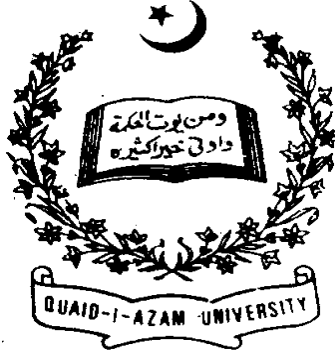## 2015

Mixed Convection Flows of Nonlinear Fluids

By

Muhammad Bilal Ashraf

Supervised By

Prof. Dr. Tasawar Hayat

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
2015

Mixed convection flows of nonlinear fluids

By

Muhammad Bilal Ashraf

A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENT
FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

Supervised By

Prof. Dr. Tasawar Hayat

**Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
2015**

Mixed Convection Flows of Nonlinear Fluids

By

Muhammad Bilal Ashraf

CERTIFICATE
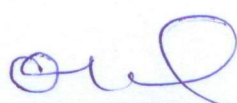
A THESIS SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE DOCTOR OF PHILOSOPHY

We accept this dissertation as conforming to the required standard

1. _____
   Prof. Dr. Tasawar Hayat
   (Supervisor)

2. _____
   Prof. Dr. Tasawar Hayat
   (Chairman)

3. _____
   Prof. Dr. Nazir Ahmed Mir
   (External Examiner)

4. _____
   Dr. Qamar Ul Haq
   (External Examiner)

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
2015

Preface

The boundary layer flows over a moving surface have vital importance due to their ever increasing usage in the industries. In such industrial processes, the kinematics of stretching and heat transfer through rate of cooling have substantial impact in the improvement of final product of better quality. No doubt, the thermal buoyancy force arising due to cooling or heating of a moving surface may alter significantly the flow and thermal fields and thereby the heat transfer behavior in the manufacturing process. In several practical applications, the order of magnitudes of buoyancy and viscous forces are comparable for moderate flow velocities and large surface temperature differences and convective heat transfer process is thus called as mixed convection. The buoyancy forces due to temperature and concentration differences are significant in mixed convection thermal and concentration diffusions. In fact the buoyancy forces causing a pressure gradient in the boundary layer modify the velocity, temperature and concentration distributions and consequently the rate of heat and mass transfer between the surface and fluid. Specifically the mixed convection flows are encountered in industrial processes like solar central receivers exposed to the wind currents, nuclear reactors called during emergency shutdown, electronic devices cooled by fans and heat exchangers etc. The mixed convection flows with heat and mass transfer are relevant to energy related engineering problems that include both metal and polymer sheets. Mostly, the fluids in industrial processes are non-Newtonian. Certain oils, paints, blood at low shear rate, shampoos, cosmetic products body fluids, pasta, ice cream, ice, mud etc are few examples of non-Newtonian fluids. Keeping all the aforementioned facts in mind, the present thesis is structured as follows.

Chapter one covers literature survey and laws of conservation of mass, linear momentum and energy. Boundary layer equations of second grade, Maxwell, Oldroyd-B and thixotropic fluids are presented. Basic idea of homotopy analysis method is also given.

Chapter two deliberates the mixed convection boundary layer flow of thixotropic fluid with thermophoresis over a stretched sheet. Fluid is electrically conducting in the presence of constant applied magnetic field. Heat and mass transfer effects are considered in the presence of Joule heating and thermal radiation. Series solutions are obtained to analyze the velocity, temperature and concentration fields. Numerical values of local Nusselt and Sherwood numbers for different values of emerging parameters are computed and analyzed. A comparative study with the previous solutions in a limiting sense is made. The leading results of this problem are published in "**Journal of Thermophysics and Heat Transfer 27 (2013) 733-740**".

Three-dimensional mixed convection flow of second grade fluid over an exponentially stretching surface are studied in chapter three. Convective boundary conditions are utilized for the heat transfer analysis. Analysis is carried out in the presence of thermal radiation. The series solutions are established through a newly developed method recognized as

the homotopy analysis method. The convergent analysis of velocity components and temperature are derived. Graphs are plotted and analyzed for interesting physical parameters. A systematic study is performed to analyze the impacts of the significant parameters on the velocity and temperature, the skin friction coefficients and the local Nusselt number. The contents of this chapter are published in "**Plos One 9 (2014) e90038**".

Chapter four reports the heat and mass transfer effects in three-dimensional mixed convection flow of viscoelastic fluid with internal heat source/sink and chemical reaction. An exponential stretching surface is employed for flow generation. Magnetic field normal to the direction of flow is taken under consideration. Convective conditions at boundary surface are also encountered. An analytical approach homotopy analysis method is used to develop the solution expressions of the problem. Impacts of different controlling parameters such as stretching ratio parameter, Hartman number, internal heat source/sink, chemical reaction, mixed convection, concentration buoyancy parameter and Biot numbers on the velocity, temperature and concentration profiles are analyzed graphically. The local Nusselt number and Sherwood numbers are sketched and examined. The results of present chapter are accepted for publication in "**Computational Mathematics and Mathematical Physics**".

Chapter 5 provides the three-dimensional mixed convection flow of viscoelastic fluid over a stretching surface in presence of thermophoresis. Soret and Dufour effects are also taken into account. Series solutions are constructed. Dimensionless velocity, temperature and concentration distributions are shown graphically for different values of involved parameters. Numerical values of local Nusselt and Sherwood numbers are computed and analyzed. The contents of this chapter are submitted for possible publication in "**International Journal of Nonlinear Sciences and Numerical Simulation**".

Three-dimensional flow of Maxwell fluid over a stretching surface is addressed in chapter six. Analysis is prepared in presence of concentration and thermal buoyancy effects. Convective boundary conditions for heat and mass transfer are explored. Series solutions of the resulting problem are established. Results are displayed to examine the influence of physical parameters on the velocity, temperature and concentration fields. Main observations of this chapter are published in "**Journal of Central South University 22 (2015) 717-726**".

Chapter seven is prepared to examine the heat and mass transfer effects in three-dimensional flow of Maxwell fluid over a stretching surface with convective boundary conditions. Mass transfer is considered in the presence of first order chemical reaction. Conservation laws of energy and concentration are based upon the Soret and Dufour effects. Convergent series solutions to the resulting nonlinear problems are developed. The relevant results are published in "**International Journal of Numerical Method for Heat and Fluid Flow 25 (2015) 98 - 120**".

Mixed convection flow by an inclined stretching surface with thermal radiation is investigated in chapter eight. The boundary layer equations of an Oldroyd-B fluid in the presence of heat transfer are used. Suitable transformations reduce the partial differential equations into the ordinary differential equations. Computational analysis is implemented

for the convergent series solutions. The values of local Nusselt number are numerically analyzed. Effects of various parameters involved in the velocity and temperature are discussed. The contents of this chapter are accepted for publication in "**Journal of Applied Mechanics and Technical Physics".**

Chapter nine provides the mixed convection flow of an Oldroyd-B fluid bounded by a stretching surface with suction/injection. Mathematical formulation is developed in the presence of heat source and power law heat flux. Velocity and temperature are computed. Numerical values of local Nusselt number are examined. Results are computed in a limiting sense with existing literature. The contents of this chapter are published in "**Journal of the Brazilian Society of Mechanical Sciences and Engineering 37 (2015) pp 423-430".**

Chapter ten investigates the effects of heat and mass transfer in the mixed convection flow of an Oldroyd-B fluid over a stretching surface with convective boundary conditions. Stress is given to the analysis of Soret and Dufour effects. Related problems are first modeled and then computed by homotopy analysis method (HAM). Velocity, temperature and concentration fields are given. In addition, the local Nusselt and Sherwood numbers are examined through numerical values. These observations are submitted for publication in "**Thermophysics and Aeromechanics".**

Falkner-Skan flow of rate type non-Newtonian fluid is analyzed in chapter eleven. Expressions of an Oldroyd-B fluid are used in the development of relevant equations. Analysis has been carried out in presence of mixed convection and thermal radiation. Expressions of flow and heat transfer are assembled. Convergence of derived nonsimilar series solutions is provided. This research is submitted for publication in "**Journal of Aerospace Engineering**".

.

Contents

Chapter 1

Literature survey and methodology

1.1   Introduction

This chapter contains the literature review related to the considered flow problems. Boundary layer equations of thixotropic, second grade, Maxwell and Oldroyd-B fluids are presented. Brief idea of homotopy analysis method (HAM) are also provided.

1.2   Background

External flows around streamlined bodies have viscous (shear and no-slip) effects confined close to the body surfaces and its wake but are nearly inviscid far from the body are termed as boundary layer flows [1] which occur in aerodynamics (airplanes, rockets, projectiles), hydrodynamics (ships, submarines, torpedoes), transportation (automobiles, trucks, cycles), wind engineering (buildings, bridges, water towers) and ocean engineering (buoys, breakwaters, cables). The boundary layer flow problem over stretching sheet have many industrial applications such as polymer sheet or filament extrusion from a dye or long thread between feed roll or wind-up roll, glass fiber and paper production, drawing of plastic films and liquid films in condensation process. Due to the high applicability of this problem in such industrial phenomena, Sakiadis [2] initiated the work for flow by moving surface. After the pioneering work of Sakiadis, researchers have studied the flow over stretching surfaces under various aspects for viscous and nonlinear fluids. A similarity solution of viscous fluid over a stretching surface which is stretched with

the velocity proportional to the distance from origin was presented by Crane [3]. Chakrabarti and Gupta [4] analyzed the hydromagnetic flow over a stretching surface. Heat transfer over a continuous stretching surface with suction and injection was analyzed by Chen and Char [5]. Vajravelu and Hadjinicolaou [6] reported the heat transfer features in the laminar boundary layer flow of viscous fluid over a linearly stretching surface with variable wall temperature. Effects of suction and injection are present. They attained the solutions of the problem in Kumar functions. Andersson et al. [7] found the solutions over a stretching surface in presence of first order chemical reactions. Similarity solutions of the boundary layer equations over stretching wall was obtained by Banks [8]. Closed form solution of magnetohydrodynamic (MHD) flow under slip condition over a permeable stretching surface was obtained by Fang et al. [9]. Mukhphadhyay et al. [10] obtained the solution of MHD boundary layer flow over a heated stretching sheet with variable viscosity. Wang [11] extended the boundary layer flow of Crane [3] for three-dimensional stretching surface. Devi et al. [12] extended the flow of ref. [11] for unsteady stretching surface in presence of heat and mass transfer effects. An approximate analytical solutions of the steady, laminar three-dimensional flow for an incompressible viscous fluid past a stretching sheet were proposed by Ariel [13,14]. Hayat and Javed [15] analyzed the three-dimensional flow of an incompressible viscous fluid over a porous stretching surface in presence of magnetic field by employing homotopy analysis method. Kumari and Nath [16] discussed the unsteady magnetohydrodynamic viscous fluid with heat transfer induced by a bilateral stretching surface. An analysis for heat transfer over a non-linearly stretching surface for a viscous fluid was provided by Vajravelu [17]. Cortell [18] extended the work of ref. [17] in presence of thermal radiation and viscous dissipation over a non-linearly stretching surface. Two-dimensional magnetohydrodynamic stagnation point flow of an incompressible micropolar fluid over a non-linearly stretching surface was explored by Hayat et al. [19]. The laminar boundary layer flow over an axisymmetric plane was provided by Afzal [20]. It has been noted by Gupta and Gupta [21] that stretching mechanism in all realistic situation is not linear. For instance the stretching is not linear in plastic and paper production industries. Besides these the flow and heat transfer by an exponentially stretching surface has been studied by Magyari and Keller [22]. In this attempt the two-dimensional flow of an incompressible viscous fluid is considered. The solutions of laminar boundary layer equations describing heat and flow in a

quiescent fluid driven by an exponentially permeable stretching surface were numerically analyzed by Elbashbashy [23]. Al- Odat et al. [24] numerically discussed the thermal boundary layer flow with an exponential temperature distribution. Here magnetohydrodynamic flow was addressed. Liu et al. [25] studied the three-dimensional boundary layer flow of a viscous fluid over an exponentially stretching surface by using the Ackroyd and Runge-Kutta methods.

Analysis of non-linear fluids is an active area of research for the last few years. In many fields such as food industry, drilling operations and bioengineering, the fluids, either synthetic or natural, are mixtures of different stuffs such as water, particle, oils, red cells and other long chain molecules. Such combination imparts strong rheological properties to the resulting liquids. The dynamic viscosity in non-linear materials varies non-linearly with the shear rate. These fluids in terms of their different rheological features cannot be described by a single constitutive relationship. Hence several relationships for the non-linear fluids are proposed. In fact the additional parameters in such non-linear fluids are the main culprit which makes the resulting systems more nonlinear, higher order and complex than the Navier-Stokes equations. The rheological fluids in general have been classified into three categories known as the differential, integral and rate types. Second grade fluid is a subclass of differential type fluids which exhibits the normal stress effects. To predict these effects many researchers studied second grade fluid under various aspects. Dandapat and Gupta [26] discussed the flow of an incompressible second-order fluid due to stretching surface under boundary layer assumptions. Chen et al. [27] delivered the temperature distribution in viscoelastic fluid of Walters' B Model over a horizontal stretching plate. The velocity of the plate is proportional to the distance from the slit. Vajravelu and Rollins [28] carried out the heat transfer effects in viscoelastic fluid over a stretching surface with frictional heating and internal heat generation or absorption. Hayat et al. [29] studied the three-dimensional flow over stretching surface in a viscoelastic fluid by applying homotopy analysis method. Liu [30] presented analytical solutions for the flow and heat transfer in steady laminar boundary flow of an electrically conducting fluid of second grade subject to transverse uniform magnetic field past a semi-infinite stretching sheet with power-law surface temperature or power-law surface heat flux. The effects of viscous dissipation, internal heat generation or absorption, work done due to deformation and Joule heating were also considered in the energy equation. Flow and heat transfer characteristics of viscoelastic fluid with porous medium over

a stretching surface with viscous dissipation was governed by Nandeppanavar et al. [31] . Differential type models do not predict the relaxation and retardation time effects while these effects can be anticipated by rate type fluids. Maxwell fluid is a simplest subclass of rate type fluids. Channel flow of an upper convected Maxwell fluid (UCM) induced by suction was presented by Choi et al. [32]. Sadeghy [33] discussed the flow of an upper-convected Maxwell (UCM) fluid above a rigid plate moving steadily. Pahlavan et al. [34] studied the MHD flows of UCM fluids above porous stretching sheets using two-auxiliary-parameter homotopy analysis method. Maxwellian MHD flow induced by a linearly stretched with thermal radiations was investigated by Aliakbar et al. [35]. Unsteady flow of generalized Maxwell fluid with fractional derivative induced by an accelerating plate was provided by Fetecau et al. [36]. Kumari and Nath [37] analyzed the Maxwell fluid over stretching sheet in presence of mixed convection. Abel et al. [38] proposed the MHD flow and heat transfer for the upper-convected Maxwell fluid over a stretching sheet. Hayat et al. [39] presented the three-dimensional Maxwell fluid over stretching surface with convective boundary conditions. Bhatanagar et al. [40] initiated to analyze the boundary layer flow of rate type fluids. They obtained the solutions of two-dimensional flow over a stretching surface with variable free stream velocity. Exact solutions of unidirectional flow of an Oldroyd-B fluid between two parallel plates was presented by Rajagopal [41]. Exact solutions for flows of an electrically conducting Oldroyd-B fluid over an infinite oscillatory plate in the presence of a transverse magnetic field when the entire system rotates about an axis normal to the plate was obtained by Hayat et al. [42]. The linear stability of the flow of an Oldroyd-B fluid through a linear array of cylinders confined in a channel was analyzed by Smith et al. [43]. They computed solutions for both the steady state and linear stability of these states by employing finite element analysis. Exact solutions for the influence of Hall current and rotation in the oscillatory flows by an infinite plate were obtained by Asghar et al. [44]. Fetecau and Fetecau [45] analyzed the unsteady flows of Oldroyd-B fluids in a channel of rectangular cross-section. A linear stability analysis determining the onset of oscillatory convection of an Oldroyd-B fluid in a two-dimensional rectangular porous medium generated by Newtonian heating was presented by Niu et al. [46].

In nonlinear fluids sometimes nonlinearity introduced by their shear-dependent viscosity and/or elasticity often gives rise to a formidable mathematical task which cannot be solved.

Obviously, the situation becomes much more complicated when the viscosity of the fluid is time-dependent. Time-dependent fluid systems are quite frequent in industrial applications with the common effect being a drop in viscosity by the progress of time. Complex fluid systems such as drilling muds, foodstuff, paints, cosmetics, pharmaceuticals, suspensions, grease, and the like belong to this class of fluids-the so-called thixotropic fluids. Physiological fluids such as blood, synovial fluid, and mucus may also exhibit thixotropic behavior depending on the time scale of the observation. A common effect among thixotropic fluids is that their viscosity is decreased even when the shear rate is constant [47,48]. Harris [49,50] tried to address boundary layer flows of thixotropic fluids. Haris presented a simple thixotropic fluid model (the so called Harris model) to investigate the effects of a fluid's thixotropic behavior on the characteristics of the momentum boundary layer formed above a fixed plate [50]. Two-dimensional flow of an incompressible thixotropic fluid obeying Harris rheological model over a fixed semi-infinite plate was investigated by Sadeqi et al. [51].

Mixed convection flows or combined free and forced convection flows occur in many technological and industrial applications and in nature for example, in solar receivers exposed to wind currents, electronic devices cooled by fans, nuclear reactors cooled during emergency shutdown, heat exchanges placed in a low-velocity environment, flows in the ocean and in the atmosphere, and many more. Finite element method was utilized for detailed analysis of mixed-convection flow in a horizontal channel heated from the side walls were computed by Sillekens [52]. Heat transfer enhancement by air injection in upward heated mixed-convection flow of water was studied by Celata et al. [53]. Barletta [54] provided the analysis of the laminar and fully developed mixed convection flow in a vertical rectangular duct with one or more isothermal walls. Magyari et al. [55] analyzed the boundary layer mixed convection flow over a permeable horizontal plate. The unsteady mixed convection boundary layer flow near the region of a stagnation point on a vertical surface embedded in a Darcian fluid-saturated porous medium was investigated by Nazar et al. [56]. They employed Keller–Box method to obtain the solutions. Laminar two-dimensional unsteady mixed convection boundary-layer flow of viscous incompressible fluid past a sharp wedge was developed by Hussain et al. [57]. Perturbation solutions were obtained for small and large dimensionless time. Experimental investigation was presented on mixed (free and forced) convection to study the local and average heat transfer

9

for hydrodynamically and thermally developed laminar air flow in a horizontal circular cylinder was analyzed by Mohammed and Salman [58]. Laminar mixed convection heat transfer for assisted and opposed air flows in the entrance region of a vertical circular tube with wall heat flux boundary condition had been experimentally investigated by Mohammed [59]. Kotouc et al. [60] also provided the loss of axisymmetry in the mixed convection assisting flow past a heated sphere. A transient laminar mixed convection flow of viscous incompressible fluid generated by thermal buoyancy force over a horizontal porous sensor surface placed inside a squeezing channel was discussed by Mahmood et al. [61]. The implicit finite difference approximation together with Keller box method was employed for the solution of small and large time regimes. Heat and mass transfer characteristics in mixed convection boundary layer flow past a linearly stretching vertical surface in a porous medium filled with a viscoelastic fluid with Dufour and Soret effects was governed by Hayat et al. [62].

Radiative mixed convection has gained much importance amongst the recent researchers due to number of applications in geophysical and energy storage problems such as in furnaces, ovens and boilers and the interest in our environment and in no conventional energy sources, such as the use of salt gradient solar ponds for energy collection and storage. Similarity equations governing steady hydromagnetic boundary-layer flow over an accelerating permeable surface in the presence of thermal radiation, thermal buoyancy and heat generation or absorption effects were obtained by Chamka [63]. Buoyancy force and thermal radiation effects in MHD boundary layer viscoelastic fluid flow over continuously moving stretching surface was investigated by Abel et al. [64]. Mukhopadhyay and Layek [65] presented the free convective boundary layer flow and heat transfer of a fluid with variable viscosity over a porous stretching vertical surface in presence of thermal radiation. Results were obtained by Lie group transformations. Also Mukhopadhyay [66] analyzed the effects of thermal radiation on unsteady boundary layer mixed convection heat transfer problem from a vertical porous stretching surface embedded in porous medium. Magnetohydrodynamic mixed convective flow and heat transfer of an electrically conducting power-law fluid past a stretching surface in the presence of heat generation/absorption and thermal radiation was studied by Chen [67]. Numerical solutions were generated by an implicit finite-difference technique for the non-similar coupled flow.

A study of utilizing heat source or sink in moving fluids has been a subject of interest of many

researchers. This is because of its possible application to geophysical sciences, astrophysical sciences, and in cosmic studies. Such flows arise either due to unsteady motion of the boundary or the boundary temperature. The study of fluctuating flow is important in the paper industry and many other technological fields. Many investigators stressed on the magnetohydrodynamic flow of an electrically conducting fluid because of numerous applications in metallurgical industry such as in drawing, annealing, in the purification of molten metals from non-metallic inclusions, electromagnetic pumps, MHD generators etc. Several studies have been presented by the authors in presence of transverse magnetic field and heat source/sink over a stretching surface. Heat transfer characteristics in an electrically conducting fluid over a stretching sheet with variable wall temperature and heat source/sink was investigated by Vajravelu and Rollins [68]. Abel et al. [69] presented the study of momentum and heat transfer characteristics in hydromagnetic flow of viscoelastic liquid over a stretching sheet with non-uniform heat source, where the flow was generated due to a linear stretching of the sheet and influenced by uniform magnetic field applied vertically. Beg et al. [70] examined the magnetohydrodynamic free convection from a sphere embedded in an electrically-conducting fluid-saturated porous regime with heat generation.

The present trend in the field of chemical reaction analysis is to give a mathematical model for the system to predict the reactor performance. A large amount of research work has been reported in this field. In particular, the study of heat and mass transfer with chemical reaction is of considerable importance in chemical and hydrometallurgical industries. MHD flow with heat and mass transfer characteristics of an incompressible viscous electrically conducting and Boussinesq fluid due to a vertical stretching surface with chemical reaction and thermal stratification effects was presented by Kandasamy et al. [71]. Mansour et al. [72] studied the effects of chemical reaction, thermal stratification, Soret and Dufour numbers on MHD free convective heat and mass transfer of a viscous incompressible and electrically conducting fluid over a vertical stretching surface embedded in a saturated porous medium. The combined effect of mixed convection with thermal radiation and chemical reaction on MHD flow of viscous and electrically conducting fluid past a vertical permeable surface in presence of porous medium was analyzed by Pal and Talukdar [73]. Mass transfer with chemical reaction in MHD mixed convective flow along a vertical stretching sheet was investigated by Singh et al. [74].

1.3 Fundamental laws

1.3.1 Law of conservation of mass

The equation of continuity (law of conservation of mass) can be represented by

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{V}) = 0, \tag{1.1}$$

where ρ is the density of fluid and $\mathbf{V}$ is the fluid velocity.

For an incompressible fluid Eq. (1.1) can be expressed as follows:

$$\boldsymbol{\nabla} \cdot \mathbf{V} = 0. \tag{1.2}$$

1.3.2 Law of conservation of linear momentum

Generalized equation of motion can be expressed as

$$\rho \frac{d\mathbf{V}}{dt} = \boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho \mathbf{b}. \tag{1.3}$$

For an incompressible flow

$$\boldsymbol{\tau} = -p\mathbf{I} + \mathbf{S} \tag{1.4}$$

in which $\boldsymbol{\tau}$ is the Cauchy stress tensor, p is the pressure, $\mathbf{I}$ is the identity tensor, $\mathbf{S}$ is the extra stress tensor, $\mathbf{b}$ is the body force and d/dt is the material time derivative. The Cauchy stress tensor and the velocity field for three-dimensional flow can be described in the forms

$$\boldsymbol{\tau} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}, \tag{1.5}$$

$$\mathbf{V} = [u(x, y, z), v(x, y, z), w(x, y, z)], \tag{1.6}$$

where σ_{xx}, σ_{yy} and σ_{zz} represent the normal stresses, τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx} and τ_{zy} show the shear stresses and u, v, w are the velocity components along the x, y and $z-$directions respectively.

Equation (1.3) in component form can be written as follows:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right) = \frac{\partial\left(\sigma_{xx}\right)}{\partial x}+\frac{\partial\left(\tau_{xy}\right)}{\partial y}+\frac{\partial\left(\tau_{xz}\right)}{\partial z}+\rho b_x, \tag{1.7}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial v}{\partial z}\right) = \frac{\partial\left(\tau_{yx}\right)}{\partial x}+\frac{\partial\left(\sigma_{yy}\right)}{\partial y}+\frac{\partial\left(\tau_{yz}\right)}{\partial z}+\rho b_y, \tag{1.8}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right) = \frac{\partial\left(\tau_{zx}\right)}{\partial x}+\frac{\partial\left(\tau_{zy}\right)}{\partial y}+\frac{\partial\left(\sigma_{zz}\right)}{\partial z}+\rho b_z, \tag{1.9}$$

where b_x, b_y and b_z show the components of body force along the x, y and $z-$axes, respectively.

The above equations for two-dimensional flow become

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = \frac{\partial\left(\sigma_{xx}\right)}{\partial x}+\frac{\partial\left(\tau_{xy}\right)}{\partial y}+\rho b_x, \tag{1.10}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = \frac{\partial\left(\tau_{yx}\right)}{\partial x}+\frac{\partial\left(\sigma_{yy}\right)}{\partial y}+\rho b_y. \tag{1.11}$$

1.3.3 Equation of heat transfer

According to first law of thermodynamics the heat transfer equation can be written as

$$\rho\frac{d\varepsilon}{dt} = \boldsymbol{\tau}.\mathbf{L}-\boldsymbol{\nabla}\cdot\mathbf{q_1}+\rho r_h, \tag{1.12}$$

where $\varepsilon = C_pT$ is the internal energy, C_p the specific heat, T the temperature, $\mathbf{L} = \boldsymbol{\nabla}\mathbf{V}$ the velocity gradient, $\mathbf{q_1} = -k\boldsymbol{\nabla}T$ the heat flux, k the thermal conductivity and r_h the radiative heating. The above equation in absence of radiative heating is given below

$$\rho C_p\frac{dT}{dt} = \boldsymbol{\tau}.\boldsymbol{\nabla}\mathbf{V}+k\boldsymbol{\nabla}^2T. \tag{1.13}$$

1.3.4 Diffusion equation

Mass transfer occurs whenever fluid flows that is some mass is transferred from one place to another. According to Fick's law

$$\frac{dC}{dt} = D_e\boldsymbol{\nabla}^2C-k_nC_n \tag{1.14}$$

where C is the concentration, D_e is the coefficient of mass diffusivity and k_n is the reaction rate diffusing species.

1.4 Boundary layer equations of nonlinear fluids

1.4.1 Second grade fluid

Extra stress tensor **S** for a second grade fluid can be expressed as

$$\mathbf{S} = \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \tag{1.15}$$

where $\mathbf{A}_1$ is the first Rivilin-Erickson tensor can be defined by

$$\mathbf{A}_1 = \operatorname{grad} \mathbf{V} + (\operatorname{grad} \mathbf{V})^{transpose}. \tag{1.16}$$

For three-dimensional flow one obtains

$$\mathbf{A}_1 = \begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} & 2\frac{\partial w}{\partial z} \end{bmatrix} \tag{1.17}$$

and the second Rivilin-Erickson tensor $\mathbf{A}_2$ can be computed through

$$\mathbf{A}_n = \frac{d\mathbf{A}_{n-1}}{dt} + \mathbf{A}_{n-1}\mathbf{L} + \mathbf{L}^{transpose}\mathbf{A}_{n-1}. \tag{1.18}$$

For thermodynamic stability the second grade model should obey the ineqaulity given below:

$$\mu \geq 0, \alpha_1 \geq 0, \alpha_1 + \alpha_2 = 0. \tag{1.19}$$

From the boundary layer theory [1], the order of u, v, x and y are 1 while the order of w and z are δ. Three-dimensional boundary layer equations for second grade fluid can be written as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2} + \frac{\alpha_1}{\rho}\left(\begin{array}{c} u\frac{\partial^3 u}{\partial x \partial z^2} + w\frac{\partial^3 u}{\partial z^3} - \\ \left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2} + 2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial x \partial z} + 2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}\right) \end{array} \right), \tag{1.20}$$

14

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \nu\frac{\partial^2 v}{\partial z^2} + \frac{\alpha_1}{\rho}\left(\begin{array}{c} v\frac{\partial^3 v}{\partial y\partial z^2} + w\frac{\partial^3 v}{\partial z^3} - \\ \left(\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial z^2} + \frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial z^2} + 2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y\partial z} + 2\frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial z^2}\right) \end{array}\right). \qquad (1.21)$$

1.4.2   Maxwell fluid

The extra stress tensor $\mathbf{S}$ for Maxwell fluid can be expressed by the following relation

$$\left(1 + \lambda_1\frac{D}{Dt}\right)\mathbf{S} = \mathbf{S} + \lambda_1\frac{D\mathbf{S}}{Dt} = \mu\mathbf{A_1}, \qquad (1.22)$$

in which λ_1 is the relaxation time, D/Dt the covariant differentiation, μ denotes the kinematic viscosity and $\mathbf{A_1}$ the first Rivlin-Erickson tensor. For a tensor $\mathbf{S}$ of rank two, a vector $\mathbf{b_1}$ and a scalar φ, we get

$$\frac{D\mathbf{S}}{Dt} = \frac{\partial\mathbf{S}}{\partial t} + (\mathbf{V}.\boldsymbol{\nabla})\mathbf{S} - \mathbf{S}(\operatorname{grad}\mathbf{V})^{transpose} - (\operatorname{grad}\mathbf{V})\mathbf{S}, \qquad (1.23)$$

$$\frac{D\mathbf{b_1}}{Dt} = \frac{\partial\mathbf{b_1}}{\partial t} + (\mathbf{V}.\boldsymbol{\nabla})\mathbf{b_1} - (\operatorname{grad}\mathbf{V})\mathbf{b_1}, \qquad (1.24)$$

$$\frac{D\varphi}{Dt} = \frac{\partial\varphi}{\partial t} + (\mathbf{V}.\boldsymbol{\nabla})\varphi. \qquad (1.25)$$

Implementation of $\left(1 + \lambda_1\frac{D}{Dt}\right)$ on Eq. (1.3), we have the following relations in the absence of body force

$$\rho\left(1 + \lambda_1\frac{D}{Dt}\right)\frac{d\mathbf{V}}{dt} = -\left(1 + \lambda_1\frac{D}{Dt}\right)\nabla p + \left(1 + \lambda_1\frac{D}{Dt}\right)(\nabla.\mathbf{S}). \qquad (1.26)$$

By adopting the procedure

$$\frac{D}{Dt}(\nabla.) = \nabla.\left(\frac{D}{Dt}\right). \qquad (1.27)$$

Hence the above relations in absence of pressure gradient is

$$\rho\left(1 + \lambda_1\frac{D}{Dt}\right)\frac{d\mathbf{V}}{dt} = \mu\left(\nabla\cdot\mathbf{A_1}\right). \qquad (1.28)$$

By using the boundary layer theory [1], the order of u, v, x and y is 1 and order of w and z is δ. The $w-$momentum equation vanishes identically because it has order δ. Hence the boundary

15

layer equations for three-dimensional flow of Maxwell fluid are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \lambda_1 \left(\begin{array}{c} u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + w^2\frac{\partial^2 u}{\partial z^2} \\ +2uv\frac{\partial^2 u}{\partial x\partial y} + 2vw\frac{\partial^2 u}{\partial y\partial z} + 2uw\frac{\partial^2 u}{\partial x\partial z} \end{array} \right) = \nu\frac{\partial^2 u}{\partial z^2}, \qquad (1.29)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \lambda_1 \left(\begin{array}{c} u^2\frac{\partial^2 v}{\partial x^2} + v^2\frac{\partial^2 v}{\partial y^2} + w^2\frac{\partial^2 v}{\partial z^2} \\ +2uv\frac{\partial^2 v}{\partial x\partial y} + 2vw\frac{\partial^2 v}{\partial y\partial z} + 2uw\frac{\partial^2 v}{\partial x\partial z} \end{array} \right) = \nu\frac{\partial^2 v}{\partial z^2}. \qquad (1.30)$$

The boundary layer equation for two-dimensional flow of Maxwell fluid are given below

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \lambda_1 \left(u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + 2uv\frac{\partial^2 u}{\partial x\partial y} \right) = \nu\frac{\partial^2 u}{\partial y^2}. \qquad (1.31)$$

1.4.3   Oldroyd-B fluid

The extra stress tensor for an Oldroyd-B fluid model can be expressed as

$$\left(1 + \lambda_1\frac{D}{Dt}\right)\mathbf{S} = \mathbf{S} + \lambda_1\frac{D\mathbf{S}}{Dt} = \mu\left(1 + \lambda_2\frac{D}{Dt}\right)\mathbf{A}_1, \qquad (1.32)$$

where λ_2 denotes the retardation time and law of conservation of momentum in absence of pressure gradient and body force can be written as follows:

$$\rho\left(1 + \lambda_1\frac{D}{Dt}\right)\frac{d\mathbf{V}}{dt} = \mu\left(1 + \lambda_2\frac{D}{Dt}\right)(\nabla.\mathbf{A}_1). \qquad (1.33)$$

The scalar forms of boundary layer equations in this case are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \lambda_1 \left(\begin{array}{c} u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + w^2\frac{\partial^2 u}{\partial z^2} \\ +2uv\frac{\partial^2 u}{\partial x\partial y} + 2vw\frac{\partial^2 u}{\partial y\partial z} + 2uw\frac{\partial^2 u}{\partial x\partial z} \end{array} \right)$$

$$= \nu\left(\frac{\partial^2 u}{\partial z^2} + \lambda_2\left(\begin{array}{c} u\frac{\partial^3 u}{\partial x\partial z^2} + v\frac{\partial^3 u}{\partial y\partial z^2} + w\frac{\partial^3 u}{\partial z^3} \\ -\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial z^2} - \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial z^2} - \frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2} \end{array} \right)\right), \qquad (1.34)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \lambda_1 \left(\begin{array}{c} u^2\frac{\partial^2 v}{\partial x^2} + v^2\frac{\partial^2 v}{\partial y^2} + w^2\frac{\partial^2 v}{\partial z^2} \\ +2uv\frac{\partial^2 v}{\partial x \partial y} + 2vw\frac{\partial^2 v}{\partial y \partial z} + 2uw\frac{\partial^2 v}{\partial x \partial z} \end{array} \right)$$

$$= \nu \left(\frac{\partial^2 u}{\partial z^2} + \lambda_2 \left(\begin{array}{c} u\frac{\partial^3 v}{\partial x \partial z^2} + v\frac{\partial^3 v}{\partial y \partial z^2} + w\frac{\partial^3 v}{\partial z^3} \\ -\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial z^2} - \frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial z^2} - \frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial z^2} \end{array} \right) \right), \tag{1.35}$$

and the governing boundary layer equation for two-dimensional flow is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \lambda_1 \left(u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + 2uv\frac{\partial^2 u}{\partial x \partial y} \right) = \nu \left(\frac{\partial^2 u}{\partial y^2} + \lambda_2 \left(\begin{array}{c} u\frac{\partial^3 u}{\partial x \partial y^2} + v\frac{\partial^3 u}{\partial y^3} \\ -\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2} \end{array} \right) \right). \tag{1.36}$$

1.4.4 Thixotropic fluid

Stress tensor $\boldsymbol{\tau}$ for thixotropic fluid model

$$\tau_{ij} = 2\mu(II_{2d}(t))d_{ij},$$

where the viscosity is allowed to be time-dependent through allowing the second invariant of the deformation-rate tensor to be time-dependent, II_{2d} is the second invariant of the deformation-rate tensor and

$$2d_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i). \tag{1.37}$$

In the simple Harris model, a quadratic form is used for the II_{2d} so that we have,

$$II_{2d} = (2d_{ij})^2 = 4\left(\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \right) > 0. \tag{1.38}$$

For the viscosity function, in the SH model we have [49,50]

$$\mu = \mu_0 - R_1 II_{2d} + R_2 \frac{dII_{2d}}{dt}, \tag{1.39}$$

where $\frac{d}{dt}$ is the material derivative defined as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (V.\nabla). \tag{1.40}$$

17

In equation (1.39) R_1 and R_2 are the material constants.

For boundary layer analysis the viscosity function becomes

$$\mu = \mu_0 - 2R_1 \left(\frac{\partial u}{\partial y}\right)^2 + 4R_2 \left(u\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y} + v\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\right). \tag{1.41}$$

By using boundary layer analysis the y-momentum equation is completely dropped for two-dimensional flow. On the other hand, the x-momentum equation is reduced to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{6R_1}{\rho}\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial^2 u}{\partial y^2}\right) + \frac{4R_2}{\rho}\left(\begin{array}{c} 2\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y^2}\right)\left(u\frac{\partial^2 u}{\partial x\partial y} + v\frac{\partial^2 u}{\partial y^2}\right) \\ + \left(\frac{\partial u}{\partial y}\right)^2\left(\begin{array}{c} u\frac{\partial^3 u}{\partial x\partial y^2} + v\frac{\partial^3 u}{\partial y^3} \\ + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2} \end{array}\right) \end{array}\right). \tag{1.42}$$

1.5   Homotopy analysis method (HAM)

In the absence of analytical solutions before the advent of computers, the researchers mainly directed their efforts at obtaining some forms of approximate solutions. One of the key issues of approximate solutions has always been the accuracy of the solutions. The accuracy, generally speaking, is measured in terms of the norm of the error in Banach space. The error being the difference of the approximate solution from the exact solution. In the absence of an exact solution (analytical or numerical) a heuristic approach consisting of the convergence of successive approximations has been chosen to judge the merit of an approximate solution. With the advent of computers the approximate solutions in fluid dynamics have lost some of their importance as more and better numerical algorithms have been developed to solve the increasingly realistic, but more complicated problems numerically. Nevertheless, approximate analytical solutions still have their relevance for the following reasons: Firstly, they give the solutions for each point within the domain of interest unlike the numerical solutions which are available for a particular run only for a set of discrete points in the domain. Secondly, compared to a numerical solution a nicely produced approximate solution, requiring a minimal effort and having a reasonable amount of accuracy is always handy for an engineer, scientist or an applied mathematician, who can obtain a solution completely, thereby gaining a valuable

insight into the essentials of the problem. Thirdly, even with most of the scientific packages, some initial guess is required for the solution, as the algorithms, in general are not globally convergent. In such situations, approximate solutions can provide an excellent starting guess, which can be rapidly refined to the exact numerical solution in a few iterations. Homotopy analysis method is proposed by Liao [75, 76] and is very useful to obtain the series solutions of the nonlinear ordinary and partial differential equations [77-85]. According to Liao [75, 76], this method distinguishes itself from other analytical methods in the following five aspects

1. HAM is not dependent on physical parameters. Therefore the technique can be used for both strong/weak nonlinear problems.

2. It is valid for strongly nonlinear problems even if a given nonlinear problem does not contain any small/large parameter.

3. It provides us with a convenient way to adjust the convergence region and rate of approximation of the series solution.

4. HAM provides freedom to chose base functions to approximate the solution of nonlinear problem.

5. This method can be coupled with many other mathematical methods such as integral transform methods, series expansion methods, numerical methods and so on.

   This technique is applicable in the development of results to numerous problems $[78-88]$. Idea behind the HAM is as follows.

Nonlinear differential equation can be written as follows:

$$C(w) + c(r) = 0, \tag{1.43}$$

where C is a nonlinear operator, $w(r)$ is an unknown function to be determined and $c(r)$ is a known function. The homotopic equation is

$$(1 - p)\mathcal{L}\left[\bar{w}(r, p) - w_0(r)\right] = p\hbar\left\{C\left[\bar{w}(r, p) - c(r)\right]\right\}, \tag{1.44}$$

19

where $w_0(r)$ is initial guess, $\mathcal{L}$ is auxiliary linear operator, $\hbar$ is auxiliary parameter or convergence control parameter, $p \in [0, 1]$ is an embedding parameter and $\bar{w}(r, p)$ is an unknown function. By expanding Taylor's series about p one obtains

$$\bar{w}(r, p) = w_0(r) + \sum_{k=1}^{\infty} w_k(r) p^k, \ u_k(r) = \frac{1}{k!} \frac{\partial^k \bar{w}(r, p)}{\partial p^k}\bigg|_{q=0}. \tag{1.45}$$

The convergence of above series firmly depends upon $\hbar$. The value of $\hbar$ is chosen in such a way that series solution is convergent at $p = 1$. Substituting $p = 1$ one obtains

$$w(r) = w_0(r) + \sum_{k=1}^{\infty} w_k(r). \tag{1.46}$$

The k-th order deformation problems are

$$\mathcal{L}\left[w_k(r) - \chi_k w_{k-1}(r)\right] = \hbar \mathcal{R}_k(r), \tag{1.47}$$

where

$$\chi_k = \begin{cases} 0, k \leq 1, \\ 1, k > 1, \end{cases} \tag{1.48}$$

$$\mathcal{R}_k(r) = \frac{1}{(k-1)!} \times \left\{ \frac{d^{k-1}}{dp^{k-1}} B\left[u_0(r) + \sum_{k=1}^{\infty} w_k(r) p^k\right]\right\}_{p=0}. \tag{1.49}$$

20

Chapter 2

MHD mixed convection flow of thixotropic fluid with thermophoresis, Joule heating and thermal radiation

This chapter deals with the magnetohydrodynamic (MHD) mixed convection flow of thixotropic fluid over a moving surface. Heat transfer is considered in the presence of thermophoresis, Joule heating and radiative effects. Dimensionless nonlinear problem is computed by homotopy analysis method (HAM). The convergent solutions are plotted and examined for various parameters of interest. Numerical values of wall shear stress and heat transfer rate are computed and discussed.

2.1 Mathematical formulation

We consider Cartesian coordinate system in such a way that $x-$axis is along the stretching surface and $y-$axis is perpendicular to it. The magnetohydrodynamic boundary layer flow of thixotropic fluid is taken into account. Heat and mass transfer characteristics are accounted in the presence of thermal radiation and thermophoresis effects. Uniform temperature of the

surface T_w is higher than the ambient fluid temperature T_∞. Further the species concentration at the surface is taken uniform $C_w = 0$ while the ambient concentration is C_∞. A constant magnetic field of strength $\mathbf{B}_0$ is applied in the y-direction. The flow is steady and the magnetic Reynolds number is taken small so that an induced magnetic field is negligible in comparison to applied magnetic field. Taking into account the Rosseland approximation for radiative heat flux, the mass, linear momentum, energy and concentration equations are simplified to the following expressions:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$
\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} =\ & \nu\frac{\partial^2 u}{\partial y^2} - \frac{6R_1}{\rho}\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial^2 u}{\partial y^2}\right) - \frac{\sigma^* B_0^2}{\rho}u \\
& + \frac{4R_2}{\rho}\left(
\begin{array}{c}
2\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y^2}\right)\left(u\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 u}{\partial y^2}\right) \\
+ \left(\frac{\partial u}{\partial y}\right)^2\left(u\frac{\partial^3 u}{\partial x \partial y^2} + v\frac{\partial^3 u}{\partial y^3} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2}\right)
\end{array}
\right) \\
& + g[\beta_T(T - T_\infty) + \beta_c(C - C_\infty)], \tag{2.2}
\end{aligned}
$$

$$
\begin{aligned}
u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} =\ & \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} + \frac{16\sigma_s T_\infty^3}{3k_e}\frac{\partial^2 T}{\partial y^2} - \frac{2\mu R_1}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^4 + \frac{4\mu R_2}{\rho c_p}u\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y}\left(\frac{\partial u}{\partial y}\right)^2 \\
& + \frac{4\mu R_2}{\rho c_p}v\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\sigma^* B_0^2}{\rho c_p}u^2, \tag{2.3}
\end{aligned}
$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial y^2} - \frac{\partial}{\partial y}(V_T(C - C_\infty)), \tag{2.4}$$

where (u, v) are the velocity components parallel to the $x-$ and $y-$axes, R_1 and R_2 are the constants, ν the dynamic viscosity of fluid, ρ the density of fluid, σ^* the electrical conductivity, g the gravitational acceleration, β_T and β_c the thermal and concentration expansion coefficients respectively, T the temperature, c_p the specific heat, σ_s the Stefan-Boltzmann constant, k_e the

mean absorption coefficient, D the diffusion coefficient and V_T the thermophoretic velocity.
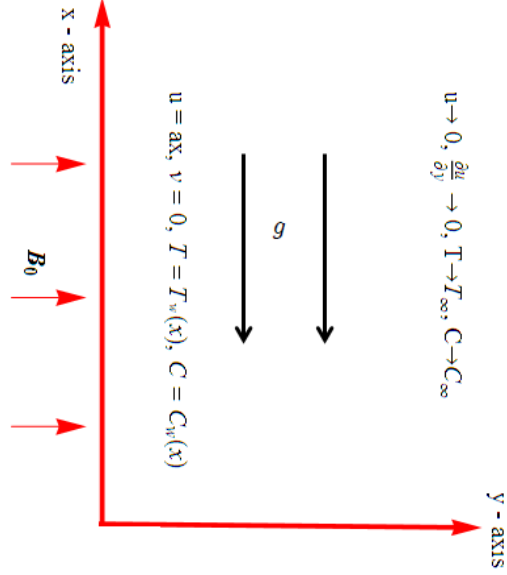
Fig. 2. 1: Physical model

Equations (2.1)-(2.4) are solved subject to the boundary conditions

$$
\begin{aligned}
u &= u_w = ax, \ \ v = 0, \ \ T = T_w(x), \ C = C_w(x) \ \text{at} \ \ y = 0, \\
u &\to 0, \ \ \frac{\partial u}{\partial y} \to 0, \ T \to T_\infty, \ C \to C_\infty \ \text{as} \ y \to \infty.
\end{aligned}
\tag{2.5}
$$

The thermophoretic term V_T in Eq. (2.4) can be defined as

$$
V_T = -k_2 \frac{\nu}{T_r} \frac{\partial T}{\partial y}.
\tag{2.6}
$$

Here k_2 is the thermophoretic coefficient and T_r is the reference temperature. A thermophoretic parameter τ is defined by the following relation

$$
\tau = -\frac{k_2(T_w - T_\infty)}{T_r}.
\tag{2.7}
$$

The wall temperature and concentration fields are

$$
T_w = T_\infty + bx, \ \ C_w = C_\infty + cx,
\tag{2.8}
$$

23

where a, b and c are the positive constants. Introducing

$$u = axf'(\eta), \ v = -\sqrt{a\nu}f(\eta), \ \eta = y\sqrt{\frac{a}{\nu}}, \ \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \quad (2.9)$$

incompressibility condition (2.1) is automatically satisfied and the Eqs. (2.2)-(2.5) become

$$f''' + ff'' - f'^2 + K_1(x)f''^2 f''' + K_2(x)\begin{pmatrix} 2f'f''^2 f''' + f''^4 \\ -2ff''f'''^2 - ff''^2 f^{iv} \end{pmatrix} - M^2 f' + \lambda(\theta + N\phi) = 0, \quad (2.10)$$

$$\left(1 + \frac{4}{3}R\right)\theta'' + Pr(f\theta' - \theta f') + PrEcf''^2 + \frac{1}{3}K_1 PrEcf''^4$$

$$+ K_2 PrEc(f'f''^4 - ff''^3 f''') + M^2 PrEcf'^2 = 0, \quad (2.11)$$

$$\phi'' + Sc(f\phi' - \phi f') - Sc\tau(\phi'\theta' - \phi\theta'') = 0, \quad (2.12)$$

$$f = 0, \ f' = 1, \ \theta = 1, \ \phi = 1 \text{ at } \eta = 0,$$

$$f' \to 0, \ f'' \to 0, \ \theta \to 0, \ \phi \to 0 \text{ as } \eta \to \infty, \quad (2.13)$$

where the non-Newtonian parameters are $K_1(x) = -\frac{6R_1 a^3 x^2}{\rho\nu^2}$ and $K_2(x) = \frac{4R_2 a^4 x^2}{\rho\nu^2}$, $M = \sigma^* B_0^2/\rho c$ the Hartman number, $\lambda = Gr_x/Re_x^2$ the local buoyancy parameter, $Gr_x = \frac{g\beta_T(T_w - T_\infty)x^3/\nu^2}{u_w^2 x^2/\nu^2}$ the local Grashof number, $N = \beta_c(C_w - C_\infty)/\beta_T(T_w - T_\infty)$ the constant dimensionless concentration buoyancy parameter, $Pr = \frac{\mu c_p}{k}$ the Prandtl number, $R = 4\sigma_s T_\infty^3/k_e k$ the radiation parameter, $Ec = u_w^2/c_p(T_w - T_\infty)$ the Eckert number and $Sc = \frac{\nu}{D}$ the Schmidt number. It is further seen that both K_1 and K_2 are functions of x. Hence the fluid flow in present situation lacks a self-similar solution. This is striking difference when compared present case with that of Blasius flow of viscous fluid i.e. in Blasius flow there is self-similar solution. Here K_2 especially indicates the fluids' thixotropic behavior.

The skin friction coefficient, local Nusselt number and local Sherwood number in non-dimensional form can be written as follows:

$$Re_x^{1/2} C_f = f''(0) - K_1/6[f''(0)]^3, \ Nu_x Re_x^{-1/2} = -\theta'(0) \text{ and } ShRe_x^{-1/2} = -\phi'(0). \quad (2.14)$$

24

Here we noticed that the wall shear stress at given $x-$location is only affected by R_1 but only indirectly by R_2 (i.e. through $f''(0)$) [51]. We have serious limitation in a simplified Harris model for representing true thixotropic fluid.

2.2   Series solutions

The homotopic solutions for f and θ in a set of base functions

$$\{\eta^k \exp(-n\eta), \ k \geq 0, n \geq 0\} \tag{2.15}$$

are given by

$$f_m(\eta) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{m,n}^k \eta^k \exp(-n\eta), \tag{2.16}$$

$$\theta_m(\eta) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} b_{m,n}^k \eta^k \exp(-n\eta) \tag{2.17}$$

$$\phi_m(\eta) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_{m,n}^k \eta^k \exp(-n\eta) \tag{2.18}$$

where $a_{m,n}^k$, $b_{m,n}^k$ and are $c_{m,n}^k$ are the coefficients. The appropriate initial approximations and auxiliary linear operators are

$$f_0(\eta) = 1 - \exp(-\eta), \ \theta_0(\eta) = \exp(-\eta), \ \phi_0(\eta) = \exp(-\eta), \tag{2.19}$$

$$\mathcal{L}_f = f''' - f', \ \ \mathcal{L}_\theta = f'' - f, \ \mathcal{L}_\phi = f'' - f \tag{2.20}$$

with

$$\mathcal{L}_f(C_1 + C_2 e^\eta + C_3 e^{-\eta}) = 0, \ \mathcal{L}_\theta(C_4 e^\eta + C_5 e^{-\eta}) = 0, \ \mathcal{L}_\phi(C_6 e^\eta + C_7 e^{-\eta}). \tag{2.21}$$

in which C_i $(i = 1-7)$ denote the arbitrary constants and the zeroth order deformation problems are

$$(1 - p)\mathcal{L}_f\left(\hat{f}(\eta; p) - f_0(\eta)\right) = p\hbar_f \mathcal{N}_f\left(\hat{f}(\eta; p), \hat{\theta}(\eta; p), \hat{\phi}(\eta; p)\right), \tag{2.22}$$

$$(1 - p)\mathcal{L}_\theta\left(\hat{\theta}(\eta; p) - \theta_0(\eta)\right) = p\hbar_\theta \mathcal{N}_\theta\left(\hat{\theta}(\eta; p), \hat{f}(\eta; p), \hat{\phi}(\eta; p)\right), \tag{2.23}$$

25

$$(1-p)\,\mathcal{L}_\phi\left(\hat{\phi}(\eta;p)-\phi_0(\eta)\right)=p\hbar_\phi\mathcal{N}_\phi\left(\hat{\theta}(\eta;p),\hat{f}(\eta;p),\hat{\phi}(\eta;p)\right),\qquad(2.24)$$

$$\begin{aligned}
\hat{f}(0;p) &= 0,\ \hat{f}'(0;p)=1,\ \hat{f}'(\infty;p)=0,\ \hat{f}''(\infty;p)=0,\\
\hat{\theta}(0;p) &= 1,\ \hat{\theta}(\infty;p)=0,\ \hat{\phi}(0;p)=1 \text{ and } \hat{\phi}(\infty;p)=0,
\end{aligned}\qquad(2.25)$$

where p shows embedding parameter, $\hbar_f$, $\hbar_\theta$ and $\hbar_\phi$ the non-zero auxiliary parameters and the nonlinear operators $\mathcal{N}_f$, $\mathcal{N}_\theta$ and $\mathcal{N}_\phi$ are

$$\begin{aligned}
\mathcal{N}_f[\hat{f}(\eta,p),\hat{\theta}(\eta;p),\hat{\phi}(\eta;p)] &= \frac{\partial^3\hat{f}(\eta,p)}{\partial\eta^3}+\hat{f}(\eta,p)\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}-\left(\frac{\partial\hat{f}(\eta,p)}{\partial\eta}\right)^2\\
&+K_1(x)\left(\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}\right)^2\frac{\partial^3\hat{f}(\eta,p)}{\partial\eta^3}\\
&+K_2(x)\left(\begin{array}{c}\frac{\partial\hat{f}(\eta,p)}{\partial\eta}\left(\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}\right)^2\frac{\partial^3\hat{f}(\eta,p)}{\partial\eta^3}+\left(\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}\right)^4-\hat{f}(\eta,p)\\\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}\left(\frac{\partial^3\hat{f}(\eta,p)}{\partial\eta^3}\right)^2-\hat{f}(\eta,q)\left(\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}\right)^2\frac{\partial^4\hat{f}(\eta,p)}{\partial\eta^4}\end{array}\right)\\
&+\lambda(\hat{\theta}(\eta,p)+N\hat{\phi}(\eta;p)-M^2\frac{\partial\hat{f}(\eta,p)}{\partial\eta},\qquad(2.26)
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}_\theta[\hat{f}(\eta,p),\hat{\theta}(\eta;p),\hat{\phi}(\eta;p)] &= \left(1+\frac{4}{3}R\right)\frac{\partial^2\hat{\theta}(\eta,p)}{\partial\eta^2}+PrEc\left(\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}\right)^2\\
&+M^2\,Pr\,Ec\left(\frac{\partial\hat{f}(\eta,p)}{\partial\eta}\right)^2+\frac{1}{3}K_1PrEc\left(\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}\right)^4\\
&-Pr\hat{\theta}(\eta,p)\frac{\partial\hat{f}(\eta,p)}{\partial\eta}+Pr\hat{f}(\eta,p)\frac{\partial\hat{\theta}(\eta,p)}{\partial\eta}\\
&+K_2PrEc\left(\begin{array}{c}\frac{\partial\hat{f}(\eta,p)}{\partial\eta}\left(\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}\right)^4\\-\hat{f}(\eta,p)\frac{\partial^3\hat{f}(\eta,p)}{\partial\eta^3}\left(\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}\right)^3\end{array}\right),\qquad(2.27)
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}_\phi[\hat{f}(\eta,p),\hat{\theta}(\eta;p),\hat{\phi}(\eta;p)] &= \frac{\partial^2\hat{\phi}(\eta,p)}{\partial\eta^2}+Sc\left(\hat{f}(\eta,p)\frac{\partial\hat{\phi}(\eta,p)}{\partial\eta}-\frac{\partial\hat{f}(\eta,p)}{\partial\eta}\hat{\phi}(\eta,p)\right)\\
&-Sc\tau\left(\frac{\partial\hat{\theta}(\eta,p)}{\partial\eta}\frac{\partial\hat{\phi}(\eta,p)}{\partial\eta}-\hat{\phi}(\eta,p)\frac{\partial^2\hat{\theta}(\eta,p)}{\partial\eta^2}\right),\qquad(2.28)
\end{aligned}$$

When $p = 0$ and $p = 1$ then

$$\hat{f}(\eta; 0) = f_0(\eta) \text{ and } \hat{f}(\eta; 1) = f(\eta), \tag{2.29}$$

$$\hat{\theta}(\eta; 0) = \theta_0(\eta) \text{ and } \hat{\theta}(\eta; 1) = \theta(\eta), \tag{2.30}$$

$$\hat{\phi}(\eta; 0) = \phi_0(\eta) \text{ and } \hat{\phi}(\eta; 1) = \phi(\eta), \tag{2.31}$$

and when p increases from 0 to 1 then $f(\eta, p)$, $\theta(\eta, p)$ and $\phi(\eta, p)$ vary from $f_0(\eta)$ to $f(\eta)$, $\theta_0(\eta)$ to $\theta(\eta)$ and $\phi_0(\eta)$ to $\phi(\eta)$. Employing the Taylor's series expansion we have

$$f(\eta, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \ f_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m f(\eta; p)}{\partial p^m} \right|_{p=0}, \tag{2.32}$$

$$\theta(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)p^m, \ \theta_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \theta(\eta; p)}{\partial p^m} \right|_{p=0}, \tag{2.33}$$

$$\phi(\eta, p) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta)p^m, \ \phi_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \phi(\eta; p)}{\partial p^m} \right|_{p=0}. \tag{2.34}$$

Convergence of series (2.32-2.34) is closely associated with $\hbar_f$, $\hbar_\theta$ and $\hbar_\phi$. The values of $\hbar_f$, $\hbar_\theta$ and $\hbar_\phi$ are chosen such that the series (2.32-2.34) converge at $p = 1$. Hence

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \tag{2.35}$$

$$\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \tag{2.36}$$

$$\phi(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta). \tag{2.37}$$

If we denote the special solutions $f_m^*(\eta)$, $\theta_m^*(\eta)$ and $\phi_m^*(\eta)$ then the general solutions $f_m(\eta)$, $\theta_m(\eta)$ and $\phi_m(\eta)$ are

$$f_m(\eta) = f_m^*(\eta) + C_1 + C_2 e^\eta + C_3 e^{-\eta}, \tag{2.38}$$

$$\theta_m(\eta) = \theta_m^*(\eta) + C_4 e^\eta + C_5 e^{-\eta}, \tag{2.39}$$

$$\phi_m(\eta) = \phi_m^*(\eta) + C_6 e^\eta + C_7 e^{-\eta}. \tag{2.40}$$

27

2.3 Convergence analysis and discussion

We recall that the auxiliary parameters $\hbar_f$, $\hbar_\theta$ and $\hbar_\phi$ are useful in controlling and adjusting the convergence of the series solutions. We draw the $\hbar-$curves at $14th$ order of approximation in obtaining the ranges for $\hbar_f$, $\hbar_\theta$ and $\hbar_\phi$. It is noticed from Fig. 2.2 that the admissible values of $\hbar_f$, $\hbar_\theta$ and $\hbar_\phi$ are $-0.7 \leq \hbar_f \leq -0.25$, $-0.95 \leq \hbar_\theta \leq -0.5$ and $-0.95 \leq \hbar_\phi \leq -0.5$. Further the series solutions converge in the whole region of η when $\hbar_f = \hbar_\theta = \hbar_\phi = -0.6$. Table 2.1 is presented to see the convergent values for different order of approximations at $\hbar_f = \hbar_\theta = \hbar_\phi = -0.6$. This Table indicates that the series solutions for velocity converge from 20th order of deformations and temperature and concentration converge from 25th order of deformations. Hence 25th order deformations are computed to find a convergent series solutions.
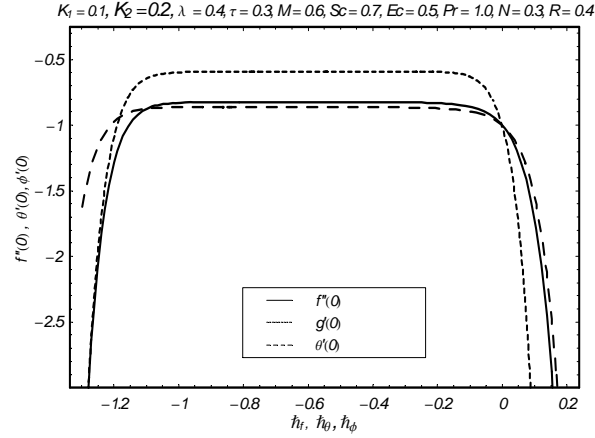

Fig. 2. 2: $\hbar-$curves for the functions f, θ and ϕ

Table 2.1: Convergence of series solutions for different order of approximations when $K_1 = 0.1$, $K_2 = 0.2$, $\lambda = 0.4$, $\tau = 0.3$, $M = 0.6$, $Sc = 0.7$, $Ec = 0.5$, $Pr = 1.0$, $N = 0.3$, $R = 0.4$

and $\hbar_f = \hbar_\theta = \hbar_\phi = -0.6$.

Order of approximations	$-f''(0)$	$-\theta'(0)$	$-\phi'(0)$
1	0.86800	0.63500	0.85000
5	0.82753	0.60365	0.71921
10	0.82461	0.60732	0.69473
20	0.82425	0.60872	0.68636
25	0.82425	0.60886	0.68578
30	0.82425	0.60886	0.68578
35	0.82425	0.60886	0.68578

Figs. 2.3- 2.7 are sketched to analyze the variations of thixotropic parameters K_1 and K_2, local buoyancy parameter λ, Hartman number M and concentration buoyancy parameter N on the fluid velocity $f'(\eta)$. Figs. 2.3 and 2.4 depict that the fluid velocity and momentum boundary layer thickness are increasing functions of thixotropic parameters. By increasing thixotropic parameters, the values of material parameters R_1 and R_2 increase. An increase in the material parameters give rise to the fluid velocity and momentum boundary layer thickness. Fig. 2.5 shows that an increase in local buoyancy parameter gives rise to the fluid velocity and its associated boundary layer thickness. Increase in buoyancy parameter corresponds to stronger buoyancy force. Buoyancy force is an agent that causes an increase in the fluid flow and its related boundary layer thickness. An increase in Hartman number reduces the fluid velocity. Hartman number involves the Lorentz force and an increase in Lorentz force reduces the fluid velocity and boundary layer thickness (see Fig. 2.6). The Lorentz force provides a resistance to flow. From Fig. 2.7 it is observed that concentration buoyancy parameter enhances the velocity.

To see the impacts of different parameters on the temperature $\theta(\eta)$, Figs. 2.8-2.15 are potrayed. From Figs. 2.8 and 2.9, we have seen that the thixotropic parameters K_1 and K_2 are decreasing functions of temperature and thermal boundary layer thicknesses. Larger values of K_1 and K_2 correspond to stronger R_1 and R_2 showing a reduction in the temperature. We also noted that the thixotropic parameters have quite opposite effects on the fluid velocity and temperature. Figs. 2.10 and 2.11 present the effects of local buoyancy parameter and

concentration buoyancy parameter on the temperature. It is found that the temperature and thermal boundary layer thickness become smaller for larger values of local buoyancy parameter and concentration buoyancy parameter. Physically, both λ and N depend on the buoyancy force. Larger values of λ and N lead to the stronger buoyancy force. Such stronger buoyancy force reduced the temperature and thermal boundary layer thickness. Fig. 2.12 is plotted to analyze the influence of Eckert number on the temperature. Here it is revealed that the temperature and thermal boundary layer thickness are increased for larger Eckert number. This is because of the reason that heat energy is always stored in the liquid due to the frictional heating. Such increase in Eckert number enhances the temperature at any point in the thermal boundary layer region. From Fig. 2.13, one can see that temperature is an increasing function of Hartman number. Larger values of Hartman number posses stronger Lorentz force. This stronger Lorentz force enhances the temperature and thermal boundary layer thickness. Impact of Prandtl number Pr on the temperature $\theta(\eta)$ is analyzed in Fig. 2.14. Thermal boundary layer thickness and temperature $\theta(\eta)$ are decreasing functions of Pr. This is due to the fact that with an enhancement in Prandtl number Pr, thermal diffusivity decreases which leads to a reduction in temperature $\theta(\eta)$. Fig. 2.15 shows that temperature $\theta(\eta)$ increases with an increase in radiation parameter R. Also thermal boundary layer thickness enhances with R which is due to the fact that as thermal radiation parameter increases, the mean absorption coefficient k_e decreases which in results give rise to the divergence of radiative heat flux. Hence the rate of radiative heat transferred to the fluid shoot up so that the fluid temperature increases.

Figs. 2.16-2.22 are plotted to see the variations of K_1, K_2, λ, N, Ec, M and τ on the concentration $\phi(\eta)$. Figs. 2.16 and 2.17 show the influence of thixotropic parameters on the concentration. From these Figs. we observed that increase in thixotropic parameters reduced the concentration and its related boundary layer thickness. We also analyzed that the effects of thixotropic parameters on the temperature and concentration are similar in a qualitative sense. A comparison of Figs. 2.8, 2.9, 2.16 and 2.17 show that the variation in temperature are dominant in comparison to variation in concentration due to thixotropic parameters. Local buoyancy parameter and concentration buoyancy parameter are decreasing functions of concentration (see Figs. 2.18 and 2.19). Fig. 2.20 shows that an increase in Eckert number leads to a decrease in the concentration and its related boundary layer thickness. For higher Eckert

number, concentration specie diffuses due to which the concentration field decreases. Fig. 2.21 illustrates the variations of Hartman number on the fluid concentration. It is revealed that concentration is a decreasing function of Hartman number. We conclude that the variations in temperature are more dominant when compared with the variations in the concentration. From Fig. 2.22 it is observed that the associated boundary layer thickness and concentration profile decrease when thermophoretic parameter τ increases.

$K_2 = 0.2, \lambda = 0.4, \tau = 0.3, M = 0.6, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3, R = 0.4$

Fig. 2.3: Influence of K_1 on $f'(\eta)$

$K_1 = 0.1, \lambda = 0.4, \tau = 0.3, M = 0.6, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3, R = 0.4$

Fig. 2.4: Influence of K_2 on $f'(\eta)$

31

$K_1 = 0.1, K_2 = 0.2, \tau = 0.3, M = 0.6, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3, R = 0.4$

Fig. 2.5: Influence of λ on $f'(\eta)$

$K_1 = 0.1, K_2 = 0.2, \lambda = 0.4, \tau = 0.3, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3, R = 0.4$

Fig. 2.6: Influence of M on $f'(\eta)$

$K_1 = 0.1, K_2 = 0.2, \lambda = 0.4, \tau = 0.3, M = 0.6, Sc = 0.7, Ec = 0.5, Pr = 1.0, R = 0.4$

Fig. 2.7: Influence of N on $f'(\eta)$

$K_2 = 0.2, \lambda = 0.4, \tau = 0.3, M = 0.6, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3, R = 0.4$

Fig. 2.8: Influence of K_1 on $\theta(\eta)$

$K_1 = 0.1, \lambda = 0.4, \tau = 0.3, M = 0.6, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3, R = 0.4$

Fig. 2.9: Influence of K_2 on $\theta(\eta)$

$K_1 = 0.1, K_2 = 0.2, \tau = 0.3, M = 0.6, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3, R = 0.4$
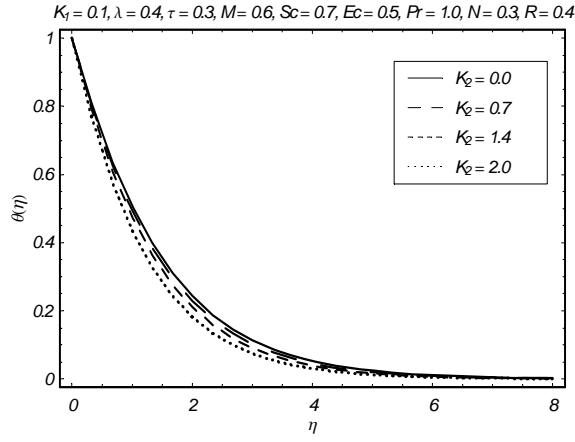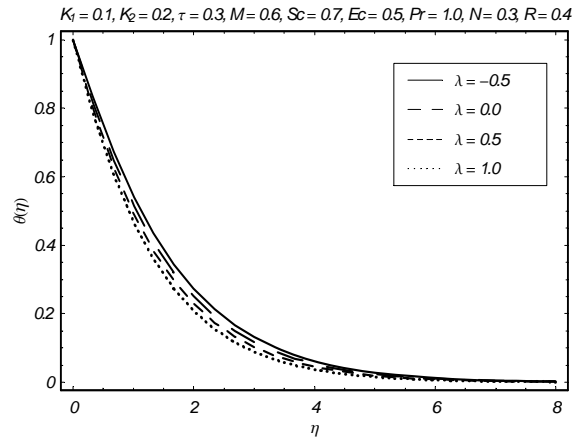
Fig. 2.10: Influence of λ on $\theta(\eta)$

Fig. 2.11: Influence of N on $\theta(\eta)$

Fig. 2.12: Influence of Ec on $\theta(\eta)$
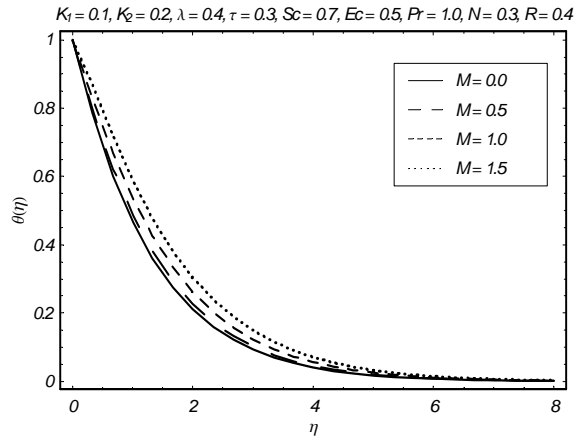
Fig. 2.13: Influence of M on $\theta(\eta)$
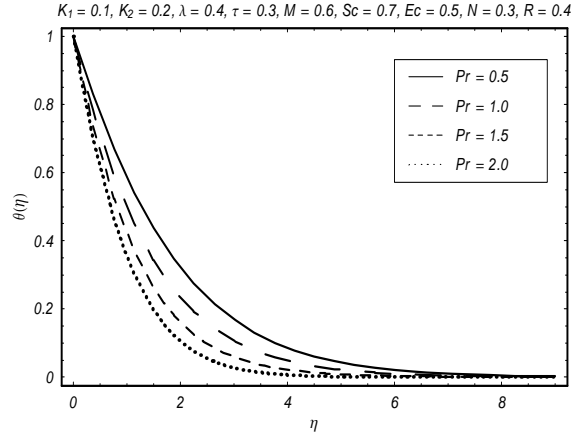
Fig. 2.14: Influence of Pr on $\theta(\eta)$
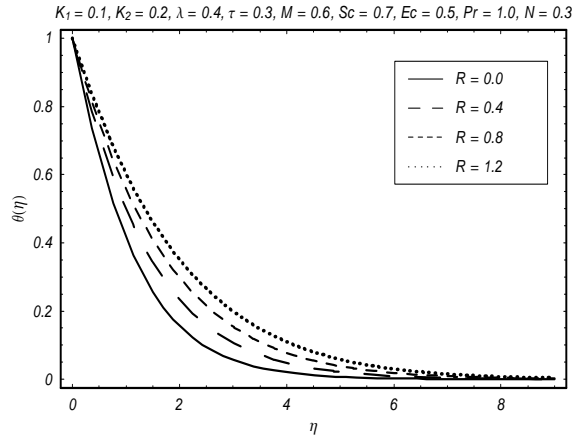
Fig. 2.15: Influence of R on $\theta(\eta)$
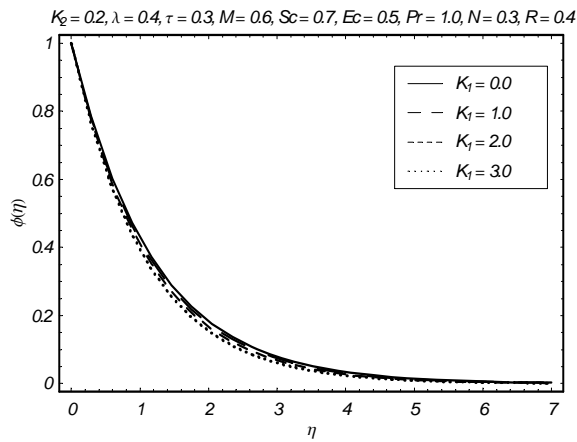
Fig. 2.16: Influence of K_1 on $\phi(\eta)$
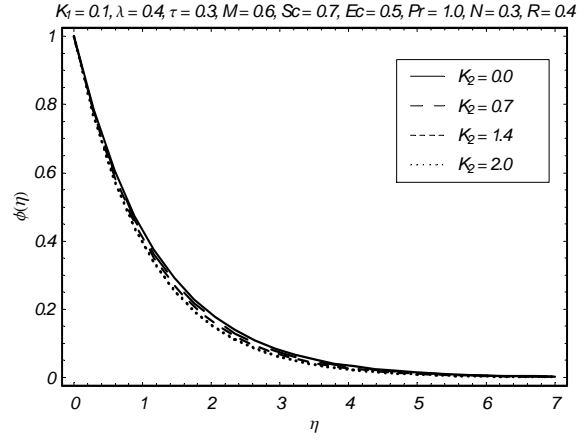
Fig. 2.17: Influence of K_2 on $\phi(\eta)$

Fig. 2.18: Influence of λ on $\phi(\eta)$

Fig. 2.19: Influence of N on $\phi(\eta)$

36

$K_1 = 0.1, K_2 = 0.2, \lambda = 0.4, \tau = 0.3, M = 0.6, Sc = 0.7, Pr = 1.0, N = 0.3, R = 0.4$

Fig. 2.20: Influence of Ec on $\phi(\eta)$

$K_1 = 0.1, K_2 = 0.2, \lambda = 0.4, \tau = 0.3, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3, R = 0.4$

Fig. 2.21: Influence of M on $\phi(\eta)$

$K_1 = 0.1, K_2 = 0.2, \lambda = 0.4, M = 0.6, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3, R = 0.4$
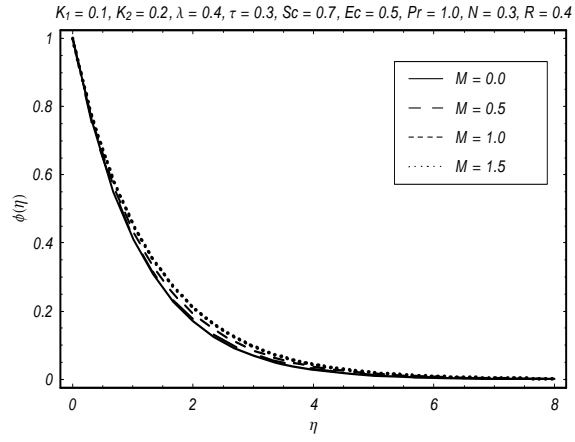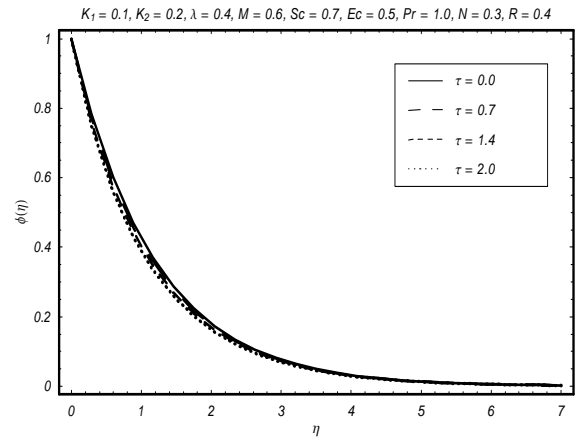
Fig. 2.22: Influence of τ on $\phi(\eta)$

Table 2.2 is made to find the numerical values of skin-friction coefficient for various values of K_1, K_2, M, λ and τ when the other parameters are fixed. We noted that the values of skin-friction coefficient are decreased by increasing K_1, K_2 and λ. However reverse situation is noted for M and τ. Table 2. 3 analyzes the numerical values of local Nusselt number and local Sherwood number for different values of emerging parameters in viscous and non-Newtonian cases. We observed that the values of local Nusselt number and local Sherwood number is large in the case of non-Newtonian fluid when compared with the case of viscous fluid. Table 2.4 is computed for the comparison of $f''(0)$ and $\theta'(0)$ through different values of Prandtl number and local buoyancy parameter when all other parameters are zero. From this Table, it is analyzed that our series solutions have a good agreement with the numerical solutions in a limiting case.

Table 2.2: Numerical values of skin-friction coefficient for different values of K_1, K_2, M, λ and τ when $N = 0.3$, $R = 0.4$, $Pr = 1.0$, $Ec = 0.5$ and $Sc = 0.7$.

K_1	K_2	M	λ	τ	$-Re_x^{1/2}C_f$
0.0	0.2	0.6	0.4	0.4	0.84799
0.3					0.79227
0.5					0.76088
0.2	0.0	0.6	0.4	0.4	0.84923
	0.3				0.79373
	0.5				0.76247
0.2	0.2	0.0	0.4	0.4	0.71906
		0.5			0.80959
		1.0			1.01672
0.2	0.2	0.5	0.0	0.4	0.95561
			0.4		0.76145
			0.8		0.56391
0.2	0.2	0.5	0.3	0.0	0.80923
				1.0	0.81003
				2.0	0.81106

Table 2.3: Numerical values of local Nusselt number and local Sherwood number for different

values of M, Ec, Pr, N and R when $\lambda = 0.3$ and $\tau = 0.4$.

M	Sc	Ec	Pr	N	R	$-\theta'(0)$	$-\phi'(0)$	$-\theta'(0)$	$-\phi'(0)$
						$K_1 = K_2 = 0.0$		$K_1 = K_2 = 0.2$	
0.0	0.7	0.5	1.0	0.3	0.4	0.67883	0.87335	0.69254	0.88088
0.5						0.58160	0.86284	0.60339	0.86671
1.0						0.29936	0.83862	0.36075	0.84058
0.5	0.5	0.5	1.0	0.3	0.4	0.58763	0.62245	0.60862	0.68707
	1.0					0.57572	1.10358	0.59829	1.10750
	1.5					0.57065	1.45081	0.59387	1.45463
0.5	0.7	0.3	1.0	0.3	0.4	0.66211	0.84754	0.67841	0.85318
		0.6				0.54179	0.87021	0.56632	0.87341
		0.8				0.46338	0.88469	0.49302	0.88618
0.5	0.7	0.5	0.5	0.3	0.4	0.41490	0.88107	0.42578	0.88426
			0.8			0.52313	0.86855	0.54006	0.87243
			1.3			0.65577	0.85452	0.68482	0.85721
0.5	0.7	0.5	1.0	0.0	0.4	0.55688	0.85559	0.58355	0.85988
				0.5		0.59650	0.86737	0.61561	0.87109
				1.0		0.62949	0.87800	0.64317	0.88142
0.5	0.7	0.5	1.0	0.3	0.0	0.70547	0.84788	0.73986	0.85133
					0.5	0.55912	0.86538	0.57908	0.86942
					1.0	0.47505	0.87469	0.48877	0.87958

Table 2.4: Comparison of $f''(0)$ and $\theta'(0)$ with Singh et al. [76] for different values of Pr and

λ.

	Singh et al. [76]		Present solutions	
	$f''(0)$	$\theta'(0)$	$f''(0)$	$\theta'(0)$
$Pr = 0.7$ $\lambda = 0.0$	-1.00	-0.79366	-1.00000	-0.79373
$Pr = 0.7$ $\lambda = 1.0$	-0.50751	-0.89613	-0.50767	-0.89614
$Pr = 0.7$ $\lambda = 10.0$	2.57771	-1.17244	2.57789	-1.17287
$Pr = 10.0$ $\lambda = 0.0$	-1.00	-3.72067	-1.00000	-3.72033
$Pr = 10.0$ $\lambda = 1.0$	-0.82568	-3.74856	-0.82529	-3.74718
$Pr = 10.0$ $\lambda = 10.0$	0.61966	-3.95235	0.61940	-3.95266

2.4   Closing remarks

Effects of Joule heating, thermophoresis and thermal radiation in MHD flow of thixotropic fluid are analyzed. The main observations are listed below.

- The non-Newtonian parameters K_1 and K_2 have similar effects on the velocity in a qualitative sense.

- The effects of M and λ on the velocity field are quite opposite.

- The variations of K_1 and K_2 on temperature and concentration are opposite in comparison to velocity.

- An increase in Schmidt number corresponds to a smaller variation in concentration field.

Chapter 3

Three-dimensional mixed convection flow of viscoelastic fluid with thermal radiation and convective conditions

The objective of this chapter is to examine the thermal radiation effect in three-dimensional mixed convection flow of viscoelastic fluid. The resulting partial differential equations are reduced into a system of nonlinear ordinary differential equations using appropriate transformations. The series solutions are developed through a modern technique known as the homotopy analysis method. The convergent expressions of the velocity components and temperature are derived. The solutions obtained are dependent on seven sundry parameters including the viscoelastic parameter, mixed convection parameter, ratio parameter, temperature exponent, Prandtl number, Biot number and radiation parameter. Discussion to these parameters is made via plots.

3.1 Mathematical analysis

Let us consider three-dimensional mixed convection flow of second grade fluid due to an exponentially stretching surface. The surface coincides with the plane $z = 0$ and the flow is confined in the region $z > 0$. The surface also possess the convective boundary condition. Influence of thermal radiation through Rosseland's approximation is taken into account. Flow configuration is given below.
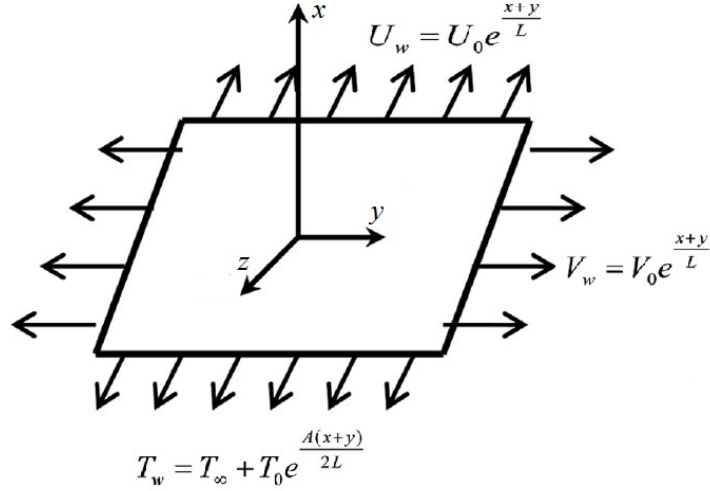
Fig. 3.1: Geometry of problem

The governing boundary layer equations for steady three-dimensional flow of viscoelastic fluid can be put into the forms:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{3.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2} + \frac{\alpha_1}{\rho}\left(u\frac{\partial^3 u}{\partial x\partial z^2} + w\frac{\partial^3 u}{\partial z^3} - \left(\begin{array}{c}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2} \\ +2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial x\partial z} + 2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}\end{array}\right)\right)$$
$$+ g\beta_T(T - T_\infty), \tag{3.2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \nu\frac{\partial^2 v}{\partial z^2} + \frac{\alpha_1}{\rho}\left(v\frac{\partial^3 v}{\partial y\partial z^2} + w\frac{\partial^3 v}{\partial z^3} - \left(\begin{array}{c}\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial z^2} + \frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial z^2} \\ +2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y\partial z} + 2\frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial z^2}\end{array}\right)\right), \tag{3.3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial z^2} - \frac{1}{\rho c_p}\frac{\partial q_r}{\partial z}, \tag{3.4}$$

42

where u, v and w are the velocity components in the $x-$, $y-$ and $z-$directions respectively, α_1 is the material fluid parameter, $\nu = (\mu/\rho)$ is the kinematic viscosity, μ is the dynamic viscosity, ρ is the fluid density, T is the fluid temperature, g is the gravitational acceleration, β_T is thermal expansion coefficient of temperature, c_p is the specific heat, k is the thermal conductivity and q_r the radiative heat flux. By using the Rosseland approximation, the radiative heat flux q_r is given by

$$q_r = -\frac{4\sigma_s}{3k_e}\frac{\partial T^4}{\partial z} \tag{3.5}$$

where σ_s is the Stefan-Boltzmann constant and k_e the mean absorption coefficient. By using the Rosseland approximation, the present analysis is limited to optically thick fluids. If the temperature differences are sufficiently small then Eq. (3.5) can be linearized by expanding T^4 into the Taylor series about T_∞, which after neglecting higher order terms takes the form:

$$T^4 = 4T_\infty^3 T - 3T_\infty^4. \tag{3.6}$$

By using Eqs. (3.5) and (3.6), Eq. (3.4) reduces to

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial z^2} - \frac{16\sigma_s T_\infty^3}{3k_e \rho c_p}\frac{\partial^2 T}{\partial z^2}.$$

The boundary conditions can be expressed as follows:

$$u = U_w, \ v = V_w, \ w = 0, \ -k\frac{\partial T}{\partial z} = h(T_f - T), \ \text{at } z = 0, \tag{3.7}$$

$$u \to 0, \ v \to 0, \ T \to T_\infty \ \text{as } z \to \infty, \tag{3.8}$$

where subscript w corresponds to the wall condition, k is the thermal conductivity, T_f is the hot fluid temperature, h is the heat transfer coefficient and T_∞ is the free stream temperature.

The velocities and temperature are taken in the following forms:

$$U_w = U_0 e^{\frac{x+y}{L}}, \ V_w = V_0 e^{\frac{x+y}{L}}, \ T_w = T_f = T_\infty + T_0 e^{\frac{A(x+y)}{2L}} \tag{3.9}$$

in which U_0, V_0 are the constants, L is the reference length and A is the temperature exponent.

The mathematical analysis of the problem is simplified by using the transformations [25]:

$$u = U_0 e^{\frac{x+y}{L}} f'(\eta), \; v = U_0 e^{\frac{x+y}{L}} g'(\eta), \; w = -\left(\frac{\nu U_0}{2L}\right)^{1/2} e^{\frac{x+y}{2L}} (f + \eta f' + g + \eta g'),$$

$$T = T_\infty + T_0 e^{\frac{A(x+y)}{2L}} \theta(\eta), \; \eta = \left(\frac{U_0}{2\nu L}\right)^{1/2} e^{\frac{x+y}{2L}} z. \tag{3.10}$$

Incompressibility condition is now clearly satisfied whereas Eqs. $(3.2) - (3.8)$ give

$$f''' + (f+g)f'' - 2(f'+g')f' + K \left(\begin{array}{c} 6f'''f' + (3g'' - 3f'' + \eta g''')f'' \\ +(4g' + 2\eta g'')f''' - (f+g+\eta g')f'''' \end{array} \right) + 2\lambda\theta = 0, \tag{3.11}$$

$$g''' + (f+g)g'' - 2(f'+g')g' + K \left(\begin{array}{c} 6g'''g' + (3f'' - 3g'' + \eta f''')g'' \\ +(4f' + 2\eta f'')g''' - (f+g+\eta f')g'''' \end{array} \right) = 0, \tag{3.12}$$

$$\left(1 + \frac{4}{3}R\right)\theta'' + Pr(f+g)\theta' - Pr\, A(f'+g')\theta = 0, \tag{3.13}$$

$$f = 0, \;\; g = 0, \;\; f' = 1, \;\; g' = \beta, \;\; \theta' = -\gamma_1(1 - \theta(0)) \;\; \text{at} \; \eta = 0, \tag{3.14}$$

$$f' \to 0, \;\; g' \to 0, \;\; \theta \to 0 \;\; \text{as} \; \eta \to \infty \tag{3.15}$$

in which K is the viscoelastic parameter, β is the ratio parameter, Pr is the Prandtl number, Gr_x is the local Grashof number, R is the radiation parameter, A is the temperature exponent, γ_1 is the Biot number, Re_x is the local Reynold number, λ is the mixed convection parameter and prime denotes the differentiation with respect to η. These can be defined as

$$K = \frac{\alpha_1 U_w}{2\mu L}, \beta = \frac{V_0}{U_0}, Pr = \frac{\mu c_p}{k}, R = \left(\frac{4\sigma^* T_\infty^3}{k_e k}\right), \gamma_1 = \frac{h}{k}\sqrt{\frac{2\nu L}{U_w}},$$

$$Re_x = \frac{U_0 L}{\nu} e^{\frac{x+y}{L}}, \lambda = \frac{Gr_x}{Re_x^2}, Gr_x = \frac{g\beta_T(T_f - T_\infty)x^3}{v^2}. \tag{3.16}$$

The skin-friction coefficients in the x and y directions are given by

$$C_{fx} = \frac{\tau_{wx}}{1/2\rho U_w^2}, \tag{3.17}$$

$$C_{fy} = \frac{\tau_{wy}}{1/2\rho U_w^2}, \tag{3.18}$$

where

$$\tau_{wx}|_{z=0} = \left(\mu\frac{\partial u}{\partial z} + k_0\left[\begin{array}{c} u\frac{\partial^2 u}{\partial x\partial z} + v\frac{\partial^2 u}{\partial y\partial z} + w\frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial x} \\ + \frac{\partial v}{\partial z}\frac{\partial v}{\partial x} + 2\frac{\partial w}{\partial z}\frac{\partial w}{\partial x} - \frac{\partial w}{\partial z}\frac{\partial u}{\partial z} \end{array}\right]\right)_{z=0},$$

$$\tau_{wy}|_{z=0} = \left(\mu\frac{\partial v}{\partial z} + k_0\left[\begin{array}{c} u\frac{\partial^2 v}{\partial x\partial z} + v\frac{\partial^2 v}{\partial y\partial z} + w\frac{\partial^2 v}{\partial z^2} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial y} \\ + \frac{\partial v}{\partial z}\frac{\partial v}{\partial y} + 2\frac{\partial w}{\partial z}\frac{\partial w}{\partial y} - \frac{\partial w}{\partial z}\frac{\partial v}{\partial z} \end{array}\right]\right)_{z=0}. \tag{3.19}$$

By using Eq. (3.19) in Eqs. (3.17) and (3.18), the non-dimensional forms of skin friction coefficients are as follows:

$$C_{fx} = \left(\frac{\mathrm{Re}}{2}\right)^{-1/2}\left[f'' + K\left(-(f+g)f''' + 5(f'+g')f'' + 2f'f'' + 2g'g''\right)\right]_{\eta=0}, \tag{3.20}$$

$$C_{fy} = \left(\frac{\mathrm{Re}}{2}\right)^{-1/2}\left[g'' + K\left(-(f+g)g''' + 5(f'+g')g'' + 2f'f'' + 2g'g''\right)\right]_{\eta=0}. \tag{3.21}$$

Further the local Nusselt number has the form

$$Nu = \frac{-\left(\frac{16\sigma_s T_\infty^3}{3k_e} + k\right)\frac{\partial T}{\partial z}}{k(T_w - T_\infty)/x} = -\frac{x}{L}\left(\frac{\mathrm{Re}}{2}\right)^{1/2}(1 + \frac{4}{3}R)\theta'(0). \tag{3.22}$$

3.2 Solutions development

The initial guesses and auxiliary linear operators in the desired HAM solutions are

$$f_0(\eta) = \left(1 - e^{-\eta}\right), \ g_0(\eta) = \beta\left(1 - e^{-\eta}\right), \ \theta_0(\eta) = \frac{\gamma_1\exp(-\eta)}{1 + \gamma_1}, \tag{3.23}$$

$$\mathcal{L}_f = f''' - f', \ \mathcal{L}_g = g''' - g', \ \mathcal{L}_\theta = \theta'' - \theta, \tag{3.24}$$

subject to the properties

$$\mathcal{L}_f(C_1 + C_2 e^{\eta} + C_3 e^{-\eta}) = 0, \ \mathcal{L}_g(C_4 + C_5 e^{\eta} + C_6 e^{-\eta}) = 0,$$

$$\mathcal{L}_\theta(C_7 e^{\eta} + C_8 e^{-\eta}) = 0. \tag{3.25}$$

in which C_i $(i = 1 - 8)$ are the arbitrary constants, $\mathcal{L}_f, \mathcal{L}_g$ and $\mathcal{L}_\theta$ are the linear operators and $f_0(\eta), g_0(\eta)$ and $\theta_0(\eta)$ are the initial guesses.

Following the idea in ref. [78] the zeroth order deformation problems are

$$(1-p)\mathcal{L}_f\left[\hat{f}(\eta;p) - f_0(\eta)\right] = p\hbar_f \mathcal{N}_f\left[\hat{f}(\eta;p), \hat{g}(\eta;p)\right], \tag{3.26}$$

$$(1-p)\mathcal{L}_g\left[\hat{g}(\eta;p) - g_0(\eta)\right] = p\hbar_g \mathcal{N}_g\left[\hat{f}(\eta;p), \hat{g}(\eta;p)\right], \tag{3.27}$$

$$(1-p)\mathcal{L}_\theta\left[\hat{\theta}(\eta;p) - \theta_0(\eta)\right] = p\hbar_\theta \mathcal{N}_\theta\left[\hat{f}(\eta;p), \hat{g}(\eta;p), \hat{\theta}(\eta,p)\right], \tag{3.28}$$

$$\hat{f}(0;p) = 0, \ \hat{f}'(0;p) = 1, \ \hat{f}'(\infty;p) = 0, \ \hat{g}(0;p) = 0, \tag{3.29}$$

$$\hat{g}'(0;p) = \alpha, \ \hat{g}'(\infty;p) = 0, \hat{\theta}'(0,p) = -\gamma_1[1 - \theta(0,p)], \ \hat{\theta}(\infty,p) = 0. \tag{3.30}$$

For $p = 0$ and $p = 1$ one has

$$\hat{f}(\eta;0) = f_0(\eta), \ \hat{g}(\eta;0) = g_0(\eta), \ \hat{\theta}(\eta,0) = \theta_0(\eta), \ \text{and} \ \hat{f}(\eta;1) = f(\eta),$$

$$\hat{g}(\eta;1) = g(\eta), \ \hat{\theta}(\eta,1) = \theta(\eta). \tag{3.31}$$

Note that when p increases from 0 to 1 then $f(\eta, p), g(\eta, p)$ and $\theta(\eta, p)$ vary from $f_0(\eta), g_0(\eta)$ and $\theta_0(\eta)$ to $f(\eta), g(\eta)$ and $\theta(\eta)$. So as the embedding parameter $p \in [0, 1]$ increases from 0 to 1, the solutions $\hat{f}(\eta; p), \hat{g}(\eta; p)$ and $\hat{\theta}(\eta; p)$ of the zeroth order deformation equations deform from the initial guesses $f_0(\eta), g_0(\eta)$ and $\theta_0(\eta)$ to the exact solutions $f(\eta), g(\eta)$ and $\theta(\eta)$ of the original nonlinear differential equations. Such kind of continuous variation is called deformation in topology and that is why the Eqs. (3.26-3.28) are called the zeroth order deformation equations.

The values of the nonlinear operators are given below:

$$\mathcal{N}_f[\hat{f}(\eta,p),\hat{g}(\eta,p)] \;=\; \frac{\partial^3 \hat{f}(\eta,p)}{\partial\eta^3} - 2\left(\frac{\partial\hat{f}(\eta,p)}{\partial\eta} + \frac{\partial\hat{g}(\eta,p)}{\partial\eta}\right)\frac{\partial\hat{f}(\eta,p)}{\partial\eta} + \left(\hat{f}(\eta,p) + \hat{g}(\eta,p)\right)\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}$$

$$+K\left(\begin{array}{c} 6\frac{\partial\hat{f}(\eta,p)}{\partial\eta}\frac{\partial^3\hat{f}(\eta,p)}{\partial\eta^3} + \left(3\frac{\partial^2\hat{g}(\eta,p)}{\partial\eta^2} - 3\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2} + \eta\frac{\partial^3\hat{g}(\eta,p)}{\partial\eta^3}\right) \\ \frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2} + \left(4\frac{\partial\hat{g}(\eta,p)}{\partial\eta} + 2\eta\frac{\partial^2\hat{g}(\eta,p)}{\partial\eta^2}\right)\frac{\partial^3\hat{f}(\eta,p)}{\partial\eta^3} \\ -\left(\hat{f}(\eta,p) + \hat{g}(\eta,p) + \eta\frac{\partial\hat{g}(\eta,p)}{\partial\eta}\right)\frac{\partial^4\hat{f}(\eta,p)}{\partial\eta^4} \end{array}\right)$$

$$+2\lambda\hat{\theta}(\eta,p), \tag{3.32}$$

$$\mathcal{N}_g[\hat{g}(\eta,p),\hat{f}(\eta,p)] \;=\; \frac{\partial^3 \hat{g}(\eta,p)}{\partial\eta^3} - 2\left(\begin{array}{c}\frac{\partial\hat{f}(\eta,p)}{\partial\eta}\\ +\frac{\partial\hat{g}(\eta,p)}{\partial\eta}\end{array}\right)\frac{\partial\hat{g}(\eta,p)}{\partial\eta} + \left(\begin{array}{c}\hat{f}(\eta,p)\\ +\hat{g}(\eta,p)\end{array}\right)\frac{\partial^2\hat{g}(\eta,p)}{\partial\eta^2}$$

$$+K\left(\begin{array}{c} 6\frac{\partial\hat{g}(\eta,p)}{\partial\eta}\frac{\partial^3\hat{g}(\eta,p)}{\partial\eta^3} + \left(3\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2} - 3\frac{\partial^2\hat{g}(\eta,p)}{\partial\eta^2} + \eta\frac{\partial^3\hat{f}(\eta,p)}{\partial\eta^3}\right) \\ \frac{\partial^2\hat{g}(\eta,p)}{\partial\eta^2} + \left(4\frac{\partial\hat{f}(\eta,p)}{\partial\eta} + 2\eta\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}\right)\frac{\partial^3\hat{g}(\eta,p)}{\partial\eta^3} \\ -\left(\hat{f}(\eta,p) + \hat{g}(\eta,p) + \eta\frac{\partial\hat{f}(\eta,p)}{\partial\eta}\right)\frac{\partial^4\hat{g}(\eta,p)}{\partial\eta^4} \end{array}\right) \tag{3.33}$$

$$\mathcal{N}_\theta[\hat{\theta}(\eta,p),\hat{f}(\eta,p),\hat{g}(\eta,p)] \;=\; (1+\frac{4}{3}R)\frac{\partial^2\hat{\theta}(\eta,p)}{\partial\eta^2} + \Pr(\hat{f}(\eta,p) + \hat{g}(\eta,p))\frac{\partial\hat{\theta}(\eta,p)}{\partial\eta}$$

$$- \Pr A\left(\frac{\partial\hat{f}(\eta,p)}{\partial\eta} + \frac{\partial\hat{g}(\eta,p)}{\partial\eta}\right)\hat{\theta}(\eta,p). \tag{3.34}$$

Here $\hbar_f$, $\hbar_g$ and $\hbar_\theta$ are the non-zero auxiliary parameters and $\mathcal{N}_f$, $\mathcal{N}_g$ and $\mathcal{N}_\theta$ the nonlinear operators. Taylor series expansion gives

$$f(\eta,p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \;\; f_m(\eta) = \frac{1}{m!}\left.\frac{\partial^m f(\eta;p)}{\partial p^m}\right|_{p=0}, \tag{3.35}$$

$$g(\eta,p) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta)p^m, \;\; g_m(\eta) = \frac{1}{m!}\left.\frac{\partial^m g(\eta;p)}{\partial p^m}\right|_{p=0}, \tag{3.36}$$

$$\theta(\eta,p) = \theta_0(\eta)\sum_{m=1}^{\infty} \theta_m(\eta)p^m, \;\; \theta_m(\eta) = \frac{1}{m!}\left.\frac{\partial^m \theta(\eta;p)}{\partial p^m}\right|_{p=0}, \tag{3.37}$$

where the convergence of above series strongly depends upon $\hbar_f$, $\hbar_g$ and $\hbar_\theta$. Considering that

47

$\hbar_f$, $\hbar_g$ and $\hbar_\theta$ are chosen in such a manner that Eqs. $(3.35) - (3.37)$ converge at $p = 1$ then

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \tag{3.38}$$

$$g(\eta) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta), \tag{3.39}$$

$$\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta). \tag{3.40}$$

The corresponding problems at mth order deformations satisfy

$$\mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar_f \mathcal{R}_f^m(\eta), \tag{3.41}$$

$$\mathcal{L}_g[g_m(\eta) - \chi_m g_{m-1}(\eta)] = \hbar_g \mathcal{R}_g^m(\eta), \tag{3.42}$$

$$\mathcal{L}_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = \hbar_\theta \mathcal{R}_\theta^m(\eta). \tag{3.43}$$

$$f_m(0) = f'_m(0) = f'_m(\infty) = 0, \ g_m(0) = g'_m(0) = g'_m(\infty) = 0,$$
$$\theta'_m(0) - \gamma_1 \theta_m(0) = \theta_m(\infty) = 0, \tag{3.44}$$

$$
\begin{aligned}
\mathcal{R}_f^m(\eta) = {}& f'''_{m-1}(\eta) - 2\sum_{k=0}^{m-1} f'_{m-1-k} f'_k - 2\sum_{k=0}^{m-1} g'_{m-1-k} f'_k + \sum_{k=0}^{m-1} (f_{m-1-k} f''_k + g_{m-1-k} f''_k) \\
& + K \left(
\begin{array}{l}
6\sum_{k=0}^{m-1} f'_{m-1-k} f'''_k + 3\sum_{k=0}^{m-1} g''_{m-1-k} f''_k - 3\sum_{k=0}^{m-1} f''_{m-1-k} f''_k \\
+ \sum_{k=0}^{m-1} \eta g'''_{m-1-k} f''_k + 4\sum_{k=0}^{m-1} g'_{m-1-k} f'''_k + 2\sum_{k=0}^{m-1} \eta g''_{m-1-k} f'''_k \\
- \sum_{k=0}^{m-1} f_{m-1-k} f''''_k - \sum_{k=0}^{m-1} g_{m-1-k} f''''_k - \sum_{k=0}^{m-1} \eta g'_{m-1-k} f''''_k
\end{array}
\right) \\
& + 2\lambda\theta,
\end{aligned}
\tag{3.45}
$$

$$\mathcal{R}_g^m(\eta) = g'''_{m-1}(\eta) - 2\sum_{k=0}^{m-1} g'_{m-1-k}g'_k - 2\sum_{k=0}^{m-1} g'_{m-1-k}f'_k + \sum_{k=0}^{m-1}(f_{m-1-k}g''_k + g_{m-1-k}g''_k)$$

$$+ K\left(\begin{array}{l} 6\sum_{k=0}^{m-1} g'_{m-1-k}g'''_k + 3\sum_{k=0}^{m-1} f''_{m-1-k}g''_k - 3\sum_{k=0}^{m-1} g''_{m-1-k}g''_k \\ + \sum_{k=0}^{m-1} \eta f'''_{m-1-k}g''_k + 4\sum_{k=0}^{m-1} f'_{m-1-k}g'''_k + 2\sum_{k=0}^{m-1} \eta f''_{m-1-k}g'''_k \\ - \sum_{k=0}^{m-1} g_{m-1-k}g''''_k - \sum_{k=0}^{m-1} f_{m-1-k}g''''_k - \sum_{k=0}^{m-1} \eta f'_{m-1-k}g''''_k \end{array}\right), \qquad (3.46)$$

$$\mathcal{R}_\theta^m(\eta) = (1 + \frac{4}{3}R)\theta''_{m-1} + \Pr\sum_{k=0}^{m-1}(\theta'_{m-1-k}f_k + \theta'_{m-1-k}g_k)$$

$$- \Pr A\sum_{k=0}^{m-1}(f'_{m-1-k}\theta_k + g'_{m-1-k}\theta_k), \qquad (3.47)$$

$$\chi_m = \left[\begin{array}{l} 0, \ \ m \le 1, \\ 1, \ \ m > 1. \end{array}\right. \qquad (3.48)$$

The mth order deformation problems have the solutions

$$f_m(\eta) = f_m^*(\eta) + C_1 + C_2 e^\eta + C_3 e^{-\eta}, \qquad (3.49)$$

$$g_m(\eta) = g_m^*(\eta) + C_4 + C_5 e^\eta + C_6 e^{-\eta}, \qquad (3.50)$$

$$\theta_m(\eta) = \theta_m^*(\eta) + C_7 e^\eta + C_8 e^{-\eta}, \qquad (3.51)$$

where the special solutions are f_m^*, g_m^* and θ_m^*.

3.3 Convergence analysis

We recall that the series $(3.38) - (3.40)$ contain the auxiliary parameters $\hbar_f$, $\hbar_g$ and $\hbar_\theta$. These parameters are useful to adjust and control the convergence of homotopic solutions. Hence the $\hbar$-curves are sketched at 15th order of approximations in order to determine the suitable ranges for $\hbar_f$, $\hbar_g$ and $\hbar_\theta$. Fig. 3.2 denotes that the ranges of admissible values of $\hbar_f$, $\hbar_g$ and $\hbar_\theta$ are $-0.7 \le \hbar_f \le -0.2$, $-0.7 \le \hbar_g \le -0.1$ and $-0.8 \le \hbar_\theta \le -0.2$. Table 3.1 presents the numerical values of $-f''(0)$, $-g''(0)$ and $-\theta'(0)$ for different order of approximations when $\hbar_f = -0.5, \hbar_g =$

-0.6 and $\hbar_\theta = -0.7$. It is seen that the values of $-f''(0)$ and $-g''(0)$ converge from 20th order of deformations whereas the values of $-\theta'(0)$ converge from 25th order approximations. Further, it is observed that we have to compute less deformations for the velocities in comparison to temperature for convergent series solutions.
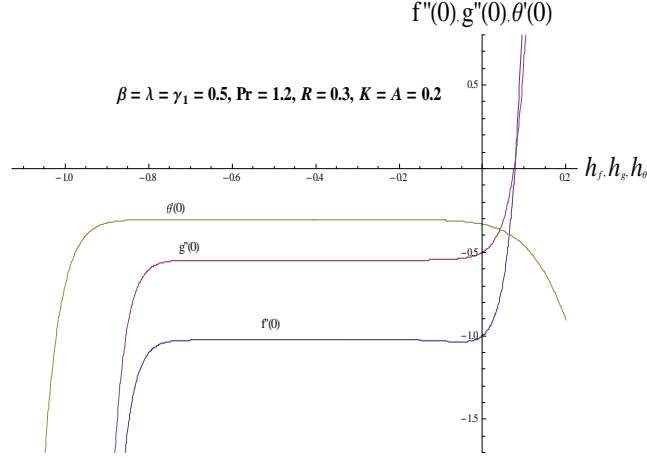
Fig. 3.2: $\hbar-$curves for the functions f, g and θ

Table 3.1: Convergence of series solutions for different order of approximations when $K = 0.1$, $A = 0.2$, $\mathrm{Pr} = 1.2$, $\gamma_1 = 0.6$, $\beta = 0.2$, $\hbar_f = -0.5, \hbar_g = -0.6$ and $\hbar_\theta = -0.7$.

order of approximations	1	5	10	15	20	25
$-f''(0)$	1.06111	1.02482	1.02609	1.02623	1.02618	1.02618
$-g''(0)$	0.544444	0.548057	0.548092	0.548043	0.548053	0.548053
$-\theta'(0)$	0.317778	0.305581	0.305729	0.305744	0.305738	0.305738

3.4 Discussion of results

The effects of ratio parameter β, viscoelastic parameter K, mixed convection parameter λ, Biot number γ_1 and radiation parameter R on the velocity component $f'(\eta)$ are shown in the Figs. 3.3-3.5. It is observed from Fig. 3.3 that velocity component $f'(\eta)$ and thermal boundary layer thickness are decreasing functions of ratio parameter β. This is due to the fact that with the increase of ratio parameter β, the x-component of velocity coefficient decreases which leads to a decrease in both the momentum boundary layer and velocity component $f'(\eta)$. Fig. 3.4

illustrates the influence of viscoelastic parameter K on the velocity component $f'(\eta)$. It is clear that both the boundary layer and velocity component $f'(\eta)$ increase when the viscoelastic parameter increases. Influence of mixed convection parameter λ on the velocity component $f'(\eta)$ is analyzed in Fig. 3.5. Increase in mixed convection parameter λ shows an increase in velocity component $f'(\eta)$. This is due to the fact that the buoyancy forces are much more effective rather than the viscous forces. Figs. 3.6 and 3.7 illustrate the variations of ratio parameter β and viscoelastic parameter K on the velocity component $g'(\eta)$. Variation of ratio parameter β is analyzed in Fig. 3.6. Through comparative study with Fig. 3.3 it is noted that $f'(\eta)$ decreases while $g'(\eta)$ increases when β increases. Physically, when β increases from zero, the lateral surface starts moving in $y-$direction and thus the velocity component $g'(\eta)$ increases and the velocity component $f'(\eta)$ decreases. Fig. 3.7 is plotted to see the variation of viscoelastic parameter K on the velocity component $g'(\eta)$. It is found that both the velocity component $g'(\eta)$ and momentum boundary layer thicknesses are increasing functions of K. It is revealed from Figs. 3.4 and 3.7 that the effect of K on both the velocities are qualitatively similar.

Figs. 3.8-3.14 are sketched to see the effects of ratio parameter β, viscoelastic parameter K, the temperature exponent A, Biot number γ_1, mixed convection parameter λ, Radiation parameter R and Prandtl number Pr on the temperature $\theta(\eta)$. Fig. 3.8 is drawn to see the impact of ratio parameter β on the temperature $\theta(\eta)$. It is noted that the temperature $\theta(\eta)$ and also the thermal boundary layer thickness decrease with increasing β. Variation of the viscoelastic parameter K on the temperature $\theta(\eta)$ is shown in Fig. 3.9. Here both the temperature and thermal boundary layer thickness are decreasing functions of K. Variation of mixed convection parameter λ is analyzed in Fig. 3.10. It is seen that both the temperature $\theta(\eta)$ and thermal boundary layer thickness are decreasing functions of mixed convection parameter λ. Fig. 3.11 presents the plots for the variation of Biot number γ_1. Note that $\theta(\eta)$ increases when γ_1 increases. The thermal boundary layer thickness is also increasing function of γ_1. It is also noted that the fluid temperature is zero when the Biot number vanishes. Influence of temperature exponent A is displayed in Fig. 3.12. It is found that both the temperature $\theta(\eta)$ and thermal boundary layer thickness decrease when A is increased. Also both the temperature $\theta(\eta)$ and thermal boundary layer thickness are increasing functions of thermal radiation parameter R

(see Fig. 3.13). It is observed that an increase in R has the ability to increase the thermal boundary layer. It is due to the fact that when the thermal radiation parameter increases, the mean absorption coefficient k_e will be decreased which in turn increases the divergence of the radiative heat flux. Hence the rate of radiative heat transfer to the fluid is increased and consequently the fluid temperature increases. Fig. 3.14 is plotted to see the effects of Pr on $\theta(\eta)$. It is noticed that both the temperature profile and thermal boundary layer thickness are decreasing functions of Pr. In fact when Pr increases then thermal diffusivity decreases. This indicates reduction in energy transfer ability and ultimate it results in the decrease of thermal boundary layer.
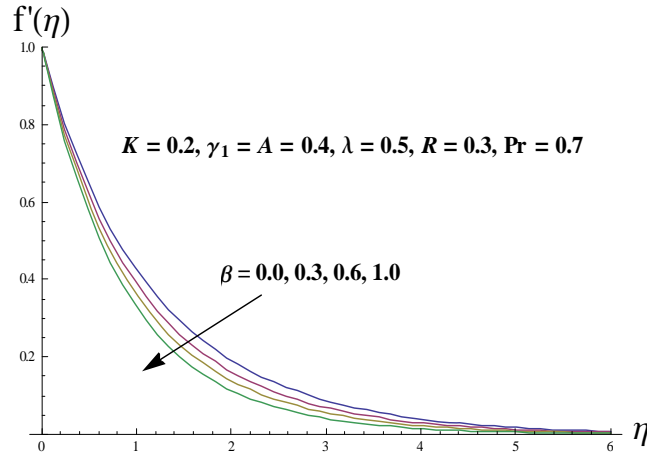
Fig. 3.3: Influence of β on the velocity $f'(\eta)$.
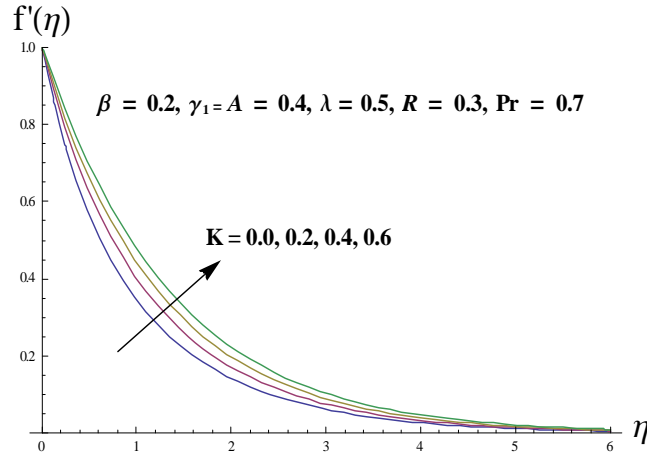
Fig. 3.4: Influence of K on the velocity $f'(\eta)$.
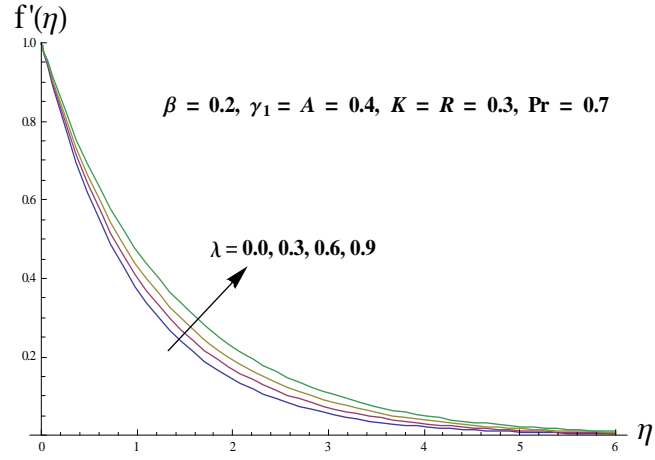
Fig. 3.5: Influence of λ on the velocity $f'(\eta)$.
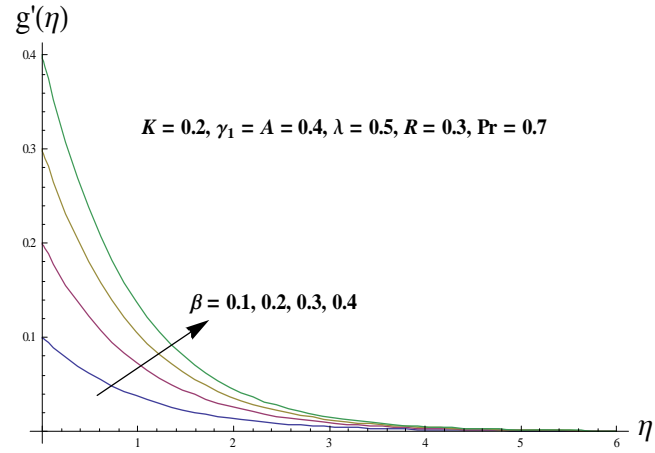
Fig. 3.6: Influence of β on the velocity $g'(\eta)$.
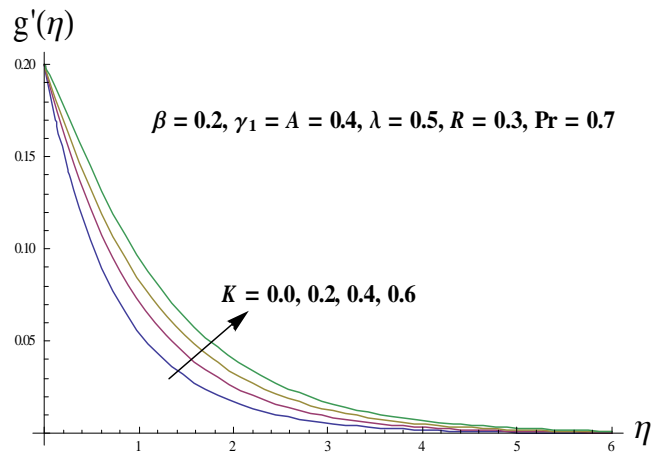
Fig. 3.7: Influence of K on the velocity $g'(\eta)$.
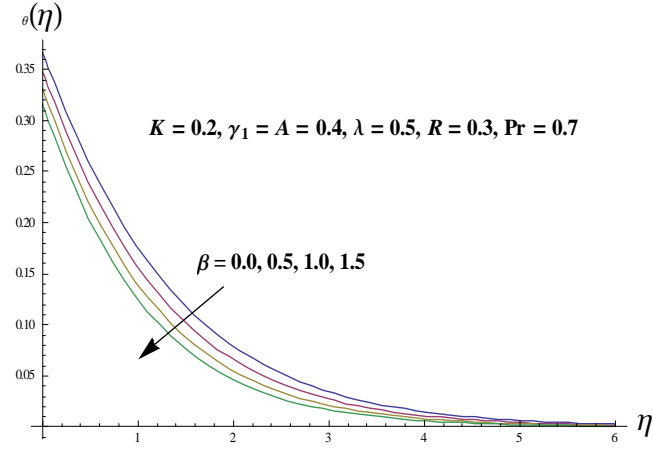
Fig. 3.8: Influence of β on the temperature $\theta(\eta)$.
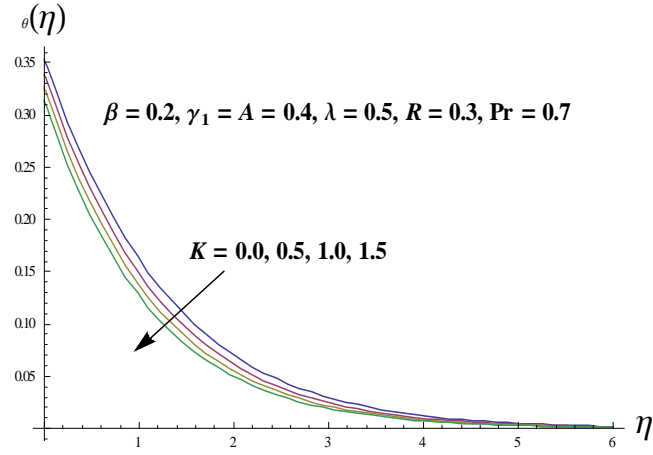
Fig. 3.9: Influence of K on the temperature $\theta(\eta)$.

Fig. 3.10: Influence of λ on the tenperature $\theta(\eta)$.

54
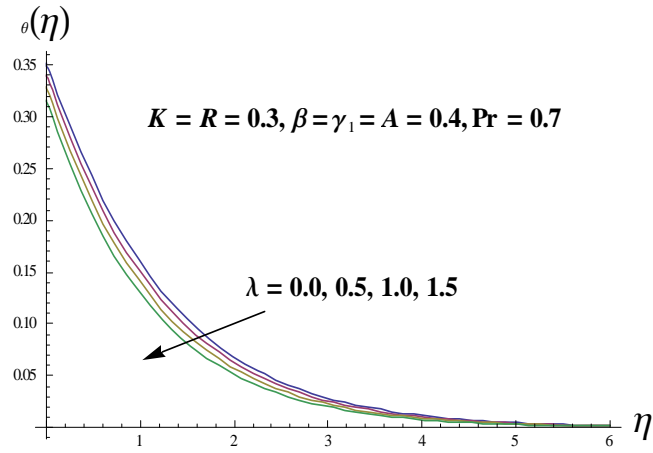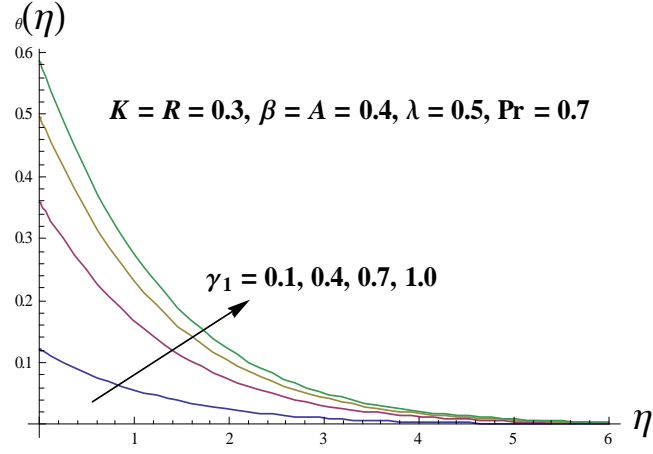
Fig. 3.11: Influence of γ_1 on the temperature $\theta(\eta)$.
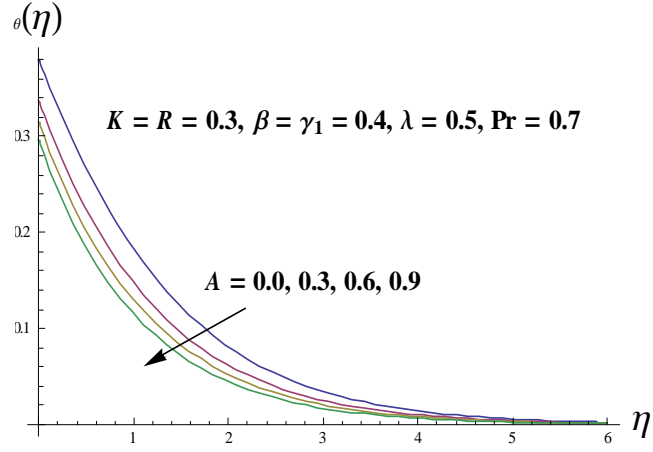
Fig. 3.12: Influence of A on the temperature $\theta(\eta)$.
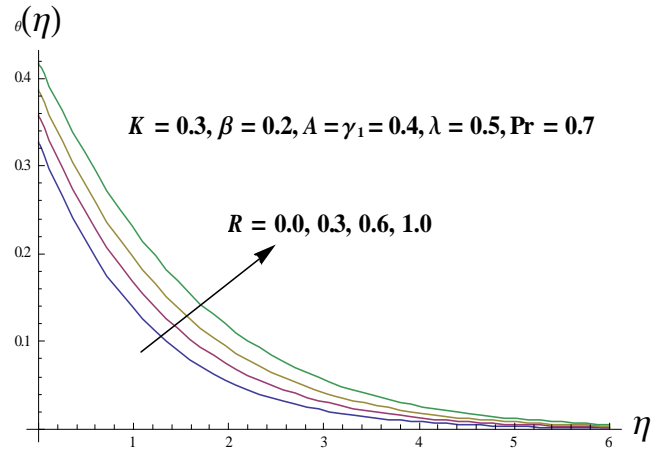
Fig. 3.13: Influence of R on the temperature $\theta(\eta)$.
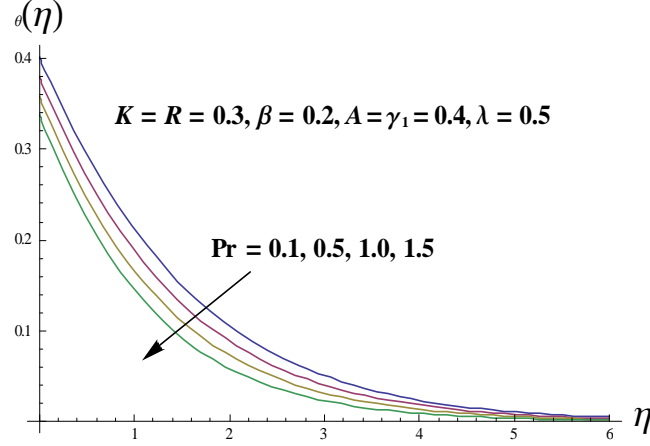
Fig. 3.14: Influence of Pr on the temperature $\theta(\eta)$.

Table 3.2 includes the values for comparison of existing solutions with the previous available solutions in a limiting case when $K = \lambda = \gamma_1 = R = 0$ and β varies. This Table presents an excellent agreement with the previous available solutions. Table 3.3 is computed to see the influences of viscoelastic parameter K and ratio parameter β on skin friction coefficients in the x and y directions. It is noted that K has quite opposite effect on skin friction coefficients while quite similar effect is seen within the increase of ratio parameter β. Table 3.4 examines the impact of viscoelastic parameter K, mixed convection parameter λ, ratio parameter β, Biot number γ_1, radiation parameter R, Prandtl number Pr and temperature exponent A on the local Nusselt number (rate of heat transfer at the wall). It is noted that the value of rate of heat transfer increases for larger viscoelastic parameter K, mixed convection parameter λ, ratio parameter β, Biot number γ_1, Prandtl number Pr and temperature exponent A while it decreases through an increase in radiation parameter R.

Table 3.2: Comparative values of $-f''(0)$, $-g''(0)$ and $f(\infty) + g(\infty)$ for different values β when $K = \lambda = \gamma_1 = R = 0$.

	Liu et al. [25]			Present results		
α	$-f''(0)$	$-g''(0)$	$f(\infty) + g(\infty)$	$-f''(0)$	$-g''(0)$	$f(\infty) + g(\infty)$
0.0	1.28180856	0	0.90564383	1.28181	0	0.90564
0.50	1.56988846	0.78494423	1.10918263	1.56989	0.78494	1.10918
1.00	1.81275105	1.81275105	1.28077378	1.81275	1.81275	1.28077

Table 3.3: Values of skin friction coefficients for different values of K and β when $\lambda = \gamma_1 = 0.5$, $R = 0.3$, $\mathrm{Pr} = 1.2$ and $A = 0.2$.

K	α	$-\left(\frac{\mathrm{Re}}{2}\right)^{1/2} C_{fx}$	$-\left(\frac{\mathrm{Re}}{2}\right)^{1/2} C_{fy}$
0.0	0.5	4.95289	4.37363
0.2		5.16586	3.97055
0.3		5.42622	3.96130
0.3	0.0	3.72170	1.65409
	0.2	4.30247	2.34617
	0.5	5.42622	3.96130

Table 3.4: Values of local Nusselt number $-(1 + \frac{4}{3}R)\theta'(0)$ for different values of the parameters K, β, λ, R, A, Pr and γ_1.

K	λ	β	γ_1	R	Pr	A	$-(1+\frac{4}{3}R)\theta'(0)$
0.0	0.5	0.5	0.5	0.3	1.2	0.2	0.297492
0.3							0.308234
0.5							0.311853
0.2	0.0						0.303062
	0.3						0.304775
	0.5						0.305738
0.2	0.5	0.0					0.282007
		0.3					0.297135
		0.5					0.305738
			0.1				0.0885730
			0.3				0.216850
			0.5				0.305738
0.2	0.5	0.5	0.5	0.0			0.329701
				0.3			0.305738
				0.5			0.292750
0.2	0.5	0.5	0.5	0.3	1.0		0.292152
					1.2		0.305738
					1.5		0.321826
0.2	0.5	0.5	0.5	0.3	1.2	0.0	0.288530
						0.2	0.305738
						0.5	0.325492

3.5   Conclusions

Three-dimensional mixed convection flow of viscoelastic fluid over an exponentially stretching surface is analyzed. The analysis is carried out in the presence of thermal radiation and convective boundary conditions. The main observations can be summarized as follows.

- Influence of ratio parameter β on the velocities $f'(\eta)$ and $g'(\eta)$ is quite opposite. However the effect of viscoelastic parameter K on the velocities $f'(\eta)$ and $g'(\eta)$ is qualitatively

similar.

- Momentum boundary layer thickness increases for $g'(\eta)$ when ratio parameter β is large. Effect of β on $f'(\eta)$ is opposite to that of $g'(\eta)$.

- Velocity component $f'(\eta)$ is increasing function of mixed convection parameter λ. However $\theta(\eta)$ decreases with an increase of mixed convection parameter λ. The impacts of Biot number γ_1 and radiation parameter R on $f'(\eta)$ and $\theta(\eta)$ are qualitatively similar.

- Momentum boundary layer is an increasing function of mixed convection parameter λ while thermal boundary layer is decreasing function of mixed convection parameter λ.

- Increase in Prandtl number decreases the temperature $\theta(\eta)$.

- Thermal boundary layer thickness decreases when ratio parameter β, viscoelastic parameter K, mixed convection parameter λ, Prandtl number Pr and temperature exponent A are increased.

- Influence of viscoelastic parameter K on the x and y directions of skin friction coefficients is opposite.

- Both components of skin friction coefficient increase through an increase in ratio parameter β.

- Local Nusselt number is an increasing function of Prandtl number Pr, ratio parameter β, viscoelastic parameter K, mixed convection parameter λ, Biot number γ_1 and temperature exponent A while it decreases for radiation parameter R.

Chapter 4

Convective heat and mass transfer in three-dimensional mixed convection flow of viscoelastic fluid with chemical reaction and heat source/sink

This chapter investigates the heat and mass transfer effects in three-dimensional mixed convection flow of viscoelastic fluid with internal heat source/sink and chemical reaction. An exponential stretching surface induces the flow. Magnetic field normal to the direction of flow is applied. Convective conditions at boundary surface are also encountered. Appropriate similarity transformations are utilized to reduce the boundary layer partial differential equations into ordinary differential equations. Analytical solutions of the resulting systems are obtained. Convergence of the obtained solutions is discussed explicitly. The local Nusselt and Sherwood numbers are sketched and examined.

4.1   Mathematical modeling

We consider the three-dimensional mixed convection boundary layer flow of viscoelastic fluid past an exponentially stretching surface. Mathematical analysis has been carried out in presence of internal heat source/sink and generative/destructive chemical reaction. Magnetic field is applied to the normal direction of flow. The surface coincides with the plane at $z = 0$ and the flow is confined in the region $z > 0$. Convective boundary conditions for both heat and mass transfer on the sheet are taken into account. The governing equations for three-dimensional flow are expressed as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{4.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2} + \frac{\alpha_1}{\rho}\left(u\frac{\partial^3 u}{\partial x\partial z^2} + w\frac{\partial^3 u}{\partial z^3} - \left(\begin{array}{c}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2} \\ +2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial x\partial z} + 2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}\end{array}\right)\right)$$
$$+g\beta_T(T - T_\infty) + g\beta_C(C - C_\infty) - \frac{\sigma^* B_0^2}{\rho}u, \tag{4.2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \nu\frac{\partial^2 v}{\partial z^2} + \frac{\alpha_1}{\rho}\left(v\frac{\partial^3 v}{\partial y\partial z^2} + w\frac{\partial^3 v}{\partial z^3} - \left(\begin{array}{c}\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial z^2} + \frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial z^2} \\ +2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y\partial z} + 2\frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial z^2}\end{array}\right)\right)$$
$$-\frac{\sigma^* B_0^2}{\rho}v, \tag{4.3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \sigma\frac{\partial^2 T}{\partial z^2} + \frac{Q}{\rho c_p}(T - T_\infty), \tag{4.4}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = D_B\frac{\partial^2 C}{\partial z^2} - k_1(C - C_\infty). \tag{4.5}$$

In the above equations, u, v and w are the velocity components in the $x-$, $y-$ and $z-$directions respectively, α_1 the material fluid parameter, β_T the thermal expansion coefficient, β_C the concentration expansion coefficient, σ^* the electrical conductivity, B_0 the magnitude of applied magnetic field, ρ the density of fluid, g the gravitational acceleration, $\nu = (\mu/\rho)$ the kinematic viscosity, μ the dynamic viscosity, σ the thermal diffusivity, T the fluid temperature, c_p the specific heat of the fluid, Q the uniform volumetric heat generation/absorption, C the con-

centration field, D the mass diffusivity and prime denotes the differentiation with respect to η.

The boundary conditions are given by

$$u = U_w, \ v = V_w, \ w = 0, \ -k\frac{\partial T}{\partial z} = h(T_f - T), \ -D\frac{\partial T}{\partial z} = h^*(C_f - C) \text{ at } z = 0, \qquad (4.6)$$

$$u \to 0, \ v \to 0, \ T \to T_\infty, \ C \to C_\infty \text{ as } z \to \infty, \qquad (4.7)$$

where subscript w corresponds to the wall condition, h is the heat transfer coefficient, h^* is the concentration transfer coefficient, T_f is the ambient fluid temperature and C_f is the ambient fluid concentration.

At wall the velocities, temperature and concentration distributions are defined as:

$$U_w = U_0 e^{\frac{x+y}{L}}, \ V_w = V_0 e^{\frac{x+y}{L}}, \ T_w = T_\infty + T_0 e^{\frac{A(x+y)}{2L}}, \ C_w = C_\infty + C_0 e^{\frac{B(x+y)}{2L}}, \qquad (4.8)$$

where U_0, V_0, W_0 are the constants, L is the reference length, T_∞ is the ambient temperature, C_∞ is the ambient concentration, A is the temperature exponent and B is the concentration exponent. By using similarity transformations [25]:

$$u = U_0 e^{\frac{x+y}{L}} f'(\eta), \ v = U_0 e^{\frac{x+y}{L}} g'(\eta), \ w = -\left(\frac{\nu U_0}{2L}\right)^{1/2} e^{\frac{x+y}{2L}} (f + \eta f' + g + \eta g'),$$

$$T = T_\infty + T_0 e^{\frac{A(x+y)}{2L}} \theta(\eta), \ C = C_\infty + C_0 e^{\frac{B(x+y)}{2L}} \phi(\eta), \ \eta = \left(\frac{U_0}{2\nu L}\right)^{1/2} e^{\frac{x+y}{2L}} z, \qquad (4.9)$$

equation (4.1) is identically satisfied and Eqs. $(4.2) - (4.9)$ give:

$$f''' + (f+g)f'' - 2(f'+g')f' + K\left(\begin{array}{c} 6f'''f' + (3g'' - 3f'' + \eta g''')f'' \\ +((4g' + 2\eta g'')f''' - (f + g + \eta g')f'''') \end{array}\right) + 2\lambda(\theta + N\phi) - Mf' = 0,$$
$$\qquad (4.10)$$

$$g''' + (f+g)g'' - 2(f'+g')g' + K\left(\begin{array}{c} 6g'''g' + (3f'' - 3g'' + \eta f''')g'' \\ +(4f' + 2\eta f'')g''' - (f + g + \eta f')g'''' \end{array}\right) - Mg' = 0, \qquad (4.11)$$

$$\theta'' + Pr(f+g)\theta' - Pr A(f'+g')\theta + Pr \beta^*\theta = 0, \qquad (4.12)$$

$$\phi'' + Sc(f+g)\phi' - ScB(f'+g')\phi - Sck^*\phi = 0, \qquad (4.13)$$

62

$$f = 0, \quad g = 0, \quad f' = 1, \quad g' = \beta, \quad \theta' = -\gamma_1(1 - \theta(0)), \quad \phi' = -\gamma_2(1 - \phi(0)), \text{ at } \eta = 0, \qquad (4.14)$$

$$f' \to 0, \quad g' \to 0, \quad \theta \to 0, \quad \phi \to 0 \text{ as } \eta \to \infty, \qquad (4.15)$$

where K is the viscoelastic parameter, λ is the mixed convection parameter, Gr_x is the local Grashof number, N is the concentration buoyancy parameter, M is the Hartman number, Pr is the Prandtl number, β^* is the heat source/sink parameter, k^* is the chemical reaction parameter, Sc is the Schmidt number, β is the ratio parameter, γ_1 is the heat transfer Biot number and γ_2 is the mass transfer Biot number. The definitions of these variables are

$$
\begin{aligned}
K &= \frac{\alpha_1 U_w}{2\nu L}, \lambda = \frac{Gr_x}{\mathrm{Re}_x^2}, \ Gr_x = \frac{g\beta_T(T_f - T_\infty)x^3}{\nu^2}, \ N = \frac{\beta_C(C_w - C_\infty)}{\beta_T(T_f - T_\infty)}, M = \frac{\sigma^* B_0^2}{\rho}, \\
Pr &= \frac{\nu}{\sigma}, \beta^* = \frac{Q}{\rho c_p}, \ k^* = \frac{k_1}{U_0}, Sc = \frac{\nu}{D}, \beta = \frac{V_0}{U_0}, \gamma_1 = \frac{h}{k}\sqrt{\frac{\nu}{a}}, \ \gamma_2 = \frac{h^*}{D}\sqrt{\frac{\nu}{a}}.
\end{aligned} \qquad (4.16)
$$

The local Nusselt and Sherwood numbers in dimensionless forms are

$$Nu/Re_x^{1/2} = -\frac{x}{2L}\theta'(0), \qquad (4.17)$$

$$Sh/Re_x^{1/2} = -\frac{x}{2L}\phi'(0), \qquad (4.18)$$

in which $Re_x = \frac{U_0 L}{\nu} e^{\frac{x+y}{L}}$ is the local Reynolds number.

4.2 Series solutions

For homotopic solutions, the initial guesses and auxiliary linear operators are chosen as follows:

$$f_0(\eta) = \left(1 - e^{-\eta}\right), \ g_0(\eta) = \beta\left(1 - e^{-\eta}\right), \ \theta_0(\eta) = \frac{\gamma_1 \exp(-\eta)}{1 + \gamma_1}, \ \phi_0(\eta) = \frac{\gamma_2 \exp(-\eta)}{1 + \gamma_2}, \qquad (4.19)$$

$$\mathcal{L}_f = f''' - f', \ \mathcal{L}_g = g''' - g', \ \mathcal{L}_\theta = \theta'' - \theta, \ \mathcal{L}_\phi = \phi'' - \phi. \qquad (4.20)$$

The above operators satisfy the following properties

$$\mathcal{L}_f(C_1 + C_2 e^{\eta} + C_3 e^{-\eta}) = 0, \ \mathcal{L}_g(C_4 + C_5 e^{\eta} + C_6 e^{-\eta}) = 0,$$

$$\mathcal{L}_\theta(C_7 e^{\eta} + C_8 e^{-\eta}) = 0, \ \mathcal{L}_\phi(C_9 e^{\eta} + C_{10} e^{-\eta}) = 0, \tag{4.21}$$

where $C_i \ (i = 1 - 10)$ are the arbitrary constants.

The problems corresponding to zeroth order are

$$(1 - p)\mathcal{L}_f \left[\hat{f}(\eta; p) - f_0(\eta) \right] = p\hbar_f \mathcal{N}_f \left[\hat{f}(\eta; p), \hat{g}(\eta; p), \hat{\theta}(\eta; p), \hat{\phi}(\eta; p) \right], \tag{4.22}$$

$$(1 - p)\mathcal{L}_g \left[\hat{g}(\eta; p) - g_0(\eta) \right] = p\hbar_g \mathcal{N}_g \left[\hat{f}(\eta; p), \hat{g}(\eta; p) \right], \tag{4.23}$$

$$(1 - p)\mathcal{L}_\theta \left[\hat{\theta}(\eta; p) - \theta_0(\eta) \right] = p\hbar_\theta \mathcal{N}_\theta \left[\hat{f}(\eta; p), \hat{g}(\eta; p), \hat{\theta}(\eta; p), \hat{\phi}(\eta; p) \right], \tag{4.24}$$

$$(1 - p)\mathcal{L}_\phi \left[\hat{\phi}(\eta; p) - \phi_0(\eta) \right] = p\hbar_\phi \mathcal{N}_\phi \left[\hat{f}(\eta; p), \hat{g}(\eta; p), \hat{\theta}(\eta; p), \hat{\phi}(\eta; p) \right], \tag{4.25}$$

$$\hat{f}(0; p) = 0, \ \hat{f}'(0; p) = 1, \ \hat{f}'(\infty; p) = 0, \ \hat{g}(0; p) = 0, \tag{4.26}$$

$$\hat{g}'(0; p) = \alpha, \hat{g}'(\infty; p) = 0, \hat{\theta}'(0, p) = -\gamma_1[1 - \theta(0, p)], \tag{4.27}$$

$$\hat{\theta}(\infty, p) = 0, \ \hat{\phi}'(0, p) = -\gamma_2[1 - \hat{\phi}(0, p)], \ \hat{\phi}(\infty, p) = 0, \tag{4.28}$$

$$\mathcal{N}_f[\hat{f}(\eta; p), \hat{g}(\eta; p), \hat{\theta}(\eta; p), \hat{\phi}(\eta; p)] = \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} - 2\left(\frac{\partial \hat{f}(\eta, p)}{\partial \eta} + \frac{\partial \hat{g}(\eta, p)}{\partial \eta} \right) \frac{\partial \hat{f}(\eta, p)}{\partial \eta}$$

$$+ \left(\begin{array}{c} \hat{f}(\eta, p) \\ +\hat{g}(\eta, p) \end{array} \right) \frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} + 2\lambda \left(\begin{array}{c} \hat{\theta}(\eta, p) \\ +N\hat{\phi}(\eta; p) \end{array} \right) - M\frac{\partial \hat{f}(\eta, p)}{\partial \eta}$$

$$+ K \left(\begin{array}{c} 6\frac{\partial \hat{f}(\eta,p)}{\partial \eta}\frac{\partial^3 \hat{f}(\eta,p)}{\partial \eta^3} + \left(\begin{array}{c} 3\frac{\partial^2 \hat{g}(\eta,p)}{\partial \eta^2} \\ -3\frac{\partial^2 \hat{f}(\eta,p)}{\partial \eta^2} + \eta\frac{\partial^3 \hat{g}(\eta,p)}{\partial \eta^3} \end{array} \right) \\ \frac{\partial^2 \hat{f}(\eta,p)}{\partial \eta^2} + \left(\begin{array}{c} 4\frac{\partial \hat{g}(\eta,p)}{\partial \eta} \\ +2\eta\frac{\partial^2 \hat{g}(\eta,p)}{\partial \eta^2} \end{array} \right)\frac{\partial^3 \hat{f}(\eta,p)}{\partial \eta^3} \\ - \left(\begin{array}{c} \hat{f}(\eta, p) + \hat{g}(\eta, p) \\ +\eta\frac{\partial \hat{g}(\eta,p)}{\partial \eta} \end{array} \right)\frac{\partial^4 \hat{f}(\eta,p)}{\partial \eta^4} \end{array} \right), \tag{4.29}$$

64

$$\mathcal{N}_g[\hat{f}(\eta;p),\hat{g}(\eta;p),\hat{\theta}(\eta;p),\hat{\phi}(\eta;p)] = \frac{\partial^3 \hat{g}(\eta,p)}{\partial \eta^3} - 2\left(\begin{array}{c}\frac{\partial \hat{f}(\eta,p)}{\partial \eta}\\+\frac{\partial \hat{g}(\eta,p)}{\partial \eta}\end{array}\right)\frac{\partial \hat{g}(\eta,p)}{\partial \eta} + \left(\begin{array}{c}\hat{f}(\eta,p)\\+\hat{g}(\eta,p)\end{array}\right)\frac{\partial^2 \hat{g}(\eta,p)}{\partial \eta^2}$$

$$+K\left(\begin{array}{c}6\frac{\partial \hat{g}(\eta,p)}{\partial \eta}\frac{\partial^3 \hat{g}(\eta,p)}{\partial \eta^3}\\+\left(\begin{array}{c}3\frac{\partial^2 \hat{f}(\eta,p)}{\partial \eta^2}-\\3\frac{\partial^2 \hat{g}(\eta,p)}{\partial \eta^2}+\eta\frac{\partial^3 \hat{f}(\eta,p)}{\partial \eta^3}\end{array}\right)\end{array}\right)$$

$$+K\left(\begin{array}{c}\frac{\partial^2 \hat{g}(\eta,p)}{\partial \eta^2}+\left(\begin{array}{c}4\frac{\partial \hat{f}(\eta,p)}{\partial \eta}\\+2\eta\frac{\partial^2 \hat{f}(\eta,p)}{\partial \eta^2}\end{array}\right)\frac{\partial^3 \hat{g}(\eta,p)}{\partial \eta^3}\\-\left(\begin{array}{c}\hat{f}(\eta,p)+\hat{g}(\eta,p)\\+\eta\frac{\partial \hat{f}(\eta,p)}{\partial \eta}\end{array}\right)\frac{\partial^4 \hat{g}(\eta,p)}{\partial \eta^4}\end{array}\right)$$

$$-M\frac{\partial \hat{g}(\eta,p)}{\partial \eta}, \tag{4.30}$$

$$\mathcal{N}_\theta[\hat{f}(\eta;p),\hat{g}(\eta;p),\hat{\theta}(\eta;p),\hat{\phi}(\eta;p)] = \frac{\partial^2 \hat{\theta}(\eta,p)}{\partial \eta^2} + \mathrm{Pr}\left(\begin{array}{c}(\hat{f}(\eta,p)\\+\hat{g}(\eta,p)\end{array}\right)\frac{\partial \hat{\theta}(\eta,p)}{\partial \eta} + \mathrm{Pr}\,\beta^*\hat{\theta}(\eta,p)$$

$$-\mathrm{Pr}\,A\left(\frac{\partial \hat{f}(\eta,p)}{\partial \eta}+\frac{\partial \hat{g}(\eta,p)}{\partial \eta}\right)\hat{\theta}(\eta,p), \tag{4.31}$$

$$\mathcal{N}_\phi[\hat{f}(\eta;p),\hat{g}(\eta;p),\hat{\theta}(\eta;p),\hat{\phi}(\eta;p)] = \frac{\partial^2 \hat{\phi}(\eta;p)}{\partial \eta^2} + Sc\left(\begin{array}{c}\hat{f}(\eta,p)\\+\hat{g}(\eta,p)\end{array}\right)\frac{\partial \hat{\phi}(\eta;p)}{\partial \eta} - Sck^*\hat{\phi}(\eta;p)$$

$$-ScB\left(\frac{\partial \hat{f}(\eta,p)}{\partial \eta}+\frac{\partial \hat{g}(\eta,p)}{\partial \eta}\right)\hat{\phi}(\eta;p). \tag{4.32}$$

Here p is an embedding parameter, the non-zero auxiliary parameters are $\hbar_f$, $\hbar_g$ $\hbar_\theta$ and $\hbar_\phi$ and the nonlinear operators are $\mathcal{N}_f$, $\mathcal{N}_g$, $\mathcal{N}_\theta$ and $\mathcal{N}_\phi$. Taking $p=0$ and $p=1$ we get

$$\hat{f}(\eta;0) = f_0(\eta),\ \hat{g}(\eta;0)=g_0(\eta),\ \hat{\theta}(\eta,0)=\theta_0(\eta),\ \hat{\phi}(\eta;0)=\phi_0(\eta)\ \text{and}\ \hat{f}(\eta;1)=f(\eta),$$

$$\hat{g}(\eta;1) = g(\eta),\ \hat{\theta}(\eta,1)=\theta(\eta),\ \hat{\phi}(\eta;1)=\phi(\eta). \tag{4.33}$$

As p enhances from 0 to 1 then $f(\eta,p)$, $g(\eta,p)$, $\theta(\eta,p)$ and $\phi(\eta,p)$ differ from $f_0(\eta)$, $g_0(\eta)$, $\theta_0(\eta)$

and $\phi_0(\eta)$ to $f(\eta)$, $g(\eta)$, $\theta(\eta)$ and $\phi(\eta)$. Applying Taylor's expansion we have

$$f(\eta, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \ f_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m f(\eta; p)}{\partial p^m} \right|_{p=0}, \tag{4.34}$$

$$g(\eta, p) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta)p^m, \ g_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m g(\eta; p)}{\partial p^m} \right|_{p=0}, \tag{4.35}$$

$$\theta(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)p^m, \ \theta_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \theta(\eta; p)}{\partial p^m} \right|_{p=0}, \tag{4.36}$$

$$\phi(\eta, p) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta)p^m, \ \phi_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \phi(\eta; p)}{\partial p^m} \right|_{p=0}. \tag{4.37}$$

The convergence of above series strongly depends upon $\hbar_f$, $\hbar_g$, $\hbar_\theta$ and $\hbar_\phi$. Considering that $\hbar_f$, $\hbar_g$, $\hbar_\theta$ and $\hbar_\phi$ are selected properly so that Eqs. $(4.34) - (4.37)$ converge at $p = 1$. Therefore

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \tag{4.38}$$

$$g(\eta) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta), \tag{4.39}$$

$$\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \tag{4.40}$$

$$\phi(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta). \tag{4.41}$$

The general solution expressions can be written as

$$f_m(\eta) = f_m^*(\eta) + C_1 + C_2 e^\eta + C_3 e^{-\eta}, \tag{4.42}$$

$$g_m(\eta) = g_m^*(\eta) + C_4 + C_5 e^\eta + C_6 e^{-\eta}, \tag{4.43}$$

$$\theta_m(\eta) = \theta_m^*(\eta) + C_7 e^\eta + C_8 e^{-\eta}, \tag{4.44}$$

$$\phi_m(\eta) = \phi_m^*(\eta) + C_9 e^\eta + C_{10} e^{-\eta}, \tag{4.45}$$

where the special solutions are f_m^*, g_m^*, θ_m^* and ϕ_m^*.

4.3　Convergence analysis and discussion

Homotopic solutions $(4.38) - (4.41)$ obviously depend on the auxiliary parameters $\hbar_f$, $\hbar_g$, $\hbar_\theta$ and $\hbar_\phi$. In order to control the convergence of series solutions' these auxiliary parameters play a central role. To obtain the convergence region, the $\hbar-$curves have been plotted at 14th order of approximations in Fig. 4.1. This Fig. clearly shows that the acceptable values of $\hbar_f$, $\hbar_g$, $\hbar_\theta$ and $\hbar_\phi$ are $-1.0 \leq \hbar_f \leq -0.4$, $-1.0 \leq \hbar_g \leq -0.2$, $-1.2 \leq \hbar_\theta \leq -0.1$ and $-1.2 \leq \hbar_\phi \leq -0.1$. Table 4.1 ensures that the series solutions converge in the whole region of η when $\hbar_f = \hbar_g = \hbar_\theta = \hbar_\phi = -0.5$.
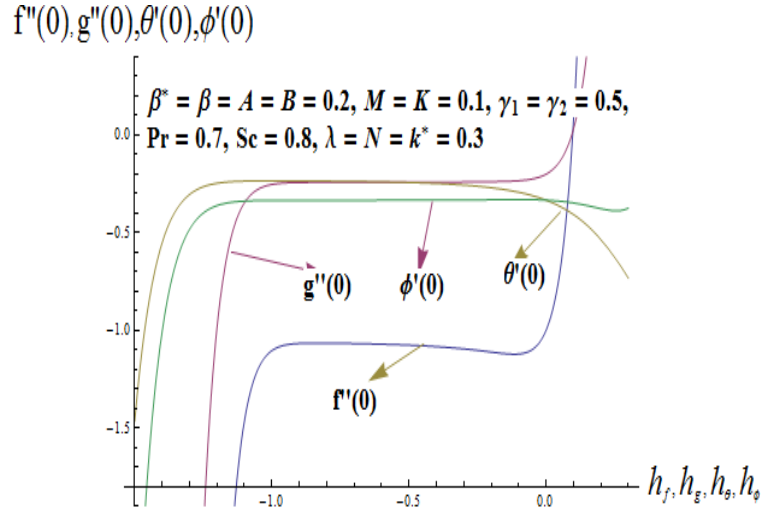
Fig. 4.1: $\hbar-$curves for the functions f, g, θ and ϕ.

Table 4.1: Convergence of series solutions for different order of approximations when $K = M = 0.1$, $A = B = \beta^* = \beta = 0.2$, $\lambda = N = k^* = 0.3, \gamma_1 = \gamma_2 = 0.5, \Pr = 0.7, Sc = 0.8$ and $\hbar_f = \hbar_g = \hbar_\theta = \hbar_\phi - 0.6$.

order of approximations	1	5	10	15	20	25	30	35
$-f''(0)$	1.155	1.104	1.078	1.068	1.065	1.064	1.064	1.064
$-g''(0)$	0.2359	0.2395	0.2414	0.2420	0.2422	0.2422	0.2422	0.2422
$-\theta'(0)$	0.3084	0.2620	0.2437	0.2373	0.2349	0.2341	0.2340	0.2340
$-\phi'(0)$	0.3300	0.3318	0.3336	0.3340	0.3341	0.3341	0.3341	0.3341

Figs. 4.2 and 4.3 are plotted to see the effects of Hartman number M on the velocity profile $f'(\eta)$ and $g'(\eta)$. The velocity profiles $f'(\eta)$ and $g'(\eta)$ are decreased when we increase the values of M. Also the momentum boundary layer thicknesses are decreasing functions of M. Fig. 4.4 is drawn to see the influence of internal heat source/ sink parameter β^* on the velocity profile $f'(\eta)$. Clearly in case of heat sink parameter $\beta^* < 0$ both momentum boundary layer thickness and $f'(\eta)$ decrease while in case of heat source parameter $\beta^* > 0$ kinetic energy of the fluid particles increases due to which the velocity profile $f'(\eta)$ increases. Outcome of mixed convection parameter λ on the velocity profile $f'(\eta)$ in both assisting and opposing flows is seen in Fig. 4.5. In case of assisting flow $\lambda > 0$ both $f'(\eta)$ and momentum boundary layer thickness are enhanced while reverse effect is observed for opposing flow $\lambda < 0$. Fig. 4.6 exhibits the variation of concentration buoyancy parameter N on the velocity profile $f'(\eta)$. It is examined that an enhancement in N gives rise to the velocity profile $f'(\eta)$.

Fig. 4.7 depicts the influence of internal heat source/sink parameter β^* on the temperature $\theta(\eta)$. With an increase in internal heat source $\beta^* > 0$ both the thermal boundary layer thickness and $\theta(\eta)$ increase while in case of heat sink parameter $\beta^* < 0$ both the thermal boundary layer thickness and $\theta(\eta)$ decrease.

Figs. 4.8-4.10 are sketched to see the variations of chemical reaction parameter k^*, concentration exponent B and mass transfer Biot number γ_2 on the concentration profile $\phi(\eta)$. Fig. 4.8 is presented to analyze the variation of chemical reaction parameter k^* on the concentration profile $\phi(\eta)$. It is noted that the associated boundary layer thickness and concentration profile $\phi(\eta)$ decrease for generative chemical reaction $k^* > 0$ while reverse phenomena is noted for destructive chemical reaction $k^* < 0$. With an enhancement in concentration exponent B both the concentration profile $\phi(\eta)$ and the boundary layer thickness decrease (see Fig. 4.9). Variation of mass transfer Biot number γ_2 on the concentration profile $\phi(\eta)$ is displayed in Fig. 4.10. Here we examined that the effect of γ_2 on the concentration profile $\phi(\eta)$ and associated boundary layer thickness are increasing.

Figs. 4.11-4.13 are displayed to see the impacts of mixed convection parameter λ, concentration buoyancy parameter N, ratio parameter β, Hartman number M, internal heat source/sink β^* and heat transfer Biot number γ_1 on the local Nusselt number $-\theta'(0)$. Fig. 4.11 shows that the heat transfer rate at the wall increases for assisting flow $\lambda > 0$ while it decreases for op-

posing flow $\lambda < 0$. It is also examined that the heat transfer rate at wall is increasing function of concentration buoyancy parameter N. Fig. 4.12 exhibited that the larger values of ratio parameter β corresponds to a higher heat transfer rate $-\theta'(0)$. Also it is to be noted that with an increase in Hartman number heat transfer rate decreases. Fig. 4.13 depicts that the heat transfer rate at the wall $-\theta'(0)$ decreases with internal heat source $\beta^* > 0$ while increases with internal heat sink $\beta^* < 0$.

Variations of mixed convection parameter λ, concentration buoyancy parameter N, ratio parameter β, Hartman number M, chemical reaction k^* and mass transfer Biot number γ_2 on Sherwood number $-\phi'(0)$ are plotted in the Figs. 4.14-4.16. Fig. 4.14 is drawn to see the influences of mixed convection parameter λ and concentration buoyancy parameter N on the Sherwood number $-\phi'(0)$. It is seen that the Sherwood number $-\phi'(0)$ is increasing function of λ and N in case of assisting flow $\lambda > 0$ while decreasing function for opposing flow case. Fig. 4.15 depicts that the Sherwood number $-\phi'(0)$ decreases with an enhancement in Hartman number M while it increases with an increase in ratio parameter β. Fig. 4.16 exhibits that the mass transfer at the wall $-\phi'(0)$ enhances with generative chemical reaction $k^* > 0$ while it reduces with destructive chemical reaction $k^* < 0$. It is also observed that the mass transfer at the wall $-\phi'(0)$ is an increasing function of mass transfer Biot number γ_2.Table 4.2 ensures the validity of present results with Liu et al. [25] in the limiting sense.
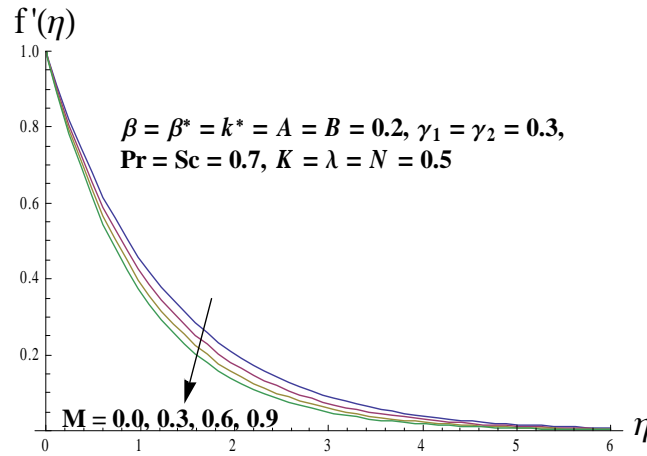
Fig. 4.2: Influence of M on velocity $f'(\eta)$.
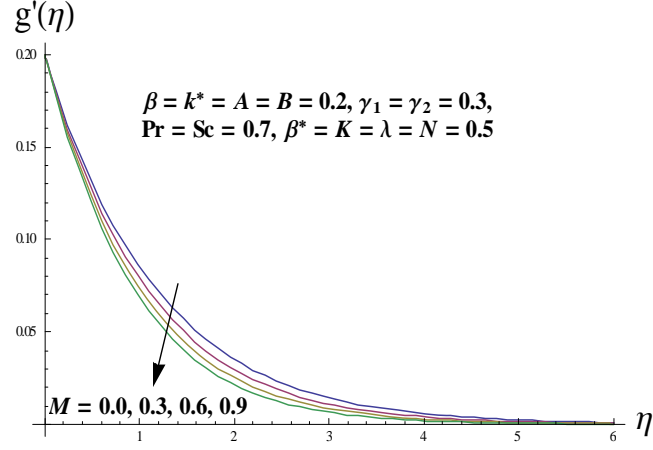
Fig. 4.3: Influence of M on velocity $g'(\eta)$.
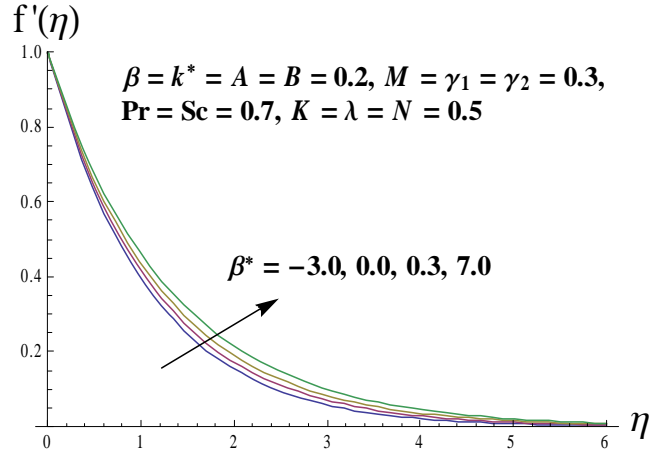
Fig. 4.4: Influence of β^* on velocity $f'(\eta)$.

Fig. 4.5: Influence of λ on velocity $f'(\eta)$.

70
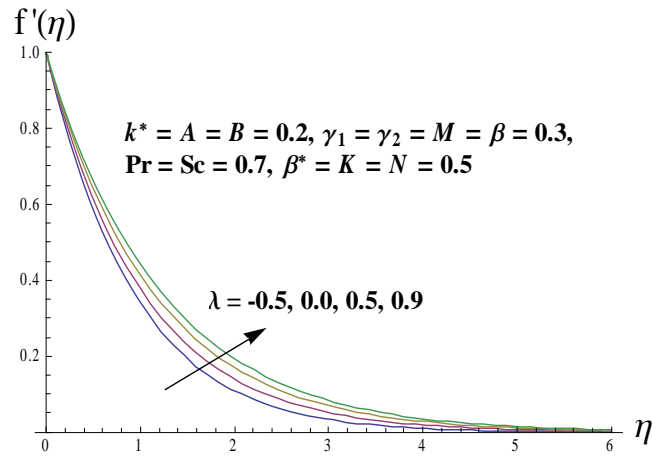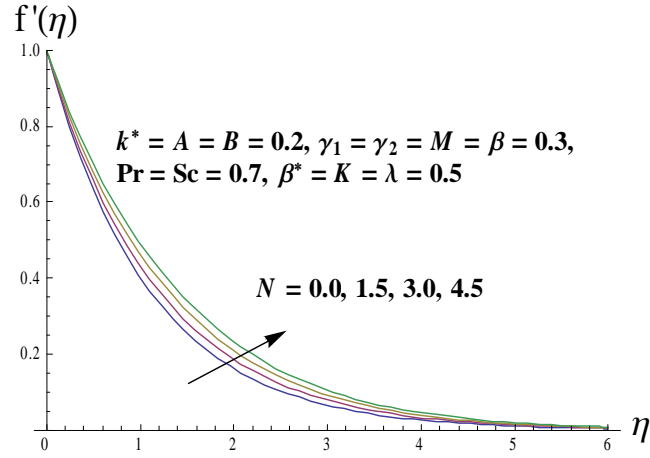
Fig. 4.6: Influence of N on velocity $f'(\eta)$.
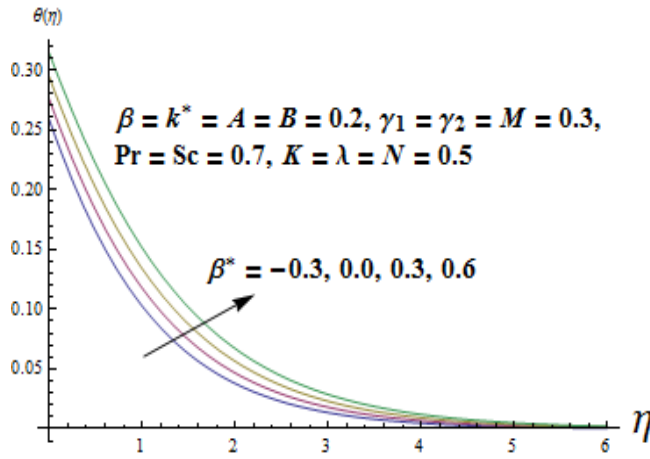
Fig. 4.7: Influence of β^* on temperature $\theta(\eta)$.
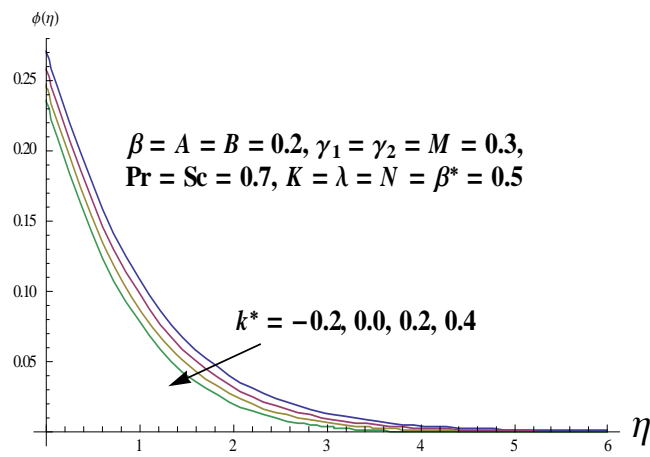
Fig. 4.8: Influence of k^* on $\phi(\eta)$.
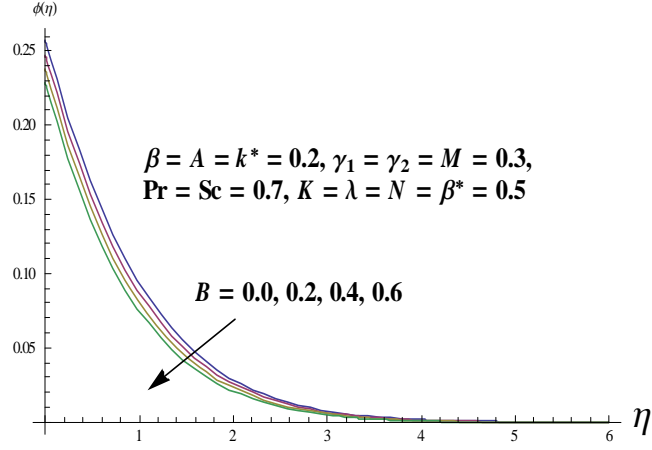
Fig. 4.9: Influence of B on concentration $\phi(\eta)$.
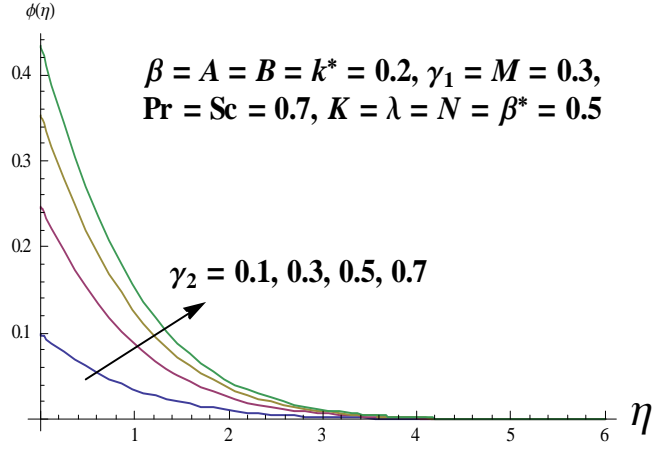
Fig. 4.10: Influence of γ_2 on $\phi(\eta)$.
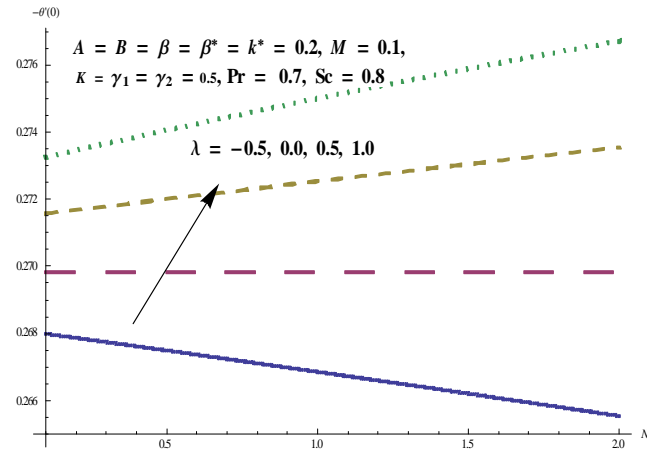
Fig. 4.11: Influence of λ and N on $-\theta'(0)$.
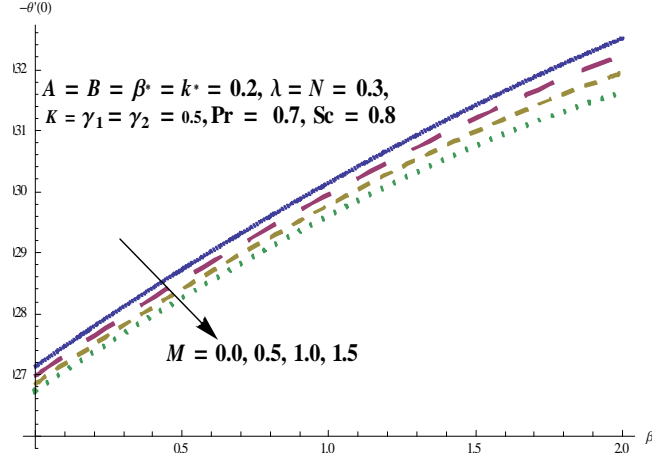
Fig. 4.11: Influence of M and β on $-\theta'(0)$.
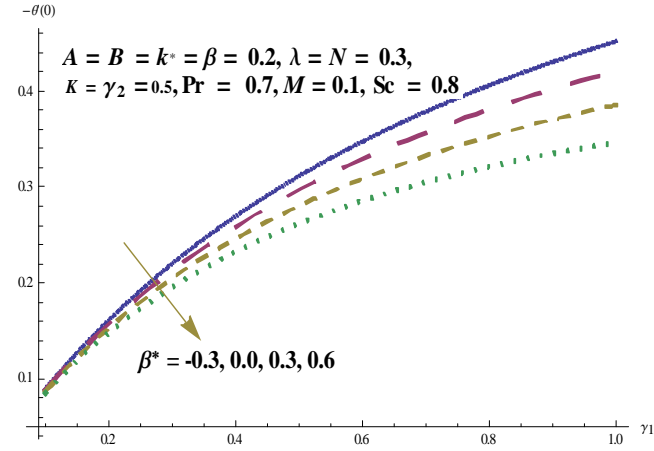
Fig. 4.13: Influence of β^* and γ_1 on $-\theta'(0)$.

Fig. 4.14: Influence of λ and N on $-\phi'(0)$.

73
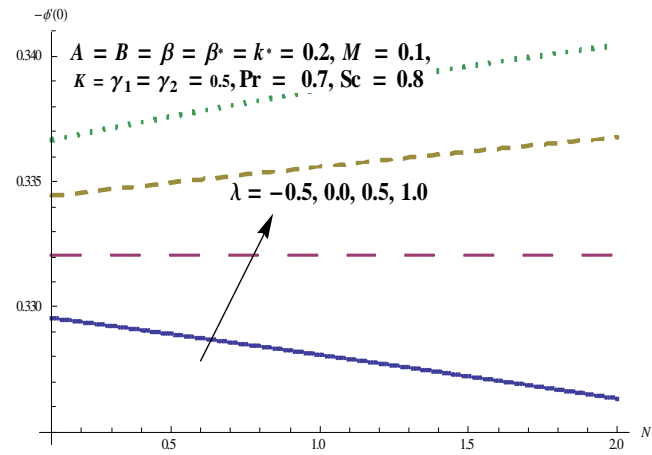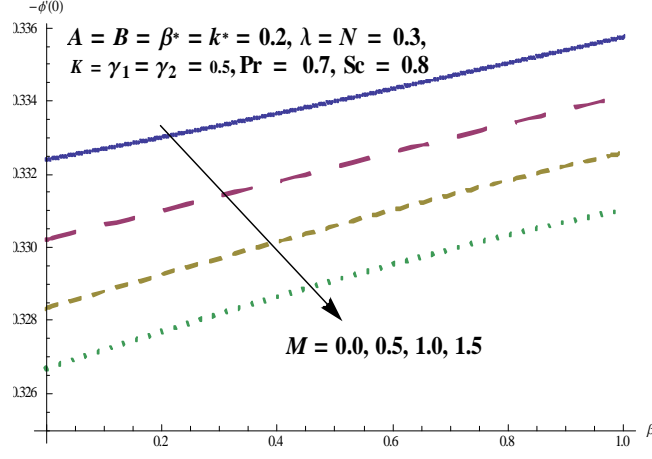
Fig. 4.15: Influence of M and β on $-\phi'(0)$.
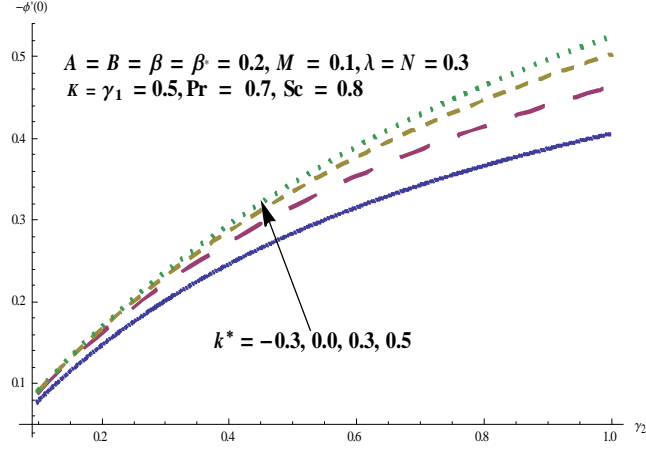
Fig. 4.16: Influence of k^* and γ_2 on $-\phi'(0)$.

Table 4.2: Comparative values of $-f''(0)$, $-g''(0)$ and $f(\infty) + g(\infty)$ for different values of β when $K = \lambda = N = \gamma_1 = \gamma_2 = \beta^* = k^* = 0$.

	Liu et al. [25]			Present results		
β	$-f''(0)$	$-g''(0)$	$f(\infty) + g(\infty)$	$-f''(0)$	$-g''(0)$	$f(\infty) + g(\infty)$
0.0	1.28180856	0	0.90564383	1.28181	0	0.90564
0.50	1.56988846	0.78494423	1.10918263	1.56989	0.78494	1.10918
1.00	1.81275105	1.81275105	1.28077378	1.81275	1.81275	1.28077

4.4   Closing remarks

The present chapter deals with the three-dimensional mixed convection flow of MHD viscoelastic fluid over an exponentially stretching surface in presence of heat source/sink and generative/destructive chemical reaction. The main outcomes are as follows.

- Velocity profiles $f'(\eta)$ and $g'(\eta)$ reduce with an increase in Hartman number M.

- Momentum boundary layer thickness decreases with an increase in ratio parameter β.

- Both the velocity profile $f'(\eta)$ and momentum boundary layer thicknesses are increasing functions of internal heat source parameter $\beta^* > 0$, assisting flow case $\lambda > 0$ and concentration buoyancy parameter N while decreasing functions of internal heat sink parameter $\beta^* < 0$ and opposing flow case $\lambda < 0$.

- Thermal boundary layer thickness and temperature $\theta(\eta)$ decrease with an increase in internal heat sink $\beta^* < 0$ while thermal boundary layer thickness and temperature $\theta(\eta)$ increase with an increase in internal heat source $\beta^* > 0$.

- With an enhancement in generative chemical reaction $k^* > 0$, concentration exponent B and Schmidt number Sc decreases the concentration profile $\phi(\eta)$. The concentration boundary layer thickness increases for larger mass transfer Biot number γ_2 and destructive chemical reaction $k^* < 0$.

- Heat transfer rate $-\theta'(0)$ boosts up in case of assisting flow $\lambda > 0$, concentration buoyancy parameter N, ratio parameter β, heat transfer Biot number γ_1 and internal heat sink parameter $\beta^* < 0$ while heat transfer rate $-\theta'(0)$ reduces with opposing flow $\lambda < 0$, Hartman number M and internal heat source $\beta^* > 0$.

- With an increase in assisting flow $\lambda > 0$, concentration buoyancy parameter N, ratio parameter β, mass transfer Biot number γ_2 and generative chemical reaction parameter $k^* > 0$, the Sherwood number $-\phi'(0)$ enhances while reverse behavior is noted in case of opposing flow $\lambda < 0$, Hartman number M and destructive chemical reaction parameter $k^* < 0$.

Chapter 5

Thermophoresis and MHD mixed convection flow with Soret and Dufour effects

This chapter investigates the heat and mass transfer effects in three-dimensional mixed convection flow of viscoelastic fluid over a stretching surface with convective boundary conditions. The fluid is electrically conducting in the presence of constant applied magnetic field. Conservation laws of energy and concentration are based upon the Soret and Dufour effects. First order chemical reaction effects are also taken into account. Dimensionless velocity, temperature and concentration distributions are shown graphically for different values of involved parameters. Numerical values of local Nusselt and Sherwood numbers are computed and analyzed.

5.1 Mathematical analysis

We consider the steady three-dimensional magnetohydrodynamic mixed convection flow of an incompressible viscoelastic fluid over a stretching surface at $z = 0$. The flow takes place in the domain $z > 0$. Heat and mass transfer characteristics are taken into account in the presence of Soret and Dufour and thermophoresis effects. The ambient fluid temperature is taken as T_∞ while the surface temperature is maintained by convective heat transfer at a certain value T_f. A constant magnetic field B_0 is applied in the z-direction. Induced magnetic field is not

considered due to small Reynolds number. In addition the effects of first order chemical reaction in mass transfer are taken into account. The governing boundary layer equations for the flow under consideration are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{5.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2} + \frac{\alpha_1}{\rho}\left(u\frac{\partial^3 u}{\partial x\partial z^2} + w\frac{\partial^3 u}{\partial z^3} - \left(\begin{array}{c}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial z^2} + \frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2} \\ +2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial x\partial z} + 2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}\end{array}\right)\right)$$
$$+g\beta_T(T - T_\infty) + g\beta_C(C - C_\infty) - \frac{\sigma^* B_0^2}{\rho}u, \tag{5.2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \nu\frac{\partial^2 v}{\partial z^2} + \frac{\alpha_1}{\rho}\left(v\frac{\partial^3 v}{\partial y\partial z^2} + w\frac{\partial^3 v}{\partial z^3} - \left(\begin{array}{c}\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial z^2} + \frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial z^2} \\ +2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y\partial z} + 2\frac{\partial w}{\partial z}\frac{\partial^2 v}{\partial z^2}\end{array}\right)\right)$$
$$-\frac{\sigma^* B_0^2}{\rho}v, \tag{5.3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \sigma\frac{\partial^2 T}{\partial z^2} + \frac{D_e k_T}{C_s C_p}\frac{\partial^2 C}{\partial z^2}, \tag{5.4}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = D_e\frac{\partial^2 C}{\partial z^2} + \frac{D_e k_T}{T_m}\frac{\partial^2 T}{\partial z^2} - k_1(C - C_\infty) - \frac{\partial}{\partial z}(V_T(C - C_\infty)). \tag{5.5}$$

In Eqs. $(5.1) - (5.5)$ the respective velocity components in the $x-$, $y-$ and $z-$directions are denoted by u, v and w, α_1 the viscoelastic parameter, σ^* the electrical conductivity, B_0 is the magnitude of applied magnetic field, ρ the density of fluid, g the gravitational acceleration, β_T the thermal expansion coefficient, β_C the concentration expansion coefficient, T the fluid temperature, σ the thermal diffusivity of fluid, ν the kinematic viscosity, μ the dynamic viscosity of fluid, C the concentration field, D_e the mass diffusivity, k_T the thermal diffusion ratio, c_p the specific heat, C_s the concentration susceptibility and V_T the thermophoretic velocity.

In Eq. (5.5) the thermophoretic term V_T can be defined as

$$V_T = -k_2\frac{\nu}{T_r}\frac{\partial T}{\partial z} \tag{5.6}$$

where k_2 is the thermophoretic coefficient and T_r is the reference temperature. A thermophoretic

parameter τ is defined as

$$\tau = -\frac{k_2(T_f - T_\infty)}{T_r} \tag{5.7}$$

The boundary conditions appropriate to the flow under consideration are given by

$$u = u_e = ax, \ v = by, \ w = 0, \ -k\frac{\partial T}{\partial z} = h(T_f - T), \ C = C_w \text{ at } z = 0,$$

$$u \to 0, \ v \to 0, \ T \to T_\infty, \ C \to C_\infty \text{ as } z \to \infty, \tag{5.8}$$

where k indicates the thermal conductivity of fluid, T_f is the hot fluid temperature, C_∞ the ambient concentration and a and b have dimension inverse of time.

We now define

$$
\begin{aligned}
u &= axf'(\eta), \ v = ayg'(\eta), \ w = -\sqrt{a\nu}(f(\eta) + g(\eta)), \\
\theta(\eta) &= \frac{T - T_\infty}{T_f - T_\infty}, \ \eta = z\sqrt{\frac{a}{\nu}}, \ \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}.
\end{aligned} \tag{5.9}
$$

Now the use of above variables satisfy Eq. (5.1) automatically while Eqs. (5.2)-(5.8) are reduced as follows:

$$f''' + (f + g)f'' - f'^2 - K\left((f + g)f'''' + (f'' - g'')f'' - 2(f' + g')f'''\right)$$

$$-M^2 f' + \lambda(\theta + N\phi) = 0, \tag{5.10}$$

$$g''' + (f + g)g'' - g'^2 - K\left((f + g)g'''' + (f'' - g'')g'' - 2(f' + g')g'''\right) - M^2 g' = 0, \tag{5.11}$$

$$\theta'' + Pr(f + g)\theta' + Pr\,D_f\phi'' = 0, \tag{5.12}$$

$$\phi'' + Sc(f + g)\phi' - Sck^*\phi + ScSr\theta'' - Sc\tau(\phi'\theta' - \phi\theta'') = 0,$$

$$f = 0, \ g = 0, \ f' = 1, \ g' = \beta, \ \theta' = -\gamma_1(1 - \theta(0)), \ \phi = 1 \text{ at } \eta = 0, \tag{5.13}$$

$$f' \to 0, \ g' \to 0, \ \theta \to 0, \ \phi \to 0 \text{ as } \eta \to \infty, \tag{5.14}$$

where K is the dimensionless viscoelastic parameter, M is the Hartman number, λ is the local buoyancy parameter, Gr_x is the local Grashof number, N is the constant dimensionless

78

concentration buoyancy parameter, Pr is the Prandtl number, D_f is the Dufour number, Sc is the Schmidt number, Sr is the Soret number, k^* is the chemical reaction parameter, τ is the thermophoretic parameter, β is ratio of rates parameters, γ_1 is the Biot number and prime shows the differentiation with respect to η. These are given by

$$K = \frac{\alpha_1 a}{\nu}, \ M = \frac{\sigma^* B_0^2}{\rho}, \ \lambda = \frac{Gr_x}{Re_x^2}, \ Gr_x = \frac{g\beta_T(T_f - T_\infty)x^3}{\upsilon^2}, \ N = \frac{\beta_C(C_w - C_\infty)}{\beta_T(T_f - T_\infty)}$$

$$Pr = \frac{\nu}{\sigma}, \ D_f = \frac{D_e k_T}{C_s C_p} \frac{(C_w - C_\infty)}{(T_f - T_\infty)\nu}, \ Sc = \frac{\nu}{D}, \ Sr = \frac{D_e k_T}{T_m \nu} \frac{(T_f - T_\infty)}{(C_w - C_\infty)}, \ k^* = \frac{k_1}{a}, \tau = -\frac{k_2(T_f - T_\infty)}{T_r},$$

$$\beta = \frac{b}{a}, \ \gamma_1 = \frac{h}{k}\sqrt{\frac{\nu}{a}}.$$

Local Nusselt and Sherwood numbers in dimensionless forms are given by

$$Nu/Re_x^{1/2} = -\theta'(0), \tag{5.15}$$

$$Sh/Re_x^{1/2} = -\phi'(0), \tag{5.16}$$

in which $Re_x = u_e x/\nu$ is the local Reynolds number.

5.2 Construction of solutions

The initial approximations and auxiliary linear operators required for homotopy analysis solutions are presented below i.e.

$$f_0(\eta) = \left(1 - e^{-\eta}\right), \ g_0(\eta) = \beta\left(1 - e^{-\eta}\right), \ \theta_0(\eta) = \frac{\gamma_1 \exp(-\eta)}{1 + \gamma_1}, \ \phi_0(\eta) = \exp(-\eta) \tag{5.17}$$

$$\mathcal{L}_f = f''' - f', \ \mathcal{L}_g = g''' - g', \ \mathcal{L}_\theta = \theta'' - \theta, \ \mathcal{L}_\phi = \phi'' - \phi, \tag{5.18}$$

with the following properties of the defined operators in Eq. (5.18) i.e.

$$\mathcal{L}_f(C_1 + C_2 e^\eta + C_3 e^{-\eta}) = 0, \ \mathcal{L}_g(C_4 + C_5 e^\eta + C_6 e^{-\eta}) = 0,$$

$$\mathcal{L}_\theta(C_7 e^\eta + C_8 e^{-\eta}) = 0, \ \mathcal{L}_\phi(C_9 e^\eta + C_{10} e^{-\eta}) = 0 \tag{5.19}$$

where $C_i \ (i = 1 - 10)$ indicate the arbitrary constants.

The corresponding problems at the zeroth order are given in the following forms:

$$(1-p)\mathcal{L}_f\left[\hat{f}(\eta;p) - f_0(\eta)\right] = p\hbar_f\mathcal{N}_f\left[\hat{f}(\eta;p), \hat{g}(\eta;p)\right], \tag{5.20}$$

$$(1-p)\mathcal{L}_g\left[\hat{g}(\eta;p) - g_0(\eta)\right] = p\hbar_g\mathcal{N}_g\left[\hat{f}(\eta;p), \hat{g}(\eta;p)\right], \tag{5.21}$$

$$(1-p)\mathcal{L}_\theta\left[\hat{\theta}(\eta;p) - \theta_0(\eta)\right] = p\hbar_\theta\mathcal{N}_\theta\left[\hat{f}(\eta;p), \hat{g}(\eta;p), \hat{\theta}(\eta,p), \hat{\phi}(\eta,p)\right], \tag{5.22}$$

$$(1-p)\mathcal{L}_\theta\left[\hat{\phi}(\eta;p) - \phi_0(\eta)\right] = p\hbar_\phi\mathcal{N}_\phi\left[\hat{f}(\eta;p), \hat{g}(\eta;p), \hat{\theta}(\eta,p), \hat{\phi}(\eta,p)\right], \tag{5.23}$$

$$\begin{aligned}
\hat{f}(0;p) &= 0,\ \hat{f}'(0;p) = 1,\ \hat{f}'(\infty;p) = 0,\ \hat{g}(0;p) = 0,\ \hat{g}'(0;p) = \beta,\ \hat{g}'(\infty;p) = 0,\\
\hat{\theta}'(0,p) &= -\gamma_1[1 - \theta(0,p)],\ \hat{\theta}(\infty,p) = 0,\ \hat{\phi}(0,p) = 1,\ \hat{\phi}(\infty,p) = 0 \tag{5.24}
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}_f[\hat{f}(\eta,p), \hat{g}(\eta,p), \hat{\theta}(\eta,p), \hat{\phi}(\eta,p)] &= \frac{\partial^3\hat{f}(\eta,p)}{\partial\eta^3} - \left(\frac{\partial\hat{f}(\eta,p)}{\partial\eta}\right)^2 + (\hat{f}(\eta,p) + \hat{g}(\eta,p))\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}\\
&\quad -K\left(\begin{array}{l}(\hat{f}(\eta,p) + \hat{g}(\eta,p))\frac{\partial^4\hat{f}(\eta,p)}{\partial\eta^4}\\ +(\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2} - \frac{\partial^2\hat{g}(\eta,p)}{\partial\eta^2})\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}\\ -2(\frac{\partial\hat{f}(\eta,p)}{\partial\eta} + \frac{\partial\hat{g}(\eta,p)}{\partial\eta})\frac{\partial^3\hat{f}(\eta,p)}{\partial\eta^3}\end{array}\right)\\
&\quad -M^2\frac{\partial\hat{f}(\eta,p)}{\partial\eta} + \lambda\hat{\theta}(\eta,p) + \lambda N\hat{\phi}(\eta,p), \tag{5.25}
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}_g[\hat{g}(\eta,p), \hat{f}(\eta,p), \hat{\theta}(\eta,p), \hat{\phi}(\eta,p)] &= \frac{\partial^3\hat{g}(\eta,p)}{\partial\eta^3} - \left(\frac{\partial\hat{g}(\eta,p)}{\partial\eta}\right)^2 + (\hat{f}(\eta,p) + \hat{g}(\eta,p))\frac{\partial^2\hat{g}(\eta,p)}{\partial\eta^2}\\
&\quad +K\left(\begin{array}{l}(\hat{f}(\eta,p) + \hat{g}(\eta,p))\frac{\partial^4\hat{g}(\eta,p)}{\partial\eta^4}\\ +(\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2} - \frac{\partial^2\hat{g}(\eta,p)}{\partial\eta^2})\frac{\partial^2\hat{g}(\eta,p)}{\partial\eta^2}\\ -2(\frac{\partial\hat{f}(\eta,p)}{\partial\eta} + \frac{\partial\hat{g}(\eta,p)}{\partial\eta})\frac{\partial^3\hat{g}(\eta,p)}{\partial\eta^3}\end{array}\right)\\
&\quad -M^2\frac{\partial\hat{g}(\eta,p)}{\partial\eta}, \tag{5.26}
\end{aligned}$$

$$\mathcal{N}_\theta[\hat\theta(\eta,p),\hat\phi(\eta,p),\hat f(\eta,p),\hat g(\eta,p)] \;=\; \frac{\partial^2\hat\theta(\eta,p)}{\partial\eta^2} + \Pr(\hat f(\eta,p)+\hat g(\eta,p))\frac{\partial\hat\theta(\eta,p)}{\partial\eta}$$
$$+ \Pr D_f\frac{\partial^2\hat\phi(\eta,p)}{\partial\eta^2}. \tag{5.27}$$

$$\mathcal{N}_\phi[\hat\phi(\eta,p),\hat\theta(\eta,p),\hat f(\eta,p),\hat g(\eta,p)] \;=\; \frac{\partial^2\hat\phi(\eta,p)}{\partial\eta^2} + Sc(\hat f(\eta,p)+\hat g(\eta,p))\frac{\partial\hat\phi(\eta,p)}{\partial\eta}$$
$$- Sck^*\hat\phi(\eta,p) + ScSr\frac{\partial^2\hat\theta(\eta,p)}{\partial\eta^2} - Sc\tau\Big(\frac{\partial\hat\phi(\eta,p)}{\partial\eta}\frac{\partial\hat\theta(\eta,p)}{\partial\eta}$$
$$- \hat\phi(\eta,p)\frac{\partial^2\hat\theta(\eta,p)}{\partial\eta^2}\Big). \tag{5.28}$$

Here p is an embedding parameter, $\hbar_f$, $\hbar_g$, $\hbar_\theta$ and $\hbar_\phi$ are the non-zero auxiliary parameters and $\mathcal{N}_f$, $\mathcal{N}_g$, $\mathcal{N}_\theta$ and $\mathcal{N}_\phi$ indicate the nonlinear operators. When $p=0$ and $p=1$ one has

$$\hat f(\eta;0) \;=\; f_0(\eta),\ \hat\theta(\eta,0)=\theta_0(\eta),\ \hat\phi(\eta,0)=\phi_0(\eta)$$
$$\hat f(\eta;1) \;=\; f(\eta),\ \hat\theta(\eta,1)=\theta(\eta),\ \hat\phi(\eta,1)=\phi_0(\eta)\ . \tag{5.29}$$

Clearly when p is increased from 0 to 1 then $f(\eta,p)$, $g(\eta,p)$, $\theta(\eta,p)$ and $\phi(\eta,p)$ vary from $f_0(\eta)$, $g_0(\eta)$, $\theta_0(\eta)$ and $\phi_0(\eta)$ to $f(\eta)$, $g(\eta)$, $\theta(\eta)$and $\phi(\eta)$. By Taylor's expansion we have

$$f(\eta,p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m,\ f_m(\eta) = \frac{1}{m!}\frac{\partial^m f(\eta;p)}{\partial p^m}\Big|_{p=0}, \tag{5.30}$$

$$g(\eta,p) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta)p^m,\ g_m(\eta) = \frac{1}{m!}\frac{\partial^m g(\eta;p)}{\partial p^m}\Big|_{p=0}, \tag{5.31}$$

$$\theta(\eta,p) = \theta_0(\eta) \sum_{m=1}^{\infty} \theta_m(\eta)p^m,\ \theta_m(\eta) = \frac{1}{m!}\frac{\partial^m \theta(\eta;p)}{\partial p^m}\Big|_{p=0}, \tag{5.32}$$

$$\phi(\eta,p) = \phi_0(\eta) \sum_{m=1}^{\infty} \phi_m(\eta)p^m,\ \phi_m(\eta) = \frac{1}{m!}\frac{\partial^m \phi(\eta;p)}{\partial p^m}\Big|_{p=0}, \tag{5.33}$$

where the convergence of above series strongly depends upon $\hbar_f$, $\hbar_g$ $\hbar_\theta$ and $\hbar_\phi$. Considering that $\hbar_f$, $\hbar_g$, $\hbar_\theta$ and $\hbar_\phi$ are selected properly so that Eqs. $(5.30)-(5.33)$ converge at $p=1$ then we can write

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \tag{5.34}$$

$$g(\eta) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta), \tag{5.35}$$

$$\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \tag{5.36}$$

$$\phi(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta). \tag{5.37}$$

The resulting problems at mth order deformation can be constructed as follows:

$$\mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar_f \mathcal{R}_f^m(\eta), \tag{5.38}$$

$$\mathcal{L}_g[g_m(\eta) - \chi_m g_{m-1}(\eta)] = \hbar_g \mathcal{R}_g^m(\eta), \tag{5.39}$$

$$\mathcal{L}_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = \hbar_\theta \mathcal{R}_\theta^m(\eta), \tag{5.40}$$

$$\mathcal{L}_\phi[\phi_m(\eta) - \chi_m \phi_{m-1}(\eta)] = \hbar_\phi \mathcal{R}_\phi^m(\eta). \tag{5.41}$$

$$f_m(0) = f'_m(0) = f'_m(\infty) = 0, \tag{5.42}$$

$$g_m(0) = g'_m(0) = g'_m(\infty) = 0, \tag{5.43}$$

$$\theta'_m(0) - \gamma_1 \theta_m(0) = \theta_m(\infty) = 0, \phi_m(0) = \phi_m(\infty) = 0. \tag{5.44}$$

$$
\begin{aligned}
\mathcal{R}_f^m(\eta) = {} & f'''_{m-1}(\eta) - \sum_{k=0}^{m-1} f'_{m-1-k} f'_k + \sum_{k=0}^{m-1} (f_{m-1-k} f''_k + g_{m-1-k} f''_k) \\
& -K \begin{pmatrix} \sum_{k=0}^{m-1} f_{m-1-k} f''''_k + \sum_{k=0}^{m-1} g_{m-1-k} f''''_k \\ + \sum_{k=0}^{m-1} f''_{m-1-k} f''_k - \sum_{k=0}^{m-1} g''_{m-1-k} f''_k \\ -2 \sum_{k=0}^{m-1} f'_{m-1-k} f'''_k - 2 \sum_{k=0}^{m-1} g'_{m-1-k} f'''_k \end{pmatrix} \\
& -M^2 f'_{m-1} + \lambda\theta + \lambda N\phi,
\end{aligned} \tag{5.45}
$$

$$\mathcal{R}_g^m(\eta) \;=\; g_{m-1}'''(\eta) - \sum_{k=0}^{m-1} g_{m-1-k}' g_k' + \sum_{k=0}^{m-1} (f_{m-1-k} g_k'' + g_{m-1-k} g_k'')$$

$$-K \left(\begin{array}{c} \displaystyle\sum_{k=0}^{m-1} f_{m-1-k} g_k'''' + \sum_{k=0}^{m-1} g_{m-1-k} g_k'''' \\[2mm] \displaystyle + \sum_{k=0}^{m-1} f_{m-1-k}'' g_k'' - \sum_{k=0}^{m-1} g_{m-1-k}'' g_k'' \\[2mm] \displaystyle -2 \sum_{k=0}^{m-1} f_{m-1-k}' g_k''' - 2 \sum_{k=0}^{m-1} g_{m-1-k}' g_k''' \end{array} \right) - M^2 g_{m-1}', \qquad (5.46)$$

$$\mathcal{R}_\theta^m(\eta) = \theta_{m-1}'' + Pr \sum_{k=0}^{m-1} (\theta_{m-1-k}' f_k + \theta_{m-1-k}' g_k) + Pr\, D_f \phi_{m-1}'', \qquad (5.47)$$

$$\mathcal{R}_\phi^m(\eta) \;=\; \phi_{m-1}'' + Sc \sum_{k=0}^{m-1} (\phi_{m-1-k}' f_k + \phi_{m-1-k}' g_k) - Sc k^* \phi + Sc Sr \theta_{m-1}''$$

$$-Sc\tau \sum_{k=0}^{m-1} (\phi_{m-1-k}' \theta_k' - \phi_{m-1-k} \theta_k''), \qquad (5.48)$$

$$\chi_m = \left[\begin{array}{l} 0, \ \ m \leq 1, \\[2mm] 1, \ \ m > 1. \end{array} \right. \qquad (5.49)$$

Solving the above mth order deformation problems we have

$$f_m(\eta) = f_m^*(\eta) + C_1 + C_2 e^\eta + C_3 e^{-\eta}, \qquad (5.50)$$

$$g_m(\eta) = g_m^*(\eta) + C_4 + C_5 e^\eta + C_6 e^{-\eta}, \qquad (5.51)$$

$$\theta_m(\eta) = \theta_m^*(\eta) + C_7 e^\eta + C_8 e^{-\eta}, \qquad (5.52)$$

$$\phi_m(\eta) = \phi_m^*(\eta) + C_9 e^\eta + C_{10} e^{-\eta}, \qquad (5.53)$$

in which the f_m^*, g_m^*, θ_m^* and ϕ_m^* indicate the special solutions.

5.3   Analysis

Obviously the series solutions $(5.34) - (5.37)$ contain the auxiliary parameters $\hbar_f$, $\hbar_g$ $\hbar_\theta$ and $\hbar_\phi$. These parameters are very important in adjusting and controlling the convergence of homotopic solutions. Hence the $\hbar-$curves are plotted at 10th order of approximations in order to find the

suitable ranges of $\hbar_f$, $\hbar_g$ $\hbar_\theta$ and $\hbar_\phi$. Fig. 5.1 indicate that the admissible values of $\hbar_f$, $\hbar_g$, $\hbar_\theta$ and $\hbar_\phi$ here are $-1.5 \leq \hbar_f \leq -0.40$, $-1.4 \leq \hbar_g \leq -0.30$, $-1.40 \leq \hbar_\theta \leq -0.25$ and $-1.40 \leq \hbar_\phi \leq -0.20$. Table 5.1 presents the convergence of homotopic solutions. It is noted that computations are sufficient for 45th order iterations of velocity and 35th order iterations of the temperature and concentration profiles for the convergent series solutions.
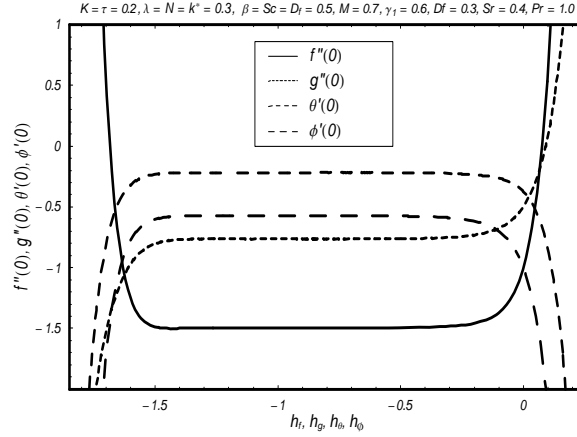
Fig. 5.1: $\hbar$−curves for the functions f, g, θ and ϕ.

Table 5.1: Convergence of series solutions for different order of approximations when $K = \tau = 0.2$, $\beta = D_f = Sc = 0.5$, $Pr = 1.0$, $Sr = 0.4$, $M = 0.7$, $\lambda = N = k^* = 0.3$, $\gamma_1 = 0.6$ and $\hbar_f = \hbar_g = \hbar_\theta = \hbar_\phi = -0.7$.

order of approximations	$-f''(0)$	$-g''(0)$	$-\theta'(0)$	$-\phi'(0)$
1	1.31063	0.632417	0.284766	0.763750
5	1.47588	0.746614	0.230340	0.599613
10	1.49331	0.762859	0.221636	0.578386
15	1.49592	0.764867	0.220665	0.575416
20	1.49650	0.765053	0.220797	0.574753
25	1.49664	0.765041	0.220916	0.574545
30	1.49667	0.765030	0.220950	0.574483
35	1.49667	0.765027	0.220953	0.574471
40	1.49667	0.765027	0.220953	0.574471

Figs. $5.2 - 5.5$ depict the behaviors of mixed convection parameter λ and concentration

buoyancy parameter N on $f'(\eta)$ and $g'(\eta)$. Figs. 5.2 and 5.3 are drawn to analyze the effects of mixed convection parameter λ on the velocity components $f'(\eta)$ and $g'(\eta)$. It is shown that $f'(\eta)$ and $g'(\eta)$ increase with an increase in λ. Effect of concentration buoyancy parameter N on the velocity components $f'(\eta)$ and $g'(\eta)$ are shown in the Figs. 5.4 and 5.5. It is examined that the concentration buoyancy parameter N shows the similar effects on momentum boundary layer thicknesses and velocity components $f'(\eta)$ and $g'(\eta)$ as we observed for mixed convection parameter λ.

Figs. $5.6 - 5.9$ examine the variation of Dufour number D_f and Soret number Sr on the temperature $\theta(\eta)$ and concentration $\phi(\eta)$. Variations of D_f on temperature $\theta(\eta)$ and concentration $\phi(\eta)$ are analyzed in the Figs. 5.6 and 5.7. It is noted from these Figs. that D_f has reverse effects on temperature $\theta(\eta)$ and concentration $\phi(\eta)$. Figs. 5.8 and 5.9 are displayed to see the variation of Sr on the temperature $\theta(\eta)$ and concentration profiles $\phi(\eta)$. We noticed that the temperature $\theta(\eta)$ and thermal boundary layer are reduced for an increase in Sr. The concentration profile $\phi(\eta)$ increases when Sr is increased. To analyze the effect of thermophoretic parameter τ on the concentration $\phi(\eta)$ profile we have sketched Fig. 5.10. It is found that an increase in thermophoretic parameter τ leads to a decrease in both concentration profile $\phi(\eta)$ and concentration boundary layer thickness.
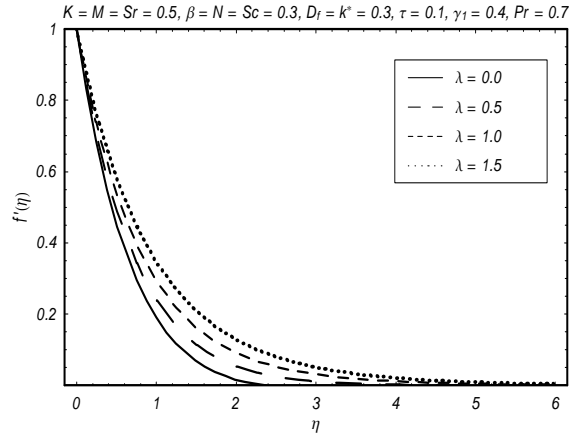
Fig. 5.2: Influence of λ on $f'(\eta)$.
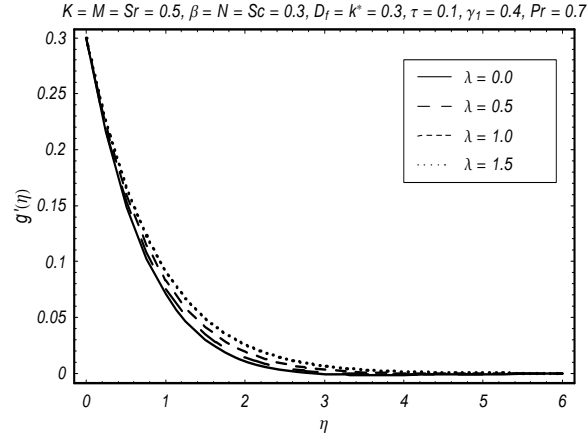
Fig. 5.3: Influence of λ on $g^{'}(\eta)$.
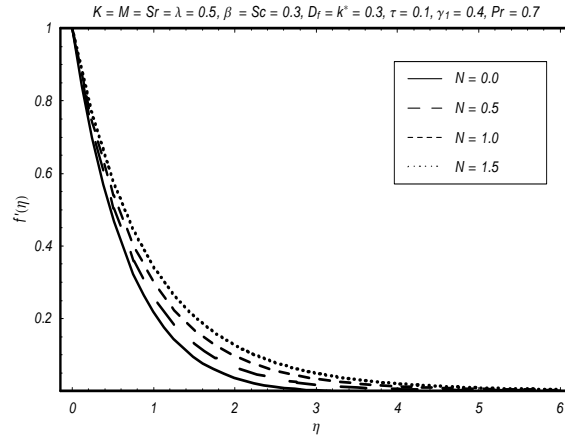
Fig. 5.4: Influence of N on $f^{'}(\eta)$.
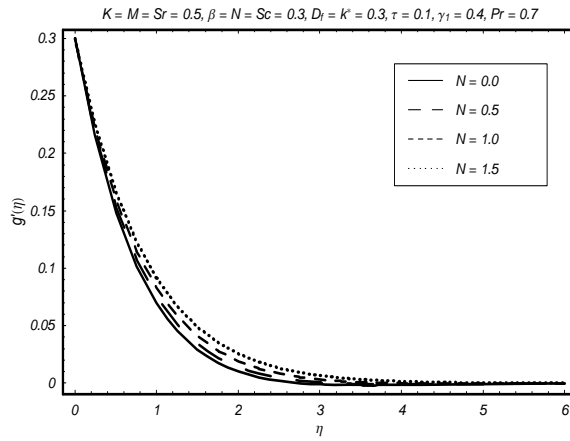
Fig. 5.5: Influence of N on $g^{'}(\eta)$.
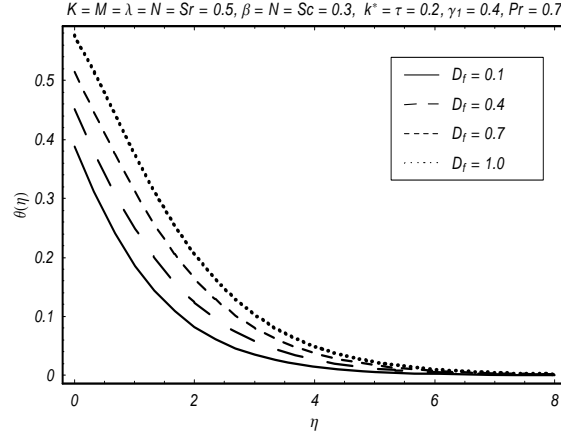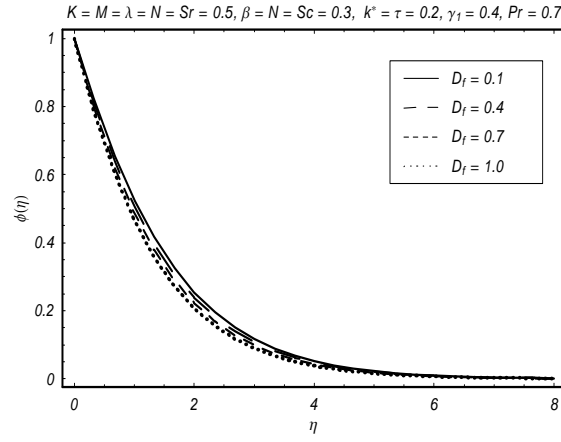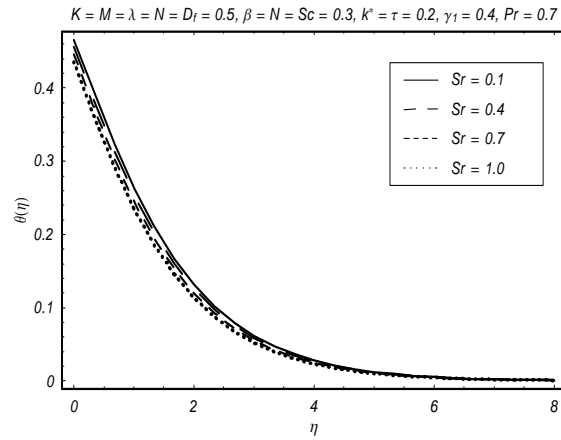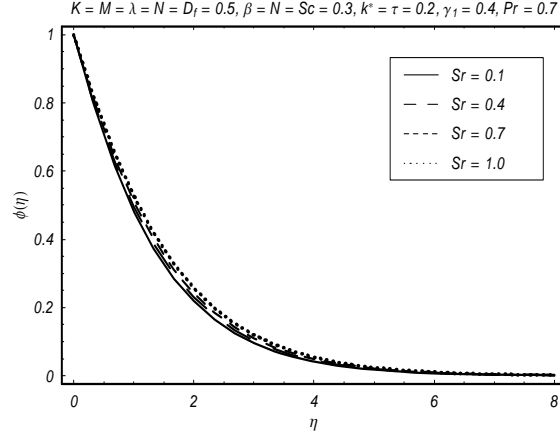
$K = M = \lambda = N = Sr = 0.5,\ \beta = N = Sc = 0.3,\ k^* = \tau = 0.2,\ \gamma_1 = 0.4,\ Pr = 0.7$

Fig. 5.6: Influence of D_f on $\theta(\eta)$.

$K = M = \lambda = N = Sr = 0.5,\ \beta = N = Sc = 0.3,\ k^* = \tau = 0.2,\ \gamma_1 = 0.4,\ Pr = 0.7$

Fig. 5.7: Influence of D_f on $\phi(\eta)$.

$K = M = \lambda = N = D_f = 0.5,\ \beta = N = Sc = 0.3,\ k^* = \tau = 0.2,\ \gamma_1 = 0.4,\ Pr = 0.7$

Fig. 5.8: Influence of Sr on $\theta(\eta)$.

87

K = M = λ = N = D_f = 0.5, β = N = Sc = 0.3, k* = τ = 0.2, γ_1 = 0.4, Pr = 0.7

Fig. 5.9: Influence of Sr on $\phi(\eta)$.

K = M = λ = N = D_f = 0.5, β = N = Sc = 0.3, k* = Sr = 0.2, γ_1 = 0.4, Pr = 0.7
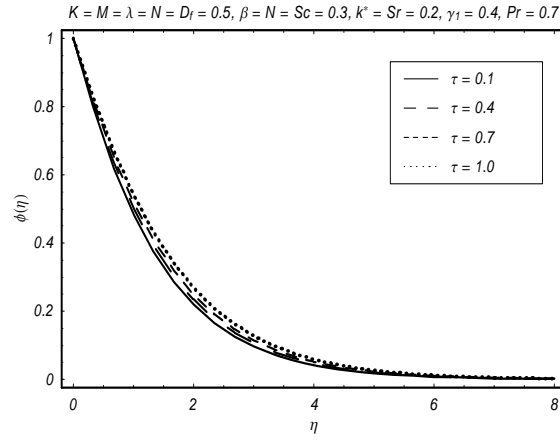
Fig. 5.10: Influence of τ on $\phi(\eta)$.

Table 5.2 is prepared to analyze numerical values of local Nusselt and Sherwood numbers. The values of $-\theta'(0)$ and $-\phi'(0)$ decrease by increasing Deborah number. Here $-\theta'(0)$ increases by increasing Prandtl and Biot numbers while reverse is the case of $-\phi'(0)$. Table 5.3 shows that local Nusselt $-\theta'(0)$ and Sherwood numbers $-\phi'(0)$ decrease with the increase in Hartman and Soret numbers. Values of local Nusselt $-\theta'(0)$ and Sherwood numbers $-\phi'(0)$ are opposite for Schmidt, Dufour, thermophoretic and chemical reaction parameters.

Table 5.2: Values of local Nusselt $-\theta'(0)$ and Sherwood numbers $-\phi'(0)$ for different values of the parameters K, β, λ, N, Pr, and γ_1 when $Sc = D_f = 0.5$, $Sr = 0.4$, $\tau = 0.2$, $k^* = 0.3$

and $M = 0.5$.

K	β	λ	N	Pr	γ_1	$-\theta'(0)$	$-\phi'(0)$
0.0	0.5	0.5	0.3	1.0	0.6	0.24700	0.61363
0.2						0.23486	0.59297
0.4						0.18422	0.51482
0.2	0.0					0.21393	0.55535
	0.2					0.22444	0.57294
	0.4					0.23200	0.58714
0.2	0.2	0.0				0.19606	0.54490
		0.3				0.21647	0.56327
		0.5				0.22444	0.57295
0.2	0.2	0.5	0.0			0.21880	0.56592
			0.3			0.22446	0.57285
			0.5			0.22772	0.57750
0.2	0.2	0.5	0.3	1.0		0.22443	0.57291
				1.5		0.24490	0.56585
				2.0		0.25725	0.56080
0.2	0.2	0.5	0.3	1.0	0.2	0.10779	0.58240
					0.5	0.20254	0.57481
					0.7	0.24341	0.57118

Table 5.3: Values of local Nusselt $-\theta'(0)$ and Sherwood numbers $-\phi'(0)$ for different values of the parameters Sc, Sr, τ, D_f, k^* and M when $\alpha = \beta = 0.2$, $\lambda = 0.5$, $N_1 = 0.3$, $Pr = 1$ and $\gamma = 0.6$.

Sc	Sr	Df	τ	k^*	M	$-\theta'(0)$	$-\phi'(0)$
0.2	0.4	0.5	0.2	0.3	0.5	0.26950	0.32824
0.5						0.22446	0.57285
0.7						0.19729	0.71482
0.5	0.2					0.22285	0.58265
	0.5					0.22530	0.56786
	0.7					0.22690	0.55779
0.5	0.4	0.2				0.27834	0.52746
		0.5				0.22695	0.55765
		0.7				0.18905	0.57984
0.5	0.4	0.5	0.4			0.22160	0.58610
			0.7			0.21705	0.60820
			1.0			0.21164	0.63434
0.5	0.4	0.5	0.2	0.5		0.20529	0.67118
				0.7		0.18854	0.75605
				1.0		0.16638	0.86737
0.5	0.4	0.5	0.2	0.3	0.6	0.22122	0.56859
					0.8	0.21345	0.55880
					1.0	0.20402	0.54895

5.4   Conclusions

MHD three-dimensional flow of viscoelastic fluid over a stretching surface is analyzed in the presence of thermophoresis and convective condition. Effects of chemical reaction and Soret and Dufour are analyzed. The main observations are listed below.

- Effects of mixed convection parameter λ and buoyancy concentration parameter N on the velocity profiles and momentum boundary layer thickness are similar.

- Effects of Sr and D_f on $\theta(\eta)$ and $\phi(\eta)$ are opposite.

- Thermal boundary layer thickness and temperature field increase when D_f increases.

- Concentration $\phi(\eta)$ and associated boundary layer thickness are decreasing functions of thermophoretic parameter τ.

- There are opposite effects of local Nusselt number and local Sherwood number when γ_1, Pr, Sc, Df, k^*, τ and Sr increase.

- Qualitative effects of local Nusselt number and local Sherwood number are similar when λ, N, M and K increase.

Chapter 6

Three-dimensional flow of Maxwell fluid over a stretching surface with heat source and convective conditions

Heat and mass transfer effects in three-dimensional flow of Maxwell fluid over a stretching surface are addressed in this chapter. Analysis has been performed in presence of internal heat generation/absorption. Concentration and thermal buoyancy effects are accounted. Convective boundary conditions for heat and mass transfer analysis are explored. Series solutions of the resulting problems are developed. Effects of mixed convection, internal heat generation/ absorption parameter and Biot numbers on the dimensionless velocity, temperature and concentration distributions are illustrated graphically. Numerical values of local Nusselt and Sherwood numbers are obtained and analyzed for all the physical parameters.

6.1 Governing problems

Here we consider the steady three-dimensional flow of an incompressible Maxwell fluid over a stretching surface at $z = 0$. The flow takes place in the domain $z > 0$. Heat and mass transfer

characteristics are taken into account in the presence of internal heat generation/absorption and mixed convection. Convective heat and mass boundary conditions are considered. The ambient fluid temperature and concentration are taken as T_∞ and C_∞ while the surface temperature and concentration are maintained by convective heat and mass transfer at certain value T_f and C_f. The governing partial differential equations subject to boundary layer flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{6.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2} - \lambda_1 \left(\begin{array}{c} u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + w^2\frac{\partial^2 u}{\partial z^2} + 2uv\frac{\partial^2 u}{\partial x\partial y} \\ +2vw\frac{\partial^2 u}{\partial y\partial z} + 2uw\frac{\partial^2 u}{\partial x\partial z} \end{array} \right)$$
$$+ g\left(\beta_T(T - T_\infty) + \beta_C(C - C_\infty)\right), \tag{6.2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \nu\frac{\partial^2 v}{\partial z^2} - \lambda_1 \left(\begin{array}{c} u^2\frac{\partial^2 v}{\partial x^2} + v^2\frac{\partial^2 v}{\partial y^2} + w^2\frac{\partial^2 v}{\partial z^2} + 2uv\frac{\partial^2 v}{\partial x\partial y} + \\ 2vw\frac{\partial^2 v}{\partial y\partial z} + 2uw\frac{\partial^2 v}{\partial x\partial z} \end{array} \right), \tag{6.3}$$

$$\rho c_p(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}) = k\frac{\partial^2 T}{\partial z^2} + Q(T - T_\infty), \tag{6.4}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = D\frac{\partial^2 C}{\partial z^2}. \tag{6.5}$$

In Eqs. $(6.1) - (6.5)$ the respective velocity components in the $x-$, $y-$ and $z-$directions are denoted by u, v and w, λ_1 shows the relaxation time, σ^* is the electrical conductivity, ρ is the density of fluid, g is the gravitational acceleration, β_T and β_C are the thermal and concentration expansion coefficients respectively, T is the fluid temperature, $\nu = (\mu/\rho)$ is the kinematic viscosity, μ is the dynamic viscosity of fluid, c_p is the specific heat, k is the thermal conductivity, Q is the uniform volumetric heat generation/absorption, C is the concentration field and D is the mass diffusivity.

The subjected boundary conditions are given by

$$u = u_e = ax, \ v = by, \ w = 0, \ -k\frac{\partial T}{\partial z} = h(T_f - T), \ -D\frac{\partial C}{\partial z} = h^*(C_f - C) \text{ at } z = 0, \tag{6.6}$$

$$u \to 0, \ v \to 0, \ T \to T_\infty, \ C \to C_\infty \text{ as } z \to \infty, \tag{6.7}$$

93

where h is the heat transfer coefficient, h^* is the concentration transfer coefficient and a and b are constants and have dimension $(time)^{-1}$.

We now define

$$
\begin{aligned}
u &= axf'(\eta), \; v = ayg'(\eta), \; w = -\sqrt{a\nu}(f(\eta) + g(\eta)), \\
\theta(\eta) &= \frac{T - T_\infty}{T_f - T_\infty}, \; \eta = z\sqrt{\frac{a}{\nu}}, \; \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}.
\end{aligned} \tag{6.8}
$$

The above variables satisfy Eq. (6.1) automatically while Eqs. (6.2)-(6.7) are converted to the following forms:

$$
f''' + (f + g)f'' - f'^2 + \beta_1[2(f + g)f'f'' - (f + g)^2 f'''] + \lambda(\theta + N\phi) = 0, \tag{6.9}
$$

$$
g''' + (f + g)g'' - g'^2 + \beta_1[2(f + g)g'g'' - (f + g)^2 g'''] = 0, \tag{6.10}
$$

$$
\theta'' + Pr(f + g)\theta' + \beta^*\theta = 0, \tag{6.11}
$$

$$
\phi'' + Sc(f + g)\phi' = 0, \tag{6.12}
$$

$$
f = 0, \; g = 0, \; f' = 1, \; g' = \beta, \; \theta' = -\gamma_1(1 - \theta(0)), \; \phi' = -\gamma_2(1 - \phi(0)), \; \text{at } \eta = 0, \tag{6.13}
$$

$$
f' \to 0, \; g' \to 0, \; \theta \to 0, \; \phi \to 0 \; \text{ as } \eta \to \infty, \tag{6.14}
$$

where β_1 is the dimensionless Deborah number, λ is the local buoyancy parameter, Gr_x is the local Grashof number, N is the concentration buoyancy parameter, Pr is the Prandtl number, β^* is the heat generation/absorption parameter, Sc is the Schmidt number, β is ratio of rates parameters, γ_1 and γ_2 are the Biot numbers and prime shows the differentiation with respect to η. These are given by

$$
\begin{aligned}
\beta_1 &= \lambda_1 a, \; \lambda = \frac{Gr_x}{\text{Re}_x^2}, \; Gr_x = \frac{g\beta_T(T_f - T_\infty)x^3}{\nu^2}, \; N = \frac{\beta_C(C_w - C_\infty)}{\beta_T(T_f - T_\infty)} \tag{6.15} \\
Pr &= \frac{\nu}{\sigma}, \; \beta^* = \frac{Q}{\rho c_p}, Sc = \frac{\nu}{D}, \beta = \frac{b}{a}, \gamma_1 = \frac{h}{k}\sqrt{\frac{\nu}{a}}, \; \gamma_2 = \frac{h^*}{D}\sqrt{\frac{\nu}{a}}.
\end{aligned}
$$

In dimensionless form the local Nusselt and local Sherwood numbers are given by

$$Nu/Re_x^{1/2} = -\theta'(0), \tag{6.16}$$

$$Sh/Re_x^{1/2} = -\phi'(0), \tag{6.17}$$

where $Re_x = u_e x/\nu$ is the local Reynolds number.

6.2 Series solutions

The initial approximations and auxiliary linear operators are required to develop homotopic solutions. We select the following initial guesses and linear operators for the present flow analysis:

$$f_0(\eta) = \left(1 - e^{-\eta}\right), \ g_0(\eta) = \beta\left(1 - e^{-\eta}\right), \ \theta_0(\eta) = \frac{\gamma_1 \exp(-\eta)}{1 + \gamma_1}, \ \phi_0(\eta) = \frac{\gamma_2 \exp(-\eta)}{1 + \gamma_2}, \tag{6.18}$$

$$\mathcal{L}_f = f''' - f', \ \mathcal{L}_g = g''' - g', \ \mathcal{L}_\theta = \theta'' - \theta, \ \mathcal{L}_\phi = \phi'' - \phi, \tag{6.19}$$

with the following properties of the defined operators in Eq. (6.19) i.e.

$$\mathcal{L}_f(C_1 + C_2 e^\eta + C_3 e^{-\eta}) = 0, \ \mathcal{L}_g(C_4 + C_5 e^\eta + C_6 e^{-\eta}) = 0,$$

$$\mathcal{L}_\theta(C_7 e^\eta + C_8 e^{-\eta}) = 0, \ \mathcal{L}_\phi(C_9 e^\eta + C_{10} e^{-\eta}) = 0, \tag{6.20}$$

where C_i $(i = 1 - 10)$ are the arbitrary constants.

The corresponding problems at the zeroth order deformations are given in the following forms:

$$(1 - p)\,\mathcal{L}_f\left[\hat{f}(\eta; p) - f_0(\eta)\right] = p\hbar_f \mathcal{N}_f\left[\hat{f}(\eta; p), \hat{g}(\eta; p), \hat{\theta}(\eta, p), \hat{\phi}(\eta, p)\right], \tag{6.21}$$

$$(1 - p)\,\mathcal{L}_g\left[\hat{g}(\eta; p) - g_0(\eta)\right] = p\hbar_g \mathcal{N}_g\left[\hat{f}(\eta; p), \hat{g}(\eta; p), \hat{\theta}(\eta, p), \hat{\phi}(\eta, p)\right], \tag{6.22}$$

$$(1 - p)\,\mathcal{L}_\theta\left[\hat{\theta}(\eta; p) - \theta_0(\eta)\right] = p\hbar_\theta \mathcal{N}_\theta\left[\hat{f}(\eta; p), \hat{g}(\eta; p), \hat{\theta}(\eta, p), \hat{\phi}(\eta, p)\right], \tag{6.23}$$

$$(1 - p)\,\mathcal{L}_\phi\left[\hat{\phi}(\eta; p) - \phi_0(\eta)\right] = p\hbar_\phi \mathcal{N}_\phi\left[\hat{f}(\eta; p), \hat{g}(\eta; p), \hat{\theta}(\eta, p), \hat{\phi}(\eta, p)\right], \tag{6.24}$$

95

$$\hat{f}(0;p) = 0, \ \hat{f}'(0;p) = 1, \ \hat{f}'(\infty;p) = 0,$$

$$\hat{g}(0;p) = 0, \ \hat{g}'(0;p) = \beta, \ \hat{g}'(\infty;p) = 0, \hat{\theta}'(0,p) = -\gamma_1[1 - \theta(0,p)],$$

$$\hat{\theta}(\infty,p) = 0, \ \hat{\phi}'(0,p) = -\gamma_2[1 - \hat{\phi}(0,p)], \ \hat{\phi}(\infty,p) = 0 \tag{6.25}$$

$$
\mathcal{N}_f[\hat{f}(\eta,p),\hat{g}(\eta,p),\hat{\theta}(\eta,p),\hat{\phi}(\eta,p)] = \frac{\partial^3 \hat{f}(\eta,p)}{\partial \eta^3} - \left(\frac{\partial \hat{f}(\eta,p)}{\partial \eta}\right)^2 + (\hat{f}(\eta,p) + \hat{g}(\eta,p))\frac{\partial^2 \hat{f}(\eta,p)}{\partial \eta^2}
$$

$$
+\beta_1 \left[\begin{array}{c} 2(\hat{f}(\eta,p) + \hat{g}(\eta,p)) \\ \frac{\partial \hat{f}(\eta,p)}{\partial \eta}\frac{\partial^2 \hat{f}(\eta,p)}{\partial \eta^2} \\ -(\hat{f}(\eta,p) + \hat{g}(\eta,p))^2\frac{\partial^3 \hat{f}(\eta,p)}{\partial \eta^2} \end{array} \right]
$$

$$
+\lambda[\hat{\theta}(\eta,p) + N_1\hat{\phi}(\eta,p)], \tag{6.26}
$$

$$
\mathcal{N}_g[\hat{g}(\eta,p),\hat{f}(\eta,p),\hat{\theta}(\eta,p),\hat{\phi}(\eta,p)] = \frac{\partial^3 \hat{g}(\eta,p)}{\partial \eta^3} - \left(\frac{\partial \hat{g}(\eta,p)}{\partial \eta}\right)^2 + (\hat{f}(\eta,p) + \hat{g}(\eta,p))\frac{\partial^2 \hat{g}(\eta,p)}{\partial \eta^2}
$$

$$
+\beta_1 \left[\begin{array}{c} 2(\hat{f}(\eta,p) + \hat{g}(\eta,p)) \\ \frac{\partial \hat{g}(\eta,p)}{\partial \eta}\frac{\partial^2 \hat{g}(\eta,p)}{\partial \eta^2} \\ -(\hat{f}(\eta,p) + \hat{g}(\eta,p))^2\frac{\partial^3 \hat{g}(\eta,p)}{\partial \eta^2} \end{array} \right], \tag{6.27}
$$

$$
\mathcal{N}_\theta[\hat{\theta}(\eta,p),\hat{\phi}(\eta,p),\hat{f}(\eta,p),\hat{g}(\eta,p)] = \frac{\partial^2 \hat{\theta}(\eta,p)}{\partial \eta^2} + \Pr(\hat{f}(\eta,p)+\hat{g}(\eta,p))\frac{\partial \hat{\theta}(\eta,p)}{\partial \eta} + \beta^*\hat{\theta}(\eta,p), \tag{6.28}
$$

$$
\mathcal{N}_\phi[\hat{\phi}(\eta,p),\hat{\theta}(\eta,p),\hat{f}(\eta,p),\hat{g}(\eta,p)] = \frac{\partial^2 \hat{\phi}(\eta,p)}{\partial \eta^2} + Sc(\hat{f}(\eta,p)+\hat{g}(\eta,p))\frac{\partial \hat{\phi}(\eta,p)}{\partial \eta}. \tag{6.29}
$$

Here p is an embedding parameter, the non-zero auxiliary parameters are $\hbar_f$, $\hbar_g$, $\hbar_\theta$ and $\hbar_\phi$ and the nonlinear operators are $\mathcal{N}_f$, $\mathcal{N}_g$, $\mathcal{N}_\theta$ and $\mathcal{N}_\phi$. When $p = 0$ and $p = 1$ one has

$$\hat{f}(\eta;0) = f_0(\eta), \ \hat{g}(\eta,0) = g_0(\eta), \ \hat{\theta}(\eta,0) = \theta_0(\eta), \ \hat{\phi}(\eta,0) = \phi_0(\eta),$$

$$\hat{f}(\eta;1) = f(\eta), \ \hat{g}(\eta,1) = g(\eta), \ \hat{\theta}(\eta,1) = \theta(\eta), \ \hat{\phi}(\eta,1) = \phi_0(\eta). \tag{6.30}$$

Clearly when p is increased from 0 to 1 then $f(\eta,p)$, $g(\eta,p)$, $\theta(\eta,p)$ and $\phi(\eta,p)$ vary from $f_0(\eta)$,

$g_0(\eta)$, $\theta_0(\eta)$ and $\phi_0(\eta)$ to $f(\eta)$, $g(\eta)$, $\theta(\eta)$ and $\phi(\eta)$. By Taylor's expansion we have

$$f(\eta, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) p^m, \;\; f_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m f(\eta; p)}{\partial p^m} \right|_{p=0}, \tag{6.31}$$

$$g(\eta, p) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta) p^m, \;\; g_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m g(\eta; p)}{\partial p^m} \right|_{p=0}, \tag{6.32}$$

$$\theta(\eta, p) = \theta_0(\eta) \sum_{m=1}^{\infty} \theta_m(\eta) p^m, \;\; \theta_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \theta(\eta; p)}{\partial p^m} \right|_{p=0}, \tag{6.33}$$

$$\phi(\eta, p) = \phi_0(\eta) \sum_{m=1}^{\infty} \phi_m(\eta) p^m, \;\; \phi_m(\eta) = \frac{1}{m!} \left. \frac{\partial^m \phi(\eta; p)}{\partial p^m} \right|_{p=0}, \tag{6.34}$$

where the convergence of above series strongly depends upon $\hbar_f$, $\hbar_g$ $\hbar_\theta$ and $\hbar_\phi$. Considering that $\hbar_f$, $\hbar_g$, $\hbar_\theta$ and $\hbar_\phi$ are selected properly so that Eqs. $(6.31) - (6.34)$ converge at $p = 1$ then we have

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \tag{6.35}$$

$$g(\eta) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta), \tag{6.36}$$

$$\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \tag{6.37}$$

$$\phi(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta). \tag{6.38}$$

The general solutions can be expressed below:

$$f_m(\eta) = f_m^*(\eta) + C_1 + C_2 e^{\eta} + C_3 e^{-\eta}, \tag{6.39}$$

$$g_m(\eta) = g_m^*(\eta) + C_4 + C_5 e^{\eta} + C_6 e^{-\eta}, \tag{6.40}$$

$$\theta_m(\eta) = \theta_m^*(\eta) + C_7 e^{\eta} + C_8 e^{-\eta}, \tag{6.41}$$

$$\phi_m(\eta) = \phi_m^*(\eta) + C_9 e^{\eta} + C_{10} e^{-\eta}, \tag{6.42}$$

in which the f_m^*, g_m^*, θ_m^* and ϕ_m^* indicate the special solutions.

6.3 Convergence analysis and discussion

Clearly the homotopic series solutions $(6.35) - (6.38)$ depend on the auxiliary parameters $\hbar_f$, $\hbar_g$, $\hbar_\theta$ and $\hbar_\phi$. These parameters have important role in the convergence of series solutions. For this purpose, the $\hbar-$curves are drawn at 15th order of approximations to determine the suitable ranges of these auxiliary parameters. Fig. 6.1 shows that the acceptable values of $\hbar_f$, $\hbar_g$, $\hbar_\theta$ and $\hbar_\phi$ are $-1.4 \leq \hbar_f \leq -0.20$, $-1.6 \leq \hbar_g \leq -0.40$ and $-1.50 \leq \hbar_\theta, \hbar_\phi \leq -0.30$. Table 6.1 ensures the convergence of homotopic series solutions in the whole region of η when $\hbar_f = \hbar_g = \hbar_\theta = \hbar_\phi = -0.5$.
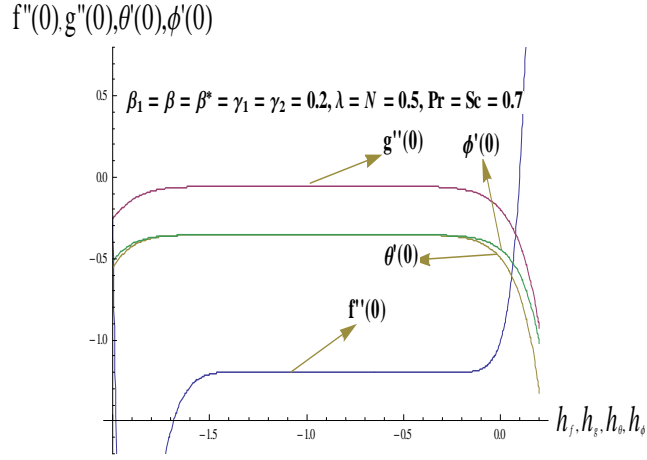
Fig. 6.1: $\hbar-$curves for the functions f, g, θ and ϕ.

Table 6.1: Convergence of series solutions for different order of approximations when $\beta = \beta_1 = \beta^* = 0.2$, $\gamma_2 = \lambda = N = 0.3$, $\gamma_1 = 0.5$, $Sc = 1.0$, $Pr = 1.2$ and $\hbar_f = \hbar_g = \hbar_\theta = \hbar_\phi = -0.5$.

order of approximations	$-f''(0)$	$-g''(0)$	$-\theta'(0)$	$-\phi'(0)$
1	1.018	0.1497	0.3133	0.2228
5	1.029	0.06217	0.2782	0.2095
10	1.028	0.03940	0.2703	0.2067
15	1.029	0.03866	0.2702	0.2066
20	1.029	0.03982	0.2705	0.2066
25	1.029	0.03993	0.2705	0.2066
30	1.029	0.03993	0.2705	0.2066

Figs. 6.2-6.7 show the effects of Deborah number β_1, mixed convection parameter λ, con-

centration buoyancy parameter N and internal heat source/sink parameter β^* on the velocity profiles $f'(\eta)$ and $g'(\eta)$. Figs. 6.2 and 6.3 are drawn to see the behavior of Deborah number β_1 on the velocity profiles $f\prime(\eta)$ and $g\prime(\eta)$. It is found that both the velocity profiles $f'(\eta)$ and $g'(\eta)$ decrease with an enhancement in β_1. It is also examined from these Figs. that associated boundary layer thicknesses are decreasing functions of β_1. This is due to the fact that β_1 depends on relaxation time. Larger relaxation time offers more resistance to the flow due to which the velocities are decreased. Figs. 6.4 and 6.5 are displayed to see the impact of mixed convection parameter λ on the velocity profiles $f'(\eta)$ and $g'(\eta)$. It is seen that both the velocity profiles $f'(\eta)$ and $g'(\eta)$ increase with an enhancement in λ. Also momentum boundary layer thicknesses are increased with an increase in λ. In fact an increase in λ enhances the buoyancy forces which are more dominant to viscous forces. Variations of concentration buoyancy parameter N on the velocity profiles $f'(\eta)$ and $g'(\eta)$ are displayed in the Figs. 6.6 and 6.7. Similar behavior of N is noted on the velocity profiles $f'(\eta)$ and $g'(\eta)$.

Figs. 6.8 and 6.9 are plotted to see the variations of internal heat source/sink parameter β^* and heat transfer Biot number γ_1 on the temperature $\theta(\eta)$. Fig. 6.8 depicts that the thermal boundary layer thickness and temperature $\theta(\eta)$ are increasing functions of internal heat source parameter $\beta^* > 0$ and decreasing functions of internal heat sink $\beta^* < 0$. With an increase in heat transfer Biot number γ_1, both the thermal boundary layer thickness and temperature $\theta(\eta)$ are enhanced (see Fig. 6.9). The reason is that as γ_1 depends on heat transfer coefficient h which leads to an increase in temperature $\theta(\eta)$.

Figs. 6.10 is displayed to analyze the behavior of concentration $\phi(\eta)$ for different values of mass transfer Biot number γ_2. It is observed that as γ_2 increases the associated boundary layer thickness and concentration profile $\phi(\eta)$ grow. As mass transfer Biot number γ_2 depends on mass transfer coefficient h^* so with an enhancement in γ_2 the mass transfer coefficient increases which leads to an increase in concentration profile $\phi(\eta)$.

Impacts of mixed convection parameter λ, concentration buoyancy parameter N, Deborah number β_1 and internal heat source/sink parameter β^* on the local Nusselt number $(-\theta'(0))$ are displayed in the Figs. 6.11 and 6.12. It is found that local Nusselt number $(-\theta'(0))$ enhances with an increase in λ and N (see Fig.6.11). Local Nusselt number $(-\theta'(0))$ reduces with internal heat source parameter $\beta^* > 0$ while it increases with internal heat sink parameter $\beta^* < 0$ (see

Fig. 6.12). It is also noticed from Fig. 6.12 that local Nusselt number $(-\theta'(0))$ decreases with an increase in β_1.

Figs. 6.13 and 6.14 are sketched to see the variations of mixed convection parameter λ, concentration buoyancy parameter N, Deborah number β_1 and internal heat source/sink parameter β^* on the Sherwood number $(-\phi'(0))$. Fig. 6.13 shows that the Sherwood number $(-\phi'(0))$ increases with an increase in λ and N. Fig. 6.14 indicates that the Sherwood number $(-\phi'(0))$ increases by increasing internal heat source $\beta^* > 0$ while reverse effect is examined with an increase in Deborah number β_1.

Fig. 6.2: Variation of β_1 on $f\prime(\eta)$.
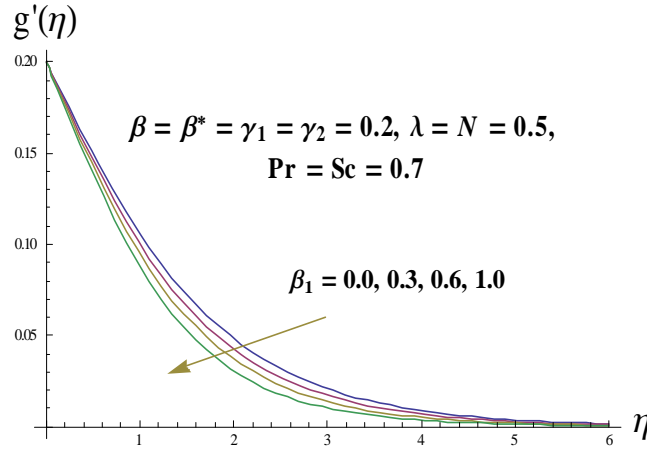
Fig. 6.3: Variation of β_1 on $g\prime(\eta)$.

100

Fig. 6.4: Variation of λ on $f\prime(\eta)$.

$\beta_1 = \beta = \beta^* = \gamma_1 = \gamma_2 = 0.2, N = 0.5,$
$Pr = Sc = 0.7$

$\lambda = 0.0, 0.5, 1.0, 1.5$

Fig. 6.5: Variation of λ on $g\prime(\eta)$.

$\beta_1 = \beta = \beta^* = \gamma_1 = \gamma_2 = 0.2, N = 0.5,$
$Pr = Sc = 0.7$

$\lambda = 0.0, 0.5, 1.0, 1.5$

Fig. 6.6: Variation of N on $f\prime(\eta)$.

$\beta_1 = \beta = \beta^* = \gamma_1 = \gamma_2 = 0.2,$
$\lambda = 0.5, Pr = Sc = 0.7$

$N = 0.0, 1.5, 3.0, 4.5$
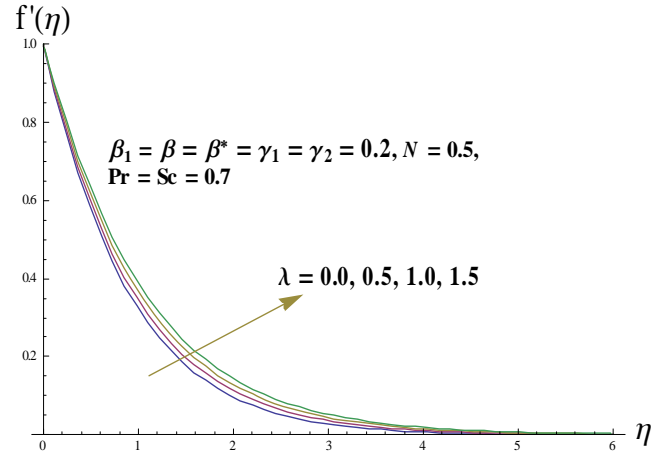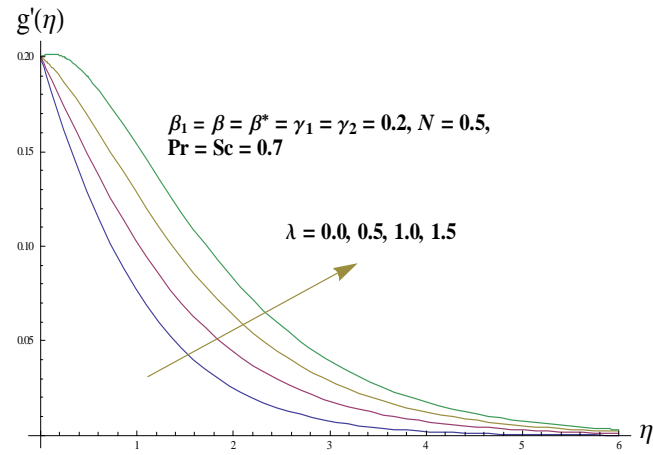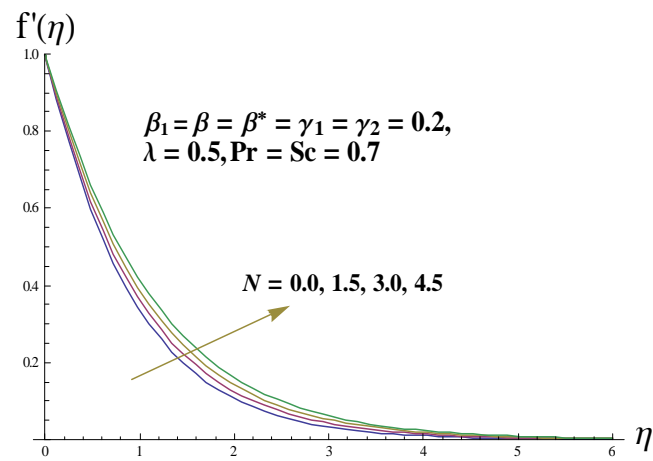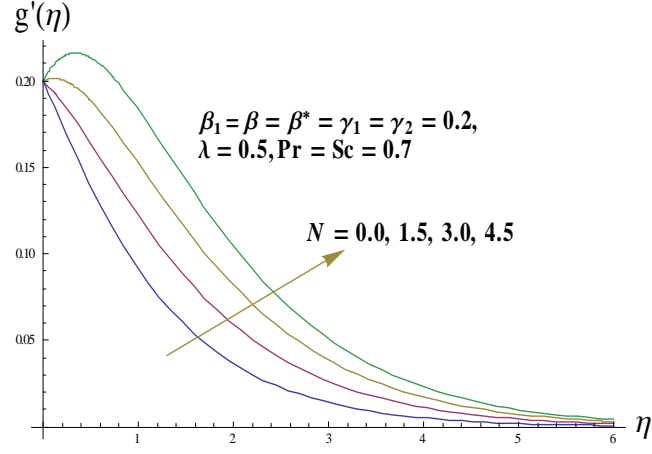
Fig. 6.7: Variation of N on $g\prime(\eta)$.
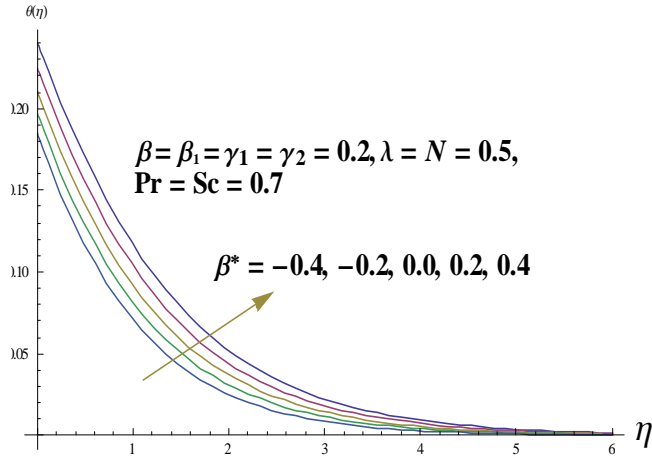
Fig. 6.8: Variation of β^* on $\theta(\eta)$.

Fig. 6.9: Variation of γ_1 on $\theta(\eta)$.

102
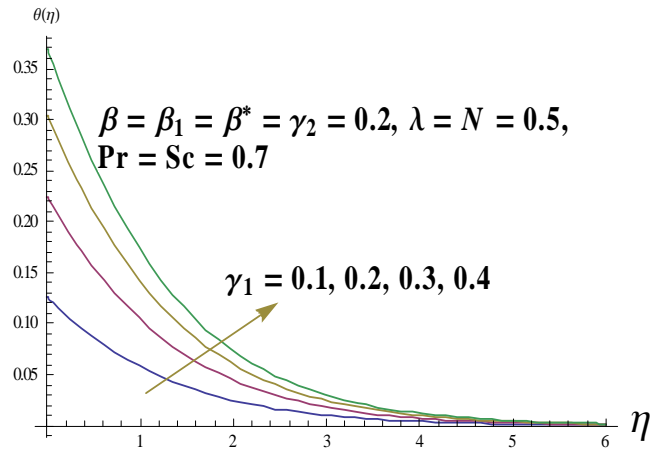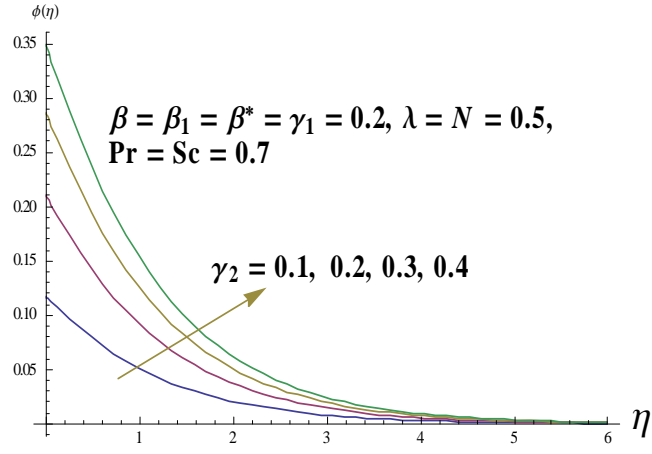
Fig. 6.10: Variation of γ_2 on $\phi(\eta)$.
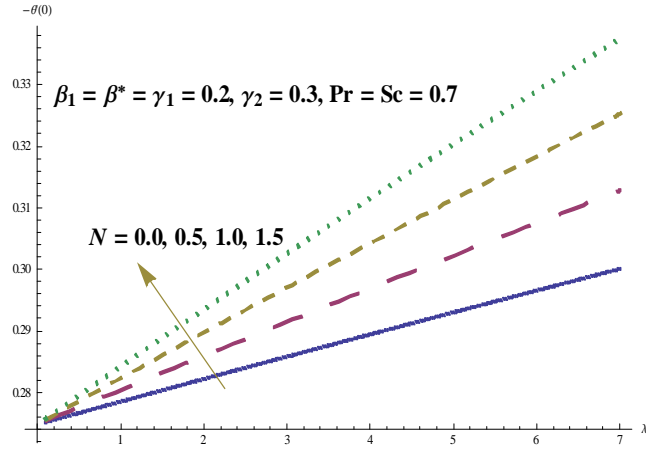
Fig. 6.11: Variations of λ and N on $-\theta'(0)$.
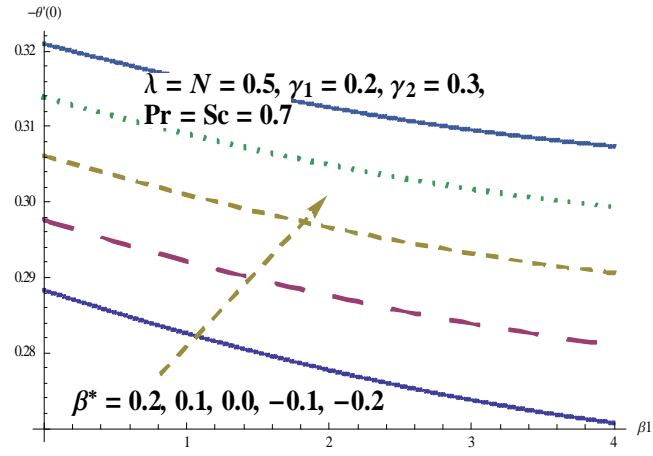
Fig. 6.12: Variations of β^* and β_1 on $-\theta'(0)$.
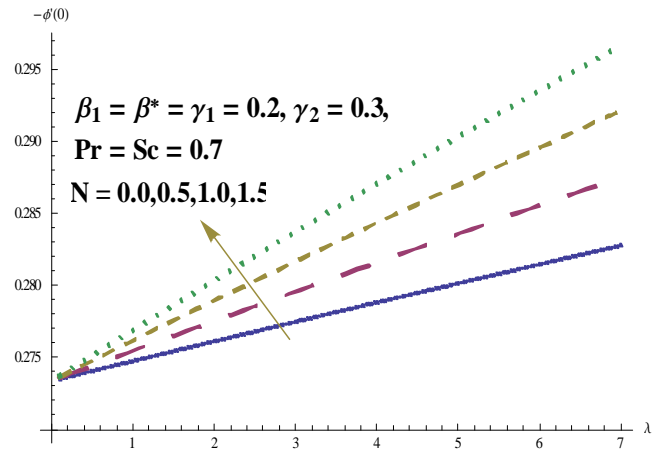
Fig. 6.13: Variations of λ and N on $-\phi'(0)$.
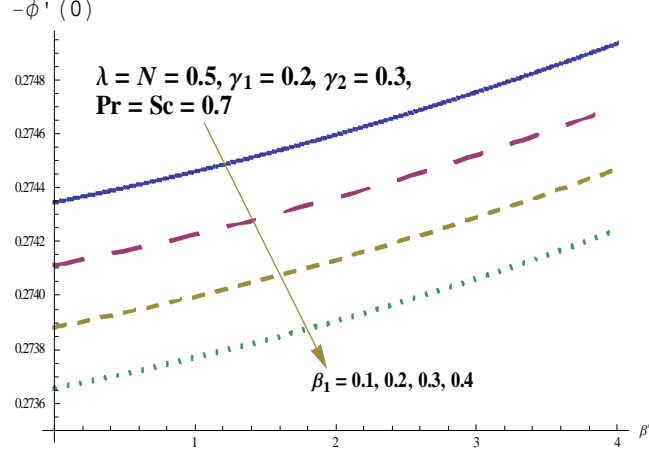
Fig. 6.14: Variations of β_1 and β^* on $-\theta'(0)$.

6.3.1 Conclusions

Three-dimensional mixed convection flow of Maxwell fluid over a stretching sheet with internal heat generation/absorption is analyzed. Convective boundary conditions for both heat and mass transfer are considered. The main observations are mentioned below.

- Variations of mixed convection parameter λ and concentration buoyancy parameter N enhance the velocity profiles and associated boundary layer thicknesses.

- Velocity profiles and temperature increase in case of internal heat source $\beta^* > 0$ while these reduce for heat sink $\beta^* < 0$.

- Heat transfer Biot number γ_1 increases the thermal boundary layer thickness and temperature. Also concentration and its associated boundary layer are enhanced with an increase in mass transfer Biot number γ_2.

- The local Nusselt and Sherwood numbers have quite similar behaviors for increasing values of mixed convection parameter λ, concentration buoyancy parameter N and Deborah number β_1.

- Larger values of heat sink parameter $\beta^* < 0$ give rise to the local Nusselt number $(-\theta'(0))$. However Sherwood number $(-\phi'(0))$ enhances with an increase in heat source $\beta^* > 0$.

Chapter 7

Soret and Dufour effects in three-dimensional flow of Maxwell fluid with chemical reaction and convective condition

This chapter addresses the heat and mass transfer effects in three-dimensional flow of Maxwell fluid over a stretching surface with convective boundary conditions. Mass transfer is considered in the presence of first order chemical reaction. Soret and Dufour effects in the conservation law of energy and concentration are considered. Convergent series solutions to the resulting nonlinear problems are developed. Plots of physical quantities of interest are analyzed.

7.1 Problems formulation

We consider the steady three-dimensional flow of an incompressible Maxwell fluid induced by a stretching surface at $z = 0$. The flow takes place in the domain $z > 0$. The ambient fluid temperature is taken as T_∞ while the surface temperature is maintained by convective heat transfer. Soret and Dufour effects in presence of mixed convection flow are taken into account. In addition the effect of first order chemical reaction in mass transfer is taken under

consideration. Flow diagram is as follows:
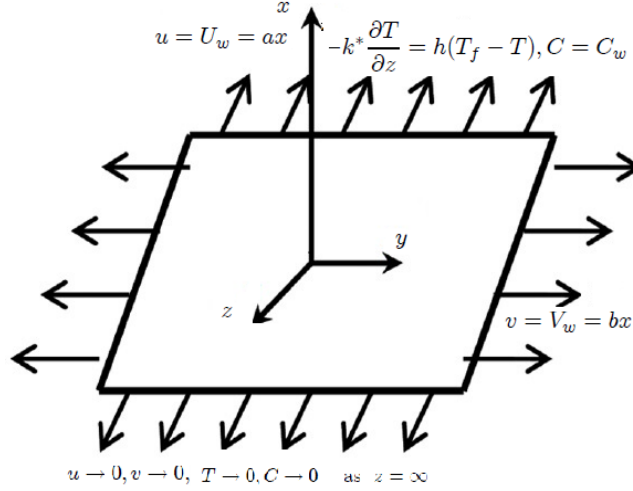
Fig. 7.1: Geometry of the problem

The governing boundary layer equations for three-dimensional flow of Maxwell fluid can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{7.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2} - \lambda_1 \left(\begin{array}{c} u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + w^2\frac{\partial^2 u}{\partial z^2} + 2uv\frac{\partial^2 u}{\partial x\partial y} \\ +2vw\frac{\partial^2 u}{\partial y\partial z} + 2uw\frac{\partial^2 u}{\partial x\partial z} \end{array} \right)$$
$$+g\beta_T(T - T_\infty) + g\beta_C(C - C_\infty), \tag{7.2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \nu\frac{\partial^2 v}{\partial z^2} - \lambda_1 \left(\begin{array}{c} u^2\frac{\partial^2 v}{\partial x^2} + v^2\frac{\partial^2 v}{\partial y^2} + w^2\frac{\partial^2 v}{\partial z^2} + 2uv\frac{\partial^2 v}{\partial x\partial y} + \\ 2vw\frac{\partial^2 v}{\partial y\partial z} + 2uw\frac{\partial^2 v}{\partial x\partial z} \end{array} \right), \tag{7.3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \sigma\frac{\partial^2 T}{\partial z^2} + \frac{D_e k_T}{C_s C_p}\frac{\partial^2 C}{\partial z^2} + \frac{\mu}{\rho c_p}\left(\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right), \tag{7.4}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = D_e\frac{\partial^2 C}{\partial z^2} + \frac{D_e k_T}{T_m}\frac{\partial^2 T}{\partial z^2} - k_1(C - C_\infty). \tag{7.5}$$

In Eqs. $(7.1) - (7.5)$ the respective velocity components in the $x-$, $y-$ and $z-$directions are denoted by u, v and w, λ_1 shows the relaxation time, T the fluid temperature, σ the thermal diffusivity of fluid, $\nu = (\mu/\rho)$ the kinematic viscosity, μ the dynamic viscosity of fluid, C the

concentration field, D_e the mass diffusivity, k_T the thermal diffusion ratio, C_p the specific heat, C_s the concentration susceptibility and ρ the density of fluid.

The boundary conditions appropriate to flow under consideration are given by

$$u = u_e = ax, \ v = by, \ w = 0, \ -k\frac{\partial T}{\partial z} = h(T_f - T), \ C = C_w \text{ at } z = 0, \qquad (7.6)$$

$$u \to 0, \ v \to 0, \ T \to T_\infty, \ C \to C_\infty \text{ as } z \to \infty, \qquad (7.7)$$

where k^* indicates the thermal conductivity of fluid, T_f is the hot fluid temperature, C_∞ the ambient concentration and a and b have dimension inverse of time.

We now define

$$
\begin{aligned}
u &= axf'(\eta), \ v = ayg'(\eta), \ w = -\sqrt{a\nu}(f(\eta) + g(\eta)), \\
\theta(\eta) &= \frac{T - T_\infty}{T_f - T_\infty}, \ \eta = z\sqrt{\frac{a}{\nu}}, \ \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}.
\end{aligned}
\qquad (7.8)
$$

Now the use of above variables satisfy Eq. (7.1) automatically while Eqs. (7.2)-(7.7) are reduced as follows:

$$f''' + (f + g)f'' - f'^2 + \beta_1[2(f + g)f'f'' - (f + g)^2 f'''] + \lambda(\theta + N\phi) = 0, \qquad (7.9)$$

$$g''' + (f + g)g'' - g'^2 + \beta_1[2(f + g)g'g'' - (f + g)^2 g'''] = 0, \qquad (7.10)$$

$$\theta'' + Pr(f + g)\theta' + Pr\, D_f\phi'' + Pr(E_1 f''^2 + E_2 g''^2) = 0, \qquad (7.11)$$

$$\phi'' + Sc(f + g)\phi' - Sck^*\phi + ScSr\theta'' = 0, \qquad (7.12)$$

$$f = 0, \ g = 0, \ f' = 1, \ g' = \beta, \ \theta' = -\gamma_1(1 - \theta(0)), \ \phi = 1 \ \text{ at } \eta = 0, \qquad (7.13)$$

$$f' \to 0, \ g' \to 0, \ \theta \to 0, \ \phi \to 0 \ \text{ as } \eta \to \infty, \qquad (7.14)$$

108

$$\begin{aligned}
\beta_1 &= \lambda_1 a, \ \lambda = \frac{Gr_x}{Re_x^2}, \ Gr_x = \frac{g\beta_T(T_f - T_\infty)x^3}{\nu^2}, \ N = \frac{\beta_C(C_w - C_\infty)}{\beta_T(T_f - T_\infty)}, Pr = \frac{\nu}{\sigma}, \\
D_f &= \frac{D_e k_T}{C_s C_p} \frac{(C_w - C_\infty)}{(T_f - T_\infty)\nu}, \ Sr = \frac{D_e k_T}{T_m \nu} \frac{(T_f - T_\infty)}{(C_w - C_\infty)}, E_1 = \frac{a^2 x^2}{C_p(T_f - T_\infty)}, E_2 = \frac{a^2 y^2}{C_p(T_f - T_\infty)}, \\
Sc &= \frac{\nu}{D_e}, k^* = \frac{k_1}{a}, \beta = \frac{b}{a}, \gamma_1 = \frac{h}{k^*}\sqrt{\frac{\nu}{a}}.
\end{aligned} \tag{7.15}$$

Here β_1 is the Deborah number, λ the mixed convection parameter, Gr_x the local Grashof number, N the concentration buoyancy parameter, Pr the Prandtl number, D_f the Dufour number, Sr the Soret number, E_1 and E_2 the Eckert numbers along the x and y directions respectively, Sc the Schmidt number, k^* the chemical reaction parameter, β the ratio of rates parameters and γ_1 the Biot number. All parameters are defined in a sequence in which they are written and prime shows the differentiation with respect to η.

Local Nusselt (dimensionless temperature gradient at the surface) and local Sherwood (dimensionless concentration gradient at the surface) numbers in dimensionless forms are given by

$$Nu/Re_x^{1/2} = -\theta'(0), \tag{7.16}$$

$$Sh/Re_x^{1/2} = -\phi'(0), \tag{7.17}$$

in which $Re_x = u_e x/\nu$ is the local Reynolds number.

7.2   Homotopy analysis solutions

The initial approximations and auxiliary linear operators required for homotopy analysis solutions are presented below i.e.

$$f_0(\eta) = \left(1 - e^{-\eta}\right), \ g_0(\eta) = \beta\left(1 - e^{-\eta}\right), \ \theta_0(\eta) = \frac{\gamma_1 \exp(-\eta)}{1 + \gamma_1}, \ \phi_0(\eta) = \exp(-\eta) \tag{7.18}$$

$$\mathcal{L}_f = f''' - f', \ \mathcal{L}_g = g''' - g', \ \mathcal{L}_\theta = \theta'' - \theta, \ \mathcal{L}_\phi = \phi'' - \phi, \tag{7.19}$$

with the following properties of the defined operators in Eq. (7.18) i.e.

$$\mathcal{L}_f(C_1 + C_2 e^{\eta} + C_3 e^{-\eta}) = 0, \ \mathcal{L}_g(C_4 + C_5 e^{\eta} + C_6 e^{-\eta}) = 0,$$

$$\mathcal{L}_\theta(C_7 e^{\eta} + C_8 e^{-\eta}) = 0, \ \mathcal{L}_\phi(C_9 e^{\eta} + C_{10} e^{-\eta}) = 0 \tag{7.20}$$

where C_i $(i = 1 - 10)$ indicate the arbitrary constants.

The corresponding problems at the zeroth order are given in the following forms:

$$(1-p)\,\mathcal{L}_f\left[\hat{f}(\eta; p) - f_0(\eta)\right] = p\hbar_f \mathcal{N}_f\left[\hat{f}(\eta; p), \hat{g}(\eta; p)\right], \tag{7.21}$$

$$(1-p)\,\mathcal{L}_g\left[\hat{g}(\eta; p) - g_0(\eta)\right] = p\hbar_g \mathcal{N}_g\left[\hat{f}(\eta; p), \hat{g}(\eta; p)\right], \tag{7.22}$$

$$(1-p)\,\mathcal{L}_\theta\left[\hat{\theta}(\eta; p) - \theta_0(\eta)\right] = p\hbar_\theta \mathcal{N}_\theta\left[\hat{f}(\eta; p), \hat{g}(\eta; p), \hat{\theta}(\eta, p), \hat{\phi}(\eta, p)\right], \tag{7.23}$$

$$(1-p)\,\mathcal{L}_\theta\left[\hat{\phi}(\eta; p) - \phi_0(\eta)\right] = p\hbar_\phi \mathcal{N}_\phi\left[\hat{f}(\eta; p), \hat{g}(\eta; p), \hat{\theta}(\eta, p), \hat{\phi}(\eta, p)\right], \tag{7.24}$$

$$\hat{f}(0; p) = 0, \ \hat{f}'(0; p) = 1, \ \hat{f}'(\infty; p) = 0, \ \hat{g}(0; p) = 0, \ \hat{g}'(0; p) = \beta, \ \hat{g}'(\infty; p) = 0,$$

$$\hat{\theta}'(0, p) = -\gamma_1[1 - \theta(0, p)], \ \hat{\theta}(\infty, p) = 0, \ \hat{\phi}(0, p) = 1, \ \hat{\phi}(\infty, p) = 0 \tag{7.25}$$

$$\mathcal{N}_f[\hat{f}(\eta, p), \hat{g}(\eta, p)] = \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} - \left(\frac{\partial \hat{f}(\eta, p)}{\partial \eta}\right)^2 + (\hat{f}(\eta, p) + \hat{g}(\eta, p))\frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2}$$

$$+\beta_1 \left[\begin{array}{c} 2(\hat{f}(\eta, p) + \hat{g}(\eta, p))\frac{\partial \hat{f}(\eta,p)}{\partial \eta}\frac{\partial^2 \hat{f}(\eta,p)}{\partial \eta^2} \\ -(\hat{f}(\eta, p) + \hat{g}(\eta, p))^2\frac{\partial^3 \hat{f}(\eta,p)}{\partial \eta^2} \end{array} \right]$$

$$+\lambda\left(\hat{\theta}(\eta, p) + N\hat{\phi}(\eta, p)\right), \tag{7.26}$$

$$\mathcal{N}_g[\hat{g}(\eta, p), \hat{f}(\eta, p)] = \frac{\partial^3 \hat{g}(\eta, p)}{\partial \eta^3} - \left(\frac{\partial \hat{g}(\eta, p)}{\partial \eta}\right)^2 + (\hat{f}(\eta, p) + \hat{g}(\eta, p))\frac{\partial^2 \hat{g}(\eta, p)}{\partial \eta^2}$$

$$+\beta_1 \left[\begin{array}{c} 2(\hat{f}(\eta, p) + \hat{g}(\eta, p))\frac{\partial \hat{g}(\eta,p)}{\partial \eta}\frac{\partial^2 \hat{g}(\eta,p)}{\partial \eta^2} \\ -(\hat{f}(\eta, p) + \hat{g}(\eta, p))^2\frac{\partial^3 \hat{g}(\eta,p)}{\partial \eta^2} \end{array} \right], \tag{7.27}$$

$$\mathcal{N}_\theta[\hat{\theta}(\eta,p),\hat{\phi}(\eta,p),\hat{f}(\eta,p),\hat{g}(\eta,p)] = \frac{\partial^2\hat{\theta}(\eta,p)}{\partial\eta^2} + \Pr(\hat{f}(\eta,p)+\hat{g}(\eta,p))\frac{\partial\hat{\theta}(\eta,p)}{\partial\eta}$$

$$+ \Pr D_f \frac{\partial^2\hat{\phi}(\eta,p)}{\partial\eta^2}$$

$$+ \Pr \left(\begin{array}{c} E_1\left(\frac{\partial^2\hat{f}(\eta,p)}{\partial\eta^2}\right)^2 \\ +E_2\left(\frac{\partial^2\hat{g}(\eta,p)}{\partial\eta^2}\right)^2 \end{array} \right). \qquad (7.28)$$

$$\mathcal{N}_\phi[\hat{\phi}(\eta,p),\hat{\theta}(\eta,p),\hat{f}(\eta,p),\hat{g}(\eta,p)] = \frac{\partial^2\hat{\phi}(\eta,p)}{\partial\eta^2} + Sc(\hat{f}(\eta,p)+\hat{g}(\eta,p))\frac{\partial\hat{\phi}(\eta,p)}{\partial\eta}$$

$$- Sck^*\hat{\phi}(\eta,p) + ScSr\frac{\partial^2\hat{\theta}(\eta,p)}{\partial\eta^2}. \qquad (7.29)$$

Here p is an embedding parameter, $\hbar_f$, $\hbar_g$, $\hbar_\theta$ and $\hbar_\phi$ are the non-zero auxiliary parameters and $\mathcal{N}_f$, $\mathcal{N}_g$, $\mathcal{N}_\theta$ and $\mathcal{N}_\phi$ indicate the nonlinear operators. When $p = 0$ and $p = 1$ one has

$$\hat{f}(\eta;0) = f_0(\eta), \ \hat{\theta}(\eta,0) = \theta_0(\eta), \ \hat{\phi}(\eta,0) = \phi_0(\eta)$$

$$\hat{f}(\eta;1) = f(\eta), \ \hat{\theta}(\eta,1) = \theta(\eta), \ \hat{\phi}(\eta,1) = \phi_0(\eta) . \qquad (7.30)$$

Clearly when p is increased from 0 to 1 then $f(\eta,p)$, $g(\eta,p)$, $\theta(\eta,p)$ and $\phi(\eta,p)$ vary from $f_0(\eta)$, $g_0(\eta)$, $\theta_0(\eta)$ and $\phi_0(\eta)$ to $f(\eta)$, $g(\eta)$, $\theta(\eta)$ and $\phi(\eta)$. By Taylor's expansion we have

$$f(\eta,p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \ f_m(\eta) = \frac{1}{m!}\left.\frac{\partial^m f(\eta;p)}{\partial p^m}\right|_{p=0}, \qquad (7.31)$$

$$g(\eta,p) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta)p^m, \ g_m(\eta) = \frac{1}{m!}\left.\frac{\partial^m g(\eta;p)}{\partial p^m}\right|_{p=0}, \qquad (7.32)$$

$$\theta(\eta,p) = \theta_0(\eta) \sum_{m=1}^{\infty} \theta_m(\eta)p^m, \ \theta_m(\eta) = \frac{1}{m!}\left.\frac{\partial^m \theta(\eta;p)}{\partial p^m}\right|_{p=0}, \qquad (7.33)$$

$$\phi(\eta,p) = \phi_0(\eta) \sum_{m=1}^{\infty} \phi_m(\eta)p^m, \ \phi_m(\eta) = \frac{1}{m!}\left.\frac{\partial^m \phi(\eta;p)}{\partial p^m}\right|_{p=0}, \qquad (7.34)$$

where the convergence of above series strongly depends upon $\hbar_f$, $\hbar_g$ $\hbar_\theta$ and $\hbar_\phi$. Considering that $\hbar_f$, $\hbar_g$, $\hbar_\theta$ and $\hbar_\phi$ are selected properly so that Eqs. $(7.31)-(7.34)$ converge at $p = 1$ then

we can write

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \qquad (7.35)$$

$$g(\eta) = g_0(\eta) + \sum_{m=1}^{\infty} g_m(\eta), \qquad (7.36)$$

$$\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \qquad (7.37)$$

$$\phi(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta). \qquad (7.38)$$

The resulting problems at mth order deformation can be constructed as follows:

$$\mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar_f \mathcal{R}_f^m(\eta), \qquad (7.39)$$

$$\mathcal{L}_g[g_m(\eta) - \chi_m g_{m-1}(\eta)] = \hbar_g \mathcal{R}_g^m(\eta), \qquad (7.40)$$

$$\mathcal{L}_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = \hbar_\theta \mathcal{R}_\theta^m(\eta), \qquad (7.41)$$

$$\mathcal{L}_\phi[\phi_m(\eta) - \chi_m \phi_{m-1}(\eta)] = \hbar_\phi \mathcal{R}_\phi^m(\eta). \qquad (7.42)$$

$$f_m(0) = f_m'(0) = f_m'(\infty) = 0, \; g_m(0) = g_m'(0) = g_m'(\infty) = 0, \; \theta_m'(0) - \gamma_1 \theta_m(0) = \theta_m(\infty) = 0,$$

$$\phi_m(0) = \phi_m(\infty) = 0. \qquad (7.43)$$

$$\begin{aligned}
\mathcal{R}_f^m(\eta) = {} & f_{m-1}'''(\eta) - \sum_{k=0}^{m-1} f_{m-1-k}' f_k' + \sum_{k=0}^{m-1} (f_{m-1-k} f_k'' + g_{m-1-k} f_k'') \\
& + \beta_1 \sum_{k=0}^{m-1} \sum_{l=0}^{k} \left(2(f_{m-1-k} + g_{m-1-k}) f_{k-l}' f_l'' - (f_{m-1-k} f_{k-l} + g_{m-1-k} g_{k-l} + 2 f_{m-1-k} g_{k-l}) f_l''' \right) \\
& + \lambda \left(\theta_{m-1} + N \phi_{m-1} \right),
\end{aligned} \qquad (7.44)$$

$$\begin{aligned}
\mathcal{R}_g^m(\eta) = {} & g_{m-1}'''(\eta) - \sum_{k=0}^{m-1} g_{m-1-k}' g_k' + \sum_{k=0}^{m-1} (f_{m-1-k} g_k'' + g_{m-1-k} g_k'') \\
& + \beta_1 \sum_{k=0}^{m-1} \sum_{l=0}^{k} [2(f_{m-1-k} + g_{m-1-k}) g_{k-l}' g_l'' \\
& - (f_{m-1-k} f_{k-l} + g_{m-1-k} g_{k-l} + 2 f_{m-1-k} g_{k-l}) g_l'''],
\end{aligned} \qquad (7.45)$$

112

$$\mathcal{R}_\theta^m(\eta) = \theta''_{m-1} + Pr \sum_{k=0}^{m-1} (\theta'_{m-1-k}f_k + \theta'_{m-1-k}g_k) + Pr\, D_f \phi''_{m-1}$$

$$+ Pr\left(E_1 \sum_{k=0}^{m-1} f''_{m-1-k}f''_k + E_2 \sum_{k=0}^{m-1} g''_{m-1-k}g''_k \right), \tag{7.46}$$

$$\mathcal{R}_\phi^m(\eta) = \phi''_{m-1} + Sc \sum_{k=0}^{m-1} (\phi'_{m-1-k}f_k + \phi'_{m-1-k}g_k) - Sck^*\phi + ScSr\theta''_{m-1}, \tag{7.47}$$

$$\chi_m = \left[\begin{array}{l} 0, \;\; m \leq 1, \\ 1, \;\; m > 1. \end{array} \right. \tag{7.48}$$

Solving the above mth order deformation problems we have

$$f_m(\eta) = f_m^*(\eta) + C_1 + C_2 e^\eta + C_3 e^{-\eta}, \tag{7.49}$$

$$g_m(\eta) = g_m^*(\eta) + C_4 + C_5 e^\eta + C_6 e^{-\eta}, \tag{7.50}$$

$$\theta_m(\eta) = \theta_m^*(\eta) + C_7 e^\eta + C_8 e^{-\eta}, \tag{7.51}$$

$$\phi_m(\eta) = \phi_m^*(\eta) + C_9 e^\eta + C_{10} e^{-\eta}, \tag{7.52}$$

in which the f_m^*, g_m^*, θ_m^* and ϕ_m^* indicate the special solutions.

7.3   Analysis and discussion

Obviously the homotopic solutions $(7.35) - (7.38)$ involve the auxiliary parameters $\hbar_f$, $\hbar_g$ $\hbar_\theta$ and $\hbar_\phi$. These parameters have pivotal role in adjusting and controlling the convergence of homotopic solutions. Hence the $\hbar-$curves are displayed at 18th order of approximations in order to determine the suitable ranges of $\hbar_f$, $\hbar_g$ $\hbar_\theta$ and $\hbar_\phi$. Fig. 7.2 witness that the admissible values of $\hbar_f$, $\hbar_g$ $\hbar_\theta$ and $\hbar_\phi$ here are $-1.25 \leq \hbar_f \leq -0.50$, $-1.25 \leq \hbar_g \leq -0.50$, $-1.40 \leq \hbar_\theta \leq -0.25$ and $-1.20 \leq \hbar_\phi \leq -0.75$. Table 7.1 ensures that the developed series solutions converge in the whole region of η when $\hbar_f = \hbar_g = \hbar_\theta = \hbar_\phi = -1$.

Table 7.1: Convergence of series solutions for different order of approximations when $\beta_1 = 0.2$, $\beta = 0.3$, $Pr = 1.0$, $Sr = 0.4$, $D_f = 0.5$, $Sc = 0.5$, $k^* = 0.3$, $\gamma_1 = 0.6$ and $\hbar_f = \hbar_g =$

$\hbar_\theta = \hbar_\phi = -1.0.$

order of approximations	1	5	10	15	20	25	30	35
$-f''(0)$	1.1215	1.1323	1.1323	1.1323	1.1323	1.1323	1.1323	1.1323
$-g''(0)$	0.25735	0.25958	0.25951	0.25951	0.25951	0.25951	0.25951	0.25951
$-\theta'(0)$	0.24141	0.22468	0.22127	0.22061	0.22045	0.22041	0.22040	0.22040
$-\phi'(0)$	0.64583	0.56841	0.55948	0.55838	0.55820	0.55817	0.55816	0.55816
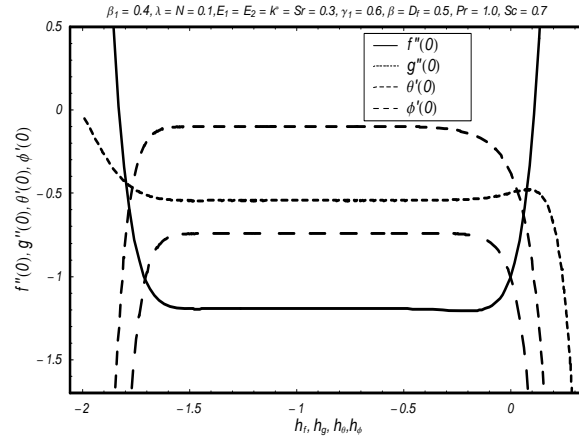
Fig.7.2: $\hbar-$curves for the function f, g, θ and ϕ.

Figs. 7.3 and 7.4 depict the behaviors of mixed convection parameter λ and concentration buoyancy parameter N on the velocity profile $f'(\eta)$. Fig. 7.3 is drawn to see the effect of mixed convection parameter λ on velocity profile $f'(\eta)$. It is noticed that momentum boundary layer thickness and velocity profile $f'(\eta)$ increase with an increase in mixed convection parameter in case of assisting flow $(\lambda > 0)$ while reverse effect is noted in case of opposing flow $(\lambda < 0)$. This is due to the fact that buoyancy forces are more dominant to viscous forces in case of assisting flow $(\lambda > 0)$ while buoyancy forces reduce for opposing flow $(\lambda < 0)$. Effect of concentration buoyancy parameter N on the velocity profile $f'(\eta)$ is analyzed in Fig. 7.4. It is seen that both the momentum boundary layer thickness and velocity profile $f'(\eta)$ are increasing functions of

114

concentration buoyancy parameter N.
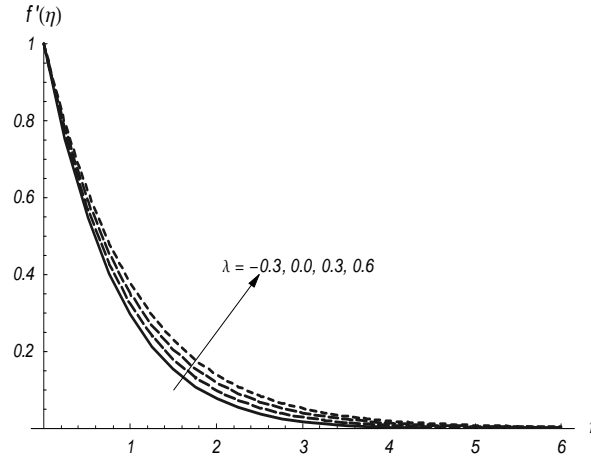
Fig. 7.3: Influence of λ on $f'(\eta)$ when $N = E_1 = E_2 = 0.3$ and $\beta_1 = 0.5$.
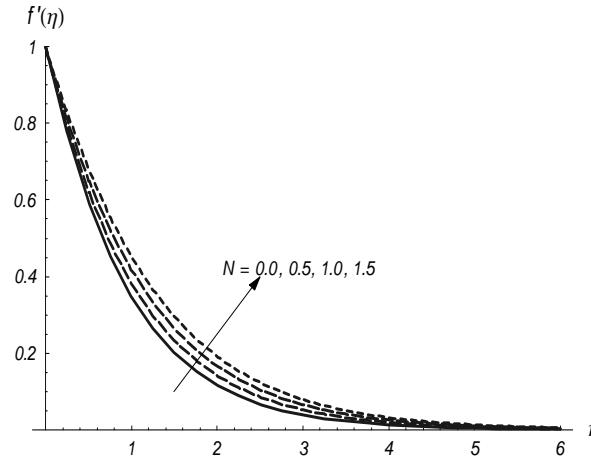
Fig. 7.4: Influence of N on $f'(\eta)$ when $\beta = \lambda = E_1 = E_2 = 0.3$ and $\beta_1 = 0.5$.

Figs. 7.5−7.9 examine the variation of Dufour number D_f, Eckert numbers E_1 and E_2, mixed convection parameter λ and concentration buoyancy parameter N on the temperature $\theta(\eta)$. Effect of Dufour number D_f on temperature profile $\theta(\eta)$ is presented in Fig. 7.5. As Dufour effect pointed out the generation of energy flux by composition gradient, then the thermal boundary layer thickness and temperature $\theta(\eta)$ increase with an increase in D_f. Figs. 7.6 and 7.7 illustrate the effect of Eckert numbers E_1 and E_2 on the temperature $\theta(\eta)$ along the x and y directions respectively. As the Eckert number expresses the relationship between the

kinetic energy in the flow and the enthalpy. It embodies the conversion of kinetic energy into internal energy by work done against the viscous fluid stresses. Greater viscous dissipative heat causes a rise in the temperature and thermal boundary layer thickness. Fig. 7.8 is drawn to see the influence of mixed convection parameter λ on the temperature $\theta(\eta)$. Thermal boundary layer thickness and temperature $\theta(\eta)$ decrease in case of assisting flow $(\lambda > 0)$ while increase in case of opposing flow $(\lambda < 0)$. This is due to the fact that in case of assisting flow $(\lambda > 0)$ the buoyancy forces are more dominant to viscous forces which causes a reduction in the temperature $\theta(\eta)$ while in case of opposing flow $(\lambda < 0)$ the viscous forces are more dominant than buoyancy forces which in results enhances the temperature $\theta(\eta)$. Influence of concentration buoyancy parameter N on the temperature $\theta(\eta)$ is seen in Fig. 7.9. It is found that the associated boundary layer and temperature $\theta(\eta)$ is decreasing function of N.

Fig. 7.5: Influence of D_f on $\theta(\eta)$ when $\beta_1 = k^* = 0.5, Pr = Sr = 0.5$

$\lambda = E_1 = E_2 = N = \beta = Sc = 0.3$ and $\gamma_1 = 0.4$.

116
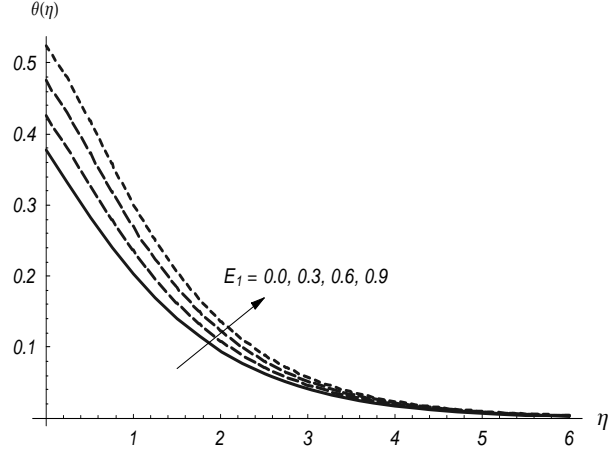
Fig. 7.6: Influence of E_1 on $\theta(\eta)$ when $\beta_1 = k^* = 0.5, Pr = Sr = 0.5,$

$\lambda = D_f = E_2 = N = \beta = Sc = 0.3$ and $\gamma_1 = 0.4.$
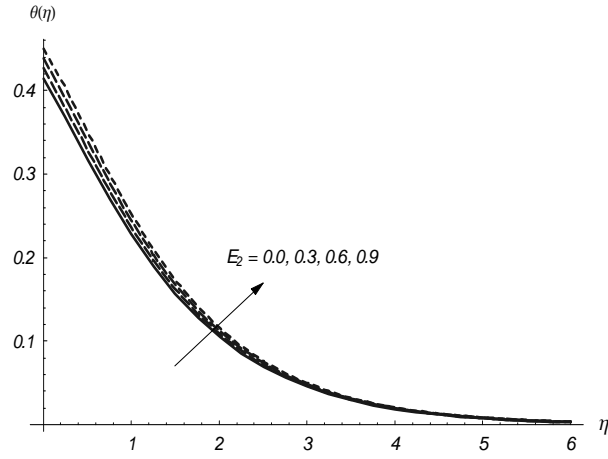
Fig. 7.7: Influence of E_2 on $\theta(\eta)$ when $\beta_1 = k^* = 0.5, \ Pr = Sr = 0.5$

$\lambda = D_f = E_1 = N = \beta = Sc = 0.3$ and $\gamma_1 = 0.4.$
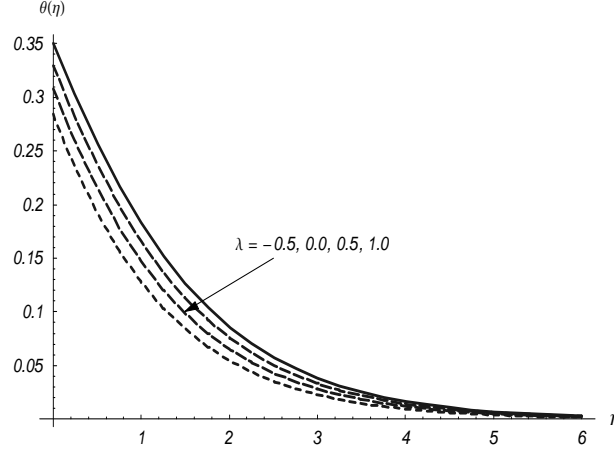
Fig. 7.8: Influence of λ on $\theta(\eta)$ when $\beta_1 = k^* = 0.5$, $Pr = Sr = 0.5$

$$D_f = E_1 = E_2 = N = \beta = Sc = 0.3 \text{ and } \gamma_1 = 0.4.$$
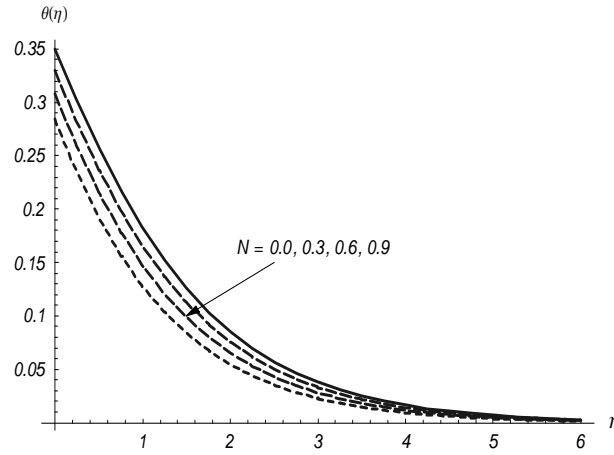
Fig.7.9: Influence of N on $\theta(\eta)$ when $\beta_1 = k^* = 0.5$, $Pr = Sr = 0.5$

$$D_f = E_1 = E_2 = \lambda = \beta = Sc = 0.3 \text{ and } \gamma_1 = 0.4.$$

Figs. 7.10-7.13 are sketched to see the effects of Soret number Sr, generative/destructive chemical reaction k^*, mixed convection parameter λ and concentration buoyancy parameter N on concentration profile $\phi(\eta)$. Fig. 7.10 depicts the influence of Soret number Sr on concentration profile $\phi(\eta)$. As in Soret effect the temperature gradient causes the mass flux which in turn enhances the concentration profile $\phi(\eta)$ and associated boundary layer thickness. Effect of destructive/generative chemical reaction k^* on the concentration profile $\phi(\eta)$ is analyzed in

118

Fig. 11. It is found that in case of generative chemical reaction ($k^* < 0$) the reduction in $\phi(\eta)$ is noted while reverse is in case of destructive chemical reaction ($k^* > 0$). Figs. 7.12 and 7.13 are presented to see the effects of mixed convection parameter λ and concentration buoyancy parameter N on the concentration profile $\phi(\eta)$. These Figs. show that $\phi(\eta)$ reduces with the increase in λ and N. Also the associated boundary layer thickness are decreasing functions of λ and N.
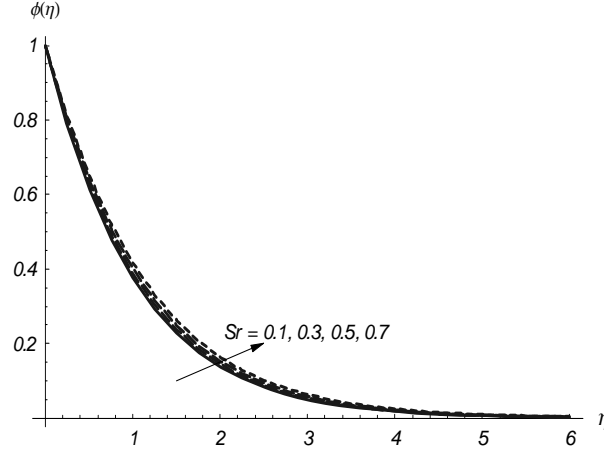
Fig. 7.10: Influence of Sr on $\phi(\eta)$ when $\beta_1 = k^* = 0.5, \Pr = 0.5, \gamma_1 = 0.4,$

$$D_f = E_1 = E_2 = \lambda = N = \beta = Sc = 0.3,$$
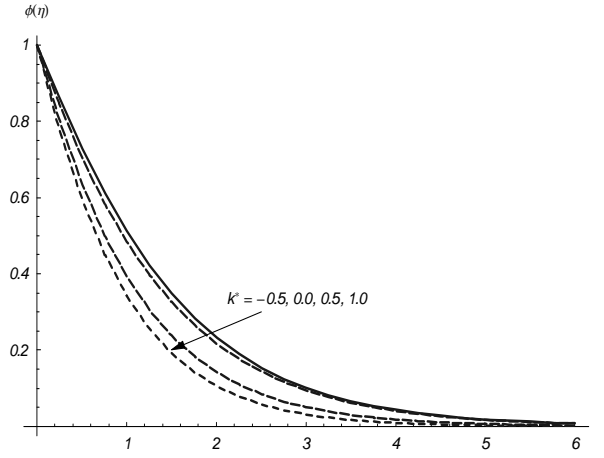
Fig. 7.11: Influence of k^* on $\phi(\eta)$ when $\beta_1 = 0.5, \ Pr = Sr = 0.5,$

$$D_f = E_1 = E_2 = \lambda = N = \beta = Sc = 0.3 \text{ and } \gamma_1 = 0.4.$$

119
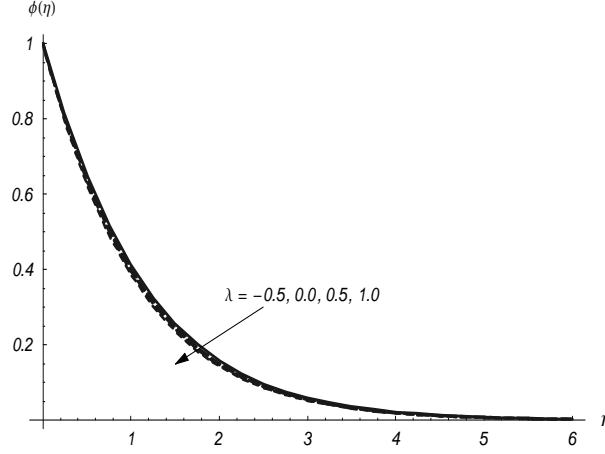
Fig. 7.12: Influence of λ on $\phi(\eta)$ when $\beta_1 = k^* = 0.5$, $Pr = Sr = 0.5$,

$$D_f = E_1 = E_2 = N = \beta = Sc = 0.3 \text{and } \gamma_1 = 0.4.$$
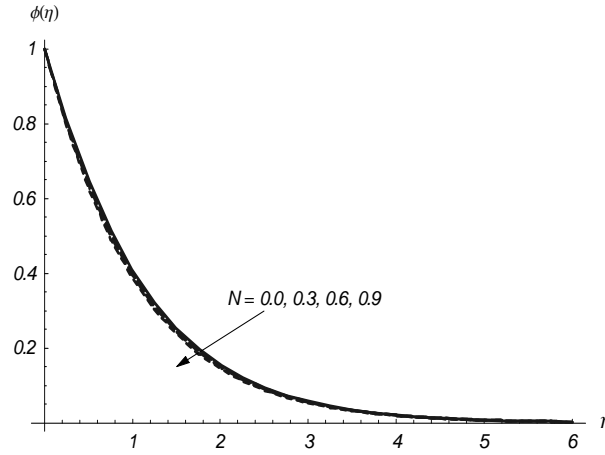
Fig. 7.13: Influence of N on $\phi(\eta)$ when $\beta_1 = k^* = 0.5$, $Pr = Sr = 0.5$,

$$D_f = E_1 = E_2 = \lambda = \beta = Sc = 0.3 \text{ and } \gamma_1 = 0.4.$$

Table 7.1 presents the convergence of homotpic solutions. It is noted that computations are sufficient for 15th order iterations of velocity and 35th order iterations of the temperature and concentration profiles for convergent series solutions. Tables 7.2 and 7.3 are prepared to analyze numerical values of local Nusselt and Sherwood numbers. The values of $-\theta'(0)$ and $-\phi'(0)$ decrease by increasing Deborah number, mixed convection parameter and concentration buoyancy parameter. Here $-\theta'(0)$ increases by increasing Prandtl number and Biot number

while reverse is the case of $-\phi'(0)$. It is also found that $-\theta'(0)$ decreases by increasing Eckert numbers while opposite behavior of $-\phi'(0)$. Table 7.4 presents the comparison of $-f''(0)$ and $-g''(0)$ for various values of β in the limiting sense with ref. [13]. Table 7.5 ensures the values of local Nusselt number $-\theta'(0)$ are in good agreement with ref. [39] in a limiting sense.

Table 7.2: Values of local Nusselt $-\theta'(0)$ and Sherwood numbers $-\phi'(0)$ for the different

values of the parameters β_1, β, Sc, Sr, D_f, k^*, Pr and γ_1 when $\lambda = N = E_1 = E_2 = 0.3$.

β_1	β	Pr	γ_1	Sc	Sr	D_f	k^*	$-\theta'(0)$	$-\phi'(0)$
0.0	0.2	0.7	0.4	0.5	0.4	0.5	0.3	0.16023	0.56411
0.2								0.15399	0.55548
0.4								0.14858	0.54795
0.2	0.0							0.14266	0.53103
	0.2							0.15399	0.55548
	0.4							0.16269	0.57666
0.2	0.2	0.4						0.13249	0.55814
		0.8						0.16009	0.55462
		1.2						0.17597	0.55212
0.2	0.2	0.7	0.2					0.098922	0.56163
			0.4					0.15319	0.55548
			0.6					0.18962	0.55150
0.2	0.2	0.7	0.4	0.0				0.20797	0.11228
				0.5				0.15399	0.55548
				1.0				0.11405	0.85681
0.2	0.2	0.7	0.4	0.5	0.0			0.15406	0.56553
					0.5			0.15457	0.55288
					1.0			0.15565	0.53984
0.2	0.2	0.7	0.4	0.5	0.5	0.2		0.19480	0.54212
						0.5		0.15469	0.55287
						1.0		0.084902	0.57214
0.2	0.2	0.7	0.4	0.5	0.5	0.5	0.0	0.18113	0.35157
							0.3	0.15469	0.55212
							0.5	0.14153	0.65090

Table 7.3: Values of local Nusselt $-\theta'(0)$ and Sherwood numbers $-\phi'(0)$ for the different values of the parameters λ, N, E_1 and E_2 when $\beta_1 = \beta = 0.2$, $Sc = D_f = 0.5$, $Sr = \gamma_1 = 0.4$,

$Pr = 0.7$ and $k^* = 0.3$.

λ	N	E_1	E_2	$-\theta'(0)$	$-\phi'(0)$
-0.3	0.3	0.3	0.3	0.0091701	0.71369
0.0				0.091253	0.73788
0.3				0.12419	0.75068
0.3	0.0			0.11837	0.74820
	0.3			0.12444	0.75042
	0.6			0.12978	0.75260
0.3	0.3	0.0		0.18952	0.73121
		0.1		0.16745	0.73772
		0.2		0.14557	0.74431
0.3	0.3	0.3	0.0	0.14414	0.74474
			0.1	0.13777	0.74657
			0.2	0.13110	0.74854

Table 7.4: Values of $-f''(0)$ and $-g''(0)$ with β when $\beta_1 = \lambda = N = 0$ with HPM (Ariel [13])

and exact solution (Ariel [13]).

β	$-f''(0)$			$-g''(0)$		
	HAM	HPM[13]	Exact[13]	HAM	HPM[13]	Exact[13]
0.0	1	1	1	0	0	0
0.1	1.020260	1.017027	1.020260	0.066847	0.073099	0.066847
0.2	1.039495	1.034587	1.039495	0.148737	0.158231	0.148737
0.3	1.057955	1.052470	1.057955	0.243360	0.254347	0.243360
0.4	1.075788	1.070529	1.075788	0.349208	0.360599	0.349209
0.5	1.093095	1.088662	1.093095	0.465205	0.476290	0.465205
0.6	1.109946	1.106797	1.109947	0.590528	0.600833	0.590529
0.7	1.126397	1.124882	1.126398	0.724532	0.733730	0.724532
0.8	1.142488	1.142879	1.142489	0.866682	0.874551	0.866683
0.9	1.158254	1.160762	1.158254	1.016539	1.022922	1.016539
1.0	1.173720	1.178511	1.173721	1.173720	1.178511	1.173721

Table 7.5: Values of local Nusselt number $-\theta'(0)$ for different values of β^*, β and Pr in a

limiting sense when $\lambda = N = E_1 = E_2 = D_f = Sr = k^* = 0.0$ and $\gamma_1 = 0.6$.

β_1	β	Pr	Present Results $-\theta'(0)$	Hayat et al [39] $-\theta'(0)$
0.0	0.5	1.0	0.330404	0.33040
0.3			0.321661	0.32160
0.8			0.308651	0.30799
1.2			0.299526	0.29873
0.4	0.0		0.287813	0.28908
	0.4		0.316638	0.31664
	0.7		0.330168	0.33017
	1.0		0.340702	0.34070
0.4	0.5	0.7	0.282787	0.28279
		1.2	0.340424	0.34042
		1.6	0.368405	0.36840
		2.0	0.388869	0.38887

7.4   Conclusions

Three dimensional mixed convection flow of Maxwell fluid over a stretching surface with convective condition is investigated. Effects of chemical reaction and Soret and Dufour are analyzed. The main observations are listed below.

- Momentum boundary layer thickness and velocity profile $f\prime(\eta)$ increase with the increase in λ and N.

- Concentration profile $\phi(\eta)$ is decreasing function of generative chemical reaction parameter $(k^* > 0)$ while increasing function of destructive chemical reaction $(k^* < 0)$ and Soret number Sr.

- Influences of λ and N on $\theta(\eta)$ and $\phi(\eta)$ are qualitatively similar.

- Thermal boundary layer thickness and temperature field increase when D_f increases.

- There are opposite effects of local Nusselt number and local Sherwood number when D_f, Sr, E_1 and E_2 increase.

- Local Nusselt and Sherwood numbers increase when k^*, D_f and Sc are enhanced.

- Effects of local Nusselt number and local Sherwood number for Pr, Sr, λ and N are similar.

Chapter 8

Radiative mixed convection flow of an Oldroyd-B fluid by an inclined stretching surface

Mixed convection flow of an Oldroyd-B fluid in the presence of thermal radiation is investigated in this chapter. Flow is induced by an inclined stretching surface. The boundary layer equations of an Oldroyd-B fluid in the presence of heat transfer are used. Appropriate transformations reduce the partial differential equations into the ordinary differential equations. Computational analysis is performed for the convergent series solutions. The values of local Nusselt number are numerically analyzed. Effects of various parameters involved in the velocity and temperature are discussed.

8.1 Mathematical analysis

Consider the steady two-dimensional mixed convection flow of an incompressible Oldroyd-B fluid by an inclined stretching surface. The heat transfer is considered in the presence of thermal radiation using Rosseland approximation. Here $x-$axis is taken along the stretching surface and $y-$axis normal to the $x-$axis. Conservation laws of mass, linear momentum and energy in absence of viscous dissipation give

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{8.1}$$

$$
\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\frac{\partial^2 u}{\partial y^2} - \lambda_1\left[u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + 2uv\frac{\partial^2 u}{\partial x \partial y}\right] \\
&\quad + \nu\lambda_2\left[u\frac{\partial^3 u}{\partial x \partial y^2} + v\frac{\partial^3 u}{\partial y^3} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}\right] + g\beta_T(T - T_\infty)\cos\alpha, \tag{8.2}
\end{aligned}
$$

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\left(\frac{16\sigma_s T_\infty^3}{3k_e} + k\right)\frac{\partial T}{\partial y}\right), \tag{8.3}$$

where u and v are the velocity components in the $x-$ and $y-$directions, λ_1 and λ_2 are the relaxation and retardation times, respectively, $\nu = (\mu/\rho)$ is the kinematic viscosity, σ_s is the Stefan-Boltzmann constant, T is the fluid temperature, ρ is the fluid density, g is the gravitational acceleration, β_T is thermal expansion coefficient of temperature, c_p is the specific heat, k_e is the mean absorption coefficient and k is the thermal conductivity.

The appropriate boundary conditions are taken as follows:

$$u = cx, \ v = 0, \ T = T_w \text{ at } y = 0, \tag{8.4}$$

$$u \to 0, \ T \to T_\infty \text{ as } y \to \infty \tag{8.5}$$

with the surface temperature T_w by

$$T_w(x,t) = T_\infty + bx, \tag{8.6}$$

where b and c are the positive constants. If ψ is the stream function then defining

$$\eta = \sqrt{\frac{c}{\nu}}y, \ \psi = \sqrt{c\nu}xf(\eta), \ \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \tag{8.7}$$

$$u = \frac{\partial \psi}{\partial y}, \ v = -\frac{\partial \psi}{\partial x}, \tag{8.8}$$

the incompressibility condition is clearly satisfied and the resulting problems for f and θ satisfy the following equations

$$f''' + ff'' - f'^2 + \beta_1(2ff'f'' - f^2f''') - \beta_2(f''^2 - ff'''') + \lambda\theta\cos\alpha = 0, \tag{8.9}$$

$$\left(1 + \frac{4}{3}R\right)\theta'' + \Pr\left(f\theta' - f'\theta\right) = 0, \tag{8.10}$$

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) \to 0, \quad \theta(0) = 1, \quad \theta(\infty) \to 0. \tag{8.11}$$

In the above expressions β_1 and β_2 are the Deborah numbers, λ is mixed convection parameter, Gr_x is the local Grashof number, Re_x is the Reynold number, $\Pr$ is the Prandtl number, R is the radiation parameter and primes indicate the differentiation with respect to η i.e.

$$\beta_1 = \lambda_1 c, \ \beta_2 = \lambda_2 c, \ \lambda = \frac{Gr_x}{Re_x^2}, \ Gr_x = \frac{g\beta(T_w - T_\infty)x^3}{\nu^2}, \ Re_x = \frac{u_w^2 x^2}{\nu^2},$$

$$\Pr = \frac{\mu c_p}{k}, R = \left(\frac{4\sigma_s T_\infty^3}{k_e k}\right). \tag{8.12}$$

Local Nusselt number Nu_x in terms of heat transfer q_w is

$$Nu_x = \frac{xq_w}{k(T_w - T_\infty)}, \quad q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0}. \tag{8.13}$$

Above equation in dimensionless variables becomes

$$Nu/Re_x^{1/2} = -(1 + \frac{4}{3}R)\,\theta'(0). \tag{8.14}$$

Considering the set of base functions

$$\left\{\eta^k \exp(-n\eta)|k \geq 0, n \geq 0\right\} \tag{8.15}$$

one can express that

$$f_m(\eta) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{m,n}^k \eta^k \exp(-n\eta), \tag{8.16}$$

$$\theta_m(\eta) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_{m,n}^k \eta^k \exp(-n\eta), \tag{8.17}$$

where $a_{m,n}$ and $b_{m,n}$ are the coefficients.

Initial guesses f_0 and θ_0 and auxiliary linear operators $\hbar_f$ and $\hbar_\theta$ are chosen as follows:

$$f_0(\eta) = 1 - \exp(-\eta), \ \ \theta_0(\eta) = \exp(-\eta), \tag{8.18}$$

$$\mathcal{L}_f = \frac{d^3 f}{d\eta^3} - \frac{df}{d\eta}, \ \ \mathcal{L}_\theta = \frac{d^2 \theta}{d\eta^2} - \theta, \tag{8.19}$$

where

$$\mathcal{L}_f \left[C_1 + C_2 \exp(\eta) + C_3 \exp(-\eta) \right] = 0, \ \ \mathcal{L}_\theta \left[C_4 \exp(\eta) + C_5 \exp(-\eta) \right] = 0 \tag{8.20}$$

in which C_i $(i = 1 - 5)$ are the arbitrary constants. Introducing $p \in [0, 1]$ as the embedding parameter and $\hbar_f$ and $\hbar_\theta$ the non-zero auxiliary parameters, the deformation problems at the zeroth order are

$$(1-p)\mathcal{L}_f[\hat{f}(\eta, p) - f_0(\eta)] = p\hbar_f \mathcal{N}_f \left[\hat{f}(\eta, p), \hat{\theta}(\eta, p) \right], \tag{8.21}$$

$$(1-p)\mathcal{L}_\theta[\hat{\theta}(\eta, p) - \theta_0(\eta)] = p\hbar_\theta \mathcal{N}_\theta \left[\hat{f}(\eta, p), \hat{\theta}(\eta, p) \right], \tag{8.22}$$

$$\hat{f}(\eta; p)\Big|_{\eta=0} = 0, \ \ \frac{\partial \hat{f}(\eta; p)}{\partial \eta}\Big|_{\eta=0} = 1, \ \ \frac{\partial \hat{f}(\eta; p)}{\partial \eta}\Big|_{\eta=\infty} = 0, \tag{8.23}$$

$$\hat{\theta}(\eta; p)\Big|_{\eta=0} = 1, \ \ \hat{\theta}(\eta; p)\Big|_{\eta=\infty} = 0, \tag{8.24}$$

$$
\begin{aligned}
\mathcal{N}_f[\hat{f}(\eta, p), \hat{\theta}(\eta, p)] \ = \ & \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} - \hat{f}(\eta, p)\frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - \left(\frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right)^2 \\
& + \beta_1 \left[2\hat{f}(\eta, p)\frac{\partial \hat{f}(\eta, p)}{\partial \eta}\frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - (\hat{f}(\eta, p))^2\frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} \right] \\
& + \beta_2 \left[\left(\frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right)^2 - \hat{f}(\eta, p)\frac{\partial^4 \hat{f}(\eta, p)}{\partial \eta^4} \right] \\
& + \lambda \hat{\theta}(\eta, p) \cos \alpha, \tag{8.25}
\end{aligned}
$$

$$\mathcal{N}_\theta \left[\hat{f}(\eta; p), \hat{\theta}(\eta; p) \right] = \left(1 + \frac{4}{3}R \right)\frac{\partial^2 \hat{\theta}(\eta, p)}{\partial \eta^2} + \Pr \left(\hat{f}(\eta; p)\frac{\partial \hat{\theta}(\eta; p)}{\partial \eta} - \hat{\theta}(\eta; p)\frac{\partial \hat{f}(\eta; p)}{\partial \eta} \right). \tag{8.26}$$

130

For $p = 0$ and $p = 1$, we have

$$\widehat{f}(\eta; 0) = f_0(\eta), \qquad \widehat{f}(\eta; 1) = f(\eta), \tag{8.27}$$

$$\widehat{\theta}(\eta; 0) = \theta_0(\eta), \qquad \widehat{\theta}(\eta; 1) = \theta(\eta) \tag{8.28}$$

and when p increases from 0 to 1 then $\widehat{f}(\eta; p)$ and $\widehat{\theta}(\eta; p)$ deform from $f_0(\eta)$ and $\theta_0(\eta)$ to $f(\eta)$ and $\theta(\eta)$ respectively. Further Taylor series expansion yields

$$\widehat{f}(\eta; p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) p^m, \tag{8.29}$$

$$\widehat{\theta}(\eta; p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) p^m, \tag{8.30}$$

$$f_m(\eta) = \frac{1}{m!} \frac{\partial^m \widehat{f}(\eta; p)}{\partial p^m}\bigg|_{p=0}, \qquad \theta_m(\eta) = \frac{1}{m!} \frac{\partial^m \widehat{\theta}(\eta; p)}{\partial p^m}\bigg|_{p=0}. \tag{8.31}$$

The auxiliary parameters $\hbar_f$ and $\hbar_\theta$ are selected such that the series (8.29) and (8.30) converge at $p = 1$. Hence

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \qquad \theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta). \tag{8.32}$$

The corresponding problems at mth order are given by

$$\mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar_f \mathcal{R}_m^f(\eta), \qquad \mathcal{L}_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = \hbar_\theta \mathcal{R}_m^\theta(\eta), \tag{8.33}$$

$$
\begin{aligned}
f_m(0) &= 0, \ f_m'(0) = 0, \ f_m'(\infty) = 0, \\
\theta_m(0) &= 0, \ \theta_m(\infty) = 0, \ \phi_m(0) = 0, \ \phi_m(\infty) = 0,
\end{aligned} \tag{8.34}
$$

131

$$\mathcal{R}_f^m(\eta) \quad = \quad f_{m-1}'''(\eta) + \sum_{k=0}^{m-1} \left[f_{m-1-k} f_k'' - f_{m-1-k}' f_k'' \right]$$

$$+\beta_1 \sum_{k=0}^{m-1} f_{m-1-k} \sum_{l=0}^{k} \{2 f_{k-l}' f_l'' - f_{k-l} f_l''' - \lambda f_{m-1}'(\eta)$$

$$+\beta_2 \{ f_{m-1-k}' \sum_{k=0}^{m-1} f_k' - f_{m-1-k} \sum_{k=0}^{m-1} f_k^{iv} \} + \lambda \theta \cos \alpha, \tag{8.35}$$

$$\mathcal{R}_m^\theta(\eta) = \left(1 + \frac{4}{3}R \right) \theta_{m-1}'' + \mathrm{Pr} \sum_{k=0}^{m-1} \left(f_{m-1-k} \theta_k' - f_{m-1-k}' \theta_k \right) \tag{8.36}$$

$$\chi_m = \begin{vmatrix} 0, & m \le 1 \\ 1, & m > 1 \end{vmatrix}. \tag{8.37}$$

The general solutions of Eq. (8.32) can be written as follows:

$$f_m(\eta) = f_m^*(\eta) + C_1 + C_2 \exp(\eta) + C_3 \exp(-\eta), \tag{8.38}$$

$$\theta_m(\eta) = \theta_m^*(\eta) + C_4 \exp(\eta) + C_5 \exp(-\eta), \tag{8.39}$$

where the special solutions are $f_m^*(\eta)$ and $\theta_m^*(\eta)$ and

$$\begin{aligned} C_2 &= C_4 = 0, \\ C_1 &= -C_3 - f_m^*(0), \quad C_3 = \left. \frac{\partial f_m^*(\eta)}{\partial \eta} \right|_{\eta=0}, \\ C_5 &= -\theta_m^*(0). \end{aligned} \tag{8.40}$$

8.2   Convergence of the series solutions

Clearly the series solutions contain the non-zero auxiliary parameters $\hbar_f$ and $\hbar_\theta$. Hence the $\hbar_f$ and $\hbar_\theta$ curves are plotted for 20th$-$order of approximation in order to find the admissible values of $\hbar_f$ and $\hbar_\theta$. It is found that the admissible values of $\hbar_f$ and $\hbar_\theta$ are $-1.4 \le \hbar_f$, $\hbar_\theta \le -0.25$ (Fig. 8.1). The series given by Eq. (8.36) converges in the whole region of η when $\hbar_f = \hbar_\theta = -0.6$.

Fig.8.1: $\hbar-$curves for the function f and θ.

8.3 Discussion

This section examines the influence of physical parameters on the velocity $f\prime(\eta)$ and temperature $\theta(\eta)$. Figs. $8.2-8.9$ depict the graphical effects of inclination α, Deborah numbers β_1 and β_2 and mixed convection parameter λ. Figs. 8.2 and 8.3 depict that fluid velocity decreases while the fluid temperature increases with the increase of inclination parameter α. It is also observed that momentum boundary layer thickness is a decreasing function of α while thermal boundary layer thickness is an increasing function of α. Figs. 8.4 and 8.5 are sketched to analyze the effect of relaxation time parameter β_1 on the velocity and temperature. As the relaxation time parameter increases the velocity profile decreases while the temperature profile is quite opposite to that of the velocity profile. Figs. 8.6 and 8.7 examine the influence of mixed convection parameter λ on the velocity and temperature. The fluid velocity and associated momentum boundary layer thickness increase by increasing λ. It is clear that the temperature has opposite effect when compared with velocity. We also noticed that the fluid velocity increases rapidly in comparison to the temperature with the increasing values of λ. Figs. 8.8 and 8.9 depict the effect of retardation time β_2 on the temperature and velocity fields. Here we observed that β_2 has quite opposite effect on the velocity and temperature. The fluid velocity increases with an

133

increase in β_2.

$\beta_1 = 0.4, \lambda = 0.5, \beta_2 = 0.3, R = 0.3, Pr = 0.7$

Fig. 8.2: Influence of sheet inclination α on $f\prime(\eta)$.

$\beta_1 = 0.4, \lambda = 1.0, \beta_2 = 0.2, R = 0.5, Pr = 0.7$

Fig. 8.3: Influence of sheet inclination α on $\theta(\eta)$.

$\alpha = \pi/4, \lambda = 1.0, \beta_2 = 0.2, R = 0.5, Pr = 0.7$

Fig.8.4: Influence of Deborah number β_1 on $f\prime(\eta)$.

134

Fig. 8.5: Influence of Deborah number β_1 on $\theta(\eta)$.

Fig. 8.6: Influence of mixed convection λ on $f\prime(\eta)$.

Fig. 8.7: Influence of mixed convection λ on $\theta(\eta)$.

135

Fig. 8.8:Influence of Deborah number β_2 on $f\prime(\eta)$.

Fig. 8.9: Influence of Deborah number β_2 on $\theta(\eta)$.

Table 8.1 also shows that the series solutions converge for 20th-order of deformations for both the velocity and temperature. Table 8.2 analyzes the numerical values of local Nusselt number for different values of β_1, β_2, λ, Pr and R. We note that the numerical values of local Nusselt number increase with the increase of β_1, β_2, Pr and such values decrease for λ and R.

Table 8.1: Convergence of homotopy solutions for different order of approximations when

$\alpha = \pi/4$, $\beta_1 = 0.2$, $\lambda = 0.5$, $\beta_2 = 0.3$, $\mathrm{Pr} = 0.7$ and $R = 0.3$.

Order of approximation	$-f''(0)$	$-\theta'(0)$
1	0.69555	0.73000
05	0.75163	0.72503
10	0.75241	0.72457
15	0.75240	0.72457
20	0.75239	0.72456
25	0.75239	0.72456
30	0.75239	0.72456

Table 8.2: Local Nusselt number and skin friction coefficient $Re_x^{-1/2}\, Nu_x$ for some values of β_1, λ, Pr, β_2 and R when $\alpha = \pi/4$.

β_1	β_2	λ	Pr	R	$-(1+\frac{4}{3}R)\,\theta'(0)$
0.1	0.2	0.5	0.7	0.3	0.73887
0.2					0.71284
0.3					0.70528
0.5	0.0				0.70729
	0.2				0.71925
	0.4				0.72951
	0.2	0.0			0.66787
		0.3			0.70324
		0.5			0.71924
		0.5	0.3		0.43757
			0.5		0.58961
			0.7	0.0	0.84020
				0.2	0.75419
				0.5	0.66085

8.4   Concluding remarks

This chapter examines the effect of thermal radiation on the mixed convection flow of an Oldroyd-B fluid over an inclined stretching sheet. The main observations of this study are as follows.

- Effect of mixed convection parameter λ on the velocity and temperature are quite opposite.

- Momentum and thermal boundary layers for λ have opposite effects.

- Decrease in temperature is more significant in comparison to velocity when Prandtl number Pr increases.

- Thermal boundary layer thickness is decreasing function of Prandtl number.

- Deborah numbers β_1 and β_2 have quite opposite effects for the velocity and temperature.

Chapter 9

Mixed convection flow of an Oldroyd-B fluid with power law heat flux and heat source

This chapter looks at the mixed convection flow of an Oldroyd-B fluid bounded by a porous stretching surface. Mathematical formulation is developed in the presence of heat source and power law heat flux. Velocity and temperature are computed. Plots for different parameters are analyzed. Numerical values of local Nusselt number are examined.

9.1   Development of problems

We consider the two-dimensional flow of an incompressible Oldroyd-B fluid over a porous surface. A Cartesian coordinate system is chosen in such a way that $x-$axis is taken along the flow direction and the $y-$axis perpendicular to the $x-$axis. The fluid fills the half space $y > 0$.

Heat source is present and power law heat flux is imposed. Flow geometry is shown in Fig. 9.1.

Fig. 9.1: Physical model of the problem

The present boundary layer flow is governed by the following expressions

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,\tag{9.1}$$

$$
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \lambda_1\left[u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + 2uv\frac{\partial^2 u}{\partial x\partial y}\right]
$$
$$
+ \nu\lambda_2\left[u\frac{\partial^3 u}{\partial x\partial y^2} + v\frac{\partial^3 u}{\partial y^3} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}\right] + g\beta_T(T - T_\infty),\tag{9.2}
$$

$$
u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} + \frac{Q}{\rho c_p}(T - T_\infty).\tag{9.3}
$$

In above equations u and v are the velocity components in the $x-$ and $y-$directions, λ_1 and λ_2 are the relaxation and retardation times respectively, g the gravitational acceleration, β_T the thermal expansion coefficient, $\nu = (\mu/\rho)$ the kinematic viscosity, T the fluid temperature, ρ the density of fluid, k the thermal conductivity of fluid, c_p the specific heat at constant pressure and Q the heat source coefficient.

The subjected boundary conditions are

$$u = cx, \quad v = -v_0, \quad \frac{\partial T}{\partial y} = Ax^2 \quad \text{at } y = 0, \tag{9.4}$$

$$u = 0, \quad T = T_\infty \quad \text{as } y \to \infty, \tag{9.5}$$

where A is the temperature coefficient and T_∞ is the ambient temperature.

The similarity transformations are given by

$$u = cxf'(\eta), \quad v = -\sqrt{c\nu}f(\eta), \quad T = T_\infty + A\sqrt{\frac{\nu}{c}}x^2\theta(\eta), \quad \eta = \sqrt{\frac{c}{\nu}}y \tag{9.6}$$

in which c is a constant and prime denotes differentiation with respect to η. Eq. (9.1) is automatically satisfied and the Eqs. (9.2-9.5) are reduced as follows

$$f''' + ff'' - f'^2 + \beta_1(2ff'f'' - f^2f''') - \beta_2(f''^2 - ff'''') + \lambda\theta = 0, \tag{9.7}$$

$$\theta'' + Prf\theta' - 2Prf'\theta + \beta^*\theta = 0, \tag{9.8}$$

$$f = S, \quad f' = 1, \quad \theta' = 1 \quad \text{at } \eta = 0, \tag{9.9}$$

$$f' = 0, \quad \theta = 0 \quad \text{as } \eta \to \infty, \tag{9.10}$$

in which $\beta_1 = \lambda_1 c$ and $\beta_2 = \lambda_2 c$ are the Deborah numbers, $\lambda = \frac{Gr_x}{Re_x^2}$ the mixed convection parameter with $Gr_x = \frac{g\beta_T(T-T_\infty)x^3}{\nu^2}$ the local Grashof number and $Re_x = \frac{Ux}{\nu}$ the local Reynolds number, $S = \frac{v_0}{\sqrt{\nu c}}$ the suction/injection parameter, $Pr = \frac{\mu c_p}{k}$ the Prandtl number and $\beta^* = \frac{\nu Q}{kc}$ a heat generation/absorption parameter.

Expression of local Nusselt number Nu_x is

$$Nu_x = \frac{xq_w}{k(T - T_\infty)}, \quad q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0}. \tag{9.11}$$

Dimensionless form of Eq. (9.11) is

$$Nu/Re_x^{1/2} = -\frac{1}{\theta(0)}. \tag{9.12}$$

9.2　Homotopy analysis solutions

Choosing the following set of base functions

$$\{\eta^k \exp(-n\eta) k \geq 0, n \geq 0\} \tag{9.13}$$

we express f and θ as follows

$$f(\eta) = a_{0,0}^0 + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{m,n}^k \eta^k \exp(-n\eta), \tag{9.14}$$

$$\theta(\eta) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_{m,n}^k \eta^k \exp(-n\eta), \tag{9.15}$$

where $a_{m,n}^k$ and $b_{m,n}^k$ are the coefficients. Initial approximations and auxiliary linear operators are taken in the following forms

$$f_0(\eta) = S + 1 - \exp(-\eta), \quad \theta_0(\eta) = -\exp(-\eta), \tag{9.16}$$

$$\mathcal{L}_f = f''' - f', \quad \mathcal{L}_\theta = \theta'' + \theta', \tag{9.17}$$

$$\mathcal{L}_f(C_1 + C_2 e^\eta + C_3 e^{-\eta}) = 0, \quad \mathcal{L}_\theta(C_4 + C_5 e^{-\eta}) = 0, \tag{9.18}$$

where C_i $(i = 1 - 5)$ are the arbitrary constants.

The corresponding zeroth order deformation problems are developed in the following fashions.

$$(1 - p)\mathcal{L}_f\left[\hat{f}(\eta; p) - f_0(\eta)\right] = ph_f\mathcal{N}_f\left[\hat{f}(\eta; p), \hat{\theta}(\eta, p)\right], \tag{9.19}$$

$$(1 - p)\mathcal{L}_\theta\left[\hat{\theta}(\eta; p) - \theta_0(\eta)\right] = ph_\theta\mathcal{N}_\theta\left[\hat{f}(\eta; p), \hat{\theta}(\eta, p)\right], \tag{9.20}$$

$$\hat{f}(0; p) = S, \ \hat{f}'(0; p) = 1, \ \hat{f}'(\infty; p) = 0, \ \hat{\theta}'(0, p) = 1, \ \hat{\theta}(\infty, p) = 0, \tag{9.21}$$

$$\mathcal{N}_f[\hat{f}(\eta,p)] = \frac{\partial^3 \hat{f}(\eta,p)}{\partial \eta^3} - \hat{f}(\eta,p)\frac{\partial^2 \hat{f}(\eta,p)}{\partial \eta^2} - \left(\frac{\partial \hat{f}(\eta,p)}{\partial \eta}\right)^2$$

$$+\beta_1 \left[2\hat{f}(\eta,p)\frac{\partial \hat{f}(\eta,p)}{\partial \eta}\frac{\partial^2 \hat{f}(\eta,p)}{\partial \eta^2} - (\hat{f}(\eta,p))^2\frac{\partial^3 \hat{f}(\eta,p)}{\partial \eta^3}\right] + \qquad (9.22)$$

$$\beta_2 \left[\left(\frac{\partial^2 \hat{f}(\eta,p)}{\partial \eta^2}\right)^2 - \hat{f}(\eta,p)\frac{\partial^4 \hat{f}(\eta,p)}{\partial \eta^4}\right] + \lambda\frac{\partial \hat{\theta}(\eta,p)}{\partial \eta}, \qquad (9.23)$$

$$\mathcal{N}_\theta[\hat{\theta}(\eta,p),\hat{f}(\eta,p)] = \frac{\partial^2 \hat{\theta}(\eta,p)}{\partial \eta^2} + Pr\hat{f}(\eta,p)\frac{\partial \hat{\theta}(\eta,p)}{\partial \eta} - 2Pr\frac{\partial \hat{f}(\eta,p)}{\partial \eta}\hat{\theta}(\eta,p) + \beta^*\hat{\theta}(\eta,p), \quad (9.24)$$

in which p is an embedding parameter, h_f and h_θ the non-zero auxiliary parameters and $\mathcal{N}_f$ and $\mathcal{N}_\theta$ the nonlinear operators. For $p = 0$ and $p = 1$ we have

$$\hat{f}(\eta;0) = f_0(\eta), \ \hat{\theta}(\eta,0) = \theta_0(\eta) \text{ and } \hat{f}(\eta;1) = f(\eta), \ \hat{\theta}(\eta,1) = \theta(\eta), \qquad (9.25)$$

and when p increases from 0 to 1 then $f(\eta,p)$ and $\theta(\eta,p)$ vary from $f_0(\eta), \theta_0(\eta)$ to $f(\eta)$ and $\theta(\eta)$. Taylor's series yields

$$f(\eta,p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \qquad (9.26)$$

$$\theta(\eta,p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)p^m, \qquad (9.27)$$

$$f_m(\eta) = \frac{1}{m!}\frac{\partial^m f(\eta;p)}{\partial p^m}\bigg|_{p=0}, \ \theta_m(\eta) = \frac{1}{m!}\frac{\partial^m \theta(\eta;p)}{\partial p^m}\bigg|_{p=0}, \qquad (9.28)$$

where the convergence of above series strongly depends upon h_f and h_θ. The auxiliary parameters h_f and h_θ are selected in such a way that (9.26) and (9.27) converge at $p = 1$ and hence

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \qquad (9.29)$$

$$\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta). \qquad (9.30)$$

The mth-order deformation problems are constructed by the following expressions

$$\mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = h_f \mathcal{R}_f^m(\eta), \qquad (9.31)$$

$$\mathcal{L}_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = h_\theta \mathcal{R}_\theta^m(\eta), \tag{9.32}$$

$$f_m(0) = f_m'(0) = f_m'(\infty) = 0, \quad \theta_m'(0) - \gamma \theta_m(0) = \theta_m(\infty) = 0, \tag{9.33}$$

$$\begin{aligned}
\mathcal{R}_f^m(\eta) &= f_{m-1}'''(\eta) + \sum_{k=0}^{m-1} \left[f_{m-1-k} f_k'' - f_{m-1-k}' f_k'' \right] \\
&\quad + \beta_1 \sum_{k=0}^{m-1} f_{m-1-k} \sum_{l=0}^{k} \{2f_{k-l}' f_l'' - f_{k-l} f_l''' - \lambda f_{m-1}'(\eta) \\
&\quad + \beta_2 \{ f_{m-1-k}' \sum_{k=0}^{m-1} f_k' - f_{m-1-k} \sum_{k=0}^{m-1} f_k^{iv}) + \lambda \theta_{m-1},
\end{aligned} \tag{9.34}$$

$$\mathcal{R}_\theta^m(\eta) = \theta_{m-1}''(\eta) + Pr \sum_{k=0}^{m-1} \theta_{m-1-k}' f_k - 2Pr \sum_{k=0}^{m-1} \theta_{m-1-k} f_k' + \beta^* \theta_{m-1}(\eta), \tag{9.35}$$

$$\chi_m = \left[\begin{array}{l} 0, \ m \leq 1, \\ 1, \ m > 1. \end{array} \right. \tag{9.36}$$

If f_m^* and θ_m^* are the special solutions then the general solutions are

$$f_m(\eta) = f_m^*(\eta) + C_1 + C_2 e^\eta + C_3 e^{-\eta}, \tag{9.37}$$

$$\theta_m(\eta) = \theta_m^*(\eta) + C_4 + C_5 e^{-\eta}. \tag{9.38}$$

9.3   Convergence of the homotopy solutions

The auxiliary parameters $\hbar_f$ and $\hbar_\theta$ have significant role in the convergence of developed series solutions. Here the $\hbar-$curves are portrayed for 18th order of approximations in order to find the values of $\hbar_f$ and $\hbar_\theta$ ensuring convergence. Figs. 9.2 depict that the range of admissible values of $\hbar_f$ and $\hbar_\theta$ are $-0.8 \leq \hbar_f \leq -0.2$ and $-0.75 \leq \hbar_\theta \leq -0.2$. The series solution converge in the whole region of η when $\hbar_f = -0.4$ and $\hbar_\theta = -0.4$. Table 9.1 depicts that 25th order

deformations are sufficient for both the velocity and temperature expressions.
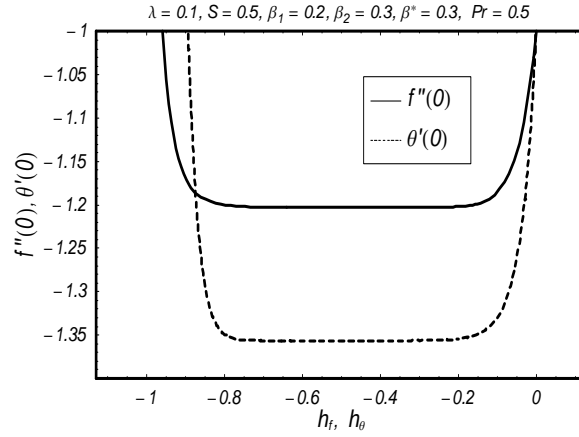
Fig.9.2: $\hbar$−curves for the function f and θ.

Table 9.1. Convergence of homotopy solutions for different order of approximations when $\beta_1 = 0.2$, $\beta_2 = \beta^* = 0.3$, $Pr = S = 0.5$, $\lambda = 0.1$, $\hbar_f = -0.4$ and $\hbar_\theta = -0.4$.

Order of approximation	$-f''(0)$	$-\theta''(0)$
1	1.10000	1.22000
5	1.19801	1.34636
10	1.20255	1.35611
15	1.20268	1.35640
20	1.20271	1.35642
25	1.20272	1.35643
30	1.20272	1.35643

9.4 Graphical results and discussion

The purpose of this section is to highlight the variations of interesting parameters through Figs. $9.3 - 9.7$ for velocity and temperature. Figs. 9.3 and 9.4 show the behaviors of suction/injection parameter S, Deborah number β_1 and mixed convection parameter λ on the velocity $f'(\eta)$. Fig. 9.3 shows the effects of suction/injection parameter S on the velocity profile $f'(\eta)$. Here $S > 0$ corresponds to suction and $S < 0$ for injection case. We observed that the velocity $f'(\eta)$ is

lower for suction case in comparison to injection phenomenon. From physical point of view suction is an agent that resists the fluid flow. Such resistance in fluid flow creates a reduction in the velocity field and associated boundary layer thickness. Fig. 9.4 illustrates that both the fluid velocity and boundary layer thickness decrease when mixed convection parameter is increased. Note that the mixed convection parameter involves the buoyancy force. Buoyancy force is stronger for the larger mixed convection parameter and weaker for the smaller mixed convection parameter. This stronger buoyancy acts as an agent to creates a reduction in the velocity profile and momentum boundary layer thickness.
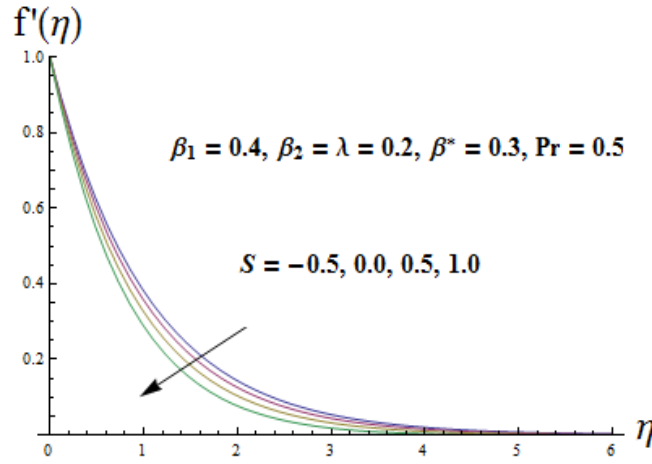
Fig. 9.3: Variation of S on $f\prime(\eta)$.
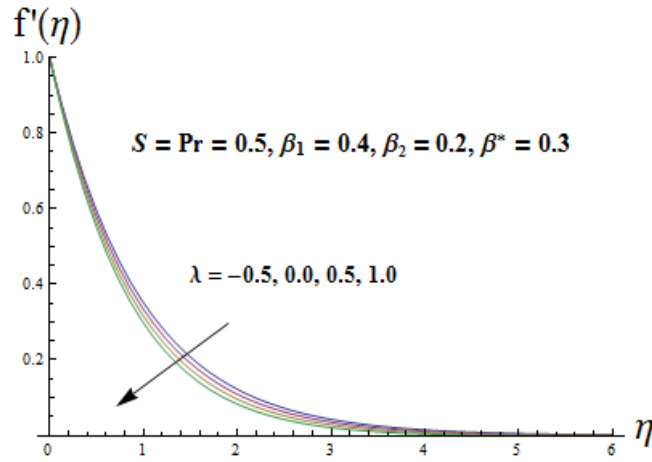
Fig. 9.4: Variation of λ on $f\prime(\eta)$.

Figs. $9.5 - 9.7$ are displayed to examine the influence of arising parameters on dimensionless

146

temperature profile $\theta(\eta)$. Fig. 9.5 presents the variations in temperature $\theta(\eta)$ for different values of suction/injection parameter S. From this Fig. it is analyzed that the temperature is higher for injection case when we compared it with suction case. The effects of suction parameter S on $\theta(\eta)$ is qualitatively similar to that of the velocity. Both the temperature $\theta(\eta)$ and thermal boundary layer thickness increase when mixed convection parameter λ increases (see Fig 9.6). Here buoyancy force is an agent that creates an enhancement in the temperature and thermal boundary layer thickness. Influence of heat source β^* on $\theta(\eta)$ is presented in Fig. 9.7. Physically $\beta^* > 0$ means that $T_w > T_\infty$ and in this case heat is supplied to the flow region from the wall. The temperature boosts with heat source parameter $\beta^* > 0$ while reduction in thermal boundary layer thickness and temperature is seen with heat sink parameter $\beta^* < 0$.
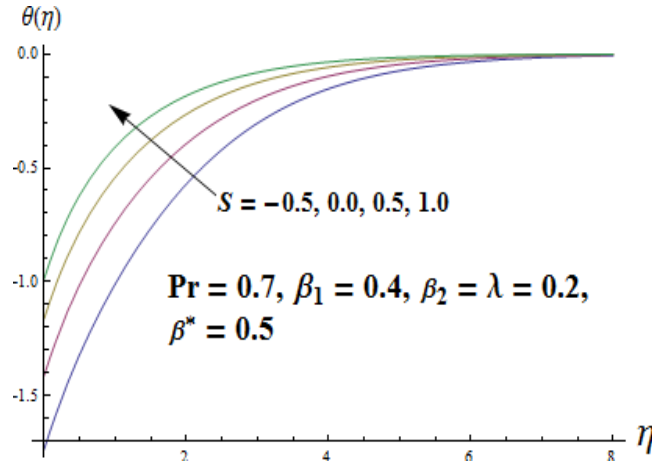
Fig. 9.5: Variation of S on $\theta(\eta)$.

Fig. 9.6: Variation of λ on $\theta(\eta)$.

Fig. 9.7: Variation of β^* on $\theta(\eta)$.

Table 9.2 shows that the local Nusselt number has quite opposite behavior for β_1 and β_2. The values of Nusselt number increases by increasing β^*, Pr and S. However it decreases by increasing λ. Table. 9.3 ensures the validity of present results for $f''(0)$ in a limiting sense.

Table 9.2: Values of local Nusselt number $Nu/Re_x^{1/2}$ for the parameters β_1, β_2, β^*, Pr, S

148

when $\lambda = 0.1$.

β_1	β_2	β^*	Pr	S	λ	$-Nu/Re_x^{1/2}$
0.0	0.3	0.3	0.5	0.5	0.1	1.20133
0.2						1.17496
0.4						1.14936
	0.4					1.18647
	0.5					1.19622
	0.6					1.20463
		−0.1				1.02500
		−0.2				1.09798
		−0.3				1.16099
			0.6			1.27668
			0.9			1.60523
			1.2			1.91014
				0.8		1.23081
				1.0		1.30323
				1.5		1.45103
					0.2	1.15381
					0.4	1.13613
					0.6	1.11294

Table 9.3: Comparison of $f''(0)$ for different values of Maxwell parameter β_1 when $\beta_2 = S =$

$\lambda = 0.$

β_1	Abel et al. [38] $-f''(0)$	Present results $-f''(0)$
0.0	1.00000	1.00000
0.2	1.051948	1.051889
0.4	1.101850	1.101903
0.6	1.150163	1.150137
0.8	1.196692	1.196711
1.2	1.285257	1.285363
1.6	1.368641	1.368758
2.0	1.447617	1.447651

9.5   Final remarks

This chapter deals with the mixed convection flow of Oldroyd-B fluid over a stretching sheet with suction/injection, heat source/sink and power law heat flux. Effects of different involved parameters such as mixed convection parameter λ and heat source/sink β^* on the flow field and temperature are analyzed. The main observations are summarized as follows:

- Velocity profile and momentum boundary layer thickness reduce in case of suction $S > 0$ while these enhance in case of injection $S < 0$.

- Thermal boundary layer thickness is increasing function of heat source parameter $\beta^* > 0$ while it reduces with heat absorption parameter $\beta^* < 0$.

- Increase in mixed convection parameter λ yields an enhancement in the temperature and thermal boundary layer thickness while reduction in heat transfer rate at wall is noted.

- Behaviors of Deborah numbers β_1 and β_2 on the heat transfer rate at the wall are opposite. Heat transfer rate at wall increases with the heat absorption $\beta^* < 0$ and suction parameters $S > 0$.

Chapter 10

Soret and Dufour effects in mixed convection flow of an Oldroyd-B fluid with convective boundary conditions

This chapter investigates the effects of heat and mass transfer in the mixed convection flow of an Oldroyd-B fluid over a stretching surface with convective boundary conditions. Emphasis is given to the analysis of Soret and Dufour effects. Relevant problems are first formulated and then computed by the homotopy analysis method (HAM). Velocity, temperature and concentration fields are computed and analyzed through plots. In addition, the local Nusselt and Sherwood numbers are examined through the numerical values.

10.1   Mathematical model

We choose $x-$axis along the stretching surface in the flow direction and y-axis is taken perpendicular to the surface. An incompressible Oldroyd-B fluid is considered. The surface satisfies the convective boundary conditions. Further, the Soret and Dufour effects are taken into account.

The resulting boundary layer equations for flow of an Oldroyd-B fluid are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{10.1}$$

$$\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\frac{\partial^2 u}{\partial y^2} - \lambda_1\left[u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + 2uv\frac{\partial^2 u}{\partial x\partial y}\right] \\
&\quad + \nu\lambda_2\left[u\frac{\partial^3 u}{\partial x\partial y^2} + v\frac{\partial^3 u}{\partial y^3} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}\right] \\
&\quad + g[\beta_T(T - T_\infty) + \beta_C(C - C_\infty)],
\end{aligned} \tag{10.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha_m\frac{\partial^2 T}{\partial y^2} + \frac{D_e k_T}{C_s C_p}\frac{\partial^2 C}{\partial y^2}, \tag{10.3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_e\frac{\partial^2 C}{\partial y^2} + \frac{D_e k_T}{T_m}\frac{\partial^2 T}{\partial y^2}, \tag{10.4}$$

where u and v denote the velocity components in the $x-$ and $y-$directions respectively, λ_1 and λ_2 the relaxation and retardation times respectively, T the fluid temperature, C the concentration field, ν the kinematic viscosity, ρ the fluid density, D_e the mass diffusivity, α_m the thermal conductivity, k_T the thermal-diffusion ratio, C_p the specific heat, C_s the concentration susceptibility, T_m the fluid mean temperature and σ the fluid electrical conductivity.

The boundary conditions are expressed in the following forms

$$u = u_w(x) = cx, \ v = 0, \ -k\frac{\partial T}{\partial y} = h(T_f - T), \ C = C_w \text{ at } y = 0, \tag{10.5}$$

$$u \to 0, \ T \to T_\infty, \ C \to C_\infty \text{ as } y \to \infty \tag{10.6}$$

in which T_f denotes the convective fluid temperature, T_∞ the ambient temperature and k the thermal conductivity.

Setting the following transformations

$$\eta = y\sqrt{\frac{c}{\nu}}, \ u = cxf'(\eta), \ v = -\sqrt{c\nu}f(\eta), \ \theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty}, \ \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}, \tag{10.7}$$

incompressibility condition (10.1) is automatically satisfied and the other equations give

$$f''' + ff'' - f'^2 + \beta_1(2ff'f'' - f^2f''') - \beta_2(f''^2 - ff'''') + \lambda\theta + \lambda N\phi = 0, \qquad (10.8)$$

$$\frac{1}{Pr}\theta'' + f\theta' + D_f\phi'' = 0, \qquad (10.9)$$

$$\phi'' + Scf\phi' + Sc\theta'' = 0, \qquad (10.10)$$

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0, \qquad (10.11)$$

$$\theta'(0) = -\gamma_1(1 - \theta(0)), \quad \theta(\infty) = 0, \qquad (10.12)$$

$$\phi(0) = 1, \quad \phi(\infty) = 0. \qquad (10.13)$$

In above expressions the prime indicates the differentiation with respect to η, $\beta_1 = \lambda_1 c$ and $\beta_2 = \lambda_2 c$ are the Deborah numbers, $\lambda = \frac{Gr_x}{Re_x^2}$ the mixed convection parameter with $Gr_x = \frac{g\beta_T(T-T_\infty)x^3}{\nu^2}$ as the Grashof number, $Re_x = \frac{Ux}{\nu}$ the local Reynold number, $Pr = \nu/\alpha_m$ the Prandtl number, $\gamma_1 = (h/k)\sqrt{\nu/c}$ the Biot number, $Sc = \nu/De$ the Schmidt number, D_f the Dufour number and Sr the Soret number. The definition of D_f and Sr are

$$D_f = \frac{D_e k_T}{C_s C_p}\frac{(C_w - C_\infty)}{(T_f - T_\infty)\nu}, \quad Sr = \frac{D_e k_T}{T_m \nu}\frac{(T_w - T_\infty)}{(C_w - C_\infty)}. \qquad (10.14)$$

Local Nusselt and local Sherwood numbers in dimensionless coordinates are expressed as follows:

$$Nu(Re_x)^{-1/2} = -\theta'(0), \qquad (10.15)$$

$$Sh(Re_x)^{-1/2} = -\phi'(0). \qquad (10.16)$$

In the next section we will develop the homotopy solutions for the resulting problems.

10.2   Series solutions

The initial guesses and auxiliary linear operators are taken as

$$f_0(\eta) = (1 - exp(-\eta)), \ \theta_0(\eta) = \frac{\gamma_1 \exp(-\eta)}{1 + \gamma_1}, \ \phi_0(\eta) = \exp(-\eta) + \frac{\eta}{2}\exp(-\eta), \qquad (10.17)$$

153

$$\mathcal{L}_f = \frac{d^3 f}{d\eta^3} - \frac{df}{d\eta}, \ \ \mathcal{L}_\theta = \frac{d^2 \theta}{d\eta^2} - \theta, \ \ \mathcal{L}_\phi = \frac{d^2 \phi}{d\eta^2} - \phi, \tag{10.18}$$

$$\mathcal{L}_f \left[C_1 + C_2 \exp(\eta) + C_3 \exp(-\eta) \right] = 0, \tag{10.19}$$

$$\mathcal{L}_\theta \left[C_4 \exp(\eta) + C_5 \exp(-\eta) \right] = 0, \ \ \mathcal{L}_\phi \left[C_6 \exp(\eta) + C_7 \exp(-\eta) \right] = 0, \tag{10.20}$$

in which C_i $(i = 1 - 7)$ denote the arbitrary constants.

10.2.1 Zeroth and mth order deformation problems

Having the non-linear operators $\mathcal{N}_f$, $\mathcal{N}_\theta$ and $\mathcal{N}_\phi$ in the forms

$$
\begin{aligned}
\mathcal{N}_f[\hat{f}(\eta, p), \hat{\theta}(\eta, p)] \ = \ & \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} - \hat{f}(\eta, p)\frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - \left(\frac{\partial \hat{f}(\eta, p)}{\partial \eta} \right)^2 \\
& + \beta_1 \left[2\hat{f}(\eta, p)\frac{\partial \hat{f}(\eta, p)}{\partial \eta}\frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} - (\hat{f}(\eta, p))^2 \frac{\partial^3 \hat{f}(\eta, p)}{\partial \eta^3} \right] \\
& + \beta_2 \left[\left(\frac{\partial^2 \hat{f}(\eta, p)}{\partial \eta^2} \right)^2 - \hat{f}(\eta, p)\frac{\partial^4 \hat{f}(\eta, p)}{\partial \eta^4} \right] \\
& + \lambda \hat{\theta}(\eta, p) + \lambda N \hat{\phi}(\eta, p), \tag{10.21}
\end{aligned}
$$

$$\mathcal{N}_\theta[\hat{f}(\eta, p), \hat{\theta}(\eta, p), \hat{\phi}(\eta, p)] = \frac{1}{Pr}\frac{\partial^2 \hat{\theta}(\eta, p)}{\partial \eta^2} + \hat{f}(\eta, p)\frac{\partial \hat{\theta}(\eta, p)}{\partial \eta} + D_f \frac{\partial^2 \hat{\phi}(\eta, p)}{\partial \eta^2}, \tag{10.22}$$

$$\mathcal{N}_\phi[\hat{f}(\eta, p), \hat{\theta}(\eta, p), \hat{\phi}(\eta, p)] = \frac{\partial^2 \hat{\phi}(\eta, p)}{\partial \eta^2} + Sc\hat{f}(\eta, p)\frac{\partial \hat{\phi}(\eta, p)}{\partial \eta} + SrSc\frac{\partial^2 \hat{\theta}(\eta, p)}{\partial \eta^2}, \tag{10.23}$$

the corresponding problems at the zeroth and mth orders can be expressed as follows:

$$(1 - p)\mathcal{L}_f[\hat{f}(\eta, p) - f_0(\eta)] = p\hbar_f \mathcal{N}_f[\hat{f}(\eta, p), \hat{\theta}(\eta, p), \hat{\phi}(\eta, p)], \tag{10.24}$$

$$(1 - p)\mathcal{L}_\theta[\hat{\theta}(\eta, p) - \theta_0(\eta)] = p\hbar_\theta \mathcal{N}_\theta[\hat{f}(\eta, p), \hat{\theta}(\eta, p), \hat{\phi}(\eta, p)], \tag{10.25}$$

$$(1 - p)\mathcal{L}_\phi[\hat{\phi}(\eta, p) - \phi_0(\eta)] = p\hbar_\phi \mathcal{N}_\phi[\hat{f}(\eta, p), \hat{\theta}(\eta, p), \hat{\phi}(\eta, p)], \tag{10.26}$$

$$\hat{f}(\eta; p)\Big|_{\eta=0} = 0, \ \ \frac{\partial \hat{f}(\eta; p)}{\partial \eta}\Big|_{\eta=0} = 1, \ \ \frac{\partial \hat{f}(\eta; p)}{\partial \eta}\Big|_{\eta=\infty} = 0, \tag{10.27}$$

$$\hat{\theta}'(\eta; p)\Big|_{\eta=0} = -\gamma[1 - \hat{\theta}(\eta; p)\Big|_{\eta=0}], \ \ \hat{\theta}(\eta; p)\Big|_{\eta=\infty} = 0, \tag{10.28}$$

$$\hat{\phi}(\eta; p)\Big|_{\eta=0} = 1, \quad \hat{\phi}(\eta; p)\Big|_{\eta=\infty} = 0, \tag{10.29}$$

$$\mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar_f \mathcal{R}_m^f(\eta), \tag{10.30}$$

$$\mathcal{L}_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = \hbar_\theta \mathcal{R}_m^\theta(\eta), \tag{10.31}$$

$$\mathcal{L}_\phi[\phi_m(\eta) - \chi_m \phi_{m-1}(\eta)] = \hbar_\phi \mathcal{R}_m^\phi(\eta), \tag{10.32}$$

$$
\begin{aligned}
f_m(0) &= 0, \ f'_m(0) = 0, \ f'_m(\infty) = 0, \ \theta'_m(0) - \gamma\theta_m(0) = 0, \\
\theta_m(\infty) &= 0, \phi_m(0) = 0, \ \phi_m(\infty) = 0,
\end{aligned}
\tag{10.33}
$$

$$
\begin{aligned}
\mathcal{R}_f^m(\eta) &= f'''_{m-1}(\eta) + \sum_{k=0}^{m-1}\left[f_{m-1-k}f''_k - f'_{m-1-k}f''_k\right] \\
&+ \beta_1 \sum_{k=0}^{m-1} f_{m-1-k} \sum_{l=0}^{k}\{2f'_{k-l}f''_l - f_{k-l}f'''_l - \lambda f'_{m-1}(\eta) \\
&+ \beta_2\{f'_{m-1-k}\sum_{k=0}^{m-1} f'_k - f_{m-1-k}\sum_{k=0}^{m-1} f_k^{iv}) + \lambda\theta'_{m-1}(\eta) + \lambda N1\phi'_{m-1}(\eta),
\end{aligned}
\tag{10.34}
$$

$$\mathcal{R}_m^\theta(\eta) = \frac{1}{Pr}\theta''_{m-1}(\eta) + \sum_{k=0}^{m-1} f_{m-1-k}\theta'_k + D_f\phi''_{m-1}(\eta), \tag{10.35}$$

$$\mathcal{R}_m^\phi(\eta) = \phi''_{m-1}(\eta) + Sc\sum_{k=0}^{m-1} f_{m-1-k}\phi'_k + Sc\theta''_{m-1}(\eta), \tag{10.36}$$

$$\chi_m = \begin{vmatrix} 0, & m \le 1 \\ 1, & m > 1, \end{vmatrix} \tag{10.37}$$

where $p \in [0, 1]$ is an embedding parameter and $\hbar_f$, $\hbar_\theta$ and $\hbar_\phi$ are the nonzero auxiliary parameters. Taylor's series gives

$$\hat{f}(\eta; p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)p^m, \quad f_m(\eta) = \frac{1}{m!}\frac{\partial^m \hat{f}(\eta; p)}{\partial p^m}\Big|_{p=0}, \tag{10.38}$$

$$\hat{\theta}(\eta; p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)p^m, \quad \theta_m(\eta) = \frac{1}{m!}\frac{\partial^m \hat{\theta}(\eta; p)}{\partial p^m}\Big|_{p=0}, \tag{10.39}$$

$$\hat{\phi}(\eta; p) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta) p^m, \quad \phi_m(\eta) = \frac{1}{m!} \frac{\partial^m \hat{\phi}(\eta; p)}{\partial p^m}\bigg|_{p=0}. \tag{10.40}$$

and when $p = 0$ and $p = 1$ then

$$\hat{f}(\eta; 0) = f_0(\eta), \;\; \hat{f}(\eta; 1) = f(\eta), \tag{10.41}$$

$$\hat{\theta}(\eta; 0) = \theta_0(\eta), \;\; \hat{\theta}(\eta; 1) = \theta(\eta), \tag{10.42}$$

$$\hat{\phi}(\eta; 0) = \phi_0(\eta), \;\; \hat{\phi}(\eta; 1) = \phi(\eta). \tag{10.43}$$

We choose the auxiliary parameters in such a way that the series solutions converge for $p = 1$ and so

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \tag{10.44}$$

$$\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \tag{10.45}$$

$$\phi(\eta) = \phi_0(\eta) + \sum_{m=1}^{\infty} \phi_m(\eta). \tag{10.46}$$

The general solutions (f_m, θ_m, ϕ_m) in terms of special solutions $(f_m^*, \theta_m^*, \phi_m^*)$ can be written as follows

$$f_m(\eta) = f_m^*(\eta) + C_1 + C_2 \exp(\eta) + C_3 \exp(-\eta), \tag{10.47}$$

$$\theta_m(\eta) = \theta_m^*(\eta) + C_4 \exp(\eta) + C_5 \exp(-\eta), \tag{10.48}$$

$$\phi_m(\eta) = \phi_m^*(\eta) + C_6 \exp(\eta) + C_7 \exp(-\eta). \tag{10.49}$$

10.3 Convergence of the homotopy solutions

The convergence analysis of the series solutions depends upon the auxiliary parameters $\hbar_f$, $\hbar_\theta$ and $\hbar_\phi$. Hence the $\hbar$−curves for the 17th order of approximations are plotted. It is found that the admissible ranges of $\hbar_f$, $\hbar_\theta$ and $\hbar_\phi$ are $-1.7 \leq \hbar_f \leq -0.20$, $-1.5 \leq \hbar_\theta \leq -0.25$ and $-1.5 \leq \hbar_\phi \leq -0.40$. The series $(10.44 - 10.46)$ converge in the whole region of η when $\hbar_f = \hbar_\theta = \hbar_\phi = -1.0$ (see Fig. 10.1). Table 10.1 indicates that how much terms for each

physical quantity are required for the convergent solution. It is noticed that less number of terms are required in the convergent expression of velocity.
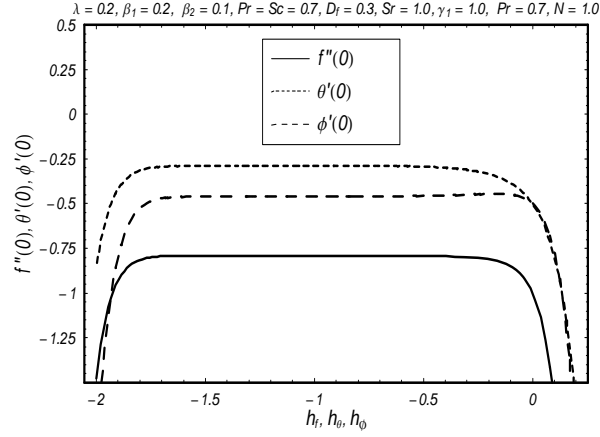
Fig.10.1 : $\hbar$−curves for the function f, θ and ϕ.

Table 10.1: Convergence of homotopy solutions for different order of approximations when $\beta_1 = 0.2$, $\beta_2 = 0.1$, $Pr = Sc = 0.7$, $\gamma_1 = 1.0$, $D_f = 0.3$, $Sr = 0.2$, $N = 1.0$, $\hbar_f = \hbar_\theta = \hbar_\phi = -1.0$.

Order of approximations	$-f''(0)$	$-\theta'(0)$	$-\phi'(0)$
1	0.82500	0.39104	0.43194
5	0.79502	0.30304	0.45255
10	0.79137	0.29030	0.46033
15	0.79129	0.28971	0.46071
20	0.79132	0.28991	0.46044
25	0.79133	0.28993	0.46042
30	0.79133	0.28993	0.46043

10.4 Discussion

Interest in this section is to analyze the variations of different emerging parameters on the physical quantities like temperature, concentration field, local Nusselt and Sherwood numbers. Figs. $10.2 − 10.9$ are displayed to see the variations of mixed convection parameter λ, concentration buoyancy parameter N, Dufour number D_f and Soret number Sr on the temperature of fluid

$\theta(\eta)$. Fig. 10.2 shows that the mixed convection parameter λ decreases the temperature $\theta(\eta)$. Also thermal boundary layer thickness reduces with an increase in mixed convection parameter λ. Fig. 10.3 illustrated that temperature and associated thermal boundary layer thickness are decreasing functions of N .Fig. 10.4 shows that temperature $\theta(\eta)$ and thermal boundary layer thickness increase with an increase in Dufour number D_f. With an increase in Soret number Sr, temperature $\theta(\eta)$ and thermal boundary layer thickness decrease as seen in Fig. 10.5.
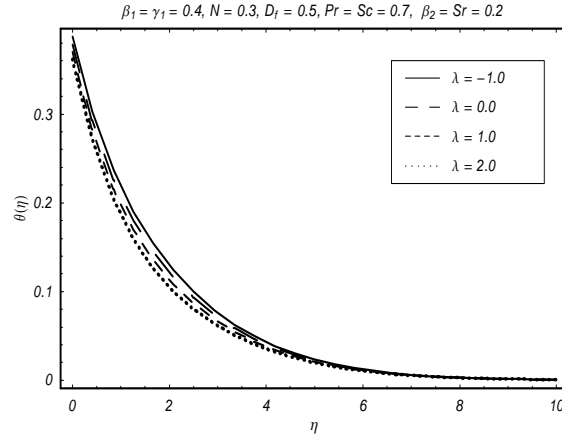
Fig. 10.2: Influence of λ on $\theta(\eta)$.
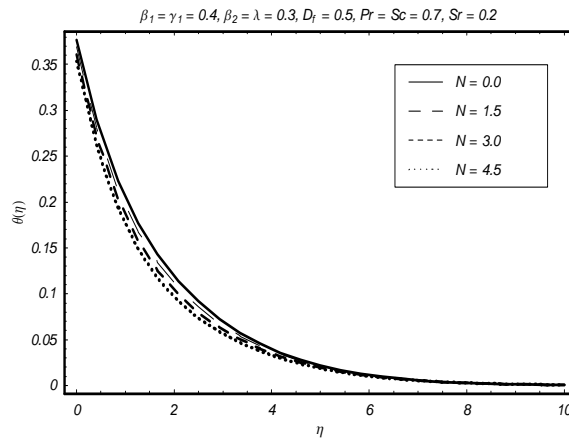
Fig. 10.3: Influence of N on $\theta(\eta)$.
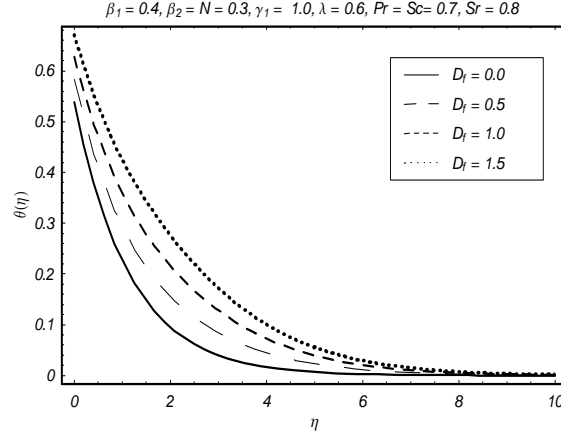
Fig. 10.4: Influence of D_f on $\theta(\eta)$.
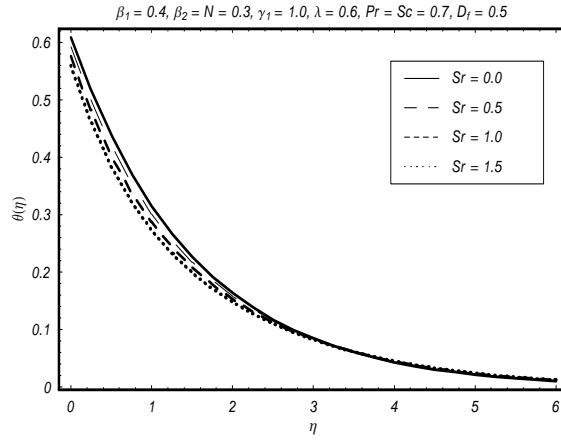
Fig. 10.5: Influence of Sr on $\theta(\eta)$.

Figs. $10.6 - 10.9$ plot the effects of λ, N, D_f and Sr on the concentration field $\phi(\eta)$. From Fig. 10.6 it is found that the concentration field and associated boundary layer thickness show a decrease when mixed convection parameter λ increases. Effects of concentration buoyancy parameter parameter N on the concentration are qualitatively similar to that of temperature (see Figs. 10.3 and 10.7). Figs. 10.4 and 10.8 show that the behaviors of the temperature and concentration profiles are quite opposite in case of Dufour number. This shows that the Dufour number corresponds to weaker concentration and stronger temperature. Fig. 10.9 pointed out

that the larger Soret number has a strong concentration.
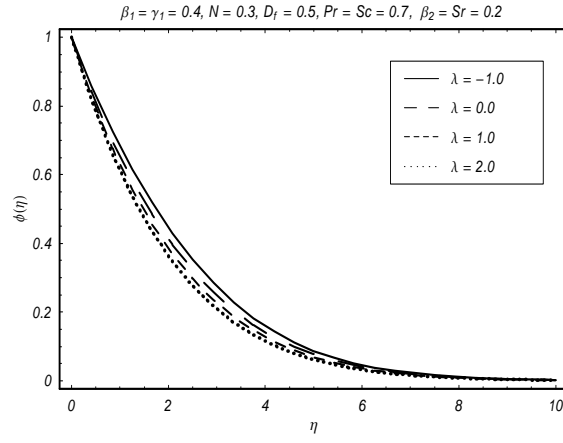
Fig. 10.6: Influence of λ on $\phi(\eta)$.
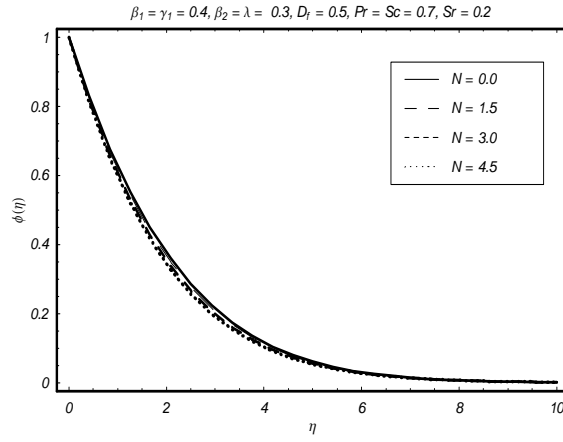
Fig. 10.7: Influence of N on $\phi(\eta)$.
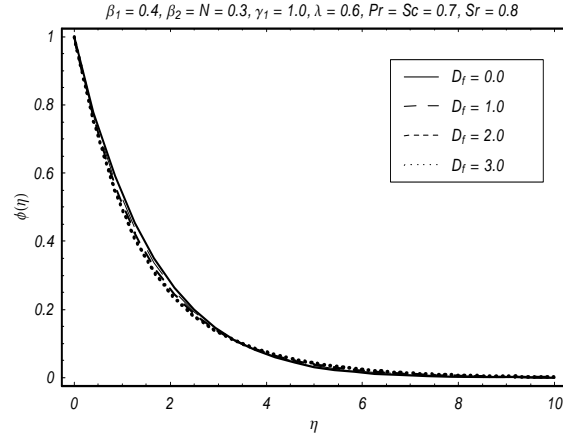
Fig. 10.8: Influence of D_f on $\phi(\eta)$.
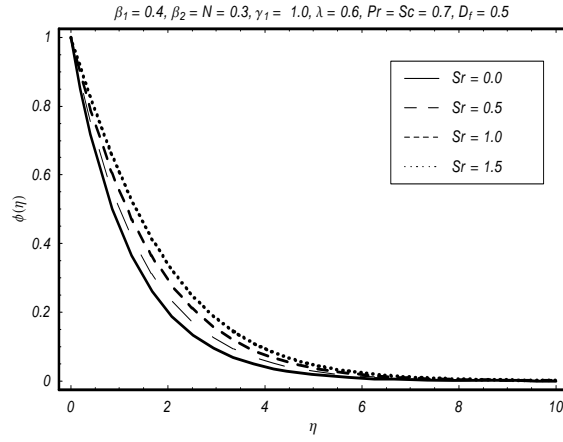
Fig. 10.9: Influence of Sr on $\phi(\eta)$.

Table 10.2: Values of local Nusselt number and Sherwood number for the parameters β_1,

β_2, Pr, Sr, D_f, γ_1 and N when $\lambda = 0.1$.

β_1	β_2	λ	Pr	Sr	D_f	Sc	γ_1	N	$-\theta'(0)$	$-\phi'(0)$
0.0	0.2	0.1	0.7	0.5	0.8	1	1	1	0.29514	0.47225
0.2									0.28993	0.46033
0.4									0.28525	0.44998
0.2	0.0								0.28797	0.45592
	0.2								0.29176	0.46462
	0.4								0.29508	0.47226
0.2	0.2	0.0							0.27062	0.41866
		0.2							0.29176	0.46463
		0.4							0.30305	0.49095
0.2	0.2	0.2	0.8						0.30136	0.50625
			0.9						0.30891	0.54602
			1.0						0.31480	0.58434
			0.7	0.0					0.28679	0.49981
				0.2					0.29176	0.46462
				0.4					0.29679	0.42800
				0.2	0.0				0.33235	0.45353
					0.2				0.30546	0.46096
					0.4				0.27788	0.46830
					0.2	1.0			0.30546	0.46094
						1.2			0.30001	0.51681
						1.4			0.29513	0.56908
						1.0	1.0		0.30546	0.46094
							1.2		0.32396	0.45920
							1.4		0.33863	0.45781
							1.0	1.0	0.30546	0.46094
								1.2	0.30700	0.46444
								1.4	0.30846	0.46779

10.5 Conclusions

Mixed convection flow of an Oldroyd-B fluid is investigated in the presence of convective boundary condition and Soret and Dufour effects. The main observations are pointed out below.

- Effects of λ and N on temperature and concentration fields are similar.

- Effects of Soret number Sr on $\theta(\eta)$ and $\phi(\eta)$ are reverse.

- Thermal boundary layer thickness and temperature field increase when D_f increases.

- There are opposite effects of local Nusselt number and Sherwood number when D_f, Sr and γ_1 increase.

- local Nusselt and Sherwood numbers decrease when β_1 increases.

- With an increase in λ and N both local Nusselt and Sherwood numbers are enhanced.

Chapter 11

Mixed convection Falkner-Skan wedge flow of an Oldroyd-B fluid in presence of thermal radiation

The present chapter examines the Falkner-Skan flow of rate type non-Newtonian fluid. Expressions of an Oldroyd-B fluid in the presence of mixed convection and thermal radiation are used in the development of relevant equations. The resulting partial differential equations are reduced into the ordinary differential equations employing appropriate transformations. Expressions of flow and heat transfer are constructed. Convergence of derived nonsimilar series solutions is guaranteed. Impact of various parameters involved in the flow and heat transfer is plotted and examined.

11.1   Problems development

Let us consider the two-dimensional Falkner-Skan flow of an Oldroyd-B fluid. We further consider the heat transfer. Cartesian coordinates (x, y) are used in such a way that x-axis is parallel to the wall and y-axis normal to it. An incompressible fluid occupies the region $y \geq 0$. The equations governing the present flow situation are based on the conservation laws of mass,

linear momentum and energy. Flow diagram of the problem is as follows:
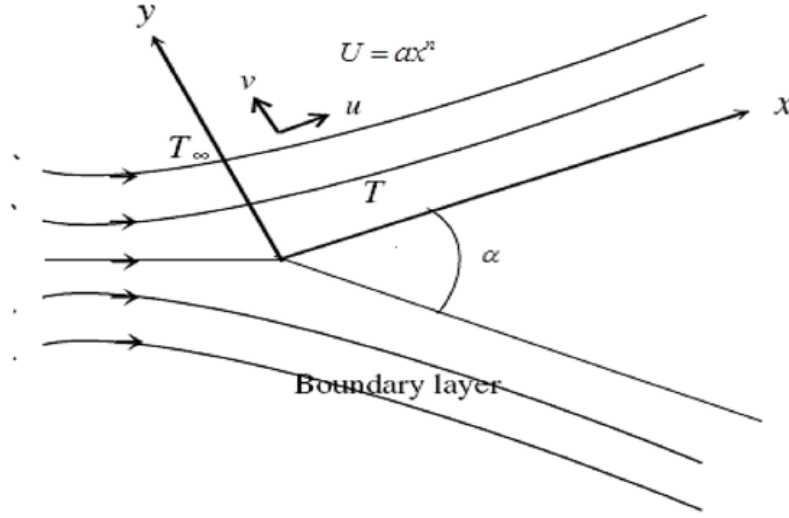
Fig. 11.1: Physical Model

Taking into account the aforementioned assumptions, the resulting boundary layer equations can be written as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{11.1}$$

$$
\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\frac{\partial^2 u}{\partial y^2} - \lambda_1\left[u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + 2uv\frac{\partial^2 u}{\partial x\partial y}\right] \\
&+ \nu\lambda_2\left[u\frac{\partial^3 u}{\partial x\partial y^2} + v\frac{\partial^3 u}{\partial y^3} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}\right] \\
&+ g\beta_T(T - T_\infty)\operatorname{Sin}\frac{\alpha}{2}, \tag{11.2}
\end{aligned}
$$

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\left(\frac{16\sigma_s T_\infty^3}{3k_e} + k\right)\frac{\partial T}{\partial y}\right). \tag{11.3}$$

The appropriate boundary conditions are

$$
\begin{aligned}
u &= U, \quad v = 0, \quad T = T_w = T_\infty + Ax^d \quad \text{at } y = 0, \\
u &\to 0, \quad T \to T_\infty \quad \text{as } y \to \infty \tag{11.4}
\end{aligned}
$$

where $U\,(= ax^n)$ is the free stream velocity, μ is the dynamic viscosity, λ_1 is the relaxation

time, λ_2 is the retardation time, α is the wedge angle, k is the thermal conductivity, d is the surface temperature exponent, T and T_∞ are the temperatures of the fluid and ambient respectively and T_w is the wall temperature. We utilize

$$u = U(x) f', \quad \eta = \sqrt{\frac{n+1}{2}} \sqrt{\frac{U}{\nu x}} y, \quad \psi = \sqrt{\frac{2}{n+1}} \sqrt{\nu x U} f(\eta),$$

$$v = -\sqrt{\frac{n+1}{2}} \sqrt{\frac{\nu U}{x}} \left[f(\eta) + \frac{n-1}{n+1} \eta f'(\eta) \right], \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \tag{11.5}$$

where η is the similarity variable, ψ is the stream function, f is the dimensionless stream function and θ is the dimensionless temperature. Now the continuity equation (11.1) is identically satisfied and Eqs. $(11.2) - (11.4)$ lead to the following expressions

$$f''' + ff'' + \beta_1 \left(-2n \left(\frac{n-1}{n+1} \right) f'^3 + (3n-1) ff'f'' - \left(\frac{n+1}{2} \right) f^2 f''' + \left(\frac{n-1}{2} \right) \eta f'^2 f'' \right)$$

$$+ \beta_2 \left(\left(\frac{3n-1}{2} \right) (f'')^2 - \left(\frac{n+1}{2} \right) ff'''' + (n-1) f' f''' \right) - \frac{2n}{n+1} f'^2 + \frac{2}{n+1} \lambda \theta \sin \frac{\alpha}{2} = 0, \tag{11.6}$$

$$\left(1 + \frac{4}{3} R \right) \theta'' + \Pr(f\theta' - \frac{2d}{n+1} f'\theta) = 0, \tag{11.7}$$

$$f(0) = 0, \ f'(0) = 1, \ \theta(0) = 1, \ f'(\infty) = 0, \ \theta(\infty) = 0. \tag{11.8}$$

Here prime denotes the differentiation with respect to η, β_1 and β_2 are the dimensionless material parameters, λ is mixed convection parameter, Gr_x is the local Grashof number, Pr is the Prandtl number and R is the radiation parameter. The definitions of these parameters are

$$\beta_1 = \frac{\lambda_1 U}{x}, \ \beta_2 = \frac{\lambda_2 U}{x}, \ \lambda = \frac{Gr_x}{\mathrm{Re}_x^2}, \ \Pr = \frac{\mu c_p}{k},$$

$$R = (\frac{4\sigma_s T_\infty^3}{k_e k}), \ Gr_x = \frac{g\beta(T_w - T_\infty)x^3}{v^2}. \tag{11.9}$$

Local Nusselt number (Nu_x) along with heat transfer rate (q_w) are

$$Nu_x = \frac{xq_w}{k(T_w - T_\infty)}, \qquad q_w = -k \left(\frac{\partial T}{\partial y} \right)_{y=0} \tag{11.10}$$

which in dimensionless form gives

$$(\text{Re}_x)^{-1/2} Nu_x = -\theta'(0).$$ (11.11)

11.2   Series solutions

The initial guesses (f_0, θ_0) and auxiliary linear operators $(L_f,\ L_\theta)$ are taken as follows

$$f_0(\eta) = 1 - e^{-\eta}, \qquad \theta_0(\eta) = e^{-\eta},$$ (11.12)

$$\mathcal{L}_f(f) = \frac{d^3 f}{d\eta^3} - \frac{df}{d\eta}, \qquad \mathcal{L}_\theta(\theta) = \frac{d^2\theta}{d\eta^2} - \theta,$$ (11.13)

with

$$\mathcal{L}_f[C_1 + C_2 \exp(\eta) + C_3 \exp(-\eta)] = 0,$$ (11.14)

$$\mathcal{L}_\theta[C_4 \exp(\eta) + C_5 \exp(-\eta)] = 0,$$ (11.15)

where C_i $(i = 1 - 5)$ are the arbitrary constants. If $p \in [0, 1]$ is the embedding parameter and $\hbar_f$ and $\hbar_\theta$ are the non-zero auxiliary parameters then the zeroth-order and mth order deformation problems are stated as follows.

11.2.1   Zeroth order problem

$$(1 - p)\mathcal{L}_f\left[\widehat{f}(\eta; p) - f_0(\eta)\right] = p\hbar_f \mathcal{N}_f\left[\widehat{f}(\eta; p), \widehat{\theta}(\eta; p)\right],$$ (11.16)

$$(1 - p)\mathcal{L}_\theta\left[\widehat{\theta}(\eta; p) - \theta_0(\eta)\right] = p\hbar_\theta \mathcal{N}_\theta\left[\widehat{\theta}(\eta; p), \widehat{f}(\eta; p)\right],$$ (11.17)

$$\widehat{f}(0; p) = 0, \widehat{f}'(0; p) = 0, \widehat{f}'(\infty; p) = 1, \widehat{\theta}(0; q) = 1, \widehat{\theta}(\infty; q) = 0,$$ (11.18)

$$\mathcal{N}_f\left[\widehat{f}(\eta,p),\widehat{\theta}(\eta;p)\right] = \frac{\partial^3 f(\eta;p)}{\partial\eta^3} + f(\eta;p)\frac{\partial^2 f(\eta;p)}{\partial\eta^2} - \frac{2n}{n+1}\left(\frac{\partial f(\eta;p)}{\partial\eta}\right)^2$$

$$+\beta_1\left(-2n\frac{n-1}{n+1}\left(\frac{\partial f(\eta;p)}{\partial\eta}\right)^3 + (3n-1)f(\eta;p)\frac{\partial f(\eta;p)}{\partial\eta}\frac{\partial^2 f(\eta;p)}{\partial\eta^2}\right)$$

$$-\beta_1\left(\frac{n+1}{2}(f(\eta;p))^2\frac{\partial^3 f(\eta;p)}{\partial\eta3} + \frac{n-1}{2}\eta\left(\frac{\partial f(\eta;p)}{\partial\eta}\right)^2\frac{\partial^2 f(\eta;p)}{\partial\eta^2}\right)$$

$$+\beta_2\left(\left(\frac{3n-1}{2}\right)\left(\frac{\partial^2 f(\eta;p)}{\partial\eta^2}\right)^2 - \left(\frac{n+1}{2}\right)f(\eta;p)\frac{\partial^4 f(\eta;p)}{\partial\eta^4}\right)$$

$$+\beta_2\left((n-1)\frac{\partial f(\eta;p)}{\partial\eta}\frac{\partial^3 f(\eta;p)}{\partial\eta^3}\right) + \frac{2}{n+1}\lambda\widehat{\theta}(\eta,p)\sin\frac{\alpha}{2}, \qquad (11.19)$$

$$\mathcal{N}_\theta\left[\widehat{\theta}(\eta;p),\widehat{f}(\eta;p)\right] = \left(1+\frac{4}{3}R\right)\frac{\partial^2\widehat{\theta}(\eta,p)}{\partial\eta^2} + \Pr\widehat{f}(\eta;p)\frac{\partial\widehat{\theta}(\eta;p)}{\partial\eta}$$

$$-\Pr\frac{2d}{n+1}\widehat{\theta}(\eta;p)\frac{\partial\widehat{f}(\eta;p)}{\partial\eta}. \qquad (11.20)$$

11.2.2 mth-order deformation problems

$$\mathcal{L}_f\left[f_m(\eta) - \chi_m f_{m-1}(\eta)\right] = \hbar_f\mathcal{R}_m^f(\eta), \qquad (11.21)$$

$$\mathcal{L}_\theta\left[\theta_m(\eta) - \chi_m\theta_{m-1}(\eta)\right] = \hbar_\theta\mathcal{R}_m^\theta(\eta), \qquad (11.22)$$

$$f_m(0) = f_m'(0) = f_m'(\infty) = f_m''(\infty) = 0,$$
$$\theta_m(0) = \theta_m(\infty) = 0, \qquad (11.23)$$

$$\mathcal{R}_m^f(\eta) = f_{m-1}'''(\eta) + \sum_{k=0}^{m-1}\left[\begin{array}{c} f_{m-1-k}f_k'' - \frac{2n}{n+1}f_{m-1-k}'f_k' \\[6pt] +\beta_1\left(\begin{array}{c} -2n\frac{n-1}{n+1}f_{m-1-k}'\sum_{l=0}^{k}f_{k-l}'f_l' + (3n-1)f_{m-1-k}\sum_{l=0}^{k}f_{k-l}'f_l'' \\[4pt] -\frac{n+1}{2}f_{m-1-k}\sum_{l=0}^{k}f_{k-l}f_l''' + \frac{n-1}{2}\eta f_{m-1-k}'\sum_{l=0}^{k}f_{k-l}'f_l'' \end{array}\right) \\[14pt] +\beta_2\left(\begin{array}{c} \frac{3n-1}{2}f_{m-1-k}''f_k'' \\[4pt] -\frac{n+1}{2}f_{m-1-k}f_k'''' \\[4pt] +(n-1)f_{m-1-k}'f_k''' \end{array}\right) \end{array}\right]$$

$$+\frac{2}{n+1}\lambda\theta_{m-1}\sin\frac{\alpha}{2}, \qquad (11.24)$$

$$\mathcal{R}_m^\theta(\eta) = \theta_{m-1}'' + \Pr \sum_{k=0}^{m-1} \left(\theta_{m-1-k}' f_k - \frac{2d}{n+1} f_{m-1-k}' \theta_k \right), \tag{11.25}$$

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}. \tag{11.26}$$

For $p = 0$ and $p = 1$, we have

$$\widehat{f}(\eta; 0) = f_0(\eta), \quad \widehat{f}(\eta; 1) = f(\eta), \tag{11.27}$$

$$\widehat{\theta}(\eta; 0) = \theta_0(\eta), \quad \widehat{\theta}(\eta; 1) = \theta(\eta), \tag{11.28}$$

and when p increases from 0 to 1 then $\widehat{f}(\eta; p)$ and $\widehat{\theta}(\eta; p)$ vary from the initial solutions $f_0(\eta)$ and $\theta_0(\eta)$ to final solutions $f(\eta)$ and $\theta(\eta)$ respectively. By Taylor's expansion one has

$$\widehat{f}(\eta; p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) p^m, \quad f_m(\eta) = \frac{1}{m!} \frac{\partial^m \widehat{f}(\eta; p)}{\partial p^m} \bigg|_{p=0}, \tag{11.29}$$

$$\widehat{\theta}(\eta; p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) p^m, \quad \theta_m(\eta) = \frac{1}{m!} \frac{\partial^m \widehat{\theta}(\eta; p)}{\partial p^m} \bigg|_{p=0}, \tag{11.30}$$

where the auxiliary parameters are so properly chosen that the series (11.29) and (11.31) converge at $p = 1$ i.e.

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta), \tag{11.31}$$

$$\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta). \tag{11.32}$$

The general solutions of Eqs. (11.32) and (11.33) are

$$f_m(\eta) = f_m^\star(\eta) + C_1 + C_2 e^\eta + C_3 e^{-\eta}, \tag{11.33}$$

$$\theta_m(\eta) = \theta_m^\star(\eta) + C_4 e^\eta + C_5 e^{-\eta}, \tag{11.34}$$

in which $f_m^\star$ and $\theta_m^\star$ are the special solutions.

11.3   Convergence

Note that the series solutions in Eqs. (11.32) and (11.33) contain two auxiliary parameters $\hbar_f$ and $\hbar_\theta$. The convergence of series solutions depend upon these auxiliary parameters. For range of values of these parameters, the $\hbar-$curves at 15th-order of approximations have been plotted in Fig. 11.2. It is found that the admissible values of $\hbar_f$ and $\hbar_\theta$ are $-1.3 \leq \hbar_f \leq -0.25$ and $-1.2 \leq \hbar_\theta \leq -0.5$. Table 11.1 further guarantees that the series solutions are convergent up to five decimal places when $\hbar_f = \hbar_\theta = -0.5$.
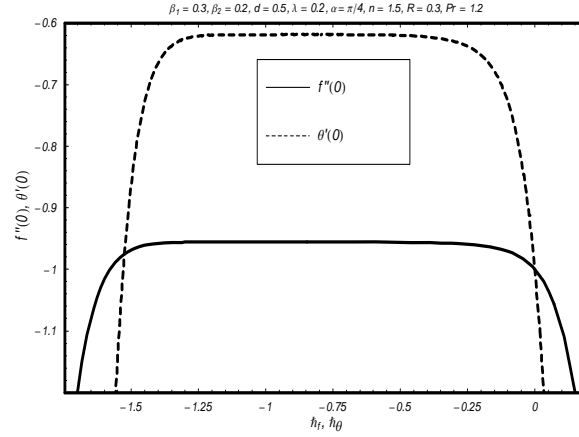
Fig.11.2 : $\hbar-$curves for the function f and θ.

11.4   Discussion

The aim of this subsection is to present the effects of pertinent parameters on the velocity, temperature and surface heat transfer. Variation of parameter n on the velocity and temperature are sketched in the Figs. 11.3 and 11.4. Clearly the effects of n on the velocity and temperature profiles are quite reverse. Influence of mixed convection parameter λ on both the velocity and temperature profiles are given in the Figs. 11.5 and 11.6. It is observed that the velocity and momentum boundary layer thickness increase with the increase of mixed convection parameter λ while the temperature and thermal boundary layer thickness decrease. Figs. 11.7 and 11.8 are drawn to see the variation of α on the velocity and temperature profiles. It is noticed that the velocity and momentum boundary layer thickness increase when α increases. It is found that the temperature and thermal boundary layer thickness are decreasing functions of α. Figs.

11.9 and 11.10 are sketched to see the variation of surface temperature parameter d on the velocity $f'(\eta)$ and the temperature $\theta(\eta)$. Both $f'(\eta)$ and $\theta(\eta)$ decrease with the increase in d. It is also observed that both the momentum and thermal boundary layer thicknesses decrease when d increases.
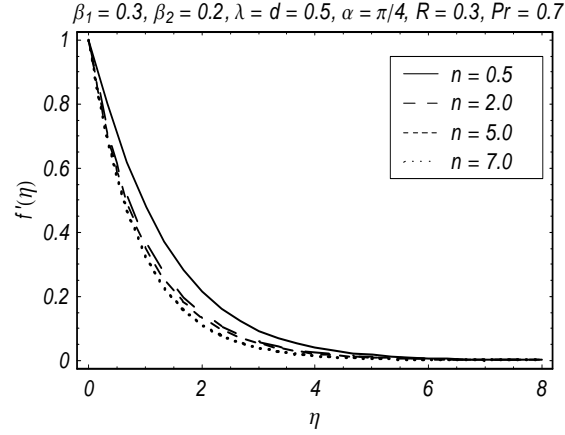
Fig. 11.3: Impact of n on $f\prime(\eta)$.
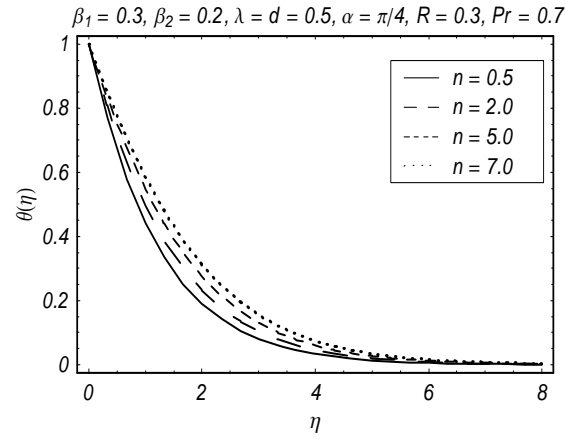
Fig. 11.4: Impact of n on $\theta(\eta)$.
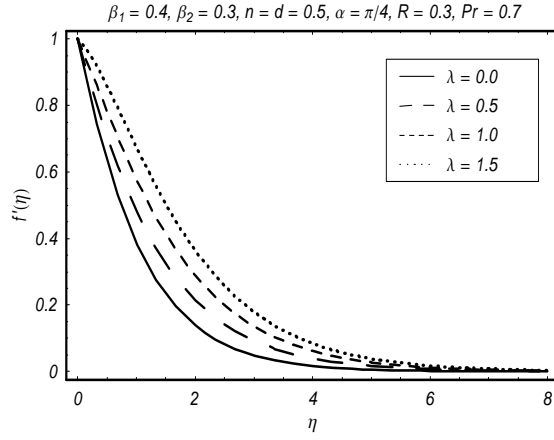
Fig. 11.5: Impact of λ on $f\prime(\eta)$.

Fig. 11.6: Impact of λ on $\theta(\eta)$.

Fig. 11.7: Impact of α on $f\prime(\eta)$.

172

Fig. 11.8: Impact of α on $\theta(\eta)$.

$\beta_1 = 0.4, \beta_2 = 0.3, n = \lambda = \ 0.5, \alpha = \pi/4, R = 0.3, Pr = 0.7$
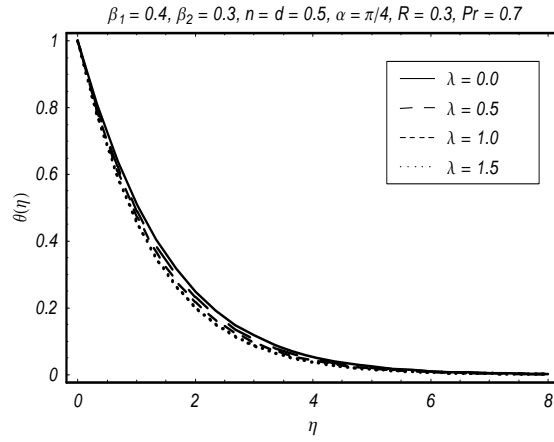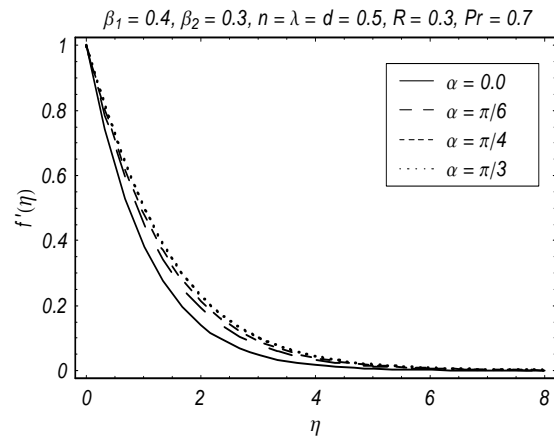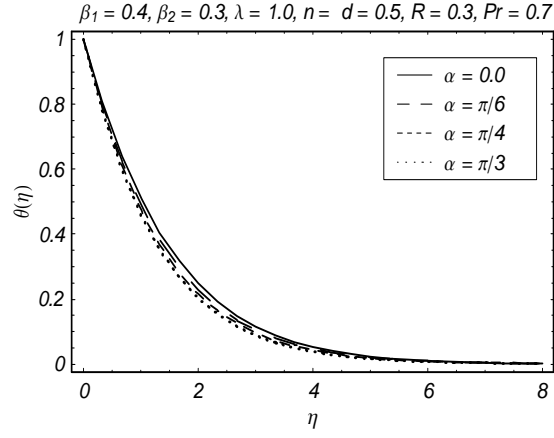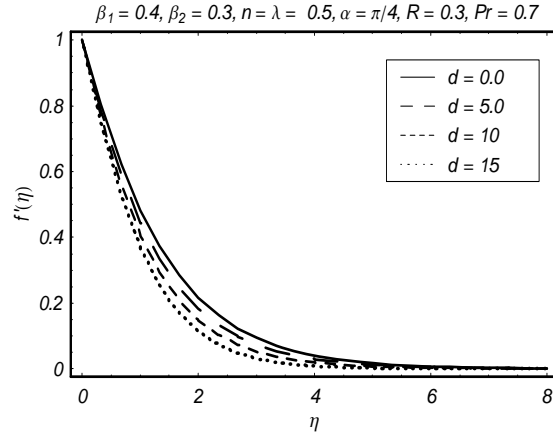
Fig. 11.9: Impact of d on $f'(\eta)$.

$\beta_1 = 0.4, \beta_2 = 0.3, n = \lambda = \ 0.5, \alpha = \pi/4, R = 0.3, Pr = 0.7$
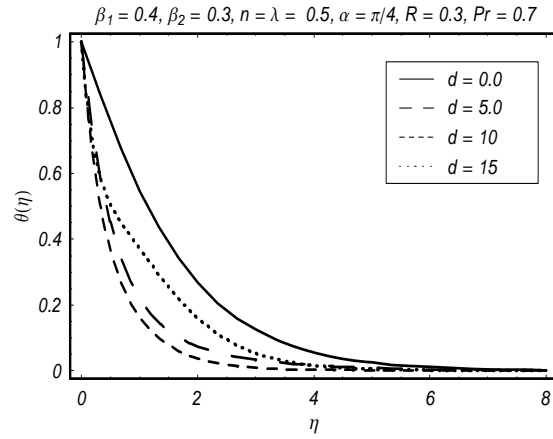
Fig. 11.10: Impact of d on $\theta(\eta)$.

Figs. 11.11 and 11.12 are drawn to see the influence of mixed convection parameter λ, wedge angle α, surface temperature parameter d and velocity index n on the local Nusselt number $-\theta'(0)$. Fig. 11.11 depicts the effects of λ and α on $-\theta'(0)$. It is noticed that $-\theta'(0)$ increases through increase of mixed convection parameter λ and wedge angle α. Fig. 11.12 depicts that variations of n and d have opposite effects on $-\theta'(0)$. A close look at Table 11.1 indicates that 25$^{\text{th}}$-order approximation gives convergent series solutions.
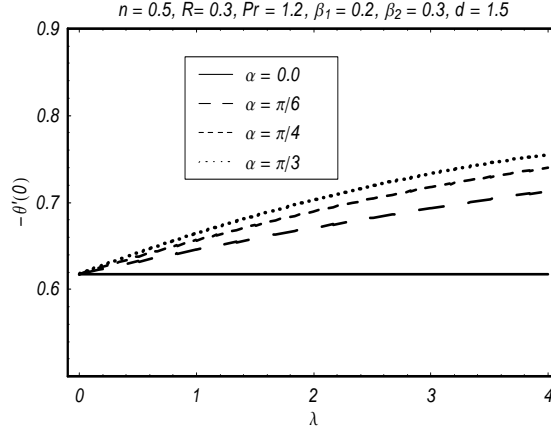
Fig. 11.11: Impacts of λ and α on $-\theta'(0)$.
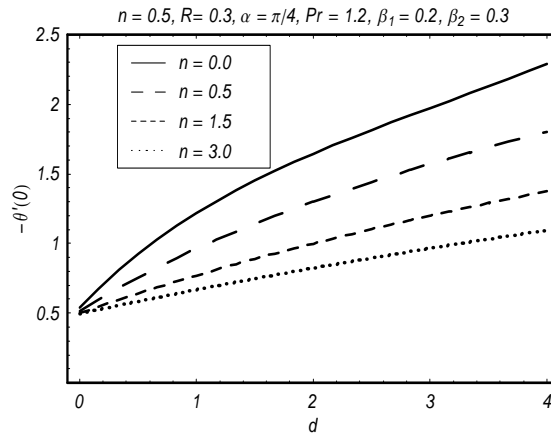
Fig. 11.12: Impacts of d and n on $-\theta'(0)$.

Table 11.1 : Convergence of the homotopy solutions for different order of approximation when Pr $= 1.0$, $\lambda = 0.3$, $n = 1.5$, $d = 0.5$, $R = 0.3$, $\alpha = \pi/4$, $\beta_1 = 0.2$, $\beta_2 = 0.3$ and

$\hbar_f = \hbar_\theta = -0.5.$

Order of approximation	$-f'''(0)$	$-\theta'(0)$
1	0.90330	0.78889
5	0.86304	0.64042
10	0.86077	0.62757
15	0.86047	0.62627
20	0.86045	0.62616
25	0.86044	0.62616
30	0.86044	0.62616

11.5   Conclusions

Mixed convection and thermal radiation effects in the Falkner-Skan wedge flow of an Oldroyd-B fluid are investigated. The following points are worth mentioning:

- Table 11.1 shows that convergence of the functions f and θ are obtained at 25th-order approximations up to five decimal places when $\hbar_f = \hbar_\theta = -0.5$.

- Influence of mixed convection parameter λ increases the velocity and momentum boundary layer thickness while it decreases the temperature and thermal boundary layer thickness.

- Influence of wedge angle α and radiation parameter R on both the temperature and velocity profiles are quite similar.

- Thermal boundary layer and momentum boundary layer thicknesses are decreasing functions of surface temperature exponent d.

- Surface heat transfer $-\theta'(0)$ increases with an increase of wedge angle α, mixed convection parameter λ and surface temperature exponent d .

Bibliography

[1] H. Schlichting, Boundary layer theory, 6th ed. Mc-Graw Hill, New York (1964).

[2] B.C. Sakiadis, Boundary layer behavior on continuous solid surfaces: II Boundary on a continuous flat surface, AIChE J. 7 (1961) 221-225.

[3] L.J. Crane, Flow past a stretching plate, Z. Angew. Math. Phys. 21 (1970) 645-647.

[4] A. Chakrabarti and A. S. Gupta, Hydromagnetic flow and heat transfer over a stretching sheet, Quarterly Appl. Math. 37 (1979) 73-78.

[5] C. K. Chen and M. I. Char, Heat transfer of a continuous stretching surface with suction or blowing, J. Math. Anal. Appl. 135 (1988) 568–580.

[6] K. Vajravelu and A. Hadjinicolaou, Heat transfer in a viscous fluid over a stretching sheet with viscous dissipation and internal heat generation, Int. Commun. Heat Mass Transf. 20 (1993) 417-430.

[7] H.I. Andersson, O.R. Hansen and B. Holmedal, Diffusion of a chemically reactive species from a stretching sheet, Int. J. Heat Mass Transf. 37 (1994) 659–664.

[8] W. H. H. Banks, Similarity solutions of the boundary-layer equations for a stretching wall, J. Méch. Théor. Appl. 2 (1983) 375–392.

[9] T. G. Fang, J. Zhang and S. Yao, Slip magnetohydrodynamic viscous flow over a permeable shrinking sheet, Chin. Phys. Lett. 27 (2010) 124702.

[10] S. Mukhopadhyay, G. C. Layek and S. A. Samad, Study of MHD boundary layer flow over a heated stretching sheet with variable viscosity, Int. J. Heat Mass Transf. 48 (2005) 4460–4466.

[11] C.Y. Wang, The three-dimensional flow due to a stretching flat surface, Phys. Fluid 27 (1984) 1915–1917.

[12] C. D. S. Devi, H. S. Takhar and G. Nath, Unsteady three-dimensional, boundary-layer flow due to a stretching surface, lnt. J. Heat Mass Transf. 29 (1986) 1996-1999.

[13] P. D. Ariel, The three-dimensional flow past a stretching sheet and the homotopy perturbation method, Computers Mathematics Appl. 54 (2007) 920–925.

[14] P. D. Ariel, On computation of the three dimensional flow past a stretching sheet, Appl. Math. Comput. 188 (2007) 1244-1250.

[15] T. Hayat and T. Javed, On analytic solution for generalized three-dimensional MHD flow over a porous stretching sheet, Phys. Lett. A 370 (2007) 243–250.

[16] M. Kumari and G. Nath, Analytic solution of unsteady three-dimensional MHD boundary layer flow and heat transfer due to impulsively stretched plane surface, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 3339-3350.

[17] K. Vajravelu, Viscous flow over a nolinear stretching sheet, Appl. Math. Comput. 124 (2001) 281–288.

[18] R. Cortell, Effects of viscous dissipation and radiation on the thermal boundary layer over a nonlinearly stretching sheet, Phys. Lett. A 372 (2008) 631–636.

[19] T. Hayat, T. Javed and Z. Abbas, MHD flow of a micropolar fluid near a stagnation-point towards a non-linear stretching surface, Nonlinear Analysis: Real World Appl. 10 (2009) 1514–1526.

[20] N. Afzal, Momentum and thermal boundary layers over a two-dimensional or axisymmetric non-linear stretching surface in a stationary fluid, Int. J. Heat Mass Transf. 53 (2010) 540-547.

[21] P. S. Gupta and A. S. Gupta, Heat and mass transfer on a stretching sheet with suction or blowing, Can. J. Chem. Eng. 55 (1977) 744-746.

[22] E. Magyari and B. Keller, Heat and mass transfer in the boundary layers on an exponentially stretching continuous surface, J. Phys. D: Appl. Phys. 32 (1999) 577—585.

[23] E. M. A. Elbashbeshy, Heat transfer over an exponentially stretching continuous surface with suction, Arch. Mech. 53 (2001) 643-651.

[24] M. Q. Al-Odat, R. A. Damesh and T. A. Al-Azab, Thermal boundary layer on an exponentially stretching continuous surface in the presence of magnetic field, Int. J. Appl. Mech. Eng. 11 (2006) 289-299.

[25] I. C. Liu, H. H. Wang and Y. F. Peng, Flow and heat transfer for three dimensional flow over an exponentially stretching surface, Chem. Eng. Commun. 200 (2013) 253-268.

[26] B. S. Dandapat and A. S. Gupta, Flow and heat transfer in a viscoelastic fluid over a stretching sheet, Int. J. Non-Linear Mech. 24 (1989) 215-219.

[27] C. K. Chen, M. I. Char and J. W. Cleaver, Temperature field in non-Newtonian flow over a stretching plate, J. Math. Anal. Appl. 151 (1990) 301-307.

[28] K. Vajravelu and D. Rollins, Heat transfer in a viscoelastic fluid over a stretching sheet, J. Math. Anal. Appl. 158 (1991) 241–255.

[29] T. Hayat, M. Sajid and I. Pop, Three-dimensional flow over a stretching surface in a viscoelastic fluid, Nonlinear Analysis: Real World Appl. 9 (2008) 1811 – 1822.

[30] I. C. Liu, Flow and heat transfer of an electrically conducting fluid of second grade over a stretching sheet subject to a transverse magnetic field, Int. J. Heat Mass Transf. 47 (2004) 4427–4437.

[31] M. M. Nandeppanavar, M. S. Abel and K. Vajravelu, Flow and heat transfer characteristics of a viscoelastic fluid in a porous medium over an impermeable stretching sheet with viscous dissipation, Int. J. Heat Mass Transf. 53 (2010) 4707–4713.

[32] J. J. Choi, Z. Rusak and J. A. Tichy, Maxwell fluid suction flow in a channel, J. Non-Newtonian Fluid Mech. 85 (1999) 165.

[33] K. Sadeghy, A. H. Najafi and M. Saffaripour, Sakiadis flow of an upper-convected Maxwell fluid, Int. J. Non-Linear Mech. 40 (2005) 1220 – 1228.

[34] A. A. Pahlavan, V. Aliakbar, F. V. Farahani and K. Sadeghy, MHD flows of UCM fluids above porous stretching sheets using two-auxiliary-parameter homotopy analysis method, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 473–488.

[35] V. Aliakbar, A. A. Pahlavan and K. Sadeghy, The influence of thermal radiation on MHD flow of Maxwellian fluids above stretching sheets, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 779–794.

[36] C. Fetecau, M. Athar and C. Fetecau, Unsteady flow of a genealized Maxwell fluid with fractional derivative due to a constantly accelerating plate, Comput. Math. Appl. 57 (2009) 596-603.

[37] M. Kumari and G. Nath, Steady mixed convection stagnation-point flow of upper convected Maxwell fluids with magnetic field, Int. J. Non-Linear Mech. 44 (2009) 1048–1055.

[38] M. S. Abel, J. V. Tawade and M. M. Nandeppanavar, MHD flow and heat transfer for the upper-convected Maxwell fluid over a stretching sheet, Meccanica 47 (2012) 385—393.

[39] T. Hayat, S. A. Shehzad and A. Alsaedi, Study on three-dimensional flow of Maxwell fluid over a stretching surface with convective boundary conditions, Int. J. Phys. Sci. 7 (2012) 761 - 768.

[40] R. K. Bhatanagar, G. Gupta and K. R. Rajagopal, Flow of an Oldroyd-B fluid due to stretching sheet in the presence of a free stream velocity, Int. J. Non-Linear Mech. 30 (1995) 391-405.

[41] K.R. Rajagopal, On an exact solution for the flow of an Oldroyd-B fluid, Bull. Tech. Univ. Istanbul 49 (1996) 617-623.

[42] T. Hayat, K. Hutter, S. Asghar and A. M. Siddiqui, MHD Flows of an Oldroyd-B Fluid, Math. Comput. Modell. 36 (2002) 987-995.

[43] M. D. Smith, Y. L. Joo, R. C. Armstrong and R. A. Brown, Linear stability analysis of flow of an Oldroyd-B fluid through a linear array of cylinders, J. Non-Newtonian Fluid Mech. 109 (2003) 13–50.

[44] S. Asghar, S. Parveen, S. Hanif, A. M. Siddiqui and T. Hayat, Hall effects on the unsteady hydromagnetic flows of an Oldroyd-B fluid, Int. J. Eng. Sci. 41 (2003) 609–619.

[45] C. Fetecau and C. Fetecau, Unsteady flows of Oldroyd-B fluids in a channel of rectangular cross-section, Int. J. Non-Linear Mech. 40 (2005) 1214 – 1219.

[46] J. Niu, C. Fu and W. Tan, Stability of thermal convection of an Oldroyd-B fluid in a porous medium with Newtonian heating, Phys. Lett. A 374 (2010) 4607-4613.

[47] J. Mewis, Thixotropy – a general review. J. Non-Newtonian Fluid Mech. 6 (1979) 1–20.

[48] H. A. Barnes, Thixotropy – a review, J. Non-Newtonian Fluid Mech. 70 (1997) 1-33.

[49] J. Harris, A continuum theory of time-dependent inelastic flow, Rheol. Acta 6 (1967) 6.

[50] J. Harris, Rheology and non-Newtonian flow, London: Longman (1977).

[51] S. Sadeqi, N. Khabazi and K. Sadeghy, Blasius flow of thixotropic fluids: A numerical study, Commun. Nonlinear Sci. Numer. Simulat. 16 (2011) 711–721.

[52] J. J. M. Sillekens, C. C. M. Rindt and A. A. V. Steenhoven, Development of laminar mixed convection in a horizontal square channel with heated side walls, Int. J. Heat and Fluid Flow 19 (1998) 270-281.

[53] G.P. Celata, A. Chiaradia, M. Cumo and F. D. Annibale, Heat transfer enhancement by air injection in upward heated mixed-convection flow of water, Int. J. Multiphase Flow 25 (1999) 1033-1052.

[54] A. Barletta, Analysis of flow reversal for laminar mixed convection in a rectangular duct with one or more isothermal walls, Int. J. Heat Mass Transf. 44 (2001) 3481-3497.

[55] E. Magyari, I. Pop and B. Keller, Mixed convection boundary layer flow past a horizontal permeable flat plate, Fluid Dynamics Research 31 (2002) 215-225.

[56] R. Nazar, N. Amin and I. Pop, Unsteady mixed convection boundary layer flow near the stagnation point on a vertical surface in a porous medium, Int. J. Heat Mass Transf. 47 (2004) 2681–2688.

[57] M. A. Hossain, S. Bhowmick and R. S. R. Gorla, Unsteady mixed-convection boundary layer flow along a symmetric wedge with variable surface temperature, Int. J. Eng. Sci. 44 (2006) 607–620.

[58] H. A. Mohammed and Y. K. Salman, Experimental investigation of mixed convection heat transfer for thermally developing flow in a horizontal circular cylinder, Appl. Therm. Eng. 27 (2007) 1522–1533.

[59] H. A. Mohammed, Laminar mixed convection heat transfer in a vertical circular tube under buoyancy-assisted and opposed flows, Energy Conv. Manag. 49 (2008) 2006–2015.

[60] M. Kotouc, G. Bouchet and J. Dusek, Loss of axisymmetry in the mixed convection, assisting flow past a heated sphere, Int. J. Heat Mass Transf. 51 (2008) 2686–2700.

[61] M. Mahmood, S. Asghar and M. A. Hossain, Transient mixed convection flow arising due to thermal diffusion over a porous sensor surface inside a squeezing horizontal channel, Int. J. Therm. Sci. 48 (2009) 1619–1626.

[62] T. Hayat, M. Mustafa and I. Pop, Heat and mass transfer for Soret and Dufour's effect on mixed convection boundary layer flow over a stretching vertical surface in a porous medium filled with a viscoelastic fluid, Commun. Nonlinear Sci. Numer. Simulat. 15 (2010) 1183–1196.

[63] A. J. Chamkha, Thermal radiation and buoyancy effects on hydromagnetic flow over an accelerating permeable surface with heat source or sink, Int. J. Eng. Sci. 38 (2000) 1699-1712.

[64] S. Abel, K. V. Prasad and A. Mahaboob, Buoyancy force and thermal radiation effects in MHD boundary layer visco-elastic fluid flow over continuously moving stretching surface, Int. J. Therm. Sci. 44 (2005) 465–476.

[65] S. Mukhopadhyay and G.C. Layek, Effects of thermal radiation and variable fluid viscosity on free convective flow and heat transfer past a porous stretching surface, Int. J. Heat Mass Transf. 51 (2008) 2167–2178.

[66] S. Mukhopadhyay, Effect of thermal radiation on unsteady mixed convection flow and heat transfer over a porous stretching surface in porous medium, Int. J. Heat Mass Transf. 52 (2009) 3261–3265.

[67] C. H. Chen, Magnetohydrodynamic mixed convection of a power-law fluid past a stretching surface in the presence of thermal radiation and internal heat generation/absorption, Int. J. Non-Linear Mech. 44 (2009) 596–603.

[68] K. Vajravellu and D. Rollins, Heat transfer in an electrically conducting fluid over a stretching surface, Int. J. Non-Linear Mech. 27 (1992) 265-277.

[69] M. S. Abel and M. M. Nandeppanavar, Heat transfer in MHD viscoelastic boundary layer flow over a stretching sheet with non-uniform heat source/sink, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 2120–2131.

[70] O. A. Beg, J. Zueco, R. Bhargava and H. S. Takhar, Magnetohydrodynamic convection flow from a sphere to a non-Darcian porous medium with heat generation or absorption effects: network simulation, Int. J. Therm. Sci. 48 (2009) 913–921.

[71] R. Kandasamy, K. Periasamy and K. K. S. Prabhu, Chemical reaction, heat and mass transfer on MHD flow over a vertical stretching surface with heat source and thermal stratification effects, Int. J. Heat Mass Transf. 48 (2005) 4557–4561.

[72] M. A. Mansour, N. F. E. Anssary and A. M. Aly, Effects of chemical reaction and thermal stratification on MHD free convective heat and mass transfer over a vertical stretching surface embedded in a porous media considering Soret and Dufour numbers, Chem. Eng. J. 145 (2008) 340–345.

[73] D. Pal and B. Talukdar, Buoyancy and chemical reaction effects on MHD mixed convection heat and mass transfer in a porous medium with thermal radiation and Ohmic heating, Commun. Nonlinear Sci. Numer. Simulat. 15 (2010) 2878–2893.

[74] G. Singh, P. R. Shrma and A. J. Chamkha, Mass transfer with chemical reaction in MHD mixed convective flow along a vertical stretching sheet, Int. J. Energy Tech. 4 (2012) 1-12.

[75] S. J. Liao, Homotopy Analysis Mathod: A new analytic technique for nonlinear problems, Commun. Nonlinear Sci. Numer. Simulat. 2 (1997) 95-100.

[76] S. J. Liao, Beyond perturbation: introduction to Homotopy Analysis Method, Chapman & Hall, Boca Raton (2003).

[77] S. Abbasbandy, Approximate solution for the nonlinear model of diffusion and reaction in porous catalysts by means of the homotopy analysis method, Chem. Eng. J. 136 (2008) 144–150.

[78] M. Sajid and T. Hayat, The application of homotopy analysis method to thin film flows of a third order fluid, Chaos Solitons Fractals 38 (2008) 506–515.

[79] S. J. Liao, A short review on the homotopy analysis method in fluid mechanics, J. Hydrodyn. 22 (2010) 882-884.

[80] M. Turkyilmazoglu, A note on the homotopy analysis method, Appl. Math. Lett. 23 (2010) 1226-1230.

[81] S. J. Liao, Homotopy analysis method in nonlinear differential equations, Higher Edu. Press, Beijing and Springer-Verlag Berlin Heidelberg (2012).

[82] S. Abbasbandy, E. Shivanian and K. Vajravelu, Mathematical properties of image-curve in the frame work of the homotopy analysis method, Commun. Nonlinear Sci. Numer. Simulat. 16 (2011) 4268-4275.

[83] M. M. Rashidi, S. A. M. Pour, T. Hayat and S. Obaidat, Analytic approximate solutions for steady flow over a rotating disk in porous medium with heat transfer by homotopy analysis method, Comput. Fluids 54 (2012) 1-9.

[84] I. Ahmad, M. Ahmed, Z. Abbas and M. Sajid, Hydromagnetic flow and heat transfer over a bidirectional stretching surface in a porous medium, Thermal Sci. 15 (2011) S205-S220.

[85] M. Turkyilmazoglu, Series solution of nonlinear two-point singularly perturbed boundary layer problems, Comput. Math. Appl. 60 (2010) 2109-2114.

Turnitin Originality Report

<u>PHD Final Thesis</u> by Bilal Ashraf

From Bilal (8899)

    Processed on 03-Jul-2014 16:44 PKT
    ID: 437813038
    Word Count: 37678

Similarity Index	Similarity by Source	
14%	Internet Sources:	3%
	Publications:	13%
	Student Papers:	0%

---

sources:

---

1 — 2% match (publications)

International Journal of Numerical Methods for Heat & Fluid Flow, Volume 24, Issue 2 (2014-03-28)

---

2 — 2% match (publications)

Hayat, T., S. A. Shehzad, and A. Alsaedi. "MHD Three-Dimensional Flow by an Exponentially Stretching Surface with Convective Boundary Condition", Journal of Aerospace Engineering, 2014.

---

3 — 2% match (publications)

International Journal of Numerical Methods for Heat & Fluid Flow, Volume 23, Issue 7 (2013-09-07)

---

4 — 1% match (publications)

International Journal of Numerical Methods for Heat & Fluid Flow, Volume 24, Issue 2 (2014-03-28)

---

5 — 1% match (publications)

Sadegi, S.. "Blasius flow of thixotropic fluids: A numerical study", Communications in Nonlinear Science and Numerical Simulation, 201102

---

6 — 1% match (publications)

Shateyi, Stanford Motsa, Sandile Sydney . "The effects of thermal radiation, hall currents, Soret, and Dufour on MHD flow by mixed convection o", Mathematical Problems in Engineering, Annual 2010 Issue

---

7 — 1% match (publications)

International Journal of Numerical Methods for Heat & Fluid Flow, Volume 24, Issue 2 (2014-03-28)

---

8 — < 1% match (publications)

Poornima, T. and Reddy, N. Bhaskar. "EFFECTS OF THERMAL RADIATION AND CHEMICAL REACTION ON MHD FREE CONVECTIVE FLOW PAST A SEMI-INFINITE VERTICAL POROUS MOVING PLATE", International Journal of Applied Mathematics & Mechanics, 2013.

---

9 < 1% match (Internet from 15-May-2009)
http://hal.physast.uga.edu/~jss/1120L/data/RCJ/orionstars.fit

10 < 1% match (Internet from 29-Aug-2008)
http://hal.physast.uga.edu/~jss/1120L/data/CCJD/Lab%20%232/Jupiter25sempty.fit

11 < 1% match (publications)
Liu, I.C.. "Flow and heat transfer of an electrically conducting fluid of second grade over a stretching sheet subject to a transverse magnetic field", International Journal of Heat and Mass Transfer, 200409

12 < 1% match (Internet from 07-Feb-2012)
http://www.rckazan.ru/upload/blank/zayav_o_vidach_pasp.rtf

13 < 1% match (Internet from 23-Jan-2013)
http://thermalscience.vinca.rs/pdfs/papers-2012/ThSci-46PQYKBUBK.pdf

14 < 1% match (Internet from 07-Aug-2013)
http://journal-enertech.eu/papers-archive/doc_download/31-paper-26-2010-17.html

15 < 1% match (publications)
Mahmoud, Mostafa A.A.. "A note on variable viscosity and chemical reaction effects on mixed convection heat and mass transfe", Mathematical Problems in Engineering, Annual 2007 Issue

16 < 1% match (publications)
Shateyi, Stanford Motsa, Sandile Sydney. "Thermal radiation effects on heat and mass transfer over an unsteady stretching surface.(Research Ar", Mathematical Problems in Engineering, Annual 2009 Issue

17 < 1% match (Internet from 01-Nov-2013)
http://permraion.ru/_res/fs/file11785.wmf

18 < 1% match (Internet from 09-May-2013)
http://www.cobrahealth.com/COBRAalternativenotice.rtf

19 < 1% match (publications)
Odat, Mohammed Q. Al- Azab, Tariq A. Al-. "Transient MHD double-diffusive natural convection over a vertical surface embedded in a non-darcy po", Mathematical Problems in Engineering, Annual 2009 Issue

20 < 1% match (publications)
Wong, Sin Wei, M. A. Omar Awang, Anuar Ishak, and Ioan Pop. "Boundary Layer Flow and Heat Transfer over an Exponentially Stretching/Shrinking Permeable Sheet with Viscous

Dissipation", Journal of Aerospace Engineering, 2014.

---

21   < 1% match (publications)

Mahmoud, Mostafa Waheed, Shimaa. "Effects of slip and heat generation/absorption on MHD mixed convection flow of a micropolar fluid ov", Mathematical Problems in Engineering, Annual 2010 Issue

---

22   < 1% match (publications)

Mahmoud, Mostafa A. A. and Waheed, Shimaa E.. "Heat and Mass Transfer for a Maxwell Fluid Flow Near a Stagnation Point with Variable Wall Heat and Mass Flux in the Presence of Heat Generation or Absorption", International Review of Physics, 2013.

---

23   < 1% match (publications)

Hsiao, Kai-Long Chen, Guan-Bang. "Conjugate heat transfer of mixed convection for viscoelastic fluid past a stretching sheet.(Research", Mathematical Problems in Engineering, Annual 2007 Issue

---

24   < 1% match (Internet from 21-May-2012)

http://q3.atdan.net/pub/maps/defrag/lick-sux.pk3

---

25   < 1% match (Internet from 10-Nov-2005)

http://unseen.free.fr/photoshop/brush/vered_useful_brush.abr

---

26   < 1% match (publications)

International Journal of Numerical Methods for Heat & Fluid Flow, Volume 23, Issue 7 (2013-09-07)

---

27   < 1% match (publications)

Aziz, Mohamed Abd El-. "The effects of variable fluid properties and viscous dissipation on forced convection of viscoelasti", Canadian Journal of Physics, August 2010 Issue

---

28   < 1% match (publications)

Ratkiewicz, Bogusław(Cieśliński, Jan Leszek). "Dyskretyzacja niektórych modeli fizycznych: od podejścia standardowego do dyskretyzacji geometrycznej", Repozytorium Uniwersytetu im Adama Mickiewicza AMUR, 2011.

---

29   < 1% match (publications)

Mdzinarishvili, Tengiz and Sherman, Simon. "Basic Equations and Computing Procedures for Frailty Modeling of Carcinogenesis: Application to Pancreatic Cancer Data", Cancer Informatics, 2013.

---

30   < 1% match (Internet from 20-Oct-2010)

http://data.astronomycamp.org/2001/SpringAdult/raw/camp_0401_0027.fits

---

31   < 1% match (publications)

Farhani, Sahbi and Rejeb, Jaleleddine Ben. "Link between Economic Growth and Energy

Consumption in Over 90 Countries", Interdisciplinary Journal of Contemporary Research in Business, 2012.

---

32 < 1% match (Internet from 20-Jun-2011)

http://data.astronomycamp.org/2005/AdvancedTeen/PrimeFocus/01_05JUL01_S03251_0004.arch

---

33 < 1% match (Internet from 26-Nov-2005)
http://pc-isa2.dacya.ucm.es/controldigitalFI/RespuesTemporal.pdf

---

34 < 1% match (publications)
Kimiaeifar, A.; Bagheri, G. H.; Barari, A.; Arabsolghar, A. R. and Rahimpour, M.. "On the Analytical Solution of Non-Orthogonal Stagnation Point Flow towards a Stretching Sheet", International Journal of Applied Mathematics, 2011.

---

35 < 1% match (publications)
Trocha, Piotr(Barnaś, Józef). "Wpływ efektów interferencyjnych i korelacji kulombowskich na transport elektronowy przez układy kropek kwantowych", Repozytorium Uniwersytetu im Adama Mickiewicza AMUR, 2011.

---

36 < 1% match (publications)
Kapcia, Konrad, Jerzy(Robaszkiewicz, Stanisław). "Uporządkowania elektronowe i ich separacje w rozszerzonych modelach Hubbarda", Repozytorium Uniwersytetu im Adama Mickiewicza AMUR, 2014.

---

37 < 1% match (publications)
Sveinsson, Oli G. B., Jose D. Salas, and Duane C. Boes. "Regional Frequency Analysis of Extreme Precipitation in Northeastern Colorado and Fort Collins Flood of 1997", Journal of Hydrologic Engineering, 2002.

---

paper text:

Preface

> 14**The boundary layer flows over a** moving **surface have**

vital importance due to their ever increasing usage in the industries. In such industrial processes, the kinematics of stretching and heat transfer through rate of cooling have substantial impact in the improvement of final product of better quality. No doubt, the thermal buoyancy force arising due to cooling or heating of a moving surface may alter significantly the flow and thermal fields and thereby the heat transfer behavior in the manufacturing process. In several practical applications, the order of magnitudes of buoyancy and viscous forces are comparable for moderate flow velocities and large surface temperature differences and convective heat transfer process is thus called as mixed convection. The buoyancy forces due to temperature and concentration differences are significant in mixed convection thermal and concentration diffusions. In fact the buoyancy forces causing a pressure gradient in the boundary layer modify

8the velocity, temperature and concentration distributions and consequently the rate of

heat and mass transfer between the surface and fluid. Specifically the mixed convection flows are encountered in industrial processes like solar central receivers exposed to the wind currents, nuclear reactors called during emergency shutdown, electronic devices cooled by fans and heat exchangers etc. The mixed convection flows with heat and mass transfer are relevant to energy related engineering problems that include both metal and polymer sheets. Mostly, the fluids in industrial processes are non-Newtonian. Certain oils, paints, blood at low shear rate, shampoos, cosmetic products body fluids, pasta, ice cream, ice, mud etc are few examples of non-Newtonian fluids. Keeping all the aforementioned facts in mind, the present thesis is structured as follows. Chapter one covers literature survey and laws of conservation of mass, linear momentum and energy. Boundary layer equations of second grade, Maxwell, Oldroyd-B and thixotropic fluids are presented. Basic idea of homotopy analysis method is also given. Chapter two deliberates the mixed convection

4boundary layer flow of thixotropic fluid with thermophoresis over a stretched sheet. Fluid is electrically conducting in the

presence of constant applied

6magnetic field. Heat and mass transfer

effects are considered

8in the presence of Joule heating and thermal radiation.

Series solutions are obtained to analyze the velocity, temperature and concentration fields.

1Numerical values of local Nusselt and Sherwood numbers for different values of emerging parameters

are computed and analyzed. A i comparative study

7with the previous solutions in a limiting sense is made. The leading results of

this problem are published in "Journal of Thermophysics and Heat Transfer 27 (2013) 733-740". Three-dimensional

2mixed convection flow of second grade fluid over an exponentially stretching

surface

are studied in chapter three. Convective boundary conditions are utilized for the heat transfer analysis. Analysis is carried out in the presence of thermal radiation. The series solutions are established through a newly developed method recognized as the homotopy analysis method. The convergent analysis of velocity components and temperature are derived. Graphs are plotted and analyzed for interesting physical parameters. A systematic study is performed to analyze the impacts of the significant

13**parameters on the velocity** and **temperature,** the **skin friction** coefficients **and** the **local Nusselt number.**

The contents of this chapter are published in "Plos One 9 (2014) e90038". Chapter four reports the

6**heat and mass transfer** effects **in** three-dimensional **mixed convection flow**

of viscoelastic fluid with internal heat source/sink and chemical reaction. An exponential stretching surface is employed for flow generation. Magnetic field normal to the direction of flow is taken under consideration. Convective conditions at boundary surface are also encountered. An analytical approach

7**homotopy analysis method is** used **to** develop **the solution** expressions **of the problem.**

Impacts of different controlling parameters such as stretching ratio parameter, Hartman number, internal heat source/sink, chemical reaction, mixed convection, concentration buoyancy parameter and Biot numbers

19**on the velocity, temperature and concentration profiles** are analyzed graphically. **The local Nusselt** number **and Sherwood numbers**

are sketched and examined. The results of present chapter are submitted for possible publication in "Applied Mathematics and Computation". Chapter 5 provides the three-dimensional mixed convection

3**flow of** viscoelastic **fluid over a stretching surface in presence of**

thermophoresis.

7**Soret and Dufour effects are also**

taken into account. Series solutions are constructed.

15**Dimensionless velocity, temperature and concentration** distributions **are shown graphically**

for different values of involved parameters.

26**Numerical values of local Nusselt and Sherwood numbers** are **computed** and analyzed. **The**

contents of this chapter are submitted for possible publication in "Engineering Applications of Computational Fluid Mechanics".

2**Three-dimensional flow of** Maxwell **fluid over** a **stretching surface is**

addressed in chapter six. Analysis is prepared in presence of concentration and thermal buoyancy effects. Convective boundary conditions for heat and mass transfer are explored. Series solutions of ii the resulting problem are established. Results are displayed to examine the influence of physical

1**parameters on the velocity, temperature and concentration** fields. Main observations **of**

this chapter are accepted for publication in "Journal of Central South University". Chapter seven is prepared to examine the

11**heat and mass transfer** effects **in** three-dimensional **flow of** Maxwell **fluid over a stretching**

2**surface with convective boundary conditions.** Mass transfer **is considered** in **the**

presence of first order chemical reaction. Conservation laws of energy and concentration are based upon the Soret and Dufour effects. Convergent series solutions to the resulting nonlinear problems are developed. The relevant results are accepted for publication in "International Journal of Numerical Method for Heat and Fluid Flow". Mixed convection

2**flow by an** inclined **stretching surface with**

thermal radiation is investigated in chapter eight. The

2boundary layer equations of an Oldroyd-B fluid

1in the presence of heat transfer are

used. Suitable transformations reduce the

2partial differential equations into the ordinary differential equations.

Computational analysis is implemented for the convergent series solutions.

4The values of local Nusselt number are

numerically analyzed. Effects of various parameters involved in the velocity and temperature are discussed. The contents of this chapter are accepted for publication in "Journal of Applied Mechanics and Technical Physics". Chapter nine provides

23the mixed convection flow of an Oldroyd-B fluid bounded by a stretching

surface with suction/injection. Mathematical formulation is developed

3in the presence of heat source and power law heat flux. Velocity and

temperature are computed. Numerical

4values of local Nusselt number are

examined. Results are computed in a limiting sense with existing literature. The contents of this chapter are published in "Journal of the Brazilian Society of Mechanical Sciences and Engineering DOI 10.1007/s40430-014- 0165-8". Chapter ten investigates the effects of

6heat and mass transfer in the mixed convection

1flow of an Oldroyd-B fluid over a stretching surface with convective boundary conditions.

Stress is given to the analysis of Soret and Dufour effects. Related problems are first modeled and then computed by homotopy analysis method (HAM).

1**Velocity, temperature and concentration** fields **are** given. **In** addition, **the**

1**local Nusselt and Sherwood numbers are**

examined through numerical values. These observations are submitted for publication in "Thermophysics and Aeromechanics". iii Falkner-Skan flow of rate type non-Newtonian fluid is analyzed in chapter eleven. Expressions of an Oldroyd-B fluid are used in the development of relevant equations. Analysis has been carried out in presence of mixed convection and thermal radiation. Expressions of flow and heat transfer are assembled.

7**Convergence of derived** nonsimilar **series solutions is**

3**mixed convection flow of** viscoelastic **fluid with** thermal **radiation and**

6**heat and mass transfer in** three-dimensional mixed **convection flow** of viscoelastic fluid **with**

> 4**Three-dimensional flow of** Maxwell **fluid over a stretching** surface with **heat** source **and**

> 2**flow of Maxwell fluid with** chemical reaction and **convective condition**

> 3**fluid with power law heat flux and heat source**

> 1**Soret and Dufour effects** in **mixed convection flow of** an Oldroyd-B **fluid**

> 2**Mixed convection Falkner-Skan** wedge **flow of** an Oldroyd-B **fluid**

Chapter 1 Literature survey and methodology 1.1 Introduction This chapter contains the literature review related to the considered flow problems. Boundary layer equations of thixotropic, second grade, Maxwell and Oldroyd-B fluids are presented. Brief idea of homotopy analysis method (HAM) are also provided. 1.2 Background External flows around streamlined bodies have viscous (shear and no-slip) effects confined close to the body surfaces and its wake but are nearly inviscid far from the body are termed as bound- ary layer flows [1] which occur in aerodynamics (airplanes, rockets, projectiles), hydrodynamics (ships, submarines, torpedoes), transportation (automobiles, trucks, cycles), wind engineering (buildings, bridges, water towers) and ocean engineering (buoys, breakwaters, cables). The boundary layer flow problem over stretching sheet have many industrial applications such as polymer sheet or filament extrusion from a dye or long thread between feed roll or wind-up roll, glass fiber and paper production, drawing of plastic films and liquid films in condensation process. Due to the high applicability of this problem in such industrial phenomena, Sakiadis [2] initiated the work for flow by moving surface. After the pioneering work of Sakiadis, researchers have studied the flow over stretching surfaces under various aspects for viscous and nonlinear fluids. A similarity solution of viscous fluid

> 14**over a stretching surface which is stretched with** the **velocity**

proportional to the distance from origin was presented by Crane [3]. Chakrabarti and Gupta [4] analyzed the hydromagnetic

> 23**flow over a stretching surface. Heat**

> 13**transfer over a** continuous **stretching surface with** suction **and injection**

was analyzed by Chen and Char [5]. Vajravelu and Hadjinicolaou [6] reported the heat transfer features

> 8**in the laminar boundary layer** flow **of viscous fluid over a** linearly **stretching** surface **with**

variable wall temperature. Ef- fects of suction and injection are present. They attained the solutions of the problem in Kumar functions. Andersson et al. [7] found the solutions

> **13** over a stretching surface in presence of first order chemical

reactions. Similarity solutions of the boundary layer equations over stretch- ing wall was obtained by Banks [8]. Closed form solution of magnetohydrodynamic (MHD) flow under slip condition

> **7** over a permeable stretching surface was obtained by Fang et al. [9]. Mukhphadhyay et al. [10] obtained the solution of

MHD

> **16** boundary layer flow over a heated stretching sheet with

variable viscosity. Wang [11] extended the boundary layer flow of Crane [3] for three-dimensional stretching surface. Devi et al. [12] extended the flow of ref. [11] for

> **16** unsteady stretching surface in presence of heat and mass transfer

effects. An approximate analytical solutions of the steady, laminar three

> **20** -dimensional flow for an incompressible viscous fluid past a stretching sheet

were proposed by Ariel [13,14]. Hayat and Javed [15] analyzed the three

> **3** -dimensional flow of an incompressible viscous fluid over a porous stretching surface in presence of

magnetic field by employing homotopy analysis method. Kumari and Nath [16] discussed the unsteady magnetohydrodynamic viscous fluid with heat transfer induced by a bilateral stretching surface. An analysis for

> **14** heat transfer over a non-linearly stretching surface for a viscous fluid

was provided by Vajravelu [17]. Cortell [18] extended the work of ref. [17] in presence of thermal radiation and viscous dissipation over a non-linearly stretching surface. Two-dimensional magnetohydrodynamic

> **3** stagnation point flow of an incompressible micropolar fluid over a non-linearly stretching surface was explored by

Hayat et al. [19]. The laminar boundary layer flow over an axisymmetric plane was provided by Afzal [20]. It has been noted by Gupta and Gupta [21] that stretching mechanism in all realistic situation is not linear. For instance the stretching is not linear in plastic and paper production industries. Besides these the

> 20 **flow and heat transfer** by **an exponentially stretching**

surface has been studied by Magyari and Keller [22].

> 20 **In this** attempt **the two-dimensional flow of an incompressible viscous fluid**

is considered. The solutions of laminar boundary layer equations describing heat and flow in a quiescent fluid driven by an exponentially permeable stretching surface were numerically ana- lyzed by Elbashbashy [23]. Al- Odat et al. [24] numerically discussed the thermal boundary layer flow with an exponential temperature distribution. Here magnetohydrodynamic flow was addressed. Liu et al. [25] studied the three-dimensional

> 13 **boundary layer flow of a viscous fluid over** an exponentially **stretching surface** by using **the**

Ackroyd and Runge-Kutta methods. Analysis of non-linear fluids

> 1 **is an active area of research**

for the last few years. In many fields such as food industry, drilling operations and bioengineering, the fluids, either synthetic or natural, are mixtures of different stuffs such as water, particle, oils, red cells and other long chain molecules. Such combination imparts strong rheological properties to the resulting liquids. The dynamic viscosity in non-linear materials varies non-linearly with the shear rate. These fluids in terms of their different rheological features cannot be described by a single constitutive relationship. Hence several relationships for the non-linear fluids are proposed. In fact the additional parameters in such non-linear fluids are the main culprit which makes the resulting systems more nonlinear, higher order and complex than the Navier-Stokes equations. The rheological fluids in general have been classified into three categories known as the differential, integral and rate types. Second grade fluid is a subclass of differential type fluids which exhibits the normal stress effects. To predict these effects many researchers studied second grade fluid under various aspects. Dandapat and Gupta [26] discussed the flow

> 23 **of an incompressible second-** order **fluid** due **to stretching**

surface under boundary layer assumptions. Chen et al. [27] delivered the temperature distribution in

11viscoelastic fluid of Walters' B Model over a horizontal stretching plate. The

velocity of the plate is proportional to the distance from the slit. Vajravelu and Rollins [28] carried out the heat transfer effects in viscoelastic

8fluid over a stretching surface with frictional heating and internal heat generation

or absorption.

2Hayat et al. [29] studied the three -dimensional flow

over stretching surface in a viscoelastic fluid by applying homotopy analysis method. Liu [30] presented

11analytical solutions for the flow and heat transfer in steady laminar boundary flow of an electrically conducting fluid of second grade subject to transverse uniform magnetic field past a semi-infinite stretching sheet with power-law surface temperature or power-law surface heat flux. The effects of viscous dissipation, internal heat generation or absorption, work done due to deformation and Joule heating were also considered in the energy equation.

1Flow and heat transfer characteristics of viscoelastic fluid with porous medium over a stretching surface with

viscous dissipation was governed by Nandeppanavar et al. [31] . Differential type models do not predict the relaxation and retardation time effects while these effects can be anticipated by rate type fluids. Maxwell fluid is a

7simplest subclass of rate type fluids.

Channel

1flow of an upper convected Maxwell fluid (UCM)

induced by suction was presented by Choi et al. [32]. Sadeghy [33] discussed the

22**flow of an upper-convected Maxwell** (UCM) **fluid above a rigid plate moving steadily.**

Pahlavan et al. [34] studied the

7**MHD flows of UCM fluids above porous stretching sheets**

using two-auxiliary-parameter homotopy analysis method. Maxwellian MHD flow induced by a linearly stretched with thermal radiations was investigated by Aliakbar et al. [35].

3**Unsteady flow of generalized Maxwell fluid with fractional derivative** induced **by**

an accelerating plate was provided by Fetecau et al. [36]. Kumari and Nath [37] analyzed the

22**Maxwell fluid over stretching sheet in presence of** mixed convection. Abel **et al.** [38] proposed **the**

MHD

22**flow and heat transfer** for the **upper-convected Maxwell fluid over a stretching sheet. Hayat et al.** [39] presented **the** three **-dimensional**

Maxwell fluid over stretching

2**surface with convective boundary** conditions. Bhatanagar **et al.**

[40] initiated to analyze the boundary layer flow of rate type fluids. They obtained the solutions of two-dimensional flow over

8**a stretching surface with** variable **free stream** velocity. Exact solutions **of**

unidirectional

2**flow of an Oldroyd-B fluid**

between two parallel plates was presented by Rajagopal [41]. Exact solutions for flows

8of an electrically conducting Oldroyd-B **fluid** over **an infinite** oscillatory **plate in**

the presence of a transverse magnetic field when the entire system rotates about an axis normal to the plate was obtained by Hayat et al. [42]. The linear stability of the

2flow of an Oldroyd-B fluid through a

linear array of cylinders confined in a channel was analyzed by Smith et al. [43]. They computed solutions for both the steady state and linear stability of these states by employing finite element analysis. Exact solutions for the influence of Hall current and rotation in the oscillatory flows by an infinite plate were obtained by Asghar et al. [44]. Fetecau and Fetecau [45] analyzed the unsteady flows of Oldroyd-B fluids in a channel of rectangular cross-section. A linear stability analysis determining the onset of oscillatory convection of an Oldroyd-B fluid in a two-dimensional rectangular porous medium generated by Newtonian heating was presented by Niu et al. [46]. In nonlinear fluids sometimes

5nonlinearity introduced by their shear-dependent viscosity and/or elasticity often gives rise to a formidable mathematical task which cannot be solved.

5Obviously, the situation becomes much more complicated when the viscosity of the fluid is time- dependent. Time-dependent fluid systems are quite frequent in industrial applications with the common effect being a drop in viscosity by the progress of time. Complex fluid systems such as drilling muds, foodstuff, paints, cosmetics, pharmaceuticals, suspensions, grease, and the like belong to this class of fluids-the so-called thixotropic fluids. Physiological fluids such as blood, synovial fluid, and mucus may also exhibit thixotropic behavior depending on the time scale of the observation.

5A common effect among thixotropic fluids is that their viscosity is decreased even when the shear rate is constant

[47,48]. Harris [49,50]

5tried to address boundary layer flows of thixotropic fluids.

Haris presented

5a simple thixotropic fluid model (the so called Harris model) to investigate the effects of a fluid's thixotropic behavior on the characteristics of the momentum boundary layer formed above a fixed plate

[50].

5Two-dimensional flow of an incompressible thixotropic fluid obeying Harris rheological model over a fixed semi-infinite plate

was investigated by Sadeqi et al. [51]. Mixed convection flows or combined free and forced convection flows occur in many techno- logical and industrial applications and in nature for example, in solar receivers exposed to wind currents, electronic devices cooled by fans, nuclear reactors cooled during emergency shutdown, heat exchanges placed in a low-velocity environment, flows in the ocean and in the atmosphere, and many more. Finite element method was utilized for detailed analysis of mixed-convection flow in a horizontal channel heated from the side walls were computed by Sillekens [52]. Heat transfer enhancement by air injection in upward heated mixed-convection flow of water was studied by Celata et al. [53]. Barletta [54] provided the analysis of the laminar and fully developed mixed convection flow in a vertical rectangular duct with one or more isothermal walls. Magyari et al. [55] analyzed the

6boundary layer mixed convection flow over a

perme- able horizontal plate. The unsteady

20mixed convection boundary layer flow near the region of a stagnation point on a vertical surface embedded in a Darcian fluid-saturated porous medium

was investigated by Nazar et al. [56]. They employed Keller—Box method to obtain the so- lutions. Laminar two-dimensional unsteady mixed convection

20boundary-layer flow of viscous incompressible fluid past a sharp wedge

was developed by Hussain et al. [57]. Perturbation solutions were obtained for small and large dimensionless time. Experimental investigation was presented on mixed (free and forced) convection to study the local and average heat transfer for hydrodynamically and thermally developed laminar air flow in a horizontal circular cylinder was analyzed by Mohammed and Salman [58]. Laminar mixed convection heat transfer for as- sisted and opposed air flows in the entrance region of a vertical circular tube with wall heat flux boundary condition had been experimentally investigated by Mohammed [59]. Kotouc et al. [60] also provided the loss of axisymmetry in the mixed convection assisting flow past a heated sphere. A transient laminar mixed convection flow of viscous incompressible fluid generated by thermal buoyancy force over a horizontal porous sensor surface placed inside a squeezing channel was discussed by Mahmood et al.

[61]. The implicit finite difference approximation together with Keller box method was employed for the solution of small and large time regimes.

6Heat and mass transfer characteristics in mixed convection boundary layer flow past a linearly stretching vertical surface in a porous medium filled with a viscoelastic fluid

with Dufour and Soret effects was governed by Hayat et al. [62]. Radiative mixed convection has gained much importance amongst the recent researchers due to number of applications in geophysical and energy storage problems such as in furnaces, ovens and boilers and the interest in our environment and in no conventional energy sources, such as the use of salt gradient solar ponds for energy collection and storage. Similarity equations governing steady hydromagnetic boundary-layer flow over an accelerating permeable surface

6in the presence of thermal radiation, thermal buoyancy and

heat generation or absorption effects were obtained by Chamka [63]. Buoyancy force and thermal radiation effects in MHD boundary layer viscoelastic fluid flow over continuously moving

4stretching surface was investigated by Abel et al.

[64]. Mukhopadhyay and Layek [65] presented the free convective boundary layer

21flow and heat transfer of a fluid with variable viscosity over a porous stretching vertical surface in presence of

thermal radiation. Results were obtained by Lie group transformations. Also Mukhopadhyay [66] analyzed the

6effects of thermal radiation on unsteady boundary layer mixed convection heat transfer problem from a vertical porous stretching surface embedded in porous medium.

Magnetohydrodynamic mixed convective flow and heat

6transfer of an electrically conducting power-law fluid past a stretching surface in the presence of heat generation/absorption and thermal

radiation was studied by Chen [67]. Numerical solutions were generated by an implicit finite-difference technique for the non-similar coupled flow. A

> 8study of utilizing heat source or sink in moving fluids

has been a subject of interest of many researchers. This is because of its possible application to geophysical sciences, astrophysical sci- ences, and in cosmic studies. Such flows arise either due to unsteady motion of the boundary or the boundary temperature. The study of fluctuating flow is important in the paper industry and many other technological fields. Many investigators stressed

> 6on the magnetohydrodynamic flow of an electrically conducting fluid

because of numerous applications in metallurgical in- dustry such as in drawing, annealing,

> 2in the purification of molten metals from non-metallic inclusions,

electromagnetic pumps, MHD generators etc. Several studies have been presented by the authors in

> 22presence of transverse magnetic field and heat

source/sink over a stretching surface.

> 8Heat transfer characteristics in an electrically conducting fluid over a stretching sheet with

variable wall temperature and heat source/sink was investigated by Vajravelu and Rollins [68]. Abel et al. [69] presented

> 4the study of momentum and heat transfer characteristics in hydromagnetic flow

of viscoelastic liquid

> 16over a stretching sheet with non-uniform heat source,

where the flow was generated due to a linear stretching of the sheet and influenced by uni- form magnetic field applied vertically. Beg et al. [70] examined the magnetohydrodynamic free

> 6convection from a sphere embedded in an electrically-conducting fluid-

saturated porous

regime with heat generation. The present trend in the field of chemical reaction analysis is to give a mathematical model for the system to predict the reactor performance. A large amount of research work has been reported in this field. In particular, the

6study of heat and mass transfer

with chemical reac- tion is of considerable importance in chemical and hydrometallurgical industries. MHD flow with

6heat and mass transfer characteristics of an incompressible viscous electrically conducting and Boussinesq fluid due to a vertical stretching

surface with chemical reaction and thermal stratification effects was presented by Kandasamy et al. [71].

6Mansour et al. [72] studied the effects of chemical reaction, thermal stratification, Soret and Dufour numbers on MHD free convective heat and mass transfer of a viscous incompressible and electrically conducting fluid over a vertical stretching surface embedded in a saturated porous medium.

The

6combined effect of mixed convection with thermal radiation and chemical reaction on MHD flow of viscous and electrically conducting fluid past a vertical permeable surface in presence of porous medium was analyzed

by Pal and Talukdar [73]. Mass transfer with

8chemical reaction in MHD mixed convective flow along a vertical

stretching sheet was investigated by Singh et al. [74]. 1.3 Fundamental laws 1.3.1 Law of conservation of mass The equation of continuity (law of conservation of mass) can be represented by ?? ?? + $\nabla \cdot$ (?V) = 0? (1.1) where ?

3is the density of fluid and $\vee$ is the fluid

velocity. For an incompressible fluid Eq. (1.1) can be expressed as follows: $\nabla \cdot$ V = 0? (1.2) 1.3.2 Law of

conservation of linear momentum Generalized equation of motion can be expressed as ? ?V ?? = ∇ · τ +? b? (1.3) For an incompressible flow τ = −pI + S (1.4) in which

> **11τ is the Cauchy stress tensor, ? is the pressure,** I is **the** identity tensor, S **is the**

extra stress tensor, b is the body force and ???? is the material time derivative. The Cauchy stress tensor and the velocity field for three-dimensional flow can be described in the forms ??? T=□ ? ?? ? ?? ??? ? ?? ? ?? □? ? ?? ? ?? ??? □ □ □ □ V = [?(?? ?? ?)? ?(?? ?? ?)? ?(?? ?? ?)]? (1.5) (1.6) where ???? ??? and ??? represent the normal stresses, ???? ???? ???? ???? ??? and ??? show the shear stresses and ?? ?? ?

> **19are the velocity components along the ?? ? and ?−directions respectively.**

Equation (1.3) in component form can be written as follows: ? ?? ?? ?? ?? + μ ?? + ??? + ? ?? + ? ?? = ? (???) ? (???) ? (???) ?? ?? + ?? + ???? (1.7) ? ?? ?? ?? ?? ¶ ?? + ??? + ? ?? + ? ?? = ? (?????) ? (?????) ? (?????) + ???? + + (1.8) μ ?? ?? ?? ?? ¶ ? = ? (???) ? (???) ? (???) ?? + + ???? (1.9) μ ?? + ? ?? + ? ?? + ? ?? ¶ ?? + ?? where ??, ?? and ?? show the components of body force along the ?? ? and ?−axes, respectively. The above equations for two-dimensional flow become ? ?? ?? ?? μ ?? +??? +??? ?? ?? ?? ¶ ? μ ?? +??? +??? ¶ 1.3.3 Equation of heat transfer = = ? (???) ? (???) ?? + + ???? (1.10) ?? ? (???) ? (???) ?? + + ???? (1.11) ?? According to first law of thermodynamics the heat transfer equation can be written as ? ?? ?? =??L−∇·q1+???? (1.12) where ? = ??? is the internal energy, ?? the specific heat, ? the temperature, L = ∇V the velocity gradient, q1 = −?∇? the heat flux, ? the thermal conductivity and ?? the radiative heating. The above equation in absence of radiative heating is given below ??????? =??∇V+?∇2?? (1.13) 1.3.4 Diffusion equation Mass transfer occurs whenever fluid flows that is some mass is transferred from one place to another. According to Fick's law ?? = ??∇2? − ???? (1.14) ?? where

> **1C is the concentration, D? is the coefficient of** mass diffusivity and ?? **is the**

reaction rate diffusing species. 1.4 Boundary layer equations of nonlinear fluids 1.4.1 Second grade fluid Extra stress tensor S for a second grade fluid can be expressed as S = ?A1 + ?1A2 + ?2A21? where A1 is the first Rivilin-Erickson tensor can be defined by A1 = grad V + (grad V)????????? ? For three-dimensional flow one obtains 2???? ???? + ???? ???? + ???? A1 = □ ???? + ?? ?? ?? 2???? ?? + ?? □ ?? □ ???? + ???? ???? + ???? 2????? □ □ □ and the second Rivilin-Erickson tensor A2 can be computed through A? = ?A???−1 + A?−1L + L????????? A?−1? (1.15) (1.16) (1.17) (1.18) For thermodynamic stability the second grade model should obey the ineqaulity given below: ? ≥ 0? ?1 ≥ 0? ?1 + ?2 = 0? (1.19) From the boundary layer theory [1], the order of ?? ?? ? and ? are 1 while the order of ? and ? are ?? Three-dimensional boundary layer equations for second grade fluid can be written as ? ???? + ? ???? + ? ???? = ? ??2??2 + ??1 □ ???? ??2??2 ???? ?????2?3??2??2++2 ????????3????3?2??−? + 2 ???? ??2??2 ? ? (1.20) + □ □³ ´□ ? ???? + ? ???? + ? ???? = ? ??2??2 + ??1 □ ?? ?2? ? (1.21) ?? ??2 + ? ???3???2 +???3??3 − ???????2??2 +2???????2??? +2???????2??2 □ □³ ´□ 1.4.2 Maxwell fluid The extra stress tensor S for Maxwell fluid can be expressed by the following relation ? ?S (1.22) μ 1 + ?1 ?? S = S + ?1 ?? = ?A1? ¶ in which ?1 is the relaxation time, ???? the covariant differentiation, ? denotes the

kinematic viscosity and A1 the first Rivlin-Erickson tensor. For a tensor S of rank two, a vector b1 and a scalar ?? we get ?S ?S ????????? ?? ?? + (V?∇)S − S(grad V) = − (grad V)S? (1.23) ??b?1 ??b?1 + (V?∇)b1 − (grad V)b1? = (1.24) ?? ?? ?? ?? + (V?∇)?? = (1.25) Implementation of 1 + ?1 ??? on Eq. (1.3), we have the following relations in the absence of body force ¡ ¢ ? 1 + ?1 ? ?V ?? = − 1 + ?1 ?? ∇? + 1 + ?1 ?? (∇?S)? ? ? (1.26) µ ?? ¶ µ ¶ µ ¶ By adopting the procedure ? (∇?) = ∇? ? ? (1.27) ?? µ ?? ¶ Hence the above relations in absence of pressure gradient is ? 1 + ?1 ?? ??V? = ? (∇?A1) ? ? (1.28) µ ¶ By using the boundary layer theory [1], the order of ?? ?? ? and ? is 1 and order of ? and ? is ?? The ? −momentum equation vanishes identically because it has order ?? Hence the boundary layer equations for three-dimensional flow of Maxwell fluid are ? ?? ?? ?? ?2 ??2??2 + ?2 ??2??2 + ?2 ?2? ??2 ?2? ?? + ? ?? + ? ?? + ?1 +2?? ???2??? + 2?? ???2??? + 2?? ???2??? □ □ =? ??2 ? (1.29) □ □ ? ?? + ? ?? + ? ?? + ?1 +2?? ???2??? + 2?? ???2??? + 2?? ???2??? ?? ?? ?? ?2 ??2??2 + ?2 ??2??2 + ?2 ??2??2 = ? ?2? □ □ ? (1.30) ??2 The boundary layer equation for

> ## 2two-dimensional flow of Maxwell fluid

are given below □ □ ? ???? + ? ???? + ?1 ?2 ??2??2 + ?2 ??2??2 + 2?? ???2??? = ? ??2??2 ? (1.31) µ ¶ 1.4.3 Oldroyd-B fluid The extra stress tensor for an Oldroyd-B fluid model can be expressed as 1 + ?1 ?? S = S + ?1 ?? = ? 1 + ?2 ?? A1? ? ?S ? (1.32) µ ¶ µ ¶ where ?2 denotes the retardation time and law of conservation of momentum in absence of pressure gradient and body force can be written as follows: ? 1 + ?1 ?? ?? = ? 1 + ?2 ?? (∇?A1)? ? ?V ? µ ¶ µ ¶ The scalar forms of boundary layer equations in this case are ?? ?? ?? ?2? ? ?? + ? ?? + ? ?? + ?1 +2?? ???2??? + 2?? ???2??? + 2?? ???2??? ?2 ??2??2 + ?2 ??2??2 + ?2 ??2 □ □ ?2? ? +?2 ???3???2 +????3???2 +???3??3 □ □ =? □??2 □ −??????2??2 − ??????2??2 − ??????2??2 □□ ? □ □ □□ (1.33) (1.34) ? ?? ?? ?? ?2 ???2??2 + ?2 ??2??2 + ?2 ??2??2 ?? + ? ?? + ? ?? + ?1 +2?? ???2??? + 2?? ???2??? + 2?? ???2??? □ □ ? ???3???2 + ? ???3???2 + ? ??3??3 □ □ =? ?2? □ ??2 + ?2 □ − ???? ??2??2 − ???? ??2??2 − ???? ??2??2 □□ ? (1.35) □ □ □□ and the governing boundary layer equation for two-dimensional flow is ? ???? + ? ???? + ?1 ?2 ??2??2 + ?2 ??2??2 + 2?? ???2??? =? + ?2 µ ?2? ? ???3???2 +???3??3 ? ¶ □ ??2 □ −??????2??2 − ??????2??2 □□ □ □ □(□1.36) 1.4.4 Thixotropic fluid Stress tensor τ for thixotropic fluid model ??? = 2?(??2? (?))???? where the

> ## 5viscosity is allowed to be time-dependent through allowing the second invariant of the deformation-rate tensor to be time-dependent, ??2? is

> ## 5the second invariant of the deformation- rate tensor

and 2??? = (??????? + ???????)? (1.37) In the simple Harris

> ## 5model, a quadratic form is used for the ??2? so that we have,

??2? = (2???)2 = 4 ???? 2 1 ?? ?? 2 ?? 2 Ã + µ ¶ 2 ?? ?? + + ?? ? 0? (1.38) µ ¶ µ ¶ ! For the

[49,50] ? = ?0 − ?1??2? + ?2 ?????2? ? (1.39) where ??? is the material derivative defined as ? = ? + (?? ▽) ? (1.40) ?? ?? In equation (1.39) ?1 and ?2 are the material constants. For boundary layer analysis the viscosity function becomes ? = ?0 − 2?1 ?? ?? 2 ?? ?2? ?? ?2? + 4?2 ? µ ¶ ?? ???? + ? ?? ??2 ? (1.41) µ ¶ By using boundary layer analysis the ?-momentum equation is completely dropped for two- dimensional flow.

?? ?? ??2??2 ? ???2??? + ? ??2??2 ? ?? + ? ?? = ? ??2 ?? ?? ?2? 6?1 ?? 2 − ? ?? ?2? 4?2 □ ??2 + ? ³ µ + □ ? ?? ´2³ ´?³???3???2 + ? ??3??3 ´ ¶ µ ¶ ?? ´ □ □ + ???? ???2??? ???? ??2??2 + □ □ ³ □ □ (□1.42) □ 1.5 Homotopy analysis method (HAM) In the absence of analytical solutions before the advent of computers, the researchers mainly directed their efforts at obtaining some forms of approximate solutions. One of the key issues of approximate solutions has always been the accuracy of the solutions. The accuracy, gen- erally speaking, is measured in terms of the norm of the error in Banach space. The error being the difference of the approximate solution from the exact solution. In the absence of an exact solution (analytical or numerical) a heuristic approach consisting of the convergence of successive approximations has been chosen to judge the merit of an approximate solution. With the advent of computers the approximate solutions in fluid dynamics have lost some of their importance as more and better numerical algorithms have been developed to solve the increasingly realistic, but more complicated problems numerically. Nevertheless, approximate analytical solutions still have their relevance for the following reasons: Firstly, they give the solutions for each point within the domain of interest unlike the numerical solutions which are available for a particular run only for a set of discrete points in the domain. Secondly, compared to a numerical solution a nicely produced approximate solution, requiring a minimal effort and having a reasonable amount of accuracy is always handy for an engineer, scientist or an applied mathematician, who can obtain a solution completely, thereby gaining a valuable insight into the essentials of the problem. Thirdly, even with most of the scientific packages, some initial guess is required for the solution, as the algorithms, in general are not globally convergent. In such situations, approximate solutions can provide an excellent starting guess, which can be rapidly refined to the exact numerical solution in a few iterations. Homotopy analysis method is proposed by Liao [75? 76] and is very useful to obtain the series solutions of the nonlinear ordinary and partial differential equations [77-85]. According to Liao [75? 76], this method distinguishes itself from other analytical methods in the following five aspects 1. HAM is not dependent on physical parameters. Therefore the technique can be used for both strong/weak nonlinear problems. 2. It is valid for strongly nonlinear problems even if a given nonlinear problem does not contain any small/large parameter. 3. It provides us with a convenient way to adjust

solution. 4. HAM provides freedom to chose base functions to approximate the solution of nonlinear problem. 5. This method can be coupled with many other mathematical methods such as integral transform methods, series expansion methods, numerical methods and so on. This technique is applicable in the development of results to numerous problems [78 − 88]? Idea behind the HAM is as follows. Nonlinear

differential equation can be written as follows: ?(?) + ?(?) = 0? (1.43) where ? is a nonlinear operator, ?(?) is an unknown function to be determined and ?(?) is a known function. The homotopic equation is (1 − ?)L [?¯(?? ?) − ?0(?)] = ?~ {? [?¯(?? ?) − ?(?)]} ? (1.44) where ?0(?) is initial guess, L is auxiliary linear operator, ~ is auxiliary parameter or con- vergence control parameter, ? ∈ [0? 1] is an embedding parameter and ?¯(?? ?) is an unknown function. By expanding Taylor's series about ? one obtains ?¯(?? ?) = ?0(?) + ??(?)??? ??(?) = ?1! ????¯(???? ?) ∞ ? (1.45) X?=1 ¯?=0 ¯ The convergence of above series firmly depends upon ~? The value ¯of ~ is chosen in such a ¯ way that series solution is convergent at ? = 1. Substituting ? = 1 one obtains The ?-th order deformation problems are ∞ ?(?) = ?0(?) + ??(?)? X?=1 (1.46) L [??(?) − ????−1(?)] = ~R?(?)? (1.47) where 0? ? ≤ 1? ?? = □ (1.48) □ 1? ? ? 1? 1 ??−□1 ∞ R?(?) = − 1)! (???−1 ? ?0(?) + X?=1 × " ?? (?)?? ? (1.49) (? #)?=0 Chapter 2

> 21**MHD mixed convection flow of** thixotropic **fluid**

with thermophoresis, Joule heating and thermal radiation This chapter deals with the magnetohydrodynamic (MHD) mixed convection

> 4**flow of** thixotropic **fluid over a** moving **surface.** Heat transfer **is** considered in **the**

presence of thermophoresis, Joule heating and radiative effects. Dimensionless nonlinear problem is computed by homotopy analy- sis method (HAM). The convergent solutions are plotted and examined for various parameters of interest. Numerical values of wall shear stress and heat transfer rate are computed and discussed. 2.1 Mathematical formulation We consider

> 3**Cartesian coordinate system in such a way that ?−axis is along the stretching surface and ?−axis** is **perpendicular to it. The**

magnetohydrodynamic

> 3**boundary layer flow of** thixotropic **fluid is**

> 1**taken into account. Heat and mass transfer** characteristics **are**

accounted

> 8**in the presence of thermal radiation** and thermophoresis **effects.**

Uniform temperature of the surface ?? is

> 8**higher than the ambient** fluid **temperature** ?∞? Further the species

concentration

at the surface is taken uniform ?? = 0 while the ambient concentration is ?∞? A constant magnetic field of strength B0

8**is applied in the ?-direction. The flow** is steady and **the**

magnetic Reynolds number is taken small so that an

8**induced magnetic field is negligible in comparison** to **applied magnetic field.** Taking into account **the**

Rosseland approximation for radiative heat flux, the mass, linear momentum, energy and concentration equations are simplified to the following expressions: ?? ?? ?? ?? = 0? + (2.1) ? ?? + ? ?? ?? ?? = ? ??2 ? 2? − 6?1 ?? 2 ?2? ?∗?02 ? ? ?? 2 − μ ? ¶ μ ¶ 4?2 ?? + ?? ??2??2 ? ???2??? + ? ??2??2 ? □ + ???? 2³ ? ?´???³3??2 +´?³???3??3 + ???? ???2??? +´ ???? ??2??2 □ +?[?? (□? − ?³∞)´+ ³??(? − ?∞)]? ´□ (2.2) ? ?? ?? ?? + ? ?? = ? ?2? 16???∞3 ?2? 2??1 ?? 4 4??2 ?? ?2? ?? 2 ??? ??2 + 3?? ??2 ??? ?? − + ? ? μ ¶ ??? ?? ???? ?? μ ¶ + 4??2 ?? ?2? ?? 2 ??? ?? ??2 ?? ? + ?? 2 ?∗?02 ?2? + (2.3) μ ¶ ??? ?? μ ¶ ??? ? ?? ?? ?? + ? ?? = ? ??2 ?? (?? (? − ?∞))?

16**?2? − ? (2.4) where (?? ?) are the velocity components** parallel to **the ?− and ?−axes,**

?1 and ?2 are the constants, ? the dynamic viscosity of fluid, ? the density of fluid, ?∗ the electrical conductivity, ? the gravitational acceleration, ?? and ?? the thermal and concentration expansion coefficients respectively, ? the temperature, ?? the specific heat, ?? the Stefan-Boltzmann constant, ?? the mean absorption coefficient, ? the diffusion coefficient and ?? the thermophoretic velocity. Fig. 2. 1: Physical model Equations (2.1)-(2.4) are solved subject to the boundary conditions ? = ?? = ??? ? = 0? ? = ??(?)? ? = ??(?) at ? = 0? ? → 0? ?? → 0? ? → ?∞? ? → ?∞ as ? → ∞? ?? The thermophoretic term ?? in Eq. (2.4) can be defined as ?? = −?2 ?? ?? ? ?? ? (2.5) (2.6) Here ?2 is the thermophoretic coefficient and ?? is the reference temperature. A thermophoretic parameter ? is defined by the following relation ? = −?2(?? − ?∞)? ?? The wall temperature and concentration fields are ?? = ?∞ + ??? ?? = ?∞ + ??? (2.7) (2.8) where ?? ? and ? are the positive constants. Introducing ? = ???0(?)? ? = − ???(?)? ? = ? ? ?(?) = ? − ?∞ ? √ ? r ? ?? − ?∞ (2.9) incompressibility condition (2.1) is automatically satisfied and the Eqs. (2.2)-(2.5) become ? 000 +? ? 00 −? 02 +?1(?)? 002? 000 +?2(?) □ −?2?0+?(?+??) = 0? (2.10) ?0?002?000 + ?004 −??00?0002 − ??002??? □ □ □ 1 + 4 ? ?00 + ? ?(? ?0 − ?? 0) + ? ???? 002 + 1 ?1? ???? 004 μ 3 ¶ 3 +?2? ???(? 0? 004 − ? ? 003? 000) + ? 2? ???? 02 = 0? (2.11) ?00 + ??(??0 − ??0) − ???(?0?0 − ?? 00) = 0? (2.12) ? = 0? ? 0 = 1? ? = 1? ? = 1 at ? = 0? ? 0 → 0? ? 00 → 0? ? → 0? ? → 0 as ? → ∞? (2.13) where the non-Newtonian parameters are ?1(?) = −6??1??23?2 and ?2(?) = 4??2??24?2 ? = ?∗?02??? the Hartman number, ? = ?????2? the local buoyancy parameter, ??? = ???(???2−??2∞?)2?3??2 the local Grashof number, ? = ????((????−−??∞∞)) the constant dimensionless concentration buoyancy parameter, ?? = ???? the Prandtl number, ? = 4??????∞3 the radiation parameter, ?? = ??2 ??(??−?∞)

the Eckert number and ?? = ?? the Schmidt number. It is further seen that both ?1 and ?2 are functions of ?? Hence the fluid flow in present situation lacks a self-similar solution. This is striking difference when compared present case with that of Blasius flow of viscous fluid i.e. in Blasius flow there is self-similar solution. Here ?2 especially indicates the fluids' thixotropic behavior. The

> 1skin friction coefficient, local Nusselt number and local Sherwood number

in non- dimensional form can be written as follows: ??1??2?? = ?00(0) − ?1?6[?00(0)]3? ?????−?1?2 = −?0(0) and ????−?1?2 = −?0(0)? (2.14) Here we noticed

> 5that the wall shear stress at given ?−location is only affected by

?1 but only indirectly by ?2 (i.e. through ?00(0)) [51]. We have serious limitation in a simplified Harris model for representing true thixotropic fluid. 2.2 Series solutions The homotopic solutions for ? and ? in a set of base functions {?? exp(−??)? ? ≥ 0? ? ≥ 0} (2.15) are given by ∞ ∞ ??(?) = ??????? exp(−??)? X? =0 X?=0 (2.16) ∞ ∞ ??(?) = ??????? exp(−??) (2.17)

> 28X?=0 X?=0∞ ∞ ??(?) = ??????? exp(−??) (2.18) X?=0 X?=0

where ?????, ????? and are ?????

> 4are the coefficients. The appropriate initial approximations and auxiliary linear operators are

?0(?) = 1 − exp(−?)? ?0(?) = exp(−?)? ?0(?) = exp(−?)? L? = ?000 − ?0? L? = ?00 − ?? L? = ?00 − ? (2.19) (2.20) with L?(?1 + ?2?? + ?3?−?) = 0, L?(?4?? + ?5?−?) = 0? L?(?6?? + ?7?−?)? (2.21) in which ?? (? = 1−7) denote

> 4the arbitrary constants and the zeroth order deformation problems

are (1 − ?) L? ?ˆ(?; ?) − ?0(?) = ?~? N?

> 37?ˆ(?; ?)? ?ˆ(?; ?)? ?ˆ(?; ?) ? (2. 22) (1 − ?) L? ˆ?(?; ?) − ?0(?)

= ?~?N?

> 10ˆ?(?; ?)? ?ˆ(?; ?)? ?ˆ(?; ?) ? ³´³´ (2.23) ³´³´ (1− ?) L? ?ˆ(?; ?) − ?0(?) = ?~?N? ?ˆ(?; ?)? ?ˆ(?; ?)? ?ˆ(?;

?) ? ³ ´ ³ ´ (2.24)

> 12?ˆ(0; ?) = 0? ?ˆ0(0; ?) = 1? **?ˆ0(∞; ?) = 0?** ?ˆ00(∞; ?) = **0? ?ˆ(0;** ?) = 1, **ˆ?(∞; ?) = 0? ?ˆ(0;** ?) = 1 and **?ˆ(∞; ?) = 0?**

(2.25) where ? shows

> 3**embedding parameter, ~? ? ~? and ~? the non-zero auxiliary parameters and the nonlinear operators** N? , N? **and**

N? are ?3

> 24**?ˆ(?? ?) + ?ˆ(?? ?) ?2?ˆ(?? ?) ??ˆ(?? ?) 2 N? [?ˆ(?? ?)? ˆ?(?; ?)? ?ˆ(?; ?)]** = − ??3 ??2 Ã ?? ! **2** +?1(?) Ã **?2?ˆ(??**

?) ?3

> 17**?ˆ(?? ?) ??2** ! ??3 **??ˆ(???) ?2?ˆ(???)** ?3 **?ˆ(???) 2** +?2(?) ☐ ?2 **???ˆ(???³) ??2** ??3 + ?2 **??ˆ(??2??) − ?ˆ(??** ?) 4 ?3 **?ˆ(???´)** ☐ **??2** ??3 2 − **?ˆ(?? ?)³ ?2??**

> 17**ˆ(??2??)´** 2 ?4 **??ˆ?(? 4??)** ☐ **+?(ˆ?(?? ?) + ??ˆ(?; ?) − ?2** ☐ ³ ´**??ˆ(?? ?) ³** ´ ☐ ☐ **?? ? (2.** 26) N **?[?ˆ(?? ?)? ?ˆ(?; ?)? ?ˆ(?;**

?)] = 4 ?2?ˆ(?? ?) ?2?ˆ(?? ?) 2 1 + ? µ 3 ¶ ??2 + ? ??? Ã ??2 ! +? 2 Pr ?? ??ˆ(?? ?) 2 1 ?2?ˆ(?? ?) 4 Ã ?? ! + ?1? ??? 3 Ã

> 9**??2 ! −? ?ˆ?(?? ?) ??ˆ(?? ?) + ? ??ˆ(?? ?) ?ˆ?(?? ?) ?? ?? ??ˆ(???)** ?2 **?ˆ(???)** 4 +?2? ??? ☐ ☐ **−?ˆ(?? ??)??3 ??ˆ³(??3??)??** 2?2 **??ˆ´(?? 2??)** **3**☐ ? (2.27) ☐ ³ ´☐ ☐ N **?[?ˆ(?? ?)? ?ˆ(?; ?)? ?ˆ(?; ?)]** = **?2?ˆ(???) +?? ?ˆ(???)??ˆ(???) ??ˆ(???)?** ˆ(???) **??2**Ã ?? − ?? ! −? ?? Ã **??ˆ(?? ?) ??ˆ(?? ?) − ?ˆ(??**

?)?2?ˆ(?? ?) ! ? (2.28) ?? ?? ??2 When ? = 0 and ? = 1 then ?ˆ(?;0) = ?0(?) and ?ˆ(?;1) = ?(?)? ?ˆ(?;0) = ?0(?) and ˆ?(?;1) = ?(?)? ?ˆ(?;0) = ?0(?) and ?ˆ(?;1) = ?(?)? (2.29) (2.30) (2.31) and when ? increases from 0 to 1 then ?(???), ?(???) and ?(???) vary from ?0(?) to ?(?), ?0(?) to ?(?) and ?0(?) to ?(?)? Employing the Taylor's series expansion we have ?(???)=?0(?)+?=1??(?)??? ??(?)= ?1! ?????(??;?)¯?=0 ∞ ? (2.32) ?(???)=?0(?)+?=1??(?)??? ??(?)= ?1! ?????(??;?)¯¯?=0 ¯ ∞ ? (2.33) P ¯ ?(?? ?) = ?0(?) + ? P=1 ??(?)??? ??(?) = ?1! ?????(??; ?)¯¯¯?=0 ∞ ¯ (2.34) ? Convergence of series (2.32-2.34) is Pclosely associated with ~? ? ~? and ~?¯? The values of ~? , ~? ¯ and ~? are chosen such that the series (2.32-

2.34) converge at ? = 1. Hence ?(?) = ?0(?) + ??(?)? ∞ (2.35) ?=1 P ?(?) = ?0(?) + ??(?)? ∞ (2.36) ?=1 P∞ ?(?) = ?0(?) + ??(?)? (2.37) ?=1 If we denote the special solutions ??∗ (?), ?∗? (?) and ?∗? (?) then the general solutions ?? (?), P ?? (?) and ?? (?) are ??(?) = ??∗(?) + ?1 + ?2?? + ?3?−?? ??(?) = ?∗?(?) + ?4?? + ?5?−?? ??(?) = ?∗?(?) + ?6?? + ?7?−?? (2.38) (2.39) (2.40) 2.3 Convergence analysis and discussion We recall that the auxiliary parameters ~?? ~? and ~? are useful in controlling and adjusting the convergence of the series solutions. We draw the ~−curves at 14?? order of approximation in obtaining the ranges for ~?? ~? and ~?? It is noticed from Fig. 2.2 that the admissible values of ~?? ~? and ~? are −0?7 ≤ ~? ≤ −0?25? −0?95 ≤ ~? ≤ −0?5 and −0?95 ≤ ~? ≤ −0?5? Further

> 3**the series** solutions **converge in the whole region of ? when** ~? = ~? = ~? = −0?
> 6? Table **2.1**

is presented to see the convergent values

> 1**for different order of approximations** at ~? = ~? = ~? = **−0?**

6? This Table indicates

> 4**that the series solutions** for velocity **converge from** 20th **order of** deformations
> **and**

temperature and concentration converge from 25th order of deformations. Hence 25th order deformations are computed to find a convergent series solutions.
K1=0.1,K2=0.2,l=0.4,t=0.3,M=0.6,Sc=0.7,Ec=0.5,Pr=1.0,N=0.3,R=0.4 -0

> 1**.5 -1 f''?0? , q'?0?, f'?0?** -1 **.5**

-2 f''?0? -2.5 g'?0? q'?0? -1.2 -1 -0.8 -0.6 -0.4 -0.2 0 0.2 Ñf, Ñq,Ñf Fig. 2. 2: ~−curves for the functions ??? and ? Table 2.1:

> 1**Convergence of** series **solutions for different order of approximations when**

?1 = 0?1? ?2 = 0?2? ? = 0?4? ? = 0?3? ? = 0?6? ?? = 0?7? ?? = 0?5? ? ? = 1?0? ? = 0?3? ? = 0?4 and ~? = ~? = ~? = −0?6? Order of approximations −? 00(0) −?0(0) −?0(0) 1 0.86800 0.63500 0.85000 5 0.82753 0.60365 0.71921 10 0.82461 0.60732 0.69473 20 0.82425 0.60872 0.68636 25 0.82425 0.60886 0.68578 30 0.82425 0.60886 0.68578 35 0.82425 0.60886 0.68578 Figs. 2.3- 2.7 are sketched to analyze the variations of thixotropic parameters ?1 and ?2? local buoyancy parameter ?? Hartman number ? and concentration buoyancy parameter ? on the fluid velocity ?0(?)? Figs. 2.3 and 2.4 depict

> 1**that the** fluid **velocity and** momentum **boundary layer thickness are** increasing

functions of

thixotropic parameters. By increasing thixotropic parameters, the values of material parameters ?1 and ?2 increase. An increase in the material parameters give rise to the fluid velocity and momentum boundary layer thickness. Fig. 2.5 shows that an increase in local buoyancy parameter gives rise to the fluid velocity and its associated boundary layer thickness. Increase in buoyancy parameter corresponds to stronger buoyancy force. Buoyancy force is an agent that causes

13**an increase in the** fluid flow **and** its related **boundary layer thickness.**

2**An increase in** Hartman **number reduces the fluid velocity.**

Hartman number involves the Lorentz force and

2**an increase in** Lorentz force **reduces the fluid velocity and boundary layer thickness**

(see Fig. 2.6). The Lorentz force provides a resistance to flow. From Fig. 2.7 it is observed that concentration buoyancy parameter enhances the velocity. To see the impacts of different parameters on the temperature ?(?)? Figs. 2.8-2.15 are potrayed. From Figs. 2.8 and 2.9, we have seen that the thixotropic parameters ?1 and ?2 are decreasing functions of

2**temperature and thermal boundary layer** thicknesses. Larger **values of**

?1 and ?2 correspond to stronger ?1 and ?2 showing a reduction in the temperature. We also noted that the thixotropic parameters have quite opposite effects on the fluid velocity and temperature. Figs. 2.10 and 2.11 present the effects of local buoyancy parameter and concentration buoyancy parameter

4**on the temperature. It is found that the temperature and thermal boundary layer thickness**

become smaller for larger values of local buoyancy parameter and concentration buoyancy parameter. Physically, both ? and ? depend on the buoyancy force. Larger values of ? and ? lead to the stronger buoyancy force. Such stronger buoyancy force reduced

13**the temperature and thermal boundary layer thickness.** Fig. 2.12 **is**

plotted to analyze the influence of Eckert number

4on the temperature. Here it is revealed that the temperature and thermal boundary layer thickness

are increased for larger Eckert number. This is because of the reason that

27heat energy is always stored in the liquid due to the frictional heating.

Such increase in Eckert number enhances the temperature at any point in the thermal boundary layer region. From Fig. 2.13, one can see that temperature is an increasing function of Hartman number. Larger values of Hartman number posses stronger Lorentz force. This stronger Lorentz force

13enhances the temperature and thermal boundary layer thickness.

Impact

2of Prandtl number ? ? on the temperature ?(?) is analyzed in Fig.

2.14.

2Thermal boundary layer thickness and temperature ?(?) are decreasing functions of

? ?? This is due to the fact that with an enhancement in Prandtl number ? ?, thermal diffusivity decreases

2which leads to a reduction in temperature ?(?)? Fig.

2.15 shows that temperature ?(?) increases with an increase in radiation parameter ?. Also thermal boundary layer thickness enhances with ? which is due to the fact that as thermal radiation parameter increases, the mean absorption coefficient ?? decreases which in results give rise to the divergence of radiative heat flux. Hence

16the rate of radiative heat transferred to the fluid

shoot up so that the fluid temperature increases. Figs. 2.16-2.22 are plotted to see the variations of ?1? ? 2? ?? ?? ??, ? and ? on the concentration ?(?)? Figs. 2.16 and 2.17 show the influence of thixotropic parameters on the concentration. From these Figs. we observed that increase in thixotropic parameters reduced the concentration and its related boundary layer thickness. We also analyzed that the effects of thixotropic parameters

26on the temperature and concentration are similar in a qualitative sense.

A comparison of Figs. 2.8, 2.9, 2.16 and 2.17 show that the variation in temperature are dominant in comparison to variation in concentration due to thixotropic parameters. Local buoyancy parameter and concentration buoyancy parameter are decreasing functions of con- centration (see Figs. 2.18 and 2.19). Fig. 2.20 shows

4that an increase in Eckert number leads to a decrease in the concentration and its related boundary layer thickness.

For higher Eckert number, concentration specie diffuses due to which the concentration field decreases. Fig. 2.21 illustrates the variations of Hartman number on the fluid concentration. It is revealed that concentration is a decreasing function of Hartman number. We conclude that the variations in temperature are more dominant when compared with the variations in the concentration.

27From Fig. 2. 22 it is observed that the associated boundary layer thickness

and concentration profile decrease when thermophoretic parameter ? increases. K2 = 0.2, l = 0.4, t = 0.3, M = 0.6, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3, R = 0.4 1 0.8 K1 = 0.0 K1 = 1.0 0.6 K1 = 2.0 K1 = 3.0 f'?h? 0.4 0.2 0 0 1 2 3 4 5 h Fig. 2.3: Influence of ?1 on ?0(?) K1=0.1,l=0.4,t=0.3,M=0.6,Sc=0.7,Ec=0.5,Pr=1.0,N=0.3,R=0.4 1 K2 = 0.0 K2 = 0.7 0.8 K2 = 1.4 K2 = 2.0

4f'?h? 0.6 0.4 0.2 0 0 1 2 3 4 5 h

Fig. 2.4: Influence of ?2 on ?0(?) 1 l = -0.5 0.8

3l = 0.0 l = 0.5 0.6 l = 1.0

f'?h? 0.4 0.2 0 0 2 4 6 8 h Fig. 2.5: Influence of ? on ?0(?) K1=0.1,K2=0.2,l=0.4,t=0.3,Sc=0.7,Ec=0.5,Pr=1.0,N=0.3,R=0.4

41 M=0.0 0.8 M = 0.5 M = 1.0 0.6 M = 1.5 f '?h? 0. 4 0. 2 0

0 2 4 6 8 h Fig. 2.6: Influence of ? on ?0(?) K1=0.1,K2=0.2,l=0.4,t=0.3,M=0.6,Sc=0.7,Ec=0.5,Pr=1.0,R=0.4 1 0.8

14N = 0. 0 N = 0.5 0.6 N = 1.0 N = 1 .5

f'?h? 0.4 0.2 0 0 2 4 6 8 h Fig. 2.7: Influence of ? on ?0(?) 1 0.8 K1 = 0.0 K1 = 1.0 0.6 K1 = 2.0 K1 = 3.0

1**q?h? 0.4 0.2 0 0 2 4** 6 8 **h**

Fig. 2.8: Influence of ?1 on ?(?) K1 = 0.1, l = 0.4, t = 0.3, M = 0.6, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3, R = 0.4 1 0.8 K2 = 0.0 K2 = 0.7 0.6 K2 = 1.4 K2 = 2.0

1**q?h? 0.4 0.2 0 0 2 4** 6 8 **h**

Fig. 2.9: Influence of ?2 on ?(?) K1=0.1,K2=0.2,t=0.3,M=0.6,Sc=0.7,Ec=0.5,Pr=1.0,N=0.3,R=0.4 1 0.8 l = -0.5

3**l = 0.0** 0.6 **l = 0.5 l = 1.0**

1**q?h? 0.4 0.2 0 0 2 4** 6 8 **h**

Fig. 2.10: Influence of ? on ?(?) 1 0.8 N = 0.0 N = 1.5 0.6 N = 3.0 N = 4.5

1**q?h? 0.4 0.2 0 0 2 4** 6 8 **h**

Fig. 2.11: Influence of ? on ?(?) K1=0.1,K2=0.2,l=0.4,t=0.3,M=0.6,Sc=0.7,Pr=1.0,N=0.3,R=0.4 1 Ec=0.0 0.8

11**Ec = 0.** 5 **Ec = 1.0** 0.6 **Ec = 1.5**

1**q?h? 0.4 0.2 0 0 2 4 h 6 8**

Fig. 2.12: Influence of ?? on ?(?) K1=0.1,K2=0.2,l=0.4,t=0.3,Sc=0.7,Ec=0.5,Pr=1.0,N=0.3,R=0.4 1 0.8

4**M=0.0 M=0.5** 0.6 **M=1.0 M=1.5** q **?h? 0.** 4 **0.** 2 **0**

0 2 4 6 8 h Fig. 2.13: Influence of ? on ?(?) 1 0.8 Pr = 0.5 Pr = 1.0 0.6 Pr = 1.5 Pr = 2.0

1**q?h? 0.4 0.2 0 0 2 4** 6 8 **h**

Fig. 2.14: Influence of Pr on ?(?) K1 = 0.1, K2 = 0.2, l = 0.4, t = 0.3, M = 0.6, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3 1 0.8 R = 0.0 R = 0.4 0.6 R = 0.8 R = 1.2

1q?h? 0.4 0.2 0 0 2 4 6 8 h

Fig. 2.15: Influence of ? on ?(?) K2 = 0.2, I = 0.4, t = 0.3, M = 0.6, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3, R = 0.4 1 0.8 K1 = 0.0 K1 = 1.0 0.6 K1 = 2.0 K1 = 3.0 f?h? 0.4 0.2 0 0 1 2 3 4 5 6 7 h Fig. 2.16: Influence of ?1 on ?(?) 1 0.8 K2 = 0.0 K2 = 0.7 0.6 K2 = 1.4 K2 = 2.0 f?h? 0.4 0.2 0 0 1 2 3 4 5 6 7 h Fig. 2.17: Influence of ?2 on ?(?) K1=0.1,K2=0.2,t=0.3,M=0.6,Sc=0.7,Ec=0.5,Pr=1.0,N=0.3,R=0.4 1 0.8 I = -0.5

3I = 0.0 0.6 I = 0.5 I = 1.0

f?h? 0.4 0.2 0 0 1 2 3 4 5 6 7 h Fig. 2.18: Influence of ? on ?(?) K1=0.1,K2=0.2,I=0.4,t=0.3,M=0.6,Sc=0.7,Ec=0.5,Pr=1.0,R=0.4 1 0.8

14N = 0. 0 N = 0.5 0.6 N = 1.0 N = 1 .5

f?h? 0.4 0.2 0 0 2 4 6 8 h Fig. 2.19: Influence of ? on ?(?) 1 Ec = 0.0 0.8

11Ec = 0. 5 Ec = 1.0 0.6 Ec = 1.5 f?h? 0.

4 0.2 0 0 2 4 6 8 h Fig. 2.20: Influence of ?? on ?(?) K1 = 0.1, K2 = 0.2, I = 0.4, t = 0.3, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3, R = 0.4

41 M = 0.0 0.8 M = 0.5 M = 1.0 0.6 M = 1.5 f?h? 0. 4 0. 2 0

0 2 4 6 8 h Fig. 2.21: Influence of ? on ?(?) K1 = 0.1, K2 = 0.2, I = 0.4, M = 0.6, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3, R = 0.4 1 t = 0.0 0.8 t = 0.7 t = 1.4 0.6 t = 2.0 f?h? 0.4 0.2 0 0 1 2 3 4 5 6 7 h Fig. 2.22: Influence of ? on ?(?) Table 2.2 is made to find

4the numerical values of skin-friction coefficient for various values of

?1? ?2? ?? ? and ? when the other parameters are fixed. We noted that the values of skin-friction coefficient are decreased by increasing ?1? ?2 and ?? However reverse situation is noted for ? and ?? Table 2. 3 analyzes the

1numerical values of local Nusselt number and local Sherwood number for different values of emerging parameters

in viscous and non-Newtonian cases. We observed that the

> 7values of local Nusselt number and local Sherwood number is large in

the case of non-Newtonian fluid when compared with the case of viscous fluid. Table 2.4 is computed for the comparison of ?00(0) and ?0(0) through different values of Prandtl number and local buoyancy parameter when all other parameters are zero. From this Table, it is analyzed that our series solutions have a good agreement with the numerical solutions in a limiting case. Table 2.2:

> 4Numerical values of skin-friction coefficient for different values of

?1? ?2? ?? ? and ? when ? = 0?3, ? = 0?4, ?? = 1?0, ?? = 0?5 and ?? = 0?7? ?1 ?2 ? ? ? −??1??2?? 0.0 0.2 0.6 0.4 0.4 0.84799 0.3 0.79227 0.5 0.76088 0.2 0.0 0.6 0.4 0.4 0.84923 0.3 0.79373 0.5 0.76247 0.2 0.2 0.0 0.4 0.4 0.71906 0.5 0.80959 1.0 1.01672 0.2 0.2 0.5 0.0 0.4 0.95561 0.4 0.76145 0.8 0.56391 0.2 0.2 0.5 0.3 0.0 0.80923 1.0 0.81003 2.0 0.81106 Table 2.3:

> 1Numerical values of local Nusselt number and local Sherwood number for
>
> different values of

?? ??? ? ?? ? and ? when ? = 0?3 and ? = 0?4? ? ?? ?? ? ? ? ? −?0(0) −?0(0) ?1 = ?2 = 0?0 −?0(0) −?0(0) ?1 = ?2 = 0?2 0.0 0.5 1.0 0.5 0.7 0.5 1.0 1.5 0.5 0.5 1.0 1.0 0.3 0.3 0.4 0.4 0.67883 0.58160 0.29936 0.58763 0.57572 0.57065 0.87335 0.86284 0.83862 0.62245 1.10358 1.45081 0.69254 0.60339 0.36075 0.60862 0.59829 0.59387 0.88088 0.86671 0.84058 0.68707 1.10750 1.45463 0.5 0.7 0.3 0.6 0.8 1.0 0.3 0.4 0.66211 0.54179 0.46338 0.84754 0.87021 0.88469 0.67841 0.56632 0.49302 0.85318 0.87341 0.88618 0.5 0.7 0.5 0.5 0.8 1.3 0.3 0.4 0.41490 0.52313 0.65577 0.88107 0.86855 0.85452 0.42578 0.54006 0.68482 0.88426 0.87243 0.85721 0.5 0.7 0.5 1.0 0.0 0.5 1.0 0.4 0.55688 0.59650 0.62949 0.85559 0.86737 0.87800 0.58355 0.61561 0.64317 0.85988 0.87109 0.88142 0.5 0.7 0.5 1.0 0.3 0.0 0.5 1.0 0.70547 0.55912 0.47505 0.84788 0.86538 0.87469 0.73986 0.57908 0.48877 0.85133 0.86942 0.87958 Table 2.4: Comparison of ? 00(0) and ?0(0) with Singh et al. [76] for different values of ? ? and ?? Singh et al. [76] Present solutions ?00(0) ?0(0) ?00(0) ?0(0) ? ? = 0?7 -1.00 -0.79366 -1.00000 -0.79373 ? = 0?0 ? ? = 0?7 -0.50751 -0.89613 -0.50767 -0.89614 ? = 1?0 ? ? = 0?7 2.57771 -1.17244 2.57789 -1.17287 ? = 10?0 ? ? = 10?0 -1.00 -3.72067 -1.00000 -3.72033 ? = 0?0 ? ? = 10?0 -0.82568 -3.74856 -0.82529 -3.74718 ? = 1?0 ? ? = 10?0 0.61966 -3.95235 0.61940 -3.95266 ? = 10?0 2.4 Closing remarks Effects of Joule heating, thermophoresis and thermal radiation in MHD flow of thixotropic fluid are analyzed. The main observations are listed below. • The non-Newtonian parameters ?1 and ?2 have similar effects on the velocity in a qual- itative sense. • The effects of ? and ? on the velocity field are quite opposite. • The variations of ?1 and ?2 on temperature and concentration are opposite in comparison to velocity. •

> 13An increase in Schmidt number corresponds to a smaller variation in

concentration field. Chapter 3 Three-dimensional

3mixed convection flow of viscoelastic fluid with thermal radiation and

convective conditions The objective of this chapter is to examine the thermal radiation effect in three-dimensional

21mixed convection flow of viscoelastic fluid. The

resulting

14partial differential equations are re- duced into a system of nonlinear ordinary differential equations

using appropriate transforma- tions. The series solutions are developed through a modern technique known as the homotopy analysis method. The convergent expressions of the velocity components and temperature are derived. The solutions obtained are dependent on seven sundry parameters including the viscoelastic parameter, mixed convection parameter, ratio parameter, temperature exponent, Prandtl number, Biot number and radiation parameter. Discussion to these parameters is made via plots. 3.1 Mathematical analysis Let us consider three-dimensional

23mixed convection flow of second grade fluid due to an expo- nentially stretching surface. The

surface coincides with the plane ? = 0 and the flow is confined in the region ? ? 0? The surface also possess the convective boundary condition. Influence of thermal radiation through Rosseland's approximation is taken into account. Flow configuration is given below. Fig. 3.1: Geometry of problem The governing boundary layer equations for steady three-dimensional flow of viscoelastic fluid can be put into the forms: ?? ?? ?? ?? ?? ?? = 0? + + (3.1) ? ???? + ? ???? + ? ???? = ? ??2??2 + ??1 □ ???3???2 ? +? ?3? − ???? ??2??2 + ???? ??2??2 ??3 □ +2???? ???2??? + 2???? ??2??2 □□ +??? (? − ?□∞)? □ □□(3.2) ?? ?? ?? ?2? ?1 ?? ?2? ?? ?2? ? ?? + ? ?? + ? ?? = ? ??2 ? □ ????2 + ? ??3 + ? ?3? ?3? − ?? ??2 + ?? ??2 □ +2???? ???2??? + 2???? ??2??2 □□ ? (3.3) ? ???? + ? ???? + ? ???? ???? ??2??2 ? 1?? ????? ? □ □ □□ = − (3.4) where ?? ? and ?

1are the velocity components in the ?−? ?− and ?−directions respectively, ?1 is the material fluid parameter, ? = (???) is the kinematic viscosity, ? is the dynamic viscosity, ? is the

fluid density, ?

1is the fluid temperature, ? is the gravitational acceleration, ?? is thermal

expansion **coefficient of**

temperature, ??

1is the specific heat, ? is the thermal conductivity and ?? **the**

radiative heat flux.

8By using the Rosseland approximation, the radiative heat flux ?? is given by

?? = −43???? ????4 (3.5)

14where ?? is the Stefan- Boltzmann **constant and ?? the** mean **absorption coefficient.**

8By using the Rosseland approximation, the present analysis is limited to optically thick fluids. If the temperature differences are sufficiently small then Eq. (3.5) **can be linearized by expanding** ? 4 **into the Taylor series** about ?∞, **which after neglecting higher order terms takes the form:**

? 4 = 4?∞3? − 3?∞4? By using Eqs. (3.5) and (3.6), Eq. (3.4) reduces to ? ?? ?? ?? ? ?2? 16???∞3 ?2? ?? + ? ?? + ? ?? ??? ??2 3????? ??2 = − ? The boundary conditions can be expressed as follows: ?=??? ?=??? ?=0? −??? =?(??−?), at?=0? ?? ? → 0? ? → 0? ? → ?∞ as ? → ∞? (3.6) (3.7) (3.8) where subscript ? corresponds to the wall condition, ?

3is the thermal conductivity, ?? is the hot **fluid** temperature, ? **is the heat**

transfer coefficient and ?∞ is the free stream temperature. The velocities and temperature are taken in the following forms: ?? = ?0? ?+?? ? ?? = ?0? ?+?? ? ?? = ?? = ?∞ + ?0? ?(?2?+?) (3.9) in which ?0? ?0 are the constants, ? is the reference length and ? is the temperature exponent. The

14mathematical analysis of the problem is simplified by using **the**

transformations [25]: ? = ?0? ?+?? ? 0(?)? ? = ?0? ?+?? ?0(?)? ? = − ?2??0 1?2 ? ?2+?? (? + ?? 0 + ? + ??0)? μ ¶ ? = ?∞ + ?0??(?2?+?) ?(?)? ? = 2?? ??2+?? ?? ? 0 1?2 (3.10) μ ¶ Incompressibility condition is now clearly satisfied whereas Eqs. (3?2) − (3?8) give ? 000 + (? + ?)? 00 − 2(? 0 + ?0)? 0 + ? 6? 000? 0 + (3?00 − 3? 00 + ??000)? 00 □ +(4?0 + 2??00)?000 − (? + ? + ??0)?0000 □ + 2?? = 0? (3.11) □ □ ?000 + (? + ?)?00 − 2(? 0 + ?0)?0 + ? 6?000?0 + (3? 00 − 3?00 + ?? 000)?00 □ +(4? 0 + 2?? 00)?000 − (? + ? +

?? 0)?0000 □ = 0? (3.12) (1 + 4 ?)?00 + ? ?(? + ?)?0 − Pr ?(? 0 + ?0)? = 0? □ □ 3 (3.13) ? = 0? ? = 0? ? 0 = 1? ?0 = ?, ?0 = −?1(1 − ?(0)) at ? = 0? (3.14) ?0 → 0? ?0 → 0? ? → 0 as ? → ∞ (3.15) in which ? is the viscoelastic parameter? ? is the ratio parameter, Pr

---

1**is the Prandtl number, ??? is the** local Grashof **number, ? is the** radiation

parameter, ? **is the**

---

temperature exponent, ?1

19**is the** Biot **number, ??? is the** local Reynold **number, ? is the** mixed convection

parameter

---

and prime denotes the differentiation with respect to ?. These can be defined as ? = ??? = ?1?? ?0 ??? 2?? 2?? ? ? = ?0 ? Pr = 4?∗?∞3 ? ? ?? = ??? ? ?1 = ? r ?? ? ?0? ? ?+?? ? ? = ?R?e2?? ? ??? = ??? (???−2 ?∞)?3 ? μ ¶ ? (3.16) The skin-friction coefficients in the ? and ? directions are given by ?? ? = ?? ? = ? ?? 1?2???2 ? 1?2???2 ? ? ?? (3.17) (3.18) where ?2? ?? ?? ???|?=0 = □ ?? + ?0 ? ?? ? ????2??? + ????2??? + ? ??2 + ?? ?? □ +???? ???? + 2???? ???? − ???? ???? □□ ? □ □ ?=0 ???|?=0 = ? ?? ? ?? ?? □□ + ?0 ???2??? + ????2??? + ????2??2 + ?? ?? . (3.19) □ ?? □ +???? ???? + 2???? ???? − ???? ???? □□ □ □ ?=0 □□ By using Eq. (3.19) in Eqs. (3.17) and (3.18), the non-dimensional forms of skin friction coefficients are as follows: ??? = ?=0 ? (3.20) μ Re −1?2 ?00 + ? −(? + ?)?000 + 5(?0 + ?0)?00 + 2?0?00 + 2?0?00 2 ¶ £ ¡ ¢¤ ??? = Re −1?2 ?00 + ? −(? + ?)?000 + 5(?0 + ?0)?00 + 2?0?00 + 2?0?00 ? μ 2 ¶ ?=0 £ ¡ ¢¤ Further the local Nusselt number has the form − 16?3????∞3 + ? ???? ? ?? = ³?(?? − ?∞)? ´? = − ? Re 1?2 (1 + 4?)?0(0)? μ 2 ¶ 3 3.2 Solutions development The initial guesses and auxiliary linear operators in the desired HAM solutions are ?0(?) = 1 − ?−? ? ?0(?) = ? 1 − ?−? ? ?0(?) = ?1 exp(−?) ? ¡ ¢ ¡ ¢ 1 + ?1 L? = ?000 − ?0? L? = ?000 − ?0? L? = ?00 − ?? (3.21) (3.22) (3.23) (3.24) subject to the properties L?(?1+?2??+?3?−?) = 0? L?(?4+?5??+?6?−?)=0? L?(?7??+?8?−?) = 0? (3.25) in which ?? (? = 1 − 8) are the arbitrary constants, L? ? L? and L? are the linear operators and ?0(?)? ?0(?) and ?0(?) are the initial guesses. Following the idea in ref. [78] the zeroth order deformation problems are (1 − ?) L? ?ˆ(?; ?) − ?0(?) = ?~? N? ?ˆ(?; ?)? ?ˆ(?; ?) ? h i h i (1 − ?) L? [?ˆ(?; ?) − ?0(?)] = ?~?N?

31**?ˆ(?; ?)? ?ˆ(?; ?) ? (1**− ?) L? **ˆ?(?; ?)** − ?0(?) = ?~?N? **?ˆ(?; ?)? ?ˆ(?;** ?)? ˆ?(?? ?)

? h **i** h **i**

---

h i

12**?ˆ(0; ?) = 0? ?ˆ0(0;** ?) = 1? **?ˆ0(∞; ?) = 0? ?ˆ(0; ?) = 0? ?ˆ0(0; ?) = ?? ?ˆ0(∞; ?) =**

0? ?ˆ0(0? ?) = −?1[1 − **?(0? ?)]? ?ˆ(∞? ?) = 0?**

For ? = 0 and ? = 1 one has

12**?ˆ(?;0) = ?0(?)? ?ˆ(?;0) = ?0(?)? ˆ?(??0) = ?0(?)?** and **?ˆ(?;** 1) = ?(?)? **?ˆ(?; 1) =**

?(?)? **ˆ?(??**

1) = ?(?)? (3.26) (3.27) (3.28) (3.29) (3.30) (3.31) Note that when ? increases from 0 to 1 then ? (?? ?)? ? (?? ?) and ?(?? ?) vary from ?0(?)? ?0(?) and ?0(?) to ? (?)? ?(?) and ?(?)? So as

34the embedding parameter $? \in [0? \ 1]$ **increases from 0 to 1,**

the solutions ?ˆ(?; ?)? ?ˆ(?; ?) and ˆ?(?; ?) of the zeroth order deformation equations deform from the initial guesses ?0(?)? ?0(?) and ?0(?) to the exact solutions ? (?)? ?(?) and ?(?) of the original nonlinear differential equations. Such kind of continuous variation is called deformation in topology and that is why the Eqs. (3.26-3.28) are called the zeroth order deformation equations. The values of the nonlinear operators are given below: N?[?ˆ(???)??ˆ(???)] = ?3??ˆ(??3??)

30−2 ??ˆ?(????) ??ˆ(?????) ??ˆ?(????) + ?ˆ(???)+?ˆ(???) ?2 **??ˆ(?? 2??) Ã + !** **6 ??ˆ?(????)?** 3 **??ˆ(?? 3??) + 3 ?2?ˆ?(??2??)−3 ?2??ˆ(?? 2??) +??3 ??ˆ(?? 3??) ³ ´ +? □ ?2 ??ˆ(?? 2??) + 4 ³??ˆ?(????)**

+2? ?2?ˆ(???) ?3

10?ˆ(???) ??2 ??3 ´□ □ − **?ˆ(?? ?)³+ ?ˆ(?? ?) + ? ??ˆ?(????)´?4 ??ˆ(??4??)** +2 **? ˆ□?(?? ?)? ³ ´ □ □ (3.32) N ?[?ˆ(?? ?)? ?ˆ(?? ?)] = ?3 ?ˆ(?? ?) ??ˆ(???)− 2 ?? ?? ˆ(?? ?) □ +??ˆ?(????) □ ?? + ?ˆ(?? ?) ?2 ?ˆ(?? ?) ??3 □ +?ˆ(?? ?) □ ??2 6 ??ˆ? (????□)?** 3 **??ˆ(?? 3??) + □3?2 ??ˆ(?? 2??) −3?□2 ??ˆ(??**

2??) +??3□??ˆ(??3??) +? □ ?2??ˆ(??2??) + 4³??ˆ?(????) + 2?

9?2??ˆ(?? 2??) ?3?ˆ?(??3??)´ □(?3.33) □ − ?ˆ(?? ?)³+ ?ˆ(?? ?) + ? ??ˆ?(????)´? 4 ?ˆ?(?? 4??) ³ ´ □ □ □ N ?[?ˆ(???)??ˆ(???)??ˆ(???)] = (1+ 4?) 3 ?2 ?ˆ(?? ?) + Pr (?ˆ(?? ?) + ?ˆ(?? ?))??ˆ(?? ?) ??2 ?? − Pr ? Ã ??ˆ(?? ?) ??ˆ(?? ?) ?ˆ(??

?)? + (3.34) ?? ?? ! Here ~? ? ~? and ~? are the

4non-zero auxiliary parameters and N? ? N? **and** N? **the nonlinear operators.**

Taylor series expansion gives ?(???)=?0(?)+?=1??(?)??? ??(?)= ?1! ?????(??;?)¯?=0 ∞ ? (3.35) ? (???)=?0(?)+?=1??(?)??? ??(?)= ?1! ?????(??;?)¯¯?=0 ∞ ¯ ? (3.36) P ¯ ?(?? ?) = ?0(?) ?=P1??(?)??? ??(?)= ?1! ?????(??;?)¯?¯¯=0 ∞ ¯ ? (3.37)

3**where the convergence of above** sePries **strongly depends upon** ~?? ~? an‾d ~?? **Considering that** ‾~? ? ~? and ~? are

chosen in such a manner that Eqs. (3?35) − (3?37) converge at ? = 1 then ?(?) = ?0(?) + ??(?)? ∞ (3.38) ?=1 P ?(?) = ?0(?) + ??(?)? ∞ ?=1 P∞ ?(?) = ?0(?) + ??(?)? ?=1 P The corresponding problems at mth order deformations satisfy L?[??(?) − ????−1(?)] = ~?R??(?)? L?[??(?) − ????−1(?)] = ~?R??(?)? L?[??(?) − ????−1(?)] = ~?R??(?)? (3.39) (3.40) (3.41) (3.42) (3.43) ??(0) = ??0(0) = ??0(∞) = 0? ??(0) = ?0?(0) = ??0(∞) = 0? ??0(0) − ?1??0(0) = ??(∞) = 0? (3.44) R??(?) = ??000−1(?) − 2 ??0−1−???0 − 2 ??0−1−???0 + (??−1−??00 + ??−1−??00) ?−1 ?−1 ?−1 ?=0 ?=0 ?=0 6 ?−P1 ??0−1−???000 + 3 ??00−1−??00 − 3 ??00−1−??00 ?−P1 ?P−1 □ ?−P1???000−1−??00 + 4 ?P−1 ??0−1−???000 + 2 ?P−1 ??0?0−1−???000 ?=0 ?=0 ?=0 □ +? + ?=0 ?=0 ?=0 − □ P?−1 ??−1−???0000 ?P−1 ??−1−???0000 − − ?=0 ?=0 ?−P1??0−1−? ??0000 +2??□? ?=0 P P P □ □ (3.45) R??(?) = ??000−1(?) − 2 ?0?−1−???0 − 2 ??0−1−???0 + ?−1 ?−1 ?−1 (??−1−??00 + ??−1−??0?0) ?=0 ?=0 ?=0 6 ?−P1 ?0?−1−???000 + 3 ??00−1−??00 − 3 ??00−1−??0?0 ?−P1 ?P−1 □ ?=0 ?=0 □ + ?−P1???000−1−??00 + 4 ?P−1 ??0−1−??0?00 + 2 ?P−1 ??00−1−???000 ?=0 +? ? (3.46) ?=0 ?=0 ?=0 − □ P?−1 ??−1−???0000 ?P−1 ??−1−???0000 ?−P1???0−1−??0000 − − □ ?=0 ?=0 ?=0 P P P □ □ R??(?) = (1 + 43?)?0?0−1 + Pr ?−1 ?=0 (?0?−1−?? + ?0?−1−???) ?−1 P − Pr ? (??0−1−??? + ??0−1−???)? (3.47) ?=0 P ?? = 0? ? ≤ 1? □ (3.48) 1? ? ? 1? The mth order deformation problems have the solutions □ ??(?) = ??∗(?) + ?1 + ?2?? + ?3?−?? (3.49) ??(?) = ??∗(?) + ?4 + ?5?? + ?6?−?? (3.50) ??(?) = ?∗?(?) + ?7?? + ?8?−?? (3.51) where the special solutions are ??∗? ??∗ and ?∗?. 3.3 Convergence analysis We recall that the series (3?38) − (3?40) contain the auxiliary parameters ~? ? ~? and ~?. These parameters are useful to adjust and control the convergence of homotopic solutions. Hence the ~−curves are sketched at 15?? order of approximations in order to determine the suitable ranges for ~? ? ~? and ~?. Fig. 3.2 denotes that the ranges of admissible values of ~? ? ~? and ~? are −0?7 ≤ ~? ≤ −0?2? −0?7 ≤ ~? ≤ −0?1 and −0?8 ≤ ~? ≤ −0?2? Table 3.1 presents the numerical values of −? 00(0)? −?00(0) and −?0(0) for different order of approximations when ~? = −0?5? ~? = −0?6 and ~? = −0?7? It is seen that the values of −?00(0) and −?00(0) converge from 20th order of deformations whereas the values of −?0(0) converge from 25th order approximations. Further, it is observed that we have to compute less deformations for the velocities in comparison to temperature for convergent series solutions. f''?0?,g''?0?,q'?0? 0.5 b = l = g1 = 0.5, Pr = 1.2, R = 0.3, K = A = 0.2 -1.0 -0.8 -0.6 -0.4 -0.2 0.2 hf,hg,hq q'(0) g''(0) -05. f''(0) -10. -15. Fig. 3.2: ~ −curves for the functions ?? ? and ? Table 3

2.**1: Convergence of** series **solutions for different order of approximations when**

? = 0?1? ? = 0?2? Pr = 1?2? ?1 = 0?6? ? = 0?2? ~? = −0?5? ~? = −0?6 and ~? = −0?7? order of approximations 1 5 10 15 20 25 −? 00 (0) 1.06111 1.02482 1.02609 1.02623 1.02618 1.02618 −?00 (0) 0.544444 0.548057 0.548092 0.548043 0.548053 0.548053 −?0(0) 0.317778 0.305581 0.305729 0.305744 0.305738 0.305738 3.4 Discussion of results The effects of ratio parameter ?? viscoelastic parameter ?? mixed convection parameter ?? Biot number ?1 and radiation parameter ? on the velocity component ?0(?) are shown in the Figs. 3.3-3.5. It is observed from Fig. 3.3 that velocity component ?0(?)

2**and thermal boundary layer thickness are decreasing functions of ratio**

parameter

?? This is due to the fact that with the increase of ratio parameter ?? the ?-component of velocity coefficient decreases which

2**leads to a decrease in** both the momentum **boundary layer and**

velocity component ?0(?)? Fig. 3.4 illustrates the influence of viscoelastic parameter ? on the velocity component ?0(?)? It is clear that both the boundary layer and velocity component ?0(?) increase when the viscoelastic parameter increases. Influence of mixed convection parameter ? on the velocity component ?0(?) is analyzed in Fig. 3.5. Increase in mixed convection parameter ? shows an increase in velocity component ?0(?). This is due to the fact that the buoyancy forces are much more effective rather than the viscous forces. Figs. 3.6 and 3.7 illustrate the variations of ratio parameter ? and viscoelastic parameter ? on the velocity component ?0(?)? Variation of ratio parameter ? is analyzed in Fig. 3.6. Through comparative study with Fig. 3.3 it is noted that ?0(?) decreases while ?0(?) increases when ? increases. Physically, when ? increases from zero, the lateral surface starts moving in ?−direction and thus the velocity component ?0(?) increases and the velocity component ?0(?) decreases. Fig. 3.7 is plotted to see the variation of viscoelastic parameter ? on the velocity component ?0(?)?

1**It is found that** both the **velocity** component ?0(?) **and** momentum **boundary layer** thicknesses **are increasing functions of**

? . It is revealed from Figs. 3.4 and 3.7 that the effect of ? on both the velocities are qualitatively similar. Figs. 3.8-3.14 are sketched to see the effects of ratio parameter ?? viscoelastic parameter ?, the temperature exponent ?? Biot number ?1? mixed convection parameter ?? Radiation para- meter ? and Prandtl number Pr on the temperature ?(?)? Fig. 3.8 is drawn to see the impact of ratio parameter ?

4**on the temperature ?(?). It is** noted **that the temperature ?(?) and** also the **thermal boundary layer thickness**

decrease with increasing ?. Variation of the viscoelastic parameter ? on the temperature ?(?) is shown in Fig. 3?9. Here both

2**the temperature and thermal boundary layer thickness are decreasing functions of**

?. Variation of mixed convection parameter ? is analyzed in Fig. 3?10. It is seen that both the

2**temperature ?(?) and thermal boundary layer thickness are decreasing functions of** mixed convection **parameter**

?? Fig. 3?11 presents the plots for the variation of Biot number ?1? Note that ?(?) increases when ?1 increases. The

> 2**thermal boundary layer thickness** is also **increasing** function **of**

?1. It is also noted that the fluid temperature is zero when the Biot number vanishes. Influence of temperature exponent ? is displayed in Fig. 3.12.

> 4**It is found that** both **the temperature ?(?) and thermal boundary layer thickness** decrease **when** A **is increased.**

Also both the

> 13**temperature ?(?) and thermal boundary layer thickness are increasing functions of**

thermal radiation parameter ? (see Fig. 3.13). It is observed that an increase in ? has the ability to increase the thermal boundary layer. It is due to the fact that when the thermal radiation parameter increases, the mean absorption coefficient ?? will be decreased which in turn increases the

> 16**divergence of the radiative heat flux.**

Hence

> 16**the rate of radiative heat** transfer **to the fluid** is increased **and consequently the fluid temperature increases.** Fig. 3?14 **is**

plotted to see the effects of Pr on ?(?).

> 4**It is** noticed **that** both **the temperature** profile **and thermal boundary layer thickness**

are decreasing functions of Pr. In fact when Pr increases then thermal diffusivity decreases. This indicates reduction in energy transfer ability and ultimate it results in the decrease of thermal boundary layer. f'?h? 1.0 0.8 K = 0.2, g1 = A = 0.4, l = 0.5, R = 0.3, Pr = 0.7 0.6 0.4 b = 0.0, 0.3, 0.6, 1.0 0.2 0 1 2 3 4 5 6 h Fig. 3.3: Influence of ? on the velocity ?0(?). f'?h? 1.0 b = 0.2, g1 = A = 0.4, l = 0.5, R = 0.3, Pr = 0.7 0.8 0.6 K = 0.0, 0.2, 0.4, 0.6 0.4 0.2 0 1 2 3 4 5 6 h Fig. 3.4: Influence of ? on the velocity ?0(?). f'?h? 1.0 0.8 b = 0.2, g1 = A = 0.4, K = R = 0.3, Pr = 0.7 0.6 l = 0.0, 0.3, 0.6, 0.9 0.4 0.2 0 1 2 3 4 5 6 h Fig. 3.5: Influence of

? on the velocity ?0(?). g'?h? 0.4 0.3 K = 0.2, g1 = A = 0.4, l = 0.5, R = 0.3, Pr = 0.7 0.2 b = 0.1, 0.2, 0.3, 0.4 0.1 1 2 3 4 5 6 h Fig. 3

> ### 2.6: Influence of ? on the velocity ?0(?). g'? h? 0.

20 b = 0.2, g1 = A = 0.4, l = 0.5, R = 0.3, Pr = 0.7 0.15 0.10 K = 0.0, 0.2, 0.4, 0.6 0.05 1 2 3 4 5 6 h Fig. 3.7: Influence of ? on the velocity ?0(?). q?h? 0.35 0.30 K = 0.2, g1 = A = 0.4, l = 0.5, R = 0.3, Pr = 0.7 0.25 0.20 0.15 b = 0.0, 0.5, 1.0, 1.5 0.10 0.05 1 2 3 4 5 6 h Fig. 3.8: Influence of ? on the temperature ? (?). q?h? 0.35 0.30 b = 0.2, g1 = A = 0.4, l = 0.5, R = 0.3, Pr = 0.7 0.25 0.20 K = 0.0, 0.5, 1.0, 1.5 0.15 0.10 0.05 1 2 3 4 5 6 h Fig. 3.9: Influence of ? on the temperature ?(?). q?h? 0.35 0.30 K = R = 0.3, b = g1 = A = 0.4, Pr = 0.7 0.25 0.20 0.15 l = 0.0, 0.5, 1.0, 1.5 0.10 0.05 1 2 3 4 5 6 h Fig. 3.10: Influence of ? on the tenperature ?(?). q?h? 0.6 0.5 K = R = 0.3,

> ### 3b = = 0. 4, l = 0. 5, Pr = 0.7 A 0. 4 0.

3 g1 = 0.1, 0.4, 0.7, 1.0 0.2 0.1 1 2 3 4 5 6 h Fig. 3.11: Influence of ?1 on the temperature ?(?). q?h? 0.3 K = R = 0.3, b = g1 = 0.4, l = 0.5, Pr = 0.7 0.2 A = 0.0, 0.3, 0.6, 0.9 0.1 1 2 3 4 5 6 h Fig. 3.12: Influence of ? on the temperature ?(?). q?h? 0.4 K = 0.3, b = 0.2, A = g1 = 0.4, l = 0.5, Pr = 0.7 0.3 R = 0.0, 0.3, 0.6, 1.0 0.2 0.1 1 2 3 4 5 6 h Fig. 3.13: Influence of ? on the temperature ?(?). q?h? 0.4 K = R = 0.3, b = 0.2, A = g1 = 0.4, l = 0.5 0.3 0.2 Pr = 0.1, 0.5, 1.0, 1.5 0.1 1 2 3 4 5 6 h Fig. 3.14: Influence of ? on the temperature ?(?). Table 3.2 includes the values for comparison of existing solutions with the previous available solutions in a limiting case when ? = ? = ?1 = ? = 0 and ? varies. This Table presents an excellent agreement with the previous available solutions. Table 3.3 is computed to see the influences of viscoelastic parameter ? and ratio parameter ? on skin friction coefficients in the ? and ? directions. It is noted that ? has quite opposite effect on skin friction coefficients while quite similar effect is seen within the increase of ratio parameter ?. Table 3.4 examines the impact of viscoelastic parameter ?, mixed convection parameter ?, ratio parameter ?, Biot number ?1, radiation parameter ?, Prandtl number Pr and temperature exponent ? on the

> ### 2local Nusselt number (rate of heat transfer at the wall).

It is noted that the value of rate of heat transfer increases for larger viscoelastic parameter ?, mixed convection parameter ?, ratio parameter ?, Biot number ?1, Prandtl number Pr and temperature exponent ? while it decreases through an increase in radiation parameter R. Table 3.2: Comparative values of −? 00(0)? −?00(0) and ?(∞) + ?(∞) for different values ? when ? = ? = ?1 = ? = 0? Liu et al. [25] Present results ? −? 00(0) −?00(0) ?(∞) + ?(∞) −?00(0) −?00(0) ?(∞) + ?(∞) 0.0 1.28180856 0 0.50 1.56988846 0.78494423 1.00 1.81275105 1.81275105 0.90564383 1.10918263 1.28077378 1.28181 0 0.90564 1.56989 0.78494 1.10918 1.81275 1.81275 1.28077 Table 3.3:

> ### 4Values of skin friction coefficients for different values of ? and ? when

? = ?1 = 0?5? ? = 0?3? Pr = 1?2 and ? = 0?2. ? ? − R2e 1?2 ??? − R2e 1?2 ??? 0.0 0.5 4.9¡528¢9

4.3¡736¢3 0.2 5.16586 3.97055 0.3 5.42622 3.96130 0.3 0.0 3.72170 1.65409 0.2 4.30247 2.34617 0.5 5.42622 3.96130 Table 3.4:

3**Values of local Nusselt number**−?0(0) **for different values**

of the parameters ?, ?? ?? ?? ?? Pr and ?1. ? ? ? ? ?1 ? Pr ? −(1 + 43?)?0(0) 0.0 0.5 0.5 0.5 0.3 1.2 0.2 0.297492 0.3 0.308234 0.5 0.311853 0.2 0.0 0.303062 0.3 0.304775 0.5 0.305738 0.2 0.5 0.0 0.282007 0.3 0.297135 0.5 0.305738 0.1 0.0885730 0.3 0.216850 0.5 0.305738 0.2 0.5 0.5 0.5 0.0 0.329701 0.3 0.305738 0.5 0.292750 0.2 0.5 0.5 0.5 0.3 1.0 0.292152 1.2 0.305738 1.5 0.321826 0.2 0.5 0.5 0.5 0.3 1.2 0.0 0.288530 0.2 0.305738 0.5 0.325492 3.5 Conclusions Three-dimensional mixed convection

2**flow of** viscoelastic **fluid over an exponentially** stretch- ing **surface is** analyzed. **The**

analysis is carried out in the presence of thermal radiation and convective boundary conditions. The main observations can be summarized as follows. • Influence of ratio parameter ? on the velocities ?0(?) and ? 0(?) is quite opposite. However the effect of viscoelastic parameter ? on the velocities ?0(?) and ?0(?) is qualitatively similar. •

2**Momentum boundary layer thickness** increases for ?0(?) **when ratio parameter ? is**

large. Effect of ? on ?0(?) is opposite to that of ?0(?)? • Velocity component ?0(?) is increasing function of mixed convection parameter ?? However ?(?)

14**decreases with** an **increase of mixed convection parameter** ?. The impacts **of**

Biot number ?1 and radiation parameter ? on ?0(?) and ?(?) are qualitatively similar. • Momentum boundary layer is an increasing function of mixed convection parameter ? while thermal boundary layer is decreasing function of mixed convection parameter ?? • Increase

4**in Prandtl number decreases the temperature ?(?). • Thermal boundary layer thickness**

decreases when ratio parameter ?? viscoelastic para- meter ?, mixed convection parameter ?? Prandtl number Pr and temperature exponent ? are increased. • Influence of viscoelastic parameter ? on the ? and ? directions of skin friction coefficients is opposite. • Both components of skin friction coefficient increase through an increase in ratio parame- ter ?? • Local Nusselt number is an increasing function of Prandtl number Pr? ratio parameter ?? viscoelastic parameter ?, mixed convection parameter ?? Biot number ?1 and temperature exponent ? while it decreases for radiation parameter ?. Chapter 4 Convective

6**heat and mass transfer in** three-dimensional mixed **convection flow** of viscoelastic fluid **with**

chemical reaction and heat source/sink This chapter investigates the

6**heat and mass transfer** effects **in** three-dimensional **mixed**

con- vection flow of viscoelastic fluid with internal heat source/sink and chemical reaction. An exponential stretching surface induces the flow. Magnetic field normal to the direction of flow is applied. Convective conditions at boundary surface are also encountered. Appropriate sim- ilarity transformations are utilized to reduce the boundary layer

2**partial differential equations into ordinary differential equations.** Analytical **solutions** of **the**

resulting systems are obtained. Convergence of the obtained solutions is discussed explicitly. The

1**local Nusselt and Sherwood** numbers **are** sketched **and**

examined. 4.1 Mathematical modeling We consider the three-dimensional mixed convection

1**boundary layer flow of** viscoelastic fluid past **an exponentially stretching**

surface. Mathematical analysis has been carried out in

1**presence of** internal **heat source/sink and**

generative/destructive chemical reaction.

6**Magnetic field is applied** to **the normal direction** of flow. **The**

surface coincides

23**with the plane** at ? = **0** and **the flow** is **confined** in the region ? ? **0?**

Convective boundary conditions for both

23heat and mass transfer on the **sheet**

are taken into account. The governing equations for three-dimensional flow are expressed as follows: ?? ?? ?? ?? ?? ?? = 0? + + (4.1) ? ?? ?? ?? ?? + ? ?? + ? ?? =? ??2??2 + ??1 □ ???3???2 + ? ? ?3? ???? ??2??2 + ???? ??2??2 ??3 − □ +2???? ???2??? + 2???? ??2??2 □□ +??? (? − ?∞) + ???(? − ?∞) −□? ∗?02 ?? □ □□ ? (4.2) ? ?? ?? ?? ?2? ?? + ? ?? + ? ?? =? ??2 + ??1 □ ???3???2 + ? ?? ?2? ?? ?2? ? ? 3? ?? ??2 ?? ??2 + ??3 − □ +2???????2??? +2??????2??2 □□ − ?∗?02 ?? □ □ □□ ? (4.3) ? ?? ?? + ? ?? + ? ?? = ? ??2 ??? (? − ?∞) ? ?? ?? ?2? + ? (4.4) ? ?? ?? ?? ?2? ?? + ? ?? + ? ?? = ?? ??2 − ?1(? − ?∞)? (4.5) In the above equations, ?? ? and ?

7are the velocity components in the ?−? ?− and ?−directions respectively, ?1 **the** material fluid parameter, ?? **the** thermal expansion coefficient, ?? **the**

concentration expansion coefficient, ?∗ the electrical conductivity, ?0 the magnitude of applied magnetic field, ? the density of fluid, ? the gravitational acceleration, ? = (???) the kinematic viscosity, ? the dynamic viscosity, ? the thermal diffusivity, ? the fluid temperature, ?? the specific heat of the fluid, ? the uniform volumetric heat generation/absorption, ? the con- centration field, ? the mass diffusivity and prime denotes the

15differentiation with respect to ?. The boundary conditions are given by

?=??? ?=??? ?=0? −????? =?(??−?), −????? =?∗(??−?)at?=0? ? → 0? ? → 0? ? → ?∞? ? → ?∞ as ? → ∞? (4.6) (4.7) where subscript ? corresponds to the wall condition, ? is the heat transfer coefficient, ?∗ is the concentration transfer coefficient, ??

16is the ambient **fluid temperature** and ?? **is the** ambient fluid **concentration.** At wall **the**

velocities, temperature and concentration distributions are defined as: ?+? ?+? ?(?+?) ?(?+?) ?? = ?0? ? ? ?? = ?0? ? ? ?? = ?∞ + ?0? 2? ? ? ?? = ?∞ + ?0? 2? ? (4.8) where ?0? ?0? ?0 are the constants, ? is the reference length, ?∞

15is the ambient temperature, ?∞ is the ambient concentration, ? is the temperature exponent and ? **is the**

concentration exponent. By using similarity transformations [25]: ? = ?0? ?+?? ? 0(?)? ? = ?0? ?+?? ? 0(?)? ? = − ??0 1?2 ? ?2+?? (? + ?? 0 + ? + ??0)? μ 2? ¶ ? ?(?+?) ?(?+?) = ?∞ + ?0? 2? ?(?)? ? = ?∞ + ?0? 2? ?(?)? ? = ? 0 1?2 2?? ? ?2+?? ?? (4.9) μ ¶ equation (4.1) is identically satisfied and Eqs. (4?2) − (4?9) give: ? 000+(? +?)? 00−2(? 0+?0)? 0+? 6? 000? 0 + (3?00 − 3? 00 + ??000)? 00 □ +((4?0 + 2?? 00)? 000 − (? + ? + ??0)? 0000 □ +2?(?+? ?)−? ? 0 = 0? □ □ (4.10) ?000 + (? + ?)?00 − 2(? 0 + ?0)?0 +

? 6?000?0 + (3? 00 − 3?00 + ?? 000)?00 ▯ +(4? 0 + 2?? 00)?000 − (? + ? + ?? 0)?0000 ▯ − ? ?0 = 0?
(4.11) ?00 + ? ?(? + ?)?0 − Pr ?(? 0 + ?0)? + Pr ?∗? = 0? ▯ ▯ (4.12) ?00 + ??(? + ?)?0 − ???(?0 + ?0)?
− ???∗? = 0? (4.13) ? = 0? ? = 0? ?0 = 1? ?0 = ?, ?0 = −?1(1 − ?(0))? ?0 = −?2(1 − ?(0))? at ? = 0?
(4.14) ?0 → 0? ?0 → 0? ? → 0? ? → 0 as ? → ∞? (4.15) where ?

> 11**is the viscoelastic parameter? ? is the** mixed convection **parameter,**

??? is the local Grashof number, ? is the concentration buoyancy parameter, ?

> 19**is the** Hartman **number, ?? is the** Prandtl **number, ?∗ is the** heat source/sink
> **parameter, ?∗ is the**

chemical reaction parameter, ??

> 1**is the Schmidt number, ? is the** ratio **parameter,**

?1 is the heat transfer Biot number and ?2 is the mass transfer Biot number. The definitions of these
variables are ?? = ???∗ = ? ? ??? ? ?∗ = ??10??? = ?? = ??00??1 = r?? ?2 = ? ? ? ?∗ ? ? ? ? r ? ?
(4.16) ?1?? ? ? = ??? ??? (?? − ?∞)?3 ? ? = ??(?? − ?∞)? ? = ?∗?02 ? ? = Re2? ? ??? = 2? ? ?2 ?? (??
− ?∞) ? The local Nusselt and Sherwood numbers in dimensionless forms are ?????1??2 = −2? ? ?0(0)?
(4.17) ?????1??2 = −2? ? ?0(0)? (4.18) in which ??? = ?0????+?? is the local Reynolds number. 4.2
Series solutions For homotopic solutions, the initial guesses and auxiliary linear operators are chosen as
follows: ?0(?) = 1 − ?−? ¢ ? ?0(?) = ? 1 − ?−? ? ?0(?) = ?11ex+p(?−1?)? ?0(?) = ?21ex+p(?−2?) ? ¡ ¡ ¢ L?
= ?000 − ?0? L? = ?000 − ?0? L? = ?00 − ?? L? = ?00 − ?? (4.19) (4.20) The above operators satisfy the
following properties L?(?1 + ?2?? + ?3?−?) = 0? L?(?4 + ?5?? + ?6?−?) = 0? L?(?7?? + ?8?−?) = 0? L?(?
9?? + ?10?−?) = 0? where ?? (? = 1 − 10) are the

> 2**arbitrary constants. The problems** corresponding to **zeroth order are (1**

− ?) L? ?ˆ(?; ?) − ?0(?) = ?~? N?

> 29**?ˆ(?; ?)? ?ˆ(?; ?)? ˆ?(?; ?)? ˆ?(?; ?) ? h** i h i **(1**

− ?) L? [?ˆ(?; ?) − ?0(?)] = ?~?N? ?ˆ(?; ?)? ?ˆ(?; ?) ? (1 − ?) L? ˆ?(?; ?) − ?0(?) = ?~?N?

> 17**?ˆ(?; ?)? ?ˆ(?; ?)? ˆ?(?; ?)? ?ˆ(?;** ?) ? h i **(1**− ?) L? **ˆ?(?;** ?) − ?0(?) = ?~?N? **?ˆ(?;**
> **?)? ?ˆ(?; ?)? ˆ?(?; ?)? ˆ?(?;**

?) ?

25**h i h i h i h i**

18**?ˆ(0; ?) = 0? ?ˆ0(0;** ?) = 1? **?ˆ0(∞; ?) = 0? ?ˆ(0; ?) = 0? ?ˆ0(0; ?) = ?? ?ˆ0(∞; ?) = 0? ˆ?0(0?** ?) = −?1[1 − ?(0? ?)]? **?ˆ(∞? ?) = 0? ˆ?0(0? ?)** = −?2[1 **?ˆ(0? ?)]? ˆ?(∞? ?) = 0?**

− (4.21) (4.22) (4.23) (4.24) (4.25) (4.26) (4.27) (4.28) N?

10**[?ˆ(?; ?)? ?ˆ(?; ?)? ˆ?(?; ?)? ?ˆ(?; ?)]** = ?3 **??ˆ(?? 3? ?) ??ˆ(?? ?) ??ˆ(?? ?) ?? ˆ(?? ?)** −2 Ã ?? + ?? ! ?? + □ **?ˆ(?? ?)** ?2 **?ˆ(?? ?)** + 2? **ˆ?(?? ?) +?ˆ(?? ?)** □ − ? ?? ˆ(?? ?) ??2 □ **+??ˆ(?; ?)** □ ?? □ □ ?2 **?ˆ(???)** 3 □ 6 **??ˆ?(????)? 3 ??ˆ(??3??)** + ??2 □ □ −3?2 **??ˆ?(??2??)** +??3 **?ˆ?(?? 3??)** □ □ +? ?2 **??ˆ(?? 2??)** + □ □ □+24 **???ˆ?(? 2 ????ˆ?(??? 2)??)** □ ?3 **??ˆ(??**

3??) ?(4.29) − □

10**?ˆ(?□??)?+?ˆ(??ˆ?(??)? ?)** ?4 **??ˆ?(? 4??)** □ □ +? ?? □ □ □ □ □ N **?[?ˆ(?; ?)? ?ˆ(?; ?)? ?ˆ(?; ?)? ?ˆ(?; ?)]** = ?3ˆ??(??3? ?) **??ˆ(???)**−2 ?? **??ˆ(?? ?) +?? ˆ(?????)** □ ?? + ?ˆ(?? ?) ?2 **?ˆ(?? ?)** □ □ **+?ˆ(??** ?) □ ??2 □6 **??ˆ?(????)** ?3 **?? ˆ(?? 3□??)** □ □ +? □ + 3 ?2 **?ˆ(???)** ??2 − □ □ □ **3** ?2 **??ˆ(??2??)+??** 3 **??ˆ(?? 3??)** □ □ □ □ ?2ˆ?(???) **??ˆ(???)**□ □ □ ??2 + 4 ?? ?3 **?ˆ(???)**

□

9**+2??2??ˆ(?? 2??)** □ ??3 □ +? **?ˆ(?□??)+?ˆ(???)** ?4 **?ˆ(???)**□ − □ □ ??4 □ +? **?? ˆ?(????) −? □??ˆ(?? ?□)** □ □ □ ?? ? (4.30) N **?[?ˆ(?; ?)? ?ˆ(?; ?)? ?ˆ(?; ?)? ?ˆ(?; ?)] = ?2ˆ?(??** ?) ??2 + Pr □ **(?ˆ(?? ?) ?ˆ?(?? ?)** + Pr **?∗ˆ?(?? ?) +?ˆ(??** ?) □ ?? − Pr ? Ã **??ˆ(??□?) ??ˆ(?? ?□) ?ˆ(??** ?)? + (4.31) ?? ?? ! N **?[?ˆ(?; ?)? ?ˆ(?; ?)? ? ˆ(?; ?)? ?ˆ(?; ?)] = ?2 ?ˆ(?; ?) ?ˆ(??** ?) ??2 + ?? □ **+?ˆ(?? ?)** □ **??ˆ(?; ?) − ???∗? ˆ(?; ?)** ?? −? ?? Ã **??ˆ(?? ?□) ??ˆ(?? ?) ?ˆ(?;**

?)? □ + ! (4.32) ?? ?? Here ?

3**is an embedding parameter, the non-zero auxiliary parameters** are ~?? ~? ~? **and ~? and the nonlinear operators**

are N?? N?, N? and N?. Taking ? = 0 and ? = 1 we get

> 12**?ˆ(?;0) = ?0(?)? ?ˆ(?;0) = ?0(?)? ˆ?(??0) = ?0(?)? ˆ?(?;0) = ?0(?)** and **?ˆ(?;** 1) = ?(?)? **?ˆ(?;** 1) = ?(?)? **ˆ?(??** 1) = ?(?)? **?ˆ(?;**

1) = ?(?)? (4.33) As ? enhances from 0 to 1 then ?(???)? ?(???), ?(???) and ?(???) differ from ?0(?)? ?0(?)? ?0(?) and ?0(?) to ?(?)? ?(?), ?(?) and ?(?)? Applying Taylor's expansion we have ?(???)=?0(?)+? =1??(?)??? ??(?)= ?1! ?????(??;?)¯?=0 ∞ ? (4.34) ?(???)=?0(?)+?=1??(?)??? ??(?)= ?1! ?????(??;?)¯?=0 ∞ ¯ ? (4.35) P ¯ ?(???)=?0(?)+?P=1??(?)??? ??(?)= ?1! ?????(??;?)‾‾?=0 ¯ ∞ ? (4.36) P ∞ ¯ ?(?? ?) = ?0(?) + (4.37) ?=1 ??(?)??? ??(?) = ?1! ?????(??; ?)‾‾?=0 ? The

> 3**convergence of above series** stroPngly **depends upon ~?, ~?, ~? and ~?**‾‾?
> **Considering that ~?,** ¯ **~?, ~? and ~? are selected properly so that**

Eqs. (4?34) − (4?37) converge at ? = 1? Therefore ?(?) = ?0(?) + ??(?)? ∞ ?=1

> 33**P ?(?) = ?0(?)** + ??(?)? ∞ ?=1 **P∞ ?(?) = ?0(?) + ??(?)? ?=1 P∞ ?(?) = ?0(?)** + ??(?)? ?=1 **P**

The general solution expressions can be written as ??(?) = ??∗(?) + ?1 + ?2?? + ?3?−?? ??(?) = ??∗(?) + ?4 + ?5?? + ?6?−?? ??(?) = ?∗?(?) + ?7?? + ?8?−?? ??(?) = ?∗?(?) + ?9?? + ?10?−?? where the special solutions are ??∗? ??∗, ?∗? and ?∗?. (4.38) (4.39) (4.40) (4.41) (4.42) (4.43) (4.44) (4.45) 4.3 Convergence analysis and discussion Homotopic solutions (4?38) − (4?41) obviously depend on the auxiliary parameters ~?? ~?, ~? and ~?. In order to control the convergence of series solutions' these auxiliary parameters play a central role. To obtain the convergence region, the ~−curves have been plotted at 14?? order of approximations in Fig. 4.1. This Fig. clearly shows that the acceptable values of ~?? ~?, ~? and ~? are −1?0 ≤ ~? ≤ −0?4? −1?0 ≤ ~? ≤ −0?2, −1?2 ≤ ~? ≤ −0?1 and −1?2 ≤ ~? ≤ −0?1? Table 4.1 ensures that the

> 1**series solutions converge in the whole region of** ? when ~? = ~? = ~? = ~? = **−0?** 5? f **"?0?,** g **"?0?,**

q'?0?,f'?0? b*=b=A=B=02.,M=K=0.1,g1=g2=0.5, Pr=0.7,Sc=0.8,l=N=k*=0.3 0.0 -0.5 g''(0)

> 1**f'(0) q'(0)** -1.0 **f''(0)** -1 **.5** -1 **.0**

-0.5 0.0 hf , hg, hq, hf Fig. 4.1: ~−curves for the functions ?? ?, ? and ?? Table 4

> 2**.1: Convergence of** series **solutions for different order of approximations**

> **when**

? = ? = 0?1? ? = ? = ?∗ = ? = 0?2? ? = ? = ?∗ = 0?3? ?1 = ?2 = 0?5? Pr = 0?7? ?? = 0?8 and ~? = ~? = ~? = ~? − 0?6? order of approximations 1 5 10 15 20 25 30 35 −? 00 (0) 1.155 1.104 1.078 1.068 1.065 1.064 1.064 1.064 −?00 (0) 0.2359 0.2395 0.2414 0.2420 0.2422 0.2422 0.2422 0.2422 −?0(0) 0.3084 0.2620 0.2437 0.2373 0.2349 0.2341 0.2340 0.2340 −?0 (0) 0.3300 0.3318 0.3336 0.3340 0.3341 0.3341 0.3341 0.3341 Figs. 4.2 and 4.3 are plotted to see the effects of Hartman number ? on the velocity profile ?0(?) and ?0(?). The velocity profiles ?0(?) and ?0(?) are decreased when we increase the values of ?. Also the

> 2**momentum boundary layer** thicknesses **are decreasing functions of ?. Fig.**

4.4 is drawn to see the influence of internal heat source/ sink parameter ?∗ on the velocity profile ?0(?)? Clearly in case of heat sink parameter ?∗ ? 0 both momentum boundary layer thickness and ?0(?) decrease while in case of heat source parameter ?∗ ? 0 kinetic energy of the fluid particles increases due to which the velocity profile ?0(?) increases. Outcome of mixed convection parameter ? on the velocity profile ?0(?) in both assisting and opposing flows is seen in Fig. 4.5. In case of assisting flow ? ? 0 both ? 0(?) and momentum boundary layer thickness are enhanced while reverse effect is observed for opposing flow ? ? 0? Fig. 4.6 exhibits the variation of concentration buoyancy parameter ? on the velocity profile ? 0(?)? It is examined that an enhancement in ? gives rise to the velocity profile ?0(?). Fig. 4.7 depicts the influence of internal heat source/sink parameter ?∗ on the temperature ?(?)? With an increase in internal heat source ?∗ ? 0 both the thermal boundary layer thickness and ?(?) increase while in case of heat sink parameter ?∗ ? 0 both the thermal boundary layer thickness and ?(?) decrease. Figs. 4.8-4.10 are sketched to see the variations of chemical reaction parameter ?∗, concen- tration exponent B and mass transfer Biot number ?2 on the concentration profile ?(?)? Fig. 4.8 is presented to analyze the variation of chemical reaction parameter ?∗ on the concentra- tion profile ?(?)? It is noted that the associated boundary layer thickness and concentration profile ?(?) decrease for generative chemical reaction ?∗ ? 0 while reverse phenomena is noted for destructive chemical reaction ?∗ ? 0. With an enhancement in concentration exponent ? both the concentration profile ?(?) and the boundary layer thickness decrease (see Fig. 4.9). Variation of mass transfer Biot number ?2 on the concentration profile ?(?) is displayed in Fig. 4.10. Here we examined that the effect of ?2 on the concentration profile ?(?) and associated boundary layer thickness are increasing? Figs. 4.11-4.13 are displayed to see the impacts of mixed convection parameter ?? concentra- tion buoyancy parameter ?, ratio parameter ?? Hartman number ?, internal heat source/sink ?∗ and heat transfer

> 2**Biot number** ?1 **on the local Nusselt number**

−?0(0)? Fig. 4.11 shows that the

> 2**heat transfer rate at the wall** increases **for**

assisting flow ? ? 0 while it decreases for op- posing flow ? ? 0? It is also examined

increasing function of concentration buoyancy parameter ?. Fig. 4.12 exhibited that the larger values of ratio parameter ? corresponds to a higher heat transfer rate −?0(0). Also it is to be noted that with an increase in Hartman number heat transfer rate decreases. Fig. 4.13 depicts that

20the heat transfer rate at the wall −?0(0) **decreases**

with internal heat source ?∗ ? 0 while increases with internal heat sink ?∗ ? 0? Variations of mixed convection parameter ?? concentration buoyancy parameter ?, ratio parameter ?? Hartman number ?, chemical reaction ?∗ and mass transfer Biot number ?2 on Sherwood number −?0(0) are plotted in the Figs. 4.14-4.16. Fig. 4.14 is drawn to see the influences of mixed convection parameter ? and concentration buoyancy parameter ? on the Sherwood number −?0(0)? It is seen that the Sherwood number −?0(0) is increasing function of ? and ? in case of assisting flow ? ? 0 while decreasing function for opposing flow case. Fig. 4.15 depicts that the Sherwood number −?0(0) decreases with an enhancement in Hartman number ? while it increases with an increase in ratio parameter ?? Fig. 4.16 exhibits that the mass transfer at the wall −?0(0) enhances with generative chemical reaction ?∗ ? 0 while it reduces with destructive chemical reaction ?∗ ? 0. It is also observed that the mass transfer at the wall −?0(0) is an increasing function of mass transfer Biot number ?2?Table 4.2 ensures the validity of present results with Liu et al. [25] in the limiting sense. f'?h? 1.0 0.8 b = b* = k* = A = B = 0.2, g1 = g2 = 0.3, 0.6 Pr = Sc = 0.7, K = l = N = 0.5 0.4 0.2 M = 0.0, 0.3, 0.6, 0.9 0 1 2 3 4 5 6 h Fig. 4.2: Influence of ? on velocity ?0(?). g'?h? 0.20 b = k* = A = B = 0.2, g1 = g2 = 0.3, Pr = Sc = 0.7, b* = K = l = N = 0.5 0.15 0.10 0.05 M = 0.0, 0.3, 0.6, 0.9 1 2 3 4 5 6 h Fig. 4.3: Influence of ? on velocity ?0(?). f '?h? 1.0 b = k* = A = B = 0.2, M = g1 = g2 = 0.3, Pr = Sc = 0.7, K = l = N = 0.5 0.8 0.6 0.4 b* = -3.0, 0.0, 0.3, 7.0 0.2 0 1 2 3 4 5 6 h

2Fig. 4. 4: **Influence of ?∗ on velocity** ?0(?). **f**

'?h? 1.0 k* = A = B = 0.2, g1 = g2 = M = b = 0.3, Pr = Sc = 0.7, b* = K = N = 0.5 0.8 0.6 0.4 l = -0.5, 0.0, 0.5, 0.9 0.2 0 1 2 3 4 5 6 h

2Fig. 4. 5: **Influence of ? on velocity** ?0(?). **f**

'?h? 1.0 0.8 k* = A = B = 0.2, g1 = g2 = M = b = 0.3, Pr = Sc = 0.7, b* = K = l = 0.5 0.6 0.4 N = 0.0, 1.5, 3.0, 4.5 0.2 0 1 2 3 4 5 6 h Fig. 4.6: Influence of ? on velocity ?0(?). q?h? 0.30 0.25 b = k* = A = B = 0.2 , g1 = g2 = M = 0.3 , Pr = Sc = 0.7 , K = l = N = 0.5 0.20 0.15 b* = -0.3 , 0.0 , 0.3 , 0.6 0.10 0.05 1 2 3 4 5 6 h Fig. 4.7: Influence of ?∗ on temperature ?(?). f?h? 0.25 0.20 b = A = B = 0.2, g1 = g2 = M = 0.3, Pr = Sc = 0.7, K = l = N = b* = 0.5 0.15 0.10 0.05 k* = -0.2, 0.0, 0.2, 0.4 1 2 3 4 5 6 h Fig. 4.8: Influence of ?∗ on ?(?). f?h? 0.25 0.20 b = A = k* = 0.2, g1 = g2 = M = 0.3, Pr = Sc = 0.7, K = l = N = b* = 0.5 0.15 0.10 0.05 B = 0.0, 0.2, 0.4, 0.6 1 2 3 4 5 6 h Fig. 4.9: Influence of ? on concentration ?(?). f?h? 0.4 b = A = B = k* = 0.2, g1 = M = 0.3, Pr = Sc = 0.7, K = l = N = b* = 0.5 0.3 0.2 g2 = 0.1, 0.3, 0.5, 0.7 0.1 1 2 3 4 5 6 h Fig. 4.10: Influence of ?2 on ?(?). -q'?0? A = B = b = b* = k* = 0.2, M = 0.1, 0.276 K = g1 = g2 = 0.5, Pr = 0.7, Sc = 0.8 l = -0.5, 0.0, 0.5, 1.0 0.274 0.272 0.270 0.268 0.266 0.5 1.0 1.5 2.0 N Fig. 4.11: Influence of ? and ?

on −?0(0). -q'?0? 0.32 A = B = b* = k* = 0.2, l = N = 0.3, K = g1 = g2 = 0.5, Pr = 0.7, Sc = 0.8 0.31 0.30 0.29 0.28 M = 0.0, 0.5, 1.0, 1.5 0.27 0.5 1.0 1.5 2.0 b Fig. 4.11: Influence of ? and ? on −?0(0). -q'?0? A = B = k* = b = 0.2, l = N = 0.3, 0.4 K = g2 = 0.5, Pr = 0.7, M = 0.1, Sc = 0.8 0.3 0.2 b* = -0.3, 0.0, 0.3, 0.6 0.1 0.2 0.4 0.6 0.8 1.0 g 1 Fig. 4.13: Influence of ?∗ and ?1 on −?0(0). -f'?0? 0.340 A = B = b = b* = k* = 0.2, M 0.1, = K = g1 = g2 = 0.5, Pr = 0.7, Sc = 0.8 0.335 0.330 l = -0.5, 0.0, 0.5, 1.0 0.5 1.0 1.5 2.0 N Fig. 4.14: Influence of ? and ? on −?0(0). -f'?0? 0.336 A = B = b* = k* = 0.2, l = N = 0.3, K = g1 = g2 = 0.5, Pr = 0.7, Sc = 0.8 0.334 0.332 0.330 0.328 M = 0.0, 0.5, 1.0, 1.5 0.326 0.2 0.4 0.6 0.8 1.0 b Fig. 4.15: Influence of ? and ? on −?0(0). -f'?0? 0.5 A = B = b = b* = 0.2, M = 0.1, l = N = 0.3 K = g1 = 0.5, Pr = 0.7, Sc = 0.8 0.4 0.3 k* = -0.3, 0.0, 0.3, 0.5 0.2 0.1 0.2 0.4 0.6 0.8 1.0 g2 Fig. 4.16: Influence of ?∗ and ?1 on −?0(0). Table 4.2: Comparative values of −?00(0)? −?00(0) and ?(∞) + ?(∞) for different values of ? when ? = ? = ? = ?1 = ?2 = ?∗ = ?∗ = 0? Liu et al. [25] Present results ? −? 00(0) −?00(0) ?(∞) + ?(∞) −?00(0) −?00(0) ?(∞) + ?(∞) 0.0 1.28180856 0 0.50 1.56988846 0.78494423 1.00 1.81275105 1.81275105 0.90564383 1.10918263 1.28077378 1.28181 0 0.90564 1.56989 0.78494 1.10918 1.81275 1.81275 1.28077 4.4 Closing remarks The present chapter deals with the three-dimensional

4**mixed convection flow of** MHD viscoelas- tic **fluid over**

an exponentially

1**stretching surface in presence of heat source/sink and**

genera- tive/destructive chemical reaction. The main outcomes are as follows. • Velocity profiles ?0(?) and ?0(?) reduce with an increase in Hartman number ?. • Momentum boundary layer thickness

2**decreases with an increase in ratio parameter** ?? • Both **the**

velocity profile ?0(?) and

2**momentum boundary layer** thicknesses **are** increasing **functions of** internal heat source **parameter**

?∗ ? 0? assisting flow case ? ? 0 and concen- tration buoyancy parameter ? while decreasing functions of internal heat sink parameter ?∗ ? 0 and opposing flow case ? ? 0? • Thermal boundary layer thickness and temperature ?(?) decrease with an increase in internal heat sink ?∗ ? 0 while thermal boundary layer thickness and temperature ?(?) increase with an increase in internal heat source ?∗ ? 0. • With an enhancement in generative chemical reaction ?∗ ? 0, concentration exponent ? and Schmidt number ?? decreases the concentration profile ?(?)? The concentration boundary layer thickness increases for larger mass transfer Biot number ?2 and destructive chemical reaction ?∗ ? 0? • Heat transfer rate −?0(0) boosts up in case of assisting flow ? ? 0? concentration buoyancy parameter ?, ratio parameter ?? heat transfer Biot number ?1 and internal heat sink parameter ?∗ ? 0 while heat transfer rate −?0(0) reduces with opposing flow ? ? 0? Hartman number ? and internal heat source ?∗ ? 0? • With an increase in assisting flow ? ? 0? concentration buoyancy parameter ?, ratio parameter ?? mass transfer Biot number ?2 and

generative chemical reaction parameter ?∗ ? 0? the Sherwood number −?0(0) enhances while reverse behavior is noted in case of opposing flow ? ? 0? Hartman number ? and destructive chemical reaction parameter ?∗ ? 0? Chapter 5 Thermophoresis and MHD mixed convection flow

> 1with Soret and Dufour effects This chapter investigates the

> 4heat and mass transfer effects in three-dimensional

mixed convec- tion flow of viscoelastic fluid over a

> 4stretching surface with convective boundary conditions. The fluid is electrically conducting in the presence of constant applied magnetic field.

Conservation laws of energy and concentration are based upon the Soret and Dufour effects. First order chemical reaction effects are also taken into account.

> 15Dimensionless velocity, temperature and concentration distributions are shown graphically

> 7for different values of involved parameters. Numerical

values of

> 1local Nusselt and Sherwood numbers are computed and analyzed. 5.1 Mathematical analysis We consider the steady

three-dimensional magnetohydrodynamic mixed convection

> 7flow of an incompressible viscoelastic fluid over a stretching surface at

? = 0? The flow takes place in the domain ? ? 0? Heat and mass transfer characteristics are taken into account

> 6in the presence of Soret and Dufour and thermophoresis effects. The

ambient fluid temperature is taken as ?∞ while the surface temperature is maintained by convective heat

transfer at a certain value ?? . A constant

> 6**magnetic field** ?0 **is applied in the ?-direction.**

> 2**Induced magnetic field is not considered** due to **small Reynolds number.**

In addition the effects of first order chemical reaction in mass transfer are taken into account. The governing boundary layer equations for the flow under consideration are ?? ?? ?? ?? ?? ?? = 0? + + (5.1) ? ?? ?? ?? ?? + ? ?? + ? ?? ?2? ?1 ?3? ?3? ?? ?2? ?? ?2? = ? ??2 ? ▯ ????2 + ???3 + ? − ?? ??2 + ?? ??2 ▯ +2???? ???2??? + 2???? ??2??2 ▯▯ +???(? − ?∞) + ???(? − ?∞) −▯?∗?02?? ▯ ▯▯ ? (5.2) ? ?? + ? ?? + ? ?? ?? ?? ?? =? ?2? + ?1 ? ?3? ?? ?2? ?? ?2? +? ?3? − ?? ??2 + ?? ??2 ??2 ? ▯ ????2 ??3 ▯ +2???????2??? +2??????2??2 ▯▯ − ?∗?02 ?? ▯ ▯ ▯▯ ? (5.3) ? ?? ?? + ? ?? + ? ?? = ? ??2 ???? ??2 ?? ?? ?2? ???? ?2? + ? (5.4) ? ?? ?? ?? + ? ?? + ? ?? = ?? ?? ??2??2 ?????? ??2??2 − ?1(? − ?∞) − ?? (?? (? − ?∞))? + ? (5.5) In Eqs. (5?1) − (5?5) the respective velocity components in the ?−? ?− and ? −directions are denoted by ?? ? and ?, ?1 the viscoelastic parameter, ?∗ the electrical conductivity, ?0 is the magnitude of applied magnetic field, ? the density of fluid, ? the gravitational acceleration, ?? the

> 19**thermal expansion coefficient, ?? the concentration expansion coefficient, ? the**

fluid temperature, ? the thermal diffusivity of fluid, ? the kinematic viscosity, ? the dynamic viscosity of fluid, ? the concentration field, ?? the mass diffusivity, ?? the thermal diffusion ratio, ?? the specific heat, C? the concentration susceptibility and ?? the thermophoretic velocity. In Eq. (5.5) the thermophoretic term ?? can be defined as ? ?? ?? = −?2 ?? ?? (5.6) where ?2 is the thermophoretic coefficient and ?? is the reference temperature. A thermophoretic parameter ? is defined as ? = −?2(?? − ?∞) ?? The boundary conditions appropriate to the flow under consideration are given by ? = ?? = ??? ? = ??? ? = 0? − ? ?? = ?(?? − ?), ? = ?? at ? = 0? ?? ? → 0? ? → 0? ? → ?∞? ? → ?∞ as ? → ∞? (5.7) (5.8) where ? indicates the thermal conductivity of fluid, ?? is the hot fluid temperature, ?∞ the ambient concentration and ? and ? have dimension inverse of time. We now define ? = ???0(?)? ?=???0(?)? ?=− ??(?(?)+?(?))? √ ?(?) = ??? −−?? ∞∞ ? ? = ?r ? ?(?) = ???−−??∞∞ ? ? ? (5.9) Now the use of above variables satisfy Eq. (5.1) automatically while Eqs. (5.2)-(5.8) are reduced as follows: ?000 + (? + ?)?00 − ?02 − ? (? + ?)?0000 + (?00 − ?00)?00 − 2(?0 + ?0)?000 ¡ ¢ −?2?0 + ?(? + ??) = 0? ?000 + (? + ?)?00 − ?02 − ? (? + ?)?0000 + (?00 − ?00)?00 − 2(?0 + ?0)?000 − ?2?0 = 0? ¡ ?00 + ? ?(? + ?)?0 + Pr ?? ?00 = 0? ¢ ?00 + ??(? + ?)?0 − ???∗? + ?????00 − ???(?0?0 − ??00) = 0? ? = 0? ? = 0? ?0 = 1? ?0 = ?, ?0 = −?1(1 − ?(0))? ? = 1 at ? = 0? ?0 → 0? ?0 → 0? ? → 0? ? → 0 as ? → ∞? (5.10) (5.11) (5.12) (5.13) (5.14) where ? is the dimensionless viscoelastic parameter? ?

> 19**is the** Hartman **number, ? is the** local buoyancy **parameter,** Gr? **is the** local Grashof **number,**

? is the constant dimensionless concentration buoyancy parameter, ? ?

1is the Prandtl **number, ?? is the** Dufour **number, ?? is the** Schmidt **number, ??**
is the Soret **number, ?∗ is the** chemical reaction **parameter, ? is the**

thermophoretic parameter, ? is ratio of rates parameters, ?1

2is the Biot number and prime shows the **differentiation with respect to**

?. These are given by ? = ?1?? ?=?∗?02? ?=?R?e2?? ???=???(???−2?∞)?3? ?=????((????−−??∞∞)) ? ? ? ? = ? ?? = ? ? ???? (?? − ?∞) ? ?? = ? ?? = ???? (?? − ?∞) ? ?∗ = ?1?? = −?2(?? − ?∞)? ? ???? (?? − ?∞)? ? ??? (?? − ?∞) ? ?? ? = ? ?? ?1 = ? ? ? ? r ? Local Nusselt and Sherwood numbers in dimensionless forms are given by ?????1??2 = −?0(0)? (5.15) ?????1??2 = −?0(0)? (5.16) in which ??? = ????? is the local Reynolds number. 5.2 Construction of solutions The initial approximations and auxiliary linear operators required for homotopy analysis solu- tions are presented below i.e. ?0(?) = 1 − ?−? ? ? 0(?) = ? 1 − ?−? ? ?0(?) = ?11ex+p(?−1?)? ?0(?) = exp(−?) (5.17) ¡ ¢ ¡ ¢ L? = ?000 − ?0? L? = ?000 − ? 0? L? = ?00 − ?? L? = ?00 − ?? with the following properties of the defined operators in Eq. (5.18) i.e. L?(? 1 + ?2?? + ?3?−?) = 0? L?(?4 + ?5?? + ?6?−?) = 0? L?(?7?? + ?8?−?) = 0? L?(?9?? + ?10?−?) = 0 where ?? (? = 1 − 10) indicate the arbitrary constants. (5.18) (5.19) The corresponding problems at the zeroth order are given in the following forms: (1 − ?) L? ?ˆ(?; ?) − ?0(?) = ?~? N? ?ˆ(?; ?)? ?ˆ(?; ?) ? (5.20) h i h i (1 − ?) L? [?ˆ(?; ?) − ?0(?)] = ?~?N? ?ˆ(?; ?)? ?ˆ(?; ?) ? (5.21) (1 − ?) L? ˆ?(?; ?) − ?0(?) = ?~?N?

29?ˆ(?; ?)? ?ˆ(?; ?)? ˆ?(?? ?)? ?ˆ(?? ?) ? h i

(5.22) (1 − ?) L? ?ˆ(?; ?) − ?0(?) = ?~?N? ?ˆ(?; ?)? ?ˆ(?; ?)? ˆ?(?? ?)? ?ˆ(?? ?) ?

25h i h i (5.23) **h i h i**

12?ˆ(0; ?) = 0? ?ˆ0(0; ?) = 1? **?ˆ0(∞; ?) = 0? ?ˆ(0; ?) = 0? ?ˆ0(0; ?) = ?? ?ˆ0(∞; ?) =**
0? ?ˆ0(0? ?) = −?1[1 − ?(0? ?)]? ˆ?(∞? ?) = 0? ?ˆ(0? ?) = 1? ?ˆ(∞? ?) = 0

(5.24) 2 N?

10[?ˆ(?? ?)? ?ˆ(?? ?)? ?ˆ(?? ?)? ?ˆ(?? ?)] = ?3 **?ˆ(?? ?) ??ˆ(?? ?) + (?ˆ(?? ?) + ?**
ˆ(?? ?))?2 **?ˆ(?? ?)** ??3 − Ã ?? ! ??2 **(?ˆ(?? ?) + ?ˆ(?? ?))** ?4 **??ˆ(?? 4??)** −? □ +(?2
??ˆ(?? 2??) − ?2 **??ˆ(?? 2??)**) ?2 **?ˆ(?? 2??)** □ −2(**??ˆ?(????) + ??ˆ?(????))** ?
3??ˆ(?? 3??) −? □2??ˆ(?? ?) + ?ˆ?(?? ?) + ?? ˆ?(?? ?)□? □ □ (5.25) ?? N **?[?ˆ(??**
?)? ?ˆ(?? ?)? ˆ?(?? ?)? ?ˆ(?? ?)] = ?3 ?ˆ(???) ??ˆ(???) 2 ??3 − +(?ˆ(???)+?
ˆ(???))? 2 **?ˆ(???)**μ ?? ¶ ??2 **(?ˆ(???)+?ˆ(???))?** 4 **??ˆ(?? 4??) +?□ +(?2 ??ˆ(??**
2??) − ?2 ??ˆ(?? 2??))?2 ??ˆ(?? 2??) □ −2 (??ˆ?(????) + ??ˆ?(????))? 3 ??ˆ(??

3??) −?□2 **??ˆ(???)**□ □ ? □ (5.26) ?? N **?[?ˆ(?? ?)? ?ˆ(?? ?)? ?ˆ(?? ?)? ?ˆ(?? ?)]**
= ?2ˆ??(??2? ?) + Pr **(?ˆ(?? ?) + ?ˆ(?? ?)) ?ˆ?(???? ?)** + Pr ?? ??2 ?2 **?ˆ(?? ?) ?**
(5.27) N **?[?ˆ(?? ?)? ˆ?(?? ?)? ?ˆ(?? ?)? ?ˆ(?? ?)]** = ?2 **?ˆ?(?? 2? ?) + ??(?ˆ(?? ?)**
+ ?ˆ(?? ?)) ??ˆ?(??? ?) −???∗?ˆ(?? ?) + ???? ?2ˆ??(??2? ?) − ???(?? **??ˆ(?? ?)**
?ˆ?(?? ?) ?? −?ˆ(??

?) ?2ˆ??(??2? ?))? (5.28) Here ?

4is an embedding parameter, **~?? ~?, ~? and** ~? are **the non-zero auxiliary**
parameters and N?? N?, N? **and** N? indicate **the nonlinear operators.** When ? = **0**
and ? = 1 **one has**

12?ˆ(?;0) = ?0(?)? ˆ?(??0) = ?0(?)? ?ˆ(??0) = ?0(?) ?ˆ(?; 1) = ?(?)? **ˆ?(?? ** 1) = ?
(?)? **?ˆ(?? ** 1) = **?0(?)**

? (5.29) Clearly when ? is increased from 0 to 1 then ?(???)? ?(???), ?(???) and ?(???) vary from ?0(?)?
?0(?)? ?0(?) and ?0(?) to ?(?)? ?(?)? ?(?)and ?(?)? By Taylor's expansion we have ?(???)=?0(?)+?=1??
(?)??? ??(?)= ?1! ?????(??;?)¯?=0 ∞ ? (5.30) ?(???)=?0(?)+?=1??(?)??? ??(?)= ?1! ?????(??;?)¯?=0
¯ ∞ ? (5.31) P ¯ ?(?? ?) = ?0(?) ?=P1??(?)??? ??(?)= ?1! ?????(??;?)¯?¯=0 ∞ ¯ ? (5.32) P ?(?? ?) = ?
0(?) ∞ ??(?)??? ??(?) = ?1! ?????(??; ?)¯¯?=0 ¯ ? (5.33) ?=1

3where the convergence of above sPeries **strongly depends upon ~?? ~?**
~¯? and ¯ ~?? Considering that ~?? ~?? ~? and ~? are selected properly so
that

Eqs. (5?30)−(5?33) converge at ? = 1 then we can write ?(?) = ?0(?) + ??(?)? ∞ (5.34) ?=1 P ?(?) = ?0(?)
+ ??(?)? ∞ ?=1 ?(?) = ?0(?) + P∞ ??(?)? ?=1 P∞ ?(?) = ?0(?) + ??(?)? ?=1 P The resulting problems at
mth order deformation can be constructed as follows: L?[??(?) − ????−1(?)] = ~?R??(?)? L?[??(?) − ????
−1(?)] = ~?R??(?)? L?[??(?) − ????−1(?)] = ~?R??(?)? L?[??(?) − ????−1(?)] = ~?R??(?)? (5.35) (5.36)
(5.37) (5.38) (5.39) (5.40) (5.41) ??(0) = ??0(0) = ??0(∞) = 0? (5.42) ??(0) = ?0?(0) = ?0?(∞) = 0? (5.43)
??0(0) − ?1??(0) = ??(∞) = 0? ??(0) = ??(∞) = 0? (5.44) R??(?) = ??000−1(?) − ??0−1−???0 + (??
−1−???00 + ??−1−???00) ?−1 ?−1 ?=0 ?=0 ?−P1 ??−1−???0000 + ??−1−???0000 ?P−1 □ ?=0 ?=0 + P?
−1 ??00−1−??00 − P?−1 □ −? ?0?0−1−??00 ?=0 ?=0 −2 ?P−1 ??0−1−? ??000 − 2P?−1 ??0−1−? ??
000 ?=0 ?=0 −? 2□??0−1 +P?? + ?? ?? □ P □ □ (5.45) R??(?) = ??000−1(?) − ?0?−1−???0 + (??−1−???
00 + ??−1−???00) ?−1 ?−1 ?=0 ?=0 ?−P1 ??−1−???0000 + ??−1−??0?000 ?P−1 □ ?=0 ?=0 P?−1 ??
00−1−??0?0 −P?−1 ?0?0−1−??00 □ −? + (5.46) ?=0 ?=0 − ?2??0−1? □ −2 ?P−1 ??0−1−???000 − 2P?
−1 ??0−1−???000 □ ?=0 ?=0 P P □ □ R??(?) = ?0?0−1 + ? ? (?0?−1−??? + ?0?−1−???) + Pr ?? ?0?
0−1? ?−1 (5.47) ?=0 P R??(?) = ?0?0−1 + ?? (?0?−1−??? + ?0?−1−???) − ???∗? + ?????0?0−1 ?−1 ?=0
−??? (?0?−1−??0? − ??−1−??0?0)? ?−1 P (5.48) ?=0 P ?? = 0? ? ≤ 1? □ 1? ? ? 1? (5.49) Solving the
above mth order deformation problems we have □ ??(?) = ??∗(?) + ?1 + ?2?? + ?3?−?? (5.50) ??(?) = ??

∗(?) + ?4 + ?5?? + ?6?−?? (5.51) ??(?) = ?∗?(?) + ?7?? + ?8?−?? (5.52) ??(?) = ?∗?(?) + ?9?? + ?10?
−?? (5.53) in which the ??∗? ??∗, ?∗? and ?∗? indicate the special solutions. 5.3 Analysis Obviously the
series solutions (5?34)−(5?37) contain the auxiliary parameters ~?? ~? ~? and ~?. These parameters are
very important in adjusting and controlling the convergence of homotopic solutions. Hence the

1~−curves are plotted at 10?? order of approximations in

order to find the suitable ranges of ~?? ~? ~? and ~?. Fig. 5.1 indicate that the admissible values of ~??
~?? ~? and ~? here are −1?5 ≤ ~? ≤ −0?40? −1?4 ≤ ~? ≤ −0?30, −1?40 ≤ ~? ≤ −0?25 and −1?40 ≤ ~? ≤
−0?20? Table 5.1 presents the convergence of homotopic solutions. It is noted that computations are
sufficient for 45?? order iterations of velocity and 35?? order iterations of the temperature and
concentration profiles for the convergent series solutions. 1 K = t = 0.2, l = N = k* = 0.3, b = Sc = Df = 0.5,
M = 0.7, g1 = 0.6, Df = 0.3, Sr = 0.4, Pr = 1.0 f''?0? 0.5 g''?0?

7q'?0? f ''?0?, g ''?0?, q '?0?, f'?0? 0 f '?0?

- 0.5 -1 - 1.5 -1.5 -1 -0.5 0 hf,hg,hq,h« Fig. 5.1: ~−curves for the functions ?? ?, ? and ?? Table 5.1:

1Convergence of series solutions for different order of approximations when

? = ? = 0?2? ? = ?? = ?? = 0?5? ?? = 1?0? ?? = 0?4? ? = 0?7? ? = ? = ?∗ = 0?3? ?1 = 0?6 and ~? = ~? =
~? = ~? = −0?7? order of approximations −? 00 (0) −?00 (0) −?0 (0) −?0 (0) 1 1.31063 0.632417
0.284766 0.763750 5 1.47588 0.746614 0.230340 0.599613 10 1.49331 0.762859 0.221636 0.578386 15
1.49592 0.764867 0.220665 0.575416 20 1.49650 0.765053 0.220797 0.574753 25 1.49664 0.765041
0.220916 0.574545 30 1.49667 0.765030 0.220950 0.574483 35 1.49667 0.765027 0.220953 0.574471
40 1.49667 0.765027 0.220953 0.574471 Figs. 5.2 − 5?5 depict the behaviors of mixed convection
parameter ? and concentration buoyancy parameter ? on ?0(?) and ?0(?)? Figs. 5.2 and 5.3 are drawn to
analyze the effects of mixed convection parameter ? on the velocity components ?0(?) and ?0(?)? It is
shown that ?0(?) and ?0(?) increase with an increase in ?. Effect of concentration buoyancy parameter ?
on the velocity components ?0(?) and ?0(?) are shown in the Figs. 5?4 and 5?5? It is examined that the
concentration buoyancy parameter ? shows the similar effects on momentum boundary layer thicknesses
and velocity components ?0(?) and ?0(?) as we observed for mixed convection parameter ?? Figs. 5?6 −
5?9 examine the variation

7of Dufour number ?? and Soret number ?? on the temperature ?(?) and
concentration

?(?)? Variations of ?? on temperature ?(?) and concentra- tion ?(?) are analyzed in the Figs. 5.6 and 5.7.
It is noted from these Figs. that ?? has reverse effects on temperature ?(?) and concentration ?(?). Figs.
5.8 and 5.9 are displayed to see the variation of ?? on the temperature ?(?) and concentration profiles ?
(?)? We noticed

2that the temperature ?(?) and thermal boundary layer are reduced for an increase in ??? The

concen- tration profile ?(?) increases when ?? is increased. To analyze the effect of thermophoretic parameter ? on the concentration ?(?) profile we have sketched Fig. 5.10.

4It is found that an increase in thermophoretic parameter ? leads to a decrease in both concentration profile ?(?) and concentration boundary layer thickness.

1 K = M = Sr = 0.5, b = N = Sc = 0.3, Df = k* = 0.3, t = 0.1, g1 = 0.4, Pr = 0.7

3I = 0.0 0.8 I = 0.5 I = 1.0 0.6 I

= 1.5 f'?h? 0.4 0.2 0 1 2 3 4 5 6 h Fig. 5.2: Influence of ? on ?0(?)? K = M = Sr = 0.5, b = N = Sc = 0.3, Df = k* = 0.3, t = 0.1, g1 = 0.4, Pr = 0.7 0.3 0.25 0.2

3I = 0.0 I = 0.5 I = 1.0 I

= 1.5 g

4'?h? 0. 15 0. 1 0. 05 0 0 1 2 3 4 5 6 h Fig. 5. 3: Influence of ? on

?0 (?)? 1 K = M = Sr = I = 0.5, b = Sc = 0.3, Df = k* = 0.3, t = 0.1, g1 = 0.4, Pr = 0.7 N = 0.0 0.8 N = 0.5 N = 1.0 0.6 N = 1.5 f '?h? 0.4 0.2 0 1 2 3 4 5 6 h Fig. 5.4: Influence of ? on ?0(?)? K = M = Sr = 0.5, b = N = Sc = 0.3, Df = k* = 0.3, t = 0.1, g1 = 0.4, Pr = 0.7 0.3 0.25

14N = 0. 0 N = 0.5 0.2 N = 1.0 N = 1 .5

g

4'?h? 0. 15 0. 1 0. 05 0 0 1 2 3 4 5 6 h Fig. 5. 5: Influence of ? on

?0(?)? K = M = I = N = Sr = 0.5, b = N = Sc = 0.3, k* = t = 0.2, g1 = 0.4, Pr = 0.7 0.5 Df = 0.1 Df = 0.4 0.4 Df = 0.7 Df = 1.0

4q?h? 0.3 0.2 0.1 0 0 2 4 6 8 h

Fig. 5.6: Influence of ?? on ?(?)? K = M = l = N = Sr = 0.5, b = N = Sc = 0.3, k* = t = 0.2, g1 = 0.4, Pr = 0.7 1 0.8 Df = 0.1 Df = 0.4 0.6 Df = 0.7 Df = 1.0 f?h? 0.4 0.2 0 0 2 4 6 8 h Fig. 5.7: Influence of ?? on ?(?)? K = M = l = N = Df = 0.5, b = N = Sc = 0.3, k* = t = 0.2, g1 = 0.4, Pr = 0.7 0.4 Sr = 0.1 Sr = 0.4 Sr = 0.7 0.3 Sr = 1.0

> **1q?h? 0.** 2 **0.** 1 **0 0 2 4** 6 8 **h**

Fig. 5.8: Influence of ?? on ?(?)? K = M = l = N = Df = 0.5, b = N = Sc = 0.3, k* = t = 0.2, g1 = 0.4, Pr = 0.7 1 0.8 Sr = 0.1 Sr = 0.4 0.6 Sr = 0.7 Sr = 1.0 f?h? 0.4 0.2 0 0 2 4 6 8 h Fig. 5.9: Influence of ?? on ?(?)? K = M = l = N = Df = 0.5, b = N = Sc = 0.3, k* = Sr = 0.2, g1 = 0.4, Pr = 0.7 1 0.8 t = 0.1 t = 0.4 0.6 t = 0.7 t = 1.0 f?h? 0.4 0.2 0 0 2 4 6 8 h Fig. 5.10: Influence of ? on ?(?)? Table 5.2 is prepared to analyze

> **1numerical values of local Nusselt and Sherwood numbers.** The **values of**

−?0(0) and −?0(0) decrease by increasing Deborah number. Here −?0(0) increases by increasing Prandtl and Biot numbers while reverse is the case of −?0(0). Table 5.3 shows that local Nusselt −?0(0) and Sherwood numbers −?0(0) decrease with the increase in Hartman and Soret numbers.

> **1Values of local Nusselt**−?0(0) **and Sherwood numbers**

−?0(0) are opposite for Schmidt, Dufour, thermophoretic and chemical reaction parameters. Table 5.2:

> **1Values of local Nusselt**−?0(0) **and Sherwood numbers**−?0(0) **for different**
> **values of** the **parameters**

?? ?? ?? ?? ? ?? and ?1 when ?? = ?? = 0?5? ?? = 0?4? ? = 0?2? ?∗ = 0?3 and ? = 0?5. ? ? ? ? ? ? ?1 −?0(0) −?0(0) 0.0 0.2 0.4 0.2 0.5 0.0 0.2 0.4 0.5 0.3 1.0 0.6 0.24700 0.23486 0.18422 0.21393 0.22444 0.23200 0.61363 0.59297 0.51482 0.55535 0.57294 0.58714 0.2 0.2 0.0 0.3 0.5 0.19606 0.21647 0.22444 0.54490 0.56327 0.57295 0.2 0.2 0.5 0.0 0.3 0.5 0.21880 0.22446 0.22772 0.56592 0.57285 0.57750 0.2 0.2 0.5 0.3 1.0 0.22443 0.57291 1.5 0.24490 0.56585 2.0 0.25725 0.56080 0.2 0.2 0.5 0.3 1.0 0.2 0.10779 0.58240 0.5 0.20254 0.57481 0.7 0.24341 0.57118 Table 5.3:

> **1Values of local Nusselt**−?0(0) **and Sherwood numbers**−?0(0) **for different**
> **values of** the **parameters**

??? ??? ?? ??? ?∗ and ? when ? = ? = 0?2? ? = 0?5? ?1 = 0?3? ?? = 1 and ? = 0?6? ?? ?? ?? ? ?∗ ? −?0(0) −?0(0) 0.2 0.4 0.5 0.2 0.3 0.5 0.26950 0.32824 0.5 0.22446 0.57285 0.7 0.19729 0.71482 0.5 0.2 0.22285 0.58265 0.5 0.22530 0.56786 0.7 0.22690 0.55779 0.5 0.4 0.2 0.27834 0.52746 0.5 0.22695 0.55765 0.7 0.18905 0.57984 0.5 0.4 0.5 0.4 0.22160 0.58610 0.7 0.21705 0.60820 1.0 0.21164 0.63434 0.5 0.4 0.5 0.2 0.5 0.20529 0.67118 0.7 0.18854 0.75605 1.0 0.16638 0.86737 0.5 0.4 0.5 0.2 0.3 0.6 0.22122 0.56859 0.8 0.21345 0.55880 1.0 0.20402 0.54895 5.4 Conclusions

2MHD three-dimensional flow of viscoelastic **fluid over** a **stretching surface is** analyzed in **the** presence of thermophoresis **and**

convective condition.

6Effects of chemical reaction and **Soret and Dufour**

are analyzed. The main observations are listed below. • Effects of mixed convection parameter ? and buoyancy concentration parameter ? on

1the velocity profiles **and** momentum **boundary layer thickness are** similar. • Effects **of**

?? and ?? on ?(?) and ?(?) are opposite. • Thermal boundary layer thickness and temperature field increase when ?? increases. • Concentration ?(?) and associated

2boundary layer thickness are decreasing functions of thermophoretic **parameter**

? ? • There are opposite effects of

1local Nusselt number and local Sherwood number when ?1? ? ?, ??, ??? ?∗? ? **and**

?? increase. • Qualitative effects of

1local Nusselt number and local Sherwood number are similar when ?, ?? ? **and**

? increase. Chapter 6

4Three-dimensional flow of Maxwell **fluid over a stretching** surface with **heat** source **and**

convective conditions

11Heat and mass transfer effects **in** three-dimensional **flow of** Maxwell **fluid over a**

stretching

sur- face are addressed in this chapter. Analysis has been performed in presence of internal heat generation/absorption. Concentration and thermal buoyancy effects are accounted. Convec- tive boundary conditions for heat and mass transfer analysis are explored. Series solutions of the resulting problems are developed. Effects of mixed convection, internal heat genera- tion/ absorption parameter and Biot numbers

15**on the dimensionless velocity, temperature and concentration** distributions **are** illustrated **graphically.**

7Numerical values of local Nusselt and Sherwood numbers are obtained and analyzed **for**

all the physical parameters. 6.1

3Governing problems Here we consider the steady three -dimensional flow of an incompressible Maxwell fluid over a stretching surface

at ? = 0? The flow takes place in the domain ? ? 0? Heat and mass transfer characteristics are taken into account

8in the presence of internal heat generation/absorption and

mixed convection. Convective heat and mass boundary conditions are considered. The ambient fluid temperature and concentration are taken as $?_\infty$ and $?_\infty$ while the surface temperature and concentration are maintained by convective heat and mass transfer at certain value ?? and ??? The governing partial differential equations subject to boundary layer flow are ?? ?? ?? ?? ?? ?? = 0? + + (6.1) ? ?? ?? ?? ?2? ? 2 ??2??2 + ?2 ??2??2 + ?2 ??2??2 + 2?? ???2??? ?? + ? ?? + ? ?? = ? ??2 − ?1 □ +2?? ???2??? + 2?? ???2??? □ +? (?? (? −□?∞) + ??(? − ?∞)) ? □ ? ?? ?? ?? ?2? ?2 ???2??2 + ?2 ??2??2 + ?2 ??2??2 + 2?? ???2??? + ?? + ? ?? + ? ?? = ? ??2 − ?1 □ 2?? ???2??? + 2?? ???2??? □ ? ? ?? ?? □ ?? ?2? □ ??? (? ?? + ? ?? + ? ??) = ? ??2 + ?(? − ?∞)? ? ?? ?? ?? ?2? ?? + ? ?? + ? ?? = ? ??2 ? (6.2) (6.3) (6.4) (6.5) In Eqs. (6?1) − (6?5) the respective velocity components in the ?−? ?− and ?−directions are denoted by ?? ? and ?, ?1 shows the relaxation time, ?∗ is the electrical conductivity, ? is the density of fluid, ? is the gravitational acceleration, ?? and ?? are the thermal and concentration expansion coefficients respectively, ? is the fluid temperature, ? = (???)

15**is the** kinematic **viscosity, ? is the** dynamic **viscosity of fluid, ?? is the** specific heat, ? **is the**

thermal conductivity, ? is the uniform volumetric heat generation/absorption, ? is the concentration field and ? is the mass diffusivity. The subjected boundary conditions are given by ? = ?? = ??? ? = ??? ? = 0? − ? ?? = ?(?? − ?), − ? ?? ???? = ?∗(?? − ?) at ? = 0? (6.6) ? → 0? ? → 0? ? → ?∞? ? → ?∞ as ? → ∞? (6.7) where ? is the heat transfer coefficient, ?∗ is the concentration transfer coefficient and ? and ? are constants and have dimension (????)−1? We now define ? = ???0(?)? ?=???0(?)? ?=− ??(?(?)+?(?))? √ ?(?) = ??? −−??∞∞ ? ? = ?r ? ?(?) = ???−−??∞∞ ? ? ? (6.8) The above variables satisfy Eq. (6.1) automatically while Eqs. (6.2)-(6.7) are converted to the following forms: ?000 + (? + ?)?00 − ?02 + ?1[2(? + ?)?0?00 − (? + ?)2?000] + ?(? + ??) = 0? ?000 + (? + ?)?00 − ?02 + ?1[2(? + ?)?0?00 − (? + ?)2?000] = 0? ?00 + ??(? + ?)?0 + ?∗? = 0? ?00 + ??(? + ?)?0 = 0? ? = 0? ? = 0? ?0 = 1? ?0 = ?, ?0 = −?1(1 − ? (0))? ?0 = −?2(1 − ?(0))? at ? = 0? ?0 → 0? ?0 → 0? ? → 0? ? → 0 as ? → ∞? (6.9) (6.10) (6.11) (6.12) (6.13) (6.14) where ?1

> 1is the dimensionless **Deborah number? ? is the** local buoyancy parameter, ??? **is the** local Grashof **number, ? is the**

concentration buoyancy parameter,

> 1**Pr is the Prandtl number, ?∗ is the** heat generation/absorption **parameter, ?? is the**

Schmidt number, ? is ratio of rates parameters, ?1 and ?2 are

> 4**the Biot** numbers **and prime** shows the **differentiation with respect to**

?. These are given by ?1 = ?1?? ? = ?R?e2?? ? ??? = ??? (???−2 ?∞)?3 ? ? = ????((???? −− ??∞∞)) (6.15) ?? = ?? ?∗=??????= ??=???1= ?? ?2= ? ? ? ? ? ? ? ? ? ?∗ ? ? ? r r In dimensionless form

> 15**the local Nusselt and local Sherwood** numbers **are** given by **?????1??**

2 = −?0(0)? (6.16) ?????1??2 = −?0(0)? (6.17) where ??? = ????? is the local Reynolds number. 6.2 Series solutions The initial approximations and auxiliary linear operators are required to develop homotopic solutions. We select the following initial guesses and linear operators for the present flow analysis: ?0(?) = 1 − ?−? ? ?0(?) = ? 1 − ?−? ? ?0(?) = ?11ex+p(?−1?)? ?0(?) = ?21ex+p(?−2?) ? ¡ ¢ ¡ ¢ L? = ?000 − ?0? L? = ?000 − ?0? L? = ?00 − ?? L? = ?00 − ?? with the following properties of the defined operators in Eq. (6.19) i.e. L?(?1 + ?2?? + ?3?−?) = 0? L?(?4 + ?5?? + ?6?−?) = 0? L?(?7?? + ?8?−?) = 0? L?(?9?? + ? 10?−?) = 0? (6.18) (6.19) (6.20) where ?? (? = 1 − 10) are the

> 2arbitrary constants. The corresponding **problems at** the **zeroth order** deformations **are**

given in the following forms: (1 − ?) L? ?ˆ(?; ?) − ?0(?) = ?~? N? ?ˆ(?; ?)?

10?ˆ(?; ?)? ?ˆ(?? ?)? ?ˆ(?? ?) ? (6.21) (1− ?) L? [?ˆ(?; ?) − ?0(?)] = ?~?N? ?ˆ(?; ?)? ?ˆ(?; ?)? ?ˆ(?? ?)? ?ˆ(??

?) ? h i h i (6.22) (1 − ?) L? ˆ?(?; ?) − ?0(?) = ?~?N?

17?ˆ(?; ?)? ?ˆ(?; ?)? ˆ?(?? ?)? ?ˆ(?? ?) ? h i (1− ?) L? ˆ?(?; ?) − ?0(?) = ?~?N? ?ˆ(?; ?)? ?ˆ(?; ?)? ˆ?(?? ?)? ˆ?(??

?) ?

25h i h i h i h i

(6.23) (6.24)

18?ˆ(0; ?) = 0? ?ˆ0(0; ?) = 1? ?ˆ0(∞; ?) = 0? ?ˆ(0; ?) = 0? ?ˆ0(0; ?) = ?? ?ˆ0(∞; ?) = 0? ˆ?0(0? ?) = −?1[1 − ?(0? ?)]? ?ˆ(∞? ?) = 0? ?ˆ0(0? ?) = −?2[1 − ?ˆ(0? ?)]? ?ˆ(∞? ?) = 0

(6.25) N?

9[?ˆ(?? ?)? ?ˆ(?? ?)? ˆ?(?? ?)? ?ˆ(?? ?)] = ?3 ?ˆ(?? ?) ??ˆ(?? ?) 2 ??3 − Ã ?? ! + (?ˆ(?? ?) + ?ˆ(?? ?)) ?2 ?ˆ(?? ?) ??2 2 (?ˆ(?? ?) + ?ˆ(?? ?)) +?1 □ ??ˆ(???) ?2 ?ˆ(???) ?? ??2 □ □ −(?ˆ(?? ?) + ?ˆ(?? ?))2 ?3 ??ˆ(?? 2??) +?[ˆ?□(?? ?) + ?1 ?ˆ(?? ?)]? □ □ □ (6.26) N ?[?ˆ(?? ?)? ?ˆ(?? ?)? ?ˆ(?? ?)? ?ˆ(?? ?)] = ?3 ?ˆ(?? ?) ??3 − ??ˆ(?? ?) 2 + (?ˆ(?? ?) + ?ˆ(?? ?))?2 ?ˆ(?? ?) μ ?? ¶ ??2 2 (?ˆ(?? ?) + ?ˆ(?? ?)) +?1 □ ??ˆ(???) ?2 ?ˆ(???) ?? ??2 □ ? (6.27) □ −(?ˆ(?? ?) + ?ˆ(?? ?))2 ?3 ??ˆ(?? 2??) □ □ □ N ?[?ˆ(?? ?)? ?ˆ(?? ?)? ?ˆ(?? ?)? ?ˆ(?? ?)] = ?2ˆ??(??2? ?) +Pr (?ˆ(?? ?)+?ˆ(?? ?))?ˆ?(???? ?) +?∗ ?ˆ(?? ?)? (6.28) N ?[?ˆ(?? ?)? ˆ?(?? ?)? ?ˆ(?? ?)? ?ˆ(?? ?)] = ?2 ?ˆ(?? ?) + ??(?ˆ(?? ?) + ?ˆ(?? ?))??ˆ(??

?) ? (6.29) ??2 ?? Here ?

3is an embedding parameter, the non-zero auxiliary parameters are ~?? ~?, ~? and ~? and the nonlinear operators

are N?? N?, N? and N?. When ? = 0 and ? = 1 one has

12?ˆ(?;0) = ?0(?)? ?ˆ(??0) = ?0(?)? ˆ?(??0) = ?0(?)? ?ˆ(??0) = ?0(?)? ?ˆ(?; 1) = ? (?)? ?ˆ(?? 1) = ?(?)? ˆ?(?? 1) = ?(?)? ?ˆ(?? 1) = ?0(?)?

(6.30) Clearly when ? is increased from 0 to 1 then ?(???)? ?(???), ?(???) and ?(???) vary from ?0(?)? ?0(?)? ?0(?) and ?0(?) to ?(?)? ?(?)? ?(?)and ?(?)? By Taylor's expansion we have ?(???)=?0(?)+?=1?? (?)??? ??(?)= ?1! ?????(??;?)⁻?=0 ∞ ? (6.31) ?(???)=?0(?)+?=1??(?)??? ??(?)= ?1! ?????(??;?)⁻?=0 ∞ ¯ ? (6.32) ¯ P ¯ ?(?? ?) = ?0(?) ?=P1??(?)??? ??(?)= ?1! ?????(??;?)⁻?¯=0 ∞ ? (6.33) P ¯ ?(?? ?) = ?0(?) ∞ ? (6.34) ?=1 ??(?)??? ??(?) = ?1! ?????(??; ?)⁻?=0

3where the convergence of above sPeries strongly depends upon ~? ? ~? ~¯? and ¯ ~?? Considering that ~?? ~?? ~? and ~? are selected properly so that

Eqs. (6?31)−(6?34) converge at ? = 1 then we have ?(?) = ?0(?) + ??(?)? ∞ ?=1 P∞ ?(?) = ?0(?) + ??(?)? ?=1 P∞ ?(?) = ?0(?) + ??(?)? ?=1 P∞ ?(?) = ?0(?) + ??(?)? ?=1 The general solutions can be expressed below: P ??(?) = ??∗(?) + ?1 + ?2?? + ?3?−?? ??(?) = ??∗(?) + ?4 + ?5?? + ?6?−?? ??(?) = ?∗?(?) + ?7?? + ?8?−?? ??(?) = ?∗?(?) + ?9?? + ?10?−?? in which the ??∗? ??∗, ?∗? and ?∗? indicate the special solutions. (6.35) (6.36) (6.37) (6.38) (6.39) (6.40) (6.41) (6.42) 6.3 Convergence analysis and discussion Clearly the homotopic series solutions (6?35) − (6?38) depend on the auxiliary parameters ~? ? ~? ? ~? and ~? ? These parameters have important role in the convergence of series solutions. For this purpose, the ~−curves are drawn at 15?? order of approximations to determine the suitable ranges of these auxiliary parameters. Fig. 6.1 shows that the acceptable values of ~? ? ~? ? ~? and ~? are −1?4 ≤ ~? ≤ −0?20? −1?6 ≤ ~? ≤ −0?40 and −1?50 ≤ ~? ? ~? ≤ −0?30. Table 6.1 ensures the convergence of homotopic

1series solutions in the whole region of ? when ~? = ~? = ~? = ~? = −0? 5? f "? 0?, g "?0?,

q'?0?,f'?0? 0.5 b1 = b = b* = g1 = g2 = 0.2, l = N = 0.5, Pr = Sc = 0.7 g''(0) f'(0) 0.0 -05. q'(0) f''(0) -10. -15. -10. -0.5 0.0 hf,hg,hq,hf Fig. 6.1: ~−curves for the functions ?? ?, ? and ?? Table 6

2.1: Convergence of series solutions for different order of approximations when

? = ?1 = ?∗ = 0?2? ?2 = ? = ? = 0?3? ?1 = 0?5? ?? = 1?0? ? ? = 1?2 and ~? = ~? = ~? = ~? = −0?5? order of approximations −?00(0) −?00(0) −?0(0) −?0(0) 1 1.018 0.1497 0.3133 0.2228 5 1.029 0.06217 0.2782 0.2095 10 1.028 0.03940 0.2703 0.2067 15 1.029 0.03866 0.2702 0.2066 20 1.029 0.03982 0.2705 0.2066 25 1.029 0.03993 0.2705 0.2066 30 1.029 0.03993 0.2705 0.2066 Figs. 6.2-6.7 show the effects of Deborah number ?1? mixed convection parameter ?? con- centration buoyancy parameter ? and

internal heat source/sink parameter ?∗ on the velocity profiles ?0(?) and ?0(?)? Figs. 6.2 and 6.3 are drawn to see the behavior of Deborah number ?1 on the velocity profiles ?0(?) and ?0(?)? It is found that both the velocity profiles ?0(?) and ?0(?) decrease with an enhancement in ?1? It is also examined from these Figs. that associ- ated boundary layer thicknesses are decreasing functions of ?1? This is due to the fact that ?1 depends on relaxation time. Larger relaxation time offers more resistance to the flow due to which the velocities are decreased. Figs. 6.4 and 6.5 are displayed to see the impact of mixed convection parameter ? on the velocity profiles ?0(?) and ?0(?)? It is seen that both the velocity profiles ?0(?) and ?0(?) increase with an enhancement in ?? Also momentum boundary layer thicknesses are increased with an increase in ?? In fact an increase in ? enhances the buoyancy forces which are more dominant to viscous forces. Variations of concentration buoyancy para- meter ? on the velocity profiles ?0(?) and ?0(?) are displayed in the Figs. 6.6 and 6.7. Similar behavior of ? is noted on the velocity profiles ?0(?) and ?0(?)? Figs. 6.8 and 6.9 are plotted to see the variations of internal heat source/sink parameter ?∗ and heat transfer

> 2**Biot number** ?1 **on the temperature ?(?)? Fig.**

6.8 depicts that the

> 13**thermal boundary layer thickness** and temperature ?(?) **are increasing functions of**

internal heat source parameter ?∗ ? 0 and decreasing functions of internal heat sink ?∗ ? 0? With an increase in heat transfer Biot number ?1? both the thermal boundary layer thickness and temperature ?(?) are enhanced (see Fig. 6.9). The reason is that as ?1 depends on heat transfer coefficient ? which leads to an increase in temperature ?(?)? Figs. 6.10 is displayed to analyze the behavior of concentration ?(?) for different values of mass transfer Biot number ?2. It is observed that as ?2 increases the associated boundary layer thickness and concentration profile ?(?) grow. As mass transfer Biot number ?2 depends on mass transfer coefficient ?∗ so with an enhancement in ?2 the mass transfer coefficient increases which leads to an increase in concentration profile ?(?)? Impacts of mixed convection parameter ?? concentration buoyancy parameter ?? Deborah number ?1 and internal heat source/sink parameter ?∗ on the local Nusselt number (−?0(0)) are displayed in the Figs. 6.11 and 6.12. It is found that local Nusselt number (−?0(0)) enhances with an increase in ? and ? (see Fig.6.11). Local Nusselt number (−?0(0)) reduces with internal heat source parameter ?∗ ? 0 while it increases with internal heat sink parameter ?∗ ? 0 (see Fig. 6.12). It is also noticed from Fig. 6.12 that local Nusselt number (−?0(0)) decreases with an increase in ?1? Figs. 6.13 and 6.14 are sketched to see the variations of mixed convection parameter ?? concentration buoyancy parameter ?? Deborah number ?1 and internal heat source/sink pa- rameter ?∗ on the Sherwood number (−?0(0)). Fig. 6.13 shows that the Sherwood number (−?0(0)) increases with an increase in ? and ?. Fig. 6.14 indicates that the Sherwood number (−?0(0)) increases by increasing internal heat source ?∗ ? 0 while reverse effect is examined with an increase in Deborah number ?1? f'?h? 1.0 0.8 b = b* = g1 = g2 = 0.2, l = N = 0.5, Pr = Sc = 0.7 0.6 0.4 b1 = 0.0, 0.3, 0.6, 1.0 0.2 0 1 2 3 4 5 6 h Fig. 6.2: Variation of ?1 on ?0(?)? g'?h? 0.20 b = b* = g1 = g2 = 0.2, l = N = 0.5, Pr = Sc = 0.7 0.15 0.10 b1 = 0.0, 0.3, 0.6, 1.0 0.05 1 2 3 4 5 6 h Fig. 6.3: Variation of ?1 on ?0(?)? f '?h? 1.0 0.8 b1 = b = b* = g1 = g2 = 0.2, N = 0.5, Pr = Sc = 0.7 0.6 0.4 l = 0.0, 0.5, 1.0, 1.5 0.2 0 1 2 3 4 5 6 h Fig. 6.4: Variation of ? on ?0(?)? g'?h? 0.20 b1 = b = b* = g1 = g2 = 0.2, N = 0.5, Pr = Sc = 0.7 0.15 0.10 l = 0.0, 0.5, 1.0, 1.5 0.05 1 2

3 4 5 6 h Fig. 6.5: Variation of ? on ?0(?)? f '?h? 1.0 0.8 b1 = b = b* = g1 = g2 = 0.2, l = 0.5, Pr = Sc = 0.7 0.6 0.4 N = 0.0, 1.5, 3.0, 4.5 0.2 0 1 2 3 4 5 6 h Fig. 6.6: Variation of ? on ?0(?)? g'?h? 0.20 b1 = b = b* = g1 = g2 = 0.2, l = 0.5, Pr = Sc = 0.7 0.15 N = 0.0, 1.5, 3.0, 4.5 0.10 0.05 1 2 3 4 5 6 h Fig. 6.7: Variation of ? on ?0(?)? q?h? 0.20 b = b1 = g1 = g2 = 0.2, l = N = 0.5, Pr = Sc = 0.7 0.15 0.10 b* = -0.4, -0.2, 0.0, 0.2, 0.4 0.05 1 2 3 4 5 6 h Fig. 6.8: Variation of ?∗ on ?(?)? q?h? 0.35 0.30 b = b1 = b* = g2 = 0.2, l = N = 0.5, Pr = Sc = 0.7 0.25 0.20 0.15 g1 = 0.1, 0.2, 0.3, 0.4 0.10 0.05 1 2 3 4 5 6 h Fig. 6.9: Variation of ?1 on ?(?)? f?h? 0.35 0.30 b = b1 = b* = g1 = 0.2, l = N = 0.5, Pr = Sc = 0.7 0.25 0.20 0.15 g2 = 0.1, 0.2, 0.3, 0.4 0.10 0.05 1 2 3 4 5 6 h Fig. 6.10: Variation of ?2 on ?(?)? -q'?0? 0.33 0.32 b1 = b* = g1 = 0.2, g2 = 0.3, Pr = Sc = 0.7 0.31 N = 0.0, 0.5, 1.0, 1.5 0.30 0.29 0.28 1 2 3 4 5 6 7 l Fig. 6.11: Variations of ? and ? on −?0(0)? -q'?0? 0.32 b* = 0.2, 0.1, 0.0, -0.1, -0.2 0.31 0.30 0.29 0.28 l = N = 0.5, g1 = 0.2, g2 = 0.3, Pr = Sc = 0.7 1 2 3 4 b1 Fig. 6.12: Variations of ?∗ and ?1 on −?0(0)? -f'?0? 0.295 b1 = b* = g1 = 0.2, g2 = 0.3, Pr = Sc = 0.7 0.290 0.285 N = 0.0,0.5,1.0,1.5 0.280 0.275 1 2 3 4 5 6 7 l Fig. 6.13: Variations of ? and ? on −?0(0)? ??'?0? 0.2748 l = N = 0.5, g1 = 0.2, g2 = 0.3, Pr = Sc = 0.7 0.2746 0.2744 0.2742 0.2740 0.2738 b1 = 0.1, 0.2, 0.3, 0.4 0.2736 * 1 2 3 4 b Fig. 6.14: Variations of ?1 and ?∗ on −?0(0)? 6.3.1 Conclusions Three-dimensional mixed

---

1**convection flow of** Maxwell **fluid over a stretching sheet**

2**with internal heat generation/** absorption is analyzed. **Convective boundary**

---

conditions for both heat and mass transfer are considered. The main observations are mentioned below. • Variations of mixed convection parameter ? and concentration buoyancy parameter ? enhance the velocity profiles and associated boundary layer thicknesses. • Velocity profiles and temperature increase in case of internal heat source ?∗ ? 0 while these reduce for heat sink ?∗ ? 0? • Heat transfer Biot number ?1

16**increases the thermal boundary layer thickness and**

---

tem- perature. Also concentration and its associated boundary layer are enhanced with an increase in mass transfer Biot number ?2? • The

---

7**local Nusselt and Sherwood numbers have** quite similar behaviors **for** increasing **values of**

---

mixed convection parameter ?? concentration buoyancy parameter ? and Deborah number ?1. • Larger values of heat sink parameter ?∗ ? 0 give rise to the local Nusselt number (−?0(0))? However Sherwood number (−?0(0)) enhances with an increase in heat source ?∗ ? 0? Chapter 7 Soret and Dufour effects in three-dimensional

---

2**flow of Maxwell fluid with** chemical reaction and **convective condition**

This chapter addresses the

11**heat and mass transfer** effects **in** three-dimensional **flow of** Maxwell **fluid over a stretching**

2**surface with convective boundary conditions.** Mass transfer **is considered** in **the**

presence of first order chemical reaction.

1**Soret and Dufour effects** in **the** conservation law **of**

energy and concentration are considered. Convergent series solutions to the resulting nonlinear problems are developed. Plots of physical quantities of interest are analyzed. 7.1 Problems formulation

3**We consider the** steady three **-dimensional flow of an incompressible Maxwell fluid** induced by **a** stretching **surface**

at ? = 0? The flow takes place in the domain ? ? 0? The ambient fluid temperature is taken as ?∞ while the surface temperature is maintained by convective heat transfer. Soret and Dufour effects in presence of mixed convection flow are taken into account. In addition the effect of first order chemical reaction in mass transfer is taken under consideration. Flow diagram is as follows: Fig. 7.1: Geometry of the problem The governing boundary layer equations for three-dimensional flow of Maxwell fluid can be written as ?? ?? ?? ?? ?? ?? = 0? + + (7.1) ? ?? ?? ?? ?2? ?2 ???2??2 + ?2 ???2??2 + ?2 ???2??2 + 2?? ????2??? ?? + ? ?? + ? ?? = ? ??2 − ?1 □ +2?? ????2??? + 2?? ????2??? □ +??? (? − □?∞) + ???(? − ?∞)? □ ? ?? ?? ?? ?2? ?? + ? ?? + ? ?? = ? ??2 − ?1 ?2 ???2??2 + ?2 ???2??2 + ?2 ???2??2 + 2?? ????2??? + □ 2?? ????2??? + 2?? ????2??? □ ? □ □ ? ?? ?? ?? ?2? ???? ?2? + ? ?? 2 ?? 2 ?? + ? ?? + ? ?? = ? ??2 ???? ??2 ??? Ãµ ?? + + ? ¶ µ ?? ¶ ! ? ?? ?? ?? ?2? ???? ?2? ?? + ? ?? + ? ?? = ?? ??2 ?? ??2 − ?1(? − ?∞)? + (7.2) (7.3) (7.4) (7.5) In Eqs. (7?1) − (7?5) the respective velocity components in the ?−? ?− and ?−directions are denoted by ?? ? and ?, ?1 shows the relaxation time, ? the fluid temperature, ? the thermal diffusivity of fluid, ? = (???) the kinematic viscosity, ? the dynamic viscosity of fluid, ? the concentration field, ?? the mass diffusivity, ?? the thermal diffusion ratio, C? the specific heat, C? the concentration susceptibility and ? the density of fluid. The boundary conditions appropriate to flow under consideration are given by ? = ?? = ??? ? = ??? ? = 0? − ? ?? = ?(?? − ?), ? = ?? at ? = 0? ?? (7.6) ? → 0? ? → 0? ? → ?∞? ? → ?∞ as ? → ∞? (7.7) where ?∗ indicates the thermal conductivity of fluid, ?? is the hot fluid temperature, ?∞ the ambient concentration and ? and ? have dimension inverse of time. We now define ? = ???0(?)? ?=??? 0(?)? ?=− ??(?(?)+?(?))? √ ?(?) = ??? −−??∞∞ ? ? = ? ? r ? ? ?(?) = ?? − ?∞ ? − ?∞ ? (7.8) Now the use of above variables satisfy Eq. (7.1) automatically while Eqs. (7.2)-(7.7) are reduced as follows: ?000 + (? + ?)?00 − ?02 + ?1[2(? + ?)?0?00 − (? + ?)2?000] + ?(? + ??) = 0? ?000 + (? + ?)?00 − ?02 + ?1[2(? + ?)?0?00 − (? + ?)2?000] = 0? ?00 + ??(? + ?)?0 + Pr???00 + Pr(?1?002 + ?2?002) = 0? ?00 + ??(? + ?)?0 − ???∗? + ?????00 = 0? ? = 0? ? = 0? ?0 = 1? ?0 = ?, ?0 = −?1(1 − ?(0))? ? = 1 at ? = 0? ?0 → 0? ?0 →

0? ? → 0? ? → 0 as ? → ∞? (7.9) (7.10) (7.11) (7.12) (7.13) (7.14) ?1 = ?1?? ?=?R?e?2?? ???=???(??? −2?∞)?3? ?=????((????−−??∞∞))???= ? ? ? ?? = ? ?? = ? ??∗ = ?1?? = ???1 = ?∗r ? ? ? ?? ? ? ? (7.15) ???????? ((????−−??∞∞)?)? ?? = ??????? ((????−−??∞∞))??1 = ??(???2?−2?∞)??2 = ??(???2? −2?∞) Here ?1 is the Deborah number, ? the mixed convection parameter, ??? the local Grashof number, ? the concentration buoyancy parameter, ? ? the Prandtl number, ?? the Dufour number, ?? the Soret number, ?1 and ?2 the Eckert numbers along the ? and ? directions respectively, ?? the Schmidt number, ?∗ the chemical reaction parameter, ? the ratio of rates parameters and ?1 the Biot number. All parameters are defined in a sequence in which they are written and prime shows the differentiation with respect to ?. Local Nusselt (dimensionless temperature gradient at the surface) and local Sherwood (di- mensionless concentration gradient at the surface) numbers in dimensionless forms are given by ????? 1??2 = −?0(0)? (7.16) ?????1??2 = −?0(0)? (7.17) in which ??? = ????? is the local Reynolds number. 7.2 Homotopy analysis solutions The initial approximations and auxiliary linear operators required for homotopy analysis solu- tions are presented below i.e. ?0(?) = 1 − ?−? ? ?0(?) = ? 1 − ?−? ? ?0(?) = ? 11ex+p(?−1?) ? ?0(?) = exp(−?) (7.18) ¡ ¢ ¡ ¢ L? = ?000 − ?0? L? = ?000 − ?0? L? = ?00 − ?? L? = ?00 − ?? (7.19) with the following properties of the defined operators in Eq. (7.18) i.e. L?(?1 + ?2?? + ?3?−?) = 0? L?(?4 + ?5?? + ?6?−?) = 0? L?(?7?? + ?8?−?) = 0? L?(?9?? + ?10?−?) = 0 (7.20) where ?? (? = 1 − 10) indicate the

2**arbitrary constants. The** corresponding **problems at** the **zeroth order are**

given in the following forms: (1 − ?) L? ?ˆ(?; ?) − ?0(?) = ?~? N? ?ˆ(?; ?)? ?ˆ(?; ?) ? (7.21) h i h i (1 − ?) L? [?ˆ(?; ?) − ?0(?)] = ?~?N? ?ˆ(?; ?)? ?ˆ(?; ?) ? (7.22) (1 − ?) L? ˆ?(?; ?) − ?0(?) = ?~?N?

29**?ˆ(?; ?)? ?ˆ(?; ?)? ˆ?(?? ?)? ?ˆ(?? ?) ? h i**

(7.23) (1 − ?) L? ?ˆ(?; ?) − ?0(?) = ?~?N? ?ˆ(?; ?)? ?ˆ(?; ?)? ˆ?(?? ?)? ?ˆ(?? ?) ?

25**h i h i** (7.24) **h i h i**

18**?ˆ(0; ?) = 0? ?ˆ0(0;** ?) = 1? **?ˆ0(∞; ?) = 0? ?ˆ(0; ?) = 0? ?ˆ0(0; ?) = ?? ?ˆ0(∞; ?) = 0? ?ˆ0(0? ?) = −?1[1 − ?(0? ?)]? ˆ?(∞? ?) = 0? ?ˆ(0? ?) = 1? ?ˆ(∞? ?) = 0** (7.25) N? **[?ˆ(?? ?)? ?ˆ(??**

?)] = N

9**?[?ˆ(?? ?)? ?ˆ(?? ?)] = ?3?ˆ(?? ?) ??ˆ(??** ?) 2 ??3 − Ã ?? ! + **(?ˆ(?? ?) + ?ˆ(?? ?)) ?2?ˆ(??** ?) ??2 +? 1 □ 2 **(?ˆ(?? ?) + ?ˆ(?? ?)) ??ˆ?(????)** ?2 **??ˆ(?? 2??) −(?ˆ(?? ?) + ?ˆ(?? ?))2 ?3 ??ˆ(?? 2??)** □ +? **?ˆ(?? ?) + ? ?ˆ(??** ?) ? □ □ (7.26) ³ ´ ?3ˆ? (?? ?) **??ˆ(??** ?) 2 + **(?ˆ(?? ?) + ?ˆ(?? ?))?2 ?ˆ(??** ?) ??3 − ?? ¶ ??2 +? 2 **(?ˆ(?? ?) + ?ˆ(?? ?))??ˆ?(????)** ?2 **??ˆ(?? 2??)** μ 1□ **−(?ˆ(?? ?) + ?ˆ(?? ?))2 ?3 ??ˆ(??**

2??) ▯ ? (7.27) ▯ ▯

32**N?[?ˆ(?? ?)? ?ˆ(?? ?)? ?ˆ(?? ?)? ?ˆ(?? ?)] = N ?[?ˆ(?? ?)? ?ˆ(?? ?)? ?ˆ(?? ?)? ?ˆ(?? ?)] = ?2ˆ?(?? ?) + Pr (?ˆ(?? ?) + ?ˆ(??**

?))

17**?ˆ?(?? ?) ??2** ?? + Pr ?? **??2** ?2 **?ˆ(?? ?) 2** + Pr ▯ **+?2³** ?2 **??ˆ(??2??)´** 2 ▯ ? ?1 ?2 **??ˆ(??2??)**▯ ▯ ³ ´ ▯ ▯ ?2ˆ?(?? ?) + **??(?ˆ(?? ?) + ?ˆ(?? ?)) ??ˆ(?? ?) ??2**

?? −???∗?ˆ(?? ?) + ???? ?2ˆ?(?? ?) ? ??2 (7.28) (7.29) Here ?

4**is an embedding parameter, ~?? ~?, ~? and** ~? are **the non-zero auxiliary parameters and** N?? N?, N? **and** N? indicate **the nonlinear operators.** When ? = **0 and** ? = 1 **one has**

12**?ˆ(?;0) = ?0(?)? ˆ?(??0) = ?0(?)? ?ˆ(??0) = ?0(?) ?ˆ(?;** 1) = ?(?)? **ˆ?(??** 1) = ? (?)? **?ˆ(??** 1) = **?0(?)**

? (7.30) Clearly when ? is increased from 0 to 1 then ?(???)? ?(???), ?(???) and ?(???) vary from ?0(?)? ?0(?)? ?0(?) and ?0(?) to ?(?)? ?(?)? ?(?)and ?(?)? By Taylor's expansion we have ?(???)=?0(?)+?=1?? (?)??? ??(?)= ?1! ?????(??;?)¯?=0 ∞ ? (7.31) ?(???)=?0(?)+?=1??(?)??? ??(?)= ?1! ?????(??;?)¯?=0 ¯ ∞ ? (7.32) P ¯ ?=P1??(?)??? ??(?)= ?1! ?????(??;?)¯?¯=0 ∞ ¯ ?(?? ?) = ?0(?) ? (7.33) P ¯ ?(?? ?) = ?0(?) ∞ ? (7.34) ?=1 ??(?)??? ??(?) = ?1! ?????(??; ?)¯¯?=0

3**where the convergence of above** sPeries **strongly depends upon ~? ? ~? ~¯? and ¯ ~?? Considering that ~?? ~?? ~? and ~? are selected properly so that**

Eqs. (7?31)−(7?34) converge at ? = 1 then we can write ?(?) = ?0(?) + ??(?)? ∞ (7.35) ?=1 ?(?) = ?0(?) + P∞ ??(?)? (7.36) ?=1 ?(?) = ?0(?) + ??(?)? (7.37) P∞ ?=1 ?(?) = ?0(?) + ??(?)? ?=1 (7.38) P∞ P The resulting problems at mth order deformation can be constructed as follows: L?[??(?) − ????−1(?)] = ~?R?? (?)? (7.39) L?[??(?) − ????−1(?)] = ~?R??(?)? (7.40) L?[??(?) − ????−1(?)] = ~?R??(?)? (7.41) L?[??(?) − ????−1(?)] = ~?R??(?)? (7.42) ??(0) = ??0(0) = ??0(∞) = 0? ??(0) = ?0?(0) = ??0(∞) = 0? ?0?(0) − ? 1??(0) = ??(∞) = 0? ??(0) = ??(∞) = 0? (7.43) ?−1 ?−1 R??(?) = ??000−1(?) − ??0−1−??0 + (??−1−??? 00 + ??−1−??0) ?=0 ?=0 +?1 ?−1 ? P 2(??−1−? + ??−1−?)??0−??00 − (??−1−???−? + ??−1−???−? + 2??−1−???−?)??000 P ?=0 ?=0 +? ?P?−1P+¡? ??−1 ? (7.44¢) ¡ ¢ R??(?) = ??000−1(?) − ?0?−1−??0? + (??−1−??0?0 + ??−1−??0?0) ?−1 ?−1 ?=0 ?=0 +?1 ?−1 ? P [2(??−1−? + ??−1−?)??0−??0?0 P ?=0 ?=0 −

(??P−1−P???−? + ??−1−???−? + 2??−1−???−?)?0?00]? (7.45) R??(?) = ?0?0−1 + ? ? (?0?−1−??? + ?0?−1−???) + Pr ?? ??−1 ?−1 00 ?=0 ?−1 + Pr ?1 ?−P1 ??00−1−??00 + ?2 ??00−1−??00 ? μ ?=0 ?=0 P P ¶ R??(?) = ?0?0−1 + ?? (?0?−1−??? + ?0?−1−???) − ???∗? + ??????−1? ?−1 00 ?=0 P ?? = 0? ? ≤ 1? □ 1? ? ? 1? Solving the above mth order deformation problems we have □ ??(?) = ??∗(?) + ?1 + ?2?? + ?3?−?? ??(?) = ??∗(?) + ?4 + ?5?? + ?6?−?? ??(?) = ?∗?(?) + ?7?? + ?8?−?? ??(?) = ?∗?(?) + ?9?? + ?10?−?? in which the ??∗? ??∗, ?∗? and ?∗? indicate the special solutions. (7.46) (7.47) (7.48) (7.49) (7.50) (7.51) (7.52) 7.3 Analysis and discussion Obviously the homotopic solutions (7?35) − (7?38) involve the auxiliary parameters ~?? ~? ~? and ~?. These parameters have pivotal role in adjusting and controlling the convergence of homotopic solutions. Hence the ~−curves are displayed at 18?? order of approximations in order to determine the suitable ranges of ~?? ~? ~? and ~?. Fig. 7.2 witness that the admissible values of ~?? ~? ~? and ~? here are −1?25 ≤ ~? ≤ −0?50? −1?25 ≤ ~? ≤ −0?50, −1?40 ≤ ~? ≤ −0?25 and −1?20 ≤ ~? ≤ −0?75? Table 7.1 ensures that the developed

1**series solutions converge in the whole region of**

? when ~? = ~? = ~? = ~? = −1? Table 7.1:

1**Convergence of** series **solutions for different order of approximations when**

?1 = 0?2? ? = 0?3? ?? = 1?0? ?? = 0?4? ?? = 0?5? ?? = 0?5? ?∗ = 0?3? ?1 = 0?6 and ~? = ~? = ~? = ~? = −1?0? order of approximations 1 5 10 15 20 25 30 35 −? 00 (0) 1.1215 1.1323 1.1323 1.1323 1.1323 1.1323 1.1323 1.1323 −?00 (0) 0.25735 0.25958 0.25951 0.25951 0.25951 0.25951 0.25951 0.25951 −?0(0) 0.24141 0.22468 0.22127 0.22061 0.22045 0.22041 0.22040 0.22040 −?0 (0) 0.64583 0.56841 0.55948 0.55838 0.55820 0.55817 0.55816 0.55816 b1 = 0.4, l = N = 0.1,E1 = E2 = k* = Sr = 0.3, g1 = 0.6, b = Df = 0.5, Pr = 1.0, Sc = 0.7 0.5 0 f''?0? g''?0? q'?0?

14**f'?0? f''?0?, g''?0?,** q **'?0?, f'?0?**

- 0.5 -1 - 1.5 -2 -1.5 -1 - 0.5 0 hf, hg, hq,hf Fig.7.2: ~−curves for the function ????? and ?? Figs. 7?3 and 7.4 depict the behaviors of mixed convection parameter ? and concentration buoyancy parameter ? on the velocity profile ?0(?)? Fig. 7?3 is drawn to see the effect of mixed convection parameter ? on velocity profile ?0(?)? It is noticed that momentum boundary layer thickness and velocity profile ?0(?) increase with an increase in mixed convection parameter in case of assisting flow (? ? 0) while reverse effect is noted in case of opposing flow (? ? 0)? This is due to the fact that buoyancy forces are more dominant to viscous forces in case of assisting flow (? ? 0) while buoyancy forces reduce for opposing flow (? ? 0). Effect of concentration buoyancy parameter ? on the velocity profile ?0(?) is analyzed in Fig. 7?4? It is seen that both the momentum boundary layer thickness and velocity profile ?0(?) are increasing functions of concentration buoyancy parameter ? . f '?h? 1 0.8 0.6 0.4 l = -0.3, 0.0, 0.3, 0.6 0.2 1 2 3 4 5 6 h Fig. 7.3: Influence of ? on ?0(?) when ? = ?1 = ?2 = 0?3 and ?1 = 0?5? f '?h? 1 0.8 0.6 0.4 N = 0.0, 0.5, 1.0, 1.5 0.2 1 2 3 4 5 6 h Fig. 7.4: Influence of ? on ?0(?) when ? = ? = ?1 = ?2 = 0?3 and ?1 = 0?5? Figs. 7.5−7?9 examine the variation of Dufour number ??, Eckert numbers E1 and E2? mixed convection parameter ? and concentration buoyancy parameter ? on the temperature ?(?)? Effect of Dufour number ?? on temperature profile ?(?) is presented in Fig. 7.5. As Dufour effect pointed out the generation of energy flux

by composition gradient, then the thermal boundary layer thickness and temperature ?(?) increase with an increase in ??. Figs. 7.6 and 7.7 illustrate the effect of Eckert numbers ?1 and ?2 on the temperature ?(?) along the ? and ? directions respectively? As the Eckert number expresses the relationship between the kinetic energy in the flow and the enthalpy. It embodies the conversion of kinetic energy into internal energy by work done against the viscous fluid stresses. Greater viscous dissipative heat causes a rise

> **13in the temperature and thermal boundary layer thickness.** Fig. 7.8 **is**

drawn to see the influence of mixed convection parameter ? on

> **2the temperature ?(?)? Thermal boundary layer thickness**

and temperature ?(?) decrease in case of assisting flow (? ? 0) while increase in case of opposing flow (? ? 0)? This is due to the fact that in case of assisting flow (? ? 0) the buoyancy forces are more dominant to viscous forces which causes a reduction in the temperature ?(?) while in case of opposing flow (? ? 0) the viscous forces are more dominant than buoyancy forces which in results enhances the temperature ?(?)? Influence of concentration buoyancy parameter ? on the temperature ?(?) is seen in Fig. 7.9. It is found that the associated boundary layer and temperature ?(?) is decreasing function of ?. q?h? 0.5 0.4 0.3 0.2 Df = 0.1, 0.3, 0.7, 1.0 0.1 1 2 3 4 5 6 h Fig. 7.5: Influence of ?? on ?(?) when ?1 = ?∗ = 0?5? ? ? = ?? = 0?5 ? = ?1 = ?2 = ? = ? = ?? = 0?3 and ?1 = 0?4? q?h? 0.5 0.4 0.3 0.2 E1 = 0.0, 0.3, 0.6, 0.9 0.1 1 2 3 4 5 6 h Fig. 7.6: Influence of ?1 on ?(?) when ?1 = ?∗ = 0?5? ? ? = ?? = 0?5, ? = ?? = ?2 = ? = ? = ?? = 0?3 and ?1 = 0?4? q?h? 0.4 0.3 0.2 E2 = 0.0, 0.3, 0.6, 0.9 0.1 1 2 3 4 5 6 h Fig. 7.7: Influence of ?2 on ?(?) when ?1 = ?∗ = 0?5? ? ? = ?? = 0?5 ? = ?? = ?1 = ? = ? = ?? = 0?3 and ?1 = 0?4? q?h? 0.35 0.3 0.25 0.2 0.15 l = -0.5, 0.0, 0.5, 1.0 0.1 0.05 1 2 3 4 5 6 h Fig. 7.8: Influence of ? on ?(?) when ?1 = ?∗ = 0?5? ? ? = ?? = 0?5 ?? = ?1 = ?2 = ? = ? = ?? = 0?3 and ?1 = 0?4? q?h? 0.35 0.3 0.25 0.2 0.15 N = 0.0, 0.3, 0.6, 0.9 0.1 0.05 1 2 3 4 5 6 h Fig.7.9: Influence of ? on ?(?) when ?1 = ?∗ = 0?5? ? ? = ?? = 0?5 ?? = ?1 = ?2 = ? = ? = ?? = 0?3 and ?1 = 0?4? Figs. 7.10-7.13 are sketched to see the effects of Soret number ??? generative/destructive chemical reaction ?∗? mixed convection parameter ? and concentration buoyancy parameter ? on concentration profile ?(?)? Fig. 7.10 depicts the influence of Soret number ?? on concentration profile ?(?)? As in Soret effect the temperature gradient causes the mass flux which in turn enhances the concentration profile ?(?) and associated boundary layer thickness. Effect of destructive/generative chemical reaction ?∗ on the concentration profile ?(?) is analyzed in Fig. 11. It is found that in case of generative chemical reaction (?∗ ? 0) the reduction in ?(?) is noted while reverse is in case of destructive chemical reaction (?∗ ? 0). Figs. 7.12 and 7.13 are presented to see the effects of mixed convection parameter ? and concentration buoyancy parameter ? on the concentration profile ?(?)? These Figs. show that ?(?) reduces with the increase in ? and ?. Also the associated boundary layer thickness are decreasing functions of ? and ? . f?h? 1 0.8 0.6 0.4 0.2 Sr = 0.1, 0.3, 0.5, 0.7 1 2 3 4 5 6 h Fig. 7.10: Influence of ?? on ?(?) when ?1 = ?∗ = 0?5? Pr = 0?5? ?1 = 0?4? ?? = ?1 = ?2 = ? = ? = ? = ?? = 0?3? f?h? 1 0.8 0.6 0.4 k* = -0.5, 0.0, 0.5, 1.0 0.2 1 2 3 4 5 6 h Fig. 7.11: Influence of ?∗ on ?(?) when ?1 = 0?5? ? ? = ?? = 0?5? ?? = ?1 = ?2 = ? = ? = ? = ?? = 0?3 and ?1 = 0?4? f?h? 1 0.8 0.6 0.4 l = -0.5, 0.0, 0.5, 1.0 0.2 1 2 3 4 5 6 h Fig. 7.12: Influence of ? on ?(?) when ?1 = ?∗ = 0?5? ? ? = ?? = 0?5? ?? = ?1 = ?2 = ? = ? = ?? = 0?3and ?1 = 0?4? f?h? 1 0.8 0.6 0.4 N = 0.0, 0.3, 0.6, 0.9 0.2 1 2 3 4 5 6 h Fig. 7.13: Influence of ? on ?(?) when ?1 = ?∗ = 0?5? ? ? = ?? = 0?5? ?? = ?1 = ?2 = ? = ? = ?? = 0?3 and ?1 = 0?4? Table 7.1 presents the convergence of homotpic solutions. It is noted that computations are sufficient for 15?? order iterations of velocity and 35?? order iterations of the temperature and

concentration profiles for convergent series solutions. Tables 7.2 and 7.3 are prepared to analyze

> **1numerical values of local Nusselt and Sherwood numbers.** The **values of**

−?0(0) and −?0(0) decrease by increasing Deborah number, mixed convection parameter and concentration buoyancy parameter. Here −?0(0) increases by increasing Prandtl number and Biot number while reverse is the case of −?0(0)? It is also found that −?0(0) decreases by increasing Eckert numbers while opposite behavior of −?0(0)? Table 7.4 presents the comparison of −?00(0) and −?00(0) for various values of ? in the limiting sense with ref. [13]. Table 7.5 ensures

> **4the values of local Nusselt number**−?0(0) **are** in good agreement **with**

ref. [39] in a limiting sense. Table 7.2:

> **1Values of local Nusselt**−?0(0) **and Sherwood numbers**−?0(0) **for** the **different values of** the **parameters**

?1? ?? ??? ??? ??? ?∗? ?? and ?1 when ? = ? = ?1 = ?2 = 0?3. ?1 ? ? ? ?1 ?? ?? ?? ?∗ −?0(0) −?0(0) 0.0 0.2 0.7 0.4 0.5 0.4 0.5 0.3 0.16023 0.56411 0.2 0.15399 0.55548 0.4 0.14858 0.54795 0.2 0.0 0.14266 0.53103 0.2 0.15399 0.55548 0.4 0.16269 0.57666 0.2 0.2 0.4 0.13249 0.55814 0.8 0.16009 0.55462 1.2 0.17597 0.55212 0.2 0.2 0.7 0.2 0.098922 0.56163 0.4 0.15319 0.55548 0.6 0.18962 0.55150 0.2 0.2 0.7 0.4 0.0 0.20797 0.11228 0.5 0.15399 0.55548 1.0 0.11405 0.85681 0.2 0.2 0.7 0.4 0.5 0.0 0.15406 0.56553 0.5 0.15457 0.55288 1.0 0.15565 0.53984 0.2 0.2 0.7 0.4 0.5 0.5 0.2 0.19480 0.54212 0.5 0.15469 0.55287 1.0 0.084902 0.57214 0.2 0.2 0.7 0.4 0.5 0.5 0.5 0.0 0.18113 0.35157 0.3 0.15469 0.55212 0.5 0.14153 0.65090 Table 7.3:

> **1Values of local Nusselt**−?0(0) **and Sherwood numbers**−?0(0) **for** the **different values of** the **parameters**

?? ?? ?1 and ?2 when ?1 = ? = 0?2? ?? = ?? = 0?5, ?? = ?1 = 0?4? ? ? = 0?7 and ?∗ = 0?3. ? -0.3 0.0 0.3 0.3 ? 0.3 0.0 0.3 0.6 ?1 0.3 ?2 0.3 −?0(0) 0.0091701 0.091253 0.12419 0.11837 0.12444 0.12978 −?0(0) 0.71369 0.73788 0.75068 0.74820 0.75042 0.75260 0.3 0.3 0.0 0.1 0.2 0.18952 0.16745 0.14557 0.73121 0.73772 0.74431 0.3 0.3 0.3 0.0 0.1 0.2 0.14414 0.13777 0.13110 0.74474 0.74657 0.74854 Table 7.4: Values of −?00(0) and −?00(0) with ? when ?1 = ? = ? = 0 with HPM (Ariel [13]) and exact solution (Ariel [13]). ? −?00(0) −?00(0) HAM HPM[13] Exact[13] HAM HPM[13] Exact[13] 0.0 1 1 1 0 0 0 0.1 1.020260 1.017027 1.020260 0.066847 0.073099 0.066847 0.2 1.039495 1.034587 1.039495 0.148737 0.158231 0.148737 0.3 1.057955 1.052470 1.057955 0.243360 0.254347 0.243360 0.4 1.075788 1.070529 1.075788 0.349208 0.360599 0.349209 0.5 1.093095 1.088662 1.093095 0.465205 0.476290 0.465205 0.6 1.109946 1.106797 1.109947 0.590528 0.600833 0.590529 0.7 1.126397 1.124882 1.126398 0.724532 0.733730 0.724532 0.8 1.142488 1.142879 1.142489 0.866682 0.874551 0.866683 0.9 1.158254 1.160762 1.158254 1.016539 1.022922 1.016539 1.0 1.173720 1.178511 1.173721 1.173720 1.178511 1.173721 Table 7.5:

3**Values of local Nusselt number**$-?0(0)$ **for different values**

of $?*??$ and Pr in a limiting sense when $? = ? = ?1 = ?2 = ?? = ?? = ?* = 0?0$ and $?1 = 0?6?$ $?1$? Pr Present Results Hayat et al [39] $-?0(0)$ $-?0(0)$ 0.0 0.5 1.0 0.330404 0.33040 0.3 0.321661 0.32160 0.8 0.308651 0.30799 1.2 0.299526 0.29873 0.4 0.0 0.287813 0.28908 0.4 0.316638 0.31664 0.7 0.330168 0.33017 1.0 0.340702 0.34070 0.4 0.5 0.7 0.282787 0.28279 1.2 0.340424 0.34042 1.6 0.368405 0.36840 2.0 0.388869 0.38887 7.4 Conclusions Three dimensional

2**mixed convection flow of** Maxwell **fluid** over **a stretching surface with**

convec- tive condition is investigated. Effects of chemical reaction and Soret and Dufour are analyzed. The main observations are listed below. • Momentum boundary layer thickness and velocity profile $?0(?)$ increase with the increase in ? and ?. • Concentration profile $?(?)$ is decreasing function of generative chemical reaction parame- ter $(?* ? 0)$ while increasing function of destructive chemical reaction $(?* ? 0)$ and Soret number $??$. • Influences of ? and ? on $?(?)$ and $?(?)$ are qualitatively similar. • Thermal boundary layer thickness and temperature field increase when $??$ increases. • There are opposite effects of

1**local Nusselt number and local Sherwood number** when $??$, ??? E1 **and**

E2 increase. • Local Nusselt and Sherwood numbers increase when $?*$, ?? and ?? are enhanced. • Effects

7**of local Nusselt number and local Sherwood number** for Pr, ??? ? and ? **are**

similar. Chapter 8 Radiative mixed convection

2**flow of an** Oldroyd-B **fluid by an** inclined **stretching surface**

Mixed convection

2**flow of an Oldroyd-B fluid**

in the presence of thermal radiation is investigated in this chapter. Flow is induced by an inclined stretching surface. The

2**boundary layer** equations **of an Oldroyd-B fluid**

1**in the presence of heat transfer are**

used. Appropriate transformations reduce the

> 2partial differential equations into the ordinary differential equations.

Computational analysis is performed for the convergent series solutions. The

> 4values of local Nusselt number are

numerically analyzed. Effects of various parameters involved in the velocity and temperature are discussed. 8.1 Mathematical analysis Consider the steady two-dimensional mixed convection

> 2flow of an incompressible Oldroyd-B fluid by an inclined stretching surface. The heat transfer is

considered

> 8in the presence of thermal radiation

using Rosseland approximation. Here

> 3?−axis is taken along the stretching surface and ?−axis normal to the ?−axis.

> 2Conservation laws of mass, linear momentum and energy

in absence of viscous dissipation give ?? ?? ?? ?? = 0? + (8.1) ? ???? + ? ???? = ? ??2??2 − ?1 ?2 ?? 2??2 + ?2 ??2??2 + 2?? ???2??? ?3? · ?3? ?? ?2? ?? ?2? ¸ +??2 ????2 + ? ??3 ?? ??2 ?? ??2 + ??? (? − ?∞) cos ?? (8.2) ? − − ·¸ ??? ? = ? ???? +????? ?? μμ16?3????∞3 +? ???? ? (8.3) μ ¶ ¶ ¶ where ? and ?

> 1are the velocity components in the ?− and ?−directions,

?1 and ?2 are the relaxation and retardation times, respectively, ? = (???)

> 1is the kinematic viscosity, ?? is the Stefan-Boltzmann constant, ? is the fluid temperature, ? is the fluid density, ? is the gravita- tional acceleration, ?? is

thermal expansion coefficient of temperature, ?? is the specific heat, ?? is the mean absorption coefficient

and ? is the thermal conductivity. The appropriate boundary conditions are taken as follows: ? = ??? ? = 0? ? = ?? at ? = 0? ? → 0? ? → ?∞ as ? → ∞ with the surface temperature ?? by ??(?? ?) = ?∞ + ??? where ? and ? are the positive constants. If ? is the stream function then defining ? = ? √ r ? ?? ? = ???? (?)? ?(?) = ?? − ?∞ ? − ?∞ ? ? = ?? , ? = − ?? ?? ?? ? (8.4) (8.5) (8.6) (8.7) (8.8) the incompressibility condition is clearly satisfied and the resulting problems for ? and ? satisfy the following equations ? 000 + ? ? 00 − ? 02 + ?1(2? ? 0? 00 − ? 2? 000) − ?2(? 002 − ? ? 0000) + ?? cos ? = 0? (8.9) 1 + 4? ?00 + Pr ??0 − ?0? = 0? (8.10) μ 3 ¶ ?(0) = 0? ?0(0) = 1? ?0(∞) → 0? ?(0) = 1? ?(∞) → 0? ¡ ¢ (8.11) In the above expressions ?1 and ?2 are the Deborah numbers, ? is mixed convection parameter, ??? is the local Grashof number, ??? is the Reynold number,

differentiation with respect to ? i.e. ?1 = ?1?? ?2 = ?2?? ? = ?R?e?2? , ??? = ??(?? ?−2 ?∞)?3 , ??? = ? 2???22 ? Pr = ????? = (4???∞3)? ? (8.12) ??? Local Nusselt number ? ?? in terms of heat transfer ?? is ??? = ?(????−??∞) ? ?? = −? ? (8.13) μ ?? ?? ¶?=0 Above equation in dimensionless variables becomes ?????1??2 = −(1 + 4?) ?0(0)? 3 (8.14) Considering the set of base functions ?? exp(−??)|? ≥ 0? ? ≥ 0 (8.15) n o one can express that ∞ ∞ ??(?) = ??????? exp(−??)? (8.16)

where ???? and ???? are the coefficients. Initial guesses ?0 and ?0 and auxiliary linear operators ~? and ~? are chosen as follows: ?0(?) = 1 − exp(−?)? ?0(?) = exp(−?)? (8.18) ?3? ?? ?2? L? = ??3 ?? ? L? = ?? 2 − ?? − (8.19) where L? [?1 + ?2 exp(?) + ?3 exp(−?)] = 0? L? [?4 exp(?) + ?5 exp(−?)] = 0 (8.20) in which ?? (? = 1 − 5)

zeroth order are (1 − ?)L?[?ˆ(?? ?) − ?0(?)] = ?~?N? ?ˆ(?? ?)? ?ˆ(?? ?) ? (8.21) (1 − ?)L?[ˆ?(?? ?) − ?0(?)] = ?~?N? ?ˆ(?? ?)? ˆ?(?? ?) ? h i (8.22) ??ˆ(?; ?) h ??ˆ(?; ?) i ?ˆ(?; ?) = 0? ?? = 1? ?? = 0? (8.23) ?=0 ¯ ?=0 ?=∞ ¯ ?ˆ(?; ?) = 1?¯ ˆ?(?; ?) = 0? ¯ ¯ ¯ (8.24) ¯ ?=0 ¯ ¯ ?=∞ ¯ ¯ ¯ ?3

?)] = ??3 ??2 − Ã ?? ! +?1

?) cos ?? ☐ (8.25) N? 4 − ? (?; ?) ?? (?; ?) ? (8.26) h ?ˆ(?; ?)? ?ˆ(?; ?) = 1 + ? ?2ˆ??(??2? ?) +Pr ? (?; ?) ?? ?(??; ?) i µ 3 ¶ Ã ?? b b b b ! For ? = 0 and ? = 1, we have ? (?; 0) = ?0 (?) ? ? (?; 1) = ? (?) ? b ? (?; 0) = ?0 (?) ? ? (?; 1) = ? (?) b (8.27) (8.28) and when ? increases from 0 tbo 1 then ? (?; ?) and ?b(?; ?) deform from ?0 (?) and ?0 (?) to ? (?) and ? (?) respectively. Further Taylor series expansion yields b b ∞ ? (?; ?) = ?0 (?) + ?? (?) ??? (8.29) b ?X=1 ∞ ? (?; ?) = ?0 (?) + ?? (?) ??? (8.30) b ?X=1 ?? (?) = ?! ??? 1 ??? (?; ?) ? ?? (?) = ?! ??? 1 ??? (?; ?) b ¯ ? (8.31) ¯ ¯?=0 b ¯?=0 ¯ The auxiliary parameters ~? and ~?

> 3**are¯ selected such that the series** (8?29)¯ **and** (8?30) **converge**

¯ at ? = 1. Hence ∞ ?(?) = ?0(?) + ??(?)? ?X=1 ∞ ?(?) = ?0(?) + ??(?)? (8.32) ?X=1 The corresponding problems at ??? order are given by L? [?? (?) − ????−1 (?)] = ~? R?? (?) ? L? [?? (?) − ????−1 (?)] = ~? R?? (?) ? (8.33) ??(0) = 0? ??0(0) = 0? ??0(∞) = 0? ??(0) = 0? ??(∞) = 0? ??(0) = 0? ??(∞) = 0? (8.34) R??(?) = ??000−1(?) + ?−1 ??−1−??00 − ??0−1−???0 00 ?=0 ?−1 P h ? i +?1 ??−1−? {2??0−??00 − ??−???000 − ???0−1(?) X?=0 X?=0 ?−1 ?−1 +?2{??0−1−? ??0 − ??−1−? ????) + ?? cos ?? (8.35) X?=0 X?=0 4 ?−1 R?? (?) = 1 + 3? ?0?0−1 + Pr ??−1−??0? − ??0−1−??? (8.36) µ ¶ ?X=0 ¡ ¢ ?? = 0? ? ≤ 1 ? (8.37) 1? ? ? 1 The general solutions of Eq. (8.32) can be ¯written as follows: ??(?) = ??∗(?) + ?1 + ?2 exp(?) + ?3 exp(−?)? (8.38) ??(?) = ?∗?(?) + ?4 exp(?) + ?5 exp(−?)? (8.39) where the special solutions are ??∗(?) and ?∗?(?) and ?2 = ?4 = 0? ?1 = −?3 − ??∗(0)? ?3 = ????∗?(?) ? ?=0 ?5 = −??∗(0)? ¯ ¯ (8.40) 8.2

> 1**Convergence of the series solutions** Clearly **the**

series solutions

> 7**contain the non-zero auxiliary parameters** ~? and ~?. Hence the ~? **and**

~? curves are plotted for 20th−order of approximation

> 1**in order to** find **the admissible values of**

~? and ~?. It is found that the admissible values of ~? and ~? are −1?4 ≤ ~?? ~? ≤ −0?25 (Fig. 8.1)? The series given by Eq. (8?36) converges in the whole region of ? when ~? = ~? = −0?6? 0 l = 0.5, b1 = 0.2, b2 = 0.3, a = p?4, R = 0.3, Pr = 0.7 -0.2 f''?0? -0.4 q'?0? f''?0?, q'?0? -0.6 -0.8 -1 -1.2 -1.4 -1.5 -1.25 -1 -0.75 -0.5 -0.25 0 Ñf, Ñq Fig.8.1: ~−curves for the function ? and ?? 8.3

> 4**Discussion This section** examines **the influence of physical parameters on the** velocity **?0(?) and**

temperature ?(?)? Figs. 8.2 − 8?9 depict the graphical effects of inclination ?? Deborah numbers ?1 and ? 2 and mixed convection parameter ?. Figs. 8.2 and 8.3 depict that fluid velocity decreases while the fluid

temperature

> 21 **increases with the increase of** inclination **parameter ?? It is** also observed **that**

> 2 **momentum boundary layer thickness** is a **decreasing** function **of**

? while

> 2 **thermal boundary layer thickness** is an **increasing** function **of**

?? Figs. 8.4 and 8.5 are sketched to analyze the effect of relaxation time parameter ?1 on the velocity and temperature. As the relaxation time parameter increases the velocity profile decreases while the temperature profile is quite opposite to that of the velocity profile. Figs. 8.6 and 8.7 examine the influence of mixed convection

> 27 **parameter ? on the velocity and temperature. The**

fluid velocity and associated momentum boundary layer thickness increase by increasing ?? It is clear that the temperature has opposite effect when compared with velocity. We also noticed that the fluid velocity increases rapidly in comparison to the temperature with the increasing values of ?? Figs. 8.8 and 8.9 depict the effect of retardation time ?2 on the temperature and velocity fields. Here we observed that ?2 has quite opposite effect on the velocity and temperature. The fluid velocity increases with an increase in ? 2? b1 = 0.4, l = 0.5, b2 = 0.3, R = 0.3, Pr = 0.7 1 0.8 a = 0.0 a = p?6 0.6 a = p?4 a = p?3 f'?h? 0.4 0.2 0 0 2 4 6 8 10 h Fig. 8.2: Influence of sheet inclination ? on ?0(?)? b1=0.4,l=1.0,b2=0.2,R=0.5,Pr=0.7 1 0.8 a = 0.0 a = p?6 0.6 a = p?4 a = p?3

> 1 **q?h? 0.4 0.2 0 0 2 4** 6 8 10 **h**

Fig. 8.3: Influence of sheet inclination ? on ?(?)? a = p?4, l = 1.0, b2 = 0.2, R = 0.5, Pr = 0.7 1 0.8 b1 = 0.0 b1 = 0.3 0.6 b1 = 0.6 b1 = 0.9 f '?h? 0.4 0.2 0 0 2 4 6 8 10 h Fig.8.4: Influence of Deborah number ?1 on ? 0(?)? a = p?4, l = 1.0, b2 = 0.2, R = 0.5, Pr = 0.7 1 0.8 b1 = 0.0 b1 = 0.3 0.6 b1 = 0.6 b1 = 0.9

> 1 **q?h? 0.4 0.2 0 0 2 4** 6 8 10 **h**

Fig. 8.5: Influence of Deborah number ?1 on ?(?)? a = p?4, R = 0.5, b1 = 0.4, b2 = 0.2, Pr = 0.7 1 0.8 l = -0.5

> 3 **l = 0.0** 0.6 **l = 0.5 l = 1.0**

f '?h? 0.4 0.2 0 0 2 4 6 8 10 h Fig. 8.6: Influence of mixed convection ? on ?0(?)? a = p?4, R = 0.5, b1 = 0.4, b2 = 0.2, Pr = 0.7 1 0.8 l = -0.5

3**l = 0.0** 0.6 **l = 0.5 l = 1.0**

1**q?h? 0.4 0.2 0 0 2 4** 6 8 10 **h**

Fig. 8.7: Influence of mixed convection ? on ?(?)? a = p?4, R = 0.5, Pr = 0.7, b1 = 0.4, l = 0.5 1 0.8 b2 = 0.0 b2 = 0.3 0.6 b2 = 0.6 b2 = 0.9 f '?h? 0.4 0.2 0 0 2 4 6 8 10 h Fig. 8.8:Influence of Deborah number ?2 on ?0(?)? a=p?4,R=0.5,Pr=0.7,b1=1.2,l=0.5 1 0.8 b2 = 0.0 b2 = 0.3 0.6 b2 = 0.7 b2 = 1.0

1**q?h? 0.4 0.2 0 0 2 4** 6 8 10 **h**

Fig. 8.9: Influence of Deborah number ?2 on ?(?)? Table 8.1 also shows

4**that the series solutions converge** for 20th **-order of** deformations **for**

both the velocity and temperature. Table 8.2 analyzes the

4**numerical values of local Nusselt number for different values of** ?1, ?2? ?? **Pr and**

?? We note that the

3**numerical values of local Nusselt number** increase **with** the **increase**

of ?1? ?2? ? ? and such values decrease for ? and ?. Table 8.1:

2**Convergence of homotopy solutions for different order of approximations when**

? = ??4? ?1 = 0?2? ? = 0?5? ?2 = 0?3? Pr = 0?7 and ? = 0?3? Order of approximation −?00(0) −?0(0) 1 0.69555 0.73000 05 0.75163 0.72503 10 0.75241 0.72457 15 0.75240 0.72457 20 0.75239 0.72456 25 0.75239 0.72456 30 0.75239 0.72456 Table 8.2: Local Nusselt number and skin friction coefficient ??−?1? 2 ??? for some values of ?1? ?? Pr? ?2 and ? when ? = ??4? ?1 ?2 ? ? ? ? −(1+ 43?) ?0(0) 0.0 0.2 0.5 0.7 0.3 0.73887 0.2 0.71284 0.4 0.72456 0.5 0.70528 0.2 0.0 0.70729 0.2 0.71925 0.4 0.72951 0.5 0.73413 0.2 0.0 0.66787 0.3 0.70324 0.5 0.71924 0.8 0.73869 0.5 0.3 0.43757 0.5 0.58961 0.7 0.71925 0.9 0.83472 0.7 0.0 0.84020 0.2 0.75419 0.5 0.66085 0.7 0.60959 8.4 Concluding remarks This chapter

examines the effect of

> **1thermal radiation on the mixed convection flow of** an Oldroyd-B **fluid over** an inclined **stretching sheet.**

The main observations of this study are as follows. • Effect of mixed convection parameter ? on the velocity and temperature are quite opposite. • Momentum and thermal boundary layers for ? have opposite effects. • Decrease in temperature is more significant in comparison to velocity when Prandtl num- ber ? ? increases? • Thermal

> **1boundary layer thickness** is **decreasing** function **of Prandtl number. • Deborah**

numbers ?1 and ?2 have quite opposite effects for the velocity and temperature. Chapter 9 Mixed convection flow of an Oldroyd-B

> **3fluid with power law heat flux and heat source**

This chapter looks at the mixed convection

> **2flow of an Oldroyd-B fluid** bounded by **a porous**

stretching surface. Mathematical formulation is developed

> **3in the presence of** heat source and **power law heat flux.** Velocity **and**

temperature are computed. Plots for different parameters are analyzed.

> **3Numerical values of local Nusselt number**

are examined. 9.1 Development of

> **3problems We consider the two-dimensional flow of an incompressible** Oldroyd-B **fluid over a porous**

sur- face.

> **3A Cartesian coordinate system is chosen in such a way that ?−axis is** taken

> **along the** flow direction **and the ?−axis perpendicular to the** ?−axis. The **fluid fills the half space**

? ? 0. Heat source is present and power law heat flux is imposed. Flow geometry is shown in Fig. 9.1. Fig. 9.1: Physical model of the problem The present boundary layer flow is governed by the following expressions ?? ?? ?? ?? = 0? + (9.1) ? ?? + ? ?? ?? ?? = ? ??2??2 − ?1 ?2 ??2??2 + ?2 ??2??2 + 2?? ???2??? ?3? · ?3? ?? ?2? ?? ?2? ¸ +??2 ????2 + ? ??3 ?? ??2 ?? ??2 + ??? (? − ?∞)? (9.2) ? − − · ¸ ? ?? ?? + ? ?? ??? ??2 ??? (? − ?∞)? ?? = ? ?2? + ? (9.3) In above equations ? and ?

> 1 **are the velocity components in the ?− and ?−directions,**

?1 and ?2

> 3 **are the relaxation and retardation times** respectively, ? **the**

gravitational acceleration, ?? the thermal expansion coefficient, ? = (???) the kinematic viscosity, ? the fluid temperature, ? the

> 3 **density of fluid, ? the thermal conductivity of fluid, ?? the specific heat at constant pressure and ? the heat source coefficient. The** subjected **boundary conditions are**

? = ??? ? = −?0? ???? = ??2 at ? = 0? ? = 0? ? = ?∞ as ? → ∞? where ?

> 3 **is the temperature coefficient and ?∞ is the** ambient **temperature. The**

similarity transformations are given by (9.4) (9.5) ? = ???0(?)? ? = − ???(?)? ? = ?∞ + ? ?2?(?)? ? = √ ? ? ? (9.6) r ? r ? in which ?

> 3 **is a constant and prime denotes differentiation with respect to**

?. Eq. (9?1) is automatically satisfied and the Eqs. (9.2-9.5) are reduced as follows ? 000 + ? ? 00 − ? 02 + ?1(2? ? 0? 00 − ? 2? 000) − ?2(? 002 − ? ? 0000) + ?? = 0? ?00 + ? ?? ?0 − 2? ?? 0? + ?∗? = 0? ? = ?? ? 0 = 1? ?0 = 1 at ? = 0? ? 0 = 0? ? = 0 as ? → ∞? (9.7) (9.8) (9.9) (9.10) in which ?1 = ?1? and ?2 = ?2? are the Deborah numbers? ? = ?R?e?2? the mixed convection parameter with ??? = ??? (??−?∞)?3 the local Grashof number and ??? = ??? the local Reynolds number, ? = √??0? the suction/injection parameter, ? ? = ???? the Prandtl number and ?∗ = ????

> 3 **a heat generation/** absorption **parameter. Expression of local Nusselt number**

? ?? is

??? = ?(??−???∞) ?? = −? ???? ¶?=0 ? ? (9.11) µ Dimensionless form of Eq. (9.11) is ? ????1??2 = − ? (0) 1 ? (9.12) 9.2 Homotopy analysis solutions Choosing the following set of base functions {?? exp(−??)? ≥ 0? ? ≥ 0} (9.13) we express ? and ? as follows ∞ ∞ ?(?) = ?00?0 + ??????? exp(−??)? (9.14)

28**X?=0 X?=0**∞ ∞ ?(?) = ??????? exp(−??)? (9.15) **X?=0 X?=0**

where ????? and ????? are the

2**coefficients. Initial approximations and auxiliary linear operators are** taken **in the** following **forms**

?0(?) = ? + 1 − exp(−?)? ?0(?) = − exp(−?)? L? = ? 000 − ? 0? L? = ?00 + ?0? L? (?1 + ?2?? + ?3?−?) = 0? L?(?4 + ?5?−?) = 0? (9.16) (9.17) (9.18) where ?? (? = 1 − 5) are the arbitrary constants. The corresponding zeroth order deformation problems are developed in the following fash- ions. (1 − ?) L? ? ˆ(?; ?) − ?0(?) = ??? N? ?ˆ(?; ?)? ˆ?(?? ?) ? (9.19) (1 − ?) L? ˆ?(?; ?) − ?0(?) = ???N? ?ˆ(?; ?)? ˆ?(?? ?) ? h i h i (9.20)

12**?ˆ(0; ?) = ?? ?ˆ0(0;** ?) = 1? **?ˆ0(∞; ?) = 0?** ˆ?0(0? ?) = 1? **ˆ?(∞? ?) = 0?**

h i h i (9.21) N?[?ˆ(???)] = ?3??ˆ(??3??)

17**−?ˆ(???)?2??ˆ(??2??) ??ˆ(?? ?) 2**− Ã ?? ! +?1 "2 **?ˆ(?? ?) ??ˆ(???? ?) ?2?? ˆ(??2? ?) − (?ˆ(?? ?))2** ?3 **??ˆ(??** 3? ?) # + (9 **.22)**

?2

31**??ˆ(??2? ?) − ?ˆ(?? ?) 2 ?2**□Ã ?4 **?ˆ(?? ?) ?ˆ?(??**

?) ! ??4 □ +? ? (9.23) ?? □ □ N

24**?[ˆ?(?? ?)? ?ˆ(?? ?)] = ?2ˆ??(??2? ?) + ? ??ˆ(?? ?)?ˆ?(???? ?) − 2? ? ?? ˆ(???? ?)ˆ?(?? ?) + ?∗ˆ?(??**

?)? (9.24)

3**in which ? is an embedding parameter, ?? and ?? the non-zero auxiliary**

parameters and N? **and** N? **the nonlinear operators. For** ? = 0 **and** ? = 1 **we have** ?ˆ **(?;0)**

= ?0(?)? ˆ?(??0) = ?0(?) and ?ˆ(?;1) = ?(?)? ˆ?(??1) = ?(?)? (9.25) and when ? increases from 0 to 1 then ?(???) and ?(???) vary from ?0(?)??0(?) to ?(?) and ?(?)? Taylor's series yields ? (?? ?) = ?0(?) + ?? (?)??? ∞ (9.26) ?=1 P∞ ?(?? ?) = ?0(?) + ??(?)??? (9.27) ?=1 1 ???(?;?) P 1 ???(?;?) ??(?) = ?! ??? ⁻? =0 ? ??(?) = (9.28) ?! ??? ⁻?=0 ?

3**where the convergence of above series** str⁻ongly **depends upon ??** **and**⁻ **???**

The auxiliary para-⁻ ⁻ meters ?? and ?? are selected in such a way that (9?26) and (9?27) converge at ? = 1 and hence ?(?) = ?0(?) + ??(?)? ∞ ?=1 P∞ ?(?) = ?0(?) + ??(?)? ?=1 P The ?th

7-**order deformation problems are constructed by the following expressions**

L? [??(?) − ????−1(?)] = ?? R??(?)? (9.29) (9.30) (9.31) L?[??(?) − ????−1(?)] = ??R??(?)? ??(0) = ?? 0(0) = ??0(∞) = 0? ?0?(0) − ???(0) = ??(∞) = 0? (9.32) (9.33) R??(?) = ??000−1(?) + ?−1 ??−1−??00 − ??0−1−???0 00 ?=0 ?−1 P h ? i +?1 ??−1−? {2??0−???00 − ??−???000 − ???0−1(?) X?=0 X?=0 ?−1 ?−1 +?2{??0−1−? ??0 − ??−1−? ????) + ???−1? (9.34) X?=0 ?X=0 ?−1 ?−1 R??(?) = ?0?0−1(?) + ? ? ?0? −1−??? − 2? ? ??−1−???0 + ?∗??−1(?)? (9.35) ?=0 ?=0 P P ?? = 0? ? ≤ 1? □ (9.36) 1? ? ? 1? If ??∗ and ?∗? are the special solutions then the general solutions are □ ??(?) = ??∗(?) + ?1 + ?2?? + ?3?−?? (9.37) ??(?) = ?∗?(?) + ?4 + ?5?−?? (9.38) 9.3

1**Convergence of the** homotopy **solutions The auxiliary parameters }? and** }? have significant role **in**

the convergence of developed series solutions. Here the }−curves are portrayed for 18?? order of approximations in order to find the values of }? and }? ensuring convergence. Figs. 9.2 depict

2**that the range of admissible values of }? and }? are −0?**

8 ≤ }? ≤ −0?2 and −0?75 ≤ }? ≤ −0?2?

3**The series** solution **converge in the whole region of ? when**

}? = −0?4 and }? = −0?4? Table 9.1 depicts that 25?? order deformations are sufficient for both the velocity and temperature expressions. -1 l = 0.1, S = 0.5, b1 = 0.2, b2 = 0.3, b* = 0.3, Pr = 0.5 -1.05 -1.1 f''?0?, q'?0? -1.15 -1.2 -1.25 -1.3 -1.35 f''?0? q'?0? -1 -0.8 -0.6 -0.4 -0.2 0 hf, hq Fig.9.2: ~−curves for the function ? and ?? Table 9

2.1. Convergence of homotopy solutions for different order of approximations when

?1 = 0?2? ?2 = ?∗ = 0?3? ?? = ? = 0?5? ? = 0?1, }? = −0?4 and }? = −0?4? Order of approximation −?00(0) 1 1.10000 5 1.19801 10 1.20255 15 1.20268 20 1.20271 25 1.20272 30 1.20272 9.4 Graphical results and discussion −?00(0) 1.22000 1.34636 1.35611 1.35640 1.35642 1.35643 1.35643 The

2**purpose of this section is to** highlight **the** variations **of** interesting **parameters**

through Figs. 9.3−9?7 for velocity and temperature. Figs. 9.3 and 9?4 show the behaviors of suction/injection parameter ?? Deborah number ?1 and mixed convection parameter ? on the velocity ?0(?). Fig. 9.3 shows the effects of suction/injection parameter ? on the velocity profile ?0(?)? Here ? ? 0 corresponds to suction and ? ? 0 for injection case. We observed that the velocity ?0(?) is lower for suction case in comparison to injection phenomenon. From physical point of view suction is an agent that resists the fluid flow. Such resistance in fluid flow creates a reduction

7**in the velocity** field **and** associated **boundary layer thickness.**

Fig. 9?4 illustrates that both the

7**fluid velocity and boundary layer thickness**

decrease when mixed convection parameter is increased. Note that the mixed convection parameter involves the buoyancy force. Buoyancy force is stronger for the larger mixed convection parameter and weaker for the smaller mixed convection parameter. This stronger buoyancy acts as an agent to creates a reduction in the velocity profile and momentum boundary layer thickness. f '?h? 1.0 0.8 0.6 b1 = 0.4, b2 = l = 0.2, b* = 0.3, Pr = 0.5 0.4 0.2 S = -0.5, 0.0, 0.5, 1.0 0 1 2 3 4 5 6 h Fig. 9.3: Variation of ? on ?0(?)? f '? h? 1.0 0.8 S=Pr=0.5,b1=0.4,b2=0.2,b*=0.3 0.6 0.4 0.2 l=-0.5,0.0,0.5,1.0 0 1 2 3 4 5 6 h Fig. 9.4: Variation of ? on ?0(?)? Figs. 9?5−9?7 are displayed to examine the influence of arising parameters on dimensionless temperature profile ?(?)? Fig. 9.5 presents the variations in temperature ?(?) for different values of suction/injection parameter ?. From this Fig. it is analyzed that the temperature is higher for injection case when we compared it with suction case. The effects of suction parameter ? on ?(?) is qualitatively similar to that of the velocity.

4**Both the temperature ?(?) and thermal boundary layer thickness**

increase when mixed convection parameter ? increases (see Fig 9?6). Here buoyancy force is an agent that creates an enhancement

13**in the temperature and thermal boundary layer thickness.**

Influence of heat source ?∗ on ?(?) is presented in Fig. 9?7. Physically ?∗ ? 0 means that ?? ? ?∞ and in this case heat is supplied to the flow region from the wall. The temperature boosts with heat source parameter ?∗ ? 0 while reduction

> 4in thermal boundary layer thickness and

temperature is seen with heat sink parameter ?∗ ? 0. q?h? 0.0 S= -0.5,0.0, 0.5,1.0 - 0.5 - 1.0 Pr=0.7,b1=0.4,b2=l=0.2,b*=0.5 - 1.5 2 4 6 8 h Fig. 9.5: Variation of ? on ?(?)? q?h? 0.0 -0.2 -0.4

> 3Pr=0.7, b1 =0. 4,b2 =0.2,S=b*=0.

5 -0.6 l=-0.5, 0.0, 0.5, 1.0 -0.8 -1.0 1 2 3 4 5 6 h Fig. 9.6: Variation of ? on ?(?)? q?h? 0.0 -0.2 S = Pr = 0.5 , b1 = 0.4 , b2 = l = 0.2 -0.4 b* =-0.2, 0.0, 0.2, 0.3 -0.6 -0.8 1 2 3 4 5 6 h Fig. 9.7: Variation of ?∗ on ? (?)? Table 9.2 shows that the local Nusselt number has quite opposite behavior for ?1 and ?2? The values of Nusselt number increases by increasing ?∗? ?? and ?? However it decreases by increasing ?? Table. 9.3 ensures the validity of present results for ?00(0) in a limiting sense. Table 9.2: Values of local Nusselt number ?????1??2 for the parameters ?1? ?2? ?∗? ??? ? when ? = 0?1? ?1 ?2 ?∗ Pr ? ? −?????1??2 0.0 0.3 0.3 0.5 0.5 0.1 1.20133 0.2 1.17496 0.4 1.14936 0?4 1?18647 0?5 1?19622 0?6 1?20463 −0?1 1?02500 −0?2 1?09798 −0?3 1?16099 0?6 1?27668 0?9 1?60523 1?2 1?91014 0?8 1?23081 1?0 1? 30323 1?5 1?45103 0?2 1?15381 0?4 1?13613 0?6 1?11294 Table 9.3: Comparison of ?00(0) for different values of Maxwell parameter ?1 when ?2 = ? = ? = 0? Abel et al. [38] Present results ?1 −? 00(0) −? 00(0) 0.0 1.00000 1.00000 0.2 1.051948 1.051889 0.4 1.101850 1.101903 0.6 1.150163 1.150137 0.8 1.196692 1.196711 1.2 1.285257 1.285363 1.6 1.368641 1.368758 2.0 1.447617 1.447651 9.5 Final remarks This chapter deals with the

> 21mixed convection flow of Oldroyd-B fluid over a stretching sheet

with suction/injection, heat source/sink and power law heat flux. Effects of different involved parameters such as mixed convection parameter ?

> 8and heat source/sink ?∗ on the flow

field and temperature are analyzed. The main observations are summarized as follows: • Velocity profile

> 2and momentum boundary layer thickness reduce in

case of suction ? ? 0 while these enhance in case of injection ? ? 0? •

> 2Thermal boundary layer thickness is increasing function of

heat source parameter ?∗ ? 0 while it reduces with heat absorption parameter ?∗ ? 0? • Increase in mixed convection parameter ? yields an enhancement

13in the temperature and thermal boundary layer thickness

while reduction in

2heat transfer rate at wall is

noted. • Behaviors of Deborah numbers ?1 and ?2 on the

2heat transfer rate at the wall are opposite? Heat transfer rate at wall

increases with the heat absorption ?∗ ? 0 and suction parame- ters ? ? 0? Chapter 10

1Soret and Dufour effects in mixed convection flow of an Oldroyd-B fluid

4with convective boundary conditions This chapter investigates the effects of

6heat and mass transfer in the mixed convection

1flow of an Oldroyd-B fluid over a stretching surface with convective boundary conditions.

Emphasis is given to the analysis of Soret and Dufour effects. Relevant problems are first formulated and then computed by the homotopy analysis method (HAM). Velocity, temperature and con- centration fields are computed and analyzed through plots. In addition, the local Nusselt and Sherwood numbers are examined through the numerical values. 10.1 Mathematical model We choose ?−axis

14along the stretching surface in the flow direction and y-axis is taken perpen- dicular to the

surface. An incompressible Oldroyd-B fluid is considered. The surface satisfies the convective boundary conditions. Further, the Soret and Dufour effects are taken into account. The resulting boundary layer equations for flow of an Oldroyd-B fluid are ?? ?? ?? ?? = 0? + (10.1) ? ?? + ? ?? = ? ?? ?? ??2??2 − ?1 ?

2 ??2??2 + ?2 ??2??2 + 2?? ?2? ???? ?3? · ?3? ?? ?2? ?? ?2? ¸ +??2 ????2 + ? ??3 ?? ??2 ?? ??2 ? − − +?[?? (? − ?∞) + ??(? − ?∞)]? · ¸ (10.2) ? ?? ?? ?2? ???? ?2? ?? + ? ?? = ?? ??2 ???? ??2 + ? (10.3) ? ?? ?? ?2? ???? ?2? ?? + ? ?? = ?? ??2 ?? ??2 + ? (10.4) where ? and ? denote the

---

---

relaxation and retardation times respectively, ? the fluid temperature, ? the con- centration field, ? the kinematic viscosity, ? the fluid density, ?? the mass diffusivity, ?? the thermal conductivity, ?? the thermal-diffusion ratio, ?? the specific heat, ?? the concentration susceptibility, ?? the fluid mean temperature and ? the fluid electrical conductivity. The boundary conditions are expressed in the following forms ? = ??(?) = ??? ? = 0? − ? ?? = ?(?? − ?)? ? = ?? at ? = 0? ?? (10.5) ? → 0? ? → ?∞? ? → ?∞ as ? → ∞ (10.6) in which ?? denotes the convective fluid temperature, ?∞ the ambient temperature and ? the thermal conductivity. Setting the following transformations ? =? ? ? ? = ???0(?)? ? = − ???(?)? ?(?) = ? − ?∞ ? ?(?) = ? − ?∞ √ ?? − ?∞ ? (10.7) r ? ?? − ?∞ incompressibility condition (10?1) is automatically satisfied and the other equations give ? 000 + ? ? 00 − ? 02 + ?1(2? ? 0? 00 − ? 2? 000) − ?2(? 002 − ? ? 0000) + ?? + ?? ? = 0? (10.8) ?1? ?00 + ? ?0 + ?? ?00 = 0? ?00 + ??? ?0 + ??00 = 0? (10.9) (10.10) ? (0) = 0? ? 0(0) = 1? ? 0(∞) = 0? (10.11) ?0(0) = −?1(1 − ?(0))? ?(∞) = 0? (10.12) ?(0) = 1? ?(∞) = 0? (10.13) In above expressions the prime indicates the differentiation with respect to ?? ?1 = ?1? and ?2 = ?2? are the Deborah numbers? ? = ?R?e?2? the mixed convection parameter with ??? = ???(??−?∞)?3 as the Grashof number, ??? = ?? the local Reynold number, ?? = ???? the ? Prandtl number, ?1 = (???) ??? the Biot number, ?? = ???? the

---

The definition of ??and ?? are ?? = ???????? ((????−−??∞∞)?) ? ?? = ???? (?? − ?∞) ??? (?? − ?∞) ? (10.14) Local Nusselt and local Sherwood numbers in dimensionless coordinates are expressed as follows: ??(???)−1?2 = −?0(0)? (10.15) ??(???)−1?2 = −?0(0)? (10.16) In the next section we will develop the homotopy solutions for the resulting problems. 10.2 Series solutions The initial guesses and auxiliary linear operators are taken as ?0(?) = (1 − ???(−?))? ?0(?) = ?11ex+p(?−1?)? ?0(?) = exp(−?) + exp(−?)? ? ? 2 (10.17) ?3? ?? ?2? L? = ??3 ?? ? L? = ??2 − ?? L? = ??2 − ?? − ?2? (10.18) L? [?1 + ?2 exp(?) + ?3 exp(−?)] = 0? (10.19) L? [?4 exp(?) + ?5 exp(−?)] = 0? L? [?6 exp(?) + ?7 exp(−?)] = 0? (10.20) in which ?? (? = 1 − 7) denote the arbitrary constants. 10.2.1 Zeroth and mth order deformation problems Having the non-linear operators N? ? N? and N? in the forms N?

??3 ??2 − Ã ?? ! +?1 "2?ˆ(?? ?) ??ˆ(???? ?) ?2??ˆ(??2? ?) − (?ˆ(?? ?))2 ?3??ˆ(??3? ?) # +? ?4?ˆ(?? ?) 2 □Ã ?2?ˆ(?? ?) − ?ˆ(?? ?) 2 ??2 ! ??4 □ +?ˆ?(□?? ?) + ?? ?ˆ(?? ?)? □ (10.21) N?[?ˆ(?? ?)? ˆ?(?? ?)? ? ˆ(?? ?)] = ?1? ?2ˆ??(??2? ?) + ?ˆ(?? ?) ?ˆ?(?? ?) ?? + ?? ??2 ?2?ˆ(?? ?) ? N?[?ˆ(?? ?)? ˆ?(?? ?)? ˆ?(?? ?)] = ?2?ˆ?(??2? ?) + ???ˆ(?? ?)??ˆ?(??? ?) ?2ˆ?(?? ?) + ???? ? ??2 (10.22) (10.23) the corresponding problems at the zeroth and ?th orders can be expressed as follows: (1 − ?)L?[?ˆ(?? ?) − ?0(?)] = ?}?N?[?ˆ(?? ?)? ˆ?(?? ?)? ?ˆ(?? ?)]? (10.24) (1 − ?)L?[ˆ?(?? ?) − ?0(?)] = ?}?N?[?ˆ(?? ?)? ˆ?(?? ?)? ˆ?(?? ?)]? (10.25) (1 − ?)L?[ˆ?(?? ?) − ?0(?)] = ?}?N?[?ˆ(?? ?)? ˆ?(?? ?)? ˆ?(?? ?)]? (10.26) ?ˆ(?; ?) = 0? ? ?) = 1? ?

?) = 0? ?=0 ?? ¯ ?=0 ?? ?=∞ ?ˆ0(?; ?) = −?[1 − ˆ?(?¯; ?)]? ˆ?(?; ?) ¯ = 0? ¯ ¯ ¯ ?=0 ?=0 ¯ ¯ ?=∞ ¯ ¯ ¯ ¯ ¯ (10.27) (10.28) ?ˆ(?; ?) = 1? ˆ?(?; ?) ?=0 ?=∞ = 0? L? [??(?¯) − ????−1(?)] =¯ }? R??(?)? ¯ ¯ ¯ L?[?? (?) − ????−1(?)] = }?R??(?)? L?[??(?) − ????−1(?)] = }?R??(?)? (10.29) (10.30) (10.31) (10.32) ??(0) = 0? ??0(0) = 0? ??0(∞) = 0? ?0?(0) − ???(0) = 0? ??(∞) = 0? ??(0) = 0? ??(∞) = 0? (10.33) R??(?) = ?? 000−1(?) + ?−1 ?=0 ??−1−???00 − ??0−1−???000 ?−1 P h ? i +?1 ??−1−? {2??0−??00 − ??−???000 − ???0−1(?) X?=0 ?−1 X?=0 ?−1 +?2{??0−1−? ??0 − ??−1−? ????) + ??0?−1(?) + ??1?0?−1(?)? (10.34) X?=0 X?=0 1 ?−1 R??(?)= ???0?0−1(?)+ ??−1−??0?+???0?0−1(?)? X?=0 ?−1 R??(?) = ?0?0−1(?) + ?? ??−1−??0? + ???0?0−1(?)? X?=0 ?? = 0? ? ≤ 1 1? ? ? 1? (10.35) (10.36) (10.37) where ? [0?1] is an embedding parameter¯and }?? }? and }? are the nonzero auxiliary para- ∈ meters. Taylor's series gives ? ˆ(?; ?) = ?0(?) + ??(?)??? ??(?) = ?1! ????ˆ?(??; ?) ∞ ¯ ? ?X=1 ¯?=0 ¯ ¯ ?ˆ(?; ?) = ?0(?) + ??(?)??? ?? (?) = ?1! ???ˆ??(??; ?) ∞ ¯ ¯ ? ?X=1 ¯?=0 ¯ ¯ ¯ (10.38) (10.39) ?ˆ(?; ?) = ?0(?) + ??(?)??? ??(?) = ?1! ????ˆ?(??; ?) ∞ ? (10.40) ?X=1 ¯?=0 ¯ and when ? = 0 and ? = 1 then ¯ ¯ ¯ ?ˆ(?;0) = ?0(?)? ?ˆ(?;1) = ? (?)? (10.41) ?ˆ(?;0) = ?0(?)? ˆ?(?;1) = ?(?)? (10.42) ?ˆ(?;0) = ?0(?)? ?ˆ(?;1) = ?(?)? (10.43) We choose the auxiliary parameters in such a way that the series solutions converge for ? = 1 and so ∞ ?(?) = ?0(?) + ??(?)? (10.44) ?X=1 ∞ ?(?) = ?0(?) + ??(?)? (10.45) ?X=1 ∞ ?(?) = ?0(?) + ??(?)? (10.46) ?X=1 The general solutions (??? ??? ??) in terms of special solutions (??∗? ?∗?? ?∗?) can be written as follows ?? (?) = ??∗(?) + ?1 + ?2 exp(?) + ?3 exp(−?)? (10.47) ??(?) = ?∗?(?) + ?4 exp(?) + ?5 exp(−?)? ??(?) = ? ∗?(?) + ?6 exp(?) + ?7 exp(−?)? 10.3 Convergence of the homotopy solutions (10.48) (10.49) The convergence analysis of the series solutions depends upon the auxiliary parameters ~?, ~? and ~?? Hence the ~−curves for the 17?? order of approximations are plotted. It is found that the admissible ranges of ~?, ~? and ~? are −1?7 ≤ ~? ≤ −0?20, −1?5 ≤ ~? ≤ −0?25 and −1?5 ≤ ~? ≤ −0?40. The series (10?44 − 10?46) converge in the whole region of ? when ~? = ~? = ~? = −1?0 (see Fig. 10.1)? Table 10.1 indicates that how much terms for each physical quantity are required for the convergent solution. It is noticed that less number of terms are required in the convergent expression of velocity. I = 0.2, b1 = 0.2, b2 = 0.1, Pr = Sc = 0.7, Df = 0.3, Sr = 1.0, g1 = 1.0, Pr = 0.7, N = 1.0 0.5 0.25 f''?0? q'?0? 0 f''?0?, q'?0?, f'?0? f'?0? -0.25 -0.5 -0.75 -1 -1.25 -2 -1 -1.5 -0.5 0 hf,hq,h« Fig?10?1 : ~−curves for the function ?, ? and ?? Table 10

2.1: Convergence of homotopy solutions for different order of approximations when

?1 = 0?2?? ?2 = 0?1? ?? = ?? = 0?7? ?1 = 1?0? ?? = 0?3? ?? = 0?2, ? = 1?0? ~? = ~? = ~? = −1?0? Order of approximations −?00(0) −?0(0) −?0(0) 1 0.82500 0.39104 0.43194 5 0.79502 0.30304 0.45255 10 0.79137 0.29030 0.46033 15 0.79129 0.28971 0.46071 20 0.79132 0.28991 0.46044 25 0.79133 0.28993 0.46042 30 0.79133 0.28993 0.46043 10.4 Discussion Interest in this section is to analyze the variations of different emerging parameters on the phys- ical quantities like temperature, concentration field, local Nusselt and Sherwood numbers. Figs. 10?2 − 10?9 are displayed to see the variations of mixed convection parameter ?? concentration buoyancy parameter ?, Dufour number ?? and Soret number ?? on the temperature of fluid ?(?)? Fig. 10?2 shows that the mixed convection parameter ? decreases

---

2the temperature ?(?)? Also thermal boundary layer thickness

---

reduces with an increase in mixed convection parameter ?? Fig. 10?3 illustrated that temperature and associated thermal

2**boundary layer thickness are decreasing functions of ? .Fig.** 10?4 **shows that**

temperature ?(?)

4**and thermal boundary layer thickness increase**

with an increase in Dufour number ??? With

13**an increase in** Soret number ??? **temperature ?(?) and thermal boundary layer thickness**

decrease as seen in Fig. 10.5. b1 = g1 = 0.4, N = 0.3, Df = 0.5, Pr = Sc = 0.7, b2 = Sr = 0.2 0.3 l = -1.0

3**l = 0.0** q?h? **0.** 2 **l = 1.0 l = 2.0**

0.1 0 0 2 4 6 8 10 h Fig. 10.2: Influence of ? on ?(?)? b1 = g1 = 0.4, b2 = l = 0.3, Df = 0.5, Pr = Sc = 0.7, Sr = 0.2 0.35 N = 0.0 0.3 N = 1.5 0.25 N = 3.0 N = 4.5 q?h? 0.2 0.15 0.1 0.05 0 0 2 4 6 8 10 h Fig. 10.3: Influence of ? on ?(?)? b1 = 0.4, b2 = N = 0.3, g1 = 1.0, l = 0.6, Pr = Sc= 0.7, Sr = 0.8 0.6 Df = 0.0 0.5 Df = 0.5 Df = 1.0 0.4 Df = 1.5

4**q?h? 0.3 0.2 0.1 0 0 2 4** 6 8 10 **h**

Fig. 10.4: Influence of ?? on ?(?)? b1 = 0.4, b2 = N = 0.3, g1 = 1.0, l = 0.6, Pr = Sc = 0.7, Df = 0.5 0.6 Sr = 0.0 0.5 Sr = 0.5 Sr = 1.0 0.4 Sr = 1.5 q?h? 0.3 0.2 0.1 0 0 1 2 3 4 5 6 h Fig. 10.5: Influence of ?? on ?(?)? Figs. 10?6 − 10?9 plot the effects of ?? ?? ?? and ?? on the concentration field ?(?)? From Fig. 10.6

4**it is found that the** concentration **field and** associated **boundary layer thickness** show a decrease **when**

mixed convection parameter ? increases? Effects of concentration buoyancy parameter parameter ? on the concentration are qualitatively similar to that of temperature (see Figs. 10?3 and 10?7). Figs. 10?4 and 10?8 show that the behaviors of the temperature and concentration profiles are quite opposite in case of Dufour number. This shows that the Dufour number corresponds to weaker concentration and stronger temperature. Fig. 10?9 pointed out that the larger Soret number has a strong concentration. b1 = g1 = 0.4, N = 0.3, Df = 0.5, Pr = Sc = 0.7, b2 = Sr = 0.2 1 0.8 l = -1.0 l = 0.0 0.6 l = 1.0 l = 2.0 f?h? 0.4 0.2 0 0 2 4 6 8 10 h Fig. 10.6: Influence of ? on ?(?)? b1 = g1 = 0.4, b2 = l = 0.3, Df = 0.5, Pr = Sc = 0.7, Sr = 0.2 1 0.8 N = 0.0 N = 1.5 0.6 N = 3.0 N = 4.5 f?h? 0.4 0.2 0 0 2 4 6 8 10 h Fig. 10.7: Influence of ? on ?(?)? b1 = 0.4, b2 = N = 0.3, g1 = 1.0, l = 0.6, Pr = Sc = 0.7, Sr = 0.8 1 0.8 Df = 0.0 Df = 1.0 0.6 Df = 2.0 Df = 3.0 f?h? 0.4 0.2 0 0 2 4 6 8 10 h Fig. 10.8: Influence of ?? on ?(?)? b1 = 0.4, b2 = N = 0.3, g1 = 1.0, l = 0.6, Pr = Sc =

0.7, Df = 0.5 1 0.8 Sr = 0.0 Sr = 0.5 0.6 Sr = 1.0 Sr = 1.5 f?h? 0.4 0.2 0 0 2 4 6 8 10 h Fig. 10.9: Influence of ?? on ?(?)? Table 10.2:

<div style="border:1px dotted #999;padding:1em;">

7**Values of local Nusselt number and Sherwood number**

</div>

for the parameters ?1? ?2? ??? ???????1 and ? when ? = 0?1? ?1 ?2 ? ? ? ?? ?? ?? ?1 ? −?0(0) −?0(0) 0.0 0.2 0.1 0.7 0.5 0.8 1 1 1 0.29514 0.47225 0.2 0.28993 0.46033 0.4 0.28525 0.44998 0.2 0.0 0.28797 0.45592 0.2 0.29176 0.46462 0.4 0.29508 0.47226 0.2 0.2 0.0 0.27062 0.41866 0.2 0.29176 0.46463 0.4 0.30305 0.49095 0.2 0.2 0.2 0.8 0.30136 0.50625 0.9 0.30891 0.54602 1.0 0.31480 0.58434 0.7 0.0 0.28679 0.49981 0.2 0.29176 0.46462 0.4 0.29679 0.42800 0.2 0.0 0.33235 0.45353 0.2 0.30546 0.46096 0.4 0.27788 0.46830 0.2 1.0 0.30546 0.46094 1.2 0.30001 0.51681 1.4 0.29513 0.56908 1.0 1.0 0.30546 0.46094 1.2 0.32396 0.45920 1.4 0.33863 0.45781 1.0 1.0 0.30546 0.46094 1.2 0.30700 0.46444 1.4 0.30846 0.46779 10.5 Conclusions Mixed convection flow of an Oldroyd-B fluid is investigated in the presence of convective bound- ary condition and Soret and Dufour effects. The main observations are pointed out below. • Effects of ? and ? on temperature and concentration fields are similar. • Effects of Soret number ?? on ?(?) and ?(?) are reverse. • Thermal boundary layer thickness and temperature field increase when ??increases. • There are opposite effects of

<div style="border:1px dotted #999;padding:1em;">

1**local Nusselt number and Sherwood number** when ??, ?? **and**

</div>

?1 increase. • local Nusselt and Sherwood numbers decrease when ?1 increases. • With an increase in ? and ?

<div style="border:1px dotted #999;padding:1em;">

1**both local Nusselt and Sherwood numbers are**

</div>

enhanced. Chapter 11

<div style="border:1px dotted #999;padding:1em;">

2**Mixed convection Falkner-Skan** wedge **flow of** an Oldroyd-B **fluid**

</div>

in presence of thermal radiation The present chapter examines the Falkner-Skan flow of rate type non-Newtonian fluid. Ex- pressions of an Oldroyd-B fluid in the presence of mixed convection and thermal radiation are used in the development of relevant equations. The resulting

<div style="border:1px dotted #999;padding:1em;">

4**partial differential equations are reduced into** the **ordinary differential**

equations employing appropriate **transformations.** Ex- pressions **of** flow **and** heat

transfer **are**

</div>

constructed. Convergence of derived nonsimilar series solutions is guaranteed. Impact of various parameters involved in the flow and heat transfer is plotted and examined. 11.1 Problems development Let us

> 3**consider the two-dimensional** Falkner-Skan **flow of an** Oldroyd-B **fluid.**

We further consider the heat transfer. Cartesian coordinates (?? ?) are used

> 3**in such a way that ?-axis is** parallel to **the** wall **and ?-axis** normal **to it.** An incompressible **fluid** occupies **the**

region ? ≥ 0? The equations governing the present flow situation are based on the

> 2**conservation laws of mass, linear momentum and energy.** Flow diagram of **the**

problem is as follows: Fig. 11.1: Physical Model Taking into account the aforementioned assumptions, the resulting boundary layer equations can be written as follows: ?? ?? ?? ?? = 0? + (11.1) ? ???? + ? ???? = ? ??2??2 − ?1 ?2 ??2??2 + ?2 ??2??2 ?2? + 2?? ???? ?3? · ?3? ?? ?2? ?? ?2? ¸ +??2 ????2 + ? ??3 ?? ??2 ?? ??2 ? − − · +??? (? − ?∞) Si n 2 ? ? ¸ (11.2) ??? ? ???? +????? ?? µµ16?3????∞3 +? ???? = ? ? (11.3) µ ¶ ¶ ¶ The appropriate boundary conditions are ? = ?? ? = 0? ? = ?? = ?∞ + ??? at ? = 0? ? → 0? ? → ?∞ as ? → ∞ (11.4) where ? (= ???) is the free stream velocity, ? is the dynamic viscosity, ?1 is the relaxation time, ?2 is the retardation time, ? is the wedge angle, ? is the thermal conductivity, ? is the surface temperature exponent, ? and ?∞ are the temperatures of the fluid and ambient respectively and ?? is the wall temperature. We utilize ? = ? (?) ? 0, ? = ? + 1 ? ?? ? = 2 √

> 35**2 ???? (?)? r r ?? ?=− ?+1 ?? r?+1 r** 2 (11.5) **r**

? ? · ? (?) + ?? −+ 11 ?? 0 (?) ? ? (?) = ???−−??∞∞ ¸ where ? is the similarity variable, ?

> 19**is the stream function, ? is the dimensionless** stream function and ? **is the**

dimensionless temperature. Now the continuity equation (11?1) is identi- cally satisfied and Eqs. (11?2) − (11?4) lead to the following expressions ? 000 + ? ? 00 + ?1 −2? ?? −+ 11 ? 03 + (3? − 1) ? ? 0? 00 − ? + 1 ? 2? 000 + ? − 1 ?? 02? 00 µ µ ¶ µ 2 ¶ µ 2 ¶ ¶ +? 2 µµ 3? − 1 ? 00 ? + 1 ? ? 0000 + (? − 1) ? 0? 000 2? ? 02 + 2 − − 2 ¶ ¡ ¢ µ 2 ¶ ¶ ?+1 1+4? ?00+Pr(??0− 2? ?0?)=0? 3 ?+1 µ ¶ 2 ? + 1 ?? sin = 0? ? 2 (11.6) (11.7) ?(0)=0? ?0(0)=1? ?(0)=1? ?0(∞)=0? ?(∞)=0? (11.8) Here prime denotes the differentiation with respect to ?, ?1 and ?2 are the dimensionless material parameters, ? is mixed convection parameter, ??? is the local Grashof number,

> 1**Pr is the Prandtl number** and ? **is the** radiation **parameter. The**

definitions of these parameters are ?1 = ?1? ?2? ??? ??? ? ? ?2 = ? ? ? = Re2? ? Pr = ? ? ? = (4?????? ∞3)? ??? = ??(?? − ?∞)?3 ?2 ? (11.9) Local Nusselt number (???) along with heat transfer rate (??) are ? ?? = ??? ? ? (?? − ?∞) ?? = −? ?? µ ?? ¶?=0 (11.10) which in dimensionless form gives (Re?)−1?2 ??? =

−?0 (0) ? (11.11) 11.2 Series solutions The initial guesses (?0? ?0) and auxiliary linear operators (??? ??) are taken as follows ?0 (?) = 1 − ?−?? ?0(?) = ?−?? (11.12) ?3? ?? ? ?2? L? (?) = ??3 ?? L? (?) = ??2 − ?? − ?? − (11.13) with L? [?1 + ?2 exp(?) + ?3 exp(−?)] = 0? (11.14) L? [?4 exp(?) + ?5 exp(−?)] = 0? (11.15) where ?? (? = 1 − 5) are the arbitrary constants. If ? ∈ [0? 1]

(1 − ?) L? ? (?; ?) − ?0 (?) = ?~? N? ? (?; ?) ? ? (?; ?) ? (1 − ?) L? ?b(?; ?) − ?0 (?) = ?~?N? ?b(?; ?) ? ? ?b(?; ?) h i h i ? ? (0; ?) = 0? ? 0 (b0; ?) = 0? ? 0 (∞; ?) = 1? ? ? (0b;?) = 1? ?b(∞; ?) = 0? h i h i b b b b b (11.16) (11.17) (11.18) N? ? (?? ?) ? ? (?; ?) = ?3? (?; ?) ?2? (?; ?) 2? ?? (?; ?) 2 h ??3 + ? (?; ?) ??2 − ? + 1 ¶ b b i 3 µ ?? +?1 −2?? + 1 ? − 1 ?? (?; ?) Ã ?? + (3? − 1) ? (?; ?) ?? (?; ?) ?2? (?; ?) µ ¶ ?? ??2 ! −?1 ? + 1 2 ?3? (?; ?) ? − 1 ?? (?; ?) 2 ?2? (?; ?) Ã 2 (? (?; ?)) ??3 + 2 ? µ ?? ¶ ??2 ! +?2 3? − 1 ?2? (?; ?) 2 ? + 1 − ? (?; ?) ?4? (?; ?) Ãµ 2 ??2 ¶ µ ¶ µ 2 ¶ ??4 ! +?2 (? − 1) ?? ?? (?; ?) ?3? (?; ?) + 2 ?ˆ?(?? ?) sin ? µ ??3 ¶ ? + 1 2 ? (11.19) N? ? (?; ?) ? ? (?; ?) = 1+ ? 4 ?2ˆ?(?? ?) ?? (?; ?) h µ 3 ¶ ??2 + Pr ? (?; ?) ?? b b i − Pr ?2+? 1 ? (?; ?) ?? ?(??; ?)b? b b b 11.2.2 ?th-order deformation problems L? [?? (?) − ????−1 (?)] = ~? R?? (?) ? L? [?? (?) − ????−1 (?)] = ~?R?? (?) ? (11.20) (11.21) (11.22) ?? (0) = ??0 (0) = ??0 (∞) = ??00(∞) = 0? ??(0) = ?? (∞) = 0? (11.23) ??−1−??00 − ?2+?1??0−1−???0 □ −2? ??−+11 ??0−1−? ?=0 ??0−???0 + (3? − 1) ? ? ?−1−? ?=0 ??0−???00 ? □ ?−1 R?? (?) = ??000−1 (?) + +?1 □ −?+21?? −1−? P??=0 ??−???000 + ?−21??0−1−? ??P=0 ??0−???00 □ X?=0 □ P 3?−1 ??00−1−? ?00 2 ? P □ +? 2 □ − ?+1 ??−1−? ??0000 2 □ 2 □ ? □ □ + (? − 1) ??0−1−? ??000 ? + 1 ???−1 sin □ □ □ □ + 2 ? (11.24) □ R?? (?) = ?0?0−1 + Pr ?X?=−01 µ?0?−1−??? − ?2+? 1??0−1−??? ? (11.25) ¶ 0? ?? = □ □ 1? For ? = 0 and ? = 1, we have □ ? (?; 0) = ?0 (?) ? ?≤1 ? (11.26) ??1 (?; 1) = ? (?) ? (11.27) ? (?; 0) = ?0 (?) ? b ? (?; 1) = ? (?) ? b (11.28) and when ? increases from 0 tob 1 then ? (?; ?) andb ? (?; ?) vary from the initial solutions ?0 (?) and ?0(?) to final solutions ? (?) and ?(?) respectively. By Taylor's expansion one has b b ? (?; ?) = ?0 (?) + ?? (?) ??? ?? (?) = ?1! ?????(??; ?) ∞ ? (11.29) b ?X=1 b ¯¯?=0 ¯ ¯ ? (?; ?) = ?0 (?) + ?? (?) ??? ?? (?) = ?1! ?????(??; ?) ∞ ¯ ? (11.30) b ?X=1 b ¯¯?=0 where

(11?2¯¯9) and (11?31) con-¯ verge at ? = 1 i.e. ∞ ? (?) = ?0 (?) + ?? (?) ? ?X=1 ∞ ? (?) = ?0 (?) + ?? (?) ? ?X=1 The general solutions of Eqs. (11?32) and (11?33) are ?? (?) = ??? (?) + ?1 + ?2?? + ?3?−?? ?? (?) = ??? (?) + ?4?? + ?5?−?? in which ??? and ??? are the special solutions. (11.31) (11.32) (11.33) (11.34) 11.3 Convergence Note that the series solutions in Eqs. (11?32) and (11?33) contain two auxiliary parameters ~? and ~?. The convergence of series solutions depend upon these auxiliary parameters. For range of

−1?3 ≤ ~? ≤ −0?25 and −1?2 ≤ ~? ≤ −0?5. Table 11.1 further guarantees that the series solutions are convergent up to five decimal places when ~? = ~? = −0?5. -0.6 b1 = 0.3, b2 = 0.2, d = 0.5, l = 0.2, a= p?4, n = 1.5, R = 0.3, Pr = 1.2 -0.7 f''?0? -0.8 f''?0?, q'?0? -0.9 q'?0? -1 -1.1 -1.5 -1.25 -1 -0.75 -0.5 -0.25 0 Ñf, Ñq Fig?11?2 : ~−curves for the function ? and ?? 11.4 Discussion The aim of this subsection is to present

> 5**the effects of** pertinent **parameters on the velocity,** tem- perature **and**

surface heat transfer. Variation of parameter ? on the velocity and temperature are sketched in the Figs. 11?3 and 11?4. Clearly the effects of ? on the velocity and temperature profiles are quite reverse. Influence of

> 21**mixed convection parameter** ? on **both the velocity and**

temperature profiles are given in the Figs. 11.5 and 11.6.

> 1**It is** observed **that** the **velocity and** momentum **boundary layer thickness**

increase

> 15**with the increase of** mixed convection **parameter ? while the** temperature and **thermal boundary layer thickness**

decrease. Figs. 11?7 and 11?8 are drawn to see the variation of ?

> 15**on the velocity** and **temperature profiles. It is**

noticed that

> 13**the velocity and** momentum **boundary layer thickness** increase when ? increases? **It is**

found

> 2**that the temperature and thermal boundary layer thickness are decreasing functions of**

?? Figs. 11.9 and 11.10 are sketched to see the variation of surface temperature parameter ? on the velocity ?0(?) and the temperature ?(?). Both ?0(?) and ?(?) decrease with the increase in ?. It is also

observed that both the momentum and thermal boundary layer thicknesses decrease when ? increases? b1 = 0.3, b2 = 0.2, l = d = 0.5, a = p?4, R = 0.3, Pr = 0.7 1 n = 0.5 0.8 n = 2.0 n = 5.0 n = 7.0 f '?h? 0.6 0.4 0.2 0 0 2 4 6 8 h Fig. 11.3: Impact of ? on ? 0(?)? b1 = 0.3, b2 = 0.2, l = d = 0.5, a = p?4, R = 0.3, Pr = 0.7 1 n = 0.5 0.8 n = 2.0 n = 5.0 0.6 n = 7.0

1**q?h? 0.4 0.2 0 0 2 4** 6 8 **h**

Fig. 11.4: Impact of ? on ?(?)? b1 = 0.4, b2 = 0.3, n = d = 0.5, a = p?4, R = 0.3, Pr = 0.7 1 0.8

3**l = 0.0 l = 0.5** 0.6 **l = 1.0 l**

= 1.5 f'?h? 0.4 0.2 0 0 2 4 6 8 h Fig. 11.5: Impact of ? on ?0(?)? b1=0.4,b2=0.3,n=d=0.5,a=p?4,R=0.3,Pr=0.7 1 0.8

3**l = 0.0 l = 0.5** 0.6 **l = 1.0 l**

= 1.5

1**q?h? 0.4 0.2 0 0 2 4** 6 8 **h**

Fig. 11.6: Impact of ? on ?(?)? b1 = 0.4, b2 = 0.3, n = l = d = 0.5, R = 0.3, Pr = 0.7 1 a = 0.0 0.8 a = p?6 a = p?4 0.6 a = p?3 f '?h? 0.4 0.2 0 0 2 4 6 8 h Fig. 11.7: Impact of ? on ? 0(?)? b1 = 0.4, b2 = 0.3, l = 1.0, n = d = 0.5, R = 0.3, Pr = 0.7 1 a = 0.0 0.8 a = p?6 a = p?4 0.6 a = p?3

1**q?h? 0.4 0.2 0 0 2 4** 6 8 **h**

Fig. 11.8: Impact of ? on ?(?)? b1 = 0.4, b2 = 0.3, n = l = 0.5, a = p?4, R = 0.3, Pr = 0.7 1 d = 0.0 0.8 d = 5.0 d = 10 0.6 d = 15 f '?h? 0.4 0.2 0 0 2 4 6 8 h Fig. 11.9: Impact of ? on ?0(?)? b1 = 0.4, b2 = 0.3, n = l = 0.5, a = p?4, R = 0.3, Pr = 0.7 1 d = 0.0 0.8 d = 5.0 d = 10 0.6 d = 15

1**q?h? 0.4 0.2 0 0 2 4** 6 8 **h**

Fig. 11.10: Impact of ? on ?(?)? Figs. 11?11 and 11.12 are drawn to see the influence of mixed convection parameter ?? wedge angle ?? surface temperature parameter ? and velocity index ? on the local Nusselt number −?0(0)? Fig. 11?11 depicts the effects of ? and ? on −?0(0). It is noticed that −?0(0) increases through increase of mixed convection parameter ? and wedge angle ?. Fig. 11?12 depicts that variations of ? and ? have opposite effects on −?0(0)? A close look at Table 11.1 indicates that 25th-order approximation gives convergent series solutions. 0.9 n = 0.5, R= 0.3, Pr = 1.2, b1 = 0.2, b2 = 0.3, d = 1.5 a = 0.0 0.8 a = p?6 a = p?4 a = p?3 -q'?0 ? 0.7 0.6 0 1 2 3 4 l Fig. 11.11: Impacts of ? and ? on −?0(0)? 2.5 n = 0.5, R= 0.3, a = p?4, Pr = 1.2, b1 = 0.2, b2 = 0

36.**3 n** = 0 **.0** 2 **n = 0.5 n = 1.**

5 n = 3.0 1.5 -q'?0? 1 0.5 0 1 2 3 4 d Fig. 11.12: Impacts of ? and ? on −?0(0)? Table 11

2.**1 : Convergence of** the **homotopy solutions for different order of**
approximation **when**

Pr = 1?0, ? = 0?3, ? = 1?5, ? = 0?5? ? = 0?3, ? = ??4? ?1 = 0?2? ?2 = 0?3 and ~? = ~? = −0?5. Order of
approximation 1 5 10 15 20 25 30 −? 000(0) 0.90330 0.86304 0.86077 0.86047 0.86045 0.86044 0.86044
−?0(0) 0.78889 0.64042 0.62757 0.62627 0.62616 0.62616 0.62616 11.5 Conclusions Mixed convection
and thermal

2**radiation effects in the** Falkner-Skan wedge **flow of**

an Oldroyd-B fluid are investigated. The following points are worth mentioning: • Table 11.1 shows

1**that convergence** of **the functions ? and ? are** obtained **at 25th-order**
approximations

up to five decimal places when ~? = ~? = −0?5? • Influence of mixed convection parameter ? increases

1**the velocity and** momentum **boundary layer thickness** while it **decreases**

the temperature and

7**thermal boundary layer thickness. • Influence of** wedge angle ? and **radiation**
parameter

? on both the temperature and velocity profiles are quite similar. • Thermal boundary layer and momentum
boundary layer thicknesses are decreasing func- tions of surface temperature exponent ?. • Surface heat
transfer −?0(0) increases with an increase of wedge angle ?? mixed convection parameter ? and surface
temperature exponent ? . Bibliography [1] H. Schlichting, Boundary layer theory, 6th ed. Mc-Graw Hill, New
York (1964). [2] B.C. Sakiadis, Boundary layer behavior on continuous solid surfaces: II Boundary on a
continuous flat surface, AIChE J. 7 (1961) 221-225. [3] L.J. Crane, Flow past a stretching plate, Z. Angew.
Math. Phys. 21 (1970) 645-647. [4] A. Chakrabarti and A. S. Gupta, Hydromagnetic flow and heat transfer
over a stretching sheet, Quarterly Appl. Math. 37 (1979) 73-78. [5] C. K. Chen and M. I. Char, Heat
transfer of a continuous stretching surface with suction or blowing, J. Math. Anal. Appl. 135 (1988) 568—
580. [6] K. Vajravelu and A. Hadjinicolaou, Heat transfer in a viscous fluid over a stretching sheet with
viscous dissipation and internal heat generation, Int. Commun. Heat Mass Transf. 20 (1993) 417-430. [7]

H.I. Andersson, O.R. Hansen and B. Holmedal, Diffusion of a chemically reactive species from a stretching sheet, Int. J. Heat Mass Transf. 37 (1994) 659—664. [8] W. H. H. Banks, Similarity solutions of the boundary-layer equations for a stretching wall, J. Méch. Théor. Appl. 2 (1983) 375—392. [9] T. G. Fang, J. Zhang and S. Yao, Slip magnetohydrodynamic viscous flow over a permeable shrinking sheet, Chin. Phys. Lett. 27 (2010) 124702. [10] S. Mukhopadhyay, G. C. Layek and S. A. Samad, Study of MHD boundary layer flow over a heated stretching sheet with variable viscosity, Int. J. Heat Mass Transf. 48 (2005) 4460—4466. [11] C.Y. Wang, The three-dimensional flow due to a stretching flat surface, Phys. Fluid 27 (1984) 1915—1917. [12] C. D. S. Devi, H. S. Takhar and G. Nath, Unsteady three-dimensional, boundary-layer flow due to a stretching surface, Int. J. Heat Mass Transf. 29 (1986) 1996-1999. [13] P. D. Ariel, The three-dimensional flow past a stretching sheet and the homotopy pertur- bation method, Computers Mathematics Appl. 54 (2007) 920—925. [14] P. D. Ariel, On computation of the three dimensional flow past a stretching sheet, Appl. Math. Comput. 188 (2007) 1244-1250. [15] T. Hayat and T. Javed, On analytic solution for generalized three-dimensional MHD flow over a porous stretching sheet, Phys. Lett. A 370 (2007) 243—250. [16] M. Kumari and G. Nath, Analytic solution of unsteady three-dimensional MHD boundary layer flow and heat transfer due to impulsively stretched plane surface, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 3339-3350. [17] K. Vajravelu, Viscous flow over a nolinear stretching sheet, Appl. Math. Comput. 124 (2001) 281—288. [18] R. Cortell, Effects of viscous dissipation and radiation on the thermal boundary layer over a nonlinearly stretching sheet, Phys. Lett. A 372 (2008) 631—636. [19] T. Hayat, T. Javed and Z. Abbas, MHD flow of a micropolar fluid near a stagnation-point towards a non-linear stretching surface, Nonlinear Analysis: Real World Appl. 10 (2009) 1514—1526. [20] N. Afzal, Momentum and thermal boundary layers over a two-dimensional or axisymmetric non-linear stretching surface in a stationary fluid, Int. J. Heat Mass Transf. 53 (2010) 540- 547. 179 [21] P. S. Gupta and A. S. Gupta, Heat and mass transfer on a stretching sheet with suction or blowing, Can. J. Chem. Eng. 55 (1977) 744-746. [22] E. Magyari and B. Keller, Heat and mass transfer in the boundary layers on an exponen- tially stretching continuous surface, J. Phys. D: Appl. Phys. 32 (1999) 577–585. [23] E. M. A. Elbashbeshy, Heat transfer over an exponentially stretching continuous surface with suction, Arch. Mech. 53 (2001) 643-651. [24] M. Q. Al-Odat, R. A. Damesh and T. A. Al-Azab, Thermal boundary layer on an exponen- tially stretching continuous surface in the presence of magnetic field, Int. J. Appl. Mech. Eng. 11 (2006) 289-299. [25] I. C. Liu, H. H. Wang and Y. F. Peng, Flow and heat transfer for three dimensional flow over an exponentially stretching surface, Chem. Eng. Commun. 200 (2013) 253-268. [26] B. S. Dandapat and A. S. Gupta, Flow and heat transfer in a viscoelastic fluid over a stretching sheet, Int. J. Non-Linear Mech. 24 (1989) 215-219. [27] C. K. Chen, M. I. Char and J. W. Cleaver, Temperature field in non-Newtonian flow over a stretching plate, J. Math. Anal. Appl. 151 (1990) 301-307. [28] K. Vajravelu and D. Rollins, Heat transfer in a viscoelastic fluid over a stretching sheet, J. Math. Anal. Appl. 158 (1991) 241—255. [29] T. Hayat, M. Sajid and I. Pop, Three-dimensional flow over a stretching surface in a viscoelastic fluid, Nonlinear Analysis: Real World Appl. 9 (2008) 1811 — 1822. [30] I. C. Liu, Flow and heat transfer of an electrically conducting fluid of second grade over a stretching sheet subject to a transverse magnetic field, Int. J. Heat Mass Transf. 47 (2004) 4427—4437. [31] M. M. Nandeppanavar, M. S. Abel and K. Vajravelu, Flow and heat transfer characteristics of a viscoelastic fluid in a porous medium over an impermeable stretching sheet with viscous dissipation, Int. J. Heat Mass Transf. 53 (2010) 4707—4713. [32] J. J. Choi, Z. Rusak and J. A. Tichy, Maxwell fluid suction flow in a channel, J. Non- Newtonian Fluid Mech. 85 (1999) 165. [33] K. Sadeghy, A. H. Najafi and M. Saffaripour, Sakiadis flow of an upper-convected Maxwell fluid, Int. J. Non-Linear Mech. 40 (2005) 1220 — 1228. [34] A. A. Pahlavan, V. Aliakbar, F. V. Farahani and K. Sadeghy, MHD flows of UCM fluids above porous stretching sheets using two-auxiliary-parameter homotopy analysis method, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 473—488. [35] V. Aliakbar, A. A. Pahlavan and K. Sadeghy, The influence of thermal radiation on MHD flow of Maxwellian fluids above stretching sheets, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 779—794. [36] C. Fetecau, M. Athar and C.

Fetecau, Unsteady flow of a genealized Maxwell fluid with fractional derivative due to a constantly accelerating plate, Comput. Math. Appl. 57 (2009) 596-603. [37] M. Kumari and G. Nath, Steady mixed convection stagnation-point flow of upper convected Maxwell fluids with magnetic field, Int. J. Non-Linear Mech. 44 (2009) 1048—1055. [38] M. S. Abel, J. V. Tawade and M. M. Nandeppanavar, MHD flow and heat transfer for the upper-convected Maxwell fluid over a stretching sheet, Meccanica 47 (2012) 385–393. [39] T. Hayat, S. A. Shehzad and A. Alsaedi, Study on three-dimensional flow of Maxwell fluid over a stretching surface with convective boundary conditions, Int. J. Phys. Sci. 7 (2012) 761 - 768. [40] R. K. Bhatanagar, G. Gupta and K. R. Rajagopal, Flow of an Oldroyd-B fluid due to stretching sheet in the presence of a free stream velocity, Int. J. Non-Linear Mech. 30 (1995) 391-405. [41] K.R. Rajagopal, On an exact solution for the flow of an Oldroyd-B fluid, Bull. Tech. Univ. Istanbul 49 (1996) 617-623. [42] T. Hayat, K. Hutter, S. Asghar and A. M. Siddiqui, MHD Flows of an Oldroyd-B Fluid, Math. Comput. Modell. 36 (2002) 987-995. 181 [43] M. D. Smith, Y. L. Joo, R. C. Armstrong and R. A. Brown, Linear stability analysis of flow of an Oldroyd-B fluid through a linear array of cylinders, J. Non-Newtonian Fluid Mech. 109 (2003) 13—50. [44] S. Asghar, S. Parveen, S. Hanif, A. M. Siddiqui and T. Hayat, Hall effects on the unsteady hydromagnetic flows of an Oldroyd-B fluid, Int. J. Eng. Sci. 41 (2003) 609—619. [45] C. Fetecau and C. Fetecau, Unsteady flows of Oldroyd-B fluids in a channel of rectangular cross-section, Int. J. Non-Linear Mech. 40 (2005) 1214 — 1219. [46] J. Niu, C. Fu and W. Tan, Stability of thermal convection of an Oldroyd-B fluid in a porous medium with Newtonian heating, Phys. Lett. A 374 (2010) 4607-4613. [47] J. Mewis, Thixotropy — a general review. J. Non-Newtonian Fluid Mech. 6 (1979) 1—20. [48] H. A. Barnes, Thixotropy — a review, J. Non-Newtonian Fluid Mech. 70 (1997) 1-33. [49] J. Harris, A continuum theory of time-dependent inelastic flow, Rheol. Acta 6 (1967) 6. [50] J. Harris, Rheology and non-Newtonian flow, London: Longman (1977). [51] S. Sadeqi, N. Khabazi and K. Sadeghy, Blasius flow of thixotropic fluids: A numerical study, Commun. Nonlinear Sci. Numer. Simulat. 16 (2011) 711—721. [52] J. J. M. Sillekens, C. C. M. Rindt and A. A. V. Steenhoven, Development of laminar mixed convection in a horizontal square channel with heated side walls, Int. J. Heat and Fluid Flow 19 (1998) 270-281. [53] G.P. Celata, A. Chiaradia, M. Cumo and F. D. Annibale, Heat transfer enhancement by air injection in upward heated mixed-convection flow of water, Int. J. Multiphase Flow 25 (1999) 1033-1052. [54] A. Barletta, Analysis of flow reversal for laminar mixed convection in a rectangular duct with one or more isothermal walls, Int. J. Heat Mass Transf. 44 (2001) 3481-3497. [55] E. Magyari, I. Pop and B. Keller, Mixed convection boundary layer flow past a horizontal permeable flat plate, Fluid Dynamics Research 31 (2002) 215-225. 182 [56] R. Nazar, N. Amin and I. Pop, Unsteady mixed convection boundary layer flow near the stagnation point on a vertical surface in a porous medium, Int. J. Heat Mass Transf. 47 (2004) 2681—2688. [57] M. A. Hossain, S. Bhowmick and R. S. R. Gorla, Unsteady mixed-convection boundary layer flow along a symmetric wedge with variable surface temperature, Int. J. Eng. Sci. 44 (2006) 607—620. [58] H. A. Mohammed and Y. K. Salman, Experimental investigation of mixed convection heat transfer for thermally developing flow in a horizontal circular cylinder, Appl. Therm. Eng. 27 (2007) 1522—1533. [59] H. A. Mohammed, Laminar mixed convection heat transfer in a vertical circular tube under buoyancy-assisted and opposed flows, Energy Conv. Manag. 49 (2008) 2006—2015. [60] M. Kotouc, G. Bouchet and J. Dusek, Loss of axisymmetry in the mixed convection, assisting flow past a heated sphere, Int. J. Heat Mass Transf. 51 (2008) 2686—2700. [61] M. Mahmood, S. Asghar and M. A. Hossain, Transient mixed convection flow arising due to thermal diffusion over a porous sensor surface inside a squeezing horizontal channel, Int. J. Therm. Sci. 48 (2009) 1619—1626. [62] T. Hayat, M. Mustafa and I. Pop, Heat and mass transfer for Soret and Dufour's effect on mixed convection boundary layer flow over a stretching vertical surface in a porous medium filled with a viscoelastic fluid, Commun. Nonlinear Sci. Numer. Simulat. 15 (2010) 1183— 1196. [63] A. J. Chamkha, Thermal radiation and buoyancy effects on hydromagnetic flow over an accelerating permeable surface with heat source or sink, Int. J. Eng. Sci. 38 (2000) 1699- 1712. [64] S. Abel, K. V. Prasad and A. Mahaboob, Buoyancy force and thermal radiation effects in MHD boundary layer visco-elastic fluid flow

over continuously moving stretching surface, Int. J. Therm. Sci. 44 (2005) 465—476. [65] S. Mukhopadhyay and G.C. Layek, Effects of thermal radiation and variable fluid viscosity on free convective flow and heat transfer past a porous stretching surface, Int. J. Heat Mass Transf. 51 (2008) 2167—2178. [66] S. Mukhopadhyay, Effect of thermal radiation on unsteady mixed convection flow and heat transfer over a porous stretching surface in porous medium, Int. J. Heat Mass Transf. 52 (2009) 3261—3265. [67] C. H. Chen, Magnetohydrodynamic mixed convection of a power-law fluid past a stretching surface in the presence of thermal radiation and internal heat generation/absorption, Int. J. Non-Linear Mech. 44 (2009) 596—603. [68] K. Vajravellu and D. Rollins, Heat transfer in an electrically conducting fluid over a stretching surface, Int. J. Non-Linear Mech. 27 (1992) 265-277. [69] M. S. Abel and M. M. Nandeppanavar, Heat transfer in MHD viscoelastic boundary layer flow over a stretching sheet with non-uniform heat source/sink, Commun. Nonlinear Sci. Numer. Simulat. 14 (2009) 2120—2131. [70] O. A. Beg, J. Zueco, R. Bhargava and H. S. Takhar, Magnetohydrodynamic convection flow from a sphere to a non-Darcian porous medium with heat generation or absorption effects: network simulation, Int. J. Therm. Sci. 48 (2009) 913—921. [71] R. Kandasamy, K. Periasamy and K. K. S. Prabhu, Chemical reaction, heat and mass transfer on MHD flow over a vertical stretching surface with heat source and thermal stratification effects, Int. J. Heat Mass Transf. 48 (2005) 4557—4561. [72] M. A. Mansour, N. F. E. Anssary and A. M. Aly, Effects of chemical reaction and thermal stratification on MHD free convective heat and mass transfer over a vertical stretching surface embedded in a porous media considering Soret and Dufour numbers, Chem. Eng. J. 145 (2008) 340—345. [73] D. Pal and B. Talukdar, Buoyancy and chemical reaction effects on MHD mixed convection heat and mass transfer in a porous medium with thermal radiation and Ohmic heating, Commun. Nonlinear Sci. Numer. Simulat. 15 (2010) 2878—2893. 184 [74] G. Singh, P. R. Shrma and A. J. Chamkha, Mass transfer with chemical reaction in MHD mixed convective flow along a vertical stretching sheet, Int. J. Energy Tech. 4 (2012) 1-12. [75] S. J. Liao, Homotopy Analysis Mathod: A new analytic technique for nonlinear problems, Commun. Nonlinear Sci. Numer. Simulat. 2 (1997) 95-100. [76] S. J. Liao, Beyond perturbation: introduction to Homotopy Analysis Method, Chapman & Hall, Boca Raton (2003). [77] S. Abbasbandy, Approximate solution for the nonlinear model of diffusion and reaction in porous catalysts by means of the homotopy analysis method, Chem. Eng. J. 136 (2008) 144—150. [78] M. Sajid and T. Hayat, The application of homotopy analysis method to thin film flows of a third order fluid, Chaos Solitons Fractals 38 (2008) 506—515. [79] S. J. Liao, A short review on the homotopy analysis method in fluid mechanics, J. Hydro- dyn. 22 (2010) 882-884. [80] M. Turkyilmazoglu, A note on the homotopy analysis method, Appl. Math. Lett. 23 (2010) 1226-1230. [81] S. J. Liao, Homotopy analysis method in nonlinear differential equations, Higher Edu. Press, Beijing and Springer-Verlag Berlin Heidelberg (2012). [82] S. Abbasbandy, E. Shivanian and K. Vajravelu, Mathematical properties of image-curve in the frame work of the homotopy analysis method, Commun. Nonlinear Sci. Numer. Simulat. 16 (2011) 4268-4275. [83] M. M. Rashidi, S. A. M. Pour, T. Hayat and S. Obaidat, Analytic approximate solutions for steady flow over a rotating disk in porous medium with heat transfer by homotopy analysis method, Comput. Fluids 54 (2012) 1-9. [84] I. Ahmad, M. Ahmed, Z. Abbas and M. Sajid, Hydromagnetic flow and heat transfer over a bidirectional stretching surface in a porous medium, Thermal Sci. 15 (2011) S205-S220. [85] M. Turkyilmazoglu, Series solution of nonlinear two-point singularly perturbed boundary layer problems, Comput. Math. Appl. 60 (2010) 2109-2114. K1 = 0.1, K2 = 0.2, t = 0.3, M = 0.6, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3, R = 0.4 K2 = 0.2, l = 0.4, t = 0.3, M = 0.6, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3, R = 0.4 K1 = 0.1, K2 = 0.2, l = 0.4, t = 0.3, M = 0.6, Sc = 0.7, Ec = 0.5, Pr = 1.0, R = 0.4 K1 = 0.1, K2 = 0.2, l = 0.4, t = 0.3, M = 0.6, Sc = 0.7, Ec = 0.5, N = 0.3, R = 0.4 K1 = 0.1, l = 0.4, t = 0.3, M = 0.6, Sc = 0.7, Ec = 0.5, Pr = 1.0, N = 0.3, R = 0.4 K1 = 0.1, K2 = 0.2, l = 0.4, t = 0.3, M = 0.6, Sc = 0.7, Pr = 1.0, N = 0.3, R = 0.4 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101

102 103 104 105 106 107 108 109 110 111 112 113 114 115 116 117 118 119 120 121 122 123 124 125 126 127 128 129 130 131 132 133 134 135 136 137 138 139 140 141 142 143 144 145 146 147 148 149 150 151 152 153 154 155 156 157 158 159 160 161 162 163 164 165 166 167 168 169 170 171 172 173 174 175 176 177 178 180 183 185 186