Some Studies on Algebraic Structures of Soft Sets

By

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بسم الله الرحمن الرحيم

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DEDEICATED to
My Mother
My Supervisor
And
My Brother

Contents

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0.2 Research Profile

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0.3 Introduction

" It is the mark of an instructed mind to rest satisfied with that degree of precision which the nature of the subject admits, and not to seek exactness where only an approximation of the truth is possible."

(Aristotle, 384–322 BC)

A great philosopher from history uttered these words and ironically, himself established a binary logic that only admits the opposites of true and false, a logic which does not admit degrees of truth in between these two extremes. In other words, Aristotelian logic does not admit imprecision in truth. However, Aristotle's quote is so appropriate today; it is a quote that admits uncertainty. It is required that we should heed; we shall have to balance the precision we seek with the uncertainty that exists. Most of the mathematical models, and solutions do not address the uncertainty in given information. Again, we quote a genius mind L. A. Zadeh saying,

" The closer one looks at a real world problem, the fuzzier becomes its solution."

(Zadeh, 1973)

We are living in a world which is becoming ever more reliant on the use of intelligent electronics to control the behavior of real-world resources. For example, an increasing amount of commerce is performed through credit card or online banking systems. Similarly, airports, large national databases, e-governments etc. are being run without ever looking out of a window. Another, more individual, example is the increasing use of personal gadgets or devices for organizing meetings and contacts and socializing purposes. All these examples share a similar structure; multiple parties (e.g. data or airplanes or people) combine together to coordinate their activities in order to attain a common goal.

Fuzzy and vague logic means approximate reasoning, information granulation, computing with words and so on. Ambiguity is always present in any realistic process. This ambiguity may arise from the interpretation of data inputs and in the rules used to describe relationships between the informative attributes. A logical view on vagueness provides an inference structure that enables the human reasoning capabilities to be applied to artificial knowledge-based systems. A logical approach provides a means for converting linguistic strategy into control actions and thus offers a high-level computation. L. A. Zadeh was the first one who introduced the concept of fuzzy sets in [46] which was proved a paradigm shift in later years. Theory of soft sets was introduced by Molodstov [34] in 1999. The purpose of this novel concept was to remove the inadequacy of parameterization tool in previously defined theories of fuzzy Mathematics. Although the theory of rough sets [39] addresses the issue of parameterization and the hybrid structure such as fuzzy rough sets can also be utilized for incorporating the fuzziness of data but no significant role of parameters can be found in operations defined on rough sets. On the other hand, the absence of any restrictions while making approximations for a given object in soft sets establishes this theory as more handy, convenient and easily applicable in practice. Since the introduction of the theory of soft sets in 1999, a lot of work has been done so far and for different applications of soft sets see [2], [3], [4], [1], [13], [12], [21], [18], [19], [20], [25], [30], [31], [32], [33], [44]. Primarily the aim of soft set theory is to provide a tool with enough parameters to deal with uncertainty associated with the data, whereas on the other hand it has the ability to represent the data in a useful manner. With the introduction of new operations on soft sets, it is felt imperative to study the underlying algebraic structures. This will give a forehand and better understanding for their applications.

In the past, studying the algebraic structure of a mathematical theory has proved itself effective in making the applications in the sciences more efficient. This is the inherent motivation for us to study the algebraic structures of these generalizations of soft set theory. Such research may not only provide more insight into soft set theory, but also hopefully develop methods for applications. Lattice theory has become a popular mathematical framework in different domains of information processing, such as fuzzy sets, formal concept analysis, mathematical morphology etc. In this work, we consider three important extensions of soft set theory, from the point of view of lattice theoretic algebraic structures. The first one deals with imprecision and vagueness in knowledge representation and information processing with one function for approximations as we have done in Chapters 2 and 3 following the notions initiated by Molodstov in [34], under the framework of crisp and fuzzy sets. The second one handles imprecision and vagueness in multi-frames of knowledge representation and information processing with more than one function for approximations as we have done in Chapters 4 and 5 following the notions initiated by Jun et al. in [18], under the framework of crisp and fuzzy sets. The third one with more than one frames deals additionally with the bipolarity of information c.f. [11], [26], [27] which occurs in several domains, such as preference modeling under some parameters, spatial reasoning, argumentation etc. In these domains, two types of information have often to be handled: (i) positive information (which is possible and desired), and (ii) negative information (which is not possible or constraint). Extension of soft set to this frameworks in crisp and fuzzy context is studied in Chapters 6 and 7.

In the study of soft sets as algebraic structures there are mainly two types of collections of soft sets. First the collection of soft sets with a fixed set of parameters, and second the collection of soft sets with different sets of parameters. These two types of collections with new operations sometimes behave similarly and sometimes differently. There are many algebras and lattice based structures associated with logic. Boolean algebras are associated with traditional two valued Aristotelean logic. MV algebras are suitable for multi-valued logic. BCI/BCK algebras generalize the notion of algebra of sets with the set subtraction as the only non-nullary operation. These algebras generalize implication algebras which is mostly based on lattice based complements and pseudocomplements. In this work, we study algebraic structures of soft sets associated with their unary and binary extended, restricted and product operations in a systematic way.

0.4 Chapter-wise Study

The present work in this thesis is written in the lattice-theoretical background of soft sets. It contains the necessary part of soft set theory and shows how to formulate in an elegant way various concepts and facts about the algebraic structures of soft sets and its generalized structures. Prerequisites are minimal and the work is self-contained.

In this thesis, we have eight chapters. In the first chapter, we have given some basic concepts and notations which will be helpful for understanding the rest of the thesis. Classical Set theory and algebraic structures, a brief introduction of fuzzy sets and bipolar fuzzy sets is included with most familiar notions as per use in literature.

In Chapter 2, definitions and operations on soft sets are given. This chapter sets forth the use of mathematical notations adapted for soft sets in our thesis in order to create a flow and understanding without any ambiguity. Definition of soft set is taken from [34] and operations on soft sets are taken from [2]. In set theory we come across with only one null set and the whole set itself as trivial cases and this holds in the case of fuzzy sets as well, but surely this is not the case in soft sets. Here we have relative null soft set and relative whole soft set over initial universe. This difference adds a new aspect to the soft set theory. Operations on soft sets are either extended or restricted based upon the choice of parameters and this property is unique for soft sets so far. No earlier vague structure addressed this problem of parametrization and therefore soft set theory is more adequate in operational use with parameters. It is important for us to get familiar with the properties of these newly defined operations on soft sets. Properties of operations defined on soft sets are discussed and examples are worked out to show way of working out with soft sets. The fact is also revealed that the distributivity of union and intersection is not following as it holds in previously defined crisp and vague set theories. A complete check for all the possible cases has been made to establish distributive laws for soft sets. In the last section of chapter 2, various algebraic structures of soft sets associated with the new operations are studied. It is seen that the collection of soft sets with fixed parameters become a Boolean algebra, MV-algebra, Stone algebra and Brouwerian and atomic lattices. Moreover, it also becomes BCK-algebra with respect to restricted difference and " $\star$ " operations.

In Chapter 3 fuzzy soft sets are discussed for their algebraic structures. Newly defined operations on fuzzy soft sets are used in this chapter in a similar way as used for soft sets in Chapter 2. Some operations of soft sets, for example extended or restricted difference are not available for fuzzy soft sets and therefore there are some properties which do not hold for fuzzy soft sets. On the other hand, we can define some operations on fuzzy soft sets which are not much meaningful in soft set theory but give interesting results in fuzzy soft context. Algebras of collections of fuzzy soft sets are studied and it is observed that the collection of fuzzy soft sets with fixed set of parameters becomes Kleene algebra, Stone algebra and Brouwerian lattice.

Chapter 4 is concerned with the study of double-framed soft sets which is a generalization of soft sets. Operations on double-framed soft sets are defined and investigated for their algebraic behaviors. After a rigorous account on the properties we have discussed the algebraic structures of double-framed soft sets. It is shown that the collection of double-framed soft sets has a different behavior than the soft sets and fuzzy soft sets and proves to be richer because we can define more operations. Collection of double-framed soft sets with fixed set of parameters becomes de Morgan algebra with " $\circ$ " operation, MV-algebra and Boolean algebra for " ^{c} " operation, pseudo-complemented lattice for " $^\diamondsuit$ " operation and Brouwerian lattice. It also becomes BCK-algebra with respect to restricted difference and " $\star$ " operations.

In Chapter 5, the concept of double-framed fuzzy soft sets is introduced as a generalization of fuzzy soft sets and double-framed soft sets. We have defined various operations on double-framed fuzzy soft sets and checked their algebraic properties. It is found that the collection of this structure with fixed set of parameters gives rise to Kleene algebra, de Morgan algebra, Stone algebra and Brouwerian lattice.

Chapter 6 introduces the idea of bipolar soft sets which is hybridization of structure of soft set, double-framed soft set and bipolarity. It is a new concept and approximates positive and negative information for available sets of choices and parameters. We have shown that the class of bipolar soft sets is a subclass of the class of double-framed soft sets. An example from psychology is also presented. Some operation of double-framed soft sets are available to bipolar soft sets while some are not. We have figured out the algebras of bipolar soft sets and obtained the results which are not simply a consequence but showing a difference of character in this newly defined structure as well. It is shown that the collection of bipolar soft sets with a fixed set of parameters becomes a Kleene algebra.

In Chapter 7, we have initiated the ideas of fuzzy bipolar soft set as a generalization of bipolar soft set and bipolar fuzzy soft set as a generalization of fuzzy soft set. We have proved that both ideas coincide with each other. We have also shown that the class of fuzzy bipolar soft sets is a subclass of the class of double-framed fuzzy soft sets. Thus the structure of fuzzy bipolar soft sets is agreeable to proceed and it is proved that the collection of fuzzy bipolar soft set with a fixed set of parameters is a de Morgan algebra for operation " $\circ$ " and Kleene algebra for operation " $*$ ".

Chapter 8 is devoted for providing a general algebraic framework for extensions in theory of soft sets in three different contexts: soft sets, multi-framed soft sets and multi-polar soft sets. A standard formula is presented for defining aggregation operators on the three types of extensions of soft sets in restricted and extended manner. The topic provides an overview of the observations made in earlier chapters and we have summarized the results in tabular form. At the end, an application of soft set theory in decision making is given with an informal algorithm and worked out example is provided for decision making with fuzzy bipolar soft sets.

Chapter 1

Preliminaries

In this chapter, theory of classical sets and theory of fuzzy sets are discussed. Various operations, their laws and properties of classical and fuzzy sets are given. The classical sets, we are going to consider, are defined by means of the crisp or definite boundaries. The concept of a set is fundamental in Mathematics and intuitively can be described as a collection of objects possibly linked through some properties. A classical set A has clear boundaries, i.e. $x \in A$ or $x \notin A$ exclude any other possibility. This implies that there is a certainty or definiteness involved in the approximation of these sets. A fuzzy set, on the other hand, is defined by its uncertain or vague properties. A fuzzy set is a class with a continuum of membership grades. So a fuzzy set A in a referential (universe of discourse) X is characterized by a membership function μ_A which associates with each element $x \in X$ a real number $\mu_A(x) \in [0, 1]$, having the interpretation $\mu_A(x)$ is the membership grade of x in the fuzzy set A. The crisp sets are sets without any ambiguity in their membership whereas fuzzy set theory is an efficient theory in dealing with the concepts of vagueness. As an extension of fuzzy sets, Lee [26] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. Bipolar-valued fuzzy sets have membership degrees that represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter property. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $(0, 1]$ indicate that elements somewhat satisfy the property, and the membership degrees on $[-1, 0)$ indicate that elements somewhat satisfy the implicit counter-property. Basic notions of bipolar fuzzy sets given after reviewing the ideas of the crisp sets and fuzzy sets.

1.1 Crisp Sets

In this section, we recall the standard definitions and main results on algebraic structure of classical crisp set theory in detail. Following definitions are taken from [7].

1.1.1 Definition

Let X be a set. An *order* $\leq$ on X is a reflexive, antisymmetric, and transitive binary relation, that is, for all $x, y, z \in X$,

1) $x \leq x$,

2) $x \leq y$ and $y \leq x$ imply $x = y$, and

3) $x \leq y$ and $y \leq z$ imply $x \leq z$.

An *ordered set* is denoted by $(X, \leq)$, where X is a non-empty set and $\leq$ an order on X.

1.1.2 Definition

Let $(X, \leq_1)$ and $(Y, \leq_2)$ be two ordered sets. A mapping $\theta : X \to Y$ such that $\theta(x_1) \leq_2 \theta(x_2)$ whenever $x \leq_1 y$ is called a *homomorphism* or an *order homomorphism or order preserving.*

1.1.3 Definition

Let X be an ordered set and let $A \subseteq X$. Then $x \in X$ is a *maximal element* of A, if $x \leq a \in A$ implies $a = x$. Further, $x \in X$ is the *greatest element* of A, if $x \geq a$ for all $a \in A$.

A *minimal element* of A and the *least element* of A are defined dually. Note that if A has a *greatest element*, it is unique. Similarly, the least element of A is unique.

1.1.4 Definition

Let P be an ordered set and $A \subseteq X$. An element $x \in X$ is an *upper bound* of A if $a \leq x$ for all $a \in A$. A *lower bound* of A is defined dually.

If there is a least element in the set of all upper bounds of A, it is called the *supremum* of A and is denoted by $\sup A$ or $\bigvee A$; dually a greatest lower bound is called *infimum* and written $\inf A$ or $\bigwedge A$. We also write $a \vee b$ for $\sup\{a, b\}$ and $a \wedge b$ for $\inf\{a, b\}$. *Supremum* and *infimum* are frequently called *join* and *meet.*

1.1.5 Definition

Let L be a non-empty ordered set. If $a \vee b$ and $a \wedge b$ exist for all $a,\ b \in L$, then L is called a *lattice*. If $\bigvee A$ and $\bigwedge A$ exist for all $A \subseteq L$, then L is called a *complete lattice*.

1.1.6 Definition

Let $(L, \leq)$ be a lattice. If $\bigvee L$ and $\bigwedge L$ exist, then L is called a *bounded lattice*. In a bounded lattice, the least element is denoted by 0 and greatest element by 1.

The definition of a lattice given with the help of a binary relation on X is a constructive approach, now, we present the algebraic definition of a lattice which is an axiomatic approach and given with the help of binary operations defined on X.

1.1.7 Definition

A binary operation " $*$ " on X is a map $* : X \times X \to X$. A set X together with a binary operation " $*$ " on it, is called a groupoid and denoted by $(X, *)$. In general $*(x, y)$ is denoted by $x * y$.

1.1.8 Definition

Let $(X, *)$ be a groupoid. Then $*$ is called

1) *Associative* if $x * (y * z) = (x * y) * z$,

2) *Commutative* if $x * y = y * x$,

3) *Idempotent* if $x * x = x$

for all $x,\ y,\ z \in X$

1.1.9 Definition

An algebraic structure $(S, *)$ is called a *semilattice* if S is a non-empty set and $*$ is a binary operation such that $*$ is commutative, associative and idempotent.

1.1.10 Definition

An algebraic structure $(L, \wedge, \vee)$ is called a *lattice* if L is a non-empty set and $\wedge$ and $\vee$ are binary operations on L, $(L, \wedge)$ and $(L, \vee)$ are semilattices and absorption laws for

$\wedge$ and $\vee$ hold i.e.

$$x \wedge (x \vee y) = x \text{ and}$$
$$x \vee (x \wedge y) = x \text{ for all } x,\ y \in L.$$

Using the basic lattice operations, an ordering can be defined as following:

1.1.11 Theorem

Let $(L, \wedge, \vee)$ be a *lattice and* $x,\ y \in L$. The binary relation $\leq$ on L is defined by:

$$x \leq y \Leftrightarrow x \vee y = y \text{ or equivalently}$$
$$x \leq y \Leftrightarrow x \wedge y = x \text{ for all } x,\ y \in L.$$

Then $(L, \leq)$ is a lattice satisfying the properties of lattice given in Definition 1.1.5.

1.1.12 Theorem

Let $(L, \leq)$ be a *lattice and* $x,\ y \in L$. The binary oprations " $\wedge$ " and " $\vee$ " on L are defined by:

$$x \wedge y = \inf\{x, y\} \text{ and}$$
$$x \vee y = \sup\{x, y\} \text{ for all } x,\ y \in L.$$

Then $(L, \wedge, \vee)$ satisfies the properties of lattice given in Definition 1.1.10.

Thus, both Definition 1.1.5 and Definition 1.1.19 are equivalent to each other. Onwards from here, we consider both notations interchangeably without stating explicitly.

1.1.13 Definition

Let $(L_1, \wedge, \vee)$ and $(L_2, \wedge, \vee)$ be two lattices. A mapping $\theta : L_1 \rightarrow L_2$ such that $\theta(x \wedge y) = \theta(x) \wedge \theta(y)$ and $\theta(x \vee y) = \theta(x) \vee \theta(y)$ is called a *homomorphism* of lattices. A one-to-one lattice homomorphism is called monomorphism. A one-to-one and onto homomorphism is called lattice isomorphism.

Next we give the definitions of various algebras of lattices:

1.1.14 Definition

Let L be a bounded lattice with a least element 0 and a greatest element 1. For an element $x \in L$, an element $y \in L$ is a *complement* of x if

$$x \vee y = 1 \text{ and } x \wedge y = 0.$$

If an element x has a unique complement, we denote it by x^c.

1.1.15 Remark

There exist bounded lattices with elements having more than one complement or no complement at all.

1.1.16 Example

Let L be a lattice given by the Figure 1.1.1. In this lattice b and e are complements of a, c has no complement, 1 has 0 as complement and 0 has 1.

1.1.17 Definition

A bounded lattice L in which every element has a complement is called a *complemented lattice.*

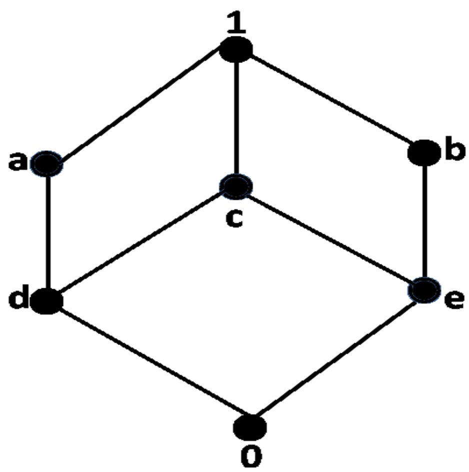

Figure 1.1.1

1.1.18 Example

Let X be a non-empty set. Then $(\mathcal{P}(X), \subseteq)$ is a complemented lattice.

1.1.19 Definition

Let L be a bounded lattice with a least element 0 and a greatest element 1. Let $\acute{}: L \to L$, mapping $x \mapsto x\acute{}$ is such that

$$(x\acute{})\acute{} = x \text{ and } x \leq y \text{ implies that } y\acute{} \leq x\acute{} \text{ for all } x, y \in L.$$

Then " ´ " is called an *involution or duality on* L.

It follows that " ´ " is bijective, and that $0\acute{}= 1$ and $1\acute{}= 0$.

1.1.20 Example

Let $I = [0, 1]$. Then $(I, \leq)$ is a bounded lattice and $': x \mapsto 1 - x$ is an involution on I.

1.1.21 Definition

Let L be a lattice with a least element 0. Then $x \in L$ is called an *atom* of L, if $0 < x$ and there is no element y in L with $0 < y < x$. The set of atoms of L is denoted by $\mathcal{A}(L)$.

1.1.22 Example

Let X be a non-empty set. Then every singleton subset of X is an atom of lattice $\mathcal{P}(X)$ and $\mathcal{A}(\mathcal{P}(X)) = \{\{x\} : x \in X\}$.

1.1.23 Definition

Let L be a bounded lattice and " $'$ " is an involution on L, the identities

$$\begin{aligned}(x \vee y)' &= x' \wedge y' \\ (x \vee y)' &= x' \wedge y'\end{aligned}$$

are called the *de Morgan Laws.*

A nice property of unions and intersections is that they distribute over each other. Therefore, it is natural to consider lattices for which joins and meets have analogous properties.

1.1.24 Definition

A lattice L satisfying the distributive laws

$$\begin{aligned}x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge y); \\ x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \quad \text{for all } x, y, z \in L\end{aligned}$$

is called a *distributive lattice.*

1.1.25 Definition

If de Morgan's laws hold for a bounded distributive lattice having an involution, then it is called a *de Morgan algebra.* Such a system is denoted by $(L, \vee, \wedge, ', 0, 1)$.

1.1.26 Definition

A bounded distributive lattice which is complemented is called a *Boolean lattice.*

1.1.27 Definition

A *de Morgan's algebra* $(L, \wedge, \vee, ', 0, 1)$ that satisfies $x \wedge x' \leqslant y \vee y'$ for all x, $y \in L$, is called a *Kleene algebra.*

1.1.28 Definition

Let L be a lattice. Then L is said to be *atomic* if every element x of L is the supremum of the atoms below it, i.e.

$$x = \bigvee \{y \in \mathcal{A}(L) | y \leq x\}.$$

1.1.29 Definition

Let L be a lattice, and $x, y \in L$. Then x is called *pseudocomplemented relative to* y if the following set:

$$T(x, y) = \{z \in L | z \wedge x \leq y\}$$

has a greatest element. This greatest element is said to be pseudocomplement of x relative to y, denoted by $x \to y$. So, $x \to y$, in case it exists, has the following property:

$$z \wedge x \leq y \text{ if and only if } z \leq x \to y.$$

1.1.30 Definition

An element $x \in L$ is said to be *relatively pseudocomplemented* if $x \to y$ exists for all $y \in L$.

1.1.31 Definition

A lattice L is said to be an *implicative lattice* or *relatively pseudocomplemented or Brouwerian*, if every element in L is *relatively pseudocomplemented.*

1.1.32 Example

Let $L(X)$ be the lattice of open sets of a topological space X. Then $L(X)$ is Brouwerian. For any open sets $A, B \in L(X)$, $A \to B = (A^c \cup B)^\circ$, the interior of the union of B and the complement of A.

1.1.33 Definition

Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and $x \in L$. Then an element x^* is called a pseudocomplement of x, if $x \wedge x^* = 0$ and $y \leqslant x^*$ whenever $x \wedge y = 0$. Note that $x \to 0 = x^*$.

1.1.34 Definition

If every element of a lattice L has a pseudocomplement then L is said to be pseudocomplemented.

1.1.35 Definition

The equation

$$x^* \vee x^{**} = 1$$

is called Stone's identity.

1.1.36 Definition

A Stone algebra is a pseudocomplemented, distributive lattice satisfying Stone's identity.

1.1.37 Definition [17]

MV-algebra is an algebraic structure $\langle M, \oplus, ^*, 0\rangle$, where $\oplus$ is a binary operation, " $*$ " is a unary operation, and 0 is a constant such that the following axioms are satisfied for any a, $b \in M$:

(MV1) $(M, \oplus, 0)$ is a commutative monoid,

(MV2) $(a^*)^* = a$,

(MV3) $0^* \oplus a = 0^*$,

(MV4) $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

1.1.38 Definition [9]

A set X with a binary operation $*$ and a constant 0 is called a BCI algebra if for any x, y, z in X, it satisfies the following conditions:

(BCI-1) $((x * y) * (x * z)) * (z * y) = 0$,

(BCI-2) $(x * (x * y)) * y = 0$,

(BCI-3) $x * x = 0$,

(BCI-4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

1.1.39 Definition [9]

A BCI-algebra $(X; *, 0)$ is called a BCK-algebra if it satisfies the following condition:

(BCK-5) $0 * x = 0$. for all $x \in X$.

1.1.40 Definition [9]

A BCK algebra X is called *bounded* if there exists some element $1 \in X$ such that $x * 1 = 0$ for all $x \in X$. For a bounded BCK algebra $(X; *, 0)$, if an element $x \in X$ satisfies $1 * (1 * x) = x$, then x is called an *involution (Different meaning from the involution given in Definition 1.1.19.*

1.2 Fuzzy Sets

The material presented in this section is taken from [46]. We give the definitions of fuzzy sets and some related terms.

Let X be a set and A be a subset of X. The characteristic function of A is the function C_A of X into {0,1} defined by $C_A(x) = 1$ if $x \in A$ and $C_A(x) = 0$ if $x \notin A$.

1.2.1 Definition

A *fuzzy subset of* X is a function from X into the unit closed interval [0,1]. The set of all fuzzy subsets of X is called the *fuzzy power set* of X, and is denoted by $\mathcal{FP}(X)$.

1.2.2 Definition

Let μ, $\upsilon \in \mathcal{FP}(X)$. If $\mu(x) \leqslant \upsilon(x)$ for all $x \in X$, then μ is said to be *contained in* υ, and we write $\mu \subseteq \upsilon($ or $\upsilon \supseteq \mu)$.

Clearly, the inclusion relation $\subseteq$ is a partial order on $\mathcal{FP}(X)$.

1.2.3 Definition

Let μ, $\upsilon \in \mathcal{FP}(X)$. Then $\mu \vee \upsilon$ and $\mu \wedge \upsilon$ are fuzzy subsets of X, defined as follows:

For all $x \in X$,

$$(\mu \vee \upsilon)(x) = \mu(x) \vee \upsilon(x),$$
$$(\mu \wedge \upsilon)(x) = \mu(x) \wedge \upsilon(x).$$

The fuzzy subsets $\mu \vee \upsilon$ and $\mu \wedge \upsilon$ are called the *union and intersection of* μ *and* υ, respectively.

1.2.4 Definition

The *complement of a fuzzy subset* μ is denoted by μ' and is defined by

$$\mu'(x) = 1 - \mu(x),$$

for all $x \in X$.

1.2.5 Definition

The fuzzy subsets of X, denoted by $\tilde{\mathbf{0}}$ and $\tilde{\mathbf{1}}$, which map every element of X onto 0 and 1 respectively, are called the empty fuzzy set or null fuzzy subset and the whole fuzzy subset of X respectively.

1.3 Bipolar Fuzzy Sets

The material presented in this section is taken from [26]. We give the definitions of bipolar fuzzy sets and some related terms. In bipolar-valued fuzzy sets, two kinds of representations are used: canonical representation and reduced representation. In the canonical representation, membership degrees are expressed with a pair of a positive membership value and a negative membership value. That is, the member ship degrees are divided into two parts: positive part in $[0, 1]$ and negative part in $[-1, 0]$. In the reduced representation, membership degrees are presented with a value in $[-1, 1]$. In our work, we use the canonical representation of a bipolar-valued fuzzy sets. For more material on this topic we refer to [26] and [27]. Let X be the universe of discourse.

1.3.1 Definition

A bipolar fuzzy set μ in X is defined as:

$$\mu = \big\{(x,\ \mu^P(x),\ \mu^N(x)) : x \in X\big\}$$

where $\mu^P : X \longrightarrow [0, 1]$ and $\mu^N : X \longrightarrow [-1, 0]$ are mappings. The positive membership degree $\mu^P(x)$ denotes the satisfaction degree of an element x to the property and the negative membership degree $\mu^N(x)$ denotes the satisfaction degree of x to some implicit counter-property. If $\mu^P(x) \neq 0$ and $\mu^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for μ. If $\mu^P(x) = 0$ and $\mu^N(x) \neq 0$, it is the situation that x does not satisfy the property of μ but somewhat satisfies the counter-property of μ. It is possible for an element x to be $\mu^N(x) \neq 0$ and $\mu^P(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain.

For example, sweetness of foods is a bipolar fuzzy set. If sweetness of foods has been given as positive membership values then bitterness of foods is for negative membership values. Other tastes like salty, sour, pungent (e.g. chili) etc. are irrelevant to the corresponding property. So these foods are taken as zero membership values.

For the sake of simplicity, we shall write $\mu = \left(\mu^P,\ \mu^N\right)$ for the bipolar fuzzy set

$$\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) : x \in X\right\}.$$

The set of all bipolar fuzzy sets of X is called the *bipolar fuzzy power set* of X, and is denoted by $\mathcal{BFP}(X)$.

1.3.2 Definition

Let $\mu,\ \upsilon \in \mathcal{BFP}(X)$. If $\mu^P(x) \leqslant \upsilon^P(x)$ and $\upsilon^N(x) \leqslant \mu^N(x)$ for all $x \in X$, then μ is said to be *contained in* υ, and we write $\mu \subseteq \upsilon($ or $\upsilon \supseteq \mu)$.

Clearly, the inclusion relation $\subseteq$ is a partial order on $\mathcal{BFP}(X)$.

1.3.3 Definition

Let $\mu,\ \upsilon \in \mathcal{BFP}(X)$. Then set operations $\mu \cup \upsilon$ and $\mu \cap \upsilon$ are bipolar fuzzy sets of X, defined as follows:

For all $x \in X$,

$$\begin{aligned}(\mu \cup \upsilon)^P(x) &= \mu^P(x) \vee \upsilon^P(x),\ (\mu \cup \upsilon)^N(x) = \mu^N(x) \wedge \upsilon^N(x) \text{ and} \\ (\mu \cap \upsilon)^P(x) &= \mu^P(x) \wedge \upsilon^P(x),\ (\mu \cap \upsilon)^N(x) = \mu^N(x) \vee \upsilon^N(x).\end{aligned}$$

The bipolar fuzzy subsets $\mu \cup \upsilon$ and $\mu \cap \upsilon$ are called the *union and intersection of* μ *and* υ, respectively.

1.3.4 Definition

The *complement of a bipolar fuzzy subset* μ is denoted by $\bar{\mu}$ and is defined by

$$(\bar{\mu})^P(x) = 1 - \mu^P(x),\ (\bar{\mu})^N(x) = -1 - \mu^N(x)$$

for all $x \in X$.

Chapter 2

Soft Sets and Their Algebraic Structures

In this chapter we will present the basic concepts of soft set theory. Soft sets have received much attention in the last decade because of their applications in decision making problems. Molodstov [34] presented the concept of soft sets to deal with uncertain type of data under a parametrized environment which is rich enough to make approximations by incorporating the previous concepts like fuzzy sets, vague sets, interval valued fuzzy sets, intuitionistic fuzzy sets, rough sets, etc. Molodstov had given the concept of soft set and introductory ideas to apply in various fields while Maji et al. defined operations on soft sets in [32], [33]. Ali et al. [2] pointed out some practical mistakes in the definition of operations by Maji et al. and defined new operations introducing the concept of extended and restricted operations for soft sets. These operations not only enriched the theory but also proved this new structure deep enough to work for further structural investigations. This gives rise to our interest in the algebraic properties of a soft set's internal structure. So here we have made our first study. Firstly the definition of a soft set and various operations are given and then, we study some important properties associated with these operations. A collection of all soft sets with respect to new operations inspires to be checked out for various lattices and algebras. Going through different axiomatic requirements we figure out the algebraic structures of soft sets and finally, we show that soft sets with a fixed set of parameters are also MV algebras and BCK algebras.

2.1 Soft Sets

Let X be an initial universe and E be a set of parameters. Let $\mathcal{P}(X)$ denotes the power set of X and A, B be non-empty subsets of E.

2.1.1 Definition [34]

A pair (α,A) is called a *soft set* over X, where α is a mapping given by $\alpha : A \to \mathcal{P}(X)$. Therefore, a soft set over X gives a parametrized family of subsets of the universe X. For $e \in A$, $\alpha(e)$ may be considered as the set of $e-$approximate elements of X by the soft set (α,A). Clearly, a soft set is not a classical set. From now onwards, we shall use the notation A_α over X to denote a soft set (α,A) over X where the meanings of α, A and X are clear in a harmony with the use of usual pair notation.

2.1.2 Definition [12]

For two soft sets A_α and B_β over X, we say that A_α is a *soft subset* of B_β if

1) $A \subseteq B$ and

2) $\alpha(e) \subseteq \beta(e)$ for all $e \in A$.

We write $A_\alpha \tilde{\subseteq} B_\beta$.

A_α is said to be a *soft super set* of B_β, if B_β is a soft subset of A_α. We denote it by $A_\alpha \tilde{\supseteq} B_\beta$.

2.1.3 Definition [12]

Two soft sets A_α and B_β over X are said to be *soft equal* if A_α and B_β are soft subsets of each other. We denote it by $A_\alpha \tilde{=} B_\beta$.

2.1.4 Example

Let X be the set of cars under consideration, and E be the set of parameters of different features in cars, $X = \{c_1,c_2,c_3,c_4,c_5\}$, $E = \{e_1,e_2,e_3,e_4,e_5\} = \{$ Seat Heater, Automatic transmission, Sunroof, Leather Seats, Navigation System$\}$. Suppose that $A = \{e_1,e_2,e_3\}$, and $B = \{e_1,e_2\}$. A soft set A_α describing the " features of cars" which Mr. X is going to consider for buying is given as follows:

$$\begin{aligned} \alpha &: A \to \mathcal{P}(X), \\ e &\longmapsto \begin{cases} \{c_2, c_3, c_4\} & \text{if } e = e_1, \\ \{c_1, c_3, c_4\} & \text{if } e = e_2, \\ \{c_2, c_3, c_4, c_5\} & \text{if } e = e_3. \end{cases} \end{aligned}$$

And the soft set B_β given by

$$\begin{aligned} \beta &: B \to \mathcal{P}(X), \\ e &\longmapsto \begin{cases} \{c_3\} & \text{if } e = e_1, \\ \{c_1, c_3, c_4\} & \text{if } e = e_2, \end{cases} \end{aligned}$$

is a soft subset of A_α which represents another look by Mr. X on his earlier choices, so $B_\beta \tilde{\subseteq} A_\alpha$.

2.2 Operations on Soft Sets

Now, we give various operations on soft sets as defined in [4]. We have made little modifications to some notations just for the convenience of reader and in order to create a unanimity in the flow of this thesis.

2.2.1 Definition

Let A_α and B_β be two soft sets over X. Then the *or-product* of A_α and B_β is defined as a soft set $(A \times B)_{\alpha \tilde{\cup} \beta}$, where $\alpha \tilde{\cup} \beta : (A \times B) \to \mathcal{P}(X)$, defined by

$$(a, b) \mapsto \alpha(a) \cup \beta(b).$$

It is denoted by $A_\alpha \vee B_\beta \tilde{=} (A \times B)_{\alpha \tilde{\cup} \beta}$.

2.2.2 Definition

Let A_α and B_β be two soft sets over X. The *and-product* of A_α and B_β is defined as a soft set $(A \times B)_{\alpha \tilde{\cap} \beta}$, where $\alpha \tilde{\cap} \beta : (A \times B) \to \mathcal{P}(X)$, defined by

$$(a, b) \mapsto \alpha(a) \cap \beta(b).$$

It is denoted by $A_\alpha \wedge B_\beta \tilde{=} (A \times B)_{\alpha \tilde{\cap} \beta}$.

2.2.3 Definition

The *extended union of two soft sets* A_α and B_β over X is defined as a soft set $(A \cup B)_{\alpha \tilde{\cup} \beta}$, where $\alpha \tilde{\cup} \beta : (A \cup B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \begin{cases} \alpha(e) & if\ e \in A - B \\ \beta(e) & if\ e \in B - A \\ \alpha(e) \cup \beta(e) & if\ e \in A \cap B \end{cases}$$

We write $A_\alpha \sqcup_\varepsilon B_\beta \tilde{=} (A \cup B)_{\alpha \tilde{\cup} \beta}$.

2.2.4 Definition

The extended intersection of two soft sets A_α and B_β over X, is defined as a soft set $(A \cup B)_{\alpha \tilde{\cap} \beta}$ where, $\alpha \tilde{\cap} \beta : (A \cup B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \begin{cases} \alpha(e) & if \ e \in A - B \\ \beta(e) & if \ e \in B - A \\ \alpha(e) \cap \beta(e) & if \ e \in A \cap B \end{cases}$$

We write $A_\alpha \sqcap_\varepsilon B_\beta \tilde{=} (A \cup B)_{\alpha \tilde{\cap} \beta}$.

2.2.5 Definition

Let A_α and B_β be two soft sets over X such that $(A \cap B) \neq \emptyset$. Then the *restricted union* of A_α and B_β is defined as a soft set $(A \cap B)_{\alpha \tilde{\cup} \beta}$ where, $\alpha \tilde{\cup} \beta : (A \cap B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \alpha(e) \cup \beta(e).$$

We write $A_\alpha \sqcup B_\beta \tilde{=} (A \cap B)_{\alpha \tilde{\cup} \beta}$.

2.2.6 Definition

Let A_α and B_β be two soft sets over X such that $(A \cap B) \neq \emptyset$. Then the *restricted intersection* of A_α and B_β is defined as a soft set $(A \cap B)_{\alpha \tilde{\cap} \beta}$ where, $\alpha \tilde{\cap} \beta : A \cap B \to \mathcal{P}(X)$, defined by

$$e \mapsto \alpha(e) \cap \beta(e).$$

We write $A_\alpha \sqcap B_\beta \tilde{=} (A \cap B)_{\alpha \tilde{\cap} \beta}$.

2.2.7 Definition

The extended difference of two soft sets A_α and B_β over X, is defined as a soft set $(A \cup B)_{\alpha \smile_\varepsilon \beta}$ where, $\alpha \smile_\varepsilon \beta : (A \cup B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \begin{cases} \alpha(e) & if \ e \in A - B \\ \beta(e) & if \ e \in B - A \\ \alpha(e) - \beta(e) & if \ e \in A \cap B. \end{cases}$$

We write $A_\alpha \smile_\varepsilon B_\beta \tilde{=} (A \cup B)_{\alpha \smile_\varepsilon \beta}$.

2.2.8 Definition

Let A_α and B_β be two soft sets over X such that $A \cap B \neq \emptyset$. Then the *restricted difference* of A_α and B_β is defined as a soft set $(A \cap B)_{\alpha \smile \beta}$ where, $\alpha \smile \beta : A \cap B \to \mathcal{P}(X)$, defined by

$$e \mapsto \alpha(e) - \beta(e).$$

We write $A_\alpha \smile B_\beta \tilde{=} (A \cap B)_{\alpha \smile \beta}$.

2.2.9 Definition

The *complement of a soft set* A_α, denoted by $(A_\alpha)^c$ and defined as $(A_\alpha)^c \tilde{=} A_{\alpha^c}$ where, $\alpha^c : A \to \mathcal{P}(X)$ is defined by

$$e \mapsto X - \alpha(e).$$

Clearly $(\alpha^c)^c$ is same as α and $((A_\alpha)^c)^c = A_\alpha$.

2.2.10 Example

Let U be the set of houses under consideration, and E be the set of parameters, $U = \{h_1,h_2,h_3,h_4,h_5\}$, $E = \{e_1,e_2,e_3,e_4,e_5,e_6\} = \{$ in the green surroundings, wooden, cheap, in good repair, furnished, traditional $\}$. Suppose that $A = \{e_1,e_2\}$, and $B = \{e_2,e_3\}$. The soft sets A_α and B_β describe the "requirements of the houses" which Mr. X and Mr. Y are going to buy respectively and is given as follows:

$$\begin{aligned} \alpha \quad &: \quad A \to \mathcal{P}(X), \text{ defined by} \\ e \quad &\longmapsto \begin{cases} \{h_2, h_3\} & \text{if } e = e_1, \\ \{h_1, h_2, h_5\} & \text{if } e = e_2, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \beta \quad &: \quad B \to \mathcal{P}(X), \text{ defined by} \\ e \quad &\longmapsto \begin{cases} \{h_2, h_5\} & \text{if } e = e_2, \\ \{h_1, h_3, h_5\} & \text{if } e = e_3. \end{cases} \end{aligned}$$

Now, we approximate the resulting soft sets obtained by applying the above mentioned operations on A_α and B_β. We have

(i) $A_\alpha \vee B_\beta \tilde{=} (A \times B)_{\alpha \tilde{\cup} \beta}$, where

$$\begin{aligned} (\alpha \tilde{\cup} \beta) \quad &: \quad (A \times B) \to \mathcal{P}(X), \text{ defined by} \\ e \quad &\longmapsto \begin{cases} \{h_2, h_3, h_5\} & \text{if } e = (e_1, e_2), \\ \{h_1, h_2, h_3, h_5\} & \text{if } e = (e_1, e_3), \\ \{h_1, h_2, h_5\} & \text{if } e = (e_2, e_2), \\ \{h_1, h_2, h_3, h_5\} & \text{if } e = (e_2, e_3). \end{cases} \end{aligned}$$

(ii) $A_\alpha \wedge B_\beta \tilde{=} (A \times B)_{\alpha \tilde{\cap} \beta}$, where

$$\begin{aligned}(\alpha\tilde{\cap}\beta) &: (A \times B) \to \mathcal{P}(X), \text{ defined by} \\ e &\longmapsto \begin{cases} \{h_2\} & \text{if } e = (e_1, e_2), \\ \{h_3\} & \text{if } e = (e_1, e_3), \\ \{h_2, h_5\} & \text{if } e = (e_2, e_2), \\ \{h_1, h_5\} & \text{if } e = (e_2, e_3). \end{cases}\end{aligned}$$

(iii) $A_\alpha \sqcup_\varepsilon B_\beta \tilde{=} (A \cup B)_{\alpha \tilde{\cup} \beta}$, where

$$\begin{aligned}(\alpha\tilde{\cup}\beta) &: (A \cup B) \to \mathcal{P}(X), \text{ defined by} \\ e &\longmapsto \begin{cases} \{h_2, h_3\} & \text{if } e = e_1, \\ \{h_1, h_2, h_5\} & \text{if } e = e_2, \\ \{h_1, h_3, h_5\} & \text{if } e = e_3, \end{cases}\end{aligned}$$

(iv) $A_\alpha \sqcap_\varepsilon B_\beta \tilde{=} (A \cup B)_{\alpha \tilde{\cap} \beta}$, where

$$\begin{aligned}(\alpha\tilde{\cap}\beta) &: (A \cup B) \to \mathcal{P}(X), \text{ defined by} \\ e &\longmapsto \begin{cases} \{h_2, h_3\} & \text{if } e = e_1, \\ \{h_2, h_5\} & \text{if } e = e_2, \\ \{h_1, h_3, h_5\} & \text{if } e = e_3, \end{cases}\end{aligned}$$

(v) $A_\alpha \sqcup B_\beta \tilde{=} (A \cap B)_{\alpha \tilde{\cup} \beta}$, where

$$\begin{aligned}(\alpha\tilde{\cup}\beta) &: (A \cap B) \to \mathcal{P}(X), \text{ defined by} \\ e_2 &\longmapsto \{h_1, h_2, h_5\}\end{aligned}$$

(vi) $A_\alpha \sqcap B_\beta \tilde{=} (A \cap B)_{\alpha \tilde{\cap} \beta}$, where

$$\begin{aligned}(\alpha\tilde{\cap}\beta) &: (A \cap B) \to \mathcal{P}(X), \text{ defined by} \\ e_2 &\longmapsto \{h_2, h_5\}\end{aligned}$$

(vii) $A_\alpha \smile_\varepsilon B_\beta \tilde{=} (A \cup B)_{\alpha \smile_\varepsilon \beta}$, where

$$\begin{aligned}\alpha \smile_\varepsilon \beta &: (A \cup B) \to \mathcal{P}(X), \text{ defined by} \\ e &\longmapsto \begin{cases} \{h_2, h_3\} & \text{if } e = e_1, \\ \{h_1\} & \text{if } e = e_2, \\ \{h_1, h_3, h_5\} & \text{if } e = e_3, \end{cases}\end{aligned}$$

(ix) $A_\alpha \smile B_\beta \tilde{=} (A \cap B)_{\alpha \smile \beta}$, where

$$\begin{aligned}\alpha \smile \beta &: (A \cap B) \to \mathcal{P}(X), \text{ defined by} \\ e_2 &\longmapsto \{h_1\}\end{aligned}$$

(x) $(A_\alpha)^c = A_{\alpha^c}$ where

$$\begin{array}{rcl} \alpha^c & : & A \to \mathcal{P}(X), \text{ where} \\ e & \longmapsto & \begin{cases} \{h_1, h_4, h_5\} & \text{if } e = e_1, \\ \{h_3, h_4\} & \text{if } e = e_2. \end{cases} \end{array}$$

2.3 Properties of Soft Sets

In this section we discuss properties and laws of soft sets with respect to operations defined on soft sets. Later on these results are utilized for the configuration of algebraic structures of soft sets. The new idea of restricted and extended operations gives rise to some different results, for example, distributive laws do not hold in general for the operations of soft sets which is an entirely new aspect in a vague structure. Associativity, absorption, distributivity, de Morgan laws are investigated for soft set theory.

2.3.1 Definition

A soft set A_α over X is called a relative null soft set, denoted by A_Φ, if $\alpha(e) = \emptyset$ for all $e \in A$.

2.3.2 Definition

A soft set A_α over X is called a relative whole or *absolute soft set*, denoted by $A_{\mathfrak{X}}$, if $\alpha(e) = X$ for all $e \in A$.

Conventionally, we take soft sets with an empty set of parameters to be equal to $\emptyset_\Phi$ and so $A_\alpha \sqcap B_\beta \widetilde{=} \emptyset_\Phi \widetilde{=} A_\alpha \sqcup B_\beta$ when $A \cap B = \emptyset$.

2.3.3 Proposition

Let A_α, A_β be any soft sets over X. Then

1) $A_\alpha \sqcup_\varepsilon A_\beta \widetilde{=} A_\alpha \sqcup A_\beta$; $A_\alpha \sqcap_\varepsilon A_\beta \widetilde{=} A_\alpha \sqcap A_\beta$,

2) $A_\alpha \lambda A_\alpha \widetilde{=} A_\alpha$, for $\lambda \in \{\sqcup, \sqcap\}$, (Idempotent)

3) $A_\alpha \sqcap A_{\mathfrak{X}} \widetilde{=} A_\alpha \widetilde{=} A_\alpha \sqcup A_\Phi$,

4) $A_\alpha \sqcup A_{\mathfrak{X}} \widetilde{=} A_{\mathfrak{X}}$; $A_\alpha \sqcap A_\Phi \widetilde{=} A_\Phi$,

5) $A_\alpha \sqcap_\varepsilon \emptyset_\Phi \widetilde{=} A_\alpha \widetilde{=} A_\alpha \sqcup_\varepsilon \emptyset_\Phi \widetilde{=} A_\alpha \sqcap E_{\mathfrak{X}}$,

6) $A_\alpha \sqcap \emptyset_\Phi \widetilde{=} \emptyset_\Phi$; $A_\alpha \sqcup_\varepsilon E_{\mathfrak{X}} \widetilde{=} E_{\mathfrak{X}}$.

Proof. Straightforward. ∎

2.3.4 Proposition

Let A_α, B_β and C_γ be any soft sets over X. Then the following are true:

1) $A_\alpha \lambda (B_\beta \lambda C_\gamma) \tilde{=} (A_\alpha \lambda B_\beta) \lambda C_\gamma$, (Associative Laws)

2) $A_\alpha \lambda B_\beta \tilde{=} B_\beta \lambda A_\alpha$, (Commutative Laws)

for all $\lambda \in \{\sqcup_\varepsilon, \sqcup, \sqcap_\varepsilon, \sqcap\}$.

Proof. Straightforward. ■

2.3.5 Proposition (Absorption Laws)

Let A_α, B_β be any soft sets over X. Then the following are true:

1) $A_\alpha \sqcup_\varepsilon (B_\beta \sqcap A_\alpha) \tilde{=} A_\alpha$,

2) $A_\alpha \sqcap (B_\beta \sqcup_\varepsilon A_\alpha) \tilde{=} A_\alpha$,

3) $A_\alpha \sqcup (B_\beta \sqcap_\varepsilon A_\alpha) \tilde{=} A_\alpha$,

4) $A_\alpha \sqcap_\varepsilon (B_\beta \sqcup A_\alpha) \tilde{=} A_\alpha$.

Proof. Straightforward. ■

2.3.6 Proposition (Distributive Laws)

Let A_α, B_β and C_γ be any soft sets over X. Then

1) $A_\alpha \sqcap (B_\beta \sqcup_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcap B_\beta) \sqcup_\varepsilon (A_\alpha \sqcap C_\gamma)$,

2) $A_\alpha \sqcap (B_\beta \sqcap_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcap B_\beta) \sqcap_\varepsilon (A_\alpha \sqcap C_\gamma)$,

3) $A_\alpha \sqcap (B_\beta \sqcup C_\gamma) \tilde{=} (A_\alpha \sqcap B_\beta) \sqcup (A_\alpha \sqcap C_\gamma)$,

4) $A_\alpha \sqcup (B_\beta \sqcup_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcup B_\beta) \sqcup_\varepsilon (A_\alpha \sqcup C_\gamma)$,

5) $A_\alpha \sqcup (B_\beta \sqcap_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcup B_\beta) \sqcap_\varepsilon (A_\alpha \sqcup C_\gamma)$,

6) $A_\alpha \sqcup (B_\beta \sqcap C_\gamma) \tilde{=} (A_\alpha \sqcup B_\beta) \sqcap (A_\alpha \sqcup C_\gamma)$,

7) $A_\alpha \sqcap_\varepsilon (B_\beta \sqcup_\varepsilon C_\gamma) \tilde{\subseteq} (A_\alpha \sqcap_\varepsilon B_\beta) \sqcup_\varepsilon (A_\alpha \sqcap_\varepsilon C_\gamma)$,

8) $A_\alpha \sqcap_\varepsilon (B_\beta \sqcup C_\gamma) \tilde{=} (A_\alpha \sqcap_\varepsilon B_\beta) \sqcup (A_\alpha \sqcap_\varepsilon C_\gamma)$,

9) $A_\alpha \sqcap_\varepsilon (B_\beta \sqcap C_\gamma) \tilde{\supseteq} (A_\alpha \sqcap_\varepsilon B_\beta) \sqcap (A_\alpha \sqcap_\varepsilon C_\gamma)$,

10) $A_\alpha \sqcup_\varepsilon (B_\beta \sqcup C_\gamma) \tilde{\subseteq} (A_\alpha \sqcup_\varepsilon B_\beta) \sqcup (A_\alpha \sqcup_\varepsilon C_\gamma)$,

11) $A_\alpha \sqcup_\varepsilon (B_\beta \sqcap_\varepsilon C_\gamma) \tilde{\supseteq} (A_\alpha \sqcup_\varepsilon B_\beta) \sqcap_\varepsilon (A_\alpha \sqcup_\varepsilon C_\gamma)$,

12) $A_\alpha \sqcup_\varepsilon (B_\beta \sqcap C_\gamma) \tilde{=} (A_\alpha \sqcup_\varepsilon B_\beta) \sqcap (A_\alpha \sqcup_\varepsilon C_\gamma)$.

Proof. We prove only one part here, the other parts can be proved in a similar way.

1) We have

$$A_\alpha \sqcap (B_\beta \sqcup_\varepsilon C_\gamma) \tilde{=} (A \cap (B \cup C))_{\alpha \tilde{\cap} (\beta \tilde{\cup} \gamma)}$$

and

$$\begin{aligned} (A_\alpha \sqcap B_\beta) \sqcup_\varepsilon (A_\alpha \sqcap C_\gamma) &\tilde{=} (A \cap B)_{(\alpha \tilde{\cap} \beta)} \sqcup_\varepsilon (A \cap C)_{(\alpha \tilde{\cap} \gamma)} \\ &\tilde{=} ((A \cap B) \cup (A \cap C))_{(\alpha \tilde{\cap} \beta) \tilde{\cup} (\alpha \tilde{\cap} \gamma)} \\ &\tilde{=} (A \cap (B \cup C))_{(\alpha \tilde{\cap} \beta) \tilde{\cup} (\alpha \tilde{\cap} \gamma)}. \end{aligned}$$

Let $e \in A \cap (B \cup C)$. Then there can be one of three cases:

(i) If $e \in A \cap (B - C)$, then

$$\begin{aligned} (\beta \tilde{\cup} \gamma)(e) &= \beta(e) \quad \text{and} \\ \{\alpha \tilde{\cap} (\beta \tilde{\cup} \gamma)\}(e) &= \alpha(e) \cap \beta(e). \end{aligned}$$

Also $A \cap (B - C) = (A \cap B) - (A \cap C)$ and hence

$$\{(\alpha \tilde{\cap} \beta) \tilde{\cup} (\alpha \tilde{\cap} \gamma)\}(e) = (\alpha \tilde{\cap} \beta)(e) = \alpha(e) \cap \beta(e).$$

(ii) If $e \in A \cap (C - B)$, then

$$\begin{aligned} (\beta \tilde{\cup} \gamma)(e) &= \gamma(e) \quad \text{and} \\ \{\alpha \tilde{\cap} (\beta \tilde{\cup} \gamma)\}(e) &= \alpha(e) \cap \gamma(e). \end{aligned}$$

Also $A \cap (C - B) = (A \cap C) - (A \cap B)$ and hence

$$\{(\alpha \tilde{\cap} \beta) \tilde{\cup} (\alpha \tilde{\cap} \gamma)\}(e) = (\alpha \tilde{\cap} \gamma)(e) = \alpha(e) \cap \gamma(e).$$

(iii) If $e \in A \cap (B \cap C)$, then

$$\begin{aligned} (\beta \tilde{\cup} \gamma)(e) &= \beta(e) \cup \gamma(e) \quad \text{and} \\ \{\alpha \tilde{\cap} (\beta \tilde{\cup} \gamma)\}(e) &= \alpha(e) \cap (\beta(e) \cup \gamma(e)). \end{aligned}$$

Also $A \cap (B \cap C) = (A \cap B) \cap (A \cap C)$ and hence

$$\begin{aligned} & \{(\alpha \tilde{\cap} \beta) \tilde{\cup} (\alpha \tilde{\cap} \gamma)\}(e) \\ = \; & (\alpha \tilde{\cap} \beta)(e) \cup (\alpha \tilde{\cap} \gamma)(e) \\ = \; & (\alpha(e) \cap \beta(e)) \cup (\alpha(e) \cap \gamma(e)) \\ = \; & \alpha(e) \cap (\beta(e) \cup \gamma(e)). \end{aligned}$$

Thus

$$\alpha \tilde{\cap} (\beta \tilde{\cup} \gamma) = (\alpha \tilde{\cap} \beta) \tilde{\cup} (\alpha \tilde{\cap} \gamma)$$

and so

$$(A \cap (B \cup C))_{\alpha \tilde{\cap} (\beta \tilde{\cup} \gamma)} \tilde{=} (A \cap (B \cup C))_{(\alpha \tilde{\cap} \beta) \tilde{\cup} (\alpha \tilde{\cap} \gamma)}$$

Similarly we can prove the remaining parts.

■

2.3.7 Example

Let X be the set of sample designs and E be the set of available colors for dresses in a boutique,

$$X = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8\}$$

$$E = \{ \text{ Red, Green, Blue, Yellow, Black, White, Pink } \}$$

Suppose that

$$\begin{aligned} A &= \{\text{Red, Green, Blue, White}\}, \; B = \{\text{Green, Blue, Yellow, Black}\} \\ \text{and} \quad C &= \{\text{Blue, Yellow, White, Pink}\}. \end{aligned}$$

Let A_α,B_β and C_γ be the soft sets over X presenting the data record for three different boutiques respectively, given as follows:

$$\begin{aligned} \alpha(\text{Red}) &= \{S_1, S_2, S_3, S_4\}; \\ \alpha(\text{Green}) &= \{S_3, S_4, S_5, S_6\}; \\ \alpha(\text{Blue}) &= \{S_1, S_2, S_4, S_7\}; \\ \alpha(\text{White}) &= \{S_2, S_3, S_4\}. \end{aligned}$$

$$\begin{aligned} \beta(\text{Green}) &= \{S_4, S_5, S_6, S_8\}; \\ \beta(\text{Blue}) &= \{S_1, S_2, S_3, S_4\}; \\ \beta(\text{Yellow}) &= \{S_4, S_5, S_6, S_7, S_8\}; \\ \beta(\text{Black}) &= \{S_1, S_2, S_4, S_7\}. \end{aligned}$$

and

$$\begin{aligned}\gamma(\text{Blue}) &= \{S_3, S_4, S_7, S_8\};\\ \gamma(\text{Yellow}) &= \{S_4, S_5, S_7\};\\ \gamma(\text{White}) &= \{S_2, S_4, S_6, S_8\};\\ \gamma(\text{Pink }) &= \{S_2, S_3, S_5, S_7\}.\end{aligned}$$

Now

$$\begin{aligned}A_\alpha \sqcup_\varepsilon (B_\beta \sqcup C_\gamma) &\tilde{=} (A \cup (B \cap C))_{\alpha\tilde{\cup}(\beta\tilde{\cup}\gamma)};\\ (A_\alpha \sqcup_\varepsilon B_\beta) \sqcup (A_\alpha \sqcup_\varepsilon C_\gamma) &\tilde{=} ((A \cup B) \cap (A \cup C))_{(\alpha\tilde{\cup}\beta)\tilde{\cup}(\alpha\tilde{\cup}\gamma)};\\ A_\alpha \sqcup_\varepsilon (B_\beta \sqcap_\varepsilon C_\gamma) &\tilde{=} (A \cup (B \cup C))_{\alpha\tilde{\cup}(\beta\tilde{\cap}\gamma)};\\ (A_\alpha \sqcup_\varepsilon B_\beta) \sqcap_\varepsilon (A_\alpha \sqcup_\varepsilon C_\gamma) &\tilde{=} ((A \cup B) \cup (B \cup C))_{(\alpha\tilde{\cup}\beta)\tilde{\cap}(\alpha\tilde{\cup}\gamma)}.\end{aligned}$$

Then

$$\begin{aligned}(\alpha\tilde{\cup}(\beta\tilde{\cup}\gamma))(\text{Green}) &= \{S_3, S_4, S_5, S_6\};\\ (\alpha\tilde{\cup}(\beta\tilde{\cup}\gamma))(\text{White}) &= \{S_2, S_3, S_4\}.\end{aligned}$$

$$\begin{aligned}((\alpha\tilde{\cup}\beta)\tilde{\cup}(\alpha\tilde{\cup}\gamma))(\text{Green}) &= \{S_3, S_4, S_5, S_6, S_8\};\\ ((\alpha\tilde{\cup}\beta)\tilde{\cup}(\alpha\tilde{\cup}\gamma))(\text{White}) &= \{S_2, S_3, S_4, S_6, S_8\}.\end{aligned}$$

Thus

$$A_\alpha \sqcup_\varepsilon (B_\beta \sqcup C_\gamma) \tilde{\neq} (A_\alpha \sqcup_\varepsilon B_\beta) \sqcup (A_\alpha \sqcup_\varepsilon C_\gamma).$$

Similarly it can be shown that

$$A_\alpha \sqcap_\varepsilon (B_\beta \sqcap C_\gamma) \tilde{\neq} (A_\alpha \sqcap_\varepsilon B_\beta) \sqcap (A_\alpha \sqcap_\varepsilon C_\gamma).$$

Again, we see that

$$\begin{aligned}(\alpha\tilde{\cup}(\beta\tilde{\cap}\gamma))(\text{Green}) &= \{S_3, S_4, S_5, S_6, S_8\};\\ (\alpha\tilde{\cup}(\beta\tilde{\cap}\gamma))(\text{White}) &= \{S_2, S_3, S_4, S_6, S_8\}\end{aligned}$$

and

$$\begin{aligned}((\alpha\tilde{\cup}\beta)\tilde{\cap}(\alpha\tilde{\cup}\gamma))(\text{Green}) &= \{S_3, S_4, S_5, S_6\};\\ ((\alpha\tilde{\cup}\beta)\tilde{\cap}(\alpha\tilde{\cup}\gamma))(\text{White}) &= \{S_2, S_3, S_4\}.\end{aligned}$$

Thus

$$A_\alpha \sqcup_\varepsilon (B_\beta \sqcap_\varepsilon C_\gamma) \neq (A_\alpha \sqcup_\varepsilon B_\beta) \sqcap_\varepsilon (A_\alpha \sqcup_\varepsilon C_\gamma).$$

Similarly it can be shown that

$$A_\alpha \sqcap_\varepsilon (B_\beta \sqcup_\varepsilon C_\gamma) \neq (A_\alpha \sqcap_\varepsilon B_\beta) \sqcup_\varepsilon (A_\alpha \sqcap_\varepsilon C_\gamma).$$

2.3.8 Proposition

Let A_α, B_β and C_γ be any *soft sets* over X. Then

1)

$$A_\alpha \sqcup_\varepsilon (B_\beta \sqcap_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcup_\varepsilon B_\beta) \sqcap_\varepsilon (A_\alpha \sqcup_\varepsilon C_\gamma)$$

if and only if

$$\begin{aligned} \alpha(e) &\subseteq \beta(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ \alpha(e) &\subseteq \gamma(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

2)

$$A_\alpha \sqcap_\varepsilon (B_\beta \sqcup_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcap_\varepsilon B_\beta) \sqcup_\varepsilon (A_\alpha \sqcap_\varepsilon C_\gamma)$$

if and only if

$$\begin{aligned} \alpha(e) &\supseteq \beta(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ \alpha(e) &\supseteq \gamma(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

Proof. Straightforward. ■

2.3.9 Corollary

Let A_α, B_β and C_γ be any *soft sets* over X. Then
$A_\alpha \sqcup_\varepsilon (B_\beta \sqcap_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcup_\varepsilon B_\beta) \sqcap_\varepsilon (A_\alpha \sqcup_\varepsilon C_\gamma)$
$A_\alpha \sqcap_\varepsilon (B_\beta \sqcup_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcap_\varepsilon B_\beta) \sqcup_\varepsilon (A_\alpha \sqcap_\varepsilon C_\gamma)$
if and only if

$$\begin{aligned} \alpha(e) &= \beta(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ \alpha(e) &= \gamma(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

2.3.10 Corollary

Let A_α, B_β and C_γ be *any soft sets* over X such that $(A \cap B) - C = (A \cap C) - B = \emptyset$. Then

1) $A_\alpha \sqcup_\varepsilon (B_\beta \sqcap_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcup_\varepsilon B_\beta) \sqcap_\varepsilon (A_\alpha \sqcup_\varepsilon C_\gamma)$,

2) $A_\alpha \sqcap_\varepsilon (B_\beta \sqcup_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcap_\varepsilon B_\beta) \sqcup_\varepsilon (A_\alpha \sqcap_\varepsilon C_\gamma)$.

2.3.11 Corollary

Let A_α, A_β and A_γ be any *soft sets* over X. Then

$$A_\alpha\lambda(A_\beta\mu A_\gamma)\tilde{=}(A_\alpha\lambda A_\beta)\mu(A_\alpha\lambda A_\gamma)$$

for distinct λ, $\mu \in \{\sqcap_\varepsilon, \sqcap, \sqcup_\varepsilon, \sqcup\}$.

2.3.12 Theorem

Let A_α and B_β be two *soft sets* over X. Then the following are true

1) $A_\alpha \sqcup_\varepsilon B_\beta$ is the smallest soft set over X which contains both A_α and B_β. (Supremum)

2) $A_\alpha \sqcap B_\beta$ is the largest soft set over X which is contained in both A_α and B_β. (Infimum)

Proof.

1) We have $A, B \subseteq (A \cup B)$ and $\alpha(e), \beta(e) \subseteq \alpha(e) \cup \beta(e)$. So $A_\alpha \tilde{\subseteq} A_\alpha \sqcup_\varepsilon B_\beta$ and $B_\beta \tilde{\subseteq} A_\alpha \sqcup_\varepsilon B_\beta$. Let C_γ be a soft set over X, such that A_α, $B_\beta \tilde{\subseteq} C_\gamma$. Then $A, B \subseteq C$ implies that $(A \cup B) \subseteq C$ and $\alpha(e), \beta(e) \subseteq \gamma(e)$ implies that $\alpha(e) \cup \beta(e) \subseteq \gamma(e)$. Thus $A_\alpha \sqcup_\varepsilon B_\beta \tilde{\subseteq} C_\gamma$. It follows that $A_\alpha \sqcup_\varepsilon B_\beta$ is the smallest soft set over X which contains both A_α and B_β.

2) We have $A \cap B \subseteq A, A \cap B \subseteq B$ and $\alpha(e) \cap \beta(e) \subseteq \alpha(e), \alpha(e) \cap \beta(e) \subseteq \beta(e)$ for all $e \in A \cap B$. So $A_\alpha \sqcap B_\beta \tilde{\subseteq} A_\alpha$ and $A_\alpha \sqcap B_\beta \tilde{\subseteq} B_\beta$. Let C_γ be a soft set over X, such that $C_\gamma \tilde{\subseteq} A_\alpha$ and $C_\gamma \tilde{\subseteq} B_\beta$. Then $C \subseteq A, C \subseteq B$ imply that $C \subseteq A \cap B$ and $\gamma(e) \subseteq \alpha(e)$, $\gamma(e) \subseteq \beta(e)$ imply that $\gamma(e) \subseteq \alpha(e) \cap \beta(e)$ for all $e \in C$. Thus $C_\gamma \tilde{\subseteq} A_\alpha \sqcap B_\beta$. It follows that $A_\alpha \sqcap B_\beta$ is the largest soft set over X which is contained in both A_α and B_β.

■

2.4 Algebras of Soft Sets

In this section, we discuss lattices and algebras for the collections of soft sets. We consider certain collections of soft sets and find their distributive lattices. The concepts of involutions, complementations and atomicity are discussed. We denote the collections as follows:

$\mathcal{SS}(X)^E$: collection of all soft sets defined over X

$\mathcal{SS}(X)_A$: collection of all soft sets defined over X with a fixed parameter set A.

Firstly, we observe that these collections are partially ordered by the relation of soft inclusion $\tilde{\subseteq}$.

2.4.1 Proposition

The structures $(\mathcal{SS}(X)^E,\sqcap_\varepsilon,\sqcup)$, $(\mathcal{SS}(X)^E,\sqcup,\sqcap_\varepsilon)$, $(\mathcal{SS}(X)^E,\sqcup_\varepsilon,\sqcap)$, $(\mathcal{SS}(X)^E,\sqcap,\sqcup_\varepsilon)$, $(\mathcal{SS}(X)_A,\sqcup,\sqcap)$, and $(\mathcal{SS}(X)_A,\sqcap,\sqcup)$ are complete lattices.

Proof. Let us consider $(\mathcal{SS}(X)^E,\sqcap_\varepsilon,\sqcup)$. Then for any soft sets $A_\alpha,B_\beta,C_\gamma \in \mathcal{SS}(X)^E$,

1) We have $A_\alpha \sqcap_\varepsilon B_\beta \tilde{=} (A\cup B)_{\alpha\tilde{\cap}\beta} \in \mathcal{SS}(X)^E$ and $A_\alpha \sqcup B_\beta \tilde{=} (A\cap B)_{\alpha\tilde{\cup}\beta} \in \mathcal{SS}(X)^E$.

2) From Proposition 2.3.3, we have

$$A_\alpha \sqcap_\varepsilon A_\alpha \tilde{=} A_\alpha \text{ and } A_\alpha \sqcup A_\alpha \tilde{=} A_\alpha.$$

3) From Proposition 2.3.4 we see that

$$\begin{aligned} A_\alpha \sqcap_\varepsilon B_\beta &\tilde{=} B_\beta \sqcap_\varepsilon A_\alpha \text{ and} \\ A_\alpha \sqcup B_\beta &\tilde{=} B_\beta \sqcup A_\alpha. \end{aligned}$$

Also

$$\begin{aligned} A_\alpha \sqcap_\varepsilon (B_\beta \sqcap_\varepsilon C_\gamma) &\tilde{=} (A_\alpha \sqcap_\varepsilon B_\beta) \sqcap_\varepsilon C_\gamma \text{ and} \\ A_\alpha \sqcup (B_\beta \sqcup C_\gamma) &\tilde{=} (A_\alpha \sqcup B_\beta) \sqcup C_\gamma. \end{aligned}$$

4) From Proposition 2.3.5,

$$A_\alpha \sqcap_\varepsilon (B_\beta \sqcup A_\alpha) \tilde{=} A_\alpha \text{ and } A_\alpha \sqcup (B_\beta \sqcap_\varepsilon A_\alpha) \tilde{=} A_\alpha.$$

So we conclude that the structure forms a lattice.

Consider a collection of soft sets $\{A_{i_{\alpha_i}} : i \in I\}$ over X. We have, $\bigcup_{i\in I} A_i \subseteq E$ and, let $\Lambda(e) = \{j : e \in A_j\}$ for any $e \in A_i$. Then $\bigcap_{i\in\Lambda(e)} \alpha_i(e) \subseteq X$. Thus $\sqcap_{\varepsilon\, i\in I} A_{i_{\alpha_i}} \in \mathcal{SS}(X)^E$. Again, we have, $\bigcap_{i\in I} A_i \subseteq E$ and for any $e \in \bigcap_{i\in I} A_i$, $\bigcup_{i\in I} \alpha_i(e) \subseteq X$. Thus $\sqcup_{i\in I} A_{i_{\alpha_i}} \in \mathcal{SS}(X)^E$.

Similarly we can show the remaining structures. ■

2.4.2 Proposition

The structures $(\mathcal{SS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_\Phi, E_\mathfrak{X})$, $(\mathcal{SS}(X)^E, \sqcup_\varepsilon, \sqcap, E_\mathfrak{X}, \emptyset_\Phi)$, $(\mathcal{SS}(X)_A, \sqcap, \sqcup, A_\Phi, A_\mathfrak{X})$ and $(\mathcal{SS}(X)_A, \sqcup, \sqcap, A_\mathfrak{X}, A_\Phi)$ are bounded distributive lattices.

Proof. From Proposition 2.3.6, we have

$$\begin{aligned} A_\alpha \sqcap (B_\beta \sqcup_\varepsilon C_\gamma) &\tilde{=} (A_\alpha \sqcap B_\beta) \sqcup_\varepsilon (A_\alpha \sqcap C_\gamma) \\ A_\alpha \sqcup_\varepsilon (B_\beta \sqcap C_\gamma) &\tilde{=} (A_\alpha \sqcup_\varepsilon B_\beta) \sqcap (A_\alpha \sqcup_\varepsilon C_\gamma) \end{aligned}$$

for all $A_\alpha, B_\beta, C_\gamma \in \mathcal{SS}(X)^E$. So $(\mathcal{SS}(X)^E, \sqcap, \sqcup_\varepsilon)$ and $(\mathcal{SS}(X)^E, \sqcup_\varepsilon, \sqcap)$ are distributive lattices. From Theorem 2.3.12, we conclude that $(\mathcal{SS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_\Phi, E_\mathfrak{X})$ is a bounded distributive lattice and $(\mathcal{SS}(X)^E, \sqcup_\varepsilon, \sqcap, E_\mathfrak{X}, \emptyset_\Phi)$ is its dual.

Now, for any soft sets $A_\alpha, A_\beta \in \mathcal{SS}(X)_A$,

$$\begin{aligned} A_\alpha \sqcap A_\beta &\tilde{=} A_{\alpha \tilde{\cap} \beta} \in \mathcal{SS}(X)_A \text{ and} \\ A_\alpha \sqcup A_\beta &\tilde{=} A_{\alpha \tilde{\cup} \beta} \in \mathcal{SS}(X)_A. \end{aligned}$$

Thus $(\mathcal{SS}(X)_A, \sqcap, \sqcup)$ is a distributive sublattice of $(\mathcal{SS}(X)^E, \sqcup_\varepsilon, \sqcap)$. Proposition 2.3.3 tells us that $A_\Phi, A_\mathfrak{X}$ are its lower and upper bounds respectively. Therefore

$(\mathcal{SS}(X)_A, \sqcap, \sqcup, A_\Phi, A_\mathfrak{X})$ is a bounded distributive lattice and $(\mathcal{SS}(X)_A, \sqcup, \sqcap, A_\mathfrak{X}, A_\Phi)$ is its dual. ■

2.4.3 Proposition

Let A_α be a *soft set* over X. Then A_{α^c} is a complement of A_α.

Proof. As $A_\alpha \sqcup A_{\alpha^c} \tilde{=} A_{(\alpha \tilde{\cup} \alpha^c)}$ so, for any $e \in A$,

$$(\alpha \tilde{\cup} \alpha^c)(e) = \alpha(e) \cup (\alpha(e))^c = X.$$

Thus $A_\alpha \sqcup A_{\alpha^c} \tilde{=} A_\mathfrak{X}$.

Also $A_\alpha \sqcap A_{\alpha^c} \tilde{=} A_{(\alpha \tilde{\cap} \alpha^c)}$, so

$$(\alpha \tilde{\cap} \alpha^c)(e) = \alpha(e) \cap (\alpha(e))^c = \emptyset.$$

Thus $A_\alpha \sqcap A_{\alpha^c} \tilde{=} A_\Phi$.

Now, we show that A_{α^c} is unique in the bounded lattice $(\mathcal{SS}(X)_A, \sqcup, \sqcap, A_\mathfrak{X}, A_\Phi)$. If there exists some $A_\beta \in \mathcal{SS}(X)_A$ such that $A_\alpha \sqcup A_\beta \tilde{=} A_\mathfrak{X}$ and $A_\alpha \sqcap A_\beta \tilde{=} A_\Phi$. For any $e \in A$,

$$\begin{aligned} &\alpha(e) \cap \beta(e) = \emptyset \\ &\Rightarrow \beta(e) \subseteq (\alpha(e))^c = \alpha^c(e) \end{aligned}$$

and

$$\alpha^c(e) \subseteq X = \alpha(e) \cup \beta(e).$$

But

$$\alpha(e) \cap \alpha^c(e) = \emptyset \text{ and so } \alpha^c(e) \subseteq \alpha(e) \cup \beta(e) \Rightarrow \alpha^c(e) \subseteq \beta(e).$$

Therefore

$$\beta(e) = \alpha^c(e) \text{ for all } e \in A \text{ and } A_\beta \tilde{=} A_{\alpha^c}.$$

Hence A_{α^c} is a complement of A_α. ■

2.4.4 Remark

We see that $(\mathcal{SS}(X)_A, \sqcap, \sqcup, A_\Phi, A_\mathfrak{X})$ and $(\mathcal{SS}(X)_A, \sqcup, \sqcap, A_\mathfrak{X}, A_\Phi)$ are dual lattices so all the properties and structural configurations hold dually in an understood manner.

2.4.5 Proposition (de Morgan Laws)

Let A_α and B_β be any soft sets over X. Then the following are true

1) $(A_\alpha \sqcup_\varepsilon B_\beta)^c \tilde{=} A_{\alpha^c} \sqcap_\varepsilon B_{\beta^c}$,

2) $(A_\alpha \sqcap_\varepsilon B_\beta)^c \tilde{=} A_{\alpha^c} \sqcup_\varepsilon B_{\beta^c}$,

3) $(A_\alpha \vee B_\beta)^c \tilde{=} A_{\alpha^c} \wedge B_{\beta^c}$,

4) $(A_\alpha \wedge B_\beta)^c \tilde{=} A_{\alpha^c} \vee B_{\beta^c}$,

5) $(A_\alpha \sqcup B_\beta)^c \tilde{=} A_{\alpha^c} \sqcap B_{\beta^c}$,

6) $(A_\alpha \sqcap B_\beta)^c \tilde{=} A_{\alpha^c} \sqcup B_{\beta^c}$.

Proof. We know that $(A_\alpha \sqcup_\varepsilon B_\beta)^c \tilde{=} ((A \cup B)_{\alpha \tilde{\cup} \beta})^c \tilde{=} (A \cup B)_{(\alpha \tilde{\cup} \beta)^c}$. Let $e \in (A \cup B)$. Then there are three cases:

(i) If $e \in A - B$, then

$$((\alpha \tilde{\cup} \beta)^c)(e) = (\alpha(e))^c = \alpha^c(e) \text{ and } (\alpha^c \tilde{\cap} \beta^c)(e) = \alpha^c(e).$$

(ii) If $e \in B - A$, then

$$(\alpha \tilde{\cup} \beta)^c(e) = (\beta(e))^c = \beta^c(e) \text{ and } (\alpha^c \tilde{\cap} \beta^c)(e) = \beta^c(e).$$

(iii) If $e \in A \cap B$, then

$$(\alpha\tilde{\cup}\beta)^c(e) = (\alpha(e) \cup \beta(e))^c = (\alpha(e))^c \cap (\beta(e))^c$$

and,

$$(\alpha^c\tilde{\cap}\beta^c)(e) = (\alpha(e))^c \cap (\beta(e))^c.$$

Therefore, in all the cases we obtain equality and thus

$$(A_\alpha \sqcup_\varepsilon B_\beta)^c \tilde{=} A_{\alpha^c} \sqcap_\varepsilon B_{\beta^c}.$$

The remaining parts can be proved in a similar way. ■

2.4.6 Proposition

$(\mathcal{SS}(X)_A, \sqcap, \sqcup, ^c, A_\Phi, A_\mathfrak{X})$ is a de Morgan algebra.

Proof. We have already seen that $(\mathcal{SS}(X)_A, \sqcap, \sqcup, A_\Phi, A_\mathfrak{X})$ is a bounded distributive lattice. Propositions 2.4.3 and 2.4.5 show that de Morgan laws hold with respect to " ^{c}" in $\mathcal{SS}(X)_A$. Thus $(\mathcal{SS}(X)_A, \sqcap, \sqcup, ^c, A_\Phi, A_\mathfrak{X})$ is a de Morgan algebra. ■

2.4.7 Proposition

$(\mathcal{SS}(X)_A, \sqcap, \sqcup, ^c, A_\Phi, A_\mathfrak{X})$ is a boolean algebra.

Proof. Follows from Propositions 2.4.2 and 2.4.3. ■

2.4.8 Proposition

Let A_α and A_β be any soft sets over X. Then $(A_\beta \sqcap A_{\beta^c}) \tilde{\subseteq} (A_\alpha \sqcup A_{\alpha^c})$ and so $(\mathcal{SS}(X)_A, \sqcap, \sqcup, ^c, A_\Phi, A_\mathfrak{X})$ is a Kleene Algebra.

Proof. We have,

$$A_\beta \sqcap A_{\beta^c} \tilde{=} A_\Phi \tilde{\subseteq} A_\mathfrak{X} \tilde{=} A_\alpha \sqcup A_{\alpha^c}$$

for all $A_\alpha, A_\beta \in \mathcal{SS}(X)_A$. We already know that $(\mathcal{SS}(X)_A, \sqcap, \sqcup, ^c, A_\Phi, A_\mathfrak{X})$ is a de Morgan algebra, so this condition assures that $(\mathcal{SS}(X)_A, \sqcap, \sqcup, ^c, A_\Phi, A_\mathfrak{X})$ is a Kleene Algebra. ■

2.4.9 Lemma

For any $x \in X$ and $A \subseteq E$. We define a soft set A_{e_x} for each $e \in A$, where $e_x : A \to \mathcal{P}(X)$ such that

$$e_x(\acute{e}) = \begin{cases} \{x\} & \text{if } \acute{e} = e \\ \emptyset & \text{if } \acute{e} \neq e \end{cases}.$$

Then A_{e_x} is an atom of lattice $(\mathcal{SS}(X)_A, \sqcap, \sqcup)$ for each $e \in A$ and $x \in X$ and we have

$$\mathcal{A}(\mathcal{SS}(X)_A) = \{A_{e_x} : e \in E \text{ and } x \in X\}.$$

Proof. Let $A_\Phi \tilde{\neq} A_\alpha \in \mathcal{SS}(X)_A$ such that $A_\alpha \tilde{\subseteq} A_{e_x}$. Then $\alpha(e) \subseteq e_x(e) = \{x\}$ and $\alpha(e') \subseteq \emptyset$ for all $(e \neq) e' \in A$. This implies that $\alpha(e') = \emptyset$ for all $(e \neq) e' \in A$ and the only possibility for $\alpha(e)$ is $\{x\}$ because $A_\Phi \tilde{\neq} A_\alpha$. Thus $A_\alpha \tilde{=} A_{e_x}$ proves that $A_{e_x} \in \mathcal{A}(\mathcal{SS}(X)_A)$. ■

2.4.10 Proposition

$(\mathcal{SS}(X)_A, \sqcap, \sqcup)$ is an atomic lattice.

Proof. Let $A_\alpha \in \mathcal{SS}(X)_A$, and take

$$\mathcal{I}_A = \{A_{e_x} \in \mathcal{A}(\mathcal{SS}(X)_A) : A_{e_x} \tilde{\subseteq} A_\alpha\}$$

the subcollection of $\mathcal{A}(\mathcal{SS}(X)_A)$ which is given in Lemma 2.4.9. Suppose that

$$A_\beta \tilde{=} \bigvee \mathcal{I}_A.$$

For any $e \in A$, $\beta(e) = \bigcup_{x \in \alpha(e)} e_x(e) = \bigcup_{x \in \alpha(e)} \{x\} = \alpha(e)$. Thus $\bigvee \mathcal{I}_A \tilde{=} A_\alpha$ and hence $(\mathcal{SS}(X)_A, \sqcap, \sqcup)$ is an atomic lattice. ■

2.4.11 Lemma

Let $A_\alpha, B_\beta \in \mathcal{SS}(X)^E$. Then the pseudocomplement of A_α relative to B_β exists in $\mathcal{SS}(X)^E$.

Proof. Consider the set

$$T(A_\alpha, B_\beta) = \{C_\gamma \in \mathcal{SS}(X)^E : C_\gamma \sqcap A_\alpha \tilde{\subseteq} B_\beta\}.$$

We define a soft set $A^c_{\alpha^c} \sqcup_\varepsilon B_\beta \tilde{=} (A^c \cup B)_{\alpha^c \tilde{\cup} \beta} \in \mathcal{SS}(X)^E$ and claim that $A_\alpha \to B_\beta \tilde{=} (A^c \cup B)_{\alpha^c \tilde{\cup} \beta}$. First of all we show that $(A^c \cup B)_{\alpha^c \tilde{\cup} \beta} \in T(A_\alpha, B_\beta)$. Consider

$$\begin{aligned} (A^c \cup B)_{\alpha^c \tilde{\cup} \beta} \sqcap A_\alpha &\tilde{=} ((A^c \cup B) \cap A)_{(\alpha^c \tilde{\cup} \beta) \tilde{\cap} \alpha} && \text{(By distributive law)} \\ &\tilde{=} ((A^c \cap A) \cup (B \cap A))_{(\alpha^c \tilde{\cap} \alpha) \tilde{\cup} (\beta \tilde{\cap} \alpha)} \\ &\tilde{=} (A \cap B)_{\alpha \tilde{\cap} \beta} \tilde{\subseteq} B_\beta. \end{aligned}$$

Thus $(A^c \cup B)_{\alpha^c \tilde{\cup} \beta} \in T(A_\alpha, B_\beta)$. For any $C_\gamma \in T(A_\alpha, B_\beta)$, we have $C_\gamma \sqcap A_\alpha \tilde{\subseteq} B_\beta$ so for any $e \in C \cap A \subseteq B$

$$\gamma(e) \cap \alpha(e) \subseteq \beta(e).$$

Now,

$$\begin{aligned} C\cap A &\subseteq B \Rightarrow (A\cap C)\cap B^c=\emptyset \\ &\Rightarrow C\subseteq (A\cap B^c)^c = A^c\cup B \end{aligned}$$

and

$$\begin{aligned} \gamma(e)\cap\alpha(e) &\subseteq \beta(e) \Rightarrow (\gamma(e)\cap\alpha(e))\cap\beta^c(e)=\emptyset \\ &\Rightarrow \gamma(e)\subseteq(\alpha(e))^c\cap\beta(e)=\alpha^c(e)\cap\beta(e) \end{aligned}$$

Thus $C_\gamma\tilde{\subseteq}(A^c\cup B)_{\alpha^c\tilde{\cup}\beta}$ and it also shows that

$$(A^c\cup B)_{\alpha^c\tilde{\cup}\beta}\tilde{=}\bigvee T(A_\alpha,B_\beta)\tilde{=}A_\alpha\to B_\beta.$$

■

2.4.12 Remark

We know that $(\mathcal{SS}(X)_A, \sqcap, \sqcup)$ is a sublattice of $(\mathcal{SS}(X)^E, \sqcap_\varepsilon, \sqcup)$. *For any* A_α, $A_\beta\in\mathcal{SS}(X)_A$, $A_\alpha\to A_\beta$ as defined in Lemma 2.4.11, is not in $\mathcal{SS}(X)_A$ because $A_\alpha\to A_\beta\tilde{=}(A^c\cup A)_{\alpha^c\tilde{\cup}\beta}\tilde{=}E_{\alpha^c\tilde{\cup}\beta}\notin\mathcal{SS}(X)_A$.

2.4.13 Lemma

Let $A_\alpha, A_\beta\in\mathcal{SS}(X)_A$. Then pseudocomplement of A_α relative to A_β exists in $\mathcal{SS}(X)^A$.

Proof. Consider the set

$$T(A_\alpha,A_\beta)=\{A_\gamma\in\mathcal{SS}(X)_A : A_\gamma\sqcap A_\alpha\tilde{\subseteq}A_\beta\}.$$

We define a soft set $A_{\alpha^c}\sqcup A_\beta\tilde{=}A_{\alpha^c\tilde{\cup}\beta}\in\mathcal{SS}(X)_A$. Consider

$$\begin{aligned} A_{\alpha^c\tilde{\cup}\beta}\sqcap A_\alpha &\tilde{=} A_{(\alpha^c\tilde{\cup}\beta)\tilde{\cap}\alpha} \\ &\tilde{=} A_{(\alpha^c\tilde{\cap}\alpha)\tilde{\cup}(\beta\tilde{\cap}\alpha)} \\ &\tilde{=} A_{\alpha\tilde{\cap}\beta}\tilde{\subseteq}A_\beta. \end{aligned}$$

Thus $A_{\alpha^c\tilde{\cup}\beta}\in T(A_\alpha,A_\beta)$. For every $A_\gamma\in T(A_\alpha,A_\beta)$, we have $A_\gamma\sqcap A_\alpha\tilde{\subseteq}A_\beta$ so for any $e\in A$,

$$\begin{aligned} \gamma(e)\cap\alpha(e) &\subseteq \beta(e) \Rightarrow (\gamma(e)\cap\alpha(e))\cap\beta^c(e)=\emptyset \\ &\Rightarrow \gamma(e)\subseteq(\alpha(e))^c\cap\beta(e)=\alpha^c(e)\cap\beta(e) \end{aligned}$$

Thus $A_\gamma\tilde{\subseteq}A_{\alpha^c\tilde{\cup}\beta}$ and it also shows that

$$A_{\alpha^c\tilde{\cup}\beta}\tilde{=}\bigvee T(A_\alpha,A_\beta)\tilde{=}A_\alpha\to_A A_\beta.$$

■

2.4.14 Proposition

$(\mathcal{SS}(X)^E,\sqcap_\varepsilon,\sqcup)$ *and* $(\mathcal{SS}(X)_A,\sqcap,\sqcup)$ are *Brouwerian lattices.*

Proof. Follows from Lemmas 2.4.11 and 2.4.13. ∎

2.4.15 Theorem

$(\mathcal{SS}(X)_A,\sqcap,^c,A_{\mathfrak{X}})$ is an MV-algebra.

Proof. $MV1$, $MV2$ and $MV3$ are straightforward. We prove $MV4$:

$$\begin{aligned}(A_{\alpha^c}\sqcap A_\beta)^c\sqcap A_\beta &\tilde{=} ((A_{\alpha^c})^c\sqcup A_{\beta^c})\sqcap A_\beta\\ &\tilde{=} (A_\alpha\sqcup A_{\beta^c})\sqcap A_\beta\\ &\tilde{=} (A_\alpha\sqcap A_\beta)\sqcup(A_{\beta^c}\sqcap A_\beta)\\ &\tilde{=} (A_\alpha\sqcap A_\beta)\sqcup A_\Phi\\ &\tilde{=} (A_\beta\sqcap A_\alpha)\sqcup(A_{\alpha^c}\sqcap A_\alpha)\\ &\tilde{=} (A_\beta\sqcup A_{\alpha^c})\sqcap A_\alpha\\ &\tilde{=} (A_{\beta^c}\sqcap A_\alpha)^c\sqcap A_\alpha.\end{aligned}$$

for all $A_\alpha,\ A_\beta\in\mathcal{SS}(X)_A$. Thus $(\mathcal{SS}(X)_A,\sqcap,^c,A_{\mathfrak{X}}\rangle$ is an MV-algebra. ∎

2.4.16 Theorem

$(\mathcal{SS}(X)_A,\sqcup,^c,A_\Phi)$ is an MV-algebra.

Proof. $MV1$, $MV2$ and $MV3$ are straightforward. We prove $MV4$:

$$\begin{aligned}(A_{\alpha^c}\sqcup A_\beta)^c\sqcup A_\beta &\tilde{=} ((A_{\alpha^c})^c\sqcap A_{\beta^c})\sqcup A_\beta\\ &\tilde{=} (A_\alpha\sqcap A_{\beta^c})\sqcup A_\beta\\ &\tilde{=} (A_\alpha\sqcup A_\beta)\sqcap(A_{\beta^c}\sqcup A_\beta)\\ &\tilde{=} (A_\alpha\sqcup A_\beta)\sqcap A_{\mathfrak{X}}\\ &\tilde{=} (A_\beta\sqcup A_\alpha)\sqcap(A_{\alpha^c}\sqcup A_\alpha)\\ &\tilde{=} (A_\beta\sqcap A_{\alpha^c})\sqcup A_\alpha\\ &\tilde{=} (A_{\beta^c}\sqcup A_\alpha)^c\sqcup A_\alpha.\end{aligned}$$

for all $A_\alpha,\ A_\beta\in\mathcal{SS}(X)_A$. Thus $(\mathcal{SS}(X)_A,\sqcup,^c,A_\Phi)$ is an MV-algebra. ∎

2.4.17 Theorem

$(\mathcal{SS}(X)_A,\smile,A_\Phi)$ is a bounded BCK-algebra whose every element is an involution.

Proof. For any $A_\alpha,\ A_\beta,\ A_\gamma\in\mathcal{SS}(X)_A$

BCI-1 $((A_\alpha \smile A_\beta) \smile (A_\alpha \smile A_\gamma)) \smile (A_\gamma \smile A_\beta)$

$\tilde{=}(A_{\alpha\smile\beta} \smile A_{\alpha\smile\gamma}) \smile A_{\gamma\smile\beta}$

$\tilde{=}A_{(\alpha\smile\beta)\smile(\alpha\smile\gamma)} \smile A_{\gamma\smile\beta}$

$\tilde{=}A_\Phi \smile A_{\gamma\smile\beta}\tilde{=}A_\Phi$.

BCI-2 $(A_\alpha \smile (A_\alpha \smile A_\beta)) \smile A_\beta$

$\tilde{=}(A_\alpha \smile A_{\alpha\smile\beta}) \smile A_\beta$

$\tilde{=}A_{(\alpha\smile(\alpha\smile\beta)} \smile A_\beta$

$\tilde{=}A_\Phi \smile A_\beta\tilde{=}A_{\Phi\smile\beta}\tilde{=}A_\Phi$.

BCI-3 $A_\alpha \smile A_\alpha\tilde{=}A_\Phi$.

BCI-4 Let $A_\alpha \smile A_\beta\tilde{=}A_\Phi$ and $A_\beta \smile A_\alpha\tilde{=}A_\Phi$. For any $e \in A$,

$$\alpha(e) - \beta(e) = \emptyset \text{ and } \beta(e) - \alpha(e) = \emptyset \text{ imply that } \alpha(e) = \beta(e).$$

Hence $A_\alpha\tilde{=}A_\beta$.

BCK-5 $A_\Phi \smile A_\alpha\tilde{=}A_{\Phi\smile\alpha}\tilde{=}A_\Phi$.

Thus $(\mathcal{SS}(X)_A,\smile,A_\Phi)$ is a BCK-algebra. Now $A_\mathfrak{X} \in \mathcal{SS}(X)_A$ is such that:

$$A_\alpha \smile A_\mathfrak{X}\tilde{=}A_{\alpha\smile\mathfrak{X}}\tilde{=}A_\Phi \text{ for all } A_\alpha \in \mathcal{SS}(X)_A.$$

Therefore $(\mathcal{SS}(X)_A,\smile,A_\Phi)$ is a bounded BCK-algebra.

For any $A_\alpha \in \mathcal{SS}(X)_A$,

$$A_\mathfrak{X} \smile (A_\mathfrak{X} \smile A_\alpha)\tilde{=}A_\mathfrak{X} \smile A_{\mathfrak{X}\smile\alpha}\tilde{=}A_\mathfrak{X} \smile A_{\alpha^c}\tilde{=}A_{\mathfrak{X}\smile\alpha^c}\tilde{=}A_{(\alpha^c)^c}\tilde{=}A_\alpha.$$

So every element of $\mathcal{SS}(X)_A$ is an involution. ■

2.4.18 Definition

Let A_α and A_β be any soft sets over X. We define

$$A_\alpha \star A_\beta\tilde{=}A_{\alpha\star\beta}\tilde{=}A_\alpha \sqcap A_{\beta^c}.$$

2.4.19 Theorem

$(\mathcal{SS}(X)_A,\star,A_\Phi)$ is a bounded BCK-algebra whose every element is an involution.

Proof. For any A_α, A_β, $A_\gamma \in \mathcal{SS}(X)_A$.

BCI-1 $((A_\alpha \star A_\beta) \star (A_\alpha \star A_\gamma)) \star (A_\gamma \star A_\beta)$

$\tilde{=}(A_{\alpha\star\beta} \star A_{\alpha\star\gamma}) \star A_{\gamma\star\beta}$

$\tilde{=}A_{((\alpha\star\beta)\star(\alpha\star\gamma))\star(\gamma\star\beta)}$

$\tilde{=}A_{((\alpha\tilde{\cap}\beta^c)\star(\alpha\tilde{\cap}\gamma^c))\star(\gamma\tilde{\cap}\beta^c)}$

$\tilde{=}A_{((\alpha\tilde{\cap}\beta^c)\tilde{\cap}(\alpha\tilde{\cap}\gamma^c)^c)\tilde{\cap}(\gamma\tilde{\cap}\beta^c)^c}$

$\tilde{=}A_{((\alpha\tilde{\cap}\beta^c)\tilde{\cap}(\alpha^c\tilde{\cup}\gamma))\tilde{\cap}(\gamma^c\tilde{\cup}\beta)}$

$\tilde{=}A_{((\alpha\tilde{\cap}\beta^c)\tilde{\cap}\gamma)\tilde{\cap}(\gamma^c\tilde{\cup}\beta)}$

$\tilde{=}A_{((\alpha\tilde{\cap}\beta^c)\tilde{\cap}\gamma)\tilde{\cap}\beta}$

$\tilde{=}A_{(\alpha\tilde{\cap}\gamma)\tilde{\cap}(\beta^c\tilde{\cap}\beta)}\tilde{=}A_\Phi$.

BCI-2 $(A_\alpha \star (A_\alpha \star A_\beta)) \star A_\beta$

$\tilde{=}(A_\alpha \star A_{\alpha\star\beta}) \star A_\beta$

$\tilde{=}A_{\alpha\star(\alpha\star\beta)} \star A_\beta$

$\tilde{=}A_{\alpha\tilde{\cap}(\alpha\tilde{\cap}\beta^c)^c} \star A_\beta$

$\tilde{=}A_{(\alpha\tilde{\cap}(\alpha^c\tilde{\cup}\beta)} \star A_\beta$

$\tilde{=}A_{\alpha\tilde{\cap}\beta} \star A_\beta\tilde{=}A_{(\alpha\tilde{\cap}\beta)\tilde{\cap}\beta^c}\tilde{=}A_\Phi$.

BCI-3 $A_\beta \star A_\beta\tilde{=}A_{\beta\tilde{\cap}\beta^c}\tilde{=}A_\Phi$.

BCI-4 Let $A_\alpha \star A_\beta\tilde{=}A_\Phi$ and $A_\beta \star A_\alpha\tilde{=}A_\Phi$. For any $e \in A$,

$$\alpha(e) \cap (\beta(e))^c = \emptyset \text{ and } \beta(e) \cap (\alpha(e))^c = \emptyset \text{ imply that } \alpha(e) = \beta(e).$$

Hence

$$A_\alpha\tilde{=}A_\beta.$$

BCK-5 $A_\Phi \star A_\alpha\tilde{=}A_{\Phi\star\alpha}\tilde{=}A_{\Phi\tilde{\cap}\alpha^c}\tilde{=}A_\Phi$.

Thus $(\mathcal{SS}(X)_A,\star,A_\Phi)$ is a BCK-algebra. Now $A_\mathfrak{X} \in \mathcal{SS}(X)_A$ is such that:

$$A_\beta \star A_\mathfrak{X}\tilde{=}A_{\alpha\star\mathfrak{X}}\tilde{=}A_{\alpha\tilde{\cap}\mathfrak{X}^c}\tilde{=}A_{\alpha\tilde{\cap}\Phi}\tilde{=}A_\Phi \quad \text{for all } A_\alpha \in \mathcal{SS}(X)_A.$$

Therefore $(\mathcal{SS}(X)_A,\star,A_\Phi)$ is a bounded BCK-algebra. ■

Chapter 3

Algebraic Structures of Fuzzy Soft Sets

In 2001, Maji and Roy proposed the concept of Fuzzy Soft Set in [30]. Different algebraic structures have also been studied in fuzzy soft context. Irfan et al. [3] pointed out some basic problems in the results related to the operations defined on fuzzy soft sets. In the paper [3], some new operations are defined for fuzzy soft sets and modified results and laws are established. In this chapter, we step forward in the same direction and check out the associativity and distributivity of these operations. First we have given preliminaries of fuzzy soft sets. We have used new and modified definitions and operations from [3] to discuss the properties of these operations on fuzzy soft sets. After accomplishing an account of algebraic properties of fuzzy soft sets, the overall algebraic structures of collections of fuzzy soft sets are studied. The two types of collections of fuzzy soft sets, one consisting of those fuzzy soft sets with a fixed set of parameters while the other containing fuzzy soft sets defined over the same universe with different set of parameters are taken into account. Both collections have some common and some different algebraic properties and therefore the algebraic structures also differ. The lattice structure of these collections is discussed and we find that the collection of all fuzzy soft sets is a bounded distributive lattice and the collection of fuzzy soft sets with a fixed set of parameters becomes a Kleene algebra. At the end we define pseudocomplement of a fuzzy soft set and with this pseudocomplement, this collection becomes a stone algebra.

3.1 Fuzzy Soft Sets

Let X be an initial universe and E be a set of parameters. Let $\mathcal{FP}(X)$ denotes the fuzzy power set of X and A, B be non-empty subsets of E.

3.1.1 Definition [30]

A pair (f,A) is called a *fuzzy soft set over* X, where f is a mapping given by $f : A \to \mathcal{FP}(X)$.

Therefore, a fuzzy soft set over X gives a parametrized family of fuzzy subsets of the universe X. For $e \in A$, $f(e)$ may be considered as the set of $e-$approximate fuzzy elements of X. From now onwards, we shall use the notation A_f over X to denote a fuzzy soft set (f,A) over X where the meanings of f, A and X are clear in a harmony with the use of usual pair notation.

3.1.2 Definition [3]

For two fuzzy soft sets A_f and B_g over a common universe X, we say that A_f is a *fuzzy soft subset* of B_g if

1) $A \subseteq B$ and

2) $f(e) \subseteq g(e)$ for all $e \in A$.

We write $A_f \tilde{\subseteq} B_g$. A_f is said to be a *fuzzy soft super set* of B_g, if B_g is a fuzzy soft subset of A_f. We denote it by $A_f \tilde{\supseteq} B_g$.

3.1.3 Definition

[3] Two fuzzy soft sets A_f and B_g over X are said to be *fuzzy soft equal* if A_f and B_g are fuzzy soft subsets of each other. We denote it by $A_f \tilde{=} B_g$.

3.1.4 Example

Let X be a set of candidates for a driver's vacant position, and E be a set of parameters, $X = \{c_1,c_2,c_3,c_4,c_5\}$, $E = \{e_1,e_2,e_3,e_4\} = \{$ knowledge about routes, driving skills, physical fitness, young$\}$. Suppose that $A = \{e_1,e_2,e_3\}$, a fuzzy soft set A_f describes the "data of candidates" which Mr. X is going to hire and is given as follows:

$$\begin{aligned} f \quad &: \quad A \to \mathcal{FP}(X), \\ e \quad &\longmapsto \begin{cases} \{c_1/0.3, c_2/0.1, c_3/0.3, c_4/0.1, c_5/0.7\} & \text{if } e = e_1, \\ \{c_1/0.1, c_2/0.9, c_3/0.3, c_4/0.8, c_5/0.2\} & \text{if } e = e_2, \\ \{c_1/0.1, c_2/0.3, c_3/0.3, c_4/0.3, c_5/0.8\} & \text{if } e = e_3, \end{cases} \end{aligned}$$

Let $B = \{e_2,e_3\}$. Then fuzzy soft set B_g given as follows:

$$\begin{aligned} g \quad &: \quad B \to \mathcal{FP}(X), \\ e \quad &\longmapsto \begin{cases} \{c_1/0.1, c_2/0.5, c_3/0.3, c_4/0.5, c_5/0.2\} & \text{if } e = e_2, \\ \{c_1/0.1, c_2/0.2, c_3/0.1, c_4/0.2, c_5/0.7\} & \text{if } e = e_3, \end{cases} \end{aligned}$$

is a fuzzy soft subset of A_f and represents a second analysis of choices made in A_f.

3.2 Operations on Fuzzy Soft Sets

Now, we define various operations on fuzzy soft sets taken from literature.

3.2.1 Definition

Let A_f and B_g be two fuzzy soft sets over X. Then the *or-product* of A_f and B_g is defined as a fuzzy soft set $(A \times B)_{f\tilde{\vee}g}$, where $f\tilde{\vee}g : (A \times B) \to \mathcal{FP}(X)$, defined by

$$(a, b) \mapsto f(a) \vee g(b).$$

It is denoted by $A_f \vee B_g \tilde{=} (A \times B)_{f\tilde{\vee}g}$.

3.2.2 Definition

Let A_f and B_g be two fuzzy soft sets over X. The *and-product* of A_f and B_g is defined as a fuzzy soft set $(A \times B)_{f\tilde{\wedge}g}$, where $f\tilde{\wedge}g : (A \times B) \to \mathcal{FP}(X)$, defined by

$$(a, b) \mapsto f(a) \wedge g(b).$$

It is denoted by $A_f \wedge B_g \tilde{=} (A \times B)_{f\tilde{\wedge}g}$.

3.2.3 Definition

The *extended union of two fuzzy soft sets* A_f and B_g over X is defined as a fuzzy soft set $(A \cup B)_{f\tilde{\vee}g}$, where $f\tilde{\vee}g : (A \cup B) \to \mathcal{FP}(X)$, defined by

$$e \mapsto \begin{cases} f(e) & if\ e \in A - B \\ g(e) & if\ e \in B - A \\ f(e) \vee g(e) & if\ e \in A \cap B \end{cases}$$

We write $A_f \sqcup_{\varepsilon} B_g \tilde{=} (A \cup B)_{f\tilde{\vee}g}$.

3.2.4 Definition

The extended intersection of two fuzzy soft sets A_f and B_g over X, is defined as a fuzzy soft set $(A \cup B)_{f\tilde{\wedge}g}$, where $f\tilde{\wedge}g : (A \cup B) \to \mathcal{FP}(X)$, defined by

$$e \mapsto \begin{cases} f(e) & if\ e \in A - B \\ g(e) & if\ e \in B - A \\ f(e) \wedge g(e) & if\ e \in A \cap B \end{cases}$$

We write $A_f \sqcap_{\varepsilon} B_g \tilde{=} (A \cup B)_{f\tilde{\wedge}g}$.

3.2.5 Definition

Let A_f and B_g be two fuzzy soft sets over X such that $A \cap B \neq \emptyset$. Then the *restricted union* of A_f and B_g is defined as a fuzzy soft set $(A \cap B)_{f\tilde{\wedge}g}$, where $f\tilde{\vee}g : A \cap B \to \mathcal{FP}(X)$,

$$e \mapsto f(e) \vee g(e).$$

We write $A_f \sqcup B_g \tilde{=} (A \cap B)_{f\tilde{\vee}g}$

3.2.6 Definition

Let A_f and B_g be two fuzzy soft sets over X such that $A \cap B \neq \emptyset$. Then the *restricted intersection* of A_f and B_g is defined as a fuzzy soft set $(A \cap B)_{f\tilde{\wedge}g}$, where $f\tilde{\wedge}g : A \cap B \to \mathcal{FP}(X)$,

$$e \mapsto f(e) \wedge g(e).$$

We write $A_f \sqcap B_g \tilde{=} (A \cap B)_{f\tilde{\wedge}g}$.

3.2.7 Definition

The *complement of a fuzzy soft set* A_f, denoted by $(A_f)'$ and defined by $(A_f)' \tilde{=} A_{f'}$, where $f' : A \to \mathcal{FP}(X)$ is given by

$$(f'(e))(x) = 1 - (f(e))(x),$$

for all $e \in A$, and for all $x \in X$.

Clearly $(f')'$ is same as f and $((A_f)')' = A_f$.

Now, we give an example to show how to apply these operations on fuzzy soft sets:

3.2.8 Example

Let X be the initial universe and E be the set of parameters,

$$X = \{x_1, x_2, x_3, x_4, x_5\},\ E = \{e_1, e_2, e_3, e_4, e_5\}.$$

Suppose

$$A = \{e_1, e_2\}, \text{ and } B = \{e_2, e_4\}.$$

Let A_f and B_g be the fuzzy soft sets over X defined by the following:

$$\begin{aligned} f &: A \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{x_1/0.1, x_2/0.2, x_3/0.3, x_4/0.7, x_5/0.4\} & \text{if } e = e_1, \\ \{x_1/0.7, x_2/0.9, x_3/0.2, x_4/0.4, x_5/0.1\} & \text{if } e = e_2, \end{cases} \end{aligned}$$

$$
\begin{aligned}
g \quad &: \quad B \rightarrow \mathcal{FP}(X),\\
e \quad &\longmapsto \begin{cases} \{x_1/0.3, x_2/0.7, x_3/0.6, x_4/0.9, x_5/0.1\} & \text{if } e = e_2,\\ \{x_1/0.4, x_2/0.2, x_3/0.7, x_4/0.8, x_5/0.7\} & \text{if } e = e_4,\end{cases}
\end{aligned}
$$

Then

(i) $A_f \sqcup_\varepsilon B_g \tilde{=} (A \cup B)_{f\tilde{\vee}g}$ where

$$
\begin{aligned}
f\tilde{\vee}g \quad &: \quad (A \cup B) \rightarrow \mathcal{FP}(X),\\
e \quad &\longmapsto \begin{cases} \{x_1/0.1, x_2/0.2, x_3/0.3, x_4/0.7, x_5/0.4\} & \text{if } e = e_1,\\ \{x_1/0.7, x_2/0.9, x_3/0.6, x_4/0.9, x_5/0.1\} & \text{if } e = e_2,\\ \{x_1/0.4, x_2/0.2, x_3/0.7, x_4/0.8, x_5/0.7\} & \text{if } e = e_4.\end{cases}
\end{aligned}
$$

(ii) $A_f \sqcap_\varepsilon B_g \tilde{=} (A \cup B)_{f\tilde{\wedge}g}$ where

$$
\begin{aligned}
f\tilde{\wedge}g \quad &: \quad (A \cup B) \rightarrow \mathcal{FP}(X),\\
e \quad &\longmapsto \begin{cases} \{x_1/0.1, x_2/0.2, x_3/0.3, x_4/0.7, x_5/0.4\} & \text{if } e = e_1,\\ \{x_1/0.3, x_2/0.7, x_3/0.2, x_4/0.4, x_5/0.1\} & \text{if } e = e_2,\\ \{x_1/0.4, x_2/0.2, x_3/0.7, x_4/0.8, x_5/0.7\} & \text{if } e = e_4.\end{cases}
\end{aligned}
$$

(iii) $A_f \sqcup B_g \tilde{=} (A \cap B)_{f\tilde{\vee}g}$ where

$$
\begin{aligned}
f\tilde{\vee}g \quad &: \quad (A \cap B) \rightarrow \mathcal{FP}(X),\\
e_2 \quad &\longmapsto \{x_1/0.7, x_2/0.9, x_3/0.6, x_4/0.9, x_5/0.1\}
\end{aligned}
$$

(iv) $A_f \sqcap B_g \tilde{=} (A \cap B)_{f\tilde{\wedge}g}$ where

$$
\begin{aligned}
f\tilde{\wedge}g \quad &: \quad (A \cap B) \rightarrow \mathcal{FP}(X),\\
e \quad &\longmapsto \begin{cases} \{x_1/0.3, x_2/0.7, x_3/0.2, x_4/0.4, x_5/0.1\} & \text{if } e = e_2,\\ \{x_1/0.3, x_2/0.7, x_3/0.3, x_4/0.2, x_5/0.5\} & \text{if } e = e_3.\end{cases}
\end{aligned}
$$

(v) $(A_f)' \tilde{=} A_{f'}$ where

$$
\begin{aligned}
f' \quad &: \quad A \rightarrow \mathcal{FP}(X),\\
e \quad &\longmapsto \begin{cases} \{x_1/0.9, x_2/0.8, x_3/0.7, x_4/0.3, x_5/0.6\} & \text{if } e = e_1,\\ \{x_1/0.3, x_2/0.1, x_3/0.8, x_4/0.6, x_5/0.9\} & \text{if } e = e_2,\end{cases}
\end{aligned}
$$

3.3 Properties of Fuzzy Soft Sets

In this section we discuss properties and laws of fuzzy soft sets with respect to operations defined on fuzzy soft sets. Later on the results will be utilized for the configuration of algebraic structures of fuzzy soft sets. Associativity, commutativity, absorption, distributivity, de Morgan laws and properties of involutions, and atomicity are investigated for collection of fuzzy soft sets.

3.3.1 Definition

A fuzzy soft set A_f over X is called a relative null fuzzy soft set, denoted by $A_{\tilde{\mathbf{0}}}$, if $f(e) = \tilde{0}$ for all $e \in A$, where $\tilde{0}$ is the fuzzy subset of X mapping every element of X on 0.

3.3.2 Definition

A fuzzy soft set A_f over X is called a relative whole or *absolute fuzzy soft set*, denoted by $A_{\tilde{\mathbf{1}}}$, if $f(e) = \tilde{\mathbf{1}}$ for all $e \in A$, where $\tilde{\mathbf{1}}$ is the fuzzy subset of X mapping every element of X on 1.

Conventionally, we take fuzzy soft sets with an empty set of parameters to be equal to $\emptyset_{\tilde{\mathbf{0}}}$ and so $A_f \sqcap B_g \tilde{=} \emptyset_{\tilde{\mathbf{0}}} \tilde{=} A_f \sqcup B_g$ when $A \cap B = \emptyset$.

3.3.3 Proposition

Let A_f, A_g be any fuzzy soft sets over X. Then

1) $A_f \lambda A_f \tilde{=} A_f$, for $\lambda \in \{\sqcup, \sqcup_\varepsilon, \sqcap, \sqcap_\varepsilon\}$, (Idempotent)

2) $A_f \sqcup_\varepsilon A_g \tilde{=} A_f \sqcup A_g$; $A_f \sqcap_\varepsilon A_g \tilde{=} A_f \sqcap A_g$,

3) $A_f \sqcap A_{\tilde{\mathbf{1}}} \tilde{=} A_f \tilde{=} A_f \sqcup A_{\tilde{\mathbf{0}}}$,

4) $A_f \sqcup A_{\tilde{\mathbf{1}}} \tilde{=} A_{\tilde{\mathbf{1}}}$; $A_f \sqcap A_{\tilde{\mathbf{0}}} \tilde{=} A_{\tilde{\mathbf{0}}}$,

5) $A_f \sqcap_\varepsilon \emptyset_{\tilde{\mathbf{0}}} \tilde{=} A_f \tilde{=} A_f \sqcup_\varepsilon \emptyset_{\tilde{\mathbf{0}}} \tilde{=} A_f \sqcap E_{\tilde{\mathbf{1}}}$,

6) $A_f \sqcap \emptyset_{\tilde{\mathbf{0}}} \tilde{=} \emptyset_{\tilde{\mathbf{0}}}$; $A_f \sqcup_\varepsilon E_{\tilde{\mathbf{1}}} \tilde{=} E_{\tilde{\mathbf{1}}}$.

Proof. Straightforward. ■

3.3.4 Proposition

Let A_f, B_g and C_h be any fuzzy soft sets over X. Then the following are true:

1) $A_f \lambda (B_g \lambda C_h) \tilde{=} (A_f \lambda B_g) \lambda C_h$, (Associative Laws)

2) $A_f \lambda B_g \tilde{=} B_g \lambda A_f$, (Commutative Laws)

for all $\lambda \in \{\sqcup_\varepsilon, \sqcup, \sqcap_\varepsilon, \sqcap\}$.
Proof. Straightforward. ■

3.3.5 Proposition (Absorption Laws)

Let A_f, B_g be any fuzzy soft sets over X. Then the following are true:

1) $A_f \sqcap_\varepsilon (B_g \sqcup A_f) \tilde{=} A_f$,

2) $A_f \sqcap (B_g \sqcup_\varepsilon A_f) \tilde{=} A_f$,

3) $A_f \sqcup (B_g \sqcap_\varepsilon A_f) \tilde{=} A_f$,

4) $A_f \sqcup_\varepsilon (B_g \sqcap A_f) \tilde{=} A_f$.

Proof. For any $e \in A$,

$$
\begin{aligned}
(f \tilde{\wedge} (f \tilde{\vee} g))(e) &= \begin{cases} f(e) & \text{if } e \in A - (A \cap B) \\ f(e) \wedge (f \tilde{\vee} g)(e) & \text{if } e \in A \cap (A \cap B) \end{cases} \\
&= \begin{cases} f(e) & \text{if } e \in A - (A \cap B) \\ f(e) \wedge (f(e) \vee g(e)) & \text{if } e \in A \cap B \end{cases} \\
&= \begin{cases} f(e) & \text{if } e \in A - (A \cap B) \\ f(e) & \text{if } e \in A \cap B \end{cases} \\
&= f(e).
\end{aligned}
$$

Thus $A_f \sqcap_\varepsilon (B \sqcup A_f) \tilde{=} A_f$. The remaining parts can also be proved similarly. ■

3.3.6 Proposition (Distributive Laws)

Let A_f, B_g and C_h be any fuzzy soft sets over X. Then

1) $A_f \sqcap (B_g \sqcup_\varepsilon C_h) \tilde{=} (A_f \sqcap B_g) \sqcup_\varepsilon (A_f \sqcap C_h)$,

2) $A_f \sqcap (B_g \sqcap_\varepsilon C_h) \tilde{=} (A_f \sqcap B_g) \sqcap_\varepsilon (A_f \sqcap C_h)$,

3) $A_f \sqcap (B_g \sqcup C_h) \tilde{=} (A_f \sqcap B_g) \sqcup (A_f \sqcap C_h)$,

4) $A_f \sqcup (B_g \sqcup_\varepsilon C_h) \tilde{=} (A_f \sqcup B_g) \sqcup_\varepsilon (A_f \sqcup C_h)$,

5) $A_f \sqcup (B_g \sqcap_\varepsilon C_h) \tilde{=} (A_f \sqcup B_g) \sqcap_\varepsilon (A_f \sqcup C_h)$,

6) $A_f \sqcup (B_g \sqcap C_h) \tilde{=} (A_f \sqcup B_g) \sqcap (A_f \sqcup C_h)$,

7) $A_f \sqcap_\varepsilon (B_g \sqcup_\varepsilon C_h) \tilde{\subseteq} (A_f \sqcap_\varepsilon B_g) \sqcup_\varepsilon (A_f \sqcap_\varepsilon C_h)$,

8) $A_f \sqcap_\varepsilon (B_g \sqcup C_h) \tilde{=} (A_f \sqcap_\varepsilon B_g) \sqcup (A_f \sqcap_\varepsilon C_h)$,

9) $A_f \sqcap_\varepsilon (B_g \sqcap C_h) \tilde{\supseteq} (A_f \sqcap_\varepsilon B_g) \sqcap (A_f \sqcap_\varepsilon C_h)$,

10) $A_f \sqcup_\varepsilon (B_g \sqcup C_h) \tilde{\subseteq} (A_f \sqcup_\varepsilon B_g) \sqcup (A_f \sqcup_\varepsilon C_h)$,

11) $A_f \sqcup_\varepsilon (B_g \sqcap_\varepsilon C_h) \tilde{\supseteq} (A_f \sqcup_\varepsilon B_g) \sqcap_\varepsilon (A_f \sqcup_\varepsilon C_h)$,

12) $A_f \sqcup_\varepsilon (B_g \sqcap C_h) \tilde{\doteq} (A_f \sqcup_\varepsilon B_g) \sqcap (A_f \sqcup_\varepsilon C_h)$.

Proof. We prove only one part here, the other parts can also be proved in a similar way.

5) We have

$$A_f \sqcup (B_g \sqcap_\varepsilon C_h) \tilde{\doteq} (A \cap (B \cup C))_{f\tilde{\vee}(g\tilde{\wedge}h)}$$

and

$$\begin{aligned}(A_f \sqcup B_g) \sqcap_\varepsilon (A_f \sqcup C_h) &\tilde{\doteq} (A \cap B)_{(f\tilde{\vee}g)} \sqcap_\varepsilon (A \cap C)_{f\tilde{\vee}h} \\ &\tilde{\doteq} ((A \cap B) \cup (A \cap C))_{(f\tilde{\vee}g)\tilde{\wedge}(f\tilde{\vee}h)} \\ &\tilde{\doteq} (A \cap (B \cup C))_{(f\tilde{\vee}g)\tilde{\wedge}(f\tilde{\vee}h)}.\end{aligned}$$

Let $e \in A \cap (B \cup C)$ then there are three possibilities:

(i) If $e \in A \cap (B - C)$ then,

$$\begin{aligned}(g\tilde{\wedge}h)(e) &= g(e) \quad \text{and} \\ \{f\tilde{\vee}(g\tilde{\wedge}h)\}(e) &= f(e) \vee g(e).\end{aligned}$$

Also $A \cap (B - C) = (A \cap B) - (A \cap C)$ and hence

$$\{(f\tilde{\vee}g)\tilde{\wedge}(f\tilde{\vee}h)\}(e) = (f\tilde{\vee}g)(e) = f(e) \vee g(e).$$

(ii) If $e \in A \cap (C - B)$ then,

$$\begin{aligned}(g\tilde{\wedge}h)(e) &= h(e) \quad \text{and} \\ \{f\tilde{\vee}(g\tilde{\wedge}h)\}(e) &= f(e) \vee h(e).\end{aligned}$$

Also $A \cap (C - B) = (A \cap C) - (A \cap B)$ and hence

$$\{(f\tilde{\vee}g)\tilde{\wedge}(f\tilde{\vee}h)\}(e) = (f\tilde{\vee}h)(e) = f(e) \vee h(e).$$

(iii) If $e \in A \cap (B \cap C)$ then,

$$\begin{aligned}(g\tilde{\wedge}h)(e) &= g(e) \wedge h(e) \quad \text{and} \\ \{f\tilde{\vee}(g\tilde{\wedge}h)\}(e) &= f(e) \vee (g(e) \wedge h(e)).\end{aligned}$$

Also $A \cap (B \cap C) = (A \cap B) \cap (A \cap C)$ and hence

$$\begin{aligned}\{(f\tilde{\vee}g)\tilde{\wedge}(f\tilde{\vee}h)\}(e) &= (f\tilde{\vee}g)(e) \wedge (f\tilde{\vee}h)(e) \\ &= (f(e) \vee g(e)) \wedge (f(e) \vee h(e)) \\ &= f(e) \vee (g(e) \wedge h(e)).\end{aligned}$$

Thus

$$f\tilde{\vee}(g\tilde{\wedge}h) = (f\tilde{\vee}g)\tilde{\wedge}(f\tilde{\vee}h)$$

and so

$$(A \cap (B \cup C))_{f\tilde{\vee}(g\tilde{\wedge}h)} \tilde{=} (A \cap (B \cup C))_{(f\tilde{\vee}g)\tilde{\wedge}(f\tilde{\vee}h)}.$$

■

3.3.7 Example

Let X be the set of houses under consideration, and E be the set of parameters,

$$X = \{h_1, h_2, h_3, h_4, h_5\},$$

$$E = \{\text{beautiful, wooden, cheap, in good repair, furnished}\}.$$

Suppose that

$$\begin{aligned}A &= \{\text{beautiful, wooden, cheap}\}, \\ B &= \{\text{wooden, cheap, in good repair}\}, \\ \text{and } C &= \{\text{cheap, in good repair, furnished}\}.\end{aligned}$$

Let A_f,B_g and C_h be the fuzzy soft sets over X defined by the following:

$$\begin{aligned}f &: A \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases}\{h_1/0.1, h_2/0.2, h_3/0.3, h_4/0.7, h_5/0.4\} & \text{if } e = e_1, \\ \{h_1/0.7, h_2/0.9, h_3/0.2, h_4/0.4, h_5/0.1\} & \text{if } e = e_2, \\ \{h_1/0.3, h_2/0.7, h_3/0.5, h_4/0.2, h_5/0.6\} & \text{if } e = e_3,\end{cases}\end{aligned}$$

$$\begin{aligned}g &: B \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases}\{h_1/0.3, h_2/0.7, h_3/0.6, h_4/0.9, h_5/0.1\} & \text{if } e = e_2, \\ \{h_1/0.6, h_2/1.0, h_3/0.3, h_4/0.2, h_5/0.5\} & \text{if } e = e_3, \\ \{h_1/0.4, h_2/0.2, h_3/0.7, h_4/0.8, h_5/0.7\} & \text{if } e = e_4,\end{cases}\end{aligned}$$

$$\begin{aligned}h &: C \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases}\{h_1/0.7, h_2/0.8, h_3/0.5, h_4/0.4, h_5/0.4\} & \text{if } e = e_3, \\ \{h_1/0.5, h_2/0.3, h_3/0.2, h_4/0.1, h_5/0.4\} & \text{if } e = e_4, \\ \{h_1/0.7, h_2/0.8, h_3/0.2, h_4/0.3, h_5/0.9\} & \text{if } e = e_5,\end{cases} .\end{aligned}$$

Now

$$\begin{aligned} A_f \sqcup_\varepsilon (B_g \sqcup C_h) &\tilde{=} (A \cup (B \cap C))_{f\tilde{\vee}(g\tilde{\vee}h)}; \\ (A_f \sqcup_\varepsilon B_g) \sqcup (A_f \sqcup_\varepsilon C_h) &\tilde{=} ((A \cup B) \cap (A \cup C))_{(f\tilde{\vee}g)\tilde{\vee}(f\tilde{\vee}h)}; \\ A_f \sqcup_\varepsilon (B_g \sqcap_\varepsilon C_h) &\tilde{=} (A \cup (B \cup C))_{f\tilde{\vee}(g\tilde{\wedge}h)}; \\ (A_f \sqcup_\varepsilon B_g) \sqcap_\varepsilon (A_f \sqcup_\varepsilon C_h) &\tilde{=} ((A \cup B) \cup (B \cup C))_{(f\tilde{\vee}g)\tilde{\wedge}(f\tilde{\vee}h)}. \end{aligned}$$

Then

$$(f\tilde{\vee}(g\tilde{\vee}h))(\text{wooden}) = \{h_1/0.7, h_2/0.9, h_3/0.2, h_4/0.4, h_5/0.1\}$$

and

$$((f\tilde{\vee}g)\tilde{\vee}(f\tilde{\vee}h))(\text{wooden}) = \{h_1/0.7, h_2/0.9, h_3/0.6, h_4/0.9, h_5/0.1\}.$$

We see that

$$(f\tilde{\vee}(g\tilde{\vee}h))(\text{wooden}) \neq ((f\tilde{\vee}g)\tilde{\vee}(f\tilde{\vee}h))(\text{wooden}).$$

Thus

$$A_f \sqcup_\varepsilon (B_g \sqcup C_h) \tilde{\neq} (A_f \sqcup_\varepsilon B_g) \sqcup (A_f \sqcup_\varepsilon C_h).$$

Again,

$$(f\tilde{\wedge}(g\tilde{\vee}h))(\text{wooden}) = \{h_1/0.3, h_2/0.7, h_3/0.2, h_4/0.4, h_5/0.1\}$$

and

$$((f\tilde{\wedge}g)\tilde{\vee}(f\tilde{\wedge}h))(\text{wooden}) = \{h_1/0.7, h_2/0.9, h_3/0.2, h_4/0.4, h_5/0.1\}.$$

We see that

$$(f\tilde{\wedge}(g\tilde{\vee}h))(\text{wooden}) \neq ((f\tilde{\wedge}g)\tilde{\vee}(f\tilde{\wedge}h))(\text{wooden}).$$

Thus

$$A_f \sqcap_\varepsilon (B_g \sqcup_\varepsilon C_h) \tilde{\neq} (A_f \sqcap_\varepsilon B_g) \sqcup_\varepsilon (A_f \sqcap_\varepsilon C_h).$$

Similarly it can be shown that

$$\begin{aligned} A_f \sqcap_\varepsilon (B_g \sqcap C_h) &\tilde{\neq} (A_f \sqcap_\varepsilon B_g) \sqcap (A_f \sqcap_\varepsilon C_h). \\ A_f \sqcup_\varepsilon (B_g \sqcap_\varepsilon C_h) &\tilde{\neq} (A_f \sqcup_\varepsilon B_g) \sqcap_\varepsilon (A_f \sqcup_\varepsilon C_h). \end{aligned}$$

3.3.8 Proposition

Let A_f, B_g and C_h be any *fuzzy soft sets* over X. Then

1)

$$A_f \sqcup_\varepsilon (B_g \sqcap_\varepsilon C_h) \tilde{=} (A_f \sqcup_\varepsilon B_g) \sqcap_\varepsilon (A_f \sqcup_\varepsilon C_h)$$

if and only if

$$\begin{aligned} f(e) &\subseteq g(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ f(e) &\subseteq h(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

2)

$$A_f \sqcap_\varepsilon (B_g \sqcup_\varepsilon C_h) \tilde{=} (A_f \sqcap_\varepsilon B_g) \sqcup_\varepsilon (A_f \sqcap_\varepsilon C_h)$$

if and only if

$$\begin{aligned} f(e) &\supseteq g(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ f(e) &\supseteq h(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

Proof. Straightforward. ■

3.3.9 Corollary

Let A_f, B_g and C_h be any *fuzzy soft set*s over X. Then

$$\begin{aligned} A_f \sqcup_\varepsilon (B_g \sqcap_\varepsilon C_h) &\tilde{=} (A_f \sqcup_\varepsilon B_g) \sqcap_\varepsilon (A_f \sqcup_\varepsilon C_h) \text{ and} \\ A_f \sqcap_\varepsilon (B_g \sqcup_\varepsilon C_h) &\tilde{=} (A_f \sqcap_\varepsilon B_g) \sqcup_\varepsilon (A_f \sqcap_\varepsilon C_h) \end{aligned}$$

hold if and only if

$$\begin{aligned} f(e) &= g(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ f(e) &= h(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

3.3.10 Corollary

Let A_f, B_g and C_h be *any fuzzy soft set*s over X such that $(A \cap B) - C = (A \cap C) - B = \emptyset$. Then

1) $A_f \sqcup_\varepsilon (B_g \sqcap_\varepsilon C_h) \tilde{=} (A_f \sqcup_\varepsilon B_g) \sqcap_\varepsilon (A_f \sqcup_\varepsilon C_h)$,

2) $A_f \sqcap_\varepsilon (B_g \sqcup_\varepsilon C_h) \tilde{=} (A_f \sqcap_\varepsilon B_g) \sqcup_\varepsilon (A_f \sqcap_\varepsilon C_h)$.

3.3.11 Corollary

Let A_f, A_g and A_h be any *fuzzy soft set*s over X. Then

$$A_f \lambda (A_g \mu A_h) \tilde{=} (A_f \lambda A_g) \mu (A_f \lambda A_h)$$

for distinct λ, $\mu \in \{\sqcap_\varepsilon, \sqcap, \sqcup_\varepsilon, \sqcup\}$.

3.3.12 Proposition

Let A_f and B_g be two *fuzzy soft set*s over X. Then the following are true

1) $A_f \sqcup_\varepsilon B_g$ is the smallest fuzzy soft set over X which contains both A_f and B_g. (Supremum)

2) $A_f \sqcap B_g$ is the largest fuzzy soft set over X which is contained in both A_f and B_g. (Infimum)

Proof.

1) $A_f \tilde{\subseteq} A_f \sqcup_\varepsilon B_g$ and $B_g \tilde{\subseteq} A_f \sqcup_\varepsilon B_g$, because $A \subseteq (A \cup B)$, $B \subseteq (A \cup B)$ and $f(e) \subseteq f(e) \vee g(e)$, $g(e) \subseteq f(e) \vee g(e)$. Let C_h be any fuzzy soft set over X, such that $A_f \tilde{\subseteq} C_h$ and $B_g \tilde{\subseteq} C_h$. Then $(A \cup B) \subseteq C$, and $f(e) \subseteq h(e)$, for all $e \in A$, $g(e) \subseteq h(e)$ for all $e \in B$ implies that $(f \tilde{\vee} g)(e) \subseteq h(e)$ for all $e \in (A \cup B)$. Thus $A_f \sqcup_\varepsilon B_g \tilde{\subseteq} C_h$.

2) $A_f \sqcap B_g \tilde{\subseteq} A_f$ and $A_f \sqcap B_g \tilde{\subseteq} B_g$, because $A \cap B \subseteq A$, $A \cap B \subseteq B$ and $f(e) \wedge g(e) \subseteq f(e)$, $f(e) \wedge g(e) \subseteq g(e)$ for all $e \in A \cap B$. Let C_h be any fuzzy soft set over X, such that $C_h \tilde{\subseteq} A_f$ and $C_h \tilde{\subseteq} B_g$. Then $C \subseteq A \cap B$, and $h(e) \subseteq f(e)$, $h(e) \subseteq g(e)$ for all $e \in C$ implies that $h(e) \subseteq f(e) \wedge g(e) = (f \tilde{\wedge} g)(e)$ for all $e \in C$. Thus $C_h \tilde{\subseteq} A_f \sqcap B_g$.

■

3.4 Algebras of Fuzzy Soft Sets

In this section, we use the ideas of lattices and algebras for fuzzy soft collections. We consider collections of fuzzy soft sets and find their distributive lattices. The collections are denoted as follows:

$\mathcal{FSS}(X)^E$: collection of all fuzzy soft sets defined over X

$\mathcal{FSS}(X)_A$: collection of all those fuzzy soft sets defined over X with a fixed parameter set A.

Firstly, we observe that these collections are partially ordered by the relation of fuzzy soft inclusion $\tilde{\subseteq}$.

3.4.1 Proposition

$(\mathcal{FSS}(X)^E, \sqcap_\varepsilon, \sqcup)$, $(\mathcal{FSS}(X)^E, \sqcup, \sqcap_\varepsilon)$, $(\mathcal{FSS}(X)^E, \sqcup_\varepsilon, \sqcap)$, $(\mathcal{FSS}(X)^E, \sqcap, \sqcup_\varepsilon)$, $(\mathcal{FSS}(X)_A, \sqcup, \sqcap)$, and $(\mathcal{FSS}(X)_A, \sqcap, \sqcup)$ are lattices.

Proof. From Propositions 3.3.3, 3.3.4 and 3.3.5 we conclude that the structures form lattices. ■

3.4.2 Proposition

Structures $(\mathcal{FSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{\tilde{\mathbf{0}}}, E_{\tilde{\mathbf{1}}})$, $(\mathcal{FSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{\tilde{\mathbf{1}}}, \emptyset_{\tilde{\mathbf{0}}})$, $(\mathcal{FSS}(X)_A, \sqcap, \sqcup, A_{\tilde{\mathbf{0}}}, A_{\tilde{\mathbf{1}}})$ and $(\mathcal{FSS}(X)_A, \sqcup, \sqcap, A_{\tilde{\mathbf{1}}}, A_{\tilde{\mathbf{0}}})$ are bounded distributive lattices.

Proof. Proposition 3.3.6 assures that $(\mathcal{FSS}(X)^E, \sqcap, \sqcup_\varepsilon)$ and $(\mathcal{FSS}(X)^E, \sqcup_\varepsilon, \sqcap)$ are distributive lattices. From Lemma 3.3.12, we conclude that $(\mathcal{FSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{\tilde{\mathbf{0}}}, E_{\tilde{\mathbf{1}}})$ is a bounded distributive lattice and $(\mathcal{FSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{\tilde{\mathbf{1}}}, \emptyset_{\tilde{\mathbf{0}}})$ is its dual. For any fuzzy soft sets $A_f, A_g \in \mathcal{FSS}(X)_A$,

$$\begin{aligned} A_f \sqcap A_g &\tilde{=} A_{f\tilde{\wedge}g} \in \mathcal{FSS}(X)_A \text{ and} \\ A_f \sqcup A_g &\tilde{=} A_{f\tilde{\vee}g} \in \mathcal{FSS}(X)_A. \end{aligned}$$

Thus $(\mathcal{FSS}(X)_A, \sqcap, \sqcup)$ is also a distributive sublattice of $(\mathcal{FSS}(X)^E, \sqcup_\varepsilon, \sqcap)$ and Proposition 3.3.3 tells us that $A_{\tilde{\mathbf{0}}}$, $A_{\tilde{\mathbf{1}}}$ are its lower and upper bounds, respectively. Therefore $(\mathcal{FSS}(X)_A, \sqcap, \sqcup, A_{\tilde{\mathbf{0}}}, A_{\tilde{\mathbf{1}}})$ is a bounded distributive lattice and $(\mathcal{FSS}(X)_A, \sqcup, \sqcap, A_{\tilde{\mathbf{1}}}, A_{\tilde{\mathbf{0}}})$ is its dual. ■

3.4.3 Proposition

Let A_f be a *fuzzy soft set* over X. Then " ´ " is an involution on $\mathcal{FSS}(X)_A$.

Proof.

(i) We have to show that $A_{(f\,')'} \tilde{=} A_f$. Now, $(A_{f\,'})' \tilde{=} A_{(f\,')'}$

$$\begin{aligned} ((f\,')'(e))(x) &= (\tilde{\mathbf{1}} - f\,'(e))(x) \\ &= 1 - (f\,'(e))(x) \\ &= 1 - (\tilde{\mathbf{1}} - f(e))(x) \\ &= 1 - 1 + (f(e))(x) \\ &= 1 - 1 + (f(e))(x) \\ &= (f(e))(x) \end{aligned}$$

for all $e \in A$, $x \in X$. Thus $(A_{f\,'})' \tilde{=} A_f$.

(ii) If $A_f \tilde{\subseteq} A_g$ then

$$\begin{aligned} (f(e))(x) &\leq (g(e))(x) \text{ and so} \\ 1 - (g(e))(x) &\leq 1 - (f(e))(x) \text{ which gives} \\ (g'(e))(x) &\leq (f'(e))(x) \text{ for all } e \in A, x \in X. \end{aligned}$$

Hence $A_{g'} \tilde{\subseteq} A_{f'}$.

Thus " ´ " is an involution on $\mathcal{FSS}(X)_A$. ■

3.4.4 Proposition (de Morgan Laws)

Let A_f and B_g be any fuzzy soft sets over X. Then the following are true

1) $(A_f \sqcup_\varepsilon B_g)' \tilde{=} A_{f'} \sqcap_\varepsilon B_{g'}$,

2) $(A_f \sqcap_\varepsilon B_g)' \tilde{=} A_{f'} \sqcup_\varepsilon B_{g'}$,

3) $(A_f \vee B_g)' \tilde{=} A_{f'} \wedge B_{g'}$,

4) $(A_f \wedge B_g)' \tilde{=} A_{f'} \vee B_{g'}$,

5) $(A_f \sqcup B_g)' \tilde{=} A_{f'} \sqcap B_{g'}$,

6) $(A_f \sqcap B_g)' \tilde{=} A_{f'} \sqcup B_{g'}$.

Proof.

1) We know that $(A_f \sqcup_\varepsilon B_g)' \tilde{=} ((A \cup B)_{f\tilde{\vee}g})' \tilde{=} ((A \cup B)_{(f\tilde{\vee}g)'}$. Let $e \in (A \cup B)$. Then there are three cases:

(i) If $e \in A - B$, then

$$((f\tilde{\vee}g)')(e) = (f(e))' = f'(e) \text{ and } (f'\tilde{\wedge}g')(e) = f'(e).$$

(ii) If $e \in B - A$, then

$$(f\tilde{\vee}g)'(e) = (g(e))' = g'(e) \text{ and } (f'\tilde{\wedge}g')(e) = g'(e).$$

(iii) If $e \in A \cap B$, then

$$(f\tilde{\vee}g)'(e) = (f(e) \vee g(e))' = (f(e))' \wedge (g(e))'$$

and,

$$(f'\tilde{\wedge}g')(e) = (f(e))' \wedge (g(e))'$$

Therefore, in all three cases we obtain equality and thus

$$(A_f \sqcup_\varepsilon B_g)' \tilde{=} A_{f'} \sqcap_\varepsilon B_{g'}.$$

The remaining parts can be proved in a similar way. ■

3.4.5 Proposition

$(\mathcal{FSS}(X)_A, \sqcap, \sqcup, ', A_{\tilde{0}}, A_{\tilde{1}})$ is a de Morgan algebra.

Proof. We have already seen that $(\mathcal{FSS}(X)_A, \sqcap, \sqcup, A_{\tilde{0}}, A_{\tilde{1}})$ is a bounded distributive lattice. Proposition 3.4.3 shows that " $'$ " is an involution on $\mathcal{FSS}(X)_A$ and Proposition 3.4.4 shows that de Morgan laws hold with respect to $'$ in $\mathcal{FSS}(X)_A$. Thus $(\mathcal{FSS}(X)_A, \sqcap, \sqcup, ', A_{\tilde{0}}, A_{\tilde{1}})$ is a de Morgan algebra. ■

3.4.6 Proposition

Let A_f and A_g be *any* fuzzy soft sets over X. Then $(A_g \sqcap A_{g'})\tilde{\subseteq}(A_f \sqcup A_{f'})$ and so $(\mathcal{FSS}(X)_A, \sqcap, \sqcup, ', A_{\tilde{\mathbf{0}}}, A_{\tilde{\mathbf{1}}})$ is a Kleene Algebra.

Proof. For any $A_f, A_g \in \mathcal{FSS}(X)_A$, such that

$$A_f \sqcap A_{f'} \tilde{\supseteq} A_g \sqcup A_{g'} \text{ where } A_f \sqcap A_{f'} \tilde{\neq} A_g \sqcup A_{g'}.$$

Then there exists some $e \in A$ such that

$$(f \sqcap f')(e)\tilde{\supseteq}(g \sqcup g')(e)$$

and so we have some $x \in X$ such that

$$\begin{aligned} ((f \sqcap f')(e))(x) &> ((g \sqcup g')(e))(x) \qquad \text{or} \\ (f(e) \sqcap f'(e))(x) &> (g(e) \sqcup g'(e))(x) \qquad \text{or} \\ (f(e))(x) \wedge (f'(e))(x) &> (g(e))(x) \vee (g'(e))(x). \end{aligned}$$

But $(f(e))(x) \wedge (f'(e))(x) \leq 0.5$ and $(g(e))(x) \vee (g'(e))(x) \geq 0.5$ which gives

$$(f(e))(x) \wedge (f'(e))(x) \leq (g(e))(x) \vee (g'(e))(x).$$

A contradiction, thus our supposition is wrong. Hence

$$A_f \sqcap A_{f'} \tilde{\subseteq} A_g \sqcup A_{g'}.$$

Therefore $(\mathcal{FSS}(X)_A, \sqcap, \sqcup, ', A_{\tilde{\mathbf{0}}}, A_{\tilde{\mathbf{1}}})$ is a Kleene algebra. ■

3.4.7 Proposition

Let $A_f, B_g \in \mathcal{FSS}(X)^E$. Then pseudocomplement of A_f relative to B_g exists in $\mathcal{FSS}(X)^E$.

Proof. Consider the set

$$T(A_f, B_g) = \{C_h \in \mathcal{FSS}(X)^E : C_h \sqcap A_f \tilde{\subseteq} B_g\}.$$

We define a fuzzy soft set $(A^c \cup B)_{f \to g} \in \mathcal{FSS}(X)^E$ where

$$\begin{aligned} ((f \to g)(e))(x) & \\ &= \begin{cases} 1 & \text{if } e \in A^c - B \\ \begin{cases} 1 & \text{if } (f(e))(x) \leq (g(e))(x) \\ (g(e))(x) & \text{if } (f(e))(x) > (g(e))(x) \end{cases} & \text{if } e \in B - A^c \\ 1 & \text{if } e \in A^c \cap B \end{cases} \end{aligned}$$

Then

$$\begin{aligned}(A^c \cup B)_{f\to g} \sqcap A_f &\tilde{\simeq} ((A^c \cup B) \cap A)_{(f\to g)\tilde{\wedge} f}\\ &\tilde{\simeq} ((A^c \cap A) \cup (B \cap A))_{(f\to g)\tilde{\wedge} f}\\ &\tilde{\simeq} (A \cap B)_{(f\to g)\tilde{\wedge} f}.\end{aligned}$$

For any $e \in A \cap B$, $x \in X$,

$$\begin{aligned}(((f \to g)\tilde{\wedge} f)(e))(x) &\\ &= \begin{cases} 1 \wedge (f(e))(x) & \text{if } (f(e))(x) \leq (g(e))(x)\\ (g(e))(x) \wedge (f(e))(x) & \text{if } (f(e))(x) > (g(e))(x)\end{cases}\\ &= \begin{cases} (f(e))(x) & \text{if } (f(e))(x) \leq (g(e))(x)\\ (g(e))(x) & \text{if } (f(e))(x) > (g(e))(x)\end{cases}\\ &\leq (g(e))(x).\end{aligned}$$

Hence,

$$(A^c \cup B)_{f\to g} \sqcap A_f \tilde{\subseteq} B_g$$

Thus $(A^c \cup B)_{f\to g} \in T(A_f, B_g)$. For all $C_h \in T(A_f, B_g)$, we have $C_h \sqcap A_f \tilde{\subseteq} B_g$ so for any $e \in C \cap A \subseteq B$

$$h(e) \wedge f(e) \subseteq g(e).$$

Now,

$$\begin{aligned}C \cap A &\subseteq B \Rightarrow (A \cap C) \cap B^c = \emptyset\\ &\Rightarrow C \subseteq (A \cap B^c)^c = A^c \cup B.\end{aligned}$$

We have following cases:

(i) If $e \in (A^c - B) \cap C$, then $h(e))(x) < 1 = ((f \to g)(e))(x)$

(ii) If $e \in (B - A^c) \cap C$, and $(f(e))(x) \leq (g(e))(x)$ then $(h(e))(x) < 1 = ((f \to g)(e))(x)$

(iii) If $e \in (B - A^c) \cap C$ and $(f(e))(x) > (g(e))(x)$, then the condition $h(e) \wedge f(e) \subseteq g(e)$ implies that $(h(e))(x) \wedge (f(e)(x)) \leq (g(e))(x)$ which is possible only if $(h(e))(x) \wedge (f(e)(x)) = (h(e))(x)$ and thus $(h(e))(x) \leq (g(e))(x) = ((f \to g)(e))(x)$

(iv) If $e \in (A^c \cap B) \cap C$, then $h(e))(x) < 1 = ((f \to g)(e))(x)$.

Thus $C_h \tilde{\subseteq} (A^c \cup B)_{f\to g}$ and it also shows that $(A^c \cup B)_{f\to g} \tilde{\simeq} \bigvee T(A_f, B_g) \tilde{\simeq} A_f \to B_g$. ■

3.4.8 Remark

We know that $(\mathcal{FSS}(X)_A, \sqcap, \sqcup)$ is a sublattice of $(\mathcal{FSS}(X)^E, \sqcap_\varepsilon, \sqcup)$. *For any* A_f, $A_g \in \mathcal{FSS}(X)_A$, $A_f \to A_g$ (as defined in Proposition 3.4.7) is not in $\mathcal{FSS}(X)_A$ because $A_f \to A_g \tilde{=} (A^c \cup A)_{f\to g} \tilde{=} E_{f\to g} \notin \mathcal{FSS}(X)_A$.

3.4.9 Proposition

Let $A_f, A_g \in \mathcal{FSS}(X)_A$. Then pseudocomplement of A_f relative to A_g exists in $\mathcal{FSS}(X)_A$.

Proof. Consider the set

$$T(A_f, A_g) = \{A_h \in \mathcal{FSS}(X)_A : A_h \sqcap A_f \tilde{\subseteq} A_g\}.$$

We define a fuzzy soft set $A_{f\to g} \in \mathcal{FSS}(X)_A$ where

$$((f \to g)(e))(x) = \begin{cases} 1 & \text{if } (f(e))(x) \leq (g(e))(x) \\ (g(e))(x) & \text{if } (f(e))(x) > (g(e))(x) \end{cases}$$

for all $e \in A$, $x \in X$. Then $A_{f\to g} \sqcap A_f \tilde{=} A_{(f\to g)\tilde{\wedge} f}$ and

$$\begin{aligned} (((f \to g)\tilde{\wedge} f)(e))(x) & \\ &= \begin{cases} 1 \wedge (f(e))(x) & \text{if } (f(e))(x) \leq (g(e))(x) \\ (g(e))(x) \wedge (f(e))(x) & \text{if } (f(e))(x) > (g(e))(x) \end{cases} \\ &= \begin{cases} (f(e))(x) & \text{if } (f(e))(x) \leq (g(e))(x) \\ (g(e))(x) & \text{if } (f(e))(x) > (g(e))(x) \end{cases} \\ &\leq (g(e))(x). \end{aligned}$$

for all $e \in A$, $x \in X$. Hence,

$$A_{f\to g} \sqcap A_f \tilde{\subseteq} A_g$$

and $A_{f\to g} \in T(A_f, A_g)$. For every $A_h \in T(A_f, A_g)$, we have $A_h \sqcap A_f \tilde{\subseteq} A_g$ so for any $e \in A$, following cases arise:

(i) If $(f(e))(x) \leq (g(e))(x)$ then $(h(e))(x) < 1 = ((f \to g)(e))(x)$

(ii) If $(f(e))(x) > (g(e))(x)$ then the condition $h(e) \wedge f(e) \subseteq g(e)$ implies that $(h(e))(x) \wedge (f(e)(x)) \leq (g(e))(x)$ and so $(h(e))(x) \leq (g(e))(x) = ((f \to g)(e))(x)$.

Thus $A_h \tilde{\subseteq} A_{f\to g}$ and it also shows that

$$A_{f\to g} \tilde{=} \bigvee T(A_f, A_g) \tilde{=} A_f \to_A A_g.$$

■

3.4.10 Proposition

$(\mathcal{FSS}(X)^E, \sqcap_\varepsilon, \sqcup)$ *and* $(\mathcal{FSS}(X)_A, \sqcap, \sqcup)$ are *Brouwerian lattices.*

Proof. Follows from Propositions 3.4.7 and 3.4.9. ■

3.4.11 Definition

For a *fuzzy soft set* A_f over X, we define a fuzzy soft set over X, which is denoted by A_{f^*} and is given by $A_{f^*} = (A_f)^*$ where

$$(f^*(e))(x) = \begin{cases} 0 & \text{if } (f(e))(x) \neq 0 \\ 1 & \text{if } (f(e))(x) = 0 \end{cases}$$

for all $x \in X$, $e \in A$.

3.4.12 Theorem

Let A_f be a *fuzzy soft set* over X. Then the following are true:

1) $A_f \sqcap A_{f^*} \tilde{=} A_{\tilde{\mathbf{0}}}$,

2) $A_g \tilde{\subseteq} A_{f^*}$ whenever $A_f \sqcap A_g \tilde{=} A_{\tilde{\mathbf{0}}}$,

3) $A_{f^*} \sqcup A_{f^{**}} \tilde{=} A_{\tilde{\mathbf{1}}}$.

Thus $(\mathcal{FSS}(X)_A, \sqcap, \sqcup, ^*, A_{\tilde{\mathbf{0}}}, A_{\tilde{\mathbf{1}}})$ is a Stone algebra.

Proof.

1) Straightforward.

2) If $A_f \sqcap A_g \tilde{=} A_\Phi$. Then for any $x \in X$, $e \in A$,

$$\text{if } (g(e))(x) = 0 \text{ then } (g(e))(x) \leq (f^*(e))(x).$$

$$\text{If } (g(e))(x) \neq 0 \text{ then } (f(e))(x) \wedge (g(e))(x) = 0$$
$$\text{implies that } (f(e))(x) = 0, \text{ so } (f^*(e))(x) = 1$$
$$\text{and hence} \quad (g(e))(x) \leq 1 = (f^*(e))(x).$$

Thus ,

$$(g(e))(x) \leq (f^*(e))(x) \text{ for all } x \in X,\ e \in A.$$

That is, $A_g \tilde{\subseteq} A_{f^*}$.

3) For any $x \in X$, $e \in A$,

$$\begin{aligned} ((f^* \sqcup f^{**})(e))(x) &= (f^*(e) \vee f^{**}(e))(x) \\ &= \max\{(f^*(e))(x),\, (f^{**}(e))(x)\} \\ &= \begin{cases} \max\{1,0\} & \text{if } (f(e))(x) \neq 0 \\ \max\{0,1\} & \text{if } (f(e))(x) = 0 \end{cases} \\ &= 1. \end{aligned}$$

Thus $A_{f^*} \sqcup A_{f^{**}} \tilde{=} A_{\tilde{\mathbf{1}}}$ and so, $(\mathcal{FSS}(X)_A, \sqcap, \sqcup, {}^*, A_{\tilde{\mathbf{0}}}, A_{\tilde{\mathbf{1}}})$ is a *Stone algebra.*

■

3.4.13 Remark

Note that $A_{f^*} \tilde{=} A_f \rightarrow_A A_{\tilde{\mathbf{0}}}$.

Chapter 4

Algebraic Structures of Double-framed Soft Sets

This chapter explores the theory of double-framed soft sets. Double-framed soft sets have been introduced by Jun et al. [19] in 2012. They discussed applications of double-framed soft sets in BCK/BCI-algebras and verified several results with uni-int concepts. Recently, some further works are presented to characterize the ideals of BCK/BCI-algebras in terms of double-framed soft sets in [20]. In our work, we have focused upon the algebraic structural properties of double-framed soft sets. New operations for double-framed soft sets are defined and their characteristics are studied. Examples are given to elaborate the concepts and to show how the ideas are utilized to work with double-framed soft sets. The lattice structure and different algebraic specifications raised by the collections of double-framed soft sets have been shown in a logical manner. Classes of MV-algebras and BCK/BCI-algebras of double-framed soft sets are presented at the end.

4.1 Double-framed Soft Sets

Let X be an initial universe and E be a set of parameters. Let $\mathcal{P}(X)$ denotes the power set of X and A, B, C are non-empty subsets of E.

4.1.1 Definition [19]

A double-framed pair $\langle(\alpha, \beta); A\rangle$ is called a double-framed soft set over X, where α and β are mappings from A to $\mathcal{P}(X)$.

From now onwards, we shall use the notation $A_{(\alpha,\beta)}$ over X to denote a double-framed soft set $\langle(\alpha, \beta); A\rangle$ over X.

4.1.2 Definition

For double-*framed* soft sets $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X, we say that $A_{(\alpha,\beta)}$ is a *double-framed* soft subset of $B_{(\gamma,\delta)}$, if

1) $A \subseteq B$ and

2) $\alpha(e) \subseteq \gamma(e)$ and $\delta(e) \subseteq \beta(e)$ for all $e \in A$.

This relationship is denoted by $A_{(\alpha,\beta)} \tilde{\subseteq} B_{(\gamma,\delta)}$.

$A_{(\alpha,\beta)}$ is said to be a *double-framed soft superset* of $B_{(\gamma,\delta)}$, if $B_{(\gamma,\delta)}$ is a *double-framed soft subset* of $A_{(\alpha,\beta)}$. We denote it by $A_{(\alpha,\beta)} \tilde{\supseteq} B_{(\gamma,\delta)}$.

4.1.3 Definition

Two *double-framed soft sets* $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X are said to be *equal* if $A_{(\alpha,\beta)}$ is a *double-framed soft subset* of $B_{(\gamma,\delta)}$ and $B_{(\gamma,\delta)}$ is a *double-framed soft subset* of $A_{(\alpha,\beta)}$. We denote it by $A_{(\alpha,\beta)} \tilde{=} B_{(\gamma,\delta)}$.

4.1.4 Example

Let X be the set of houses under consideration, and E be the set of parameters, $X = \{h_1,h_2,h_3,h_4,h_5\}$, $E = \{e_1,e_2,e_3,e_4,e_5,e_6\} = \{$ in the green surroundings, wooden, cheap, in good repair, furnished, traditional $\}$. Suppose that $A = \{e_1,e_2,e_3,e_6\}$, a double-framed soft set $A_{(\alpha,\beta)}$ describes the data for "requirements of the houses" where function α approximates the houses with a high level of appreciation and β approximates the houses with a high level of critique by two different groups of experts and given as follows:

$$\alpha \ : \ A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_2, h_3, h_4\} & \text{if } e = e_1, \\ \{h_3, h_4\} & \text{if } e = e_2, \\ X & \text{if } e = e_3, \\ \{h_2, h_3, h_4, h_5\} & \text{if } e = e_6, \end{cases}$$

$$\beta \ : \ A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_2, h_4, h_5\} & \text{if } e = e_1, \\ \{h_1, h_2, h_3\} & \text{if } e = e_2, \\ \{h_3, h_4, h_5\} & \text{if } e = e_3, \\ \{h_1, h_3, h_4, h_5\} & \text{if } e = e_6. \end{cases}$$

Let $B = \{e_2,e_3,e_6\}$. The double-framed soft set $B_{(\gamma,\delta)}$ given by

$$\gamma \ : \ B \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_3\} & \text{if } e = e_2, \\ \{h_1, h_4, h_5\} & \text{if } e = e_3, \\ \{h_2, h_3, h_4\} & \text{if } e = e_6, \end{cases}$$

$$\delta \ : \ B \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_1, h_2, h_3, h_5\} & \text{if } e = e_2, \\ \{h_1, h_3, h_4, h_5\} & \text{if } e = e_3, \\ X & \text{if } e = e_6. \end{cases}$$

is a double-framed soft subset of $A_{(\alpha,\beta)}$ so $A_{(\alpha,\beta)} \tilde{\subseteq} B_{(\gamma,\delta)}$. Here, we can see that γ approximates less houses than α being less appreciating, while δ approximates more houses than β being less critical. This justifies our definition of inclusion for double-framed soft sets.

4.2 Operations on Double-framed Soft Sets

4.2.1 Definition [19]

Let $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ be *double-framed* soft sets over X. The int-uni product of $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ is defined as a double-framed soft set $(A \times B)_{(\alpha\wedge\gamma,\beta\vee\delta)}$ over X in which $\alpha \wedge \gamma : (A \times B) \to \mathcal{P}(X)$, $\beta \vee \delta : (A \times B) \to \mathcal{P}(X)$, defined by

$$(a, b) \mapsto \alpha(a) \cap \gamma(b),\ (a, b) \mapsto \beta(a) \cup \delta(b).$$

It is denoted by $A_{(\alpha,\beta)} \wedge B_{(\gamma,\delta)} \tilde{=} (A \times B)_{(\alpha\wedge\gamma,\beta\vee\delta)}$.

4.2.2 Definition [19]

Let $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ be *double-framed* soft sets over X. The uni-int product of $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ is defined as a double-framed soft set $(A \times B)_{(\alpha\vee\gamma,\beta\wedge\delta)}$ over X in which $\alpha \vee \gamma : (A \times B) \to \mathcal{P}(X)$, $\beta \wedge \delta : (A \times B) \to \mathcal{P}(X)$, defined by

$$(a, b) \mapsto \alpha(a) \cup \gamma(b),\ (a, b) \mapsto \beta(a) \cap \delta(b).$$

It is denoted by $A_{(\alpha,\beta)} \vee B_{(\gamma,\delta)} \tilde{=} (A \times B)_{(\alpha\vee\gamma,\beta\wedge\delta)}$.

4.2.3 Definition

For double-*framed soft sets* $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X, the *extended int-uni double-framed soft set* of $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ is defined as a *double-framed soft set* $(A{\cup}B)_{(\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta)}$

where $\alpha\tilde{\cap}\gamma : (A \cup B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \begin{cases} \alpha(e) & if\ e \in A - B \\ \gamma(e) & if\ e \in B - A \\ \alpha(e) \cap \gamma(e) & if\ e \in A \cap B \end{cases}$$

and $\beta\tilde{\cup}\delta : (A \cup B) \to \mathcal{P}(X)$,

$$e \mapsto \begin{cases} \beta(e) & if\ e \in A - B \\ \delta(e) & if\ e \in B - A \\ \beta(e) \cup \delta(e) & if\ e \in A \cap B \end{cases}.$$

It is denoted by $A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)} \tilde{=} (A \cup B)_{(\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta)}$.

4.2.4 Definition

For double-*framed soft sets* $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X, the *extended uni-int set double-framed soft* of $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ is defined as a *double-framed soft set* $(A \cup B)_{(\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta)}$ where $\alpha\tilde{\cup}\gamma : (A \cup B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \begin{cases} \alpha(e) & if\ e \in A - B \\ \gamma(e) & if\ e \in B - A \\ \alpha(e) \cup \gamma(e) & if\ e \in A \cap B \end{cases}$$

and $\beta\tilde{\cap}\delta : (A \cup B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \begin{cases} \beta(e) & if\ e \in A - B \\ \delta(e) & if\ e \in B - A \\ \beta(e) \cap \delta(e) & if\ e \in A \cap B \end{cases}.$$

It is denoted by $A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)} \tilde{=} (A \cup B)_{(\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta)}$.

4.2.5 Definition

For double-framed soft sets $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X, the *extended difference double-framed soft set* of $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ is defined as a double-framed soft set $(A \cup B)_{(\alpha \smile_{\varepsilon} \gamma,\beta \smile_{\varepsilon} \delta)}$ where

$$\alpha \smile_{\varepsilon} \gamma : (A \cup B) \to \mathcal{P}(X),\ e \mapsto \begin{cases} \alpha(e) & \text{if } e \in A - B \\ \gamma(e) & \text{if } e \in B - A \\ \alpha(e) - \gamma(e) & \text{if } e \in A \cap B \end{cases}$$

$$\beta \smile_{\varepsilon} \delta : (A \cup B) \to \mathcal{P}(X),\ e \mapsto \begin{cases} \beta(e) & \text{if } e \in A - B \\ \delta(e) & \text{if } e \in B - A \\ \beta(e) - \delta(e) & \text{if } e \in A \cap B \end{cases}.$$

It is denoted by $A_{(\alpha,\beta)} \smile_{\varepsilon} B_{(\gamma,\delta)} \tilde{=} (A \cup B)_{(\alpha \smile_{\varepsilon} \gamma,\beta \smile_{\varepsilon} \delta)}$.

4.2.6 Definition

For double-*framed soft sets* $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X with $A \cap B \neq \emptyset$, the *restricted int-uni double-framed soft set* of $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ is defined as a *double-framed soft set* $(A \cap B)_{(\alpha \tilde{\cap} \gamma, \beta \tilde{\cup} \delta)}$ where $\alpha \tilde{\cap} \gamma : (A \cap B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \alpha(e) \cap \gamma(e),$$

and $\beta \tilde{\cup} \delta : (A \cap B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \beta(e) \cup \delta(e).$$

It is denoted by $A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)} \tilde{=} (A \cap B)_{(\alpha \tilde{\cap} \gamma, \beta \tilde{\cup} \delta)}$.

4.2.7 Definition

For double-*framed soft sets* $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X with $(A \cap B) \neq \emptyset$, the *restricted uni-int double-framed soft set* of $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ is defined as a *double-framed soft set* $(A \cap B)_{(\alpha \tilde{\cup} \gamma, \beta \tilde{\cap} \delta)}$ where $\alpha \tilde{\cup} \gamma : (A \cap B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \alpha(e) \cup \gamma(e),$$

and $\beta \tilde{\cap} \delta : (A \cap B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \beta(e) \cap \delta(e).$$

It is denoted by $A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)} \tilde{=} (A \cap B)_{(\alpha \tilde{\cup} \gamma, \beta \tilde{\cap} \delta)}$.

4.2.8 Definition

For double-*framed soft sets* $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X with $(A \cap B) \neq \emptyset$, the *restricted difference double-framed soft set* of $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ is defined as a *double-framed soft set* $(A \cap B)_{(\alpha \smile \gamma, \beta \smile \delta)}$ where $\alpha \smile \gamma : (A \cap B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \alpha(e) - \gamma(e),$$

and $\beta \smile \delta : (A \cap B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \beta(e) - \delta(e).$$

It is denoted by $A_{(\alpha,\beta)} \smile B_{(\gamma,\delta)} \tilde{=} (A \cap B)_{(\alpha \smile \gamma, \beta \smile \delta)}$.

4.2.9 Definition

Let $A_{(\alpha,\beta)}$ be a *double-framed* soft set over X. The *complement of a double-framed soft set* $A_{(\alpha,\beta)}$ is defined as a *double-framed soft set* $A_{(\alpha^c,\beta^c)}$ where

$$\alpha^c : A \to \mathcal{P}(X),\ e \mapsto (\alpha\,(e))^c \text{ and } \beta^c : A \to \mathcal{P}(X),\ e \mapsto (\beta\,(e))^c.$$

It is denoted by $A_{(\alpha,\beta)^c} \tilde{=} A_{(\alpha^c,\beta^c)}$.

4.2.10 Example

Let X be the initial universe and E be the set of parameters, where $X = \{x_1,x_2,x_3,x_4,x_5,x_6\}$, $E = \{e_1,e_2,e_3,e_4,e_5\}$. Suppose that $A = \{e_2,e_3\}$, and $B = \{e_3,e_4,\}$. The double-framed soft sets $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X are given as follows:

$$\begin{aligned} \alpha &: A \to \mathcal{P}(X), \\ e &\longmapsto \begin{cases} \{x_2, x_5, x_6\} & \text{if } e = e_2, \\ \{x_1, x_3, x_4, x_5\} & \text{if } e = e_3, \end{cases} \\ \beta &: A \to \mathcal{P}(X), \\ e &\longmapsto \begin{cases} \{x_1\} & \text{if } e = e_2, \\ X & \text{if } e = e_3, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \gamma &: B \to \mathcal{P}(X), \\ e &\longmapsto \begin{cases} X & \text{if } e = e_3, \\ \{x_1, x_4, x_6\} & \text{if } e = e_4, \end{cases} \\ \delta &: B \to \mathcal{P}(X), \\ e &\longmapsto \begin{cases} \{x_1, x_4, x_5, x_6\} & \text{if } e = e_3, \\ \{x_1, x_2, x_5\} & \text{if } e = e_4. \end{cases} \end{aligned}$$

Now, we apply various operations on $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$. Then

(i) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)} \tilde{=} (A \cup B)_{(\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta)}$, where

$$\begin{aligned} (\alpha\tilde{\cup}\gamma) &: (A \cup B) \to \mathcal{P}(X), \\ e &\longmapsto \begin{cases} \{x_2, x_5, x_6\} & \text{if } e = e_2, \\ X & \text{if } e = e_3, \\ \{x_1, x_4, x_6\} & \text{if } e = e_4, \end{cases} \\ (\beta\tilde{\cap}\delta) &: (A \cup B) \to \mathcal{P}(X), \\ e &\longmapsto \begin{cases} \{x_1\} & \text{if } e = e_2, \\ \{x_1, x_4, x_5, x_6\} & \text{if } e = e_3, \\ \{x_1, x_2, x_5\} & \text{if } e = e_4, \end{cases} \end{aligned}$$

(ii) $A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)} \tilde{=} (A \cap B)_{(\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta)}$, where

$$\begin{aligned} (\alpha\tilde{\cap}\beta) &: (A \cap B) \to \mathcal{P}(X), \\ e_3 &\longmapsto \{x_1, x_3, x_4, x_5\} \\ (\beta\tilde{\cup}\delta) &: (A \cap B) \to \mathcal{P}(X), \\ e_3 &\longmapsto X \end{aligned}$$

(iii) $A_{(\alpha,\beta)} \smile_{\varepsilon} B_{(\gamma,\delta)} \tilde{=} (A \cup B)_{(\alpha \smile_{\varepsilon} \gamma, \beta \smile_{\varepsilon} \delta)}$, where

$$\begin{aligned} \alpha \smile_{\varepsilon} \gamma &: (A \cup B) \to \mathcal{P}(X), \\ e &\longmapsto \begin{cases} \{x_2, x_5, x_6\} & \text{if } e = e_2, \\ \{\} & \text{if } e = e_3, \\ \{x_1, x_4, x_6\} & \text{if } e = e_4, \end{cases} \\ \beta \smile_{\varepsilon} \delta &: (A \cup B) \to \mathcal{P}(X), \\ e &\longmapsto \begin{cases} \{x_1\} & \text{if } e = e_2, \\ \{x_2, x_3\} & \text{if } e = e_3, \\ \{x_1, x_2, x_5\} & \text{if } e = e_4, \end{cases} \end{aligned}$$

(iv) $A_{(\alpha,\beta)^c} \tilde{=} A_{(\alpha^c,\beta^c)}$, where

$$\begin{aligned} \alpha^c &: A \to \mathcal{P}(X), \\ e &\longmapsto \begin{cases} \{x_1, x_3, x_4\} & \text{if } e = e_2, \\ \{x_2, x_6\} & \text{if } e = e_3, \end{cases} \\ \beta^c &: A \to \mathcal{P}(X), \\ e &\longmapsto \begin{cases} \{x_2, x_3, x_4, x_5, x_6\} & \text{if } e = e_2, \\ \{\} & \text{if } e = e_3. \end{cases} \end{aligned}$$

4.3 Properties of Double-framed Soft Sets

In this section we discuss properties and laws of double-framed soft sets with respect to their operations. Associativity, absorption, distributivity, de Morgan laws and properties of involutions, complementations and atomicity are investigated for double-framed soft set theory.

4.3.1 Definition

A *double-framed soft set* over X is said to be a *relative null double-framed soft set,* denoted by $A_{(\Phi,\mathfrak{X})}$ where

$$\Phi : A \to \mathcal{P}(X),\ e \mapsto \emptyset \ \text{ and } \mathfrak{X} : A \to \mathcal{P}(X),\ e \mapsto X.$$

4.3.2 Definition

A *double-framed* soft set over X is said to be a *relative absolute double-framed soft set,* denoted by $A_{(\mathfrak{X},\Phi)}$ where

$$\mathfrak{X} : A \to \mathcal{P}(X),\ e \mapsto X \text{ and } \ \Phi : A \to \mathcal{P}(X),\ e \mapsto \emptyset.$$

Conventionally, we take the *double-framed* soft sets with empty set of parameters to be equal to $\emptyset_{(\Phi,\mathfrak{X})}$ and so $A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)} \tilde{=} A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)} \tilde{=} \emptyset_{(\Phi,\mathfrak{X})}$ whenever $(A \cap B) = \emptyset$.

4.3.3 Proposition

If $A_{(\Phi,\mathfrak{X})}$ is a null *double-framed* soft set, $A_{(\mathfrak{X},\Phi)}$ an absolute *double-framed* soft set, and $A_{(\alpha,\beta)}$, $A_{(\gamma,\delta)}$ are *double-framed* soft sets over X, then

1) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} A_{(\gamma,\delta)} \tilde{=} A_{(\alpha,\beta)} \sqcup A_{(\gamma,\delta)}$,

2) $A_{(\alpha,\beta)} \sqcap_{\varepsilon} A_{(\gamma,\delta)} \tilde{=} A_{(\alpha,\beta)} \sqcap A_{(\gamma,\delta)}$,

3) $A_{(\alpha,\beta)} \sqcap A_{(\alpha,\beta)} \tilde{=} A_{(\alpha,\beta)} \tilde{=} A_{(\alpha,\beta)} \sqcup A_{(\alpha,\beta)}$,

4) $A_{(\alpha,\beta)} \sqcup A_{(\Phi,\mathfrak{X})} \tilde{=} A_{(\alpha,\beta)} \tilde{=} A_{(\alpha,\beta)} \sqcap A_{(\mathfrak{X},\Phi)}$,

5) $A_{(\alpha,\beta)} \sqcup A_{(\mathfrak{X},\Phi)} \tilde{=} A_{(\mathfrak{X},\Phi)}$; $A_{(\alpha,\beta)} \sqcap A_{(\Phi,\mathfrak{X})} \tilde{=} A_{(\Phi,\mathfrak{X})}$.

Proof. Proofs of 1), 2) and 3) are straightforward.

4) As $A_{(\alpha,\beta)} \sqcup A_{(\Phi,\mathfrak{X})} \tilde{=} A_{(\alpha\tilde{\cup}\Phi,\beta\tilde{\cap}\mathfrak{X})}$. Therefore for any $e \in A$,

$$(\alpha\tilde{\cup}\Phi)(e) = \alpha(e) \cup \Phi(e) = \alpha(e) \text{ and } (\beta\tilde{\cap}\mathfrak{X})(e) = \beta(e) \cap \mathfrak{X}(e) = \beta(e).$$

Thus $A_{(\alpha,\beta)} \sqcup A_{(\Phi,\mathfrak{X})} \tilde{=} A_{(\alpha,\beta)}$.

Again, $A_{(\alpha,\beta)} \sqcap A_{(\mathfrak{X},\Phi)} \tilde{=} A_{(\alpha\tilde{\cap}\mathfrak{X},\beta\tilde{\cup}\Phi)}$. For any $e \in A$,

$$(\alpha\tilde{\cap}\mathfrak{X})(e) = \alpha(e) \cap \mathfrak{X}(e) = \alpha(e) \text{ and } (\beta\tilde{\cup}\Phi)(e) = \beta(e) \cup \Phi(e) = \beta(e).$$

So $A_{(\alpha,\beta)} \sqcap A_{(\mathfrak{X},\Phi)} \tilde{=} A_{(\alpha,\beta)}$.

Part 5) can be proved in a similar way. ■

4.3.4 Proposition

Let $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}$ and $C_{(\zeta,\eta)}$ be any *double-framed soft set*s over X. Then the following are true

1) $A_{(\alpha,\beta)}\lambda(B_{(\gamma,\delta)}\lambda C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)}\lambda B_{(\gamma,\delta)})\lambda C_{(\zeta,\eta)}$, (Associative Laws)

2) $A_{(\alpha,\beta)}\lambda B_{(\gamma,\delta)} \tilde{=} B_{(\gamma,\delta)}\lambda A_{(\alpha,\beta)}$, (Commutative Laws)

for all $\lambda \in \{\sqcup_{\varepsilon}, \sqcup, \sqcap_{\varepsilon}, \sqcap\}$.
Proof.

1) Since $A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A \cup (B \cup C))_{(\alpha\tilde{\cup}(\gamma\tilde{\cup}\zeta),\beta\tilde{\cap}(\delta\tilde{\cap}\eta))}$, we have for any $e \in A \cup (B \cup C)$:

(i) If $e \in A - (B \cup C)$, then

$$\begin{aligned}(\alpha\tilde{\cup}(\gamma\tilde{\cup}\zeta))(e) &= \alpha(e) = ((\alpha\tilde{\cup}\gamma)\tilde{\cup}\zeta)(e)\\ (\beta\tilde{\cap}(\delta\tilde{\cap}\eta))(e) &= \beta(e) = ((\beta\tilde{\cap}\delta)\tilde{\cap}\eta)(e)\end{aligned}$$

(ii) If $e \in B - (A \cup C)$

$$\begin{aligned}(\alpha\tilde{\cup}(\gamma\tilde{\cup}\zeta))(e) &= \gamma(e) = ((\alpha\tilde{\cup}\gamma)\tilde{\cup}\zeta)(e)\\ (\beta\tilde{\cap}(\delta\tilde{\cap}\eta))(e) &= \delta(e) = ((\beta\tilde{\cap}\delta)\tilde{\cap}\eta)(e)\end{aligned}$$

(iii) If $e \in C - (A \cup B)$, then

$$\begin{aligned}\alpha\tilde{\cup}(\gamma\tilde{\cup}\zeta)(e) &= \zeta(e) = (\alpha\tilde{\cup}\gamma)\tilde{\cup}\zeta(e)\\ \beta\tilde{\cap}(\delta\tilde{\cap}\eta)(e) &= \eta(e) = (\beta\tilde{\cap}\delta)\tilde{\cap}\eta(e)\end{aligned}$$

(iv) If $e \in (A \cap B) - C$, then

$$\begin{aligned}\alpha\tilde{\cup}(\gamma\tilde{\cup}\zeta)(e) &= \alpha(e) \cup \gamma(e) = (\alpha\tilde{\cup}\gamma)(e) = (\alpha\tilde{\cup}\gamma)\tilde{\cup}\zeta(e)\\ \beta\tilde{\cap}(\delta\tilde{\cap}\eta)(e) &= \beta(e) \cap \delta(e) = (\beta\tilde{\cap}\delta)(e) = (\beta\tilde{\cap}\delta)\tilde{\cap}\eta(e)\end{aligned}$$

(v) If $e \in (A \cap C) - B$, then

$$\begin{aligned}\alpha\tilde{\cup}(\gamma\tilde{\cup}\zeta)(e) &= \alpha(e) \cup \zeta(e) = (\alpha\tilde{\cup}\gamma)\tilde{\cup}\zeta(e)\\ \beta\tilde{\cap}(\delta\tilde{\cap}\eta)(e) &= \beta(e) \cap \eta(e) = (\beta\tilde{\cap}\delta)\tilde{\cap}\eta(e)\end{aligned}$$

(vi) If $e \in (B \cap C) - A$, then

$$\begin{aligned}\alpha\tilde{\cup}(\gamma\tilde{\cup}\zeta)(e) &= \gamma(e) \cup \zeta(e) = (\alpha\tilde{\cup}\gamma)\tilde{\cup}\zeta(e)\\ \beta\tilde{\cap}(\delta\tilde{\cap}\eta)(e) &= \delta(e) \cap \eta(e) = (\beta\tilde{\cap}\delta)\tilde{\cap}\eta(e)\end{aligned}$$

(vii) If $e \in (A \cap B) \cap C$, then

$$\begin{aligned}\alpha\tilde{\cup}(\gamma\tilde{\cup}\zeta)(e) &= \alpha(e) \cup (\gamma(e) \cup \zeta(e)) = (\alpha(e) \cup \gamma(e)) \cup \zeta(e) = (\alpha\tilde{\cup}\gamma)\tilde{\cup}\zeta(e)\\ \beta\tilde{\cap}(\delta\tilde{\cap}\eta)(e) &= \beta(e) \cap (\delta(e) \cap \eta(e)) = (\beta(e) \cap \delta(e)) \cap \eta(e) = (\beta\tilde{\cap}\delta)\tilde{\cap}\eta(e)\end{aligned}$$

Thus $A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}) \sqcup_{\varepsilon} C_{(\zeta,\eta)}$. Similarly, we can prove for $\lambda \in \{\sqcup, \sqcap_{\varepsilon}, \sqcap\}$.

2) This is straightforward.

■

4.3.5 Proposition (Absorption Laws)

Let $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}$ be any *double-framed* soft sets over X. Then the following are true:

1) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcap A_{(\alpha,\beta)}) \tilde{=} A_{(\alpha,\beta)}$,

2) $A_{(\alpha,\beta)} \sqcap (B_{(\gamma,\delta)} \sqcup_{\varepsilon} A_{(\alpha,\beta)}) \tilde{=} A_{(\alpha,\beta)}$,

3) $A_{(\alpha,\beta)} \sqcup (B_{(\gamma,\delta)} \sqcap_{\varepsilon} A_{(\alpha,\beta)}) \tilde{=} A_{(\alpha,\beta)}$,

4) $A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcup A_{(\alpha,\beta)}) \tilde{=} A_{(\alpha,\beta)}$.

Proof. Straightforward. ■

4.3.6 Proposition (Distributive Laws)

Let $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}$ and $C_{(\zeta,\eta)}$ be any *double-framed soft set*s over X. Then

1) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcup C_{(\zeta,\eta)}) \tilde{\subseteq} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}) \sqcup (A_{(\alpha,\beta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)})$,

2) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}) \tilde{\supseteq} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}) \sqcap_{\varepsilon} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)})$,

3) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcap C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}) \sqcap (A_{(\alpha,\beta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)})$,

4) $A_{(\alpha,\beta)} \sqcup (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcup C_{(\zeta,\eta)})$,

5) $A_{(\alpha,\beta)} \sqcup (B_{(\gamma,\delta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)}) \sqcap_{\varepsilon} (A_{(\alpha,\beta)} \sqcup C_{(\zeta,\eta)})$,

6) $A_{(\alpha,\beta)} \sqcup (B_{(\gamma,\delta)} \sqcap C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)}) \sqcap (A_{(\alpha,\beta)} \sqcup C_{(\zeta,\eta)})$,

7) $A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \tilde{\subseteq} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)})$,

8) $A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcup C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)}) \sqcup (A_{(\alpha,\beta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)})$,

9) $A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcap C_{(\zeta,\eta)}) \tilde{\supseteq} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)}) \sqcap (A_{(\alpha,\beta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)})$,

10) $A_{(\alpha,\beta)} \sqcap (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcap C_{(\zeta,\eta)})$,

11) $A_{(\alpha,\beta)} \sqcap (B_{(\gamma,\delta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)}) \sqcap_{\varepsilon} (A_{(\alpha,\beta)} \sqcap C_{(\zeta,\eta)})$,

12) $A_{(\alpha,\beta)} \sqcap (B_{(\gamma,\delta)} \sqcup C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)}) \sqcup (A_{(\alpha,\beta)} \sqcap C_{(\zeta,\eta)})$.

Proof. Consider 10)

$$A_{(\alpha,\beta)} \sqcap (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcap C_{(\zeta,\eta)}).$$

For any $e \in A \cap (B \cup C)$, we have following three disjoint cases:

(i) If $e \in A \cap (B - C)$, then

$$(\alpha\tilde{\cap}(\gamma\tilde{\cup}\zeta))(e) = \alpha(e) \cap \gamma(e) \text{ and } (\beta\tilde{\cup}(\delta\tilde{\cap}\eta))(e) = \beta(e) \cup \delta(e)$$

and

$$\begin{aligned}((\alpha\tilde{\cap}\gamma)\tilde{\cup}(\alpha\tilde{\cap}\zeta))(e) &= (\alpha\tilde{\cap}\gamma)(e) \cup \emptyset = \alpha(e) \cap \gamma(e) \text{ and} \\ ((\beta\tilde{\cup}\delta)\tilde{\cap}(\beta\tilde{\cup}\eta))(e) &= (\beta\tilde{\cup}\delta)(e) \cap X = \beta(e) \cup \delta(e).\end{aligned}$$

(ii) If $e \in A \cap (C - B)$, then

$$(\alpha\tilde{\cap}(\gamma\tilde{\cup}\zeta))(e) = \alpha(e) \cap \zeta(e) \text{ and } (\beta\tilde{\cup}(\delta\tilde{\cap}\eta))(e) = \beta(e) \cup \eta(e)$$

and

$$\begin{aligned}((\alpha\tilde{\cap}\gamma)\tilde{\cup}(\alpha\tilde{\cap}\zeta))(e) &= \emptyset \cup (\alpha\tilde{\cap}\zeta)(e) = \alpha(e) \cap \zeta(e) \text{ and} \\ ((\beta\tilde{\cup}\delta)\tilde{\cap}(\beta\tilde{\cup}\eta))(e) &= X \cap (\beta\tilde{\cap}\eta)(e) = \beta(e) \cup \eta(e).\end{aligned}$$

(iii) If $e \in A \cap (B \cap C)$, then

$$\begin{aligned}(\alpha\tilde{\cap}(\gamma\tilde{\cup}\zeta))(e) &= \alpha(e) \cap (\gamma(e) \cup \zeta(e)) \text{ and} \\ (\beta\tilde{\cup}(\delta\tilde{\cap}\eta))(e) &= \beta(e) \cup (\delta(e) \cap \eta(e))\end{aligned}$$

and

$$\begin{aligned}((\alpha\tilde{\cap}\gamma)\tilde{\cup}(\alpha\tilde{\cap}\zeta))(e) &= (\alpha\tilde{\cap}\gamma)(e) \cup (\alpha\tilde{\cap}\zeta)(e) \\ &= (\alpha(e) \cap \gamma(e)) \cup (\alpha(e) \cap \zeta(e)) \\ &= \alpha(e) \cap (\gamma(e) \cup \zeta(e)) \qquad \text{and} \\ ((\beta\tilde{\cup}\delta)\tilde{\cap}(\beta\tilde{\cup}\eta))(e) &= (\beta\tilde{\cup}\delta)(e) \cap (\beta\tilde{\cup}\eta)(e) \\ &= (\beta(e) \cup \delta(e)) \cap (\beta(e) \cup \eta(e)) \\ &= \beta(e) \cup (\delta(e) \cap \eta(e)).\end{aligned}$$

Thus

$$A_{(\alpha,\beta)} \sqcap (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcap C_{(\zeta,\eta)}),$$

Similarly we can prove the remaining parts. ■

4.3.7 Example

Let $X = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8\}$ be the set of candidates who have applied for a job position of Office Representative in Customer Care Centre of a company. Let $E = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\} =$ {Hard Working, Optimism, Enthusiasm, Individualism, Imaginative, Flexibility, Decisiveness, Self-confidence, Politeness}. Suppose that $A = \{x_1,x_2,x_3,x_6,x_7,x_9\}$, $B = \{x_2,x_4,x_5,x_7,x_8\}$, $C = \{x_3,x_5,x_7,x_9\}$, the double-framed soft sets $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}$, $C_{(\zeta,\eta)}$ describes the " Personality Analysis of Candidates" for three different positions. The company has recorded this data obtained through interview and practical sessions conducted by a panel of experts which is presented by mappings α, γ, ζ and β, δ, η for three positions respectively. The double-framed soft sets $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}$, $C_{(\zeta,\eta)}$ over X be given as follows:

$$\alpha : A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{m_1, m_4, m_5, m_6, m_8\} & \text{if } e = x_1, \\ \{m_1, m_2, m_3, m_4, m_7, m_8\} & \text{if } e = x_2, \\ \{m_2, m_4, m_6, m_7, m_8\} & \text{if } e = x_3, \\ \{m_4, m_5, m_6, m_7\} & \text{if } e = x_6, \\ \{m_5, m_6, m_8\} & \text{if } e = x_7, \\ \{m_2, m_3, m_4, m_6, m_7\} & \text{if } e = x_9, \end{cases}$$

$$\beta : A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{m_1, m_2, m_3, m_5, m_7, m_8\} & \text{if } e = x_1, \\ \{m_2, m_5, m_6, m_7\} & \text{if } e = x_2, \\ \{m_1, m_2, m_3, m_4, m_6, m_8\} & \text{if } e = x_3, \\ \{m_3, m_4, m_5, m_6, m_7\} & \text{if } e = x_6, \\ \{m_1, m_2, m_3\} & \text{if } e = x_7, \\ \{m_4, m_5, m_6, m_7, m_8\} & \text{if } e = x_9. \end{cases}$$

$$\gamma : B \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{m_1, m_2, m_3, m_6, m_7\} & \text{if } e = x_2, \\ \{m_2, m_3, m_4, m_8\} & \text{if } e = x_4, \\ \{m_1, m_2, m_4, m_6, m_7, m_8\} & \text{if } e = x_5, \\ \{m_2, m_4, m_6, m_8\} & \text{if } e = x_7, \\ \{m_1, m_2, m_3, m_5, m_6, m_7\} & \text{if } e = x_8, \end{cases}$$

$$\delta : B \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{m_2, m_3, m_4, m_5, m_6\} & \text{if } e = x_2, \\ \{m_4, m_6, m_7, m_8\} & \text{if } e = x_4, \\ \{m_3, m_4, m_5, m_7\} & \text{if } e = x_5, \\ \{m_1, m_2, m_3\} & \text{if } e = x_7, \\ \{m_3, m_4, m_5, m_6, m_7, m_8\} & \text{if } e = x_8. \end{cases}$$

$$\zeta : C \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{m_5, m_7, m_8\} & \text{if } e = x_3, \\ \{m_1, m_2, m_4, m_5, m_6, m_7\} & \text{if } e = x_5, \\ \{m_6, m_7\} & \text{if } e = x_7, \\ \{m_1, m_2, m_3, m_4, m_5\} & \text{if } e = x_9, \end{cases}$$

$$\eta : C \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{m_1, m_2, m_3, m_4, m_5, m_8\} & \text{if } e = x_3, \\ \{m_3, m_4, m_5, m_6\} & \text{if } e = x_5, \\ \{m_2, m_3, m_6\} & \text{if } e = x_7, \\ \{m_2, m_3, m_5, m_6, m_7, m_8\} & \text{if } e = x_9. \end{cases}$$

Now

$$A_{(\alpha,\beta)} \sqcap_\varepsilon (B_{(\gamma,\delta)} \sqcap C_{(\zeta,\eta)}) \tilde{=} (A \cup (B \cap C))_{(\alpha \tilde{\cap} (\gamma \tilde{\cap} \zeta), \beta \tilde{\cup} (\delta \tilde{\cup} \eta))}$$

and

$$(A_{(\alpha,\beta)} \sqcap_\varepsilon B_{(\gamma,\delta)}) \sqcap (A_{(\alpha,\beta)} \sqcap_\varepsilon C_{(\zeta,\eta)}) \tilde{=} ((A \cup B) \cap (A \cup C))_{((\alpha \tilde{\cap} \gamma) \tilde{\cap} (\alpha \tilde{\cap} \zeta), (\beta \tilde{\cup} \delta) \tilde{\cup} (\beta \tilde{\cup} \eta))}.$$

Then the approximations for parameter x_2 are not same on both sides e.g.

$$\begin{aligned} (\alpha \tilde{\cap} (\gamma \tilde{\cap} \zeta))(x_2) &= \{m_1, m_2, m_3, m_4, m_7, m_8\} \\ &\neq \{m_1, m_2, m_3, m_7\} = ((\alpha \tilde{\cap} \gamma) \tilde{\cap} (\alpha \tilde{\cap} \zeta))(x_2) \text{ and} \\ (\beta \tilde{\cup} (\delta \tilde{\cup} \eta))(x_2) &= \{m_2, m_5, m_6, m_7\} \\ &\neq \{m_2, m_3, m_4, m_5, m_6, m_7\} = ((\beta \tilde{\cup} \delta) \tilde{\cup} (\beta \tilde{\cup} \eta))(x_2). \end{aligned}$$

Thus

$$A_{(\alpha,\beta)} \sqcap_\varepsilon (B_{(\gamma,\delta)} \sqcap C_{(\zeta,\eta)}) \tilde{\neq} (A_{(\alpha,\beta)} \sqcap_\varepsilon B_{(\gamma,\delta)}) \sqcap (A_{(\alpha,\beta)} \sqcap_\varepsilon C_{(\zeta,\eta)}).$$

Now, consider

$$A_{(\alpha,\beta)} \sqcap_\varepsilon (B_{(\gamma,\delta)} \sqcup_\varepsilon C_{(\zeta,\eta)}) \tilde{=} (A \cup (B \cup C))_{(\alpha \tilde{\cap} (\gamma \tilde{\cup} \zeta), \beta \tilde{\cup} (\delta \tilde{\cap} \eta))}$$

and

$$\begin{aligned} (A_{(\alpha,\beta)} \sqcap_\varepsilon B_{(\gamma,\delta)}) \sqcup_\varepsilon (A_{(\alpha,\beta)} \sqcap_\varepsilon C_{(\zeta,\eta)}) &\tilde{=} (A \cup B)_{(\alpha \tilde{\cap} \gamma, \beta \tilde{\cup} \delta)} \sqcup_\varepsilon (A \cup C)_{(\alpha \tilde{\cap} \zeta, \beta \tilde{\cup} \eta)} \\ &\tilde{=} ((A \cup B) \cup (A \cup C))_{((\alpha \tilde{\cap} \gamma) \tilde{\cup} (\alpha \tilde{\cap} \zeta), (\beta \tilde{\cup} \delta) \tilde{\cap} (\beta \tilde{\cup} \eta))}. \end{aligned}$$

Then the approximations for parameter x_2 are not same on both sides e.g.

$$\begin{aligned} (\alpha \tilde{\cap} (\gamma \tilde{\cup} \zeta))(x_2) &= \{m_1, m_2, m_3, m_7\} \\ &\neq \{m_1, m_2, m_3, m_4, m_7, m_8\} = ((\alpha \tilde{\cap} \gamma) \tilde{\cup} (\alpha \tilde{\cap} \zeta))(x_2) \text{ and} \\ (\beta \tilde{\cup} (\delta \tilde{\cap} \eta))(x_2) &= \{m_2, m_3, m_4, m_5, m_6, m_7, m_8\} \\ &\neq \{m_2, m_5, m_6, m_7\} = (\beta \tilde{\cup} \delta) \tilde{\cap} (\beta \tilde{\cup} \eta))(x_2). \end{aligned}$$

Thus

$$A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \tilde{\neq} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}).$$

Similarly it can be shown that

$$A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcup C_{(\zeta,\eta)}) \tilde{\neq} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}) \sqcup (A_{(\alpha,\beta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}).$$

$$A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}) \tilde{\neq} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}) \sqcap_{\varepsilon} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}).$$

4.3.8 Proposition

Let $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}$ and $C_{(\zeta,\eta)}$ be any *double-framed soft sets* over X. Then

1) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}) \sqcap_{\varepsilon} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)})$ if and only if

$$\begin{aligned} \alpha(e) &\subseteq \gamma(e) \text{ and } \beta(e) \supseteq \delta(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ \alpha(e) &\subseteq \zeta(e) \text{ and } \beta(e) \supseteq \eta(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

2) $A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)})$ if and only if

$$\begin{aligned} \alpha(e) &\supseteq \gamma(e) \text{ and } \beta(e) \subseteq \delta(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ \alpha(e) &\supseteq \zeta(e) \text{ and } \beta(e) \subseteq \eta(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

Proof. Straightforward. ■

4.3.9 Corollary

Let $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}$ and $C_{(\zeta,\eta)}$ are three *double-framed soft sets* over X. Then

1) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}) \sqcap_{\varepsilon} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)})$

2) $A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)})$

if and only if

$$\begin{aligned} \alpha(e) &= \gamma(e) \text{ and } \beta(e) = \delta(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ \alpha(e) &= \zeta(e) \text{ and } \beta(e) = \eta(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

4.3.10 Corollary

Let $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}$ and $C_{(\zeta,\eta)}$ are three *double-framed soft sets* over X such that $(A \cap B) - C = (A \cap C) - B = \emptyset$. Then

1) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}) \sqcap_{\varepsilon} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)})$,

2) $A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)})$.

4.3.11 Corollary

Let $A_{(\alpha,\beta)}$, $A_{(\gamma,\delta)}$ and $A_{(\zeta,\eta)}$ are three *double-framed soft set*s over X. Then

$$A_{(\alpha,\beta)}\lambda(A_{(\gamma,\delta)}\mu A_{(\zeta,\eta)})\tilde{=}(A_{(\alpha,\beta)}\lambda A_{(\gamma,\delta)})\mu(A_{(\alpha,\beta)}\lambda A_{(\zeta,\eta)})$$

for distinct λ, $\mu \in \{\sqcap_\varepsilon, \sqcap, \sqcup_\varepsilon, \sqcup\}$.

4.3.12 Theorem

Let $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ be double-framed soft sets over X. Then the following are true

1) $A_{(\alpha,\beta)} \sqcup_\varepsilon B_{(\gamma,\delta)}$ is the smallest double-framed soft set over X which contains both $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$. (Supremum)

2) $A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)}$ is the largest double-framed soft set over X which is contained in both $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$. (Infimum)

Proof.

1) We have $A, B \subseteq (A \cup B)$ and $\alpha(e), \gamma(e) \subseteq \alpha(e) \cup \gamma(e)$ and $\beta(e) \cap \delta(e) \subseteq \beta(e)$, $\beta(e)\cap\delta(e) \subseteq \delta(e)$. So $A_{(\alpha,\beta)}\tilde{\subseteq}A_{(\alpha,\beta)}\sqcup_\varepsilon B_{(\gamma,\delta)}$ and $B_{(\gamma,\delta)}\tilde{\subseteq}A_{(\alpha,\beta)}\sqcup_\varepsilon B_{(\gamma,\delta)}$. Let $C_{(\zeta,\eta)}$ be a double-framed soft set over X, such that $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}\tilde{\subseteq}C_{(\zeta,\eta)}$. Then $A, B \subseteq C$ implies that $(A\cup B) \subseteq C$ and $\alpha(e), \gamma(e) \subseteq \zeta(e)$ implies that $\alpha(e)\cup\gamma(e) \subseteq \zeta(e)$. Also $\eta(e) \subseteq \beta(e)$, $\eta(e) \subseteq \delta(e)$ imply that $\eta(e) \subseteq \beta(e) \cap \delta(e)$ for all $e \in A \cup B$. Thus $A_{(\alpha,\beta)}\sqcup_\varepsilon B_{(\gamma,\delta)}\tilde{\subseteq}C_{(\zeta,\eta)}$. It follows that $A_{(\alpha,\beta)}\sqcup_\varepsilon B_{(\gamma,\delta)}$ is the smallest double-framed soft set over X which contains both $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$.

2) We have $A \cap B \subseteq A, A \cap B \subseteq B$ and $\alpha(e) \cap \gamma(e) \subseteq \alpha(e), \alpha(e) \cap \gamma(e) \subseteq \gamma(e)$ and $\beta(e) \subseteq \beta(e) \cup \delta(e), \delta(e) \subseteq \beta(e) \cup \delta(e)$ for all $e \in A\cap B$. So $A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)}\tilde{\subseteq}A_{(\alpha,\beta)}$ and $A_{(\alpha,\beta)}\sqcap B_{(\gamma,\delta)}\tilde{\subseteq}B_{(\gamma,\delta)}$. Let $C_{(\zeta,\eta)}$ be a double-framed soft set over X, such that $C_{(\zeta,\eta)}\tilde{\subseteq}A_{(\alpha,\beta)}$ and $C_{(\zeta,\eta)}\tilde{\subseteq}B_{(\gamma,\delta)}$. Then $C \subseteq A$, $C \subseteq B$ implies that $C \subseteq A \cap B$ and $\zeta(e) \subseteq \alpha(e)$, $\zeta(e) \subseteq \beta(e)$ imply that $\zeta(e) \subseteq \alpha(e) \cap \beta(e)$, and $\beta(e) \subseteq \eta(e)$, $\delta(e) \subseteq \eta(e)$ imply that $\beta(e)\cup\delta(e) \subseteq \eta(e)$ for all $e \in C$. Thus $C_{(\zeta,\eta)}\tilde{\subseteq}A_{(\alpha,\beta)}\sqcap B_{(\gamma,\delta)}$. It follows that $A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)}$ is the largest double-framed soft set over X which is contained in both $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$.

■

4.4 Algebras of Double-framed Soft Sets

In this section, we discuss the ideas of lattices and algebras for the collections of double-framed soft sets. Let $\mathcal{DSS}(X)^E$ be the collection of all double-framed soft sets over X and $\mathcal{DSS}(X)_A$ be its subcollection of all double-framed soft sets over X with fixed set of parameters A. We note that these collections are partially ordered by the relation of soft inclusion $\tilde{\subseteq}$ given in Definition 4.1.2.

4.4.1 Theorem

$(\mathcal{DSS}(X)^E, \sqcap_\varepsilon, \sqcup)$, $(\mathcal{DSS}(X)^E, \sqcup, \sqcap_\varepsilon)$, $(\mathcal{DSS}(X)^E, \sqcup_\varepsilon, \sqcap)$, $(\mathcal{DSS}(X)^E, \sqcap, \sqcup_\varepsilon)$, $(\mathcal{DSS}(X)_A, \sqcup, \sqcap)$, and $(\mathcal{DSS}(X)_A, \sqcap, \sqcup)$ are complete lattices.

Proof. Let us consider $(\mathcal{DSS}(X)^E, \sqcap_\varepsilon, \sqcup)$. Then for any double-framed soft sets $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}$, $C_{(\zeta,\eta)} \in \mathcal{DSS}(X)^E$, we have

1) $A_{(\alpha,\beta)} \sqcap_\varepsilon B_{(\gamma,\delta)} \tilde{=} (A \cup B)_{(\alpha \tilde{\cap} \gamma, \beta \tilde{\cup} \delta)} \in \mathcal{DSS}(X)^E$ and $A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)} \tilde{=} (A \cap B)_{(\alpha \tilde{\cup} \gamma, \beta \tilde{\cap} \delta)} \in \mathcal{DSS}(X)^E$.

2) From Proposition 4.3.3, we have $A_{(\alpha,\beta)} \sqcap_\varepsilon A_{(\alpha,\beta)} \tilde{=} A_{(\alpha,\beta)}$ and $A_{(\alpha,\beta)} \sqcup A_{(\alpha,\beta)} \tilde{=} A_{(\alpha,\beta)}$.

3) From Proposition 4.3.4 we see that $A_{(\alpha,\beta)} \sqcap_\varepsilon B_{(\gamma,\delta)} \tilde{=} B_{(\gamma,\delta)} \sqcap_\varepsilon A_{(\alpha,\beta)}$ and $A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)} \tilde{=} B_{(\gamma,\delta)} \sqcup A_{(\alpha,\beta)}$. Also $A_{(\alpha,\beta)} \sqcap_\varepsilon (B_{(\gamma,\delta)} \sqcap_\varepsilon C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcap_\varepsilon B_{(\gamma,\delta)}) \sqcap_\varepsilon C_{(\zeta,\eta)}$ and $A_{(\alpha,\beta)} \sqcup (B_{(\gamma,\delta)} \sqcup C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)}) \sqcup C_{(\zeta,\eta)}$.

4) From Proposition 4.3.5,

$$A_{(\alpha,\beta)} \sqcap_\varepsilon (B_{(\gamma,\delta)} \sqcup A_{(\alpha,\beta)}) \tilde{=} A_{(\alpha,\beta)} \text{ and } A_{(\alpha,\beta)} \sqcup (B_{(\gamma,\delta)} \sqcap_\varepsilon A_{(\alpha,\beta)}) \tilde{=} A_{(\alpha,\beta)}.$$

So we conclude that the structure forms a lattice.

Consider a collection of double-framed soft sets $\{A_{i_{(\alpha_i,\beta_i)}} : i \in I\}$ over X. We have, $\bigcup\limits_{i \in I} A_i \subseteq E$ and, let $\Lambda(e) = \{j : e \in A_j\}$ for any $e \in A_i$. Then $\bigcap\limits_{i \in \Lambda(e)} \alpha_i(e) \subseteq X$ and $\bigcup\limits_{i \in I} \beta_i(e) \subseteq X$. Thus $\sqcap_{\varepsilon\, i \in I} A_{i_{(\alpha_i,\beta_i)}} \in \mathcal{DSS}(X)^E$. Again, we have, $\bigcap\limits_{i \in I} A_i \subseteq E$ and for any $e \in \bigcap\limits_{i \in I} A_i$, $\bigcup\limits_{i \in I} \alpha_i(e) \subseteq X$ and $\bigcap\limits_{i \in I} \beta_i(e) \subseteq X$. Thus $\sqcup_{i \in I} A_{i_{(\alpha_i,\beta_i)}} \in \mathcal{DSS}(X)^E$.

Similarly we can show for the remaining structures. ■

4.4.2 Theorem

$(\mathcal{DSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{(\Phi,\mathfrak{X})}, E_{(\mathfrak{X},\Phi)})$, $(\mathcal{DSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{(\mathfrak{X},\Phi)}, \emptyset_{(\Phi,\mathfrak{X})})$, $(\mathcal{DSS}(X)_A, \sqcap, \sqcup, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ and $(\mathcal{DSS}(X)_A, \sqcup, \sqcap, A_{(\mathfrak{X},\Phi)}, A_{(\Phi,\mathfrak{X})})$ are bounded distributive lattices.

Proof. Proposition 4.3.6 assures that $(\mathcal{DSS}(X)^E, \sqcap, \sqcup_\varepsilon)$ and $(\mathcal{DSS}(X)^E, \sqcup_\varepsilon, \sqcap)$ are distributive lattices. From Theorem 4.3.12, we conclude that $(\mathcal{DSS}(X)^E$, $\sqcap$, $\sqcup_\varepsilon$,

$\emptyset_{(\Phi,\mathfrak{X})}$, $E_{(\mathfrak{X},\Phi)}$) is a bounded distributive lattice and $(\mathcal{DSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{(\mathfrak{X},\Phi)}, \emptyset_{(\Phi,\mathfrak{X})})$ is its dual. For any double-framed soft sets $A_{(\alpha,\beta)}$, $A_{(\gamma,\delta)} \in \mathcal{DSS}(X)_A$,

$$\begin{aligned} A_{(\alpha,\beta)} \sqcap A_{(\gamma,\delta)} &\tilde{=} A_{(\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta)} \in \mathcal{DSS}(X)_A \text{ and} \\ A_{(\alpha,\beta)} \sqcup A_{(\gamma,\delta)} &\tilde{=} A_{(\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta)} \in \mathcal{DSS}(X)_A. \end{aligned}$$

Thus $(\mathcal{DSS}(X)_A, \sqcap, \sqcup)$ is a distributive sublattice of $(\mathcal{DSS}(X)^E, \sqcup_\varepsilon, \sqcap)$ and Proposition 4.3.3 tells us that $A_{(\Phi,\mathfrak{X})}$, $A_{(\mathfrak{X},\Phi)}$ are its lower and upper bounds respectively. Therefore $(\mathcal{DSS}(X)_A, \sqcap, \sqcup, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is a bounded distributive lattice and $(\mathcal{DSS}(X)_A, \sqcup, \sqcap, A_{(\mathfrak{X},\Phi)}, A_{(\Phi,\mathfrak{X})})$ is its dual. ■

4.4.3 Proposition

Let $A_{(\alpha,\beta)}$ be a *double-framed soft set* over X. Then $A_{(\alpha,\beta)^c}$ is a complement of $A_{(\alpha,\beta)}$.

Proof. As $A_{(\alpha,\beta)} \sqcup A_{(\alpha,\beta)^c} \tilde{=} A_{(\alpha\tilde{\cup}\alpha^c,\beta\tilde{\cap}\beta^c)}$. Now, for any $e \in A$,

$$\begin{aligned} (\alpha\tilde{\cup}\alpha^c)(e) &= \alpha(e) \cup (\alpha(e))^c = X \quad \text{and} \\ (\beta\tilde{\cap}\beta^c)(e) &= \beta(e) \cap (\beta(e))^c = \emptyset. \end{aligned}$$

Thus $A_{(\alpha,\beta)} \sqcup A_{(\alpha,\beta)^c} \tilde{=} A_{(\mathfrak{X},\Phi)}$.

Again, we have $A_{(\alpha,\beta)} \sqcap A_{(\alpha,\beta)^c} \tilde{=} A_{(\alpha\tilde{\cap}\alpha^c,\beta\tilde{\cup}\beta^c)}$, so for any $e \in A$,

$$\begin{aligned} (\alpha\tilde{\cap}\alpha^c)(e) &= \alpha(e) \cap (\alpha(e))^c = \emptyset \quad \text{and} \\ (\beta\tilde{\cup}\beta^c)(e) &= \beta(e) \cup (\beta(e))^c = X. \end{aligned}$$

Thus $A_{(\alpha,\beta)} \sqcap A_{(\alpha,\beta)^c} \tilde{=} A_{(\Phi,\mathfrak{X})}$. From $A_{(\alpha,\beta)} \sqcup A_{(\alpha,\beta)^c} \tilde{=} A_{(\mathfrak{X},\Phi)}$ and $A_{(\alpha,\beta)} \sqcap A_{(\alpha,\beta)^c} \tilde{=} A_{(\Phi,\mathfrak{X})}$, we conclude that $A_{(\alpha,\beta)^c}$ is a complement of $A_{(\alpha,\beta)}$.

Now, we show that $A_{(\alpha,\beta)^c}$ is unique in the bounded lattice $(\mathcal{DSS}(X)_A, \sqcup, \sqcap, A_{(\mathfrak{X},\Phi)}, A_{(\Phi,\mathfrak{X})})$. If there exists some $A_{(\gamma,\delta)} \in \mathcal{DSS}(X)_A$ such that $A_{(\alpha,\beta)} \sqcup A_{(\gamma,\delta)} \tilde{=} A_{(\mathfrak{X},\Phi)}$ and $A_{(\alpha,\beta)} \sqcap A_{(\gamma,\delta)} \tilde{=} A_{(\Phi,\mathfrak{X})}$. Then for any $e \in A$,

$$\alpha(e) \cap \gamma(e) = \emptyset \text{ and } \beta(e) \cap \delta(e) = \emptyset$$
$$\Rightarrow \gamma(e) \subseteq (\alpha(e))^c = \alpha^c(e) \text{ and } \delta(e) \subseteq (\beta(e))^c = \beta^c(e)$$

and

$$\alpha^c(e) \subseteq X = \alpha(e) \cup \gamma(e) \text{ and } \beta^c(e) \subseteq X = \beta(e) \cup \delta(e).$$

But

$$\alpha(e) \cap \alpha^c(e) = \emptyset \text{ and } \beta(e) \cap \beta^c(e) = \emptyset \text{ so}$$
$$\alpha^c(e) \subseteq \alpha(e) \cup \gamma(e) \Rightarrow \alpha^c(e) \subseteq \gamma(e) \text{ and } \beta^c(e) \subseteq \beta(e) \cup \delta(e) \Rightarrow \beta^c(e) \subseteq \delta(e).$$

Therefore

$$\gamma(e) = \alpha^c(e) \text{ and } \delta(e) = \beta^c(e) \text{ for all } e \in A \text{ and } A_{(\gamma,\delta)} \tilde{=} A_{(\alpha,\beta)^c}.$$

Hence $A_{(\alpha,\beta)^c}$ is unique complement of $A_{(\alpha,\beta)}$. ■

4.4.4 Proposition (de Morgan Laws)

Let $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ be double-*framed* soft sets over X. Then the following are true:

1) $(A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)})^c \tilde{=} A_{(\alpha,\beta)^c} \sqcap_{\varepsilon} B_{(\gamma,\delta)^c}$,

2) $(A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)})^c \tilde{=} A_{(\alpha,\beta)^c} \sqcup_{\varepsilon} B_{(\gamma,\delta)^c}$,

3) $(A_{(\alpha,\beta)} \vee B_{(\gamma,\delta)})^c \tilde{=} A_{(\alpha,\beta)^c} \wedge B_{(\gamma,\delta)^c}$,

4) $(A_{(\alpha,\beta)} \wedge B_{(\gamma,\delta)})^c \tilde{=} A_{(\alpha,\beta)^c} \vee B_{(\gamma,\delta)^c}$,

5) $(A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)})^c \tilde{=} A_{(\alpha,\beta)^c} \sqcap B_{(\gamma,\delta)^c}$,

6) $(A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)})^c \tilde{=} A_{(\alpha,\beta)^c} \sqcup B_{(\gamma,\delta)^c}$.

Proof.

1) We know that $(A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)})^c \tilde{=} (A \cup B)_{(\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta)^c} \tilde{=} (A \cup B)_{((\alpha\tilde{\cup}\gamma)^c,(\beta\tilde{\cap}\delta)^c)}$. Let $e \in (A \cup B)$. Then there are three cases:

(i) If $e \in A - B$, then

$$\begin{aligned}(\alpha\tilde{\cup}\gamma)^c(e) &= (\alpha(e))^c = \alpha^c(e) \text{ and } (\alpha^c\tilde{\cap}\gamma^c)(e) = \alpha^c(e) \text{ and}\\ (\beta\tilde{\cap}\delta)^c(e) &= (\beta(e))^c = \beta^c(e) \text{ and } (\beta^c\tilde{\cup}\delta^c)(e) = \beta^c(e).\end{aligned}$$

Thus

$$\begin{aligned}(\alpha\tilde{\cup}\gamma)^c(e) &= (\alpha^c\tilde{\cap}\gamma^c)(e) \text{ and}\\ (\beta\tilde{\cap}\delta)^c(e) &= (\beta^c\tilde{\cup}\delta^c)(e).\end{aligned}$$

(ii) If $e \in B - A$, then

$$\begin{aligned}(\alpha\tilde{\cup}\gamma)^c(e) &= (\gamma(e))^c = \gamma^c(e) \text{ and } (\alpha^c\tilde{\cap}\gamma^c)(e) = \gamma^c(e) \text{ and}\\ (\beta\tilde{\cap}\delta)^c(e) &= (\delta(e))^c = \delta^c(e) \text{ and } (\beta^c\tilde{\cup}\delta^c)(e) = \delta^c(e).\end{aligned}$$

Thus

$$\begin{aligned}(\alpha\tilde{\cup}\gamma)^c(e) &= (\alpha^c\tilde{\cap}\gamma^c)(e) \text{ and}\\ (\beta\tilde{\cap}\delta)^c(e) &= (\beta^c\tilde{\cup}\delta^c)(e).\end{aligned}$$

(iii) If $e \in A \cap B$, then

$$\begin{aligned}(\alpha\tilde{\cup}\gamma)^c(e) &= (\alpha(e) \cup \gamma(e))^c = (\alpha(e))^c \cap (\gamma(e))^c \text{ and}\\ (\beta\tilde{\cup}\delta)^c(e) &= (\beta(e) \cap \delta(e))^c = (\beta(e))^c \cup (\delta(e))^c,\end{aligned}$$

and

$$(\alpha^c \tilde{\cap} \gamma^c)(e) = (\alpha(e))^c \cap (\gamma(e))^c = (\alpha \tilde{\cup} \gamma)^c(e) \text{ and}$$
$$(\beta^c \tilde{\cap} \delta^c)(e) = (\beta(e))^c \cup (\delta(e))^c = (\beta \tilde{\cup} \delta)^c(e).$$

Therefore, in all three cases we obtain equality and thus

$$(A_{(\alpha,\beta)} \sqcup_\varepsilon B_{(\gamma,\delta)})^c \tilde{=} A_{(\alpha,\beta)^c} \sqcap_\varepsilon B_{(\gamma,\delta)^c}.$$

The remaining parts can be proved in a similar way.

■

4.4.5 Proposition

$(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^c, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is a de Morgan algebra.

Proof. We have already seen that $(\mathcal{DSS}(X)_A, \sqcap, \sqcup, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is a bounded distributive lattice. Proposition 4.4.3 show that " ^{c}" is a complementation and hence an involution on $\mathcal{DSS}(X)_A$ and Proposition 4.4.4 shows that de Morgan laws hold with respect to " ^{c}" in $\mathcal{DSS}(X)_A$. Thus $(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^c, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is a de Morgan algebra. ■

4.4.6 Proposition

$(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^c, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is a boolean algebra.

Proof. Proof follows from Propositions 4.4.4 and 4.4.3. ■

4.4.7 Proposition

$(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^c, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is a Kleene Algebra.

Proof. Note that, $A_{(\gamma,\delta)} \sqcap A_{(\gamma,\delta)^c} \tilde{=} \emptyset_{(\Phi,\mathfrak{X})} \tilde{\subset} A_{(\mathfrak{X},\Phi)} \tilde{=} A_{(\alpha,\beta)} \sqcup A_{(\alpha,\beta)^c}$. We already know that $(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^c, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is a de Morgan algebra, so this condition assures that $(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^c, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is also a Kleene Algebra. ■

4.4.8 Definition

Let $A_{(\alpha,\beta)}$ be a *double-framed* soft set over X. We define

$$(A_{(\alpha,\beta)})^\circ \tilde{=} A_{(\alpha,\beta)^\circ} \tilde{=} A_{(\beta,\alpha)}.$$

4.4.9 Proposition

Let $A_{(\alpha,\beta)}$ be a *double-framed soft set* over X. Then $A_{(\alpha,\beta)} \tilde{=} (A_{(\alpha,\beta)^\circ})^\circ$, $A_{(\mathfrak{X},\Phi)^\circ} \tilde{=} A_{(\Phi,\mathfrak{X})}$ and $A_{(\Phi,\mathfrak{X})^\circ} \tilde{=} A_{(\mathfrak{X},\Phi)}$.

Proof. Straightforward. ■

4.4.10 Proposition (de Morgan Laws)

Let $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ be double-*framed* soft sets over X. Then the following are true

1) $(A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)})^{\circ} \tilde{=} A_{(\alpha,\beta)^{\circ}} \sqcap_{\varepsilon} B_{(\gamma,\delta)^{\circ}}$,

2) $(A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)})^{\circ} \tilde{=} A_{(\alpha,\beta)^{\circ}} \sqcup_{\varepsilon} B_{(\gamma,\delta)^{\circ}}$,

3) $(A_{(\alpha,\beta)} \vee B_{(\gamma,\delta)})^{\circ} \tilde{=} A_{(\alpha,\beta)^{\circ}} \wedge B_{(\gamma,\delta)^{\circ}}$,

4) $(A_{(\alpha,\beta)} \wedge B_{(\gamma,\delta)})^{\circ} \tilde{=} A_{(\alpha,\beta)^{\circ}} \vee B_{(\gamma,\delta)^{\circ}}$,

5) $(A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)})^{\circ} \tilde{=} A_{(\alpha,\beta)^{\circ}} \sqcap B_{(\gamma,\delta)^{\circ}}$,

6) $(A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)})^{\circ} \tilde{=} A_{(\alpha,\beta)^{\circ}} \sqcup B_{(\gamma,\delta)^{\circ}}$.

Proof.

1) We have $(A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)})^{\circ} \tilde{=} ((A \cup B)_{(\alpha \tilde{\cup} \gamma, \beta \tilde{\cap} \delta)})^{\circ} \tilde{=} (A \cup B)_{(\beta \tilde{\cap} \delta, \alpha \tilde{\cup} \gamma)}$ and

$$A_{(\alpha,\beta)^{\circ}} \sqcap_{\varepsilon} B_{(\gamma,\delta)^{\circ}} \tilde{=} A_{(\beta,\alpha)} \sqcap_{\varepsilon} B_{(\delta,\gamma)} \tilde{=} (A \cup B)_{(\beta \tilde{\cap} \delta, \alpha \tilde{\cup} \gamma)}.$$

Thus $(A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)})^{\circ} \tilde{=} A_{(\alpha,\beta)^{\circ}} \sqcap_{\varepsilon} B_{(\gamma,\delta)^{\circ}}$.

The remaining parts can be proved in a similar way. ■

4.4.11 Proposition

$(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^{\circ}, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is a de Morgan algebra.

Proof. Proof follows from Propositions 4.4.9 and 4.4.10. ■

4.4.12 Definition

Let $A_{(\alpha,\beta)}$ be a *double-framed* soft set over X. We define $A_{(\alpha,\beta)^{\diamond}}$ as a *double-framed soft set* $A_{(\alpha^c,\mathfrak{X})}$ where

$$\begin{aligned} \alpha^c &: A \to \mathcal{P}(X),\ e \mapsto (\alpha(e))^c \\ \mathfrak{X} &: A \to \mathcal{P}(X),\ e \mapsto X. \end{aligned}$$

4.4.13 Proposition

Let $A_{(\alpha,\beta)}$ and $A_{(\gamma,\delta)}$ be double-framed soft sets over X. Then

1) $A_{(\gamma,\delta)} \sqcap A_{(\gamma,\delta)^{\diamond}} \tilde{=} A_{(\Phi,\mathfrak{X})}$,

2) $A_{(\gamma,\delta)} \tilde{\subset} A_{(\alpha,\beta)^{\diamond}}$ whenever $A_{(\gamma,\delta)} \sqcap A_{(\alpha,\beta)} \tilde{=} A_{(\Phi,\mathfrak{X})}$.

Proof.

1) For any $e \in A$,

$$\begin{aligned}(\gamma\tilde{\cap}\gamma^c)(e) &= \gamma(e) \cap (\gamma(e))^c = \emptyset = \Phi(e) \quad \text{and} \\ (\delta\tilde{\cup}\mathfrak{X})(e) &= \delta(e) \cup X = X = \mathfrak{X}(e).\end{aligned}$$

Thus $A_{(\gamma,\delta)} \sqcap A_{(\gamma,\delta)^\diamond} \tilde{=} A_{(\Phi,\mathfrak{X})}$.

2) Assume $A_{(\gamma,\delta)} \sqcap A_{(\alpha,\beta)} \tilde{=} A_{(\Phi,\mathfrak{X})}$. Now, for any $e \in A$,

$$\begin{aligned}\gamma(e) \cap \alpha(e) &= (\gamma\tilde{\cap}\alpha)(e) = \Phi(e) = \emptyset \text{ and so } \gamma(e) \subseteq (\alpha(e))^c = \alpha^c(e). \\ \text{Also } \delta(e) &\subseteq X = \mathfrak{X}(e).\end{aligned}$$

Therefore $A_{(\gamma,\delta)} \tilde{\subset} A_{(\alpha,\beta)^\diamond}$. So, we conclude that $(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^\diamond, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is pseudocomplemented.

■

4.4.14 Proposition

Let $A_{(\alpha,\beta)}, B_{(\gamma,\delta)} \in \mathcal{DSS}(X)^E$. Then pseudocomplement of $A_{(\alpha,\beta)}$ relative to $B_{(\gamma,\delta)}$ exists in $(\mathcal{DSS}(X)^E, \sqcap, \sqcup_\varepsilon)$.

Proof. Consider the set

$$T(A_{(\alpha,\beta)}, B_{(\gamma,\delta)}) = \{C_{(\zeta,\eta)} \in \mathcal{SS}(X)^E : C_{(\zeta,\eta)} \sqcap A_{(\alpha,\beta)} \tilde{\subseteq} B_{(\gamma,\delta)}\}.$$

We define a double-framed soft set $A^c_{(\alpha^c,\beta^c)} \sqcup_\varepsilon B_{(\gamma,\delta)} \tilde{=} (A^c \cup B)_{(\alpha^c\tilde{\cup}\gamma, \beta^c\tilde{\cap}\delta)} \in \mathcal{DSS}(X)^E$. Then

$$\begin{aligned}(A^c \cup B)_{(\alpha^c\tilde{\cup}\gamma,\beta^c\tilde{\cap}\delta)} \sqcap A_{(\alpha,\beta)} &\tilde{=} ((A^c \cup B) \cap A)_{((\alpha^c\tilde{\cup}\gamma)\tilde{\cap}\alpha, (\beta^c\tilde{\cap}\delta)\tilde{\cup}\beta)} \\ &\tilde{=} ((A^c \cap A) \cup (B \cap A))_{((\alpha^c\tilde{\cap}\alpha)\tilde{\cup}(\gamma\tilde{\cap}\alpha), (\beta^c\tilde{\cup}\beta)\tilde{\cap}(\delta\tilde{\cup}\beta))} \\ &\tilde{=} (A \cap B)_{(\gamma\tilde{\cap}\alpha, \delta\tilde{\cup}\beta)} \tilde{\subseteq} B_{(\gamma,\delta)}.\end{aligned}$$

Thus $(A^c \cup B)_{(\alpha^c\tilde{\cup}\gamma,\beta^c\tilde{\cap}\delta)} \in T(A_{(\alpha,\beta)}, B_{(\gamma,\delta)})$. For any $C_{(\zeta,\eta)} \in T(A_{(\alpha,\beta)}, B_{(\gamma,\delta)})$, we have $C_{(\zeta,\eta)} \sqcap A_{(\alpha,\beta)} \tilde{\subseteq} B_{(\gamma,\delta)}$ so for any $e \in C \cap A \subseteq B$

$$\zeta(e) \cap \alpha(e) \subseteq \gamma(e) \text{ and } \eta(e) \cup \beta(e) \supseteq \delta(e)$$

Now,

$$\begin{aligned}C \cap A \subseteq B &\Rightarrow (A \cap C) \cap B^c = \emptyset \\ &\Rightarrow C \subseteq (A \cap B^c)^c = A^c \cup B\end{aligned}$$

and

$$\zeta(e)\cap\alpha(e)\subseteq\gamma(e) \text{ and } \eta(e)\cup\beta(e)\supseteq\delta(e)$$
$$\Rightarrow \zeta(e)\cap\alpha(e)\cap\gamma^c(e)=\emptyset \text{ and } \eta^c(e)\cap\beta^c(e)\subseteq\delta^c(e)$$
$$\Rightarrow \zeta(e)\subseteq\alpha^c(e)\cup\gamma(e) \quad \text{and} \quad \eta^c(e)\cap\beta^c(e)\cap\delta(e)=\emptyset$$
$$\Rightarrow \zeta(e)\subseteq\alpha^c(e)\cup\gamma(e) \quad \text{and} \quad \beta^c(e)\cap\delta(e)\subseteq\eta(e).$$

Thus $C_{(\zeta,\eta)}\tilde{\subseteq}(A^c\cup B)_{(\alpha^c\tilde{\cup}\gamma,\beta^c\tilde{\cap}\delta)}$, also

$$(A^c\cup B)_{(\alpha^c\tilde{\cup}\gamma,\beta^c\tilde{\cap}\delta)}\tilde{=}\bigvee T(A_{(\alpha,\beta)},B_{(\gamma,\delta)})\tilde{=}A_{(\alpha,\beta)}\rightarrow B_{(\gamma,\delta)}.$$

■

4.4.15 Remark

We know that $(\mathcal{DSS}(X)_A,\sqcap,\sqcup)$ is a sublattice of $(\mathcal{DSS}(X)^E,\sqcap_\varepsilon,\sqcup)$. *For any* $A_{(\alpha,\beta)}$, $A_{(\gamma,\delta)}\in\mathcal{DSS}(X)_A$, $A_{(\alpha,\beta)}\rightarrow A_{(\gamma,\delta)}$ as defined in Lemma 4.4.14, is not in $\mathcal{DSS}(X)_A$ because $A_{(\alpha,\beta)}\rightarrow A_{(\gamma,\delta)}\tilde{=}(A^c\cup A)_{(\alpha^c\tilde{\cup}\gamma,\beta^c\tilde{\cap}\delta)}\tilde{=}E_{(\alpha^c\tilde{\cup}\gamma,\beta^c\tilde{\cap}\delta)}\notin\mathcal{DSS}(X)_A$.

4.4.16 Lemma

Let $A_{(\alpha,\beta)},A_{(\gamma,\delta)}\in\mathcal{DSS}(X)_A$. Then pseudocomplement of $A_{(\alpha,\beta)}$ relative to $A_{(\gamma,\delta)}$ exists in $\mathcal{DSS}(X)^A$.

Proof. Consider the set

$$T(A_{(\alpha,\beta)},A_{(\gamma,\delta)})=\{A_{(\zeta,\eta)}\in\mathcal{DSS}(X)_A : A_{(\zeta,\eta)}\sqcap A_{(\alpha,\beta)}\tilde{\subseteq}A_{(\gamma,\delta)}\}.$$

We define a double-framed soft set $A_{(\alpha^c,\beta^c)}\sqcup A_{(\gamma,\delta)}\tilde{=}A_{(\alpha^c\tilde{\cup}\gamma,\beta^c\tilde{\cap}\delta)}\in\mathcal{DSS}(X)_A$. Consider

$$\begin{aligned}A_{(\alpha^c\tilde{\cup}\gamma,\beta^c\tilde{\cap}\delta)}\sqcap A_{(\alpha,\beta)} &\tilde{=} A_{((\alpha^c\tilde{\cup}\gamma)\tilde{\cap}\alpha,(\beta^c\tilde{\cap}\delta)\tilde{\cup}\beta)}\\ &\tilde{=} A_{((\alpha^c\tilde{\cap}\alpha)\tilde{\cup}(\gamma\tilde{\cap}\alpha),(\beta^c\tilde{\cup}\beta)\tilde{\cap}(\delta\tilde{\cup}\beta))}\\ &\tilde{=} A_{((\gamma\tilde{\cap}\alpha),(\delta\tilde{\cup}\beta))}\tilde{\subseteq}A_{(\gamma,\delta)}.\end{aligned}$$

Thus $A_{(\alpha^c\tilde{\cup}\gamma,\beta^c\tilde{\cap}\delta)}\in T(A_{(\alpha,\beta)},A_{(\gamma,\delta)})$. For every $A_{(\zeta,\eta)}\in T(A_{(\alpha,\beta)},A_{(\gamma,\delta)})$, we have $A_{(\zeta,\eta)}\sqcap A_{(\alpha,\beta)}\tilde{\subseteq}A_{(\gamma,\delta)}$ so for any $e\in A$,

$$\zeta(e)\cap\alpha(e)\subseteq\gamma(e) \text{ and } \eta(e)\cup\beta(e)\supseteq\delta(e)$$
$$\Rightarrow \zeta(e)\cap\alpha(e)\cap\gamma^c(e)=\emptyset \text{ and } \eta^c(e)\cap\beta^c(e)\subseteq\delta^c(e)$$
$$\Rightarrow \zeta(e)\subseteq\alpha^c(e)\cup\gamma(e) \quad \text{and} \quad \eta^c(e)\cap\beta^c(e)\cap\delta(e)=\emptyset$$
$$\Rightarrow \zeta(e)\subseteq\alpha^c(e)\cup\gamma(e) \quad \text{and} \quad \beta^c(e)\cap\delta(e)\subseteq\eta(e).$$

Thus $A_{(\zeta,\eta)}\tilde{\subseteq}A_{(\alpha^c\tilde{\cup}\gamma,\beta^c\tilde{\cap}\delta)}$ and also

$$A_{(\alpha^c\tilde{\cup}\gamma,\beta^c\tilde{\cap}\delta)}\tilde{=}\bigvee T(A_{(\alpha,\beta)},A_{(\gamma,\delta)})\tilde{=}A_{(\alpha,\beta)}\rightarrow_A A_{(\gamma,\delta)}.$$

■

4.4.17 Proposition

$(\mathcal{DSS}(X)^E,\sqcap_\varepsilon,\sqcup)$ *and* $(\mathcal{DSS}(X)_A,\sqcap,\sqcup)$ are *Brouwerian lattices.*

Proof. Follows from Lemmas 4.4.14 and 4.4.16. ■

4.4.18 Theorem

$(\mathcal{DSS}(X)_A,\sqcap,^c,A_{(\mathfrak{X},\Phi)})$ is an MV-algebra.

Proof.

(MV1) $(\mathcal{DSS}(X)_A,\sqcap,A_{(\mathfrak{X},\Phi)})$ is a commutative monoid.

(MV2) $(A_{(\gamma,\delta)^c})^c\tilde{=}A_{(\gamma,\delta)}$.

(MV3) $A_{(\mathfrak{X},\Phi)^c}\sqcap A_{(\gamma,\delta)}\tilde{=}A_{(\Phi,\mathfrak{X})}\sqcap A_{(\gamma,\delta)}\tilde{=}A_{(\Phi,\mathfrak{X})}\tilde{=}A_{(\mathfrak{X},\Phi)^c}$.

(MV4) $(A_{(\alpha,\beta)^c}\sqcap A_{(\gamma,\delta)})^c\sqcap A_{(\gamma,\delta)}$

$\tilde{=}(A_{(\alpha^c,\beta^c)}\sqcap A_{(\gamma,\delta)})^c\sqcap A_{(\gamma,\delta)}$

$\tilde{=}(A_{(\alpha^c,\beta^c)^c}\sqcup A_{(\gamma,\delta)^c})\sqcap A_{(\gamma,\delta)}$

$\tilde{=}(A_{(\alpha,\beta)}\sqcup A_{(\gamma^c,\delta^c)})\sqcap A_{(\gamma,\delta)}$

$\tilde{=}(A_{(\alpha,\beta)}\sqcap A_{(\gamma,\delta)})\sqcup(A_{(\gamma^c,\delta^c)}\sqcap A_{(\gamma,\delta)})$

$\tilde{=}(A_{(\alpha,\beta)}\sqcap A_{(\gamma,\delta)})\sqcup A_{(\Phi,\mathfrak{X})}$

$\tilde{=}(A_{(\gamma,\delta)}\sqcap A_{(\alpha,\beta)})\sqcup(A_{(\alpha,\beta)^c}\sqcap A_{(\alpha,\beta)})$

$\tilde{=}(A_{(\gamma,\delta)}\sqcup A_{(\alpha,\beta)^c})\sqcap A_{(\alpha,\beta)}$

$\tilde{=}(A_{(\gamma,\delta)^c}\sqcap A_{(\alpha,\beta)})^c\sqcap A_{(\alpha,\beta)}$

for all $A_{(\alpha,\beta)}$, $A_{(\gamma,\delta)}\in\mathcal{DSS}(X)_A$. Thus $(\mathcal{DSS}(X)_A,\sqcap,^c,A_{(\mathfrak{X},\Phi)})$ is an MV-algebra.

■

4.4.19 Theorem

$(\mathcal{DSS}(X)_A, \sqcup, ^c, A_{(\Phi,\mathfrak{X})})$ is an MV-algebra.

Proof.

(MV1) $(\mathcal{DSS}(X)_A, \sqcup, A_{(\Phi,\mathfrak{X})})$ is a commutative monoid.

(MV2) $(A_{(\gamma,\delta)^c})^c \tilde{=} A_{(\gamma,\delta)}$.

(MV3) $A_{(\Phi,\mathfrak{X})^c} \sqcup A_{(\gamma,\delta)} \tilde{=} A_{(\mathfrak{X},\Phi)} \sqcup A_{(\gamma,\delta)} \tilde{=} A_{(\mathfrak{X},\Phi)} \tilde{=} A_{(\Phi,\mathfrak{X})^c}$.

(MV4) $(A_{(\alpha,\beta)^c} \sqcup A_{(\gamma,\delta)})^c \sqcup A_{(\gamma,\delta)}$

$\tilde{=}(A_{(\alpha^c,\beta^c)} \sqcup A_{(\gamma,\delta)})^c \sqcup A_{(\gamma,\delta)}$

$\tilde{=}(A_{(\alpha^c,\beta^c)^c} \sqcap A_{(\gamma,\delta)^c}) \sqcup A_{(\gamma,\delta)}$

$\tilde{=}(A_{(\alpha,\beta)} \sqcap A_{(\gamma^c,\delta^c)}) \sqcup A_{(\gamma,\delta)}$

$\tilde{=}(A_{(\alpha,\beta)} \sqcup A_{(\gamma,\delta)}) \sqcap (A_{(\gamma^c,\delta^c)} \sqcup A_{(\gamma,\delta)})$

$\tilde{=}(A_{(\alpha,\beta)} \sqcup A_{(\gamma,\delta)}) \sqcap A_{(\mathfrak{X},\Phi)}$

$\tilde{=}(A_{(\gamma,\delta)} \sqcup A_{(\alpha,\beta)}) \sqcap (A_{(\alpha,\beta)^c} \sqcup A_{(\alpha,\beta)})$

$\tilde{=}(A_{(\gamma,\delta)} \sqcap A_{(\alpha,\beta)^c}) \sqcup A_{(\alpha,\beta)}$

$\tilde{=}(A_{(\gamma,\delta)^c} \sqcup A_{(\alpha,\beta)})^c \sqcup A_{(\alpha,\beta)}$

for all $A_{(\alpha,\beta)}$, $A_{(\gamma,\delta)} \in \mathcal{DSS}(X)_A$. Thus $(\mathcal{DSS}(X)_A, \sqcup, ^c, A_{(\Phi,\mathfrak{X})})$ is an MV-algebra. ■

4.4.20 Theorem

$(\mathcal{DSS}(X)_A, \smile, A_{(\Phi,\Phi)})$ is a bounded BCK-algebra whose every element is an involution.

Proof. For any $A_{(\alpha,\beta)}$, $A_{(\gamma,\delta)}$, $A_{(\zeta,\eta)} \in \mathcal{DSS}(X)_A$.

BCI-1 $((A_{(\alpha,\beta)} \smile A_{(\gamma,\delta)}) \smile (A_{(\alpha,\beta)} \smile A_{(\zeta,\eta)})) \smile (A_{(\zeta,\eta)} \smile A_{(\gamma,\delta)})$

$\tilde{=}(A_{(\alpha\smile\gamma,\beta\smile\delta)} \smile A_{(\alpha\smile\zeta,\beta\smile\eta)}) \smile A_{(\zeta\smile\gamma,\eta\smile\delta)}$

$\tilde{=}A_{(((\alpha\smile\gamma)\smile(\alpha\smile\zeta))\smile(\zeta\smile\gamma),((\beta\smile\delta)\smile(\beta\smile\eta))\smile(\eta\smile\delta))}$

$\tilde{=}A_{(\Phi\smile(\zeta\smile\gamma),\Phi\smile(\eta\smile\delta))} \tilde{=} A_{(\Phi,\Phi)}$.

BCI-2 $(A_{(\alpha,\beta)} \smile (A_{(\alpha,\beta)} \smile A_{(\gamma,\delta)})) \smile A_{(\gamma,\delta)}$

$\tilde{=}(A_{(\alpha,\beta)} \smile A_{(\alpha\smile\gamma,\beta\smile\delta)}) \smile A_{(\gamma,\delta)}$

$\tilde{=}A_{(\alpha\smile(\alpha\smile\gamma),\beta\smile(\beta\smile\delta))} \smile A_{(\gamma,\delta)}\tilde{=}A_{(\Phi\smile\gamma,\Phi\smile\delta)}\tilde{=}A_{(\Phi,\Phi)}$.

BCI-3 $A_{(\alpha,\beta)} \smile A_{(\alpha,\beta)}\tilde{=}A_{(\Phi,\Phi)}$.

BCI-4 Let

$$A_{(\alpha,\beta)} \smile A_{(\gamma,\delta)}\tilde{=}A_{(\Phi,\Phi)} \text{ and}$$
$$A_{(\gamma,\delta)} \smile A_{(\alpha,\beta)}\tilde{=}A_{(\Phi,\Phi)}.$$

For any $e \in A$,

$$\alpha(e) - \gamma(e) = \emptyset \text{ and } \gamma(e) - \alpha(e) = \emptyset \text{ imply that } \alpha(e) = \gamma(e),$$

also

$$\beta(e) - \delta(e) = \emptyset \text{ and } \delta(e) - \beta(e) = \emptyset \text{ imply that } \beta(e) = \delta(e).$$

Hence

$$A_{(\alpha,\beta)}\tilde{=}A_{(\gamma,\delta)}.$$

BCK-5 $A_{(\Phi,\Phi)} \smile A_{(\alpha,\beta)}\tilde{=}A_{(\Phi\smile\alpha,\Phi\smile\beta)}\tilde{=}A_{(\Phi,\Phi)}$.Thus $(\mathcal{DSS}(X)_A, \smile, A_{(\Phi,\Phi)})$ is a BCK-algebra.

Now $A_{(\mathfrak{X},\mathfrak{X})} \in \mathcal{DSS}(X)_A$ is such that:

$$A_{(\alpha,\beta)} \smile A_{(\mathfrak{X},\mathfrak{X})}\tilde{=}A_{(\alpha\smile\mathfrak{X},\beta\smile\mathfrak{X})}\tilde{=}A_{(\Phi,\Phi)}$$

for all $A_{(\alpha,\beta)} \in \mathcal{DSS}(X)_A$. Therefore $(\mathcal{DSS}(X)_A, \smile, A_{(\Phi,\Phi)})$ is a bounded BCK-algebra.

For any $A_{(\alpha,\beta)} \in \mathcal{DSS}(X)_A$,

$$\begin{aligned} A_{(\mathfrak{X},\mathfrak{X})} &\smile (A_{(\mathfrak{X},\mathfrak{X})} \smile A_{(\alpha,\beta)}) \\ &\tilde{=} A_{(\mathfrak{X},\mathfrak{X})} \smile A_{(\mathfrak{X}\smile\alpha,\mathfrak{X}\smile\beta)} \\ &\tilde{=} A_{(\mathfrak{X},\mathfrak{X})} \smile A_{(\alpha^c,\beta^c)} \\ &\tilde{=} A_{(\mathfrak{X}\smile\alpha^c,\mathfrak{X}\smile\beta^c)} \\ &\tilde{=} A_{((\alpha^c)^c,(\beta^c)^c)}\tilde{=}A_{(\alpha,\beta)}. \end{aligned}$$

So every element of $\mathcal{DSS}(X)_A$ is an involution. ■

4.4.21 Definition

Let $A_{(\alpha,\beta)}$ and $A_{(\gamma,\delta)}$ be double-*framed* soft sets over X. We define

$$A_{(\alpha,\beta)} \star A_{(\gamma,\delta)}\tilde{=}A_{(\alpha,\beta)} \sqcap A_{(\gamma,\delta)^c}.$$

4.4.22 Theorem

$(\mathcal{DSS}(X)_A, \star, A_{(\Phi,\mathfrak{X})})$ is a bounded BCK-algebra whose every element is an involution.

Proof. For any $A_{(\alpha,\beta)}$, $A_{(\gamma,\delta)}$, $A_{(\zeta,\eta)} \in \mathcal{DSS}(X)_A$.

BCI-1 $((A_{(\alpha,\beta)} \star A_{(\gamma,\delta)}) \star (A_{(\alpha,\beta)} \star A_{(\zeta,\eta)})) \star (A_{(\zeta,\eta)} \star A_{(\gamma,\delta)})$

$\tilde{=}(A_{(\alpha\star\gamma,\beta\star\delta);A\star\langle(\alpha\star\zeta,\beta\star\eta)}) \star A_{(\zeta\star\gamma,\eta\star\delta)}$

$\tilde{=}A_{(((\alpha\star\gamma)\star(\alpha\star\zeta))\star(\zeta\star\gamma),((\beta\star\delta)\star(\beta\star\eta))\star(\eta\star\delta))}$

$\tilde{=}A_{(((\alpha\tilde{\cap}\gamma^c)\star(\alpha\tilde{\cap}\zeta^c))\star(\zeta\tilde{\cap}\gamma^c),((\beta\tilde{\cup}\delta^c)\star(\beta\tilde{\cup}\eta^c))\star(\eta\tilde{\cup}\delta^c))}$

$\tilde{=}A_{(((\alpha\tilde{\cap}\gamma^c)\tilde{\cap}(\alpha\tilde{\cap}\zeta^c)^c)\tilde{\cap}(\zeta\tilde{\cap}\gamma^c)^c,((\beta\tilde{\cup}\delta^c)\tilde{\cup}(\beta\tilde{\cup}\eta^c)^c)\tilde{\cup}(\eta\tilde{\cup}\delta^c)^c)}$

$\tilde{=}A_{(((\alpha\tilde{\cap}\gamma^c)\tilde{\cap}(\alpha^c\tilde{\cup}\zeta))\tilde{\cap}(\zeta^c\tilde{\cup}\gamma),((\beta\tilde{\cup}\delta^c)\tilde{\cup}(\beta^c\tilde{\cap}\eta))\tilde{\cup}(\eta^c\tilde{\cap}\delta))}$

$\tilde{=}A_{((\alpha\tilde{\cap}\zeta)\tilde{\cap}(\gamma^c\tilde{\cap}\zeta^c),(\beta\tilde{\cup}\eta)\tilde{\cup}(\delta^c\tilde{\cup}\eta^c))}\tilde{=}A_{(\Phi,\mathfrak{X})}$.

BCI-2 $(A_{(\alpha,\beta)} \star (A_{(\alpha,\beta)} \star A_{(\gamma,\delta)})) \star A_{(\gamma,\delta)}$

$\tilde{=}A_{(\alpha\tilde{\cap}(\alpha\tilde{\cap}\gamma^c)^c,\beta\tilde{\cup}(\beta\tilde{\cup}\delta^c)^c)} \star A_{(\gamma,\delta)}$

$\tilde{=}A_{(\alpha\tilde{\cap}(\alpha^c\tilde{\cup}\gamma),\beta\tilde{\cup}(\beta^c\tilde{\cap}\delta))} \star A_{(\gamma,\delta)}$

$\tilde{=}A_{(\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta)} \star A_{(\gamma,\delta)}$

$\tilde{=}A_{((\alpha\tilde{\cap}\gamma)\tilde{\cap}\gamma^c,(\beta\tilde{\cup}\delta)\tilde{\cup}\delta^c)}\tilde{=}A_{(\Phi,\mathfrak{X})}$.

BCI-3 $A_{(\alpha,\beta)} \star A_{(\alpha,\beta)}\tilde{=}A_{(\alpha\tilde{\cap}\alpha^c,\beta\tilde{\cup}\beta^c)}\tilde{=}A_{(\Phi,\mathfrak{X})}$.

BCI-4 Let $A_{(\alpha,\beta)} \star A_{(\gamma,\delta)}\tilde{=}A_{(\Phi,\mathfrak{X})}$ and $A_{(\gamma,\delta)} \star A_{(\alpha,\beta)}\tilde{=}A_{(\Phi,\mathfrak{X})}$. For any $e \in A$,

$$\alpha(e) \cap (\gamma(e))^c = \emptyset \text{ and } \gamma(e) \cap (\alpha(e))^c = \emptyset \text{ imply that } \alpha(e) = \gamma(e),$$

also

$$\begin{aligned} \beta(e) \cup (\delta(e))^c &= X \text{ and } \delta(e) \cup (\beta(e))^c = X \\ &\Rightarrow \beta(e) \cap (\delta(e))^c = \emptyset \text{ and } \delta(e) \cap (\beta(e))^c = \emptyset \\ &\Rightarrow \beta(e) = \delta(e). \end{aligned}$$

Hence $A_{(\alpha,\beta)}\tilde{=}A_{(\gamma,\delta)}$.

BCK-5 $A_{(\Phi,\mathfrak{X})} \star A_{(\alpha,\beta)}\tilde{=}A_{(\Phi\star\alpha,\mathfrak{X}\star\beta)}\tilde{=}A_{(\Phi\tilde{\cap}\alpha^c,\mathfrak{X}\tilde{\cup}\beta^c)}\tilde{=}A_{(\Phi,\mathfrak{X})}$.

Thus $(\mathcal{DSS}(X)_A, \star, A_{(\Phi,\mathfrak{X})})$ is a BCK-algebra.

Now $A_{(\mathfrak{X},\Phi)} \in \mathcal{DSS}(X)_A$ is such that:

$$\begin{aligned} A_{(\alpha,\beta)} \star A_{(\mathfrak{X},\Phi)} &\tilde{=} A_{(\alpha\star\mathfrak{X},\beta\star\Phi)} \\ &\tilde{=} A_{(\alpha\tilde{\cap}\mathfrak{X}^c,\beta\tilde{\cup}\Phi^c)} \\ &\tilde{=} A_{(\alpha\tilde{\cap}\Phi,\beta\tilde{\cup}\mathfrak{X})} \\ &\tilde{=} A_{(\Phi,\mathfrak{X})} \qquad \text{for all } A_{(\alpha,\beta)} \in \mathcal{DSS}(X)_A. \end{aligned}$$

Therefore $(\mathcal{DSS}(X)_A, \star, A_{(\Phi,\mathfrak{X})})$ is a bounded BCK-algebra.

For any $A_{(\alpha,\beta)} \in \mathcal{DSS}(X)_A$,

$$\begin{aligned} A_{(\mathfrak{X},\Phi)} \star (A_{(\mathfrak{X},\Phi)} \star A_{(\alpha,\beta)}) &\tilde{=} A_{(\mathfrak{X},\Phi)} \star A_{(\mathfrak{X}\star\alpha,\Phi\star\beta)} \\ &\tilde{=} A_{(\mathfrak{X},\Phi)} \star A_{(\mathfrak{X}\tilde{\cap}\alpha^c,\Phi\tilde{\cup}\beta^c)} \\ &\tilde{=} A_{(\mathfrak{X},\Phi)} \star A_{(\alpha^c,\beta^c)} \\ &\tilde{=} A_{(\mathfrak{X}\tilde{\cap}(\alpha^c)^c,\Phi\tilde{\cup}(\beta^c)^c)} \\ &\tilde{=} A_{(\mathfrak{X}\tilde{\cap}\alpha,\Phi\tilde{\cup}\beta)} \tilde{=} A_{(\alpha,\beta)}. \end{aligned}$$

So every element of $\mathcal{DSS}(X)_A$ is an involution. ■

Chapter 5

Double-framed Fuzzy Soft Sets and Their Algebraic Structures

This chapter explores the theory of double-framed fuzzy soft sets which is a generalization of double-framed soft sets and most generalized structure in our work. Double-framed fuzzy soft sets and their operations are defined and their characteristics are studied. Examples are given to elaborate the concepts and to show how the ideas are utilized to work with double-framed fuzzy soft sets. We see from examples that the cases for double-framed fuzzy soft sets are of more generalized nature and we cannot model those with double-framed soft sets.

5.1 Double-framed Fuzzy Soft Sets

Let X be an initial universe and E be a set of parameters. Let $\mathcal{FP}(X)$ denotes the fuzzy power set of X and A, B, C are non-empty subsets of E.

5.1.1 Definition

A double-framed pair $\langle (f, g)\,; A\rangle$ is called a double-framed fuzzy soft set over X, where f and g are mappings from A to $\mathcal{FP}(X)$.

From here, we shall use the notation $A_{(f,g)}$ over X to denote a double-framed fuzzy soft set $\langle (f, g); A\rangle$ over X where the meanings of f, g, A and X are clear.

5.1.2 Definition

For double-*framed fuzzy* soft sets $A_{(f,g)}$ and $B_{(h,i)}$ over X, we say that $A_{(f,g)}$ is a *double-framed* fuzzy soft subset of $B_{(h,i)}$, if

1) $A \subseteq B$ and

2) $f(e) \subseteq h(e)$ and $i(e) \subseteq g(e)$ for all $e \in A$.

This relationship is denoted by $A_{(f,g)} \tilde{\subseteq} B_{(h,i)}$. Also $A_{(f,g)}$ is said to be a *double-framed fuzzy soft superset* of $B_{(h,i)}$, if $B_{(h,i)}$ is a *double-framed fuzzy soft subset* of $A_{(f,g)}$. We denote it by $A_{(f,g)} \tilde{\supseteq} B_{(h,i)}$.

5.1.3 Definition

Two *double-framed fuzzy soft sets* $A_{(f,g)}$ and $B_{(h,i)}$ over X are said to be *equal* if $A_{(f,g)}$ is a *double-framed fuzzy soft subset* of $B_{(h,i)}$ and $B_{(h,i)}$ is a *double-framed fuzzy soft subset* of $A_{(f,g)}$. We denote it by $A_{(f,g)} \tilde{=} B_{(h,i)}$.

5.1.4 Example

Let X be the set of houses under consideration, and E be the set of parameters, $X = \{h_1,h_2,h_3,h_4,h_5\}$, $E = \{e_1,e_2,e_3,e_4,e_5,e_6\} = \{$ in the green surroundings, wooden, cheap, in good repair, furnished, traditional $\}$. Suppose that $A = \{e_1,e_2,e_3,e_6\}$, a double-framed fuzzy soft set $A_{(f,g)}$ describes the "highest and lowest budget ratings of the houses under consideration" given by f and g respectively. The double-framed fuzzy soft set $A_{(f,g)}$ over X is given as follows:

$$\begin{aligned} f &: A \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{h_1/0.9, h_2/0.3, h_3/0.8, h_4/0.7, h_5/0.5\} & \text{if } e = e_1, \\ \{h_1/0.1, h_2/0.5, h_3/0.1, h_4/0.8, h_5/0.6\} & \text{if } e = e_2, \\ \{h_1/0.2, h_2/0.5, h_3/0.2, h_4/0.9, h_5/0.9\} & \text{if } e = e_3, \\ \{h_1/0.7, h_2/0.4, h_3/0.2, h_4/0.1, h_5/0\} & \text{if } e = e_6, \end{cases} \end{aligned}$$

$$\begin{aligned} g &: A \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{h_1/0.2, h_2/0.3, h_3/0.3, h_4/0.4, h_5/0.8\} & \text{if } e = e_1, \\ \{h_1/0.7, h_2/0.4, h_3/0.8, h_4/0.7, h_5/0.9\} & \text{if } e = e_2, \\ \{h_1/0.6, h_2/0.4, h_3/0.6, h_4/0.6, h_5/0.7\} & \text{if } e = e_3, \\ \{h_1/0.9, h_2/0.3, h_3/0.8, h_4/0.7, h_5/0.5\} & \text{if } e = e_6. \end{cases} \end{aligned}$$

Let $B = \{e_2,e_6\}$. Then the double-framed fuzzy soft set $B_{(h,i)}$ given by

$$h : B \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_1/0.1, h_2/0.5, h_3/0.1, h_4/0.8, h_5/0.6\} & \text{if } e = e_2, \\ \{h_1/0.7, h_2/0.4, h_3/0.2, h_4/0.1, h_5/0\} & \text{if } e = e_6, \end{cases}$$

$$i : B \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_1/0.1, h_2/0.2, h_3/0.4, h_4/0.3, h_5/0.5\} & \text{if } e = e_2, \\ \{h_1/0.9, h_2/0.4, h_3/0.9, h_4/0.8, h_5/0.7\} & \text{if } e = e_6. \end{cases}$$

is a double-framed fuzzy soft subset of $A_{(f,g)}$ which represents a finer data analysis and so $B_{(h,i)} \tilde{\subseteq} A_{(f,g)}$.

5.2 Operations on Double-framed Fuzzy Soft Sets

In this section, we define various operations on *double-framed fuzzy* soft sets:

5.2.1 Definition

Let $A_{(f,g)}$ and $B_{(h,i)}$ be *double-framed fuzzy* soft sets over X. The *int-uni product* of $A_{(f,g)}$ and $B_{(h,i)}$ is defined as a double-framed fuzzy soft set $(A \times B)_{(f\tilde{\wedge}h,g\tilde{\vee}i)}$ over X in which $f\tilde{\wedge}h : (A \times B) \to \mathcal{FP}(X)$, where

$$(a,b) \mapsto f(a) \wedge h(b),$$

and $g\tilde{\vee}i : (A \times B) \to \mathcal{FP}(X)$, where

$$(a,b) \mapsto g(a) \vee i(b).$$

It is denoted by $A_{(f,g)} \wedge B_{(h,i)} \tilde{=} (A \times B)_{(f\tilde{\wedge}h,g\tilde{\vee}i)}$.

5.2.2 Definition

Let $A_{(f,g)}$ and $B_{(h,i)}$ be *double-framed fuzzy* soft sets over X. The *uni-int product* of $A_{(f,g)}$ and $B_{(h,i)}$ is defined as a double-framed fuzzy soft set $(A \times B)_{(f\tilde{\vee}h,g\tilde{\wedge}i)}$ over X in which $f\tilde{\vee}h : (A \times B) \to \mathcal{FP}(X)$, where

$$(a,b) \mapsto f(a) \vee h(b),$$

and $g\tilde{\wedge}i : (A \times B) \to \mathcal{FP}(X)$, where

$$(a,b) \mapsto g(a) \wedge i(b).$$

It is denoted by $A_{(f,g)} \vee B_{(h,i)} \tilde{=} (A \times B)_{(f\tilde{\vee}h,g\tilde{\wedge}i)}$.

5.2.3 Definition

For double-*framed fuzzy soft sets* $A_{(f,g)}$ and $B_{(h,i)}$ over X, *the extended int-uni double-framed fuzzy soft set* of $A_{(f,g)}$ and $B_{(h,i)}$ is defined as a *double-framed fuzzy soft set* $(A \cup B)_{(f\tilde{\wedge}h,g\tilde{\vee}i)}$ where $f\tilde{\wedge}h : (A \cup B) \to \mathcal{FP}(X)$, given by

$$e \mapsto \begin{cases} f(e) & if\ e \in A - B \\ h(e) & if\ e \in B - A \\ f(e) \wedge h(e) & if\ e \in A \cap B \end{cases}$$

and $g\tilde{\vee}i : (A \cup B) \to \mathcal{FP}(X)$, given by

$$e \mapsto \begin{cases} g(e) & if\ e \in A - B \\ i(e) & if\ e \in B - A \\ g(e) \vee i(e) & if\ e \in A \cap B \end{cases}.$$

It is denoted by $A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)} \tilde{=} (A \cup B)_{(f\tilde{\wedge}h,g\tilde{\vee}i)}$.

5.2.4 Definition

For double-*framed fuzzy soft sets* $A_{(f,g)}$ and $B_{(h,i)}$ over X, *the extended uni-int double-framed fuzzy soft set* of $A_{(f,g)}$ and $B_{(h,i)}$ is defined as a *double-framed fuzzy soft set* $(A \cup B)_{(f\tilde{\vee}h, g\tilde{\wedge}i)}$ where $f\tilde{\vee}h : (A \cup B) \to \mathcal{FP}(X)$, given by

$$e \mapsto \begin{cases} f(e) & if\ e \in A - B \\ h(e) & if\ e \in B - A \\ f(e) \vee h(e) & if\ e \in A \cap B \end{cases}$$

and $g\tilde{\wedge}i : (A \cup B) \to \mathcal{FP}(X)$, given by

$$e \mapsto \begin{cases} g(e) & if\ e \in A - B \\ i(e) & if\ e \in B - A \\ g(e) \wedge i(e) & if\ e \in A \cap B \end{cases} .$$

It is denoted by $A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)} \tilde{=} (A \cup B)_{(f\tilde{\vee}h, g\tilde{\wedge}i)}$.

5.2.5 Definition

For double-*framed fuzzy soft sets* $A_{(f,g)}$ and $B_{(h,i)}$ over X with $(A \cap B) \neq \emptyset$, the *restricted int-uni double-framed fuzzy soft set* of $A_{(f,g)}$ and $B_{(h,i)}$ is defined as a *double-framed fuzzy soft set* $(A \cap B)_{(f\tilde{\wedge}h, g\tilde{\vee}i)}$ where $f\tilde{\wedge}h : (A \cap B) \to \mathcal{FP}(X)$,

$$e \mapsto f(e) \wedge h(e),$$

and $g\tilde{\vee}i : (A \cap B) \to \mathcal{FP}(X)$,

$$e \mapsto g(e) \vee i(e).$$

It is denoted by $A_{(f,g)} \sqcap B_{(h,i)} \tilde{=} (A \cap B)_{(f\tilde{\wedge}h, g\tilde{\vee}i)}$.

5.2.6 Definition

For double-*framed fuzzy soft sets* $A_{(f,g)}$ and $B_{(h,i)}$ over X with $(A \cap B) \neq \emptyset$, the *restricted uni-int double-framed fuzzy soft set* of $A_{(f,g)}$ and $B_{(h,i)}$ is defined as a *double-framed fuzzy soft set* $(A \cap B)_{(f\tilde{\vee}h, g\tilde{\wedge}i)}$ where $f\tilde{\vee}h : (A \cap B) \to \mathcal{FP}(X)$, given by

$$e \mapsto f(e) \vee h(e),$$

and $g\tilde{\wedge}i : (A \cap B) \to \mathcal{FP}(X)$,

$$e \mapsto g(e) \wedge i(e).$$

It is denoted by $A_{(f,g)} \sqcup B_{(h,i)} \tilde{=} (A \cap B)_{(f\tilde{\vee}h, g\tilde{\wedge}i)}$.

5.2.7 Definition

Let $A_{(f,g)}$ be a *double-framed fuzzy* soft set over X. The *complement of a double-framed fuzzy soft set* $A_{(f,g)}$ over X is defined as a *double-framed fuzzy soft set* $A_{(f\,',g\,')}$ over X where $f\,': A \rightarrow \mathcal{FP}(X)$, given by

$$e \mapsto (f\,(e))'$$

and $g\,': A \rightarrow \mathcal{FP}(X)$,

$$e \mapsto (g\,(e))'.$$

It is denoted by $A_{(f,g)}\,'\tilde{=}A_{(f\,',g\,')}$.

5.3 Properties of Double-framed Fuzzy Soft Sets

In this section we discuss properties and laws of double-framed fuzzy soft sets with respect to their operations. Associativity, commutativity, absorption, distributivity and properties of double-framed fuzzy soft sets are investigated.

5.3.1 Definition

A *double-framed fuzzy soft set* over X is said to be a *relative null double-framed fuzzy soft set,* denoted by $A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$ where

$$\begin{aligned} \tilde{\mathbf{0}} &: A \rightarrow \mathcal{FP}(X),\, e \mapsto \tilde{\mathbf{0}}, \text{ where } \tilde{\mathbf{0}} \text{ maps every element of } X \text{ onto } 0 \\ \tilde{\mathbf{1}} &: A \rightarrow \mathcal{FP}(X),\, e \mapsto \tilde{\mathbf{1}}, \text{ where } \tilde{\mathbf{1}} \text{ maps every element of } X \text{ onto } 1 \end{aligned}$$

5.3.2 Definition

A *double-framed fuzzy* soft set over X is said to be a *relative absolute double-framed fuzzy soft set,* denoted by $A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}$ where

$$\begin{aligned} \tilde{\mathbf{1}} &: A \rightarrow \mathcal{FP}(X),\, e \mapsto \tilde{\mathbf{1}}, \\ \tilde{\mathbf{0}} &: A \rightarrow \mathcal{FP}(X),\, e \mapsto \tilde{\mathbf{0}}. \end{aligned}$$

Conventionally, we take the *double-framed fuzzy* soft sets with empty set of parameters to be equal to $\emptyset_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$ and so $A_{(f,g)} \sqcap B_{(h,i)} \tilde{=} A_{(f,g)} \sqcup B_{(h,i)} \tilde{=} \emptyset_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$ where $(A \cap B) = \emptyset$.

5.3.3 Proposition

If $A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$ is a null *double-framed fuzzy* soft set, $A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}$ an absolute *double-framed fuzzy* soft set, and $A_{(f,g)}$, $A_{(h,i)}$ are *double-framed fuzzy* soft sets over X, then

1) $A_{(f,g)} \sqcup_{\varepsilon} A_{(h,i)} \tilde{\tilde{=}} A_{(f,g)} \sqcup A_{(h,i)}$,

2) $A_{(f,g)} \sqcap_{\varepsilon} A_{(h,i)} \tilde{\tilde{=}} A_{(f,g)} \sqcap A_{(h,i)}$,

3) $A_{(f,g)} \sqcap A_{(f,g)} \tilde{\tilde{=}} A_{(f,g)} \tilde{\tilde{=}} A_{(f,g)} \sqcup A_{(f,g)}$, (Idempotent)

4) $A_{(f,g)} \sqcup A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})} \tilde{\tilde{=}} A_{(f,g)} \tilde{\tilde{=}} A_{(f,g)} \sqcap A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}$,

5) $A_{(f,g)} \sqcup A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})} \tilde{\tilde{=}} A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}$; $A_{(f,g)} \sqcap A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})} \tilde{\tilde{=}} A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$.

Proof. Proofs of 1), 2) and 3) are straightforward.

4) As $A_{(f,g)} \sqcup A_{(\Phi,\tilde{\mathbf{1}})} \tilde{\tilde{=}} A_{(f\tilde{\vee}\tilde{\mathbf{0}},g\tilde{\wedge}\tilde{\mathbf{1}})}$. Therefore for any $e \in A$,

$$(f\tilde{\vee}\tilde{\mathbf{0}})(e) = f(e) \vee \tilde{\mathbf{0}}(e) = f(e) \text{ and } (g\tilde{\wedge}\tilde{\mathbf{1}})(e) = g(e) \wedge \tilde{\mathbf{1}}(e) = g(e).$$

Thus $A_{(f,g)} \sqcup A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})} \tilde{\tilde{=}} A_{(f,g)}$.

Again, $A_{(f,g)} \sqcap A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})} \tilde{\tilde{=}} A_{(f\tilde{\wedge}\tilde{\mathbf{1}},g\tilde{\vee}\tilde{\mathbf{0}})}$. For any $e \in A$,

$$(f\tilde{\wedge}\tilde{\mathbf{1}})(e) = f(e) \wedge \tilde{\mathbf{1}}(e) = f(e) \text{ and } (g\tilde{\vee}\tilde{\mathbf{0}})(e) = g(e) \vee \tilde{\mathbf{0}}(e) = g(e).$$

So $A_{(f,g)} \sqcap A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})} \tilde{\tilde{=}} A_{(f,g)}$.

Part 5) can be proved in a similar way. ■

5.3.4 Proposition

Let $A_{(f,g)}$, $B_{(h,i)}$ and $C_{(j,k)}$be any *double-framed fuzzy soft set*s over a common universe X. Then the following are true

1) $A_{(f,g)}\lambda(B_{(h,i)}\lambda C_{(j,k)}) \tilde{\tilde{=}} (A_{(f,g)}\lambda B_{(h,i)})\lambda C_{(j,k)}$, (Associative Laws)

2) $A_{(f,g)}\lambda B_{(h,i)} \tilde{\tilde{=}} B_{(h,i)}\lambda A_{(f,g)}$, (Commutative Laws)

for all $\lambda \in \{\sqcup_{\varepsilon}, \sqcup, \sqcap_{\varepsilon}, \sqcap\}$.
Proof.

1) Since $A_{(f,g)} \sqcup_{\varepsilon} (B_{(h,i)} \sqcup_{\varepsilon} C_{(j,k)}) \tilde{\tilde{=}} A \cup (B \cup C)_{(f\tilde{\vee}(h\tilde{\vee}j),g\tilde{\wedge}(g\tilde{\wedge}k))}$, we have for any $e \in A \cup (B \cup C)$:

(i) If $e \in A - (B \cup C)$, then

$$\begin{aligned}(f\tilde{\vee}(h\tilde{\vee}j))(e) &= f(e) = ((f\tilde{\vee}h)\tilde{\vee}j)(e)\\(g\tilde{\wedge}(i\tilde{\wedge}k))(e) &= g(e) = ((g\tilde{\wedge}i)\tilde{\wedge}k)(e)\end{aligned}$$

(ii) If $e \in B - (A \cup C)$, then

$$\begin{aligned}(f\tilde{\vee}(h\tilde{\vee}j))(e) &= h(e) = ((f\tilde{\vee}h)\tilde{\vee}j)(e)\\(g\tilde{\wedge}(i\tilde{\wedge}k))(e) &= g(e) = ((g\tilde{\wedge}i)\tilde{\wedge}k)(e)\end{aligned}$$

(iii) If $e \in C - (A \cup B)$, then

$$\begin{aligned}(f\tilde{\vee}(h\tilde{\vee}j))(e) &= j(e) = ((f\tilde{\vee}h)\tilde{\vee}j)(e)\\(g\tilde{\wedge}(i\tilde{\wedge}k))(e) &= k(e) = ((g\tilde{\wedge}i)\tilde{\wedge}k)(e)\end{aligned}$$

(iv) If $e \in (A \cap B) - C$, then

$$\begin{aligned}(f\tilde{\vee}(h\tilde{\vee}j))(e) &= f(e) \vee h(e) = (f\tilde{\vee}h)(e) = ((f\tilde{\vee}h)\tilde{\vee}j)(e)\\(g\tilde{\wedge}(i\tilde{\wedge}k))(e) &= g(e) \wedge i(e) = (g\tilde{\wedge}i)(e) = ((g\tilde{\wedge}i)\tilde{\wedge}k)(e)\end{aligned}$$

(v) If $e \in (A \cap C) - B$, then

$$\begin{aligned}(f\tilde{\vee}(h\tilde{\vee}j))(e) &= f(e) \vee j(e) = ((f\tilde{\vee}h)\tilde{\vee}j)(e)\\(g\tilde{\wedge}(i\tilde{\wedge}k))(e) &= g(e) \wedge k(e) = ((g\tilde{\wedge}i)\tilde{\wedge}k)(e)\end{aligned}$$

(vi) If $e \in (B \cap C) - A$, then

$$\begin{aligned}(f\tilde{\vee}(h\tilde{\vee}j))(e) &= h(e) \vee j(e) = (f\tilde{\vee}h)\tilde{\vee}j(e)\\(g\tilde{\wedge}(i\tilde{\wedge}k))(e) &= g(e) \wedge k(e) = (g\tilde{\wedge}i)\tilde{\wedge}k(e)\end{aligned}$$

(vii) If $e \in (A \cap B) \cap C$, then

$$\begin{aligned}(f\tilde{\vee}(h\tilde{\vee}j))(e) &= f(e) \vee (h(e) \vee j(e)) = (f(e) \vee h(e)) \vee j(e) = ((f\tilde{\vee}h)\tilde{\vee}j)(e)\\(g\tilde{\wedge}(i\tilde{\wedge}k))(e) &= g(e) \wedge (i(e) \wedge k(e)) = (g(e) \wedge i(e)) \wedge k(e) = ((g\tilde{\wedge}i)\tilde{\wedge}k)(e)\end{aligned}$$

Thus $A_{(f,g)} \sqcup_\varepsilon (B_{(h,i)} \sqcup_\varepsilon C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcup_\varepsilon B_{(h,i)}) \sqcup_\varepsilon C_{(j,k)}$.

Similarly, we can prove for $\lambda \in \{\sqcup, \sqcap_\varepsilon, \sqcap\}$

2) This is straightforward.

■

5.3.5 Proposition (Absorption Laws)

Let $A_{(f,g)}$, $B_{(h,i)}$ be any *double-framed fuzzy* soft sets over X. Then the following are true:

1) $A_{(f,g)} \sqcup_{\varepsilon} (B_{(h,i)} \sqcap A_{(f,g)}) \tilde{=} A_{(f,g)}$,

2) $A_{(f,g)} \sqcap (B_{(h,i)} \sqcup_{\varepsilon} A_{(f,g)}) \tilde{=} A_{(f,g)}$,

3) $A_{(f,g)} \sqcup (B_{(h,i)} \sqcap_{\varepsilon} A_{(f,g)}) \tilde{=} A_{(f,g)}$,

4) $A_{(f,g)} \sqcap_{\varepsilon} (B_{(h,i)} \sqcup A_{(f,g)}) \tilde{=} A_{(f,g)}$.

Proof. Straightforward. ■

5.3.6 Proposition (Distributive Laws)

Let $A_{(f,g)}$, $B_{(h,i)}$ and $C_{(j,k)}$ be any *double-framed fuzzy soft set*s over X. Then

1) $A_{(f,g)} \sqcap (B_{(h,i)} \sqcup_{\varepsilon} C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcap B_{(h,i)}) \sqcup_{\varepsilon} (A_{(f,g)} \sqcap C_{(j,k)})$,

2) $A_{(f,g)} \sqcap (B_{(h,i)} \sqcap_{\varepsilon} C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcap B_{(h,i)}) \sqcap_{\varepsilon} (A_{(f,g)} \sqcap C_{(j,k)})$,

3) $A_{(f,g)} \sqcap (B_{(h,i)} \sqcup C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcap B_{(h,i)}) \sqcup (A_{(f,g)} \sqcap C_{(j,k)})$,

4) $A_{(f,g)} \sqcup (B_{(h,i)} \sqcup_{\varepsilon} C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcup B_{(h,i)}) \sqcup_{\varepsilon} (A_{(f,g)} \sqcup C_{(j,k)})$,

5) $A_{(f,g)} \sqcup (B_{(h,i)} \sqcap_{\varepsilon} C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcup B_{(h,i)}) \sqcap_{\varepsilon} (A_{(f,g)} \sqcup C_{(j,k)})$,

6) $A_{(f,g)} \sqcup (B_{(h,i)} \sqcap C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcup B_{(h,i)}) \sqcap (A_{(f,g)} \sqcup C_{(j,k)})$,

7) $A_{(f,g)} \sqcap_{\varepsilon} (B_{(h,i)} \sqcup_{\varepsilon} C_{(j,k)}) \tilde{\subseteq} (A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)}) \sqcup_{\varepsilon} (A_{(f,g)} \sqcap_{\varepsilon} C_{(j,k)})$,

8) $A_{(f,g)} \sqcap_{\varepsilon} (B_{(h,i)} \sqcup C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)}) \sqcup (A_{(f,g)} \sqcap_{\varepsilon} C_{(j,k)})$,

9) $A_{(f,g)} \sqcap_{\varepsilon} (B_{(h,i)} \sqcap C_{(j,k)}) \tilde{\supseteq} (A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)}) \sqcap (A_{(f,g)} \sqcap_{\varepsilon} C_{(j,k)})$,

10) $A_{(f,g)} \sqcup_{\varepsilon} (B_{(h,i)} \sqcup C_{(j,k)}) \tilde{\subseteq} (A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)}) \sqcup (A_{(f,g)} \sqcup_{\varepsilon} C_{(j,k)})$,

11) $A_{(f,g)} \sqcup_{\varepsilon} (B_{(h,i)} \sqcap_{\varepsilon} C_{(j,k)}) \tilde{\supseteq} (A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)}) \sqcap_{\varepsilon} (A_{(f,g)} \sqcup_{\varepsilon} C_{(j,k)})$,

12) $A_{(f,g)} \sqcup_{\varepsilon} (B_{(h,i)} \sqcap C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)}) \sqcap (A_{(f,g)} \sqcup_{\varepsilon} C_{(j,k)})$.

Proof. We prove only one part here and remaining parts can be proved in a similar way.

1) Consider $A_{(f,g)} \sqcap (B_{(h,i)} \sqcup_{\varepsilon} C_{(j,k)})$. For any $e \in A \cap (B \cup C)$, we have following three disjoint cases:

(i) If $e \in A \cap (B - C)$, then

$$(f\tilde{\wedge}(h\tilde{\vee}j))(e) = f(e) \wedge h(e) \text{ and } (g\tilde{\vee}(i\tilde{\wedge}k))(e) = g(e) \vee i(e)$$

and

$$\begin{aligned}((f\tilde{\wedge}h)\tilde{\vee}(f\tilde{\wedge}j))(e) &= (f\tilde{\wedge}h)(e) = f(e) \wedge h(e) \text{ and} \\ ((g\tilde{\vee}i)\tilde{\wedge}(g\tilde{\vee}k))(e) &= (g\tilde{\vee}i)(e) = g(e) \vee i(e).\end{aligned}$$

(ii) If $e \in A \cap (C - B)$, then

$$(f\tilde{\wedge}(h\tilde{\vee}j))(e) = f(e) \wedge j(e) \text{ and } (g\tilde{\vee}(i\tilde{\wedge}k))(e) = g(e) \vee k(e)$$

and

$$\begin{aligned}((f\tilde{\wedge}h)\tilde{\vee}(f\tilde{\wedge}j))(e) &= (f\tilde{\wedge}j)(e) = f(e) \wedge j(e) \text{ and} \\ ((g\tilde{\vee}i)\tilde{\wedge}(g\tilde{\vee}k))(e) &= (g\tilde{\vee}k)(e) = g(e) \vee k(e).\end{aligned}$$

(iii) If $e \in A \cap (B \cap C)$, then

$$\begin{aligned}(f\tilde{\wedge}(h\tilde{\vee}j))(e) &= f(e) \wedge (h(e) \vee j(e)) \text{ and} \\ (g\tilde{\vee}(i\tilde{\wedge}k))(e) &= g(e) \vee (i(e) \wedge k(e))\end{aligned}$$

and

$$\begin{aligned}((f\tilde{\wedge}h)\tilde{\vee}(f\tilde{\wedge}j))(e) &= (f\tilde{\wedge}h)(e)\tilde{\vee}(f\tilde{\wedge}j)(e) \\ &= (f(e) \wedge h(e)) \vee (f(e) \wedge j(e)) \\ &= f(e) \wedge (h(e) \vee j(e)) \qquad \text{and} \\ ((g\tilde{\vee}i)\tilde{\wedge}(g\tilde{\vee}k))(e) &= (g\tilde{\vee}i)(e) \wedge (g\tilde{\vee}k)(e) \\ &= (g(e) \vee i(e)) \wedge (g(e) \vee k(e)) \\ &= g(e) \vee (i(e) \wedge k(e)).\end{aligned}$$

Thus

$$A_{(f,g)} \sqcap (B_{(h,i)} \sqcup_{\varepsilon} C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcap B_{(h,i)}) \sqcup_{\varepsilon} (A_{(f,g)} \sqcap C_{(j,k)}).$$

■

5.3.7 Example

Let X be the set of cars of different models, and E be the set of colors, $X = \{x_1, x_2, x_3, x_4, x_5\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\} = \{$ green, red, blue, black, white, silver

}. Suppose that $A = \{e_1, e_2, e_3\}$, $B = \{e_2, e_3, e_4\}$, and $C = \{e_3, e_4, e_5\}$. The double-framed fuzzy soft sets $A_{(f,g)}$, $B_{(h,i)}$ and $C_{(j,k)}$ over X describe the level of appreciation from customers based upon the annual survey reports of three different showrooms respectively. Here $\{f,\, h,\, j\}$ and $\{g,\, i,\, k\}$ collect results for positive and negative aspects respectively. We have

$$\begin{aligned}
f &: A \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases} \{x_1/0.3, x_2/0.1, x_3/0.3, x_4/0.1, x_5/0.7\} & \text{if } e = e_1,\\ \{x_1/0.1, x_2/0.9, x_3/0.3, x_4/0.8, x_5/0.2\} & \text{if } e = e_2,\\ \{x_1/0.1, x_2/0.3, x_3/0.3, x_4/0.3, x_5/0.8\} & \text{if } e = e_3,\end{cases}\\
g &: A \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases} \{x_1/0.4, x_2/0.7, x_3/0.7, x_4/0.7, x_5/0.1\} & \text{if } e = e_1,\\ \{x_1/0.8, x_2/0, x_3/0.5, x_4/0.1, x_5/0.6\} & \text{if } e = e_2,\\ \{x_1/0.7, x_2/0.5, x_3/0.7, x_4/0.6, x_5/0.1\} & \text{if } e = e_3.\end{cases}
\end{aligned}$$

$$\begin{aligned}
h &: B \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases} \{x_1/0.1, x_2/0.3, x_3/0.6, x_4/0.2, x_5/0.3\} & \text{if } e = e_2,\\ \{x_1/0.8, x_2/0.9, x_3/0.5, x_4/0.4, x_5/0.2\} & \text{if } e = e_3,\\ \{x_1/0.1, x_2/0.4, x_3/0.3, x_4/0.6, x_5/0.9\} & \text{if } e = e_4,\end{cases}\\
g &: B \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases} \{x_1/0.1, x_2/0.3, x_3/0.3, x_4/0.6, x_5/0.6\} & \text{if } e = e_2,\\ \{x_1/0.1, x_2/0, x_3/0.3, x_4/0.4, x_5/0.6\} & \text{if } e = e_3,\\ \{x_1/0.9, x_2/0.5, x_3/0.5, x_4/0.3, x_5/0.1\} & \text{if } e = e_4.\end{cases}\\
j &: C \to \mathcal{FP}(X),
\end{aligned}$$

$$\begin{aligned}
e &\longmapsto \begin{cases} \{x_1/0.1, x_2/0.2, x_3/0.3, x_4/0.1, x_5/0.1\} & \text{if } e = e_3,\\ \{x_1/0.2, x_2/0.2, x_3/0.3, x_4/0.3, x_5/0.2\} & \text{if } e = e_4,\\ \{x_1/0.1, x_2/0.1, x_3/0.3, x_4/0.5, x_5/0.7\} & \text{if } e = e_5,\end{cases}\\
k &: C \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases} \{x_1/0.7, x_2/0.7, x_3/0.4, x_4/0.7, x_5/0.4\} & \text{if } e = e_3,\\ \{x_1/0.6, x_2/0.5, x_3/0.6, x_4/0.1, x_5/0.6\} & \text{if } e = e_4,\\ \{x_1/0.3, x_2/0.4, x_3/0.4, x_4/0.3, x_5/0.1\} & \text{if } e = e_5.\end{cases}
\end{aligned}$$

We know that

$$A_{(f,g)} \sqcup_\varepsilon (B_{(h,i)} \sqcap_\varepsilon C_{(j,k)}) \tilde{=} ((A \cup B) \cup C)_{(f \tilde{\vee} (h \tilde{\wedge} j), g \tilde{\wedge} (i \tilde{\vee} k))}$$

and

$$(A_{(f,g)} \sqcup_\varepsilon B_{\langle h,g \rangle}) \sqcap_\varepsilon (A_{(f,g)} \sqcup_\varepsilon C_{(j,k)}) \tilde{=} ((A \cup B) \cup C)_{((f \tilde{\vee} h) \tilde{\wedge} (f \tilde{\vee} j))}.$$

Then

$$\begin{aligned}(f\tilde{\vee}(h\tilde{\wedge}j))(e_2) &= \{x_1/0.1, x_2/0.9, x_3/0.6, x_4/0.8, x_5/0.3\}\\ &\neq \{x_1/0.1, x_2/0.9, x_3/0.3, x_4/0.8, x_5/0.2\}\\ &= ((f\tilde{\vee}h)\tilde{\wedge}(f\tilde{\vee}j))(e_2) \text{ and}\\ (g\tilde{\wedge}(i\tilde{\vee}k))(e_2) &= \{x_1/0.1, x_2/0.0, x_3/0.3, x_4/0.1, x_5/0.6\}\\ &\neq \{x_1/0.8, x_2/0.0, x_3/0.5, x_4/0.1, x_5/0.6\}\\ &= ((g\tilde{\wedge}i)\tilde{\vee}(g\tilde{\wedge}k))(e_2),\end{aligned}$$

so that

$$A_{(f,g)} \sqcup_{\varepsilon} (B_{(h,i)} \sqcap_{\varepsilon} C_{(j,k)})\tilde{\neq}(A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)}) \sqcap_{\varepsilon} (A_{(f,g)} \sqcup_{\varepsilon} C_{(j,k)}).$$

Now,

$$A_{(f,g)} \sqcap_{\varepsilon} (B_{(h,i)} \sqcup_{\varepsilon} C_{(j,k)})\tilde{=}((A \cup B) \cup C)_{(f\tilde{\wedge}(h\tilde{\vee}j), g\tilde{\vee}(i\tilde{\wedge}k))}$$

and

$$(A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)}) \sqcup_{\varepsilon} (A_{(f,g)} \sqcap_{\varepsilon} C_{(j,k)})\tilde{=}((A \cup B) \cup C)_{((f\tilde{\wedge}h)\tilde{\vee}(f\tilde{\wedge}j), (g\tilde{\vee}i)\tilde{\wedge}(g\tilde{\vee}k))}.$$

Then,

$$\begin{aligned}(f\tilde{\wedge}(h\tilde{\vee}j))(e_2) &= \{x_1/0.1, x_2/0.3, x_3/0.3, x_4/0.2, x_5/0.2\}\\ &\neq \{x_1/0.1, x_2/0.9, x_3/0.3, x_4/0.8, x_5/0.2\}\\ &= ((f\tilde{\wedge}h)\tilde{\vee}(f\tilde{\wedge}j))(e_2)\end{aligned}$$

and

$$\begin{aligned}(g\tilde{\vee}(i\tilde{\wedge}k))(e_2) &= \{x_1/0.8, x_2/0.3, x_3/0.5, x_4/0.6, x_5/0.6\}\\ &\neq \{x_1/0.8, x_2/0.0, x_3/0.5, x_4/0.1, x_5/0.6\}\\ &= ((g\tilde{\vee}i)\tilde{\wedge}(g\tilde{\vee}k))(e_2).\end{aligned}$$

So

$$A_{(f,g)} \sqcap_{\varepsilon} (B_{(h,i)} \sqcup_{\varepsilon} C_{(j,k)})\tilde{\neq}(A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)}) \sqcup_{\varepsilon} (A_{(f,g)} \sqcap_{\varepsilon} C_{(j,k)}).$$

Similarly we can show that

$$A_{(f,g)} \sqcup_{\varepsilon} (B_{(h,i)} \sqcup C_{(j,k)})\tilde{\neq}(A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)}) \sqcup (A_{(f,g)} \sqcup_{\varepsilon} C_{(j,k)}),$$

and

$$A_{(f,g)} \sqcap_{\varepsilon} (B_{(h,i)} \sqcap C_{(j,k)})\tilde{\neq}(A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)}) \sqcap (A_{(f,g)} \sqcap_{\varepsilon} C_{(j,k)}).$$

5.3.8 Proposition

Let $A_{(f,g)}$, $B_{(h,i)}$ and $C_{(j,k)}$ be any *double-framed fuzzy soft sets* over X. Then

1) $A_{(f,g)} \sqcup_\varepsilon (B_{(h,i)} \sqcap_\varepsilon C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcup_\varepsilon B_{(h,i)}) \sqcap_\varepsilon (A_{(f,g)} \sqcup_\varepsilon C_{(j,k)})$ if and only if

$$\begin{aligned} f(e) &\subseteq h(e) \text{ and } g(e) \supseteq i(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ f(e) &\subseteq j(e) \text{ and } g(e) \supseteq k(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

2) $A_{(f,g)} \sqcap_\varepsilon (B_{(h,i)} \sqcup_\varepsilon C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcap_\varepsilon B_{(h,i)}) \sqcup_\varepsilon (A_{(f,g)} \sqcap_\varepsilon C_{(j,k)})$ if and only if

$$\begin{aligned} f(e) &\supseteq h(e) \text{ and } g(e) \subseteq i(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ f(e) &\supseteq j(e) \text{ and } g(e) \subseteq k(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

Proof. Straightforward. ■

5.3.9 Corollary

Let $A_{(f,g)}$, $B_{(h,i)}$ and $C_{(j,k)}$ be three *double-framed fuzzy soft sets* over X such that $(A \cap B) - C = (A \cap C) - B = \emptyset$. Then

1) $A_{(f,g)} \sqcup_\varepsilon (B_{(h,i)} \sqcap_\varepsilon C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcup_\varepsilon B_{(h,i)}) \sqcap_\varepsilon (A_{(f,g)} \sqcup_\varepsilon C_{(j,k)})$,

2) $A_{(f,g)} \sqcap_\varepsilon (B_{(h,i)} \sqcup_\varepsilon C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcap_\varepsilon B_{(h,i)}) \sqcup_\varepsilon (A_{(f,g)} \sqcap_\varepsilon C_{(j,k)})$.

5.3.10 Corollary

Let $A_{(f,g)}$, $A_{(h,i)}$ and $A_{(j,k)}$ be three *double-framed fuzzy soft sets* over X. Then

$$A_{(f,g)} \zeta (A_{(h,i)} \rho A_{(j,k)}) \tilde{=} (A_{(f,g)} \zeta A_{(h,i)}) \rho (A_{(f,g)} \zeta A_{(j,k)})$$

for distinct ζ, $\rho \in \{\sqcap_\varepsilon, \sqcap, \sqcup_\varepsilon, \sqcup\}$.

5.3.11 Theorem

Let $A_{(f,g)}$ and $B_{(h,i)}$ be double-*framed fuzzy soft sets* over X. Then the following are true

1) $A_{(f,g)} \sqcup_\varepsilon B_{(h,i)}$ is the smallest *double-framed fuzzy* soft set over X which contains both $A_{(f,g)}$ and $B_{(h,i)}$. (Supremum)

2) $A_{(f,g)} \sqcap B_{(h,i)}$ is the largest *double-framed* fuzzy soft set over X which is contained in both $A_{(f,g)}$ and $B_{(h,i)}$. (Infimum)

Proof.

1) We have $A, B \subseteq (A \cup B)$ and $f(e), h(e) \subseteq f(e) \vee h(e)$ and $g(e) \wedge i(e) \subseteq g(e)$, $g(e) \wedge i(e) \subseteq i(e)$. So $A_{(f,g)} \tilde{\subseteq} A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)}$ and $B_{(h,i)} \tilde{\subseteq} A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)}$. Let $C_{(j,k)}$ be a double-framed fuzzy soft set over X, such that $A_{(f,g)}$, $B_{(h,i)} \tilde{\subseteq} C_{(j,k)}$. Then A, $B \subseteq C$ implies that $(A \cup B) \subseteq C$ and $f(e)$, $h(e) \subseteq j(e)$ implies that $f(e) \vee h(e) \subseteq j(e)$. Also $k(e) \subseteq g(e)$, $k(e) \subseteq i(e)$ imply that $k(e) \subseteq g(e) \wedge i(e)$ for all $e \in A \cup B$. Thus $A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)} \tilde{\subseteq} C_{(j,k)}$. It follows that $A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)}$ is the smallest double-framed fuzzy soft set over X which contains both $A_{(f,g)}$ and $B_{(h,i)}$.

2) We have $A \cap B \subseteq A$, $A \cap B \subseteq B$ and $f(e) \wedge h(e) \subseteq f(e)$, $f(e) \wedge h(e) \subseteq h(e)$ and $g(e) \subseteq g(e) \vee i(e)$, $i(e) \subseteq g(e) \vee i(e)$ for all $e \in A \cap B$. So $A_{(f,g)} \sqcap B_{(h,i)} \tilde{\subseteq} A_{(f,g)}$ and $A_{(f,g)} \sqcap B_{(h,i)} \tilde{\subseteq} B_{(h,i)}$. Let $C_{(j,k)}$ be a double-framed fuzzy soft set over X, such that $C_{(j,k)} \tilde{\subseteq} A_{(f,g)}$ and $C_{(j,k)} \tilde{\subseteq} B_{(h,i)}$. Then $C \subseteq A$, $C \subseteq B$ implies that $C \subseteq A \cap B$ and $j(e) \subseteq f(e)$, $j(e) \subseteq g(e)$ imply that $j(e) \subseteq f(e) \wedge g(e)$, and $g(e) \subseteq k(e)$, $i(e) \subseteq k(e)$ imply that $g(e) \vee i(e) \subseteq k(e)$ for all $e \in C$. Thus $C_{(j,k)} \tilde{\subseteq} A_{(f,g)} \sqcap B_{(h,i)}$. It follows that $A_{(f,g)} \sqcap B_{(h,i)}$ is the largest double-framed fuzzy soft set over X which is contained in both $A_{(f,g)}$ and $B_{(h,i)}$.

■

5.4 Algebras of Double-framed Fuzzy Soft Sets

In this section, we discuss the concepts of lattices and algebras for the collections of double-framed fuzzy soft sets. Let $\mathcal{DFSS}(X)^E$ be the collection of all double-framed fuzzy soft sets over X and $\mathcal{DFSS}(X)_A$ be its sub collection of all double-framed fuzzy soft sets over X with a fixed set of parameters A. We note that these collections are partially ordered by the relation of soft inclusion $\tilde{\subseteq}$ given in Definition 5.1.2.

5.4.1 Proposition

$(\mathcal{DFSS}(X)^E, \sqcap_{\varepsilon}, \sqcup)$, $(\mathcal{DFSS}(X)^E, \sqcup, \sqcap_{\varepsilon})$, $(\mathcal{DFSS}(X)^E, \sqcup_{\varepsilon}, \sqcap)$, $(\mathcal{DFSS}(X)^E, \sqcap, \sqcup_{\varepsilon})$ $(\mathcal{DFSS}(X)_A, \sqcup, \sqcap)$, and $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup)$ are complete lattices.

Proof. Let us consider $(\mathcal{DFSS}(X)^E, \sqcap_{\varepsilon}, \sqcup)$. Then for any double-framed fuzzy soft sets $A_{(f,g)}$, $B_{(h,i)}$, $C_{(j,k)} \in \mathcal{DFSS}(X)^E$,

1) We have

$$\begin{aligned} A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)} &\tilde{=} (A \cup B)_{(f \tilde{\wedge} h, g \tilde{\vee} i)} \in \mathcal{DFSS}(X)^E \text{ and} \\ A_{(f,g)} \sqcup B_{(h,i)} &\tilde{=} (A \cap B)_{(f \tilde{\vee} h, g \tilde{\wedge} i)} \in \mathcal{DFSS}(X)^E. \end{aligned}$$

2) From Proposition 5.3.3, we have

$$A_{(f,g)} \sqcap_{\varepsilon} A_{(f,g)} \tilde{=} A_{(f,g)} \text{ and } A_{(f,g)} \sqcup A_{(f,g)} \tilde{=} A_{(f,g)}.$$

3) From Proposition 5.3.4 we see that

$$\begin{aligned} A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)} &\tilde{=} B_{(h,i)} \sqcap_{\varepsilon} A_{(f,g)} \text{ and} \\ A_{(f,g)} \sqcup B_{(h,i)} &\tilde{=} B_{(h,i)} \sqcup A_{(f,g)}. \end{aligned}$$

Also

$$\begin{aligned} A_{(f,g)} \sqcap_{\varepsilon} (B_{(h,i)} \sqcap_{\varepsilon} C_{(j,k)}) &\tilde{=} (A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)}) \sqcap_{\varepsilon} C_{(j,k)} \text{ and} \\ A_{(f,g)} \sqcup (B_{(h,i)} \sqcup C_{(j,k)}) &\tilde{=} (A_{(f,g)} \sqcup B_{(h,i)}) \sqcup C_{(j,k)}. \end{aligned}$$

4) From Proposition 5.3.5,

$$A_{(f,g)} \sqcap_{\varepsilon} (B_{(h,i)} \sqcup A_{(f,g)}) \tilde{=} A_{(f,g)} \text{ and } A_{(f,g)} \sqcup (B_{(h,i)} \sqcap_{\varepsilon} A_{(f,g)}) \tilde{=} A_{(f,g)}.$$

So we conclude that the structure forms a lattice. Consider a collection of double-framed fuzzy soft sets $\{A_{i_{(f_i,g_i)}} : i \in I\}$ over X. We have, $\bigcup\limits_{i\in I} A_i \subseteq E$ and, let $\Lambda(e) = \{j : e \in A_j\}$ for any $e \in A_i$. Then $\left(\bigwedge_{i\in\Lambda(e)} f_i(e)\right)(x) \in [0,1]$ and $\left(\bigvee\limits_{i\in\Lambda(e)} g_i(e)\right)(x) \in [0,1]$ for all $x \in X$. Thus $\underset{i\in I}{\sqcap_{\varepsilon}} A_{i_{(f_i,g_i)}} \in \mathcal{DFSS}(X)^E$.

Again, we have, $\bigcap\limits_{i\in I} A_i \subseteq E$ and for any $e \in \bigcap\limits_{i\in I} A_i$, $\left(\bigvee\limits_{i\in I} f_i(e)\right)(x) \in [0,1]$ and $\left(\bigwedge_{i\in I} g_i(e)\right)(x) \in [0,1]$ for all $x \in X$. Thus $\underset{i\in I}{\sqcup} A_{i_{(f_i,g_i)}} \in \mathcal{DFSS}(X)^E$.

Similarly we can show for the remaining structures. ■

5.4.2 Proposition

The structures $(\mathcal{DFSS}(X)^E, \sqcap, \sqcup_{\varepsilon}, \emptyset_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, E_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$, $(\mathcal{DFSS}(X)^E, \sqcup_{\varepsilon}, \sqcap, E_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}, \emptyset_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})})$, $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$ and $(\mathcal{DFSS}(X)_A, \sqcup, \sqcap, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}, A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})})$ are bounded distributive lattices.

Proof. Proposition 5.3.6 assures the distributivity of $(\mathcal{DFSS}(X)^E, \sqcap, \sqcup_{\varepsilon})$ and $(\mathcal{DFSS}(X)^E, \sqcup_{\varepsilon}, \sqcap)$. From Theorem 5.3.11, we conclude that $(\mathcal{DFSS}(X)^E, \sqcap, \sqcup_{\varepsilon}, \emptyset_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, E_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$ is a bounded distributive lattice and $(\mathcal{DFSS}(X)^E, \sqcup_{\varepsilon}, \sqcap, E_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}, \emptyset_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})})$ is its dual. For any double-framed fuzzy soft sets $A_{(f,g)}$, $A_{(h,i)} \in \mathcal{DFSS}(X)_A$,

$$\begin{aligned} A_{(f,g)} \sqcap A_{(h,i)} &\tilde{=} A_{(f\tilde{\wedge}h, g\tilde{\vee}i)} \in \mathcal{DFSS}(X)_A \text{ and} \\ A_{(f,g)} \sqcup A_{(h,i)} &\tilde{=} A_{(f\tilde{\vee}h, g\tilde{\wedge}i)} \in \mathcal{DFSS}(X)_A. \end{aligned}$$

Thus $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup)$ is also a distributive sublattice of $(\mathcal{DFSS}(X)^E, \sqcup_\varepsilon, \sqcap)$ and Theorem 5.3.3 tells us that $A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$, $A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}$ are its lower and upper bounds respectively. Therefore $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$ is a bounded distributive lattice and $(\mathcal{DFSS}(X)_A, \sqcup, \sqcap, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}, A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})})$ is its dual. ■

5.4.3 Proposition

Let $A_{(f,g)}$ be a *double-framed fuzzy soft set* over X. Then the operation $A_{(f,g)} \mapsto A_{(f,g)^{\prime}}$ on $\mathcal{DFSS}(X)^E$ which is given in Definition 5.2.7 satisfies:

1) $(A_{(f,g)^{\prime}})^{\prime} \tilde{=} A_{(f,g)}$ and $A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})^{\prime}} \tilde{=} A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$, $A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})^{\prime}} \tilde{=} A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}$,

2) if $A_{(h,i)}$ is a *double-framed fuzzy* soft set over X then $A_{(f,g)} \tilde{\subseteq} A_{(h,i)}$ if and only if $A_{(h,i)^{\prime}} \tilde{\subseteq} A_{(f,g)^{\prime}}$.

Proof.

1) The proof follows from the fact that, for any $e \in A$

$$\begin{aligned} ((f^{\,\prime})^{\prime})(e) &= (f^{\,\prime}(e))^{\prime} = ((f(e))^{\,\prime})^{\prime} = f(e) \quad \text{and} \\ ((g^{\,\prime})^{\prime})(e) &= (g^{\,\prime}(e))^{\prime} = ((g(e))^{\,\prime})^{\prime} = g(e). \end{aligned}$$

Also

$$\begin{aligned} A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})^{\prime}} &\tilde{=} A_{(\tilde{\mathbf{1}}^{\prime},\tilde{\mathbf{0}}^{\prime})} \tilde{=} A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, \\ A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})^{\prime}} &\tilde{=} A_{(\tilde{\mathbf{0}}^{\prime},\tilde{\mathbf{1}}^{\prime})} \tilde{=} A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}. \end{aligned}$$

2) Let $e \in A$. If $A_{(f,g)} \tilde{\subseteq} A_{(h,i)}$ then $f(e) \subseteq h(e)$ and $i(e) \subseteq g(e)$.

Now,

$$\begin{aligned} (f^{\,\prime}(e))(x) &= (f(e))^{\prime}(x) \\ &= 1 - (f(e))(x) \\ &\geq 1 - (h(e))(x) \\ &= (h(e))^{\prime}(x) = (h^{\prime}(e))(x) \quad \text{and} \\ (g^{\prime}(e))(x) &= (g(e))^{\prime}(x) \\ &= 1 - (g(e))(x) \\ &\leq 1 - (i(e))(x) \\ &= (i(e))^{\prime}(x) = (i^{\prime}(e))(x) \end{aligned}$$

for all $x \in X$. Thus $A_{(h,i)^{\prime}} \tilde{\subseteq} A_{(f,g)^{\prime}}$. Conversely, if $A_{(h,i)^{\prime}} \tilde{\subseteq} A_{(f,g)^{\prime}}$ then $(A_{(f,g)^{\prime}})^{\prime} \tilde{\subseteq} (A_{(h,i)^{\prime}})^{\prime}$ implies $A_{(f,g)} \tilde{\subseteq} A_{(h,i)}$.

■

5.4.4 Proposition (de Morgan Laws)

Let $A_{(f,g)}$ and $B_{(h,i)}$ be double-*framed fuzzy* soft sets over X. Then the following are true

1) $(A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)})' \tilde{=} A_{(f,g)'} \sqcap_{\varepsilon} B_{(h,i)'}$,

2) $(A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)})' \tilde{=} A_{(f,g)'} \sqcup_{\varepsilon} B_{(h,i)'}$,

3) $(A_{(f,g)} \vee B_{(h,i)})' \tilde{=} A_{(f,g)'} \wedge B_{(h,i)'}$,

4) $(A_{(f,g)} \wedge B_{(h,i)})' \tilde{=} A_{(f,g)'} \vee B_{(h,i)'}$,

5) $(A_{(f,g)} \sqcup B_{(h,i)})' \tilde{=} A_{(f,g)'} \sqcap B_{(h,i)'}$,

6) $(A_{(f,g)} \sqcap B_{(h,i)})' \tilde{=} A_{(f,g)'} \sqcup B_{(h,i)'}$.

Proof. 1) We have $(A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)})' \tilde{=} ((A \cup B)_{(f \tilde{\vee} h, g \tilde{\wedge} g)})' \tilde{=} (A \cup B)_{((f \tilde{\vee} h)', (g \tilde{\wedge} g)')}$. Let $e \in (A \cup B)$. There are three cases:

(i) If $e \in A - B$, then

$$\begin{aligned} (f \tilde{\vee} h)'(e) &= (f(e))' = f'(e) = (f' \tilde{\wedge} h')(e) \\ (g \tilde{\wedge} i)'(e) &= (g(e))' = g'(e) = (g' \tilde{\vee} i')(e), \end{aligned}$$

(ii) If $e \in B - A$, then

$$\begin{aligned} (f \tilde{\vee} h)'(e) &= (h(e))' = h'(e) = (f' \tilde{\wedge} h')(e) \\ (g \tilde{\wedge} i)'(e) &= (i(e))' = i'(e) = (g' \tilde{\vee} i')(e), \end{aligned}$$

(iii) If $e \in (A \cap B)$, then

$$\begin{aligned} (f \tilde{\vee} h)'(e) &= (f(e) \vee h(e))' = (f(e))' \wedge (h(e))' \\ (g \tilde{\vee} i)'(e) &= (g(e) \wedge i(e))' = (g(e))' \vee (i(e))', \end{aligned}$$

and,

$$\begin{aligned} (f' \tilde{\wedge} h')(e) &= (f(e))' \wedge (h(e))' = (f \tilde{\vee} h)'(e) \\ (g' \tilde{\wedge} i')(e) &= (g(e))' \vee (i(e))' = (g \tilde{\vee} i)'(e). \end{aligned}$$

Therefore, in all three cases we obtain equality and thus

$$(A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)})' \tilde{=} A_{(f,g)'} \sqcap_{\varepsilon} B_{(h,i)'}.$$

The remaining parts can also be proved in a similar way. ■

5.4.5 Proposition

$(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ', A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$ is a de Morgan algebra.

Proof. We have already seen that $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$ is a bounded distributive lattice. Proposition 5.4.3 shows that " $'$ " is an involution on $\mathcal{DFSS}(X)_A$ and Proposition 5.4.4 shows that de Morgan laws hold with respect to " $'$ " in $\mathcal{DFSS}(X)_A$. Thus $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ', A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$ is a de Morgan algebra. ■

5.4.6 Proposition

Let $A_{(f,g)}$ and $A_{(h,i)}$ be double-*framed fuzzy* soft sets over X. Then $A_{(h,i)} \sqcap A_{(h,i)'} \tilde{\subseteq} A_{(f,g)} \sqcup A_{(f,g)'}$ and so $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ', A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$ is a Kleene Algebra.

Proof. We have already seen that $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ', A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$ is a *de Morgan algebra*. Now, suppose that for some $A_{(f,g)}$, $A_{(h,i)} \in \mathcal{DFSS}(X)_A$ we have

$$A_{(h,i)} \sqcap A_{(h,i)'} \tilde{\supseteq} A_{(f,g)} \sqcup A_{(f,g)'} \text{ where } A_{(h,i)} \sqcap A_{(h,i)'} \tilde{\neq} A_{(f,g)} \sqcup A_{(f,g)'}.$$

Then there exists some $e \in A$ such that

$$(h \tilde{\wedge} h\ ')(e) \supset (f \tilde{\vee} f\ ')(e) \text{ or } (g \tilde{\vee} g')(e) \subset (g \tilde{\wedge} g')(e)$$

and so there exists some $x \in X$ such that

$$\begin{aligned}
((h\tilde{\wedge}h')(e))(x) &> ((f\tilde{\vee}f\ ')(e))(x) \\
&\Rightarrow (h(e)\tilde{\wedge}h'(e))(x) > (f(e)\tilde{\vee}f\ '(e))(x) \\
&\Rightarrow (h(e))(x) \wedge (h'(e))(x) > (f(e))(x) \vee (f\ '(e))(x)
\end{aligned}$$

or

$$\begin{aligned}
((i\tilde{\vee}i')(e))(x) &< ((g\tilde{\wedge}g')(e))(x) \\
&\Rightarrow (i(e)\tilde{\vee}i'(e))(x) < (g(e)\tilde{\wedge}g'(e))(x) \\
&\Rightarrow (i(e))(x) \vee (i'(e))(x) < (g(e))(x) \wedge (g'(e))(x).
\end{aligned}$$

But

$$\begin{aligned}
(h(e))(x) \wedge (h'(e))(x) &\leq 0.5 \text{ and} \\
(g(e))(x) \wedge (g'(e))(x) &\leq 0.5
\end{aligned}$$

and

$$\begin{aligned}
(f(e))(x) \vee (f'(e))(x) &\geq 0.5 \text{ and} \\
(i(e))(x) \vee (i'(e))(x) &\geq 0.5.
\end{aligned}$$

which gives

$$\begin{aligned}(h(e))(x) \wedge (h\acute{}(e))(x) &\leq (f(e))(x) \vee (f\ \acute{}(e))(x) \text{ or} \\ (g(e))(x) \wedge (g\acute{}(e))(x) &\leq (i(e))(x) \vee (i\acute{}(e))(x).\end{aligned}$$

A contradiction. Thus our supposition is wrong and

$$A_{(h,i)} \sqcap A_{(h,i)'} \tilde{\subseteq} A_{(f,g)} \sqcup A_{(f,g)'}.$$

Therefore $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ', A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$ is a Kleene Algebra. ■

5.4.7 Lemma

Let $A_{(f,g)}, B_{(h,i)} \in \mathcal{DFSS}(X)^E$. Then pseudocomplement of $A_{(f,g)}$ relative to $B_{(h,i)}$ exists in $\mathcal{DFSS}(X)^E$.

Proof. Consider the set

$$T(A_{(f,g)}, A_{(h,i)}) = \{C_{(j,k)} \in \mathcal{DFSS}(X)^E : C_{(j,k)} \sqcap A_{(f,g)} \tilde{\subseteq} B_{(h,i)}\}.$$

We define a double-framed fuzzy soft set $(A^c \cup B)_{(f,g)\to(h,i)} \tilde{=} (A^c \cup B)_{(f\to h, g\to i)} \in \mathcal{DFSS}(X)^E$ where

$$\begin{aligned}((f &\to h)(e))(x) \\ &= \begin{cases} 1 & \text{if } e \in A^c - B \\ \begin{cases} 1 & \text{if } (f(e))(x) \leq (h(e))(x) \\ (h(e))(x) & \text{if } (f(e))(x) > (h(e))(x) \end{cases} & \text{if } e \in B - A^c \\ 1 & \text{if } e \in A^c \cap B \end{cases} \\ &\text{and} \\ ((g &\to i)(e))(x) \\ &= \begin{cases} 0 & \text{if } e \in A^c - B \\ \begin{cases} 0 & \text{if } (i(e))(x) < (g(e))(x) \\ (i(e))(x) & \text{if } (i(e))(x) \geq (g(e))(x) \end{cases} & \text{if } e \in B - A^c \\ 0 & \text{if } e \in A^c \cap B \end{cases}\end{aligned}$$

Then

$$\begin{aligned}(A^c \cup B)_{(f\to h, g\to i)} \sqcap A_{(f,g)} &\tilde{=} ((A^c \cup B) \cap A)_{((f\to h)\tilde{\wedge} f, (g\to i)\tilde{\vee} g)} \\ &\tilde{=} ((A^c \cap A) \cup (B \cap A))_{((f\to h)\tilde{\wedge} f, (g\to i)\tilde{\vee} g)} \\ &\tilde{=} (A \cap B)_{((f\to h)\tilde{\wedge} f, (g\to i)\tilde{\vee} g)}.\end{aligned}$$

For any $e \in A \cap B$, $x \in X$,

$$\begin{aligned}(((f \to h)\tilde{\wedge} f))(e))(x) &= \begin{cases} 1 \wedge (f(e))(x) & \text{if } (f(e))(x) \leq (h(e))(x) \\ (h(e))(x) \wedge (f(e))(x) & \text{if } (f(e))(x) > (h(e))(x) \end{cases} \\ &= \begin{cases} (f(e))(x) & \text{if } (f(e))(x) \leq (h(e))(x) \\ (h(e))(x) & \text{if } (f(e))(x) > (h(e))(x) \end{cases} \\ &\leq (h(e))(x).\end{aligned}$$

and

$$\begin{aligned}(((g \to i)\tilde{\vee} g)(e))(x) &= \begin{cases} 0 \vee (g(e))(x) & \text{if } (i(e))(x) < (g(e))(x) \\ (i(e))(x) \vee (g(e))(x) & \text{if } (i(e))(x) \geq (g(e))(x) \end{cases} \\ &= \begin{cases} (g(e))(x) & \text{if } (i(e))(x) \leq (g(e))(x) \\ (i(e))(x) & \text{if } (i(e))(x) > (g(e))(x) \end{cases} \\ &\geq (i(e))(x).\end{aligned}$$

Hence,

$$(A^c \cup B)_{(f \to h, g \to i)} \sqcap A_{(f,g)} \tilde{\subseteq} B_{(h,i)}$$

Thus $(A^c \cup B)_{(f \to h, g \to i)} \in T(A_{(f,g)}, A_{(h,i)})$. For all $C_{(j,k)} \in T(A_{(f,g)}, A_{(h,i)})$, we have $C_{(j,k)} \sqcap A_{(f,g)} \tilde{\subseteq} A_{(h,i)}$ so for any $e \in C \cap A \subseteq B$

$$j(e) \wedge f(e) \subseteq h(e) \text{ and } k(e) \vee g(e) \supseteq i(e)$$

Now,

$$\begin{aligned}C \cap A &\subseteq B \Rightarrow (A \cap C) \cap B^c = \emptyset \\ &\Rightarrow C \subseteq (A \cap B^c)^c = A^c \cup B.\end{aligned}$$

We have following cases:

(i) If $e \in (A^c - B) \cap C$, then $j(e))(x) \leq 1 = ((f \to h)(e))(x)$ and $k(e))(x) \geq 0 = ((g \to i)\ (e))\ (x)$

(ii) If $e \in (B - A^c) \cap C$, and $(i(e))(x) < (g(e))(x)$ then $(k(e))(x) \geq 0 = ((g \to i)(e))(x)$

(iii) If $e \in (B - A^c) \cap C$, and $(f(e))(x) \leq (h(e))(x)$ then $(j(e))(x) < 1 = ((h \to i)(e))(x)$

(iv) If $e \in (B - A^c) \cap C$ and $(i(e))(x) \geq (g(e))(x)$, then the condition $k(e) \vee g(e) \supseteq i(e)$ implies that $(k(e))(x) \geq (i(e))(x) = ((h \to i)(e))(x)$

(v) If $e \in (B - A^c) \cap C$ and $(f(e))(x) > (h(e))(x)$, then the condition $j(e) \wedge f(e) \subseteq h(e)$ implies that $(j(e))(x) \leq (h(e))(x) = ((h \to i)(e))(x)$

(vi) If $e \in (A^c \cap B) \cap C$, then $j(e))(x) < 1 = ((h \to i)(e))(x)$ and $k(e))(x) \geq 0 = ((g \to i)\ (e))\ (x)$.

Thus $C_{(j,k)} \tilde{\subseteq} (A^c \cup B)_{(f \to h, g \to g)}$ and it also shows that

$$(A^c \cup B)_{(f \to h, g \to g)} \tilde{=} \bigvee T(A_{(f,g)}, A_{(h,i)}) \tilde{=} A_{(f,g)} \to A_{(h,i)}.$$

■

5.4.8 Remark

We know that $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup)$ is a sublattice of $(\mathcal{DFSS}(X)^E, \sqcap_\varepsilon, \sqcup)$. *For any* $A_{(f,g)}$, $A_{(h,i)} \in \mathcal{DFSS}(X)_A$, $A_{(f,g)} \to A_{(h,i)}$ as defined in Lemma 5.4.7, is not in $\mathcal{DFSS}(X)_A$ because $A_{(f,g)} \to A_{(h,i)} \tilde{=} (A^c \cup A)_{(f \to h, g \to i)} \tilde{=} E_{(f \to h, g \to i)} \notin \mathcal{DFSS}(X)_A$.

5.4.9 Lemma

Let $A_{(f,g)}, A_{(h,i)} \in \mathcal{DFSS}(X)_A$. Then pseudocomplement of $A_{(f,g)}$ relative to $A_{(h,i)}$ exists in $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup)$.

Proof. Consider the set

$$T(A_{(f,g)}, A_{(h,i)}) = \{A_{(j,k)} \in \mathcal{DFSS}(X)_A : A_{(j,k)} \sqcap A_{(f,g)} \tilde{\subseteq} A_{(h,i)}\}.$$

We define a double-framed fuzzy soft set $A_{(f \to h, g \to i)} \in \mathcal{DFSS}(X)_A$ where

$$((f \to h)(e))(x) = \begin{cases} 1 & \text{if } (f(e))(x) \leq (h(e))(x) \\ (h(e))(x) & \text{if } (f(e))(x) > (h(e))(x) \end{cases}$$

and

$$((g \to i)(e))(x) = \begin{cases} 0 & \text{if } (i(e))(x) < (g(e))(x) \\ (i(e))(x) & \text{if } (i(e))(x) \geq (g(e))(x) \end{cases}$$

for all $e \in A$, $x \in X$. Then $A_{(f \to h, g \to i)} \sqcap A_{(f,g)} \tilde{=} A_{(f \to h, g \to i) \tilde{\wedge} h}$ and

$$\begin{aligned} &(((f \to h) \tilde{\wedge} f))(e))(x) \\ &= \begin{cases} 1 \wedge (f(e))(x) & \text{if } (f(e))(x) \leq (h(e))(x) \\ (h(e))(x) \wedge (f(e))(x) & \text{if } (f(e))(x) > (h(e))(x) \end{cases} \\ &= \begin{cases} (f(e))(x) & \text{if } (f(e))(x) \leq (h(e))(x) \\ (h(e))(x) & \text{if } (f(e))(x) > (h(e))(x) \end{cases} \\ &\leq (h(e))(x). \end{aligned}$$

and

$$\begin{aligned}(((g \;\to\; i)\tilde{\vee} g)(e))(x) \\ &= \begin{cases} 0 \vee (g(e))(x) & \text{if } (i(e))(x) < (g(e))(x) \\ (i(e))(x) \vee (g(e))(x) & \text{if } (i(e))(x) \geq (g(e))(x) \end{cases} \\ &= \begin{cases} (g(e))(x) & \text{if } (i(e))(x) \leq (g(e))(x) \\ (i(e))(x) & \text{if } (i(e))(x) > (g(e))(x) \end{cases} \\ &\geq (i(e))(x).\end{aligned}$$

for all $e \in A$, $x \in X$. Hence,

$$A_{(f\to h,g\to i)} \sqcap A_{(f,g)} \tilde{\subseteq} A_{(h,i)}$$

and $A_{(f\to h,g\to i)} \in T(A_{(f,g)}, A_{(h,i)})$. For every $A_{(j,k)} \in T(A_{(f,g)}, A_{(h,i)})$, we have $A_{(j,k)} \sqcap A_{(f,g)} \tilde{\subseteq} A_{(h,i)}$ so for any $e \in A$, following cases arise:

(i) If $(i(e))(x) < (g(e))(x)$ then $(k(e))(x) \geq 0 = ((g \to i)(e))(x)$

(ii) If $(f(e))(x) \leq (h(e))(x)$ then $(j(e))(x) < 1 = ((h \to i)(e))(x)$

(iii) If $(i(e))(x) \geq (g(e))(x)$, then the condition $k(e) \vee g(e) \supseteq i(e)$ implies that $(k(e))(x) \geq (i(e))(x) = ((h \to i)(e))(x)$

(iv) If $(f(e))(x) > (h(e))(x)$, then the condition $j(e) \wedge f(e) \subseteq h(e)$ implies that $(j(e))(x) \leq (h(e))(x) = ((h \to i)(e))(x)$.

Thus $A_{(j,k)} \tilde{\subseteq} A_{(f\to h,g\to i)}$ and it also shows that

$$A_{(f\to h,g\to i)} \tilde{=} \bigvee T(A_{(f,g)}, A_{(h,i)}) \tilde{=} A_{(f,g)} \to_A A_{(h,i)}.$$

■

5.4.10 Proposition

$(\mathcal{DFSS}(X)^E, \sqcap_\varepsilon, \sqcup)$ *and* $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup)$ are *Brouwerian lattices.*

Proof. Follows from Lemmas 5.4.7 and 5.4.9. ■

5.4.11 Definition

Let $A_{(f,g)}$ be a *double-framed fuzzy* soft set over X. We define $A_{(f,g)^*}$ as a *double-framed fuzzy soft set* $A_{(f^*,g^*)}$ where

$$f^* : A \to \mathcal{FP}(X),\ e \mapsto (f(e))^*,$$

$$(f(e))^*(x) = \begin{cases} 0 & \text{if } (f(e))^*(x) \neq 0 \\ 1 & \text{if } (f(e))^*(x) = 0 \end{cases}$$

$$g^* : A \to \mathcal{FP}(X),\, e \mapsto (g\,(e))^*,$$
$$(g\,(e))^*(x) = \begin{cases} 1 & \text{if } (g\,(e))^*(x) \neq 1 \\ 0 & \text{if } (g\,(e))^*(x) = 1 \end{cases} \qquad \text{for } x \in X.$$

5.4.12 Theorem

Let $A_{(f,g)}$ and $A_{(h,i)}$ be double-framed fuzzy soft sets over X. Then

1) $A_{(f,g)} \sqcap A_{(f,g)^*} \tilde{=} A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$,

2) $A_{(f,g)} \tilde{\subseteq} A_{(h,i)^*}$ whenever $A_{(f,g)} \sqcap A_{(h.i)} \tilde{=} A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$,

3) $A_{(f,g)^*} \sqcup A_{((f,g)^*)^*} \tilde{=} A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}$.

Thus $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ^*, A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$ is a Stone algebra.

Proof.

1) Consider $A_{(f,g)} \sqcap A_{(f,g)^*}$. For any $e \in A$

$$(f \tilde{\wedge} f^*)(e) = f(e) \wedge f^*(e) \text{ and } (g \tilde{\vee} g^*)(e) = g(e) \vee g^*(e).$$

$\Longrightarrow$

$$\begin{aligned} ((f \tilde{\wedge} f^*)(e))(x) &= \begin{cases} (f(e))(x) \wedge 0 & \text{if } (f(e))(x) \neq 0 \\ 0 \wedge 1 & \text{if } (f(e))(x) = 0 \end{cases} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} ((g \tilde{\vee} g^*)(e))(x) &= \begin{cases} (g(e))(x) \vee 1 & \text{if } (g(e))(x) \neq 1 \\ 1 \vee 0 & \text{if } (g(e))(x) = 1 \end{cases} \\ &= 1 \end{aligned}$$

for all $x \in X$. Thus $A_{(f,g)} \sqcap A_{(f,g)^*} \tilde{=} A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$.

2) If $A_{(f,g)} \sqcap A_{(h,i)} \tilde{=} A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$, then

$$(f(e))(x) \wedge (h(e))(x) = 0 \tag{b}$$

and

$$(g(e))(x) \vee (i(e))(x) = 1 \tag{c}$$

for all $x \in X$, $e \in A$. From Equation (b) we have two cases :

$$\text{If } (h(e))(x) = 0 \text{ then } (h^*(e))(x) = 1 \geq (f(e))(x)$$
$$\text{and}$$
$$\text{if } (h(e))(x) \neq 0 \text{ then } (f(e))(x) = 0 \leq (h^*(e))(x).$$

Thus $(f(e))(x) \leq (h^*(e))(x)$ for all $x \in X$.

From Equation (c), there are two cases:

$$\begin{aligned}\text{If } (i(e))(x) &= 1 \text{ then } (i^*(e))(x) = 0 \leq (g(e))(x)\\ &\text{and}\\ \text{if } (i(e))(x) &\neq 1 \text{ then } (g(e))(x) = 1 \geq (i^*(e))(x).\end{aligned}$$

So $(i^*(e))(x) \leq (g(e))(x)$ for all $x \in X$. This implies that

$$f(e) \subseteq h^*(e) \text{ and } i^*(e) \subseteq g(e) \text{ for all } e \in A.$$

Therefore $A_{(f,g)} \tilde{\subseteq} A_{(h,i)^*}$.

3) Consider $A_{(f,g)^*} \sqcup A_{((f,g)^*)^*}$. For any $e \in A$

$$(f^* \tilde{\vee} f^{**})(e) = f^*(e) \vee f^{**}(e)$$

and

$$(g^* \tilde{\wedge} g^{**})(e) = g^*(e) \wedge g^{**}(e).$$

$\Longrightarrow$

$$\begin{aligned}((f^*(e))(x) \vee (f^{**}(e))(x) &= \begin{cases} 0 \vee 1 & \text{if } (f(e))(x) \neq 0 \\ 1 \vee 0 & \text{if } (f(e))(x) = 0 \end{cases}\\ &= 1\end{aligned}$$

and

$$\begin{aligned}((g^*(e))(x) \wedge (g^{**}(e))(x) &= \begin{cases} 1 \wedge 0 & \text{if } (g(e))(x) \neq 1 \\ 0 \wedge 1 & \text{if } (g(e))(x) = 1 \end{cases}\\ &= 0\end{aligned}$$

for all $x \in X$. Thus $A_{(f,g)^*} \sqcup A_{((f,g)^*)^*} \tilde{=} A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}$.

■

5.4.13 Definition

Let $A_{(f,g)}$ be a *double-framed* fuzzy soft set over X. We define

$$(A_{(f,g)})^\circ \tilde{=} A_{(f,g)^\circ} \tilde{=} A_{(g,f)}.$$

5.4.14 Proposition (Involution)

Let $A_{(f,g)}$ be a *double-framed fuzzy soft set* over X. Then $(A_{(f,g)^\circ})^\circ \tilde{=} A_{(f,g)}$, $A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})^\circ} \tilde{=} A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$ and $A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})^\circ} \tilde{=} A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}$

Proof. It is straightforward that $A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})^\circ} \tilde{=} A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$ and $A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})^\circ} \tilde{=} A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}$. We have

$$(A_{(f,g)^\circ})^\circ \tilde{=} A_{(g,f)^\circ} \tilde{=} A_{(f,g)}.$$

■

5.4.15 Proposition (de Morgan Laws)

Let $A_{(f,g)}$ and $B_{(h,i)}$ be double-*framed* fuzzy soft sets over X. Then the following are true

1) $(A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)})^{\circ} \tilde{=} A_{(f,g)^{\circ}} \sqcap_{\varepsilon} B_{(h,i)^{\circ}}$,

2) $(A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)})^{\circ} \tilde{=} A_{(f,g)^{\circ}} \sqcup_{\varepsilon} B_{(h,i)^{\circ}}$,

3) $(A_{(f,g)} \vee B_{(h,i)})^{\circ} \tilde{=} A_{(f,g)^{\circ}} \wedge B_{(h,i)^{\circ}}$,

4) $(A_{(f,g)} \wedge B_{(h,i)})^{\circ} \tilde{=} A_{(f,g)^{\circ}} \vee B_{(h,i)^{\circ}}$,

5) $(A_{(f,g)} \sqcup B_{(h,i)})^{\circ} \tilde{=} A_{(f,g)^{\circ}} \sqcap B_{(h,i)^{\circ}}$,

6) $(A_{(f,g)} \sqcap B_{(h,i)})^{\circ} \tilde{=} A_{(f,g)^{\circ}} \sqcup B_{(h,i)^{\circ}}$.

Proof.

1) We have

$$(A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)})^{\circ} \tilde{=} ((A \cup B)_{(f \tilde{\vee} h, g \tilde{\wedge} i)})^{\circ} \tilde{=} (A \cup B)_{(g \tilde{\wedge} i, f \tilde{\vee} h)}$$

and

$$A_{(f,g)^{\circ}} \sqcap_{\varepsilon} B_{(h,i)^{\circ}} \tilde{=} A_{(g,f)} \sqcap_{\varepsilon} B_{(i,h)} \tilde{=} (A \cup B)_{(g \tilde{\wedge} i, f \tilde{\vee} h)}.$$

Thus

$$(A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)})^{\circ} \tilde{=} A_{(f,g)^{\circ}} \sqcap_{\varepsilon} B_{(h,i)^{\circ}}.$$

The remaining parts can be proved in a similar way.

■

5.4.16 Theorem

$(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ^{\circ}, A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$ is a de Morgan algebra.

Proof. Follows from Propositions 5.4.14 and 5.4.15. ■

Chapter 6

Algebraic Structures of Bipolar Soft Sets

Bipolarity refers to an explicit handling of positive and negative sides of information. Three types of bipolarity were discussed in [11] but we are using a rather generalized bipolarity here, dealing with the positive and negative impacts in information associated with a soft set and its representation. This chapter introduces the concept of a bipolar soft set. A bipolar soft set is obtained by considering not only a carefully chosen set of parameters but also an allied set of oppositely meaning parameters named as "Not set of parameters". Structure of a bipolar soft set is managed by two functions, say $\alpha : A \to \mathcal{P}(X)$ and $\beta : \neg A \to \mathcal{P}(X)$ where $\neg A$ stands for the "not set of A" and β describes somewhat an opposite or negative approximation for the attractiveness of a houses of X, relative to the approximation computed by α. Maji et al. [33] had used the "not set" to define complement of a soft set. The *complement of a soft set* simply gives the complements of the approximations. The above mentioned soft function β is rather more generalized than soft complement function and $(\beta,\neg A)$ can be any soft subset of $(\alpha,A)^c$. The difference is the gray area of choice, that is, we may find some houses which do not satisfy any criteria properly e.g. A house may not be highly expensive but it does not assure its cheapness either. Thus, we must be careful while making our considerations for the parameterization of data keeping in view that, during approximations, there might be some indifferent elements in X. This gives us a motivation to define the idea of bipolar soft sets. We have defined operations of union and intersection for bipolar soft sets by taking restricted, extended and product sets of parameters. The algebraic structures of bipolar soft sets are discussed with the properties of operations.

6.1 Bipolar Soft Sets

Let X be an initial universe and E be a set of parameters. Let $\mathcal{P}(X)$ denotes the power set of X and A, B, C be non-empty subsets of E.

6.1.1 Definition

A triplet $(\alpha,\beta : A)$ is called a *bipolar* soft set over X, where α and β are mappings, given by $\alpha : A \to \mathcal{P}(X)$ and $\beta : \neg A \to \mathcal{P}(X)$ such that $\alpha(e) \cap \beta(\neg e) = \emptyset$ (Empty Set) for all $e \in A$.

In other words, a *bipolar* soft set over X gives two parametrized families of subsets of the universe X and the condition $\alpha(e) \cap \beta(\neg e) = \emptyset$ for all $e \in A$, is imposed as a consistency constraint. For each $e \in A$, $\alpha(e)$ and $\beta(\neg e)$ are regarded as the set of e-approximate elements of *bipolar* soft set $(\alpha,\beta : A)$. It is also observed that the relationship between a complement function and the defining function of a soft set becomes a particular case for the defining functions of a bipolar soft set, that is, $(\alpha,\alpha^c : A)$ is a bipolar soft set over X. The difference occurs due to the presence of uncertainty or hesitation or lack of knowledge in defining the membership function. We name this uncertainty or gray area as the approximation for the degree of hesitation. Thus the union of three approximations, that is, e-approximation, $\neg e$-approximation, and approximation of hesitation is X. We note that $\emptyset \subseteq X - \{\alpha(e) \cup \beta(\neg e)\} \subseteq X$, for each $e \in A$. So, we may approximate the degree of hesitation in $(\alpha,\beta : A)$ by an allied soft set A_h defined over X, where $h(e) = X - \{\alpha(e) \cup \beta(\neg e)\}$ for all $e \in A$.

6.1.2 Definition

For two *bipolar* soft sets $(\alpha,\beta : A)$ and $(\gamma,\delta : B)$ over a universe X, we say that $(\alpha,\beta : A)$ is a *bipolar* soft subset of $(\gamma,\delta : B)$, if

1) $A \subseteq B$ and

2) $\alpha(e) \subseteq \gamma(e)$ and $\delta(\neg e) \subseteq \beta(\neg e)$ for all $e \in A$.

This relationship is denoted by $(\alpha,\beta : A)\tilde{\subseteq}(\gamma,\delta : B)$. Similarly $(\alpha,\beta : A)$ is said to be a *bipolar* soft superset of $(\gamma,\delta : B)$, if $(\gamma,\delta : B)$ is a *bipolar soft subset* of $(\alpha,\beta : A)$. We denote it by $(\alpha,\beta : A)\tilde{\supseteq}(\gamma,\delta : B)$.

6.1.3 Definition

Two *bipolar* soft sets $(\alpha,\beta : A)$ and $(\gamma,\delta : B)$ over X are said to be *equal* if $(\alpha,\beta : A)$ is a *bipolar* soft subset of $(\gamma,\delta : B)$ and $(\gamma,\delta : B)$ is a *bipolar* soft subset of $(\alpha,\beta : A)$.

Let $\mathcal{BSS}(X)^E$ denotes the set of all bipolar soft sets defined over X with set of parameters E ordered by the relation of inclusion $\tilde{\subseteq}$ as defined in Definition 6.1.2.

Now we claim that every bipolar soft set is equivalent to a double-framed soft set and give the following theorem:

6.1.4 Theorem

The mapping $\theta : \mathcal{BSS}(X)^E \to \mathcal{DSS}(X)^E$, $(\alpha,\beta : A) \mapsto A_{(\alpha_1,\beta_1)}$ is a monomorphism of lattices where

$$\alpha(e) = \alpha_1(e), \text{ and } \ \beta(e) = \beta_1(\neg e) \text{ for all } e \in A.$$

Proof. Clearly θ is well-defined. If

$$\theta((\alpha, \beta : A))\tilde{=}\theta((\gamma, \delta : B))$$

where

$$\theta((\alpha, \beta : A))\tilde{=}A_{(\alpha_1,\beta_1)} \text{ and } \theta((\gamma, \delta : B))\tilde{=}B_{(\gamma_1,\delta_1)}$$

then $A = B$ and

$$\alpha(e) = \alpha_1(e),\, \gamma(e) = \gamma_1(e) \ \text{ and } \ \beta(e) = \beta_1(\neg e),\, \delta(e) = \delta_1(\neg e) \ \text{ for all } e \in A.$$

Now,

$$\alpha(e) = \alpha_1(e) = \gamma_1(e) = \gamma(e) \text{ and } \beta(e) = \beta_1(\neg e) = \delta_1(\neg e) = \delta(e) \ \text{ for all } e \in A.$$

Thus

$$(\alpha, \beta : A)\tilde{=}(\gamma, \delta : B)$$

shows that θ is one-to-one. Clearly θ preserves the order of inclusion. ■

6.1.5 Remark

Note that θ is not onto because of the extra condition of consistency constraint for defining bipolar soft sets.

By Theorem 6.1.4, we can equate every bipolar soft set with a double-framed soft set with the consistency constraint and so, from onwards, we shall denote a bipolar soft set $(\alpha,\beta : A)$ by its image $\theta((\alpha,\beta : A))\tilde{=}A_{\langle\alpha,\beta\rangle}$ where the meanings of A, α and β are clear.

6.1.6 Example

Let X be the set of houses under consideration, and E be the set of parameters, $X = \{h_1,h_2,h_3,h_4,h_5\}$, $E = \{e_1,e_2,e_3,e_4,e_5,e_6\} = \{$ in the green surroundings, wooden, cheap, in good repair, furnished, traditional $\}$. Suppose that $A = \{e_1,e_2,e_3,e_6\}$, a bipolar soft set $A_{\langle\alpha,\beta\rangle}$ describes the "requirements of the houses" which Mr. Y is going to buy. The bipolar soft set $A_{\langle\alpha,\beta\rangle}$ over X, where α and β represent the classification under high and low investment respectively, is given as follows:

$$\alpha \;:\; A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_1, h_2, h_4\} & \text{if } e = e_1, \\ \{h_3, h_4\} & \text{if } e = e_2, \\ X & \text{if } e = e_3, \\ \{h_2, h_3, h_5\} & \text{if } e = e_6, \end{cases}$$

$$\beta \;:\; A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_3, h_5\} & \text{if } e = e_1, \\ \{h_1, h_2, h_5\} & \text{if } e = e_2, \\ \{\} & \text{if } e = e_3, \\ \{h_1\} & \text{if } e = e_6. \end{cases}$$

Let $B = \{e_2,e_3\}$. Then bipolar soft set $B_{\langle\gamma,\delta\rangle}$ given by

$$\gamma \;:\; A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_3\} & \text{if } e = e_2, \\ \{h_1, h_4, h_5\} & \text{if } e = e_3, \end{cases}$$

$$\delta \;:\; A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} X & \text{if } e = e_2, \\ \{h_1\} & \text{if } e = e_3, \end{cases}$$

is a bipolar soft subset of $A_{\langle\alpha,\beta\rangle}$ and represents the data under a strict set of parameters B following A.

6.2 Operations on Bipolar Soft Sets

This section gives various operations defined on bipolar soft sets:

6.2.1 Definition

If $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ are two *bipolar* soft sets over X. The int-uni product of $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$is defined to be a bipolar soft set $(A \times B)_{\langle\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta\rangle}$ over X in which $\alpha\tilde{\cap}\gamma :$ $(A \times B) \to \mathcal{P}(X)$, where

$$(a, b) \mapsto \alpha(a) \cap \gamma(b),$$

and $\beta\tilde{\cup}\delta : (A \times B) \to \mathcal{P}(X)$, where

$$(a, b) \mapsto \beta(a) \cup \delta(b).$$

It is denoted by $A_{\langle\alpha,\beta\rangle} \wedge B_{\langle\gamma,\delta\rangle} \tilde{=} (A \times B)_{\langle\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta\rangle}$.

6.2.2 Definition

If $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ are two *bipolar* soft sets over X then uni-int product of $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ is defined as a bipolar soft set $(A\times B)_{\langle\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta\rangle}$ over X in which $\alpha\tilde{\cup}\gamma : (A\times B) \to \mathcal{P}(X)$, where

$$(a,b) \mapsto \alpha(a) \cup \gamma(b),$$

and $\beta\tilde{\cap}\delta : (A \times B) \to \mathcal{P}(X)$, where

$$(a,b) \mapsto \beta(a) \cap \delta(b).$$

It is denoted by $A_{\langle\alpha,\beta\rangle} \vee B_{\langle\gamma,\delta\rangle} \tilde{=} (A \times B)_{\langle\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta\rangle}$.

6.2.3 Definition

For two *bipolar soft sets* $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ over X, the *extended int-uni bipolar soft set* of $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ is defined as a *bipolar soft set* $(A \cup B)_{\langle\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta\rangle}$ over X in which $\alpha\tilde{\cap}\gamma : (A \cup B) \to \mathcal{P}(X)$, where

$$e \mapsto \begin{cases} \alpha(e) & \text{if } e \in A - B \\ \gamma(e) & \text{if } e \in B - A \\ \alpha(e) \cap \gamma(e) & \text{if } e \in (A \cap B) \end{cases}$$

and $\beta\tilde{\cap}\delta : (A \cup B) \to \mathcal{P}(X)$,

$$e \mapsto \begin{cases} \beta(e) & \text{if } e \in A - B \\ \delta(e) & \text{if } e \in B - A \\ \beta(e) \cup \delta(e) & \text{if } e \in (A \cap B) \end{cases}.$$

It is denoted by $A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} B_{\langle\gamma,\delta\rangle} \tilde{=} (A \cup B)_{\langle\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta\rangle}$.

6.2.4 Definition

For two *bipolar soft sets* $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ over X, the *extended uni-int bipolar soft set* of $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ is defined as a *bipolar soft set* $(A \cup B)_{\langle\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta\rangle}$ over X in which $\alpha\tilde{\cup}\gamma : (A \cup B) \to \mathcal{P}(X)$, where

$$e \mapsto \begin{cases} \alpha(e) & \text{if } e \in A - B \\ \gamma(e) & \text{if } e \in B - A \\ \alpha(e) \cup \gamma(e) & \text{if } e \in (A \cap B) \end{cases}$$

and $\beta\tilde{\cap}\delta : (A \cup B) \to \mathcal{P}(X)$, where

$$e \mapsto \begin{cases} \beta(e) & \text{if } e \in A - B \\ \delta(e) & \text{if } e \in B - A \\ \beta(e) \cap \delta(e) & \text{if } e \in (A \cap B) \end{cases}.$$

It is denoted by $A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} B_{\langle\gamma,\delta\rangle} \tilde{=} (A \cup B)_{\langle\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta\rangle}$.

6.2.5 Definition

For two *bipolar soft sets* $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ over X, the *extended difference bipolar soft set* of $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ is defined as a *bipolar soft set* $(A\cup B)_{\langle\alpha\smile_{\varepsilon}\gamma,\beta\smile_{\varepsilon}\delta\rangle}$ over X in which $\alpha\smile_{\varepsilon}\gamma:(A\cup B)\to\mathcal{P}(X)$, where

$$e\mapsto\begin{cases}\alpha(e) & \text{if } e\in A-B\\ \gamma(e) & \text{if } e\in B-A\\ \alpha(e)-\gamma(e) & \text{if } e\in(A\cap B)\end{cases}$$

and $\beta\smile_{\varepsilon}\delta:(A\cup B)\to\mathcal{P}(X)$, where

$$e\mapsto\begin{cases}\beta(e) & \text{if } e\in A-B\\ \delta(e) & \text{if } e\in B-A\\ \beta(e)-\delta(e) & \text{if } e\in(A\cap B).\end{cases}$$

It is denoted by $A_{\langle\alpha,\beta\rangle}\smile_{\varepsilon}B_{\langle\gamma,\delta\rangle}\tilde{=}(A\cup B)_{\langle\alpha\smile_{\varepsilon}\gamma,\beta\smile_{\varepsilon}\delta\rangle}$.

6.2.6 Definition

For two *bipolar soft sets* $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ over X with $(A\cap B)\neq\emptyset$, the *restricted int-uni bipolar soft set* of $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ is defined as a *bipolar soft set* $(A\cap B)_{\langle\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta\rangle}$ over X in which $\alpha\tilde{\cap}\gamma:(A\cap B)\to\mathcal{P}(X)$, where

$$e\mapsto\alpha(e)\cap\gamma(e),$$

and $\beta\tilde{\cup}\delta:(A\cap B)\to\mathcal{P}(X)$, where

$$e\mapsto\beta(e)\cup\delta(e).$$

It is denoted by $A_{\langle\alpha,\beta\rangle}\sqcap B_{\langle\gamma,\delta\rangle}\tilde{=}(A\cap B)_{\langle\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta\rangle}$.

6.2.7 Definition

For two *bipolar soft sets* $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ over X with $(A\cap B)\neq\emptyset$, the *restricted uni-int bipolar soft set* of $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ is defined as a *bipolar soft set* $(A\cap B)_{\langle\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta\rangle}$ over X in which $\alpha\tilde{\cup}\gamma:(A\cap B)\to\mathcal{P}(X)$, where

$$e\mapsto\alpha(e)\cup\gamma(e),$$

and $\beta\tilde{\cap}\delta:(A\cap B)\to\mathcal{P}(X)$, where

$$e\mapsto\beta(e)\cap\delta(e).$$

It is denoted by $A_{\langle\alpha,\beta\rangle}\sqcup B_{\langle\gamma,\delta\rangle}\tilde{=}(A\cap B)_{\langle\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta\rangle}$.

6.2.8 Definition

For two *bipolar soft sets* $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ over X with $(A \cap B) \neq \emptyset$, the *restricted difference bipolar soft set* of $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ is defined as a *bipolar soft set* $(A \cap B)_{\langle\alpha\smile\gamma,\beta\smile\delta\rangle}$ over X in which $\alpha \smile \gamma : (A \cap B) \to \mathcal{P}(X)$, where

$$e \mapsto \alpha(e) - \gamma(e),$$

and $\beta \smile \delta : (A \cap B) \to \mathcal{P}(X)$, where

$$e \mapsto \beta(e) - \delta(e).$$

It is denoted by $A_{\langle\alpha,\beta\rangle} \smile B_{\langle\gamma,\delta\rangle} \tilde{=} (A \cap B)_{\langle\alpha\smile\gamma,\beta\smile\delta\rangle}$.

6.2.9 Proposition

The mapping $\theta : \mathcal{BSS}(X)^E \to \mathcal{DSS}(X)^E$ as defined in Theorem 6.1.4 preserves the product, extended and restricted uni-int and int-uni operations.

Proof. Straightforward. ■

6.2.10 Remark

The operation of complementation as defined in Definition 4.2.9 for double-framed soft sets is no more valid for bipolar soft sets because $(A_{\langle\alpha,\beta\rangle})^c \tilde{=} A_{(\alpha^c,\beta^c)}$ which may not satisfy the consistency constraint as shown by the following example:

6.2.11 Example

Let E, A, X and bipolar soft set $A_{\langle\alpha,\beta\rangle}$ over X be taken as in Example 6.1.6. Then $(A_{\langle\alpha,\beta\rangle})^c$ is given as follows:

$$\alpha^c \;:\; A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_3, h_5\} & \text{if } e = e_1, \\ \{h_1, h_2, h_5\} & \text{if } e = e_2, \\ \{\} & \text{if } e = e_3, \\ \{h_1, h_2\} & \text{if } e = e_6, \end{cases}$$

$$\beta^c \;:\; A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_1, h_2, h_4\} & \text{if } e = e_1, \\ \{h_3, h_4\} & \text{if } e = e_2, \\ X & \text{if } e = e_3, \\ \{h_2, h_3, h_5\} & \text{if } e = e_6. \end{cases}$$

but

$$\alpha^c(e_6) \cap \beta^c(e_6) \neq \emptyset$$

so $(A_{\langle\alpha,\beta\rangle})^c \notin \mathcal{BSS}(X)^E$. Thus "^{c} " is not defined on $\mathcal{BSS}(X)^E$.

6.2.12 Proposition

Let $A_{\langle\alpha,\beta\rangle}$ be a *bipolar* soft set over X. Then $^\circ : \mathcal{BSS}(X)^E \to \mathcal{BSS}(X)^E$ is defined and we denote $(A_{\langle\alpha,\beta\rangle})^\circ$ by $A_{\langle\alpha,\beta\rangle^\circ}$.

Proof. If $A_{\langle\alpha,\beta\rangle} \in \mathcal{BSS}(X)^E$ then

$$\begin{aligned} A_{\langle\alpha,\beta\rangle^\circ} &\tilde{=} A_{\langle\alpha^\circ,\beta^\circ\rangle} \qquad \text{where} \\ \alpha^\circ &: A \to \mathcal{P}(X),\ e \mapsto \beta(e) \text{ and } \beta^\circ : A \to \mathcal{P}(X),\ e \mapsto \alpha(e). \end{aligned}$$

Clearly

$$\alpha^\circ(e) \cap \beta^\circ(e) = \beta(e) \cap \alpha(e) = \emptyset.$$

Thus $A_{\langle\alpha,\beta\rangle^\circ} \in \mathcal{BSS}(X)^E$. ■

6.3 Properties of Bipolar Soft Sets

In this section we check the properties and associative, commutative, distributive and absorption laws of bipolar soft sets with respect to their operations.

6.3.1 Definition

A *bipolar soft set* over X is said to be a *relative null bipolar soft set,* denoted by $A_{\langle\Phi,\mathfrak{X}\rangle}$ where

$$\Phi : A \to \mathcal{P}(X),\ e \mapsto \emptyset \text{ and } \mathfrak{X} : A \to \mathcal{P}(X),\ e \mapsto X.$$

6.3.2 Definition

A bipolar soft set over X is said to be a *relative absolute bipolar soft set,* denoted by $A_{\langle\mathfrak{X},\Phi\rangle}$ where

$$\mathfrak{X} : A \to \mathcal{P}(X),\ e \mapsto X \text{ and } \Phi : A \to \mathcal{P}(X),\ e \mapsto \emptyset.$$

Conventionally, we take the bipolar soft sets with empty set of parameters to be equal to $\emptyset_{\langle\Phi,\mathfrak{X}\rangle}$ and so $A_{\langle\alpha,\beta\rangle} \sqcap B_{\langle\gamma,\delta\rangle} \tilde{=} \emptyset_{\langle\Phi,\mathfrak{X}\rangle} \tilde{=} A_{\langle\alpha,\beta\rangle} \sqcup B_{\langle\gamma,\delta\rangle}$ whenever $(A \cap B) = \emptyset$.

6.3.3 Proposition

If $A_{\langle\Phi,\mathfrak{X}\rangle}$ is a null bipolar soft set, $A_{\langle\mathfrak{X},\Phi\rangle}$ an absolute bipolar soft set, and $A_{\langle\alpha,\beta\rangle}$, $A_{\langle\gamma,\delta\rangle}$ are bipolar soft sets over X, then

1) $A_{\langle\alpha,\beta\rangle} \sqcup_\varepsilon A_{\langle\gamma,\delta\rangle} \tilde{=} A_{\langle\alpha,\beta\rangle} \sqcup A_{\langle\gamma,\delta\rangle}$,

2) $A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon A_{\langle\gamma,\delta\rangle} \tilde{=} A_{\langle\alpha,\beta\rangle} \sqcap A_{\langle\gamma,\delta\rangle}$,

3) $A_{\langle\alpha,\beta\rangle} \sqcap A_{\langle\alpha,\beta\rangle} \tilde{=} A_{\langle\alpha,\beta\rangle} \tilde{=} A_{\langle\alpha,\beta\rangle} \sqcup A_{\langle\alpha,\beta\rangle}$,

4) $A_{\langle\alpha,\beta\rangle} \sqcup A_{\langle\Phi,\mathfrak{X}\rangle} \tilde{=} A_{\langle\alpha,\beta\rangle} \tilde{=} A_{\langle\alpha,\beta\rangle} \sqcap A_{\langle\mathfrak{X},\Phi\rangle}$,

5) $A_{\langle\alpha,\beta\rangle} \sqcup A_{\langle\mathfrak{X},\Phi\rangle} \tilde{=} A_{\langle\mathfrak{X},\Phi\rangle}$; $A_{\langle\alpha,\beta\rangle} \sqcap A_{\langle\Phi,\mathfrak{X}\rangle} \tilde{=} A_{\langle\Phi,\mathfrak{X}\rangle}$.

Proof. Straightforward. ■

6.3.4 Proposition

Let $A_{\langle\alpha,\beta\rangle}$, $B_{\langle\gamma,\delta\rangle}$ and $C_{\langle\zeta,\eta\rangle}$ be any *bipolar soft set*s over X. Then the following are true

1) (Absorption Laws)

(i) $A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcap A_{\langle\alpha,\beta\rangle}) \tilde{=} A_{\langle\alpha,\beta\rangle}$,

(ii) $A_{\langle\alpha,\beta\rangle} \sqcap (B_{\langle\gamma,\delta\rangle} \sqcup_{\varepsilon} A_{\langle\alpha,\beta\rangle}) \tilde{=} A_{\langle\alpha,\beta\rangle}$,

(iii) $A_{\langle\alpha,\beta\rangle} \sqcup (B_{\langle\gamma,\delta\rangle} \sqcap_{\varepsilon} A_{\langle\alpha,\beta\rangle}) \tilde{=} A_{\langle\alpha,\beta\rangle}$,

(iv) $A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcup A_{\langle\alpha,\beta\rangle}) \tilde{=} A_{\langle\alpha,\beta\rangle}$.

2) (Associative Laws) $A_{\langle\alpha,\beta\rangle} \lambda (B_{\langle\gamma,\delta\rangle} \lambda C_{\langle\zeta,\eta\rangle}) \tilde{=} (A_{\langle\alpha,\beta\rangle} \lambda B_{\langle\gamma,\delta\rangle}) \lambda C_{\langle\zeta,\eta\rangle}$,

3) (Commutative Laws) $A_{\langle\alpha,\beta\rangle} \lambda B_{\langle\gamma,\delta\rangle} \tilde{=} B_{\langle\gamma,\delta\rangle} \lambda A_{\langle\alpha,\beta\rangle}$,

4) (Distributive Laws)

(i) $A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcup C_{\langle\zeta,\eta\rangle}) \tilde{\subseteq} (A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} B_{\langle\gamma,\delta\rangle}) \sqcup (A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} C_{\langle\zeta,\eta\rangle})$,

(ii) $A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcap_{\varepsilon} C_{\langle\zeta,\eta\rangle}) \tilde{\supseteq} (A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} B_{\langle\gamma,\delta\rangle}) \sqcap_{\varepsilon} (A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} C_{\langle\zeta,\eta\rangle})$,

(iii) $A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcap C_{\langle\zeta,\eta\rangle}) \tilde{=} (A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} B_{\langle\gamma,\delta\rangle}) \sqcap (A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} C_{\langle\zeta,\eta\rangle})$,

(iv) $A_{\langle\alpha,\beta\rangle} \sqcup (B_{\langle\gamma,\delta\rangle} \sqcup_{\varepsilon} C_{\langle\zeta,\eta\rangle}) \tilde{=} (A_{\langle\alpha,\beta\rangle} \sqcup B_{\langle\gamma,\delta\rangle}) \sqcup_{\varepsilon} (A_{\langle\alpha,\beta\rangle} \sqcup C_{\langle\zeta,\eta\rangle})$,

(v) $A_{\langle\alpha,\beta\rangle} \sqcup (B_{\langle\gamma,\delta\rangle} \sqcap_{\varepsilon} C_{\langle\zeta,\eta\rangle}) \tilde{=} (A_{\langle\alpha,\beta\rangle} \sqcup B_{\langle\gamma,\delta\rangle}) \sqcap_{\varepsilon} (A_{\langle\alpha,\beta\rangle} \sqcup C_{\langle\zeta,\eta\rangle})$,

(vi) $A_{\langle\alpha,\beta\rangle} \sqcup (B_{\langle\gamma,\delta\rangle} \sqcap C_{\langle\zeta,\eta\rangle}) \tilde{=} (A_{\langle\alpha,\beta\rangle} \sqcup B_{\langle\gamma,\delta\rangle}) \sqcap (A_{\langle\alpha,\beta\rangle} \sqcup C_{\langle\zeta,\eta\rangle})$,

(vii) $A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcup_{\varepsilon} C_{\langle\zeta,\eta\rangle}) \tilde{\subseteq} (A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} B_{\langle\gamma,\delta\rangle}) \sqcup_{\varepsilon} (A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} C_{\langle\zeta,\eta\rangle})$,

(viii) $A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcup C_{\langle\zeta,\eta\rangle}) \tilde{=} (A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} B_{\langle\gamma,\delta\rangle}) \sqcup (A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} C_{\langle\zeta,\eta\rangle})$,

(ix) $A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcap C_{\langle\zeta,\eta\rangle}) \tilde{\supseteq} (A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} B_{\langle\gamma,\delta\rangle}) \sqcap (A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} C_{\langle\zeta,\eta\rangle})$,

(x) $A_{\langle\alpha,\beta\rangle} \sqcup (B_{\langle\gamma,\delta\rangle} \sqcup_{\varepsilon} C_{\langle\zeta,\eta\rangle}) \tilde{=} (A_{\langle\alpha,\beta\rangle} \sqcup B_{\langle\gamma,\delta\rangle}) \sqcup_{\varepsilon} (A_{\langle\alpha,\beta\rangle} \sqcup C_{\langle\zeta,\eta\rangle})$,

(xi) $A_{\langle\alpha,\beta\rangle} \sqcup (B_{\langle\gamma,\delta\rangle} \sqcap_{\varepsilon} C_{\langle\zeta,\eta\rangle}) \tilde{=} (A_{\langle\alpha,\beta\rangle} \sqcup B_{\langle\gamma,\delta\rangle}) \sqcap_{\varepsilon} (A_{\langle\alpha,\beta\rangle} \sqcup C_{\langle\zeta,\eta\rangle})$,

(xii) $A_{\langle\alpha,\beta\rangle} \sqcup (B_{\langle\gamma,\delta\rangle} \sqcap C_{\langle\zeta,\eta\rangle}) \tilde{=} (A_{\langle\alpha,\beta\rangle} \sqcup B_{\langle\gamma,\delta\rangle}) \sqcap (A_{\langle\alpha,\beta\rangle} \sqcup C_{\langle\zeta,\eta\rangle})$.

Proof. It follows from Theorem 6.1.4 and Proposition 6.2.9 in a straightforward manner. ■

6.3.5 Example

Bipolar disorder is a serious psychological illness that can lead to dangerous behavior, problematic careers and relationships, and suicidal tendencies, especially if not treated early. Let $X = \{1,2,3,4,5,6,7\}$ be the set of days in which the record has been maintained i.e. $i = ith$ day of patient under observation, for $1 \leq i \leq 7$. Let $E = \{e_1,e_2,e_3,e_4,e_5\}$ = {Severe Mania, Severe Depression, Anxiety, Medication, Side effects} and $\neg E = \{\neg e_1,\neg e_2,\neg e_3,\neg e_4,\neg e_5\}$ = {Mild Mania, Mild Depression, No Anxiety, No Medication, No Side effects}. Here the gray area is obviously the moderate form of parameters. Suppose that $A = \{e_1,e_2,e_3\}$, $B = \{e_2,e_4,e_5\}$, $C = \{e_1,e_3,e_5\}$. Let the bipolar soft sets $A_{\langle\alpha,\beta\rangle}$, $B_{\langle\gamma,\delta\rangle}$ and $C_{\langle\zeta,\eta\rangle}$ over X describe the "daily record of the behavior" of P_1, P_2, and P_3. Suppose that

$$\alpha : A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{1,4,5,6\} & \text{if } e = e_1, \\ \{1,2,3,4,5,7\} & \text{if } e = e_2, \\ \{2,4,6,7\} & \text{if } e = e_3, \end{cases}$$

$$\beta : A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{2,3,7\} & \text{if } e = e_1, \\ \{6\} & \text{if } e = e_2, \\ \{3\} & \text{if } e = e_3, \end{cases}$$

$$\gamma : A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{3,5,6\} & \text{if } e = e_2, \\ \{1,5,7\} & \text{if } e = e_4, \\ \{2,3,4,5,6\} & \text{if } e = e_5, \end{cases}$$

$$\delta : A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{1,4,7\} & \text{if } e = e_2, \\ \{3,6\} & \text{if } e = e_4, \\ \{\} & \text{if } e = e_5, \end{cases}$$

$$\zeta : A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} X & \text{if } e = e_1, \\ \{1,2\} & \text{if } e = e_3, \\ \{4,5,6\} & \text{if } e = e_5, \end{cases}$$

$$\eta : A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{\} & \text{if } e = e_1, \\ \{3,4\} & \text{if } e = e_3, \\ \{1,2\} & \text{if } e = e_5, \end{cases}$$

We have

$$A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcap C_{\langle\zeta,\eta\rangle}) \tilde{\hat{=}} (A \cup (B \cap C))_{\langle \alpha \tilde{\cap} (\gamma \tilde{\cap} \zeta), \beta \tilde{\cup} (\delta \tilde{\cup} \eta)\rangle}$$

and

$$(A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} B_{\langle\gamma,\delta\rangle}) \sqcap (A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} C_{\langle\zeta,\eta\rangle}) \tilde{\hat{=}} (A \cup B) \cap (A \cup C)_{\langle(\alpha\tilde{\cap}\gamma)\tilde{\cap}(\alpha\tilde{\cap}\zeta), (\beta\tilde{\cup}\beta)\tilde{\cup}(\beta\tilde{\cup}\eta)\rangle}.$$

Then the approximations for parameter e_2 are not same on both sides

$$\begin{aligned} (\alpha\tilde{\cap}(\gamma\tilde{\cap}\zeta))(e_2) &= \{1,2,3,4,5,7\} \neq \{3,5\} = ((\alpha\tilde{\cap}\gamma)\tilde{\cap}(\alpha\tilde{\cap}\zeta))(e_2) \\ \text{and } (\beta\tilde{\cup}(\delta\tilde{\cup}\eta))(e_2) &= \{6\} \neq \{1,4,7,6\} = ((\beta\tilde{\cup}\delta)\tilde{\cup}(\beta\tilde{\cup}\eta))(e_2). \end{aligned}$$

Thus

$$A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcap C_{\langle\zeta,\eta\rangle}) \tilde{\hat{\neq}} (A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} B_{\langle\gamma,\delta\rangle}) \sqcap (A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} C_{\langle\zeta,\eta\rangle}).$$

Now, consider

$$A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcup_{\varepsilon} C_{\langle\zeta,\eta\rangle}) \tilde{\hat{=}} (A \cup (B \cup C))_{\langle \alpha \tilde{\cap} (\gamma \tilde{\cup} \zeta), \beta \tilde{\cup} (\delta \tilde{\cap} \eta)\rangle}$$

and

$$\begin{aligned} (A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} B_{\langle\gamma,\delta\rangle}) \sqcup_{\varepsilon} (A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} C_{\langle\zeta,\eta\rangle}) &\tilde{\hat{=}} (A \cup B)_{\langle\alpha\tilde{\cap}\gamma, \beta\tilde{\cup}\delta\rangle} \sqcup_{\varepsilon} (A \cup C)_{\langle\alpha\tilde{\cap}\zeta, \beta\tilde{\cup}\eta\rangle} \\ &\tilde{\hat{=}} (A \cup B) \cup (A \cup C)_{\langle(\alpha\tilde{\cap}\gamma)\tilde{\cup}(\alpha\tilde{\cap}\zeta), (\beta\tilde{\cup}\delta)\tilde{\cap}(\beta\tilde{\cup}\eta)\rangle}. \end{aligned}$$

Then the approximations for parameter e_2 are not same on both sides

$$\begin{aligned} (\alpha\tilde{\cap}(\gamma\tilde{\cup}\zeta))(e_2) &= \{5\} \neq \{1,2,3,4,5,7\} = ((\alpha\tilde{\cap}\gamma)\tilde{\cup}(\alpha\tilde{\cap}\zeta))(e_2) \\ \text{and } (\beta\tilde{\cup}(\delta\tilde{\cap}\eta))(e_2) &= \{1,4,7,6\} \neq \{6\} = (\beta\tilde{\cup}\delta)\tilde{\cap}(\beta\tilde{\cup}\eta))(e_2). \end{aligned}$$

Thus

$$A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcup_{\varepsilon} C_{\langle\zeta,\eta\rangle}) \tilde{\hat{\neq}} (A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} B_{\langle\gamma,\delta\rangle}) \sqcup_{\varepsilon} (A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} C_{\langle\zeta,\eta\rangle}).$$

Similarly it can be shown that

$$A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcup C_{\langle\zeta,\eta\rangle}) \tilde{\hat{\neq}} (A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} B_{\langle\gamma,\delta\rangle}) \sqcup (A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} C_{\langle\zeta,\eta\rangle}).$$

$$A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcap_{\varepsilon} C_{\langle\zeta,\eta\rangle}) \tilde{\hat{\neq}} (A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} B_{\langle\gamma,\delta\rangle}) \sqcap_{\varepsilon} (A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} C_{\langle\zeta,\eta\rangle}).$$

6.3.6 Corollary

Let $A_{\langle\alpha,\beta\rangle}$, $B_{\langle\gamma,\delta\rangle}$ and $A_{\langle\zeta,\eta\rangle}$ be any *bipolar soft set*s over X. Then

$$\begin{aligned} A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcap_{\varepsilon} A_{\langle\zeta,\eta\rangle}) &\tilde{\hat{=}} (A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} B_{\langle\gamma,\delta\rangle}) \sqcap_{\varepsilon} (A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} A_{\langle\zeta,\eta\rangle}) \text{ and} \\ A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcup_{\varepsilon} A_{\langle\zeta,\eta\rangle}) &\tilde{\hat{=}} (A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} B_{\langle\gamma,\delta\rangle}) \sqcup_{\varepsilon} (A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} A_{\langle\zeta,\eta\rangle}) \end{aligned}$$

if and only if

$$\begin{aligned} \alpha(e) &= \gamma(e) \text{ and } \beta(e) = \delta(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ \alpha(e) &= \zeta(e) \text{ and } \beta(e) = \eta(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

6.3.7 Corollary

Let $A_{\langle\alpha,\beta\rangle}$, $A_{\langle\gamma,\delta\rangle}$ and $A_{\langle\zeta,\eta\rangle}$ are three *bipolar soft sets* over X. Then

$$A_{\langle\alpha,\beta\rangle}\lambda(A_{\langle\gamma,\delta\rangle}\rho A_{(\zeta,\eta)})\tilde{=}(A_{\langle\alpha,\beta\rangle}\lambda A_{\langle\gamma,\delta\rangle})\rho(A_{\langle\alpha,\beta\rangle}\lambda A_{\langle\zeta,\eta\rangle})$$

for distinct λ, $\rho \in \{\sqcap_\varepsilon,\sqcap,\sqcup_\varepsilon,\sqcup\}$.

A *bipolar mood chart* is a simple and yet effective means of tracking and representing patient's condition every month. Bipolar mood charts help patients, their families and their doctors to see probable patterns that might have been very difficult to determine. Bipolar children and their families will greatly benefit from mood charting and can expect early detection of symptoms and determination of proper treatments by their doctors. We construct a mood chart based upon a bipolar soft set as follows:

A bipolar soft set $A_{\langle\alpha,\beta\rangle}$ over X may be represented by a pair of binary tables, one for each of the functions α and β respectively. In both tables, rows and columns are labeled by the elements of X and parameters respectively. We use following key for tables of α and β respectively:

$$a_{ij} = \begin{cases} 1 & \text{if } x_i \in \alpha(e_j) \\ 0 & \text{if } x_i \notin \alpha(e_j) \end{cases}$$

$$a_{ij} = \begin{cases} 1 & \text{if } x_i \in \beta(e_j) \\ 0 & \text{if } x_i \notin \beta(e_j) \end{cases}$$

where a_{ij} is the ith entry of jth column of each table. We can also represent a bipolar soft set with the help of a single table by putting

$$a_{ij} = \begin{cases} 1 & \text{if } h_i \in \alpha(e_j) \\ 0 & \text{if } h_i \in X - \{\alpha(e_j) \cup \beta(e_j)\} \\ -1 & \text{if } h_i \in \beta(e_j) \end{cases}$$

where a_{ij} is the ith entry of jth column of table whose rows and columns are labeled by elements of X and parameters respectively. The tabular representations of bipolar soft set $A_{\langle\alpha,\beta\rangle}$ as given in Example 6.3.5 are given by Table 6.1 and Table 6.2.

Both Tables 6.1 and Table 6.2 can be used as Mood Chart of patient P_1 for a week.

6.4 Algebras of Bipolar Soft Sets

In this section, we discuss the lattices and algebras for collections of bipolar soft sets. Let $\mathcal{BSS}(X)^E$ be the collection of all bipolar soft sets over X and $\mathcal{DSS}(X)_A$ be its subcollection of all bipolar soft sets over X with fixed set of parameters A. We note that these collections are partially ordered by the relation of soft inclusion $\tilde{\subseteq}$ given in Definition 6.1.2. We conclude from above results that:

α	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
1	1	1	0
2	0	1	1
3	0	1	0
4	1	1	1
5	1	1	0
6	1	0	1
7	0	1	1

β	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
1	0	0	0
2	1	0	0
3	1	0	1
4	0	0	0
5	0	0	0
6	0	1	0
7	1	0	0

Table 6.1: Tabular Representaion Using a Pair of Tables

$A_{\langle\alpha,\beta\rangle}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
1	1	1	0
2	-1	1	1
3	-1	1	-1
4	1	1	1
5	1	1	0
6	1	-1	1
7	-1	1	1

Table 6.2: Tabular Representaion Using Only One Table

6.4.1 Proposition

$(\mathcal{BSS}(X)^E, \sqcap_\varepsilon, \sqcup)$, $(\mathcal{BSS}(X)^E, \sqcup, \sqcap_\varepsilon)$, $(\mathcal{BSS}(X)^E, \sqcup_\varepsilon, \sqcap)$, $(\mathcal{BSS}(X)^E, \sqcap, \sqcup_\varepsilon)$, $(\mathcal{BSS}(X)_A, \sqcup, \sqcap)$, and $(\mathcal{BSS}(X)_A, \sqcap, \sqcup)$ are lattices.

Proof. From Propositions 6.3.3 and 6.3.4, we conclude that the structures form lattices. ■

6.4.2 Proposition

Let $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ be two *bipolar soft set*s over X. Then the following are true

1) $A_{\langle\alpha,\beta\rangle} \sqcup_\varepsilon B_{\langle\gamma,\delta\rangle}$ is the smallest *bipolar* soft set over X which contains both $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$.

2) $A_{\langle\alpha,\beta\rangle} \sqcap B_{\langle\gamma,\delta\rangle}$ is the largest *bipolar* soft set over X which is contained in both $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$.

Proof. Straightforward. ■

6.4.3 Proposition

$(\mathcal{BSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{\langle\Phi,\mathfrak{X}\rangle}, E_{\langle\mathfrak{X},\Phi\rangle})$, $(\mathcal{BSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{\langle\mathfrak{X},\Phi\rangle}, \emptyset_{\langle\Phi,\mathfrak{X}\rangle})$, $(\mathcal{BSS}(X)_A, \sqcap, \sqcup, A_{\langle\Phi,\mathfrak{X}\rangle}, A_{\langle\mathfrak{X},\Phi\rangle})$ and $(\mathcal{BSS}(X)_A, \sqcup, \sqcap, A_{\langle\mathfrak{X},\Phi\rangle}, A_{\langle\Phi,\mathfrak{X}\rangle})$ are bounded distributive lattices.

Proof. From Proposition 6.3.4 and Lemma 6.4.2, we conclude that $(\mathcal{BSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{\langle\Phi,\mathfrak{X}\rangle}, E_{\langle\mathfrak{X},\Phi\rangle})$ is a bounded distributive lattice and $(\mathcal{BSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{\langle\mathfrak{X},\Phi\rangle}, \emptyset_{\langle\Phi,\mathfrak{X}\rangle})$ is its dual. For bipolar soft sets $A_{\langle\alpha,\beta\rangle}, A_{\langle\gamma,\delta\rangle} \in \mathcal{BSS}(X)_A$,

$$\begin{aligned} A_{\langle\alpha,\beta\rangle} \sqcap A_{\langle\gamma,\delta\rangle} &\tilde{=} A_{\langle\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta\rangle} \in \mathcal{BSS}(X)_A \text{ and} \\ A_{\langle\alpha,\beta\rangle} \sqcup A_{\langle\gamma,\delta\rangle} &\tilde{=} A_{\langle\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta\rangle} \in \mathcal{BSS}(X)_A. \end{aligned}$$

Thus $(\mathcal{BSS}(X)_A,\sqcap,\sqcup)$ is also a distributive sublattice of $(\mathcal{BSS}(X)^E,\sqcup_\varepsilon,\sqcap)$ and Proposition 6.3.3 tells us that $A_{\langle\Phi,\mathfrak{X}\rangle}$, $A_{\langle\mathfrak{X},\Phi\rangle}$ are its lower and upper bounds respectively. Therefore $(\mathcal{BSS}(X)_A, \sqcap, \sqcup, A_{\langle\Phi,\mathfrak{X}\rangle}, A_{\langle\mathfrak{X},\Phi\rangle})$ is a bounded distributive lattice and $(\mathcal{BSS}(X)_A, \sqcup, \sqcap, A_{\langle\mathfrak{X},\Phi\rangle}, A_{\langle\Phi,\mathfrak{X}\rangle})$ is its dual. ■

6.4.4 Proposition

Let $A_{\langle\alpha,\beta\rangle}$ and $A_{\langle\gamma,\delta\rangle}$ be two *bipolar soft set*s over X. Then

1) $(A_{\langle\alpha,\beta\rangle^\circ})^\circ = A_{\langle\alpha,\beta\rangle}$,

2) $A_{\langle\alpha,\beta\rangle}\tilde{\subseteq}A_{\langle\gamma,\delta\rangle}$ if and only if $A_{\langle\gamma,\delta\rangle^\circ}\tilde{\subseteq}A_{\langle\alpha,\beta\rangle^\circ}$.

Proof.

1) Straightforward

2) If $A_{\langle\alpha,\beta\rangle}\tilde{\subseteq}A_{\langle\gamma,\delta\rangle}$ then

$$\alpha(e) \subseteq \gamma(e) \text{ and } \delta(e) \subseteq \beta(e) \text{ for all } e \in A$$

implies that

$$A_{\langle\gamma,\delta\rangle}\tilde{\subseteq}A_{\langle\alpha,\beta\rangle}.$$

Hence $A_{\langle\gamma,\delta\rangle^\circ}\tilde{\subseteq}A_{\langle\alpha,\beta\rangle^\circ}$. If $A_{\langle\gamma,\delta\rangle^\circ}\tilde{\subseteq}A_{\langle\alpha,\beta\rangle^\circ}$ then

$$A_{\langle\alpha,\beta\rangle}\tilde{=}(A_{\langle\alpha,\beta\rangle^\circ})^\circ\tilde{\subseteq}(A_{\langle\gamma,\delta\rangle^\circ})^\circ\tilde{=}A_{\langle\gamma,\delta\rangle}.$$

■

6.4.5 Proposition (de Morgan Laws)

Let $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ be two *bipolar* soft sets over X. Then the following are true:

1) $(A_{\langle\alpha,\beta\rangle} \sqcup_\varepsilon B_{\langle\gamma,\delta\rangle})^\circ\tilde{=}A_{\langle\alpha,\beta\rangle^\circ} \sqcap_\varepsilon B_{\langle\gamma,\delta\rangle^\circ}$,

2) $(A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon B_{\langle\gamma,\delta\rangle})^\circ\tilde{=}A_{\langle\alpha,\beta\rangle^\circ} \sqcup_\varepsilon B_{\langle\gamma,\delta\rangle^\circ}$,

3) $(A_{\langle\alpha,\beta\rangle} \vee B_{\langle\gamma,\delta\rangle})^{\circ} \tilde{=} A_{\langle\alpha,\beta\rangle^{\circ}} \wedge B_{\langle\gamma,\delta\rangle^{\circ}}$,

4) $(A_{\langle\alpha,\beta\rangle} \wedge B_{\langle\gamma,\delta\rangle})^{\circ} \tilde{=} A_{\langle\alpha,\beta\rangle^{\circ}} \vee B_{\langle\gamma,\delta\rangle^{\circ}}$,

5) $(A_{\langle\alpha,\beta\rangle} \sqcup B_{\langle\gamma,\delta\rangle})^{\circ} \tilde{=} A_{\langle\alpha,\beta\rangle^{\circ}} \sqcap B_{\langle\gamma,\delta\rangle^{\circ}}$,

6) $(A_{\langle\alpha,\beta\rangle} \sqcap B_{\langle\gamma,\delta\rangle})^{\circ} \tilde{=} A_{\langle\alpha,\beta\rangle^{\circ}} \sqcup B_{\langle\gamma,\delta\rangle^{\circ}}$.

Proof.

1) We have

$$(A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} B_{\langle\gamma,\delta\rangle})^{\circ} \tilde{=} ((A \cup B)_{\langle\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta\rangle})^{\circ} \tilde{=} (A \cup B)_{\langle\beta\tilde{\cap}\delta,\alpha\tilde{\cup}\gamma\rangle}$$

and

$$A_{\langle\alpha,\beta\rangle^{\circ}} \sqcap_{\varepsilon} B_{\langle\gamma,\delta\rangle^{\circ}} \tilde{=} A_{(\beta,\alpha)} \sqcap_{\varepsilon} B_{(\delta,\gamma)} \tilde{=} (A \cup B)_{\langle\beta\tilde{\cap}\delta,\alpha\tilde{\cup}\gamma\rangle}.$$

Thus

$$(A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} B_{\langle\gamma,\delta\rangle})^{\circ} \tilde{=} A_{\langle\alpha,\beta\rangle^{\circ}} \sqcap_{\varepsilon} B_{\langle\gamma,\delta\rangle^{\circ}}.$$

The remaining parts can also be proved in a similar way.

■

6.4.6 Proposition

$(\mathcal{BSS}(X)_A, \sqcap, \sqcup, ^{\circ}, A_{\langle\Phi,\mathfrak{X}\rangle}, A_{\langle\mathfrak{X},\Phi\rangle})$ is a de Morgan algebra.

Proof. Proof follows from Propositions 6.4.4 and 6.4.5. ■

6.4.7 Proposition

$(\mathcal{BSS}(X)_A, \sqcap, \sqcup, ^{\circ}, A_{\langle\Phi,\mathfrak{X}\rangle}, A_{\langle\mathfrak{X},\Phi\rangle})$ is a Kleene algebra.

Proof. For $A_{\langle\alpha,\beta\rangle}$, $A_{\langle\gamma,\delta\rangle} \in \mathcal{BSS}(X)_A$

$$A_{\langle\alpha,\beta\rangle} \sqcap A_{\langle\alpha,\beta\rangle^{\circ}} \tilde{=} A_{\langle\alpha,\beta\rangle} \sqcap A_{\langle\beta,\alpha\rangle} \tilde{=} A_{\langle\alpha\tilde{\cap}\beta,\beta\tilde{\cup}\alpha\rangle} \tilde{=} A_{\langle\Phi,\beta\tilde{\cup}\alpha\rangle} \quad \text{and}$$
$$A_{\langle\gamma,\delta\rangle} \sqcup A_{\langle\gamma,\delta\rangle^{\circ}} \tilde{=} A_{\langle\gamma,\delta\rangle} \sqcup A_{\langle\delta,\gamma\rangle} \tilde{=} A_{\langle\gamma\tilde{\cup}\delta,\delta\tilde{\cap}\gamma\rangle} \tilde{=} A_{\langle\gamma\tilde{\cup}\delta,\Phi\rangle}.$$

Clearly $A_{\langle\alpha,\beta\rangle} \sqcap A_{\langle\alpha,\beta\rangle^{\circ}} \tilde{\subseteq} A_{\langle\gamma,\delta\rangle} \sqcup A_{\langle\gamma,\delta\rangle^{\circ}}$.

We already know that $(\mathcal{BSS}(X)_A, \sqcap, \sqcup, ^{\circ}, A_{\langle\Phi,\mathfrak{X}\rangle}, A_{\langle\mathfrak{X},\Phi\rangle})$ is a de Morgan algebra, so this condition assures that $(\mathcal{BSS}(X)_A, \sqcap, \sqcup, ^{\circ}, A_{\langle\Phi,\mathfrak{X}\rangle}, A_{\langle\mathfrak{X},\Phi\rangle})$ is also a Kleene algebra. ■

6.4.8 Remark

We have seen that $(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^{\circ}, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is a de Morgan algebra but not a Kleene algebra whereas $(\mathcal{BSS}(X)_A, \sqcap, \sqcup, ^{\circ}, A_{\langle\Phi,\mathfrak{X}\rangle}, A_{\langle\mathfrak{X},\Phi\rangle})$ is its de Morgan subalgebra and also a Kleene subalgebra.

Chapter 7

Algebraic Structures of Fuzzy Bipolar Soft Sets

In this chapter, we have initiated a concept of fuzzy bipolar soft sets. The idea is generated with the motivation of bipolarity of parameters and then the fuzziness of data comes into play. A fuzzy bipolar soft set is defined with the help of two mappings, one for approximating the degree of fuzziness of the positivity or presence of a certain parameter in the objects of initial universal set and the other one is to approximate a relative degree of fuzziness of the negativity or absence of same parameter. In this way, we have combined these three concepts of bipolarity, fuzziness and parameterization and thus it is shown through examples that we have found a very easy to use way of modeling the phenomena where all these three factors are involved. To move further, we have defined the basic algebra for the fuzzy bipolar soft sets and discussed their algebraic properties in detail. It is also shown that the collection of fuzzy bipolar soft sets forms a stone algebra.

7.1 Fuzzy Bipolar Soft Sets

Let X be an initial universe and E be a set of parameters. Let $\mathcal{FP}(X)$ denotes the collection of all fuzzy subsets of X and A, B, C are non-empty subsets of E. Now, we define

7.1.1 Definition

A triplet $(f,g : A)$ is called a fuzzy *bipolar* soft set over X, where f and g are mappings, given by $f : A \to \mathcal{FP}(X)$ and $g : \neg A \to \mathcal{FP}(X)$ such that $0 \leq (f(e))(x)+(g(\neg e))(x) \leq 1$ for all $e \in A$.

In other words, a *fuzzy bipolar soft set* over X gives two parametrized families of fuzzy subsets of the universe X and the condition $0 \leq (f(e))(x) + (g(\neg e))(x) \leq 1$ for all.$e \in A$, is imposed as a consistency constraint. For each $e \in A$, $f(e)$ and $g(\neg e)$ are regarded as the set of e-approximate elements of the *fuzzy bipolar soft set* $A_{\langle f,g\rangle}$.

Note that, from now on, we shall use the notation $A_{\langle f,g\rangle}$ over X to denote a *fuzzy bipolar soft set* $(f,g : A)$ over X where the meanings of f, g, A and X are clear.

7.1.2 Definition

For a *fuzzy bipolar soft set* $A_{\langle f,g\rangle}$ over X, we define a fuzzy soft set A_h over X for the approximation of the degree of hesitation in $A_{\langle f,g\rangle}$ as $h : A \rightarrow \mathcal{FP}(X)$ defined by $(h(e))(x) = 1 - (f(e))(x) - (g(\neg e))(x)$ for all $x \in X$, $e \in A$. Clearly, A_h approximates the lack of knowledge about the objects of X while considering the presence or absence of a particular parameter from A.

7.1.3 Definition

For two *fuzzy bipolar soft sets* $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ over X, we say that $A_{\langle f,g\rangle}$ is a *fuzzy bipolar* soft subset of $B_{\langle h,i\rangle}$, if

1) $A \subseteq B$ and

2) $f(e) \subseteq h(e)$ and $i(\neg e) \subseteq g(\neg e)$ for all $e \in A$.

This relationship is denoted by $A_{\langle f,g\rangle} \tilde{\subseteq} B_{\langle h,i\rangle}$.

Similarly $A_{\langle f,g\rangle}$ is said to be a *fuzzy bipolar* soft superset of $B_{\langle h,i\rangle}$, if $B_{\langle h,i\rangle}$ is a *fuzzy bipolar soft subset* of $A_{\langle f,g\rangle}$. We denote it by $A_{\langle f,g\rangle} \tilde{\supseteq} B_{\langle h,i\rangle}$.

7.1.4 Definition

Two *fuzzy bipolar soft sets* $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ over X are said to be *equal denoted as* $A_{\langle f,g\rangle} \tilde{=} B_{\langle h,i\rangle}$ if $A_{\langle f,g\rangle}$ is a *fuzzy bipolar* soft subset of $B_{\langle h,i\rangle}$ and $B_{\langle h,i\rangle}$ is a *fuzzy bipolar* soft subset of $A_{\langle f,g\rangle}$.

7.1.5 Example

Let X be a set of different books, and E be the set of parameters where, $X = \{b_1,b_2,b_3,b_4,b_5\}$, $E = \{e_1,e_2,e_3,e_4,e_5,e_6\}$ = { Simple, Logical, Orderly, Concise, Varied, Appealing}, $\neg E = \{\neg e_1,\neg e_2,\neg e_3,\neg e_4,\neg e_5\}$ = {Complicated, Illogical, Chaotic, Wordy, Monotonous, Distant}. Suppose that $A = \{e_1,e_2,e_3,e_6\}$, a fuzzy bipolar soft set $A_{\langle f,g\rangle}$

describes the "reader ratings of books under consideration". The fuzzy bipolar soft set $A_{\langle f,g\rangle}$ over X is given as follows:

$$\begin{aligned} f &: A \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{b_1/0.9, b_2/0.3, b_3/0.8, b_4/0.7, b_5/0.5\} & \text{if } e = e_1, \\ \{b_1/0.1, b_2/0.5, b_3/0.1, b_4/0.8, b_5/0.6\} & \text{if } e = e_2, \\ \{b_1/0.2, b_2/0.5, b_3/0.2, b_4/0.8, b_5/0.7\} & \text{if } e = e_3, \\ \{b_1/0.7, b_2/0.4, b_3/0.2, b_4/0.1, b_5/0.1\} & \text{if } e = e_6, \end{cases} \\ g &: \neg A \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{b_1/0.1, b_2/0.3, b_3/0.1, b_4/0.2, b_5/0.3\} & \text{if } e = \neg e_1, \\ \{b_1/0.7, b_2/0.4, b_3/0.8, b_4/0.1, b_5/0.2\} & \text{if } e = \neg e_2, \\ \{b_1/0.6, b_2/0.4, b_3/0.6, b_4/0.1, b_5/0.3\} & \text{if } e = \neg e_3, \\ \{b_1/0.2, b_2/0.3, b_3/0.8, b_4/0.7, b_5/0.5\} & \text{if } e = \neg e_6. \end{cases} \end{aligned}$$

Let $B = \{e_2,e_6\}$. Then a second approximations with respect to the earlier approximations by $A_{\langle f,g\rangle}$ is represented by a fuzzy bipolar soft subset $B_{\langle h,i\rangle}$.of $A_{\langle f,g\rangle}$ and given by:

$$\begin{aligned} h &: B \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{b_1/0.1, b_2/0.5, b_3/0.1, b_4/0.8, b_5/0.6\} & \text{if } e = e_2, \\ \{b_1/0.7, b_2/0.4, b_3/0.2, b_4/0.1, b_5/0.1\} & \text{if } e = e_6, \end{cases} \\ i &: \neg B \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{b_1/0.7, b_2/0.4, b_3/0.8, b_4/0.1, b_5/0.2\} & \text{if } e = \neg e_2, \\ \{b_1/0.2, b_2/0.3, b_3/0.8, b_4/0.7, b_5/0.5\} & \text{if } e = \neg e_6. \end{cases} \end{aligned}$$

7.2 Bipolar fuzzy Soft Sets

We present the concept of bipolar fuzzy soft sets as a generalization of soft sets in bipolar fuzzy context. Let $\mathcal{BFP}(X)$ denotes the set of all bipolar fuzzy subsets of X.

7.2.1 Definition

A pair (f,A) is called a *bipolar fuzzy soft set over* X, where f is a mapping given by $f : A \to \mathcal{BFP}(X)$.

Thus a bipolar fuzzy soft set over X gives a parametrized family of bipolar fuzzy subsets of the universe X. For any $e \in A$, $f(e) = \{(x,f(e)^P,f(e)^N) : x \in X\}$ where $f(e)^P : X \to [0,1]$ and $f(e)^N : X \to [-1,0]$ are mappings.

Before proceeding to the further development of theory of bipolar fuzzy soft sets, we give following interpretations:

7.2.2 Proposition

A fuzzy bipolar soft set over X is equivalent to a bipolar fuzzy soft set over X and vice versa.

Proof. Let $A_{\langle f,g\rangle}$ be a given fuzzy bipolar soft set defined over X. We define a bipolar fuzzy soft set (h,A) over X as:

$$h(e) = \{(x, f(e), -(g(\neg e)(x)) : x \in X\}$$

for all $e \in A$. Then $(x, f(e), -(g(\neg e)(x)) \in \mathcal{BFP}(X)$.

Conversely assume that we are given a bipolar fuzzy soft set (h,A) over X. We can define a fuzzy bipolar soft set $A_{\langle f,g\rangle}$ over X in the following manner:

$$\begin{aligned} f(e) &= h(e)^P \\ g(\neg e) &= -(h(e)^N) \end{aligned}$$

for all $e \in A$.

Thus both definitions are equivalent and may be used interchangeably. ■

Consider the following example:

7.2.3 Example

Let $X = \{m_1,m_2,m_3,m_4,m_5\}$ be the set of candidates who have applied for a job position of Office Representative in Customer Care Centre of a company. Let $E = \{e_1,e_2,e_3,e_4,e_5,e_6,e_7\}$ = {Hard Working, Optimism, Enthusiasm, Individualism, Imaginative, Decisiveness, Self-confidence} and $\neg E = \{\neg e_1,\neg e_2,\neg e_3, \neg e_4, \neg e_5, \neg e_6, \neg e_7\}$ = {Negligent, Pessimism, Half-hearted, Dependence, Unimaginative, Indecisiveness, Shyness}. Here the gray area is obviously a moderate form of parameters. Let us suppose that the *fuzzy bipolar soft set* $E_{\langle f,g\rangle}$ describes " Personality Analysis of Candidates" as:

$$\begin{aligned} f &: E \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{m_1/0.5, m_2/0.7, m_3/0.6, m_4/0.7, m_5/0.5\} & \text{if } e = e_1, \\ \{m_1/0.6, m_2/0.7, m_3/0.8, m_4/0.8, m_5/0.4\} & \text{if } e = e_2, \\ \{m_1/0.8, m_2/0.8, m_3/0.4, m_4/0.6, m_5/0.5\} & \text{if } e = e_3, \\ \{m_1/0.7, m_2/0.6, m_3/0.1, m_4/0.7, m_5/0.6\} & \text{if } e = e_4, \\ \{m_1/0.5, m_2/0.8, m_3/0.6, m_4/0.5, m_5/0.7\} & \text{if } e = e_5, \\ \{m_1/0.4, m_2/0.9, m_3/0.5, m_4/0.4, m_5/0.7\} & \text{if } e = e_6, \\ \{m_1/0.3, m_2/0.8, m_3/0.4, m_4/0.6, m_5/0.8\} & \text{if } e = e_7, \end{cases} \\ g &: \neg E \to \mathcal{FP}(X), \end{aligned}$$

$$e \longmapsto \begin{cases} \{m_1/0.3, m_2/0.2, m_3/0.4, m_4/0.1, m_5/0.3\} & \text{if } e = \neg e_1, \\ \{m_1/0.4, m_2/0.1, m_3/0.2, m_4/0.1, m_5/0.5\} & \text{if } e = \neg e_2, \\ \{m_1/0.05, m_2/0.1, m_3/0.5, m_4/0.33, m_5/0.4\} & \text{if } e = \neg e_3, \\ \{m_1/0.23, m_2/0.3, m_3/0.6, m_4/0.2, m_5/0.3\} & \text{if } e = \neg e_4, \\ \{m_1/0.4, m_2/0.2, m_3/0.35, m_4/0.4, m_5/0.1\} & \text{if } e = \neg e_5, \\ \{m_1/0.4, m_2/0.2, m_3/0.3, m_4/0.3, m_5/0.2\} & \text{if } e = \neg e_6, \\ \{m_1/0.7, m_2/0.08, m_3/0.5, m_4/0.3, m_5/0.18\} & \text{if } e = \neg e_7, \end{cases}$$

Now let's see the corresponding bipolar fuzzy soft set:

$$\begin{aligned} h(e_1) &= \{(m_1, 0.5, -0.3), (m_2, 0.7, -0.2), (m_3, 0.6, -0.4), (m_4, 0.7, -0.1), (m_5, 0.5, -0.3)\}, \\ h(e_2) &= \{(m_1, 0.6, -0.4), (m_2, 0.7, -0.1), (m_3, 0.8, -0.2), (m_4, 0.8, -0.1), (m_5, 0.4, -0.5)\}, \\ h(e_3) &= \{(m_1, 0.8, -0.05), (m_2, 0.8, -0.1), (m_3, 0.4, -0.5), (m_4, 0.6, -0.33), (m_5, 0.5, -0.4)\}, \\ h(e_4) &= \{(m_1, 0.7, -0.23), (m_2, 0.6, -0.3), (m_3, 0.1, -0.4), (m_4, 0.7, -0.2), (m_5, 0.6, -0.3)\}, \\ h(e_5) &= \{(m_1, 0.5, -0.4), (m_2, 0.8, -0.2), (m_3, 0.6, -0.35), (m_4, 0.5, -0.4), (m_5, 0.7, -0.1)\}, \\ h(e_6) &= \{(m_1, 0.4, -0.4), (m_2, 0.9, -0.2), (m_3, 0.5, -0.3), (m_4, 0.4, -0.3), (m_5, 0.7, -0.2)\}, \\ h(e_7) &= \{(m_1, 0.3, -0.7), (m_2, 0.8, -0.08), (m_3, 0.4, -0.5), (m_4, 0.6, -0.3), (m_5, 0.8, -0.18)\}. \end{aligned}$$

It is clear that fuzzy bipolar soft set depicts the information in a better and comprehensive way than bipolar fuzzy soft set. For example, if we read the data of candidate m_1 with fuzzy bipolar soft set $A_{\langle f,g \rangle}$ then he is having 0.6 fuzzy value for optimism and 0.4 fuzzy value for pessimism and if we use the bipolar fuzzy soft set (h,E) then m_1 is having 0.6 fuzzy value for optimism and -0.4 shows the degree where m_1 is showing pessimism.

Let $\mathcal{FBSS}(X)^E$ denotes the set of all fuzzy bipolar soft sets defined over X with set of parameters E, ordered by the relation of inclusion $\tilde{\subseteq}$ as defined in Definition 7.1.3. We show that every fuzzy bipolar soft set is equivalent to a double-framed fuzzy soft set and give the following theorem:

7.2.4 Theorem

The mapping $\theta : \mathcal{FBSS}(X)^E \to \mathcal{DFSS}(X)^E$, $A_{\langle f,g \rangle} \mapsto A_{(f_1,g_1)}$ is a monomorphism of lattices where

$$f_1(e) = f(e), \text{ and } \; g_1(e) = g(\neg e) \text{ for all } e \in A.$$

Proof. Clearly θ is well-defined. If

$$\theta(A_{\langle f,g \rangle}) \tilde{=} \theta(B_{\langle h,i \rangle})$$

where

$$\theta(A_{\langle f,g\rangle})\tilde{=}A_{(f_1,g_1)} \text{ and } \theta(B_{\langle h,i\rangle})\tilde{=}B_{(h_1,i_1)}$$

then

$$f_1(e)=f(e),\, h_1(e)=h(e) \;\text{ and }\; g_1(e)=g(\neg e),\, i_1(e)=i(\neg e) \;\text{ for all } e\in A.$$

Now,

$$f(e)=f_1(e)=h_1(e)=h(e) \text{ and } g(\neg e)=g_1(e)=i_1(e)=i(\neg e) \;\text{ for all } e\in A.$$

Thus

$$A_{\langle f,g\rangle}\tilde{=}B_{\langle h,i\rangle}$$

shows that θ is one-to-one. Clearly θ is order preserving. ■

7.2.5 Remark

Note that θ is not onto because of the consistency constraint for defining fuzzy bipolar soft sets and $\mathcal{FBSS}(X)^E\tilde{=}\mathcal{BFSS}(X)^E \hookrightarrow \mathcal{DFSS}(X)^E$.

By Theorem 7.2.4, we can equate every fuzzy bipolar soft set $A_{\langle f,g\rangle}$ over X with a double-framed fuzzy soft set and so, we can take f and g as mappings from A to $\mathcal{BFP}(X)$ where the meanings of A, f and g are clear in this context.

7.3 Operations on Fuzzy Bipolar Soft Sets

This section provides some operations defined on fuzzy bipolar soft sets:

7.3.1 Definition

Let $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ be fuzzy bipolar soft sets over X. The *int-uni product* of $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ is defined as a fuzzy bipolar soft set $(A\times B)_{\langle f\tilde{\wedge}h,g\tilde{\vee}i\rangle}$ over X in which

$$\begin{aligned} f\tilde{\wedge}h &: (A\times B)\to \mathcal{FP}(X),\,(a,b)\mapsto f(a)\wedge h(b),\\ g\tilde{\vee}i &: (A\times B)\to \mathcal{FP}(X),\,(a,b)\mapsto g(a)\vee i(b).\end{aligned}$$

It is denoted by $A_{\langle f,g\rangle}\wedge B_{\langle h,i\rangle}\tilde{=}(A\times B)_{\langle f\tilde{\wedge}h,g\tilde{\vee}i\rangle}$.

7.3.2 Definition

Let $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ be *fuzzy bipolar* soft sets over X. The *uni-int product* of $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ is defined as the fuzzy bipolar soft soft set $(A\times B)_{\langle f\tilde{\vee}h,\beta\tilde{\wedge}i\rangle}$ over X in which $f\tilde{\vee}h:(A\times B)\to\mathcal{FP}(X)$, where

$$(a,b)\mapsto f(a)\vee h(b),$$

and $g\tilde{\wedge}i : (A \times B) \to \mathcal{FP}(X)$, where

$$(a, b) \mapsto g(a) \wedge i(b).$$

It is denoted by $A_{\langle f,g \rangle} \vee B_{\langle h,i \rangle} \tilde{=} (A \times B)_{\langle f\tilde{\vee}h, g\tilde{\wedge}i \rangle}$.

7.3.3 Definition

For two *fuzzy bipolar soft sets* $A_{\langle f,g \rangle}$ and $B_{\langle h,i \rangle}$ over X, *the extended int-uni fuzzy bipolar soft set* of $A_{\langle f,g \rangle}$ and $B_{\langle h,i \rangle}$ is defined as the *fuzzy bipolar soft set* $(A \cup B)_{\langle f\tilde{\wedge}h, g\tilde{\vee}i \rangle}$ where $f\tilde{\wedge}h : (A \cup B) \to \mathcal{FP}(X)$,

$$e \mapsto \begin{cases} f(e) & \text{if } e \in A - B \\ h(e) & \text{if } e \in B - A \\ f(e) \wedge h(e) & \text{if } e \in (A \cap B) \end{cases}$$

and $g\tilde{\vee}i : (A \cup B) \to \mathcal{FP}(X)$, where

$$e \mapsto \begin{cases} g(e) & \text{if } e \in A - B \\ i(e) & \text{if } e \in B - A \\ g(e) \vee i(e) & \text{if } e \in (A \cap B) \end{cases}.$$

It is denoted by $A_{\langle f,g \rangle} \sqcap_{\varepsilon} B_{\langle h,i \rangle} \tilde{=} (A \cup B)_{\langle f\tilde{\wedge}h, g\tilde{\vee}i \rangle}$.

7.3.4 Definition

For two *fuzzy bipolar soft sets* $A_{\langle f,g \rangle}$ and $B_{\langle h,i \rangle}$ over X, *the extended uni-int fuzzy bipolar soft set* of $A_{\langle f,g \rangle}$ and $B_{\langle h,i \rangle}$ is defined as the *fuzzy bipolar soft set* $(A \cup B)_{\langle f\tilde{\vee}h, g\tilde{\wedge}i \rangle}$ where $f\tilde{\vee}h : (A \cup B) \to \mathcal{FP}(X)$,

$$e \mapsto \begin{cases} f(e) & \text{if } e \in A - B \\ h(e) & \text{if } e \in B - A \\ f(e) \vee h(e) & \text{if } e \in (A \cap B) \end{cases}$$

and $g\tilde{\wedge}i : (A \cup B) \to \mathcal{FP}(X)$, where

$$e \mapsto \begin{cases} g(e) & \text{if } e \in A - B \\ i(e) & \text{if } e \in B - A \\ g(e) \wedge i(e) & \text{if } e \in (A \cap B) \end{cases}.$$

It is denoted by $A_{\langle f,g \rangle} \sqcup_{\varepsilon} B_{\langle h,i \rangle} \tilde{=} (A \cup B)_{\langle f\tilde{\vee}h, g\tilde{\wedge}i \rangle}$.

7.3.5 Definition

For two *fuzzy bipolar soft sets* $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ over X with $(A\cap B)\neq\emptyset$, the *restricted int-uni fuzzy bipolar soft set* of $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ is defined as the *fuzzy bipolar soft set* $(A\cap B)_{\langle f\tilde{\wedge}h,g\tilde{\vee}i\rangle}$ where $f\tilde{\wedge}h:(A\cap B)\to\mathcal{FP}(X)$,

$$e\mapsto f(e)\wedge h(e),$$

and $g\tilde{\vee}i:(A\cap B)\to\mathcal{FP}(X)$, where

$$e\mapsto g(e)\vee i(e).$$

It is denoted by $A_{\langle f,g\rangle}\sqcap B_{\langle h,i\rangle}\tilde{=}(A\cap B)_{\langle f\tilde{\wedge}h,g\tilde{\vee}i\rangle}$.

7.3.6 Definition

For two *fuzzy bipolar soft sets* $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ over X with $(A\cap B)\neq\emptyset$, the *restricted uni-int fuzzy bipolar soft set* of $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ is defined as the *fuzzy bipolar soft set* $(A\cap B)_{\langle f\tilde{\vee}h,g\tilde{\wedge}i\rangle}$ where, $f\tilde{\vee}h:(A\cap B)\to\mathcal{FP}(X)$

$$e\mapsto f(e)\vee h(e),$$

and $g\tilde{\wedge}i:(A\cap B)\to\mathcal{FP}(X)$,

$$e\mapsto g(e)\wedge i(e).$$

It is denoted by $A_{\langle f,g\rangle}\sqcup B_{\langle h,i\rangle}\tilde{=}(A\cap B)_{\langle f\tilde{\vee}h,g\tilde{\wedge}i\rangle}$.

7.3.7 Remark

The operation of complementation as defined in Definition 5.2.7 for double-framed fuzzy soft sets is no more valid for fuzzy bipolar soft sets because $(A_{\langle f,g\rangle})\acute{}\,\tilde{=}A_{\langle \acute{f},\acute{g}\rangle}$ may not satisfy the consistency constraint as shown by the following example:

7.3.8 Example

Let E, A, X and fuzzy bipolar soft set $A_{\langle f,g\rangle}$ over X be taken as in Example 7.1.5. Then $(A_{\langle f,g\rangle})\acute{}$ is given as follows:

$$\begin{aligned}
\acute{f} \quad &: \quad A\to\mathcal{FP}(X),\\
e \quad &\longmapsto \begin{cases}\{b_1/0.1,b_2/0.7,b_3/0.2,b_4/0.3,b_5/0.5\} & \text{if } e=e_1,\\ \{b_1/0.9,b_2/0.5,b_3/0.9,b_4/0.2,b_5/0.4\} & \text{if } e=e_2,\\ \{b_1/0.8,b_2/0.5,b_3/0.8,b_4/0.1,b_5/0.1\} & \text{if } e=e_3,\\ \{b_1/0.3,b_2/0.6,b_3/0.8,b_4/0.9,b_5/1.0\} & \text{if } e=e_6,\end{cases}\\
\acute{g} \quad &: \quad A\to\mathcal{FP}(X),
\end{aligned}$$

$$e \longmapsto \begin{cases} \{b_1/0.8, b_2/0.7, b_3/0.7, b_4/0.6, b_5/0.2\} & \text{if } e = e_1, \\ \{b_1/0.3, b_2/0.6, b_3/0.2, b_4/0.3, b_5/0.1\} & \text{if } e = e_2, \\ \{b_1/0.4, b_2/0.6, b_3/0.4, b_4/0.4, b_5/0.3\} & \text{if } e = e_3, \\ \{b_1/0.1, b_2/0.7, b_3/0.2, b_4/0.3, b_5/0.5\} & \text{if } e = e_6. \end{cases}$$

but

$$(f'(e_1))(b_2) + (g'(e_1))(b_2) = 0.7 + 0.7 = 1.4 > 1$$

so $(A_{\langle f,g\rangle})' \notin \mathcal{FBSS}(X)^E$. Thus " $'$ " is not defined on $\mathcal{FBSS}(X)^E$.

7.3.9 Proposition

Let $A_{\langle f,g\rangle}$ be a fuzzy *bipolar* soft set over X. Then $^\circ : \mathcal{FBSS}(X)^E \to \mathcal{FBSS}(X)^E$ is defined and we denote $(A_{\langle f,g\rangle})^\circ$ by $A_{\langle f,g\rangle^\circ}$.

Proof. If $A_{\langle f,g\rangle} \in \mathcal{FBSS}(X)^E$ then

$A_{\langle f,g\rangle^\circ} \tilde{=} A_{\langle f^\circ,g^\circ\rangle}$ where $f^\circ : A \to \mathcal{FP}(X),\ e \mapsto g(e)$ and $g^\circ : A \to \mathcal{FP}(X),\ e \mapsto f(e)$.

Clearly

$$0 \leq (f^\circ(e))(x) + (g^\circ(\neg e))(x) \leq 1$$

Thus $A_{\langle f,g\rangle^\circ} \in \mathcal{FBSS}(X)^E$. ■

7.4 Properties of Fuzzy Bipolar Soft Sets

In this section we discuss properties of fuzzy bipolar soft sets with respect to their operations. Associativity, commutativity, absorption, distributivity and properties of fuzzy bipolar soft sets are investigated.

7.4.1 Definition

A *fuzzy bipolar* soft set over X is said to be a *relative absolute fuzzy bipolar soft set,* denoted by $A_{\langle \tilde{\mathbf{1}},\tilde{\mathbf{0}}\rangle}$ where

$$\tilde{\mathbf{1}} : A \to \mathcal{FP}(X),\ e \mapsto \tilde{\mathbf{1}} \text{ and } \tilde{\mathbf{0}} : A \to \mathcal{FP}(X),\ e \mapsto \tilde{\mathbf{0}}.$$

7.4.2 Definition

A *fuzzy bipolar soft set* over X is said to be a *relative null fuzzy bipolar soft set,* denoted by $A_{\langle \tilde{\mathbf{0}},\tilde{\mathbf{1}}\rangle}$ where

$$\tilde{\mathbf{0}} : A \to \mathcal{FP}(X),\ e \mapsto \tilde{\mathbf{0}} \text{ and } \tilde{\mathbf{1}} : A \to \mathcal{FP}(X),\ e \mapsto \tilde{\mathbf{1}}.$$

Conventionally, we take the fuzzy bipolar soft sets with empty set of parameters to be equal to $\emptyset_{\langle \tilde{\mathbf{0}},\tilde{\mathbf{1}}\rangle}$ and so $A_{\langle f,g\rangle} \sqcap B_{\langle h,i\rangle} \tilde{=} A_{\langle f,g\rangle} \sqcup B_{\langle h,i\rangle} \tilde{=} \emptyset_{\langle \tilde{\mathbf{0}},\tilde{\mathbf{1}}\rangle}$ whenever $(A \cap B) = \emptyset$.

7.4.3 Proposition

If $A_{\langle \tilde{\mathbf{0}}, \tilde{\mathbf{1}} \rangle}$ is a null fuzzy bipolar soft set, $A_{\langle \tilde{\mathbf{1}}, \tilde{\mathbf{0}} \rangle}$ an absolute fuzzy bipolar soft set, and $A_{\langle f,g \rangle}$, $A_{\langle h,i \rangle}$ are fuzzy bipolar soft sets over X, then

1) $A_{\langle f,g \rangle} \sqcup_{\varepsilon} A_{\langle h,i \rangle} \tilde{=} A_{\langle f,g \rangle} \sqcup A_{\langle h,i \rangle}$,

2) $A_{\langle f,g \rangle} \sqcap_{\varepsilon} A_{\langle h,i \rangle} \tilde{=} A_{\langle f,g \rangle} \sqcap A_{\langle h,i \rangle}$,

3) $A_{\langle f,g \rangle} \sqcap A_{\langle f,g \rangle} \tilde{=} A_{\langle f,g \rangle} \tilde{=} A_{\langle f,g \rangle} \sqcup A_{\langle f,g \rangle}$,

4) $A_{\langle f,g \rangle} \sqcup A_{\langle \tilde{\mathbf{0}}, \tilde{\mathbf{1}} \rangle} \tilde{=} A_{\langle f,g \rangle} \tilde{=} A_{\langle f,g \rangle} \sqcap A_{\langle \tilde{\mathbf{1}}, \tilde{\mathbf{0}} \rangle}$,

5) $A_{\langle f,g \rangle} \sqcup A_{\langle \tilde{\mathbf{1}}, \tilde{\mathbf{0}} \rangle} \tilde{=} A_{\langle \tilde{\mathbf{1}}, \tilde{\mathbf{0}} \rangle}$; $A_{\langle f,g \rangle} \sqcap A_{\langle \tilde{\mathbf{0}}, \tilde{\mathbf{1}} \rangle} \tilde{=} A_{\langle \tilde{\mathbf{0}}, \tilde{\mathbf{1}} \rangle}$.

Proof. Straightforward. ■

7.4.4 Proposition

Let $A_{\langle f,g \rangle}$, $B_{\langle h,i \rangle}$ and $C_{\langle j,k \rangle}$ be any *fuzzy bipolar soft set*s over X. Then the following are true

1) (Absorption Laws)

- **(i)** $A_{\langle f,g \rangle} \sqcup_{\varepsilon} (B_{\langle h,i \rangle} \sqcap A_{\langle f,g \rangle}) \tilde{=} A_{\langle f,g \rangle}$,
- **(ii)** $A_{\langle f,g \rangle} \sqcap (B_{\langle h,i \rangle} \sqcup_{\varepsilon} A_{\langle f,g \rangle}) \tilde{=} A_{\langle f,g \rangle}$,
- **(iii)** $A_{\langle f,g \rangle} \sqcup (B_{\langle h,i \rangle} \sqcap_{\varepsilon} A_{\langle f,g \rangle}) \tilde{=} A_{\langle f,g \rangle}$,
- **(iv)** $A_{\langle f,g \rangle} \sqcap_{\varepsilon} (B_{\langle h,i \rangle} \sqcup A_{\langle f,g \rangle}) \tilde{=} A_{\langle f,g \rangle}$.

2) (Associative Laws) $A_{\langle f,g \rangle} \lambda (B_{\langle h,i \rangle} \lambda C_{\langle j,k \rangle}) \tilde{=} (A_{\langle f,g \rangle} \lambda B_{\langle h,i \rangle}) \lambda C_{\langle j,k \rangle}$,

3) (Commutative Laws) $A_{\langle f,g \rangle} \lambda B_{\langle h,i \rangle} \tilde{=} B_{\langle h,i \rangle} \lambda A_{\langle f,g \rangle}$,

4) (Distributive Laws)(Distributive Laws)

- **(i)** $A_{\langle f,g \rangle} \sqcup_{\varepsilon} (B_{\langle h,i \rangle} \sqcup C_{\langle j,k \rangle}) \tilde{\subseteq} (A_{\langle f,g \rangle} \sqcup_{\varepsilon} B_{\langle h,i \rangle}) \sqcup (A_{\langle f,g \rangle} \sqcup_{\varepsilon} C_{\langle j,k \rangle})$,
- **(ii)** $A_{\langle f,g \rangle} \sqcup_{\varepsilon} (B_{\langle h,i \rangle} \sqcap_{\varepsilon} C_{\langle j,k \rangle}) \tilde{\supseteq} (A_{\langle f,g \rangle} \sqcup_{\varepsilon} B_{\langle h,i \rangle}) \sqcap_{\varepsilon} (A_{\langle f,g \rangle} \sqcup_{\varepsilon} C_{\langle j,k \rangle})$,
- **(iii)** $A_{\langle f,g \rangle} \sqcup_{\varepsilon} (B_{\langle h,i \rangle} \sqcap C_{\langle j,k \rangle}) \tilde{=} (A_{\langle f,g \rangle} \sqcup_{\varepsilon} B_{\langle h,i \rangle}) \sqcap (A_{\langle f,g \rangle} \sqcup_{\varepsilon} C_{\langle j,k \rangle})$,
- **(iv)** $A_{\langle f,g \rangle} \sqcup (B_{\langle h,i \rangle} \sqcup_{\varepsilon} C_{\langle j,k \rangle}) \tilde{=} (A_{\langle f,g \rangle} \sqcup B_{\langle h,i \rangle}) \sqcup_{\varepsilon} (A_{\langle f,g \rangle} \sqcup C_{\langle j,k \rangle})$,
- **(v)** $A_{\langle f,g \rangle} \sqcup (B_{\langle h,i \rangle} \sqcap_{\varepsilon} C_{\langle j,k \rangle}) \tilde{=} (A_{\langle f,g \rangle} \sqcup B_{\langle h,i \rangle}) \sqcap_{\varepsilon} (A_{\langle f,g \rangle} \sqcup C_{\langle j,k \rangle})$,
- **(vi)** $A_{\langle f,g \rangle} \sqcup (B_{\langle h,i \rangle} \sqcap C_{\langle j,k \rangle}) \tilde{=} (A_{\langle f,g \rangle} \sqcup B_{\langle h,i \rangle}) \sqcap (A_{\langle f,g \rangle} \sqcup C_{\langle j,k \rangle})$,
- **(vii)** $A_{\langle f,g \rangle} \sqcap_{\varepsilon} (B_{\langle h,i \rangle} \sqcup_{\varepsilon} C_{\langle j,k \rangle}) \tilde{\subseteq} (A_{\langle f,g \rangle} \sqcap_{\varepsilon} B_{\langle h,i \rangle}) \sqcup_{\varepsilon} (A_{\langle f,g \rangle} \sqcap_{\varepsilon} C_{\langle j,k \rangle})$,

(viii) $A_{\langle f,g\rangle} \sqcap_{\varepsilon} (B_{\langle h,i\rangle} \sqcup C_{\langle j,k\rangle}) \tilde{=} (A_{\langle f,g\rangle} \sqcap_{\varepsilon} B_{\langle h,i\rangle}) \sqcup (A_{\langle f,g\rangle} \sqcap_{\varepsilon} C_{\langle j,k\rangle})$,

(ix) $A_{\langle f,g\rangle} \sqcap_{\varepsilon} (B_{\langle h,i\rangle} \sqcap C_{\langle j,k\rangle}) \tilde{\supseteq} (A_{\langle f,g\rangle} \sqcap_{\varepsilon} B_{\langle h,i\rangle}) \sqcap (A_{\langle f,g\rangle} \sqcap_{\varepsilon} C_{\langle j,k\rangle})$,

(x) $A_{\langle f,g\rangle} \sqcap (B_{\langle h,i\rangle} \sqcup_{\varepsilon} C_{\langle j,k\rangle}) \tilde{=} (A_{\langle f,g\rangle} \sqcap B_{\langle h,i\rangle}) \sqcup_{\varepsilon} (A_{\langle f,g\rangle} \sqcap C_{\langle j,k\rangle})$,

(xi) $A_{\langle f,g\rangle} \sqcap (B_{\langle h,i\rangle} \sqcap_{\varepsilon} C_{\langle j,k\rangle}) \tilde{=} (A_{\langle f,g\rangle} \sqcap B_{\langle h,i\rangle}) \sqcap_{\varepsilon} (A_{\langle f,g\rangle} \sqcap C_{\langle j,k\rangle})$,

(xii) $A_{\langle f,g\rangle} \sqcap (B_{\langle h,i\rangle} \sqcup C_{\langle j,k\rangle}) \tilde{=} (A_{\langle f,g\rangle} \sqcap B_{\langle h,i\rangle}) \sqcup (A_{\langle f,g\rangle} \sqcap C_{\langle j,k\rangle})$.

Proof. From Theorem 7.2.4, it is easy to see that these properties hold as for double-framed fuzzy soft sets ■

7.4.5 Example

Let X be the set of houses under consideration, and E be the set of parameters, $X = \{h_1,h_2,h_3,h_4,h_5\}$, $E = \{e_1,e_2,e_3,e_4,e_5\} = \{$ in the green surroundings, cheap, in good repair, furnished, traditional $\}$. Let $\neg E = \{\neg e_1,\neg e_2,\neg e_3,\neg e_4,\neg e_5\} = \{$ in the commercial area, expensive, in bad repair, non-furnished, modern $\}$. Suppose that $A = \{e_1,e_2,e_3\}$, $B = \{e_2,e_3,e_4\}$, and $C = \{e_3,e_4,e_5\}$. The *fuzzy bipolar soft sets* $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ and $C_{\langle j,k\rangle}$ describe the "requirements of the houses" which Mr. X, Mr. Y and Mr. Z are going to buy respectively. Suppose that

$$
\begin{aligned}
f &: A \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases} \{x_1/0.4, x_2/0.7, x_3/0.7, x_4/0.7, x_5/0.1\} & \text{if } e = e_1,\\ \{x_1/0.8, x_2/0.0, x_3/0.5, x_4/0.1, x_5/0.6\} & \text{if } e = e_2,\\ \{x_1/0.7, x_2/0.5, x_3/0.7, x_4/0.6, x_5/0.1\} & \text{if } e = e_3.\end{cases}\\
g &: A \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases} \{x_1/0.3, x_2/0.1, x_3/0.3, x_4/0.1, x_5/0.7\} & \text{if } e = e_1,\\ \{x_1/0.1, x_2/0.9, x_3/0.3, x_4/0.8, x_5/0.2\} & \text{if } e = e_2,\\ \{x_1/0.1, x_2/0.3, x_3/0.3, x_4/0.3, x_5/0.8\} & \text{if } e = e_3,\end{cases}\\
h &: B \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases} \{x_1/0.1, x_2/0.3, x_3/0.3, x_4/0.6, x_5/0.6\} & \text{if } e = e_2,\\ \{x_1/0.1, x_2/0, x_3/0.3, x_4/0.4, x_5/0.6\} & \text{if } e = e_3,\\ \{x_1/0.9, x_2/0.5, x_3/0.5, x_4/0.3, x_5/0.1\} & \text{if } e = e_4.\end{cases}\\
i &: B \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases} \{x_1/0.1, x_2/0.3, x_3/0.6, x_4/0.2, x_5/0.3\} & \text{if } e = e_2,\\ \{x_1/0.8, x_2/0.9, x_3/0.5, x_4/0.4, x_5/0.2\} & \text{if } e = e_3,\\ \{x_1/0.1, x_2/0.4, x_3/0.3, x_4/0.6, x_5/0.9\} & \text{if } e = e_4,\end{cases}
\end{aligned}
$$

$$\begin{aligned}
j &: C \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases} \{x_1/0.7, x_2/0.7, x_3/0.4, x_4/0.7, x_5/0.4\} & \text{if } e = e_3,\\ \{x_1/0.6, x_2/0.5, x_3/0.6, x_4/0.1, x_5/0.6\} & \text{if } e = e_4,\\ \{x_1/0.3, x_2/0.4, x_3/0.4, x_4/0.3, x_5/0.1\} & \text{if } e = e_5.\end{cases}\\
k &: C \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases} \{x_1/0.1, x_2/0.2, x_3/0.3, x_4/0.1, x_5/0.1\} & \text{if } e = e_3,\\ \{x_1/0.2, x_2/0.2, x_3/0.3, x_4/0.3, x_5/0.2\} & \text{if } e = e_4,\\ \{x_1/0.1, x_2/0.1, x_3/0.3, x_4/0.5, x_5/0.7\} & \text{if } e = e_5,\end{cases}
\end{aligned}$$

Let

$$A_{\langle f,g\rangle} \sqcup_\varepsilon (B_{\langle h,i\rangle} \sqcap_\varepsilon C_{\langle j,k\rangle}) \tilde{=} (A \cup B) \cup C_{\langle f\tilde{\vee}(h\tilde{\wedge}j), g\tilde{\wedge}(i\tilde{\vee}k)\rangle}$$

and

$$(A_{\langle f,g\rangle} \sqcup_\varepsilon B_{\langle h,i\rangle}) \sqcap_\varepsilon (A_{\langle f,g\rangle} \sqcup_\varepsilon C_{\langle j,k\rangle}) \tilde{=} (A \cup B) \cup C_{\langle (f\tilde{\vee}h)\tilde{\wedge}(f\tilde{\vee}j)\rangle}.$$

Then

$$\begin{aligned}
(f\tilde{\vee}(h\tilde{\wedge}j))(e_2) &= \{x_1/0.1, x_2/0.0, x_3/0.3, x_4/0.1, x_5/0.6\}\\
&\neq \{x_1/0.8, x_2/0.0, x_3/0.5, x_4/0.1, x_5/0.6\}\\
&= ((f\tilde{\vee}h)\tilde{\wedge}(f\tilde{\vee}j))(e_2) \qquad \text{and}\\
(g\tilde{\wedge}(i\tilde{\vee}k))(e_2) &= \{x_1/0.1, x_2/0.9, x_3/0.6, x_4/0.8, x_5/0.3\}\\
&\neq \{x_1/0.1, x_2/0.9, x_3/0.3, x_4/0.8, x_5/0.2\}\\
&= ((g\tilde{\wedge}i)\tilde{\vee}(g\tilde{\wedge}k))(e_2),
\end{aligned}$$

so that

$$A_{\langle f,g\rangle} \sqcup_\varepsilon (B_{\langle h,i\rangle} \sqcap_\varepsilon C_{\langle j,k\rangle}) \tilde{\neq} (A_{\langle f,g\rangle} \sqcup_\varepsilon B_{\langle h,i\rangle}) \sqcap_\varepsilon (A_{\langle f,g\rangle} \sqcup_\varepsilon C_{\langle j,k\rangle}).$$

Now,

$$A_{\langle f,g\rangle} \sqcap_\varepsilon (B_{\langle h,i\rangle} \sqcup_\varepsilon C_{\langle j,k\rangle}) \tilde{=} (A \cup B) \cup C_{\langle f\tilde{\wedge}(h\tilde{\vee}j), g\tilde{\vee}(i\tilde{\wedge}k)\rangle}$$

and

$$(A_{\langle f,g\rangle} \sqcap_\varepsilon B_{\langle h,i\rangle}) \sqcup_\varepsilon (A_{\langle f,g\rangle} \sqcap_\varepsilon C_{\langle j,k\rangle}) \tilde{=} (A \cup B) \cup C_{\langle (f\tilde{\wedge}h)\tilde{\vee}(f\tilde{\wedge}j), (g\tilde{\vee}i)\tilde{\wedge}(g\tilde{\vee}k)\rangle}.$$

Then,

$$\begin{aligned}
(f\tilde{\wedge}(h\tilde{\vee}j))(e_2) &= \{x_1/0.8, x_2/0.3, x_3/0.5, x_4/0.6, x_5/0.6\}\\
&\neq \{x_1/0.8, x_2/0.0, x_3/0.5, x_4/0.1, x_5/0.6\}\\
&= ((f\tilde{\wedge}h)\tilde{\vee}(f\tilde{\wedge}j))(e_2)
\end{aligned}$$

and

$$\begin{aligned}(g\tilde{\vee}(i\tilde{\wedge}k))(e_2) &= \{x_1/0.1, x_2/0.3, x_3/0.3, x_4/0.2, x_5/0.2\}\\ &\neq \{x_1/0.1, x_2/0.9, x_3/0.3, x_4/0.8, x_5/0.2\}\\ &= ((g\tilde{\vee}i)\tilde{\wedge}(g\tilde{\vee}k))(e_2).\end{aligned}$$

So that

$$A_{\langle f,g\rangle} \sqcap_{\varepsilon} (B_{\langle h,i\rangle} \sqcup_{\varepsilon} C_{\langle j,k\rangle})\tilde{\neq}(A_{\langle f,g\rangle} \sqcap_{\varepsilon} B_{\langle h,i\rangle}) \sqcup_{\varepsilon} (A_{\langle f,g\rangle} \sqcap_{\varepsilon} C_{\langle j,k\rangle}).$$

Similarly we can show that

$$A_{\langle f,g\rangle} \sqcup_{\varepsilon} (B_{\langle h,i\rangle} \sqcup C_{\langle j,k\rangle})\tilde{\neq}(A_{\langle f,g\rangle} \sqcup_{\varepsilon} B_{\langle h,i\rangle}) \sqcup (A_{\langle f,g\rangle} \sqcup_{\varepsilon} C_{\langle j,k\rangle}),$$

and

$$A_{\langle f,g\rangle} \sqcap_{\varepsilon} (B_{\langle h,i\rangle} \sqcap C_{\langle j,k\rangle})\tilde{\neq}(A_{\langle f,g\rangle} \sqcap_{\varepsilon} B_{\langle h,i\rangle}) \sqcap (A_{\langle f,g\rangle} \sqcap_{\varepsilon} C_{\langle j,k\rangle}).$$

7.4.6 Corollary

Let $A_{\langle f,g\rangle}$, $B_{\langle h,i\rangle}$ and $C_{\langle j,k\rangle}$ be three *fuzzy bipolar soft set*s over X such that $(A \cap B) - C = (A \cap C) - B = \emptyset$. Then

1)

$$A_{\langle f,g\rangle} \sqcup_{\varepsilon} (B_{\langle h,i\rangle} \sqcap_{\varepsilon} C_{\langle j,k\rangle})\tilde{=}(A_{\langle f,g\rangle} \sqcup_{\varepsilon} B_{\langle h,i\rangle}) \sqcap_{\varepsilon} (A_{\langle f,g\rangle} \sqcup_{\varepsilon} C_{\langle j,k\rangle}),$$

2)

$$A_{\langle f,g\rangle} \sqcap_{\varepsilon} (B_{\langle h,i\rangle} \sqcup_{\varepsilon} C_{\langle j,k\rangle})\tilde{=}(A_{\langle f,g\rangle} \sqcap_{\varepsilon} B_{\langle h,i\rangle}) \sqcup_{\varepsilon} (A_{\langle f,g\rangle} \sqcap_{\varepsilon} C_{\langle j,k\rangle}).$$

7.4.7 Corollary

Let $A_{\langle f,g\rangle}$, $A_{\langle h,i\rangle}$ and $A_{\langle j,k\rangle}$ be any *fuzzy bipolar soft set*s over X. Then

$$A_{\langle f,g\rangle}\lambda(A_{\langle h,i\rangle}\rho A_{\langle j,k\rangle})\tilde{=}(A_{\langle f,g\rangle}\lambda A_{\langle h,i\rangle})\rho(A_{\langle f,g\rangle}\lambda A_{\langle j,k\rangle})$$

for distinct λ, $\rho \in \{\sqcap_{\varepsilon}, \sqcap, \sqcup_{\varepsilon}, \sqcup\}$.

7.4.8 Proposition

Let $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ be two *fuzzy bipolar soft set*s over X. Then the following are true

1) $A_{\langle f,g\rangle} \sqcup_{\varepsilon} B_{\langle h,i\rangle}$ is the smallest *fuzzy bipolar* soft set over X which contains both $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$. (Supremum)

2) $A_{\langle f,g\rangle} \sqcap B_{\langle h,i\rangle}$ is the largest *fuzzy bipolar* soft set over X which is contained in both $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$. (Infimum)

Proof. Straightforward. ■

7.5 Algebras of Fuzzy Bipolar Soft Sets

Now we consider the collection of all fuzzy bipolar soft sets over X and denote it by $\mathcal{FBSS}(X)^E$ and let us denote its sub collection of all fuzzy bipolar soft sets over X with fixed set of parameters A by $\mathcal{FBSS}(X)_A$. We note that this collection is partially ordered by inclusion. We conclude from above results that:

7.5.1 Proposition

$(\mathcal{FBSS}(X)^E, \sqcap_\varepsilon, \sqcup)$ and $(\mathcal{FBSS}(X)^E, \sqcup_\varepsilon, \sqcap)$ are distributive lattices and $(\mathcal{FBSS}(X)^E, \sqcup, \sqcap_\varepsilon)$ and $(\mathcal{FBSS}(X)^E, \sqcap, \sqcup_\varepsilon)$ are their duals, respectively.

Proof. Follows from Propositions 7.4.3 and 7.4.4. ■

7.5.2 Proposition

$(\mathcal{FBSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{\langle \Phi, \mathfrak{X} \rangle}, E_{\langle \mathfrak{X}, \Phi \rangle})$, $(\mathcal{FBSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{\langle \mathfrak{X}, \Phi \rangle}, \emptyset_{\langle \Phi, \mathfrak{X} \rangle})$, $(\mathcal{FBSS}(X)_A, \sqcap, \sqcup, A_{\langle \Phi, \mathfrak{X} \rangle}, A_{\langle \mathfrak{X}, \Phi \rangle})$ and $(\mathcal{FBSS}(X)_A, \sqcup, \sqcap, A_{\langle \mathfrak{X}, \Phi \rangle}, A_{\langle \Phi, \mathfrak{X} \rangle})$ are bounded distributive lattices.

Proof. From Proposition 7.4.8, we know that $(\mathcal{FBSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{\langle \Phi, \mathfrak{X} \rangle}, E_{\langle \mathfrak{X}, \Phi \rangle})$ is a bounded distributive lattice and $(\mathcal{FBSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{\langle \mathfrak{X}, \Phi \rangle}, \emptyset_{\langle \Phi, \mathfrak{X} \rangle})$ is its dual. For any fuzzy bipolar soft sets $A_{\langle f,g \rangle}, A_{\langle h,i \rangle} \in \mathcal{FBSS}(X)_A$,

$$\begin{aligned} A_{\langle f,g \rangle} \sqcap A_{\langle h,i \rangle} &\tilde{=} A_{\langle f \tilde{\wedge} h, g \tilde{\vee} i \rangle} \in \mathcal{FBSS}(X)_A \text{ and} \\ A_{\langle f,g \rangle} \sqcup A_{\langle h,i \rangle} &\tilde{=} A_{\langle f \tilde{\vee} h, g \tilde{\wedge} i \rangle} \in \mathcal{FBSS}(X)_A. \end{aligned}$$

Thus $(\mathcal{FBSS}(X)_A, \sqcap, \sqcup)$ is also a distributive sublattice of $(\mathcal{FBSS}(X)^E, \sqcup_\varepsilon, \sqcap)$ and Proposition 7.4.3 shows that $(\mathcal{FBSS}(X)_A, \sqcap, \sqcup, A_{\langle \Phi, \mathfrak{X} \rangle}, A_{\langle \mathfrak{X}, \Phi \rangle})$ is a bounded distributive lattice and $(\mathcal{FBSS}(X)_A, \sqcup, \sqcap, A_{\langle \mathfrak{X}, \Phi \rangle}, A_{\langle \Phi, \mathfrak{X} \rangle})$ is its dual. ■

7.5.3 Proposition (de Morgan Laws)

Let $A_{\langle f,g \rangle}$ and $B_{\langle h,i \rangle}$ be two *fuzzy bipolar* soft sets over X. Then the following are true

1) $(A_{\langle f,g \rangle} \sqcup_\varepsilon B_{\langle h,i \rangle})^\circ \tilde{=} A_{\langle f,g \rangle^\circ} \sqcap_\varepsilon B_{\langle h,i \rangle^\circ}$,

2) $(A_{\langle f,g \rangle} \sqcap_\varepsilon B_{\langle h,i \rangle})^\circ \tilde{=} A_{\langle f,g \rangle^\circ} \sqcup_\varepsilon B_{\langle h,i \rangle^\circ}$,

3) $(A_{\langle f,g \rangle} \vee B_{\langle h,i \rangle})^\circ \tilde{=} A_{\langle f,g \rangle^\circ} \wedge B_{\langle h,i \rangle^\circ}$,

4) $(A_{\langle f,g \rangle} \wedge B_{\langle h,i \rangle})^\circ \tilde{=} A_{\langle f,g \rangle^\circ} \vee B_{\langle h,i \rangle^\circ}$,

5) $(A_{\langle f,g \rangle} \sqcup B_{\langle h,i \rangle})^\circ \tilde{=} A_{\langle f,g \rangle^\circ} \sqcap B_{\langle h,i \rangle^\circ}$,

6) $(A_{\langle f,g\rangle} \sqcap B_{\langle h,i\rangle})^{\circ} \tilde{=} A_{\langle f,g\rangle^{\circ}} \sqcup B_{\langle h,i\rangle^{\circ}}$.

Proof.

1) We have

$$(A_{\langle f,g\rangle} \sqcup_{\varepsilon} B_{\langle h,i\rangle})^{\circ} \tilde{=} ((A \cup B)_{\langle f\tilde{\vee}h, g\tilde{\wedge}i\rangle})^{\circ} \tilde{=} (A \cup B)_{\langle g\tilde{\wedge}i, f\tilde{\vee}h\rangle}$$

and

$$A_{\langle f,g\rangle^{\circ}} \sqcap_{\varepsilon} B_{\langle h,i\rangle^{\circ}} \tilde{=} A_{\langle g,f\rangle} \sqcap_{\varepsilon} B_{\langle i,h\rangle} \tilde{=} (A \cup B)_{\langle g\tilde{\wedge}i, f\tilde{\vee}h\rangle}.$$

Thus

$$(A_{\langle f,g\rangle} \sqcup_{\varepsilon} B_{\langle h,i\rangle})^{\circ} \tilde{=} A_{\langle f,g\rangle^{\circ}} \sqcap_{\varepsilon} B_{\langle h,i\rangle^{\circ}}.$$

The remaining parts can be proved in a similar way.

■

7.5.4 Proposition

$(\mathcal{FBSS}(X)_A, \sqcap, \sqcup, ^{\circ}, A_{\langle \Phi, \mathfrak{X}\rangle}, A_{\langle \mathfrak{X}, \Phi\rangle})$ is a de Morgan algebra.

Proof. Proof follows from Propositions 7.3.9 and 7.5.3. ■

7.5.5 Definition

Let $A_{\langle f,g\rangle}$ be a *fuzzy bipolar* soft set over X. We define $A_{\langle f,g\rangle^*}$ as a *fuzzy bipolar soft set* $A_{\langle f^*,g^*\rangle}$ where

$$\begin{aligned} f^* &: A \to \mathcal{FP}(X),\ e \mapsto (f(e))^*, \\ (f(e))^*(x) &= \begin{cases} 0 & \text{if } (f(e))^*(x) \neq 0 \\ 1 & \text{if } (f(e))^*(x) = 0 \end{cases} \\ g^* &: A \to \mathcal{FP}(X),\ e \mapsto (g(e))^*, \\ (g(e))^*(x) &= \begin{cases} 1 & \text{if } (g(e))^*(x) \neq 1 \\ 0 & \text{if } (g(e))^*(x) = 1 \end{cases} \quad \text{for } x \in X. \end{aligned}$$

7.2.4.

7.5.6 Theorem

$(\mathcal{FBSS}(X)_A, \sqcap, \sqcup, ^*, A_{\langle \Phi, \mathfrak{X}\rangle}, A_{\langle \mathfrak{X}, \Phi\rangle})$ is a Stone algebra.

Proof. From Proposition 7.5.2 it is evident that $(\mathcal{FBSS}(X)_A, \sqcap, \sqcup, A_{\langle \Phi, \mathfrak{X}\rangle}, A_{\langle \mathfrak{X}, \Phi\rangle})$ is a bounded distributive lattice and $A_{\langle f,g\rangle^*} \tilde{=} \theta(A_{(f,g)^*})$ where θ is mapping defined in Theorem 7.2.4 assures that * is a pseudocomplementing function satisfying Stone's identity. Thus $(\mathcal{FBSS}(X)_A, \sqcap, \sqcup, ^*, A_{\langle \Phi, \mathfrak{X}\rangle}, A_{\langle \mathfrak{X}, \Phi\rangle})$ is a Stone algebra. ■

Chapter 8

A Generalized Framework for Soft Set Theory

This chapter is more of a collective nature than the previous ones and not only summarizes the main results but also provides a general framework to deal with soft sets in a logical manner. We have given an over all review of various kinds of soft sets. A brief discussion about defining ideas of extended soft sets and their operations, a summary of algebraic structures and an application of soft sets in decision making problems has been made in this chapter to conclude thesis here. We initiate discussion with definition of soft sets.

8.1 General Definition of Soft Set and its Extensions:

Let X be an initial universe and E be a set of parameters. Let $\lambda\mathcal{P}(X)$ be a generalized fuzzy power set of X where $\lambda\mathcal{P}(X)$ may be a collection of all crisp or fuzzy or type-2 fuzzy or n-fuzzy or hesitant fuzzy or interval-valued fuzzy or vague or intuitionistic fuzzy or bipolar fuzzy subsets of X and, say, λ stands for a fuzzy criteria of collection $\lambda\mathcal{P}(X)$.

- A mapping $f : A \to \lambda\mathcal{P}(X)$ is called a $\lambda-$*soft set* over X denoted by A_f where $A \subseteq E$. We note that parameters in E can be a specific criteria for which an approximation of elements of X is made by f, so a $\lambda-$*soft set* over X gives a parameterized family of $\lambda-$subsets of X.
- In our next step towards a general framework for soft sets, we allow to consider more than one frames of reference for X within the context of each parameter. This consideration requires some modifications in the ongoing soft set based model and so, this requirement is fulfilled by introducing a set of functions f_i :

$A \to \lambda\mathcal{P}(X)$, $i = 1, 2, ..., n$ and denote it by $A_{(f_1,f_2,...,f_n)}$ and call it an *n-framed* $\lambda-$*soft set over* X. Clearly, an *n-framed* $\lambda-$*soft set* gives n parametrized families of $\lambda-$subsets of X.

- Now, if the frames of references are mutually exclusive or obeying some other mutual relation which is causing a polarity among those, then we incorporate the idea by imposing a suitably chosen set of consistency constraints $\mathcal{C}$. Hence we give the concept of λ *n-polar soft set* over X comprising of functions $f_i : A \to \lambda\mathcal{P}(X)$, $(f_i \in \mathcal{C})$ $\quad i = 1, 2, ..., n$ denoted by $A_{\langle f_1,f_2,...,f_n\rangle}$.

In a natural way, all λ *multi-polar soft sets* are *multi-framed* $\lambda-$*soft sets* over X but the converse is not true. It is also interesting to observe that *multi-polar* λ *soft sets* can be presented in an equivalent and better way by using λ *multi-polar soft sets*. A particular case for $n = 2$ is already discussed in Chapter 7 for fuzzy subsets of X.

8.2 Aggregation Operators for Soft Sets in General Form

We need to apply a process for aggregation where the number of inputs are grouped together in order to get a single output that is easier to use for further computations. Usually when an object or an alternative is characterized by several numbers or values describing its various parameters or is given evaluations from several experts and one has to aggregate these values in order to describe the object by just one meaningful value or set of values. Aggregation operators are an important tool that is used in many domains [6], [8]. For a soft set and its hybrid generalizations and extensions, an input space for aggregation is a bit unconventional because it is required to deal each object in a parametrized context. Therefore a soft aggregation operator is a function working on a particular number of inputs for each parameter, with output lying again in a parametrized manner. We define soft aggregation operators in either restricted or extended context. A restricted soft aggregation operator joins two soft sets with a restricted set of parameters, that is, only those parameters which are combined to both and mathematically the set of parameters is taken as the intersection of parameters sets in input soft sets. On the other hand, an extended soft aggregation operator joins two soft sets with an extended set of parameters, that is, all those parameters apparent are taken into consideration and mathematically the set of parameters in output is union of parameters sets in input soft sets. Let m be a positive integer and K be a set of various operations defined for λ fuzzy subsets of X.

- Let $A_i, B \subseteq E$ and $A_{i_{f_i}}$ be $\lambda-$*soft sets* over X, where $i = 1, 2, ..., m$. Then an aggregation operator is a mapping $(A_{1_{f_1}}, A_{2_{f_2}}, ..., A_{m_{f_m}}) \mapsto B_g$. We have two cases:

(i) For the case of restricted aggregation operators, we have $B = \bigcap_{i=1}^{m} A_i$ and

$$g(e) = k\{f_i(e) : i = 1, 2, ..., m\}$$

for all $e \in B$.

(ii) For the case of extended aggregation operators, we have $B = \bigcup_{i=1}^{m} A_i$ and we define the set $\Lambda(e) = \{j : e \in A_j\}$

$$g(e) = k\{f_i(e) : i \in \Lambda(e)\}$$

for all $e \in B$.

- Let $A_i, B \subseteq E$ and $A_{i_{(f_{i1}, f_{i2}, \cdots, f_{in})}}$ be *n-framed* $\lambda-$*soft sets* over X, where $i = 1$, 2, ..., m, and $(k_1, k_2, ..., k_n) \in K^n$. Then an aggregation operator is a mapping $(A_{1_{(f_{11}, f_{12}, \cdots, f_{1n})}}, A_{2_{(f_{21}, f_{22}, \cdots, f_{2n})}}, ..., A_{m_{(f_{m1}, f_{m2}, \cdots, f_{mn})}}) \mapsto B_{(g_1, g_2, \cdots, g_n)}$. We have two cases:

(i) For the case of restricted aggregation operators, we have $B = \bigcap_{i=1}^{m} A_i$ and

$$g_j(e) = k_j\{f_{ij}(e) : i = 1, 2, ..., m\},\ j = 1, 2, ..., n$$

for all $e \in B$.

(ii) For the case of extended aggregation operators, we have $B = \bigcup_{i=1}^{m} A_i$ and we define the set $\Lambda(e) = \{j : e \in A_j\}$

$$g_j(e) = k_j\{f_{ij}(e) : i \in \Lambda(e)\},\ j = 1, 2, ..., n$$

for all $e \in B$.

- Let $A_i, B \subseteq E$ and $A_{i_{\langle f_{i1}, f_{i2}, \cdots, f_{in} \rangle}}$ $(f_{ij} \in \mathcal{C})$ be λ *n-polar soft sets* over X where $i = 1$, 2, ..., m, and $(k_1, k_2, ..., k_n) \in K^n$. Then an aggregation operator is a mapping $(A_{1_{\langle f_{11}, f_{12}, \cdots, f_{1n} \rangle}}, A_{2_{\langle f_{21}, f_{22}, \cdots, f_{2n} \rangle}}, ..., A_{m_{\langle f_{m1}, f_{m2}, \cdots, f_{mn} \rangle}}) \mapsto B_{\langle g_1, g_2, \cdots, g_n \rangle}$ $(g_j \in \mathcal{C})$. We have two cases:

(i) For the case of restricted aggregation operators, we have $B = \bigcap_{i=1}^{m} A_i$ and

$$g_j(e) = k_j\{f_{ij}(e) : i = 1, 2, ..., m\},\ j = 1, 2, ..., n$$

for all $e \in B$.

(ii) For the case of extended aggregation operators, we have $B = \bigcup_{i=1}^{m} A_i$ and we define the set $\Lambda(e) = \{j : e \in A_j\}$

$$g_j(e) = k_j\{f_{ij}(e) : i \in \Lambda(e)\},\ j = 1, 2, ..., n$$

for all $e \in B$.

All aggregation operators defined for *n-framed* $\lambda-$*soft sets* over X can be used to define aggregation operators for λ *n-polar soft sets* over X as except where consistency constraints are absent. We have seen an example of complement operation defined for double-framed soft sets which is no more available for bipolar soft sets due to hazard of consistency constraint. Thus the set of aggregation operators for λ *n-polar soft sets* is contained in the set of aggregation operators for *n-framed* $\lambda-$*soft sets.*

8.3 New Examples of Logical Algebraic Structures

In this section we present a summary of results that we have found in our research regarding different types of soft sets and their collections and thus new examples of these algebras are contributed through our work. Following table gives an overview of the algebraic structures of soft sets:

1	Lattices:
	$(\mathcal{SS}(X)^E, \sqcap_\varepsilon, \sqcup), (\mathcal{SS}(X)^E, \sqcup, \sqcap_\varepsilon), (\mathcal{FSS}(X)^E, \sqcap_\varepsilon, \sqcup),$ $(\mathcal{FSS}(X)^E, \sqcup, \sqcap_\varepsilon), (\mathcal{DSS}(X)^E, \sqcap_\varepsilon, \sqcup), (\mathcal{DSS}(X)^E, \sqcup, \sqcap_\varepsilon),$ $(\mathcal{DFSS}(X)^E, \sqcap_\varepsilon, \sqcup), (\mathcal{DFSS}(X)^E, \sqcup, \sqcap_\varepsilon), (\mathcal{BSS}(X)^E, \sqcap_\varepsilon, \sqcup),$ $(\mathcal{BSS}(X)^E, \sqcup, \sqcap_\varepsilon), (\mathcal{FBSS}(X)^E, \sqcap_\varepsilon, \sqcup), (\mathcal{FBSS}(X)^E, \sqcup, \sqcap_\varepsilon)$
2	Bounded Distributive Lattices:
	$(\mathcal{SS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{\mathbf{\Phi}}, E_{\mathfrak{X}}), (\mathcal{SS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{\mathfrak{X}}, \emptyset_{\mathbf{\Phi}}),$ $(\mathcal{FSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{\tilde{\mathbf{0}}}, E_{\tilde{\mathbf{1}}}), (\mathcal{FSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{\tilde{\mathbf{1}}}, \emptyset_{\tilde{\mathbf{0}}}),$ $(\mathcal{DSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{(\mathbf{\Phi},\mathfrak{X})}, E_{(\mathfrak{X},\mathbf{\Phi})}), (\mathcal{DSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{(\mathfrak{X},\mathbf{\Phi})}, \emptyset_{(\mathbf{\Phi},\mathfrak{X})}),$ $(\mathcal{DFSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, E_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}), (\mathcal{DFSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}, \emptyset_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}),$ $(\mathcal{BSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{\langle\mathbf{\Phi},\mathfrak{X}\rangle}, E_{\langle\mathfrak{X},\mathbf{\Phi}\rangle}), (\mathcal{BSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{\langle\mathfrak{X},\mathbf{\Phi}\rangle}, \emptyset_{\langle\mathbf{\Phi},\mathfrak{X}\rangle}),$ $(\mathcal{FBSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{\langle\tilde{\mathbf{0}},\tilde{\mathbf{1}}\rangle}, E_{\langle\tilde{\mathbf{1}},\tilde{\mathbf{0}}\rangle}), (\mathcal{FBSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{\langle\tilde{\mathbf{1}},\tilde{\mathbf{0}}\rangle}, \emptyset_{\langle\tilde{\mathbf{0}},\tilde{\mathbf{1}}\rangle})$
3	De Morgan Algebras:
	$(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^\circ, A_{(\mathbf{\Phi},\mathfrak{X})}, A_{(\mathfrak{X},\mathbf{\Phi})}), (\mathcal{DSS}(X)_A, \sqcup, \sqcap, ^\circ, A_{(\mathfrak{X},\mathbf{\Phi})}, A_{(\mathbf{\Phi},\mathfrak{X})})$ $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ^\circ, A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}), (\mathcal{DFSS}(X)_A, \sqcup, \sqcap, ^\circ, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}, A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})})$ $(\mathcal{FBSS}(X)_A, \sqcap, \sqcup, ^\circ, A_{\langle\tilde{\mathbf{0}},\tilde{\mathbf{1}}\rangle}, A_{\langle\tilde{\mathbf{1}},\tilde{\mathbf{0}}\rangle}), (\mathcal{FBSS}(X)_A, \sqcup, \sqcap, ^\circ, A_{\langle\tilde{\mathbf{1}},\tilde{\mathbf{0}}\rangle}, A_{\langle\tilde{\mathbf{0}},\tilde{\mathbf{1}}\rangle}),$

4	Boolean Algebras:
	$(\mathcal{SS}(X)_A, \sqcap, \sqcup, ^c, A_\Phi, A_\mathfrak{X}), (\mathcal{SS}(X)_A, \sqcup, \sqcap, ^c, A_\mathfrak{X}, A_\Phi)$, $(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^c, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)}), (\mathcal{DSS}(X)_A, \sqcup, \sqcap, ^c, A_{(\mathfrak{X},\Phi)}, A_{(\Phi,\mathfrak{X})})$,
5	Kleene Algebras:
	$(\mathcal{FSS}(X)_A, \sqcap, \sqcup, ', A_{\tilde{\mathbf{0}}}, A_{\tilde{\mathbf{1}}}), (\mathcal{FSS}(X)_A, \sqcup, \sqcap, ', A_{\tilde{\mathbf{1}}}, A_{\tilde{\mathbf{0}}})$, $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ', A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}), (\mathcal{DFSS}(X)_A, \sqcup, \sqcap, ', A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}, A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})})$, $(\mathcal{BSS}(X)_A, \sqcap, \sqcup, ^\circ, A_{\langle\Phi,\mathfrak{X}\rangle}, A_{\langle\mathfrak{X},\Phi\rangle}), (\mathcal{BSS}(X)_A, \sqcup, \sqcap, ^\circ, A_{\langle\mathfrak{X},\Phi\rangle}, A_{\langle\Phi,\mathfrak{X}\rangle})$,
6	Pseudocomplemented Lattices:
	$(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^\diamondsuit, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$
7	Stone Algebras:
	$(\mathcal{FSS}(X)_A, \sqcap, \sqcup, ^*, A_{\tilde{\mathbf{0}}}, A_{\tilde{\mathbf{1}}}), (\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ^*, A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$, $(\mathcal{FBSS}(X)_A, \sqcap, \sqcup, ^*, A_{\langle\tilde{\mathbf{0}},\tilde{\mathbf{1}}\rangle}, A_{\langle\tilde{\mathbf{1}},\tilde{\mathbf{0}}\rangle})$
8	Atomic Lattices:
	$(\mathcal{SS}(X)_A, \sqcap, \sqcup)$
9	Brouwerian lattices:
	$(\mathcal{SS}(X)^E, \sqcap, \sqcup_\varepsilon), (\mathcal{SS}(X)_A, \sqcap, \sqcup), (\mathcal{FSS}(X)^E, \sqcap, \sqcup_\varepsilon), (\mathcal{FSS}(X)_A, \sqcap, \sqcup)$ $(\mathcal{DSS}(X)^E, \sqcap, \sqcup_\varepsilon), (\mathcal{DSS}(X)_A, \sqcap, \sqcup), (\mathcal{DFSS}(X)^E, \sqcap, \sqcup_\varepsilon)$, $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup)$
10	MV-algebras:
	$(\mathcal{SS}(X)_A, \sqcap, ^c, A_\mathfrak{X}), (\mathcal{SS}(X)_A, \sqcup, ^c, A_\Phi), (\mathcal{DSS}(X)_A, \sqcap, ^c, A_{(\mathfrak{X},\Phi)})$, $(\mathcal{DSS}(X)_A, \sqcup, ^c, A_{(\Phi,\mathfrak{X})})$
11	BCK-algebras:
	$(\mathcal{SS}(X)_A, \smile, A_\Phi), (\mathcal{SS}(X)_A, \star, A_\Phi), (\mathcal{DSS}(X)_A, \smile, A_{(\Phi,\Phi)})$, $(\mathcal{DSS}(X)_A, \star, A_{(\Phi,\mathfrak{X})})$

8.4 Application of Soft Sets in a Decision Making Problem

Decision making is an important factor of all scientific professions where experts apply their knowledge in that area to make decisions wisely. Many researchers have applied soft set theory in various decision making problems using different algorithms. A general algorithm for the decision of best object using soft sets is given as follows:

8.4.1 Algorithm

Let X be an initial universal set of available objects and E be the set of parameters. The algorithm for the selection of the best choice among the objects of X is given as:

1. Input $A_{(f_1, f_2, \ldots, f_n)}$, an *n-framed* $\lambda-$*soft set over* X where $A \subseteq E$.

2. Input the set of choice parameters $P \subseteq E$ and find the reduced *n-framed* $\lambda-$*soft set over* X which is reduct of $A_{(f_1, f_2, ..., f_n)}$.

3. Compute the comparison tables for functions $f_1, f_2, ..., f_n$ by using the predefined rule or Aggregation operator.

4. Compute the scores for each object.

5. Compute the final score S_i for each object $x_i \in X$.

6. Find k, for which $S_k = \max S_i$.

Then h_k is the optimal choice object. If k has more than one values, then any one of h_k's can be chosen.

Now, we apply the concept of *fuzzy bipolar soft set*s for modelling a given problem and then, we give an algorithm for the choice of optimal object based upon the available sets of information. Let X be the initial universe and E be a set of parameters. We shall adapt the following terminology afterwards:

8.4.2 Definition

Let $E_{\langle f,g \rangle}$ be a fuzzy bipolar soft set defined over X. A Comparison table for f is a square table in which the number of rows and number of columns are equal, rows and columns both are labelled by the object names h_1, h_2, h_3, ..., h_n of the initial universe X, and the entries are t_{ij}, i, $j = 1, 2, ..., n$, given by

$$t_{ij} = \text{the number of parameters for which the membership value of } h_i \text{ exceeds or equal to the membership value of } h_j$$

Clearly, $0 \leq t_{ij} \leq k$, and $t_{ii} = k$, for all i, j where k is the number of parameters present in E. Thus, t_{ij} indicates a numerical measure, which is an integer. A Comparison table for g is a square table in which the number of rows and number of columns are equal, rows and columns both are labelled by the object names h_1, h_2, h_3, ..., h_n of the initial universe X, and the entries are s_{ij}, i, $j = 1, 2, ..., n$, given by

$$s_{ij} = \text{the number of parameters for which the membership value of } h_i \text{ dominates or equal to the membership value of } h_j$$

Clearly, $0 \leq s_{ij} \leq k$, and $s_{ii} = k$, for all i, j where k is the number of parameters present in E. Thus, s_{ij} also indicates a numerical measure, which is an integer.

8.4.3 Definition

The positive row sum and column of an object h_i, denoted by r_i and c_i are calculated by using the formulae,

$$r_i = \sum_{j=1}^{n} t_{ij}, \quad c_j = \sum_{i=1}^{n} t_{ij},$$

The negative row sum and column sum of an object h_i, denoted by $r\acute{}_i$ and $c\acute{}_j$ are calculated by using the formulae,

$$\acute{r_i} = \sum_{j=1}^{n} s_{ij}, \quad \acute{c_j} = \sum_{i=1}^{n} s_{ij}.$$

The positive score P_i of object h_i will be given by:

$$P_i = r_i - c_i$$

while the negative score N_i will be given by:

$$N_i = \acute{r_i} - \acute{c_i}.$$

The final score S_i of object h_i will be given by:

$$S_i = P_i - N_i$$

for all $i = 1, 2, ..., n$.

We wish to find an object from the set of choice parameters A. We are now giving an algorithm for the choice of best object according to the specifications made by observer and recorded data with the help of a fuzzy bipolar soft set.

8.4.4 Algorithm

The algorithm for the selection of the best choice is given as:

1. Input the *fuzzy bipolar soft set* $E_{\langle f,g \rangle}$.
2. Input the set of choice parameters $P \subseteq E$ and find the reduced fuzzy bipolar soft set $P_{\langle f,g \rangle}$.
3. Compute the comparison tables for functions f and g respectively.
4. Compute the positive and negative scores for each object.
5. Compute the final score.
6. Find k, for which $S_k = \max S_i$.

Then h_k is the optimal choice object. If k has more than one values, then any one of h_k's can be chosen

8.4.5 Example

Let $X = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8\}$ be the set of candidates who have applied for a job position of Office Representative in Customer Care Centre of a company. Let $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$ = {Hard Working, Optimism, Enthusiasm, Individualism, Imaginative, Flexibility, Decisiveness, Self-confidence, Politeness} and $\neg E = \{\neg e_1, \neg e_2, \neg e_3, \neg e_4, \neg e_5, \neg e_6, \neg e_7, \neg e_8, \neg e_9\}$ = {Negligent, Pessimism, Half-hearted, Dependence, Unimaginative, Rigidity, Indecisiveness, Shyness, Harshness }. Here the gray area is obviously the moderate form of parameters. Let the *fuzzy bipolar soft sets* $E_{\langle f,g\rangle}$ describes the " Personality Analysis of Candidates" as:

$$
\begin{aligned}
f &: E \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases}
\{m_1/.5, m_2/.7, m_3/.6, m_4/.7, m_5/.5, m_6/.5, m_7/.4, m_8/.8\} & \text{if } e = e_1,\\
\{m_1/.6, m_2/.7, m_3/.8, m_4/.8, m_5/.4, m_6/.4, m_7/.2, m_8/.7\} & \text{if } e = e_2,\\
\{m_1/.8, m_2/.8, m_3/.4, m_4/.6, m_5/.5, m_6/.5, m_7/.4, m_8/.8\} & \text{if } e = e_3,\\
\{m_1/.7, m_2/.6, m_3/.1, m_4/.7, m_5/.6, m_6/.6, m_7/.6, m_8/.9\} & \text{if } e = e_4,\\
\{m_1/.5, m_2/.8, m_3/.6, m_4/.5, m_5/.7, m_6/.3, m_7/.7, m_8/.6\} & \text{if } e = e_5,\\
\{m_1/.4, m_2/.9, m_3/.5, m_4/.4, m_5/.7, m_6/.3, m_7/.6, m_8/.5\} & \text{if } e = e_6,\\
\{m_1/.3, m_2/.8, m_3/.4, m_4/.6, m_5/.8, m_6/.2, m_7/.5, m_8/.4\} & \text{if } e = e_7,\\
\{m_1/.6, m_2/.7, m_3/.5, m_4/.5, m_5/.6, m_6/.4, m_7/.3, m_8/.6\} & \text{if } e = e_8,\\
\{m_1/.8, m_2/.5, m_3/.6, m_4/.6, m_5/.7, m_6/.4, m_7/.2, m_8/.7\} & \text{if } e = e_9,
\end{cases}\\
g &: E \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases}
\{m_1/.3, m_2/.2, m_3/.4, m_4/.1, m_5/.3, m_6/.5, m_7/.4, m_8/.2\} & \text{if } e = e_1,\\
\{m_1/.4, m_2/.1, m_3/.2, m_4/.1, m_5/.5, m_6/.5, m_7/.7, m_8/.1\} & \text{if } e = e_2,\\
\{m_1/.05, m_2/.1, m_3/.5, m_4/.33, m_5/.4, m_6/.3, m_7/.6, m_8/.15\} & \text{if } e = e_3,
\end{cases}
\end{aligned}
$$

$$
e \longmapsto \begin{cases}
\{m_1/.23, m_2/.3, m_3/.6, m_4/.2, m_5/.3, m_6/.33, m_7/.2, m_8/.1\} & \text{if } e = e_4,\\
\{m_1/.4, m_2/.2, m_3/.35, m_4/.4, m_5/.1, m_6/.6, m_7/.2, m_8/.35\} & \text{if } e = e_5,\\
\{m_1/.4, m_2/.2, m_3/.3, m_4/.3, m_5/.2, m_6/.5, m_7/.25, m_8/.31\} & \text{if } e = e_6,\\
\{m_1/.7, m_2/.08, m_3/.5, m_4/.3, m_5/.18, m_6/.78, m_7/.4, m_8/.4\} & \text{if } e = e_7,\\
\{m_1/.4, m_2/.2, m_3/.3, m_4/.45, m_5/.4, m_6/.4, m_7/.6, m_8/.26\} & \text{if } e = e_8,\\
\{m_1/.1, m_2/.4, m_3/.36, m_4/.27, m_5/.2, m_6/.5, m_7/.8, m_8/.2\} & \text{if } e = e_9.
\end{cases}
$$

1. Input the *fuzzy bipolar soft set* $E_{\langle f,g\rangle}$.

2. Input the set of choice parameters $P = \{e_1, e_3, e_4, e_5, e_7, e_8\} \subseteq E$ and find the

reduced fuzzy bipolar soft set $P_{\langle f,g\rangle}$ given as:

$$f \quad : \quad P \rightarrow \mathcal{FP}(X),$$

$$e \longmapsto \begin{cases} \{m_1/.5, m_2/.7, m_3/.6, m_4/.7, m_5/.5, m_6/.5, m_7/.4, m_8/.8\} & \text{if } e = e_1, \\ \{m_1/.8, m_2/.8, m_3/.4, m_4/.6, m_5/.5, m_6/.5, m_7/.4, m_8/.8\} & \text{if } e = e_3, \\ \{m_1/.7, m_2/.6, m_3/.1, m_4/.7, m_5/.6, m_6/.6, m_7/.6, m_8/.9\} & \text{if } e = e_4, \\ \{m_1/.5, m_2/.8, m_3/.6, m_4/.5, m_5/.7, m_6/.3, m_7/.7, m_8/.6\} & \text{if } e = e_5, \\ \{m_1/.3, m_2/.8, m_3/.4, m_4/.6, m_5/.8, m_6/.2, m_7/.5, m_8/.4\} & \text{if } e = e_7, \\ \{m_1/.6, m_2/.7, m_3/.5, m_4/.5, m_5/.6, m_6/.4, m_7/.3, m_8/.6\} & \text{if } e = e_8, \end{cases}$$

$$g \quad : \quad P \rightarrow \mathcal{FP}(X),$$

$$e \longmapsto \begin{cases} \{m_1/.3, m_2/.2, m_3/.4, m_4/.1, m_5/.3, m_6/.5, m_7/.4, m_8/.2\} & \text{if } e = e_1 \\ \{m_1/.05, m_2/.1, m_3/.5, m_4/.33, m_5/.4, m_6/.3, m_7/.6, m_8/.15\} & \text{if } e = e_3 \\ \{m_1/.23, m_2/.3, m_3/.6, m_4/.2, m_5/.3, m_6/.33, m_7/.2, m_8/.1\} & \text{if } e = e_4 \\ \{m_1/.4, m_2/.2, m_3/.35, m_4/.4, m_5/.1, m_6/.6, m_7/.2, m_8/.35\} & \text{if } e = e_5 \\ \{m_1/.7, m_2/.08, m_3/.5, m_4/.3, m_5/.18, m_6/.78, m_7/.4, m_8/.4\} & \text{if } e = e_7 \\ \{m_1/.4, m_2/.2, m_3/.3, m_4/.45, m_5/.4, m_6/.4, m_7/.6, m_8/.26\} & \text{if } e = e_8 \end{cases}$$

3. Compute the comparison tables for functions f and g respectively

f	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8
m_1	6	2	3	4	4	6	4	2
m_2	5	6	6	5	6	6	6	3
m_3	3	0	6	2	1	4	3	2
m_4	4	2	5	6	3	6	5	1
m_5	4	2	5	3	6	6	6	3
m_6	1	1	2	0	3	6	4	0
m_7	2	1	4	1	2	3	6	2
m_8	6	3	6	5	4	6	4	6

Table 8.1: Comparison Table for f

4. Compute the positive and negative scores for each object as given by Table 8.3 and Table 8.4.

5. Compute the final score given by Table 8.5.

6. From Table 8.5 we find $k = 4$.

Thus m_4 is the best candidate for the position. In case that m_4 can not join the position either m_3 or m_8 may be selected.

g	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8
m_1	6	2	5	3	4	6	3	1
m_2	4	6	6	4	5	6	5	5
m_3	3	0	6	2	1	4	3	1
m_4	2	2	4	6	3	4	5	1
m_5	4	2	5	3	6	6	5	2
m_6	2	0	2	2	2	6	2	0
m_7	3	2	4	2	1	4	6	2
m_8	5	2	6	4	3	6	5	6

Table 8.2: Comparison Table for g

	Row Sum: r_i	Column Sum: c_i	Positive Score: P_i
m_1	31	31	0
m_2	43	17	26
m_3	21	37	-16
m_4	32	26	6
m_5	35	29	6
m_6	17	43	-26
m_7	21	38	-17
m_8	40	19	21

Table 8.3: Positive Score

	Row Sum: r'_i	Column Sum: c'_i	Negative Score: N_i
m_1	30	29	1
m_2	41	16	25
m_3	20	38	-18
m_4	27	26	1
m_5	33	25	8
m_6	16	42	-26
m_7	24	34	-10
m_8	37	18	19

Table 8.4: Negative Score

	Final Score
m_1	-1
m_2	1
m_3	2
m_4	5
m_5	-2
m_6	0
m_7	-7
m_8	2

Table 8.5: Final Score

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On fuzzy bipolar soft sets, their algebraic structures and applications

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Abstract. We have defined fuzzy bipolar soft sets and basic operations of union, intersection and complementation for fuzzy bipolar soft sets. The algebraic properties of fuzzy bipolar soft sets are discussed. The concept of bipolar fuzzy soft set is also given and the equivalence of both structures is established. An application of fuzzy bipolar soft sets in decision making problems is presented with the help of an example.

Keywords: Bipolarity, bipolar fuzzy sets, soft sets, extended union, extended intersection, restricted union, restricted intersection

1. Introduction

While we talk about the modeling of real world problems which are ranging from engineering to medical and medical to social fields, we come across with the presence of uncertainty in data. L.A. Zadeh [21] was the first one to introduce the theory of fuzzy sets that yielded a whole field of fuzzy mathematics. The nature of data is an important factor in the process of developing mathematical models in various fields like engineering, life sciences, pattern recognition, neural networks, artificial intelligence, behavioral and social sciences. There are also some other factors which may affect our considerations related to the nature of data and an obvious one is the bipolarity of data. It is evidently observed that every information about a particular phenomenon has two aspects i.e. presence of a property or its absence [5]. There are models that are developed through the tools (e.g. bipolar fuzzy sets [8, 9]) in which a positive measure has been used to approximate the presence of a particular attribute and a negative measure is used to approximate the degree of absence of that same attribute. There is always a possibility of gray areas where we get uncertain to decide whether a phenomenon possesses a property or not. Some other theories which are capable of handling these kinds of situations include intuitionistic fuzzy sets, interval valued fuzzy sets, vague sets etc [4, 7].

Theory of soft sets was introduced by Molodstov in 1999 [15]. The purpose of the novel concept was to remove the inadequacy of parameterization tool in previously defined theories of fuzzy Mathematics. Although the theory of rough sets [10, 16] addresses the issue of parameterization and the hybrid structure such as fuzzy rough sets can also be utilized for incorporating the fuzziness of data but the addition of any further factor such as bipolarity of information makes it too complicated to use. On the other hand, the absence of any restrictions while making approximations for a given object in soft sets establishes this theory as more handy, convenient and easily applicable in practice. Since the introduction of the theory of soft sets in 1999, a lot of work has been done so far. We can find the studies on structure as well as on the applications of soft sets in various fields [1–3, 6, 11–14, 17–20].

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In this paper, we have initiated a concept of fuzzy bipolar soft sets. The idea is generated with the motivation of bipolarity of parameters and then the fuzziness of data comes into play. We have considered a set of parameters and its negative set i.e. the absence of these parameters and denote this set by "not set", for each parameter$e$, not $e = \neg e$ is the absence of e. A fuzzy bipolar soft set is defined with the help of two mappings, one for approximating the degree of fuzziness of the positivity or presence of a certain parameter in the objects of initial universal set and the other one is to approximate the relative degree of fuzziness of the negativity or absence of the same parameter. In this way, we have combined these three concepts of bipolarity, fuzziness and parameterization and thus it is shown through examples that we have found a very easy to use way of modeling the phenomena where all these three factors are involved. To move further, we have defined the basic algebra for the fuzzy bipolar soft sets and discussed their algebraic properties in detail. It is also shown that the collection of fuzzy bipolar soft sets forms a stone algebra. At the end, an application of fuzzy bipolar soft sets in the decision making problems is presented along with the algorithm.

2. Preliminaries

Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice with least element 0 and maximum element 1. An involution μ on L is a mapping $\mu : L \to L$ such that $\mu(\mu(x)) = x$, $\mu(0) = 1$ and $\mu(1) = 0$. A bounded lattice is called distributive if the distributive laws hold with respect to $\vee$ and $\wedge$. If De Morgan's laws hold for a bounded distributive lattice having an involution μ, then it is called De Morgan algebra. Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice and $x \in L$, then an element x^* is called a pseudo complement of x, if $x \wedge x^* = 0$ and $y \leq x^*$ whenever $x \wedge y = 0$. If every element has a pseudo complement then L is pseudo complemented. The equation $x^* \vee x^{**} = 1$ is called Stone's identity. A Stone algebra is a pseudo complemented distributive lattice satisfying Stone's identity.

Now we define fuzzy sets. Let X be a given set.

Definition 1. [21] A fuzzy subset of X is a function from X into the unit closed interval [0, 1]. The set of all fuzzy subsets of X is called the fuzzy power set of X, and is denoted by $FP(X)$.

Definition 2. [21] Let $\mu, \nu \in FP(X)$. If $\mu(x) \leq \nu(x)$ for all $x \in X$, then μ is said to be contained in ν, and we write $\mu \subseteq \nu($ or $\nu \supseteq \mu)$.

Clearly, the inclusion relation $\subseteq$ is a partial order on $FP(X)$.

Definition 3. [21] Let $\mu, \nu \in FP(X)$. Then $\mu \vee \nu$ and $\mu \wedge \nu$ are fuzzy subsets of X, defined as follows:

For all $x \in X$,

$$(\mu \vee \nu)\ (x) = \mu\,(x) \vee \nu\,(x)\ ,$$

$$(\mu \wedge \nu)\ (x) = \mu\,(x) \wedge \nu\,(x)\ .$$

The fuzzy subsets $\mu \vee \nu$ and $\mu \wedge \nu$ are called the union and intersection of μ and ν, respectively.

Definition 4. [21] Two fuzzy subsets of X are denoted by $\emptyset$ and X which map every element of onto 0 and 1 respectively. We call $\emptyset$ as the empty set or null fuzzy subset and X as the whole fuzzy subset of X.

Definition 5. [8] A bipolar fuzzy set μ in X is defined as:

$$\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) :\ x \in X\right\}$$

where $\mu^P : X \to [0, 1]$ and $\mu^N : X \to [-1, 0]$ are mappings. The positive membership degree $\mu^P(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy set

$$\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) :\ x \in X\right\}$$

and the negative membership degree $\mu^N(x)$ denotes the satisfaction degree of x to some implicit counter-property of

$$\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) :\ x \in X\right\}.$$

if $\mu^P(x) \neq 0$ and $\mu^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for

$$\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) :\ x \in X\right\}.$$

if $\mu^P(x) = 0$ and $\mu^N(x) \neq 0$, it is the situation that x does not satisfy the property of

$$\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) :\ x \in X\right\},$$

but somewhat satisfies the counter-property of

$$\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) :\ x \in X\right\}.$$

it is possible for an element x to be $\mu^P(x) \neq 0$ and $\mu^N(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain. For the sake of simplicity, we shall write $\mu = \left(\mu^P,\ \mu^N\right)$ for the bipolar fuzzy set

$$\mu = \left\{\left(x,\ \mu^P(x),\ \mu^N(x)\right)\ :\ x \in X\right\}$$

3. Fuzzy bipolar soft sets

Let U be an initial universe and E be a set of parameters. Let $FP(X)$ denotes the collection of all fuzzy subsets of U and A, B, C are non-empty subsets of E. Now, we define

Definition 6. A triplet (F, G, A) is called a fuzzy bipolar soft set over U, where F and G are mappings, given by $F : A \to FP(U)$ and $G : \neg A \to FP(U)$ such that

$$0 \leq (F(e))(x) + (G(\neg e))(x) \leq 1$$

for all $e \in A$.

In other words, a fuzzy bipolar soft set over U gives two parameterized families of subsets of the universe U and the condition

$$0 \leq (F(e))(x) + (G(\neg e))(x) \leq 1$$

for all $e \in A$, is imposed as a consistency constraint. For each $e \in A$, $F(e)$ and $G(\neg e)$ are regarded as the set of e -approximate elements of the fuzzy bipolar soft set (F, G, A).

Definition 7. For a fuzzy bipolar soft set (F, G, A) over U, we define a fuzzy soft set $(H_{(F,G)}, A)$ over U for the approximation of the degree of hesitation in (F, G, A) as follows:

$H_{(F,G)} \;:\; A \to FP(U)$ defined by

$$(H_{(F,G)}(e))(x) = 1 - (F(e))(x) - (G(\neg e))(x)$$

for all $x \in U, e \in A$. Clearly, $(H_{(F,G)}, A)$ approximates the lack of knowledge about the objects of U while considering the presence or absence of a particular parameter of A.

Definition 8. For two fuzzy bipolar soft sets (F, G, A) and (F_1, G_1, A) over a universe U, we say that (F, G, A) is a fuzzy bipolar soft subset of (F_1, G_1, A), if,

1. $A \subseteq B$ and
   $F(e) \subseteq F_1(e)$ and $G_1(\neg e) \subseteq G(\neg e)$ for all $e \in A$.

This relationship is denoted by $(F, G, A) \tilde{\subseteq} (F_1, G_1, A)$. Similarly (F, G, A) is said to be a fuzzy bipolar soft superset of (F_1, G_1, A), if (F_1, G_1, A) is a fuzzy bipolar soft subset of (F, G, A). We denote it by $(F, G, A)\tilde{\supseteq}(F_1, G_1, A)$.

Definition 9. Two fuzzy bipolar soft sets (F, G, A) and (F_1, G_1, A) over a universe U are said to be equal if (F, G, A) is a fuzzy bipolar soft subset of (F_1, G_1, A) and (F_1, G_1, A) is a fuzzy bipolar soft subset of (F, G, A).

Definition 10. The complement of a fuzzy bipolar soft set (F, G, A) is denoted by $(F, G, A)^c$ and defined by $(F, G, A)^c = (F^c, G^c, A)$ where F^c and G^c are mappings given by $F^c(e) = G(\neg e)$ and $G^c(\neg e) = F(e)$ for all $e \in A$.

Definition 11. A fuzzy bipolar soft set over U is said to be a relative null fuzzy bipolar soft set, denoted by (Φ, U, A) if for all $e \in A$, $\Phi(e) = \emptyset$ and $U(\neg e) = U$, for all $e \in A$.

Definition 12. A fuzzy bipolar soft set over U is said to be a relative absolute fuzzy bipolar soft set, denoted by (Φ, U, A), if for all $e \in A$, $U(e) = U$ and $\Phi(\neg e) = \emptyset$, for all $e \in A$.

Definition 13. If (F, G, A) and (F_1, G_1, B) are two fuzzy bipolar soft sets over U then "(F, G, A) and (F_1, G_1, B)" denoted by $(F, G, A) \wedge (F_1, G_1, B)$ is defined by $(F, G, A) \wedge (F_1, G_1, B) = (H, I, A \times B)$ where $H(a, b) = F(a) \wedge F_1(b)$ and $I(\neg a, \neg b) = G(\neg a) \vee G_1(\neg b)$ for all $(a, b) \in A \times B$.

Definition 14. If (F, G, A) and (F_1, G_1, B) are two fuzzy bipolar soft sets over U then "(F, G, A) or (F_1, G_1, B)" denoted by $(F, G, A) \vee (F_1, G_1, B)$ is defined by $(F, G, A) \vee (F_1, G_1, B) = (H, I, A \times B)$ where $H(a, b) = F(a) \vee F_1(b)$ and $I(\neg a, \neg b) = G(\neg a) \wedge G_1(\neg b)$ for all $(a, b) \in A \times B$.

Proposition 1. *If (F, G, A) and (F_1, G_1, B) are two fuzzy bipolar soft sets over U then*

1. $((F, G, A) \vee (F_1, G_1, B))^c = (F, G, A)^c \wedge (F_1, G_1, B)^c$
2. $((F, G, A) \wedge (F_1, G_1, B))^c = (F, G, A)^c \vee (F_1, G_1, B)^c$

Proof. Straightforward.

Definition 15. Extended Union of two fuzzy bipolar soft sets (F, G, A) and (F_1, G_1, B) over the common universe U is the fuzzy bipolar soft set (H, I, C) over U where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & if\ e \in A - B \\ F_1(e) & if\ e \in B - A \\ F(e) \vee F_1(e) & if\ e \in A \cap B \end{cases}$$

$$I(\neg e) = \begin{cases} G(\neg e) & if\ e \in (\neg A) - (\neg B) \\ G_1(\neg e) & if\ e \in (\neg B) - (\neg A) \\ G(\neg e) \wedge G_1(\neg e) & if\ e \in (\neg A) \cap (\neg B) \end{cases}$$

we denote it by $(F,\ G,\ A)\,\tilde{\cup}\,(F_1,\ G_1,\ B) = (H,\ I,\ C)$.

Definition 16. Extended Intersection of two fuzzy bipolar soft sets $(F,\ G,\ A)$ and $(F_1,\ G_1,\ B)$ over the common universe U is the fuzzy bipolar soft set $(H,\ I,\ C)$ over U where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & if\ e \in A - B \\ F_1(e) & if\ e \in B - A \\ F(e) \wedge F_1(e) & if\ e \in A \cap B \end{cases}$$

$$I(\neg e) = \begin{cases} G(\neg e) & if\ e \in (\neg A) - (\neg B) \\ G_1(\neg e) & if\ e \in (\neg B) - (\neg A) \\ G(\neg e) \vee G_1(\neg e) & if\ e \in (\neg A) \cap (\neg B) \end{cases}$$

we denote it by $(F,\ G,\ A)\,\tilde{\cap}\,(F_1,\ G_1,\ B) = (H,\ I,\ C)$.

Definition 17. Restricted Union of two fuzzy bipolar soft sets $(F,\ G,\ A)$ and $(F_1,\ G_1,\ B)$ over the common universe U is the fuzzy bipolar soft set $(H,\ I,\ C)$, where $C = A \cap B$ is non-empty and for all $e \in C$

$H(e) = F(e) \vee G(e)$ and $I(\neg e) = F_1(\neg e) \wedge G_1(\neg e)$. We denote it by $(F,\ G,\ A) \cup_R (F_1,\ G_1,\ B) = (H,\ I,\ C)$.

Definition 18. Restricted Intersection of two fuzzy bipolar soft sets $(F,\ G,\ A)$ and $(F_1,\ G_1,\ B)$ over the common universe U is the fuzzy bipolar soft set $(H,\ I,\ C)$, where $C = A \cap B$ is non-empty and for all $e \in C$:

$H(e) = F(e) \wedge G(e)$ and $I(\neg e) = F_1(\neg e) \vee G_1(\neg e)$. We denote it by $(F,\ G,\ A) \cap_R (F_1,\ G_1,\ B) = (H,\ I,\ C)$.

Conventionally we assume that $(F, G, A) \cap_R (F_1,\ G_1,\ B) = (\Phi,\ U, \emptyset) = (F,\ G,\ A) \cup_R (F_1,\ G_1,\ B)$ whenever $A \cap B = \emptyset$.

Lemma 1. *Let (F, G, A), (F_1, G_1, B) and (F_2, G_2, C) be any fuzzy bipolar soft sets over a common universe U. Then the following are true:*

1. $(F,\ G,\ A)\alpha((F_1,\ G_1,\ B)\alpha(F_2,\ G_2,\ C)) = ((F,\ G,\ A)\alpha(F_1,\ G_1,\ B))\alpha(F_2,\ G_2,\ C)$
2. $(F,\ G,\ A)\alpha(F_1,\ G_1,\ B) = (F,\ G,\ A)\alpha(F_1, G_1, B)$ *for all* $\alpha \in \{\tilde{\cap}, \cap_R,\ \tilde{\cup}, \cup_R\}$.

Proof. Straightforward.

Lemma 2. *If $(\Phi,\ U,\ A)$ is a null fuzzy bipolar soft set $(U,\ \Phi,\ A)$ an absolute fuzzy bipolar soft set, and $(F,\ G,\ A)$, $(F_1,\ G_1,\ A)$ are fuzzy bipolar soft sets over U. Then*

1. $(F,\ G,\ A)\,\tilde{\cup}\,(F_1,\ G_1,\ A) = (F,\ G,\ A) \cup_R (F_1,\ G_1,\ A)$,
2. $(F,\ G,\ A)\,\tilde{\cap}\,(F_1,\ G_1,\ A) = (F,\ G,\ A) \cap_R (F_1,\ G_1,\ A)$,
3. $(F,\ G,\ A)\,\tilde{\cup}\,(F,\ G,\ A) = (F,\ G,\ A) \cup_R (F,\ G,\ A) = (F,\ G,\ A)$,
4. $(F,\ G,\ A)\,\tilde{\cap}\,(F,\ G,\ A) = (F,\ G,\ A) \cap_R (F,\ G,\ A) = (F,\ G,\ A)$,
5. $(F,\ G,\ A)\,\tilde{\cup}\,(\Phi,\ U,\ A) = (F,\ G,\ A) \cup_R (F,\ G,\ A) = (F,\ G,\ A)$,
6. $(F,\ G,\ A)\,\tilde{\cap}\,(U,\ \Phi,\ A) = (F,\ G,\ A) \cap_R (U,\ \Phi,\ A) = (F,\ G,\ A)$.

Proof. Straightforward.

Lemma 3. *Let $(F,\ G,\ A)$ and $(F_1,\ G_1,\ B)$ be two fuzzy bipolar soft sets over a common universe U. Then the following are true:*

1. *$(F,\ G,\ A)\,\tilde{\cup}\,(F_1,\ G_1,\ B)$ is the smallest fuzzy bipolar soft set over U which contains both $(F,\ G,\ A)$ and $(F_1,\ G_1,\ B)$.*
2. *$(F,\ G,\ A) \cap_R (F_1,\ G_1,\ B)$ is the largest fuzzy bipolar soft set over U which is contained in both $(F,\ G,\ A)$ and $(F_1,\ G_1,\ B)$.*

Proof. Straightforward.

Lemma 4. Let $(F,\ G,\ A)$ and $(F_1,\ G_1,\ B)$ be two fuzzy bipolar soft sets over a common universe U. Then

1. $((F,\ G,\ A)\,\tilde{\cup}\,(F_1,\ G_1,\ B))^c = (F,\ G,\ A)^c\tilde{\cap}(F_1,\ G_1,\ B)^c$,
2. $((F,\ G,\ A)\,\tilde{\cap}\,(F_1,\ G_1,\ B))^c = (F,\ G,\ A)^c\tilde{\cup}(F_1,\ G_1,\ B)^c$,
3. $((F,\ G,\ A) \cup_R (F_1,\ G_1,\ B))^c = (F,\ G,\ A)^c \cap_R (F_1,\ G_1,\ B)^c$,
4. $((F,\ G,\ A) \cap_R (F_1,\ G_1,\ B))^c = (F,\ G,\ A)^c \cup_R (F_1,\ G_1,\ B)^c$.

Proof. Straightforward.

Lemma 5. *Let (F, G, A), (F_1, G_1, B) and (F_2, G_2, C) be any fuzzy bipolar soft sets over a common universe U. Then*

1. $(F, G, A)\alpha((F_1, G_1, B)\beta(F_2, G_2, C)) = ((F, G, A)\alpha(F_1, G_1, B))\beta((F, G, A)\alpha(F_2, G_2, C))$ where $\alpha \neq \beta$, $\alpha \in \{\cap_R, \cup_R\}$ and $\beta \in \{\cap_R, \cup_R, \tilde{\cup}, \tilde{\cap}\}$
2. $(F, G, A)\tilde{\cup}((F_1, G_1, B)\tilde{\cap}(F_2, G_2, C)) \tilde{\supset}((F, G, A)\tilde{\cup}(F_1, G_1, B))\tilde{\cap}((F, G, A)\tilde{\cup}(F_2, G_2, C))$
3. $(F, G, A)\tilde{\cup}((F_1, G_1, B)\cup_R(F_2, G_2, C)) \tilde{\subset}((F, G, A)\tilde{\cup}(F_1, G_1, B))\cup_R((F, G, A)\tilde{\cup}(F_2, G_2, C))$
4. $(F, G, A)\tilde{\cup}((F_1, G_1, B)\cap_R(F_2, G_2, C)) = ((F, G, A)\tilde{\cup}(F_1, G_1, B))\cap_R((F, G, A)\tilde{\cup}(F_2, G_2, C))$
5. $(F, G, A)\tilde{\cap}((F_1, G_1, B)\tilde{\cup}(F_2, G_2, C)) \tilde{\subset}((F, G, A)\tilde{\cap}(F_1, G_1, B))\tilde{\cup}((F, G, A)\tilde{\cap}(F_2, G_2, C))$
6. $(F, G, A)\tilde{\cap}((F_1, G_1, B)\cup_R(F_2, G_2, C)) = ((F, G, A)\tilde{\cap}(F_1, G_1, B))\cup_R((F, G, A)\tilde{\cap}(F_2, G_2, C))$
7. $(F, G, A)\tilde{\cap}((F_1, G_1, B)\cap_R(F_2, G_2, C))\tilde{\supset}((F, G, A)\tilde{\cap}(F_1, G_1, B))\cap_R((F, G, A)\tilde{\cap}(F_2, G_2, C))$.

Proof.

1) For any $e \in A \cap (B \cup C)$, we have following three disjoint cases:

(i) If $e \in A \cap (B - C)$, then

$$(F \cap_R (F_1 \tilde{\cup} F_2))(e) = F(e) \wedge F_1(e)$$
$$(G \cap_R (G_1 \tilde{\cup} G_2))(\neg e) = G(\neg e) \vee G_1(\neg e)$$

and

$$((F \cap_R F_1)\tilde{\cup}(F \cap_R F_2))(e) = (F \cap_R F_1)(e) \vee \emptyset = F(e) \wedge F_1(e)$$
$$((G \cap_R G_1)\tilde{\cup}(G \cap_R G_2))(\neg e) = (G \cap_R G_1)(\neg e) \wedge U = G(\neg e) \vee G_1(\neg e).$$

(ii) *If* $e \in A \cap (C - B)$, *then*

$$(F \cap_R (F_1 \tilde{\cup} F_2))(e) = F(e) \wedge F_2(e)$$
$$(G \cap_R (G_1 \tilde{\cup} G_2))(\neg e) = G(\neg e) \vee G_2(\neg e)$$

and

$$((F \cap_R F_1)\tilde{\cup}(F \cap_R F_2))(e) = \emptyset \vee (F \cap_R F_2)(e) = F(e) \wedge F_2(e)$$
$$((G \cap_R G_1)\tilde{\cup}(G \cap_R G_2))(\neg e) = U \wedge (G \cap_R G_2)(\neg e) = G(\neg e) \vee G_2(\neg e).$$

(iii) *If* $e \in A \cap (B \cap C)$, *then*

$$(F \cap_R (F_1 \tilde{\cup} F_2))(e) = F(e) \wedge (F_1(e) \vee F_2(e))$$
$$(G \cap_R (G_1 \tilde{\cup} G_2))(\neg e) = G(\neg e) \vee (G_1(\neg e) \wedge G_2(\neg e))$$

and

$$((F \cap_R F_1)\tilde{\cup}(F \cap_R F_2))(e) = (F \cap_R F_1)(e) \vee (F \cap_R F_2)(e) = (F(e) \wedge F_1(e)) \vee (F(e) \wedge F_2(e)) = F(e) \wedge (F_1(e) \vee F_2(e))$$
$$((G \cap_R G_1)\tilde{\cup}(G \cap_R G_2))(\neg e) = (G \cap_R G_1)(\neg e) \wedge (G \cap_R G_2)(\neg e) = (G(\neg e) \vee G_1(\neg e)) \wedge (G(\neg e) \vee G_2(\neg e)) = G(\neg e) \vee (G_1(\neg e) \wedge G_2(\neg e)).$$

thus

$$(F, G, A) \cap_R ((F_1, G_1, B)\tilde{\cup}(F_2, G_2, C)) = ((F, G, A) \cap_R (F_1, G_1, B))\tilde{\cup}((F, G, A) \cap_R (F_2, G_2, C))$$

Similarly, we can check for the remaining parts.

Example 1. Let U be the set of houses under consideration, and E be the set of parameters, $U = \{h_1, h_2, h_3, h_4, h_5\}$ $E = \{e_1, e_2, e_3, e_4, e_5\} =$ {in the green surroundings, cheap, in good repair, furnished, traditional}. Let $\neg E = \{\neg e_1, \neg e_2, \neg e_3, \neg e_4, \neg e_5\} =$ { in the commercial area, expensive, in bad repair, non-furnished, modern}.

Suppose that $A = \{e_1, e_2, e_3\}$, $B = \{e_2, e_3, e_4\}$ and $C = \{e_3, e_4, e_5\}$. The fuzzy bipolar soft sets (F, G, A), (F_1, G_1, B) and (F_2, G_2, C) describe the requirements of the houses which Mr. X, Mr. Y and Mr. Z are going to buy respectively.

suppose that

$$F(e_1) = \{h_1/0.3, h_2/0.1, h_3/0.3, h_4/0.1, h_5/0.7\},$$
$$F(e_2) = \{h_1/0.1, h_2/0.9, h_3/0.3, h_4/0.8, h_5/0.2\},$$
$$F(e_3) = \{h_1/0.1, h_2/0.3, h_3/0.3, h_4/0.3, h_5/0.8\},$$
$$G(\neg e_1) = \{h_1/0.4, h_2/0.7, h_3/0.7, h_4/0.7, h_5/0.1\},$$
$$G(\neg e_2) = \{h_1/0.8, h_2/0, h_3/0.5, h_4/0.1, h_5/0.6\},$$
$$G(\neg e_3) = \{h_1/0.7, h_2/0.5, h_3/0.7, h_4/0.6, h_5/0.1\},$$

and

$$F_1(e_2) = \{h_1/0.1, h_2/0.3, h_3/0.6, h_4/0.2, h_5/0.3\},$$
$$F_1(e_3) = \{h_1/0.8, h_2/0.9, h_3/0.5, h_4/0.4, h_5/0.2\},$$
$$F_1(e_4) = \{h_1/0.1, h_2/0.4, h_3/0.3, h_4/0.6, h_5/0.9\},$$

$G_1(\neg e_2) = \{h_1/0.1,\ h_2/0.3,\ h_3/0.3,\ h_4/0.6,\ h_5/0.6\}$,
$G_1(\neg e_3) = \{h_1/0.1,\ h_2/0,\ h_3/0.3,\ h_4/0.4,\ h_5/0.6\}$,
$G_1(\neg e_4) = \{h_1/0.9,\ h_2/0.5,\ h_3/0.5,\ h_4/0.3,\ h_5/0.1\}$.

and

$F_2(e_3)=\{h_1/0.1,\ h_2/0.2,\ h_3/0.3,\ h_4/0.1,\ h_5/0.1\}$,
$F_2(e_4)=\{h_1/0.2,\ h_2/0.2,\ h_3/0.3,\ h_4/0.3,\ h_5/0.2\}$,
$F_2(e_5)=\{h_1/0.1,\ h_2/0.1,\ h_3/0.3,\ h_4/0.5,\ h_5/0.7\}$,
$G_2(\neg e_3)=\{h_1/0.7,\ h_2/0.7,\ h_3/0.4,\ h_4/0.7,\ h_5/0.4\}$,
$G_2(\neg e_4)=\{h_1/0.6,\ h_2/0.5,\ h_3/0.6,\ h_4/0.1,\ h_5/0.6\}$,
$G_2(\neg e_5)=\{h_1/0.3,\ h_2/0.4,\ h_3/0.4,\ h_4/0.3,\ h_5/0.1\}$.

let

$$(F,\ G,\ A)\tilde{\cup}((F_1,\ G_1,\ B)\tilde{\cap}(F_2,\ G_2,\ C)) = (H_1,\ I_1,\ A\cup B\cup C)$$

and

$$((F,\ G,\ A)\tilde{\cup}(F_1,\ G_1,\ B))\tilde{\cap}((F,\ G,\ A)\tilde{\cup}(F_2,\ G_2,\ C)) = (H_2,\ I_2,\ A\cup B\cup C).$$

then

$H_1(e_1) = \{h_1/0.3,\ h_2/0.1,\ h_3/0.3,\ h_4/0.1,\ h_5/0.7\}$,
$H_1(e_2) = \{h_1/0.1,\ h_2/0.9,\ h_3/0.6,\ h_4/0.8,\ h_5/0.3\}$,
$H_1(e_3) = \{h_1/0.1,\ h_2/0.3,\ h_3/0.3,\ h_4/0.3,\ h_5/0.8\}$,
$H_1(e_4) = \{h_1/0.1,\ h_2/0.2,\ h_3/0.3,\ h_4/0.3,\ h_5/0.2\}$,
$H_1(e_5) = \{h_1/0.1,\ h_2/0.1,\ h_3/0.3,\ h_4/0.5,\ h_5/0.7\}$,

and

$I_1(\neg e_1) = \{h_1/0.4,\ h_2/0.7,\ h_3/0.7,\ h_4/0.7,\ h_5/0.1\}$,
$I_1(\neg e_2) = \{h_1/0.1,\ h_2/0.0,\ h_3/0.3,\ h_4/0.1,\ h_5/0.6\}$,
$I_1(\neg e_3) = \{h_1/0.7,\ h_2/0.5,\ h_3/0.4,\ h_4/0.6,\ h_5/0.1\}$,
$I_1(\neg e_4) = \{h_1/0.9,\ h_2/0.5,\ h_3/0.6,\ h_4/0.3,\ h_5/0.6\}$,
$I_1(\neg e_5) = \{h_1/0.3,\ h_2/0.4,\ h_3/0.4,\ h_4/0.3,\ h_5/0.1\}$.

also

$H_2(e_1) = \{h_1/0.3,\ h_2/0.1,\ h_3/0.3,\ h_4/0.1,\ h_5/0.7\}$,
$H_2(e_2) = \{h_1/0.1,\ h_2/0.9,\ h_3/0.3,\ h_4/0.8,\ h_5/0.2\}$,
$H_2(e_3) = \{h_1/0.1,\ h_2/0.3,\ h_3/0.3,\ h_4/0.3,\ h_5/0.8\}$,
$H_2(e_4) = \{h_1/0.1,\ h_2/0.2,\ h_3/0.3,\ h_4/0.3,\ h_5/0.2\}$,
$H_2(e_5) = \{h_1/0.1,\ h_2/0.1,\ h_3/0.3,\ h_4/0.5,\ h_5/0.7\}$,

and

$I_2(\neg e_1) = \{h_1/0.4,\ h_2/0.7,\ h_3/0.7,\ h_4/0.7,\ h_5/0.1\}$,
$I_2(\neg e_2) = \{h_1/0.8,\ h_2/0.0,\ h_3/0.5,\ h_4/0.1,\ h_5/0.6\}$,
$I_2(\neg e_3) = \{h_1/0.7,\ h_2/0.5,\ h_3/0.4,\ h_4/0.6,\ h_5/0.1\}$,
$I_2(\neg e_4) = \{h_1/0.9,\ h_2/0.5,\ h_3/0.6,\ h_4/0.3,\ h_5/0.6\}$,
$I_2(\neg e_5) = \{h_1/0.3,\ h_2/0.4,\ h_3/0.4,\ h_4/0.3,\ h_5/0.1\}$.

Clearly $H_1(e_2) \neq H_2(e_2)$ and $I_1(\neg e_2) \neq I_2(\neg e_2)$, so that

$$(F,\ G,\ A)\tilde{\cup}((F_1,\ G_1,\ B)\tilde{\cap}(F_2,\ G_2,\ C)) \neq ((F,\ G,\ A)\tilde{\cup}(F_1,\ G_1,\ B))\tilde{\cap}((F,\ G,\ A)\tilde{\cup}(F_2,\ G_2,\ C)).$$

now, if we take

$$(F,\ G,\ A)\tilde{\cap}((F_1,\ G_1,\ B)\tilde{\cup}(F_2,\ G_2,\ C)) = (H_3,\ I_3,\ A\cup B\cup C)$$

and

$$((F, G, A)\tilde{\cap}(F_1, G_1, B))\tilde{\cup}((F, G, A)\tilde{\cap}(F_2, G_2, C)) = (H_4,\ I_4,\ A\cup B\cup C)$$

then

$H_3(e_1) = \{h_1/0.3,\ h_2/0.1,\ h_3/0.3,\ h_4/0.1,\ h_5/0.7\}$,
$H_3(e_2) = \{h_1/0.1,\ h_2/0.3,\ h_3/0.3,\ h_4/0.2,\ h_5/0.2\}$,
$H_3(e_3) = \{h_1/0.1,\ h_2/0.3,\ h_3/0.3,\ h_4/0.3,\ h_5/0.2\}$,
$H_3(e_4) = \{h_1/0.2,\ h_2/0.4,\ h_3/0.3,\ h_4/0.6,\ h_5/0.9\}$,
$H_3(e_5) = \{h_1/0.1,\ h_2/0.1,\ h_3/0.3,\ h_4/0.5,\ h_5/0.7\}$,

and

$I_3(\neg e_1) = \{h_1/0.4,\ h_2/0.7,\ h_3/0.7,\ h_4/0.7,\ h_5/0.1\}$,
$I_3(\neg e_2) = \{h_1/0.8,\ h_2/0.3,\ h_3/0.5,\ h_4/0.6,\ h_5/0.6\}$,
$I_3(\neg e_3) = \{h_1/0.7,\ h_2/0.5,\ h_3/0.7,\ h_4/0.6,\ h_5/0.4\}$,
$I_3(\neg e_4) = \{h_1/0.6,\ h_2/0.5,\ h_3/0.5,\ h_4/0.1,\ h_5/0.1\}$,
$I_3(\neg e_5) = \{h_1/0.3,\ h_2/0.4,\ h_3/0.4,\ h_4/0.3,\ h_5/0.1\}$.

also

$H_4(e_1) = \{h_1/0.3,\ h_2/0.1,\ h_3/0.3,\ h_4/0.1,\ h_5/0.7\}$,
$H_4(e_2) = \{h_1/0.1,\ h_2/0.9,\ h_3/0.3,\ h_4/0.8,\ h_5/0.2\}$,
$H_4(e_3) = \{h_1/0.1,\ h_2/0.3,\ h_3/0.3,\ h_4/0.3,\ h_5/0.2\}$,

$$H_4(e_4) = \{h_1/0.2,\ h_2/0.4,\ h_3/0.3,\ h_4/0.6,\ h_5/0.9\},$$
$$H_4(e_5) = \{h_1/0.1,\ h_2/0.1,\ h_3/0.3,\ h_4/0.5,\ h_5/0.7\},$$

and

$$I_4(\neg e_1) = \{h_1/0.4,\ h_2/0.7,\ h_3/0.7,\ h_4/0.7,\ h_5/0.1\},$$
$$I_4(\neg e_2) = \{h_1/0.8,\ h_2/0.0,\ h_3/0.5,\ h_4/0.1,\ h_5/0.6\},$$
$$I_4(\neg e_3) = \{h_1/0.7,\ h_2/0.5,\ h_3/0.7,\ h_4/0.6,\ h_5/0.4\},$$
$$I_4(\neg e_4) = \{h_1/0.6,\ h_2/0.5,\ h_3/0.5,\ h_4/0.1,\ h_5/0.1\},$$
$$I_4(\neg e_5) = \{h_1/0.3,\ h_2/0.4,\ h_3/0.4,\ h_4/0.3,\ h_5/0.1\}.$$

Clearly, $H_3(e_2) \neq H_4(e_2)$ and $I_3(\neg e_2) \neq I_4(\neg e_2)$, so that

$$(F, G, A)\,\tilde{\cap}\,((F_1, G_1, B)\,\tilde{\cup}\,(F_2, G_2, C)) \neq ((F, G, A)\,\tilde{\cap}\,(F_1, G_1, B))\,\tilde{\cup}\,((F, G, A)\,\tilde{\cap}\,(F_2, G_2, C)).$$

similarly we can show that

$$(F, G, A)\,\tilde{\cup}\,((F_1, G_1, B)\cup_R (F_2, G_2, C)) \neq ((F, G, A)\,\tilde{\cup}\,(F_1, G_1, B))\cup_R ((F, G, A)\,\tilde{\cup}\,(F_2, G_2, C))$$

and

$$(F, G, A)\,\tilde{\cap}\,((F_1, G_1, B)\cap_R (F_2, G_2, C)) \neq ((F, G, A)\,\tilde{\cap}\,(F_1, G_1, B))\cap_R ((F, G, A)\,\tilde{\cup}\,(F_2, G_2, C))$$

Now we consider the collection of all fuzzy bipolar soft sets over U and denote it by $FBSS(U)^E$ and let us denote its sub collection of all fuzzy bipolar soft sets over U with fixed set of parameters A by $FBSS(U)_A$. We note that this collection is partially ordered by inclusion. We conclude from above results that:

Proposition 2. *$(FBSS(U)^E, \tilde{\cap}, \cup_R)$ and $(FBSS(U)^E, \tilde{\cup}, \cap_R)$ are distributive lattices and $(FBSS(U)^E, \cup_R, \tilde{\cap})$ and $(FBSS(U)^E, \cap_R, \tilde{\cup})$ are their duals respectively.*

Proof. Follows from above results.

Proposition 3. *$(FBSS(U)^E, \cap_R, \tilde{\cup})$ is a bounded distributive lattice, with least element $(\Phi, U, \emptyset)$ and greatest element (U, Φ, E), while $(FBSS(U)^E, \tilde{\cup}, \cap_R, (U, \Phi, E), (\Phi, U, \emptyset))$ is its dual.*

Proof. Follows from above results.

Proposition 4. *$(FBSS(U)_A, \cap_R, \tilde{\cup}) = (FBSS(U)_A, \tilde{\cap}, \cup_R)$ is a bounded distributive lattice, with least element (Φ, U, A) and greatest element (U, Φ, A).*

Proof. Follows from above results.

Proposition 5. *Let (F, G, A) and (F_1, G_1, A) be two fuzzy bipolar soft sets over a common universe U. Then*

1. $((F, G, A)^c)^c = (F, G, A)$,
2. $(F, G, A)\tilde{\subseteq}(F_1, G_1, A)$ *implies* $(F_1, G_1, A)^c\tilde{\subseteq}(F, G, A)^c$.

Proof.

1. is straightforward.
2. If $(F, G, A)\tilde{\subseteq}(F_1, G_1, A)$ then

$F(e) \subseteq F_1(e)$ and $G_1(\neg e) \subseteq G(\neg e)$ for all $e \in A$ implies that $(G_1, F_1, A)\tilde{\subseteq}(G, F, A)$.
Hence $(F_1, G_1, A)^c\tilde{\subseteq}(F, G, A)^c$.

Proposition 6. *$(FBSS(U)_A, \cap_R, \cup_R, ^c, (U, \Phi, A), (\Phi, U, A))$ is a De Morgan algebra.*

Proof. Straightforward.

Definition 19. For a fuzzy bipolar soft set (F, G, A) over U, we define a fuzzy bipolar soft set over U, which is denoted by $(F, G, A)^*$ and given by $(F, G, A)^* = (F^*, G^*, A)$ where

$$(F^*(e))(u) = \begin{cases} 0 & \text{if } (F(e))(u) \neq 0 \\ 1 & \text{if } (F(e))(u) = 0 \end{cases}$$

and

$$(G^*(e))(\neg u) = \begin{cases} 1 & \text{if } (G(\neg e))(u) \neq 1 \\ 0 & \text{if } (G(\neg e))(u) = 1 \end{cases}$$

for all $u \in U$ and for all $e \in A$.

Theorem 1. *Let (F, G, A) be a fuzzy bipolar soft set over U, then the following are true:*

1. $(F, G, A) \cap_R (F, G, A)^* = (\Phi, U, A)$,
2. $(F_1, G_1, A)\tilde{\subseteq}(F, G, A)^*$ *whenever* $(F, G, A) \cap_R (F_1, G_1, A) = (\Phi, U, A)$,
3. $(F, G, A)^* \cup_R (F, G, A)^{**} = (U, \Phi, A)$.

Thus $(FBSS(U)_A, \cap_R, \cup_R, ^*, (U, \Phi, A), (\Phi, U, A))$ is a Stone algebra.

Proof.

(1) Consider $(F, G, A) \cap_R (F, G, A)^*$. For any $e \in A$

$$(F \cap_R F^*)(e) = F(e) \wedge F^*(e)$$

and

$$(G \cap_R G^*)(\neg e) = G(\neg e) \vee G^*(\neg e).$$

$$\Rightarrow$$

$$((F \cap_R F^*)(e))(u) = \begin{cases} (F(e))(u) \wedge 0 & \text{if } (F(e))(u) \neq 0 \\ 0 \wedge 1 & \text{if } (F(e))(u) = 0 \end{cases} = 0$$

and

$$((G \cap_R G^*))(\neg e)(u) = \begin{cases} (G(\neg e))(u) \vee 1 & \text{if } (G(\neg e))(u) \neq 1 \\ 1 \vee 0 & \text{if } (G(\neg e))(u) = 1 \end{cases} = 1$$

for all $u \in U$.

Thus $(F, G, A) \cap_R (F, G, A)^* = (\Phi, U, A)$.

(2) If $(F, G, A) \cap_R (F_1, G_1, A) = (\Phi, U, A)$, then $(F(e))(u) \wedge (F_1(e))(u) = 0$ *and* $(G(\neg e))(u) \vee (G_1(\neg e))(u) = 1$ *for all* $u \in U$ $e \in A$. *We have two cases here:*

(i) If (F(e))(u) = 0 then

$$(F^*(e))(u) = 1 \geq (F_1(e))(u) \text{ and}$$

(ii) If (F(e))(u) $\neq$ 0 then

$$(F_1(e))(u) = 0 \leq (F^*(e))(u).$$

Thus $(F_1(e))(u) \leq (F^*(e))(u)$ *for all* $u \in U$.

Again there are two cases:

(i) If (G(¬ e))(u) = 1 then

$$(G^*(\neg e))(u) = 0 \leq (G_1(\neg e))(u) \text{ and}$$

(ii) If (G(¬ e))(u) $\neq$ 1 then

$$(G_1(\neg e))(u) = 1 \geq (G^*(\neg e))(u).$$

So $(G^*(\neg e))(u) \leq (G_1(\neg e))(u)$ for all $u \in U$. This implies that

$$F_1(e) \subseteq F^*(e) \text{ and } G^*(\neg e) \subseteq G_1(\neg e)$$

for all $e \in A$.

Therefore, $(F_1, G_1, A) \tilde{\subseteq} (F, G, A)^*$.

(3) Consider $(F, G, A)^* \cup_R (F, G, A)^{**}$. For any $e \in A$

$$(F^* \cup_R F^{**})(e) = F^*(e) \vee F^{**}(e)$$

and

$$(G^* \cup_R G^{**})(\neg e) = G^*(\neg e) \wedge G^{**}(\neg e).$$

$$\Rightarrow$$

$$((F^*(e))(u) \vee (F^{**}(e))(u) = \begin{cases} 0 \vee 1 & \text{if } (F(e))(u) \neq 0 \\ 1 \vee 0 & \text{if } (F(e))(u) = 0 \end{cases} = 1$$

and

$$((G^*(e))(u) \wedge (G^{**}(e))(u) = \begin{cases} 1 \wedge 0 & \text{if } (G(\neg e))(u) \neq 1 \\ 0 \wedge 1 & \text{if } (G(\neg e))(u) = 1 \end{cases} = 0$$

for all $u \in U$.

Thus $(F, G, A)^* \cup_R (F, G, A)^{**} = (U, \Phi, A)$.

4. Application of fuzzy bipolar soft sets in a decision making problem

Decision making is an important factor of all scientific professions where experts apply their knowledge in that area to make decisions wisely. We apply the concept of *fuzzy bipolar soft sets* for modeling of a given problem and then we give an algorithm for the choice of optimal object based upon the available sets of information. Let U be the initial universe and E be a set of parameters. We shall adapt the following terminology afterwards:

Definition 20. Let (F, G, E) be a fuzzy bipolar soft set defined over U. A Comparison table for F is a square table in which the number of rows and number of columns are equal, rows and columns both are labeled by the object names $h_1, h_2, h_3, \ldots, h_n$ of the initial universe U, and the entries are t_{ij}, $i, j = 1, 2, ..., n$, given by

$t_{ij} =$ the number of parameters for which the membership value of h_i exceeds or equal to the membership value of h_j

Clearly, $0 \leq t_{ij} \leq k$, and $t_{ii} = k$, for all i, j where k is the number of parameters present in E. Thus t_{ij} indicates a numerical measure, which is an integer. A Comparison table for G is a square table in which the number of rows and number of columns are equal, rows and columns both are labeled by the object names $h_1, h_2, h_3, \ldots, h_n$ of the initial universe U, and the entries are s_{ij}, $i, j = 1, 2, \ldots, n$, given by

$s_{ij} =$ the number of parameters for which the membership value of h_i dominates or equal to the membership value of h_j

Clearly, $0 \leq s_{ij} \leq k$, and $s_{ii} = k$, for all i, j where k is the number of parameters present in E. Thus s_{ij} also indicates a numerical measure, which is an integer.

Definition 21. The positive row sum and column of an object h_i, denoted by r_i and c_i are calculated by using the formulae,

$$r_i = \sum_{j=1}^{n} t_{ij},\ c_j = \sum_{i=1}^{n} t_{ij},$$

The negative row sum and column sum of an object h_i, denoted by r'_i and c'_j are calculated by using the formulae,

$$r'_i = \sum_{j=1}^{n} s_{ij},\ c'_j = \sum_{i=1}^{n} s_{ij}.$$

Definition 22. The positive score P_i of object h_i will be given by:

$$P_i = r_i - c_i$$

while the negative score N_i will be given by:

$$N_i = r'_i - c'_i.$$

The final score S_i of object h_i will be given by:

$$S_i = P_i - N_i$$

for all $j = 1, 2, ..., n$.

We wish to find an object from the set of choice parameters A. We are now giving an algorithm for the choice of best object according to the specifications made by observer and recorded data with the help of a fuzzy bipolar soft set.

Algorithm. The algorithm for the selection of the best choice is given as:

(1) Input the fuzzy bipolar soft set (F, G, E).
(2) Input the set of choice parameters $P \subseteq E$ *and find the reduced fuzzy bipolar soft set* (F, G, P).
(3) Compute the comparison tables for functions F *and* G *respectively*
(4) Compute the positive and negative scores for each object.
(5) Compute the final score.
(6) Find k, *for which* $S_k = \max S_i$.
(7) Then h_k *is the optimal choice object. If* k *has more than one values, then any one of* h_k *'s can be chosen.*

Example 2. Let $U = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8\}$ be the set of candidates who have applied for a job position of Office Representative in Customer Care Centre of a company. Let $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$ = Hard Working, Optimism, Enthusiasm, Individualism, Imaginative, Flexibility, Decisiveness, Self-confidence, Politeness and $\neg E = \{\neg e_1, \neg e_2, \neg e_3, \neg e_4, \neg e_5, \neg e_6, \neg e_7, \neg e_8, \neg e_9\}$ = Negligent, Pessimism, Half-hearted, Dependence, Unimaginative, Rigidity, Indecisiveness, Shyness, Harshness. Here the gray area is obviously the moderate form of parameters. Let the fuzzy bipolar soft sets (F, G, E) describes the Personality Analysis of Candidates as:

F	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
m_1	0.5	0.6	0.8	0.7	0.5	0.4	0.3	0.6	0.8
m_2	0.7	0.7	0.8	0.6	0.8	0.9	0.8	0.7	0.5
m_3	0.6	0.8	0.4	0.1	0.6	0.5	0.4	0.5	0.6
m_4	0.7	0.8	0.6	0.7	0.5	0.4	0.6	0.5	0.6
m_5	0.5	0.4	0.5	0.6	0.7	0.7	0.8	0.6	0.7
m_6	0.5	0.4	0.5	0.6	0.3	0.3	0.2	0.4	0.4
m_7	0.4	0.2	0.4	0.6	0.7	0.6	0.5	0.3	0.2
m_8	0.8	0.7	0.8	0.9	0.6	0.5	0.4	0.6	0.7

And

G	$\neg e_1$	$\neg e_2$	$\neg e_3$	$\neg e_4$	$\neg e_5$	$\neg e_6$	$\neg e_7$	$\neg e_8$	$\neg e_9$
m_1	0.3	0.4	0.1	0.2	0.4	0.4	0.7	0.4	0.1
m_2	0.2	0.1	0.1	0.3	0.2	0.2	0.1	0.2	0.4
m_3	0.4	0.2	0.5	0.6	0.3	0.3	0.5	0.3	0.4
m_4	0.1	0.1	0.3	0.2	0.4	0.3	0.3	0.4	0.3
m_5	0.3	0.5	0.4	0.3	0.1	0.2	0.2	0.4	0.2
m_6	0.5	0.5	0.3	0.3	0.6	0.5	0.8	0.4	0.5
m_7	0.4	0.7	0.6	0.2	0.2	0.2	0.4	0.6	0.8
m_8	0.2	0.1	0.1	0.1	0.3	0.3	0.4	0.3	0.2

(1) Input the fuzzy bipolar soft set (F, G, E).
(2) Input the set of choice parameters $P = \{e_1, e_3, e_4, e_5, e_7, e_8\} \subseteq E$ and find the reduced fuzzy bipolar soft set (F, G, P) given as:

F	e_1	e_3	e_4	e_5	e_7	e_8
m_1	0.5	0.8	0.7	0.5	0.3	0.6
m_2	0.7	0.8	0.6	0.8	0.8	0.7
m_3	0.6	0.4	0.1	0.6	0.4	0.5
m_4	0.7	0.6	0.7	0.5	0.6	0.5
m_5	0.5	0.5	0.6	0.7	0.8	0.6
m_6	0.5	0.5	0.6	0.3	0.2	0.4
m_7	0.4	0.4	0.6	0.7	0.5	0.3
m_8	0.8	0.8	0.9	0.6	0.4	0.6

G	$\neg e_1$	$\neg e_3$	$\neg e_4$	$\neg e_5$	$\neg e_7$	$\neg e_8$
m_1	0.3	0.1	0.2	0.4	0.7	0.4
m_2	0.2	0.1	0.3	0.2	0.1	0.2
m_3	0.4	0.5	0.6	0.3	0.5	0.3
m_4	0.1	0.3	0.2	0.4	0.3	0.4
m_5	0.3	0.4	0.3	0.1	0.2	0.4
m_6	0.5	0.3	0.3	0.6	0.8	0.4
m_7	0.4	0.6	0.2	0.2	0.4	0.6
m_8	0.2	0.1	0.1	0.3	0.4	0.3

(3) Compute the comparison tables for functions F and G respectively.
(4) Compute the positive and negative scores for each object as given by Tables 3 and 4.
(5) Compute the final score given by Table 5.

From Table 5 we find $k = 5$.

Thus m_5 is the best candidate for the position. In case that m_5 can not join the position m_2 may be selected.

5. Bipolar fuzzy soft sets

Let U be an initial universe and E be a set of parameters. Let $BFP(U)$ denotes the set of all bipolar fuzzy sets of U and A, B, C be non-empty subsets of E.

Definition 23. A pair (F, A) is called a bipolar fuzzy soft set over U, where F is a mapping given by $F : A \to BFP(U)$.

Thus a bipolar fuzzy soft set over U gives a parameterized family of bipolar fuzzy subsets of the universe U. For any $e \in A$,

$F(e) = \{(x, \mu^P_{F(e)}, \mu^N_{F(e)}) : x \in U\}$ where $\mu^P_{F(e)} : U \to [0, 1]$ and $\mu^N_{F(e)} : U \to [-1, 0]$ are mappings.

Before proceeding to the further development of theory of bipolar fuzzy soft sets, we give the following interpretations:

Proposition 7. *Let (F, G, A) and (F_1, A) be the fuzzy bipolar and bipolar fuzzy soft sets defined over U respectively. Then (F, G, A) and (F_1, A) are equivalent.*

Table 1

F	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8
m_1	6	2	3	4	4	6	4	2
m_2	5	6	6	5	6	6	6	4
m_3	3	0	6	2	1	4	3	2
m_4	4	2	5	6	3	6	5	1
m_5	4	2	5	3	6	6	6	3
m_6	1	1	2	0	3	6	4	0
m_7	2	1	4	1	2	3	6	2
m_8	6	3	6	5	4	6	4	6

Table 2

G	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8
m_1	6	2	3	4	4	6	4	1
m_2	5	6	6	4	5	5	5	5
m_3	3	0	6	2	1	4	3	2
m_4	4	2	4	6	4	6	5	2
m_5	4	1	5	3	6	5	5	2
m_6	1	2	2	2	3	6	2	0
m_7	2	2	4	2	2	4	6	2
m_8	6	2	6	4	4	6	5	6

Table 3

	Row sum: r_i	Column sum: c_i	Positive score: P_i
m_1	31	31	0
m_2	44	17	27
m_3	21	37	−16
m_4	32	26	6
m_5	35	29	6
m_6	17	43	−26
m_7	21	38	−17
m_8	40	20	20

Table 4

	Row sum: r'_i	Column sum: c'_i	Negative score: N_i
m_1	30	32	2
m_2	41	17	24
m_3	21	36	−15
m_4	33	27	6
m_5	31	29	2
m_6	18	42	−24
m_7	25	35	−10
m_8	39	20	19

Table 5

	Final Score
m_1	−2
m_2	3
m_3	−1
m_4	0
m_5	4
m_6	−2
m_7	−7
m_8	1

Proof. Let (F, G, A) be a given fuzzy bipolar soft set defined over U. We define a bipolar fuzzy soft set (F_1, A) over U as:

$F_1(e) = \{(x, F(e), -G(\neg e)) : x \in U\}$ where $-G(\neg e)(x) = -(G(\neg e)(x))$ for all $e \in A$.

Conversely assume that we are given a bipolar fuzzy soft set (F_1, A) over U. We can define a fuzzy bipolar soft set (F, G, A) over U in the following manner:

$F(e) = \mu^P_{F_1(e)}$, $G(\neg e) = -\mu^N_{F_1(e)}$ for all $e \in A$.

Thus both definitions are equivalent and may be used interchangeably.

Consider the following example:

Example 3. Let $U = \{m_1, m_2, m_3, m_4, m_5\}$ be the set of candidates who have applied for a job position of Office Representative in Customer Care Centre of a company. Let $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$={Hard Working, Optimism, Enthusiasm, Individualism, Imaginative, Decisiveness, Self-confidence} and $\neg E = \{\neg e_1, \neg e_2, \neg e_3, \neg e_4, \neg e_5, \neg e_6, \neg e_7\}$={Negligent, Pessimism, Half-hearted, Dependence, Unimaginative, Indecisiveness, Shyness}. Here the gray area is obviously the moderate form of parameters. Let the fuzzy bipolar soft sets (F, G, E) describes the Personality Analysis of Candidates as:

$$F(e_1) = \{m_1/0.5,\ m_2/0.7,\ m_3/0.6,\ m_4/0.7\},$$
$$F(e_2) = \{m_1/0.6,\ m_2/0.7,\ m_3/0.8,\ m_4/0.8\},$$
$$F(e_3) = \{m_1/0.8,\ m_2/0.8,\ m_3/0.4,\ m_4/0.6\},$$
$$F(e_4) = \{m_1/0.7,\ m_2/0.6,\ m_3/0.1,\ m_4/0.7\},$$
$$F(e_5) = \{m_1/0.5,\ m_2/0.8,\ m_3/0.6,\ m_4/0.5\},$$
$$F(e_6) = \{m_1/0.4,\ m_2/0.9,\ m_3/0.5,\ m_4/0.4\},$$
$$F(e_7) = \{m_1/0.3,\ m_2/0.8,\ m_3/0.4,\ m_4/0.6\},$$

and

$$G(\neg e_1) = \{m_1/0.3,\ m_2/0.2,\ m_3/0.4,\ m_4/0.1\},$$
$$G(\neg e_2) = \{m_1/0.4,\ m_2/0.1,\ m_3/0.2,\ m_4/0.1\},$$
$$G(\neg e_3) = \{m_1/0,\ m_2/0.1,\ m_3/0.5,\ m_4/0.3\},$$
$$G(\neg e_4) = \{m_1/0.2,\ m_2/0.3,\ m_3/0.6,\ m_4/0.2\},$$
$$G(\neg e_5) = \{m_1/0.4,\ m_2/0.2,\ m_3/0.3,\ m_4/0.4\},$$
$$G(\neg e_6) = \{m_1/0.4,\ m_2/0.2,\ m_3/0.3,\ m_4/0.3\},$$
$$G(\neg e_7) = \{m_1/0.7,\ m_2/0.1,\ m_3/0.5,\ m_4/0.3\}.$$

Now let's see the corresponding bipolar fuzzy soft set:

$$F_1(e_1) = \{(m_1, 0.5, -0.3), (m_2, 0.7, -0.2), (m_3, 0.6, -0.4), (m_4, 0.7, -0.1)\},$$
$$F_1(e_2) = \{(m_1, 0.6, -0.4), (m_2, 0.7, -0.1), (m_3, 0.8, -0.2), (m_4, 0.8, -0.1)\},$$
$$F_1(e_3) = \{(m_1, 0.8, -0), (m_2, 0.8, -0.1), (m_3, 0.4, -0.5), (m_4, 0.6, -0.3)\},$$
$$F_1(e_4) = \{(m_1, 0.7, -0.2), (m_2, 0.6, -0.3), (m_3, 0.1, -0.6), (m_4, 0.7, -0.2)\},$$
$$F_1(e_5) = \{(m_1, 0.5, -0.4), (m_2, 0.8, -0.2), (m_3, 0.6, -0.3), (m_4, 0.5, -0.4)\},$$
$$F_1(e_6) = \{(m_1, 0.4, -0.4), (m_2, 0.9, -0.2), (m_3, 0.5, -0.3), (m_4, 0.4, -0.3)\},$$
$$F_1(e_7) = \{(m_1, 0.3, -0.7), (m_2, 0.8, -0.1), (m_3, 0.4, -0.5), (m_4, 0.6, -0.3)\}.$$

It is clear that fuzzy bipolar soft set depicts the information in a better and comprehensive way than bipolar fuzzy soft set. For example, if we read the data of candidate m_1 with fuzzy bipolar soft set (F, G, E) then he is having 0.6 fuzzy value for optimism and 0.4 fuzzy value for pessimism and if we use the bipolar fuzzy soft set (F_1, E) then m_1 is having 0.6 fuzzy value for optimism and -0.4 shows the degree where m_1 is lacking optimism.

6. Conclusion

Our approach in this paper combines the bipolarity, fuzziness and parameterization for defining the fuzzy bipolar soft sets. The idea of fuzzy bipolarity of soft sets has been given. We have also given the definition of bipolar fuzzy soft sets in which the parameterization is done through a single mapping from the set of parameters to the collection of all bipolar fuzzy sets of initial universal set. We have shown through a formation that the two ideas actually coincide with each other and the fuzzy bipolar soft set is similar in working as bipolar fuzzy soft set. Both definitions are equivalent but it is easier and straightforward to model the phenomenon using fuzzy bipolar soft sets because it is a more logical and suitable approach according to the nature of the modeling problems. Future research may be done to explore further aspects of this newly defined structure. Modeling of supported physical phenomenon is our next goal. Another prospective direction is to study the topological structure and similarity measures of fuzzy bipolar soft sets in order to explore for a solid foundation of the research work and development of working methodologies.

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Some Studies on Algebraic Structures of Soft Sets

by Munazza Naz

Chapter 1

Preliminaries

In this chapter, theory of classical sets and theory of fuzzy sets are discussed. Various operations, their laws and properties of classical and fuzzy sets are given. The classical sets, we are going to consider, are defined by means of the crisp or definite boundaries. The concept of a set is fundamental in Mathematics and intuitively can be described as a collection of objects possibly linked through some properties. A classical set A has clear boundaries, i.e. $x \in A$ or $x \notin A$ exclude any other possibility. This implies that there is a certainty or definiteness involved in the approximation of these sets. A fuzzy set, on the other hand, is defined by its uncertain or vague properties. A fuzzy set is a class with a continuum of membership grades. So a fuzzy set A in a referential (universe of discourse) X is characterized by a membership function μ_A which associates with each element $x \in X$ a real number $\mu_A(x) \in [0, 1]$, having the interpretation $\mu_A(x)$ is the membership grade of x in the fuzzy set A. The crisp sets are sets without any ambiguity in their membership whereas fuzzy set theory is an efficient theory in dealing with the concepts of vagueness. As an extension of fuzzy sets, Lee [26] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. Bipolar-valued fuzzy sets have membership degrees that represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter property. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $(0, 1]$ indicate that elements somewhat satisfy the property, and the membership degrees on $[-1, 0)$ indicate that elements somewhat satisfy the implicit counter-property. Basic notions of bipolar fuzzy sets given after reviewing the ideas of the crisp sets and fuzzy sets.

1.1 Crisp Sets

In this section, we recall the standard definitions and main results on algebraic structure of classical crisp set theory in detail. Following definitions are taken from [7].

1.1.1 Definition

Let X be a set. An *order* $\leq$ on X is a reflexive, antisymmetric, and transitive binary relation, that is, for all $x, y, z \in X$,

1) $x \leq x$,

2) $x \leq y$ and $y \leq x$ imply $x = y$, and

3) $x \leq y$ and $y \leq z$ imply $x \leq z$.

An *ordered set* is denoted by $(X, \leq)$, where X is a non-empty set and $\leq$ an order on X.

1.1.2 Definition

Let $(X, \leq_1)$ and $(Y, \leq_2)$ be two ordered sets. A mapping $\theta : X \to Y$ such that $\theta(x_1) \leq_2 \theta(x_2)$ whenever $x \leq_1 y$ is called a *homomorphism* or an *order homomorphism or order preserving.*

1.1.3 Definition

Let X be an ordered set and let $A \subseteq X$. Then $x \in X$ is a *maximal element* of A, if $x \leq a \in A$ implies $a = x$. Further, $x \in X$ is the *greatest element* of A, if $x \geq a$ for all $a \in A$.

A *minimal element* of A and the *least element* of A are defined dually. Note that if A has a *greatest element*, it is unique. Similarly, the least element of A is unique.

1.1.4 Definition

Let P be an ordered set and $A \subseteq X$. An element $x \in X$ is an *upper bound* of A if $a \leq x$ for all $a \in A$. A *lower bound* of A is defined dually.

If there is a least element in the set of all upper bounds of A, it is called the *supremum* of A and is denoted by $\sup A$ or $\bigvee A$; dually a greatest lower bound is called *infimum* and written $\inf A$ or $\bigwedge A$. We also write $a \vee b$ for $\sup\{a, b\}$ and $a \wedge b$ for $\inf\{a, b\}$. *Supremum* and *infimum* are frequently called *join* and *meet.*

1.1.5 Definition

Let L be a non-empty ordered set. If $a \vee b$ and $a \wedge b$ exist for all a, $b \in L$, then L is called a *lattice*. If $\bigvee A$ and $\bigwedge A$ exist for all $A \subseteq L$, then L is called a *complete lattice.*

1.1.6 Definition

Let $(L, \leq)$ be a lattice. If $\bigvee L$ and $\bigwedge L$ exist, then L is called a *bounded lattice*. In a bounded lattice, the least element is denoted by 0 and greatest element by 1.

The definition of a lattice given with the help of a binary relation on X is a constructive approach, now, we present the algebraic definition of a lattice which is an axiomatic approach and given with the help of binary operations defined on X.

1.1.7 Definition

A binary operation " $*$ " on X is a map $* : X \times X \to X$. A set X together with a binary operation " $*$ " on it, is called a groupoid and denoted by $(X, *)$. In general $*(x, y)$ is denoted by $x * y$.

1.1.8 Definition

Let $(X, *)$ be a groupoid. Then $*$ is called

1) *Associative* if $x * (y * z) = (x * y) * z$,

2) *Commutative* if $x * y = y * x$,

3) *Idempotent* if $x * x = x$

for all $x, y, z \in X$

1.1.9 Definition

"An algebraic structure $(S, *)$ is called a *semilattice* if S is a non-empty set and $*$ is a binary operation such that $*$ is commutative, associative and idempotent."

1.1.10 Definition

"An algebraic structure $(L, \wedge, \vee)$ is called a *lattice* if L is a non-empty set and $\wedge$ and $\vee$ are binary operations on L, $(L, \wedge)$ and $(L, \vee)$ are semilattices and absorption laws for $\wedge$ and $\vee$ hold i.e.

$$\begin{aligned} x \wedge (x \vee y) &= x \text{ and} \\ x \vee (x \wedge y) &= x \text{ for all } x,\ y \in L. \end{aligned}$$

Using the basic lattice operations, an ordering can be defined as following:

1.1.11 Theorem

" Let $(L, \wedge, \vee)$ be a *lattice and* x, $y \in L$. The binary relation $\leq$ on L is defined by:

$$\begin{aligned} x &\leq y \Leftrightarrow x \vee y = y \text{ or equivalently} \\ x &\leq y \Leftrightarrow x \wedge y = x \text{ for all } x,\ y \in L. \end{aligned}$$

Then $(L, \leq)$ is a lattice satisfying the properties of lattice given in Definition 1.1.5."

1.1.12 Theorem

"Let $(L, \leq)$ be a *lattice and* $x,\ y \in L$. The binary oprations " $\wedge$ " and " $\vee$ " on L are defined by:

$$\begin{aligned} x \wedge y &= \inf\{x, y\} \text{ and} \\ x \vee y &= \sup\{x, y\} \text{ for all } x,\ y \in L. \end{aligned}$$

Then $(L, \wedge, \vee)$ satisfies the properties of lattice given in Definition 1.1.10."

Thus, both Definition 1.1.5 and Definition 1.1.19 are equivalent to each other. Onwards from here, we consider both notations interchangeably without stating explicitly.

1.1.13 Definition

"Let $(L_1, \wedge, \vee)$ and $(L_2, \wedge, \vee)$ be two lattices. A mapping $\theta : L_1 \to L_2$ such that $\theta(x \wedge y) = \theta(x) \wedge \theta(y)$ and $\theta(x \vee y) = \theta(x) \vee \theta(y)$ is called a *homomorphism* of lattices. A one-to-one lattice homomorphism is called monomorphism. A one-to-one and onto homomorphism is called lattice isomorphism."

Next we give the definitions of various algebras of lattices:

1.1.14 Definition

"Let L be a bounded lattice with a least element 0 and a greatest element 1. For an element $x \in L$, an element $y \in L$ is a *complement* of x if

$$x \vee y = 1 \text{ and } x \wedge y = 0.$$

If an element x has a unique complement, we denote it by x^c."

1.1.15 Remark

There exist bounded lattices with elements having more than one complement or no complement at all.

1.1.16 Example

"Let L be a lattice given by the Figure 1.1.1. In this lattice b and e are complements of a, c has no complement, 1 has 0 as complement and 0 has 1."

1.1.17 Definition

"A bounded lattice L in which every element has a complement is called a *complemented lattice.*"

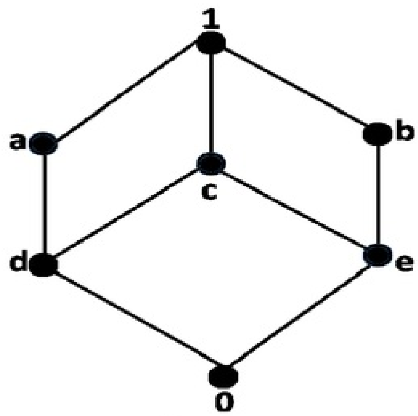

Figure 1.1.1

1.1.18 Example

Let X be a non-empty set. Then $(\mathcal{P}(X), \subseteq)$ is a complemented lattice.

1.1.19 Definition

Let L be a bounded lattice with a least element 0 and a greatest element 1. Let $': L \to L$, mapping $x \mapsto x'$ is such that

$$(x')' = x \text{ and } x \leq y \text{ implies that } y' \leq x' \text{ for all } x, y \in L.$$

Then " $'$ " is called an *involution or duality on* L.

It follows that " $'$ " is bijective, and that $0' = 1$ and $1' = 0$.

1.1.20 Example

Let $I = [0, 1]$. Then $(I, \leq)$ is a bounded lattice and $': x \mapsto 1 - x$ is an involution on I.

1.1.21 Definition

"Let L be a lattice with a least element 0. Then $x \in L$ is called an *atom* of L, if $0 < x$ and there is no element y in L with $0 < y < x$. The set of atoms of L is denoted by $\mathcal{A}(L)$."

1.1.22 Example

Let X be a non-empty set. Then every singleton subset of X is an atom of lattice $\mathcal{P}(X)$ and $\mathcal{A}(\mathcal{P}(X)) = \{\{x\} : x \in X\}$.

1.1.23 Definition

"Let L be a bounded lattice and " $'$ " is an involution on L, the identities

$$\begin{aligned}(x \vee y)' &= x' \wedge y' \\ (x \vee y)' &= x' \wedge y'\end{aligned}$$

are called the *de Morgan Laws.*"

A nice property of unions and intersections is that they distribute over each other. Therefore, it is natural to consider lattices for which joins and meets have analogous properties.

1.1.24 Definition

"A lattice L satisfying the distributive laws

$$\begin{aligned}x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge y); \\ x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \quad \text{for all } x, y, z \in L\end{aligned}$$

is called a *distributive lattice.*"

1.1.25 Definition

"If de Morgan's laws hold for a bounded distributive lattice having an involution, then it is called a *de Morgan algebra.* Such a system is denoted by $(L, \vee, \wedge, ', 0, 1)$."

1.1.26 Definition

"A bounded distributive lattice which is complemented is called a *Boolean lattice.*"

1.1.27 Definition

"A *de Morgan's algebra* $(L, \wedge, \vee, ', 0, 1)$ that satisfies $x \wedge x' \leqslant y \vee y'$ for all $x,\ y \in L$, is called a *Kleene algebra.*"

1.1.28 Definition

"Let L be a lattice. Then L is said to be *atomic* if every element x of L is the supremum of the atoms below it, i.e.

$$x = \bigvee \{y \in \mathcal{A}(L) | y \leq x\}."$$

1.1.29 Definition

"Let L be a lattice, and $x, y \in L$. Then x is called *pseudocomplemented relative to y* if the following set:

$$T(x, y) = \{z \in L | z \wedge x \leq y\}$$

has a greatest element. This greatest element is said to be pseudocomplement of x relative to y, denoted by $x \to y$. So, $x \to y$, in case it exists, has the following property:

$$z \wedge x \leq y \text{ if and only if } z \leq x \to y."$$

1.1.30 Definition

"An element $x \in L$ is said to be *relatively pseudocomplemented* if $x \to y$ exists for all $y \in L$."

1.1.31 Definition

"A lattice L is said to be an *implicative lattice* or *relatively pseudocomplemented or Brouwerian*, if every element in L is *relatively pseudocomplemented.*"

1.1.32 Example

"Let $L(X)$ be the lattice of open sets of a topological space X. Then $L(X)$ is Brouwerian. For any open sets $A, B \in L(X)$, $A \to B = (A^c \cup B)^\circ$, the interior of the union of B and the complement of A."

1.1.33 Definition

"Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and $x \in L$. Then an element x^* is called a pseudocomplement of x, if $x \wedge x^* = 0$ and $y \leqslant x^*$ whenever $x \wedge y = 0$. Note that $x \to 0 = x^*$."

1.1.34 Definition

"If every element of a lattice L has a pseudocomplement then L is said to be pseudocomplemented."

1.1.35 Definition

"The equation

$$x^* \vee x^{**} = 1$$

is called Stone's identity."

1.1.36 Definition

"A Stone algebra is a pseudocomplemented, distributive lattice satisfying Stone's identity."

1.1.37 Definition [17]

"*MV-algebra* is an algebraic structure $\langle M, \oplus, ^*, 0\rangle$, where $\oplus$ is a binary operation, " * " is a unary operation, and 0 is a constant such that the following axioms are satisfied for any a, $b \in M$:

(MV1) $(M, \oplus, 0)$ is a commutative monoid,

(MV2) $(a^*)^* = a$,

(MV3) $0^* \oplus a = 0^*$,

(MV4) $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$."

1.1.38 Definition [9]

"A set X with a binary operation $*$ and a constant 0 is called a BCI algebra if for any x, y, z in X, it satisfies the following conditions:

(BCI-1) $((x * y) * (x * z)) * (z * y) = 0$,

(BCI-2) $(x * (x * y)) * y = 0$,

(BCI-3) $x * x = 0$,

(BCI-4) $x * y = 0$ and $y * x = 0$ imply $x = y$."

1.1.39 Definition [9]

"A BCI-algebra $(X; *, 0)$ is called a BCK-algebra if it satisfies the following condition:

(BCK-5) $0 * x = 0$. for all $x \in X$."

1.1.40 Definition [9]

"A BCK algebra X is called *bounded* if there exists some element $1 \in X$ such that $x * 1 = 0$ for all $x \in X$. For a bounded BCK algebra $(X; *, 0)$, if an element $x \in X$ satisfies $1 * (1 * x) = x$, then x is called an *involution (Different meaning from the involution given in Definition 1.1.19.*"

1.2 Fuzzy Sets

"The material presented in this section is taken from [46]. We give the definitions of fuzzy sets and some related terms.

Let X be a set and A be a subset of X. The characteristic function of A is the function C_A of X into {0,1} defined by $C_A(x) = 1$ if $x \in A$ and $C_A(x) = 0$ if $x \notin A$."

1.2.1 Definition

"A *fuzzy subset of* X is a function from X into the unit closed interval [0,1]. The set of all fuzzy subsets of X is called the *fuzzy power set* of X, and is denoted by $\mathcal{FP}(X)$."

1.2.2 Definition

"Let μ, $v \in \mathcal{FP}(X)$. If $\mu(x) \leqslant v(x)$ for all $x \in X$, then μ is said to be *contained in* v, and we write $\mu \subseteq v$(or $v \supseteq \mu$).

Clearly, the inclusion relation $\subseteq$ is a partial order on $\mathcal{FP}(X)$."

1.2.3 Definition

"Let $\mu, v \in \mathcal{FP}(X)$. Then $\mu \vee v$ and $\mu \wedge v$ are fuzzy subsets of X, defined as follows: For all $x \in X$,

$$\begin{aligned} (\mu \vee v)(x) &= \mu(x) \vee v(x), \\ (\mu \wedge v)(x) &= \mu(x) \wedge v(x). \end{aligned}$$

The fuzzy subsets $\mu \vee v$ and $\mu \wedge v$ are called the *union and intersection of* μ *and* v, respectively."

1.2.4 Definition

"The *complement of a fuzzy subset* μ is denoted by μ' and is defined by

$$\mu'(x) = 1 - \mu(x),$$

for all $x \in X$."

1.2.5 Definition

"The fuzzy subsets of X, denoted by $\bar{\mathbf{0}}$ and $\bar{\mathbf{1}}$, which map every element of X onto 0 and 1 respectively, are called the empty fuzzy set or null fuzzy subset and the whole fuzzy subset of X respectively."

1.3 Bipolar Fuzzy Sets

The material presented in this section is taken from [26]. We give the definitions of bipolar fuzzy sets and some related terms. In bipolar-valued fuzzy sets, two kinds of representations are used: canonical representation and reduced representation. In the canonical representation, membership degrees are expressed with a pair of a positive membership value and a negative membership value. That is, the member ship degrees are divided into two parts: positive part in $[0, 1]$ and negative part in $[-1, 0]$. In the reduced representation, membership degrees are presented with a value in $[-1, 1]$. In our work, we use the canonical representation of a bipolar-valued fuzzy sets. For more material on this topic we refer to [26] and [27]. Let X be the universe of discourse.

1.3.1 Definition

"A bipolar fuzzy set μ in X is defined as:

$$\mu = \left\{ (x,\ \mu^P(x),\ \mu^N(x)) : x \in X \right\}$$

where $\mu^P : X \longrightarrow [0, 1]$ and $\mu^N : X \longrightarrow [-1, 0]$ are mappings. The positive membership degree $\mu^P(x)$ denotes the satisfaction degree of an element x to the property and the negative membership degree $\mu^N(x)$ denotes the satisfaction degree of x to some implicit counter-property. If $\mu^P(x) \neq 0$ and $\mu^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for μ. If $\mu^P(x) = 0$ and $\mu^N(x) \neq 0$, it is the situation that x does not satisfy the property of μ but somewhat satisfies the counter-property of μ. It is possible for an element x to be $\mu^N(x) \neq 0$ and $\mu^P(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain.

For example, sweetness of foods is a bipolar fuzzy set. If sweetness of foods has been given as positive membership values then bitterness of foods is for negative membership values. Other tastes like salty, sour, pungent (e.g. chili) etc. are irrelevant to the corresponding property. So these foods are taken as zero membership values.

For the sake of simplicity, we shall write $\mu = \left(\mu^P,\ \mu^N\right)$ for the bipolar fuzzy set

$$\mu = \left\{(x,\ \mu^P(x),\ \mu^N(x)) : x \in X\right\}.$$

The set of all bipolar fuzzy sets of X is called the *bipolar fuzzy power set* of X, and is denoted by $\mathcal{BFP}(X)$."

1.3.2 Definition

"Let $\mu,\ v \in \mathcal{BFP}(X)$. If $\mu^P(x) \leqslant v^P(x)$ and $v^N(x) \leqslant \mu^N(x)$ for all $x \in X$, then μ is said to be *contained in* v, and we write $\mu \subseteq v($ or $v \supseteq \mu)$.

Clearly, the inclusion relation $\subseteq$ is a partial order on $\mathcal{BFP}(X)$."

1.3.3 Definition

"Let $\mu,\ v \in \mathcal{BFP}(X)$. Then set operations $\mu \cup v$ and $\mu \cap v$ are bipolar fuzzy sets of X, defined as follows:

For all $x \in X$,

$$\begin{aligned}(\mu \cup v)^P(x) &= \mu^P(x) \vee v^P(x),\ (\mu \cup v)^N(x) = \mu^N(x) \wedge v^N(x) \text{ and} \\ (\mu \cap v)^P(x) &= \mu^P(x) \wedge v^P(x),\ (\mu \cap v)^N(x) = \mu^N(x) \vee v^N(x).\end{aligned}$$

The bipolar fuzzy subsets $\mu \cup v$ and $\mu \cap v$ are called the *union and intersection of* μ *and* v, respectively."

1.3.4 Definition

"The *complement of a bipolar fuzzy subset* μ is denoted by $\bar{\mu}$ and is defined by

$$(\bar{\mu})^P(x) = 1 - \mu^P(x),\ (\bar{\mu})^N(x) = -1 - \mu^N(x)$$

for all $x \in X$."

Chapter 2

Soft Sets and Their Algebraic Structures

In this chapter we will present the basic concepts of soft set theory. Soft sets have received much attention in the last decade because of their applications in decision making problems. Molodstov [34] presented the concept of soft sets to deal with uncertain type of data under a parametrized environment which is rich enough to make approximations by incorporating the previous concepts like fuzzy sets, vague sets, interval valued fuzzy sets, intuitionistic fuzzy sets, rough sets, etc. Molodstov had given the concept of soft set and introductory ideas to apply in various fields while Maji et al. defined operations on soft sets in [32], [33]. Ali et al. [2] pointed out some practical mistakes in the definition of operations by Maji et al. and defined new operations introducing the concept of extended and restricted operations for soft sets. These operations not only enriched the theory but also proved this new structure deep enough to work for further structural investigations. This gives rise to our interest in the algebraic properties of a soft set's internal structure. So here we have made our first study. Firstly the definition of a soft set and various operations are given and then, we study some important properties associated with these operations. A collection of all soft sets with respect to new operations inspires to be checked out for various lattices and algebras. Going through different axiomatic requirements we figure out the algebraic structures of soft sets and finally, we show that soft sets with a fixed set of parameters are also MV algebras and BCK algebras.

2.1 Soft Sets

Let X be an initial universe and E be a set of parameters. Let $\mathcal{P}(X)$ denotes the power set of X and A, B be non-empty subsets of E.

2.1.1 Definition [34]

A pair (α,A) is called a *soft set* over X, where α is a mapping given by $\alpha : A \rightarrow \mathcal{P}(X)$. Therefore, a soft set over X gives a parametrized family of subsets of the universe X. For $e \in A$, $\alpha(e)$ may be considered as the set of $e-$approximate elements of X by the soft set (α,A). Clearly, a soft set is not a classical set. From now onwards, we shall use the notation A_α over X to denote a soft set (α,A) over X where the meanings of α, A and X are clear in a harmony with the use of usual pair notation.

2.1.2 Definition [12]

For two soft sets A_α and B_β over X, we say that A_α is a *soft subset* of B_β if

1) $A \subseteq B$ and

2) $\alpha(e) \subseteq \beta(e)$ for all $e \in A$.

We write $A_\alpha \tilde{\subseteq} B_\beta$.

A_α is said to be a *soft super set* of B_β, if B_β is a soft subset of A_α. We denote it by $A_\alpha \tilde{\supseteq} B_\beta$.

2.1.3 Definition [12]

Two soft sets A_α and B_β over X are said to be *soft equal* if A_α and B_β are soft subsets of each other. We denote it by $A_\alpha \tilde{=} B_\beta$.

2.1.4 Example

Let X be the set of cars under consideration, and E be the set of parameters of different features in cars, $X = \{c_1,c_2,c_3,c_4,c_5\}$, $E = \{e_1,e_2,e_3,e_4,e_5\} = \{$ Seat Heater, Automatic transmission, Sunroof, Leather Seats, Navigation System$\}$. Suppose that $A = \{e_1,e_2,e_3\}$, and $B = \{e_1,e_2\}$. A soft set A_α describing the " features of cars" which Mr. X is going to consider for buying is given as follows:

$$\alpha : A \to \mathcal{P}(X),$$
$$e \longmapsto \begin{cases} \{c_2,c_3,c_4\} & \text{if } e = e_1, \\ \{c_1,c_3,c_4\} & \text{if } e = e_2, \\ \{c_2,c_3,c_4,c_5\} & \text{if } e = e_3. \end{cases}$$

And the soft set B_β given by

$$\beta : B \to \mathcal{P}(X),$$
$$e \longmapsto \begin{cases} \{c_3\} & \text{if } e = e_1, \\ \{c_1,c_3,c_4\} & \text{if } e = e_2, \end{cases}$$

is a soft subset of A_α which represents another look by Mr. X on his earlier choices, so $B_\beta \tilde{\subseteq} A_\alpha$.

2.2 Operations on Soft Sets

Now, we give various operations on soft sets as defined in [4]. We have made little modifications to some notations just for the convenience of reader and in order to create a unanimity in the flow of this thesis.

2.2.1 Definition

Let A_α and B_β be two soft sets over X. Then the *or-product* of A_α and B_β is defined as a soft set $(A \times B)_{\alpha \tilde{\cup} \beta}$, where $\alpha \tilde{\cup} \beta : (A \times B) \to \mathcal{P}(X)$, defined by

$$(a, b) \mapsto \alpha(a) \cup \beta(b).$$

It is denoted by $A_\alpha \vee B_\beta \tilde{=} (A \times B)_{\alpha \tilde{\cup} \beta}$.

2.2.2 Definition

Let A_α and B_β be two soft sets over X. The *and-product* of A_α and B_β is defined as a soft set $(A \times B)_{\alpha \tilde{\cap} \beta}$, where $\alpha \tilde{\cap} \beta : (A \times B) \to \mathcal{P}(X)$, defined by

$$(a, b) \mapsto \alpha(a) \cap \beta(b).$$

It is denoted by $A_\alpha \wedge B_\beta \tilde{=} (A \times B)_{\alpha \tilde{\cap} \beta}$.

2.2.3 Definition

The *extended union of two soft sets* A_α and B_β over X is defined as a soft set $(A \cup B)_{\alpha \tilde{\cup} \beta}$, where $\alpha \tilde{\cup} \beta : (A \cup B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \begin{cases} \alpha(e) & if\ e \in A - B \\ \beta(e) & if\ e \in B - A \\ \alpha(e) \cup \beta(e) & if\ e \in A \cap B \end{cases}$$

We write $A_\alpha \sqcup_\varepsilon B_\beta \tilde{=} (A \cup B)_{\alpha \tilde{\cup} \beta}$.

2.2.4 Definition

The extended intersection of two soft sets A_α and B_β over X, is defined as a soft set $(A \cup B)_{\alpha \tilde{\cap} \beta}$ where, $\alpha \tilde{\cap} \beta : (A \cup B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \begin{cases} \alpha(e) & if\ e \in A - B \\ \beta(e) & if\ e \in B - A \\ \alpha(e) \cap \beta(e) & if\ e \in A \cap B \end{cases}$$

We write $A_\alpha \sqcap_\varepsilon B_\beta \tilde{=} (A \cup B)_{\alpha \tilde{\cap} \beta}$.

2.2.5 Definition

Let A_α and B_β be two soft sets over X such that $(A \cap B) \neq \emptyset$. Then the *restricted union* of A_α and B_β is defined as a soft set $(A \cap B)_{\alpha \tilde{\cap} \beta}$ where, $\alpha \tilde{\cup} \beta : (A \cap B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \alpha(e) \cup \beta(e).$$

We write $A_\alpha \sqcup B_\beta \tilde{=} (A \cap B)_{\alpha \tilde{\cup} \beta}$.

2.2.6 Definition

Let A_α and B_β be two soft sets over X such that $(A \cap B) \neq \emptyset$. Then the *restricted intersection* of A_α and B_β is defined as a soft set $(A \cap B)_{\alpha \tilde{\cap} \beta}$ where, $\alpha \tilde{\cap} \beta : A \cap B \to \mathcal{P}(X)$, defined by

$$e \mapsto \alpha(e) \cap \beta(e).$$

We write $A_\alpha \sqcap B_\beta \tilde{=} (A \cap B)_{\alpha \tilde{\cap} \beta}$.

2.2.7 Definition

The extended difference of two soft sets A_α and B_β over X, is defined as a soft set $(A \cup B)_{\alpha \smile_\varepsilon \beta}$ where, $\alpha \smile_\varepsilon \beta : (A \cup B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \begin{cases} \alpha(e) & if\ e \in A - B \\ \beta(e) & if\ e \in B - A \\ \alpha(e) - \beta(e) & if\ e \in A \cap B. \end{cases}$$

We write $A_\alpha \smile_\varepsilon B_\beta \tilde{=} (A \cup B)_{\alpha \smile_\varepsilon \beta}$.

2.2.8 Definition

Let A_α and B_β be two soft sets over X such that $A \cap B \neq \emptyset$. Then the *restricted difference* of A_α and B_β is defined as a soft set $(A \cap B)_{\alpha \smile \beta}$ where, $\alpha \smile \beta : A \cap B \to \mathcal{P}(X)$, defined by

$$e \mapsto \alpha(e) - \beta(e).$$

We write $A_\alpha \smile B_\beta \tilde{=} (A \cap B)_{\alpha \smile \beta}$.

2.2.9 Definition

The *complement of a soft set* A_α, denoted by $(A_\alpha)^c$ and defined as $(A_\alpha)^c \tilde{=} A_{\alpha^c}$ where, $\alpha^c : A \to \mathcal{P}(X)$ is defined by

$$e \mapsto X - \alpha(e).$$

Clearly $(\alpha^c)^c$ is same as α and $((A_\alpha)^c)^c = A_\alpha$.

2.2.10 Example

Let U be the set of houses under consideration, and E be the set of parameters, $U = \{h_1, h_2, h_3, h_4, h_5\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\} = \{$ in the green surroundings, wooden, cheap, in good repair, furnished, traditional $\}$. Suppose that $A = \{e_1, e_2\}$, and $B = \{e_2, e_3\}$. The soft sets A_α and B_β describe the "requirements of the houses" which Mr. X and Mr. Y are going to buy respectively and is given as follows:

$$\alpha \quad : \quad A \to \mathcal{P}(X), \text{ defined by}$$

$$e \longmapsto \begin{cases} \{h_2, h_3\} & \text{if } e = e_1, \\ \{h_1, h_2, h_5\} & \text{if } e = e_2, \end{cases}$$

and

$$\begin{array}{rcl} \beta & : & B \to \mathcal{P}(X), \text{ defined by} \\ e & \longmapsto & \begin{cases} \{h_2, h_5\} & \text{if } e = e_2, \\ \{h_1, h_3, h_5\} & \text{if } e = e_3. \end{cases} \end{array}$$

Now, we approximate the resulting soft sets obtained by applying the above mentioned operations on A_α and B_β. We have

(i) $A_\alpha \vee B_\beta \tilde{=} (A \times B)_{\alpha \tilde{\cup} \beta}$, where

$$\begin{array}{rcl} (\alpha \tilde{\cup} \beta) & : & (A \times B) \to \mathcal{P}(X), \text{ defined by} \\ e & \longmapsto & \begin{cases} \{h_2, h_3, h_5\} & \text{if } e = (e_1, e_2), \\ \{h_1, h_2, h_3, h_5\} & \text{if } e = (e_1, e_3), \\ \{h_1, h_2, h_5\} & \text{if } e = (e_2, e_2), \\ \{h_1, h_2, h_3, h_5\} & \text{if } e = (e_2, e_3). \end{cases} \end{array}$$

(ii) $A_\alpha \wedge B_\beta \tilde{=} (A \times B)_{\alpha \tilde{\cap} \beta}$, where

$$\begin{array}{rcl} (\alpha \tilde{\cap} \beta) & : & (A \times B) \to \mathcal{P}(X), \text{ defined by} \\ e & \longmapsto & \begin{cases} \{h_2\} & \text{if } e = (e_1, e_2), \\ \{h_3\} & \text{if } e = (e_1, e_3), \\ \{h_2, h_5\} & \text{if } e = (e_2, e_2), \\ \{h_1, h_5\} & \text{if } e = (e_2, e_3). \end{cases} \end{array}$$

(iii) $A_\alpha \sqcup_\varepsilon B_\beta \tilde{=} (A \cup B)_{\alpha \tilde{\cup} \beta}$, where

$$\begin{array}{rcl} (\alpha \tilde{\cup} \beta) & : & (A \cup B) \to \mathcal{P}(X), \text{ defined by} \\ e & \longmapsto & \begin{cases} \{h_2, h_3\} & \text{if } e = e_1, \\ \{h_1, h_2, h_5\} & \text{if } e = e_2, \\ \{h_1, h_3, h_5\} & \text{if } e = e_3, \end{cases} \end{array}$$

(iv) $A_\alpha \sqcap_\varepsilon B_\beta \tilde{=} (A \cup B)_{\alpha \tilde{\cap} \beta}$, where

$$\begin{array}{rcl} (\alpha \tilde{\cap} \beta) & : & (A \cup B) \to \mathcal{P}(X), \text{ defined by} \\ e & \longmapsto & \begin{cases} \{h_2, h_3\} & \text{if } e = e_1, \\ \{h_2, h_5\} & \text{if } e = e_2, \\ \{h_1, h_3, h_5\} & \text{if } e = e_3, \end{cases} \end{array}$$

(v) $A_\alpha \sqcup B_\beta \tilde{=} (A \cap B)_{\alpha \tilde{\cup} \beta}$, where

$$\begin{array}{rcl} (\alpha \tilde{\cup} \beta) & : & (A \cap B) \to \mathcal{P}(X), \text{ defined by} \\ e_2 & \longmapsto & \{h_1, h_2, h_5\} \end{array}$$

(vi) $A_\alpha \sqcap B_\beta \tilde{=} (A \cap B)_{\alpha \tilde{\cap} \beta}$, where

$$\begin{array}{rcl} (\alpha \tilde{\cap} \beta) & : & (A \cap B) \to \mathcal{P}(X), \text{ defined by} \\ e_2 & \longmapsto & \{h_2, h_5\} \end{array}$$

(vii) $A_\alpha \smallsmile_\varepsilon B_\beta \tilde{=} (A \cup B)_{\alpha \smallsmile_\varepsilon \beta}$, where

$$\begin{aligned} \alpha \quad \smallsmile \quad _\varepsilon\beta &: (A \cup B) \to \mathcal{P}(X), \text{ defined by} \\ e \quad \longmapsto &\begin{cases} \{h_2, h_3\} & \text{if } e = e_1, \\ \{h_1\} & \text{if } e = e_2, \\ \{h_1, h_3, h_5\} & \text{if } e = e_3, \end{cases} \end{aligned}$$

(ix) $A_\alpha \smallsmile B_\beta \tilde{=} (A \cap B)_{\alpha \smallsmile \beta}$, where

$$\begin{aligned} \alpha \quad \smallsmile \quad \beta &: (A \cap B) \to \mathcal{P}(X), \text{ defined by} \\ e_2 \quad \longmapsto &\quad \{h_1\} \end{aligned}$$

(x) $(A_\alpha)^c = A_{\alpha^c}$ where

$$\begin{aligned} \alpha^c \quad &: \quad A \to \mathcal{P}(X), \text{ where} \\ e \quad \longmapsto &\begin{cases} \{h_1, h_4, h_5\} & \text{if } e = e_1, \\ \{h_3, h_4\} & \text{if } e = e_2. \end{cases} \end{aligned}$$

2.3 Properties of Soft Sets

In this section we discuss properties and laws of soft sets with respect to operations defined on soft sets. Later on these results are utilized for the configuration of algebraic structures of soft sets. The new idea of restricted and extended operations gives rise to some different results, for example, distributive laws do not hold in general for the operations of soft sets which is an entirely new aspect in a vague structure. Associativity, absorption, distributivity, de Morgan laws are investigated for soft set theory.

2.3.1 Definition

A soft set A_α over X is called a relative null soft set, denoted by A_Φ, if $\alpha(e) = \emptyset$ for all $e \in A$.

2.3.2 Definition

A soft set A_α over X is called a relative whole or *absolute soft set*, denoted by $A_{\mathfrak{X}}$, if $\alpha(e) = X$ for all $e \in A$.

Conventionally, we take soft sets with an empty set of parameters to be equal to $\emptyset_\Phi$ and so $A_\alpha \sqcap B_\beta \tilde{=} \emptyset_\Phi \tilde{=} A_\alpha \sqcup B_\beta$ when $A \cap B = \emptyset$.

2.3.3 Proposition

Let A_α, A_β be any soft sets over X. Then

1) $A_\alpha \sqcup_\varepsilon A_\beta \tilde{=} A_\alpha \sqcup A_\beta$; $A_\alpha \sqcap_\varepsilon A_\beta \tilde{=} A_\alpha \sqcap A_\beta$,

2) $A_\alpha \lambda A_\alpha \tilde{=} A_\alpha$, for $\lambda \in \{\sqcup, \sqcap\}$, (Idempotent)

3) $A_\alpha \sqcap A_{\mathfrak{X}} \tilde{=} A_\alpha \tilde{=} A_\alpha \sqcup A_\Phi$,

4) $A_\alpha \sqcup A_{\mathfrak{X}} \tilde{=} A_{\mathfrak{X}}$; $A_\alpha \sqcap A_\Phi \tilde{=} A_\Phi$,

5) $A_\alpha \sqcap_\varepsilon \emptyset_\Phi \tilde{=} A_\alpha \tilde{=} A_\alpha \sqcup_\varepsilon \emptyset_\Phi \tilde{=} A_\alpha \sqcap E_{\mathfrak{X}}$,

6) $A_\alpha \sqcap \emptyset_\Phi \tilde{=} \emptyset_\Phi$; $A_\alpha \sqcup_\varepsilon E_{\mathfrak{X}} \tilde{=} E_{\mathfrak{X}}$.

Proof. Straightforward. ∎

2.3.4 Proposition

Let A_α, B_β and C_γ be any soft sets over X. Then the following are true:

1) $A_\alpha \lambda (B_\beta \lambda C_\gamma) \tilde{=} (A_\alpha \lambda B_\beta) \lambda C_\gamma$, (Associative Laws)

2) $A_\alpha \lambda B_\beta \tilde{=} B_\beta \lambda A_\alpha$, (Commutative Laws)

for all $\lambda \in \{\sqcup_\varepsilon, \sqcup, \sqcap_\varepsilon, \sqcap\}$.
Proof. Straightforward. ∎

2.3.5 Proposition (Absorption Laws)

Let A_α, B_β be any soft sets over X. Then the following are true:

1) $A_\alpha \sqcup_\varepsilon (B_\beta \sqcap A_\alpha) \tilde{=} A_\alpha$,

2) $A_\alpha \sqcap (B_\beta \sqcup_\varepsilon A_\alpha) \tilde{=} A_\alpha$,

3) $A_\alpha \sqcup (B_\beta \sqcap_\varepsilon A_\alpha) \tilde{=} A_\alpha$,

4) $A_\alpha \sqcap_\varepsilon (B_\beta \sqcup A_\alpha) \tilde{=} A_\alpha$.

Proof. Straightforward. ∎

2.3.6 Proposition (Distributive Laws)

Let A_α, B_β and C_γ be any soft sets over X. Then

1) $A_\alpha \sqcap (B_\beta \sqcup_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcap B_\beta) \sqcup_\varepsilon (A_\alpha \sqcap C_\gamma)$,

2) $A_\alpha \sqcap (B_\beta \sqcap_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcap B_\beta) \sqcap_\varepsilon (A_\alpha \sqcap C_\gamma)$,

3) $A_\alpha \sqcap (B_\beta \sqcup C_\gamma) \tilde{=} (A_\alpha \sqcap B_\beta) \sqcup (A_\alpha \sqcap C_\gamma)$,

4) $A_\alpha \sqcup (B_\beta \sqcup_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcup B_\beta) \sqcup_\varepsilon (A_\alpha \sqcup C_\gamma)$,

5) $A_\alpha \sqcup (B_\beta \sqcap_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcup B_\beta) \sqcap_\varepsilon (A_\alpha \sqcup C_\gamma)$,

6) $A_\alpha \sqcup (B_\beta \sqcap C_\gamma) \tilde{=} (A_\alpha \sqcup B_\beta) \sqcap (A_\alpha \sqcup C_\gamma)$,

7) $A_\alpha \sqcap_\varepsilon (B_\beta \sqcup_\varepsilon C_\gamma) \tilde{\sqsubseteq} (A_\alpha \sqcap_\varepsilon B_\beta) \sqcup_\varepsilon (A_\alpha \sqcap_\varepsilon C_\gamma)$,

8) $A_\alpha \sqcap_\varepsilon (B_\beta \sqcup C_\gamma) \tilde{=} (A_\alpha \sqcap_\varepsilon B_\beta) \sqcup (A_\alpha \sqcap_\varepsilon C_\gamma)$,

9) $A_\alpha \sqcap_\varepsilon (B_\beta \sqcap C_\gamma) \tilde{\supseteq} (A_\alpha \sqcap_\varepsilon B_\beta) \sqcap (A_\alpha \sqcap_\varepsilon C_\gamma)$,

10) $A_\alpha \sqcup_\varepsilon (B_\beta \sqcup C_\gamma) \tilde{\subseteq} (A_\alpha \sqcup_\varepsilon B_\beta) \sqcup (A_\alpha \sqcup_\varepsilon C_\gamma)$,

11) $A_\alpha \sqcup_\varepsilon (B_\beta \sqcap_\varepsilon C_\gamma) \tilde{\supseteq} (A_\alpha \sqcup_\varepsilon B_\beta) \sqcap_\varepsilon (A_\alpha \sqcup_\varepsilon C_\gamma)$,

12) $A_\alpha \sqcup_\varepsilon (B_\beta \sqcap C_\gamma) \tilde{=} (A_\alpha \sqcup_\varepsilon B_\beta) \sqcap (A_\alpha \sqcup_\varepsilon C_\gamma)$.

Proof. We prove only one part here, the other parts can be proved in a similar way.

1) We have

$$A_\alpha \sqcap (B_\beta \sqcup_\varepsilon C_\gamma) \tilde{=} (A \cap (B \cup C))_{\alpha \tilde{\cap} (\beta \tilde{\cup} \gamma)}$$

and

$$\begin{aligned}(A_\alpha \sqcap B_\beta) \sqcup_\varepsilon (A_\alpha \sqcap C_\gamma) &\tilde{=} (A \cap B)_{(\alpha \tilde{\cap} \beta)} \sqcup_\varepsilon (A \cap C)_{(\alpha \tilde{\cap} \gamma)} \\ &\tilde{=} ((A \cap B) \cup (A \cap C))_{(\alpha \tilde{\cap} \beta) \tilde{\cup} (\alpha \tilde{\cap} \gamma)} \\ &\tilde{=} (A \cap (B \cup C))_{(\alpha \tilde{\cap} \beta) \tilde{\cup} (\alpha \tilde{\cap} \gamma)}.\end{aligned}$$

Let $e \in A \cap (B \cup C)$. Then there can be one of three cases:

(i) If $e \in A \cap (B - C)$, then

$$\begin{aligned}(\beta \tilde{\cup} \gamma)(e) &= \beta(e) \quad \text{and} \\ \{\alpha \tilde{\cap} (\beta \tilde{\cup} \gamma)\}(e) &= \alpha(e) \cap \beta(e).\end{aligned}$$

Also $A \cap (B - C) = (A \cap B) - (A \cap C)$ and hence

$$\{(\alpha \tilde{\cap} \beta) \tilde{\cup} (\alpha \tilde{\cap} \gamma)\}(e) = (\alpha \tilde{\cap} \beta)(e) = \alpha(e) \cap \beta(e).$$

(ii) If $e \in A \cap (C - B)$, then

$$\begin{aligned}(\beta \tilde{\cup} \gamma)(e) &= \gamma(e) \quad \text{and} \\ \{\alpha \tilde{\cap} (\beta \tilde{\cup} \gamma)\}(e) &= \alpha(e) \cap \gamma(e).\end{aligned}$$

Also $A \cap (C - B) = (A \cap C) - (A \cap B)$ and hence

$$\{(\alpha \tilde{\cap} \beta) \tilde{\cup} (\alpha \tilde{\cap} \gamma)\}(e) = (\alpha \tilde{\cap} \gamma)(e) = \alpha(e) \cap \gamma(e).$$

(iii) If $e \in A \cap (B \cap C)$, then

$$\begin{aligned}(\beta \tilde{\cup} \gamma)(e) &= \beta(e) \cup \gamma(e) \quad \text{and} \\ \{\alpha \tilde{\cap} (\beta \tilde{\cup} \gamma)\}(e) &= \alpha(e) \cap (\beta(e) \cup \gamma(e)).\end{aligned}$$

Also $A \cap (B \cap C) = (A \cap B) \cap (A \cap C)$ and hence

$$\begin{aligned} & \{(\alpha \tilde{\cap} \beta) \tilde{\cup} (\alpha \tilde{\cap} \gamma)\}(e) \\ = \ & (\alpha \tilde{\cap} \beta)(e) \cup (\alpha \tilde{\cap} \gamma)(e) \\ = \ & (\alpha (e) \cap \beta (e)) \cup (\alpha (e) \cap \gamma (e)) \\ = \ & \alpha (e) \cap (\beta (e) \cup \gamma (e)). \end{aligned}$$

Thus

$$\alpha \tilde{\cap} (\beta \tilde{\cup} \gamma) = (\alpha \tilde{\cap} \beta) \tilde{\cup} (\alpha \tilde{\cap} \gamma)$$

and so

$$(A \cap (B \cup C))_{\alpha \tilde{\cap} (\beta \tilde{\cup} \gamma)} \doteq (A \cap (B \cup C))_{(\alpha \tilde{\cap} \beta) \tilde{\cup} (\alpha \tilde{\cap} \gamma)}$$

Similarly we can prove the remaining parts.

■

2.3.7 Example

Let X be the set of sample designs and E be the set of available colors for dresses in a boutique,

$$X = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8\}$$

$$E = \{ \text{Red, Green, Blue, Yellow, Black, White, Pink} \}$$

Suppose that

$$\begin{aligned} A &= \{\text{Red, Green, Blue, White}\},\ B = \{\text{Green, Blue, Yellow, Black}\} \\ \text{and} \quad C &= \{\text{Blue, Yellow, White, Pink}\}. \end{aligned}$$

Let A_α,B_β and C_γ be the soft sets over X presenting the data record for three different boutiques respectively, given as follows:

$$\begin{aligned} \alpha(\text{Red}) &= \{S_1, S_2, S_3, S_4\}; \\ \alpha(\text{Green}) &= \{S_3, S_4, S_5, S_6\}; \\ \alpha(\text{Blue}) &= \{S_1, S_2, S_4, S_7\}; \\ \alpha(\text{White}) &= \{S_2, S_3, S_4\}. \end{aligned}$$

$$\begin{aligned} \beta(\text{Green}) &= \{S_4, S_5, S_6, S_8\}; \\ \beta(\text{Blue}) &= \{S_1, S_2, S_3, S_4\}; \\ \beta(\text{Yellow}) &= \{S_4, S_5, S_6, S_7, S_8\}; \\ \beta(\text{Black}) &= \{S_1, S_2, S_4, S_7\}. \end{aligned}$$

and

$$\begin{aligned} \gamma(\text{Blue}) &= \{S_3, S_4, S_7, S_8\}; \\ \gamma(\text{Yellow}) &= \{S_4, S_5, S_7\}; \\ \gamma(\text{White}) &= \{S_2, S_4, S_6, S_8\}; \\ \gamma(\text{Pink }) &= \{S_2, S_3, S_5, S_7\}. \end{aligned}$$

Now

$$\begin{aligned}
A_\alpha \sqcup_\varepsilon (B_\beta \sqcup C_\gamma) &\tilde{=} (A \cup (B \cap C))_{\alpha \tilde{\cup} (\beta \tilde{\cup} \gamma)};\\
(A_\alpha \sqcup_\varepsilon B_\beta) \sqcup (A_\alpha \sqcup_\varepsilon C_\gamma) &\tilde{=} ((A \cup B) \cap (A \cup C))_{(\alpha \tilde{\cup} \beta) \tilde{\cup} (\alpha \tilde{\cup} \gamma)};\\
A_\alpha \sqcup_\varepsilon (B_\beta \sqcap_\varepsilon C_\gamma) &\tilde{=} (A \cup (B \cup C))_{\alpha \tilde{\cup} (\beta \tilde{\cap} \gamma)};\\
(A_\alpha \sqcup_\varepsilon B_\beta) \sqcap_\varepsilon (A_\alpha \sqcup_\varepsilon C_\gamma) &\tilde{=} ((A \cup B) \cup (B \cup C))_{(\alpha \tilde{\cup} \beta) \tilde{\cap} (\alpha \tilde{\cup} \gamma)}.
\end{aligned}$$

Then

$$\begin{aligned}
(\alpha \tilde{\cup} (\beta \tilde{\cup} \gamma))(\text{Green}) &= \{S_3, S_4, S_5, S_6\};\\
(\alpha \tilde{\cup} (\beta \tilde{\cup} \gamma))(\text{White}) &= \{S_2, S_3, S_4\}.
\end{aligned}$$

$$\begin{aligned}
((\alpha \tilde{\cup} \beta) \tilde{\cup} (\alpha \tilde{\cup} \gamma))(\text{Green}) &= \{S_3, S_4, S_5, S_6, S_8\};\\
((\alpha \tilde{\cup} \beta) \tilde{\cup} (\alpha \tilde{\cup} \gamma))(\text{White}) &= \{S_2, S_3, S_4, S_6, S_8\}.
\end{aligned}$$

Thus

$$A_\alpha \sqcup_\varepsilon (B_\beta \sqcup C_\gamma) \tilde{\neq} (A_\alpha \sqcup_\varepsilon B_\beta) \sqcup (A_\alpha \sqcup_\varepsilon C_\gamma).$$

Similarly it can be shown that

$$A_\alpha \sqcap_\varepsilon (B_\beta \sqcap C_\gamma) \tilde{\neq} (A_\alpha \sqcap_\varepsilon B_\beta) \sqcap (A_\alpha \sqcap_\varepsilon C_\gamma).$$

Again, we see that

$$\begin{aligned}
(\alpha \tilde{\cup} (\beta \tilde{\cap} \gamma))(\text{Green}) &= \{S_3, S_4, S_5, S_6, S_8\};\\
(\alpha \tilde{\cup} (\beta \tilde{\cap} \gamma))(\text{White}) &= \{S_2, S_3, S_4, S_6, S_8\}
\end{aligned}$$

and

$$\begin{aligned}
((\alpha \tilde{\cup} \beta) \tilde{\cap} (\alpha \tilde{\cup} \gamma))(\text{Green}) &= \{S_3, S_4, S_5, S_6\};\\
((\alpha \tilde{\cup} \beta) \tilde{\cap} (\alpha \tilde{\cup} \gamma))(\text{White}) &= \{S_2, S_3, S_4\}.
\end{aligned}$$

Thus

$$A_\alpha \sqcup_\varepsilon (B_\beta \sqcap_\varepsilon C_\gamma) \neq (A_\alpha \sqcup_\varepsilon B_\beta) \sqcap_\varepsilon (A_\alpha \sqcup_\varepsilon C_\gamma).$$

Similarly it can be shown that

$$A_\alpha \sqcap_\varepsilon (B_\beta \sqcup_\varepsilon C_\gamma) \neq (A_\alpha \sqcap_\varepsilon B_\beta) \sqcup_\varepsilon (A_\alpha \sqcap_\varepsilon C_\gamma).$$

2.3.8 Proposition

Let A_α, B_β and C_γ be any *soft sets* over X. Then

1)

$$A_\alpha \sqcup_\varepsilon (B_\beta \sqcap_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcup_\varepsilon B_\beta) \sqcap_\varepsilon (A_\alpha \sqcup_\varepsilon C_\gamma)$$

if and only if

$$\begin{aligned}
\alpha(e) &\subseteq \beta(e) \text{ for all } e \in (A \cap B) - C \text{ and}\\
\alpha(e) &\subseteq \gamma(e) \text{ for all } e \in (A \cap C) - B.
\end{aligned}$$

2)

$$A_\alpha \sqcap_\varepsilon (B_\beta \sqcup_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcap_\varepsilon B_\beta) \sqcup_\varepsilon (A_\alpha \sqcap_\varepsilon C_\gamma)$$

if and only if

$$\begin{aligned} \alpha(e) &\supseteq \beta(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ \alpha(e) &\supseteq \gamma(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

Proof. Straightforward. ■

2.3.9 Corollary

Let A_α, B_β and C_γ be any *soft sets* over X. Then
$A_\alpha \sqcup_\varepsilon (B_\beta \sqcap_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcup_\varepsilon B_\beta) \sqcap_\varepsilon (A_\alpha \sqcup_\varepsilon C_\gamma)$
$A_\alpha \sqcap_\varepsilon (B_\beta \sqcup_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcap_\varepsilon B_\beta) \sqcup_\varepsilon (A_\alpha \sqcap_\varepsilon C_\gamma)$
if and only if

$$\begin{aligned} \alpha(e) &= \beta(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ \alpha(e) &= \gamma(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

2.3.10 Corollary

Let A_α, B_β and C_γ be *any soft sets* over X such that $(A \cap B) - C = (A \cap C) - B = \emptyset$. Then

1) $A_\alpha \sqcup_\varepsilon (B_\beta \sqcap_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcup_\varepsilon B_\beta) \sqcap_\varepsilon (A_\alpha \sqcup_\varepsilon C_\gamma)$,

2) $A_\alpha \sqcap_\varepsilon (B_\beta \sqcup_\varepsilon C_\gamma) \tilde{=} (A_\alpha \sqcap_\varepsilon B_\beta) \sqcup_\varepsilon (A_\alpha \sqcap_\varepsilon C_\gamma)$.

2.3.11 Corollary

Let A_α, A_β and A_γ be any *soft sets* over X. Then

$$A_\alpha \lambda (A_\beta \mu A_\gamma) \tilde{=} (A_\alpha \lambda A_\beta) \mu (A_\alpha \lambda A_\gamma)$$

for distinct λ, $\mu \in \{\sqcap_\varepsilon, \sqcap, \sqcup_\varepsilon, \sqcup\}$.

2.3.12 Theorem

Let A_α and B_β be two *soft sets* over X. Then the following are true

1) $A_\alpha \sqcup_\varepsilon B_\beta$ is the smallest soft set over X which contains both A_α and B_β. (Supremum)

2) $A_\alpha \sqcap B_\beta$ is the largest soft set over X which is contained in both A_α and B_β. (Infimum)

Proof.

1) We have $A, B \subseteq (A \cup B)$ and $\alpha(e), \beta(e) \subseteq \alpha(e) \cup \beta(e)$. So $A_\alpha \tilde{\subseteq} A_\alpha \sqcup_\varepsilon B_\beta$ and $B_\beta \tilde{\subseteq} A_\alpha \sqcup_\varepsilon B_\beta$. Let C_γ be a soft set over X, such that $A_\alpha, B_\beta \tilde{\subseteq} C_\gamma$. Then $A, B \subseteq C$ implies that $(A \cup B) \subseteq C$ and $\alpha(e), \beta(e) \subseteq \gamma(e)$ implies that $\alpha(e) \cup \beta(e) \subseteq \gamma(e)$. Thus $A_\alpha \sqcup_\varepsilon B_\beta \tilde{\subseteq} C_\gamma$. It follows that $A_\alpha \sqcup_\varepsilon B_\beta$ is the smallest soft set over X which contains both A_α and B_β.

2) We have $A \cap B \subseteq A, A \cap B \subseteq B$ and $\alpha(e) \cap \beta(e) \subseteq \alpha(e), \alpha(e) \cap \beta(e) \subseteq \beta(e)$ for all $e \in A \cap B$. So $A_\alpha \sqcap B_\beta \tilde{\subseteq} A_\alpha$ and $A_\alpha \sqcap B_\beta \tilde{\subseteq} B_\beta$. Let C_γ be a soft set over X, such that $C_\gamma \tilde{\subseteq} A_\alpha$ and $C_\gamma \tilde{\subseteq} B_\beta$. Then $C \subseteq A, C \subseteq B$ imply that $C \subseteq A \cap B$ and $\gamma(e) \subseteq \alpha(e), \gamma(e) \subseteq \beta(e)$ imply that $\gamma(e) \subseteq \alpha(e) \cap \beta(e)$ for all $e \in C$. Thus $C_\gamma \tilde{\subseteq} A_\alpha \sqcap B_\beta$. It follows that $A_\alpha \sqcap B_\beta$ is the largest soft set over X which is contained in both A_α and B_β.

■

2.4 Algebras of Soft Sets

In this section, we discuss lattices and algebras for the collections of soft sets. We consider certain collections of soft sets and find their distributive lattices. The concepts of involutions, complementations and atomicity are discussed. We denote the collections as follows:

$\mathcal{SS}(X)^E$: collection of all soft sets defined over X

$\mathcal{SS}(X)_A$: collection of all soft sets defined over X with a fixed parameter set A.

Firstly, we observe that these collections are partially ordered by the relation of soft inclusion $\tilde{\subseteq}$.

2.4.1 Proposition

The structures $(\mathcal{SS}(X)^E, \sqcap_\varepsilon, \sqcup)$, $(\mathcal{SS}(X)^E, \sqcup, \sqcap_\varepsilon)$, $(\mathcal{SS}(X)^E, \sqcup_\varepsilon, \sqcap)$, $(\mathcal{SS}(X)^E, \sqcap, \sqcup_\varepsilon)$, $(\mathcal{SS}(X)_A, \sqcup, \sqcap)$, and $(\mathcal{SS}(X)_A, \sqcap, \sqcup)$ are complete lattices.

Proof. Let us consider $(\mathcal{SS}(X)^E, \sqcap_\varepsilon, \sqcup)$. Then for any soft sets $A_\alpha, B_\beta, C_\gamma \in \mathcal{SS}(X)^E$,

1) We have $A_\alpha \sqcap_\varepsilon B_\beta \tilde{=} (A \cup B)_{\alpha \tilde{\cap} \beta} \in \mathcal{SS}(X)^E$ and $A_\alpha \sqcup B_\beta \tilde{=} (A \cap B)_{\alpha \tilde{\cup} \beta} \in \mathcal{SS}(X)^E$.

2) From Proposition 2.3.3, we have

$$A_\alpha \sqcap_\varepsilon A_\alpha \tilde{=} A_\alpha \text{ and } A_\alpha \sqcup A_\alpha \tilde{=} A_\alpha.$$

3) From Proposition 2.3.4 we see that

$$\begin{aligned} A_\alpha \sqcap_\varepsilon B_\beta &\tilde{=} B_\beta \sqcap_\varepsilon A_\alpha \text{ and} \\ A_\alpha \sqcup B_\beta &\tilde{=} B_\beta \sqcup A_\alpha. \end{aligned}$$

Also

$$\begin{aligned} A_\alpha \sqcap_\varepsilon (B_\beta \sqcap_\varepsilon C_\gamma) &\tilde{=} (A_\alpha \sqcap_\varepsilon B_\beta) \sqcap_\varepsilon C_\gamma \text{ and} \\ A_\alpha \sqcup (B_\beta \sqcup C_\gamma) &\tilde{=} (A_\alpha \sqcup B_\beta) \sqcup C_\gamma. \end{aligned}$$

4) From Proposition 2.3.5,

$$A_\alpha \sqcap_\varepsilon (B_\beta \sqcup A_\alpha) \tilde{=} A_\alpha \text{ and } A_\alpha \sqcup (B_\beta \sqcap_\varepsilon A_\alpha) \tilde{=} A_\alpha.$$

So we conclude that the structure forms a lattice.

Consider a collection of soft sets $\{A_{i_{\alpha_i}} : i \in I\}$ over X. We have, $\bigcup_{i\in I} A_i \subseteq E$ and, let $\Lambda(e) = \{j : e \in A_j\}$ for any $e \in A_i$. Then $\bigcap_{i\in\Lambda(e)} \alpha_i(e) \subseteq X$. Thus $\sqcap_{\varepsilon\, i\in I} A_{i_{\alpha_i}} \in \mathcal{SS}(X)^E$. Again, we have, $\bigcap_{i\in I} A_i \subseteq E$ and for any $e \in \bigcap_{i\in I} A_i$, $\bigcup_{i\in I} \alpha_i(e) \subseteq X$. Thus $\sqcup_{i\in I} A_{i_{\alpha_i}} \in \mathcal{SS}(X)^E$.

Similarly we can show the remaining structures. ∎

2.4.2 Proposition

The structures $(\mathcal{SS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_\Phi, E_\mathfrak{X})$, $(\mathcal{SS}(X)^E, \sqcup_\varepsilon, \sqcap, E_\mathfrak{X}, \emptyset_\Phi)$, $(\mathcal{SS}(X)_A, \sqcap, \sqcup, A_\Phi, A_\mathfrak{X})$ and $(\mathcal{SS}(X)_A, \sqcup, \sqcap, A_\mathfrak{X}, A_\Phi)$ are bounded distributive lattices.

Proof. From Proposition 2.3.6, we have

$$\begin{aligned} A_\alpha \sqcap (B_\beta \sqcup_\varepsilon C_\gamma) &\tilde{=} (A_\alpha \sqcap B_\beta) \sqcup_\varepsilon (A_\alpha \sqcap C_\gamma) \\ A_\alpha \sqcup_\varepsilon (B_\beta \sqcap C_\gamma) &\tilde{=} (A_\alpha \sqcup_\varepsilon B_\beta) \sqcap (A_\alpha \sqcup_\varepsilon C_\gamma) \end{aligned}$$

for all $A_\alpha, B_\beta, C_\gamma \in \mathcal{SS}(X)^E$. So $(\mathcal{SS}(X)^E, \sqcap, \sqcup_\varepsilon)$ and $(\mathcal{SS}(X)^E, \sqcup_\varepsilon, \sqcap)$ are distributive lattices. From Theorem 2.3.12, we conclude that $(\mathcal{SS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_\Phi, E_\mathfrak{X})$ is a bounded distributive lattice and $(\mathcal{SS}(X)^E, \sqcup_\varepsilon, \sqcap, E_\mathfrak{X}, \emptyset_\Phi)$ is its dual.

Now, for any soft sets $A_\alpha, A_\beta \in \mathcal{SS}(X)_A$,

$$\begin{aligned} A_\alpha \sqcap A_\beta &\tilde{=} A_{\alpha\tilde{\cap}\beta} \in \mathcal{SS}(X)_A \text{ and} \\ A_\alpha \sqcup A_\beta &\tilde{=} A_{\alpha\tilde{\cup}\beta} \in \mathcal{SS}(X)_A. \end{aligned}$$

Thus $(\mathcal{SS}(X)_A, \sqcap, \sqcup)$ is a distributive sublattice of $(\mathcal{SS}(X)^E, \sqcup_\varepsilon, \sqcap)$. Proposition 2.3.3 tells us that $A_\Phi, A_\mathfrak{X}$ are its lower and upper bounds respectively. Therefore

$(\mathcal{SS}(X)_A, \sqcap, \sqcup, A_\Phi, A_\mathfrak{X})$ is a bounded distributive lattice and $(\mathcal{SS}(X)_A, \sqcup, \sqcap, A_\mathfrak{X}, A_\Phi)$ is its dual. ∎

2.4.3 Proposition

Let A_α be a *soft set* over X. Then A_{α^c} is a complement of A_α.

Proof. As $A_\alpha \sqcup A_{\alpha^c} \tilde{=} A_{(\alpha\tilde{\cup}\alpha^c)}$ so, for any $e \in A$,

$$(\alpha\tilde{\cup}\alpha^c)(e) = \alpha(e) \cup (\alpha(e))^c = X.$$

Thus $A_\alpha \sqcup A_{\alpha^c} \tilde{=} A_\mathfrak{X}$.

Also $A_\alpha \sqcap A_{\alpha^c} \tilde{=} A_{(\alpha\tilde{\cap}\alpha^c)}$, so

$$(\alpha\tilde{\cap}\alpha^c)(e) = \alpha(e) \cap (\alpha(e))^c = \emptyset.$$

Thus $A_\alpha \sqcap A_{\alpha^c} \tilde{=} A_\Phi$.

Now, we show that A_{α^c} is unique in the bounded lattice $(\mathcal{SS}(X)_A, \sqcup, \sqcap, A_\mathfrak{X}, A_\Phi)$. If there exists some $A_\beta \in \mathcal{SS}(X)_A$ such that $A_\alpha \sqcup A_\beta \tilde{=} A_\mathfrak{X}$ and $A_\alpha \sqcap A_\beta \tilde{=} A_\Phi$. For any $e \in A$,

$$\alpha(e) \cap \beta(e) = \emptyset$$
$$\Rightarrow \beta(e) \subseteq (\alpha(e))^c = \alpha^c(e)$$

and

$$\alpha^c(e) \subseteq X = \alpha(e) \cup \beta(e).$$

But

$$\alpha(e) \cap \alpha^c(e) = \emptyset \text{ and so } \alpha^c(e) \subseteq \alpha(e) \cup \beta(e) \Rightarrow \alpha^c(e) \subseteq \beta(e).$$

Therefore

$$\beta(e) = \alpha^c(e) \text{ for all } e \in A \text{ and } A_\beta \tilde{=} A_{\alpha^c}.$$

Hence A_{α^c} is a complement of A_α. ■

2.4.4 Remark

We see that $(\mathcal{SS}(X)_A, \sqcap, \sqcup, A_\Phi, A_\mathfrak{X})$ and $(\mathcal{SS}(X)_A, \sqcup, \sqcap, A_\mathfrak{X}, A_\Phi)$ are dual lattices so all the properties and structural configurations hold dually in an understood manner.

2.4.5 Proposition (de Morgan Laws)

Let A_α and B_β be any soft sets over X. Then the following are true

1) $(A_\alpha \sqcup_\varepsilon B_\beta)^c \tilde{=} A_{\alpha^c} \sqcap_\varepsilon B_{\beta^c}$,

2) $(A_\alpha \sqcap_\varepsilon B_\beta)^c \tilde{=} A_{\alpha^c} \sqcup_\varepsilon B_{\beta^c}$,

3) $(A_\alpha \vee B_\beta)^c \tilde{=} A_{\alpha^c} \wedge B_{\beta^c}$,

4) $(A_\alpha \wedge B_\beta)^c \tilde{=} A_{\alpha^c} \vee B_{\beta^c}$,

5) $(A_\alpha \sqcup B_\beta)^c \tilde{=} A_{\alpha^c} \sqcap B_{\beta^c}$,

6) $(A_\alpha \sqcap B_\beta)^c \tilde{=} A_{\alpha^c} \sqcup B_{\beta^c}$.

Proof. We know that $(A_\alpha \sqcup_\varepsilon B_\beta)^c \tilde{=} ((A \cup B)_{\alpha \tilde{\cup} \beta})^c \tilde{=} (A \cup B)_{(\alpha \tilde{\cup} \beta)^c}$. Let $e \in (A \cup B)$. Then there are three cases:

(i) If $e \in A - B$, then

$$((\alpha \tilde{\cup} \beta)^c)(e) = (\alpha(e))^c = \alpha^c(e) \text{ and } (\alpha^c \tilde{\cap} \beta^c)(e) = \alpha^c(e).$$

(ii) If $e \in B - A$, then

$$(\alpha \tilde{\cup} \beta)^c(e) = (\beta(e))^c = \beta^c(e) \text{ and } (\alpha^c \tilde{\cap} \beta^c)(e) = \beta^c(e).$$

(iii) If $e \in A \cap B$, then

$$(\alpha \tilde{\cup} \beta)^c(e) = (\alpha(e) \cup \beta(e))^c = (\alpha(e))^c \cap (\beta(e))^c$$

and,

$$(\alpha^c \tilde{\cap} \beta^c)(e) = (\alpha(e))^c \cap (\beta(e))^c.$$

Therefore, in all the cases we obtain equality and thus

$$(A_\alpha \sqcup_\varepsilon B_\beta)^c \tilde{=} A_{\alpha^c} \sqcap_\varepsilon B_{\beta^c}.$$

The remaining parts can be proved in a similar way. ■

2.4.6 Proposition

$(\mathcal{SS}(X)_A, \sqcap, \sqcup, {}^c, A_\Phi, A_\mathfrak{X})$ is a de Morgan algebra.

Proof. We have already seen that $(\mathcal{SS}(X)_A, \sqcap, \sqcup, A_\Phi, A_\mathfrak{X})$ is a bounded distributive lattice. Propositions 2.4.3 and 2.4.5 show that de Morgan laws hold with respect to "^{c}" in $\mathcal{SS}(X)_A$. Thus $(\mathcal{SS}(X)_A, \sqcap, \sqcup, {}^c, A_\Phi, A_\mathfrak{X})$ is a de Morgan algebra. ■

2.4.7 Proposition

$(\mathcal{SS}(X)_A, \sqcap, \sqcup, {}^c, A_\Phi, A_\mathfrak{X})$ is a boolean algebra.

Proof. Follows from Propositions 2.4.2 and 2.4.3. ■

2.4.8 Proposition

Let A_α and A_β be any soft sets over X. Then $(A_\beta \sqcap A_{\beta^c}) \tilde{\subseteq} (A_\alpha \sqcup A_{\alpha^c})$ and so $(\mathcal{SS}(X)_A, \sqcap, \sqcup, {}^c, A_\Phi, A_\mathfrak{X})$ is a Kleene Algebra.

Proof. We have,

$$A_\beta \sqcap A_{\beta^c} \tilde{=} A_\Phi \tilde{\subseteq} A_\mathfrak{X} \tilde{=} A_\alpha \sqcup A_{\alpha^c}$$

for all $A_\alpha, A_\beta \in \mathcal{SS}(X)_A$. We already know that $(\mathcal{SS}(X)_A, \sqcap, \sqcup, {}^c, A_\Phi, A_\mathfrak{X})$ is a de Morgan algebra, so this condition assures that $(\mathcal{SS}(X)_A, \sqcap, \sqcup, {}^c, A_\Phi, A_\mathfrak{X})$ is a Kleene Algebra. ■

2.4.9 Lemma

For any $x \in X$ and $A \subseteq E$. We define a soft set A_{e_x} for each $e \in A$, where $e_x : A \to \mathcal{P}(X)$ such that

$$e_x(e') = \begin{cases} \{x\} & \text{if } e' = e \\ \emptyset & \text{if } e' \neq e \end{cases}.$$

Then A_{e_x} is an atom of lattice $(\mathcal{SS}(X)_A, \sqcap, \sqcup)$ for each $e \in A$ and $x \in X$ and we have

$$\mathcal{A}(\mathcal{SS}(X)_A) = \{A_{e_x} : e \in E \text{ and } x \in X\}.$$

Proof. Let $A_\Phi \tilde{\neq} A_\alpha \in \mathcal{SS}(X)_A$ such that $A_\alpha \tilde{\subseteq} A_{e_x}$. Then $\alpha(e) \subseteq e_x(e) = \{x\}$ and $\alpha(e') \subseteq \emptyset$ for all $(e \neq) e' \in A$. This implies that $\alpha(e') = \emptyset$ for all $(e \neq) e' \in A$ and the only possibility for $\alpha(e)$ is $\{x\}$ because $A_\Phi \tilde{\neq} A_\alpha$. Thus $A_\alpha \tilde{=} A_{e_x}$ proves that $A_{e_x} \in \mathcal{A}(\mathcal{SS}(X)_A)$. ■

2.4.10 Proposition

$(\mathcal{SS}(X)_A, \sqcap, \sqcup)$ is an atomic lattice.

Proof. Let $A_\alpha \in \mathcal{SS}(X)_A$, and take

$$\mathcal{I}_A = \{A_{e_x} \in \mathcal{A}(\mathcal{SS}(X)_A) : A_{e_x} \tilde{\subseteq} A_\alpha\}$$

the subcollection of $\mathcal{A}(\mathcal{SS}(X)_A)$ which is given in Lemma 2.4.9. Suppose that

$$A_\beta \tilde{=} \bigvee \mathcal{I}_A.$$

For any $e \in A$, $\beta(e) = \bigcup_{x \in \alpha(e)} e_x(e) = \bigcup_{x \in \alpha(e)} \{x\} = \alpha(e)$. Thus $\bigvee \mathcal{I}_A \tilde{=} A_\alpha$ and hence $(\mathcal{SS}(X)_A, \sqcap, \sqcup)$ is an atomic lattice. ■

2.4.11 Lemma

Let $A_\alpha, B_\beta \in \mathcal{SS}(X)^E$. Then the pseudocomplement of A_α relative to B_β exists in $\mathcal{SS}(X)^E$.

Proof. Consider the set

$$T(A_\alpha, B_\beta) = \{C_\gamma \in \mathcal{SS}(X)^E : C_\gamma \sqcap A_\alpha \tilde{\subseteq} B_\beta\}.$$

We define a soft set $A^c_{\alpha^c} \sqcup_\varepsilon B_\beta \tilde{=} (A^c \cup B)_{\alpha^c \tilde{\cup} \beta} \in \mathcal{SS}(X)^E$ and claim that $A_\alpha \to B_\beta \tilde{=} (A^c \cup B)_{\alpha^c \tilde{\cup} \beta}$. First of all we show that $(A^c \cup B)_{\alpha^c \tilde{\cup} \beta} \in T(A_\alpha, B_\beta)$. Consider

$$\begin{aligned} (A^c \cup B)_{\alpha^c \tilde{\cup} \beta} \sqcap A_\alpha &\tilde{=} ((A^c \cup B) \cap A)_{(\alpha^c \tilde{\cup} \beta) \tilde{\cap} \alpha} && \text{(By distributive law)} \\ &\tilde{=} ((A^c \cap A) \cup (B \cap A))_{(\alpha^c \tilde{\cap} \alpha) \tilde{\cup} (\beta \tilde{\cap} \alpha)} \\ &\tilde{=} (A \cap B)_{\alpha \tilde{\cap} \beta} \tilde{\subseteq} B_\beta. \end{aligned}$$

Thus $(A^c \cup B)_{\alpha^c \tilde{\cup} \beta} \in T(A_\alpha, B_\beta)$. For any $C_\gamma \in T(A_\alpha, B_\beta)$, we have $C_\gamma \sqcap A_\alpha \tilde{\subseteq} B_\beta$ so for any $e \in C \cap A \subseteq B$

$$\gamma(e) \cap \alpha(e) \subseteq \beta(e).$$

Now,

$$\begin{aligned} C \cap A &\subseteq B \Rightarrow (A \cap C) \cap B^c = \emptyset \\ &\Rightarrow C \subseteq (A \cap B^c)^c = A^c \cup B \end{aligned}$$

and

$$\begin{aligned} \gamma(e) \cap \alpha(e) &\subseteq \beta(e) \Rightarrow (\gamma(e) \cap \alpha(e)) \cap \beta^c(e) = \emptyset \\ &\Rightarrow \gamma(e) \subseteq (\alpha(e))^c \cap \beta(e) = \alpha^c(e) \cap \beta(e) \end{aligned}$$

Thus $C_\gamma \tilde{\subseteq} (A^c \cup B)_{\alpha^c \tilde{\cup} \beta}$ and it also shows that

$$(A^c \cup B)_{\alpha^c \tilde{\cup} \beta} \tilde{=} \bigvee T(A_\alpha, B_\beta) \tilde{=} A_\alpha \to B_\beta.$$

■

2.4.12 Remark

We know that $(\mathcal{SS}(X)_A, \sqcap, \sqcup)$ is a sublattice of $(\mathcal{SS}(X)^E, \sqcap_\varepsilon, \sqcup)$. *For any* $A_\alpha, A_\beta \in \mathcal{SS}(X)_A$, $A_\alpha \to A_\beta$ as defined in Lemma 2.4.11, is not in $\mathcal{SS}(X)_A$ because $A_\alpha \to A_\beta \tilde{=} (A^c \cup A)_{\alpha^c \tilde{\cup} \beta} \tilde{=} E_{\alpha^c \tilde{\cup} \beta} \notin \mathcal{SS}(X)_A$.

2.4.13 Lemma

Let $A_\alpha, A_\beta \in \mathcal{SS}(X)_A$. Then pseudocomplement of A_α relative to A_β exists in $\mathcal{SS}(X)^A$.

Proof. Consider the set

$$T(A_\alpha, A_\beta) = \{A_\gamma \in \mathcal{SS}(X)_A : A_\gamma \sqcap A_\alpha \tilde{\subseteq} A_\beta\}.$$

We define a soft set $A_{\alpha^c} \sqcup A_\beta \tilde{=} A_{\alpha^c \tilde{\cup} \beta} \in \mathcal{SS}(X)_A$. Consider

$$\begin{aligned} A_{\alpha^c \tilde{\cup} \beta} \sqcap A_\alpha &\tilde{=} A_{(\alpha^c \tilde{\cup} \beta) \tilde{\cap} \alpha} \\ &\tilde{=} A_{(\alpha^c \tilde{\cap} \alpha) \tilde{\cup} (\beta \tilde{\cap} \alpha)} \\ &\tilde{=} A_{\alpha \tilde{\cap} \beta} \tilde{\subseteq} A_\beta. \end{aligned}$$

Thus $A_{\alpha^c \tilde{\cup} \beta} \in T(A_\alpha, A_\beta)$. For every $A_\gamma \in T(A_\alpha, A_\beta)$, we have $A_\gamma \sqcap A_\alpha \tilde{\subseteq} A_\beta$ so for any $e \in A$,

$$\begin{aligned} \gamma(e) \cap \alpha(e) &\subseteq \beta(e) \Rightarrow (\gamma(e) \cap \alpha(e)) \cap \beta^c(e) = \emptyset \\ &\Rightarrow \gamma(e) \subseteq (\alpha(e))^c \cap \beta(e) = \alpha^c(e) \cap \beta(e) \end{aligned}$$

Thus $A_\gamma \tilde{\subseteq} A_{\alpha^c \tilde{\cup} \beta}$ and it also shows that

$$A_{\alpha^c \tilde{\cup} \beta} \tilde{=} \bigvee T(A_\alpha, A_\beta) \tilde{=} A_\alpha \to_A A_\beta.$$

■

2.4.14 Proposition

$(\mathcal{SS}(X)^E, \sqcap_\varepsilon, \sqcup)$ *and* $(\mathcal{SS}(X)_A, \sqcap, \sqcup)$ are *Brouwerian lattices.*

Proof. Follows from Lemmas 2.4.11 and 2.4.13. ■

2.4.15 Theorem

$(\mathcal{SS}(X)_A, \sqcap, ^c, A_{\mathfrak{X}})$ is an MV-algebra.

Proof. $MV1$, $MV2$ and $MV3$ are straightforward. We prove $MV4$:

$$\begin{aligned} (A_{\alpha^c} \sqcap A_\beta)^c \sqcap A_\beta &\tilde{=} ((A_{\alpha^c})^c \sqcup A_{\beta^c}) \sqcap A_\beta \\ &\tilde{=} (A_\alpha \sqcup A_{\beta^c}) \sqcap A_\beta \\ &\tilde{=} (A_\alpha \sqcap A_\beta) \sqcup (A_{\beta^c} \sqcap A_\beta) \\ &\tilde{=} (A_\alpha \sqcap A_\beta) \sqcup A_\Phi \\ &\tilde{=} (A_\beta \sqcap A_\alpha) \sqcup (A_{\alpha^c} \sqcap A_\alpha) \\ &\tilde{=} (A_\beta \sqcup A_{\alpha^c}) \sqcap A_\alpha \\ &\tilde{=} (A_{\beta^c} \sqcap A_\alpha)^c \sqcap A_\alpha. \end{aligned}$$

for all $A_\alpha, A_\beta \in \mathcal{SS}(X)_A$. Thus $(\mathcal{SS}(X)_A, \sqcap, ^c, A_{\mathfrak{X}})$ is an MV-algebra. ■

2.4.16 Theorem

$(\mathcal{SS}(X)_A, \sqcup, ^c, A_\Phi)$ is an MV-algebra.

Proof. $MV1$, $MV2$ and $MV3$ are straightforward. We prove $MV4$:

$$\begin{aligned}(A_{\alpha^c} \sqcup A_\beta)^c \sqcup A_\beta &\tilde{=} ((A_{\alpha^c})^c \sqcap A_{\beta^c}) \sqcup A_\beta \\ &\tilde{=} (A_\alpha \sqcap A_{\beta^c}) \sqcup A_\beta \\ &\tilde{=} (A_\alpha \sqcup A_\beta) \sqcap (A_{\beta^c} \sqcup A_\beta) \\ &\tilde{=} (A_\alpha \sqcup A_\beta) \sqcap A_{\mathfrak{X}} \\ &\tilde{=} (A_\beta \sqcup A_\alpha) \sqcap (A_{\alpha^c} \sqcup A_\alpha) \\ &\tilde{=} (A_\beta \sqcap A_{\alpha^c}) \sqcup A_\alpha \\ &\tilde{=} (A_{\beta^c} \sqcup A_\alpha)^c \sqcup A_\alpha.\end{aligned}$$

for all $A_\alpha, A_\beta \in \mathcal{SS}(X)_A$. Thus $(\mathcal{SS}(X)_A, \sqcup, ^c, A_\Phi)$ is an MV-algebra. ■

2.4.17 Theorem

$(\mathcal{SS}(X)_A, \smile, A_\Phi)$ is a bounded BCK-algebra whose every element is an involution.

Proof. For any $A_\alpha, A_\beta, A_\gamma \in \mathcal{SS}(X)_A$

BCI-1 $((A_\alpha \smile A_\beta) \smile (A_\alpha \smile A_\gamma)) \smile (A_\gamma \smile A_\beta)$

$\tilde{=}(A_{\alpha\smile\beta} \smile A_{\alpha\smile\gamma}) \smile A_{\gamma\smile\beta}$

$\tilde{=}A_{(\alpha\smile\beta)\smile(\alpha\smile\gamma)} \smile A_{\gamma\smile\beta}$

$\tilde{=}A_\Phi \smile A_{\gamma\smile\beta} \tilde{=} A_\Phi$.

BCI-2 $(A_\alpha \smile (A_\alpha \smile A_\beta)) \smile A_\beta$

$\tilde{=}(A_\alpha \smile A_{\alpha\smile\beta}) \smile A_\beta$

$\tilde{=}A_{(\alpha\smile(\alpha\smile\beta)} \smile A_\beta$

$\tilde{=}A_\Phi \smile A_\beta \tilde{=} A_{\Phi\smile\beta} \tilde{=} A_\Phi$.

BCI-3 $A_\alpha \smile A_\alpha \tilde{=} A_\Phi$.

BCI-4 Let $A_\alpha \smile A_\beta \tilde{=} A_\Phi$ and $A_\beta \smile A_\alpha \tilde{=} A_\Phi$. For any $e \in A$,

$$\alpha(e) - \beta(e) = \emptyset \text{ and } \beta(e) - \alpha(e) = \emptyset \text{ imply that } \alpha(e) = \beta(e).$$

Hence $A_\alpha \tilde{=} A_\beta$.

BCK-5 $A_\Phi \smile A_\alpha \tilde{=} A_{\Phi\smile\alpha} \tilde{=} A_\Phi$.

Thus $(\mathcal{SS}(X)_A, \smile, A_\Phi)$ is a BCK-algebra. Now $A_{\mathfrak{X}} \in \mathcal{SS}(X)_A$ is such that:

$$A_\alpha \smile A_{\mathfrak{X}} \tilde{=} A_{\alpha\smile\mathfrak{X}} \tilde{=} A_\Phi \text{ for all } A_\alpha \in \mathcal{SS}(X)_A.$$

Therefore $(\mathcal{SS}(X)_A, \smile, A_\Phi)$ is a bounded BCK-algebra.

For any $A_\alpha \in \mathcal{SS}(X)_A$,

$$A_{\mathfrak{X}} \smile (A_{\mathfrak{X}} \smile A_\alpha) \tilde{=} A_{\mathfrak{X}} \smile A_{\mathfrak{X}\smile\alpha} \tilde{=} A_{\mathfrak{X}} \smile A_{\alpha^c} \tilde{=} A_{\mathfrak{X}\smile\alpha^c} \tilde{=} A_{(\alpha^c)^c} \tilde{=} A_\alpha.$$

So every element of $\mathcal{SS}(X)_A$ is an involution. ■

2.4.18 Definition

Let A_α and A_β be any soft sets over X. We define

$$A_\alpha \star A_\beta \tilde{=} A_{\alpha\star\beta} \tilde{=} A_\alpha \sqcap A_{\beta^c}.$$

2.4.19 Theorem

$(\mathcal{SS}(X)_A, \star, A_\Phi)$ is a bounded BCK-algebra whose every element is an involution.

Proof. For any A_α, A_β, $A_\gamma \in \mathcal{SS}(X)_A$.

BCI-1 $((A_\alpha \star A_\beta) \star (A_\alpha \star A_\gamma)) \star (A_\gamma \star A_\beta)$

$\tilde{=}(A_{\alpha\star\beta} \star A_{\alpha\star\gamma}) \star A_{\gamma\star\beta}$

$\tilde{=}A_{((\alpha\star\beta)\star(\alpha\star\gamma))\star(\gamma\star\beta)}$

$\tilde{=}A_{((\alpha\tilde{\cap}\beta^c)\star(\alpha\tilde{\cap}\gamma^c))\star(\gamma\tilde{\cap}\beta^c)}$

$\tilde{=}A_{((\alpha\tilde{\cap}\beta^c)\tilde{\cap}(\alpha\tilde{\cap}\gamma^c)^c)\tilde{\cap}(\gamma\tilde{\cap}\beta^c)^c}$

$\tilde{=}A_{((\alpha\tilde{\cap}\beta^c)\tilde{\cap}(\alpha^c\tilde{\cup}\gamma))\tilde{\cap}(\gamma^c\tilde{\cup}\beta)}$

$\tilde{=}A_{((\alpha\tilde{\cap}\beta^c)\tilde{\cap}\gamma)\tilde{\cap}(\gamma^c\tilde{\cup}\beta)}$

$\tilde{=}A_{((\alpha\tilde{\cap}\beta^c)\tilde{\cap}\gamma)\tilde{\cap}\beta}$

$\tilde{=}A_{(\alpha\tilde{\cap}\gamma)\tilde{\cap}(\beta^c\tilde{\cap}\beta)} \tilde{=} A_\Phi.$

BCI-2 $(A_\alpha \star (A_\alpha \star A_\beta)) \star A_\beta$

$\tilde{=}(A_\alpha \star A_{\alpha\star\beta}) \star A_\beta$

$\tilde{=}A_{\alpha\star(\alpha\star\beta)} \star A_\beta$

$\tilde{=}A_{\alpha\tilde{\cap}(\alpha\tilde{\cap}\beta^c)^c} \star A_\beta$

$\tilde{=}A_{(\alpha\tilde{\cap}(\alpha^c\tilde{\cup}\beta)} \star A_\beta$

$\tilde{=}A_{\alpha\tilde{\cap}\beta} \star A_\beta \tilde{=} A_{(\alpha\tilde{\cap}\beta)\tilde{\cap}\beta^c} \tilde{=} A_\Phi.$

BCI-3 $A_\beta \star A_\beta \tilde{=} A_{\beta\tilde{\cap}\beta^c} \tilde{=} A_\Phi.$

BCI-4 Let $A_\alpha \star A_\beta \tilde{=} A_\Phi$ and $A_\beta \star A_\alpha \tilde{=} A_\Phi$. For any $e \in A$,

$$\alpha(e) \cap (\beta(e))^c = \emptyset \text{ and } \beta(e) \cap (\alpha(e))^c = \emptyset \text{ imply that } \alpha(e) = \beta(e).$$

Hence

$$A_\alpha \tilde{=} A_\beta.$$

BCK-5 $A_\Phi \star A_\alpha \tilde{=} A_{\Phi\star\alpha} \tilde{=} A_{\Phi\tilde{\cap}\alpha^c} \tilde{=} A_\Phi.$

Thus $(\mathcal{SS}(X)_A, \star, A_\Phi)$ is a BCK-algebra. Now $A_\mathfrak{X} \in \mathcal{SS}(X)_A$ is such that:

$$A_\beta \star A_\mathfrak{X} \tilde{=} A_{\alpha\star\mathfrak{X}} \tilde{=} A_{\alpha\tilde{\cap}\mathfrak{X}^c} \tilde{=} A_{\alpha\tilde{\cap}\Phi} \tilde{=} A_\Phi \quad \text{for all } A_\alpha \in \mathcal{SS}(X)_A.$$

Therefore $(\mathcal{SS}(X)_A, \star, A_\Phi)$ is a bounded BCK-algebra. ∎

Chapter 3

Algebraic Structures of Fuzzy Soft Sets

In 2001, Maji and Roy proposed the concept of Fuzzy Soft Set in [30]. Different algebraic structures have also been studied in fuzzy soft context. Irfan et al. [3] pointed out some basic problems in the results related to the operations defined on fuzzy soft sets. In the paper [3], some new operations are defined for fuzzy soft sets and modified results and laws are established. In this chapter, we step forward in the same direction and check out the associativity and distributivity of these operations. First we have given preliminaries of fuzzy soft sets. We have used new and modified definitions and operations from [3] to discuss the properties of these operations on fuzzy soft sets. After accomplishing an account of algebraic properties of fuzzy soft sets, the overall algebraic structures of collections of fuzzy soft sets are studied. The two types of collections of fuzzy soft sets, one consisting of those fuzzy soft sets with a fixed set of parameters while the other containing fuzzy soft sets defined over the same universe with different set of parameters are taken into account. Both collections have some common and some different algebraic properties and therefore the algebraic structures also differ. The lattice structure of these collections is discussed and we find that the collection of all fuzzy soft sets is a bounded distributive lattice and the collection of fuzzy soft sets with a fixed set of parameters becomes a Kleene algebra. At the end we define pseudocomplement of a fuzzy soft set and with this pseudocomplement, this collection becomes a stone algebra.

3.1 Fuzzy Soft Sets

Let X be an initial universe and E be a set of parameters. Let $\mathcal{FP}(X)$ denotes the fuzzy power set of X and A, B be non-empty subsets of E.

3.1.1 Definition [30]

A pair (f,A) is called a *fuzzy soft set over* X, where f is a mapping given by $f : A \to \mathcal{FP}(X)$.

Therefore, a fuzzy soft set over X gives a parametrized family of fuzzy subsets of the universe X. For $e \in A$, $f(e)$ may be considered as the set of $e-$approximate fuzzy elements of X. From now onwards, we shall use the notation A_f over X to denote a fuzzy soft set (f,A) over X where the meanings of f, A and X are clear in a harmony with the use of usual pair notation.

3.1.2 Definition [3]

For two fuzzy soft sets A_f and B_g over a common universe X, we say that A_f is a *fuzzy soft subset* of B_g if

1) $A \subseteq B$ and

2) $f(e) \subseteq g(e)$ for all $e \in A$.

We write $A_f \tilde{\subseteq} B_g$. A_f is said to be a *fuzzy soft super set* of B_g, if B_g is a fuzzy soft subset of A_f. We denote it by $A_f \tilde{\supseteq} B_g$.

3.1.3 Definition

[3] Two fuzzy soft sets A_f and B_g over X are said to be *fuzzy soft equal* if A_f and B_g are fuzzy soft subsets of each other. We denote it by $A_f \tilde{=} B_g$.

3.1.4 Example

Let X be a set of candidates for a driver's vacant position, and E be a set of parameters, $X = \{c_1,c_2,c_3,c_4,c_5\}$, $E = \{e_1,e_2,e_3,e_4\} = \{$ knowledge about routes, driving skills, physical fitness, young$\}$. Suppose that $A = \{e_1,e_2,e_3\}$, a fuzzy soft set A_f describes the "data of candidates" which Mr. X is going to hire and is given as follows:

$$\begin{aligned} f \quad &: \quad A \to \mathcal{FP}(X), \\ e \quad &\longmapsto \quad \begin{cases} \{c_1/0.3, c_2/0.1, c_3/0.3, c_4/0.1, c_5/0.7\} & \text{if } e = e_1, \\ \{c_1/0.1, c_2/0.9, c_3/0.3, c_4/0.8, c_5/0.2\} & \text{if } e = e_2, \\ \{c_1/0.1, c_2/0.3, c_3/0.3, c_4/0.3, c_5/0.8\} & \text{if } e = e_3, \end{cases} \end{aligned}$$

Let $B = \{e_2,e_3\}$. Then fuzzy soft set B_g given as follows:

$$\begin{aligned} g \quad &: \quad B \to \mathcal{FP}(X), \\ e \quad &\longmapsto \quad \begin{cases} \{c_1/0.1, c_2/0.5, c_3/0.3, c_4/0.5, c_5/0.2\} & \text{if } e = e_2, \\ \{c_1/0.1, c_2/0.2, c_3/0.1, c_4/0.2, c_5/0.7\} & \text{if } e = e_3, \end{cases} \end{aligned}$$

is a fuzzy soft subset of A_f and represents a second analysis of choices made in A_f.

3.2 Operations on Fuzzy Soft Sets

Now, we define various operations on fuzzy soft sets taken from literature.

3.2.1 Definition

Let A_f and B_g be two fuzzy soft sets over X. Then the *or-product* of A_f and B_g is defined as a fuzzy soft set $(A \times B)_{f\tilde{\vee}g}$, where $f\tilde{\vee}g : (A \times B) \to \mathcal{FP}(X)$, defined by

$$(a, b) \mapsto f(a) \vee g(b).$$

It is denoted by $A_f \vee B_g \tilde{=} (A \times B)_{f\tilde{\vee}g}$.

3.2.2 Definition

Let A_f and B_g be two fuzzy soft sets over X. The *and-product* of A_f and B_g is defined as a fuzzy soft set $(A \times B)_{f\tilde{\wedge}g}$, where $f\tilde{\wedge}g : (A \times B) \to \mathcal{FP}(X)$, defined by

$$(a, b) \mapsto f(a) \wedge g(b).$$

It is denoted by $A_f \wedge B_g \tilde{=} (A \times B)_{f\tilde{\wedge}g}$.

3.2.3 Definition

The *extended union of two fuzzy soft sets* A_f and B_g over X is defined as a fuzzy soft set $(A \cup B)_{f\tilde{\vee}g}$, where $f\tilde{\vee}g : (A \cup B) \to \mathcal{FP}(X)$, defined by

$$e \mapsto \begin{cases} f(e) & if\ e \in A - B \\ g(e) & if\ e \in B - A \\ f(e) \vee g(e) & if\ e \in A \cap B \end{cases}$$

We write $A_f \sqcup_\varepsilon B_g \tilde{=} (A \cup B)_{f\tilde{\vee}g}$.

3.2.4 Definition

The extended intersection of two fuzzy soft sets A_f and B_g over X, is defined as a fuzzy soft set $(A \cup B)_{f\tilde{\wedge}g}$, where $f\tilde{\wedge}g : (A \cup B) \to \mathcal{FP}(X)$, defined by

$$e \mapsto \begin{cases} f(e) & if\ e \in A - B \\ g(e) & if\ e \in B - A \\ f(e) \wedge g(e) & if\ e \in A \cap B \end{cases}$$

We write $A_f \sqcap_\varepsilon B_g \tilde{=} (A \cup B)_{f\tilde{\wedge}g}$.

3.2.5 Definition

Let A_f and B_g be two fuzzy soft sets over X such that $A \cap B \neq \emptyset$. Then the *restricted union* of A_f and B_g is defined as a fuzzy soft set $(A \cap B)_{f\tilde{\wedge}g}$, where $f\tilde{\vee}g : A \cap B \to \mathcal{FP}(X)$,

$$e \mapsto f(e) \vee g(e).$$

We write $A_f \sqcup B_g \tilde{=} (A \cap B)_{f\tilde{\vee}g}$

3.2.6 Definition

Let A_f and B_g be two fuzzy soft sets over X such that $A \cap B \neq \emptyset$. Then the *restricted intersection* of A_f and B_g is defined as a fuzzy soft set $(A \cap B)_{f\tilde{\wedge}g}$, where $f\tilde{\wedge}g : A \cap B \to \mathcal{FP}(X)$,

$$e \mapsto f(e) \wedge g(e).$$

We write $A_f \sqcap B_g \hat{=} (A \cap B)_{f\tilde{\wedge}g}$.

3.2.7 Definition

The *complement of a fuzzy soft set* A_f, denoted by $(A_f)'$ and defined by $(A_f)' \hat{=} A_{f'}$, where $f' : A \to \mathcal{FP}(X)$ is given by

$$(f'(e))(x) = 1 - (f(e))(x),$$

for all $e \in A$, and for all $x \in X$.

Clearly $(f')'$ is same as f and $((A_f)')' = A_f$.

Now, we give an example to show how to apply these operations on fuzzy soft sets:

3.2.8 Example

Let X be the initial universe and E be the set of parameters,

$$X = \{x_1, x_2, x_3, x_4, x_5\},\ E = \{e_1, e_2, e_3, e_4, e_5\}.$$

Suppose

$$A = \{e_1, e_2\}, \text{ and } B = \{e_2, e_4\}.$$

Let A_f and B_g be the fuzzy soft sets over X defined by the following:

$$\begin{aligned} f &: A \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{x_1/0.1, x_2/0.2, x_3/0.3, x_4/0.7, x_5/0.4\} & \text{if } e = e_1, \\ \{x_1/0.7, x_2/0.9, x_3/0.2, x_4/0.4, x_5/0.1\} & \text{if } e = e_2, \end{cases} \end{aligned}$$

$$\begin{aligned} g &: B \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{x_1/0.3, x_2/0.7, x_3/0.6, x_4/0.9, x_5/0.1\} & \text{if } e = e_2, \\ \{x_1/0.4, x_2/0.2, x_3/0.7, x_4/0.8, x_5/0.7\} & \text{if } e = e_4, \end{cases} \end{aligned}$$

Then

(i) $A_f \sqcup_\varepsilon B_g \hat{=} (A \cup B)_{f\tilde{\vee}g}$ where

$$\begin{aligned} f\tilde{\vee}g &: (A \cup B) \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{x_1/0.1, x_2/0.2, x_3/0.3, x_4/0.7, x_5/0.4\} & \text{if } e = e_1, \\ \{x_1/0.7, x_2/0.9, x_3/0.6, x_4/0.9, x_5/0.1\} & \text{if } e = e_2, \\ \{x_1/0.4, x_2/0.2, x_3/0.7, x_4/0.8, x_5/0.7\} & \text{if } e = e_4. \end{cases} \end{aligned}$$

(ii) $A_f \sqcap_\varepsilon B_g \tilde{=} (A \cup B)_{f\tilde{\wedge}g}$ where

$$f\tilde{\wedge}g \quad : \quad (A \cup B) \to \mathcal{FP}(X),$$
$$e \longmapsto \begin{cases} \{x_1/0.1, x_2/0.2, x_3/0.3, x_4/0.7, x_5/0.4\} & \text{if } e = e_1, \\ \{x_1/0.3, x_2/0.7, x_3/0.2, x_4/0.4, x_5/0.1\} & \text{if } e = e_2, \\ \{x_1/0.4, x_2/0.2, x_3/0.7, x_4/0.8, x_5/0.7\} & \text{if } e = e_4. \end{cases}$$

(iii) $A_f \sqcup B_g \tilde{=} (A \cap B)_{f\tilde{\vee}g}$ where

$$f\tilde{\vee}g \quad : \quad (A \cap B) \to \mathcal{FP}(X),$$
$$e_2 \longmapsto \{x_1/0.7, x_2/0.9, x_3/0.6, x_4/0.9, x_5/0.1\}$$

(iv) $A_f \sqcap B_g \tilde{=} (A \cap B)_{f\tilde{\wedge}g}$ where

$$f\tilde{\wedge}g \quad : \quad (A \cap B) \to \mathcal{FP}(X),$$
$$e \longmapsto \begin{cases} \{x_1/0.3, x_2/0.7, x_3/0.2, x_4/0.4, x_5/0.1\} & \text{if } e = e_2, \\ \{x_1/0.3, x_2/0.7, x_3/0.3, x_4/0.2, x_5/0.5\} & \text{if } e = e_3. \end{cases}$$

(v) $(A_f)' \tilde{=} A_{f'}$ where

$$f' \quad : \quad A \to \mathcal{FP}(X),$$
$$e \longmapsto \begin{cases} \{x_1/0.9, x_2/0.8, x_3/0.7, x_4/0.3, x_5/0.6\} & \text{if } e = e_1, \\ \{x_1/0.3, x_2/0.1, x_3/0.8, x_4/0.6, x_5/0.9\} & \text{if } e = e_2, \end{cases}$$

3.3 Properties of Fuzzy Soft Sets

In this section we discuss properties and laws of fuzzy soft sets with respect to operations defined on fuzzy soft sets. Later on the results will be utilized for the configuration of algebraic structures of fuzzy soft sets. Associativity, commutativity, absorption, distributivity, de Morgan laws and properties of involutions, and atomicity are investigated for collection of fuzzy soft sets.

3.3.1 Definition

A fuzzy soft set A_f over X is called a relative null fuzzy soft set, denoted by $A_{\tilde{\mathbf{0}}}$, if $f(e) = \tilde{0}$ for all $e \in A$, where $\tilde{0}$ is the fuzzy subset of X mapping every element of X on 0.

3.3.2 Definition

A fuzzy soft set A_f over X is called a relative whole or *absolute fuzzy soft set*, denoted by $A_{\tilde{\mathbf{1}}}$, if $f(e) = \tilde{\mathbf{1}}$ for all $e \in A$, where $\tilde{\mathbf{1}}$ is the fuzzy subset of X mapping every element of X on 1.

Conventionally, we take fuzzy soft sets with an empty set of parameters to be equal to $\emptyset_{\tilde{\mathbf{0}}}$ and so $A_f \sqcap B_g \tilde{=} \emptyset_{\tilde{\mathbf{0}}} \tilde{=} A_f \sqcup B_g$ when $A \cap B = \emptyset$.

3.3.3 Proposition

Let A_f, A_g be any fuzzy soft sets over X. Then

1) $A_f \lambda A_f \tilde{=} A_f$, for $\lambda \in \{\sqcup, \sqcup_\varepsilon, \sqcap, \sqcap_\varepsilon\}$, (Idempotent)

2) $A_f \sqcup_\varepsilon A_g \tilde{=} A_f \sqcup A_g$; $A_f \sqcap_\varepsilon A_g \tilde{=} A_f \sqcap A_g$,

3) $A_f \sqcap A_{\tilde{\mathbf{1}}} \tilde{=} A_f \tilde{=} A_f \sqcup A_{\tilde{\mathbf{0}}}$,

4) $A_f \sqcup A_{\tilde{\mathbf{1}}} \tilde{=} A_{\tilde{\mathbf{1}}}$; $A_f \sqcap A_{\tilde{\mathbf{0}}} \tilde{=} A_{\tilde{\mathbf{0}}}$,

5) $A_f \sqcap_\varepsilon \emptyset_{\tilde{\mathbf{0}}} \tilde{=} A_f \tilde{=} A_f \sqcup_\varepsilon \emptyset_{\tilde{\mathbf{0}}} \tilde{=} A_f \sqcap E_{\tilde{\mathbf{1}}}$,

6) $A_f \sqcap \emptyset_{\tilde{\mathbf{0}}} \tilde{=} \emptyset_{\tilde{\mathbf{0}}}$; $A_f \sqcup_\varepsilon E_{\tilde{\mathbf{1}}} \tilde{=} E_{\tilde{\mathbf{1}}}$.

Proof. Straightforward. ■

3.3.4 Proposition

Let A_f, B_g and C_h be any fuzzy soft sets over X. Then the following are true:

1) $A_f \lambda (B_g \lambda C_h) \tilde{=} (A_f \lambda B_g) \lambda C_h$, (Associative Laws)

2) $A_f \lambda B_g \tilde{=} B_g \lambda A_f$, (Commutative Laws)

for all $\lambda \in \{\sqcup_\varepsilon, \sqcup, \sqcap_\varepsilon, \sqcap\}$.
Proof. Straightforward. ■

3.3.5 Proposition (Absorption Laws)

Let A_f, B_g be any fuzzy soft sets over X. Then the following are true:

1) $A_f \sqcap_\varepsilon (B_g \sqcup A_f) \tilde{=} A_f$,

2) $A_f \sqcap (B_g \sqcup_\varepsilon A_f) \tilde{=} A_f$,

3) $A_f \sqcup (B_g \sqcap_\varepsilon A_f) \tilde{=} A_f$,

4) $A_f \sqcup_\varepsilon (B_g \sqcap A_f) \tilde{=} A_f$.

Proof. For any $e \in A$,

$$
\begin{aligned}
(f \tilde{\wedge} (f \tilde{\vee} g))(e) &= \begin{cases} f(e) & \text{if } e \in A - (A \cap B) \\ f(e) \wedge (f \tilde{\vee} g)(e) & \text{if } e \in A \cap (A \cap B) \end{cases} \\
&= \begin{cases} f(e) & \text{if } e \in A - (A \cap B) \\ f(e) \wedge (f(e) \vee g(e)) & \text{if } e \in A \cap B \end{cases} \\
&= \begin{cases} f(e) & \text{if } e \in A - (A \cap B) \\ f(e) & \text{if } e \in A \cap B \end{cases} \\
&= f(e).
\end{aligned}
$$

Thus $A_f \sqcap_\varepsilon (B \sqcup A_f) \tilde{=} A_f$. The remaining parts can also be proved similarly. ■

3.3.6 Proposition (Distributive Laws)

Let A_f, B_g and C_h be any fuzzy soft sets over X. Then

1) $A_f \sqcap (B_g \sqcup_\varepsilon C_h) \tilde{=} (A_f \sqcap B_g) \sqcup_\varepsilon (A_f \sqcap C_h)$,

2) $A_f \sqcap (B_g \sqcap_\varepsilon C_h) \tilde{=} (A_f \sqcap B_g) \sqcap_\varepsilon (A_f \sqcap C_h)$,

3) $A_f \sqcap (B_g \sqcup C_h) \tilde{=} (A_f \sqcap B_g) \sqcup (A_f \sqcap C_h)$,

4) $A_f \sqcup (B_g \sqcup_\varepsilon C_h) \tilde{=} (A_f \sqcup B_g) \sqcup_\varepsilon (A_f \sqcup C_h)$,

5) $A_f \sqcup (B_g \sqcap_\varepsilon C_h) \tilde{=} (A_f \sqcup B_g) \sqcap_\varepsilon (A_f \sqcup C_h)$,

6) $A_f \sqcup (B_g \sqcap C_h) \tilde{=} (A_f \sqcup B_g) \sqcap (A_f \sqcup C_h)$,

7) $A_f \sqcap_\varepsilon (B_g \sqcup_\varepsilon C_h) \tilde{\subseteq} (A_f \sqcap_\varepsilon B_g) \sqcup_\varepsilon (A_f \sqcap_\varepsilon C_h)$,

8) $A_f \sqcap_\varepsilon (B_g \sqcup C_h) \tilde{=} (A_f \sqcap_\varepsilon B_g) \sqcup (A_f \sqcap_\varepsilon C_h)$,

9) $A_f \sqcap_\varepsilon (B_g \sqcap C_h) \tilde{\supseteq} (A_f \sqcap_\varepsilon B_g) \sqcap (A_f \sqcap_\varepsilon C_h)$,

10) $A_f \sqcup_\varepsilon (B_g \sqcup C_h) \tilde{\subseteq} (A_f \sqcup_\varepsilon B_g) \sqcup (A_f \sqcup_\varepsilon C_h)$,

11) $A_f \sqcup_\varepsilon (B_g \sqcap_\varepsilon C_h) \tilde{\supseteq} (A_f \sqcup_\varepsilon B_g) \sqcap_\varepsilon (A_f \sqcup_\varepsilon C_h)$,

12) $A_f \sqcup_\varepsilon (B_g \sqcap C_h) \tilde{=} (A_f \sqcup_\varepsilon B_g) \sqcap (A_f \sqcup_\varepsilon C_h)$.

Proof. We prove only one part here, the other parts can also be proved in a similar way.

5) We have

$$A_f \sqcup (B_g \sqcap_\varepsilon C_h) \tilde{=} (A \cap (B \cup C))_{f\tilde{\vee}(g\tilde{\wedge}h)}$$

and

$$\begin{aligned}(A_f \sqcup B_g) \sqcap_\varepsilon (A_f \sqcup C_h) &\tilde{=} (A \cap B)_{(f\tilde{\vee}g)} \sqcap_\varepsilon (A \cap C)_{f\tilde{\vee}h} \\ &\tilde{=} ((A \cap B) \cup (A \cap C))_{(f\tilde{\vee}g)\tilde{\wedge}(f\tilde{\vee}h)} \\ &\tilde{=} (A \cap (B \cup C))_{(f\tilde{\vee}g)\tilde{\wedge}(f\tilde{\vee}h)}.\end{aligned}$$

Let $e \in A \cap (B \cup C)$ then there are three possibilities:

(i) If $e \in A \cap (B - C)$ then,

$$\begin{aligned}(g\tilde{\wedge}h)(e) &= g(e) \quad \text{and} \\ \{f\tilde{\vee}(g\tilde{\wedge}h)\}(e) &= f(e) \vee g(e).\end{aligned}$$

Also $A \cap (B - C) = (A \cap B) - (A \cap C)$ and hence

$$\{(f\tilde{\vee}g)\tilde{\wedge}(f\tilde{\vee}h)\}(e) = (f\tilde{\vee}g)(e) = f(e) \vee g(e).$$

(ii) If $e \in A \cap (C - B)$ then,

$$\begin{aligned} (g\tilde{\wedge}h)(e) &= h(e) \quad \text{and} \\ \{f\tilde{\vee}(g\tilde{\wedge}h)\}(e) &= f(e) \vee h(e). \end{aligned}$$

Also $A \cap (C - B) = (A \cap C) - (A \cap B)$ and hence

$$\{(f\tilde{\vee}g)\tilde{\wedge}(f\tilde{\vee}h)\}(e) = (f\tilde{\vee}h)(e) = f(e) \vee h(e).$$

(iii) If $e \in A \cap (B \cap C)$ then,

$$\begin{aligned} (g\tilde{\wedge}h)(e) &= g(e) \wedge h(e) \quad \text{and} \\ \{f\tilde{\vee}(g\tilde{\wedge}h)\}(e) &= f(e) \vee (g(e) \wedge h(e)). \end{aligned}$$

Also $A \cap (B \cap C) = (A \cap B) \cap (A \cap C)$ and hence

$$\begin{aligned} \{(f\tilde{\vee}g)\tilde{\wedge}(f\tilde{\vee}h)\}(e) &= (f\tilde{\vee}g)(e) \wedge (f\tilde{\vee}h)(e) \\ &= (f(e) \vee g(e)) \wedge (f(e) \vee h(e)) \\ &= f(e) \vee (g(e) \wedge h(e)). \end{aligned}$$

Thus

$$f\tilde{\vee}(g\tilde{\wedge}h) = (f\tilde{\vee}g)\tilde{\wedge}(f\tilde{\vee}h)$$

and so

$$(A \cap (B \cup C))_{f\tilde{\vee}(g\tilde{\wedge}h)} \tilde{=} (A \cap (B \cup C))_{(f\tilde{\vee}g)\tilde{\wedge}(f\tilde{\vee}h)}.$$

■

3.3.7 Example

Let X be the set of houses under consideration, and E be the set of parameters,

$$X = \{h_1, h_2, h_3, h_4, h_5\},$$

$$E = \{\text{beautiful, wooden, cheap, in good repair, furnished}\}.$$

Suppose that

$$\begin{aligned} A &= \{\text{beautiful, wooden, cheap}\}, \\ B &= \{\text{wooden, cheap, in good repair}\}, \\ \text{and } C &= \{\text{cheap, in good repair, furnished}\}. \end{aligned}$$

Let A_f,B_g and C_h be the fuzzy soft sets over X defined by the following:

$$\begin{aligned} f &: A \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{h_1/0.1, h_2/0.2, h_3/0.3, h_4/0.7, h_5/0.4\} & \text{if } e = e_1, \\ \{h_1/0.7, h_2/0.9, h_3/0.2, h_4/0.4, h_5/0.1\} & \text{if } e = e_2, \\ \{h_1/0.3, h_2/0.7, h_3/0.5, h_4/0.2, h_5/0.6\} & \text{if } e = e_3, \end{cases} \end{aligned}$$

$$g : B \to \mathcal{FP}(X),$$
$$e \longmapsto \begin{cases} \{h_1/0.3, h_2/0.7, h_3/0.6, h_4/0.9, h_5/0.1\} & \text{if } e = e_2, \\ \{h_1/0.6, h_2/1.0, h_3/0.3, h_4/0.2, h_5/0.5\} & \text{if } e = e_3, \\ \{h_1/0.4, h_2/0.2, h_3/0.7, h_4/0.8, h_5/0.7\} & \text{if } e = e_4, \end{cases}$$

$$h : C \to \mathcal{FP}(X),$$
$$e \longmapsto \begin{cases} \{h_1/0.7, h_2/0.8, h_3/0.5, h_4/0.4, h_5/0.4\} & \text{if } e = e_3, \\ \{h_1/0.5, h_2/0.3, h_3/0.2, h_4/0.1, h_5/0.4\} & \text{if } e = e_4, \\ \{h_1/0.7, h_2/0.8, h_3/0.2, h_4/0.3, h_5/0.9\} & \text{if } e = e_5, \end{cases} .$$

Now

$$\begin{aligned} A_f \sqcup_\varepsilon (B_g \sqcup C_h) &\tilde{=} (A \cup (B \cap C))_{f\tilde{\vee}(g\tilde{\vee}h)}; \\ (A_f \sqcup_\varepsilon B_g) \sqcup (A_f \sqcup_\varepsilon C_h) &\tilde{=} ((A \cup B) \cap (A \cup C))_{(f\tilde{\vee}g)\tilde{\vee}(f\tilde{\vee}h)}; \\ A_f \sqcup_\varepsilon (B_g \sqcap_\varepsilon C_h) &\tilde{=} (A \cup (B \cup C))_{f\tilde{\vee}(g\tilde{\wedge}h)}; \\ (A_f \sqcup_\varepsilon B_g) \sqcap_\varepsilon (A_f \sqcup_\varepsilon C_h) &\tilde{=} ((A \cup B) \cup (B \cup C))_{(f\tilde{\vee}g)\tilde{\wedge}(f\tilde{\vee}h)}. \end{aligned}$$

Then

$$(f\tilde{\vee}(g\tilde{\vee}h))(\text{wooden}) = \{h_1/0.7, h_2/0.9, h_3/0.2, h_4/0.4, h_5/0.1\}$$

and

$$((f\tilde{\vee}g)\tilde{\vee}(f\tilde{\vee}h))(\text{wooden}) = \{h_1/0.7, h_2/0.9, h_3/0.6, h_4/0.9, h_5/0.1\}.$$

We see that

$$(f\tilde{\vee}(g\tilde{\vee}h))(\text{wooden}) \neq ((f\tilde{\vee}g)\tilde{\vee}(f\tilde{\vee}h))(\text{wooden}).$$

Thus

$$A_f \sqcup_\varepsilon (B_g \sqcup C_h) \tilde{\neq} (A_f \sqcup_\varepsilon B_g) \sqcup (A_f \sqcup_\varepsilon C_h).$$

Again,

$$(f\tilde{\wedge}(g\tilde{\vee}h))(\text{wooden}) = \{h_1/0.3, h_2/0.7, h_3/0.2, h_4/0.4, h_5/0.1\}$$

and

$$((f\tilde{\wedge}g)\tilde{\vee}(f\tilde{\wedge}h))(\text{wooden}) = \{h_1/0.7, h_2/0.9, h_3/0.2, h_4/0.4, h_5/0.1\}.$$

We see that

$$(f\tilde{\wedge}(g\tilde{\vee}h))(\text{wooden}) \neq ((f\tilde{\wedge}g)\tilde{\vee}(f\tilde{\wedge}h))(\text{wooden}).$$

Thus

$$A_f \sqcap_\varepsilon (B_g \sqcup_\varepsilon C_h) \tilde{\neq} (A_f \sqcap_\varepsilon B_g) \sqcup_\varepsilon (A_f \sqcap_\varepsilon C_h).$$

Similarly it can be shown that

$$\begin{aligned} A_f \sqcap_\varepsilon (B_g \sqcap C_h) &\tilde{\neq} (A_f \sqcap_\varepsilon B_g) \sqcap (A_f \sqcap_\varepsilon C_h). \\ A_f \sqcup_\varepsilon (B_g \sqcap_\varepsilon C_h) &\tilde{\neq} (A_f \sqcup_\varepsilon B_g) \sqcap_\varepsilon (A_f \sqcup_\varepsilon C_h). \end{aligned}$$

3.3.8 Proposition

Let A_f, B_g and C_h be any *fuzzy soft sets* over X. Then

1)

$$A_f \sqcup_\varepsilon (B_g \sqcap_\varepsilon C_h) \tilde{=} (A_f \sqcup_\varepsilon B_g) \sqcap_\varepsilon (A_f \sqcup_\varepsilon C_h)$$

if and only if

$$\begin{aligned} f(e) &\subseteq g(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ f(e) &\subseteq h(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

2)

$$A_f \sqcap_\varepsilon (B_g \sqcup_\varepsilon C_h) \tilde{=} (A_f \sqcap_\varepsilon B_g) \sqcup_\varepsilon (A_f \sqcap_\varepsilon C_h)$$

if and only if

$$\begin{aligned} f(e) &\supseteq g(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ f(e) &\supseteq h(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

Proof. Straightforward. ■

3.3.9 Corollary

Let A_f, B_g and C_h be any *fuzzy soft sets* over X. Then

$$\begin{aligned} A_f \sqcup_\varepsilon (B_g \sqcap_\varepsilon C_h) &\tilde{=} (A_f \sqcup_\varepsilon B_g) \sqcap_\varepsilon (A_f \sqcup_\varepsilon C_h) \text{ and} \\ A_f \sqcap_\varepsilon (B_g \sqcup_\varepsilon C_h) &\tilde{=} (A_f \sqcap_\varepsilon B_g) \sqcup_\varepsilon (A_f \sqcap_\varepsilon C_h) \end{aligned}$$

hold if and only if

$$\begin{aligned} f(e) &= g(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ f(e) &= h(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

3.3.10 Corollary

Let A_f, B_g and C_h be any *fuzzy soft sets* over X such that $(A \cap B) - C = (A \cap C) - B = \emptyset$. Then

1) $A_f \sqcup_\varepsilon (B_g \sqcap_\varepsilon C_h) \tilde{=} (A_f \sqcup_\varepsilon B_g) \sqcap_\varepsilon (A_f \sqcup_\varepsilon C_h)$,

2) $A_f \sqcap_\varepsilon (B_g \sqcup_\varepsilon C_h) \tilde{=} (A_f \sqcap_\varepsilon B_g) \sqcup_\varepsilon (A_f \sqcap_\varepsilon C_h)$.

3.3.11 Corollary

Let A_f, A_g and A_h be any *fuzzy soft sets* over X. Then

$$A_f \lambda (A_g \mu A_h) \tilde{=} (A_f \lambda A_g) \mu (A_f \lambda A_h)$$

for distinct λ, $\mu \in \{\sqcap_\varepsilon, \sqcap, \sqcup_\varepsilon, \sqcup\}$.

3.3.12 Proposition

Let A_f and B_g be two *fuzzy soft sets* over X. Then the following are true

1) $A_f \sqcup_\varepsilon B_g$ is the smallest fuzzy soft set over X which contains both A_f and B_g. (Supremum)

2) $A_f \sqcap B_g$ is the largest fuzzy soft set over X which is contained in both A_f and B_g. (Infimum)

Proof.

1) $A_f \tilde{\subseteq} A_f \sqcup_\varepsilon B_g$ and $B_g \tilde{\subseteq} A_f \sqcup_\varepsilon B_g$, because $A \subseteq (A \cup B)$, $B \subseteq (A \cup B)$ and $f(e) \subseteq f(e) \vee g(e)$, $g(e) \subseteq f(e) \vee g(e)$. Let C_h be any fuzzy soft set over X, such that $A_f \tilde{\subseteq} C_h$ and $B_g \tilde{\subseteq} C_h$. Then $(A \cup B) \subseteq C$, and $f(e) \subseteq h(e)$, for all $e \in A$, $g(e) \subseteq h(e)$ for all $e \in B$ implies that $(f \tilde{\vee} g)(e) \subseteq h(e)$ for all $e \in (A \cup B)$. Thus $A_f \sqcup_\varepsilon B_g \tilde{\subseteq} C_h$.

2) $A_f \sqcap B_g \tilde{\subseteq} A_f$ and $A_f \sqcap B_g \tilde{\subseteq} B_g$, because $A \cap B \subseteq A$, $A \cap B \subseteq B$ and $f(e) \wedge g(e) \subseteq f(e)$, $f(e) \wedge g(e) \subseteq g(e)$ for all $e \in A \cap B$. Let C_h be any fuzzy soft set over X, such that $C_h \tilde{\subseteq} A_f$ and $C_h \tilde{\subseteq} B_g$. Then $C \subseteq A \cap B$, and $h(e) \subseteq f(e)$, $h(e) \subseteq g(e)$ for all $e \in C$ implies that $h(e) \subseteq f(e) \wedge g(e) = (f \tilde{\wedge} g)(e)$ for all $e \in C$. Thus $C_h \tilde{\subseteq} A_f \sqcap B_g$.

■

3.4 Algebras of Fuzzy Soft Sets

In this section, we use the ideas of lattices and algebras for fuzzy soft collections. We consider collections of fuzzy soft sets and find their distributive lattices. The collections are denoted as follows:

$\mathcal{FSS}(X)^E$: collection of all fuzzy soft sets defined over X

$\mathcal{FSS}(X)_A$: collection of all those fuzzy soft sets defined over X with a fixed parameter set A.

Firstly, we observe that these collections are partially ordered by the relation of fuzzy soft inclusion $\tilde{\subseteq}$.

3.4.1 Proposition

$(\mathcal{FSS}(X)^E, \sqcap_\varepsilon, \sqcup)$, $(\mathcal{FSS}(X)^E, \sqcup, \sqcap_\varepsilon)$, $(\mathcal{FSS}(X)^E, \sqcup_\varepsilon, \sqcap)$, $(\mathcal{FSS}(X)^E, \sqcap, \sqcup_\varepsilon)$, $(\mathcal{FSS}(X)_A, \sqcup, \sqcap)$, and $(\mathcal{FSS}(X)_A, \sqcap, \sqcup)$ are lattices.

Proof. From Propositions 3.3.3, 3.3.4 and 3.3.5 we conclude that the structures form lattices. ■

3.4.2 Proposition

Structures $(\mathcal{FSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{\tilde{\mathbf{0}}}, E_{\tilde{\mathbf{1}}})$, $(\mathcal{FSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{\tilde{\mathbf{1}}}, \emptyset_{\tilde{\mathbf{0}}})$, $(\mathcal{FSS}(X)_A, \sqcap, \sqcup, A_{\tilde{\mathbf{0}}}, A_{\tilde{\mathbf{1}}})$ and $(\mathcal{FSS}(X)_A, \sqcup, \sqcap, A_{\tilde{\mathbf{1}}}, A_{\tilde{\mathbf{0}}})$ are bounded distributive lattices.

Proof. Proposition 3.3.6 assures that $(\mathcal{FSS}(X)^E, \sqcap, \sqcup_\varepsilon)$ and $(\mathcal{FSS}(X)^E, \sqcup_\varepsilon, \sqcap)$ are distributive lattices. From Lemma 3.3.12, we conclude that $(\mathcal{FSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{\tilde{\mathbf{0}}}, E_{\tilde{\mathbf{1}}})$ is a bounded distributive lattice and $(\mathcal{FSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{\tilde{\mathbf{1}}}, \emptyset_{\tilde{\mathbf{0}}})$ is its dual. For any fuzzy soft sets $A_f, A_g \in \mathcal{FSS}(X)_A$,

$$\begin{aligned} A_f \sqcap A_g &\,\tilde{=}\, A_{f \tilde{\wedge} g} \in \mathcal{FSS}(X)_A \text{ and} \\ A_f \sqcup A_g &\,\tilde{=}\, A_{f \tilde{\vee} g} \in \mathcal{FSS}(X)_A. \end{aligned}$$

Thus $(\mathcal{FSS}(X)_A, \sqcap, \sqcup)$ is also a distributive sublattice of $(\mathcal{FSS}(X)^E, \sqcup_\varepsilon, \sqcap)$ and Proposition 3.3.3 tells us that $A_{\tilde{\mathbf{0}}}$, $A_{\tilde{\mathbf{1}}}$ are its lower and upper bounds, respectively. Therefore $(\mathcal{FSS}(X)_A, \sqcap, \sqcup, A_{\tilde{\mathbf{0}}}, A_{\tilde{\mathbf{1}}})$ is a bounded distributive lattice and $(\mathcal{FSS}(X)_A, \sqcup, \sqcap, A_{\tilde{\mathbf{1}}}, A_{\tilde{\mathbf{0}}})$ is its dual. ■

3.4.3 Proposition

Let A_f be a *fuzzy soft set* over X. Then " ´ " is an involution on $\mathcal{FSS}(X)_A$.

Proof.

(i) We have to show that $A_{(f\,')'} \tilde{=} A_f$. Now, $(A_f\,')' \tilde{=} A_{(f\,')'}$.

$$\begin{aligned} ((f\,')'(e))(x) &= (\bar{\mathbf{1}} - f\,'(e))(x) \\ &= 1 - (f\,'(e))(x) \\ &= 1 - (\bar{\mathbf{1}} - f(e))(x) \\ &= 1 - 1 + (f(e))(x) \\ &= 1 - 1 + (f(e))(x) \\ &= (f(e))(x) \end{aligned}$$

for all $e \in A$, $x \in X$. Thus $(A_f\,')' \tilde{=} A_f$.

(ii) If $A_f \tilde{\subseteq} A_g$ then

$$\begin{aligned} (f(e))(x) &\le (g(e))(x) \text{ and so} \\ 1 - (g(e))(x) &\le 1 - (f(e))(x) \text{ which gives} \\ (g(e))(x) &\le (f(e))(x) \text{ for all } e \in A, x \in X. \end{aligned}$$

Hence $A_g \tilde{\subseteq} A_f$.

Thus " ´ " is an involution on $\mathcal{FSS}(X)_A$. ■

3.4.4 Proposition (de Morgan Laws)

Let A_f and B_g be any fuzzy soft sets over X. Then the following are true

1) $(A_f \sqcup_\varepsilon B_g)' \tilde{=} A_{f'} \sqcap_\varepsilon B_{g'}$,

2) $(A_f \sqcap_\varepsilon B_g)' \tilde{=} A_{f'} \sqcup_\varepsilon B_{g'}$,

3) $(A_f \vee B_g)' \tilde{=} A_{f'} \wedge B_{g'}$,

4) $(A_f \wedge B_g)' \tilde{=} A_{f'} \vee B_{g'}$,

5) $(A_f \sqcup B_g)' \tilde{=} A_{f'} \sqcap B_{g'}$,

6) $(A_f \sqcap B_g)' \tilde{=} A_{f'} \sqcup B_{g'}$.

Proof.

1) We know that $(A_f \sqcup_\varepsilon B_g)' \tilde{=} ((A \cup B)_{f \tilde{\vee} g})' \tilde{=} ((A \cup B)_{(f \tilde{\vee} g)'}$. Let $e \in (A \cup B)$. Then there are three cases:

(i) If $e \in A - B$, then

$$((f \tilde{\vee} g)')(e) = (f(e))' = f'(e) \text{ and } (f' \tilde{\wedge} g')(e) = f'(e).$$

(ii) If $e \in B - A$, then

$$(f \tilde{\vee} g)'(e) = (g(e))' = g'(e) \text{ and } (f' \tilde{\wedge} g')(e) = g'(e).$$

(iii) If $e \in A \cap B$, then

$$(f \tilde{\vee} g)'(e) = (f(e) \vee g(e))' = (f(e))' \wedge (g(e))'$$

and,

$$(f' \tilde{\wedge} g')(e) = (f(e))' \wedge (g(e))'$$

Therefore, in all three cases we obtain equality and thus

$$(A_f \sqcup_\varepsilon B_g)' \tilde{=} A_{f'} \sqcap_\varepsilon B_{g'}.$$

The remaining parts can be proved in a similar way. ■

3.4.5 Proposition

$(\mathcal{FSS}(X)_A, \sqcap, \sqcup, ', A_{\tilde{0}}, A_{\tilde{1}})$ is a de Morgan algebra.

Proof. We have already seen that $(\mathcal{FSS}(X)_A, \sqcap, \sqcup, A_{\tilde{0}}, A_{\tilde{1}})$ is a bounded distributive lattice. Proposition 3.4.3 shows that " $'$ " is an involution on $\mathcal{FSS}(X)_A$ and Proposition 3.4.4 shows that de Morgan laws hold with respect to $'$ in $\mathcal{FSS}(X)_A$. Thus $(\mathcal{FSS}(X)_A, \sqcap, \sqcup, ', A_{\tilde{0}}, A_{\tilde{1}})$ is a de Morgan algebra. ■

3.4.6 Proposition

Let A_f and A_g be *any* fuzzy soft sets over X. Then $(A_g \sqcap A_{g'})\tilde{\subseteq}(A_f \sqcup A_{f'})$ and so $(\mathcal{FSS}(X)_A, \sqcap, \sqcup, ', A_{\tilde{\mathbf{0}}}, A_{\tilde{\mathbf{1}}})$ is a Kleene Algebra.

Proof. For any $A_f, A_g \in \mathcal{FSS}(X)_A$, such that

$$A_f \sqcap A_{f'} \tilde{\supseteq} A_g \sqcup A_{g'} \text{ where } A_f \sqcap A_{f'} \tilde{\neq} A_g \sqcup A_{g'}.$$

Then there exists some $e \in A$ such that

$$(f \sqcap f')(e)\tilde{\supseteq}(g \sqcup g')(e)$$

and so we have some $x \in X$ such that

$$\begin{aligned} ((f \sqcap f')(e))(x) &> ((g \sqcup g')(e))(x) && \text{or} \\ (f(e) \sqcap f'(e))(x) &> (g(e) \sqcup g'(e))(x) && \text{or} \\ (f(e))(x) \wedge (f'(e))(x) &> (g(e))(x) \vee (g'(e))(x). \end{aligned}$$

But $(f(e))(x) \wedge (f'(e))(x) \leq 0.5$ and $(g(e))(x) \vee (g'(e))(x) \geq 0.5$ which gives

$$(f(e))(x) \wedge (f'(e))(x) \leq (g(e))(x) \vee (g'(e))(x).$$

A contradiction, thus our supposition is wrong. Hence

$$A_f \sqcap A_{f'} \tilde{\subseteq} A_g \sqcup A_{g'}.$$

Therefore $(\mathcal{FSS}(X)_A, \sqcap, \sqcup, ', A_{\tilde{\mathbf{0}}}, A_{\tilde{\mathbf{1}}})$ is a Kleene algebra. ■

3.4.7 Proposition

Let $A_f, B_g \in \mathcal{FSS}(X)^E$. Then pseudocomplement of A_f relative to B_g exists in $\mathcal{FSS}(X)^E$.

Proof. Consider the set

$$T(A_f, B_g) = \{C_h \in \mathcal{FSS}(X)^E : C_h \sqcap A_f \tilde{\subseteq} B_g\}.$$

We define a fuzzy soft set $(A^c \cup B)_{f \to g} \in \mathcal{FSS}(X)^E$ where

$$((f \to g)(e))(x) = \begin{cases} 1 & \text{if } e \in A^c - B \\ \begin{cases} 1 & \text{if } (f(e))(x) \leq (g(e))(x) \\ (g(e))(x) & \text{if } (f(e))(x) > (g(e))(x) \end{cases} & \text{if } e \in B - A^c \\ 1 & \text{if } e \in A^c \cap B \end{cases}$$

Then

$$\begin{aligned} (A^c \cup B)_{f \to g} \sqcap A_f &\tilde{=} ((A^c \cup B) \cap A)_{(f \to g)\tilde{\wedge} f} \\ &\tilde{=} ((A^c \cap A) \cup (B \cap A))_{(f \to g)\tilde{\wedge} f} \\ &\tilde{=} (A \cap B)_{(f \to g)\tilde{\wedge} f}. \end{aligned}$$

For any $e \in A \cap B$, $x \in X$,

$$\begin{aligned}
(((f &\to g)\tilde{\wedge} f)(e))(x) \\
&= \begin{cases} 1 \wedge (f(e))(x) & \text{if } (f(e))(x) \leq (g(e))(x) \\ (g(e))(x) \wedge (f(e))(x) & \text{if } (f(e))(x) > (g(e))(x) \end{cases} \\
&= \begin{cases} (f(e))(x) & \text{if } (f(e))(x) \leq (g(e))(x) \\ (g(e))(x) & \text{if } (f(e))(x) > (g(e))(x) \end{cases} \\
&\leq (g(e))(x).
\end{aligned}$$

Hence,

$$(A^c \cup B)_{f\to g} \sqcap A_f \tilde{\subseteq} B_g$$

Thus $(A^c \cup B)_{f\to g} \in T(A_f, B_g)$. For all $C_h \in T(A_f, B_g)$, we have $C_h \sqcap A_f \tilde{\subseteq} B_g$ so for any $e \in C \cap A \subseteq B$

$$h(e) \wedge f(e) \subseteq g(e).$$

Now,

$$\begin{aligned}
C \cap A &\subseteq B \Rightarrow (A \cap C) \cap B^c = \emptyset \\
&\Rightarrow C \subseteq (A \cap B^c)^c = A^c \cup B.
\end{aligned}$$

We have following cases:

(i) If $e \in (A^c - B) \cap C$, then $h(e))(x) < 1 = ((f \to g)(e))(x)$

(ii) If $e \in (B - A^c) \cap C$, and $(f(e))(x) \leq (g(e))(x)$ then $(h(e))(x) < 1 = ((f \to g)(e))(x)$

(iii) If $e \in (B - A^c) \cap C$ and $(f(e))(x) > (g(e))(x)$, then the condition $h(e) \wedge f(e) \subseteq g(e)$ implies that $(h(e))(x) \wedge (f(e)(x)) \leq (g(e))(x)$ which is possible only if $(h(e))(x) \wedge (f(e)(x)) = (h(e))(x)$ and thus $(h(e))(x) \leq (g(e))(x) = ((f \to g)(e))(x)$

(iv) If $e \in (A^c \cap B) \cap C$, then $h(e))(x) < 1 = ((f \to g)(e))(x)$.

Thus $C_h \tilde{\subseteq} (A^c \cup B)_{f\to g}$ and it also shows that $(A^c \cup B)_{f\to g} \tilde{=} \bigvee T(A_f, B_g) \tilde{=} A_f \to B_g$. ■

3.4.8 Remark

We know that $(\mathcal{FSS}(X)_A, \sqcap, \sqcup)$ is a sublattice of $(\mathcal{FSS}(X)^E, \sqcap_\varepsilon, \sqcup)$. *For any* A_f, $A_g \in \mathcal{FSS}(X)_A$, $A_f \to A_g$ (as defined in Proposition 3.4.7) is not in $\mathcal{FSS}(X)_A$ because $A_f \to A_g \tilde{=} (A^c \cup A)_{f\to g} \tilde{=} E_{f\to g} \notin \mathcal{FSS}(X)_A$.

3.4.9 Proposition

Let $A_f, A_g \in \mathcal{FSS}(X)_A$. Then pseudocomplement of A_f relative to A_g exists in $\mathcal{FSS}(X)_A$.

Proof. Consider the set

$$T(A_f, A_g) = \{A_h \in \mathcal{FSS}(X)_A : A_h \sqcap A_f \tilde{\subseteq} A_g\}.$$

We define a fuzzy soft set $A_{f\to g} \in \mathcal{FSS}(X)_A$ where

$$((f \to g)(e))(x) = \begin{cases} 1 & \text{if } (f(e))(x) \leq (g(e))(x) \\ (g(e))(x) & \text{if } (f(e))(x) > (g(e))(x) \end{cases}$$

for all $e \in A$, $x \in X$. Then $A_{f\to g} \sqcap A_f \tilde{=} A_{(f\to g)\tilde{\wedge} f}$ and

$$\begin{aligned} (((f \to g)\tilde{\wedge} f)(e))(x) &= \begin{cases} 1 \wedge (f(e))(x) & \text{if } (f(e))(x) \leq (g(e))(x) \\ (g(e))(x) \wedge (f(e))(x) & \text{if } (f(e))(x) > (g(e))(x) \end{cases} \\ &= \begin{cases} (f(e))(x) & \text{if } (f(e))(x) \leq (g(e))(x) \\ (g(e))(x) & \text{if } (f(e))(x) > (g(e))(x) \end{cases} \\ &\leq (g(e))(x). \end{aligned}$$

for all $e \in A$, $x \in X$. Hence,

$$A_{f\to g} \sqcap A_f \tilde{\subseteq} A_g$$

and $A_{f\to g} \in T(A_f, A_g)$. For every $A_h \in T(A_f, A_g)$, we have $A_h \sqcap A_f \tilde{\subseteq} A_g$ so for any $e \in A$, following cases arise:

(i) If $(f(e))(x) \leq (g(e))(x)$ then $(h(e))(x) < 1 = ((f \to g)(e))(x)$

(ii) If $(f(e))(x) > (g(e))(x)$ then the condition $h(e) \wedge f(e) \subseteq g(e)$ implies that $(h(e))(x) \wedge (f(e)(x)) \leq (g(e))(x)$ and so $(h(e))(x) \leq (g(e))(x) = ((f \to g)(e))(x)$.

Thus $A_h \tilde{\subseteq} A_{f\to g}$ and it also shows that

$$A_{f\to g} \tilde{=} \bigvee T(A_f, A_g) \tilde{=} A_f \to_A A_g.$$

■

3.4.10 Proposition

$(\mathcal{FSS}(X)^E, \sqcap_\varepsilon, \sqcup)$ *and* $(\mathcal{FSS}(X)_A, \sqcap, \sqcup)$ *are Brouwerian lattices.*

Proof. Follows from Propositions 3.4.7 and 3.4.9. ■

3.4.11 Definition

For a *fuzzy soft set* A_f over X, we define a fuzzy soft set over X, which is denoted by A_{f^*} and is given by $A_{f^*} = (A_f)^*$ where

$$(f^*(e))(x) = \begin{cases} 0 & \text{if } (f(e))(x) \neq 0 \\ 1 & \text{if } (f(e))(x) = 0 \end{cases}$$

for all $x \in X$, $e \in A$.

3.4.12 Theorem

Let A_f be a *fuzzy soft set* over X. Then the following are true:

1) $A_f \sqcap A_{f^*} \stackrel{\sim}{=} A_{\tilde{0}}$,

2) $A_g \tilde{\subseteq} A_{f^*}$ whenever $A_f \sqcap A_g \stackrel{\sim}{=} A_{\tilde{0}}$,

3) $A_{f^*} \sqcup A_{f^{**}} \stackrel{\sim}{=} A_{\tilde{1}}$.

Thus $(\mathcal{FSS}(X)_A, \sqcap, \sqcup, ^*, A_{\tilde{0}}, A_{\tilde{1}})$ is a Stone algebra.
Proof.

1) Straightforward.

2) If $A_f \sqcap A_g \stackrel{\sim}{=} A_\Phi$. Then for any $x \in X$, $e \in A$,

$$\text{if } (g(e))(x) = 0 \text{ then } (g(e))(x) \leq (f^*(e))(x).$$

$$\begin{array}{c} \text{If } (g(e))(x) \neq 0 \text{ then } (f(e))(x) \wedge (g(e))(x) = 0 \\ \text{implies that } (f(e))(x) = 0, \text{ so } (f^*(e))(x) = 1 \\ \text{and hence } \quad (g(e))(x) \leq 1 = (f^*(e))(x). \end{array}$$

Thus ,

$$(g(e))(x) \leq (f^*(e))(x) \text{ for all } x \in X,\ e \in A.$$

That is, $A_g \tilde{\subseteq} A_{f^*}$.

3) For any $x \in X$, $e \in A$,

$$\begin{aligned} ((f^* \sqcup f^{**})(e))(x) &= (f^*(e) \vee f^{**}(e))(x) \\ &= \max\{(f^*(e))(x), (f^{**}(e))(x)\} \\ &= \begin{cases} \max\{1,0\} & \text{if } (f(e))(x) \neq 0 \\ \max\{0,1\} & \text{if } (f(e))(x) = 0 \end{cases} \\ &= 1. \end{aligned}$$

Thus $A_{f^*} \sqcup A_{f^{**}} \stackrel{\sim}{=} A_{\tilde{1}}$ and so, $(\mathcal{FSS}(X)_A, \sqcap, \sqcup, ^*, A_{\tilde{0}}, A_{\tilde{1}})$ is a *Stone algebra*.

■

3.4.13 Remark

Note that $A_{f^*} \stackrel{\sim}{=} A_f \rightarrow_A A_{\tilde{0}}$.

Chapter 4

Algebraic Structures of Double-framed Soft Sets

This chapter explores the theory of double-framed soft sets. Double-framed soft sets have been introduced by Jun et al. [19] in 2012. They discussed applications of double-framed soft sets in BCK/BCI-algebras and verified several results with uni-int concepts. Recently, some further works are presented to characterize the ideals of BCK/BCI-algebras in terms of double-framed soft sets in [20]. In our work, we have focused upon the algebraic structural properties of double-framed soft sets. New operations for double-framed soft sets are defined and their characteristics are studied. Examples are given to elaborate the concepts and to show how the ideas are utilized to work with double-framed soft sets. The lattice structure and different algebraic specifications raised by the collections of double-framed soft sets have been shown in a logical manner. Classes of MV-algebras and BCK/BCI-algebras of double-framed soft sets are presented at the end.

4.1 Double-framed Soft Sets

Let X be an initial universe and E be a set of parameters. Let $\mathcal{P}(X)$ denotes the power set of X and A, B, C are non-empty subsets of E.

4.1.1 Definition [19]

A double-framed pair $\langle(\alpha, \beta); A\rangle$ is called a double-framed soft set over X, where α and β are mappings from A to $\mathcal{P}(X)$.

From now onwards, we shall use the notation $A_{(\alpha,\beta)}$ over X to denote a double-framed soft set $\langle(\alpha, \beta); A\rangle$ over X.

4.1.2 Definition

For double-*framed* soft sets $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X, we say that $A_{(\alpha,\beta)}$ is a *double-framed* soft subset of $B_{(\gamma,\delta)}$, if

1) $A \subseteq B$ and

2) $\alpha(e) \subseteq \gamma(e)$ and $\delta(e) \subseteq \beta(e)$ for all $e \in A$.

This relationship is denoted by $A_{(\alpha,\beta)} \tilde{\subseteq} B_{(\gamma,\delta)}$.

$A_{(\alpha,\beta)}$ is said to be a *double-framed soft superset* of $B_{(\gamma,\delta)}$, if $B_{(\gamma,\delta)}$ is a *double-framed soft subset* of $A_{(\alpha,\beta)}$. We denote it by $A_{(\alpha,\beta)} \tilde{\supseteq} B_{(\gamma,\delta)}$.

4.1.3 Definition

Two *double-framed soft sets* $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X are said to be *equal* if $A_{(\alpha,\beta)}$ is a *double-framed soft subset* of $B_{(\gamma,\delta)}$ and $B_{(\gamma,\delta)}$ is a *double-framed soft subset* of $A_{(\alpha,\beta)}$. We denote it by $A_{(\alpha,\beta)} \tilde{=} B_{(\gamma,\delta)}$.

4.1.4 Example

Let X be the set of houses under consideration, and E be the set of parameters, $X = \{h_1,h_2,h_3,h_4,h_5\}$, $E = \{e_1,e_2,e_3,e_4,e_5,e_6\} = \{$ in the green surroundings, wooden, cheap, in good repair, furnished, traditional $\}$. Suppose that $A = \{e_1,e_2,e_3,e_6\}$, a double-framed soft set $A_{(\alpha,\beta)}$ describes the data for "requirements of the houses" where function α approximates the houses with a high level of appreciation and β approximates the houses with a high level of critique by two different groups of experts and given as follows:

$$\alpha \ : \ A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_2, h_3, h_4\} & \text{if } e = e_1, \\ \{h_3, h_4\} & \text{if } e = e_2, \\ X & \text{if } e = e_3, \\ \{h_2, h_3, h_4, h_5\} & \text{if } e = e_6, \end{cases}$$

$$\beta \ : \ A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_2, h_4, h_5\} & \text{if } e = e_1, \\ \{h_1, h_2, h_3\} & \text{if } e = e_2, \\ \{h_3, h_4, h_5\} & \text{if } e = e_3, \\ \{h_1, h_3, h_4, h_5\} & \text{if } e = e_6. \end{cases}$$

Let $B = \{e_2,e_3,e_6\}$. The double-framed soft set $B_{(\gamma,\delta)}$ given by

$$\gamma \ : \ B \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_3\} & \text{if } e = e_2, \\ \{h_1, h_4, h_5\} & \text{if } e = e_3, \\ \{h_2, h_3, h_4\} & \text{if } e = e_6, \end{cases}$$

$$\delta \ : \ B \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_1, h_2, h_3, h_5\} & \text{if } e = e_2, \\ \{h_1, h_3, h_4, h_5\} & \text{if } e = e_3, \\ X & \text{if } e = e_6. \end{cases}$$

is a double-framed soft subset of $A_{(\alpha,\beta)}$ so $A_{(\alpha,\beta)} \tilde{\subseteq} B_{(\gamma,\delta)}$. Here, we can see that γ approximates less houses than α being less appreciating, while δ approximates more houses than β being less critical. This justifies our definition of inclusion for double-framed soft sets.

4.2 Operations on Double-framed Soft Sets

4.2.1 Definition [19]

Let $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ be *double-framed* soft sets over X. The int-uni product of $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ is defined as a double-framed soft set $(A \times B)_{(\alpha\wedge\gamma,\beta\vee\delta)}$ over X in which $\alpha \wedge \gamma : (A \times B) \to \mathcal{P}(X)$, $\beta \vee \delta : (A \times B) \to \mathcal{P}(X)$, defined by

$$(a, b) \mapsto \alpha(a) \cap \gamma(b),\ (a, b) \mapsto \beta(a) \cup \delta(b).$$

It is denoted by $A_{(\alpha,\beta)} \wedge B_{(\gamma,\delta)} \hat{=} (A \times B)_{(\alpha\wedge\gamma,\beta\vee\delta)}$.

4.2.2 Definition [19]

Let $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ be *double-framed* soft sets over X. The uni-int product of $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ is defined as a double-framed soft set $(A \times B)_{(\alpha\vee\gamma,\beta\wedge\delta)}$ over X in which $\alpha \vee \gamma : (A \times B) \to \mathcal{P}(X)$, $\beta \wedge \delta : (A \times B) \to \mathcal{P}(X)$, defined by

$$(a, b) \mapsto \alpha(a) \cup \gamma(b),\ (a, b) \mapsto \beta(a) \cap \delta(b).$$

It is denoted by $A_{(\alpha,\beta)} \vee B_{(\gamma,\delta)} \hat{=} (A \times B)_{(\alpha\vee\gamma,\beta\wedge\delta)}$.

4.2.3 Definition

For double-*framed soft sets* $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X, the *extended int-uni double-framed soft set* of $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ is defined as a *double-framed soft set* $(A \cup B)_{(\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta)}$ where $\alpha\tilde{\cap}\gamma : (A \cup B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \begin{cases} \alpha(e) & if\ e \in A - B \\ \gamma(e) & if\ e \in B - A \\ \alpha(e) \cap \gamma(e) & if\ e \in A \cap B \end{cases}$$

and $\beta\tilde{\cup}\delta : (A \cup B) \to \mathcal{P}(X)$,

$$e \mapsto \begin{cases} \beta(e) & if\ e \in A - B \\ \delta(e) & if\ e \in B - A \\ \beta(e) \cup \delta(e) & if\ e \in A \cap B \end{cases}.$$

It is denoted by $A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)} \hat{=} (A \cup B)_{(\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta)}$.

4.2.4 Definition

For double-*framed soft sets* $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X, the *extended uni-int set double-framed soft* of $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ is defined as a *double-framed soft set* $(A \cup B)_{(\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta)}$ where $\alpha\tilde{\cup}\gamma : (A \cup B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \begin{cases} \alpha(e) & if\ e \in A - B \\ \gamma(e) & if\ e \in B - A \\ \alpha(e) \cup \gamma(e) & if\ e \in A \cap B \end{cases}$$

and $\beta \tilde{\cap} \delta : (A \cup B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \begin{cases} \beta(e) & if\ e \in A - B \\ \delta(e) & if\ e \in B - A \\ \beta(e) \cap \delta(e) & if\ e \in A \cap B \end{cases}.$$

It is denoted by $A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)} \tilde{=} (A \cup B)_{(\alpha \tilde{\cup} \gamma, \beta \tilde{\cap} \delta)}$.

4.2.5 Definition

For double-framed soft sets $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X, the *extended difference double-framed soft set* of $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ is defined as a double-framed soft set $(A \cup B)_{(\alpha \smile_{\varepsilon} \gamma, \beta \smile_{\varepsilon} \delta)}$ where

$$\alpha \smile_{\varepsilon} \gamma : (A \cup B) \to \mathcal{P}(X),\ e \mapsto \begin{cases} \alpha(e) & \text{if } e \in A - B \\ \gamma(e) & \text{if } e \in B - A \\ \alpha(e) - \gamma(e) & \text{if } e \in A \cap B \end{cases}$$

$$\beta \smile_{\varepsilon} \delta : (A \cup B) \to \mathcal{P}(X),\ e \mapsto \begin{cases} \beta(e) & \text{if } e \in A - B \\ \delta(e) & \text{if } e \in B - A \\ \beta(e) - \delta(e) & \text{if } e \in A \cap B \end{cases}.$$

It is denoted by $A_{(\alpha,\beta)} \smile_{\varepsilon} B_{(\gamma,\delta)} \tilde{=} (A \cup B)_{(\alpha \smile_{\varepsilon} \gamma, \beta \smile_{\varepsilon} \delta)}$.

4.2.6 Definition

For double-*framed soft sets* $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X with $A \cap B \neq \emptyset$, the *restricted int-uni double-framed soft set* of $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ is defined as a *double-framed soft set* $(A \cap B)_{(\alpha \tilde{\cap} \gamma, \beta \tilde{\cup} \delta)}$ where $\alpha \tilde{\cap} \gamma : (A \cap B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \alpha(e) \cap \gamma(e),$$

and $\beta \tilde{\cup} \delta : (A \cap B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \beta(e) \cup \delta(e).$$

It is denoted by $A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)} \tilde{=} (A \cap B)_{(\alpha \tilde{\cap} \gamma, \beta \tilde{\cup} \delta)}$.

4.2.7 Definition

For double-*framed soft sets* $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X with $(A \cap B) \neq \emptyset$, the *restricted uni-int double-framed soft set* of $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ is defined as a *double-framed soft set* $(A \cap B)_{(\alpha \tilde{\cup} \gamma, \beta \tilde{\cap} \delta)}$ where $\alpha \tilde{\cup} \gamma : (A \cap B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \alpha(e) \cup \gamma(e),$$

and $\beta \tilde{\cap} \delta : (A \cap B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \beta(e) \cap \delta(e).$$

It is denoted by $A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)} \tilde{=} (A \cap B)_{(\alpha \tilde{\cup} \gamma, \beta \tilde{\cap} \delta)}$.

4.2.8 Definition

For double-*framed soft sets* $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X with $(A \cap B) \neq \emptyset$, the *restricted difference double-framed soft set* of $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ is defined as a *double-framed soft set* $(A \cap B)_{(\alpha \smile \gamma, \beta \smile \delta)}$ where $\alpha \smile \gamma : (A \cap B) \to \mathcal{P}(X)$, defined by

$$e \mapsto \alpha(e) - \gamma(e),$$

and $\beta \smile \delta : (A \cap B) \to \mathcal{P}(X)$, ddefined by

$$e \mapsto \beta(e) - \delta(e).$$

It is denoted by $A_{(\alpha,\beta)} \smile B_{(\gamma,\delta)} \tilde{=} (A \cap B)_{(\alpha \smile \gamma, \beta \smile \delta)}$.

4.2.9 Definition

Let $A_{(\alpha,\beta)}$ be a *double-framed* soft set over X. The *complement of a double-framed soft set* $A_{(\alpha,\beta)}$ is defined as a *double-framed soft set* $A_{(\alpha^c,\beta^c)}$ where

$$\alpha^c : A \to \mathcal{P}(X),\ e \mapsto (\alpha(e))^c \text{ and } \beta^c : A \to \mathcal{P}(X),\ e \mapsto (\beta(e))^c.$$

It is denoted by $A_{(\alpha,\beta)^c} \tilde{=} A_{(\alpha^c,\beta^c)}$.

4.2.10 Example

Let X be the initial universe and E be the set of parameters, where $X = \{x_1,x_2,x_3,x_4,x_5,x_6\}$, $E = \{e_1,e_2,e_3,e_4,e_5\}$. Suppose that $A = \{e_2,e_3\}$, and $B = \{e_3,e_4,\}$. The double-framed soft sets $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ over X are given as follows:

$$\begin{aligned}
\alpha &: A \to \mathcal{P}(X), \\
e &\longmapsto \begin{cases} \{x_2, x_5, x_6\} & \text{if } e = e_2, \\ \{x_1, x_3, x_4, x_5\} & \text{if } e = e_3, \end{cases} \\
\beta &: A \to \mathcal{P}(X), \\
e &\longmapsto \begin{cases} \{x_1\} & \text{if } e = e_2, \\ X & \text{if } e = e_3, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\gamma &: B \to \mathcal{P}(X), \\
e &\longmapsto \begin{cases} X & \text{if } e = e_3, \\ \{x_1, x_4, x_6\} & \text{if } e = e_4, \end{cases} \\
\delta &: B \to \mathcal{P}(X), \\
e &\longmapsto \begin{cases} \{x_1, x_4, x_5, x_6\} & \text{if } e = e_3, \\ \{x_1, x_2, x_5\} & \text{if } e = e_4. \end{cases}
\end{aligned}$$

Now, we apply various operations on $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$. Then

(i) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)} \tilde{=} (A \cup B)_{(\alpha\tilde{\cup}\gamma, \beta\tilde{\cap}\delta)}$, where

$$\begin{aligned}
(\alpha\tilde{\cup}\gamma) &: (A \cup B) \to \mathcal{P}(X), \\
e &\longmapsto \begin{cases} \{x_2, x_5, x_6\} & \text{if } e = e_2, \\ X & \text{if } e = e_3, \\ \{x_1, x_4, x_6\} & \text{if } e = e_4, \end{cases} \\
(\beta\tilde{\cap}\delta) &: (A \cup B) \to \mathcal{P}(X), \\
e &\longmapsto \begin{cases} \{x_1\} & \text{if } e = e_2, \\ \{x_1, x_4, x_5, x_6\} & \text{if } e = e_3, \\ \{x_1, x_2, x_5\} & \text{if } e = e_4, \end{cases}
\end{aligned}$$

(ii) $A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)} \tilde{=} (A \cap B)_{(\alpha\tilde{\cap}\gamma, \beta\tilde{\cup}\delta)}$, where

$$\begin{aligned}
(\alpha\tilde{\cap}\beta) &: (A \cap B) \to \mathcal{P}(X), \\
e_3 &\longmapsto \{x_1, x_3, x_4, x_5\} \\
(\beta\tilde{\cup}\delta) &: (A \cap B) \to \mathcal{P}(X), \\
e_3 &\longmapsto X
\end{aligned}$$

(iii) $A_{(\alpha,\beta)} \smile_{\varepsilon} B_{(\gamma,\delta)} \tilde{=} (A \cup B)_{(\alpha\smile_{\varepsilon}\gamma, \beta\smile_{\varepsilon}\delta)}$, where

$$\begin{aligned}
\alpha &\smile {}_{\varepsilon}\gamma : (A \cup B) \to \mathcal{P}(X), \\
e &\longmapsto \begin{cases} \{x_2, x_5, x_6\} & \text{if } e = e_2, \\ \{\} & \text{if } e = e_3, \\ \{x_1, x_4, x_6\} & \text{if } e = e_4, \end{cases} \\
\beta &\smile {}_{\varepsilon}\delta : (A \cup B) \to \mathcal{P}(X), \\
e &\longmapsto \begin{cases} \{x_1\} & \text{if } e = e_2, \\ \{x_2, x_3\} & \text{if } e = e_3, \\ \{x_1, x_2, x_5\} & \text{if } e = e_4, \end{cases}
\end{aligned}$$

(iv) $A_{(\alpha,\beta)^c} \tilde{=} A_{(\alpha^c,\beta^c)}$, where

$$\begin{aligned}
\alpha^c &: A \to \mathcal{P}(X), \\
e &\longmapsto \begin{cases} \{x_1, x_3, x_4\} & \text{if } e = e_2, \\ \{x_2, x_6\} & \text{if } e = e_3, \end{cases} \\
\beta^c &: A \to \mathcal{P}(X), \\
e &\longmapsto \begin{cases} \{x_2, x_3, x_4, x_5, x_6\} & \text{if } e = e_2, \\ \{\} & \text{if } e = e_3. \end{cases}
\end{aligned}$$

4.3 Properties of Double-framed Soft Sets

In this section we discuss properties and laws of double-framed soft sets with respect to their operations. Associativity, absorption, distributivity, de Morgan laws and properties of involutions, complementations and atomicity are investigated for double-framed soft set theory.

4.3.1 Definition

A *double-framed soft set* over X is said to be a *relative null double-framed soft set*, denoted by $A_{(\Phi,\mathfrak{X})}$ where

$$\Phi : A \to \mathcal{P}(X),\ e \mapsto \emptyset \text{ and } \mathfrak{X} : A \to \mathcal{P}(X),\ e \mapsto X.$$

4.3.2 Definition

A *double-framed* soft set over X is said to be a *relative absolute double-framed soft set*, denoted by $A_{(\mathfrak{X},\Phi)}$ where

$$\mathfrak{X} : A \to \mathcal{P}(X),\ e \mapsto X \text{ and } \Phi : A \to \mathcal{P}(X),\ e \mapsto \emptyset.$$

Conventionally, we take the *double-framed* soft sets with empty set of parameters to be equal to $\emptyset_{(\Phi,\mathfrak{X})}$ and so $A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)} \tilde{=} A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)} \tilde{=} \emptyset_{(\Phi,\mathfrak{X})}$ whenever $(A \cap B) = \emptyset$.

4.3.3 Proposition

If $A_{(\Phi,\mathfrak{X})}$ is a null *double-framed* soft set, $A_{(\mathfrak{X},\Phi)}$ an absolute *double-framed* soft set, and $A_{(\alpha,\beta)}$, $A_{(\gamma,\delta)}$ are *double-framed* soft sets over X, then

1) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} A_{(\gamma,\delta)} \tilde{=} A_{(\alpha,\beta)} \sqcup A_{(\gamma,\delta)}$,

2) $A_{(\alpha,\beta)} \sqcap_{\varepsilon} A_{(\gamma,\delta)} \tilde{=} A_{(\alpha,\beta)} \sqcap A_{(\gamma,\delta)}$,

3) $A_{(\alpha,\beta)} \sqcap A_{(\alpha,\beta)} \tilde{=} A_{(\alpha,\beta)} \tilde{=} A_{(\alpha,\beta)} \sqcup A_{(\alpha,\beta)}$,

4) $A_{(\alpha,\beta)} \sqcup A_{(\Phi,\mathfrak{X})} \tilde{=} A_{(\alpha,\beta)} \tilde{=} A_{(\alpha,\beta)} \sqcap A_{(\mathfrak{X},\Phi)}$,

5) $A_{(\alpha,\beta)} \sqcup A_{(\mathfrak{X},\Phi)} \tilde{=} A_{(\mathfrak{X},\Phi)}$; $A_{(\alpha,\beta)} \sqcap A_{(\Phi,\mathfrak{X})} \tilde{=} A_{(\Phi,\mathfrak{X})}$.

Proof. Proofs of 1), 2) and 3) are straightforward.

4) As $A_{(\alpha,\beta)} \sqcup A_{(\Phi,\mathfrak{X})} \tilde{=} A_{(\alpha\tilde{\cup}\Phi,\beta\tilde{\cap}\mathfrak{X})}$. Therefore for any $e \in A$,

$$(\alpha\tilde{\cup}\Phi)(e) = \alpha(e) \cup \Phi(e) = \alpha(e) \text{ and } (\beta\tilde{\cap}\mathfrak{X})(e) = \beta(e) \cap \mathfrak{X}(e) = \beta(e).$$

Thus $A_{(\alpha,\beta)} \sqcup A_{(\Phi,\mathfrak{X})} \tilde{=} A_{(\alpha,\beta)}$.

Again, $A_{(\alpha,\beta)} \sqcap A_{(\mathfrak{X},\Phi)} \tilde{=} A_{(\alpha\tilde{\cap}\mathfrak{X},\beta\tilde{\cup}\Phi)}$. For any $e \in A$,

$$(\alpha\tilde{\cap}\mathfrak{X})(e) = \alpha(e) \cap \mathfrak{X}(e) = \alpha(e) \text{ and } (\beta\tilde{\cup}\Phi)(e) = \beta(e) \cup \Phi(e) = \beta(e).$$

So $A_{(\alpha,\beta)} \sqcap A_{(\mathfrak{X},\Phi)} \tilde{=} A_{(\alpha,\beta)}$.

Part 5) can be proved in a similar way. ■

4.3.4 Proposition

Let $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}$ and $C_{(\zeta,\eta)}$ be any *double-framed soft sets* over X. Then the following are true

1) $A_{(\alpha,\beta)}\lambda(B_{(\gamma,\delta)}\lambda C_{(\zeta,\eta)})\doteq(A_{(\alpha,\beta)}\lambda B_{(\gamma,\delta)})\lambda C_{(\zeta,\eta)}$, (Associative Laws)

2) $A_{(\alpha,\beta)}\lambda B_{(\gamma,\delta)}\doteq B_{(\gamma,\delta)}\lambda A_{(\alpha,\beta)}$, (Commutative Laws)

for all $\lambda \in \{\sqcup_\varepsilon, \sqcup, \sqcap_\varepsilon, \sqcap\}$.

Proof.

1) Since $A_{(\alpha,\beta)} \sqcup_\varepsilon (B_{(\gamma,\delta)} \sqcup_\varepsilon C_{(\zeta,\eta)})\doteq(A \cup (B \cup C))_{(\alpha\tilde{\cup}(\gamma\tilde{\cup}\zeta),\beta\tilde{\cap}(\delta\tilde{\cap}\eta))}$, we have for any $e \in A \cup (B \cup C)$:

(i) If $e \in A - (B \cup C)$, then

$$(\alpha\tilde{\cup}(\gamma\tilde{\cup}\zeta))(e) = \alpha(e) = ((\alpha\tilde{\cup}\gamma)\tilde{\cup}\zeta)(e)$$
$$(\beta\tilde{\cap}(\delta\tilde{\cap}\eta))(e) = \beta(e) = ((\beta\tilde{\cap}\delta)\tilde{\cap}\eta)(e)$$

(ii) If $e \in B - (A \cup C)$

$$(\alpha\tilde{\cup}(\gamma\tilde{\cup}\zeta))(e) = \gamma(e) = ((\alpha\tilde{\cup}\gamma)\tilde{\cup}\zeta)(e)$$
$$(\beta\tilde{\cap}(\delta\tilde{\cap}\eta))(e) = \delta(e) = ((\beta\tilde{\cap}\delta)\tilde{\cap}\eta)(e)$$

(iii) If $e \in C - (A \cup B)$, then

$$\alpha\tilde{\cup}(\gamma\tilde{\cup}\zeta)(e) = \zeta(e) = (\alpha\tilde{\cup}\gamma)\tilde{\cup}\zeta(e)$$
$$\beta\tilde{\cap}(\delta\tilde{\cap}\eta)(e) = \eta(e) = (\beta\tilde{\cap}\delta)\tilde{\cap}\eta(e)$$

(iv) If $e \in (A \cap B) - C$, then

$$\alpha\tilde{\cup}(\gamma\tilde{\cup}\zeta)(e) = \alpha(e) \cup \gamma(e) = (\alpha\tilde{\cup}\gamma)(e) = (\alpha\tilde{\cup}\gamma)\tilde{\cup}\zeta(e)$$
$$\beta\tilde{\cap}(\delta\tilde{\cap}\eta)(e) = \beta(e) \cap \delta(e) = (\beta\tilde{\cap}\delta)(e) = (\beta\tilde{\cap}\delta)\tilde{\cap}\eta(e)$$

(v) If $e \in (A \cap C) - B$, then

$$\alpha\tilde{\cup}(\gamma\tilde{\cup}\zeta)(e) = \alpha(e) \cup \zeta(e) = (\alpha\tilde{\cup}\gamma)\tilde{\cup}\zeta(e)$$
$$\beta\tilde{\cap}(\delta\tilde{\cap}\eta)(e) = \beta(e) \cap \eta(e) = (\beta\tilde{\cap}\delta)\tilde{\cap}\eta(e)$$

(vi) If $e \in (B \cap C) - A$, then

$$\alpha\tilde{\cup}(\gamma\tilde{\cup}\zeta)(e) = \gamma(e) \cup \zeta(e) = (\alpha\tilde{\cup}\gamma)\tilde{\cup}\zeta(e)$$
$$\beta\tilde{\cap}(\delta\tilde{\cap}\eta)(e) = \delta(e) \cap \eta(e) = (\beta\tilde{\cap}\delta)\tilde{\cap}\eta(e)$$

(vii) If $e \in (A \cap B) \cap C$, then

$$\alpha\tilde{\cup}(\gamma\tilde{\cup}\zeta)(e) = \alpha(e) \cup (\gamma(e) \cup \zeta(e)) = (\alpha(e) \cup \gamma(e)) \cup \zeta(e) = (\alpha\tilde{\cup}\gamma)\tilde{\cup}\zeta(e)$$
$$\beta\tilde{\cap}(\delta\tilde{\cap}\eta)(e) = \beta(e) \cap (\delta(e) \cap \eta(e)) = (\beta(e) \cap \delta(e)) \cap \eta(e) = (\beta\tilde{\cap}\delta)\tilde{\cap}\eta(e)$$

Thus $A_{(\alpha,\beta)} \sqcup_\varepsilon (B_{(\gamma,\delta)} \sqcup_\varepsilon C_{(\zeta,\eta)})\doteq(A_{(\alpha,\beta)} \sqcup_\varepsilon B_{(\gamma,\delta)}) \sqcup_\varepsilon C_{(\zeta,\eta)}$. Similarly, we can prove for $\lambda \in \{\sqcup, \sqcap_\varepsilon, \sqcap\}$.

2) This is straightforward.

■

4.3.5 Proposition (Absorption Laws)

Let $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}$ be any *double-framed* soft sets over X. Then the following are true:

1) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcap A_{(\alpha,\beta)}) \breve{=} A_{(\alpha,\beta)}$,

2) $A_{(\alpha,\beta)} \sqcap (B_{(\gamma,\delta)} \sqcup_{\varepsilon} A_{(\alpha,\beta)}) \breve{=} A_{(\alpha,\beta)}$,

3) $A_{(\alpha,\beta)} \sqcup (B_{(\gamma,\delta)} \sqcap_{\varepsilon} A_{(\alpha,\beta)}) \breve{=} A_{(\alpha,\beta)}$,

4) $A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcup A_{(\alpha,\beta)}) \breve{=} A_{(\alpha,\beta)}$.

Proof. Straightforward. ■

4.3.6 Proposition (Distributive Laws)

Let $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}$ and $C_{(\zeta,\eta)}$ be any *double-framed soft sets* over X. Then

1) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcup C_{(\zeta,\eta)}) \breve{\subseteq} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}) \sqcup (A_{(\alpha,\beta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)})$,

2) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}) \breve{\supseteq} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}) \sqcap_{\varepsilon} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)})$,

3) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcap C_{(\zeta,\eta)}) \breve{=} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}) \sqcap (A_{(\alpha,\beta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)})$,

4) $A_{(\alpha,\beta)} \sqcup (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \breve{=} (A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcup C_{(\zeta,\eta)})$,

5) $A_{(\alpha,\beta)} \sqcup (B_{(\gamma,\delta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}) \breve{=} (A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)}) \sqcap_{\varepsilon} (A_{(\alpha,\beta)} \sqcup C_{(\zeta,\eta)})$,

6) $A_{(\alpha,\beta)} \sqcup (B_{(\gamma,\delta)} \sqcap C_{(\zeta,\eta)}) \breve{=} (A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)}) \sqcap (A_{(\alpha,\beta)} \sqcup C_{(\zeta,\eta)})$,

7) $A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \breve{\subseteq} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)})$,

8) $A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcup C_{(\zeta,\eta)}) \breve{=} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)}) \sqcup (A_{(\alpha,\beta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)})$,

9) $A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcap C_{(\zeta,\eta)}) \breve{\supseteq} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)}) \sqcap (A_{(\alpha,\beta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)})$,

10) $A_{(\alpha,\beta)} \sqcap (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \breve{=} (A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcap C_{(\zeta,\eta)})$,

11) $A_{(\alpha,\beta)} \sqcap (B_{(\gamma,\delta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}) \breve{=} (A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)}) \sqcap_{\varepsilon} (A_{(\alpha,\beta)} \sqcap C_{(\zeta,\eta)})$,

12) $A_{(\alpha,\beta)} \sqcap (B_{(\gamma,\delta)} \sqcup C_{(\zeta,\eta)}) \breve{=} (A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)}) \sqcup (A_{(\alpha,\beta)} \sqcap C_{(\zeta,\eta)})$.

Proof. Consider 10)

$$A_{(\alpha,\beta)} \sqcap (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \breve{=} (A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcap C_{(\zeta,\eta)}).$$

For any $e \in A \cap (B \cup C)$, we have following three disjoint cases:

(i) If $e \in A \cap (B - C)$, then

$$(\alpha \tilde{\cap} (\gamma \tilde{\cup} \zeta))(e) = \alpha(e) \cap \gamma(e) \quad \text{and} \quad (\beta \tilde{\cup} (\delta \tilde{\cap} \eta))(e) = \beta(e) \cup \delta(e)$$

and

$$\begin{aligned} ((\alpha \tilde{\cap} \gamma) \tilde{\cup} (\alpha \tilde{\cap} \zeta))(e) &= (\alpha \tilde{\cap} \gamma)(e) \cup \emptyset = \alpha(e) \cap \gamma(e) \quad \text{and} \\ ((\beta \tilde{\cup} \delta) \tilde{\cap} (\beta \tilde{\cup} \eta))(e) &= (\beta \tilde{\cup} \delta)(e) \cap X = \beta(e) \cup \delta(e). \end{aligned}$$

(ii) If $e \in A \cap (C - B)$, then

$$(\alpha \tilde{\cap} (\gamma \tilde{\cup} \zeta))(e) = \alpha(e) \cap \zeta(e) \text{ and } (\beta \tilde{\cup} (\delta \tilde{\cap} \eta))(e) = \beta(e) \cup \eta(e)$$

and

$$\begin{aligned} ((\alpha \tilde{\cap} \gamma) \tilde{\cup} (\alpha \tilde{\cap} \zeta))(e) &= \emptyset \cup (\alpha \tilde{\cap} \zeta)(e) = \alpha(e) \cap \zeta(e) \text{ and} \\ ((\beta \tilde{\cup} \delta) \tilde{\cap} (\beta \tilde{\cup} \eta))(e) &= X \cap (\beta \tilde{\cap} \eta)(e) = \beta(e) \cup \eta(e). \end{aligned}$$

(iii) If $e \in A \cap (B \cap C)$, then

$$\begin{aligned} (\alpha \tilde{\cap} (\gamma \tilde{\cup} \zeta))(e) &= \alpha(e) \cap (\gamma(e) \cup \zeta(e)) \text{ and} \\ (\beta \tilde{\cup} (\delta \tilde{\cap} \eta))(e) &= \beta(e) \cup (\delta(e) \cap \eta(e)) \end{aligned}$$

and

$$\begin{aligned} ((\alpha \tilde{\cap} \gamma) \tilde{\cup} (\alpha \tilde{\cap} \zeta))(e) &= (\alpha \tilde{\cap} \gamma)(e) \cup (\alpha \tilde{\cap} \zeta)(e) \\ &= (\alpha(e) \cap \gamma(e)) \cup (\alpha(e) \cap \zeta(e)) \\ &= \alpha(e) \cap (\gamma(e) \cup \zeta(e)) \qquad \text{and} \\ ((\beta \tilde{\cup} \delta) \tilde{\cap} (\beta \tilde{\cup} \eta))(e) &= (\beta \tilde{\cup} \delta)(e) \cap (\beta \tilde{\cup} \eta)(e) \\ &= (\beta(e) \cup \delta(e)) \cap (\beta(e) \cup \eta(e)) \\ &= \beta(e) \cup (\delta(e) \cap \eta(e)). \end{aligned}$$

Thus

$$A_{(\alpha,\beta)} \sqcap (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \doteq (A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcap C_{(\zeta,\eta)}),$$

Similarly we can prove the remaining parts. ■

4.3.7 Example

Let $X = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8\}$ be the set of candidates who have applied for a job position of Office Representative in Customer Care Centre of a company. Let $E = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ = {Hard Working, Optimism, Enthusiasm, Individualism, Imaginative, Flexibility, Decisiveness, Self-confidence, Politeness}. Suppose that $A = \{x_1, x_2, x_3, x_6, x_7, x_9\}$, $B = \{x_2, x_4, x_5, x_7, x_8\}$, $C = \{x_3, x_5, x_7, x_9\}$, the double-framed soft sets $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}$, $C_{(\zeta,\eta)}$ describes the " Personality Analysis of Candidates" for three different positions. The company has recorded this data obtained through interview and practical sessions conducted by a panel of experts which is presented by mappings α, γ, ζ and β, δ, η for three positions respectively. The double-framed soft sets $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}$, $C_{(\zeta,\eta)}$ over X be given as follows:

$$\alpha : A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{m_1, m_4, m_5, m_6, m_8\} & \text{if } e = x_1, \\ \{m_1, m_2, m_3, m_4, m_7, m_8\} & \text{if } e = x_2, \\ \{m_2, m_4, m_6, m_7, m_8\} & \text{if } e = x_3, \\ \{m_4, m_5, m_6, m_7\} & \text{if } e = x_6, \\ \{m_5, m_6, m_8\} & \text{if } e = x_7, \\ \{m_2, m_3, m_4, m_6, m_7\} & \text{if } e = x_9, \end{cases}$$

$$\beta : A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{m_1, m_2, m_3, m_5, m_7, m_8\} & \text{if } e = x_1, \\ \{m_2, m_5, m_6, m_7\} & \text{if } e = x_2, \\ \{m_1, m_2, m_3, m_4, m_6, m_8\} & \text{if } e = x_3, \\ \{m_3, m_4, m_5, m_6, m_7\} & \text{if } e = x_6, \\ \{m_1, m_2, m_3\} & \text{if } e = x_7, \\ \{m_4, m_5, m_6, m_7, m_8\} & \text{if } e = x_9. \end{cases}$$

$$\gamma : B \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{m_1, m_2, m_3, m_6, m_7\} & \text{if } e = x_2, \\ \{m_2, m_3, m_4, m_8\} & \text{if } e = x_4, \\ \{m_1, m_2, m_4, m_6, m_7, m_8\} & \text{if } e = x_5, \\ \{m_2, m_4, m_6, m_8\} & \text{if } e = x_7, \\ \{m_1, m_2, m_3, m_5, m_6, m_7\} & \text{if } e = x_8, \end{cases}$$

$$\delta : B \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{m_2, m_3, m_4, m_5, m_6\} & \text{if } e = x_2, \\ \{m_4, m_6, m_7, m_8\} & \text{if } e = x_4, \\ \{m_3, m_4, m_5, m_7\} & \text{if } e = x_5, \\ \{m_1, m_2, m_3\} & \text{if } e = x_7, \\ \{m_3, m_4, m_5, m_6, m_7, m_8\} & \text{if } e = x_8. \end{cases}$$

$$\zeta : C \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{m_5, m_7, m_8\} & \text{if } e = x_3, \\ \{m_1, m_2, m_4, m_5, m_6, m_7\} & \text{if } e = x_5, \\ \{m_6, m_7\} & \text{if } e = x_7, \\ \{m_1, m_2, m_3, m_4, m_5\} & \text{if } e = x_9, \end{cases}$$

$$\eta : C \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{m_1, m_2, m_3, m_4, m_5, m_8\} & \text{if } e = x_3, \\ \{m_3, m_4, m_5, m_6\} & \text{if } e = x_5, \\ \{m_2, m_3, m_6\} & \text{if } e = x_7, \\ \{m_2, m_3, m_5, m_6, m_7, m_8\} & \text{if } e = x_9. \end{cases}$$

Now

$$A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcap C_{(\zeta,\eta)}) \doteq (A \cup (B \cap C))_{(\alpha \tilde{\cap} (\gamma \tilde{\cap} \zeta), \beta \tilde{\cup} (\delta \tilde{\cup} \eta))}$$

and

$$(A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)}) \sqcap (A_{(\alpha,\beta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}) \doteq ((A \cup B) \cap (A \cup C))_{((\alpha \tilde{\cap} \gamma) \tilde{\cap} (\alpha \tilde{\cap} \zeta), (\beta \tilde{\cup} \delta) \tilde{\cup} (\beta \tilde{\cup} \eta))}.$$

Then the approximations for parameter x_2 are not same on both sides e.g.

$$\begin{aligned} (\alpha \tilde{\cap} (\gamma \tilde{\cap} \zeta))(x_2) &= \{m_1, m_2, m_3, m_4, m_7, m_8\} \\ &\neq \{m_1, m_2, m_3, m_7\} = ((\alpha \tilde{\cap} \gamma) \tilde{\cap} (\alpha \tilde{\cap} \zeta))(x_2) \text{ and} \\ (\beta \tilde{\cup} (\delta \tilde{\cup} \eta))(x_2) &= \{m_2, m_5, m_6, m_7\} \\ &\neq \{m_2, m_3, m_4, m_5, m_6, m_7\} = ((\beta \tilde{\cup} \delta) \tilde{\cup} (\beta \tilde{\cup} \eta))(x_2). \end{aligned}$$

Thus

$$A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcap C_{(\zeta,\eta)}) \tilde{\neq} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)}) \sqcap (A_{(\alpha,\beta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}).$$

Now, consider

$$A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A \cup (B \cup C))_{(\alpha\tilde{\cap}(\gamma\tilde{\cup}\zeta),\beta\tilde{\cup}(\delta\tilde{\cap}\eta))}$$

and

$$\begin{aligned}(A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}) &\tilde{=} (A \cup B)_{(\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta)} \sqcup_{\varepsilon} (A \cup C)_{(\alpha\tilde{\cap}\zeta,\beta\tilde{\cup}\eta)} \\ &\tilde{=} ((A \cup B) \cup (A \cup C))_{((\alpha\tilde{\cap}\gamma)\tilde{\cup}(\alpha\tilde{\cap}\zeta),(\beta\tilde{\cup}\delta)\tilde{\cap}(\beta\tilde{\cup}\eta))}.\end{aligned}$$

Then the approximations for parameter x_2 are not same on both sides e.g.

$$\begin{aligned}(\alpha\tilde{\cap}(\gamma\tilde{\cup}\zeta))(x_2) &= \{m_1, m_2, m_3, m_7\} \\ &\neq \{m_1, m_2, m_3, m_4, m_7, m_8\} = ((\alpha\tilde{\cap}\gamma)\tilde{\cup}(\alpha\tilde{\cap}\zeta))(x_2) \text{ and} \\ (\beta\tilde{\cup}(\delta\tilde{\cap}\eta))(x_2) &= \{m_2, m_3, m_4, m_5, m_6, m_7, m_8\} \\ &\neq \{m_2, m_5, m_6, m_7\} = (\beta\tilde{\cup}\delta)\tilde{\cap}(\beta\tilde{\cup}\eta))(x_2).\end{aligned}$$

Thus

$$A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \tilde{\neq} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}).$$

Similarly it can be shown that

$$A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcup C_{(\zeta,\eta)}) \tilde{\neq} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}) \sqcup (A_{(\alpha,\beta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}).$$

$$A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}) \tilde{\neq} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}) \sqcap_{\varepsilon} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}).$$

4.3.8 Proposition

Let $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}$ and $C_{(\zeta,\eta)}$ be any *double-framed soft sets* over X. Then

1) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}) \sqcap_{\varepsilon} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)})$ if and only if

$$\begin{aligned}\alpha(e) &\subseteq \gamma(e) \text{ and } \beta(e) \supseteq \delta(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ \alpha(e) &\subseteq \zeta(e) \text{ and } \beta(e) \supseteq \eta(e) \text{ for all } e \in (A \cap C) - B.\end{aligned}$$

2) $A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)})$ if and only if

$$\begin{aligned}\alpha(e) &\supseteq \gamma(e) \text{ and } \beta(e) \subseteq \delta(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ \alpha(e) &\supseteq \zeta(e) \text{ and } \beta(e) \subseteq \eta(e) \text{ for all } e \in (A \cap C) - B.\end{aligned}$$

Proof. Straightforward. ■

4.3.9 Corollary

Let $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}$ and $C_{(\zeta,\eta)}$ are three *double-framed soft sets* over X. Then

1) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}) \sqcap_{\varepsilon} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)})$

2) $A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)})$

if and only if

$$\begin{aligned} \alpha(e) &= \gamma(e) \text{ and } \beta(e) = \delta(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ \alpha(e) &= \zeta(e) \text{ and } \beta(e) = \eta(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

4.3.10 Corollary

Let $A_{(\alpha,\beta)}$, $B_{(\gamma,\delta)}$ and $C_{(\zeta,\eta)}$ are three *double-framed soft sets* over X such that $(A \cap B) - C = (A \cap C) - B = \emptyset$. Then

1) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} (B_{(\gamma,\delta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}) \sqcap_{\varepsilon} (A_{(\alpha,\beta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)})$,

2) $A_{(\alpha,\beta)} \sqcap_{\varepsilon} (B_{(\gamma,\delta)} \sqcup_{\varepsilon} C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)}) \sqcup_{\varepsilon} (A_{(\alpha,\beta)} \sqcap_{\varepsilon} C_{(\zeta,\eta)})$.

4.3.11 Corollary

Let $A_{(\alpha,\beta)}$, $A_{(\gamma,\delta)}$ and $A_{(\zeta,\eta)}$ are three *double-framed soft sets* over X. Then

$$A_{(\alpha,\beta)} \lambda (A_{(\gamma,\delta)} \mu A_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \lambda A_{(\gamma,\delta)}) \mu (A_{(\alpha,\beta)} \lambda A_{(\zeta,\eta)})$$

for distinct λ, $\mu \in \{\sqcap_{\varepsilon}, \sqcap, \sqcup_{\varepsilon}, \sqcup\}$.

4.3.12 Theorem

Let $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ be double-framed soft sets over X. Then the following are true

1) $A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}$ is the smallest double-framed soft set over X which contains both $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$. (Supremum)

2) $A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)}$ is the largest double-framed soft set over X which is contained in both $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$. (Infimum)

Proof.

1) We have $A, B \subseteq (A \cup B)$ and $\alpha(e), \gamma(e) \subseteq \alpha(e) \cup \gamma(e)$ and $\beta(e) \cap \delta(e) \subseteq \beta(e)$, $\beta(e) \cap \delta(e) \subseteq \delta(e)$. So $A_{(\alpha,\beta)} \tilde{\subseteq} A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}$ and $B_{(\gamma,\delta)} \tilde{\subseteq} A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}$. Let $C_{(\zeta,\eta)}$ be a double-framed soft set over X, such that $A_{(\alpha,\beta)}, B_{(\gamma,\delta)} \tilde{\subseteq} C_{(\zeta,\eta)}$. Then $A, B \subseteq C$ implies that $(A \cup B) \subseteq C$ and $\alpha(e), \gamma(e) \subseteq \zeta(e)$ implies that $\alpha(e) \cup \gamma(e) \subseteq \zeta(e)$. Also $\eta(e) \subseteq \beta(e)$, $\eta(e) \subseteq \delta(e)$ imply that $\eta(e) \subseteq \beta(e) \cap \delta(e)$ for all $e \in A \cup B$. Thus $A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)} \tilde{\subseteq} C_{(\zeta,\eta)}$. It follows that $A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)}$ is the smallest double-framed soft set over X which contains both $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$.

2) We have $A \cap B \subseteq A, A \cap B \subseteq B$ and $\alpha(e) \cap \gamma(e) \subseteq \alpha(e), \alpha(e) \cap \gamma(e) \subseteq \gamma(e)$ and $\beta(e) \subseteq \beta(e) \cup \delta(e), \delta(e) \subseteq \beta(e) \cup \delta(e)$ for all $e \in A \cap B$. So $A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)} \tilde{\subseteq} A_{(\alpha,\beta)}$ and $A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)} \tilde{\subseteq} B_{(\gamma,\delta)}$. Let $C_{(\zeta,\eta)}$ be a double-framed soft set over X, such that $C_{(\zeta,\eta)} \tilde{\subseteq} A_{(\alpha,\beta)}$ and $C_{(\zeta,\eta)} \tilde{\subseteq} B_{(\gamma,\delta)}$. Then $C \subseteq A$, $C \subseteq B$ implies that $C \subseteq A \cap B$ and $\zeta(e) \subseteq \alpha(e)$, $\zeta(e) \subseteq \beta(e)$ imply that $\zeta(e) \subseteq \alpha(e) \cap \beta(e)$, and $\beta(e) \subseteq \eta(e)$, $\delta(e) \subseteq \eta(e)$ imply that $\beta(e) \cup \delta(e) \subseteq \eta(e)$ for all $e \in C$. Thus $C_{(\zeta,\eta)} \tilde{\subseteq} A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)}$. It follows that $A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)}$ is the largest double-framed soft set over X which is contained in both $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$.

■

4.4 Algebras of Double-framed Soft Sets

In this section, we discuss the ideas of lattices and algebras for the collections of double-framed soft sets. Let $\mathcal{DSS}(X)^E$ be the collection of all double-framed soft sets over X and $\mathcal{DSS}(X)_A$ be its subcollection of all double-framed soft sets over X with fixed set of parameters A. We note that these collections are partially ordered by the relation of soft inclusion $\tilde{\subseteq}$ given in Definition 4.1.2.

4.4.1 Theorem

$(\mathcal{DSS}(X)^E, \sqcap_\varepsilon, \sqcup)$, $(\mathcal{DSS}(X)^E, \sqcup, \sqcap_\varepsilon)$, $(\mathcal{DSS}(X)^E, \sqcup_\varepsilon, \sqcap)$, $(\mathcal{DSS}(X)^E, \sqcap, \sqcup_\varepsilon)$, $(\mathcal{DSS}(X)_A, \sqcup, \sqcap)$, and $(\mathcal{DSS}(X)_A, \sqcap, \sqcup)$ are complete lattices.

Proof. Let us consider $(\mathcal{DSS}(X)^E, \sqcap_\varepsilon, \sqcup)$. Then for any double-framed soft sets $A_{(\alpha,\beta)}, B_{(\gamma,\delta)}, C_{(\zeta,\eta)} \in \mathcal{DSS}(X)^E$, we have

1) $A_{(\alpha,\beta)} \sqcap_\varepsilon B_{(\gamma,\delta)} \tilde{=} (A \cup B)_{(\alpha \tilde{\cap} \gamma, \beta \tilde{\cup} \delta)} \in \mathcal{DSS}(X)^E$ and $A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)} \tilde{=} (A \cap B)_{(\alpha \tilde{\cup} \gamma, \beta \tilde{\cap} \delta)} \in \mathcal{DSS}(X)^E$.

2) From Proposition 4.3.3, we have $A_{(\alpha,\beta)} \sqcap_\varepsilon A_{(\alpha,\beta)} \tilde{=} A_{(\alpha,\beta)}$ and $A_{(\alpha,\beta)} \sqcup A_{(\alpha,\beta)} \tilde{=} A_{(\alpha,\beta)}$.

3) From Proposition 4.3.4 we see that $A_{(\alpha,\beta)} \sqcap_\varepsilon B_{(\gamma,\delta)} \tilde{=} B_{(\gamma,\delta)} \sqcap_\varepsilon A_{(\alpha,\beta)}$ and $A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)} \tilde{=} B_{(\gamma,\delta)} \sqcup A_{(\alpha,\beta)}$. Also $A_{(\alpha,\beta)} \sqcap_\varepsilon (B_{(\gamma,\delta)} \sqcap_\varepsilon C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcap_\varepsilon B_{(\gamma,\delta)}) \sqcap_\varepsilon C_{(\zeta,\eta)}$ and $A_{(\alpha,\beta)} \sqcup (B_{(\gamma,\delta)} \sqcup C_{(\zeta,\eta)}) \tilde{=} (A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)}) \sqcup C_{(\zeta,\eta)}$.

4) From Proposition 4.3.5,

$$A_{(\alpha,\beta)} \sqcap_\varepsilon (B_{(\gamma,\delta)} \sqcup A_{(\alpha,\beta)}) \tilde{=} A_{(\alpha,\beta)} \text{ and } A_{(\alpha,\beta)} \sqcup (B_{(\gamma,\delta)} \sqcap_\varepsilon A_{(\alpha,\beta)}) \tilde{=} A_{(\alpha,\beta)}.$$

So we conclude that the structure forms a lattice.

Consider a collection of double-framed soft sets $\{A_{i_{(\alpha_i,\beta_i)}} : i \in I\}$ over X. We have, $\bigcup_{i \in I} A_i \subseteq E$ and, let $\Lambda(e) = \{j : e \in A_j\}$ for any $e \in A_i$. Then $\bigcap_{i \in \Lambda(e)} \alpha_i(e) \subseteq X$ and $\bigcup_{i \in I} \beta_i(e) \subseteq X$. Thus $\sqcap_{\varepsilon\, i \in I} A_{i_{(\alpha_i,\beta_i)}} \in \mathcal{DSS}(X)^E$. Again, we have, $\bigcap_{i \in I} A_i \subseteq E$ and for any $e \in \bigcap_{i \in I} A_i$, $\bigcup_{i \in I} \alpha_i(e) \subseteq X$ and $\bigcap_{i \in I} \beta_i(e) \subseteq X$. Thus $\sqcup_{i \in I} A_{i_{(\alpha_i,\beta_i)}} \in \mathcal{DSS}(X)^E$.

Similarly we can show for the remaining structures. ■

4.4.2 Theorem

$(\mathcal{DSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{(\Phi,\mathfrak{X})}, E_{(\mathfrak{X},\Phi)})$, $(\mathcal{DSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{(\mathfrak{X},\Phi)}, \emptyset_{(\Phi,\mathfrak{X})})$, $(\mathcal{DSS}(X)_A, \sqcap, \sqcup, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ and $(\mathcal{DSS}(X)_A, \sqcup, \sqcap, A_{(\mathfrak{X},\Phi)}, A_{(\Phi,\mathfrak{X})})$ are bounded distributive lattices.

Proof. Proposition 4.3.6 assures that $(\mathcal{DSS}(X)^E, \sqcap, \sqcup_\varepsilon)$ and $(\mathcal{DSS}(X)^E, \sqcup_\varepsilon, \sqcap)$ are distributive lattices. From Theorem 4.3.12, we conclude that $(\mathcal{DSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{(\Phi,\mathfrak{X})}, E_{(\mathfrak{X},\Phi)})$ is a bounded distributive lattice and $(\mathcal{DSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{(\mathfrak{X},\Phi)}, \emptyset_{(\Phi,\mathfrak{X})})$ is its dual. For any double-framed soft sets $A_{(\alpha,\beta)}, A_{(\gamma,\delta)} \in \mathcal{DSS}(X)_A$,

$$\begin{aligned} A_{(\alpha,\beta)} \sqcap A_{(\gamma,\delta)} &\tilde{=} A_{(\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta)} \in \mathcal{DSS}(X)_A \text{ and} \\ A_{(\alpha,\beta)} \sqcup A_{(\gamma,\delta)} &\tilde{=} A_{(\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta)} \in \mathcal{DSS}(X)_A. \end{aligned}$$

Thus $(\mathcal{DSS}(X)_A, \sqcap, \sqcup)$ is a distributive sublattice of $(\mathcal{DSS}(X)^E, \sqcup_\varepsilon, \sqcap)$ and Proposition 4.3.3 tells us that $A_{(\Phi,\mathfrak{X})}$, $A_{(\mathfrak{X},\Phi)}$ are its lower and upper bounds respectively. Therefore $(\mathcal{DSS}(X)_A, \sqcap, \sqcup, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is a bounded distributive lattice and $(\mathcal{DSS}(X)_A, \sqcup, \sqcap, A_{(\mathfrak{X},\Phi)}, A_{(\Phi,\mathfrak{X})})$ is its dual. ■

4.4.3 Proposition

Let $A_{(\alpha,\beta)}$ be a *double-framed soft set* over X. Then $A_{(\alpha,\beta)^c}$ is a complement of $A_{(\alpha,\beta)}$.

Proof. As $A_{(\alpha,\beta)} \sqcup A_{(\alpha,\beta)^c} \tilde{=} A_{(\alpha\tilde{\cup}\alpha^c,\beta\tilde{\cap}\beta^c)}$. Now, for any $e \in A$,

$$\begin{aligned} (\alpha\tilde{\cup}\alpha^c)(e) &= \alpha(e) \cup (\alpha(e))^c = X \quad \text{and} \\ (\beta\tilde{\cap}\beta^c)(e) &= \beta(e) \cap (\beta(e))^c = \emptyset. \end{aligned}$$

Thus $A_{(\alpha,\beta)} \sqcup A_{(\alpha,\beta)^c} \tilde{=} A_{(\mathfrak{X},\Phi)}$.

Again, we have $A_{(\alpha,\beta)} \sqcap A_{(\alpha,\beta)^c} \tilde{=} A_{(\alpha\tilde{\cap}\alpha^c,\beta\tilde{\cup}\beta^c)}$, so for any $e \in A$,

$$\begin{aligned} (\alpha\tilde{\cap}\alpha^c)(e) &= \alpha(e) \cap (\alpha(e))^c = \emptyset \quad \text{and} \\ (\beta\tilde{\cup}\beta^c)(e) &= \beta(e) \cup (\beta(e))^c = X. \end{aligned}$$

Thus $A_{(\alpha,\beta)} \sqcap A_{(\alpha,\beta)^c} \tilde{=} A_{(\Phi,\mathfrak{X})}$. From $A_{(\alpha,\beta)} \sqcup A_{(\alpha,\beta)^c} \tilde{=} A_{(\mathfrak{X},\Phi)}$ and $A_{(\alpha,\beta)} \sqcap A_{(\alpha,\beta)^c} \tilde{=} A_{(\Phi,\mathfrak{X})}$, we conclude that $A_{(\alpha,\beta)^c}$ is a complement of $A_{(\alpha,\beta)}$.

Now, we show that $A_{(\alpha,\beta)^c}$ is unique in the bounded lattice $(\mathcal{DSS}(X)_A, \sqcup, \sqcap, A_{(\mathfrak{X},\Phi)}, A_{(\Phi,\mathfrak{X})})$. If there exists some $A_{(\gamma,\delta)} \in \mathcal{DSS}(X)_A$ such that $A_{(\alpha,\beta)} \sqcup A_{(\gamma,\delta)} \tilde{=} A_{(\mathfrak{X},\Phi)}$ and $A_{(\alpha,\beta)} \sqcap A_{(\gamma,\delta)} \tilde{=} A_{(\Phi,\mathfrak{X})}$. Then for any $e \in A$,

$$\begin{gathered} \alpha(e) \cap \gamma(e) = \emptyset \text{ and } \beta(e) \cap \delta(e) = \emptyset \\ \Rightarrow \gamma(e) \subseteq (\alpha(e))^c = \alpha^c(e) \text{ and } \delta(e) \subseteq (\beta(e))^c = \beta^c(e) \end{gathered}$$

and

$$\alpha^c(e) \subseteq X = \alpha(e) \cup \gamma(e) \text{ and } \beta^c(e) \subseteq X = \beta(e) \cup \delta(e).$$

But

$$\begin{gathered} \alpha(e) \cap \alpha^c(e) = \emptyset \text{ and } \beta(e) \cap \beta^c(e) = \emptyset \text{ so} \\ \alpha^c(e) \subseteq \alpha(e) \cup \gamma(e) \Rightarrow \alpha^c(e) \subseteq \gamma(e) \text{ and } \beta^c(e) \subseteq \beta(e) \cup \delta(e) \Rightarrow \beta^c(e) \subseteq \delta(e). \end{gathered}$$

Therefore

$$\gamma(e) = \alpha^c(e) \text{ and } \delta(e) = \beta^c(e) \text{ for all } e \in A \text{ and } A_{(\gamma,\delta)} \tilde{=} A_{(\alpha,\beta)^c}.$$

Hence $A_{(\alpha,\beta)^c}$ is unique complement of $A_{(\alpha,\beta)}$. ■

4.4.4 Proposition (de Morgan Laws)

Let $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ be double-*framed* soft sets over X. Then the following are true:

1) $(A_{(\alpha,\beta)} \sqcup_\varepsilon B_{(\gamma,\delta)})^c \tilde{=} A_{(\alpha,\beta)^c} \sqcap_\varepsilon B_{(\gamma,\delta)^c}$,

2) $(A_{(\alpha,\beta)} \sqcap_\varepsilon B_{(\gamma,\delta)})^c \tilde{=} A_{(\alpha,\beta)^c} \sqcup_\varepsilon B_{(\gamma,\delta)^c}$,

3) $(A_{(\alpha,\beta)} \vee B_{(\gamma,\delta)})^c \tilde{=} A_{(\alpha,\beta)^c} \wedge B_{(\gamma,\delta)^c}$,

4) $(A_{(\alpha,\beta)} \wedge B_{(\gamma,\delta)})^c \tilde{=} A_{(\alpha,\beta)^c} \vee B_{(\gamma,\delta)^c}$,

5) $(A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)})^c \tilde{=} A_{(\alpha,\beta)^c} \sqcap B_{(\gamma,\delta)^c}$,

6) $(A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)})^c \tilde{=} A_{(\alpha,\beta)^c} \sqcup B_{(\gamma,\delta)^c}$.

Proof.

1) We know that $(A_{(\alpha,\beta)} \sqcup_\varepsilon B_{(\gamma,\delta)})^c \tilde{=} (A \cup B)_{(\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta)^c} \tilde{=} (A \cup B)_{((\alpha\tilde{\cup}\gamma)^c,(\beta\tilde{\cap}\delta)^c)}$. Let $e \in (A \cup B)$. Then there are three cases:

(i) If $e \in A - B$, then

$$\begin{aligned} (\alpha\tilde{\cup}\gamma)^c(e) &= (\alpha(e))^c = \alpha^c(e) \text{ and } (\alpha^c\tilde{\cap}\gamma^c)(e) = \alpha^c(e) \text{ and} \\ (\beta\tilde{\cap}\delta)^c(e) &= (\beta(e))^c = \beta^c(e) \text{ and } (\beta^c\tilde{\cup}\delta^c)(e) = \beta^c(e). \end{aligned}$$

Thus

$$\begin{aligned} (\alpha\tilde{\cup}\gamma)^c(e) &= (\alpha^c\tilde{\cap}\gamma^c)(e) \text{ and} \\ (\beta\tilde{\cap}\delta)^c(e) &= (\beta^c\tilde{\cup}\delta^c)(e). \end{aligned}$$

(ii) If $e \in B - A$, then

$$\begin{aligned} (\alpha\tilde{\cup}\gamma)^c(e) &= (\gamma(e))^c = \gamma^c(e) \text{ and } (\alpha^c\tilde{\cap}\gamma^c)(e) = \gamma^c(e) \text{ and} \\ (\beta\tilde{\cap}\delta)^c(e) &= (\delta(e))^c = \delta^c(e) \text{ and } (\beta^c\tilde{\cup}\delta^c)(e) = \delta^c(e). \end{aligned}$$

Thus

$$\begin{aligned} (\alpha\tilde{\cup}\gamma)^c(e) &= (\alpha^c\tilde{\cap}\gamma^c)(e) \text{ and} \\ (\beta\tilde{\cap}\delta)^c(e) &= (\beta^c\tilde{\cup}\delta^c)(e). \end{aligned}$$

(iii) If $e \in A \cap B$, then

$$\begin{aligned}(\alpha\tilde{\cup}\gamma)^c(e) &= (\alpha(e) \cup \gamma(e))^c = (\alpha(e))^c \cap (\gamma(e))^c \text{ and} \\ (\beta\tilde{\cup}\delta)^c(e) &= (\beta(e) \cap \delta(e))^c = (\beta(e))^c \cup (\delta(e))^c,\end{aligned}$$

and

$$\begin{aligned}(\alpha^c\tilde{\cap}\gamma^c)(e) &= (\alpha(e))^c \cap (\gamma(e))^c = (\alpha\tilde{\cup}\gamma)^c(e) \text{ and} \\ (\beta^c\tilde{\cap}\delta^c)(e) &= (\beta(e))^c \cup (\delta(e))^c = (\beta\tilde{\cup}\delta)^c(e).\end{aligned}$$

Therefore, in all three cases we obtain equality and thus

$$(A_{(\alpha,\beta)} \sqcup_\varepsilon B_{(\gamma,\delta)})^c \doteq A_{(\alpha,\beta)^c} \sqcap_\varepsilon B_{(\gamma,\delta)^c}.$$

The remaining parts can be proved in a similar way.

■

4.4.5 Proposition

$(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^c, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is a de Morgan algebra.

Proof. We have already seen that $(\mathcal{DSS}(X)_A, \sqcap, \sqcup, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is a bounded distributive lattice. Proposition 4.4.3 show that " ^{c}" is a complementation and hence an involution on $\mathcal{DSS}(X)_A$ and Proposition 4.4.4 shows that de Morgan laws hold with respect to " ^{c}" in $\mathcal{DSS}(X)_A$. Thus $(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^c, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is a de Morgan algebra. ■

4.4.6 Proposition

$(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^c, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is a boolean algebra.

Proof. Proof follows from Propositions 4.4.4 and 4.4.3. ■

4.4.7 Proposition

$(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^c, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is a Kleene Algebra.

Proof. Note that, $A_{(\gamma,\delta)} \sqcap A_{(\gamma,\delta)^c} \doteq \emptyset_{(\Phi,\mathfrak{X})} \tilde{\subset} A_{(\mathfrak{X},\Phi)} \doteq A_{(\alpha,\beta)} \sqcup A_{(\alpha,\beta)^c}$. We already know that $(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^c, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is a de Morgan algebra, so this condition assures that $(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^c, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is also a Kleene Algebra. ■

4.4.8 Definition

Let $A_{(\alpha,\beta)}$ be a *double-framed* soft set over X. We define

$$(A_{(\alpha,\beta)})^\circ \doteq A_{(\alpha,\beta)^\circ} \doteq A_{(\beta,\alpha)}.$$

4.4.9 Proposition

Let $A_{(\alpha,\beta)}$ be a *double-framed soft set* over X. Then $A_{(\alpha,\beta)} \doteq (A_{(\alpha,\beta)^\circ})^\circ$, $A_{(\mathfrak{X},\Phi)^\circ} \doteq A_{(\Phi,\mathfrak{X})}$ and $A_{(\Phi,\mathfrak{X})^\circ} \doteq A_{(\mathfrak{X},\Phi)}$.

Proof. Straightforward. ■

4.4.10 Proposition (de Morgan Laws)

Let $A_{(\alpha,\beta)}$ and $B_{(\gamma,\delta)}$ be double-*framed* soft sets over X. Then the following are true

1) $(A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)})^{\circ} \tilde{=} A_{(\alpha,\beta)^{\circ}} \sqcap_{\varepsilon} B_{(\gamma,\delta)^{\circ}}$,

2) $(A_{(\alpha,\beta)} \sqcap_{\varepsilon} B_{(\gamma,\delta)})^{\circ} \tilde{=} A_{(\alpha,\beta)^{\circ}} \sqcup_{\varepsilon} B_{(\gamma,\delta)^{\circ}}$,

3) $(A_{(\alpha,\beta)} \vee B_{(\gamma,\delta)})^{\circ} \tilde{=} A_{(\alpha,\beta)^{\circ}} \wedge B_{(\gamma,\delta)^{\circ}}$,

4) $(A_{(\alpha,\beta)} \wedge B_{(\gamma,\delta)})^{\circ} \tilde{=} A_{(\alpha,\beta)^{\circ}} \vee B_{(\gamma,\delta)^{\circ}}$,

5) $(A_{(\alpha,\beta)} \sqcup B_{(\gamma,\delta)})^{\circ} \tilde{=} A_{(\alpha,\beta)^{\circ}} \sqcap B_{(\gamma,\delta)^{\circ}}$,

6) $(A_{(\alpha,\beta)} \sqcap B_{(\gamma,\delta)})^{\circ} \tilde{=} A_{(\alpha,\beta)^{\circ}} \sqcup B_{(\gamma,\delta)^{\circ}}$.

Proof.

1) We have $(A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)})^{\circ} \tilde{=} ((A \cup B)_{(\alpha \tilde{\cup} \gamma, \beta \tilde{\cap} \delta)})^{\circ} \tilde{=} (A \cup B)_{(\beta \tilde{\cap} \delta, \alpha \tilde{\cup} \gamma)}$ and

$$A_{(\alpha,\beta)^{\circ}} \sqcap_{\varepsilon} B_{(\gamma,\delta)^{\circ}} \tilde{=} A_{(\beta,\alpha)} \sqcap_{\varepsilon} B_{(\delta,\gamma)} \tilde{=} (A \cup B)_{(\beta \tilde{\cap} \delta, \alpha \tilde{\cup} \gamma)}.$$

Thus $(A_{(\alpha,\beta)} \sqcup_{\varepsilon} B_{(\gamma,\delta)})^{\circ} \tilde{=} A_{(\alpha,\beta)^{\circ}} \sqcap_{\varepsilon} B_{(\gamma,\delta)^{\circ}}$.

The remaining parts can be proved in a similar way. ■

4.4.11 Proposition

$(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^{\circ}, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is a de Morgan algebra.

Proof. Proof follows from Propositions 4.4.9 and 4.4.10. ■

4.4.12 Definition

Let $A_{(\alpha,\beta)}$ be a *double-framed* soft set over X. We define $A_{(\alpha,\beta)^{\diamond}}$ as a *double-framed soft set* $A_{(\alpha^c,\mathfrak{X})}$ where

$$\alpha^c : A \to \mathcal{P}(X),\ e \mapsto (\alpha(e))^c$$
$$\mathfrak{X} : A \to \mathcal{P}(X),\ e \mapsto X.$$

4.4.13 Proposition

Let $A_{(\alpha,\beta)}$ and $A_{(\gamma,\delta)}$ be double-framed soft sets over X. Then

1) $A_{(\gamma,\delta)} \sqcap A_{(\gamma,\delta)^{\diamond}} \tilde{=} A_{(\Phi,\mathfrak{X})}$,

2) $A_{(\gamma,\delta)} \tilde{\subset} A_{(\alpha,\beta)^{\diamond}}$ whenever $A_{(\gamma,\delta)} \sqcap A_{(\alpha,\beta)} \tilde{=} A_{(\Phi,\mathfrak{X})}$.

Proof.

1) For any $e \in A$,

$$(\gamma \tilde{\cap} \gamma^c)(e) = \gamma(e) \cap (\gamma(e))^c = \emptyset = \Phi(e) \quad \text{and}$$
$$(\delta \tilde{\cup} \mathfrak{X})(e) = \delta(e) \cup X = X = \mathfrak{X}(e).$$

Thus $A_{(\gamma,\delta)} \sqcap A_{(\gamma,\delta)^\diamond} \doteq A_{(\Phi,\mathfrak{X})}$.

2) Assume $A_{(\gamma,\delta)} \sqcap A_{(\alpha,\beta)} \doteq A_{(\Phi,\mathfrak{X})}$. Now, for any $e \in A$,

$$\gamma(e) \cap \alpha(e) = (\gamma \tilde{\cap} \alpha)(e) = \Phi(e) = \emptyset \text{ and so } \gamma(e) \subseteq (\alpha(e))^c = \alpha^c(e).$$
$$\text{Also } \delta(e) \subseteq X = \mathfrak{X}(e).$$

Therefore $A_{(\gamma,\delta)} \tilde{\subset} A_{(\alpha,\beta)^\diamond}$. So, we conclude that $(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^\diamond, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ is pseudocomplemented.

■

4.4.14 Proposition

Let $A_{(\alpha,\beta)}, B_{(\gamma,\delta)} \in \mathcal{DSS}(X)^E$. Then pseudocomplement of $A_{(\alpha,\beta)}$ relative to $B_{(\gamma,\delta)}$ exists in $(\mathcal{DSS}(X)^E, \sqcap, \sqcup_\varepsilon)$.

Proof. Consider the set

$$T(A_{(\alpha,\beta)}, B_{(\gamma,\delta)}) = \{C_{(\zeta,\eta)} \in \mathcal{SS}(X)^E : C_{(\zeta,\eta)} \sqcap A_{(\alpha,\beta)} \tilde{\subseteq} B_{(\gamma,\delta)}\}.$$

We define a double-framed soft set $A^c_{(\alpha^c,\beta^c)} \sqcup_\varepsilon B_{(\gamma,\delta)} \doteq (A^c \cup B)_{(\alpha^c \tilde{\cup} \gamma, \beta^c \tilde{\cap} \delta)} \in \mathcal{DSS}(X)^E$. Then

$$\begin{aligned}
(A^c \cup B)_{(\alpha^c \tilde{\cup} \gamma, \beta^c \tilde{\cap} \delta)} \sqcap A_{(\alpha,\beta)} &\doteq ((A^c \cup B) \cap A)_{((\alpha^c \tilde{\cup} \gamma) \tilde{\cap} \alpha, (\beta^c \tilde{\cap} \delta) \tilde{\cup} \beta)} \\
&\doteq ((A^c \cap A) \cup (B \cap A))_{((\alpha^c \tilde{\cap} \alpha) \tilde{\cup} (\gamma \tilde{\cap} \alpha), (\beta^c \tilde{\cup} \beta) \tilde{\cap} (\delta \tilde{\cup} \beta))} \\
&\doteq (A \cap B)_{(\gamma \tilde{\cap} \alpha, \delta \tilde{\cup} \beta)} \tilde{\subseteq} B_{(\gamma,\delta)}.
\end{aligned}$$

Thus $(A^c \cup B)_{(\alpha^c \tilde{\cup} \gamma, \beta^c \tilde{\cap} \delta)} \in T(A_{(\alpha,\beta)}, B_{(\gamma,\delta)})$. For any $C_{(\zeta,\eta)} \in T(A_{(\alpha,\beta)}, B_{(\gamma,\delta)})$, we have $C_{(\zeta,\eta)} \sqcap A_{(\alpha,\beta)} \tilde{\subseteq} B_{(\gamma,\delta)}$ so for any $e \in C \cap A \subseteq B$

$$\zeta(e) \cap \alpha(e) \subseteq \gamma(e) \text{ and } \eta(e) \cup \beta(e) \supseteq \delta(e)$$

Now,

$$C \cap A \subseteq B \Rightarrow (A \cap C) \cap B^c = \emptyset$$
$$\Rightarrow C \subseteq (A \cap B^c)^c = A^c \cup B$$

and

$$\begin{aligned}
&\zeta(e) \cap \alpha(e) \subseteq \gamma(e) \text{ and } \eta(e) \cup \beta(e) \supseteq \delta(e) \\
&\Rightarrow \zeta(e) \cap \alpha(e) \cap \gamma^c(e) = \emptyset \text{ and } \eta^c(e) \cap \beta^c(e) \subseteq \delta^c(e) \\
&\Rightarrow \zeta(e) \subseteq \alpha^c(e) \cup \gamma(e) \quad \text{and} \quad \eta^c(e) \cap \beta^c(e) \cap \delta(e) = \emptyset \\
&\Rightarrow \zeta(e) \subseteq \alpha^c(e) \cup \gamma(e) \quad \text{and} \quad \beta^c(e) \cap \delta(e) \subseteq \eta(e).
\end{aligned}$$

Thus $C_{(\zeta,\eta)} \tilde{\subseteq} (A^c \cup B)_{(\alpha^c \tilde{\cup} \gamma, \beta^c \tilde{\cap} \delta)}$, also

$$(A^c \cup B)_{(\alpha^c \tilde{\cup} \gamma, \beta^c \tilde{\cap} \delta)} \doteq \bigvee T(A_{(\alpha,\beta)}, B_{(\gamma,\delta)}) \doteq A_{(\alpha,\beta)} \to B_{(\gamma,\delta)}.$$

■

4.4.15 Remark

We know that $(\mathcal{DSS}(X)_A, \sqcap, \sqcup)$ is a sublattice of $(\mathcal{DSS}(X)^E, \sqcap_\varepsilon, \sqcup)$. *For any* $A_{(\alpha,\beta)}$, $A_{(\gamma,\delta)} \in \mathcal{DSS}(X)_A$, $A_{(\alpha,\beta)} \to A_{(\gamma,\delta)}$ as defined in Lemma 4.4.14, is not in $\mathcal{DSS}(X)_A$ because $A_{(\alpha,\beta)} \to A_{(\gamma,\delta)} \tilde{=} (A^c \cup A)_{(\alpha^c \tilde{\cup} \gamma, \beta^c \tilde{\cap} \delta)} \tilde{=} E_{(\alpha^c \tilde{\cup} \gamma, \beta^c \tilde{\cap} \delta)} \notin \mathcal{DSS}(X)_A$.

4.4.16 Lemma

Let $A_{(\alpha,\beta)}, A_{(\gamma,\delta)} \in \mathcal{DSS}(X)_A$. Then pseudocomplement of $A_{(\alpha,\beta)}$ relative to $A_{(\gamma,\delta)}$ exists in $\mathcal{DSS}(X)^A$.

Proof. Consider the set

$$T(A_{(\alpha,\beta)}, A_{(\gamma,\delta)}) = \{A_{(\zeta,\eta)} \in \mathcal{DSS}(X)_A : A_{(\zeta,\eta)} \sqcap A_{(\alpha,\beta)} \tilde{\subseteq} A_{(\gamma,\delta)}\}.$$

We define a double-framed soft set $A_{(\alpha^c,\beta^c)} \sqcup A_{(\gamma,\delta)} \tilde{=} A_{(\alpha^c \tilde{\cup} \gamma, \beta^c \tilde{\cap} \delta)} \in \mathcal{DSS}(X)_A$. Consider

$$\begin{aligned} A_{(\alpha^c \tilde{\cup} \gamma, \beta^c \tilde{\cap} \delta)} \sqcap A_{(\alpha,\beta)} &\tilde{=} A_{((\alpha^c \tilde{\cup} \gamma) \tilde{\cap} \alpha, (\beta^c \tilde{\cap} \delta) \tilde{\cup} \beta)} \\ &\tilde{=} A_{((\alpha^c \tilde{\cap} \alpha) \tilde{\cup} (\gamma \tilde{\cap} \alpha), (\beta^c \tilde{\cup} \beta) \tilde{\cap} (\delta \tilde{\cup} \beta))} \\ &\tilde{=} A_{((\gamma \tilde{\cap} \alpha), (\delta \tilde{\cup} \beta))} \tilde{\subseteq} A_{(\gamma,\delta)}. \end{aligned}$$

Thus $A_{(\alpha^c \tilde{\cup} \gamma, \beta^c \tilde{\cap} \delta)} \in T(A_{(\alpha,\beta)}, A_{(\gamma,\delta)})$. For every $A_{(\zeta,\eta)} \in T(A_{(\alpha,\beta)}, A_{(\gamma,\delta)})$, we have $A_{(\zeta,\eta)} \sqcap A_{(\alpha,\beta)} \tilde{\subseteq} A_{(\gamma,\delta)}$ so for any $e \in A$,

$$\begin{gathered} \zeta(e) \cap \alpha(e) \subseteq \gamma(e) \text{ and } \eta(e) \cup \beta(e) \supseteq \delta(e) \\ \Rightarrow \zeta(e) \cap \alpha(e) \cap \gamma^c(e) = \emptyset \text{ and } \eta^c(e) \cap \beta^c(e) \subseteq \delta^c(e) \\ \Rightarrow \zeta(e) \subseteq \alpha^c(e) \cup \gamma(e) \quad \text{and} \quad \eta^c(e) \cap \beta^c(e) \cap \delta(e) = \emptyset \\ \Rightarrow \zeta(e) \subseteq \alpha^c(e) \cup \gamma(e) \quad \text{and} \quad \beta^c(e) \cap \delta(e) \subseteq \eta(e). \end{gathered}$$

Thus $A_{(\zeta,\eta)} \tilde{\subseteq} A_{(\alpha^c \tilde{\cup} \gamma, \beta^c \tilde{\cap} \delta)}$ and also

$$A_{(\alpha^c \tilde{\cup} \gamma, \beta^c \tilde{\cap} \delta)} \tilde{=} \bigvee T(A_{(\alpha,\beta)}, A_{(\gamma,\delta)}) \tilde{=} A_{(\alpha,\beta)} \to_A A_{(\gamma,\delta)}.$$

■

4.4.17 Proposition

$(\mathcal{DSS}(X)^E, \sqcap_\varepsilon, \sqcup)$ *and* $(\mathcal{DSS}(X)_A, \sqcap, \sqcup)$ are *Brouwerian lattices.*

Proof. Follows from Lemmas 4.4.14 and 4.4.16. ■

4.4.18 Theorem

$(\mathcal{DSS}(X)_A, \sqcap, ^c, A_{(\mathfrak{X},\Phi)})$ is an MV-algebra.

Proof.

(MV1) $(\mathcal{DSS}(X)_A, \sqcap, A_{(\mathfrak{X},\Phi)})$ is a commutative monoid.

(MV2) $(A_{(\gamma,\delta)^c})^c \tilde{=} A_{(\gamma,\delta)}$.

(MV3) $A_{(\mathfrak{X},\Phi)^c} \sqcap A_{(\gamma,\delta)} \tilde{=} A_{(\Phi,\mathfrak{X})} \sqcap A_{(\gamma,\delta)} \tilde{=} A_{(\Phi,\mathfrak{X})} \tilde{=} A_{(\mathfrak{X},\Phi)^c}$.

(MV4) $(A_{(\alpha,\beta)^c} \sqcap A_{(\gamma,\delta)})^c \sqcap A_{(\gamma,\delta)}$

$\tilde{=}(A_{(\alpha^c,\beta^c)} \sqcap A_{(\gamma,\delta)})^c \sqcap A_{(\gamma,\delta)}$

$\tilde{=}(A_{(\alpha^c,\beta^c)^c} \sqcup A_{(\gamma,\delta)^c}) \sqcap A_{(\gamma,\delta)}$

$\tilde{=}(A_{(\alpha,\beta)} \sqcup A_{(\gamma^c,\delta^c)}) \sqcap A_{(\gamma,\delta)}$

$\tilde{=}(A_{(\alpha,\beta)} \sqcap A_{(\gamma,\delta)}) \sqcup (A_{(\gamma^c,\delta^c)} \sqcap A_{(\gamma,\delta)})$

$\tilde{=}(A_{(\alpha,\beta)} \sqcap A_{(\gamma,\delta)}) \sqcup A_{(\Phi,\mathfrak{X})}$

$\tilde{=}(A_{(\gamma,\delta)} \sqcap A_{(\alpha,\beta)}) \sqcup (A_{(\alpha,\beta)^c} \sqcap A_{(\alpha,\beta)})$

$\tilde{=}(A_{(\gamma,\delta)} \sqcup A_{(\alpha,\beta)^c}) \sqcap A_{(\alpha,\beta)}$

$\tilde{=}(A_{(\gamma,\delta)^c} \sqcap A_{(\alpha,\beta)})^c \sqcap A_{(\alpha,\beta)}$

for all $A_{(\alpha,\beta)}, A_{(\gamma,\delta)} \in \mathcal{DSS}(X)_A$. Thus $(\mathcal{DSS}(X)_A, \sqcap, ^c, A_{(\mathfrak{X},\Phi)})$ is an MV-algebra.

■

4.4.19 Theorem

$(\mathcal{DSS}(X)_A, \sqcup, ^c, A_{(\Phi,\mathfrak{X})})$ is an MV-algebra.

Proof.

(MV1) $(\mathcal{DSS}(X)_A, \sqcup, A_{(\Phi,\mathfrak{X})})$ is a commutative monoid.

(MV2) $(A_{(\gamma,\delta)^c})^c \tilde{=} A_{(\gamma,\delta)}$.

(MV3) $A_{(\Phi,\mathfrak{X})^c} \sqcup A_{(\gamma,\delta)} \tilde{=} A_{(\mathfrak{X},\Phi)} \sqcup A_{(\gamma,\delta)} \tilde{=} A_{(\mathfrak{X},\Phi)} \tilde{=} A_{(\Phi,\mathfrak{X})^c}$.

(MV4) $(A_{(\alpha,\beta)^c} \sqcup A_{(\gamma,\delta)})^c \sqcup A_{(\gamma,\delta)}$

$\tilde{=}(A_{(\alpha^c,\beta^c)} \sqcup A_{(\gamma,\delta)})^c \sqcup A_{(\gamma,\delta)}$

$\tilde{=}(A_{(\alpha^c,\beta^c)^c} \sqcap A_{(\gamma,\delta)^c}) \sqcup A_{(\gamma,\delta)}$

$\tilde{=}(A_{(\alpha,\beta)} \sqcap A_{(\gamma^c,\delta^c)}) \sqcup A_{(\gamma,\delta)}$

$\tilde{=}(A_{(\alpha,\beta)} \sqcup A_{(\gamma,\delta)}) \sqcap (A_{(\gamma^c,\delta^c)} \sqcup A_{(\gamma,\delta)})$

$\tilde{=}(A_{(\alpha,\beta)} \sqcup A_{(\gamma,\delta)}) \sqcap A_{(\mathfrak{X},\Phi)}$

$\tilde{=}(A_{(\gamma,\delta)} \sqcup A_{(\alpha,\beta)}) \sqcap (A_{(\alpha,\beta)^c} \sqcup A_{(\alpha,\beta)})$

$\tilde{=}(A_{(\gamma,\delta)} \sqcap A_{(\alpha,\beta)^c}) \sqcup A_{(\alpha,\beta)}$

$\tilde{=}(A_{(\gamma,\delta)^c} \sqcup A_{(\alpha,\beta)})^c \sqcup A_{(\alpha,\beta)}$

for all $A_{(\alpha,\beta)}, A_{(\gamma,\delta)} \in \mathcal{DSS}(X)_A$. Thus $(\mathcal{DSS}(X)_A, \sqcup, ^c, A_{(\Phi,\mathfrak{X})})$ is an MV-algebra.

■

4.4.20 Theorem

$(\mathcal{DSS}(X)_A, \smile, A_{(\Phi,\Phi)})$ is a bounded BCK-algebra whose every element is an involution.

Proof. For any $A_{(\alpha,\beta)}$, $A_{(\gamma,\delta)}$, $A_{(\zeta,\eta)} \in \mathcal{DSS}(X)_A$.

BCI-1 $((A_{(\alpha,\beta)} \smile A_{(\gamma,\delta)}) \smile (A_{(\alpha,\beta)} \smile A_{(\zeta,\eta)})) \smile (A_{(\zeta,\eta)} \smile A_{(\gamma,\delta)})$

$\stackrel{\sim}{=}(A_{(\alpha\smile\gamma,\beta\smile\delta)} \smile A_{(\alpha\smile\zeta,\beta\smile\eta)}) \smile A_{(\zeta\smile\gamma,\eta\smile\delta)}$

$\stackrel{\sim}{=}A_{(((\alpha\smile\gamma)\smile(\alpha\smile\zeta))\smile(\zeta\smile\gamma),((\beta\smile\delta)\smile(\beta\smile\eta))\smile(\eta\smile\delta))}$

$\stackrel{\sim}{=}A_{(\Phi\smile(\zeta\smile\gamma),\Phi\smile(\eta\smile\delta))}\stackrel{\sim}{=}A_{(\Phi,\Phi)}.$

BCI-2 $(A_{(\alpha,\beta)} \smile (A_{(\alpha,\beta)} \smile A_{(\gamma,\delta)})) \smile A_{(\gamma,\delta)}$

$\stackrel{\sim}{=}(A_{(\alpha,\beta)} \smile A_{(\alpha\smile\gamma,\beta\smile\delta)}) \smile A_{(\gamma,\delta)}$

$\stackrel{\sim}{=}A_{(\alpha\smile(\alpha\smile\gamma),\beta\smile(\beta\smile\delta))} \smile A_{(\gamma,\delta)}\stackrel{\sim}{=}A_{(\Phi\smile\gamma,\Phi\smile\delta)}\stackrel{\sim}{=}A_{(\Phi,\Phi)}.$

BCI-3 $A_{(\alpha,\beta)} \smile A_{(\alpha,\beta)}\stackrel{\sim}{=}A_{(\Phi,\Phi)}.$

BCI-4 Let

$$A_{(\alpha,\beta)} \quad\smile\quad A_{(\gamma,\delta)}\stackrel{\sim}{=}A_{(\Phi,\Phi)} \text{ and}$$
$$A_{(\gamma,\delta)} \quad\smile\quad A_{(\alpha,\beta)}\stackrel{\sim}{=}A_{(\Phi,\Phi)}.$$

For any $e \in A$,

$$\alpha(e) - \gamma(e) = \emptyset \text{ and } \gamma(e) - \alpha(e) = \emptyset \text{ imply that } \alpha(e) = \gamma(e),$$

also

$$\beta(e) - \delta(e) = \emptyset \text{ and } \delta(e) - \beta(e) = \emptyset \text{ imply that } \beta(e) = \delta(e).$$

Hence

$$A_{(\alpha,\beta)}\stackrel{\sim}{=}A_{(\gamma,\delta)}.$$

BCK-5 $A_{(\Phi,\Phi)} \smile A_{(\alpha,\beta)}\stackrel{\sim}{=}A_{(\Phi\smile\alpha,\Phi\smile\beta)}\stackrel{\sim}{=}A_{(\Phi,\Phi)}$. Thus $(\mathcal{DSS}(X)_A, \smile, A_{(\Phi,\Phi)})$ is a BCK-algebra.

Now $A_{(\mathfrak{x},\mathfrak{x})} \in \mathcal{DSS}(X)_A$ is such that:

$$A_{(\alpha,\beta)} \smile A_{(\mathfrak{x},\mathfrak{x})}\stackrel{\sim}{=}A_{(\alpha\smile\mathfrak{x},\beta\smile\mathfrak{x})}\stackrel{\sim}{=}A_{(\Phi,\Phi)}$$

for all $A_{(\alpha,\beta)} \in \mathcal{DSS}(X)_A$. Therefore $(\mathcal{DSS}(X)_A, \smile, A_{(\Phi,\Phi)})$ is a bounded BCK-algebra.

For any $A_{(\alpha,\beta)} \in \mathcal{DSS}(X)_A$,

$$\begin{aligned}
A_{(\mathfrak{x},\mathfrak{x})} &\smile (A_{(\mathfrak{x},\mathfrak{x})} \smile A_{(\alpha,\beta)}) \\
&\stackrel{\sim}{=} A_{(\mathfrak{x},\mathfrak{x})} \smile A_{(\mathfrak{x}\smile\alpha,\mathfrak{x}\smile\beta)} \\
&\stackrel{\sim}{=} A_{(\mathfrak{x},\mathfrak{x})} \smile A_{(\alpha^c,\beta^c)} \\
&\stackrel{\sim}{=} A_{(\mathfrak{x}\smile\alpha^c,\mathfrak{x}\smile\beta^c)} \\
&\stackrel{\sim}{=} A_{((\alpha^c)^c,(\beta^c)^c)}\stackrel{\sim}{=}A_{(\alpha,\beta)}.
\end{aligned}$$

So every element of $\mathcal{DSS}(X)_A$ is an involution. ■

4.4.21 Definition

Let $A_{(\alpha,\beta)}$ and $A_{(\gamma,\delta)}$ be double-*framed* soft sets over X. We define

$$A_{(\alpha,\beta)} \star A_{(\gamma,\delta)} \tilde{=} A_{(\alpha,\beta)} \sqcap A_{(\gamma,\delta)^c}.$$

4.4.22 Theorem

$(\mathcal{DSS}(X)_A, \star, A_{(\Phi,\mathfrak{X})})$ is a bounded BCK-algebra whose every element is an involution.

Proof. For any $A_{(\alpha,\beta)}$, $A_{(\gamma,\delta)}$, $A_{(\zeta,\eta)} \in \mathcal{DSS}(X)_A$.

BCI-1 $((A_{(\alpha,\beta)} \star A_{(\gamma,\delta)}) \star (A_{(\alpha,\beta)} \star A_{(\zeta,\eta)})) \star (A_{(\zeta,\eta)} \star A_{(\gamma,\delta)})$

$$\tilde{=}(A_{(\alpha\star\gamma,\beta\star\delta);A\star((\alpha\star\zeta,\beta\star\eta)}) \star A_{(\zeta\star\gamma,\eta\star\delta)}$$

$$\tilde{=}A_{(((\alpha\star\gamma)\star(\alpha\star\zeta))\star(\zeta\star\gamma),((\beta\star\delta)\star(\beta\star\eta))\star(\eta\star\delta))}$$

$$\tilde{=}A_{(((\alpha\tilde{\cap}\gamma^c)\star(\alpha\tilde{\cap}\zeta^c))\star(\zeta\tilde{\cap}\gamma^c),((\beta\tilde{\cup}\delta^c)\star(\beta\tilde{\cup}\eta^c))\star(\eta\tilde{\cup}\delta^c))}$$

$$\tilde{=}A_{(((\alpha\tilde{\cap}\gamma^c)\tilde{\cap}(\alpha\tilde{\cap}\zeta^c)^c)\tilde{\cap}(\zeta\tilde{\cap}\gamma^c)^c,((\beta\tilde{\cup}\delta^c)\tilde{\cup}(\beta\tilde{\cup}\eta^c)^c)\tilde{\cup}(\eta\tilde{\cup}\delta^c)^c)}$$

$$\tilde{=}A_{(((\alpha\tilde{\cap}\gamma^c)\tilde{\cap}(\alpha^c\tilde{\cup}\zeta))\tilde{\cap}(\zeta^c\tilde{\cup}\gamma),((\beta\tilde{\cup}\delta^c)\tilde{\cup}(\beta^c\tilde{\cap}\eta))\tilde{\cup}(\eta^c\tilde{\cap}\delta))}$$

$$\tilde{=}A_{((\alpha\tilde{\cap}\zeta)\tilde{\cap}(\gamma^c\tilde{\cap}\zeta^c),(\beta\tilde{\cup}\eta)\tilde{\cup}(\delta^c\tilde{\cup}\eta^c))} \tilde{=} A_{(\Phi,\mathfrak{X})}.$$

BCI-2 $(A_{(\alpha,\beta)} \star (A_{(\alpha,\beta)} \star A_{(\gamma,\delta)})) \star A_{(\gamma,\delta)}$

$$\tilde{=}A_{(\alpha\tilde{\cap}(\alpha\tilde{\cap}\gamma^c)^c,\beta\tilde{\cup}(\beta\tilde{\cup}\delta^c)^c)} \star A_{(\gamma,\delta)}$$

$$\tilde{=}A_{(\alpha\tilde{\cap}(\alpha^c\tilde{\cup}\gamma),\beta\tilde{\cup}(\beta^c\tilde{\cap}\delta))} \star A_{(\gamma,\delta)}$$

$$\tilde{=}A_{(\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta)} \star A_{(\gamma,\delta)}$$

$$\tilde{=}A_{((\alpha\tilde{\cap}\gamma)\tilde{\cap}\gamma^c,(\beta\tilde{\cup}\delta)\tilde{\cup}\delta^c)} \tilde{=} A_{(\Phi,\mathfrak{X})}.$$

BCI-3 $A_{(\alpha,\beta)} \star A_{(\alpha,\beta)} \tilde{=} A_{(\alpha\tilde{\cap}\alpha^c,\beta\tilde{\cup}\beta^c)} \tilde{=} A_{(\Phi,\mathfrak{X})}$.

BCI-4 Let $A_{(\alpha,\beta)} \star A_{(\gamma,\delta)} \tilde{=} A_{(\Phi,\mathfrak{X})}$ and $A_{(\gamma,\delta)} \star A_{(\alpha,\beta)} \tilde{=} A_{(\Phi,\mathfrak{X})}$. For any $e \in A$,

$$\alpha(e) \cap (\gamma(e))^c = \emptyset \text{ and } \gamma(e) \cap (\alpha(e))^c = \emptyset \text{ imply that } \alpha(e) = \gamma(e),$$

also

$$\begin{aligned} \beta(e) \cup (\delta(e))^c &= X \text{ and } \delta(e) \cup (\beta(e))^c = X \\ &\Rightarrow \beta(e) \cap (\delta(e))^c = \emptyset \text{ and } \delta(e) \cap (\beta(e))^c = \emptyset \\ &\Rightarrow \beta(e) = \delta(e). \end{aligned}$$

Hence $A_{(\alpha,\beta)} \tilde{=} A_{(\gamma,\delta)}$.

BCK-5 $A_{(\Phi,\mathfrak{X})} \star A_{(\alpha,\beta)} \tilde{=} A_{(\Phi\star\alpha,\mathfrak{X}\star\beta)} \tilde{=} A_{(\Phi\tilde{\cap}\alpha^c,\mathfrak{X}\tilde{\cup}\beta^c)} \tilde{=} A_{(\Phi,\mathfrak{X})}$.

Thus $(\mathcal{DSS}(X)_A, \star, A_{(\Phi,\mathfrak{X})})$ is a BCK-algebra.

Now $A_{(\mathfrak{X},\Phi)} \in \mathcal{DSS}(X)_A$ is such that:

$$\begin{aligned} A_{(\alpha,\beta)} \star A_{(\mathfrak{X},\Phi)} &\tilde{=} A_{(\alpha\star\mathfrak{X},\beta\star\Phi)} \\ &\tilde{=} A_{(\alpha\tilde{\cap}\mathfrak{X}^c,\beta\tilde{\cup}\Phi^c)} \\ &\tilde{=} A_{(\alpha\tilde{\cap}\Phi,\beta\tilde{\cup}\mathfrak{X})} \\ &\tilde{=} A_{(\Phi,\mathfrak{X})} \qquad \text{for all } A_{(\alpha,\beta)} \in \mathcal{DSS}(X)_A. \end{aligned}$$

Therefore $(\mathcal{DSS}(X)_A, \star, A_{(\Phi,\mathfrak{X})})$ is a bounded BCK-algebra.

For any $A_{(\alpha,\beta)} \in \mathcal{DSS}(X)_A$,

$$\begin{aligned} A_{(\mathfrak{X},\Phi)} \star (A_{(\mathfrak{X},\Phi)} \star A_{(\alpha,\beta)}) &\tilde{=} A_{(\mathfrak{X},\Phi)} \star A_{(\mathfrak{X}\star\alpha,\Phi\star\beta)} \\ &\tilde{=} A_{(\mathfrak{X},\Phi)} \star A_{(\mathfrak{X}\tilde{\cap}\alpha^c,\Phi\tilde{\cup}\beta^c)} \\ &\tilde{=} A_{(\mathfrak{X},\Phi)} \star A_{(\alpha^c,\beta^c)} \\ &\tilde{=} A_{(\mathfrak{X}\tilde{\cap}(\alpha^c)^c,\Phi\tilde{\cup}(\beta^c)^c)} \\ &\tilde{=} A_{(\mathfrak{X}\tilde{\cap}\alpha,\Phi\tilde{\cup}\beta)} \tilde{=} A_{(\alpha,\beta)}. \end{aligned}$$

So every element of $\mathcal{DSS}(X)_A$ is an involution. ∎

Chapter 5

Double-framed Fuzzy Soft Sets and Their Algebraic Structures

This chapter explores the theory of double-framed fuzzy soft sets which is a generalization of double-framed soft sets and most generalized structure in our work. Double-framed fuzzy soft sets and their operations are defined and their characteristics are studied. Examples are given to elaborate the concepts and to show how the ideas are utilized to work with double-framed fuzzy soft sets. We see from examples that the cases for double-framed fuzzy soft sets are of more generalized nature and we cannot model those with double-framed soft sets.

5.1 Double-framed Fuzzy Soft Sets

Let X be an initial universe and E be a set of parameters. Let $\mathcal{FP}(X)$ denotes the fuzzy power set of X and A, B, C are non-empty subsets of E.

5.1.1 Definition

A double-framed pair $\langle (f, g); A \rangle$ is called a double-framed fuzzy soft set over X, where f and g are mappings from A to $\mathcal{FP}(X)$.

From here, we shall use the notation $A_{(f,g)}$ over X to denote a double-framed fuzzy soft set $\langle (f, g); A \rangle$ over X where the meanings of f, g, A and X are clear.

5.1.2 Definition

For double-*framed* *fuzzy* soft sets $A_{(f,g)}$ and $B_{(h,i)}$ over X, we say that $A_{(f,g)}$ is a *double-framed* fuzzy soft subset of $B_{(h,i)}$, if

1) $A \subseteq B$ and

2) $f(e) \subseteq h(e)$ and $i(e) \subseteq g(e)$ for all $e \in A$.

This relationship is denoted by $A_{(f,g)} \tilde{\subseteq} B_{(h,i)}$. Also $A_{(f,g)}$ is said to be a *double-framed fuzzy soft superset* of $B_{(h,i)}$, if $B_{(h,i)}$ is a *double-framed fuzzy soft subset* of $A_{(f,g)}$. We denote it by $A_{(f,g)} \tilde{\supseteq} B_{(h,i)}$.

5.1.3 Definition

Two *double-framed fuzzy soft sets* $A_{(f,g)}$ and $B_{(h,i)}$ over X are said to be *equal* if $A_{(f,g)}$ is a *double-framed fuzzy soft subset* of $B_{(h,i)}$ and $B_{(h,i)}$ is a *double-framed fuzzy soft subset* of $A_{(f,g)}$. We denote it by $A_{(f,g)} \tilde{=} B_{(h,i)}$.

5.1.4 Example

Let X be the set of houses under consideration, and E be the set of parameters, $X = \{h_1,h_2,h_3,h_4,h_5\}$, $E = \{e_1,e_2,e_3,e_4,e_5,e_6\} = \{$ in the green surroundings, wooden, cheap, in good repair, furnished, traditional $\}$. Suppose that $A = \{e_1,e_2,e_3,e_6\}$, a double-framed fuzzy soft set $A_{(f,g)}$ describes the "highest and lowest budget ratings of the houses under consideration" given by f and g respectively. The double-framed fuzzy soft set $A_{(f,g)}$ over X is given as follows:

$$\begin{aligned} f &: A \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{h_1/0.9, h_2/0.3, h_3/0.8, h_4/0.7, h_5/0.5\} & \text{if } e = e_1, \\ \{h_1/0.1, h_2/0.5, h_3/0.1, h_4/0.8, h_5/0.6\} & \text{if } e = e_2, \\ \{h_1/0.2, h_2/0.5, h_3/0.2, h_4/0.9, h_5/0.9\} & \text{if } e = e_3, \\ \{h_1/0.7, h_2/0.4, h_3/0.2, h_4/0.1, h_5/0\} & \text{if } e = e_6, \end{cases} \end{aligned}$$

$$\begin{aligned} g &: A \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{h_1/0.2, h_2/0.3, h_3/0.3, h_4/0.4, h_5/0.8\} & \text{if } e = e_1, \\ \{h_1/0.7, h_2/0.4, h_3/0.8, h_4/0.7, h_5/0.9\} & \text{if } e = e_2, \\ \{h_1/0.6, h_2/0.4, h_3/0.6, h_4/0.6, h_5/0.7\} & \text{if } e = e_3, \\ \{h_1/0.9, h_2/0.3, h_3/0.8, h_4/0.7, h_5/0.5\} & \text{if } e = e_6. \end{cases} \end{aligned}$$

Let $B = \{e_2,e_6\}$. Then the double-framed fuzzy soft set $B_{(h,i)}$ given by

$$h : B \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_1/0.1, h_2/0.5, h_3/0.1, h_4/0.8, h_5/0.6\} & \text{if } e = e_2, \\ \{h_1/0.7, h_2/0.4, h_3/0.2, h_4/0.1, h_5/0\} & \text{if } e = e_6, \end{cases}$$

$$i : B \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_1/0.1, h_2/0.2, h_3/0.4, h_4/0.3, h_5/0.5\} & \text{if } e = e_2, \\ \{h_1/0.9, h_2/0.4, h_3/0.9, h_4/0.8, h_5/0.7\} & \text{if } e = e_6. \end{cases}$$

is a double-framed fuzzy soft subset of $A_{(f,g)}$ which represents a finer data analysis and so $B_{(h,i)} \tilde{\subseteq} A_{(f,g)}$.

5.2 Operations on Double-framed Fuzzy Soft Sets

In this section, we define various operations on *double-framed fuzzy* soft sets:

5.2.1 Definition

Let $A_{(f,g)}$ and $B_{(h,i)}$ be *double-framed fuzzy* soft sets over X. The *int-uni product* of $A_{(f,g)}$ and $B_{(h,i)}$ is defined as a double-framed fuzzy soft set $(A \times B)_{(f\tilde{\wedge}h, g\tilde{\vee}i)}$ over X in which $f\tilde{\wedge}h : (A \times B) \to \mathcal{FP}(X)$, where

$$(a,b) \mapsto f(a) \wedge h(b),$$

and $g\tilde{\vee}i : (A \times B) \to \mathcal{FP}(X)$, where

$$(a, b) \mapsto g(a) \vee i(b).$$

It is denoted by $A_{(f,g)} \wedge B_{(h,i)} \tilde{=} (A \times B)_{(f\tilde{\wedge}h, g\tilde{\vee}i)}$.

5.2.2 Definition

Let $A_{(f,g)}$ and $B_{(h,i)}$ be *double-framed fuzzy* soft sets over X. The *uni-int product* of $A_{(f,g)}$ and $B_{(h,i)}$ is defined as a double-framed fuzzy soft set $(A \times B)_{(f\tilde{\vee}h, g\tilde{\wedge}i)}$ over X in which $f\tilde{\vee}h : (A \times B) \to \mathcal{FP}(X)$, where

$$(a, b) \mapsto f(a) \vee h(b),$$

and $g\tilde{\wedge}i : (A \times B) \to \mathcal{FP}(X)$, where

$$(a, b) \mapsto g(a) \wedge i(b).$$

It is denoted by $A_{(f,g)} \vee B_{(h,i)} \tilde{=} (A \times B)_{(f\tilde{\vee}h, g\tilde{\wedge}i)}$.

5.2.3 Definition

For double-*framed fuzzy soft sets* $A_{(f,g)}$ and $B_{(h,i)}$ over X, *the extended int-uni double-framed fuzzy soft set* of $A_{(f,g)}$ and $B_{(h,i)}$ is defined as a *double-framed fuzzy soft set* $(A \cup B)_{(f\tilde{\wedge}h, g\tilde{\vee}i)}$ where $f\tilde{\wedge}h : (A \cup B) \to \mathcal{FP}(X)$, given by

$$e \mapsto \begin{cases} f(e) & if\ e \in A - B \\ h(e) & if\ e \in B - A \\ f(e) \wedge h(e) & if\ e \in A \cap B \end{cases}$$

and $g\tilde{\vee}i : (A \cup B) \to \mathcal{FP}(X)$, given by

$$e \mapsto \begin{cases} g(e) & if\ e \in A - B \\ i(e) & if\ e \in B - A \\ g(e) \vee i(e) & if\ e \in A \cap B \end{cases}.$$

It is denoted by $A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)} \tilde{=} (A \cup B)_{(f\tilde{\wedge}h, g\tilde{\vee}i)}$.

5.2.4 Definition

For double-*framed fuzzy soft sets* $A_{(f,g)}$ and $B_{(h,i)}$ over X, *the extended uni-int double-framed fuzzy soft set* of $A_{(f,g)}$ and $B_{(h,i)}$ is defined as a *double-framed fuzzy soft set* $(A \cup B)_{(f\tilde{\vee}h, g\tilde{\wedge}i)}$ where $f\tilde{\vee}h : (A \cup B) \to \mathcal{FP}(X)$, given by

$$e \mapsto \begin{cases} f(e) & if\ e \in A - B \\ h(e) & if\ e \in B - A \\ f(e) \vee h(e) & if\ e \in A \cap B \end{cases}$$

and $g\tilde{\wedge}i : (A \cup B) \to \mathcal{FP}(X)$, given by

$$e \mapsto \begin{cases} g(e) & if\ e \in A - B \\ i(e) & if\ e \in B - A \\ g(e) \wedge i(e) & if\ e \in A \cap B \end{cases}.$$

It is denoted by $A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)} \tilde{=} (A \cup B)_{(f\tilde{\vee}h, g\tilde{\wedge}i)}$.

5.2.5 Definition

For double-*framed* *fuzzy soft sets* $A_{(f,g)}$ and $B_{(h,i)}$ over X with $(A \cap B) \neq \emptyset$, the *restricted int-uni double-framed fuzzy soft set* of $A_{(f,g)}$ and $B_{(h,i)}$ is defined as a *double-framed fuzzy soft set* $(A \cap B)_{(f\tilde{\wedge}h, g\tilde{\vee}i)}$ where $f\tilde{\wedge}h : (A \cap B) \to \mathcal{FP}(X)$,

$$e \mapsto f(e) \wedge h(e),$$

and $g\tilde{\vee}i : (A \cap B) \to \mathcal{FP}(X)$,

$$e \mapsto g(e) \vee i(e).$$

It is denoted by $A_{(f,g)} \sqcap B_{(h,i)} \tilde{=} (A \cap B)_{(f\tilde{\wedge}h, g\tilde{\vee}i)}$.

5.2.6 Definition

For double-*framed* *fuzzy soft sets* $A_{(f,g)}$ and $B_{(h,i)}$ over X with $(A \cap B) \neq \emptyset$, the *restricted uni-int double-framed fuzzy soft set* of $A_{(f,g)}$ and $B_{(h,i)}$ is defined as a *double-framed fuzzy soft set* $(A \cap B)_{(f\tilde{\vee}h, g\tilde{\wedge}i)}$ where $f\tilde{\vee}h : (A \cap B) \to \mathcal{FP}(X)$, given by

$$e \mapsto f(e) \vee h(e),$$

and $g\tilde{\wedge}i : (A \cap B) \to \mathcal{FP}(X)$,

$$e \mapsto g(e) \wedge i(e).$$

It is denoted by $A_{(f,g)} \sqcup B_{(h,i)} \tilde{=} (A \cap B)_{(f\tilde{\vee}h, g\tilde{\wedge}i)}$.

5.2.7 Definition

Let $A_{(f,g)}$ be a *double-framed fuzzy* soft set over X. The *complement of a double-framed fuzzy soft set* $A_{(f,g)}$ over X is defined as a *double-framed fuzzy soft set* $A_{(f\,',g\,')}$ over X where $f\,' : A \to \mathcal{FP}(X)$, given by

$$e \mapsto (f(e))'$$

and $g\,' : A \to \mathcal{FP}(X)$,

$$e \mapsto (g(e))'.$$

It is denoted by $A_{(f,g)}{}' \tilde{=} A_{(f\,',g\,')}$.

5.3 Properties of Double-framed Fuzzy Soft Sets

In this section we discuss properties and laws of double-framed fuzzy soft sets with respect to their operations. Associativity, commutativity, absorption, distributivity and properties of double-framed fuzzy soft sets are investigated.

5.3.1 Definition

A *double-framed fuzzy soft set* over X is said to be a *relative null double-framed fuzzy soft set*, denoted by $A_{(\bar{0},\bar{1})}$ where

$$\bar{0} : A \to \mathcal{FP}(X),\ e \mapsto \bar{0}, \text{ where } \bar{0} \text{ maps every element of } X \text{ onto } 0$$
$$\bar{1} : A \to \mathcal{FP}(X),\ e \mapsto \bar{1}, \text{ where } \bar{1} \text{ maps every element of } X \text{ onto } 1$$

5.3.2 Definition

A *double-framed fuzzy* soft set over X is said to be a *relative absolute double-framed fuzzy soft set*, denoted by $A_{(\bar{\mathbf{1}},\bar{\mathbf{0}})}$ where

$$\begin{aligned}\bar{\mathbf{1}} &: A \to \mathcal{FP}(X),\ e \mapsto \bar{\mathbf{1}},\\ \bar{\mathbf{0}} &: A \to \mathcal{FP}(X),\ e \mapsto \bar{\mathbf{0}}.\end{aligned}$$

Conventionally, we take the *double-framed fuzzy* soft sets with empty set of parameters to be equal to $\emptyset_{(\bar{\mathbf{0}},\bar{\mathbf{1}})}$ and so $A_{(f,g)} \sqcap B_{(h,i)} \tilde{=} A_{(f,g)} \sqcup B_{(h,i)} \tilde{=} \emptyset_{(\bar{\mathbf{0}},\bar{\mathbf{1}})}$ where $(A \cap B) = \emptyset$.

5.3.3 Proposition

If $A_{(\bar{\mathbf{0}},\bar{\mathbf{1}})}$ is a null *double-framed fuzzy* soft set, $A_{(\bar{\mathbf{1}},\bar{\mathbf{0}})}$ an absolute *double-framed fuzzy* soft set, and $A_{(f,g)}$, $A_{(h,i)}$ are *double-framed fuzzy* soft sets over X, then

1) $A_{(f,g)} \sqcup_{\varepsilon} A_{(h,i)} \tilde{=} A_{(f,g)} \sqcup A_{(h,i)}$,

2) $A_{(f,g)} \sqcap_{\varepsilon} A_{(h,i)} \tilde{=} A_{(f,g)} \sqcap A_{(h,i)}$,

3) $A_{(f,g)} \sqcap A_{(f,g)} \tilde{=} A_{(f,g)} \tilde{=} A_{(f,g)} \sqcup A_{(f,g)}$, (Idempotent)

4) $A_{(f,g)} \sqcup A_{(\bar{\mathbf{0}},\bar{\mathbf{1}})} \tilde{=} A_{(f,g)} \tilde{=} A_{(f,g)} \sqcap A_{(\bar{\mathbf{1}},\bar{\mathbf{0}})}$,

5) $A_{(f,g)} \sqcup A_{(\bar{\mathbf{1}},\bar{\mathbf{0}})} \tilde{=} A_{(\bar{\mathbf{1}},\bar{\mathbf{0}})}$; $A_{(f,g)} \sqcap A_{(\bar{\mathbf{0}},\bar{\mathbf{1}})} \tilde{=} A_{(\bar{\mathbf{0}},\bar{\mathbf{1}})}$.

Proof. Proofs of 1), 2) and 3) are straightforward.

4) As $A_{(f,g)} \sqcup A_{(\Phi,\bar{\mathbf{1}})} \tilde{=} A_{(f \tilde{\vee} \bar{\mathbf{0}}, g \tilde{\wedge} \bar{\mathbf{1}})}$. Therefore for any $e \in A$,

$$(f \tilde{\vee} \bar{\mathbf{0}})(e) = f(e) \vee \bar{\mathbf{0}}(e) = f(e) \text{ and } (g \tilde{\wedge} \bar{\mathbf{1}})(e) = g(e) \wedge \bar{\mathbf{1}}(e) = g(e).$$

Thus $A_{(f,g)} \sqcup A_{(\bar{\mathbf{0}},\bar{\mathbf{1}})} \tilde{=} A_{(f,g)}$.

Again, $A_{(f,g)} \sqcap A_{(\bar{\mathbf{1}},\bar{\mathbf{0}})} \tilde{=} A_{(f \tilde{\wedge} \bar{\mathbf{1}}, g \tilde{\vee} \bar{\mathbf{0}})}$. For any $e \in A$,

$$(f \tilde{\wedge} \bar{\mathbf{1}})(e) = f(e) \wedge \bar{\mathbf{1}}(e) = f(e) \text{ and } (g \tilde{\vee} \bar{\mathbf{0}})(e) = g(e) \vee \bar{\mathbf{0}}(e) = g(e).$$

So $A_{(f,g)} \sqcap A_{(\bar{\mathbf{1}},\bar{\mathbf{0}})} \tilde{=} A_{(f,g)}$.

Part 5) can be proved in a similar way. ■

5.3.4 Proposition

Let $A_{(f,g)}$, $B_{(h,i)}$ and $C_{(j,k)}$ be any *double-framed fuzzy soft sets* over a common universe X. Then the following are true

1) $A_{(f,g)} \lambda (B_{(h,i)} \lambda C_{(j,k)}) \tilde{=} (A_{(f,g)} \lambda B_{(h,i)}) \lambda C_{(j,k)}$, (Associative Laws)

2) $A_{(f,g)} \lambda B_{(h,i)} \tilde{=} B_{(h,i)} \lambda A_{(f,g)}$, (Commutative Laws) for all $\lambda \in \{\sqcup_\varepsilon, \sqcup, \sqcap_\varepsilon, \sqcap\}$.

Proof.

1) Since $A_{(f,g)} \sqcup_\varepsilon (B_{(h,i)} \sqcup_\varepsilon C_{(j,k)}) \doteq A \cup (B \cup C)_{(f\tilde{\vee}(h\tilde{\vee}j), g\tilde{\wedge}(g\tilde{\wedge}k))}$, we have for any $e \in A \cup (B \cup C)$:

(i) If $e \in A - (B \cup C)$, then

$$\begin{aligned}(f\tilde{\vee}(h\tilde{\vee}j))(e) &= f(e) = ((f\tilde{\vee}h)\tilde{\vee}j)(e)\\ (g\tilde{\wedge}(i\tilde{\wedge}k))(e) &= g(e) = ((g\tilde{\wedge}i)\tilde{\wedge}k)(e)\end{aligned}$$

(ii) If $e \in B - (A \cup C)$, then

$$\begin{aligned}(f\tilde{\vee}(h\tilde{\vee}j))(e) &= h(e) = ((f\tilde{\vee}h)\tilde{\vee}j)(e)\\ (g\tilde{\wedge}(i\tilde{\wedge}k))(e) &= g(e) = ((g\tilde{\wedge}i)\tilde{\wedge}k)(e)\end{aligned}$$

(iii) If $e \in C - (A \cup B)$, then

$$\begin{aligned}(f\tilde{\vee}(h\tilde{\vee}j))(e) &= j(e) = ((f\tilde{\vee}h)\tilde{\vee}j)(e)\\ (g\tilde{\wedge}(i\tilde{\wedge}k))(e) &= k(e) = ((g\tilde{\wedge}i)\tilde{\wedge}k)(e)\end{aligned}$$

(iv) If $e \in (A \cap B) - C$, then

$$\begin{aligned}(f\tilde{\vee}(h\tilde{\vee}j))(e) &= f(e) \vee h(e) = (f\tilde{\vee}h)(e) = ((f\tilde{\vee}h)\tilde{\vee}j)(e)\\ (g\tilde{\wedge}(i\tilde{\wedge}k))(e) &= g(e) \wedge i(e) = (g\tilde{\wedge}i)(e) = ((g\tilde{\wedge}i)\tilde{\wedge}k)(e)\end{aligned}$$

(v) If $e \in (A \cap C) - B$, then

$$\begin{aligned}(f\tilde{\vee}(h\tilde{\vee}j))(e) &= f(e) \vee j(e) = ((f\tilde{\vee}h)\tilde{\vee}j)(e)\\ (g\tilde{\wedge}(i\tilde{\wedge}k))(e) &= g(e) \wedge k(e) = ((g\tilde{\wedge}i)\tilde{\wedge}k)(e)\end{aligned}$$

(vi) If $e \in (B \cap C) - A$, then

$$\begin{aligned}(f\tilde{\vee}(h\tilde{\vee}j))(e) &= h(e) \vee j(e) = (f\tilde{\vee}h)\tilde{\vee}j(e)\\ (g\tilde{\wedge}(i\tilde{\wedge}k))(e) &= g(e) \wedge k(e) = (g\tilde{\wedge}i)\tilde{\wedge}k(e)\end{aligned}$$

(vii) If $e \in (A \cap B) \cap C$, then

$$\begin{aligned}(f\tilde{\vee}(h\tilde{\vee}j))(e) &= f(e) \vee (h(e) \vee j(e)) = (f(e) \vee h(e)) \vee j(e) = ((f\tilde{\vee}h)\tilde{\vee}j)(e)\\ (g\tilde{\wedge}(i\tilde{\wedge}k))(e) &= g(e) \wedge (i(e) \wedge k(e)) = (g(e) \wedge i(e)) \wedge k(e) = ((g\tilde{\wedge}i)\tilde{\wedge}k)(e)\end{aligned}$$

Thus $A_{(f,g)} \sqcup_\varepsilon (B_{(h,i)} \sqcup_\varepsilon C_{(j,k)}) \doteq (A_{(f,g)} \sqcup_\varepsilon B_{(h,i)}) \sqcup_\varepsilon C_{(j,k)}$.

Similarly, we can prove for $\lambda \in \{\sqcup, \sqcap_\varepsilon, \sqcap\}$

2) This is straightforward.

■

5.3.5 Proposition (Absorption Laws)

Let $A_{(f,g)}$, $B_{(h,i)}$ be any *double-framed fuzzy* soft sets over X. Then the following are true:

1) $A_{(f,g)} \sqcup_{\varepsilon} (B_{(h,i)} \sqcap A_{(f,g)}) \tilde{=} A_{(f,g)}$,

2) $A_{(f,g)} \sqcap (B_{(h,i)} \sqcup_{\varepsilon} A_{(f,g)}) \tilde{=} A_{(f,g)}$,

3) $A_{(f,g)} \sqcup (B_{(h,i)} \sqcap_{\varepsilon} A_{(f,g)}) \tilde{=} A_{(f,g)}$,

4) $A_{(f,g)} \sqcap_{\varepsilon} (B_{(h,i)} \sqcup A_{(f,g)}) \tilde{=} A_{(f,g)}$.

Proof. Straightforward. ∎

5.3.6 Proposition (Distributive Laws)

Let $A_{(f,g)}$, $B_{(h,i)}$ and $C_{(j,k)}$ be any *double-framed fuzzy soft sets* over X. Then

1) $A_{(f,g)} \sqcap (B_{(h,i)} \sqcup_{\varepsilon} C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcap B_{(h,i)}) \sqcup_{\varepsilon} (A_{(f,g)} \sqcap C_{(j,k)})$,

2) $A_{(f,g)} \sqcap (B_{(h,i)} \sqcap_{\varepsilon} C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcap B_{(h,i)}) \sqcap_{\varepsilon} (A_{(f,g)} \sqcap C_{(j,k)})$,

3) $A_{(f,g)} \sqcap (B_{(h,i)} \sqcup C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcap B_{(h,i)}) \sqcup (A_{(f,g)} \sqcap C_{(j,k)})$,

4) $A_{(f,g)} \sqcup (B_{(h,i)} \sqcup_{\varepsilon} C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcup B_{(h,i)}) \sqcup_{\varepsilon} (A_{(f,g)} \sqcup C_{(j,k)})$,

5) $A_{(f,g)} \sqcup (B_{(h,i)} \sqcap_{\varepsilon} C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcup B_{(h,i)}) \sqcap_{\varepsilon} (A_{(f,g)} \sqcup C_{(j,k)})$,

6) $A_{(f,g)} \sqcup (B_{(h,i)} \sqcap C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcup B_{(h,i)}) \sqcap (A_{(f,g)} \sqcup C_{(j,k)})$,

7) $A_{(f,g)} \sqcap_{\varepsilon} (B_{(h,i)} \sqcup_{\varepsilon} C_{(j,k)}) \tilde{\subseteq} (A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)}) \sqcup_{\varepsilon} (A_{(f,g)} \sqcap_{\varepsilon} C_{(j,k)})$,

8) $A_{(f,g)} \sqcap_{\varepsilon} (B_{(h,i)} \sqcup C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)}) \sqcup (A_{(f,g)} \sqcap_{\varepsilon} C_{(j,k)})$,

9) $A_{(f,g)} \sqcap_{\varepsilon} (B_{(h,i)} \sqcap C_{(j,k)}) \tilde{\supseteq} (A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)}) \sqcap (A_{(f,g)} \sqcap_{\varepsilon} C_{(j,k)})$,

10) $A_{(f,g)} \sqcup_{\varepsilon} (B_{(h,i)} \sqcup C_{(j,k)}) \tilde{\subseteq} (A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)}) \sqcup (A_{(f,g)} \sqcup_{\varepsilon} C_{(j,k)})$,

11) $A_{(f,g)} \sqcup_{\varepsilon} (B_{(h,i)} \sqcap_{\varepsilon} C_{(j,k)}) \tilde{\supseteq} (A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)}) \sqcap_{\varepsilon} (A_{(f,g)} \sqcup_{\varepsilon} C_{(j,k)})$,

12) $A_{(f,g)} \sqcup_{\varepsilon} (B_{(h,i)} \sqcap C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)}) \sqcap (A_{(f,g)} \sqcup_{\varepsilon} C_{(j,k)})$.

Proof. We prove only one part here and remaining parts can be proved in a similar way.

1) Consider $A_{(f,g)} \sqcap (B_{(h,i)} \sqcup_{\varepsilon} C_{(j,k)})$. For any $e \in A \cap (B \cup C)$, we have following three disjoint cases:

(i) If $e \in A \cap (B - C)$, then

$$(f\tilde{\wedge}(h\tilde{\vee}j))(e) = f(e) \wedge h(e) \text{ and } (g\tilde{\vee}(i\tilde{\wedge}k))(e) = g(e) \vee i(e)$$

and

$$\begin{aligned} ((f\tilde{\wedge}h)\tilde{\vee}(f\tilde{\wedge}j))(e) &= (f\tilde{\wedge}h)(e) = f(e) \wedge h(e) \text{ and} \\ ((g\tilde{\vee}i)\tilde{\wedge}(g\tilde{\vee}k))(e) &= (g\tilde{\vee}i)(e) = g(e) \vee i(e). \end{aligned}$$

(ii) If $e \in A \cap (C - B)$, then

$$(f\tilde{\wedge}(h\tilde{\vee}j))(e) = f(e) \wedge j(e) \text{ and } (g\tilde{\vee}(i\tilde{\wedge}k))(e) = g(e) \vee k(e)$$

and

$$\begin{aligned} ((f\tilde{\wedge}h)\tilde{\vee}(f\tilde{\wedge}j))(e) &= (f\tilde{\wedge}j)(e) = f(e) \wedge j(e) \text{ and} \\ ((g\tilde{\vee}i)\tilde{\wedge}(g\tilde{\vee}k))(e) &= (g\tilde{\vee}k)(e) = g(e) \vee k(e). \end{aligned}$$

(iii) If $e \in A \cap (B \cap C)$, then

$$\begin{aligned} (f\tilde{\wedge}(h\tilde{\vee}j))(e) &= f(e) \wedge (h(e) \vee j(e)) \text{ and} \\ (g\tilde{\vee}(i\tilde{\wedge}k))(e) &= g(e) \vee (i(e) \wedge k(e)) \end{aligned}$$

and

$$\begin{aligned} ((f\tilde{\wedge}h)\tilde{\vee}(f\tilde{\wedge}j))(e) &= (f\tilde{\wedge}h)(e)\tilde{\vee}(f\tilde{\wedge}j)(e) \\ &= (f(e) \wedge h(e)) \vee (f(e) \wedge j(e)) \\ &= f(e) \wedge (h(e) \vee j(e)) \qquad \text{and} \\ ((g\tilde{\vee}i)\tilde{\wedge}(g\tilde{\vee}k))(e) &= (g\tilde{\vee}i)(e) \wedge (g\tilde{\vee}k)(e) \\ &= (g(e) \vee i(e)) \wedge (g(e) \vee k(e)) \\ &= g(e) \vee (i(e) \wedge k(e)). \end{aligned}$$

Thus

$$A_{(f,g)} \sqcap (B_{(h,i)} \sqcup_{\varepsilon} C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcap B_{(h,i)}) \sqcup_{\varepsilon} (A_{(f,g)} \sqcap C_{(j,k)}).$$

■

5.3.7 Example

Let X be the set of cars of different models, and E be the set of colors, $X = \{x_1,x_2,x_3,x_4,x_5\}$, $E = \{e_1,e_2,e_3,e_4,e_5,e_6\} = \{$ green, red, blue, black, white, silver $\}$. Suppose that $A = \{e_1, e_2, e_3\}$, $B = \{e_2, e_3, e_4\}$, and $C = \{e_3, e_4, e_5\}$. The double-framed fuzzy soft sets $A_{(f,g)}$, $B_{(h,i)}$ and $C_{(j,k)}$ over X describe the level of appreciation from customers based upon the annual survey reports of three different showrooms respectively. Here $\{f, h, j\}$ and $\{g, i, k\}$ collect results for positive and negative aspects respectively. We have

$$\begin{aligned}
f &: A \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases} \{x_1/0.3, x_2/0.1, x_3/0.3, x_4/0.1, x_5/0.7\} & \text{if } e = e_1,\\ \{x_1/0.1, x_2/0.9, x_3/0.3, x_4/0.8, x_5/0.2\} & \text{if } e = e_2,\\ \{x_1/0.1, x_2/0.3, x_3/0.3, x_4/0.3, x_5/0.8\} & \text{if } e = e_3, \end{cases}\\
g &: A \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases} \{x_1/0.4, x_2/0.7, x_3/0.7, x_4/0.7, x_5/0.1\} & \text{if } e = e_1,\\ \{x_1/0.8, x_2/0, x_3/0.5, x_4/0.1, x_5/0.6\} & \text{if } e = e_2,\\ \{x_1/0.7, x_2/0.5, x_3/0.7, x_4/0.6, x_5/0.1\} & \text{if } e = e_3. \end{cases}
\end{aligned}$$

$$\begin{aligned}
h &: B \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases} \{x_1/0.1, x_2/0.3, x_3/0.6, x_4/0.2, x_5/0.3\} & \text{if } e = e_2,\\ \{x_1/0.8, x_2/0.9, x_3/0.5, x_4/0.4, x_5/0.2\} & \text{if } e = e_3,\\ \{x_1/0.1, x_2/0.4, x_3/0.3, x_4/0.6, x_5/0.9\} & \text{if } e = e_4, \end{cases}\\
g &: B \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases} \{x_1/0.1, x_2/0.3, x_3/0.3, x_4/0.6, x_5/0.6\} & \text{if } e = e_2,\\ \{x_1/0.1, x_2/0, x_3/0.3, x_4/0.4, x_5/0.6\} & \text{if } e = e_3,\\ \{x_1/0.9, x_2/0.5, x_3/0.5, x_4/0.3, x_5/0.1\} & \text{if } e = e_4. \end{cases}\\
j &: C \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases} \{x_1/0.1, x_2/0.2, x_3/0.3, x_4/0.1, x_5/0.1\} & \text{if } e = e_3,\\ \{x_1/0.2, x_2/0.2, x_3/0.3, x_4/0.3, x_5/0.2\} & \text{if } e = e_4,\\ \{x_1/0.1, x_2/0.1, x_3/0.3, x_4/0.5, x_5/0.7\} & \text{if } e = e_5, \end{cases}\\
k &: C \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases} \{x_1/0.7, x_2/0.7, x_3/0.4, x_4/0.7, x_5/0.4\} & \text{if } e = e_3,\\ \{x_1/0.6, x_2/0.5, x_3/0.6, x_4/0.1, x_5/0.6\} & \text{if } e = e_4,\\ \{x_1/0.3, x_2/0.4, x_3/0.4, x_4/0.3, x_5/0.1\} & \text{if } e = e_5. \end{cases}
\end{aligned}$$

We know that

$$A_{(f,g)} \sqcup_\varepsilon (B_{(h,i)} \sqcap_\varepsilon C_{(j,k)}) \tilde{=} ((A \cup B) \cup C)_{(f \tilde{\vee} (h \tilde{\wedge} j), g \tilde{\wedge} (i \tilde{\vee} k))}$$

and

$$(A_{(f,g)} \sqcup_\varepsilon B_{(h,g)}) \sqcap_\varepsilon (A_{(f,g)} \sqcup_\varepsilon C_{(j,k)}) \tilde{=} ((A \cup B) \cup C)_{((f \tilde{\vee} h) \tilde{\wedge} (f \tilde{\vee} j))}.$$

Then

$$\begin{aligned}
(f \tilde{\vee} (h \tilde{\wedge} j))(e_2) &= \{x_1/0.1, x_2/0.9, x_3/0.6, x_4/0.8, x_5/0.3\}\\
&\neq \{x_1/0.1, x_2/0.9, x_3/0.3, x_4/0.8, x_5/0.2\}\\
&= ((f \tilde{\vee} h) \tilde{\wedge} (f \tilde{\vee} j))(e_2) \text{ and}\\
(g \tilde{\wedge} (i \tilde{\vee} k))(e_2) &= \{x_1/0.1, x_2/0.0, x_3/0.3, x_4/0.1, x_5/0.6\}\\
&\neq \{x_1/0.8, x_2/0.0, x_3/0.5, x_4/0.1, x_5/0.6\}\\
&= ((g \tilde{\wedge} i) \tilde{\vee} (g \tilde{\wedge} k))(e_2),
\end{aligned}$$

so that

$$A_{(f,g)} \sqcup_\varepsilon (B_{(h,i)} \sqcap_\varepsilon C_{(j,k)}) \tilde{\neq} (A_{(f,g)} \sqcup_\varepsilon B_{(h,i)}) \sqcap_\varepsilon (A_{(f,g)} \sqcup_\varepsilon C_{(j,k)}).$$

Now,

$$A_{(f,g)} \sqcap_\varepsilon (B_{(h,i)} \sqcup_\varepsilon C_{(j,k)}) \tilde{=} ((A \cup B) \cup C)_{(f\tilde{\wedge}(h\tilde{\vee}j), g\tilde{\vee}(i\tilde{\wedge}k))}$$

and

$$(A_{(f,g)} \sqcap_\varepsilon B_{(h,i)}) \sqcup_\varepsilon (A_{(f,g)} \sqcap_\varepsilon C_{(j,k)}) \tilde{=} ((A \cup B) \cup C)_{((f\tilde{\wedge}h)\tilde{\vee}(f\tilde{\wedge}j), (g\tilde{\vee}i)\tilde{\wedge}(g\tilde{\vee}k))}.$$

Then,

$$\begin{aligned} (f\tilde{\wedge}(h\tilde{\vee}j))(e_2) &= \{x_1/0.1, x_2/0.3, x_3/0.3, x_4/0.2, x_5/0.2\} \\ &\neq \{x_1/0.1, x_2/0.9, x_3/0.3, x_4/0.8, x_5/0.2\} \\ &= ((f\tilde{\wedge}h)\tilde{\vee}(f\tilde{\wedge}j))(e_2) \end{aligned}$$

and

$$\begin{aligned} (g\tilde{\vee}(i\tilde{\wedge}k))(e_2) &= \{x_1/0.8, x_2/0.3, x_3/0.5, x_4/0.6, x_5/0.6\} \\ &\neq \{x_1/0.8, x_2/0.0, x_3/0.5, x_4/0.1, x_5/0.6\} \\ &= ((g\tilde{\vee}i)\tilde{\wedge}(g\tilde{\vee}k))(e_2). \end{aligned}$$

So

$$A_{(f,g)} \sqcap_\varepsilon (B_{(h,i)} \sqcup_\varepsilon C_{(j,k)}) \tilde{\neq} (A_{(f,g)} \sqcap_\varepsilon B_{(h,i)}) \sqcup_\varepsilon (A_{(f,g)} \sqcap_\varepsilon C_{(j,k)}).$$

Similarly we can show that

$$A_{(f,g)} \sqcup_\varepsilon (B_{(h,i)} \sqcup C_{(j,k)}) \tilde{\neq} (A_{(f,g)} \sqcup_\varepsilon B_{(h,i)}) \sqcup (A_{(f,g)} \sqcup_\varepsilon C_{(j,k)}),$$

and

$$A_{(f,g)} \sqcap_\varepsilon (B_{(h,i)} \sqcap C_{(j,k)}) \tilde{\neq} (A_{(f,g)} \sqcap_\varepsilon B_{(h,i)}) \sqcap (A_{(f,g)} \sqcap_\varepsilon C_{(j,k)}).$$

5.3.8 Proposition

Let $A_{(f,g)}$, $B_{(h,i)}$ and $C_{(j,k)}$ be any *double-framed fuzzy soft sets* over X. Then

1) $A_{(f,g)} \sqcup_\varepsilon (B_{(h,i)} \sqcap_\varepsilon C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcup_\varepsilon B_{(h,i)}) \sqcap_\varepsilon (A_{(f,g)} \sqcup_\varepsilon C_{(j,k)})$ if and only if

$$\begin{aligned} f(e) &\subseteq h(e) \text{ and } g(e) \supseteq i(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ f(e) &\subseteq j(e) \text{ and } g(e) \supseteq k(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

2) $A_{(f,g)} \sqcap_\varepsilon (B_{(h,i)} \sqcup_\varepsilon C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcap_\varepsilon B_{(h,i)}) \sqcup_\varepsilon (A_{(f,g)} \sqcap_\varepsilon C_{(j,k)})$ if and only if

$$\begin{aligned} f(e) &\supseteq h(e) \text{ and } g(e) \subseteq i(e) \text{ for all } e \in (A \cap B) - C \text{ and} \\ f(e) &\supseteq j(e) \text{ and } g(e) \subseteq k(e) \text{ for all } e \in (A \cap C) - B. \end{aligned}$$

Proof. Straightforward. ■

5.3.9 Corollary

Let $A_{(f,g)}$, $B_{(h,i)}$ and $C_{(j,k)}$ be three *double-framed fuzzy soft sets* over X such that $(A \cap B) - C = (A \cap C) - B = \emptyset$. Then

1) $A_{(f,g)} \sqcup_{\varepsilon} (B_{(h,i)} \sqcap_{\varepsilon} C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)}) \sqcap_{\varepsilon} (A_{(f,g)} \sqcup_{\varepsilon} C_{(j,k)})$,

2) $A_{(f,g)} \sqcap_{\varepsilon} (B_{(h,i)} \sqcup_{\varepsilon} C_{(j,k)}) \tilde{=} (A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)}) \sqcup_{\varepsilon} (A_{(f,g)} \sqcap_{\varepsilon} C_{(j,k)})$.

5.3.10 Corollary

Let $A_{(f,g)}$, $A_{(h,i)}$ and $A_{(j,k)}$ be three *double-framed fuzzy soft sets* over X. Then

$$A_{(f,g)} \zeta (A_{(h,i)} \rho A_{(j,k)}) \tilde{=} (A_{(f,g)} \zeta A_{(h,i)}) \rho (A_{(f,g)} \zeta A_{(j,k)})$$

for distinct $\zeta, \rho \in \{\sqcap_{\varepsilon}, \sqcap, \sqcup_{\varepsilon}, \sqcup\}$.

5.3.11 Theorem

Let $A_{(f,g)}$ and $B_{(h,i)}$ be double-*framed fuzzy soft sets* over X. Then the following are true

1) $A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)}$ is the smallest *double-framed fuzzy* soft set over X which contains both $A_{(f,g)}$ and $B_{(h,i)}$. (Supremum)

2) $A_{(f,g)} \sqcap B_{(h,i)}$ is the largest *double-framed* fuzzy soft set over X which is contained in both $A_{(f,g)}$ and $B_{(h,i)}$. (Infimum)

Proof.

1) We have $A, B \subseteq (A \cup B)$ and $f(e), h(e) \subseteq f(e) \vee h(e)$ and $g(e) \wedge i(e) \subseteq g(e)$, $g(e) \wedge i(e) \subseteq i(e)$. So $A_{(f,g)} \tilde{\subseteq} A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)}$ and $B_{(h,i)} \tilde{\subseteq} A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)}$. Let $C_{(j,k)}$ be a double-framed fuzzy soft set over X, such that $A_{(f,g)}, B_{(h,i)} \tilde{\subseteq} C_{(j,k)}$. Then $A, B \subseteq C$ implies that $(A \cup B) \subseteq C$ and $f(e), h(e) \subseteq j(e)$ implies that $f(e) \vee h(e) \subseteq j(e)$. Also $k(e) \subseteq g(e)$, $k(e) \subseteq i(e)$ imply that $k(e) \subseteq g(e) \wedge i(e)$ for all $e \in A \cup B$. Thus $A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)} \tilde{\subseteq} C_{(j,k)}$. It follows that $A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)}$ is the smallest double-framed fuzzy soft set over X which contains both $A_{(f,g)}$ and $B_{(h,i)}$.

2) We have $A \cap B \subseteq A$, $A \cap B \subseteq B$ and $f(e) \wedge h(e) \subseteq f(e)$, $f(e) \wedge h(e) \subseteq h(e)$ and $g(e) \subseteq g(e) \vee i(e)$, $i(e) \subseteq g(e) \vee i(e)$ for all $e \in A \cap B$. So $A_{(f,g)} \sqcap B_{(h,i)} \tilde{\subseteq} A_{(f,g)}$ and $A_{(f,g)} \sqcap B_{(h,i)} \tilde{\subseteq} B_{(h,i)}$. Let $C_{(j,k)}$ be a double-framed fuzzy soft set over X, such that $C_{(j,k)} \tilde{\subseteq} A_{(f,g)}$ and $C_{(j,k)} \tilde{\subseteq} B_{(h,i)}$. Then $C \subseteq A$, $C \subseteq B$ implies that $C \subseteq A \cap B$ and $j(e) \subseteq f(e)$, $j(e) \subseteq g(e)$ imply that $j(e) \subseteq f(e) \wedge g(e)$, and $g(e) \subseteq k(e)$, $i(e) \subseteq k(e)$ imply that $g(e) \vee i(e) \subseteq k(e)$ for all $e \in C$. Thus $C_{(j,k)} \tilde{\subseteq} A_{(f,g)} \sqcap B_{(h,i)}$. It follows that $A_{(f,g)} \sqcap B_{(h,i)}$ is the largest double-framed fuzzy soft set over X which is contained in both $A_{(f,g)}$ and $B_{(h,i)}$.

■

5.4 Algebras of Double-framed Fuzzy Soft Sets

In this section, we discuss the concepts of lattices and algebras for the collections of double-framed fuzzy soft sets. Let $\mathcal{DFSS}(X)^E$ be the collection of all double-framed fuzzy soft sets over X and $\mathcal{DFSS}(X)_A$ be its sub collection of all double-framed fuzzy soft sets over X with a fixed set of parameters A. We note that these collections are partially ordered by the relation of soft inclusion $\tilde{\subseteq}$ given in Definition 5.1.2.

5.4.1 Proposition

$(\mathcal{DFSS}(X)^E, \sqcap_\varepsilon, \sqcup)$, $(\mathcal{DFSS}(X)^E, \sqcup, \sqcap_\varepsilon)$, $(\mathcal{DFSS}(X)^E, \sqcup_\varepsilon, \sqcap)$, $(\mathcal{DFSS}(X)^E, \sqcap, \sqcup_\varepsilon)$ $(\mathcal{DFSS}(X)_A, \sqcup, \sqcap)$, and $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup)$ are complete lattices.

Proof. Let us consider $(\mathcal{DFSS}(X)^E, \sqcap_\varepsilon, \sqcup)$. Then for any double-framed fuzzy soft sets $A_{(f,g)}$, $B_{(h,i)}$, $C_{(j,k)} \in \mathcal{DFSS}(X)^E$,

1) We have

$$\begin{aligned} A_{(f,g)} \sqcap_\varepsilon B_{(h,i)} &\tilde{=} (A \cup B)_{(f \tilde{\wedge} h, g \tilde{\vee} i)} \in \mathcal{DFSS}(X)^E \text{ and} \\ A_{(f,g)} \sqcup B_{(h,i)} &\tilde{=} (A \cap B)_{(f \tilde{\vee} h, g \tilde{\wedge} i)} \in \mathcal{DFSS}(X)^E. \end{aligned}$$

2) From Proposition 5.3.3, we have

$$A_{(f,g)} \sqcap_\varepsilon A_{(f,g)} \tilde{=} A_{(f,g)} \text{ and } A_{(f,g)} \sqcup A_{(f,g)} \tilde{=} A_{(f,g)}.$$

3) From Proposition 5.3.4 we see that

$$\begin{aligned} A_{(f,g)} \sqcap_\varepsilon B_{(h,i)} &\tilde{=} B_{(h,i)} \sqcap_\varepsilon A_{(f,g)} \text{ and} \\ A_{(f,g)} \sqcup B_{(h,i)} &\tilde{=} B_{(h,i)} \sqcup A_{(f,g)}. \end{aligned}$$

Also

$$\begin{aligned} A_{(f,g)} \sqcap_\varepsilon (B_{(h,i)} \sqcap_\varepsilon C_{(j,k)}) &\tilde{=} (A_{(f,g)} \sqcap_\varepsilon B_{(h,i)}) \sqcap_\varepsilon C_{(j,k)} \text{ and} \\ A_{(f,g)} \sqcup (B_{(h,i)} \sqcup C_{(j,k)}) &\tilde{=} (A_{(f,g)} \sqcup B_{(h,i)}) \sqcup C_{(j,k)}. \end{aligned}$$

4) From Proposition 5.3.5,

$$A_{(f,g)} \sqcap_\varepsilon (B_{(h,i)} \sqcup A_{(f,g)}) \tilde{=} A_{(f,g)} \text{ and } A_{(f,g)} \sqcup (B_{(h,i)} \sqcap_\varepsilon A_{(f,g)}) \tilde{=} A_{(f,g)}.$$

So we conclude that the structure forms a lattice. Consider a collection of double-framed fuzzy soft sets $\{A_{i_{(f_i,g_i)}} : i \in I\}$ over X. We have, $\bigcup_{i \in I} A_i \subseteq E$ and, let $\Lambda(e) = \{j : e \in A_j\}$ for any $e \in A_i$. Then $\left(\bigwedge_{i \in \Lambda(e)} f_i(e)\right)(x) \in [0,1]$ and $\left(\bigvee_{i \in \Lambda(e)} g_i(e)\right)(x) \in [0,1]$ for all $x \in X$. Thus $\sqcap_{\varepsilon\, i \in I} A_{i_{(f_i,g_i)}} \in \mathcal{DFSS}(X)^E$.

Again, we have, $\bigcap_{i \in I} A_i \subseteq E$ and for any $e \in \bigcap_{i \in I} A_i$, $\left(\bigvee_{i \in I} f_i(e)\right)(x) \in [0,1]$ and $\left(\bigwedge_{i \in I} g_i(e)\right)(x) \in [0,1]$ for all $x \in X$. Thus $\sqcup_{i \in I} A_{i_{(f_i,g_i)}} \in \mathcal{DFSS}(X)^E$.

Similarly we can show for the remaining structures. ■

5.4.2 Proposition

The structures $(\mathcal{DFSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, E_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$, $(\mathcal{DFSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}, \emptyset_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})})$, $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$ and $(\mathcal{DFSS}(X)_A, \sqcup, \sqcap, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}, A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})})$ are bounded distributive lattices.

Proof. Proposition 5.3.6 assures the distributivity of $(\mathcal{DFSS}(X)^E, \sqcap, \sqcup_\varepsilon)$ and $(\mathcal{DFSS}(X)^E, \sqcup_\varepsilon, \sqcap)$. From Theorem 5.3.11, we conclude that $(\mathcal{DFSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, E_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$ is a bounded distributive lattice and $(\mathcal{DFSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}, \emptyset_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})})$ is its dual. For any double-framed fuzzy soft sets $A_{(f,g)}$, $A_{(h,i)} \in \mathcal{DFSS}(X)_A$,

$$\begin{aligned} A_{(f,g)} \sqcap A_{(h,i)} &\tilde{=} A_{(f\tilde{\wedge}h, g\tilde{\vee}i)} \in \mathcal{DFSS}(X)_A \text{ and} \\ A_{(f,g)} \sqcup A_{(h,i)} &\tilde{=} A_{(f\tilde{\vee}h, g\tilde{\wedge}i)} \in \mathcal{DFSS}(X)_A. \end{aligned}$$

Thus $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup)$ is also a distributive sublattice of $(\mathcal{DFSS}(X)^E, \sqcup_\varepsilon, \sqcap)$ and Theorem 5.3.3 tells us that $A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$, $A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}$ are its lower and upper bounds respectively. Therefore $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$ is a bounded distributive lattice and $(\mathcal{DFSS}(X)_A, \sqcup, \sqcap, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}, A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})})$ is its dual. ■

5.4.3 Proposition

Let $A_{(f,g)}$ be a *double-framed fuzzy soft set* over X. Then the operation $A_{(f,g)} \mapsto A_{(f,g)^\prime}$ on $\mathcal{DFSS}(X)^E$ which is given in Definition 5.2.7 satisfies:

1) $(A_{(f,g)^\prime})^\prime \tilde{=} A_{(f,g)}$ and $A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})^\prime} \tilde{=} A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$, $A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})^\prime} \tilde{=} A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}$,

2) if $A_{(h,i)}$ is a *double-framed* fuzzy soft set over X then $A_{(f,g)} \tilde{\subseteq} A_{(h,i)}$ if and only if $A_{(h,i)^\prime} \tilde{\subseteq} A_{(f,g)^\prime}$.

Proof.

1) The proof follows from the fact that, for any $e \in A$

$$\begin{aligned} ((f^\prime)^\prime)(e) &= (f^\prime(e))^\prime = ((f(e))^\prime)^\prime = f(e) \quad \text{and} \\ ((g^\prime)^\prime)(e) &= (g^\prime(e))^\prime = ((g(e))^\prime)^\prime = g(e). \end{aligned}$$

Also

$$\begin{aligned} A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})^\prime} &\tilde{=} A_{(\tilde{\mathbf{1}}^\prime,\tilde{\mathbf{0}}^\prime)} \tilde{=} A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, \\ A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})^\prime} &\tilde{=} A_{(\tilde{\mathbf{0}}^\prime,\tilde{\mathbf{1}}^\prime)} \tilde{=} A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}. \end{aligned}$$

2) Let $e \in A$. If $A_{(f,g)} \tilde{\subseteq} A_{(h,i)}$ then $f(e) \subseteq h(e)$ and $i(e) \subseteq g(e)$.

Now,

$$\begin{aligned}(f\,\acute{}(e))(x) &= (f(e))\acute{}(x)\\ &= 1-(f(e))(x)\\ &\geq 1-(h(e))(x)\\ &= (h(e))\acute{}(x) = (h\acute{}(e))(x) \quad \text{and}\\ (g\acute{}(e))(x) &= (g(e))\acute{}(x)\\ &= 1-(g(e))(x)\\ &\leq 1-(i(e))(x)\\ &= (i(e))\acute{}(x) = (i\acute{}(e))(x)\end{aligned}$$

for all $x \in X$. Thus $A_{(h,i)\acute{}}\tilde{\subseteq}A_{(f,g)\acute{}}$. Conversely, if $A_{(h,i)\acute{}}\tilde{\subseteq}A_{(f,g)\acute{}}$ then $(A_{(f,g)})\acute{}\tilde{\subseteq}(A_{(h,i)})\acute{}$ implies $A_{(f,g)}\tilde{\subseteq}A_{(h,i)}$.

■

5.4.4 Proposition (de Morgan Laws)

Let $A_{(f,g)}$ and $B_{(h,i)}$ be double-*framed fuzzy* soft sets over X. Then the following are true

1) $(A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)})\acute{}\,\tilde{=}A_{(f,g)\acute{}} \sqcap_{\varepsilon} B_{(h,i)\acute{}}$,

2) $(A_{(f,g)} \sqcap_{\varepsilon} B_{(h,i)})\acute{}\,\tilde{=}A_{(f,g)\acute{}} \sqcup_{\varepsilon} B_{(h,i)\acute{}}$,

3) $(A_{(f,g)} \vee B_{(h,i)})\acute{}\,\tilde{=}A_{(f,g)\acute{}} \wedge B_{(h,i)\acute{}}$,

4) $(A_{(f,g)} \wedge B_{(h,i)})\acute{}\,\tilde{=}A_{(f,g)\acute{}} \vee B_{(h,i)\acute{}}$,

5) $(A_{(f,g)} \sqcup B_{(h,i)})\acute{}\,\tilde{=}A_{(f,g)\acute{}} \sqcap B_{(h,i)\acute{}}$,

6) $(A_{(f,g)} \sqcap B_{(h,i)})\acute{}\,\tilde{=}A_{(f,g)\acute{}} \sqcup B_{(h,i)\acute{}}$.

Proof. **1)** We have $(A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)})\acute{}\,\tilde{=}((A \cup B)_{(f\tilde{\vee}h,g\tilde{\wedge}g)})\acute{}\,\tilde{=}(A \cup B)_{((f\tilde{\vee}h)\acute{},(g\tilde{\wedge}g)\acute{})}$. Let $e \in (A \cup B)$. There are three cases:

(i) If $e \in A - B$, then

$$\begin{aligned}(f\tilde{\vee}h)\acute{}(e) &= (f(e))\acute{} = f\acute{}(e) = (f\acute{}\tilde{\wedge}h\acute{})(e)\\ (g\tilde{\wedge}i)\acute{}(e) &= (g(e))\acute{} = g\acute{}(e) = (g\acute{}\tilde{\vee}i\acute{})(e),\end{aligned}$$

(ii) If $e \in B - A$, then

$$\begin{aligned}(f\tilde{\vee}h)\acute{}(e) &= (h(e))\acute{} = h\acute{}(e) = (f\acute{}\tilde{\wedge}h\acute{})(e)\\ (g\tilde{\wedge}i)\acute{}(e) &= (i(e))\acute{} = i\acute{}(e) = (g\acute{}\tilde{\vee}i\acute{})(e),\end{aligned}$$

(iii) If $e \in (A \cap B)$, then

$$\begin{aligned}(f\tilde{\vee}h)'(e) &= (f(e) \vee h(e))' = (f(e))' \wedge (h(e))' \\ (g\tilde{\vee}i)'(e) &= (g(e) \wedge i(e))' = (g(e))' \vee (i(e))',\end{aligned}$$

and,

$$\begin{aligned}(f'\tilde{\wedge}h')(e) &= (f(e))' \wedge (h(e))' = (f\tilde{\vee}h)'(e) \\ (g'\tilde{\wedge}i')(e) &= (g(e))' \vee (i(e))' = (g\tilde{\vee}i)'(e).\end{aligned}$$

Therefore, in all three cases we obtain equality and thus

$$(A_{(f,g)} \sqcup_{\varepsilon} B_{(h,i)})' \tilde{=} A_{(f,g)'} \sqcap_{\varepsilon} B_{(h,i)'}.$$

The remaining parts can also be proved in a similar way. ■

5.4.5 Proposition

$(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ', A_{(\bar{\mathbf{0}},\bar{\mathbf{1}})}, A_{(\bar{\mathbf{1}},\bar{\mathbf{0}})})$ is a de Morgan algebra.

Proof. We have already seen that $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, A_{(\bar{\mathbf{0}},\bar{\mathbf{1}})}, A_{(\bar{\mathbf{1}},\bar{\mathbf{0}})})$ is a bounded distributive lattice. Proposition 5.4.3 shows that " ′ " is an involution on $\mathcal{DFSS}(X)_A$ and Proposition 5.4.4 shows that de Morgan laws hold with respect to " ′ " in $\mathcal{DFSS}(X)_A$. Thus $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ', A_{(\bar{\mathbf{0}},\bar{\mathbf{1}})}, A_{(\bar{\mathbf{1}},\bar{\mathbf{0}})})$ is a de Morgan algebra. ■

5.4.6 Proposition

Let $A_{(f,g)}$ and $A_{(h,i)}$ be double-*framed fuzzy* soft sets over X. Then $A_{(h,i)} \sqcap A_{(h,i)'} \tilde{\subseteq} A_{(f,g)} \sqcup A_{(f,g)'}$ and so $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ', A_{(\bar{\mathbf{0}},\bar{\mathbf{1}})}, A_{(\bar{\mathbf{1}},\bar{\mathbf{0}})})$ is a Kleene Algebra.

Proof. We have already seen that $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ', A_{(\bar{\mathbf{0}},\bar{\mathbf{1}})}, A_{(\bar{\mathbf{1}},\bar{\mathbf{0}})})$ is a *de Morgan algebra*. Now, suppose that for some $A_{(f,g)}$, $A_{(h,i)} \in \mathcal{DFSS}(X)_A$ we have

$$A_{(h,i)} \sqcap A_{(h,i)'} \tilde{\supseteq} A_{(f,g)} \sqcup A_{(f,g)'} \text{ where } A_{(h,i)} \sqcap A_{(h,i)'} \tilde{\neq} A_{(f,g)} \sqcup A_{(f,g)'}.$$

Then there exists some $e \in A$ such that

$$(h\tilde{\wedge}h\ ')(e) \supset (f\tilde{\vee}f\ ')(e) \text{ or } (g\tilde{\vee}g\ ')(e) \subset (g\tilde{\wedge}g\ ')(e)$$

and so there exists some $x \in X$ such that

$$\begin{aligned}((h\tilde{\wedge}h\ ')(e))(x) &> ((f\tilde{\vee}f\ ')(e))(x) \\ &\Rightarrow (h(e)\tilde{\wedge}h\ '(e))(x) > (f(e)\tilde{\vee}f\ '(e))(x) \\ &\Rightarrow (h(e))(x) \wedge (h'(e))(x) > (f(e))(x) \vee (f\ '(e))(x)\end{aligned}$$

or

$$\begin{aligned}((i\tilde{\vee}i')(e))(x) &< ((g\tilde{\wedge}g')(e))(x) \\ &\Rightarrow (i(e)\tilde{\vee}i'(e))(x) < (g(e)\tilde{\wedge}g'(e))(x) \\ &\Rightarrow (i(e))(x) \vee (i'(e))(x) < (g(e))(x) \wedge (g'(e))(x).\end{aligned}$$

But

$$\begin{aligned}(h(e))(x)\wedge(h'(e))(x) &\leq 0.5 \text{ and}\\(g(e))(x)\wedge(g'(e))(x) &\leq 0.5\end{aligned}$$

and

$$\begin{aligned}(f(e))(x)\vee(f'(e))(x) &\geq 0.5 \text{ and}\\(i(e))(x)\vee(i'(e))(x) &\geq 0.5.\end{aligned}$$

which gives

$$\begin{aligned}(h(e))(x)\wedge(h'(e))(x) &\leq (f(e))(x)\vee(f\ '(e))(x) \text{ or}\\(g(e))(x)\wedge(g'(e))(x) &\leq (i(e))(x)\vee(i'(e))(x).\end{aligned}$$

A contradiction. Thus our supposition is wrong and

$$A_{(h,i)}\sqcap A_{(h,i)'}\tilde{\subseteq}A_{(f,g)}\sqcup A_{(f,g)'}.$$

Therefore $(\mathcal{DFSS}(X)_A,\sqcap,\sqcup,',A_{(\bar{\mathbf{0}},\bar{\mathbf{1}})},A_{(\bar{\mathbf{1}},\bar{\mathbf{0}})})$ is a Kleene Algebra. ■

5.4.7 Lemma

Let $A_{(f,g)},B_{(h,i)}\in\mathcal{DFSS}(X)^E$. Then pseudocomplement of $A_{(f,g)}$ relative to $B_{(h,i)}$ exists in $\mathcal{DFSS}(X)^E$.

Proof. Consider the set

$$T(A_{(f,g)},A_{(h,i)})=\{C_{(j,k)}\in\mathcal{DFSS}(X)^E : C_{(j,k)}\sqcap A_{(f,g)}\tilde{\subseteq}B_{(h,i)}\}.$$

We define a double-framed fuzzy soft set $(A^c\cup B)_{(f,g)\to(h,i)}\tilde{=}(A^c\cup B)_{(f\to h,g\to i)}\in\mathcal{DFSS}(X)^E$ where

$$\begin{aligned}&((f\to h)(e))(x)\\&=\begin{cases}1 & \text{if } e\in A^c-B\\ \begin{cases}1 & \text{if } (f(e))(x)\leq(h(e))(x)\\ (h(e))(x) & \text{if } (f(e))(x)>(h(e))(x)\end{cases} & \text{if } e\in B-A^c\\ 1 & \text{if } e\in A^c\cap B\end{cases}\\&\text{and}\\&((g\to i)(e))(x)\\&=\begin{cases}0 & \text{if } e\in A^c-B\\ \begin{cases}0 & \text{if } (i(e))(x)<(g(e))(x)\\ (i(e))(x) & \text{if } (i(e))(x)\geq(g(e))(x)\end{cases} & \text{if } e\in B-A^c\\ 0 & \text{if } e\in A^c\cap B\end{cases}\end{aligned}$$

Then

$$\begin{aligned}(A^c\cup B)_{(f\to h,g\to i)}\sqcap A_{(f,g)} &\tilde{=} ((A^c\cup B)\cap A)_{((f\to h)\tilde{\wedge}f,(g\to i)\tilde{\vee}g)}\\&\tilde{=} ((A^c\cap A)\cup(B\cap A))_{((f\to h)\tilde{\wedge}f,(g\to i)\tilde{\vee}g)}\\&\tilde{=} (A\cap B)_{((f\to h)\tilde{\wedge}f,(g\to i)\tilde{\vee}g)}.\end{aligned}$$

For any $e \in A \cap B$, $x \in X$,

$$\begin{aligned}
(((f \quad &\rightarrow \quad h)\tilde{\wedge} f))(e))(x) \\
&= \begin{cases} 1 \wedge (f(e))(x) & \text{if } (f(e))(x) \leq (h(e))(x) \\ (h(e))(x) \wedge (f(e))(x) & \text{if } (f(e))(x) > (h(e))(x) \end{cases} \\
&= \begin{cases} (f(e))(x) & \text{if } (f(e))(x) \leq (h(e))(x) \\ (h(e))(x) & \text{if } (f(e))(x) > (h(e))(x) \end{cases} \\
&\leq (h(e))(x).
\end{aligned}$$

and

$$\begin{aligned}
(((g \quad &\rightarrow \quad i)\tilde{\vee} g)(e))(x) \\
&= \begin{cases} 0 \vee (g(e))(x) & \text{if } (i(e))(x) < (g(e))(x) \\ (i(e))(x) \vee (g(e))(x) & \text{if } (i(e))(x) \geq (g(e))(x) \end{cases} \\
&= \begin{cases} (g(e))(x) & \text{if } (i(e))(x) \leq (g(e))(x) \\ (i(e))(x) & \text{if } (i(e))(x) > (g(e))(x) \end{cases} \\
&\geq (i(e))(x).
\end{aligned}$$

Hence,

$$(A^c \cup B)_{(f \rightarrow h, g \rightarrow i)} \sqcap A_{(f,g)} \tilde{\subseteq} B_{(h,i)}$$

Thus $(A^c \cup B)_{(f \rightarrow h, g \rightarrow i)} \in T(A_{(f,g)}, A_{(h,i)})$. For all $C_{(j,k)} \in T(A_{(f,g)}, A_{(h,i)})$, we have $C_{(j,k)} \sqcap A_{(f,g)} \tilde{\subseteq} A_{(h,i)}$ so for any $e \in C \cap A \subseteq B$

$$j(e) \wedge f(e) \subseteq h(e) \text{ and } k(e) \vee g(e) \supseteq i(e)$$

Now,

$$\begin{aligned}
C \cap A \quad &\subseteq \quad B \Rightarrow (A \cap C) \cap B^c = \emptyset \\
&\Rightarrow \quad C \subseteq (A \cap B^c)^c = A^c \cup B.
\end{aligned}$$

We have following cases:

(i) If $e \in (A^c - B) \cap C$, then $j(e))(x) \leq 1 = ((f \rightarrow h)(e))(x)$ and $k(e))(x) \geq 0 = ((g \rightarrow i)\ (e))\ (x)$

(ii) If $e \in (B - A^c) \cap C$, and $(i(e))(x) < (g(e))(x)$ then $(k(e))(x) \geq 0 = ((g \rightarrow i)(e))(x)$

(iii) If $e \in (B - A^c) \cap C$, and $(f(e))(x) \leq (h(e))(x)$ then $(j(e))(x) < 1 = ((h \rightarrow i)(e))(x)$

(iv) If $e \in (B - A^c) \cap C$ and $(i(e))(x) \geq (g(e))(x)$, then the condition $k(e) \vee g(e) \supseteq i(e)$ implies that $(k(e))(x) \geq (i(e))(x) = ((h \rightarrow i)(e))(x)$

(v) If $e \in (B - A^c) \cap C$ and $(f(e))(x) > (h(e))(x)$, then the condition $j(e) \wedge f(e) \subseteq h(e)$ implies that $(j(e))(x) \leq (h(e))(x) = ((h \rightarrow i)(e))(x)$

(vi) If $e \in (A^c \cap B) \cap C$, then $j(e))(x) < 1 = ((h \rightarrow i)(e))(x)$ and $k(e))(x) \geq 0 = ((g \rightarrow i)\ (e))\ (x)$.

Thus $C_{(j,k)}\tilde{\subseteq}(A^c \cup B)_{(f\to h,g\to g)}$ and it also shows that

$$(A^c \cup B)_{(f\to h,g\to g)}\tilde{=}\bigvee T(A_{(f,g)},A_{(h,i)})\tilde{=}A_{(f,g)} \to A_{(h,i)}.$$

■

5.4.8 Remark

We know that $(\mathcal{DFSS}(X)_A,\sqcap,\sqcup)$ is a sublattice of $(\mathcal{DFSS}(X)^E,\sqcap_\varepsilon,\sqcup)$. *For any* $A_{(f,g)}$, $A_{(h,i)} \in \mathcal{DFSS}(X)_A$, $A_{(f,g)} \to A_{(h,i)}$ as defined in Lemma 5.4.7, is not in $\mathcal{DFSS}(X)_A$ because $A_{(f,g)} \to A_{(h,i)}\tilde{=}(A^c \cup A)_{(f\to h,g\to i)}\tilde{=}E_{(f\to h,g\to i)} \notin \mathcal{DFSS}(X)_A$.

5.4.9 Lemma

Let $A_{(f,g)}, A_{(h,i)} \in \mathcal{DFSS}(X)_A$. Then pseudocomplement of $A_{(f,g)}$ relative to $A_{(h,i)}$ exists in $(\mathcal{DFSS}(X)_A,\sqcap,\sqcup)$.

Proof. Consider the set

$$T(A_{(f,g)},A_{(h,i)}) = \{A_{(j,k)} \in \mathcal{DFSS}(X)_A : A_{(j,k)} \sqcap A_{(f,g)}\tilde{\subseteq}A_{(h,i)}\}.$$

We define a double-framed fuzzy soft set $A_{(f\to h,g\to i)} \in \mathcal{DFSS}(X)_A$ where

$$((f\to h)(e))(x) = \begin{cases} 1 & \text{if } (f(e))(x) \le (h(e))(x) \\ (h(e))(x) & \text{if } (f(e))(x) > (h(e))(x) \end{cases}$$

and

$$((g\to i)(e))(x) = \begin{cases} 0 & \text{if } (i(e))(x) < (g(e))(x) \\ (i(e))(x) & \text{if } (i(e))(x) \ge (g(e))(x) \end{cases}$$

for all $e \in A$, $x \in X$. Then $A_{(f\to h,g\to i)} \sqcap A_{(f,g)}\tilde{=}A_{(f\to h,g\to i)\tilde{\wedge}h}$ and

$$\begin{aligned}
&(((f \to h)\tilde{\wedge}f))(e))(x) \\
&\quad= \begin{cases} 1 \wedge (f(e))(x) & \text{if } (f(e))(x) \le (h(e))(x) \\ (h(e))(x) \wedge (f(e))(x) & \text{if } (f(e))(x) > (h(e))(x) \end{cases} \\
&\quad= \begin{cases} (f(e))(x) & \text{if } (f(e))(x) \le (h(e))(x) \\ (h(e))(x) & \text{if } (f(e))(x) > (h(e))(x) \end{cases} \\
&\quad\le (h(e))(x).
\end{aligned}$$

and

$$\begin{aligned}
&(((g \to i)\tilde{\vee}g)(e))(x) \\
&\quad= \begin{cases} 0 \vee (g(e))(x) & \text{if } (i(e))(x) < (g(e))(x) \\ (i(e))(x) \vee (g(e))(x) & \text{if } (i(e))(x) \ge (g(e))(x) \end{cases} \\
&\quad= \begin{cases} (g(e))(x) & \text{if } (i(e))(x) \le (g(e))(x) \\ (i(e))(x) & \text{if } (i(e))(x) > (g(e))(x) \end{cases} \\
&\quad\ge (i(e))(x).
\end{aligned}$$

for all $e \in A$, $x \in X$. Hence,

$$A_{(f\to h,g\to i)} \sqcap A_{(f,g)}\tilde{\subseteq}A_{(h,i)}$$

and $A_{(f\to h,g\to i)} \in T(A_{(f,g)}, A_{(h,i)})$. For every $A_{(j,k)} \in T(A_{(f,g)}, A_{(h,i)})$, we have $A_{(j,k)} \sqcap A_{(f,g)} \tilde{\subseteq} A_{(h,i)}$ so for any $e \in A$, following cases arise:

(i) If $(i(e))(x) < (g(e))(x)$ then $(k(e))(x) \geq 0 = ((g \to i)(e))(x)$

(ii) If $(f(e))(x) \leq (h(e))(x)$ then $(j(e))(x) < 1 = ((h \to i)(e))(x)$

(iii) If $(i(e))(x) \geq (g(e))(x)$, then the condition $k(e) \vee g(e) \supseteq i(e)$ implies that $(k(e))(x) \geq (i(e))(x) = ((h \to i)(e))(x)$

(iv) If $(f(e))(x) > (h(e))(x)$, then the condition $j(e) \wedge f(e) \subseteq h(e)$ implies that $(j(e))(x) \leq (h(e))(x) = ((h \to i)(e))(x)$.

Thus $A_{(j,k)} \tilde{\subseteq} A_{(f\to h,g\to i)}$ and it also shows that

$$A_{(f\to h,g\to i)} \tilde{=} \bigvee T(A_{(f,g)}, A_{(h,i)}) \tilde{=} A_{(f,g)} \to_A A_{(h,i)}.$$

■

5.4.10 Proposition

$(\mathcal{DFSS}(X)^E, \sqcap_\varepsilon, \sqcup)$ *and* $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup)$ are *Brouwerian lattices.*

Proof. Follows from Lemmas 5.4.7 and 5.4.9. ■

5.4.11 Definition

Let $A_{(f,g)}$ be a *double-framed fuzzy* soft set over X. We define $A_{(f,g)^*}$ as a *double-framed fuzzy soft set* $A_{(f^*,g^*)}$ where

$$f^* : A \to \mathcal{FP}(X),\ e \mapsto (f(e))^*,$$
$$(f(e))^*(x) = \begin{cases} 0 & \text{if } (f(e))^*(x) \neq 0 \\ 1 & \text{if } (f(e))^*(x) = 0 \end{cases}$$

$$g^* : A \to \mathcal{FP}(X),\ e \mapsto (g(e))^*,$$
$$(g(e))^*(x) = \begin{cases} 1 & \text{if } (g(e))^*(x) \neq 1 \\ 0 & \text{if } (g(e))^*(x) = 1 \end{cases} \quad \text{for } x \in X.$$

5.4.12 Theorem

Let $A_{(f,g)}$ and $A_{(h,i)}$ be double-framed fuzzy soft sets over X. Then

1) $A_{(f,g)} \sqcap A_{(f,g)^*} \tilde{=} A_{(\bar{\mathbf{0}},\bar{\mathbf{1}})}$,

2) $A_{(f,g)} \tilde{\subseteq} A_{(h,i)^*}$ whenever $A_{(f,g)} \sqcap A_{(h.i)} \tilde{=} A_{(\bar{\mathbf{0}},\bar{\mathbf{1}})}$,

3) $A_{(f,g)^*} \sqcup A_{((f,g)^*)^*} \tilde{=} A_{(\bar{\mathbf{1}},\bar{\mathbf{0}})}$.

Thus $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ^*, A_{(\bar{\mathbf{0}},\bar{\mathbf{1}})}, A_{(\bar{\mathbf{1}},\bar{\mathbf{0}})})$ is a Stone algebra.

Proof.

1) Consider $A_{(f,g)} \sqcap A_{(f,g)^*}$. For any $e \in A$

$$(f\tilde{\wedge}f^*)(e) = f(e) \wedge f^*(e) \text{ and } (g\tilde{\vee}g^*)(e) = g(e) \vee g^*(e).$$

$\Longrightarrow$

$$\begin{aligned} ((f\tilde{\wedge}f^*)(e))(x) &= \begin{cases} (f(e))(x) \wedge 0 & \text{if } (f(e))(x) \neq 0 \\ 0 \wedge 1 & \text{if } (f(e))(x) = 0 \end{cases} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} ((g\tilde{\vee}g^*)(e))(x) &= \begin{cases} (g(e))(x) \vee 1 & \text{if } (g(e))(x) \neq 1 \\ 1 \vee 0 & \text{if } (g(e))(x) = 1 \end{cases} \\ &= 1 \end{aligned}$$

for all $x \in X$. Thus $A_{(f,g)} \sqcap A_{(f,g)^*} \tilde{=} A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$.

2) If $A_{(f,g)} \sqcap A_{(h,i)} \tilde{=} A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$, then

$$(f(e))(x) \wedge (h(e))(x) = 0 \tag{b}$$

and

$$(g(e))(x) \vee (i(e))(x) = 1 \tag{c}$$

for all $x \in X$, $e \in A$. From Equation (b) we have two cases :

$$\text{If } (h(e))(x) = 0 \text{ then } (h^*(e))(x) = 1 \geq (f(e))(x)$$
$$\text{and}$$
$$\text{if } (h(e))(x) \neq 0 \text{ then } (f(e))(x) = 0 \leq (h^*(e))(x).$$

Thus $(f(e))(x) \leq (h^*(e))(x)$ for all $x \in X$.

From Equation (c), there are two cases:

$$\begin{aligned} \text{If } (i(e))(x) &= 1 \text{ then } (i^*(e))(x) = 0 \leq (g(e))(x) \\ &\text{and} \\ \text{if } (i(e))(x) &\neq 1 \text{ then } (g(e))(x) = 1 \geq (i^*(e))(x). \end{aligned}$$

So $(i^*(e))(x) \leq (g(e))(x)$ for all $x \in X$. This implies that

$$f(e) \subseteq h^*(e) \text{ and } i^*(e) \subseteq g(e) \text{ for all } e \in A.$$

Therefore $A_{(f,g)} \tilde{\subseteq} A_{(h,i)^*}$.

3) Consider $A_{(f,g)^*} \sqcup A_{((f,g)^*)^*}$. For any $e \in A$

$$(f^*\tilde{\vee}f^{**})(e) = f^*(e) \vee f^{**}(e)$$

and

$$(g^*\tilde{\wedge}g^{**})(e) = g^*(e) \wedge g^{**}(e).$$

$\Longrightarrow$

$$((f^*(e))(x) \vee (f^{**}(e))(x) = \begin{cases} 0 \vee 1 & \text{if } (f(e))(x) \neq 0 \\ 1 \vee 0 & \text{if } (f(e))(x) = 0 \end{cases} = 1$$

and

$$((g^*(e))(x) \wedge (g^{**}(e))(x) = \begin{cases} 1 \wedge 0 & \text{if } (g(e))(x) \neq 1 \\ 0 \wedge 1 & \text{if } (g(e))(x) = 1 \end{cases} = 0$$

for all $x \in X$. Thus $A_{(f,g)^*} \sqcup A_{((f,g)^*)^*} \tilde{=} A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}$.

■

5.4.13 Definition

Let $A_{(f,g)}$ be a *double-framed* fuzzy soft set over X. We define

$$(A_{(f,g)})^\circ \tilde{=} A_{(f,g)^\circ} \tilde{=} A_{(g,f)}.$$

5.4.14 Proposition (Involution)

Let $A_{(f,g)}$ be a *double-framed fuzzy soft set* over X. Then $(A_{(f,g)^\circ})^\circ \tilde{=} A_{(f,g)}$, $A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})^\circ} \tilde{=} A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$ and $A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})^\circ} \tilde{=} A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}$

Proof. It is straightforward that $A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})^\circ} \tilde{=} A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}$ and $A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})^\circ} \tilde{=} A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})}$. We have

$$(A_{(f,g)^\circ})^\circ \tilde{=} A_{(g,f)^\circ} \tilde{=} A_{(f,g)}.$$

■

5.4.15 Proposition (de Morgan Laws)

Let $A_{(f,g)}$ and $B_{(h,i)}$ be double-*framed* fuzzy soft sets over X. Then the following are true

1) $(A_{(f,g)} \sqcup_\varepsilon B_{(h,i)})^\circ \tilde{=} A_{(f,g)^\circ} \sqcap_\varepsilon B_{(h,i)^\circ}$,

2) $(A_{(f,g)} \sqcap_\varepsilon B_{(h,i)})^\circ \tilde{=} A_{(f,g)^\circ} \sqcup_\varepsilon B_{(h,i)^\circ}$,

3) $(A_{(f,g)} \vee B_{(h,i)})^\circ \tilde{=} A_{(f,g)^\circ} \wedge B_{(h,i)^\circ}$,

4) $(A_{(f,g)} \wedge B_{(h,i)})^\circ \tilde{=} A_{(f,g)^\circ} \vee B_{(h,i)^\circ}$,

5) $(A_{(f,g)} \sqcup B_{(h,i)})^\circ \tilde{=} A_{(f,g)^\circ} \sqcap B_{(h,i)^\circ}$,

6) $(A_{(f,g)} \sqcap B_{(h,i)})^\circ \tilde{=} A_{(f,g)^\circ} \sqcup B_{(h,i)^\circ}$.

Proof.

1) We have

$$(A_{(f,g)} \sqcup_\varepsilon B_{(h,i)})^\circ \tilde{=} ((A \cup B)_{(f\tilde{\vee}h, g\tilde{\wedge}i)})^\circ \tilde{=} (A \cup B)_{(g\tilde{\wedge}i, f\tilde{\vee}h)}$$

and

$$A_{(f,g)^\circ} \sqcap_\varepsilon B_{(h,i)^\circ} \tilde{=} A_{(g,f)} \sqcap_\varepsilon B_{(i,h)} \tilde{=} (A \cup B)_{(g\tilde{\wedge}i, f\tilde{\vee}h)}.$$

Thus

$$(A_{(f,g)} \sqcup_\varepsilon B_{(h,i)})^\circ \tilde{=} A_{(f,g)^\circ} \sqcap_\varepsilon B_{(h,i)^\circ}.$$

The remaining parts can be proved in a similar way.

■

5.4.16 Theorem

$(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ^\circ, A_{(\tilde{\mathbf{0}},\tilde{\mathbf{1}})}, A_{(\tilde{\mathbf{1}},\tilde{\mathbf{0}})})$ is a de Morgan algebra.

Proof. Follows from Propositions 5.4.14 and 5.4.15. ■

Chapter 6

Algebraic Structures of Bipolar Soft Sets

Bipolarity refers to an explicit handling of positive and negative sides of information. Three types of bipolarity were discussed in [11] but we are using a rather generalized bipolarity here, dealing with the positive and negative impacts in information associated with a soft set and its representation. This chapter introduces the concept of a bipolar soft set. A bipolar soft set is obtained by considering not only a carefully chosen set of parameters but also an allied set of oppositely meaning parameters named as "Not set of parameters". Structure of a bipolar soft set is managed by two functions, say $\alpha : A \to \mathcal{P}(X)$ and $\beta : \neg A \to \mathcal{P}(X)$ where $\neg A$ stands for the "not set of A" and β describes somewhat an opposite or negative approximation for the attractiveness of a houses of X, relative to the approximation computed by α. Maji et al. [33] had used the "not set" to define complement of a soft set. The *complement of a soft set* simply gives the complements of the approximations. The above mentioned soft function β is rather more generalized than soft complement function and $(\beta,\neg A)$ can be any soft subset of $(\alpha,A)^c$. The difference is the gray area of choice, that is, we may find some houses which do not satisfy any criteria properly e.g. A house may not be highly expensive but it does not assure its cheapness either. Thus, we must be careful while making our considerations for the parameterization of data keeping in view that, during approximations, there might be some indifferent elements in X. This gives us a motivation to define the idea of bipolar soft sets. We have defined operations of union and intersection for bipolar soft sets by taking restricted, extended and product sets of parameters. The algebraic structures of bipolar soft sets are discussed with the properties of operations.

6.1 Bipolar Soft Sets

Let X be an initial universe and E be a set of parameters. Let $\mathcal{P}(X)$ denotes the power set of X and A, B, C be non-empty subsets of E.

6.1.1 Definition

A triplet $(\alpha,\beta : A)$ is called a *bipolar* soft set over X, where α and β are mappings, given by $\alpha : A \to \mathcal{P}(X)$ and $\beta : \neg A \to \mathcal{P}(X)$ such that $\alpha(e) \cap \beta(\neg e) = \emptyset$ (Empty Set) for all $e \in A$.

In other words, a *bipolar* soft set over X gives two parametrized families of subsets of the universe X and the condition $\alpha(e) \cap \beta(\neg e) = \emptyset$ for all $e \in A$, is imposed as a consistency constraint. For each $e \in A$, $\alpha(e)$ and $\beta(\neg e)$ are regarded as the set of e-approximate elements of *bipolar* soft set $(\alpha,\beta : A)$. It is also observed that the relationship between a complement function and the defining function of a soft set becomes a particular case for the defining functions of a bipolar soft set, that is, $(\alpha,\alpha^c : A)$ is a bipolar soft set over X. The difference occurs due to the presence of uncertainty or hesitation or lack of knowledge in defining the membership function. We name this uncertainty or gray area as the approximation for the degree of hesitation. Thus the union of three approximations, that is, e-approximation, $\neg e$-approximation, and approximation of hesitation is X. We note that $\emptyset \subseteq X - \{\alpha(e) \cup \beta(\neg e)\} \subseteq X$, for each $e \in A$. So, we may approximate the degree of hesitation in $(\alpha,\beta : A)$ by an allied soft set A_h defined over X, where $h(e) = X - \{\alpha(e) \cup \beta(\neg e)\}$ for all $e \in A$.

6.1.2 Definition

For two *bipolar* soft sets $(\alpha,\beta : A)$ and $(\gamma,\delta : B)$ over a universe X, we say that $(\alpha,\beta : A)$ is a *bipolar* soft subset of $(\gamma,\delta : B)$, if

1) $A \subseteq B$ and

2) $\alpha(e) \subseteq \gamma(e)$ and $\delta(\neg e) \subseteq \beta(\neg e)$ for all $e \in A$.

This relationship is denoted by $(\alpha,\beta : A)\tilde{\subseteq}(\gamma,\delta : B)$. Similarly $(\alpha,\beta : A)$ is said to be a *bipolar* soft superset of $(\gamma,\delta : B)$, if $(\gamma,\delta : B)$ is a *bipolar soft subset* of $(\alpha,\beta : A)$. We denote it by $(\alpha,\beta : A)\tilde{\supseteq}(\gamma,\delta : B)$.

6.1.3 Definition

Two *bipolar* soft sets $(\alpha,\beta : A)$ and $(\gamma,\delta : B)$ over X are said to be *equal* if $(\alpha,\beta : A)$ is a *bipolar* soft subset of $(\gamma,\delta : B)$ and $(\gamma,\delta : B)$ is a *bipolar* soft subset of $(\alpha,\beta : A)$.

Let $\mathcal{BSS}(X)^E$ denotes the set of all bipolar soft sets defined over X with set of parameters E ordered by the relation of inclusion $\tilde{\subseteq}$ as defined in Definition 6.1.2.

Now we claim that every bipolar soft set is equivalent to a double-framed soft set and give the following theorem:

6.1.4 Theorem

The mapping $\theta : \mathcal{BSS}(X)^E \to \mathcal{DSS}(X)^E$, $(\alpha,\beta : A) \mapsto A_{(\alpha_1,\beta_1)}$ is a monomorphism of lattices where

$$\alpha(e) = \alpha_1(e), \text{ and } \beta(e) = \beta_1(\neg e) \text{ for all } e \in A.$$

Proof. Clearly θ is well-defined. If

$$\theta((\alpha,\beta:A))\tilde{=}\theta((\gamma,\delta:B))$$

where

$$\theta((\alpha,\beta:A))\tilde{=}A_{(\alpha_1,\beta_1)} \text{ and } \theta((\gamma,\delta:B))\tilde{=}B_{(\gamma_1,\delta_1)}$$

then $A=B$ and

$$\alpha(e)=\alpha_1(e),\ \gamma(e)=\gamma_1(e) \text{ and } \beta(e)=\beta_1(\neg e),\ \delta(e)=\delta_1(\neg e) \text{ for all } e\in A.$$

Now,

$$\alpha(e)=\alpha_1(e)=\gamma_1(e)=\gamma(e) \text{ and } \beta(e)=\beta_1(\neg e)=\delta_1(\neg e)=\delta(e) \text{ for all } e\in A.$$

Thus

$$(\alpha,\beta:A)\tilde{=}(\gamma,\delta:B)$$

shows that θ is one-to-one. Clearly θ preserves the order of inclusion. ■

6.1.5 Remark

Note that θ is not onto because of the extra condition of consistency constraint for defining bipolar soft sets.

By Theorem 6.1.4, we can equate every bipolar soft set with a double-framed soft set with the consistency constraint and so, from onwards, we shall denote a bipolar soft set $(\alpha,\beta:A)$ by its image $\theta((\alpha,\beta:A))\tilde{=}A_{\langle\alpha,\beta\rangle}$ where the meanings of A, α and β are clear.

6.1.6 Example

Let X be the set of houses under consideration, and E be the set of parameters, $X=\{h_1,h_2,h_3,h_4,h_5\}$, $E=\{e_1,e_2,e_3,e_4,e_5,e_6\}=\{$ in the green surroundings, wooden, cheap, in good repair, furnished, traditional $\}$. Suppose that $A=\{e_1,e_2,e_3,e_6\}$, a bipolar soft set $A_{\langle\alpha,\beta\rangle}$ describes the "requirements of the houses" which Mr. Y is going to buy. The bipolar soft set $A_{\langle\alpha,\beta\rangle}$ over X, where α and β represent the classification under high and low investment respectively, is given as follows:

$$\alpha : A\to\mathcal{P}(X),\quad e\longmapsto\begin{cases}\{h_1,h_2,h_4\} & \text{if } e=e_1,\\ \{h_3,h_4\} & \text{if } e=e_2,\\ X & \text{if } e=e_3,\\ \{h_2,h_3,h_5\} & \text{if } e=e_6,\end{cases}$$

$$\beta : A\to\mathcal{P}(X),\quad e\longmapsto\begin{cases}\{h_3,h_5\} & \text{if } e=e_1,\\ \{h_1,h_2,h_5\} & \text{if } e=e_2,\\ \{\} & \text{if } e=e_3,\\ \{h_1\} & \text{if } e=e_6.\end{cases}$$

Let $B = \{e_2, e_3\}$. Then bipolar soft set $B_{\langle\gamma,\delta\rangle}$ given by

$$\gamma \;:\; A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_3\} & \text{if } e = e_2, \\ \{h_1, h_4, h_5\} & \text{if } e = e_3, \end{cases}$$

$$\delta \;:\; A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} X & \text{if } e = e_2, \\ \{h_1\} & \text{if } e = e_3, \end{cases}$$

is a bipolar soft subset of $A_{\langle\alpha,\beta\rangle}$ and represents the data under a strict set of parameters B following A.

6.2 Operations on Bipolar Soft Sets

This section gives various operations defined on bipolar soft sets:

6.2.1 Definition

If $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ are two *bipolar* soft sets over X. The int-uni product of $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ is defined to be a bipolar soft set $(A \times B)_{\langle\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta\rangle}$ over X in which $\alpha\tilde{\cap}\gamma : (A \times B) \to \mathcal{P}(X)$, where

$$(a, b) \mapsto \alpha(a) \cap \gamma(b),$$

and $\beta\tilde{\cup}\delta : (A \times B) \to \mathcal{P}(X)$, where

$$(a, b) \mapsto \beta(a) \cup \delta(b).$$

It is denoted by $A_{\langle\alpha,\beta\rangle} \wedge B_{\langle\gamma,\delta\rangle} \hat{=} (A \times B)_{\langle\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta\rangle}$.

6.2.2 Definition

If $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ are two *bipolar* soft sets over X then uni-int product of $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ is defined as a bipolar soft set $(A \times B)_{\langle\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta\rangle}$ over X in which $\alpha\tilde{\cup}\gamma : (A \times B) \to \mathcal{P}(X)$, where

$$(a, b) \mapsto \alpha(a) \cup \gamma(b),$$

and $\beta\tilde{\cap}\delta : (A \times B) \to \mathcal{P}(X)$, where

$$(a, b) \mapsto \beta(a) \cap \delta(b).$$

It is denoted by $A_{\langle\alpha,\beta\rangle} \vee B_{\langle\gamma,\delta\rangle} \hat{=} (A \times B)_{\langle\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta\rangle}$.

6.2.3 Definition

For two *bipolar soft sets* $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ over X, the *extended int-uni bipolar soft set* of $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ is defined as a *bipolar soft set* $(A \cup B)_{\langle\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta\rangle}$ over X in which $\alpha\tilde{\cap}\gamma : (A \cup B) \to \mathcal{P}(X)$, where

$$e \mapsto \begin{cases} \alpha(e) & \text{if } e \in A - B \\ \gamma(e) & \text{if } e \in B - A \\ \alpha(e) \cap \gamma(e) & \text{if } e \in (A \cap B) \end{cases}$$

and $\beta\tilde{\cap}\delta : (A \cup B) \to \mathcal{P}(X)$,

$$e \mapsto \begin{cases} \beta(e) & \text{if } e \in A - B \\ \delta(e) & \text{if } e \in B - A \\ \beta(e) \cup \delta(e) & \text{if } e \in (A \cap B) \end{cases}.$$

It is denoted by $A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} B_{\langle\gamma,\delta\rangle} \tilde{=} (A \cup B)_{\langle\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta\rangle}$.

6.2.4 Definition

For two *bipolar soft sets* $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ over X, the *extended uni-int bipolar soft set* of $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ is defined as a *bipolar soft set* $(A \cup B)_{\langle\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta\rangle}$ over X in which $\alpha\tilde{\cup}\gamma : (A \cup B) \to \mathcal{P}(X)$, where

$$e \mapsto \begin{cases} \alpha(e) & \text{if } e \in A - B \\ \gamma(e) & \text{if } e \in B - A \\ \alpha(e) \cup \gamma(e) & \text{if } e \in (A \cap B) \end{cases}$$

and $\beta\tilde{\cap}\delta : (A \cup B) \to \mathcal{P}(X)$, where

$$e \mapsto \begin{cases} \beta(e) & \text{if } e \in A - B \\ \delta(e) & \text{if } e \in B - A \\ \beta(e) \cap \delta(e) & \text{if } e \in (A \cap B) \end{cases}.$$

It is denoted by $A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} B_{\langle\gamma,\delta\rangle} \tilde{=} (A \cup B)_{\langle\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta\rangle}$.

6.2.5 Definition

For two *bipolar soft sets* $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ over X, the *extended difference bipolar soft set* of $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ is defined as a *bipolar soft set* $(A \cup B)_{\langle\alpha\smile_{\varepsilon}\gamma,\beta\smile_{\varepsilon}\delta\rangle}$ over X in which $\alpha \smile_{\varepsilon} \gamma : (A \cup B) \to \mathcal{P}(X)$, where

$$e \mapsto \begin{cases} \alpha(e) & \text{if } e \in A - B \\ \gamma(e) & \text{if } e \in B - A \\ \alpha(e) - \gamma(e) & \text{if } e \in (A \cap B) \end{cases}$$

and $\beta \smile_{\varepsilon} \delta : (A \cup B) \to \mathcal{P}(X)$, where

$$e \mapsto \begin{cases} \beta(e) & \text{if } e \in A - B \\ \delta(e) & \text{if } e \in B - A \\ \beta(e) - \delta(e) & \text{if } e \in (A \cap B). \end{cases}$$

It is denoted by $A_{\langle\alpha,\beta\rangle} \smile_{\varepsilon} B_{\langle\gamma,\delta\rangle} \tilde{=} (A \cup B)_{\langle\alpha\smile_{\varepsilon}\gamma,\beta\smile_{\varepsilon}\delta\rangle}$.

6.2.6 Definition

For two *bipolar soft sets* $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ over X with $(A \cap B) \neq \emptyset$, the *restricted int-uni bipolar soft set* of $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ is defined as a *bipolar soft set* $(A \cap B)_{\langle\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta\rangle}$ over X in which $\alpha\tilde{\cap}\gamma : (A \cap B) \to \mathcal{P}(X)$, where

$$e \mapsto \alpha(e) \cap \gamma(e),$$

and $\beta \tilde{\cup} \delta : (A \cap B) \to \mathcal{P}(X)$, where

$$e \mapsto \beta(e) \cup \delta(e).$$

It is denoted by $A_{\langle \alpha,\beta \rangle} \sqcap B_{\langle \gamma,\delta \rangle} \tilde{=} (A \cap B)_{\langle \alpha \tilde{\cap} \gamma, \beta \tilde{\cup} \delta \rangle}$.

6.2.7 Definition

For two *bipolar soft sets* $A_{\langle \alpha,\beta \rangle}$ and $B_{\langle \gamma,\delta \rangle}$ over X with $(A \cap B) \neq \emptyset$, the *restricted uni-int bipolar soft set* of $A_{\langle \alpha,\beta \rangle}$ and $B_{\langle \gamma,\delta \rangle}$ is defined as a *bipolar soft set* $(A \cap B)_{\langle \alpha \tilde{\cup} \gamma, \beta \tilde{\cap} \delta \rangle}$ over X in which $\alpha \tilde{\cup} \gamma : (A \cap B) \to \mathcal{P}(X)$, where

$$e \mapsto \alpha(e) \cup \gamma(e),$$

and $\beta \tilde{\cap} \delta : (A \cap B) \to \mathcal{P}(X)$, where

$$e \mapsto \beta(e) \cap \delta(e).$$

It is denoted by $A_{\langle \alpha,\beta \rangle} \sqcup B_{\langle \gamma,\delta \rangle} \tilde{=} (A \cap B)_{\langle \alpha \tilde{\cup} \gamma, \beta \tilde{\cap} \delta \rangle}$.

6.2.8 Definition

For two *bipolar soft sets* $A_{\langle \alpha,\beta \rangle}$ and $B_{\langle \gamma,\delta \rangle}$ over X with $(A \cap B) \neq \emptyset$, the *restricted difference bipolar soft set* of $A_{\langle \alpha,\beta \rangle}$ and $B_{\langle \gamma,\delta \rangle}$ is defined as a *bipolar soft set* $(A \cap B)_{\langle \alpha \smile \gamma, \beta \smile \delta \rangle}$ over X in which $\alpha \smile \gamma : (A \cap B) \to \mathcal{P}(X)$, where

$$e \mapsto \alpha(e) - \gamma(e),$$

and $\beta \smile \delta : (A \cap B) \to \mathcal{P}(X)$, where

$$e \mapsto \beta(e) - \delta(e).$$

It is denoted by $A_{\langle \alpha,\beta \rangle} \smile B_{\langle \gamma,\delta \rangle} \tilde{=} (A \cap B)_{\langle \alpha \smile \gamma, \beta \smile \delta \rangle}$.

6.2.9 Proposition

The mapping $\theta : \mathcal{BSS}(X)^E \to \mathcal{DSS}(X)^E$ as defined in Theorem 6.1.4 preserves the product, extended and restricted uni-int and int-uni operations.

Proof. Straightforward. ■

6.2.10 Remark

The operation of complementation as defined in Definition 4.2.9 for double-framed soft sets is no more valid for bipolar soft sets because $(A_{\langle \alpha,\beta \rangle})^c \tilde{=} A_{\langle \alpha^c,\beta^c \rangle}$ which may not satisfy the consistency constraint as shown by the following example:

6.2.11 Example

Let E, A, X and bipolar soft set $A_{\langle\alpha,\beta\rangle}$ over X be taken as in Example 6.1.6. Then $(A_{\langle\alpha,\beta\rangle})^c$ is given as follows:

$$\alpha^c \quad : \quad A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_3, h_5\} & \text{if } e = e_1, \\ \{h_1, h_2, h_5\} & \text{if } e = e_2, \\ \{\} & \text{if } e = e_3, \\ \{h_1, h_2\} & \text{if } e = e_6, \end{cases}$$

$$\beta^c \quad : \quad A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{h_1, h_2, h_4\} & \text{if } e = e_1, \\ \{h_3, h_4\} & \text{if } e = e_2, \\ X & \text{if } e = e_3, \\ \{h_2, h_3, h_5\} & \text{if } e = e_6. \end{cases}$$

but

$$\alpha^c(e_6) \cap \beta^c(e_6) \neq \emptyset$$

so $(A_{\langle\alpha,\beta\rangle})^c \notin \mathcal{BSS}(X)^E$. Thus "^{c} " is not defined on $\mathcal{BSS}(X)^E$.

6.2.12 Proposition

Let $A_{\langle\alpha,\beta\rangle}$ be a *bipolar* soft set over X. Then $^\circ : \mathcal{BSS}(X)^E \to \mathcal{BSS}(X)^E$ is defined and we denote $(A_{\langle\alpha,\beta\rangle})^\circ$ by $A_{\langle\alpha,\beta\rangle^\circ}$.

Proof. If $A_{\langle\alpha,\beta\rangle} \in \mathcal{BSS}(X)^E$ then

$$A_{\langle\alpha,\beta\rangle^\circ} \doteq A_{\langle\alpha^\circ,\beta^\circ\rangle} \qquad \text{where}$$
$$\alpha^\circ \quad : \quad A \to \mathcal{P}(X),\ e \mapsto \beta(e) \ \text{ and } \beta^\circ : A \to \mathcal{P}(X),\ e \mapsto \alpha(e).$$

Clearly

$$\alpha^\circ(e) \cap \beta^\circ(e) = \beta(e) \cap \alpha(e) = \emptyset.$$

Thus $A_{\langle\alpha,\beta\rangle^\circ} \in \mathcal{BSS}(X)^E$. ■

6.3 Properties of Bipolar Soft Sets

In this section we check the properties and associative, commutative, distributive and absorption laws of bipolar soft sets with respect to their operations.

6.3.1 Definition

A *bipolar soft set* over X is said to be a *relative null bipolar soft set,* denoted by $A_{\langle\Phi,\mathfrak{X}\rangle}$ where

$$\Phi : A \to \mathcal{P}(X),\ e \mapsto \emptyset \ \text{ and } \mathfrak{X} : A \to \mathcal{P}(X),\ e \mapsto X.$$

6.3.2 Definition

A bipolar soft set over X is said to be a *relative absolute bipolar soft set*, denoted by $A_{\langle \mathfrak{X}, \Phi \rangle}$ where

$$\mathfrak{X} : A \to \mathcal{P}(X),\ e \mapsto X \quad \text{and} \quad \Phi : A \to \mathcal{P}(X),\ e \mapsto \emptyset.$$

Conventionally, we take the bipolar soft sets with empty set of parameters to be equal to $\emptyset_{\langle \Phi, \mathfrak{X} \rangle}$ and so $A_{\langle \alpha, \beta \rangle} \sqcap B_{\langle \gamma, \delta \rangle} \tilde{=} \emptyset_{\langle \Phi, \mathfrak{X} \rangle} \tilde{=} A_{\langle \alpha, \beta \rangle} \sqcup B_{\langle \gamma, \delta \rangle}$ whenever $(A \cap B) = \emptyset$.

6.3.3 Proposition

If $A_{\langle \Phi, \mathfrak{X} \rangle}$ is a null bipolar soft set, $A_{\langle \mathfrak{X}, \Phi \rangle}$ an absolute bipolar soft set, and $A_{\langle \alpha, \beta \rangle}$, $A_{\langle \gamma, \delta \rangle}$ are bipolar soft sets over X, then

1) $A_{\langle \alpha, \beta \rangle} \sqcup_{\varepsilon} A_{\langle \gamma, \delta \rangle} \tilde{=} A_{\langle \alpha, \beta \rangle} \sqcup A_{\langle \gamma, \delta \rangle}$,

2) $A_{\langle \alpha, \beta \rangle} \sqcap_{\varepsilon} A_{\langle \gamma, \delta \rangle} \tilde{=} A_{\langle \alpha, \beta \rangle} \sqcap A_{\langle \gamma, \delta \rangle}$,

3) $A_{\langle \alpha, \beta \rangle} \sqcap A_{\langle \alpha, \beta \rangle} \tilde{=} A_{\langle \alpha, \beta \rangle} \tilde{=} A_{\langle \alpha, \beta \rangle} \sqcup A_{\langle \alpha, \beta \rangle}$,

4) $A_{\langle \alpha, \beta \rangle} \sqcup A_{\langle \Phi, \mathfrak{X} \rangle} \tilde{=} A_{\langle \alpha, \beta \rangle} \tilde{=} A_{\langle \alpha, \beta \rangle} \sqcap A_{\langle \mathfrak{X}, \Phi \rangle}$,

5) $A_{\langle \alpha, \beta \rangle} \sqcup A_{\langle \mathfrak{X}, \Phi \rangle} \tilde{=} A_{\langle \mathfrak{X}, \Phi \rangle}$; $A_{\langle \alpha, \beta \rangle} \sqcap A_{\langle \Phi, \mathfrak{X} \rangle} \tilde{=} A_{\langle \Phi, \mathfrak{X} \rangle}$.

Proof. Straightforward. ■

6.3.4 Proposition

Let $A_{\langle \alpha, \beta \rangle}$, $B_{\langle \gamma, \delta \rangle}$ and $C_{\langle \zeta, \eta \rangle}$ be any *bipolar soft sets* over X. Then the following are true

1) (Absorption Laws)

(i) $A_{\langle \alpha, \beta \rangle} \sqcup_{\varepsilon} (B_{\langle \gamma, \delta \rangle} \sqcap A_{\langle \alpha, \beta \rangle}) \tilde{=} A_{\langle \alpha, \beta \rangle}$,

(ii) $A_{\langle \alpha, \beta \rangle} \sqcap (B_{\langle \gamma, \delta \rangle} \sqcup_{\varepsilon} A_{\langle \alpha, \beta \rangle}) \tilde{=} A_{\langle \alpha, \beta \rangle}$,

(iii) $A_{\langle \alpha, \beta \rangle} \sqcup (B_{\langle \gamma, \delta \rangle} \sqcap_{\varepsilon} A_{\langle \alpha, \beta \rangle}) \tilde{=} A_{\langle \alpha, \beta \rangle}$,

(iv) $A_{\langle \alpha, \beta \rangle} \sqcap_{\varepsilon} (B_{\langle \gamma, \delta \rangle} \sqcup A_{\langle \alpha, \beta \rangle}) \tilde{=} A_{\langle \alpha, \beta \rangle}$.

2) (Associative Laws) $A_{\langle \alpha, \beta \rangle} \lambda (B_{\langle \gamma, \delta \rangle} \lambda C_{\langle \zeta, \eta \rangle}) \tilde{=} (A_{\langle \alpha, \beta \rangle} \lambda B_{\langle \gamma, \delta \rangle}) \lambda C_{\langle \zeta, \eta \rangle}$,

3) (Commutative Laws) $A_{\langle \alpha, \beta \rangle} \lambda B_{\langle \gamma, \delta \rangle} \tilde{=} B_{\langle \gamma, \delta \rangle} \lambda A_{\langle \alpha, \beta \rangle}$,

4) (Distributive Laws)

(i) $A_{\langle \alpha, \beta \rangle} \sqcup_{\varepsilon} (B_{\langle \gamma, \delta \rangle} \sqcup C_{\langle \zeta, \eta \rangle}) \tilde{\subseteq} (A_{\langle \alpha, \beta \rangle} \sqcup_{\varepsilon} B_{\langle \gamma, \delta \rangle}) \sqcup (A_{\langle \alpha, \beta \rangle} \sqcup_{\varepsilon} C_{\langle \zeta, \eta \rangle})$,

(ii) $A_{\langle \alpha, \beta \rangle} \sqcup_{\varepsilon} (B_{\langle \gamma, \delta \rangle} \sqcap_{\varepsilon} C_{\langle \zeta, \eta \rangle}) \tilde{\supseteq} (A_{\langle \alpha, \beta \rangle} \sqcup_{\varepsilon} B_{\langle \gamma, \delta \rangle}) \sqcap_{\varepsilon} (A_{\langle \alpha, \beta \rangle} \sqcup_{\varepsilon} C_{\langle \zeta, \eta \rangle})$,

(iii) $A_{\langle \alpha, \beta \rangle} \sqcup_{\varepsilon} (B_{\langle \gamma, \delta \rangle} \sqcap C_{\langle \zeta, \eta \rangle}) \tilde{=} (A_{\langle \alpha, \beta \rangle} \sqcup_{\varepsilon} B_{\langle \gamma, \delta \rangle}) \sqcap (A_{\langle \alpha, \beta \rangle} \sqcup_{\varepsilon} C_{\langle \zeta, \eta \rangle})$,

(iv) $A_{\langle\alpha,\beta\rangle} \sqcup (B_{\langle\gamma,\delta\rangle} \sqcup_\varepsilon C_{\langle\zeta,\eta\rangle}) \tilde{=} (A_{\langle\alpha,\beta\rangle} \sqcup B_{\langle\gamma,\delta\rangle}) \sqcup_\varepsilon (A_{\langle\alpha,\beta\rangle} \sqcup C_{\langle\zeta,\eta\rangle})$,

(v) $A_{\langle\alpha,\beta\rangle} \sqcup (B_{\langle\gamma,\delta\rangle} \sqcap_\varepsilon C_{\langle\zeta,\eta\rangle}) \tilde{=} (A_{\langle\alpha,\beta\rangle} \sqcup B_{\langle\gamma,\delta\rangle}) \sqcap_\varepsilon (A_{\langle\alpha,\beta\rangle} \sqcup C_{\langle\zeta,\eta\rangle})$,

(vi) $A_{\langle\alpha,\beta\rangle} \sqcup (B_{\langle\gamma,\delta\rangle} \sqcap C_{\langle\zeta,\eta\rangle}) \tilde{=} (A_{\langle\alpha,\beta\rangle} \sqcup B_{\langle\gamma,\delta\rangle}) \sqcap (A_{\langle\alpha,\beta\rangle} \sqcup C_{\langle\zeta,\eta\rangle})$,

(vii) $A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon (B_{\langle\gamma,\delta\rangle} \sqcup_\varepsilon C_{\langle\zeta,\eta\rangle}) \tilde{\subseteq} (A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon B_{\langle\gamma,\delta\rangle}) \sqcup_\varepsilon (A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon C_{\langle\zeta,\eta\rangle})$,

(viii) $A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon (B_{\langle\gamma,\delta\rangle} \sqcup C_{\langle\zeta,\eta\rangle}) \tilde{=} (A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon B_{\langle\gamma,\delta\rangle}) \sqcup (A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon C_{\langle\zeta,\eta\rangle})$,

(ix) $A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon (B_{\langle\gamma,\delta\rangle} \sqcap C_{\langle\zeta,\eta\rangle}) \tilde{\supseteq} (A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon B_{\langle\gamma,\delta\rangle}) \sqcap (A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon C_{\langle\zeta,\eta\rangle})$,

(x) $A_{\langle\alpha,\beta\rangle} \sqcup (B_{\langle\gamma,\delta\rangle} \sqcup_\varepsilon C_{\langle\zeta,\eta\rangle}) \tilde{=} (A_{\langle\alpha,\beta\rangle} \sqcup B_{\langle\gamma,\delta\rangle}) \sqcup_\varepsilon (A_{\langle\alpha,\beta\rangle} \sqcup C_{\langle\zeta,\eta\rangle})$,

(xi) $A_{\langle\alpha,\beta\rangle} \sqcup (B_{\langle\gamma,\delta\rangle} \sqcap_\varepsilon C_{\langle\zeta,\eta\rangle}) \tilde{=} (A_{\langle\alpha,\beta\rangle} \sqcup B_{\langle\gamma,\delta\rangle}) \sqcap_\varepsilon (A_{\langle\alpha,\beta\rangle} \sqcup C_{\langle\zeta,\eta\rangle})$,

(xii) $A_{\langle\alpha,\beta\rangle} \sqcup (B_{\langle\gamma,\delta\rangle} \sqcap C_{\langle\zeta,\eta\rangle}) \tilde{=} (A_{\langle\alpha,\beta\rangle} \sqcup B_{\langle\gamma,\delta\rangle}) \sqcap (A_{\langle\alpha,\beta\rangle} \sqcup C_{\langle\zeta,\eta\rangle})$.

Proof. It follows from Theorem 6.1.4 and Proposition 6.2.9 in a straightforward manner. ■

6.3.5 Example

Bipolar disorder is a serious psychological illness that can lead to dangerous behavior, problematic careers and relationships, and suicidal tendencies, especially if not treated early. Let $X = \{1,2,3,4,5,6,7\}$ be the set of days in which the record has been maintained i.e. $i = ith$ day of patient under observation, for $1 \leq i \leq 7$. Let $E = \{e_1,e_2,e_3,e_4,e_5\}$ = {Severe Mania, Severe Depression, Anxiety, Medication, Side effects} and $\neg E = \{\neg e_1,\neg e_2,\neg e_3,\neg e_4,\neg e_5\}$ = {Mild Mania, Mild Depression, No Anxiety, No Medication, No Side effects}. Here the gray area is obviously the moderate form of parameters. Suppose that $A = \{e_1,e_2,e_3\}$, $B = \{e_2,e_4,e_5\}$, $C = \{e_1,e_3,e_5\}$. Let the bipolar soft sets $A_{\langle\alpha,\beta\rangle}$, $B_{\langle\gamma,\delta\rangle}$ and $C_{\langle\zeta,\eta\rangle}$ over X describe the "daily record of the behavior" of P_1, P_2, and P_3. Suppose that

$$\alpha \;:\; A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{1,4,5,6\} & \text{if } e = e_1, \\ \{1,2,3,4,5,7\} & \text{if } e = e_2, \\ \{2,4,6,7\} & \text{if } e = e_3, \end{cases}$$

$$\beta \;:\; A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{2,3,7\} & \text{if } e = e_1, \\ \{6\} & \text{if } e = e_2, \\ \{3\} & \text{if } e = e_3, \end{cases}$$

$$\gamma \;:\; A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{3,5,6\} & \text{if } e = e_2, \\ \{1,5,7\} & \text{if } e = e_4, \\ \{2,3,4,5,6\} & \text{if } e = e_5, \end{cases}$$

$$\delta : A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{1,4,7\} & \text{if } e = e_2, \\ \{3,6\} & \text{if } e = e_4, \\ \{\} & \text{if } e = e_5, \end{cases}$$
$$\zeta : A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} X & \text{if } e = e_1, \\ \{1,2\} & \text{if } e = e_3, \\ \{4,5,6\} & \text{if } e = e_5, \end{cases}$$
$$\eta : A \to \mathcal{P}(X), \quad e \longmapsto \begin{cases} \{\} & \text{if } e = e_1, \\ \{3,4\} & \text{if } e = e_3, \\ \{1,2\} & \text{if } e = e_5, \end{cases}$$

We have

$$A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon (B_{\langle\gamma,\delta\rangle} \sqcap C_{\langle\zeta,\eta\rangle}) \tilde{=} (A \cup (B \cap C))_{\langle\alpha\tilde{\cap}(\gamma\tilde{\cap}\zeta),\beta\tilde{\cup}(\delta\tilde{\cup}\eta)\rangle}$$

and

$$(A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon B_{\langle\gamma,\delta\rangle}) \sqcap (A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon C_{\langle\zeta,\eta\rangle}) \tilde{=} (A \cup B) \cap (A \cup C)_{\langle(\alpha\tilde{\cap}\gamma)\tilde{\cap}(\alpha\tilde{\cap}\zeta),(\beta\tilde{\cup}\beta)\tilde{\cup}(\beta\tilde{\cup}\eta)\rangle}.$$

Then the approximations for parameter e_2 are not same on both sides

$$\begin{aligned} (\alpha\tilde{\cap}(\gamma\tilde{\cap}\zeta))(e_2) &= \{1,2,3,4,5,7\} \neq \{3,5\} = ((\alpha\tilde{\cap}\gamma)\tilde{\cap}(\alpha\tilde{\cap}\zeta))(e_2) \\ \text{and } (\beta\tilde{\cup}(\delta\tilde{\cup}\eta))(e_2) &= \{6\} \neq \{1,4,7,6\} = ((\beta\tilde{\cup}\delta)\tilde{\cup}(\beta\tilde{\cup}\eta))(e_2). \end{aligned}$$

Thus

$$A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon (B_{\langle\gamma,\delta\rangle} \sqcap C_{\langle\zeta,\eta\rangle}) \tilde{\neq} (A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon B_{\langle\gamma,\delta\rangle}) \sqcap (A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon C_{\langle\zeta,\eta\rangle}).$$

Now, consider

$$A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon (B_{\langle\gamma,\delta\rangle} \sqcup_\varepsilon C_{\langle\zeta,\eta\rangle}) \tilde{=} (A \cup (B \cup C))_{\langle\alpha\tilde{\cap}(\gamma\tilde{\cup}\zeta),\beta\tilde{\cup}(\delta\tilde{\cap}\eta)\rangle}$$

and

$$\begin{aligned} (A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon B_{\langle\gamma,\delta\rangle}) \sqcup_\varepsilon (A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon C_{\langle\zeta,\eta\rangle}) &\tilde{=} (A \cup B)_{\langle\alpha\tilde{\cap}\gamma,\beta\tilde{\cup}\delta\rangle} \sqcup_\varepsilon (A \cup C)_{\langle\alpha\tilde{\cap}\zeta,\beta\tilde{\cup}\eta\rangle} \\ &\tilde{=} (A \cup B) \cup (A \cup C)_{\langle(\alpha\tilde{\cap}\gamma)\tilde{\cup}(\alpha\tilde{\cap}\zeta),(\beta\tilde{\cup}\delta)\tilde{\cap}(\beta\tilde{\cup}\eta)\rangle}. \end{aligned}$$

Then the approximations for parameter e_2 are not same on both sides

$$\begin{aligned} (\alpha\tilde{\cap}(\gamma\tilde{\cup}\zeta))(e_2) &= \{5\} \neq \{1,2,3,4,5,7\} = ((\alpha\tilde{\cap}\gamma)\tilde{\cup}(\alpha\tilde{\cap}\zeta))(e_2) \\ \text{and } (\beta\tilde{\cup}(\delta\tilde{\cap}\eta))(e_2) &= \{1,4,7,6\} \neq \{6\} = (\beta\tilde{\cup}\delta)\tilde{\cap}(\beta\tilde{\cup}\eta))(e_2). \end{aligned}$$

Thus

$$A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon (B_{\langle\gamma,\delta\rangle} \sqcup_\varepsilon C_{\langle\zeta,\eta\rangle}) \tilde{\neq} (A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon B_{\langle\gamma,\delta\rangle}) \sqcup_\varepsilon (A_{\langle\alpha,\beta\rangle} \sqcap_\varepsilon C_{\langle\zeta,\eta\rangle}).$$

Similarly it can be shown that

$$A_{\langle\alpha,\beta\rangle} \sqcup_\varepsilon (B_{\langle\gamma,\delta\rangle} \sqcup C_{\langle\zeta,\eta\rangle}) \tilde{\neq} (A_{\langle\alpha,\beta\rangle} \sqcup_\varepsilon B_{\langle\gamma,\delta\rangle}) \sqcup (A_{\langle\alpha,\beta\rangle} \sqcup_\varepsilon C_{\langle\zeta,\eta\rangle}).$$

$$A_{\langle\alpha,\beta\rangle} \sqcup_\varepsilon (B_{\langle\gamma,\delta\rangle} \sqcap_\varepsilon C_{\langle\zeta,\eta\rangle}) \tilde{\neq} (A_{\langle\alpha,\beta\rangle} \sqcup_\varepsilon B_{\langle\gamma,\delta\rangle}) \sqcap_\varepsilon (A_{\langle\alpha,\beta\rangle} \sqcup_\varepsilon C_{\langle\zeta,\eta\rangle}).$$

6.3.6 Corollary

Let $A_{\langle\alpha,\beta\rangle}$, $B_{\langle\gamma,\delta\rangle}$ and $A_{\langle\zeta,\eta\rangle}$ be any *bipolar soft sets* over X. Then

$$A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcap_{\varepsilon} A_{\langle\zeta,\eta\rangle}) \;\tilde{=}\; (A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} B_{\langle\gamma,\delta\rangle}) \sqcap_{\varepsilon} (A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} A_{\langle\zeta,\eta\rangle}) \text{ and}$$
$$A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} (B_{\langle\gamma,\delta\rangle} \sqcup_{\varepsilon} A_{\langle\zeta,\eta\rangle}) \;\tilde{=}\; (A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} B_{\langle\gamma,\delta\rangle}) \sqcup_{\varepsilon} (A_{\langle\alpha,\beta\rangle} \sqcap_{\varepsilon} A_{\langle\zeta,\eta\rangle})$$

if and only if

$$\alpha(e) = \gamma(e) \text{ and } \beta(e) = \delta(e) \text{ for all } e \in (A \cap B) - C \text{ and}$$
$$\alpha(e) = \zeta(e) \text{ and } \beta(e) = \eta(e) \text{ for all } e \in (A \cap C) - B.$$

6.3.7 Corollary

Let $A_{\langle\alpha,\beta\rangle}$, $A_{\langle\gamma,\delta\rangle}$ and $A_{\langle\zeta,\eta\rangle}$ are three *bipolar soft sets* over X. Then

$$A_{\langle\alpha,\beta\rangle}\lambda(A_{\langle\gamma,\delta\rangle}\rho A_{\langle\zeta,\eta\rangle})\tilde{=}(A_{\langle\alpha,\beta\rangle}\lambda A_{\langle\gamma,\delta\rangle})\rho(A_{\langle\alpha,\beta\rangle}\lambda A_{\langle\zeta,\eta\rangle})$$

for distinct λ, $\rho \in \{\sqcap_{\varepsilon},\sqcap,\sqcup_{\varepsilon},\sqcup\}$.

A *bipolar mood chart* is a simple and yet effective means of tracking and representing patient's condition every month. Bipolar mood charts help patients, their families and their doctors to see probable patterns that might have been very difficult to determine. Bipolar children and their families will greatly benefit from mood charting and can expect early detection of symptoms and determination of proper treatments by their doctors. We construct a mood chart based upon a bipolar soft set as follows:

A bipolar soft set $A_{\langle\alpha,\beta\rangle}$ over X may be represented by a pair of binary tables, one for each of the functions α and β respectively. In both tables, rows and columns are labeled by the elements of X and parameters respectively. We use following key for tables of α and β respectively:

$$a_{ij} = \begin{cases} 1 & \text{if } x_i \in \alpha(e_j) \\ 0 & \text{if } x_i \notin \alpha(e_j) \end{cases}$$
$$a_{ij} = \begin{cases} 1 & \text{if } x_i \in \beta(e_j) \\ 0 & \text{if } x_i \notin \beta(e_j) \end{cases}$$

where a_{ij} is the ith entry of jth column of each table. We can also represent a bipolar soft set with the help of a single table by putting

$$a_{ij} = \begin{cases} 1 & \text{if } h_i \in \alpha(e_j) \\ 0 & \text{if } h_i \in X - \{\alpha(e_j) \cup \beta(e_j)\} \\ -1 & \text{if } h_i \in \beta(e_j) \end{cases}$$

where a_{ij} is the ith entry of jth column of table whose rows and columns are labeled by elements of X and parameters respectively. The tabular representations of bipolar soft set $A_{\langle\alpha,\beta\rangle}$ as given in Example 6.3.5 are given by Table 6.1 and Table 6.2.

Both Tables 6.1 and Table 6.2 can be used as Mood Chart of patient P_1 for a week.

α	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
1	1	1	0
2	0	1	1
3	0	1	0
4	1	1	1
5	1	1	0
6	1	0	1
7	0	1	1

β	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
1	0	0	0
2	1	0	0
3	1	0	1
4	0	0	0
5	0	0	0
6	0	1	0
7	1	0	0

Table 6.1: Tabular Representaion Using a Pair of Tables

$A_{\langle\alpha,\beta\rangle}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
1	1	1	0
2	-1	1	1
3	-1	1	-1
4	1	1	1
5	1	1	0
6	1	-1	1
7	-1	1	1

Table 6.2: Tabular Representaion Using Only One Table

6.4 Algebras of Bipolar Soft Sets

In this section, we discuss the lattices and algebras for collections of bipolar soft sets. Let $\mathcal{BSS}(X)^E$ be the collection of all bipolar soft sets over X and $\mathcal{DSS}(X)_A$ be its subcollection of all bipolar soft sets over X with fixed set of parameters A. We note that these collections are partially ordered by the relation of soft inclusion $\tilde{\subseteq}$ given in Definition 6.1.2. We conclude from above results that:

6.4.1 Proposition

$(\mathcal{BSS}(X)^E, \sqcap_\varepsilon, \sqcup)$, $(\mathcal{BSS}(X)^E, \sqcup, \sqcap_\varepsilon)$, $(\mathcal{BSS}(X)^E, \sqcup_\varepsilon, \sqcap)$, $(\mathcal{BSS}(X)^E, \sqcap, \sqcup_\varepsilon)$, $(\mathcal{BSS}(X)_A, \sqcup, \sqcap)$, and $(\mathcal{BSS}(X)_A, \sqcap, \sqcup)$ are lattices.

Proof. From Propositions 6.3.3 and 6.3.4, we conclude that the structures form lattices. ■

6.4.2 Proposition

Let $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$ be two *bipolar soft sets* over X. Then the following are true

1) $A_{\langle\alpha,\beta\rangle} \sqcup_\varepsilon B_{\langle\gamma,\delta\rangle}$ is the smallest *bipolar* soft set over X which contains both $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$.

2) $A_{\langle\alpha,\beta\rangle} \sqcap B_{\langle\gamma,\delta\rangle}$ is the largest *bipolar* soft set over X which is contained in both $A_{\langle\alpha,\beta\rangle}$ and $B_{\langle\gamma,\delta\rangle}$.

Proof. Straightforward. ■

6.4.3 Proposition

$(\mathcal{BSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{\langle \Phi, \mathfrak{X} \rangle}, E_{\langle \mathfrak{X}, \Phi \rangle})$, $(\mathcal{BSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{\langle \mathfrak{X}, \Phi \rangle}, \emptyset_{\langle \Phi, \mathfrak{X} \rangle})$, $(\mathcal{BSS}(X)_A, \sqcap, \sqcup, A_{\langle \Phi, \mathfrak{X} \rangle}, A_{\langle \mathfrak{X}, \Phi \rangle})$ and $(\mathcal{BSS}(X)_A, \sqcup, \sqcap, A_{\langle \mathfrak{X}, \Phi \rangle}, A_{\langle \Phi, \mathfrak{X} \rangle})$ are bounded distributive lattices.

Proof. From Proposition 6.3.4 and Lemma 6.4.2, we conclude that $(\mathcal{BSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{\langle \Phi, \mathfrak{X} \rangle}, E_{\langle \mathfrak{X}, \Phi \rangle})$ is a bounded distributive lattice and $(\mathcal{BSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{\langle \mathfrak{X}, \Phi \rangle}, \emptyset_{\langle \Phi, \mathfrak{X} \rangle})$ is its dual. For bipolar soft sets $A_{\langle \alpha, \beta \rangle}, A_{\langle \gamma, \delta \rangle} \in \mathcal{BSS}(X)_A$,

$$A_{\langle \alpha, \beta \rangle} \sqcap A_{\langle \gamma, \delta \rangle} \tilde{=} A_{\langle \alpha \tilde{\cap} \gamma, \beta \tilde{\cup} \delta \rangle} \in \mathcal{BSS}(X)_A \text{ and}$$
$$A_{\langle \alpha, \beta \rangle} \sqcup A_{\langle \gamma, \delta \rangle} \tilde{=} A_{\langle \alpha \tilde{\cup} \gamma, \beta \tilde{\cap} \delta \rangle} \in \mathcal{BSS}(X)_A.$$

Thus $(\mathcal{BSS}(X)_A, \sqcap, \sqcup)$ is also a distributive sublattice of $(\mathcal{BSS}(X)^E, \sqcup_\varepsilon, \sqcap)$ and Proposition 6.3.3 tells us that $A_{\langle \Phi, \mathfrak{X} \rangle}$, $A_{\langle \mathfrak{X}, \Phi \rangle}$ are its lower and upper bounds respectively. Therefore $(\mathcal{BSS}(X)_A, \sqcap, \sqcup, A_{\langle \Phi, \mathfrak{X} \rangle}, A_{\langle \mathfrak{X}, \Phi \rangle})$ is a bounded distributive lattice and $(\mathcal{BSS}(X)_A, \sqcup, \sqcap, A_{\langle \mathfrak{X}, \Phi \rangle}, A_{\langle \Phi, \mathfrak{X} \rangle})$ is its dual. ■

6.4.4 Proposition

Let $A_{\langle \alpha, \beta \rangle}$ and $A_{\langle \gamma, \delta \rangle}$ be two *bipolar soft sets* over X. Then

1) $(A_{\langle \alpha, \beta \rangle^\circ})^\circ = A_{\langle \alpha, \beta \rangle}$,

2) $A_{\langle \alpha, \beta \rangle} \tilde{\subseteq} A_{\langle \gamma, \delta \rangle}$ if and only if $A_{\langle \gamma, \delta \rangle^\circ} \tilde{\subseteq} A_{\langle \alpha, \beta \rangle^\circ}$.

Proof.

1) Straightforward

2) If $A_{\langle \alpha, \beta \rangle} \tilde{\subseteq} A_{\langle \gamma, \delta \rangle}$ then

$$\alpha(e) \subseteq \gamma(e) \text{ and } \delta(e) \subseteq \beta(e) \text{ for all } e \in A$$

implies that
$$A_{\langle \gamma, \delta \rangle} \tilde{\subseteq} A_{\langle \alpha, \beta \rangle}.$$

Hence $A_{\langle \gamma, \delta \rangle^\circ} \tilde{\subseteq} A_{\langle \alpha, \beta \rangle^\circ}$. If $A_{\langle \gamma, \delta \rangle^\circ} \tilde{\subseteq} A_{\langle \alpha, \beta \rangle^\circ}$ then

$$A_{\langle \alpha, \beta \rangle} \tilde{=} (A_{\langle \alpha, \beta \rangle^\circ})^\circ \tilde{\subseteq} (A_{\langle \gamma, \delta \rangle^\circ})^\circ \tilde{=} A_{\langle \gamma, \delta \rangle}.$$

■

6.4.5 Proposition (de Morgan Laws)

Let $A_{\langle \alpha, \beta \rangle}$ and $B_{\langle \gamma, \delta \rangle}$ be two *bipolar* soft sets over X. Then the following are true:

1) $(A_{\langle \alpha, \beta \rangle} \sqcup_\varepsilon B_{\langle \gamma, \delta \rangle})^\circ \tilde{=} A_{\langle \alpha, \beta \rangle^\circ} \sqcap_\varepsilon B_{\langle \gamma, \delta \rangle^\circ}$,

2) $(A_{\langle \alpha, \beta \rangle} \sqcap_\varepsilon B_{\langle \gamma, \delta \rangle})^\circ \tilde{=} A_{\langle \alpha, \beta \rangle^\circ} \sqcup_\varepsilon B_{\langle \gamma, \delta \rangle^\circ}$,

3) $(A_{\langle \alpha, \beta \rangle} \vee B_{\langle \gamma, \delta \rangle})^\circ \tilde{=} A_{\langle \alpha, \beta \rangle^\circ} \wedge B_{\langle \gamma, \delta \rangle^\circ}$,

4) $(A_{\langle\alpha,\beta\rangle} \wedge B_{\langle\gamma,\delta\rangle})^{\circ} \tilde{=} A_{\langle\alpha,\beta\rangle^{\circ}} \vee B_{\langle\gamma,\delta\rangle^{\circ}}$,

5) $(A_{\langle\alpha,\beta\rangle} \sqcup B_{\langle\gamma,\delta\rangle})^{\circ} \tilde{=} A_{\langle\alpha,\beta\rangle^{\circ}} \sqcap B_{\langle\gamma,\delta\rangle^{\circ}}$,

6) $(A_{\langle\alpha,\beta\rangle} \sqcap B_{\langle\gamma,\delta\rangle})^{\circ} \tilde{=} A_{\langle\alpha,\beta\rangle^{\circ}} \sqcup B_{\langle\gamma,\delta\rangle^{\circ}}$.

Proof.

1) We have

$$(A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} B_{\langle\gamma,\delta\rangle})^{\circ} \tilde{=} ((A \cup B)_{\langle\alpha\tilde{\cup}\gamma,\beta\tilde{\cap}\delta\rangle})^{\circ} \tilde{=} (A \cup B)_{\langle\beta\tilde{\cap}\delta,\alpha\tilde{\cup}\gamma\rangle}$$

and

$$A_{\langle\alpha,\beta\rangle^{\circ}} \sqcap_{\varepsilon} B_{\langle\gamma,\delta\rangle^{\circ}} \tilde{=} A_{(\beta,\alpha)} \sqcap_{\varepsilon} B_{(\delta,\gamma)} \tilde{=} (A \cup B)_{\langle\beta\tilde{\cap}\delta,\alpha\tilde{\cup}\gamma\rangle}.$$

Thus

$$(A_{\langle\alpha,\beta\rangle} \sqcup_{\varepsilon} B_{\langle\gamma,\delta\rangle})^{\circ} \tilde{=} A_{\langle\alpha,\beta\rangle^{\circ}} \sqcap_{\varepsilon} B_{\langle\gamma,\delta\rangle^{\circ}}.$$

The remaining parts can also be proved in a similar way.

■

6.4.6 Proposition

$(\mathcal{BSS}(X)_A,\sqcap,\sqcup,^{\circ},A_{\langle\Phi,\mathfrak{X}\rangle},A_{\langle\mathfrak{X},\Phi\rangle})$ is a de Morgan algebra.

Proof. Proof follows from Propositions 6.4.4 and 6.4.5. ■

6.4.7 Proposition

$(\mathcal{BSS}(X)_A,\sqcap,\sqcup,^{\circ},A_{\langle\Phi,\mathfrak{X}\rangle},A_{\langle\mathfrak{X},\Phi\rangle})$ is a Kleene algebra.

Proof. For $A_{\langle\alpha,\beta\rangle}$, $A_{\langle\gamma,\delta\rangle} \in \mathcal{BSS}(X)_A$

$$A_{\langle\alpha,\beta\rangle} \sqcap A_{\langle\alpha,\beta\rangle^{\circ}} \tilde{=} A_{\langle\alpha,\beta\rangle} \sqcap A_{(\beta,\alpha)} \tilde{=} A_{\langle\alpha\tilde{\cap}\beta,\beta\tilde{\cup}\alpha\rangle} \tilde{=} A_{\langle\Phi,\beta\tilde{\cup}\alpha\rangle} \quad \text{and}$$

$$A_{\langle\gamma,\delta\rangle} \sqcup A_{\langle\gamma,\delta\rangle^{\circ}} \tilde{=} A_{\langle\gamma,\delta\rangle} \sqcup A_{(\delta,\gamma)} \tilde{=} A_{\langle\gamma\tilde{\cup}\delta,\delta\tilde{\cap}\gamma\rangle} \tilde{=} A_{\langle\gamma\tilde{\cup}\delta,\Phi\rangle}.$$

Clearly $$A_{\langle\alpha,\beta\rangle} \sqcap A_{\langle\alpha,\beta\rangle^{\circ}} \tilde{\subseteq} A_{\langle\gamma,\delta\rangle} \sqcup A_{\langle\gamma,\delta\rangle^{\circ}}.$$

We already know that $(\mathcal{BSS}(X)_A,\sqcap,\sqcup,^{\circ},A_{\langle\Phi,\mathfrak{X}\rangle},A_{\langle\mathfrak{X},\Phi\rangle})$ is a de Morgan algebra, so this condition assures that $(\mathcal{BSS}(X)_A,\sqcap,\sqcup,^{\circ},A_{\langle\Phi,\mathfrak{X}\rangle},A_{\langle\mathfrak{X},\Phi\rangle})$ is also a Kleene algebra. ■

6.4.8 Remark

We have seen that $(\mathcal{DSS}(X)_A,\sqcap,\sqcup,^{\circ},A_{(\Phi,\mathfrak{X})},A_{(\mathfrak{X},\Phi)})$ is a de Morgan algebra but not a Kleene algebra whereas $(\mathcal{BSS}(X)_A,\sqcap,\sqcup,^{\circ},A_{\langle\Phi,\mathfrak{X}\rangle},A_{\langle\mathfrak{X},\Phi\rangle})$ is its de Morgan subalgebra and also a Kleene subalgebra.

Chapter 7

Algebraic Structures of Fuzzy Bipolar Soft Sets

In this chapter, we have initiated a concept of fuzzy bipolar soft sets. The idea is generated with the motivation of bipolarity of parameters and then the fuzziness of data comes into play. A fuzzy bipolar soft set is defined with the help of two mappings, one for approximating the degree of fuzziness of the positivity or presence of a certain parameter in the objects of initial universal set and the other one is to approximate a relative degree of fuzziness of the negativity or absence of same parameter. In this way, we have combined these three concepts of bipolarity, fuzziness and parameterization and thus it is shown through examples that we have found a very easy to use way of modeling the phenomena where all these three factors are involved. To move further, we have defined the basic algebra for the fuzzy bipolar soft sets and discussed their algebraic properties in detail. It is also shown that the collection of fuzzy bipolar soft sets forms a stone algebra.

7.1 Fuzzy Bipolar Soft Sets

Let X be an initial universe and E be a set of parameters. Let $\mathcal{FP}(X)$ denotes the collection of all fuzzy subsets of X and A, B, C are non-empty subsets of E. Now, we define

7.1.1 Definition

A triplet $(f,g : A)$ is called a fuzzy *bipolar* soft set over X, where f and g are mappings, given by $f : A \to \mathcal{FP}(X)$ and $g : \neg A \to \mathcal{FP}(X)$ such that $0 \leq (f(e))(x)+(g(\neg e))(x) \leq 1$ for all $e \in A$.

In other words, a *fuzzy bipolar soft set* over X gives two parametrized families of fuzzy subsets of the universe X and the condition $0 \leq (f(e))(x) + (g(\neg e))(x) \leq 1$ for all.$e \in A$, is imposed as a consistency constraint. For each $e \in A$, $f(e)$ and $g(\neg e)$ are regarded as the set of e-approximate elements of the *fuzzy bipolar soft set* $A_{(f,g)}$.

Note that, from now on, we shall use the notation $A_{(f,g)}$ over X to denote a *fuzzy bipolar soft set* $(f,g : A)$ over X where the meanings of f, g, A and X are clear.

7.1.2 Definition

For a *fuzzy bipolar soft set* $A_{\langle f,g\rangle}$ over X, we define a fuzzy soft set A_h over X for the approximation of the degree of hesitation in $A_{\langle f,g\rangle}$ as $h : A \to \mathcal{FP}(X)$ defined by $(h(e))(x) = 1 - (f(e))(x) - (g(\neg e))(x)$ for all $x \in X$, $e \in A$. Clearly, A_h approximates the lack of knowledge about the objects of X while considering the presence or absence of a particular parameter from A.

7.1.3 Definition

For two *fuzzy bipolar soft sets* $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ over X, we say that $A_{\langle f,g\rangle}$ is a *fuzzy bipolar* soft subset of $B_{\langle h,i\rangle}$, if

1) $A \subseteq B$ and

2) $f(e) \subseteq h(e)$ and $i(\neg e) \subseteq g(\neg e)$ for all $e \in A$.

This relationship is denoted by $A_{\langle f,g\rangle}\tilde{\subseteq}B_{\langle h,i\rangle}$.

Similarly $A_{\langle f,g\rangle}$ is said to be a *fuzzy bipolar* soft superset of $B_{\langle h,i\rangle}$, if $B_{\langle h,i\rangle}$ is a *fuzzy bipolar soft subset* of $A_{\langle f,g\rangle}$. We denote it by $A_{\langle f,g\rangle}\tilde{\supseteq}B_{\langle h,i\rangle}$.

7.1.4 Definition

Two *fuzzy bipolar soft sets* $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ over X are said to be *equal denoted as* $A_{\langle f,g\rangle}\tilde{=}B_{\langle h,i\rangle}$ if $A_{\langle f,g\rangle}$ is a *fuzzy bipolar* soft subset of $B_{\langle h,i\rangle}$ and $B_{\langle h,i\rangle}$ is a *fuzzy bipolar* soft subset of $A_{\langle f,g\rangle}$.

7.1.5 Example

Let X be a set of different books, and E be the set of parameters where, $X = \{b_1,b_2,b_3,b_4,b_5\}$, $E = \{e_1,e_2,e_3,e_4,e_5,e_6\} = \{$ Simple, Logical, Orderly, Concise, Varied, Appealing$\}$, $\neg E = \{\neg e_1,\neg e_2,\neg e_3,\neg e_4,\neg e_5\} = \{$Complicated, Illogical, Chaotic, Wordy, Monotonous, Distant$\}$. Suppose that $A = \{e_1,e_2,e_3,e_6\}$, a fuzzy bipolar soft set $A_{\langle f,g\rangle}$ describes the "reader ratings of books under consideration". The fuzzy bipolar soft set $A_{\langle f,g\rangle}$ over X is given as follows:

$$\begin{aligned}
f &: A \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases}
\{b_1/0.9, b_2/0.3, b_3/0.8, b_4/0.7, b_5/0.5\} & \text{if } e = e_1,\\
\{b_1/0.1, b_2/0.5, b_3/0.1, b_4/0.8, b_5/0.6\} & \text{if } e = e_2,\\
\{b_1/0.2, b_2/0.5, b_3/0.2, b_4/0.8, b_5/0.7\} & \text{if } e = e_3,\\
\{b_1/0.7, b_2/0.4, b_3/0.2, b_4/0.1, b_5/0.1\} & \text{if } e = e_6,
\end{cases}\\
g &: \neg A \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases}
\{b_1/0.1, b_2/0.3, b_3/0.1, b_4/0.2, b_5/0.3\} & \text{if } e = \neg e_1,\\
\{b_1/0.7, b_2/0.4, b_3/0.8, b_4/0.1, b_5/0.2\} & \text{if } e = \neg e_2,\\
\{b_1/0.6, b_2/0.4, b_3/0.6, b_4/0.1, b_5/0.3\} & \text{if } e = \neg e_3,\\
\{b_1/0.2, b_2/0.3, b_3/0.8, b_4/0.7, b_5/0.5\} & \text{if } e = \neg e_6.
\end{cases}
\end{aligned}$$

Let $B = \{e_2,e_6\}$. Then a second approximations with respect to the earlier approximations by $A_{\langle f,g\rangle}$ is represented by a fuzzy bipolar soft subset $B_{\langle h,i\rangle}$ .of $A_{\langle f,g\rangle}$ and given by:

$$\begin{array}{lcl} h & : & B \to \mathcal{FP}(X), \\ e & \longmapsto & \begin{cases} \{b_1/0.1, b_2/0.5, b_3/0.1, b_4/0.8, b_5/0.6\} & \text{if } e = e_2, \\ \{b_1/0.7, b_2/0.4, b_3/0.2, b_4/0.1, b_5/0.1\} & \text{if } e = e_6, \end{cases} \\ i & : & \neg B \to \mathcal{FP}(X), \\ e & \longmapsto & \begin{cases} \{b_1/0.7, b_2/0.4, b_3/0.8, b_4/0.1, b_5/0.2\} & \text{if } e = \neg e_2, \\ \{b_1/0.2, b_2/0.3, b_3/0.8, b_4/0.7, b_5/0.5\} & \text{if } e = \neg e_6. \end{cases} \end{array}$$

7.2 Bipolar fuzzy Soft Sets

We present the concept of bipolar fuzzy soft sets as a generalization of soft sets in bipolar fuzzy context. Let $\mathcal{BFP}(X)$ denotes the set of all bipolar fuzzy subsets of X.

7.2.1 Definition

A pair (f,A) is called a *bipolar fuzzy soft set over* X, where f is a mapping given by $f : A \to \mathcal{BFP}(X)$.

Thus a bipolar fuzzy soft set over X gives a parametrized family of bipolar fuzzy subsets of the universe X. For any $e \in A$, $f(e) = \{(x,f(e)^P,f(e)^N) : x \in X\}$ where $f(e)^P : X \to [0,1]$ and $f(e)^N : X \to [-1,0]$ are mappings.

Before proceeding to the further development of theory of bipolar fuzzy soft sets, we give following interpretations:

7.2.2 Proposition

A fuzzy bipolar soft set over X is equivalent to a bipolar fuzzy soft set over X and vice versa.

Proof. Let $A_{\langle f,g\rangle}$ be a given fuzzy bipolar soft set defined over X. We define a bipolar fuzzy soft set (h,A) over X as:

$$h(e) = \{(x, f(e), -(g(\neg e)(x)) : x \in X\}$$

for all $e \in A$. Then $(x, f(e), -(g(\neg e)(x)) \in \mathcal{BFP}(X)$.

Conversely assume that we are given a bipolar fuzzy soft set (h,A) over X. We can define a fuzzy bipolar soft set $A_{\langle f,g\rangle}$ over X in the following manner:

$$\begin{aligned} f(e) &= h(e)^P \\ g(\neg e) &= -(h(e)^N) \end{aligned}$$

for all $e \in A$.

Thus both definitions are equivalent and may be used interchangeably. ■

Consider the following example:

7.2.3 Example

Let $X = \{m_1,m_2,m_3,m_4,m_5\}$ be the set of candidates who have applied for a job position of Office Representative in Customer Care Centre of a company. Let $E = \{e_1,e_2,e_3,e_4,e_5,e_6,e_7\}$ = {Hard Working, Optimism, Enthusiasm, Individualism, Imaginative, Decisiveness, Self-confidence} and $\neg E = \{\neg e_1,\neg e_2,\neg e_3, \neg e_4, \neg e_5, \neg e_6, \neg e_7\}$ = {Negligent, Pessimism, Half-hearted, Dependence, Unimaginative, Indecisiveness, Shyness}. Here the gray area is obviously a moderate form of parameters. Let us suppose that the *fuzzy bipolar soft set* $E_{\langle f,g\rangle}$ describes " Personality Analysis of Candidates" as:

$$
\begin{aligned}
f &: E \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases}
\{m_1/0.5, m_2/0.7, m_3/0.6, m_4/0.7, m_5/0.5\} & \text{if } e = e_1,\\
\{m_1/0.6, m_2/0.7, m_3/0.8, m_4/0.8, m_5/0.4\} & \text{if } e = e_2,\\
\{m_1/0.8, m_2/0.8, m_3/0.4, m_4/0.6, m_5/0.5\} & \text{if } e = e_3,\\
\{m_1/0.7, m_2/0.6, m_3/0.1, m_4/0.7, m_5/0.6\} & \text{if } e = e_4,\\
\{m_1/0.5, m_2/0.8, m_3/0.6, m_4/0.5, m_5/0.7\} & \text{if } e = e_5,\\
\{m_1/0.4, m_2/0.9, m_3/0.5, m_4/0.4, m_5/0.7\} & \text{if } e = e_6,\\
\{m_1/0.3, m_2/0.8, m_3/0.4, m_4/0.6, m_5/0.8\} & \text{if } e = e_7,
\end{cases}\\
g &: \neg E \to \mathcal{FP}(X),\\
e &\longmapsto \begin{cases}
\{m_1/0.3, m_2/0.2, m_3/0.4, m_4/0.1, m_5/0.3\} & \text{if } e = \neg e_1,\\
\{m_1/0.4, m_2/0.1, m_3/0.2, m_4/0.1, m_5/0.5\} & \text{if } e = \neg e_2,\\
\{m_1/0.05, m_2/0.1, m_3/0.5, m_4/0.33, m_5/0.4\} & \text{if } e = \neg e_3,\\
\{m_1/0.23, m_2/0.3, m_3/0.6, m_4/0.2, m_5/0.3\} & \text{if } e = \neg e_4,\\
\{m_1/0.4, m_2/0.2, m_3/0.35, m_4/0.4, m_5/0.1\} & \text{if } e = \neg e_5,\\
\{m_1/0.4, m_2/0.2, m_3/0.3, m_4/0.3, m_5/0.2\} & \text{if } e = \neg e_6,\\
\{m_1/0.7, m_2/0.08, m_3/0.5, m_4/0.3, m_5/0.18\} & \text{if } e = \neg e_7,
\end{cases}
\end{aligned}
$$

Now let's see the corresponding bipolar fuzzy soft set:

$$
\begin{aligned}
h(e_1) &= \{(m_1, 0.5, -0.3), (m_2, 0.7, -0.2), (m_3, 0.6, -0.4), (m_4, 0.7, -0.1), (m_5, 0.5, -0.3)\},\\
h(e_2) &= \{(m_1, 0.6, -0.4), (m_2, 0.7, -0.1), (m_3, 0.8, -0.2), (m_4, 0.8, -0.1), (m_5, 0.4, -0.5)\},\\
h(e_3) &= \{(m_1, 0.8, -0.05), (m_2, 0.8, -0.1), (m_3, 0.4, -0.5), (m_4, 0.6, -0.33), (m_5, 0.5, -0.4)\},\\
h(e_4) &= \{(m_1, 0.7, -0.23), (m_2, 0.6, -0.3), (m_3, 0.1, -0.4), (m_4, 0.7, -0.2), (m_5, 0.6, -0.3)\},\\
h(e_5) &= \{(m_1, 0.5, -0.4), (m_2, 0.8, -0.2), (m_3, 0.6, -0.35), (m_4, 0.5, -0.4), (m_5, 0.7, -0.1)\},\\
h(e_6) &= \{(m_1, 0.4, -0.4), (m_2, 0.9, -0.2), (m_3, 0.5, -0.3), (m_4, 0.4, -0.3), (m_5, 0.7, -0.2)\},\\
h(e_7) &= \{(m_1, 0.3, -0.7), (m_2, 0.8, -0.08), (m_3, 0.4, -0.5), (m_4, 0.6, -0.3), (m_5, 0.8, -0.18)\}.
\end{aligned}
$$

It is clear that fuzzy bipolar soft set depicts the information in a better and comprehensive way than bipolar fuzzy soft set. For example, if we read the data of candidate m_1 with fuzzy bipolar soft set $A_{\langle f,g\rangle}$ then he is having 0.6 fuzzy value for optimism and 0.4 fuzzy value for pessimism and if we use the bipolar fuzzy soft set (h,E) then m_1 is having 0.6 fuzzy value for optimism and -0.4 shows the degree where m_1 is showing pessimism.

Let $\mathcal{FBSS}(X)^E$ denotes the set of all fuzzy bipolar soft sets defined over X with set of parameters E, ordered by the relation of inclusion $\tilde{\subseteq}$ as defined in Definition 7.1.3. We show that every fuzzy bipolar soft set is equivalent to a double-framed fuzzy soft set and give the following theorem:

7.2.4 Theorem

The mapping $\theta : \mathcal{FBSS}(X)^E \to \mathcal{DFSS}(X)^E$, $A_{\langle f,g\rangle} \mapsto A_{(f_1,g_1)}$ is a monomorphism of lattices where

$$f_1(e) = f(e), \text{ and } \; g_1(e) = g(\neg e) \text{ for all } e \in A.$$

Proof. Clearly θ is well-defined. If

$$\theta(A_{\langle f,g\rangle}) \tilde{=} \theta(B_{\langle h,i\rangle})$$

where

$$\theta(A_{\langle f,g\rangle}) \tilde{=} A_{(f_1,g_1)} \text{ and } \theta(B_{\langle h,i\rangle}) \tilde{=} B_{(h_1,i_1)}$$

then

$$f_1(e) = f(e),\, h_1(e) = h(e) \;\text{ and }\; g_1(e) = g(\neg e),\, i_1(e) = i(\neg e) \;\text{ for all } e \in A.$$

Now,

$$f(e) = f_1(e) = h_1(e) = h(e) \text{ and } g(\neg e) = g_1(e) = i_1(e) = i(\neg e) \;\text{ for all } e \in A.$$

Thus

$$A_{\langle f,g\rangle} \tilde{=} B_{\langle h,i\rangle}$$

shows that θ is one-to-one. Clearly θ is order preserving. ■

7.2.5 Remark

Note that θ is not onto because of the consistency constraint for defining fuzzy bipolar soft sets and $\mathcal{FBSS}(X)^E \tilde{=} \mathcal{BFSS}(X)^E \hookrightarrow \mathcal{DFSS}(X)^E$.

By Theorem 7.2.4, we can equate every fuzzy bipolar soft set $A_{\langle f,g\rangle}$ over X with a double-framed fuzzy soft set and so, we can take f and g as mappings from A to $\mathcal{BFP}(X)$ where the meanings of A, f and g are clear in this context.

7.3 Operations on Fuzzy Bipolar Soft Sets

This section provides some operations defined on fuzzy bipolar soft sets:

7.3.1 Definition

Let $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ be fuzzy bipolar soft sets over X. The *int-uni product* of $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ is defined as a fuzzy bipolar soft set $(A \times B)_{\langle f\tilde{\wedge}h, g\tilde{\vee}i\rangle}$ over X in which

$$\begin{aligned} f\tilde{\wedge}h &: (A \times B) \to \mathcal{FP}(X),\, (a,b) \mapsto f(a) \wedge h(b), \\ g\tilde{\vee}i &: (A \times B) \to \mathcal{FP}(X),\, (a,b) \mapsto g(a) \vee i(b). \end{aligned}$$

It is denoted by $A_{\langle f,g\rangle} \wedge B_{\langle h,i\rangle} \tilde{=} (A \times B)_{\langle f\tilde{\wedge}h, g\tilde{\vee}i\rangle}$.

7.3.2 Definition

Let $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ be *fuzzy bipolar* soft sets over X. The *uni-int product* of $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ is defined as the fuzzy bipolar soft soft set $(A\times B)_{\langle f\tilde{\vee}h,\beta\tilde{\wedge}i\rangle}$ over X in which $f\tilde{\vee}h:(A\times B)\to\mathcal{FP}(X)$, where

$$(a,b)\mapsto f(a)\vee h(b),$$

and $g\tilde{\wedge}i:(A\times B)\to\mathcal{FP}(X)$, where

$$(a,b)\mapsto g(a)\wedge i(b).$$

It is denoted by $A_{\langle f,g\rangle}\vee B_{\langle h,i\rangle}\tilde{=}(A\times B)_{\langle f\tilde{\vee}h,g\tilde{\wedge}i\rangle}$.

7.3.3 Definition

For two *fuzzy bipolar soft sets* $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ over X, *the extended int-uni fuzzy bipolar soft set* of $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ is defined as the *fuzzy bipolar soft set* $(A\cup B)_{\langle f\tilde{\wedge}h,g\tilde{\vee}i\rangle}$ where $f\tilde{\wedge}h:(A\cup B)\to\mathcal{FP}(X)$,

$$e\mapsto\begin{cases}f(e) & \text{if } e\in A-B\\ h(e) & \text{if } e\in B-A\\ f(e)\wedge h(e) & \text{if } e\in(A\cap B)\end{cases}$$

and $g\tilde{\vee}i:(A\cup B)\to\mathcal{FP}(X)$, where

$$e\mapsto\begin{cases}g(e) & \text{if } e\in A-B\\ i(e) & \text{if } e\in B-A\\ g(e)\vee i(e) & \text{if } e\in(A\cap B)\end{cases}.$$

It is denoted by $A_{\langle f,g\rangle}\sqcap_{\varepsilon}B_{\langle h,i\rangle}\tilde{=}(A\cup B)_{\langle f\tilde{\wedge}h,g\tilde{\vee}i\rangle}$.

7.3.4 Definition

For two *fuzzy bipolar soft sets* $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ over X, *the extended uni-int fuzzy bipolar soft set* of $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ is defined as the *fuzzy bipolar soft set* $(A\cup B)_{\langle f\tilde{\vee}h,g\tilde{\wedge}i\rangle}$ where $f\tilde{\vee}h:(A\cup B)\to\mathcal{FP}(X)$,

$$e\mapsto\begin{cases}f(e) & \text{if } e\in A-B\\ h(e) & \text{if } e\in B-A\\ f(e)\vee h(e) & \text{if } e\in(A\cap B)\end{cases}$$

and $g\tilde{\wedge}i:(A\cup B)\to\mathcal{FP}(X)$, where

$$e\mapsto\begin{cases}g(e) & \text{if } e\in A-B\\ i(e) & \text{if } e\in B-A\\ g(e)\wedge i(e) & \text{if } e\in(A\cap B)\end{cases}.$$

It is denoted by $A_{\langle f,g\rangle}\sqcup_{\varepsilon}B_{\langle h,i\rangle}\tilde{=}(A\cup B)_{\langle f\tilde{\vee}h,g\tilde{\wedge}i\rangle}$.

7.3.5 Definition

For two *fuzzy bipolar soft sets* $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ over X with $(A \cap B) \neq \emptyset$, the *restricted int-uni fuzzy bipolar soft set* of $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ is defined as the *fuzzy bipolar soft set* $(A \cap B)_{\langle f\tilde{\wedge}h, g\tilde{\vee}i\rangle}$ where $f\tilde{\wedge}h : (A \cap B) \to \mathcal{FP}(X)$,

$$e \mapsto f(e) \wedge h(e),$$

and $g\tilde{\vee}i : (A \cap B) \to \mathcal{FP}(X)$, where

$$e \mapsto g(e) \vee i(e).$$

It is denoted by $A_{\langle f,g\rangle} \sqcap B_{\langle h,i\rangle} \tilde{=} (A \cap B)_{\langle f\tilde{\wedge}h, g\tilde{\vee}i\rangle}$.

7.3.6 Definition

For two *fuzzy bipolar soft sets* $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ over X with $(A \cap B) \neq \emptyset$, the *restricted uni-int fuzzy bipolar soft set* of $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ is defined as the *fuzzy bipolar soft set* $(A \cap B)_{\langle f\tilde{\vee}h, g\tilde{\wedge}i\rangle}$ where, $f\tilde{\vee}h : (A \cap B) \to \mathcal{FP}(X)$

$$e \mapsto f(e) \vee h(e),$$

and $g\tilde{\wedge}i : (A \cap B) \to \mathcal{FP}(X)$,

$$e \mapsto g(e) \wedge i(e).$$

It is denoted by $A_{\langle f,g\rangle} \sqcup B_{\langle h,i\rangle} \tilde{=} (A \cap B)_{\langle f\tilde{\vee}h, g\tilde{\wedge}i\rangle}$.

7.3.7 Remark

The operation of complementation as defined in Definition 5.2.7 for double-framed fuzzy soft sets is no more valid for fuzzy bipolar soft sets because $(A_{\langle f,g\rangle})' \tilde{=} A_{\langle f',g'\rangle}$ may not satisfy the consistency constraint as shown by the following example:

7.3.8 Example

Let E, A, X and fuzzy bipolar soft set $A_{\langle f,g\rangle}$ over X be taken as in Example 7.1.5. Then $(A_{\langle f,g\rangle})'$ is given as follows:

$$\begin{aligned} f' \quad &: \quad A \to \mathcal{FP}(X), \\ e \quad &\longmapsto \begin{cases} \{b_1/0.1, b_2/0.7, b_3/0.2, b_4/0.3, b_5/0.5\} & \text{if } e = e_1, \\ \{b_1/0.9, b_2/0.5, b_3/0.9, b_4/0.2, b_5/0.4\} & \text{if } e = e_2, \\ \{b_1/0.8, b_2/0.5, b_3/0.8, b_4/0.1, b_5/0.1\} & \text{if } e = e_3, \\ \{b_1/0.3, b_2/0.6, b_3/0.8, b_4/0.9, b_5/1.0\} & \text{if } e = e_6, \end{cases} \\ g' \quad &: \quad A \to \mathcal{FP}(X), \\ e \quad &\longmapsto \begin{cases} \{b_1/0.8, b_2/0.7, b_3/0.7, b_4/0.6, b_5/0.2\} & \text{if } e = e_1, \\ \{b_1/0.3, b_2/0.6, b_3/0.2, b_4/0.3, b_5/0.1\} & \text{if } e = e_2, \\ \{b_1/0.4, b_2/0.6, b_3/0.4, b_4/0.4, b_5/0.3\} & \text{if } e = e_3, \\ \{b_1/0.1, b_2/0.7, b_3/0.2, b_4/0.3, b_5/0.5\} & \text{if } e = e_6. \end{cases} \end{aligned}$$

but

$$(f'(e_1))(b_2) + (g'(e_1))(b_2) = 0.7 + 0.7 = 1.4 > 1$$

so $(A_{\langle f,g\rangle})' \notin \mathcal{FBSS}(X)^E$. Thus " $'$ " is not defined on $\mathcal{FBSS}(X)^E$.

7.3.9 Proposition

Let $A_{\langle f,g\rangle}$ be a fuzzy *bipolar* soft set over X. Then $^\circ : \mathcal{FBSS}(X)^E \to \mathcal{FBSS}(X)^E$ is defined and we denote $(A_{\langle f,g\rangle})^\circ$ by $A_{\langle f,g\rangle^\circ}$.

Proof. If $A_{\langle f,g\rangle} \in \mathcal{FBSS}(X)^E$ then

$A_{\langle f,g\rangle^\circ} \tilde{=} A_{\langle f^\circ,g^\circ\rangle}$ where $f^\circ : A \to \mathcal{FP}(X)$, $e \mapsto g(e)$ and $g^\circ : A \to \mathcal{FP}(X)$, $e \mapsto f(e)$.

Clearly

$$0 \leq (f^\circ(e))(x) + (g^\circ(\neg e))(x) \leq 1$$

Thus $A_{\langle f,g\rangle^\circ} \in \mathcal{FBSS}(X)^E$. ■

7.4 Properties of Fuzzy Bipolar Soft Sets

In this section we discuss properties of fuzzy bipolar soft sets with respect to their operations. Associativity, commutativity, absorption, distributivity and properties of fuzzy bipolar soft sets are investigated.

7.4.1 Definition

A *fuzzy bipolar* soft set over X is said to be a *relative absolute fuzzy bipolar soft set,* denoted by $A_{\langle \bar{\mathbf{1}},\bar{\mathbf{0}}\rangle}$ where

$$\bar{\mathbf{1}} : A \to \mathcal{FP}(X),\ e \mapsto \bar{\mathbf{1}} \ \text{ and } \ \bar{\mathbf{0}} : A \to \mathcal{FP}(X),\ e \mapsto \bar{\mathbf{0}}.$$

7.4.2 Definition

A *fuzzy bipolar soft set* over X is said to be a *relative null fuzzy bipolar soft set,* denoted by $A_{\langle \bar{\mathbf{0}},\bar{\mathbf{1}}\rangle}$ where

$$\bar{\mathbf{0}} : A \to \mathcal{FP}(X),\ e \mapsto \bar{\mathbf{0}} \ \text{ and } \ \bar{\mathbf{1}} : A \to \mathcal{FP}(X),\ e \mapsto \bar{\mathbf{1}}.$$

Conventionally, we take the fuzzy bipolar soft sets with empty set of parameters to be equal to $\emptyset_{\langle \bar{\mathbf{0}},\bar{\mathbf{1}}\rangle}$ and so $A_{\langle f,g\rangle} \sqcap B_{\langle h,i\rangle} \tilde{=} A_{\langle f,g\rangle} \sqcup B_{\langle h,i\rangle} \tilde{=} \emptyset_{\langle \bar{\mathbf{0}},\bar{\mathbf{1}}\rangle}$ whenever $(A \cap B) = \emptyset$.

7.4.3 Proposition

If $A_{\langle \bar{\mathbf{0}},\bar{\mathbf{1}}\rangle}$ is a null fuzzy bipolar soft set, $A_{\langle \bar{\mathbf{1}},\bar{\mathbf{0}}\rangle}$ an absolute fuzzy bipolar soft set, and $A_{\langle f,g\rangle}$, $A_{\langle h,i\rangle}$ are fuzzy bipolar soft sets over X, then

1) $A_{\langle f,g\rangle} \sqcup_\varepsilon A_{\langle h,i\rangle} \tilde{=} A_{\langle f,g\rangle} \sqcup A_{\langle h,i\rangle}$,

2) $A_{\langle f,g\rangle} \sqcap_\varepsilon A_{\langle h,i\rangle} \tilde{=} A_{\langle f,g\rangle} \sqcap A_{\langle h,i\rangle}$,

3) $A_{\langle f,g\rangle} \sqcap A_{\langle f,g\rangle} \tilde{=} A_{\langle f,g\rangle} \tilde{=} A_{\langle f,g\rangle} \sqcup A_{\langle f,g\rangle}$,

4) $A_{\langle f,g\rangle} \sqcup A_{\langle \bar{\mathbf{0}},\bar{\mathbf{1}}\rangle} \tilde{=} A_{\langle f,g\rangle} \tilde{=} A_{\langle f,g\rangle} \sqcap A_{\langle \bar{\mathbf{1}},\bar{\mathbf{0}}\rangle}$,

5) $A_{\langle f,g\rangle} \sqcup A_{\langle \bar{\mathbf{1}},\bar{\mathbf{0}}\rangle} \tilde{=} A_{\langle \bar{\mathbf{1}},\bar{\mathbf{0}}\rangle}$; $A_{\langle f,g\rangle} \sqcap A_{\langle \bar{\mathbf{0}},\bar{\mathbf{1}}\rangle} \tilde{=} A_{\langle \bar{\mathbf{0}},\bar{\mathbf{1}}\rangle}$.

Proof. Straightforward. ■

7.4.4 Proposition

Let $A_{\langle f,g\rangle}$, $B_{\langle h,i\rangle}$ and $C_{\langle j,k\rangle}$ be any *fuzzy bipolar soft sets* over X. Then the following are true

1) (Absorption Laws)

(i) $A_{\langle f,g\rangle} \sqcup_{\varepsilon} (B_{\langle h,i\rangle} \sqcap A_{\langle f,g\rangle}) \tilde{=} A_{\langle f,g\rangle}$,
(ii) $A_{\langle f,g\rangle} \sqcap (B_{\langle h,i\rangle} \sqcup_{\varepsilon} A_{\langle f,g\rangle}) \tilde{=} A_{\langle f,g\rangle}$,
(iii) $A_{\langle f,g\rangle} \sqcup (B_{\langle h,i\rangle} \sqcap_{\varepsilon} A_{\langle f,g\rangle}) \tilde{=} A_{\langle f,g\rangle}$,
(iv) $A_{\langle f,g\rangle} \sqcap_{\varepsilon} (B_{\langle h,i\rangle} \sqcup A_{\langle f,g\rangle}) \tilde{=} A_{\langle f,g\rangle}$.

2) (Associative Laws) $A_{\langle f,g\rangle} \lambda (B_{\langle h,i\rangle} \lambda C_{\langle j,k\rangle}) \tilde{=} (A_{\langle f,g\rangle} \lambda B_{\langle h,i\rangle}) \lambda C_{\langle j,k\rangle}$,

3) (Commutative Laws) $A_{\langle f,g\rangle} \lambda B_{\langle h,i\rangle} \tilde{=} B_{\langle h,i\rangle} \lambda A_{\langle f,g\rangle}$,

4) (Distributive Laws)(Distributive Laws)

(i) $A_{\langle f,g\rangle} \sqcup_{\varepsilon} (B_{\langle h,i\rangle} \sqcup C_{\langle j,k\rangle}) \tilde{\subseteq} (A_{\langle f,g\rangle} \sqcup_{\varepsilon} B_{\langle h,i\rangle}) \sqcup (A_{\langle f,g\rangle} \sqcup_{\varepsilon} C_{\langle j,k\rangle})$,
(ii) $A_{\langle f,g\rangle} \sqcup_{\varepsilon} (B_{\langle h,i\rangle} \sqcap_{\varepsilon} C_{\langle j,k\rangle}) \tilde{\supseteq} (A_{\langle f,g\rangle} \sqcup_{\varepsilon} B_{\langle h,i\rangle}) \sqcap_{\varepsilon} (A_{\langle f,g\rangle} \sqcup_{\varepsilon} C_{\langle j,k\rangle})$,
(iii) $A_{\langle f,g\rangle} \sqcup_{\varepsilon} (B_{\langle h,i\rangle} \sqcap C_{\langle j,k\rangle}) \tilde{=} (A_{\langle f,g\rangle} \sqcup_{\varepsilon} B_{\langle h,i\rangle}) \sqcap (A_{\langle f,g\rangle} \sqcup_{\varepsilon} C_{\langle j,k\rangle})$,
(iv) $A_{\langle f,g\rangle} \sqcup (B_{\langle h,i\rangle} \sqcup_{\varepsilon} C_{\langle j,k\rangle}) \tilde{=} (A_{\langle f,g\rangle} \sqcup B_{\langle h,i\rangle}) \sqcup_{\varepsilon} (A_{\langle f,g\rangle} \sqcup C_{\langle j,k\rangle})$,
(v) $A_{\langle f,g\rangle} \sqcup (B_{\langle h,i\rangle} \sqcap_{\varepsilon} C_{\langle j,k\rangle}) \tilde{=} (A_{\langle f,g\rangle} \sqcup B_{\langle h,i\rangle}) \sqcap_{\varepsilon} (A_{\langle f,g\rangle} \sqcup C_{\langle j,k\rangle})$,
(vi) $A_{\langle f,g\rangle} \sqcup (B_{\langle h,i\rangle} \sqcap C_{\langle j,k\rangle}) \tilde{=} (A_{\langle f,g\rangle} \sqcup B_{\langle h,i\rangle}) \sqcap (A_{\langle f,g\rangle} \sqcup C_{\langle j,k\rangle})$,
(vii) $A_{\langle f,g\rangle} \sqcap_{\varepsilon} (B_{\langle h,i\rangle} \sqcup_{\varepsilon} C_{\langle j,k\rangle}) \tilde{\subseteq} (A_{\langle f,g\rangle} \sqcap_{\varepsilon} B_{\langle h,i\rangle}) \sqcup_{\varepsilon} (A_{\langle f,g\rangle} \sqcap_{\varepsilon} C_{\langle j,k\rangle})$,
(viii) $A_{\langle f,g\rangle} \sqcap_{\varepsilon} (B_{\langle h,i\rangle} \sqcup C_{\langle j,k\rangle}) \tilde{=} (A_{\langle f,g\rangle} \sqcap_{\varepsilon} B_{\langle h,i\rangle}) \sqcup (A_{\langle f,g\rangle} \sqcap_{\varepsilon} C_{\langle j,k\rangle})$,
(ix) $A_{\langle f,g\rangle} \sqcap_{\varepsilon} (B_{\langle h,i\rangle} \sqcap C_{\langle j,k\rangle}) \tilde{\supseteq} (A_{\langle f,g\rangle} \sqcap_{\varepsilon} B_{\langle h,i\rangle}) \sqcap (A_{\langle f,g\rangle} \sqcap_{\varepsilon} C_{\langle j,k\rangle})$,
(x) $A_{\langle f,g\rangle} \sqcap (B_{\langle h,i\rangle} \sqcup_{\varepsilon} C_{\langle j,k\rangle}) \tilde{=} (A_{\langle f,g\rangle} \sqcap B_{\langle h,i\rangle}) \sqcup_{\varepsilon} (A_{\langle f,g\rangle} \sqcap C_{\langle j,k\rangle})$,
(xi) $A_{\langle f,g\rangle} \sqcap (B_{\langle h,i\rangle} \sqcap_{\varepsilon} C_{\langle j,k\rangle}) \tilde{=} (A_{\langle f,g\rangle} \sqcap B_{\langle h,i\rangle}) \sqcap_{\varepsilon} (A_{\langle f,g\rangle} \sqcap C_{\langle j,k\rangle})$,
(xii) $A_{\langle f,g\rangle} \sqcap (B_{\langle h,i\rangle} \sqcup C_{\langle j,k\rangle}) \tilde{=} (A_{\langle f,g\rangle} \sqcap B_{\langle h,i\rangle}) \sqcup (A_{\langle f,g\rangle} \sqcap C_{\langle j,k\rangle})$.

Proof. From Theorem 7.2.4, it is easy to see that these properties hold as for double-framed fuzzy soft sets ■

7.4.5 Example

Let X be the set of houses under consideration, and E be the set of parameters, $X = \{h_1,h_2,h_3,h_4,h_5\}$, $E = \{e_1,e_2,e_3,e_4,e_5\}$ = { in the green surroundings, cheap, in good repair, furnished, traditional }. Let $\neg E = \{\neg e_1,\neg e_2,\neg e_3,\neg e_4,\neg e_5\}$ = { in the commercial area, expensive, in bad repair, non-furnished, modern }. Suppose that $A = \{e_1,e_2,e_3\}$, $B = \{e_2,e_3,e_4\}$, and $C = \{e_3,e_4,e_5\}$. The *fuzzy bipolar soft sets* $A_{\langle f,g\rangle}$

and $B_{\langle h,i\rangle}$ and $C_{\langle j,k\rangle}$ describe the "requirements of the houses" which Mr. X, Mr. Y and Mr. Z are going to buy respectively. Suppose that

$$\begin{aligned} f &: A \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{x_1/0.4, x_2/0.7, x_3/0.7, x_4/0.7, x_5/0.1\} & \text{if } e = e_1, \\ \{x_1/0.8, x_2/0.0, x_3/0.5, x_4/0.1, x_5/0.6\} & \text{if } e = e_2, \\ \{x_1/0.7, x_2/0.5, x_3/0.7, x_4/0.6, x_5/0.1\} & \text{if } e = e_3. \end{cases} \\ g &: A \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{x_1/0.3, x_2/0.1, x_3/0.3, x_4/0.1, x_5/0.7\} & \text{if } e = e_1, \\ \{x_1/0.1, x_2/0.9, x_3/0.3, x_4/0.8, x_5/0.2\} & \text{if } e = e_2, \\ \{x_1/0.1, x_2/0.3, x_3/0.3, x_4/0.3, x_5/0.8\} & \text{if } e = e_3, \end{cases} \end{aligned}$$

$$\begin{aligned} h &: B \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{x_1/0.1, x_2/0.3, x_3/0.3, x_4/0.6, x_5/0.6\} & \text{if } e = e_2, \\ \{x_1/0.1, x_2/0, x_3/0.3, x_4/0.4, x_5/0.6\} & \text{if } e = e_3, \\ \{x_1/0.9, x_2/0.5, x_3/0.5, x_4/0.3, x_5/0.1\} & \text{if } e = e_4. \end{cases} \\ i &: B \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{x_1/0.1, x_2/0.3, x_3/0.6, x_4/0.2, x_5/0.3\} & \text{if } e = e_2, \\ \{x_1/0.8, x_2/0.9, x_3/0.5, x_4/0.4, x_5/0.2\} & \text{if } e = e_3, \\ \{x_1/0.1, x_2/0.4, x_3/0.3, x_4/0.6, x_5/0.9\} & \text{if } e = e_4, \end{cases} \\ j &: C \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{x_1/0.7, x_2/0.7, x_3/0.4, x_4/0.7, x_5/0.4\} & \text{if } e = e_3, \\ \{x_1/0.6, x_2/0.5, x_3/0.6, x_4/0.1, x_5/0.6\} & \text{if } e = e_4, \\ \{x_1/0.3, x_2/0.4, x_3/0.4, x_4/0.3, x_5/0.1\} & \text{if } e = e_5. \end{cases} \\ k &: C \to \mathcal{FP}(X), \\ e &\longmapsto \begin{cases} \{x_1/0.1, x_2/0.2, x_3/0.3, x_4/0.1, x_5/0.1\} & \text{if } e = e_3, \\ \{x_1/0.2, x_2/0.2, x_3/0.3, x_4/0.3, x_5/0.2\} & \text{if } e = e_4, \\ \{x_1/0.1, x_2/0.1, x_3/0.3, x_4/0.5, x_5/0.7\} & \text{if } e = e_5, \end{cases} \end{aligned}$$

Let

$$A_{\langle f,g\rangle} \sqcup_\varepsilon (B_{\langle h,i\rangle} \sqcap_\varepsilon C_{\langle j,k\rangle}) \doteq (A \cup B) \cup C_{\langle f\tilde{\vee}(h\tilde{\wedge}j), g\tilde{\wedge}(i\tilde{\vee}k)\rangle}$$

and

$$(A_{\langle f,g\rangle} \sqcup_\varepsilon B_{\langle h,i\rangle}) \sqcap_\varepsilon (A_{\langle f,g\rangle} \sqcup_\varepsilon C_{\langle j,k\rangle}) \doteq (A \cup B) \cup C_{\langle (f\tilde{\vee}h)\tilde{\wedge}(f\tilde{\vee}j)\rangle}.$$

Then

$$\begin{aligned} (f\tilde{\vee}(h\tilde{\wedge}j))(e_2) &= \{x_1/0.1, x_2/0.0, x_3/0.3, x_4/0.1, x_5/0.6\} \\ &\neq \{x_1/0.8, x_2/0.0, x_3/0.5, x_4/0.1, x_5/0.6\} \\ &= ((f\tilde{\vee}h)\tilde{\wedge}(f\tilde{\vee}j))(e_2) \qquad \text{and} \\ (g\tilde{\wedge}(i\tilde{\vee}k))(e_2) &= \{x_1/0.1, x_2/0.9, x_3/0.6, x_4/0.8, x_5/0.3\} \\ &\neq \{x_1/0.1, x_2/0.9, x_3/0.3, x_4/0.8, x_5/0.2\} \\ &= ((g\tilde{\wedge}i)\tilde{\vee}(g\tilde{\wedge}k))(e_2), \end{aligned}$$

so that

$$A_{\langle f,g\rangle} \sqcup_\varepsilon (B_{\langle h,i\rangle} \sqcap_\varepsilon C_{\langle j,k\rangle}) \tilde{\neq} (A_{\langle f,g\rangle} \sqcup_\varepsilon B_{\langle h,i\rangle}) \sqcap_\varepsilon (A_{\langle f,g\rangle} \sqcup_\varepsilon C_{\langle j,k\rangle}).$$

Now,

$$A_{\langle f,g\rangle} \sqcap_\varepsilon (B_{\langle h,i\rangle} \sqcup_\varepsilon C_{\langle j,k\rangle}) \tilde{=} (A \cup B) \cup C_{\langle f\tilde{\wedge}(h\tilde{\vee}j), g\tilde{\vee}(i\tilde{\wedge}k)\rangle}$$

and

$$(A_{\langle f,g\rangle} \sqcap_\varepsilon B_{\langle h,i\rangle}) \sqcup_\varepsilon (A_{\langle f,g\rangle} \sqcap_\varepsilon C_{\langle j,k\rangle}) \tilde{=} (A \cup B) \cup C_{\langle (f\tilde{\wedge}h)\tilde{\vee}(f\tilde{\wedge}j), (g\tilde{\vee}i)\tilde{\wedge}(g\tilde{\vee}k)\rangle}.$$

Then,

$$\begin{aligned} (f\tilde{\wedge}(h\tilde{\vee}j))(e_2) &= \{x_1/0.8, x_2/0.3, x_3/0.5, x_4/0.6, x_5/0.6\} \\ &\neq \{x_1/0.8, x_2/0.0, x_3/0.5, x_4/0.1, x_5/0.6\} \\ &= ((f\tilde{\wedge}h)\tilde{\vee}(f\tilde{\wedge}j))(e_2) \end{aligned}$$

and

$$\begin{aligned} (g\tilde{\vee}(i\tilde{\wedge}k))(e_2) &= \{x_1/0.1, x_2/0.3, x_3/0.3, x_4/0.2, x_5/0.2\} \\ &\neq \{x_1/0.1, x_2/0.9, x_3/0.3, x_4/0.8, x_5/0.2\} \\ &= ((g\tilde{\vee}i)\tilde{\wedge}(g\tilde{\vee}k))(e_2). \end{aligned}$$

So that

$$A_{\langle f,g\rangle} \sqcap_\varepsilon (B_{\langle h,i\rangle} \sqcup_\varepsilon C_{\langle j,k\rangle}) \tilde{\neq} (A_{\langle f,g\rangle} \sqcap_\varepsilon B_{\langle h,i\rangle}) \sqcup_\varepsilon (A_{\langle f,g\rangle} \sqcap_\varepsilon C_{\langle j,k\rangle}).$$

Similarly we can show that

$$A_{\langle f,g\rangle} \sqcup_\varepsilon (B_{\langle h,i\rangle} \sqcup C_{\langle j,k\rangle}) \tilde{\neq} (A_{\langle f,g\rangle} \sqcup_\varepsilon B_{\langle h,i\rangle}) \sqcup (A_{\langle f,g\rangle} \sqcup_\varepsilon C_{\langle j,k\rangle}),$$

and

$$A_{\langle f,g\rangle} \sqcap_\varepsilon (B_{\langle h,i\rangle} \sqcap C_{\langle j,k\rangle}) \tilde{\neq} (A_{\langle f,g\rangle} \sqcap_\varepsilon B_{\langle h,i\rangle}) \sqcap (A_{\langle f,g\rangle} \sqcap_\varepsilon C_{\langle j,k\rangle}).$$

7.4.6 Corollary

Let $A_{\langle f,g\rangle}$, $B_{\langle h,i\rangle}$ and $C_{\langle j,k\rangle}$ be three *fuzzy bipolar soft sets* over X such that $(A \cap B) - C = (A \cap C) - B = \emptyset$. Then

1)

$$A_{\langle f,g\rangle} \sqcup_\varepsilon (B_{\langle h,i\rangle} \sqcap_\varepsilon C_{\langle j,k\rangle}) \tilde{=} (A_{\langle f,g\rangle} \sqcup_\varepsilon B_{\langle h,i\rangle}) \sqcap_\varepsilon (A_{\langle f,g\rangle} \sqcup_\varepsilon C_{\langle j,k\rangle}),$$

2)

$$A_{\langle f,g\rangle} \sqcap_\varepsilon (B_{\langle h,i\rangle} \sqcup_\varepsilon C_{\langle j,k\rangle}) \tilde{=} (A_{\langle f,g\rangle} \sqcap_\varepsilon B_{\langle h,i\rangle}) \sqcup_\varepsilon (A_{\langle f,g\rangle} \sqcap_\varepsilon C_{\langle j,k\rangle}).$$

7.4.7 Corollary

Let $A_{\langle f,g\rangle}$, $A_{\langle h,i\rangle}$ and $A_{\langle j,k\rangle}$ be any *fuzzy bipolar soft sets* over X. Then

$$A_{\langle f,g\rangle} \lambda (A_{\langle h,i\rangle} \rho A_{\langle j,k\rangle}) \tilde{=} (A_{\langle f,g\rangle} \lambda A_{\langle h,i\rangle}) \rho (A_{\langle f,g\rangle} \lambda A_{\langle j,k\rangle})$$

for distinct λ, $\rho \in \{\sqcap_\varepsilon, \sqcap, \sqcup_\varepsilon, \sqcup\}$.

7.4.8 Proposition

Let $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ be two *fuzzy bipolar soft sets* over X. Then the following are true

1) $A_{\langle f,g\rangle} \sqcup_{\varepsilon} B_{\langle h,i\rangle}$ is the smallest *fuzzy bipolar* soft set over X which contains both $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$. (Supremum)

2) $A_{\langle f,g\rangle} \sqcap B_{\langle h,i\rangle}$ is the largest *fuzzy bipolar* soft set over X which is contained in both $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$. (Infimum)

Proof. Straightforward. ■

7.5 Algebras of Fuzzy Bipolar Soft Sets

Now we consider the collection of all fuzzy bipolar soft sets over X and denote it by $\mathcal{FBSS}(X)^E$ and let us denote its sub collection of all fuzzy bipolar soft sets over X with fixed set of parameters A by $\mathcal{FBSS}(X)_A$. We note that this collection is partially ordered by inclusion. We conclude from above results that:

7.5.1 Proposition

$(\mathcal{FBSS}(X)^E, \sqcap_{\varepsilon}, \sqcup)$ and $(\mathcal{FBSS}(X)^E, \sqcup_{\varepsilon}, \sqcap)$ are distributive lattices and $(\mathcal{FBSS}(X)^E, \sqcup, \sqcap_{\varepsilon})$ and $(\mathcal{FBSS}(X)^E, \sqcap, \sqcup_{\varepsilon})$ are their duals, respectively.

Proof. Follows from Propositions 7.4.3 and 7.4.4. ■

7.5.2 Proposition

$(\mathcal{FBSS}(X)^E, \sqcap, \sqcup_{\varepsilon}, \emptyset_{\langle \Phi,\mathfrak{x}\rangle}, E_{\langle \mathfrak{x},\Phi\rangle})$, $(\mathcal{FBSS}(X)^E, \sqcup_{\varepsilon}, \sqcap, E_{\langle \mathfrak{x},\Phi\rangle}, \emptyset_{\langle \Phi,\mathfrak{x}\rangle})$, $(\mathcal{FBSS}(X)_A, \sqcap, \sqcup, A_{\langle \Phi,\mathfrak{x}\rangle}, A_{\langle \mathfrak{x},\Phi\rangle})$ and $(\mathcal{FBSS}(X)_A, \sqcup, \sqcap, A_{\langle \mathfrak{x},\Phi\rangle}, A_{\langle \Phi,\mathfrak{x}\rangle})$ are bounded distributive lattices.

Proof. From Proposition 7.4.8, we know that $(\mathcal{FBSS}(X)^E, \sqcap, \sqcup_{\varepsilon}, \emptyset_{\langle \Phi,\mathfrak{x}\rangle}, E_{\langle \mathfrak{x},\Phi\rangle})$ is a bounded distributive lattice and $(\mathcal{FBSS}(X)^E, \sqcup_{\varepsilon}, \sqcap, E_{\langle \mathfrak{x},\Phi\rangle}, \emptyset_{\langle \Phi,\mathfrak{x}\rangle})$ is its dual. For any fuzzy bipolar soft sets $A_{\langle f,g\rangle}, A_{\langle h,i\rangle} \in \mathcal{FBSS}(X)_A$,

$$
\begin{aligned}
A_{\langle f,g\rangle} \sqcap A_{\langle h,i\rangle} &\hat{=} A_{\langle f\tilde{\wedge}h, g\tilde{\vee}i\rangle} \in \mathcal{FBSS}(X)_A \text{ and} \\
A_{\langle f,g\rangle} \sqcup A_{\langle h,i\rangle} &\hat{=} A_{\langle f\tilde{\vee}h, g\tilde{\wedge}i\rangle} \in \mathcal{FBSS}(X)_A.
\end{aligned}
$$

Thus $(\mathcal{FBSS}(X)_A, \sqcap, \sqcup)$ is also a distributive sublattice of $(\mathcal{FBSS}(X)^E, \sqcup_{\varepsilon}, \sqcap)$ and Proposition 7.4.3 shows that $(\mathcal{FBSS}(X)_A, \sqcap, \sqcup, A_{\langle \Phi,\mathfrak{x}\rangle}, A_{\langle \mathfrak{x},\Phi\rangle})$ is a bounded distributive lattice and $(\mathcal{FBSS}(X)_A, \sqcup, \sqcap, A_{\langle \mathfrak{x},\Phi\rangle}, A_{\langle \Phi,\mathfrak{x}\rangle})$ is its dual. ■

7.5.3 Proposition (de Morgan Laws)

Let $A_{\langle f,g\rangle}$ and $B_{\langle h,i\rangle}$ be two *fuzzy bipolar* soft sets over X. Then the following are true

1) $(A_{\langle f,g\rangle} \sqcup_{\varepsilon} B_{\langle h,i\rangle})^{\circ} \hat{=} A_{\langle f,g\rangle^{\circ}} \sqcap_{\varepsilon} B_{\langle h,i\rangle^{\circ}}$,

2) $(A_{\langle f,g\rangle} \sqcap_{\varepsilon} B_{\langle h,i\rangle})^{\circ} \hat{=} A_{\langle f,g\rangle^{\circ}} \sqcup_{\varepsilon} B_{\langle h,i\rangle^{\circ}}$,

3) $(A_{\langle f,g\rangle} \vee B_{\langle h,i\rangle})^{\circ} \tilde{=} A_{\langle f,g\rangle^{\circ}} \wedge B_{\langle h,i\rangle^{\circ}}$,

4) $(A_{\langle f,g\rangle} \wedge B_{\langle h,i\rangle})^{\circ} \tilde{=} A_{\langle f,g\rangle^{\circ}} \vee B_{\langle h,i\rangle^{\circ}}$,

5) $(A_{\langle f,g\rangle} \sqcup B_{\langle h,i\rangle})^{\circ} \tilde{=} A_{\langle f,g\rangle^{\circ}} \sqcap B_{\langle h,i\rangle^{\circ}}$,

6) $(A_{\langle f,g\rangle} \sqcap B_{\langle h,i\rangle})^{\circ} \tilde{=} A_{\langle f,g\rangle^{\circ}} \sqcup B_{\langle h,i\rangle^{\circ}}$.

Proof.

1) We have

$$(A_{\langle f,g\rangle} \sqcup_{\varepsilon} B_{\langle h,i\rangle})^{\circ} \tilde{=} ((A \cup B)_{\langle f \tilde{\vee} h, g \tilde{\wedge} i\rangle})^{\circ} \tilde{=} (A \cup B)_{\langle g \tilde{\wedge} i, f \tilde{\vee} h\rangle}$$

and

$$A_{\langle f,g\rangle^{\circ}} \sqcap_{\varepsilon} B_{\langle h,i\rangle^{\circ}} \tilde{=} A_{\langle g,f\rangle} \sqcap_{\varepsilon} B_{\langle i,h\rangle} \tilde{=} (A \cup B)_{\langle g \tilde{\wedge} i, f \tilde{\vee} h\rangle}.$$

Thus

$$(A_{\langle f,g\rangle} \sqcup_{\varepsilon} B_{\langle h,i\rangle})^{\circ} \tilde{=} A_{\langle f,g\rangle^{\circ}} \sqcap_{\varepsilon} B_{\langle h,i\rangle^{\circ}}.$$

The remaining parts can be proved in a similar way.

■

7.5.4 Proposition

$(\mathcal{FBSS}(X)_A, \sqcap, \sqcup, {}^{\circ}, A_{\langle \Phi, \mathfrak{X}\rangle}, A_{\langle \mathfrak{X}, \Phi\rangle})$ is a de Morgan algebra.

Proof. Proof follows from Propositions 7.3.9 and 7.5.3. ■

7.5.5 Definition

Let $A_{\langle f,g\rangle}$ be a *fuzzy bipolar* soft set over X. We define $A_{\langle f,g\rangle^*}$ as a *fuzzy bipolar soft set* $A_{\langle f^*,g^*\rangle}$ where

$$\begin{aligned}
f^* &: A \to \mathcal{FP}(X),\ e \mapsto (f(e))^*, \\
(f(e))^*(x) &= \begin{cases} 0 & \text{if } (f(e))^*(x) \neq 0 \\ 1 & \text{if } (f(e))^*(x) = 0 \end{cases} \\
g^* &: A \to \mathcal{FP}(X),\ e \mapsto (g(e))^*, \\
(g(e))^*(x) &= \begin{cases} 1 & \text{if } (g(e))^*(x) \neq 1 \\ 0 & \text{if } (g(e))^*(x) = 1 \end{cases} \quad \text{for } x \in X.
\end{aligned}$$

7.2.4.

7.5.6 Theorem

$(\mathcal{FBSS}(X)_A, \sqcap, \sqcup, {}^*, A_{\langle \Phi, \mathfrak{X}\rangle}, A_{\langle \mathfrak{X}, \Phi\rangle})$ is a Stone algebra.

Proof. From Proposition 7.5.2 it is evident that $(\mathcal{FBSS}(X)_A, \sqcap, \sqcup, A_{\langle \Phi, \mathfrak{X}\rangle}, A_{\langle \mathfrak{X}, \Phi\rangle})$ is a bounded distributive lattice and $A_{\langle f,g\rangle^*} \tilde{=} \theta(A_{\langle f,g\rangle^*})$ where θ is mapping defined in Theorem 7.2.4 assures that * is a pseudocomplementing function satisfying Stone's identity. Thus $(\mathcal{FBSS}(X)_A, \sqcap, \sqcup, {}^*, A_{\langle \Phi, \mathfrak{X}\rangle}, A_{\langle \mathfrak{X}, \Phi\rangle})$ is a Stone algebra. ■

Chapter 8

A Generalized Framework for Soft Set Theory

This chapter is more of a collective nature than the previous ones and not only summarizes the main results but also provides a general framework to deal with soft sets in a logical manner. We have given an over all review of various kinds of soft sets. A brief discussion about defining ideas of extended soft sets and their operations, a summary of algebraic structures and an application of soft sets in decision making problems has been made in this chapter to conclude thesis here. We initiate discussion with definition of soft sets.

8.1 General Definition of Soft Set and its Extensions:

Let X be an initial universe and E be a set of parameters. Let $\lambda\mathcal{P}(X)$ be a generalized fuzzy power set of X where $\lambda\mathcal{P}(X)$ may be a collection of all crisp or fuzzy or type-2 fuzzy or n-fuzzy or hesitant fuzzy or interval-valued fuzzy or vague or intuitionistic fuzzy or bipolar fuzzy subsets of X and, say, λ stands for a fuzzy criteria of collection $\lambda\mathcal{P}(X)$.

- A mapping $f : A \rightarrow \lambda\mathcal{P}(X)$ is called a $\lambda-$*soft set* over X denoted by A_f where $A \subseteq E$. We note that parameters in E can be a specific criteria for which an approximation of elements of X is made by f, so a $\lambda-$*soft set* over X gives a parameterized family of $\lambda-$subsets of X.

- In our next step towards a general framework for soft sets, we allow to consider more than one frames of reference for X within the context of each parameter. This consideration requires some modifications in the ongoing soft set based model and so, this requirement is fulfilled by introducing a set of functions $f_i : A \rightarrow \lambda\mathcal{P}(X)$, $i = 1, 2, ..., n$ and denote it by $A_{(f_1,f_2,...,f_n)}$ and call it an n-*framed* $\lambda-$*soft set over* X. Clearly, an n-*framed* $\lambda-$*soft set* gives n parametrized families of $\lambda-$subsets of X.

- Now, if the frames of references are mutually exclusive or obeying some other mutual relation which is causing a polarity among those, then we incorporate the idea by imposing a suitably chosen set of consistency constraints $\mathcal{C}$. Hence we give the concept of λ *n-polar soft set* over X comprising of functions $f_i : A \to \lambda\mathcal{P}(X)$, $(f_i \in \mathcal{C}) \quad i = 1, 2, ..., n$ denoted by $A_{\langle f_1, f_2, ..., f_n\rangle}$.

In a natural way, all λ *multi-polar soft sets* are *multi-framed $\lambda-$soft sets* over X but the converse is not true. It is also interesting to observe that *multi-polar λ soft sets* can be presented in an equivalent and better way by using λ *multi-polar soft sets.* A particular case for $n = 2$ is already discussed in Chapter 7 for fuzzy subsets of X.

8.2 Aggregation Operators for Soft Sets in General Form

We need to apply a process for aggregation where the number of inputs are grouped together in order to get a single output that is easier to use for further computations. Usually when an object or an alternative is characterized by several numbers or values describing its various parameters or is given evaluations from several experts and one has to aggregate these values in order to describe the object by just one meaningful value or set of values. Aggregation operators are an important tool that is used in many domains [6], [8]. For a soft set and its hybrid generalizations and extensions, an input space for aggregation is a bit unconventional because it is required to deal each object in a parametrized context. Therefore a soft aggregation operator is a function working on a particular number of inputs for each parameter, with output lying again in a parametrized manner. We define soft aggregation operators in either restricted or extended context. A restricted soft aggregation operator joins two soft sets with a restricted set of parameters, that is, only those parameters which are combined to both and mathematically the set of parameters is taken as the intersection of parameters sets in input soft sets. On the other hand, an extended soft aggregation operator joins two soft sets with an extended set of parameters, that is, all those parameters apparent are taken into consideration and mathematically the set of parameters in output is union of parameters sets in input soft sets. Let m be a positive integer and K be a set of various operations defined for λ fuzzy subsets of X.

- Let $A_i, B \subseteq E$ and $A_{i_{f_i}}$ be $\lambda-$*soft sets* over X, where $i = 1, 2, ..., m$. Then an aggregation operator is a mapping $(A_{1_{f_1}}, A_{2_{f_2}}, ..., A_{m_{f_m}}) \mapsto B_g$. We have two cases:

  (i) For the case of restricted aggregation operators, we have $B = \bigcap_{i=1}^{m} A_i$ and
  $$g(e) = k\{f_i(e) : i = 1, 2, ..., m\}$$
  for all $e \in B$.

  (ii) For the case of extended aggregation operators, we have $B = \bigcup_{i=1}^{m} A_i$ and we define the set $\Lambda(e) = \{j : e \in A_j\}$
  $$g(e) = k\{f_i(e) : i \in \Lambda(e)\}$$
  for all $e \in B$.

- Let $A_i, B \subseteq E$ and $A_{i_{(f_{i1},f_{i2},...,f_{in})}}$ be *n-framed $\lambda-$soft sets* over X, where $i = 1$, 2, ..., m, and $(k_1, k_2, ..., k_n) \in K^n$. Then an aggregation operator is a mapping $(A_{1_{(f_{11},f_{12},...,f_{1n})}}, A_{2_{(f_{21},f_{22},...,f_{2n})}}, ..., A_{m_{(f_{m1},f_{m2},...,f_{mn})}}) \mapsto B_{(g_1,g_2,...,g_n)}$. We have two cases:

  (i) For the case of restricted aggregation operators, we have $B = \bigcap_{i=1}^{m} A_i$ and

  $$g_j(e) = k_j\{f_{ij}(e) : i = 1, 2, ..., m\},\ j = 1, 2, ..., n$$

  for all $e \in B$.

  (ii) For the case of extended aggregation operators, we have $B = \bigcup_{i=1}^{m} A_i$ and we define the set $\Lambda(e) = \{j : e \in A_j\}$

  $$g_j(e) = k_j\{f_{ij}(e) : i \in \Lambda(e)\},\ j = 1, 2, ..., n$$

  for all $e \in B$.

- Let $A_i, B \subseteq E$ and $A_{i_{\langle f_{i1},f_{i2},...,f_{in}\rangle}}$ $(f_{ij} \in \mathcal{C})$ be *λ n-polar soft sets* over X where $i = 1$, 2, ..., m, and $(k_1, k_2, ..., k_n) \in K^n$. Then an aggregation operator is a mapping $(A_{1_{\langle f_{11},f_{12},...,f_{1n}\rangle}}, A_{2_{\langle f_{21},f_{22},...,f_{2n}\rangle}}, ..., A_{m_{\langle f_{m1},f_{m2},...,f_{mn}\rangle}}) \mapsto B_{\langle g_1,g_2,...,g_n\rangle}$ $(g_j \in \mathcal{C})$. We have two cases:

  (i) For the case of restricted aggregation operators, we have $B = \bigcap_{i=1}^{m} A_i$ and

  $$g_j(e) = k_j\{f_{ij}(e) : i = 1, 2, ..., m\},\ j = 1, 2, ..., n$$

  for all $e \in B$.

  (ii) For the case of extended aggregation operators, we have $B = \bigcup_{i=1}^{m} A_i$ and we define the set $\Lambda(e) = \{j : e \in A_j\}$

  $$g_j(e) = k_j\{f_{ij}(e) : i \in \Lambda(e)\},\ j = 1, 2, ..., n$$

  for all $e \in B$.

All aggregation operators defined for *n-framed $\lambda-$soft sets* over X can be used to define aggregation operators for *λ n-polar soft sets* over X as except where consistency constraints are absent. We have seen an example of complement operation defined for double-framed soft sets which is no more available for bipolar soft sets due to hazard of consistency constraint. Thus the set of aggregation operators for *λ n-polar soft sets* is contained in the set of aggregation operators for *n-framed $\lambda-$soft sets*.

8.3 New Examples of Logical Algebraic Structures

In this section we present a summary of results that we have found in our research regarding different types of soft sets and their collections and thus new examples of these algebras are contributed through our work. Following table gives an overview of the algebraic structures of soft sets:

1	Lattices:
	$(\mathcal{SS}(X)^E, \sqcap_\varepsilon, \sqcup), (\mathcal{SS}(X)^E, \sqcup, \sqcap_\varepsilon), (\mathcal{FSS}(X)^E, \sqcap_\varepsilon, \sqcup),$ $(\mathcal{FSS}(X)^E, \sqcup, \sqcap_\varepsilon), (\mathcal{DSS}(X)^E, \sqcap_\varepsilon, \sqcup), (\mathcal{DSS}(X)^E, \sqcup, \sqcap_\varepsilon),$ $(\mathcal{DFSS}(X)^E, \sqcap_\varepsilon, \sqcup), (\mathcal{DFSS}(X)^E, \sqcup, \sqcap_\varepsilon), (\mathcal{BSS}(X)^E, \sqcap_\varepsilon, \sqcup),$ $(\mathcal{BSS}(X)^E, \sqcup, \sqcap_\varepsilon), (\mathcal{FBSS}(X)^E, \sqcap_\varepsilon, \sqcup), (\mathcal{FBSS}(X)^E, \sqcup, \sqcap_\varepsilon)$
2	Bounded Distributive Lattices:
	$(\mathcal{SS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_\Phi, E_\mathfrak{X}), (\mathcal{SS}(X)^E, \sqcup_\varepsilon, \sqcap, E_\mathfrak{X}, \emptyset_\Phi),$ $(\mathcal{FSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_\Phi, E_\mathfrak{X}), (\mathcal{FSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_\mathfrak{X}, \emptyset_\Phi),$ $(\mathcal{DSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{(\Phi,\mathfrak{X})}, E_{(\mathfrak{X},\Phi)}), (\mathcal{DSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{(\mathfrak{X},\Phi)}, \emptyset_{(\Phi,\mathfrak{X})}),$ $(\mathcal{DFSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{(\Phi,\mathfrak{X})}, E_{(\mathfrak{X},\Phi)}), (\mathcal{DSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{(\mathfrak{X},\Phi)}, \emptyset_{(\Phi,\mathfrak{X})}),$ $(\mathcal{BSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{\langle\Phi,\mathfrak{X}\rangle}, E_{\langle\mathfrak{X},\Phi\rangle}), (\mathcal{BSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{\langle\mathfrak{X},\Phi\rangle}, \emptyset_{\langle\Phi,\mathfrak{X}\rangle}),$ $(\mathcal{FBSS}(X)^E, \sqcap, \sqcup_\varepsilon, \emptyset_{\langle\Phi,\mathfrak{X}\rangle}, E_{\langle\mathfrak{X},\Phi\rangle}), (\mathcal{FBSS}(X)^E, \sqcup_\varepsilon, \sqcap, E_{\langle\mathfrak{X},\Phi\rangle}, \emptyset_{\langle\Phi,\mathfrak{X}\rangle})$
3	De Morgan Algebras:
	$(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^\circ, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)}), (\mathcal{DSS}(X)_A, \sqcup, \sqcap, ^\circ, A_{(\mathfrak{X},\Phi)}, A_{(\Phi,\mathfrak{X})})$ $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ^\circ, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)}), (\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ^\circ, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$ $(\mathcal{FBSS}(X)_A, \sqcap, \sqcup, ^\circ, A_{\langle\Phi,\mathfrak{X}\rangle}, A_{\langle\mathfrak{X},\Phi\rangle}), (\mathcal{FBSS}(X)_A, \sqcup, \sqcap, ^\circ, A_{\langle\mathfrak{X},\Phi\rangle}, A_{\langle\Phi,\mathfrak{X}\rangle}),$
4	Boolean Algebras:
	$(\mathcal{SS}(X)_A, \sqcap, \sqcup, ^c, A_\Phi, A_\mathfrak{X}), (\mathcal{SS}(X)_A, \sqcup, \sqcap, ^c, A_\mathfrak{X}, A_\Phi),$ $(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^c, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)}), (\mathcal{DSS}(X)_A, \sqcup, \sqcap, ^c, A_{(\mathfrak{X},\Phi)}, A_{(\Phi,\mathfrak{X})}),$
5	Kleene Algebras:
	$(\mathcal{FSS}(X)_A, \sqcap, \sqcup, ', A_\Phi, A_\mathfrak{X}), (\mathcal{FSS}(X)_A, \sqcup, \sqcap, ', A_\mathfrak{X}, A_\Phi),$ $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ', A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)}), (\mathcal{DFSS}(X)_A, \sqcup, \sqcap, ', A_{(\mathfrak{X},\Phi)}, A_{(\Phi,\mathfrak{X})}),$ $(\mathcal{BSS}(X)_A, \sqcap, \sqcup, ^\circ, A_{\langle\Phi,\mathfrak{X}\rangle}, A_{\langle\mathfrak{X},\Phi\rangle}), (\mathcal{BSS}(X)_A, \sqcup, \sqcap, ^\circ, A_{\langle\mathfrak{X},\Phi\rangle}, A_{\langle\Phi,\mathfrak{X}\rangle}),$
6	Pseudocomplemented Lattices:
	$(\mathcal{DSS}(X)_A, \sqcap, \sqcup, ^\diamond, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)})$
7	Stone Algebras:
	$(\mathcal{FSS}(X)_A, \sqcap, \sqcup, ^*, A_\Phi, A_\mathfrak{X}), (\mathcal{DFSS}(X)_A, \sqcap, \sqcup, ^*, A_{(\Phi,\mathfrak{X})}, A_{(\mathfrak{X},\Phi)}),$ $(\mathcal{FBSS}(X)_A, \sqcap, \sqcup, ^*, A_{\langle\Phi,\mathfrak{X}\rangle}, A_{\langle\mathfrak{X},\Phi\rangle})$
8	Atomic Lattices:
	$(\mathcal{SS}(X)_A, \sqcap, \sqcup)$
9	Brouwerian lattices:
	$(\mathcal{SS}(X)^E, \sqcap, \sqcup_\varepsilon), (\mathcal{SS}(X)_A, \sqcap, \sqcup), (\mathcal{FSS}(X)^E, \sqcap, \sqcup_\varepsilon), (\mathcal{FSS}(X)_A, \sqcap, \sqcup)$ $(\mathcal{DSS}(X)^E, \sqcap, \sqcup_\varepsilon), (\mathcal{DSS}(X)_A, \sqcap, \sqcup), (\mathcal{DFSS}(X)^E, \sqcap, \sqcup_\varepsilon),$ $(\mathcal{DFSS}(X)_A, \sqcap, \sqcup)$

10	MV-algebras:
	$(\mathcal{SS}(X)_A, \sqcap, ^c, A_{\mathfrak{X}}), (\mathcal{SS}(X)_A, \sqcup, ^c, A_\Phi), (\mathcal{DSS}(X)_A, \sqcap, ^c, A_{(\mathfrak{X},\Phi)}),$ $(\mathcal{DSS}(X)_A, \sqcup, ^c, A_{(\Phi,\mathfrak{X})})$
11	BCK-algebras:
	$(\mathcal{SS}(X)_A, \smile, A_\Phi), (\mathcal{SS}(X)_A, \star, A_\Phi), (\mathcal{DSS}(X)_A, \smile, A_{(\Phi,\Phi)}),$ $(\mathcal{DSS}(X)_A, \star, A_{(\Phi,\mathfrak{X})})$

8.4 Application of Soft Sets in a Decision Making Problem

Decision making is an important factor of all scientific professions where experts apply their knowledge in that area to make decisions wisely. Many researchers have applied soft set theory in various decision making problems using different algorithms. A general algorithm for the decision of best object using soft sets is given as follows:

8.4.1 Algorithm

Let X be an initial universal set of available objects and E be the set of parameters. The algorithm for the selection of the best choice among the objects of X is given as:

1. Input $A_{(f_1,f_2,\ldots,f_n)}$, an *n-framed* $\lambda-$*soft set over* X where $A \subseteq E$.
2. Input the set of choice parameters $P \subseteq E$ and find the reduced *n-framed* $\lambda-$*soft set over* X which is reduct of $A_{(f_1,f_2,\ldots,f_n)}$.
3. Compute the comparison tables for functions $f_1, f_2, \ldots, f_n$ by using the predefined rule or Aggregation operator.
4. Compute the scores for each object.
5. Compute the final score S_i for each object $x_i \in X$.
6. Find k, for which $S_k = \max S_i$.

Then h_k is the optimal choice object. If k has more than one values, then any one of h_k's can be chosen.

Now, we apply the concept of *fuzzy bipolar soft sets* for modelling a given problem and then, we give an algorithm for the choice of optimal object based upon the available sets of information. Let X be the initial universe and E be a set of parameters. We shall adapt the following terminology afterwards:

8.4.2 Definition

Let $E_{\langle f,g\rangle}$ be a fuzzy bipolar soft set defined over X. A Comparison table for f is a square table in which the number of rows and number of columns are equal, rows and columns both are labelled by the object names h_1, h_2, h_3, ..., h_n of the initial universe X, and the entries are t_{ij}, i, $j = 1, 2, ..., n$, given by

t_{ij} = the number of parameters for which the membership value of h_i exceeds or equal to the membership value of h_j

Clearly, $0 \leq t_{ij} \leq k$, and $t_{ii} = k$, for all i, j where k is the number of parameters present in E. Thus, t_{ij} indicates a numerical measure, which is an integer. A Comparison table for g is a square table in which the number of rows and number of columns are equal, rows and columns both are labelled by the object names h_1, h_2, h_3, ..., h_n of the initial universe X, and the entries are s_{ij}, i, $j = 1, 2, ..., n$, given by

s_{ij} = the number of parameters for which the membership value of h_i dominates or equal to the membership value of h_j

Clearly, $0 \leq s_{ij} \leq k$, and $s_{ii} = k$, for all i, j where k is the number of parameters present in E. Thus, s_{ij} also indicates a numerical measure, which is an integer.

8.4.3 Definition

The positive row sum and column of an object h_i, denoted by r_i and c_i are calculated by using the formulae,

$$r_i = \sum_{j=1}^{n} t_{ij}, \quad c_j = \sum_{i=1}^{n} t_{ij},$$

The negative row sum and column sum of an object h_i, denoted by r'_i and c'_j are calculated by using the formulae,

$$r'_i = \sum_{j=1}^{n} s_{ij}, \quad c'_j = \sum_{i=1}^{n} s_{ij}.$$

The positive score P_i of object h_i will be given by:

$$P_i = r_i - c_i$$

while the negative score N_i will be given by:

$$N_i = r'_i - c'_i.$$

The final score S_i of object h_i will be given by:

$$S_i = P_i - N_i$$

for all $i = 1, 2, ..., n$.

We wish to find an object from the set of choice parameters A. We are now giving an algorithm for the choice of best object according to the specifications made by observer and recorded data with the help of a fuzzy bipolar soft set.

8.4.4 Algorithm

The algorithm for the selection of the best choice is given as:

1. Input the *fuzzy bipolar soft set* $E_{\langle f,g\rangle}$.
2. Input the set of choice parameters $P \subseteq E$ and find the reduced fuzzy bipolar soft set $P_{\langle f,g\rangle}$.
3. Compute the comparison tables for functions f and g respectively.
4. Compute the positive and negative scores for each object.
5. Compute the final score.
6. Find k, for which $S_k = \max S_i$.

Then h_k is the optimal choice object. If k has more than one values, then any one of h_k's can be chosen

8.4.5 Example

Let $X = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8\}$ be the set of candidates who have applied for a job position of Office Representative in Customer Care Centre of a company. Let $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$ = {Hard Working, Optimism, Enthusiasm, Individualism, Imaginative, Flexibility, Decisiveness, Self-confidence, Politeness} and $\neg E = \{\neg e_1, \neg e_2, \neg e_3, \neg e_4, \neg e_5, \neg e_6, \neg e_7, \neg e_8, \neg e_9\}$ = {Negligent, Pessimism, Half-hearted, Dependence, Unimaginative, Rigidity, Indecisiveness, Shyness, Harshness }. Here the gray area is obviously the moderate form of parameters. Let the *fuzzy bipolar soft sets* $E_{\langle f,g\rangle}$ describes the " Personality Analysis of Candidates" as:

$$f \quad : \quad E \rightarrow \mathcal{FP}(X),$$

$$e \quad \longmapsto \quad \begin{cases} \{m_1/.5, m_2/.7, m_3/.6, m_4/.7, m_5/.5, m_6/.5, m_7/.4, m_8/.8\} & \text{if } e = e_1, \\ \{m_1/.6, m_2/.7, m_3/.8, m_4/.8, m_5/.4, m_6/.4, m_7/.2, m_8/.7\} & \text{if } e = e_2, \\ \{m_1/.8, m_2/.8, m_3/.4, m_4/.6, m_5/.5, m_6/.5, m_7/.4, m_8/.8\} & \text{if } e = e_3, \\ \{m_1/.7, m_2/.6, m_3/.1, m_4/.7, m_5/.6, m_6/.6, m_7/.6, m_8/.9\} & \text{if } e = e_4, \\ \{m_1/.5, m_2/.8, m_3/.6, m_4/.5, m_5/.7, m_6/.3, m_7/.7, m_8/.6\} & \text{if } e = e_5, \\ \{m_1/.4, m_2/.9, m_3/.5, m_4/.4, m_5/.7, m_6/.3, m_7/.6, m_8/.5\} & \text{if } e = e_6, \\ \{m_1/.3, m_2/.8, m_3/.4, m_4/.6, m_5/.8, m_6/.2, m_7/.5, m_8/.4\} & \text{if } e = e_7, \\ \{m_1/.6, m_2/.7, m_3/.5, m_4/.5, m_5/.6, m_6/.4, m_7/.3, m_8/.6\} & \text{if } e = e_8, \\ \{m_1/.8, m_2/.5, m_3/.6, m_4/.6, m_5/.7, m_6/.4, m_7/.2, m_8/.7\} & \text{if } e = e_9, \end{cases}$$

$$g \quad : \quad E \rightarrow \mathcal{FP}(X),$$

$$e \quad \longmapsto \quad \begin{cases} \{m_1/.3, m_2/.2, m_3/.4, m_4/.1, m_5/.3, m_6/.5, m_7/.4, m_8/.2\} & \text{if } e = e_1, \\ \{m_1/.4, m_2/.1, m_3/.2, m_4/.1, m_5/.5, m_6/.5, m_7/.7, m_8/.1\} & \text{if } e = e_2, \\ \{m_1/.05, m_2/.1, m_3/.5, m_4/.33, m_5/.4, m_6/.3, m_7/.6, m_8/.15\} & \text{if } e = e_3, \end{cases}$$

$$e \longmapsto \begin{cases} \{m_1/.23, m_2/.3, m_3/.6, m_4/.2, m_5/.3, m_6/.33, m_7/.2, m_8/.1\} & \text{if } e = e_4, \\ \{m_1/.4, m_2/.2, m_3/.35, m_4/.4, m_5/.1, m_6/.6, m_7/.2, m_8/.35\} & \text{if } e = e_5, \\ \{m_1/.4, m_2/.2, m_3/.3, m_4/.3, m_5/.2, m_6/.5, m_7/.25, m_8/.31\} & \text{if } e = e_6, \\ \{m_1/.7, m_2/.08, m_3/.5, m_4/.3, m_5/.18, m_6/.78, m_7/.4, m_8/.4\} & \text{if } e = e_7, \\ \{m_1/.4, m_2/.2, m_3/.3, m_4/.45, m_5/.4, m_6/.4, m_7/.6, m_8/.26\} & \text{if } e = e_8, \\ \{m_1/.1, m_2/.4, m_3/.36, m_4/.27, m_5/.2, m_6/.5, m_7/.8, m_8/.2\} & \text{if } e = e_9. \end{cases}$$

1. Input the *fuzzy bipolar soft set* $E_{\langle f,g \rangle}$.

2. Input the set of choice parameters $P = \{e_1, e_3, e_4, e_5, e_7, e_8\} \subseteq E$ and find the reduced fuzzy bipolar soft set $P_{\langle f,g \rangle}$ given as:

$$f \quad : \quad P \to \mathcal{FP}(X),$$

$$e \quad \longmapsto \begin{cases} \{m_1/.5, m_2/.7, m_3/.6, m_4/.7, m_5/.5, m_6/.5, m_7/.4, m_8/.8\} & \text{if } e = e_1, \\ \{m_1/.8, m_2/.8, m_3/.4, m_4/.6, m_5/.5, m_6/.5, m_7/.4, m_8/.8\} & \text{if } e = e_3, \\ \{m_1/.7, m_2/.6, m_3/.1, m_4/.7, m_5/.6, m_6/.6, m_7/.6, m_8/.9\} & \text{if } e = e_4, \\ \{m_1/.5, m_2/.8, m_3/.6, m_4/.5, m_5/.7, m_6/.3, m_7/.7, m_8/.6\} & \text{if } e = e_5, \\ \{m_1/.3, m_2/.8, m_3/.4, m_4/.6, m_5/.8, m_6/.2, m_7/.5, m_8/.4\} & \text{if } e = e_7, \\ \{m_1/.6, m_2/.7, m_3/.5, m_4/.5, m_5/.6, m_6/.4, m_7/.3, m_8/.6\} & \text{if } e = e_8, \end{cases}$$

$$g \quad : \quad P \to \mathcal{FP}(X),$$

$$e \quad \longmapsto \begin{cases} \{m_1/.3, m_2/.2, m_3/.4, m_4/.1, m_5/.3, m_6/.5, m_7/.4, m_8/.2\} & \text{if } e = e_1 \\ \{m_1/.05, m_2/.1, m_3/.5, m_4/.33, m_5/.4, m_6/.3, m_7/.6, m_8/.15\} & \text{if } e = e_3 \\ \{m_1/.23, m_2/.3, m_3/.6, m_4/.2, m_5/.3, m_6/.33, m_7/.2, m_8/.1\} & \text{if } e = e_4 \\ \{m_1/.4, m_2/.2, m_3/.35, m_4/.4, m_5/.1, m_6/.6, m_7/.2, m_8/.35\} & \text{if } e = e_5 \\ \{m_1/.7, m_2/.08, m_3/.5, m_4/.3, m_5/.18, m_6/.78, m_7/.4, m_8/.4\} & \text{if } e = e_7 \\ \{m_1/.4, m_2/.2, m_3/.3, m_4/.45, m_5/.4, m_6/.4, m_7/.6, m_8/.26\} & \text{if } e = e_8 \end{cases}$$

3. Compute the comparison tables for functions f and g respectively

f	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8
m_1	6	2	3	4	4	6	4	2
m_2	5	6	6	5	6	6	6	3
m_3	3	0	6	2	1	4	3	2
m_4	4	2	5	6	3	6	5	1
m_5	4	2	5	3	6	6	6	3
m_6	1	1	2	0	3	6	4	0
m_7	2	1	4	1	2	3	6	2
m_8	6	3	6	5	4	6	4	6

Table 8.1: Comparison Table for f

4. Compute the positive and negative scores for each object as given by Table 8.3 and Table

   8.4.

g	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8
m_1	6	2	5	3	4	6	3	1
m_2	4	6	6	4	5	6	5	5
m_3	3	0	6	2	1	4	3	1
m_4	2	2	4	6	3	4	5	1
m_5	4	2	5	3	6	6	5	2
m_6	2	0	2	2	2	6	2	0
m_7	3	2	4	2	1	4	6	2
m_8	5	2	6	4	3	6	5	6

Table 8.2: Comparison Table for g

	Row Sum: r_i	Column Sum: c_i	Positive Score: P_i
m_1	31	31	0
m_2	43	17	26
m_3	21	37	-16
m_4	32	26	6
m_5	35	29	6
m_6	17	43	-26
m_7	21	38	-17
m_8	40	19	21

Table 8.3: Positive Score

5. Compute the final score given by Table 8.5.

6. From Table 8.5 we find $k = 4$.

Thus m_4 is the best candidate for the position. In case that m_4 can not join the position either m_3 or m_8 may be selected.

	Row Sum: r'_i	Column Sum: c'_i	Negative Score: N_i
m_1	30	29	1
m_2	41	16	25
m_3	20	38	-18
m_4	27	26	1
m_5	33	25	8
m_6	16	42	-26
m_7	24	34	-10
m_8	37	18	19

Table 8.4: Negative Score

	Final Score
m_1	-1
m_2	1
m_3	2
m_4	5
m_5	-2
m_6	0
m_7	-7
m_8	2

Table 8.5: Final Score

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