Fuzzy Subalgebraic Structures of Ternary Semigroups based on Fuzzy Point

ISLAMABAD

Ph.D. Thesis

By

Noor Rehman

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
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A THESIS SUBMITTED IN THE PARTIAL FULFILMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
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IN
MATHEMATICS

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Certificate

A thesis submitted in the partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics

We accept this thesis as conforming to the required standard.

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Contents

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Noor Rehman

Introduction

The idea of investigation of n-ary algebraic structures, that is, sets with one n-ary operation dated back to Kasner's lectures [23] at the 53rd annual meeting of the American Association of the Advancement of Science in 1904. But the first paper concerning the theory of n-ary groups was written by Dornte in 1928 (see [11]). Since then numerous papers concerning various n-ary algebras have appeared in the literature (for example see [12]). Sets with one n-ary operation having different properties were investigated by many authors. Such systems have many applications in different branches. For example, in the theory of authomata [14] n-ary systems satisfying some associative law are used, some other n-ary systems are applied in the theory of quantum groups [39] and combinatorics [62, 63].

Ternary and n-ary generalizations of algebraic structures are the most natural ways for further development and deeper understanding of their fundamental properties. Firstly, ternary algebraic operations were already introduced in the nineteenth century by A. Cayley. As the development of Cayley's ideas it were considered n-ary generalization of matrices and their determinants [22, 61] and general theory of n-ary algebras [8, 31, 40], n-group rings [69-71] and ternary rings [2, 33]. The notion of ternay semigroups was also known to Banach (cf. [34]) who is credited with an example of a ternary semigroup which does not reduce to a semigroup. Sioson studied ternary semigroups with a special reference to ideals and radicals. He also introduced the notion of regular ternary semigroups and characterized them by using the notion of quasi-ideals [60].

The fundamental concept of fuzzy set, introduced by Zadeh in his seminal paper [68] of 1965, provides a natural framework for generalizing several basic notions of algebra. Many papers on fuzzy sets appeared showing the importance of the concept and its application to logic, set theory, ring theory, group theory, semigroup theory, topology etc. Rosenfeld in [53] inspired the fuzzification of algebraic structures and introduced the notion of fuzzy subgroup. Kuroki initiated the theory of fuzzy semigroups (see [26-30]). A systematic exposition of fuzzy semigroups by Mordeson et al. appeared in [37], where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. Bhakat and Das (see [3-7]) gave the concept of (α,β)-fuzzy subgroups of a group using the concept of `belongs to' and `quasi-coincident with' between a fuzzy point and a fuzzy set which is mentioned in [41]. They studied $(\in,\in\vee q)$-fuzzy subgroup of a group. In fact $(\in,\in\vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. Shabir et al. in [56] characterized regular semigroups by the properties of $(\in,\in\vee q)$-fuzzy ideals. Dudek et al. in [13] and Ma and Zhan in [35] defined (α,β)-fuzzy ideals in hemirings, and investigated some related properties of hemirings. In [16], Jun and Song initiated the study of (α,β)-fuzzy interior ideals of a semigroup. Kazanci and Yamak in [24], studied $(\in,\in\vee q)$-fuzzy bi-ideals of a semigroup. In [72], Zhan and Jun studied $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior ideals of semigroups. Generalizing the concept of quasi-coincidence of a fuzzy point with a fuzzy set Jun in [16] defined $(\in,\in\vee q_k)$-fuzzy subalgebras in BCK/BCI-algebras. In [18] Jun et al. discussed $(\in,\in\vee q_k)$-fuzzy h-ideals and $(\in,\in\vee q_k)$-fuzzy k-ideals of a hemiring. More detailed results on $(\in,\in\vee q_k)$-type fuzzy ideals in hemirings can be seen in [1]. Shabir et al. in [57] characterized different classes of semigroups by $(\in,\in\vee q_k)$-fuzzy ideals and $(\in,\in\vee q_k)$-fuzzy bi-ideals. Ma et al. in [36], introduced the concept of $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy ideals of BCI-algebras. Recently, Shabir and Ali characterized semigroups by the properties of their $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy ideals [54].

Research Profile

The thesis is based on the following research papers.

1. N. Rehman, M. Shabir, *Characterizations of ternary semigroups by* (α,β)*-fuzzy ideals*, World Appl. Sci. J. **18** (11) (2012), 1556-1570.

2. M. Shabir, N. Rehman, $(\overline{\alpha},\overline{\beta})$*-fuzzy ideals of ternary semigroups,* World Appl. Sci. J. 17 (2012), 1736-1758.

3. M. Shabir, N. Rehman, *Characterizations of ternary semigroups by* $(\in,\in\vee q_k)$*-fuzzy ideals*, Iranian Journal of Science and Tech. Trans. A , 36 (Special issue-Mathematics) (2012), 395-410

4. N. Rehman, M. Shabir, *New types of ternary semigroups,* U. P. B. Scientific Bull. Ser. A **75** (2013), 67-80

5. N. Rehman, M. Shabir, *Some kinds of* $(\in_\gamma,\in_\gamma\vee q_\delta)$*-fuzzy ideals of ternary semigroups,* Iranian Journal of Science and Tech. Trans. A , 37 (Special issue-Mathematics) (2013), 365-378.

6. N. Rehman, M. Shabir, *Some characterizations of ternary semigroups by the properties of their* $(\in_\gamma,\in_\gamma\vee q_\delta)$*-fuzzy ideals,* Journal of Intelligent and Fuzzy Systems, 26 (2014), 2107-2117.

7. N. Rehman, M. Shabir, Y. B. Jun, *Classifications and properties of* (α,β)*-fuzzy ternary subsemigroups in ternary semigroups,* submitted.

8. N. Rehman, M. Shabir, Y. B. Jun, *Classifications and properties of* (α,β)*-fuzzy ideals in ternary semigroups,* Appl. Math. Inf. Sci., 9 (3) (2015), 1575-1585.

9. N. Rehman, M. Shabir, Y. B. Jun, *Classifications and properties of* (α,β)*-fuzzy bi-ideals in ternary semigroups,* submitted.

10. N. Rehman, M. Shabir, Y. B. Jun, *Characterizations of ternary semigroups in terms of* $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy ideals, submitted.

11. N. Rehman, M. Shabir, Y. B. Jun, *Further results on* $(\in,\in\vee q_k)$*-fuzzy ternary subsemigroups and ideals in ternary semigroups,* submitted.

12. N. Rehman, M. Shabir, Y. B. Jun, *More on* $(\in,\in\vee q_k)$*-fuzzy bi-ideals in ternary semigroups,* submitted.

13. N. Rehman, M. Shabir, Y. B. Jun, *Implication-based fuzzy ternary subsemigroups in ternary semigroups,* submitted.

Chapter-wise study

This thesis consists of seven chapters. Througout the thesis, S denotes a ternary semigroup, unless otherwise specified.

Chapter one, which is of introductory nature, provides basic definitions and results, which are needed for the subsequent chapters.

In chapter two, using the combined notions of `belongs to' and `quasi-coincidence' we introduce and classify (α,β)-type fuzzy ternary subsemigroups (left ideals, right ideals, bi-ideals) in ternary semigroups, where α,β are any of $\in, q, \in\vee q$ or $\in\wedge q$ with $\alpha\neq\in\wedge q$. Special attention is paid to $(\in,\in\vee q)$-type fuzzy ternary subsemigroups (left ideals, right ideals, quasi-ideals, prime ideals, bi-ideals). We give numerouse examples to illustrate the concepts.

In chapter three, we study $(\in,\in\vee q_k)$-type fuzzy ternary subsemigroups (left ideals, right ideals, quasi-ideals, bi-ideals, generalized bi-ideals) in ternary semigroups. The classes of regular ternary semigroups and weakly regular ternary semigroups are characterized in terms of $(\in,\in\vee q_k)$-fuzzy left ideals (right ideals, quasi-ideals, bi-ideals, generalized bi-ideals).

Chapter four is devoted to the study of $(\overline{\alpha},\overline{\beta})$-type fuzzy ternary subsemigroups (left ideals, right ideals, bi-ideals) in ternary semigroups. We mainly focused on $(\overline{\in},\overline{\in}\vee\overline{q})$-type fuzzy ternary subsemigroups (left ideals, right ideals, lateral ideals, quasi-ideals, prime ideals, bi-ideals). We conclude this chapter by characterizing reguar ternary semigroups and weakly regular ternary semigroups in terms of these notions.

In chapter five, we focused our attention to highlight the study of $(\overline{\in},\overline{\in}\vee\overline{q_k})$-type fuzzy ternary subsemigroups (left ideals, right ideals, lateral ideals, quasi-ideals, bi-ideals, generalized bi-ideals) in ternary semigroups and investigate some related properties. Regular ternary semigroups and right weakly regular ternary semigroups are characterized in terms of these fuzzy substructures.

Chapter six presents the study of $(\in_\gamma,\in_\gamma\vee q_\delta)$-type fuzzy ternary subsemigroups (left ideals, right ideals, bi-ideals, generalized bi-ideals) in ternary semigroups. Several characterizations of regular ternary semigroups and right weakly regular ternary semigroups in terms of $(\in_\gamma,\in_\gamma\vee q_\delta)$-type fuzzy substructures are established.

The goal of chapter seven is to study the implication-based fuzzy ternary subsemigroups in ternary semigroups. Using the four implication operators, that is, Gaines-Rescher implication operator, Gödel implication operator, the contraposition of Gödel implication operator and the Łuckasiewicz implication operator, the implication-based fuzzy ternary subsemigroups are considered. Relations between fuzzy (resp. $(\in, \in \vee q)$-fuzzy) ternary subsemigroups are discussed. Conditions for a fuzzy ternary subsemigroup with thresholds 0 and 0.5 to be an implication-based fuzzy ternary subsemigroups under the Łuckasiewicz implication operator are provided.

Chapter 1

Preliminary Notions

This chapter aims to recall basic concepts of ternary semigroups and fuzzy set theory including fuzzy substructures of ternary semigroups which will serve as the base of remaining chapters in the thesis.

1.1 Ternary Semigroups; Basic Definitions and Examples

Definition 1 *A ternary semigroup is an algebraic structure (S, f) such that S is a nonempty set and $f : S^3 \to S$ is a ternary operation, satisfying the following associative law:*

$$f\left(f\left(a, b, c\right), d, e\right) = f\left(a, f\left(b, c, d\right), e\right) = f\left(a, b, f\left(c, d, e\right)\right) \tag{1.1}$$

for all $a, b, c, d, e \in S$.

This notation is rather cumbersome and we shall follow the usual algebraic practice of writing the ternary operation as multiplication. Thus $f(a, b, c)$ becomes $a \cdot b \cdot c$ or (more usually) abc and the above associative law takes the simple form $((abc)\,de) = (a\,(bcd)\,e) = (ab\,(cde))$.

1.1.1 Examples

1. Let $S = \mathbb{Z}^-$ be the set of all negative integers. Then $\mathbb{Z}^-$ is a ternary semigroup with respect to ternary multiplication of numbers.
2. The set of all odd permutations under composition.
3. Any semigroup S can be made into a ternary semigroup by defining the ternary product to be $f(a, b, c) = abc$, for all $a, b, c \in S$.
4. Let $\mathbb{Q}^-$ be the set of all negative rational numbers and $S = \left\{r\sqrt{2} : r \in \mathbb{Q}^-\right\}$. Then under usual ternary multiplication, S forms a ternary semigroup.

5. Let $\mathbb{R}$ be the set of all real numbers and $S = \left\{ ri : r \in \mathbb{R},\ i = \sqrt{-1} \right\}$. Then with the usual ternary multiplication, S forms a ternary semigroup.

6. Let S be the set of all $n \times n$ square matrices over $\mathbb{Z}^-$. Then with ternary multiplication of matrices, S forms a ternary semigroup.

7. Let $S = \mathbb{N}$. Then S is a ternary semigroup with respect to each of the following operations:

$$\begin{aligned} \oplus(abc) &= \text{gcd } (a, b, c) \quad \text{for all } a, b, c \in S, \\ \ominus(abc) &= \text{lcm } (a, b, c) \quad \text{for all } a, b, c \in S, \\ \otimes(abc) &= \max\ (a, b, c) \quad \text{for all } a, b, c \in S, \\ \odot(abc) &= \min\ (a, b, c) \quad \text{for all } a, b, c \in S. \end{aligned}$$

Definition 2 *If A, B and C are nonempty subsets of a ternary semigroup S. Then we write*

$$ABC = \{abc : a \in A, b \in B, c \in C\}.$$

Definition 3 *A ternary semigroup S is said to be a left zero ternary semigroup if*

$$abc = a \text{ for all } a, b, c \in S.$$

Right zero ternary semigroup can be defined analogously.

1.2 Ternary Subsemigroups and Ideals

In this section we define ternary subsemigroup and left (right, lateral) ideals of a ternary semigroup. We will also give some basic properties of these substructures.

A nonempty subset A of a ternary semigroup S is called a ternary subsemigroup of S if $AAA \subseteq A$, that is, $abc \in A$ for all $a, b, c \in A$.

A nonempty subset A of a ternary semigroup S is called a left (resp. right, lateral) ideal of S if $SSA \subseteq A$ (resp. $ASS \subseteq A$, $SAS \subseteq A$). If a nonempty subset of a ternary semigroup S is a left and right ideal of S, then it is called a two sided ideal of S.

If a nonempty subset of a ternary semigroup S is a left ideal, right ideal and lateral ideal of S, then it is called an ideal of S.

A nonempty subset Q of a ternary semigroup S is called a quasi-ideal of S if

(1) $(QSS) \cap (SQS) \cap (SSQ) \subseteq Q$.

(2) $(QSS) \cap (SSQSS) \cap (SSQ) \subseteq Q$.

Every left, right and lateral ideal in a ternary semigroup is a quasi-ideal. But the converse is not true in general.

A ternary subsemigroup B of a ternary semigroup S is called a bi-ideal of S if $BSBSB \subseteq B$. A nonempty subset B of a ternary semigroup S is called a generalized bi-ideal of S if $BSBSB \subseteq B$.

A proper ideal P of S is called a prime ideal if $ABC \subseteq P$ implies $A \subseteq P$, $B \subseteq P$ or $C \subseteq P$ for all ideals A, B, C of S.

The intersection of any number of ternary subsemigroups of a ternary semigroup S is either empty or a ternary subsemigroup of S.

Proposition 4 *Let R be a right ideal, M a lateral ideal and L a left ideal of a ternary semigroup S. Then*

$$RML \subseteq R \cap M \cap L.$$

1.3 Regular Ternary Semigroups

An element a in a ternary semigroup S is called regular if there exists an element x in S such that $axa = a$, that is, $a \in aSa$. A ternary semigroup S is called regular if its all elements are regular.

Theorem 5 *[60] A ternary semigroup S is regular if and only if $R \cap M \cap L = RML$ for every right ideal R, lateral ideal M and a left ideal L of S.*

Theorem 6 *[55] The following conditions on a ternary semigroup S are equivalent:*

(1) *S is regular;*

(2) *$R \cap L = RSL$ for every right ideal R and every left ideal L of S.*

Theorem 7 *[55] The following assertions on a ternary semigroup S are equivalent:*

(1) *S is regular ;*

(2) *$B = BSB$ for every bi-ideal B of S;*

(3) *$Q = QSQ$ for every quasi-ideal Q of S;*

(4) *$B = BSBSB$ for every bi-ideal B of S;*

(5) *$Q = QSQSQ$ for every quasi-ideal Q of S.*

Definition 8 *[55] A ternary semigroup S is said to be right (resp. left) weakly regular, if $x \in (xSS)^3$ $\left(\text{resp. } x \in (SSx)^3\right)$ for all $x \in S$. A ternary semigroup S is right weakly regular if every right ideal R of S is idempotent, that is, $R^3 = R$.*

Lemma 9 *[55] A ternary semigroup S is right weakly regular if and only if $R \cap I = RII$, for every right ideal R and every two sided ideal I of S.*

1.4 Fuzzy sets

Let X be a nonempty set. For $A \subseteq X$, the characteristic function $\chi_A : X \longrightarrow \{0,1\}$ of A is defined by

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

A fuzzy set μ of a universe X is a function from X into the unit closed interval $[0,1]$, that is, $\mu : X \to [0,1]$. A fuzzy set μ in X of the form

$$\mu(y) := \begin{cases} t \in (0,1] & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t. For a fuzzy point x_t and fuzzy set μ in a set X, Pu and Liu [41] gave meaning to the symbol $x_t\alpha\mu$, where $\alpha \in \{\in, q, \in \vee q, \in \wedge q\}$. A fuzzy point x_t is said to belongs to (resp. be quasi-coincident with) a fuzzy set μ written $x_t \in \mu$ (resp. $x_tq\mu$) if $\mu(x) \geqslant t$ (resp. $\mu(x) + t > 1$) and in this case, $x_t \in \vee q\mu$ (resp. $x_t \in \wedge q\mu$) means that $x_t \in \mu$ or $x_tq\mu$ (resp. $x_t \in \mu$ and $x_tq\mu$). To say that $x_t\bar{\alpha}\mu$ means that $x_t\alpha\mu$ does not hold.

If μ and λ are two fuzzy sets in X then $\lambda \leq \mu$ means that $\lambda(x) \leq \mu(x)$ for all $x \in X$. The fuzzy sets $\mu \cap \lambda$ and $\mu \cup \lambda$ of X are called intersection and union of μ and λ, respectively and are defined as follows:

$$\begin{aligned} (\mu \cap \lambda)(x) &= \min\{\mu(x), \lambda(x)\} = \mu(x) \wedge \lambda(x) \\ (\mu \cup \lambda)(x) &= \max\{\mu(x), \lambda(x)\} = \mu(x) \vee \lambda(x) \end{aligned}$$

for all $x \in S$. Similarly, for any family of fuzzy sets $\{\mu_i\}_{i\in\Lambda}$ of S we define $\left(\bigcap_{i\in\Lambda} \mu_i\right)(x) = \bigwedge_{i\in\Lambda} \mu_i(x)$ and $\left(\bigcup_{i\in\Lambda} \mu_i\right)(x) = \bigvee_{i\in\Lambda} \mu_i(x)$ for all $x \in S$.

Definition 10 *Let μ, λ and ν be three fuzzy sets in a ternary semigroup S. The product $\mu \circ \lambda \circ \nu$ is defined by:*

$$(\mu \circ \lambda \circ \nu)(a) = \begin{cases} \bigvee_{a=xyz} \{\mu(x) \wedge \lambda(y) \wedge \nu(z)\} & \text{if there exist } x, y, z \in S \text{ such that } a = xyz \\ 0 & \text{otherwise.} \end{cases}$$

Definition 11 *A fuzzy set μ in a ternary semigroup S is said to be a fuzzy ternary subsemigroup of S if it satisfies the following conditions:*

$$\mu(xyz) \geq \min\{\mu(x),\ \mu(y),\ \mu(z)\} \tag{1.2}$$

for all $x, y, z \in S$.

Definition 12 *A fuzzy set μ in a ternary semigroup S is called a fuzzy left (resp. right and lateral) ideal of S (see [20]) if it satisfies:*

$$\mu(xyz) \geq \mu(z) (\textit{resp. } \mu(x) \textit{ and } \mu(y)) \tag{1.3}$$

for all $x, y, z \in S$.

Definition 13 *[21] A fuzzy ternary subsemigroup μ in a ternary semigroup S is said to be a fuzzy bi-ideal of S if it satisfies the following condition:*

$$\mu(xuyvz) \geq \min\{\mu(x), \ \mu(y), \ \mu(z)\} \tag{1.4}$$

for all $x, u, y, v, z \in S$.

Definition 14 *[21] A fuzzy set μ in a ternary semigroup S is a fuzzy quasi-ideal of S if*

$$\begin{aligned} (i) \ \ \mu(x) &\geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\}, \\ (ii) \ \ \mu(x) &\geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\}, \end{aligned}$$

for all $x \in S$, where $\mathcal{S}$ is the fuzzy set in S mapping every element of S on 1.

Obviously, every one sided fuzzy ideal of a ternary semigroup S is a fuzzy quasi-ideal, every fuzzy quasi-ideal is fuzzy bi-ideal and every fuzzy bi-ideal is fuzzy generalized bi-ideal of S.

Chapter 2

Classifications and properties of (α, β)-fuzzy ternary subsemigroups and ideals in ternary semigroups

In this chapter we classify (α, β)-fuzzy ternary subsemigroups, ideals and bi-ideals in ternary semigroups and investigate the related properties. Different classes of ternary semigroups are characterized in terms of $(\in, \in \vee q)$-fuzzy left (right, lateral) ideals, $(\in, \in \vee q)$-fuzzy quasi-ideals, $(\in, \in \vee q)$-fuzzy bi (generalized bi) ideals.

2.1 Classifications and properties of (α, β)-fuzzy ternary subsemigroups

In what follows, let S denote a ternary semigroup and $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ unless otherwise specified.

Definition 15 *A fuzzy set μ in S is said to be an* (α, β)-fuzzy ternary subsemigroup *of S, where $\alpha \neq \in \wedge\, q$, if it satisfies the following condition:*

$$x_{t_1} \alpha\, \mu,\ y_{t_2} \alpha\, \mu \text{ and } z_{t_3} \alpha\, \mu \text{ imply } (xyz)_{\min\{t_1,t_2,t_3\}}\, \beta\, \mu \tag{2.1}$$

for all $x, y, z \in S$ and $t_1, t_2, t_3 \in (0, 1]$.

The case $\alpha = \in \wedge q$ in Definition 15 is omitted, because for a fuzzy set μ in S if $\mu(x) \leq 0.5$ for all $x \in S$, then for the case $x_t \in \wedge q \mu$, $t \in (0, 1]$ we have $\mu(x) \geq t$ and

$\mu(x) + t > 1$. Thus $1 < \mu(x) + t \leq \mu(x) + \mu(x) = 2\mu(x)$, which implies $\mu(x) > 0.5$. This means that $\{x_t \mid x_t \in \wedge q\mu\} = \emptyset$.

Example 16 *Consider $S = \{-i, 0, i\}$, where S is a ternary semigroup under the usual multiplication of complex numbers. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], x \to \begin{cases} 0.5 & \text{if } x = 0, \\ 0.7 & \text{if } x = i, \\ 0.6 & \text{if } x = -i. \end{cases}$$

Then simple calculations show that μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S.

Theorem 17 *A fuzzy set μ in S is a fuzzy ternary subsemigroup of S if and only if μ is an $(\in, \in)$-fuzzy ternary subsemigroup of S.*

Proof. Suppose μ is a fuzzy ternary subsemigroup of S. Let $x, y, z \in S$ and $t_1, t_2, t_3 \in (0, 1]$ be such that $x_{t_1} \in \mu$, $y_{t_2} \in \mu$ and $z_{t_3} \in \mu$. Then $\mu(x) \geqslant t_1$, $\mu(y) \geqslant t_2$ and $\mu(z) \geqslant t_3$. Since μ is a fuzzy ternary subsemigroup of S, it follows that

$$\mu(xyz) \geqslant \min\{\mu(x), \mu(y), \mu(z)\} \geqslant \min\{t_1, t_2, t_3\}.$$

Hence $(xyz)_{\min\{t_1, t_2, t_3\}} \in \mu$.

Conversely, assume that μ is an $(\in, \in)$-fuzzy ternary subsemigroup of S. If there exist $x, y, z \in S$ such that

$$\mu(xyz) < \mu(x) \wedge \mu(y) \wedge \mu(z),$$

then we can choose $t \in (0, 1]$ such that

$$\mu(xyz) < t \leq \mu(x) \wedge \mu(y) \wedge \mu(z).$$

Then $x_t \in \mu$, $y_t \in \mu$ and $z_t \in \mu$ but $(xyz)_t \overline{\in} \mu$, which is a contradiction. Hence $\mu(xyz) \geqslant \mu(x) \wedge \mu(y) \wedge \mu(z)$. This completes the proof. ■

Theorem 18 *Every $(\in, \in)$-fuzzy ternary subsemigroup is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup.*

Proof. Straightforward. ■

Remark 19 *The converse of Theorem 18 is not true in general. In fact, the $(\in, \in \vee q)$-fuzzy ternary subsemigroup μ of S in Example 16 is not an $(\in, \in)$-fuzzy ternary subsemigroup of S since $i_{0.7} \in \mu$ but $(iii)_{0.7} \overline{\in} \mu$.*

Proposition 20 *In a left zero ternary semigroup, every fuzzy set is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup.*

Proof. Straightforward. ■

For a fuzzy set μ in S, we denote $S_0 := \{x \in S \mid \mu(x) > 0\}$.

Theorem 21 *If μ is one of the following:*
(i) an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S,
(ii) an $(\in, \in)$-fuzzy ternary subsemigroup of S,
(iii) an $(\in, q)$-fuzzy ternary subsemigroup of S,
(iv) a $(q, \in)$-fuzzy ternary subsemigroup of S,
(v) a (q, q)-fuzzy ternary subsemigroup of S,
(vi) an $(\in, \in \wedge q)$-fuzzy ternary subsemigroup of S,
(vii) an $(\in \vee q, q)$-fuzzy ternary subsemigroup of S,
(viii) an $(\in \vee q, \in)$-fuzzy ternary subsemigroup of S,
(ix) an $(\in \vee q, \in \wedge q)$-fuzzy ternary subsemigroup of S,
(x) a $(q, \in \wedge q)$-fuzzy ternary subsemigroup of S,
(xi) a $(q, \in \vee q)$-fuzzy ternary subsemigroup of S,
then the set S_0 is a ternary subsemigroup of S.

Proof. (i) Let $x,\ y,\ z \in S_0$. Then $\mu(x) > 0, \mu(y) > 0, \mu(z) > 0$. Assume that $xyz \notin S_0$. Then

$$\mu(xyz) = 0 < \min\{\mu(x), \mu(y), \mu(z)\}.$$

If we take $t = \min\{\mu(x), \mu(y), \mu(z)\}$, then $t \in (0, 1]$, $x_t \in \mu$, $y_t \in \mu$ and $z_t \in \mu$. Since $\mu(xyz) = 0 < t$ and $\mu(xyz) + t = t \leq 1$, we have $(xyz)_t \overline{\in \vee q} \mu$. This is a contradiction. Hence $xyz \in S_0$. Consequently S_0 is a ternary subsemigroup of S.

(ii) It is a corollary of (i).

(iii) Let $x, y,\ z \in S_0$. Then $\mu(x) > 0, \mu(y) > 0, \mu(z) > 0$. If $\mu(xyz) = 0$, then

$$\mu(xyz) + \min\{\mu(x), \mu(y), \mu(z)\} = \min\{\mu(x), \mu(y), \mu(z)\} \leq 1.$$

Hence $(xyz)_{\min\{\mu(x),\mu(y),\mu(z)\}} \overline{q} \mu$, which is contradiction. Thus $\mu(xyz) > 0$, and so $xyz \in S_0$. Therefore S_0 is a ternary subsemigroup of S.

(iv) Let $x, y,\ z \in S_0$. Then $\mu(x) > 0, \mu(y) > 0$ and $\mu(z) > 0$. Thus $\mu(x) + 1 > 0$, $\mu(y) + 1 > 0$ and $\mu(z) + 1 > 0$, which imply that $x_1 q\mu$, $y_1 q\mu$ and $z_1 q\mu$. If $\mu(xyz) = 0$, then $\mu(xyz) < 1 = \min\{1, 1, 1\}$. Therefore $(xyz)_{\min\{1,1,1\}} \overline{\in} \mu$, which is a contradiction. It follows that $\mu(xyz) > 0$, so $xyz \in S_0$. This completes the proof.

(v) Let $x, y, z \in S_0$. Then $\mu(x) > 0, \mu(y) > 0$ and $\mu(z) > 0$. Thus $\mu(x) + 1 > 1$, $\mu(y) + 1 > 1$ and $\mu(z) + 1 > 1$, and therefore $x_1 q\mu$, $y_1 q\mu$ and $z_1 q\mu$. If $\mu(xyz) = 0$, then $\mu(xyz) + \min\{1,1,1\} = 0 + 1 = 1$, and so $(xyz)_{\min\{1,1,1\}}\, \overline{q}\mu$. This is impossible, hence $\mu(xyz) > 0$, that is, $xyz \in S_0$. This completes the proof.

The proofs of the remaining parts are similar. ■

Theorem 22 *Let A be a non-empty subset of S and $\alpha \in \{\in, q, \in \vee q\}$. Then A is a ternary subsemigroup of S if and only if the fuzzy set μ in S defined by:*

$$\mu(x) = \begin{cases} \geq 0.5 & \text{for all } x \in A, \\ 0 & \text{for all } x \in S \setminus A, \end{cases}$$

is an $(\alpha, \in \vee q)$-fuzzy ternary subsemigroup of S.

Proof. Suppose A is a ternary subsemigroup of S.

(a) In this part we show that μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S. Let x, y, $z \in S$ and $t, r, s \in (0,1]$ be such that x_t, y_r, $z_s \in \mu$. Then $\mu(x) \geq t$, $\mu(y) \geq r$, $\mu(z) \geq s$. Thus $x, y, z \in A$, so $xyz \in A$. Thus $\mu(xyz) \geq 0.5$. If $\min\{t,r,s\} \leq 0.5$, then $\mu(xyz) \geq 0.5 \geq \min\{t,r,s\}$ implies $(xyz)_{\min\{t,r,s\}} \in \mu$. If $\min\{t,r,s\} > 0.5$, then $\mu(xyz) + \min\{t,r,s\} > 0.5 + 0.5 = 1$, so $(xyz)_{\min\{t,r,s\}}\, q\mu$. Thus $(xyz)_{\min\{t,r,s\}} \in \vee q\mu$. Hence μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S.

(b) In this part we show that μ is a $(q, \in \vee q)$-fuzzy ternary subsemigroup of S. Let x, y, $z \in S$ and $t, r, s \in (0,1]$ be such that x_t, y_r, $z_s q\mu$. Then $\mu(x) + t > 1$, $\mu(y) + r > 1$, $\mu(z) + s > 1$. Thus $x, y, z \in A$, so $xyz \in A$. Thus $\mu(xyz) \geq 0.5$. If $\min\{t,r,s\} \leq 0.5$, then $\mu(xyz) \geq 0.5 \geq \min\{t,r,s\}$. Hence $(xyz)_{\min\{t,r,s\}} \in \mu$. If $\min\{t,r,s\} > 0.5$, then $\mu(xyz) + \min\{t,r,s\} > 0.5 + 0.5 = 1$, so $(xyz)_{\min\{t,r,s\}}\, q\mu$. Thus $(xyz)_{\min\{t,r,s\}} \in \vee q\mu$. Hence μ is a $(q, \in \vee q)$-fuzzy ternary subsemigroup of S.

(c) In this part we show that μ is an $(\in \vee q, \in \vee q)$-fuzzy ternary subsemigroup of S. Let $x, y, z \in S$ and $t, r, s \in (0,1]$ be such that x_t, y_r, $z_s \in \vee q\mu$. Then $x_t \in \mu$ or $x_t q\mu$, $y_r \in \mu$ or $y_r q\mu$, $z_s \in \mu$ or $z_s q\mu$. This implies that $\mu(x) \geq t$ or $\mu(x) + t > 1$, $\mu(y) \geq r$ or $\mu(y) + r > 1$, $\mu(z) \geq s$ or $\mu(z) + s > 1$.

Now there are eight possible cases, each case implies that $x, y, z \in A$ so $xyz \in A$. Analogous to (a) and (b) we obtain $(xyz)_{\min\{t,r,s\}} \in \vee q\mu$.

Hence μ is an $(\in \vee q, \in \vee q)$-fuzzy ternary subsemigroup of S.

Conversely, assume that μ is an $(\alpha, \in \vee q)$-fuzzy ternary subsemigroup of S. Then $A = S_0$. Just like Theorem 21 we can prove that in any case A is a ternary subsemigroup of S. Hence by Theorem 21, A is a ternary subsemigroup of S. ■

Example 23 *Consider the ternary semigroup S of Example 16. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], x \to \begin{cases} 0.50 & \text{if } x = 0, \\ 0.72 & \text{if } x = i, \\ 0.61 & \text{if } x = -i. \end{cases}$$

Then μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S. But
(1) *μ is not an $(\in, \in)$-fuzzy ternary subsemigroup of S, since $i_{0.71} \in \mu$ but*

$$(iii)_{\min\{0.71,0.71,0.71\}} = -i_{0.71} \overline{\in} \mu.$$

(2) *μ is not an $(\in, q)$-fuzzy ternary subsemigroup of S, since $i_{0.1} \in \mu$ but*

$$(iii)_{\min\{0.1,0.1,0.1\}} = -i_{0.1} \overline{q} \mu.$$

(3) *μ is not a (q, q)-fuzzy ternary subsemigroup of S, since $i_{0.3} q \mu$ but*

$$(iii)_{\min\{0.3,0.3,0.3\}} = -i_{0.3} \overline{q} \mu.$$

(4) *μ is not a $(q, \in)$-fuzzy ternary subsemigroup of S, since $i_{0.68} q\mu$ but*

$$(iii)_{\min\{0.68,0.68,0.68\}} = -i_{0.68} \overline{\in} \mu.$$

(5) *μ is not an $(\in, \in \wedge q)$-fuzzy ternary subsemigroup of S, since $i_{0.27} \in \mu$ but*

$$(iii)_{0.27} \overline{q} \mu \text{ and so } (iii)_{0.27} \overline{\in \wedge q} \mu.$$

(6) *μ is not a $(q, \in \wedge q)$-fuzzy ternary subsemigroup of S, since $i_{0.29} q\mu$ but*

$$(iii)_{\min\{0.29,0.29,0.29\}} = -i_{0.29} \overline{q} \mu \text{ and so } (iii)_{0.29} \overline{\in \wedge q} \mu.$$

(7) *μ is not an $(\in \vee q, \in)$-fuzzy ternary subsemigroup of S, since $i_{0.65} \in \vee q\mu$ but*

$$(iii)_{\min\{0.65,0.65,0.65\}} = -i_{0.65} \overline{\in} \mu.$$

(8) *μ is not an $(\in \vee q, q)$-fuzzy ternary subsemigroup of S, since $i_{0.31} \in \vee q\mu$, $i_{0.38} \in \vee q\mu$ and $i_{0.32} \in \vee q\mu$ but*

$$(iii)_{\min\{0.31,0.38,0.32\}} = -i_{0.31} \overline{q} \mu.$$

(9) *μ is not an $(\in \vee q, \in \wedge q)$-fuzzy ternary subsemigroup of S, since $i_{0.34} \in \vee q\mu$ but*

$$(iii)_{0.34} \overline{q} \mu \text{ and so } (iii)_{0.34} \overline{\in \wedge q} \mu.$$

Theorem 24 *Every $(\in \vee q, \in \vee q)$-fuzzy ternary subsemigroup of a ternary semigroup S is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S.*

Proof. Straightforward. $\blacksquare$

Remark 25 *In general, it is not true that $(\in, \in\vee q)$-type implies $(\in\vee q, \in\vee q)$-type as seen in the following example.*

Example 26 *Let $S = \{a, b, c, d\}$ be a left zero ternary semigroup, that is, $xyz = x$ for all $x, y, z \in S$. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.31 & \text{if } x = a, \\ 0.61 & \text{if } x = b, \\ 0.78 & \text{if } x = c, \\ 0.88 & \text{if } x = d. \end{cases}$$

Then μ is an $(\in, \in\vee q)$-fuzzy ternary subsemigroup of S by Proposition 20. But it is not an $(\in\vee q, \in\vee q)$-fuzzy ternary subsemigroup of S since $a_{0.7} \in\vee q\, \mu$, $c_{0.5} \in\vee q\, \mu$ and $d_{0.4} \in\vee q\, \mu$, but $(acd)_{\min\{0.7,0.5,0.4\}} \overline{\in\vee q}\, \mu$.

Now, we investigate relations between $(\in, \in\vee q)$-fuzzy ternary subsemigroups and $(q, \in\vee q)$-fuzzy ternary subsemigroups.

Theorem 27 *Every $(q, \in\vee q)$-fuzzy ternary subsemigroup of a ternary semigroup S is an $(\in, \in\vee q)$-fuzzy ternary subsemigroup of S.*

Proof. Let μ be a $(q, \in\vee q)$-fuzzy ternary subsemigroup of S, $x, y, z \in S$ and $t_1, t_2, t_3 \in (0, 1]$ be such that $x_{t_1} \in \mu, y_{t_2} \in \mu$ and $z_{t_3} \in \mu$. Then $\mu(x) \geq t_1, \mu(y) \geq t_2$ and $\mu(z) \geq t_3$. Suppose $(xyz)_{\min\{t_1,t_2,t_3\}} \overline{\in\vee q}\mu$. Then

$$\mu(xyz) < \min\{t_1, t_2, t_3\},$$

and

$$\mu(xyz) + \min\{t_1, t_2, t_3\} \leq 1.$$

It follows that

$$\mu(xyz) < 0.5.$$

Thus we have

$$\mu(xyz) < \min\{t_1, t_2, t_3, 0.5\}$$

and so

$$\begin{aligned} 1 - \mu(xyz) &> 1 - \min\{t_1, t_2, t_3, 0.5\} \\ &= \max\{1 - t_1, 1 - t_2, 1 - t_3, 0.5\} \\ &\geq \max\{1 - \mu(x), 1 - \mu(y), 1 - \mu(z), 0.5\}. \end{aligned}$$

Hence there exists $\delta \in (0, 1]$ such that

$$1 - \mu(xyz) \geq \delta > \max\{1 - \mu(x), 1 - \mu(y), 1 - \mu(z), 0.5\}. \tag{2.2}$$

From the right inequality in (2.2) we have $\mu(x) + \delta > 1$, $\mu(y) + \delta > 1$, and $\mu(z) + \delta > 1$, that is, $x_\delta q\mu$, $y_\delta q\mu$ and $z_\delta q\mu$. Since μ is a $(q, \in \vee q)$-fuzzy ternary subsemigroup of S, it follows that $(xyz)_\delta \in \vee q\mu$. But from the left inequality in (2.2) we have $\mu(xyz) + \delta \leq 1$, that is, $(xyz)_\delta \overline{q}\mu$ and $\mu(xyz) \leq 1 - \delta < 1 - 0.5 = 0.5 < \delta$, that is, $(xyz)_\delta \overline{\in}\mu$. Thus $(xyz)_\delta \overline{\in \vee q}\mu$, a contradiction. Hence, $(xyz)_{\min\{t_1,t_2,t_3\}} \in \vee q\mu$, and therefore μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S. ■

In general, an $(\in, \in \vee q)$-fuzzy ternary subsemigroup may not be a $(q, \in \vee q)$-fuzzy ternary subsemigroup as seen in the following example.

Example 28 *Consider the $(\in, \in \vee q)$-fuzzy ternary subsemigroup μ of Example 26. We know that $a_{0.7}\, q\, \mu$, $c_{0.5}\, q\, \mu$ and $d_{0.4}\, q\, \mu$, but $(acd)_{\min\{0.7,0.5,0.4\}} \overline{\in \vee q}\, \mu$. Hence μ is not a $(q, \in \vee q)$-fuzzy ternary subsemigroup of S.*

Based on above discussion, we now classify (α, β)-fuzzy ternary subsemigroups in a ternary semigroup S.

In considering (α, β)-fuzzy ternary subsemigroups in a ternary semigroup S, we have twelve different types of such structures, that is, (α, β) is any one of $(\in, \in)$, $(\in, q)$, $(\in, \in \wedge q)$, $(\in, \in \vee q)$, $(q, \in)$, (q, q), $(q, \in \wedge q)$, $(q, \in \vee q)$, $(\in \vee q, \in)$, $(\in \vee q, q)$, $(\in \vee q, \in \wedge q)$, and $(\in \vee q, \in \vee q)$. Clearly, we have relations among these types which are described in the following theorems.

Theorem 29 *We have the following relations:*

$$\begin{tikzcd}
(\in, \in) \arrow[dr, Rightarrow] & (\in, \in \wedge q) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (\in, q) \arrow[dl, Rightarrow] \\
 & (\in, \in \vee q) & \\
 & (\in \vee q, \in \vee q) \arrow[u, Rightarrow] & \\
(\in \vee q, \in) \arrow[ur, Rightarrow] & (\in \vee q, \in \wedge q) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[u, Rightarrow] & (\in \vee q, q) \arrow[ul, Rightarrow]
\end{tikzcd} \tag{2.3}$$

and

$$\begin{tikzcd}
(q, \in) \arrow[dr, Rightarrow] & (q, \in \wedge q) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (q, q) \arrow[dl, Rightarrow] \\
 & (q, \in \vee q) &
\end{tikzcd} \tag{2.4}$$

Theorem 30 *If $\mu(x) > 0.5$ for all $x \in S$, then we have the following relation:*

$$\begin{tikzcd}
(\in\wedge q,\in) \arrow[dr, Rightarrow] & (\in\wedge q,\in\wedge q) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (\in\wedge q,q) \arrow[dl, Rightarrow] \\
& (\in\wedge q,\in\vee q) & \\
& (\in,\in\vee q) \arrow[u, Rightarrow] &
\end{tikzcd} \tag{2.5}$$

Combining Theorem 27 and (2.4) in Theorem 29, we have

Theorem 31 *We have the following relations.*

$$\begin{tikzcd}
(q,\in) \arrow[dr, Rightarrow] & (q,\in\wedge q) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (q,q) \arrow[dl, Rightarrow] \\
& (q,\in\vee q) \arrow[d, Rightarrow] & \\
& (\in,\in\vee q) &
\end{tikzcd} \tag{2.6}$$

The following theorem is a characterization of $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S.

Theorem 32 *A fuzzy set μ in S is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S if and only if μ satisfies:*

$$\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\} \tag{2.7}$$

for all $x, y, z \in S$.

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S. Let $x, y, z \in S$ be such that $\mu(xyz) < \min\{\mu(x), \mu(y), \mu(z), 0.5\}$. First we consider the case when $\min\{\mu(x), \mu(y), \mu(z)\} < 0.5$. Then $\mu(xyz) < \min\{\mu(x), \mu(y), \mu(z), 0.5\} = \min\{\mu(x), \mu(y), \mu(z)\}$. Choose $t \in (0, 1]$ such that $\mu(xyz) < t \leq \min\{\mu(x), \mu(y), \mu(z)\}$. Then $x_t, y_t, z_t \in \mu$ but $(xyz)_t \overline{\in} \mu$. Also $\mu(xyz) + t \leq 1$. This implies $(xyz)_t \overline{q} \mu$. Thus $(xyz)_t \overline{\in} \wedge \overline{q} \mu$, which is a contradiction.

Now we consider the second case when $\min\{\mu(x), \mu(y), \mu(z)\} \geq 0.5$. Then $\mu(xyz) < \min\{\mu(x), \mu(y), \mu(z), 0.5\} = 0.5$. Thus $x_{0.5}$, $y_{0.5}$, $z_{0.5} \in \mu$ but $(xyz)_{0.5} \overline{\in} \wedge \overline{q}\mu$, which is a contradiction. Hence $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\}$.

Conversely, assume that $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\}$. Let x_t, y_r, $z_s \in \mu$. Then $\mu(x) \geq t, \mu(y) \geq r, \mu(z) \geq s$. Now, $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\} \geq \min\{t, r, s, 0.5\}$. If $\min\{t, r, s\} \leq 0.5$, then $\mu(xyz) \geq \min\{t, r, s\}$. Thus $(xyz)_{\min\{t,r,s\}} \in \mu$. If $\min\{t, r, s\} > 0.5$, then $\mu(xyz) > 0.5$. This implies $\mu(xyz) + \min\{t, r, s\} > 0.5 + 0.5 = 1$. Thus $(xyz)_{\min\{t,r,s\}}\, q\mu$. Hence $(xyz)_{\min\{t,r,s\}} \in \vee q\mu$. Therefore μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S. ■

Theorem 33 *Let μ be a fuzzy set in S. Then μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S if and only if the nonempty level set*

$$U(\mu; t) := \{x \in S \mid \mu(x) \geq t\}$$

is a ternary subsemigroup of S for all $t \in (0, 0.5]$.

Proof. Suppose μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S and $U(\mu; t) \neq \emptyset$ for $t \in (0, 0.5]$. Let x, y, $z \in U(\mu; t)$. Then $\mu(x) \geq t, \mu(y) \geq t, \mu(z) \geq t$. Since $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\} \geq \min\{t, 0.5\} = t$. Thus $\mu(xyz) \geq t$, so $xyz \in U(\mu; t)$. Hence $U(\mu; t)$ is a ternary subsemigroup of S.

Conversely, assume that $U(\mu; t)\,(\neq \emptyset)$ is a ternary subsemigroup of S for all $t \in (0, 0.5]$. Suppose there exist $x, y, z \in S$ such that $\mu(xyz) < \min\{\mu(x), \mu(y), \mu(z), 0.5\}$. Choose $t \in (0, 0.5]$ such that $\mu(xyz) < t \leq \min\{\mu(x), \mu(y), \mu(z), 0.5\}$. Then $x, y, z \in U(\mu; t)$ but $xyz \notin U(\mu; t)$, which is a contradiction. Thus

$$\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\}.$$

Hence μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S. ■

Corollary 34 *Let μ be an (α, β)-fuzzy ternary subsemigroup of S where (α, β) is any one of $(\in, \in)$, $(\in, q)$, $(\in, \in \wedge q)$, $(\in \vee q, \in)$, $(\in \vee q, q)$, $(\in \vee q, \in \wedge q)$, and $(\in \vee q, \in \vee q)$. Then the set*

$$U(\mu; t) := \{x \in S \mid \mu(x) \geq t\}$$

is a ternary subsemigroup of S for all $t \in (0, 0.5]$.

Corollary 35 *For any subset A of S, the characteristic function χ_A of A is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S if and only if A is a ternary subsemigroup of S.*

In the following theorem we provide condition for $(\in, \in \vee q)$-fuzzy ternary subsemigroup to be an $(\in, \in)$-fuzzy ternary subsemigroup.

Theorem 36 *If μ is an $(\in, \in\vee q)$-fuzzy ternary subsemigroup of S such that $\mu(x) < 0.5$ for all $x \in S$, then μ is an $(\in, \in)$-fuzzy ternary subsemigroup of S.*

Proof. Let $x, y, z \in S$ and $t, r, s \in (0, 1]$ be such that $x_t \in \mu$, $y_r \in \mu$ and $z_s \in \mu$. Then $\mu(x) \geq t$, $\mu(y) \geq r$ and $\mu(z) \geq s$. Since $0.5 > \mu(a)$ for all $a \in S$, we have $t, r, s \in (0, 0.5]$. Since

$$\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\} \geq \min\{t, r, s, 0.5\} = \min\{t, r, s\}$$

we have $(xyz)_{\min\{t,r,s\}} \in \mu$. Hence μ is an $(\in, \in)$-fuzzy ternary subsemigroup of S. ■

Theorem 37 *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\in, \in\vee q)$-fuzzy ternary subsemigroups of S. Then $\mu := \bigcap_{i\in\Lambda} \mu_i$ is an $(\in, \in\vee q)$-fuzzy ternary subsemigroup of S.*

Proof. Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\in, \in\vee q)$-fuzzy ternary subsemigroups of a ternary semigroup S. Let $x, y, z \in S$. Then

$$\mu(xyz) = \left(\bigcap_{i\in\Lambda} \mu_i\right)(xyz) = \bigwedge_{i\in\Lambda} \mu_i(xyz)$$

Since each μ_i is an $(\in, \in\vee q)$-fuzzy ternary subsemigroup of S. It follows that

$$\begin{aligned}
\mu(xyz) &= \bigwedge_{i\in\Lambda} \mu_i(xyz) \geq \bigwedge_{i\in\Lambda} \{\mu_i(x) \wedge \mu_i(y) \wedge \mu_i(z) \wedge 0.5\} \\
&= \min\left\{\bigwedge_{i\in\Lambda} \mu_i(x),\ \bigwedge_{i\in\Lambda} \mu_i(y),\ \bigwedge_{i\in\Lambda} \mu_i(z),\ 0.5\right\} \\
&= \min\left\{\left(\bigcap_{i\in\Lambda} \mu_i\right)(x),\ \left(\bigcap_{i\in\Lambda} \mu_i\right)(y),\ \left(\bigcap_{i\in\Lambda} \mu_i\right)(z),\ 0.5\right\} \\
&= \min\{\mu(x),\ \mu(y),\ \mu(z),\ 0.5\}.
\end{aligned}$$

Hence $\mu := \bigcap_{i\in\Lambda} \mu_i$ is an $(\in, \in\vee q)$-fuzzy ternary subsemigroup of S. ■

For a fuzzy set μ in S and $t \in (0, 1]$, consider the q-set and $\in\vee q$-set with respect to t (briefly, t-q-set and t-$\in\vee q$-set, respectively) as follows:

$$S_q^t := \{x \in X \mid x_t\, q\, \mu\} \text{ and } S_{\in\vee q}^t := \{x \in X \mid x_t \in\vee q\, \mu\}.$$

Note that, for any $t, r \in (0, 1]$, if $t \geq r$ then every r-q-set is contained in the t-q-set, that is, $S_q^r \subseteq S_q^t$. Obviously, $S_{\in\vee q}^t = U(\mu; t) \cup S_q^t$.

Theorem 38 *If μ is an $(\in, \in)$-fuzzy ternary subsemigroup of S, then the t-q-set S_q^t is a ternary subsemigroup of S for all $t \in (0, 1]$ whenever it is nonempty.*

Proof. Let $x, y, z \in S_q^t$. Then $x_t q\mu$, $y_t q\mu$ and $z_t q\mu$, that is, $\mu(x)+t > 1$, $\mu(y)+t > 1$ and $\mu(z) + t > 1$. It follows that

$$\begin{aligned} \mu(xyz) + t &\geq \min\{\mu(x), \mu(y), \mu(z)\} + t \\ &= \min\{\mu(x) + t, \mu(y) + t, \mu(z) + t\} > 1 \end{aligned}$$

and so $(xyz)_t\, q\mu$. Hence $xyz \in S_q^t$, and therefore S_q^t is a ternary subsemigroup of S. ■

Definition 39 *A fuzzy set μ in S is said to be an $(\in, q)^{\max}$-fuzzy ternary subsemigroup of S if μ satisfies the following condition:*

$$x_{t_1} \in \mu,\ y_{t_2} \in \mu,\ z_{t_3} \in \mu \implies (xyz)_{\max\{t_1,t_2,t_3\}}\, q\, \mu \tag{2.8}$$

for all $x, y, z \in S$ and $t_1, t_2, t_3 \in (0, 1]$.

Obviously, every $(\in, q)$-fuzzy ternary subsemigroup is an $(\in, q)^{\max}$-fuzzy ternary subsemigroup.

Theorem 40 *For a fuzzy set μ in S, if the t-q-set S_q^t is a ternary subsemigroup of S for all $t \in (0.5, 1]$, then μ is an $(\in, q)^{\max}$-fuzzy ternary subsemigroup of S.*

Proof. Let $x, y, z \in S$ and $t_1, t_2, t_3 \in (0.5, 1]$ be such that $x_{t_1} \in \mu$, $y_{t_2} \in \mu$ and $z_{t_3} \in \mu$. Then $\mu(x) \geq t_1 > 1 - t_1$, $\mu(y) \geq t_2 > 1 - t_2$, $\mu(z) \geq t_3 > 1 - t_3$, that is $x \in S_q^{t_1}$, $y \in S_q^{t_2}$ and $z \in S_q^{t_3}$. It follows that $x, y, z \in S_q^{\max\{t_1,t_2,t_3\}}$ and $\max\{t_1, t_2, t_3\} \in (0.5, 1]$. By hypothesis, we have $xyz \in S_q^{\max\{t_1,t_2,t_3\}}$ and so $(xyz)_{\max\{t_1,t_2,t_3\}}\, q\mu$. Therefore μ is an $(\in, q)^{\max}$-fuzzy ternary subsemigroup of S. ■

Definition 41 *A fuzzy set μ in S is said to be a $(q, \in)^{\max}$-fuzzy ternary subsemigroup of S if it satisfies the following condition:*

$$x_{t_1}\, q\, \mu,\ y_{t_2}\, q\, \mu,\ z_{t_3}\, q\, \mu \Rightarrow (xyz)_{\max\{t_1,t_2,t_3\}} \in \mu \tag{2.9}$$

for all $x, y, z \in S$ and $t_1, t_2, t_3 \in (0, 1]$.

Obviously, every $(q, \in)^{\max}$-fuzzy ternary subsemigroup is a $(q, \in)$-fuzzy ternary subsemigroup.

Theorem 42 *For a fuzzy set μ in S, if the t-q-set S_q^t is a ternary subsemigroup of S for all $t \in (0, 0.5]$, then μ is a $(q, \in)^{\max}$-fuzzy ternary subsemigroup of S.*

Proof. Let $x, y, z \in S$ and $t_1, t_2, t_3 \in (0, 0.5]$ be such that $x_{t_1} q\mu$, $y_{t_2} q\mu$ and $z_{t_3} q\mu$. Then $x \in S_q^{t_1}$, $y \in S_q^{t_2}$ and $z \in S_q^{t_3}$. It follows that $x, y, z \in S_q^{\max\{t_1,t_2,t_3\}}$ and

$\max\{t_1, t_2, t_3\} \in (0, 0.5]$. Hence $xyz \in S_q^{\max\{t_1,t_2,t_3\}}$, which implies that $\mu(xyz) + \max\{t_1, t_2, t_3\} > 1$. Thus

$$\mu(xyz) > 1 - \max\{t_1, t_2, t_3\} \geq \max\{t_1, t_2, t_3\},$$

and so $(xyz)_{\max\{t_1,t_2,t_3\}} \in \mu$. Therefore μ is a $(q, \in)^{\max}$-fuzzy ternary subsemigroup of S. ∎

Theorem 43 *If μ is a $(q, \in\vee q)$-fuzzy ternary subsemigroup of S, then the t-q-set S_q^t is a ternary subsemigroup of S for all $t \in (0.5, 1]$ whenever it is nonempty.*

Proof. Let $x, y, z \in S_q^t$. Then $x_t q\mu$, $y_t q\mu$ and $z_t q\mu$. Since μ is $(q, \in\vee q)$-fuzzy ternary subsemigroup of S, we have $(xyz)_t \in \vee q\mu$, that is, $(xyz)_t \in \mu$ or $(xyz)_t\, q\mu$. If $(xyz)_t\, q\mu$, then $xyz \in S_q^t$. If $(xyz)_t \in \mu$, then $\mu(xyz) \geq t > 1 - t$, since $t > 0.5$. Hence $(xyz)_t\, q\mu$ and so $xyz \in S_q^t$. Therefore S_q^t is a ternary subsemigroup of S. ∎

Corollary 44 *If μ is an (α, β)-fuzzy ternary subsemigroup of S where (α, β) is one of $(q, \in)$, (q, q) and $(q, \in\wedge q)$, then the t-q-set S_q^t is a ternary subsemigroup of S for all $t \in (0.5, 1]$ whenever it is nonempty.*

Using Theorems 22 and 43, we have the following result.

Theorem 45 *For a ternary subsemigroup A of S, if μ is a fuzzy set in S such that*

$$\mu(x) = \begin{cases} \geq 0.5 & \text{for all } x \in A, \\ 0 & \text{for all } x \in S \setminus A, \end{cases}$$

then the nonempty t-q-set S_q^t is a ternary subsemigroup of S for all $t \in (0.5, 1]$.

Definition 46 *A fuzzy set μ in S is said to be a $(q, \in\vee q)^{\max}$-fuzzy ternary subsemigroup of S, if it satisfies the following condition:*

$$x_{t_1}\, q\, \mu,\ y_{t_2}\, q\, \mu, z_{t_3}\, q\, \mu \Rightarrow (xyz)_{\max\{t_1,t_2,t_3\}} \in\vee q\, \mu. \tag{2.10}$$

for all $x, y, z \in S$ and $t_1, t_2, t_3 \in (0, 1]$.

Theorem 47 *For a fuzzy set μ in S, if the nonempty t-$\in\vee q$-set $S_{\in\vee q}^t$ is a ternary subsemigroup of S for all $t \in (0, 1]$, then μ is a $(q, \in\vee q)^{\max}$-fuzzy ternary subsemigroup of S.*

Proof. Let $x, y, z \in S$ and $t_1, t_2, t_3 \in (0, 1]$ be such that $x_{t_1} q\mu$, $y_{t_2} q\mu$ and $z_{t_3} q\mu$. Then $x \in S_q^{t_1} \subseteq S_{\in\vee q}^{t_1}$, $y \in S_q^{t_2} \subseteq S_{\in\vee q}^{t_2}$ and $z \in S_q^{t_3} \subseteq S_{\in\vee q}^{t_3}$. It follows that $x, y, z \in S_{\in\vee q}^{\max\{t_1,t_2,t_3\}}$ and so from the hypothesis $xyz \in S_{\in\vee q}^{\max\{t_1,t_2,t_3\}}$. Hence $(xyz)_{\max\{t_1,t_2,t_3\}} \in \vee q\mu$. Therefore μ is a $(q, \in\vee q)^{\max}$-fuzzy ternary subsemigroup of S over $(0, 1]$. ∎

Theorem 48 *If μ is an $(\in, \in\vee q)$-fuzzy ternary subsemigroup of S, then the nonempty t-q-set S_q^t is a ternary subsemigroup of S for all $t \in (0.5,1]$.*

Proof. Assume that $S_q^t \neq \emptyset$ for all $t \in (0.5,1]$. Let $x,y,z \in S_q^t$. Then $x_t q\mu, y_t q\mu, z_t q\mu$, that is, $\mu(x)+t>1, \mu(y)+t>1, \mu(z)+t>1$. It follows from Theorem 32 that

$$\begin{aligned}\mu(xyz)+t &\geq \min\{\mu(x),\mu(y),\mu(z),0.5\}+t\\ &= \min\{\mu(x)+t,\mu(y)+t,\mu(z)+t,0.5+t\}\\ &> 1.\end{aligned}$$

So $(xyz)_t\, q\mu$. Hence $xyz \in S_q^t$, and therefore S_q^t is a ternary subsemigroup of S. ■

Corollary 49 *If μ is an (α,β)-fuzzy ternary subsemigroup of S where (α,β) is any one of $(\in,\in)$, $(\in,q)$, $(\in,\in\wedge q)$, $(\in\vee q,\in)$, $(\in\vee q,q)$, $(\in\vee q,\in\wedge q)$, $(\in\vee q,\in\vee q)$, $(q,\in)$, (q,q), $(q,\in\vee q)$, and $(q,\in\wedge q)$, then the nonempty t-q-set S_q^t is a ternary subsemigroup of S for all $t\in(0.5,1]$.*

Theorem 50 *Let μ be a fuzzy set in S. Then μ is an $(\in,\in\vee q)$-fuzzy ternary subsemigroup of S, if and only if the nonempty t-$\in\vee q$-set $S_{\in\vee q}^t$ is a ternary subsemigroup of S for all $t\in(0,1]$.*

Proof. Let μ be a fuzzy set in S and $t\in(0,1]$ be such that $S_{\in\vee q}^t$ is a ternary subsemigroup of S. If possible, let

$$\mu(xyz) < t \leq \min\{\mu(x),\mu(y),\mu(z),0.5\}$$

for some $t\in(0,0.5)$ and $x,y,z\in S$. Then $x,y,z\in U(\mu;t)\subseteq S_{\in\vee q}^t$, which implies that $xyz\in S_{\in\vee q}^t$. Hence $\mu(xyz)\geq t$ or $\mu(xyz)+t>1$, a contradiction. Therefore

$$\mu(xyz)\geq \min\{\mu(x),\mu(y),\mu(z),0.5\}$$

for all $x,y,z\in S$. It follows from Theorem 32 that μ is an $(\in,\in\vee q)$-fuzzy ternary subsemigroup of S.

Conversely, suppose that μ is an $(\in,\in\vee q)$-fuzzy ternary subsemigroup of S. Let $x,y,z\in S_{\in\vee q}^t$. Then $x_t\in\vee q\mu$, $y_t\in\vee q\mu$, and $z_t\in\vee q\mu$, that is,

(i) $\mu(x)\geq t$ or $\mu(x)+t>1$,

(ii) $\mu(y)\geq t$ or $\mu(y)+t>1$,

(iii) $\mu(z)\geq t$ or $\mu(z)+t>1$.

Since μ is an $(\in,\in\vee q)$-fuzzy ternary subsemigroup of S, we have

$$\mu(xyz)\geq \min\{\mu(x),\mu(y),\mu(z),0.5\}.$$

We consider four cases.

Case 1. $\mu(x) \geq t$, $\mu(y) \geq t$ and $\mu(z) \geq t$.

Case 2. Any two of $\mu(x)$, $\mu(y)$ and $\mu(z)$ are greater than or equal to t, and the remaining is greater than $1-t$.

Case 3. Any two of $\mu(x)$, $\mu(y)$ and $\mu(z)$ are greater than $1-t$, and the remaining is greater than or equal to t.

Case 4. $\mu(x)+t>1$, $\mu(y)+t>1$ and $\mu(z)+t>1$.

For the first case, if $t>0.5$, then

$$\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\} = 0.5,$$

and so $(xyz)_t\, q\mu$. If $t \leq 0.5$, then

$$\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\} \geq t,$$

and thus $(xyz)_t \in \mu$. Therefore $(xyz)_t \in \vee q\mu$.

For the Case 2, it is sufficient to consider the case: $\mu(x) \geq t$, $\mu(y) \geq t$ and $\mu(z)+t>1$. If $t>0.5$, then $1-t<0.5<t$, and so

$$\begin{aligned}
\mu(xyz) &\geq \min\{\mu(x), \mu(y), \mu(z), 0.5\} \\
&\geq \min\{t, \mu(z), 0.5\} \\
&= \begin{cases} \mu(z) & \text{if } \mu(z)<0.5 \\ 0.5 & \text{if } \mu(z) \geq 0.5 \end{cases} \\
&> 1-t,
\end{aligned}$$

which implies $(xyz)_t\, q\mu$. If $t \leq 0.5$, then

$$\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\} \geq \min\{t, 1-t, 0.5\} = t.$$

Thus $(xyz)_t \in \mu$. Hence $(xyz)_t \in \vee q\mu$.

For the Case 3, it is sufficient to consider the case: $\mu(x) \geq t > 1$, $\mu(y)+t>1$ and $\mu(z)+t>1$. If $t>0.5$, then $1-t<0.5<t$ and we have the following cases:

(i) $\mu(y) \geq \mu(z) \geq 0.5$,

(ii) $\mu(z) \geq \mu(y) \geq 0.5$,

(iii) $\mu(y) \geq 0.5 \geq \mu(z)$,

(iv) $\mu(z) \geq 0.5 \geq \mu(y)$,

(v) $0.5 \geq \mu(y) \geq \mu(z)$,

(vi) $0.5 \geq \mu(z) \geq \mu(y)$.

It follows that

$$\begin{aligned}
\mu(xyz) &\geq \min\{\mu(x), \mu(y), \mu(z), 0.5\} \\
&= \min\{\mu(y), \mu(z), 0.5\} \\
&= \begin{cases} 0.5 & \text{for the cases } (i) \text{ and } (ii), \\ \mu(y) & \text{for the cases } (iv) \text{ and } (vi), \\ \mu(z) & \text{for the cases } (iii) \text{ and } (v), \end{cases} \\
&> 1-t,
\end{aligned}$$

and thus $(xyz)_t\, q\mu$. If $t \leq 0.5$, then

$$\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\} \geq \min\{t, 1-t, 0.5\} = 0.5 = t,$$

and so $(xyz)_t \in \mu$. Hence $(xyz)_t \in \vee q\mu$.

For the case 4, if $t > 0.5$, then

$$\begin{aligned}
\mu(xyz) &\geq \min\{\mu(x), \mu(y), \mu(z), 0.5\} \\
&> \min\{1-t, 0.5\} = 1-t
\end{aligned}$$

and thus $(xyz)_t\, q\mu$. If $t \leq 0.5$, then

$$\begin{aligned}
\mu(xyz) &\geq \min\{\mu(x), \mu(y), \mu(z), 0.5\} \\
&\geq \min\{1-t, 0.5\} = 0.5 \geq t,
\end{aligned}$$

and so $(xyz)_t \in \mu$. Hence $(xyz)_t \in \vee q\mu$. Therefore $S^t_{\in\vee q}$ is a ternary subsemigroup of S. ■

Corollary 51 *If μ is an (α, β)-fuzzy ternary subsemigroup of S where (α, β) is any one of $(\in, \in)$, $(\in, q)$, $(\in, \in\wedge q)$, $(\in\vee q, \in)$, $(\in\vee q, q)$, $(\in\vee q, \in\wedge q)$, and $(\in\vee q, \in\vee q)$, then the nonempty t-$\in\vee q$-set $S^t_{\in\vee q}$ is a ternary subsemigroup of S for all $t \in (0, 1]$.*

2.2 Classifications and properties of (α, β)-fuzzy ideals

We start this section with the following definition.

Definition 52 *A fuzzy set μ in S is called an (α, β)-fuzzy left (resp. right, lateral) ideal of S if it satisfies:*

$$z_t \alpha\mu \text{ implies } (xyz)_t\, \beta\mu \ (\text{resp. } (zxy)_t\, \beta\mu, \quad (xzy)_t\, \beta\mu) \tag{2.11}$$

for all $x, y, z \in S$ and $t \in (0, 1]$.

A fuzzy set μ in S is called an (α, β)-fuzzy two sided ideal if it is both (α, β)-fuzzy left ideal and (α, β)-fuzzy right ideal of S and is called an (α, β)-fuzzy ideal if it is (α, β)-fuzzy left ideal, (α, β)-fuzzy lateral ideal and (α, β)-fuzzy right ideal of S.

Example 53 *Let $S = \{a, b, c, d, e\}$ and $xyz = (x * y) * z = x * (y * z)$ for all $x, y, z \in S$, where $*$ is defined by the following table:*

$*$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

Then S is a ternary semigroup. Define a fuzzy set μ in S as follows:

$$\mu : S \to [0, 1], \quad x \mapsto \begin{cases} 0.50 & \text{if } x = a, \\ 0.70 & \text{if } x = b, \\ 0.20 & \text{if } x = c, \\ 0.55 & \text{if } x = d, \\ 0.60 & \text{if } x = e. \end{cases}$$

Then simple calculations show that μ is an $(\in, \in \vee q)$-fuzzy left ideal of S, but neither an $(\in, \in \vee q)$-fuzzy right ideal, nor an $(\in, \in \vee q)$-fuzzy lateral ideal of S, since $e_{0.3} \in \mu$, but

$$(ecc)_{0.3} \overline{\in \vee q} \mu,$$

and

$$(cec)_{0.3} \overline{\in \vee q} \mu.$$

Moreover we see that:

(i) μ is not an $(\in, \in)$-fuzzy left ideal of S, since $b_{0.65} \in \mu$ but

$$(dcb)_{0.65} \overline{\in} \mu.$$

(ii) μ is not an $(\in, q)$-fuzzy left ideal of S, since $b_{0.3} \in \mu$, but

$$(ddb)_{0.3} \overline{q} \mu.$$

(iii) μ is not a (q, q)-fuzzy left ideal of S, since $b_{0.32} q \mu$, but

$$(ddb)_{0.32} \overline{q} \mu.$$

(iv) μ is not a $(q, \in)$-fuzzy left ideal of S, since $b_{0.65} q \mu$, but

$$(bad)_{0.65} \overline{\in} \mu.$$

(v) μ is not an $(\in, \in \wedge q)$-fuzzy left ideal of S, since $b_{0.31} \in \mu$, but

$$(ddb)_{0.31} \overline{q} \mu \text{ and so } (ddb)_{0.31} \overline{\in \wedge q} \mu.$$

(vi) μ *is not a* $(q, \in \wedge q)$*-fuzzy left ideal of* S*, since* $b_{0.33} q\mu$*, but*

$$(ddb)_{0.33}\, \overline{q}\mu \text{ and so } (ddb)_{0.33}\, \overline{\in \wedge q}\mu.$$

(vii) μ *is not an* $(\in \vee q, \in)$*-fuzzy left ideal of* S*, since* $b_{0.64} \in \vee q\mu$*, but*

$$(bad)_{0.64}\, \overline{\in}\mu.$$

(viii) μ *is not an* $(\in \vee q, q)$*-fuzzy left ideal of* S*, since* $b_{0.27} \in \vee q\mu$*, but*

$$(ddb)_{0.27}\, \overline{q}\mu.$$

(ix) μ *is not an* $(\in \vee q, \in \wedge q)$*-fuzzy left ideal of* S*, since* $b_{0.38} \in \vee q\mu$*, but*

$$(adb)_{0.38}\, \overline{q}\mu \text{ and so } (adb)_{0.38}\, \overline{\in \wedge q}\mu.$$

Example 54 *Let* $S = \{0, a, b, c, 1\}$ *and* $xyz = (x * y) * z = x * (y * z)$ *for all* $x, y, z \in S$*, where* $*$ *is defined by the following table:*

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	b	b	b
c	0	0	b	c	c
1	0	a	b	c	1

Then S *is a ternary semigroup. Define a fuzzy set* λ *in* S *as follows:*

$$\lambda : S \to [0,1], \quad x \mapsto \begin{cases} 0.8 & \text{if } x = 0, \\ 0.2 & \text{if } x = a, \\ 0.5 & \text{if } x = b, \\ 0.7 & \text{if } x = c, \\ 0.2 & \text{if } x = 1. \end{cases}$$

Then simple calculations show that λ *is an* $(\in, \in \vee q)$*-fuzzy right ideal of* S*, but neither an* $(\in, \in \vee q)$*-fuzzy left ideal nor an* $(\in, \in \vee q)$*-fuzzy lateral ideal of* S*, since* $c_{0.1} \in \lambda$*, but*

$$(1ac)_{0.1}\, \overline{\in \vee q}\lambda$$

and

$$(ac1)_{0.1}\, \overline{\in \vee q}\lambda.$$

Moreover we see that:

(i) λ *is not an* $(\in, \in)$*-fuzzy right ideal of* S, *since* $c_{0.6} \in \lambda$ *but*

$$(c1b)_{0.6} \overline{\in} \lambda.$$

(ii) λ *is not an* $(\in, q)$*-fuzzy right ideal of* S, *since* $c_{0.3} \in \lambda$, *but*

$$(c1b)_{0.3} \overline{q} \lambda.$$

(iii) λ *is not a* (q, q)*-fuzzy right ideal of* S, *since* $c_{0.31} q \lambda$, *but*

$$(cbb)_{0.31} \overline{q} \mu.$$

(iv) λ *is not a* $(q, \in)$*-fuzzy right ideal of* S, *since* $a_{0.9} q \lambda$, *but*

$$(a00)_{0.9} \overline{\in} \lambda.$$

(v) λ *is not an* $(\in, \in \wedge q)$*-fuzzy right ideal of* S, *since* $c_{0.51} \in \lambda$, *but*

$$(c1b)_{0.51} \overline{\in} \lambda \text{ and so } (bca)_{0.31} \overline{\in \wedge q} \lambda.$$

(vi) λ *is not a* $(q, \in \wedge q)$*-fuzzy right ideal of* S, *since* $c_{0.35} q \lambda$, *but*

$$(cbb)_{0.35} \overline{q} \lambda \text{ and so } (cbb)_{0.35} \overline{\in \wedge q} \lambda.$$

(vii) λ *is not an* $(\in \vee q, \in)$*-fuzzy right ideal of* S, *since* $c_{0.59} \in \vee q \lambda$, *but*

$$(c1b)_{0.59} \overline{\in} \lambda.$$

(viii) λ *is not an* $(\in \vee q, q)$*-fuzzy right ideal of* S, *since* $c_{0.34} \in \vee q \lambda$, *but*

$$(abb)_{0.34} \overline{q} \lambda.$$

(ix) λ *is not an* $(\in \vee q, \in \wedge q)$*-fuzzy right ideal of* S, *since* $c_{0.39} \in \vee q \lambda$, *but*

$$(cbb)_{0.39} \overline{q} \lambda \text{ and so } (cbb)_{0.39} \overline{\in \wedge q} \lambda.$$

Remark 55 *In a ternary semigroup* S, *every* $(\in, \in \vee q)$*-fuzzy left (right, lateral) ideal of* S *is an* $(\in, \in \vee q)$*-fuzzy ternary subsemigroup of* S, *but the converse is not true in general. In fact, the* $(\in, \in \vee q)$*-fuzzy ternary subsemigroup of* S *given in Example 23 is neither an* $(\in, \in \vee q)$*-fuzzy left ideal nor an* $(\in, \in \vee q)$*-fuzzy lateral ideal nor an* $(\in, \in \vee q)$*-fuzzy right ideal of* S.

Theorem 56 *A fuzzy set* μ *in* S *is a fuzzy left (resp. right, lateral) ideal of* S *if and only if* μ *is an* $(\in, \in)$*-fuzzy left (resp. right, lateral) ideal of* S.

Proof. The proof is similar to the proof of Theorem 17. ■

Theorem 57 *Every $(\in, \in)$-fuzzy ideal of S is an $(\in, \in\vee q)$-fuzzy ideal of S.*

Proof. Straightforward. ■

Remark 58 *The converse of Theorem 57 is not true in general as seen in the following examples:*

Example 59 *Let $S = \mathbb{Z}^-$ be the set of all negative integers, where $\mathbb{Z}^-$ is a ternary semigroup with respect to ternary multiplication of numbers. Let A be the subset of S such that $A = \{-2, -4, -6, ...\}$. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1],\ x \to \begin{cases} 0.5 & \text{if } a = -2x \textit{ for some positive integer } x, \\ 0.6 & \text{if } a \neq -2x \quad \textit{and } a \neq -1, \\ 0.7 & \text{if } a = -1. \end{cases}$$

Clearly μ is an $(\in, \in \vee q)$-fuzzy ideal of S but not an $(\in, \in)$- fuzzy ideal of S.

Example 60 *Consider the left zero ternary semigroup $S = \{a, b, c, d\}$. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.5 & \text{if } x = a, \\ 0.6 & \text{if } x = b, \\ 0.8 & \text{if } x = c, \\ 0.9 & \text{if } x = d. \end{cases}$$

Then simple calculations show that μ is an $(\in, \in \vee q)$-fuzzy ideal of S but not a fuzzy ideal of S because $\mu(acd) \ngeq \mu(c)$.

Theorem 61 *Let μ be a fuzzy set in S. Then μ is an $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideal of S if and only if*

$$\mu(xyz) \geq \min\{\mu(z), 0.5\}$$

$$(\textit{resp. } \mu(xyz) \geq \min\{\mu(x), 0.5\},\ \mu(xyz) \geq \min\{\mu(y), 0.5\}).$$

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy left ideal of S. Suppose there exist $x, y, z \in S$ such that $\mu(xyz) < \min\{\mu(z), 0.5\}$. Choose $t \in (0,1]$ such that $\mu(xyz) < t \leq \min\{\mu(z), 0.5\}$. Then $z_t \in \mu$ but $(xyz)_t \overline{\in \vee q}\mu$, which is a contradiction. Hence $\mu(xyz) \geq \min\{\mu(z), 0.5\}$.

Conversely, assume that $\mu(xyz) \geq \min\{\mu(z), 0.5\}$. Let $z_t \in \mu$. Then $\mu(z) \geq t$. Now, $\mu(xyz) \geq \min\{\mu(z), 0.5\} \geq \min\{t, 0.5\}$. If $t \leq 0.5$, then $\mu(xyz) \geq t$ so $(xyz)_t \in \mu$. If $t > 0.5$, then $\mu(xyz) > 0.5$. This implies that $\mu(xyz) + t > 0.5 + 0.5 = 1$. Thus $(xyz)_t\, q\mu$. Hence $(xyz)_t \in \vee q\mu$.

Similarly we can prove the other cases. ■

Corollary 62 *Let μ be a fuzzy set in S. Then μ is an $(\in, \in \vee q)$-fuzzy two sided ideal of S if and only if $\mu(xyz) \geq \min\{\mu(z), 0.5\}$ and $\mu(xyz) \geq \min\{\mu(x), 0.5\}$.*

Theorem 63 *A fuzzy set μ in S is an $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideal of S if and only if the nonempty level set $U(\mu; t)$ is a left (resp. right, lateral) ideal of S for all $t \in (0, 0.5]$.*

Proof. The proof is similar to the proo of Theorem 33. ■

Corollary 64 *Let μ be an (α, β)-fuzzy left (resp. right, lateral) ideal of S where (α, β) is any one of $(\in, \in)$, $(\in, q)$, $(\in, \in \wedge q)$, $(\in \vee q, \in)$, $(\in \vee q, q)$, $(\in \vee q, \in \wedge q)$, and $(\in \vee q, \in \vee q)$. Then the set $U(\mu; t)$ is a left (resp. right, lateral) ideal of S for all $t \in (0, 0.5]$.*

Corollary 65 *Let μ be a nonzero $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideal of S. Then the set S_0 is a left (resp. right, lateral) ideal of S.*

Theorem 66 *Let L be a nonempty subset of S and $\alpha \in \{\in, q, \in \vee q\}$. Then L is a left (resp. right, lateral) ideal of S if and only if the fuzzy set μ in S defined by:*

$$\mu(x) = \begin{cases} \geq 0.5 & \text{for all } x \in L, \\ 0 & \text{for all } x \in S \setminus L, \end{cases}$$

is an $(\alpha, \in \vee q)$-fuzzy left (resp. right, lateral) ideal of S.

Proof. The proof is similar to the proof of Theorem 22. ■

Corollary 67 *The nonempty subset A of S is a right (resp. left, lateral) ideal of S if and only if the characteristic function χ_A of A is an $(\alpha, \in \vee q)$-fuzzy right (resp. left, lateral) ideal of S.*

Proposition 68 *In a left (resp. right) zero ternary semigroup, every fuzzy set is an $(\in, \in \vee q)$-fuzzy right (resp. left) ideal.*

Proof. Straightforward. ■

Remark 69 *In a left zero ternary semigroup S, there exists a fuzzy set which is neither an $(\in, \in \vee q)$-fuzzy left ideal nor an $(\in, \in \vee q)$-fuzzy lateral ideal of S as seen in the following example.*

Example 70 *Let $S = \{a, b, c, d\}$ be the left zero ternary semigroup. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0, 1], \quad x \mapsto \begin{cases} 0.32 & \text{if } x = a, \\ 0.63 & \text{if } x = b, \\ 0.79 & \text{if } x = c, \\ 0.87 & \text{if } x = d. \end{cases}$$

Then μ is neither an $(\in, \in \vee q)$-fuzzy left ideal nor an $(\in, \in \vee q)$-fuzzy lateral ideal of S, since

$$\mu(acd) \ngeq \min\{\mu(d), 0.5\}$$

and

$$\mu(acd) \ngeq \min\{\mu(c), 0.5\}.$$

Now, we investigate relations between $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideals and $(q, \in \vee q)$-fuzzy left (resp. right, lateral) ideals.

Theorem 71 *Every $(q, \in \vee q)$-fuzzy left (resp. right, lateral) ideal of S is an $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideal of S.*

Proof. The proof is similar to the proof of Theorem 27. ■

In the following theorem we provide condition for an $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideal of S to be an $(\in, \in)$-fuzzy left (resp. right, lateral) ideal of S.

Theorem 72 *Let μ be an $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideal of S such that $\mu(x) < 0.5$ for all $x \in S$. Then μ is an $(\in, \in)$-fuzzy left (resp. right, lateral) ideal of S.*

Proof. It is straightforward. ■

The proofs of the following theorems are straightforward and we omit the details.

Theorem 73 *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideals of S. Then $\mu := \bigcap_{i \in \Lambda} \mu_i$ is an $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideal of S.*

Theorem 74 *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideals of S. Then $\mu := \bigcup_{i \in \Lambda} \mu_i$ is an $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideal of S.*

Based on above discussion, we now classify (α, β)-fuzzy left (right, lateral) ideals in a ternary semigroup S.

In considering (α, β)-fuzzy left (right, lateral) ideals in a ternary semigroup S, we have twelve different types of such structures, that is, (α, β) is any one of $(\in, \in)$, $(\in, q)$, $(\in, \in \wedge q)$, $(\in, \in \vee q)$, $(q, \in)$, (q, q), $(q, \in \wedge q)$, $(q, \in \vee q)$, $(\in \vee q, \in)$, $(\in \vee q, q)$, $(\in \vee q, \in \wedge q)$, and $(\in \vee q, \in \vee q)$. Clearly, we have relations among these types which are described in Theorems 29, 30 and 31.

The proofs of the following theorems are similar to the corresponding theorems proved in Section 2.1, so we omit the detail.

Theorem 75 *If μ is an $(\in, \in)$-fuzzy left (resp. right, lateral) ideal of S, then the t-q-set S_q^t is a left (resp. right, lateral) ideal of S for all $t \in (0, 1]$, whenever it is nonempty.*

Theorem 76 *For a fuzzy set μ in S, if the t-q-set S_q^t is a left (resp. right, lateral) ideal of S for all $t \in (0.5, 1]$, then μ is an $(\in, q)$-fuzzy left (resp. right, lateral) ideal of S.*

Theorem 77 *For a fuzzy set μ in S, if the t-q-set S_q^t is a left (resp. right, lateral) ideal of S for all $t \in (0, 0.5]$, then μ is a $(q, \in)$-fuzzy left (resp. right, lateral) ideal of S.*

Theorem 78 *If μ is a $(q, \in\vee q)$-fuzzy left (resp. right, lateral) ideal of S, then the t-q-set S_q^t is a left (resp. right, lateral) ideal of S for all $t \in (0.5, 1]$, whenever it is nonempty.*

Corollary 79 *If μ is an (α, β)-fuzzy left (resp. right, lateral) ideal of S where (α, β) is one of $(q, \in)$, (q, q) and $(q, \in\wedge q)$, then the t-q-set S_q^t is a left (resp. right, lateral) ideal of S for all $t \in (0.5, 1]$ whenever it is nonempty.*

Using Theorems 66 and 78, we have the following result.

Theorem 80 *For a left (resp. right, lateral) ideal A of S, if μ is a fuzzy set in S such that*

$$\mu(x) = \begin{cases} \geq 0.5 & \text{for all } x \in A, \\ 0 & \text{for all } x \in S \setminus A, \end{cases}$$

then the nonempty t-q-set S_q^t is a left (resp. right, lateral) ideal of S for all $t \in (0.5, 1]$.

Theorem 81 *For a fuzzy set μ in S, if the nonempty t-$\in\vee q$-set $S_{\in\vee q}^t$ is a left (resp. right, lateral) ideal of S for all $t \in (0, 1]$, then μ is a $(q, \in\vee q)$-fuzzy left (resp. right, lateral) ideal of S.*

One naturally asks the following interesting question:

Question: If μ is an $(\in, \in \vee q)$-fuzzy fuzzy left (resp. right, lateral) ideal of S, then is the t-q-set S_q^t a left (resp. right, lateral) ideal of S?

The answer to the above question is negative (for $t \leq 0.5$) as seen in the following example:

Example 82 *Consider the ternary semigroup S of Example 53. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.87 & \text{if } x = a, \\ 0.74 & \text{if } x = b, \\ 0.25 & \text{if } x = c, \\ 0.62 & \text{if } x = d, \\ 0.41 & \text{if } x = e. \end{cases}$$

Then simple calculations show that μ is an $(\in, \in \vee q)$-fuzzy left ideal of S, but the set

$$S_q^{0.27} = \{a, b\}$$

is not a left ideal of S because $aab = d \notin S_q^{0.27}$.

But the following theorem answers the above question affirmatively:

Theorem 83 *If μ is an $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideal of S, then the nonempty t-q-set S_q^t is a left (resp. right, lateral) ideal of S for all $t \in (0.5, 1]$.*

Corollary 84 *If μ is an (α, β)-fuzzy left (resp. right, lateral) ideal of S where (α, β) is any one of $(\in, \in)$, $(\in, q)$, $(\in, \in \wedge q)$, $(\in \vee q, \in)$, $(\in \vee q, q)$, $(\in \vee q, \in \wedge q)$, $(\in \vee q, \in \vee q)$, $(q, \in)$, (q, q), $(q, \in \vee q)$, and $(q, \in \wedge q)$, then the nonempty t-q-set S_q^t is a left (resp. right, lateral) ideal of S for all $t \in (0.5, 1]$.*

Theorem 85 *Let μ be a fuzzy set in S. Then μ is an $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideal of S, if and only if the nonempty t-$\in \vee q$-set $S_{\in \vee q}^t$ is a left (resp. right, lateral) ideal of S for all $t \in (0, 1]$.*

Corollary 86 *If μ is an (α, β)-fuzzy left (resp. right, lateral) ideal of S where (α, β) is any one of $(\in, \in)$, $(\in, q)$, $(\in, \in \wedge q)$, $(\in \vee q, \in)$, $(\in \vee q, q)$, $(\in \vee q, \in \wedge q)$, $(\in \vee q, \in \vee q)$, $(q, \in)$, (q, q), $(q, \in \vee q)$, and $(q, \in \wedge q)$, then the nonempty t-$\in \vee q$-set $S_{\in \vee q}^t$ is a left (resp. right, lateral) ideal of S for all $t \in (0, 0.1]$.*

2.3 Properties of fuzzy prime ideals of type $(\in, \in \vee q)$

Definition 87 *An $(\in, \in \vee q)$-fuzzy ideal μ of S is called an $(\in, \in \vee q)$-fuzzy prime ideal if it satisfies:*

$$(xyz)_t \in \mu \text{ implies } x_t \in \vee q\mu, \ y_t \in \vee q\mu \text{ or } z_t \in \vee q\mu \tag{2.12}$$

for all $x, y, z \in S$ and $t \in (0, 1]$.

Example 88 *Let $S=\overline{\mathbb{Z}}_0$, where $\overline{\mathbb{Z}}_0$ is the set of all negative integers with zero. Then S is a ternary semigroup under the ternary multiplication of numbers. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.9 & \text{if } x=0, \\ 0.8 & \text{if } x \in \left\{9k \mid k \in \overline{\mathbb{Z}}\right\}, \\ 0.7 & \text{if } x \in \left\{6k \mid k \in \overline{\mathbb{Z}}\right\}, \\ 0.6 & \text{if } x \in \left\{3k \mid k \in \overline{\mathbb{Z}}\right\}, \\ 0.2 & \text{otherwise.} \end{cases}$$

Then simple calculations show that μ is an $(\in,\in\vee q)$-fuzzy prime ideal of S.

Theorem 89 *Let μ be an $(\in,\in\vee q)$-fuzzy ideal of S. Then μ is an $(\in,\in\vee q)$-fuzzy prime ideal of S if and only if it satisfies the following condition:*

$$\max\left\{\mu(x),\mu(y),\mu(z)\right\} \geq \min\left\{\mu(xyz),0.5\right\}$$

for all $x,y,z\in S$.

Proof. Let μ be an $(\in,\in\vee q)$-fuzzy prime ideal of S and $x,y,z\in S$. Then clearly $\max\left\{\mu(x),\mu(y),\mu(z)\right\} \geq \min\left\{\mu(xyz),0.5\right\}$. For if

$$\max\left\{\mu(x),\mu(y),\mu(z)\right\} < \min\left\{\mu(xyz),0.5\right\}$$

for some $x,y,z\in S$, then we can choose $t\in(0,0.5]$ such that

$$\max\left\{\mu(x),\mu(y),\mu(z)\right\} < t < \min\left\{\mu(xyz),0.5\right\}.$$

Then $(xyz)_t\in\mu$, but $\mu(x)<t$, $\mu(y)<t$ and $\mu(z)<t$. Also $\mu(x)+t<0.5+0.5=1$, that is, $x_t\overline{q}\mu$. Similarly $y_t\overline{q}\mu$ and $z_t\overline{q}\mu$, that is, $x_t\overline{\in\vee q}\mu$, $y_t\overline{\in\vee q}\mu$ and $z_t\overline{\in\vee q}\mu$, a contradiction. Hence $\max\left\{\mu(x),\mu(y),\mu(z)\right\} \geq \min\left\{\mu(xyz),0.5\right\}$.

Conversely, assume that $\max\left\{\mu(x),\mu(y),\mu(z)\right\} \geq \min\left\{\mu(xyz),0.5\right\}$ for all $x,y,z\in S$. Let $(xyz)_t\in\mu$. Then $\mu(xyz)\geq t$ and so

$$\begin{aligned} \max\left\{\mu(x),\mu(y),\mu(z)\right\} &\geq \min\left\{\mu(xyz),0.5\right\} \\ &\geq \min\left\{t,0.5\right\}. \end{aligned}$$

If $t\leq 0.5$, then $\max\left\{\mu(x),\mu(y),\mu(z)\right\}\geq t$ and so $x_t\in\mu$, $y_t\in\mu$ or $z_t\in\mu$, that is, $x_t\in\vee q\mu$, $y_t\in\vee q\mu$ or $z_t\in\vee q\mu$. If $t>0.5$, then $\max\left\{\mu(x),\mu(y),\mu(z)\right\}\geq 0.5$. If $\mu(x)=\max\left\{\mu(x),\mu(y),\mu(z)\right\}$, then $\mu(x)+t>0.5+0.5=1$, that is $x_tq\mu$. Similarly $y_tq\mu$ or $z_tq\mu$, that is $x_t\in\vee q\mu$, $y_t\in\vee q\mu$ or $z_t\in\vee q\mu$. Therefore μ is an $(\in,\in\vee q)$-fuzzy prime ideal of S. ■

Lemma 90 *A nonempty subset A of S is a prime ideal of S if and only if the characteristic function χ_A of A is an $(\in, \in \vee q)$-fuzzy prime ideal of S.*

Proof. Straightforward. ■

Theorem 91 *A fuzzy set μ is S is an $(\in, \in \vee q)$-fuzzy prime ideal of S if and only if the set $U(\mu; t)$ is a prime ideal of S for all $t \in (0, 0.5]$.*

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy prime ideal of S and $t \in (0, 0.5]$. Then by Theorem 63, $U(\mu; t)$ is an ideal of S. Let $xyz \in U(\mu; t)$. Then by Theorem 89

$$\begin{aligned} \max\{\mu(x), \mu(y), \mu(z)\} &\geq \min\{\mu(xyz), 0.5\} \\ &\geq \min\{t, 0.5\} = t, \end{aligned}$$

and so $\mu(x) \geq 0.5$, $\mu(y) \geq 0.5$ or $\mu(z) \geq 0.5$. Then $x \in U(\mu; t)$, $y \in U(\mu; t)$ or $z \in U(\mu; t)$. Hence $U(\mu; t)$ is prime.

Conversely assume that $U(\mu; t)$ is a prime ideal of S. Then by Theorem 63, μ is an $(\in, \in \vee q)$-fuzzy ideal of S. Let $(xyz)_t \in \mu$. Then $xyz \in U(\mu; t)$. Since $U(\mu; t)$ is prime, so $x \in U(\mu; t)$, $y \in U(\mu; t)$ or $z \in U(\mu; t)$. Thus $x_t \in \vee q\mu$, $y_t \in \vee q\mu$ or $z_t \in \vee q\mu$. Hence μ is an $(\in, \in \vee q)$-fuzzy prime ideal of S. ■

Theorem 92 *If μ is an $(\in, \in \vee q)$-fuzzy prime ideal of S, then $\mu \wedge 0.5$ is an $(\in, \in \vee q)$-fuzzy prime ideal of S, where $(\mu \wedge 0.5)(x) = \mu(x) \wedge 0.5$ for all $x \in S$.*

Proof. Straightforward. ■

Theorem 93 *Let μ be a fuzzy set in S. Then μ is an $(\in, \in \vee q)$-fuzzy prime ideal of S if and only if the nonempty t-$\in \vee q$-set $S^t_{\in \vee q}$ is a prime ideal of S for all $t \in (0, 1]$.*

Proof. Assume that μ is an $(\in, \in \vee q)$-fuzzy prime ideal of S. Then by Theorem 85, we have $S^t_{\in \vee q}$ is an ideal of S for all $t \in (0, 1]$. To prove $S^t_{\in \vee q}$ is prime, let $xyz \in S^t_{\in \vee q}$. Since $S^t_{\in \vee q} = S^t_q \cup U(\mu; t)$, we have $xyz \in S^t_q$ or $xyz \in U(\mu; t)$.

Case 1. $xyz \in S^t_q - U(\mu; t)$. Then $\mu(xyz) + t > 1$ and $\mu(xyz) < t$.

(a) If $\mu(xyz) \leq 0.5$, then $\max\{\mu(x), \mu(y), \mu(z)\} + t \geq \min\{\mu(xyz), 0.5\} + t = \mu(xyz) + t > 1$, which implies that $\mu(x) + t > 1, \mu(y) + t > 1$ or $\mu(z) + t > 1$, that is, $x \in S^t_q \subseteq S^t_{\in \vee q}$, $y \in S^t_q \subseteq S^t_{\in \vee q}$ or $z \in S^t_q \subseteq S^t_{\in \vee q}$.

(b) If $\mu(xyz) > 0.5$, then $0.5 < \mu(xyz) < t$. Thus, $\max\{\mu(x), \mu(y), \mu(z)\} + t \geq \min\{\mu(xyz), 0.5\} + t = 0.5 + t > 1$. Hence $x \in S^t_q \subseteq S^t_{\in \vee q}$, $y \in S^t_q \subseteq S^t_{\in \vee q}$ or $z \in S^t_q \subseteq S^t_{\in \vee q}$.

Case 2. $xyz \in U(\mu; t)$. Then $\mu(xyz) \geq t$.

(a) If $t \leq 0.5$, then $\max\{\mu(x), \mu(y), \mu(z)\} \geq \min\{\mu(xyz), 0.5\} \geq t$, which implies that, $x \in U(\mu; t) \subseteq S^t_{\in\vee q}$, $y \in U(\mu; t) \subseteq S^t_{\in\vee q}$ or $z \in U(\mu; t) \subseteq S^t_{\in\vee q}$.

(b) If $t > 0.5$, then $\max\{\mu(x), \mu(y), \mu(z)\} \geq \min\{t, 0.5\} = 0.5$, which implies that, $\max\{\mu(x), \mu(y), \mu(z)\} + t > 1$. Hence $x \in S^t_q \subseteq S^t_{\in\vee q}$, $y \in S^t_q \subseteq S^t_{\in\vee q}$ or $z \in S^t_q \subseteq S^t_{\in\vee q}$. Therefore $S^t_{\in\vee q}$ is prime ideal of S.

Conversely assume that the nonempty t-$\in\vee q$-set $S^t_{\in\vee q}$ is a prime ideal of S for all $t \in (0, 1]$. Then by Theorem 85, μ is an $(\in, \in\vee q)$-fuzzy ideal of S. Let $(xyz)_t \in \mu$. Then $xyz \in U(\mu; t) \subseteq S^t_{\in\vee q}$. Since $S^t_{\in\vee q}$ is prime, we have $x \in S^t_{\in\vee q}$, $y \in S^t_{\in\vee q}$ or $z \in S^t_{\in\vee q}$. This implies that $x_t \in\vee q\mu$, $y_t \in\vee q\mu$ or $z_t \in\vee q\mu$. Therefore μ is an $(\in, \in\vee q)$-fuzzy prime ideal of S. ■

2.4 Properties of fuzzy quasi-ideals of type $(\in, \in\vee q)$

Definition 94 *A fuzzy set μ in S is said to be an $(\in, \in\vee q)$-fuzzy quasi-ideal of S if it satisfies the following conditions:*

$$
\begin{aligned}
(i)\ \ \mu(x) &\geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x), 0.5\}, \\
(ii)\ \ \mu(x) &\geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x), 0.5\},
\end{aligned}
$$

for all $x \in S$, where $\mathcal{S}$ is the fuzzy set in S mapping every element of S on 1.

Example 95 *Let* $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $d = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. *Then $S = \{0, a, b, c, d\}$ is a ternary semigroup under matrix multiplication. Define fuzzy sets μ and λ in S as follows:*

$$
\mu : S \to [0, 1], \quad x \mapsto \begin{cases} 0.9 & \text{if } x = 0, \\ 0.6 & \text{if } x = a, \\ 0.4 & \text{if } x = b, \\ 0.2 & \text{if } x = c, \\ 0.2 & \text{if } x = d. \end{cases}
$$

and

$$
\lambda : S \to [0, 1], \quad x \mapsto \begin{cases} 0.7 & \text{if } x = 0, \\ 0.5 & \text{if } x = a, \\ 0.3 & \text{if } x = b, \\ 0.2 & \text{if } x = c, \\ 0.1 & \text{if } x = d. \end{cases}
$$

Then simple calculations show that μ and λ are $(\in, \in\vee q)$-fuzzy quasi-ideals of S.

Theorem 96 *If μ is an $(\in,\in\vee q)$-fuzzy quasi-ideal of S, then the set S_0 is a quasi-ideal of S.*

Proof. We show that $SSS_0\cap SS_0S\cap S_0SS\subseteq S_0$ and $SS\ S_0\cap SSS_0SS\cap S_0SS\subseteq S_0$.
Let $a\in SSS_0\cap SS_0S\cap S_0SS$. Then $a\in SSS_0$, $a\in SS_0S$ and $a\in S_0SS$. This implies that there exist $x,y,z\in S_0$ and $s_1,s_2,s_3,t_1,t_2,t_3\in S$ such that $a=s_1t_1x$, $a=s_2yt_2$, $a=zs_3t_3$. Since μ is an $(\in,\in\vee q)$-fuzzy quasi-ideal, it follows that

$$\mu(a)\geq \min\{(\mu\circ\mathcal{S}\circ\mathcal{S})(a),(\mathcal{S}\circ\mu\circ\mathcal{S})(a),(\mathcal{S}\circ\mathcal{S}\circ\mu)(a),0.5\}.$$

Consider

$$\begin{aligned}(\mathcal{S}\circ\mathcal{S}\circ\mu)(a) &= \bigvee_{a=pqr}\{\mathcal{S}(p)\wedge\mathcal{S}(q)\wedge\mu(r)\}\\ &\geq \mathcal{S}(s_1)\wedge\mathcal{S}(t_1)\wedge\mu(x)\\ &= 1\wedge 1\wedge\mu(x)=\mu(x).\end{aligned}$$

Similarly $(\mathcal{S}\circ\mu\circ\mathcal{S})(a)\geq\mu(y)$ and $(\mu\circ\mathcal{S}\circ\mathcal{S})(a)\geq\mu(z)$. It follows that

$$\begin{aligned}\mu(a) &\geq \min\{(\mu\circ\mathcal{S}\circ\mathcal{S})(a),(\mathcal{S}\circ\mu\circ\mathcal{S})(a),(\mathcal{S}\circ\mathcal{S}\circ\mu)(a),0.5\}\\ &\geq \min\{\mu(z),\mu(y),\mu(x),0.5\}\\ &> 0\quad(\text{since }\ \mu(x)>0,\ \mu(y)>0,\ \mu(z)>0).\end{aligned}$$

So $a\in S_0$. Thus $SSS_0\cap S\mu_0S\cap\ S_0SS\subseteq S_0$. Next suppose that $a\in SSS_0\cap SSS_0SS\cap S_0SS$. Then $a\in SSS_0$, $a\in SSS_0SS$ and $a\in S_0SS$. Therefore $a=s_1t_1x$, $a=zs_3t_3$, $a=s_2t_2ys_4t_4$ for some x, y, $z\in S_0$ and $s_1,s_2,s_3,s_4,t_1,t_2,t_3,t_4\in S$. For $a=s_1t_1x$, $a=zs_3t_3$ discussed above.

Now, $\mu(a)\geq\min\{(\mu\circ\mathcal{S}\circ\mathcal{S})(a),(\mathcal{S}\circ\mathcal{S}\circ\mu\circ\mathcal{S}\circ\mathcal{S})(a),(\mathcal{S}\circ\mathcal{S}\circ\mu)(a),0.5\}$ and by above arguments $(\mathcal{S}\circ\mu\circ\mathcal{S})(a)\geq\mu(y)$, $(\mu\circ\mathcal{S}\circ\mathcal{S})(a)\geq\mu(z)$ and

$$\begin{aligned}(\mathcal{S}\circ\mathcal{S}\circ\mu\circ\mathcal{S}\circ\mathcal{S})(a) &= \bigvee_{a=rst}\{(\mathcal{S}\circ\mathcal{S}\circ\mu)(r)\wedge\mathcal{S}(s)\wedge\mathcal{S}(t)\}\\ &= \bigvee_{a=rst}\left\{\left\{\bigvee_{r=jlm}(\mathcal{S}(j)\wedge\mathcal{S}(l)\wedge\mu(m))\right\}\wedge\mathcal{S}(s)\wedge\mathcal{S}(t)\right\}\\ &= \bigvee_{a=(jlm)st}\{\mathcal{S}(j)\wedge\mathcal{S}(l)\wedge\mu(m)\wedge\mathcal{S}(s)\wedge\mathcal{S}(t)\}\\ &\geq \mathcal{S}(s_2)\wedge\mathcal{S}(t_2)\wedge\mu(y)\wedge\mathcal{S}(s_4)\wedge\mathcal{S}(t_4)\\ &= 1\wedge1\wedge\mu(y)\wedge1\wedge1=\mu(y).\end{aligned}$$

We have

$$\begin{aligned}\mu(a) &\geq \min\{(\mu\circ\mathcal{S}\circ\mathcal{S})(a),(\mathcal{S}\circ\mathcal{S}\circ\mu\circ\mathcal{S}\circ\mathcal{S})(a),(\mathcal{S}\circ\mathcal{S}\circ\mu)(a),0.5\}\\ &\geq \{\mu(z),\mu(y),\mu(x),0.5\}\\ &> 0\quad(\text{since }\mu(x)>0,\ \mu(y)>0,\ \mu(z)>0).\end{aligned}$$

Thus, $a \in S_0$ and hence $SSS_0 \cap SSS_0SS \cap S_0SS \subseteq S_0$. Therefore S_0 is a quasi-ideal of S. ■

Now we shall prove

Lemma 97 *A nonempty subset Q of S is a quasi-ideal of S if and only if the characteristic function χ_Q of Q is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S.*

Proof. Assume that Q is a quasi-ideal of S, χ_Q the characteristic function of Q and $x \in S$. If $x \notin Q$, then $x \notin SSQ$, $x \notin QSS$ or $x \notin SQS$. If $x \notin SSQ$, then $\left(\mathcal{S} \circ \mathcal{S} \circ \chi_Q\right)(x) = 0$. It follows that

$$\min\left\{\left(\mathcal{S} \circ \mathcal{S} \circ \chi_Q\right)(x), \left(\mathcal{S} \circ \chi_Q \circ \mathcal{S}\right)(x), \left(\chi_Q \circ \mathcal{S} \circ \mathcal{S}\right)(x), 0.5\right\} = 0 = \chi_Q(x) = 0.$$

Similarly for other cases. If $x \in Q$, then

$$\chi_Q(x) = 1 \geq \min\left\{\left(\mathcal{S} \circ \mathcal{S} \circ \chi_Q\right)(x), \left(\mathcal{S} \circ \chi_Q \circ \mathcal{S}\right)(x), \left(\chi_Q \circ \mathcal{S} \circ \mathcal{S}\right)(x), 0.5\right\}.$$

Similar arguments leads to

$$\chi_Q(x) \geq \min\left\{\left(\mathcal{S} \circ \mathcal{S} \circ \chi_Q\right)(x), \left(\mathcal{S} \circ \mathcal{S} \circ \chi_Q \circ \mathcal{S} \circ \mathcal{S}\right)(x), \left(\chi_Q \circ \mathcal{S} \circ \mathcal{S}\right)(x), 0.5\right\}.$$

Hence χ_Q is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S.

Conversely, assume that χ_Q is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S. Let $a \in SSQ \cap SQS \cap QSS$. Then $a \in SSQ$, $a \in SQS$ and $a \in QSS$. This implies that there exist $x, y, z \in Q$ and $s_1, s_2, s_3, t_1, t_2, t_3 \in S$ such that $a = s_1t_1x$ and $a = s_2yt_2$ and $a = zs_3t_3$. We have

$$\begin{aligned}
\left(\chi_Q \circ \mathcal{S} \circ \mathcal{S}\right)(a) &= \bigvee_{a=pqr} \left\{\chi_Q(p) \wedge \mathcal{S}(q) \wedge \mathcal{S}(r)\right\} \\
&\geq \chi_Q(z) \wedge \mathcal{S}(s_3) \wedge \mathcal{S}(t_3) = 1 \wedge 1 \wedge 1 = 1.
\end{aligned}$$

So $\left(\chi_Q \circ \mathcal{S} \circ \mathcal{S}\right)(a) = 1$. Similarly $\left(\mathcal{S} \circ \mathcal{S} \circ \chi_Q\right)(a) = 1$ and $\left(\mathcal{S} \circ \chi_Q \circ \mathcal{S}\right)(a) = 1$. It follows that

$$\chi_Q(a) \geq \min\left\{\left(\mathcal{S} \circ \mathcal{S} \circ \chi_Q\right)(a), \left(\mathcal{S} \circ \chi_Q \circ \mathcal{S}\right)(a), \left(\chi_Q \circ \mathcal{S} \circ \mathcal{S}\right)(a), 0.5\right\} = 0.5.$$

This implies that $\chi_Q(a) = 1$. Hence $a \in Q$ so $SSQ \cap SQS \cap QSS \subseteq Q$.

Next suppose $a \in SSQ \cap SSQSS \cap QSS$. Then $a \in SSQ$ and $a \in SSQSS$ and $a \in QSS$. This implies that there exist $s_1, s_2, s_3, s_4, t_1, t_2, t_3, t_4 \in S$, $y \in Q$ such that $a = s_1t_1x$, $a = zs_3t_3$ and $a = s_2t_2ys_4t_4$. For $a = s_1t_1x$ and $a = zs_3t_3$ discussed above. Consider

$$\begin{aligned}
\left(\mathcal{S} \circ \mathcal{S} \circ C_Q \circ \mathcal{S} \circ \mathcal{S}\right)(a) &= \bigvee_{a=pqrst} \left\{\mathcal{S}(p) \wedge \mathcal{S}(q) \wedge C_Q(r) \wedge \mathcal{S}(s) \wedge \mathcal{S}(t)\right\} \\
&\geq \mathcal{S}(s_2) \wedge \mathcal{S}(t_2) \wedge C_Q(y) \wedge \mathcal{S}(s_4) \wedge \mathcal{S}(t_4) \\
&= 1 \wedge 1 \wedge 1 \wedge 1 \wedge 1 = 1.
\end{aligned}$$

It follows that

$$C_Q(a) \geq \min\{(\mathcal{S} \circ \mathcal{S} \circ C_Q)(a), (\mathcal{S} \circ \mathcal{S} \circ C_Q \circ \mathcal{S} \circ \mathcal{S})(a), (C_Q \circ \mathcal{S} \circ \mathcal{S})(a), 0.5\} = 0.5.$$

This implies that $C_Q(a) = 1$. Hence $a \in Q$ so $SSQ \cap SSQSS \cap QSS \subseteq Q$. Therefore Q is a quasi-ideal of S. ■

Proposition 98 *Every $(\in, \in \vee q)$-fuzzy left (right, lateral) ideal of S is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S.*

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy left ideal of S and $a \in S$. Then

$$(\mathcal{S} \circ \mathcal{S} \circ \mu)(a) = \bigvee_{a=xyz} \{\mathcal{S}(x) \wedge \mathcal{S}(y) \wedge \mu(z)\} = \bigvee_{a=xyz} \mu(z).$$

This implies that

$$(\mathcal{S} \circ \mathcal{S} \circ \mu)(a) \wedge 0.5 = \left(\bigvee_{a=xyz} \mu(z)\right) \wedge 0.5 \leq \bigvee_{a=xyz} \mu(xyz) = \mu(a).$$

It follows that

$$\begin{aligned} \mu(a) &\geq (\mathcal{S} \circ \mathcal{S} \circ \mu)(a) \wedge 0.5 \\ &\geq \min\{(\mathcal{S} \circ \mathcal{S} \circ \mu)(a), (\mathcal{S} \circ \mu \circ \mathcal{S})(a), (\mu \circ \mathcal{S} \circ \mathcal{S})(a), 0.5\}. \end{aligned}$$

Hence

$$\mu(a) \geq \min\{(\mathcal{S} \circ \mathcal{S} \circ \mu)(a), (\mathcal{S} \circ \mu \circ \mathcal{S})(a), (\mu \circ \mathcal{S} \circ \mathcal{S})(a), 0.5\}.$$

Again

$$\begin{aligned} \mu(a) &\geq (\mathcal{S} \circ \mathcal{S} \circ \mu)(a) \wedge 0.5 \\ &\geq \min\{(\mathcal{S} \circ \mathcal{S} \circ \mu)(a), (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(a), (\mu \circ \mathcal{S} \circ \mathcal{S})(a), 0.5\}. \end{aligned}$$

Hence

$$\mu(a) \geq \min\{(\mathcal{S} \circ \mathcal{S} \circ \mu)(a), (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(a), (\mu \circ \mathcal{S} \circ \mathcal{S})(a), 0.5\}.$$

Therefore μ is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S. ■

Remark 99 *The converse of Proposition 98 is not true in general, as shown in the following example.*

Example 100 *Consider the ternary semigroup S of Example 95. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.5 & \text{if } x = 0, \\ 0.3 & \text{if } x = a, \\ 0.8 & \text{if } x = b, \\ 0.3 & \text{if } x = c, \\ 0.1 & \text{if } x = d. \end{cases}$$

Then simple calculations show that μ is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S, which is neither an $(\in, \in \vee q)$-fuzzy left ideal, nor an $(\in, \in \vee q)$-fuzzy right ideal nor an $(\in, \in \vee q)$-fuzzy lateral ideal of S.

2.5 Classifications and properties of (α, β)-fuzzy bi-ideals

This section is devoted to the study of (α, β)-fuzzy bi-ideals. We begin with

Definition 101 *A fuzzy set μ in S is said to be an* (α, β)-fuzzy bi-deal *of S, where $\alpha \neq \in \wedge q$, if it satisfies the following conditions:*

$$x_{t_1} \alpha \mu, y_{t_2} \alpha \mu \text{ and } z_{t_3} \alpha \mu \text{ imply } (xyz)_{\min\{t_1, t_2, t_3\}} \beta \mu, \tag{2.13}$$

and

$$x_{t_4} \alpha \mu, y_{t_5} \alpha \mu \text{ and } z_{t_6} \alpha \mu \text{ imply } (xuyvz)_{\min\{t_4, t_5, t_6\}} \beta \mu, \tag{2.14}$$

for all $u, v, x, y, z \in S$ and $t_1, t_2, t_3, t_4, t_5, t_6 \in (0, 1]$.

A fuzzy set μ in S satisfying condition (2.14) is called (α, β)*-fuzzy generalized bi-deal* of S.

Example 102 *Let $S = \{0, 1, 2, 3, 4, 5\}$ and $xyz = (x * y) * z = x * (y * z)$ for all $x, y, z \in S$, where $*$ is defined by the following table:*

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	3	1	1
3	0	1	1	1	2	3
4	0	1	4	5	1	1
5	0	1	1	1	4	5

Then S is a ternary semigroup with bi-ideals: $\{0\}, \{0, 1\}, \{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4, \}$, $\{0, 1, 5\}, \{0, 1, 2, 4\}, \{0, 1, 3, 5\}, \{0, 1, 2, 3\}, \{0, 1, 4, 5\}$ and S (see [55]). Define a fuzzy *set μ in S as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.5 & \text{if } x = 0, \\ 0.6 & \text{if } x = 1, \\ 0.7 & \text{if } x = 2, \\ 0.8 & \text{if } x = 3, \\ 0.3 & \text{if } x = 4, \\ 0.3 & \text{if } x = 5. \end{cases}$$

Then simple calculations show that μ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. But

(i) μ is not an $(\in, \in)$-fuzzy bi-ideal of S, since $2_{0.7} \in \mu$ but

$$(20202)_{0.7} = 0_{0.7} \overline{\in} \mu.$$

(ii) μ is not an $(\in, q)$-fuzzy bi-ideal of S, since $1_{0.3} \in \mu$ but

$$(13101)_{0.3} = 0_{0.3} \, \overline{q} \, \mu.$$

(iii) μ is not a (q, q)-fuzzy bi-ideal of S, since $2_{0.35} \, q \, \mu$ but

$$(23202)_{0.35} = 0_{0.35} \, \overline{q} \, \mu.$$

(iv) μ is not a $(q, \in)$-fuzzy bi-ideal of S, since $3_{0.8} q \mu$ but

$$(333)_{0.8} = 1_{0.8} \overline{\in} \mu.$$

(v) μ is not an $(\in, \in \wedge q)$-fuzzy bi-ideal of S, since $1_{0.41} \in \mu$ but

$$(14101)_{0.41} = 0_{0.41} \overline{\in} \mu \textit{ and so } 0_{0.41} \overline{\in \wedge q} \mu.$$

(vi) μ is not a $(q, \in \wedge q)$-fuzzy bi-ideal of S, since $3_{0.28} q \mu$ but

$$(31313)_{0.28} = 1_{0.28} \, \overline{q} \, \mu \textit{ and so } 1_{0.28} \overline{\in \wedge q} \mu.$$

(vii) μ is not an $(\in \vee q, \in)$-fuzzy bi-ideal of S, since $3_{0.68} \in \vee q \mu$ but

$$(333)_{0.68} = 1_{0.68} \overline{\in} \mu.$$

(viii) μ is not an $(\in \vee q, q)$-fuzzy bi-ideal of S, since $3_{0.24} \in \vee q \mu$ but

$$(35343)_{0.24} = 1_{0.24} \overline{q} \mu.$$

(ix) μ is not an $(\in \vee q, \in \wedge q)$-fuzzy bi-ideal of S, since $2_{0.32} \in \vee q \mu$ but

$$(21212)_{0.32} = 1 \overline{q} \mu \textit{ and so } 1_{0.32} \overline{\in \wedge q} \mu.$$

Theorem 103 *A fuzzy set μ in S is a fuzzy bi-ideal of S if and only if it satisfies:*

$$x_{t_1} \in \mu, y_{t_2} \in \mu \text{ and } z_{t_3} \in \mu \text{ imply } (xyz)_{\min\{t_1,t_2,t_3\}} \in \mu, \tag{2.15}$$

and

$$x_{t_4} \in \mu, y_{t_5} \in \mu \text{ and } z_{t_6} \in \mu \text{ imply } (xuyvz)_{\min\{t_4,t_5,t_6\}} \in \mu, \tag{2.16}$$

for all $u, v, x, y, z \in S$ and $t_1, t_2, t_3, t_4, t_5, t_6 \in (0, 1]$.

Proof. The proof is similar to the proof of Theorem 17. ■

Remark 104 *Theorem 103 shows that every $(\in, \in)$-fuzzy bi-ideal is precisely a fuzzy bi-ideal and vice versa.*

Theorem 105 *Every $(\in, \in)$-fuzzy bi-ideal is an $(\in, \in\vee q)$-fuzzy bi-ideal.*

Proof. Straightforward. ■

Remark 106 *The converse of Theorem 105 is not true in general, as seen in Example 102 (i).*

Theorem 107 *Let B be a nonempty subset of S and $\alpha \in \{\in, q, \in\vee q\}$. Then B is a bi-ideal (resp. generalized bi-ideal) of S if and only if the fuzzy set μ in S defined by:*

$$\mu(x) = \begin{cases} \geq 0.5 & \text{for all } x \in B, \\ 0 & \text{for all } x \in S \setminus B, \end{cases}$$

is an $(\alpha, \in\vee q)$-fuzzy bi-ideal (resp. generalized bi-ideal) of S.

Proof. The proof is similar to the proof of Theorem 22. ■

Proposition 108 *In a left (right) zero ternary semigroup, every fuzzy set is an $(\in, \in\vee q)$-fuzzy bi-ideal.*

Proof. Straightforward. ■

Remark 109 *For a ternary semigroup S, every $(\in, \in\vee q)$-fuzzy left (or right) ideal is an $(\in, \in\vee q)$-fuzzy bi-ideal of S.*

The converse of Remark 109 may not be true as seen in the following examples:

Example 110 *Let $S = \{a, b, c, d, e\}$ be a left zero ternary semigroup. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0, 1], \quad x \mapsto \begin{cases} 0.39 & \text{if } x = a, \\ 0.52 & \text{if } x = b, \\ 0.65 & \text{if } x = c, \\ 0.77 & \text{if } x = d, \\ 0.83 & \text{if } x = e. \end{cases}$$

Clearly μ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. But μ is not an $(\in, \in\vee q)$-fuzzy left ideal of S, since $d_{0.5} \in \mu$ but $(acd)_{0.5} \overline{\in \vee q}\mu$.

Example 111 *Let $S = \{a, b, c, d, e\}$ be a right zero ternary semigroup. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0, 1], \quad x \mapsto \begin{cases} 0.38 & \text{if } x = a, \\ 0.62 & \text{if } x = b, \\ 0.55 & \text{if } x = c, \\ 0.67 & \text{if } x = d, \\ 0.93 & \text{if } x = e. \end{cases}$$

Clearly μ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. But μ is not an $(\in, \in\vee q)$-fuzzy right ideal of S since $d_{0.45} \in \mu$ but $(dca)_{0.45} \overline{\in \vee q}\mu$.

As a special case we have:

Proposition 112 *(1) In a left zero ternary semigroup the concepts of $(\in, \in\vee q)$-fuzzy bi-ideal and $(\in, \in\vee q)$-fuzzy right ideal coincide.*

(2) In a right zero ternary semigroup the concepts of $(\in, \in\vee q)$-fuzzy bi-ideal and $(\in, \in\vee q)$-fuzzy left ideal coincide.

Theorem 113 *Every $(\in \vee q, \in \vee q)$-fuzzy bi-ideal is an $(\in, \in\vee q)$-fuzzy bi-ideal.*

Proof. The proof is similar to the proof of Theorem 24. ■

In general, it is not true that $(\in, \in\vee q)$-type implies $(\in \vee q, \in \vee q)$-type as seen in the following example.

Example 114 *Consider the ternary semigroup S and $(\in, \in\vee q)$-fuzzy bi-ideal μ of S as given in Example 110. Then μ is not an $(\in \vee q, \in \vee q)$-fuzzy bi-ideal of S, since $a_{0.79} \in \vee q\mu$, $c_{0.42} \in \vee q\mu$ and $e_{0.5} \in \vee q\mu$ but $(ace)_{\min\{0.79, 0.42, 0.5\}} \overline{\in \vee q}\mu$.*

Now, we investigate relations between $(\in, \in\vee q)$-fuzzy bi-ideal and $(q, \in\vee q)$-fuzzy bi-ideal.

Theorem 115 *Every $(q, \in\vee q)$-fuzzy bi-ideal of S is an $(\in, \in\vee q)$-fuzzy bi-ideal of S.*

Proof. The proof is similar to the proof of Theorem 27. ■

Remark 116 *In general, an $(\in, \in\vee q)$-fuzzy bi-ideal may not be a $(q, \in\vee q)$-fuzzy bi-ideal as seen in the following example.*

Example 117 *Consider the $(\in, \in\vee q)$-fuzzy bi-ideal as given in Example 110. Then μ is not a $(q, \in\vee q)$-fuzzy bi-ideal of S since $a_{0.8}q\mu$, $c_{0.43}q\mu$ and $e_{0.5}q\mu$ but*

$$(aec)_{\min\{0.8,0.5,0.43\}} \overline{\in \vee q}\mu.$$

In considering (α, β)-fuzzy bi-ideals in a ternary semigroup S, we have twelve different types of such structures, that is, (α, β) is any one of $(\in, \in)$, $(\in, q)$, $(\in, \in\wedge q)$, $(\in, \in\vee q)$, $(q, \in)$, (q, q), $(q, \in\wedge q)$, $(q, \in\vee q)$, $(\in\vee q, \in)$, $(\in\vee q, q)$, $(\in\vee q, \in\wedge q)$, and $(\in\vee q, \in\vee q)$. Clearly, we have relations among these types which are described in Theorems 29, 30 and 31.

The following theorem is a characterization of $(\in, \in\vee q)$-fuzzy bi-ideal.

Theorem 118 *A fuzzy set μ in S is an $(\in, \in\vee q)$-*fuzzy bi-ideal *of S if and only if it satisfies the following conditions:*

$$\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\}, \tag{2.17}$$

and

$$\mu(xuyvz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\}, \tag{2.18}$$

for all $u, v, x, y, z \in S$.

Proof. The proof is similar to the proof of Theorem 32. ■

Corollary 119 *A fuzzy set μ in S is an $(\in, \in\vee q)$-*fuzzy generalized bi-ideal *of S if and only if condition (2.18) is valid.*

Lemma 120 *A nonempty subset B of S is a bi-ideal (resp. generalized bi-ideal) of S if and only if the characteristic function χ_B of B is an $(\in, \in\vee q)$-fuzzy bi-ideal (resp. generalized bi-ideal) of S.*

Proof. It is straightforward. ■

Theorem 121 *A fuzzy set μ in S is an $(\in, \in\vee q)$-fuzzy bi-ideal (resp. generalized bi-ideal) of S if and only if the set $U(\mu; t)$ is a bi-ideal (resp. generalized bi-ideal) of S for all $t \in (0, 0.5]$.*

Proof. The proof is similar to the proof of Theorem 33. ■

Corollary 122 *Let μ be an (α, β)-fuzzy bi-ideal (resp. generalized bi-ideal) of S. Then the set $U(\mu; t)$ is a bi-ideal (resp. generalized bi-ideal) of S for all $t \in (0, 0.5]$.*

Theorem 123 *If μ is an $(\in, \in)$-fuzzy bi-ideal (resp. generalized bi-ideal) of S, then the t-q-set S_q^t is a bi-ideal (resp. generalized bi-ideal) of S for all $t \in (0, 1]$ whenever it is nonempty.*

Proof. The proof is similar to the proof of Theorem 38. ■

Theorem 124 *If μ is a $(q, \in \vee\, q)$-fuzzy bi-ideal (resp. generalized bi-ideal) of S, then the t-q-set S_q^t is a bi-ideal (resp. generalized bi-ideal) of S for all $t \in (0.5, 1]$ whenever it is nonempty.*

Proof. The proof is similar to the proof of Theorem 43. ■

Using Theorems 107 and 124, we have the following result.

Theorem 125 *For a bi-ideal B of S, if μ is a fuzzy set in S such that*

$$\mu(x) = \begin{cases} \geq 0.5 & \textit{for all } x \in B, \\ 0 & \textit{for all } x \in S \setminus B, \end{cases}$$

then the nonempty t-q-set S_q^t is a bi-ideal of S for all $t \in (0.5, 1]$.

One naturally asks the following interesting question:

Question: If μ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S, then is the t-q-set S_q^t a bi-ideal of S?

The answer to the above question is negative (for $t \leq 0.5$) as seen in the following example:

Example 126 *Consider the $(\in, \in \vee q)$-fuzzy bi-ideal of S as given in Example 102. Then the set*

$$S_q^{0.31} = \{2, 3\}$$

is not a bi-ideal of S because $333 = 1 \notin S_q^{0.31}$.

But the following theorem answers the above question affirmatively:

Theorem 127 *If μ is an $(\in, \in \vee\, q)$-fuzzy bi-ideal of S, then the nonempty t-q-set S_q^t is a bi-ideal of S for all $t \in (0.5, 1]$.*

Proof. The proof is similar to the proof of Theorem 48. ■

Corollary 128 *If μ is an (α,β)-fuzzy bi-ideal of S, then the nonempty t-q-set S_q^t is a bi-ideal of S for all $t \in (0.5, 1]$.*

Theorem 129 *Let μ be a fuzzy set in S. Then μ is an $(\in, \in\vee q)$-fuzzy bi-ideal of S if and only if the nonempty t-$\in\vee q$-set $S_{\in\vee q}^t$ is a bi-ideal of S for all $t \in (0, 1]$.*

Proof. The proof is similar to the proof of Theorem 50. ■

Corollary 130 *If μ is an (α,β)-fuzzy bi-ideal of S, then the nonempty t-$\in\vee q$-set $S_{\in\vee q}^t$ is a bi-ideal of S for all $t \in (0, 1]$.*

Proposition 131 *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\in, \in\vee q)$-fuzzy bi-ideals of S. Then $\mu := \bigcap_{i\in\Lambda} \mu_i$ is an $(\in, \in\vee q)$-fuzzy bi-ideal of S.*

Proof. It is straightforward. ■

Remark 132 *The union of $(\in, \in\vee q)$-fuzzy bi-ideals of S may not be an $(\in, \in\vee q)$-fuzzy bi-ideal of S, as seen in the following example.*

Example 133 *Consider the ternary semigroup S of Example 102. Define fuzzy sets μ and λ in S as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.90 & \text{if } x = 0, \\ 0.89 & \text{if } x = 1, \\ 0.61 & \text{if } x = 2, \\ 0.51 & \text{if } x = 3, \\ 0.31 & \text{if } x = 4, \\ 0.32 & \text{if } x = 5. \end{cases}$$

$$\lambda : S \to [0,1], \quad x \mapsto \begin{cases} 0.81 & \text{if } x = 0, \\ 0.71 & \text{if } x = 1, \\ 0.70 & \text{if } x = 2, \\ 0.41 & \text{if } x = 3, \\ 0.50 & \text{if } x = 4, \\ 0.40 & \text{if } x = 5. \end{cases}$$

Then simple calculations show that μ and λ are $(\in, \in\vee q)$-fuzzy bi-ideal of S. If $t \in (0.4, 0.5]$, then $U(\mu \cup \lambda; t) = \{0, 1, 2, 3, 4\}$, which is not a bi-ideal of S. It follows from Theorem 121 that $\mu \cup \lambda$ is not an $(\in, \in\vee q)$-fuzzy bi-ideal of S.

Theorem 134 *If μ is a nonzero (α,β)-fuzzy generalized bi-ideal of S, then the set S_0 is a generalized bi-ideal of S.*

Proof. Let $x,\ y,\ z \in S_0$ and $u,\ v \in S$. Then $\mu(x) > 0,\ \mu(y) > 0,\ \mu(z) > 0$. Let $\mu(xuyvz) = 0$. If $\alpha \in \{\in, \in\vee q\}$, then $x_{\mu(x)}\alpha\mu,\ y_{\mu(y)}\alpha\mu,\ z_{\mu(z)}\alpha\mu$ but $\mu(xuyvz) = 0 < \min\{\mu(x), \mu(y), \mu(z)\}$ and $\mu(xuyvz) + \min\{\mu(x), \mu(y), \mu(z)\} \leq 0 + 1 = 1$. This implies that $(xuyvz)_{\min\{\mu(x),\mu(y),\mu(z)\}}\overline{\beta}\mu$ for every $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$, which is a contradiction. Hence $\mu(xuyvz) > 0$, that is, $xuyvz \in S_0$. Also $x_1 q\mu$, $y_1 q\mu$ and $z_1 q\mu$. But $(xuyvz)_1\overline{\beta}\mu$ for every $\beta \in \{\in, q, \in\vee q, \in\wedge q\}$. Hence $\mu(xuyvz) > 0$, that is, $xuyvz \in S_0$. Therefore S_0 is a generalized bi-ideal of S. ■

Similarly we can prove that:

Theorem 135 *If μ is a nonzero (α, β)-fuzzy bi-ideal of S, then the set S_0 is a bi-ideal of S.*

Theorem 136 *Every $(\in, \in\vee q)$-fuzzy quasi-ideal of S is an $(\in, \in\vee q)$-fuzzy bi-ideal of S.*

Proof. Assume that μ is an $(\in, \in\vee q)$-fuzzy quasi-ideal of S. Then

$$\begin{aligned}
\mu(xyz) &\geq (\mu\circ\mathcal{S}\circ\mathcal{S})(xyz) \wedge (\mathcal{S}\circ\mu\circ\mathcal{S})(xyz) \wedge (\mathcal{S}\circ\mathcal{S}\circ\mu)(xyz) \wedge 0.5 \\
&= \left\{\bigvee_{xyz=abc}\{\mu(a)\wedge\mathcal{S}(b)\wedge\mathcal{S}(c)\}\right\} \wedge \left\{\bigvee_{xyz=pqr}\{\mathcal{S}(p)\wedge\mu(q)\wedge\mathcal{S}(r)\}\right\} \\
&\quad \wedge \left\{\bigvee_{xyz=uvw}\{\mathcal{S}(u)\wedge\mathcal{S}(v)\wedge\mu(w)\}\right\} \wedge 0.5 \\
&\geq \{\mu(x)\wedge\mathcal{S}(y)\wedge\mathcal{S}(z)\} \wedge \{\mathcal{S}(x)\wedge\mu(y)\wedge\mathcal{S}(z)\} \\
&\quad \wedge \{\mathcal{S}(x)\wedge\mathcal{S}(y)\wedge\mu(z)\} \wedge 0.5 \\
&= \{\mu(x)\wedge 1\wedge 1\} \wedge \{1\wedge\mu(y)\wedge 1\} \wedge \{1\wedge 1\wedge\mu(z)\} \wedge 0.5 \\
&= \mu(x)\wedge\mu(y)\wedge\mu(z)\wedge 0.5.
\end{aligned}$$

Thus $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\}$. Also

$$\mu(xuyvz) \geq \left\{\begin{array}{c} (\mu\circ\mathcal{S}\circ\mathcal{S})(xuyvz) \wedge (\mathcal{S}\circ\mathcal{S}\circ\mu\circ\mathcal{S}\circ\mathcal{S})(xuyvz) \\ \wedge(\mathcal{S}\circ\mathcal{S}\circ\mu)(xuyvz) \end{array}\right\} \wedge 0.5$$

$$= \left\{\begin{array}{c} \left\{\bigvee_{xuyvz=abc}\{\mu(a)\wedge\mathcal{S}(b)\wedge\mathcal{S}(c)\}\right\} \\ \wedge\left\{\bigvee_{xuyvz=rst}(\mathcal{S}\circ\mathcal{S}\circ\mu)(r)\wedge\mathcal{S}(s)\wedge\mathcal{S}(t)\right\} \\ \wedge\left\{\bigvee_{xuyvz=lmn}\{\mathcal{S}(l)\wedge\mathcal{S}(m)\wedge\mu(n)\}\right\} \end{array}\right\} \wedge 0.5$$

$$
= \left\{ \begin{array}{c}
\left\{ \bigvee_{xuyvz=abc} \{\mu(a) \wedge \mathcal{S}(b) \wedge \mathcal{S}(c)\} \right\} \\
\wedge \left\{ \bigvee_{xuyvz=rst} \left\{ \bigvee_{r=ijk} \mathcal{S}(i) \wedge \mathcal{S}(j) \wedge \mu(k) \right\} \wedge \mathcal{S}(s) \wedge \mathcal{S}(t) \right\} \\
\wedge \left\{ \bigvee_{xuyvz=lmn} \{\mathcal{S}(l) \wedge \mathcal{S}(m) \wedge \mu(n)\} \right\}
\end{array} \right\} \wedge 0.5
$$

$$
\geq \left\{ \begin{array}{c}
(\mu(x) \wedge \mathcal{S}(uyv) \wedge \mathcal{S}(z)) \\
\wedge (\mathcal{S}(x) \wedge \mathcal{S}(u) \wedge \mu(y) \wedge \mathcal{S}(v) \wedge \mathcal{S}(z)) \\
\wedge (\mathcal{S}(xuy) \wedge \mathcal{S}(v) \wedge \mu(z))
\end{array} \right\} \wedge 0.5
$$

$$
= \{\{\mu(x) \wedge 1 \wedge 1\} \wedge \{1 \wedge 1 \wedge \mu(y) \wedge 1 \wedge 1\} \wedge \{1 \wedge 1 \wedge \mu(z)\}\} \wedge 0.5
$$

Thus $\mu(xuyvz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\}$. Hence μ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. ■

2.6 Lower parts of $(\in, \in \vee q)$-fuzzy ideals

Let μ be a fuzzy set in S. Define the fuzzy set μ^- in S as follows:
$\mu^-(x) = \mu(x) \wedge 0.5$ for all $x \in S$.

Lemma 137 *Let μ, λ and ν be fuzzy sets in S. Then the following hold:*
(1) $(\mu \wedge \lambda)^- = (\mu^- \wedge \lambda^-)$
(2) $(\mu \vee \lambda)^- = (\mu^- \vee \lambda^-)$
(3) $(\mu \circ \lambda \circ \nu)^- = (\mu^- \circ \lambda^- \circ \nu^-)$.

Proof. The proofs of (1) and (2) are straightforward.

(3) Let $a \in S$. If a is not expressible as $a = bcd$ for some $b, c, d \in S$, then $(\mu \circ \lambda \circ \nu)(a) = 0$. It follows that

$$
\begin{aligned}
(\mu \circ \lambda \circ \nu)^-(a) &= (\mu \circ \lambda \circ \nu)(a) \wedge 0.5 \\
&= 0 \wedge 0.5 = 0.
\end{aligned}
$$

Since a is not expressible as $a = bcd$ so $(\mu^- \circ \lambda^- \circ \nu^-)(a) = 0$. Thus, in this case

$(\mu \circ \lambda \circ \nu)^{-} = \left(\mu^{-} \circ \lambda^{-} \circ \nu^{-}\right)$. If a is expressible as $a = xyz$, then

$$\begin{aligned}
(\mu \circ \lambda \circ \nu)^{-}(a) &= (\mu \circ \lambda \circ \nu)(a) \wedge 0.5 \\
&= \left\{ \bigvee_{a=xyz} \{\mu(x) \wedge \mu(y) \wedge \mu(z)\} \right\} \wedge 0.5 \\
&= \bigvee_{a=xyz} \{(\mu(x) \wedge 0.5) \wedge (\mu(y) \wedge 0.5) \wedge (\mu(z) \wedge 0.5)\} \\
&= \bigvee_{a=xyz} \left\{\mu^{-}(x) \wedge \lambda^{-}(y) \wedge \nu^{-}(z)\right\} \\
&= \left(\mu^{-} \circ \lambda^{-} \circ \nu^{-}\right)(a).
\end{aligned}$$

Hence $(\mu \circ \lambda \circ \nu)^{-} = \left(\mu^{-} \circ \lambda^{-} \circ \nu^{-}\right)$. ■

Definition 138 *Let A be a nonempty subset of S. Then χ_A^{-} is defined as follows:*

$$\chi_A^{-}(a) = \begin{cases} 0.5 & \text{if } a \in A \\ 0 & \text{if } a \notin A. \end{cases}$$

Lemma 139 *Let A, B and C be nonempty subsets of S. Then the following hold:*

(1) $(\chi_A \wedge \chi_B)^{-} = \chi_{A\cap B}^{-}$

(2) $(\chi_A \vee \chi_B)^{-} = \chi_{A\cup B}^{-}$

(3) $(\chi_A \circ \chi_B \circ \chi_C)^{-} = \chi_{ABC}^{-}$.

Lemma 140 *The lower part of the characteristic function, that is, χ_L^{-} is an $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideal of S if and only if L is a left (resp. right, lateral) ideal of S.*

Proof. We prove only for left ideal. The cases of right ideal and lateral ideal can be dealt with similarly. Let L be the left ideal of S. Then by Theorem 66, χ_L^{-} is an $(\in, \in \vee q)$-fuzzy left ideal of S.

Conversely assume that χ_L^{-} is an $(\in, \in \vee q)$-fuzzy left ideal of S. Let $z \in L$. Then $\chi_L^{-}(z) = 0.5$ and so $z_{0.5} \in \chi_L^{-}$. Since χ_L^{-} is an $(\in, \in \vee q)$-fuzzy left ideal of S, so $(xyz)_{0.5} \in \vee q \chi_L^{-}$. This implies that $(xyz)_{0.5} \in \chi_L^{-}$ or $(xyz)_{0.5}\, q \chi_L^{-}$. Thus $\chi_L^{-}(xyz) \geq 0.5$ or $\chi_L^{-}(xyz) + 0.5 > 1$. If $\chi_L^{-}(xyz) + 0.5 > 1$ which $\left(\text{by definition of } \chi_L^{-}\right)$ is impossible. Thus $\chi_L^{-}(xyz) \geq 0.5$ which implies that $\chi_L^{-}(xyz) = 0.5$. Thus $xyz \in L$. Hence L is a left ideal of S. ■

Similarly we can prove that

Lemma 141 *Let Q be a non-empty subset of S. Then Q is a quasi-ideal of S if and only if the lower part of the characteristic function, that is, χ_Q^{-} is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S.*

Proposition 142 *Let μ be an $(\in, \in \vee q)$-fuzzy left (resp. right, lateral, bi-, generalized bi-) ideal of S. Then μ^{-} is a fuzzy left (resp. right, lateral, bi-, generalized bi-) ideal of S.*

Proof. We prove only for left ideal. Others follows in an analogus way. Let μ be an $(\in, \in \vee q)$-fuzzy left ideal of S. Then for all $x, y, z \in S$, we have $\mu(xyz) \geq \mu(z) \wedge 0.5$. Thus $\mu(xyz) \wedge 0.5 \geq (\mu(z) \wedge 0.5) \wedge 0.5$. Hence $\mu^{-}(abc) \geq \mu^{-}(c)$. Therefore μ^{-} is a fuzzy left ideal of S. ■

2.7 Regular ternary semigroups

In this section we characterize regular ternary semigroups in terms of lower parts of $(\in, \in \vee q)$-fuzzy left (right and lateral) ideals, $(\in, \in \vee q)$-fuzzy quasi-ideals, $(\in, \in \vee q)$-fuzzy bi- (generalized bi-) ideals.

Theorem 143 *For a ternary semigroup S the following conditions are equivalent:*

(1) *S is regular;*

(2) $(\mu \wedge \lambda \wedge \nu)^{-} = (\mu \circ \lambda \circ \nu)^{-}$ *for every $(\in, \in \vee q)$-fuzzy right ideal μ, every $(\in, \in \vee q)$-fuzzy lateral ideal λ and every $(\in, \in \vee q)$-fuzzy left ideal ν of S.*

Proof. $(1) \Rightarrow (2)$: Let μ be an $(\in, \in \vee q)$-fuzzy right ideal, λ an $(\in, \in \vee q)$-fuzzy lateral ideal and ν an $(\in, \in \vee q)$-fuzzy left ideal of S and $a \in S$. Then

$$\begin{aligned}
(\mu \circ \lambda \circ \nu)^{-}(a) &= (\mu \circ \lambda \circ \nu)(a) \wedge 0.5 \\
&= \left\{ \bigvee_{a=xyz} \{\mu(x) \wedge \lambda(y) \wedge \nu(z)\} \right\} \wedge 0.5 \\
&\leq \bigvee_{a=xyz} \{\mu(xyz) \wedge \lambda(xyz) \wedge \nu(xyz)\} \wedge 0.5 \\
&= \{\mu(a) \wedge \lambda(a) \wedge \nu(a)\} \wedge 0.5 \\
&= (\mu \wedge \lambda \wedge \nu)(a) \wedge 0.5 = (\mu \wedge \lambda \wedge \nu)^{-}.
\end{aligned}$$

Thus $(\mu \circ \lambda \circ \nu)^{-} \leq (\mu \wedge \lambda \wedge \nu)^{-}$.

Since S is regular, so for any $a \in S$ there exists $x \in S$ such that $a = axa = a(xax)a$. It follows that

$$\begin{aligned}
(\mu \circ \lambda \circ \nu)^{-}(a) &= (\mu \circ \lambda \circ \nu)(a) \wedge 0.5 \\
&= \left\{ \bigvee_{a=pqr} (\mu(p) \wedge \lambda(q) \wedge \nu(r)) \right\} \wedge 0.5 \\
&\geq \{\mu(a) \wedge \lambda(xax) \wedge \nu(a)\} \wedge 0.5 \\
&\geq \{\mu(a) \wedge \lambda(a) \wedge \nu(a)\} \wedge 0.5 \\
&= (\mu \wedge \lambda \wedge \nu)(a) \wedge 0.5 = (\mu \wedge \lambda \wedge \nu)^{-}(a).
\end{aligned}$$

Thus $(\mu \circ \lambda \circ \nu)^{-} \geq (\mu \wedge \lambda \wedge \nu)^{-}$. Hence $(\mu \circ \lambda \circ \nu)^{-} = (\mu \wedge \lambda \wedge \nu)^{-}$.

(2) $\Rightarrow$ (1) : Let R, M and L be the right, lateral and left ideals of S, respectively. Then by Lemma 140, the lower part of the characteristic functions, that is, χ_R^{-}, χ_M^{-} and χ_L^{-} are $(\in, \in \vee q)$-fuzzy right ideal, $(\in, \in \vee q)$-fuzzy lateral ideal and $(\in, \in \vee q)$-fuzzy left ideal of S, respectively. Thus by hypothesis

$$\begin{aligned} (\chi_R \wedge \chi_M \wedge \chi_L)^{-} &= (\chi_R \circ \chi_M \circ \chi_L)^{-} \\ \chi_{R \cap M \cap L}^{-} &= \chi_{RML}^{-}. \end{aligned}$$

Hence $R \cap M \cap L = RML$. Therefore by Theorem 5, S is regular. ■

Theorem 144 *For a ternary semigroup S the following conditions are equivalent:*

(1) *S is regular;*

(2) $(\mu \wedge \lambda)^{-} = (\mu \circ \mathcal{S} \circ \lambda)^{-}$ *for every $(\in, \in \vee q)$-fuzzy right ideal μ and every $(\in, \in \vee q)$-fuzzy left ideal λ of S.*

Proof. (1) $\Rightarrow$ (2) : Let μ be an $(\in, \in \vee q)$-fuzzy right ideal and λ an $(\in, \in \vee q)$-fuzzy left ideal of S. As $\mathcal{S}$ is an $(\in, \in \vee q)$-fuzzy lateral ideal of S, we have from Theorem 143,

$$(\mu \circ \mathcal{S} \circ \lambda)^{-} = (\mu \wedge \mathcal{S} \wedge \lambda)^{-} = (\mu \wedge \lambda)^{-}.$$

(2) $\Rightarrow$ (1) : Let R and L be the right and left ideals of S, respectively. Then by Lemma 140, the lower part of the characteristic functions, that is, χ_R^{-} and χ_L^{-} are $(\in, \in \vee q)$-fuzzy right ideal and $(\in, \in \vee q)$-fuzzy left ideal of S, respectively. Thus by hypothesis

$$\begin{aligned} (\chi_R \wedge \chi_L)^{-} &= (\chi_R \circ \mathcal{S} \circ \chi_L)^{-} \\ \chi_{R \cap L}^{-} &= (\chi_{RSL})^{-}. \end{aligned}$$

Thus $R \cap L = RSL$. Hence by Theorem 6, S is regular. ■

Theorem 145 *For a ternary semigroup S, the following conditions are equivalent:*

(1) *S is regular;*

(2) $\mu^{-} = (\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)^{-}$ *for every $(\in, \in \vee q)$-fuzzy generalized bi-ideal μ of S;*

(3) $\mu^{-} = (\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)^{-}$ *for every $(\in, \in \vee q)$-fuzzy bi-ideal μ of S;*

(4) $\mu^{-} = (\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)^{-}$ *for every $(\in, \in \vee q)$-fuzzy quasi-ideal μ of S.*

Proof. $(1) \Rightarrow (2)$: Let μ be an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S and $a \in S$. Since S is regular so there exists $x \in S$ such that $a = axa = axaxa$. It follows that

$$\begin{aligned}
(\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)^{-}(a) &= (\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)(a) \wedge 0.5 \\
&= \left\{ \bigvee_{a=pqr} \{(\mu \circ \mathcal{S} \circ \mu)(p) \wedge \mathcal{S}(q) \wedge \mu(r)\} \right\} \wedge 0.5 \\
&\geq \{(\mu \circ \mathcal{S} \circ \mu)(a) \wedge \mathcal{S}(x) \wedge \mu(a)\} \wedge 0.5 \\
&= \left\{ \left\{ \bigvee_{a=ijk} \mu(i) \wedge \mathcal{S}(j) \wedge \mu(k) \right\} \wedge \mu(a) \right\} \wedge 0.5 \\
&\geq \{\mu(a) \wedge \mathcal{S}(x) \wedge \mu(a) \wedge \mu(a)\} \wedge 0.5 \\
&= \mu(a) \wedge 0.5 = \mu^{-}(a).
\end{aligned}$$

Thus $(\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)^{-} \geq \mu^{-}$.

Since μ is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S, so

$$\begin{aligned}
(\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)^{-}(a) &= (\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)(a) \wedge 0.5 \\
&= \left\{ \bigvee_{a=pvz} (\mu \circ \mathcal{S} \circ \mu)(p) \wedge \mathcal{S}(v) \wedge \mu(z) \right\} \wedge 0.5 \\
&= \left\{ \bigvee_{a=pvz} \left\{ \bigvee_{p=xuy} \mu(x) \wedge \mathcal{S}(u) \wedge \mu(y) \right\} \wedge \mathcal{S}(v) \wedge \mu(z) \right\} \wedge 0.5 \\
&\leq \left\{ \bigvee_{a=xuyvz} \{\mu(x) \wedge \mathcal{S}(u) \wedge \mu(y) \wedge \mathcal{S}(v) \wedge \mu(z)\} \right\} \wedge 0.5 \\
&= \left\{ \bigvee_{a=xuyvz} \{\mu(x) \wedge \mu(y) \wedge \mu(z)\} \right\} \wedge 0.5 \\
&\leq \bigvee_{a=xuyvz} \mu(xuyvz) \wedge 0.5 = \mu(a) \wedge 0.5 = \mu^{-}(a).
\end{aligned}$$

Thus $(\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)^{-} \leq \mu^{-}$. Hence $\mu^{-} = (\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)^{-}$.

$(2) \Rightarrow (3) \Rightarrow (4)$ are straightforward.

$(4) \Rightarrow (1)$: Let Q be a quasi-ideal of S. Then by Lemma 97, the characteristic function χ_Q of Q is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S. Thus by hypothesis

$$\begin{aligned}
\chi_Q^{-} &= (\chi_Q \circ \mathcal{S} \circ \chi_Q \circ \mathcal{S} \circ \chi_Q)^{-} \\
\chi_Q^{-} &= \chi_{QSQSQ}^{-}.
\end{aligned}$$

Thus $Q = QSQSQ$. Hence it follows from Theorem 7, that S is regular. ■

Theorem 146 *For a ternary semigroup S, the following assertions are equivalent:*

(1) *S is regular;*

(2) $(\mu \wedge \lambda)^{-} \leq (\mu \circ \mathcal{S} \circ \lambda)^{-}$ *for every* $(\in, \in \vee q)$*-fuzzy quasi-ideal* μ *and every* $(\in, \in \vee q)$*-fuzzy left ideal* λ *of* S;

(3) $(\mu \wedge \lambda)^{-} \leq (\mu \circ \mathcal{S} \circ \lambda)^{-}$ *for every* $(\in, \in \vee q)$*-fuzzy bi-ideal* μ *and every* $(\in, \in \vee q)$*-fuzzy left ideal* λ *of* S;

(4) $(\mu \wedge \lambda)^{-} \leq (\mu \circ \mathcal{S} \circ \lambda)^{-}$ *for every* $(\in, \in \vee q)$*-fuzzy generalized bi-ideal* μ *and every* $(\in, \in \vee q)$*-fuzzy left ideal* λ *of* S.

Proof. (1) $\Rightarrow$ (4) : Let μ be an $(\in, \in \vee q)$-fuzzy generalized bi-ideal and λ an $(\in, \in \vee q)$-fuzzy left ideal of S. Since S is regular so for all $a \in S$ there exists $x \in S$ such that $a = axa$. It follows that

$$\begin{aligned}
(\mu \circ \mathcal{S} \circ \lambda)^{-}(a) &= (\mu \circ \mathcal{S} \circ \lambda)(a) \wedge 0.5 \\
&= \left\{ \bigvee_{a=pqr} \{\mu(p) \wedge \mathcal{S}(q) \wedge \lambda(r)\} \right\} \wedge 0.5 \\
&\geq \{\mu(a) \wedge \mathcal{S}(x) \wedge \lambda(a)\} \wedge 0.5 \\
&= \{\mu(a) \wedge 1 \wedge \lambda(a)\} \wedge 0.5 \\
&= \{\mu(a) \wedge \lambda(a)\} \wedge 0.5 \\
&= (\mu \wedge \lambda)(a) \wedge 0.5 = (\mu \wedge \lambda)^{-}(a).
\end{aligned}$$

Thus $(\mu \circ \mathcal{S} \circ \lambda)^{-} \geq (\mu \wedge \lambda)^{-}$.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) : Straightforward.

(2) $\Rightarrow$ (1) : Let μ be an $(\in, \in \vee q)$-fuzzy right ideal and λ an $(\in, \in \vee q)$-fuzzy left ideal of S. Since every $(\in, \in \vee q)$-fuzzy right ideal of S is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S, so $(\mu \wedge \lambda)^{-} \leq (\mu \circ \mathcal{S} \circ \lambda)^{-}$. It follows that

$$\begin{aligned}
(\mu \circ \mathcal{S} \circ \lambda)^{-}(a) &= (\mu \circ \mathcal{S} \circ \lambda)(a) \wedge 0.5 \\
&= \left\{ \bigvee_{a=xyz} \{\mu(x) \wedge \mathcal{S}(y) \wedge \lambda(z)\} \right\} \wedge 0.5 \\
&= \left\{ \bigvee_{a=xyz} \mu\{(x) \wedge \lambda(z)\} \right\} \wedge 0.5 \\
&\leq \left\{ \bigvee_{a=xyz} \{\mu(xyz) \wedge \lambda(xyz)\} \right\} \wedge 0.5 \\
&= \{\mu(a) \wedge \lambda(a)\} \wedge 0.5 \\
&= (\mu \wedge \lambda)(a) \wedge 0.5 = (\mu \wedge \lambda)^{-}(a).
\end{aligned}$$

Thus $(\mu \wedge \lambda)^{-} \geq (\mu \circ \mathcal{S} \circ \lambda)^{-}$. Hence $(\mu \wedge \lambda)^{-} = (\mu \circ \mathcal{S} \circ \lambda)^{-}$. Therefore by Theorem 144, S is regular. ■

Theorem 147 *For a ternary semigroup* S*, the following conditions are equivalent:*

(1) *S is regular;*

(2) $(\mu \wedge \lambda)^{-} \leq (\lambda \circ \mathcal{S} \circ \mu)^{-}$ *for every* $(\in, \in \vee q)$*-fuzzy quasi-ideal* μ *and every* $(\in, \in \vee q)$*-fuzzy right ideal* λ *of* S;

(3) $(\mu \wedge \lambda)^{-} \leq (\lambda \circ \mathcal{S} \circ \mu)^{-}$ *for every* $(\in, \in \vee q)$*-fuzzy bi-ideal* μ *and every* $(\in, \in \vee q)$*-fuzzy right ideal* λ *of* S.

(4) $(\mu \wedge \lambda)^{-} \leq (\lambda \circ \mathcal{S} \circ \mu)^{-}$ *for every* $(\in, \in \vee q)$*-fuzzy generalized bi-ideal* μ *and every* $(\in, \in \vee q)$*-fuzzy right ideal* λ *of* S.

Proof. This is the dual of Theorem 146. ■

Theorem 148 *For a ternary semigroup* S, *the following statements are equivalent:*

(1) *S is regular;*

(2) $(\mu \wedge \lambda)^{-} = (\mu \circ \lambda \circ \mu)^{-}$ *for every* $(\in, \in \vee q)$*-fuzzy bi-ideal* μ *and every* $(\in, \in \vee q)$*-fuzzy lateral ideal* λ *of* S;

(3) $(\mu \wedge \lambda)^{-} = (\mu \circ \lambda \circ \mu)^{-}$ *for every* $(\in, \in \vee q)$*-fuzzy quasi-ideal* μ *and every* $(\in, \in \vee q)$*-fuzzy lateral ideal* λ *of* S.

Proof. $(1) \Rightarrow (2)$: Let μ be an $(\in, \in \vee q)$-fuzzy bi-ideal, λ an $(\in, \in \vee q)$-fuzzy lateral ideal of S and $a \in S$. Since S is regular so there exists $x \in S$ such that $a = axa = a(xax)a$. It follows that

$$\begin{aligned}
(\mu \circ \lambda \circ \mu)^{-}(a) &= (\mu \circ \lambda \circ \mu)(a) \wedge 0.5 \\
&= \left\{ \bigvee_{a=pqr} \{\mu(p) \wedge \lambda(q) \wedge \mu(r)\} \right\} \wedge 0.5 \\
&\geq \{\mu(a) \wedge \lambda(xax) \wedge \mu(a)\} \wedge 0.5 \\
&\geq \{\mu(a) \wedge \lambda(a) \wedge \mu(a)\} \wedge 0.5 \\
&= \{\mu(a) \wedge \lambda(a)\} \wedge 0.5 \\
&= (\mu \wedge \lambda)(a) \wedge 0.5 = (\mu \wedge \lambda)^{-}(a).
\end{aligned}$$

Thus $(\mu \circ \lambda \circ \mu)^{-} \geq (\mu \wedge \lambda)^{-}$.

On the other hand $(\mu \circ \lambda \circ \mu)^{-} \leq (\mu \wedge \lambda)^{-}$ always hold. Hence $(\mu \wedge \lambda)^{-} = (\mu \circ \lambda \circ \mu)^{-}$.

$(2) \Rightarrow (3)$: Straightforward.

$(3) \Rightarrow (1)$ Let Q be a quasi-ideal of S. Then by Lemma 97, the characteristic function χ_Q of Q is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S. Thus by hypothesis

$$\begin{aligned}
\chi_Q^{-} &= (\chi_Q \circ \mathcal{S} \circ \chi_Q)^{-} \\
\chi_Q^{-} &= \chi_{QSQ}^{-}.
\end{aligned}$$

Thus $Q = QSQ$. Hence it follows from Theorem 7, that S is regular. ■

2.8 Weakly regular ternary semigroups

In this section we characterize right weakly regular ternary semigroups in terms of $(\in, \in \vee q)$-fuzzy right ideals, $(\in, \in \vee q)$-fuzzy two sided ideals and $(\in, \in \vee q)$-fuzzy generalized bi-ideals.

Theorem 149 *For a ternary semigroup S, the following assertions are equivalent:*

(1) *S is right weakly regular;*

(2) $(\mu \wedge \nu)^{-} = (\mu \circ \nu \circ \nu)^{-}$ *for every $(\in, \in \vee q)$-fuzzy right ideal μ and every $(\in, \in \vee q)$-fuzzy two sided ideal ν of S;*

(3) $(\mu \wedge \nu)^{-} = (\mu \circ \nu \circ \nu)^{-}$ *for every $(\in, \in \vee q)$-fuzzy right ideal μ and every $(\in, \in \vee q)$-fuzzy ideal ν of S.*

Proof. $(1) \Rightarrow (2)$: Let μ be an $(\in, \in \vee q)$-fuzzy right ideal and ν an $(\in, \in \vee q)$-fuzzy two sided ideal of S. Now for any $a \in S$, there exist $s_1, s_2, s_3, t_1, t_2, t_3 \in S$ such that $a = (as_1t_1)(as_2t_2)(as_3t_3)$. It follows that

$$\begin{aligned}
(\mu \circ \nu \circ \nu)^{-}(a) &= (\mu \circ \nu \circ \nu)(a) \wedge 0.5 \\
&= \left\{ \bigvee_{a=pqr} \{\mu(p) \wedge \nu(q) \wedge \nu(r)\} \right\} \wedge 0.5 \\
&\leq \left\{ \bigvee_{a=pqr} \{\mu(pqr) \wedge 0.5\} \wedge \nu(q) \wedge \{\nu(pqr) \wedge 0.5\} \right\} \wedge 0.5 \\
&\leq \left\{ \bigvee_{a=pqr} \{\mu(pqr) \wedge \nu(pqr)\} \right\} \wedge 0.5 \\
&\leq \{\mu(a) \wedge \nu(a)\} \wedge 0.5 = (\mu \wedge \nu)^{-}(a)
\end{aligned}$$

Thus $(\mu \circ \nu \circ \nu)^{-} \leq (\mu \wedge \nu)^{-}$.

On the other hand

$$\begin{aligned}
(\mu \wedge \nu)^{-}(a) &= (\mu \wedge \nu)(a) \wedge 0.5 \\
&= \{\mu(a) \wedge \nu(a) \wedge \nu(a)\} \wedge 0.5 \\
&\leq \{\mu(as_1t_1) \wedge \nu(as_2t_2) \wedge \nu(as_3t_3)\} \wedge 0.5 \\
&\leq \left\{ \bigvee_{a=xyz} (\mu(x) \wedge \nu(y) \wedge \nu(z)) \right\} \wedge 0.5 \\
&= (\mu \circ \nu \circ \nu)(a) \wedge 0.5 = (\mu \circ \nu \circ \nu)^{-}(a).
\end{aligned}$$

Thus $(\mu \wedge \nu)^{-} \leq (\mu \circ \nu \circ \nu)^{-}$. Consequently $(\mu \wedge \nu)^{-} = (\mu \circ \nu \circ \nu)^{-}$.

$(2) \Rightarrow (3)$: This is obvious, because every $(\in, \in \vee q)$-fuzzy ideal is an $(\in, \in \vee q)$-fuzzy two sided ideal of S.

(3) $\Rightarrow$ (1) : Let R be a right ideal and I an ideal of S. Then by Corollary 67, χ_R and χ_I are $(\in, \in \vee q)$-fuzzy right ideal and $(\in, \in \vee q)$-fuzzy ideal of S, respectively. By hypothesis it follows that

$$\begin{aligned}(\chi_R \wedge \chi_I)^- &= (\chi_R \circ \chi_I \circ \chi_I)^- \\ \chi_{R\cap I}^- &= (\chi_{RII})^- .\end{aligned}$$

Thus $R \cap I = RII$. Hence by Lemma 9, S is right weakly regular. ■

Corollary 150 *If $(\mu \wedge \nu)^- = (\mu \circ \nu \circ \nu)^-$ for every (α, β)-fuzzy right ideal μ and every (α, β)-fuzzy two sided ideal ν of S, where (α, β) is any one of $(\in, \in)$, $(\in, q)$, $(\in, \in \wedge q)$, $(\in \vee q, \in)$, $(\in \vee q, q)$, $(\in \vee q, \in \wedge q)$, and $(\in \vee q, \in \vee q)$ then S is a right weakly regular ternary semigroup.*

Theorem 151 *For a ternary semigroup S, the following assertions are equivalent:*

(1) *S is right weakly regular;*

(2) *Each $(\in, \in \vee q)$-fuzzy right ideal μ of S is idempotent.*

Proof. (1) $\Rightarrow$ (2) : Let μ be an $(\in, \in \vee q)$-fuzzy right ideal of S. Now for any $a \in S$, there exist $s_1, s_2, s_3, t_1, t_2, t_3 \in S$ such that $a = (as_1t_1)(as_2t_2)(as_3t_3)$. It follows that

$$\begin{aligned}(\mu \circ \mu \circ \mu)^- (a) &= (\mu \circ \mu \circ \mu)(a) \wedge 0.5 \\ &= \left\{ \bigvee_{a=pqr} \{\mu(p) \wedge \mu(q) \wedge \mu(r)\} \right\} \wedge 0.5 \\ &\leq \left\{ \bigvee_{a=pqr} \{\{\mu(pqr) \wedge 0.5\} \wedge \mu(q) \wedge \mu(r)\} \right\} \wedge 0.5 \\ &\leq \bigvee_{a=pqr} \mu(pqr) \wedge 0.5 \\ &\leq \mu(a) \wedge 0.5 = \mu^-(a) .\end{aligned}$$

Thus $(\mu \circ \mu \circ \mu)^- \leq \mu^-$.

On the other hand

$$\begin{aligned}\mu^-(a) &= \mu(a) \wedge 0.5 \\ &= \{\mu(a) \wedge \mu(a) \wedge \mu(a)\} \wedge 0.5 \\ &\leq \left\{ \begin{array}{c} \{\mu(as_1t_1) \wedge 0.5\} \wedge \{\mu(as_2t_2) \wedge 0.5\} \\ \wedge \{\mu(as_3t_3) \wedge 0.5\} \end{array} \right\} \wedge 0.5 \\ &\leq \left\{ \bigvee_{a=pqr} \{\mu(p) \wedge \mu(q) \wedge \mu(r)\} \right\} \wedge 0.5 \\ &= (\mu \circ \mu \circ \mu)(a) \wedge 0.5 = (\mu \circ \mu \circ \mu)^-(a) .\end{aligned}$$

Thus $\mu^- \leq (\mu \circ \mu \circ \mu)^-$. Hence $\mu^- = (\mu \circ \mu \circ \mu)^-$.

(2) $\Rightarrow$ (1) : Let A be a ight ideal and let χ_A be the characteristic function of A. Then by Lemma 186, χ_A is an $(\in, \in \vee q)$-fuzzy right ideal of S. Thus by hypothesis

$$\begin{aligned} \chi_A^- &= (\chi_A \circ \chi_A \circ \chi_A)^- \\ &= \chi_{A^3}^-. \end{aligned}$$

This implies that $A = A^3$. Hence S is right weakly regular. ■

Corollary 152 *If every (α, β)-fuzzy right ideal μ of S is idempotent, where (α, β) is any one of $(\in, \in)$, $(\in, q)$, $(\in, \in \wedge q)$, $(\in \vee q, \in)$, $(\in \vee q, q)$, $(\in \vee q, \in \wedge q)$, and $(\in \vee q, \in \vee q)$, then S is a right weakly regular ternary semigroup.*

Theorem 153 *For a ternary semigroup S, the following assertions are equivalent:*

(1) *S is right weakly regular;*

(2) *$(\mu \wedge \nu \wedge \lambda)^- = (\mu \circ \nu \circ \lambda)^-$ for every $(\in, \in \vee q)$-fuzzy bi-ideal μ, every $(\in, \in \vee q)$-fuzzy two sided ideal ν and every $(\in, \in \vee q)$-fuzzy right ideal λ of S;*

(3) *$(\mu \wedge \nu \wedge \lambda)^- = (\mu \circ \nu \circ \lambda)^-$ for every $(\in, \in \vee q)$-fuzzy quasi-ideal μ, every $(\in, \in \vee q)$-fuzzy two sided ideal ν and every $(\in, \in \vee q)$-fuzzy right ideal λ of S.*

Proof. (1) $\Rightarrow$ (2) : Let μ be an $(\in, \in \vee q)$-fuzzy bi-ideal, ν an $(\in, \in \vee q)$-fuzzy two sided ideal and λ an $(\in, \in \vee q)$-fuzzy right ideal of S. Now for $a \in S$, there exist $s_1, s_2, s_3, t_1, t_2, t_3 \in S$ such that $a = (as_1t_1)(as_2t_2)(as_3t_3) = a(s_1t_1as_2t_2)(as_3t_3)$. It follows that

$$\begin{aligned} (\mu \wedge \nu \wedge \lambda)^-(a) &= (\mu \wedge \nu \wedge \lambda)(a) \wedge 0.5 \\ &= \{\mu(a) \wedge \nu(a) \wedge \lambda(a)\} \wedge 0.5 \\ &\leq \{\{\mu(a) \wedge \nu(s_1t_1as_2t_2) \wedge 0.5\} \wedge \{\lambda(as_3t_3) \wedge 0.5\}\} \wedge 0.5 \\ &= \left\{ \bigvee_{a=pqr} \{\mu(p) \wedge \nu(q) \wedge \lambda(r)\} \right\} \wedge 0.5 \\ &= (\mu \circ \nu \circ \lambda)(a) \wedge 0.5 = (\mu \circ \nu \circ \lambda)^-(a). \end{aligned}$$

Thus $(\mu \wedge \nu \wedge \lambda)^- \leq (\mu \circ \nu \circ \lambda)^-$.

(2) $\Rightarrow$ (3) : This is obvious, because every $(\in, \in \vee q)$-fuzzy quasi-ideal is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.

(3) $\Rightarrow$ (1) : Let μ be an $(\in, \in \vee q)$-fuzzy right ideal and ν an $(\in, \in \vee q)$-fuzzy two sided ideal of S. Take $\lambda = \nu$. Since every $(\in, \in \vee q)$-fuzzy right ideal is an $(\in, \in \vee q)$-fuzzy quasi-ideal so by hypothesis $(\mu \wedge \nu \wedge \nu)^- \leq (\mu \circ \nu \circ \nu)^-$. Thus $(\mu \wedge \nu)^- \leq (\mu \circ \nu \circ \nu)^-$. But $(\mu \circ \nu \circ \nu)^- \leq (\mu \wedge \nu)^-$ always holds. Hence $(\mu \wedge \nu)^- = (\mu \circ \nu \circ \nu)^-$. Therefore by Theorem 149, S is right weakly regular. ■

Corollary 154 *If* $(\mu \wedge \nu \wedge \lambda)^{-} = (\mu \circ \nu \circ \lambda)^{-}$ *for every* (α, β)*-fuzzy bi-ideal (quasi-ideal)* μ*, every* (α, β)*-fuzzy two sided ideal* ν *and every* (α, β)*-fuzzy right ideal* λ *of* S*, where* (α, β) *is any one of* $(\in, \in)$, $(\in, q)$, $(\in, \in \wedge q)$, $(\in \vee q, \in)$, $(\in \vee q, q)$, $(\in \vee q, \in \wedge q)$, *and* $(\in \vee q, \in \vee q)$*, then* S *is a right weakly regular ternary semigroup.*

Theorem 155 *For a ternary semigroup* S*, the following assertions are equivalent:*

(1) S *is right weakly regular;*

(2) $(\mu \wedge \nu)^{-} \leq (\mu \circ \nu \circ \nu)^{-}$ *for every* $(\in, \in \vee q)$*-fuzzy bi-ideal* μ *and every* $(\in, \in \vee q)$*-fuzzy two sided ideal* ν *of* S;

(3) $(\mu \wedge \nu)^{-} \leq (\mu \circ \nu \circ \nu)^{-}$ *for every* $(\in, \in \vee q)$*-fuzzy quasi-ideal* μ *and every* $(\in, \in \vee q)$*-fuzzy two sided ideal* ν *of* S.

Proof. $(1) \Rightarrow (2)$: Let μ be an $(\in, \in \vee q)$-fuzzy bi-ideal and ν an $(\in, \in \vee q)$-fuzzy two sided ideal of S. Now for any $a \in S$, there exist $s_1, s_2, s_3, t_1, t_2, t_3 \in S$ such that $a = (as_1t_1)(as_2t_2)(as_3t_3) = a(s_1t_1as_2t_2)(as_3t_3)$. It follows that

$$\begin{aligned}
(\mu \wedge \nu)^{-}(a) &= (\mu \wedge \nu)(a) \wedge 0.5 \\
&= \{\mu(a) \wedge \nu(a) \wedge \nu(a)\} \wedge 0.5 \\
&\leq \{\{\mu(a) \wedge \nu(s_1t_1as_2t_2) \wedge 0.5\} \wedge \{\nu(as_3t_3) \wedge 0.5\}\} \wedge 0.5 \\
&= \left\{\bigvee_{a=lmn} \{\mu(l) \wedge \nu(m) \wedge \nu(n)\}\right\} \wedge 0.5 \\
&= (\mu \circ \nu \circ \nu)(a) \wedge 0.5 = (\mu \circ \nu \circ \nu)^{-}(a).
\end{aligned}$$

Thus $(\mu \wedge \nu)^{-} \leq (\mu \circ \nu \circ \nu)^{-}$.

$(2) \Rightarrow (3)$: This is obvious, because every $(\in, \in \vee q)$-fuzzy quasi-ideal is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.

$(3) \Rightarrow (1)$: Let μ be an $(\in, \in \vee q)$-fuzzy right ideal and ν an $(\in, \in \vee q)$-fuzzy two sided ideal of S. Since every $(\in, \in \vee q)$-fuzzy right ideal is an $(\in, \in \vee q)$-fuzzy quasi-ideal of S. Thus by hypothesis $(\mu \wedge \nu)^{-} \leq (\mu \circ \nu \circ \nu)^{-}$. Also

$$\begin{aligned}
(\mu \circ \nu \circ \nu)^{-}(a) &= (\mu \circ \nu \circ \nu)(a) \wedge 0.5 \\
&= \left\{\bigvee_{a=lmn} \{\mu(l) \wedge \nu(m) \wedge \nu(n)\}\right\} \wedge 0.5 \\
&\leq \left\{\bigvee_{a=lmn} \{\mu(lmn) \wedge 0.5\} \wedge \nu(m) \wedge \{\nu(lmn) \wedge 0.5\}\right\} \wedge 0.5 \\
&\leq \bigvee_{a=lmn} \mu(lmn) \wedge \nu(lmn) \wedge 0.5 \\
&= (\mu \wedge \nu)^{-}(a).
\end{aligned}$$

Thus $(\mu \circ \nu \circ \nu)^{-} \leq (\mu \wedge \nu)^{-}$. Hence $(\mu \wedge \nu)^{-} = (\mu \circ \nu \circ \nu)^{-}$. Therefore by Theorem 149, S is right weakly regular. ■

Corollary 156 *If* $(\mu \wedge \nu)^{-} \leq (\mu \circ \nu \circ \nu)^{-}$ *for every* (α, β)*-fuzzy bi-ideal (quasi-ideal)* μ *and every* (α, β)*-fuzzy two sided ideal* ν *of* S*, where* (α, β) *is any one of* $(\in, \in)$, $(\in, q)$, $(\in, \in \wedge q)$, $(\in \vee q, \in)$, $(\in \vee q, q)$, $(\in \vee q, \in \wedge q)$, *and* $(\in \vee q, \in \vee q)$, *then* S *is a right weakly regular ternary semigroup.*

Chapter 3

$(\in, \in \vee q_k)$-fuzzy ternary subsemigroups and ideals in ternary semigroups

In this chapter we study $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup and ideals in ternary semigroups which is the generalizations of $(\in, \in \vee q)$-fuzzy ternary subsemigroup and ideals in ternary semigroups. The classes of regular ternary semigroups and right weakly regular ternary semigroups are characterized in terms of $(\in, \in \vee q_k)$-fuzzy left (right and lateral) ideals, $(\in, \in \vee q_k)$-fuzzy quasi-ideals and $(\in, \in \vee q_k)$-fuzzy bi- (generalized bi-) ideals.

3.1 $(\in, \in \vee q_k)$-fuzzy ternary subsemigroups

In what follows, let S denote a ternary semigroup and k an arbitrary element of $[0, 1)$ unless otherwise specified. Jun initiated the concept of $(\in, \in \vee q_k)$-fuzzy subalgebras in BCK/BCI-algebras in [17]. For a fuzzy point x_t and a fuzzy set μ in S, we say that

(i) $x_t q_k \mu$ if $\mu(x) + t + k > 1$.

(ii) $x_t \in \vee q_k \mu$ if $x_t \in \mu$ or $x_t q_k \mu$.

(iii) $x_t \in \wedge q_k \mu$ if $x_t \in \mu$ and $x_t q_k \mu$.

(iv) $x_t \overline{\alpha} \mu$ if $x_t \alpha \mu$ does not hold for $\alpha \in \{q_k, \in \vee q_k, \in \wedge q_k\}$.

Definition 157 *Let $\alpha \in \{\in, q, \in \vee q\}$. A fuzzy set μ in S is said to be an $(\alpha, \in \vee q_k)$-*fuzzy ternary subsemigroup *of S, if it satisfies the following condition:*

$$x_t \alpha \mu, \; y_r \alpha \mu \text{ and } z_s \alpha \mu \;\; \text{imply } (xyz)_{\min\{t,\, r,\, s\}} \in \vee q_k \mu, \tag{3.1}$$

for all $x, y, z \in S$ and $t, r, s \in (0, 1]$.

Example 158 *Consider the ternary semigroup* $S = \{-i, 0, i\}$ *of Example 16. Define a fuzzy set* μ *in* S *as follows:*

$$\mu : S \to [0,1]\,,\; x \to \begin{cases} 0.80 & \text{if } x = 0, \\ 0.70 & \text{if } x = i, \\ 0.35 & \text{if } x = -i. \end{cases}$$

Then simple calculations show that μ *is an* $(\in, \in \vee q_k)$*-fuzzy ternary subsemigroup of* S *for* $k = 0.3$*. But*

(1) μ *is not an* $(\in, \in)$*-fuzzy ternary subsemigroup of* S*, since* $i_{0.63} \in \mu$*, but* $(iii)_{0.63} \overline{\in} \mu$*.*

(2) μ *is not an* $(\in, \in \vee q)$*-fuzzy ternary subsemigroup of* S*, since* $i_{0.7} \in \mu$*, but* $(iii)_{0.7} \overline{\in} \wedge \overline{q} \mu$*.*

(3) μ *is not an* $(\in, \in \vee q_{0.1})$*-fuzzy ternary subsemigroup of* S*, since* $i_{0.4} \in \mu$*, but* $(iii)_{0.4} \overline{\in \vee q_{0.1}} \mu$*.*

Example 159 *Consider the ternary semigroup* S *of Example 54. Define a fuzzy set* μ *in* S *as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.40 & \text{if } x = 0, \\ 0.50 & \text{if } x = a, \\ 0.60 & \text{if } x = b, \\ 0.20 & \text{if } x = c, \\ 0.30 & \text{if } x = 1. \end{cases}$$

Then simple calculations show that μ *is an* $(\in, \in \vee q_{0.2})$*-fuzzy ternary subsemigroup of* S*.*

Theorem 160 *Every* $(\in, q_k)$*-fuzzy ternary subsemigroup of* S *is an* $(\in, \in \vee q_k)$*-fuzzy ternary subsemigroup of* S*.*

Proof. Straightforward ■

Taking $k = 0$ in Theorem 160 we have the following corollary.

Corollary 161 *Every* $(\in, q)$*-fuzzy ternary subsemigroup of* S *is an* $(\in, \in \vee q)$*-fuzzy ternary subsemigroup of* S*.*

Theorem 162 *If* $0 \leq k < r < 1$*, then every* $(\in, \in \vee q_k)$*-fuzzy ternary subsemigroup of* S *is an* $(\in, \in \vee q_r)$*-fuzzy ternary subsemigroup of* S*.*

Proof. Straightforward. ■

The following example shows that if $0 \leq k < r < 1$, then an $(\in, \in \vee q_r)$-fuzzy ternary subsemigroup of S may not be an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S.

Example 163 *Consider the* $(\in, \in \vee\, q_{0.2}\,)$*-fuzzy ternary subsemigroup* μ *of* S *given in Example 159. Then* μ *is not an* $(\in, \in \vee\, q_{0.1}\,)$*-fuzzy ternary subsemigroup of* S*, because* $a_{0.42} \in \mu$*, but* $(aaa)_{0.42}\, \overline{\in \vee q_{0.1}} \mu$*.*

Theorem 164 *Every* $(\in, \in)$*-fuzzy ternary subsemigroup of* S *is an* $(\in, \in \vee\, q_k\,)$*-fuzzy ternary subsemigroup of* S*.*

Remark 165 *The converse of Theorem 164 is not true in general as seen in Examples 158 and 159.*

Theorem 166 *Every* $(\in, \in \vee\, q\,)$*-fuzzy ternary subsemigroup of* S *is an* $(\in, \in \vee\, q_k\,)$*-fuzzy ternary subsemigroup of* S *for every* $k \in (0, 1]$*.*

Proof. Straightforward. ■

Theorem 167 *Let* A *be a ternary subsemigroup of* S *and* $\alpha \in \{\in, q, \in \vee q\}$*. Then the fuzzy set* μ *in* S *defined by:*

$$\mu(x) = \begin{cases} \geq \frac{1-k}{2} & \text{for all } x \in A, \\ 0 & \text{for all } x \in S \setminus A, \end{cases}$$

is an $(\alpha, \in \vee q_k)$*-fuzzy ternary subsemigroup of* S*.*

Proof. Let A be a ternary subsemigroup of S.

(a) In this part we show that μ is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S. Let x, y, $z \in S$ and t, r, $s \in (0, 1]$ be such that $x_t \in \mu$, $y_r \in \mu$ and $z_s \in \mu$. Then $\mu(x) \geq t > 0$, $\mu(y) \geq r > 0$ and $\mu(z) \geq s > 0$, implies $\mu(x) \geq \frac{1-k}{2}$, $\mu(y) \geq \frac{1-k}{2}$, $\mu(z) \geq \frac{1-k}{2}$. Thus x, y, $z \in A$, so $xyz \in A$. Thus $\mu(xyz) \geq \frac{1-k}{2}$. If $\min\{t, r, s\} \leq \frac{1-k}{2}$, then $\mu(xyz) \geq \frac{1-k}{2} \geq \min\{t, r, s\}$. Hence $(xyz)_{\min\{t,r,s\}} \in \mu$. If $\min\{t, r, s\} > \frac{1-k}{2}$, then $\mu(xyz) + \min\{t, r, s\} + k > \frac{1-k}{2} + \frac{1-k}{2} + k > 1$, so $(xyz)_{\min\{t,r,s\}}\, q_k \mu$. Thus $(xyz)_{\min\{t,r,s\}} \in \vee q_k \mu$. Hence μ is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S.

(b) In this part we show that μ is a $(q, \in \vee q_k)$-fuzzy ternary subsemigroup of S. Let $x, y, z \in S$ and $t, r, s \in (0, 1]$ be such that $x_t q\mu$, $y_r q\mu$, and $z_s q\mu$. Then $\mu(x) + t > 1$, $\mu(y) + r > 1$ and $\mu(z) + s > 1$. Thus x, y, $z \in A$, so $xyz \in A$. Thus $\mu(xyz) \geq \frac{1-k}{2}$. If $\min\{t, r, s\} \leq \frac{1-k}{2}$, then $\mu(xyz) \geq \frac{1-k}{2} \geq \min\{t, r, s\}$. Hence $(xyz)_{\min\{t,r,s\}} \in \mu$. If $\min\{t, r, s\} > \frac{1-k}{2}$, then $\mu(xyz) + \min\{t, r, s\} + k > 1$, so $(xyz)_{\min\{t,r,s\}}\, q_k \mu$. Thus $(xyz)_{\min\{t,r,s\}} \in \vee q_k \mu$. Hence μ is an $(q, \in \vee q_k)$-fuzzy ternary subsemigroup of S.

(c) In this part we show that μ is an $(\in \vee q, \in \vee q_k)$-fuzzy ternary subsemigroup of S. Let $x, y, z \in S$ and $t, r, s \in (0, 1]$ be such that $x_t \in \vee q\mu$, $y_r \in \vee q\mu$, $z_s \in \vee q\mu$. Then $x_t \in \mu$ or $x_t q\mu$, $y_r \in \mu$ or $y_r q\mu$, $z_s \in \mu$ or $z_s q\mu$. Now there are eight possible cases, each case implies that $x, y, z \in A$, so $xyz \in A$. Hence $\mu(xyz) \geq \frac{1-k}{2}$. Analogous to (a) and (b) we obtain $(xyz)_{\min\{t,r,s\}} \in \vee q_k \mu$. Therefore μ is an $(\in \vee q, \in \vee q_k)$-fuzzy ternary subsemigroup of S. ■

Corollary 168 *Let A be a ternary subsemigroup of S and $\alpha \in \{\in, q, \in \vee q\}$. Then the fuzzy set μ in S defined by:*

$$\mu(x) = \begin{cases} \geq 0.5 & \text{for all } x \in A, \\ 0 & \text{for all } x \in S \setminus A, \end{cases}$$

is an $(\alpha, \in \vee q)$-fuzzy ternary subsemigroup of S.

Characterization of $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup is given in the following theorem.

Theorem 169 *Let μ be a fuzzy set in S. Then μ is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S if and only if μ satisfies the following condition:*

$$\mu(xyz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\}, \tag{3.2}$$

for all $x, y, z \in S$.

Proof. Let μ be an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S. If there exist $x, y, z \in S$ such that $\mu(xyz) < \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\}$, then we can choose $t \in (0, 1]$ such that

$$\mu(xyz) < t \leq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\},$$

and so $x_t \in \mu$, $y_t \in \mu$ and $z_t \in \mu$. But $\mu(xyz) < t \Rightarrow (xyz)_t \overline{\in} \mu$ and $\mu(xyz) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. It follows that $(xyz)_{\min\{t,t,t\}} \overline{\in \vee q_k} \mu = (xyz)_t \overline{\in \vee q_k} \mu$, which is a contradiction. Hence $\mu(xyz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\}$.

Conversely, assume that $\mu(xyz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\}$. Let $x_t \in \mu$, $y_r \in \mu$, $z_s \in \mu$ for $t, r, s \in (0, 1]$. Then $\mu(x) \geq t$, $\mu(y) \geq r$, $\mu(z) \geq s$. It follows that

$$\mu(xyz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\} \geq \min\left\{t, r, s, \frac{1-k}{2}\right\}.$$

If $\min\{t, r, s\} \leq \frac{1-k}{2}$, then $\mu(xyz) \geq \min\{t, r, s\}$ and so $(xyz)_{\min\{t,r,s\}} \in \mu$. If $\min\{t, r, s\} > \frac{1-k}{2}$, then $\mu(xyz) + \min\{t, r, s\} + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$, so $(xyz)_{\min\{t,r,s\}} q_k \mu$. Hence $(xyz)_{\min\{t,r,s\}} \in \vee q_k \mu$. Therefore μ is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S. ■

Taking $k = 0$ in Theorem 169 we have the following corollary.

Corollary 170 *Let μ be a fuzzy set in S. Then μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S if and only if μ satisfies the following condition:*

$$\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\}, \tag{3.3}$$

for all $x, y, z \in S$.

Lemma 171 *A nonempty subset A of S is a ternary subsemigroup of S if and only if the characteristic function χ_A of A is an $(\in, \in\vee q_k)$-fuzzy ternary subsemigroup of S.*

Proof. Let A be a ternary subsemigroup of S and $x, y, z \in A$. Then $\chi_A(x) = \chi_A(y) = \chi_A(z) = 1$. Since A is a ternary subsemigroup of S, so $xyz \in A$ and $\chi_A(xyz) = 1$. It follows that

$$\chi_A(xyz) \geq \min\left\{\chi_A(x), \chi_A(y), \chi_A(z), \frac{1-k}{2}\right\}.$$

If one of x, y, z is not in A, then $\min\left\{\chi_A(x), \chi_A(y), \chi_A(z), \frac{1-k}{2}\right\} = 0 \leq \chi_A(xyz)$. Hence by Theorem 169, χ_A is an $(\in, \in\vee q_k)$-fuzzy ternary subsemigroup of S.

Conversely, assume that χ_A is an $(\in, \in\vee q_k)$-fuzzy ternary subsemigroup of S. Let $x, y, z \in A$. Then

$$\chi_A(xyz) \geq \min\left\{\chi_A(x), \chi_A(y), \chi_A(z), \frac{1-k}{2}\right\} = \frac{1-k}{2}.$$

Since $k \in [0, 1)$, so $\chi_A(xyz) = 1$ and $xyz \in A$. Hence A is a ternary subsemigroup of S. ■

Theorem 172 *A fuzzy set μ in S is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S if and only if the the nonempty level set $U(\mu; t)$ is a ternary subsemigroup of S, for all $t \in (0, \frac{1-k}{2}]$.*

Proof. Let μ be an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S and $x, y, z \in U(\mu; t)$ for some $t \in (0, \frac{1-k}{2}]$. Then $\mu(x) \geq t$, $\mu(y) \geq t$, $\mu(z) \geq t$. By Theorem 169 it follows that

$$\mu(xyz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\} = t,$$

so $\mu(xyz) \geq t$. This implies that $xyz \in U(\mu; t)$. Hence $U(\mu; t)$ is a ternary subsemigroup of S.

Conversely, assume that $U(\mu; t)$ is a ternary subsemigroup of S for all $t \in (0, \frac{1-k}{2}]$. If there exist $x, y, z \in S$ such that $\mu(xyz) < \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\}$, then $\mu(xyz) < t \leq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\}$ for some $t \in (0, \frac{1-k}{2}]$, and so $x, y, z \in U(\mu; t)$ but $xyz \notin U(\mu; t)$. This is a contradiction. Hence

$$\mu(xyz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\}.$$

It follows from Theorem 169 that μ is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S. ■

Taking $k = 0$ in Theorem 172, we have the following corollary.

Corollary 173 *A fuzzy set μ in S is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S if and only if the the nonempty level set $U(\mu; t)$ is a ternary subsemigroup of S, for all $t \in (0, 0.5]$.*

In the following theorem we provide condition for an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup to be an $(\in, \in)$-fuzzy ternary subsemigroup.

Theorem 174 *Let μ be an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S such that $\mu(x) < \frac{1-k}{2}$ for all $x \in S$. Then μ is an $(\in, \in)$-fuzzy ternary subsemigroup of S.*

Proof. Straightforward. ■

If we take $k = 0$ in Theorem 174, then we have the following corollary.

Corollary 175 *Let μ be an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S such that $\mu(x) < 0.5$ for all $x \in S$. Then μ is an $(\in, \in)$-fuzzy ternary subsemigroup of S.*

Theorem 176 *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\in, \in \vee q_k)$-fuzzy ternary subsemigroups of S. Then $\mu := \bigcap_{i \in \Lambda} \mu_i$ is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S.*

Proof. Straightforward. ■

For a fuzzy set μ in S and $t \in (0, 1]$, consider the t-q_k-set $S^t_{q_k}$ and $\in \vee q_k$-set $S^t_{\in \vee q_k}$ with respect to t (briefly, t-q_k-set and t-$\in \vee q_k$-set, respectively) as follows:

$$S^t_{q_k} := \{x \in X \mid x_t \, q_k \, \mu\} \text{ and } S^t_{\in \vee q_k} := \{x \in X \mid x_t \in \vee q_k \, \mu\}.$$

Note that, for any $t, r \in (0, 1]$, if $t \geq r$ then every r-q_k-set is contained in the t-q_k-set, that is, $S^r_{q_k} \subseteq S^t_{q_k}$, and $U(\mu; t) \cup S^r_{q_k} \subseteq U(\mu; r) \cup S^t_{q_k}$. Obviously, $S^t_{\in \vee q_k} = U(\mu; t) \cup S^t_{q_k}$.

Theorem 177 *If μ is an $(\in, \in)$-fuzzy ternary subsemigroup of S, then the t-q_k-set $S^t_{q_k}$ is a ternary subsemigroup of S for all $t \in (0, 1]$, whenever it is nonempty.*

Proof. Assume that $S^t_{q_k} \neq \emptyset$ for $t \in (0, 1]$. Let $x, y, z \in S^t_{q_k}$. Then $x_t q_k \mu$, $y_t q_k \mu$, $z_t q_k \mu$, that is, $\mu(x) + t + k > 1$, $\mu(y) + t + k > 1$, $\mu(z) + t + k > 1$. It follows that

$$\begin{aligned} \mu(xyz) + t + k &\geq \min\{\mu(x),\ \mu(y),\ \mu(z)\} + t + k \\ &= \min\{\mu(x) + t + k,\ \mu(y) + t + k,\ \mu(z) + t + k\} > 1 \end{aligned}$$

and so $(xyz)_t \, q_k \mu$. Hence $xyz \in S^t_{q_k}$, therefore $S^t_{q_k}$ is a ternary subsemigroup of S. ■

The following theorem characterizes the ternary subsemigroup of ternary semigroup in terms of $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup.

Theorem 178 *Let μ be a fuzzy set in S. Then μ is an $(\in, \in \vee\, q_k)$-fuzzy ternary subsemigroup of S if and only if $S^t_{\in\vee\, q_k}$ is a ternary subsemigroup of S for all $t \in (0,1]$.*

Proof. Assume that μ is an $(\in, \in \vee\, q_k)$-fuzzy ternary subsemigroup of S and let $x, y, z \in S^t_{\in\vee\, q_k}$ for $t \in (0,1]$. Then $x_t \in \vee\, q_k\, \mu$, $y_t \in \vee\, q_k\, \mu$ and $z_t \in \vee\, q_k\, \mu$, that is, $\mu(x) \geq t$ or $\mu(x) + t > 1 - k$, $\mu(y) \geq t$ or $\mu(y) + t > 1 - k$, and $\mu(z) \geq t$ or $\mu(z) + t > 1 - k$. Using Theorem 169, $\mu(xyz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\}$,

Case 1. $\mu(x) \geq t$, $\mu(y) \geq t$, and $\mu(z) \geq t$. If $t > \frac{1-k}{2}$, then

$$\mu(xyz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\} = \frac{1-k}{2}.$$

Hence $\mu(xyz) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$, and so $(xyz)_t\, q_k\, \mu$.

If $t \leq \frac{1-k}{2}$, then

$$\mu(xyz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\} \geq t,$$

and thus $(xyz)_t \in \mu$. Therefore $(xyz)_t \in \vee q_k\, \mu$, that is, $xyz \in S^t_{\in\vee\, q_k}$.

Case 2. $\mu(x) \geq t$, $\mu(y) \geq t$, and $\mu(z) + t > 1 - k$. If $t > \frac{1-k}{2}$, then

$$\begin{aligned} \mu(xyz) &\geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\} = \mu(z) \wedge \frac{1-k}{2} \\ &> (1-k-t) \wedge \frac{1-k}{2} = 1-k-t, \end{aligned}$$

and so $(xyz)_t\, q_k\, \mu$. If $t \leq \frac{1-k}{2}$, then

$$\mu(xyz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\} \geq \min\left\{t, t, 1-k-t, \frac{1-k}{2}\right\} = t.$$

Hence $(xyz)_t \in \mu$ and thus $(xyz)_t \in \vee q_k\, \mu$, that is, $xyz \in S^t_{\in\vee\, q_k}$.

Case 3. $\mu(x) \geq t$, $\mu(y) + t > 1 - k$, and $\mu(z) + t > 1 - k$. If $t > \frac{1-k}{2}$, then

$$\begin{aligned} \mu(xyz) &\geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\} = \mu(y) \wedge \mu(z) \wedge \frac{1-k}{2} \\ &> (1-k-t) \wedge (1-k-t) \wedge \frac{1-k}{2} = 1-k-t, \end{aligned}$$

and so $(xyz)_t\, q_k\, \mu$. If $t \leq \frac{1-k}{2}$, then

$$\mu(xyz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\} \geq \min\left\{t, 1-k-t, 1-k-t, \frac{1-k}{2}\right\} = t.$$

Hence $(xyz)_t \in \mu$ and thus $(xyz)_t \in \vee q_k\, \mu$, that is, $xyz \in S^t_{\in\vee\, q_k}$.

Case 4. $\mu(x)+t > 1-k$, $\mu(y)+t > 1-k$, and $\mu(z)+t > 1-k$. If $t > \frac{1-k}{2}$, then

$$\mu(xyz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\} > (1-k-t) \wedge \frac{1-k}{2} = 1-k-t.$$

Thus $(xyz)_t\, q_k\, \mu$. If $t \leq \frac{1-k}{2}$, then

$$\mu(xyz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\} \geq (1-k-t) \wedge \frac{1-k}{2} = \frac{1-k}{2} \geq t,$$

and so $(xyz)_t \in \mu$. Hence $(xyz)_t \in \vee q_k\, \mu$, that is, $xyz \in S^t_{\in\vee\, q_k}$. Consequently, $S^t_{\in\vee\, q_k}$ is a ternary subsemigroup of S.

Conversely assume that μ is a fuzzy set in S and $t \in (0,1]$ be such that $S^t_{\in\vee\, q_k}$ is a ternary subsemigroup of S. If possible, let

$$\mu(xyz) < t \leq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\}$$

for some $t \in (0, \frac{1-k}{2}]$. Then $x, y, z \in U(\mu; t) \subseteq S^t_{\in\vee\, q_k}$, which implies that $xyz \in S^t_{\in\vee\, q_k}$. Hence $\mu(xyz) \geq t$ or $\mu(xyz) + t + k > 1$, a contradiction. Therefore $\mu(xyz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\}$ for all $x, y, z \in S$. Therefore μ is an $(\in, \in \vee\, q_k)$-fuzzy ternary subsemigroup of S. ■

3.2 $(\in, \in \vee q_k)$-fuzzy ideals

We begin this section with the following definition.

Definition 179 *Let $\alpha \in \{\in,\ q, \in \vee\, q\}$. A fuzzy set μ in S is said to be an $(\alpha, \in \vee q_k)$-*fuzzy left (resp. right, lateral) ideal *of S, if it satisfies the following condition:*

$$z_t \alpha\, \mu \ \ \textit{implies}\ (xyz)_t \in\vee\, q_k\, \mu\ (\textit{resp.}\ (zxy)_t \in \vee\, q_k\, \mu,\ xzy)_t \in \vee\, q_k\, \mu) \tag{3.4}$$

for all $x, y, z \in S$ and $t \in (0,1]$.

Moreover a fuzzy set μ in S is called an $(\alpha, \in \vee q_k)$-fuzzy two sided ideal of S if it is both $(\alpha, \in \vee q_k)$-fuzzy left ideal and $(\alpha, \in \vee q_k)$-fuzzy right ideal of S. A fuzzy set μ in S is called an $(\alpha, \in \vee q_k)$-fuzzy ideal of S if it is an $(\alpha, \in \vee q_k)$-fuzzy left ideal, $(\alpha, \in \vee q_k)$-fuzzy right ideal and $(\alpha, \in \vee q_k)$-fuzzy lateral ideal of S.

Example 180 *Consider the ternary semigroup $S = \{a, b, c, d\}$, with the ternary operation $[xyz] = x$ for all $x, y, z \in S$. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.70 & \text{if } x = a, \\ 0.60 & \text{if } x = b, \\ 0.80 & \text{if } x = c, \\ 0.55 & \text{if } x = d. \end{cases}$$

Then simple calculations show that μ is an $(\in, \in \vee q_k)$-fuzzy left ideal of S for $k = 0.2$. But μ is not an $(\in, \in)$-fuzzy left ideal of S since $a_{0.6} \in \mu$, but $(dba)_{0.6} = d_{0.6}\overline{\in}\mu$.

Example 181 *Consider the ternary semigroup S of Example 54.*

(1) *Let μ be a fuzzy set in S defined by:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.30 & \text{if } x = 0, \\ 0.35 & \text{if } x = a, \\ 0.41 & \text{if } x = b, \\ 0.62 & \text{if } x = c, \\ 0.75 & \text{if } x = 1. \end{cases}$$

Then simple calculations show that μ is an $(\in, \in \vee q_{0.3})$-fuzzy left ideal of S. But μ is neither an $(\in, \in \vee q_{0.3})$-fuzzy right ideal of S, nor an $(\in, \in \vee q_{0.3})$-fuzzy lateral ideal of S. Since $c_{0.35} \in \mu$, but

$$(cab)_{0.35} = 0_{0.35}\overline{\in \vee q_{0.3}}\mu,$$

and

$$(baa)_{0.35} = 0_{0.35}\overline{\in \vee q_{0.3}}\mu.$$

Moreover, we see that:

(*i*) *μ is not an $(\in, \in)$-fuzzy left ideal of S, since $c_{0.33} \in \mu$, but $(abc)_{0.33} = 0_{0.33}\overline{\in}\mu$.*

(*ii*) *μ is not an $(\in, \in \vee q)$-fuzzy left ideal of S, since $c_{0.37} \in \mu$, but $(cab)_{0.37} = 0_{0.37}\overline{\in \vee q}\mu$.*

(2) *Let λ be a fuzzy set in S defined by:*

$$\lambda : S \to [0,1], \quad x \mapsto \begin{cases} 0.30 & \text{if } x = 0, \\ 0.19 & \text{if } x = a, \\ 0.52 & \text{if } x = b, \\ 0.71 & \text{if } x = c, \\ 0.20 & \text{if } x = 1. \end{cases}$$

Then simple calculations show that λ is an $(\in, \in \vee q_{0.4})$-fuzzy right ideal of S. But λ is neither an $(\in, \in \vee q_{0.4})$-fuzzy left ideal of S, nor an $(\in, \in \vee q_{0.4})$-fuzzy lateral ideal of S. Since $b_{0.4} \in \mu$, but

$$(aab)_{0.4} = 0_{0.4}\overline{\in \vee q_{0.4}}\lambda,$$

and

$$(cba)_{0.4} = 0_{0.4}\overline{\in \vee q_{0.4}}\lambda.$$

Moreover, we see that:

(*i*) *λ is not an $(\in, \in)$-fuzzy right ideal of S, since $c_{0.36} \in \mu$, but $(cab)_{0.36} = 0_{0.36}\overline{\in}\mu$.*

(*ii*) *λ is not an $(\in, \in \vee q)$-fuzzy right ideal of S, since $c_{0.39} \in \lambda$, but $(cab)_{0.39} = 0_{0.39}\overline{\in \vee q}\lambda$.*

Theorem 182 *Assume that μ is a fuzzy set in S. Then*

(1) *If μ is an $(\in, \in)$-fuzzy left (resp. right, lateral) ideal of S, then μ is an $(\in, \in \vee q_k)$-fuzzy left (resp. right, lateral) ideal of S.*

(2) *If μ is an $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideal of S, then μ is an $(\in, \in \vee q_k)$-fuzzy left (resp. right, lateral) ideal of S.*

Proof. Straightforward. ■

Remark 183 *The converse of Theorem 182 is not true in general as seen in Example 181.*

Theorem 184 *A fuzzy set μ in S is an $(\in, \in \vee q_k)$-fuzzy left (resp. right, lateral) ideal of S if and only if it satisfies:*

$$\mu(xyz) \geq \min\left\{\mu(z), \frac{1-k}{2}\right\}$$

$$\left(\textit{resp. } \mu(xyz) \geq \min\left\{\mu(x), \frac{1-k}{2}\right\} \textit{ and } \mu(xyz) \geq \min\left\{\mu(y), \frac{1-k}{2}\right\}\right)$$

for all $x, y, z \in S$.

Proof. Let μ be an $(\in, \in \vee q_k)$-fuzzy left ideal of S. If there exist $x, y, z \in S$ such that $\mu(xyz) < \min\left\{\mu(z), \frac{1-k}{2}\right\}$, then we can choose $t \in (0, 1]$ such that $\mu(xyz) < t \leq \min\left\{\mu(z), \frac{1-k}{2}\right\}$. Then $z_t \in \mu$ but $(xyz)_t \overline{\in} \mu$ and $\mu(xyz) + t + k < \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. This implies that $(xyz)_t \overline{\in \vee q_k} \mu$, a contradiction. Hence $\mu(xyz) \geq \min\left\{\mu(z), \frac{1-k}{2}\right\}$.

Conversely, assume that $\mu(xyz) \geq \min\left\{\mu(z), \frac{1-k}{2}\right\}$. Let $x, y, z \in S$ and $t \in (0, 1]$ be such that $z_t \in \mu$. Then $\mu(z) \geq t$, which implies that $\mu(xyz) \geq \min\left\{\mu(z), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\}$. If $t > \frac{1-k}{2}$, then $\mu(xyz) \geq \frac{1-k}{2}$ and so $\mu(xyz) + t + k > \frac{1-k}{2} + \frac{1-k}{2} + k = 1$. This implies that $(xyz)_t \, q_k \mu$. If $t \leq \frac{1-k}{2}$, then $\mu(xyz) \geq t$. Thus $(xyz)_t \in \mu$. Hence $(xyz)_t \in \vee q_k \mu$. Therefore μ is an $(\in, \in \vee q_k)$-fuzzy left ideal of S. ■

If we take $k = 0$ in Theorem 184, we obtain the following corollary.

Corollary 185 *A fuzzy set μ in S is an $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideal of S if and only if $\mu(xyz) \geq \min\{\mu(z), 0.5\}$ (resp. $\mu(xyz) \geq \min\{\mu(x), 0.5\}$, $\mu(xyz) \geq \min\{\mu(y), 0.5\}$).*

Lemma 186 *A nonempty subset A of S is a right (resp. left, lateral) ideal of S if and only if the characteristic function χ_A of A is an $(\in, \in \vee q_k)$-fuzzy right (resp. left, lateral) ideal of S.*

Proof. Let A be a right ideal of S, χ_A the characteristic function of A and $x \in S$. If $x \notin A$, then $\chi_A(x) = 0$ and so $\chi_A(xyz) \geq \min\left\{\chi_A(x), \frac{1-k}{2}\right\}$. If $x \in A$, then $\chi_A(x) = 1$. Since A is a right ideal of S, so $xyz \in A$ and $\chi_A(xyz) = 1$. It follows that

$$\chi_A(xyz) \geq \min\left\{\chi_A(x), \frac{1-k}{2}\right\}.$$

Thus χ_A is an $(\in, \in \vee q_k)$-fuzzy right ideal of S.

Conversely assume that the characteristic function χ_A of A is an $(\in, \in \vee q_k)$-fuzzy right ideal of S. Let $x \in ASS$. Then $x = auv$ for some $u, v \in S$ and $a \in A$. Therefore $\chi_A(a) = 1$. It follows that

$$\chi_A(x) = \chi_A(auv) \geq \min\left\{\chi_A(a), \frac{1-k}{2}\right\}.$$

Thus $\chi_A(x) = 1$. Hence $x \in A$. Therefore A is a right ideal of S. ■

Theorem 187 *Let μ be a fuzzy set in S. Then μ is an $(\in, \in \vee q_k)$-fuzzy left (resp. right, lateral) ideal of S if and only if $U(\mu; t)$ is a left (resp. right, lateral) ideal of S for all $t \in (0, \frac{1-k}{2}]$, whenever it is nonempty.*

Proof. The proof is similar to the proof of Theorem 172. ■

Theorem 188 *If μ is an $(\in, \in \vee q_k)$- fuzzy left ideal, λ an $(\in, \in \vee q_k)$-fuzzy lateral ideal and ν an $(\in, \in \vee q_k)$-fuzzy right ideal of S, then $\mu \circ \lambda \circ \nu$ is an $(\in, \in \vee q_k)$-fuzzy two sided ideal of S.*

Proof. The proof is straightforward. ■

Lemma 189 *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\in, \in \vee q_k)$-fuzzy left (resp. right, lateral) ideals of S. Then*

(i) $\bigcap_{i \in \Lambda} \mu_i$ is an $(\in, \in \vee q_k)$-fuzzy left (resp. right, lateral) ideal of S.

(ii) $\bigcup_{i \in \Lambda} \mu_i$ is an $(\in, \in \vee q_k)$-fuzzy left (resp. right, lateral) ideal of S.

Proof. Straightforward. ■

Theorem 190 *If μ is an $(\in, \in)$-fuzzy left (resp. right, lateral) ideal of S, then the t-q_k-set $S_{q_k}^t$ is a left (resp. right, lateral) ideal of S for all $t \in (0, 1]$, whenever it is nonempty.*

Proof. The proof is similar to the proof of Theorem 177. ■

Theorem 191 *For a fuzzy set μ in S, if the t-q_k-set $S_{q_k}^t$ is a left (resp. right, lateral) ideal of S for all $t \in (\frac{1-k}{2}, 1]$, then μ is an $(\in, q_k)$-fuzzy left (resp. right, lateral) ideal of S.*

Proof. Let $x, y, z \in S$ and $t \in (\frac{1-k}{2}, 1]$ be such that $z_t \in \mu$. Then $\mu(z) \geq t > \frac{1-k}{2}$. It follows that $\mu(z) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$, that is, $z_t q_k \mu$ and so $z \in S_{q_k}^t$. By hypothesis we have $xyz \in S_{q_k}^t$ and so $(xyz)_t\, q_k \mu$. Therefore μ is an $(\in, q_k)$-fuzzy left ideal of S.

Similarly we can prove the cases of right ideal and lateral ideal of S. ■

Theorem 192 *For a fuzzy set μ in S, if the t-q_k-set $S_{q_k}^t$ is a left (resp. right, lateral) ideal of S for all $t \in (0, \frac{1-k}{2}]$, then μ is a $(q, \in)$-fuzzy left (resp. right, lateral) ideal of S.*

Proof. Let $x, y, z \in S$ and $t \in (0, \frac{1-k}{2}]$ be such that $z_t q \mu$. Then $z \in S_{q_k}^t$. By hypothesis we have $xyz \in S_{q_k}^t$ and so $(xyz)_t\, q_k \mu$. It follows that $\mu(xyz) + t + k > 1$, that is, $\mu(xyz) > 1 - t - k \geq t$, and so $\mu(xyz) \geq t$. Hence $(xyz)_t \in \mu$. Therefore μ is a $(q, \in)$-fuzzy left ideal of S.

The cases of right ideal and lateral ideal of S may be dealt with similarly. ■

Similarly we can prove that:

Theorem 193 *For a fuzzy set μ in S, if the t-q_k-set $S_{q_k}^t$ is a left (resp. right, lateral) ideal of S for all $t \in (0, \frac{1-k}{2}]$, then μ is a $(q, \in \vee q_k)$-fuzzy left (resp. right, lateral) ideal of S.*

Theorem 194 *For a fuzzy set μ in S, if the nonempty t-$\in \vee\, q_k$ -set $S_{\in \vee\, q_k}^t$ is a left (resp. right, lateral) ideal of S for all $t \in (0, 1]$, then μ is a $(q, \in \vee\, q_k\,)$-fuzzy left (resp. right, lateral) ideal of S.*

Proof. We prove only for left ideal. Let $x, y, z \in S$ and $t \in (0, 1]$ be such that $z_t q \mu$. Then $z \in S_q^t \subseteq S_{\in \vee\, q_k}^t$. By hypothesis it follows that $xyz \in S_{\in \vee\, q_k}^t$. Hence $(xyz)_t \in \vee\, q_k \mu$. Therefore μ is a $(q, \in \vee\, q_k\,)$-fuzzy left ideal of S. ■

One naturally asks the following interesting question:

Question: If μ is an $(\in, \in \vee q_k)$-fuzzy left (resp. right, lateral) ideal of S, then is the t-q_k-set $S_{q_k}^t$ a left (resp. right, lateral) ideal of S?

The answer to the above question is negative (for $t \leq \frac{1-k}{2}$) as seen in the following example:

Example 195 *Consider the ternary semigroup S of Example 54. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.40 & \text{if } x = 0, \\ 0.62 & \text{if } x = a, \\ 0.71 & \text{if } x = b, \\ 0.82 & \text{if } x = c, \\ 0.30 & \text{if } x = 1. \end{cases}$$

Then

$$U(\mu; t) = \begin{cases} S & \text{if } t \in (0, 0.30], \\ \{0, a, b, c\} & \text{if } t \in (0.30, 0.40], \\ \{a, b, c\} & \text{if } t \in (0.40, 0.62] \\ \{b, c\} & \text{if } t \in (0.62, 0.71], \\ \{c\} & \text{if } t \in (0.71, 0.82], \\ \emptyset & \text{if } t \in (0.82, 1]. \end{cases}$$

Then by Theorem 187, μ is an $(\in, \in \vee q_{0.2})$-fuzzy left ideal of S, but the set

$$S_{q_{0.2}}^{0.30} = \{a, b, c\},$$

is not a left ideal of S, because $0ab = 0 \notin S_{q_{0.2}}^{0.30}$.

But the following theorem answers the above question affirmatively:

Theorem 196 *If μ is an $(\in, \in \vee q_k)$-fuzzy left (resp. right, lateral) ideal of S, then the nonempty t-q_k-set $S_{q_k}^t$ is a left (resp. right, lateral) ideal of S for all $t \in (\frac{1-k}{2}, 1]$.*

Proof. Assume that $S_{q_k}^t \neq \emptyset$ for $t \in (\frac{1-k}{2}, 1]$. Let $x, y, z \in S$ and $z \in S_{q_k}^t$. Then $z_t q_k \mu$, that is, $\mu(z) + t + k > 1$. By Theorem 184 it follows that

$$\begin{aligned} \mu(xyz) + t + k &\geq \min\left\{\mu(z), \frac{1-k}{2}\right\} + t + k \\ &= \min\left\{\mu(z) + t + k, \frac{1-k}{2} + t + k\right\} > 1. \end{aligned}$$

So $(xyz)_t\, q_k \mu$. Hence $xyz \in S_{q_k}^t$. Therefore $S_{q_k}^t$ is a left ideal of S.

The same argument leads to the proof of the cases of right and lateral ideal. ■

Theorem 197 *A fuzzy set μ in S is an $(\in, \in \vee q_k)$-fuzzy left (resp. right, lateral) ideal of S, if and only if the nonempty t-$\in \vee q_k$-set $S_{\in \vee q_k}^t$ is a left (resp. right, lateral) ideal of S for all $t \in (0, 1]$.*

Proof. The proof is similar to the proof of Theorem 178. ■

3.2.1 $(\in, \in \vee q_k)$-fuzzy quasi-ideals

Definition 198 *A fuzzy set μ in S is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S if it satisfies:*

$$(1)\ \mu(x) \geq \min\left\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x), \frac{1-k}{2}\right\}$$

$$(2)\ \mu(x) \geq \min\left\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x), \frac{1-k}{2}\right\},$$

where $\mathcal{S}$ is the fuzzy set in S mapping every element of S on 1.

Theorem 199 *If μ is an $(\in, \in \vee q_k)$-fuzzy quasi ideal of S, then S_0 is a quasi-ideal of S.*

Proof. Let $a \in SSS_0 \cap SS_0S \cap S_0SS$. Then $a \in SSS_0$, $a \in SS_0S$ and $a \in S_0SS$. Thus there exist $x, y, z \in S_0$ and $s_1, s_2, s_3, t_1, t_2, t_3 \in S$ such that $a = xs_1t_1$, $a = s_2yt_2$, $a = s_3t_3z$. It follows that

$$\begin{aligned} (\mathcal{S} \circ \mathcal{S} \circ \mu)(a) &= \bigvee_{a=pqr} \{\mathcal{S}(p) \wedge \mathcal{S}(q) \wedge \mu(r)\} \\ &\geq \mathcal{S}(s_3) \wedge \mathcal{S}(t_3) \wedge \mu(z) = \mu(z). \end{aligned}$$

Similarly we can show that $(\mathcal{S} \circ \mu \circ \mathcal{S})(a) \geq \mu(y)$ and $(\mu \circ \mathcal{S} \circ \mathcal{S})(a) \geq \mu(x)$. It follows that

$$\begin{aligned} \mu(a) &\geq \min\left\{(\mu \circ \mathcal{S} \circ \mathcal{S})(a), (\mathcal{S} \circ \mu \circ \mathcal{S})(a), (\mathcal{S} \circ \mathcal{S} \circ \mu)(a), \frac{1-k}{2}\right\} \\ &\geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\} > 0 \end{aligned}$$

This implies that $a \in S_0$. Hence $SSS_0 \cap SS_0S \cap S_0SS \subseteq S_0$.

Again suppose $a \in SSS_0 \cap SSS_0SS \cap S_0SS$. Then $a \in SSS_0$, $a \in SSS_0SS$ and $a \in S_0SS$. Thus $a = xs_1t_1$, $a = s_2t_2ys_4t_4$, $a = s_3t_3z$ for some $x, y, z, s_1, s_2, s_3, s_4, t_1, t_2, t_3, t_4 \in S$. For $a = s_3t_3z$ and $a = xs_1t_1$, discussed above. We have

$$\mu(a) \geq \min\left\{(\mu \circ \mathcal{S} \circ \mathcal{S})(a), (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(a), (\mathcal{S} \circ \mathcal{S} \circ \mu)(a), \frac{1-k}{2}\right\},$$

and by the above arguments $(\mathcal{S} \circ \mathcal{S} \circ \mu)(a) \geq \mu(z)$, $(\mu \circ \mathcal{S} \circ \mathcal{S})(a) \geq \mu(x)$. It follows that

$$\begin{aligned} (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(a) &= \bigvee_{a=pqr} ((\mathcal{S} \circ \mathcal{S} \circ \mu)(p) \wedge \mathcal{S}(q) \wedge \mathcal{S}(r)) \\ &= \left\{\bigvee_{a=pqr} \left\{\bigvee_{p=uvw} \mathcal{S}(u) \wedge \mathcal{S}(v) \wedge \mu(w)\right\} \wedge \mathcal{S}(q) \wedge \mathcal{S}(r)\right\} \\ &= \left\{\bigvee_{a=(uvw)qr} \mathcal{S}(u) \wedge \mathcal{S}(v) \wedge \mu(w) \wedge \mathcal{S}(q) \wedge \mathcal{S}(r)\right\} \\ &\geq \mathcal{S}(s_2) \wedge \mathcal{S}(t_2) \wedge \mu(y) \wedge \mathcal{S}(s_4) \wedge \mathcal{S}(t_4) = \mu(y). \end{aligned}$$

Now

$$\begin{aligned}\mu(a) &\geq \min\left\{(\mu \circ \mathcal{S} \circ \mathcal{S})(a), (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(a), (\mathcal{S} \circ \mathcal{S} \circ \mu)(a), \frac{1-k}{2}\right\} \\ &\geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\} \\ &> 0 \qquad \because \mu(x), \mu(y), \mu(z) > 0, \frac{1-k}{2} > 0.\end{aligned}$$

Thus $a \in S_0$. Hence $SSS_0 \cap SSS_0SS \cap S_0SS \subseteq S_0$. Therefore S_0 is a quasi-ideal of S. ■

Lemma 200 *A nonempty subset Q of S is a quasi-ideal of S if and only if the characteristic function χ_Q of Q is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S.*

Proof. The proof is straightforward. ■

Theorem 201 *Every $(\in, \in \vee q_k)$-fuzzy left (right, lateral) ideal of S is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S.*

Proof. Straightforward. ■

Remark 202 *The converse of the Theorem 201 may not be true in general as seen in the following example.*

Example 203 *Let*

$$S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

be a ternary semigroup under ternary matrix multiplication. Then

$$Q = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

is the quasi-ideal of S which is neither left, nor right, nor a lateral ideal of S (see [10]). Define a fuzzy set μ in S as follows:

$$\mu : S \to [0,1], \; x \to \begin{cases} 0.70 & \text{if } x = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ 0.51 & \text{if } x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ 0.35 & \text{otherwise.} \end{cases}$$

Then simple calculations show that μ is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S for $k \in [0,1)$, which is neither $(\in, \in \vee q_k)$-fuzzy left, nor $(\in, \in \vee q_k)$-fuzzy right, nor $(\in, \in \vee q_k)$-fuzzy lateral ideal of S.

3.3 $(\in, \in \vee q_k)$-fuzzy bi-ideals

Definition 204 *A fuzzy set μ in S is said to be an* $(\in, \in \vee q_k)$-fuzzy bi-ideal *of S, if it satisfies the following conditions:*

$$x_{t_1} \in \mu,\ y_{t_2} \in \mu \text{ and } z_{t_3} \in \mu \text{ imply } (xyz)_{\min\{t_1,t_2,t_3\}} \in\vee\, q_k\, \mu, \tag{3.5}$$

and

$$x_{t_4} \in \mu,\ y_{t_5} \in \mu \text{ and } z_{t_6} \in \mu \text{ imply } (xuyvz)_{\min\{t_4,t_5,t_6\}} \in\vee\, q_k\, \mu, \tag{3.6}$$

for all $x, u, y, v, z \in S$ and $t_1, t_2, t_3, t_4, t_5, t_6 \in (0, 1]$.

A fuzzy set μ in S satisfying condition (3.6) is called an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S.

An $(\in, \in\vee q_k)$-fuzzy bi-ideal of S with $k = 0$ is an $(\in, \in\vee q)$-fuzzy bi-ideal of S.

Example 205 *Consider the ternary semigroup S as given in Example 102. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0, 1], \quad x \mapsto \begin{cases} 0.4 & \text{if } x = 0, \\ 0.6 & \text{if } x = 1, \\ 0.7 & \text{if } x = 2, \\ 0.8 & \text{if } x = 3, \\ 0.3 & \text{if } x = 4, \\ 0.3 & \text{if } x = 5. \end{cases}$$

Then simple calculations show that μ is an $(\in, \in \vee q_{0.2})$-fuzzy bi-ideal of S. But

(i) μ is not an $(\in, \in)$-fuzzy bi-ideal of S, because $2_{0.6} \in \mu$ but $(20202)_{0.6} = 0_{0.6} \overline{\in} \mu$.

(ii) μ is not an $(\in, \in \vee q)$-fuzzy bi-ideal of S, because $2_{0.5} \in \mu$ but $(20202)_{0.5} = 0_{0.5} \overline{\in \vee q} \mu$.

Theorem 206 *Let B be a bi-ideal (resp. generalized bi-ideal) of S and $\alpha \in \{\in, q, \in \vee q\}$. Then the fuzzy set μ in S defined by:*

$$\mu(x) = \begin{cases} \geq 0.5 & \text{for all } x \in B, \\ 0 & \text{for all } x \in S \setminus B, \end{cases}$$

is an $(\alpha, \in \vee q_k)$-fuzzy bi-ideal (resp. generalized bi-ideal) of S.

Proof. The proof is similar to the proof of Theorem 167. ■

Corollary 207 *A nonempty subset B of S is a bi-ideal of S if and only if the characteristic function χ_B of B is an $(\in, \in\vee q_k)$-fuzzy bi-ideal of S.*

Theorem 208 *Let μ be a fuzzy set in S. Then μ is an $(\in, \in \vee\, q_k\,)$-fuzzy bi-ideal of S if and only if it satisfies:*

$$\mu(xyz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\} \tag{3.7}$$

$$\mu(xuyvz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\} \tag{3.8}$$

for all $x, y, z, u, v \in S$.

Proof. The proof is similar to the proof of Theorem 169. ■

If we take $k = 0$ in Theorem 208, then we have the following corollary.

Corollary 209 *Let μ be a fuzzy set in S. Then μ is an $(\in, \in \vee\, q)$-fuzzy bi-ideal of S if and only if it satisfies:*

$$\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\}$$

and

$$\mu(xuyvz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\}$$

for all $x, y, z, u, v \in S$.

Theorem 210 *Let μ be a fuzzy set in S. Then μ is an $(\in, \in \vee\, q_k\,)$-fuzzy bi-ideal (resp. generalized bi-ideal) of S if and only if $U(\mu; t)$ is a bi-ideal (resp. generalized bi-ideal) of S for all $t \in (0, \frac{1-k}{2}]$, whenever it is nonempty.*

Proof. The proof is similar to the proof of Theorem 172. ■

Theorem 211 *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\in, \in \vee\, q_k\,)$-fuzzy bi-ideals (resp. generalized bi-ideals) of S. Then $\mu := \bigcap_{i\in\Lambda} \mu_i$ is an $(\in, \in \vee\, q_k\,)$-fuzzy bi-ideal (resp. generalized bi-ideal) of S.*

Proof. Let $x, u, y, v, z \in S$. Then

$$\begin{aligned}
\mu(xuyvz) &= \left(\bigcap_{i\in\Lambda} \mu_i\right)(xuyvz) = \bigwedge_{i\in\Lambda} \mu_i(xuyvz) \\
&\geq \bigwedge_{i\in\Lambda} \left\{\min\left\{\mu_i(x), \mu_i(y), \mu_i(z), \frac{1-k}{2}\right\}\right\} \\
&= \min\left\{\bigwedge_{i\in\Lambda} \mu_i(x),\ \bigwedge_{i\in\Lambda} \mu_i(y),\ \bigwedge_{i\in x\Lambda} \mu_i(z), \frac{1-k}{2}\right\} \\
&= \min\left\{\left(\bigcap_{i\in\Lambda} \mu_i\right)(x),\ \left(\bigcap_{i\in\Lambda} \mu_i\right)(y),\ \left(\bigcap_{i\in\Lambda} \mu_i\right)(z), \frac{1-k}{2}\right\} \\
&= \min\left\{\mu(x),\ \mu(y),\ \mu(z),\ \frac{1-k}{2}\right\}.
\end{aligned}$$

Similarly we have $\mu(xyz) \geq \min\left\{\mu(x),\ \mu(y),\ \mu(z),\ \frac{1-k}{2}\right\}$. Therefore $\mu := \bigcap\limits_{i\in\Lambda} \mu_i$ is an $(\in, \in\vee q_k)$-fuzzy bi-ideal of S. ■

Using Theorem 208, we provide a condition for an $(\in, \in\vee q_k)$-fuzzy bi-ideal to be an $(\in, \in)$-fuzzy bi-ideal.

Theorem 212 *Let μ be an $(\in, \in\vee q_k)$-fuzzy bi-ideal of S such that $\mu(x) < \frac{1-k}{2}$ for all $x \in S$. Then μ is an $(\in, \in)$-fuzzy bi-ideal of S.*

Proof. Straightforward by using Theorem 208. ■

Theorem 213 *If $0 \leq k < r < 1$, then every $(\in, \in\vee q_k)$-fuzzy bi-ideal of S is an $(\in, \in\vee q_r)$-fuzzy bi-ideal of S.*

Proof. Straightforward. ■

Remark 214 *If $0 \leq k < r < 1$, then an $(\in, \in\vee q_r)$-fuzzy bi-ideal of S may not be an $(\in, \in\vee q_k)$-fuzzy bi-ideal of S. In fact, the $(\in, \in \vee q_{0.2})$-fuzzy bi-ideal μ of S given in Example 102 is not an $(\in, \in \vee q_{0.1})$-fuzzy bi-ideal of S, since $2_{0.42} \in \mu$, but $(20202)_{0.42} = 0_{0.42}\overline{\in \vee q}_{0.1}\mu$.*

Theorem 215 *Every $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S.*

Proof. Let μ be an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S and $u, v, x, y, z \in S$. Then

$$\begin{aligned}
\mu(xyz) &\geq (\mu \circ \mathcal{S} \circ \mathcal{S})(xyz) \wedge (\mathcal{S} \circ \mu \circ \mathcal{S})(xyz) \wedge (\mathcal{S} \circ \mathcal{S} \circ \mu)(xyz) \wedge \frac{1-k}{2} \\
&= \left\{\bigvee_{xyz=abc} \{\mu(a) \wedge \mathcal{S}(b) \wedge \mathcal{S}(c)\}\right\} \wedge \left\{\bigvee_{xyz=pqr} \{\mathcal{S}(p) \wedge \mu(q) \wedge \mathcal{S}(r)\}\right\} \\
&\quad \wedge \left\{\bigvee_{xyz=lmn} \{\mathcal{S}(l) \wedge \mathcal{S}(m) \wedge \mu(n)\}\right\} \wedge \frac{1-k}{2} \\
&\geq \{\mu(x) \wedge \mathcal{S}(y) \wedge \mathcal{S}(z)\} \wedge \{\mathcal{S}(x) \wedge \mu(y) \wedge \mathcal{S}(z)\} \\
&\quad \wedge \{\mathcal{S}(x) \wedge \mathcal{S}(y) \wedge \mu(z)\} \wedge \frac{1-k}{2} = \mu(x) \wedge \mu(y) \wedge \mu(z) \wedge \frac{1-k}{2}.
\end{aligned}$$

Hence $\mu(xyz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\}$. Also

$$\begin{aligned}
\mu(xuyvz) &\geq (\mu \circ \mathcal{S} \circ \mathcal{S})(xuyvz) \wedge (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(xuyvz) \\
&\quad \wedge (\mathcal{S} \circ \mathcal{S} \circ \mu)(xuyvz) \wedge \frac{1-k}{2} \\
&= \left\{\bigvee_{xuyvz=abc} \{\mu(a) \wedge \mathcal{S}(b) \wedge \mathcal{S}(c)\}\right\} \\
&\quad \wedge \left\{\bigvee_{xuyvz=rst} \left\{\bigvee_{r=s_1s_2s_3} \mathcal{S}(s_1) \wedge \mathcal{S}(s_2) \wedge \mu(s_3)\right\} \wedge \mathcal{S}(s) \wedge \mathcal{S}(t)\}\right\} \\
&\quad \wedge \left\{\bigvee_{xuyvz=lmn} \{\mathcal{S}(l) \wedge \mathcal{S}(m) \wedge \mu(n)\}\right\} \wedge \frac{1-k}{2}
\end{aligned}$$

$$\begin{aligned}
&\geq \{\mu(x) \wedge \mathcal{S}(uyv) \wedge \mathcal{S}(z)\} \wedge \{\mathcal{S}(x) \wedge \mathcal{S}(u) \wedge \mu(y) \wedge \mathcal{S}(v) \wedge \mathcal{S}(z)\} \\
&\quad \wedge \{\mathcal{S}(x) \wedge \mathcal{S}(uyv) \wedge \mu(z)\} \wedge \frac{1-k}{2} \\
&= \mu(x) \wedge \mu(y) \wedge \mu(z) \wedge \frac{1-k}{2}.
\end{aligned}$$

Hence $\mu(xuyvz) \geq \min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\}$. Therefore μ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. ■

Theorem 216 *If μ is an $(\in, \in)$-fuzzy bi-ideal of S, then the t-q_k-set $S^t_{q_k}$ is a bi-ideal of S for all $t \in (0, 1]$, whenever it is nonempty.*

Proof. It is similar to the proof of Theorem 177. ■

The following theorem characterizes the bi-ideal of ternary semigroup in terms of $(\in, \in \vee q_k)$-fuzzy bi-ideal.

Theorem 217 *Let μ be a fuzzy set in S. Then μ is an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S if and only if $S^t_{\in \vee q_k}$ is a bi-ideal of S for all $t \in (0, 1]$.*

Proof. It is similar to the proof of Theorem 178. ■

3.4 Regular Ternary Semigroups

In this section we characterize regular ternary semigroups in terms of $(\in, \in \vee q_k)$-fuzzy ideals , $(\in, \in \vee q_k)$-fuzzy quasi-ideals, $(\in, \in \vee q_k)$-fuzzy bi-ideals.

Definition 218 *Let μ, λ and ν be fuzzy sets in S. Define the fuzzy sets μ_k, $\mu \wedge_k \lambda$, $\mu \vee_k \lambda$ and $\mu \circ_k \lambda \circ_k \nu$ in S as follows:*

(1) $\mu_k(x) = \mu(x) \wedge \frac{1-k}{2}$;

(2) $(\mu \wedge_k \lambda)(x) = (\mu \wedge \lambda)(x) \wedge \frac{1-k}{2}$;
(3) $(\mu \vee_k \lambda)(x) = (\mu \vee \lambda)(x) \wedge \frac{1-k}{2}$;
(4) $(\mu \circ_k \lambda \circ_k \nu)(x) = (\mu \circ \lambda \circ \nu)(x) \wedge \frac{1-k}{2}$;
for all $x \in S$.

Lemma 219 *Let* μ, λ *and* ν *be fuzzy sets in* S. *Then the following hold:*
(1) $\mu \wedge_k \lambda = \mu_k \wedge \lambda_k$;
(2) $\mu \vee_k \lambda = \mu_k \vee \lambda_k$;
(3) $\mu \circ_k \lambda \circ_k \nu = \mu_k \circ \lambda_k \circ \nu_k$.

Proof. Straightforward. ■

Theorem 220 *Let* μ *be a fuzzy set in* S. *If*

(i) μ *is an* $(\in, \in \vee q_k)$*-fuzzy ternary subsemigroup of* S, *then* μ_k *is an* $(\in, \in \vee q_k)$*-fuzzy ternary subsemigroup of* S.

(ii) μ *is an* $(\in, \in \vee q_k)$*-fuzzy left (resp. right, lateral) ideal of* S, *then* μ_k *is an* $(\in, \in \vee q_k)$*-fuzzy left (resp. right, lateral) ideal of* S.

(iii) μ *is an* $(\in, \in \vee q_k)$*-fuzzy quasi-ideal of* S, *then* μ_k *is an* $(\in, \in \vee q_k)$*-fuzzy quasi-ideal of* S.

(iv) μ *is an* $(\in, \in \vee q_k)$*-fuzzy bi-ideal of* S, *then* μ_k *is an* $(\in, \in \vee q_k)$*-fuzzy bi-ideal of* S.

Proof. We prove only (i). Assume that μ is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S and $x, y, z \in S$. Then

$$\begin{aligned}
(\mu_k)(xyz) &= \mu(xyz) \wedge \frac{1-k}{2} \geq \left\{\min\left\{\mu(x), \mu(y), \mu(z), \frac{1-k}{2}\right\}\right\} \wedge \frac{1-k}{2} \\
&= \min\left\{\mu(x) \wedge \frac{1-k}{2}, \mu(y) \wedge \frac{1-k}{2}, \mu(z) \wedge \frac{1-k}{2}, \frac{1-k}{2}\right\} \\
&= \min\left\{\mu_k(x), \mu_k(y), \mu_k(z), \frac{1-k}{2}\right\}.
\end{aligned}$$

Hence μ_k is an $(\in, \in \vee q_k)$-fuzzy ternary subsemigroup of S. ■

Lemma 221 *Let* A, B *and* C *be nonempty subsets of* S. *Then the following hold:*
(1) $(\chi_A \wedge_k \chi_B) = (\chi_{A \cap B})_k$;
(2) $(\chi_A \vee_k C_B) = (\chi_{A \cup B})_k$;
(3) $(\chi_A \circ_k \chi_B \circ_k \chi_c) = (\chi_{ABC})_k$;
where χ_A *is the characteristic function of* A.

Proof. Straightforward. ■

Lemma 222 *A nonempty subset A of S is a left (resp. right, lateral) ideal of S if and only if $(\chi_A)_k$ is an $(\in, \in \vee q_k)$-fuzzy left (resp. right, lateral) ideal of S.*

Proof. Let A be a left ideal of S. Then by Theorem 167 it follows that, $(\chi_A)_k$ is an $(\in, \in \vee q_k)$-fuzzy left ideal of S.

Conversely assume that $(\chi_A)_k$ is an $(\in, \in \vee q_k)$-fuzzy left ideal of S and $z \in A$. Then $(\chi_A)_k(z) = \frac{1-k}{2}$, which implies that $z_{\frac{1-k}{2}} \in (\chi_A)_k$. Since $(\chi_A)_k$ is an $(\in, \in \vee q_k)$-fuzzy left ideal of S, so, $(xyz)_{\frac{1-k}{2}} \in \vee q_k (\chi_A)_k$. This implies that $(xyz)_{\frac{1-k}{2}} \in (\chi_A)_k$ or $(xyz)_{\frac{1-k}{2}} q_k (\chi_A)_k$. Thus $(\chi_A)_k(xyz) \geq \frac{1-k}{2}$ or $(\chi_A)_k(xyz) + \frac{1-k}{2} + k > 1$.

If $(\chi_A)_k(xyz) + \frac{1-k}{2} + k > 1$, then $(\chi_A)_k(xyz) > \frac{1-k}{2}$. Thus $(\chi_A)_k(xyz) \geq \frac{1-k}{2}$, so $(\chi_A)_k(xyz) = \frac{1-k}{2}$. Hence $xyz \in A$. Therefore A is a left ideal of S. ■

Definition 223 *An $(\in, \in \vee q_k)$-fuzzy ideal μ of S is called idempotent if $\mu \circ_k \mu \circ_k \mu = \mu_k$.*

Lemma 224 *A nonempty subset Q of S is a quasi-ideal of S if and only if $(\chi_Q)_k$ is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S.*

Proof. The proof is similar to the proof of Lemma 222. ■

Next we show that if μ is an $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideal of S, then μ_k is a fuzzy left (resp. right) ideal of S.

Proposition 225 *Let μ be an $(\in, \in \vee q_k)$-fuzzy left (resp. right) ideal of S. Then μ_k is a fuzzy left (resp. right) ideal of S.*

Proof. Let μ be an $(\in, \in \vee q_k)$-fuzzy left ideal of S and $x, y, z \in S$. Then $\mu(xyz) \geq \mu(z) \wedge \frac{1-k}{2}$. It follows that $\mu(xyz) \wedge \frac{1-k}{2} \geq \mu(z) \wedge \frac{1-k}{2} \wedge \frac{1-k}{2}$, which implies that $\mu_k(xyz) \geq \mu_k(z)$. Hence μ_k is a fuzzy left ideal of S.

Case of right ideal may be dealt with similarly. ■

The following example shows that there exists a fuzzy left ideal of S which is not of the form μ_k for some $(\in, \in \vee q_k)$-fuzzy left ideal μ of S.

Example 226 *Let $S = \{a, b, c, d\}$ and define the ternary operation on S as $[xyz] = a$ for all $x, y, z \in S$. Then S is a ternary semigroup. Define a fuzzy set μ in S by:*

$\mu(a) = 1, \mu(b), \mu(c), \mu(d) > \frac{1-k}{2}$. Then μ is a fuzzy left ideal of S, for all $k \in [0, 1)$, but this is not of the form μ_k for some $(\in, \in \vee q_k)$-fuzzy left ideal μ of S.

Lemma 227 *Let μ be an $(\in, \in \vee q_k)$-fuzzy right ideal and λ an $(\in, \in \vee q_k)$-fuzzy left ideal of S. Then $\mu \circ_k \mathcal{S} \circ_k \lambda \leq \mu \wedge_k \lambda$.*

Proof. Let μ be an $(\in, \in \vee q_k)$-fuzzy right ideal and λ an $(\in, \in \vee q_k)$-fuzzy left ideal of S and $a \in S$. Then

$$\begin{aligned}
(\mu \circ_k \mathcal{S} \circ_k \lambda)(a) &= (\mu \circ \mathcal{S} \circ \lambda)(a) \wedge \frac{1-k}{2} \\
&= \left\{ \bigvee_{a=lmn} \mu(l) \wedge \mathcal{S}(m) \wedge \lambda(n) \right\} \wedge \frac{1-k}{2} \\
&= \left\{ \bigvee_{a=lmn} \mu(l) \wedge \lambda(n) \right\} \wedge \frac{1-k}{2} \\
&\leq \left\{ \bigvee_{a=lmn} \mu(lmn) \wedge \lambda(lmn) \right\} \wedge \frac{1-k}{2} = (\mu \wedge_k \lambda)(a).
\end{aligned}$$

Hence $\mu \circ_k \mathcal{S} \circ_k \lambda \leq \mu \wedge_k \lambda$. ∎

Theorem 228 *For a ternary semigroup S, the following assertions are equivalent:*

(1) *S is regular;*

(2) *$\mu \wedge_k \lambda \wedge_k \nu = \mu \circ_k \lambda \circ_k \nu$ for every $(\in, \in \vee q_k)$-fuzzy right ideal μ, every $(\in, \in \vee q_k)$-fuzzy lateral ideal λ and every $(\in, \in \vee q_k)$-fuzzy left ideal ν of S.*

Proof. (1) $\Rightarrow$ (2) : Let μ be an $(\in, \in \vee q_k)$-fuzzy right ideal, λ an $(\in, \in \vee q_k)$-fuzzy lateral ideal, ν an $(\in, \in \vee q_k)$-fuzzy left ideal of S and $a \in S$. Then

$$\begin{aligned}
(\mu \circ_k \lambda \circ_k \nu)(a) &= (\mu \circ \lambda \circ \nu)(a) \wedge \frac{1-k}{2} \\
&= \left\{ \bigvee_{a=pqr} \mu(p) \wedge \lambda(q) \wedge \nu(r) \right\} \wedge \frac{1-k}{2} \\
&= \bigvee_{a=pqr} \left\{ \begin{array}{c} \left(\mu(p) \wedge \frac{1-k}{2}\right) \wedge \left(\lambda(q) \wedge \frac{1-k}{2}\right) \\ \wedge \left(\nu(r) \wedge \frac{1-k}{2}\right) \end{array} \right\} \wedge \frac{1-k}{2} \\
&\leq \bigvee_{a=pqr} \{\mu(pqr) \wedge \lambda(pqr) \wedge \nu(pqr)\} \wedge \frac{1-k}{2} \\
&= \mu(a) \wedge \lambda(a) \wedge \nu(a) \wedge \frac{1-k}{2} = (\mu \wedge_k \lambda \wedge_k \nu)(a).
\end{aligned}$$

Hence $\mu \circ_k \lambda \circ_k \nu \leq \mu \wedge_k \lambda \wedge_k \nu$. Now since S is regular, so for any $a \in S$ there exists $x \in S$ such that $a = axa = a(xax)a$. It follows that

$$\begin{aligned}
(\mu \circ_k \lambda \circ_k \nu)(a) &= (\mu \circ \lambda \circ \nu)(a) \wedge \frac{1-k}{2} \\
&= \left\{ \bigvee_{a=pqr} \mu(p) \wedge \lambda(q) \wedge \nu(r) \right\} \wedge \frac{1-k}{2} \\
&\geq \{\mu(a) \wedge \lambda(xax) \wedge \nu(a)\} \wedge \frac{1-k}{2}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{\mu(a) \wedge \left\{\lambda(xax) \wedge \frac{1-k}{2}\right\} \wedge \nu(a)\right\} \wedge \frac{1-k}{2} \\
&\geq \{\mu(a) \wedge \lambda(a) \wedge \nu(a)\} \wedge \frac{1-k}{2} = (\mu \wedge_k \lambda \wedge_k \nu)(a).
\end{aligned}
$$

Thus $\mu \circ_k \lambda \circ_k \nu \geq \mu \wedge_k \lambda \wedge_k \nu$. Hence $\mu \wedge_k \lambda \wedge_k \nu = \mu \circ_k \lambda \circ_k \nu$.

(2) $\Rightarrow$ (1) : Let R, M and L be the right ideal, lateral ideal and left ideal of S respectively. Then by Lemma 186, χ_R, χ_M and χ_L are $(\in, \in \vee q_k)$-fuzzy right ideal, $(\in, \in \vee q_k)$-fuzzy lateral ideal and $(\in, \in \vee q_k)$-fuzzy left ideal of S, respectively. Thus by hypothesis

$$
\begin{aligned}
\chi_R \wedge_k \chi_M \wedge_k \chi_L &= (\chi_R \circ_k \chi_M \circ_k \chi_L) \\
(\chi_{R \cap M \cap L})_k &= (\chi_{RML})_k .
\end{aligned}
$$

Hence $R \cap M \cap L = RML$. Therefore by Theorem 5, S is regular. ■

Theorem 229 *The following assertions are equivalent for a ternary semigroup S :*

(1) *S is regular;*

(2) *$\mu \wedge_k \lambda = \mu \circ_k \mathcal{S} \circ_k \lambda$ for every $(\in, \in \vee q_k)$-fuzzy right ideal μ and every $(\in, \in \vee q_k)$-fuzzy left ideal λ of S.*

Proof. The proof follows from Theorem 6 and Theorem 228. ■

Theorem 230 *For a ternary semigroup S, the following conditions are equivalent:*

(1) *S is regular;*

(2) *$\mu_k = \mu \circ_k \mathcal{S} \circ_k \mu \circ_k \mathcal{S} \circ_k \mu$ for every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal μ of S;*

(3) *$\mu_k = \mu \circ_k \mathcal{S} \circ_k \mu \circ_k \mathcal{S} \circ_k \mu$ for every $(\in, \in \vee q_k)$-fuzzy bi-ideal μ of S;*

(4) *$\mu_k = \mu \circ_k \mathcal{S} \circ_k \mu \circ_k \mathcal{S} \circ_k \mu$ for every $(\in, \in \vee q_k)$-fuzzy quasi-ideal μ of S.*

Proof. (1) $\Rightarrow$ (2) : Let μ be an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S and $a \in S$. Since S is regular, so there exists $x \in S$ such that $a = axa = axaxa$. It follows that

$$
\begin{aligned}
(\mu \circ_k \mathcal{S} \circ_k \mu \circ_k \mathcal{S} \circ_k \mu)(a) &= (\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)(a) \wedge \frac{1-k}{2} \\
&= \left\{\bigvee_{a=rst} (\mu \circ \mathcal{S} \circ \mu)(r) \wedge \mathcal{S}(s) \wedge \mu(t)\right\} \wedge \frac{1-k}{2} \\
&\geq \{(\mu \circ \mathcal{S} \circ \mu)(a) \wedge \mathcal{S}(x) \wedge \mu(a)\} \wedge \frac{1-k}{2} \\
&= \left\{\left\{\bigvee_{a=lmn} \mu(l) \wedge \mathcal{S}(m) \wedge \mu(n)\right\} \wedge \mathcal{S}(x) \wedge \mu(a)\right\} \wedge \frac{1-k}{2} \\
&\geq \{\mu(a) \wedge \mathcal{S}(x) \wedge \mu(a) \wedge \mathcal{S}(x) \wedge \mu(a)\} \wedge \frac{1-k}{2} \\
&= \mu(a) \wedge \frac{1-k}{2} = \mu_k(a).
\end{aligned}
$$

Thus $\mu \circ_k \mathcal{S} \circ_k \mu \circ_k \mathcal{S} \circ_k \mu \geq \mu_k$.

Since μ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S, it follows that

$$\begin{aligned}
(\mu \circ_k \mathcal{S} \circ_k \mu \circ_k \mathcal{S} \circ_k \mu)(a) &= (\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)(a) \wedge \frac{1-k}{2} \\
&= \left\{ \bigvee_{a=pvz} (\mu \circ \mathcal{S} \circ \mu)(p) \wedge \mathcal{S}(v) \wedge \mu(z) \right\} \wedge \frac{1-k}{2} \\
&= \left\{ \begin{array}{c} \bigvee_{a=pvz} \left\{ \bigvee_{p=xuy} \mu(x) \wedge \mathcal{S}(u) \wedge \mu(y) \right\} \\ \wedge \mathcal{S}(v) \wedge \mu(z) \end{array} \right\} \wedge \frac{1-k}{2} \\
&\leq \left\{ \bigvee_{a=xuyvz} \mu(x) \wedge \mathcal{S}(u) \wedge \mu(y) \wedge \mathcal{S}(v) \wedge \mu(z) \right\} \wedge \frac{1-k}{2} \\
&= \left\{ \bigvee_{a=xuyvz} \mu(x) \wedge \mu(y) \wedge \mu(z) \right\} \wedge \frac{1-k}{2} \\
&\leq \bigvee_{a=xuyvz} \mu(xuyvz) \wedge \frac{1-k}{2} = \mu(a) \wedge \frac{1-k}{2} = \mu_k(a).
\end{aligned}$$

Thus $\mu \circ_k \mathcal{S} \circ_k \mu \circ_k \mathcal{S} \circ_k \mu \leq \mu_k$. Hence $\mu_k = \mu \circ_k \mathcal{S} \circ_k \mu \circ_k \mathcal{S} \circ_k \mu$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) : Straightforward, because every $(\in, \in \vee q_k)$-fuzzy quasi-ideal is an $(\in, \in \vee q_k)$-fuzzy bi-ideal and every $(\in, \in \vee q_k)$-fuzzy bi-ideal is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S.

(4) $\Rightarrow$ (1) : Let Q be any quasi-ideal of S. Then by Lemma 200, χ_Q is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S. Thus by hypothesis

$$(\chi_Q)_k = (\chi_Q \circ_k \chi_S \circ_k \chi_Q \circ_k \chi_S \circ_k \chi_Q).$$

This implies that $(\chi_Q)_k = (\chi_{QSQSQ})_k$. Thus $Q = QSQSQ$. Hence by Theorem 7, S is regular. ■

Theorem 231 *The following conditions are equivalent for a ternary semigroup S :*

(1) *S is regular;*

(2) *$\mu \wedge_k \lambda \leq (\mu \circ_k \mathcal{S} \circ_k \lambda) \wedge (\lambda \circ_k \mathcal{S} \circ_k \mu)$ for every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals μ and λ of S;*

(3) *$\mu \wedge_k \lambda \leq (\mu \circ_k \mathcal{S} \circ_k \lambda) \wedge (\lambda \circ_k \mathcal{S} \circ_k \mu)$ for every $(\in, \in \vee q_k)$-fuzzy bi-ideals μ and λ of S;*

(4) *$\mu \wedge_k \lambda \leq (\mu \circ_k \mathcal{S} \circ_k \lambda) \wedge (\lambda \circ_k \mathcal{S} \circ_k \mu)$ for every $(\in, \in \vee q_k)$-fuzzy bi-ideal μ and every $(\in, \in \vee q_k)$-fuzzy quasi-ideal λ of S;*

(5) *$\mu \wedge_k \lambda \leq (\mu \circ_k \mathcal{S} \circ_k \lambda) \wedge (\lambda \circ_k \mathcal{S} \circ_k \mu)$ for every $(\in, \in \vee q_k)$-fuzzy bi-ideal μ and every $(\in, \in \vee q_k)$-fuzzy left ideal λ of S;*

(6) $\mu \wedge_k \lambda \leq (\mu \circ_k \mathcal{S} \circ_k \lambda) \wedge (\lambda \circ_k \mathcal{S} \circ_k \mu)$ *for every* $(\in, \in \vee q_k)$*-fuzzy quasi-ideal* μ *and every* $(\in, \in \vee q_k)$*-fuzzy left ideal* λ *of* S;

(7) $\mu \wedge_k \lambda \leq (\mu \circ_k \mathcal{S} \circ_k \lambda) \wedge (\lambda \circ_k \mathcal{S} \circ_k \mu)$ *for every* $(\in, \in \vee q_k)$*-fuzzy right ideal* μ *and every* $(\in, \in \vee q_k)$*-fuzzy left ideal* λ *of* S.

Proof. $(1) \Rightarrow (2)$: Let μ and λ be $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals of S and $a \in S$. Since S is regular, so there exists $x \in S$ such that $a = axa$. It follows that

$$\begin{aligned}
(\mu \circ_k \mathcal{S} \circ_k \lambda)(a) &= (\mu \circ \mathcal{S} \circ \lambda)(a) \wedge \frac{1-k}{2} \\
&= \left\{ \bigvee_{a=rst} \mu(r) \wedge \mathcal{S}(s) \wedge \lambda(t) \right\} \wedge \frac{1-k}{2} \\
&\geq \{\mu(a) \wedge \mathcal{S}(x) \wedge \lambda(a)\} \wedge \frac{1-k}{2} \\
&= (\mu \wedge \lambda)(a) \wedge \frac{1-k}{2} = (\mu \wedge_k \lambda)(a).
\end{aligned}$$

Thus $\mu \wedge_k \lambda \leq \mu \circ_k \mathcal{S} \circ_k \lambda$. On the other hand

$$\begin{aligned}
(\lambda \circ_k \mathcal{S} \circ_k \mu)(a) &= (\lambda \circ \mathcal{S} \circ \mu)(a) \wedge \frac{1-k}{2} \\
&= \left\{ \bigvee_{a=lmn} (\lambda(l) \wedge \mathcal{S}(m) \wedge \mu(n) \right\} \wedge \frac{1-k}{2} \\
&\geq \{\lambda(a) \wedge \mathcal{S}(x) \wedge \mu(a)\} \wedge \frac{1-k}{2} = (\mu \wedge_k \lambda)(a).
\end{aligned}$$

Thus $\mu \wedge_k \lambda \leq \lambda \circ_k \mathcal{S} \circ_k \mu$. Hence $\mu \wedge_k \lambda \leq (\mu \circ_k \mathcal{S} \circ_k \lambda) \wedge (\lambda \circ_k \mathcal{S} \circ_k \mu)$.

It is clear that $(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7)$.

$(7) \Rightarrow (1)$: Let $\mu \wedge_k \lambda \leq (\mu \circ_k \mathcal{S} \circ_k \lambda) \wedge (\lambda \circ_k \mathcal{S} \circ_k \mu)$ for every $(\in, \in \vee q_k)$-fuzzy right ideal μ and every $(\in, \in \vee q_k)$-fuzzy left ideal λ of S. Now, (7) implies $\mu \wedge_k \lambda \leq (\mu \circ_k \mathcal{S} \circ_k \lambda) \wedge (\lambda \circ_k \mathcal{S} \circ_k \mu) \leq (\mu \circ_k \mathcal{S} \circ_k \lambda)$ and by Lemma 227, $(\mu \circ_k \mathcal{S} \circ_k \lambda) \leq \mu \wedge_k \lambda$. Consequently, $\mu \wedge_k \lambda = (\mu \circ_k \mathcal{S} \circ_k \lambda)$. Therefore by Theorem 229, S is regular. ∎

Theorem 232 *The following assertions are equivalent for a ternary semigroup* S :

(1) S *is regular;*

(2) $\mu \wedge_k \lambda \leq (\mu \circ_k \mathcal{S} \circ_k \lambda)$ *for every* $(\in, \in \vee q_k)$*-fuzzy generalized bi-ideal* μ *and every* $(\in, \in \vee q_k)$*-fuzzy left ideal* λ *of* S;

(3) $\mu \wedge_k \lambda \leq (\mu \circ_k \mathcal{S} \circ_k \lambda)$ *for every* $(\in, \in \vee q_k)$*-fuzzy bi-ideal* μ *and every* $(\in, \in \vee q_k)$*-fuzzy left ideal* λ *of* S;

(4) $\mu \wedge_k \lambda \leq (\mu \circ_k \mathcal{S} \circ_k \lambda)$ *for every* $(\in, \in \vee q_k)$*-fuzzy quasi-ideal* μ *and every* $(\in, \in \vee q_k)$*-fuzzy left ideal* λ *of* S;

(5) $\mu \wedge_k \lambda \leq (\mu \circ_k \mathcal{S} \circ_k \lambda)$ *for every* $(\in, \in \vee q_k)$*-fuzzy right ideal* μ *and every* $(\in, \in \vee q_k)$*-fuzzy bi-ideal* λ *of* S;

(6) $\mu \wedge_k \lambda \leq (\mu \circ_k \mathcal{S} \circ_k \lambda)$ *for every* $(\in, \in \vee q_k)$*-fuzzy right ideal* μ *and every* $(\in, \in \vee q_k)$*-fuzzy quasi-ideal* λ *of* S.

Proof. $(1) \Rightarrow (2)$: Assume that μ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal and λ an $(\in, \in \vee q_k)$-fuzzy left ideal of S. Since S is regular, so for any $a \in S$ there exists $x \in S$ such that $a = axa$. It follows that

$$\begin{aligned} (\mu \circ_k \mathcal{S} \circ_k \lambda)(a) &= (\mu \circ \mathcal{S} \circ \lambda)(a) \wedge \frac{1-k}{2} \\ &= \left\{ \bigvee_{a=pqr} \mu(p) \wedge \mathcal{S}(q) \wedge \lambda(r) \right\} \wedge \frac{1-k}{2} \\ &\geq \{\mu(a) \wedge \mathcal{S}(x) \wedge \lambda(a)\} \wedge \frac{1-k}{2} \\ &= (\mu \wedge \lambda)(a) \wedge \frac{1-k}{2} = (\mu \wedge_k \lambda)(a). \end{aligned}$$

Hence $\mu \wedge_k \lambda \leq \mu \circ_k \mathcal{S} \circ_k \lambda$.

It is clear that $(2) \Rightarrow (3) \Rightarrow (4)$.

$(4) \Rightarrow (1)$: Suppose that $\mu \wedge_k \lambda \leq \mu \circ_k \mathcal{S} \circ_k \lambda$ for every $(\in, \in \vee q_k)$-fuzzy quasi-ideal μ and every $(\in, \in \vee q_k)$-fuzzy left ideal λ of S. Let ν be any $(\in, \in \vee q_k)$-fuzzy right ideal of S. Take $\mu = \nu$, then by (4), $\nu \wedge_k \lambda \leq \nu \circ_k \mathcal{S} \circ_k \lambda$ and by Lemma 227, $\nu \circ_k \mathcal{S} \circ_k \lambda \leq \nu \wedge_k \lambda$. Thus $\nu \wedge_k \lambda = \nu \circ_k \mathcal{S} \circ_k \lambda$. Hence by Theorem 229, S is regular.

$(1) \Rightarrow (5)$: Let μ be an $(\in, \in \vee q_k)$-fuzzy right ideal and λ an $(\in, \in \vee q_k)$-fuzzy bi-ideal of S. Since S is regular, so for any $a \in S$ there exists $x \in S$ such that $a = axa$. It follows that

$$\begin{aligned} (\mu \circ_k \mathcal{S} \circ_k \lambda)(a) &= (\mu \circ \mathcal{S} \circ \lambda)(a) \wedge \frac{1-k}{2} = \left\{ \bigvee_{a=pqr} \mu(p) \wedge \mathcal{S}(q) \wedge \lambda(r) \right\} \wedge \frac{1-k}{2} \\ &\geq \mu(a) \wedge \mathcal{S}(x) \wedge \lambda(a) \wedge \frac{1-k}{2} = (\mu(a) \wedge \lambda(a)) \wedge \frac{1-k}{2} \\ &= (\mu \wedge_k \lambda)(a). \end{aligned}$$

Thus $\mu \wedge_k \lambda \leq \mu \circ_k \mathcal{S} \circ_k \lambda$.

It is clear that $(5) \Rightarrow (6)$.

$(6) \Rightarrow (1)$: Suppose that $\mu \wedge_k \lambda \leq \mu \circ_k \mathcal{S} \circ_k \lambda$ for every $(\in, \in \vee q_k)$-fuzzy right ideal μ and every $(\in, \in \vee q_k)$-fuzzy quasi-ideal λ of S. Let ν be any $(\in, \in \vee q_k)$-fuzzy left ideal of S. Take $\nu = \lambda$, then by (6), $\mu \wedge_k \nu \leq \mu \circ_k \mathcal{S} \circ_k \nu$ and by Lemma 227, $\mu \circ_k \mathcal{S} \circ_k \nu \leq \mu \wedge_k \nu$. Hence $\mu \wedge_k \nu = \mu \circ_k \mathcal{S} \circ_k \nu$. Therefore by Theorem 229, S is regular. ■

3.5 Weakly regular ternary semigroups

In this section we characterize right weakly regular ternary semigroups in terms of $(\in, \in \vee q_k)$-fuzzy right ideals, $(\in, \in \vee q_k)$-fuzzy two sided ideals and $(\in, \in \vee q_k)$-fuzzy generalized bi-ideals.

Recall that a ternary semigroup S is said to be right (resp. left) weakly regular, if $x \in (xSS)^3$ $\left(\text{resp. } x \in (SSx)^3\right)$ for all $x \in S$.

Example 233 *Let X be a countably infinite set and S be the set of one-one maps $\alpha : X \to X$ with the property that $X - \alpha(X)$ is infinite. Then S is a ternary semigroup with respect to the composition of functions, that is, for $\alpha, \beta, \gamma \in S$, $[\alpha\beta\gamma] = \alpha \circ \beta \circ \gamma$.*

This ternary semigroup is right weakly regular but not regular.

Theorem 234 *For a ternary semigroup S, the following assertions are equivalent:*

(1) S is right weakly regular;

(2) $\mu \wedge_k \lambda = \mu \circ_k \lambda \circ_k \lambda$ for every $(\in, \in \vee q_k)$-fuzzy right ideal μ and every $(\in, \in \vee q_k)$-fuzzy two sided ideal λ of S.

Proof. $(1) \Rightarrow (2)$: Let μ be an $(\in, \in \vee q_k)$-fuzzy right ideal, λ an $(\in, \in \vee q_k)$-fuzzy two sided ideal of S and $a \in S$. Then

$$\begin{aligned}
(\mu \circ_k \lambda \circ_k \lambda)(a) &= (\mu \circ \lambda \circ \lambda)(a) \wedge \frac{1-k}{2} \\
&= \left\{ \bigvee_{a=pqr} \mu(p) \wedge \lambda(q) \wedge \lambda(r) \right\} \wedge \frac{1-k}{2} \\
&\leq \bigvee_{a=pqr} \{\mu(pqr) \wedge \lambda(q) \wedge \lambda(pqr)\} \wedge \frac{1-k}{2} \\
&\leq \bigvee_{a=pqr} \{\mu(pqr) \wedge \lambda(pqr)\} \wedge \frac{1-k}{2} \\
&\leq (\mu(a) \wedge \lambda(a)) \wedge \frac{1-k}{2} = (\mu \wedge_k \lambda)(a).
\end{aligned}$$

Thus $\mu \circ_k \lambda \circ_k \lambda \leq \mu \wedge_k \lambda$. Now we show $\mu \wedge_k \lambda \leq \mu \circ_k \lambda \circ_k \lambda$. Let $a \in S$. Then there exist $s_1, s_2, s_3, t_1, t_2, t_3 \in S$ such that $a = (as_1t_1)(as_2t_2)(as_3t_3)$. It follows that

$$\begin{aligned}
(\mu \wedge_k \lambda)(a) &= (\mu \wedge \lambda)(a) \wedge \frac{1-k}{2} \\
&= (\mu \wedge \lambda \wedge \lambda)(a) \wedge \frac{1-k}{2} \\
&\leq \mu(as_1t_1) \wedge \lambda(as_2t_2) \wedge \lambda(as_3t_3) \wedge \frac{1-k}{2} \\
&\leq \left\{ \bigvee_{a=xyz} \mu(x) \wedge \lambda(y) \wedge \lambda(z) \right\} \wedge \frac{1-k}{2} \\
&= (\mu \circ \lambda \circ \lambda)(a) \wedge \frac{1-k}{2} = (\mu \circ_k \lambda \circ_k \lambda)(a).
\end{aligned}$$

Thus $\mu \wedge_k \lambda \leq \mu \circ_k \lambda \circ_k \lambda$. Hence $\mu \wedge_k \lambda = \mu \circ_k \lambda \circ_k \lambda$.

$(2)\Rightarrow(1)$: Let R be a right ideal and I a two sided ideal of S. Then by Lemma 186, χ_R and χ_I are $(\in,\in\vee q_k)$-fuzzy right ideal and $(\in,\in\vee q_k)$-fuzzy two sided ideal of S, respectively. Thus by hypothesis

$$\begin{aligned}\chi_R\wedge_k\chi_I &= \chi_R\circ_k\chi_I\circ_k\chi_I\\ (\chi_{R\cap I})_k &= (\chi_{RII})_k\,.\end{aligned}$$

Hence $R\cap I=RII$. Therefore by Lemma 9, S is right weakly regular. ■

Theorem 235 *For a ternary semigroup S, the following assertions are equivalent:*

(1) *S is right weakly regular;*

(2) *Each $(\in,\in\vee q_k)$-fuzzy right ideal μ of S is idempotent.*

Proof. Straightforward. ■

Theorem 236 *For a ternary semigroup S, the following assertions are equivalent:*

(1) *S is right weakly regular;*

(2) *$\mu\wedge_k\lambda\wedge_k\nu=\mu\circ_k\lambda\circ_k\nu$ for every $(\in,\in\vee q_k)$-fuzzy bi-ideal μ, every $(\in,\in\vee q_k)$-fuzzy two sided ideal λ and every $(\in,\in\vee q_k)$-fuzzy right ideal ν of S;*

(3) *$\mu\wedge_k\lambda\wedge_k\nu=\mu\circ_k\lambda\circ_k\nu$ for every $(\in,\in\vee q_k)$-fuzzy quasi-ideal μ, every $(\in,\in\vee q_k)$-fuzzy two sided ideal λ and every $(\in,\in\vee q_k)$-fuzzy right ideal ν of S.*

Proof. $(1)\Rightarrow(2)$: Assume that μ is an $(\in,\in\vee q_k)$-fuzzy bi-ideal, λ an $(\in,\in\vee q_k)$-fuzzy two sided ideal and ν an $(\in,\in\vee q_k)$-fuzzy right ideal of S. Since S is right weakly regular, so for each $a\in S$ there exist $s_1,s_2,s_3,t_1,t_2,t_3\in S$ such that $a=(as_1t_1)(as_2t_2)(as_3t_3)=a(s_1t_1as_2t_2)(as_3t_3)$. It follows that

$$\begin{aligned}(\mu\wedge_k\lambda\wedge_k\nu)(a) &= (\mu\wedge\lambda\wedge\nu)(a)\wedge\frac{1-k}{2}\\ &= \{\mu(a)\wedge\lambda(a)\wedge\nu(a)\}\wedge\frac{1-k}{2}\\ &\le \{\mu(a)\wedge\lambda(s_1t_1as_2t_2)\wedge\nu(as_3t_3)\}\wedge\frac{1-k}{2}\\ &= \left\{\bigvee_{a=pqr}(\mu(p)\wedge\lambda(q)\wedge\nu(r))\right\}\wedge\frac{1-k}{2}\\ &= (\mu\circ\lambda\circ\nu)(a)\wedge\frac{1-k}{2}=(\mu\circ_k\lambda\circ_k\nu)(a)\,.\end{aligned}$$

Thus $\mu\wedge_k\lambda\wedge_k\nu\le\mu\circ_k\lambda\circ_k\nu$.

$(2)\Rightarrow(3)$: Straightforward, because every $(\in,\in\vee q_k)$-fuzzy quasi-ideal is an $(\in,\in\vee q_k)$-fuzzy bi-ideal of S.

$(3)\Rightarrow(1)$: Let μ be an $(\in,\in\vee q_k)$-fuzzy right ideal and λ an $(\in,\in\vee q_k)$-fuzzy two sided ideal of S. Take $\nu=\lambda$. Since every $(\in,\in\vee q_k)$-fuzzy right ideal is also an

$(\in, \in \vee q_k)$-fuzzy quasi-ideal. Thus by hypothesis $\mu \wedge_k \lambda \wedge_k \lambda \leq \mu \circ_k \lambda \circ_k \lambda$. This implies that $\mu \wedge_k \lambda \leq \mu \circ_k \lambda \circ_k \lambda$ and $\mu \circ_k \lambda \circ_k \lambda \leq \mu \wedge_k \lambda$ is straightforward. Hence $\mu \wedge_k \lambda = \mu \circ_k \lambda \circ_k \lambda$. Therefore by Theorem 234, S is right weakly regular. ■

Theorem 237 *For a ternary semigroup S, the following assertions are equivalent:*

(1) *S is right weakly regular;*

(2) $\mu \wedge_k \lambda \leq \mu \circ_k \lambda \circ_k \lambda$ *for every $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal μ and every $(\in, \in \vee q_k)$-fuzzy two sided ideal λ of S;*

(3) $\mu \wedge_k \lambda \leq \mu \circ_k \lambda \circ_k \lambda$ *for every $(\in, \in \vee q_k)$-fuzzy bi-ideal μ and every $(\in, \in \vee q_k)$-fuzzy two sided ideal λ of S;*

(4) $\mu \wedge_k \lambda \leq \mu \circ_k \lambda \circ_k \lambda$ *for every $(\in, \in \vee q_k)$-fuzzy quasi-ideal μ and every $(\in, \in \vee q_k)$-fuzzy two sided ideal λ of S.*

Proof. (1) $\Rightarrow$ (2) : Assume that μ is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal and λ an $(\in, \in \vee q_k)$-fuzzy two sided ideal of S. Since S is right weakly regular, so for each $a \in S$, there exist $s_1, s_2, s_3, t_1, t_2, t_3 \in S$ such that $a = (as_1t_1)(as_2t_2)(as_3t_3) = a(s_1t_1as_2t_2)(as_3t_3)$. It follows that

$$\begin{aligned}
(\mu \wedge_k \lambda)(a) &= (\mu \wedge \lambda)(a) \wedge \frac{1-k}{2} = (\mu(a) \wedge \lambda(a) \wedge \lambda(a)) \wedge \frac{1-k}{2} \\
&\leq (\mu(a) \wedge \lambda(s_1t_1as_2t_2) \wedge \lambda(as_3t_3)) \wedge \frac{1-k}{2} \\
&= \left\{ \bigvee_{a=lmn} (\mu(l) \wedge \lambda(m) \wedge \lambda(n)) \right\} \wedge \frac{1-k}{2} \\
&= (\mu \circ \lambda \circ \lambda)(a) \wedge \frac{1-k}{2} = (\mu \circ_k \lambda \circ_k \lambda)(a).
\end{aligned}$$

Thus $\mu \wedge_k \lambda \leq \mu \circ_k \lambda \circ_k \lambda$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) : Straightforward because every $(\in, \in \vee q_k)$-fuzzy quasi-ideal is an $(\in, \in \vee q_k)$-fuzzy bi-ideal and every $(\in, \in \vee q_k)$-fuzzy bi-ideal is an $(\in, \in \vee q_k)$-fuzzy generalized bi-ideal of S.

(4) $\Rightarrow$ (1) : Let μ be an $(\in, \in \vee q_k)$-fuzzy right ideal and λ an $(\in, \in \vee q_k)$-fuzzy two sided ideal of S. Since every $(\in, \in \vee q_k)$-fuzzy right ideal is an $(\in, \in \vee q_k)$-fuzzy quasi-ideal of S. Thus by hypothesis $\mu \wedge_k \lambda \leq \mu \circ_k \lambda \circ_k \lambda$ and $\mu \circ_k \lambda \circ_k \lambda \leq \mu \wedge_k \lambda$ is straightforward. Hence $\mu \wedge_k \lambda = \mu \circ_k \lambda \circ_k \lambda$. Therefore by Theorem 234, S is right weakly regular. ■

Chapter 4

$(\overline{\alpha}, \overline{\beta})$-fuzzy ideals in ternary semigroups

In this chapter we study $(\overline{\alpha}, \overline{\beta})$-fuzzy ternary subsemigroup, $(\overline{\alpha}, \overline{\beta})$-fuzzy left (right and lateral) ideals in ternary semigroups. Special attention is paid to $(\overline{\in}, \overline{\in} \vee \overline{q})$-type fuzzy ternary subsemigroups and ideals. Some classes of ternary semigroups are characterized in terms of $(\overline{\alpha}, \overline{\beta})$-fuzzy left (right and lateral) ideals, $(\overline{\alpha}, \overline{\beta})$-fuzzy quasi-ideals and $(\overline{\alpha}, \overline{\beta})$-fuzzy bi- (generalized bi-) ideals. Throughout this chapter S will denote a ternary semigroup and $\overline{\alpha}, \overline{\beta}$ are any two of $\overline{\in}, \overline{q}, \overline{\in} \vee \overline{q}, \overline{\in} \wedge \overline{q}$ unless otherwise mentioned.

4.1 $(\overline{\alpha}, \overline{\beta})$-fuzzy ternary subsemigroups

We start this section with the following definition.

Definition 238 *A fuzzy set μ in S is said to be an $(\overline{\alpha}, \overline{\beta})$-fuzzy ternary subsemigroup of S, where $\overline{\alpha} \neq \overline{\in} \wedge \overline{q}$, if it satisfies the following condition:*

$$(xyz)_{\min\{t,r,s\}} \overline{\alpha}\mu \text{ implies } x_t \overline{\beta}\mu, \; y_r \overline{\beta}\mu \text{ or } z_s \overline{\beta}\mu \tag{4.1}$$

for all $x, y, z \in S$ and $t, r, s \in (0, 1]$.

Let μ be a fuzzy set in S such that $\mu(x) \geq 0.5$ for all $x \in S$. Let $x \in S$ and $t \in (0, 1]$ be such that $x_t \overline{\in} \wedge \overline{q}\mu$. Then $\mu(x) < t$ and $\mu(x) + t \leq 1$. It follows that $2\mu(x) = \mu(x) + \mu(x) < \mu(x) + t \leq 1$. This implies that $\mu(x) < 0.5$. This means that $\{x_t : x_t \overline{\in} \wedge \overline{q}\mu\} = \emptyset$. Therefore, the case $\overline{\alpha} = \overline{\in} \wedge \overline{q}$ in the above definition is omitted.

Example 239 *Consider the ternary semigroup $S=\{-i,0,i\}$ of Example 16. Define a fuzzy set μ in S as follows:*

$$\mu : S \rightarrow [0,1], x \rightarrow \begin{cases} 0.6 & \text{if } x=0, \\ 0.3 & \text{if } x=i, \\ 0.4 & \text{if } x=-i. \end{cases}$$

Then simple calculations show that μ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ternary subsemigroup of S.

Theorem 240 *A fuzzy set μ in S is a fuzzy ternary subsemigroup of S if and only if μ is an $(\overline{\in},\overline{\in})$-fuzzy ternary subsemigroup of S.*

Proof. Let μ be a fuzzy ternary subsemigroup of S. Let $x,y,z\in S$ and $t,r,s\in(0,1]$ be such that $(xyz)_{\min\{t,r,s\}}\overline{\in}\mu$. Then $\mu(xyz)<\min\{t,r,s\}$. Since μ is a fuzzy ternary subsemigroup of S, it follows that

$$\min\{t,r,s\}>\mu(xyz)\geq\min\{\mu(x),\mu(y),\mu(z)\}.$$

This implies that $t>\mu(x)$, $r>\mu(y)$ or $s>\mu(z)$, that is, $x_t\overline{\in}\mu$, $y_r\overline{\in}\mu$ or $z_s\overline{\in}\mu$. Hence μ is an $(\overline{\in},\overline{\in})$-fuzzy ternary subsemigroup of S.

Conversely, assume that μ is an $(\overline{\in},\overline{\in})$-fuzzy ternary subsemigroup of S. If there exist x, y, $z\in S$ such that

$$\mu(xyz)<\min\{\mu(x),\mu(y),\mu(z)\},$$

then we can choose $t\in(0,1]$ such that

$$\mu(xyz)<t\leq\min\{\mu(x),\mu(y),\mu(z)\}.$$

Thus $(xyz)_t\overline{\in}\mu$ but $x_t\in\mu$, $y_t\in\mu$ and $z_t\in\mu$, which is a contradiction. Hence $\mu(xyz)\geq\min\{\mu(x),\mu(y),\mu(z)\}$. ■

Theorem 241 *Every $(\overline{\in},\overline{\in})$-fuzzy ternary subsemigroup is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ternary subsemigroup.*

Proof. Obvious. ■

Remark 242 *The converse of Theorem 241 may not be true. In fact, the $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ternary subsemigroup μ in Example 239 is not an $(\overline{\in},\overline{\in})$-fuzzy ternary subsemigroup.*

Recall that for a fuzzy set μ in S, we denote $S_0:=\{x\in S:\mu(x)>0\}$.

Theorem 243 *If μ is one of the following:*

(i) an $(\overline{\in},\overline{\in})$-fuzzy ternary subsemigroup of S;

(ii) an $(\overline{\in},\overline{q})$-fuzzy ternary subsemigroup of S;

(iii) a $(\overline{q},\overline{\in})$-fuzzy ternary subsemigroup of S;

(iv) a $(\overline{q},\overline{q})$-fuzzy ternary subsemigroup of S;

(v) an $(\overline{\in},\overline{\in\wedge q})$-fuzzy ternary subsemigroup of S;

(vi) an $(\overline{\in},\overline{\in\vee q})$-fuzzy ternary subsemigroup of S;

(vii) an $(\overline{\in\vee q},\ \overline{q})$-fuzzy ternary subsemigroup of S;

(viii) an $(\overline{\in\vee q},\overline{\in})$-fuzzy ternary subsemigroup of S;

(ix) an $(\overline{\in\vee q},\overline{\in\wedge q})$-fuzzy ternary subsemigroup of S;

(x) a $(\overline{q},\overline{\in\wedge q})$-fuzzy ternary subsemigroup of S;

(xi) a $(\overline{q},\overline{\in\vee q})$-fuzzy ternary subsemigroup of S;

then the set S_0 is a ternary subsemigroup of S.

Proof. (i) Straightforward.

(ii) Let $x,\ y,\ z\in S_0$. Then $\mu(x)>0$, $\mu(y)>0$, $\mu(z)>0$. If $\mu(xyz)=0$, then $(xyz)_{\min\{1,1,1\}}\overline{\in}\mu$ but $x_1q\mu$, $y_1q\mu$ and $z_1q\mu$, which is a contradiction. Thus $\mu(xyz)>0$, and so $xyz\in S_0$. Therefore S_0 is a ternary subsemigroup of S.

(iii) Let $x,\ y,\ z\in S_0$. Then $\mu(x)>0$, $\mu(y)>0$, $\mu(z)>0$. If $\mu(xyz)=0$, then $(xyz)_{\min\{\mu(x),\mu(y),\mu(z)\}}\overline{q}\mu$ but $x_{\mu(x)}\in\mu$, $y_{\mu(y)}\in\mu$ and $z_{\mu(z)}\in\mu$, which is a contradiction. It follows that $\mu(xyz)>0$ so that $xyz\in S_0$. Therefore S_0 is a ternary subsemigroup of S. This completes the proof.

(iv) Let $x,\ y,\ z\in S_0$. Then $\mu(x)>0$, $\mu(y)>0$, $\mu(z)>0$. Suppose that $xyz\notin S_0$. Then $\mu(xyz)=0$. Note that $(xyz)_{\min\{1,1,1\}}\overline{q}\mu$ but $x_1q\mu$, $y_1q\mu$ and $z_1q\mu$, because $\mu(x)+1>1$, $\mu(y)+1>1$ and $\mu(z)+1>1$. This is a contradiction, and thus $\mu(xyz)>0$, which shows that $xyz\in S_0$. Consequently S_0 is a ternary subsemigroup of S.

Similarly we can prove the remaining parts. ■

Theorem 244 *Every $(\overline{\in\vee q},\overline{\in\vee q})$-fuzzy ternary subsemigroup is an $(\overline{\in},\overline{\in\vee q})$-fuzzy ternary subsemigroup.*

Proof. Straightforward. ■

The following is a characterization of $(\overline{\in},\overline{\in\vee q})$-fuzzy ternary subsemigroup.

Theorem 245 *A fuzzy set μ in S is an $(\overline{\in},\overline{\in\vee q})$-fuzzy ternary subsemigroup of S if and only if*

$$\max\{\mu(xyz),\ 0.5\}\geq\min\{\mu(x),\ \mu(y),\ \mu(z)\}\tag{4.2}$$

for all $x,y,z\in S$.

Proof. Suppose μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ternary subsemigroup of S. Let $x, y, z \in S$ be such that $\max\{\mu(xyz), 0.5\} < \min\{\mu(x), \mu(y), \mu(z)\}$. Choose $t \in (0.5, 1]$ such that

$$\max\{\mu(xyz), 0.5\} < t = \min\{\mu(x), \mu(y), \mu(z)\}.$$

Then $(xyz)_t \overline{\in} \mu$, but $x_t \in \wedge q\mu$, $y_t \in \wedge q\mu$ and $z_t \in \wedge q\mu$, which is a contradiction. Hence $\max\{\mu(xyz), 0.5\} \geq \min\{\mu(x), \mu(y), \mu(z)\}$.

Conversely, assume that $x, y, z \in S$, and $t, r, s \in (0, 1]$ are such that $(xyz)_{\min\{t,r,s\}} \overline{\in} \mu$. Then $\mu(xyz) < \min\{t, r, s\}$. If $\max\{\mu(xyz), 0.5\} = \mu(xyz)$, then

$$\min\{\mu(x), \mu(y), \mu(z)\} \leq \mu(xyz) < \min\{t, r, s\}.$$

This implies that $\min\{\mu(x), \mu(y), \mu(z)\} < \min\{t, r, s\}$ and consequently, $\mu(x) < t$, $\mu(y) < r$ or $\mu(z) < s$. It follows that $x_t \overline{\in} \mu$, $y_r \overline{\in} \mu$ and $z_s \overline{\in} \mu$. Thus $x_t \overline{\in} \vee \overline{q}\mu$, $y_r \overline{\in} \vee \overline{q}\mu$ or $z_s \overline{\in} \vee \overline{q}\mu$. If $\max\{\mu(xyz), 0.5\} = 0.5$, then $\min\{\mu(x), \mu(y), \mu(z)\} \leq 0.5$. Suppose $x_t \in \mu$, $y_r \in \mu$, and $z_s \in \mu$. Then $t \leq \mu(x) \leq 0.5$, $r \leq \mu(y) \leq 0.5$ or $s \leq \mu(z) \leq 0.5$. It follows that $x_t \overline{q}\mu$, $y_r \overline{q}\mu$ or $z_s \overline{q}\mu$. Hence $x_t \overline{\in} \vee \overline{q}\mu$, $y_r \overline{\in} \vee \overline{q}\mu$ or $z_s \overline{\in} \vee \overline{q}\mu$. ■

Corollary 246 *A nonempty subset A of S is a ternary subsemigroup of S if and only if the characteristic function χ_A of A is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ternary subsemigroup of S.*

Theorem 247 *A fuzzy set μ in S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ternary subsemigroup of S if and only if $U(\mu; t)$ is a ternary subsemigroup of S for all $t \in (0.5, 1]$, whenever it is nonempty.*

Proof. Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ternary subsemigroup of S and let $x, y, z \in U(\mu; t)$ for some $t \in (0.5, 1]$. Then $\mu(x) \geq t$, $\mu(y) \geq t$ and $\mu(z) \geq t$. It follows that

$$0.5 < t \leq \min\{\mu(x), \mu(y), \mu(z)\} \leq \max\{\mu(xyz), 0.5\}.$$

Thus $\mu(xyz) \geq t$, and so $xyz \in U(\mu; t)$. Hence $U(\mu; t)$ is a ternary subsemigroup of S.

Conversely, assume that $U(\mu; t) (\neq \emptyset)$ is a ternary subsemigroup of S for all $t \in (0.5, 1]$. Suppose there exist $x, y, z \in S$ such that

$$\max\{\mu(xyz), 0.5\} < \min\{\mu(x), \mu(y), \mu(z)\} = t.$$

Then $t \in (0.5, 1]$, $x, y, z \in U(\mu; t)$ but $xyz \notin U(\mu; t)$. This is a contradiction. Hence $\max\{\mu(xyz), 0.5\} \geq \min\{\mu(x), \mu(y), \mu(z)\}$. Therefore μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy ternary subsemigroup of S. ■

Theorem 248 *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\overline{\in},\overline{\in} \vee \overline{q})$-fuzzy ternary subsemigroups of S. Then $\mu := \bigcap_{i\in\Lambda} \mu_i$ is an $(\overline{\in},\overline{\in} \vee \overline{q})$-fuzzy ternary subsemigroup of S.*

Proof. Straightforward. ■

Theorem 249 *If μ is an $(\overline{\in},\overline{\in}\vee \overline{q})$-fuzzy ternary subsemigroup of S, then the nonempty t-q-set S_q^t is a ternary subsemigroup of S for all $t \in (0, 0.5]$.*

Proof. Let $t \in (0, 0.5]$ and $x, y, z \in S_q^t$. Then $x_t q\mu, y_t q\mu$ and $z_t q\mu$, that is, $\mu(x) + t > 1, \mu(y) + t > 1$ and $\mu(z) + t > 1$. It follows from Theorem 245 that

$$\begin{aligned} \max\{\mu(xyz),\ 0.5\} + t &\geq \min\{\mu(x), \mu(y), \mu(z)\} + t \\ &= \min\{\mu(x) + t, \mu(y) + t, \mu(z) + t\} \\ &> 1. \end{aligned}$$

So $(xyz)_t\, q\mu$. Hence $xyz \in S_q^t$, and therefore S_q^t is a ternary subsemigroup of S. ■

4.2 $(\overline{\alpha},\overline{\beta})$-fuzzy ideals

Definition 250 *A fuzzy set μ in S is said to be an $(\overline{\alpha},\overline{\beta})$-fuzzy left (resp. right, lateral) ideal of S, where $\overline{\alpha} \neq \overline{\in} \wedge \overline{q}$, if it satisfies:*

$$(xyz)_t\, \overline{\alpha}\mu \text{ implies } z_t\overline{\beta}\mu \ \left(\text{resp. } x_t\overline{\beta}\mu,\ y_t\overline{\beta}\mu\right) \tag{4.3}$$

for all $x, y, z \in S$ and $t \in (0, 1]$.

Moreover a fuzzy set μ in S is said to be an $(\overline{\alpha},\overline{\beta})$-fuzzy two sided ideal of S if it is an $(\overline{\alpha},\overline{\beta})$-fuzzy left ideal and an $(\overline{\alpha},\overline{\beta})$-fuzzy right ideal of S. μ is called an $(\overline{\alpha},\overline{\beta})$-fuzzy ideal of S if it is an $(\overline{\alpha},\overline{\beta})$-fuzzy left ideal, $(\overline{\alpha},\overline{\beta})$-fuzzy right ideal and $(\overline{\alpha},\overline{\beta})$-fuzzy lateral ideal of S.

Example 251 *Consider the ternary semigroup S of Example 53. Define fuzzy sets μ_1 and μ_2 in S as follows:*

$$\mu_1 : S \to [0,1], \quad x \mapsto \begin{cases} 0.7 & \text{if } x = a, \\ 0.3 & \text{if } x = b, \\ 0.2 & \text{if } x = c, \\ 0.7 & \text{if } x = d, \\ 0.7 & \text{if } x = e. \end{cases}$$

$$\mu_2 : S \to [0,1], \quad x \mapsto \begin{cases} 0.7 & \text{if } x = a, \\ 0.5 & \text{if } x = b, \\ 0.6 & \text{if } x = c, \\ 0.9 & \text{if } x = d, \\ 0.3 & \text{if } x = e. \end{cases}$$

By routine calculations, we know that μ_1 and μ_2 are $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideals of S. Moreover, μ_1 is neither an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal, nor an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy lateral ideal of S, since $(ecc)_{0.5} \overline{\in} \mu_1$, but $e_{0.5} \in \wedge q \mu_1$ and $(cec)_{0.62} \overline{\in} \mu_1$, but $e_{0.62} \in \wedge q \mu_1$.

Example 252 *Consider the ternary semigroup S as given in Example 54. Define fuzzy sets λ_1 and λ_2 in S as follows:*

$$\lambda_1 : S \to [0,1], \quad x \mapsto \begin{cases} 0.9 & \text{if } x = 0, \\ 0.8 & \text{if } x = a, \\ 0.7 & \text{if } x = b, \\ 0.6 & \text{if } x = c, \\ 0.2 & \text{if } x = 1. \end{cases}$$

$$\lambda_2 : S \to [0,1], \quad x \mapsto \begin{cases} 0.9 & \text{if } x = 0, \\ 0.4 & \text{if } x = a, \\ 0.8 & \text{if } x = b, \\ 0.7 & \text{if } x = c, \\ 0.3 & \text{if } x = 1. \end{cases}$$

By routine calculations, we know that λ_1 and λ_2 are $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideals of S. Moreover, λ_2 is neither an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal, nor an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy lateral ideal of S, since $(1ac)_{0.35} \overline{\in} \lambda_2$, but $c_{0.35} \in \wedge q \lambda_2$ and $(ac1)_{0.62} \overline{\in} \mu_1$, but $c_{0.62} \in \wedge q \lambda_2$.

Theorem 253 *A fuzzy set μ in S is a fuzzy left (resp. right, lateral) ideal of S if and only if it is an $(\overline{\in}, \overline{\in})$-fuzzy left (resp. right, lateral) ideal of S.*

Proof. The proof is similar to the proof of Theorem 240. ■

Theorem 254 *Let μ be a fuzzy set in S. If μ is one of the following:*

(i) an $(\overline{\in}, \overline{\in})$-fuzzy left (resp. right, lateral) ideal of S;
(ii) an $(\overline{\in}, \overline{q})$-fuzzy left (resp. right, lateral) ideal of S;
(iii) a $(\overline{q}, \overline{\in})$-fuzzy left (resp. right, lateral) ideal of S;
(iv) a $(\overline{q}, \overline{q})$-fuzzy left (resp. right, lateral) ideal of S;
(v) an $(\overline{\in}, \overline{\in} \wedge \overline{q})$-fuzzy left (resp. right, lateral) ideal of S;
(vi) an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp. right, lateral) ideal of S;

(vii) an $(\overline{\in}\vee\overline{q},\ \overline{q})$*-fuzzy left (resp. right, lateral) ideal of* S;

(viii) an $(\overline{\in}\vee\overline{q},\overline{\in})$*-fuzzy left (resp. right, lateral) ideal of* S;

(ix) an $(\overline{\in}\vee\overline{q},\overline{\in}\wedge\overline{q})$*-fuzzy left (resp. right, lateral) ideal of* S;

(x) a $(\overline{q},\overline{\in}\wedge\overline{q})$*-fuzzy left (resp. right, lateral) ideal of* S;

(xi) a $(\overline{q},\overline{\in}\vee\overline{q})$*-fuzzy left (resp. right, lateral) ideal of* S;

then the set S_0 *is a left (resp. right, lateral) ideal of* S.

Proof. The proof is similar to the proof of Theorem 243. ■

The following theorem is a characterization of left (right, lateral) ideal of S.

Theorem 255 *Let* μ *be a fuzzy set in* S. *Then* μ *is an* $(\overline{\in},\overline{\in}\vee\overline{q})$*-fuzzy left (resp. right, lateral) ideal of* S *if and only if*

$$\max\{\mu(xyz),\ 0.5\}\geq\mu(z) \tag{4.4}$$

$$(\textit{resp. } \max\{\mu(xyz),\ 0.5\}\geq\mu(y),\ \ \max\{\mu(xyz),\ 0.5\}\geq\mu(x))$$

for all $x,y,z\in S$.

Proof. The proof is similar to the proof of Theorem 245. ■

Theorem 256 *Let* A *be a nonempty subset of* S. *Then* A *is a left (resp. right, lateral) ideal of* S *if and only if the fuzzy set* μ *in* S *defined by:*

$$\mu(x)=\begin{cases}\leq 0.5 & \text{for all } S\setminus A,\\ 1 & \text{for all } x\in A,\end{cases}$$

is an $(\overline{\alpha},\overline{\in}\vee\overline{q})$*-fuzzy left (resp. right, lateral) ideal of* S, *where* $\overline{\alpha}\in\{\overline{\in},\overline{q},\overline{\in}\vee\overline{q}\}$.

Proof. Let A be a left ideal of S.

(a) In this part we show that μ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal of S. Let $x,y,z\in S$ and $t\in(0,1]$ be such that $(xyz)_t\overline{\in}\mu$. Then $\mu(xyz)<t$, so $xyz\notin A$. Thus $z\notin A$. If $t>0.5$, then $\mu(z)\leq 0.5<t$, implies, $\mu(z)<t$. Hence $z_t\overline{\in}\mu$. If $t\leq 0.5$, then $\mu(z)+t\leq 0.5+0.5=1$, so $z_t\overline{q}\mu$. Thus $z_t\overline{\in}\vee\overline{q}\mu$. Hence μ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal of S.

(b) In this part we show that μ is a $(\overline{q},\overline{\in}\vee\overline{q})$-fuzzy left ideal of S. Let $x,y,z\in S$ and $t\in(0,1]$ be such that $(xyz)_t\overline{q}\mu$. Then $\mu(xyz)+t\leq 1$, so $xyz\notin A$. Therefore $z\notin A$. If $t>0.5$, then $\mu(z)\leq 0.5<t$. Hence $z_t\overline{\in}\mu$. If $t\leq 0.5$, then $\mu(z)+t\leq 0.5+0.5=1$, so $z_t\overline{q}\mu$. Thus $z_t\overline{\in}\vee\overline{q}\mu$. Hence μ is a $(\overline{q},\overline{\in}\vee\overline{q})$-fuzzy left ideal of S.

(c) In this part we show that μ is an $(\overline{\in}\vee\overline{q},\overline{\in}\vee\overline{q})$-fuzzy left ideal of S. Let $x,y,z\in S$ and $t\in(0,1]$ be such that $(xyz)_t\overline{\in}\vee\overline{q}\mu$. Then $(xyz)_t\overline{\in}\mu$ or $(xyz)_t\overline{q}\mu$. The rest of the proof is a consequence of (a) and (b).

Conversely assume that μ is an $(\overline{\alpha},\overline{\in}\vee\overline{q})$-fuzzy left ideal of S. Suppose $\overline{\alpha}=\overline{\in}$. Let $z\in A$. Then $\mu(z)=1$. Since μ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal of S, so $\mu(xyz)\vee 0.5\geq \mu(z)=1$. This implies that $\mu(xyz)=1$. Hence $xyz\in A$. Similarly we can prove for $\overline{\alpha}=\overline{q}$ and $\overline{\alpha}=\overline{\in}\vee\overline{q}$. Therefore A is a left ideal of S. ■

Corollary 257 *A nonempty subset A of S is a left (resp. right, lateral) ideal of S if and only if the characteristic function χ_A of A is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left (resp. right, lateral) ideal of S.*

Using level set, we can characterize the $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left (right and lateral) ideal as follows:

Theorem 258 *Let μ be a fuzzy set in S. Then μ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left (resp. right, lateral) ideal of S if and only if $U(\mu;t)$ is a left (resp. right, lateral) ideal of S for all $t\in(0.5,1]$ whenever it is nonempty.*

Proof. The proof is similar to the proof of Theorem 247. ■

Theorem 259 *Let $\{\mu_i \mid i\in\Lambda\}$ be a family of $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left (resp. right, lateral) ideals of S. Then $\mu:=\bigcap_{i\in\Lambda}\mu_i$ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left (resp. right, lateral) ideal of S.*

Proof. Straightforward. ■

Theorem 260 *Let $\{\mu_i \mid i\in\Lambda\}$ be a family of $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left (resp. right, lateral) ideals of S. Then $\mu:=\bigcup_{i\in\Lambda}\mu_i$ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left (resp. right, lateral) ideal of S.*

Proof. Straightforward. ■

Theorem 261 *If μ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal of S, λ a fuzzy set in S and ν is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal of S, then $\mu\circ\lambda\circ\nu$ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy two sided ideal of S.*

Proof. Let $x,y,z\in S$. Then $(\mu\circ\lambda\circ\nu)(z)=\bigvee_{z=pqr}\{\mu(p)\wedge\lambda(q)\wedge\nu(r)\}$. If $z=pqr$, then $xyz=(xyp)qr$. Since μ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal of S, so by Theorem 255, $\mu(xyp)\vee 0.5\geq\mu(p)$. It follows that

$$\begin{aligned}
(\mu\circ\lambda\circ\nu)(z) &= \bigvee_{z=pqr}\{\mu(p)\wedge\lambda(q)\wedge\nu(r)\}\\
&\leq \bigvee_{z=(xyp)qr}\{\mu(xyp)\vee 0.5\wedge\lambda(q)\wedge\nu(r)\}\\
&\leq \bigvee_{xyz=abc}\{(\mu(a)\vee 0.5)\wedge\lambda(b)\wedge\nu(c)\}\\
&= \bigvee_{xyz=abc}\{\mu(a)\wedge\lambda(b)\wedge\nu(c)\}\vee 0.5\\
&= (\mu\wedge\lambda\wedge\nu)(xyz)\vee 0.5.
\end{aligned}$$

This implies that $(\mu \circ \lambda \circ \nu)(z) \leq (\mu \circ \lambda \circ \nu)(xyz) \vee 0.5$. It follows from Theorem 255 that $\mu \circ \lambda \circ \nu$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S. Similarly we can prove that $\mu \circ \lambda \circ \nu$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal of S. Hence $\mu \circ \lambda \circ \nu$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy two sided ideal of S. ■

4.3 Fuzzy quasi-ideals of type $(\overline{\in}, \overline{\in} \vee \overline{q})$

Definition 262 *A fuzzy set μ in S is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of S if it satisfies:*

$$x_t \overline{\in} \mu \text{ implies } x_t \overline{\in} \vee \overline{q} \left((\mu \circ \mathcal{S} \circ \mathcal{S}) \wedge (\mathcal{S} \circ \mu \circ \mathcal{S}) \wedge (\mathcal{S} \circ \mathcal{S} \circ \mu) \right), \tag{4.5}$$

$$x_t \overline{\in} \mu \text{ implies } x_t \overline{\in} \vee \overline{q} \left((\mu \circ \mathcal{S} \circ \mathcal{S}) \wedge (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S}) \wedge (\mathcal{S} \circ \mathcal{S} \circ \mu) \right), \tag{4.6}$$

for all $x \in S$, where $\mathcal{S}$ is the fuzzy set in S mapping every element of S on 1.

Example 263 *Consider the ternary semigroup S of Example 95. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.9 & \text{if } x = 0, \\ 0.3 & \text{if } x = a, \\ 0.7 & \text{if } x = b, \\ 0.3 & \text{if } x = c, \\ 0.3 & \text{if } x = d. \end{cases}$$

By routine calculations, we know that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of S.

Theorem 264 *For a fuzzy set μ in S, the conditions (4.5) and (4.6) are equivalent to the conditions:*

$$(i) \max\{\mu(x), 0.5\} \geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\},$$

$$(ii) \max\{\mu(x), 0.5\} \geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\},$$

respectively.

Proof. Assume that (4.5) holds. Suppose there exists $x \in S$ such that

$$\max\{\mu(x), 0.5\} < \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\}.$$

Then we can choose $t \in (0,1]$ such that

$$\max\{\mu(x), 0.5\} < t = \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\}.$$

Then $x_t\overline{\in}\mu$ but

$$x_t \in \wedge q\left(\left(\mu\circ\mathcal{S}\circ\mathcal{S}\right)\wedge\left(\mathcal{S}\circ\mu\circ\mathcal{S}\right)\wedge\left(\mathcal{S}\circ\mathcal{S}\circ\mu\right)\right),$$

which is a contradiction. Hence

$$\max\left\{\mu\left(x\right),\,0.5\right\}\geq\min\left\{\left(\mu\circ\mathcal{S}\circ\mathcal{S}\right)\left(x\right),\left(\mathcal{S}\circ\mu\circ\mathcal{S}\right)\left(x\right),\left(\mathcal{S}\circ\mathcal{S}\circ\mu\right)\left(x\right)\right\}.$$

Conversely suppose that (i) is valid. Let $x_t\overline{\in}\mu$ for $t\in(0,1]$. If $\max\left\{\mu\left(x\right),\,0.5\right\}=\mu\left(x\right)$, then

$$t>\mu\left(x\right)\geq\left(\left(\mu\circ\mathcal{S}\circ\mathcal{S}\right)\wedge\left(\mathcal{S}\circ\mu\circ\mathcal{S}\right)\wedge\left(\mathcal{S}\circ\mathcal{S}\circ\mu\right)\right)\left(x\right),$$

which implies that $x_t\overline{\in}\left(\left(\mu\circ\mathcal{S}\circ\mathcal{S}\right)\wedge\left(\mathcal{S}\circ\mu\circ\mathcal{S}\right)\wedge\left(\mathcal{S}\circ\mathcal{S}\circ\mu\right)\right)$.

If $\max\,\left\{\mu\left(x\right),\,0.5\right\}=0.5$, then

$$\left(\left(\mu\circ\mathcal{S}\circ\mathcal{S}\right)\wedge\left(\mathcal{S}\circ\mu\circ\mathcal{S}\right)\wedge\left(\mathcal{S}\circ\mathcal{S}\circ\mu\right)\right)\left(x\right)+t\leq0.5+0.5=1.$$

Hence $x_t\overline{q}\left(\left(\mu\circ\mathcal{S}\circ\mathcal{S}\right)\wedge\left(\mathcal{S}\circ\mu\circ\mathcal{S}\right)\wedge\left(\mathcal{S}\circ\mathcal{S}\circ\mu\right)\right)$ and so
$x_t\overline{\in}\vee\overline{q}\left(\left(\mu\circ\mathcal{S}\circ\mathcal{S}\right)\wedge\left(\mathcal{S}\circ\mu\circ\mathcal{S}\right)\wedge\left(\mathcal{S}\circ\mathcal{S}\circ\mu\right)\right)$.

Similarly we can show that $(4.6)\Leftrightarrow(ii)$. ■

Lemma 265 *A nonempty subset Q of S is a quasi-ideal of S if and only if the characteristic function χ_Q of Q is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal of S.*

Proof. Let Q be a quasi-ideal of S and χ_Q the characteristic function of Q. Let $x\in S$. If $x\notin Q$, then $x\notin SSQ$, $x\notin QSS$ or $x\notin SQS$. If $x\notin SSQ$, then $\left(\mathcal{S}\circ\mathcal{S}\circ\chi_Q\right)(x)=0$. It follows that

$$\min\left\{\left(\mathcal{S}\circ\mathcal{S}\circ\chi_Q\right)(x),\left(\mathcal{S}\circ\chi_Q\circ\mathcal{S}\right)(x),\left(\chi_Q\circ\mathcal{S}\circ\mathcal{S}\right)(x)\right\}=0\leq\chi_Q(x)\vee0.5$$

If $x\in Q$, then

$$\chi_Q(x)\vee0.5=1\geq\min\left\{\left(\mathcal{S}\circ\mathcal{S}\circ\chi_Q\right)(x),\left(\mathcal{S}\circ\chi_Q\circ\mathcal{S}\right)(x),\left(\chi_Q\circ\mathcal{S}\circ\mathcal{S}\right)(x)\right\}.$$

Similarly we can prove that

$$\chi_Q(x)\vee0.5\geq\min\left\{\left(\mathcal{S}\circ\mathcal{S}\circ\chi_Q\right)(x),\left(\mathcal{S}\circ\mathcal{S}\circ\chi_Q\circ\mathcal{S}\circ\mathcal{S}\right)(x),\left(\chi_Q\circ\mathcal{S}\circ\mathcal{S}\right)(x)\right\}.$$

Hence it follows by Theorem 264 that χ_Q is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal of S.

Conversely assume that χ_Q is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal of S. Let $a\in SSQ\cap SQS\cap QSS$. Then $a\in SSQ$, $a\in SQS$ and $a\in QSS$. This implies there exist $x,y,z\in Q$ and $s_1,s_2,s_3,t_1,t_2,t_3\in S$ such that $a=s_1t_1x$, $a=s_2yt_2$, $a=zs_3t_3$. It follows that

$$\begin{aligned}\left(\chi_Q\circ\mathcal{S}\circ\mathcal{S}\right)(a) &= \bigvee_{a=pqr}\left\{\chi_Q(p)\wedge\mathcal{S}(q)\wedge\mathcal{S}(r)\right\}\\ &\geq \chi_Q(z)\wedge\mathcal{S}(s_3)\wedge\mathcal{S}(t_3)=\chi_Q(z)=1.\end{aligned}$$

This implies that $(\chi_Q \circ \mathcal{S} \circ \mathcal{S})(a) = 1$. Similar arguments lead to $(\mathcal{S} \circ \mathcal{S} \circ \chi_Q)(a) = 1$ and $(\mathcal{S} \circ \chi_Q \circ \mathcal{S})(a) = 1$. Now since

$$\chi_Q(a) \vee 0.5 \geq \min\left\{(\mathcal{S} \circ \mathcal{S} \circ \chi_Q)(a), (\mathcal{S} \circ \chi_Q \circ \mathcal{S})(a), (\chi_Q \circ \mathcal{S} \circ \mathcal{S})(a)\right\} = 1.$$

This implies that $\chi_Q(a) \vee 0.5 = 1$, and so $\chi_Q(a) = 1$. This implies that $a \in Q$. Hence $SSQ \cap SQS \cap QSS \subseteq Q$. Next let $a \in SSQ \cap SSQSS \cap QSS$. Then $a \in SSQ$, $a \in SSQSS$ and $a \in QSS$. This implies there exist $s_1, s_2, s_3, s_4, t_1, t_2, t_3, t_4 \in S$ and $y \in Q$ such that $a = s_1 t_1 x$, $a = z s_3 t_3$ and $a = s_2 t_2 y s_4 t_4$. For $a = s_1 t_1 x$ and $a = z s_3 t_3$ discussed above. Now

$$\begin{aligned}
(\mathcal{S} \circ \mathcal{S} \circ \chi_Q \circ \mathcal{S} \circ \mathcal{S})(a) &= \bigvee_{a=pqr} \left\{(\mathcal{S} \circ \mathcal{S} \circ \chi_Q)(p) \wedge \mathcal{S}(q) \wedge \mathcal{S}(r)\right\} \\
& \bigvee_{a=pqr} \left\{\left\{\bigvee_{r=stu} \left(\mathcal{S}(s) \wedge \mathcal{S}(t) \wedge \chi_Q(u)\right)\right\}(p) \wedge \mathcal{S}(q) \wedge \mathcal{S}(r)\right\} \\
&\geq \left(\mathcal{S}(s_2) \wedge \mathcal{S}(t_2) \wedge \chi_Q(y) \wedge \mathcal{S}(s_4) \wedge \mathcal{S}(t_4)\right) = 1.
\end{aligned}$$

This implies that $(\mathcal{S} \circ \mathcal{S} \circ \chi_Q \circ \mathcal{S} \circ \mathcal{S})(a) = 1$. Since

$$\chi_Q(a) \vee 0.5 \geq \min\left\{(\mathcal{S} \circ \mathcal{S} \circ \chi_Q)(a), (\mathcal{S} \circ \mathcal{S} \circ \chi_Q \circ \mathcal{S} \circ \mathcal{S})(a), (\chi_Q \circ \mathcal{S} \circ \mathcal{S})(a)\right\} = 1.$$

Thus $\chi_Q(a) \vee 0.5 = 1$. This implies that $\chi_Q(a) = 1$, $a \in Q$. Hence $SSQ \cap SSQSS \cap QSS \subseteq Q$. Therefore Q is a quasi-ideal of S. ■

Theorem 266 *Every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right, lateral) ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of S.*

Proof. Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S and $a \in S$. It follows that

$$\begin{aligned}
(\mathcal{S} \circ \mathcal{S} \circ \mu)(a) &= \bigvee_{a=xyz} \{\mathcal{S}(x) \wedge \mathcal{S}(y) \wedge \mu(z)\} \\
&= \bigvee_{a=xyz} \mu(z) \leq \bigvee_{a=xyz} \mu(xyz) \vee 0.5 = \mu(a) \vee 0.5.
\end{aligned}$$

and so $(\mathcal{S} \circ \mathcal{S} \circ \mu)(a) \leq \mu(a) \vee 0.5$. Hence

$$\mu(a) \vee 0.5 \geq (\mathcal{S} \circ \mathcal{S} \circ \mu)(a) \geq \min\{(\mathcal{S} \circ \mathcal{S} \circ \mu)(a), (\mathcal{S} \circ \mu \circ \mathcal{S})(a), (\mu \circ \mathcal{S} \circ \mathcal{S})(a)\}.$$

It follows that

$$\mu(a) \vee 0.5 \geq \min\{(\mathcal{S} \circ \mathcal{S} \circ \mu)(a), (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(a), (\mu \circ \mathcal{S} \circ \mathcal{S})(a)\}.$$

Hence μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of S. ■

Remark 267 *The converse of Theorem 266 may not be true. In fact, the $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of S given in Example 263 is neither an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal, nor an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal, nor an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy lateral ideal of S.*

Theorem 268 *Let* $\{\mu_i \mid i \in \Lambda\}$ *be a family of* $(\overline{\in},\overline{\in} \vee \overline{q})$*-fuzzy quasi-ideals of* S*. Then* $\mu := \bigcap_{i\in\Lambda} \mu_i$ *is an* $(\overline{\in},\overline{\in} \vee \overline{q})$*-fuzzy quasi-ideal of* S*.*

Proof. Straightforward. ■

4.4 Fuzzy bi-ideals of type $(\overline{\in},\overline{\in} \vee \overline{q})$

Definition 269 *An* $(\overline{\alpha},\overline{\beta})$*-fuzzy ternary subsemigroup* μ *of* S *is called an* $(\overline{\alpha},\overline{\beta})$*-fuzzy bi-ideal of* S*, where* $\overline{\alpha} \neq \overline{\in} \wedge \overline{q}$*, if it satisfies the following condition:*

$$(xuyvz)_{\min\{r,s,t\}}\,\overline{\alpha}\mu \text{ implies } x_r\overline{\beta}\mu,\ y_s\overline{\beta}\mu \text{ or } z_t\overline{\beta}\mu \tag{4.7}$$

for all $u,v,x,y,z \in S$ *and* $r,s,t \in (0,1]$*.*

Example 270 *Consider the ternary semigroup* S *of Example 102. Define a fuzzy set* μ *in* S *as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.9 & \text{if } x = 0, \\ 0.8 & \text{if } x = 1, \\ 0.1 & \text{if } x = 2, \\ 0.7 & \text{if } x = 3, \\ 0.1 & \text{if } x = 4, \\ 0.5 & \text{if } x = 5. \end{cases}$$

By routine calculations, we know that μ *is an* $(\overline{\in},\overline{\in} \vee \overline{q})$*-fuzzy bi-ideal of* S*.*

Lemma 271 *A fuzzy set* μ *in* S *is a fuzzy bi-ideal of* S *if and only if* μ *is an* $(\overline{\in},\overline{\in})$*-fuzzy bi-ideal of* S*.*

Proof. The proof is similar to the proof of Theorem 240. ■

Definition 272 *A fuzzy set* μ *in* S *is called an* $(\overline{\alpha},\overline{\beta})$*-fuzzy generalized bi-ideal of* S *if it satisfies* (4.7)*.*

Theorem 273 *If* μ *is a nonzero* $(\overline{\alpha},\overline{\beta})$*-fuzzy generalized bi-ideal (bi-ideal) of* S*, then the set* S_0 *is a generalized bi-ideal (bi-ideal) of* S*.*

Proof. The proof is similar to the proof of Theorem 243. ■

Similarly we can prove that

Theorem 274 *Let* G *be a nonempty subset of* S*. Then* G *is a bi-ideal (resp. generalized bi-ideal) of* S *if and only if the fuzzy set* μ *in* S *defined by:*

$$\mu(x) = \begin{cases} \leq 0.5 & \textit{for all } S \setminus G, \\ 1 & \textit{for all } x \in G \end{cases}$$

is an $(\overline{\alpha},\overline{\in} \vee \overline{q})$*-fuzzy bi-ideal (resp. generalized bi-ideal) of* S*, where* $\overline{\alpha} \in \{\overline{\in},\overline{q},\overline{\in} \vee \overline{q}\}$*.*

Proof. The proof is similar to the proof of Theorem 256. ■

Corollary 275 *A nonempty subset A of S is a bi-ideal (resp. generalized bi-ideal) of S if and only if the characteristic function χ_A of A is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal (resp. generalized bi-ideal) of S.*

Lemma 276 *Every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal of S is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal of S.*

Proof. Let μ be an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal of S and $u,v,x,y,z\in S$. Then

$$\begin{aligned}
\mu(xyz)\vee 0.5 &\geq \{(\mathcal{S}\circ\mathcal{S}\circ\mu)(xyz)\wedge(\mathcal{S}\circ\mu\circ\mathcal{S})(xyz)\wedge(\mu\circ\mathcal{S}\circ\mathcal{S})(xyz)\} \\
&= \left\{\bigvee_{xyz=abc}\{\mathcal{S}(a)\wedge\mathcal{S}(b)\wedge\mu(c)\}\right\}\wedge\left\{\bigvee_{xyz=pqr}\mathcal{S}(p)\wedge\mu(q)\wedge\mathcal{S}(r)\right\} \\
&\quad\wedge\left\{\bigvee_{xyz=lmn}\mu(l)\wedge\mathcal{S}(m)\wedge\mathcal{S}(n)\right\} \\
&\geq (\mathcal{S}(x)\wedge\mathcal{S}(y)\wedge\mu(z))\wedge(\mathcal{S}(x)\wedge\mu(y)\wedge\mathcal{S}(z))\wedge(\mu(x)\wedge\mathcal{S}(y)\wedge\mathcal{S}(z)) \\
&= \mu(x)\wedge\mu(y)\wedge\mu(z).
\end{aligned}$$

Thus $\mu(xyz)\vee 0.5\geq\min\{\mu(x),\mu(y),\mu(z)\}$.

Also

$$\begin{aligned}
\mu(xuyvz)\vee 0.5 &\geq \left\{\begin{array}{c}(\mu\circ\mathcal{S}\circ\mathcal{S})(xuyvz)\wedge(\mathcal{S}\circ\mathcal{S}\circ\mu\circ\mathcal{S}\circ\mathcal{S})(xuyvz)\\ \wedge(\mathcal{S}\circ\mathcal{S}\circ\mu)(xuyvz)\end{array}\right\} \\
&= \left\{\bigvee_{xuyvz=abc}\{\mu(a)\wedge\mathcal{S}(b)\wedge\mathcal{S}(c)\}\right\} \\
&\quad\wedge\left\{\bigvee_{xuyvz=pqr}(\mathcal{S}\circ\mathcal{S}\circ\mu)(p)\wedge\mathcal{S}(q)\wedge\mathcal{S}(r)\right\} \\
&\quad\wedge\left\{\bigvee_{xuyvz=lmn}\{\mathcal{S}(l)\wedge\mathcal{S}(m)\wedge\mu(n)\}\right\} \\
&= \left\{\bigvee_{xuyvz=abc}\{\mu(a)\wedge\mathcal{S}(b)\wedge\mathcal{S}(c)\}\right\} \\
&\quad\wedge\left\{\bigvee_{xuyvz=pqr}\left\{\bigvee_{p=uvw}(\mathcal{S}(u)\wedge\mathcal{S}(v)\wedge\mu(w))\right\}\wedge\mathcal{S}(q)\wedge\mathcal{S}(r)\right\} \\
&\quad\wedge\left\{\bigvee_{xuyvz=lmn}\{\mathcal{S}(l)\wedge\mathcal{S}(m)\wedge\mu(n)\}\right\}
\end{aligned}$$

$$\begin{aligned}
&= \left\{ \bigvee_{xuyvz=abc} \{\mu(a) \wedge \mathcal{S}(b) \wedge \mathcal{S}(c)\} \right\} \\
&\quad \wedge \left\{ \bigvee_{xuyvz=(uvw)qr} (\mathcal{S}(u) \wedge \mathcal{S}(v) \wedge \mu(w)) \wedge \mathcal{S}(q) \wedge \mathcal{S}(r) \right\} \\
&\quad \wedge \left\{ \bigvee_{xuyvz=lmn} \{\mathcal{S}(l) \wedge \mathcal{S}(m) \wedge \mu(n)\} \right\} \\
&\geq \{\mu(x) \wedge \mathcal{S}(uyv) \wedge \mathcal{S}(z)\} \wedge \{\mathcal{S}(x) \wedge \mathcal{S}(u) \wedge \mu(y) \wedge \mathcal{S}(v) \wedge \mathcal{S}(z)\} \\
&\quad \wedge \{\mathcal{S}(xuy) \wedge \mathcal{S}(v) \wedge \mu(z)\} \\
&= \mu(x) \wedge \mu(y) \wedge \mu(z).
\end{aligned}$$

Hence $\mu(xuyvz) \vee 0.5 \geq \min\{\mu(x), \mu(y), \mu(z)\}$. Therefore μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S. ■

Lemma 277 *If μ, λ and ν are $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideals of S, then $\mu \circ \lambda \circ \nu$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S.*

Proof. The proof is similar to the proof of Theorem 261. ■

Corollary 278 *If μ, λ and ν are $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideals of S, then $\mu \circ \lambda \circ \nu$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S.*

Theorem 279 *A fuzzy set μ in S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal (resp. generalized bi-ideal) of S if and only if $U(\mu; t)$ is a bi-ideal (resp. generalized bi-ideal) of S for all $t \in (0.5, 1]$ whenever it is nonempty.*

Proof. The proof is similar to the proof of Theorem 258. ■

Theorem 280 *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideals of S. Then $\mu := \bigcap_{i \in \Lambda} \mu_i$ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S.*

Proof. Straightforward. ■

The following example shows that the union of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideals of S may not be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S.

Example 281 *Consider the ternary semigroup S of Example 102. Define fuzzy sets μ_1 and μ_2 in S as follows:*

$$\mu_1 : S \to [0,1], \quad x \mapsto \begin{cases} 0.9 & \text{if } x = 0, \\ 0.7 & \text{if } x = 1, \\ 0.2 & \text{if } x = 2, \\ 0.5 & \text{if } x = 3, \\ 0.2 & \text{if } x = 4, \\ 0.2 & \text{if } x = 5. \end{cases}$$

$$\mu_2 : S \to [0,1], \quad x \mapsto \begin{cases} 0.8 & \text{if } x = 0, \\ 0.8 & \text{if } x = 1, \\ 0.4 & \text{if } x = 2, \\ 0.6 & \text{if } x = 3, \\ 0.6 & \text{if } x = 4, \\ 0.4 & \text{if } x = 5. \end{cases}$$

By routine calculations, we know that μ_1 and μ_2 are $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideals of S. Now $\mu := \mu_1 \cup \mu_2$ is given by:

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.9 & \text{if } x = 0, \\ 0.8 & \text{if } x = 1, \\ 0.4 & \text{if } x = 2, \\ 0.6 & \text{if } x = 3, \\ 0.6 & \text{if } x = 4, \\ 0.4 & \text{if } x = 5. \end{cases}$$

But $\mu := \mu_1 \cup \mu_2$ is not an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal of S. If $t \in (0.5, 0.6]$, then $U(\mu_1 \cup \mu_2; t) = \{0,1,3,4\}$, which is not a bi-ideal of S. It follows from Theorem 279 that $\mu_1 \cup \mu_2$ is not an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal of S.

4.5 Upper parts of $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy ideals

Definition 282 *Let μ be a fuzzy set in S. Define the upper part μ^+ of μ as follows:*
$\mu^+(x) = \mu(x) \vee 0.5$ *for all $x \in S$.*

Lemma 283 *Let μ, λ and ν be fuzzy sets in S. Then the following hold:*
(1) $(\mu \wedge \lambda)^+ = (\mu^+ \wedge \lambda^+)$
(2) $(\mu \vee \lambda)^+ = (\mu^+ \vee \lambda^+)$
(3) $(\mu \circ \lambda \circ \nu)^+ \geq (\mu^+ \circ \lambda^+ \circ \nu^+)$
If every element $x \in S$ is expressible as $x = abc$ for some $a, b, c \in S$, then $(\mu \circ \lambda \circ \nu)^+ = (\mu^+ \circ \lambda^+ \circ \nu^+)$.

Proof. The proofs of (1) and (2) are straightforward.

(3) Let $a \in S$. If a is not expressible as $a = bcd$ for some $b, c, d \in S$, then $(\mu \circ \lambda \circ \nu) = 0$. It follows that

$$\begin{aligned} (\mu \circ \lambda \circ \nu)^+(a) &= (\mu \circ \lambda \circ \nu)(a) \vee 0.5 \\ &= 0 \vee 0.5 = 0.5, \end{aligned}$$

and $(\mu^+ \circ \lambda^+ \circ \nu^+) = 0$. So $(\mu \circ \lambda \circ \nu)^+ \geq (\mu^+ \circ \lambda^+ \circ \nu^+)$.

If a is expressible as $a = bcd$, then

$$\begin{aligned}
(\mu \circ \lambda \circ \nu)^{+}(a) &= (\mu \circ \lambda \circ \nu)(a) \vee 0.5 \\
&= \left\{ \bigvee_{a=xyz} \{\mu(x) \wedge \lambda(y) \wedge \nu(z)\} \right\} \vee 0.5 \\
&= \bigvee_{a=xyz} \{(\mu(x) \vee 0.5) \wedge (\lambda(y) \vee 0.5) \wedge (\nu(z) \vee 0.5)\} \\
&= \bigvee_{a=xyz} \{\mu^{+}(x) \wedge \lambda^{+}(y) \wedge \nu^{+}(z)\} \\
&= \left(\mu^{+} \circ \lambda^{+} \circ \nu^{+}\right)(a).
\end{aligned}$$

Hence $(\mu \circ \lambda \circ \nu)^{+} = \left(\mu^{+} \circ \lambda^{+} \circ \nu^{+}\right)$. ■

Definition 284 *Let A be a nonempty subset of S. Then χ_A^{+} is defined as follows:*

$$\chi_A^{+}(a) = \begin{cases} 1 & \text{if } a \in A \\ 0.5 & \text{if } a \notin A. \end{cases}$$

Lemma 285 *Let A, B and C be nonempty subsets of S. Then*
(1) $(\chi_A \wedge \chi_B)^{+} = \chi_{A \cap B}^{+}$
(2) $(\chi_A \vee \chi_B)^{+} = \chi_{A \cup B}^{+}$
(3) $(\chi_A \circ \chi_B \circ \chi_C)^{+} = \chi_{ABC}^{+}$
where χ_A is the characteristic function of A.

Proof. The proofs of (1) and (2) are straightforward.

(3) Suppose $a \in ABC$. Then $a = xyz$ for some $x \in A$, $y \in B$ and $z \in C$. It follows that

$$\begin{aligned}
(\chi_A \circ \chi_B \circ \chi_C)^{+}(a) &= (\chi_A \circ \chi_B \circ \chi_C)(a) \vee 0.5 \\
&= \left\{ \bigvee_{a=uvw} \{\chi_A(u) \wedge \chi_B(v) \wedge \chi_C(w)\} \right\} \vee 0.5 \\
&\geq \{\chi_A(x) \wedge \chi_B(y) \wedge \chi_C(z)\} \vee 0.5 = 1.
\end{aligned}$$

Hence $(\chi_A \circ \chi_B \circ \chi_C)^{+}(a) = 1$. Since $a \in ABC$, so $\chi_{ABC}^{+}(a) = 1$. In the case when $a \notin ABC$, we have $a \neq xyz$ for all $x \in A$, $y \in B$, $z \in C$. If $a = uvw$ for some $u, v, w \in S$, then

$$\begin{aligned}
(\chi_A \circ \chi_B \circ \chi_C)^{+}(a) &= (\chi_A \circ \chi_B \circ \chi_C)(a) \vee 0.5 \\
&= \left\{ \bigvee_{a=uvw} \{\chi_A(u) \wedge \chi_B(v) \wedge \chi_C(w)\} \right\} \vee 0.5 \\
&= 0 \vee 0.5 = 0.5 = \chi_{ABC}^{+}(a).
\end{aligned}$$

If $a \neq uvw$ for all $u, v, w \in S$, then $(\chi_A \circ \chi_B \circ \chi_C)^{+}(a) = 0.5 = \chi_{ABC}^{+}(a)$. Thus in any case $(\chi_A \circ \chi_B \circ \chi_C)^{+} = \chi_{ABC}^{+}$. ■

Proposition 286 *Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp. right, lateral, bi-ideal, generalized bi-ideal) ideal of S. Then μ^+ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp. right, lateral, bi-ideal, generalized bi-ideal) ideal of S.*

Proof. Assume that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S and $x, y, z \in S$. Then

$$\begin{aligned} \max\left\{\mu^+(xyz), 0.5\right\} &= \max\{\max(\mu(xyz), 0.5), 0.5\} \\ &= \max\{\mu(xyz), 0.5\} \geq \mu(z). \end{aligned}$$

Thus $\max\{\mu^+(xyz), 0.5\} \geq \mu(z)$. Also $\max\{\mu^+(xyz), 0.5\} \geq 0.5$. Hence

$$\max\left\{\mu^+(xyz), 0.5\right\} \geq \mu(z) \vee 0.5 = \mu^+(z).$$

Therefore μ^+ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S.

The remaining cases can be dealt similarly. ■

Lemma 287 *(i) A nonempty subset A of S is a left (resp. right, lateral) ideal of S if and only if the upper part of the characteristic function χ_A^+ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp. right, lateral) ideal of S.*

(ii) A nonempty subset Q of S is a quasi-ideal (resp. bi-ideal, generalized bi-ideal) of S if and only if the upper part χ_Q^+ of the characteristic function χ_Q is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal (resp. bi-ideal, generalized bi-ideal) of S.

Proof. (i) Suppose A is a left ideal of S. Then by Corollary 257 and Proposition 286, χ_A^+ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S.

Conversely, assume that χ_A^+ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S. Let $z \in A$, $x, y \in S$. Then $\chi_A(z) = 1$. Since χ_A^+ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S, so $\max\left\{\chi_A^+(xyz), 0.5\right\} \geq \chi_A^+(z) = 1$. Thus $\chi_A^+(xyz) = 1$ and hence $xyz \in A$. Therefore A is a left ideal of S.

Similarly we can prove (ii). ■

Lemma 288 *If μ, λ and ν are $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal, $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy lateral ideal and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S, respectively, then $(\mu \circ \lambda \circ \nu)^+ \leq (\mu \wedge \lambda \wedge \nu)^+$.*

Proof. Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal, λ an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy lateral ideal, ν an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S, respectively and $a \in S$. Then

$$\begin{aligned} (\mu \circ \lambda \circ \nu)^+(a) &= (\mu \circ \lambda \circ \nu)(a) \vee 0.5 \\ &= \left\{\bigvee_{a=xyz} \{\mu(x) \wedge \lambda(y) \wedge \nu(z)\}\right\} \vee 0.5 \\ &\leq \left\{\bigvee_{a=xyz} \{(\mu(xyz) \vee 0.5) \wedge (\lambda(xyz) \vee 0.5) \wedge (\nu(xyz) \vee 0.5)\}\right\} \vee 0.5 \\ &= \{\mu(a) \wedge \lambda(a) \wedge \nu(a)\} \vee 0.5 = (\mu \wedge \lambda \wedge \nu)^+(a). \end{aligned}$$

This implies that $(\mu \circ \lambda \circ \nu)^+ \leq (\mu \wedge \lambda \wedge \nu)^+$.

If a is not expressible as $a = xyz$, then

$$\begin{aligned}(\mu \circ \lambda \circ \nu)^+ (a) &= (\mu \circ \lambda \circ \nu)(a) \vee 0.5 \\ &= 0 \vee 0.5 = 0.5 \leq (\mu \wedge \lambda \wedge \nu)(a) \vee 0.5 = (\mu \wedge \lambda \wedge \nu)^+ (a).\end{aligned}$$

Hence $(\mu \circ \lambda \circ \nu)^+ \leq (\mu \wedge \lambda \wedge \nu)^+$. ■

Corollary 289 *If μ and λ are $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal of S, respectively, then $(\lambda \circ \mathcal{S} \circ \mu)^+ \leq (\mu \wedge \lambda)^+$.*

4.6 Regular ternary semigroups

In this section we characterize regular ternary semigroups in terms of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (right, lateral) ideals, $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideals, $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi- (generalized bi-) ideals.

Theorem 290 *The following assertions are equivalent for a ternary semigroup S:*

(1) *S is regular;*

(2) *$(\mu \wedge \lambda \wedge \nu)^+ = (\mu \circ \lambda \circ \nu)^+$ for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal μ, every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy lateral ideal λ and every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal ν of S.*

Proof. (1) $\Rightarrow$ (2) : Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal of S, λ an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy lateral ideal of S and ν an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S. Then by Lemma 288, we have

$$(\mu \circ \lambda \circ \nu)^+ \leq (\mu \wedge \lambda \wedge \nu)^+.$$

On the other hand, since S is regular, so for any $a \in S$ there exists $x \in S$ such that $a = axa = a(xax)a$. It follows that

$$\begin{aligned}(\mu \circ \lambda \circ \nu)^+ (a) &= (\mu \circ \lambda \circ \nu)(a) \vee 0.5 \\ &= \left\{ \bigvee_{a=pqr} \{\mu(p) \wedge \lambda(q) \wedge \nu(r)\} \right\} \vee 0.5 \\ &\geq \{\mu(a) \wedge (\lambda(xax) \vee 0.5) \wedge \nu(a)\} \vee 0.5 \\ &\geq \{\mu(a) \wedge \lambda(a) \wedge \nu(a)\} \vee 0.5 \\ &= (\mu \wedge \lambda \wedge \nu)(a) \vee 0.5 = (\mu \wedge \lambda \wedge \nu)^+ (a).\end{aligned}$$

Thus $(\mu \circ \lambda \circ \nu)^+ \geq (\mu \wedge \lambda \wedge \nu)^+$. Hence $(\mu \circ \lambda \circ \nu)^+ = (\mu \wedge \lambda \wedge \nu)^+$.

(2) $\Rightarrow$ (1) : Let L, M and R be the left ideal, lateral ideal and right ideal of S, respectively. Then by Lemma 257, χ_L, χ_M and χ_R, are $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal,

$(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy lateral ideal and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal of S, respectively. Thus by hypothesis

$$\begin{aligned} (\chi_R \wedge \chi_M \wedge \chi_L)^+ &= (\chi_R \circ \chi_M \circ \chi_L)^+ \\ \chi^+_{R \cap M \cap L} &= \chi^+_{RML}. \end{aligned}$$

Hence $R \cap M \cap L = RML$. Therefore by Theorem 5, S is regular. ∎

Theorem 291 *For a ternary semigroup S, the following assertions are equivalent:*

(1) *S is regular;*

(2) $(\mu \wedge \lambda)^+ = (\mu \circ \mathcal{S} \circ \lambda)^+$ *for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal λ and every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal μ of S.*

Proof. It is similar to the proof of Theorem 290. ∎

Theorem 292 *The following assertions are equivalent for a ternary semigroup S :*

(1) *S is regular;*

(2) $\mu^+ = (\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)^+$ *for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal μ of S;*

(3) $\mu^+ = (\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)^+$ *for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal μ of S;*

(4) $\mu^+ = (\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)^+$ *for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal μ of S.*

Proof. $(1) \Rightarrow (2)$: Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal of S and $a \in S$. Since S is regular so there exists $x \in S$ such that $a = axa = axaxa$. It follows that

$$\begin{aligned} (\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)^+ (a) &= (\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu) \vee 0.5 \\ &= \left\{ \bigvee_{a=pqr} (\mu \circ \mathcal{S} \circ \mu)(p) \wedge \mathcal{S}(q) \wedge \mu(r) \right\} \vee 0.5 \\ &\geq \{(\mu \circ \mathcal{S} \circ \mu)(a) \wedge \mathcal{S}(x) \wedge \mu(a)\} \vee 0.5 \\ &= \left[\left\{ \bigvee_{a=uvw} \mu(u) \wedge \mathcal{S}(v) \wedge \mu(w) \right\} \wedge \mu(a) \right] \vee 0.5 \\ &\geq \{\mu(a) \wedge \mathcal{S}(x) \wedge \mu(a) \wedge \mu(a)\} \vee 0.5 \\ &= \mu(a) \vee 0.5 = \mu^+(a). \end{aligned}$$

Thus $(\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)^+ \geq \mu^+$. Now, since μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal of S, we have

$$\begin{aligned} (\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)^+ (a) &= (\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)(a) \vee 0.5 \\ &= \left\{ \bigvee_{a=xyz} (\mu \circ \mathcal{S} \circ \mu)(x) \wedge \mathcal{S}(y) \wedge \mu(z) \right\} \vee 0.5 \\ &= \left\{ \bigvee_{a=xyz} \left\{ \bigvee_{x=uvw} (\mu(u) \wedge \mathcal{S}(v) \wedge \mu(w)) \right\} \wedge \mathcal{S}(y) \wedge \mu(z) \right\} \vee 0.5 \\ &\leq \bigvee_{a=(uvw)yz} \mu(uvwyz) \vee 0.5 = \mu(a) \vee 0.5 = \mu^+(a). \end{aligned}$$

Thus $(\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)^{+} \leq \mu^{+}$. Hence $\mu^{+} = (\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu)^{+}$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) : Straightforward, because every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S and every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal of S.

(4) $\Rightarrow$ (1) : Let Q be a quasi-ideal of S. Then by Lemma 265, χ_Q is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of S. Thus by hypothesis

$$
\begin{aligned}
\chi_Q^{+} &= \left(\chi_Q \circ \mathcal{S} \circ \chi_Q \circ \mathcal{S} \circ \chi_Q\right)^{+} \\
&= \chi_{QSQSQ}^{+}.
\end{aligned}
$$

Thus $Q = QSQSQ$. Hence by Theorem 7, S is regular. ■

Theorem 293 *For a ternary semigroup S, the following assertions are equivalent:*

(1) S is regular;

(2) $(\mu \wedge \lambda)^{+} \leq (\mu \circ \mathcal{S} \circ \lambda)^{+}$ for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal λ of S;

(3) $(\mu \wedge \lambda)^{+} \leq (\mu \circ \mathcal{S} \circ \lambda)^{+}$ for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal λ of S;

(4) $(\mu \wedge \lambda)^{+} \leq (\mu \circ \mathcal{S} \circ \lambda)^{+}$ for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal λ of S;

(5) $(\mu \wedge \lambda)^{+} \leq (\mu \circ \mathcal{S} \circ \lambda)^{+}$ for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal λ of S;

(6) $(\mu \wedge \lambda)^{+} \leq (\mu \circ \mathcal{S} \circ \lambda)^{+}$ for every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal λ of S.

Proof. (1) $\Rightarrow$ (2) : Assume that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal and λ an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left ideal of S. Since S is regular, so for any $a \in S$ there exists $x \in S$ such that $a = axa$. It follows that

$$
\begin{aligned}
(\mu \circ \mathcal{S} \circ \lambda)^{+}(a) &= (\mu \circ \mathcal{S} \circ \lambda)(a) \vee 0.5 \\
&= \left\{ \bigvee_{a=lmn} \mu(l) \wedge \mathcal{S}(m) \wedge \lambda(n) \right\} \vee 0.5 \\
&\geq \{\mu(a) \wedge \mathcal{S}(x) \wedge \lambda(a)\} \vee 0.5 \\
&= (\mu \wedge \lambda)(a) \vee 0.5 = (\mu \wedge \lambda)^{+}(a).
\end{aligned}
$$

Hence $(\mu \wedge \lambda)^{+} \leq (\mu \circ \mathcal{S} \circ \lambda)^{+}$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) : Straightforward, because every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S and every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy generalized bi-ideal of S.

$(4)\Rightarrow(1)$: Let $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal μ and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal λ of S. Let ν be any $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal of S. Take $\mu=\nu$, then by (3), $(\nu\wedge\lambda)^{+}\leq(\nu\circ\mathcal{S}\circ\lambda)^{+}$. But $(\nu\circ\mathcal{S}\circ\lambda)^{+}\leq(\nu\wedge\lambda)^{+}$ always holds. Hence $(\nu\wedge\lambda)^{+}=(\nu\circ\mathcal{S}\circ\lambda)^{+}$. Therefore by Theorem 291, S is regular.

$(1)\Rightarrow(5)$: Suppose μ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal and λ an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal of S. Since S is regular so for any $a\in S$ there exists $x\in S$ such that $a=axa$. It follows that

$$\begin{aligned}(\mu\circ\mathcal{S}\circ\lambda)^{+}(a) &= (\mu\circ\mathcal{S}\circ\lambda)(a)\vee0.5=\left\{\bigvee_{a=pqr}\mu(p)\wedge\mathcal{S}(q)\wedge\lambda(r)\right\}\vee0.5\\ &\geq \mu(a)\wedge\mathcal{S}(x)\wedge\lambda(a)\vee0.5=\{\mu(a)\wedge\lambda(a)\}\vee0.5\\ &= (\mu\wedge\lambda)^{+}(a).\end{aligned}$$

Hence $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}$.

$(5)\Rightarrow(6)$: Straightforward.

$(6)\Rightarrow(1)$: Suppose that $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal μ and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal λ of S. Let ν be any $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal of S. Take $\nu=\lambda$, then by (5), $(\mu\wedge\nu)^{+}\leq(\mu\circ\mathcal{S}\circ\nu)^{+}$. But by Lemma 289, $(\mu\circ\mathcal{S}\circ\nu)^{+}\leq(\mu\wedge\nu)^{+}$. Hence $(\mu\wedge\nu)^{+}=(\mu\circ\mathcal{S}\circ\nu)^{+}$. Therefore by Theorem 291, S is regular. ■

The dual of Theorem 293 is:

Theorem 294 *The following assertions are equivalent for a ternary semigroup S :*

(1) *S is regular;*

(2) $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}$ *for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy generalized bi-ideal μ and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal λ of S;*

(3) $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}$ *for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal μ and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal λ of S;*

(4) $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}$ *for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal μ and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal λ of S;*

(5) $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}$ *for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal μ and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal λ of S;*

(6) $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}$ *for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal μ and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal λ of S.*

Theorem 295 *The following assertions are equivalent for a ternary semigroup S :*

(1) *S is regular;*

(2) $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}\wedge(\lambda\circ\mathcal{S}\circ\mu)^{+}$ *for all $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy generalized bi-ideals μ and λ of S;*

(3) $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}\wedge(\lambda\circ\mathcal{S}\circ\mu)^{+}$ *for all* $(\overline{\in},\overline{\in}\vee\overline{q})$*-fuzzy bi-ideals* μ *and* λ *of* S;

(4) $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}\wedge(\lambda\circ\mathcal{S}\circ\mu)^{+}$ *for every* $(\overline{\in},\overline{\in}\vee\overline{q})$*-fuzzy bi-ideal* μ *and every* $(\overline{\in},\overline{\in}\vee\overline{q})$*-fuzzy quasi-ideal* λ *of* S;

(5) $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}\wedge(\lambda\circ\mathcal{S}\circ\mu)^{+}$ *for every* $(\overline{\in},\overline{\in}\vee\overline{q})$*-fuzzy bi-ideal* μ *and every* $(\overline{\in},\overline{\in}\vee\overline{q})$*-fuzzy left ideal* λ *of* S;

(6) $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}\wedge(\lambda\circ\mathcal{S}\circ\mu)^{+}$ *for every* $(\overline{\in},\overline{\in}\vee\overline{q})$*-fuzzy quasi-ideal* μ *and every* $(\overline{\in},\overline{\in}\vee\overline{q})$*-fuzzy left ideal* λ *of* S;

(7) $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}\wedge(\lambda\circ\mathcal{S}\circ\mu)^{+}$ *for every* $(\overline{\in},\overline{\in}\vee\overline{q})$*-fuzzy right ideal* μ *and every* $(\overline{\in},\overline{\in}\vee\overline{q})$*-fuzzy left ideal* λ *of* S.

Proof. $(1)\Rightarrow(2)$: Assume that μ and λ are $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy generalized bi-ideals of S and $a\in S$. Since S is regular so there exists $x\in S$ such that $a=axa$. It follows that

$$\begin{aligned}(\mu\circ\mathcal{S}\circ\lambda)^{+}(a) &= (\mu\circ\mathcal{S}\circ\lambda)(a)\vee 0.5\\ &= \left\{\bigvee_{a=pqr}\mu(p)\wedge\mathcal{S}(q)\wedge\lambda(r)\right\}\vee 0.5\\ &\geq \{\mu(a)\wedge\mathcal{S}(x)\wedge\lambda(a)\}\vee 0.5\\ &= (\mu\wedge\lambda)(a)\vee 0.5=(\mu\wedge\lambda)^{+}(a).\end{aligned}$$

Thus $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}$. On the other hand

$$\begin{aligned}(\lambda\circ\mathcal{S}\circ\mu)^{+}(a) &= (\lambda\circ\mathcal{S}\circ\mu)(a)\vee 0.5\\ &= \left\{\bigvee_{a=lmn}(\lambda(l)\wedge\mathcal{S}(m)\wedge\mu(n)\right\}\vee 0.5\\ &\geq \{\lambda(a)\wedge\mathcal{S}(x)\wedge\mu(a)\}\vee 0.5=(\mu\wedge\lambda)^{+}(a).\end{aligned}$$

Thus $(\mu\wedge\lambda)^{+}\leq(\lambda\circ\mathcal{S}\circ\mu)^{+}$. Hence $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}\wedge(\lambda\circ\mathcal{S}\circ\mu)^{+}$.

It is clear that $(2)\Rightarrow(3)\Rightarrow(4)\Rightarrow(5)\Rightarrow(6)\Rightarrow(7)$.

$(7)\Rightarrow(1)$: Let $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}\wedge(\lambda\circ\mathcal{S}\circ\mu)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal μ and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left ideal λ of S. This implies that $(\mu\wedge\lambda)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}\wedge(\lambda\circ\mathcal{S}\circ\mu)^{+}\leq(\mu\circ\mathcal{S}\circ\lambda)^{+}$ and by Lemma 289, $(\mu\circ\mathcal{S}\circ\lambda)^{+}\leq(\mu\wedge\lambda)$. Consequently $(\mu\wedge\lambda)^{+}=(\mu\circ\mathcal{S}\circ\lambda)^{+}$. Therefore by Theorem 291, S is regular. ■

4.7 Weakly regular ternary semigroups

In this section we characterize right weakly regular ternary semigroups in terms of $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left (right and lateral) ideals, $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideals, $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi- (generalized bi-) ideals.

Theorem 296 *For a ternary semigroup S, the following assertions are equivalent:*

(1) *S is right weakly regular;*

(2) *$(\mu\wedge\lambda)^{+} = (\mu\circ\lambda\circ\lambda)^{+}$ for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal μ and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy two sided ideal λ of S.*

Proof. (1) $\Rightarrow$ (2) : Let μ be an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal and λ an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy two sided ideal of S and $a\in S$. It follows that

$$\begin{aligned}
(\mu\circ\lambda\circ\lambda)^{+}(a) &= (\mu\circ\lambda\circ\lambda)(a)\vee 0.5\\
&= \left\{\bigvee_{a=pqr}\mu(p)\wedge\lambda(q)\wedge\lambda(r)\right\}\vee 0.5\\
&\leq \left\{\bigvee_{a=pqr}(\mu(pqr)\vee 0.5)\wedge\lambda(q)\wedge(\lambda(pqr)\vee 0.5)\right\}\vee 0.5\\
&\leq \bigvee_{a=pqr}(\mu(pqr)\wedge\lambda(pqr))\vee 0.5\\
&\leq (\mu(a)\wedge\lambda(a))\vee 0.5 = (\mu\wedge\lambda)^{+}(a).
\end{aligned}$$

Thus $(\mu\circ\lambda\circ\lambda)^{+}\leq(\mu\wedge\lambda)^{+}$.

Now we show that $(\mu\wedge\lambda)^{+}\leq(\mu\circ\lambda\circ\lambda)^{+}$. Since S is right weakly regular so for any $a\in S$ there exist $s_1,s_2,s_3,t_1,t_2,t_3\in S$ such that $a=(as_1t_1)(as_2t_2)(as_3t_3)$. It follows that

$$\begin{aligned}
(\mu\wedge\lambda)^{+}(a) &= (\mu\wedge\lambda)(a)\vee 0.5\\
&= (\mu(a)\wedge\lambda(a)\wedge\lambda(a))\vee 0.5\\
&\leq (\mu(as_1t_1)\wedge\lambda(as_2t_2)\wedge\lambda(as_3t_3))\vee 0.5\\
&\leq \left\{\bigvee_{a=xyz}(\mu(x)\wedge\lambda(y)\wedge\lambda(z))\right\}\vee 0.5\\
&= (\mu\circ\lambda\circ\lambda)(a)\vee 0.5 = (\mu\circ\lambda\circ\lambda)^{+}(a).
\end{aligned}$$

Thus $(\mu\wedge\lambda)^{+}\leq(\mu\circ\lambda\circ\lambda)^{+}$. Hence $(\mu\wedge\lambda)^{+}=(\mu\circ\lambda\circ\lambda)^{+}$.

(2) $\Rightarrow$ (1) : Let R be a right ideal and I two sided ideal of S. Then by Corollary 257, χ_R and χ_I are $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal and $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy two sided ideal of S, respectively. Thus by hypothesis

$$\begin{aligned}
(\chi_R\wedge\chi_I)^{+} &= (\chi_R\circ\chi_I\circ\chi_I)^{+}\\
\chi^{+}_{R\cap I} &= (\chi_{RII})^{+}.
\end{aligned}$$

Thus $R\cap I=RII$. Hence by Lemma 9, S is right weakly regular. ■

Theorem 297 *For a ternary semigroup S, the following assertions are equivalent:*

(1) *S is right weakly regular;*

(2) *Each $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal μ of S is idempotent.*

Proof. (1) $\Rightarrow$ (2) : Let μ be an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal of S. We prove $(\mu\circ\mu\circ\mu)^{+}=\mu^{+}$. Let $a\in S$. Then

$$\begin{aligned}
(\mu\circ\mu\circ\mu)^{+}(a) &= (\mu\circ\mu\circ\mu)(a)\vee 0.5\\
&= \left\{\bigvee_{a=pqr}(\mu(p)\wedge\mu(q)\wedge\mu(r))\right\}\vee 0.5\\
&\leq \left\{\bigvee_{a=pqr}(\mu(pqr)\vee 0.5)\wedge\mu(q)\wedge(\mu r)\right\}\vee 0.5\\
&\leq \bigvee_{a=pqr}\mu(pqr)\wedge\vee 0.5\\
&\leq \mu(a)\vee 0.5=\mu^{+}(a).
\end{aligned}$$

Thus $(\mu\circ\mu\circ\mu)^{+}\leq\mu^{+}$. On the other hand since S is right weakly regular so for any $a\in S$, there exist $s_1,s_2,s_3,t_1,t_2,t_3\in S$ such that $a=(as_1t_1)(as_2t_2)(as_3t_3)$. It follows that

$$\begin{aligned}
\mu^{+}(a) &= (\mu(a)\wedge\mu(a)\wedge\mu(a))\vee 0.5\\
&= (\mu(a)\wedge\mu(a)\wedge\mu(a))\vee 0.5\\
&\leq \{(\mu(as_1t_1)\vee 0.5)\wedge(\mu(as_2t_2)\vee 0.5)\wedge(\mu(as_3t_3)\vee 0.5)\}\vee 0.5\\
&\leq \left\{\bigvee_{a=pqr}(\mu(p)\wedge\mu(q)\wedge\mu(r))\right\}\vee 0.5\\
&= (\mu\circ\mu\circ\mu)(a)\vee 0.5=(\mu\circ\mu\circ\mu)^{+}(a).
\end{aligned}$$

Thus $\mu^{+}\leq(\mu\circ\mu\circ\mu)^{+}$. Hence $\mu^{+}=(\mu\circ\mu\circ\mu)^{+}$.

(2) $\Rightarrow$ (1) : Let A be a right ideal of S. Then χ_A is the characteristic function of A. Now by Corollary 257, χ_A is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal of S. Thus by (2)

$$\begin{aligned}
\chi_A^{+} &= (\chi_A\circ\chi_A\circ\chi_A)^{+}\\
&= C_{A^3}^{+}.
\end{aligned}$$

This implies that $A=A^3$. Hence S is right weakly regular. ■

Theorem 298 *For a ternary semigroup S, the following assertions are equivalent:*

(1) *S is right weakly regular;*

(2) $(\mu\wedge\lambda\wedge\nu)^{+}=(\mu\circ\lambda\circ\nu)^{+}$ *for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal μ, every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy two sided ideal λ and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal ν of S;*

(3) $(\mu\wedge\lambda\wedge\nu)^{+}=(\mu\circ\lambda\circ\nu)^{+}$ *for every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy quasi-ideal μ, every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy two sided ideal λ and every $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal ν of S.*

Proof. (1) $\Rightarrow$ (2) : Let μ be an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal, λ an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy two sided ideal and ν an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy right ideal of S. Since S is right weakly regular so for each $a \in S$ there exist $s_1, s_2, s_3, t_1, t_2, t_3 \in S$ such that $a = (as_1t_1)(as_2t_2)(as_3t_3) = a(s_1t_1as_2t_2)(as_3t_3)$. It follows that

$$\begin{aligned}
(\mu \wedge \lambda \wedge \nu)^+(a) &= (\mu \wedge \lambda \wedge \nu)(a) \vee 0.5 \\
&= \{\mu(a) \wedge \lambda(a) \wedge \nu(a)\} \vee 0.5 \\
&\leq \{\{\mu(a) \wedge \lambda(s_1t_1as_2t_2) \vee 0.5\} \wedge \{\nu(as_3t_3) \vee 0.5\}\} \vee 0.5 \\
&= \left\{\bigvee_{a=pqr} (\mu(p) \wedge \lambda(q) \wedge \nu(r))\right\} \vee 0.5 \\
&= (\mu \circ \lambda \circ \nu)(a) \vee 0.5 = (\mu \circ \lambda \circ \nu)^+(a).
\end{aligned}$$

Thus $(\mu \wedge \lambda \wedge \nu)^+ \leq (\mu \circ \lambda \circ \nu)$.

(2) $\Rightarrow$ (3) : Straightforward, because every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S.

(3) $\Rightarrow$ (1) : Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal and λ an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy two sided ideal of S. Take $\nu = \lambda$. Since every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of S, so by hypothesis $(\mu \wedge \lambda \wedge \lambda)^+ \leq (\mu \circ \lambda \circ \lambda)^+$. This implies that $(\mu \wedge \lambda)^+ \leq (\mu \circ \lambda \circ \lambda)^+$. But $(\mu \circ \lambda \circ \lambda)^+ \leq (\mu \wedge \lambda)^+$ is straightforward. Hence $(\mu \wedge \lambda)^+ = (\mu \circ \lambda \circ \lambda)^+$. Therefore by Theorem 296, S is right weakly regular. ■

Theorem 299 *For a ternary semigroup S, the following assertions are equivalent:*

(1) *S is right weakly regular;*

(2) $(\mu \wedge \lambda)^+ \leq (\mu \circ \lambda \circ \lambda)^+$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy bi-ideal* μ *and every* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy two sided ideal* λ *of* S;

(3) $(\mu \wedge \lambda)^+ \leq (\mu \circ \lambda \circ \lambda)^+$ *for every* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy quasi-ideal* μ *and every* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy two sided ideal* λ *of* S.

Proof. (1) $\Rightarrow$ (2) : Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal, λ an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy two sided ideal of S and $a \in S$. Since S is right weakly regular so $a = (as_1t_1)(as_2t_2)(as_3t_3) = a(s_1t_1as_2t_2)(as_3t_3)$ for some $s_1, s_2, s_3, t_1, t_2, t_3 \in S$. It follows that

$$\begin{aligned}
(\mu \wedge \lambda)^+(a) &= (\mu \wedge \lambda)(a) \vee 0.5 \\
&= (\mu(a) \wedge \lambda(a) \wedge \lambda(a)) \vee 0.5 \\
&\leq \{\{\mu(a) \wedge \lambda(s_1t_1as_2t_2) \vee 0.5\} \wedge \{\lambda(as_3t_3) \vee 0.5\}\} \vee 0.5 \\
&= \left\{\bigvee_{a=lmn} (\mu(l) \wedge \lambda(m) \wedge \lambda(n))\right\} \vee 0.5 \\
&= (\mu \circ \lambda \circ \lambda)(a) \vee 0.5 = (\mu \circ \lambda \circ \lambda)^+(a).
\end{aligned}$$

Thus $(\mu \wedge \lambda)^+ \leq (\mu \circ \lambda \circ \lambda)^+$.

(2) $\Rightarrow$ (3) : Straightforward, because every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy bi-ideal of S.

(3) $\Rightarrow$ (1) : Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal and λ an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy two sided ideal of S. Since every $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy right ideal is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy quasi-ideal of S. Thus by hypothesis $(\mu \wedge \lambda)^+ \leq (\mu \circ \lambda \circ \lambda)^+$. Now we have

$$\begin{aligned}
(\mu \circ \lambda \circ \lambda)^+ (a) &= (\mu \circ \lambda \circ \lambda)(a) \vee 0.5 \\
&= \left\{ \bigvee_{a=lmn} \mu(l) \wedge \lambda(m) \wedge \lambda(n) \right\} \vee 0.5 \\
&\leq \left\{ \bigvee_{a=lmn} (\mu(lmn) \vee 0.5) \wedge \lambda(m) \wedge (\lambda(lmn) \vee 0.5) \right\} \vee 0.5 \\
&\leq \bigvee_{a=lmn} \mu(lmn) \wedge \lambda(lmn) \vee 0.5 \\
&= (\mu \wedge \lambda)^+ (a) .
\end{aligned}$$

Thus $(\mu \circ \lambda \circ \lambda)^+ \leq (\mu \wedge \lambda)^+$. Hence $(\mu \wedge \lambda)^+ = (\mu \circ \lambda \circ \lambda)^+$. Therefore by Theorem 296, S is right weakly regular. ■

Chapter 5

$(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals in ternary semigroups

This chapter deals with the study of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (right, lateral) ideals, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi (bi-, generalized bi-) ideals in ternary semigroups. The classes of regular ternary semigroups and right weakly regular ternary semigroups are characterized in terms of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (right, lateral) ideals, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi- (generalized bi-) ideals. Throughout this chapter S will denote a ternary semigroup and $k \in [0, 1)$ unless stated otherwise.

5.1 $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroups

Definition 300 *A fuzzy set μ in S is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup of S, if it satisfies the following condition:*

$$(xyz)_{\min\{t,r,s\}} \overline{\in} \mu \text{ implies } x_t \overline{\in} \vee \overline{q_k} \mu, \; y_r \overline{\in} \vee \overline{q_k} \mu \text{ or } z_s \overline{\in} \vee \overline{q_k} \mu, \tag{5.1}$$

for all $x, y, z \in S$ and $t, r, s \in (0, 1]$.

Example 301 *Consider the ternary semigroup $S = \{0, a, b, c, 1\}$ of Example 54. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0, 1], \quad x \mapsto \begin{cases} 0.20 & \text{if } x = 0, \\ 0.20 & \text{if } x = a, \\ 0.35 & \text{if } x = b, \\ 0.40 & \text{if } x = c, \\ 0.80 & \text{if } x = 1. \end{cases}$$

Then simple calculations show that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_{0.3}})$-fuzzy ternary subsemigroup of S.

Theorem 302 *A fuzzy set μ in S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup of S if and only if it satisfies the following condition:*

$$\max\{\mu(xyz),\ \frac{1-k}{2}\} \geq \min\{\mu(x),\ \mu(y),\ \mu(z)\}, \tag{5.2}$$

for all $x, y, z \in S$ and $t, r, s \in (0, 1]$.

Proof. Assume that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup of S. If there exist $x, y, z \in S$ such that $\max\{\mu(xyz),\ \frac{1-k}{2}\} < \min\{\mu(x),\ \mu(y),\ \mu(z)\}$, then we can choose $t \in (\frac{1-k}{2}, 1]$ such that $\max\{\mu(xyz),\ \frac{1-k}{2}\} < t = \min\{\mu(x),\ \mu(y),\ \mu(z)\}$. Then $(xyz)_t \overline{\in} \mu$ but $x_t \in \wedge q_k \mu$, $y_t \in \wedge q_k \mu$ and $z_t \in \wedge q_k \mu$, which is a contradiction. Hence $\max\{\mu(xyz), \frac{1-k}{2}\} \geq \min\{\mu(x), \mu(y), \mu(z)\}$.

Conversely, suppose that condition (5.2) is valid. Let $(xyz)_{\min\{t,r,s\}} \overline{\in} \mu$. Then $\mu(xyz) < \min\{t, r, s\}$. If $\max\{\mu(xyz),\ \frac{1-k}{2}\} = \mu(xyz)$, then $\min\{\mu(x), \mu(y), \mu(z)\} \leq \mu(xyz) < \min\{t, r, s\}$, which implies that $\mu(x) < t$, $\mu(y) < r$ or $\mu(z) < s$. Thus $x_t \overline{\in} \mu$, $y_r \overline{\in} \mu$ or $z_s \overline{\in} \mu$. On the other hand if $\max\{\mu(xyz),\ \frac{1-k}{2}\} = \frac{1-k}{2}$, then $\min\{\mu(x),\ \mu(y),\ \mu(z)\} \leq \frac{1-k}{2}$. Suppose $x_t \in \mu$, $y_r \in \mu$ and $z_s \in \mu$. Then $t \leq \mu(x) \leq \frac{1-k}{2}$, $r \leq \mu(y) \leq \frac{1-k}{2}$ or $s \leq \mu(z) \leq \frac{1-k}{2}$. This implies that $x_t \overline{q_k} \mu$, $y_r \overline{q_k} \mu$ or $z_s \overline{q_k} \mu$. Hence $x_t \overline{\in} \vee \overline{q_k} \mu$, $y_r \overline{\in} \vee \overline{q_k} \mu$ or $z_s \overline{\in} \vee \overline{q_k} \mu$. ■

In the next theorem we describe the relationship among $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup and the crisp ternary subsemigroup of S.

Theorem 303 *A fuzzy set μ in S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup of S if and only if the nonempty level set $U(\mu; t)$ is a ternary subsemigroup of S for all $t \in (\frac{1-k}{2}, 1]$.*

Proof. Assume that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup of S. Let $t \in (\frac{1-k}{2}, 1]$ and $x, y, z \in U(\mu; t)$. Then $\mu(x) \geq t$, $\mu(y) \geq t$ and $\mu(z) \geq t$. It follows from (5.2) that

$$\frac{1-k}{2} < t \leq \min\{\mu(x),\ \mu(y),\ \mu(z)\} \leq \max\{\mu(xyz),\ \frac{1-k}{2}\}.$$

Thus $\mu(xyz) \geq t$ and so $xyz \in U(\mu; t)$. Hence $U(\mu; t)$ is a ternary subsemigroup of S.

Conversely, suppose that $U(\mu; t)(\neq \emptyset)$ is a ternary subsemigroup of S for all $t \in (\frac{1-k}{2}, 1]$. If there exist $x, y, z \in S$ such that

$$\max\{\mu(xyz), \frac{1-k}{2}\} < \min\{\mu(x), \mu(y), \mu(z)\},$$

then we can choose $t \in (\frac{1-k}{2}, 1]$ such that $\max\{\mu(xyz), \frac{1-k}{2}\} < t \leq \min\{\mu(x), \mu(y), \mu(z)\}$. Thus $x, y, z \in U(\mu; t)$, but $xyz \notin U(\mu; t)$, which is a contradiction. Hence

$$\max\{\mu(xyz), \frac{1-k}{2}\} \geq \min\{\mu(x), \mu(y), \mu(z)\}.$$

Therefore μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup of S. ■

Theorem 304 *Let A be a nonempty subset of S. Then A is a ternary subsemigroup of S if and only if the fuzzy set μ in S defined by:*

$$\mu(x) = \begin{cases} \leq \frac{1-k}{2} & \text{for all } x \in S \setminus A, \\ 1 & \text{for all } x \in A, \end{cases}$$

is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup of S.

Proof. Assume that A is a ternary subsemigroup of S and $x, y, z \in S$. If $x, y, z \in A$, then $\mu(x) = \mu(y) = \mu(z) = 1$. Since A is a ternary subsemigroup of S, so $xyz \in A$ and $\mu(xyz) = 1$. Hence

$$\max\left\{\mu(xyz), \frac{1-k}{2}\right\} = 1 = \min\{\mu(x), \mu(y), \mu(z)\}.$$

If $x \notin A$, $y \notin A$ or $z \notin A$, then $\mu(x) \leq \frac{1-k}{2}$, $\mu(y) \leq \frac{1-k}{2}$ or $\mu(z) \leq \frac{1-k}{2}$. It follows that

$$\min\{\mu(x), \mu(y), \mu(z)\} \leq \frac{1-k}{2} \leq \max\left\{\mu(xyz), \frac{1-k}{2}\right\}.$$

Hence in any case $\max\left\{\mu(xyz), \frac{1-k}{2}\right\} \geq \min\{\mu(x), \mu(y), \mu(z)\}$. Consequently μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup of S.

Conversely, suppose that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup of S and $x, y, z \in A$. Then $\mu(x) = \mu(y) = \mu(z) = 1$. It follows that $\max\left\{\mu(xyz), \frac{1-k}{2}\right\} \geq \min\{\mu(x), \mu(y), \mu(z)\} = 1$. Hence $\mu(xyz) = 1$, that is, $xyz \in A$. Therefore A is a ternary subsemigroup of S. ■

Corollary 305 *A nonempty subset A of S is a ternary subsemigroup of S if and only if the characteristic function χ_A of A, is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup of S.*

Theorem 306 *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroups of S. Then $\mu := \bigcap_{i \in \Lambda} \mu_i$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup of S.*

Proof. It is straightforward. ■

The following example shows that the union of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroups of S may not be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroups of S.

Example 307 *Consider the ternary semigroup S and the $(\overline{\in}, \overline{\in} \vee \overline{q_{0.3}})$-fuzzy ternary subsemigroup μ of S given in Example 301. Define a fuzzy set λ in S as follows:*

$$\lambda : S \to [0,1], \quad x \mapsto \begin{cases} 0.30 & \text{if } x = 0, \\ 0.70 & \text{if } x = a, \\ 0.30 & \text{if } x = b, \\ 0.20 & \text{if } x = c, \\ 0.10 & \text{if } x = 1. \end{cases}$$

By routine calculations, we know that λ is an $(\overline{\in}, \overline{\in} \vee \overline{q_{0.3}})$-fuzzy ternary subsemigroup of S. Note that $\nu := \mu \cup \lambda$ is given by

$$\nu : S \to [0,1], \quad x \mapsto \begin{cases} 0.30 & \text{if } x = 0, \\ 0.70 & \text{if } x = a, \\ 0.35 & \text{if } x = b, \\ 0.40 & \text{if } x = c, \\ 0.80 & \text{if } x = 1. \end{cases}$$

But $\nu := \mu \cup \lambda$ is not an $(\overline{\in}, \overline{\in} \vee \overline{q_{0.3}})$-fuzzy ternary subsemigroup of S. If $t \in (0.35, 0.40]$, then $U(\mu \cup \lambda; t) = \{a, c, 1\}$, which is not a ternary subsemigroup of S. It follows from Theorem 303, that $\nu := \mu \cup \lambda$ is not an $(\overline{\in}, \overline{\in} \vee \overline{q_{0.3}})$-fuzzy ternary subsemigroup of S.

5.2 $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideals

We begin this section with

Definition 308 *A fuzzy set μ in S is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right, lateral) ideal of S if it satisfies the following condition:*

$$(xyz)_t \overline{\in} \mu \text{ implies } z_t \overline{\in} \vee \overline{q_k} \mu \text{ (resp. } x_t \overline{\in} \vee \overline{q_k} \mu, \ y_t \overline{\in} \vee \overline{q_k} \mu) \tag{5.3}$$

for all $x, y, z \in S$ and $t \in (0, 1]$.

A fuzzy set μ in S is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy two sided ideal of S if it is both an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal of S. A fuzzy set μ in S is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of S if it is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy lateral ideal and an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal of S.

Example 309 *Consider the ternary semigroup $S = \{a, b, c, d, e\}$ of Example 53. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.9 & \text{if } x = a, \\ 0.7 & \text{if } x = b, \\ 0.3 & \text{if } x = c, \\ 0.8 & \text{if } x = d, \\ 0.3 & \text{if } x = e. \end{cases}$$

Routine calculations show that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_{0.4}})$-fuzzy left ideal of S. But μ is neither an $(\overline{\in}, \overline{\in} \vee \overline{q_{0.4}})$-fuzzy right ideal, nor an $(\overline{\in}, \overline{\in} \vee \overline{q_{0.4}})$-fuzzy lateral ideal of S. Since $(ecc)_{0.4} \overline{\in} \mu$, but $e_{0.4} \in \wedge q_{0.4} \mu$. Also, $(cec)_{0.39} \overline{\in} \mu$, but $e_{0.39} \in \wedge q_{0.4} \mu$.

Example 310 *Consider the ternary semigroup S of Example 54. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.9 & \text{if } x = 0, \\ 0.4 & \text{if } x = a, \\ 0.8 & \text{if } x = b, \\ 0.5 & \text{if } x = c, \\ 0.3 & \text{if } x = 1. \end{cases}$$

Routine calculations show that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_{0.2}})$-fuzzy right ideal of S. But μ is neither an $(\overline{\in}, \overline{\in} \vee \overline{q_{0.2}})$-fuzzy left ideal, nor an $(\overline{\in}, \overline{\in} \vee \overline{q_{0.2}})$-fuzzy lateral ideal of S. Since $(1ac)_{0.45} \overline{\in} \mu$, but $c_{0.45} \in \wedge q_{0.2} \mu$. Also, $(ac1)_{0.44} \overline{\in} \mu$, but $c_{0.44} \in \wedge q_{0.2} \mu$.

Theorem 311 *Let μ be a fuzzy set in S and $x, y, z \in S$, $t \in (0,1]$. Then*

$$(1)\ (xyz)_t \overline{\in} \mu \text{ implies } z_t \overline{\in} \vee \overline{q_k} \mu;$$
$$(2)\ (xyz)_t \overline{\in} \mu \text{ implies } x_t \overline{\in} \vee \overline{q_k} \mu;$$
$$(3)\ (xyz)_t \overline{\in} \mu \text{ implies } y_t \overline{\in} \vee \overline{q_k} \mu;$$

are respectively equivalent to:

$$\begin{aligned} (1')\ \max\{\mu(xyz), \frac{1-k}{2}\} &\geq \mu(z); \\ (2')\ \max\{\mu(xyz), \frac{1-k}{2}\} &\geq \mu(x); \\ (3')\ \max\{\mu(xyz), \frac{1-k}{2}\} &\geq \mu(y); \end{aligned}$$

for all $x, y, z \in S$.

Proof. The proof is similar to the proof of Theorem 302. ■

Corollary 312 *A fuzzy set μ in S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right, lateral) ideal of S if it satisfies condition $(1')$ (resp. $(2')$, $(3')$)*

Corollary 313 *A fuzzy set μ in S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of S if it satisfies conditions $(1')$, $(2')$ and $(3')$.*

In the next theorem we describe the relationship among $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right, lateral) ideal and the crisp left (resp. right, lateral) ideal of S.

Theorem 314 *A fuzzy set μ in S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right, lateral) ideal of S if and only if the level set $U(\mu; t)$ is a left (resp. right, lateral) ideal of S for all $t \in (\frac{1-k}{2}, 1]$, whenever it is nonempty.*

Proof. The proof is similar to the proof of Theorem 303. ■

Corollary 315 *A fuzzy set μ in S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ideal of S if and only if $U(\mu; t)(\neq \emptyset)$ is an ideal of S for all $t \in (\frac{1-k}{2}, 1]$.*

Theorem 316 *Let A be a nonempty subset of S. Then A is a left (resp. right, lateral) ideal of S if and only if the fuzzy set μ in S defined by:*

$$\mu(x) = \begin{cases} \leq \frac{1-k}{2} & \text{for all } x \in S \setminus A, \\ 1 & \text{for all } x \in A, \end{cases}$$

is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right, lateral) ideal of S.

Proof. The proof is similar to the proof of Theorem 304. ■

Corollary 317 *A nonempty subset A of S is a left (resp. right, lateral) ideal of S if and only if the characteristic function χ_A of A is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right, lateral) ideal of S.*

Theorem 318 *If μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal, λ a fuzzy set and ν an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal of S, then $\mu \circ \lambda \circ \nu$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy two sided ideal of S.*

Proof. Assume that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal, λ a fuzzy set in S and ν an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal of S, $x, y, z \in S$. Then

$$(\mu \circ \lambda \circ \nu)(z) = \bigvee_{z=uvw} \{\mu(u) \wedge \lambda(v) \wedge \nu(w)\}.$$

(If $z = uvw$, then $xyz = xy(uvw) = (xyu)vw$. Since μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S, so $\mu(xyu) \vee \frac{1-k}{2} \geq \mu(u)$). It follows that

$$\begin{aligned}
(\mu \circ \lambda \circ \nu)(z) &= \bigvee_{z=uvw} \{\mu(u) \wedge \lambda(v) \wedge \nu(w)\} \\
&\leq \bigvee_{z=uvw} \left\{ \left\{ \mu(xyu) \vee \frac{1-k}{2} \right\} \wedge \lambda(v) \wedge \nu(w) \right\} \\
&\leq \bigvee_{xyz=abc} \left\{ \{\mu(a) \wedge \lambda(b) \wedge \nu(c)\} \vee \frac{1-k}{2} \right\} \\
&= \left\{ \bigvee_{xyz=abc} \{\mu(a) \wedge \lambda(b) \wedge \nu(c)\} \right\} \vee \frac{1-k}{2} \\
&= (\mu \circ \lambda \circ \nu)(xyz) \vee \frac{1-k}{2}.
\end{aligned}$$

Thus $(\mu \circ \lambda \circ \nu)(xyz) \vee \frac{1-k}{2} \geq (\mu \circ \lambda \circ \nu)(z)$.

If $(\mu \circ \lambda \circ \nu)(z) = 0$, then $(\mu \circ \lambda \circ \nu)(z) = 0 \leq (\mu \circ \lambda \circ \nu)(xyz) \vee \frac{1-k}{2}$. Hence $\mu \circ \lambda \circ \nu$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S.

Similarly we can show that $\mu \circ \lambda \circ \nu$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal of S. Therefore $\mu \circ \lambda \circ \nu$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy two sided ideal of S. ■

Lemma 319 *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right, lateral) ideals of S. Then*

(i) $\bigcap_{i\in\Lambda} \mu_i$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right, lateral) ideal of S.

(ii) $\bigcup_{i\in\Lambda} \mu_i$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right, lateral) ideal of S.

Proof. It is straightforward. ■

5.3 Fuzzy quasi-ideals of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$

Definition 320 *A fuzzy set μ in S is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S if it satisfies:*

$$(1)\quad \max\{\mu(x), \frac{1-k}{2}\} \geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\} \text{ and}$$

$$(2)\quad \max\{\mu(x), \frac{1-k}{2}\} \geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\}$$

for all $x \in S$.

Theorem 321 *A fuzzy set μ in S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S if and only if the level set $U(\mu; t)$ is a quasi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$, whenever it is nonempty.*

Proof. Assume that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. Let $x \in SSU(\mu; t) \cap SU(\mu; t)S \cap U(\mu; t)SS$. Then there exist $t_1, t_2, t_3, t_4, t_5, t_6 \in S$ and $s_1, s_2, s_3 \in U(\mu; t)$ such that $x = t_1t_2s_1$, $x = t_3s_2t_4$ and $x = s_3t_5t_6$. It follows that

$$\begin{aligned}
(\mu \circ \mathcal{S} \circ \mathcal{S})(x) &= \bigvee_{x=pqr} \{\mu(p) \wedge \mathcal{S}(q) \wedge \mathcal{S}(r)\} \\
&\geq \mu(s_3)\mathcal{S}(t_5) \wedge \mathcal{S}(t_6)) = \mu(s_3) \geq t.
\end{aligned}$$

Also,

$$\begin{aligned}
(\mathcal{S} \circ \mathcal{S} \circ \mu)(x) &= \bigvee_{x=uvw} \{\mathcal{S}(u) \wedge \mathcal{S}(v) \wedge \mu(w)\} \\
&\geq \mathcal{S}(t_1) \wedge \mathcal{S}(t_2) \wedge \mu(s_1) = \mu(s_1) \geq t,
\end{aligned}$$

and

$$\begin{aligned}
(\mathcal{S} \circ \mu \circ \mathcal{S})(x) &= \bigvee_{x=lmn} \{\mathcal{S}(l) \wedge \mu(m) \wedge \mathcal{S}(n)\} \\
&\geq \mathcal{S}(t_3) \wedge \mu(s_2) \wedge \mathcal{S}(t_4) = \mu(s_2) \geq t.
\end{aligned}$$

Thus by hypothesis

$$\begin{aligned}\max\{\mu(x), \frac{1-k}{2}\} &\geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\} \\ &\geq t > \frac{1-k}{2}.\end{aligned}$$

Thus $\mu(x) \geq t$, that is $x \in U(\mu; t)$, which implies $SSU(\mu; t) \cap SU(\mu; t)S \cap U(\mu; t)SS \subseteq U(\mu; t)$. Similarly $SSU(\mu; t) \cap SSU(\mu; t)SS \cap U(\mu; t)SS \subseteq U(\mu; t)$. Hence $U(\mu; t)$ is a quasi-ideal of S.

Conversely, suppose that $U(\mu; t)$ is a quasi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$ and $x \in S$. If possible, let

$$\max\{\mu(x), \frac{1-k}{2}\} < \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\}.$$

Choose $t \in (\frac{1-k}{2}, 1]$ such that

$$\max\{\mu(x), \frac{1-k}{2}\} < t \leq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\},$$

which implies that $\mu(x) < t$ and $(\mu \circ \mathcal{S} \circ \mathcal{S})(x) \geq t$, $(\mathcal{S} \circ \mu \circ \mathcal{S})(x) \geq t$ and $(\mathcal{S} \circ \mathcal{S} \circ \mu)(x) \geq t$. If $(\mu \circ \mathcal{S} \circ \mathcal{S})(x) \geq t$, then

$$(\mu \circ \mathcal{S} \circ \mathcal{S})(x) = \bigvee_{x=uvw} \{\mu(u) \wedge \mathcal{S}(v) \wedge \mathcal{S}(w)\} = \bigvee_{x=uvw} \mu(u) \geq t.$$

Thus there exists $z \in U(\mu; t)$ such that $x = abz$ for some $a, b \in S$. Similarly $(\mathcal{S} \circ \mu \circ \mathcal{S})(x) \geq t$ implies $m \in U(\mu; t)$ such that $x = lmn$ for some $l, n \in S$, and $(\mathcal{S} \circ \mathcal{S} \circ \mu)(x) \geq t$ implies $r \in U(\mu; t)$ such that $x = pqr$ for some $p, q \in S$. Hence $x \in SSU(\mu; t) \cap SU(\mu; t)S \cap U(\mu; t)SS \subseteq U(\mu; t)$. This implies that $\mu(x) \geq t$, which is a contradiction, because $\mu(x) < t$. Therefore

$$\max\{\mu(x), \frac{1-k}{2}\} \geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\}.$$

Similarly, $\max\{\mu(x), \frac{1-k}{2}\} \geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\}$.

Consequently μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. ■

Theorem 322 *Let A be a nonempty subset of S. Then A is a quasi-ideal of S if and only if the fuzzy set μ in S defined by:*

$$\mu(x) = \begin{cases} \leq \frac{1-k}{2} & \text{for all } x \in S \setminus A, \\ 1 & \text{for all } x \in A, \end{cases}$$

is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S.

Proof. Assume that A is a quasi-ideal of S and $x \in S$. If $x \in A$, then $\mu(x) = 1$, which implies that

$$\max\{\mu(x), \frac{1-k}{2}\} = 1 \geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\}.$$

If $x \notin A$, then $x \notin SSA \cap SAS \cap ASS$ and $x \notin SSA \cap SSASS \cap ASS$ and so $\mu(x) \leq \frac{1-k}{2}$. Since $x \notin SSA \cap SAS \cap ASS$, so $\min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\} \neq 1$. It follows that

$$\min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\} \leq \frac{1-k}{2} = \max\{\mu(x), \frac{1-k}{2}\}.$$

Similarly, since $x \notin SSA \cap SSASS \cap ASS$, so $\min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\} \neq 1$. Hence in any case

$$\max\{\mu(x), \frac{1-k}{2}\} \geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\},$$

and

$$\max\{\mu(x), \frac{1-k}{2}\} \geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\}.$$

Consequently μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S.

Conversely, suppose that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. Let $x \in SSA \cap SAS \cap ASS$. Then there exist $a, b, c \in A$ and $t_1, t_2, t_3, t_4, t_5, t_6 \in S$ such that $x = at_1t_2$, $x = t_3bt_4$, and $x = t_5t_6c$. It follows that

$$\begin{aligned} (\mu \circ \mathcal{S} \circ \mathcal{S})(x) &= \bigvee_{x=pqr} \{\mu(p) \wedge \mathcal{S}(q) \wedge \mathcal{S}(r)\} \\ &\geq \mu(a) \wedge \mathcal{S}(t_1) \wedge \mathcal{S}(t_2) = 1. \end{aligned}$$

This implies that $(\mu \circ \mathcal{S} \circ \mathcal{S})(x) = 1$. Similarly $(\mathcal{S} \circ \mu \circ \mathcal{S})(x) = 1$ and $(\mathcal{S} \circ \mathcal{S} \circ \mu)(x) = 1$. Thus by hypothesis

$$\max\{\mu(x), \frac{1-k}{2}\} \geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\} = 1.$$

This implies that $\mu(x) = 1$, that is, $x \in A$. This shows that $SSA \cap SAS \cap ASS \subseteq A$. Similarly, $SSA \cap SSASS \cap ASS \subseteq A$. Hence A is a quasi-ideal of S. ■

Corollary 323 *A nonempty subset A of S is a quasi-ideal of S if and only if the characteristic function χ_A of A is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S.*

Lemma 324 *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals of S. Then $\mu := \bigcap_{i \in \Lambda} \mu_i$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S.*

Proof. The proof is straightforward. ■

Theorem 325 *Every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (right, lateral) ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S.*

Proof. Assume that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S and $x \in S$. Then

$$\begin{aligned}(\mathcal{S} \circ \mathcal{S} \circ \mu)(x) &= \bigvee_{x=abc} \{\mathcal{S}(a) \wedge \mathcal{S}(b) \wedge \mu(c)\} = \bigvee_{x=abc} \mu(c) \leq \bigvee_{x=abc} \mu(abc) \vee \frac{1-k}{2} \\ &= \mu(x) \vee \frac{1-k}{2},\end{aligned}$$

and so $(\mathcal{S} \circ \mathcal{S} \circ \mu)(x) \leq \mu(x) \vee \frac{1-k}{2}$. Hence $\max\{\mu(x), \frac{1-k}{2}\} \geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)$. Similarly

$$\max\{\mu(x), \frac{1-k}{2}\} \geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x)\}.$$

Hence μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. ■

The converse of the above theorem is not true in general.

Example 326 *Let* $S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ *be the ternary semigroup under matrix multiplication. Then* $Q = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$ *is the quasi-ideal of S, which is neither a left ideal, nor a right ideal, nor a lateral ideal of S. Then by Corollary 323, the characteristic function χ_Q of Q is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S, but neither $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal, nor $(\overline{\in} \vee \overline{q_k})$-fuzzy right ideal, nor $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy lateral ideal of S.*

5.4 Fuzzy bi-ideals of type $(\overline{\in}, \overline{\in} \vee \overline{q_k})$

Definition 327 *An $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy ternary subsemigroup μ of S is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S if it satisfies the following condition:*

$$(xuyvz)_{\min\{t,r,s\}} \overline{\in} \mu \text{ implies } x_t \overline{\in} \vee \overline{q_k} \mu, \ y_r \overline{\in} \vee \overline{q_k} \mu \text{ or } z_s \overline{\in} \vee \overline{q_k} \mu, \tag{5.4}$$

for all $x, y, z, u, v \in S$ and $t, r, s \in (0, 1]$.

A fuzzy set μ in S is said to be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal of S if it satisfies condition (5.4).

Obviously every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal of S but the converse may not be true.

The following theorem is a characterization of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal.

Theorem 328 *Let μ be a fuzzy set in S. Then μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal of S if and only if it satisfies:*

$$\max\{\mu(xuyvz), \frac{1-k}{2}\} \geq \min\{\mu(x),\ \mu(y),\ \mu(z)\}, \tag{5.5}$$

for all $x, y, z, u, v \in S$ and $t, r, s \in (0, 1]$.

Proof. The proof is similar to the proof of Theorem 302. ■

Theorem 329 *Let μ be a fuzzy set in S. Then μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal (resp. generalized bi-ideal) of S if and only if the nonempty level set $U(\mu; t)$ is a bi-ideal (resp. generalized bi-ideal) of S for all $t \in (\frac{1-k}{2}, 1]$.*

Proof. The proof is similar to the proof of Theorem 303. ■

Theorem 330 *Let A be a nonempty subset of S. Then A is a bi-ideal (resp. generalized bi-ideal) of S if and only if the fuzzy set μ in S defined by:*

$$\mu(x) = \begin{cases} \leq \frac{1-k}{2} & \text{for all } x \in S \setminus A, \\ 1 & \text{for all } x \in A, \end{cases}$$

is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal (resp. generalized bi-ideal) of S.

Proof. The proof is similar to the proof of Theorem 304. ■

Corollary 331 *A nonempty subset A of S is a bi-ideal (resp. generalized bi-ideal) of S if and only if the characteristic function χ_A of A is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal (resp. generalized bi-ideal) of S.*

Lemma 332 *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideals of S. Then $\mu := \bigcap_{i \in \Lambda} \mu_i$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S.*

Proof. The proof is similar to the proof of Lemma 319. ■

Lemma 333 *Let μ, λ and ν be $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideals of S. Then $\mu \circ \lambda \circ \nu$ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S.*

Proof. The proof is straightforward. ■

5.5 Regular ternary semigroups

In this section we characterize regular ternary semigroups in terms of $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left (right) ideals, $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideals and $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy bi-(generalized bi-) ideals.

Definition 334 *Let μ, λ and ν be fuzzy sets in S. Define fuzzy sets $\mu^k, \mu\wedge^k\lambda, \mu\vee^k\lambda$ and $\mu\circ^k\lambda\circ^k\nu$ in S as follows:*

(1) $\mu^k(x)=\mu(x)\vee\frac{1-k}{2}$;

(2) $(\mu\wedge^k\lambda)(x)=(\mu\wedge\lambda)(x)\vee\frac{1-k}{2}$;

(3) $(\mu\vee^k\lambda)(x)=(\mu\vee\lambda)(x)\vee\frac{1-k}{2}$;

(4) $(\mu\circ^k\lambda\circ^k\nu)(x)=(\mu\circ\lambda\circ\nu)(x)\vee\frac{1-k}{2}$;

for all $x\in S$.

Lemma 335 *Let μ,λ and ν be fuzzy sets in S. Then the following hold:*

(1) $\mu\wedge^k\lambda=\mu^k\wedge\lambda^k$;

(2) $\mu\vee^k\lambda=\mu^k\vee\lambda^k$;

(3) $\mu\circ^k\lambda\circ^k\nu\geq\mu^k\circ\lambda^k\circ\nu^k$;

and if x is expressible as $x=abc$ for all $x\in S$, then $\mu\circ^k\lambda\circ^k\nu=\mu^k\circ\lambda^k\circ\nu^k$.

Proof. The proof is straightforward. ■

Lemma 336 *Let A, B and C be nonempty subsets of S. Then the following hold:*

(1) $\chi_A\wedge^k\chi_B=\chi^k_{A\cap B}$;

(2) $\chi_A\vee^k\chi_B=\chi^k_{A\cup B}$;

(3) $\chi_A\circ^k\chi_B\circ^k\chi_C=\chi^k_{ABC}$;

where χ_A is the characteristic function of A.

Proof. The proof is straightforward. ■

Proposition 337 *If μ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left (resp. right, lateral, Quasi-, bi-, generalized bi-) ideal of S, then μ^k is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left (resp. right, lateral, Quasi-, bi-, generalized bi-) ideal of S.*

Proof. Assume that μ is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left ideal of S and $x,y,z\in S$. Then

$$\begin{aligned}\max\{\mu^k(xyz),\frac{1-k}{2}\} &= \max\{\max(\mu(xyz),\frac{1-k}{2}),\frac{1-k}{2}\}\\ &\geq \max\{\mu(z),\frac{1-k}{2}\}=\mu^k(z).\end{aligned}$$

Thus μ^k is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy left ideal of S. ■

Proposition 338 *A nonempty subset A of S is a quasi-ideal (resp. bi-ideal, generalized bi-ideal) of S if and only if χ_A^k is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal (resp. bi-ideal, generalized bi-ideal) of S.*

Proof. Straightforward. ■

Proposition 339 *Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal, λ an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy lateral ideal and ν an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S. Then $\mu \circ^k \lambda \circ^k \nu \leq \mu \wedge^k \lambda \wedge^k \nu$.*

Proof. Assume that μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal, λ an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy lateral ideal and ν an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S and $a \in S$. Then

$$\begin{aligned}
\left(\mu \circ^k \lambda \circ^k \nu\right)(a) &= (\mu \circ \lambda \circ \nu)(a) \vee \frac{1-k}{2} \\
&= \left\{\bigvee_{a=pqr} \{\mu(p) \wedge \lambda(q) \wedge \nu(r)\}\right\} \vee \frac{1-k}{2} \\
&\leq \left\{\bigvee_{a=pqr} \left\{ \begin{array}{c} \left(\mu(pqr) \vee \frac{1-k}{2}\right) \wedge \left(\lambda(pqr) \vee \frac{1-k}{2}\right) \\ \wedge \left(\nu(pqr) \vee \frac{1-k}{2}\right) \end{array} \right\}\right\} \vee \frac{1-k}{2} \\
&= \left\{(\mu(a) \wedge \lambda(a) \wedge \nu(a)) \vee \frac{1-k}{2}\right\} \vee \frac{1-k}{2} \\
&= (\mu \wedge \lambda \wedge \nu)(a) \vee \frac{1-k}{2} = \left(\mu \wedge^k \lambda \wedge^k \nu\right)(a).
\end{aligned}$$

If a is not expressible as $a = pqr$, then

$$\begin{aligned}
\left(\mu \circ^k \lambda \circ^k \nu\right)(a) &= (\mu \circ \lambda \circ \nu)(a) \vee \frac{1-k}{2} \\
&= 0 \vee \frac{1-k}{2} = \frac{1-k}{2} \\
&\leq (\mu \wedge \lambda \wedge \nu)(a) \vee \frac{1-k}{2}.
\end{aligned}$$

Hence $\mu \circ^k \lambda \circ^k \nu \leq \mu \wedge^k \lambda \wedge^k \nu$. ■

Corollary 340 *Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal and λ an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S. Then $\mu \circ^k \mathcal{S} \circ^k \lambda \leq \mu \wedge^k \lambda$.*

Next we characterize regular ternary semigroups in terms of $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (right, lateral) ideals, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideals, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-(generalized bi-) ideals.

Theorem 341 *The following assertions are equivalent for a ternary semigroup S :*

(1) *S is regular;*

(2) *$\mu \wedge^k \lambda \wedge^k \nu = \mu \circ^k \lambda \circ^k \nu$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal μ, every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy lateral ideal λ and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal ν of S.*

Proof. (1) $\Rightarrow$ (2) : Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal, λ an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy lateral ideal and ν an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S. Then by Proposition 339, $\mu \circ^k \lambda \circ^k \nu \leq \mu \wedge^k \lambda \wedge^k \nu$. Since S is regular, so for any $s \in S$ there exists $x \in S$ such that $s = sxs = sxsxs$. It follows that

$$\begin{aligned}
\left(\mu \circ^k \lambda \circ^k \nu\right)(s) &= (\mu \circ \lambda \circ \nu)(s) \vee \frac{1-k}{2} \\
&= \bigvee_{s=pqr} \{\mu(p) \wedge \lambda(q) \wedge \nu(r)\} \vee \frac{1-k}{2} \\
&\geq \{\mu(s) \wedge \lambda(xsx) \wedge \nu(s)\} \vee \frac{1-k}{2} \\
&\geq \{\mu(s) \wedge \lambda(s) \wedge \nu(s)\} \vee \frac{1-k}{2} \\
&= (\mu \wedge \lambda \wedge \nu)(s) \vee \frac{1-k}{2} = \left(\mu \wedge^k \lambda \wedge^k \nu\right)(s).
\end{aligned}$$

Thus $\mu \wedge^k \lambda \wedge^k \nu = \mu \circ^k \lambda \circ^k \nu$.

(2) $\Rightarrow$ (1) : Let R, M and L be the right ideal, lateral ideal and left ideal of S, respectively. Then by Corollary 317, χ_R, χ_M and χ_L are $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal, $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy lateral ideal and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S, respectively. Thus by hypothesis

$$\begin{aligned}
\chi_R \wedge^k \chi_M \wedge^k \chi_L &= \chi_R \circ^k \chi_M \circ^k \chi_L \\
\chi^k_{R \cap M \cap L} &= \chi^k_{RML}.
\end{aligned}$$

Thus $R \cap M \cap L = RML$. Hence by Theorem 5, S is regular. ∎

Theorem 342 *The following assertions are equivalent for a ternary semigroup S :*

(1) S is regular;

(2) $\mu \wedge^k \lambda = \mu \circ^k \mathcal{S} \circ^k \lambda$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal λ and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal μ of S.

Proof. (1) $\Rightarrow$ (2) : Suppose μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal and λ an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal of S. Then by Corollary 340, $\mu \circ^k \mathcal{S} \circ^k \lambda \leq \mu \wedge^k \lambda$. Since S is regular, so for any element $s \in S$ there exists $x \in S$ such that $s = sxs = s(xsx)s$. It follows that

$$\begin{aligned}
\left(\mu \circ^k \mathcal{S} \circ^k \lambda\right)(s) &= (\mu \circ \mathcal{S} \circ \lambda)(s) \vee \frac{1-k}{2} \\
&= \left\{\bigvee_{s=pqr} (\mu(p) \wedge \mathcal{S}(q) \wedge \lambda(r))\right\} \vee \frac{1-k}{2} \\
&\geq (\mu(s) \wedge \mathcal{S}(xsx) \wedge \lambda(s)) \vee \frac{1-k}{2} \\
&= (\mu(s) \wedge \lambda(s)) \vee \frac{1-k}{2} = \left(\mu \wedge^k \lambda\right)(s).
\end{aligned}$$

Thus $\left(\mu \circ^k \mathcal{S} \circ^k \lambda\right) \geq \left(\mu \wedge^k \lambda\right)$. Hence $\left(\mu \wedge^k \lambda\right) = \left(\mu \circ^k \mathcal{S} \circ^k \lambda\right)$.

(2) $\Rightarrow$ (1) : Let L and R be left ideal and right ideal of S, respectively. Then by Corollary 317, χ_L and χ_R are $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal of S, respectively. Thus by hypothesis

$$\begin{aligned}
\left(\chi_R \wedge^k \chi_L\right) &= \left(\chi_R \circ^k \mathcal{S} \circ^k \chi_L\right) \\
\chi^k_{R\cap L} &= \chi^k_{RSL}.
\end{aligned}$$

Hence $R \cap L = RSL$. Therefore by Theorem 6, S is regular. ■

Theorem 343 *The following assertions are equivalent for a ternary semigroup S:*

(1) *S is regular;*

(2) *$\mu^k = \mu \circ^k \mathcal{S} \circ^k \mu \circ^k \mathcal{S} \circ^k \mu$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal μ of S;*

(3) *$\mu^k = \mu \circ^k \mathcal{S} \circ^k \mu \circ^k \mathcal{S} \circ^k \mu$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal μ of S;*

(4) *$\mu^k = \mu \circ^k \mathcal{S} \circ^k \mu \circ^k \mathcal{S} \circ^k \mu$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal μ of S.*

Proof. (1) $\Rightarrow$ (2) : Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal of S and $s \in S$. Since S is regular, so there exists $x \in S$ such that $s = sxs$. It follows that

$$\begin{aligned}
\left(\mu \circ^k \mathcal{S} \circ^k \mu \circ^k \mathcal{S} \circ^k \mu\right)(s) &= \left(\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu\right)(s) \vee \frac{1-k}{2} \\
&= \left\{ \bigvee_{s=pqr} \left(\mu \circ \mathcal{S} \circ \mu\right)(p) \wedge \mathcal{S}(q) \wedge \mu(r) \right\} \vee \frac{1-k}{2} \\
&\geq \left\{ \left(\mu \circ \mathcal{S} \circ \mu\right)(s) \wedge \mathcal{S}(x) \wedge \mu(s) \right\} \vee \frac{1-k}{2} \\
&= \left\{ \left\{ \bigvee_{s=pqr} \left(\mu(p) \wedge \mathcal{S}(q) \wedge \mu(r)\right) \right\} \wedge \mu(s) \right\} \vee \frac{1-k}{2} \\
&\geq \left\{ \left(\mu(s) \wedge \mathcal{S}(x) \wedge \mu(s)\right) \wedge \mu(s) \right\} \vee \frac{1-k}{2} \\
&= \mu(s) \vee \frac{1-k}{2} = \mu^k(s).
\end{aligned}$$

Thus $\mu \circ^k \mathcal{S} \circ^k \mu \circ^k \mathcal{S} \circ^k \mu \geq \mu^k$. Now, since μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal of S, it follows that

$$\begin{aligned}
\left(\mu \circ^k \mathcal{S} \circ^k \mu \circ^k \mathcal{S} \circ^k \mu\right)(s) &= \left(\mu \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mu\right)(s) \vee \frac{1-k}{2} \\
&= \left\{ \bigvee_{s=xyz} \left(\mu \circ \mathcal{S} \circ \mu\right)(x) \wedge \mathcal{S}(y) \wedge \mu(z) \right\} \vee \frac{1-k}{2} \\
&= \left\{ \begin{array}{c} \bigvee_{s=xyz} \left\{ \bigvee_{x=uvw} \left(\mu(u) \wedge \mathcal{S}(v) \wedge \mu(w)\right) \right\} \\ \wedge \mathcal{S}(y) \wedge \mu(z) \end{array} \right\} \vee \frac{1-k}{2} \\
&\leq \bigvee_{s=(uvw)yz} \mu(uvwyz) \vee \frac{1-k}{2} = \mu(s) \vee \frac{1-k}{2} = \mu^k(s).
\end{aligned}$$

Thus $\left(\mu \circ^k \mathcal{S} \circ^k \mu \circ^k \mathcal{S} \circ^k \mu\right) \leq \mu^k$. Hence $\mu^k = \mu \circ^k \mathcal{S} \circ^k \mu \circ^k \mathcal{S} \circ^k \mu$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) : Straightforward, because every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal of S is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal of S.

(4) $\Rightarrow$ (1) : Let Q be any quasi-ideal of S. Then by Corollary 323, χ_Q is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal of S. Thus by hypothesis

$$\begin{aligned} \chi_Q^k &= \left(\chi_Q \circ^k \mathcal{S} \circ^k \chi_Q \circ^k \mathcal{S} \circ^k \chi_Q\right) \\ &= \chi_{QSQSQ}^k. \end{aligned}$$

This implies that $Q = QSQSQ$. Hence by Theorem 7, S is regular. ■

Theorem 344 *The following assertions are equivalent for a ternary semigroup S :*

(1) *S is regular;*

(2) $\mu \wedge^k \lambda \leq \left(\mu \circ^k \mathcal{S} \circ^k \lambda\right) \wedge \left(\lambda \circ^k \mathcal{S} \circ^k \mu\right)$ *for all $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideals μ and λ of S;*

(3) $\mu \wedge^k \lambda \leq \left(\mu \circ^k \mathcal{S} \circ^k \lambda\right) \wedge \left(\lambda \circ^k \mathcal{S} \circ^k \mu\right)$ *for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal λ of S;*

(4) $\mu \wedge^k \lambda \leq \left(\mu \circ^k \mathcal{S} \circ^k \lambda\right) \wedge \left(\lambda \circ^k \mathcal{S} \circ^k \mu\right)$ *for all $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideals μ and λ of S;*

(5) $\mu \wedge^k \lambda \leq \left(\mu \circ^k \mathcal{S} \circ^k \lambda\right) \wedge \left(\lambda \circ^k \mathcal{S} \circ^k \mu\right)$ *for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal λ of S;*

(6) $\mu \wedge^k \lambda \leq \left(\mu \circ^k \mathcal{S} \circ^k \lambda\right) \wedge \left(\lambda \circ^k \mathcal{S} \circ^k \mu\right)$ *for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal λ of S;*

(7) $\mu \wedge^k \lambda \leq \left(\mu \circ^k \mathcal{S} \circ^k \lambda\right) \wedge \left(\lambda \circ^k \mathcal{S} \circ^k \mu\right)$ *for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal λ of S;*

(8) $\mu \wedge^k \lambda \leq \left(\mu \circ^k \mathcal{S} \circ^k \lambda\right) \wedge \left(\lambda \circ^k \mathcal{S} \circ^k \mu\right)$ *for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal λ of S.*

Proof. (1) $\Rightarrow$ (2) : Let μ and λ be $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideals of S. Since S is regular, so for any $s \in S$ there exists $x \in S$ such that $s = sxs$. It follows that

$$\begin{aligned} \left(\mu \circ^k \mathcal{S} \circ^k \lambda\right)(s) &= (\mu \circ \mathcal{S} \circ \lambda)(s) \vee \frac{1-k}{2} \\ &= \left\{\bigvee_{s=abc} \mu(a) \wedge \mathcal{S}(b) \wedge \lambda(c)\right\} \vee \frac{1-k}{2} \\ &\geq \{\mu(s) \wedge \mathcal{S}(x) \wedge \lambda(s)\} \vee \frac{1-k}{2} \\ &= (\mu \wedge \lambda)(s) \vee \frac{1-k}{2} = \left(\mu \wedge^k \lambda\right)(s) \end{aligned}$$

Thus $\mu \wedge^k \lambda \leq \mu \circ^k \mathcal{S} \circ^k \lambda$. Also,

$$\begin{aligned}\left(\lambda \circ^k \mathcal{S} \circ^k \mu\right)(s) &= (\lambda \circ \mathcal{S} \circ \mu)(s) \vee \frac{1-k}{2} \\ &= \left\{\bigvee_{s=lmn} (\lambda(l) \wedge \mathcal{S}(m) \wedge \mu(n)\right\} \vee \frac{1-k}{2} \\ &\geq \{\lambda(s) \wedge \mathcal{S}(x) \wedge \mu(s)\} \vee \frac{1-k}{2} = \left(\mu \wedge^k \lambda\right)(s).\end{aligned}$$

Thus $\mu \wedge^k \lambda \leq \lambda \circ^k \mathcal{S} \circ^k \mu$. Hence $\mu \wedge^k \lambda \leq \left(\mu \circ^k \mathcal{S} \circ^k \lambda\right) \wedge \left(\lambda \circ^k \mathcal{S} \circ^k \mu\right)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) : are straightforward.

(8) $\Rightarrow$ (1) : Let $\mu \wedge^k \lambda \leq \left(\mu \circ^k \mathcal{S} \circ^k \lambda\right) \wedge \left(\lambda \circ^k \mathcal{S} \circ^k \mu\right)$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal λ of S. This implies that $\mu \wedge^k \lambda \leq \left(\mu \circ^k \mathcal{S} \circ^k \lambda\right) \wedge \left(\lambda \circ^k \mathcal{S} \circ^k \mu\right) \leq \mu \circ^k \mathcal{S} \circ^k \lambda$ and by Corollary 340, $\mu \circ^k \mathcal{S} \circ^k \lambda \leq \mu \wedge^k \lambda$. Hence, $\mu \wedge^k \lambda = \mu \circ^k \mathcal{S} \circ^k \lambda$. Therefore by Theorem 342, S is regular. ■

Theorem 345 *The following assertions are equivalent for a ternary semigroup S :*

(1) *S is regular;*

(2) *$\mu \wedge^k \lambda \leq \mu \circ^k \mathcal{S} \circ^k \lambda$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized bi-ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal λ of S;*

(3) *$\mu \wedge^k \lambda \leq \mu \circ^k \mathcal{S} \circ^k \lambda$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal λ of S;*

(4) *$\mu \wedge^k \lambda \leq \mu \circ^k \mathcal{S} \circ^k \lambda$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left ideal λ of S;*

(5) *$\mu \wedge^k \lambda \leq \mu \circ^k \mathcal{S} \circ^k \lambda$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy bi-ideal λ of S;*

(6) *$\mu \wedge^k \lambda \leq \mu \circ^k \mathcal{S} \circ^k \lambda$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal λ of S;*

(7) *$\mu \wedge^k \lambda \leq \mu \circ^k \mathcal{S} \circ^k \lambda$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal λ of S;*

(8) *$\mu \wedge^k \lambda \leq \mu \circ^k \mathcal{S} \circ^k \lambda$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy quasi-ideal λ of S;*

(9) *$\mu \wedge^k \lambda \leq \mu \circ^k \mathcal{S} \circ^k \lambda$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy generalized-ideal λ of S.*

Proof. The proof is similar to the proof of Theorem 344. ■

5.6 Weakly regular ternary semigroups

This section is devoted to the study of weakly regular ternary semigroups.

Theorem 346 *The following assertions are equivalent for a ternary semigroup S :*

(1) S is right weakly regular;

(2) $\mu \wedge^k \lambda = \mu \circ^k \lambda \circ^k \lambda$ for every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal μ and every $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy two sided ideal λ of S.

Proof. (1) $\Rightarrow$ (2) : Let μ be an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal, λ an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy two sided ideal of S and $s \in S$. Then

$$\begin{aligned}
\left(\mu \circ^k \lambda \circ^k \lambda\right)(s) &= (\mu \circ \lambda \circ \lambda)(s) \vee \frac{1-k}{2} \\
&= \left\{\bigvee_{s=pqr} \mu(p) \wedge \lambda(q) \wedge \lambda(r)\right\} \vee \frac{1-k}{2} \\
&\leq \left\{\bigvee_{s=pqr} \left\{\mu(pqr) \vee \frac{1-k}{2}\right\} \wedge \lambda(q) \wedge \left\{\lambda(pqr) \vee \frac{1-k}{2}\right\}\right\} \vee \frac{1-k}{2} \\
&\leq \bigvee_{s=pqr} \{\mu(pqr) \wedge \lambda(pqr)\} \vee \frac{1-k}{2} \\
&\leq (\mu(s) \wedge \lambda(s)) \vee \frac{1-k}{2} = \left(\mu \wedge^k \lambda\right)(s).
\end{aligned}$$

Thus $\mu \circ^k \lambda \circ^k \lambda \leq \mu \wedge^k \lambda$. On the other hand, since S is right weakly regular, so for any $s \in S$, there exist $x_1, x_2, x_3, y_1, y_2, y_3 \in S$ such that $s = (sx_1y_1)(sx_2y_2)(sx_3y_3)$. It follows that

$$\begin{aligned}
\left(\mu \wedge^k \lambda\right)(s) &= (\mu \wedge \lambda)(s) \vee \frac{1-k}{2} \\
&= \{\mu(s) \wedge \lambda(s) \wedge \lambda(s)\} \vee \frac{1-k}{2} \\
&\leq \{\mu(sx_1y_1) \wedge \lambda(sx_2y_2) \wedge \lambda(sx_3y_3)\} \vee \frac{1-k}{2} \\
&\leq \left\{\bigvee_{s=xyz} (\mu(x) \wedge \lambda(y) \wedge \lambda(z))\right\} \vee \frac{1-k}{2} \\
&= (\mu \circ \lambda \circ \lambda)(s) \vee \frac{1-k}{2} = \left(\mu \circ^k \lambda \circ^k \lambda\right)(s).
\end{aligned}$$

Thus $\mu \wedge^k \lambda \leq \mu \circ^k \lambda \circ^k \lambda$. Hence $\mu \wedge^k \lambda = \mu \circ^k \lambda \circ^k \lambda$.

(2) $\Rightarrow$ (1) : Let R be a right ideal and I a two sided ideal of S. Then by Corollary 317, χ_R and χ_I are $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy right ideal and $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy two sided ideal of S, respectively. Thus by hypothesis

$$\begin{aligned}
\left(\chi_R \wedge^k \chi_I\right) &= \left(\chi_R \circ^k \chi_I \circ^k \chi_I\right) \\
\chi^k_{R\cap I} &= \chi^k_{RII}.
\end{aligned}$$

Thus $R \cap I = RII$. Hence by Lemma 9, S is right weakly regular. ■

Theorem 347 *For a ternary semigroup S, the following assertions are equivalent:*

(1) *S is right weakly regular;*

(2) $\left(\mu\wedge^k\lambda\wedge^k\nu\right)\leq\left(\mu\circ^k\lambda\circ^k\nu\right)$ *for every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy bi-ideal μ, every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy two sided ideal λ and every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right ideal ν of S;*

(3) $\left(\mu\wedge^k\lambda\wedge^k\nu\right)\leq\left(\mu\circ^k\lambda\circ^k\nu\right)$ *for every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideal μ, every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy two sided ideal λ and every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right ideal ν of S.*

Proof. (1) $\Rightarrow$ (2) : Let μ be an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy bi-ideal, λ an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy two sided ideal and ν an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right ideal of S. Since S is right weakly regular, so for any $s\in S$, there exist $x_1,x_2,x_3,y_1,y_2,y_3\in S$ such that $s=(sx_1y_1)(sx_2y_2)(sx_3y_3)=s(x_1y_1sx_2y_2)(sx_3y_3)$. It follows that

$$\begin{aligned}\left(\mu\wedge^k\lambda\wedge^k\nu\right)(s) &= (\mu\wedge\lambda\wedge\nu)(s)\vee\frac{1-k}{2}\\ &= \{\mu(s)\wedge\lambda(s)\wedge\nu(s)\}\vee\frac{1-k}{2}\\ &\leq \left\{\mu(s)\wedge\left\{\lambda(x_1y_1sx_2y_2)\vee\frac{1-k}{2}\right\}\wedge\left\{\nu(sx_3y_3)\vee\frac{1-k}{2}\right\}\right\}\\ &\quad\vee\frac{1-k}{2}\\ &= \left\{\bigvee_{s=pqr}(\mu(p)\wedge\lambda(q)\wedge\nu(r))\right\}\vee\frac{1-k}{2}\\ &= (\mu\circ\lambda\circ\nu)(s)\vee\frac{1-k}{2}=\left(\mu\circ^k\lambda\circ^k\nu\right)(s).\end{aligned}$$

Thus $\mu\wedge^k\lambda\wedge^k\nu\leq\mu\circ^k\lambda\circ^k\nu$.

(2) $\Rightarrow$ (3) : Straightforward, because every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideal is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy bi-ideal of S.

(3) $\Rightarrow$ (1) : Let μ be an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right ideal and λ an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy two sided ideal of S. Take $\nu=\lambda$. Since every $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy right ideal is an $(\overline{\in},\overline{\in}\vee\overline{q_k})$-fuzzy quasi-ideal. Thus by hypothesis $\mu\wedge^k\lambda\wedge^k\lambda\leq\mu\circ^k\lambda\circ^k\lambda$. This implies that $\mu\wedge^k\lambda\leq\mu\circ^k\lambda\circ^k\lambda$. But $\mu\circ^k\lambda\circ^k\lambda\leq\mu\wedge^k\lambda$ is straightforward. Hence $\mu\wedge^k\lambda=\mu\circ^k\lambda\circ^k\lambda$. Therefore by Theorem 346, S is right weakly regular. ∎

Chapter 6

$\left(\in_\gamma, \in_\gamma \vee q_\delta\right)$-fuzzy ternary subsemigroups and ideals in ternary semigroups

The aim of this chapter is to study $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroups, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right, lateral) ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi- (generalized bi-) ideals of ternary semigroups. The characterizations of regular and right weakly regular ternary semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right, lateral) ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quais-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi- (generalized bi-) ideals are also established.

6.1 $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroups

Let γ, $\delta \in [0,1]$ be such that $\gamma < \delta$, and α', $\beta' \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$ with $\alpha' \neq \in_\gamma \wedge q_\delta$. For a fuzzy point x_t and a fuzzy set μ in S, we say that:

(1) $x_t \in_\gamma \mu$ if $\mu(x) \geq t > \gamma$

(2) $x_t q_\delta \mu$ if $\mu(x) + t > 2\delta$

(3) $x_t \in_\gamma \vee q_\delta \mu$ if $x_t \in_\gamma \mu$ or $x_t q_\delta \mu$

(4) $x_t \in_\gamma \wedge q_\delta \mu$ if $x_t \in_\gamma \mu$ and $x_t q_\delta \mu$

(5) $x_t \overline{\alpha} \mu$ means that $x_t \alpha \mu$ does not hold.

Definition 348 *A fuzzy set μ in S is said to be an $\left(\alpha', \beta'\right)$-fuzzy ternary subsemigroup of S, where $\alpha' \neq \in_\gamma \wedge q_\delta$, if μ satisfies the following condition:*

$$x_t \alpha' \mu,\ y_r \alpha' \mu \text{ and } z_s \alpha' \mu \text{ imply } (xyz)_{\min\{t,r,s\}} \beta' \mu \tag{6.1}$$

for all $x, y, z \in S$ and $t, r, s \in (0,1]$.

The case $\alpha' = \in_\gamma \wedge q_\delta$ is omitted because for a fuzzy set μ in S such that $\mu(x) \leq \delta$ for any $x \in S$, in this case $x_t \in_\gamma \wedge q_\delta \mu$, implies $\mu(x) \geq t > \gamma$ and $\mu(x) + t > 2\delta$. Thus $\mu(x) + \mu(x) > \mu(x) + t > 2\delta$, which implies $\mu(x) > \delta$. This means that $\{x_t : x_t \in_\gamma \wedge q_\delta \mu\} = \emptyset$.

Example 349 *Consider the ternary semigroup S of Example 16. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.43 & \text{if } x = -i, \\ 0.60 & \text{if } x = 0, \\ 0.35 & \text{if } x = i. \end{cases}$$

Then routine calculations show that μ is an $(\in_{0.5}, \in_{0.5} \vee q_{0.7})$-fuzzy ternary subsemigroup of S.

Theorem 350 *Every $(\in, \in)$-fuzzy ternary subsemigroup of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S.*

Proof. Straightforward. ■

Remark 351 *The converse of Theorem 350 may not be true in general. In fact, the $(\in_{0.5}, \in_{0.5} \vee q_{0.7})$-fuzzy ternary subsemigroup of S given in Example 349 is not an $(\in, \in)$-fuzzy ternary subsemigroup of S, since $(-i)_{0.4} \in \mu$, but*

$$((-i)(-i)(-i))_{0.4} = i_{0.4} \overline{\in} \mu.$$

Theorem 352 *Let $2\delta = 1 + \gamma$ and μ be an (α', β')-fuzzy ternary subsemigroup of S. Then the set*

$$S_\gamma := \{x \in S : \mu(x) > \gamma\},$$

is a ternary subsemigroup of S.

Proof. Suppose $x, y, z \in S_\gamma$. Then $\mu(x) > \gamma$, $\mu(y) > \gamma$ and $\mu(z) > \gamma$. Assume that $\mu(xyz) \leq \gamma$. If $\alpha' \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$, then $x_{\mu(x)} \alpha' \mu$, $y_{\mu(y)} \alpha' \mu$, and $z_{\mu(z)} \alpha' \mu$ but $\mu(xyz) \leq \gamma < \min\{\mu(x), \mu(y), \mu(z)\}$ and $\mu(xyz) + \min\{\mu(x), \mu(y), \mu(z)\} \leq \gamma + 1 = 2\delta$. This implies that $(xyz)_{\min\{\mu(x), \mu(y), \mu(z)\}} \overline{\beta'} \mu$ for every $\beta' \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, which is a contradiction. Hence $\mu(xyz) > \gamma$, that is, $xyz \in S_\gamma$. Also $\mu(x) + 1 > \gamma + 1 = 2\delta$, $\mu(y) + 1 > \gamma + 1 = 2\delta$ and $\mu(z) + 1 > \gamma + 1 = 2\delta$. This implies that $x_1 q_\delta \mu$, $y_1 q_\delta \mu$ and $z_1 q_\delta \mu$, but $\mu(xyz) \leq \gamma$ so $(xyz)_1 \overline{\in}_\gamma \mu$ and $\mu(xyz) + 1 \leq \gamma + 1 = 2\delta$, so $(xyz)_1 \overline{q}_\delta \mu$. This is a contradiction. Hence $\mu(xyz) > \gamma$, that is, $xyz \in S_\gamma$. Therefore S_γ is a ternary subsemigroup of S. ■

Theorem 353 *Let $2\delta = 1+\gamma$ and A be a nonempty subset of S. Then A is a ternary subsemigroup of S if and only if the fuzzy set μ in S defined by:*

$$\mu(x) = \begin{cases} \leq \gamma & \text{for all } x \in S \setminus A, \\ \geq \delta & \text{for all } x \in A, \end{cases}$$

is an $(\alpha', \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S, where $\alpha' \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta\}$.

Proof. Assume that A is a ternary subsemigroup of S.

(a) In this part we show that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S. Let $x, y, z \in S$ and $t, r, s \in (\gamma, 1]$ be such that $x_t \in_\gamma \mu$, $y_r \in_\gamma \mu$ and $z_s \in_\gamma \mu$. Then $\mu(x) \geq t > \gamma$, $\mu(y) \geq r > \gamma$ and $\mu(z) \geq s > \gamma$. Thus $x, y, z \in A$, and so $xyz \in A$, that is, $\mu(xyz) \geq \delta$. If $\min\{t, r, s\} \leq \delta$, then $\mu(xyz) \geq \delta \geq \min\{t, r, s\} > \gamma$, implies that $(xyz)_{\min\{t,r,s\}} \in_\gamma \mu$. If $\min\{t, r, s\} > \delta$, then $\mu(xyz) + \min\{t, r, s\} > \delta + \delta = 2\delta$, so $(xyz)_{\min\{t,r,s\}}\, q_\delta \mu$. Thus $(xyz)_{\min\{t,r,s\}} \in_\gamma \vee q_\delta \mu$. Hence μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S.

(b) In this part we show that μ is a $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S. Let $x, y, z \in S$ and $t, r, s \in (\gamma, 1]$ be such that $x_t q_\delta \mu$, $y_r q_\delta \mu$ and $z_s q_\delta \mu$. Then $\mu(x) + t > 2\delta$, $\mu(y) + r > 2\delta$ and $\mu(z) + s > 2\delta$. This implies that $\mu(x) > 2\delta - t \geq 2\delta - 1 = \gamma$, also $\mu(y) > 2\delta - r \geq 2\delta - 1 = \gamma$ and $\mu(z) > 2\delta - s \geq 2\delta - 1 = \gamma$. Thus $x, y, z \in A$ and so $xyz \in A$. This implies that $\mu(xyz) \geq \delta$. If $\min\{t, r, s\} \leq \delta$, then $\mu(xyz) \geq \delta \geq \min\{t, r, s\} > \gamma$. Thus $(xyz)_{\min\{t,r,s\}} \in_\gamma \mu$. If $\min\{t, r, s\} > \delta$, then $\mu(xyz) + \min\{t, r, s\} > \delta + \delta = 2\delta$, and so that $(xyz)_{\min\{t,r,s\}}\, q_\delta \mu$. Hence μ is a $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S.

(c) In this part we show that μ is an $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S. Let $x, y, z \in S$ and $t, r, s \in (\gamma, 1]$ be such that $x_t \in_\gamma \vee q_\delta \mu$, $y_r \in_\gamma \vee q_\delta \mu$ and $z_s \in_\gamma \vee q_\delta \mu$. Then $x_t \in_\gamma \mu$ or $x_t q_\delta \mu$, $y_r \in_\gamma \mu$ or $y_r q_\delta \mu$ and $z_s \in_\gamma \mu$ or $z_s q_\delta \mu$. There are eight possible cases, each case implies that $x, y, z \in A$ and so $xyz \in A$. Analogous to (a) and (b) we obtain $(xyz)_{\min\{t,r,s\}} \in_\gamma \vee q_\delta \mu$. Hence μ is an $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S.

Conversely, suppose that μ is an $(\alpha', \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S, for $\alpha' \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta\}$. Then $A = S_\gamma$. Hence by Theorem 352, A is a ternary subsemigroup of S. ■

Corollary 354 *Let $2\delta = 1+\gamma$ and $\alpha' \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta\}$. Then a nonempty subset A of S is a ternary subsemigroup of S if and only if the characteristic function χ_A of A is an $(\alpha', \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S.*

Theorem 355 *Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S.*

Proof. Assume that μ is a $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S. Let $x, y, z \in S$ and $t, r, s \in (\gamma, 1]$ be such that $x_t \in_\gamma \mu$, $y_r \in_\gamma \mu$ and $z_s \in_\gamma \mu$. Then $\mu(x) \geq t > \gamma$, $\mu(y) \geq r > \gamma$ and $\mu(z) \geq s > \gamma$. If possible, let $(xyz)_{\min\{t,r,s\}} \overline{\in_\gamma \vee q_\delta} \mu$, then $\mu(xyz) < \min\{t, r, s\}$ and $\mu(xyz) + \min\{t, r, s\} \leq 2\delta$. This implies that $\mu(xyz) + \mu(xyz) < \mu(xyz) + \min\{t, r, s\} \leq 2\delta$. Thus $\mu(xyz) < \delta$. It follows that $\max\{\mu(xyz), \gamma\} < \min\{\mu(x), \mu(y), \mu(z), \delta\}$. Choose $t_1 \in (\gamma, 1]$ such that $2\delta - \max\{\mu(xyz), \gamma\} \geq t_1 > 2\delta - \min\{\mu(x), \mu(y), \mu(z), \delta\}$, which implies

$$\begin{aligned} 2\delta - \max\{\mu(xyz), \gamma\} &= \min\{2\delta - \mu(xyz), 2\delta - \gamma\} \geq t_1 \\ &> \max\{2\delta - \mu(x), 2\delta - \mu(y), 2\delta - \mu(z), \delta\}. \end{aligned}$$

It follows that $t_1 > 2\delta - \mu(x)$, $t_1 > 2\delta - \mu(y)$, $t_1 > 2\delta - \mu(z)$. Thus $\mu(x) + t_1 > 2\delta$, $\mu(y) + t_1 > 2\delta$, $\mu(z) + t_1 > 2\delta$ and $2\delta - \mu(xyz) > t_1$. This implies that $\mu(xyz) + t_1 < 2\delta$ and $\mu(xyz) < \delta < t_1$. Thus $x_{t_1} q_\delta \mu$, $y_{t_1} q_\delta \mu$ and $z_{t_1} q_\delta \mu$ but $(xyz)_{t_1} \overline{\in_\gamma \vee q_\delta} \mu$, which is a contradiction. Hence μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S. ■

Theorem 356 *Every $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S.*

Proof. The proof follows from the fact that if $x_t \in_\gamma \mu$, then $x_t \in_\gamma \vee q_\delta \mu$. ■

Theorem 357 *Every $(\in_\gamma, \in_\gamma)$-fuzzy ternary subsemigroup of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S.*

Proof. Starightforward. ■

Theorem 358 *Let μ be a fuzzy set in S. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S if and only if*

$$\max\{\mu(xyz),\ \gamma\} \geq \min\{\mu(x),\ \mu(y),\ \mu(z),\ \delta\} \tag{6.2}$$

for all $x, y, z \in S$.

Proof. Assume that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S. If there exist $x, y, z \in S$ such that

$$\max\{\mu(xyz), \gamma\} < \min\{\mu(x), \mu(y), \mu(z), \delta\},$$

then we can choose $t \in (\gamma, 1]$ such that $\max\{\mu(xyz), \gamma\} < t \leq \min\{\mu(x), \mu(y), \mu(z), \delta\}$. This implies that $x_t \in_\gamma \mu$, $y_t \in_\gamma \mu$ and $z_t \in_\gamma \mu$, but $(xyz)_t \overline{\in_\gamma} \mu$ and $\mu(xyz) + t < \delta + \delta = 2\delta$. Thus $(xyz)_t \overline{\in_\gamma \vee q_\delta} \mu$, which is a contradiction. Hence $\max\{\mu(xyz), \gamma\} \geq \min\{\mu(x), \mu(y), \mu(z), \delta\}$.

Conversely, suppose that $\max\{\mu(xyz), \gamma\} \geq \min\{\mu(x), \mu(y), \mu(z), \delta\}$. Let $x_t \in_\gamma \mu$, $y_r \in_\gamma \mu$ and $z_s \in_\gamma \mu$. Then $\mu(x) \geq t > \gamma, \mu(y) \geq r > \gamma$ and $\mu(z) \geq s > \gamma$. It follows that

$$\begin{aligned}\max\{\mu(xyz), \gamma\} &\geq \min\{\mu(x),\ \mu(y),\ \mu(z),\ \delta\}\\ &\geq \min\{t, r, s,\ \delta\} = \min\{\min\{t, r, s\},\ \delta\}.\end{aligned}$$

If $\min\{t, r, s\} \leq \delta$, then $\max\{\mu(xyz), \gamma\} \geq \min\{t, r, s\}$. But $\min\{t, r, s\} > \gamma$, so $\mu(xyz) \geq \min\{t, r, s\} > \gamma$. Thus $(xyz)_{\min\{t,r,s\}} \in_\gamma \mu$. If $\min\{t, r, s\} > \delta$, then $\max\{\mu(xyz), \gamma\} \geq \delta$. But $\gamma < \delta$, so $\mu(xyz) \geq \delta$. Thus $\mu(xyz) + \min\{t, r, s\} > \delta + \delta = 2\delta$. Hence $(xyz)_{\min\{t,r,s\}}\, q_\delta \mu$. Therefore μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S. ■

Taking $\gamma = 0$ and $\delta = 0.5$ in Theorem 358, we get the following corollary.

Corollary 359 *A fuzzy set μ in S is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S if and only if $\mu(xyz) \geq \min\{\mu(x), \mu(y), \mu(z), 0.5\}$, for all $x, y, z \in S$.*

The next theorem provides the relationship between $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup and the crisp ternary subsemigroup of S.

Theorem 360 *Let μ be a fuzzy set in S. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S if and only if the nonempty level set $U(\mu; t)$ is a ternary subsemigroup of S for all $t \in (\gamma, \delta]$.*

Proof. Assume that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S and $x, y, z \in U(\mu; t)$ for some $t \in (\gamma, \delta]$. Then $\mu(x) \geq t > \gamma$, $\mu(y) \geq t > \gamma$ and $\mu(z) \geq t > \gamma$. By hypothesis it follows that

$$\max\{\mu(xyz),\ \gamma\} \geq \min\{\mu(x),\ \mu(y),\ \mu(z),\ \delta\} \geq \min\{t, \delta\} = t.$$

This implies that $\max\{\mu(xyz), \gamma\} \geq t$. But $t > \gamma$, so $\mu(xyz) \geq t$ and $xyz \in U(\mu; t)$. Hence $U(\mu; t)$ is a ternary subsemigroup of S.

Conversely, suppose that $U(\mu; t)$ is a ternary subsemigroup of S for all $t \in (\gamma, \delta]$. Suppose there exist $x, y, z \in S$ such that

$$\max\{\mu(xyz),\ \gamma\} < \min\{\mu(x),\ \mu(y),\ \mu(z),\ \delta\}.$$

Then we can choose $t \in (\gamma, \delta]$ such that

$$\max\{\mu(xyz), \gamma\} < t \leq \min\{\mu(x),\ \mu(y),\ \mu(z),\ \delta\}.$$

Thus $x, y, z \in U(\mu; t)$ but, $xyz \notin U(\mu; t)$, which is a contradiction. Hence

$$\max\{\mu(xyz),\ \gamma\} \geq \min\{\mu(x),\ \mu(y),\ \mu(z),\ \delta\}.$$

Therefore μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S. ■

Theorem 361 *Let* $\{\mu_i \mid i \in \Lambda\}$ *be a family of* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy ternary subsemigroups of* S*. Then* $\mu := \bigcap\limits_{i\in\Lambda} \mu_i$ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy ternary subsemigroup of* S*.*

Proof. Suppose that $\{\mu_i \mid i \in \Lambda\}$ is a family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroups of S and x, y, $z \in S$. Then

$$\begin{aligned}
\mu(xyz) \vee \gamma &= \left(\bigcap_{i\in\Lambda} \mu_i\right)(xyz) \vee \gamma \\
&= \bigwedge_{i\in\Lambda} \mu_i(xyz) \vee \gamma \\
&= \bigwedge_{i\in\Lambda} (\mu_i(xyz) \vee \gamma).
\end{aligned}$$

Since each μ_i is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroups of S. It follows that

$$\begin{aligned}
\mu(xyz) \vee \gamma &\geq \bigwedge_{i\in\Lambda} \{\mu_i(x) \wedge \mu_i(y) \wedge \mu_i(z) \wedge \delta\} \\
&= \min\left\{\bigwedge_{i\in\Lambda} \mu_i(x),\ \bigwedge_{i\in\Lambda} \mu_i(y),\ \bigwedge_{i\in\Lambda} \mu_i(z), \delta\right\} \\
&= \min\left\{\left(\bigcap_{i\in\Lambda} \mu_i\right)(x),\ \left(\bigcap_{i\in\Lambda} \mu_i\right)(y),\ \left(\bigcap_{i\in\Lambda} \mu_i\right)(z), \delta\right\} \\
&= \min\{\mu(x),\ \mu(y),\ \mu(z), \delta\}.
\end{aligned}$$

Hence $\mu := \bigcap\limits_{i\in\Lambda} \mu_i$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S. ■

Remark 362 *For any* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy ternary subsemigroups* μ *of* S*, we can conclude that:*

(i) if $\gamma = 0$ *and* $\delta = 1$*, then* μ *is an* $(\in, \in)$*-fuzzy ternary subsemigroup of* S*,*

(ii) if $\gamma = 0$ *and* $\delta = 0.5$*, then* μ *is an* $(\in, \in \vee q)$*-fuzzy ternary subsemigroup of* S*, which is discussed in Chapter 2.*

(iii) if $\gamma = 0$ *and* $\delta = \frac{1-k}{2}$*, then* μ *is an* $(\in, \in \vee q_k)$*-fuzzy ternary subsemigroup of* S*, which is discussed in Chapter 3.*

(iv) if $\gamma = 0.5$ *and* $\delta = 1$*, then* μ *is an* $(\overline{\in}, \overline{\in} \vee \overline{q})$*-fuzzy ternary subsemigroup of* S*, which is discussed in Chapter 4.*

(v) if $\gamma = \frac{1-k}{2}$ *and* $\delta = 1$*, then* μ *is an* $(\overline{\in}, \overline{\in} \vee \overline{q_k})$*-fuzzy ternary subsemigroup of* S*, which is discussed in Chapter 5.*

For a fuzzy set μ in S, and γ, $\delta \in [0,1]$ with $\gamma < \delta$, consider the q_δ-set and $\in_\gamma \vee q_\delta$-set with respect to t (briefly, t-q_δ-set and t-$\in_\gamma \vee q_\delta$-set, respectively) as follows:

$$S^t_{q_\delta} := \{x \in S : x_t q_\delta \mu\} \text{ and } S^t_{\in_\gamma \vee q_\delta} := \{x \in S : x_t \in_\gamma \vee q_\delta \mu\},$$

for $t \in (\gamma, 1]$. Clearly, $S^t_{\in_\gamma \vee q_\delta} = S^t_{\in_\gamma} \cup S^t_{q_\delta}$.

Theorem 363 *Let $2\delta = 1 + \gamma$ and μ be a fuzzy set in S. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S if and only if $S^t_{q_\delta}$ is a ternary subsemigroup of S for all $t \in (\delta, 1]$, whenever it is nonempty.*

Proof. Assume that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S and $x, y, z \in S^t_{q_\delta}$. Then $x_t q_\delta \mu$, $y_t q_\delta \mu$, $z_t q_\delta \mu$. This implies that $\mu(x) + t > 2\delta$, $\mu(y) + t > 2\delta$, $\mu(z) + t > 2\delta$. This implies that $\mu(x) > 2\delta - t \geq 2\delta - 1 = \gamma$ and so $\mu(x) > \gamma$. Similarly, $\mu(y) > \gamma$ and $\mu(z) > \gamma$. By hypothesis it follows that

$$\max\{\mu(xyz), \gamma\} \geq \min\{\mu(x),\ \mu(y),\ \mu(z),\ \delta\} > \min\{2\delta - t, \delta\} = 2\delta - t,$$

which implies that $\mu(xyz) + t > 2\delta$. Thus $xyz \in S^t_{q_\delta}$. Hence $S^t_{q_\delta}$ is a ternary subsemigroup of S.

Conversely, suppose that $S^t_{q_\delta}$ is a ternary subsemigroup of S for all $t \in (\delta, 1]$. Let $x, y, z \in S$ be such that $\max\{\mu(xyz), \gamma\} < \min\{\mu(x),\ \mu(y),\ \mu(z),\ \delta\}$. Then $2\delta - \min\{\mu(x),\ \mu(y),\ \mu(z),\ \delta\} < 2\delta - \max\{\mu(xyz), \gamma\}$, which implies

$$\max\{2\delta - \mu(x), 2\delta - \mu(y), 2\delta - \mu(z), \delta\} < \min\{2\delta - \mu(xyz), 2\delta - \gamma\}.$$

Taking $r \in (\delta, 1]$ such that

$$\max\{2\delta - \mu(x), 2\delta - \mu(y), 2\delta - \mu(z), \delta\} < r \leq \min\{2\delta - \mu(xyz), 2\delta - \gamma\}.$$

Then $2\delta - \mu(x) < r, 2\delta - \mu(y) < r$, $2\delta - \mu(z) < r$, and $r \leq 2\delta - \mu(xyz)$. This implies that $\mu(x) + r > 2\delta$, $\mu(y) + r > 2\delta$, $\mu(z) + r > 2\delta$ and $\mu(xyz) + r \leq 2\delta$. This implies that $x_r q_\delta \mu$, $y_r q_\delta \mu$ and $z_r q_\delta \mu$ but $(xyz)_r\, \overline{q_\delta} \mu$, which is a contradiction. Thus $\max\{\mu(xyz), \gamma\} \geq \min\{\mu(x),\ \mu(y),\ \mu(z),\ \delta\}$. Hence μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S. ■

Theorem 364 *Let $2\delta = 1 + \gamma$ and μ be a fuzzy set in S. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S if and only if $S^t_{\in_\gamma \vee q_\delta}$ is a ternary subsemigroup of S for all $t \in (\gamma, 1]$, whenever it is nonempty.*

Proof. Assume that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S and $x, y, z \in S^t_{\in_\gamma \vee q_\delta}$. Then $x_t \in_\gamma \vee q_\delta \mu$, $y_t \in_\gamma \vee q_\delta \mu$ and $z_t \in_\gamma \vee q_\delta \mu$. This implies that $\mu(x) \geq t > \gamma$ or $\mu(x) + t > 2\delta$, $\mu(y) \geq t > \gamma$ or $\mu(y) + t > 2\delta$, and $\mu(z) \geq t > \gamma$ or $\mu(z) + t > 2\delta$. Thus $\mu(x) \geq t > \gamma$ or $\mu(x) > 2\delta - t \geq 2\delta - 1 = \gamma$. Similarly, $\mu(y) \geq t > \gamma$ or $\mu(y) > 2\delta - t \geq 2\delta - 1 = \gamma$ and $\mu(z) \geq t > \gamma$ or $\mu(z) > 2\delta - t \geq 2\delta - 1 = \gamma$. If $t \in (\gamma, \delta]$, then $2\delta - t \geq \delta \geq t$. By hypothesis it follows that

$$\max\{\mu(xyz), \gamma\} \geq \min\{\mu(x),\ \mu(y),\ \mu(z),\ \delta\} \geq \min\{t,\ \delta\} = t.$$

This implies that $\mu(xyz) \geq t > \gamma$. Thus $(xyz)_t \in_\gamma \mu$. Hence $xyz \in S^t_{\in_\gamma \vee q_\delta}$. If $t \in (\delta, 1]$, then $2\delta - t < \delta < t$ and so $\mu(x) > 2\delta - t$, $\mu(y) > 2\delta - t$ and $\mu(z) > 2\delta - t$. By hypothesis $\max\{\mu(xyz), \gamma\} \geq \min\{\mu(x), \ \mu(y), \ \mu(z), \ \delta\}$. It follows that

$$\mu(xyz) \geq \min\{\mu(x), \ \mu(y), \ \mu(z), \ \delta\} > \min\{2\delta - t, \ \delta\} = 2\delta - t.$$

Thus $\mu(xyz) + t > 2\delta$ and so $(xyz)_t \, q_\delta \mu$. Hence $xyz \in S^t_{\in_\gamma \vee q_\delta}$. Therefore $S^t_{\in_\gamma \vee q_\delta}$ is a ternary subsemigroup of S.

Conversely, suppose that $S^t_{\in_\gamma \vee q_\delta}$ is a ternary subsemigroup of S for all $t \in (\gamma, 1]$. Let $x, y, z \in S$ be such that $\max\{\mu(xyz), \gamma\} < \min\{\mu(x), \ \mu(y), \ \mu(z), \ \delta\}$. Then we can choose $t \in (\gamma, 1]$ such that $\max\{\mu(xyz), \gamma\} < t \leq \min\{\mu(x), \ \mu(y), \ \mu(z), \ \delta\}$. This implies that $x_t \in_\gamma \mu$, $y_t \in_\gamma \mu$ and $z_t \in_\gamma \mu$, but $(xyz)_t \overline{\in_\gamma \vee q_\delta} \mu$, which is a contradiction. Hence μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S. ■

As a direct consequence of Theorems 363 and 364 we have the following result.

Corollary 365 *Let $\gamma, \gamma', \delta, \delta' \in [0, 1]$ be such that $\gamma < \delta, \gamma' < \delta'$, $\gamma < \gamma'$ and $\delta' < \delta$. Then every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S is an $\left(\in_{\gamma'}, \in_{\gamma'} \vee q_{\delta'}\right)$-fuzzy ternary subsemigroup of S.*

6.2 $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals

We start this section with the following definition.

Definition 366 *A fuzzy set μ in S is said to be an $\left(\alpha', \beta'\right)$-fuzzy left (resp. right, lateral) ideal of S, where $\alpha' \neq \in_\gamma \wedge q_\delta$, if μ satisfies the following condition:*

$$z_t \alpha' \mu \text{ implies } (xyz)_t \, \beta' \mu \ \left(\text{resp. } (zxy)_t \, \beta' \mu, \ (xzy)_t \, \beta' \mu\right), \tag{6.3}$$

for all $x, y, z \in S$ and $t \in (\gamma, 1]$.

Example 367 *Consider the ternary semigroup S of Example 53. Define a fuzzy set μ in S as follows:*

$$\mu : S \rightarrow [0, 1], \quad x \mapsto \begin{cases} 0.8 & \text{if } x = a, \\ 0.1 & \text{if } x = b, \\ 0.2 & \text{if } x = c, \\ 0.7 & \text{if } x = d, \\ 0.5 & \text{if } x = e. \end{cases}$$

Routine calculations show that μ is an $(\in_{0.3}, \in_{0.3} \vee q_{0.8})$-fuzzy left ideal of S. But μ is neither an $(\in_{0.3}, \in_{0.3} \vee q_{0.8})$-fuzzy right ideal, nor an $(\in_{0.3}, \in_{0.3} \vee q_{0.8})$-fuzzy lateral ideal of S. Since $e_{0.31} \in_{0.3} \mu$, but

$$(ecc)_{0.31} \overline{\in_{0.3} \vee q_{0.8}} \mu,$$

and

$$(cec)_{0.31}\overline{\in_{0.3} \vee q_{0.8}}\mu.$$

Example 368 *Consider the ternary semigroup* $S = \{0, a, b, c, 1\}$ *of Example 54. Define a fuzzy set* μ *in* S *as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.7 & \text{if } x = 0, \\ 0.2 & \text{if } x = a, \\ 0.6 & \text{if } x = b, \\ 0.4 & \text{if } x = c, \\ 0.3 & \text{if } x = 1. \end{cases}$$

Routine calculations show that μ *is an* $(\in_{0.3}, \in_{0.3} \vee q_{0.7})$*-fuzzy right ideal of* S*. But* μ *is neither an* $(\in_{0.3}, \in_{0.3} \vee q_{0.7})$*-fuzzy left ideal, nor an* $(\in_{0.3}, \in_{0.3} \vee q_{0.7})$*-fuzzy lateral ideal of* S*. Since* $c_{0.32} \in_{0.3} \mu$*, but,*

$$(1ac)_{0.32}\overline{\in_{0.3} \vee q_{0.7}}\mu,$$

and

$$(ac1)_{0.32}\overline{\in_{0.3} \vee q_{0.7}}\mu.$$

Theorem 369 *Every* $(\in, \in)$*-fuzzy left (resp. right, lateral) ideal of* S *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left (resp. right, lateral) ideal of* S*;*

Proof. Straightforward. ■

Theorem 370 *Let* $2\delta = 1 + \gamma$ *and* μ *be an* (α', β')*-fuzzy left (resp. right, lateral) ideal of* S*. Then the set* S_γ *is a left (resp. right, lateral) ideal of* S*.*

Proof. The proof is similar to the proof of Theorem 352. ■

Theorem 371 *Let* $2\delta = 1 + \gamma$ *and* A *be a nonempty subset of* S*. Then* A *is a left (resp. right, lateral) ideal of* S *if and only if the fuzzy set* μ *in* S *defined by:*

$$\mu(x) = \begin{cases} \leq \gamma & \text{for all } x \in S \setminus A, \\ \geq \delta & \text{for all } x \in A, \end{cases}$$

is an $(\alpha', \in_\gamma \vee q_\delta)$*-fuzzy left (resp. right, lateral) ideal of* S*, where* $\alpha' \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta\}$.

Proof. The proof is similar to the proof of Theorem 353. ■

Corollary 372 *Let* $2\delta = 1 + \gamma$ *and* $\alpha' \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta\}$ *. Then a nonempty subset* A *of* S *is a left (resp. right, lateral) ideal of* S *if and only if the characteristic function* χ_A *of* A *is an* $(\alpha', \in_\gamma \vee q_\delta)$*-fuzzy left (resp. right, lateral) ideal of* S*.*

Corollary 373 *A nonempty subset L of S is a left (resp. right, lateral) ideal of S if and only if the characteristic function χ_L of L is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right, lateral) ideal of S.*

Theorem 374 *Every $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right, lateral) ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right, lateral) ideal of S.*

Proof. The proof is similar to the proof of Theorem 355. ■

Theorem 375 *Every $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right, lateral) ideal S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right, lateral) ideal of S.*

Proof. The proof follows from the fact that if $x_t \in_\gamma \mu$, then $x_t \in_\gamma \vee q_\delta \mu$. ■

Theorem 376 *Let μ be a a fuzzy in S. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right, lateral) ideal of S if and only if*

$$\max\{\mu(xyz),\ \gamma\} \geq \min\{\mu(z),\ \delta\} \tag{6.4}$$

$$(\textit{resp. } \max\{\mu(xyz),\ \gamma\} \geq \min\{\mu(x),\ \delta\},\ \ \max\{\mu(xyz),\ \gamma\} \geq \min\{\mu(y),\ \delta\})$$

for all $x, y, z \in S$.

Proof. The proof is similar to the proof of Theorem 358. ■

If we put $\gamma = 0$ and $\delta = 0.5$ in Theorem 376 we get the following corollary.

Corollary 377 *A fuzzy set μ in S is an $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideal of S if and only if*

$$\mu(xyz) \geq \mu(z) \wedge 0.5\,(\textit{resp. } \mu(xyz) \geq \mu(x) \wedge 0.5,\ \mu(xyz) \geq \mu(y) \wedge 0.5)$$

for all $x, y, z \in S$.

Theorem 378 *Let μ be a fuzzy set in S. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right, lateral) ideal of S if and only if the nonempty level set $U(\mu; t)$ is a left (resp. right, lateral) ideal of S for all $t \in (\gamma, \delta]$.*

Proof. The proof is similar to the proof of Theorem 360. ■

Remark 379 *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ternary subsemigroup of S but the converse is not true in general as seen in the following example.*

Example 380 *Consider the ternary semigroup S of Example 16. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.60 & \text{if } x = -i, \\ 0.40 & \text{if } x = 0, \\ 0.87 & \text{if } x = i. \end{cases}$$

It is now routine to verify that μ is an $(\in_{0.48}, \in_{0.48} \vee q_{0.59})$-fuzzy ternary subsemigroup but not an $(\in_{0.48}, \in_{0.48} \vee q_{0.59})$-fuzzy ideal of S.

Theorem 381 *Let $\{\mu_i \mid i \in \Lambda\}$ be a family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right, lateral) ideals of S. Then*

(i) $\mu := \bigcap\limits_{i\in\Lambda} \mu_i$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right, lateral) ideal of S.

(ii) $\mu := \bigcup\limits_{i\in\Lambda} \mu_i$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right, lateral) ideal of S.

Proof. We prove only (i). The other part follows in an analogus way. Suppose that $\{\mu_i \mid i \in \Lambda\}$ is a family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals of S and x, y, $z \in S$. Then

$$\begin{aligned} \mu(xyz) \vee \gamma &= \left(\bigcap_{i\in\Lambda} \mu_i\right)(xyz) \vee \gamma \\ &= \bigwedge_{i\in\Lambda} \mu_i(xyz) \vee \gamma \\ &= \bigwedge_{i\in\Lambda} \left(\mu_i(xyz) \vee \gamma\right). \end{aligned}$$

Since each μ_i is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. It follows that

$$\begin{aligned} \mu(xyz) \vee \gamma &\geq \bigwedge_{i\in\Lambda} \{\mu_i(z) \wedge \delta\} \\ &= \min\left\{\bigwedge_{i\in\Lambda} \mu_i(z), \delta\right\} \\ &= \min\left\{\left(\bigcap_{i\in\Lambda} \mu_i\right)(z), \delta\right\} \\ &= \min\{\mu(z), \delta\}. \end{aligned}$$

Hence $\mu := \bigcap\limits_{i\in\Lambda} \mu_i$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. ■

Theorem 382 *Let $2\delta = 1 + \gamma$ and μ be a fuzzy set in S. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right, lateral) ideal of S if and only if $S^t_{q_\delta}$ is a left (resp. right, lateral) ideal of S for all $t \in (\delta, 1]$, whenever it is nonempty.*

Proof. The proof is similar to the proof of Theorem 363. ■

Theorem 383 *Let $2\delta = 1+\gamma$ and μ be a fuzzy set in S. Then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right, lateral) ideal of S if and only if $S^t_{\in_\gamma \vee q_\delta}$ is a left (resp. right, lateral) ideal of S for all $t \in (\gamma, 1]$, whenever it is nonempty.*

Proof. The proof is similar to the proof of Theorem 364. ■

As a direct consequence of Theorems 382 and 383 we have the following result.

Corollary 384 *Let $\gamma, \gamma', \delta, \delta' \in [0,1]$ be such that $\gamma < \delta, \gamma' < \delta'$, $\gamma < \gamma'$ and $\delta' < \delta$. Then every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right, lateral) ideal of S is an $\left(\in_{\gamma'}, \in_{\gamma'} \vee q_{\delta'}\right)$-fuzzy left (resp. right, lateral) ideal of S.*

Remark 385 *For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right, lateral) ideal μ of S, we can conclude that:*

(i) if $\gamma = 0$ and $\delta = 1$, then μ is an $(\in, \in)$-fuzzy left (resp. right, lateral) ideal of S,

(ii) if $\gamma = 0$ and $\delta = 0.5$, then μ is an $(\in, \in \vee q)$-fuzzy left (resp. right, lateral) ideal of S, which is discussed in Chapter 2.

(iii) if $\gamma = 0$ and $\delta = \frac{1-k}{2}$, then μ is an $(\in, \in \vee q_k)$-fuzzy left (resp. right, lateral) ideal of S, which is discussed in Chapter 3.

(iv) if $\gamma = 0.5$ and $\delta = 1$, then μ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy left (resp. right, lateral) ideal of S, which is discussed in Chapter 4.

(v) if $\gamma = \frac{1-k}{2}$ and $\delta = 1$, then μ is an $(\overline{\in}, \overline{\in} \vee \overline{q_k})$-fuzzy left (resp. right, lateral) ideal of S, which is discussed in Chapter 5.

6.3 Fuzzy quasi-ideals of type $(\in_\gamma, \in_\gamma \vee q_\delta)$

Definition 386 *A fuzzy set μ in S is said to be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S if it satisfies the following conditions:*

$$(i) \max\{\mu(x), \gamma\} \geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x), \delta\}$$

$$(ii) \max\{\mu(x), \gamma\} \geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x),\ \delta\}$$

for all $x \in S$.

Lemma 387 *A nonempty subset Q of S is a quasi-ideal of S if and only if the characteristic function χ_Q of Q is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S.*

Proof. Assume that Q is a quasi-ideal of S, χ_Q the characteristic function of Q and $x \in S$. If $x \notin Q$, then $x \notin SSQ$, $x \notin SQS$ or $x \notin QSS$ and so $(\mathcal{S} \circ \mathcal{S} \circ C_Q)(x) = 0$, $(\mathcal{S} \circ C_Q \circ \mathcal{S})(x) = 0$ or $(C_Q \circ \mathcal{S} \circ \mathcal{S})(x) = 0$. It follows that

$$\min\left\{\left(\mathcal{S} \circ \mathcal{S} \circ \chi_Q\right)(x), \left(\mathcal{S} \circ \chi_Q \circ \mathcal{S}\right)(x), \left(\chi_Q \circ \mathcal{S} \circ \mathcal{S}\right)(x), \delta\right\} = 0 \leq \max\left\{\chi_Q(x), \gamma\right\}.$$

If $x \in Q$, then

$$\max\left\{\chi_Q(x), \gamma\right\} = 1 \geq \min\left\{\left(\mathcal{S} \circ \mathcal{S} \circ \chi_Q\right)(x), \left(\mathcal{S} \circ \chi_Q \circ \mathcal{S}\right)(x), \left(\chi_Q \circ \mathcal{S} \circ \mathcal{S}\right)(x), \delta\right\}.$$

Similarly,

$$\begin{aligned} \max\left\{\chi_Q(x), \gamma\right\} &= 1 \\ &\geq \min\left\{\begin{array}{c} \left(\mathcal{S} \circ \mathcal{S} \circ \chi_Q\right)(x), \left(\mathcal{S} \circ \mathcal{S} \circ \chi_Q \circ \mathcal{S} \circ \mathcal{S}\right)(x), \\ \left(\chi_Q \circ \mathcal{S} \circ \mathcal{S}\right)(x), \delta \end{array}\right\}. \end{aligned}$$

Hence χ_Q is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S.

Conversely, suppose that χ_Q is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S. Let $a \in SSQ \cap SQS \cap QSS$. Then $a \in SSQ$, $a \in SQS$ and $a \in QSS$. This implies that there exist $x, y, z \in Q$ and $s_1, s_2, s_{3,} t_1, t_2, t_3 \in S$ such that $a = s_1 t_1 x$ and $a = s_2 y t_2$ and $a = z s_3 t_3$. It follows that

$$\begin{aligned} \left(\chi_Q \circ \mathcal{S} \circ \mathcal{S}\right)(a) &= \bigvee_{a=pqr} \{C_Q(p) \wedge \mathcal{S}(q) \wedge \mathcal{S}(r)\} \\ &\geq \chi_Q(z) \wedge \mathcal{S}(s_3) \wedge \mathcal{S}(S_3) \\ &= \chi_Q(z) \wedge 1 \wedge 1 = \chi_Q(z) = 1. \end{aligned}$$

So $\left(\chi_Q \circ \mathcal{S} \circ \mathcal{S}\right)(a) = 1$. Similarly $\left(\mathcal{S} \circ \mathcal{S} \circ \chi_Q\right)(a) = 1$ and $\left(\mathcal{S} \circ \chi_Q \circ \mathcal{S}\right)(a) = 1$. It follows that

$$\begin{aligned} \max\left\{\chi_Q(a), \gamma\right\} &\geq \min\left\{\left(\mathcal{S} \circ \mathcal{S} \circ \chi_Q\right)(a), \left(\mathcal{S} \circ \mathcal{S} \circ \chi_Q \circ \mathcal{S} \circ \mathcal{S}\right)(a), \left(\chi_Q \circ \mathcal{S} \circ \mathcal{S}\right)(a), \delta\right\} \\ &= \delta. \end{aligned}$$

Thus $\chi_Q(a) = 1$. Hence $a \in Q$ and so $SSQ \cap SQS \cap QSS \subseteq Q$. Similarly we can show that $SSQ \cap SSQSS \cap QSS \subseteq Q$. Hence Q is a quasi-ideal of S. ■

Theorem 388 *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right (left, lateral) ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S.*

Proof. Assume that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S and $x \in S$. Then

$$(\mu \circ \mathcal{S} \circ \mathcal{S})(x) = \bigvee_{x=uvw} (\mu(u) \wedge \mathcal{S}(v) \wedge \mathcal{S}(w)) = \bigvee_{x=uvw} \mu(u).$$

It follows that

$$\begin{aligned} (\mu \circ \mathcal{S} \circ \mathcal{S})(x) \wedge \delta &= \left\{\bigvee_{x=uvw} \mu(u)\right\} \wedge \delta = \bigvee_{x=uvw} (\mu(u) \wedge \delta) \\ &\leq \bigvee_{x=uvw} \mu(uvw) \vee \gamma = \mu(x) \vee \gamma. \end{aligned}$$

Thus $\mu(x) \vee \gamma \geq (\mu \circ \mathcal{S} \circ \mathcal{S})(x) \wedge \delta$, which implies

$$\max\{\mu(x), \gamma\} \geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mu \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x),\ \delta\}.$$

Similarly,

$$\max\{\mu(x), \gamma\} \geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(x), (\mathcal{S} \circ \mathcal{S} \circ \mu)(x),\ \delta\}.$$

Hence μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S. ■

The converse of Theorem 388 may not be true in general, as seen in the following example.

Example 389 *Let* $a = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$, $d = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$, $e = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$. *Then* $S = \{a, b, c, d, e\}$ *is a ternary semigroup under ternary matrix multiplication. Define a fuzzy set* μ *in* S *as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.7 & \text{if } x = a, \\ 0.4 & \text{if } x = b, \\ 0.5 & \text{if } x = c, \\ 0.3 & \text{if } x = d, \\ 0.2 & \text{if } x = e. \end{cases}$$

Then

$$U(\mu; t) = \begin{cases} S & \text{if } t \in (0, 0.2], \\ \{a, b, c, d\} & \text{if } t \in (0.2, 0.3], \\ \{a, b, c\} & \text{if } t \in (0.3, 0.4], \\ \{a, c\} & \text{if } t \in (0.4, 0.5], \\ \{a\} & \text{if } t \in (0.5, 0.7], \\ \emptyset & \text{if } t \in (0.7, 1]. \end{cases}$$

Clearly, μ *is an* $(\in_{0.4}, \in_{0.4} \vee q_{0.7})$*-fuzzy quasi-ideal of* S *which is neither* $(\in_{0.4}, \in_{0.4} \vee q_{0.7})$*-fuzzy left ideal nor* $(\in_{0.4}, \in_{0.4} \vee q_{0.7})$*-fuzzy right ideal nor* $(\in_{0.4}, \in_{0.4} \vee q_{0.7})$*-fuzzy lateral ideal of* S*.*

6.4 $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals

Definition 390 *An* (α', β')*-fuzzy ternary subsemigroup* μ *of* S *is said to be an* (α', β')*-fuzzy bi-ideal of* S*, where* $\alpha' \neq \in_\gamma \wedge q_\delta$*, if* μ *satisfies the following condition:*

$$x_t \alpha' \mu,\ y_r \alpha' \mu \text{ and } z_s \alpha' \mu \text{ imply } (xuyvz)_{\min\{t,r,s\}}\, \beta' \mu \tag{6.5}$$

for all $u, v, x, y, z \in S$ *and* $t \in (\gamma, 1]$.

A fuzzy set μ *in* S *is called an* (α', β')*-fuzzy generalized bi-ideal of* S, *if it satisfies condition (6.5).*

Theorem 391 *Let* $2\delta = 1 + \gamma$ *and* μ *be an* (α', β')*-fuzzy bi-ideal (resp. generalized bi-ideal) of* S. *Then the set* S_γ *is a bi-ideal (resp. generalized bi-ideal) of* S.

Proof. The proof is similar to the proof of Theorem 352. ■

Theorem 392 *Let* $2\delta = 1 + \gamma$ *and* A *be a nonempty subset of* S. *Then* A *is a bi-ideal (resp. generalized bi-ideal) of* S *if and only if the fuzzy set* μ *in* S *defined by:*

$$\mu(x) = \begin{cases} \leq \gamma & \text{for all } x \in S \setminus A, \\ \geq \delta & \text{for all } x \in A, \end{cases}$$

is an $(\alpha', \in_\gamma \vee q_\delta)$*-fuzzy bi-ideal (resp. generalized bi-ideal) of* S, *where* $\alpha' \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta\}$.

Proof. The proof is similar to the proof of Theorem 353. ■

Corollary 393 *Let* $2\delta = 1 + \gamma$ *and* $\alpha' \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta\}$. *Then a nonempty subset* A *of* S *is a bi-ideal (resp. generalized bi-ideal) of* S *if and only if the characteristic function* χ_A *of* A *is an* $(\alpha', \in_\gamma \vee q_\delta)$*-fuzzy bi-ideal (resp. generalized bi-ideal) ideal of* S.

Theorem 394 *Every* $(q_\delta, \in_\gamma \vee q_\delta)$*-fuzzy bi-ideal (resp. generalized bi-ideal) of* S *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy bi-ideal (resp. generalized bi-ideal) of* S.

Proof. The proof is similar to the proof of Theorem 355. ■

Theorem 395 *Every* $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$*-fuzzy bi-ideal (resp. generalized bi-ideal) of* S *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy bi-ideal (resp. generalized bi-ideal) of* S.

Proof. The proof follows from the fact that if $x_t \in_\gamma \mu$, then $x_t \in_\gamma \vee q_\delta \mu$. ■

Theorem 396 *Let* $2\delta = 1 + \gamma$ *and* μ *be a fuzzy set in* S. *Then* μ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy bi-ideal (resp. generalized bi-ideal) of* S *if and only if* $S^t_{q_\delta}$ *is a bi-ideal (resp. generalized bi-ideal) of* S *for all* $t \in (\delta, 1]$, *whenever it is nonempty.*

Proof. The proof is similar to the proof of Theorem 363. ■

Theorem 397 *Let* $2\delta = 1 + \gamma$ *and* μ *be a fuzzy set in* S. *Then* μ *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy bi-ideal (resp. generalized bi-ideal) of* S *if and only if* $S^t_{\in_\gamma \vee q_\delta}$ *is a bi-ideal (resp. generalized bi-ideal) of* S *for all* $t \in (\gamma, 1]$, *whenever it is nonempty.*

Proof. The proof is similar to the proof of Theorem 364. ■

As a direct consequence of Theorems 396 and 397 we have the following result.

Corollary 398 *Let* $\gamma, \gamma', \delta, \delta' \in [0,1]$ *be such that* $\gamma < \delta, \gamma' < \delta'$, $\gamma < \gamma'$ *and* $\delta' < \delta$. *Then every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy bi-ideal (resp. generalized bi-ideal) of* S *is an* $\left(\in_{\gamma'}, \in_{\gamma'} \vee q_{\delta'}\right)$*-fuzzy bi-ideal (resp. generalized bi-ideal) of* S.

Theorem 399 *Every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy quasi-ideal of* S *is an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy bi-ideal of* S.

Proof. Assume that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S and $u, v, x, y, z \in S$. Then

$$\begin{aligned}
\max\{\mu(xyz), \gamma\} &\geq \min\{(\mu \circ \mathcal{S} \circ \mathcal{S})(xyz), (\mathcal{S} \circ \mu \circ \mathcal{S})(xyz), (\mathcal{S} \circ \mathcal{S} \circ \mu)(xyz), \delta\} \\
&= (\mu \circ \mathcal{S} \circ \mathcal{S})(xyz) \wedge (\mathcal{S} \circ \mu \circ \mathcal{S})(xyz) \wedge (\mathcal{S} \circ \mathcal{S} \circ \mu)(xyz) \wedge \delta \\
&= \left\{\bigvee_{xyz=uvw} \{\mu(u) \wedge \mathcal{S}(v) \wedge \mathcal{S}(w)\}\right\} \\
&\quad \wedge \left\{\bigvee_{xyz=pqr} \{\mathcal{S}(p) \wedge \mu(q) \wedge \mathcal{S}(r)\}\right\} \\
&\quad \wedge \left\{\bigvee_{xyz=abc} \{\mathcal{S}(a) \wedge \mathcal{S}(b) \wedge \mu(c)\}\right\} \wedge \delta \\
&\geq \{\mu(x) \wedge \mathcal{S}(y) \wedge \mathcal{S}(z)\} \\
&\quad \wedge \{\mathcal{S}(x) \wedge \mu(y) \wedge \mathcal{S}(z)\} \wedge \{\mathcal{S}(x) \wedge \mathcal{S}(y) \wedge \mu(z)\} \wedge \delta \\
&= \mu(x) \wedge \mu(y) \wedge \mu(z) \wedge \delta.
\end{aligned}$$

Thus $\max\{\mu(xyz), \gamma\} \geq \min\{\mu(x), \mu(y), \mu(z), \delta\}$. Also,

$$\max\{\mu(xuyvz), \gamma\} \geq \min\left\{\begin{array}{c}(\mu \circ \mathcal{S} \circ \mathcal{S})(xuyvz), (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(xuyvz), \\ (\mathcal{S} \circ \mathcal{S} \circ \mu)(xuyvz), \delta\end{array}\right\}$$

$$\begin{aligned}
\max\{\mu(xuyvz), \gamma\} &\geq (\mu \circ \mathcal{S} \circ \mathcal{S})(xuyvz) \wedge (\mathcal{S} \circ \mathcal{S} \circ \mu \circ \mathcal{S} \circ \mathcal{S})(xuyvz) \\
&\quad \wedge (\mathcal{S} \circ \mathcal{S} \circ \mu)(xuyvz) \wedge \delta \\
&= \left\{\bigvee_{xuyvz=s_1s_2s_3} \{\mu(s_1) \wedge \mathcal{S}(s_2) \wedge \mathcal{S}(s_3)\}\right\} \\
&\quad \wedge \left\{\bigvee_{xuyvz=s_4s_5s_6} \{(\mathcal{S} \circ \mathcal{S} \circ \mu)(s_4) \wedge \mathcal{S}(s_5) \wedge \mathcal{S}(s_6)\}\right\} \\
&\quad \wedge \left\{\bigvee_{xuyvz=r_1r_2r_3} \{\mathcal{S}(r_1) \wedge \mathcal{S}(r_2) \wedge \mu(r_3)\}\right\} \wedge \delta \\
&= \left\{\bigvee_{xuyvz=s_1s_2s_3} \{\mu(s_1) \wedge \mathcal{S}(s_2) \wedge \mathcal{S}(s_3)\}\right\} \\
&\quad \wedge \left\{\bigvee_{xuyvz=s_4s_5s_6} \{\bigvee_{s_4=abc} (\mathcal{S}(a) \wedge \mathcal{S}(b) \wedge \mu(c)) \wedge \mathcal{S}(s_5) \wedge \mathcal{S}(s_6)\}\right\} \\
&\quad \wedge \left\{\bigvee_{xuyvz=r_1r_2r_3} \{\mathcal{S}(r_1) \wedge \mathcal{S}(r_2) \wedge \mu(r_3)\}\right\} \wedge \delta
\end{aligned}$$

$$\begin{aligned}
&= \left\{\bigvee_{xuyvz=s_1s_2s_3} \{\mu(s_1) \wedge \mathcal{S}(s_2) \wedge \mathcal{S}(s_3)\}\right\} \\
&\quad \wedge \left\{\bigvee_{xuyvz=(abc)s_5s_6} (\mathcal{S}(a) \wedge \mathcal{S}(b) \wedge \mu(c)) \wedge \mathcal{S}(s_5) \wedge \mathcal{S}(s_6)\right\} \\
&\quad \wedge \left\{\bigvee_{xuyvz=r_1r_2r_3} \{\mathcal{S}(r_1) \wedge \mathcal{S}(r_2) \wedge \mu(r_3)\}\right\} \wedge \delta \\
&\geq \{\mu(x) \wedge \mathcal{S}(uyv) \wedge \mathcal{S}(z)\} \wedge \{\mathcal{S}(x) \wedge \mathcal{S}(u) \wedge \mu(y) \wedge \mathcal{S}(v) \wedge \mathcal{S}(z)\} \\
&\quad \wedge \{\mathcal{S}(x) \wedge \mathcal{S}(uyv) \wedge \mu(z)\} \wedge \delta \\
&= \mu(x) \wedge \mu(y) \wedge \mu(z) \wedge \delta.
\end{aligned}$$

This implies that $\max\{\mu(xuyvz), \gamma\} \geq \min\{\mu(x), \mu(y), \mu(z), \delta\}$. Hence μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S. ■

Definition 400 *Let μ, λ and ν be fuzzy sets in S. Define the fuzzy sets $(\mu)_\gamma^\delta, \mu \wedge_\gamma^\delta \lambda$, $\mu \vee_\gamma^\delta \lambda$ and $\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \nu$ in S as follows:*

(1) $(\mu)_\gamma^\delta(x) = (\mu(x) \vee \gamma) \wedge \delta$;

(2) $\left(\mu \wedge_\gamma^\delta \lambda\right)(x) = ((\mu(x) \wedge \lambda(x)) \vee \gamma) \wedge \delta$;

(3) $\left(\mu \vee_\gamma^\delta \lambda\right)(x) = ((\mu(x) \vee \lambda(x)) \vee \gamma) \wedge \delta$;

(4) $\left(\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \nu\right)(x) = ((\mu \circ \lambda \circ \nu)(x) \vee \gamma) \wedge \delta$

for all $x \in S$.

Lemma 401 *Let* μ, λ *and* ν *be fuzzy sets in* S. *Then the following hold:*
(1) $\left(\mu \wedge_\gamma^\delta \lambda\right) = (\mu)_\gamma^\delta \wedge (\lambda)_\gamma^\delta$
(2) $\left(\mu \vee_\gamma^\delta \lambda\right) = (\mu)_\gamma^\delta \vee (\lambda)_\gamma^\delta$
(3) $\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \nu = (\mu)_\gamma^\delta \circ (\lambda)_\gamma^\delta \circ (\nu)_\gamma^\delta$.

Proof. The proofs of (1) and (2) are obvious.

(3) Let $x \in S$. Then

$$\begin{aligned}
\left(\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \nu\right)(x) &= ((\mu \circ \lambda \circ \nu)(x) \vee \gamma) \wedge \delta \\
&= \left\{\left\{\bigvee_{x=pqr} (\mu(p) \wedge \lambda(q) \wedge \nu(r))\right\} \vee \gamma\right\} \wedge \delta \\
&= \left\{\bigvee_{x=pqr} \{(\mu(p) \vee \gamma) \wedge \delta\} \wedge \{(\lambda(q) \vee \gamma) \wedge \delta\} \wedge \{(\nu(r) \vee \gamma) \wedge \delta\}\right\} \\
&= \left\{\bigvee_{x=pqr} (\mu)_\gamma^\delta(p) \wedge (\lambda)_\gamma^\delta(q) \wedge (\nu)_\gamma^\delta(r)\right\} \\
&= \left((\mu)_\gamma^\delta \circ (\lambda)_\gamma^\delta \circ (\nu)_\gamma^\delta\right)(x).
\end{aligned}$$

Hence $\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \nu = (\mu)_\gamma^\delta \circ (\lambda)_\gamma^\delta \circ (\nu)_\gamma^\delta$. ■

Lemma 402 *Let* μ *be an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy right ideal,* λ *an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy lateral ideal and* ν *an* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy left ideal of* S. *Then* $\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \nu \leq \mu \wedge_\gamma^\delta \lambda \wedge_\gamma^\delta \nu$.

Proof. Assume that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal, λ an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy lateral ideal and ν an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S and $x \in S$. Then

$$\begin{aligned}
\left(\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \nu\right)(x) &= \{(\mu \circ \lambda \circ \nu)(x) \vee \gamma\} \wedge \delta \\
&= \left\{\left\{\bigvee_{x=uvw} (\mu(u) \wedge \lambda(v) \wedge \nu(w))\right\} \vee \gamma\right\} \wedge \delta \\
&= \left\{\left\{\bigvee_{x=uvw} (\mu(u) \wedge \delta) \wedge (\lambda(v) \wedge \delta) \wedge (\nu(w) \wedge \delta)\right\} \vee \gamma\right\} \wedge \delta \\
&\leq \left\{\left\{\bigvee_{x=uvw} (\mu(uvw) \vee \gamma) \wedge (\lambda(uvw) \vee \gamma) \wedge (\nu(uvw) \vee \gamma)\right\} \vee \gamma\right\} \wedge \delta \\
&= \left\{\left\{\bigvee_{x=uvw} \mu(uvw) \wedge \lambda(uvw) \wedge \nu(uvw)\right\} \vee \gamma\right\} \wedge \delta \\
&= \{(\mu(x) \wedge \lambda(x) \wedge \nu(x)) \vee \gamma\} \wedge \delta \\
&= \left(\mu \wedge_\gamma^\delta \lambda \wedge_\gamma^\delta \nu\right)(x).
\end{aligned}$$

Hence $\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \nu \leq \mu \wedge_\gamma^\delta \lambda \wedge_\gamma^\delta \nu$. ■

Lemma 403 *Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and λ an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. Then $\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda \leq \mu \wedge_\gamma^\delta \lambda$.*

Proof. Assume that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and λ an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S and $x \in S$. Then

$$\begin{aligned}
\left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda\right)(x) &= \{(\mu \circ \mathcal{S} \circ \lambda) \vee \gamma\} \wedge \delta \\
&= \left\{\bigvee_{x=uvw} (\mu(u) \wedge \mathcal{S}(v) \wedge \lambda(w)) \vee \gamma\right\} \wedge \delta \\
&= \left\{\left\{\bigvee_{x=uvw} (\mu(u) \vee \gamma) \wedge (\lambda(w) \vee \gamma)\right\} \vee \gamma\right\} \wedge \delta \\
&\leq \left\{\left\{\bigvee_{x=uvw} (\mu(uvw) \wedge \delta) \wedge (\lambda(uvw) \wedge \delta)\right\} \vee \gamma\right\} \wedge \delta \\
&= \{(\mu(x) \wedge \lambda(x)) \vee \gamma\} \wedge \delta = \left(\mu \wedge_\gamma^\delta \lambda\right)(x).
\end{aligned}$$

Hence $\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda \leq \mu \wedge_\gamma^\delta \lambda$. ■

The proof of the following theorem is straightforward and we omitt the detail.

Theorem 404 *A nonempty subset A of S is a left (resp. right, lateral) ideal of S if and only if $(\chi_A)_\gamma^\delta$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right, lateral) ideal of S, where χ_A is the characteristic function of A.*

6.5 Regular ternary semigroups

In this section we characterize regular ternary semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (right) ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals (generalized bi-ideals).

Theorem 405 *The following assertions are equivalent for a ternary semigroup S :*

(1) *S is regular;*

(2) *$\mu \wedge_\gamma^\delta \lambda \wedge_\gamma^\delta \nu = \mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \nu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal μ, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy lateral ideal λ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal ν of S.*

Proof. (1) $\Rightarrow$ (2) : Assume that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal, λ an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy lateral ideal and ν an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. Then by Lemma 402, $\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \nu \leq \mu \wedge_\gamma^\delta \lambda \wedge_\gamma^\delta \nu$. Now, since S is regular, so for any $x \in S$ there exists $a \in S$ such that $x = xax$. It follows that

$$\begin{aligned}\left(\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \nu\right)(x) &= \{(\mu \circ \lambda \circ \nu)(x) \vee \gamma\} \wedge \delta \\ &= \left\{\left\{\bigvee_{x=lmn} (\mu(l) \wedge \lambda(m) \wedge \nu(n))\right\} \vee \gamma\right\} \wedge \delta \\ &\geq \{(\mu(x) \wedge \lambda(axa) \wedge \nu(x)) \vee \gamma\} \wedge \delta \\ &= \{(\mu(x) \wedge (\lambda(axa) \vee \gamma) \wedge \nu(x)) \vee \gamma\} \wedge \delta \\ &\geq \{(\mu(x) \wedge \lambda(x) \wedge \nu(x)) \vee \gamma\} \wedge \delta \\ &= \left(\mu \wedge_\gamma^\delta \lambda \wedge_\gamma^\delta \nu\right)(x).\end{aligned}$$

Thus $\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \nu \geq \mu \wedge_\gamma^\delta \lambda \wedge_\gamma^\delta \nu$. Hence $\mu \wedge_\gamma^\delta \lambda \wedge_\gamma^\delta \nu = \mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \nu$.

(2) $\Rightarrow$ (1) : Let R, M and L be the right ideal, lateral ideal and left ideal of S, respectively. Then χ_R, χ_M and χ_L are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy lateral ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S, respectively. By hypothesis, it follows that

$$\begin{aligned}\chi_R \wedge_\gamma^\delta \chi_M \wedge_\gamma^\delta \chi_L &= \chi_R \circ_\gamma^\delta \chi_M \circ_\gamma^\delta \chi_L \\ (\chi_{R\cap M\cap L})_\gamma^\delta &= (\chi_{RML})_\gamma^\delta.\end{aligned}$$

Thus $R \cap M \cap L = RML$. Hence by Theorem 5, S is regular. ■

Theorem 406 *The following assertions are equivalent for a ternary semigroup S :*

(1) *S is regular;*

(2) *$\mu \wedge_\gamma^\delta \lambda = \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal λ of S.*

Proof. (1) $\Rightarrow$ (2) : Assume that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal, λ an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S and $x \in S$. Then by Lemma 403 $\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda \leq \mu \wedge_\gamma^\delta \lambda$.

Now, since S is regular, so for any $x \in S$, there exists $a \in S$ such that $x = xax = x(axa)x$. It follows that

$$\begin{aligned}\left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda\right)(x) &= \{(\mu \circ \mathcal{S} \circ \lambda)(x) \vee \gamma\} \wedge \delta \\ &= \left\{\left\{\bigvee_{x=uvw} (\mu(u) \wedge \mathcal{S}(v) \wedge \lambda(w))\right\} \vee \gamma\right\} \wedge \delta \\ &\geq \{(\mu(x) \wedge \mathcal{S}(axa) \wedge \lambda(x)) \vee \gamma\} \wedge \delta \\ &= \{(\mu(x) \wedge \lambda(x)) \vee \gamma\} \wedge \delta = \left(\mu \wedge_\gamma^\delta \lambda\right)(x).\end{aligned}$$

Thus $\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda \geq \mu \wedge_\gamma^\delta \lambda$. Hence $\mu \wedge_\gamma^\delta \lambda = \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda$.

$(2) \Rightarrow (1)$: Let R and L be right and left ideals of S, respectively. Then χ_R and χ_L are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S, respectively. By hypothesis, it follows that

$$\begin{aligned} \chi_R \wedge_\gamma^\delta \chi_L &= \chi_R \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \chi_L \\ (\chi_{R\cap L})_\gamma^\delta &= (\chi_{RSL})_\gamma^\delta . \end{aligned}$$

Thus $R \cap L = RSL$. Hence by Theorem 6, S is regular. ■

Theorem 407 *The following assertions are equivalent for a ternary semigroup* S :

(1) S *is regular;*

(2) $(\mu)_\gamma^\delta = \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu$ *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy generalized bi-ideal* μ *of* S;

(3) $(\mu)_\gamma^\delta = \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu$ *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy bi-ideal* μ *of* S;

(4) $(\mu)_\gamma^\delta = \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu$ *for every* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy quasi-ideal* μ *of* S.

Proof. $(1) \Rightarrow (2)$: Assume that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of S. Since S is regular, so for any $x \in S$ there exists $a \in S$ such that $x = xax$. It follows that

$$\begin{aligned} \left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu\right)(x) &= \left\{\left\{\bigvee_{x=uvw} \left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu\right)(u) \wedge \mathcal{S}(v) \wedge \mu(w)\right\} \vee \gamma\right\} \wedge \delta \\ &\geq \left\{\left\{\left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu\right)(x) \wedge \mathcal{S}(a) \wedge \mu(x)\right\}\right\} \\ &= \left\{\left\{\left\{\bigvee_{x=pqr} \mu(p) \wedge \mathcal{S}(q) \wedge \mu(r)\right\} \wedge \mu(x)\right\} \vee \gamma\right\} \wedge \delta \\ &\geq \{\{(\mu(x) \wedge \mathcal{S}(a) \wedge \mu(x)) \wedge \mu(x)\} \vee \gamma\} \wedge \delta \\ &= (\mu(x) \vee \gamma) \wedge \delta = (\mu)_\gamma^\delta(x) . \end{aligned}$$

Thus $(\mu)_\gamma^\delta \leq \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu$. Now, since μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of S, it follows that

$$\begin{aligned}
& \left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu\right)(x) \\
= & \left\{\left\{\bigvee_{x=lmn} \left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu\right)(l) \wedge \mathcal{S}(m) \wedge \mu(n)\right\} \vee \gamma\right\} \wedge \delta \\
= & \left\{\left\{\begin{array}{c} \bigvee_{x=lmn} \left\{\left\{\left\{\bigvee_{l=pqr} \mu(p) \wedge \mathcal{S}(q) \wedge \mu(r)\right\} \vee \gamma\right\} \wedge \delta\right\} \\ \wedge \mathcal{S}(m) \wedge \mu(n) \end{array}\right\} \vee \gamma\right\} \wedge \delta \\
= & \left\{\begin{array}{c} \left\{\bigvee_{x=lmn} \left(\bigvee_{l=pqr} \mu(p) \wedge \mathcal{S}(q) \wedge \mu(r)\right) \wedge \mathcal{S}(m) \wedge \mu(n)\right\} \\ \vee\gamma \end{array}\right\} \wedge \delta \\
= & \left\{\begin{array}{c} \bigvee_{x=lmn} \bigvee_{l=pqr} (\mu(p) \wedge \mathcal{S}(q) \wedge \mu(r) \wedge \mathcal{S}(m) \wedge \mu(n)) \\ \vee\gamma \end{array}\right\} \wedge \delta \\
= & \left\{\left\{\left\{\bigvee_{x=lmn} \left\{\bigvee_{l=pqr} (\mu(p) \wedge \mu(r) \wedge \mu(n) \wedge \delta)\right\}\right\}\right\} \vee \gamma\right\} \wedge \delta \\
\leq & \left\{\left\{\bigvee_{x=lmn} \left\{\bigvee_{l=pqr} \mu(pqrmn) \vee \gamma\right\}\right\} \vee \gamma\right\} \wedge \delta \\
= & (\mu(x) \vee \gamma) \wedge \delta = (\mu)_\gamma^\delta(x).
\end{aligned}$$

Thus $(\mu)_\gamma^\delta \geq \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu$. Hence $(\mu)_\gamma^\delta = \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) : are obvious.

(4) $\Rightarrow$ (1) : Let Q be a quasi-ideal of S. Then by Lemma 387, χ_Q is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S. By hypothesis, it follows that

$$\begin{aligned}
(\chi_Q)_\gamma^\delta &= \left(\chi_Q \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \chi_Q \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \chi_Q\right) \\
(\chi_Q \vee \gamma) \wedge \delta &= \left\{\left(\chi_Q \circ \mathcal{S} \circ \chi_Q \circ \mathcal{S} \circ \chi_Q\right) \vee \gamma\right\} \wedge \delta \\
&= \left\{\left(\chi_{QSQSQ}\right) \vee \gamma\right\} \wedge \delta.
\end{aligned}$$

Thus $Q = QSQSQ$. Hence by Theorem 7, S is regular. ■

Theorem 408 *The following assertions are equivalent for a ternary semigroup S :*

(1) *S is regular;*

(2) *$\mu \wedge_\gamma^\delta \lambda \leq \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal λ of S;*

(3) *$\mu \wedge_\gamma^\delta \lambda \leq \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal λ of S;*

(4) *$\mu \wedge_\gamma^\delta \lambda \leq \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal λ of S.*

Proof. (1) $\Rightarrow$ (2) : Assume that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal and λ an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. Since S is regular, so for any $x \in S$, there exists $a \in S$ such that $x = xax$. It follows that

$$\begin{aligned}
\left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda\right)(x) &= \{(\mu \circ \mathcal{S} \circ \lambda)(x) \vee \gamma\} \wedge \delta \\
&= \left\{\left\{\bigvee_{x=uvw} \mu(u) \wedge \mathcal{S}(v) \wedge \lambda(w)\right\} \vee \gamma\right\} \wedge \delta \\
&\geq \{(\mu(x) \wedge \mathcal{S}(a) \wedge \lambda(x)) \vee \gamma\} \wedge \delta \\
&= ((\mu \wedge \lambda)(x) \vee \gamma) \wedge \delta = \left(\mu \wedge_\gamma^\delta \lambda\right)(x).
\end{aligned}$$

Thus $\mu \wedge_\gamma^\delta \lambda \leq \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) : are obvious.

(4) $\Rightarrow$ (1) : Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and λ an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of S. Since every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S, so $\mu \wedge_\gamma^\delta \lambda \leq \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda$. It follows that

$$\begin{aligned}
\left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda\right)(x) &= \{(\mu \circ \mathcal{S} \circ \lambda)(x) \vee \gamma\} \wedge \delta \\
&= \left\{\left\{\bigvee_{x=pqr} \mu(p) \wedge \mathcal{S}(q) \wedge \lambda(r)\right\} \vee \gamma\right\} \wedge \delta \\
&= \left\{\bigvee_{x=pqr} \{(\mu(p) \wedge \delta) \wedge (\lambda(r) \wedge \delta)\} \vee \gamma\right\} \wedge \delta \\
&\leq \left\{\bigvee_{x=pqr} \{(\mu(pqr) \vee \gamma) \wedge (\lambda(pqr) \vee \gamma)\} \vee \gamma\right\} \wedge \delta \\
&= \left\{\bigvee_{x=pqr} (\mu(pqr) \wedge \lambda(pqr)) \vee \gamma\right\} \wedge \delta \\
&= \left(\mu \wedge_\gamma^\delta \lambda\right)(x).
\end{aligned}$$

Thus $\mu \wedge_\gamma^\delta \lambda \geq \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda$. Hence $\mu \wedge_\gamma^\delta \lambda = \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda$. Therefore by Theorem 406, S is regular. ■

In a similar manner we can prove the following theorem.

Theorem 409 *The following assertions are equivalent for a ternary semigroup S :*

(1) *S is regular;*

(2) *$\lambda \wedge_\gamma^\delta \mu \leq \lambda \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal λ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal μ of S;*

(3) *$\lambda \wedge_\gamma^\delta \mu \leq \lambda \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal λ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal μ of S;*

(4) *$\lambda \wedge_\gamma^\delta \mu \leq \lambda \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal λ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal μ of S.*

Theorem 410 *The following assertions are equivalent for a ternary semigroup S :*

(1) *S is regular;*

(2) $\mu \wedge_\gamma^\delta \lambda \leq \left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda\right) \wedge \left(\lambda \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu\right)$ *for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideals μ and λ of S;*

(3) $\mu \wedge_\gamma^\delta \lambda \leq \left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda\right) \wedge \left(\lambda \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu\right)$ *for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals μ and λ of S;*

(4) $\mu \wedge_\gamma^\delta \lambda \leq \left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda\right) \wedge \left(\lambda \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu\right)$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal λ of S;*

(5) $\mu \wedge_\gamma^\delta \lambda \leq \left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda\right) \wedge \left(\lambda \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu\right)$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal λ of S;*

(6) $\mu \wedge_\gamma^\delta \lambda \leq \left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda\right) \wedge \left(\lambda \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu\right)$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal λ of S;*

(7) $\mu \wedge_\gamma^\delta \lambda \leq \left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda\right) \wedge \left(\lambda \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu\right)$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal λ of S.*

Proof. (1) $\Rightarrow$ (2) : Assume that μ and λ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideals of S. Since S is regular, so for any $x \in S$, there exists $a \in S$ such that $x = xax$. It follows that

$$\begin{aligned}
\left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda\right)(x) &= \{(\mu \circ \mathcal{S} \circ \lambda)(x) \vee \gamma\} \wedge \delta \\
&= \left\{\left\{\bigvee_{x=uvw} \mu(u) \wedge \mathcal{S}(v) \wedge \lambda(w)\right\} \vee \gamma\right\} \wedge \delta \\
&\geq \{(\mu(x) \wedge \mathcal{S}(a) \wedge \lambda(x)) \vee \gamma\} \wedge \delta \\
&= \{(\mu(x) \wedge \lambda(x)) \vee \gamma\} \wedge \delta = \left(\mu \wedge_\gamma^\delta \lambda\right)(x).
\end{aligned}$$

Thus $\mu \wedge_\gamma^\delta \lambda \leq \mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda$. Also we have

$$\begin{aligned}
\left(\lambda \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu\right)(x) &= \{(\lambda \circ \mathcal{S} \circ \mu)(x) \vee \gamma\} \wedge \delta \\
&= \left\{\bigvee_{x=pqr} (\lambda(p) \wedge \mathcal{S}(q) \wedge \mu(r)) \vee \gamma\right\} \wedge \delta \\
&\geq \{(\lambda(x) \wedge \mathcal{S}(a) \wedge \mu(x)) \vee \gamma\} \wedge \delta \\
&= \{(\lambda(x) \wedge \mu(x)) \vee \gamma\} \wedge \delta \\
&= \left(\mu \wedge_\gamma^\delta \lambda\right)(x).
\end{aligned}$$

This implies that $\mu \wedge_\gamma^\delta \lambda \leq \lambda \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu$. Therefore, $\mu \wedge_\gamma^\delta \lambda \leq \left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda\right) \wedge \left(\lambda \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu\right)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) : are obvious.

(7) $\Rightarrow$ (1) : Suppose $\mu \wedge_\gamma^\delta \lambda \leq \left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda\right) \wedge \left(\lambda \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu\right)$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal λ of S. Now by hypothesis $\mu \wedge_\gamma^\delta \lambda \leq \left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda\right) \wedge \left(\lambda \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \mu\right) \leq \left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda\right)$ and by Lemma 403,

$\left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda\right) \leq \mu \wedge_\gamma^\delta \lambda$. Hence $\mu \wedge_\gamma^\delta \lambda = \left(\mu \circ_\gamma^\delta \mathcal{S} \circ_\gamma^\delta \lambda\right)$. Therefore, by Theorem 406, S is regular. ■

6.6 Weakly regular ternary semigroups

In the present section we characterize weakly regular ternary semigroups in terms of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right (quasi-, bi-, generalized bi-, two sided) ideals.

Theorem 411 *The following assertions are equivalent for a ternary semigroup S :*

(1) *S is right weakly regular;*

(2) *$\mu \wedge_\gamma^\delta \lambda = \mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal λ of S.*

Proof. (1) $\Rightarrow$ (2) : Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and λ an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal of S and $x \in S$. It follows that

$$\begin{aligned}
\left(\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda\right)(x) &= \{(\mu \circ \lambda \circ \lambda)(x) \vee \gamma\} \wedge \delta \\
&= \left\{\bigvee_{x=pqr} \{\mu(p) \wedge \lambda(q) \wedge \lambda(r)\} \vee \gamma\right\} \wedge \delta \\
&= \left\{\bigvee_{x=pqr} (\mu(p) \wedge \delta) \wedge \lambda(q) \wedge (\lambda(r) \wedge \delta) \vee \gamma\right\} \wedge \delta \\
&\leq \left\{\bigvee_{x=pqr} (\mu(pqr) \vee \gamma) \wedge \lambda(q) \wedge (\lambda(pqr) \vee \gamma) \vee \gamma\right\} \wedge \delta \\
&\leq \left\{\bigvee_{x=pqr} (\mu(pqr) \wedge \lambda(pqr)) \vee \gamma\right\} \wedge \delta \\
&\leq \{(\mu(x) \wedge \lambda(x)) \vee \gamma\} \wedge \delta = \left(\mu \wedge_\gamma^\delta \lambda\right)(x).
\end{aligned}$$

Thus $\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda \leq \mu \wedge_\gamma^\delta \lambda$. Now, since S is right weakly regular, so for any $x \in S$, there exist $s_1, s_2, s_3, t_1, t_2, t_3 \in S$ such that $x = (xs_1t_1)(xs_2t_2)(xs_3t_3)$. It follows that

$$\begin{aligned}
\left(\mu \wedge_\gamma^\delta \lambda\right)(x) &= \{(\mu \wedge \lambda \wedge \lambda)(x) \vee \gamma\} \wedge \delta \\
&= \{(\mu(x) \wedge \lambda(x) \wedge \lambda(x)) \vee \gamma\} \wedge \delta \\
&\leq \{(\mu(xs_1t_1) \wedge \lambda(xs_2t_2) \wedge \lambda(xs_3t_3)) \vee \gamma\} \wedge \delta \\
&\leq \left\{\bigvee_{x=uvw} \{(\mu(u) \wedge \lambda(v) \wedge \lambda(w)) \vee \gamma\}\right\} \wedge \delta \\
&= \{(\mu \circ \lambda \circ \lambda)(x) \vee \gamma\} \wedge \delta = \left(\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda\right)(x).
\end{aligned}$$

Thus $\mu \wedge_\gamma^\delta \lambda \leq \mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda$. Hence $\mu \wedge_\gamma^\delta \lambda = \mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda$.

$(2) \Rightarrow (1)$: Let R be a right ideal and I a two sided ideal of S. Then χ_R and χ_I are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal of S, respectively. By hypothesis, it follows that

$$\begin{aligned} \chi_R \wedge_\gamma^\delta \chi_I &= \chi_R \circ_\gamma^\delta \chi_I \circ_\gamma^\delta \chi_I \\ (\chi_{R\cap I})_\gamma^\delta &= (\chi_{RII})_\gamma^\delta . \end{aligned}$$

Thus $R \cap I = RII$. Hence by Lemma 9, S is right weakly regular. ■

Theorem 412 *The following assertions are equivalent for a ternary semigroup* S :
(1) S *is right weakly regular;*
(2) *Each* $(\in_\gamma, \in_\gamma \vee q_\delta)$*-fuzzy right ideal* μ *of* S *is idempotent.*

Proof. $(1) \Rightarrow (2)$: Assume that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal. We prove $\mu \circ_\gamma^\delta \mu \circ_\gamma^\delta \mu = (\mu)_\gamma^\delta$. Let $x \in S$. Then

$$\begin{aligned} \left(\mu \circ_\gamma^\delta \mu \circ_\gamma^\delta \mu\right)(x) &= \{(\mu \circ \mu \circ \mu)(x) \vee \gamma\} \wedge \delta \\ &= \left\{ \bigvee_{x=pqr} \{\mu(p) \wedge \mu(q) \wedge \mu(r)\} \vee \gamma \right\} \wedge \delta \\ &= \left\{ \bigvee_{x=pqr} \{(\mu(p) \wedge \delta) \wedge \mu(q) \wedge \mu(r)\} \vee \gamma \right\} \wedge \delta \\ &\leq \left\{ \bigvee_{x=pqr} \{(\mu(pqr) \vee \gamma) \wedge \mu(q) \wedge \mu(r)\} \vee \gamma \right\} \wedge \delta \\ &= \left\{ \bigvee_{x=pqr} \{\mu(pqr) \wedge \mu(q) \wedge \mu(r)\} \vee \gamma \right\} \wedge \delta \\ &\leq \bigvee_{x=pqr} \{(\mu(pqr)) \vee \gamma\} \wedge \delta \leq \{\mu(x) \vee \gamma\} \wedge \delta \\ &= (\mu)_\gamma^\delta (x) . \end{aligned}$$

Thus $\mu \circ_\gamma^\delta \mu \circ_\gamma^\delta \mu \leq (\mu)_\gamma^\delta$. Now since S is right weakly regular, so for any $x \in S$, there exist $s_1, s_2, s_3, t_1, t_2, t_3 \in S$ such that $x = (xs_1t_1)(xs_2t_2)(xs_3t_3)$. It follows that

$$\begin{aligned} (\mu)_\gamma^\delta (x) &= \{(\mu(x) \wedge \mu(x) \wedge \mu(x)) \vee \gamma\} \wedge \delta \\ &= [\{((\mu(x) \wedge \delta) \wedge (\mu(x) \wedge \delta) \wedge (\mu(x) \wedge \delta)) \vee \gamma\}] \wedge \delta \\ &\leq \{(\mu(xs_1S_1) \vee \gamma) \wedge (\mu(xs_2S_2) \wedge \delta) \wedge (\mu(xs_3S_3) \wedge \delta) \vee \gamma\} \wedge \delta \\ &= \left\{ \bigvee_{x=pqr} (\mu(p) \wedge \mu(q) \wedge \mu(r)) \vee \gamma \right\} \wedge \delta \\ &= \left\{ \bigvee_{x=pqr} (\mu(p) \circ \mu(q) \circ \mu(r)) \vee \gamma \right\} \wedge \delta = \left(\mu \circ_\gamma^\delta \mu \circ_\gamma^\delta \mu\right)(x) . \end{aligned}$$

Thus $(\mu)_\gamma^\delta \leq \mu \circ_\gamma^\delta \mu \circ_\gamma^\delta \mu$. Hence $(\mu)_\gamma^\delta = \mu \circ_\gamma^\delta \mu \circ_\gamma^\delta \mu$.

(2) $\Rightarrow$ (1) Let A be the right ideal of S. Then by Theorem 404, $(\chi_A)_\gamma^\delta$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S. Thus by hypothesis

$$\begin{aligned}(\chi_A)_\gamma^\delta &= \chi_A \circ_\gamma^\delta \chi_A \circ_\gamma^\delta \chi_A \\ &= \{(\chi_A \circ \chi_A \circ \chi_A) \vee \gamma\} \wedge \delta \\ &= (\chi_{A^3})_\gamma^\delta .\end{aligned}$$

This implies that $A = A^3$. Hence S is right weakly regular. ■

Theorem 413 *The following assertions are equivalent for a ternary semigroup S :*

(1) *S is right weakly regular;*

(2) $\mu \wedge_\gamma^\delta \lambda \wedge_\gamma^\delta \nu \leq \mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \nu$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal μ, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal λ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal ν of S;*

(3) $\mu \wedge_\gamma^\delta \lambda \wedge_\gamma^\delta \nu \leq \mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \nu$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal μ, every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal λ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal ν of S.*

Proof. (1) $\Rightarrow$ (2) : Assume that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal, λ an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal and ν an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal of S. Since S is right weakly regular so for each $x \in S$ there exist $a_1, a_2, a_3, b_1, b_2, b_3 \in S$ such that $x = (xa_1b_1)(xa_2b_2)(xa_3b_3) = x(a_1b_1xa_2b_2)(xa_3b_3)$. It follows that

$$\begin{aligned}\left(\mu \wedge_\gamma^\delta \lambda \wedge_\gamma^\delta \nu\right)(x) &= \{(\mu \wedge \lambda \wedge \nu)(x) \vee \gamma\} \wedge \delta \\ &= \{(\mu(x) \wedge \lambda(x) \wedge \nu(x)) \vee \gamma\} \wedge \delta \\ &\leq \{(\mu(x) \wedge \lambda(a_1b_1xa_2b_2) \wedge \nu(xa_3b_3)) \vee \gamma\} \wedge \delta \\ &= \left\{\bigvee_{x=pqr} \{\mu(p) \wedge \lambda(q) \wedge \nu(r)\} \vee \gamma\right\} \wedge \delta \\ &= \left(\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \nu\right)(x).\end{aligned}$$

Hence $\mu \wedge_\gamma^\delta \lambda \wedge_\gamma^\delta \nu \leq \mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \nu$.

(2) $\Rightarrow$ (3) : Straightforward, because every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of S.

(3) $\Rightarrow$ (1) : Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and λ an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal of S. Put $\nu = \lambda$. Since every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal, so by hypothesis $\mu \wedge_\gamma^\delta \lambda \wedge_\gamma^\delta \lambda \leq \mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda$. It follows

that

$$\begin{aligned}
\left(\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda\right)(x) &= \{(\mu \circ \lambda \circ \lambda)(x) \vee \gamma\} \wedge \delta \\
&= \left\{ \bigvee_{x=pqr} ((\mu(p) \wedge \delta) \wedge \lambda(q) \wedge (\lambda(r) \wedge \delta)) \vee \gamma \right\} \wedge \delta \\
&\leq \left\{ \bigvee_{x=pqr} \{(\mu(pqr) \vee \gamma) \wedge \lambda(q) \wedge (\lambda(pqr) \wedge \delta)\} \vee \gamma \right\} \wedge \delta \\
&\leq \left\{ \bigvee_{x=pqr} (\mu(pqr) \wedge \lambda(pqr)) \vee \gamma \right\} \wedge \delta \\
&\leq \{(\mu(x) \wedge \lambda(x)) \vee \gamma\} \wedge \delta = \{(\mu \wedge \lambda)(x) \vee \gamma\} \wedge \delta.
\end{aligned}$$

Thus $\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda \leq \mu \wedge_\gamma^\delta \lambda$. Hence $\mu \wedge_\gamma^\delta \lambda = \mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda$. Therefore by Theorem 411, S is right weakly regular. ■

Theorem 414 *The following assertions are equivalent for a ternary semigroup S :*

(1) *S is right weakly regular;*

(2) $\mu \wedge_\gamma^\delta \lambda \leq \mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal λ of S;*

(3) $\mu \wedge_\gamma^\delta \lambda \leq \mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal λ of S;*

(4) $\mu \wedge_\gamma^\delta \lambda \leq \mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda$ *for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy quasi-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal λ of S.*

Proof. (1) $\Rightarrow$ (2) : Assume that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal and λ an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal of S. Since S is right weakly regular, so for any $x \in S$. there exist $a_1, a_2, a_3, b_1, b_2, b_3 \in S$ such that $x = (xa_1b_1)(xa_2b_2)(xa_3b_3) = x(a_1b_1xa_2b_2)(xa_3b_3)$. It follows that

$$\begin{aligned}
\left(\mu \wedge_\gamma^\delta \lambda\right)(x) &= \{(\mu \wedge \lambda)(x) \vee \gamma\} \wedge \delta \\
&= \{(\mu(x) \wedge \lambda(x) \wedge \lambda(x)) \vee \gamma\} \wedge \delta \\
&\leq \{(\mu(x) \wedge \lambda(a_1b_1xa_2b_2) \wedge \lambda(xa_3b_3)) \vee \gamma\} \wedge \delta \\
&= \left\{ \bigvee_{x=uvw} \{(\mu(u) \wedge \lambda(v) \wedge \lambda(w)) \vee \gamma\} \wedge \delta \right\} \\
&= \left(\mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda\right)(x).
\end{aligned}$$

Thus $\mu \wedge_\gamma^\delta \lambda \leq \mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) : are obvious.

(4) $\Rightarrow$ (1) : Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal and λ an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy two sided ideal of S. Since every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right ideal is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy

quasi-ideal of S. Then by hypothesis $\mu \wedge_\gamma^\delta \lambda \leq \mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda$. On the other hand by Theorem 413, $\mu \wedge_\gamma^\delta \lambda \geq \mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda$. Hence $\mu \wedge_\gamma^\delta \lambda = \mu \circ_\gamma^\delta \lambda \circ_\gamma^\delta \lambda$. Therefore by Theorem 411, S is right weakly regular. ∎

Chapter 7

Implication-based fuzzy ternary subsemigroups

In this last chapter by using four implication operators, that is, Gaines-Rescher implication operator, Gödel implication operator, the contraposition of Gödel implication operator and the Łuckasiewicz implication operator, the implication-based fuzzy ternary subsemigroups are considered. Relations between fuzzy (resp. $(\in, \in \vee q)$-fuzzy) ternary subsemigroups are discussed. Conditions for a fuzzy ternary subsemigroup with thresholds 0 and 0.5 to be an implication-based fuzzy ternary subsemigroups under the Łuckasiewicz implication operator are provided.

7.1 Implication-based fuzzy ternary subsemigroups

Fuzzy logic is an extension of set theoretic multivalued logic in which the truth values are linguistic variables or terms of the linguistic variable truth. Some operators, for example $\wedge, \vee$, $\neg$, $\rightarrow$ in fuzzy logic are also defined by using truth tables and the extension principle can be applied to derive definitions of the operators. In fuzzy logic, the truth value of fuzzy proposition Φ is denoted by $[\Phi]$. For a universe of discourse U, we display the fuzzy logical and corresponding set-theoretical notations used in this thesis

$$[x \in \mu] = \mu(x), \tag{7.1}$$

$$[\Phi \wedge \Psi] = \min\{[\Phi],\ [\Psi]\}, \tag{7.2}$$

$$[\Phi \rightarrow \Psi] = \min\{1,\ 1 - [\Phi] + [\Psi]\}, \tag{7.3}$$

$$[\forall x\ \Phi(x)] = \inf_{x \in U} [\Phi(x)], \tag{7.4}$$

$$\models \Phi \text{ if and only if } [\Phi] = 1 \text{ for all valuations.} \tag{7.5}$$

The truth valuation rules given in (7.3) are those in the Łuckasiewicz system of continuous-valued logic. Of course, various implication operators have been defined. We show only a section of them in the following

(a) Gaines-Rescher implication operator (I_{GR}):

$$I_{GR}(a,b) = \begin{cases} 1 & \text{if } a \leq b, \\ 0 & \text{otherwise} \end{cases}$$

(b) Gödel implication operator (I_G):

$$I_G(a,b) = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise} \end{cases}$$

(c) The contraposition of Gödel implication operator (I_{cG}):

$$I_{cG}(a,b) = \begin{cases} 1 & \text{if } a \leq b, \\ 1-a & \text{otherwise} \end{cases}$$

(d) The Łuckasiewicz implication operator (I_{LI}):

$$I_{LI}(a,b) = \begin{cases} 1 & \text{if } a \leq b, \\ 1-a+b & \text{otherwise} \end{cases}$$

for all $a, b \in [0,1]$. Ying [65] introduced the concept of fuzzifying topology. We can expand his/her idea to ternary semigroups, and we define fuzzifying ternary subsemigroup as follows:

Definition 415 *A fuzzy set μ in S is called a fuzzifying ternary subsemigroup of S if it satisfies the following condition:*

$$\models [x \in \mu] \wedge [y \in \mu] \wedge [x \in \mu] \to [xyz \in \mu] \tag{7.6}$$

for all $x, y, z \in S$.

Obviously, condition (7.6) is equivalent to (11). Therefore a fuzzifying ternary subsemigroup is an ordinary fuzzy ternary subsemigroup. In [66] the concept of t-tautology is introduced, that is,

$$\models_t \Phi \text{ if and only if } [\Phi] \geq t \text{ for all valuations.}$$

Now we extend the concept of implication-based fuzzy ternary subsemigroups in the following way:

Definition 416 *Let μ be a fuzzy set in S and $t \in (0,1]$. Then μ is called a t-implication-based fuzzy ternary subsemigroup of S if and only if it satisfies*

$$\models_t [x \in \mu] \wedge [y \in \mu] \wedge [x \in \mu] \rightarrow [xyz \in \mu] \tag{7.7}$$

for all $x, y, z \in S$.

Let I be an implication operator. Clearly, μ is a t-implication-based fuzzy ternary subsemigroup of S if and only if it satisfies

$$I\left(\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\}, \mu\left(xyz\right)\right) \geq t$$

for all $x, y, z \in S$.

Example 417 *Consider the ternary semigroup S of Example 16 Define a fuzzy set μ in S as follows:*

$$\mu : S \rightarrow [0,1], x \rightarrow \begin{cases} 0.6 & \text{if } x = -i \\ 0.2 & \text{if } x = 0 \\ 0.7 & \text{if } x = i. \end{cases}$$

Then μ is a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, 0.6]$ under the Gödel implication operator I_G. Also μ is a 0.3-implication-based fuzzy ternary subsemigroup of S under the contraposition of Gödel implication operator I_{cG}. We also see that μ is a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, 0.9]$ under the Luckasiewicz implication operator I_{LI}.

Example 418 *Consider the ternary semigroup S of Example 16. Define a fuzzy set μ in S as follows:*

$$\mu : S \rightarrow [0,1], x \rightarrow \begin{cases} 0.9 & \text{if } x = -i \\ 0.3 & \text{if } x = 0 \\ 0.7 & \text{if } x = i. \end{cases}$$

By routine calculations, we know that μ is a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, 0.7]$ under the Gödel implication operator I_G. Also μ is a 0.1-implication-based fuzzy ternary subsemigroup of S under the contraposition of Gödel implication operator I_{cG}. We also see that μ is a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, 0.8]$ under the Luckasiewicz implication operator I_{LI}.

Note that if $t_1, t_2 \in (0,1]$ with $t_1 > t_2$, then every t_1-implication-based fuzzy ternary subsemigroup of S is a t_2-implication-based fuzzy ternary subsemigroup of S. But the converse is false. In fact, in Example 418, the t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, 0.7]$ under the Gödel implication operator I_G is not a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0.7, 1]$ under the Gödel implication operator I_G since

$$I_G\left(\min\left\{\mu\left(-i\right),\ \mu\left(-i\right),\ \mu\left(-i\right)\right\}, \mu\left(\left(-i\right)\left(-i\right)\left(-i\right)\right)\right) = I_G\left(0.9, 0.7\right) = 0.7 \ngeq t$$

for $t \in (0.7, 1]$.

We recall

Lemma 419 *A fuzzy set μ in S is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S if and only if it satisfies:*

$$(\forall x, y, z \in S)\left(\mu\left(xyz\right) \geq \min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right),\ 0.5\right\}\right). \tag{7.8}$$

Theorem 420 *For any fuzzy set μ in S, if $I = I_G$ and μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S, then μ is a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, 0.5]$.*

Proof. Let $t \in (0, 0.5]$ and assume that μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S. Then

$$\mu\left(xyz\right) \geq \min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right),\ 0.5\right\}$$

for all $x, y, z \in S$. If $\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\} \leq 0.5$, then

$$\mu\left(xyz\right) \geq \min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\}$$

and so

$$I_G\left(\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\}, \mu\left(xyz\right)\right) = 1 \geq t.$$

Now suppose that $\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\} > 0.5$. Then $\mu\left(xyz\right) \geq 0.5$, and either $\mu\left(xyz\right) \geq \min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\}$ or $\mu\left(xyz\right) < \min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\}$. If $\mu\left(xyz\right) \geq \min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\}$, then

$$I_G\left(\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\}, \mu\left(xyz\right)\right) = 1 \geq t.$$

If $\mu\left(xyz\right) < \min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\}$, then

$$I_G\left(\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\}, \mu\left(xyz\right)\right) = \mu\left(xyz\right) \geq 0.5 \geq t.$$

Therefore μ is a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, 0.5]$.

■

Corollary 421 *For any fuzzy set μ in S if the level set*

$$U(\mu;t) := \{x \in S \mid \mu(x) \geq t\}$$

is a ternary subsemigroup of S, then μ is a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, 0.5]$ under the Gödel implication operator.

Proof. Straightforward. ■

In the following example we show that there exists a fuzzy set μ in S which is a t-implication-based fuzzy ternary subsemigroup of S for $t \in (0, 0.4]$ under the Gödel implication operator I_G. But μ is not an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S. This shows that the partial converse of Theorem 420 is not true.

Example 422 *Consider the ternary semigroup S of Example 54. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.4 & \text{if } x = 0, \\ 0.9 & \text{if } x = a, \\ 0.6 & \text{if } x = b, \\ 0.7 & \text{if } x = c, \\ 0.2 & \text{if } x = 1. \end{cases}$$

Routine calculations show that μ is a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, 0.4]$ under the Gödel implication operator, but μ is not an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S.

Theorem 423 *For any fuzzy set μ in S and $I = I_G$, if there exists $t \in [0.5, 1]$ such that μ is a t-implication-based fuzzy ternary subsemigroup of S, then μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S.*

Proof. Let $t \in [0.5, 1]$ be such that μ is a t-implication-based fuzzy ternary subsemigroup of S. Then

$$I_G(\min\{\mu(x), \ \mu(y), \ \mu(z)\}, \mu(xyz)) \geq t$$

for all $x, y, z \in S$, and so either $I_G(\min\{\mu(x), \ \mu(y), \ \mu(z)\}, \mu(xyz)) = 1$, that is,

$$\mu(xyz) \geq \min\{\mu(x), \ \mu(y), \ \mu(z)\}$$

or $I_G(\min\{\mu(x), \ \mu(y), \ \mu(z)\}, \mu(xyz)) = \mu(xyz) \geq t \geq 0.5$. Hence

$$\mu(xyz) \geq \min\{\mu(x), \ \mu(y), \ \mu(z), 0.5\}.$$

Using Lemma 419, we know that μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S. ■

Corollary 424 *For any $t \in [0.5, 1]$, if μ is a t-implication-based fuzzy ternary subsemigroup of S under the Gödel implication operator I_G, then μ is a fuzzy ternary subsemigroup of S with thresholds $\gamma = 0$ and $\delta \in (0, 0.5]$.*

Proof. Straightforward. ■

Corollary 425 *For any $t \in [0.5, 1]$, if μ is a t-implication-based fuzzy ternary subsemigroup of S under the Gödel implication operator I_G, then the level set*

$$U(\mu; k) := \{x \in S \mid \mu(x) \geq k\}$$

is a ternary subsemigroup of S for all $k \in (0, 0.5]$.

Proof. Straightforward. ■

If $t \in (0.5, 1]$, then the converse of Theorem 423 may not be true in general as seen in the following example.

Example 426 *Consider the ternary semigroup S of Example 417. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], x \to \begin{cases} 0.8 & \text{if } x = -i \\ 0.3 & \text{if } x = 0 \\ 0.5 & \text{if } x = i \end{cases}$$

Routine calculations show that μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S. But μ is not a t-implication-based fuzzy ternary subsemigroup of S for any $t \in (0.5, 1]$ under the Gödel implication operator I_G, since

$$\begin{aligned} I_G(\min\{\mu(-i),\ \mu(-i),\ \mu(-i)\}, \mu((-i)(-i)(-i))) &= I_G(0.8, 0.5) \\ &= 0.5 \ngeq t \end{aligned}$$

for any $t \in (0.5, 1]$.

Combining Theorems 420 and 423 we have the following corollary.

Corollary 427 *For any fuzzy set μ in S, if $I = I_G$, then μ is a 0.5-implication-based fuzzy ternary subsemigroup of S if and only if μ is a fuzzy ternary subsemigroup of S with thresholds $\gamma = 0$ and $\delta = 0.5$, that is, μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S.*

Theorem 428 *Consider $I = I_{cG}$ and let $t \in [0.5, 1]$. If μ is a t-implication-based fuzzy ternary subsemigroup of S, then μ is a fuzzy ternary subsemigroup with thresholds $\gamma = t$ and δ, where $\delta = \sup_{x \in S} \mu(x)$.*

Proof. Let $t \in [0.5, 1]$ and assume that μ is a t-implication-based fuzzy ternary subsemigroup of S. Then

$$I_{cG}\left(\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\}, \mu\left(xyz\right)\right) \geq t$$

for all $x, y, z \in S$, and so either $I_{cG}\left(\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\}, \mu\left(xyz\right)\right) = 1$, that is,

$$\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\} \leq \mu\left(xyz\right)$$

or $1 - \min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\} = I_{cG}\left(\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\}, \mu\left(xyz\right)\right) \geq t$, that is,

$$\min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\} \leq 1 - t \leq t$$

since $t \in [0.5, 1]$. It follows that

$$\max\left\{\mu\left(xyz\right), t\right\} \geq \min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right)\right\} = \min\left\{\mu\left(x\right),\ \mu\left(y\right),\ \mu\left(z\right),\ \delta\right\}.$$

Therefore μ is a fuzzy ternary subsemigroup of S with thresholds $\gamma = t$ and $\delta = \sup_{x \in S} \mu\left(x\right)$. ■

If $t \in (0.5,\ 1]$, then the converse of Theorem 428 may not be true in general as seen in the following example.

Example 429 *Consider the ternary semigroup S as given in Example 16. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0, 1], x \to \begin{cases} 0.4 & \text{if } x = -i \\ 0.6 & \text{if } x = 0 \\ 0.3 & \text{if } x = i. \end{cases}$$

Routine calculations show that μ is a fuzzy ternary subsemigroup of S with thresholds $\gamma = t$ and $\delta = \sup_{x \in S} \mu\left(x\right)$, for all $t \in (0.5,\ 1]$. But μ is not a t-implication-based fuzzy ternary subsemigroup of S for $t \in (0.6, 1]$. Since

$$\begin{aligned} I_{cG}\left(\min\left\{\mu\left(-i\right),\ \mu\left(-i\right),\ \mu\left(-i\right)\right\}, \mu\left(\left(-i\right)x\left(-i\right)y\left(-i\right)\right)\right) &= I_{cG}\left(0.4, 0.3\right) \\ &= 0.6 \ngeq t \end{aligned}$$

for all $t \in (0.6, 1]$.

Now we prove:

Theorem 430 *Consider $I = I_{cG}$ and let μ be a fuzzy set in S. For every $t \in (0, 0.5]$, if μ is a t-implication-based fuzzy ternary subsemigroup of S, then μ is a fuzzy ternary subsemigroup with thresholds $\gamma = 1 - t$ and $\delta = \sup_{x \in S} \mu\left(x\right)$.*

Proof. Assume that μ is a t-implication-based fuzzy ternary subsemigroup of S for $t \in (0, 0.5]$. Then

$$I_{cG}(\min\{\mu(x),\ \mu(y),\ \mu(z)\}, \mu(xyz)) \geq t$$

for all $x, y, z \in S$, which implies that either $\min\{\mu(x),\ \mu(y),\ \mu(z)\} \leq \mu(xyz)$ or

$$1 - \min\{\mu(x),\ \mu(y),\ \mu(z)\} = I_{cG}(\min\{\mu(x),\ \mu(y),\ \mu(z)\}, \mu(xyz)) \geq t$$

and so $\min\{\mu(x),\ \mu(y),\ \mu(z)\} \leq 1 - t$. It follows that

$$\max\{\mu(xyz), 1 - t\} \geq \min\{\mu(x),\ \mu(y),\ \mu(z)\} = \min\{\mu(x),\ \mu(y),\ \mu(z),\ \delta\}.$$

Therefore μ is a fuzzy ternary subsemigroup of S with thresholds $\gamma = 1 - t$ and $\delta = \sup_{x \in S} \mu(x)$. ■

Corollary 431 *For every $t \in (0, 0.5]$, if μ is a t-implication-based fuzzy ternary subsemigroup of S under the contraposition of Gödel implication operator I_{cG}, then μ is a fuzzy ternary subsemigroup with thresholds $\gamma = 1 - t$ and $\delta = 1$.*

For the converse of Theorem 428, we have the following theorem.

Theorem 432 *Consider $I = I_{cG}$ and let μ be a fuzzy set in S. For every $t \in (0, 0.5]$, if μ is a fuzzy ternary subsemigroup of S with thresholds $\gamma = t$ and $\delta = \sup_{x \in S} \mu(x)$, then μ is a t-implication-based fuzzy ternary subsemigroup of S.*

Proof. Let $t \in (0, 0.5]$ and assume that μ is a fuzzy ternary subsemigroup of S with thresholds $\gamma = t$ and $\delta = \sup_{x \in S} \mu(x)$. Then, for all $x, y, z \in S$, we have

$$\max\{\mu(xyz), t\} \geq \min\{\mu(x),\ \mu(y),\ \mu(z), \delta\} = \min\{\mu(x),\ \mu(y),\ \mu(z)\}.$$

If $\mu(xyz) \geq t$, then $\mu(xyz) \geq \min\{\mu(x),\ \mu(y),\ \mu(z)\}$ and so

$$I_{cG}(\min\{\mu(x),\ \mu(y),\ \mu(z)\}, \mu(xyz)) = 1 \geq t.$$

If $\mu(xyz) < t$, then $\min\{\mu(x),\ \mu(y),\ \mu(z)\} \leq t$. Hence if $\min\{\mu(x),\ \mu(y),\ \mu(z)\} \leq \mu(xyz)$, then

$$I_{cG}(\min\{\mu(x),\ \mu(y),\ \mu(z)\}, \mu(xyz)) = 1 \geq t.$$

If $\min\{\mu(x),\ \mu(y),\ \mu(z)\} > \mu(xyz)$, then

$$I_{cG}(\min\{\mu(x),\ \mu(y),\ \mu(z)\}, \mu(xyz)) = 1 - \min\{\mu(x),\ \mu(y),\ \mu(z)\} \geq 1 - t \geq t.$$

Consequently μ is a t-implication-based fuzzy ternary subsemigroup of S for every $t \in (0, 0.5]$. ■

Corollary 433 *For every $t \in (0, 0.5]$, if μ is a fuzzy ternary subsemigroup of S with thresholds $\gamma = t$ and $\delta = 1$, then μ is a t-implication-based fuzzy ternary subsemigroup of S under the contraposition of Gödel implication operator I_{cG}.*

Combining Corollaries 431 and 433, we have the following corollary.

Corollary 434 *For any fuzzy set μ in S, if $I = I_{cG}$, then μ is a 0.5-implication-based fuzzy ternary subsemigroup of S if and only if μ is a fuzzy ternary subsemigroup of S with thresholds $\gamma = t$ and $\delta = 1$.*

If $t \in (0, 0.5)$, then the converse of Theorem 432 may not be true in general as seen in the following example.

Example 435 *Consider the ternary semigroup S as given in Example 16. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], x \to \begin{cases} 0.3 & \text{if } x = -i \\ 0.5 & \text{if } x = 0 \\ 0.4 & \text{if } x = i. \end{cases}$$

Routine calculations show that μ is a t-implication-based fuzzy ternary subsemigroup of S for $t \in (0, 0.5)$. But if $t \in (0, 0.4)$, then

$$\max\{\mu(iii), t\} \ngeq \min\{\mu(i), \mu(i), \mu(i)\}.$$

Theorem 436 *Consider $I = I_{GR}$ and let $t \in (0, 1]$. If μ is a t-implication-based fuzzy ternary subsemigroup of S, then μ is a fuzzy ternary subsemigroup of S.*

Proof. Let $t \in (0, 1]$ be such that μ is a t-implication-based fuzzy ternary subsemigroup of S under the Gaines-Rescher implication operator I_{GR}. Then

$$I_{GR}(\min\{\mu(x),\ \mu(y),\ \mu(z)\}, \mu(xyz)) \geq t.$$

Since $t \neq (0, 1]$, it follows that $I_{GR}(\min\{\mu(x),\ \mu(y),\ \mu(z)\}, \mu(xyz)) = 1$ and so that $\mu(xyz) \geq \min\{\mu(x),\ \mu(y),\ \mu(z)\}$. Therefore μ is a fuzzy teranry subsemigroup of S. ■

Corollary 437 *For any $t \in (0, 1]$, if μ is a t-implication-based fuzzy ternary subsemigroup of S under the Gaines-Rescher implication operator I_{GR}, then the set*

$$U(\mu; t) := \{x \in S \mid \mu(x) \geq t\}$$

is a ternary subsemigroup of S.

Proof. Straightforward. ■

Theorem 438 *Every fuzzy ternary subsemigroup of S is a t-implication-based fuzzy ternary subsemigroup for all $t \in (0,1]$ under the Gaines-Rescher implication operator I_{GR}.*

Proof. Straightforward. ■

The following corollary is by Theorems 436 and 438.

Corollary 439 *A fuzzy set in S is a 0.5-implication-based fuzzy ternary subsemigroup of S under the Gaines-Rescher implication operator I_{GR} if and only if it is a fuzzy ternary subsemigroup of S.*

Theorem 440 *Every fuzzy ternary subsemigroup of S is a t-implication-based fuzzy ternary subsemigroup for all $t \in (0,1]$ under the Łuckasiewicz implication operator I_{LI}.*

Proof. Straightforward. ■

The following example shows that for a fuzzy set μ in S there exists $t \in (0,1]$ such that

(1) μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S.

(2) μ is not a t-implication-based fuzzy ternary subsemigroup of S under the Łuckasiewicz implication operator I_{LI}.

Example 441 *Consider the ternary semigroup S of Example 54. Define a fuzzy set μ in S as follows:*

$$\mu : S \to [0,1], \quad x \mapsto \begin{cases} 0.6 & \text{if } x = 0, \\ 0.9 & \text{if } x = a, \\ 0.8 & \text{if } x = b, \\ 0.7 & \text{if } x = c, \\ 0.3 & \text{if } x = 1. \end{cases}$$

Routine calculations show that μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S. But μ is not a 0.91-implication-based fuzzy ternary subsemigroup of S under the Łuckasiewicz implication operator I_{LI} since

$$\begin{aligned} I_{LI}\left(\min\left\{\mu(a),\ \mu(b),\ \mu(c)\right\}, \mu(abc)\right) &= I_{LI}(0.7, 0.6) \\ &= 1 - 0.7 + 0.6 \\ &= 0.9 \ngeq 0.91. \end{aligned}$$

We provide consitions for an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S to be a t-implication-based fuzzy ternary subsemigroup of S under the Łuckasiewicz implication operator I_{LI}.

Theorem 442 *Let μ be an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S. If there exist $x, y, z \in S$ such that* $\min\{\mu(x), \ \mu(y), \ \mu(z)\} > \mu(xyz)$, *and let*
$\omega = 1 - \min\{\mu(x), \ \mu(y), \ \mu(z)\} + \mu(xyz)$. *Then μ is a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, \omega]$.*

Proof. Assume that μ is an $(\in, \in \vee q)$-fuzzy ternary subsemigroup of S. Then

$$\mu(xyz) \geq \min\{\mu(x), \ \mu(y), \ \mu(z), \ 0.5\} = \min\{\min\{\mu(x), \ \mu(y), \ \mu(z)\}, \ 0.5\},$$

for all $x, y, z \in S$. Suppose that $\min\{\mu(x), \ \mu(y), \ \mu(z)\} \leq 0.5$. Then $\mu(xyz) \geq \min\{\mu(x), \ \mu(y), \ \mu(z)\}$ and so

$$I_{LI}(\min\{\mu(x), \ \mu(y), \ \mu(z)\}, \mu(xyz)) = 1 \geq t$$

for all $t \in (0, \omega]$. Assume that $\min\{\mu(x), \ \mu(y), \ \mu(z)\} > 0.5$ for all $x, y, z \in S$. Then $\mu(xyz) \geq 0.5$. Thus we have two cases:

(1) $\mu(xyz) \geq \min\{\mu(x), \ \mu(y), \ \mu(z)\}$,

(2) $\mu(xyz) < \min\{\mu(x), \ \mu(y), \ \mu(z)\}$.

First case implies that

$$I_{LI}(\min\{\mu(x), \ \mu(y), \ \mu(z)\}, \mu(xyz)) = 1 \geq t$$

for all $t \in (0, \omega]$. The second case induces

$$\begin{aligned} I_{LI}(\min\{\mu(x), \ \mu(y), \ \mu(z)\}, \mu(xyz)) &= 1 - \min\{\mu(x), \ \mu(y), \ \mu(z)\} + \mu(xyz) \\ &= \omega \geq t \end{aligned}$$

for all $t \in (0, \omega]$. Therefore μ is a t-implication-based fuzzy ternary subsemigroup of S for all $t \in (0, \omega]$ under the Łuckasiewicz implication operator I_{LI}. ■

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