Dedicated

To

My Parents

&

My Family

GENERALIZATION OF FUZZY HYPERIDEALS IN SEMIHYPERRINGS

By

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Department of Mathematics
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0.1 Acknowledgement

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0.2 Introduction

The concept of hyperstructure was first introduced by Marty [55] at the eighth Congress of Scandinavian Mathematicians in 1934, when he defined hypergroups and started to analyze their properties. Now, the theory of algebraic hyperstructures has become a well-established branch in algebraic theory and it has extensive applications in many branches of mathematics and applied science. Later on, people have developed semi-hypergroups, which are the simplest algebraic hyperstructures having closure and associative properties. Many books have been written on different algebraic structures by different authors [23, 24, 40, 64]. In [24], the author discussed some applications of theory of hyperstructures in rough set theory, coding theory, lattices, binary relations, cryptography, automata, probability, geometry, graphs and hypergraphs. Some basic concepts and notions of semihypergroups theory can be found in [21, 35]. A comprehensive review of the theory of hyperstructures can be found in [23, 24, 64]. The canonical hypergroup is a special type of hypergroups which was introduced by Mittas [57]. The theory of hypermodules in which their additive structure is just a canonical hypergroup has been studied by several authors, for example, Massouros [56], Corsini [24], Davvaz [27, 29], Zhan et al. [70], Ameri [8], Zahedi and Ameri [73]. A semiring in which addition is a hyperoperation is called a semihyperring [30]. In [65], Vougiouklis generalized the concept of hyperring $(\mathcal{R}, \oplus, \odot)$ by dropping the reproduction axiom where $\oplus$ and $\odot$ are associative hyperoperations and $\odot$ distributive over $\oplus$, and named it as semihyperring. In [22], Chaopraknoi et al studied semihyperring with zero. In [34], Davvaz and Poursalavati introduced the matrix representation of poly-groups over hyperring and also over semihyperring. Ameri and Hedayati studied semihyperring and its hyperideals in [10]. Dehkordi and Davvaz introduced the notion of Γ-semihyperrings and discussed roughness as a type of strong regular equivalence relations on Γ-semihyperrings.

The study of fuzzy hyperstructure is an interesting research topic of fuzzy sets. There is a lot of work that has been done on the connections between fuzzy sets and hyperstructures [24]. The theory of fuzzy sets, introduced by Zadeh [71] in 1965, has provided a useful mathematical tool for describing the behavior of systems that are too complex or ill-defined to admit precise mathematical analysis by classical methods and tools. In [1, 3, 7, 52, 53, 60, 74, 75], some applications of this theory in algebraic structures and hyperstructures can be seen. Some of them concern the fuzzy hyperalgebras. This is a direct extension of the concept of fuzzy algebras (fuzzy subgroups, groups, fuzzy lattices, fuzzy rings etc). This approach can be extended to fuzzy hypergroups. Nowadays, Fuzzy hyperstructures is a fascinating research area. Davvaz introduced the notion of fuzzy subhypergroups in [28], Ameri and Nozari [7] defined fuzzy regular relations and fuzzy strongly regular relations of fuzzy hyperalgebras and also established a connection between fuzzy hyperalgebras and algebras. Fuzzy subhypergroup has also been studied by Cristea [26]. Fuzzy hyperideals of semihyperrings have been studied by Davvaz et al [9, 30, 33]. Recently, Sen et al have introduced and analyzed semihypergroups [61]. Leoreanu introduced and studied the notion of fuzzy hypermodules of hyperrings.

Davvaz et. al. initiated the concept of fuzzy krasner (m, n)-hyperrings, (α, β)-fuzzy H_V-ideals of H_V-rings and (α, β)-fuzzy ideals of ternary semigroups [31, 32, 41]. In [58], Murali proposed the concept of a fuzzy point belonging to a fuzzy subset under natural equivalence on fuzzy subset. Bhakat and Das introduced the concepts of $(\alpha,\ \beta)$-fuzzy subgroups by using the "*belong to*" relation $(\in)$ and "*quasi-coincident with*" relation (q) between a fuzzy point and a fuzzy subgroup, and defined an $(\in,\ \in \vee q)$-fuzzy subgroup of a group [20]. In [54], Tariq et. al. introduced the concept of (α, β)-fuzzy hyperideals and $(\in, \in \vee q)$-fuzzy hyperideals in semihypergroups. Y. Yin et. al. defined the general forms of (α, β)-fuzzy subhypergroups of hypergroups [67].

Zadeh also introduced the idea of interval valued fuzzy sets, in 1975 [72]. The three dimensional sets are defined by Li et al [8]. Shang et al introduced n-dimensional fuzzy sets in 2010. In 1984, Atanassov introduced in [16], the concept of intuitionistic fuzzy sets on a non-empty set X, which gives both a membership degree and a non-membership degree. The relations between intuitionistic fuzzy sets and algebraic structures have been already considered by many mathematicians. In [39], using Atanassov's idea, Davvaz established the intuitionistic fuzzification of the concept of hyperideals in a semihypergroup and investigated some of their properties. Recently, in [15, 47], the authors have initiated a study on intuitionistic fuzzy sets in semihypergroups.

0.3 Research profile

The following research papers are the outcome of my Ph.D, thesis.

1. A. Ahmed, M. Aslam and S. Abdullah, (α, β)-fuzzy hyperideals in semihyperrings, World Applied Sciences Journal, 18, 11, (2012), 1501-1511.

2. A. Ahmed, M. Aslam and S. Abdullah, Interval valued (α, β)-fuzzy hyperideals in semihyperrings, U.P.B. Sci., Bull., 75, 2, (2013), 69-86.

3. A. Ahmed, M. Aslam and S. Abdullah, n-dimensional fuzzy hyperideals of semihyperrings, Int. J. Mach. Learn. and Cyber, (2014),1-8.

4. A. Ahmed, M. Aslam and S. Abdullah, Fuzzy semihyperring, submitted for publication.

5. A. Ahmed, M. Aslam and S. Abdullah, n-dimensional fuzzy k-hyperideals of semihyperrings, submitted for publication.

6. A. Ahmed, M. Aslam and S. Abdullah, n-dimensional (α, β)-fuzzy hyperideals of semihyperrings, submitted for publication.

7. A. Ahmed, M. Aslam and S. Abdullah, (α, β)-intuitionistic fuzzy bi-hyperideals in semihyperrings, submitted for publication.

8. A. Ahmed, M. Aslam and S. Abdullah, interval valued (α, β)-intuitionistic fuzzy bi-hyperideals in semihyperrings, submitted for publication.

0.4 Chapter-wise study

This thesis consists of six chapters. Throughout in this thesis, $\mathcal{R}$ will represent a semihyperring.

Chapter one, which is of introductory nature, provides basic definitions and results, which are necessary for the subsequent chapters.

In chapter two, we pursue an algebraic approach to investigate the concept of fuzzy semihyperring and related notions in order to set the ground for future work. First, we provide basic definitions and establish some preliminary results. After that we investigate fully idempotent semihyperrings, that is, semihyperrings all of whose hyperideals are idempotent. It is proved that such semihyperrings are characterized by the property that each proper fuzzy hyperideal is the intersection of fuzzy prime hyperideals containing it. Finally, we construct the fuzzy prime spectrum of fully idempotent semihyperrings in a manner analogous to the construction of the prime spectrum in classical semiring theory.

In chapter three, we concentrate on the concept of quasi-coincidence of fuzzy point with a fuzzy subset. By using this idea, the notion of (α, β)-fuzzy hyperideal in a semihyperring is introduced and, consequently, a generalization of fuzzy hyperideal is defined. We also study the related properties of the (α, β)-fuzzy hyperideals and in particular, an $(\in, \in \vee q)$-fuzzy hyperideal in semihyperring is investigated. Moreover, we also considered the concept of implication-based fuzzy hyperideals in a semihyperring and obtained some useful results. In what follows, α, β will denote one of $\in, q, \in \vee q$, or $\in \wedge q$ unless otherwise specified. Also $\overline{\alpha}$ means α does not hold.

In chapter four, we concentrate on the concept of quasi-coincidence of interval valued fuzzy point with an interval valued fuzzy subset. By using this idea, the notion of interval valued (α, β)-fuzzy hyperideal in a semihyperring is introduced and consequently, a generalization of interval valued fuzzy hyperideal is defined. We study the related properties of the interval valued (α, β)-fuzzy hyperideals and in particular, an interval valued $(\in, \in \vee q)$-fuzzy hyperideals in semihyperrings are investigated. Moreover, we also consider the concept of implication-based interval valued fuzzy hyperideals in a semihyperring and obtained some results.

In chapter five, we introduce the generalization of intuitionistic fuzzy bi-hyperideals of a semihyperring. First, we discuss the notion of intuitionistic fuzzy hyperideal (bi-hyperideal) in semihyperrings. We also define the (α, β)-intuitionistic fuzzy bi-hyperideal of a semihyperring. This concept is a new generalization of the notion of intuitionistic fuzzy bi-hyperideal of a semihyperring. We give some interesting results as well as examples of this notion. In the last section we discuss intuitionistic fuzzy bi-hyperideal of type $(\in, \in \vee q)$.

In chapter six, we introduce the notion of n-dimensional fuzzy sets, fuzzy hyperideals and fuzzy prime hyperideals in semihyperrings with identity. We also discuss some basic properties of n-dimensional fuzzy prime hyperideals and characterize the n-dimensional fuzzy prime hyperideals. We also investigate the topology on n-dimensional fuzzy hyperideals and fuzzy prime hyperideals. Furthermore, we introduce the notion of n-dimensional weak (resp. strong) fuzzy k-hyperideals and the behavior of them under homomorphisms of semihyperrings. Also we define the quotient of fuzzy weak (resp. strong) k-hyperideals by regular relation of semihyperring and obtain some results. In last section, we combine the notions of n-dimensional fuzzy set and n-dimensional fuzzy point to introduce a new notion called n-dimensional (α, β)-fuzzy hyperideal in semihyperring. We also introduce the characterization of n-dimensional (α, β)-fuzzy hyperideal in semihyperring by upper level set. Moreover, we define n-dimensional prime (α, β)-fuzzy hyperideal in semihyperring.

Chapter 1

Fundamental Concepts

The aim of this chapter is to provide a brief summary of basic definitions and preliminary results, concerning semihyperrings, which will be of great help in further pursuits.

1.1 Basic definitions and notations

A hyperstructure is a non-empty set H together with a mapping " $\circ$": $H \times H \to P^*(H)$, where $P^*(H)$ is the set of all the non-empty subsets of H and " $\circ$ " is called a hyperoperation. If $x \in H$ and $A, B \in P^*(H)$, then by $A \circ B$, $A \circ x$, and $x \circ B$, we mean $A \circ B = \cup\{a \circ b : a \in A$ and $b \in B\}$, $A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$, respectively. A hypergroupoid is a set H with a binary hyperoperation " $\circ$ ". If ' $\circ$ ' is associative, that is, $x \circ (y \circ z) = (x \circ y) \circ z$, for all $x, y, z \in H$, then it is called a semihypergroup. A hypergroup is a semihypergroup, such that, for all $x \in H$, we have $x \circ H = H = H \circ x$, which is called reproduction axiom. If H is a hypergroup and K is a non-empty subset of H, then K is a subhypergroup of H if K itself a hypergroup under the same hyperoperation, defined in H. Hence it is clear that a subset K of H is a subhypergroup if and only if, for all $a \in K$, $a \circ K = K \circ a = K$, under the hyperoperation on H.

A set H together with a hyperoperation "$\circ$" is called a polygroup, if the following conditions are satisfied:

(1) $x \circ (y \circ z) = (x \circ y) \circ z$, for all $x, y, z \in H$.

(2) There exists a unique element, $e \in H$ such that $e \circ x = x \circ e = \{x\}$, for all $x \in H$.

(3) For all $x \in H$, there exists a unique element, say $x' \in H$ such that $e \in x \circ x' \cap x' \circ x$ (where $x' = x^{-1}$).

(4) For all $x, y, z \in H$, $z \in x \circ y \Rightarrow x \in z \circ y^{-1} \Rightarrow y \in x^{-1} \circ z$.

A non-empty subset K of a polygroup $(H, \circ)$ is called a subpolygroup if $(K, \circ)$ is itself a polygroup. In this case, we write $K <_p H$.

A commutative polygroup is called a canonical hypergroup.

Definition 1 *[10] A semihyperring is an algebraic hypersystem $(\mathcal{R}, \oplus, \cdot)$ consisting of a non-empty set $\mathcal{R}$ together with one hyperoperation "$\oplus$" and one binary operation "$\cdot$" on $\mathcal{R}$, such that $(\mathcal{R}, \oplus)$ is a commutative semihypergroup and $(\mathcal{R}, .)$ is a semigroup. For all $x, y, z \in \mathcal{R}$, the binary operation of multiplication is distributive over hyperoperation from both sides, that is, $x.(y \oplus z) = x.y \oplus x.z$ and $(x \oplus y).z = x.z \oplus y.z$.*

Definition 2 *[10] A semihyperring $(\mathcal{R}, \oplus, .)$ has a zero element, if there exists a unique element $0 \in \mathcal{R}$, such that $x \in 0 \oplus x = \{x\} = x \oplus 0$, and $0.x = 0 = x.0$, for all $x \in \mathcal{R}$. Element 0 is also called an absorbing element.*

A semihyperring $(\mathcal{R}, \oplus, \cdot)$ is called a hyperring if $(\mathcal{R}, \oplus)$ is a canonical hypergroup and $(\mathcal{R}, .)$ is a semigroup.

Definition 3 *[43] A non-empty subset S of a semihyperring $(\mathcal{R}, \oplus, .)$ is called a subsemihyperring of $\mathcal{R}$ if $\forall\ x, y \in S$*

(i) $x \oplus y \subseteq S$,

(ii) $x.y \in S$.

Example 4 *On four elements semihyperring $(\mathcal{R}, \oplus, .)$ defined by the following two tables:*

$\oplus$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{a\}$	$\{b\}$	$\{c\}$
b	$\{b\}$	$\{b\}$	$\{b\}$	$\{c\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

By routine calculation $\{0\}$, $\{0,b\}$, $\{0,c\}$, $\{0,a,b\}$, $\{0,b,c\}$, $\mathcal{R}$, are subsemihyperrings of $\mathcal{R}$.

Example 5 *Consider $\mathbb{Z}$, the set of integers. Define a hyperoperation "$\oplus$" and a binary operation "$\cdot$" on $\mathbb{Z}$ as follow $m \oplus n = \{m, n\}$ and $mn = mn$ $\forall$ $m, n \in \mathbb{Z}$. Clearly $(\mathbb{Z}, \oplus, \cdot)$ is a semihyperring.*

Example 6 *Let $\mathcal{R} = \{0, a, b\}$ be a set with hyperoperation "$\oplus$" and binary operation "$\cdot$" on $\mathcal{R}$, defined by the following two tables:*

$\oplus$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a,b\}$
b	$\{b\}$	$\{0,a,b\}$	$\{0,b\}$

$\cdot$	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

Then $\mathcal{R}$ is a semihyperring.

Example 7 *Consider the semihyperring $\mathcal{R}$, defined by the following two tables:*

$\oplus$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a,b\}$	$\{0,a,c\}$
b	$\{b\}$	$\{0,a,b\}$	$\{0,b\}$	$\{0,b,c\}$
c	$\{c\}$	$\{0,a,c\}$	$\{0,b,c\}$	$\{0,c\}$

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

For every $x \in \mathcal{R}$ there exists one and only one $-x \in \mathcal{R}$ such that $0 \in x \oplus (-x) = (-x) \oplus x$. We shall write $\hat{x}$ for $-x$ and call it the opposite of x.

Denote the set all opposite elements of a semihyperring $\mathcal{R}$, by $V(\mathcal{R})$,

$V(\mathcal{R}) = \{a \in \mathcal{R} \mid \exists\, b \in \mathcal{R}$ and $0 \in a \oplus b\}$.

Definition 8 *[10] A semihyperring $(\mathcal{R}, \oplus, .)$ is called commutative if $a.b = b.a$, for all $a, b \in \mathcal{R}$.*

Definition 9 *[10] A semihyperring $(\mathcal{R}, \oplus, .)$ is called a semihyperring with identity if $1_{\mathcal{R}} \in \mathcal{R}$, such that $x.1_{\mathcal{R}} = 1_{\mathcal{R}}.x = x$, for all $x \in \mathcal{R}$.*

Definition 10 *An element $x \in \mathcal{R}$ is called a unit if and only if there exists $y \in \mathcal{R}$ such that, $1_{\mathcal{R}} = x.y = y.x$. The set of all unit elements of a semihyperring $\mathcal{R}$ is denoted by $U(\mathcal{R})$.*

1.2 Hyperideals

Definition 11 *[13] By a left (right) hyperideal of* $\mathcal{R}$*, we mean a subsemihyperring* I *of* $\mathcal{R}$ *such that for all* $r \in \mathcal{R}$ *and* $x \in I$*, we have* $rx \in I$ $(xr \in I)$ *or a non-empty subset* I *of a semihyperring* $\mathcal{R}$ *is called a left (resp., right) hyperideal of* $\mathcal{R}$ *if it satisfies* $I + I \subseteq I$ *and* $\mathcal{R}I \subseteq I$ *(resp.,* $I\mathcal{R} \subseteq I$*).*

By a hyperideal, we mean a subset of $\mathcal{R}$ which is both a left and a right hyperideal of $\mathcal{R}$. An absorbing element $0 \in \mathcal{R}$ also belong to any hyperideal I of $\mathcal{R}$, since $0 \in \mathcal{R}$ so for any $x \in I$, we have $0.x = x.0 = 0 \in I$.

Example 12 *On four elements semihyperring* $(\mathcal{R}, \oplus, .)$ *defined by the example 4.*

These all $\{0, b\}$, $\{0, c\}$, $\{0, a, b\}$, $\{0, b, c\}$, are right hyperideals of semihyperring $\mathcal{R}$.

Example 13 *By example 5, it is easy to verify that* $I =< 2 >= \{2k | k \in \mathbb{Z}\}$*, is a hyperideal of* $\mathbb{Z}$*.*

Proposition 14 *A left (resp. right) hyperideal of a semihyperring is a subsemihyperring.*

Proof. Let I be a left hyperideal of a semihyperring. For any $a, b \in I$, we have $a + b \subseteq I$. Also, for any $a, b \in I$, since $a \in I \subseteq \mathcal{R}$. We have $a.b \in I$. Hence I is a subsemihyperring of the semihyperring $\mathcal{R}$. ■

Lemma 15 *[13] The intersection of any collection of hyperideals in a semihyperring* $\mathcal{R}$ *is also a hyperideal of* $\mathcal{R}$*.*

If I and J are two hyperideals of a semihyperring $\mathcal{R}$, then the sum and product of two hyperideals are also a hyperideal and defined as respectively:

$$I \oplus J = \bigcup_{\substack{a_i \in I \\ b_i \in J}} (a_i \oplus b_j)$$

and

$$IJ = \cup\{ \sum_{finite} a_i b_j ; a_i \in I, b_j \in J \}$$

We can see that IJ is a hyperideal of $\mathcal{R}$ contained in $I \cap J$.

Lemma 16 *[13] If $\mathcal{R}$ is a semihyperring with unity and $x \in \mathcal{R}$ then*

$$x.\mathcal{R} = \bigcup_{r\in\mathcal{R}} \left\{ \sum_{finite} x.r \right\} \left(\mathcal{R}.x = \bigcup_{r\in\mathcal{R}} \left\{ \sum_{finite} r.x \right\} \right)$$

is the smallest hyperideal of $\mathcal{R}$ containing x.

Definition 17 *[11] Let A be a non-empty subset of a semihyperring $\mathcal{R}$. Then a hyperideal generated by A and denoted by $\langle A \rangle$, is the intersection of all hyperideals of $\mathcal{R}$, which contains A;*

$$< A > \quad = \cap\{I \text{ is a hyperideal of } \mathcal{R} : A \subseteq I\}$$

It is smallest hyperideal of $\mathcal{R}$ containing A.

Lemma 18 *[11] If $\mathcal{R}$ is a commutative semihyperring with unity and $x \in \mathcal{R}\backslash\{0\}$, then*

$$< x > \quad = \bigcup_{r\in\mathcal{R}} \left\{ \sum_{finite} x.r \right\}$$

Proposition 19 *[43] If S is a subsemihyperring of a semihyperring $\mathcal{R}$ and I is a hyperideal of $\mathcal{R}$ then;*

(i) $S + I$ is a subsemihyperring of $\mathcal{R}$.

(ii) $S \cap I$ is a hyperideal of S.

Definition 20 *[13] Let $(\mathcal{R}, \oplus, .)$ be a semihyperring and $\{I_i\}_{i\in\Lambda}$ be a family of hyperideals of $\mathcal{R}$. Then $\bigcap_{i\in\Lambda} I_i$ is also a hyperideal of $\mathcal{R}$.*

Theorem 21 *[40] If I and J are hyperideals of a semihyperring $\mathcal{R}$, then $I + J$ is the smallest hyperideal of $\mathcal{R}$, containing both I and J. Where*

$$I \oplus J = \bigcup_{\substack{a_i \in I \\ b_i \in J}} (a_i \oplus b_j)$$

Definition 22 *[6] A hyperideal I is idempotent if $I^2 = I$.*

Definition 23 *[6] A semihyperring $\mathcal{R}$ is called fully idempotent if each hyperideal of $\mathcal{R}$ is idempotent.*

Definition 24 *[6] A semihyperring $\mathcal{R}$ is said to be regular if for each $x \in \mathcal{R}$, there exists $a \in \mathcal{R}$ such that $x = xax$.*

Definition 25 *[8] A non-empty set M, which is a commutative semihypergroup with respect to addition, with an absorbing element 0 is called a right, R-semihypermodule M_R, if R is a semihyperring and there is a function $\alpha : M \times R \longrightarrow P^*(M)$, where $P^*(M) = P(M) \diagdown \{0\}$, such that if $\alpha(m, x)$ is denoted by mx and $mx \subseteq M$, for all $x \in R$ and $m \in M$. Then the following conditions hold, for all $x, y \in R$ and $m_1, m_2, m \in M$:*

(i) $(m_1 \oplus m_2)x = m_1x \oplus m_2x$

(ii) $m(x \oplus y) = mx \oplus my$

(iii) $m(xy) = (mx)y$

(iv) $0.x = m.0 = 0$.

Similarly, we can define a left R-semihypermodule $_RM$. A semihyperring R is a right semihypermodule over itself which will be denoted by R_R. A non-empty subset N of a right R-semihypermodule M is called a subsemihypermodule of M, if $(N, \oplus)$ is a subsemihypergroup of $(M, \oplus)$ and $RN \subseteq P^*(N)$. Also note that, the right (left) subsemihypermodules R_R ($_RR$) are right (left) hyperideals of R.

Remark 26 *Every hyperideal of a semihyperring R is a semihypermodule of R.*

Definition 27 *[6] A hyperideal I of a semihyperring $\mathcal{R}$ is called a prime hyperideal of $\mathcal{R}$ if for any two hyperideals A, B of $\mathcal{R}$, $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$.*

Proposition 28 *[6] The following conditions on hyperideal I of a semihyperring $\mathcal{R}$ with identity are equivalent.*

(i) I is prime hyperideal of $\mathcal{R}$.

(ii) $\{arb : r \in \mathcal{R}\} \subseteq I$ if and only if, either $a \in I$ or $b \in I$.

(iii) If a and b are elements of $\mathcal{R}$ satisfying; $\langle a \rangle \langle b \rangle \subseteq I$, then either $a \in I$ or $b \in I$.

Corollary 29 *[13] If a and b are elements of a semihyperring $\mathcal{R}$, then the following conditions on prime hyperideal I of semihyperring $\mathcal{R}$ are equivalent.*

(i) If $a.b \in I$, then $a \in I$ or $b \in I$.

(ii) If $a.b \in I$, then $b.a \in I$.

Definition 30 *[6] A hyperideal I of a semihyperring $\mathcal{R}$ is called semiprime hyperideal if for any hyperideal J of $\mathcal{R}$ satisfying $J^2 \subseteq I$ implies $J \subseteq I$.*

Remark 31 *Prime hyperideals are surely semiprime hyperideals.*

Proposition 32 *The following conditions on a hyperideal I of a semihyperring $\mathcal{R}$ with unity are equivalent.*

(i) I is semiprime.

(ii) $\{ara : r \in \mathcal{R}\} \subseteq I$ if and only if $a \in I$.

Definition 33 *[6] A hyperideal I of a semihyperring $\mathcal{R}$ is called irreducible if for all hyperideals A, B of $\mathcal{R}$, $A \cap B = I$, implies $A = I$ or $B = I$.*

Definition 34 *[6] A hyperideal I of a semihyperring $\mathcal{R}$ is called strongly irreducible if for all hyperideals A, B of $\mathcal{R}$, $A \cap B \subseteq I$, implies $A \subseteq I$ or $B \subseteq I$.*

Remark 35 *Every strongly irreducible hyperideal is an irreducible hyperideal.*

Remark 36 *Any prime hyperideal is strongly irreducible.*

Proposition 37 *[6] Any hyperideal of a semihyperring $\mathcal{R}$ is the intersection of all irreducible hyperideals of $\mathcal{R}$ containing it.*

Proposition 38 *[6] A hyperideal I of a semihyperring $\mathcal{R}$ is prime if and only if it is semiprime and strongly irreducible.*

Definition 39 *A non-empty subset B of a semihyperring $\mathcal{R}$ is called a bi-hyperideal of $\mathcal{R}$ if it satisfies $B + B \subseteq B$, $BB \subseteq B$ and $B\mathcal{R}B \subseteq B$.*

Definition 40 *A non-empty subset Q of a semihyperring $\mathcal{R}$ is called a quasi-hyperideal of $\mathcal{R}$ if it satisfies $Q + Q \subseteq Q$, $Q\mathcal{R} \cap \mathcal{R}Q \subseteq Q$.*

1.3 k-Hyperideals

Definition 41 *[44] A non-empty subset I of $\mathcal{R}$ is called a left (resp. right) weak k-hyperideal of $\mathcal{R}$ iff*

(i) $(I, +)$ is a semihypergroup of $(\mathcal{R}, +)$;

(ii) $rx \in I$ (resp. $xr \in I$), $\forall\ x \in I$ and $r \in \mathcal{R}$.

(iii) $r + x \subseteq I$ or $x + r \subseteq I \Longrightarrow r \in I$, for $x \in I$ and $r \in \mathcal{R}$.

And I is called a left (resp. right) strong k-hyperideal of $\mathcal{R}$ iff

(i) $(I, +)$ is a semihypergroup of $(\mathcal{R}, +)$;

(ii) $rx \in I$ (resp. $xr \in I$), $\forall\ x \in I$ and $r \in \mathcal{R}$.

(iii) $r + x \approx I$ or $x + r \approx I \Longrightarrow r \in I$, $\forall\ x \in I$ and $r \in \mathcal{R}$.

where by $A \approx B$, we mean $A \cap B \neq \varphi$, $\forall$ non-empty subsets A and B of $\mathcal{R}$.

A two sided weak (resp. strong) k-hyperideal is called a weak (resp. strong) k-hyperideal of $\mathcal{R}$. As $(\mathcal{R}, +)$ is a commutative semihypergroup, so $r + x = x + r$, $\forall$ elements of I or $\mathcal{R}$.

1.4 Fuzzy sets

Definition 42 *If X is a universe and $A \subseteq X$, then characteristic function of A is a function $\chi_A : X \to \{0, 1\}$, defined by*

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

A fuzzy subset λ of X is a function $\lambda : X \to [0, 1]$. We write $\lambda(x) \in [0, 1]$, for all $x \in X$, where λ is a fuzzy subset of X such that for each $x \in X$, $0 \leq \lambda(x) \leq 1$.

A fuzzy set μ in a set X of the form

$$\mu(y) = \begin{cases} t \neq 0, & if\ \ y = x \\ 0, & if\ \ y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by $x_{t.}$ A fuzzy point x_t is said to be belong to (resp. quasi-coincident with) a fuzzy set μ, written as $x_t \in \mu$ (*resp.* $x_t q\mu$) if $\mu(x) \geq t$ $(\mu(x) + t > 1)$. If $x_t \in \mu$ or $x_t q\mu$, then we write $x_t \in \vee q\mu$. If $x_t \in \mu$ and $x_t q\mu$, then we write $x_t \in \wedge q\mu$. The symbol $\overline{\in \vee q}$ means neither $\in$ nor q holds. The symbol $\overline{\in \wedge q}$ means $\in$ or q does not hold.

For any two fuzzy subsets λ and μ of X, $\lambda \leq \mu$ if and only if $\lambda(x) \leq \mu(x)$, for all $x \in X$. The symbols $\lambda \wedge \mu$, and $\lambda \vee \mu$ will mean the following fuzzy subsets of X, for all $x \in X$.

$$\begin{aligned}(\lambda \wedge \mu)(x) &= \min\{\lambda(x), \mu(x)\} \\ (\lambda \vee \mu)(x) &= \max\{\lambda(x), \mu(x)\}\end{aligned}$$

More generally, if $\{\lambda_i : i \in \Lambda\}$ is a family of fuzzy subsets of X, then $\wedge_{i\in\Lambda}\lambda_i$ and $\vee_{i\in\Lambda}\lambda_i$ are defined by

$$\begin{aligned}(\wedge_{i\in\Lambda}\lambda_i)(x) &= \min_{i\in\Lambda}(\lambda_i(x) \\ (\vee_{i\in\Lambda}\lambda_i)(x) &= \max_{i\in\Lambda}(\lambda_i(x)\end{aligned}$$

and will called the intersection and union of the family $\{\lambda_i : i \in \Lambda\}$ of fuzzy subsets of X.

Let λ be a fuzzy subset of X and $t \in (0, 1]$. Then the set $U_t^\lambda = \{x \in X : \lambda(x) \geq t\}$ is called the level subset of X.

1.5 Fuzzy hyperideals

Definition 43 *[44] Let $\mathcal{R}$ be a semihyperring and μ a fuzzy set in $\mathcal{R}$. Then, μ is said to be a fuzzy hyperideal of $\mathcal{R}$ if for all $r, x, y \in \mathcal{R}$ the following axioms hold:*

(i) $\inf\limits_{z\in x\oplus y} \mu(z) \geq \mu(x) \wedge \mu(y)$, for all $x, y \in \mathcal{R}$.

(ii) $\mu(xr) \geq \mu(x)$ and $\mu(rx) \geq \mu(x)$ for all $r, x \in \mathcal{R}$.

Example 44 *Let μ be a fuzzy set in a semihyperring $(\mathbb{Z}, \oplus, \cdot)$, where $\mathbb{Z}$ is the set of integers defined by*

$$\mu(x) = \begin{cases} 0.2 & \text{if } x \text{ is odd,} \\ 0.6 & \text{if } x \text{ is non-zero even,} \\ 1 & \text{if } x = 0. \end{cases}$$

Then it is easy to show that μ is a fuzzy hyperideal of $\mathbb{Z}$.

Example 45 *Let μ be a fuzzy set in a semihyperring $(\mathbb{Z}, \oplus, \cdot)$, where $\mathbb{Z}$ is the set of integers defined by*

$$\mu(x) = \begin{cases} 1 & \text{if } 8 < x, \\ 0.5 & \text{if } 5 \leq x \leq 8, \\ 0 & \text{if } x < 5. \end{cases}$$

Then it is easy to show that μ is a fuzzy hyperideal of $\mathbb{Z}$.

Lemma 46 *[44] Let μ be a fuzzy set in a semihyperring $\mathcal{R}$. If $\mathcal{R}$ has zero element, then $\mu(0) \geq \mu(x)$ for all $x \in \mathcal{R}$.*

Lemma 47 *[12] Let μ be a fuzzy set of a semihyperring $\mathcal{R}$. Then, μ_* defined as*

$$\mu_* = \{x \in \mathcal{R} | \mu(x) = \mu(0)\}.$$

If μ is a fuzzy hyperideal of $\mathcal{R}$, then μ_* is a hyperideal of $\mathcal{R}$.

1.6 Interval valued fuzzy sets

By an interval number $\widetilde{a}$, we mean an interval $[a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$. The set of all interval numbers is denoted by $D[0,1]$. We now identify the interval $[a\ , a]$ with the number $a \in [0,1]$.

For the interval number $\widetilde{a}_i = [a_i^-, a_i^+] \in D\ [0,1]$, $i \in I$, we define the following notations:
$\text{rmax}\{\widetilde{a}_i, \widetilde{b}_i\} = [\max\{a_i^-, b_i^-\}, \max\{a_i^+, b_i^+\}]$,
$\text{rmin}\{\widetilde{a}_i, \widetilde{b}_i\} = [\min\{a_i^-, b_i^-\}, \min\{a_i^+, b_i^+\}]$,
$\text{rinf}\ \widetilde{a}_i = [\wedge a_i^-, \wedge a_i^+]$, $\text{rsup}\ \widetilde{a}_i = [\vee a_i^-, \vee a_i^+]$,
and then, we put
(1) $\widetilde{a}_1 \leq \widetilde{a}_2 \Longleftrightarrow a_1^- \leq a_2^-$ and $a_1^+ \leq a_2^+$,
(2) $\widetilde{a}_1 = \widetilde{a}_2 \Longleftrightarrow a_1^- = a_2^-$ and $a_1^+ = a_2^+$,
(3) $\widetilde{a}_1 < \widetilde{a}_2 \Longleftrightarrow \widetilde{a}_1 \leq \widetilde{a}_2$ and $\widetilde{a}_1 \neq \widetilde{a}_2$,

(4) $k\widetilde{a} = [ka^-, ka^+]$, whenever $0 \leq k \leq 1$.

It is clear that $(D[0,1], \leq, \vee, \wedge)$ is a complete lattice with $\tilde{0} = [0,0]$ as the least element and $\tilde{1} = [1,1]$ as the greatest element. By an interval valued fuzzy set F on X, we mean the set,

$$F = \{(x,\ [\mu_F^-(x), \mu_F^+(x)]) \mid x \in X\},$$

where μ_F^- and μ_F^+ are two fuzzy subsets of X such that $\mu_F^-(x) \leq \mu_F^+(x)$ for all $x \in X$. Putting $\widetilde{\mu_F}(x) = [\mu_F^-(x), \mu_F^+(x)]$, then we see that

$$F = \{(x, \widetilde{\mu_F}(x)) \mid x \in X\},$$

where $\widetilde{\mu_F} : X \longrightarrow D[0,1]$.

Theorem 48 *Let $\widetilde{\mu}$ be an interval valued fuzzy set in a semihyperring $\mathcal{R}$. Then, $\widetilde{\mu}$ is a fuzzy hyperideal of $\mathcal{R}$ if and only if for every $t \in (0,1]$, the level subset $\widetilde{\mu}_t = \{x \in \mathcal{R} | \widetilde{\mu}(x) \geq \tilde{t}\} \neq \varphi$, is a hyperideal of $\mathcal{R}$.*

Lemma 49 *Let $\widetilde{\mu}$ be an interval valued fuzzy set in a semihyperring $\mathcal{R}$. If $\mathcal{R}$ has zero element, then $\widetilde{\mu}([0,0]) \geq \widetilde{\mu}([x^-, x^+])$ for all $x \in \mathcal{R}$.*

1.7 Intuitionistic Fuzzy sets

The concept of intuitionistic fuzzy set was introduced and studied by Atanassov [16]. Intuitionistic fuzzy sets are extensions of the standard fuzzy sets. An intuitionistic fuzzy set A in a non-empty set X has the form $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$. Here, $\mu_A : X \longrightarrow [0,1]$ is the degree of membership of the element $x \in X$ to the set A, and $\nu_A : X \longrightarrow [0,1]$, is the degree of nonmembership of the element $x \in X$ to the set A. We have also $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$.

Example 50 *Consider the universe* $A = \{< 10, 0.01, 0.9 >, < 100, 0.1, 0.88 >, < 500, 0.4, 0.05 >, < 1000, 0.8, 0.1 >, < 1200, 1, 0 >\}$.

For simplicity, we can use the symbol $A = (\mu_A, \nu_A)$ instead of

$$A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}.$$

Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets of X. Then the following expressions are defined as follows:

(1) $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for all $x \in X$,

(2) $A^c = \{(x, \nu_A(x), \mu_A(x)) \mid x \in X\}$,

(3) $A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}) \mid \text{for all } x \in X\}$,

(4) $A \cup B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}) \mid \text{for all } x \in X\}$,

(5) $\circ A = \{(x, \mu_A(x), \mu_A^c(x)) \mid \text{for all } x \in X\}$,

(6) $\diamond A = \{(x, \nu_A^c(x), \nu_B(x)) \mid \text{for all } x \in X\}$.

An interval valued intuitionistic fuzzy set A in a non-empty set X has the form

$A = \{(x, \widetilde{\mu}_A(x), \widetilde{\nu}_A(x)) \mid x \in X\}$. Here, $\mu_A : X \longrightarrow [0,1]$ is the degree of membership of the element $x \in X$ to the set A and $\widetilde{\nu}_A : X \longrightarrow [0,1]$ is the degree of nonmembership of the element $x \in X$ to the set A. We have also $0 \leq \widetilde{\mu}_A(x) + \widetilde{\nu}_A(x) \leq 1$, for all $x \in X$, where $\widetilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$, $\widetilde{\nu}_A(x) = [\nu_A^-(x), \nu_A^+(x)]$.

Definition 51 *[5] Let c be a point in a non-empty set X. If $t \in (0,1]$ and $s \in [0,1)$ are two numbers such that $0 \leq t + s \leq 1$ and at the same time both values t and s are not less than 0.5. Then, the IFS*

$$c(t,s) = \langle x, c_t, 1 - c_{1-s} \rangle$$

is called an intuitionistic fuzzy point in X, where t (resp. s) is the degree of membership (resp. non-membership) of $c(t,s)$ and $c \in X$ is the support of $c(t,s)$. Let $c(t,s)$ be an intuitionistic fuzzy point in X and let $A = \langle x, \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy set in X. Then, $c(t,s)$ is said to belong to A written $c(t,s) \in A$, if $\mu_A(c) \geq t$ and $\lambda_A(c) \leq s$. We say that $c(t,s)$ is quasi-coincident with A, written $c(t,s)qA$, if $\mu_A(c) + t > 1$ and $\lambda_A(c) + s < 1$. To say that $c(t,s) \in \vee qA$ (resp. $c(t,s) \in \wedge qA$) means that $c(t,s) \in A$ or $c(t,s)qA$ (resp. $c(t,s) \in A$ and $c(t,s)qA$) and $c(t,s)\overline{\in \vee q}A$ means that $c(t,s) \in \vee qA$ does not hold.

1.8 n-Dimensional Fuzzy sets

Let $L_n = \{(a_1, a_2, ..., a_n) \mid 0 \leq a_1 \leq a_2 \leq ... \leq a_n \leq 1\}$. We set operations in L_n as follow:

1. $(a_1, a_2, ..., a_n) \leq (b_1, b_2, ..., b_n) \Leftrightarrow a_i \leq b_i (i = 1, 2, ..., \text{n})$,

   $(a_1, a_2, ..., a_n) < (b_1, b_2, ..., b_n) \Leftrightarrow a_i \leq b_i (i = 1, 2, ..., \text{n})$, where at least one $a_i \neq b_j$.

2. Let $\alpha_t = (a_1^t, a_2^t, ..., a_n^t) \in L_n (t \in T = [0,1])$. Then

$$\underset{t\in T}{\vee} a_t = (\underset{t\in T}{\vee} a_1^t, \underset{t\in T}{\vee} a_2^t, ..., \underset{t\in T}{\vee} a_n^t), \quad , \underset{t\in T}{\wedge} a_t = (\underset{t\in T}{\wedge} a_1^t, \underset{t\in T}{\wedge} a_2^t, ..., \underset{t\in T}{\wedge} a_n^t)$$

3. Let $\alpha = (a_1, a_2, ..., a_n) \in L_n$. Then $\alpha^c = (1 - a_n, 1 - a_{n-1}, ..., 1 - a_1)$.

4. $\overline{1} = (1, 1, ..., 1)$ and $\overline{0} = (0, 0, ..., 0)$.

Then $(L_n, \vee, \wedge, \overline{1}, \overline{0})$ form a De Morgan algebra.

Definition 52 *[2, 62, 69] Let X be a set and $L_n^x = \{A \mid A : X \longrightarrow L_n$ is a mapping$\}$. $A \in L_n^x$ is called an n-dimensional fuzzy set of X and denoted as*

$$A(x) = (A_1(x), A_2(x), ..., A_n(x)), \forall\, x \in X\ .$$

Definition 53 *Let A be a subset of $\mathcal{R}$ and $\hat{\mu}$ be an n-dimensional fuzzy set in a semihyperring $\mathcal{R}$ is defined by*

$$\hat{\mu}_A(x) = \begin{cases} \widehat{t} = (t_1, t_2, ..., t_n) & \textit{if } x \in A \\ \overline{0} = (0, 0, ..., 0) & \textit{otherwise} \end{cases}$$

In particular, if $A = \{x\}$, we denote $\hat{\mu}_{\{x\}}$ by $\hat{\mu}_x$ and call it a fuzzy point of $\mathcal{R}$.

If $\hat{\mu}, \hat{\nu}$ are two n-dimensional fuzzy sets, then $\hat{\mu} \subseteq \hat{\nu}$ if $\hat{\mu}(x) \subseteq \hat{\nu}(x)$ for all $x \in \mathcal{R}$. The intersection and union of two n-dimensional fuzzy sets $\hat{\mu}, \hat{\nu}$ are defined respectively as

$(\hat{\mu} \cap \hat{\nu})(x) = \hat{\mu}(x) \wedge \hat{\nu}(x)$ and $(\hat{\mu} \cup \hat{\nu})(x) = \hat{\mu}(x) \vee \hat{\nu}(x)$, for all $x \in \mathcal{R}$.

Definition 54 *Let A be a non-empty subset of a semihyperring $\mathcal{R}$. Then the n-dimensional characteristic function of A denoted and defined by*

$$\widehat{\chi}_A = \begin{cases} \overline{1} = (1, 1, ..., 1) & \textit{if } x \in A \\ \overline{0} = (0, 0, ..., 0) & \textit{otherwise} \end{cases}$$

Clearly, the n-dimensional characteristic function of any subset of $\mathcal{R}$ is an n-dimensional fuzzy subset of $\mathcal{R}$.

1.9 Fuzzy k-hyperideals

Definition 55 *[44] Let $\mathcal{R}$ be a semihyperring and μ a fuzzy set in $\mathcal{R}$. Then, μ is said to be a weak fuzzy k-hyperideal of $\mathcal{R}$ iff $\forall\, r, x, y \in \mathcal{R}$, the following axioms hold:*

(i) $\mu(z) \geq \mu(x) \wedge \mu(y),\ \forall\ z \in x + y,$

(ii) $\mu(rx) \geq \mu(x)$ and $\mu(xr) \geq \mu(x),$

(iii) $\mu(z) \geq [(\inf_{u \in z+y} \mu(u)) \vee (\inf_{v \in y+z} \mu(v))] \wedge \mu(y),\ \forall\ y, z \in \mathcal{R}.$

And a strong fuzzy k-hyperideal of $\mathcal{R}$ iff

(i) $\mu(z) \geq \mu(x) \wedge \mu(y),\ \forall\ z \in x + y,$

(ii) $\mu(rx) \geq \mu(x)$ and $\mu(xr) \geq \mu(x),$

(iii) $\mu(z) \geq [\mu(x) \vee \mu(x')] \wedge \mu(y),\ \forall\ x \in z + y$ and $x' \in y + z.$

Note that, $(\mathcal{R}, +)$ is commutative semihypergroup, therefore above conditions of weak and strong fuzzy k-hyperideal of $\mathcal{R}$ are reduced to the following conditions:

(i) $\mu(z) \geq \mu(x) \wedge \mu(y),\ \forall\ z \in x + y,$

(ii) $\mu(rx) \geq \mu(x)$ and $\mu(xr) \geq \mu(x),$

(iii) $\mu(z) \geq \mu(x) \wedge \mu(y),\ \forall\ x \in z + y.$

Proposition 56 *[44] Let μ be a fuzzy set in a semihyperring $\mathcal{R}$. Then,*

(i) μ is a fuzzy hyperideal of $\mathcal{R}$ if and only if for every $t \in (0, 1]$, the level subset $\mu_t (\neq \varphi)$ is a hyperideal of $\mathcal{R}$, where $\mu_t = \{x \in \mathcal{R} |\ \mu(x) \geq t\}$.

(ii) μ is a weak fuzzy k-hyperideal of $\mathcal{R}$ if and only if for every $t \in (0, 1]$, the level subset $\mu_t (\neq \varphi)$ is a weak k-hyperideal of $\mathcal{R}$, where $\mu_t = \{x \in \mathcal{R} |\ \mu(x) \geq t\}$.

(iii) μ is a strong fuzzy k-hyperideal of $\mathcal{R}$ if and only if for every $t \in (0, 1]$, the level subset $\mu_t (\neq \varphi)$ is a strong k-hyperideal of $\mathcal{R}$, where $\mu_t = \{x \in \mathcal{R} |\ \mu(x) \geq t\}$.

Lemma 57 *Let μ be a fuzzy hyperideal of a semihyperring $\mathcal{R}$. If $\mathcal{R}$ has a zero element, then $\mu(0) \geq \mu(x)\ \forall\ x \in \mathcal{R}$.*

Chapter 2

Fuzzy Hyperideals in Semihyperrings

2.1 Introduction

In this chapter, we pursue an algebraic approach to investigate the concept of fuzzy semihyperring and related notions in order to set the ground for future work. First, we provide basic definitions and establish some preliminary results. Then we investigate fully idempotent semihyperrings, that is, semihyperrings all of whose hyperideals are idempotent. It is proved that such semihyperrings are characterized by the property that each proper fuzzy hyperideal is the intersection of fuzzy prime hyperideals containing it. Finally, we construct the fuzzy prime spectrum of fully idempotent semihyperrings in a manner analogous to the construction of the prime spectrum in classical semiring theory.

2.2 Fuzzy hyperideals

In [44], the concept of fuzzy hyperideal of semihyperring was introduced by Hedayati and Ameri. They discussed some basic properties of fuzzy hyperideals. We extend this concept for further studies. We discuss some more general properties of fuzzy hyperideals of semihyperrings.

Theorem 58 *Let μ be a fuzzy set in a semihyperring $\mathcal{R}$. Then, μ is a fuzzy hyperideal of $\mathcal{R}$ if* *and only if for every* $t \in (0,1]$, *the level subset*

$$\mu_t = \{x \in \mathcal{R} | \mu(x) \geq t\} \neq \varphi$$

is a hyperideal of $\mathcal{R}$.

Proof. Suppose that μ is an fuzzy hyperideal of a semihyperring $\mathcal{R}$ and $t \in (0,1]$ such that $\mu_t \neq \varphi$. Let $x, y \in \mu_t$ then $\mu(x) \geq t$ and $\mu(y) \geq t$. As $\inf\limits_{z \in x \oplus y} \mu(z) \geq \mu(x) \wedge \mu(y)$, so $\inf\limits_{z \in x \oplus y} \mu(z) \geq t \Longrightarrow z \in \mu_t$, for all $z \in x \oplus y$.

For $r \in \mathcal{R}$, $\mu(rx) \geq \mu(x) \geq t$ so $\mu(rx) \geq t$. This implies $rx \in \mu_t$. Hence μ_t is a hyperideal of $\mathcal{R}$.

Conversely, assume μ_t is a hyperideal of $\mathcal{R}$, let $x, y \in \mathcal{R}$ be such that $\inf\limits_{z \in x \oplus y} \mu(z) < \mu(x) \wedge \mu(y)$. We take $t \in (0,1]$ such that $\inf\limits_{z \in x \oplus y} \mu(z) < t \leq \mu(x) \wedge \mu(y)$, then $x, y \in \mu_t$ but $z \notin \mu_t$, for all $z \in x \oplus y$. Which is contradiction, hence $\inf\limits_{z \in x \oplus y} \mu(z) \geq \mu(x) \wedge \mu(y)$. Similarly, we can show that $\mu(xr) \geq \mu(x)$ and $\mu(rx) \geq \mu(x)$, for all $r, x, y \in \mathcal{R}$. Thus μ is a fuzzy hyperideal of a semihyperring $\mathcal{R}$. ■

Now, we define fuzzy subsemihypermodules of semihypermodules and discussed some basic properties with fuzzy hyperideals.

Definition 59 *Let* M *be a right (left)* $\mathcal{R}$*-semihypermodule. A function* $\mu : M \to [0,1]$*, is called a fuzzy subsemihypermodule of* $M_{\mathcal{R}}$ $({}_{\mathcal{R}}M)$*, if the following conditions hold for all* $m_1, m_2, m \in M$*:*

(i) $\mu(0_M) = 1$

(ii) $\inf\limits_{m' \in m_1 \oplus m_2} \mu(m') \geq \mu(m_1) \wedge \mu(m_2)$*, for all* $m_1, m_2 \in M$*,*

(iii) $\mu(mr) \geq \mu(m)$*,* $(\mu(rm) \geq \mu(m))$*, for all* $r \in \mathcal{R}$ *and* $m \in M$*.*

Also note that, fuzzy subsemihypermodules of $\mathcal{R}_{\mathcal{R}}$ $({}_{\mathcal{R}}\mathcal{R})$ are called fuzzy hyperideals of $\mathcal{R}$.

Generalizing the notion of a fuzzy hypermodule [1, 53, 59], we formulate the following definition.

Definition 60 *Let* λ *be a fuzzy subsemihypermodule of a right semihypermodule* $M_{\mathcal{R}}$ *and* μ *a* *fuzzy hyperideal of $\mathcal{R}$. Then the fuzzy subset $\lambda\mu$ of M is defined by*

$$(\lambda\mu)(x) = \bigvee_{x\in\Sigma_{i=1}^{p} y_i z_i} [\bigwedge_{1\leq i\leq p} [\lambda(y_i) \wedge \mu(z_i)]]$$

where $x \in M$, $y_i \in M$, $z_i \in \mathcal{R}$ and $p \in n$.

Proposition 61 *If λ is a fuzzy subsemihypermodule of $M_{\mathcal{R}}$ and μ a fuzzy hyperideal of $\mathcal{R}$, then the fuzzy subset $\lambda\mu$ is a fuzzy subsemihypermodule of M.*

Proof. We have

$$(\lambda\mu)(0_M) = \bigvee_{0\in\Sigma_{i=1}^{p} y_i z_i} [\bigwedge_{1\leq i\leq p} [\lambda(y_i) \wedge \mu(z_i)]] \geq \lambda(0_M) \wedge \mu(0) = 1.$$

Thus $(\lambda\mu)(0_M) = 1$. Also

$$(\lambda\mu)(m) = \bigvee_{m\in\Sigma_{j=1}^{q} y'_j z'_j} [\bigwedge_{1\leq j\leq q} [\lambda(y'_j) \wedge \mu(z'_j)]],$$

and

$(\lambda\mu)(m') = \bigvee_{m'\in\Sigma_{k=1}^{r} y''_k z''_k} [\bigwedge_{1\leq k\leq r} [\lambda(y''_k) \wedge \mu(z''_k)]]$, where $m, m' \in M$.

Thus

$(\lambda\mu)(m) \wedge (\lambda\mu)(m')$

$= [\bigvee_{m\in\Sigma_{j=1}^{q} y'_j z'_j} [\bigwedge_{1\leq j\leq q} [\lambda(y'_j) \wedge \mu(z'_j)]]] \wedge [\bigvee_{m'\in\Sigma_{k=1}^{r} y''_k z''_k} [\bigwedge_{1\leq k\leq r} [\lambda(y''_k) \wedge \mu(z''_k)]]]$

(using the infinite meet distributive law)

$= \bigvee_{m\in\Sigma_{j=1}^{q} y'_j z'_j} \bigvee_{m'\in\Sigma_{k=1}^{r} y''_k z''_k} [\bigwedge_{1\leq j\leq q} (\lambda(y'_j) \wedge \mu(z'_j))] \wedge [\bigwedge_{1\leq k\leq r} (\lambda(y''_k) \wedge \mu(z''_k))]$

$\leq \bigvee_{m\oplus m'\subseteq\Sigma_{l=1}^{s} y'''_l z'''_l} [\bigwedge_{1\leq l\leq s} (\lambda(y'''_l) \wedge \mu(z'''_l))] = \inf_{m''\in m\oplus m'} \lambda\mu(m'')$

and

$(\lambda\mu)(m) = \bigvee_{m\in\Sigma_{j=1}^{q} y'_j z'_j} [\bigwedge_{1\leq j\leq q} [\lambda(y'_j) \wedge \mu(z'_j)]],$

$\leq \bigvee_{m\in\Sigma_{j=1}^{q} y'_j z'_j} [\bigwedge_{1\leq j\leq q} [\lambda(y'_j) \wedge \mu(z'_j a)]]$, where a is any element of $\mathcal{R}$.

$\leq \bigvee_{ma\subseteq\Sigma_{j=1}^{t} y''_j z''_j} [\bigwedge_{1\leq n\leq t} [\lambda(y''_n) \wedge \mu(z''_n)]] = \lambda\mu(ma)$. ■

Corollary 62 *If λ and μ are fuzzy hyperideals of $\mathcal{R}$, then $\lambda\mu$ is a fuzzy hyperideal of $\mathcal{R}$, called the product of λ and μ.*

Remark 63 *If λ and μ are fuzzy hyperideals of $\mathcal{R}$, then $\lambda \wedge \mu$ is clearly a fuzzy hyperideal of $\mathcal{R}$. In general, $\lambda \wedge \mu \neq \lambda\mu$.*

Definition 64 *If λ and μ are fuzzy hyperideals of $\mathcal{R}$. The fuzzy subset $\lambda \oplus \mu$ of $\mathcal{R}$ is defined by*

$$(\lambda \oplus \mu)(x) = \bigvee_{x \in y \oplus z} [\lambda(y) \wedge \mu(z)],$$

for $x \in \mathcal{R}$.

Proposition 65 *For fuzzy hyperideals λ and μ of $\mathcal{R}$, $\lambda \oplus \mu$ is a fuzzy hyperideal of $\mathcal{R}$ (called the sum of λ and μ).*

Proof. For any x, $x' \in \mathcal{R}$

$$\begin{aligned}
(\lambda \oplus \mu)(x) \wedge (\lambda \oplus \mu)(x') &= [\bigvee_{x \in y \oplus z} (\lambda(y) \wedge \mu(z))] \wedge [\bigvee_{x' \in y' \oplus z'} (\lambda(y') \wedge \mu(z'))] \\
&= \bigvee_{\substack{x \in y \oplus z \\ x' \in y' \oplus z'}} [[(\lambda(y) \wedge \mu(z))] \wedge [\lambda(y') \wedge \mu(z')]] \\
&= \bigvee_{\substack{x \in y \oplus z \\ x' \in y' \oplus z'}} [[(\lambda(y) \wedge \lambda(y'))] \wedge [\mu(z) \wedge \mu(z')]] \\
&\leq \bigvee_{x'' \in y'' \oplus z''} [\inf_{y'' \in y \oplus y'} \lambda(y'') \wedge \inf_{z'' \in z \oplus z'} \mu(z'')] \\
&\leq \inf_{x'' \in x \oplus x'} (\lambda \oplus \mu)(x'').
\end{aligned}$$

Again

$$\begin{aligned}
(\lambda \oplus \mu)(x) &= \bigvee_{x \in y \oplus z} [(\lambda(y) \wedge \mu(z))] \\
&\leq \bigvee_{xa \subseteq ya \oplus za} [\lambda(ya) \wedge \mu(za)] \text{ (where } a \text{ is any element of } \mathcal{R}) \\
&\leq \bigvee_{xa \subseteq y' \oplus z'} [\lambda(y') \wedge \mu(z')] = \inf_{b \in xa} (\lambda \oplus \mu)(b).
\end{aligned}$$

Hence $\lambda \oplus \mu$ is a fuzzy hyperideal of $\mathcal{R}$. ■

2.3 Fully idempotent semihyperrings

A semihyperring $\mathcal{R}$ is called fully idempotent if each hyperideal of $\mathcal{R}$ is idempotent (a hyperideal I is idempotent if $I^2 = I$), and a semihyperring $\mathcal{R}$ is said to be regular if for each $x \in \mathcal{R}$, there exist $a \in \mathcal{R}$ such that $x = xax$.

For these semihyperrings, we prove the following characterization theorem.

Theorem 66 *The following conditions for a semihyperring $\mathcal{R}$, are equivalent:*

(1) $\mathcal{R}$ is fully idempotent,

(2) Fuzzy hyperideal of $\mathcal{R}$ is idempotent.

(3) For each pair of fuzzy hyperideals λ and μ of $\mathcal{R}$, $\lambda \wedge \mu = \lambda\mu$.

If $\mathcal{R}$ is assumed to be commutative (that is, $xy = yx$ for all $x, y \in \mathcal{R}$), then the above conditions are equivalent to:

(4) $\mathcal{R}$ is regular.

Proof. (1)$\Rightarrow$(2). Let δ be a fuzzy hyperideal of $\mathcal{R}$. For any $x \in \mathcal{R}$,

$$\begin{aligned}
\delta^2(x) &= (\delta\delta)(x) \\
&= \bigvee_{x\in\Sigma_{i=1}^{p} y_i z_i} [\bigwedge_{1\leq i\leq p} (\delta(y_i) \wedge \delta(z_i))] \\
&\leq \bigvee_{x\in\Sigma_{i=1}^{p} y_i z_i} [\bigwedge_{1\leq i\leq p} (\delta(y_i z_i) \wedge \delta(y_i z_i))] \\
&= \bigvee_{x\in\Sigma_{i=1}^{p} y_i z_i} [[\bigwedge_{1\leq i\leq p} \delta(y_i z_i)] \wedge [\bigwedge_{1\leq i\leq p} \delta(y_i z_i)]] \\
&\leq \bigvee_{x\in\Sigma_{i=1}^{p} y_i z_i} [\delta(x) \wedge \delta(x)] = \delta(x).
\end{aligned}$$

Since each hyperideal of $\mathcal{R}$ is idempotent, therefore, $(x) = (x)^2$, for each $x \in \mathcal{R}$. Since $x \in (x)$ it follows that $x \in (x)^2 = \mathcal{R}x\mathcal{R}\mathcal{R}x\mathcal{R}$. Hence, $x = \Sigma_{i=1}^{q} a_i x a'_i b_i x b'_i$ where $a_i, a'_i, b_i, b'_i \in \mathcal{R}$ and $q \in n$. Now, $\delta(x) = \delta(x) \wedge \delta(x) \leq \delta(a_i x a'_i) \wedge \delta(b_i x b'_i) \quad (1 \leq i \leq q)$.

Therefore,

$$\begin{aligned}
\delta(x) &\leq \bigwedge_{1\leq i\leq q} [\delta(a_i x a'_i) \wedge \delta(b_i x b'_i)] \\
&\leq \bigvee_{x\in\Sigma_{i=1}^{q} a_i x a'_i b_i x b'_i} [\bigwedge_{1\leq i\leq q} [\delta(a_i x a'_i) \wedge \delta(b_i x b'_i)]] \\
&\leq \bigvee_{x\in\Sigma_{i=1}^{q} y_i z_i} [\bigwedge_{1\leq j\leq r} [\delta(y_i) \wedge \delta(z_i)]] \\
&= (\delta\delta)(x) = \delta^2(x).
\end{aligned}$$

Thus $\delta^2 = \delta$.

(2)$\Rightarrow$(1). Let I be a hyperideal of $\mathcal{R}$. Thus δ_I, the characteristic function of I, is a fuzzy hyperideal of $\mathcal{R}$. Hence $\delta_I^2 = \delta_I$. Therefore, $\delta_I \delta_I = \delta_I$, hence $\delta_{I^2} = \delta_I$. It follows that $I^2 = I$. Hence (2)$\Leftrightarrow$(1).

(1)$\Rightarrow$(3). Let λ and μ be any pair of fuzzy hyperideals of $\mathcal{R}$. For any $x \in \mathcal{R}$.

$$\begin{aligned}
&(\lambda\mu)(x) = \bigvee_{x\in\Sigma_{i=1}^{p} y_i z_i} [\bigwedge_{1\leq i\leq p} (\lambda(y_i) \wedge \mu(z_i))] \\
&\leq \bigvee_{x\in\Sigma_{i=1}^{p} y_i z_i} [\bigwedge_{1\leq i\leq p} (\lambda(y_i z_i) \wedge \mu(y_i z_i))] \\
&\leq \bigvee_{x\in\Sigma_{i=1}^{p} y_i z_i} [[\bigwedge_{1\leq i\leq p} \lambda(y_i z_i)] \wedge [\bigwedge_{1\leq i\leq p} \mu(y_i z_i)]] \\
&\leq \bigvee_{x\in\Sigma_{i=1}^{p} y_i z_i} [\lambda(x) \wedge \mu(x)] \\
&= \lambda(x) \wedge \mu(x) = (\lambda \wedge \mu)(x).
\end{aligned}$$

Again, since $\mathcal{R}$ is fully idempotent, $(x) = (x)^2$, for any $x \in \mathcal{R}$. Hence, as argued in the first part of the proof of this theorem, we have

$$\begin{aligned}
&(\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x) \\
&\leq \bigvee_{x\in\Sigma_{i=1}^{p} y_i z_i} [\bigwedge_{1\leq i\leq p} (\lambda(y_i) \wedge \mu(z_i))] \\
&= (\lambda\mu)(x).
\end{aligned}$$

Thus $\lambda \wedge \mu = \lambda\mu$.

(3)$\Rightarrow$(1). Let λ and μ be any pair of fuzzy hyperideals of $\mathcal{R}$. We have, $\lambda \wedge \mu = \lambda\mu$. take $\mu = \lambda$.

Thus $\lambda \wedge \lambda = \lambda^2$, that is, $\lambda = \lambda^2$, where λ is any fuzzy hyperideal of $\mathcal{R}$. Hence, (3)$\Rightarrow$(2). Since we already proved that (1) and (2) are equivalent, hence it follows that (3)$\Rightarrow$(1) and (1)$\Rightarrow$(3). This establishes (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3). Finally, if $\mathcal{R}$ is commutative then it is easy to verify that (1)$\Leftrightarrow$(4).

Next, we prove another characterization theorem for fully idempotent semihyperrings. ■

Theorem 67 *Let $\mathcal{R}$ be a semihyperring, Then, the following conditions are equivalent;*

(1) $\mathcal{R}$ is fully idempotent,

(2) The set of all fuzzy hyperideals of $\mathcal{R}$ (ordered by $\leq$) forms a distributive lattice $FI_{\mathcal{R}}$ under the sum and intersection of fuzzy hyperideals with $\lambda \wedge \mu = \lambda\mu$, for each pair of fuzzy hyperideals λ, μ of $\mathcal{R}$.

Proof. (1)$\Leftrightarrow$(2). The set $FI_{\mathcal{R}}$ of all fuzzy hyperideals of $\mathcal{R}$ (ordered by $\leq$) is clearly a lattice under the sum and intersection of fuzzy hyperideals. Moreover, since $\mathcal{R}$ is a fully idempotent semihyperring, it follows from above Theorem that $\lambda \wedge \mu = \lambda\mu$, for each pair of fuzzy hyperideals λ, μ of $\mathcal{R}$. We now show that $FI_{\mathcal{R}}$ is a distributive lattice, that is, for fuzzy hyperideals λ, δ and η of $\mathcal{R}$, we have

$[(\lambda \wedge \delta) \oplus \eta)] = [(\lambda \oplus \eta) \wedge (\delta \oplus \eta)]$.

For any $x \in \mathcal{R}$,

$[(\lambda \wedge \delta) \oplus \eta)](x) = \bigvee_{x \in y \oplus z} [(\lambda \wedge \delta)(y) \wedge \eta(z)]$

$= \bigvee_{x \in y \oplus z} [\lambda(y) \wedge \delta(y) \wedge \eta(z)]$

$= \bigvee_{x \in y \oplus z} [\lambda(y) \wedge \delta(y) \wedge \eta(z) \wedge \eta(z)]$

$= \bigvee_{x \in y \oplus z} [\lambda(y) \wedge \eta(z)] \wedge [\delta(y) \wedge \eta(z)]$

$= \bigvee_{x \in y \oplus z} [(\lambda \oplus \eta)(x) \wedge (\delta \oplus \eta)(x)]$

because, for $x \in y \oplus z$, $\lambda(y) \oplus \eta(z) \leq (\lambda \oplus \eta)(x)$ and, similarly, $\delta(y) \wedge \eta(z) \leq (\delta \oplus \eta)(x)$

$= (\lambda \oplus \eta)(x) \wedge (\delta \oplus \eta)(x)$

$= [(\lambda \oplus \eta)(x) \wedge (\delta \oplus \eta)](x)$

Again,

$[(\lambda \oplus \eta) \wedge (\delta \oplus \eta)](x)$

$= [(\lambda \oplus \eta)(\delta \oplus \eta)](x)$

$= \bigvee_{x \in \Sigma_{i=1}^{p} y_i z_i} [\bigwedge_{1 \leq i \leq p} [(\lambda \oplus \eta)(y_i) \wedge (\delta \oplus \eta)(z_i)]]$

$= \bigvee_{x \in \Sigma_{i=1}^{p} y_i z_i} [\bigwedge_{1 \leq i \leq p} [[\bigvee_{y_i \in r_i \oplus s_i} (\lambda(r_i) \wedge \eta(s_i))] \wedge [\bigvee_{z_i \in t_i \oplus u_i} (\delta(t_i) \wedge \eta(u_i)]]]$

$= \bigvee_{x \in \Sigma_{i=1}^{p} y_i z_i} [\bigwedge_{1 \leq i \leq p} [\bigvee_{\substack{y_i \in r_i \oplus s_i \\ z_i \in t_i \oplus u_i}} [(\lambda(r_i) \wedge \eta(s_i) \wedge (\delta(t_i) \wedge \eta(u_i))]]]$

using infinite meet distributive law

$= \bigvee_{x \in \Sigma_{i=1}^{p} y_i z_i} [\bigwedge_{1 \leq i \leq p} [\bigvee_{\substack{y_i \in r_i \oplus s_i \\ z_i \in t_i \oplus u_i}} [(\lambda(r_i) \wedge \eta(s_i) \wedge \eta(s_i) \wedge (\delta(t_i) \wedge \eta(u_i)]]]$

$\leq \bigvee_{x \in \Sigma_{i=1}^{p} y_i z_i} [\bigwedge_{1 \leq i \leq p} [\bigvee_{\substack{y_i \in r_i \oplus s_i \\ z_i \in t_i \oplus u_i}} [(\lambda(r_i t_i) \wedge \delta(r_i t_i) \wedge \eta(s_i t_i) \wedge \eta(s_i u_i) \wedge \eta(r_i u_i)]]]$

$\leq \bigvee_{x \in \Sigma_{i=1}^{p} y_i z_i} [\bigwedge_{1 \leq i \leq p} [\bigvee_{\substack{y_i \in r_i \oplus s_i \\ z_i \in t_i \oplus u_i}} [(\lambda \wedge \delta)(r_i t_i) \wedge \eta(s_i t_i \oplus s_i u_i \oplus r_i u_i)]]]$

$\leq \bigvee_{x \in \Sigma_{i=1}^{p} y_i z_i} [\bigwedge_{1 \leq i \leq p} [(\lambda \wedge \delta) \wedge \eta](y_i z_i)]$

$\leq \bigvee_{x \in \Sigma_{i=1}^{p} y_i z_i} [(\lambda \wedge \delta) \wedge \eta](x)]$

$= [(\lambda \wedge \delta) \wedge \eta](x)$.

$[(\lambda \oplus \eta) \wedge (\delta \oplus \eta)] = [(\lambda \wedge \delta) \wedge \eta]$.

(2)$\Leftrightarrow$(1). Suppose that the set $FI_{\mathcal{R}}$, of all the fuzzy hyperideals of $\mathcal{R}$ (ordered by $\leq$) is a distributive lattice under the sum and intersection of fuzzy hyperideals with $\lambda \wedge \mu = \lambda\mu$, for each pair of fuzzy hyperideals λ, μ of $\mathcal{R}$.

Then for any fuzzy hyperideals λ of $\mathcal{R}$, we have $\lambda^2 = \lambda.\lambda = \lambda \wedge \lambda =$g.l.b. of $\{\lambda, \lambda\} = \lambda$. Hence $\mathcal{R}$ is fully idempotent. ■

Definition 68 *A hyperideal I of a semihyperring $\mathcal{R}$ is called a prime hyperideal of $\mathcal{R}$ if for all hyperideals A, B of $\mathcal{R}$, $AB \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$.*

Definition 69 *A hyperideal I of a semihyperring $\mathcal{R}$ is called an irreducible if for all hyperideals A, B of $\mathcal{R}$, $A \cap B = I$, implies $A = I$ or $B = I$.*

Definition 70 *A fuzzy hyperideal η of a semihyperring $\mathcal{R}$ is called a fuzzy prime hyperideal of $\mathcal{R}$ if for fuzzy hyperideals λ and μ of $\mathcal{R}$, $\lambda\mu \leq \eta \Rightarrow \lambda \leq \eta$ or $\mu \leq \eta$; η is called fuzzy irreducible if for fuzzy hyperideals λ, μ of $\mathcal{R}$, $\lambda \wedge \mu = \eta \Rightarrow \lambda = \eta$ or $\mu = \eta$.*

Theorem 71 *Let $\mathcal{R}$ be a fully idempotent semihyperring. For a fuzzy hyperideal η of $\mathcal{R}$, the following conditions are equivalent:*

(1) η is a fuzzy prime hyperideal,

(2) η is a fuzzy irreducible hyperideal.

Proof. (1) Assume that η is a fuzzy prime hyperideal. We show that η is fuzzy irreducible, that is , for fuzzy hyperideals λ, μ of $\mathcal{R}$, $\lambda \wedge \mu = \eta \Rightarrow \lambda = \eta$ or $\mu = \eta$. Since $\mathcal{R}$ is a fully idempotent semihyperring, the set of fuzzy hyperideals of $\mathcal{R}$ (ordered by $\leq$) is a distributive lattice under the sum and intersection of fuzzy hyperideals by Theorem 67

This implies that $\eta = g.l.b.$ of $\{\lambda, \mu\}$, since $\eta = \lambda \wedge \mu$. Thus it follows that $\lambda \leq \eta$ and $\eta \leq \mu$. On the other hand, as $\lambda \leq \mathcal{R}$ is fully idempotent, it follows from Theorem 66 (3) that $\lambda \wedge \mu = \lambda\mu$. Hence $\eta = \lambda \wedge \mu = \lambda\mu$. Since η is a fuzzy prime hyperideal, by the above definition, either $\lambda \leq \eta$ or $\mu \leq \eta$. As already noted, $\eta \leq \lambda$ and $\eta \leq \mu$; so it follows that either $\lambda = \eta$ or $\mu = \eta$. Hence η is a fuzzy irreducible hyperideal.

(2) Conversely, assume that η is a fuzzy irreducible hyperideal. We show that η is a fuzzy prime hyperideal. Suppose there exist fuzzy hyperideals λ and μ such that $\lambda\mu \leq \eta$. Since $\mathcal{R}$ is assumed to be a fully idempotent semihyperring, it follows from Theorem 66 (3) that the set of fuzzy hyperideals of $\mathcal{R}$ (ordered by $\leq$) is a distributive lattice with respect to the sum and intersection of fuzzy hyperideals. Hence the inequality $\lambda \wedge \mu \leq \eta \Rightarrow (\lambda \wedge \mu) \oplus \eta = \eta$, and using

the distributivity of this lattice, we have $\eta = (\lambda \wedge \mu) \oplus \eta = (\lambda \oplus \eta) \wedge (\mu \oplus \eta)]$. Since η is fuzzy irreducible, it follows that either $\lambda \oplus \eta = \eta$ or $\mu \oplus \eta = \eta$. This implies that either $\lambda \leq \eta$ or $\mu \leq \eta$. Hence η is a fuzzy prime hyperideal. ■

Lemma 72 *Let $\mathcal{R}$ be a fully idempotent semihyperring. If λ is a fuzzy hyperideal of $\mathcal{R}$ with $\lambda(a) = \alpha$, where a is any element of $\mathcal{R}$ and $\alpha \in [0,1]$, then there exists a fuzzy prime hyperideal η of $\mathcal{R}$ such that $\lambda \leq \eta$ and $\eta(a) = \alpha$.*

Proof. Let $X = \{\mu : \mu$ is a fuzzy hyperideal of $\mathcal{R}$, $\mu(a) = \alpha$, and $\lambda \leq \mu\}$. Then $X \neq \phi$, since $\lambda \in X$. Let τ be a totally ordered subset of X, say $\tau = \{\lambda_i : i \in I\}$. We claim that $\vee_{i \in I} \lambda_i$ is a fuzzy hyperideal of $\mathcal{R}$. Clearly, $\vee_{i \in I} \lambda_i(x) = 1$. Also, for any $x, r \in \mathcal{R}$, we have

$\underset{i}{\vee} \lambda_i(x) = \underset{i}{\vee} \lambda_i(x) \leq [\vee(\lambda_i(xr))] = \underset{i}{\vee} \lambda_i(xr)$.

Similarly, $\underset{i}{\vee} \lambda_i(x) \leq \vee \underset{i}{\lambda_i}(rx)$. Finally, we show that $\inf\limits_{z \in x \oplus y} \underset{i}{\vee} \lambda_i(z) \geq \underset{i}{\vee} \lambda_i(x) \wedge \underset{i}{\vee} \lambda_i(y)$, for any $x, y \in \mathcal{R}$. Consider

$$\begin{aligned}
&\underset{i \in I}{\vee} \lambda_i(x) \wedge \underset{i \in I}{\vee} \lambda_i(y) = \underset{i \in I}{\vee} \lambda_i(x) \wedge \underset{j \in I}{\vee} \lambda_j(y) \\
&= [\underset{i \in I}{\vee} \lambda_i(x)] \wedge \underset{j \in I}{\vee} [\lambda_j(y)] \\
&= \underset{j \in I}{\vee} [[\underset{i \in I}{\vee} \lambda_i(x)] \wedge \lambda_j(y)] \\
&= \underset{j \in I}{\vee} [\underset{i \in I}{\vee} [\lambda_i(x) \wedge \lambda_j(y)]] \\
&\leq \underset{j \in I}{\vee} [\underset{i \in I}{\vee} [\lambda_i^j(x) \wedge \lambda_i^j(y)]] \text{ where } \lambda_i^j \in \{\lambda_i : i \in I\} \\
&\leq \underset{j \in I}{\vee} [\underset{i \in I}{\vee} [\inf_{z \in x \oplus y} \lambda_i^j(z)]] \\
&\leq \underset{i,j \in I}{\vee} [\inf_{z \in x \oplus y} \lambda_i^j(z)] \\
&\leq \underset{i \in I}{\vee} [\inf_{z \in x \oplus y} \lambda_i^j(z)] \\
&= \underset{i \in I}{\vee} \inf_{z \in x \oplus y} \lambda_i^j(z).
\end{aligned}$$

Thus $\underset{i \in I}{\vee}$ is a fuzzy hyperideal of $\mathcal{R}$. Clearly $\lambda \leq \underset{i \in I}{\vee} \lambda_i$ and $\underset{i \in I}{\vee} \lambda_i(a) = \underset{i \in I}{\vee} \lambda_i(a) = \alpha$. Thus $\underset{i \in I}{\vee} \lambda_i$ is the l.u.b of τ.

Hence, by Zorn's lemma, there exists a fuzzy hyperideal η of $\mathcal{R}$ which is maximal with respect to the property that $\lambda \leq \eta$ and $\eta(a) = \alpha$. We now show that η is a fuzzy irreducible hyperideal of $\mathcal{R}$. Suppose $\eta = \delta_1 \wedge \delta_2$, where δ_1 and δ_2 are fuzzy hyperideal of $\mathcal{R}$. Since $\mathcal{R}$ is assumed to be a fully idempotent semihyperring, so by Theorem 67, the set of fuzzy hyperideals of $\mathcal{R}$ (ordered by $\leq$) is a distributive lattice under the sum and intersection of fuzzy hyperideals. Hence $\eta = \delta_1 \wedge \delta_2 = g.l.b.\{\delta_1, \delta_2\}$. This implies that $\eta \leq \delta_1$ and $\eta \leq \delta_2$. We claim that either

$\eta = \delta_1$ or $\eta = \delta_2$. Suppose, on the contrary, $\eta \neq \delta_1$ and $\eta \neq \delta_2$, it follows that $\delta_1(a) \neq \alpha$ and $\delta_2(a) \neq \alpha$. Hence $\alpha = \eta(a) = (\delta_1 \wedge \delta_2)(a) = \{\delta_1(a) \wedge \delta_2(a)\} \neq \alpha$, which is contradiction. Hence either $\eta = \delta_1$ or $\eta = \delta_2$.This proves that η is a fuzzy irreducible hyperideal. Hence by Theorem 71, η is a fuzzy prime hyperideal. ■

Now, we prove the main characterization theorem for fully idempotent semihyperrings.

Theorem 73 *Let $\mathcal{R}$ be a semihyperring. Then the following conditions are equivalent:*

(1) $\mathcal{R}$ is fully idempotent,

(2) The lattice of all fuzzy hyperideals of $\mathcal{R}$ (ordered by $\leq$) is a distributive lattice under the sum and intersection of fuzzy hyperideals with $\lambda \wedge \mu = \lambda\mu$, for each pair of fuzzy hyperideals λ, μ of $\mathcal{R}$.

(3) Each fuzzy hyperideal is the intersection of those fuzzy prime hyperideals of $\mathcal{R}$ which contain it. If, in addition, $\mathcal{R}$ is assumed to be commutative, then the above conditions are equivalent to:

(4) $\mathcal{R}$ is regular.

Proof. (1)$\Rightarrow$(2). This follows from Theorem 67.

(2)$\Rightarrow$(3). Let λ be a fuzzy hyperideal of $\mathcal{R}$ and let $\{\lambda_s : s \in \Omega\}$ be the family of all fuzzy prime hyperideals of $\mathcal{R}$ which contain λ. Obviously, $\lambda \leq \wedge_{s\in\Omega}\lambda_s$. We now prove that $\wedge_{s\in\Omega}\lambda_s \leq \lambda$. Let a be any element of $\mathcal{R}$. By Lemma 72, there exists a fuzzy prime hyperideal λ_t (say) such that $\lambda \leq \lambda_t$ and $\lambda(a) = \lambda_t(a)$. Thus $\lambda_t \in \{\lambda_s : s \in \Omega\}$. Hence $\wedge_{s\in\Omega}\lambda_s \leq \lambda_t$, so $\wedge_{s\in\Omega}\lambda_s(a) \leq \lambda_t(a) = \lambda(a)$. This implies that $\wedge_{s\in\Omega}\lambda_s \leq \lambda$, so $\wedge_{s\in\Omega}\lambda_s = \lambda$.

(3)$\Rightarrow$(1). Let λ be any fuzzy hyperideal of $\mathcal{R}$. Then λ^2 is also a fuzzy hyperideal of $\mathcal{R}$. Hence, according to statement (3), λ^2 can be written as $\lambda^2 = \wedge_{s\in\Omega}\lambda_s$, where $\{\lambda_s : s \in \Omega\}$ is the family of all fuzzy prime hyperideals of $\mathcal{R}$ which contains λ^2. Now $\lambda^2 \leq \lambda$ is always true. Hence, $\lambda^2 = \lambda$. Therefore, $\mathcal{R}$ is fully idempotent. Finally, if $\mathcal{R}$ is assumed to be commutaive, then as noted in Theorem 66, (1)$\Leftrightarrow$(4). This completes the proof of the theorem. ■

At the end of this section, we prove the following fuzzy theoretic characterization of regular semihyperring. First we recall the following definition.

Definition 74 *Let λ and μ be fuzzy subsets of a semihyperring $\mathcal{R}$. Then the fuzzy subset $\lambda \circ \mu$ is defined by* $(\lambda \circ \mu)(x) = \bigvee_{x=yz}[(\lambda(y) \wedge \mu(z))]$, *for all $x \in \mathcal{R}$.*

Theorem 75 *The following conditions for a semihyperring $\mathcal{R}$ are equivalent:*

(1) $\mathcal{R}$ is regular,

(2) For any right hyperideal of $\mathcal{R}$ and any left hyperideal L of $\mathcal{R}$, $\mathcal{R} \cap L = \mathcal{R}L$,

(3) For any fuzzy right hyperideal λ and any fuzzy left hyperideal μ of $\mathcal{R}$, $\lambda \wedge \mu = \lambda \circ \mu$.

Proof. For (1)$\Rightarrow$(2), we refer to Golan [43], proposition 5.27, p.63]. So we have to prove only (1)$\Rightarrow$(3). Suppose that $\mathcal{R}$ is regular. Let δ be any fuzzy right hyperideal and μ any fuzzy left hyperideal of $\mathcal{R}$. We show that $\lambda \wedge \mu = \lambda \circ \mu$. Let $x \in \mathcal{R}$. Then

$(\lambda \circ \mu)(x) = \underset{x=yz}{\vee}[\lambda(y) \wedge \mu(z)]$
$\leq \underset{x=yz}{\vee}[\lambda(yz) \wedge \mu(yz)] = \underset{x=yz}{\vee}[\lambda(x) \wedge \mu(x)]$
$= \underset{x=yz}{\vee}[\lambda(x) \wedge \mu(x)] = (\lambda \wedge \mu)(x)$.

Thus $(\lambda \circ \mu) \leq (\lambda \wedge \mu)$. This does not depend upon the hypothesis. We now show that $(\lambda \wedge \mu) \leq (\lambda \circ \mu)$. Let $x \in \mathcal{R}$. Since $\mathcal{R}$ is von Neumann regular, there exists $a \in \mathcal{R}$ such that $x = xax$. Thus

$(\lambda \wedge \mu)(x) = (\lambda(x) \wedge \mu(x)) \leq (\lambda(xa) \wedge \mu(x) \leq \underset{x=yz}{\vee}(\lambda(y) \wedge \mu(z)) = (\lambda \circ \mu)(x)$.

Hence $\lambda \circ \mu = \lambda \wedge \mu$. Conversely, assume that $\lambda \wedge \mu = \lambda \circ \mu$ for any fuzzy right hyperideal λ and any left hyperideal μ of $\mathcal{R}$. We show that $\mathcal{R}$ is regular. Let $x \in \mathcal{R}$, $x\mathcal{R}$ and $\mathcal{R}x$ are the principal right and left hyperideals of $\mathcal{R}$, respectively, which are generated by x. Thus, if $\delta_{x\mathcal{R}}$ and $\delta_{\mathcal{R}x}$ denote, respectively, the characteristic functions of $x\mathcal{R}$ and $\mathcal{R}x$, then clearly $\delta_{x\mathcal{R}}$ and $\delta_{\mathcal{R}x}$ are fuzzy right and left hyperideals of $\mathcal{R}$. Hence, by the assumption $\delta_{x\mathcal{R}} \wedge \delta_{\mathcal{R}x} = \delta_{x\mathcal{R}} \circ \delta_{\mathcal{R}x}$. This implies that $x\mathcal{R} \cap \mathcal{R}x = x\mathcal{R}\mathcal{R}x$. Thus $x \in x\mathcal{R} \cap \mathcal{R}x = x\mathcal{R}\mathcal{R}x \subseteq x\mathcal{R}x$. Hence, there exists $x \in \mathcal{R}$ such that $x = xax$, thus showing that $\mathcal{R}$ is regular. ■

2.4 Fuzzy prime spectrum of a fully idempotent semihyperring

In this section $\mathcal{R}$ will denote a fully idempotent semihyperring, $FI_{\mathcal{R}}$ will denote the lattice of fuzzy hyperideals of $\mathcal{R}$, and $FP_{\mathcal{R}}$ the set of all proper fuzzy prime hyperideals of $\mathcal{R}$. For any fuzzy hyperideal λ of $\mathcal{R}$, we define $O_\lambda = \{\mu \in FP_{\mathcal{R}} : \lambda \nleq \mu\}$ and $\tau(FP_{\mathcal{R}}) = \{O_\lambda : \lambda \in FI_{\mathcal{R}}\}$. A fuzzy hyperideal λ of $\mathcal{R}$ is called proper if $\lambda \neq A$, where the fuzzy hyperideal A of $\mathcal{R}$ is defined by $A(x) = 1$, for all $x \in \mathcal{R}$. We prove the following:

Theorem 76 *The set $\tau(FP_{\mathcal{R}})$ forms a topology on the set $FP_{\mathcal{R}}$. The assignment $\lambda \longmapsto O_\lambda$ is an isomorphism between the lattice $FI_{\mathcal{R}}$ of fuzzy hyperideals of $\mathcal{R}$ and the lattice of open subsets of $FP_{\mathcal{R}}$.*

Proof. First we show that $\tau(FP_{\mathcal{R}})$ forms a topology on the set $FP_{\mathcal{R}}$. Note that $O_\varphi = \{\mu \in FP_{\mathcal{R}} : \varphi \nleq \mu\} = \varphi$, where φ is the usual empty set and φ is the fuzzy zero hyperideal of $\mathcal{R}$ defined by $\varphi(a) = 0$ for all $a \in \mathcal{R}$. This follows since φ is contained in every fuzzy (prime) hyperideal of $\mathcal{R}$. Thus O_φ is the empty subset of $\tau(FP_{\mathcal{R}})$. On the other hand, we have $O_A = \{\mu \in FP_{\mathcal{R}} : A \nleq \mu\} = FP_{\mathcal{R}}$. This is true, since $FP_{\mathcal{R}}$ is the set of proper fuzzy prime hyperideals of $\mathcal{R}$. So $O_A = FP_{\mathcal{R}}$ is an element of $\tau(FP_{\mathcal{R}})$. Now, let $O_{\delta_1}, O_{\delta_2} \in FP_{\mathcal{R}}$ with δ_1 and δ_2 in $FI_{\mathcal{R}}$. Then $O_{\delta_1} \cap O_{\delta_2} = \{\mu \in FP_{\mathcal{R}} : \delta_1 \nleq \mu$ and $\delta_2 \nleq \mu\}$. Since $\mathcal{R}$ is fully idempotent, therefore, $\delta_1\delta_2 = \delta_1 \wedge \delta_2$, by Theorem 66. Since μ is fuzzy prime, so $\delta_1\delta_2 \leq \mu$ implies that $\delta_1 \leq \mu$ or $\delta_2 \leq \mu$. Hence, it follows that $\delta_1\delta_2 \nleq \mu$, that is, $\delta_1 \wedge \delta_2 \nleq \mu$. Conversely, $\delta_1 \wedge \delta_2 \nleq \mu$, obviously, implies that $\delta_1 \nleq \mu$ and $\delta_2 \nleq \mu$. Thus the statements $\delta_1 \nleq \mu$ and $\delta_2 \nleq \mu$, and $\delta_1 \wedge \delta_2 \nleq \mu$, are equivalent. Hence

$O_{\delta_1} \cap O_{\delta_2} = \{\mu \in FP_{\mathcal{R}} : \delta_1 \wedge \delta_2 \nleq \mu\} = O_{\delta_1 \wedge \delta_2}$.

Let us now consider an arbitrary family

$\{\eta_i\}_{i \in I}$ of fuzzy hyperideals of $\mathcal{R}$. Since

$\underset{i\in I}{\cup} O_{\eta_i} = \underset{i\in I}{\cup} \{\mu \in FP_{\mathcal{R}} : \eta_i \nleq \mu\} = \{\mu \in FP_{\mathcal{R}} : \exists\ k \in I$ so that $\eta_i \nleq \mu\}$.

Note that

$(\underset{i\in I}{\Sigma} \eta_i)(x) = \underset{x \in x_1 \oplus x_2 \oplus x_3 \oplus \ldots}{\vee} (\eta_1(x_1) \wedge \eta_2(x_2) \wedge \eta_3(x_3) \wedge \ldots)$

where only a finite number of the $x_i's$ are not 0. Thus, since $\eta_i(0) = 1$, therefore, we are considering the infimum of a finite number of terms, because the $1's$ are effectively not being considered.

Now, if for some $k \in I$, $\eta_k \nleq \mu$, then there exists $x \in \mathcal{R}$ such that $\eta_k > \mu$. Consider the particular factorization of x for which $x_k = x$ and $x_i = 0$ for all $i \neq k$. We see that $\eta_k(x)$ is an element of the set whose supremum is defined to be $(\underset{i\in I}{\Sigma} \eta_i)(x)$. Thus, $(\underset{i\in I}{\Sigma} \eta_i)(x) \geq \eta_k(x) > \mu(x)$. Thus $(\underset{i\in I}{\Sigma} \eta_i)(x) > \mu(x)$. Hence, we have $\underset{i\in I}{\Sigma} \eta_i \nleq \mu$.

Hence, $\eta_k \nleq \mu$ for some $k \in I$ implies that $\underset{i\in I}{\Sigma} \eta_i \nleq \mu$.

Conversely, suppose that $\underset{i\in I}{\Sigma} \eta_i \nleq \mu$. Therefore, there exists an element $x \in \mathcal{R}$ such that $(O)(x) > \mu(x)$. This means that $\underset{x \in x_1 \oplus x_2 \oplus x_3 \oplus \ldots}{\vee} (\eta_1(x_1) \wedge \eta_2(x_2) \wedge \eta_3(x_3) \wedge \ldots) > \mu(x)$.

Now, if all the elements of the set, whose supremum we are taking, are individually less than or equal to $\mu(x)$, then we have

$$(\sum_{i\in I}\eta_i)(x) = \bigvee_{x\in x_1\oplus x_2\oplus x_3\oplus ...}(\eta_1(x_1)\wedge\eta_2(x_2)\wedge\eta_3(x_3)\wedge ...)\leq\mu(x)$$

which does not agree with what we have assumed. Thus, there is at least one element of the set (whose supremum we are taking), say, $\eta_1(x_1')\wedge\eta_2(x_2')\wedge\eta_3(x_3')\wedge$ ...which is greater than $\mu(x)(x\in x_1'\oplus x_2'\oplus x_3'\oplus$ ...being the corresponding breakup of x, where only a finite number of the $x_s'{}'s$ are nonzero). Thus $\eta_1(x_1')\wedge\eta_2(x_2')\wedge\eta_3(x_3')\wedge ... > \mu(x)\geq\mu(x_1')\wedge\mu(x_2')\wedge\mu(x_3')\wedge ...$

That is, $\eta_1(x_1')\wedge\eta_2(x_2')\wedge\eta_3(x_3')\wedge ... > \mu(x_1')\wedge\mu(x_2')\wedge\mu(x_3')\wedge ...$

That is, $\eta_1(x_1')\wedge\eta_2(x_2')\wedge\eta_3(x_3')\wedge ... > \mu(x_p')$

where $\mu(x_p') = \mu(x_1')\wedge\mu(x_2')\wedge\mu(x_3')\wedge ...(p\in I)$.

Hence $\eta_1(x_1') > \mu(x_p')$. It follows that $\eta_p\nleq\mu$ for some $p\in$n. Hence, $\sum_{i\in I}\eta_i\nleq\mu\Rightarrow\eta_p\nleq\mu$ for some $p\in$n. Hence, the two statements, that is,

(i) $\sum_{i\in I}\eta_i\nleq\mu$, and

(ii) $\eta_p\nleq\mu$ for some $p\in I$ are equivalent. Hence

$\bigcup_{i\in I}O_{\eta_i} = \bigvee_{i\in I}\{\mu\in FP_{\mathcal{R}}:\eta_i\nleq\mu\}$

$= \bigcup_{i\in I}\{\mu\in FP_{\mathcal{R}}:\sum_{i\in I}\eta_i\nleq\mu\}$

$= O_{\sum_{i\in I}\eta_i}$

because, $\sum_{i\in I}\eta_i$ is also a fuzzy hyperideal of $\mathcal{R}$. Thus, $\bigcup_{i\in I}O_{\eta_i}\in\tau(FP_{\mathcal{R}})$. Hence it follows that $\tau(FP_{\mathcal{R}})$ forms a topology on the set $FP_{\mathcal{R}}$. Let $\phi: FI_{\mathcal{R}}\longrightarrow FP_{\mathcal{R}}$ be the mapping defined by $\lambda\longrightarrow O_\lambda$. It follows from the above that the prescription $\phi(\lambda)=O_\lambda$ preserves finite intersection and arbitrary union. Thus ϕ is a lattice homomorphism. To conclude the proof, we must show that ϕ is bijective. In fact, we need to prove the equivalence $\delta_1=\delta_2$, if and only if $O_{\delta_1}=O_{\delta_2}$, for δ_1, δ_2 in L_A. Suppose that $O_{\delta_1}=O_{\delta_2}$. If $\delta_1\neq\delta_2$, then there exists $x\in\mathcal{R}$ such that $\delta_1(x)\neq\delta_2(x)$. Thus, either $\delta_1(x)>\delta_2(x)$ or $\delta_2(x)>\delta_1(x)$. Suppose that $\delta_1(x)>\delta_2(x)$. Using Lemma 72, there exists a fuzzy prime hyperideal μ of $\mathcal{R}$ such that $\delta_2\leq\mu$ and $\delta_2(x)=\mu(x)$. Hence, $\delta_1\nleq\mu$, because $\delta_1(x)>\delta_2(x)=\mu(x)$. Therefore, $\delta_1(x)>\mu(x)$. Thus, $\mu\in O_{\delta_1}$. Our assumption is that $O_{\delta_1}=O_{\delta_2}$. Hence, we have $\mu\in O_{\delta_2}$. Hence $\delta_2\nleq\mu$. this is a contradiction. If, in the beginning, we had assumed that $\delta_2(x)>\delta_1(x)$ then, again, by similar reasoning, we get a contradiction. Thus, $O_{\delta_1} = O_{\delta_2}$ implies that $\delta_1 = \delta_2$. Conversely, if $\delta_1 = \delta_2$, then, by definition, it is obvious that $O_{\delta_1} = O_{\delta_2}$. Thus, we have proved that $\delta_1 = \delta_2$ if and only if $O_{\delta_1} = O_{\delta_2}$ for $\delta_1$and δ_2 in L_A. This completes the proof of the theorem.

The set $FP_{\mathcal{R}}$ will be called the fuzzy prime spectrum of $\mathcal{R}$ and the topology $\tau(FP_{\mathcal{R}})$ constructed in the above theorem will be called the fuzzy spectral topology on $FP_{\mathcal{R}}$. The associated topological space will be called the fuzzy spectral space of $\mathcal{R}$. ∎

Chapter 3

On (α, β)-Fuzzy Hyperideals of Semihyperrings

In this chapter we concentrate on the concept of quasi-coincidence of fuzzy point with a fuzzy subset. By using this idea, the notion of (α, β)-fuzzy hyperideal in a semihyperring introduced and consequently, a generalization of fuzzy hyperideals is defined. We also study the related properties of the (α, β)-fuzzy hyperideals and in particular, an $(\in, \in \vee q)$-fuzzy hyperideals in semihyperrings will be investigated. Moreover, we also consider the concept of implication-based fuzzy hyperideals in a semihyperring and obtained some useful results. In what follows, $\mathcal{R}$ will denote a semihyperring, and α, β will denote one of $\in, q, \in \vee q$, or $\in \wedge q$ unless otherwise specified. Also $\overline{\alpha}$ means α does not hold.

Definition 77 *A fuzzy set μ in $\mathcal{R}$ is called an (α, β)-fuzzy hyperideal of $\mathcal{R}$, where $\alpha \neq \in \wedge q$, if for all $r, x, y \in \mathcal{R}$ and $t_1,\ t_2 \in (0, 1]$, the following conditions hold:*

(i) $x_{t_1} \alpha \mu$ and $y_{t_2} \alpha \mu$ imply $(z)_{t_1 \wedge t_2} \beta \mu$, for all $z \in x + y$,

(ii) $x_{t_1} \alpha \mu$ implies $(rx)_{t_1} \beta \mu$ and $(xr)_{t_1} \beta \mu$.

where $t_1 \wedge t_2 = \min\{t, t_2\}$. Let μ be a fuzzy set in $\mathcal{R}$ such that $\mu(x) \leq 0.5$ for all $x \in \mathcal{R}$. Suppose that $x \in \mathcal{R}$ and $t \in (0, 1]$, such that $x_t \in \wedge q\mu$. Then $\mu(x) \geq t$ and $(\mu(x) + t > 1)$. It follows that $1 < \mu(x) + t \leq \mu(x) + \mu(x) = 2\mu(x)$, so that $\mu(x) > \frac{1}{2}$. This means that $\{x_t |\ x_t \in \wedge q\mu\} = \varphi$. Therefore, the case $\alpha = \in \wedge q$ in Definition 77 is omitted.

In the next theorem , by an (α, β)-fuzzy hyperideal of $\mathcal{R}$, we construct an ordinary hyperideal of $\mathcal{R}$.

Theorem 78 *Let μ be a non-zero (α, β)-fuzzy hyperideal of $\mathcal{R}$. Then, the set* *$supp(\mu) = \{x \in \mathcal{R} | \ \mu(x) > 0\}$ is a hyperideal of $\mathcal{R}$.*

Proof. Suppose that $x, y \in \text{supp}(\mu)$. Then $\mu(x) > 0$ and $\mu(y) > 0$. Assume that $\mu(z) = 0$, for all $z \in x+y$. If $\alpha \in \{\in, \in \vee q\}$ then $x_{\mu(x)} \alpha \mu$ and $y_{\mu(y)} \alpha \mu$. But, for all $z \in x+y$, $(z)_{\mu(x) \wedge \mu(y)} \overline{\beta} \mu$, for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Note that $x_{t_1} q \mu$ and $y_{t_2} q \mu$ but, for all $z \in x+y$, $(z)_{1 \wedge 1} = (z)_1 \overline{\beta} \mu$, for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Hence, $\mu(z) > 0$, for all $z \in x+y$, that is, for all $z \in x+y$, $z \in \text{supp}(\mu)$. Also, let there exists $r \in \mathcal{R}$ such that $\mu(xr) = 0$. If $\alpha \in \{\in, \in \vee q\}$, then $x_{\mu(x)} \alpha \mu$. But $(xr)_{t_1} \overline{\beta} \mu$, for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. We know that $x_1 q \mu$. But $(xr)_1 \overline{\beta} \mu$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Hence, $\mu(xr) > 0$, that is $xr \in \text{supp}(\mu)$. Similarly, we can show that $rx \in \text{supp}(\mu)$. Therefore, $\text{supp}(\mu)$ is a hyperideal of $\mathcal{R}$.

In the next theorem, we see that a (q, q)-fuzzy hyperideal is constant under suitable condition. ■

Theorem 79 *Let $\mathcal{R}$ have a zero element and μ be a non-zero (q, q)-fuzzy hyperideal of $\mathcal{R}$, then, μ is constant on $supp(\mu)$.*

Proof. By Lemma 46, we know that $\mu(0) = \vee\{\mu(x) | x \in \mathcal{R}\}$. Suppose that there exists $x \in \text{supp}(\mu)$ such that $t_x = \mu(x) \neq t_0$, then $t_x < t_0$. Choose $t_1, t_2 \in (0, 1]$ such that $1 - t_0 < t_1 < 1 - t_x < t_2$. Then $0_{t_1} q \mu$ and $x_{t_2} q \mu$ but $(z)_{t_1 \wedge t_2} = x_{t_1} \overline{q} \mu$ for all $z \in 0 + x$ and $(z)_{t_1 \wedge t_2} = x_{t_1} \overline{q} \mu$ for all $z \in x + 0$, which is a contradiction. Thus, $\mu(x) = \mu(0)$ for all $x \in \text{supp}(\mu)$. Therefore, μ is constant on $\text{supp}(\mu)$. In the following theorem, we investigate some conditions that make a fuzzy set μ in $\mathcal{R}$ as a $(q, \in \vee q)$-fuzzy hyperideal. ■

Theorem 80 *Let I be a hyperideal of $\mathcal{R}$ and μ a fuzzy set in $\mathcal{R}$ such that*

(i) $\forall \ x \in \mathcal{R} \setminus I$, $\mu(x) = 0$,

(ii) $\forall \ x \in I$, $\mu(x) \geqslant 0.5$.

Then, μ is an $(q, \in \vee q)-$fuzzy hyperideal of $\mathcal{R}$.

Proof. Suppose that $x, y \in \mathcal{R}$ and $t_1, t_2 \in (0, 1]$ such that $x_{t_1} q\mu$ and $y_{t_2} q\mu$. Then $x, y \in I$, and so $z \subseteq I$ for all $z \in x + y$. We can consider the following cases:

(1) $t_1 \wedge t_2 \leq 0.5$, then $\mu(z) \geq 0.5 \geq t_1 \wedge t_2$, for all $z \in x + y$ and hence $(z)_{t_1 \wedge t_2} \in \mu$ for all $z \in x + y$.

(2) $t_1 \wedge t_2 > 0.5$, then $\mu(z) + t_1 \wedge t_2 > 0.5 + 0.5 = 1$ and so $(z)_{t_1 \wedge t_2} q\mu$. Therefore, $(z)_{t_1 \wedge t_2} \in \vee q\mu$ for all $z \in x + y$.

Now, suppose that $r \in \mathcal{R}$ and $t \in (0, 1]$ such that $x_t q\mu$. Then, $x \in I$, and so $rx \in I$. We can see two following cases:

(1) $t \leq 0.5$, then $\mu(rx) \geq 0.5 \geq t$ and hence $(rx)_t \in \mu$. Similarly, $(xr)_t \in \mu$.

(2) $t > 0.5$, then $\mu(rx) + 1 > 0.5 + 0.5 = 1$ and so $(rx)_t q\mu$. Similarly, $(xr)_t q\mu$. Therefore, $(rx)_t \in \vee q\mu$ and $(xr)_t \in \vee q\mu$. This completes the proof.

Also, we have the converse of this Theorem as follows: ■

Theorem 81 *Let $\mathcal{R}$ be a semihyperring with zero and μ an $(q, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$, such that μ is not constant on supp(μ). Then, $\mu(x) \geq 0.5$, for all $x \in$ supp(μ).*

Proof. By Lemma 46, we know that $\mu(0) = \vee\{\mu(x) | x \in \mathcal{R}\}$. Assume that $\mu(x) < 0.5$ for all $x \in \mathcal{R}$. Since μ is not constant on supp (μ), there exists $x \in \text{supp}(\mu)$ such that $t_x = \mu(x) \neq \mu(0) = t_0$, then $t_x < t_0$. Choose $t_1 > 0.5$ such that $t_x + t_1 < 1 < t_0 + t_1$. Then $0_{t_1} q\mu$ and $x_{t_1} q\mu$. Since $\mu(x) + t_1 = t_x + t_1 < 1$, we have $x_{t_1} \overline{q}\mu$ and so $(z)_{1 \wedge t_1} = x_{t_1} \overline{\in \vee q}\mu$ for all $z \in 0 + x$ or $z \in x + 0$. This contradicts μ is a $(q, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$. Therefore, $\mu(y) \geq 0.5$ for some $y \in \mathcal{R}$. Also, since $\mu(0) \geq \mu(y)$, then $\mu(0) \geq 0.5$. Finally, let $t_x = \mu(x) < 0.5$ for some $x \in \text{supp}(\mu)$. Take $t_1 > 0$ such that $t_x + t_1 < 0.5$, then $x_1 q\mu$ and $0_{0.5+t_1} q\mu$. But $\mu(x) + 0.5 + t_1 = t_x + 0.5 + t_1 < 0.5 + 0.5 = 1$, which implies $x_{0.5+t_1} \overline{q}\mu$.Thus, $(z)_{1 \wedge 0.5+t_1} = x_{0.5+t_1} \overline{\in \vee q}\mu$, for all $z \in 0+x$ or $z \in x+0$, which is a contradiction. Therefore, $\mu(x) \geq 0.5$ for all $x \in \text{supp}(\mu)$. ■

A fuzzy set μ in $\mathcal{R}$ is said to be proper if Im(μ) has at least two elements. Two fuzzy sets are said to be equivalent if they have the same family of level subsets. Otherwise, they are said to be non-equivalent. Now, we can discuss on $(\in, \in)$-fuzzy hyperideal of $\mathcal{R}$ which can be expressed as the union of two proper non-equivalent $(\in, \in)$-fuzzy hyperideals.

Theorem 82 *Let $\mathcal{R}$ have proper hyperideals. A proper $(\in, \in)$-fuzzy hyperideal μ of $\mathcal{R}$ such* *that* $3 \leq \mid \mathrm{Im}(\mu) \mid < \infty$, *can be expressed as the union of two proper non-equivalent* $(\in, \in)$*-fuzzy hyperideals of* $\mathcal{R}$.

Proof. Let μ be a proper $(\in, \in)$-fuzzy hyperideal of $\mathcal{R}$ with $\mathrm{Im}(\mu) = \{t_0, t_{1,...,}t_n\}$, where $t_0 > t_1 > ... > t_n$ and n$\geq$ 2. Then, $\mu_{t_0} \subseteq \mu_{t_1} \subseteq \mu_{t_2} \subseteq ... \subseteq \mu_{t_n} = \mathcal{R}$ is the chain of level hyperideals of $\mathcal{R}$. Define fuzzy sets ν and θ in $\mathcal{R}$ by

$$\nu(x) = \left\{ \begin{array}{cc} r_1 & \text{if } x \in \mu_{t_1}, \\ t_2 & \text{if } x \in \mu_{t_2} \setminus \mu_{t_1}, \\ & . \\ & . \\ & . \\ t_n & \text{if } x \in \mu_{t_n} \setminus \mu_{t_{n-1}}, \end{array} \right\}$$

and

$$\theta(x) = \left\{ \begin{array}{cc} t_0 & \text{if } x \in \mu_{t_0}\ , \\ t_1 & \text{if } x \in \mu_{t_1} \setminus \mu_{t_0}, \\ r_2 & \text{if } x \in \mu_{t_3} \setminus \mu_{t_1}, \\ t_4 & \text{if } x \in \mu_{t_4} \setminus \mu_{t_3}, \\ & . \\ & . \\ & . \\ t_n & \text{if } x \in \mu_{t_n} \setminus \mu_{n-1}, \end{array} \right\}$$

where $t_2 < r_1 < t_1$ and $t_4 < r_2 < t_2$. Then, ν and θ are $(\in, \in)$-fuzzy hyperideal of $\mathcal{R}$,

$\mu_{t_1} \subseteq \mu_{t_2} \subseteq \mu_{t_3} \subseteq ... \subseteq \mu_{t_n} = \mathcal{R}$, and $\mu_{t_0} \subseteq \mu_{t_1} \subseteq \mu_{t_3} \subseteq ... \subseteq \mu_{t_n} = \mathcal{R}$, are the chains of level hyperideals respectively, and ν, $\theta \leq \mu$. Therefore, μ and θ are non-equivalent, and clearly, $\mu = \nu \vee \theta$. ■

3.1 Fuzzy hyperideals of type $(\in, \in \vee q)$

In this section, we investigate some results and properties of (α, β)-fuzzy hyperideals (specifically, ($\in, \in \vee q$)-fuzzy hyperideals) of $\mathcal{R}$.

Definition 83 *A fuzzy set* μ *in* $\mathcal{R}$ *is called an* $(\in, \in \vee q)$*-fuzzy hyperideal of* $\mathcal{R}$*, if for all* $r, x, y \in \mathcal{R}$ *and* t_1, $t_2 \in (0, 1]$ *the following conditions hold:*

(i) $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$ imply $(z)_{t_1 \wedge t_2} \in \vee q\mu$, for all $z \in x + y$,

(ii) $x_{t_1} \in \mu$ implies $(rx)_{t_1} \in \vee q\mu$ and $(xr)_{t_1} \in \vee q\mu$.

Theorem 84 *Every $(\in \vee q, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$ is an $(\in, \in \vee q)$-fuzzy hyperideals of $\mathcal{R}$.*

Proof. Let μ be an $(\in \vee q, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$. Suppose that $x, y \in \mathcal{R}$ and $t_1, t_2 \in (0, 1]$ such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then, $x_{t_1} \in \vee q\mu$ and $y_{t_1} \in \vee q\mu$. By the hypothesis, if it follows that $(z)_{t_1 \wedge t_2} \in \vee q\mu$, for all $z \in x + y$. Now, let $r, x \in \mathcal{R}$ and $t \in (0, 1]$ such that $x_t \in \mu$. Then, $x_t \in \vee q\mu$, so by hypothesis $(rx)_t \in \vee q\mu$ and $(xr)_t \in \vee q\mu$. Therefore, μ is an $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$. ■

Theorem 85 *Every $(\in, \in)$-fuzzy hyperideal of $\mathcal{R}$ is an $(\in, \in \vee q)$-fuzzy hyperideal.*

Proof. The proof is straight forward. ■

Proposition 86 *If I is a hyperideal of $\mathcal{R}$, then X_I (the characteristic function of I) is an $(\in, \in)$-fuzzy hyperideal of $\mathcal{R}$.*

Proof. Suppose that $x, y \in \mathcal{R}$ and $t_1, t_2 \in (0, 1]$ such that $x_{t_1} \in X_I$ and $y_{t_2} \in X_I$. Then, $X_I(x) \geq t_1 > 0$ and $X_I(x) \geq t_2 > 0$, which imply that $X_I(x) = 1 = X_I(y)$. Hence $x, y \in I$ and so $z \in I$ for all $z \in x+y$, it follows that $X_I(z) = 1 \geq t_1 \wedge t_2$, for all $z \in x+y$, that is $(z)_{t_1 \wedge t_2} \in X_I$ for all $z \in x + y$. Now, let $r, x \in \mathcal{R}$ and $t \in (0, 1]$ such that $x_t \in X_I$. Then $X_I(x) \geq t > 0$ which implies $X_I(x) = 1$. Hence, $x \in I$ so $rx \in I$ and $xr \in I$. It follows $X_I(rx) = X_I(xr) = 1 \geq t$, that is $(rx)_t \in X_I$ and $(xr)_t \in X_I$. Therefore, X_I is an $(\in, \in)$-fuzzy hyperideal of $\mathcal{R}$. ■

Conversely, if I is an ideal of $\mathcal{R}$, then by proposition 86, X_I is an $(\in, \in)$-fuzzy hyperideal of $\mathcal{R}$. Therefore, by Theorem 84, X_I is an $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$.

Example 87 *On four element semihyperring $(\mathcal{R}, +, .)$ defined by the following two tables:*

$\oplus$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{a\}$	$\{b\}$	$\{c\}$
b	$\{b\}$	$\{b\}$	$\{b\}$	$\{c\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

Consider a fuzzy set μ as follows:

$$\mu(x) = \begin{cases} 0.6, & \text{if} \quad x = 0 \\ 0.7, & \text{if} \quad x \neq 0. \end{cases}$$

It is easy to see that μ is an $(\in, \in \vee q)$-fuzzy hyperideal of $(\mathcal{R}, +, .)$.

Example 88 *Consider $\mathbb{Z}$, the set of integers. Define a hyperoperation "$\oplus$" and a binary operation "$\cdot$" on $\mathbb{Z}$ as follow $m \oplus n = \{m, n\}$ and $mn = mn \ \forall\ m, n \in \mathbb{Z}$. Clearly $(\mathbb{Z}, \oplus, \cdot)$ is a semihyperring. Now define*

$$\mu(x) = \begin{cases} 0.6, & \text{if } x \in \langle 4 \rangle, \\ 0.8, & \text{if } x \in \langle 2 \rangle \backslash \langle 4 \rangle, \\ 0.7, & \text{otherwise,} \end{cases}$$

where $\langle n \rangle$ denotes the set of all integers divisible by n, it is a routine work to calculate that μ is an $(\in, \in \vee q)$-fuzzy hyperideal of $(\mathbb{Z}, +, .)$.

In the next theorem, we prove an equivalent condition for $(\in, \in \vee q)$-fuzzy hyperideals.

Theorem 89 *A fuzzy set μ in $\mathcal{R}$ is an $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$ if and only if for all $r, x, y \in \mathcal{R}$ the following two conditions hold:*

(i) $\mu(z) \geq \mu(x) \wedge \mu(y) \wedge 0.5$, *for all $z \in x + y$.*

(ii) $\mu(rx) \geq \mu(x) \wedge 0.5$ *and* $\mu(xr) \geq \mu(x) \wedge 0.5$.

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$ and $x, y \in \mathcal{R}$. We can consider the following cases:

(1) $\mu(x) \wedge \mu(y) < 0.5$. In this case, if $\mu(z) < \mu(x) \wedge \mu(y)$, for all $z \in x + y$. We can choose $t \in (0, 0.5)$ such that, for all $z \in x + y$, $\mu(z) < t < \mu(x) \wedge \mu(y)$. Then $x_t \in \mu$ and $y_t \in \mu$, but $(z)_t \overline{\in \vee q} \mu$, for all $z \in x + y$, which is a contradiction. Thus, for all $z \in x + y$, $\mu(z) \geq \mu(x) \wedge \mu(y) = \mu(x) \wedge \mu(y) \wedge 0.5$.

(2) $\mu(x) \wedge \mu(y) \geq 0.5$. In this case, we have $\mu_{0.5} \in \mu$ and $y_{0.5} \in \mu$. If for all $z \in x + y$, $\mu(z) < 0.5$, then, for all $z \in x + y$, $(z)_{0.5} \overline{\in} \mu$ and $\mu(z) + 0.5 < 1$ (or $(z)_{0.5} \overline{q} \mu$, for all $z \in x + y$). Hence, $(z)_{0.5 \wedge 0.5} \overline{\in \vee q} \mu$, for all $z \in x + y$, which is a contradiction. Thus, $\mu(z) \geq 0.5 = \mu(x) \wedge \mu(y) \wedge 0.5$, for all $z \in x + y$. Also, if $r, x \in \mathcal{R}$, we can consider two following cases:

(1) $\mu(x) \geq 0.5$. In this case, if $\mu(xr) < \mu(x)$, we can choose $t \in (0, 0.5)$ such that $\mu(rx) < t < \mu(x)$. Then $x_t \in \mu$, but $(rx)_t \overline{\in \vee q\mu}$, which is a contradiction. Thus, $\mu(rx) \geq \mu(x) = \mu(x) \wedge 0.5$. Similarly, $\mu(xr) \geq \mu(x) = \mu(x) \wedge 0.5$.

(2) $\mu(x) \geq 0.5$. In this case, we have $\mu_{0.5} \in \mu$. If $\mu(rx) < 0.5$, then $(rx)_{0.5} \overline{\in} \mu$ and $\mu(rx) + 0.5 < 1$ (or $(rx)_{0.5} \overline{q} \mu$). Hence $(rx)_{0.5} \overline{\in \vee q\mu}$, which is a contradiction. Thus, $\mu(rx) \geq 0.5 = \mu(x) \wedge 0.5$. Similarly, $\mu(xr) \geq \mu(x) \wedge 0.5$.

Conversely, suppose that μ satisfies condition (i) and (ii). Let $x, y \in \mathcal{R}$ and $t_1, t_2 \in (0, 1]$ such that $x_{t_1} \in \mu$ and $y_{t_2} \in \mu$. Then, $\mu(x) \geq t_1$ and $\mu(y) \geq t_2$. Suppose that $\mu(z) < t_1 \wedge t_2$ for all $z \in x + y$. If $\mu(x) \wedge \mu(y) < 0.5$, then, for all $z \in x + y$, $\mu(z) \geq \mu(x) \wedge \mu(y) \wedge 0.5 = \mu(x) \wedge \mu(y) > t_1 \wedge t_2$, which is a contradiction, so $\mu(x) \wedge \mu(y) \geq 0.5$. It follows that, for all $z \in x + y$, $\mu(z) + (t_1 \wedge t_2) > 2\mu(z) \geq 2(\mu(x) \wedge \mu(y) \wedge 0.5) = 1$. Hence, $(z)_{t_1 \wedge t_1} q\mu$, for all $z \in x + y$, which implies $(z)_{t_1 \wedge t_1} \in \vee q\mu$, for all $z \in x + y$. Also, let $r, x \in \mathcal{R}$ and $t \in (0, 1]$ such that $x_t \in \mu$, then , $\mu(x) \geq t$. Suppose that $\mu(rx) < t$. If $\mu(x) < 0.5$, then $\mu(rx) \geq \mu(x) \wedge 0.5 = \mu(x) \geq t$, which is contradiction, and so $\mu(x) \geq 0.5$. It follows that $\mu(rx) + t > 2\mu(rx) \geq 2(\mu(x) \wedge 0.5) = 1$. Hence, $(rx)_t q\mu$, which implies $(rx)_t \in \vee q\mu$. Similarly, $(xr)_t \in \vee q\mu$. Therefore, μ is an $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$. ■

In the following theorem, we characterize $(\in, \in \vee q)$-fuzzy hyperideals based on level subsets.

Theorem 90 *Let μ be a fuzzy set in $\mathcal{R}$. If μ is an $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$, then for all $0 < t \leq 0.5$, $\mu_t = \varphi$ or μ_t is a hyperideal of $\mathcal{R}$.*

Conversely, if $\mu_t (\neq \varphi)$ is a hyperideal of $\mathcal{R}$ for all $0 < t \leq 0.5$, then μ is an $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$.

Proof. Let μ be an $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$ and $0 < t \leq 0.5$. If $x, y \in \mu_t$, then $\mu(x) \geq t$ and $\mu(y) \geq t$. Hence, for all $z \in x + y$, $\mu(z) \geq \mu(x) \wedge \mu(y) \wedge 0.5 \geq t \wedge 0.5 = t$, which implies that $\mu(z) \geq t$, for all $z \in x + y$. That is $z \in \mu_t$ for all $z \in x + y$. Now, suppose that $x \in \mu_t$ and $r \in \mathcal{R}$. Then, $\mu(x) \geq t$, and hence $\mu(rx) \geq \mu(x) \wedge 0.5 \geq t \wedge 0.5 = t$. It implies $\mu(rx) \geq t$, that is $rx \in \mu_t$. Similarly, $xr \in \mu_t$. Therefore, μ_t is a hyperideal of $\mathcal{R}$.

Conversely, let μ be a fuzzy set in $\mathcal{R}$ such that $\mu_t (\neq \varphi)$ is a hyperideal of $\mathcal{R}$, for all $0 < t \leq 0.5$. If $x, y \in \mathcal{R}$, we have $\mu(x) \geq \mu(x) \wedge \mu(y) \wedge 0.5 = t_0$, $\mu(y) \geq \mu(x) \wedge \mu(y) \wedge 0.5 = t_0$, then $x, y \in \mu_{t_0}$, and so $z \in \mu_{t_0}$ for all $z \in x + y$. Now, we have $\mu(z) \geq t_0 = \mu(x) \wedge \mu(y) \wedge 0.5$, for all $z \in x+y$. Hence, condition (i) of the Theorem 89 is verified. Now, if $x \in \mathcal{R}$, we have $\mu(x) \geq \mu(x) \wedge 0.5 = t_0^{`}$. Then, $x \in \mu_{t_0}$, so $rx \in \mu_{t_0}$ for all $r \in \mathcal{R}$. Hence, $\mu(x) \geq t_0^{`} = \mu(a) \wedge 0.5$. Similarly, $\mu(xr) \geq t_0^{`} = \mu(a) \wedge 0.5$. This shows condition (ii) of the Theorem 89 holds. Therefore, μ is an $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$. In next Theorem , we discuss on level subsets in the interval $(0, 0.5]$. In the next theorem, we see what happen to the subsets in interval $(0.5, 1]$. ■

Theorem 91 *Let μ be a fuzzy set in $\mathcal{R}$. Then, $\mu_t (\neq \varphi)$ is a hyperideal of $\mathcal{R}$ for all $t \in (0.5, 1]$ if and only if for all $x, y \in \mathcal{R}$.*

(i) $\mu(x) \wedge \mu(y) \leq \mu(z) \vee 0.5$, for all $z \in x+y$,

(ii) $\mu(x) \leq \mu(rx) \vee 0.5$ and $\mu(x) \leq \mu(xr) \vee 0.5$.

Proof. Let μ be a hyperideal of $\mathcal{R}$ for all $t \in (0.5, 1]$. If there exists $x, y \in \mathcal{R}$ such that $\mu(z) \vee 0.5 < \mu\ (x) \wedge \mu(y) = t$, for all $z \in x+y$, then $t \in (0.5, 1]$, $\mu(z) < t$, $x \in \mu_t$ and $y \in \mu_t$ for all $z \in x+y$. Hence, $z \in \mu_t$, for all $z \in x+y$ and so $\mu(z) \geq t$, for all $z \in x+y$, which is a contradiction. Therefore, for all $x, y \in \mathcal{R}$, we have $\mu(z) \vee 0.5 \geq \mu\ (x) \wedge \mu(y)$, for all $z \in x+y$. Thus, (1) is proved. Also, if there exist $r, x \in \mathcal{R}$ such that $\mu(rx) \vee 0.5 < \mu(x) = t$ ($\mu(xr) \vee 0.5 < \mu(x) = t$), then $t \in (0.5, 1]$, $\mu(rx) < t$ and $x \in \mu_t$. Hence, $rx \in \mu_t$ and so $\mu(rx) \geq t$, which is a contradiction. Therefore, for all r, $x \in \mathcal{R}$, we have $\mu(rx) \vee 0.5 \geq \mu(x)$ and $\mu(xr) \vee 0.5 \geq \mu(x)$. Thus, (2) is proved.

Conversely, let (1) and (2) hold. Assume that $t \in (0.5, 1]$ and $x, y \in \mu_t$. Then, by (1) we have $0.5 < t \leq \mu(x) \wedge \mu(y) \leq \mu(z) \vee 0.5$, for all $z \in x+y$. It implies that $0.5 < t \leq \mu(z) \vee 0.5$ for all $z \in x+y$. Hence, $\mu(z) \geq t$ for all $z \in x+y$, which means $z \in \mu_t$, for all $z \in x+y$. Also, suppose that $t \in (0.5, 1]$, $x \in \mu_t$ and $r \in \mathcal{R}$. Then, by (2) we have $0.5 < t \leq \mu(x) \leq \mu(rx) \vee 0.5$. It implies $0.5 < t \leq \mu(rx) \vee 0.5$. Hence, $\mu(rx) \geq t$,which means $rx \in \mu_t$. Similarly, $xr \in \mu_t$. Therefore, μ_t is a hyperideal of $\mathcal{R}$. Let μ be a fuzzy set in $\mathcal{R}$ and J be the set of $t \in (0, 1]$ such that $\mu_t = \Phi$ or μ_t is a hyperideal of $\mathcal{R}$. If $J = (0, 1]$, then by Theorem 90, μ is an $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$. Naturally, a corresponding result should be considered when $J = (0.5, 1]$. ■

Definition 92 *A fuzzy set μ in $\mathcal{R}$ is called $(\overline{\in}, \overline{\in \wedge q})$- fuzzy hyperideal of $\mathcal{R}$ if for all $t_1, t_2 \in (0, 1]$ and r, $x, y \in \mathcal{R}$, the following condition hold:*

(i) $(z)_{t_1 \wedge t_1} \overline{\in} \mu$, for all $z \in x+y$, implies $x_{t_1} \overline{\in \wedge q \mu}$ or $y_{t_2} \overline{\in \wedge q \mu}$.

(ii) $(rx)_t \overline{\in} \mu$ or $(xr)_t \overline{\in} \mu$ implies $x_{t_1} \overline{\in \wedge q \mu}$.

Theorem 93 *Let μ be a fuzzy set in $\mathcal{R}$. Then, μ is an $(\overline{\in}, \overline{\in \wedge q})$- fuzzy hyperideal of $\mathcal{R}$ if and only if for all $r, x, y \in \mathcal{R}$,*

the following conditions hold:

(i) $\mu(z) \vee 0.5 \geq \mu(x) \wedge \mu(y)$ for all $z \in x + y$,

(ii) $\mu(rx) \vee 0.5 \geq \mu(x)$ and $\mu(xr) \vee 0.5 \geq \mu(x)$.

Proof. Let μ be an $(\overline{\in}, \overline{\in \wedge q})$-fuzzy hyperideal of $\mathcal{R}$. If there exist $x, y \in \mathcal{R}$ such that $\mu(z) \vee 0.5 < \mu\ (x) \wedge \mu(y) = t$, for all $z \in x + y$. Then $t \in (0.5, 1], (z)_t \overline{\in} \mu$, for all $z \in x + y$ and $x_t, y_t \in \mu$. By definition 92, it follows that $x_t \overline{q} \mu$ or $y_t \overline{q} \mu$. Then, ($\mu(x) \geq t$ and $\mu(x) + t \leq 1$) or ($\mu(y) \geq t$ and $\mu(y) + t \leq 1$). It follows that $t \leq 0.5$, which is a contradiction. Hence, (1) holds. Also, if there exist $r, x \in \mathcal{R}$ such that $\mu(rx) \vee 0.5 < \mu(x) = t$, (or $\mu(xr) \vee 0.5 < \mu(x) = t$), then $t \in (0.5, 1]$, $(rx)_t \overline{\in} \mu$ and $x_t \in \mu$. By definition 92, it follows that $x_t \overline{q} \mu$. Then, $\mu(x) \geq t$ and $\mu(x) + t \leq 1$. It concludes that $t \leq 0.5$, which is a contradiction. Thus, (2) holds

Conversely, let conditions (1) and (2) hold. Also, let $x, y \in \mathcal{R}$ such that $(z)_{t_1 \wedge t_1} \overline{\in} \mu$ for all $z \in x + y$, then $\mu(z) < t_1 \wedge t_2$ for all $z \in x + y$ we can consider the following cases:(a) If $\mu(z) \geq \mu(x) \wedge \mu(y)$, for all $z \in x + y$, then $\mu(x) \wedge \mu(y) < t_1 \wedge t_2$ and so $\mu(x) < t_1$, $\mu(y) < t_2$. It follows that $x_{t_1} \overline{\in} \mu$ or $y_{t_2} \overline{\in} \mu$, which implies that $x_{t_1} \overline{\in \wedge q \mu}$ or $y_{t_2} \overline{\in \wedge q \mu}$.

(b) If $\mu\ (z) < \mu(x) \wedge \mu(y)$ for all $z \in x + y$, then by (i) we have $0.5 \geq \mu(x) \wedge \mu(y)$. Hence $\mu(\ z) \vee 0.5 \geq \mu(x) \wedge \mu(y)$ for all $z \in x + y$. Now, if x_{t_1}, $y_{t_2} \in \mu$, then $t_1 \leq \mu(x) \leq 0.5$ or $t_2 \leq \mu(y) \leq 0.5$. It follows that $x_t \overline{q} \mu$ or $y_t \overline{q} \mu$, which implies that $x_{t_1} \overline{\in \wedge q \mu}$ or $y_{t_2} \overline{\in \wedge q \mu}$. Now, let $r, x, y \in \mathcal{R}$ such that $(rx)_t \overline{\in} \mu$ or $(xr)_t \overline{\in} \mu$, then $\mu(rx) < t$ or $\mu(xr) < t$. We can consider two following cases:

(a) If $\mu(rx) \geq \mu(x)$, then $\mu(x) < t$. It follows that $x_t \overline{\in} \mu$, which implies that $x_{t_1} \overline{\in \wedge q \mu}$.

(b) If $\mu(rx) \geq \mu(x)$, then by (2) we have $0.5 \geq \mu(x)$. Hence, $\mu(rx) \vee 0.5 \geq \mu(x)$. Now if $x_t \in \mu$, then $t \leq \mu(x) \leq 0.5$. It follows that $x_t \overline{q} \mu$, which implies that $x_t \overline{\in \wedge q \mu}$. Therefore, μ is an $(\overline{\in}, \overline{\in \wedge q})$- fuzzy hyperideal of $\mathcal{R}$. ∎

In the following theorem, we characterize $(\overline{\in}, \overline{\in \wedge q})$- fuzzy hyperideals based on level subsets.

Theorem 94 *A fuzzy set μ in $\mathcal{R}$ is an $(\overline{\in}, \overline{\in \wedge q})$- fuzzy hyperideal of $\mathcal{R}$ if and only if $\mu_t (\neq \Phi)$ is a hyperideal of $\mathcal{R}$ for all $t \in (0.5, 1]$.*

Proof. It follows by Theorem 90 and Theorem 93. For any fuzzy set μ in $\mathcal{R}$ and $t \in (0,1]$, we put $\overline{\mu}_t = \{x \in \mathcal{R} \ / \ x_t q \mu\}$, $\overline{\mid \mu \mid}_t = \{x \in \mathcal{R} | x_t \in \vee q \mu\}$. Clearly $\overline{\mid \mu \mid}_t = \mu_t \cup \overline{\mu}_t$. In fact, $\overline{\mu}_t$ and $\overline{\mid \mu \mid}_t$ are generalized level subsets. Now, we can characterize $(\in, \in \vee q)$-fuzzy hyperideals based on generalized level subsets. ■

Theorem 95 *A fuzzy set μ in $\mathcal{R}$ is an $(\in, \in \vee q)$- fuzzy hyperideal of $\mathcal{R}$ if and only if $\overline{\mid \mu \mid}_t$ is a hyperideal of $\mathcal{R}$ for all $t \in (0.5, 1]$.*

Proof. Let μ be an $(\in, \in \vee q)$- fuzzy hyperideal of $\mathcal{R}$ and $x, y \in \overline{\mid \mu \mid}_t$ for $t \in (0.5, 1]$. Then, $x_t \in \vee q \mu$ and $y_t \in \vee q \mu$, which means $\mu(x) \geq t$ or $\mu(x) + t > 1$, and $\mu(y) \geq t$ or $\mu(y) + t > 1$. On the other hand, by Theorem 89, we know $\mu\ (z) \geq \mu(x) \wedge \mu(y) \wedge 0.5$ for all $z \in x + y$, so $\mu\ (z) \geq t \wedge 0.5$, for all $z \in x + y$ (since, if $\mu\ (z) < t \wedge 0.5$, for all $z \in x + y$, then $\mu(x) \wedge \mu(y) \wedge 0.5 \leq \mu\ (z) < t \wedge 0.5$, for all $z \in x+y$, which implies $\mu(x) \wedge \mu(y) \wedge 0.5 < t \wedge 0.5$). Hence, $\mu(x) < t$ or $\mu(y) < t$, that is $x_t \overline{\in} \mu$ or $y_t \overline{\in} \mu$. Thus, $x_t \overline{\in \vee q \mu}$ or $y_t \overline{\in \vee q \mu}$ which is contradiction. We know $t \in 0.5$, then $\mu\ (z) \geq t \wedge 0.5 = 1$ and so $z \in \mu_t \subseteq \overline{\mid \mu \mid}_t$ for all $z \in x + y$. Also, let $r \in \mathcal{R}$ and $x \in \overline{\mid \mu \mid}_t$ for $t \in (0, 0.5]$. Then, $x_t \in \vee q \mu$ which means $\mu(x) \geq t$ or $\mu(x) + t > 1$. On the other hand, by Theorem 89, we know that $\mu(rx) \geq \mu(x) \wedge 0.5$, so $\mu(rx) \geq t \wedge 0.5$ (since if $\mu(rx) < t \wedge 0.5$, then $\mu(x) \wedge 0.5 \leq \mu(rx) < t \wedge 0.5$). Hence, $\mu(x) < t$, that is $x_t \overline{\in} \mu$, thus $x_{t_1} \overline{\in \wedge q \mu}$, which is a contradiction. We know $t \leq 0.5$, then $\mu(rx) \geq t \wedge 0.5 = t$ and so $rx \in \mu_t \subseteq \overline{\mid \mu \mid}_t$. Similarly, $xr \in \overline{\mid \mu \mid}_t$, therefore, $\overline{\mid \mu \mid}_t$ is a hyperideal of $\mathcal{R}$.

Conversely, let $\overline{\mid \mu \mid}_t$ be a hyperideal of $\mathcal{R}$ for $t \in (0, 0.5]$. Suppose $x, y \in \mathcal{R}$ such that $\mu\ (z) < \mu(x) \wedge \mu(y) \wedge 0.5$, for all $z \in x + y$. Then, there exists $t \in (0, 0.5)$ such that $\mu(z) < t < \mu(x) \wedge \mu(y) \wedge 0.5$, for all $z \in x + y$. It follows $x, y \in \mu_t \subseteq \overline{\mid \mu \mid}_t$, which implies $z \in \overline{\mid \mu \mid}_t$, for all $z \in x+y$. Hence, $\mu\ (z) \geq t$ or $\mu\ (z) + t > 1$ for all $z \in x+y$, which is a contradiction. Therefore, $\mu\ (z) \geq \mu(x) \wedge \mu(y) \wedge 0.5$ for all $z \in x+y$. Also, suppose r, $x \in \mathcal{R}$ such that $\mu\ (rx) < \mu(x) \wedge 0.5$, then there exists $t \in (0, 0.5)$ such that $\mu\ (rx) < t < \mu(x) \wedge 0.5$. It follows $x \in \mu_t \subseteq \overline{\mid \mu \mid}_t$, which implies $rx \in \overline{\mid \mu \mid}_t$. Hence, $\mu(rx) \geq t$ or $\mu(rx) + t > 1$, which is a contradiction. Thus, $\mu(rx) \geq \mu(x) \wedge 0.5$. Similarly, $\mu(xr) \geq \mu(x) \wedge 0.5$, therefore, the proof is completed. ■

In the next theorem, we discuss on $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$ which can be expressed as the union of two proper non-equivalent $(\in, \in \vee q)$- fuzzy hyperideals.

Theorem 96 *Let μ be a proper $(\in, \in \vee q)$- fuzzy hyperideal of $\mathcal{R}$ such that*

$$2 \leq \mid \{\mu(x) \mid \mu(x) < 0.5\} \mid < \infty.$$

Then, there exist two proper non-equivalent $(\in, \in \vee q)$- fuzzy hyperideal of $\mathcal{R}$ such that μ can be expressed as the union of them.

Proof. Let $\{\mu(x) \mid \mu(x) < 0.5\} = \{t_0, t_{1,...,}t_n\}$, where $t_1 > t_2 > ... > t_r$ and $r \geq 2$. Then, the chain of $(\in \vee q)$-level hyperideals of $\mathcal{R}$ is $\overline{\mid \mu \mid}_{0.5} \subseteq \overline{\mid \mu \mid}_{t_1} \subseteq \overline{\mid \mu \mid}_{t_2} ... \subseteq \overline{\mid \mu \mid}_{t_r} = \mathcal{R}$. Let ν and θ be fuzzy sets in $\mathcal{R}$ defined by

$$\nu(x) = \left\{ \begin{array}{ll} t_1 & \text{if } x \in \overline{\mid \mu \mid}_{t_1}, \\ t_2 & \text{if } x \in \overline{\mid \mu \mid}_{t_2} \setminus \overline{\mid \mu \mid}_{t_1}, \\ & . \\ & . \\ & . \\ t_r & \text{if } x \in \overline{\mid \mu \mid}_{t_r} \setminus \overline{\mid \mu \mid}_{t_r - 1}, \end{array} \right\}$$

and

$$\theta\,(x) = \left\{ \begin{array}{ll} \mu(x) & \text{if } x \in \overline{\mid \mu \mid}_{0.5}, \\ k & \text{if } x \in \overline{\mid \mu \mid}_{t_2} \setminus \overline{\mid \mu \mid}_{0.5}, \\ t_3 & \text{if } x \in \overline{\mid \mu \mid}_{t_3} \setminus \overline{\mid \mu \mid}_{t_2}, \\ t_4 & \text{if } x \in \overline{\mid \mu \mid}_{t_4} \setminus \overline{\mid \mu \mid}_{t_3}, \\ & . \\ & . \\ & . \\ t_r & \text{if } x \in \overline{\mid \mu \mid}_{t_r} \setminus \overline{\mid \mu \mid}_{t_r - 1}, \end{array} \right\}$$

where $t_3 < k < t_{2.}$ The, ν and θ are $(\in, \in \vee q)$- fuzzy hyperideals of $\mathcal{R}$, and $\nu, \theta \leq \mu$. The chains of $(\in \vee q)$-level ideals of ν and θ are, respectively, given by $\overline{\mid \mu \mid}_{0.5} \subseteq \overline{\mid \mu \mid}_{t_1} \subseteq \overline{\mid \mu \mid}_{t_2} ... \subseteq \overline{\mid \mu \mid}_{t_r}$ and $\overline{\mid \mu \mid}_{0.5} \subseteq \overline{\mid \mu \mid}_{t_1} \subseteq \overline{\mid \mu \mid}_{t_2} ... \subseteq \overline{\mid \mu \mid}_{t_r}$. Thus, ν and θ are non-equivalent and clearly $\mu = \nu \vee \theta$. Therefore, μ be expressed as the union of two proper non-equivalent $(\in, \in \vee q)$- fuzzy hyperideals of $\mathcal{R}$. ■

3.2 t-Implication-based fuzzy hyperideals of semihyperrings

In this section, we generalize the notion of ordinary fuzzy hyperideals, $(\in, \in \vee q)$-fuzzy hyperideals and $(\overline{\in}, \overline{\in \vee q})$- fuzzy hyperideals. Specially, we characterize fuzzy hyperideals, $(\in, \in \vee q)$-fuzzy hyperideals and $(\overline{\in}, \overline{\in \vee q})$- fuzzy hyperideals based on implication operators.

Definition 97 *Let $m, n \in [0,1]$, $m < n$ and μ be a fuzzy set in $\mathcal{R}$. Then, μ is said to be a fuzzy hyperideal with thresholds (m, n) of $\mathcal{R}$, if for all $r, x, y \in \mathcal{R}$, the following conditions hold:*

(i) $\mu(x) \wedge \mu(y) \wedge n \leq \mu(z) \vee m$, for all $z \in x + y$.

(ii) $\mu(x) \wedge n \leq \mu(rx) \vee m$ and $\mu(x) \wedge n \leq \mu(xr) \vee m$.

Clearly, every fuzzy hyperideal with thresholds (m, n) of $\mathcal{R}$ is an ordinary fuzzy hyperideal when $m = 0$ and $n = 1$ (see definition 1). Also, it is an $(\in, \in \vee q)$- fuzzy (resp. $(\overline{\in}, \overline{\in \vee q})$- fuzzy) hyperideals when $m = 0$ and n= 0.5 (resp. $m = 0$ and $n = 0.5$) (see Theorem 96).

Theorem 98 *A fuzzy set μ in $\mathcal{R}$ is a fuzzy hyperideal with threshold (m, n) of $\mathcal{R}$ if and only if $\mu_t (\neq \Phi)$ is a hyperideal of $\mathcal{R}$ for all $t \in (m, n]$.*

Proof. Suppose that μ is a fuzzy hyperideal with thresholds (m, n) of $\mathcal{R}$ and $t \in (m, n]$. If $x, y \in \mu_t$, then $\mu\ (x) \geq t$ and $\mu\ (y) \geq t$. We have $\mu(z) \vee m \geq \mu(x) \wedge \mu(y) \wedge n \geq t \wedge n = t > m$, for all $z \in x + y$. Hence, $\mu(z) \vee m \geq t > m$, for all $z \in x + y$, Which implies $\mu(z) \geq t$, for all $z \in x + y$, that is $z \in \mu_t$ for all $z \in x + y$. Now, if $x \in \mu_t$ and $r \in \mathcal{R}$, then $\mu(x) \geq t$. We have $\mu(rx) \vee m \geq \mu(x) \wedge n \geq t \wedge n = t > m$. Hence, $\mu(rx) \vee m \geq t > m$, Which implies $\mu(rx) \geq t$, that is $rx \in \mu_t$. Similarly, $xr \in \mu_t$. Therefore, μ_t is a hyperideal of $\mathcal{R}$.

Conversely, let μ be a fuzzy set in $\mathcal{R}$. If there exist $x, y \in \mathcal{R}$ such that $\mu(z) \vee m < \mu(x) \wedge \mu(y) \wedge n = t$, for all $z \in x + y$. then $t \in (m, n]$, $\mu(z) < t$, $x \in \mu$ and $y \in \mu_t$, for all $z \in x + y$. Since μ_t is a hyperideal of $\mathcal{R}$, we have $z \in \mu_t$ for all $z \in x + y$. Thus, $z \subseteq \mu_t$, for all $z \in x+y$. Hence, $\mu(z) \geq t$ for all $z \in x+y$, which is a contradiction. Therefore, for all $x, y \in \mathcal{R}$, we have $\mu(x) \wedge \mu(y) \wedge \leq \mu(z) \vee m$, for all $z \in x + y$. Also, if there exist $r, x \in \mathcal{R}$ such that $\mu(rx) \vee m \geq \mu(x) \wedge n \geq t \wedge n = t$, then $t \in (m, n]$, $\mu(rx) \geq t$, which is a contradiction. Thus, for all $r, x \in \mathcal{R}$, we have $\mu\ (x) \wedge n \leq \mu(rx) \vee m$. Similarly, $\mu\ (x) \wedge n \leq \mu(xr) \vee m$. Therefore, μ is a fuzzy hyperideal with thresholds (m, n) of $\mathcal{R}$. ∎

Set theoretic multivalued logic is a special case of fuzzy logic such that the truth values are linguistic variables (or terms of the linguistic variables truth). By using extension principal some operators like $\wedge, \vee, \neg, \longrightarrow$ can be applied in fuzzy logic. In fuzzy logic, $[P]$ means the truth value of fuzzy proposition P. In the following, we show a correspondence between fuzzy logic and set-theoretical notions.

$[x \in A] = A(x)$, $\qquad [x \neq A] = 1 - A(x)$,

$[P \wedge Q] = \min\{[P], [Q]\}$, $\qquad [P \vee Q] = \max\{[P], [Q]\}$,

$[P \longrightarrow Q] = \min\{1, 1 - [P] + [Q]\}$,

$[\forall x \in P(x)] = \inf[P(x)]$, $\qquad \models P$ if and only if $[P] = 1$ for all valuations.

We show some of important implication operators, where α denotes the degree of membership of the premise and β is the degree of membership of the consequence, and I the resulting degree of truth for the implication.

Early Zadeh $\qquad I_m(\alpha, \beta) = \max\{1 - \alpha, \min\{\alpha, \beta\}\}$,

Lukasiewicz $\qquad I_\alpha(\alpha, \beta) = \min\{1 - \alpha + \beta\}$,

Standard Star (Godel) $\qquad I_g(\alpha, \beta) = \left\{ \begin{array}{ll} 1 & \alpha \leq \beta \\ \beta & \text{otherwise} \end{array} \right\}$,

Contraposition of (Godel) $\qquad I_{cg}(\alpha, \beta) = \left\{ \begin{array}{ll} 1 & \alpha \leq \beta \\ 1 - \alpha & \text{otherwise} \end{array} \right\}$

Gaines-Rescher $\qquad I_{gr}(\alpha, \beta) = \left\{ \begin{array}{ll} 1 & \alpha \leq \beta \\ 0 & \text{otherwise} \end{array} \right\}$

kleene-dienes $\qquad I_b(\alpha, \beta) = \max\{1 - \alpha, \beta\}$.

Definition 99 *A fuzzy set μ in $\mathcal{R}$ is called fuzzifying hyperideal of $\mathcal{R}$, if and only if for all $r, x, y \in \mathcal{R}$ it satisfies:*

(1) $\quad \models [[x \in \mu] \wedge [y \in \mu] \longrightarrow [x + y \in \mu]]$,

(2) $\quad \models [[x \in \mu] \wedge [rx \in \mu]$ and $\models [[x \in \mu] \longrightarrow [xr \in \mu]]$.

Clearly, definition 99 is equivalent to definition 43. Therefore, a fuzzifying hyperideal is an ordinary fuzzy hyperideal. We have the notion of $t-$tautology. In fact $\models_t P$, if and only if $[P] \geq t$ ([68]).

Definition 100 *A fuzzy set μ in $\mathcal{R}$ is said to be a t-implication-based fuzzy hyperideal of $\mathcal{R}$ with respect to the implication $\longrightarrow$ if the following conditions hold for all $r, x, y \in \mathcal{R}$:*

(1) $\models_t [[x \in \mu] \wedge [y \in \mu] \longrightarrow [z \in \mu]]$, *for all* $z \in x + y$,

(2) $\models_t [[x \in \mu] \wedge [rx \in \mu]$ *and* $\models_t [[x \in \mu] \longrightarrow [xr \in \mu]]$.

Example 101 *(1) Consider the semihyperring $(\mathcal{R}, +, .)$ defined by the following two tables:*

$+$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{a\}$	$\{b\}$	$\{c\}$
b	$\{b\}$	$\{b\}$	$\{b\}$	$\{c\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{b\}$

$.$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	a	a
c	0	a	a	a

Define a fuzzy set μ in $\mathcal{R}$ as follows:

$$\mu(x) = \left\{ \begin{array}{cc} 0.85 & \text{if } x = 0 \\ 0.8 & \text{if } x \neq 0 \end{array} \right\}$$

Then, μ is a fuzzifying hyperideal of $\mathcal{R}$. Also, μ is a 0.5-implication-based fuzzy hyperideal of $\mathcal{R}$ with respect to the

Gaines-Rescher implication operator

(2) Consider the semihyperring $(\mathbb{N}, \oplus, .)$ defined by a hyperoperation " $\oplus$ " and a binary operation " $\cdot$ " on $\mathbb{N}$ as follow $m \oplus n = \{m, n\}$ and $mn = mn$, $\forall\ m, n \in \mathbb{N}$. Clearly $(\mathbb{N}, \oplus, \cdot)$ is a semihyperring. Define a fuzzy set μ as follows:

$$\mu(x) = \left\{ \begin{array}{lr} 0.52, & \text{if } x \in \langle 4 \rangle, \\ 0.55, & \text{if } x \in \langle 2 \rangle \backslash \langle 4 \rangle, \\ 0.57, & \text{otherwise,} \end{array} \right\}$$

Then, μ is an 0.5-implication-based fuzzy hyperideal of $\mathcal{R}$ with respect to the Godel implication operator.

(3) *Consider the above semihyperring* $(\mathbb{N}, +, .)$ *and fuzzy set* μ *as follows:*

$$\mu(x) = \left\{ \begin{array}{ll} 0.2, & \textit{if } 1 \leq \ x \leq 5, \\ 0.35, & \textit{if } 5 \leq x < 7, \\ 0.37, & \textit{if } x \geq 7. \end{array} \right\}$$

Then, μ *is a* 0.5*-implication-based fuzzy hyperideal of* $\mathcal{R}$ *with respect to the contraposition of Godel implication operator.*

Proposition 102 *A fuzzy set* μ *of* $\mathcal{R}$ *is a* t*-implication-based fuzzy hyperideal of* $\mathcal{R}$ *with respect to the implication operator* I *if and only if for all* $r, x, y \in \mathcal{R}$.

(i) $I(\mu(x)) \wedge \mu(y),\ \mu(x+y) \geq t$, for all $z \in x \oplus y$,

(ii) $I(\mu(x)),\ \mu(rx) \geq t$ and $I(\mu(x)),\ \mu(xr) \geq t$.

Proof. The proof is clear by considering the definitions. ■

Theorem 103 *(1)* *Let* $I = I_{gr}$ *(Gaines-Rescher). Then,* μ *is a* 0.5*-implication-based fuzzy hyperideal of* $\mathcal{R}$ *if and only if* μ *is a fuzzy hyperideal with thresholds* $m = 0$ *and* $n = 1$ *of* $\mathcal{R}$ *(or equivalent,* μ *is an ordinary fuzzy hyperideal of* $\mathcal{R}$*).*

(2) Let $I = I_{gr}$ (Godel). Then, μ is a 0.5-implication-based fuzzy hyperideal of $\mathcal{R}$ if and only if μ is a fuzzy hyperideal with thresholds $m = 0$ and n= 0.5 of $\mathcal{R}$ (or equivalent, μ is an $(\in, \in \vee q)-$fuzzy hyperideal of $\mathcal{R}$).

(3) Let $I = I_{cg}$ (Contraposition of Godel). Then, μ is an 0.5-implication-based fuzzy hyperideal of $\mathcal{R}$ if and only if μ is a fuzzy hyperideal with thresholds $m = 0.5$ and $n = 1$ of $\mathcal{R}$ (or equivalent, μ is an $(\overline{\in}, \overline{\in \vee q})-$fuzzy hyperideal of $\mathcal{R}$).

Proof. (1) Let μ be a 0.5-implication-based fuzzy hyperideal of $\mathcal{R}$. Then $I_{gr}((\mu(x) \wedge \mu(y), \mu(z)) \geq 0.5$, for all $z \in x + y$. Which implies $\mu(z) \geq \mu(x)) \wedge \mu(y)$, for all $z \in x + y$. Also, $I_{gr}((\mu(x), \mu(rx)) \geq 0.5$, which implication $\mu(rx) \geq \mu(x)$. Similarly, $\mu(xr) \geq \mu(x)$. Therefore, μ is a fuzzy hyperideal with threshold $m = 0$ and n= 1 of $\mathcal{R}$.

Conversely, let μ be a fuzzy hyperideal with threshold $m = 0$ and $n = 1$ of $\mathcal{R}$. Then, for all $r, x, y \in R$,

$$\begin{aligned} \mu(z) &\geq \mu(x) \wedge \mu(y), \text{ for all } z \in x + y, \\ \mu(rx) &\geq \mu(x), \ \mu(xr) \geq \mu(x). \end{aligned}$$

Hence, $I_{gr}(\mu(x) \wedge \mu(y),\ \mu(z)) = 1$, for all $z \in x + y$, $I_{gr}(\mu(x),\ \mu(xr)) = 1 = I_{gr}(\mu(x),$ $\mu(rx))$. Thus, $I_{gr}((\mu(x) \wedge \mu(y),\ \mu(z)) \geq 0.5$, for all $z \in x + y$, $I_{gr}(\mu(x),\ \mu(rx)) \geq 0.5$, and $I_{gr}((\mu(x), \mu(xr)) \geq 0.5$. Therefore, μ is a 0.5-implication-based fuzzy hyperideal of $\mathcal{R}$.

(2) Let μ be a 0.5-implication-based fuzzy hyperideal of $\mathcal{R}$. Then, for all $r, x, y \in \mathcal{R}$ we have $I_g((\mu(x) \wedge \mu(y), \mu(z)) \geq 0.5$, for all $z \in x + y$, $I_g(\mu(x),\ \mu(rx) \geq 0.5$, and $I_g((\mu(x), \mu(xr)) \geq 0.5$. By the definition of I_g, we can consider the following cases:

(a) $I_g(\mu(x) \wedge \mu(y),\ \mu(z)) = 1$, for all $z \in x + y$, then $\mu(x) \wedge \mu(y) \leq \mu(z)$, for all $z \in x + y$, which implies $\mu(x) \wedge \mu(y) \wedge 0.5 \leq \mu(z)$, for all $z \in x + y$.

(b) $I_g(\mu(x) \wedge \mu(y),\ \mu(x+y)) = 1$, then $\mu(z) = \mu(z)$, for all $z \in x+y$ then $\mu(z) \geq 0.5$, for all $z \in x + y$. Which implies $\mu(x) \wedge \mu(y) \wedge 0.5 \leq \mu(z)$, for all $z \in x + y$. Similarly, we can show that $\mu(x) \wedge 0.5 \leq \mu(rx)$ and $\mu(x) \wedge 0.5 \leq \mu(xr)$. Therefore, μ is a fuzzy hyperideal with thresholds $m = 0$ and $n = 0.5$ of $\mathcal{R}$.

Conversely, let μ is a fuzzy hyperideal with thresholds $m = 0$ and $n = 0.5$ of $\mathcal{R}$. Then, for all $r, x, y \in \mathcal{R}$, by Definition 97 $\mu(x) \wedge \mu(y) \wedge 0.5 \leq \mu(z)$, for all $z \in x+y$, and $\mu(x) \wedge 0.5 \leq \mu(rx)$ and $\mu(x) \wedge 0.5 \leq \mu(xr)$. Hence, in each case, $I_g(\mu(x) \wedge \mu(y),\ \mu(z)) \geq 0.5$, for all $z \in x + y$, $I_g(\mu(x),\ \mu(rx)) \geq 0.5$, and $I_g(\mu(x),\ \mu(xr)) \geq 0.5$. Therefore, μ is an $0.5-$implication-based fuzzy hyperideal of $\mathcal{R}$.

(3) Let μ be a 0.5-implication-based fuzzy hyperideal of $\mathcal{R}$. Then, for all $r, x, y \in \mathcal{R}$, we have $I_{cg}(\mu(x) \wedge \mu(y),\ \mu(x + y)) \geq 0.5$, $I_{cg}(\mu(x),\ \mu(rx)) \geq 0.5$ and $I_{cg}(\mu(x),\ \mu(xr)) \geq 0.5$. By definition of I_{cg}, we can consider the following cases:

(a) $I_{cg}(\mu(x) \wedge \mu(y), \mu(z)) = 1$,for all $z \in x + y$. Then $\mu(x) \wedge \mu(y) \leq \mu(z)$, for all $z \in x + y$, which implies that $\mu(x) \wedge \mu(y) \leq \mu(z) \vee 0.5$, for all $z \in x + y$.

(b) $I_{cg}(\mu(x) \wedge \mu(y), \mu(z)) = 1 - (\mu(x) \wedge \mu(y))$, for all $z \in x + y$. Then $1 - (\mu(x) \wedge \mu(y) \geq 0.5$, it implies that $\mu(x) \wedge \mu(y) \leq 0.5$ and hence $\mu(x) \wedge \mu(y) \leq \mu(z) \vee 0.5$, for all $z \in x + y$. Similarly, we can show that $\mu(x) \leq \mu(rx) \vee 0.5$ and $\mu(x) \leq \mu(xr) \vee 0.5$. Therefore, μ is a fuzzy hyperideal with threshold $m = 0.5$ and $n = 1$ of $\mathcal{R}$.

Conversely, let μ be a fuzzy hyperideal with threshold $m = 0.5$ and $n = 1$ of $\mathcal{R}$. Then, for all $r, x, y \in \mathcal{R}$, we have $\mu(x) \wedge \mu(y) \leq \mu(z) \vee 0.5$, for all $z \in x + y$, $\mu(x) \leq \mu(rx) \vee 0.5$ and $\mu(x) \leq \mu(xr) \vee 0.5$. Now, we can consider two following cases:(a) $\mu(x) \wedge \mu(y) \leq \mu(z)$, for all $z \in x + y$, which implies $I_{cg}(\mu(x) \wedge \mu(y), \mu(z)) = 1 \geq 0.5$, for all $z \in x + y$.

(b) $\mu(x) \wedge \mu(y) > \mu(z)$, for all $z \in x + y$, which implies $\mu(x) \wedge \mu(y) \geq 0.5$.

Hence, $1 - (\mu(x) \wedge \mu(y)) \geq 0.5$. Thus, $I_{cg}(\mu(x) \wedge \mu(y), \mu(z)) = 1 - (\mu(x) \wedge \mu(y)) \geq 0.5$, for all $z \in x + y$. Similarly, we can prove that $I_{cg}(\mu(x), \mu(rx)) \geq 0.5$ and $I_{cg}(\mu(x), \mu(xr)) \geq 0.5$. Therefore, μ is a 0.5-implication-based fuzzy hyperideal of $\mathcal{R}$. ■

Chapter 4

Interval valued (α, β)-Fuzzy Hyperideals of Semihyperrings

In this chapter, we concentrate on the concept of quasi-coincidence of interval valued fuzzy point with an interval valued fuzzy subset. By using this idea, the notion of interval valued (α, β)-fuzzy hyperideal in a semihyperring is introduced and consequently, a generalization of interval valued fuzzy hyperideal is defined. We study the related properties of the interval valued (α, β)-fuzzy hyperideals and in particular, an interval valued $(\in, \in \vee q)$-fuzzy hyperideal in semihyperring are investigated. Moreover, we also consider the concept of implication-based interval valued fuzzy hyperideals in a semihyperring and obtain some results.

4.1 Interval valued (α, β)-fuzzy hyperideals

An interval valued fuzzy set $\tilde{\mu}$ of a semihyperring $\mathcal{R}$ of the form

$$\tilde{\mu}(y) = \begin{cases} \tilde{t} \neq [0,0] & \text{if } y = x, \\ [0,0] & \text{if } y \neq x, \end{cases}$$

is said to be a interval valued fuzzy point with support x, interval value $\tilde{t}$ and is denoted by $\mu(x;\tilde{t})$. An interval valued fuzzy point $\mu(x;\tilde{t})$ is said to be belong to (resp. quasi-coincident with) an interval valued fuzzy set $\tilde{\mu}$, written as

$\mu(x;\tilde{t}) \in \tilde{\mu}$(resp. $\mu(x;\tilde{t})q\tilde{\mu}$) if $\tilde{\mu}(x) \geq \tilde{t}$(resp. $\tilde{\mu}(x) + \tilde{t} > [1,1]$).

If $\mu(x;\tilde{t}) \in \tilde{\mu}$ or (resp. $\mu(x;\tilde{t})q\tilde{\mu}$), then we write $\mu(x;\tilde{t}) \in \vee q\tilde{\mu}$. If $\mu(x;\tilde{t}) \in \tilde{\mu}$ and $\mu(x;\tilde{t})q\tilde{\mu}$, then we write $\mu(x;\tilde{t}) \in \wedge q\tilde{\mu}$. The symbol $\overline{\in \vee q}$ means neither $\in$ nor q holds. The symbol $\overline{\in \wedge q}$ means $\in$ or q does not hold.

In what follows, we let $\mathcal{R}$ be a semihyperring. Then we use α, β to denote any one of the " $\in, q, \in \vee q$ or $\in \wedge q$" unless otherwise specified. We also emphasis that $\tilde{\mu}(x) = [\mu^-(x), \mu^+(x)]$, must satisfy the following conditions:

(1) Any two elements of $D[0,1]$ are comparable;

(2) $[\mu^-(x), \mu^+(x)] \leq [0.5, 0.5]$ or $[0.5, 0.5] < [\mu^-(x), \mu^+(x)]$, for all $x \in \mathcal{R}$.

Definition 104 *An interval valued fuzzy set $\tilde{\mu}$ in $\mathcal{R}$ is called an interval valued (α, β)-fuzzy hyperideal of $\mathcal{R}$, where $\alpha \neq \in \wedge q$, if for all $r, x, y \in \mathcal{R}$ and the following conditions hold:*

(i) $\mu(x;\tilde{t})\alpha\tilde{\mu}$ and $\mu(y;\tilde{t})\alpha\tilde{\mu}$ imply $u(z; r\min\{\tilde{t}, r\})\beta\tilde{\mu}$, for all $z \in x + y$,

(ii) $\mu(x;\tilde{t})\alpha\tilde{\mu}$ implies $\mu(rx;\tilde{t})\beta\tilde{\mu}$ and $\mu(xr;\tilde{t})\beta\tilde{\mu}$.

Let $\tilde{\mu}$ be an interval valued fuzzy set in $\mathcal{R}$ such that $\tilde{\mu}(x) \leq [0.5, 0.5]$ for all $x \in \mathcal{R}$. Suppose that $x \in \mathcal{R}$ and $t \in D(0,1]$, such that $\mu(x;\tilde{t}) \in \wedge q\tilde{\mu}$. Then $\tilde{\mu}(x) \geq \tilde{t}$ and $\tilde{\mu}(x) + \tilde{t} > [1,1]$. It follows that

$$[1,1] < \tilde{\mu}(x) + \tilde{t} \leq \tilde{\mu}(x) + \tilde{\mu}(x) = 2\tilde{\mu}(x),$$

so that $\widetilde{\mu}(x) > [0.5, 0.5]$. This means that

$$\left\{\mu(x;\tilde{t}) | \mu(x;\tilde{t}) \in \wedge q\tilde{\mu}\right\} = \varphi.$$

Therefore, the case $\alpha = \in \wedge q$ in this Definition is omitted.

In the next theorem, by an interval valued (α, β)-fuzzy hyperideal of $\mathcal{R}$, we construct an ordinary hyperideal of $\mathcal{R}$.

Theorem 105 *Let $\tilde{\mu}$ be a non-zero interval valued (α, β)-fuzzy hyperideal of $\mathcal{R}$. Then, the set $supp(\tilde{\mu}) = \{x \in \mathcal{R} | \tilde{\mu}(x) > [0,0]\}$ is a hyperideal of $\mathcal{R}$.*

Proof. Suppose that $x, y \in \text{supp}(\tilde{\mu})$ and $t, r \in (0,1]$. Then $\mu(x;\tilde{t}) > [0,0]$ and $\tilde{\mu}(y;r) > [0,0]$. Assume that $\tilde{\mu}(z) = [0,0]$ for all $z \in x + y$. If $\alpha \in \{\in, \in \vee q\}$ then $\mu(x;\tilde{t})\alpha\tilde{\mu}$ and $\mu(y;r)\alpha\tilde{\mu}$. But, for all $z \in x + y$, $\mu(z; r\min\{\tilde{t}, r\})\overline{\beta}\tilde{\mu}$, for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Note that $\mu(x;\tilde{t})\alpha\tilde{\mu}$ and $\mu(y;r)\alpha\tilde{\mu}$ but, for all $z \in x+y$,

$$\mu(z; r\min\{[1,1],[1,1]\}) = \mu(z;[1,1])\overline{\beta}\tilde{\mu},$$

for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Hence, for all $z \in x+y$, $\tilde{\mu}(z) > [0,0]$, that is, for all $z \in x+y$, $z \in$supp$(\tilde{\mu})$. Also, let there exists $r \in \mathcal{R}$ such that $\widetilde{\mu}(xr;\tilde{t}) = [0,0]$. If $\alpha \in \{\in, \in \vee q\}$, then $\mu(x;\tilde{t})\alpha\tilde{\mu}$. But $\mu(xr;\tilde{t})\overline{\beta}\tilde{\mu}$, for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. We know that $\mu(x;[1,1])q\tilde{\mu}$. But $\mu(xr\ ;\tilde{t})\overline{\beta}\tilde{\mu}$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Hence, $\mu(xr\ ;\tilde{t}) > [0,0]$ that is $xr \in$ supp$(\tilde{\mu})$. Similarly, we can show that $rx \in$supp$(\tilde{\mu})$. Therefore, supp $(\tilde{\mu})$ is a hyperideal of $\mathcal{R}$. ■

In the next theorem, we see that an interval valued (q,q)-fuzzy hyperideal is constant under suitable condition.

Theorem 106 *Let $\mathcal{R}$ have zero element and $\tilde{\mu}$ be a non-zero interval valued (q,q)-fuzzy hyperideal of $\mathcal{R}$. Then, $\tilde{\mu}$ is a constant on supp$(\tilde{\mu})$.*

Proof. By Lemma 49, we know that

$$\mu(\tilde{\mu};[0,0]) = \vee\{\tilde{\mu}(x) > [0,0] | x \in \mathcal{R}\}.$$

Suppose that there exists $x \in$supp$(\tilde{\mu})$ such that $\tilde{t}_x = \tilde{\mu}(x) \neq \tilde{t}_0$, then

$\tilde{t}_x < \tilde{t}_0$. Choose $\tilde{t}_1$, $\tilde{t}_2 \in D(0,1]$ such that

$$1 - \tilde{t}_0 < \tilde{t}_1 < 1 - \tilde{t}_x < \tilde{t}_2.$$

Then $\mu([0,0];\tilde{t}_1)q\tilde{\mu}$ and $\mu(x;\tilde{t}_2)q\tilde{\mu}$ but for all $z \in 0+x$,

$$\mu(z; r\min\{\tilde{t}_1,\ \tilde{t}_2\} = \mu(x;\tilde{t}_1)\overline{q}\tilde{\mu}$$

and for all $z \in x+0$,

$$\mu(z; r\min\{\tilde{t}_1,\ \tilde{t}_2\}) = \mu(x;\tilde{t}_1)\overline{q}\tilde{\mu}$$

which is a contradiction. Thus, $\tilde{\mu}(x) = \tilde{\mu}(0)$, for all $x \in$ supp$(\tilde{\mu})$. Therefore, $\tilde{\mu}$ is constant on supp$(\tilde{\mu})$. ■

In the following theorem, we investigate some conditions that make an interval valued fuzzy set $\tilde{\mu}$ in $\mathcal{R}$ as an interval valued $(q, \in \vee q)$-fuzzy hyperideal.

Theorem 107 *Let I be a hyperideal of $\mathcal{R}$ and $\tilde{\mu}$ is an interval valued fuzzy set in $\mathcal{R}$ such that*

(i) $\forall\ x \in \mathcal{R} \backslash I$, $\tilde{\mu}(x) = [0, 0]$,

(ii) $\forall\ x \in I$, $\tilde{\mu}(x) \geqslant [0.5, 0.5]$,

Then, $\tilde{\mu}$ is an interval valued $(q, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$.

Proof. Suppose that $x, y \in \mathcal{R}$ and $\tilde{t}_1,\ \tilde{t}_2 \in D(0, 1]$ such that $\mu(x; \tilde{t}_1)q\tilde{\mu}$ and $(y; \tilde{t}_2)q\tilde{\mu}$. Then $x, y \in I$, and so $z \subseteq I$ for all $z \in x + y$. We can consider the following cases:

(1) In case of $\tilde{t}_1 \wedge \tilde{t}_2 \leq [0.5, 0.5]$, then

$$\tilde{\mu}(z) \geq [0.5, 0.5] \geq \tilde{t}_1 \wedge \tilde{t}_2,$$

for all $z \in x + y$ and hence $(z; r\min\{\tilde{t}_1,\ \tilde{t}_2\}) \in \tilde{\mu}$ for all $z \in x + y$.

(2) In case of $\tilde{t}_1 \wedge \tilde{t}_2 > [0.5, 0.5]$, then

$$\begin{aligned} \tilde{\mu}(z) + \tilde{t}_1 \wedge \tilde{t}_2 &> [0.5, 0.5] + [0.5, 0.5] = [1, 1] \\ & \mu(z; r\min\{\tilde{t}_1, \tilde{t}_2\})q\tilde{\mu}. \end{aligned}$$

Therefore,

$$\mu(z; r\min\{\tilde{t}_1, \tilde{t}_2\} \in \vee q\tilde{\mu}, \text{ for all } z \in x + y.$$

Now, suppose that $r \in \mathcal{R}$ and $\tilde{t} \in D\ (0, 1]$ such that $\mu(x; \tilde{t})q\tilde{\mu}$, then, $x \in I$, and so $rx \subseteq I$. We can see two following cases:

(1) In case of $\tilde{t} \leq [0.5, 0.5]$, then

$$\tilde{\mu}(rx) \geq [0.5, 0.5] \geq \tilde{t}$$

and hence $\mu(rx; \tilde{t}) \in \tilde{\mu}$. Similarly, $\mu(xr; \tilde{t}) \in \tilde{\mu}$.

(2) In case of $\tilde{t} > [0.5, 0.5]$, then

$$\tilde{\mu}(rx) + [1, 1] > [0.5, 0.5] + [0.5, 0.5] = [1, 1]$$

and so $\mu(rx;\tilde{t})q\tilde{\mu}$. Similarly, $\mu(xr;\tilde{t})q\tilde{\mu}$. Therefore, $\mu(rx;\tilde{t}) \in \vee q\tilde{\mu}$ and $\mu(xr;\tilde{t}) \in \vee q\tilde{\mu}$. This completes the proof. ■

Also, we have the converse of above Theorem as follows:

Theorem 108 *Let $\mathcal{R}$ be a semihyperring with zero and $\tilde{\mu}$ be an interval valued $(q, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$, such that $\tilde{\mu}$ is not constant on supp$(\tilde{\mu})$. Then, $\tilde{\mu}(x) \geq [0.5, 0.5]$ for all $x \in$supp$(\tilde{\mu})$.*

Proof. By Lemma 49, we know that

$$\tilde{\mu}(0) = \vee\{\tilde{\mu}(x)|x \in \mathcal{R}\}.$$

Assume that

$$\tilde{\mu}(x) < [0.5, 0.5]$$

for all $x \in \mathcal{R}$. Since $\tilde{\mu}$ is not constant on supp$(\tilde{\mu})$, there exists $x \in$ supp$(\tilde{\mu})$ such that

$$\tilde{t}_x = \tilde{\mu}(x) \neq \tilde{\mu}(0) = \tilde{t}_0,$$

then $\tilde{t}_x < \tilde{t}_0$. Choose $\tilde{t}_1 > [0.5, 0.5]$ such that

$$\tilde{t}_x + \tilde{t}_1 < [1, 1] < \tilde{t}_0 + \tilde{t}_1.$$

Then $\mu(0;\tilde{t}_1)q\tilde{\mu}$ and $\mu(x;\tilde{t}_1)q\tilde{\mu}$. Since

$$\tilde{\mu}(x) + \tilde{t}_1 = \tilde{t}_x + \tilde{t}_1 < [1, 1],$$

we have $\mu(x;\tilde{t}_1)\overline{q}\tilde{\mu}$ and so

$$\mu(z; r\min\{[1, 1], \tilde{t}_1]) = \mu(x;\tilde{t})\overline{\in \vee q}\tilde{\mu} \text{ for all } z \in 0 + x \text{ or } z \in x + 0.$$

This contradicts $\tilde{\mu}$ is an interval valued $(q, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$. Therefore, for some $y \in \mathcal{R}$.

$$\tilde{\mu}(y) \geq [0.5, 0.5]$$

Also, since

$$\tilde{\mu}(0) \geq \tilde{\mu}(y) \Rightarrow \tilde{\mu}(0) \geq [0.5, 0.5]$$

Finally, let $\tilde{t}_x = \tilde{\mu}(x) < [0.5, 0.5]$ for some $x \in$supp$(\tilde{\mu})$. Take $\tilde{t}_1 > [0, 0]$ such that $\tilde{t}_x + \tilde{t}_1 <$ $[0.5, 0.5]$ then

$$\mu(x; [1, 1])q\tilde{\mu} \text{ and } \mu(0; \{[0.5, 0.5] + \tilde{t}_1\})q\tilde{\mu}.$$

But

$$\tilde{\mu}(x) + [0.5, 0.5] + \tilde{t}_1 = \tilde{t}_x + [0.5, 0.5] + \tilde{t}_1 < [0.5, 0.5] + [0.5, 0.5] = [1, 1]$$

which implies

$$\mu(x; \{[0.5, 0.5] + \tilde{t}_1\})\overline{q}\tilde{\mu}.$$

Thus,

$$\mu(z; r \min\{[1, 1], [0.5, 0.5] + \tilde{t}_1\}) = \mu(x; \{[0.5, 0.5] + \tilde{t}_1\})\overline{\in \vee q}\tilde{\mu},$$

for all $z \in 0 + x$ or $z \in x + 0$, which is a contradiction. Therefore, $\tilde{\mu}(x) \geq [0.5, 0.5]$ for all $x \in$ supp$(\tilde{\mu})$. ■

An interval valued fuzzy set $\tilde{\mu}$ in $\mathcal{R}$ is said to be proper if Im$(\tilde{\mu})$ has at least two elements. The two interval valued fuzzy sets are said to be equivalent if they have same family of interval valued level subsets. 0therwise, they are said to be non-equivalent. Now, we can discuss an interval valued $(\in, \in)$-fuzzy hyperideal of $\mathcal{R}$ which can be expressed as the union of two proper non-equivalent interval valued $(\in, \in)$-fuzzy hyperideals.

Theorem 109 *Let $\mathcal{R}$ have some proper hyperideals. Then a proper interval valued $(\in, \in)$-fuzzy hyperideal $\tilde{\mu}$ of $\mathcal{R}$ such that $3 \leq|$ Im$(\tilde{\mu})$ $|< \infty$, can be expressed as the union of two proper non-equivalent interval valued $(\in, \in)$-fuzzy hyperideals of $\mathcal{R}$.*

Proof. Let $\tilde{\mu}$ be a proper interval valued $(\in, \in)$-fuzzy hyperideal of $\mathcal{R}$ with Im$(\tilde{\mu})$ = $\{\tilde{t}_0,\ \tilde{t}_{1,...,}\ \tilde{t}_n\}$, where $\tilde{t}_0 > \tilde{t}_1 > ... > \tilde{t}_n$ and n$\geq$ 2. Then,

$$\mu(\tilde{\mu}; \widetilde{t_0}) \subseteq \mu(\tilde{\mu}; \widetilde{t_1}) \subseteq \mu(\tilde{\mu}; \widetilde{t_2}) \subseteq ... \subseteq \mu(\tilde{\mu}; \widetilde{t_n}) = \mathcal{R}$$

is the chain of interval valued level hyperideals of $\mathcal{R}$. Define two interval valued fuzzy sets A

and B in $\mathcal{R}$ by and

$\widetilde{\mu_A}(x) = \tilde{r}_1$, for $x \in \mu(\tilde{\mu}; \widetilde{t_1})$ and $\widetilde{\mu_A}(x) = \tilde{r}_k$, for $x \in \mu(\tilde{\mu}; \widetilde{t_k}) \setminus \mu(\tilde{\mu}; \widetilde{t_{k-1}})$, where ($k = 2, ...,$n).

$\widetilde{\mu_B}(x) = \widetilde{t_0}$, for $x \in \mu(\tilde{\mu}; \widetilde{t_1})$; $\widetilde{\mu_B}(x) = \widetilde{t_1}$, for $x \in \mu(\tilde{\mu}; \widetilde{t_1}) \setminus \mu(\tilde{\mu}; \widetilde{t_0})$; $\widetilde{\mu_B}(x) = \widetilde{r_2}$, for $x \in \mu(\tilde{\mu}; \widetilde{t_3}) \setminus \mu(\tilde{\mu}; \widetilde{t_1})$ and $\widetilde{\mu_B}(x) = \widetilde{t_k}$, for $x \in \mu(\tilde{\mu}; \widetilde{t_k}) \setminus \mu(\tilde{\mu}; \widetilde{t_{k-1}})$, where ($k = 4, ...,$n).

$$\mu(\tilde{\mu}; \widetilde{t_1}) \subseteq \mu(\tilde{\mu}; \widetilde{t_2}) \subseteq ... \subseteq \mu(\tilde{\mu}; \widetilde{t_n}) = \mathcal{R}$$

and

$$\mu(\tilde{\mu}; \widetilde{t_0}) \subseteq \mu(\tilde{\mu}; \widetilde{t_1}) \subseteq ... \subseteq \mu(\tilde{\mu}; \widetilde{t_n}) = \mathcal{R}$$

are respectively the chain of level hyperideals of $\mathcal{R}$, and $A, B \leq \tilde{\mu}$. Thus A and B are non-equivalent, and it is obvious that $A \cup B = \tilde{\mu}$. This completes the proof. ■

4.2 Interval valued $(\in, \in \vee q)$-fuzzy hyperideals

In this section, we investigate some results and properties of interval valued (α, β)-fuzzy hyperideals (specifically, ($\in, \in \vee q$)-fuzzy hyperideals) of $\mathcal{R}$.

Definition 110 *An interval valued fuzzy set $\tilde{\mu}$ in $\mathcal{R}$ is called an interval valued $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$, if for all $r, x, y \in \mathcal{R}$ and the following conditions hold:*

(i) $\mu(x; \tilde{t}) \in \tilde{\mu}$ and $\mu(y; \tilde{t}) \in \tilde{\mu}$ imply $u(z; r \min\{\tilde{t}, r\}) \in \vee q\tilde{\mu}$, for all $z \in x + y$,

(ii) $\mu(x; \tilde{t}) \in \tilde{\mu}$ implies $\mu(rx; \tilde{t}) \in \vee q\tilde{\mu}$ and $\mu(xr; \tilde{t}) \in \vee q\tilde{\mu}$.

Theorem 111 *Every interval valued $(\in \vee q, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$ is an interval valued $(\in, \in \vee q)$-fuzzy hyperideals of $\mathcal{R}$.*

Proof. Let $\tilde{\mu}$ be an interval valued ($\in \vee q$, $\in \vee q$)-fuzzy hyperideal of $\mathcal{R}$. Suppose that $x, y \in \mathcal{R}$ and $\tilde{t}_1$, $\tilde{t}_2 \in D$ $(0, 1]$ such that $\mu(x; \tilde{t}_1) \in \tilde{\mu}$ and $\mu(y; \tilde{t}_1) \in \tilde{\mu}$. Then, $\mu(x; \tilde{t}_1) \in \vee q\tilde{\mu}$ and $\mu(x; \tilde{t}_1) \in \vee q\tilde{\mu}$. By the hypothesis, if it follows that $\mu(z; r \min\{\tilde{t}_1, \tilde{t}_2\}) \in \vee q\tilde{\mu}$, for all $z \in x + y$. Now, let $r, x \in \mathcal{R}$ and $\tilde{t} \in D(0, 1]$ such that $\mu(x; \tilde{t}) \in \tilde{\mu}$. Then, $\mu(x; \tilde{t}) \in \vee q\tilde{\mu}$, so by hypothesis $\mu(rx; \tilde{t}) \in \vee q\tilde{\mu}$ and $\mu(xr; \tilde{t}) \in \vee q\tilde{\mu}$. Therefore, $\tilde{\mu}$ is an interval valued ($\in, \in \vee q$)-fuzzy hyperideal of $\mathcal{R}$. ■

Theorem 112 *Every interval valued* $(\in, \in)$*-fuzzy hyperideal of* $\mathcal{R}$ *is an interval valued* $(\in, \in \vee q)$*-fuzzy hyperideal.*

Proof. The proof is straight forward. ■

Proposition 113 *If* I *is a hyperideal of* $\mathcal{R}$*, then* X_I*(the characteristic function of* I *is an interval valued* $(\in, \in)$*-fuzzy hyperideal of* $\mathcal{R}$*.*

Proof. Suppose that $x, y \in \mathcal{R}$ and $\tilde{t}_1,\ \tilde{t}_2 \in D\ (0,1]$ such that $\mu(x;\tilde{t}_1) \in X_I$ and $\mu(y;\tilde{t}_2) \in X_I$. Then, $X_I(x) \geq \tilde{t}_1 > [0,0]$ and $X_I(x) \geq \tilde{t}_2 > [0,0]$, which imply that $X_I(x) = [1,1] = X_I(y)$. Hence $x, y \in I$ and so $z \in I$ for all $z \in x+y$, it follows that $X_I\ (z) = [1,1] \geq \tilde{t}_1 \wedge \tilde{t}_2$, for all $z \in x+y$, that is $\mu(z; r\min\{\tilde{t}_1,\ \tilde{t}_2\} \in X_I$ for all $z \in x+y$. Now, let $r, x \in \mathcal{R}$ and $\tilde{t} \in D(0,1]$ such that $\mu(x;\tilde{t}) \in X_I$. Then $X_I(x) \geq \tilde{t} > [0,0]$ which implies $X_I(x) = [1,1]$. Hence, $x \in I$ so $\mu(rx;\tilde{t}) \in I$ and $xr \in I$. It follows

$$X_I(rx) = X_I(xr) = [1,1] \geq \tilde{t},$$

that is $\mu(rx;\tilde{t}) \in X_I$ and $\mu(xr;\tilde{t}) \in X_I$. Therefore, X_I is an interval valued $(\in, \in)$-fuzzy hyperideal of $\mathcal{R}$.

Conversely, if I is a hyperideal of $\mathcal{R}$, then by Proposition 113, X_I is an interval valued $(\in, \in)$-fuzzy hyperideal of $\mathcal{R}$. Therefore, by Theorem 112, X_I is an interval valued $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$. ■

Example 114 *On four element semihyperring* $(\mathcal{R}, +, .)$ *defined by the following two tables:*

$+$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{a\}$	$\{b\}$	$\{c\}$
b	$\{b\}$	$\{b\}$	$\{b\}$	$\{c\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$

$.$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

Consider a fuzzy set $\tilde{\mu}$ *as follows:*

$$\tilde{u}_{\tilde{\mu}}(x) = \begin{cases} [0.8, 0.9] & \textit{if } x = 0 \\ [0.6, 0.7] & \textit{if } x = a, b \\ [0.2, 0.3] & \textit{if } x = c \end{cases}$$

It is easy to see that $\tilde{\mu}$ *is an interval valued* $(\in, \in \vee q)$*-fuzzy hyperideal of* $(\mathcal{R}, +, .)$.

In the next theorem, we prove an equivalent condition for interval valued $(\in, \in \vee q)$-fuzzy hyperideals.

Theorem 115 *An interval valued fuzzy set* $\tilde{\mu}$ *in* $\mathcal{R}$ *is an interval valued* $(\in, \in \vee q)$*-fuzzy hyperideal of* $\mathcal{R}$ *if and only if for all* $t,\ r \in (0, 1]$ *and* $x, y \in \mathcal{R}$ *the following two conditions hold:*

(i) $r\min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} \leq r\inf\{\tilde{\mu}(z) \backslash$ *for all* $z \in x + y\}$, *for all* $x, y \in \mathcal{R}$.

(ii) $\tilde{\mu}(rx) \geq r\min\{\tilde{\mu}(x), [0.5, 0.5]\}$ *and* $\tilde{\mu}(xr) \geq r\min\{\tilde{\mu}(x), [0.5, 0.5]\}$.

Proof. Let $\tilde{\mu}$ be an interval valued $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$ and $x, y \in \mathcal{R}$. We can consider the following cases:

(1) $r\inf\{\tilde{\mu}(x), \tilde{\mu}(y)\} < [0.5, 0.5]$. In this case, $r\min\{\tilde{\mu}(z)\backslash$ if for all $z \in x+y\} < r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\}$. We can choose $\tilde{t} \in (0, 0.5)$ such that, for all $z \in x + y$,

$$\tilde{\mu}(z) < \tilde{t} < \{\tilde{\mu}(x), \tilde{\mu}(y)\}.$$

Then $\mu(x; \tilde{t}) \in \tilde{\mu}$ and $\mu(y; \tilde{t}) \in \tilde{\mu}$, but $\mu(z; \tilde{t})\overline{\in \vee q\tilde{\mu}}$, for all $z \in x + y$,

which is a contradiction. Thus, for all $z \in x + y$,

$$\tilde{\mu}(z) \geq r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\} = r\min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\}.$$

(2) $r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\} \geq [0.5, 0.5]$. In this case, we have $\widetilde{\mu}_{[0.5,0.5]} \in \tilde{\mu}$ and $\mu(y; [0.5, 0.5]) \in \tilde{\mu}$. If for all $z \in x + y$, we have

$$\begin{aligned} \tilde{\mu}(z) &< [0.5, 0.5] \Rightarrow \mu(z; [0.5, 0.5])\overline{\in}\tilde{\mu} \\ \tilde{\mu}(z) + [0.5, 0.5] &< [1, 1] \text{ (or } \mu(z; [0.5, 0.5])\overline{q}\tilde{\mu}, \text{ for all } z \in x + y). \end{aligned}$$

Hence, $\mu(z; r\min\{[0.5,0.5],[0.5,0.5]\})\overline{\in \vee q\tilde{\mu}}$, for all $z \in x+y$, which is a contradiction. Thus, for all $z \in x+y$

$$\tilde{\mu}(z) \geq [0.5,0.5] = r\min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5,0.5]\},$$

Also, if $r, x \in \mathcal{R}$ we can consider two following cases:

(1) $\tilde{\mu}(x) \geq [0.5,0.5]$. In this case, if $\tilde{\mu}(xr) < \tilde{\mu}(x)$, we can choose $\tilde{t} \in (0,0.5)$ such that $\tilde{\mu}(rx) < \tilde{t} < \tilde{\mu}(x)$. Then $\mu(x;\tilde{t}) \in \tilde{\mu}$, but $\mu(rx;\tilde{t})\overline{\in \vee q\tilde{\mu}}$, which is a contradiction. Thus,

$$\tilde{\mu}(rx) \geq \tilde{\mu}(x) = r\min\{\widetilde{\mu}(x), [0.5,0.5]\}.$$

Similarly,

$$\tilde{\mu}(xr) \geq \tilde{\mu}(x) = r\min\{\widetilde{\mu}(x), [0.5,0.5]\}.$$

(2) $\tilde{\mu}(x) \geq [0.5,0.5]$. In this case, we have $\widetilde{\mu}_{[0.5,0.5]} \in \tilde{\mu}$. If $\tilde{\mu}(rx) < [0.5,0.5]$ then $\mu(rx;[0.5,0.5])\overline{\in}\tilde{\mu}$ and $\tilde{\mu}(rx) + [0.5,0.5] < [1,1]$ (or $\mu(rx;[0.5,0.5])\overline{q}\tilde{\mu}$).

Hence $\mu(rx;[0.5,0.5])\overline{\in \vee q\tilde{\mu}}$ which is a contradiction. Thus,

$$\tilde{\mu}(rx) \geq [0.5,0.5] = r\min\{\tilde{\mu}(x), [0.5,0.5]\}.$$

Similarly,

$$\tilde{\mu}(xr) \geq r\min\{\tilde{\mu}(x), [0.5,0.5]\}.$$

Conversely, suppose that $\tilde{\mu}$ satisfies condition (i) and (ii). Let $x, y \in \mathcal{R}$ and $\tilde{t}_1,\ \tilde{t}_2 \in D\ (0,1]$ such that

$$\begin{aligned} \mu(x;\tilde{t}_1) &\in \tilde{\mu} \text{ and } \mu(y;\tilde{t}_2) \in \tilde{\mu} \\ \tilde{\mu}(x) &\geq \tilde{t}_1 \text{ and } \tilde{\mu}(y) \geq \tilde{t}_2 \end{aligned}$$

Suppose that for all $z \in x+y$,

$$\tilde{\mu}(z) < r\min\{\tilde{t}_1, \tilde{t}_2\}.$$

If

$$r\min\{\tilde{\mu}(x), \tilde{\mu}(y) < [0.5,0.5],$$

then, for all $z \in x + y$,

$$\tilde{\mu}(z) \geq r \min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} = r \min\{\tilde{\mu}(x), \tilde{\mu}(y)\} > r \min\{\tilde{t}_1, \tilde{t}_2\},$$

which is a contradiction. So

$$r \min\{\tilde{\mu}(x), \tilde{\mu}(y)\} \geq [0.5, 0.5].$$

It follows that, for all $z \in x + y$,

$$\tilde{\mu}(z) + r \min(\tilde{t}_1, \tilde{t}_2) > 2\tilde{\mu}(z) \geq 2r \min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} = [1, 1].$$

Hence, $\mu(z; r \min\{\tilde{t}_1, \tilde{t}_2\}) q\tilde{\mu}$, for all $z \in x + y$, which implies $\mu(z; r \min\{\tilde{t}_1, \tilde{t}_2\} \in \vee q\tilde{\mu}$, for all $z \in x + y$. Also, let $r, x \in \mathcal{R}$ and $\tilde{t} \in D\ (0, 1]$ such that $\mu(x; \tilde{t}) \in \tilde{\mu}$, then, $\tilde{\mu}(x) \geq \tilde{t}$. Suppose that $\tilde{\mu}(rx) < \tilde{t}$. If $\tilde{\mu}(x) < [0.5, 0.5]$, then

$$\tilde{\mu}(rx) \geq r \min\{\tilde{\mu}(x), [0.5, 0.5]\} = \tilde{\mu}(x) \geq \tilde{t},$$

which is a contradiction, and so $\tilde{\mu}(x) \geq [0.5, 0.5]$. It follows that

$$\tilde{\mu}(rx) + \tilde{t} > 2\tilde{\mu}(rx) \geq 2r \min\{\tilde{\mu}(x), [0.5, 0.5]\} = [1, 1].$$

Hence, $\mu(rx; \tilde{t}) q\tilde{\mu}$, which implies $\mu(rx; \tilde{t}) \in \vee q\tilde{\mu}$. Similarly, $\mu(xr; \tilde{t}) \in \vee q\tilde{\mu}$. Therefore, $\tilde{\mu}$ is an interval valued $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$. ■

In the following theorem, we characterize interval valued $(\in, \in \vee q)$-fuzzy hyperideals based on level subsets.

Theorem 116 *Let $\tilde{\mu}$ be an interval valued fuzzy set in $\mathcal{R}$. If $\tilde{\mu}$ is an interval valued $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$, then for all $[0, 0] < \tilde{t} \leq [0.5, 0.5]$, $\mu(\tilde{\mu}; \tilde{t}) = \varphi$ or $\mu(\tilde{\mu}; \tilde{t})$ is a hyperideal of $\mathcal{R}$.*

Conversely, if $\mu(\tilde{\mu}; \tilde{t})(\neq \varphi)$ is a hyperideal of $\mathcal{R}$ for all $[0, 0] < \tilde{t} \leq [0.5, 0, 5]$, then $\tilde{\mu}$ is an interval valued $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$.

Proof. Let $\tilde{\mu}$ be an interval valued $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$ and $[0,0] < \tilde{t} \leq [0.5, 0.5]$. If $x, y \in \mu(\tilde{\mu}; \tilde{t})$, then $\tilde{\mu}(x) \geq \tilde{t}$ and $\tilde{\mu}(y) \geq \tilde{t}$. Hence, for all $z \in x + y$,

$$\tilde{\mu}(z) \geq r\min\{\widetilde{\mu}(x), \widetilde{\mu}(y), [0.5, 0.5]\} \geq r\min\{\tilde{t}, [0.5, 0.5]\} = \tilde{t},$$

which implies that $\tilde{\mu}(z) \geq \tilde{t}$, for all $z \in x+y$. That is $z \in \mu(\tilde{\mu}; \tilde{t})$ for all $z \in x+y$. Now, suppose that $x \in \mu(\tilde{\mu}; \tilde{t})$ and $r \in \mathcal{R}$. Then, $\tilde{\mu}(x) \geq \tilde{t}$, and hence

$$\tilde{\mu}(rx) \geq r\min\{\tilde{\mu}(x), [0.5, 0.5]\} \geq r\min\{\tilde{t}, [0.5, 0.5]\} = \tilde{t}.$$

It implies $\tilde{\mu}(rx) \geq \tilde{t}$, that is $rx \in \mu(\tilde{\mu}; \tilde{t})$. Similarly, $xr \in \mu(\tilde{\mu}; \tilde{t})$. Therefore, $\mu(\tilde{\mu}; \tilde{t})$ is a hyperideal of $\mathcal{R}$.

Conversely**,** Let $\tilde{\mu}$ be an interval valued fuzzy set in $\mathcal{R}$ such that $\mu(\tilde{\mu}; \tilde{t}) \neq \varphi)$ is a hyperideal of $\mathcal{R}$ for all $[0,0] < \tilde{t} \leq [0.5, 0.5]$. If $x, y \in \mathcal{R}$, we have

$$\begin{aligned}\tilde{\mu}(x) &\geq r\min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} = \tilde{t}_0 \\ \tilde{\mu}(y) &\geq r\min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} = \tilde{t}_{0,}\end{aligned}$$

then $x, y \in \mu(\tilde{\mu}; \tilde{t}_0)$, and so $z \in \mu(\tilde{\mu}; \tilde{t}_0)$ for all $z \in x + y$. Now, we have

$$\tilde{\mu}(z) \geq \tilde{t}_0 = r\min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\},$$

for all $z \in x + y$. Hence, condition (i) of Theorem 115 is verified. Now, if $x \in \mathcal{R}$, we have

$$\tilde{\mu}(x) \geq r\min\{\tilde{\mu}(x), [0.5, 0.5]\} = \tilde{t}_0'.$$

Then, $x \in \mu(\tilde{\mu}; \tilde{t}_0)$, so $rx \in \mu(\tilde{\mu}; \tilde{t}_0)$, for all $r \in \mathcal{R}$. Hence,

$$\tilde{\mu}(x) \geq \tilde{t}_0' = r\min\{\tilde{\mu}(a), [0.5, 0.5]\}.$$

Similarly,

$$\tilde{\mu}(xr) \geq \tilde{t}_0' = r\min\{\tilde{\mu}(a), [0.5, 0.5]\}.$$

This shows condition (ii) of Theorem 115 holds. Therefore, $\tilde{\mu}$ is an interval valued $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$. ■

In Theorem 116, we discuss on level subsets in the interval $(0, 0.5]$. In the next theorem, we see what happen to the subsets in interval $(0.5, 1]$.

Theorem 117 *Let $\tilde{\mu}$ be an interval valued fuzzy set in $\mathcal{R}$. Then, $\mu(\tilde{\mu}; \tilde{t}_0)(\neq \varphi)$ is a hyperideal of $\mathcal{R}$ for all $\tilde{t} \in (0.5, 1]$ if and only if for all $x, y \in \mathcal{R}$.*

(i) $r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\} \leq r\max\{\tilde{\mu}(z, [0.5, 0.5]\}$, for all $z \in x + y$.

(ii) $\tilde{\mu}(x) \leq r\max\{\widetilde{\mu}(rx), [0.5, 0.5]\}$ and $\tilde{\mu}(x) \leq r\max\{\widetilde{\mu}(xr), [0.5, 0.5]\}$.

Proof. Let $\mu(\tilde{\mu}; \tilde{t}_0)(\neq \varphi)$ be a hyperideal of $\mathcal{R}$ for all $\tilde{t} \in (0.5, 1]$. If there exists $x, y \in \mathcal{R}$ such that, for all $z \in x + y$,

$$r\max\{\tilde{\mu}(z), [0.5.0.5]\} < r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\} = \tilde{t}$$

then $\tilde{t} \in (0.5, 1]$, $\tilde{\mu}(z) < \tilde{t}$, $x \in \mu(\tilde{\mu}; \tilde{t})$ and $y \in \mu(\tilde{\mu}; \tilde{t}_0)$ for all $z \in x + y$. Hence, $z \in \mu(\tilde{\mu}; \tilde{t})$, for all $z \in x + y$ and so $\tilde{\mu}(z) \geq \tilde{t}$, for all $z \in x + y$, which is a contradiction. Therefore, for all $x, y \in \mathcal{R}$, we have for all $z \in x + y$,

$$r\max\{\tilde{\mu}(z), [0.5, 0.5]\} \geq r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\},$$

Thus, (1) is proved. Also, if there exist $r, x \in \mathcal{R}$ such that

$$r\max\{\tilde{\mu}(rx), [0.5, 0.5]\} < \tilde{\mu}(x) = \tilde{t}(r\max\{\tilde{\mu}(xr), [0.5, 0.5]\} < \tilde{\mu}(x) = \tilde{t}),$$

then $\tilde{t} \in (0.5, 1], \tilde{\mu}(rx) < \tilde{t}$ and $x \in \mu(\tilde{\mu}; \tilde{t})$ Hence, $rx \in \mu(\tilde{\mu}; \tilde{t})$ and so $\tilde{\mu}(rx) \geq \tilde{t}$, which is a contradiction. Therefore, for all $r, x \in \mathcal{R}$, we have

$$r\max\{\tilde{\mu}(rx), [0.5, 0.5]\} \geq \tilde{\mu}(x)$$

and

$$r\max\{\tilde{\mu}(xr), [0.5, 0.5]\} \geq \tilde{\mu}(x).$$

Thus, (2) is proved.

Conversely, let (1) and (2) hold. Assume that $\tilde{t} \in (0.5, 1]$ and $x, y \in \mu(\tilde{\mu}; \tilde{t})$. Then, by (1) we have

$$[0.5, 0.5] < \tilde{t} \leq r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\} \leq r\max\{\tilde{\mu}(z), [0.5, 0.5]\}$$

for all $z \in x + y$. It implies that

$$[0.5, 0.5] < \tilde{t} \leq r\max\{\tilde{\mu}(z), [0.5, 0.5]\}$$

for all $z \in x + y$. Hence, $\tilde{\mu}(z) \geq \tilde{t}$, for all $z \in x + y$, which means $z \in \mu(\tilde{\mu}; \tilde{t})$, for all $z \in x + y$. Also, suppose that $\tilde{t} \in (0.5, 1]$, $x \in \mu(\tilde{\mu}; \tilde{t})$ and $r \in \mathcal{R}$. Then, by (2) we have

$$[0.5, 0.5] < \tilde{t} \leq \tilde{\mu}(x) \leq r\max\{\tilde{\mu}(rx), [0.5, 0.5]\}.$$

It implies

$$[0.5, 0.5] < \tilde{t} \leq r\max\{\tilde{\mu}(rx), [0.5, 0.5]\}.$$

Hence, $\tilde{\mu}(rx) \geq \tilde{t}$, which means $rx \in \mu(\tilde{\mu}; \tilde{t})$, similarly, $xr \in \mu(\tilde{\mu}; \tilde{t})$. Therefore, $\mu(\tilde{\mu}; \tilde{t})$ is a hyperideal of $\mathcal{R}$. Let $\tilde{\mu}$ be an interval valued fuzzy set in $\mathcal{R}$ and J be the set of $\tilde{t} \in D(0, 1]$ such that $\mu(\tilde{\mu}; \tilde{t}) = \Phi$ or $\mu(\tilde{\mu}; \tilde{t})$ is a hyperideal of $\mathcal{R}$. If $J = D(0, 1]$, then by Theorem 116, $\tilde{\mu}$ is an interval valued $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$. Naturally, a corresponding result should be considered when $J = (0.5, 1]$. ■

Definition 118 *An interval valued fuzzy set $\tilde{\mu}$ in $\mathcal{R}$ is called interval valued $(\overline{\in}, \overline{\in \wedge q})$- fuzzy hyperideal of $\mathcal{R}$ if for all $\tilde{t}_1, \tilde{t}_2 \in D(0, 1]$ and $r, x, y \in \mathcal{R}$, the following condition hold:*

(i) $\mu(z; r\min\{\tilde{t}_1, \tilde{t}_2\})\overline{\in}\tilde{\mu}$, for all $z \in x + y$, implies $\mu(x; \tilde{t}_1)\overline{\in \wedge q\tilde{\mu}}$ or $\mu(y; \tilde{t}_2)\overline{\in \wedge q\tilde{\mu}}$.

(ii) $\mu(rx; \tilde{t})\overline{\in}\tilde{\mu}$ or $\mu(xr; \tilde{t})\overline{\in}\tilde{\mu}$ implies $\mu(x; \tilde{t})$ $\overline{\in \wedge q\tilde{\mu}}$.

In the next theorem, we prove an equivalent condition for interval valued $(\overline{\in}, \overline{\in \wedge q})$- fuzzy hyperideals.

Theorem 119 *Let $\tilde{\mu}$ be an interval valued fuzzy set in $\mathcal{R}$. Then, $\tilde{\mu}$ is an interval valued $(\overline{\in}, \overline{\in \wedge q})$- fuzzy hyperideal of $\mathcal{R}$ if and only if for all $r, x, y \in \mathcal{R}$, the following conditions hold:*

(i) $r\max\{\tilde{\mu}(z), [0.5, 0.5]\} \geq r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\}$, for all $z \in x + y$.

(ii) $r\max\{\tilde{\mu}(rx),[0.5,0.5]\} \geq \tilde{\mu}(x)$ *and* $r\max\{\widetilde{\mu}(xr),[0.5,0.5]\} \geq \tilde{\mu}(x)$.

Proof. Let $\tilde{\mu}$ be an interval valued $(\overline{\in}, \overline{\in \wedge q})$- fuzzy hyperideal of $\mathcal{R}$. If there exist $x, y \in \mathcal{R}$ such that

$$r\max\{\tilde{\mu}(z),[0.5,0.5]\} < r\min\{\tilde{\mu}(x),\tilde{\mu}(y)\} = \tilde{t}$$

for all $z \in x+y$, then $\tilde{t} \in (0.5,1]$, $\mu(z;\tilde{t})\overline{\in}\tilde{\mu}$ for all $z \in x+y$ and $\mu(x;\tilde{t})$, $\mu(y;\tilde{t}) \in \tilde{\mu}$. By Definition 118, it follows that $\mu(x;\tilde{t})\overline{q}\tilde{\mu}$ or $\mu(y;\tilde{t})\overline{q}\tilde{\mu}$. Then, ($\tilde{\mu}(x) \geq \tilde{t}$ and $\tilde{\mu}(x)+\tilde{t} \leq [1,1]$) or ($\tilde{\mu}(y) \geq \tilde{t}$ and $\tilde{\mu}(y)+\tilde{t} \leq [1,1]$). It follows that $\tilde{t} \leq [0.5,0.5]$ which is a contradiction. Hence, (i) holds. Also, if there exist $r, x \in \mathcal{R}$ such that

$$r\max(\tilde{\mu}(rx),[0.5,0.5]) < \tilde{\mu}(x) = \tilde{t},$$

or

$$r\max(\tilde{\mu}(xr),[0.5,0.5]) < \tilde{\mu}(x) = \tilde{t},$$

then $\tilde{t} \in (0.5,1]$, $\mu(rx;\tilde{t})\overline{\in}\tilde{\mu}$ and $\mu(x;\tilde{t}) \in \tilde{\mu}$. By Definition 118, it follows that $\mu(x;\tilde{t})\overline{q}\tilde{\mu}$. Then, $\tilde{\mu}(x) \geq \tilde{t}$ and $\tilde{\mu}(x)+\tilde{t} \leq 1$. It concludes that $\tilde{t} \leq [0.5,0.5]$ which is a contradiction. Thus, (2) holds

Conversely, let conditions (i) and (ii) hold. Also, let $x, y \in \mathcal{R}$ such that for all $z \in x+y$

$$\mu(z; r\min\{\tilde{t}_1,\ \tilde{t}_2\})\overline{\in}\tilde{\mu}$$

$\tilde{\mu}(z) < r\min\{\tilde{t}_1,\ \tilde{t}_2\}$. Then, we can consider the following cases:

(a) If for all $z \in x+y$

$$\tilde{\mu}(z) \geq r\min\{\tilde{\mu}(x),\tilde{\mu}(y)\},$$

then

$$r\min\{\tilde{\mu}(x),\tilde{\mu}(y)\} < r\min\{\tilde{t}_1,\tilde{t}_2\}$$

and so

$$\begin{aligned}\tilde{\mu}(x) &< \tilde{t}_1 \text{ or } \tilde{\mu}(y) < \tilde{t}_2 \\ & \mu(x;\tilde{t}_1)\overline{\in}\tilde{\mu} \text{ or } \mu(y;\tilde{t}_2)\overline{\in}\tilde{\mu} \\ & \mu(x;\tilde{t}_1)\overline{\in \wedge q\tilde{\mu}} \text{ or } \mu(x;\tilde{t}_2)\overline{\in \wedge q\tilde{\mu}}.\end{aligned}$$

(b) If

$$\tilde{\mu}(z) < r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\} \text{ for all } z \in x + y,$$

then by (i), we have

$$[0.5, 0.5] \geq r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\}.$$

Hence for all $z \in x + y$,

$$\tilde{\mu}(z) \vee [0.5, 0.5] \geq \tilde{\mu}(x) \wedge \tilde{\mu}(y)$$

Now, if $\mu(x;\tilde{t}_1)$, $\mu(y;\tilde{t}_2) \in \tilde{\mu}$, then

$$\tilde{t}_1 \leq \tilde{\mu}(x) \leq [0.5, 0.5]$$

or

$$\tilde{t}_2 \leq \tilde{\mu}(y) \leq [0.5, 0.5].$$

It follows that $\mu(x;\tilde{t})\overline{q}\tilde{\mu}$ or $\mu(x;\tilde{t})\overline{q}\tilde{\mu}$.

Which implies that $\mu(x;\tilde{t}_1)\overline{\in \wedge q\tilde{\mu}}$ or $\mu(y;\tilde{t}_2)\overline{\in \wedge q\tilde{\mu}}$. Now, let $r, x, y \in \mathcal{R}$ such that $\mu(rx;\tilde{t})\overline{\in}\tilde{\mu}$ or $\mu(xr;\tilde{t})\overline{\in}\tilde{\mu}$, then $\tilde{\mu}(rx)\tilde{t}$ or $\tilde{\mu}(xr) < \tilde{t}$. We can consider two following cases:

(a) If $\tilde{\mu}(rx) \geq \tilde{\mu}(x)$, then $\tilde{\mu}(x) < \tilde{t}$. It follows that $\mu(x;\tilde{t})\overline{\in}\tilde{\mu}$, which implies that $\mu(x;\tilde{t}_1)\overline{\in \wedge q\tilde{\mu}}$.

(b) If $\tilde{\mu}(rx) \geq \tilde{\mu}(x)$, then by (ii) we have $[0.5, 0.5] \geq \tilde{\mu}(x)$. Hence,

$$r\max(\tilde{\mu}(rx), [0.5, 0.5]) \geq \tilde{\mu}(x).$$

Now if $\mu(x;\tilde{t}) \in \tilde{\mu}$, then

$$\tilde{t} \leq \tilde{\mu}(x) \leq [0.5, 0.5].$$

It follows that $\mu(x;\tilde{t})\overline{q}\tilde{\mu}$, which implies that $\mu(x;\tilde{t})\overline{\in \wedge q\tilde{\mu}}$. Therefore, $\tilde{\mu}$ is an interval valued $(\overline{\in}, \overline{\in \wedge q})$-fuzzy hyperideal of $\mathcal{R}$. ∎

In the following theorem, we characterize interval valued $(\overline{\in}, \overline{\in \wedge q})$-fuzzy hyperideals based on level subsets.

Theorem 120 *An interval valued fuzzy set $\tilde{\mu}$ in $\mathcal{R}$ is an interval valued $(\overline{\in}, \overline{\in \wedge q})$- fuzzy hyperideal of $\mathcal{R}$ if and only if $\mu(x;\tilde{t})(\neq \Phi)$ is a hyperideal of $\mathcal{R}$ for all $\tilde{t} \in (0.5, 1]$.*

Proof. It follows by Theorem 117 and 119. For any interval valued fuzzy set $\widetilde{\mu}$ in $\mathcal{R}$ and $\tilde{t} \in D(0,1]$, we put

$$\bar{\tilde{\mu}}_{\tilde{t}} = \mu(x;\tilde{t}) = \{x \in \mathcal{R}/\mu(x;\tilde{t})q\tilde{\mu}\},$$

and

$$\overline{\mid \tilde{\mu} \mid}_{\tilde{t}} = \{x \in \mathcal{R}|\mu(x;\tilde{t}) \in \vee q\tilde{\mu}\}.$$

Clearly $\overline{\mid \tilde{\mu} \mid}_{\tilde{t}} = \tilde{\mu}_{\ \tilde{t}} \cup \bar{\tilde{\mu}}_{\tilde{t}}$. In fact, $\bar{\tilde{\mu}}_{\tilde{t}}$and $\overline{\mid \tilde{\mu} \mid}_{\tilde{t}}$ are generalized level subsets. Now, we can characterize interval valued $(\in, \in \vee q)$-fuzzy hyperideals based on generalized level subsets. ∎

Theorem 121 *An interval valued fuzzy set $\tilde{\mu}$ in $\mathcal{R}$ is an interval valued $(\in, \in \vee q)$- fuzzy hyperideal of $\mathcal{R}$ if and only if $\overline{\mid \tilde{\mu} \mid}_{\tilde{t}}$is a hyperideal of $\mathcal{R}$ for all $\tilde{t} \in (0.5, 1]$.*

Proof. Let $\tilde{\mu}$ be an interval valued $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$ and $x, y \in \overline{\mid \tilde{\mu} \mid}_{\tilde{t}}$ for $\tilde{t} \in (0.5, 1]$. Then, $\mu(x;\tilde{t}) \in \vee q\tilde{\mu}$ and $\mu(y;\tilde{t}) \in \vee q\tilde{\mu}$, which means $\tilde{\mu}(x) \geq \tilde{t}$ or $\tilde{\mu}(x) + \tilde{t} > 1$, and $\tilde{\mu}(y) \geq \tilde{t}$ or $\tilde{\mu}(y) + \tilde{t} > [1, 1]$. On the other hand, by Theorem 115, we know, for all $z \in x + y$,

$$\tilde{\mu}(z) \geq r\min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5.0.5]\}$$

so for all $z \in x + y$,

$$\tilde{\mu}(z) \geq r\min\{\tilde{t}, [0.5, 0.5]).$$

Since, if for all $z \in x + y$,

$$\tilde{\mu}(z) < r\min\{\tilde{t}, [0.5, 0.5]\}$$

then for all $z \in x + y$,

$$r\min\{\tilde{\mu}(x), \widetilde{\mu}(y), [0.5, 0.5]) \leq \tilde{\mu}(z) < r\min\{\tilde{t}, [0.5, 0.5]\},$$

which implies,

$$r\min\{\widetilde{\mu}(x), \widetilde{\mu}(y), [0.5, 0.5]\} < r\min\{\tilde{t}, [0.5, 0.5]).$$

Hence, $\tilde{\mu}(x) < \tilde{t}$ or $\tilde{\mu}(y) < \tilde{t}$, that is $\mu(x;\tilde{t})\overline{\in}\tilde{\mu}$ or $\mu(y;\tilde{t})\overline{\in}\tilde{\mu}$. Thus, $\mu(x;\tilde{t})\overline{\in \vee q\tilde{\mu}}$ or $\mu(y;\tilde{t})\overline{\in \vee q\tilde{\mu}}$ which is a contradiction. We know $\tilde{t} \in [0.5, 0.5]$ then

$$\tilde{\mu}(z) \geq r\min\{\tilde{t}, [0.5, 0.5]\} = [1, 1]$$

and so $z \in \tilde{\mu}_{\tilde{t}} \subseteq \overline{\mid \tilde{\mu} \mid}_{\tilde{t}}$ for all $z \in x + y$. Also, let $r \in \mathcal{R}$ and $x \in \overline{\mid \tilde{\mu} \mid}_{\tilde{t}}$ for $\tilde{t} \in (0, 0.5]$. Then, $\mu(x;\tilde{t}) \in \vee q\tilde{\mu}$ which means $\tilde{\mu}(x) \geq \tilde{t}$ or $\tilde{\mu}(x) + \tilde{t} > 1$. On the other hand, by Theorem 115, we know that

$$\tilde{\mu}(rx) \geq r\min\{\tilde{\mu}(x), [0.5, 0.5]\}$$

so

$$\tilde{\mu}(rx) \geq r\min\{\tilde{t}, [0.5, 0.5]\}$$

Since if,

$$\tilde{\mu}(rx) < r\min\{\tilde{t}, [0.5, 0.5]\}$$

then,

$$r\min\{\tilde{\mu}(x), [0.5, 0.5]\} \leq \tilde{\mu}(rx) < r\min\{\tilde{t}, [0.5, 0.5]).$$

Hence, $\tilde{\mu}(x) < \tilde{t}$, that is $\mu(x;\tilde{t})\overline{\in}\tilde{\mu}$, thus $\mu(x;\tilde{t})\overline{\in \wedge q\tilde{\mu}}$, which is a contradiction. We know $\tilde{t} \leq [0.5, 0.5]$ then

$$\tilde{\mu}(rx) \geq r\min\{\tilde{t}, [0.5, 0.5]\} = \tilde{t}$$

and so $rx \in \tilde{\mu}_{\tilde{t}} \subseteq \overline{\mid \tilde{\mu} \mid}_{\tilde{t}}$. Similarly, $xr \in \overline{\mid \tilde{\mu} \mid}_{\tilde{t}}$, therefore, $\overline{\mid \tilde{\mu} \mid}_{\tilde{t}}$ is a hyperideal of $\mathcal{R}$.

Conversely, let $\overline{\mid \tilde{\mu} \mid}_{\tilde{t}}$ be a hyperideal of $\mathcal{R}$ for $\tilde{t} \in (0, 0.5]$. Suppose $x, y \in \mathcal{R}$ such that, for all $z \in x + y$,

$$\tilde{\mu}(z) < r\min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\}$$

Then, there exists $\tilde{t} \in (0, 0.5)$ such that for all $z \in x + y$

$$\tilde{\mu}(z) < \tilde{t} < \tilde{\mu}(x) \wedge \tilde{\mu}(y) \wedge [0.5, 0.5]\}$$

It follows $x, y \in \tilde{\mu}_{\tilde{t}} \subseteq \overline{|\, \tilde{\mu} \,|}_{\tilde{t}}$, which implies $z \in \overline{|\, \tilde{\mu} \,|}_{\tilde{t}}$, for all $z \in x + y$. Hence, $\tilde{\mu}(z) \geq \tilde{t}$ or $\tilde{\mu}(z) + \tilde{t} > [1, 1]$, for all $z \in x + y$, which is a contradiction. Therefore, for all $z \in x + y$,

$$\tilde{\mu}(z) \geq r \min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\}$$

Also, suppose r, $x \in \mathcal{R}$ such that

$$\tilde{\mu}(rx) < r \min\{\tilde{\mu}(x), [0.5, 0.5]\}$$

then there exists $\tilde{t} \in (0, 0.5)$ such that

$$\tilde{\mu}(rx) < \tilde{t} < r \min\{\tilde{\mu}(x), [0.5, 0.5]\}.$$

It follows $x \in \tilde{\mu}_{\ \tilde{t}} \subseteq \overline{|\, \tilde{\mu} \,|}_{\tilde{t}}$, which implies $rx \in \overline{|\, \tilde{\mu} \,|}_{\tilde{t}}$.

Hence, $\tilde{\mu}(rx) \geq \tilde{t}$ or $\tilde{\mu}(rx) + \tilde{t} > [1, 1]$ which is a contradiction. Thus,

$$\tilde{\mu}(rx) \geq r \min\{\tilde{\mu}(x), [0.5, 0.5]\}.$$

Similarly,

$$\tilde{\mu}(xr) \geq r \min\{\tilde{\mu}(x), [0.5, 0.5]\}$$

therefore, the proof is completed. ∎

In the next theorem, we discuss an interval valued $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$ which can be expressed as the union of two proper non-equivalent interval valued $(\in, \in \vee q)$- fuzzy hyperideals.

Theorem 122 *Let $\tilde{\mu}$ be a proper interval valued $(\in, \in \vee q)$-fuzzy hyperideal of $\mathcal{R}$ such that*

$$2 \leq \mid \{\tilde{\mu}(x) \mid \tilde{\mu}(x) < [0.5, 0.5]\} \mid < \infty.$$

Then, there exist two proper non-equivalent interval valued $(\in, \in \vee q)$- fuzzy hyperideal of $\mathcal{R}$ such that $\tilde{\mu}$ can be expressed as their union.

Proof. Let

$$\{\tilde{\mu}(x) \mid \tilde{\mu}(x) < [0.5, 0.5]\} = \{\tilde{t}_0,\ \tilde{t}_{1,...,}\ \tilde{t}_n\},$$

where $\tilde{t}_1 > \tilde{t}_2 > ... > \tilde{t}_r$ and $r \geq 2$. Then, the chain of interval valued $(\in \vee q)$-level hyperideals of $\mathcal{R}$ is

$$\overline{\mid \tilde{\mu} \mid}_{[0.5,0.5]} \subseteq \overline{\mid \tilde{\mu} \mid}_{\tilde{t}_1} \subseteq \overline{\mid \tilde{\mu} \mid}_{\tilde{t}_2} ... \subseteq \overline{\mid \tilde{\mu} \mid}_{\tilde{t}_r} = \mathcal{R}.$$

Let $\bar{\nu}$ and $\bar{\theta}$ be fuzzy sets in $\mathcal{R}$ defined by

$$\bar{\nu}(x) = \begin{cases} \tilde{t}_1 & \text{if } x \in \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_1}, \\ \tilde{t}_2 & \text{if } x \in \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_2} \backslash \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_1}, \\ & . \\ & . \\ & . \\ \tilde{t}_r & \text{if } x \in \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_r} \backslash \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_r - 1}, \end{cases}$$

and

$$\bar{\theta}\,(x) = \begin{cases} \widetilde{\mu}(x) & \text{if } x \in \overline{\mid \widetilde{\mu} \mid}_{[0.5,0.5]}\ , \\ k & \text{if } x \in \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_2} \backslash \overline{\mid \widetilde{\mu} \mid}_{[0.5,0.5]}, \\ \tilde{t}_3 & \text{if } x \in \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_3} \backslash \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_2}, \\ \tilde{t}_4 & \text{if } x \in \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_4} \backslash \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_3}, \\ & . \\ & . \\ & . \\ \tilde{t}_r & \text{if } x \in \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_r} \backslash \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_r - 1}, \end{cases}$$

where $\tilde{t}_3 < k < \tilde{t}_{2.}$ The, $\bar{\nu}$ and $\bar{\theta}$ are interval valued $(\in, \in \vee q)$- fuzzy hyperideals of $\mathcal{R}$, and $\bar{\nu}, \bar{\theta} \leq \widetilde{\mu}$. The chains of interval valued $(\in \vee q)$-level hyperideals of $\bar{\nu}$ and $\bar{\theta}$ are, respectively, given by

$$\overline{\mid \widetilde{\mu} \mid}_{[0.5,0.5]} \subseteq \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_1} \subseteq \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_2} ... \subseteq \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_r}$$

and

$$\overline{\mid \widetilde{\mu} \mid}_{[0.5,0.5]} \subseteq \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_1} \subseteq \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_2} ... \subseteq \overline{\mid \widetilde{\mu} \mid}_{\tilde{t}_r}.$$

Thus, $\bar{\nu}$ and $\bar{\theta}$ are non-equivalent and clearly $\widetilde{\mu} = \bar{\nu} \vee \bar{\theta}$. Therefore, $\widetilde{\mu}$ can be expressed as the union of two proper non-equivalent interval valued $(\in, \in \vee q)$- fuzzy hyperideal of $\mathcal{R}$. ∎

4.3 t-Implication-based interval valued fuzzy hyperideals of semi-hyperrings

In this section, we generalize the notion of ordinary fuzzy hyperideals, interval valued $(\in, \in \vee q)$-fuzzy hyperideals and interval valued $(\overline{\in}, \overline{\in \vee q})$-fuzzy hyperideals. Specially, we characterize fuzzy hyperideals, interval valued $(\in, \in \vee q)$-fuzzy hyperideals and interval valued $(\overline{\in}, \overline{\in \vee q})$-fuzzy hyperideals based on implication operators.

Definition 123 *Let $\tilde{m}, \tilde{n} \in [0,1]$, $\tilde{m} < \tilde{n}$ and $\tilde{\mu}$ be an interval valued fuzzy set in $\mathcal{R}$. Then, $\tilde{\mu}$ is said to be a fuzzy hyperideal with threshold $(\tilde{m}, \tilde{n})$ of $\mathcal{R}$, if for all $r, x, y \in \mathcal{R}$, the following conditions hold:*

(i) $r\min\{\tilde{\mu}(x), \tilde{\mu}(y), \tilde{n}\} \leq r\max\{\widetilde{\mu}(z), \tilde{m}\}$, for all $z \in x + y$,

(ii) $r\min\{\tilde{\mu}(x), \tilde{n}\} \leq r\max\{\tilde{\mu}(rx, \tilde{m}\}$ and $r\min\{\tilde{\mu}(x), \tilde{n}\} \leq r\max\{\tilde{\mu}(xr), \tilde{m}\}$.

Clearly, every interval valued fuzzy hyperideal with thresholds $(\tilde{m}, \tilde{n})$ of $\mathcal{R}$ is an ordinary interval valued fuzzy hyperideal when $\tilde{m} = [0,0]$ and $\tilde{n} = [1,1]$. Also, it is an interval valued $(\in, \in \vee q)$-fuzzy (resp. interval valued $(\overline{\in}, \overline{\in \vee q})$-fuzzy) hyperideal when $\tilde{m} = [0,0]$ and $\tilde{n} = [0.5, 0.5]$ (resp. $\tilde{m} = [0,0]$ and $\tilde{n} = [0.5, 0.5]$) (see Theorem 119).

Theorem 124 *An interval valued fuzzy set $\tilde{\mu}$ in $\mathcal{R}$ is a interval valued fuzzy hyperideal with threshold $(\tilde{m}, \tilde{n})$ of $\mathcal{R}$ if and only if $\tilde{\mu}_{\tilde{t}} (\neq \Phi)$ is a hyperideal of $\mathcal{R}$ for all $\tilde{t} \in (\tilde{m}, \tilde{n}]$.*

Proof. Suppose that $\tilde{\mu}$ is an interval valued fuzzy hyperideal with thresholds $(\tilde{m}, \tilde{n})$ of $\mathcal{R}$ and $\tilde{t} \in (\tilde{m}, \tilde{n}]$. If $x, y \in \tilde{\mu}_{\tilde{t}}$, then $\tilde{\mu}(x) \geq \tilde{t}$ and $\tilde{\mu}(y) \geq \tilde{t}$. We have, for all $z \in x + y$,

$$r\max\{\tilde{\mu}(z), \tilde{m}\} \geq r\min\{\tilde{\mu}(x), \tilde{\mu}(y), \tilde{n}\} \geq r\min\{\tilde{t}, \tilde{n}\} = \tilde{t} > \tilde{m},$$

Hence, $r\max\{\tilde{\mu}(z), \tilde{m}\} \geq \tilde{t} > \tilde{m}$, for all $z \in x + y$, which implies $\tilde{\mu}(z) \geq \tilde{t}$, for all $z \in x+y$, that is $z \in \tilde{\mu}_{\tilde{t}}$ for all $z \in x+y$. Now, if $x \in \tilde{\mu}_{\tilde{t}}$ and $r \in \mathcal{R}$, then $\tilde{\mu}(x) \geq \tilde{t}$. We have

$$r\max\{\tilde{\mu}(rx), \tilde{m}\} \geq r\min\{\tilde{\mu}(x), \tilde{n}\} \geq r\min\{\tilde{t}, \tilde{n}\} = \tilde{t} > \tilde{m}.$$

Hence,

$$r\max\{\tilde{\mu}(rx), \tilde{m}\} \geq \tilde{t} > \tilde{m},$$

which implies $\tilde{\mu}(rx) \geq \tilde{t}$, that is $rx \in \tilde{\mu}_{\tilde{t}}$. Similarly, $xr \in \tilde{\mu}_{\tilde{t}}$. Therefore, $\tilde{\mu}_{\tilde{t}}$ is a hyperideal of $\mathcal{R}$.

Conversely, let $\tilde{\mu}$ be an interval valued fuzzy set in $\mathcal{R}$. If there exist $x, y \in \mathcal{R}$ such that

$$r\max\{\tilde{\mu}(z), \tilde{m}\} < r\min\{\tilde{\mu}(x), \tilde{\mu}(y), \tilde{n}\} = \tilde{t},$$

for all $z \in x+y$, then $\tilde{t} \in (\tilde{m}, \tilde{n}]$, $\tilde{\mu}(z) < \tilde{t}$, $x \in \tilde{\mu}$ and $y \in \tilde{\mu}_{\tilde{t}}$, for all $z \in x+y$. Since $\tilde{\mu}_{\tilde{t}}$ is a hyperideal of $\mathcal{R}$, we have $z \in \tilde{\mu}_{\tilde{t}}$ for all $z \in x+y$. Thus, $z \subseteq \tilde{\mu}_{\tilde{t}}$, for all $z \in x+y$. Hence, $\tilde{\mu}(z) \geq \tilde{t}$ for all $z \in x+y$, which is a contradiction. Therefore, for all $x, y \in \mathcal{R}$, we have for all $z \in x+y$

$$r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\} \leq r\max\{\tilde{\mu}(z), \tilde{m}\},$$

Also, if there exist $r, x \in \mathcal{R}$ such that

$$r\max\{\tilde{\mu}(rx), \tilde{m}\} \geq r\min\{\tilde{\mu}(x), \tilde{n}\} \geq r\min\{\tilde{t}, \tilde{n}\} = \tilde{t},$$

then $\tilde{t} \in (\tilde{m}, \tilde{n}]$, $\tilde{\mu}(rx) \geq \tilde{t}$, which is a contradiction. Thus, for all $r, x \in \mathcal{R}$, we have

$$r\min\{\tilde{\mu}(x), \tilde{n}\} \leq r\max\{\tilde{\mu}(rx), \tilde{m}\}.$$

Similarly,

$$r\min\{\tilde{\mu}(x), \tilde{n}\} \leq r\max\{\tilde{\mu}(xr), \tilde{m}\},$$

therefore, $\tilde{\mu}$ is an interval valued fuzzy hyperideal with thresholds $(\tilde{m}, \tilde{n})$ of $\mathcal{R}$. ■

Set theoretic multivalued logic is a special case of fuzzy logic such that the truth values are linguistic variables (or terms of the linguistic variables truth). By using extension principal some operators like $\wedge, \vee, \neg, \longrightarrow$ can be applied in fuzzy logic. In fuzzy logic, $[P]$ means the truth value of fuzzy proposition P. In the following, we show a correspondence between fuzzy logic and set-theoretical notions.

$[x \in \tilde{\mu}] = \tilde{\mu}(x)$, $\qquad [x \neq \tilde{\mu}] = [1,1] - \tilde{\mu}(x)$,

$[P \wedge Q] = \min\{[P], [Q]\}$, $\qquad [P \vee Q] = \max\{[P], [Q]\}$,

$[P \longrightarrow Q] = \min\{[1,1], [1,1] - [P] + [Q]\}$,

$[\forall x \in P(x)] = \inf[P(x)]$, $\qquad \models P$ if and only if $[P] = [1,1]$for all valuations.

We show some of important implication operators, where α denotes the degree of membership of the premise and β is the degree of membership of the consequence, and I the resulting degree of truth for the implication.

Early Zadeh $\qquad I_m(\bar{\alpha}, \bar{\beta}) = \max\{[1,1] - \bar{\alpha}, \min\{\bar{\alpha}, \bar{\beta}\}\}$,

Lukasiewicz $\qquad I_\alpha(\bar{\alpha}, \bar{\beta}) = \min\{[1,1] - \bar{\alpha} + \bar{\beta}\}$,

Standard Star (Godel)

$$I_g(\bar{\alpha}, \bar{\beta}) = \begin{cases} [1,1] & \bar{\alpha} \leq \bar{\beta} \\ \bar{\beta} & \text{otherwise} \end{cases},$$

Contraposition of (Godel)

$$I_{cg}(\bar{\alpha}, \bar{\beta}) = \begin{cases} [1,1] & \bar{\alpha} \leq \bar{\beta} \\ [1,1] - \bar{\alpha} & \text{otherwise} \end{cases}$$

Gaines-Rescher

$$I_{gr}(\bar{\alpha}, \bar{\beta}) = \begin{cases} [1,1] & \bar{\alpha} \leq \bar{\beta} \\ [0,0] & \text{otherwise} \end{cases}$$

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$$I_b(\bar{\alpha}, \bar{\beta}) = \max\{[1,1] - \bar{\alpha}, \bar{\beta}\}.$$

Definition 125 *An interval valued fuzzy set $\tilde{\mu}$ in $\mathcal{R}$ is called fuzzifying hyperideal of $\mathcal{R}$, if and only if for all $r, x, y \in \mathcal{R}$ it satisfies:*

(1) $\qquad \models [[x \in \tilde{\mu}] \wedge [y \in \tilde{\mu}] \longrightarrow [z \in \tilde{\mu}]]$, for all $z \in x + y$,

(2) $\qquad \models [[x \in \tilde{\mu}] \wedge [rx \in \tilde{\mu}]$ and $\models [[x \in \tilde{\mu}] \longrightarrow [xr \in \tilde{\mu}]]$.

Clearly, Definition 125 is equivalent to Definition 43. Therefore, a fuzzifying hyperideal is an ordinary fuzzy hyperideal. We have the notion of $\tilde{t}$-tautology. In fact $\models_{\tilde{t}} P$, if and only if $[P] \geq \tilde{t}$ (see [68]).

Definition 126 *An interval valued fuzzy set $\tilde{\mu}$ in $\mathcal{R}$ is said to be $\tilde{t}$-implication-based fuzzy left (resp. right) interval valued hyperideal of $\mathcal{R}$ with respect to the implication $\longrightarrow$ if the following conditions hold for all $r, x, y \in \mathcal{R}$:*

(1) $\models_{\tilde{t}} [[x \in \tilde{\mu}] \wedge [y \in \tilde{\mu}] \longrightarrow [z \in \tilde{\mu}]]$, for all $z \in x + y$,

(2) $\models_{\tilde{t}} [[x \in \tilde{\mu}] \wedge [rx \in \tilde{\mu}]$ (resp. $\models_{\tilde{t}} [[x \in \tilde{\mu}] \longrightarrow [xr \in \tilde{\mu}]].$).

An interval valued fuzzy set $\tilde{\mu}$ in $\mathcal{R}$ is said to be $\tilde{t}$-implication-based interval valued fuzzy hyperideal of $\mathcal{R}$ with respect to the implication $\longrightarrow$ if $\tilde{\mu}$ is both $\tilde{t}$-implication-based interval valued fuzzy left and right hyperideal of $\mathcal{R}$ with respect to the implication $\longrightarrow$.

Proposition 127 *An interval valued fuzzy set $\tilde{\mu}$ of $\mathcal{R}$ is a $\tilde{t}$-implication-based interval valued fuzzy hyperideal of $\mathcal{R}$ with respect to the implication operator I if and only if for all $r, x, y \in \mathcal{R}$.*

(i) $I\left(r \min\{\tilde{\mu}(x), \tilde{\mu}(y)\}, \tilde{\mu}(z)\right) \geq \tilde{t}$ for all $z \in x + y$,

(ii) $I\left(\tilde{\mu}(x), \tilde{\mu}(rx)\right) \geq \tilde{t}$ and $I(\tilde{\mu}(x), \tilde{\mu}(xr)) \geq \tilde{t}$.

Proof. The proof is clear by considering the definitions. ■

Theorem 128 *(1) Let $I = I_{gr}$ (Gaines-Rescher). Then, $\tilde{\mu}$ is an $[0.5, 0.5]$-implication-based interval valued fuzzy hyperideal of $\mathcal{R}$ if and only if $\tilde{\mu}$ is an interval valued fuzzy hyperideal with thresholds $m = [0, 0]$ and $n = [1, 1]$ of $\mathcal{R}$ (or equivalent, $\tilde{\mu}$ is an ordinary interval valued fuzzy hyperideal of $\mathcal{R}$).*

(2) Let $I = I_{gr}$ (Godel). Then, $\tilde{\mu}$ is an $[0.5, 0.5]$-implication-based interval valued fuzzy hyperideal of $\mathcal{R}$ if and only if $\tilde{\mu}$ is an interval valued fuzzy hyperideal with thresholds $m = [0, 0]$ and $n = [0.5, 0.5]$ of $\mathcal{R}$ (or equivalent, $\tilde{\mu}$ is an interval valued $(\in, \in \vee q)-$fuzzy hyperideal of $\mathcal{R}$).

(3) Let $I = I_{cg}$ (Contraposition of Godel). Then, $\tilde{\mu}$ is an $[0.5, 0.5]$-implication-based interval valued fuzzy hyperideal of $\mathcal{R}$ if and only if $\tilde{\mu}$ is an interval valued fuzzy hyperideal with thresholds $m = [0.5, 0.5]$ and $n = [1, 1]$ of $\mathcal{R}$ (or equivalent, $\tilde{\mu}$ is an interval valued $(\overline{\in}, \overline{\in \vee q})$- fuzzy hyperideal of $\mathcal{R}$).

Proof. (1) Let $\tilde{\mu}$ be an $[0.5, 0.5]$-implication-based interval valued fuzzy hyperideal of $\mathcal{R}$. Then for all $z \in x + y$

$$I_{gr}(r \min\{\tilde{\mu}(x), \tilde{\mu}(y)\}, \tilde{\mu}(z)) \geq [0.5, 0, 5],$$

Which implies for all $z \in x + y$

$$\tilde{\mu}(z) \geq r \min\{\tilde{\mu}(x)), \tilde{\mu}(y)\},$$

Also,

$$I_{gr}(\widetilde{\mu}(x), \widetilde{\mu}(rx)) \geq [0.5, 0.5]$$

which implies $\tilde{\mu}(rx) \geq \tilde{\mu}(x)$. Similarly, $\tilde{\mu}(xr) \geq \tilde{\mu}(x)$. Therefore, $\tilde{\mu}$ is an interval valued fuzzy hyperideal with threshold $m = [0, 0]$ and $n = [1, 1]$ of $\mathcal{R}$.

Conversely, let $\tilde{\mu}$ be an interval valued fuzzy hyperideal with threshold $m = [0, 0]$ and $n = [1, 1]$ of $\mathcal{R}$. Then,

$$\begin{aligned}
\tilde{\mu}(z) &\geq r \min\{\tilde{\mu}(x), \tilde{\mu}(y)\}, \text{ for all } z \in x + y, \\
\tilde{\mu}(rx) &\geq \tilde{\mu}(x), \tilde{\mu}(xr) \geq \tilde{\mu}(x), \text{ for all } r, x, y \in \mathcal{R}.
\end{aligned}$$

Hence, for all $z \in x + y$

$$I_{gr}(r \min\{\tilde{\mu}(x), \tilde{\mu}(y)\}, \tilde{\mu}(z)) = [1, 1]$$

$$I_{gr}(\tilde{\mu}(x), \tilde{\mu}(xr)) = [1, 1] = I_{gr}(\tilde{\mu}(x), \tilde{\mu}(rx)).$$

Thus, for all $z \in x + y$,

$$I_{gr}(r \min\{\tilde{\mu}(x), \tilde{\mu}(y)\}, \tilde{\mu}(z)) \geq [0.5, 0.5],$$

$$I_{gr}(\tilde{\mu}(x), \tilde{\mu}(rx)) \geq [0.5, 0.5]$$

and

$$I_{gr}(\tilde{\mu}(x), \tilde{\mu}(xr)) \geq [0.5, 0.5].$$

Therefore, is a $[0.5, 0.5]$-implication-based interval valued fuzzy hyperideal of $\mathcal{R}$.

(2) Let $\tilde{\mu}$ be an $[0.5, 0.5]$-implication-based interval valued fuzzy $\tilde{\mu}$ hyperideal of $\mathcal{R}$. Then, for all $r, x, y \in \mathcal{R}$, we have, for all $z \in x + y$,

$$I_g(r \min\{\tilde{\mu}(x), \tilde{\mu}(y)\}, \tilde{\mu}(z)) \geq [0.5, 0.5]$$

$$I_g(\tilde{\mu}(x), \tilde{\mu}(rx) \geq [0.5, 0.5]$$

and

$$I_g((\tilde{\mu}(x), \tilde{\mu}(xr)) \geq [0.5, 0.5].$$

By the definition of I_g, we can consider the following cases:

(a) $I_g(r \min\{\tilde{\mu}(x), \tilde{\mu}(y)\}, \tilde{\mu}(z)) = [1, 1]$ for all $z \in x + y$, then $r \min\{\tilde{\mu}(x), \tilde{\mu}(y)\} \leq \tilde{\mu}(z)$, for all $z \in x + y$, which implies, for all $z \in x + y$

$$r \min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} \leq \tilde{\mu}(z),$$

(b) $I_g(r \min\{\tilde{\mu}(x), \tilde{\mu}(y)\}, \widetilde{\mu}(z)) = \tilde{\mu}(z)$, for all $z \in x + y$. Then $\tilde{\mu}(z) \geq [0.5, 0.5]$ for all $z \in x + y$. Which implies, for all $z \in x + y$

$$r \min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} \leq \tilde{\mu}(z)$$

Similarly, we can show that

$$r \min\{\tilde{\mu}(x), [0.5, 0.5]\} \leq \tilde{\mu}(rx)$$

and

$$r \min\{\tilde{\mu}(x), [0.5, 0.5]\} \leq \tilde{\mu}(xr).$$

Therefore, $\tilde{\mu}$ is an interval valued fuzzy hyperideal with thresholds $m = [0, 0]$ and $n = [0.5, 0.5]$ of $\mathcal{R}$.

Conversely, let $\tilde{\mu}$ is an interval valued fuzzy hyperideal with thresholds $m = [0, 0]$ and $n = [0.5, 0.5]$ of $\mathcal{R}$. Then, for all $r, x, y \in \mathcal{R}$, by Definition 123 for all $z \in x + y$,

$$r \min\{\tilde{\mu}(x), \tilde{\mu}(y), [0.5, 0.5]\} \leq \tilde{\mu}(z),$$

and

$$r\min\{\tilde{\mu}(x), [0.5, 0.5]\} \leq \tilde{\mu}(rx)$$

$$r\min\{\tilde{\mu}(x), [0.5, 0.5]\} \leq \tilde{\mu}(xr).$$

Hence, in each case, for all $z \in x + y$

$$I_g(r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\}, \tilde{\mu}(z)) \geq [0.5, 0.5]$$

$$I_g(\tilde{\mu}(x), \tilde{\mu}(rx)) \geq [0.5, 0.5]$$

and

$$I_g(\tilde{\mu}(x), \tilde{\mu}(xr)) \geq [0.5, 0.5].$$

Therefore, $\tilde{\mu}$ is an $[0.5, 0.5]$-implication-based interval valued fuzzy hyperideal of $\mathcal{R}$.

(3) Let $\tilde{\mu}$ be an $[0.5, 0.5]$-implication-based interval valued fuzzy hyperideal of $\mathcal{R}$. Then, for all $r, x, y \in \mathcal{R}$, we have for all $z \in x + y$,

$$I_{cg}(r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\}, \tilde{\mu}(z)) \geq [0.5, 0.5],$$

$$I_{cg}(\tilde{\mu}(x), \tilde{\mu}(rx)) \geq [0.5, 0.5]$$

and

$$I_{cg}(\tilde{\mu}(x), \tilde{\mu}(xr)) \geq [0.5, 0.5].$$

By definition of I_{cg}, we can consider the following cases:

(a) If $I_{cg}(r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\}, \tilde{\mu}(z)) = [1, 1]$, for all $z \in x + y$,

then for all $z \in x + y$,

$$r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\} \leq \tilde{\mu}(z),$$

which implies that for all $z \in x + y$.

$$r\min\{\tilde{\mu}(x), \tilde{\mu}(y)\} \leq r\max\{\tilde{\mu}(z), [0.5, 0.5]\}$$

(b) If for all $z \in x+y$,

$$I_{cg}(r\min\{\tilde{\mu}(x),\tilde{\mu}(y)\},\tilde{\mu}(z)) = [1,1]-(\tilde{\mu}(x)\wedge\tilde{\mu}(y)),$$

then

$$[1,1]-r\min\{\tilde{\mu}(x),\tilde{\mu}(y)\} \geq [0.5,0.5]$$

it implies that

$$r\min\{\tilde{\mu}(x),\tilde{\mu}(y)\} \leq [0.5,0.5]$$

and hence for all $z \in x+y$

$$r\min\{\tilde{\mu}(x),\tilde{\mu}(y)\} \leq r\max\{\tilde{\mu}(z),[0.5,0.5]\}.$$

Similarly, we can show that

$$\tilde{\mu}(x) \leq r\max\{\tilde{\mu}(rx),[0.5,0.5]\}$$

and

$$\tilde{\mu}(x) \leq r\max\{\tilde{\mu}(xr),[0.5,0,5]\}.$$

Therefore, $\tilde{\mu}$ is an interval valued fuzzy hyperideal with threshold $m=[0.5,0.5]$ and $n=[1,1]$ of $\mathcal{R}$.

Conversely, let $\tilde{\mu}$ be an interval valued fuzzy hyperideal with threshold $m=[0.5,0.5]$ and n= $[1,1]$ of $\mathcal{R}$. Then, for all $r,x,y \in \mathcal{R}$,

we have, for all $z \in x+y$

$$r\min\{\tilde{\mu}(x),\tilde{\mu}(y)\} \leq r\max\{\tilde{\mu}(z),[0.5,0.5]\}$$

$$\tilde{\mu}(x) \leq r\max\{\tilde{\mu}(rx),[0.5,0.5]\}$$

and

$$\tilde{\mu}(x) \leq r\max\{\tilde{\mu}(xr),[0.5,0.5]\}$$

Now, we can consider two following cases:

(a) For all $z \in x + y$,

$$r \min\{\tilde{\mu}(x), \tilde{\mu}(y)\} \leq \tilde{\mu}(z),$$

which implies, for all $z \in x + y$,

$$I_{cg}(r \min\{\tilde{\mu}(x), \tilde{\mu}(y)\}, \tilde{\mu}(z)) = [1, 1] \geq [0.5, 0.5]$$

(b) If $r \min\{\tilde{\mu}(x), \tilde{\mu}(y)\} > \tilde{\mu}(z)$, for all $z \in x + y$, which implies

$$r \min\{\tilde{\mu}(x), \tilde{\mu}(y)\} \geq [0.5, 0.5].$$

Hence,

$$[1, 1] - r \min\{\tilde{\mu}(x), \tilde{\mu}(y)\} \geq [0.5, 0.5].$$

Thus, for all $z \in x + y$

$$I_{cg}(\tilde{\mu}(x) \wedge \tilde{\mu}(y), \tilde{\mu}(z)) = [1, 1] - (\tilde{\mu}(x) \wedge \tilde{\mu}(y)) \geq [0.5, 0.5]$$

Similarly, we can prove that

$$I_{cg}(\tilde{\mu}(x), \tilde{\mu}(rx)) \geq [0.5, 0.5]$$

and

$$I_{cg}(\tilde{\mu}(x), \tilde{\mu}(xr)) \geq [0.5, 0.5].$$

Therefore, $\tilde{\mu}$ is an $[0.5, 0.5]$-implication-based interval valued fuzzy hyperideal of $\mathcal{R}$. ■

Chapter 5

(α, β)-Intuitionistic Fuzzy Bi-hyperideals

The aim of this chapter is to introduce the generalization of intuitionistic fuzzy bi-hyperideals of a semihyperrings. First, we discuss the notion of intuitionistic fuzzy hyperideals (bi-hyperideals) in semihyperring and we then define the (α, β)-intuitionistic fuzzy bi-hyperideals of a semihyperring. This concept is a new generalization of the notion of intuitionistic fuzzy bi-hyperideals of a semihyperring. We give some interesting results and as well as examples of this notion. In the last section we discuss the intuitionistic fuzzy bi-hyperideals of type $(\in, \in \vee q)$.

5.1 Intuitionistic fuzzy hyperideal

Definition 129 *Let $\mathcal{R}$ be a semihyperring. An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ in $\mathcal{R}$ is called a left (resp. right) intuitionistic fuzzy hyperideal of a semihyperring $\mathcal{R}$ if*

(i) $\inf_{z\in x\oplus y} \mu_A(z) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\sup_{z\in x\oplus y} \nu_A(z) \leq \max\{\nu_A(x), \nu_A(y)\}$.

(ii) $\mu_A(xy) \geq \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(y)$ (resp. right $\mu_A(xy) \geq \mu_A(x)$ and $\nu_A(xy) \leq \nu_A(x)$, for all $x, y \in \mathcal{R}$.

An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ in $\mathcal{R}$ is called an intuitionistic fuzzy hyperideal of semihyperring $\mathcal{R}$ if it is both left and right intuitionistic fuzzy hyperideal of semihyperring $\mathcal{R}$.

Theorem 130 *Let A be an intuitionistic fuzzy set in a semihyperring $\mathcal{R}$. Then, A is an intu-*

itionistic fuzzy hyperideal of $\mathcal{R}$ if and only if for every $t, s \in (0,1]$, the level subset $U(\mu_A; t) = \{x \in \mathcal{R} | \mu_A(x) \geq t\} \neq \varphi$ and $V(\nu_A; s) = \{x \in \mathcal{R} | \nu_A(x) \leq s\} \neq \varphi$ are hyperideals of $\mathcal{R}$.

Definition 131 *Let $\mathcal{R}$ be a semihyperring. An intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ in $\mathcal{R}$ is called an intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$ if for all $x, y, z \in \mathcal{R}$,*

(i) $\inf_{z \in x \oplus y} \mu_A(z) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\sup_{z \in x \oplus y} \nu_A(z) \leq \max\{\nu_A(x), \nu_A(y)\}$.

(ii) $\mu_A(xy) \geq \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(y)$ (resp. right $\mu_A(xy) \geq \mu_A(x)$ and $\nu_A(xy) \leq \nu_A(x)$, for all $x, y \in \mathcal{R}$.

(iii) $\mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(z)\}$ and $\nu_A(xyz) \leq \max\{\nu_A(x), \nu_A(z)\}$.

5.2 (α, β)-Intuitionistic fuzzy bi-hyperideals

In the above section we introduced the notion of an intuitionistic fuzzy hyperideal, intuitionistic fuzzy bi-hyperideal in a semihyperring and studied some fundamental properties.

In this section, we introduce the concept of an (α, β)-intuitionistic fuzzy bi-hyperideal of a semihyperring by using the notion of intuitionsitic fuzzy point to intuitionistic fuzzy set. The notion of an (α, β)-intuitionistic fuzzy bi-hyperideal of a semihyperring is a generalization of ordinary intuitionitic fuzzy bi-hyperideal.

Definition 132 *An IFS $A = \langle \mu_A, \lambda_A \rangle$ in a semihypergroup $\mathcal{R}$ is said to be an (α, β)-intuitionistic fuzzy hyperideal of a semihyperring $\mathcal{R}$, where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, if for all $x, y, z \in \mathcal{R}$, $t_1, t_2 \in [0, 0.5]$ and $s_1, s_2 \in (0.5, 1]$ or $t_1, t_2 \in (0.5, 1]$ and $s_1, s_2 \in [0, 0.5]$ the following conditions hold:*

(i) $x(t_1, s_1)\alpha A$ and $y(t_2, s_2)\alpha A \Longrightarrow (z)(m\{t_1, t_2\}, M\{s_1, s_2\})\beta A$, for each $z \in x \oplus y$,

(ii) $x(t_1, s_1)\alpha A$ and $y(t_2, s_2)\alpha A \Longrightarrow (xy)(m\{t_1, t_2\}, M\{s_1, s_2\})\beta A$.

Definition 133 *An IFS $A = \langle \mu_A, \lambda_A \rangle$ in a semihypergroup $\mathcal{R}$ is said to be an (α, β)-intuitionistic fuzzy bi-hyperideal of a semihyperring $\mathcal{R}$, where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, if for all $x, y, z \in \mathcal{R}$, $t_1, t_2 \in [0, 0.5]$ and $s_1, s_2 \in (0.5, 1]$ or $t_1, t_2 \in (0.5, 1]$ and $s_1, s_2 \in [0, 0.5]$, the following conditions hold:*

(i) $x(t_1,s_1)\alpha A$ and $y(t_2,s_2)\alpha A \Longrightarrow (z)(m\{t_1,t_2\},M\{s_1,s_2\})\beta A$, for each $z\in x\oplus y$,

(ii) $x(t_1,s_1)\alpha A$ and $y(t_2,s_2)\alpha A \Longrightarrow (xy)(m\{t_1,t_2\},M\{s_1,s_2\})\beta A$,

(iii) $x(t_1,s_1)\alpha A$ and $z(t_2,s_2)\alpha A \Longrightarrow (xyz)(m\{t_1,t_2\},M\{s_1,s_2\})\beta A$.

Theorem 134 *Let $A=\langle\mu_A,\nu_A\rangle$ be a non-zero (α,β)-intuitionistic fuzzy subsemihyperring of $\mathcal{R}$. Then, the set $A_{(0,1)}=\{x\in\mathcal{R}:\mu_A(x)>0$ and $\nu_A(x)<1\}$ is a subsemihyperring of $\mathcal{R}$.*

Proof. (i) Let $x,y\in A_{(0,1)}$. Then, $\mu_A(x)>0$ and $\nu_A(x)<1$, and $\mu_A(y)>0$ and $\nu_A(y)<1$. Let us suppose that $\mu_A(z)=0$ and $\nu_A(z)=1$, for all $z\in x\oplus y$. If $\alpha\in\{\in,\in\vee q\}$, then

$$\begin{aligned} & x\left(\mu_A(x),\nu_A(x)\right)\alpha A \text{ and } y\left(\mu_A(y),\nu_A(y)\right)\alpha A \text{ but} \\ \mu_A(z) &= 0<m\left\{\mu_A(x),\mu_A(y)\right\} \text{ and } \nu_A(z)=1>M\left\{\nu_A(x),\nu_A(y)\right\}, \text{ for all } z\in x\oplus y, \end{aligned}$$

So, for all $z\in x\oplus y$, $(z)\left(m\left\{\mu_A(x),\mu_A(y)\right\},M\left\{\nu_A(x),\nu_A(y)\right\}\right)\overline{\beta}A$ for $\beta\in\{\in,q,\in\vee q,\in\wedge q\}$, which is a contradiction. Now, let $x(1,0)\,qA$ and $y(1,0)\,qA$ but $(z)(1,0)\,\overline{\beta}A$, for all $z\in x\oplus y$, for $\beta\in\{\in,q,\in\vee q,\in\wedge q\}$, which is a contradiction. Hence $\mu_A(z)>0$ and $\nu_A(z)<1$, for all $z\in x\oplus y$, that is $z\in A_{(0,1)}$, for all $z\in x\oplus y$.

(ii) Let $x,y\in A_{(0,1)}$. Then, $\mu_A(x)>0$ and $\nu_A(x)<1$, and $\mu_A(y)>0$ and $\nu_A(y)<1$. Let us suppose that $\mu_A(xy)=0$ and $\nu_A(xy)=1$. If $\alpha\in\{\in,\in\vee q\}$, then

$$\begin{aligned} & x\left(\mu_A(x),\nu_A(x)\right)\alpha A \text{ and } y\left(\mu_A(y),\nu_A(y)\right)\alpha A \text{ but} \\ & \mu_A(xy)=0<m\left\{\mu_A(x),\mu_A(y)\right\} \text{ and } \nu_A(xy)=1>M\left\{\nu_A(x),\nu_A(y)\right\} \end{aligned}$$

So, $(xy)\left(m\left\{\mu_A(x),\mu_A(y)\right\},M\left\{\nu_A(x),\nu_A(y)\right\}\right)\overline{\beta}A$ for $\beta\in\{\in,q,\in\vee q,\in\wedge q\}$, which is a contradiction. Now, let $x(1,0)\,qA$ and $y(1,0)\,qA$ but $(xy)(1,0)\,\overline{\beta}A$, for $\beta\in\{\in,q,\in\vee q,\in\wedge q\}$, which is a contradiction. Hence $\mu_A(xy)>0$ and $\nu_A(xy)<1$, that is $xy\in A_{(0,1)}$. Thus, $A_{(0,1)}$ is a subsemihyperring of $\mathcal{R}$. ■

Theorem 135 *Let $A=\langle\mu_A,\nu_A\rangle$ be a non-zero (α,β)-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$. Then, the set $A_{(0,1)}=\{x\in\mathcal{R}:\mu_A(x)>0$ and $\nu_A(x)<1\}$ is a bi-hyperideal of $\mathcal{R}$.*

Proof. Let $A=\langle\mu_A,\nu_A\rangle$ be a non-zero (α,β)-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$. Then, by Theorem 134, $A_{(0,1)}$ is a subsemihyperring of $\mathcal{R}$. Now, let $x, z \in A_{(0,1)}$, $y \in \mathcal{R}$ and. Then, $\mu_A(x) > 0$ and $\nu_A(x) < 1$, and $\mu_A(z) > 0$ and $\nu_A(z) < 1$. Suppose that $\mu_A(xyz) = 0$ and $\nu_A(xyz) = 1$. If $\alpha \in \{\in, \in\vee q\}$, then

$$x(\mu_A(x), \nu_A(x))\,\alpha A \text{ and } z(\mu_A(z), \nu_A(z))\,\alpha A \text{ but}$$
$$\mu_A(xyz) = 0 < m\{\mu_A(x), \mu_A(z)\} \text{ and } \nu_A(xyz) = 1 > M\{\nu_A(x), \nu_A(z)\},$$

which implies that, $(xyz)(m\{\mu_A(x), \mu_A(z)\}, M\{\nu_A(x), \nu_A(z)\})\,\overline{\beta}A$ for $\beta \in \{\in,\ q,\ \in\vee q,\ \in\wedge q\}$, this is a contradiction. Now, let $x(1,0)\,qA$ and $z(1,0)\,qA$ but, $(xyz)(1,0)\,\overline{\beta}A$ for $\beta \in \{\in,\ q,\ \in\vee q,\ \in\wedge q\}$, which is again contradiction. Hence, $\mu_A(xyz) > 0$ and $\nu_A(xyz) < 1$, that is, $xyz \subseteq A_{(0,1)}$. Thus, $A_{(0,1)}$ is a bi-hyperideal of $\mathcal{R}$. ■

Theorem 136 *Let L be a left (resp. right)-hyperideal of $\mathcal{R}$ and let $A = \langle \mu_A, \nu_A \rangle$ be an IFS such that*

(a) $(\forall x \in \mathcal{R} \setminus L)$ $(\mu_A(x) = 0$ *and* $\nu_A(x) = 1)$,

(b) $(\forall x \in L)$ $(\mu_A(x) \geq 0.5$ *and* $\nu_A(x) \leq 0.5)$.

Then, $A = \langle \mu_A, \nu_A \rangle$ is an $(\alpha, \in\vee q)$-intuitionistic fuzzy left (resp. right)-hyperideal of $\mathcal{R}$.

Proof. (i) (For $\alpha = q$), let $x, y \in \mathcal{R}$, $t \in [0, 0.5]$ and $s \in (0.5, 1]$ or $s \in [0, 0.5]$ and $t \in (0.5, 1]$ be such that $y(t,s)\,qA$. Then, $\mu_A(y) + t > 1$ and $\nu_A(y) + s < 1$. So, $y \in L$. Therefore, $x \oplus y \subseteq L$. Thus, if $t \leq 0.5$ and $s \geq 0.5$, then for all $z \in x \oplus y$, $\mu_A(z) \geq 0.5 \geq t$ and $\nu_A(z) \leq 0.5 \leq s$. So, $(z)(t,s) \in A$, for all $z \in x \oplus y$. If $t > 0.5$ and $s < 0.5$, then for all $z \in x \oplus y$ $\mu_A(z) + t > 0.5 + 0.5 = 1$ and $\nu_A(z) + s < 0.5 + 0.5 = 1$. This implies $(z)(t,s)\,qA$, for all $z \in x \oplus y$.

(ii) Let $x, y \in \mathcal{R}$, $t \in [0, 0.5]$ and $s \in (0.5, 1]$ or $s \in [0, 0.5]$ and $t \in (0.5, 1]$ be such that $y(t,s)\,qA$. Then, $\mu_A(y) + t > 1$ and $\nu_A(y) + s < 1$. So, $y \in L$. Therefore, $xy \in L$. Thus, if $t \leq 0.5$ and $s \geq 0.5$, then $\mu_A(xy) \geq 0.5 \geq t$ and $\nu_A(xy) \leq 0.5 \leq s$ and so $(xy)(t,s) \in A$. If $t > 0.5$ and $s < 0.5$, then $\mu_A(xy) + t > 0.5 + 0.5 = 1$ and $\nu_A(xy) + s < 0.5 + 0.5 = 1$, this implies $(xy)(t,s)\,qA$. Since $0 \leq t + s \leq 1$, the case $t > 0.5$ and $s > 0.5$ does not occur. From the fact that at the same time all values of t and s are not less than 0.5, thus the case $t < 0.5$ and $s < 0.5$ are not occur. Therefore, $(xyz)(t,s) \in \vee qA$. Hence, $A = \langle \mu_A, \nu_A \rangle$ is an $(q, \in\vee q)$-intuitionistic fuzzy left hyperideal of $\mathcal{R}$.

(ii) (For $\alpha = \in$), let $x, y \in \mathcal{R}$, $t \in (0,1]$ and $s \in [0,1)$ or $s \in [0,0.5]$ and $t \in (0.5,1]$ be such that $y(t,s) \in A$. Then, $\mu_A(y) \geq t$ and $\nu_A(y) \leq s$. So, $y \in L$. Therefore, $z \in L$, for all $z \in x \oplus y$. Thus, if $t \leq 0.5$ and $s \geq 0.5$, then $\mu_A(z) \geq 0.5 \geq t$ and $\nu_A(z) \leq 0.5 \leq s$, for all $z \in x \oplus y$, this implies $(z)(t,s) \in A$, for all $z \in x \oplus y$. If $t > 0.5$ and $s < 0.5$, then $\mu_A(z) + t > 0.5 + 0.5 = 1$ and $\nu_A(z) + s < 0.5 + 0.5 = 1$, for all $z \in x \oplus y$, this implies $(z)(t,s)\, qA$, for all $z \in x \oplus y$. Since $0 \leq t + s \leq 1$, the case $t > 0.5$ and $s > 0.5$ are not occur. From the fact that at the same time all values of t, s are not less than 0.5, thus the case $t < 0.5$ and $s < 0.5$ are not occur. Therefore, $(z)(t,s) \in \vee qA$, for all $z \in x \oplus y$.

Now again let $x, y \in \mathcal{R}$, $t \in (0,1]$ and $s \in [0,1)$ or $s \in [0,0.5]$ and $t \in (0.5,1]$ be such that $y(t,s) \in A$. Then, $\mu_A(y) \geq t$ and $\nu_A(y) \leq s$. So, $y \in L$. Therefore, $xy \in L$. Thus, if $t \leq 0.5$ and $s \geq 0.5$, then $\mu_A(xy) \geq 0.5 \geq t$ and $\nu_A(xy) \leq 0.5 \leq s$, this implies $(xy)(t,s) \in A$. If $t > 0.5$ and $s < 0.5$, then $\mu_A(xy) + t > 0.5 + 0.5 = 1$ and $\nu_A(xy) + s < 0.5 + 0.5 = 1$ this implies $(xy)(t,s)\, qA$. Since $0 \leq t + s \leq 1$, the case $t > 0.5$ and $s > 0.5$ are not occur. From the fact that at the same time all values of t, s are not less than 0.5, thus the case $t < 0.5$ and $s < 0.5$ are not occur. Therefore, $(xy)(t,s) \in \vee qA$. Hence, $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy left-hyperideal of $\mathcal{R}$.

(iii) (For $\alpha = \in \vee q$), follows from (i) and (ii). ■

Theorem 137 *Let L be a subsemihyperring of $\mathcal{R}$ and let $A = \langle \mu_A, \nu_A \rangle$ be an IFS such that*

(a) $(\forall x \in \mathcal{R} \setminus L)$ $(\mu_A(x) = 0$ and $\nu_A(x) = 1)$,

(b) $(\forall x \in L)$ $(\mu_A(x) \geq 0.5$ and $\nu_A(x) \leq 0.5)$.

Then, $A = \langle \mu_A, \nu_A \rangle$ is an $(\alpha, \in \vee q)$-intuitionistic fuzzy subsemihyperring of $\mathcal{R}$.

Proof. As in the previous pages. ■

Theorem 138 *Let L be a bi-hyperideal of a semihyperring $\mathcal{R}$ and let $A = \langle \mu_A, \nu_A \rangle$ be an IFS of $\mathcal{R}$ such that*

(a) $(\forall x \in \mathcal{R} \setminus L)$ $(\mu_A(x) = 0$ and $\nu_A(x) = 1)$,

(b) $(\forall x \in L)$ $(\mu_A(x) \geq 0.5$ and $\nu_A(x) \leq 0.5)$.

Then, $A = \langle \mu_A, \nu_A \rangle$ is an $(\alpha, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$.

Proof. (i) (For $\alpha = q$), let $x, y \in \mathcal{R}$, $t_1, t_2 \in [0,0.5]$ and $s_1, s_2 \in (0.5,1]$ or $s_1, s_2 \in [0,0.5]$ and $t_1, t_2 \in (0.5,1]$ be such that $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A$. Then, $x, y \in B$. Since

B is a subsemihyperring of $\mathcal{R}$. So, $z \in B$, for all $z \in x \oplus y$. If $m(t_1, t_2) > 0.5$ and $M(s_1, s_2) < 0.5$, then $\mu_A(z) + m(t_1, t_2) > 1$ and $\nu_A(z) + M(s_1, s_2) < 1$, for all $z \in x \oplus y$. Thus, $(z)(m(t_1, t_2), M(s_1, s_2))qA$, for all $z \in x \oplus y$. If $m(t_1, t_2) \leq 0.5$ and $M(s_1, s_2) \geq 0.5$, then $(z)(m(t_1, t_2),\ M(s_1, s_2)) \in A$, for all $z \in x \oplus y$. Since $0 \leq t_1 + s_1 \leq 1$ and $0 \leq t_2 + s_2 \leq 1$, the case

$\begin{cases} m(t_1, t_2) > 0.5 \\ M(s_1, s_2) > 0.5 \end{cases}$ does not occur.

From the fact that at same time all values of t_i, s_i are not less than 0.5, the case

$\begin{cases} m(t_1, t_2) < 0.5 \\ M(s_1, s_2) < 0.5 \end{cases}$ does not occur.

And again, let $x, y \in \mathcal{R}$, $t_1, t_2 \in [0, 0.5]$ and $s_1, s_2 \in (0.5, 1)$ or $s_1, s_2 \in [0, 0.5]$ and $t_1, t_2 \in (0.5, 1]$ be such that $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A$. Then, $x, y \in B$. Since B is a subsemihyperring of $\mathcal{R}$. So, $xy \in B$. If $m(t_1, t_2) > 0.5$ and $M(s_1, s_2) < 0.5$, then $\mu_A(xy) + m(t_1, t_2) > 1$ and $\nu_A(xy) + M(s_1, s_2) < 1$. Thus, $(xy)(m(t_1, t_2), M(s_1, s_2))qA$. If $m(t_1, t_2) \leq 0.5$ and $M(s_1, s_2) \geq 0.5$, then $(xy)(m(t_1, t_2), M(s_1, s_2)) \in A$. Since $0 \leq t_1 + s_1 \leq 1$ and $0 \leq t_2 + s_2 \leq 1$, the case

$\begin{cases} m(t_1, t_2) > 0.5 \\ M(s_1, s_2) > 0.5 \end{cases}$ does not occur. From the fact that at the same time all values of t_i, s_i are not less than 0.5, thus the case

$\begin{cases} m(t_1, t_2) < 0.5 \\ M(s_1, s_2) < 0.5 \end{cases}$ does not occur. Hence, $A = \langle \mu_A, \nu_A \rangle$ is a $(q, \in \vee q)$-intuitionistic fuzzy subsemihyperring of $\mathcal{R}$. Let $x, y, z \in A$, $t_1, t_2 \in [0, 0.5]$ and $s_1, s_2 \in (0.5, 1]$ or $s_1, s_2 \in [0, 0.5]$ and $t_1, t_2 \in (0.5, 1]$ be such that $x(t_1, s_1) \in A$ and $z(t_2, s_2) \in A$. Then, $x, z \in B$. Since B is a bi-hyperideal of $\mathcal{R}$. So, $xyz \in B$. If $m(t_1, t_2) > 0.5$ and $M(s_1, s_2) < 0.5$, then $\mu_A(xyz) + m(t_1, t_2) > 1$ and $\nu_A(xyz) + M(s_1, s_2) < 1$. So, $(xyz)((m(t_1, t_2), M(s_1, s_2))qA$. If $m(t_1, t_2) \leq 0.5$ and $M(s_1, s_2) \geq 0.5$, then $(xyz)(m(t_1, t_2), M(s_1, s_2)) \in A$. Therefore, $(xyz)(m(t_1, t_2), M(s_1, s_2)) \in \vee qA$. Hence, $A = \langle \mu_A, \nu_A \rangle$ is a $(q, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$.

(ii) (For $\alpha = \in$ and $\in \vee q$), the case is straightforward. ■

5.3 Intuitionistic fuzzy bi-hyperideal of type $(\in, \in \vee q)$

The concept of $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideals in a semihyperring plays a vital rule in the theory of (α, β)-intuitionistic fuzzy bi-hyperideals. We give some different characterization of $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideals in a semihyperring.

Definition 139 *An IFS $A = \langle \mu_A, \nu_A \rangle$ in semihyperring $\mathcal{R}$ is said to be an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of a semihyperring S if $\forall\ x, y, a \in \mathcal{R}$, $t_1, t_2 \in [0, 0.5]$ and $s_1, s_2 \in (0.5, 1]$ or $s_1, s_2 \in [0, 0.5]$ and $t_1, t_2 \in (0.5, 1]$ the following conditions hold.*

(i) $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A \Longrightarrow (z)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee qA$ for each $z \in x \oplus y$,

(ii) $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A \Longrightarrow (xy)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee qA$,

(iii) $x(t_1, s_1) \in A$ and $z(t_2, s_2) \in A \Longrightarrow (xyz)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee qA$.

Proposition 140 *An IFS $A = \langle \mu_A, \nu_A \rangle$ of a semihyperring $\mathcal{R}$ is an intuitionistic fuzzy subsemihyperring if and only if it satisfy for all $x, y \in \mathcal{R}$, $t_1, t_2 \in [0, 0.5]$ and $s_1, s_2 \in (0.5, 1]$ or $s_1, s_2 \in [0, 0.5]$ and $t_1, t_2 \in (0.5, 1]$.*

(i) $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A \Longrightarrow (z)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in A$, for each $z \in x \oplus y$,

(ii) $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A \Longrightarrow (xy)(m\{t_1, t_2\},\ M\{s_1, s_2\}) \in A$.

Proof. Let us suppose that $A = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy subsemihyperring of $\mathcal{R}$. Let $x, y \in \mathcal{R}$, $t_1, t_2 \in [0, 0.5]$ and $s_1, s_2 \in (0.5, 1]$ or $s_1, s_2 \in [0, 0.5]$ and $t_1, t_2 \in (0.5, 1]$ and let $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A$. Then, $\mu_A(x) \geq t_1$ and $\nu_A(x) \leq s_1$ and $\mu_A(y) \geq t_2$ and $\nu_A(y) \leq s_2$. Since by given condition, for each $z \in x \oplus y$,

$$
\begin{aligned}
\inf_{z \in x \oplus y} \mu_A(z) &\geq \min\{\mu_A(x), \mu_A(y)\} \text{ and } \sup_{z \in x \oplus y} \nu_A(z) \leq \max\{\nu_A(x), \nu_A(y)\} \\
\inf_{z \in x \oplus y} \mu_A(z) &\geq m\{t_1, t_2\} \text{ and } \sup_{z \in x \oplus y} \nu_A(z) \leq \{s_1, s_2\}.
\end{aligned}
$$

So, $(z)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in A$, for all $z \in x \oplus y$.

Let $x, y \in \mathcal{R}$, $t_1, t_2 \in [0, 0.5]$ and $s_1, s_2 \in (0.5, 1]$ or $s_1, s_2 \in [0, 0.5]$ and $t_1, t_2 \in (0.5, 1]$ and let $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A$. Then, $\mu_A(x) \geq t_1$ and $\nu_A(x) \leq s_1$, and $\mu_A(y) \geq t_2$ and $\nu_A(y) \leq s_2$. Since by given condition,

$$\begin{aligned}\mu_A(xy) &\geq \min\{\mu_A(x), \mu_A(y)\} \text{ and } \nu_A(xy) \leq \max\{\nu_A(x), \nu_A(y)\} \\ \mu_A(xy) &\geq m\{t_1, t_2\} \text{ and } \nu_A(xy) \leq \{s_1, s_2\}.\end{aligned}$$

So, $(xy)(m\{t_1,t_2\}, M\{s_1,s_2\}) \in A$. Thus, $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in)$-intuitionistic fuzzy subsemihyperring of $\mathcal{R}$.

Conversely, suppose that $A = \langle \mu_A, \nu_A \rangle$ is satisfies the given conditions. We show that $\inf_{z\in x\oplus y} \mu_A(z) \geq \min\{\mu_A(x), \mu_A(y)\}$, $\sup_{z\in x\oplus y} \nu_A(z) \leq \max\{\nu_A(x), \nu_A(y)\}$ and $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$, $\nu_A(xy) \leq \max\{\nu_A(x), \nu_A(y)\}$. On the contrary we first assume that there exist $x, y \in \mathcal{R}$ such that, for all $z \in x \oplus y$, $\inf_{z\in x\oplus y} \mu_A(z) < \min\{\mu_A(x), \mu_A(y)\}$ and $\sup_{z\in x\oplus y} \nu_A(z) > \max\{\nu_A(x), \nu_A(y)\}$. Let $t \in [0, 0.5]$ and $s \in (0.5, 1]$ be such that $\inf_{z\in x\oplus y} \mu_A(z) < t < \min\{\mu_A(x), \mu_A(y)\}$ and $\sup_{z\in x\oplus y} \nu_A(z) > s > \max\{\nu_A(x), \nu_A(y)\}$, for all $z \in x \oplus y$,. Then, $x(t,s) \in A$ and $y(t,s) \in A$ but $(z)(t,s)\overline{\in}A$, for all $z \in x \oplus y$, which contradicts to our hypothesis.

Similarly, if there exist $x, y \in \mathcal{R}$ such that $\mu_A(xy) < \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(xy) > \max\{\nu_A(x), \nu_A(y)\}$. Let $t \in [0, 0.5]$ and $s \in (0.5, 1]$ be such that $\mu_A(xy) < t < \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(xy) > s > \max\{\nu_A(x), \nu_A(y)\}$. Then, $x(t,s) \in A$ and $y(t,s) \in A$ but $(xy)(t,s)\overline{\in}A$, which contradicts to our hypothesis. Hence, $\inf_{z\in x\oplus y} \mu_A(z) \geq \min\{\mu_A(x), \mu_A(y)\}$, $\sup_{z\in x\oplus y} \nu_A(z) \leq \max\{\nu_A(x), \nu_A(y)\}$ and $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\}$, $\nu_A(xy) \leq \max\{\nu_A(x), \nu_A(y)\} \Longrightarrow$ $A = \langle \mu_A, \nu_A \rangle$ is an intuitionistic fuzzy subsemihyperring of $\mathcal{R}$. ■

Proposition 141 *An IFS $A = \langle \mu_A, \nu_A \rangle$ of a semihyperring $\mathcal{R}$ is an intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$ if and only if it satisfy for all $x, y, z \in \mathcal{R}$ and $t_1, t_2 \in [0, 0.5]$ and $s_1, s_2 \in (0.5, 1]$ or $s_1, s_2 \in [0, 0.5]$ and $t_1, t_2 \in (0.5, 1]$.*

(a) $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A \Longrightarrow (z)(m\{t_1,t_2\}, M\{s_1,s_2\}) \in A$, for each $z \in x \oplus y$,

(b) $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A \Longrightarrow (xy)(m\{t_1,t_2\}, M\{s_1,s_2\}) \in A$,

(c) $x(t_1, s_1) \in A$ and $z(t_2, s_2) \in A \Longrightarrow (xyz)(m\{t_1,t_2\}, M\{s_1,s_2\}) \in A$.

Proof. Proof follows from Proposition 140. ■

Theorem 142 *Let $A = \langle \mu_A, \nu_A \rangle$ be IFS in semihyperring $\mathcal{R}$. Then, $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of a semihyperring $\mathcal{R}$ if and only if the following conditions hold;*

(a) $\inf_{z \in x \oplus y} \mu_A(z) \geq \min\{\mu_A(x), \mu_A(y), 0.5\}$ *and* $\sup_{z \in x \oplus y} \nu_A(z) \leq \max\{\nu_A(x), \nu_A(y), 0.5\}$.

(b) $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y), 0.5\}$ *and* $\nu_A(xy) \leq \max\{\nu_A(x), \nu_A(y), 0.5\}$.

(c) $\mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(y), 0.5\}$ *and* $\nu_A(xyz) \leq \max\{\nu_A(x), \nu_A(y), 0.5\}$.

Proof. Suppose that $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of a semihyperring $\mathcal{R}$.

(a) Let $x, y \in \mathcal{R}$. We consider the following cases:

(1) $\min\{\mu_A(x), \mu_A(y)\} < 0.5$ and $\max\{\nu_A(x), \nu_A(y)\} > 0.5$,

(2) $\min\{\mu_A(x), \mu_A(y)\} \geq 0.5$ and $\max\{\nu_A(x), \nu_A(y)\} \leq 0.5$,

(3) $\min\{\mu_A(x), \mu_A(y)\} < 0.5$ and $\max\{\nu_A(x), \nu_A(y)\} < 0.5$.

(1) First assume that, for all $z \in x \oplus y$,

$$\inf_{z \in x \oplus y} \mu_A(z) < \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \sup_{z \in x \oplus y} \nu_A(z) > \max\{\nu_A(x), \nu_A(y), 0.5\}.$$

Then, $\inf_{z \in x \oplus y} \mu_A(z) < \min\{\mu_A(x), \mu_A(y)\}$ and $\sup_{z \in x \oplus y} \nu_A(z) > \max\{\nu_A(x), \nu_A(y)\}$.

Choose $t \in [0, 0.5]$ and $s \in (0.5, 1]$ or $s \in [0, 0.5]$ and $t \in (0.5, 1]$ such that, for all $z \in x \oplus y$,

$$\inf_{z \in x \oplus y} \mu_A(z) < t < \min\{\mu_A(x), \mu_A(y)\} \text{ and } \sup_{z \in x \oplus y} \nu_A(z) > s > \max\{\nu_A(x), \nu_A(y)\}$$

If $\min\{\mu_A(x), \mu_A(y)\} < 0.5$ and $\max\{\nu_A(x), \nu_A(y)\} > 0.5$, then $x(t,s) \in A$ and $y(t,s) \in A$, but $(z)(t,s)\overline{\in \vee q}A$, a contradicts, for all $z \in x \oplus y$.

And if we take,

$$\mu_A(xy) < \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \nu_A(xy) > \max\{\nu_A(x), \nu_A(y), 0.5\}.$$

Then, $\mu_A(xy) < \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(xy) > \max\{\nu_A(x), \nu_A(y)\}$

Choose $t \in [0, 0.5]$ and $s \in (0.5, 1]$ or $s \in [0, 0.5]$ and $t \in (0.5, 1]$ such that

$$\mu_A(xy) < t < \min\{\mu_A(x), \mu_A(y)\} \text{ and } \nu_A(xy) > s > \max\{\nu_A(x), \nu_A(y)\}$$

If $\min\left\{\mu_A(x),\mu_A(y)\right\}<0.5$ and $\max\left\{\nu_A(x),\nu_A(y)\right\}>0.5$, then $x(t,s)\in A$ and $y(t,s)\in A$, but $(xy)(t,s)\overline{\in\vee q}A$, a contradicts.

(2) If $\min\left\{\mu_A(x),\mu_A(y)\right\}\geq 0.5$ and $\max\left\{\nu_A(x),\nu_A(y)\right\}\leq 0.5$, then $\inf_{z\in x\oplus y}\mu_A(z)<0.5$ and $\sup_{z\in x\oplus y}\nu_A(z)>0.5$. Thus, $x(0.5,0.5)\in A$ and $y(0.5,0.5)\in A$, but $(z)(0.5,0.5)\overline{\in\vee q}A$, for all $z\in x\oplus y$, a contradicts. And similarly if $\min\left\{\mu_A(x),\mu_A(y)\right\}\geq 0.5$ and $\max\left\{\nu_A(x),\nu_A(y)\right\}\leq 0.5$, then $\mu_A(xy)<0.5$ and $\nu_A(xy)>0.5$. Thus, $x(0.5,0.5)\in A$ and $y(0.5,0.5)\in A$, but $(xy)(0.5,0.5)\overline{\in\vee q}A$, a contradicts.

(3) If $\min\left\{\mu_A(x),\mu_A(y)\right\}<0.5$ and $\max\left\{\nu_A(x),\nu_A(y)\right\}<0.5$, then for all $z\in x\oplus y$, $\inf_{z\in x\oplus y}\mu_A(z)<\min\left\{\mu_A(x),\mu_A(y)\right\}$ and $\sup_{z\in x\oplus y}\nu_A(z)>0.5$. Thus, $x(t,s)\in A$ and $y(t,s)\in A$, but $(z)(t,s)\overline{\in\vee q}A$, for all $z\in x\oplus y$, a contradicts. Therefore, $\inf_{z\in x\oplus y}\mu_A(z)\geq\min\left\{\mu_A(x),\mu_A(y),0.5\right\}$ and $\sup_{z\in x\oplus y}\nu_A(z)\leq\max\left\{\nu_A(x),\nu_A(y),0.5\right\}$. And similarly if $\min\left\{\mu_A(x),\mu_A(y)\right\}<0.5$ and $\max\left\{\nu_A(x),\nu_A(y)\right\}<0.5$, then $\mu_A(xy)<\min\left\{\mu_A(x),\mu_A(y)\right\}$ and $\nu_A(xy)>0.5$. Thus, $x(t,s)\in A$ and $y(t,s)\in A$, but $(xy)(t,s)\overline{\in\vee q}A$, a contradicts. Therefore, $\mu_A(xy)\geq\min\left\{\mu_A(x),\mu_A(y),0.5\right\}$ and $\nu_A(xy)\leq\max\left\{\nu_A(x),\nu_A(y),0.5\right\}$.

(b) Now, let $x,y,z\in\mathcal{R}$. We consider the following case's

(1) $\min\left\{\mu_A(x),\mu_A(y)\right\}<0.5$ and $\max\left\{\nu_A(x),\nu_A(y)\right\}>0.5$,

(2) $\min\left\{\mu_A(x),\mu_A(y)\right\}\geq 0.5$ and $\max\left\{\nu_A(x),\nu_A(y)\right\}\leq 0.5$,

(3) $\min\left\{\mu_A(x),\mu_A(y)\right\}<0.5$ and $\max\left\{\nu_A(x),\nu_A(y)\right\}<0.5$.

(1) Assume that for some

$$\begin{aligned}\mu_A(xyz) &< \min\left\{\mu_A(x),\mu_A(y),0.5\right\} \text{ and } \nu_A(xyz)>\max\left\{\nu_A(x),\nu_A(y),0.5\right\}\\ \mu_A(xyz) &< \min\left\{\mu_A(x),\mu_A(y)\right\} \text{ and } \nu_A(xyz)>\max\left\{\nu_A(x),\nu_A(y)\right\}\end{aligned}$$

Choose $t\in[0,0.5]$ and $s\in(0.5,1]$ or $s\in[0,0.5]$ and $t\in(0.5,1]$ such that

$$\mu_A(xyz)<t<\min\left\{\mu_A(x),\mu_A(y)\right\} \text{ and } \nu_A(xyz)>s>\max\left\{\nu_A(x),\nu_A(y)\right\}$$

Then $x(t,s)\in A$ and $y(t,s)\in A$, but $(xyz)(t,s)\overline{\in\vee q}A$, which is a contradiction.

(2) If $\min\left\{\mu_A(x),\mu_A(y)\right\}\geq 0.5$ and $\max\left\{\nu_A(x),\nu_A(y)\right\}\leq 0.5$, then $\mu_A(xyz)<0.5$ and $\nu_A(xyz)>0.5$. Since, $x(0.5,0.5)\in A$ and $y(0.5,0.5)\in A$, but $(xyz)(0.5,0.5)\overline{\in\vee q}A$, which is a contradiction.

(3) If $\min\{\mu_A(x), \mu_A(y)\} < 0.5$ and $\max\{\nu_A(x), \nu_A(y)\} < 0.5$, then $\mu_A(xyz) < \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(xyz) > 0.5$. Thus, $x(t,s) \in A$ and $y(t,s) \in A$, but $(xyz)(t,s)\overline{\in \vee q}A$, which is again a contradiction. Therefore,

$$\mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \nu_A(xyz) \leq \max\{\nu_A(x), \nu_A(y), 0.5\}.$$

Conversely, assume that $A = \langle \mu_A, \nu_A \rangle$ satisfy (a) and (b). Let for any $x, y \in S$ and $t_1, t_2 \in (0,1]$ and $s_1, s_2 \in [0,1)$, such that $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A$. Then, $\mu_A(x) \geq t_1$ and $\nu_A(x) \leq s_1$, and $\mu_A(y) \geq t_2$ and $\nu_A(y) \leq s_2$.

$$\begin{aligned}
\mu_A(xyz) &\geq \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \nu_A(xyz) \leq \max\{\nu_A(x), \nu_A(y), 0.5\} \\
\mu_A(xyz) &\geq \min\{t_1, t_2, 0.5\} \text{ and } \nu_A(xyz) \leq \max\{s_1, s_2, 0.5\}.
\end{aligned}$$

Then, we have the following cases:

(1) $\min\{t_1, t_2\} \leq 0.5$ and $\max\{s_1, s_2\} \geq 0.5$,

(2) $\min\{t_1, t_2\} > 0.5$ and $\max\{s_1, s_2\} < 0.5$.

(1) If $\min\{t_1, t_2\} \leq 0.5$ and $\max\{s_1, s_2\} \geq 0.5$. Then, $\mu_A(xyz) \geq \min\{t_1, t_2\}$ and $\nu_A(xyz) \leq \max\{s_1, s_2\}$, which implies that $(xyz)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in A$.

(2) If $\min\{t_1, t_2\} > 0.5$ and $\max\{s_1, s_2\} < 0.5$, then $\mu_A(xyz) \geq 0.5$ and $\nu_A(xyz) \leq 0.5$, which implies that $\mu_A(xyz) + \min\{t_1, t_2\} > 0.5 + 0.5 = 1$ and $\nu_A(xyz) + \max\{s_1, s_2\} < 0.5 + 0.5 = 1$. Therefore, $(xyz)(m\{t_1, t_2\}, M\{s_1, s_2\})qA$. Hence, $(xyz)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee qA$.

Let $x, y, z \in \mathcal{R}$, $t_1, t_2 \in [0, 0.5]$ and $s_1, s_2 \in (0.5, 1]$ or $s_1, s_2 \in [0, 0.5]$ and $t_1, t_2 \in (0.5, 1]$.such that $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A$. Then, $\mu_A(x) \geq t_1$ and $\nu_A(x) \leq s_1$, and $\mu_A(y) \geq t_2$ and $\nu_A(y) \leq s_2$. Now we have

$$\begin{aligned}
\mu_A(xyz) &\geq \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \nu_A(xyz) \leq \max\{\nu_A(x), \nu_A(y), 0.5\} \\
\mu_A(xyz) &\geq \min\{t_1, t_2, 0.5\} \text{ and } \nu_A(xyz) \leq \max\{s_1, s_2, 0.5\}.
\end{aligned}$$

Then, we have the following cases:

(3) $\min\{t_1, t_2\} \leq 0.5$ and $\max\{s_1, s_2\} \geq 0.5$,

(4) $\min\{t_1, t_2\} > 0.5$ and $\max\{s_1, s_2\} < 0.5$.

(3) If $\min\{t_1, t_2\} \leq 0.5$ and $\max\{s_1, s_2\} \geq 0.5$, then $\mu_A(xyz) \geq \min\{t_1, t_2\}$ and $\nu_A(xyz) \leq \max\{s_1, s_2\}$, which implies that $(xyz)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in A$.

(4) If $\min\{t_1, t_2\} > 0.5$ and $\max\{s_1, s_2\} < 0.5$, then $\mu_A(xyz) \geq 0.5$ and $\nu_A(xyz) \leq 0.5$, which implies that $\mu_A(xyz) + \min\{t_1, t_2\} > 0.5 + 0.5 = 1$ and $\nu_A(xyz) + \max\{s_1, s_2\} < 0.5 + 0.5 = 1$. Therefore, $(xyz)(m\{t_1, t_2\}, M\{s_1, s_2\})qA$. Hence, $(xyz)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee qA$. This completes the proof. ■

Remark 143 *Every intuitionistic fuzzy bi-hyperideal of a semihyperring $\mathcal{R}$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$. The converse is not true, as in the example.*

Example 144 *Let $\mathcal{R} = [-1, 1]$ be a non-empty set with hyperoperation $\oplus$ and a binary operation" $\cdot$ " defined for all $x, y \in \mathcal{R}$ as follows:*

$$x \oplus y = \begin{cases} [\frac{x+y}{2}, 0] & \text{if } x < 0 \text{ and } y < 0, \\ [0, \frac{x+y}{2}] & \text{if } x > 0 \text{ and } y > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $x.y = xy$. Then clearly $\mathcal{R}$ satisfies all conditions of a semihyperring. Let $A = \langle \mu_A, \nu_A \rangle$ be an IFS in a semihyperring $\mathcal{R}$ defined by.

$$\mu_A(x) = \begin{cases} 0.6 & \text{if } x \in [-0.5, 0) \cup (0, 0.5], \\ 0.5 & x = 0 \text{ or } x \in [-1, -0.5) \cup (0.5, 1], \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0.3 & \text{if } x \in [-0.5, 0) \cup (0, 0.5], \\ 0.5 & x = 0 \text{ or } x \in [-1, -0.5) \cup (0.5, 1]. \end{cases}$$

Hence, by routine calculation $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of a semihyperring $\mathcal{R}$. But $A = \langle \mu_A, \nu_A \rangle$ is not an intuitionistic fuzzy bi-hyperideal of a semihy-

perring $\mathcal{R}$. If $x, y \in [-0.5, 0) \cup (0, 0.5]$, then we have

$$\begin{aligned}
\inf \mu_A (x \oplus y) &= 0.5 \ngeqslant \min \{\mu_A (x), \mu_A (y)\} = 0.6, \\
\sup \nu_A (x \oplus y) &= 0.5 \nleqslant \max \{\nu_A (x), \nu_A(y)\} = 0.3.
\end{aligned}$$

Proposition 145 (1) *Every $(\in \vee q, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$.*

(2) *Every $(\in, \in)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$.*

Proof. The proof is straightforward. ■

Example 144 shows that the converse of Proposition 145, is not true in general.

Theorem 146 *If $\{A\}_{i\in\Lambda}$ is a family of $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideals of $\mathcal{R}$, then $\bigcap_{i\in\Lambda} A_i$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$, where $\bigcap_{i\in\Lambda} A_i = \langle \bigwedge_{i\in\Lambda} \mu_{A_i}, \bigvee_{i\in\Lambda} \nu_{A_i} \rangle$.*

Proof. Let $x, y \in \mathcal{R}$ and for each $z \in x \oplus y$. Then, we have

$$\begin{aligned}
\left(\bigwedge_{i\in\Lambda} \mu_{A_i}\right)(z) &= \bigwedge_{i\in\Lambda} \left(\mu_{A_i}(z)\right) \geq \bigwedge_{i\in\Lambda} \left(\min\left\{\mu_{A_i}(x), \mu_{A_i}(y), 0.5\right\}\right) \\
&= \min\left\{\bigwedge_{i\in\Lambda} \mu_{A_i}(x), \bigwedge_{i\in\Lambda} \mu_{A_i}(y), 0.5\right\} \\
\left(\bigwedge_{i\in\Lambda} \mu_{A_i}\right)(z) &\geq \min\left\{\left(\bigwedge_{i\in\Lambda} \mu_{A_i}\right)(x), \left(\bigwedge_{i\in\Lambda} \mu_{A_i}\right)(y), 0.5\right\} \text{ and} \\
\left(\bigvee_{i\in\Lambda} \nu_{A_i}\right)(z) &= \bigvee_{i\in\Lambda} \left(\nu_{A_i}(z)\right) \leq \bigvee_{i\in\Lambda} \left(\max\left\{\nu_{A_i}(x), \nu_{A_i}(y), 0.5\right\}\right) \\
&= \max\left\{\bigvee_{i\in\Lambda} \nu_{A_i}(x), \bigvee_{i\in\Lambda} \nu_{A_i}(y), 0.5\right\} \\
\left(\bigvee_{i\in\Lambda} \nu_{A_i}\right)(z) &\leq \max\left\{\left(\bigvee_{i\in\Lambda} \nu_{A_i}\right)(x), \left(\bigvee_{i\in\Lambda} \nu_{A_i}\right)(y), 0.5\right\}
\end{aligned}$$

Let $x, y \in \mathcal{R}$. Then, we have

$$\begin{aligned}
\left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(xy) &= \bigwedge_{i\in\Lambda}\left(\mu_{A_i}(xy)\right) \geq \bigwedge_{i\in\Lambda}\left(\min\left\{\mu_{A_i}(x), \mu_{A_i}(y), 0.5\right\}\right) \\
&= \min\left\{\bigwedge_{i\in\Lambda}\mu_{A_i}(x), \bigwedge_{i\in\Lambda}\mu_{A_i}(y), 0.5\right\} \\
\left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(xy) &\geq \min\left\{\left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(x), \left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(y), 0.5\right\} \text{ and} \\
\left(\bigvee_{i\in\Lambda}\nu_{A_i}\right)(xy) &= \bigvee_{i\in\Lambda}\left(\nu_{A_i}(xy)\right) \leq \bigvee_{i\in\Lambda}\left(\max\left\{\nu_{A_i}(x), \nu_{A_i}(y), 0.5\right\}\right) \\
&= \max\left\{\bigvee_{i\in\Lambda}\nu_{A_i}(x), \bigvee_{i\in\Lambda}\nu_{A_i}(y), 0.5\right\} \\
\left(\bigvee_{i\in\Lambda}\nu_{A_i}\right)(xy) &\leq \max\left\{\left(\bigvee_{i\in\Lambda}\nu_{A_i}\right)(x), \left(\bigvee_{i\in\Lambda}\nu_{A_i}\right)(y), 0.5\right\}
\end{aligned}$$

Now, let $x, y, y \in \mathcal{R}$. Then, we have

$$\begin{aligned}
\left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(xyz) &= \bigwedge_{i\in\Lambda}\left(\mu_{A_i}(xyz)\right) \geq \bigwedge_{i\in\Lambda}\left(\min\left\{\mu_{A_i}(x), \mu_{A_i}(y), 0.5\right\}\right) \\
&= \min\left\{\bigwedge_{i\in\Lambda}\mu_{A_i}(x), \bigwedge_{i\in\Lambda}\mu_{A_i}(y), 0.5\right\} \\
\left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(xyz) &\geq \min\left\{\left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(x), \left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(y), 0.5\right\} \text{ and} \\
\left(\bigvee_{i\in\Lambda}\nu_{A_i}\right)(xyz) &= \bigvee_{i\in\Lambda}\left(\nu_{A_i}(z)\right) \leq \bigvee_{i\in\Lambda}\left(\max\left\{\nu_{A_i}(x), \nu_{A_i}(y), 0.5\right\}\right) \\
&= \max\left\{\bigvee_{i\in\Lambda}\nu_{A_i}(x), \bigvee_{i\in\Lambda}\nu_{A_i}(y), 0.5\right\} \\
\left(\bigvee_{i\in\Lambda}\nu_{A_i}\right)(xyz) &\leq \max\left\{\left(\bigvee_{i\in\Lambda}\nu_{A_i}\right)(x), \left(\bigvee_{i\in\Lambda}\nu_{A_i}\right)(y), 0.5\right\}
\end{aligned}$$

Hence $\bigcap_{i\in\Lambda} A_i = \langle \bigwedge_{i\in\Lambda}\mu_{A_i}, \bigvee_{i\in\Lambda}\nu_{A_i}\rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$. ■

Remark 147 *If* $\{A\}_{i\in\Lambda}$ *is a family of* $(\in, \in \vee q)$*-intuitionistic fuzzy bi-hyperideals of* $\mathcal{R}$*, then*

$\bigcup_{i\in\Lambda} A_i$ *is not an* $(\in, \in\vee q)$*-intuitionistic fuzzy bi-hyperideal of* $\mathcal{R}$*, where* $\bigcup_{i\in\Lambda} A_i = \langle \bigvee_{i\in\Lambda} \mu_{A_i}, \bigwedge_{i\in\Lambda} \nu_{A_i}\rangle$*.*

Theorem 148 *If* $\{A_i\}_{i\in\Lambda}$ *is a family of* $(\in, \in\vee q)$*-intuitionistic fuzzy bi-hyperidealof* $\mathcal{R}$ *such that* $A_i \subseteq A_j$ *or* $A_j \subseteq A_i$ *for all* i*,* $j \in I$*, then* $\bigcap_{i\in\Lambda} A_i = \langle \bigvee_{i\in\Lambda} \mu_{A_i}, \bigwedge_{i\in\Lambda} \nu_{A_i}\rangle$ *is an* $(\in, \in\vee q)$*-intuitionistic fuzzy bi-hyperideal of* $\mathcal{R}$*.*

Proof. For all $x, y \in \mathcal{R}$ and for each $z \in x \oplus y$ we have

$$\begin{aligned}
\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(z) &= \bigvee_{i\in\Lambda} \left(\mu_{A_i}(z)\right) \geq \bigvee_{i\in\Lambda} \left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5\}\right] \\
&= \left[\bigvee_{i\in\Lambda} \mu_{A_i}(x) \wedge \bigvee_{i\in\Lambda} \mu_{A_i}(y) \wedge 0.5\right] \\
&= \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right] \\
\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(z) &\geq \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right].
\end{aligned}$$

It is clear that

$$\bigvee_{i\in\Lambda} \left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5\right] \leq \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right].$$

Assume that

$$\bigvee_{i\in\Lambda} \left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5\right] \neq \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right].$$

Then, there exists t such that

$$\bigvee_{i\in\Lambda} \left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5\right] < t < \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right]$$

Since $\mu_{A_i} \subseteq \mu_{A_j}$ or $\mu_{A_j} \subseteq \mu_{A_i}$ for all i, $j \in I$, so there exists $k \in I$ such that $t < \mu_{A_k}(x) \wedge \mu_{A_k}(y) \wedge 0.5$. On other hand $\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5 < t$ for all $i \in I$, a contradiction. Hence,

$$\bigvee_{i\in\Lambda} \left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5\right] = \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right]$$

and also for each $z \in x \oplus y$,

$$\begin{aligned}
\left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(z) &= \bigwedge_{i\in\Lambda} \left(\nu_{A_i}(z)\right) \leq \bigwedge_{i\in\Lambda} [\nu_{A_i}(x) \vee \nu_{A_i}(y) \vee 0.5\}] \\
&= \left[\bigwedge_{i\in\Lambda} \nu_{A_i}(x) \vee \bigwedge_{i\in\Lambda} \nu_{A_i}(y) \vee 0.5\right] \\
&= \left[\left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(y) \vee 0.5\right] \\
\left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(z) &\leq \left[\left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(y) \vee 0.5\right].
\end{aligned}$$

It is clear that

$$\bigwedge_{i\in\Lambda} [\nu_{A_i}(x) \vee \nu_{A_i}(y) \vee 0.5] \geq \left[\left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda} \mu_{A_i}\right)(y) \vee 0.5\right].$$

Assume that

$$\bigwedge_{i\in\Lambda} [\nu_{A_i}(x) \vee \nu_{A_i}(y) \vee 0.5] \neq \left[\left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(y) \vee 0.5\right].$$

Then there exists t such that

$$\bigwedge_{i\in\Lambda} [\nu_{A_i}(x) \vee \nu_{A_i}(y) \vee 0.5] > t > \left[\left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(y) \vee 0.5\right]$$

Since $\nu_{A_i} \subseteq \nu_{A_j}$ or $\nu_{A_j} \subseteq \nu_{A_i}$ for all i, $j \in I$, so there exists $k \in I$ such that k$> \mu_{A_k}(x) \wedge \mu_{A_k}(y) \wedge 0.5$. On other hand $\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5 > t$ for all $i \in I$, a contradiction. Hence,

$$\bigwedge_{i\in\Lambda} [\nu_{A_i}(x) \wedge \nu_{A_i}(y) \wedge 0.5] = \left[\left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(x) \wedge \left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(y) \wedge 0.5\right]$$

Similarly, for all $x, y \in \mathcal{R}$ we have

$$\begin{aligned}
\left(\bigvee_{i\in\Lambda}\mu_{A_i}\right)(xy) &= \bigvee_{i\in\Lambda}\left(\mu_{A_i}(xy)\right) \geq \bigvee_{i\in\Lambda}\left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5\}\right] \\
&= \left[\bigvee_{i\in\Lambda}\mu_{A_i}(x) \wedge \bigvee_{i\in\Lambda}\mu_{A_i}(y) \wedge 0.5\right] \\
&= \left[\left(\bigvee_{i\in\Lambda}\mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda}\mu_{A_i}\right)(y) \wedge 0.5\right] \\
\left(\bigvee_{i\in\Lambda}\mu_{A_i}\right)(xy) &\geq \left[\left(\bigvee_{i\in\Lambda}\mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda}\mu_{A_i}\right)(y) \wedge 0.5\right].
\end{aligned}$$

It is clear that

$$\bigvee_{i\in\Lambda}\left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5\right] \leq \left[\left(\bigvee_{i\in\Lambda}\mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda}\mu_{A_i}\right)(y) \wedge 0.5\right].$$

Assume that

$$\bigvee_{i\in\Lambda}\left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5\right] \neq \left[\left(\bigvee_{i\in\Lambda}\mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda}\mu_{A_i}\right)(y) \wedge 0.5\right].$$

Then, there exists t such that

$$\bigvee_{i\in\Lambda}\left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5\right] < t < \left[\left(\bigvee_{i\in\Lambda}\mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda}\mu_{A_i}\right)(y) \wedge 0.5\right]$$

Since $\mu_{A_i} \subseteq \mu_{A_j}$ or $\mu_{A_j} \subseteq \mu_{A_i}$ for all i, $j \in I$, so there exists $k \in I$ such that $t < \mu_{A_k}(x) \wedge \mu_{A_k}(y) \wedge 0.5$. On other hand $\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5 < t$ for all $i \in I$, a contradiction. Hence,

$$\bigvee_{i\in\Lambda}\left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5\right] = \left[\left(\bigvee_{i\in\Lambda}\mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda}\mu_{A_i}\right)(y) \wedge 0.5\right]$$

and

$$
\begin{aligned}
\left(\bigwedge_{i\in\Lambda}\nu_{A_i}\right)(xy) &= \bigwedge_{i\in\Lambda}\left(\nu_{A_i}(z)\right) \leq \bigwedge_{i\in\Lambda}[\nu_{A_i}(x) \vee \nu_{A_i}(y) \vee 0.5\}] \\
&= \left[\bigwedge_{i\in\Lambda}\nu_{A_i}(x) \vee \bigwedge_{i\in\Lambda}\nu_{A_i}(y) \vee 0.5\right] \\
&= \left[\left(\bigwedge_{i\in\Lambda}\nu_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda}\nu_{A_i}\right)(y) \vee 0.5\right] \\
\left(\bigwedge_{i\in\Lambda}\nu_{A_i}\right)(xy) &\leq \left[\left(\bigwedge_{i\in\Lambda}\nu_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda}\nu_{A_i}\right)(y) \vee 0.5\right].
\end{aligned}
$$

It is clear that

$$\bigwedge_{i\in\Lambda}[\nu_{A_i}(x) \vee \nu_{A_i}(y) \vee 0.5] \geq \left[\left(\bigwedge_{i\in\Lambda}\nu_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(y) \vee 0.5\right].$$

Assume that

$$\bigwedge_{i\in\Lambda}[\nu_{A_i}(x) \vee \nu_{A_i}(y) \vee 0.5] \neq \left[\left(\bigwedge_{i\in\Lambda}\nu_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda}\nu_{A_i}\right)(y) \vee 0.5\right].$$

Then there exists t such that

$$\bigwedge_{i\in\Lambda}[\nu_{A_i}(x) \vee \nu_{A_i}(y) \vee 0.5] > t > \left[\left(\bigwedge_{i\in\Lambda}\nu_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda}\nu_{A_i}\right)(y) \vee 0.5\right]$$

Since $\nu_{A_i} \subseteq \nu_{A_j}$ or $\nu_{A_j} \subseteq \nu_{A_i}$ for all i, $j \in I$, so there exists $k \in I$ such that $k > \mu_{A_k}(x) \wedge \mu_{A_k}(y) \wedge 0.5$. On other hand $\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5 > t$ for all $i \in I$, a contradiction. Hence,

$$\bigwedge_{i\in\Lambda}[\nu_{A_i}(x) \wedge \nu_{A_i}(y) \wedge 0.5] = \left[\left(\bigwedge_{i\in\Lambda}\nu_{A_i}\right)(x) \wedge \left(\bigwedge_{i\in\Lambda}\nu_{A_i}\right)(y) \wedge 0.5\right]$$

For all $x, y, z \in \mathcal{R}$, we obtain

$$\begin{aligned}
\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(xyz) &= \bigvee_{i\in\Lambda} \left(\mu_{A_i}(xyz)\right) \geq \bigvee_{i\in\Lambda} \left[\mu_{A_i}(x) \wedge \mu_{A_i}(z) \wedge 0.5\}\right] \\
&= \left[\bigvee_{i\in\Lambda} \mu_{A_i}(x) \wedge \bigvee_{i\in\Lambda} \mu_{A_i}(z) \wedge 0.5\right] \\
&= \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(z) \wedge 0.5\right] \\
\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(xyz) &\geq \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(z) \wedge 0.5\right]
\end{aligned}$$

and

$$\begin{aligned}
\left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(xyz) &= \bigwedge_{i\in\Lambda} \left(\nu_{A_i}(xaz)\right) \leq \bigwedge_{i\in\Lambda} \left[\nu_{A_i}(x) \vee \nu_{A_i}(z) \vee 0.5\}\right] \\
&= \left[\bigwedge_{i\in\Lambda} \nu_{A_i}(x) \vee \bigwedge_{i\in\Lambda} \nu_{A_i}(z) \vee 0.5\right] \\
&= \left[\left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(z) \vee 0.5\right] \\
\left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(xyz) &\leq \left[\left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda} \nu_{A_i}\right)(z) \vee 0.5\right]
\end{aligned}$$

Hence, $\bigcap_{i\in\Lambda} A_i = \langle \bigvee_{i\in\Lambda} \mu_{A_i}, \bigwedge_{i\in\Lambda} \nu_{A_i} \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$.
■

Definition 149 *Let $\mathcal{R}$ be a semihyperring and $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ be IFSs of $\mathcal{R}$. Then, the 0.5-sum of A and B is defined by:*

$$\begin{aligned}
A_{\oplus 0.5} B &= \langle \mu_A \oplus_{0.5} \mu_B, \nu_A \oplus_{0.5} \nu_B \rangle \\
(\mu_A \oplus_{0.5} \mu_B)(z) &= \left\{ \begin{array}{ll} \bigvee_{z\in x\oplus y} \{\mu_A(x) \wedge \mu_B(y) \wedge 0.5\} & \text{if } z \in x \oplus y \\ 0 & \text{if } z \notin x \oplus y \end{array} \right\} \\
(\nu_A \oplus_{0.5} \nu_B)(z) &= \left\{ \begin{array}{ll} \bigwedge_{z\in x\oplus y} \{\nu_A(x) \vee \nu_B(y) \vee 0.5\} & \text{if } z \in x \oplus y \\ 1 & \text{if } z \notin x \oplus y \end{array} \right\}.
\end{aligned}$$

Definition 150 *Let $\mathcal{R}$ be a semihyperring and $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ be IFSs of $\mathcal{R}$. Then, the 0.5 product of A and B is defined by:*

$$\begin{aligned}
A \cdot_{0.5} B &= \langle \mu_A \cdot_{0.5} \mu_B, \nu_A \cdot_{0.5} \nu_B \rangle \\
(\mu_A \cdot_{0.5} \mu_B)(z) &= \left\{ \begin{array}{ll} \bigvee\limits_{z=xy} \{\mu_A(x) \wedge \mu_B(y) \wedge 0.5\} & \text{if } z = xy \\ 0 & \text{if } z \neq xy \end{array} \right\} \\
(\nu_A \cdot_{0.5} \nu_B)(z) &= \left\{ \begin{array}{ll} \bigwedge\limits_{x=yz} \{\nu_A(x) \vee \nu_B(y) \vee 0.5\} & \text{if } z = xy \\ 1 & \text{if } z \neq xy \end{array} \right\}.
\end{aligned}$$

Let $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ be $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperidealof $\mathcal{R}$. Then,

$$\begin{aligned}
A \cap_{0.5} B &= \langle \mu_A \wedge_{0.5} \mu_B, \nu_A \vee_{0.5} \nu_B \rangle \\
(\mu_A \wedge_{0.5} \mu_B)(x) &= \mu_A(x) \wedge \mu_B(x) \wedge 0.5 \text{ and} \\
(\nu_A \vee_{0.5} \nu_B)(x) &= \nu_A(x) \vee \nu_B(x) \vee 0.5.
\end{aligned}$$

Remark 151 *If $\mathcal{R}$ is a semihyperring and A, B, C, D are IFSs of $\mathcal{R}$ such that $A \subseteq B$ and $C \subseteq D$, then $A \cdot_{0.5} C \subseteq B \cdot_{0.5} D$.*

Proposition 152 *Let $\mathcal{R}$ be a semihyperring, $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ be $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperidealof $\mathcal{R}$. Then, $A \cap_{0.5} B$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$.*

Proof. The proof is straightforward. ■

Definition 153 *An $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$ is called 0.5 idempotent if $A \cdot_{0.5} A = A$.*

Theorem 154 *Let $\mathcal{R}$ be a semihyperring and $A = \langle \mu_A, \nu_A \rangle$ be an IFS of $\mathcal{R}$. Then, $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$ if and only if the following hold*

(a) $A \oplus_{0.5} A \subseteq A$

(b) $A \cdot_{0.5} A \subseteq A$,

(c) $A \cdot_{0.5} \mathcal{R} \cdot_{0.5} A \subseteq A$.

Proof. Let $A = \langle \mu_A, \nu_A \rangle$ be an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$. (a) For each $x \in \mathcal{R}$, we have two cases: (1) If $z \notin x \oplus y$. (2) If $z \in x \oplus y$.

Case 1 : If $z \notin x \oplus y$, then clearly

$$\begin{aligned}
(\mu_A \oplus_{0.5} \mu_A)(z) &= 0 \leq \mu_A(z) \text{ and } (\nu_A \oplus_{0.5} \nu_A)(z) = 1 \geq \nu_A \\
\text{Thus, } A \oplus_{0.5} A &\subseteq A.
\end{aligned}$$

Case 2 : If $z \in x \oplus y$, then

$$\begin{aligned}
(\mu_A \oplus_{0.5} \mu_A)(z) &= \bigvee_{z \in x \oplus y} \{\min\{\mu_A(x), \mu_A(y), 0.5\}\} \\
&\leq \bigvee_{z \in x \oplus y} \left\{ \inf_{z \in x \oplus y} \mu_A(z) \right\} \\
(\mu_A \oplus_{0.5} \mu_A)(z) &\leq \mu_A(z), \text{ for all } z \in x \oplus y.
\end{aligned}$$

and

$$\begin{aligned}
(\nu_A \oplus_{0.5} \nu_A)(z) &= \bigwedge_{z \in x \oplus y} \{\max\{\nu_A(x), \nu_A(y), 0.5\}\} \\
&\geq \bigwedge_{z \in x \oplus y} \left\{ \sup_{z \in x \oplus y} \nu_A(z) \right\} \\
(\nu_A \oplus_{0.5} \nu_A)(z) &\geq \nu_A(z), \text{ for all } z \in x \oplus y.
\end{aligned}$$

Thus, $A \oplus_{0.5} A \subseteq A$.

Similarly, for (b) we have cases for each $x \in \mathcal{R}$: (1) If $z \neq xy$. (2) If $z = xy$.

Case 1 : If $z \neq xy$, then clearly

$$(\mu_A \cdot_{0.5} \mu_A)(z) = 0 \leq \mu_A(z) \text{ and } (\nu_A \cdot_{0.5} \nu_A)(z) = 1 \geq \nu_A$$

Thus, $A \circ_{0.5} A \subseteq A$.

Case 2 : If $z = xy$, then

$$\begin{aligned}
(\mu_A \cdot_{0.5} \mu_A)(z) &= \bigvee_{z=xy} \{\min\{\mu_A(x), \mu_A(y), 0.5\}\} \\
&\leq \bigvee_{z=xy} \{\mu_A(xy)\} = \mu_A(xy) \\
(\mu_A \cdot_{0.5} \mu_A)(z) &\leq \mu_A(yz)
\end{aligned}$$

and

$$\begin{aligned}
(\nu_A \cdot_{0.5} \nu_A)(z) &= \bigwedge_{z=xy} \{\max\{\nu_A(x), \nu_A(y), 0.5\}\} \\
&\geq \bigwedge_{z=xy} \{\nu_A(xy)\} = \nu_A(xy) \\
(\nu_A \cdot_{0.5} \nu_A)(z) &\geq \nu_A(xy)
\end{aligned}$$

Thus, $A \cdot_{0.5} A \subseteq A$.

Now, for (3) we have also two cases, for each $x \in \mathcal{R}$. (1) If $x \neq yz$ for every $y, z \in \mathcal{R}$. (2) If $x = yz$ for some $y, z \in \mathcal{R}$.

Case 1 : If $x \neq yz$, then clearly

$$(\mu_A \circ_{0.5} 1 \circ_{0.5} \mu_A)(x) = 0 \leq \mu_A(x) \text{ and } (\nu_A \circ_{0.5} 0 \circ_{0.5} \nu_A) = 1 \geq \nu_A$$

Thus, $A \circ_{0.5} \mathcal{R} \circ_{0.5} A \subseteq A$.

Case 2 :If $x = yz$ for some $y, z \in \mathcal{R}$, then we have

$$\begin{aligned}
(\mu_A \circ_{0.5} 1 \circ_{0.5} \mu_A)(x) &= \bigvee_{x=yz} \{\min\{\mu_A(y), (1 \circ_{0.5} \mu_A)(z), 0.5\}\} \\
&= \bigvee_{x=yz} \left\{\min\left\{\mu_A(y), \bigvee_{z=tr} \{\min\{1(t), \mu_A(r), 0.5\}\}, 0.5\right\}\right\} \\
&= \bigvee_{x=yz} \bigvee_{z=tr} \{\min\{\mu_A(y), 1, \mu_A(r), 0.5\}\} \\
&= \bigvee_{x=ytr} \{\min\{\mu_A(y), \mu_A(r), 0.5\}\}
\end{aligned}$$

Since $x = yz = y(tr) = ytr$ and $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$, so we have $\mu_A(ytr) \geq \min\{\mu_A(y), \mu_A(r), 0.5\}$. Thus,

$$\begin{aligned}
\bigvee_{x=ytr} \{\min\{\mu_A(y), \mu_A(r), 0.5\}\} &\leq \bigvee_{x=ytr} \{\mu_A(x)\} = \mu_A(ytr) \\
(\mu_A \circ_{0.5} 1 \circ_{0.5} \mu_A)(ytr) &\leq \mu_A(ytr)
\end{aligned}$$

and

$$\begin{aligned}
(\nu_A \circ_{0.5} 0 \circ_{0.5} \nu_A)(x) &= \bigwedge_{x=yz} \{\max\{\nu_A(y), (0 \circ_{0.5} \nu_A)(z), 0.5\}\} \\
&= \bigwedge_{x=yz} \left\{\max\left\{\mu_A(y), \bigwedge_{z=tr} \{\max\{0(t), \nu_A(r), 0.5\}\}, 0.5\right\}\right\} \\
&= \bigwedge_{x=yz} \bigwedge_{z=tr} \{\max\{\nu_A(y), 0, \nu_A(r), 0.5\}\} \\
&= \bigwedge_{x=ytr} \{\max\{\nu_A(y), \nu_A(r), 0.5\}\}.
\end{aligned}$$

$A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$, so we have $\sup_{x=ytr} \nu_A(x) \leq \max\{\nu_A(y), \nu_A(r), 0.5\}$. Thus,

$$\begin{aligned}
\bigwedge_{x=ytr} \{\min\{\nu_A(y), \nu_A(r), 0.5\}\} &\geq \bigwedge_{x=ytr} \{\nu_A(x)\} = \nu_A(ytr) \\
(\nu_A \circ_{0.5} 0 \circ_{0.5} \nu_A)(x) &\leq \nu_A(ytr).
\end{aligned}$$

Hence, $A \circ_{0.5} \mathcal{R} \circ_{0.5} A \subseteq A$. Then, by Proposition 152 and Theorem 148, we have $A \oplus_{0.5} A \subseteq A$, $A \circ_{0.5} A \subseteq A$ and $A \circ_{0.5} \mathcal{R} \circ_{0.5} A \subseteq A$.

Conversely, suppose that the given conditions hold. Let $x, y \in \mathcal{R}$ such that $z \in x \oplus y$. Then, we have

$$\begin{aligned}
\inf_{z \in x \oplus y} \mu_A(z) &\geq (\mu_A \circ_{0.5} \mu_A)(z) = \bigvee_{z \in st} \{\min\{\mu_A(s), \mu_A(t), 0.5\}\} \\
&\geq \min\{\mu_A(x), \mu_A(y), 0.5\} \\
\inf_{z \in x \oplus y} \mu_A(z) &\geq \min\{\mu_A(x), \mu_A(y), 0.5\}
\end{aligned}$$

and

$$\begin{aligned}
\sup_{z\in x\oplus y} \nu_A(z) &\leq (\nu_A \circ_{0.5} \nu_A)(z) = \bigwedge_{z\in s\oplus t} \{\max\{\nu_A(s), \nu_A(t), 0.5\}\} \\
&\leq \max\{\nu_A(x), \nu_A(y), 0.5\} \\
\sup_{z\in x\oplus y} \nu_A(z) &\leq \min\{\nu_A(x), \nu_A(y), 0.5\}.
\end{aligned}$$

Now, let $x, y \in \mathcal{R}$ such that $z = xy$. Then, we have

$$\begin{aligned}
\inf_{z=xy} \mu_A(z) &\geq (\mu_A \circ_{0.5} \mu_A)(z) = \bigvee_{z=st} \{\min\{\mu_A(s), \mu_A(t), 0.5\}\} \\
&\geq \min\{\mu_A(x), \mu_A(y), 0.5\} \\
\inf_{z=xy} \mu_A(z) &\geq \min\{\mu_A(x), \mu_A(y), 0.5\}
\end{aligned}$$

and

$$\begin{aligned}
\sup_{z=xy} \nu_A(z) &\leq (\nu_A \circ_{0.5} \nu_A)(a) = \bigwedge_{z=st} \{\max\{\nu_A(s), \nu_A(t), 0.5\}\} \\
&\leq \max\{\nu_A(x), \nu_A(y), 0.5\} \\
\sup_{z=xy} \nu_A(z) &\leq \min\{\nu_A(x), \nu_A(y), 0.5\}.
\end{aligned}$$

Now, let $x, y, z \in \mathcal{R}$ such that $w = xyz$. Then, we have

$$\begin{aligned}
\mu_A(w) &\geq (\mu_A \circ_{0.5} 1 \circ_{0.5} \mu_A)(w) = \bigvee_{w=st} \{\min\{\mu_A(s), (1 \circ_{0.5} \mu_A)(t), 0.5\}\} \\
&= \bigvee_{w=st} \left\{\min\left\{\mu_A(s), \bigvee_{t=pq} \{\min\{1(p), \mu_A(q), 0.5\}\}, 0.5\right\}\right\} \\
&= \bigvee_{w=st} \left\{\min\left\{\mu_A(s), \bigvee_{t=pq} \{\min\{1, \mu_A(q), 0.5\}\}, 0.5\right\}\right\} \\
&\geq \bigvee_{w=st} \bigvee_{t=pq} \{\min\{\mu_A(s), \mu_A(q), 0.5\}\} \geq \bigvee_{w=spq} \{\min\{\mu_A(s), \mu_A(q), 0.5\}\} \\
&\geq \min\{\mu_A(x), \mu_A(z), 0.5\} \\
\mu_A(w) &\geq \min\{\mu_A(x), \mu_A(z), 0.5\}.
\end{aligned}$$

and

$$\begin{aligned}
\nu_A(w) &\geq (\nu_A \circ_{0.5} 0 \circ_{0.5} \nu_A)(w) = \bigwedge_{w=st} \{\max\{\nu_A(s), (0 \circ_{0.5} \nu_A)(t), 0.5\}\} \\
&= \bigwedge_{w=st} \left\{\max\left\{\nu_A(s), \bigwedge_{t=pq} \{\max\{0(p), \nu_A(q), 0.5\}\}, 0.5\right\}\right\} \\
&= \bigwedge_{w=st} \left\{\max\left\{\nu_A(s), \bigwedge_{t=pq} \{\max\{0, \nu_A(q), 0.5\}\}, 0.5\right\}\right\} \\
&\leq \bigwedge_{w=st} \bigwedge_{t=pq} \{\max\{\nu_A(s), \nu_A(q), 0.5\}\} \geq \bigwedge_{w=spq} \{\max\{\nu_A(s), \nu_A(q), 0.5\}\} \\
&\leq \max\{\nu_A(x), \nu_A(z), 0.5\} \\
\nu_A(w) &\leq \max\{\nu_A(x), \nu_A(z), 0.5\}.
\end{aligned}$$

Hence, $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$. ■

Theorem 155 *Let $\mathcal{R}$ be a semihyperring and $A = \langle \mu_A, \nu_A \rangle$ be an IFS of $\mathcal{R}$. Then, $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy left (resp. right, two sided) hyperideal of $\mathcal{R}$ if and only if the following hold: $\mathcal{R} \circ_{0.5} A \subseteq A$ (resp. $A \circ_{0.5} \mathcal{R} \subseteq A$, $A \circ_{0.5} \mathcal{R} \subseteq A$ and $\mathcal{R} \circ_{0.5} A \subseteq A$).*

Proof. Let $A = \langle \mu_A, \nu_A \rangle$ be an $(\in, \in \vee q)$-intuitionistic fuzzy left hyperideal of $\mathcal{R}$ and let $x \in \mathcal{R}$. Then, we have two cases (1) If $x \neq yz$ for every $x, y \in \mathcal{R}$ and (2) If $x = yz$ for some $x, y \in \mathcal{R}$.

Case (1) If $x \neq yz$ for every $x, y \in \mathcal{R}$, then clearly $(1 \circ_{0.5} \mu_A)(x) = 0 \leq \mu_A(x)$ and $(0 \circ_{0.5} \nu_A)(x) = 1 \leq \mu_A(x)$.

Case (2) If $x = yz$ for some $x, y \in \mathcal{R}$, then we have

$$\begin{aligned}
(1 \circ_{0.5} \mu_A)(x) &= \bigvee_{x=yz} \{\min\{1(y), \mu_A(z), 0.5\}\} \\
&= \bigvee_{x=yz} \{\min\{1, \mu_A(z), 0.5\}\} \\
&= \bigvee_{x=yz} \{\min\{\mu_A(z), 0.5\}\}
\end{aligned}$$

Since $A = \langle \mu_A, \nu_A \rangle$ be an $(\in, \in \vee q)$-intuitionistic fuzzy left hyperideal of $\mathcal{R}$, so

$$\inf_{x=yz} \mu_A(yz) \geq \min\{\mu_A(z), 0.5\}$$

Thus,

$$\begin{aligned}
\bigvee_{x=yz} \{\min\{\mu_A(z), 0.5\}\} &\leq \bigvee_{x=yz} \left(\inf_{x=yz} \mu_A(x) \right) = \mu_A(yz) \\
(1 \circ_{0.5} \mu_A)(x) &\leq \inf_{x \in yz} \mu_A(x) \leq \mu_A(yz)
\end{aligned}$$

and

$$\begin{aligned}
(0 \circ_{0.5} \nu_A)(x) &= \bigwedge_{x=yz} \{\max\{0(y), \nu_A(z), 0.5\}\} \\
&= \bigwedge_{x=yz} \{\max\{0, \nu_A(z), 0.5\}\} \\
&= \bigwedge_{x=yz} \{\max\{\nu_A(z), 0.5\}\}
\end{aligned}$$

Since $A = \langle \mu_A, \nu_A \rangle$ be an $(\in, \in \vee q)$-intuitionistic fuzzy left hyperideal of $\mathcal{R}$, so

$$\nu_A(x) \leq \max\{\nu_A(z), 0.5\}.$$

Thus,

$$\begin{aligned}
\bigwedge_{x=yz} \{\max\{\nu_A(z), 0.5\}\} &\geq \bigwedge_{x=yz} \left(\sup_{x=yz} \nu_A(x) \right) = \nu_A(yz) \\
(0 \circ_{0.5} \nu_A)(x) &\geq \sup_{x=yz} \nu_A(x) \geq \nu_A(yz)
\end{aligned}$$

Hence, $\mathcal{R} \circ_{0.5} A \subseteq A$.

Conversely, suppose that the given condition holds and let $x, y \in \mathcal{R}$ such that $z = xy$. Then,

$$\begin{aligned}
\inf_{z=xy} \mu_A(z) &\geq (1 \circ_{0.5} \mu_A)(z) = \bigvee_{x=pq} \{\min\{1(p), \mu_A(q), 0.5\}\} \\
&\geq \bigvee_{x=pq} \{\min\{1, \mu_A(q), 0.5\}\} = \bigvee_{x=pq} \{\min\{\mu_A(q), 0.5\}\} \\
&\geq \min\{\mu_A(y), 0.5\}, \text{ because } z = xy.
\end{aligned}$$

and

$$\begin{aligned}
\sup_{z=xy} \nu_A(z) &\geq (0 \circ_{0.5} \nu_A)(z) = \bigwedge_{x=pq} \{\max\{0(p), \nu_A(q), 0.5\}\} \\
&\leq \bigwedge_{x=pq} \{\max\{0, \nu_A(q), 0.5\}\} = \bigwedge_{x=pq} \{\max\{\nu_A(q), 0.5\}\} \\
&\leq \max\{\mu_A(y), 0.5\}, \text{ because } z = xy.
\end{aligned}$$

Hence, $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy left hyperideal of $\mathcal{R}$. This completes the proof. ■

Theorem 156 *Let $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ be $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideals of $\mathcal{R}$. Then, $A \circ_{0.5} B$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$.*

Proof. Let $A = \langle \mu_A, \nu_A \rangle$ and $B = \langle \mu_B, \nu_B \rangle$ be $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideals of $\mathcal{R}$ and let $x \in \mathcal{R}$. Then, we have two cases (1) If $x \neq yz$ for any $y, z \in \mathcal{R}$ (2) If $x = yz$ for some $y, z \in \mathcal{R}$.

Case 1: If $x \neq yz$ for any $y, z \in \mathcal{R}$, then

$$((\mu_A \circ_{0.5} \mu_B) \circ_{0.5} (\mu_A \circ_{0.5} \mu_B))(x) = 0 \leq (\mu_A \circ_{0.5} \mu_B)(x)$$

and

$$((\nu_A \circ_{0.5} \nu_B) \circ_{0.5} (\nu_A \circ_{0.5} \nu_B))(x) = 1 \geq (\nu_A \circ_{0.5} \nu_B)(x)$$

Thus, $A \circ_{0.5} A \subseteq A$ in this case.

Case 2: If $x = yz$ for some $y, z \in \mathcal{R}$, then

$$\begin{aligned}
\left(\left(\mu_A \circ_{0.5} \mu_B\right) \circ_{0.5} \left(\mu_A \circ_{0.5} \mu_B\right)\right)(x) &= \bigvee_{x=yz} \left\{ \begin{array}{c} \left(\mu_A \circ_{0.5} \mu_B\right)(y) \wedge \left(\mu_A \circ_{0.5} \mu_B\right)(z) \\ \wedge 0.5 \end{array} \right\} \\
&= \bigvee_{x=yz} \left\{ \begin{array}{c} \bigvee_{y=ab} \{\mu_A(a) \wedge \mu_B(b) \wedge 0.5\} \\ \wedge \bigvee_{z=pq} \{\mu_A(p) \wedge \mu_B(q) \wedge 0.5\} \end{array} \right\} \\
&= \bigvee_{x=yz} \bigvee_{y=ab} \bigvee_{z=pq} \left\{ \begin{array}{c} \{\mu_A(a) \wedge \mu_B(b) \wedge 0.5\} \\ \wedge \{\mu_A(p) \wedge \mu_B(q) \wedge 0.5\} \end{array} \right\} \\
&= \bigvee_{x=yz} \bigvee_{y=ab} \bigvee_{z=pq} \left\{ \begin{array}{c} \{\mu_A(a) \wedge \mu_A(p) \\ \wedge \mu_B(b) \wedge \mu_B(q) \wedge 0.5\} \end{array} \right\} \\
&\leq \bigvee_{x=yz} \bigvee_{y=ab} \bigvee_{z=pq} \{\mu_A(a) \wedge \mu_A(p) \wedge 0.5 \wedge \mu_B(q)\}
\end{aligned}$$

Since $x = yz$, $y = ab$ and $z = pq$. So, $x = (ab)(pq) = (abp)q$ and we have

$$\begin{aligned}
& \bigvee_{x=yz} \bigvee_{y=ab} \bigvee_{z=pq} \{\mu_A(a) \wedge \mu_A(p) \wedge 0.5 \wedge \mu_B(q)\} \\
\leq & \bigvee_{x \in (abp)q} \{\mu_A(a) \wedge \mu_A(p) \wedge 0.5 \wedge \mu_B(q) \wedge 0.5\}
\end{aligned}$$

Since $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$ we have

$$\inf_{m=abp} \mu_A(x) \geq \mu_A(a) \wedge \mu_A(p) \wedge 0.5.$$

So,

$$\begin{aligned}
& \bigvee_{x=mq} \{\mu_A(a) \wedge \mu_A(p) \wedge 0.5 \wedge \mu_B(q) \wedge 0.5\} \\
\leq & \bigvee_{x=mq} \left\{ \inf_{m=abp} \mu_A(m) \wedge \mu_B(q) \wedge 0.5 \right\} \\
\leq & \bigvee_{x=mbq} \{\mu_A(m) \wedge \mu_B(q) \wedge 0.5\} = \left(\mu_A \circ_{0.5} \mu_B\right)(mbq) = \left(\mu_A \circ_{0.5} \mu_B\right)(x)
\end{aligned}$$

Therefore, $\left(\left(\mu_A \circ_{0.5} \mu_B\right) \circ_{0.5} \left(\mu_A \circ_{0.5} \mu_B\right)\right)(x) \leq \left(\mu_A \circ_{0.5} \mu_B\right)(x)$. Now,

$$\begin{aligned}
\left(\left(\nu_A \circ_{0.5} \nu_B\right) \circ_{0.5} \left(\nu_A \circ_{0.5} \nu_B\right)\right)(x) &= \bigwedge_{x=yz} \left\{ \begin{array}{c} \left(\nu_A \circ_{0.5} \nu_B\right)(y) \vee \left(\nu_A \circ_{0.5} \nu_B\right)(z) \\ \vee 0.5 \end{array} \right\} \\
&= \bigwedge_{x=yz} \left\{ \begin{array}{c} \bigwedge\limits_{y=ab} \left\{\nu_A(a) \vee \nu_B(b) \vee 0.5\right\} \\ \vee \bigwedge\limits_{z=pq} \left\{\nu_A(p) \vee \nu_B(q) \vee 0.5\right\} \end{array} \right\} \\
&= \bigwedge_{x=yz} \bigwedge_{y=ab} \bigwedge_{z=pq} \left\{ \begin{array}{c} \left\{\nu_A(a) \vee \nu_B(b) \vee 0.5\right\} \\ \vee \left\{\nu_A(p) \vee \nu_B(q) \vee 0.5\right\} \end{array} \right\} \\
&= \bigwedge_{x=yz} \bigwedge_{y=ab} \bigwedge_{z=pq} \left\{ \begin{array}{c} \nu_A(a) \vee \nu_A(p) \\ \vee \nu_B(b) \vee \nu_B(q) \vee 0.5 \end{array} \right\} \\
&\geq \bigwedge_{x=yz} \bigwedge_{y=ab} \bigwedge_{z=pq} \left\{\nu_A(a) \vee \nu_A(p) \vee 0.5 \vee \nu_B(q)\right\}
\end{aligned}$$

Since $x \in y\alpha z$, $y \in a\gamma b$ and $z \in p\beta q$. So, $x \in (a\gamma b)\,\alpha\,(p\beta q) = (a\gamma b\alpha p)\,\beta q$ and we have

$$\begin{aligned}
& \bigwedge_{x\in y\alpha z} \bigwedge_{y\in a\gamma b} \bigwedge_{z\in p\beta q} \left\{\nu_A(a) \vee \nu_A(p) \vee 0.5 \vee \nu_B(q)\right\} \\
\geq & \bigwedge_{x=(a\gamma b\alpha p)\beta q} \left\{\nu_A(a) \vee \nu_A(p) \vee 0.5 \vee \nu_B(q)\right\}
\end{aligned}$$

Since $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$ we have

$$\sup_{w=abp} \nu_A(w) \leq \nu_A(a) \vee \nu_A(p) \vee 0.5.$$

So,

$$\begin{aligned}
& \bigwedge_{x=(w)q} \left\{\nu_A(a) \vee \nu_A(p) \vee 0.5 \vee \nu_B(q)\right\} \\
\geq & \bigwedge_{x=(w)q} \left\{ \sup_{w=abp} \nu_A(w) \vee \nu_B(q) \vee 0.5 \right\} \\
\geq & \bigwedge_{x=wq} \left\{\nu_A(w) \vee \nu_B(q) \vee 0.5\right\} = \left(\nu_A \circ_{0.5} \nu_B\right)(wq) = \left(\nu_A \circ_{0.5} \nu_B\right)(x).
\end{aligned}$$

Therefore, $((\nu_A \circ_{0.5} \nu_B) \circ_{0.5} (\nu_A \circ_{0.5} \nu_B))(x) \geq (\nu_A \circ_{0.5} \nu_B)(x)$ and so $A \circ_{0.5} A \subseteq A$. Thus, $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy subsemihyperring of $\mathcal{R}$.

Now, let $x, y, z \in \mathcal{R}$. Then,

$$\begin{aligned}
(\mu_A \circ_{0.5} \mu_B)(x) \wedge (\mu_A \circ_{0.5} \mu_B)(z) \wedge 0.5 &= \begin{array}{c} \left[\bigvee_{z=ab} \{\mu_A(a) \wedge \mu_B(b) \wedge 0.5\}\right] \wedge \\ \left[\bigvee_{z=pq} \{\mu_A(p) \wedge \mu_B(q) \wedge 0.5\}\right] \wedge 0.5 \end{array} \\
&= \bigvee_{z=ab} \bigvee_{z=pq} \left[\begin{array}{c} \{\mu_A(a) \wedge \mu_B(b) \wedge 0.5\} \\ \wedge \{\mu_A(p) \wedge \mu_B(q) \wedge 0.5\} \\ \wedge 0.5 \end{array}\right] \\
&\leq \bigvee_{z=ab} \bigvee_{z=pq} \left[\begin{array}{c} \mu_A(a) \wedge \mu_A(p) \wedge \mu_B(b) \\ \wedge \mu_B(q) \wedge 0.5 \end{array}\right] \\
&\leq \bigvee_{z=ab} \bigvee_{z=pq} [\mu_A(a) \wedge \mu_A(p) \wedge \mu_B(q) \wedge 0.5]
\end{aligned}$$

Since $x \in yz$, $y \in ab$ and $z \in pq$. So, $x \in (ab)(pq) = (abp)q$ and we have

$$\begin{aligned}
&\bigvee_{x=ab} \bigvee_{z=pq} [\mu_A(a) \wedge \mu_A(p) \wedge \mu_B(q) \wedge 0.5] \\
\leq &\bigvee_{xyz=(a(by)p)q} [\{\mu_A(a) \wedge \mu_A(p) \wedge 0.5\} \wedge \mu_B(q)]
\end{aligned}$$

Since $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$ we have

$$\mu_A(a(by)p) \geq \mu_A(a) \wedge \mu_A(p) \wedge 0.5.$$

So,

$$\begin{aligned}
&\bigvee_{xyz=(a(by)p)q} [\{\mu_A(a) \wedge \mu_A(p) \wedge 0.5\} \wedge \mu_B(q)] \\
\leq &\bigvee_{xyz=(a(by)p)q} [\mu_A(a(by)p) \wedge \mu_B(q)] = (\mu_A \circ_{0.5} \mu_B)(xyz).
\end{aligned}$$

Thus,

$$(\mu_A \circ_{0.5} \mu_B)(xyz) \geq (\mu_A \circ_{0.5} \mu_B)(x) \wedge (\mu_A \circ_{0.5} \mu_B)(z) \wedge 0.5$$

and

$$
\begin{aligned}
(\nu_A \circ_{0.5} \nu_B)(x) \vee (\nu_A \circ_{0.5} \nu_B)(z) \vee 0.5 &= \begin{array}{c} \left[\bigwedge_{x=ab} \{\nu_A(a) \vee \nu_B(b) \vee 0.5\}\right] \vee \\ \left[\bigwedge_{z=pq} \{\nu_A(p) \vee \nu_B(q) \vee 0.5\}\right] \vee 0.5 \end{array} \\
&= \bigwedge_{x=ab} \bigwedge_{z=pq} \left[\begin{array}{c} \{\nu_A(a) \vee \nu_B(b) \vee 0.5\} \\ \vee \{\nu_A(p) \vee \nu_B(q) \vee 0.5\} \\ \vee 0.5 \end{array}\right] \\
&\geq \bigwedge_{x=ab} \bigwedge_{z=pq} \left[\begin{array}{c} \nu_A(a) \vee \nu_A(p) \vee \nu_B(b) \\ \vee \nu_B(q) \vee 0.5 \end{array}\right] \\
&\geq \bigwedge_{x=ab} \bigwedge_{z=pq} [\nu_A(a) \vee \nu_A(p) \vee \nu_B(q) \vee 0.5]
\end{aligned}
$$

Since $x = ab$ and $z = pq$, so $xyz = (ab)\, y\, (pq) = (a\,(by)\,p)\, q$ and we have

$$
\begin{aligned}
&\bigwedge_{x=ab} \bigwedge_{z=pq} [\nu_A(a) \vee \nu_A(p) \vee \nu_B(q) \vee 0.5] \\
\geq &\bigwedge_{xyz=(a(by)p)q} [\{\nu_A(a) \vee \nu_A(p) \vee 0.5\} \vee \nu_B(q)]
\end{aligned}
$$

Since $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$, we have

$$\nu_A(a\,(by)\,p) \geq \nu_A(a) \vee \nu_A(p) \vee 0.5.$$

So,

$$
\begin{aligned}
&\bigwedge_{xyz=(a(by)p)q} [\{\nu_A(a) \vee \nu_A(p) \vee 0.5\} \vee \nu_B(q)] \\
\geq &\bigwedge_{xyz=(a(by)p)q} [\nu_A(a\,(by)\,p) \vee \nu_B(q)] = (\nu_A \circ_{0.5} \nu_B)(xyz).
\end{aligned}
$$

Thus,

$$(\nu_A \circ_{0.5} \nu_B)(xyz) \leq (\nu_A \circ_{0.5} \nu_B)(x) \vee (\nu_A \circ_{0.5} \nu_B)(z) \vee 0.5.$$

Hence, $A \circ_{0.5} B$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$. ■

For any intuitionistic fuzzy set $A = \langle \mu_A, \nu_A \rangle$ in $\mathcal{R}$ and $t \in (0.5, 1]$, $s \in [0, 0.5]$ or $s \in (0.5, 1]$, $t \in [0, 0.5]$, we denote $U_{(t,s)} = \{x \in \mathcal{R} : x\,(t, s) \in A\}$, $A_{(t,s)} = \{x \in \mathcal{R} : x\,(t, s)\, qA\}$ and $[A]_{(t,s)} = \{x \in \mathcal{R} : x\,(t, s) \in \vee qA\}$. Obviously, $[A]_{(t,s)} = A_{(t,s)} \cup U_{(t,s)}$, where $U_{(t,s)}$, $A_{(t,s)}$ and $[A]_{(t,s)}$ are called $\in$-level set, q-level set and $\in \vee q$-level set of $A = \langle \mu_A, \nu_A \rangle$, respectively ([4]).

Theorem 157 *Let $\mathcal{R}$ be a semihyperring and $A = \langle \mu_A, \nu_A \rangle$ an IFS of $\mathcal{R}$. Then, $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy left (resp. right) hyperideal of $\mathcal{R}$ if and only if for all $t \in (0, 0.5]$ and $s \in (0.5, 1)$ or $s \in (0, 0.5]$ and $t \in (0.5, 1)$, the set $U_{(t,s)} \neq \varphi$ is a left (resp. right) hyperideal of $\mathcal{R}$.*

Proof. Let $A = \langle \mu_A, \nu_A \rangle$ be an $(\in, \in \vee q)$-intuitionistic fuzzy left hyperideal of $\mathcal{R}$ and $U_{(t,s)} \neq \varphi$ for any $t \in (0, 0.5]$ and $s \in [0.5, 1)$ or $s \in (0, 0.5]$ and $t \in (0.5, 1)$. Let $y \in U_{(t,s)} \neq \varphi$ and $x \in \mathcal{R}$. Then, $\mu_A(y) \geq t$ and $\nu_A(y) \leq s$. Since $z \in U_{(t,s)}$ for all $z \in x \oplus y$ because

$$\inf_{z \in x \oplus y} \mu_A(z) \geq \mu_A(y) \wedge 0.5 \geq t \wedge 0.5 \geq t, \quad \sup_{z \in x \oplus y} \nu_A(z) \leq \nu_A(y) \vee 0.5 \leq t \vee 0.5 \leq s.$$

and

$$\mu_A(xy) \geq \mu_A(y) \wedge 0.5 \geq t \wedge 0.5 \geq t, \;\; \nu_A(xy) \leq \nu_A(y) \vee 0.5 \leq t \vee 0.5 \leq s.$$

So, also $xy \in U_{(t,s)}$. Hence, $U_{(t,s)}$ is a left hyperideal of $\mathcal{R}$.

Conversely, let us suppose that $A = \langle \mu_A, \nu_A \rangle$ is an IFS of $\mathcal{R}$ such that $U_{(t,s)} \neq \varphi$ is a left hyperideal of $\mathcal{R}$. Suppose on the contrary there exist $x, y \in \mathcal{R}$ such that

$$\inf_{z \in x \oplus y} \mu_A(z) < \mu_A(y) \wedge 0.5, \quad \sup_{z \in x \oplus y} \nu_A(z) > \nu_A(y) \wedge 0.5.$$

and

$$\mu_A(xy) < \mu_A(y) \wedge 0.5, \;\; \nu_A(xy) > \nu_A(y) \wedge 0.5.$$

Let us choose $t \in (0, 0.5]$ and $s \in (0.5, 1]$ or $s \in (0, 0.5]$ and $t \in (0.5, 1]$. Then,

$$\inf_{z \in x \oplus y} \mu_A(z) < t < \mu_A(y) \wedge 0.5, \quad \sup_{z \in x \oplus y} \nu_A(z) > s > \nu_A(y) \wedge 0.5.$$

and

$$\mu_A(xy) < t < \mu_A(y) \wedge 0.5,\ \nu_A(xy) > s > \nu_A(y) \wedge 0.5.$$

Thus, $y \in U_{(t,s)}$ but $z \notin U_{(t,s)}$ for all $z \in x \oplus y$ and $x \oplus y \subseteq U_{(t,s)}$, which is a contradiction. Hence,

$$\inf_{z\in x\oplus y} \mu_A(z) \geq \mu_A(y) \wedge 0.5,\quad \sup_{z\in x\oplus y} \nu_A(z) \leq \nu_A(y) \wedge 0.5.$$

and

$$\mu_A(xy) \geq \mu_A(y) \wedge 0.5 \geq t \wedge 0.5 \geq t,\ \nu_A(xy) \leq \nu_A(y) \vee 0.5 \leq t \vee 0.5 \leq s.$$

This completes the proof. ■

Theorem 158 *Let $\mathcal{R}$ be a semihyperring and $A = \langle \mu_A, \nu_A \rangle$ an IFS of $\mathcal{R}$. Then, $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of $\mathcal{R}$ if and only if for all $t \in (0, 0.5]$ and $s \in (0.5, 1)$ or $s \in (0, 0.5]$ and $t \in (0.5, 1)$, the set $U_{(t,s)} \neq \varphi$ is a bi-hyperideal of $\mathcal{R}$.*

Proof. The proof follows from Theorem 157. ■

Theorem 159 *Let $\mathcal{R}$ be a semihyperring and $A = \langle \mu_A, \nu_A \rangle$ an IFS of $\mathcal{R}$. Then, $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy subsemihyperring of $\mathcal{R}$ if and only if for all $t \in (0, 0.5]$ and $s \in (0.5, 1)$ or $s \in (0, 0.5]$ and $t \in (0.5, 1)$, the set $[A]_{(t,s)} \neq \varphi$ is a subsemihyperring of $\mathcal{R}$.*

Proof. Let $x, y \in [A]_{(t,s)}$. Then, $\mu_A(x) \geq t$ and $\nu_A(x) \leq s$ or $\mu_A(x) + t > 1$ and $\nu_A(x) + s < 1$, and $\mu_A(y) \geq t$ and $\nu_A(y) \leq s$ or $\mu_A(y) + t > 1$ and $\nu_A(y) + s < 1$. We can consider four cases:

(*i*) $\mu_A(x) \geq t$ and $\nu_A(x) \leq s$, and $\mu_A(y) \geq t$ and $\nu_A(y) \leq s$,

(*ii*) $\mu_A(x) \geq t$ and $\nu_A(x) \leq s$, and $\mu_A(y) + t > 1$ and $\nu_A(y) + s < 1$,

(*iii*) $\mu_A(x) + t > 1$ and $\nu_A(x) + s < 1$, and $\mu_A(y) \geq t$ and $\nu_A(y) \leq s$,

(*iv*) $\mu_A(x) + t > 1$ and $\nu_A(x) + s < 1$, and $\mu_A(y) + t > 1$ and $\nu_A(y) + s < 1$.

For the first case, by Theorem 142 (a), it implies that, for all $z \in x \oplus y$

$$\inf_{z\in x\oplus y} \mu_A(z) \geq \min\{\mu_A(x), \mu_A(y), 0.5\} = \min\{t, 0.5\} = \begin{cases} 0.5 & \text{if } t > 0.5 \\ t & \text{if } t \leq 0.5 \end{cases}$$

and

$$\sup_{z\in x\oplus y}\nu_A(z)\leq\max\{\nu_A(x),\nu_A(y),0.5\}=\max\{s,0.5\}=\begin{cases}0.5 & \text{if } s<0.5\\ s & \text{if } s\geq 0.5\end{cases}$$

and so $\inf_{z\in x\oplus y}\mu_A(z)+t>0.5+0.5=1$ and $\sup_{z\in x\oplus y}\nu_A(z)+s<0.5+0.5=1$, *i.e.*, $(z)(s,t)\,qA$, or $z\in A_{(t,s)}$ for all $z\in x\oplus y$. Therefore, for all $z\in x\oplus y$, $z\in U_{(t,s)}\cup A_{(t,s)}=[A]_{(t,s)}$.

Similarly

$$\mu_A(xy)\geq\min\{\mu_A(x),\mu_A(y),0.5\}=\min\{t,0.5\}=\begin{cases}0.5 & \text{if } t>0.5\\ t & \text{if } t\leq 0.5\end{cases}$$

and

$$\nu_A(xy)\leq\max\{\nu_A(x),\nu_A(y),0.5\}=\max\{s,0.5\}=\begin{cases}0.5 & \text{if } s<0.5\\ s & \text{if } s\geq 0.5\end{cases}$$

and so $\mu_A(xy)+t>0.5+0.5=1$ and $\nu_A(xy)+s<0.5+0.5=1$, *i.e.*, $(xy)(s,t)\,qA$, or $xy\in A_{(t,s)}$. Therefore, $xy\in U_{(t,s)}\cup A_{(t,s)}=[A]_{(t,s)}$.

For the case (ii), assume that $t>0.5$ and $s<0.5$. Then, $1-t<0.5$ and $1-s>0.5$. If $\min\{\mu_A(y),0.5\}\leq\mu_A(x)$ and $\max\{\nu_A(y),0.5\}\geq\nu_A(x)$, then

$$\inf_{z\in x\oplus y}\mu_A(z)\geq\min\{\mu_A(y),0.5\}>1-t \text{ and } \sup_{z\in x\oplus y}\nu_A(z)\leq\max\{\nu_A(y),0.5\}<1-s$$

and if $\min\{\mu_A(y),0.5\}>\mu_A(x)$ and $\max\{\nu_A(y),0.5\}<\nu_A(x)$, then $\inf_{z\in x\oplus y}\mu_A(z)\geq\mu_A(x)\geq t$ and $\sup_{z\in x\oplus y}\nu_A(z)\leq\nu_A(x)\leq s$, for all $z\in x\oplus y$. Hence, $z\in U_{(t,s)}\cup A_{(t,s)}=[A]_{(t,s)}$, for all $z\in x\oplus y$. Therefore, for all $z\in x\oplus y$, $z\in U_{(t,s)}\cup A_{(t,s)}=[A]_{(t,s)}$ for $t>0.5$ and $s<0.5$. Similarly

$$\mu_A(xy)\geq\min\{\mu_A(y),0.5\}>1-t \text{ and } \nu_A(xy)\leq\max\{\nu_A(y),0.5\}<1-s$$

and if $\min\{\mu_A(y),0.5\}>\mu_A(x)$ and $\max\{\nu_A(y),0.5\}<\nu_A(x)$, then $\mu_A(xy)\geq\mu_A(x)\geq t$ and $\nu_A(xy)\leq\nu_A(x)\leq s$. Hence, $xy\in U_{(t,s)}\cup A_{(t,s)}=[A]_{(t,s)}$. Therefore, $xy\in U_{(t,s)}\cup A_{(t,s)}=[A]_{(t,s)}$ for $t>0.5$ and $s<0.5$.

Suppose that $t \leq 0.5$ and $s \geq 0.5$. Then, $1-t \geq 0.5$ and $1-s \leq 0.5$. If $\min\{\mu_A(x), 0.5\} \leq \mu_A(y)$ and $\max\{\nu_A(x), 0.5\} \geq \nu_A(y)$. Then, for all $z \in x \oplus y$,

$$\begin{aligned}
\inf_{z\in x\oplus y} \mu_A(z) &\geq \min\{\mu_A(x), 0.5\} \geq t \text{ and} \\
\sup_{z\in x\oplus y} \nu_A(z) &\leq \max\{\nu_A(x), 0.5\} \leq s
\end{aligned}$$

and if $\min\{\mu_A(x), 0.5\} > \mu_A(y)$ and $\max\{\nu_A(x), 0.5\} < \nu_A(y)$, then $\inf_{z\in x\oplus y} \mu_A(z) \geq \mu_A(y) > 1-t$ and $\sup_{z\in x\oplus y} \nu_A(z) \leq \nu_A(y) < 1-s$, for all $z \in x \oplus y$. Thus, $z \in U_{(t,s)} \cup A_{(t,s)} = [A]_{(t,s)}$, for all $z \in x \oplus y$. Therefore, for all $z \in x \oplus y$, $z \in U_{(t,s)} \cup A_{(t,s)} = [A]_{(t,s)}$ for $t \leq 0.5$ and $s \geq 0.5$. Similarly

$$\begin{aligned}
\mu_A(xy) &\geq \min\{\mu_A(x), 0.5\} \geq t \text{ and} \\
\nu_A(xy) &\leq \max\{\nu_A(x), 0.5\} \leq s
\end{aligned}$$

and if $\min\{\mu_A(x), 0.5\} > \mu_A(y)$ and $\max\{\nu_A(x), 0.5\} < \nu_A(y)$, then $\mu_A(xy) \geq \mu_A(y) > 1-t$ and $\nu_A(xy) \leq \nu_A(y) < 1-s$. Thus, $xy \in U_{(t,s)} \cup A_{(t,s)} = [A]_{(t,s)}$. Therefore, $xy \in U_{(t,s)} \cup A_{(t,s)} = [A]_{(t,s)}$ for $t \leq 0.5$ and $s \geq 0.5$. We have similar result for the case (iii). For final case, if $t > 0.5$ and $s < 0.5$, then $1-t < 0.5$ and $1-s > 0.5$. Hence, for all $z \in x \oplus y$,

$$\begin{aligned}
\inf_{z\in x\oplus y} \mu_A(z) &\geq \min\{\mu_A(x), \mu_A(y), 0.5\} \\
&= \begin{cases} 0.5 > 1-t & \text{if } \min\{\mu_A(x), \mu_A(y)\} \geq 0.5, \\ \min\{\mu_A(x), \mu_A(y)\} > 1-t & \text{if } \min\{\mu_A(x), \mu_A(y)\} < 0.5, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\sup_{z\in x\oplus y} \nu_A(z) &\leq \max\{\nu_A(x), \nu_A(y), 0.5\} \\
&= \begin{cases} 0.5 < 1-s & \text{if } \max\{\nu_A(x), \nu_A(y)\} \leq 0.5, \\ \max\{\nu_A(x), \nu_A(y)\} < 1-s & \text{if } \max\{\nu_A(x), \nu_A(y)\} > 0.5, \end{cases}
\end{aligned}$$

and so $z \in A_{(t,s)} \subseteq [A]_{(t,s)}$, for all $z \in x \oplus y$. Similarly

$$\begin{aligned}\mu_A(xy) &\geq \min\{\mu_A(x), \mu_A(y), 0.5\} \\ &= \begin{cases} 0.5 > 1-t & \text{if } \min\{\mu_A(x), \mu_A(y)\} \geq 0.5, \\ \min\{\mu_A(x), \mu_A(y)\} > 1-t & \text{if } \min\{\mu_A(x), \mu_A(y)\} < 0.5, \end{cases}\end{aligned}$$

and

$$\begin{aligned}\nu_A(xy) &\leq \max\{\nu_A(x), \nu_A(y), 0.5\} \\ &= \begin{cases} 0.5 < 1-s & \text{if } \max\{\nu_A(x), \nu_A(y)\} \leq 0.5, \\ \max\{\nu_A(x), \nu_A(y)\} < 1-s & \text{if } \max\{\nu_A(x), \nu_A(y)\} > 0.5, \end{cases}\end{aligned}$$

and so $xy \in A_{(t,s)} \subseteq [A]_{(t,s)}$.

If $t \leq 0.5$ and $s \geq 0.5$, then $1-t \geq 0.5$ and $1-s \leq 0.5$. Thus, for all $z \in x \oplus y$,

$$\begin{aligned}\inf_{z \in x \oplus y} \mu_A(z) &\geq \min\{\mu_A(x), \mu_A(y), 0.5\} \\ &= \begin{cases} 0.5 > 1-t & \text{if } \min\{\mu_A(x), \mu_A(y)\} \geq 0.5, \\ \min\{\mu_A(x), \mu_A(y)\} > 1-t & \text{if } \min\{\mu_A(x), \mu_A(y)\} < 0.5, \end{cases}\end{aligned}$$

and

$$\begin{aligned}\sup_{z \in x \oplus y} \nu_A(z) &\leq \max\{\nu_A(x), \nu_A(y), 0.5\} \\ &= \begin{cases} 0.5 < 1-s & \text{if } \max\{\nu_A(x), \nu_A(y)\} \leq 0.5, \\ \max\{\nu_A(x), \nu_A(y)\} < 1-s & \text{if } \max\{\nu_A(x), \nu_A(y)\} > 0.5, \end{cases}\end{aligned}$$

and so $z \in A_{(t,s)} \subseteq [A]_{(t,s)}$, for all $z \in x \oplus y$. Similarly

$$\begin{aligned}\mu_A(xy) &\geq \min\{\mu_A(x), \mu_A(y), 0.5\} \\ &= \begin{cases} 0.5 \geq t & \text{if } \min\{\mu_A(x), \mu_A(y)\} \geq 0.5, \\ \min\{\mu_A(x), \mu_A(y)\} > 1-t & \text{if } \min\{\mu_A(x), \mu_A(y)\} < 0.5, \end{cases}\end{aligned}$$

and

$$
\begin{aligned}
\nu_A(xy) &\leq \max\{\nu_A(x), \nu_A(y), 0.5\} \\
&= \begin{cases} 0.5 \leq s & \text{if } \max\{\nu_A(x), \nu_A(y)\} \leq 0.5, \\ \max\{\nu_A(x), \nu_A(y)\} < 1 - s & \text{if } \max\{\nu_A(x), \nu_A(y)\} > 0.5, \end{cases}
\end{aligned}
$$

which implies that $xy \in U_{(t,s)} \cup A_{(t,s)} = [A]_{(t,s)}$.

Conversely, suppose that $A = \langle \mu_A, \nu_A \rangle$ is an IFS in $\mathcal{R}$ such that $[A]_{(t,s)}$ is a subsemihyperring of $\mathcal{R}$. Suppose that $A = \langle \mu_A, \nu_A \rangle$ is not an $(\in, \in \vee q)$-intuitionistic fuzzy subsemihyperring of $\mathcal{R}$. Then, there exist $x, y \in \mathcal{R}$ such that, for all $z \in x \oplus y$,

$$
\inf_{z \in x \oplus y} \mu_A(z) < \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \sup_{z \in x \oplus y} \nu_A(z) > \max\{\nu_A(x), \nu_A(y), 0.5\}.
$$

and

$$
\mu_A(xy) < \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \nu_A(xy) > \max\{\nu_A(x), \nu_A(y), 0.5\}.
$$

Let, for all $z \in x \oplus y$,

$$
\begin{aligned}
t &= \frac{1}{2}\left[\inf_{z \in x \oplus y} \mu_A(z) + \min\{\mu_A(x), \mu_A(y), 0.5\}\right] \text{ and} \\
s &= \frac{1}{2}\left[\sup_{z \in x \oplus y} \nu_A(z) + \max\{\nu_A(x), \nu_A(y), 0.5\}\right].
\end{aligned}
$$

Then, for all $z \in x \oplus y$,

$$
\begin{aligned}
\inf_{z \in x \oplus y} \mu_A(z) &< t < \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and} \\
\sup_{z \in x \oplus y} \nu_A(z) &> s > \max\{\nu_A(x), \nu_A(y), 0.5\}.
\end{aligned}
$$

this implies that $x, y \in [A]_{(t,s)}$ and $z \in [A]_{(t,s)}$, for all $z \in x \oplus y$. Hence, $\inf_{z \in x \oplus y} \mu_A(z) \geq t$ and $\sup_{z \in x \oplus y} \nu_A(z) \leq s$ or $\inf_{z \in x \oplus y} \mu_A(z) + t > 1$ and $\sup_{z \in x \oplus y} \nu_A(z) + s < 1$, for all $z \in x \oplus y$. Which is a contradiction. Therefore, we have, for all $z \in x \oplus y$,

$$
\inf_{z \in x \oplus y} \mu_A(z) \geq \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \sup_{z \in x \oplus y} \nu_A(z) \leq \max\{\nu_A(x), \nu_A(y), 0.5\}.
$$

Similarly we can prove for $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y), 0.5\}$ and $\nu_A(xy) \leq \max\{\nu_A(x), \nu_A(y), 0.5\}$.

Thus, $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy subsemihyperring of $\mathcal{R}$. ■

Theorem 160 *Let $\mathcal{R}$ be a semihyperring and $A = \langle \mu_A, \nu_A \rangle$ an IFS of $\mathcal{R}$. Then, $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy left (resp. right) hyperideal of $\mathcal{R}$ if and only if for all $t \in (0, 0.5]$ and $s \in (0.5, 1)$ or $s \in (0, 0.5]$ and $t \in (0.5, 1)$, the set $[A]_{(t,s)} \neq \varphi$ is a left (resp. right) hyperideal of $\mathcal{R}$.*

Proof. Assume that $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy left hyperideal of $\mathcal{R}$ and let $t \in (0, 0.5]$ and $s \in (0.5, 1)$ or $s \in (0, 0.5]$ and $t \in (0.5, 1)$, be such that $[A]_{(t,s)} \neq \varphi$. Let $y \in [A]_{(t,s)}$ and $x \in \mathcal{R}$. Then, $\mu_A(y) \geq t$ and $\nu_A(y) \leq s$ or $\mu_A(y) + t > 1$ and $\nu_A(y) + s < 1$. Assume that $\mu_A(y) \geq t$ and $\nu_A(y) \leq s$ by Theorem 142 (a), implies that for all $z \in x \oplus y$,

$$\inf_{z \in x \oplus y} \mu_A(z) \geq \min\{\mu_A(y), 0.5\} \geq \min\{t, 0.5\} = \begin{cases} t & \text{if } t \leq 0.5, \\ 0.5 > 1 - t & \text{if } t > 0.5, \end{cases}$$

and

$$\sup_{z \in x \oplus y} \nu_A(z) \leq \max\{\nu_A(y), 0.5\} \geq \max\{s, 0.5\} = \begin{cases} s & \text{if } s \geq 0.5, \\ 0.5 < 1 - s & \text{if } s < 0.5, \end{cases}$$

so that, for all $z \in x \oplus y$, $z \in U_{(t,s)} \cup A_{(t,s)} = [A]_{(t,s)}$.

Similarly

$$\mu_A(xy) \geq \min\{\mu_A(y), 0.5\} \geq \min\{t, 0.5\} = \begin{cases} t & \text{if } t \leq 0.5, \\ 0.5 > 1 - t & \text{if } t > 0.5, \end{cases}$$

and

$$\nu_A(xy) \leq \max\{\nu_A(y), 0.5\} \geq \max\{s, 0.5\} = \begin{cases} s & \text{if } s \geq 0.5, \\ 0.5 < 1 - s & \text{if } s < 0.5, \end{cases}$$

so that $xy \in U_{(t,s)} \cup A_{(t,s)} = [A]_{(t,s)}$. Suppose that $\mu_A(y) + t > 1$ and $\nu_A(y) + s < 1$. If $t > 0.5$ and $s < 0.5$, then for all $z \in x \oplus y$,

$$\inf_{z \in x \oplus y} \mu_A(z) \geq \min\{\mu_A(y), 0.5\} = \begin{cases} 0.5 > 1 - t & \text{if } t \leq 0.5, \\ \mu_A(y) > 1 - t & \text{if } t > 0.5, \end{cases}$$

and

$$\sup_{z \in x \oplus y} \nu_A(z) \geq \max\{\nu_A(y), 0.5\} = \begin{cases} 0.5 < 1 - s & \text{if } s \geq 0.5, \\ \nu_A(y) < 1 - s & \text{if } s < 0.5, \end{cases}$$

and thus $z \in A_{(t,s)} \subseteq [A]_{(t,s)}$, for all $z \in x \oplus y$. Similarly

$$\mu_A(xy) \geq \min\{\mu_A(y), 0.5\} = \begin{cases} 0.5 > 1 - t & \text{if } t \leq 0.5, \\ \mu_A(y) > 1 - t & \text{if } t > 0.5, \end{cases}$$

and

$$\nu_A(xy) \geq \max\{\nu_A(y), 0.5\} = \begin{cases} 0.5 < 1 - s & \text{if } s \geq 0.5, \\ \nu_A(y) < 1 - s & \text{if } s < 0.5, \end{cases}$$

and thus $xy \in A_{(t,s)} \subseteq [A]_{(t,s)}$. Consequently, $[A]_{(t,s)}$ is a left hyperideal of $\mathcal{R}$.

Conversely, suppose that $A = \langle \mu_A, \nu_A \rangle$ is an IFS in $\mathcal{R}$ such that $[A]_{(t,s)}$ is a left hyperideal of $\mathcal{R}$. Suppose that $A = \langle \mu_A, \nu_A \rangle$ is not an $(\in, \in \vee q)$-intuitionistic fuzzy hyperideal of $\mathcal{R}$. Then, there exist $x, y \in \mathcal{R}$ such that, for all $z \in x \oplus y$,

$$\inf_{z \in x \oplus y} \mu_A(z) < \min\{\mu_A(y), 0.5\} \text{ and } \nu_A(xy) > \max\{\nu_A(y), 0.5\}.$$

and

$$\mu_A(xy) < \min\{\mu_A(y), 0.5\} \text{ and } \nu_A(xy) > \max\{\nu_A(y), 0.5\}.$$

Let for all $z \in x \oplus y$,

$$t = \frac{1}{2}\left[\inf_{z \in x \oplus y} \mu_A(z) + \min\{\mu_A(y), 0.5\}\right] \text{ and } s = \frac{1}{2}\left[\sup_{z \in x \oplus y} \nu_A(z) + \max\{\nu_A(y), 0.5\}\right].$$

$$\inf_{z \in x \oplus y} \mu_A(z) < t < \min\{\mu_A(y), 0.5\} \text{ and } \sup_{z \in x \oplus y} \nu_A(z) > s > \max\{\nu_A(y), 0.5\}.$$

this implies that $x, y \in U_{(t,s)} \subseteq [A]_{(t,s)}$, so that $z \in [A]_{(t,s)}$ for all $z \in x \oplus y$. Thus, for all $z \in x \oplus y$, $\inf_{z\in x\oplus y} \mu_A(z) \geq t$ and $\sup_{z\in x\oplus y} \nu_A(z) \leq s$ or $\inf_{z\in x\oplus y} \mu_A(z) + t > 1$ and $\sup_{z\in x\oplus y} \nu_A(z) + s < 1$, which is a contradiction. Therefore, we have

$$\inf_{z\in x\oplus y} \mu_A(z) \geq \min\{\mu_A(y), 0.5\} \text{ and } \sup_{z\in x\oplus y} \nu_A(z) \leq \max\{\nu_A(y), 0.5\},$$

for all $z \in x \oplus y$. Similarly we can prove for $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y), 0.5\}$ and $\nu_A(xy) \leq \max\{\nu_A(x), \nu_A(y), 0.5\}$. Hence, $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy left hyperideal of $\mathcal{R}$. Similarly, the right case also follows. This completes the proof. ■

Theorem 161 *Let $\mathcal{R}$ be a semihyperring and $A = \langle \mu_A, \nu_A \rangle$ an IFS of $\mathcal{R}$. Then, $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of S if and only if for all $t \in (0, 0.5]$ and $s \in (0.5, 1)$ or $s \in (0, 0.5]$ and $t \in (0.5, 1)$, the set $[A]_{(t,s)} \neq \varphi$ is a subsemihyperring of $\mathcal{R}$.*

Theorem 162 *Every $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of a semihyperring $\mathcal{R}$ is an $(\in, \in \vee q)$-intuitionistic fuzzy hyperideal of a semihyperring $\mathcal{R}$.*

Proof. Let $A = \langle \mu_A, \nu_A \rangle$ be an $(\in, \in \vee q)$-intuitionistic fuzzy bi-hyperideal of a semihyperring $\mathcal{R}$. Then

$$\inf_{z\in x\oplus y} \mu_A(z) \geq \min\{\mu_A(x), \mu_A(y), 0.5\}, \quad \sup_{z\in x\oplus y} \nu_A(z) \leq \max\{\nu_A(x), \nu_A(y), 0.5\}$$

and $\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y), 0.5\}$, $\nu_A(xy) \leq \max\{\nu_A(x), \nu_A(y), 0.5\}$

Now let for any $x, y, z, a \in \mathcal{R}$. Then

$$\begin{aligned}
\mu_A(xa(yz)) &\geq \min\{\mu_A(xay), \mu_A(z), 0.5\} \\
\mu_A((xay)z) &\geq \min\{\mu_A(xay), \mu_A(z), 0.5\} \\
\mu_A(xa(yz)) &\geq \min\{\mu_A(x), \mu_A(y), \mu_A(z), 0.5\}. \\
\nu_A(xa(yz)) &\leq \max\{\nu_A(xay), \nu_A(z), 0.5\} \\
\nu_A(xa(yz)) &\leq \max\{\nu_A(xay), \nu_A(z), 0.5\} \\
\nu_A(xa(yz)) &\leq \max\{\nu_A(x), \nu_A(y), \nu_A(z), 0.5\}
\end{aligned}$$

Therefor $A = \langle \mu_A, \nu_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy hyperideal of a semihyperring $\mathcal{R}$. ■

Chapter 6

n-Dimensional Fuzzy Hyperideals in Semihyperrings

The defined notion is a generalization of fuzzy hyperideals, fuzzy prime hyperideals and topological space of fuzzy prime hyperideals of a semihyperring. In this chapter, we introduce the notion of n-dimensional fuzzy sets, fuzzy hyperideals and fuzzy prime hyperideals in semihyperring with identity. We also discuss some basic properties of n-dimensional fuzzy prime hyperideals and characterize the n-dimensional fuzzy prime hyperideals. We will also investigate the topology on n-dimensional fuzzy hyperideals and fuzzy prime hyperideals. In the second section, we study and describe the behavior of n-dimensional fuzzy weak (strong) k-hyperideals of semihyperrings under homomorphism of semihyperrings and some different characterization of fuzzy weak (strong) k-hyperideals of semihyperrings. In the last section, we study n-dimensional (α, β)-fuzzy hyperideals in semihyperring and their characterization.

6.1 n-Dimensional fuzzy hyperideals

This section deals with some basic definitions of an n-dimensional fuzzy set in a semihyperring $\mathcal{R}$. We also discuss some relative results.

Definition 163 *Let $\mathcal{R}$ be a semihyperring. A map $\hat{\mu} : \mathcal{R} \longrightarrow I_n$ is called an n-dimensional fuzzy subset of $\mathcal{R}$ and denoted as*

$\hat{\mu}(x) = (\mu_1(x), \mu_2(x), ...\mu_n(x)), \forall\, x \in \mathcal{R}$.

Definition 164 *Let A be a subset of $\mathcal{R}$ and $\hat{\mu}$ be an n-dimensional fuzzy set in a semihyperring $\mathcal{R}$ is defined by*

$$\hat{\mu}_A(x) = \begin{cases} \widehat{t} = (t_1, t_2, ..., t_n) & \text{if } x \in A \\ \hat{0} = (0, 0, ..., 0) & \text{otherwise} \end{cases}$$

In particular, if $A = \{x\}$, we denote $\hat{\mu}_{\{x\}}$ by $\hat{\mu}_x$ and call it a fuzzy point of $\mathcal{R}$.

If $\hat{\mu}, \hat{\nu}$ are two n-dimensional fuzzy sets, then $\hat{\mu} \subseteq \hat{\nu}$ if $\hat{\mu}(x) \subseteq \hat{\nu}(x)$, for all $x \in \mathcal{R}$. The intersection and union of two n-dimensional fuzzy sets $\hat{\mu}, \hat{\nu}$ are defined respectively as $(\hat{\mu} \cap \hat{\nu})(x) = \hat{\mu}(x) \wedge \hat{\nu}(x)$ and $(\hat{\mu} \cup \hat{\nu})(x) = \hat{\mu}(x) \vee \hat{\nu}(x)$, for all $x \in \mathcal{R}$.

Definition 165 *Let A be a non-empty subset of a semihyperring $\mathcal{R}$. Then the n-dimensional characteristic function of A denoted and defined by*

$$\widehat{\chi}_A = \begin{cases} \hat{1} = (1, 1, ..., 1) & \text{if } x \in A \\ \hat{0} = (0, 0, ..., 0) & \text{otherwise} \end{cases}$$

Clearly, the n-dimensional characteristic function of any subset of $\mathcal{R}$ is an n-dimensional fuzzy subset of $\mathcal{R}$.

Definition 166 *An n-dimensional fuzzy subset $\hat{\mu}$ of a semihyperring $\mathcal{R}$ is called an n-dimensional fuzzy subsemihypering of $\mathcal{R}$ if it satisfies the following conditions:*

(i) $\inf\limits_{z \in x \oplus y} \hat{\mu}(z) \geq \hat{\mu}(x) \wedge \hat{\mu}(y)$, for all $x, y \in \mathcal{R}$.

(ii) $\hat{\mu}(xy) \geq \hat{\mu}(x) \wedge \hat{\mu}(y)$ for all $x, y \in \mathcal{R}$, where each $\hat{\mu} = \mu_i$, $i = 1, 2, 3, ..., n$, is a fuzzy subsemihyperring of $\mathcal{R}$.

Definition 167 *An n-dimensional fuzzy subset $\hat{\mu}$ of a semihyperring $\mathcal{R}$ is called an n-dimensional fuzzy left (right) hyperideal of $\mathcal{R}$ if it satisfies the following conditions:*

(i) $\inf\limits_{z \in x \oplus y} \hat{\mu}(z) \geq \hat{\mu}(x) \wedge \hat{\mu}(y)$, for all $x, y \in \mathcal{R}$.

(ii) $\hat{\mu}(xy) \geq \hat{\mu}(x)$ and $\hat{\mu}(xy) \geq \hat{\mu}(y)$, for all $x, y \in \mathcal{R}$, where each μ_i, $i = 1, 2, 3, ..., n$, is a fuzzy left (right) hyperideal of $\mathcal{R}$.

Example 168 *By example 5, let* $(\mathbb{Z}, \oplus, \cdot)$, *be a semihyperring, where* $\mathbb{Z}$ *is the set of integers defined by a hyperoperation* "$\oplus$" *and a binary operation* "$\cdot$" *on* $\mathbb{Z}$ *as follows* $m \oplus n = \{m, n\}$ *and* $mn = mn \ \forall \ m, n \in \mathbb{Z}$.

Consider a 7-dimensional fuzzy set as follows:

$$\hat{\mu}(x) = \begin{cases} (0.2, 0.22, 0.23, 0.24, 0.25, 0.26, 0.27) & \text{if } x \text{ is odd} \\ (0.6, 0.61, 0.62, 0.63, 0.64, 0.65, 0.67) & \text{if } x \text{ is non-zero even} \\ (0.9, 0.91, 0.92, 0.93, 0.94, 0.95, 0.96) & \text{if } x = 0 \end{cases}$$

Then it is easy to show that $\hat{\mu}$ is a 7-dimensional fuzzy hyperideal of $\mathbb{Z}$.

Proposition 169 *A non-empty subset* A *of a semihyperring* $\mathcal{R}$ *is a subsemihyperring of* $\mathcal{R}$ *iff the* n*-dimensional characteristic function of* A *is an* n*-dimensional fuzzy subsemihyperring of* $\mathcal{R}$.

Proof. Suppose A is a subsemihyperring of $\mathcal{R}$ and $x, y \in \mathcal{R}$. If $x, y \in A$, then for all $z \in x \oplus y$, $z \subseteq A$ and $xy \in A$. (i) As $\widehat{\chi}_A(x) = \widehat{\chi}_A(y) = \overline{1} = (1, 1, ..., 1)$ and for all $z \in x \oplus y \subseteq A \Rightarrow \inf\limits_{z \in x \oplus y} \widehat{\chi}_A(z) = \overline{1} = (1, 1, ..., 1)$. For all $x, y \in A$,

$$\inf_{z \in x \oplus y} \widehat{\chi}_A(z) \geq \widehat{\chi}_A(x) \wedge \widehat{\chi}_A(y)$$

(ii) For all $x, y \in A$ $\widehat{\chi}_A(x) = \widehat{\chi}_A(y) = \widehat{\chi}_A(xy) = \overline{1} = (1, 1, ..., 1)$

$$\widehat{\chi}_A(xy) \geq \widehat{\chi}_A(x) \wedge \widehat{\chi}_A(y)$$

If $x, y \notin A$, then (i) $\widehat{\chi}_A(x) = \widehat{\chi}_A(y) = \overline{0} = (0, 0, ..., 0)$ and for all $z \in x \oplus y \nsubseteq A \Rightarrow \inf\limits_{z \in x \oplus y} \widehat{\chi}_A(z) = \overline{0} = (0, 0, ..., 0)$. This implies $\inf\limits_{z \in x \oplus y} \widehat{\chi}_A(z) \geq \widehat{\chi}_A(x) \wedge \widehat{\chi}_A(y)$, for all $x, y \notin A$. (ii) $\widehat{\chi}_A(x) = \widehat{\chi}_A(y) = \widehat{\chi}_A(xy) = \overline{0} = (0, 0, ..., 0)$. This implies $\widehat{\chi}_A(xy) \geq \widehat{\chi}_A(x) \wedge \widehat{\chi}_A(y)$, for all $x, y \notin A$. This shows that $\widehat{\chi}_A$ is an n-dimensional fuzzy subsemihyperring of $\mathcal{R}$.

Conversely, assume that $\widehat{\chi}_A$ is an n-dimensional fuzzy subsemihyperring of $\mathcal{R}$. Let $x, y \in A$, then for all $x, y \in A$, $\widehat{\chi}_A(x) = \widehat{\chi}_A(y) = \overline{1} = (1, 1, ..., 1)$ and as $\inf\limits_{z \in x \oplus y} \widehat{\chi}_A(z) \geq \widehat{\chi}_A(x) \wedge \widehat{\chi}_A(y) = \overline{1} = (1, 1, ..., 1)$. This implies $\inf\limits_{z \in x \oplus y} \widehat{\chi}_A(z) = \overline{1} = (1, 1, ..., 1) \Rightarrow x \oplus y \subseteq A$. Similarly, $\widehat{\chi}_A(xy) \geq$

$\widehat{\chi}_A(x) \wedge \widehat{\chi}_A(y)$, for all $x, y \in A$. This implies $\widehat{\chi}_A(x) = \widehat{\chi}_A(y) = \widehat{\chi}_A(xy) = \overline{1} = (1, 1, ..., 1)$. Thus, $xy \in A$. Hence A is a subsemihyperring of $\mathcal{R}$. ■

Theorem 170 *Let $\hat{\mu}$ be an n-dimensional fuzzy subset of a semihyperring $\mathcal{R}$. Then, $\hat{\mu}$ is an n-dimensional fuzzy hyperideal of $\mathcal{R}$ if and only if for every $t \in [0, 1]$, the level subset or t-cut,*

$$\hat{\mu}_t = \{x \in \mathcal{R} | \hat{\mu}(x) \geq t\} \neq \varphi$$

is a hyperideal of $\mathcal{R}$.

Proof. Suppose that $\hat{\mu}$ is an n-dimensional fuzzy hyperideal of a semihyperring $\mathcal{R}$ and $t \in I_n$ such that $\hat{\mu}_t \neq \varphi$. Let $x, y \in \hat{\mu}_t$ then $\hat{\mu}(x) \geq t$ and $\hat{\mu}(y) \geq t$. As $\inf\limits_{z \in x \oplus y} \hat{\mu}(z) \geq \hat{\mu}(x) \wedge \hat{\mu}(y)$, so $\inf\limits_{z \in x \oplus y} \hat{\mu}(z) \geq t \Longrightarrow x \oplus y \in \hat{\mu}_t$.

For $r \in \mathcal{R}$, $\hat{\mu}(rx) \geq \hat{\mu}(x) \geq t$ so $\hat{\mu}(rx) \geq t$. This implies $rx \in \hat{\mu}_t$. Hence $\hat{\mu}_t$ is a hyperideal of $\mathcal{R}$.

Conversely, assume $\hat{\mu}_t$ is a hyperideal of $\mathcal{R}$, let $x, y \in \mathcal{R}$ be such that $\inf\limits_{z \in x \oplus y} \hat{\mu}(z) < \hat{\mu}(x) \wedge \hat{\mu}(y)$. We take $t \in (0, 1]$ such that $\inf\limits_{z \in x \oplus y} \hat{\mu}(z) < t \leq \hat{\mu}(x) \wedge \hat{\mu}(y)$, then $x, y \in \hat{\mu}_t$ but $z \notin \hat{\mu}_t$, for all $z \in x \oplus y$. Which is contradiction, hence $\inf\limits_{z \in x \oplus y} \hat{\mu}(z) \geq \hat{\mu}(x) \wedge \hat{\mu}(y)$. Similarly, we can show that $\hat{\mu}(xy) \geq \hat{\mu}(x)$ and $\hat{\mu}(xy) \geq \hat{\mu}(y)$, for all $x, y \in \mathcal{R}$. Thus $\hat{\mu}$ is an n-dimensional fuzzy hyperideal of a semihyperring $\mathcal{R}$. ■

Proposition 171 *A non-empty subset of a semihyperring $\mathcal{R}$ is a left (right) hyperideal of $\mathcal{R}$ iff the n-dimensional characteristic function of A is an n-dimensional fuzzy left (right) hyperideal of $\mathcal{R}$.*

Theorem 172 *If $\hat{\mu}, \hat{\nu}$ are two n-dimensional fuzzy left (right) hyperideals of a semihyperring $\mathcal{R}$, then $\hat{\mu} \cap \hat{\nu}$ is also an n-dimensional fuzzy left (right) hyperideal of a semihyperring $\mathcal{R}$.*

Proof. (i) $\inf\limits_{z \in x \oplus y} (\hat{\mu} \cap \hat{\nu})(z) = \inf\limits_{z \in x \oplus y} \hat{\mu}(z) \wedge \inf\limits_{z \in x \oplus y} \hat{\nu}(z) \geq \{\hat{\mu}(x) \wedge \hat{\mu}(y)\} \wedge \{\hat{\nu}(x) \wedge \hat{\nu}(y)\} = \{\hat{\mu}(x) \wedge \hat{\nu}(x)\} \wedge \{\hat{\mu}(y) \wedge \hat{\nu}(y)\}$

$= (\hat{\mu} \cap \hat{\nu})(x) \wedge (\hat{\mu} \cap \hat{\nu})(y)$

(ii) $(\hat{\mu} \cap \hat{\nu})(xy) = \hat{\mu}(xy) \wedge \hat{\nu}(xy) \geq \hat{\mu}(x) \wedge \hat{\nu}(x) \geq (\hat{\mu} \cap \hat{\nu})(x)$, which proves the result. ■

Definition 173 *Let $\hat{\mu}$ be an n-dimensional fuzzy subset of a semihyperring $\mathcal{R}$. Then, $\hat{\mu}_*$ is defined as*

$$\hat{\mu}_* = \{x \in \mathcal{R} | \hat{\mu}(x) = \hat{\mu}(0)\}.$$

If $\hat{\mu}$ is a hyperideal of $\mathcal{R}$, then $\hat{\mu}_*$ is a hyperideal of $\mathcal{R}$.

Definition 174 *If $\hat{\mu}, \hat{\nu}$ are two n-dimensional fuzzy left (right) hyperideals of a semihyperring $\mathcal{R}$. Then fuzzy subset $\hat{\mu} \oplus \hat{\nu}$ of $\mathcal{R}$ is defined by*

$$(\hat{\mu} \oplus \hat{\nu})(x) = \bigvee_{x \in y \oplus z} [\hat{\mu}(y) \wedge \hat{\nu}(z)], \textit{ for some } y, z \in \mathcal{R}.$$

Definition 175 *If $\hat{\mu}, \hat{\nu}$ are two n-dimensional fuzzy left (right) hyperideals of a semihyperring $\mathcal{R}$. Then product of two n-dimensional fuzzy subsets $\hat{\mu}$ and $\hat{\nu}$ of $\mathcal{R}$ is defined by*

$$(\hat{\mu}\hat{\nu})(x) = \bigvee_{x = yz} [\hat{\mu}(y) \wedge \hat{\nu}(z)], \textit{ for some } y, z \in \mathcal{R}.$$

Lemma 176 *Let $x, y \in \mathcal{R}$, then $< x >< y > \subseteq < xy >$.*

Proof. Let $z \in < x >< y >$, then $z \in \sum_{i=1}^{n} x_i y_i$ for $x_i \in < x >$ and $y_i \in < y >$. But for each $1 \leq i \leq n$ there exist $r_i, r'_i \in \mathcal{R}$, such that $x_i = r_i x$ and $y_i = r'_i y$. So $z \in \sum_{i=1}^{n} (r_i x)(r'_i y) = \sum_{i=1}^{n} (r_i r'_i)(xy)$.

But $\sum_{i=1}^{n} (r_i r'_i)(xy) = \cup\{c.b | c \in \bigcup_{a_i \in r_i r'_i} a_i . b = x.y\}$. Now there exist $c \in \bigcup_{a_i \in r_i r'_i} a_i$ and $b = x.y$, such that $z = c.b$. Since $c \in \bigcup_{a_i \in r_i r'_i} a_i$, then for $i = 1, 2, ...,$n, there exist $a_i = r_i r'_i$, such that $c \in \sum_{i=1}^{n} a_i$. Therefore $z \in (\sum_{i=1}^{n} a_i).b = \sum_{i=1}^{n} a_i.b$, this means that $z \in < x.y >$. Thus $< x > . < y > \subseteq < x.y >$. ■

Definition 177 *Let $\mathcal{R}$ be a semihyperring and $\hat{\mu}$ be an n-dimensional fuzzy subset of $\mathcal{R}$, then*

$$< \hat{\mu} > = \cap\{\hat{\nu}; \hat{\nu} \textit{ is an n-dimensional fuzzy subset of } \mathcal{R} \mid \hat{\mu} \subseteq \hat{\nu}\}.$$

then $< \hat{\mu} >$ is called an n-dimensional fuzzy hyperideal generated by $\hat{\mu}$.

Lemma 178 *Let* $\{\hat{\mu}_i\}_{i\in I}$ *is set of an* n*-dimensional fuzzy hyperideals of* $\mathcal{R}$*, then* $\underset{i\in I}{\cap}\hat{\mu}$ *is also an* n*-dimensional fuzzy hyperideal of* $\mathcal{R}$*.*

Proof. For each $i \in I$ we have for all $z \in x + y$,

$$\hat{\mu}_i(z) \geq \hat{\mu}_i(x) \wedge \hat{\mu}_i(y) \geq \underset{i\in I}{\cap}\hat{\mu}_i(x) \wedge \underset{i\in I}{\cap}\hat{\mu}_i(y).$$

So for all $z \in x + y$,

$$\underset{i\in I}{\cap}\hat{\mu}_i(z) \geq \underset{i\in I}{\cap}\hat{\mu}_i(x) \wedge \underset{i\in I}{\cap}\hat{\mu}_i(y).$$

Similarly we can prove $\underset{i\in I}{\cap}\hat{\mu}(xy) \geq \underset{i\in I}{\cap}\hat{\mu}(x)$ or $\underset{i\in I}{\cap}\hat{\mu}(xy) \geq \underset{i\in I}{\cap}\hat{\mu}(y)$. ■

Corollary 179 *Let* $\hat{\mu}$ *is an* n*-dimensional fuzzy subset of* $\mathcal{R}$*, then* $<\hat{\mu}>$ *is an* n*-dimensional fuzzy hyperideal of* $\mathcal{R}$*.*

Theorem 180 *Let* $\mathcal{R}$ *be a semihyperring and* $A \subseteq \mathcal{R}$ *and* $\hat{t} \in I_n$*. Then* $\hat{t}_{<A>} = <\hat{t}_A>$ *.*

Proof. Since $A \subseteq <A>$, so $\hat{t}_A \subseteq \hat{t}_{<A>}$ and then

$$<\hat{t}_A> \subseteq \hat{t}_{<A>} \tag{1}$$

Now let $\hat{\mu}$ be an n-dimensional fuzzy hyperideal of $\mathcal{R}$ such that $\hat{t}_A \subseteq \hat{\mu}$, so for each $a \in A$, we have $\hat{t} \leq \hat{\mu}(a)$. Let $y \in <A>$, then $y \in \Sigma_{i=1}^{n} r_i a_i$ for $r_i \in \mathcal{R}$ and $a_i \in A$ and n$\in \mathbb{N}$. Therefore, for $i = 1, 2, ...,$n there exist $s_i = r_i a_i$, such that $y \in \Sigma_{i=1}^{n} s_i$. Now

$$\begin{aligned}
\hat{\mu}(y) &\geq \overset{n}{\underset{i=1}{\wedge}}\hat{\mu}(s_i) \geq \overset{n}{\underset{i=1}{\wedge}}(\hat{\mu}(r_i) \vee \hat{\mu}(a_i)) \\
&\geq \overset{n}{\underset{i=1}{\wedge}}(\hat{\mu}(r_i) \vee \hat{t}) \\
&= (\overset{n}{\underset{i=1}{\wedge}}(\hat{\mu}(r_i)) \vee \hat{t} \\
&\geq \hat{t} = \hat{t}_{<A>}(y)
\end{aligned}$$

and if $y \notin <A>$ then $\hat{0} = \hat{t}_{<A>}(y) \leq \hat{\mu}(y)$. Hence $\hat{t}_{<A>} \leq \hat{\mu}$ and then

$$\hat{t}_{<A>} \subseteq \cap\{\hat{\mu} | \hat{t}_A \subseteq \hat{\mu}\} = <\hat{t}_A> \tag{2}$$

By (1), (2) we have $< \hat{t}_A >= \hat{t}_{<A>}$. ■

Corollary 181 *$< \hat{t}_A >= \hat{t}_{<A>}$, for $\hat{t} \in I_n$ and $x \in \mathcal{R}$.*

Theorem 182 *Let $\hat{\mu}$, $\hat{\nu}$ be n-dimensional fuzzy subsets of $\mathcal{R}$, if $\hat{\mu}$ is an n-dimensional fuzzy hyperideal of $\mathcal{R}$, then $\hat{\mu}\hat{\nu} \subseteq \hat{\mu}$.*

Proof. Suppose that $x = yz$, for $y, z \in \mathcal{R}$, so

$$\hat{\mu}(x) \geq \bigwedge_{x=yz} \hat{\mu}(x) \geq \hat{\mu}(y) \vee \hat{\mu}(z) \geq \hat{\mu}(y) \geq \hat{\mu}(y) \wedge \hat{\nu}(z)$$

Then for all $y, z \in \mathcal{R}$, such that $x = yz$, we have

$$\hat{\mu}(x) \geq \vee\{\hat{\mu}(y) \vee \hat{\nu}(z) \mid x = yz\} = (\hat{\mu}\hat{\nu})(x)$$

and so $\hat{\mu}\hat{\nu} \subseteq \hat{\mu}$. ■

Corollary 183 *Let $\hat{\mu}$, $\hat{\nu}$ are n-dimensional fuzzy hyperideals of $\mathcal{R}$, then $\hat{\mu}\hat{\nu} \subseteq \hat{\mu} \cap \hat{\nu}$.*

6.2 n-Dimensional prime fuzzy hyperideals

In this section, we discuss n-dimensional fuzzy prime hyperideals of a semihyperring and corresponding results.

Definition 184 *An n-dimensional fuzzy hyperideals $\hat{p}$ of $\mathcal{R}$ is called an n-dimensional fuzzy prime hyperideal if $\hat{p}$ is non-constant and for any n-dimensional fuzzy hyperideals $\hat{\mu}, \hat{\nu}$ of $\mathcal{R}$ if $\hat{\mu}\hat{\nu} \subseteq \hat{p}$, then either $\hat{\mu} \subseteq \hat{p}$ or $\hat{\nu} \subseteq \hat{p}$.*

Lemma 185 *Let $x, y \in \mathcal{R}$ and $\hat{a}, \hat{b} \in I_n$, then $(\hat{a}_{<x>}.\hat{b}_{<y>})(z) \subseteq < (\hat{a} \wedge \hat{b})_{xy} >$.*

Proof. By Theorem 180 and Corollary 181 it is sufficient to show that $a_{<x>}.b_{<y>} \subseteq (a \wedge b)_{<xy>}$. Let $z \in \mathcal{R}$. Then,

$$(\hat{a}_{<x>}.\hat{b}_{<y>})(z) = \vee\{\hat{a}_{<x>}(r) \wedge \hat{b}_{<y>}(s) | z = rs\}.$$

If there exist $r \in< x >$ and $s \in< y >$ such that $z = r.s$, then $(\hat{a}_{<x>}.\hat{b}_{<y>})(z) = \hat{a} \wedge \hat{b}$. Since $r \in< x >$ and $s \in< y >$, $z \in< x >< y >$. By lemma 176 $< x >< y >\subseteq< xy >$, so $z \in< xy >$ and hence $(\hat{a} \wedge \hat{b})_{<xy>}(z) = \hat{a} \wedge \hat{b}$. If for each $r \in< x >$ and $s \in< y >$, $z \neq r.s$, then $(\hat{a}_{<x>}.\hat{b}_{<y>})(z) = \hat{0}$. Hence $(\hat{a}_{<x>}.\hat{b}_{<y>})(z) \subseteq< (\hat{a} \wedge \hat{b})_{xy} >$. ■

Theorem 186 *If $\hat{p}$ is an n-dimensional fuzzy prime hyperideal of $\mathcal{R}$, then $\hat{p}_*$ is a prime hyperideal of $\mathcal{R}$.*

Proof. Let x, $y \in \mathcal{R}$ and $x.y \in \hat{p}_*$. Then, $< xy >\subseteq \hat{p}_*$. Let $\hat{\mu} = \hat{1}_{<x>}$, $\hat{v} = \hat{1}_{<y>}$. By Corollary 181, $\hat{\mu} =< \hat{1}_x >, \hat{v} =< \hat{1}_y >$ and $\widehat{\mu v} \subseteq< \hat{1}_{xy} >$, again by theorem 180, $< \hat{1}_{xy} >= \hat{1}_{<xy>}$. Therefore $\widehat{\mu v} \subseteq \hat{1}_{<xy>} \subseteq \hat{1}_{\hat{p}_*} \subseteq \hat{p}$. Since $\hat{p}$ is an n-dimensional fuzzy prime hyperideal either $\hat{\mu} \subseteq \hat{p}$ or $\widehat{v} \subseteq \hat{p}$. So either $\hat{1}_{<x>} \subseteq \hat{p}$ or $\hat{1}_{<y>} \subseteq \hat{p} \Longrightarrow$ either $< x >\subseteq \hat{p}_*$ or $< y >\subseteq \hat{p}_*$ and finally either $x \in \hat{p}_*$ or $y \in \hat{p}_*$. Hence $\hat{p}_*$ is a prime. ■

Theorem 187 *Suppose that $\hat{p}$ is an n-dimensional fuzzy hyperideal of $\mathcal{R}$. Then $\hat{p}$ is an n-dimensional fuzzy prime hyperideal of $\mathcal{R}$ if and only if $\hat{p}(0) = \hat{1}$, $\hat{p}_*$ is a prime hyperideal of $\mathcal{R}$ and $\hat{p} = \hat{1}_{\hat{p}_*} \cup \hat{t}_{\mathcal{R}}$ for some $\hat{t} \in I_n$.*

Proof. Suppose that $\hat{p}$ is an n-dimensional fuzzy prime hyperideal. By theorem 186 $\hat{p}_*$ is a prime hyperideal of $\mathcal{R}$. Now we show that $\hat{p}(0) = \hat{1}$. Suppose that $\hat{p}(0) < \hat{1}$, since $\hat{p}$ is non-constant, then there exist $x \in \mathcal{R}$ such that $\hat{p}(x) < \hat{p}(0)$. Let $\hat{\mu}$, $\hat{v}$ are n-dimensional fuzzy sets of $\mathcal{R}$ and defined by

$$\hat{\mu}(x) = \begin{cases} \hat{1} = (1, 1, 1, ..., 1) & x \in \hat{p}_* \\ \hat{0} = (0, 0, 0, ..., 0) & x \notin \hat{p}_* \end{cases} \text{ and } \hat{v}(x) = \hat{p}(0).$$

So $\hat{\mu}$, $\hat{v}$ are n-dimensional fuzzy hyperideals of $\mathcal{R}$ and $\hat{\mu}$, $\hat{v} \subseteq \hat{p}$. But $\hat{\mu}(0) = \hat{1} > \hat{p}(0)$ and for $x \in \mathcal{R}$. If $x \notin \hat{p}_*$, $\hat{v}(x) = \hat{p}(0) > \hat{p}(x)$, so $\hat{\mu} \nsubseteq \hat{p}$, $\hat{v} \nsubseteq \hat{p}$ which is a contradiction. Therefore $\hat{p}(0) = \hat{1}$. Since $\hat{1} \subseteq \hat{p}(\mathcal{R})$ and $\hat{p}$ is non-constant, $\mid \hat{p}(\mathcal{R}) \mid\geq 2$. Let $x, y \in \mathcal{R}\backslash\hat{p}_*$. We shall show that $\hat{p}(x) = \hat{p}(y)$. Let $\hat{p}(x) = \hat{t}$. Then by Corollary 181 $\hat{t}_{<\hat{a}>} =< \hat{t}_{\hat{a}} >\subseteq \hat{p}$. Now $\hat{1}_{<x>}, \hat{t}_{\mathcal{R}}$ are n-dimensional fuzzy hyperideals of $\mathcal{R}$ and by Corollary 183, we have

$$\hat{1}_{<x>}, \hat{t}_{\mathcal{R}} \subseteq \hat{1}_{<x>} \cap \hat{t}_{\mathcal{R}} = \hat{t}_{<\hat{a}>} \subseteq \hat{p}.$$

and $\hat{1}_{<x>} \nsubseteq \hat{p}$. Since $\hat{p}$ is an n-dimensional fuzzy prime hyperideal of $\mathcal{R}$, $\hat{t}_{\mathcal{R}} \subseteq \hat{p}$. Thus $\hat{p}(x) = \hat{t} = \hat{t}_{\mathcal{R}}(y) \leq \hat{p}(y)$. Similarly it can be shown that $\hat{p}(y) \leq \hat{p}(x)$. Hence $\hat{p}(x) = \hat{p}(y)$. This means that $\mid \hat{p}(\mathcal{R}) \mid \geq \hat{2}$. Therefore, $\hat{p} = \hat{1}_{\hat{p}_*} \cup \hat{t}_{\mathcal{R},}$ where $\hat{p}_*$ is an n-dimensional fuzzy prime hyperideal of $\mathcal{R}$ and $\hat{t} \in I_n$.

Conversely, Suppose that $\hat{p}(0) = \hat{1}$, $\hat{p}_*$ is a prime hyperideal of $\mathcal{R}$ and $\hat{p} = \hat{1}_{\hat{p}_*} \cup \hat{t}_{\mathcal{R},}$ for $\hat{t} \in I_n$. Since $\hat{p}(\mathcal{R}) = \{\hat{1}, \hat{t}\}$ then $\hat{p}$ is non-constant. Let for $\hat{\mu}$, $\hat{v}$ be an n-dimensional fuzzy hyperideals of $\mathcal{R}$, $\hat{\mu}\hat{v} \subseteq \hat{p}$ but $\hat{\mu} \nsubseteq \hat{p}$ and $\hat{v} \nsubseteq \hat{p}$. Then there exist $x, y \in \mathcal{R}$, such that $\hat{\mu}(x) > \hat{p}(x)$ and $\hat{v}(y) > \hat{p}(y)$. It means that $x, y \in \mathcal{R} \backslash \hat{p}_*$, then $\hat{p}(x) = \hat{p}(y) = \hat{t}$. Thus $\hat{\mu}(x) > \hat{t}$ and $\hat{v}(y) > \hat{t}$. Since $x, y \in \mathcal{R} \backslash \hat{p}_*$ and $\hat{p}_*$ is a prime hyperideal of $\mathcal{R}$. Therefore there exists $s = xy$ such that $s \in \hat{p}_*$ and then $\hat{p}(s) = \hat{t}$. Now since $s = xy$ then :

$$\hat{t} = \hat{p}(s) \geq (\hat{\mu}\hat{v})(s) \geq \hat{\mu}(x) \wedge \hat{v}(y).$$

Hence $\hat{\mu}(x) > \hat{t}$ or $\hat{v}(y) > \hat{t}$, which is a contradiction. Thus $\hat{p}$ is an n-dimensional fuzzy prime hyperideal of $\mathcal{R}$. ■

6.3 Topology on n-dimensional fuzzy spectrum of $\mathcal{R}$

Let $\mathcal{R}$ be a commutative semihyperring. We mean by an n-dimensional fuzzy spectrum of $\mathcal{R}$, the set of all n-dimensional fuzzy prime hyperideals of $\mathcal{R}$. Let $X = \{n$-dimensional fuzzy spectrum of $\mathcal{R}\}$. If $\hat{\mu}$ is an n-dimensional fuzzy hyperideal of $\mathcal{R}$, then we define $V(\hat{\mu}) = \{p \in X \mid \hat{\mu} \subseteq p\}$, $E(\hat{\mu}) = X \backslash V(\hat{\mu})$.

Lemma 188 *Let $\hat{\mu}$ be an n-dimensional fuzzy set of $\mathcal{R}$. Then, $V(< \hat{\mu} >) = V(\hat{\mu})$.*

Proof. The proof is straight forward ■

Let $T = \{E(\hat{\mu}) | \hat{\mu} \in n$-dimensional fuzzy hyperideals of $\mathcal{R}\}$. Next we will prove that the pair (X, T) is a topological space.

Theorem 189 *The pair (X, T) is a topological space, where $T = \{E(\hat{\mu}) | \hat{\mu} \in n$-dimensional fuzzy hyperideal of $\mathcal{R}\}$.*

Proof. (1) Let $\hat{\mu} = 1_{\mathcal{R}}$ and $\hat{\nu} = 0_{\mathcal{R}}$. Then $V(\hat{\mu}) = \phi$ and $V(\hat{\mu}) = X$, therefore $E(\hat{\mu}) = X$ and $E(\hat{\nu}) = \phi$. therefore X, $\phi \in T$.

(2) Suppose that $\hat{\mu}_1$, $\hat{\mu}_2$ are n-dimensional fuzzy hyperideals of $\mathcal{R}$, we have to show

$$E(\hat{\mu}_1) \cap E(\hat{\mu}_2) = E(\hat{\mu}_1 \cap \hat{\mu}_2).$$

Let $\hat{\nu} \in E(\hat{\mu}_1) \cap E(\hat{\mu}_2)$, then $\hat{\mu}_1 \nsubseteq \hat{\nu}$and $\hat{\mu}_2 \nsubseteq \hat{\nu}$. Since $\hat{\nu}$ is an n-dimensional fuzzy prime hyperideal, $\hat{\mu}_1\hat{\mu}_2 \nsubseteq \hat{\nu}$. Using Corollary 183, $\hat{\mu}_1\hat{\mu}_2 \subseteq \hat{\mu}_1 \cap \hat{\mu}_2$. Therefore $\hat{\mu}_1 \cap \hat{\mu}_2 \nsubseteq \hat{\nu}$ and hence $\hat{\nu} \in E(\hat{\mu}_1 \cap \hat{\mu}_2)$. Thus

$$E(\hat{\mu}_1) \cap E(\hat{\mu}_2) \subseteq E(\hat{\mu}_1 \cap \hat{\mu}_2) \qquad (1)$$

Now let $\hat{\nu} \in E(\hat{\mu}_1 \cap \hat{\mu}_2)$, then $\hat{\mu}_1 \cap \hat{\mu}_2 \nsubseteq \hat{\nu}$, so $\hat{\mu}_1 \nsubseteq \hat{\nu}$and $\hat{\mu}_2 \nsubseteq \hat{\nu}$. Hence $\hat{\nu} \in E(\hat{\mu}_1)$ and $\hat{\nu} \in E(\hat{\mu}_2)$. Thus $\hat{\nu} \in E(\hat{\mu}_1) \cap E(\hat{\mu}_2)$, so

$$E(\hat{\mu}_1 \cap \hat{\mu}_2) \subseteq E(\hat{\mu}_1) \cap E(\hat{\mu}_2) \qquad (2)$$

From (1) and (2) it follows

$$E(\hat{\mu}_1) \cap E(\hat{\mu}_2) = E(\hat{\mu}_1 \cap \hat{\mu}_2).$$

(3) Suppose that $\{\hat{\mu}_i | i \in I\}$ is a family of n-dimensional fuzzy hyperideals of $\mathcal{R}$. We show that

$$\underset{i\in I}{\cup} E(\hat{\mu}_i) = E(< \underset{i\in I}{\cup} \hat{\mu}_i >).$$

Let $\hat{\mu} \notin \underset{i\in I}{\cup} E(\hat{\mu}_i)$, then for each $i \in I$, $\hat{\mu} \notin E(\hat{\mu}_i)$ and for each $i \in I$, $\hat{\mu}_i \subseteq \hat{\mu}$. Then $\underset{i\in I}{\cup} \hat{\mu}_i \subseteq \hat{\mu}$. Now we have

$$< \underset{i\in I}{\cup} \hat{\mu}_i > \subseteq \mu \Longrightarrow \mu \in V(< \underset{i\in I}{\cup} \hat{\mu}_i >) \Longrightarrow \hat{\mu} \notin E(< \underset{i\in I}{\cup} \hat{\mu}_i >).$$

It means that

$$E(< \underset{i\in I}{\cup} \hat{\mu}_i >) \subseteq \underset{i\in I}{\cup} E(\hat{\mu}_i)$$

Let $\hat{\mu} \notin E(< \underset{i\in I}{\cup} \hat{\mu}_i >)$, then for $\hat{\mu} \in V(< \underset{i\in I}{\cup} \hat{\mu}_i >)$ therefore $< \underset{i\in I}{\cup} \hat{\mu}_i > \subseteq \hat{\mu}$. For each $i \in I$, $\hat{\mu}_i \subseteq \underset{i\in I}{\cup} \hat{\mu}_i = < \underset{i\in I}{\cup} \hat{\mu}_i > \subseteq \hat{\mu} \Longrightarrow$ for each $i \in I$, $\hat{\mu}_i \subseteq \hat{\mu}$. Thus

$$\mu \in V(\hat{\mu}_i), \text{ for all } i \in I \Longrightarrow \hat{\mu} \notin \underset{i\in I}{\cup} E(\hat{\mu}_i).$$

Thus

$$\underset{i\in I}{\cup} E(\hat{\mu}_i) \subseteq E(< \underset{i\in I}{\cup} \hat{\mu}_i >).$$

Using (1) and (2)

$$\underset{i\in I}{\cup} E(\hat{\mu}_i) = E(< \underset{i\in I}{\cup} \hat{\mu}_i >).$$

Hence the pair (X, T) is a topological space. ■

Theorem 190 *Let $x, y \in \mathcal{R}$ and $\hat{a}, \hat{b} \in I_n$. Then,*

$$E(x_{\hat{a}}) \cap E(y_{\hat{b}}) = \underset{t=xy}{\cup} (\hat{t}_{(\hat{a}\wedge\hat{b})}).$$

Proof. Suppose that $\hat{\mu} \in E(x_{\hat{a}}) \cap E(y_{\hat{b}})$ then $\hat{\mu} \in E(x_{\hat{a}}) \cap E(y_{\hat{b}})$ then $\hat{\mu} \in E(x_{\hat{a}})$ and $\hat{\mu} \in E(y_{\hat{b}}) \Longrightarrow x_{\hat{a}} \nsubseteq \hat{\mu}$ and $y_{\hat{b}} \nsubseteq \hat{\mu} \Longrightarrow \hat{a} > \hat{\mu}(x)$ and $\hat{b} > \hat{\mu}(y)$. Thus $\hat{\mu}(x) \neq \hat{1}$and $\hat{\mu}(y) \neq \hat{1}$. Since $\hat{\mu}$ is an n-dimensional fuzzy prime hyperideal and $|\operatorname{Im}(u)| = 2$ then $\hat{\mu}(x) = \hat{\mu}(y)$ and $x, y \notin \hat{\mu}_*$. But $\hat{\mu}_*$ is a prime hyperideal and hence $xy \notin \hat{\mu}_*$, then there exists $t = xy$, such that $t \notin \hat{\mu}_*$, hence $\hat{\mu}(t) = \hat{\mu}(x) = \hat{\mu}(y)$ and we have

$$\hat{\mu}(t) < \hat{a}, \hat{b} \Longrightarrow \hat{\mu}(t) < \hat{a} \wedge \hat{b} \Longrightarrow \hat{t}_{\hat{a}\wedge\hat{b}} \nsubseteq \hat{\mu} \Longrightarrow \hat{\mu} \in E(\hat{t}_{(\hat{a}\wedge\hat{b})}) \Longrightarrow \hat{\mu} \in \underset{i\in I}{\cup} (\hat{t}_{(\hat{a}\wedge\hat{b})}).$$

Thus

$$E(x_{\hat{a}}) \cap E(y_{\hat{b}}) \subseteq \underset{t=xy}{\cup} (\hat{t}_{(\hat{a}\wedge\hat{b})}) \tag{1}$$

Now suppose that $\hat{\mu} \in \underset{i\in I}{\cup} E(\hat{t}_{(\hat{a}\wedge\hat{b})})$, then

$$\hat{t}_{(\hat{a}\wedge\hat{b})} \nsubseteq \hat{\mu} \Longrightarrow \hat{\mu}(t) < \hat{a} \wedge \hat{b} \leq \hat{a}, \hat{b}.$$

But $\hat{\mu}(t) \neq \hat{1}$ and so $t \notin \hat{\mu}_*$. Now we have

$$\hat{a}, \hat{b} > \hat{\mu}(t) \geq \underset{t\in xy}{\vee} \hat{\mu}(t) \geq \hat{\mu}(x) \vee \hat{\mu}(y) \geq \hat{\mu}(x), \hat{\mu}(y).$$

So

$$\hat{\mu}(x) < \hat{a},\ \hat{\mu}(y) < \hat{b} \Longrightarrow x_{\hat{a}} \nsubseteq \hat{\mu} \text{ and } y_{\hat{b}} \nsubseteq \hat{\mu} \Longrightarrow \hat{\mu} \in E(x_{\hat{a}}),\ E(y_{\hat{b}}) \Longrightarrow \hat{\mu} \in E(x_{\hat{a}}) \cap E(y_{\hat{b}}).$$

Therefore,

$$\underset{i\in I}{\cup} E(\hat{t}_{(\hat{a}\wedge\hat{b})}) \subseteq E(x_{\hat{a}}) \cap E(y_{\hat{b}}) \qquad (2).$$

From (1) and (2), $\underset{i\in I}{\cup} E(\hat{t}_{(\hat{a}\wedge\hat{b})}) = E(x_{\hat{a}}) \cap E(y_{\hat{b}})$. ■

Theorem 191 *The set $B = \{E(x_{\hat{a}}) | x \in \mathcal{R},\ \hat{a} \in I_n,$ where $\hat{a} \neq \hat{0}\}$, forms a base for (X, T).*

Proof. Let $E(\hat{\mu})$ be an open set in T and let $\hat{\nu} \in E(\hat{\mu})$, then $\hat{\mu} \nsubseteq \hat{\nu}$ and so for some $x \in \mathcal{R}$, $\hat{\mu}(x) > \hat{\nu}(y)$. Letting $\alpha = \hat{\mu}(x)$ then $x_{\hat{a}} \nsubseteq \hat{\nu}$, therefore $\hat{\nu} \in E(x_{\hat{a}})$. Now we show that $V(\hat{\mu}) \subseteq V(x_{\hat{a}})$. Let $\hat{\theta} \in V(\hat{\mu})$, then $\hat{\mu} \subseteq \hat{\theta} \Longrightarrow x_{\hat{a}}(x) = \hat{\mu}(x) \leq \hat{\theta}(x) \Longrightarrow x_{\hat{a}} \subseteq \hat{\theta} \Longrightarrow \hat{\theta} \in V(x_{\hat{a}})$. Hence $V(\hat{\mu}) \subseteq V(x_{\hat{a}})$ and so $E(x_{\hat{a}}) \subseteq E(\hat{\mu})$. Thus $\hat{\nu} \in E(x_{\hat{a}}) \subseteq E(\hat{\mu})$. It means B is a base for (X, T). ■

Lemma 192 *Let $\hat{a}, \hat{b} \in I_n$, where $\hat{a}, \hat{b} \neq \hat{0}$ and $\hat{a} \leq \hat{b}$. Then $E(x_{\hat{a}}) \subseteq E(x_{\hat{b}})$ for $x \in \mathcal{R}$.*

Proof. Suppose that $\hat{\mu} \in E(x_{\hat{a}})$ then

$$x_{\hat{a}} \nsubseteq \hat{\mu} \Longrightarrow \hat{a} > \hat{\mu}(x).$$

But $\hat{b} \geq \hat{a}$ then $\hat{b} > \hat{\mu}(x)$ and hence

$$x_{\hat{b}} \nsubseteq \hat{\mu} \Longrightarrow \hat{\mu} \in E(x_{\hat{b}}).$$

Therefore $E(x_{\hat{a}}) \subseteq E(x_{\hat{b}})$. ■

Lemma 193 *Let $k \subseteq I_n$ where $k \notin \hat{0}$ and let $X = \cup\{E((x_i)_{\hat{t}}) | i \in I, \hat{t} \in k,\ x_i \in \mathcal{R}\}$. Then $\vee\{t | t \in k\} = 1$.*

Proof. The proof is same as in [50]. ■

Theorem 194 *The topological space* (X, T) *is compact.*

Proof. Since the set $B = \{E(x_{\hat{a}}) | x \in \mathcal{R},\ \hat{a} \in I_n$, where $\hat{a} \neq \hat{0}\}$ is a base for (X, T), we assume that the set $\{E((x_i)_{\hat{t}}) | i \in I,\ \hat{t} \in k \subseteq I_n$, where $k \notin \hat{0},\ x_i \in \mathcal{R}\}$ is a cover for X. Let $\hat{\alpha} = \vee\{\hat{t} | \hat{t} \in k\}$. By Lemma 192, $\hat{\alpha} = \hat{1}$ and by Lemma 193, the set $\{E((x_i)_{\hat{t}}) | i \in I\}$ is a covering X. Now we know $\{E((x_i)_{\hat{1}}) | i \in I,\ X = \underset{i\in I}{\cup} E((x_i)_{\hat{1}}) = E(< \underset{i\in I}{\cup} x_i >) = X \backslash V(< \underset{i\in I}{\cup} (x_i)_{\hat{1}} >)$. On the other hand we have $V(< \underset{i\in I}{\cup} (x_i)_{\hat{1}} >) = V(\underset{i\in I}{\cup} (x_i)_{\hat{1}})$, so $X = X \backslash V(< \underset{i\in I}{\cup} (x_i)_{\hat{1}} >)$, and hence $V(\underset{i\in I}{\cup} (x_i)_{\hat{1}}) = \phi$. Let $\hat{P}$ be any n-dimensional fuzzy prime hyperideal of $\mathcal{R}$ and let

$$\hat{\mu}(x) = \begin{cases} \hat{1} & x \in \hat{p} \\ \hat{0} & x \notin \hat{p} \end{cases}$$

Clearly $\hat{\mu}$ is an n-dimensional fuzzy prime hyperideal of $\mathcal{R}$ and $\hat{\mu} \in V(\underset{i\in I}{\cup} (x_i)_{\hat{1}})$, then $\underset{i\in I}{\cup} (x_i)_{\hat{1}} \nsubseteq \hat{\mu}$, so there exists $j \in I$, such that $(x_j)_i \nsubseteq \hat{\mu}$. Therefore $\hat{\mu}(x_j) < \hat{1}$ and hence $x_j \notin p$.Thus there is no any n-dimensional fuzzy prime hyperideal consisting the set $\{x_i | i \in I\}$ and then there is no n-dimensional hyperideal consisting the set $\{x_i | i \in I\}$, otherwise $I \subseteq m$ for some maximal n-dimensional hyperideal m and so m is prime. Which is contradiction, hence $< \{x_i | i \in I\} >= \mathcal{R}$. Since $\hat{1}_{\mathcal{R}} \in \mathcal{R}$, then $\hat{1}_{\mathcal{R}} \in \sum_{i=1}^{n} (r_i x_i)$ for $r_i \in \mathcal{R}$ and $n \in \mathbb{N}$. Now we show that $V(\underset{i\in I}{\cup} (x_i)_{\hat{1}}) = \phi$. Let $\hat{\mu} \in V(\underset{i\in I}{\cup} (x_i)_{\hat{1}})$. Then, $\underset{i\in I}{\cup} (x_i)_{\hat{1}} \subseteq \hat{\mu}$. So for each $i = 1, 2, ...$,n, $(x_i)_{\hat{1}} \subseteq \hat{\mu}$, therefore for each $i = 1, 2, ...$,n, $\hat{1} \leq \hat{\mu}(x_i)$ and so $\hat{\mu}(x_i) = \hat{1}$ for each $i = 1, 2, ...$,n. Thus for each $i = 1, 2, ...$,n, $x_i \in \hat{\mu}_*$ and hence $\sum_{i=1}^{n} r_i x_i \subseteq \hat{\mu}_*$, therefore $\hat{1}_{\mathcal{R}} \in \hat{\mu}_*$, which is a contradiction. Thus, $V(\overset{n}{\underset{i=1}{\cup}} (x_i)_1) = \phi$ and so $X = X \backslash V(\overset{n}{\underset{i=1}{\cup}} (x_i)_{\hat{1}}) = X \backslash V(< \overset{n}{\underset{i=1}{\cup}} (x_i)_{\hat{1}} >) = E(< \overset{n}{\underset{i=1}{\cup}} (x_i)_{\hat{1}} >) = \overset{n}{\underset{i=1}{\cup}} E((x_i)_{\hat{1}})$. This shows that X is compact. ■

6.4 Homomorphism of n-dimensional fuzzy k-hyperideals

In this section, we study and describe the behavior of n-dimensional fuzzy weak (strong) k-hyperideals of semihyperrings under homomorphism of semihyperrings. We give some different results of fuzzy weak (strong) k-hyperideals related to homomorphism of semihyperings. We also study some different characterization of fuzzy weak (strong) k-hyperideals of semihyperrings.

Definition 195 *Let* $\mathcal{R}$ *be a semihyperring and* $\hat{\mu}$ *an* n*-dimensional fuzzy set in* $\mathcal{R}$*. Then,* $\hat{\mu}$ *is* *said to be a weak n-dimensional fuzzy k-hyperideal of $\mathcal{R}$ iff $\forall\ r, x, y \in \mathcal{R}$, the following axioms hold:*

(i) $\hat{\mu}(z) \geq \hat{\mu}(x) \wedge \hat{\mu}(y),\ \forall\ z \in x \oplus y$,

(ii) $\hat{\mu}(rx) \geq \hat{\mu}(x)$ and $\hat{\mu}(xr) \geq \hat{\mu}(x)$,

(iii) $\hat{\mu}(x) \geq [(\inf_{u \in x+y} \hat{\mu}(u)) \vee (\inf_{v \in y+x} \hat{\mu}(v))] \wedge \hat{\mu}(y),\ \forall\ x, y \in \mathcal{R}$.

And a strong n-dimensional fuzzy k-hyperideal of $\mathcal{R}$ iff

(i) $\hat{\mu}(z) \geq \hat{\mu}(x) \wedge \hat{\mu}(y),\ \forall\ z \in x \oplus y$,

(ii) $\hat{\mu}(rx) \geq \hat{\mu}(x)$ and $\hat{\mu}(xr) \geq \hat{\mu}(x)$,

(iii) $\hat{\mu}(x) \geq [\hat{\mu}(z) \vee \hat{\mu}(z')] \wedge \hat{\mu}(y),\ \forall\ z \in x \oplus y$ and $z' \in y \oplus x$.

Note that, $(\mathcal{R}, +)$ is commutative semihypergroup, therefore above conditions of weak and strong n-dimensional fuzzy k-hyperideal of $\mathcal{R}$ are reduced to the following conditions:

(i) $\hat{\mu}(z) \geq \hat{\mu}(x) \wedge \hat{\mu}(y),\ \forall\ z \in x \oplus y$,

(ii) $\hat{\mu}(rx) \geq \hat{\mu}(x)$ and $\hat{\mu}(xr) \geq \hat{\mu}(x)$,

(iii) $\hat{\mu}(x) \geq \hat{\mu}(z) \wedge \hat{\mu}(y),\ \forall\ z \in x \oplus y$.

Proposition 196 *Let $\hat{\mu}$ be an n-dimensional fuzzy set in a semihyperring $\mathcal{R}$. Then,*

(i) $\hat{\mu}$ is an n-dimensional fuzzy hyperideal of $\mathcal{R}$ if and only if for every $\hat{t} \in I_n$, the level subset $\hat{\mu}_{\hat{t}} (\neq \Phi)$ is a hyperideal of $\mathcal{R}$, where $\hat{\mu}_{\hat{t}} = \{x \in \mathcal{R} |\ \hat{\mu}(x) \geq \hat{t}\}$.

(ii) $\hat{\mu}$ is a weak n-dimensional fuzzy k-hyperideal of $\mathcal{R}$ if and only if for every $\hat{t} \in I_n$, the level subset $\hat{\mu}_{\hat{t}} (\neq \Phi)$ is a weak k-hyperideal of $\mathcal{R}$, where $\hat{\mu}_{\hat{t}} = \{x \in \mathcal{R} |\ \hat{\mu}(x) \geq \hat{t}\}$.

(iii) $\hat{\mu}$ is a strong n-dimensional fuzzy k-hyperideal of $\mathcal{R}$ if and only if for every $\hat{t} \in I_n$, the level subset $\hat{\mu}_{\hat{t}} (\neq \Phi)$ is a strong k-hyperideal of $\mathcal{R}$, where $\hat{\mu}_{\hat{t}} = \{x \in \mathcal{R} |\ \hat{\mu}(x) \geq \hat{t}\}$.

Lemma 197 *Let $\hat{\mu}$ be an n-dimensional fuzzy hyperideal of a semihyperring $\mathcal{R}$. If $\mathcal{R}$ has zero element, then $\hat{\mu}(0) \geq \hat{\mu}(x),\ \forall\ x \in \mathcal{R}$.*

Definition 198 *[23] Let $\mathcal{R}$ and S be semihyperrings. A mapping $f : \mathcal{R} \to S$ is said to be*

(i) homomorphism if and only if $f(x \oplus y) \subseteq f(x) \oplus f(y)$ and $f(xy) = f(x)f(y)\ \forall\ x, y \in \mathcal{R}$.

(ii) good homomorphism if and only if $f(x \oplus y) = f(x) \oplus f(y)$ and $f(xy) = f(x)f(y)\ \forall\ x, y \in \mathcal{R}$.

Proposition 199 *Let $\hat{f} : \mathcal{R} \to \mathcal{R}'$ be a homomorphism of semihyperrings. If $\hat{\nu}$ is an n-dimensional strong fuzzy k-hyperideal of $\mathcal{R}'$, then $\hat{f}^{-1}(\hat{v})$ is an n-dimensional strong fuzzy k-hyperideal of $\mathcal{R}$.*

Proof. We know that $\hat{f}^{-1}(\hat{v})(x) = (\hat{v})(\hat{f}(x))$. Let $x, y \in \mathcal{R}$ and $z \in x \oplus y$, then we have

$$\hat{f}(z) \in \hat{f}(x \oplus y) \subseteq \hat{f}(x) + \hat{f}(y),$$

and since $\hat{\nu}$ is an n-dimensional strong fuzzy k-hyperideal of $\mathcal{R}'$, it implies, for all $z \in x \oplus y$

$$\hat{v}(\hat{f}(z)) \geq \hat{v}(\hat{f}(x)) \wedge \hat{v}(\hat{f}(y)).$$

Also

$$\hat{v}(\hat{f}(xy)) \geq \hat{v}(\hat{f}(x)\hat{f}(y)) \geq \hat{v}(\hat{f}(x)) \wedge \hat{v}(\hat{f}(y)).$$

Therefore $\hat{f}^{-1}(\hat{v})$ is an n-dimensional fuzzy hyperideal of $\mathcal{R}$. Now let $z \in x \oplus y$ and $z' \in x \oplus y$, thus $\hat{f}(z) \in \hat{f}(x) \oplus \hat{f}(y)$ and $\hat{f}(z') \in \hat{f}(y) \oplus \hat{f}(x)$, then $\hat{\nu}$ is an n-dimensional strong fuzzy k-hyperideal of $\mathcal{R}'$ implies that

$$\hat{v}(\hat{f}(x)) \geq [\hat{v}(\hat{f}(z)) \vee \hat{v}(\hat{f}(z'))] \wedge \hat{v}(\hat{f}(y)).$$

Hence proved. ■

Proposition 200 *Let $\hat{f} : \mathcal{R} \to \mathcal{R}'$ be a good homomorphism of semihyperrings. If $\hat{\nu}$ is an n-dimensional weak fuzzy k-hyperideal of $\mathcal{R}'$, then $\hat{f}^{-1}(\hat{v})$ is an n-dimensional weak fuzzy k-hyperideal of $\mathcal{R}$.*

Proof. We know that $\hat{f}^{-1}(\hat{v})(x) = (\hat{v})(\hat{f}(x))$. First we prove that $\hat{f}^{-1}(\hat{v})$ is an n-dimensional fuzzy hyperideal of $\mathcal{R}$. Let $x, y \in \mathcal{R}$ and $z \in x \oplus y$, since $\hat{\nu}$ is an n-dimensional weak fuzzy k-hyperideal of $\mathcal{R}'$ and $\hat{f}$ is a good homomorphism, then for all $z \in x \oplus y$

$$\hat{f}(z) \in \hat{f}(x \oplus y) = \hat{f}(x) \oplus \hat{f}(y).$$

Then, we have

$$\hat{v}(\hat{f}(z)) \geq \hat{v}(\hat{f}(x)) \wedge \hat{v}(\hat{f}(y)).$$

Also,

$$\hat{v}(\hat{f}(xy)) \geq \hat{v}(\hat{f}(x)\hat{f}(y)) \geq \hat{v}(\hat{f}(x)) \wedge \hat{v}(\hat{f}(y)).$$

Therefore $\hat{f}^{-1}(\hat{v})$ is an n-dimensional weak fuzzy k-hyperideal of $\mathcal{R}$. As $\hat{f}$ is a good homomorphism, then for all $z \in x \oplus y$ if and only if $\hat{f}(z) \in \hat{f}(x \oplus y) = \hat{f}(x) \oplus \hat{f}(y)$ and also $\hat{\nu}$ is an n-dimensional weak fuzzy k-hyperideal of $\mathcal{R}'$, we have

$$\hat{v}(\hat{f}(x)) \geq [(\bigwedge_{\hat{f}(z)\in\hat{f}(x)\oplus\hat{f}(y)} (\hat{v})\hat{f}(z)) \vee (\bigwedge_{\hat{f}(z')\in\hat{f}(y)\oplus\hat{f}(x)} (\hat{v})\hat{f}(z'))] \wedge (\hat{v})\hat{f}(y)$$

Now, we prove that $\hat{f}^{-1}(\hat{v})$ is an n-dimensional fuzzy hyperideal of $\mathcal{R}$. That is,

$$\hat{f}^{-1}(\hat{v})(x)) \geq [(\bigwedge_{z\in x\oplus y} \hat{f}^{-1}(\hat{v})(z)) \vee (\bigwedge_{z'\in y\oplus x} \hat{f}^{-1}(\hat{v})(z'))] \wedge \hat{f}^{-1}(\hat{v})(y).$$

Hence complete the proof. ■

Proposition 201 *Let $\hat{f} : \mathcal{R} \to \mathcal{R}'$ be a good homomorphism of semihyperrings. If $\hat{\mu}$ is an n-dimensional weak (resp. strong) fuzzy k-hyperideal of $\mathcal{R}$ and $\hat{\mu}$ be a $\hat{f}$-invariant, then $\hat{f}(\hat{\mu})$ is an n-dimensional weak (resp. strong) fuzzy k-hyperideal of $\mathcal{R}'$.*

Proof. First we show that $\hat{f}(\hat{\mu})$ is an n-dimensional fuzzy hyperideal of $\mathcal{R}'$. Let $a, b \in \mathcal{R}$ and $c \in a + b$, we should prove that

$$\hat{f}(\hat{\mu}(c)) \geq \hat{f}(\hat{\mu}(a)) \oplus \hat{f}(\hat{\mu}(b)).$$

We have

$$\hat{f}(\hat{\mu}(c)) \geq \bigvee_{z\in\hat{f}^{-1}(c)} \hat{\mu}(z),$$

$$\hat{f}(\hat{\mu}(a)) \geq \bigvee_{x\in\hat{f}^{-1}(a)} \hat{\mu}(x),$$

$$\hat{f}(\hat{\mu}(b)) \geq \bigvee_{y\in\hat{f}^{-1}(b)} \hat{\mu}(y).$$

Since $\hat{\mu}$ be a $\hat{f}$-invariant, then

$$\exists\ z_0 \in \hat{f}^{-1}(c),\ \hat{f}(\hat{\mu}(c)) = \hat{\mu}(z_0),$$

$$\exists\ x_0 \in \hat{f}^{-1}(a),\ \hat{f}(\hat{\mu}(a)) = \hat{\mu}(x_0),$$

$$\exists\ y_0 \in \hat{f}^{-1}(b),\ \hat{f}(\hat{\mu}(b)) = \hat{\mu}(y_0).$$

Therefore, $\hat{f}(z_0) = c$, $\hat{f}(x_0) = a$, $\hat{f}(y_0) = b$

$$\Rightarrow \hat{f}(z_0) \in \hat{f}(x_0) \oplus \hat{f}(y_0),$$

$\Rightarrow z_0 \in x_0 \oplus y_0$ ($\hat{f}$ is a good homomorphism and $\hat{\mu}$ is an n-dimensional fuzzy hyperideal)

$$\Rightarrow \hat{\mu}(z_0) \in \hat{\mu}(x_0) \wedge \hat{\mu}(y_0),$$

$$\Rightarrow \hat{f}(\hat{\mu}(c)) \in \hat{f}(\hat{\mu}(a)) \wedge \hat{f}(\hat{\mu}(b)).$$

Now for the second condition of a fuzzy hyperideal, we have to prove that

$$\hat{f}(\hat{\mu})(r'x') \in \hat{f}(\hat{\mu})(x') \vee \hat{f}(\hat{\mu})(r'),\ \forall\ r', x' \in \mathcal{R}'.$$

Since $\hat{f}$ is onto, then $r' = \hat{f}(r)$ and $x' = \hat{f}(x)$ for some r and x in $\mathcal{R}$. Thus

$$\begin{aligned}
\hat{f}(\hat{\mu})(r'x') &= \bigvee_{rx \in \hat{f}^{-1}(r'x')} \hat{\mu}(rx) \\
&= \hat{\mu}(r_0 x_0), \exists r_0 \in \hat{f}^{-1}(r'), x_0 \in \hat{f}^{-1}(x'), \qquad (\hat{\mu} \text{ is a } \hat{f}\text{-invariant}) \\
&\geq \hat{\mu}(x_0) \vee \hat{\mu}(r_0), \qquad (\hat{\mu} \text{ is an } n\text{-dimensional fuzzy hyperideal of } \mathcal{R}) \\
&= \hat{f}(\hat{\mu})(x') \vee \hat{f}(\hat{\mu})(r') \qquad (\hat{\mu} \text{ is a } \hat{f}\text{-invariant}).
\end{aligned}$$

Therefore, we have

$$\hat{f}(\hat{\mu})(r'x') \geq \hat{f}(\hat{\mu})(x') \vee \hat{f}(\hat{\mu})(r').$$

Now we prove that $\hat{f}(\hat{\mu})$ is an n-dimensional weak fuzzy k-hyperideal of $\mathcal{R}'$. Let a, $b \in \mathcal{R}$, we show that

$$\hat{f}(\hat{\mu})(a) \in [(\bigwedge_{z\in a\oplus b} \hat{f}(\hat{\mu})(z)) \vee (\bigwedge_{z'\in b\oplus a} \hat{f}(\hat{\mu})(z'))] \wedge \hat{f}(\hat{\mu})(b), \tag{1}$$

Since $\hat{f}$ is onto and $\hat{\mu}$ is $\hat{f}$-invariant, then $\hat{f}(\hat{\mu})(a) = \hat{\mu}(x_0)$, $\hat{f}(\hat{\mu})(z) = \hat{\mu}(z_0)$, $\hat{f}(\hat{\mu})(z') = \hat{\mu}(z'_0)$, $\hat{f}(\hat{\mu})(b) = \hat{\mu}(y_0)$, where $z'_0 \in \hat{f}^{-1}(z')$, $z_0 \in \hat{f}^{-1}(z)$, $x_0 \in \hat{f}^{-1}(a)$, $y_0 \in \hat{f}^{-1}(b)$. Hence (1) reduced to the form

$$\hat{f}(\hat{\mu})(x_0) \in [(\bigwedge_{z\in a\oplus b} \hat{f}(\hat{\mu})(z_0)) \vee (\bigwedge_{z'\in b\oplus a} \hat{f}(\hat{\mu})(z'_0))] \wedge \hat{f}(\hat{\mu})(y_0), \tag{2}$$

On the other hand from above discussion and $\hat{f}$ is a good homomorphism $z \in a+b$ if and only if

$$\hat{f}(z_0) \in \hat{f}(x_0) \oplus \hat{f}(y_0) \Leftrightarrow z_0 \in x_0 \oplus y_0.$$

Similarly, $z' \in b+a$ if and only if $z'_0 \in x_0 \oplus y_0$. Therefore by (2), it is enough that we prove that

$$\hat{\mu}(x_0) \geq [(\bigwedge_{z_0\in x_0\oplus y_0} \hat{\mu}(z_0)) \vee (\bigwedge_{z'_0\in y_0\oplus x_0} \hat{\mu}(z'_0))] \wedge \hat{\mu}(y_0),$$

it is clearly from the last statement is true, since $\hat{\mu}$ is an n-dimensional weak fuzzy k-hyperideal of $\mathcal{R}$. This complete the proof. ■

In next part we define the quotient of fuzzy weak (strong) n-dimensional k-hyperideals by regular relation of semihyperring.

Definition 202 *Let $\mathcal{R}$ be a semihyperring and ρ be an equivalence relation on $\mathcal{R}$. Naturally we can extend ρ to $\bar{\rho}$ to the subsets of $\mathcal{R}$ as follow:*

For A, B be non-empty subsets of $\mathcal{R}$. Define $A\bar{\rho}B \Leftrightarrow \forall\ a \in A\ \exists\ b \in B : a\rho b$, $\forall\ b \in B\ \exists\ a \in A : b\rho a$.

An equivalence relation ρ on $\mathcal{R}$ is said to be regular if for all $a, b, x \in \mathcal{R}$, we have

(i) $a\rho b \Rightarrow (a + x)\bar{\rho}(b + x)$ and $(x + a)\bar{\rho}(x + b)$,

(ii) $a\rho b \Rightarrow (ax)\bar{\rho}(bx)$ and $(xa)\bar{\rho}(xb)$.

By $\mathcal{R} : \rho$ we mean the set of all equivalence classes with respect to ρ, that is

$$\mathcal{R} : \rho = \{r_\rho | r \in \mathcal{R}\}$$

Remark 203 *We know that if $\mathcal{R}$ is a semihyperring and ρ is a regular equivalence relation on $\mathcal{R}$, then $\mathcal{R} : \rho$ by a hyperoperation "$\oplus$" and a binary operation "$\cdot$" is defined as follow*

$$x_\rho \oplus y_\rho = \{x_\rho | z \in x \oplus y\}$$

$$x_\rho \cdot y_\rho = (xy)_\rho$$

is a semihyperring. For $\hat{\mu} \in FS(\mathcal{R})$, define $(\hat{\mu} : \rho)(x_\rho) = \bigvee_{y \in x_\rho} \hat{\mu}(y)$. Also we know that the mapping $\psi : \mathcal{R} \to \mathcal{R} : \rho$ define by $\psi(a) = a_\rho$ is a good epimorphism. Now if $\hat{\mu}$ is an n-dimensional weak fuzzy k-hyperideal of $\mathcal{R}$ and $\hat{\mu}$ be ψ-invariant then by proposition 201 it concludes that $\psi(a) = \hat{\mu} : \rho$ is an n-dimensional weak fuzzy k-hyperideal of $\mathcal{R} : \rho$.

Proposition 204 *If $\hat{\mu}$ is an n-dimensional weak fuzzy k-hyperideal of $\mathcal{R}$ and $\mathcal{R}$ has zero, then $\hat{\mu}_* = \{x \in \mathcal{R} | \hat{\mu}(x) = \hat{\mu}(0)\}$ is an n-dimensional weak k-hyperideal of $\mathcal{R}$.*

Proof. First we prove that $\hat{\mu}_*$ is an n-dimensional hyperideal of $\mathcal{R}$. For $x, y \in \hat{\mu}_*$ and $z \in x + y$, then $\hat{\mu}(z) \geq \hat{\mu}(x) \wedge \hat{\mu}(y) = \hat{\mu}(0)$, therefore by Lemma 197 $\hat{\mu}(z) = \hat{\mu}(0)$, hence $z \in \hat{\mu}_*$.

Let $r \in \mathcal{R}$ and $x \in \hat{\mu}_*$, then we have

$\hat{\mu}(rx) \geq \hat{\mu}(r) \vee \hat{\mu}(x)$

$= \hat{\mu}(r) \vee \hat{\mu}(0) \quad (x \in \hat{\mu}_*)$

$= \hat{\mu}(0) \quad$ (by Lemma 197)

$\Rightarrow rx \in \hat{\mu}_*$

Now suppose $r + x \subseteq \hat{\mu}_*$ or $x + r \subseteq \hat{\mu}_*$ and $x \in \hat{\mu}_*$, we show that $r \in \hat{\mu}_*$.

Since $\hat{\mu}$ is an n-dimensional weak fuzzy k-hyperideal of $\mathcal{R}$ then we have,

$$\hat{\mu}(r) \geq [(\bigwedge_{z \in r \oplus x} \hat{\mu}(z)) \vee (\bigwedge_{z' \in x \oplus r} \hat{\mu}(z'))] \wedge \hat{\mu}(x).$$

Since $\hat{\mu}(x) = \hat{\mu}(0)$ and $\bigwedge_{z \in r \oplus x} \hat{\mu}(z) = \hat{\mu}(0)$ and $\bigwedge_{z' \in r \oplus x} \hat{\mu}(z') = \hat{\mu}(0)$, then $\hat{\mu}(r) \geq \hat{\mu}(0)$ and then by Lemma 197, $\hat{\mu}(r) \geq \hat{\mu}(0)$. Therefore is an n-dimensional weak k-hyperideal of $\mathcal{R}$. ■

Proposition 205 *If $\hat{\mu}$ is an n-dimensional strong fuzzy k-hyperideal of $\mathcal{R}$ and $\mathcal{R}$ has zero, then $\hat{\mu}^* = \{x \in \mathcal{R} | \hat{\mu}(x) = \hat{\mu}(0)\}$ is an n-dimensional strong k-hyperideal of $\mathcal{R}$.*

Proof. First we prove that $\hat{\mu}^*$ is an n-dimensional hyperideal of $\mathcal{R}$. For $x, y \in \hat{\mu}^*$ and $z \in x + y$, then $\hat{\mu}(z) \geq \hat{\mu}(x) \wedge \hat{\mu}(y) > 0$, thus $z \in \hat{\mu}^*$.

If $r \in \mathcal{R}$ and $x \in \hat{\mu}^*$, then we have

$\hat{\mu}(rx) \geq \hat{\mu}(r) \vee \hat{\mu}(x) \geq \hat{\mu}(x) > 0$,

$\Rightarrow rx \in \hat{\mu}^*$. Similarly $\Rightarrow xr \in \hat{\mu}^*$. Thus $\hat{\mu}^*$ is an n-dimensional hyperideal of $\mathcal{R}$.

Now if $r + x \approx \hat{\mu}^*$ or $x + r \approx \hat{\mu}^*$ and $x \in \hat{\mu}^*$.

By hypothesis for all $z \in r + x \approx \hat{\mu}^*$ or $z' \in x + r \approx \hat{\mu}^*$ we have

$$\hat{\mu}(r) \geq [\hat{\mu}(z) \vee \hat{\mu}(z')] \wedge \hat{\mu}(x).$$

that is $r \in \hat{\mu}^*$and hence $\hat{\mu}^*$ is an n-dimensional strong k-hyperideal of $\mathcal{R}$. ■

Proposition 206 *Let $\mathcal{R}$ be a semihyperring with zero and $x, y \in \mathcal{R}$:*

(i) If $\hat{\mu}$ is an n-dimensional weak fuzzy k-hyperideal of $\mathcal{R}$ and $\hat{\mu}(z) = \hat{\mu}(0) = \hat{\mu}(z')$ for all $z \in x + y$ and $z' \in y + x$, then $\hat{\mu}(x) = \hat{\mu}(y)$.

(ii) If $\hat{\mu}$ is an n-dimensional strong fuzzy k-hyperideal of $\mathcal{R}$ and $\hat{\mu}(u) = \hat{\mu}(0) = \hat{\mu}(v)$ for all $u \in x + y$ and $v \in y + x$, then $\hat{\mu}(x) = \hat{\mu}(y)$.

Proof. (i) Since $\hat{\mu}$ is an n-dimensional weak fuzzy k-hyperideal of $\mathcal{R}$ and $\hat{\mu}(z) = \hat{\mu}(0) = \hat{\mu}(z')$ for all $z \in x + y$ and $z' \in y + x$, then $\underset{z \in x \oplus y}{\wedge} \hat{\mu}(z) = \hat{\mu}(0) = \underset{z \in y \oplus x}{\wedge} \hat{\mu}(z')$, thus

$$\begin{aligned} \hat{\mu}(x) &\geq [(\underset{z \in x \oplus y}{\wedge} \hat{\mu}(z)) \vee (\underset{z' \in y \oplus x}{\wedge} \hat{\mu}(z'))] \wedge \hat{\mu}(y) \\ &= \hat{\mu}(0) \wedge \hat{\mu}(y) \\ &= \hat{\mu}(y) \quad \text{(by Lemma 197)} \end{aligned}$$

$\Rightarrow \hat{\mu}(x) \geq \hat{\mu}(y)$.

Similarly, we conclude that $\hat{\mu}(y) \geq \hat{\mu}(x)$. Therefore $\hat{\mu}(x) = \hat{\mu}(y)$.

(ii) Suppose $u \in x + y$ and $v \in y + x$, such that $\hat{\mu}(u) = \hat{\mu}(o) = \hat{\mu}(o)$, since $\hat{\mu}$ is an n-dimensional strong fuzzy k-hyperideal of $\mathcal{R}$, then

$$\begin{aligned} \hat{\mu}(y) &\geq (\hat{\mu}(u) \vee \hat{\mu}(v)) \wedge \hat{\mu}(x) \\ &= \hat{\mu}(0) \wedge \hat{\mu}(x) \end{aligned}$$

$= \hat{\mu}(x) \quad$ (by Lemma 197)

$\Rightarrow \hat{\mu}(y) \geq \hat{\mu}(x)$.

Similarly, we obtain $\hat{\mu}(x) \geq \hat{\mu}(y)$. Therefore $\hat{\mu}(x) = \hat{\mu}(y)$. ■

6.5 n-Dimensional (α, β)-fuzzy hyperideals in semihyperrings

We combine the notion of n-dimensional fuzzy set and n-dimensional fuzzy point to introduce a new notion called n-dimensional (α, β)-fuzzy hyperideals in semihyperring. We also introduce the characterization of n-dimensional (α, β)-fuzzy hyperideals in semihyperrings by upper level set. More over we define n-dimensional prime (α, β)-fuzzy hyperideals in semihyperrings. In what follows, let R denote a semihyperring, and α, β denote any one of $\in, q, \in \vee q$ or $\in \wedge q$ unless otherwise specified.

An n-dimensional fuzzy subset $\widehat{\lambda}$ of the form

$$\widehat{\lambda}(y) = \begin{cases} \widehat{t} = (t_1, t_2, ...t_n) \in I_n & \text{if } y = x \\ \widehat{0} = (0, 0, ...0) & \text{if } y \neq x \end{cases}$$

is called an n-dimensional fuzzy point with support x and value $\widehat{t} = (t_1, t_2, ...t_n)$ and is denoted by $x_{\widehat{t}}$. An n-dimensional fuzzy point $x_{\widehat{t}}$ is said to belong to (resp. quasi-coincident with) an n-dimensional fuzzy subset $\widehat{\lambda}$, written $x_{\widehat{t}} \in \widehat{\lambda} \left(\text{resp. } x_t q \widehat{\lambda}\right)$ if $\widehat{\lambda}(x) \geq \widehat{t}$ i.e., $\widehat{\lambda}_i(x) \geq t_i$ for $i = 1, 2, ...n$ (resp. $\widehat{\lambda}(x) + \widehat{t} > \widehat{1}$ i.e., $\widehat{\lambda}_i(x) + t_i > 1$, for $i = 1, 2, ...$n). To say that $x_{\widehat{t}} \in \vee q \widehat{\lambda}$ (resp. $\in \wedge q$) means that $x_{\widehat{t}} \in \widehat{\widehat{\lambda}}$ or $x_{\widehat{t}} q \widehat{\widehat{\lambda}}$ (resp. $x_{\widehat{t}} \in \widehat{\lambda}$ and $x_{\widehat{t}} q \widehat{\lambda}$). To say that $x_{\widehat{t}} \overline{\alpha} \widehat{\lambda}$ means that $x_{\widehat{t}} \alpha \widehat{\lambda}$ does not hold.

Definition 207 *An n-dimensional fuzzy subset $\widehat{\lambda}$ of R is called an n-dimensional (α, β)-fuzzy hyperideal of R, where $\alpha \neq \in \wedge q$, if the following conditions satisfy*

1. $x_{\widehat{t}} \alpha \widehat{\lambda}, y_{\widehat{s}} \alpha \widehat{\lambda}$ implies $z_{\min(\widehat{t}, \widehat{s})} \beta \widehat{\lambda}$ for each $z \in x + y$

2. $x_{\widehat{t}} \alpha \widehat{\lambda}$ implies $(xy)_{\widehat{t}} \beta \widehat{\lambda}$ and $(yx)_{\widehat{t}} \beta \widehat{\lambda}$ for all $x, y \in$R and $t, s \in I_n$.

Fuzzy set is a special case of an n-dimensional fuzzy set as shown in [63]. The concept of an n-dimensional (α, β)-fuzzy hyperideal in a semihyperring a generalization of an (α,β)-fuzzy hyperideal in a semihyperring so an (α,β)-fuzzy hyperideal in a semihyperring is a special case of an n-dimensional (α, β)-fuzzy hyperideal in a semihyperring.

Theorem 208 *Let $\widehat{\lambda}$ be a non-zero n-dimensional (α, β)-fuzzy hyperideal of R. Then, the set $\widehat{\lambda}_0 = \left\{x \in \mathcal{R} : \widehat{\lambda}(x) > \widehat{0}\right\}$ is a hyperideal of R.*

Proof. Let $x, y \in \widehat{\lambda}_{\widehat{0}}$. Then, $\widehat{\lambda}(x) > \widehat{0}$ and $\widehat{\lambda}(y) > \widehat{0}$. Assume that $\widehat{\lambda}(z) = \widehat{0}$ for some $z \in x+y$. If $\alpha \in \{\in, \in \vee q\}$, then $x_{\widehat{\lambda}(x)}\alpha\widehat{\lambda}$ and $y_{\widehat{\lambda}(y)}\alpha\widehat{\lambda}$ but for some $z \in x+y$, $(z)_{\min\left(\widehat{\lambda}(x),\widehat{\lambda}(y)\right)}\overline{\beta}\widehat{\lambda}$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, a contradiction. Also, $x_{\widehat{1}}q\widehat{\lambda}$ and $y_{\widehat{1}}q\widehat{\lambda}$ but for some $z \in x+y$, $(z)_{\widehat{1}}\overline{\beta}\widehat{\lambda}$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$. Hence, for each $z \in x+y$, $\widehat{\lambda}(z) > \widehat{0}$, this implies, $z \in \widehat{\lambda}_0$, that is, $x + y \subseteq \widehat{\lambda}_0$

Let $x \in \widehat{\lambda}_{\widehat{0}}$ and $y \in$R. Suppose that $\widehat{\lambda}(xy) = \widehat{0}$ and let $\alpha \in \{\in, \in \vee q\}$. Then, $x_{\widehat{\lambda}(x)}\alpha\widehat{\lambda}$ but $(xy)_{\widehat{\lambda}(x)}\overline{\beta}\widehat{\lambda}$ and $(yx)_{\widehat{\lambda}(x)}\overline{\beta}\widehat{\lambda}$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Also, $x_{\widehat{1}}q\widehat{\lambda}$ but $(xy)_{\widehat{1}}\overline{\beta}\widehat{\lambda}$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which contradicts to our hypothesis. Therefore, $\widehat{\lambda}(xy) > \widehat{0}$ and so $xy \in \widehat{\lambda}_{\widehat{0}}$. Similarly, we can prove that $yx \in \widehat{\lambda}_{\widehat{0}}$. Hence, $\widehat{\lambda}_{\widehat{0}}$ is a hyperideal of R. ■

Definition 209 *An n-dimensional fuzzy subset $\widehat{\lambda}$ of R is called an n-dimensional (α, β)-fuzzy weak k-hyperideal of R, where $\alpha \neq \in \wedge q$, if the following conditions satisfy*

1. $\widehat{\lambda}$ is an n-dimensional (α, β)-fuzzy weak k-hyperideal of R.
2. If for each $u \in x+y$ or $v \in x+y$, $u_{\widehat{s_1}}\alpha\widehat{\lambda}$, $v_{\widehat{s_2}}\alpha\widehat{\lambda}$ and $y_{\widehat{s_3}}\alpha\widehat{\lambda}$, then $x_{\min\{\max\{\widehat{s_1},\ \widehat{s_2}\},\ \widehat{s_3}\}}\beta\widehat{\lambda}$ for all $u, v, x, y \in$ R and for all $\widehat{s_1}, \widehat{s_2}, \widehat{s_3}, \in I_n$.

Theorem 210 *Let $\widehat{\lambda}$ be a non-zero n-dimensional (α, β)-fuzzy k-hyperideal of R. Then, the set $\widehat{\lambda}_{\widehat{0}} = \left\{x \in \mathcal{R} : \widehat{\lambda}(x) > \widehat{0}\right\}$ is a k-hyperideal of R.*

Proof. By Theorem 208 $\widehat{\lambda}_{\widehat{0}}$ is a hyperideal of R. Next we show that if for $y \in \widehat{\lambda}_{\widehat{0}}$ and $x \in$R we have $x+y \subseteq \widehat{\lambda}_{\widehat{0}}$ or $x+y \subseteq \widehat{\lambda}_{\widehat{0}}$ this implies $x \in \widehat{\lambda}_{\widehat{0}}$. Let $y \in \widehat{\lambda}_{\widehat{0}}$ and $x \in$R such that $x+y \subseteq \widehat{\lambda}_{\widehat{0}}$ or $x+y \subseteq \widehat{\lambda}_{\widehat{0}}$. Then, $\widehat{\lambda}(y) > \widehat{0}$, $\inf_{u \in x+y}\widehat{\lambda}(u) > 0$ and $\inf_{v \in y+x}\widehat{\lambda}(v) > 0$. Suppose $\widehat{\lambda}(x) = \widehat{0}$. Let $\alpha \in \{\in, \in \vee q\}$. Then, $y_{\widehat{\lambda}(y)}\alpha\widehat{\lambda}$, $u_{\widehat{\lambda}(u)}\alpha\widehat{\lambda}$ and $v_{\widehat{\lambda}(v)}\alpha\widehat{\lambda}$ but $x_{\min\left\{\widehat{\lambda}(y),\max\left\{\widehat{\lambda}(u),\widehat{\lambda}(u)\right\}\right\}}\overline{\beta}\widehat{\lambda}$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, this is a contradiction to the fact. Also, $y_{\widehat{1}}q\widehat{\lambda}$, $u_{\widehat{1}}q\widehat{\lambda}$ and $v_{\widehat{1}}q\widehat{\lambda}$ but $x_{\widehat{1}}\overline{\beta}\widehat{\lambda}$ for every $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, this is again contradiction to the fact. Thus, our supposition

$\widehat{\lambda}(x) = \widehat{0}$ was wrong. Therefore, $\widehat{\lambda}(x) > \widehat{0}$ and so $x \in \widehat{\lambda}_{\widehat{0}}$. Hence $\widehat{\lambda}_{\widehat{0}}$ is a weak k-hyperideal of R. ■

Theorem 211 *Let A be a hyperideal of R and let $\widehat{\lambda}$ be an n-dimensional fuzzy subset of R such that*

$$\widehat{\lambda}(x) = \begin{cases} \widehat{0} & \text{for all } x \in \mathcal{R} \backslash A, \\ \geq \widehat{0.5} & \text{for all } x \in A. \end{cases}$$

Then, $\widehat{\lambda}$ is an n-dimensional $(\alpha, \in \vee q)$-fuzzy hyperideal of R.

Proof. (a) (For $\alpha = q$) Let $x, y \in$R and $\widehat{t}, \widehat{s} \in I_n$ be such that $x_{\widehat{t}} q \widehat{\lambda}$ and $y_{\widehat{s}} q \widehat{\lambda}$. Then, $\widehat{\lambda}(x) + \widehat{t} > \widehat{1}$ and $\widehat{\lambda}(y) + \widehat{s} > \widehat{1}$. It mean that $x, y \in A$. Since, A is a hyperideal of R, so by definition, $x + y \subseteq A$. Thus, $\widehat{\lambda}(z) \geq 0.5$ for each $z \in x + y$. If $\min\{\widehat{t}, \widehat{s}\} \leq 0.5$, then $\widehat{\lambda}(z) \geq 0.5 \geq \min\{\widehat{t}, \ \widehat{s}\}$ for each $z \in x + y$. Hence, $(z)_{\min\{\widehat{t}, \widehat{s}\}} \in \widehat{\lambda}$ for each $z \in x + y$. If $\min\{\widehat{t}, \ \widehat{s}\} > 0.5$, then $\widehat{\lambda}(z) + \min\{\widehat{t}, \widehat{s}\} > \widehat{0.5} + \widehat{0.5} = \widehat{1}$ for each $z \in x + y$ and so $(z)_{\min\{\widehat{t}, \widehat{s}\}} q \widehat{\lambda}$ for each $z \in x + y$. Therefore, $(z)_{\min\{\widehat{t}, \widehat{s}\}} \in \vee q \widehat{\lambda}$ for each $z \in x + y$.

Now, let $x, y \in$R and $\widehat{t} \in I_n$ such that $x_{\widehat{t}} q \widehat{\lambda}$, which implies that $\widehat{\lambda}(x) + \widehat{t} > \widehat{1}$. Its mean that $x \in A$. Since, A is a hyperideal of R, so by definition, xy and yx are in A. It follows that $\widehat{\lambda}(xy) \geq \widehat{0.5}$ and $\widehat{\lambda}(yx) \geq \widehat{0.5}$. If $\widehat{t} \leq \widehat{0.5}$, then $\widehat{\lambda}(xy) \geq \widehat{0.5} \geq \widehat{t}$ and $\widehat{\lambda}(yx) \geq \widehat{0.5} \geq \widehat{t}$. Hence, $(xy)_{\widehat{t}} \in \widehat{\lambda}$ and $(yx)_{\widehat{t}} \in \widehat{\lambda}$.

If $\widehat{t} > \widehat{0.5}$, then $\widehat{\lambda}(xy) + \widehat{t} > \widehat{0.5} + \widehat{0.5} = \widehat{1}$ and $\widehat{\lambda}(yx) + \widehat{t} > \widehat{0.5} + \widehat{0.5} = \widehat{1}$. This implies $(xy)_{\widehat{t}} q \widehat{\lambda}$ and $(yx)_{\widehat{t}} q \widehat{\lambda}$. Thus, $(xy)_{\widehat{t}} \in \vee q \widehat{\lambda}$ and $(yx)_{\widehat{t}} \in \vee q \widehat{\lambda}$. Hence, $\widehat{\lambda}$ is an n-dimensional $(q, \in \vee q)$-fuzzy hyperideal of R.

(b) (For $\alpha = \in$) Let $x, y \in$R and $\widehat{t}, \widehat{s} \in I_n$ be such that $x_{\widehat{t}} \in \widehat{\lambda}$ and $y_{\widehat{s}} \in \widehat{\lambda}$. Then, $\widehat{\lambda}(x) \geq \widehat{t}$ and $\widehat{\lambda}(y) \geq \widehat{s}$. Thus, $x, y \in A$. Since, A is a hyperideal of R, so by definition, $x + y \subseteq A$, that is, $\widehat{\lambda}(z) \geq \widehat{0.5}$ for each $z \in x + y$. If $\min\{\widehat{t}, \widehat{s}\} \leq \widehat{0.5}$, then $\widehat{\lambda}(z) \geq \widehat{0.5} \geq \min\{\widehat{t}, \widehat{s}\}$ for each $z \in x + y$. Hence, $(z)_{\min\{\widehat{t}, \widehat{s}\}} \in \widehat{\lambda}$ for each $z \in x + y$. If $\min\{\widehat{t}, \widehat{s}\} > \widehat{0.5}$, then $\widehat{\lambda}(z) + \min\{\widehat{t}, \widehat{s}\} > \widehat{0.5} + \widehat{0.5} = \widehat{1}$ and so $(x + y)_{\min\{\widehat{t}, \widehat{s}\}} q \widehat{\lambda}$ for each $z \in x + y$. Therefore, $(z)_{\min\{\widehat{t}, \widehat{s}\}} \in \vee q \widehat{\lambda}$ for each $z \in x + y$. Now, let $x, y \in$R and $\widehat{t} \in I_n$ such that $x_{\widehat{t}} \in \widehat{\lambda}$, which implies that $\widehat{\lambda}(x) \geq \widehat{t}$, that is, $x \in A$. Since, A is a hyperideal of R, so by definition, $xy \in A$ and $yx \in A$, it follows that $\widehat{\lambda}(xy) \geq \widehat{0.5}$ and $\widehat{\lambda}(yx) \geq \widehat{0.5}$. If $\widehat{t} \leq \widehat{0.5}$, then $\widehat{\lambda}(xy) \geq \widehat{0.5} \geq \widehat{t}$ and $\widehat{\lambda}(yx) \geq \widehat{0.5} \geq \widehat{t}$. Hence, $(xy)_{\widehat{t}} \in \widehat{\lambda}$ and $(yx)_{\widehat{t}} \in \widehat{\lambda}$.

If $t > 0.5$, then $\widehat{\lambda}(xy) + \widehat{t} > \widehat{0.5} + \widehat{0.5} = \widehat{1}$ and $\widehat{\lambda}(yx) + \widehat{t} > \widehat{0.5} + \widehat{0.5} = \widehat{1}$. Thus, $(xy)_{\widehat{t}} q\widehat{\lambda}$ and $(yx)_{\widehat{t}} q\widehat{\lambda}$. Thus, $(xy)_{\widehat{t}} \in \vee q\widehat{\lambda}$ and $(yx)_{\widehat{t}} \in \vee q\widehat{\lambda}$. Hence, $\widehat{\lambda}$ is an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal of R.

(c) (For $\alpha = \in \vee q$) It follows from (a) and (b). This completes the proof. ■

Theorem 212 *Let A be a k-hyperideal of R and let $\widehat{\lambda}$ be an n-dimensional fuzzy subset of R such that*

$$\widehat{\lambda}(x) = \begin{cases} \widehat{0} & \text{for all } x \in \mathcal{R} \backslash A, \\ \geq \widehat{0.5} & \text{for all } x \in A. \end{cases}$$

Then, $\widehat{\lambda}$ is an n-dimensional $(\alpha, \in \vee q)$-fuzzy k-hyperideal of R.

Proof. (a) (For $\alpha = q$) By Theorem 211 $\widehat{\lambda}$ is an n-dimensional $(q, \in \vee q)$-fuzzy hyperideal of R. Now, for each $u \in x + y$ or $v \in y + x$, for $u, v, x, y \in$R and $\widehat{s_1}, \widehat{s_2}, \widehat{s_3} \in I_n$ be such that $u_{\widehat{t_1}} q\widehat{\lambda}$ $v_{\widehat{s_2}} q\widehat{\lambda}$and $y_{\widehat{s_3}} q\widehat{\lambda}$. Then, $\widehat{\lambda}(u) + \widehat{t} > \widehat{1}$, $\widehat{\lambda}(v) + \widehat{s} > \widehat{1}$ and $\widehat{\lambda}(y) + \widehat{s} > \widehat{1}$. This mean that $u, v, y \in A$, it follows $x \in \mathcal{R}$ and $y \in A$ such that $x + y \subseteq A$ or $y + x \subseteq A$. Since A is a k-hyperideal, so $x \in A$. Thus $\widehat{\lambda}(x) \geq \widehat{0.5}$. If $\min\{\max\{\widehat{s_1},\ \widehat{s_2}\},\ \widehat{s_3}\} \leq \widehat{0.5}$, then $\widehat{\lambda}(x) \geq \widehat{0.5} \geq \min\{\max\{\widehat{s_1},\ \widehat{s_2}\},\ \widehat{s_3}\}$. So, $x_{\min\{\max\{\widehat{s_1},\ \widehat{s_2}\},\ \widehat{s_3}\}} \in \widehat{\lambda}$. If $\min\{\max\{\widehat{s_1},\ \widehat{s_2}\},\ \widehat{s_3}\} > \widehat{0.5}$, then $\widehat{\lambda}(x) + \min\{\max\{\widehat{s_1},\ \widehat{s_2}\},$ $\widehat{s_3}\} > \widehat{0.5} + \widehat{0.5} = \widehat{1}$ and so $x_{\min\{\max\{\widehat{s_1},\ \widehat{s_2}\},\ \widehat{s_3}\}} q\widehat{\lambda}$. Therefore, $x_{\min\{\max\{\widehat{s_1},\ \widehat{s_2}\},\ \widehat{s_3}\}} \in \vee q\widehat{\lambda}$. Hence, $\widehat{\lambda}$ is an n-dimensional $(q, \in \vee q)$-fuzzy k-hyperideal of R.

(b) (For $\alpha = \in, \in \vee q$) Proof is similar to part (a). This completes the proof. ■

From the above Theorem it is clear that the n-dimensional characteristic function of (k-hyperideal) is an n-dimensional $(\alpha, \in \vee q)$-fuzzy (k-hyperideal, h-hyperideal) hyperideal of R, where $\alpha \in \{\in, q\}$.

6.6 n-Dimensional $(\in, \in \vee q)$-fuzzy hyperideals

In this section we define a special type of n-dimensional fuzzy hyperideal (k-hyperideal) of semihyperring so called n-dimensional $(\in, \in \vee q)$-fuzzy hyperideals (k-hyperideals). We will give some different characterization of n-dimensional $(\in, \in \vee q)$-fuzzy hyperideals (k-hyperideals) by their level sets.

Definition 213 *An n-dimensional fuzzy subset $\widehat{\lambda}$ of R is called an n-dimensional* $(\in, \in \vee q)$*-fuzzy hyperideal of R, if the following conditions satisfy*

1. $x_{\widehat{t}} \in \widehat{\lambda}, y_{\widehat{s}} \in \widehat{\lambda}$ implies $z_{\min(\widehat{t},\, \widehat{s})} \in \vee q\widehat{\lambda}$ for each $z \in x + y$
2. $x_{\widehat{t}} \in \widehat{\lambda}$ implies $(xy)_{\widehat{t}} \in \vee q\widehat{\lambda}$ and $(yx)_{\widehat{t}} \in \vee q\widehat{\lambda}$ for all $x, y \in$R and $\widehat{t} \in I_n$.

The concept of an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal in a semihyperring is a generalization of an $(\in, \in \vee q)$-fuzzy hyperideal in a semihyperring so an $(\in, \in \vee q)$-fuzzy hyperideal in a semihyperring is a special case of an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal in a semihyperring.

Definition 214 *An n-dimensional fuzzy subset $\widehat{\lambda}$ of R is called an n-dimensional* $(\in, \in \vee q)$*-fuzzy weak k-hyperideal of R, if the following conditions are satisfied*

1. $\widehat{\lambda}$ is an n-dimensional $(\in, \in \vee q)$-fuzzy weak k-hyperideal of R.
2. If for each $u \in x+y$ or $v \in x+y$, $u_{\widehat{s_1}} \in \widehat{\lambda}$, $v_{\widehat{s_2}} \alpha \widehat{\lambda}$ and $y_{\widehat{s_3}} \in \widehat{\lambda}$, then $x_{\min\{\max\{\widehat{s_1},\, \widehat{s_2}\},\, \widehat{s_3}\}} \in \vee q\widehat{\lambda}$ for all $u, v, x, y \in$R and for all $\widehat{s_1}, \widehat{s_2}, \widehat{s_3}, \in I_n$.

Example 215 *Consider the semihyperring* $(\mathbb{N}, \oplus, .)$ *defined by a hyperoperation* "$\oplus$" *and a binary operation* "$\cdot$" *on* $\mathbb{N}$ *as follow* $m \oplus n = \{m, n\}$ *and* $mn = mn$, $\forall\ m, n \in \mathbb{N}$. *Clearly* $(\mathbb{N}, \oplus, \cdot)$ *is a semihyperring.*

Define a 5*-dimensional fuzzy set* $\widetilde{\lambda}$ *of* $\mathbb{N}$ *by*

$$\widetilde{\lambda}(x) = \begin{cases} (0.5, 0.55, 0.6, 0.7, 0.8) & \text{if } x \in \langle 4 \rangle \\ (0.7, 0.75, 0.8, 0.85, 0.9) & \text{if } x \in \langle 2 \rangle - \langle 4 \rangle \\ (0.4, 0.45, 0.5, 0.55, 0.6) & \text{otherwise} \end{cases}$$

One can easily check that $\widetilde{\lambda}$ *is a* 5*-dimensional* $(\in, \in \vee q)$ *fuzzy left hyperideal of* $(\mathbb{N}, \oplus, \cdot)$

Example 216 *On four elements semihyperring $(\mathcal{R}, \oplus, .)$ defined by the following two tables:*

$\oplus$	0	a	b	c
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{a\}$	$\{b\}$	$\{c\}$
b	$\{b\}$	$\{b\}$	$\{b\}$	$\{c\}$
c	$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
c	0	c	c	c

Define a 4-dimensional fuzzy set $\widetilde{\lambda}$ of R by

$$\widetilde{\lambda}(x) = \begin{cases} (0.8, 0.84, 0.88, 0.9) & \text{if } x \in a \\ (0.6, 0.64, 0.68, 0.7) & \text{if } x \in a, b \\ (0.2, 0.24, 0.28, 0.3) & \text{if } x \in c \end{cases}$$

One can easily check that $\widetilde{\lambda}$ is a 4-dimensional $(\in, \in \vee q)$ fuzzy left hyperideal of $(\mathcal{R}, \oplus, \cdot)$.

Theorem 217 *Let $\widehat{\lambda}$ be an n-dimensional fuzzy subset of R. Then, $\widehat{\lambda}$ is an n-dimensional $(\in, \in)$-fuzzy hyperideal of R if and only if $\widehat{\lambda}$ is an n-dimensional fuzzy hyperideal of R.*

Corollary 218 *Let $\widehat{\lambda}$ be an n-dimensional fuzzy subset of a semihyperring R. Then, $\widehat{\lambda}$ is an n-dimensional fuzzy k-hyperideal of R if and only if $\widehat{\lambda}$ is an n-dimensional $(\in, \in)$-fuzzy k-hyperideal of R.*

Corollary 219 *Let $\widehat{\lambda}$ be an n-dimensional fuzzy subset of a semihyperring R. Then, $\widehat{\lambda}$ is an n-dimensional fuzzy h-hyperideal of R if and only if $\widehat{\lambda}$ is an n-dimensional $(\in, \in)$-fuzzy h-hyperideal of R.*

Proof. Proof is similar to the proof of Corollary 218. ■

The above Theorem and Corollaries show that the concept of n-dimensional $(\in, \in)$-fuzzy (k-hyperideal) hyperideal is the same as n-dimensional fuzzy (k-hyperideal) hyperideal of R.

Lemma 220 *Let $\widehat{\lambda}$ be an n-dimensional fuzzy subset of R. Then, the following are equivalent*

1. $x_{\widehat{t}}, y_{\widehat{s}} \in \widehat{\lambda}$ implies $(z)_{\min\{\widehat{t},\, \widehat{s}\}} \in \vee q\widehat{\lambda}$ for each $z \in x + y$.

2. $\inf_{z\in x+y}\widehat{\lambda}(z)\geq\min\left\{\widehat{\lambda}(x),\widehat{\lambda}(y),\widehat{0.5}\right\}$ for all $x,y\in$R.

Proof. $(1)\Rightarrow(2)$ Let $\widehat{\lambda}$ be an n-dimensional fuzzy subset of R and $x,y\in$R such that $\widehat{\lambda}(z)<\min\left\{\widehat{\lambda}(x),\widehat{\lambda}(y),\widehat{0.5}\right\}$ for some $z\in x+y$. If $\min\left\{\widehat{\lambda}(x),\widehat{\lambda}(y)\right\}<\widehat{0.5}$, then $\widehat{\lambda}(z)<\min\left\{\widehat{\lambda}(x),\widehat{\lambda}(y)\right\}$. Choose a $\widehat{t}\in I_n$ such that $\widehat{\lambda}(x+y)<\widehat{t}<\min\left\{\widehat{\lambda}(x),\widehat{\lambda}(y)\right\}$, that is, $(x)_{\widehat{t}}\in\widehat{\lambda}$ and $(y)_{\widehat{t}}\in\widehat{\lambda}$ but $(z)_{\widehat{t}}\overline{\in\vee q}\widehat{\lambda}$ for some $z\in x+y$, this is a contradiction to (1). Next, if $\min\left\{\widehat{\lambda}(x),\widehat{\lambda}(y)\right\}\geq\widehat{0.5}$, then $\widehat{\lambda}(z)<\widehat{0.5}$ for some $z\in x+y$. This implies $x_{\widehat{0.5}},y_{\widehat{0.5}}\in\widehat{\lambda}$ but $(z)_{\widehat{0.5}}\overline{\in\vee q}\widehat{\lambda}$ for some $z\in x+y$, again this is a contradiction to (1). Hence, $\inf_{z\in x+y}\widehat{\lambda}(z)\geq\min\left\{\widehat{\lambda}(x),\widehat{\lambda}(y),\widehat{0.5}\right\}$.

$(2)\Rightarrow(1)$ Let $x_{\widehat{t}},y_{\widehat{s}}\in\widehat{\lambda}$. Then, $\widehat{\lambda}(x)\geq\widehat{t}$ and $\widehat{\lambda}(y)\geq\widehat{s}$. By hypothesis

$$\inf_{z\in x+y}\widehat{\lambda}(x+y)\geq\min\left\{\widehat{\lambda}(x),\widehat{\lambda}(y),\widehat{0.5}\right\}\geq\min\left\{\widehat{t},\widehat{s},\widehat{0.5}\right\}.$$

Thus,

$$\inf_{z\in x+y}\widehat{\lambda}(z)\geq\min\left\{\widehat{t},\widehat{s}\right\}\qquad\text{if }\min\left\{\widehat{t},\widehat{s}\right\}\leq\widehat{0.5}$$

and

$$\inf_{z\in x+y}\widehat{\lambda}(z)\geq\widehat{0.5}\qquad\text{if }\min\left\{\hat{t},\hat{r}\right\}>\widehat{0.5}.$$

Hence, in both case we get $(z)_{\min\{\widehat{t},\,\widehat{s}\}}\in\vee q\widehat{\lambda}$ for some $z\in x+y$. This completes the proof. ■

Lemma 221 *Let $\widehat{\lambda}$ be an n-dimensional fuzzy subset of R. Then, the following are equivalent*

1. $x_{\widehat{t}}\in\widehat{\lambda}$ and $y\in$R implies $(xy)_{\widehat{t}}\in\vee q\widehat{\lambda}$
2. $\widehat{\lambda}(xy)\geq\min\left(\widehat{\lambda}(x),\widehat{0.5}\right)$.

Proof. $(1)\Rightarrow(2)$ Let $x,y\in$R, and $\widehat{\lambda}(x)<0.5$. Assume that $\widehat{\lambda}(xy)<\widehat{\lambda}(x)$. Choose a $\widehat{t}\in I_n$ such that $\widehat{\lambda}(xy)<\widehat{t}<\widehat{\lambda}(x)$. Then, $(x)_{\widehat{t}}\in\widehat{\lambda}$ but $(xy)_{\widehat{t}}\overline{\in\vee q}\widehat{\lambda}$, which is a contradiction to (1). Now, let $\widehat{\lambda}(x)\geq\widehat{0.5}$. Assume that $\widehat{\lambda}(xy)<\widehat{0.5}$. Then, $x_{\widehat{0.5}}\in\widehat{\lambda}$ but $(xy)_{\widehat{0.5}}\overline{\in\vee q}\widehat{\lambda}$, which is again a contradiction to (1). Hence, $\widehat{\lambda}(xy)\geq\min\left(\widehat{\lambda}(x),\widehat{0.5}\right)$.

$(2)\Rightarrow(1)$ Let $x_{\widehat{t}}\in\widehat{\lambda},y\in$R. Then, $\widehat{\lambda}(x)\geq\widehat{t}$. By hypothesis

$$\widehat{\lambda}(xy)\geq\min\left(\widehat{\lambda}(x),\widehat{0.5}\right)\geq\min\left(\widehat{t},\widehat{0.5}\right).$$

Thus,

$$\widehat{\lambda}(xy) \geq \widehat{t} \text{ if } t < \widehat{0.5}$$

and

$$\widehat{\lambda}(xy) \geq \widehat{0.5} \text{ if } t \geq \widehat{0.5}.$$

Hence, in both cases we get $(xy)_t \in \vee q\widehat{\lambda}$. ■

Similarly we can show that the following are equivalent:

1. $x_{\widehat{t}} \in \widehat{\lambda}$ and $y \in$R implies $(yx)_{\widehat{t}} \in \vee q\widehat{\lambda}$
2. $\widehat{\lambda}(yx) \geq \min\left(\widehat{\lambda}(x), \widehat{0.5}\right)$ for all $x, y \in$R and $\widehat{t} \in I_n$.

Lemma 222 *Let $\widehat{\lambda}$ be an n-dimensional fuzzy subset of R and $x, y, u, v \in$R. Then, for all $\widehat{s_1}, \widehat{s_2}, \widehat{s_3}, \in I_n$ the following are equivalent*

1. If for each $u \in x+y$ or $v \in x+y$, $u_{\widehat{s_1}} \in \widehat{\lambda}$, $v_{\widehat{s_2}} \in \widehat{\lambda}$and $y_{\widehat{s_3}} \in \widehat{\lambda}$, then $x_{\min\{\max\{\widehat{s_1},\ \widehat{s_2}\},\ \widehat{s_3}\}} \in \vee q\widehat{\lambda}$ for all $u, v, x, y \in$R and for all $\widehat{s_1}, \widehat{s_2}, \widehat{s_3}, \in I_n$.
2. $\widehat{\lambda}(x) \geq \left(\inf_{u \in x+y} \widehat{\lambda}(u) \vee \inf_{v \in y+x} \widehat{\lambda}(v)\right) \wedge \widehat{\lambda}(v) \wedge \widehat{0.5}$ for all $x, y, u, v \in$R.

Proof. Let $x, y \in$R and assume that $\widehat{\lambda}(x) < \left(\inf_{u \in x+y} \widehat{\lambda}(a) \vee \inf_{v \in y+x} \widehat{\lambda}(v)\right) \wedge \widehat{\lambda}(v) \wedge \widehat{0.5}$. Choose a $\widehat{t} \in I_n$ such that $\widehat{\lambda}(x) < \widehat{t} < \left(\inf_{u \in x+y} \widehat{\lambda}(u) \vee \inf_{v \in y+x} \widehat{\lambda}(v)\right) \wedge \widehat{\lambda}(v) \wedge \widehat{0.5}$. If $\left(\inf_{u \in x+y} \widehat{\lambda}(u) \vee \inf_{v \in y+x} \widehat{\lambda}(v)\right) \wedge \widehat{\lambda}(y) < \widehat{0.5}$, then $(u)_{\widehat{t}}, (v)_{\widehat{t}}, (y)_{\widehat{t}}, \in \widehat{\lambda}$ for each $u \in x+y$ or $v \in x+y$ but $(x)_{\widehat{t}} \overline{\in \vee q}\widehat{\lambda}$. This gives a contradiction to (1). Now, let $\left(\inf_{u \in x+y} \widehat{\lambda}(u) \vee \inf_{v \in y+x} \widehat{\lambda}(v)\right) \wedge \widehat{\lambda}(y) \geq \widehat{0.5}$, then $\widehat{\lambda}(x) < \widehat{0.5}$ then $(u)_{\widehat{0.5}}, (v)_{\widehat{0.5}}, (y)_{\widehat{0.5}}, \in \widehat{\lambda}$ for each $u \in x+y$ or $v \in x+y \in \widehat{\lambda}$ but $(x)_{\widehat{0.5}} \overline{\in \vee q}\widehat{\lambda}$. Again this gives a contradiction to (1). Hence, $\widehat{\lambda}(x) \geq \left(\inf_{u \in x+y} \widehat{\lambda}(u) \vee \inf_{v \in y+x} \widehat{\lambda}(v)\right) \wedge \widehat{\lambda}(v) \wedge \widehat{0.5}$ for all $x, y, u, v \in$R.

(2) $\Rightarrow$ (1) Let $x, y, u, v \in$R such that $u \in x+y$ and $v \in y+x$. If $u_{\widehat{s_1}} \in \widehat{\lambda}$, $v_{\widehat{s_2}} \in \widehat{\lambda}$and $y_{\widehat{s_2}} \in \widehat{\lambda}$ for $\widehat{s_1}, \widehat{s_2}, \widehat{s_3}, \in I_n$, then $\widehat{\lambda}(u) \geq \widehat{s_1}$, $\widehat{\lambda}(v) \geq \widehat{s_2}$ and $\widehat{\lambda}(y) \geq \widehat{s_3}$ by hypothesis

$$\widehat{\lambda}(x) \geq \left(\inf_{u \in x+y} \widehat{\lambda}(u) \vee \inf_{v \in y+x} \widehat{\lambda}(v)\right) \wedge \widehat{\lambda}(v) \wedge \widehat{0.5} \geq \min\{\max\{\widehat{s_1},\ \widehat{s_2}\}, \widehat{s_3}, 0.5\}.$$

Thus

$$\widehat{\lambda}(x) \geq \min\{\max\{\widehat{s_1},\ \widehat{s_2}\}, \widehat{s_3}\} \qquad \text{if } \min\{\max\{\widehat{s_1},\ \widehat{s_2}\}, \widehat{s_3}\} \leq \widehat{0.5}$$

and

$$\widehat{\lambda}(x) \geq \widehat{0.5} \qquad\qquad \text{if } \min\{\max\{\widehat{s_1},\ \widehat{s_2}\},\ \widehat{s_3}\} > \widehat{0.5}.$$

Hence, from both cases, we get $(x)_{\min\{\max\{\widehat{s_1},\ \widehat{s_2}\},\ \widehat{s_3}\}} \in \vee q\widehat{\lambda}$. ■

From Lemma 220 and Lemma 221 we deduce that

Theorem 223 *An n-dimensional fuzzy subset $\widehat{\lambda}$ of R is an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal of R if and only if*

1. $\inf_{z\in x+y} \widehat{\lambda}(z) \geq \min\left\{\widehat{\lambda}(x), \widehat{\lambda}(y), \widehat{0.5}\right\}$
2. $\widehat{\lambda}(xy) \geq \min\left(\widehat{\lambda}(x), \widehat{0.5}\right)$ and $\widehat{\lambda}(xy) \geq \min\left(\widehat{\lambda}(y), \widehat{0.5}\right)$ for all $x, y \in$R.

Proof. The proof is straightforward. ■

From Theorem 223 and Lemma 222 we deduce that

Theorem 224 *An n-dimensional fuzzy subset $\widehat{\lambda}$ of R is an n-dimensional $(\in, \in \vee q)$-fuzzy k-hyperideal of R if and only if*

1. $\inf_{z\in x+y} \widehat{\lambda}(z) \geq \min\left\{\widehat{\lambda}(x), \widehat{\lambda}(y), \widehat{0.5}\right\}$
2. $\widehat{\lambda}(xy) \geq \min\left(\widehat{\lambda}(x), 0.5\right)$ and $\widehat{\lambda}(xy) \geq \min\left(\widehat{\lambda}(y), \widehat{0.5}\right)$ for all $x, y \in$R.
3. $\widehat{\lambda}(x) \geq \left(\inf_{u\in x+y} \widehat{\lambda}(u) \vee \inf_{v\in y+x} \widehat{\lambda}(v)\right) \wedge \widehat{\lambda}(v) \wedge \widehat{0.5}$ for all $x, y, u, v \in$R.

Theorem 225 *An n-dimensional fuzzy subset $\widehat{\lambda}$ of R is an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal of R if and only if $\widehat{\lambda}_{\widehat{t}} \neq \varphi$ is a hyperideal of R for all $\widehat{0} < \widehat{t} \leq \widehat{0.5}$.*

Proof. Let $\widehat{\lambda}$ be an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal of R. Let $\widehat{0} < \widehat{t} \leq \widehat{0.5}$ and $x, y \in \widehat{\lambda}_t$. Then, $\widehat{\lambda}(x) \geq \widehat{t_1}$ and $\widehat{\lambda}(y) \geq \widehat{t_2}$. Since

$$\inf_{z\in x+y} \widehat{\lambda}(z) \geq \min\left\{\widehat{\lambda}(x), \widehat{\lambda}(y), \widehat{0.5}\right\} \geq \min\left\{\widehat{t_1}, \widehat{t_2}, \widehat{0.5}\right\} = \min\left\{\widehat{t_1}, \widehat{t_2}\right\}.$$

Thus, $\inf_{z\in x+y} \widehat{\lambda}(z) \geq \min\left\{\widehat{t_1}, \widehat{t_2}\right\}$. This implies $z \in \widehat{\lambda}_{\widehat{t}}$ for each $z \in x+y$. Hence, $x+y \subseteq \widehat{\lambda}_{\widehat{t}}$. Now, let $x \in \widehat{\lambda}_{\widehat{t}}$, then $\widehat{\lambda}(x) \geq \widehat{t}$. Since $\widehat{\lambda}(xr), \widehat{\lambda}(rx) \geq \min\left(\widehat{\lambda}(x), \widehat{0.5}\right) \geq \min\left\{\widehat{t}, \widehat{0.5}\right\} = \widehat{t}$ for all $r \in$R. Therefore, $xr, rx \in \widehat{\lambda}_t$. This shows that $\widehat{\lambda}_{\widehat{t}}$ is a hyperideal of R.

Conversely, let $\widehat{\lambda}_{\widehat{t}}$ be a hyperideal of R for all $\widehat{0} < \widehat{t} \leq \widehat{0.5}$. We show that $\widehat{\lambda}$ is an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal of R. If possible, let there exist some $x, y \in$R such that $\widehat{\lambda}(z) < \min\left\{\widehat{\lambda}(x), \widehat{\lambda}(y), \widehat{0.5}\right\}$ for some $z \in x+y$. Choose a $\widehat{t} \in I_n$ such that $\widehat{\lambda}(z) < \widehat{t} < \min\left\{\widehat{\lambda}(x), \widehat{\lambda}(y), \widehat{0.5}\right\}$ for some $z \in x+y$, then $x, y \in \widehat{\lambda}_{\widehat{t}}$ but $z \notin \widehat{\lambda}_{\widehat{t}}$ This means that $x+y \nsubseteq \widehat{\lambda}_{\widehat{t}}$ This gives us a contradiction. Therefore, $\inf_{z \in x+y} \widehat{\lambda}(z) \geq \min\left\{\widehat{\lambda}(x), \widehat{\lambda}(y), \widehat{0.5}\right\}$ for all $x, y \in$R. Similarly, it can be shown that $\widehat{\lambda}(xy) \geq \min\left(\widehat{\lambda}(x), \widehat{0.5}\right)$ and $\widehat{\lambda}(xy) \geq \min\left(\widehat{\lambda}(y), \widehat{0.5}\right)$ for all $x, y \in$R. ■

Corollary 226 *An n-dimensional fuzzy subset $\widehat{\lambda}$ of R is an n-dimensional $(\in, \in \vee q)$-fuzzy k-hyperideal of R if and only if $\widehat{\lambda}_{\widehat{t}}$ is a k-hyperideal of R for all $\widehat{0} < \widehat{t} \leq \widehat{0.5}$.*

Proof. Let $\widehat{\lambda}$ be an n-dimensional $(\in, \in \vee q)$-fuzzy k-hyperideal of R. Then, by Theorem 225 $\widehat{\lambda}_{\widehat{t}}$ is a hyperideal of R. Next we show that if for $y \in \widehat{\lambda}_{\widehat{t}}$ and $x \in$R we have $x+y \subseteq \widehat{\lambda}_{\widehat{t}}$ or $x+y \subseteq \widehat{\lambda}_{\widehat{t}}$ this implies $x \in \widehat{\lambda}_{\widehat{0}}$ Now, let $x \in$R and $y \in \widehat{\lambda}_{\widehat{t}}$ such that $x+y \subseteq \widehat{\lambda}_{\widehat{t}}$ or $y+x \subseteq \widehat{\lambda}_{\widehat{t}}$. Then, we have $\widehat{\lambda}(u) \geq \widehat{t}$, $\widehat{\lambda}(v) \geq \widehat{t}$ and $\widehat{\lambda}(y) \geq \widehat{t}$ for each $u \in x+y$ or $v \in y+x$. Since

$$\begin{aligned}
\widehat{\lambda}(x) &\geq \left(\inf_{u \in x+y} \widehat{\lambda}(u) \vee \inf_{v \in y+x} \widehat{\lambda}(v)\right) \wedge \widehat{\lambda}(v) \wedge \widehat{0.5} \geq \min\left\{\max\left\{\widehat{t},\ \widehat{t}\right\}, \widehat{t}, 0.5\right\} \\
\widehat{\lambda}(x) &\geq t.
\end{aligned}$$

Therefore, $x \in \widehat{\lambda}_{\widehat{t}}$. Hence, $\widehat{\lambda}_{\widehat{t}}$ is a k-hyperideal of R.

Conversely, let $\widehat{\lambda}_t$ be a k-hyperideal of R for all $\widehat{0} < \widehat{t} \leq \widehat{0.5}$. By Theorem 225 $\widehat{\lambda}$ is an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal of R. Now, let $x, y, u, v \in$R, if possible let

$$\widehat{\lambda}(x) < \left(\inf_{u \in x+y} \widehat{\lambda}(u) \vee \inf_{v \in y+x} \widehat{\lambda}(v)\right) \wedge \widehat{\lambda}(v) \wedge \widehat{0.5}$$

Choose a $\widehat{t} \in I_n$ such that $\widehat{\lambda}(x) < \widehat{t} < \left(\inf_{u \in x+y} \widehat{\lambda}(u) \vee \inf_{v \in y+x} \widehat{\lambda}(v)\right) \wedge \widehat{\lambda}(v) \wedge \widehat{0.5}$. This implies $x+y \subseteq \widehat{\lambda}_{\widehat{t}}, y+x \subseteq \widehat{\lambda}_{\widehat{t}}$ for $y \in \widehat{\lambda}_{\widehat{t}}$ and $x \in$R but $x \notin \widehat{\lambda}_{\widehat{t}}$, This gives us a contradiction. Therefore, $\widehat{\lambda}(x) \geq \left(\inf_{u \in x+y} \widehat{\lambda}(a) \vee \inf_{v \in y+x} \widehat{\lambda}(v)\right) \wedge \widehat{\lambda}(v) \wedge \widehat{0.5}$. This completes the proof. ■

Theorem 227 *Let $\left\{\widehat{\lambda}_i : i \in \Lambda\right\}$ be a family of an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideals of a semihyperring R. Then, $\widehat{\lambda} = \bigcap_{i \in \Lambda} \widehat{\lambda}_i$ is an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal of R.*

Proof. Let $x, y \in$R$, t_1, t_2 \in I_n$ such that $x_{t_1}, y_{t_2} \in \widehat{\lambda}$. Assume that $(z)_{\min\{\widehat{t},\ \widehat{s}\}} \overline{\in \vee q}\widehat{\lambda}$ for some $z \in x+y$, then $\widehat{\lambda}(z) < \min\{\widehat{t},\ \widehat{s}\}$ and $\widehat{\lambda}(z) + \min\{\widehat{t},\ \widehat{s}\} \leq \widehat{1}$ for some $z \in x+y$. Thus, for some $z \in x+y$

$$\widehat{\lambda}(z) < \widehat{0.5} \qquad (1)$$

Let $\Omega_1 = \left\{i \in \Lambda : \inf_{z \in x+y} \widehat{\lambda}_i(z) \geq \min\{\widehat{t},\ \widehat{s}\}\right\}$ and

$$\Omega_2 = \left\{i \in \Lambda : \widehat{\lambda}_i(z) + \min\{\widehat{t},\ \widehat{s}\} > \widehat{1} \text{ and } \widehat{\lambda}_i(z) < \min\{\widehat{t},\ \widehat{s}\} \text{ for each } z \in x+y\right\}$$

Then $\Lambda = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. If $\Omega_2 = \emptyset$, then for each $z \in x+y$, $\widehat{\lambda}_i(z) \geq \min\{\widehat{t},\ \widehat{s}\}$ for all $i \in \Lambda$ and so $\widehat{\lambda}(z) \geq \min\{\widehat{t},\ \widehat{s}\}$ which is a contradiction. Thus $\Omega_2 \neq \emptyset$ and hence for each $i \in \Omega_2$ we have $\widehat{\lambda}_i(z) + \min\{\widehat{t},\ \widehat{s}\} > \widehat{1}$ and $\widehat{\lambda}_i(z) < \min\{\widehat{t},\ \widehat{s}\}$ for each $z \in x+y$. It follows that $\min\{\widehat{t},\ \widehat{s}\} > \widehat{0.5}$, so that $\widehat{\lambda}_i(x) \geq \widehat{\lambda}(x) \geq t_1 \geq \min\{\widehat{t},\ \widehat{s}\} > 0.5$ for all $i \in \Lambda$ Similarly we have $\widehat{\lambda}_i(y) > 0.5$ for all $i \in \Lambda$. Now suppose that $t = \widehat{\lambda}_i(z) < 0.5$ for some $i \in \Lambda$ and for some $z \in x+y$. Let $\widehat{t'} \in I_n^{0.5}$ be such that $\widehat{t} < \widehat{t'}$, then $\widehat{\lambda}_i(x) > \widehat{0.5} > \widehat{t'}$ and $\widehat{\lambda}_i(y) > 0.5 > t'$ that is $x_{t'} \in \widehat{\lambda}_i$ and $y_{t'} \in \widehat{\lambda}_i$ but $\widehat{\lambda}_i(z) = \widehat{t} < \widehat{t'}$ and $\widehat{\lambda}_i(z) + t' < \widehat{1}$ that is $(z)_{t'} \overline{\in \vee q}\widehat{\lambda}_i$ for some $z \in x+y$. This is a contradiction to $\widehat{\lambda}_i$ is an n-dimensional $(\in, \in \vee q)$ fuzzy hyperideal of R. Hence, $\widehat{\lambda}_i(z) \geq \widehat{0.5}$ for all $i \in \Lambda$ and thus $\widehat{\lambda}(z) \geq \widehat{0.5}$ this contradicts with (1). Therefore $(z)_{\min\{\widehat{t},\ \widehat{s}\}} \in \vee q\widehat{\lambda}$ for each $z \in x+y$. Similarly we can show that if $x, y \in$R$, \widehat{t} \in I_n$, such that $x_{\widehat{t}} \in \widehat{\lambda}$ then

$$(xy)_{\widehat{t}} \text{ and } (yx)_{\widehat{t}} \in \vee q\widehat{\lambda}.$$

This completes the proof. ■

Corollary 228 *Let $\left\{\widehat{\lambda}_i : i \in \Lambda\right\}$ be a family of n-dimensional $(\in, \in \vee q)$-fuzzy k-hyperideals of a semihyperring R. Then, $\widehat{\lambda} = \bigcap_{i \in \Lambda} \widehat{\lambda}_i$ is an n-dimensional $(\in, \in \vee q)$-fuzzy k-hyperideal of R.*

Proof. By Theorem 227 $\widehat{\lambda} = \bigcap_{i \in \Lambda} \widehat{\lambda}_i$ is an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal of R. Remaining part follows from Theorem 227. ■

Theorem 229 *Let $\widehat{\lambda}$ be an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal of R such that $\widehat{\lambda}(x) < \widehat{0.5}$ for all $x \in$R. Then, $\widehat{\lambda}$ is an n-dimensional $(\in, \in)$-fuzzy hyperideal of R.*

Proof. Follows from Theorem 217 and Theorem 223 ■

Corollary 230 *Let $\widehat{\lambda}$ be an n-dimensional $(\in, \in \vee q)$-fuzzy k-hyperideal of R such that $\widehat{\lambda}(x) < \widehat{0.5}$ for all $x \in R$. Then $\widehat{\lambda}$ is an n-dimensional $(\in, \in)$-fuzzy k-hyperideal of R.*

Proof. The proof follows from Corollary 218 and Theorem 224 ■

For any n-dimensional fuzzy subset $\widehat{\lambda}$ of R and $\widehat{t} \in I_n$, we denote $Q\left(\widehat{\lambda}, \widehat{t}\right) = \left\{x \in \mathcal{R} : x_{\widehat{t}} q \widehat{\lambda}\right\}, \widehat{\lambda}_{\widehat{t}} = \left\{x \in \mathcal{R} : x_{\widehat{t}} \in \widehat{\lambda}\right\}, \left[\widehat{\lambda}\right]_{\widehat{t}} = \left\{x \in \mathcal{R} : x_{\widehat{t}} \in \vee q \widehat{\lambda}\right\}$. Then it is clear that $\left[\widehat{\lambda}\right]_{\widehat{t}} = Q\left(\widehat{\lambda}, \widehat{t}\right) \cup \widehat{\lambda}_{\widehat{t}}$. The set $\widehat{\lambda}_{\widehat{t}}$ is called $\in$-level subset of $\widehat{\lambda}$. The set $Q\left(\widehat{\lambda}, \widehat{t}\right)$ is called q-level subset of $\widehat{\lambda}$. The set $\left[\widehat{\lambda}\right]_{\widehat{t}}$ is called an $(\in, \in \vee q)$-level subset of R determined by $\widehat{\lambda}$ and t. In Theorem 225 we have shown that an n-dimensional fuzzy subset $\widehat{\lambda}$ of R is an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal of R if and only if $\widehat{\lambda}_{\widehat{t}} \neq \emptyset$ is a hyperideal of R for all $\widehat{0} < \widehat{t} \leq \widehat{0.5}$. Now we show that the following result for $(\in, \in \vee q)$-level subset.

Theorem 231 *An n-dimensional fuzzy subset $\widehat{\lambda}$ of R is an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal of R if and only if $\left[\widehat{\lambda}\right]_{\widehat{t}}$ is a hyperideal of R for all $\widehat{t} \in I_n$.*

Proof. Assume that $\widehat{\lambda}$ is an n-dimensional $(\in, \in \vee q)$-fuzzy left hyperideal of R and let $\widehat{t} \in I_n$ be such that $\left[\widehat{\lambda}\right]_{\widehat{t}} \neq \varnothing$. Let $x, y \in [A]_{\widehat{t}}$. Then, $\widehat{\lambda}(x) \geq \widehat{t}$ or $\widehat{\lambda}(x) + \widehat{t} > \widehat{1}$ and $\widehat{\lambda}(y) \geq \widehat{t}$ or $\widehat{\lambda}(y) + \widehat{t} > \widehat{1}$. We can consider four cases for each $\in x + y$:

(i) $\widehat{\lambda}(x) \geq \widehat{t}$ and $\widehat{\lambda}(y) \geq \widehat{t}$

(ii) $\widehat{\lambda}(x) \geq \widehat{t}$ and $\widehat{\lambda}(y) + \widehat{t} > \widehat{1}$,

(iii) $\widehat{\lambda}(x) + \widehat{t} > \widehat{1}$ and $\widehat{\lambda}(y) \geq \widehat{t}$

(iv) $\widehat{\lambda}(x) + \widehat{t} > \widehat{1}$ and $\widehat{\lambda}(y) + \widehat{t} > \widehat{1}$.

For the first case, by Theorem 223 (a), implies that

$$\widehat{\lambda}(z) \geq \min\{\widehat{\lambda}(x), \widehat{\lambda}(y), \widehat{0.5}\} = \min\{\widehat{t}, \widehat{0.5}\} = \begin{cases} \widehat{0.5} & \text{if } \widehat{t} > \widehat{0.5} \\ \widehat{t} & \text{if } \widehat{t} \leq \widehat{0.5} \end{cases}$$

and so $\widehat{\lambda}(z) + \widehat{t} > \widehat{0.5} + \widehat{0.5} = \widetilde{1}$, *i.e.*, $(z)\left(\widetilde{s}, \widehat{t}\right) q \widehat{\lambda}$, or $x + y \subseteq \widehat{\lambda}_{\widehat{t}}$. Therefore, $x + y \subseteq \lambda_{\widehat{t}} \cup Q\left(\widehat{\lambda}, \widehat{t}\right) = \left[\widehat{\lambda}\right]_{\widehat{t}}$. For the case (ii), assume that $\widehat{t} > \widehat{0.5}$. Then, $\widetilde{1} - \widehat{t} < \widehat{0.5}$. If $\min\{\widehat{\lambda}(y), \widehat{0.5}\} \leq \widehat{\lambda}(x)$, then

$$\widehat{\lambda}(z) \geq \min\{\widehat{\lambda}(y), \widehat{0.5}\} > \widetilde{1} - \widehat{t}$$

and if $\min\{\widehat{\lambda}(y), \widehat{0.5}\} > \widehat{\lambda}(x)$, then $\widehat{\lambda}(z) \geq \widehat{\lambda}(x) \geq \widehat{t}$. Hence, $x + y \subseteq \lambda_{\widehat{t}} \cup Q\left(\widehat{\lambda}, \widehat{t}\right) = \left[\widehat{\lambda}\right]_{\widehat{t}}$ for $\widehat{t} > \widehat{0.5}$. Suppose that $\widehat{t} \leq \widehat{0.5}$. Then, $\widetilde{1} - \widehat{t} \geq \widehat{0.5}$. If $\min\{\widehat{\lambda}(x), 0.5\} \leq \widehat{\lambda}(y)$, then

$$\widehat{\lambda}(z) \geq \min\{\widehat{\lambda}(x), \widehat{0.5}\} \geq \widehat{t} \text{ and}$$

and if $\min\{\widehat{\lambda}(x), \widehat{0.5}\} > \widehat{\lambda}(y)$, then $\widehat{\lambda}(z) \geq \widehat{\lambda}(y) > \widetilde{1} - \widehat{t}$. Thus, $x + y \subseteq \lambda_{\widehat{t}} \cup Q\left(\widehat{\lambda}, \widehat{t}\right) = \left[\widehat{\lambda}\right]_{\widehat{t}}$ for $\widehat{t} \leq \widehat{0.5}$. We have similar result for the case (iii). For final case, if $\widehat{t} > \widehat{0.5}$, then $\widetilde{1} - \widehat{t} < \widehat{0.5}$. Hence,

$$\begin{aligned}
\widehat{\lambda}(z) &\geq \min\{\widehat{\lambda}(x), \widehat{\lambda}(y), \widehat{0.5}\} \\
&= \begin{cases} \widehat{0.5} > \widetilde{1} - \widehat{t} & \text{if } \min\left\{\widehat{\lambda}(x), \widehat{\lambda}(y)\right\} \geq \widehat{0.5}, \\ \min\left\{\widehat{\lambda}(x), \widehat{\lambda}(y)\right\} > \widetilde{1} - \widehat{t} & \text{if } \min\left\{\widehat{\lambda}(x), \widehat{\lambda}(y)\right\} < \widehat{0.5}, \end{cases}
\end{aligned}$$

and so $(x + y) \subseteq Q\left(\widehat{\lambda}, \widehat{t}\right) \subseteq \left[\widehat{\lambda}\right]_{\widehat{t}}$. If $\widehat{t} \leq \widehat{0.5}$, then $\widetilde{1} - \widehat{t} \geq \widehat{0.5}$. Thus,

$$\begin{aligned}
\widehat{\lambda}(z) &\geq \min\{\widehat{\lambda}(x), \widehat{\lambda}(y), \widehat{0.5}\} \\
&= \begin{cases} \widehat{0.5} \geq \widehat{t} & \text{if } \min\left\{\widehat{\lambda}(x), \widehat{\lambda}(y)\right\} \geq \widehat{0.5}, \\ \min\left\{\widehat{\lambda}(x), \widehat{\lambda}(y)\right\} > \widetilde{1} - \widehat{t} & \text{if } \min\left\{\widehat{\lambda}(x), \widehat{\lambda}(y)\right\} < \widehat{0.5}, \end{cases}
\end{aligned}$$

which implies that $x + y \subseteq \lambda_{\widehat{t}} \cup Q\left(\widehat{\lambda}, \widehat{t}\right) = \left[\widehat{\lambda}\right]_{\widehat{t}}$.

Now, let $x \in [A]_{\widehat{t}}$ and $y \in$R. Then, $\widehat{\lambda}(x) \geq \widehat{t}$ or $\widehat{\lambda}(x) + \widehat{t} > \widetilde{1}$. Assume that $\widehat{\lambda}(x) \geq \widehat{t}$ by Theorem 224 (b), implies that

$$\widehat{\lambda}(yx) \geq \min\{\widehat{\lambda}(x), \widehat{0.5}\} \geq \min\{\widehat{t}, \widehat{0.5}\} = \begin{cases} \widehat{t} & \text{if } \widehat{t} \leq \widehat{0.5}, \\ \widehat{0.5} > \widetilde{1} - \widehat{t} & \text{if } \widehat{t} > \widehat{0.5}, \end{cases}$$

so that $yx \in \lambda_{\widehat{t}} \cup Q\left(\widehat{\lambda}, \widehat{t}\right) = \left[\widehat{\lambda}\right]_{\widehat{t}}$. Suppose that $\widehat{\lambda}(x) + \widehat{t} > \widetilde{1}$. If $\widehat{t} > \widehat{0.5}$, then

$$\widehat{\lambda}(yx) \geq \min\{\widehat{\lambda}(x), \widehat{0.5}\} = \begin{cases} \widehat{0.5} > \widetilde{1} - \widehat{t} & \text{if } \widehat{t} \leq \widehat{0.5}, \\ \widehat{\lambda}(x) > \widetilde{1} - \widehat{t} & \text{if } \widehat{t} > \widehat{0.5}, \end{cases}$$

and thus $yx \in Q\left(\widehat{\lambda}, \widehat{t}\right) \subseteq \left[\widehat{\lambda}\right]_{\widehat{t}}$. Similarly, $xy \in \left[\widehat{\lambda}\right]_{\widehat{t}}$. Consequently, $[A]_{\widehat{t}}$ is a hyperideal of R.

Conversely, suppose that $\widehat{\lambda}$ is an n-dimensional fuzzy set in R such that $\left[\widehat{\lambda}\right]_{\widehat{t}}$ is a hyperideal of R. Suppose that $\widehat{\lambda}$ is not an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal of R. Then, there exist $x, y \in$R such that

$$\inf_{z \in x+y} \widehat{\lambda}(z) < \min\{\widehat{\lambda}(x), \widehat{\lambda}(y), \widehat{0.5}\}.$$

Let

$$\widehat{t} = \frac{1}{2}\left[\inf_{z \in x+y} \widehat{\lambda}(z) + \min\{\widehat{\lambda}(x), \widehat{\lambda}(y), \widehat{0.5}\}\right].$$

Then,

$$\inf_{z \in x+y} \widehat{\lambda}(z) < \widehat{t} < \min\{\widehat{\lambda}(x), \widehat{\lambda}(y), \widehat{0.5}\}.$$

This implies that $x, y \in [A]_{\widehat{t}}$ and $(x+y) \subseteq \left[\widehat{\lambda}\right]_{\widehat{t}}$. Hence, $\widehat{\lambda}(z) \geq \widehat{t}$ or $\widehat{\lambda}(x+y) + \widehat{t} > \widetilde{1}$ for each $z \in x+y$, which is a contradiction. Therefore, we have

$$\inf_{z \in x+y} \widehat{\lambda}(z) \geq \min\{\widehat{\lambda}(x), \widehat{\lambda}(y), \widehat{0.5}\},$$

for all $x, y \in$R. Similarly, we can show that

$$\widehat{\lambda}(xy) \geq \min\{\widehat{\lambda}(y), \widehat{0.5}\} \text{ and } \widehat{\lambda}(yx) \leq \max\{\widetilde{\lambda}_A(y), \widehat{0.5}\},$$

for all $x, y \in$R. Hence, $\widehat{\lambda}$ is an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal of R. This completes the proof. ■

Corollary 232 *An n-dimensional fuzzy subset $\widehat{\lambda}$ of R is an n-dimensional $(\in, \in \vee q)$-fuzzy k-hyperideal of R if and only if $\left[\widehat{\lambda}\right]_{\widehat{t}}$ is a k-hyperideal of R for all $\widehat{t} \in I_n$*

Proof. Suppose that $\widehat{\lambda}$ is an n-dimensional $(\in, \in \vee q)$-fuzzy k-hyperideal of R. Then, this follows from Theorem 231 and Corollary 226 $\left[\widehat{\lambda}\right]_{\widehat{t}}$ is a hyperideal of R.

Conversely, suppose that $\widehat{\lambda}$ is an n-dimensional fuzzy set of R such that $\left[\widehat{\lambda}\right]_{\widehat{t}}$ is a hyperideal of R. Then, by using Theorem 231 and Corollary 226, $\widehat{\lambda}$ is an n-dimensional $(\in, \in \vee q)$-fuzzy k-hyperideal of R. ■

6.7 n-Dimensional (α, β)-fuzzy prime hyperideals in semihyperrings

Definition 233 *An n-dimensional (α, β)-fuzzy hyperideal $\widehat{\lambda}$ of R is called semiprime if for all $x \in R$, $\widehat{t} \in I_n, x^2_{\widehat{t}} \alpha \widehat{\lambda}$ implies that $x_{\widehat{t}} \beta \widehat{\lambda}$, where $\alpha \neq \in \wedge q$. An n-dimensional (α, β)-fuzzy hyperideal $\widehat{\lambda}$ of R is called prime if for all $x, y \in R$ and $\widehat{t} \in I_n$, $(xy)_{\widehat{t}} \alpha \widehat{\lambda}$ implies that $x_{\widehat{t}} \beta \widehat{\lambda}$ or $y_{\widehat{t}} \beta \widehat{\lambda}$. An n-dimensional (α, β)-fuzzy k-hyperideal $\widehat{\lambda}$ of R is called prime (semiprime) if it is prime (semiprime).*

We put $\alpha = \in$ and $\beta = \in \vee q$ in Definition 233, we get the definition of prime (semiprime) n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal (k-hyperideal) $\widehat{\lambda}$ of R.

In Theorem shows the condition of a prime n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal $\widehat{\lambda}$ of R.

Theorem 234 *An n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal $\widehat{\lambda}$ of R is prime if and only if* $\max\left(\widehat{\lambda}(x), \widehat{\lambda}(y)\right) \geq \min\left(\widehat{\lambda}(xy), \widehat{0.5}\right)$ *for all $x, y \in R$.*

Proof. Let $\widehat{\lambda}$ be an n-dimensional $(\in, \in \vee q)$-fuzzy prime hyperideal of R. Let $x, y \in$R, such that $\max\left(\widehat{\lambda}(x), \widehat{\lambda}(y)\right) < \min\left(\widehat{\lambda}(xy), \widehat{0.5}\right)$. Choose a $\widehat{t} \in I_n^{0.5}$ such that

$$\max\left(\widehat{\lambda}(x), \widehat{\lambda}(y)\right) < \widehat{t} < \min\left(\widehat{\lambda}(xy), \widehat{0.5}\right).$$

Then $(xy)_{\widehat{t}} \in \widehat{\lambda}$ but $x_{\widehat{t}} \overline{\in \vee q} \widehat{\lambda}$, and $y_{\widehat{t}} \overline{\in \vee q} \widehat{\lambda}$. This is a contradiction to our supposition. Hence $\max\left(\widehat{\lambda}(x), \widehat{\lambda}(y)\right) \geq \min\left(\widehat{\lambda}(xy), \widehat{0.5}\right)$.

Conversely, assume that $\max\left\{\widehat{\lambda}(x), \widehat{\lambda}(y)\right\} \geq \min\left\{\widehat{\lambda}(xy), \widehat{0.5}\right\}$ for all $x, y \in$R. Let $(xy)_{\widehat{t}} \in \widehat{\lambda}$. Then,

$$\max\left\{\widehat{\lambda}(x), \widehat{\lambda}(y), \widehat{0.5}\right\} \geq \min\left\{\widehat{\lambda}(xy), \widehat{0.5}\right\} \geq \min\left\{\widehat{t}, \widehat{0.5}\right\}.$$

If $\widehat{t} \leq \widehat{0.5}$, then $\max\left\{\widehat{\lambda}(x), \widehat{\lambda}(y)\right\} \geq \widehat{t}$. Thus either $\widehat{\lambda}(x) \geq \widehat{t}$ or $\widehat{\lambda}(y) \geq \widehat{t}$, that is either $x_{\widehat{t}} \in \widehat{\lambda}$ or $y_{\widehat{t}} \in \widehat{\lambda}$. If $\widehat{t} > \widehat{0.5}$, then $\max\left\{\widehat{\lambda}(x), \widehat{\lambda}(y)\right\} \geq \widehat{0.5}$. Thus either $\widehat{\lambda}(x) + \widehat{t} > \widehat{0.5} + \widehat{0.5} = \widehat{1}$ or $\widehat{\lambda}(y) + \widehat{t} > \widehat{0.5} + \widehat{0.5} = \widehat{1}$. Hence, either $x_{\widehat{t}} \in \vee q \widehat{\lambda}$ or $y_{\widehat{t}} \in \vee q \widehat{\lambda}$. Therefore $\widehat{\lambda}$ is an n-dimensional $(\in, \in \vee q)$-fuzzy prime hyperideal. ■

Corollary 235 *An n-dimensional $(\in, \in \vee q)$-fuzzy k-hyperideal $\widehat{\lambda}$ of R is prime if and only if* $\max\left\{\widehat{\lambda}(x), \widehat{\lambda}(y)\right\} \geq \min\left\{\widehat{\lambda}(xy), \widehat{0.5}\right\}$ *for all $x, y \in R$.*

Proof. Proof follows from Theorem 234. ■

Proposition 236 *An n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal $\widehat{\lambda}$ of R is semiprime if and only if $\widehat{\lambda}(x) \geq \min\left\{\widehat{\lambda}\left(x^2\right), \widehat{0.5}\right\}$ for all $x \in R$.*

Proof. Similar to the proof of the above Theorem 234. ■

Corollary 237 *An n-dimensional $(\in, \in \vee q)$-fuzzy k-hyperideal $\widehat{\lambda}$ of R is semiprime if and only if $\widehat{\lambda}(x) \geq \min\left(\widehat{\lambda}\left(x^2\right), \widehat{0.5}\right)$ for all $x \in R$.*

Theorem 238 *An n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal $\widehat{\lambda}$ of R is prime if and only if $\widehat{\lambda}_{\widehat{t}} \neq \varphi$ is a prime hyperideal of R for all $\widehat{0} < \widehat{t} \leq \widehat{0.5}$.*

Proof. Let $\widehat{\lambda}$ be an n-dimensional $(\in, \in \vee q)$-fuzzy prime hyperideal of R and $\widehat{t} \in I_n$. Then by Theorem 225 $\widehat{\lambda}_t$ is a hyperideal of R. Let $x, y \in$R, such that $xy \in \widehat{\lambda}_t$. Since $\widehat{\lambda}$ is an $(\in, \in \vee q)$-fuzzy prime hyperideal, therefore by Proposition 234 $\max\left\{\widehat{\lambda}(x), \widehat{\lambda}(y)\right\} \geq \min\left\{\widehat{\lambda}(xy), \widehat{0.5}\right\} \geq \min\left\{\widehat{t}, \widehat{0.5}\right\} = \widehat{t}$, so $\widehat{\lambda}(x) \geq \widehat{t}$ or $\widehat{\lambda}(y) \geq \widehat{t}$. Thus, $x \in \widehat{\lambda}_{\widehat{t}}$ or $y \in \widehat{\lambda}_{\widehat{t}}$. Hence $\widehat{\lambda}_{\widehat{t}}$ is a prime hyperideal of R.

Conversely: assume that $\widehat{\lambda}_{\widehat{t}}$ is a prime hyperideal of R for each $\widehat{t} \in I_n$. Then by Theorem 225, $\widehat{\lambda}$ is an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal of R. Let $\widehat{t} \leq \widehat{0.5}$ and $(xy)_{\widehat{t}} \in \widehat{\lambda}$. Then $xy \in \widehat{\lambda}_{\widehat{t}}$ so either $x \in \widehat{\lambda}_{\widehat{t}}$ or $y \in \widehat{\lambda}_{\widehat{t}}$ that is $x_{\widehat{t}} \in \widehat{\lambda}$ or $y_{\widehat{t}} \in \widehat{\lambda}$. If $\widehat{t} > \widehat{0.5}$ and $(xy)_{\widehat{t}} \in \widehat{\lambda}$. Then $\widehat{\lambda}(xy) \geq \widehat{t} > \widehat{0.5}$. Thus, $xy \in \widehat{\lambda}_{\widehat{0.5}}$ and so $x \in \widehat{\lambda}_{\widehat{0.5}}$ or $y \in \widehat{\lambda}_{\widehat{0.5}}$. This implies either $x_{\widehat{t}} q \widehat{\lambda}$ or $y_{\widehat{t}} q \widehat{\lambda}$. This shows that $\widehat{\lambda}$ is an n-dimensional $(\in, \in \vee q)$-fuzzy prime hyperideal of R. ■

Corollary 239 *An n-dimensional $(\in, \in \vee q)$-fuzzy k-hyperideal $\widehat{\lambda}$ of R is prime if and only if $\widehat{\lambda}_t \neq \varphi$ is a prime k-hyperideal of R for all $\widehat{0} < \widehat{t} \leq \widehat{0.5}$.*

Theorem 240 *An n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal $\widehat{\lambda}$ of R is semiprime if and only if $\widehat{\lambda}_{\widehat{t}} \neq \varphi$ is a semiprime hyperideal of R for all $\widehat{0} < \widehat{t} \leq \widehat{0.5}$.*

Proof. Similar to the proof of the above Theorem. ■

Corollary 241 *An n-dimensional $(\in, \in \vee q)$-fuzzy k-hyperideal $\widehat{\lambda}$ of R is semiprime if and only if $\widehat{\lambda}_{\widehat{t}} \neq \varphi$ is a semiprime k-hyperideal of R for all $\widehat{0} < \widehat{t} \leq \widehat{0.5}$.*

Theorem 242 *An non empty subset I of R is a prime hyperideal if and only if $\mathcal{X}_I$ is an n-dimensional $(\in, \in \vee q)$-fuzzy prime hyperideal of R.*

Proof. Let I be a prime hyperideal of R. Then, by Theorem 211, $\mathcal{X}_I$ is an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal of R. Let $(xy)_{\widehat{t}} \in \mathcal{X}_I$ this implies that $\mathcal{X}_I(xy) \geq \widehat{t}$ this implies that $xy \in I$. Since I is a prime hyperideal, so $x \in I$ or $y \in I$ this implies that $\mathcal{X}_I(x) = \widehat{1}$ or $\delta_I(y) = \widehat{1}$.Hence, $x_{\widehat{t}} \in \vee q\mathcal{X}_I$ or $y_{\widehat{t}} \in \vee q\mathcal{X}_I$.

Conversely: Let $\mathcal{X}_I$ be an n-dimensional $(\in, \in \vee q)$-fuzzy prime hyperideal of R. and $xy \in I$ this implies that $\mathcal{X}_I(xy) = \widehat{1} \geq \widehat{t}$ this implies $(xy)_{\widehat{t}} \in \delta_I$ therefore $x_{\widehat{t}} \in \vee q\mathcal{X}_I$ or $y_{\widehat{t}} \in \vee q\mathcal{X}_I$ this implies that $\mathcal{X}_I(x) \geq \widehat{t}$ or $\mathcal{X}_I(x) + \widehat{t} > \widehat{1}$ or $\mathcal{X}_I(y) \geq \widehat{t}$ or $\mathcal{X}_I(y) + \widehat{t} > \widehat{1}$ this implies that $x_{\widehat{t}} \in \mathcal{X}_I$ or $y_{\widehat{t}} \in \mathcal{X}_I$ this implies that $\mathcal{X}_I(x) = \widehat{1}$ or $\mathcal{X}_I(y) = \widehat{1}$. Hence, $x \in I$ or $y \in I$, this shows that I is a prime hyperideal of R. ■

Theorem 243 *Let $\left\{\widehat{\lambda}_i : i \in \Lambda\right\}$ be a family of n-dimensional $(\in, \in \vee q)$-fuzzy prime hyperideals of a semihyperring R. Then, $\widehat{\lambda} = \bigcap\limits_{i \in \Lambda} \widehat{\lambda}_i$ is an n-dimensional $(\in, \in \vee q)$-fuzzy semi-prime hyperideal of R.*

Proof. By Theorem 227 $\widehat{\lambda} = \bigcap\limits_{i \in \Lambda} \widehat{\lambda}_i$ is an n-dimensional $(\in, \in \vee q)$-fuzzy hyperideal of R. Let $x \in$R$, \widehat{t} \in I_n$ such that $\left(x^2\right)_{\widehat{t}} \in \widehat{\lambda}$. Assume that $x_{\widehat{t}} \overline{\in \vee q} \widehat{\lambda}$, then $\widehat{\lambda}(x) < \widehat{t}$ and $\widehat{\lambda}(x) + \widehat{t} \leq \widehat{1}$. Thus

$$\widehat{\lambda}(x) < \widehat{0.5} \qquad (1)$$

Let $\Omega_1 = \left\{i \in \Lambda : \widehat{\lambda}_i(x) \geq \widehat{t}\right\}$ and $\Omega_2 = \left\{i \in \Lambda : \widehat{\lambda}_i(x) + \widehat{t} > \widehat{1} \text{ and } \widehat{\lambda}_i(x) < \widehat{t}\right\}$ Then $\Lambda = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$ if $\Omega_2 = \emptyset$, then $\widehat{\lambda}_i(x) \geq \widehat{t}$ for all $i \in \Lambda$ and so $\widehat{\lambda}(x) \geq \widehat{t}$ which is a contradiction to (1). Thus $\Omega_2 \neq \emptyset$ and hence for each $i \in \Omega_2$ we have $\widehat{\lambda}_i(x) + \widehat{t} > \widehat{1}$ and $\widehat{\lambda}_i(x) < \widehat{t}$. It follows that $t > 0.5$, so $\widehat{\lambda}_i(x) \geq \widehat{\lambda}(x) > t > 0.5$ for all $i \in \Lambda$.

Now suppose that $s = \widehat{\lambda}_i(x) < \widehat{0.5}$ for some $i \in \Lambda$. Let $\widehat{s^{/}} \in I_n$ be such that $\widehat{s} < \widehat{s^{/}}$, then we get $\widehat{\lambda}_i(x) = \widehat{s} < \widehat{s^{/}}$ and $\widehat{\lambda}_i(x) + \widehat{s^{/}} < \widehat{1}$ that is $x_{\widehat{s^{/}}} \overline{\in \vee q} \widehat{\lambda}_i$. This is a contradiction to hypothesis that $\widehat{\lambda}_i$ is an n-dimensional $(\in, \in \vee q)$ fuzzy prime hyperideal of R. Hence, $\widehat{\lambda}_i(x) \geq \widehat{0.5}$ for all $i \in \Lambda$ and thus $\widehat{\lambda}(x) \geq \widehat{0.5}$ this contradicts (1), therefore $x_{\widehat{t}} \in \vee q\widehat{\lambda}$. ■

Corollary 244 *Let* $\left\{\widehat{\lambda}_i : i \in \Lambda\right\}$ *be a family of* n*-dimensional* $(\in, \in \vee q)$*-fuzzy prime* k*-hyperideals (h-hyperideal) of a semihyperring* R*, then* $\widehat{\lambda} = \bigcap\limits_{i \in \Lambda} \widehat{\lambda}_i$ *is an* $(\in, \in \vee q)$*-fuzzy semiprime* k*-hyperideal of* R*.*

Theorem 245 *A fuzzy subset* $\widehat{\lambda}$ *of* R *is an* $(\in, \in \vee q)$*-fuzzy prime hyperideal of* R *if and only if* $\left[\widehat{\lambda}\right]_{\widehat{t}}$ *is a prime hyperideal of* R*, for all* $t \in I_n$*.*

Proof. Proof follows from Theorem 231. ■

Conclusion 246 *In this thesis, we have investigate some new characterization of some kinds of semihyperrings. Semihyperring owe their importance to the fact that so many models arising in the solutions of specific problems turn out to be semihyperrings. For this reason, the basic concepts introduced here have exhibited some universality and are applicable in so many diverse contexts. These concepts are important and effective tools in hyperalgebraic systems, automata and artificial intelligence. Our future work on this topic will focus on studying h-hyperideals, qausi-hyperideals, intuitionistic and their (soft)or interval valued fuzzy hyperideals,* n*-dimensional fuzzy* k*-hyperideals, bi-*k*-hyperideals and their* (α, β)*-fuzzy hyperideals etc.*

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