More results on almost cohen macaulay modules

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DEDICATION

This Thesis is dedicated to:

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For their love, patience and unfailing support to me.

A strong and gentle soul who always taught me to trust

in God Almighty, believe in hard work and that so much

could be done with little.

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Chapter 1

Preliminaries

In this chapter we provide a brief and compact glance on important definitions and some basic results used in the succeeding chapters of this dissertation. We attempt to makeup some background informations regarding to rings, modules and many of their properties. Details of this literature are provided in [2], [4], [9].

1.1 Rings

Here we shall describe basic definitions of ring, ideal, prime ideal, maximal ideal, quotient ring, integral domain and their properties with some examples.

Definition 1.1.1.

A set T which is non empty with two binary operations addition "+" and multiplication "." is known as a ring, if following postulates are true in T given below.

(1) T under "+" becomes an abelian group.

(2) Associative law under "." holds in T.

$\acute{f}.(\acute{g}.\acute{h}) = (\acute{f}.\acute{g}).\acute{h}, \forall\ \acute{f}, \acute{g}, \acute{h} \in T.$

(3)Distributive laws of multiplication over addition satisfies in T.

$\acute{f}.(\acute{g} + \acute{h}) = (\acute{f}.\acute{g}) + (\acute{f}.\acute{h})$ (left distributive law of "." over "+") and

$(\acute{f} + \acute{g}).\acute{h} = (\acute{f}.\acute{h}) + (\acute{g}.\acute{h})$ (right distributive law of "." over "+") $\forall\ \acute{f}, \acute{g}, \acute{h} \in T.$

Example 1.1.2.

$(M_r(R), +, .)$ becomes a ring over set of real numbers, where $r > 1$. We call it as ring of square matrices having order "r" over real numbers.

Definition 1.1.3.

If multiplication "." is commutative in any ring T and $\exists$ an element $\acute{e}$ $\in$T for which $\acute{f}.\acute{e} = \acute{f} = \acute{e}.\acute{f}$, for each $\acute{f}$ $\in$T, then T is a commutative ring with unity $\acute{e}$.

Example 1.1.4.

($\mathbb{Z}$,+,.), the ring of integers is a commutative with 1 as unity.

Ring ($\mathbb{Q}$,+,.) of rational numbers is a commutative with unity 1.

Ring ($\mathbb{R}$,+,.) of real numbers is a commutative having unity 1.

Proposition 1.1.5. [4]

Following below statements hold in any ring T.

(i) $0\acute{f} = 0 = \acute{f}0$, for all $\acute{f}$ $\in$T.

(ii) $(-\acute{g})(\acute{h}) = -(\acute{h}\acute{g}) = (\acute{f})(-\acute{g})$, $\forall\ \acute{g}, \acute{h}$ $\in$T. (recall $-\acute{g}$ denotes additive inverse of $\acute{b}$)

(iii) $(-\acute{f})(-\acute{g}) = \acute{f}\acute{g}$, $\forall\ \acute{f}, \acute{g} \in$ T.

(iv) If a ring T contains an identity element say $\acute{e}$, then this identity $\acute{e}$ is unique & $-\acute{d} = (-1)\acute{d}$, for each $\acute{d} \in$ T.

Definition 1.1.6.

An element t∈T is a zero-divisor of T, if

(i) $t \neq 0$.

(ii) $\exists\ 0 \neq t'$ in T, such that we have $tt' = 0 = t't$.

An element in T is a non zero-divisor if it is not zero-divisor.

Note :

There may be more than one zero-divisors in a ring.

Example 1.1.7.

2 and 3 are both zero-divisors in ring $\mathbb{Z}_6$, because $\overline{2} \times \overline{3} = \overline{3} \times \overline{2} = \overline{6} = \overline{0}$.

Definition 1.1.8.

In a commutative ring T with unity, if all elements of T are non zero-divisors, then T becomes an integral domain(I.D).

Example 1.1.9.

Consider $\mathbb{Z}_7 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}$ which is a commutative ring containing identity $\overline{1}$. Now we see $\overline{1} \times \overline{2} = \overline{2} \neq \overline{0}$

$\overline{1} \times \overline{3} = \overline{3} \neq \overline{0}$

$\overline{2} \times \overline{3} = \overline{6} \neq \overline{0}$

$\overline{2} \times \overline{4} = \overline{1} \neq \overline{0}$

$\overline{2} \times \overline{5} = \overline{3} \neq \overline{0}$

$\overline{2} \times \overline{6} = \overline{5} \neq \overline{0}$

$\overline{3} \times \overline{4} = \overline{5} \neq \overline{0}$

The otheres possible multiplication of elements with each other can be seen. So, $\mathbb{Z}_7$ is an I.D.

$\mathbb{Z}$, the ring of integers is also an I.D.

Note :

Ring $\mathbb{Z}_r = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, ..., \overline{r}\}$ *of modulo "r" is an I.D iff "r" is a prime number.*

Definition 1.1.10.

Consider a commutative ring T with unity 1. Then T is called a field, if each element m∈ T has a multiplicative inverse in T i.e for every $m' \in T$ there exists $m'' \in T$ such that $m'm'' = 1 = m''m'$.

Example 1.1.11.

$\mathbb{R}$ (ring of real numbers) is a field.

Corollary 1.1.12. [1]

A finite I.D is a field.

Definition 1.1.13.

Consider T as commutative ring with unity. A non-empty subset $\mathcal{B}$ is known an ideal of T, if it full fills

(i) $\acute{n} - \acute{l} \in \mathcal{B}$, for all $\acute{n}, \acute{l} \in \mathcal{B}$.

(ii) tm, mt $\in \mathcal{B}$, for all t $\in$ T and $m \in \mathcal{B}$.

Example 1.1.14.

$4\mathbb{Z}$ is ideal in $\mathbb{Z}$.

Definition 1.1.15.

An ideal W becomes a maximal ideal in a ring T, if

(1) $W \subsetneq T$.

(2) There exists no ideal K in T such that $W \subsetneq K \subsetneq T$.

Equivalently we can say that

Maximal ideal W of T is that ideal which has the property that if for any ideal K of T such that $W \subseteq K \subseteq T$, then either W = K or K = T.

Example 1.1.16.

$r\mathbb{Z}$ is a maximal ideal in $\mathbb{Z} \Leftrightarrow r$ is prime.

Remark 1.1.17.

If T is a field, then $\{0\}$ is the only maximal ideal of T.

Definition 1.1.18.

A proper ideal A in T is prime if $\acute{a}\acute{b} \in A$, then either $\acute{a} \in A$ or $\acute{b} \in A$.

Example 1.1.19.

$2\mathbb{Z}$ is a prime ideal for ring of integers $\mathbb{Z}$.

Definition 1.1.20.

An ideal of ring which is generated by single element is called a principle ideal.

Every ideal in ring of integers is principle.

Definition 1.1.21.

Suppose T is a ring and W is ideal of T, then $T/W = \{i + W : i \in T\}$ represents family of co-sets of ideal W in T. Under the operations of "+" and "." defined below, T/W forms a ring called quotient ring. Where "+" and "." are given as follows:

$(i + W) + (\acute{i} + W) = (i + \acute{i}) + W$, for all i, $\acute{i} \in$ T.

$(i + W)(\acute{i} + W) = (i\acute{i}) + W$, for all i, $\acute{i} \in T$.

Example 1.1.22.

Consider $\mathbb{Z}$ as the ring of integers and $6\mathbb{Z}$ as its ideal. Then $\mathbb{Z}/6\mathbb{Z}$ becomes the quotient ring.

Lemma 1.1.23. [2]

Consider T as ring and W is its ideal. Then T/W is an I.D iff W becomes prime ideal of T.

Definition 1.1.24.

The intersection of all maximal ideals in T is known as Jacobson radical in T and denoted as J(T). For example $J(\mathbb{Z}/12\mathbb{Z}) = 6\mathbb{Z}/12\mathbb{Z} = 2\mathbb{Z}/12\mathbb{Z} \cap 3\mathbb{Z}/12\mathbb{Z}$.

Definition 1.1.25.

T is known as a Notherian if ascending chain condition for ideals holds in T i.e each infinite ascending chain of ideals $W_1 \subset W_2 \subset ... \subset W_r \subset ...$ stabilizes i.e $\exists$ natural number r for which $W_m = W_r$ for every $m \geq r$.

Theorem 1.1.26. [9]

A ring T is Notherian iff each ideal in T is finitely generated.

Example 1.1.27.

1. $\mathbb{Z}$ is Notherian because its each ideal is generated by one element.

2. The ring of polynomials $R[Y_1, Y_2, ...Y_r]$ over real numbers is also a Nothrian ring.

3.The ring of polynomials $R[Y_1, Y_2, ...Y_r....]$ over real numbers is not a Nothrian ring.

Definition 1.1.28.

T is called a local ring if T contains only one maximal ideal.

Example 1.1.29.

As $4\mathbb{Z}$ is ideal of $\mathbb{Z}$. The quotient ring $\mathbb{Z}/4\mathbb{Z}$ is a finite local ring.

Definition 1.1.30.

Consider S and T as rings. A mapping $\gamma : S \to T$ becomes ring homomorphism if it agrees

(1). $\gamma(s^* + s^{**}) = \gamma(s^*) \dot{+} \gamma(s^{**})$.

(2). $\gamma(s^*.s^{**}) = \gamma(s^*) \circ \gamma(s^{**})$

for any $s^*, s^{**} \in S$.

Example 1.1.31.

The mapping $\mathbb{Z} \to \mathbb{Z}_r$ between ring of integers and modulo ring defined as $z \to z(modr)$ is a ring homomorphism.

Definition 1.1.32.

A ring homomorphism $\gamma : S \to T$ between rings S and T is called surjective if for each $t \in T$, $\exists\ s \in S$ for which $\gamma(s) = t$.

Example 1.1.33.

The mapping $\mathbb{Z} \to \mathbb{Z}_r$ between ring of integers and modulo ring defined as $z \to z(mod r)$ is a surjective ring homomorphism.

Definition 1.1.34.

A ring homomorphism $\gamma : S \to T$ between rings S and T is called injective if each element of S has distinct image in T.

Example 1.1.35.

The mapping $S \to S \times T$ defined by $s \to (s, 0)$ between rings S and T is an injective ring homomorphism, where $S \times T = \{(s^*, t^*) : s^* \in S, t^* \in T\}$ is ring with component wise multiplication and addition.

Definition 1.1.36.

Let $\gamma : S \to T$ be a ring homomorphism between rings S and T. We define kernal and image of γ as follows:

$$ker(\gamma) = \{s \in S : \gamma(s) = 0\},$$

and

$$Im(\gamma) = \{\gamma(s) : s \in S\} = \gamma(S).$$

Example 1.1.37.

1. Kernal of mapping $\mathbb{Z} \to \mathbb{Z}_r$ defined as $z \to z(modr)$ is $r\mathbb{Z}$.

2. Image of mapping $\mathbb{Z} \to \mathbb{Z}_r$ defined as $z \to z(modr)$ is $\mathbb{Z}_r$.

Remark 1.1.38. [12]

Suppose $\gamma : S \to T$ is a ring homomorphism between rings S and T. Then $ker(\gamma)$ becomes an ideal of S.

Definition 1.1.39.

Let (S,$\underline{\text{a}}$) and (T,$\underline{b}$) be two local rings. A ring homomorphism $\beta : S \to T$ becomes a local morphism when $\beta(\underline{a}) \subset \underline{b}$.

1.2 Modules

In this section we shall describe basic concept of modules, submodules, quotient modules, category, functor and their basis characteristics with some examples. Details are given in literature [4], [9], [10], [11], [12].

Definition 1.2.1.

Suppose T is a ring (not necessarily commutative or having unity). A non-empty set F is said to be a left module over T (Or called T-module), if it satisfies the following postulates .

(a) (F,+) becomes an abelian group.

(b) Defines action of T on F (map; $T \times F \to F$) which is indicated by $tf' \ \forall \ t \in T$ and $f' \in F$ fullfilling the following conditions:

(i) $(t' + t'')\acute{f} = t'\acute{f} + t''\acute{f}$, for all $t', t'' \in T$ and $\acute{f} \in F$.

(ii) $t'(\acute{f} + f'') = t'\acute{f} + t'f''$, for any $t' \in T$ and $\acute{f}, f'' \in F$.

(iii) $(t't'')f' = t'(t''f')$, $\forall \ t', t'' \in T$ & $f' \in F$.

If a ring has unity say 1, then we define the following condition

(iv) $1f' = f'$, $\forall \ f' \in$ F.

Note

For a commutative ring T, if F is a left T-module, then F can be made a right T-module, by defining multiplication as $f''t'' = t''f''$, for all $f'' \in F$ and $t'' \in T$.

Example 1.2.2.

(1) Any abelian group Q over $\mathbb{Z}$ is module.

(2) Each ring is a module over itself.

(3) Matrices $M_{i,j}(T)$ is also a module over T.

Definition 1.2.3.

Suppose E and F are two modules over T, then the map $\alpha : E \to F$ becomes a T-module homomorphism, if the two conditions are fullfilled.

(1) $\alpha(0) = 0^*$ and $\alpha(e_2 + e_1) = \alpha(e_2) + \alpha(e_1)$, for each $e_1, e_2 \in E$.

(2) $\alpha(te) = t\alpha(e)$, for $t \in T$ and $e \in E$.

Example 1.2.4.

(1) Consider ring of integers $\mathbb{Z}$ and take module also $\mathbb{Z}$. Then the map $\rho : \mathbb{Z} \to \mathbb{Z}$ defined as $\rho(c) = 2c$ is a $\mathbb{Z}$-module homomorphism.

(2) A zero homomorphism always exists between any two T-modules E and F. Zero morphism is defined as $0(e) = 0_F$ for all $e \in E$.

Definition 1.2.5.

Suppose E and F are two modules over T and $\alpha : E \to F$ is T-module homomorphism. Then kernal and image of α are defined as follows

$ker(\alpha) = \{w \in E : \alpha(w) = 0\}$ and $Im(\alpha) = \{\alpha(w) : w \in E\} = \alpha(E)$.

Remark 1.2.6. [12]

Suppose E and F are two modules over T and $\alpha : E \to F$ is T-module homomorphism. Then

(1) $ker(\alpha)$ is submodule of E.

(2) $Im(\alpha)$ is submodule of F.

Definition 1.2.7.

Suppose F is a module over T. A non empty subset F^* of F becomes sub module of F if these two conditions holds in F^*.

1. $e_1^* - e_2^* \in F^*$ for each $e_1^* - e_2^* \in F^*$.

2. For any $t' \in T$ and $e^* \in F^* \Rightarrow t'e^* \in F^*$.

Example 1.2.8.

$\mathbb{Z}$ is a commutative ring having unity 1. Then $r\mathbb{Z}$ are its sub modules for $r \in \mathbb{Z}$.

Definition 1.2.9.

Suppose F is T-module. Also suppose that F^* is a sub-module of F, then $F/F^* = \{f + F^* : f \in F\}$ is abelian group where addition is given as

$(f_1 + F^*) + (f_2 + F^*) = (f_1 + f_2) + F^*$, for every $f_1, f_2 \in F$. For every $t \in T$ & $f \in F$, define $t(f + F^*) = tf + F^*$. These operations give us a well defined action of T on F/F^* and makes F/F^* a T-module. We call F/F^* a quotient module.

Example 1.2.10.

$\mathbb{Z}/r\mathbb{Z}$ are quotient modules for $r \in \mathbb{Z}$.

Proposition 1.2.11. [12]

Suppose F is a module over T. Also consider F_1 and F_2 as sub modules of F. Then $F_1 \cap F_2$ is a T-sub module of F.

Remark 1.2.12. [10]

Suppose F is a module over T then F^* is a sub module of F $\Leftrightarrow$ for each $x, y \in F^*$ and $t', t'' \in T$ $\Rightarrow t'x + t''y \in F^*$.

Theorem 1.2.13. [4]

(1) Suppose F is a T-module . Suppose F^* and E^* are sub-modules of F, then

$$(F^* + E^*)/F^* \cong E^*/(F^* \cap E^*).$$

(2) If $F \supseteq E \supseteq G$ are all T-modules, then we have

$$(F/G)/(E/G) \cong F/E.$$

Definition 1.2.14.

Let F^* and E^* be sub-modules of F. We define sum of F^* and E^* as

$$F^* + E^* = \{f^* + e^* : f^* \in F^*, e^* \in E^*\}.$$

Definition 1.2.15.

A module F becomes a finitely generated, if it contains finite number of generators i.e $F = (f_1, f_2, ..., f_r) = f_1T + f_2T + ...f_rT,\ f_i \in F$.

Example 1.2.16.

(1) A finite dimensional vector space over a field say T is a finitely generated T-module.

(2) An abelian group which contains a finite set of generators is considered as finitely generated $\mathbb{Z}$-module. In particular, abelian groups of finite orders are finitely generated $\mathbb{Z}$-module.

Definition 1.2.17.

Consider a ring T. The sequence of T-modules and with T-modules homomorphisms

$$\cdots F_{i-1} \xrightarrow{g_{i-1}} F_i \xrightarrow{g_i} F_{i+1} \rightarrow \cdots$$

is exact if $img(g_{i-1}) = ker(g_i)$ for all i.

Example 1.2.18.

Consider $\mathbb{Z}$ and r as a prime number, then sequence

$$\cdots \mathbb{Z}_{r^2} \xrightarrow{.r} \mathbb{Z}_{r^2} \xrightarrow{.r} \mathbb{Z}_{r^2} \rightarrow \cdots$$

is exact because $img(.r) = ker(.r) = r\mathbb{Z}_{r^2}$.

Definition 1.2.19.

Let E be a right and F be a left T-modules, then a mapping $\alpha : E \times F \rightarrow G^*$, where G^* is any abelian group under addition, is said to be T-biadditive if

(a) $\alpha(e_1 + e_2, f) = \alpha(e_1, f) + \alpha(e_2, f)$.

(b) $\alpha(e, f_1 + f_2) = \alpha(e, f_1) + \alpha(e, f_2)$.

(c) $\alpha(e.t, f) = \alpha(e, t.f)$ for all $e, e_1, e_2 \in E,\ f, f_1, f_2 \in F$ and $t \in T$.

Definition 1.2.20.

Suppose E is a right and F is a left T-modules. Also G^* is any abelian group under addition. Then G^* with T-biadditive map say $\alpha : E \times F \to G^*$ is said to be a tensor product of modules E and F if for any abelian group A^* and T-biadditive map say $\beta : E \times F \to A^*$, there exists a group map $\beta^* : G^* \to A^*$ such that $\beta^* \circ \alpha = \beta$. We denote $E \otimes F$ as tensor product.

Theorem 1.2.21. [11]

Let T be a commutative ring, then $E \otimes_T F$ becomes a T-module.

Definition 1.2.22.

Consider $F_1 \to F_2 \to F_3$ as exact sequence of modules over T. Then T-module E is called flat, if the sequence $E \otimes_T F_1 \to E \otimes_T F_2 \to E \otimes_T F_3$ remains exact.

Example 1.2.23.

Any vector space over a field is a flat module.

Definition 1.2.24.

A collection of objects and morphisms between objects is called a category $\mathcal{C}$ if it satisfies following axioms

(i) For any $A^*, B^*, D^* \in \mathcal{C}$ we have a composition

$$mor(A^*, B^*) \times mor(B^*, D^*) \to mor(A^*, D^*)$$

$$(f^*, g^*) \to g^* \circ f^*$$

satisfying associative law i.e

$(f^* \circ g^*) \circ h^* = f^* \circ (g^* \circ h^*)$ where $h^* \in mor(D^*, E^*)$, where $f^* \in mor(A^*, B^*)$ and $g^* \in mor(B^*, D^*)$.

(ii) For any $A^* \in \mathcal{C}$, there exists a unique $1_{A^*} \in mor(A^*, A^*)$ satisfying $1_{A^*} \circ f^* = f^*$ and also $g^* \circ 1_{A^*} = g^*$ where $f^* \in mor(B^*, A^*)$ and $g^* \in mor(D^*, A^*)$.

Example 1.2.25.

Rings with morphisms as ring maps is an example of category called category of rings.

Definition 1.2.26.

Consider two categories $\mathcal{C}^*$ and $\mathcal{C}$. Then $F\hat{:}\mathcal{C}^* \to \mathcal{C}$ becomes functor if it full fills following properties:

(a) For each $A^* \in \mathcal{C}^*$, there exists a unique $F\hat{(}A^*) \in \mathcal{C}$.

(b) If $\alpha : A^* \to B^*$ is any morphism in $\mathcal{C}^*$, then there exists a unique morphism $F\hat{(}\alpha) : F\hat{(}A^*) \to F\hat{(}B^*)$ in $\mathcal{C}$ such that if $A^* \xrightarrow{\alpha} B^* \xrightarrow{\beta} D^*$ in $\mathcal{C}^*$ then it implies that $F\hat{(}A^*) \xrightarrow{F\hat{(}\alpha)} F\hat{(}B^*) \xrightarrow{F\hat{(}\beta)} F\hat{(}D^*)$ with $F\hat{(}\beta) \circ F\hat{(}\alpha) = F\hat{(}\beta \circ \alpha)$.

(c) $F\hat{(}1_{A^*}) = 1_{F\hat{(}A^*)}$ for all $A^* \in \mathcal{C}^*$.

Example 1.2.27.

$E \otimes F$ is a functor(here E is a right and F is a left fixed modules) from category of left modules to category of abelian groups.

Definition 1.2.28.

A functor $\hat{F}$ is an exact functor if exactness of sequence $0 \to B_1 \xrightarrow{\alpha} B_2 \xrightarrow{\beta} B_3 \to 0$ gives the exactness of sequence $0 \to F(\hat{B_1}) \xrightarrow{F(\hat{\alpha})} F(\hat{B_2}) \xrightarrow{F(\hat{\beta})} F(\hat{B_3}) \to 0$.

Example 1.2.29.

$E \otimes \bullet$ is a right exact functor for any right module E.

Definition 1.2.30.

Suppose T is a commutative ring having unity. A subset S^* is said to be a multiplicative set if following axioms are satisfied.

(a) S^* contains unity.

(b) S^* should close under multiplication i.e $j^*k^* \in S^*$ for each $j^*, k^* \in S^*$.

Example 1.2.31.

As $\mathbb{Z}$ is I.D. Therefore $S^* = Z - \{0\}$ is a multiplicative set.

Note *[11]*

(1) Consider a commutative ring T with unity and ideal P^*. Then $S^* = T - P^*$ is a multiplicative set iff P^* is prime. In such case $S^{*-1}T = T_{P^*}$ becomes a local ring.

(2) Suppose T is a commutative ring having unity. If F is a T-module, then $S^{*-1}F$ is an $S^{*-1}T$ module. The addition and multiplication of the elements of $S^{*-1}T$ are discussed in [11].

Definition 1.2.32.

Suppose F is a T-module. The set $\{t \in T : Tf = 0, where f \in F\}$ is annihilator of F and is denoted as Ann(F). It is an ideal of T.

Definition 1.2.33.

(1) The set of all prime ideals of T is known as Spectrum of T. Mathematically we write $Spec(T) = \{$A: A is a prime ideal in T$\}$.

(2) The set of all maximal ideals of T is called Maximum of T. Mathematically $Max(T) = \{$W: W is a maximal ideal in T$\}$.

(3) Consider a T-module F. Then Support of F consists of those prime ideals A of T for which $F_A \neq 0$. It is written as $Supp(F)$. Note that $Supp(F) \subset Spec(T)$.

Definition 1.2.34.

Consider A as prime ideal in T. Then A becomes an associated prime of F if $\exists\ f \in F$ such that $A = Ann(f)$ and denoted by Ass(F).

Equivalently a prime ideal $A \in Ass(F)$ iff the T-module morphism $\alpha : T/A \to F$ is injective. Also note that $Ass(F) \subset Supp(F)$ ([3]).

Example 1.2.35.

Consider module as $\mathbb{Z}$ and $2\mathbb{Z}$ a sub module of $\mathbb{Z}$. Then $Ass(\mathbb{Z}/2\mathbb{Z}) = \{2\mathbb{Z}\}$.

1.3 Depth and Regular sequences

In this section we define and explain basic concepts about regular sequences, depth, grade, height, dimension and projective dimension etc. The details are discussed in literature [3].

Definition 1.3.1.

(1) The dimension of a ring T is the maximum length say r in the chain $A_1 \subset A_2 \subset ... \subset A_r$ of prime ideals of T. If there is no upper bound of above chain, then T is infinite dimensional. We denote dimension of T simply as $dim(T)$.

Example 1.3.2.

(1)The polynomial ring $k^*[Y_1, Y_2, ...]$ over the field k^* is an infinite dimensional because the chain $(Y_1) \subset (Y_1, Y_2) \subset ...$ of prime ideals is infinite.

(2)While the dimension of $\mathbb{Z}$ is 1.

Definition 1.3.3.

(1) Consider A as prime ideal in T. The height of A is the maximum length say r in the chain $A_0 \subset A_1 \subset ... \subset A_r = A$ of prime ideals in T and we represent it as $ht(A)$. In other words we have $ht(A) = dim(T_A)$.

(2) If I^* is any proper ideal in T. The minimum of the heights of prime ideals $A \supset I^*$ is said to be height of I^*. Mathematically we can write $ht(I^*) = min_{A \supset I^*} ht(A)$, where A carries over prime ideals of T.

Example 1.3.4.

Consider polynomial ring $T = k^*[Y]$ over the field k^*. Then $I^* = (Y^2)$ is a proper ideal in T and $A = (Y)$ is the only prime ideal in T such that $A \supset I^*$. So we have $ht(I^*) = 1$.

Definition 1.3.5.

The dimension of a finitely generated module F is the maximum length of chains of prime ideals such that all prime ideals of chains belong to $Supp(F)$. In particular, we define dimension of F in two ways $dim(F) = dim(Supp(F))$ or $dim(F) = dim(T/AnnF)$.

Definition 1.3.6.

Consider a module F over T. An element $t \in T$ is an F-regular if for non-zero element $f \in F$ and $t.f = 0$ implies $f = 0$ or the map $F \xrightarrow{t} F$(multiplication by t) is injective or simply t is not a zero divisor on F.

Definition 1.3.7.

Consider a T-module F. A finite sequence $y = (y_1, y_2, ...y_r)$ containing the elements of T is known as an F-regular sequence if it satisfies axioms described below:

(a) y_j is not a zero divisor in $F/(y_1, y_2, ...y_{j-1})F$ for all $j = 1, 2, ..., r$.

(b) $F/yF \neq 0$.

The integer r is length of F-regular sequence.

Example 1.3.8.

(1)In the polynomial ring $\mathbb{Z}[y]$ both sequences 4, y and y, y-1 are regulars.

(2) Elements orders in regular sequence matter except in case of commutative Notherian local ring and finitely generated module. The elements $x^*, y^*(1 - x^*), z^*(1 - x^*)$ form a regular

sequence in polynomial ring $k[x^*, y^*, z^*]$ over a field k but the elements $y^*(1-x^*), z^*(1-x^*), x^*$ can not.

Definition 1.3.9.

Consider a module F over Notherian ring T. Also suppose I^* is proper ideal in T. An F-regular sequence $y = (y_1, y_2, ...y_r)$ contained in I^* becomes a maximal F-regular sequence in I^* if there does not exist $y_{r+1} \in I^*$ for which $(y_1, y_2, ...y_{r+1})$ becomes an F-regular sequence.

Theorem 1.3.10. [3]

Suppose T is a Notherian ring, F is its finite module and J is its ideal for which $JF \neq F$. Then all maximal F-sequences contained in J will have same length.

Definition 1.3.11.

Consider a Notherian ring T, a finite T-module F and I^* a proper ideal in T such that $I^*F \neq F$. The common length of maximal F-regular sequences which is contained in I^* becomes grade of I^* on F and is represented by $grade(I^*, F)$.

We take $grade(I^*, F) = \infty$ in the case when $I^*F = F$.

We can also define grade of a non-zero module in terms of Ext functor as follows.

$$gradeF = min\{j \in N : Ext_T^j(F, T) \neq 0\}$$

We have $gradeF = \infty$, when $F = 0$. For the definition and properties of Ext functor see [11].

Example 1.3.12.

(1) Let T=$\mathbb{Z}$=F. Take A=3$\mathbb{Z}$ a prime ideal of $\mathbb{Z}$. Then $grade(A, F) \leq dim(T) = 1$. So $grade(3\mathbb{Z}, \mathbb{Z}) = 1$ because T is an I.D.

We can generalize above as If $A = pZ$ is any ideal of $\mathbb{Z}$, then we have

$$grade(p\mathbb{Z}, \mathbb{Z}) = 1.$$

(2) Consider $T = \mathbb{Z}[\mathbb{X}]$. Take $A = (X^3+X+1, 5)$ a prime ideal where X^3+X+1 is irreducible in $\mathbb{Z}_5$. Also X^3+X+1 is irreducible in $\mathbb{Z}$. Therefore X^3+X+1 is not a zero divisor of $\mathbb{Z}[\mathbb{X}]$. Hence X^3+X+1 is regular element in $\mathbb{Z}[\mathbb{X}]$. Now we show that 5 is not a zero divisor of $\mathbb{Z}[\mathbb{X}]/(X^3+X+1)\mathbb{Z}[\mathbb{X}]$. Contrary suppose that 5 is a zero divisor of $\mathbb{Z}[\mathbb{X}]/(X^3+X+1)\mathbb{Z}[\mathbb{X}]$. This means $5(f+I) = I$, where $f \in \mathbb{Z}[\mathbb{X}]$, $I = (X^3+X+1)\mathbb{Z}[\mathbb{X}]$ and $f \notin I$. This means

$$5f + I = I$$

$$(5+I)(f+I) = I.$$

As $f \notin I$, therefore $f + I \neq I$. Hence $5 + I = I$. Which yields $5 \in I$, a contradiction. Thus 5 is not a zero divisor of $\mathbb{Z}[\mathbb{X}]/(X^3+X+1)\mathbb{Z}[\mathbb{X}]$ and 5 is regular in $\mathbb{Z}[\mathbb{X}]/(X^3+X+1)\mathbb{Z}[\mathbb{X}]$. So we have

$$2 = dim(T) \geq grade(A, T) \geq 2.$$

This can be generalized as follows: Let p be a prime number and g be an irreducible polynomial in $\mathbb{Z}_p$, ring of modulo p. If $A = (g, p)$ is a prime ideal of $T = \mathbb{Z}[\mathbb{X}]$. Then we can have

$$grade(A, T) = 2.$$

Definition 1.3.13.

Consider a Notherian local ring $(T, \underline{a}, k)$, where $\underline{a}$ is a maximal ideal of T and $k = T/\underline{a}$ its residue field. Also suppose that F is a finite T-module. Then the grade of maximal ideal $\underline{a}$ on F becomes the depth of F and is represented by $depth(F)$. In term of Ext Functor, we have

$$depthF = inf\{j \in N : Ext^j_T(k, F) \neq 0\}.$$

The value of depth belongs to the set $\mathbb{N} \cup \{\infty\}$. We have $depth(F) = \infty$, when $F = 0$.

Chapter 2

Introduction To Almost Cohen Macaulay Module

In this chapter first of all we shall define almost Cohen Macaulay (aCM) module. We shall also check the properties of aCM module in terms of theorems, corollaries and lemmas.

2.1 Almost Cohen Macaulay Modules

Definition 2.1.1.

Let T be a commutative Noetherian ring having non-zero identity and A be its prime ideal. Also consider a T-module F. Then F becomes an aCM if for each $A \in Supp(F)$, we have $grade(A, F) = grade(AT_A, F_A)$.

Example 2.1.2.

Consider T=$\mathbb{Z}$ and A=3$\mathbb{Z}$ its prime ideal. The length of maximal sequence $(0) \subset A \subset T$ is 1. We have $dim(T) = 1$. Now $grade(A, T) = 1$ because T is an I.D. Now we have following relation

$$1 = grade(A, T) \leq grade(AT_A, T_A) \leq dim(T_A) \leq dim(T) = 1.$$

This gives the equation

$$grade(A, T) = grade(AT_A, T_A) = 1.$$

So $\mathbb{Z}$ is an aCM.

Note :

The more general result is that $\mathbb{Z}$ is an aCM ring if we take its any prime ideal satisfying definition of aCM ring.

Lemma 2.1.3.

F is an aCM iff for all prime ideals A of T such that $A \in Supp(F)$ and for each F-regular sequence $(y_1, y_2, ...y_n)$ contained in A, A is associated prime of $F/(y_1, y_2, ...y_n)F$.

Proof.

Consider A as prime ideal such that $A \in Supp(F)$ and F be an aCM.

Let us take $grade(A, F) = 0$. As F is aCM. So we have $depth(AT_A, F_A) = 0$ iff $AT_A \in Ass_{T_A}(\mathrm{F}_A) = 0$ iff $A \in Ass_T(F)$ for all $A \in Supp(F)$ as required.

Now we consider the case when $grade(A, T) = n \geq 1$ and an F-regular sequence $(y_1, y_2, ...y_n)$ contained in A, then $depth(AT_A, F_A) = n$ for all $A \in Supp(F)$. From here we get

$depth(AT_A/(y_1, y_2, ...y_n)T_A, F_A/(y_1, y_2, ...y_n)F_A) = 0$

Iff $AT_A/(y_1, y_2, ...y_n)T_A \in Ass_{T_A/(y_1,y_2,....y_n)T_A}(F_A/(y_1, y_2, ...y_n)F_A)$

Iff $A/(y_1, y_2, ...y_n) \in Ass_{T/(y_1,y_2,....y_n)T}(F/(y_1, y_2, ...y_n)F)$

Iff $A \in Ass_T(F/(y_1, y_2, ...y_n)F)$.

The last relation holds because of [8, (9, A)].

Lemma 2.1.4.

Consider T as a commutative Notherian ring and its T-module F. F is aCM iff for every prime ideal A such that $A \in Supp(F)$, we have $dim(F_A) \leq depth(A, F) + 1$.

Proof.

Suppose contrary that $dim(F_A) \geq depth(A, F) + 1$ for some prime ideal A such that $A \in Supp(F)$ and F is an aCM. We have therefore $depth(A, M) = depth(AT_A, F_A) = n$, where n is finite positive number.

Now by Lemma 2.1.3, we can take a maximal F-sequence say $(y_1, y_2, ...y_n)$ contained in A.

Now let $B = Ann(F) + (y_1, y_2, ...y_n)$, where Ann(F) represents annihilator of F. So $A/B \in Ass_{T_B}(F/BF)$ by Lemma 2.1.3. Also we have $dim(F_A) = dim(R/Ann(F))_A \geq$ n+2. So $ht(A/B)$ in T/B is at least two. So [7, Theorem 144, page 107] implies that infinitely many prime ideals in T/B contained in A/B. As F is an aCM and $(y_1, y_2, ...y_n)$ is F-regular. Now we prove $F/(y_1, y_2, ...y_n)F \cong F/BF$ is an aCM over $T/(y_1, y_2, ...y_n)$. We have $grade(A/(y_1, y_2, ...y_n), AF/(y_1, y_2, ...y_n)F) = grade(A, F) - n$.

Also $grade(A/(y_1, y_2, ...y_n)T_A, (AF/(y_1, y_2, ...y_n)F)_A) = grade(AT_A, F_A) - n$. Since F is an aCM i.e $grade(A, F) = grade(AT_A, F_A)$, therefore $F/(y_1, y_2, ...y_n)F \cong F/BF$ is an aCM. So all conditions of Lemma 2.1.3 are satisfied for F/BF. Thus $Ass_{T_B}(F/BF)$ is not finite. Which contradicts to the fact that n is finite. Hence F is not aCM. This gives us that if F is an aCM, then $dim(F_A) \leq depth(A, F) + 1$ for all prime ideal $A \in Supp(F)$.

Conversely suppose $dim(F_A) \leq depth(A, F) + 1$ for all prime ideal $A \in Supp(F)$ is true. We prove that F is an aCM. Let $(y_1, y_2, ...y_n)$ be any maximal F-regular sequence contained in any prime ideal $Q \in Supp(F)$. We only need to show that $Q \in Ass_T(F/(y_1, y_2, ...y_n)F)$. For this again suppose $B = Ann(F) + (y_1, y_2, ...y_n)$ and Q/B as prime ideal of T/B. Now we have $depth(Q/B, F/B) = 0$ and $dim(F_Q) \leq n + 1$. This gives $ht(Q/B) \leq 1$ in T/B.

Now if we take $ht(Q/B) = 0$. This implies that $Q/B \in Ass_{T_B}(F/BF)$.

Now take $ht(Q/B) = 1$, there exists prime ideal $Q_0 \in Spec(T)$ containing Q and we have $Q_0/B \in Ass_{T_B}(F/BF)$. Hence $depth(Q_0, F) = n$ and since $dim(F_{Q_0}) \leq depth(Q_0, F) + 1$ was true. It implies $Q_0 = Q$. Thus $Q/B \in Ass_{T_B}(F/BF)$. In both cases we obtained that $Q \in Ass_T(F/(y_1, y_2, ...y_n)F)$ from [4, (9, A)] and by isomorphism $F/(y_1, y_2, ...y_n)F \cong F/BF$. Thus F is an aCM module by Lemma 2.1.3.

Lemma 2.1.5.

A module F over a local ring (T, Q_0) such that $dim(F) \leq depth(Q_0, F) + 1$, then F becomes an aCM.

Proof.

We apply induction on $depth(Q_0, F) - depth(A, F)$ to prove $dim(F_A) \leq depth(A, F) + 1$ for all prime ideal $A \in Supp(F)$.

Firstly if A and Q_0 are distinct such that $depth(Q_0, F) = depth(A, F)$. By using given fact that $dim(F) \leq depth(Q_0, F) + 1$, we obtain the following relation

$dim(F_A) \leq dim(F) \leq depth(Q_0, F) + 1 = depth(A, F) + 1$. Thus F is an aCM by Lemma 2.1.4.

Secondly we consider the case when $depth(A, F) \leq depth(Q_0, F)$. So $\exists$ a prime ideal A_0 such that $depth(A_0, F) = depth(A, F) + 1$ and $A_0 \subset A \subset Q_0$ from [7, Theorem 128, page 93]. Also the inequality $dim(F_{A_0}) \leq depth(A_0, F) + 1$ holds due to Induction hypothesis. By using all above facts we get finally $dim(F_A) \leq dim(F_{A_0}) \leq depth(A_0, F) + 1 = depth(A, F) + 2$. This gives us $dim(F_A) \leq depth(A, F) + 1$. So F becomes an aCM by Lemma 2.1.4.

Note :

Remember that F over a commutative Notherian ring T is Cohen Macaulay (CM) if

$$dim(F) = depth(F).$$

Corollary 2.1.6.

A CM module over a local ring (T, Q_0) is aCM.

Proof.

Let F be CM module over a local ring (T, Q_0), then $dim(F) = depth(Q_0, F)$.

Now consider $dim(F) - depth(Q_0, F) = dim(F) - dim(F) = 0 \leq 1$. So F is an aCM T-module from Lemma 2.1.5.

Corollary 2.1.7.

Let F be a module over a local ring (T, Q_0) such that $dim(F) \leq 1$, then F becomes an aCM T-module.

Proof.

We have a relation $depth(F) \leq dim(F)$ [7]. Also given that $dim(F) \leq 1$. Combining above inequalities, we get $depth(F) \leq dim(F) \leq 1$. This implies that $depth(F) \leq 1$.

Now we also have $dim(F) - depth(F) \leq 1$. Thus from Lemma 2.1.5, T-module F becomes an aCM.

Lemma 2.1.8.

Consider (T, Q_0) as local ring and F a aCM. Also suppose that an element $y \in T$ which is not a zero divisor of F. Then F/yF becomes an aCM T/y-module.

Proof.

Since F is an aCM T-module. So by Lemma 2.1.5, we have $dim(F) \leq depth(Q_0, F) + 1$. Now $dim(F/yF) = dim(F) - 1$. Combining these two relations we get

$dim(F/yF) = dim(F) - 1 \leq depth(Q_0, F) + 1 - 1 = depth(Q_0, F) - 1 + 1 = depth(F/yF) + 1$.

This means we have $dim(F/yF) \leq depth(F/yF) + 1$. Finally by Lemma 2.1.5, F/yF is an aCM T/y-module.

Lemma 2.1.9.

Almost cohen-macaulayness is preserved by localization.

Proof.

Consider (T, Q_0) as local and $U \subset T$ as multiplicatively closed set. Also suppose that F is an aCM T-module. We have to show that the module $U^{-1}F$ becomes an aCM over $U^{-1}T$. Since F is an aCM T-module, So we have from Lemma 2.1.5 $dim(F) \leq depth(F)+1$. Also we have the following inequalities $dim(U^{-1}F) \leq dim(F)$ and $depth(F) \leq depth(U^{-1}F)$. So we obtain the following relation by combining all these inequalities

$dim(U^{-1}F) \leq dim(F) \leq depth(F) + 1 \leq depth(U^{-1}F) + 1$.

Or $dim(U^{-1}F) \leq depth(U^{-1}F)+1$. So $U^{-1}F$ is an aCM over $U^{-1}T$ from Lemma 2.1.5. Hence almost cohen-macaulayness is preserved by localization.

Lemma 2.1.10.

Suppose F is a module over a commutative Notherian T and A is its any prime ideal. Then F is an aCM iff F_A is an aCM over AT_A for all $A \in Supp(F)$.

Proof.

Suppose F is an aCM T-module. So by definition of aCM module, we have

$grade(A, F) = grade(AT_A, F_A)$ for all $A \in Supp(F)$. We have to show that F_A is an aCM AT_A- module. For this we have to show that $grade(AT_A, F_A) = grade(A(T_A)_{BT_A}, (F_A)_{BT_A})$ for all $B \in Supp(F_A)$. Now we have the following isomorphisms.$(T_A)_{BT_A} \cong T_B$ and also $(F_A)_{BT_A} \cong F_B$. Therefore we have $grade(AT_B, F_B) = grade(AT_A, F_A) = grade(A, F)$ because F is aCM. Thus F_A becomes an aCM AT_A- module.

Conversely suppose that F_A is an aCM. So for all $B \in Supp(F_A)$ and for all $A \in Supp(F)$ we have $grade(AT_A, F_A) = grade(A(T_A)_{BT_A}, (F_A)_{BT_A})$ and also by Lemma 2.1.4, we have the following inequality

$dim(F_B)_{AT_B} \leq depth(AT_A, F_A) + 1$. Since $(F_B)_{AF_B} \cong F_A$. Therefore the above inequality yields $dim(F_A) \leq depth(AT - A, F_A) + 1$. But $depth(A, F) \leq depth(AT_A, F_A)$. The last two inequalities yield us $dim(F_A) \leq depth(A, F) + 1$. Hence F is an aCM by Lemma 2.1.4.

Lemma 2.1.11.

Suppose F is a module over a commutative Notherian T and A is any prime ideal of T. F is aCM iff F_A is an aCM over AT_A for all $A \in Supp(F) \cap Max(T)$.

Proof.

The almost cohen-macaulayness of F implies the almost cohen-macaulayness of F_A for all $A \in Supp(F) \cap Max(T)$ by Lemma 2.1.10.

For converse suppose that F_A is an aCM for all $A \in Supp(F) \cap Max(T)$. Now we can choose a maximal ideal A such that $B \subset A$ for each $B \in Supp(F)$ and $depth(B, F) = depth(BT_A, F_A)$ from [7, Theorem 135, page 96]. Since F_A is an aCM. By Lemma 2.1.4, we have $dim(F_A)_{BT_A} \leq depth(BT_A, F_A)+1$. Since $(F_A)_{BF_A} \cong F_B$. Therefore the above inequality yields $dim(F_B) \leq depth(B, F) + 1$. Hence F is an aCM by Lemma 2.1.4.

Lemma 2.1.12.

Suppose F is a module over a commutative Notherian T. Then F_A is an aCM if and only if

$$dim(F_A) \leq depth(AT_A, F_A) + 1.$$

For all $A \in Supp(F) \cap Max(T)$.

Proof.

Suppose F_A is an aCM. So we have $dim(F_A)_{AT_A} \leq depth(AT_A, F_A)+1$ by Lemma 2.1.4. Since $(F_A)_{AF_A} \cong F_A$. Therefore above inequality yields us

$$dim(F_A) \leq depth(AT_A, F_A) + 1.$$

The converse is trivial from Lemma 2.1.4.

Remark :

From Lemma 2.1.11 and Lemma 2.1.12 we can conclude that

A module F over a commutative Notherian T is an aCM.

Iff F_A *is an aCM over* AT_A *for all* $A \in Supp(F) \cap Max(T)$.

Iff $dim(F_A) \leq depth(AT_A, F_A) + 1$ *for all* $A \in Supp(F) \cap Max(T)$.

2.2 More results for almost Cohen-Macaulay Module

In this section we prove more results for aCM modules and aCM rings with the help of the results proved in Section 2.1. We also prove almost cohen-macaulayness of polynomial ring as well as almost cohen-macaulayness of power series ring.

Theorem 2.2.1.

- 1. Consider a commutative Notherian ring T. T is an aCM if $dim(T) \leq 1$.
- 2. If T is CM, then T becomes an aCM.
- 3. T is an aCM iff for each maximal regular sequence $(k_1, k_2, ...k_n)$ contained in prime ideal A for all $A \in Spec(T)$, then $A \in Ass(T/(k_1, k_2, ...k_n))$.

Proof.

1. The proof is similar as Corollary 2.1.7 in Section 2.1.

2. The proof is same as Corollary 2.1.6 in Section 2.1.

3.The proof can be done according as Lemma 2.1.3 by just replacing T with F.

Lemma 2.2.2.

Suppose a commutative Notherian ring T and a T-module F. Suppose an element $q \in T$ is in $Jrad(T)$ and q is non-zero divisor for T and F. Also suppose that if F/qF is an aCM T/(q)-module, then so is F over T.

Proof.

Let $A \in Supp(F) \cap Max(T)$ be any ideal of T. Now we have $dimF_A = 1 + dim(\frac{F}{qF})_{A/q}$. Since F/qF is an aCM T/(q)-module, by Lemma 2.1.4 we have $dim(\frac{F}{qF})_{A/q} \leq depth(A/q, F/qF) + 1$. Or $dimF_A \leq depth(A/q, F/qF) + 2 = depth(A, F) - 1 + 2 = depth(A, F) + 1$. Thus by Lemma 2.1.12 in Section 2.1, F becomes an aCM over T.

Theorem 2.2.3.

Suppose a commutative Notherian ring T. Then T is an aCM iff $T[[Y_1, Y_2, ..., Y_r]]$ is an aCM ring for all $r \geq 1$.

Proof.

Let T be an aCM. We have to prove that $T[[Y_1, Y_2, ..., Y_r]]$ is an aCM for all $r \geq 1$. It is enough to show that $T[[Y_1]]$ is an aCM . Here Y_1 is not a zero divisor for T. We have isomorphism $T \cong T[[Y_1]]/(Y_1)$. Since T is an aCM , therefore $T[[Y_1]]/(Y_1)$ is also an aCM. Thus $T[[Y_1]]$ is an aCM by Lemma 2.2.2.

Conversely let $T[[Y_1]]$ be an aCM and Y_1 is not a zero divisor for T. By Lemma 2.1.8 in Section 2.1, we have $T[[Y_1]]/(Y_1)$ is also an aCM. Since $T \cong T[[Y_1]]/(Y_1)$ and $T[[Y_1]]/(Y_1)$ is an aCM. Therefore T is an aCM .

Thus in general we have a commutative Notherian ring T is an aCM iff $T[[Y_1, Y_2, ..., Y_r]]$, the power series ring is an aCM for all $r \geq 1$.

Theorem 2.2.4.

Consider a flat morphism $\gamma : S \to T$ of commutative Notherian rings. Suppose that $AT \neq T$ for all $A \in Spec(S)$, i.e the mapping $\gamma^* : Spec(T) \to Spec(S)$ is a surjective.

- 1. The almost cohen-macaulayness of T implies the almost cohen-macaulayness of S and the fibre ring $T \otimes_S k(A)$ for all $A \in Spec(S)$.

- 2. Suppose S is an aCM and $T \otimes_S k(A)$ is a CM for all $A \in Spec(S)$, then T becomes an aCM.

Proof.

(1). For any $A \in Spec(S)$ and any prime ideal in $T \otimes_S k(A)$, there exists a unique $B \in Spec(T)$ such that $B \cap S = A$ and $BT \otimes_S k(A)$ is a prime ideal in $T \otimes_S k(A)$. We have following relations by [8, (21.A) and (21.B), page 153] $dimT_B = dimS_A + dimT_B \otimes_S k(A)$.

Also we have $depth(BT_B, T_B) = depth(AS_A, S_A) + depth(BT_B \otimes_S k(A), T_B \otimes_S k(A))$.

Now consider the difference

$dimT_B - depth(BT_B, T_B) = dimS_A - depth(AS_A, S_A) + dimT_B \otimes_S k(A) - depth(BT_B \otimes_S k(A), T_B \otimes_S k(A)) \leq 1$ because of the almost cohen-macaulayness of T. This yields us $dimS_A - depth(AS_A, S_A) \leq 1$ and also $dimT_B \otimes_S k(A) - depth(BT_B \otimes_S k(A), T_B \otimes_S k(A)) \leq 1$. So both S and $T \otimes_S k(A)$ are aCM rings for all $A \in Spec(S)$.

(2). If S is an aCM and $T \otimes_S k(A)$ is CM for all $A \in Spec(S)$, then we have $dimS_A - depth(AS_A, S_A) \leq 1$ and $dimT_B \otimes_S k(A) = depth(BT_B \otimes_S k(A), T_B \otimes_S k(A))$.

Now we have $dimT_B - depth(BT_B, T_B) = dimS_A - depth(AS_A, S_A) \leq 1$. Thus T is an aCM.

Theorem 2.2.5.

Consider a commutative Notherian ring T. Then T is an aCM iff $T[Y_1, Y_2, ...Y_r]$, the ring of polynomials is an aCM for all $r \geq 1$.

Proof.

It is enough to show that T is an aCM iff $T[Y_1]$ is an aCM. Suppose that if T is an aCM for all $A \in Spec(T)$, then the fibre ring $T[Y_1] \otimes_T k(A)$ is a polynomial ring over the field k(A). So $T[Y_1] \otimes_T k(A))$ is CM. Hence $T[Y_1]$ is an aCM by part2 of Theorem 2.2.4

Conversely suppose $T[Y_1]$ is an aCM. For any $A \in Spec(T)$, we have $A[Y_1] \in Spec(T[Y_1])$. Also $dim T_A \leq dim T[Y_1]_{A[Y_1]}$ and $depth(A, T) \leq depth(A[Y_1], T[Y_1])$. Since $T[Y_1]$ is an aCM, therefore $dim T[Y_1]_{A[Y_1]} - depth(A[Y_1], T[Y_1]) \leq 1$. By combining all above facts we obtain

$$dim T_A - depth(A, T) \leq dim T[Y_1]_{A[Y_1]} - depth(A[Y_1], T[Y_1]) \leq 1.$$

Hence T is an aCM.

In general a commutative Notherian ring T is an aCM iff $T[Y_1, Y_2, ...Y_r]$,the ring of polynomials is an aCM for all $r \geq 1$.

Chapter 3

Basic Results on Almost Cohen-Macaulay Modules

In this chapter, we will prove more results on aCM module. Furthermore we prove if T is aCM and F is perfect having finite projective dimension. Then F becomes aCM. Next we check the nature of aCM modules according with flat morphisms. Also we will provide a sufficient condition for a T-module F to become an aCM.

3.1 Some Basic Results related to aCM Modules

In this section we define perfect module. We also discuss under what conditions a perfect module becomes an aCM module.

Note :

A finitely generated T-module F is called an aCM if for each $A \in Supp(F)$, we have

$$grade(A, F) = grade(AT_A, F_A).$$

Definition 3.1.1. Consider a non-zero finite module F over a Notherian ring T. Then F is a perfect if

$$projdim(F) = grade(F).$$

Lemma 3.1.2.

Suppose F is an aCM T-module over a local ring (T,$\underline{g}$) with maximal ideal $\underline{g}$. Then

$$dim(F) - dimT/A \leq 1.$$

For each $A \in Ass(F)$.

Proof.

We have the following inequality by [3 , Proposition 1.2.13] $depth(F) \leq dim(T/A)$ for each A$\in Ass(F)$. Also we have $dim(T/A) \leq dim(F)$. Since F is an aCM. Therefore from Remark(3), it satisfies $dim(F) - depth(F) \leq 1$. By combining above results we get $dim(F) - dim(T/A) \leq 1$. So the lemma is proved.

Lemma 3.1.3.

Let J be any proper ideal of a local ring (T,$\underline{g}$) and F be a T-module. The T-module F is an aCM if

$$grade(J,F) = grade(JT_A, F_A).$$

For any A$\in$ $Supp(F/JF)$.

Proof.

The proof is simply obtained by using the definition and properties of grade.

Theorem 3.1.4.

Consider (T,$\underline{g}$) as local. Suppose F is aCM T-module. For all ideals $\underline{b}$ contained in $\underline{g}$, we have

$$dim(F) - dimF/\underline{b}F \leq grade(\underline{b}, F) + 1.$$

Proof.

Suppose $grade(\underline{b}, F) = 0$. It means that $\exists$ A in $Ass(F)$ such that $\underline{b}$ is contained in A and we have $dimT/A \leq dimF/\underline{b}F$. So from lemma(3.1.2), we get $dim(F) - dimF/\underline{b}F \leq 1$. Now we consider the case when grade($\underline{b}$,F) is greater than zero. So there exists an element y in $\underline{b}$ and y is regular on F. We have

$$grade(\underline{b}, F/yF) = grade(\underline{b}, F) - 1$$

and also

$$dimF/yF = dim(F) - 1.$$

Hence the result is proved by induction.

Corollary 3.1.5. [1]

Let F be an aCM module over (T,g) with maximal ideal $\underline{g}$. Then we have

$$dim(F) - dim(F/AF) \leq dimF_A + 1$$

for every $A \in Supp(F)$.

Proof.

We obtain $dim(F) - dimF/$AF$\leq grade(A, F) + 1$ by using theorem(3.1.4). We also have $grade(A, F) \leq ht(A) = dimF_A$. By combining these inequalities, we can obtain

$$dim(F) - dim(F/AF) \leq dimF_A + 1.$$

Corollary 3.1.6. [2]

Suppose J is a proper ideal in (T,g). If T is an aCM, then we have

$$ht(J) \leq dim(T) - dimT/J \leq ht(J) + 1.$$

Proof.

As grade(J)≤ht(J). From the definition of height, we have $ht(J) \leq dim(T) - dimT/J$. Now by Theorem(3.1.4) and above facts yield

$$ht(J) \leq dim(T) - dimT/J \leq ht(J) + 1.$$

Remark 3.1.7. Suppose T is a Notherian ring. We have already proved in chapter 2.

1.T is an aCM module iff

$ht(J) \leq 1 + depth(J,T)$ for every J $\in$ Spec(T)

2.For any J $\in$ Spec(T), T is an aCM iff T_U is an aCM for any U in Max(T)

iff $ht(U) \leq 1 + depth(T,U)$ for each U $\in$ Max(T).

3.Moreover if T is a local, then from 2 we can have T is an aCM

iff $dim(T) \leq 1 + depth(T)$.

The next theorem is an extension of the result in [13, 1.9].

Theorem 3.1.8.

Consider T as an aCM, and F as perfect T- module whose projective dimension is finite. Then F becomes an aCM.

Proof.

We have following inequalities for any A $\in$ Supp(F)

$$grade(F) \leq gradeF_A \leq pdF_A \leq pd(F).$$

As F is perfect i.e $pd(F) = grade(F)$. Therefore F_A also becomes a perfect T_A-module by above inequality. So we may suppose that T is a local ring. Now by Auslander-Buchsbaum formula we have $pd(F) - depth(F) = depth(T)$. By using Corollary(3.1.6), the given fact that T is an aCM and Remark 3.1.7(3). We obtain

$$dim(F) - depth(F) \leq dim(T) - depth(T) \leq 1.$$

Hence F becomes an aCM.

The next proposition extends the result in [6, Proposition 2.2].

Proposition 3.1.9.

Consider a local homomorphism $\gamma : (X, \underline{a}) \to (Y, \underline{b})$ of Notherian local rings. Consider E as finitely-generated X-module and F as an X-flate finitely-generated Y-module.

- (1) The almost cohen-macaulazness of $E \otimes_X F$ as Y-module implies the almost cohen-macaulazness of E as X-module and $F/\underline{a}F$ as Y-module.

- (2)Furthermore if E and F are both aCM as X-module and Y-module respectively and if any one of them is CM. Then $E \otimes_X F$ is an aCM Y-module.

Proof.

(1)

We have the following relations from [4, Proposition1.2.16(a) and Theorem A.11(b)]

$$dim_Y(E \otimes_X F) = dim_X(E) + dim_Y(F/\underline{a}F)$$

and

$$depth_Y(E \otimes_X F) = depth_X(E) + depth_Y(F/\underline{a}F)$$

Now consider the difference

$$0 \leq dim_Y(E \otimes_X F) - dept_Y(E \otimes_X F) = dim_X(E) - depth_X(E) + dim_Y(F/\underline{a}F) - depth_Y(F/\underline{a}F) \leq 1$$

because due to given fact that E$\otimes_X$ F over X is an aCM Y-module. The last inequality implies

$$dim_X(E) - depth_X(E) \leq 1$$

as well as

$$dim_Y(F/\underline{a}F) - depth_Y(F/\underline{a}F) \leq 1.$$

Thus E as X-module and F as Y-module are aCM.

(2).Now if E and F are both aCM as X-module and Y-module respectively and if we take E as CM. Then $dim_X(E) = depth_X(E)$ and almost cohen-macaulayness of F gives $dim_Y(F/\underline{a}F) - depth_Y(F/\underline{a}F) \leq 1$. Now the difference given below becomes

$$dim_Y(E \otimes_X F) - dept_Y(E \otimes_X F) = dim_Y(F/\underline{a}F) - depth_Y(F/\underline{a}F) \leq 1.$$

Thus $E \otimes_X F$ becomes an aCM Y-module.

3.2 Almost Cohen-Macaulay modules and Ext Functors

We will establish here a necessary and sufficient condition for a T-module F to become an aCM with the help of Exact Functors.

Definition 3.2.1.

Suppose (T,$\underline{g}$) is a local ring with a maximal ideal $\underline{g}$ and F is a finite T-module. F is callad a maximal aCM over T if

$$depth(F) = dim(T).$$

Remark 3.2.2.

From the above definition we can conclude.

1. A CM module F over a Notherian local ring T is called a maximal CM if

$$dim(F) = dim(T)$$

2. More precisely a CM module F over a Notherian local ring T becomes a maximal CM if

$$Supp((F) = Spec(T).$$

Definition 3.2.3.

Let (T,$\underline{g}$) be a CM local ring having finite dimension q. Then a T-module G is said to be a canonical if G is a maximal CM of type 1 with finite injective dimension.

Proposition 3.2.4.

Suppose F is an aCM over a CM local ring (T,$\underline{g}$) and G is a canonical module over T. Let $depth(F) = r$ and $dim(T) = q$. Also if $dim(F) - depth(F) = 1$, then we have $\text{Ext}^j_T(F, G) \neq 0$ only either j=q - r or j=q - r - 1.

Proof.

By [10, Proposition 3.1(b)] we have grade$_G$F=grade(F). Also from [5,Proposition 1.2(g), (i)] we have $depth(T) = depth(F) + \sup\{j : \text{Ext}^j_T(F, G) \neq 0\}$. From here we get the equation $\sup\{j : \text{Ext}^j_T(F, G) \neq 0\} = q - r$ and $depth(T) \leq dim(F) + depth(F) \leq dim(T)$. Since T is a CM local ring, therefore $depth(T) = dim(T)$. So the above inequality yields $grade(F) + dim(F) = dim(T)$. By using $dim(F) - depth(F) = 1$ and $grade_G F = grade(F)$. We get $grade_G F = dim(T) - depth(F) - 1$ which yields $grade_G F = q - r - 1$. By definition of grade we have, $\inf\{j : \text{Ext}^j_T(F, G) \neq 0\} = q - r - 1$. The proof is completed.

In the next theorem, we will check the necessary and sufficient condition for a module to become an aCM over a CM local ring with the help of Exact Functor.

Theorem 3.2.5.

Suppose that (T,$\underline{g}$) is a CM local ring of dimension q and G is a canonical module over T. Then a non CM module F over T with $depth(F) = r$ is an aCM iff $\text{Ext}^j_T(F, G) \neq 0$ only when either j=q - r or j=q - r- 1.

Proof.

Suppose that F over T with $depth(F) = r$ is an aCM module. Then by Proposition 3.2.4, we get the required condition.

Conversely suppose that $\mathrm{Ext}^j_T(F, G) \neq 0$ only when either j=q - r or j=q - r - 1. Which implies that $\mathrm{Ext}^j_T(F, G) \neq 0$ when j= q - r and $\mathrm{Ext}^j_T(F, G) = 0$ for every j greater than q - r. So we obtain

$\sup\{j : \mathrm{Ext}^j_T(F, G) \neq 0\} = q - r$. Now from [5, Proposition 1.2(g), (i)] we get the equation $depth(T) = grade(F) + \sup\{j : \mathrm{Ext}^j_T(F, G) \neq 0\}$ which yiels that $q = depth(F) + q - r$. From here we get $depth(F) = r$. Also by applying [5, Proposition 1.2(h) and 3.1(b)] we can get the relation

$grade_G F = grade(F) = \inf\{j : \mathrm{Ext}^j_T(F, G) \neq 0\} = q - r - 1$. Now by using [5, Proposition 1.2(i)] and the fact that T is a CM ring i.e $dim(T) = depth(T)$, we get the following equation $grade(F) + dim(F) = dim(T)$. If we use $grade(F) = q - r - 1$ and $dim(T) = q$. Then the last equation yields $dim(F) = r + 1 = depth(F) + 1$ which gives

$$dim(F) - depth(F) = 1.$$

Thus F becomes an aCM T-module.

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