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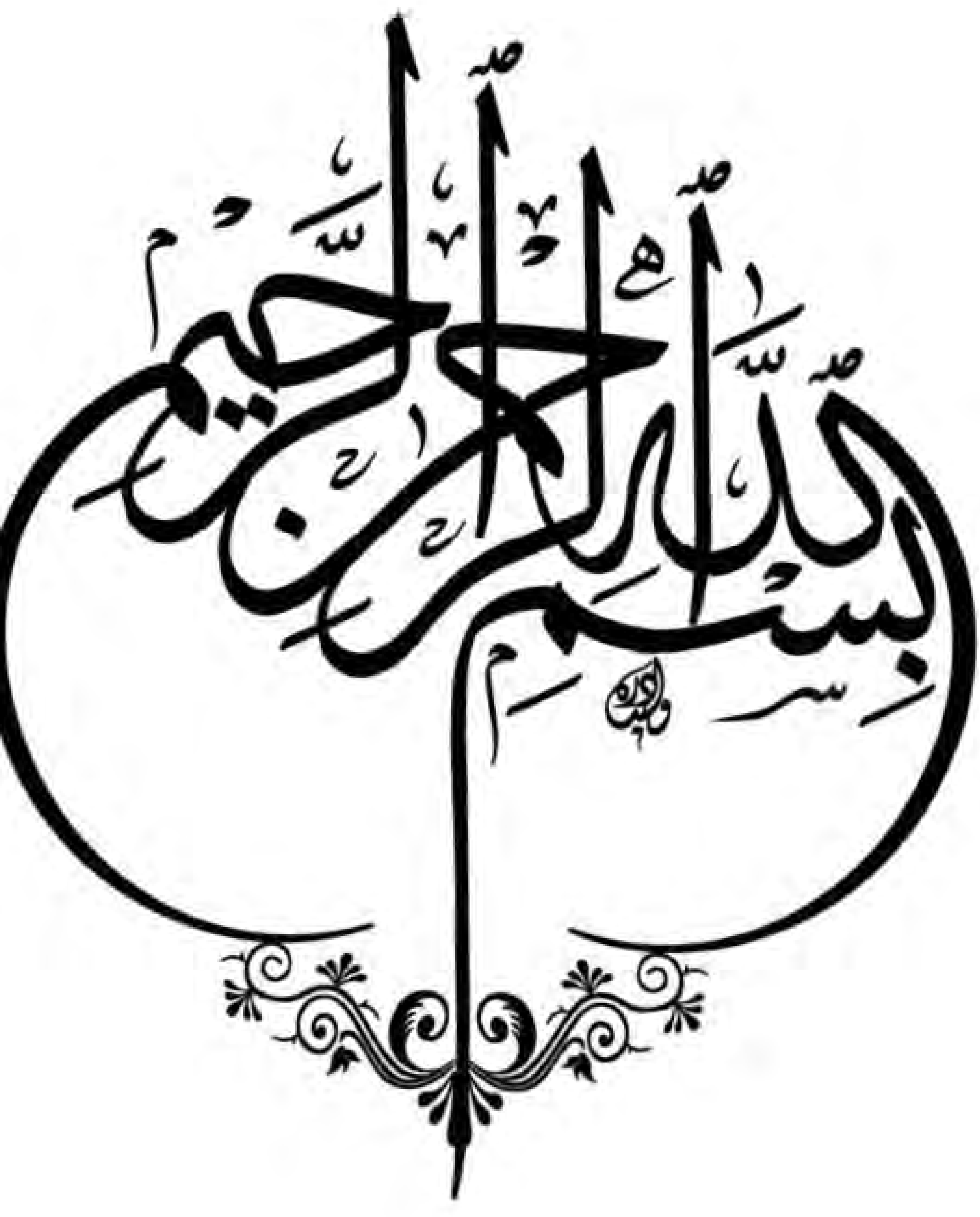

Study of lipschitz functions over metric measure spaces

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Abstract

The motive of this thesis is to review generalization of characteristic of lipschitz function over metric measure spaces. We also review the results in the class of metric-measure spaces which satisfy version(strong) of doubling condition(Bishop-Gromov regularity)[10].In fact, we set up a necessary and sufficient condition in the direction that any measurable function that assure integrability condition is to be essentially Lipschitzian. As well as we review the generalized version of differentiability property (having derivative zero) of functions [14] and [13], after that we review about the characterization of constant function [1]

Contents

Chapter 1

PREPARATORY INFORMATION

This chapter is based on some basic definitions,which terms are used in the lemmas and theorem of chapter 2 and chapter 3.

1.1 Prerequisite

Definition 1.1.1. *[6]**Metric space***
A metric space is a pair $(\mathcal{Y}, \mathcal{D})$, where $\mathcal{Y}$ is a set and $\mathcal{D}$ is a metric on $\mathcal{Y}$ or distance function on $\mathcal{Y}$,a function defined on $\mathcal{Y} \times \mathcal{Y}$ such that for all a,b,c $\in \mathcal{Y}$ we have

i. *$\mathcal{D}$ is real-valued, finite and nonnegative.*
ii. *$\mathcal{D}$ (e,f)= 0   if and only if   e=f.*
iii. *$\mathcal{D}(e, f) = \mathcal{D}$(f,e)   (symmetry).*
iv. *$\mathcal{D}(e, f) \leqq \mathcal{D}(e, c)$+$\mathcal{D}(c, f)$ (Triangle inequality).*

Example 1.1.2. *[6]**Real line*** $\mathbb{R}$
The set of all real numbers, taken with the usual metric defined by

$$\mathcal{D}(e, f) = |e - f|.$$

Example 1.1.3. *[6]**Euclidean plane*** $\mathbb{R}^2$
Let e,f $\in \mathbb{R}^2$ such that e=(e_1, e_2) and f=(f_1, f_2), then the Euclidean metric is defined by

$$\mathcal{D}(e, f) = \sqrt{(e_1 - f_1)^2 + (e_2 - f_2)^2}.$$

Example 1.1.4. *[6]**Function space C[u,v]***
*Let $\mathcal{Y}$ be a set with the real valued function e,f,.... which are defined and continuous on closed interval I=[u,v], then we choose the metric on **C[u,v]** defined by*

$$\mathcal{D}(e,f) = \max_{t \in I} |e(t) - f(t)|,$$

where a and b are the functions of independent variable t.

Definition 1.1.5. *[7]**Open ball***
Let $(\mathcal{Y}, \mathcal{D})$ is metric space and $a_0 \in \mathcal{Y}$ and $r >$ 0, then the open ball is defined as

$$\mathcal{B}(e_0, r) = \{e \in \mathcal{Y} \quad | \quad d(e, e_0) < r\}.$$

Definition 1.1.6. *[7]**Closed ball***
Let $(\mathcal{Y}, \mathcal{D})$ is metric space and $e_0 \in \mathcal{Y}$ and $r >$ 0, then the closed ball is defined as

$$\mathcal{B}(e_0, r) = \{e \in \mathcal{Y} \quad | \quad d(e, e_0) \leq r\}.$$

Definition 1.1.7. *[7]**Sphere***
Let $(\mathcal{Y}, \mathcal{D})$ is metric space and $e_0 \in \mathcal{Y}$ and $r >$ 0, then the sphere is defined as

$$\mathcal{B}(e_0, r) = \{e \in \mathcal{Y} \quad | \quad d(e, e_0) = r\},$$

here in metric space $(\mathcal{Y}, \mathcal{D})$, we designate an open ball with a radius of r $>$ 0 and a centre at y as $\mathcal{B}(y, r)$.

Definition 1.1.8. *[16]**Algebra***
Let $\mathcal{Y}$ be a set and $\mathcal{A} \subset \mathcal{P}(\mathcal{Y})$. Then $\mathcal{A}$ is called algebra if

i. $A^c \in \mathcal{A}$ *for* $A \in \mathcal{A}$.

ii. $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$ *for* $A_1, A_2 ... A_n \in \mathcal{A}$.

Definition 1.1.9. *[16]**σ-Algebra***
Let $\mathcal{Y}$ be a set and $\mathcal{A} \subset \mathcal{P}(\mathcal{Y})$. Then $\mathcal{A}$ is called σ- algebra if

i. $A^c \in \mathcal{A}$ *for* $A \in \mathcal{A}$.

ii. $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ *for* $A_1, A_2 .... \in \mathcal{A}$, *that is closed under countable union.*

Note: Every algebra is σ-algebra but converse is not true.

Definition 1.1.10. *[16]**Borel σ-Algebra***
consider ($\mathcal{Y}$,$\mathfrak{T}$) be a topological space and D is the collection of all open sets that is $\mathfrak{T} = D$ then the smallest σ- algebra is called Borel σ-algebra on $\mathcal{Y}$, we can denote it by $B(\mathcal{Y})$ or $B_{\mathcal{Y}}$.

Definition 1.1.11. *[16]**Borel set***
Let $B(\mathcal{Y})$ is Borel σ- algebra then the members or elements of $B(\mathcal{Y})$ are called Borel sets.

Definition 1.1.12. *[16]**Measure***
consider $\mathcal{Y}$ is non empty set and $\mathcal{A}$ is the σ-algebra on $\mathcal{Y}$. Then the set function

$$\eta : \mathcal{A} \to [0, \infty],$$

is said to be measure if

i. *$\eta(\phi)$=0,*

ii. *If $\{A_1,A_2,....\}$ is a disjoint sequence in $\mathcal{A}$, then*

$$\eta\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \eta(A_i),$$

that is η is countably additive.

Example 1.1.13. *[2]**Measure on** $\mathbb{R}$*
Consider $\mathbb{R}$ be the set of real numbers and $B_{\mathbb{R}}$ is the Borel σ-algebra on $\mathbb{R}$, then set function

$$\eta : B_{\mathbb{R}} \to [0, \infty],$$

define the measure by

$$\eta(A) = |A| = \textit{number of elements in } A.$$

Definition 1.1.14. *[2]**Finite Measure***
Let $\mathcal{Y}$ be the non empty set and $\mathcal{A}$ is σ-algebra on $\mathcal{Y}$. A measure

$$\eta : \mathcal{A} \to [0, \infty],$$

is said to be finite measure if

$$\eta(\mathcal{Y}) < \infty.$$

Definition 1.1.15. *[2]**σ-Finite Measure***
Let $\mathcal{Y}$ be the non empty set and $\mathcal{A}$ is σ-algebra on $\mathcal{Y}$. A measure

$$\eta : \mathcal{A} \to [0, \infty],$$

is said to be σ-Finite measure if $\exists$ a sequence $\{A_1,A_2,....\}$ in $\mathcal{A}$ such that

$$\mathcal{Y} = \bigcup_{i=1}^{\infty} A_i \quad \textit{and} \quad \eta(A_i) < \infty.$$

Definition 1.1.16. *[2]**Measrable space***
Let $\mathcal{Y}$ be non empty set and σ-algebra on $\mathcal{Y}$ is $\mathcal{A}$ and η:$\mathcal{A} \rightarrow$ [0,∞] is measure on $\mathcal{A}$. Then ($\mathcal{Y}$,$\mathcal{A}$) is called measurable space.

Definition 1.1.17. *[2]**Measure space***
Let $\mathcal{Y}$ be non empty set and σ-algebra on $\mathcal{Y}$ is $\mathcal{A}$ and η:$\mathcal{A} \rightarrow$ [0,∞] is measure on $\mathcal{A}$. Then the triplet ($\mathcal{Y}$,$\mathcal{A}$,η) is called measure space.

Definition 1.1.18. *[4]**Metric measure space***
The triple ($\mathcal{Y}$,$\mathcal{D}$,η) is claimed to be metric measure space, if $(\mathcal{Y}, \mathcal{D})$ is a metric space and η is (Borel)measure on $\mathcal{Y}$.

Definition 1.1.19. *[2]**Finite Measure space***
The triplet ($\mathcal{Y}$,$\mathcal{A}$,η) be the measure space, it is called Finite measure space if

$$\eta : \mathcal{A} \rightarrow [0, \infty],$$

is the finite measure that is $\eta(\mathcal{Y}) < \infty$.

Definition 1.1.20. *[16]σ**-Finite measure space***
A triplet ($\mathcal{Y}$,$\mathcal{A}$,η) be the measure space, it is called μ-Finite measure space if

$$\eta : \mathcal{A} \rightarrow [0, \infty],$$

is the σ-finite measure that is $\exists$ a sequence $\{A_1,A_2,....\}$ in $\mathcal{A}$ such that

$$\mathcal{Y} = \bigcup_{i=1}^{\infty} A_i \quad \textit{with} \quad \eta(A_i) < \infty.$$

Definition 1.1.21. *[16]$\mathcal{A}$**-Measurable set***
Consider the measurable space ($\mathcal{Y}$,$\mathcal{A}$) then the members of $\mathcal{A}$ are called $\mathcal{A}$-Measurable set.

Definition 1.1.22. *[16]σ**-Finite set***
Consider the measure space ($\mathcal{Y}$,$\mathcal{A}$,η), a set $A \in \mathcal{A}$ is said to be σ-Finite set if $\exists$ a sequence $\{A_1,A_2,....\}$ in $\mathcal{A}$ such that

$$A = \bigcup_{n=1}^{\infty} A_n, \quad \textit{with} \quad \eta(A_n) < \infty, \quad \forall \quad n \in \mathbb{N}.$$

Definition 1.1.23. *[16]**Null set***
Consider the triplet ($\mathcal{Y}$,$\mathcal{A}$,η) be the measure space , The subset A of $\mathcal{Y}$ is called null set if η(A)=0.

Example 1.1.24. *As $\eta(\phi)$=0 so ϕ is null set.*

Note: In every measure space ϕ is null set but a null set need not to be ϕ. There may be any other set whose measure is zero.

Definition 1.1.25. *[16]**Complete σ-algebra***
Consider the triplet ($\mathcal{Y}$,$\mathcal{A}$,η) be the measure space.The σ-algebra which is denoted by $\mathcal{A}$ is said to be complete σ-algebra if each subset of null set is the member of $\mathcal{A}$.

Definition 1.1.26. *[16]**Complete Measure space***
Consider the triplet ($\mathcal{Y}$,$\mathcal{A}$,η) be the measure space. It is complete measure space if σ-algebra $\mathcal{A}$ is complete.

Definition 1.1.27. *[16]**Set of Extended real numbers***
If we add the symbols $-\infty$ and ∞ in the set of real numbers $\mathbb{R}$ so we call this set of extended real numbers that is

$$\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}.$$

Definition 1.1.28. *[16]**Outer measure***
Consider that $\mathcal{Y}$ is non empty, a set function

$$\eta^* : \mathcal{P}(\mathcal{Y}) \to [0, \infty],$$

is said to be outer measure it satisfies these three axioms

i. η^* (ϕ)=0.

ii. If Y_1,Y_2, $\in \mathcal{P}(\mathcal{Y})$ such that $Y_1 \subseteq Y_2$
$\Rightarrow$ η^* (Y_1) $\leq \eta^*$ (Y_2),
that is Property of monotonicity satisfied.

iii. For sequence $\{Y_1, Y_2, ...\}$ in $\mathcal{P}(\mathcal{Y})$ such that

$$\eta^*(\bigcup_{i=1}^{\infty} Y_i) \leq \sum_{i}^{\infty} \eta^*(Y_i),$$

That is η^* has countably sub additive property.

Definition 1.1.29. *[16]**Measureable set***
Consider ($\mathcal{Y}$,$\mathcal{A}$) be the measurable space ,$G \in \mathcal{A}$. A function

$$g : G \to \overline{\mathbb{R}},$$

is called $\mathcal{A}$-measurable function on G, if

$$\{y \in G \quad | \quad g(y) < \beta\} \in \mathcal{A},$$

for every β, where β is real number
Correspondingly if

$$\{y \in G \quad | \quad g(y) \in [-\infty, \beta)\} \in \mathcal{A},$$

OR

$$g^{-1}([-\infty, \beta)\beta) \in \mathcal{A}.$$

Lemma 1.1.30. *Assume measurable space be ($\mathcal{Y}$,$\mathcal{A}$),and function*

$$g : G \to \overline{\mathbb{R}},$$

defined on $G \in \mathcal{A}$. Then given conditions are alike

$$\begin{aligned}
a.\quad & \{y \in G \quad | \quad g(y) \leq \beta\} = g^{-1}([-\infty, \beta]) \in \mathcal{A}, \quad \forall \quad \beta \in \mathcal{A},\\
b.\quad & \{y \in G \quad | \quad g(y) > \beta\} = g^{-1}((\beta, \infty]) \in \mathcal{A}, \quad \forall \quad \beta \in \mathcal{A},\\
c.\quad & \{y \in G \quad | \quad g(y) \geq \beta\} = g^{-1}([\beta, \infty]) \in \mathcal{A}, \quad \forall \quad \beta \in \mathcal{A},\\
d.\quad & \{y \in G \quad | \quad g(y) < \beta\} = g^{-1}([\infty, \beta)) \in \mathcal{A}, \quad \forall \quad \beta \in \mathcal{A}.
\end{aligned}$$

Definition 1.1.31. ***Characteristic function***
Let $\mathcal{Y} \neq \phi$, and $H \subseteq \mathcal{Y}$ then the function

$$\chi_H : \mathcal{Y} \to \{0, 1\},$$

defined as

$$\chi_H(e) = \begin{cases} 0 \quad ; \quad if \quad e \notin H, \\ 1 \quad ; \quad if \quad e \in H. \end{cases}$$

Definition 1.1.32. ***Almost every where Property***
Consider the measure space ($\mathcal{Y}$,$\mathcal{A}$,η). P property holds almost everywhere in $\mathcal{Y}$ iff $\exists$ set $L \in \mathcal{A}$ such that $\eta(L)$=0 and P satisfies $\forall$ $y \in \mathcal{Y} \setminus L$.

Definition 1.1.33. *[12]**Doubling Measure***
Consider the Borel regular measure η on metric space $(\mathcal{Y}, \mathcal{D})$, if each ball in $\mathcal{Y}$ possess finite

and positive measure and $\exists\ C_1 \geq 1$ *such that*

$$\eta(B(e,2R)) \leq C_1 \eta(\mathcal{B}(e,R)),$$

holds for each $e \in \mathcal{Y}$ *and* $r > 0$*, then it is named as doubling measure.Where* C_1 *is doubling constant.*

Definition 1.1.34. *[12]**Locally doubling space***
The triplet ($\mathcal{Y}$,$\mathcal{D}$,η) *be the metric-measure space, if for each* $r > 0\ \exists\ C_r > 1$*, and*

$$\eta(\mathcal{B}(e,2R)) \leq C_r \eta(\mathcal{B}(e,R)),$$

holds for $e \in \mathcal{Y}$ *and* $R \leq r$*, then we call this space as locally doubling. Where* C_r *is locally doubling constant of* $\mathcal{Y}$.

Definition 1.1.35. *[12]**Doubling space***
Assume $\mathcal{Y}$ *is locally doubling space, if there is* $C_1 \geq 1$ *so that* $C_r \leq C_1$, $\forall\ r > 0$*, then we call this space is doubling space.*

Definition 1.1.36. *[12]**Locally Bishop-Gromov regular***
We call $\mathcal{Y}$ *is Locally Bishop-Gromov regular space of dimension m,where* $m > 0$ *,*
if for each $r > 0\ \exists\ U_r \geq 1$ *s.t*

$$\frac{\eta(\mathcal{B}(e,S_1))}{{S_1}^m} \leq U_r \frac{\eta(\mathcal{B}(f,R_1))}{{R_1}^m},$$

$\forall\ 0 < R_1 < S_1 \leq r$ *and e,f* $\in \mathcal{Y}$*, and also for* $\mathcal{D}(e,f) \leq S_1$.

Definition 1.1.37. *[12]**Bishop-Gromov regular***
Let $\mathcal{Y}$ *is Locally Bishop-Gromov regular space, if* $\exists\ U \geq 1$ *so that*

$$U_r \leq U,$$

$\forall\ r > 0$ *, then we say this space is Bishop-Gromov regular space.*

Lemma 1.1.38. *If* $\mathcal{Y}$ *is Bishop-Gromov regular space whose dimension is m, then we call* $\mathcal{Y}$ *be Bishop-Gromov regular space whose dimension is n, in any case* $n \geq m$.
Similarly
If $\mathcal{Y}$ *is Locally Bishop-Gromov regular space whose dimension is m, then we call* $\mathcal{Y}$ *is Locally Bishop-Gromov regular space whose dimension be n, in any case* $n \geq m$.

Note: If $\mathcal{Y}$ be the doubling metric space having constant C_1, then we call $\mathcal{Y}$ is Bishop-Gromov regular space having dimension

$$m = \log_2 C_1.$$

Similarly

If $\mathcal{Y}$ be the locally doubling metric space having constant C_1, then we call $\mathcal{Y}$ is Locally Bishop-Gromov regular space having dimension,

$$m = \log_2 C_1.$$

Definition 1.1.39. *[7]**Compact set***
Let E be the set of real numbers, we say this set is compact, if $\{y_n\}$ is sequence in E. And $\{y_{nk}\}$ is subsequence in $\{y_n\}$ such that

$$lim_{n\to\infty}\{y_{nk}\} = y \in E.$$

Corospondingly,
E is compact iff it is closed and bounded.

Theorem 1.1.40. *[7]**Heine-Borel Theorem***
Consider E be the set of real numbers it is compact iff each open cover D of E has finite subcover.

Definition 1.1.41. *[7]**Locally compact***
consider $\mathcal{Y}$ be the topological space, if every point has compact neighborhood then we say $\mathcal{Y}$ is locally compact.

Definition 1.1.42. ***Radon measure***
Consider $\mathcal{Y}$ be the metric space and η is the Borel-regular measure, we say this measure is redon measure if

$$\eta(W) < \infty \quad \textit{for each compact set} \quad W \subset Y,$$

and if

$$\eta(V) = \inf\{\eta(U) : \quad V \subset U, \quad U \subset \mathcal{Y} \quad open\},$$

holds and also

$$\eta(U) = \sup\{\eta(W) : W \subset U, W \subset \mathcal{Y} \quad Compact\},$$

for each open set $U \subset \mathcal{Y}$.

Definition 1.1.43. ***Geodesic space***
Assume metric space is ($\mathcal{Y}$,$\mathcal{D}$), if for each e,f $\in \mathcal{Y}$, $\exists$ a geodesic

$$\gamma_{e,f} : [0,1] \to \mathcal{Y},$$

(having $\mathcal{D}(e,f)$ be velocity) from e to f, i.e

$$\mathcal{D}(\gamma_{e,f}(s), \gamma_{e,f}(t)) = |s-t|\mathcal{D}(e,f).$$

Also,

$$\gamma_{e,f}(0) = e \quad and \quad \gamma_{e,f}(1) = f,$$

then we say this space is geodesic space.

Definition 1.1.44. *[4]**Lipschitz function***
Assume the function

$$g : \mathcal{Y} \to \mathcal{X},$$

if $\exists$ constant C_1 such that,

$$|g(e) - g(f)| \leq C_1|e-f|,$$

for each e,f $\in \mathcal{Y}$ and C_1 is not depending on e and f,
then we call that function is lipschitz function.

Definition 1.1.45. *[4]**L-Lipschitz function***
Take the function,

$$g : \mathcal{Y} \to \mathcal{X},$$

where $\mathcal{Y}$ and $\mathcal{X}$ are metric spaces, if $\exists$ constant $\mathbf{L} > 0$ so that

$$\mathcal{D}_X(g(e), g(f)) \leq \mathbf{L}\mathcal{D}_Y(e,f),$$

for every e,f $\in \mathcal{Y}$, ten we call this function is L-Lipschitz function.

Note: If the function is L-Lipschitz then we say that it is also Lipschitz function.

Definition 1.1.46. *[4]**Bilipschitz function***
Consider the map,

$$g : \mathcal{Y} \to \mathcal{X},$$

we call the map is bilipschitz and Y and X be bilipschitz equivalent if g^{-1} is Lipschitz .

Definition 1.1.47. *[4]**L-bilipschitz function***
Consider the map,

$$g : \mathcal{Y} \to \mathcal{X},$$

we call the map is L-bilipschitz and Y and X be L-bilipschitz equivalent if g^{-1} is L-Lipschitz.

Definition 1.1.48. *[4]**Locally Lipschitz function***
Consider the function,

$$\mathcal{G} : \mathcal{Y} \to \mathcal{X},$$

it is called locally Lipschitz if for each point if X $\exists$ a neighborhood V_ϵ such that restriction of $\mathcal{G}$ to V_ϵ is Lipschitz, we denote it by $\mathcal{G}|_{V_\epsilon}$.

Chapter 2

An integral type charcterization of constant function

In this chapter we review the generalized version of differentiability property (having derivative zero) of functions [14] and [13], after that we review about the characterization of constant function [1]

Definition 2.0.1. *Let ($\mathcal{Y}$,$\mathcal{D}$,η) be a metric-measure space. Let $\mathcal{Y}$ be a geodesic space. Take ψ:$\mathcal{Y}$ $\times$ $\mathcal{Y}$ $\times$ [0,1] $\to$ y is Borel measurable function such that*

$$\psi(e, f, s) = \psi(f, e, 1 - s) = \gamma_{e,f}(s)$$
$$\psi(e, \psi(e, f, t), s) = \psi(e, f, st) = \gamma_{e,f}(st)$$

$\forall$ e,f $\in$ $\mathcal{Y}$ and s,t $\in$ [0,1], where $\gamma_{e,f}$ is a geodesic from e to f.We may denote $\psi(e, f, s)$ by $\psi_s(e, f)$.

Now, we provide the definition of strongly Bishop-Gromov regular space.

Definition 2.0.2. ***Strongly Bishop-Gromov regular metric-measure space***
Let $\mathcal{Y}$ and ψ be same as definition 2.0.1. If for every $r > 0$ there is x_r such that for almost everywhere $e \in \mathcal{Y}$ and any Borel measure(open) subset $P \subset \mathcal{B}$(e,r), we have

$$\eta(c \in \mathcal{B}(e, r) : \psi_t(e, c) \in P) \leq \frac{x_r}{t^m}\eta(P),$$

$\forall$ 0<t$\leq$ 1. Then we call $\mathcal{Y}$ is Locally Bishop-Gromov regular space of dimension m, for some positive constant m.

A locally strongly Bishop-Gromov regular space $\mathcal{Y}$ is named to be strongly Bishop-Gromov regular space, if $\exists\ x > 0$ such that $x_r \leq x$, $\forall\ r > 0$.

If $\mathcal{Y}$ is (locally) strongly Bishop-Gromov regular space of dimension m, then $\mathcal{Y}$ is (locally) strongly Bishop-Gromov regular space of dimension n, in any case of $n \geq m$, in addition.

Definition 2.0.3. *$\mathcal{Y}$ is said to be locally strongly doubling, if the inequality*

$$\eta(c \in \mathcal{B}(e, r) : \psi_t(e, c) \in P) \leq \frac{x_r}{t^m}\eta(P),$$

hold $\forall\ 1/2 < t \leq 1$.

Every (locally) strongly Bishop-Gromov regular space is (locally) strongly doubling space as well.

Moreover,if $\mathcal{Y}$ is (locally) strongly doubling space, then its also (locally) Bishop-Gromov regular space of dimension m, for some positive number m.

Definition 2.0.4. ***Mean value integral***
Consider $\mathcal{F}$ be measurable set in $\mathcal{Y}$ and we have measurable function $\mathcal{G} : \mathcal{Y} \to \mathbb{R}$, here we can write the mean-value integral of $\mathcal{G}$

$$⨍_{\mathcal{F}} \mathcal{G}(f)d\eta(f) = \frac{1}{\eta(\mathcal{F})}\int_{\mathcal{F}} \mathcal{G}(f)d\eta(f),$$

with $\eta(\mathcal{F})$ is non zero and $\mathcal{G}$ is integrable.

Definition 2.0.5. ***Point of density one for $\mathcal{F}$***
Consider $\mathcal{F}$ be measurable set in $\mathcal{Y}$ and we have measurable function $\mathcal{G} : \mathcal{Y} \to \mathbb{R}$, we can say that $a \in \mathcal{F}$ be the point of density one for $\mathcal{F}$, if

$$\lim_{R_1 \to 0^+} \frac{\eta(\mathcal{B}(e, R_1)) \cap \mathcal{F}}{\eta(\mathcal{B}(e, R_1))} = 1,$$

in $\mathcal{Y}$ (Doubling space).

Definition 2.0.6. ***Lebesgue point***
For doubling space $\mathcal{Y}$,For each integrable function $\mathcal{G} : \mathcal{Y} \to \mathbb{R}$, then

$$\lim_{R_1 \to 0^+} ⨍_{\mathcal{B}(e,R_1)} \mathcal{G}(f)d\eta(f) = \mathcal{G}(e),$$

for almost everywhere $e \in \mathcal{Y}$, then we can say that Lebesgue point of $\mathcal{G}$ and $\mathcal{L}(\mathcal{G})$ is e.

Lemma 2.0.7. *Assume triplet $(\mathcal{Y}, \mathcal{D}, \eta)$ is metric-measure space and $q > 1$. Also $\mathcal{Y}$ be the Bishop-Gromov regular space whose dimension is m, consider the measurable function*

$$\mathcal{G} : \mathcal{Y} \to \mathbb{R},$$

for $e \in \mathcal{Y}$, $r > 0$, assume

$$\int_{\mathcal{B}(e,r)} \frac{|\mathcal{G}(f) - \mathcal{G}(e)|^q}{\mathcal{D}^{m+\beta q}(f,e)} d\eta(f) < \infty,$$

then

$$\lim_{R \to 0^+} \frac{1}{\eta(\mathcal{F})} \int_{\mathcal{B}(e,R)} \frac{|\mathcal{G}(f) - \mathcal{G}(e)|^q}{R^{\beta q}} d\eta(f) = \lim_{R \to 0^+} \int_{\mathcal{B}(e,R)} \frac{|\mathcal{G}(f) - \mathcal{G}(e)|^q}{\mathcal{D}^{m+\beta q}(f,e)} d\eta(f) = 0,$$

here

$$\frac{|\mathcal{G}(f) - \mathcal{G}(e)|^q}{\mathcal{D}^{m+\beta q}(f,e)} = 0 \quad for \quad f = e. \tag{2.0.1}$$

Proof. Assume the decreasing sequence $\{u_1, u_2, ...\}$ of positive number such that

$$\lim_{j \to 0} \{u_j\} = 0,$$

as it is decreasing sequence so we have

$$\mathcal{B}(e, u_1) \supset \mathcal{B}(e, u_2) \supset, ...$$

And

$$\bigcap_{j=1}^{\infty} \mathcal{B}(e, u_j) = \{e\},$$

thus we've,

$$\lim_{j \to \infty} \int_{\mathcal{B}(e,u_n)} \frac{|\mathcal{G}(f) - \mathcal{G}(e)|^q}{\mathcal{D}^{m+\beta q}(f,e)} d\eta(f) = \int_{\{e\}} \frac{|\mathcal{G}(f) - \mathcal{G}(e)|^q}{\mathcal{D}^{m+\beta q}(f,e)} d\eta(f) = 0, \tag{2.0.2}$$

because we have given in 2.0.1, now for any c $\in \mathcal{Y}$ assume $\mathcal{B}(c,r)$ then,

$$\begin{aligned}
&\lim_{R\to 0^+} \fint_{\mathcal{B}(e,R)} \frac{|\mathcal{G}(f)-\mathcal{G}(e)|^q}{R^{\beta q}} d\eta(f) \\
&= \lim_{R\to 0^+} \frac{1}{\eta(\mathcal{F})} \int_{\mathcal{B}(e,R)} \frac{|\mathcal{G}(f)-\mathcal{G}(e)|^q}{R^{\beta q}} d\eta(f) \\
&\leq \left(\frac{U_r r^m}{\eta(\mathcal{B}(c,r))R^m}\right) \lim_{R\to 0^+} \int_{\mathcal{B}(e,R)} \frac{|\mathcal{G}(f)-\mathcal{G}(e)|^q}{R^{\beta q}} d\eta(f) \\
&= \left(\frac{U_r r^m}{\eta(\mathcal{B}(c,r))}\right) \lim_{R\to 0^+} \int_{\mathcal{B}(e,R)} \frac{|\mathcal{G}(f)-\mathcal{G}(e)|^q}{R^{m+\beta q}} d\eta(f) \\
&\leq \lim_{R\to 0^+} \int_{\mathcal{B}(e,R)} \frac{|\mathcal{G}(f)-\mathcal{G}(e)|^q}{\mathcal{D}^{m+\beta q}(f,e)} d\eta(f) \\
&= 0. \qquad\qquad \text{by using equation 2.0.2}
\end{aligned}$$

So we have proved the required condition. □

Lemma 2.0.8. *Assume the triplet* ($\mathcal{Y}$,$\mathcal{D}$,η) *is the metric-measure space and* $q \geq 1$. *Assume* $\mathcal{Y}$ *be the Bishop Grom regular space whose dimension is* m, *consider the measurable function*

$$\mathcal{G} : \mathcal{Y} \to \mathbb{R},$$

and $\beta > 0$. *here we have* U, V, r *are positive numbers and* $c \in \mathcal{Y}$, *also here we have the set* $\mathcal{F}$ *as*

$$\mathcal{F} = \mathcal{L}(\mathcal{G}) \cap \left\{ e \in \mathcal{B}(c,U) : \int_{\mathcal{B}(e,r)} \frac{|\mathcal{G}(f)-\mathcal{G}(e)|^q}{\mathcal{D}^{m+\beta q}(f,e)} d\eta(b) \leq V^q \right\}, \tag{2.0.3}$$

here we have

$$\frac{|\mathcal{G}(f)-\mathcal{G}(e)|^q}{\mathcal{D}^{m+\beta q}(f,e)} = 0 \quad \text{for} \quad f = e,$$

then,

i. $\mathcal{G}|_{\mathcal{F}}$=$\beta$-*Holder continuous function.*

ii. $\mathcal{G}|_{\mathcal{F}}$=$\beta$-*Holder continuous function and its domain is whole* $\mathcal{Y}$.

iii. For ε *positive,* $\exists\ \mathcal{G}_\varepsilon : \mathcal{Y} \to \mathbb{R}$ *is* β-*Holder continuous function, and* $\mathcal{F}_\varepsilon$ *be the subset which is contained in* $\mathcal{B}(c,U)$ *such that*

$$\mathcal{G} = \mathcal{G}_\varepsilon \quad on \quad \mathcal{F}_\varepsilon, \quad \eta(\mathcal{F}/\mathcal{F}_\varepsilon),$$

also,

$$\limsup_{f\to e} \frac{|\mathcal{G}_\varepsilon(f)-\mathcal{G}_\varepsilon(e)|}{\mathcal{D}^\beta(f,e)} \le \lambda\varepsilon,$$

for almost everywhere, e belongs to $\mathcal{F}_\varepsilon$ and here λ be the constant also here we denote $\mathcal{G}|_{\mathcal{F}}$ is restriction of $\mathcal{G}$ to $\mathcal{F}$.

Proof. i. consider here.

$$P_R(\tau) = \int_{\mathcal{B}(\tau,R_j)} \mathcal{G}(c)d\eta(c),$$

here $\tau \in \mathcal{Y}$ with R positive.
Assume that $R_j = r/2^j$ where j ≥ 0, Let e $\in \mathcal{Y}$,so we have

$$\begin{aligned}
&|P_{R_j}(e) - P_{R_{j+1}}(e)|,\\
&\le C_1 ⨍_{\mathcal{B}(e,R_j)} ⨍_{\mathcal{B}(e,R_j)} |\mathcal{G}(c)-\mathcal{G}(\tau)|d\eta(c)d\eta(\tau)\\
&= C_1 ⨍_{\mathcal{B}(e,R_j)} ⨍_{\mathcal{B}(e,R_j)} |\mathcal{G}(c)-\mathcal{G}(\tau)+\mathcal{G}(e)-\mathcal{G}(e)|d\eta(c)d\eta(\tau)\\
&\le C_1 ⨍_{\mathcal{B}(e,R_j)} ⨍_{\mathcal{B}(e,R_j)} |\mathcal{G}(c)-\mathcal{G}(e)|+|\mathcal{G}(\tau)-\mathcal{G}(e)|d\eta(c)d\eta(\tau)\\
&\le 2C_1 ⨍_{\mathcal{B}(e,R_j)} |\mathcal{G}(c)-\mathcal{G}(\tau)|d\eta(c)\\
&\le 2C_1 \left(⨍_{\mathcal{B}(e,R_j)} |\mathcal{G}(c)-\mathcal{G}(\tau)|^q \right)^{1/q} d\eta(c)\\
&\le 2C_1 \left(\frac{1}{\eta(\mathcal{B}(e,R_j))} \int_{\mathcal{B}(e,R_j)} |\mathcal{G}(c)-\mathcal{G}(\tau)|^q \right)^{1/q} d\eta(c)
\end{aligned}$$

$$\leq 2C_1\left(\frac{U_r r^m}{\eta(\mathcal{B}(d,r))R_j^m}\int_{\mathcal{B}(a,R_j)}|\mathcal{G}(c)-\mathcal{G}(\tau)|^q\right)^{1/q} d\eta(c)$$

$$\leq 2C_1\left(\frac{U_r r^m}{\eta(\mathcal{B}(d,r))R_j^{m+q\beta-q\beta}}\int_{\mathcal{B}(e,R_j)}|\mathcal{G}(c)-\mathcal{G}(\tau)|^q\right)^{1/q} d\eta(c)$$

$$\leq 2C_1\left(\frac{U_r r^m}{\eta(\mathcal{B}(d,r))R_j^{-q\beta}}\int_{\mathcal{B}(e,R_j)}\frac{|\mathcal{G}(c)-\mathcal{G}(\tau)|^q}{\mathcal{D}^{m+q\beta}(c,e)}\right) d\eta(c)$$

$$\leq C_1 V\left(\frac{r^m}{\eta(\mathcal{B}(d,r))}\right)^{1/q} R_j^{\beta}.$$

∀ j ≥ 0 C_1 depending upon U_r and m,
then ∀ k $>$ j ≥ 0,

$$|P_{R_k}(e)-P_{R_j}(e)| \leq \sum_{n=j}^{k-1}|P_n(e)-P_{n+1}(e)| \leq C_1 V\left(\frac{r^m}{\eta(\mathcal{B}(d,r))}\right)^{1/q}\sum_{n=j}^{k-1}R_n^{\beta}, \qquad (2.0.4)$$

as e $\in \mathcal{L}$ $(\mathcal{G})$ so

$$\mathcal{G}(e) = \lim_{j\to\infty} P_{R_j}(e),$$

now using 2.0.4 ,for j $\geq$0 we have,

$$|\mathcal{G}(e)-P_{R_j}(e)| \leq C_1 V\left(\frac{r^m}{\eta(\mathcal{B}(d,r))}\right)^{1/q}\sum_{n=j}^{k-1}R_n^{\beta}. \qquad (2.0.5)$$

Let us consider,for e,f $\in \mathcal{F}$ and $j_0 \geq 0$(integer) with

$$R_{j_0+1} \leq \mathcal{D}(e,f) \leq R_{j_0},$$

then we have,

$$
\begin{aligned}
|P_{R_{j_0}}(e) - P_{R_{j_0}}(f)| &\le C_1 \fint_{\mathcal{B}(e,2R_{j_0})} \fint_{\mathcal{B}(e,2R_0)} |\mathcal{G}(c) - \mathcal{G}(\tau)| d\eta(c) d\eta(\tau) \\
&= C_1 \fint_{\mathcal{B}(e,2R_0)} \fint_{\mathcal{B}(e,2R_{j_0})} |\mathcal{G}(c) - \mathcal{G}(\tau) + \mathcal{G}(e) - \mathcal{G}(e)| d\eta(c) d\eta(\tau) \\
&\le C_1 \fint_{\mathcal{B}(e,2R_{j_0})} \fint_{\mathcal{B}(e,2R_{j_0})} |\mathcal{G}(c) - \mathcal{G}(e)| + |\mathcal{G}(\tau) - \mathcal{G}(e)| d\eta(c) d\eta(\tau) \\
&\le 2C_1 \fint_{\mathcal{B}(e,2R_{j_0})} |\mathcal{G}(c) - \mathcal{G}(\tau)| d\eta(c) \\
&\le 2C_1 \left(\fint_{\mathcal{B}(e,2R_{j_0})} |\mathcal{G}(c) - \mathcal{G}(\tau)|^q \right)^{1/q} d\eta(c) \\
&\le 2C_1 \left(\frac{1}{\mathcal{B}(e,2R_{j_0})} \int_{\mathcal{B}(e,R_j)} |\mathcal{G}(c) - \mathcal{G}(\tau)|^q \right)^{1/q} d\eta(c) \\
&\le 2C_1 \left(\frac{U_r r^m}{\eta(\mathcal{B}(d,r))(2R_{j_0})^m} \int_{\mathcal{B}(e,2R_{j_0})} |\mathcal{G}(c) - \mathcal{G}(\tau)|^q \right)^{1/q} d\eta(c) \\
&\le 2C_1 \left(\frac{U_r r^m}{\eta(\mathcal{B}(d,r))(2R_{j_0})^{m+q\beta-q\beta}} \int_{\mathcal{B}(e,2R_{j_0})} |\mathcal{G}(c) - \mathcal{G}(\tau)|^q \right)^{1/q} d\eta(c) \\
&\le 2C_1 \left(\frac{U_r r^m}{\eta(\mathcal{B}(d,r))(2R_{j_0})^{-q\beta}} \int_{\mathcal{B}(e,2R_{j_0})} \frac{|\mathcal{G}(c) - \mathcal{G}(\tau)|^q}{\mathcal{D}^{m+q\beta}(c,e)} \right) d\eta(c) \\
&\le C_1 V \left(\frac{r^m}{\eta(\mathcal{B}(d,r))} \right)^{1/q} (2R_{j_0})^\beta,
\end{aligned}
$$

so,

$$
|P_{R_{j_0}}(e) - P_{R_{j_0}}(f)| \le C_1 V \left(\frac{r^m}{\eta(\mathcal{B}(d,r))} \right)^{1/q} (2R_{j_0})^\beta, \tag{2.0.6}
$$

finally, by using (2.0.5) and (2.0.6),we get

$$
|\mathcal{G}(e) - \mathcal{G}(f)| \le 2^\beta C_1 V \left(\frac{r^m}{\eta(\mathcal{B}(d,r))} \right)^{1/q} \sum_{n=j_0}^{\infty} R_m^\beta, \tag{2.0.7}
$$

as,

$$
\frac{r}{2^{j_0+1}} = R_{j_0+1} \le \mathcal{D}(e,f) \le R_{j_0} = \frac{r}{2^{j_0}}, \tag{2.0.8}
$$

$\exists$ constant K_β with

$$|\mathcal{G}(e)-\mathcal{G}(f)| \le K_\beta C_1 V \left(\frac{r^m}{\eta(\mathcal{B}(d,r))}\right)^{1/q} \mathcal{D}^\beta(e,f), \tag{2.0.9}$$

$\forall$ e,f $\in \mathcal{F}$ when $\mathcal{D}(e,f) \le r/2$, hence (i) is proved.

ii. Its the result of the McShane′s Theorem, as in the above part we have proved whose domain is $\mathcal{F}$ so it is also hold for the whole domain $\mathcal{Y}$.

iii. As we see in the Lemma (2.0.7) that,

$$\lim_{R\to 0^+} \int_{\mathcal{B}(e,R)} \frac{|\mathcal{G}(f)-\mathcal{G}(e)|^q}{\mathcal{D}^{m+\beta q}(f,e)} d\eta(f) = 0,$$

$\forall$ e $\in \mathcal{F}$,here we may consider $0 < \varepsilon$ V,$\exists$ $0 < R_0 < r$ such that

$$\eta(\mathcal{F}\backslash\mathcal{F}_\varepsilon) < \varepsilon,$$

here we define $\mathcal{F}_\varepsilon$ by

$$\mathcal{F} = \mathcal{L}(\mathcal{G}) \cap \left\{ e \in \mathcal{B}(d,U) : \int_{\mathcal{B}(e,r)} \frac{|\mathcal{G}(f)-\mathcal{G}(e)|^q}{\mathcal{D}^{m+\beta q}(f,e)} d\eta(f) \le \varepsilon^q \right\},$$

using (2.0.9), we get $\forall \in \mathcal{F}_\varepsilon$

$$\limsup_{f\to e, f\in\mathcal{F}_\varepsilon} \frac{|\mathcal{G}(e)-\mathcal{G}(f)|}{\mathcal{D}^\beta(e,f)} \le K_\beta C_1 \left(\frac{r^m}{\eta(\mathcal{B}(d,r))}\right)^{1/q} \varepsilon, \tag{2.0.10}$$

with part (ii), $\exists$ β-Holder continuous function $\mathcal{G}_\varepsilon$ and its domain is $\mathcal{Y}$ so that $\mathcal{G}|_{\mathcal{F}_\varepsilon}$=$\mathcal{G}_\varepsilon$, we have to show that for the point of density one for $\mathcal{F}_\varepsilon$ such that,

$$\limsup_{f\to e} \frac{|\mathcal{G}_\varepsilon(e)-\mathcal{G}_\varepsilon(f)|}{\mathcal{D}^\beta(e,f)} \le K_\beta C_1 \left(\frac{r^m}{\eta(\mathcal{B}(d,r))}\right)^{1/q} \varepsilon, \tag{2.0.11}$$

assume that ρ is positive and too small,as a is point of density one for $\mathcal{F}_\varepsilon$,here we have $0 < \delta_1 < R_0$ depends upon ρ,so that

$$\frac{\eta(\mathcal{B}(e,R) \cap \mathcal{F}_\varepsilon^c)}{\eta(\mathcal{B}(e,R))} < \rho,$$

$\forall$ 0 < R $\le \delta_1$, as we have given that $\mathcal{Y}$ be the Bishop-Gromov regular whose dimension is m, so $\exists$ $U_r \ge 1$, we got,

$$\begin{aligned}\frac{\eta(\mathcal{B}(c,r))}{r^m} &\leq U_r \frac{\eta(\mathcal{B}(\tau,R))}{R^m}\\ \frac{\eta(\mathcal{B}(c,r))}{\eta(\mathcal{B}(\tau,R))} &\leq U_r \left(\frac{r}{R}\right)^m\\ U_r^{-1}\left(\frac{R}{r}\right)^m &\leq \frac{\eta(\mathcal{B}(\tau,R))}{\eta(\mathcal{B}(c,r))},\end{aligned}$$

$\forall$ 0 < R < r and c,$\tau \in \mathcal{Y}$ when $\mathcal{D}(c,\tau) \leq r$.
Now we define δ_2 and δ for $0 < \rho < (4U_r)^{-1}$ and $0 < \delta_1 < R_0$ such that,

$$\delta_2 = (U_r\rho)^{1/m}\delta_1 \quad \text{and} \quad \delta = \delta_1 - \delta_2,$$

then we will show for f $\in \mathcal{B}(e,\delta)$,

$$\mathcal{B}(f,\delta_2) \cap \mathcal{F}_\varepsilon \neq \phi, \tag{2.0.12}$$

on contrary we assume that$\mathcal{B}(f,\delta_2) \cap \mathcal{F}_\varepsilon = \phi$, then

$$\begin{aligned}\rho &> \frac{\eta(\mathcal{B}(e,\delta_1) \cap \mathcal{F}_\varepsilon^c)}{\eta(\mathcal{B}(e,\delta_1))}\\ &\geq \frac{\eta(\mathcal{B}(e,\delta_2)}{\eta(\mathcal{B}(e,\delta_1))}\\ &\geq U_r^{-1}\left(\frac{\delta_2}{\delta_1}\right)^m\\ &= U_r^{-1}\left(\frac{(U_r\rho)^{1/m}}{\delta_1}\right)^m\\ &= \rho,\end{aligned}$$

so it contradict to our assumptions.
So for each f $\in \mathcal{Y}$, $\exists$ c $\in \mathcal{B}(f,\delta_2) \cap \mathcal{F}_\varepsilon$ so Equation (2.0.12) holds, so we have

$$\begin{aligned}|\mathcal{G}_\varepsilon(f) - \mathcal{G}_\varepsilon(e)| &\leq |\mathcal{G}_\varepsilon(f) - \mathcal{G}_\varepsilon(c)| + |\mathcal{G}_\varepsilon(c) - \mathcal{G}_\varepsilon(e)|\\ &\leq L_1\mathcal{D}^\beta(f,c) + |\mathcal{G}(c) - \mathcal{G}(e)|\\ &\leq L_1\delta_2^\beta + |\mathcal{G}(c) - \mathcal{G}(e)|,\end{aligned}$$

here L_1 is β-Holder constant, now take $\rho \to 0$ (2.0.10) implies

$$\limsup_{f\to e}\frac{|\mathcal{G}_\varepsilon(e)-\mathcal{G}_\varepsilon(f)|}{\mathcal{D}^\beta(e,f)}\le K_\beta C_1\left(\frac{r^m}{\eta(\mathcal{B}(d,r))}\right)^{1/q}\varepsilon,$$

$\forall$ e $\in \mathcal{F}_\varepsilon$ whenever e be the point of density one for $\mathcal{F}_\varepsilon$,as $\mathcal{Y}$ be the doubling space hence we have proved the required result.

□

Here we provide basic definition which terms are used in next theorems.

Definition 2.0.9. ***Lebesgue Space***
Consider $\mathcal{Y}$ denote the open set in $\mathbb{R}$ and Consider the lebesgue meaurable function,

$$h:\mathcal{Y}\to\mathbb{C}.$$

Then we named $\mathcal{L}^q$ is Lebesgue space

$$\mathcal{L}^q=\{h\in\mathcal{L}^1(\mathcal{Y}):\int_Y|h|^q<+\infty\}.$$

Definition 2.0.10. ***Connected set***
Consider the set $\mathcal{F}\subset\mathbb{R}$, it is said to be a connected set if $\nexists$ two disjoint open set such that

$$\mathcal{F}\nsubseteq\mathcal{C}\cup\mathcal{H}\quad \textit{and}\quad \mathcal{C}\cap\mathcal{H}=\Phi.$$

Definition 2.0.11. ***Convex set***
Assume that $\mathcal{Y}$ be the convex subset, the function,

$$h:\mathcal{Y}\to\mathbb{R},$$

It is said to be convex function if $\forall$ s $\in$ [0,1] it satisfies

$$h(sy_1+(1-s)y_2)\le sh(y_1)+(1-s)h(y_2),\qquad \textit{for}\quad y_1,y_2\in\mathcal{Y}.$$

Theorem 2.0.12. *Let we take function*

$$h:\mho\to\mathbb{R},$$

which is (Lebesgue) measurable and defined on connected open subset $\mho$ from $\mathbb{R}^n$ such that

$$\int_{\mho\times\mho}\frac{|h(f)-h(e)|}{|f-e|^{n+1}}d(f)d(e)<\infty.$$

Then the function h will be constant almost everywhere.

Here we will prove the generalization of Theorem 2.0.12

Theorem 2.0.13. *Assume that the (Lebesgue) measurable function is*

$$h : \mho \to \mathbb{R},$$

which is defined on connected open subset $\mho$ *in* $\mathbb{R}^n$
and we have the convex function

$$\varrho : [0, \infty[\to [0, \infty[,$$

such that

$$\varrho(s) = 0 \quad \textit{if and only if} \quad s = 0,$$

also assume that

$$\int_{\mho\times\mho} \varrho\left(\frac{|h(b) - h(a)|}{|b - a|}\right)\frac{d(b)d(a)}{|b - a|^n} < \infty,$$

then h is constant function almost everywhere.

Proof. We consider without loss of generality that $\mho$ be the open ball.Here we consider the function

$$g :]0, \infty[\to \mathbb{R},$$

and it is defined as,

$$g(u) := \int_{\mho^2(u)} \varrho\left(\frac{|h(f) - h(e)|}{|f - e|}\right)\frac{d(f)d(e)}{|f - e|^n},$$

here we consider as

$$\rho = \frac{e + f}{2} \quad \text{and} \quad \mho^2(u) = \{(e, f) \in \mho \times \mho \quad : \quad |f - e| < u\},$$

thus we've

$$
\begin{aligned}
g(u) &= \int_{\mho^2(u)} \varrho\Big(\frac{|h(f)-h(e)|}{|f-e|}\Big)\frac{d(f)d(e)}{|f-e|^n} \\
&= \int_{\mho^2(u)} \varrho\Big(\frac{|h(f)-h(e)+h(\rho)-h(\rho)|}{|f-e|}\Big)\frac{d(f)d(e)}{|f-e|^n} \\
&\leq \int_{\mho^2(u)} \varrho\Big(\frac{|h(f)-h(\rho)|+|h(\rho)-h(e)|}{|f-e|}\Big)\frac{d(f)d(e)}{|f-e|^n} \qquad \because \text{Triangular inequality,} \\
&= \frac{1}{2}\int_{\mho^2(u)} \varrho\Big(\frac{2|h(f)-h(\rho)|}{|f-e|}\Big)\frac{d(f)d(e)}{|f-e|^n} + \frac{1}{2}\int_{\mho^2(u)} \varrho\Big(\frac{2|h(\rho)-h(e)|}{|f-e|}\Big)\frac{d(f)d(e)}{|f-e|^n} \\
&= \frac{1}{2}\int_{\mho^2(u)} \varrho\Big(\frac{2|h(2\rho-e)-h(\rho)|}{|2\rho-e-e|}\Big)\frac{d(f)d(e)}{|2\rho-e-e|^n} + \frac{1}{2}\int_{\mho^2(u)} \varrho\Big(\frac{2|h(\rho)-h(e)|}{|2\rho-e-e|}\Big)\frac{d(e)d(f)}{|2\rho-e-e|^n} \\
&= \frac{1}{2}\int_{\mho^2(u)} \varrho\Big(\frac{2|2h(\rho)-h(e)-h(\rho)|}{|2(\rho-e)|}\Big)\frac{d(f)d(e)}{|2(\rho-e)|^n} + \frac{1}{2}\int_{\mho^2(u)} \varrho\Big(\frac{2|h(\rho)-h(e)|}{|2(\rho-e)|}\Big)\frac{d(f)d(e)}{|2(\rho-e)|^n},
\end{aligned}
$$

now we use convexity which is h($2\rho - $e) $\leq$ 2h(ρ) $-$ h(e),then we have

$$
\begin{aligned}
g(u) &\leq \frac{1}{2}\int_{\mho^2(u)} \varrho\Big(\frac{|2h(\rho)-h(e)-h(\rho)|}{|(\rho-e)|}\Big)\frac{d(f)d(e)}{|2(\rho-e)|^n} + \frac{1}{2}\int_{\mho^2(u)} \varrho\Big(\frac{|h(\rho)-h(e)|}{|(\rho-e)|}\Big)\frac{d(f)d(e)}{|2(\rho-e)|^n} \\
&= \int_{\mho^2(u)} \varrho\Big(\frac{|h(\rho)-h(e)|}{|(\rho-e)|}\Big)\frac{d(f)d(e)}{|2(\rho-e)|^n},
\end{aligned}
$$

$\forall$ positive u.
Here we have changed the variables as

$$(e, f) \to (e, \rho),$$

then we have

$$\{(e, f) \in \mho \times \mho : |2\rho - e - e| < u\} = \{(e, f) \in \mho \times \mho : |\rho - e| < \frac{u}{2}\} = \mho^2(u/2),$$

so,

$$g(u) \leq \int_{\mho^2(u/2)} \varrho\Big(\frac{|h(\rho)-h(e)|}{|(\rho-e)|}\Big)\frac{d(f)d(e)}{|2(\rho-e)|^n} = g(u/2),$$

which implies that g(u)$\leq$ g(u/2).Similarly in this way we get g(u) $\leq g(u/2^j)$, now if j$\to \infty$ $\forall$ u $>$ 0, then g(u)=0

$$
\begin{aligned}
g(u) := \int_{\mho^2(u)} \varrho\left(\frac{|h(f) - h(e)|}{|f - e|}\right)\frac{d(f)d(e)}{|f - e|^n} = 0 \\
h(f) - h(e) = 0 \\
h(f) = h(e).
\end{aligned}
$$

Which implies that h is constant function almost everywhere. □

Chapter 3

An integral type characterization of Lipschitz functions

In this part we will review about generalised change of variables formula for functions over(locally) strongly doubling metric-measure spaces, actually we will review that how we extend the characterization of Lipschitz function to general abstract metric-measure space,i.e $(\mathcal{Y}, \mathcal{D})$ be the matric space with η measure ([2], [5]).At the end we will review results in the class of metric measure space such that it satisfies the condition of Strongly Bishop-Gromov regularity.

3.1 Generalised change of variables

Definition 3.1.1. ***Essentially Lipschitzian***
Take a measurable function,

$$\zeta : [0,1] \to \mathbb{R},$$

It is essentially Lipschitzian iff $\exists\ L_1 > 0$ *with*

$$\int_o^1 \int_o^1 \exp\left[\frac{|\zeta(f) - \zeta(e)|}{L_1|f-e|} \ln \frac{1}{|f-e|}\right] d(f)d(e) \prec \infty.$$

Theorem 3.1.2. ***Generalised change of variables:*** *Assume that* $(\mathcal{Y},\mathcal{D},\eta)$ *be a locally strongly Bishop-Gromov Regular Space with dimension m. Let* ψ *be a Borel measurable function as in the definition 2.0.1 and*

$$f : \mathcal{Y} \times \mathcal{Y} \to [0, \infty[,$$

be a measurable function. Then

$$\int_{\mathcal{B}(y,r)^2} f(\psi_s(e,f),\psi_t(e,f))d\eta(f)d\eta(e) \leq \frac{N}{|s-t|^m}\int_{B(y,r)^2} f(c,v)d\eta(v)d\eta(c),$$

where N is constant depending on x_r and $\mathcal{D}(y,r)^2$=B(y,r)× $\mathcal{D}$(y,r),for almost everywhere s,t ∈ [0,1].

Proof. Without sacrificing generality, We may consider that ψ_t : Y × Y → Y, is measurable function ∀ t ∈ [0,1]. We will prove that for every r > 0, there is $\tilde{x_r} > 0$ such that for almost everywhere y ∈ $\mathcal{Y}$. And any Borel measurable subset Q ⊂ $\mathcal{D}(y,r)^2$, we have

$$\eta\times\eta(\{(e,f)\in\mathcal{B}(y,r)^2 : (\psi_s(e,f),\psi_t(e,f))\in Q\}) \leq \frac{\tilde{x_r}}{|s-t|^m}\eta\times\eta(Q), \tag{3.1.1}$$

∀ s,t ∈ [0,1]. We may consider the product measure $\eta \times \eta$ on $\mathcal{Y} \times \mathcal{Y}$ and the metric is

$$\mathcal{D}((e_1,f_1),(e_2,f_2)) := \max\{\mathcal{D}(e_1,e_2),\mathcal{D}(f_1,f_2)\},$$

It is, in fact, both compulsory and sufficient to demonstrate that the 3.1.1 exist for subset Q = $\mathcal{S} \times \mathcal{T}$. Here $\mathcal{S}$ and $\mathcal{T}$ are Borel measurable subset within $\mathcal{B}(y,r)$.

Firstly, consider s = 0, namely, $\psi_s(e,f) \equiv$ e, so we get that

$$\begin{aligned}
&\eta\times\eta(\{(e,f)\in\mathcal{B}(y,r)^2 : (e,\psi_t(e,f)\in S\times T\})\\
&= \int_S \eta(\{f\in\mathcal{B}(e,r) : \psi_t(e,f)\in T\})\\
&\leq \int_S \eta(\{f\in\mathcal{B}(e,2r) : \psi_t(e,f)\in T\}) \quad \because \mathcal{B}(e,r)\subseteq\mathcal{B}(e,2r)\\
&\leq \int_S \frac{x_{2r}}{t^m}\eta(T)d\eta(e) \qquad \because \mathcal{Y} \text{ is locally Bishop-Gromov}\\
&= \left[\int_S d\eta(e)\right]\frac{x_{2r}}{t^m}\eta(T)\\
&= \eta(S)\frac{x_{2r}}{t^m}\eta(T)\\
&= \frac{x_{2r}}{t^m}\eta(S)\eta(T)\\
&= \frac{x_{2r}}{t^m}\eta\times\eta(S\times T)\\
&= \frac{x_{2r}}{t^m}\eta(Q) \qquad \because S\times T = Q,
\end{aligned}$$

thus
For s = 0 and Q = S × T the inequality (3.1.1) holds ,And It is applicable to any Borel measurable subset Q in $B(e,r)^2$.
Now, For the general case we will prove the inequality (3.1.1).
We may assume Without sacrificing generality that 0 < s < t < 1.
Define

$$\Lambda_1 := (e, \psi_t(e,f))$$
$$\Lambda_2 := (\psi_{s/t}(e,f), f) = (\psi_{1-s/t}(f,e), f),$$

as

$$\Lambda_2 o \Lambda_1 = (\psi_s(e,f), \psi_t(e,f)),$$

Since

$$\begin{aligned}
\Lambda_2 o \Lambda_1 &= \Lambda_2(\Lambda_1(e,f)) \\
&= \Lambda_2(e, \psi_t(e,f)) \\
&= (\psi_{s/t}(e, \psi_t(e,f)), \psi_t(e,f)) \\
&= (\psi_{s/t}(e, \psi(e,f,t)), \psi_t(e,f)) \\
&= (\psi(e, \psi(e,f,t), s/t), \psi_t(e,f)) \\
&= (\psi(e,f,\frac{s}{t}.t), \psi_t(e,f)) \\
&= (\psi_s(e,f), \psi_t(e,f)).
\end{aligned}$$

Now we prove following containment,

$$\{(a,b) \in \mathcal{B}(y,r)^2 : \Lambda_2 o \Lambda_1(a,b) \in Q\} \subseteq \{(a,b) \in \mathcal{B}(y,r)^2 : \Lambda_1(a,b) \in \Lambda_2^{-1}(Q)\},$$

let

$$\begin{aligned}
&(a_1,b_1) \in \{(a,b) \in \mathcal{B}(y,r)^2 : \Lambda_2 o \Lambda_1(a,b) \in Q\} \\
&\Rightarrow (a_1,b_1) \in \mathcal{B}(y,r)^2 \quad \text{such that} \quad \Lambda_2 o \Lambda_1(a,b) \in Q \\
&\Rightarrow (a_1,b_1) \in \mathcal{B}(y,r)^2 \quad \text{such that} \quad \Lambda_2(\Lambda_1(a_1,b_1)) \in Q \\
&\Rightarrow (a_1,b_2) \in \mathcal{B}(y,r)^2 \quad \text{such that} \quad \Lambda_1(a_1,b_1) \in \Lambda_2^{-1}(Q).
\end{aligned}$$

So the containment holds in any case of Borel measurable subset Q in $B(y,r)^2$. Similarly by using the above computations, we obtain

$$
\begin{aligned}
&\eta \times \eta(\{(a,b) \in B(y,r)^2 : \Lambda_2 o \Lambda_1(a,b) \in Q\}) \\
&\leq \eta \times \eta(\{(a,b) \in B(y,r)^2 : \Lambda_1(a,b) \in \Lambda_2^{-1}(Q)\}) \\
&\leq \frac{x_{2r}}{t^m} \eta \times \eta(\Lambda_2^{-1}(Q) \cap \mathcal{B}(y,r)^2) \qquad \text{By using Definition (2.0.2)} \\
&\leq \frac{x_{2r}}{t^m} \frac{x_{3r}}{(1-s/t)^m} \eta \times \eta(Q) \\
&\leq \frac{(x_{2r})(x_{3r})}{|t-s|^m} \eta \times \eta(Q),
\end{aligned}
$$

in any case of Borel measurable subset Q in $\mathcal{B}(y,r) \times \mathcal{B}(y,r)$. Which completes the proof for inequality (3.1.1). □

Corollary 3.1.3. *Assume that assumptions and notation are same as in Theorem (3.1.2). Then we have the following inequality*

$$
\int_{\mathcal{B}(y,r)^2} g(\psi_s(e,f), \psi_t(e,f)) d\eta(f) d\eta(e) \leq \frac{N}{|s-t|^m} \int_{\mathcal{B}(y,r)^2 \cap \nabla_{s,t}} g(c,\nu) d\eta(\nu) d\eta(c),
$$

where,

$$
\nabla_{s,t} = \{(c,\nu) \in \mathcal{Y} \times \mathcal{Y} : \mathcal{D}(c,\nu) \leq |t-s|\}.
$$

3.2 Results

Theorem 3.2.1. *Let ($\mathcal{Y}$,$\mathcal{D}$,η) be the locally strongly BGRS of dimension m, for some positive number m.Also assume that*

$$
g : \mathcal{Y} \to \mathbb{R},
$$

is a measurable function so that

$$
\int_{\mathcal{Y} \times \mathcal{Y}} \exp\left(\frac{|g(f) - g(e)|}{\mathcal{D}(f,e)})| \ln \mathcal{D}^m(f,e)| \right) d\eta(f) d\eta(e) < \infty
$$

Then g is essentially 1-Lipschitz, that is $|g(f) - g(e)| \leq \mathcal{D}(f,e)$, for almost everywhere e,f $\in \mathcal{Y}$.

Proof. consider $\mathcal{B}$(y,r) in $\mathcal{Y}$ also $\vartheta > 1$, we define the function

$$
\zeta :]0,\infty[\to]0,\infty[,
$$

As

$$\zeta(u) = \int_{E_u} \exp\left(\frac{|g(f) - g(e)|}{\mathcal{D}(f,e)} |\ln \mathcal{D}^m(f,e)|\right) d\eta(f) d\eta(e),$$

where

$$E_u = \{(e,f) \in \mathcal{B}(y,r) \times \mathcal{B}(y,r) : |g(f) - g(e)| \geq \vartheta \mathcal{D}(f,e), \quad \& \quad \mathcal{D}(f,e) \leq u\}.$$

Without loss of generality, now we may assume that

$$\psi_t : \mathcal{Y} \times \mathcal{Y} \to \mathcal{Y},$$

is measurable, for all t $\in$ [0,1]. consider positive integer n and e,f $\in \mathcal{Y}$,
set

$$e_i := \psi(e, f, i/p),$$

we have

$$\begin{aligned}
\zeta(u) &= \int_{E_u} \exp\left(\frac{|g(f) - g(e)|}{\mathcal{D}(f,e)} |\ln \mathcal{D}^m(f,e)|\right) d\eta(f) d\eta(e) \\
&= \int_{E_u} \exp\left(\frac{|g(f) - g(e)|}{\mathcal{D}(f,e)} |\ln \mathcal{D}^m(f,e) - \ln p^m + \ln n^m|\right) d\eta(f) d\eta(e) \\
&= \int_{E_u} \exp\left(\frac{|g(f) - g(e)|}{\mathcal{D}(f,e)} |\ln \frac{\mathcal{D}^m(f,e)}{p^m} + \ln p^m|\right) d\eta(f) d\eta(e) \quad \because \ln(e/f) = \ln(e) - \ln(f) \\
&= \int_{E_u} \exp\left(\frac{|g(f) - g(e)|}{\mathcal{D}(f,e)} |\ln(\frac{\mathcal{D}(f,e)}{p})^m + \ln p^m|\right) d\eta(f) d\eta(e) \\
&= \int_{E_u} \exp\left(\frac{|g(f) - g(e)|}{\mathcal{D}(f,e)} |\ln(\frac{\mathcal{D}(f,e)}{p})^m| - \ln p^m\right) d\eta(f) d\eta(e) \quad \because \text{n is too large} \\
&= \int_{E_u} \exp\left(\frac{|g(f) - g(e)|}{\mathcal{D}(f,e)} \ln\left(\frac{\mathcal{D}(f,e)}{p}\right)^m\right) \exp\left(\frac{|g(f) - g(e)|}{\mathcal{D}(f,e)} \ln p^{-m}\right) d\eta(f) d\eta(e),
\end{aligned}$$

since $\vartheta > \frac{|g(f)-g(e)|}{\mathcal{D}(f,e)}$,

$$
\begin{aligned}
&\le \int_{E_u} \exp\left(\frac{|g(f)-g(e)|}{\mathcal{D}(f,e)}\left|\ln\left(\frac{\mathcal{D}(f,e)}{p}\right)^m\right|\right)e^{\vartheta(-\ln p^m)}d\eta(f)d\eta(e)\\
&\le \int_{E_u} \exp\left(\frac{|g(f)-g(e)|}{\mathcal{D}(f,e)}\left|\ln\left(\frac{\mathcal{D}(f,e)}{p}\right)^m\right|\right)e^{\ln(n^{-\vartheta m})}d\eta(f)d\eta(e)\\
&= p^{-m\vartheta}\int_{E_u} \exp\left(\frac{|g(f)-g(e)|}{\mathcal{D}(f,e)}\left|\ln\left(\frac{\mathcal{D}(f,e)}{p}\right)^m\right|\right)d\eta(f)d\eta(e)\\
&= p^{-m\vartheta}\int_{E_u} \exp\left(\frac{|g(f)-g(e)|}{n(\mathcal{D}(f,e)/p)}\left|\ln\left(\frac{\mathcal{D}(f,e)}{p}\right)^m\right|\right)d\eta(f)d\eta(e)\\
&\le p^{-m\vartheta}\int_{E_u} \exp\left(\sum_{i=0}^{p-1}\frac{|g(e_{i+1})-g(e_i)|}{n(\mathcal{D}(f,e)/p)}\left|\ln\left(\frac{\mathcal{D}(f,e)}{p}\right)^m\right|\right)d\eta(f)d\eta(e),
\end{aligned}
$$

since

$$
a_i = \psi(e,f,i/p) = \psi_{i/p}(e,f) = \gamma_{e,f}(i/p)
$$

$$
\begin{aligned}
\mathcal{D}(e_i,e_{i+1}) &= \mathcal{D}(\gamma_{e,f}(i/p),\gamma_{e,f}((i+1)/p))\\
&= \left|\frac{i}{p}-\frac{i+1}{p}\right|\mathcal{D}(e,f),\\
&= \frac{1}{p}\mathcal{D}(e,f),
\end{aligned}
$$

then

$$
\zeta(u) < p^{-m\vartheta}\int_{E_u} \exp\left(\sum_{i=0}^{p-1}\frac{|g(e_{i+1})-g(e_i)|}{n(\mathcal{D}(e_{i+1},e_i)}|\ln \mathcal{D}^m(e_{i+1},e_i)|d\eta(f)d\eta(e),
$$

now by using the following inequality

$$
\exp(\sum_{i=0}^{p-1}\alpha_i) \le \frac{1}{p}\sum_{i=0}^{p-1}\exp(p\alpha_i),
$$

then we have

$$
\zeta(u) \le p^{-m\vartheta}\frac{1}{p}\sum_{i=0}^{p-1}\int_{E_u} p\frac{|g(e_{i+1})-g(e_i)|}{p(\mathcal{D}(e_{i+1},e_i)}|\ln \mathcal{D}^m(e_{i+1},e_i)|d\eta(f)d\eta(e),
$$

$$\zeta(u) \leq p^{-m\vartheta}\frac{1}{p}\sum_{i=0}^{p-1}\int_{E_u}\frac{|g(e_{i+1})-g(e_i)|}{(\mathcal{D}(e_{i+1},e_i)}|\ln \mathcal{D}^m(e_{i+1},e_i)|d\eta(f)d\eta(e),$$

As we know that

$$e_i = \psi(e,f,i/p) = \psi_{i/p}(e,f),$$

So the above exponential function is the function of variable $\psi_{i/p}(e,f)$, so by the Lemma (3.1.2)
Here

$$s = \frac{i+1}{p}, \qquad t = \frac{i}{p},$$

So for u $\in$]0,1], we get

$$\begin{aligned}
\zeta(u) &\leq p^{-m\vartheta-1}\sum_{i=0}^{n-1}\frac{N}{|\frac{i+1}{n}-\frac{i}{n}|^m}\int_{Y\times Y}\exp\left(\frac{|g(e_{i+1})-g(e_i)|}{\mathcal{D}(e_{i+1},e_i)}|\ln \mathcal{D}^m(e_{i+1},e_i)|\right)d\eta(e_{i+1})d\eta(e_i)\\
&\leq p^{-m\vartheta-1}\sum_{i=0}^{n-1}\frac{N}{n^{-m}}\int_{Y\times Y}\exp\left(\frac{|g(c)-g(\nu)|}{d(c,\nu)}|\ln \mathcal{D}^m(c,\nu)|\right)d\eta(c)d\eta(\nu)\\
&= p^{-m\vartheta-1-m}N\int_{\mathcal{Y}\times\mathcal{Y}}\exp\left(\frac{|g(c)-g(\nu)|}{\mathcal{D}(c,\nu)}|\ln \mathcal{D}^m(c,\nu)|\right)d\eta(c)d\eta(\nu)\\
&\leq p^{m(1-\vartheta)}N\int_{\mathcal{Y}\times\mathcal{Y}}\exp\left(\frac{|g(c)-g(\nu)|}{\mathcal{D}(c,\nu)}|\ln \mathcal{D}^m(c,\nu)|\right)d\eta(c)d\eta(\nu).
\end{aligned}$$

For r $\in$]0,1], and constant N be depends upon x_r.
Taking p $\to \infty$ then we have

$$\zeta(u) \leq 0 \qquad \text{for} \quad \vartheta > 1,$$

Which implies that h(u) = 0 while u belongs]0,1] which gives that g is essentially ϑ-Lipschitz,while taking $\vartheta > 1$ and $\mathcal{Y}$ is also geodesic space, so at last if $\vartheta \to 1^+$ then function g is essentially 1-Lipschitz. □

Corollary 3.2.2. *Assume triplet ($\mathcal{Y}$,$\mathcal{D}$,η) is locally strongly doubling space,also assume the function*

$$g : \mathcal{Y} \to \mathbb{R},$$

is measurable and **L** *is greater than zero such that,*

$$\int_{\mathcal{Y}\times\mathcal{Y}} \exp\left(\frac{|g(f)-g(e)|}{\mathbf{L}\mathcal{D}(f,e)}|\ln \mathcal{D}(f,e)|\right) d\eta(f)d\eta(e) < \infty,$$

then we say g be the essentially lipschitzian.

Proposition 3.2.3. *Assume the triplet ($\mathcal{Y}$,$\mathcal{D}$,η) is metric-measure space, and take n > 0, r > 0,also assume*

$$\eta(\mathcal{B}(e,R)) \leq D_r R^n, \tag{3.2.1}$$

∀ 0 < $R \leq r$ and $e \in \mathcal{Y}$, Where $D_r > 0$.
Assume that,

$$g : \mathcal{Y} \to \mathbb{R},$$

be 1-Lipschitz, then

$$\int_{\mathcal{B}(y,r)\times\mathcal{B}(y,r)} \exp\left(\frac{|g(f)-g(e)|}{\mathcal{D}(f,e)}|\ln \mathcal{D}^p(f,e)|\right) d\eta(f)d\eta(e) < \infty,$$

∀ 0 < p < n and $y \in \mathcal{Y}$.

Proof. Without loss of generality , suppose that r ≤ 1, let $R_l = r/2^l$.
∀ a ∈ $\mathcal{Y}$,then

$$\begin{aligned}
&\int_{\mathcal{B}(e,r)} \exp\left(\frac{|g(f)-g(e)|}{\mathcal{D}(f,e)}|\ln \mathcal{D}^p(f,e)|\right) d\eta(f) \\
&\leq \int_{\mathcal{B}(e,r)} \exp\left(\frac{|f-e|}{\mathcal{D}(f,e)}|\ln \mathcal{D}^p(f,e)|\right) d\eta(f) \qquad \text{(since g is 1-Lipschitz)} \\
&= \int_{\mathcal{B}(e,r)} \exp\left(\frac{\mathcal{D}(f,e)}{\mathcal{D}(f,e)}|\ln \mathcal{D}^p(f,e)|\right) d\eta(f) \\
&= \int_{\mathcal{B}(e,r)} \exp(|\ln \mathcal{D}^q(f,e)|) d\eta(f) \\
&\leq \int_{\mathcal{B}(e,r)} \mathcal{D}^{-q}(f,e) d\eta(b) \\
&\leq \sum_{l=0}^{\infty} \int_{\mathcal{B}(e,R_l)\setminus\mathcal{B}(e,R_{l+1})} \mathcal{D}^{-q}(f,e) d\eta(f) \\
&\leq \sum_{l=0}^{\infty} \int_{\mathcal{B}(e,R_l)\setminus\mathcal{B}(e,R_{l+1})} \frac{1}{R_j} d\eta(f)
\end{aligned}$$

$$\leq \sum_{l=0}^{\infty} \int_{\mathcal{B}(e,R_l)\backslash\mathcal{B}(e,R_{l+1})} \frac{1}{R_{l+1}^q} d\eta(f) \qquad \because R_l \geq R_{l+1}$$

$$= \sum_{l=0}^{\infty} \eta(\mathcal{B}(e,R_l)\backslash\mathcal{B}(e,R_{l+1})) \frac{1}{R_{l+1}^q}$$

$$= \sum_{l=0}^{\infty} \eta(\mathcal{B}(e,R_l)\backslash\mathcal{B}(e,R_{l+1})) \frac{2^{(l+1)q}}{r^q} \qquad \because R_l = r/2^j$$

$$\leq r^{-q} \sum_{l=0}^{\infty} \eta(\mathcal{B}(e,R_l)) 2^{(l+1)q}$$

$$\leq r^{-q} \sum_{l=0}^{\infty} D_R R_l^n 2^{(l+1)q} \qquad \text{By using Equation} \quad (3.2.1)$$

$$\leq r^{-q} \sum_{l=0}^{\infty} D_R \left(\frac{r}{2^l}\right)^n 2^{(l+1)q}$$

$$= 2^q r^{n-q} D_R \sum_{j=0}^{\infty} 2^{l(q-n)}.$$

Since for n $\succ$ q the series

$$\sum_{l=0}^{\infty} 2^{l(q-n)} = 1 + \frac{1}{2^{n-q}} + \frac{1}{2^{2(n-q)}} + ...$$

is geometric series which converges, hence

$$2^q r^{n-q} D_R \sum_{l=0}^{\infty} 2^{l(q-n)} < \infty,$$

finally

$$\int_{B(e,r)} \exp\left(\frac{|g(f)-g(e)|}{\mathcal{D}(f,e)} |\ln \mathcal{D}^p(f,e)|\right) d\eta(f) < \infty,$$

so it is the required result. □

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