ON LASKERIAN RINGS

By

Muhammad Saeed

Department of Mathematics
Quaid-i-Azam University, Islamabad
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Muhammad Saeed

Supervised

By

Dr. Tariq Shah

Department of Mathematics
Quaid-i-Azam University, Islamabad
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Preface

In 1905, Emanuel Lasker introduced the notion of the primary ideal, which corresponds to an irreducible variety and plays a role similar to prime powers in the prime decomposition of an integer. He proved the primary decomposition theorem for an ideal of a polynomial ring in terms of primary ideals [47]. The decomposition of an ideal into primary ideals is a conventional pillar of ideal theory. It provides the algebraic foundation for decomposing an algebraic variety into its irreducible components. Primary decomposition provides a generalization of the factorization of an integer as a product of prime powers. Developing on the findings of Lasker in 1921 Emmy Noether, in her seminal paper [61], proved that in a commutative ring satisfying the ascending chain conditions on ideals, every ideal is the intersection of finite number of irreducible ideals (an irreducible ideal of a Noetherian ring is a primary ideal). She established several intersection decompositions in rings. A ring which is characterized by the primary decomposition property of the ideals is called "a Laskerian ring" (in the honor of Emanuel Lasker). In the terminology of Bourbaki [11], a ring R is Laskerian if each ideal of R is a finite intersection of primary ideals and R is strongly Laskerian if each ideal of R is a finite intersection of strongly primary ideals.

Lasker's theory of primary decomposition of ideals has influence on the theory of Noetherian rings, which are by far the most important class of rings in commutative algebra. It is well known that:

Noetherian⇒Strongly Laskerian⇒Laskerian

A ring R satisfies *ACC* on principal ideals (*ACCP*) if there is no existence of an infinite strict ascending chain of principal ideals. Any Noetherian ring, in particular, any Dedekind domain satisfies *ACCP*.

For any type of abstract algebra, a generalization is a defined class of such algebra. Of course, a generalization of a concept is an extension of the concept to less specific criteria. Generalization plays a vital role in enhancing mathematical concepts and walking around the tracks which lead to achieving new goals.

Since the introduction of the concept of Noetherian rings, much progress has been made by many researchers in the development of this notion through generalization.

The subject of Laskerian rings continues to generate considerable interest. Laskerian rings are known to possess many properties that one finds in a Noetherian ring, regarding ideals. However, some rather desirable properties do not carry over; most notably Laskerian rings need not satisfy ascending chain condition on ideals. In [31], Heinzer and Ohm proved that for a commutative ring R, the ring $R[X]$ is Laskerian if and only if it is a ZD-ring if and only if R is Noetherian. In [25] Gilmer and Heinzer proved the equivalence: $R[[X]]$ is Laskerian if and only if R is Noetherian but they gave an example of a non-Noetherian ring such that *R[[X]]* is a ZD ring. They have also proved that a Laskerian ring has a Noetherian spectrum. A prime ideal *P* of a ring *R* is called MPD (minimal prime divisor) of the ideal *I*, if *P* is minimal among the prime ideals containing *I*. If *R* is Laskerian, then each ideal of *R* has only finitely many minimal prime divisors. A ring with latter property has Noetherian spectrum if and only if the ascending chain condition for prime ideals is satisfied in *R*. Barucci and Fontana have studied the transfer of Laskerian and strongly Laskerian properties in *(D+M)* type constructions in [6]. In continuation to [6], Hizem has studied Laskerian rings of the form $A + XB[X]$ and $A + XB[[X]]$, *where* $A \subseteq B$, in [35]. Recently, in [36] it is established that if *R* is strongly Laskerian, then *R[[X]]* is strongly Hopfian.

Other researchers who studied the properties of Laskerian rings include I. Armeano [3] 1977, N. Radu [65] 1980, H. A. Hussain [38] 1980, Heinzer and Ohm [31] 1972, Heinzer and Lantz [33] 1986, S. Visweswaran ([70], [71]) in 1989 and 2007.

In this dissertation, we initiate to create due space for Laskerian rings parallel to Noetherian rings. In doing this, in some cases, we generalize some results where Noetherian hypothesis is not necessary, and in other cases, we impose extra conditions on Laskerian rings and see their impact on the neighboring area. This activity enables us to develop a linkage of domains with focus on Laskerian domains. Moreover, taking inspiration from rapidly growing research on fuzzy concepts in various areas of ring theory and other algebraic structure, it is our intention/goal to contribute to Laskerian settings.

This study comprises two parts. The first part is further subdivided into three parts. In first subpart of Section One, we have discussed the overrings, integral closure and complete integral closure of an integral domain in Laskerian perspective.

It is well known that Noetherian domains are Laskerian. In Noetherian domains, every ascending chain of ideals stabilizes; in particular, Noetherian rings satisfy *ACCP*. The property of *ACCP* does not carry over to Laskerian ring, for example, a non discrete valuation ring is a Laskerian which does not satisfy *ACCP*.

Second part of Section One, mainly deals with a strongly Laskerian domain. We see that a strongly Laskerian domain *D* satisfies *ACCP* and every non zero ideal of *D* (strongly Laskerian domain) can be uniquely expressed as a product of primary ideals whose radicals are all distinct, if *D* is one dimensional. Furthermore, if each overring (respectively valuation overring) of an integral domain *D* is strongly Laskerian then *D* has a Dedekind (respectively an almost Dedekind) integral closure. Following [24], an ideal *A* of a domain *D* is a valuation ideal if there exists a valuation ring $D_v \supset D$ and an ideal A_v of D_v such that $A_v \cap D = A$.

In third part of Section One, we have characterized the Laskerian domains with the property that every primary ideal is a valuation ideal. We conclude first section by developing a chart connecting various domains with focus on Laskerian domain.

Second section of this study deals with fuzzy concepts. In 1965 L. A. Zadeh [73], proposed theory of fuzzy sets, which provides a useful mathematical tool for describing the behavior of the multifaceted or distracted systems to admit accurate mathematical analysis by classical technique. The study of fuzzy algebraic structure was started by Rosenfeld [66] and since then, this concept has been applied to various algebraic structures. Liu introduced the concept of a fuzzy ideal of a ring in [49]. The notions of prime fuzzy ideals, maximal fuzzy ideals, and primary fuzzy ideals were introduced in [52], [53] and [54]. Malik discussed Fuzzy ideals of Artinian rings in [51]. Mukerjee and Sen [56] studied rings with chain conditions with the help of fuzzy ideals. They considered Primary fuzzy ideals and radical of fuzzy ideals in [57]. The notion of fuzzy quotient ring was introduced by Kumar [45], Kuroaka and Kuroki [46]. In [48] K. H. Lee examined some properties of fuzzy quotient rings and used them to characterize Artinian and Noetherian rings.

In this study, we discuss strongly irreducible fuzzy ideals in Laskerian rings. Moreover, we show that: A ring R is Laskerian (respectively strongly Laskerian) if and only if R_μ (quotient ring of R by fuzzy ideal μ) is Laskerian (respectively strongly Laskerian) for every fuzzy ideal μ of R. Besides this, we extend the idea of anti-homomorphism of fuzzy ideals in rings, property proposed by Sheikabdullah and Jeyaraman in [67], to semiprime, strongly primary, irreducible and strongly irreducible fuzzy ideals of a ring. We also prove that: For a surjective ring anti-homomorphism $f: R \to R^{\prime}$, if every fuzzy ideal of R is f-invariant and has a fuzzy primary (respectively, strongly primary) decomposition in R, then every fuzzy ideal of $R^{\prime}$ has a fuzzy primary (respectively, strongly primary) decomposition in $R^{\prime}$.

This dissertation consists of six chapters.

In Chapter 1, a brief history of Laskerian rings is given and strongly Laskerian rings have been discussed. Moreover, some basics of fuzzy concepts have been provided. We have also included the fundamental information about these structures which are directly related to our study. Whereas in Chapter 2, we have established: For an integral domain D with quotient field K and $\overline{D}$ as its integral closure. (1) If $\overline{D}$ is a one dimensional Laskerian ring such that each primary ideal of $\overline{D}$ is a valuation ideal, then each overring of D is Archimedean. (2) If D is not a field, then D is a Dedekind domain if and only if D is a Laskerian almost Dedekind domain. (3) $\overline{D}$ is a one dimensional Laskerian and each primary ideal of $\overline{D}$ is a valuation ideal if and only if $\overline{D}$ is a one dimensional Prufer and $\overline{D}$ has finite character. In this case D is Laskerian. (4) $\overline{D}$ is a one dimensional Prufer (respectively almost Dedekind) if and only if every valuation ring of K lying over D is Laskerian (respectively strongly Laskerian). (5) The complete integral closure of a pseudo-valuation domain (D, M) is Laskerian of dimension at the most one.

In Chapter 3, we have explained that a strongly Laskerian domain D satisfies *ACCP*, every non zero ideal of D can be uniquely expressed as a product of primary ideals whose radicals are all distinct, if D is one dimensional, D is completely integrally closed if it is a *QR*-domain. Furthermore, if each overring (respectively valuation overring) of an integral domain D is strongly Laskerian then integral closure of D is a Dedekind domain (respectively an almost Dedekind domain).

In Chapter 4, we characterize Laskerian domains with the property that every primary ideal is a valuation ideal.

Following [21], $T(I)=U_{n\geq 1}\ (D: I^n)$. An integral domain D is said to satisfy trace formula for ideals if for all ideals I and J of D,

$$T(IJ)=T(I)+T(J)$$

It is well known that transform formula holds for finitely generated ideals in Prufer domains and all ideals in Dedekind domains. We extend the previous result by weakening the Noetherian hypothesis as: If D is a Laskerian domain in which every primary ideal is a valuation ideal (of course not Dedekind), then the transform formula holds for all ideals of D.

In Chapter 5, we have developed linkage among domains possessing chain condition on ideals, domains having factorization properties and domains having primary decomposition on ideals with focus on Laskerian domains. We have presented this linkage in the form of a table.

In Chapter 6, we move towards fuzzy concepts. We have divided our work into two sections. In Section 1, we have generalized some results which were established for Artinian and Noetherian rings in [48]. We have introduced strongly primary fuzzy ideals and strongly irreducible fuzzy ideals in a unitary commutative ring and fixed their role in a Laskerian ring. We established that: A finite intersection of prime fuzzy ideals (respectively primary fuzzy ideals, irreducible fuzzy ideals and strongly irreducible fuzzy ideals) is a prime fuzzy ideal (respectively primary fuzzy ideal, irreducible fuzzy ideal and strongly irreducible fuzzy ideal). We have also observed that a fuzzy ideal of a ring is a prime if and only if it is semiprime and strongly irreducible. Furthermore, we have shown that every nonzero fuzzy ideal of a one dimensional Laskerian domain can be uniquely expressed as a product of primary fuzzy ideals with distinct radicals. We have characterized Laskerian rings by fuzzy quotient rings. In particular, we prove that: a unitary commutative ring is (strongly) Laskerian if and only if its localization is (strongly) Laskerian with respect to every fuzzy ideal.

In section 2, we have investigated anti-homomorphic images and pre images of semiprime, strongly primary, irreducible and strongly irreducible fuzzy ideals of a ring. We have also proved that: for a surjective ring anti-homomorphism $f: R\rightarrow R^{/}$, if every fuzzy ideal of R is f-invariant and has a fuzzy primary (respectively strongly primary) decomposition in R, then every fuzzy ideal of $R^{/}$ has a fuzzy primary (respectively strongly primary) decomposition in $R^{/}$.

Research Profile

[1] A note on Laskerian rings (with T. Shah), Proceedings of Pakistan Academy of Sciences 48(1) (2011).
[2] A note on strongly Laskerian domains (with T. Shah), Advances in Algebra. 3(1) (2010), pp. 79–86.
[3] A non-Noetherian Laskerian domain in which every primary ideal is a valuation ideal (with T. Shah), International Electronic Journal of Pure and Applied Mathematics. 2(4) (2010), pp. 211-217.
[4] Factorization properties and chain conditions on ideals: A linkage (with T. Shah), to appear.
[5] Complete integral closure of a domain in an extension field and Laskerian valuation domains (with T. Shah), submitted.
[6] Fuzzy ideals in Laskerian rings (with T. Shah), submitted.
[7] On fuzzy ideals in rings and anti-homomorphism (with T. Shah), submitted.

Titles and Abstracts

[1] Title: A note on Laskerian rings

Abstract: Let $\overline{D}$ be an integral domain with quotient field *K* and $\overline{D}$ is its integral closure. (1) If $\overline{D}$ is a one dimensional Laskerian ring such that each primary ideal of $\overline{D}$ is a valuation ideal, then each overring of *D* is Archimedean. (2) If *D* is not a field, then *D* is a Dedekind domain if and only if *D* is a Laskerian almost Dedekind domain. (3) $\overline{D}$ is one dimensional Laskerian and each primary ideal of $\overline{D}$ is a valuation ideal if and only if $\overline{D}$ is one dimensional Prufer and $\overline{D}$ has finite character. In this case *D* is Laskerian. (4) $\overline{D}$ is one dimensional Prufer (respectively almost Dedekind) if and only if every valuation ring of *K* lying over *D* is Laskerian (respectively strongly Laskerian). (5) The complete integral closure of a pseudo-valuation domain *(D, M)* is Laskerian of dimension at most one.

[2] Title: A note on strongly Laskerian domains

Abstract: This study explains that a strongly Laskerian domain *D* satisfies *ACCP*, every non zero ideal of *D* can be uniquely expressed as a product of primary ideals whose radicals are all distinct if *D* is one dimensional, *D* is completely integrally closed if it is a QR-domain. Furthermore if each overring (respectively valuation overring) of an integral domain *D* is strongly Laskerian then integral closure of *D* is a Dedekind domain (respectively an almost Dedekind domain).

[3] Title: A non-Noetherian Laskerian domain in which every Primary ideal is a valuation ideal

Abstract: The purpose of this study is to characterize Laskerian domains with the property that every primary ideal is a valuation ideal.

[4] Title: Factorization properties and chain conditions on ideals: A linkage

Abstract: The purpose of this study is to find relationship among the various domains. In particular, the domains possessing factorization properties and the domains which hold different chain conditions on ideals.

[5] Title: Complete Integral Closure of a Domain in an Extension Field and Laskerian Valuation Domains

Abstract: Suppose D is an integral domain with quotient field K, and assume that the complete integral closure of D is an intersection of Laskerian valuation domains on K. If L is an extension field of K, then the complete integral closure of D in L is an intersection of Laskerian valuation domains on L.

[6] Title: Polynomial and formal power series extensions of a strongly Laskerian domain and chain conditions

Abstract: In this paper we study different chain conditions in polynomial and formal series extensions of a strongly Laskerian ring.

[7] Title: Fuzzy ideals in Laskerian rings

Abstract: The aim of this paper is to introduce strongly primary fuzzy ideals and strongly irreducible fuzzy ideals in rings. We examine finite intersection property of some fuzzy ideals. That is finite intersection of prime fuzzy ideals, primary fuzzy ideals, irreducible fuzzy ideals and strongly irreducible fuzzy ideal is prime fuzzy ideal, primary fuzzy ideal, irreducible fuzzy ideal and strongly irreducible fuzzy ideal respectively. We discussed strongly irreducible fuzzy ideals in Laskerian rings. Moreover we showed that: A ring R is Laskerian if and only if R_μ is Laskerian for every fuzzy ideal μ of R.

[8] Title: On fuzzy ideals in rings and anti-homomorphism

Abstract: We investigate anti- homomorphic images and pre images of semiprime, strongly primary, irreducible and strongly irreducible fuzzy ideals of a ring. We also prove that: For a surjective anti-homomorphism $f: R \to R'$, if every fuzzy ideal of R is f-invariant and has a fuzzy primary (respectively, strongly primary) decomposition in R, then every fuzzy ideal of R' has a fuzzy primary (respectively, strongly primary) decomposition in R'.

Notations

Mostly, symbols and notations used in this dissertation are standard; however some repeatedly notations and symbols are listed below

R represent a (commutative) ring with identity, it is also used for integral domains (we mentioned whenever feel necessary).

D is for integral domain

D^0 nonzero elements of D

$\overline{D}$ integral closure of D

D^*complete integral closure of D

F or K for a field

F^*or K^* a field without zero

I, J for an ideal of a ring

μ, θ and σ are used to represent fuzzy ideals

$A \subseteq B$ means B contains A or $A=B$

$\geq$ stands for greater than or equal to, it has also been present a partial ordering (we mentioned it in thesis)

D_v Represents valuation overring of D

A_v Denotes valuation ideal

$dim\ D$ represents Krull dimension of D

$dim_v\ D$ means D has valuative dimension n, defined as: if each valuation overring of D has dimension at most n and if there exists a valuation overring of D with dimension n.

$x|y$ is for x divides y.

$\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, represents the set of complex numbers, the set of real numbers, the set of rational numbers, the set of integers and the set of natural numbers respectively.

Contents

Chapter 1

BRIEF HISTORY AND PRELIMINARIES

Introduction

This chapter contains some fundamental concepts, basic definitions; examples and preliminaries which are related to our study. We have divided this chapter into two sections. In first section, we have provided definitions of some basic terms of commutative ring theory which will be frequently used throughout this tract. In second section of this chapter, we have given a brief history of fuzzy ideals in rings with some definitions.

1.1 Commutative Rings

We begin this section with the fundamentals of ring theory.

A ring is a set R together with two binary operations, addition and multiplication such that; $(R, +)$ is an abelian group, $(R, \cdot)$ is a semigroup and the multiplication is distributive over addition. An element say 1 is called identity element if $1 \cdot x = x = x \cdot 1$ for all $x \in R$. The identity element is also called unity and a ring with 1 is known as ring with unity or unitary ring. An element $x \in R$ is said to be invertible (or unit), whenever x posses a two sided inverse with respect to multiplication i.e., there exist $x^{-1} \in R$ such that $xx^{-1} = 1 = x^{-1}x$.

The set, $U(R)$ represents units of a ring R. A ring R is commutative if $xy = yx$ for all x, $y \in R$. A nonzero nonunit element p of a commutative ring R is said to be prime (respectively primary) if whenever p divides xy then p divides x or p divides y (respectively some power of y), for x, $y \in R$. Whereas a nonzero nonunit element q of a unitary commutative ring R is said to be irreducible (or non-factorable) if there exist any factorization $q = yz$ with y, $z \in R$, then either y is invertible or z is invertible. A nonzero element x of a unitary commutative ring R is said to be a zero divisor of R, if there exist $0 \neq y \in R$ such that $xy = 0$. A commutative ring with identity $1 \neq 0$ is said to be an integral domain if it has no zero divisors. A unitary commutative ring R is said to be a field if $U(R) = R - \{0\}$.

1.1.1 Subrings and Ideals

Let $(R, +, \cdot)$ be a ring and $S \subseteq R$ be a non empty subset of R. If the system $(S, +, \cdot)$ is itself a ring under the induced operation, then $(S, +, \cdot)$ is said to be a subring of $(R, +, \cdot)$. Ideals of rings (integral domains) play an important role in their characterization, so we give a bit introduction about prime ideal, primary ideal, principal ideal and their role in the determination of an integral domain. A subset I of a commutative ring R is said to be an ideal if for all $x, y \in I$ and $r \in R$; $x - y \in I$ and $rx \in I$. An ideal P of a unitary commutative ring R such that $P \neq R$, is a prime ideal of R if for all $x, y \in R$, $xy \in P$ then $x \in P$ or $y \in P$. In other words an ideal generated by a prime element is a prime ideal. The set of all prime ideals in a ring R is called the spectrum of R and is represented as $Spec(R)$. An ideal Q of a unitary commutative ring R is called primary if for all $x, y \in R$, $xy \in Q \implies x \in Q$ or $y^n \in Q$ for some positive integer n. In $\mathbb{Z}$ primary ideals are precisely of the form (p^e) where p is a prime integer and $e \in \mathbb{Z}^+$. See that a prime ideal is a primary ideal but the converse is not true. An ideal I of a unitary ring R is principal, if it is generated by a single element that is $I =< x >$ for some $x \in R$. An ideal I of a unitary commutative ring is said to be maximal if $I \neq R$ and for every ideal J such that $I \subseteq J \subseteq R$, either $J = R$ or $J = I$. In other words an ideal generated by irreducible element is a maximal ideal. The set of all maximal ideals of R is represented as $Max(R)$. Note that $Max(R) \subseteq Spec(R)$ If we take $(\mathbb{Z}, +, \cdot)$ a ring of integers, then the maximal ideals of $\mathbb{Z}$ correspond to the prime numbers. More precisely, the principal ideal (x), $x > 1$, is maximal if x is prime. For an ideal I of a unitary commutative ring R, radical of I is denoted by $\sqrt{I}$ or $Rad(I)$ and is defined as:

$Rad\ (I)= \{r \in R/\ r^n \in I$ for some $n \in \mathbb{Z}^+\}$. If D is an integral domain and K its field of fractions, we say that a D-submodule I of K is a fractional ideal of D if $I \neq 0$, and there exist a non-zero element $x \in D$ such that $xI \subset D$. Most famous result which classifies rings to integral domains and fields with prime and maximal ideal is: A proper ideal I of the ring R is a prime (respectively maximal) ideal if and only if the quotient ring R/I is an integral domain (respectively Field). Furthermore, in a unitary commutative ring R, every maximal ideal is a prime ideal.

1.1.2 Integral Dependence

Integral dependence is one of the very powerful and useful tools of multiplicative ideal theory. This concept was introduced by Emmy Noether in [62], and has been investigated over a period of many years. Here we give definitions of some basic terms.

Let S be a ring, R a subring of S (so that $1 \in R$). An element x of S is said to be integral over R if x is a root of a monic polynomial with coefficients in R, that is if x satisfies an equation of the form

$$\mathrm{x}^n + \mathrm{a}_1\mathrm{x}^{n-1} + \ldots + \mathrm{a}_n = 0$$

where a_i are elements of R. Every element of R is integral over itself.

Let the set of integral elements of S over R is denoted by R^*. R^* is called integral closure of R in S. If $R^* = R$ then R is integrally closed in S. If S is total quotient ring of R, R^* is usually referred as integral closure of R, rather than the integral closure of R in S.

The concept of almost integrality was introduced by Krull in his famous 1932 paper Allgemeine Bewertungs theorie. Following [12], if K is quotient field of an integral domain

D and $x \in K$, we say that x is "almost integral" over D provided there exists $d \in D$ such that $d \neq 0$ and $dx^n \in D$ for each positive integer n. The set of almost integral elements of K over D is called complete integral closure of D in K. If the set of almost integral elements of K over D is equal to D, then D is called "completely integrally closed" in K.

In chapter 2 and 3, we have discussed integral closures and complete integral closure of domains in Laskerian perspective.

1.1.3 Krull Dimension of a Ring

Following [24, 11.7, Page 105], If R is a ring (not necessarily containing an identity), a finite chain $P_1 \subset P_2 \subset ... \subset P_n \subset P_{n+1}$ of $n+1$ proper prime ideals of R will be said to have length n. Krull dimension of R is defined in terms of this concept.

Two cases arise, namely: There is a nonnegative integer n such that R contains a chain of proper prime ideals of length n but no such chain of length $n+1$, or no such integer exist. In the first case we say that R has dimension n, and we write dim$R = n$. In the second case, we say that R is infinite dimensional. A field has dimension 0; $\mathbb{Z}$ has dimension 1; more generally a PID which is not a field has dimension 1. For a field K, the polynomial ring $K[X_1, ..., X_n]$ and the power series ring $K[[X_1, ..., X_n]]$ are $n-$dimensional. An important property of integral dependence is that it preserves dimension.

1.1.4 Valuation Rings

Following [39, Page 12], an integral domain D with quotient field K, is said to be a valuation domain if it satisfies either of the (equivalent) conditions:

(i) For any two elements $x, y \in D$, either x divides y or y divides x.

(ii) For any element $x \in K$, either $x \in D$ or $x^{-1} \in D$.

A valuation ring is said to be a discrete valuation ring (DVR) if its value group is isomorphic to $\mathbb{Z}$. A Noetherian valuation ring is a DVR.

Throughout this study all rings are commutative with identity (unless mentioned otherwise). The letter D denotes an integral domain with quotient field K. By an overring of D

we mean a ring between D and K. We use $\overline{D}$ to denote integral closure of D in K, $\dim D$ to represent Krull dimension of D. By [24, Page 360], D is said to have valuative dimension n, represented as $\dim_v D = n$, if each valuation overring of D has dimension at most n and if there exists a valuation overring of D with dimension n.

1.1.5 Noetherian Rings

A Noetherian ring, named after the famous mathematician Emmy Noether, is a ring in which every ascending chain of ideals has a maximal element. The class of Noetherian rings has important role in commutative ring theory and algebraic geometry because Noetherian property is the ring-theoretic analogue of finiteness in some sense. They are useful in the specification of the Krull rings (Noetherian and integrally closed).

Let R be a unitary commutative ring. Then R is Noetherian if and only if every prime ideal of R is finitely generated. Following are the equivalent conditions for a ring R to be Noetherian:

(i) Every non-empty set of ideals in R has maximal element.

(ii) Every ascending chain of ideals in R is stationary.

(iii) Every ideal in R is finitely generated. The well known examples of Noetherian rings are fields, PIDs, polynomial extension of field over finite number of indeterminates etc.

According to Hilbert basis theorem, the polynomial extension of a Noetherian ring is again Noetherian ring. Consequently $R[X_1, X_2, ..., X_n]$ is Noetherian whenever R is Noetherian. A Noetherian ring with only one maximal ideal (finite number of maximal ideals) is known as a local (semilocal) ring. However non-Noetherian unitary commutative local ring is called quasi-local (quasi-semilocal) if it has only one maximal ideal (finite number of maximal ideals).

In chapters, 2, 3, 5 and 6, we have generalized some results, already existed with Noetherian hypothesis, for Laskerian rings/domains.

Primary decomposition

One of the most valuable characteristics of a Noetherian ring R is that, in it, every ideal has a finite primary decomposition. Which means that every ideal in R can be written as a finite intersection of primary ideals.

In $\mathbb{Z}$, we can write primary decomposition of an ideal (n) as:

$(n) = \cap_{i=1}^{m}(p_i^{e_i})$ where $\pm 1 \neq n \in \mathbb{Z}$ and $(p_i^{e_i})$ are primary ideals in $\mathbb{Z}$ generated by prime powers.

1.1.6 Laskerian Rings

Emanuel Lasker was a German mathematician. Except chess genius, he is known for his contributions to commutative algebra. In 1905, Lasker initiated the concept of primary ideal, which match ups to an irreducible variety and plays a role analogous to prime powers in the prime decomposition of an integer. He established the primary decomposition theorem for an ideal of a polynomial ring in terms of primary ideals [47]. On this foundation Emmy Noether presented the influential paper in 1921 [61]. In this paper she established an abstract theory which developed ring theory into major mathematical topic and provided the foundation of modern algebraic geometry. She proved that in a commutative ring fulfilling the ascending chain conditions on ideals, every ideal is the intersection of finite number of irreducible ideals (an irreducible ideal is a primary ideal). She established a number of intersection decompositions.

Rings which are characterized by the property of primary decomposition of ideals are called Laskerian rings (in the honour of Emanuel Lasker). A commutative ring R with identity is Laskerian if each ideal of R admits a shortest primary representation; R is strongly Laskerian, if R is Laskerian and each primary ideal of R contains a power of its radical [24, Page 455]. It is equivalent to say that a commutative ring R with identity is Laskerian (respectively strongly Laskerian) if every ideal of R can be represented as finite intersection of primary ideals (respectively strongly primary ideals). Whereas an ideal Q of R is primary if each zero divisor of the ring R/Q is nilpotent and Q is strongly primary if Q is primary and satisfies

$(\sqrt{Q})^n \subset Q$. Following [20, Page 505], R is a zero divisor ring (ZD ring), if $Z_R(R/I)$, the set of zero divisors of R/I, is a finite union of prime ideals for all ideals I of R.

In general,

$$\text{Artenian} \Longrightarrow \text{Noetherian} \Longrightarrow \text{Strongly Las}\ker\text{ian} \Longrightarrow \text{Las}\ker\text{ian} \Longrightarrow \text{ZD ring}$$

but none of the above implication is reversible.

An integral domain D is said to be a Prüfer domain if D_P is a valuation ring for every prime ideal P of D. By [24, Page 434], a domain D is an almost Dedekind domain if D_M is a Noetherian valuation ring for each maximal ideal M of D and its dimension is at most one. If D is a Prüfer domain, then D is Laskerian if and only if dim $D \leq 1$ and each nonzero element of D belongs to only a finitely many maximal ideals of D (see [24, Page 456]). In chapters, 2, 3 and 4, we continued to develop on Laskerian Prüfer domains.

Some Examples of Laskerian Rings

Example 1 (a) *A finite ring is Laskerian.*

(b) *A ring R is said to have finite quotients if for any ideal $I \neq 0$ in R, R/I is finite.*

A ring with finite quotients is Laskerian.

Of course examples (a) and (b) are Noetherian.

Easy examples of non-Noetherian Laskerian are hard to find. However we present some examples of non-Noetherian Laskerian rings and domains from [10], [22] and [35].

The following examples anticipate most of the results in the first part of the thesis.

Example 2 *[6, Example 1] Let X be an indeterminate over a field K. Consider $R = K[X]/(X^2) = K[\in]$, where $\in = X + (X^2)$, and define the canonical surjection $\varphi : R \to K$, mapping $\in$ to 0. For every subring D of K, we take the subring $A = D + \in K[\in]$ of $K[\in]$. Keeping the field $K = \mathbb{C}$ fixed and changing the subring D in the ring A, we get different classes of rings. If $D = \mathbb{Q}$, then A is a zero dimensional non Noetherian strongly Laskerian* *ring. However if $D = \mathbb{R}$, then A is a zero dimensional Noetherian ring. In case if $D = \mathbb{Z}$, then A is a one dimensional non-Laskerian ring. In addition to this, if $D = \mathbb{Z}_{(p)}$, then A is a one dimensional non-Laskerian local ring. However if the field $K = K(Y)$ and subring $D = K$, where K is a field and Y is an indeterminate over K, then A is a zero dimensional non-Noetherian ring which is strongly Laskerian. Furthermore this ring is integrally closed in its total ring of fractions (that is R).*

Example 3 *[6, Example 2] Let D be a subring of a field K and X be an indeterminate over K. Take $R = K[X]_{(X)}$ and define the canonical homomorphism $\varphi : R \to K$, mapping X to 0. Now consider the ring $A = D + XK[X]_{(X)}$. At first fix the field $K = \mathbb{C}$ and make changes in the subring D and see the corresponding effect on R. If the subring $D = \mathbb{Q}$ (the field of rational numbers), then the ring A is a one dimensional non-Noetherian, strongly Laskerian local integral domain. Furthermore if $D = \mathbb{R}$ (the field of real numbers), then A is a one dimensional Noetherian local integral domain. However, if $D = \mathbb{Z}$ (the ring of integers), then A a two dimensional Prüfer domain (of course integrally closed) which is not Laskerian. If we change the field in A as $K = \mathbb{Q}$ and $D = \mathbb{Z}$, then in this case A is a two dimensional non-Laskerian Prüfer domain. In case of $D = \mathbb{Z}_{(p)}$, the ring A is a two dimensional valuation ring which is not Laskerian. If we take $K = K(Y)$ where K is a field and Y an indeterminate over K and $D = K$, then A is a 1-dimensional integrally closed non-Noetherian, strongly Laskerian PVD. W. Krull, has used this example to prove that a one dimensional local integrally closed domain is not a valuation ring.*

Example 4 *[22] (Non-Noetherian Laskerian domain). Let K be a field and X, Y are indeterminates over K. Take R as the set equivalence classes of elements of the form $f(X,Y)/g(X,Y)$ where $f, g \in K[X,Y], X$ does not divide g (in $K[X,Y]$) and $f(0,Y)/g(0,Y) \in K$. Under usual addition and multiplication of rational functions R becomes a ring. It can be shown that R is a non-Noetherian commutative ring but every ideal of R has a primary decomposition.*

Example 5 *[35, Proposition 5.7] Let $A \subset B$ be an extension of integral domains. Then* *the ring $A+XB[X]$ (respectively $A+XB[[X]]$) is of Krull dimension 1 if and only if $A \subset B$ is an extension of fields .In this case $A+XB[X]$ and $A+XB[[X]]$ are Laskerian.*

Proposition 6 *Let R be a Laskerian ring.*

(1) *If I is any ideal of R, then quotient ring R/I is Laskerian.*

(2) *If $S \subset R$ is any multiplicative subset, then localization $S^{-1}R$ is Laskerian.*

Proof. Let I, J be ideals of a ring R. Then any ideal of R/I is of the form J/I. Since R is Laskerian J has finite primary decomposition, therefore a fortiori J/I has finite primary decomposition,so R/I is Laskerian. A very similar argument holds for the localization. ■

For definitions not given in the thesis the reader may refer to [24], [11] and [4].

1.2 Fuzzy Concepts in Rings

1.2.1 A Brief History

In 1965, L. A. Zadeh [73], proposed theory of fuzzy sets, which provides a useful mathematical tool for describing the behavior of the multifaceted or distracted systems to admit accurate mathematical analysis by classical technique. The study of fuzzy algebraic structure has started by Rosenfeld [66] and since then this concept has been applied to various algebraic structures. Liu introduced the concept of a fuzzy ideal of a ring in [49]. The notions of prime fuzzy ideals, maximal fuzzy ideals, primary fuzzy ideals were introduced in [52], [53] and [54]. Also, Malik [51], Mukererjee and Sen [56] studied rings with chain conditions with the help of fuzzy ideals. The notion of fuzzy quotient ring was introduced by Kumar [45], Kuroaka and Kuroki [46]. In [48] K. H. Lee examined some properties of fuzzy quotient rings and used them to characterize Artinian and Noetherian rings.

1.2.2 Some Fundamental Concepts

Following [73] a fuzzy subset of a non-empty set X is a function $\mu : X \longrightarrow [0,1]$. By [72], the set of all fuzzy subsets of a set X with the relation $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$, $\forall x \in X$ is a complete lattice. Whereas for a non empty family $\{\mu_i : i \in I\}$ of fuzzy subsets of X,

$$\begin{aligned} sup\{\mu_i &: i \in I\} : X \to [0,1], x \mapsto sup\{\mu_i(x) : i \in I\}, \forall x \in X \text{ and} \\ inf\{\mu_i &: i \in I\} : X \to [0,1], x \mapsto inf\{\mu_i(x) : i \in I\} \end{aligned}$$

A fuzzy subset μ of a ring R is a fuzzy left (respectively right) ideal of R if for every x, $y \in R$;

$\mu(x-y) \geq$ min $\{\mu(x), \mu(y)\}$ and $\mu(xy) \geq \mu(y)$ (respectively $\mu(xy) \geq \mu(x)$).

If μ is a fuzzy subset of R, then for $t \in [0,1]$ the set $\mu_t = \{x \in R : \mu(x) \geq t\}$ is called a level subset of R with respect to μ. A fuzzy subset is a fuzzy left ideal if and only if $\mu(0) \geq \mu(x)$ $\forall x \in R$ and μ_t is a left ideal of R, $\forall t \in [0, \mu(0)]$. We denote $\mu_* = \{x \in R : \mu(x) = \mu(0)\}$. A fuzzy subset μ of R is a fuzzy ideal of R if it is a left and right fuzzy ideal. Following [67], a fuzzy ideal μ of a ring R is called a prime fuzzy ideal if for any two fuzzy ideals σ and θ of R the condition $\sigma\theta \subseteq \mu$ implies that $\sigma \subseteq \mu$ or $\theta \subseteq \mu$.

According to [53], [54], for a fuzzy ideal μ of a ring R, the fuzzy radical of μ, denoted by $\sqrt{\mu}$, is defined by $\sqrt{\mu} = \cap\{\sigma : \sigma$ is a fuzzy prime ideal of $R, \sigma \subseteq \mu, \sigma_* \subseteq \mu_*\}$. In [58], V. Murali and B. Makaba has discussed concepts of primary decomposition of fuzzy ideals and the radicals of such ideals over a Noetherian ring.

The operations of intersection, union,sums and products on fuzzy subsets μ and ν of R are defined as follows:

$$\begin{aligned} (\mu \cap \nu)(x) &= (\mu \Lambda \nu)(x) = \min(\mu(x) + \nu(x)) \\ (\mu \cup \nu)(x) &= (\mu \vee \nu)(x) = \max(\mu(x) + \nu(x)) \\ (\mu + \nu)(x) &= \left\{ \begin{array}{cc} sup_{x=r+s}\{\min\{\mu(r), \nu(s)\} & \text{if } x \text{ is expressed as } x = r+s \in R \\ 0 & \text{otherwise} \end{array} \right\} \\ (\mu o \nu)(x) &= \left\{ \begin{array}{cc} sup_{x=rs}\{\min\ (\mu(r), \nu(s))\} & \text{if } x \text{ can be expressed as } x = rs,\ r, s \in R \\ 0 & \text{otherwise} \end{array} \right\} \\ (\mu\nu)(x) &= \left\{ \begin{array}{c} sup\{\Lambda_{i=1}^n(\mu(r_i)\Lambda\nu(s_i)) : x = \sum_{i=1}^n r_i s_i, r_i, s_i \in R, n \in \mathbb{N}\}. \\ 0 \end{array} \right\} \end{aligned}$$

Let μ be a fuzzy ideal of a ring R and $x \in R$. By [48], the fuzzy subset of R defined as $\mu_x^*(r) = \{\mu(r - x)$ for all $r \in R\}$ is termed as the fuzzy coset determined by x and μ. The set of all cosets of μ in R is a ring under the binary operations $\mu_x^* + \mu_y^* = \mu_{x+y}^*$and $\mu_x^* \mu_y^* = \mu_{xy}^*$ for all x, $y \in R$ and it is denoted by R_μ, and known as fuzzy quotient ring of R induced by the fuzzy ideal μ.

In [48] K. H. Lee has examined some properties of fuzzy quotient rings and utilized them to characterize Artinian and Noetherian rings. On his foundation, we have characterized Laskerian and strongly Laskerian rings in chapter 6.

Other important definitions and related results will be recalled in due course of time.

Chapter 2

INTEGRAL DEPENDENCE AND LASKERIAN RINGS

Introduction

In this chapter we have discussed integral closure of a domain, complete integral closure of PVD and overrings of a domain in Laskerian perspective. We have proved the necessary and sufficient condition for an almost Dedekind domain to be a Dedekind domain through Laskerian property. We have divided this chapter in three sections. In first section we have proved that for an integral domain D with quotient field K and $\overline{D}$ as its integral closure: (1) If $\overline{D}$ is a one dimensional Laskerian ring such that each primary ideal of $\overline{D}$ is a valuation ideal, then each overring of D is Archimedean. (2) If D is not a field, then D is a Dedekind domain if and only if D is Laskerian almost Dedekind domain. (3) $\overline{D}$ is a one dimensional Laskerian and each primary ideal of $\overline{D}$ is a valuation ideal if and only if $\overline{D}$ is a one dimensional Prüfer and $\overline{D}$ has finite character. In this case D is a Laskerian. In second section we proved that: $\overline{D}$ is a one dimensional Prüfer (respectively almost Dedekind) if and only if every valuation ring of K lying over D is Laskerian (respectively strongly Laskerian). Third section is focused on complete integral closure of a pseudo-valuation domain.

2.1 The Case of Laskerian Integral Closure

In this section, we transformed $(\Rightarrow)$ of : "Integral closure $\overline{D}$ of an integral domain D is a Dedekind domain $\Longleftrightarrow$ every overring of D satisfies $ACCP$ (or atomic) [19, Theorem 1]", as: "If integral closure $\overline{D}$ of an integral domain D is a one dimensional Laskerian ring such that each primary ideal of $\overline{D}$ is a valuation ideal, then each overring of D is Archimedean". We have proved that: $\overline{D}$ is a one dimensional Laskerian and each primary ideal of $\overline{D}$ is a valuation ideal if and only if $\overline{D}$ is a one dimensional Prüfer and $\overline{D}$ has finite character. In this case D is a Laskerian".

Following [68], an integral domain D is Archimedean in case $\cap_{n\geq 1} Dr^n = 0$ for each nonunit $r \in D$. The most natural examples of Archimedean domains are arbitrary completely integrally closed domains, arbitrary one dimensional integral domains and arbitrary Noetherian integral domains. An ideal I of a domain D is a valuation ideal if there exist a valuation ring $D_v \supset D$ and an ideal I_v of D_v such that $I_v \cap D = I$.

Lemma 7 *Let D be an integral domain. If D is Laskerian such that every primary ideal is a valuation ideal then D is Prüfer.*

Proof. Since D is Laskerian, it is clear that each ideal of D has finitely many minimal prime divisors. Since a ring with later property has Noetherian spectrum if and only if ascending chain condition for prime ideals is satisfied in D. Therefore by [23, Theorem 3.8], D is a Prüfer domain. ■

Proposition 8 *Let D be an integral domain such that its integral closure $\overline{D}$ is a one dimensional Laskerian ring and each primary ideal of $\overline{D}$ is a valuation ideal, then each overring of D is Archimedean.*

Proof. If integral closure $\overline{D}$ of an integral domain D is one dimensional Laskerian ring, such that each primary ideal of $\overline{D}$ is a valuation ideal, then by [24, Theorems 36.2 and Theorem 30.8], $dim_v D = dim_v \overline{D} = dim \overline{D} = dim D \leq 1$. So by [63, Corollary 1.4], each overring of D is Archimedean. ■

Remark 9 *Each overring of an integral domain D is Noetherian if and only if D is Noetherian and dim$D \leq 1$ [24, Page 493, Exercise 16]. This cannot be generalized in Laskerian domains. Because, if D is a non Noetherian Laskerian integral domain such as a one dimensional valuation ring (the case when Archimedean and Laskerian behave alike), then overrings of D do not satisfy $ACCP$ [9, Theorem 2.1].*

It is easy to demonstrate that a Dedekind domain is a Laskerian domain. We prove in the next proposition that an almost Dedekind domain which is not Dedekind is not a Laskerian domain. This then demonstrates a clear difference between Dedekind domains and non Noetherian almost Dedekind domain. Here the concept of Laskerian domain is expanded to provide a way of measuring how close an almost Dedekind domain which is not Dedekind is to being Dedekind

Proposition 10 *In an integral domain D with identity which is not a field, the following conditions are equivalent:*

(1) D is a Dedekind domain.

(2) D is Laskerian almost Dedekind domain.

Proof. (1) $\Longrightarrow$ (2): As D is a Dedekind domain$\Rightarrow$ D is Noetherian. If M is a maximal ideal of D, then D_M is a nontrivial Noetherian valuation ring. Therefore by [24, Theorem 17.5] D_M is of rank one discrete and D is almost Dedekind.

(2) $\Longrightarrow$ (1): If D is an almost Dedekind then dim(D)$= 1$. In a one dimensional integral domain, the Laskerian property in D is equivalent to the condition that each principal ideal of D is decomposable, which in turn is equivalent to the condition that each non zero element of D belongs to only finitely many maximal ideals of D. Hence by [24, Theorem 37.2] D is Dedekind. ■

Mori and Nagata have proved that if D is Noetherian and one or two dimensional, then $\overline{D}$ is Noetherian [60]. Hence if D is Noetherian and one dimensional, then $\overline{D}$ is a Dedekind domain. On the other hand the integral closure $\overline{D}$ of D is Dedekind, it is not necessary that D is Noetherian. For example if $R = \mathbb{Q} + X\overline{\mathbb{Q}}[X] = \left\{a_0 + \sum a_i X^i \mid a_0 \in \mathbb{Q},\ a_i \in \overline{\mathbb{Q}}\right\}$, where $\mathbb{Q}$ is the field of rationals and $\overline{\mathbb{Q}}$ is algebraic closure of $\mathbb{Q}$, then $\overline{R} = \overline{\mathbb{Q}}[X]$ is a PID but R is not Noetherian. In [12, Lemma 2] it is proved that if integral closure $\overline{D}$ of an integral domain D is a Dedekind domain then D is Laskerian. In the next proposition we observe that D remains Laskerian if integral closure of D is one dimensional Prüfer with finite character.

Proposition 11 *Let $\overline{D}$ be integral closure of an integral domain D. $\overline{D}$ is a one dimensional Laskerian and each primary ideal of $\overline{D}$ is a valuation ideal $\Longleftrightarrow$ $\overline{D}$ is a one dimensional Prüfer and $\overline{D}$ has finite character. In this case D is a Laskerian.*

Proof. ($\Longrightarrow$) By Lemma 7, $\overline{D}$ is Prüfer. Since $\overline{D}$ is Laskerian (by [25, Theorem 4]) it has Noetherian spectrum which means that every non zero element of $\overline{D}$ belongs to finite number of maximal ideals of $\overline{D}$, that is, $\overline{D}$ has finite character.

($\Longleftarrow$) Since $\overline{D}$ is one dimensional Prüfer and has finite character, therefore by [24, Page 456, Exercise 9], $\overline{D}$ is Laskerian and every primary ideal of $\overline{D}$ is a valuation ideal [23]. Next we show that D is Laskerian. By Proposition 8, dim(D)≤ 1. Proper prime ideals are maximal in D, then every ideal of D is equal to its kernel [44]. Since every proper ideal in D has a finite number of prime divisors, therefore every ideal in D is an intersection of a finite number of pairwise comaximal primary ideals in D. ■

Remark 12 *If $\overline{D}$ is a one dimensional Laskerian and each primary ideal of $\overline{D}$ is a valuation ideal, then, D is Laskerian.*

2.2 Laskerian Valuation Overrings and Integral Closure

With the inspiration; "a valuation ring V is a Laskerian ring (respectively a strongly Laskerian ring) if and only if V has rank at most one (respectively V is discrete of rank at most one) (see [24, Page 456])", we established that every valuation ring of K lying over D is Laskerian (respectively strongly Laskerian) if and only if $\overline{D}$ is one dimensional Prüfer domain (respectively an almost Dedekind domain).

Lemma 13 *Let V be a non trivial valuation ring on the field K, then V is completely integrally closed $\iff$ V is one dimensional $\iff$ V is Laskerian $\iff$ V is Archimedean.*

Proof. A valuation ring V is completely integrally closed if and only if V is one dimensional (cf. [24, Theorem 17.5]). By [24, Page 456], a valuation ring V is Laskerian if and only if V has rank at most one. Since all valuation rings are GCD domains. By (cf. [9, Theorem 3.1]) V is Archimedean if and only if V is completely integrally closed. ∎

Remark 14 *A rank 2 valuation ring is not Laskerian (see [11, Page 170, Exercise 19]) and*

$$A = \mathbb{Z}_{(p)} + X\mathbb{Q}[X]_{(X)}$$

Here A is a 2-dimensional valuation ring which is non-Laskerian ring (see Example 3). However any valuation ring is $Z.D$ $(Zero\ divisor)$ ring (see [20, Page 507]).

Theorem 15 *Let D be an integral domain with quotient field K. Then the integral closure $\overline{D}$ of D is a one dimensional Prüfer domain $\iff$ every valuation ring of K lying over D is Laskerian.*

Proof. Let $\overline{D}$ be a one dimensional Prüfer domain and let D_v is any valuation overring of D. So $\overline{D} \subset D_v \subset K$. Let P_v be centre of D_v in $\overline{D}$. Since $\overline{D}$ is a one dimensional Prüfer domain, therefore $\overline{D}_{P_v}$ is a rank one valuation ring (and hence maximal ring in K) and hence $\overline{D}_{P_v} = D_v$. Since every valuation ring lying over D is rank one, therefore it is Laskerian.

Conversely, suppose that every valuation ring of K lying over D is Laskerian, therefore by Lemma 13, it is a one dimensional. Hence by [12, Theorem 1] $\overline{D}$ is a Prüfer domain. ∎

Theorem 16 *Every valuation ring of K lying over an integral domain D is strongly Laskerian $\iff$ $\overline{D}$ is an almost Dedekind domain.*

Proof. Suppose every valuation ring of K lying over D is strongly Laskerian, therefore it is discrete and rank one (see [24, Page 456]). By [24, Theorem 36.2], $\overline{D}$ is an almost Dedekind domain.

Conversely, suppose $\overline{D}$ is an almost Dedekind domain (hence one dimensional) and let D_v be a valuation overring of D (ring between D and K). Then $\overline{D} \subset D_v \subset K$. If P is the centre of D_v in $\overline{D}$, then $\overline{D}_P \subset D_v$; since $\overline{D}_P$ is discrete and rank one, it follows that $\overline{D}_P = D_v$. Therefore every valuation ring of K lying over D is strongly Laskerian (see [24, Page 456]). ■

Remark 17 *Theorems 15 and 16 respectively generalize [19, Theorem 1] as follows: Let D be an integral domain. Then the integral closure of D is a Dedekind domain (respectively Prüfer 1-dimensional domain, respectively an almost Dedekind domain) if and only if every overring of D satisfies $ACCP$ (respectively every valuation overring of D is Laskerian, respectively every valuation overring of D is strongly Laskerian).*

2.3 Complete Integral Closure of a PVD

In this section, we have observed that the complete integral closure D^* of a PVD (pseudo-valuation domain) (D, M) (i.e. in D every prime ideal is strongly prime) is Laskerian of dimension ≤ 1.

By [30], an integral domain D with quotient field K, is said to be a pseudo-valuation domain (PVD), if whenever P is a prime ideal in D and $xy \in P$, where $x, y \in K$, then $x \in P$ or $y \in P$ (i.e. in a PVD every prime ideal is strongly prime). Equivalently an integral domain D with quotient field K, is said to be a PVD if for every nonzero $x \in K$, either $x \in D$ or $ax^{-1} \in D$ for every nonunit $a \in D$. It is well known that a PVD is a local ring. A valuation domain is a PVD but converse is not true, for example the non integrally closed domain $\mathbb{R} + X\mathbb{C}[[X]]$ is a PVD, which is not a valuation domain.

Theorem 18 *Let (D, M) be a PVD. The complete integral closure D^* of D is a quasilocal Laskerian ring of dim≤ 1.*

Proof. If (D, M) is a PVD with quotient field K, then there is a valuation overring V of D such that $spec(D) = spec(V)$, and hence D^* is integral closure of V. Thus (i) $D^* = V_p$

if V has a height-one prime ideal P or (ii) $D^* = K$ if V does not have a height one prime ideal. In both cases D^* is Laskerian ring. ■

In the following we restate theorem [27, Theorem 4] by replacing Archimedean valuation domain with Laskerian valuation domain.

Theorem 19 *Suppose D is an integral domain with quotient field K and let L be an extension field of K. If the complete integral closure of D is an intersection of Laskerian valuation domains on K, then the complete integral closure of D in L is an intersection of Laskerian valuation domains on L.*

Proof. By Lemma 13, a valuation ring is Laskerian if and only if it is Archimedean. The rest follows from [27, Theorem 4]. ■

Chapter 3

ON STRONGLY LASKERIAN DOMAINS

Introduction

It is well known that Noetherian rings satisfy $ACCP$. From chapter 2, Lemma 13, one can see that a one dimensional valuation ring is Laskerian. And one dimensional non discrete valuation rings don't satisfy $ACCP$. This information leads us to conclude that Laskerian rings don't satisfy $ACCP$. However we observed that a strongly Laskerian domain (class of domains lying between Noetherian and Laskerian domains) D satisfies $ACCP$. We have proved that, like a Noetherian domain, every non zero ideal of a strongly Laskerian domain D can be uniquely expressed as a product of primary ideals whose radicals are all distinct if D is a one dimensional. In general a strongly Laskerian domain D is not necessarily completely integrally closed, however we have proved that if we consider D as a QR-domain, then D is completely integrally closed. Furthermore if each overring (respectively valuation overring) of an integral domain D is strongly Laskerian then integral closure of D is a Dedekind domain (respectively an almost Dedekind domain).

Following [68], an integral domain D is Archimedean in case $\cap_{n\geq 1} Dr^n = 0$ for each non unit $r \in D$. The most natural examples of Archimedean domains are completely integrally closed domains, one dimensional integral domains and Noetherian integral domains.

3.1 Strongly Laskerian Domain and $ACCP$

We start this section with a well known lemma.

Lemma 20 *Let D is a strongly Laskerian domain and I be an ideal of a ring D. Then for any sequence (x_k) of elements of D the increasing sequence of ideals $I : (x_1) \subseteq I : (x_2) \subseteq I : (x_3) \subseteq ...$ stabilizes.*

Proposition 21 *A strongly Laskerian domain D satisfies $ACCP$.*

Proof. Since D is strongly Laskerian domain therefore every ideal of D can be written as finite intersection of strongly primary ideals. If I be an ideal of D then $I = \cap_{i=1}^{n} P_i$ where each P_i is strongly primary $i.e$ $(\sqrt{P_i})^n \subset P_i$. Suppose D does not have ACC on principal ideals and there exist non terminating ascending chain $(x_1) \subset (x_2) \subset (x_3) \subset$ ...of principal ideals of D. There exist non units $y_i \in D$ such that $x_i = y_i x_{i+1}$ for $i = 1, 2, 3, ...$thus $x_1 = y_1...y_n x_{n+1}$ for $n = 1, 2, 3, ...$ Since D is strongly Laskerian, then by Lemma 20 increasing chain $(x_1) : y_1 \subset (x_1) : y_1 y_2 \subset (x_1) : y_1 y_2 y_3 \subset$ ....terminates. Therefore $(x_1) : y_1 y_2 ... y_m = (x_1) : y_1 y_2 ... y_n$ for $m \geq n$. $\Rightarrow x_1 = y_1...y_n x_{n+1} = y_1...y_n y_{n+1} x_{n+2}$. Thus $x_{n+2} \in (x_1) : y_1.y_2....y_{n+1} = (x_1) : y_1.y_2....y_n$. Hence we obtain $x_{n+2} y_1.y_2...y_n \in (x_1)$. Thus $x_1 = y.y...y_{n+1} x_{n+2} = y_{n+1}(y_1.y_2...y_n x_{n+2}) = y_{n+1} d x_1$, for some $d \in D$. $\Rightarrow 1 = y_{n+1} d$. Which is a contradiction because y_{n+1} is not a unit in D. Thus D has ACC on principal ideals. ■

Corollary 22 *A strongly Laskerian domain D is Archimedean.*

Proof. Assume that D is not Archimedean. Then there exist nonzero x, $y \in D$ and x is nonunit such that $y \in \cap_{n=1}^{\infty} x^n D$. For each $n = 1, 2, 3, ...$ let d_n be the unique element of D such that $y = x^n d_n$. See that $d_n = x d_{n+1}$ for each n, therefore $(d_1) \subseteq (d_2) \subseteq (d_3) \subseteq$ is an ascending chain of principal ideals of D. Furthermore, since x is nonunit of D,this chain is strictly ascending and no d_n is zero. This shows that D does not have $ACCP$. But by Proposition 21, D has $ACCP$, a contradiction. Therefore D is Archimedean. ■

Combination of Proposition 21, and [9, Theorem 2.1] allows us to write:

Proposition 23 *Let D be an integral domain.*

(1) D is strongly Laskerian.

(2) D has ACC on principal ideals.

(3) If $x, y \in D^o$ (monoid of non zero elements of domain D) are such that x, y^2, x^3, y^4...,divide next and if $x \mid y$ or $y \mid x$, then a and b are associate in D.

(4) D is Archimedean.

Then $(1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4)$.

Proof. $(1) \Rightarrow (2)$ Follow from Proposition 21.

$(2) \Rightarrow (3) \Longleftrightarrow (4)$ Follows from [9, Theorem 2.1]. ■

Remark 24 *(i) $(2) \nRightarrow (1)$ In [8, Proposition 1.1] it has been shown that the domain $R = A + XB[X]$, where B is a domain, X is an indeterminate over B, A is a subring of B that is a field, and B is not integral over A ,satisfies $ACCP$. But by [35, Proposition 5.4 and 5.18], R is not Laskerian.*

(ii) $(4) \nRightarrow (1)$ or (2) One dimensional non discrete valuation domain is Archimedean (Laskerian). It is neither strongly Laskerian nor satisfies $ACCP$.

(iii) $D = \mathbb{Q} + X\mathbb{R}[X]$ is non- Noetherian strongly Laskerian domain, but satisfies $ACCP$, and hence is atomic (i.e. each non zero non unit is a product of finite number of irreducible elements (atoms)).

Corollary 25 *[9, Lemma 4.2] Let D be a strongly Laskerian domain (by Proposition 21, D satisfies $ACCP$) and S be a saturated multiplicative subset of D generated by set of prime elements of D and $U(D)$ (unit elements of D) Then*

(1) Each $a \in D^0$(monoid of nonzero elements of D) may be expressed as $a = a_0 s_0,\ Where\ s_0 \in S$ and $a_0 \in D$ is relatively prime to S (in the sense that S contains no nonunit factor of a_0).

(2) If $a_0 s_0 = a_1 s_1$ with $s_0,\ s_1 \in S$ and $0 \neq a_0,\ a_1$ relatively prime to S, then $Da_0 = Da_1$(and so $Ds_0 = Ds_1$).

(3) If $a_0 \in D^0$ is relatively prime to S, then $D_{sa_0} \cap D = D_{a_0}$

Following [2] an integral domain D is strongly atomic if for each a, $b \in D^0$ (non zero elements of D), we can write $a = a_1....a_s c$ and $b = a_1....a_s d$ where $a_1, ..., a_s \in D$ $(s \geq 0)$ are irreducible and $c, d \in D$ satisfy gcd(c, d)= 1. D is a weak GCD domain if for each $a, b \in D^0$, there are $c, a', b' \in D$ so that $a = ca'$ and $b = cb'$, where gcd(a', b')= 1

Remark 26 *D strongly Laskerian $\Longrightarrow$ D satisfies ACCP $\Longrightarrow$ D is atomic $\Longrightarrow$ D is semi rigid.*

$\Rightarrow$ strongly Laskerian GCD domain is a generalization of Krull domains [74, Theorem 5].

Remark 27 *A strongly Laskerian GCD domain is UFD. This remark generalizes [69, Corollary 3.4].*

Corollary 28 *If D is strongly Laskerian domain then D is an atomic weak GCD-domain.*

Proof. By proposition 23, D is strongly Laskerian domain $\Longrightarrow$ D satisfies $ACCP$ $\Longrightarrow$ $D[X]$ satisfies $ACCP$ $\Longrightarrow$ $D[X]$ is atomic $\Longrightarrow$ D is strongly atomic $\Longrightarrow$ D is a weak GCD domain [2, Theorem 1.3].

Corollary 29 *If D is strongly Laskerian domain then $D[X]$ satisfies $ACCP$.*

■

We have the following implications for a commutative ring:

$$\begin{array}{ccccccc}
\text{Noetherian} & \Longrightarrow & \text{strongly Laskerian} & \Longrightarrow & \text{Laskerian} & \Longrightarrow & ZD \text{ ring} \\
 & & \Downarrow & & \Downarrow & & \\
\text{strongly atomic} & \Longleftarrow & ACCP & & ACCR & \Longrightarrow & \text{strongly Hopfian}
\end{array}$$

In fact all the above implications are well known except strongly Laskerian$\Longrightarrow$ $ACCP$. For the non irreversibility of above implications we have provided examples in chapter five along with more implications.

3.1.1 Summary of polynomial extensions

The following table summarizes findings of Hilbert, [36], [37], [2, Theorem 1.3] and Corollary 29 for polynomial extensions of some generalizations of Noetherian domains.

Integral domain D		Polynomial extension $D[X]$
Noetherian	$\Longleftrightarrow$	Noetherian
Noetherian	$\Longleftarrow$	$\left\{\begin{array}{c}\text{Strongly Laskerian}\\ \text{Laskerian}\\ \text{ZD}\\ \text{Has ACCR}\end{array}\right\}$
Strongly Laskerian	$\Longrightarrow$	Satisfies $ACCP$
Strongly Laskerian	$\Longrightarrow$	Strongly Hopfian [36, Proposition 1.12]
Laskerian	$\Longrightarrow$	Has Noetherian spectrum
Has Noetherian spectrum	$\Longleftrightarrow$	Has Noetherian spectrum
Strongly Hopfian	$\Longleftrightarrow$	Strongly Hopfian [37]
Strongly atomic	$\Longleftarrow$	Atomic [2, Theorem 1.3]

Remark 30 *From the above table we can see that if D is a strongly Laskerian domain then $D[X]$ is strongly Hopfian with $ACCP$ (atomic strongly Hopfian). Actually this remark sharpens [36, Proposition 1.12] in one way.*

Lemma 31 *[18, Remark 1.1] Let D be an integral domain then D satisfies $ACCP$ if and only if for any sequence (a_n) of non invertible elements of D, $\cap_{n\geq 1} a_1 ... a_n D = (0)$.*

Lemma 32 *For a strongly Laskerian domain D, any sequence (a_n) of non invertible elements of D, $\cap_{n\geq 1} a_1 ... a_n D = (0)$.*

Corollary 33 *If I is a proper ideal of a strongly Laskerian domain D then by Proposition 21 and [35, Proposition 4.6]:*

(1) $D +(X_1, ..., X_n)I[X_1, ..., X_n]$ satisfies $ACCP$.

(2) $D +(X_1, ..., X_n)I[[X_1, ..., X_n]]$ satisfies $ACCP$.

Recall that an atomic integral domain D is a half factorial domain (HFD) if for any irreducible elements $a_1, ..., a_n$, $b_1, ..., b_m$ of D with $a_1...a_n = b_1...b_m$, then $m = n$. Clearly a UFD (unique factorization domain) is also an HFD. However the converse is not true.

Remark 34 *Let $K \subseteq B$, where K is a field and B a domain, then $K + XB[X]$ is*

(i) Laskerian HFD if B is integrally closed and Noetherian. It follows from [35, Proposition 5.18] and [13, Theorem 2.1].

(ii) Laskerian HFD if B is a Krull domain. Furthermore if $Cl(B) \simeq Z_2$, then B is an HFD.

Lemma 35 *(2nd uniqueness theorem) "Let I be a decomposable ideal, let $I = \cap_{i=1}^{n} Q_i$ be a minimal primary decomposition of I, and let $\{P_{1_1}, ..., p_{i_m}\}$ be an isolated set of prime ideals of I. Then $Q_{i_1} \cap ... \cap Q_{i_m}$ is independent of decomposition". In particular "the isolated primary components (i.e., the primary components Q_i corresponding to minimal prime ideals P_i) are uniquely determined by I.*

Lemma 36 *If I_m and I_n are coprime ideals ($I_m + I_n = (1)$) whenever $m \neq n$, then $\Pi I_m = \cap I_n$.*

The following Proposition states that in [4, Proposition 9.1] there is no need to assume that the domain D is Noetherian. Here in the hypothesis, one can consider domain to be a Laskerian.

Proposition 37 *Let D be a Laskerian domain of dimension 1. Then every non zero ideal I of D can be uniquely expressed as a product of primary ideals whose radicals are all distinct.*

Proof. Let I be a non zero ideal of a Laskerian domain D. Then, $I = Q_1 \cap Q_2 \cap ... \cap Q_n$, where each Q_i is P_i primary ideal of D. The ideals $P_{1,}..., P_n$ are maximal since they stem from reduced primary decomposition, they are pairwise different and therefore pairwise coprime. Therefore $Q_1, ..., Q_n$ are pairwise coprime and by Lemma 36 we have $\Pi Q_i = \cap Q_i$. Hence $I = \Pi Q_i$.

Conversely, if $I = \Pi Q_i$, the same argument shows that $I = \cap Q_i$; this is a minimal primary decomposition of I, in which each Q_i is isolated primary component, and by Lemma 35 is therefore unique. ■

3.2 A Strongly Laskerian Domain and CIC

Definition 38 *A domain D with the property that each overring of D is a quotient ring of D is said to have the QR-property. Any QR-domain is a Prüfer domain and, hence, integrally closed.*

Remark 39 *[71] If D is a Noetherian domain, then it is well known that D is integrally closed if and only if D is completely integrally closed. However there are examples of integrally closed strongly Laskerian domains which are not completely integrally closed. The domain D in [71, Example 1.5] is one such example.*

Proposition 40 *A Strongly Laskerian QR-domain is completely integrally closed.*

Proof. Let D be a strongly Laskerian domain. By Proposition 23, D is Archimedean. Since D has QR-property, therefore by [16, Corollary 2.4]. D is completely integrally closed. ■

Following [9], a commutative integral domain D satisfies $(*)$ in case, whenever nonzero elements a and b in D are such that each element in the sequence $a, b^2, a^3, b^4, \ldots$divides the next, then a and b are associates in D (that $a = bu$ for some unit u in D)

Corollary 41 *A Strongly Laskerian QR-domain satisfies $(*)$.*

Proof. Combine Proposition 40 with [9, Theorem 2.1]. ■

Proposition 42 *Let D be Laskerian domain with identity which is not a field. Then D is a Dedekind domain if one of the following equivalent conditions hold in D;*

1) D is almost Dedekind.

2) D is one dimensional and primary ideals of D are prime powers.

3) The cancellation law for ideals hold in D.

4) D is a one dimensional Prüfer domain and D contains no idempotent maximal ideal.

Proof. (1) If D is an almost Dedekind then by Proposition 10 D is Dedekind.

(2) Let D be a Laskerian integral domain and I be an ideal of D. Then $I = \cap_{i=1}^{n} Q_i$,where each Q_i is say P_i primary. Since dim$D = 1$ each non zero prime ideal of D is maximal, hence P_i are distinct maximal ideals (since $P_i \supseteq Q_i \supseteq I \neq 0$), and are therefore pairwise comaximal. Hence by [4, Proposition 1.16] the Q_i are pairwise comaximal . But then by Chinese Remainder Theorem $I = Q_1 \cap ... \cap Q_n = Q_1 ... Q_n$. By assumption every primary ideal of D is a prime power; hence there are natural numbers e_i with $Q_i = P_i^{e_i}$. This yields $I = P^{e_1}.P^{e_2}...P^{e_n}$.

Equivalence of (1), (2), (3) and (4) follow from [24, Theorem 36.4 and 36.5]. ■

3.3 Integral Closure and Complete Integral Closure

In proving the next results we were frequently using techniques that resembled those encountered in dealing with Archimedean domains.

Proposition 43 *Let D be strongly Laskerian domain then,*

(1) $U(T) \cap D = U(D)$ for each overring T of D which is contained in D^*.

(2) $U(D^*) \cap D = U(D)$.

Proof. (1) Let $x \in U(T) \cap D$. As $T \subset D^*$, x^{-1} is almost integral over D; it means, there exist $0 \neq d \in D$ such that $d(x^{-1})^n \in D$ for each $n \geq 0$. Since $d \in \cap Dx^n$ (By Proposition 23 D is Archimedean this implies $x \in U(D)$. Other inclusion is trivial.

(2) Direct application of (1). ■

Following [15], an integral domain D is conducive if, for each overring T of D other than K, the conductor $(D : T)$= $\{a \in K : aT \subset D\}$ is non zero. Familiar examples of conducive integral domains are arbitrary valuations and arbitrary $D + M$ constructions. A valuation domain is divided and conducive. In [15, Corollary 2.7] it is proved that a Noetherian conducive integral domain must be local and of dimension at most 1. We generalize this result in the next proposition by replacing Noetherian hypothesis by weaker, strongly Laskerian, structure.

Proposition 44 *A strongly Laskerian conducive domain is quasilocal with dimension at most* 1.

Proof. Assume D is strongly Laskerian. It is sufficient to prove that if P is a non zero prime ideal of D, then $D\backslash P = U(D)$. Since D is conducive, it shares a non zero ideal with some conducive overring [5, Theorem 4.1], $(D : D_P) \neq 0$. Thus $D_P \subset D^*$ (cf. [24, Lemma 26.5]). Now Proposition 43 yields $U(D_P) \cap D = U(D)$. Since $U(D_P) \cap D = D\backslash P$, this completes the proof. ■

Proposition 45 *Each non trivial overring of conducive strongly Laskerian domain D is almost integral over D.*

Proof. By Proposition 44, D is quasilocal domain of dimension at most 1 therefore D is a divided domain. By [17, Corollary 2.6], each non trivial overring of D is almost integral over D. ■

Lemma 46 *The Integral closure $\overline{D}$ of an integral domain D is Prüfer domain if each proper valuation overring of D is strongly Laskerian. In fact, (at least) one of the following conditions holds:*

(a) D is a Laskerian valuation domain.

(b) D is a valuation domain, dim$D = 2$, and D_P is a DVR,where P denotes height 1 prime ideal of D.

(c) $\overline{D}$ is an almost Dedekind.

Proof. By [24, Page 456, Exercise 7], a valuation domain is strongly Laskerian if and only if it is a one dimensional discrete valuation ring. So, if D is not a valuation domain, [24, Theorem 26.1] $\overline{D}$ is an almost Dedekind domain. On the other hand if D is a valuation domain then by [24, Theorem 26.1], proper overrings of D are the rings D_P, P ranging over the non maximal primes of D, then either (a) or (b). ■

By [34] a simple overring of a domain D, we mean a ring of the form $D[\frac{a}{b}]$, where $a, b \in D$ such that $b \neq 0$.

Proposition 47 *Let D be an integral domain which is not integrally closed. If each proper simple overring of D is strongly Laskerian, then integral closure and complete integral closure of D coincide.*

Proof. Indeed $\overline{D} \subseteq D^*$. Conversely if $D^* \nsubseteq \overline{D}$ then there exist $x \in D^* \backslash \overline{D}$, and set $T = D[x^{-1}]$. We claim that T is a proper overring of D. Otherwise, select $y \in \overline{D} \backslash D$, set $S = D[y]$. Since S is strongly Laskerian, by Proposition 43 $x^{-1} \in U(S^*) \cap S = U(S)$, therefore $x \in \overline{D}$, a contradiction. ■

We conclude the chapter by generalizing the converse of Krull - Akizuki theorem (cf [43, Page 64, Exercise 20]) in the following way.

Theorem 48 *A domain whose each overring is strongly Laskerian has Dedekind integral closure.*

Proof. Since each overring of D is strongly Laskerian. By Proposition 23, overrings of D satisfy ACCP. Therefore by [19, Theorem 1], $\overline{D}$ is a Dedekind domain. ■

Remark 49 *If each overring of an integral domain D is strongly Laskerian then D is Laskerian and dim$D \leq 1$. Indeed it follows by Theorem 48 and Proposition 11.*

Chapter 4

A NON-NOETHERIAN LASKERIAN DOMAIN

Introduction

The purpose of this chapter is to characterize the Laskerian domain with the property that every primary ideal is a valuation ideal. An integral domain D is said to satisfy the ascending chain condition for prime ideals provided any strictly ascending chain of prime ideals $P_1 \subset P_2 \subset$ ...stabilizes. This is equivalent to say that every non empty family of prime ideals contains a maximal element. If spec(D) is Noetherian, then radical ideals of D have ACC. Here we observe that a Laskerian domain with the property that each primary ideal is a valuation ideal forms a well behaved sub class of Prüfer domains.

Recall that an ideal I of a domain D is a valuation ideal if there exists a valuation ring $D_v \supset D$ and an ideal I_v of D_v such that $I_v \cap D = I$. When we want to specify the particular valuation ring D_v, we shall say I is a v-ideal. If I is a v- ideal, then $ID_v \cap D = I$. By [24, Page 304, Exercise 6] if D is a Prüfer domain then each primary ideal of D is a valuation ideal. Following [24, Page 305, Exercise 7] if D is Noetherian and each primary ideal of D is a valuation ideal, then D is a Prüfer domain.

4.1 Laskerian Prüfer Domain

Lemma 50 *Let D be a Laskerian integral domain. Then the following are equivalent:*

(1) Every primary ideal is a valuation ideal.

(2) D is Prüfer and every proper prime ideal is maximal.

Proof. $(2) \Longrightarrow (1)$: Holds even without hypothesis on D to be Laskerian [24, Page 304, Exercise 6].

$(1) \Longrightarrow (2)$: Follows from Lemma 7. ■

Definition 51 *If D is an integral domain with identity, D is an almost Dedekind if D_M is a Noetherian valuation ring for each maximal ideal M of D. An almost Dedekind domain has dimension at most 1, and is of course, a Prüfer domain.*

Corollary 52 *Let D be a Laskerian integral domain. If every primary ideal of D is a valuation ideal then every valuation ideal is primary.*

Proof. By Lemma50 dim$D \leq 1$. By [24, Page 305, Exercise 9], every valuation ideal of D is a primary ideal. ■

Corollary 53 *Let D be a Laskerian domain. If every primary ideal of D is a valuation ideal, then prime ideals of D are linearly ordered.*

Proof. Since D is Laskerian. D has Noetherian spectrum. D satisfies ACC for prime ideals. Therefore, by [23, Theorem 3.4] the prime ideals of D are linearly ordered. ■

Proposition 54 *Let D be a Laskerian domain such that each primary ideal of D is a valuation ideal then following are equivalent:*

(1) D is almost Dedekind.

(2) Primary ideals of D are prime powers.

Proof. $(1) \Longrightarrow (2)$: Trivial.

$(2) \Longrightarrow (1)$: By Lemma 50, nonzero proper prime ideals of D are maximal. Apply [23, Theorem 1]. ■

Corollary 55 *Let D be a Laskerian domain, and suppose every primary ideal of D is a valuation ideal. Then D is a valuation ring.*

Proof. By Lemma 50, D is a Prüfer domain with one maximal ideal M and hence $D = D_M$. ■

Lemma 56 *Let D be a Laskerian domain. If every primary ideal is a valuation ideal. Then D is completely integrally closed.*

Proof. By Lemma 50, D is Prüfer of dimension at most 1.Therefore by [24, Page 333, Exercise 23], D is completely integrally closed. ■

Following [21, Page 1], a fractional ideal I of an integral domain D is a $D-$submodule of K where there exist $0 \neq d \in D$ such that $dI \subseteq D$. A fractional ideal I of D is a divisorial or $v-$ideal of D in case $I = I_v$; and I is an invertible ideal of D provided $II^{-1} = D$.

Proposition 57 *Let D be a Laskerian domain, and suppose every primary ideal of D is a valuation ideal, then each non zero ideal of D is divisorial if and only if D is a Dedekind domain.*

Proof. Suppose that each nonzero ideal of D is divisorial. By Lemma 50, D is 1-dimensional Prüfer, consequently D is CIC. Therefore by [24, Theorem 34.3], divisorial fractional ideals of D form a group under the operation $I * J = (IJ)_v$ with D as an identity element. For every ideal I of D there exist a fractional ideal J of D such that $(IJ)_v = D$. But $IJ = (IJ)_v$, so I is invertible. Hence D is Dedekind.

Conversely, in a Dedekind domain each nonzero ideal is invertible therefore divisorial. ■

Remark 58 *Combining Propositions 10 and 57, see that for a Laskerian domain D, with the property that every primary ideal of D is a valuation ideal, each non zero ideal of D is divisorial even if D is an almost Dedekind domain.*

It is well known that overrings of Prüfer (respectively almost Dedekind, respectively Dedekind) domains are Prüfer (respectively almost Dedekind, respectively Dedekind). If D is a one-dimensional Noetherian Prüfer domain (Dedekind domain) then each overring of D is 1-dimensional Noetherian Prüfer domain (Dedekind domain). We have observed that if D is a Laskerian Prüfer domain then each overring of D is a one-dimensional Laskerian integrally closed (Laskerian Prüfer).

Proposition 59 *Let D be Laskerian integral domain. If every primary ideal of D is a valuation ideal then each overring of D is 1-dimensional Laskerian integrally closed.*

Proof. By Lemma 50 D is a 1-dimensional Prüfer domain. Therefore each overring of D is 1-dimensional integrally closed Prüfer. Furthermore by Corollary 55 D is a valuation ring. This ensures that each overring of D is a valuation ring. Therefore by [24, Exercise 9 , Page 456], overrings of D are Laskerian. ∎

Remark 60 *Since D is a one dimensional Prüfer domain, D is an almost Dedekind if and only if each non trivial valuation overring of D is strongly Laskerian.*

Theorem 61 *Assume every primary ideal of a domain D is a valuation ideal, then following are equivalent:*

(1) D is Laskerian.

(2) D is strongly Laskerian.

(3) D is Noetherian.

Proof. $(1) \Rightarrow (2)$: By Corollary 55 D is a valuation ring. To prove D is strongly Laskerian it is sufficient to show that every primary ideal P of D is strongly primary. Let $a, b \in K^*$where K^* is quotient field of D and $ab \in P$ further suppose that $a \notin P$. Since D is a valuation domain, if $a \notin D$ then $a^{-1} \in D$ and we have $b = a^{-1}ab \in P$. Hence we may as well assume that $a \in D$. Since $a = b^{-1}ab \notin P$, it follows that $b \in D$. Now since $a, b \in D$ with P primary, we have $b^n \in P$ for some $n \geq 1$, hence P is strongly primary. Hence D is a strongly Laskerian domain.

$(2) \Rightarrow (3)$: By [24, Exercise 8, Page 456], in case of valuation ring strongly Laskerian and Noetherian properties coincide. Since D is a valuation ring, it is obvious.

$(3) \Rightarrow (2) \Rightarrow (1)$ trivial. ■

Following [21, Page 33] for an integral domain D with quotient field K and for nonzero ideal I we define $(I : I) = \{x \in K : xI \subseteq I\}$ and $I^{-1} = (D : I) = \{x \in K : xI \subseteq D\}$.

Proposition 62 *Let D be a Laskerian domain in which every primary ideal is a valuation ideal, and let I be a non zero ideal of D. Assume I^{-1} is a ring. The following conditions are equivalent:*

(1) $I^{-1} = (I : I)$;

(2) $I =$ Rad(I).

The minimal prime ideals of I in $(I : I)$ are all maximal ideals.

Proof. By Lemma 50 D is Laskerian Prüfer domain. The rest follows from [21, Theorem 3.1.12, Page 42]. ■

Proposition 63 *Let D be a Laskerian domain such that every primary ideal is a valuation ideal. Let I be a non zero ideal of D. If I is a primary ideal that is not prime, then I^{-1} is not a ring.*

Proof. On contrary suppose I^{-1} is a ring. By 50 and [21, Lemma 3.1.13] $I^{-1} = (I : I)$, by [21, Theorem 3.1.12] $I =$Rad(I). which is a contradiction to the supposition that every primary ideal is prime. Hence I^{-1} is not a ring. ■

4.1.1 The Transform Formula for Ideals

Following [21, Page 33], $T(A) = \cup_{n\geq 1}(D : A^n)$. An integral domain D is said to satisfy trace formula for ideals if for all ideals A and B

$$T(AB) = T(A) + T(B)$$

It is well known that transform formula for finitely generated ideals hold in Prüfer domains (see [24, Page 331, Exercise 10]) and holds for all ideals in Dedekind domains. We extend the previous result by weakening the Noetherian hypothesis as:

If D is a Laskerian domain in which every primary ideal is a valuation ideal (of course not Dedekind), then the transform formula holds for all ideals of D.

Before proving this we provide alternate proofs of the following well known lemmas.

Lemma 64 *Let D be Laskerian domain. Then each prime ideal of D is the radical of a finitely generated ideal.*

Proof. Assume that P is a non zero prime ideal of a domain D. Since D is a Laskerian domain, D has ACC on prime ideals. This implies, there exists a prime ideal $Q \subset P$ such that each prime ideal of D properly contained in P is contained in Q. Take an element $p \in P \backslash Q$ and note that P is minimal over (p). Being Laskerian (p) has only finitely many primes. Now by [21, Lemma 4.2.29], P is the radical of a finitely generated ideal. ■

Lemma 65 *Let D be a Laskerian commutative ring, then radical of each ideal of D is equal to radical of some finitely generated ideal.*

Proof. On contrary assume there exist an ideal that is not radical of a finitely generated ideal of D. Take Γ as a set of all such ideals in D, then Γ is inductive. Further suppose that P is a maximal ideal in Γ. We aim to prove that P is a prime ideal of D.

If P is not prime ideal of D, then there exist elements $x, y \in D \backslash P$ such that $xy \in P$. Take $I = (P, x)$ and $J = (P, y)$ and see that $I,\ J \notin \Gamma$. As Rad(IJ) =Rad(I)Rad(J) note that $IJ \notin \Gamma$. On the other hand Rad(IJ) =Rad(P), therefore $P \notin \Gamma$. This contradiction shows that P is a prime ideal. But by Lemma 64 each prime ideal is radical of some finitely generated ideal. This completes the proof. ■

For the sake of definiteness we state and prove the following theorem.

Theorem 66 *If D is a Laskerian domain in which every primary ideal is a valuation ideal, then the transform formula holds for all ideals of D.*

Proof. Let A be an ideal of D. Since D is Laskerian, D has Noetherian spectrum. Therefore every ideal of D has finitely many minimal prime divisors, and hence $T(A) =$

$T(\mathsf{Rad}(A))$. Also by Lemma 65 for ideals A and B of D, there exist finitely generated ideals A_0 and B_0 such that $\mathsf{Rad}(A) = \mathsf{Rad}(A_0)$ and $\mathsf{Rad}(B) = \mathsf{Rad}(B_0)$. ■

$$
\begin{aligned}
T(AB) &= T(\mathrm{Rad}(AB)) = T(\mathrm{Rad}(A) \cap \mathrm{Rad}(B)) \text{ [4, Exercise 1.13 (iii)]} \\
&= T(\mathrm{Rad}(A_0) \cap \mathrm{Rad}(B_0)) = T(\mathrm{Rad}(A_0 B_0)) \\
&= T(A_0 B_0) = T(A_0) + T(B_0) \text{ [lemma 50 and ([21, Theorem 4.5.4])]} \\
&= T(A) + T(B)
\end{aligned}
$$

The proof is complete.

Chapter 5

FACTORIZATION PROPERTIES AND CHAIN CONDITIONS ON IDEALS: A LINKAGE

Introduction

The purpose of this study is to find relationship among the various domains. In particular, the domains possessing factorization properties and the domains which hold different chain conditions on ideals, some of which have been discussed in earlier chapters 2,3 and 4.

Techniques used in literature to explore different properties in (unitary) commutative rings (respectively integral domains) include factorization properties among elements (e.g., UFD, FFD, HFD, BFD, Atomic domains, etc.), decomposition of ideals (e.g., Noetherian domains, (Strongly) Laskerian domains) and chain conditions on ideals (e.g., $ACCI$, Domains satisfying $ACCP$).

Chain condition on ideals in (unitary) commutative rings plays an important role in commutative algebra. Initially there was two classes of such rings, i.e., Noetherian rings and Artinian rings which are satisfying ACC (ascending chain condition) and DCC (descending chain condition) for ideals, respectively. It is known that Artinian rings are Noetherian. Some of other domains satisfying chain condition on ideals include, domains satisfying ACC on principal ideals, Mori domains (ACC on divisorial ideals), Laskerian domains (ACC on prime ideals), strongly Laskerian domains (ACC on principal ideals) and strongly Hopfian domains (ACC on ann(a)$\subseteq$ ann(a^2)$\subseteq$ ...).

5.1 Discovering links

The motivating factor behind this study is the work of D. D. Anderson, D. F. Anderson and M. Zafrullah [2], Kabbaj and Mimouni [40, Figure 1] and Bourbaki [11]. We have combined the work of [2], [11] and [40] . We have added a few more implications.

An integral domain has ACC on principal ideals if every ascending chain of principal ideals stabilizes. It is well known that if an integral domain has ACC on principals then it is atomic. It is not necessary for an atomic domain to satisfy $ACCP$. For instance if F is a field and T is an additive submonoid of $\mathbb{Q}^+$ generated by $\{1/3, 1/(2.5), \ldots, 1/(2^j p_j), \ldots\}$, where $p_0 = 3$ and $p_1 = 5$,... is the sequence of odd primes. Let R be the monoid domain $F[X;T]$ and $N = \{f \in R$ such that f has nonzero constant term$\}$. Then $A = F[X;T]_N$ is an atomic domain which does not satisfy ACC on principal ideals (see [28]).

By [76], an integral domain D is a Mori domain if it satisfies ACC on v-ideals. Obviously a Noetherian domain is a Mori domain. A Mori domain has the ascending chain condition on principal ideals; they are Archimedean (cf. [7, Page 353]).

A (commutative) ring R is Noetherian if every ideal of R is finitely generated or any ascending chain of ideals in R is stationary, i.e. R has $ACCI$ (ascending chain condition on ideals). In the terminology of Bourbaki [11, Ch. 4, Pages 295 and 298] a ring R is Laskerian if each ideal of R can be written as a finite intersection of primary ideals. In [61], Emmy Noether has proved that every ideal in a Noetherian ring is a finite intersection of primary ideals. In fact she has proved that every Noetherian ring is Laskerian. However a ring R is strongly Laskerian if each ideal of R can be written as a finite intersection of strongly primary ideals. It is well known that:

Noetherian $\Longrightarrow$ strongly Laskerian $\Longrightarrow$ Laskerian

The non irreversibility of the above implications can be seen from Examples (2 ,3, and 4).

Let K be a field and let Y, Z be indeterminate over K. Let $V = K(Z)[Y])(Y) = K+M$, where M is the unique maximal ideal of V. Let $D = K + M$. Then D is non Noetherian strongly Laskerian (see [75, Example 1.5]).

Also a non discrete one dimensional valuation ring is a Laskerian ring but not strongly Laskerian. Notice that this ring does not have ACC on principal ideals.

Following [20], a ring R is a zero divisor ring, ZD-ring, if $Z_R(R/I)$ is a finite union of prime ideals for all ideals I of R. Further in [20, Proposition 7], Evans has shown that Laskerian rings are ZD-rings. The converse is false, (see [11, Page 170, Exercise 19]), which shows that a rank 2 valuation ring is not a Laskerian. Any valuation ring is ZD ring as; any union of prime ideals is again a prime ideal. Recall [50], that a ring R satisfies the ascending chain condition on residuals ($ACCR$) when for each ideal I of R and each element $b \in R$ the chain $I : b \subseteq I : b^2 \subseteq ...$ stabilizes. Following [36], a commutative ring R is said to be strongly Hopfian if the chain of annihilators ann$(a) \subseteq$ ann$(a^2) \subseteq \dots$ stabilizes for each $a \in R$. Laskerian satisfy $ACCR$ and $ACCR$ imply strongly Hopfian, but the converse is not true (see [36, Remark 1.17]).

A (commutative) ring R with unity is called an N-ring if for each ideal I of R there exists a Noetherian ring T containing R as a unitary subring with $IT \cap R = I$ (See [26], [32] and [33]). N-rings are Laskerian [26, Proposition 2.14]. By [1] R is said to be a Q -ring, if every ideal is a finite product of primary ideals. By [1, Theorem 13], R is a Q -ring if and only if R is a Laskerian ring in which every non-maximal prime ideal is quasi-principal. The ring R is I_1 if, given any proper ideal H and any prime P, for some f not in P. The saturations of H through $R\backslash P$ is the same as the conductor ideal that drives f into H and a ring R is I_2 if, given any descending chain of multiplicatively closed sets, and an ideal J, the saturations of J become constant. I_1 and I_2 are generalizations of Laskerian rings. By [24, Page 456, Exercise 7], a valuation ring is Laskerian (respectively strongly Laskerian) if and only if it is 1-dimensional (respectivelyDVR of dimension 1). Following [24, Page 79] an integral domain D is a Bezout if every finitely generated ideal of D is principal. By [24, Theorem 17.1] every valuation ring is a Bezout domain. It is well known that a Bezout domain is a GCD domain but a GCD domain is Bezout if and only if it is a Prüfer domain. Any Bezout domain is QR, in the sense that each of its overrings is localization. Indeed, a Bezout domain may be characterized as a GCD domain which is QR. An integral domain D is said to be a Prüfer domain if D_M is a valuation ring for each maximal ideal M of D and a domain D with the property that each overring of D is a quotient ring of D is said to have the QR -property. A domain which has the QR-property is necessarily Prüfer. (A Prüfer domain need not be a Bezout domain. $\mathbb{Z}[\sqrt(-5)]$ is a Noetherian Prüfer domain which is not a Bezout domain (see [24, Page 278]). Pendleton [64, Page 500] has shown that a Prüfer domain D has the QR- property if and only if the radical of each finitely generated ideal of D is the radical of a principal ideal. Recall that a domain D is a QQR -domain if each overring of D is an intersection of localizations at prime ideals of D. Davis [14] showed that a Prüfer domain must have the QQR-property.

An integral domain D is an atomic domain if every nonzero nonunit of D can be factored as a product of irreducible elements of D. Following [2] an integral domain D is strongly atomic if for each $a, b \in D^0$ (non zero elements of D), we can write $a = a_1...a_s c$ and $b = a_1...a_s d$ where $a_1, ..., a_s \in D$ ($s \geq 0$) are irreducible and $c, d \in D$ satisfy $gcd(c, d) = 1$.

An integral domain D (with or without unity) is called a Euclidean domain (ED) if there is a map $d : D^0 \to \mathbb{Z}^+$ (where D^0 is set of nonzero elements of D) such that

(i) $\forall$ $a, b \in D^0$, $a|b$ implies that $d(a) \leq d(b)$ or equivalently, $d(x) \leq d(xy)$for all $x, y \in D^0$.

(ii) Given $a \in D$, $b \in D^0$ there exist $q, r \in D$ such that $a = bq + r$ with either $r = 0$ or else $d(r) < d(b)$.

Recall that a principal ideal domain (PID) is an integral domain such that every ideal can be generated by a single element (i.e., every ideal is a principal ideal). By [59, Theorem 4.3.1] every ED is a PID, but the converse is not true because the ring $R = \mathbb{Z}[(1 + i\sqrt{}19)/2]$ or the ring of even integers $R = 2\mathbb{Z}$ are $PIDs$ which are not EDs. A PID is a UFD but converse is not true, however if every nontrivial prime ideal is maximal, a UFD is a PID.

In the terminology of Kaplansky [42] a GCD- domain is an integral domain in which each pair of elements has a greatest common divisor. A GCD- domain is a generalization of unique factorization domain (UFD). An integral domain D is a weak GCD domain if for each $a, b \in D^0$ (nonzero elements of D) there are $c,\ a^/\ b^/ \in D$ so that $a = ca^/$and $b = cb^/$, where $gcd(a, b) = 1$.

In Proposition 21, authors have proved that a strongly Laskerian domain is a weak GCD-domain. The terminology of half-factorial domain (HFD) was introduced by Zaks in [76] and [77] as a generalization of unique factorization domain (UFD). An atomic integral domain D is a half-factorial domain (HFD) if for any irreducible elements $a_1, ..., a_n$ and $b_1, \ldots, b_m$ of D with $a_1 ... a_n = b_1 \ldots b_m$, then $m = n$. Clearly a UFD is also an HFD. Call an integral domain D a finite factorization domain (FFD) if every nonzero nonunit element of D is either irreducible or a product of irreducible elements. FFD is a much weaker concept than UFD. In [29] Grams and Warner introduced idf –domains (for irreducible-divisor- finite) as; a domain D is an idf-domain if every nonzero element of D has at most a finite number of non associate irreducible divisors. An idf -domain need not to be atomic. The idf- property does not imply any other factorization property. Following [2] a domain D is a bounded factorization domain (BFD) if D is atomic and for each nonzero nonunit of D there is a bound on length of factorization into product of irreducible elements. The following non reversible implications are taken from [2, Page 2].

$$\begin{array}{ccccccccc} & & HFD & & & & & & \\ & \nearrow & \downarrow & \searrow & & & & & \\ UFD & \rightarrow & FFD & \rightarrow & BFD & \rightarrow & ACCP & \rightarrow & Atomic \\ & \searrow & \downarrow & \nearrow & & & & & \\ & & idf-domain & & & & & & \end{array}$$

A Noetherian domain is BFD (cf. [2, Proposition 2.2]. But converse is not true.

For any pair of fields $K_1 \subseteq K_2$, $K_1 + XK_2[X]$ (respectively $K_1 + XK_2[[X]]$) are $BFDs$ but not Noetherian if $[K_2 : K_1]$ is infinite (see [2, Page 10]). However a BFD satisfies ACC

on principal ideals but the converse is not true (cf. [2, Example 2.1]).

In Proposition 21, we have proved that a strongly Laskerian domain satisfy $ACCP$ (respectively. strongly atomic). Following [68], an integral domain D is Archimedean in case $\cap_{n\geq 1} Dr^n = 0$ for each non unit $r \in D$. The most natural examples of Archimedean domains are arbitrary completely integrally closed domains, arbitrary one dimensional integral domains and arbitrary Noetherian integral domains. By [9, Theorem 2.1] or by [34, Proposition 2.2] if a ring satisfy $ACCP$, then it is Archimedean.

Figures display a diagram of implications summarizing the relations for some well known integral domains. Note that none of the implications in the diagram is reversible.

5.2 Table of Implications

Chapter 6

FUZZY IDEALS IN LASKERIAN RINGS

Introduction

Taking motivation from swiftly mounting literature on fuzzy concepts in various areas of ring theory and other algebraic structure, we intended to continue our study in fuzzy settings.

The theory of fuzzy sets, proposed by Zadeh [73], has provided a useful mathematical device for unfolding the behavior of the systems that are too complex or nonspecific to admit precise mathematical analysis by conventional methods and tools. The study of fuzzy algebraic structure has started by Rosenfeld [66] and since then this concept has been applied to a variety of algebraic structures. Liu introduced the concept of a fuzzy ideal of a ring in [49]. The concepts of prime fuzzy ideals, maximal fuzzy ideals, primary fuzzy ideals were introduced in [52], [53] and [54]. Also, Malik [51], Mukererjee and Sen [56], studied rings with chain conditions with the help of fuzzy ideals. The notion of fuzzy quotient ring was introduced by Kumar [45], Kuroaka and Kuroki [46]. In [48], K. H. Lee examined some properties of fuzzy quotient rings and used them to characterize Artinian and Noetherian rings.

This chapter consists of three sections. In the first section, we introduced strongly primary fuzzy ideals and strongly irreducible fuzzy ideals in rings. We examined finite intersection property of some fuzzy ideals in chained rings.

Second section of this chapter deals with fuzzy ideals in Laskerian rings. We have proved that for a 1-dimensional Laskerian domain D, every non zero fuzzy ideal μ of D can be uniquely expressed as a product of primary fuzzy ideals whose radicals are all distinct. Moreover we showed that: A ring R is Laskerian (respectively strongly Laskerian) if and only if R_μ is Laskerian (respectively strongly Laskerian) for every fuzzy ideal μ of R.

In the third section, we have investigated anti- homomorphic images and pre images of semiprime, strongly primary, irreducible and strongly irreducible fuzzy ideals of a ring. We also proved that: For a surjective anti-homomorphism $f : R \to R^/$, if every fuzzy ideal of R is f-invariant and has a fuzzy primary (respectively, strongly primary) decomposition in R, then every fuzzy ideal of $R^/$ has a fuzzy primary (respectively, strongly primary) decomposition in $R^/$.

6.1 Finite Intersection of Fuzzy Ideals

In this section we have proved that for a chained ring (a ring in which any two fuzzy ideals are comparable), finite intersection of prime fuzzy ideals, primary fuzzy ideals, irreducible fuzzy ideals and strongly irreducible fuzzy ideal is prime fuzzy ideal, primary fuzzy ideal, irreducible fuzzy ideal and strongly irreducible fuzzy ideal respectively.

Definition 67 *Following [55], a fuzzy ideal μ of a ring R is a prime fuzzy ideal of R if*

(i) μ is not a constant function and

(ii) for any fuzzy ideals μ_1, μ_2 in R if $\mu_1\mu_2 \subseteq \mu$, then $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$.

Note: $\mu_1 \subseteq \mu$ means $\mu_1(x) \leq \mu(x)$ for all $x \in R$.

Lemma 68 *If μ_1 and μ_2 are fuzzy prime (respectively primary, respectively strongly primary) ideals of a ring R, then $\mu_1 \cap \mu_2$ is a fuzzy prime (respectively primary, respectively strongly primary) ideal of R if and only if $\mu_1 \subseteq \mu_2$ or $\mu_2 \subseteq \mu_1$.*

Proof. It is obvious from the fact that $\mu_1\mu_2 \subseteq \mu_1 \cap \mu_2$. ■

We initiate with the following proposition.

Proposition 69 *If prime fuzzy ideals of a ring form a proper chain, then finite intersection of prime fuzzy ideals of a ring is a prime fuzzy ideal.*

Proof. Let $\{\mu_i : 1 \leq i \leq n\}$ be a family of prime fuzzy ideals of a ring R and let $\theta\sigma \subseteq \Lambda_{i=1}^{n}\mu_i$ for any two fuzzy ideals θ and σ of R.This means $\theta\sigma \subseteq \mu_i$ for all $1 \leq i \leq n$. Since μ_i are prime fuzzy ideals, therefore if $\theta \nsubseteq \mu_i$ for all $1 \leq i \leq n$, then $\sigma \subseteq \mu_i$, for all $1 \leq i \leq n$. Similarly, if $\sigma \nsubseteq \mu_i$ for all $1 \leq i \leq n$, then $\sigma \subseteq \mu_i$, for all $1 \leq i \leq n$. This implies that if $\theta \nsubseteq \Lambda_{i=1}^{n}\mu_i$, then $\sigma \subseteq \Lambda_{i=1}^{n}\mu_i$.Hence $\Lambda_{i=1}^{n}\mu_i$ is a prime fuzzy ideal of R. ■

Following [67] a fuzzy ideal μ of a ring R is known as a primary fuzzy ideal if for any two fuzzy ideals σ and θ of R the conditions $\sigma\theta \subseteq \mu$ and $\sigma \nsubseteq \mu$ together imply that $\theta \subseteq \sqrt{\mu}$.

Proposition 70 *If primary fuzzy ideals of a ring form a proper chain, then finite intersection of primary fuzzy ideals of a ring is a primary fuzzy ideal.*

Proof. Let $\{\mu_i : 1 \leq i \leq n\}$ be a family of primary fuzzy ideals of a ring R and let $\theta\sigma \subseteq \Lambda_{i=1}^{n}\mu_i$ for any two fuzzy ideals θ and σ.This implies that $\theta\sigma \subseteq \mu_i$ for all $1 \leq i \leq n$. Since μ_i are primary fuzzy ideals therefore if $\theta \nsubseteq \mu_i$ for all $1 \leq i \leq n$ then $\sigma \subseteq \sqrt{\mu_i}$ for all $1 \leq i \leq n$. So, if $\theta \nsubseteq \Lambda_{i=1}^{n}\mu_i$ for some i then $\sigma \subseteq \sqrt{\Lambda_{i=1}^{n}\mu_i}$. Consequently $\Lambda_{i=1}^{n}\mu_i$ is a primary fuzzy ideal of R. ■

Remark 71 *Every prime fuzzy ideal of a ring is a primary fuzzy ideal.*

We give the following definition.

Definition 72 *A fuzzy ideal μ of a ring R is called strongly primary fuzzy ideal in R if μ is a primary fuzzy ideal and $(\sqrt{\mu})^n \subset \mu$ for some $n \in \mathbb{N}$.*

Proposition 73 *If strongly primary fuzzy ideals of a ring form a proper chain, then finite intersection of strongly primary fuzzy ideals of a ring is a primary fuzzy ideal.*

Proof. It can be proved on the same lines as in Proposition 70. ■

Following [58, Definition 4.1] a fuzzy ideal μ of a Noetherian ring R is said to be irreducible if $\mu \neq R$ and whenever $\mu = \mu_1 \Lambda \mu_2$, where μ_1, μ_2 are fuzzy ideals of R, then $\mu = \mu_1$ or $\mu = \mu_2$. A prime fuzzy ideal is necessarily irreducible; however, the converse is not true (see [58, Example 4.6]).

In the rest of text instead of a Noetherian ring we consider an arbitrary ring.

We give the following definition.

Definition 74 *A proper fuzzy ideal μ of a ring R is said to be strongly irreducible if for each pair of fuzzy ideals θ and σ of R, if $\theta \Lambda \sigma \subseteq \mu$, then either $\theta \subseteq \mu$ or $\sigma \subseteq \mu$.*

Remark 75 *A strongly irreducible fuzzy ideal is irreducible.*

Proposition 76 *If strongly primary fuzzy ideals of a ring form a proper chain, then finite intersection of strongly irreducible fuzzy ideals of a ring is a strongly irreducible fuzzy ideal.*

Proof. Let $\{\mu_i : 1 \leq i \leq n\}$ be a family of strongly irreducible fuzzy ideals of a ring R and let $\theta \Lambda \sigma \subseteq \Lambda_{i=1}^{n} \mu_i$ for any two fuzzy ideals θ and σ. This means $\theta \Lambda \sigma \subseteq \mu_i$ for all $1 \leq i \leq n$. This implies $\theta \subseteq \mu_i$ or $\sigma \subseteq \mu_i$ for all $1 \leq i \leq n$. Since μ_i are strongly irreducible fuzzy ideals, therefore if $\theta \subseteq \mu_i$ or $\sigma \subseteq \mu_i$ for all $1 \leq i \leq n$, then $\theta \subseteq \Lambda_{i=1}^{n} \mu_i$ or $\sigma \subseteq \Lambda_{i=1}^{n} \mu_i$ for all i. Hence $\Lambda_{i=1}^{n} \mu_i$ is a strongly irreducible fuzzy ideal of R. ■

Recall that a fuzzy ideal μ of a ring is known as radical ideal if $\mu = \sqrt{\mu}$.

Proposition 77 *A strongly irreducible fuzzy ideal in a ring is a prime fuzzy ideal if and only if it is a radical ideal.*

Proof. If μ is a prime fuzzy ideal of a ring R, then by [10, Theorem 5.10] $\mu = \sqrt{\mu}$. Conversely assume that $\mu = \sqrt{\mu}$. Let $\mu_1 \Lambda \mu_2 \subseteq \mu$, where μ_1 and μ_2 are fuzzy ideals of R. Then $\mu_1 \mu_2 \subseteq \mu_1 \Lambda \mu_2 \subseteq \sqrt{\mu_1 \Lambda \mu_2} = \sqrt{\mu_1 \mu_2} \subseteq \sqrt{\mu} = \mu$, and since μ is a strongly irreducible fuzzy ideal of R. Hence $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$. ■

Following [41], a fuzzy ideal μ of a ring R is called semiprime if $\mu^2(x) = \mu(x)$ for all $x \in R$.

Proposition 78 *A fuzzy ideal μ of a ring R is prime if and only if it is semiprime and strongly irreducible.*

Proof. Suppose μ is a fuzzy prime ideal of R. Obviously μ is fuzzy semiprime ideal. Moreover, if σ and θ are fuzzy ideals of R, satisfying $\sigma\Lambda\theta \subseteq \mu$, then $\sigma\theta \subseteq \mu$, since $\sigma\theta \subseteq \sigma\Lambda\theta$. This implies $\sigma \subseteq \mu$ or $\theta \subseteq \mu$. Hence μ is strongly irreducible. Conversely, assume that μ is a strongly irreducible semiprime fuzzy ideal of R. Suppose σ and θ be two fuzzy ideals of R with $\sigma\theta \subseteq \mu$. Consider $(\sigma\Lambda\theta)^2(x) = ((\sigma\Lambda\theta)(\sigma\Lambda\theta))(x)$

$= sup\{\Lambda_{i=1}^{n}((\sigma\Lambda\theta)(r_i)\Lambda(\sigma\Lambda\theta)(s_i)) : x = \sum_{i=1}^{n} r_i s_i, r_i, s_i \in R, n \in \mathbb{N}\}$

$= sup\{\Lambda_{i=1}^{n} min\{(\sigma\Lambda\theta)(r_i), (\sigma\Lambda\theta)(s_i)\}\} : x=\sum_{i=1}^{n} r_i s_i, r_i, s_i \in \mathsf{R}, \mathsf{n} \in \mathbb{N}\}$

$= sup\{\Lambda_{i=1}^{n} min\{min\{\sigma(r_i), \theta(r_i)\}, min\{\sigma(s_i), \theta(s_i)\}\}\} : x = \sum_{i=1}^{n} r_i s_i, \mathsf{r}_i, \mathsf{s}_i \in \mathsf{R}, \mathsf{n} \in \mathbb{N}\}$

$\leq sup\{\Lambda_{i=1}^{n} min\{\sigma(r_i), \theta(s_i)\}\} : x = \sum_{i=1}^{n} r_i s_i, r_i, s_i \in R, n \in \mathbb{N}\}$

$= (\sigma\theta)(\mathsf{x})$.

Therefore $(\sigma\Lambda\theta)^2 \subseteq \sigma\theta \subseteq \mu$. But, since μ is semiprime, this means $\sigma\Lambda\theta \subseteq \mu$. Hence $\sigma \subseteq \mu$ or $\theta \subseteq \mu$, as μ is strongly irreducible. ■

6.2 Some Fuzzy Ideals in Laskerian Rings

Following [58, Definition 3.1 and 3.2], if for a collection $\{v_i : i = 1, 2, ..., n\}$ of primary fuzzy ideals of R and $\{\mu_i : i = 1, 2, ..., n\}$ a finite collection of v_i-primary fuzzy ideals of R,then, $\mu = \Lambda_{i=1}^{n}\mu_i$ is called a primary decomposition of μ. This decomposition is said to be reduced or irredundant if

(i) the $v_1, v_2, ..., v_n$ are all distinct and

(ii) $\mu_j \not\supseteq \Lambda_{i=1, i\neq j}^{n}\mu_i$, for all $j = 1, 2, ..., n$.

Every irreducible fuzzy ideal of a Noetherian ring is a primary fuzzy ideal (cf. [58, Proposition 4.7]). In the following proposition we generalize [58, Proposition 4.7], in case of Laskerian rings.

Proposition 79 *In a Laskerian ring an irreducible fuzzy ideal is a primary fuzzy ideal.*

Proof. By passage to the quotient rings, it is sufficient to consider that the (0) ideal is an irreducible fuzzy ideal and prove that it is primary. So, suppose that $\mu\upsilon = 0$ and $\mu \neq 0$, where μ and υ are fuzzy ideals of R. Consider the chain of ideals $ann(\upsilon) \subseteq ann(\upsilon^2) \subseteq ... \subseteq ann(\upsilon^n)...$. Since R is Laskerian (strongly Hopfian), this chain stabilizes: there exists a positive integer n such that $ann(\upsilon^n) = ann(\upsilon^{n+k})$ for all k. It follows that $\mu\Lambda\ \upsilon^n = 0$. Indeed, if $a \in \mu$, then $a\mu = 0$, and if $a \in \upsilon^n$, then $a = b\upsilon^n$ for some $b \in R$. Hence $b\upsilon^{n+1} = 0$, so $b \in ann(\upsilon^{n+1}) = \ ann(\upsilon^n)$. Hence $b\upsilon^n = 0$; that is, $a = 0$. Since the (0) ideal is irreducible and $\mu \neq 0$, we must then have $\upsilon^n = 0$, and this shows that (0) is primary. ■

Proposition 80 *If every primary fuzzy ideal of a ring R is a strongly irreducible fuzzy ideal, then every fuzzy minimal primary decomposition for each fuzzy ideal of R is unique.*

Proof. Let a fuzzy ideal μ of R has two fuzzy minimal primary decompositions $\Lambda_{i=1}^n \mu_i$ and $\Lambda_{i=1}^m \sigma_i$. For $n \leq m$, we have $\Lambda_{i=1}^n \mu_i \subseteq \Lambda_{i=1}^m \sigma_i$ and since σ_1 is strongly irreducible fuzzy ideal for some j, $1 \leq j \leq n$, therefore $\mu_j \subseteq \sigma_1$. On the other hand, $\Lambda_{i=1}^m \sigma_i \subseteq \mu_j$. Since μ_j is strongly irreducible fuzzy ideal, for some k, $1 \leq k \leq m$, we have $\sigma_k \subseteq \mu_j \subseteq \sigma_1$. Since $\Lambda_{i=1}^m \sigma_i$ is a fuzzy minimal primary decomposition, $\sigma_k = \sigma_1$ and so $k = 1$. Hence $\sigma_1 = \mu_j$. Without loss of generality, let $\sigma_1 = \mu_1$. Similarly we can show that $\sigma_2 = \mu_t$ for some t, $1 \leq t \leq n$, and since $\sigma_2 \neq \sigma_1$ $\mu_t \neq \mu_1$. That is, $t \neq 1$. Therefore, without loss of generality, we can assume that $\sigma_2 = \mu_2$. The same argument will show that for each t, $1 \leq t \leq m$, $\sigma_i = \mu_i$ and $n = m$. ■

Two ideals I and J in a ring are said to be coprime (or comaximal) if $I + J = (1)$ [4, Page 7].

Proposition 81 *Let D be a Laskerian domain of dimension 1. Then every nonzero fuzzy ideal μ of D can be uniquely expressed as a product of primary fuzzy ideals with distinct radicals.*

Proof. Let μ be a nonzero fuzzy ideal of a Laskerian domain D. Then $\mu = \Lambda_{i=1}^n \mu_i$, where each μ_i is $P-$primary fuzzy ideal of D. The ideals $P_{1,...,}P_n$ are maximal since they stem from reduced primary decomposition, they are pair-wise different and therefore pair-wise coprime. Therefore $\mu_1, ..., \mu_n$ are pair-wise coprime and by [4, Proposition 1.10.], $\Pi\mu_i = \Lambda\mu_i$. Hence $\mu = \Pi\mu_i$.

Conversely, if $\mu = \Pi\mu_i$, the same argument shows that $\mu = \Lambda\mu_i$; this is a minimal primary decomposition of μ, in which each μ_i is isolated primary component, and by [4, Theorem 4.10], is therefore unique. ■

The following proposition will help us to find the rings for which every primary fuzzy ideal is a strongly irreducible fuzzy ideal.

Proposition 82 *In a ring R, the following are equivalent.*

(1) Every fuzzy ideal of R is a strongly irreducible fuzzy ideal.

(2) Every two fuzzy ideals of R are comparable.

Proof. (1) $\Rightarrow$ (2). Let μ and σ be two fuzzy ideals of R. Note that $\mu\Lambda\sigma$ is a strongly irreducible fuzzy ideal, and $\mu\Lambda\sigma \subseteq \mu\Lambda\sigma$. So $\mu \subseteq \mu\Lambda\sigma \subseteq \sigma$ or $\sigma \subseteq \mu\Lambda\sigma \subseteq \mu$.

(2) $\Rightarrow$ (1) The proof is obvious. ■

Let μ be a fuzzy ideal of a ring R and $x \in R$. By [48] the fuzzy subset of R defined as $\mu_x^*(r) = \{\mu(r - x)$ for all $r \in R\}$ is termed as the fuzzy coset determined by x and μ. The set of all cosets of μ in R is a ring under the binary operations $\mu_x^* + \mu_y^* = \mu_{x+y}^*$and $\mu_x^*\mu_y^* = \mu_{xy}^*$ for all x, $y \in R$ and it is denoted by R_μ, and known as fuzzy quotient ring of R induced by the fuzzy ideal μ.

In [48] K. H. Lee has characterized Artinian and Noetherian rings, respectively, using fuzzy quotient rings (see [48, Proposition 3.9 and Proposition 3.10]). In the following proposition we characterize Laskerian rings by fuzzy quotient ring using the same technique of [48].

Proposition 83 *A ring R is strongly Laskerian if and only if R_μ is strongly Laskerian for every fuzzy ideal μ of R.*

Proof. Let $\overline{\nu} : R_\mu \to [0, 1]$ be a fuzzy ideal of R_μ. To show that $\overline{\nu}$ has a finite reduced strongly primary decomposition, define a map $\theta : R \to [0, 1]$ by $\theta(x) = \overline{\nu}(\mu_x^*)$ for every $x \in R$.

Then θ is a fuzzy ideal of R and it has a finite reduced strongly primary decomposition. Since the set of values of θ is same to the set of values of $\overline{\nu}$ therefore $\overline{\nu}$ also has a finite reduced strongly primary decomposition. Hence R_μ is strongly Laskerian.

Conversely, let μ be a fuzzy ideal of a ring R. Then the fuzzy ideal $\overline{\nu}$ of R_μ defined by $\overline{\nu}(\mu_x^*) = \mu(x)$ for every $x \in R$ has a finite reduced primary decomposition, so that μ also has finite reduced strongly primary decomposition. Hence R is strongly Laskerian. ■

Proposition 84 *A ring R is Laskerian if and only if R_μ is Laskerian for every fuzzy ideal μ of R.*

Proof. Let $\overline{\nu}$ be a fuzzy ideal of R_μ. To show that $\overline{\nu}$ has a finite reduced primary decomposition, define a map $\theta : R \rightarrow [0,1]$ by $\theta(x) = \overline{\nu}(\mu_x^*)$ for every $x \in R$. Then θ is a fuzzy ideal of R and it has finite primary decomposition. Since the set of values of θ is same to the set of values of $\overline{\nu}$ therefore $\overline{\nu}$ also has a finite reduced primary decomposition. This implies that R_μ is Laskerian.

Conversely, let μ be a fuzzy ideal of a ring R. Then the fuzzy ideal $\overline{\nu}$ of R_μ defined by $\overline{\nu}(\mu_x^*) = \mu(x)$ for every $x \in R$ has a finite reduced primary decomposition, so that μ also has finite fuzzy primary decomposition. Hence R is Laskerian. ■

The following table summarizes findings of [48, Proposition 3.9 and Proposition 3.10] and Propositions (83 and 84).

R		R_μ
Artinian	$\Longleftrightarrow$	Artinian
Noetherian	$\Longleftrightarrow$	Noetherian
Strongly Laskerian	$\Longleftrightarrow$	Strongly Laskerian
Laskerian	$\Longleftrightarrow$	Laskerian
Q-ring	$\Longrightarrow$	Laskerian

6.3 The Anti-homomorphism

A. Sheikabdullah and K. Jeyaraman introduced the property of anti homomorphism in rings in fuzzy settings (see [67]). According to them:

A mapping $f : R \to R^{/}$ is called a fuzzy anti-homomorphism of rings, if

$$f(\mu + \sigma) = f(\mu) + f(\sigma) \text{ and } f(\mu\sigma) = f(\sigma)f(\mu).$$

where μ and σ are fuzzy ideals of R.

In this section we tried to observe impact of anti homomorphism on some other classes of fuzzy ideals.

Following [67], a fuzzy ideal μ of a ring R is known as a primary fuzzy ideal if for any two fuzzy ideals σ and θ of R the conditions $\sigma\theta \subseteq \mu$ and $\sigma \nsubseteq \mu$ together imply that $\theta \subseteq \sqrt{\mu}$.

Lemma 85 *If μ is a primary fuzzy ideal of R, then $\sqrt{\mu}$ is a prime fuzzy ideal of R.*

Proof. Let σ and θ be two fuzzy ideal of R such that $\sigma\theta \subseteq \sqrt{\mu}$ and $\sigma \nsubseteq \sqrt{\mu}$. Then $\sigma \nsubseteq \mu$ and so $\theta \subseteq \sqrt{\mu}$. Hence $\sqrt{\mu}$ is a prime fuzzy ideal of R. ∎

Remark 86 *In a Laskerian ring an irreducible fuzzy ideal is primary (see Proposition 79), therefore if μ is an irreducible fuzzy ideal of a Laskerian ring, then $\sqrt{\mu}$ is a prime fuzzy ideal.*

Let X and Y be two non empty sets, $f : X \to Y$, μ and σ be fuzzy subsets of X and Y respectively. Then $f(\mu)$, the image of μ under f is a fuzzy subset of Y denoted by

$$(f(\mu))(y) = \left\{ \begin{array}{cc} \sup_{f(x)=y} \mu(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{array} \right\}$$

$f^{-1}(\sigma)$, the pre image of σ under f is a fuzzy subset of X defined by $(f^{-1}(\sigma))(x) = \sigma(f(x))$ for all $x \in X$.

Proposition 87 *Let $f : R \to R^{/}$ be a surjective ring anti-homomorphism, if $\mu^{/}$ is a fuzzy primary ideal of $R^{/}$, then $f^{-1}(\mu^{/})$ is a primary fuzzy ideal of R.*

Proof. *Let θ and σ be any two fuzzy ideals of R, such that $\theta\sigma \subset f^{-1}(\mu^{/})$. This implies that $f(\theta\sigma) \subset ff^{-1}(\mu^{/}) = \mu^{/}$ which means $f(\sigma)f(\theta) \subset \mu^{/}$. Because $\mu^{/}$ is a primary fuzzy ideal of $R^{/}$ therefore, $f(\sigma) \subset \mu^{/}$ or $f(\theta) \subset \sqrt{\mu^{/}}$. So $f^{-1}(f(\sigma)) \subset f^{-1}(\mu^{/})$ or $f^{-1}(f(\theta))$ $\subset f^{-1}(\sqrt{\mu^{/}})$. That is, $\sigma \subset f^{-1}(\mu^{/})$ or $\theta \subset f^{-1}(\sqrt{\mu^{/}}) = \sqrt{f^{-1}(\mu^{/})}$. As a result $f^{-1}(\mu^{/})$ is a primary fuzzy ideal of $R^{/}$.* ■

Recall that, a fuzzy ideal μ of a ring R is called strongly primary fuzzy ideal of R if μ is a primary fuzzy ideal and $(\sqrt{\mu})^n \subset \mu$ for some $n \in \mathbb{N}$.

For a function $f : R \to R^{/}$, a fuzzy subset μ of a ring R is called f-invariant if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$, x, $y \in R$. Clearly, if μ is f-invariant then $f^{-1}(f(\mu)) = \mu$.

Proposition 88 *For a surjective ring anti-homomorphism (homomorphism) $f : R \to R^{/}$, if μ is an f-invariant strongly primary fuzzy ideal of R, then $f(\mu)$ is a strongly primary fuzzy ideal of $R^{/}$.*

Proof. In [67, Proposition 3.5], it is proved that $f(\mu)$ is a primary fuzzy ideal of $R^{/}$. We need to prove that $[\sqrt{f(\mu)}]^n \subset f(\mu)$ for some $n \in \mathbb{N}$. By Lemma 85, $\sqrt{f(\mu)}$ is a prime fuzzy ideal of $R^{/}$, therefore $[\sqrt{f(\mu)}]^n \subset f(\mu)$ for some $n \in \mathbb{N}$. Hence $f(\mu)$ is a strongly primary ideal of $R^{/}$. ■

Following [41], a fuzzy ideal μ of a ring R is called semiprime if $\mu^2(x) = \mu(x)$ for all $x \in R$.

Proposition 89 *For a surjective ring anti-homomorphism (homomorphism) $f : R \to R^{/}$, if $\mu^{/}$ is a semiprime fuzzy ideal of $R^{/}$, then $f^{-1}(\mu^{/})$ is a semiprime fuzzy ideal of R.*

Proof. If $\mu^{/}$ is a semiprime fuzzy ideal of $R^{/}$, then by [67, Proposition 3.2], $f^{-1}(\mu^{/})$ is a fuzzy ideal of R. We only need to show that $[f^{-1}(\mu^{/})]^2 = f^{-1}(\mu^{/})$. For this let $f^{-1}(\mu^{/}) = \mu$ this implies that $\mu^{/} = f(\mu)$. Now $\mu^{/} = \mu^{/}\mu^{/} = f(\mu)f(\mu) = f(\mu\mu) = f(\mu^2)$

$\Rightarrow \mu^2 = f^{-1}(\mu^{/}) = \mu$. Hence proved. ■

Proposition 90 *Let $f : R \to R^{/}$ be a ring anti-homomorphism (homomorphism). If μ is any f-invariant semiprime fuzzy ideal of R, then $f(\mu)$ is a semiprime fuzzy ideal of $R^{/}$.*

Proof. If μ is an f-invariant semiprime fuzzy ideal of R, then by [67, Proposition 3.1], $f(\mu)$ is a fuzzy ideal of $R^{/}$. We only need to show that $[f(\mu)]^2 = f(\mu)$. For this consider

$[f(\mu)]^2 = f(\mu)f(\mu) = f(\mu^2) = f(\mu)$. ■

Following [58, Definition 4.1], a fuzzy ideal μ of a Noetherian ring R is said to be irreducible if $\mu \neq R$ and whenever $\mu = \mu_1 \Lambda \mu_2$, where μ_1, μ_2 are fuzzy ideals of R, then $\mu = \mu_1$ or $\mu = \mu_2$. A prime fuzzy ideal is irreducible;

In the rest of text we consider ring to be a Laskerian ring.

Recall that, a proper fuzzy ideal μ of a ring R is said to be strongly irreducible if for each pair of fuzzy ideals θ and σ of R, if $\theta \Lambda \sigma \subseteq \mu$, then either $\theta \subseteq \mu$ or $\sigma \subseteq \mu$.

Remark 91 *A strongly irreducible fuzzy ideal is irreducible.*

Definition 92 *An element a of a ring R is called regular element of R if there exists an element x of R such that $a = axa$. A ring R is called regular if each element of R is regular.*

Proposition 93 *If R is a regular ring, then $\mu \circ \sigma = \mu \cap \sigma$, where μ is a fuzzy right ideal and σ is a fuzzy left ideal.*

Proof. Since $\mu \circ \sigma \subseteq \mu \cap \sigma$. Let $a \in R$. This implies there exists an element $x \in R$ such that $a = axa$, since R is regular. Now

$$
\begin{aligned}
(\mu \circ \sigma)(a) &= \vee_a \text{ where } a = \sum_{i=1}^{n} a_i b_i \{\Lambda_{i=1}^{n} \{\mu(a_i) \wedge \sigma(b_i)\}\} \\
&\geq \mu(ax)\Lambda\sigma(a) \geq \mu(a)\Lambda\sigma(a) = (\mu \cap \sigma)(a) \\
&\Longrightarrow \mu \cap \sigma \subseteq \mu \circ \sigma
\end{aligned}
$$

Hence $\mu \circ \sigma = \mu \cap \sigma$. ■

Note: For Propositions 94, 95 and 98, we consider the rings to be regular.

Proposition 94 *Let $f : R \to R^{/}$ be a surjective ring anti-homomorphism. If $\mu^{/}$ is an irreducible fuzzy ideal of $R^{/}$, then $f^{-1}(\mu^{/})$ is an irreducible fuzzy ideal of R.*

Proof. Let μ and σ be any two fuzzy ideals of R, such that $\mu\Lambda\sigma = f^{-1}(\mu^{/})$. This implies that $f(\mu\Lambda\sigma) = ff^{-1}(\mu^{/}) = \mu^{/}$. Because f is a surjective anti-homomorphism, therefore $f(\sigma)\Lambda f(\mu) = \mu^{/}$. As $\mu^{/}$ is irreducible fuzzy ideal of $R^{/}$ therefore $f(\sigma) = \mu^{/}$ or $f(\mu) = \mu^{/}$. So $f^{-1}(f(\sigma)) = f^{-1}(\mu)$ or $f^{-1}(f(\mu)) = f^{-1}(\mu^{/})$, this means $\sigma = f^{-1}(\mu^{/})$ or $\mu = f^{-1}(\mu^{/})$. Hence $f^{-1}(\mu^{/})$ is a fuzzy irreducible ideal of $R^{/}$. ■

Proposition 95 *Let $f : R \to R^{/}$ be a surjective ring anti-homomorphism. If $\mu^{/}$ is a strongly irreducible fuzzy ideal of $R^{/}$, then $f^{-1}(\mu^{/})$ is a strongly irreducible fuzzy ideal of R.*

Proof. Let μ and σ be any two fuzzy ideals of R, such that $\mu\Lambda\sigma \subseteq f^{-1}(\mu^{/})$. This implies that $f(\mu\Lambda\sigma) \subseteq ff^{-1}(\mu^{/}) = \mu^{/}$. Because f is a surjective anti-homomorphism $f(\sigma)\Lambda f(\mu) \subseteq \mu^{/}$. This means that $inf\{f(\sigma), f(\mu)\} \subseteq \mu^{/}$ so, $f(\sigma) \subseteq \mu^{/}$ or $f(\mu) \subseteq \mu^{/}$. Therefore $f^{-1}(f(\sigma)) \subseteq f^{-1}(\mu^{/})$ or $f^{-1}(f(\mu)) \subseteq f^{-1}(\mu^{/})$, consequently $\sigma \subseteq f^{-1}(\mu^{/})$ or $\mu \subseteq f^{-1}(\mu^{/})$. Hence $f^{-1}(\mu^{/})$ is a strongly irreducible fuzzy ideal of $R^{/}$. ■

Proposition 96 *For a ring anti-homomorphism $f : R \to R^{/}$, if μ is an f-invariant strongly irreducible fuzzy ideal of R, then $f(\mu)$ is a strongly irreducible fuzzy ideal of $R^{/}$.*

Proof. Let $\sigma^{/}$ and $\theta^{/}$ be any two fuzzy ideals of R, such that $\sigma^{/}\Lambda\theta^{/} \subseteq f(\mu)$. This implies that $f^{-1}(\sigma^{/}\Lambda\theta^{/}) \subseteq f^{-1}f(\mu) = \mu$. therefore $f^{-1}(\theta^{/})\Lambda f^{-1}(\sigma^{/}) \subseteq \mu$. Since μ is strongly irreducible fuzzy ideal of R therefore $f^{-1}(\theta^{/}) \subseteq \mu$ or $f^{-1}(\sigma^{/}) \subseteq \mu$. So $ff^{-1}(\theta^{/}) \subseteq f(\mu)$ or $ff^{-1}(\sigma^{/}) \subseteq f(\mu)$ consequently $\theta^{/} \subseteq f(\mu)$ or $\sigma^{/} \subseteq f(\mu)$ that is $f(\mu)$ is strongly irreducible fuzzy ideal of $R^{/}$. ■

Proposition 97 *Let $f : R \to R^{/}$ be a ring anti-homomorphism. If μ is an f-invariant irreducible fuzzy ideal of R, then $f(\mu)$ is an irreducible fuzzy ideal of $R^{/}$.*

Proof. Let $\sigma^{/}$ and $\theta^{/}$ be any two fuzzy ideals of R, such that $\sigma^{/}\Lambda\theta^{/} = f(\mu)$. This implies that $f^{-1}(\sigma^{/}\Lambda\theta^{/}) = f^{-1}f(\mu) = \mu = f^{-1}(\theta^{/})\Lambda f^{-1}(\sigma^{/})$. Since μ is irreducible fuzzy ideal of

R, therefore, either $f^{-1}(\theta^/) = \mu$ or $f^{-1}(\sigma^/) = \mu$. So $ff^{-1}(\theta^/) = f(\mu)$ or $ff^{-1}(\sigma^/) = f(\mu)$. This means $\theta^/ = f(\mu)$ or $\sigma^/ = f(\mu)$. Hence $f(\mu)$ is an irreducible fuzzy ideal of $R^/$. ■

Proposition 98 *For a surjective ring anti-homomorphism $f : R \to R^/$, if every fuzzy ideal of R is f-invariant and has a fuzzy primary (respectively, strongly primary) decomposition in R, then every fuzzy ideal of $R^/$ has a fuzzy primary (respectively, strongly primary) decomposition in $R^/$.*

Proof. If $\mu^/$ is a fuzzy ideal of $R^/$, then $f^{-1}(\mu^/)$ is a fuzzy ideal of R. Take $f^{-1}(\mu') = \Lambda_{i=1}^{n}\mu_i$ where μ_i are primary (respectively, strongly primary) fuzzy ideals of R and $i = 1, 2, ..., n$. This implies that $ff^{-1}(\mu^/) = f(\Lambda_{i=1}^{n}\mu_i)$ that is, $\mu^/ = \Lambda_{i=1}^{n} f(\mu_i)$ by [67, Proposition 3.5], $f(\mu_i)$ are primary (respectively, strongly primary (by Proposition 88)) fuzzy ideals for all $i = 1, 2, ..., n$. Hence $\mu^/$ has a fuzzy primary (respectively, strongly primary) decomposition in $R^/$. ■

The following tables summarizes results proved in chapter 6 and [67]:

Let $f : R \to R^/$ be a ring anti − homomorphism. If $\mu^/$ is a fuzzy ideal of R$^/$, then $f^{-1}(\mu^/)$ is a fuzzy id

if $\mu^/$ is	then $f^{-1}(\mu^/)$ is
prime	prime
primary	primary
semiprime	semiprime
irreducible	irreducible
strongly irreducible	strongly irreducible

Similarly,

Let $f : R \to R^/$ be a surjective ring anti − homomorphism. If μ is an f − invariant fuzzy ideal of R, then, $f(\mu)$ is a fuzzy ideal of $R^/$. In addition

if μ is	then $f(\mu)$ is
Pr ime	Pr ime
Pr imary	Pr imary
Semiprime	Semiprime
Irreducible (i)	Irreducible
Strongly irreducible (ii)	Strongly irreducible
Fuzzy primary decomposible in R (iii)	Fuzzy primary decomposable in $R^{/}$

As mentioned earlier (i), (ii) and (iii) hold for regular rings (at least).

Chapter 7

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