Contributions to L-Fuzzy Soft Nearrings

By

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بسم الله الرحمن الرحيم

CERTIFICATE

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A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF THE MASTER OF PHILOSOPHY

We accept this dissertation as conforming to the required standard.

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Dedicated to that enlightened treasure who brings hope to my life !

# *Abstract:*

To solve complicated problems in economics, engineering and environment sciences, the classical methods cannot be successfully used due to various uncertainties for those problems. There are theories viz, theory of probability, theory of fuzzy sets , theory of intuitionistic fuzzy sets , theory of vague sets, theory of interval mathematics and theory of rough sets which can be considered as mathematical tools for dealing with uncertainties but all these theories have their inherent difficulties. Molodtsov [10] initiated the concept of soft set as mathematical tool for dealing with uncertainties which is free from above difficulties. Maji et al. [16] defined some operations on soft sets. Ali et al.introduced several new operations of soft sets. The theory has also seen a wide-ranging applications in the mean of algebraic structures such as groups, semirings, rings, BCK/BCI-algebras , nearrings and soft substructures and union soft substructures.

The fundamental concept of fuzzy set was introduced by Zadeh [27] in 1965. Rosenfeld inspired the fuzzification of algebraic structures and introduced the notions of fuzzy subgroups. Das [7] characterized fuzzy subgroups by their level subgroups. W. Liu [8] studied fuzzy ideals of rings.Abou-Zaid introduced the notion of a fuzzy subnearring and studied fuzzy ideals of a nearring.The concept of fuzzy subnearring and fuzzy ideal was discussed further by many researchers. Davvaz for a complete lattice L, introduced interval-valued L-fuzzy ideal (prime ideal) of a nearring which is an extended notion of a fuzzy ideal (prime ideal) of a nearring.

This dissertation is devoted to the discussion of algebraic structures of L-fuzzy soft sets and basic concepts of lattices and L-fuzzy sets. This dissertation consists of three chapters. Chapter one consists of some basic definitions and examples of Nearrings and basic concept of soft sets, fuzzy sets and L-fuzzy soft sets. In Chapter two, We initiated the study of L-fuzzy soft ideals along with L-fuzzy soft nearrings. In Chapter three, We introduced L-fuzzy soft prime and semiprime ideals . Moreover, We have done the characterization of nearrings by the properties of their L-fuzzy soft ideals. We have characterized those nearrings for which each L-fuzzy soft ideal is Prime and also those nearrings for which each L-fuzzy soft ideal is idempotent.

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All praises to Almighty Allah, the most beneficent and most merciful. I'm highly obliged to Almighty Allah for His countless blessings and guidance.

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Rani Sumaira Kanwal

Contents

Chapter 1

Fundamental Concepts

The aim of this chapter is to present a brief summary of basic definitions and preliminary results which will be of value for later pursuits. First we start with the basic definitions and examples of nearrings. For terms and notations which are not defined here, we refer to [13].

1.1 Nearrings: Basic Definitions and Examples

Definition 1 *A nearring is a non-empty set N together with two binary operations "+" and "·" such that*

(i) $(N,+)$ *is a group (not necessarily abelian).*

(ii) $(N,\cdot)$ *is a semigroup.*

(iii) *For all* $n_1, n_2, n_3 \in N : n_1(n_2+n_3) = n_1n_2 + n_1n_3$ *(left distributive law).*

Remark 2 *In view of (iii), one speaks more precisely of a "left nearring", postulating* $(iii)'$ *for all* $n_1, n_2, n_3 \in N : (n_1+n_2)n_3 = n_1n_3 + n_2n_3$ *instead of (iii), one gets " right nearring ". The theory runs completely parallel in both cases. In this dissertation we will use left nearring. By N we shall mean a left nearring unless explicitly mentioned.*

Example 3 *Let $(G,+)$ be a group (not necessarily abelian). Then the set* $M(G) = \{f \mid f : G \to G\} = G^G$ *of all mappings (functions) from G into G is a right nearring under pointwise addition and composition of functions if we write image of $x \in G$ under $f \in M(G)$ as $f(x)$ and is a left nearring if we write image of $x \in G$ under $f \in M(G)$ as $(x)f$.*

Example 4 *Let $D(\mathbb{R})$ be the set of all differentiable functions on $\mathbb{R}$ (the set of real numbers). Define $(f+g)(x) = f(x) + g(x)$ and $(f \circ g)(x) = f(g(x))$. Then*

(i) $(D(\mathbb{R}), +)$ *is an abelian group.*

(ii) $(D(\mathbb{R}), \circ)$ *is a semigroup.*

(iii) $h \circ (f+g) \neq h \circ f + h \circ g$ *but* $(f+g) \circ h = f \circ h + g \circ h$.

Thus $(D(\mathbb{R}), +, \circ)$ is a right nearring but not a left nearring and hence not a ring.

Example 5 *Let $(G, +)$ be a group (not necessarily abelian). Then*

(i) $(G, +, \cdot)$ with $g_1 \cdot g_2 = 0$ for all $g_1, g_2 \in G$ is both a left and a right nearring.

(ii) $(G, +, \circledast)$ with $g_{1\circledast}g_2 = g_1$ for all $g_1, g_2 \in G$ is a right nearring but not a left nearring.

Note that $n0 = 0$ and $n\left(-n^{/}\right) = -nn^{/}$ but in general $0n \neq 0$ and $n\left(-n^{/}\right) = -nn^{/}$ for $n, n^{/} \in N$.

Example 6 *Every ring is a nearring.*

Definition 7 *$N_\circ = \{n \in N : 0n = 0\}$ is called the Zero Symmetric part of N.*

$N_c = \{n \in N : 0n = n\}$ is called the constant part of N.

$N_\circ$ and N_c both are nearrings.

Definition 8 *A nearring N is called Zero Symmetric (Constant) nearring if*

$N = N_\circ$ $(N = N_c)$.

Definition 9 *An element $d \in N$ is called distributive if for all $n_1, n_2 \in N$, $d(n_1 + n_2) = dn_1 + dn_2$.*

Definition 10 *Let $N_d = \{d \in N : d$ is distributive$\}$. A nearring N is called distributively generated (d.g.) if there is a semigroup D of $(N_d, \cdot)$ which generates $(N, +)$.*

Definition 11 *Let N be a nearring. If $(N, +)$ is an abelian group, we call N an abelian nearring.*

Definition 12 *If $(N, \cdot)$ is commutative, we call N a commutative nearring. If $N = N_d$, N is said to be distributive.*

Definition 13 *If all non-zero elements of N are left (right) cancellable, we say that N fulfills the left (right) cancellation laws. N is integral if N has no non-zero divisors of zero. If $N^{\star} = N \backslash \{0\}$ is a group, N is called a near-field.*

1.2 Ideals in Nearrings

Definition 14 *Let N be a nearring. A subgroup $(M, +)$ of $(N, +)$ is called a subnearring of N if $m_1 m_2 \in M$ for all $m_1, m_2 \in M$.*

Definition 15 *Let N be a nearring. A normal subgroup I of $(N, +)$ is called an ideal of N if*

(i) $NI \subseteq I$, that is $ni \in I$ for all $i \in I$ and $n \in N$.

(ii) For all $n, n_2 \in N$ and $i \in I : (n_1 + i)\, n_2 - n_1 n_2 \in I$.

Normal subgroup I of $(N, +)$ with (i) is called left ideal of N while normal subgroup I of $(N, +)$ with (ii) is called right ideal of N.

Proposition 16 *The intersection of any family of (left, right) ideals of a nearring N is a (left, right) ideal of N.*

Theorem 17 *Let $\{I_k\}$ be a family of ideals of a nearring N. Then the following sets are equivalent:*

(i) The set of all finite sums of elements of $I_k^{'}s$;

(ii) The set of all finite sums of elements of different $I_k^{'}s$;

(iii) The sum of normal subgroups $(I_k, +)$;

(iv) The subgroup of $(N, +)$ generated by $\underset{k \in K}{\cup} I_k$;

(v) The normal subgroup of $(N, +)$ generated by $\underset{k \in K}{\cup} I_k$;

(vi) The ideal of $(N, +)$ generated by $\underset{k \in K}{\cup} I_k$.

Definition 18 *The set $(i) - (vi)$ above is called the sum of the ideals $I_k (k \in K)$ and is denoted by $\sum\limits_{k \in K} I_k$.*

The sum of ideals of N is again an ideal of N.

1.3 Prime and Semiprime Ideals

If A and B are non-empty subsets of a nearring N, then the product of A and B, AB is defined by $AB = \{ab : a \in A \text{ and } b \in B\}$.

Clearly, If A, B, C are non-empty subsets of a nearring N then $A(BC) = (AB)C$. Note that AB has no particular structure in general. Even if A, B are ideals then AB is not even subsemigroup of $(N, +)$. If A is a non-empty subset of a nearring N then the smallest ideal of N containing A is denoted by $< A >$ and is called the ideal generated by A. If $A = \{n\}$, then the ideal generated by A is denoted by $< n >$ instead of $< \{n\} >$.

Definition 19 *An ideal P of a nearring N is called Prime if for all ideals I, J of N: $IJ \subseteq P$ implies that either $I \subseteq P$ or $J \subseteq P$.*

Proposition 20 *Let P be an ideal of a nearring N. Then the following are equivalent:*

(a) P is a prime ideal.

(b) For all ideals I, J of N: $< IJ > \subseteq P \Longrightarrow I \subseteq P$ or $J \subseteq P$.

(c) For all i, j in N, $i \notin P$ and $j \notin P \Longrightarrow < i >< j > \nsubseteq P$.

(d) For all ideals I, J of N such that $I \supset P$ and $J \supset P \Rightarrow IJ \nsubseteq P$.

(e) For all ideals I, J of N such that $I \nsubseteq P$ and $J \nsubseteq P \Longrightarrow IJ \nsubseteq P$.

Proposition 21 *Let $\{P_\alpha\}$ be a family of prime ideals of a nearring N, totally ordered by inclusion. Then $\cap P_\alpha = P$ is a prime ideal of N.*

Definition 22 *An ideal S of a nearring N is called semiprime if for all ideals I of N, $I^2 \subseteq S \Longrightarrow I \subseteq S$.*

Each prime ideal of a nearring N is a Semiprime ideal of N.

Proposition 23 *For an ideal S of a nearring N, the following conditions are equivalent:*

(a) S is Semiprime.

(b) For all ideals I of N, $< I^2 > \subseteq S \Longrightarrow I \subseteq S$.

(c) For all $n \in N$, $< n >^2 \subseteq S \Longrightarrow n \in S$.

(d) For all ideals I of N, $I \supset S \Longrightarrow I^2 \supset S$.

(e) For all ideals I of N, $I \nsubseteq S \Longrightarrow I^2 \nsubseteq S$.

Definition 24 *An ideal I of a nearring N is called Completely Prime if $ab \in I \Longrightarrow a \in I$ or $b \in I$.*

Definition 25 *An ideal J of a nearring N is called irreducible (resp. strongly irreducible) if $A \cap B = J \Longrightarrow A = J$ or $B = J$ (resp. $A \cap B \subseteq J \Longrightarrow A \subseteq J$ or $B \subseteq J$) for all ideals A, B of N.*

Definition 26 *If A and B are ideals of a nearring N then generally $AB \nsubseteq A \cap B$. However if N is Zero Symmetric, then $AB \subseteq A \cap B$.*

Proposition 27 *For a zero symmetric nearring N, every prime ideal is strongly irreducible.*

Proof. *Let P be a prime ideal of N and $A \cap B \subseteq P$ for ideals A and B of N. As N is zerosymmetric, we have $AB \subseteq A \cap B \subseteq P$. Since P is a prime ideal, so either $A \subseteq P$ or $B \subseteq P$. Thus P is strongly irreducible.* ■

Proposition 28 *Every strongly irreducible ideal is irreducible.*

1.4 Fully Idempotent Nearrings

A ring N is fully idempotent if each ideal I of N is idempotent, that is if $I = I^2$ [1]. J. Ahsan and G. Mason examined the nearring analogue of fully idempotent rings. In this section, N denotes the zerosymmeric nearring. All the results given in this section are from [1].

Definition 29 *A nearring N is fully idempotent if each ideal I of N is the ideal generated by I^2 that is if $I =< I^2 >$.*

Proposition 30 *The follwing assertions for a nearring N are equivalent:*

(1) *N is fully idempotent.*

(2) *For each pair of ideals I, J of N, $I \cap J =< IJ >$.*

(3) *The set of ideals L_N of N (ordered by inclusion) forms a lattice $(L_N, \vee, \wedge)$ with $I \vee J = I + J$ and $I \wedge J =< IJ >$ for each pair of ideals I, J of N.*

Proof. (1) $\Longrightarrow$ (2)

For each pair of ideals I, J of N, we always have $IJ \subseteq I \cap J$. Hence $< IJ > \subseteq I \cap J$. For the reverse inclusion, let $a \in I \cap J$ and let $< a >$ be the (two-sided) ideal of N generated by a. Then $a \in < a > = << a >< a >> \subseteq < IJ >$. Thus $I \cap J \subseteq < IJ >$. Hence $I \cap J = < IJ >$.

(2) $\Longrightarrow$ (3)

The set of ideals of a nearring N ordered by inclusion forms a lattice under the sum and intersection of ideals [13]. Thus for each pair of ideals I, J of N, $I \vee J = I + J$ and by assumption, $I \wedge J = I \cap J = < IJ >$.

(3) $\Longrightarrow$ (2)

For each pair of ideals I, J of N, $I \cap J = < IJ >$.

(2) $\Longrightarrow$ (1)

Taking $I = J$ in the hypothesis, we have $I = < I^2 >$ for each ideal I of N. Hence N is fully idempotent. ■

Lemma 31 *If I is an ideal of a nearring N and $a \notin I$, then there exists an irreducible ideal K of N such that $I \subseteq K$ and $a \notin K$.*

Proof. Let $\mathcal{A} = \{L : L$ is an ideal of N, $I \subseteq L$ and $a \notin L\}$. Then $\mathcal{A}$ is non-empty because $I \in \mathcal{A}$. $\mathcal{A}$ is a partially ordered set by inclusion. If $\{L_\alpha\}$ is a chain in $\mathcal{A}$, then $\cup L_\alpha$ is an ideal of N containing I but not containing a. Hence by Zorn's Lemma, $\mathcal{A}$ has a maximal element. Let K be such one. Let $K = B \cap C$, where B and C are ideals of N. If both B and C properly contain I, then by maximality of K they both contain a. But $a \in B \cap C = K$, a contradiction. Hence K is an irreducible ideal. ■

Corollary 32 *Every proper ideal of a nearring N is contained in a proper irreducible ideal of N.*

Proposition 33 *Let N be a fully idempotent nearring and let P be an ideal of N. Then the following assertions are equivalent:*

(1) *P is irreducible.*

(2) *P is strongly irreducible.*

(3) *P is prime.*

Proof. (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) is clear. It suffices to show that (1) $\Longrightarrow$ (3). Suppose $IJ \subseteq P$ for ideals I, J of N. Since N is fully idempotent, $I \cap J =< IJ >$. On the other hand, $IJ \subseteq P$, implies that $(I \cap J) + P = P$. Since N is fully idempotent, so the ideal lattice of N is distributive. Hence $P = (I \cap J) + P = (I + P) \cap (J + P)$. Since P is irreducible, we have $I + P = P$ or $J + P = P$. This implies that $I \subseteq P$ or $J \subseteq P$. Hence P is a prime ideal. ■

Theorem 34 *The following are equivalent for a nearring N :*

(1) *N is fully idempotent.*

(2) *Every proper ideal of N is the intersection of all prime ideals of N containing it.*

Proof. (1) $\Longrightarrow$ (2)

First note that if N is fully idempotent then every ideal is contained in some prime ideal. Let $\{P_\alpha\}$ be the family of prime ideals of N containing I, so $I \subseteq \cap P_\alpha$. For reverse inclusion let $a \notin I$. Then there exists a prime ideal P with $I \subseteq P$ and $a \notin P$. Hence $\cap P_\alpha \subseteq I$. Thus $I = \cap P_\alpha$.

(2) $\Longrightarrow$ (1)

Let I be an ideal of N. If $< I^2 >= N$. Then $< I^2 >= I$. If $< I^2 > \neq N$, then $I^2 \subseteq < I^2 > = \cap_\alpha P_\alpha \subseteq P_\alpha$, so $I \subseteq P_\alpha$ for all α. Thus $I \subseteq \cap P_\alpha =< I^2 >$. Since $< I^2 > \subseteq I$, we are done. ■

Corollary 35 *N is fully idempotent if and only if each ideal of N is semiprime [13].*

1.5 Soft Sets

Definition 36 *[4, **?**] A soft set f_A of a set N over U is a function defined by $f_A : N \longrightarrow P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$, where $A \subseteq N$. The set of all soft sets of a set N over U is denoted by $S(U)$.*

Definition 37 *[4] Let $f_A, f_B \in S(U)$. Then f_A is called a soft subset of f_B, denoted by $f_A \tilde{\subseteq} f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in N$.*

Definition 38 *[4] Let $f_A, f_B \in S(U)$. Then the union of f_A and f_B, denoted by $f_A \tilde{\cup} f_B$, is defined as $f_A \tilde{\cup} f_B = f_{A \cup B}$, where $(f_A \cup f_B)(x) = f_A(x) \cup f_B(x)$ for all $x \in N$.*

Definition 39 *[4] Let* $f_A, f_B \in S(U)$*. Then the intersection of* f_A *and* f_B*, denoted by* $f_A \tilde{\cap} f_B$*, is defined as* $f_A \tilde{\cap} f_B = f_{A \cap B}$*, where* $(f_A \cap f_B)(x) = f_A(x) \cap f_B(x)$ *for all* $x \in N$*.*

Definition 40 *[4] Let* $f_A, f_B \in S(U)$*. Then* $f_A \wedge f_B$*, is defined as* $f_{A \wedge B}(x,\ y) = f_A(x) \wedge f_B(y)$ *for all* $(x, y) \in N \times N$*.*

1.6 Fuzzy set

Let X be a non-empty set. By a fuzzy subset f of X, we mean a membership function $f : X \to [0, 1]$ which associates with each element in X a real number from the unit closed interval $[0, 1]$, the value $f(x)$ represents the "grade of membership" of x in f.

A fuzzy subset $f : X \to [0, 1]$ is called non-empty if f is not a constant map which assumes the value 0. For any fuzzy subsets f, g of X, $f \leqslant g$ means that for all $x \in X$, $f(x) \leqslant g(x)$. The fuzzy subsets $f \wedge g$ and $f \vee g$ will mean the following fuzzy subsets of X:

$$(f \wedge g)(x) = f(x) \wedge g(x)$$

$$(f \vee g)(x) = f(x) \vee g(x)$$

for all $x \in X$.

More generally, if $\{(f)_i : i \in I\}$ is a family of fuzzy subsets of X, then $\underset{i \in I}{\wedge} f_i$ and $\underset{i \in I}{\vee} f_i$ are defined by

$$\left(\underset{i \in I}{\wedge} f_i\right)(x) = \underset{i \in I}{\wedge}(f_i(x));$$

$$\left(\underset{i \in I}{\vee} f_i\right)(x) = \underset{i \in I}{\vee}(f_i(x))$$

respectively.

1.7 Fuzzy Ideals of Nearrings

Let f and g be two fuzzy subsets of a nearring N.

Then the product fog is defined by

$$(fog)(x) = \begin{cases} \underset{x=yz}{\vee} (f(y) \wedge g(z)) & \text{if } x \text{ is expressible as } x = yz \\ 0 & \text{otherwise} \end{cases}$$

A fuzzy subset f of a nearring N is called a fuzzy subnearring of N if

(1) $f(x-y) \geqslant f(x) \wedge f(y)$

(2) $f(xy) \geqslant f(x) \wedge f(y)$, for all $x, y \in N$.

A fuzzy subset f of a nearring N is called a fuzzy ideal of N if f is a subnearrng of N and

(3) $f(x) = f(y + x - y)$

(4) $f(xy) \geqslant f(y)$

(5) $f((x+i)y - xy) \geqslant f(i)$, for any $x, y, i \in N$.

f is a fuzzy left ideal of N if it satisfies $(1), (3)$ and (4); f is a fuzzy right ideal of N if it satisfies $(1), (2), (3)$ and (5).

Example 41 *Let $N = \{a, b, c, d\}$ be a nearring with the following two binary operations*:

$+$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

$\bullet$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	a	b

Define a fuzzy subset $f : N \to [0,1]$ by $f(c) = f(d) < f(b) < f(a)$. Then f is a fuzzy ideal of N.

1.8 L-fuzzy set

A partially ordered set (poset) $(L, \leq)$ is called

1) a lattice, if $a \vee b \in L$, $a \wedge b \in L$ for any $a, b \in L$.

2) a complete lattice, if $\vee N \in L$, $\wedge N \in L$ for any $N \subseteq L$.

3) a lattice is called distributive, if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$; $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for any $a, b, c \in L$.

Definition 42 *Let L be a lattice with top element 1_L and bottom element 0_L and let $a, b \in L$.*

Then b is called a complement of a, if $a \vee b = 1_L$ and $a \wedge b = 0_L$. If $a \in L$ has a complement, then it is unique. It is denoted by a'.

Definition 43 *A lattice L is called a Boolean lattice, if*

(i) L is distributive,

(ii) L has 0_L and 1_L,

(iii) each $a \in L$ has the complement $a' \in L$.

Definition 44 *[7] Let U be a set and L be a complete distributive lattice with 1_L and 0_L. An L-fuzzy set A in U is a map $A : U \to L$. We denote the family of all L-fuzzy sets in U by L^U. For $A, B \in L^X$, $A \subseteq B$ if $A(x) \leq B(x)$ for every $x \in U$. For L-fuzzy sets A and B, new L-fuzzy sets can be constructed as follows: $(A \cap B)(x) = A(x) \cap B(x)$; $(A \cup B)(x) = A(x) \cup B(x)$ for all $x \in U$.*

Chapter 2

L-fuzzy Soft Ideals

In this chapter we define L-fuzzy soft subnearring, L-fuzzy soft left (right) ideal, L-fuzzy soft N-subgroup over a universe U. We study some of their properties.

2.1 L-fuzzy Soft sets

In this section we define sum and product of L-fuzzy soft subsets of a nearring over a universe U and study some properties of these operations.

An L- fuzzy set A in a nonempty set X is a function $A : X \to L$, where L is a complete distributive lattice with 1 and 0. We denote by L^X the set of all L-fuzzy sets in X.

Let $A, B \in L^X$. Then their union and intersection are L-fuzzy sets in X, defined as

$(A \cup B)(x) = A(x) \vee B(x)$ and $(A \cap B)(x) = A(x) \wedge B(x)$ for all $x \in X$.

$A \subseteq B$ if and only if $A(x) \leq B(x)$ for all $x \in X$.

The L-fuzzy sets $\widetilde{0}$ and $\widetilde{1}$ of X are defined as $\widetilde{0}(x) = 0$ and $\widetilde{1}\ (x) = 1$ for all $x \in X$. Obviously $\widetilde{0} \subseteq A \subseteq \widetilde{1}$ for all $A \in L^X$.

Definition 45 *[12] A pair (F, E) is called a soft set $($over $U)$ if F is a mapping of E into the power set of U, that is*

$F : E \longrightarrow P(U)$. In other words, the soft set is a parametrized family of subsets of the set U.

Definition 46 *[8] Let E be a set of parameters, U be an initial universe, L be a complete* *dstributive bounded lattice and* $A \subseteq E$. *An L-fuzzy soft set* f_A *over* U *is a mapping defined by* $f_A : E \longrightarrow L(U)$, *such that* $f_A(x) = \widetilde{0}$ *if* $x \notin A$.

The following operations on L-fuzzy soft sets are defined as

1) Let f_A and g_B be two L-fuzzy soft sets over U. Then f_A is contained in g_B denoted by $f_A \ \widetilde{\subseteq}\ g_B$ if $f_A(e) \subseteq g_B(e)$ for all $e \in E$, that is $(f_A(e))(u) \leq (g_B(e))(u)$ for all $u \in U$.

Two L-fuzzy soft sets f_A and g_B over U are said to be equal, denoted by $f_A \widetilde{=} g_B$ if $f_A \ \widetilde{\subseteq}\ g_B$ and $g_B \ \widetilde{\subseteq}\ f_A$.

2) Let f_A and g_B be two L-fuzzy soft sets over U. Then their union $f_A \widetilde{\cup} g_B \widetilde{=} h_{A\cup B}$, where $h_{A\cup B}(e) = f_A(e) \cup g_B(e)$ for all $e \in E$.

3) Let f_A and g_B be two L-fuzzy soft sets over U. Then their intersection $f_A \widetilde{\cap} g_B \widetilde{=} h_{A\cap B}$, where $h_{A\cap B}(e) = f_A(e) \cap g_B(e)$ for all $e \in E$.

Proposition 47 *Let* $A, B, C \subseteq E$ *and* f_A, g_B, h_C *be three L-fuzzy soft sets over* U. *Then*

(1) $f_A \widetilde{\cup} f_A \widetilde{=} f_A$

(2) $f_A \widetilde{\cup} g_B \widetilde{=} g_B \widetilde{\cup} f_A$

(3) $\left(f_A \widetilde{\cup} g_B\right) \widetilde{\cup} h_C \widetilde{=} f_A \widetilde{\cup} \left(g_B \widetilde{\cup} h_C\right)$.

Proposition 48 *Let* $A, B, C \subseteq E$ *and let* f_A, g_B, h_C *be three L-fuzzy soft sets over* X. *Then*

(1) $f_A \widetilde{\cap} f_A \widetilde{=} f_A$

(2) $f_A \widetilde{\cap} g_B \widetilde{=} g_B \widetilde{\cap} f_A$

(3) $\left(f_A \widetilde{\cap} g_B\right) \widetilde{\cap} h_C \widetilde{=} f_A \widetilde{\cap} \left(g_B \widetilde{\cap} h_C\right)$.

In the next definition $E = N$, a nearring. We call an L-fuzzy soft set over U as an L-fuzzy soft set of N over U.

Definition 49 *Let* f_A *and* g_B *be two L-fuzzy soft sets of a nearring* N *over the common universe* U. *Then the soft product* $f_A \odot g_B$ *is an L-fuzzy soft set of* N *over* U *defined by*

$$(f_A \odot g_B)(x) = \begin{cases} \bigcup_{x=yz} f_A(y) \cap g_B(z) & \text{if } \exists \ y, z \in N \text{ such that } x = yz \\ \widetilde{0} & \text{otherwise} \end{cases} \quad \forall x \in N.$$

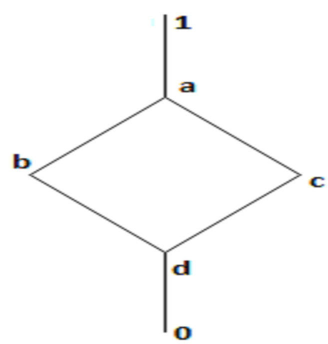
We next show that if f_A, g_B are L-fuzzy soft sets of N over U, then $f_A \odot g_B \neq g_B \odot f_A$.

Example 50 *Let $N = \{0, x, y, z\}$ be a nearring with the binary operations as defined below:*

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

$\bullet$	0	x	y	z
0	0	0	0	0
x	0	x	0	x
y	0	0	0	0
z	0	z	0	z

Consider a complete bounded distributive lattice $L = \{1, a, b, c, d, 0\}$. Let $U = \{p, q\}$ and $A = B = \{x, y, z\}$.



Let f_A and g_B be two L-fuzzy soft sets of N over U as follows:

	p	q
$f_A(x)$	b	d
$f_A(y)$	a	c
$f_A(z)$	a	b

	p	q
$g_B(x)$	1	0
$g_B(y)$	1	c
$g_B(z)$	0	b

Now, for $x \in N$,

$$\begin{aligned}
(f_A \odot g_B)(x) &= \underset{x=yz}{\cup} \{f_A(y) \cap g_B(z)\} \\
&= \underset{x=yz}{\cup} \{f_A(x) \cap g_B(x), f_A(x) \cap g_B(z)\} = \{(b\ d) \cap (1\ 0)\} \cup \{(b\ d) \cap (0\ b)\} \\
&= (b\ 0) \cup (0\ d) = (b\ d).
\end{aligned}$$

And,

$$\begin{aligned}
(g_B \odot f_A)(x) &= \underset{x=yz}{\cup} g_B(y) \cap f_A(z) \\
&= \underset{x=yz}{\cup} \{g_B(x) \cap f_A(x), g_B(x) \cap f_A(z)\} = \{(1\ 0) \cap (b\ d)\} \cup \{(1\ 0) \cap (a\ b)\} \\
&= (b\ 0) \cup (a\ 0) = (a\ 0).
\end{aligned}$$

Hence,

$$f_A \odot g_B \neq g_B \odot f_A.$$

Proposition 51 *Let $f_A, g_B, h_C \in S(U)$, where $S(U)$ is the collection of all L-fuzzy soft sets of a nearring N over U. Then*

(i) $(f_A \odot g_B) \odot h_C \widetilde{=} f_A \odot (g_B \odot h_C)$.

(ii) $f_A \widetilde{\subseteq} g_B \Rightarrow (f_A \odot h_C)\ \widetilde{\subseteq} (g_B \odot h_C)$ *and* $(h_C \odot f_A)\ \widetilde{\subseteq} (h_C \odot g_B)$.

(iii) $f_A \odot \left(g_B \widetilde{\cup} h_C\right) \widetilde{=} (f_A \odot g_B) \widetilde{\cup} (f_A \odot h_C)$ *and* $\left(f_A\ \widetilde{\cup}\ g_B\right) \odot h_C \widetilde{=} (f_A \odot h_C) \widetilde{\cup} (g_B \odot h_C)$.

(iv) $f_A \odot \left(g_B \widetilde{\cap} h_C\right) \widetilde{\subseteq} (f_A \odot g_B) \widetilde{\cap} (f_A \odot h_C)$ *and* $\left(f_A \widetilde{\cap} g_B\right) \odot h_C \widetilde{\subseteq} (f_A \odot h_C) \widetilde{\cap} (g_B \odot h_C))$.

Proof. (i) Let $x \in N$. Then

$$\begin{aligned}
((f_A \odot g_B) \odot h_C)(x) &= \underset{x=yz}{\cup} \{(f_A \odot g_B)(y) \cap h_C(z)\} \\
&= \underset{x=yz}{\cup} \{\{\underset{y=st}{\cup} (g_B(s) \cap f_A(t))\} \cap h_C(z)\} = \underset{x=yz}{\cup} \underset{y=st}{\cup} \{(g_B(s) \cap f_A(t)) \cap h_C(z)\} \\
&= \underset{x=(st)z}{\cup} \{(f_A(s) \cap g_B(t)) \cap h_C(z)\}
\end{aligned}$$

$$\begin{aligned}
&= \underset{x=s(tz)}{\cup} \{f_A(s) \cap (g_B(t) \cap h_C(z))\} \subseteq \underset{x=sp}{\cup} \{f_A(s) \cap (g_B \odot h_C)(p)\} \\
&= \underset{x=sp}{\cup} \{(f_A \odot (g_B \odot h_C))(x).
\end{aligned}$$

This implies .

$$(f_A \odot g_B) \odot h_C \widetilde{\subseteq} f_A \odot (g_B \odot h_C)$$

Similarly, we can show that

$$f_A \odot (g_B \odot h_C) \widetilde{\subseteq} (f_A \odot g_B) \odot h_C.$$

Hence ,

$$(f_A \odot g_B) \odot h_C = f_A \odot (g_B \odot h_C)$$

(ii) As

$$f_A \widetilde{\subseteq} g_B \Rightarrow f_A (y) \widetilde{\subseteq} g_B (y)$$

for all $y \in N$.

Let $x \in N$. If $x \neq yz$ for all $y, z \in N$ then

$$(f_A \odot h_C)(x) = 0 = (g_B \odot h_C)(x).$$

Otherwise

$$(f_A \odot g_B)(x) = \underset{x=yz}{\cup} \{f_A (y) \cap h_C (z)\} \subseteq \underset{x=yz}{\cup} \{g_B (y) \cap h_C (z)\} = (g_B \odot h_C)(x).$$

Hence,

$$f_A \widetilde{\subseteq} g_B \Rightarrow (f_A \odot h_C) \ \widetilde{\subseteq} (g_B \odot h_C).$$

(iii) Let $x \in N$. If x is not expressible as $x = yz$ for all $y, z \in N$, then

$$(f_A \odot (g_B \widetilde{\cup} h_C)(x) = \widetilde{0} = (f_A \odot g_B)(x) \cup (f_A \odot h_C)(x).$$

Otherwise

$$
\begin{aligned}
(f_A \odot (g_B \widetilde{\cup} h_C)(x) &= \underset{x=yz}{\cup} \{f_A(y) \cap \left(g_B \widetilde{\cup} h_C\right)(z) \\
&= \underset{x=yz}{\cup} \{f_A(y) \cap (g_B(z) \cup h_C(z)\} = \underset{x=yz}{\cup} \{(f_A(y) \cap (g_B(z)) \cup (f_A(y) \cap (h_C(z) \\
&= \{\underset{x=yz}{\cup} (f_A(y) \cap (g_B(z))\} \cup \{\underset{x=yz}{\cup} (f_A(y) \cap (h_C(z))\} \\
&= (f_A \odot g_B)(x) \cup (f_A \odot h_C)(x).
\end{aligned}
$$

This implies that

$$
f_A \odot \left(g_B \ \widetilde{\cup} \ h_C\right) = (f_A \odot g_B) \widetilde{\cup} (f_A \odot h_C).
$$

Hence,

$$
f_A \odot \left(g_B \widetilde{\cup} h_C\right) = (f_A \odot g_B) \widetilde{\cup} (f_A \odot h_C).
$$

(iv) Let $x \in N$. If x is not expressible as $x = yz$ for all $y, z \in N$, then

$$
(f_A \odot (g_B \widetilde{\cap} h_C)(x) = \widetilde{0} = (f_A \odot g_B)(x) \widetilde{\cap} (f_A \odot h_C)(x).
$$

Otherwise

$$
\begin{aligned}
(f_A \odot (g_B \widetilde{\cap} h_C)(x) &= \underset{x=yz}{\cup} \{f_A(y) \cap \left(g_B \widetilde{\cap} h_C\right)(z) \\
&= \underset{x=yz}{\cup} \{f_A(y) \cap (g_B(z) \cap h_C(z)\} = \underset{x=yz}{\cup} \{(f_A(y) \cap (g_B(z)) \cap (f_A(y) \cap (h_C(z)\} \\
&\subseteq \{\underset{x=yz}{\cup} (f_A(y) \cap (g_B(z))\} \cap \{\underset{x=yz}{\cup} (f_A(y) \cap (h_C(z))\} \\
&= (f_A \odot g_B)(x) \cap (f_A \odot h_C)(x).
\end{aligned}
$$

This implies that

$$
f_A \odot \left(g_B \widetilde{\cap} h_C\right) \ \widetilde{\subseteq} (f_A \odot g_B) \widetilde{\cap} (f_A \odot h_C)
$$

Similarly, we can prove that

$$
\left(f_A \widetilde{\cap} g_B\right) \odot h_C \widetilde{\subseteq} (f_A \odot h_C) \widetilde{\cap} (g_B \odot h_C)
$$

■

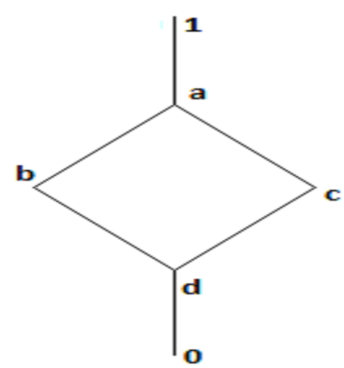
Equality does not hold in (iv) which is shown in the following example.

Example 52 *Let $N = \{0, x, y, z\}$ be the nearring with binary operations as defined below:*

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

$\bullet$	0	x	y	z
0	0	0	0	0
x	0	x	0	x
y	0	0	0	0
z	0	z	0	z

Consider a complete bounded distributive lattice $L = \{1, a, b, c, d, 0\}$ and $U = \{p, q\}$, $A = B = C = \{x, y, z\}$



Define f_A, g_B and h_C be the L-fuzzy soft sets of N over U as follows:

	p	q
$f_A(x)$	a	d
$f_A(y)$	a	b
$f_A(z)$	c	a

	p	q
$g_B(x)$	b	0
$g_B(y)$	1	b
$g_B(z)$	c	b

	p	q
$h_C(x)$	0	1
$h_C(y)$	1	0
$h_C(z)$	0	d

Now

$$\begin{aligned}(f_A \odot (g_B \widetilde{\cap} h_C)(z) &= \underset{z=xy}{\cup}\{f_A(x) \cap \left(g_B \widetilde{\cap} h_C\right)(y) = \underset{z=xy}{\cup}\{f_A(x) \cap ((g_B(y) \cap h_C(y))\} \\ &= \underset{z=xy}{\cup}\{f_A(z) \cap ((g_B(x) \cap h_C(x)), f_A(z) \cap ((g_B(z) \cap h_C(z))\} \\ &= ((c\ \ a) \cap ((b\ 0) \cap (0\ 1))) \cup ((c\ a) \cap ((c\ b) \cap (0\ d))) \\ &= ((c\ a) \cap (0\ 0)) \cup ((c\ a) \cap (0\ d)) = (0\ 0) \cup (0\ d) = (0\ d)\end{aligned}$$

And

$$\begin{aligned}(f_A \odot g_B)(z) \widetilde{\cap} (f_A \odot h_C)(z) &= ((f_A(z) \cap g_B(x)) \cup (f_A(z) \cap g_B(z))) \cap ((f_A(z) \cap h_C(x)) \cup (f_A(z) \cap h_C(z) \\ &= (((c\ a) \cap b\ 0) \cup ((c\ a) \cap (c\ b))) \cap (((c\ a) \cap (0\ 1)) \cup ((c\ a) \cap (0\ d))) \\ &= ((0\ 0) \cup (c\ b)) \cap ((0\ a) \cup (0\ d)) = (c\ b) \cap (0\ a) = (0\ b).\end{aligned}$$

Hence .

$$f_A \odot \left(g_B \widetilde{\cap} h_C\right) \widetilde{\subseteq} (f_A \odot g_B) \widetilde{\cap} (f_A \odot h_C)$$

Definition 53 *Let f_A and g_B be soft sets of a nearring N over the common universe U. Then the soft sum $f_A \oplus g_B$ is defined by*

$(f_A \oplus g_B)(x) = \underset{x=y+z}{\cup} f_A(y) \cap g_B(z) \qquad \forall x \in N.$

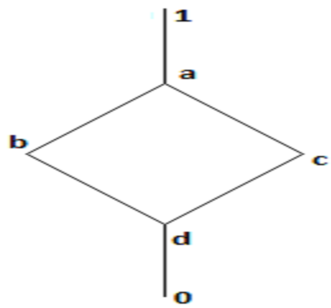
Next we show that $f_A \oplus g_B \neq g_B \oplus f_A$ for L-fuzzy soft sets f_A, g_B of a nearring N over U.

Example 54 *Consider $S_3 = \{1, a, b, a^2, ab, a^2b\}$ with the binary operations addition and multiplication as defined below:*

$\oplus$	1	a	a^2	b	ab	a^2b
1	1	a	a^2	b	ab	a^2b
a	a	a^2	1	ab	a^2b	b
a^2	a^2	1	a	a^2b	b	ab
b	b	a^2b	ab	1	a^2	a
ab	ab	b	a^2b	a	1	a^2
a^2b	a^2b	ab	b	a^2	a	1

$\odot$	1	a	a^2	b	ab	a^2b
1	1	1	1	1	1	1
a	1	1	1	1	1	1
a^2	1	1	1	1	1	1
b	1	1	1	1	1	1
ab	1	1	1	1	1	1
a^2b	1	1	1	1	1	1

Then $(S_3, \oplus, \odot)$ *is a left nearring. Consider the complete bounded distributive lattice* $L = \{1, a, b, c, d, 0\}$ *and* $U = \{p, q\}$, $A = B = S_3$



Define two L-fuzzy soft sets f_A *and* g_B *of* N *over* U *as follows*:

	p	q
$f_A(1)$	1	b
$f_A(a)$	c	d
$f_A(a^2)$	a	1
$f_A(b)$	d	c
$f_A(ab)$	b	1
$f_A(a^2b)$	0	a

	p	q
$g_B(1)$	1	a
$g_B(a)$	b	0
$g_B(a^2)$	d	b
$g_B(b)$	0	c
$g_B(ab)$	c	d
$g_B(a^2b)$	a	1

Then

$$\begin{aligned}
(f_A \oplus g_B)(a) &= \underset{a=x+y}{\cup}\{f_A(x) \cap g_B(y)\} \\
&= \cup\{f_A(1) \cap g_B(a), f_A(a) \cap g_B(1), f_A(a^2) \cap g_B(a^2), \\
&\quad f_A(b) \cap g_B(a^2b), f_A(ab) \cap g_B(b), f_A(a^2b) \cap g_B(ab)\} \\
&= \cup\left\{\begin{array}{c} (1\ b) \cap (b\ 0), (c\ d) \cap (1\ a), (a\ 1) \cap (d\ b), \\ (d\ c) \cap (a\ 1), (b\ 1) \cap (0\ c), (0\ a) \cap (c\ d) \end{array}\right\} \\
&= \{(b\ 0) \cup (c\ d) \cup (d\ b) \cup (d\ c) \cup (0\ c) \cup (0\ d)\} = (a\ 1)
\end{aligned}$$

Similarly,

$$
\begin{aligned}
(g_B \oplus f_A)(a) &= \underset{a=x+y}{\cup}\{g_B(x) \cap f_A(y)\} \\
&= \cup\{g_B(1) \cap f_A(a), g_B(a) \cap f_A(1), g_B(a^2) \cap f_A(a^2), g_B(b) \cap f_A(a^2 b), g_B(ab) \cap f_A(b), g_B \\
&= \cup\{(1\ a) \cap (c\ d), (b\ 0) \cap (1\ b), (c\ d) \cap (b\ 1), (d\ b) \cap (0\ a), (0\ c) \cap (a\ 1), (a\ 1) \cap (d\ c)\} \\
&= \{(c\ d) \cup (b\ 0) \cup (d\ d) \cup (0\ b) \cup (0\ c) \cup (d\ c)\} = (a\ a).
\end{aligned}
$$

Hence,

$$f_A \oplus g_B \neq g_B \oplus f_A$$

because $(a\ 1) \neq (a\ a)$.

Proposition 55 *Let* $f_A, g_B, h_C \in S(U)$. *Then*

(i) $(f_A \oplus g_B) \oplus h_C \widetilde{=} f_A \oplus (g_B \oplus h_C)$.

(ii) $f_A \widetilde{\subseteq} g_B \Rightarrow (f_A \oplus h_C) \widetilde{\subseteq} (g_B \oplus h_C)$.

(iii) $f_A \oplus (g_B \widetilde{\cup} h_C) \widetilde{=} (f_A \oplus g_B) \widetilde{\cup} (f_A \oplus h_C)$ *and* $(f_A \widetilde{\cup} g_B) \oplus h_C = (f_A \oplus h_C) \widetilde{\cup} (g_B \oplus h_C)$.

(iv) $f_A \oplus (g_B \widetilde{\cap} h_C) \widetilde{\subseteq} (f_A \oplus g_B) \widetilde{\cap} (f_A \oplus h_C)$ *and* $(f_A \widetilde{\cap} g_B) \oplus h_C \widetilde{\subseteq} (f_A \oplus h_C) \widetilde{\cap} (g_B \oplus h_C))$.

Proof. (i) Let $x \in N$. Then

$$
\begin{aligned}
((f_A \oplus g_B) \oplus h_C)(x) &= \underset{x=y+z}{\cup}\{(f_A \oplus g_B)(y) \cap h_C(z)\} \\
&= \underset{x=y+z}{\cup}\{\{\underset{y=s+t}{\cup}(g_B(s) \cap f_A(t))\} \cap h_C(z)\} \\
&= \underset{x=y+z}{\cup}\ \underset{y=s+t}{\cup}(g_B(s) \cap f_A(t))\} \cap h_C(z)\} \\
&= \underset{x=(s+t)+z}{\cup}\{(f_A(s) \cap g_B(t)) \cap h_C(z)\} \\
&\subseteq \underset{x=s+(t+z)}{\cup}\{f_A(s) \cap (g_B(t) \cap h_C(z))\} \\
&\subseteq \underset{x=s+p}{\cup}\{f_A(s) \cap (g_B \oplus h_C)(p)\} \\
&= \underset{x=s+p}{\cup}\{(f_A \oplus (g_B \oplus h_C))(x)\}.
\end{aligned}
$$

This implies

$$(f_A \oplus g_B) \oplus h_C \widetilde{\subseteq} f_A \oplus (g_B \oplus h_C).$$

Similarly, we can show that

$$f_A \oplus (g_B \oplus h_C) \widetilde{\subseteq} (f_A \oplus g_B) \oplus h_C.$$

Hence

$$(f_A \oplus g_B) \oplus h_C \widetilde{=} f_A \oplus (g_B \oplus h_C).$$

(ii) As $f_A \widetilde{\subseteq} g_B \Rightarrow f_A(y) \widetilde{\subseteq} g_B(y)$ for all $y \in N$.
Let $x \in N$. Then

$$(f_A \oplus g_B)(x) = \underset{x=y+z}{\cup} \{f_A(y) \cap h_C(z)\} \subseteq \underset{x=y+z}{\cup} \{g_B(y) \cap h_C(z)\} = (g_B \oplus h_C)(x).$$

Hence,

$$f_A \widetilde{\subseteq} g_B \Rightarrow (f_A \oplus h_C) \widetilde{\subseteq} (g_B \oplus h_C).$$

(iii) Let $x \in N$. Then

$$\begin{aligned}
(f_A \oplus (g_B \widetilde{\cup} h_C)(x) &= \underset{x=y+z}{\cup} \{f_A(y) \cap \left(g_B \widetilde{\cup} h_C\right)(z) \\
&= \underset{x=y+z}{\cup} \{f_A(y) \cap (g_B(z) \cup h_C(z)\} \\
&= \underset{x=y+z}{\cup} \{(f_A(y) \cap (g_B(z)) \cup (f_A(y) \cap (h_C(z)\} \\
&= \{\underset{x=y+z}{\cup} (f_A(y) \cap (g_B(z))\} \cup \{\underset{x=y+z}{\cup} (f_A(y) \cap (h_C(z))\} \\
&= (f_A \oplus g_B)(x) \cup (f_A \oplus h_C)(x).
\end{aligned}$$

This implies that

$$f_A \oplus \left(g_B \widetilde{\cup} h_C\right) \widetilde{=} (f_A \oplus g_B) \widetilde{\cup} (f_A \oplus h_C).$$

Similarly, we can show that

$$\left(f_A \widetilde{\cup} g_B\right) \oplus h_C = (f_A \oplus h_C) \widetilde{\cup} (g_B \oplus h_C).$$

(iv) Let $x \in N$. Then

$$
\begin{aligned}
\left(f_A \oplus \left(g_B \widetilde{\cap} h_C\right)(x)\right. &= \underset{x=yz}{\cup}\left\{f_A(y) \cap \left(g_B \widetilde{\cap} h_C\right)(z)\right) \\
&= \underset{x=yz}{\cup}\left\{f_A(y) \cap \left(g_B(z) \cap h_C(z)\right)\right\} \\
&= \underset{x=yz}{\cup}\left\{\left(f_A(y) \cap \left(g_B(z)\right) \cap \left(f_A(y) \cap \left(h_C(z)\right)\right\}\right.\right. \\
&\subseteq \left\{\underset{x=yz}{\cup}\left(f_A(y) \cap \left(g_B(z)\right)\right)\right\} \cap \left\{\underset{x=yz}{\cup}\left(f_A(y) \cap \left(h_C(z)\right)\right)\right\} \\
&= \left(f_A \oplus g_B\right)(x) \cap \left(f_A \oplus h_C\right)(x).
\end{aligned}
$$

This implies that .

$$f_A \oplus \left(g_B \widetilde{\cap} h_C\right) \widetilde{\subseteq} \left(f_A \oplus g_B\right) \widetilde{\cap} \left(f_A \oplus h_C\right).$$

Similarly, we can show that

$$\left(f_A \widetilde{\cap} g_B\right) \oplus h_C \widetilde{\subseteq} \left(f_A \oplus h_C\right) \widetilde{\cap} \left(g_B \oplus h_C\right).$$

■

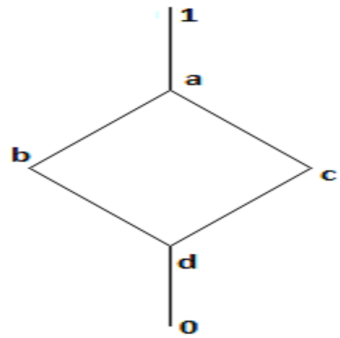
Next we show that equality does not hold in (iv).

Example 56 *Let $N = \{0, x, y, z\}$ be the nearring with the binary operations as defined below:*

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

$\bullet$	0	x	y	z
0	0	0	0	0
x	0	x	0	x
y	0	0	0	0
z	0	z	0	z

Consider a complete bounded distributive lattice $L = \{1, a, b, c, d, 0\}$, $U = \{p, q\}$ and $A = B = C = \{0, x, y, z\}$.



Define f_A, g_B *and* h_C *the L-fuzzy soft sets of* N *over* U *as follows*:

	p	q
$f_A(0)$	1	1
$f_A(x)$	b	d
$f_A(y)$	a	b
$f_A(z)$	c	a

	p	q
$g_B(0)$	1	0
$g_B(x)$	b	1
$g_B(y)$	0	b
$g_B(z)$	0	b

	p	q
$h_C(0)$	a	1
$h_C(x)$	0	1
$h_C(y)$	b	0
$h_C(z)$	0	d

Now

$$\begin{aligned}
(f_A \oplus (g_B \widetilde{\cap} h_C)(z) &= \underset{z=x+y}{\cup}\{f_A(x) \cap \left(g_B \widetilde{\cap} h_C\right)(y) = \underset{z=x+y}{\cup}\{f_A(x) \cap ((g_B(y) \cap h_C(y))\} \\
&= \underset{z=x+y}{\cup}\{f_A(0) \cap ((g_B(x) \cap h_C(x)), f_A(x) \cap ((g_B(0) \cap h_C(0)), \\
&\quad f_A(y) \cap ((g_B(z) \cap h_C(z)), f_A(z) \cap ((g_B(y) \cap h_C(y))\} \\
&= \cup \left\{ \begin{array}{c} (1\ 1) \cap ((b\ 1) \cap (0\ 1)), (b\ d) \cap ((1\ 0) \cap (a\ 1)), \\ (a\ b) \cap ((0\ b) \cap (0\ d)), (c\ a) \cap ((0\ b) \cap (b\ 0)) \end{array} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \cup \left\{ \begin{array}{c} ((1\ 1) \cap (0\ 1)), ((b\ d) \cap (b\ 0)), \\ ((a\ b) \cap (0\ b)), ((c\ a) \cap (0\ 0)) \end{array} \right\} \\
&= (0\ 1) \cup (b\ 0) \cup (0\ b) \cup (0\ 0) = (b\ 1).
\end{aligned}$$

Now,

$$
\begin{aligned}
\{(f_A \oplus g_B) \widetilde{\cap} (f_A \oplus h_C)\}(z) &= (f_A \oplus g_B)(z) \cap (f_A \oplus h_C)(z) \\
&= \{\underset{z=x+y}{\cup}((f_A(y) \cap (g_B(z))\} \cap \{\underset{z=x+y}{\cup}(f_A(y) \cap (h_C(z))\} \\
&= \{\underset{z=x+y}{\cup}(\ (f_A(0) \cap (g_B(x)), (f_A(x) \cap (g_B(0)), (f_A(y) \cap (g_B(z)), (f_A(z) \cap (g_B(y \\
&\quad \cap\{\underset{z=x+y}{\cup}(\ (f_A(0) \cap (h_C(x)), (f_A(x) \cap (h_C(0)), (f_A(y) \cap (h_C(z)), (f_A(z) \cap (h
\end{aligned}
$$

$$
\begin{aligned}
&= \{\cup \left\{ \begin{array}{c} ((1\ 1) \cap (b\ 1)), ((b\ d) \cap (1\ 0)), \\ ((a\ b) \cap (0\ b)), ((c\ a) \cap (0\ b)) \end{array} \right\}\} \\
&\quad \cap \left\{ \cup \left\{ \begin{array}{c} ((1\ 1) \cap (0\ 1)), ((b\ d) \cap (a\ 1)), \\ ((a\ b) \cap (0\ d)), ((c\ a) \cap (b\ 0)) \end{array} \right\} \right\} \\
&= \{(b\ 1) \cup (b\ 0) \cup (0\ b) \cup (0\ b)\} \cap \{(0\ 1) \cup (b\ d) \cup (0\ d) \cup (0\ 0)\} \\
&= (b\ 1) \cap (b\ 1) = (b\ 1).
\end{aligned}
$$

Hence,

$$f_A \oplus \left(g_B \ \widetilde{\cap} \ h_C\right) \widetilde{\subseteq} (f_A \oplus g_B) \widetilde{\cap} (f_A \oplus h_C).$$

2.2 L-fuzzy Soft Sub-nearring and L-fuzzy soft Ideals

In this section we define L-fuzzy soft sub-nearring and L-fuzzy soft ideal of a nearring N over U and prove some related results.

Definition 57 *An L-fuzzy soft subset f_A of a nearring N over U is called an L-fuzzy soft subnearring of N if*

(1) $f_A(x-y) \supseteq f_A(x) \cap f_A(y)$

(2) $f_A(xy) \supseteq f_A(x) \cap f_A(y)$, *for all* $x, y \in N$, *where* $A \subseteq N$.

Definition 58 *An L-fuzzy soft subset f_A of a nearring N over U is called an L- fuzzy soft ideal of N if f_A is an L-fuzzy subnearring of N and*

(3) $f_A(x) = f_A(y + x - y)$

(4) $f_A(xy) \supseteq f_A(y)$

(5) $f_A((x+i)y - xy) \supseteq f_A(i)$, *for any* $x, y, i \in N$, *where* $A \subseteq N$.

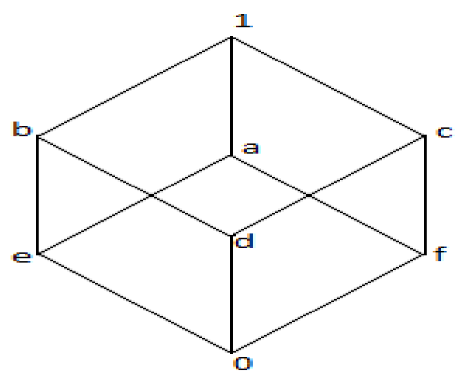
f_A *is an L- fuzzy soft left ideal of* N *if it satisfies* $(1), (3)$ *and* (4); f_A *is an L- fuzzy soft right ideal of* N *if it satisfies* $(1), (2), (3)$ *and* (5).

Example 59 *Let* $N = \{0, x, y, z\}$ *be the nearring with binary operations as defined below:*

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

$\bullet$	0	x	y	z
0	0	0	0	0
x	0	x	0	x
y	0	0	0	0
z	0	z	0	z

Consider the complete bounded distributive lattice $L = \{0, a, b, c, d, e, f, 1\}$ *and* $U = \{j, k\}$, $A = N$.



Define an L-fuzzy soft set f_A *of* N *over* U *as follows:*

	j	k
$f_A(0)$	1	a
$f_A(x)$	b	f
$f_A(y)$	c	0
$f_A(z)$	a	e

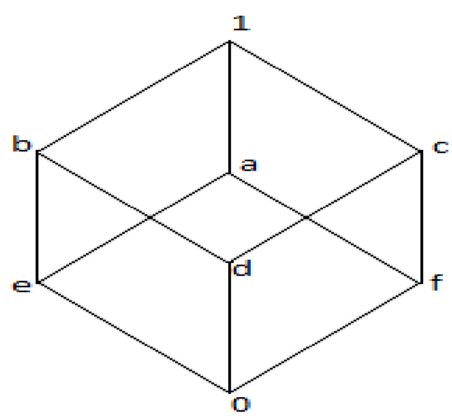
Simple calculations show that f_A *is an L-fuzzy soft subnearring of* N *over* U *but* $f_A(xz) \ngeq f_A(z)$ *because* $(b\ f) \ngeq (a\ e)$ *and* $f_A((x+z)x - xx) \nsupseteq f_A(z)$ *because* $(b\ f) \ngeq (a\ e)$. *Thus it is neither an L-fuzzy soft left ideal nor an L-fuzzy soft right ideal of* N *and hence not an L-fuzzy soft ideal of* N.

Example 60 *Let* $N = \{0, x, y, z\}$ *be the nearring with binary operations as defined below:*

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	0	x
z	z	y	x	0

$\bullet$	0	x	y	z
0	0	0	0	0
x	0	x	0	x
y	0	0	0	0
z	0	z	0	z

Consider the complete Boolean lattice $L = \{0, a, b, c, d, e, f, 1\}$ *and* $U = \{j, k\}$, $A = N$.



Define an L-fuzzy soft set f_A *of* N *over* U *as follows:*

	j	k
$f_A(0)$	1	f
$f_A(x)$	a	0
$f_A(y)$	a	0
$f_A(z)$	1	f

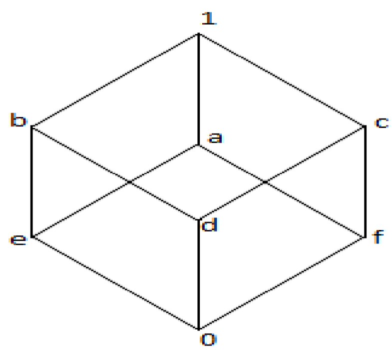
Simple calculations show that f_A *is an L-fuzzy soft subnearring of* N *over* U *and an L-fuzzy* *soft left ideal but* $f_A((x+z)x - xx) \nsupseteq f_A(z)$ *because* $(a\ 0) \ngeq (1\ f)$. *Thus it is not an L-fuzzy soft right ideal and hence not an L-fuzzy soft ideal of* N.

Example 61 *Let* $N = \{0, x, y, z\}$ *be the nearring with binary operations as defined below:*

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	x	0
z	z	y	0	x

$\bullet$	0	x	y	z
0	0	0	0	0
x	0	0	0	0
y	0	0	0	0
z	0	0	x	x

Consider the complete Boolean lattice $L = \{0, a, b, c, d, e, f, 1\}$ *and* $U = \{j, k\}$, $A = N$.



Define an L-fuzzy soft set f_A *of* N *over* U *as follows:*

	j	k
$f_A(0)$	1	1
$f_A(x)$	a	b
$f_A(y)$	a	e
$f_A(z)$	a	e

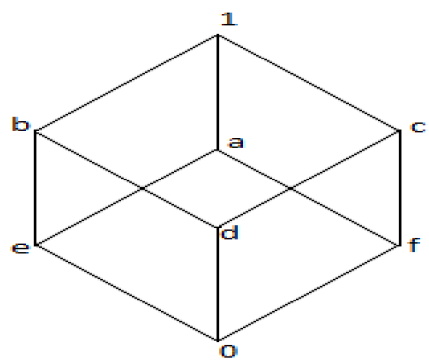
Simple calculations show that f_A *is an L-fuzzy soft subnearring of* N *over* U, *an L-fuzzy soft left ideal, an L-fuzzy soft right ideal and hence an L-fuzzy soft ideal of* N.

Example 62 *Let* $N = \{0, x, y, z\}$ *be the nearring with binary operations as defined below:*

+	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	x	0
z	z	y	0	x

$\bullet$	0	x	y	z
0	0	0	0	0
x	0	0	0	0
y	0	0	0	0
z	0	0	x	x

Consider the complete Boolean lattice $L = \{0, a, b, c, d, e, f, 1\}$ *and* $U = \{j, k\}$, $A = N$.



Define an L-fuzzy soft set g_B *of* N *over* U *as follows*:

	j	k
$g_B(0)$	1	1
$g_B(x)$	e	d
$g_B(y)$	e	0
$g_B(z)$	e	0

Simple calculations show that g_B *is an L-fuzzy soft subnearring of* N *over* U, *an L-fuzzy soft left ideal, an L-fuzzy soft right ideal and hence an L-fuzzy soft ideal of* N.

Lemma 63 *The intersection of two L-fuzzy soft (left, right) ideals of a nearring* N *over* U *is again an L-fuzzy soft (left, right) ideal of* N *over* U.

Proof. Let f_A and g_B be two L-fuzzy soft ideals of a nearring N over U and $x, y \in N$. Then

(1)

$$\begin{aligned}
& \left(f_A \widetilde{\cap} g_B\right)(x-y) = f_A(x-y) \cap g_B(x-y) \\
\supseteq \quad & \left\{f_A(x) \cap f_A(y)\right\} \cap \left\{g_B(x) \cap g_B(y)\right\} \\
= \quad & \left\{f_A(x) \cap g_B(x)\right\} \cap \left\{f_A(y) \cap g_B(y)\right\} \\
= \quad & \left\{\left(f_A \widetilde{\cap} g_B\right)(x)\right\} \cap \left\{\left(f_A \widetilde{\cap} g_B\right)(y)\right\}.
\end{aligned}$$

Hence,

$$\left(f_A \widetilde{\cap} g_B\right)(x-y) \supseteq \left\{\left(f_A \widetilde{\cap} g_B\right)(x)\right\} \cap \left\{\left(f_A \widetilde{\cap} g_B\right)(y)\right\}.$$

(2)

$$\begin{aligned}
& \left(f_A \widetilde{\cap} g_B\right)(xy) = f_A(xy) \cap g_B(xy) \\
\supseteq \quad & \left\{f_A(x) \cap f_A(y)\right\} \cap \left\{g_B(x) \cap g_B(y)\right\} \\
= \quad & \left\{f_A(x) \cap g_B(x)\right\} \cap \left\{f_A(y) \cap g_B(y)\right\} \\
= \quad & \left\{\left(f_A \widetilde{\cap} g_B\right)(x)\right\} \cap \left\{\left(f_A \widetilde{\cap} g_B\right)(y)\right\}.
\end{aligned}$$

Hence,

$$\left(f_A \widetilde{\cap} g_B\right)(xy) \supseteq \left\{\left(f_A \widetilde{\cap} g_B\right)(x)\right\} \cap \left\{\left(f_A \widetilde{\cap} g_B\right)(y)\right\}.$$

(3)

$$\begin{aligned}
& \left(f_A \widetilde{\cap} g_B\right)(y+x-y) = f_A(y+x-y) \cap g_B(y+x-y) \\
\supseteq \quad & f_A(x) \cap g_B(x) = \left(f_A \widetilde{\cap} g_B\right)(x).
\end{aligned}$$

Hence,

$$\left(f_A \widetilde{\cap} g_B\right)(y+x-y) \supseteq \left(f_A \widetilde{\cap} g_B\right)(x).$$

(4)

$$\left(f_A \widetilde{\cap} g_B\right)(xy) = f_A(xy) \cap g_B(xy) \supseteq f_A(x) \cap g_B(y) = \left(f_A \widetilde{\cap} g_B\right)(y).$$

Hence,

$$\left(f_A\widetilde{\cap}g_B\right)(xy) \supseteq \left(f_A\widetilde{\cap}g_B\right)(y).$$

(5)

$$\begin{aligned}\left(f_A\widetilde{\cap}g_B\right)((x+i)\,y-xy) &= f_A\left((x+i)\,y-xy\right)\cap g_B\left((x+i)\,y-xy\right)\\ \supseteq \quad & f_A(i)\cap g_B(i) = \left(f_A\widetilde{\cap}g_B\right)(i).\end{aligned}$$

Hence,

$$\left(f_A\widetilde{\cap}g_B\right)((x+i)\,y-xy) \supseteq \left(f_A\widetilde{\cap}g_B\right)(i).$$

for all $x, y, i \in N$.

Consequently, $\left(f_A\widetilde{\cap}g_B\right)$ is an L-fuzzy soft ideal of N. ■

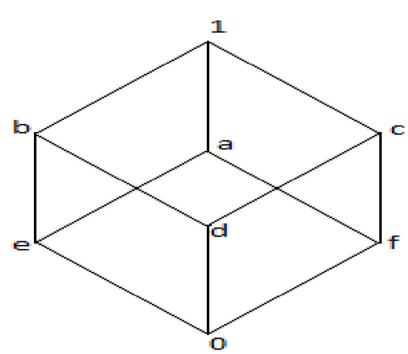
Next we show that the union of two L-fuzzy soft ideals of a nearring N is not necessarily an L-fuzzy soft ideal of N.

Example 64 *Let $N = \{0, x, y, z\}$ be the nearring with binary operations as defined below:*

$+$	0	x	y	z
0	0	x	y	z
x	x	0	z	y
y	y	z	x	0
z	z	y	0	x

$\bullet$	0	x	y	z
0	0	0	0	0
x	0	0	0	0
y	0	0	0	0
z	0	0	x	x

Consider the complete Boolean lattice $L = \{0, a, b, c, d, e, f, 1\}$ and $U = \{j, k\}$, $A = B = N$.



Let f_A and g_B be two L-fuzzy soft ideals (left, right) of a nearring N over U.

	j	k
$f_A(0)$	1	1
$f_A(x)$	a	b
$f_A(y)$	a	e
$f_A(z)$	a	e

	j	k
$g_B(0)$	1	1
$g_B(x)$	e	d
$g_B(y)$	e	0
$g_B(z)$	e	0

Simple calculations show that f_A and g_B are L-fuzzy soft ideals of N over U. Now

$$\left(f_A \widetilde{\cup} g_B\right)(y-x) = (a\ e)$$

and

$$\left(f_A \widetilde{\cup} g_B\right)(y) \cap \left(f_A \widetilde{\cup} g_B\right)(x) = (a\ e\ ) \cap (a\ b) = (a\ b)$$

and $e \not\supseteq b$. Hence,

$$\left(f_A \widetilde{\cup} g_B\right)(y-x) \not\supseteq \left(f_A \widetilde{\cup} g_B\right)(y) \cap \left(f_A \widetilde{\cup} g_B\right)(x).$$

Hence, $f_A \widetilde{\cup} g_B$ is not an L-fuzzy ideal of N over U.

Definition 65 *Let f_A be an L- fuzzy soft subset of a nearring N over U. For $\alpha \in L^U$, the set $f_A^\alpha = \{x \in N : f_A(x) \supseteq \alpha\}$ is called a level subset of f_A.*

Theorem 66 *Let N be a nearring and f_A be an L- fuzzy soft subset of N over U. Then f_A is an L- fuzzy soft subnearing (ideal) of N over U if and only if the level subset $f_A^\alpha \neq \emptyset$ is a subnearring (ideal) of N for all $\alpha \in L^U$.*

Proof. Let f_A be an L-fuzzy soft left ideal of N. Let $x, y \in f_A^\alpha$. Then $f_A(x) \supseteq \alpha$ and $f_A(y) \supseteq \alpha$. Now, as

$$f_A(x-y) \supseteq f_A(x) \cap f_A(y) \supseteq \alpha \cap \alpha = \alpha \Rightarrow f_A(x-y) \supseteq \alpha \Rightarrow x - y \in f_A^\alpha.$$

Now, let $x \in f_A^\alpha$ and $y \in N$. Then $f_A(x) \supseteq \alpha$. As

$$f_A(x) = f_A(y+x-y) \supseteq \alpha \Rightarrow y + x - y \in f_A^\alpha.$$

Finally let $y \in f_A^{\alpha}$ and $x \in N$. Then $f_A(y) \supseteq \alpha$. As

$$f_A(xy) \supseteq f_A(y) \supseteq \alpha \Rightarrow xy \in f_A^{\alpha}.$$

Hence, f_A^{α} is a left ideal of N.

Conversely, suppose that f_A^{α} is a left ideal of N. Let $x, y \in N$ be such that $f_A(x-y) \subset f_A(x) \cap f_A(y)$. Then there exists an $\alpha \in L^U$ such that

$$f_A(x-y) \subset \alpha \subseteq f_A(x) \cap f_A(y) \Rightarrow f_A(x) \cap f_A(y) \supseteq \alpha \Rightarrow f_A(x) \supseteq \alpha$$

and $f_A(y) \supseteq \alpha \Rightarrow x \in f_A^{\alpha}$ and $y \in f_A^{\alpha}$ but $x - y \notin f_A^{\alpha}$, which is a contradiction. Hence,

$$f_A(x-y) \supseteq f_A(x) \cap f_A(y)$$

for all $x, y \in N$.

Again assume that there exist $x, y \in N$ such that

$$f_A(y+x-y) \subset f_A(x),$$

so there exists $\alpha \in L^U$ such that

$$f_A(y+x-y) \subset \alpha \subseteq f_A(x) \Rightarrow f_A(x) \supseteq \alpha \Rightarrow x \in f_A^{\alpha},$$

but $y + x - y \notin f_A^{\alpha}$, which is a contradiction. Hence,

$$f_A(y+x-y) \supseteq f_A(x).$$

Finally suppose that there exist $x, y \in N$ such that $f_A(xy) \subset f_A(y)$, so there exists $\alpha \in L^U$ such that

$$f_A(xy) \subset \alpha \subseteq f_A(y) \Rightarrow f_A(y) \supseteq \alpha \Rightarrow y \in f_A^{\alpha},$$

but $xy \notin f_A^{\alpha}$, which is a contradiction. Hence,

$$f_A(xy) \supseteq f_A(y)$$

. Hence, f_A an L-fuzzy soft left ideal of N. ■

Lemma 67 *If an L-fuzzy soft set f_A of a nearring N over U satisfies the property $f_A(x-y) \supseteq f_A(x) \cap f_A(y)$ for all $x, y \in N$, then*

(i) $f_A(0_N) \supseteq f_A(x)$

(ii) $f_A(-x) = f_A(x)$ for all $x, y \in N$.

Proof. (i) For any $x \in N$,

$$f_A(0_N) = f_A(x-x) \supseteq f_A(x) \cap f_A(x) = f_A(x).$$

Hence, $f_A(0_N) \supseteq f_A(x)$.

(ii) For all $x \in N$,

$$f_A(-x) = f_A(0_N - x) \supseteq f_A(0_N) \cap f_A(x) = f_A(x).$$

Since x is arbitrary, we conclude that

$$f_A(-x) = f_A(x).$$

■

Proposition 68 *Let f_A be an L-fuzzy soft ideal of of a nearring N over U. If $f_A(x-y) = f_A(0_N)$, then $f_A(x) = f_A(y)$ for all $x, y \in N$.*

Proof. Assume that

$$f_A(x-y) = f_A(0_N)$$

for all $x, y \in N$. Then

$$f_A(x) = f_A(x-y+y) \supseteq f_A(x-y) \cap f_A(y) = f_A(0_N) \cap f_A(y) = f_A(y)$$

Similarly, using

$$f_A(y-x) = f_A(x-y) = f_A(0_N)$$

we have

$$f_A(y) \supseteq f_A(x).$$

Hence,

$$f_A(x) = f_A(y).$$

■

Theorem 69 *Let I be a left (right) ideal of a nearring N. Then for any $\alpha \in L(U)$, there exists an L-fuzzy soft left (right) ideal f_A of N such that $f_A^\alpha = I$.*

Proof. Let $f_A : N \to L(U)$ be an L- fuzzy soft set of N over U defined by $f_A(x) = \begin{cases} \alpha & \text{if } x \in I \\ \widetilde{0} & \text{if } x \notin I \end{cases}$

where $\widetilde{0}$ is the zero L-fuzzy set and α is a fixed L-fuzzy set in $L(U)$. Then clearly $f_A^\alpha = I$. By Theorem 79 f_A is an L-fuzzy soft ideal of N over U. ■

Definition 70 *Let I be a non-empty subset of a nearring N. Define an L-fuzzy soft subset χ_I of N over U as following:*

$$\chi_I(x) = \begin{cases} \widetilde{1} & \text{if } x \in I \\ \widetilde{0} & \text{if } x \notin I \end{cases}$$

This is called an L-fuzzy soft characteristic function of I.

Theorem 71 *The characteristic function χ_I of I is an L-fuzzy soft left (right) ideal of N over U if and only if I is a left (right) ideal of N.*

Proof. Assume that χ_I is an L-fuzzy soft ideal of N. Let $x, y \in I$. Then $\chi_I(x) = \widetilde{1}$ and $\chi_I(y) = \widetilde{1}$. Now,

$$\chi_I(x-y) \supseteq \chi_I(x) \cap \chi_I(y) = \widetilde{1} \cap \widetilde{1} = \widetilde{1},$$

so

$$\chi_I(x-y) \supseteq \widetilde{1}.$$

This means that $x - y \in I$. Also

$$\chi_I(xy) \supseteq \chi_I(x) \cap \chi_I(y) = \widetilde{1} \cap \widetilde{1} = \widetilde{1},$$

so

$$\chi_I\ (xy) \supseteq \widetilde{1}.$$

This means that

$$xy \in I.$$

Now, let $x \in N$ and $y \in I$. Then

$$\chi_I\ (xy) \supseteq \chi_I\ (y) = \widetilde{1} \Rightarrow \chi_I\ (xy) \supseteq \widetilde{1},$$

that is

$$\chi_I\ (xy) = \widetilde{1} \Rightarrow xy \in I.$$

Also,

$$\chi_I(y + x - y) \supseteq \chi_I(x) = \widetilde{1} \Rightarrow \chi_I\ (y + x - y) = \widetilde{1} \Rightarrow y + x - y \in I.$$

And finally, assume that $x, y \in N$ and $i \in I$, then

$$\chi_I((x + i)\, y - xy) \supseteq \chi_I(i) = \widetilde{1},$$

implies $(x + i)\, y - xy \in I$. Hence I is an ideal of N.

Conversely, suppose that I is an ideal of N. Let $x, y \in I$. Then

$$x - y \in I.$$

Thus

$$\chi_I(x) = \widetilde{1} = \chi_I\ (y) = \chi_I(x - y).$$

Hence,

$$\chi_I\ (x - y) = \chi_I\ (x) \cap \chi_I\ (y)$$

If one of x, y is not in I then

$$\chi_I\ (x) \cap \chi_I\ (y) = \widetilde{0}.$$

So

$$\chi_I\ (x - y) \supseteq \chi_I\,(x) \cap \chi_I\,(y)$$

Let $x, y \in I$. Then

$$xy \in I.$$

Thus $\chi_I\,(x) = \widetilde{1} = \chi_I\,(y) = \chi_I\,(xy)$. Hence,

$$\chi_I\ (xy) = \chi_I\,(x) \cap \chi_I\,(y)$$

If one of x, y is not in I then

$$\chi_I\ (x) \cap \chi_I\ (y) = \widetilde{0}$$

So

$$\chi_I\,(xy) \supseteq \chi_I\,(x) \cap \chi_I\,(y)\,.$$

Now, let $x \in I$ and $y \in N$. Then

$$y + x - y \in I$$

Thus

$$\chi_I\ (y + x - y) = \widetilde{1} = \chi_I\,(x)\,.$$

If $x \notin I$ then

$$\chi_I\,(x) = \widetilde{0}.$$

So

$$\chi_I\ (y + x - y) \supseteq \chi_I\,(x)\,.$$

Finally, if $i \in I$ then

$$(x + i)\,y - xy \in I.$$

Thus

$$\chi_I\ ((x + i)\,y - xy) = \widetilde{1} = \chi_I\ (i)\,.$$

Now, if $i \notin I$. Then

$$\chi_I(x) = \widetilde{0}.$$

Thus

$$\chi_I\ ((x+i)\,y - xy) \supseteq \chi_I\ (i)$$

. Hence, the characteristic function of I is an L-fuzzy soft ideal of N. This completes the proof. ■

Chapter 3

Prime and Semiprime ideals

In this chapter we define L-fuzzy prime and semiprime soft ideals of a nearring. We also characterize those nearrings for which each L-fuzzy soft ideal is prime.

3.1 Product of L-fuzzy soft sets

Definition 72 *Let f_A and g_B be two L-fuzzy soft sets of a nearring N over the common universe U. Then the soft product $f_A \odot g_B$ is an L-fuzzy soft set of N over U defined by*

$$(f_A \odot g_B)(x) = \begin{cases} \underset{x=yz}{\cup} f_A(y) \cap g_B(z) & \text{if } \exists\ y, z \in N \text{ such that } x = yz \\ \widetilde{0} & \text{otherwise} \end{cases}$$

for all $x \in N$.

Proposition 73 *Let A, B be non-empty subsets of a nearring N. Then $\chi_A \odot \chi_B = \chi_{AB}$.*

Proof. Let $x \in N$. If $x \in AB$ then there exist $a \in A$ and $b \in B$ such that $x = ab$. In this case

$$\begin{aligned} (\chi_A \odot \chi_B)(x) &= \underset{x=yz}{\cup} \chi_A(y) \cap \chi_B(z) \\ &\supseteq \chi_A(a) \cap \chi_B(b) \\ &= \widetilde{1} \cap \widetilde{1} = \widetilde{1}. \end{aligned}$$

Hence $(\chi_A \odot \chi_B)(x) = \widetilde{1} = \chi_{AB}(x)$. If $x \notin AB$ then there do not exist $a \in A$ and $b \in B$ such that $x = ab$. Hence

$$\begin{aligned} \chi_{AB}(x) &= \widetilde{0} \\ &= \widetilde{0} \cap \widetilde{0} \\ &= (\chi_A \odot \chi_B)(x). \end{aligned}$$

This shows that $\chi_A \odot \chi_B = \chi_{AB}$. ■

Definition 74 *An L-fuzzy soft ideal f_A of a nearring N over U is called prime if f_A is not a constant function and for any L-fuzzy soft ideals g_B, h_C of N over U,*

$$g_B \odot h_C \widetilde{\subseteq} f_A \Rightarrow g_B \widetilde{\subseteq} f_A \text{ or } h_C \widetilde{\subseteq} f_A.$$

Theorem 75 *An ideal P of a nearring N is prime if and only if χ_P is an L-fuzzy prime soft ideal of N over U.*

Proof. Suppose that the characteristic function χ_P of P is an L-fuzzy prime soft ideal of N over U. Then P is an ideal of N. Let A, B be any ideals of N such that $AB \subseteq P$. Then χ_A and χ_B are L-fuzzy soft ideals of N and $\chi_A \chi_B \widetilde{\subseteq} \chi_P$. Since χ_P is prime, so $\chi_A \widetilde{\subseteq} \chi_P$ or $\chi_B \widetilde{\subseteq} \chi_P$. This implies $A \subseteq P$ or $B \subseteq P$.

Conversely, assume that P is a prime ideal of N. Then by Theorem 74, χ_P is an L-fuzzy soft ideal of N over$U$. Let g_B, h_C be L-fuzzy soft ideals of N such that $g_B \odot h_C \widetilde{\subseteq} \chi_P$. Suppose $g_B \nsubseteq \chi_P$ and $h_C \nsubseteq \chi_P$. Then there exist $x, y \in N$ and $u \in U$ such that $(g_B(x))(u) \supseteq (\chi_P(x))(u)$ and $(h_C(y))(u) \supseteq (\chi_P(y))(u)$. Now $(\chi_P(x))(u) \cap (\chi_P(y))(u) \leq (g_B(x))(u) \cap (h_C(y))(u)$, this implies $\chi_P(xy) \nsupseteq g_B(x) \cap h_C(y)$, which is a contradiction. Hence $g_B \subseteq \chi_P$ or $h_C \subseteq \chi_P$. ■

Definition 76 *An L-fuzzy soft ideal f_A of a nearring N over U is called semiprime if f_A is not a constant function and for any L-fuzzy soft ideal g_B of N, $g_B \odot g_B \widetilde{\subseteq} f_A$ implies $g_B \widetilde{\subseteq} f_A$.*

Theorem 77 *An ideal P of a nearring N is semiprime if and only if χ_P is an L-fuzzy soft semiprime ideal of N over U.*

Proof. Suppose that the characteristic function χ_P of P is an L-fuzzy soft semiprime ideal of N over U. Then P is an ideal of N. Let A be any ideals of N such that $AA \subseteq P$. Then by Theorem 74, χ_A is an L-fuzzy soft ideals of N and $\chi_A \chi_A \tilde{\subseteq} \chi_P$. Since χ_P is semiprime, so $\chi_A \tilde{\subseteq} \chi_P$. This implies $A \subseteq P$.

Conversely, assume that P is a semiprime ideal of N. Then χ_P is an L-fuzzy soft ideal of N over U. Let g_B be an L-fuzzy soft ideal of N such that

$$g_B \odot g_B \tilde{\subseteq} \chi_P.$$

Suppose $g_B \nsubseteq \chi_P$. Then there exist $x \in N$ and $u \in U$ such that

$$\left(g_B\left(x\right)\right)\left(u\right) \geq \left(\chi_P\left(x\right)\right)\left(u\right).$$

Now

$$\left(\chi_P\left(x\right)\right)\left(u\right) = \left(\chi_P\left(x\right)\right)\left(u\right) \cap \left(\chi_P\left(y\right)\right)\left(u\right) \subseteq \left(g_B\left(x\right)\right)\left(u\right) \cap \left(g_B\left(x\right)\right)\left(u\right),$$

this implies

$$\chi_P\left(x\right) \nsupseteq g_B\left(x\right) \cap g_B\left(x\right),$$

which is a contradiction. Hence

$$g_B \tilde{\subseteq} \chi_P.$$

■

3.2 Characterization of nearrings by the properties of their L-fuzzy soft ideals

In this section, we characterize those nearrings for which each L-fuzzy soft ideal is prime and also those nearrings for which each L-fuzzy soft ideal is idempotent.

Definition 78 *Let f_A and g_B be two L-fuzzy soft sets of a nearring N over the common universe U. The L-fuzzy soft subset $f_A \oplus g_B$ of N is defined as*

$$(f_A \oplus g_B)(x) = \underset{x=y+z}{Sup} \{f_A(y) \cap g_B(z)\} \text{ where } y, z \in N \text{ such that } x = y + z.$$

Proposition 79 *Let f_A and g_B be two L-fuzzy soft ideals of a nearring N. Then $f_A \oplus g_B$ is the smallest L-fuzzy soft ideal of N containing both f_A and g_B.*

Proof. For any $x, y \in N$,

$$\begin{aligned}
(f_A \oplus g_B)(x) \cap (f_A \oplus g_B)(y) &= \left[\bigcup_{x=a+b}[f_A(a) \cap g_B(b)]\right] \cap \left[\bigcup_{y=c+d}[f_A(c) \cap g_B(d)]\right] \\
&= \bigcup_{\substack{x=a+b \\ y=c+d}}[[f_A(a) \cap g_B(b)] \cap [f_A(c) \cap g_B(d)]] \\
&= \bigcup_{\substack{x=a+b \\ y=c+d}}[[f_A(a) \cap f_A(c)] \cap [g_B(b) \cap g_B(d)]]
\end{aligned}$$

$$\left(\begin{array}{c}
\text{Since } x - y = a + b - (c + d) = a + b - d - c = a - c + (c + b - c) + (c - d - c) \\
\text{and } g_B(c + b - c) = g_B(b), \\
g_B(c - d - c) = g_B(-d) = g_B(d) \text{ we have}
\end{array}\right)$$

$$\begin{aligned}
&= \bigcup_{\substack{x=a+b \\ y=c+d}}[[f_A(a) \cap f_A(c)] \cap [g_B(c + b - c) \cap g_B(c - d - c)]] \\
&\subseteq \bigcup_{x-y=e+f}[f_A(e) \cap g_B(f) = (f_A \oplus g_B)(x - y).
\end{aligned}$$

Thus, $(f_A \oplus g_B)(x - y) \supseteq (f_A \oplus g_B)(x) \cap (f_A \oplus g_B)(y)$.

Now,

$$\begin{aligned}
(f_A \oplus g_B)(y) &= \bigcup_{y=a+b}[f_A(a) \cap g_B(b)] \\
&\subseteq \bigcup_{y=a+b}[f_A(xa) \cap g_B(xb) \\
&\subseteq \bigcup_{xy=c+d}[f_A(c) \cap g_B(d) = (f_A \oplus g_B)(xy).
\end{aligned}$$

Thus, $(f_A \oplus g_B)(xy) \supseteq (f_A \oplus g_B)(y)$.

Hence, $(f_A \oplus g_B)(xy) \supseteq (f_A \oplus g_B)(x) \cap (f_A \oplus g_B)(y)$.

Now,

$$\begin{aligned}(f_A \oplus g_B)(x) &= \underset{x=a+b}{\cup}[f_A(a) \cap g_B(b)] \\ &= \underset{x=a+b}{\cup}[f_A(y+a-y) \cap g_B(y+b-y)] \\ &= \underset{y+x-y=c+d}{\cup}[f_A(c) \cap g_B(d)]\end{aligned}$$

$$\left(\begin{array}{c}\text{Because for each } x=a+b, \text{ we have } y+x-y=y+a-y+y+b-y=(y+a-y)+(y+b-y) \\ \text{and for each } y+x-y=c+d, \text{ we have } x=-y+c+d+y=(-y+c+y)+(-y+d+y).\end{array}\right)$$

$$= (f_A \oplus g_B)(y+x-y).$$

Hence, $(f_A \oplus g_B)(x) = (f_A \oplus g_B)(y+x-y)$.

Let $i = a+b$. Then $i = a+b = b-b+a+b$ and $g_B(-b+a+b) = g_B(a)$. Hence whenever $f_A(a) \cap g_B(b)$ is present,

then $f_A(b) \cap g_B(a)$ is also present. Now

$$(x+i)y - xy = (x+(a+b))y - xy = (x+(a+b))y - (x+a)y + (x+a)y - xy.$$

Thus $f_A(b) \subseteq f_A((x+a)y+b) - (x+a)y)$.

Now,

$$\begin{aligned}(f_A \oplus g_B)(i) &= \underset{i=a+b}{\cup}[f_A(a) \cap g_B(b) \\ &= \underset{i=a+b}{\cup}[f_A(b) \cap g_B(a) \\ &\subseteq \underset{i=a+b}{\cup}[f_A((x+(a+b))y-(x+a)y) \cap g_B((x+a)y-xy) \\ &\subseteq \underset{(x+i)y-xy=c+d}{\cup}[f_A(c) \cap g_B(d)] \\ &= (f_A \oplus g_B)((x+i)y-xy).\end{aligned}$$

Hence, $f_A \oplus g_B$ is an L-fuzzy soft ideal of N.

Now, $(f_A \oplus g_B)(x) = \underset{x=a+b}{\cup}[f_A(a) \cap g_B(b)]$

As $x = x + 0$ and $x = 0 + x$, so $(f_A \oplus g_B)(x) \supseteq f_A(x)$ and also $(f_A \oplus g_B)(x) \supseteq g_B(x)$. If h_C is an L-fuzzy soft ideal of N such that $h_C(x) \supseteq g_B(x)$ and $h_C(x) \supseteq f_A(x)$ for all $x \in N$, then

$$\begin{aligned}(f_A \oplus g_B)(x) &= \underset{x=a+b}{\cup}[f_A(a) \cap g_B(b)] \\ &\subseteq \underset{x=a+b}{\cup}[h_C(a) \cap h_C(b)] \\ &= \underset{x=a+b}{\cup}[h_C(a) \cap h_C(-b)] \\ &\subseteq \underset{x=a+b}{\cup} h_C(a+b) = h_C(x).\end{aligned}$$

Thus, $f_A \oplus g_B \widetilde{\subseteq} h_C$. ■

Proposition 80 *Let N be a zero-symmetric nearring and f_A and g_B be L-fuzzy soft ideals of N over U. Then $f_A \odot g_B \widetilde{\subseteq} f_A \cap g_B$.*

Proof. Let f_A and g_B be L-fuzzy soft ideals of N and $x \in N$. Then

$$(f_A \odot g_B)(x) = \begin{cases} \underset{x=yz}{\cup} f_A(y) \cap g_B(z) & \text{if } \exists\ y, z \in N \text{ such that } x = yz \\ \widetilde{0} & \text{otherwise} \end{cases}.$$

As f_A is an L-fuzzy soft ideal, so $f_A(z) \subseteq f_A(yz) = f_A(x)$. As N is a zerosymmetric nearring, so $yz = (0 + y)z - 0z$. Hence, $g_B(x) = g_B(yz) = g_B((0 + y)z - 0z) \supseteq g_B(y)$. Thus,

$$\begin{aligned}(f_A \odot g_B)(x) &= \begin{cases} \underset{x=yz}{\cup} f_A(yz) \cap g_B(yz) & \text{if } \exists\ y, z \in N \text{ such that } x = yz \\ \widetilde{0} & \text{otherwise} \end{cases} \\ &\subseteq (f_A \cap g_B)(x).\end{aligned}$$

■

Let N be a nearring. Let $F(N)$ denote the set of all L-fuzzy soft subsets of N over U. Let $F^{\star}(N)$ be the set of all L-fuzzy soft ideals of N. Let $f_A \in F(N)$. Then the L-fuzzy soft ideal generated by f_A, denoted by $< f_A >$, is the intersection of all L-fuzzy soft ideals of N which contain f_A. Now, onwards N will denote a zerosymmetric left nearring.

Definition 81 *A nearring N is called fully L-fuzzy soft idempotent if for each L-fuzzy soft ideal f_A of N, $f_A \widetilde{=} < f_A^2 >$.*

Proposition 82 *The following assertions for a nearring N are equivalent:*

(i) N is fully L-fuzzy soft idempotent.

(ii) For each pair of L-fuzzy soft ideals f_A, g_B of N, $f_A \cap g_B \widetilde{=} < f_A \odot g_B >$.

(iii) The set of L-fuzzy soft ideals of N form a lattice $(F^\star(N), \cup, \cap)$ with $f_A \cup g_B \widetilde{=} f_A \oplus g_B$ and $f_A \widetilde{\cap} g_B \widetilde{=} < f_A \odot g_B >$ for each pair of L-fuzzy soft ideals f_A, g_B of N.

Proof. $(i) \Rightarrow (ii)$ For each pair of L-fuzzy soft ideals f_A, g_B of N

$$f_A \odot g_B \widetilde{\subseteq} f_A \cap g_B,$$

thus

$$< f_A \odot g_B > \widetilde{\subseteq} f_A \cap g_B.$$

For reverse inclusion, as $f_A \widetilde{\cap} g_B$ is an L-fuzzy soft ideal and

$$f_A \widetilde{\cap} g_B \widetilde{\subseteq} f_A$$

and

$$f_A \widetilde{\cap} g_B \widetilde{\subseteq} g_B, \text{ we have } (f_A \widetilde{\cap} g_B)^2 \widetilde{\subseteq} f_A \odot g_B,$$

This implies

$$f_A \widetilde{\cap} g_B \widetilde{=} < (f_A \widetilde{\cap} g_B)^2 > \widetilde{\subseteq} < f_A \odot g_B > .$$

Thus,

$$f_A \widetilde{\cap} g_B \widetilde{=} < f_A \odot g_B > .$$

$(ii) \Rightarrow (iii)$ The set of all L-fuzzy soft ideals of a nearring N ordered by inclusion forms a lattice under the sum and intersection of L-fuzzy soft ideals. Thus for each pair of L-fuzzy soft ideals f_A, g_B of N,

$$f_A \widetilde{\cup} g_B \widetilde{=} f_A \oplus g_B$$

and

$$f_A \widetilde{\cap} g_B \widetilde{=} < f_A \odot g_B > .$$

$(iii) \Rightarrow (i)$ By assumption

$$f_A \widetilde{\cap} g_B \widetilde{=} < f_A \odot g_B > .$$

Now taking $f_A \widetilde{=} g_B$, we have

$$f_A \widetilde{=} < f_A \odot f_A > \widetilde{=} < (f_A)^2 > .$$

Hence, N is fully L-fuzzy soft idempotent. ■

Theorem 83 *The set of all L-fuzzy soft ideals of a zerosymmetric fully L-fuzzy soft idempotent nearring N (ordered by inclusion) forms a distributive lattice under the sum and intersection of ideals.*

Proof. Straight forward. ■

3.3 Fully L-fuzzy soft Prime nearrings

Definition 84 *An L-fuzzy soft ideal f_A of a nearring N is an L-fuzzy soft irreducible (resp. L-fuzzy soft strongly irreducible) ideal if for any L-fuzzy soft ideals g_B, h_C of N, if $g_B \widetilde{\cap} h_C \widetilde{=} f_A$ implies $g_B \widetilde{=} f_A$ or $h_C \widetilde{=} f_A$.(resp. $g_B \widetilde{\cap} h_C \widetilde{\subseteq} f_A$ implies $g_B \widetilde{\subseteq} f_A$ or $h_C \widetilde{\subseteq} f_A$).*

Proposition 85 *An L-fuzzy soft prime ideal of a zerosymmetric nearring N is L-fuzzy soft semiprime and strongly irreducible.*

Proof. Let f_A be an L-fuzzy soft prime ideal of N and g_B, h_C be any L-fuzzy soft ideals of N. Clearly, f_A is an L-fuzzy soft semiprime ideal of N. Let $g_B \widetilde{\cap} h_C \widetilde{\subseteq} f_A$. As $g_B \odot h_C \widetilde{\subseteq} g_B \widetilde{\cap} h_C$, we have $g_B \odot h_C \widetilde{\subseteq} f_A$. As f_A is an L-fuzzy soft prime ideal so either $g_B \widetilde{\subseteq} f_A$ or $h_C \widetilde{\subseteq} f_A$. ■

Lemma 86 ?? *If f_A is an L-fuzzy soft ideal of a nearring N and $f_A(a) \widetilde{=} \alpha$ where a is any element of N and $\alpha \in L(U)$. Then there exists an L-fuzzy soft irreducible ideal h_C of N such that $f_A \widetilde{\subseteq} h_C$ and $h_C(a) = \alpha$.*

Proof. Let $\chi = \left\{ g_B : g_B \text{ is an } L\text{-fuzzy soft ideal of } N,\ g_B(a) = \alpha \text{ and } f_A \widetilde{\subseteq} g_B \right\}$. Then $\chi \neq \varphi$ because $f_A \in \chi$.

Let ξ be a totally ordered subset of χ, say $\xi = \{(f_A)_i : i \in I\}$. We will show that $\underset{i \in I}{\cup} (f_A)_i$ is an L-fuzzy soft ideal of N. Let $x, y \in N$. Then

$$\begin{aligned} \left(\underset{i \in I}{\widetilde{\cup}} (f_A)_i \right)(x-y) &= \underset{i \in I}{\cup} ((f_A)_i (x-y)) \\ &\supseteq \underset{i \in I}{\cup} ((f_A)_i (x) \cap (f_A)_i (y)) \\ &\supseteq \left(\underset{i \in I}{\widetilde{\cup}} (f_A)_i \right)(x) \cap \left(\underset{i \in I}{\widetilde{\cup}} (f_A)_i \right)(y) \end{aligned}$$

And,

$$\begin{aligned} \left(\underset{i \in I}{\widetilde{\cup}} (f_A)_i \right)(xy) &= \underset{i \in I}{\cup} ((f_A)_i (xy)) \\ &\supseteq \underset{i \in I}{\cup} ((f_A)_i (x) \cap (f_A)_i (y)) \\ &\supseteq \left(\underset{i \in I}{\widetilde{\cup}} (f_A)_i \right)(x) \cap \left(\underset{i \in I}{\widetilde{\cup}} (f_A)_i \right)(y). \end{aligned}$$

Now,

$$\begin{aligned} \left(\underset{i \in I}{\widetilde{\cup}} (f_A)_i \right)(x) &= \underset{i \in I}{\cup} ((f_A)_i (x)) \\ &\widetilde{=} \underset{i \in I}{\cup} ((f_A)_i (y + x - y)) \\ &= \left(\underset{i \in I}{\widetilde{\cup}} (f_A)_i \right)(y + x - y). \end{aligned}$$

And,

$$\begin{aligned} \left(\underset{i \in I}{\widetilde{\cup}} (f_A)_i \right)(xy) &= \underset{i \in I}{\cup} ((f_A)_i (xy)) \\ &\supseteq \underset{i \in I}{\cup} ((f_A)_i x) = \left(\underset{i \in I}{\widetilde{\cup}} (f_A)_i \right)(x) \end{aligned}$$

Also,

$$\left(\underset{i\in I}{\widetilde{\cup}}(f_A)_i\right)((x+a)y-xy) = \underset{i\in I}{\cup}((f_A)_i((x+a)y-xy))$$
$$\supseteq \underset{i\in I}{\cup}((f_A)_i(a) = \left(\underset{i\in I}{\widetilde{\cup}}(f_A)_i\right)(a)$$

for any $x, y, a \in N$.

Hence, $\underset{i\in I}{\cup}(f_A)_i$ is an L-fuzzy soft ideal of N. As each $(f_A)_i$ satisfies $(f_A)_i(a) = \alpha$, so we have

$$\left(\underset{i\in I}{\widetilde{\cup}}(f_A)_i\right)(a) = \underset{i\in I}{\cup}((f_A)_i(a)) = \underset{i\in I}{\cup}\alpha = \alpha.$$

Also, as $f_A \widetilde{\subseteq} (f_A)_i$ for each i, so $f_A \widetilde{\subseteq} \underset{i\in I}{\cup}(f_A)_i$. Hence, ξ is bounded above. Thus, by Zorn's lemma, there exists an L-fuzzy soft ideal h_C of N which is maximal in χ. We now show that h_C is an L-fuzzy irreducible ideal of N.

Let $(h_C)_1$ and $(h_C)_2$ be two L-fuzzy soft ideals of N such that $h_C \widetilde{=} (h_C)_1 \widetilde{\cap} (h_C)_2$. This implies that $h_C \widetilde{\subseteq} (h_C)_1$ and $h_C \widetilde{\subseteq} (h_C)_2$. We claim that either $h_C \widetilde{=} (h_C)_1$ or $h_C \widetilde{=} (h_C)_2$. Suppose on contrary that $h_C \neq (h_C)_1$ and $h_C \neq (h_C)_2$. Since h_C is maximal with respect to the property that $h_C(a) = \alpha$ and $h_C \widetilde{\nsubseteq} (h_C)_1$ and $h_C \widetilde{\nsubseteq} (h_C)_2$, it follows that $(h_C)_1(a) \neq \alpha$ and $(h_C)_2(a) \neq \alpha$.

Hence, $\alpha = h_C(a) = ((h_C)_1 \cap (h_C)_2)(a) = (h_C)_1(a) \widetilde{\cap} (h_C)_2(a) \neq \alpha$ which is impossible. Hence, either $h_C \widetilde{=} (h_C)_1$ or $h_C \widetilde{=} (h_C)_2$. Thus, h_C is an irreducible L-fuzzy soft ideal of N. ■

Proposition 87 *Every proper L-fuzzy soft ideal of N is the intersection of all those L-fuzzy soft irreducible ideals of N which contain it.*

Proof. Let f_A be an L-fuzzy soft proper ideal of N and let $A = \{(f_A)_\alpha : \alpha \in \Omega\}$ be a family of L-fuzzy soft irreducible ideals of N which contains f_A. where A is a non-empty set. Obviously $f_A \widetilde{\subseteq} \underset{\alpha\in\Omega}{\cap}(f_A)_\alpha$. We now show that $\underset{\alpha\in\Omega}{\cap}(f_A)_\alpha \widetilde{\subseteq} f_A$. Let a be an element of N. Then there exists an L-fuzzy soft irreducible ideal $(f_A)_\beta$ of N such that $(f_A)_\beta(a) = f_A(a)$ and $f_A \subseteq (f_A)_\beta$. Thus, $(f_A)_\beta \in A$. Hence, $\underset{\alpha\in\Omega}{\widetilde{\cap}}(f_A)_\alpha \widetilde{\subseteq} (f_A)_\beta$. So $\underset{\alpha\in\Omega}{\cap}(f_A)_\alpha(a) \subseteq (f_A)_\beta(a) = f_A(a) \Rightarrow \underset{\alpha\in\Omega}{\widetilde{\cap}}(f_A)_\alpha \widetilde{\subseteq} f_A$. Hence, $\underset{\alpha\in\Omega}{\widetilde{\cap}}(f_A)_\alpha \widetilde{=} f_A$. ■

Proposition 88 *Let N be a fully idempotent zerosymmetric nearring and f_A be an L-fuzzy soft ideal of N. Then the following assertions are equivalent*:

(i) *f_A is L-fuzzy soft prime.*

(ii) *f_A is L-fuzzy soft strongly irreducible.*

(iii) *f_A is L-fuzzy soft irreducible.*

Proof. $(i) \Rightarrow (ii)$ Suppose f_A is an L-fuzzy soft prime ideal of N and g_B, h_C are L-fuzzy soft ideals of N such that $g_B \widetilde{\cap} h_C \widetilde{\subseteq} f_A$. As

$$g_B \odot h_C \widetilde{\subseteq} g_B \widetilde{\cap} h_C \widetilde{\subseteq} f_A,$$

and f_A is an L-fuzzy soft prime ideal, so $g_B \widetilde{\subseteq} f_A$ or $h_C \widetilde{\subseteq} f_A$. Thus, f_A is an L-fuzzy soft strongly irreducible ideal.

$(ii) \Rightarrow (iii)$ Suppose f_A is an L-fuzzy soft strongly irreducible ideal and g_B, h_C are L-fuzzy soft ideals of N such that $g_B \widetilde{\cap} h_C \widetilde{=} f_A$. Then as f_A is an L-fuzzy soft strongly irreducible, so $g_B \widetilde{\subseteq} f_A$ or $h_C \widetilde{\subseteq} f_A$. But $f_A \widetilde{\subseteq} g_B$ and $f_A \widetilde{\subseteq} h_C$ so either $f_A \widetilde{=} g_B$ or $f_A \widetilde{=} h_C$ that is f_A is an L-fuzzy soft irreducible ideal of N.

$(iii) \Rightarrow (i)$ Suppose f_A is an L-fuzzy soft irreducible ideal of N and g_B, h_C are L-fuzzy soft ideals of N such that

$$g_B \odot h_C \widetilde{\subseteq} f_A \Rightarrow < g_B \odot h_C > \widetilde{\subseteq} f_A \Rightarrow g_B \widetilde{\cap} h_C \widetilde{\subseteq} f_A.$$

Since the set of all L-fuzzy soft ideals of N forms a distributive lattice under the sum and intersection of L-fuzzy soft ideals, we have

$$(g_B \cap h_C) \oplus f_A \widetilde{=} f_A \Rightarrow (g_B \oplus f_A) \cap (h_C \oplus f_A) \widetilde{=} f_A.$$

Since f_A is an L-fuzzy soft irreducible so $g_B \oplus f_A \widetilde{=} f_A$ or $h_C \oplus f_A \widetilde{=} f_A \Rightarrow g_B \widetilde{\subseteq} f_A$ or $h_C \widetilde{\subseteq} f_A$. Hence, f_A is an L-fuzzy soft prime. ■

Theorem 89 *Let N be a zerosymmetric nearring. Then the following assertions are equivalent*:

(i) *N is fully L-fuzzy soft idempotent nearring.*

(*ii*) *Each L-fuzzy soft ideal of N is the intersection of those L-fuzzy soft prime ideals of N which contain it.*

(*iii*) *Each L-fuzzy soft ideal of N is L-fuzzy soft semiprime.*

Proof. $(i) \Rightarrow (ii)$The concept of irreducibility and primeness for L-fuzzy soft ideals coincide in a fully L-fuzzy soft idempotent nearring. By Proposition 87, every proper L-fuzzy soft ideal of N is the intersection of all those L-fuzzy soft irreducible ideals of N which contain it. Hence every ideal is the intersection of L-fuzzy soft prime ideals of N which contain it.

$(ii) \Rightarrow (iii)$ Since the intersection of L-fuzzy soft prime ideals of N is an L-fuzzy soft semiprime ideal, so each L-fuzzy soft ideal of N is an L-fuzzy soft semiprime ideal.

$(iii) \Rightarrow (i)$ Let f_A be an L-fuzzy soft ideal of N. As $(f_A)^2 \,\widetilde{\subseteq} < (f_A)^2 >$. By (iii), $< (f_A)^2 >$ is semiprime so $f_A \widetilde{\subseteq} < (f_A)^2 >$. But $< (f_A)^2 > \widetilde{\subseteq} f_A$ always. Hence, $(f_A) \widetilde{=} < (f_A)^2 >$. ■

Theorem 90 *Let N be a zerosymmetric nearring. Then the following assertions are equivalent*:

(*i*) *N is fully L-fuzzy soft idempotent and the set of all L-fuzzy soft ideals of N is totally ordered.*

(*ii*) *N is fully L-fuzzy soft prime that is every L-fuzzy soft ideal of N is prime.*

Proof. $(i) \Rightarrow (ii)$ Let f_A, g_B, h_C be L-fuzzy soft ideals of N such that $f_A \odot g_B \widetilde{\subseteq} h_C \Rightarrow < f_A \odot g_B > \widetilde{\subseteq} h_C$. Since N is fully L-fuzzy soft idempotent, so

$$f_A \widetilde{\cap} g_B \widetilde{=} < f_A \odot g_B > \widetilde{\subseteq} h_C.$$

Since the set of L-fuzzy soft ideals of N is totally ordered, so either $f_A \widetilde{\subseteq} g_B$ or $g_B \widetilde{\subseteq} f_A$ that is either $f_A \widetilde{=} f_A \widetilde{\cap} g_B$ or $g_B \widetilde{=} f_A \widetilde{\cap} g_B$.

Thus, either $f_A \widetilde{\subseteq} h_C$ or $g_B \widetilde{\subseteq} h_C$.

$(ii) \Rightarrow (i)$ Suppose each L-fuzzy soft ideal of N is prime. Let f_A be an L-fuzzy soft ideal of N. As $(f_A)^2 \,\widetilde{\subseteq} < (f_A)^2 >$ this implies that $f_A \widetilde{\subseteq} < (f_A)^2 >$, but $< (f_A)^2 > \widetilde{\subseteq} f_A$ always. Hence, $f_A \widetilde{=} < (f_A)^2 >$ that is each L-fuzzy soft ideal is idempotent.

Let g_B, h_C be L-fuzzy soft ideals of N. As $g_B \cap h_C$ is an L-fuzzy soft ideal of N and $g_B \odot h_C \widetilde{\subseteq} g_B \widetilde{\cap} h_C$. Thus , either $g_B \widetilde{\subseteq} g_B \widetilde{\cap} h_C$ or $h_C \widetilde{\subseteq} g_B \widetilde{\cap} h_C \Rightarrow g_B \widetilde{\subseteq} h_C$ or $h_C \widetilde{\subseteq} g_B$. Hence, the set of L-fuzzy soft ideals of N is totally ordered. ■

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