Bound Muon To Bound Electron And a Majoron Decay in position space

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Asif Ali

Department of Physics
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Department of Physics
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A dissertation submitted in partial fulfillment of the requirements for the degree of
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Certificate

This is to certify that **ASIF ALI** Roll no. **02181913030** has carried out the work contained in this dissertation under my supervision and is accepted by the Department of Physics, Quaid-i-Azam University, Islamabad as satisfying the dissertation requirement for the degree of Master of Philosophy in Physics.

Supervisor:

Dr.M. Jamil Aslam
Department of Physics
Quaid-i-Azam University

Sumbmitted through:

Chairman:
Dr. Nawazish Ali Khan
Department of Physics
Quaid-i-Azam University,
Islamabad, Pakistan.

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Asif Ali

Contents

List of Notations and Abbreviations

QED - Quantum Electrodynamics

SM-Standard Model

CLFV- Charge leptop flavor violation

Z- Atomic number

α- Fine structure constant

$\epsilon = 0.01$

b=0.99

m_e-mass of electron

m_μ-mass of muon

$g_1 and, g_2$-Coupling constants

$g_{\mu\nu}$ - Minkowski metric

p (Roman style) - 4-vectors

p (Bold face) - 3-vectors

$p = |\mathbf{p}|$ (Italic style) - scalars

$\not{p} = \mathrm{p}_\mu \gamma^\mu$ - Operators contracted with gamma matrices

$\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$ - Dirac matrices

$a = \iota\sqrt{\frac{1-\gamma}{1+\gamma}}$

Indices:

Latin letters (i, j) run over $1, 2, 3$

Greek letters (μ, ν) run over $0, 1, 2, 3$.

Abstract

Like an electron, the muon can be localized under the central potential of a nuclei to form a muonic atom. Its decay into an electron gives continuous as well as the discrete energy spectrum. To study muon decay, a lot of experiments have been performed to find the possible new physics, also known as physics beyond the SM. In this dissertation, we studied bound muon dec'-ytfday to a bound electron and neutrinos and calculated the branching ratio for different values of Z . We have also studied neutrino-less conversion of bound muon into a bound electron by taking a scalar Majoron instead of neutrinos. We hope that the present work is useful to hunt for the CLFV in muon decays.

b

Part I

Introduction

The muon is an elementary particle, similar to an electron in many respects, but its mass is 207 times greater than an electron, therefore, having small Bohr radius (it is inversely proportional to mass $r \backsim \frac{1}{m}$) and has a finite lifetime (about $2.2\mu s$). It was first discovered in cosmic-ray interactions in 1937, the study of its characteristics and decay rate contributed a vital role to developed and test the SM. A muonic atom can be formed by bombarding a high-energy muon beam on an atom. The atom captures muons, which fall into a $1S$ state because of heavy weight, ejecting all electrons in the atom. Within SM the muon decay into electron as

$$\mu \rightarrow e + \nu_\mu + \bar{\nu_e},$$

where ν_μ and $\bar{\nu_e}$ are neutrino and antineutrino of muon and electron, respectively, which are considered to be mass less. The SM is not complete yet. The nature of the flavors of elementary particles is mysterious, their characteristics and structure let us think and research beyond the SM. Experiments are conducted at Fermi lab [1] and Coherent Muon to Electron Transition (COMET) [2] to find physics beyond SM. Both experiments focus on studying muon to electron conversion without emission of neutrinos, which led to occur CLFV. To explain the CLFV, neutrino oscillation is considered, which states that, while propagating in space the neutrino beam no longer contains initial charge flavor (ν_e, ν_μ,ν_τ) i.e., it continuously converts from one flavor to another flavor. It means the neutrinos masses are non-zero and distinct, the flavor of neutrino is determined as a superposition of so-called mass eigenstates. Suppose the mass eigenstates are ν_1, ν_2 and ν_3 then the neutrinos flavor is expressed as:

$$\begin{aligned}
\nu_e &= a_{e1}\nu_1 + a_{e2}\nu_2 + a_{e3}\nu_3 \\
\nu_\mu &= a_{\mu1}\nu_1 + a_{\mu2}\nu_2 + a_{\mu3}\nu_3 \\
\nu_\tau &= a_{\tau1}\nu_1 + a_{\tau2}\nu_2 + a_{\tau3}\nu_3.
\end{aligned}$$

In above expressions a_1,a_2 and a_3 represent normalization constants. However, even considering the neutrino oscillation the predicted branching ratio of CLFV processes are less than 10^{-50} [3] which is beyond the sensitivity of any ongoing or future experiments. However, if we find the experimental signatures CLFV, it would be a hint of a new physics.

The purpose of this work is to study the SM decay rate of a bound muon into a bound electron in position space. First of all we will focus on the SM decay $(Z\mu) \rightarrow (Ze)\nu_\mu\bar{\nu}_e$, where $(Z\mu)$ and (Ze) correspond to the bound muon and bound electron, respectively. Later we find the branching ratio of bound muon decaying to a bound electron with emission of scalar Majoron (J), i.e., $(Z\mu) \rightarrow (Ze)J$, which is purely a lepton flavor violating decay. In both decays, we considered the nucleus to be spin-less and there is only a muon in the $1S$ state.

This dissertation is organized as follows: In Chapter 2 we find the exact solutions for a bound muon and electron [4] , comprising of solving Dirac equation in the presence of central potential. We derived wave-function in the point nucleus approximation by considering the Coulomb potential $V(r) = -\frac{Ze}{r}$, and obtained wave functions for the $1S$ state.

Chapter 3 is based on detailed calculation of bound muon decay into a bound electron, neutrino and antineutrino: $(Z\mu) \rightarrow (Ze)\nu_\mu\bar{\nu}_e$ in position space. We use the standard Casimir's trick based method [5], to find the transition amplitude. In order to find the dependence of the decay rate on $Z\alpha$, we consider the equal muon and electron mass. Later, by considering the actual masses of these particles, we will calculate the dependence of decay rate on the $Z\alpha$.

Chapter 4 comprises of complete calculation of decay rate of $(Z\mu) \rightarrow (Ze)J$ decay. We followed the same method as we developed in Chapter 3, except by changing the the coupling at $\mu \rightarrow e$ vertex, which in this case is just $1-\gamma^5$. We find that in the equal muon to electron mass limit, the branching ratio goes as the third power of $\gamma = \sqrt{1-(Z\alpha)^2}$, which is quite different from the $(Z\mu) \rightarrow (Ze)\nu_\mu\bar{\nu}_e$. The main findings of the work are concluded in the Chapter 5.

Chapter II

Dirac Equation in External Electromagnetic Field

In order to study bound muon decay, first of all, we have to obtain relativistic wave function of an initial state of muon and final state of electron. Therefore, we solve Dirac equation in central field, which can be solved analytically for Coulomb potential [4].

1 Central Field Dirac Equation for Relativistic Electron

The total angular momentum for an electron moving in a spherically symmetric field is given by

$$\mathbf{J} = \mathbf{L} + \mathbf{S}, \tag{1.1}$$

where

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \tag{1.2}$$

is the orbital momentum and spherical harmonics are its eigen functions $Y_l^m(\hat{\mathbf{r}})$:

$$\mathbf{L}^2 Y_l^m(\hat{\mathbf{r}}) = l(l+1) Y_l^m(\hat{\mathbf{r}}), \tag{1.3}$$

$$L_z Y_l^m(\hat{\mathbf{r}}) = m Y_l^m(\hat{\mathbf{r}}). \tag{1.4}$$

The operator

$$\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}, \tag{1.5}$$

is the spin momentum whose eigen functions are two-component spinors η_μ

$$\mathbf{S}^2 \eta_\mu = \frac{1}{2}\left(\frac{1}{2}+1\right)\eta_\mu = \frac{3}{4}\eta_\mu, \tag{1.6}$$

$$S_z \eta_\mu = \mu \eta_\mu, \tag{1.7}$$

where $\mu = \pm\frac{1}{2}$. In the presence of an electromagnetic field, the stationary Dirac equation can be defined as

$$[\gamma^\mu(\mathrm{p}_\mu - e\mathrm{A}_\mu) - m]\,\Phi = 0. \tag{1.8}$$

Using this equation we have derived the Dirac-Coulomb Hamiltonian $\mathcal{H}_{DC}(\mathbf{r})$ which satisfies the following equation given below

$$\mathcal{H}_{DC}(\mathbf{r})\,\Phi(\mathbf{r}) = E\Phi(\mathbf{r}),$$

where

$$\mathcal{H}_{DC}(\mathbf{r}) = \boldsymbol{\alpha}\cdot\mathbf{p} - \frac{e^2 Z}{|\mathbf{r}|} + m\beta. \tag{1.9}$$

The wave function Φ in the component form is

$$\Phi(\mathbf{r}) = \begin{pmatrix} \Phi^u(\mathbf{r}) \\ \Phi^\iota(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} g_{Ejl}(r)\,\chi_{jlM}(\hat{\mathbf{r}}) \\ i f_{Ejl}(r)\,\chi_{j\bar{l}M}(\hat{\mathbf{r}}) \end{pmatrix}, \tag{1.10}$$

where the quantum number l defines the orbital angular momentum and $\bar{l}$ will be defined below. The functions $g_{Ejl}(r)$ and $f_{Ejl}(r)$ are the radial wave functions corresponding to upper and lower components, respectively. The two-component functions $\chi_{jlM}(\hat{\mathbf{r}})$ have only the angular dependence. As $\mathcal{H}_{DC}$ commutes with both operators $\mathbf{J}^2$ and J_z, so wave function $\Phi(\mathbf{r})$, or their angular parts $\chi_{jlM}(\hat{\mathbf{r}})$, must be their eigen function as well:

$$\begin{cases} [\mathcal{H}_{DC}(\mathbf{r}), \mathbf{J}^2] = 0 \\ [\mathcal{H}_{DC}(\mathbf{r}), J_z] \ = 0 \end{cases} \Rightarrow \begin{cases} \mathbf{J}^2\Phi(\mathbf{r}) = j(j+1)\,\Phi(\mathbf{r}) \\ J_z\Phi(\mathbf{r}) = M\Phi(\mathbf{r}) \end{cases}. \tag{1.11}$$

The angular wave functions $\chi_{jlM}(\hat{\mathbf{r}})$ satisfy the set of relations (1.11)

$$\begin{aligned} \mathbf{J}^2\chi_{jlM}(\hat{\mathbf{r}}) &= j(j+1)\,\chi_{jlM}(\hat{\mathbf{r}}) \\ \chi_{jlM}(\hat{\mathbf{r}}) &= M\chi_{jlM}(\hat{\mathbf{r}}), \end{aligned}$$

$$. \tag{1.12}$$

therefore, the functions $\chi_{jlM}(\hat{\mathbf{r}})$ can be described as a linear combination of the spherical harmonics $Y_l^m(\hat{\mathbf{r}})$ and two-component spinor η_μ

$$\chi_{jlM}(\hat{\mathbf{r}}) = \sum_{m\mu} C^{jM}_{lm\frac{1}{2}\mu} Y_l^m(\hat{\mathbf{r}})\,\eta_\mu. \tag{1.13}$$

Here $C^{jm}_{j_1m_1j_2m_2}$ are the Clebsch-Gordan coefficient for which the following identities are satisfied

$$|j_1 - j_2| \leq j \leq j_1 + j_2. \tag{1.14}$$

The spherical spinors form a complete set of ortho-normalized wave functions

$$\int d\Omega \left(\chi_{jlM}\right)^{\dagger} \chi_{j'l'M'} = \delta_{jj'}\delta_{ll'}\delta_{MM'}. \tag{1.15}$$

Thus using Eq. (1.13) in Eqs. (1.3) and (1.6), we get

$$\begin{aligned}
\mathbf{L}^2\chi_{jlM}\left(\hat{\mathbf{r}}\right) &= \sum_{m\mu} C^{jM}_{lm\frac{1}{2}\mu}\left[\mathbf{L}^2 Y_l^m\left(\hat{\mathbf{r}}\right)\right]\eta_\mu \\
&= l\left(l+1\right)\sum_{m\mu} C^{jM}_{lm\frac{1}{2}\mu} Y_l^m\left(\hat{\mathbf{r}}\right)\eta_\mu = l\left(l+1\right)\chi_{jlM}\left(\hat{\mathbf{r}}\right), \qquad (1.16)\\
\mathbf{S}^2\chi_{jlM}\left(\hat{\mathbf{r}}\right) &= \sum_{m\mu} C^{jM}_{lm\frac{1}{2}\mu}\left[\mathbf{S}^2 Y_l^m\left(\hat{\mathbf{r}}\right)\right]\eta_\mu \\
&= \frac{3}{4}\sum_{m\mu} C^{jM}_{lm\frac{1}{2}\mu} Y_l^m\left(\hat{\mathbf{r}}\right)\eta_\mu = \frac{3}{4}\chi_{jlM}\left(\hat{\mathbf{r}}\right). \qquad (1.17)
\end{aligned}$$

The quantum number l appearing in Eq. (1.10) represents the orbital momentum of the particle along with its parity. Now take the space inversion $\mathbf{P}:\mathbf{r}\rightarrow-\mathbf{r}$ in the Dirac equation (1.8). Such transformation will directly act on the position space on which the wave function (1.10) is defined as

$$\Phi\left(t,\mathbf{r}\right) \rightarrow \Phi'\left(t,\mathbf{P}\mathbf{r}\right) = \mathcal{P}\Phi\left(t,\mathbf{r}\right), \tag{1.18}$$

where $\mathcal{P}$ is the linear operator which should preserve the invariance of the Dirac equation

$$\mathbf{P}\left[\gamma^\mu\left(\mathrm{p}_\mu - e\mathrm{A}_\mu\right) - m\right]\Phi'\left(t,\mathbf{P}\mathbf{r}\right) = 0. \tag{1.19}$$

Thus

$$\begin{aligned}
&\mathbf{P}\left[\gamma^\mu\left(\mathrm{p}_\mu - e\mathrm{A}_\mu\right) - m\right]\mathcal{P}\Phi\left(t,\mathbf{r}\right) \\
&= \left[\mathbf{P}\left\{\gamma^0\left(\mathrm{p}_0 - eV\right)\right\} - \mathbf{P}\left\{\boldsymbol{\gamma}\cdot\left(\mathbf{p}-e\mathbf{A}\right) - m\right\}\right]\mathcal{P}\Phi\left(t,\mathbf{r}\right) \\
&= \left[\gamma^0\left(\mathrm{p}_0 - eV\right) + \boldsymbol{\gamma}\cdot\left(\mathbf{p}-e\mathbf{A}\right) - m\right]\mathcal{P}\Phi\left(t,\mathbf{r}\right) = 0. \qquad (1.20)
\end{aligned}$$

Since the last expression can be expressed as $\left[\gamma^\mu\left(\mathrm{p}_\mu - e\mathrm{A}_\mu\right) - m\right]\Phi\left(t,\mathbf{r}\right) = 0$, it means that

$$\gamma^0\mathcal{P} = \mathcal{P}\gamma^0,\ \boldsymbol{\gamma}\mathcal{P} = -\mathcal{P}\boldsymbol{\gamma}, \tag{1.21}$$

and we can satisfy Eq. (1.21) if we choose

$$\mathcal{P} = c_p\gamma^0, \tag{1.22}$$

where c_p is some c-number, which actually depends on the particle's intrinsic parity. Now

$$\mathcal{P}\Phi\left(t,\mathbf{Pr}\right)=c_p\gamma^0\Phi\left(t,-\mathbf{r}\right)=c_p\begin{pmatrix} g_{Ejl}\left(r\right)\chi_{jlM}\left(-\hat{\mathbf{r}}\right) \\ -if_{Ejl}\left(r\right)\chi_{j\bar{l}M}\left(-\hat{\mathbf{r}}\right)\end{pmatrix}. \tag{1.23}$$

The space inversion only affects the spherical harmonics $Y_l^m\left(\hat{\mathbf{r}}\right)=Y_l^m\left(\theta,\phi\right)$ in the spherical polar coordinates in the following manner

$$\mathbf{P}:\begin{cases}\theta\rightarrow\pi-\theta \\ \phi\rightarrow\pi+\phi\end{cases}\Rightarrow\mathbf{P}Y_l^m\left(\theta,\phi\right)=\left(-1\right)^l Y_l^m\left(\theta,\phi\right). \tag{1.24}$$

Therefore,

$$\chi_{jlM}\left(-\hat{\mathbf{r}}\right)=\sum_{m\mu}C^{jM}_{lm\frac{1}{2}\mu}Y_l^m\left(-\hat{\mathbf{r}}\right)\eta_\mu=\left(-1\right)^l\chi_{jlM}\left(\hat{\mathbf{r}}\right). \tag{1.25}$$

Using the above result in Eq. (1.23) we get

$$\mathcal{P}\Phi\left(t,\mathbf{Pr}\right)=c_p\begin{pmatrix} g_{Ejl}\left(r\right)\left(-1\right)^l\chi_{jlM}\left(\hat{\mathbf{r}}\right) \\ if_{Ejl}\left(r\right)\left(-1\right)^{\bar{l}+1}\chi_{j\bar{l}M}\left(\hat{\mathbf{r}}\right)\end{pmatrix}, \tag{1.26}$$

where the components of this wave function should have the same parity as they have in Eq. (1.10). It follows that

$$l=\bar{l}+1. \tag{1.27}$$

From the system of two equations for the upper and lower components of the bispinor (for detailed derivation see Appendix A of [4]), it follows

$$\left(E+m\right)\Phi^\iota\left(\mathbf{p}\right)=\left(\boldsymbol{\sigma}\cdot\mathbf{p}\right)\Phi^u\left(\mathbf{p}\right), \tag{1.28}$$

$$\left(E-m\right)\Phi^u\left(\mathbf{p}\right)=\left(\boldsymbol{\sigma}\cdot\mathbf{p}\right)\Phi^\iota\left(\mathbf{p}\right). \tag{1.29}$$

After substituting the explicit form of upper and lower components of equation (1.10) in Eq. (1.28), it becomes

$$\left(E+m\right)if_{Ejl}\left(r\right)\chi_{j\bar{l}M}\left(\hat{\mathbf{r}}\right)=p\left(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\right)g_{Ejl}\left(r\right)\chi_{jlM}\left(\hat{\mathbf{r}}\right). \tag{1.30}$$

Under the spatial rotation $(\boldsymbol{\sigma}.\mathbf{p})$ acts in a similar way as $(\boldsymbol{\sigma}.\hat{\boldsymbol{r}})$. Thus

$$\left(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\right)\chi_{jlM}\left(\hat{\mathbf{r}}\right)=\mathrm{c}\chi_{j\bar{l}M}\left(\hat{\mathbf{r}}\right). \tag{1.31}$$

Using orthogonality condition $\int d\Omega \left(X_{jlM}\right)^{\dagger} X_{j'l'm'} = \delta_{jj'}\delta_{ll'}\delta_{mm'}$, the constant becomes

$$\mathrm{c} = \int \chi^{\dagger}_{j\bar{l}M}\left(\hat{\mathbf{r}}\right)\left(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}\right)\chi_{jlM}\left(\hat{\mathbf{r}}\right)d\Omega. \tag{1.32}$$

To evaluate this integral it will be useful to define unit vectors in terms of spherical coordinates

$$\begin{aligned}
\hat{\mathrm{r}}_{\mathrm{x}} &= \sqrt{\frac{2\pi}{3}}\left(Y_1^{-1} - Y_1^{1}\right),\\
\hat{\mathrm{r}}_{\mathrm{y}} &= i\sqrt{\frac{2\pi}{3}}\left(Y_1^{-1} + Y_1^{1}\right),\\
\hat{\mathrm{r}}_{\mathrm{z}} &= 2\sqrt{\frac{\pi}{3}}Y_1^{0},
\end{aligned} \tag{1.33}$$

where $\mathrm{x} = \mathrm{r}\sin\theta\cos\phi$, $\mathrm{y} = \mathrm{r}\sin\theta\sin\phi$, $\mathrm{z} = \mathrm{r}\cos\theta$ and $Y_1^{-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{-i\phi}\sin\theta, Y_1^{1} = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}e^{i\phi}\sin\theta$,$Y_1^{0} = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$. Then use the formula for the integration of three spherical harmonics

$$\int d\Omega Y_{l_1}^{m_1 *} Y_{l_2}^{m_2} Y_{l_3}^{m_3} = \sqrt{\frac{\left(2l_2+1\right)\left(2l_3+1\right)}{4\pi\left(2l_1+1\right)}} C^{l_1 m_1}_{l_2 m_2 l_3 m_3} C^{l_1 0}_{l_2 0 l_3 0}. \tag{1.34}$$

Also the Pauli matrices act on spinors in a following manner

$$\eta^{\dagger}_{\mu_1}\sigma^{x}\eta_{\mu_2} = \delta_{\mu_1,-\mu_2}, \tag{1.35}$$
$$\eta^{\dagger}_{\mu_1}\sigma^{\mathrm{y}}\eta_{\mu_2} = \left(-1\right)^{1-\mu_1}\delta_{\mu_1,-\mu_2}, \tag{1.36}$$
$$\eta^{\dagger}_{\mu_1}\sigma^{z}\eta_{\mu_2} = \left(-1\right)^{\frac{1}{2}-\mu_1}\delta_{\mu_1,\mu_2}. \tag{1.37}$$

Inserting everything in Eq. (1.32), we obtain the coefficient c that is equal to -1. Thus, Eq. (1.31) can be rewritten as

$$X_{j\bar{l}M}(\hat{\mathbf{r}}) = -\left(\boldsymbol{\sigma}.\hat{\mathbf{r}}\right)X_{jlM}(\hat{\mathbf{r}}).$$

Substitute these result into the set of equations (1.28) and (1.29) with Coulomb Potential

$$\begin{aligned}
\left(E - eV - m\right)\Phi^{u}\left(\mathbf{p}\right) - \left(\boldsymbol{\sigma}\cdot\mathbf{p}\right)\Phi^{\iota}\left(\mathbf{p}\right) = 0,\\
\left(E - eV + m\right)\Phi^{\iota}\left(\mathbf{p}\right) - \left(\boldsymbol{\sigma}\cdot\mathbf{p}\right)\Phi^{u}\left(\mathbf{p}\right) = 0,
\end{aligned} \tag{1.38}$$

we get the following equation for the lower component of the Dirac bispinor

$$(\boldsymbol{\sigma}\cdot\mathbf{p})\,\Phi^{\iota}(\mathbf{p}) = i\,(\boldsymbol{\sigma}\cdot\mathbf{p})\,f_{Ejl}(r)\,\chi_{j\bar{l}M}(\hat{\mathbf{r}}) = -i\,(\boldsymbol{\sigma}\cdot\mathbf{p})\,(\boldsymbol{\sigma}\cdot\hat{\mathbf{r}})\,f_{Ejl}(r)\,\chi_{jlM}(\hat{\mathbf{r}})\,. \tag{1.39}$$

Using Pauli matrices identity $(\boldsymbol{\sigma}\cdot\mathbf{p})(\boldsymbol{\sigma}.\mathbf{r}) = (\mathbf{p}.\mathbf{r}) + i\sigma.[\mathbf{p}\times\mathbf{r}]$, and $\hat{\mathbf{r}} = \frac{\vec{\mathrm{r}}}{|\mathrm{r}|}$, Eq. (1.39) becomes

$$\begin{aligned}(\boldsymbol{\sigma}\cdot\mathbf{p})\,\Phi^{\iota}(\mathbf{p}) &= -\left\{i\,(\mathbf{p}\cdot\mathbf{r}) - \boldsymbol{\sigma}\cdot[\mathbf{p}\times\mathbf{r}]\right\}\frac{f_{Ejl}(\mathrm{r})}{\mathrm{r}}\chi_{jlM}(\hat{\mathbf{r}}) \\ &= -\left\{(\nabla\cdot\mathbf{r})\,\frac{f_{Ejl}(\mathrm{r})}{\mathrm{r}} - \boldsymbol{\sigma}\cdot[\mathbf{p}\times\mathbf{r}]\,\frac{f_{Ejl}(\mathrm{r})}{\mathrm{r}}\right\}\chi_{jlM}(\hat{\mathbf{r}}) \\ &= -\left\{\mathbf{r}\nabla\left(\frac{f_{Ejl}(\mathrm{r})}{\mathrm{r}}\right) + \frac{f_{Ejl}(\mathrm{r})}{\mathrm{r}}\mathrm{div}\,(\mathbf{r}) + (\boldsymbol{\sigma}\cdot\mathbf{L})\,\frac{f_{Ejl}(\mathrm{r})}{\mathrm{r}}\right\}\chi_{jlM}(\hat{\mathbf{r}})\,.\end{aligned} \tag{1.40}$$

With the differential calculus identity

$$\nabla.\,(f\boldsymbol{A}) = f\,(\nabla.\boldsymbol{A}) + \boldsymbol{A}.\,(\nabla f)\,,$$

Eq. (1.40) leads to

$$(\nabla.\mathbf{r})\,\frac{f_{Ejl}(\mathrm{r})}{\mathrm{r}} = \mathrm{r}\nabla.\left(\frac{f_{Ejl}(\mathrm{r})}{\mathrm{r}}\right) + \frac{f_{Ejl}(\mathrm{r})}{\mathrm{r}}div\,(\mathbf{r})\,,$$

$$(\nabla.\mathbf{r})\,\frac{f_{Ejl}(\mathrm{r})}{\mathrm{r}} = \mathrm{r}\nabla.\left(\frac{1}{\mathrm{r}}\right)f_{Ejl}(\mathrm{r}) + \frac{df_{Ejl}(\mathrm{r})}{d\mathrm{r}}. \tag{1.41}$$

As we know $div\,(\mathbf{r}) = 3$, $\mathrm{r}\nabla.\left(\frac{1}{\mathrm{r}}\right) = -\frac{1}{\mathrm{r}}$, hence Eq. (1.41) becomes

$$(\nabla.\mathbf{r})\,\frac{f_{Ejl}(\mathrm{r})}{\mathrm{r}} = -\frac{f_{Ejl}(\mathrm{r})}{\mathrm{r}} + 3\frac{f_{Ejl}(\mathrm{r})}{\mathrm{r}} + \frac{df_{Ejl}(\mathrm{r})}{d\mathrm{r}} = \frac{2}{\mathrm{r}}f_{Ejl}(\mathrm{r}) + \frac{df_{Ejl}(\mathrm{r})}{d\mathrm{r}},$$

$$(\boldsymbol{\sigma}.\mathbf{p})\,\varPhi^{l}(\mathbf{p}) = -\left\{\frac{2}{\mathrm{r}}f_{Ejl}(\mathrm{r}) + \frac{df_{Ejl}(\mathrm{r})}{d\mathrm{r}} + \frac{1}{\mathrm{r}}\,(\sigma.\mathbf{L})\,f_{Ejl}(\mathrm{r})\right\}X_{jlM}(\hat{\mathbf{r}}). \tag{1.42}$$

Now consider the operator identity

$$\mathbf{J}^2 = (\mathbf{L}+\mathbf{S})^2 = \mathbf{L}^2 + 2\mathbf{S}\cdot\mathbf{L} + \mathbf{S}^2 \Rightarrow 2\mathbf{S}\cdot\mathbf{L} = \boldsymbol{\sigma}\cdot\mathbf{L} = \mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2, \tag{1.43}$$

which upon acting on a spherical spinor $\chi_{jlM}\left(\hat{\mathbf{r}}\right)$ gives

$$\begin{aligned}
\left(\boldsymbol{\sigma}\cdot\mathbf{L}\right)\chi_{jlM}\left(\hat{\mathbf{r}}\right) = \left(\mathbf{J}^2-\mathbf{L}^2-\mathbf{S}^2\right)\chi_{jlM}\left(\hat{\mathbf{r}}\right) = \left[j\left(j+1\right)-l\left(l+1\right)-\frac{3}{4}\right]\chi_{jlM}\left(\hat{\mathbf{r}}\right),\\
\equiv -\left(1+\kappa_{jl}\right)\chi_{jlM}\left(\hat{\mathbf{r}}\right), \qquad (1.44)
\end{aligned}$$

where the quantum number κ_{jl} can be defined as

$$\kappa_{jl} = l\left(l+1\right)-j\left(j+1\right)-\frac{1}{4}. \qquad (1.45)$$

If $j = l-\frac{1}{2}$ then

$$\kappa_{jl} = l\left(l+1\right)-\left(l-\frac{1}{2}\right)\left(l+\frac{1}{2}\right)-\frac{1}{4} = l^2+l-l^2+\frac{1}{4}-\frac{1}{4} = l \qquad (1.46)$$

and if $j = l+\frac{1}{2}$

$$\kappa_{jl} = l\left(l+1\right)-\left(l+\frac{1}{2}\right)\left(l+\frac{3}{2}\right)-\frac{1}{4} = l^2+l-l^2-2l-\frac{3}{4}-\frac{1}{4} = -\left(l+1\right). \qquad (1.47)$$

To sum up

$$\kappa_{jl} = \begin{cases} l, & \text{if } j = l-\frac{1}{2} \\ -\left(l+1\right), & \text{if } j = l+\frac{1}{2} \end{cases}, \text{ or } \kappa_{jl} = \begin{cases} j+\frac{1}{2}, & \text{if } j = l-\frac{1}{2} \\ -\left(j+\frac{1}{2}\right), & \text{if } j = l+\frac{1}{2} \end{cases}, \qquad (1.48)$$

and

$$\begin{aligned}
\kappa_{jl} =& -\kappa_{j\bar{l}}, \qquad (1.49)\\
\bar{l} =& l-1. \qquad (1.50)
\end{aligned}$$

Now Eq. (1.42) can be expressed in terms of the newly defined quantum number κ_{jl}

$$\left(\boldsymbol{\sigma}.\mathbf{p}\right)\mathit{\Phi}^l\left(\mathbf{p}\right) = -\left\{\frac{df_{Ejl}\left(\mathrm{r}\right)}{d\mathrm{r}}+\frac{\left(2-1-k_{jl}\right)}{\mathrm{r}}f_{Ejl}\left(\mathrm{r}\right)\right\}X_{jlM}(\hat{\mathbf{r}}),$$

$$\left(\boldsymbol{\sigma}\cdot\mathbf{p}\right)\Phi^{\iota}\left(\mathbf{p}\right) = -\left\{\frac{df_{Ejl}\left(\mathrm{r}\right)}{d\mathrm{r}}+\frac{1-\kappa_{jl}}{\mathrm{r}}f_{Ejl}\left(\mathrm{r}\right)\right\}\chi_{jlM}\left(\hat{\mathbf{r}}\right). \qquad (1.51)$$

Similarly, for the upper component of the Dirac bispinor

$$(\boldsymbol{\sigma}\cdot\mathbf{p})\,\Phi^{u}(\mathbf{p}) = (\boldsymbol{\sigma}\cdot\mathbf{p})\,g_{Ejl}(\mathrm{r})\,\chi_{jlM}(\hat{\mathbf{r}}) = -(\boldsymbol{\sigma}\cdot\mathbf{p})\,(\boldsymbol{\sigma}\cdot\mathbf{r})\,\frac{g_{Ejl}(\mathrm{r})}{\mathrm{r}}\chi_{j\bar{l}M}(\hat{\mathbf{r}})$$
$$= -\left\{-i(\nabla\cdot\mathbf{r}) + i\boldsymbol{\sigma}\cdot[\mathbf{p}\times\mathbf{r}]\right\}\frac{g_{Ejl}(\mathrm{r})}{\mathrm{r}}\chi_{j\bar{l}M}(\hat{\mathbf{r}})$$
$$= -\left\{-i(\nabla.\mathbf{r}) - i\sigma.\mathbf{L}\right\}\frac{g_{Ejl}(\mathrm{r})}{\mathrm{r}}X_{j\bar{l}M}(\hat{\boldsymbol{r}})\chi_{j\bar{l}M}(\hat{\mathbf{r}})\,.$$

using $k_{jl} = -k_{j\bar{l}}$

$$(\boldsymbol{\sigma}.\mathbf{p})\,\mathit{\Phi}^{u}(\mathbf{p}) = i\left\{(\nabla\cdot\mathbf{r})\,\frac{g_{Ejl}(\mathrm{r})}{\mathrm{r}} - \frac{(1-k_{jl})}{\mathrm{r}}g_{Ejl}\right\}X_{j\bar{l}M}(\hat{\mathbf{r}}),$$

$$(\boldsymbol{\sigma}.\mathbf{p})\,\mathit{\Phi}^{u}(\mathbf{p}) = i\left\{\frac{2}{\mathrm{r}}g_{Ejl}(\mathrm{r}) + \frac{dg_{Ejl}(\mathrm{r})}{d\mathrm{r}} - \frac{(1-k_{jl})}{\mathrm{r}}g_{Ejl}(\mathrm{r})\right\}X_{j\bar{l}M}(\hat{\mathbf{r}}),$$

$$(\boldsymbol{\sigma}.\mathbf{p})\,\mathit{\Phi}^{u}(\mathbf{p}) = i\left\{\frac{dg_{Ejl}(\mathrm{r})}{d\mathrm{r}} + \frac{(1+k_{jl})}{\mathrm{r}}g_{Ejl}(\mathrm{r})\right\}X_{j\bar{l}M}(\hat{\mathbf{r}}). \tag{1.52}$$

Using the above results in Eq. (1.38) and after simplifying we get

$$\begin{gathered}\left(\tfrac{d}{d\mathrm{r}} + \tfrac{1+\kappa_{jl}}{\mathrm{r}}\right)g_{Ejl}(\mathrm{r}) - (E - eV + m)\,f_{Ejl}(\mathrm{r}) = 0,\\ \left(\tfrac{d}{d\mathrm{r}} + \tfrac{1-\kappa_{jl}}{\mathrm{r}}\right)f_{Ejl}(\mathrm{r}) + (E - eV - m)\,g_{Ejl}(\mathrm{r}) = 0.\end{gathered} \tag{1.53}$$

2 Electron in Coulomb Field of Point Nucleus

Now we will derive wave functions in the point nucleus approximation, which is valid only for $Z\alpha \ll 1$. Consider the Coulomb potential $V(\mathrm{r}) = -\frac{Ze}{\mathrm{r}}$ for the set of equations (1.53). In the limit $\mathrm{r}\to 0$ the set of equations (1.53) reduces to the following form

$$\mathrm{r}\frac{d}{dr}f_{Ejl}(\mathrm{r}) + (1-k_{jl})\,f_{Ejl}(\mathrm{r}) + \mathrm{r}\,(E - eV - m)\,g_{Ejl}(\mathrm{r}) = 0, \tag{2.1}$$
$$\mathrm{r}\frac{d}{dr}g_{Ejl}(\mathrm{r}) + (1+k_{jl})\,g_{Ejl}(\mathrm{r}) - \mathrm{r}\,(E - eV + m)\,f_{Ejl}(\mathrm{r}) = 0, \tag{2.2}$$
$$\mathrm{r}\frac{d}{d\mathrm{r}}f_{Ejl}(\mathrm{r}) + f_{Ejl}(\mathrm{r}) - \frac{k_{jl}}{\mathrm{r}}\,(\mathrm{r}f_{Ejl}(\mathrm{r})) + \left(E + \frac{Ze^2}{\mathrm{r}} - m\right)(\mathrm{r}g_{Ejl}(\mathrm{r})) = 0, \tag{2.3}$$
$$\mathrm{r}\frac{d}{dr}g_{Ejl}(\mathrm{r}) + g_{Ejl}(\mathrm{r}) + \frac{k_{jl}}{\mathrm{r}}\,(\mathrm{r}g_{Ejl}(\mathrm{r})) - \left(E + \frac{Ze^2}{\mathrm{r}} + m\right)(\mathrm{r}f_{Ejl}(\mathrm{r})) = 0. \tag{2.4}$$

Now the terms proportional to $E \pm m$ will be neglected, giving

$$\begin{array}{l} \frac{d}{dr}\left(\mathrm{r}f_{Ejl}\left(\mathrm{r}\right)\right) - \frac{k}{\mathrm{r}}\left(\mathrm{r}f_{Ejl}\left(\mathrm{r}\right)\right) + \frac{Ze^2}{\mathrm{r}}\left(\mathrm{r}g_{Ejl}\left(\mathrm{r}\right)\right) = 0, \\ \frac{d}{d\mathrm{r}}\left(\mathrm{r}g_{Ejl}\left(\mathrm{r}\right)\right) + \frac{k}{\mathrm{r}}\left(\mathrm{r}g_{Ejl}\left(\mathrm{r}\right)\right) - \frac{Ze^2}{\mathrm{r}}\left(\mathrm{r}f_{Ejl}\left(\mathrm{r}\right)\right) = 0. \end{array} \tag{2.5}$$

As we know $\alpha = \frac{e^2}{\hbar c}$, we will make following change of variable and indices are dropped here for brevity

$$G\left(\mathrm{r}\right) = \mathrm{r}g_{Ejl}\left(\mathrm{r}\right), \; F\left(\mathrm{r}\right) = \mathrm{r}f_{Ejl}\left(\mathrm{r}\right), \tag{2.6}$$

$$\begin{array}{l} \left(\frac{d}{d\mathrm{r}} + \frac{\kappa}{\mathrm{r}}\right) G\left(\mathrm{r}\right) - \frac{Z\alpha}{\mathrm{r}} F\left(\mathrm{r}\right) = 0, \\ \left(\frac{d}{d\mathrm{r}} - \frac{\kappa}{\mathrm{r}}\right) F\left(\mathrm{r}\right) + \frac{Z\alpha}{\mathrm{r}} G\left(\mathrm{r}\right) = 0. \end{array} \tag{2.7}$$

In Eq. (2.7) the terms proportional to $E \pm m$ were neglected. Let's consider that the solutions of Eqs. (2.7) are of the form

$$G\left(\mathrm{r}\right) = G_0 \mathrm{r}^{\gamma}, \; F\left(\mathrm{r}\right) = F_0 \mathrm{r}^{\gamma}. \tag{2.8}$$

Upon substitution in Eq. (2.7), we have

$$\begin{array}{c} F_0 \gamma \mathrm{r}^{\gamma-1} - kF_0 \mathrm{r}^{\gamma-1} + Z\alpha G_0 \mathrm{r}^{\gamma-1} = 0, \\ G_0 \gamma \mathrm{r}^{\gamma-1} + kG_0 \mathrm{r}^{\gamma-1} - Z\alpha F_0 \mathrm{r}^{\gamma-1} = 0. \end{array} \tag{2.9}$$

As $\mathrm{r}^{\gamma-1} \neq 0$, we get

$$\begin{array}{l} G_0\left(\gamma + \kappa\right) - F_0 Z\alpha = 0, \\ G_0 Z\alpha + F_0\left(\gamma - \kappa\right) = 0. \end{array} \tag{2.10}$$

This system has non-trivial solutions only when

$$\begin{vmatrix} (\gamma + \kappa) & -Z\alpha \\ Z\alpha & (\gamma - \kappa) \end{vmatrix} = 0 \Rightarrow \gamma^2 = \kappa^2 - \left(Z\alpha\right)^2. \tag{2.11}$$

2.1 Solution For Radial Equation:

Let the solutions for the radial wave functions in Eq. (1.53) be of the form

$$g\left(x\right) = \sqrt{m + E} e^{-\frac{1}{2}x} x^{\gamma-1}\left[W_1\left(x\right) + W_2\left(x\right)\right], \tag{2.12}$$

$$f\left(x\right) = -\sqrt{m - E} e^{-\frac{1}{2}x} x^{\gamma-1}\left[W_1\left(x\right) - W_2\left(x\right)\right], \tag{2.13}$$

where the indices E_{jl} were dropped and some variables were redefined as

$$x = 2\lambda \mathrm{r},\ \lambda = \sqrt{m^2 - E^2}, \Longrightarrow \frac{d}{d\mathrm{r}} = 2\lambda \frac{d}{dx}. \tag{2.14}$$

Now we will express Eq. (1.53) in terms of Eq. (2.14), as follows

$$2\lambda \left(\frac{d}{dx} + \frac{(1+k)}{x} \right) g(x) - (E+m) f(x) - \frac{2Z\lambda\alpha}{x} f(x) = 0. \tag{2.15}$$

Using the radial functions given in Eq. (2.12) and Eq. (2.13) in above equation, it becomes

$$\begin{aligned}
&2\lambda \left(\frac{d}{dx} + \frac{1+\kappa}{x} \right) g(x) - (E+m) f(x) - \frac{2Z\alpha\lambda}{x} f = 0, \\
&2\lambda \left(\frac{d}{dx} + \frac{(1+k)}{x} \right) \left(\sqrt{m+E} e^{-\frac{1}{2}x} x^{\gamma-1} \left[W_1(x) + W_2(x) \right] \right) \\
&- \left(E + m + \frac{2Z\lambda\alpha}{x} \right) \left(-\sqrt{m-E} e^{-\frac{1}{2}x} x^{\gamma-1} \left[W_1(x) - W_2(x) \right] \right) = 0, \\
&\sqrt{m+E} e^{-\frac{1}{2}x} x^{\gamma-1} \left\{ W_1(x) + W_2(x) \right\} \left[\left(-\frac{1}{2} \right) + \frac{(\gamma-1)}{x} \right] \\
&+ e^{-\frac{1}{2}x} x^{\gamma-1} \sqrt{m+E} \left[\frac{d}{dx} \left(W_1(x) + W_2(x) \right) + \frac{(1+k)}{x} \left[W_1(x) + W_2(x) \right] \right] \\
&\sqrt{m-E} e^{-\frac{1}{2}x} x^{\gamma-1} \left\{ W_1(x) - W_2(x) \right\} \left[\frac{1}{2\lambda} (E+m) + \frac{Z\alpha}{x} \right] = 0.
\end{aligned}$$

As $\sqrt{m+E} e^{-\frac{1}{2}x} x^{\gamma-1} \neq 0$, after simplifications we get

$$\begin{aligned}
&\left[-\frac{1}{2} + \frac{(\gamma-1)}{x} + \frac{(1+k)}{x} \right] \left[W_1(x) + W_2(x) \right] + \frac{d}{dx} \left(W_1(x) + W_2(x) \right) \\
&\left(\frac{Z\alpha}{x} \right) \sqrt{\frac{m-E}{m+E}} \left[W_1(x) - W_2(x) \right] + \left[W_1(x) - W_2(x) \right] \frac{\sqrt{m^2-E^2}}{2\lambda} = 0, \\
&\left[-\frac{x}{2} + \gamma + k \right] \left[W_1 + W_2 \right] + x \frac{d}{dx} \left[W_1 + W_2 \right] + \left(\frac{x}{2} + Z\alpha \sqrt{\frac{m-E}{m+E}} \right) \left[W_1 - W_2 \right] = 0.
\end{aligned} \tag{2.16}$$

Rearrangement of Eq. (2.16) gives

$$x \frac{d}{dx} (W_1 + W_2) + (\gamma + \kappa)(W_1 + W_2) - x W_2 + Z\alpha \sqrt{\frac{m-E}{m+E}} (W_1 - W_2) = 0. \tag{2.17}$$

Following the same procedure for the second equation of (1.53), as a first step we get

$$2\lambda\left(\frac{d}{dx}+\frac{1-k}{x}\right)f(x)+(E-m)\,g(x)+\frac{2\lambda Z\alpha}{x}g(x)=0.$$

Again using the radial functions defined in Eq. (2.12) and Eq. (2.13), we obtain

$$\begin{aligned}
&2\lambda\left(\frac{d}{dx}+\frac{1-k}{x}\right)\sqrt{m-E}e^{-\frac{1}{2}x}x^{\gamma-1}\left[W_1(x)-W_2(x)\right]\\
&-\left(E-m+\frac{2\lambda Z\alpha}{x}\right)\sqrt{m+E}e^{-\frac{1}{2}x}x^{\gamma-1}\left[W_1(x)+W_2(x)\right]=0,\\
&\sqrt{m-E}e^{-\frac{1}{2}x}x^{\gamma-1}\left\{W_1(x)-W_2(x)\right\}\left[\left(-\frac{1}{2}\right)+\frac{(\gamma-1)}{x}\right]\\
&+e^{-\frac{1}{2}x}x^{\gamma-1}\sqrt{m-E}\left[\frac{d}{dx}\left(W_1(x)-W_2(x)\right)+\frac{(1-k)}{x}\left[W_1(x)-W_2(x)\right]\right]\\
&\sqrt{m+E}e^{-\frac{1}{2}x}x^{\gamma-1}\left\{W_1(x)+W_2(x)\right\}\left[\frac{1}{2\lambda}(m-E)-\frac{Z\alpha}{x}\right]=0.
\end{aligned}$$

After some simplification, it will lead to

$$(W_1-W_2)\left(\gamma-k-\frac{x}{2}\right)+x\frac{d}{dx}(W_1-W_2)+\frac{x}{2}(W_1+W_2)-Z\alpha\sqrt{\frac{m+E}{m-E}}(W_1+W_2)=0,$$

$$x\frac{d}{dx}(W_1-W_2)+(\gamma-\kappa)(W_1-W_2)+xW_2-Z\alpha\sqrt{\frac{m+E}{m-E}}(W_1+W_2)=0. \tag{2.18}$$

Adding Eqs. (2.17) and (2.18), we have

$$2\left[x\frac{d}{dx}W_1+\gamma W_1+kW_2\right]+(Z\alpha)\,W_1\left[\sqrt{\frac{m-E}{m+E}}-\sqrt{\frac{m+E}{m-E}}\right]-Z\alpha W_2\left[\sqrt{\frac{m-E}{m+E}}+\sqrt{\frac{m+E}{m-E}}\right]=0,$$

$$x\frac{d}{dx}W_1+\gamma W_1+kW_2+\frac{Z\alpha W_1}{2}\left[-\frac{2E}{\lambda}\right]-\frac{Z\alpha W_2}{2}\frac{2m}{\lambda}=0,$$

$$x\frac{dW_1}{dx}+\left(\gamma-\frac{Z\alpha E}{\lambda}\right)W_1+\left(\kappa-\frac{Z\alpha m}{\lambda}\right)W_2=0. \tag{2.19}$$

And in case of subtraction of Eq. (2.17) and Eq. (2.18), we are left with

$$2\left[x\frac{d}{dx}W_2+\gamma W_2+kW_1-xW_2\right]+Z\alpha W_1\left[\sqrt{\frac{m-E}{m+E}}+\sqrt{\frac{m+E}{m-E}}\right]+Z\alpha W_2\left[\sqrt{\frac{m+E}{m-E}}-\sqrt{\frac{m-E}{m+E}}\right]=0,$$

$$x\frac{d}{dx}W_2+\gamma W_2+kW_1-xW_2+\frac{Z\alpha m}{\lambda}W_1+\frac{Z\alpha E}{\lambda}W_2=0,$$

$$x\frac{dW_2}{dx}+\left(\gamma+\frac{Z\alpha E}{\lambda}-x\right)W_2+\left(\kappa+\frac{Z\alpha m}{\lambda}\right)W_1=0. \tag{2.20}$$

From Eq. (2.19), W_2 is given by

$$W_2=\left[\left(\frac{Z\alpha E}{\lambda}-\gamma\right)W_1-x\frac{dW_1}{dx}\right]\left[\kappa-\frac{Z\alpha m}{\lambda}\right]^{-1}. \tag{2.21}$$

Differentiate W_2 with respect to x gives

$$\frac{dW_2}{dx}=\left[\left(\frac{Z\alpha E}{\lambda}-\gamma-1\right)\frac{dW_1}{dx}-x\frac{d^2W_1}{dx^2}\right]\left[\kappa-\frac{Z\alpha m}{\lambda}\right]^{-1}. \tag{2.22}$$

Putting these expression into Eq. (2.20), gives

$$x\left[\left(\frac{Z\alpha E}{\lambda}-\gamma-1\right)\frac{dW_1}{dx}-x\frac{d^2W_1}{dx^2}\right]\left[k-\frac{Z\alpha m}{\lambda}\right]^{-1}$$

$$+\left(\gamma-x+\frac{Z\alpha E}{\lambda}\right)\left[\left(\frac{Z\alpha E}{\lambda}-\gamma\right)W_1-x\frac{dW_1}{dx}\right]\left[k-\frac{Z\alpha m}{\lambda}\right]^{-1}+\left(k+\frac{Z\alpha m}{\lambda}\right)W_1=0,$$

$$x\left[\left(\frac{Z\alpha E}{\lambda}-\gamma-1\right)\frac{dW_1}{dx}-x\frac{d^2W_1}{dx^2}\right]+\left(\gamma-x+\frac{Z\alpha E}{\lambda}\right)\left[\left(\frac{Z\alpha E}{\lambda}-\gamma\right)W_1-x\frac{dW_1}{dx}\right]$$

$$\left[k^2-\left(\frac{Z\alpha m}{\lambda}\right)^2\right]W_1=0,$$

$$-x^2\frac{d^2W_1}{dx^2}+x\frac{dW_1}{dx}\left(-2\gamma-1+x\right)+\left[k^2-\gamma^2-x\left(\frac{Z\alpha E}{\lambda}-\gamma\right)+\left(\frac{Z\alpha E}{\lambda}\right)^2-\left(\frac{Z\alpha m}{\lambda}\right)^2\right]W_1=0,$$

$$x\frac{d^2W_1}{dx^2}+\left(2\gamma+1-x\right)\frac{dW_1}{dx}-\left[k^2-\gamma^2-x\left(\frac{Z\alpha E}{\lambda}-\gamma\right)+\left(\frac{Z\alpha E}{\lambda}\right)^2-\left(\frac{Z\alpha m}{\lambda}\right)^2\right]\frac{W_1}{x}=0.$$

It could be notices that

$$k^2-\left(\frac{Z\alpha m}{\lambda}\right)^2-\gamma^2+\left(\frac{Z\alpha E}{\lambda}\right)^2=k^2-\left(\frac{Z\alpha}{\lambda}\right)^2\left(m^2-E^2\right)-\gamma^2.$$

Substituting the values of λ and γ, we can see that

$$\kappa^2-\left(\frac{Z\alpha m}{\lambda}\right)^2-\gamma^2+\left(\frac{Z\alpha E}{\lambda}\right)^2=0. \tag{2.23}$$

Thus

$$x\frac{d^2W_1}{dx^2}+\left(2\gamma+1-x\right)\frac{dW_1}{dx}-\left(\gamma-\frac{Z\alpha E}{\lambda}\right)W_1=0. \tag{2.24}$$

Using Eq. (2.20), W_1 can be defined as

$$W_1=-\left[x\frac{dW_2}{dx}+\left(\gamma+\frac{Z\alpha E}{\lambda}-x\right)W_2\right]\left[\kappa+\frac{Z\alpha m}{\lambda}\right]^{-1}, \tag{2.25}$$

and after differentiating it with respect to x, we get

$$\frac{dW_1}{dx}=-\left[\frac{dW_2}{dx}+x\frac{d^2W_2}{dx^2}+\left(\gamma+\frac{Z\alpha E}{\lambda}-x\right)\frac{dW_2}{dx}-W_2\right]\left[\kappa+\frac{Z\alpha m}{\lambda}\right]^{-1}. \tag{2.26}$$

Now substitute the value of W_1 and $\frac{dW_1}{dx}$ in Eq. (2.19), gives

$$-x\left[\frac{dW_2}{dx}+x\frac{d^2W_2}{dx^2}+\left(\gamma+\frac{Z\alpha E}{\lambda}-x\right)\frac{dW_2}{dx}-W_2\right]\left[k+\frac{Z\alpha m}{\lambda}\right]^{-1}$$
$$-\left(\gamma-\frac{Z\alpha E}{\lambda}\right)\left[x\frac{dW_2}{dx}+\left(\gamma+\frac{Z\alpha E}{\lambda}-x\right)W_2\right]\left[k+\frac{Z\alpha m}{\lambda}\right]^{-1}+\left(k-\frac{Z\alpha m}{\lambda}\right)W_2=0,$$

$$\left[\frac{dW_2}{dx} + x\frac{d^2W_2}{dx^2} + \left(\gamma + \frac{Z\alpha E}{\lambda} - x\right)\frac{dW_2}{dx} - W_2\right] + \frac{\left(\gamma - \frac{Z\alpha E}{\lambda}\right)}{x}\left[x\frac{dW_2}{dx} + \left(\gamma + \frac{Z\alpha E}{\lambda} - x\right)W_2\right]$$

$$\left(k - \frac{Z\alpha m}{\lambda}\right)\frac{W_2}{x} = 0,$$

$$x\frac{d^2W_2}{dx^2} + (2\gamma - x + 1)\frac{dW_2}{dx} + \left[-1 + \frac{1}{x}\left(\gamma^2 - \gamma x - \left(\frac{Z\alpha E}{\lambda}\right)^2 + \frac{Z\alpha Ex}{\lambda} - k^2 + \left(\frac{Z\alpha m}{\lambda}\right)^2\right)\right]W_2 = 0.$$

Using the definition of γ and λ, it becomes

$$x\frac{d^2W_2}{dx^2} + (2\gamma - x + 1)\frac{dW_2}{dx} - \left(1 + \gamma - \frac{Z\alpha E}{\lambda}\right)W_2 = 0. \tag{2.27}$$

As we can see that Eq. (2.24) and Eq. (2.27) are of the form of the Kummer's equation [6]

$$z\frac{d^2w}{dz^2} + (b - z)\frac{dw}{dz} - aw = 0, \tag{2.28}$$

which contains a confluent hyper-geometric function

$$F(a, b; z) = 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + ... \tag{2.29}$$

as their solution. Therefore W_1 and W_2 can be expressed as confluent hyper-geometric functions

$$W_1(x) = \alpha_0 F\left(\gamma - \frac{Z\alpha E}{\lambda}, 2\gamma + 1; x\right), \tag{2.30}$$

$$W_2(x) = \beta_0 F\left(1 + \gamma - \frac{Z\alpha E}{\lambda}, 2\gamma + 1; x\right). \tag{2.31}$$

Using them in Eq. (2.19) and setting $x = 0$ gives the condition for coefficient α_0 and β_0

$$\left(\kappa - \frac{Z\alpha m}{\lambda}\right)\beta_0 = -\left(\gamma - \frac{Z\alpha E}{\lambda}\right)\alpha_0. \tag{2.32}$$

It follows that from explicit form of hyper-geometric functions given in Eq. (2.29) the function

W_1 and W_2 will go to infinity for limit $x \to \infty$. Therefore, we will impose certain condition to make the series convergent

$$\gamma - \frac{Z\alpha}{\lambda} = -n_{\mathrm{r}},\ n_{\mathrm{r}} = \begin{cases} 0, 1, 2, ... \text{ if } \kappa < 0 \\ 1, 2, 3, ... \text{ if } \kappa > 0 \end{cases}. \tag{2.33}$$

From the continuity equation

$$\frac{\partial \mathrm{j}_\mu(\mathrm{x})}{\partial \mathrm{x}_\mu} = 0, \tag{2.34}$$

the normalization for the stationary bound states can be done as

$$\int \rho(\mathbf{r})\, d^3\mathbf{r} = \int \Phi^\dagger(\mathbf{r})\, \Phi(\mathbf{r})\, d^3\mathbf{r} = 1, \tag{2.35}$$

and radial functions are therefore normalized as

$$\int d\mathrm{r}\, \mathrm{r}^2 \left[g^2(\mathrm{r}) + f^2(\mathrm{r})\right] = 1. \tag{2.36}$$

Using this condition together with Eq. (2.32) gives explicit expression for radial wave functions.

Finally, the Dirac-Coulomb wave function can be expressed as,

$$\Phi(\mathbf{r}) = \begin{pmatrix} g_{nlj}(\mathrm{r})\chi_{ljM}(\hat{\mathbf{r}}) \\ if_{nlj}(\mathrm{r})\chi_{\bar{l}jM}(\hat{\mathbf{r}}) \end{pmatrix}, \tag{2.37}$$

where $\bar{l} = 2j - l = l \pm 1$ and the radial wave functions

$$\begin{aligned} g_{nlj}(\mathrm{r}) =& \frac{(2\lambda_n)^{\frac{3}{2}}}{\Gamma(2\gamma_n + 1)} \left[\frac{\left(1 + \frac{E_n}{m}\right)\Gamma(2\gamma_n + n_{\mathrm{r}} + 1)}{4N_n(N_n - \kappa_n)\, n_{\mathrm{r}}!}\right]^{\frac{1}{2}} (2\lambda\mathrm{r})^{\gamma_n - 1} e^{-\lambda_n \mathrm{r}} \\ & \times \{(N_n - \kappa_n)\, F(-n_{\mathrm{r}}, 2\gamma_n + 1; 2\lambda_n \mathrm{r}) - n_{\mathrm{r}} F(1 - n_{\mathrm{r}}, 2\gamma_n + 1; 2\lambda_n \mathrm{r})\}, \end{aligned} \tag{2.38}$$

$$\begin{aligned} f_{nlj}(\mathrm{r}) =& \frac{-(2\lambda_n)^{\frac{3}{2}}}{\Gamma(2\gamma_n + 1)} \left[\frac{\left(1 - \frac{E_n}{m}\right)\Gamma(2\gamma_n + n_{\mathrm{r}} + 1)}{4N_n(N_n - \kappa_n)\, n_{\mathrm{r}}!}\right]^{\frac{1}{2}} (2\lambda\mathrm{r})^{\gamma_n - 1} e^{-\lambda_n \mathrm{r}} \\ & \times \{(N_n - \kappa_n)\, F(-n_{\mathrm{r}}, 2\gamma_n + 1; 2\lambda_n \mathrm{r}) + n_{\mathrm{r}} F(1 - n_{\mathrm{r}}, 2\gamma_n + 1; 2\lambda_n \mathrm{r})\}. \end{aligned} \tag{2.39}$$

Here n is the principle quantum number and the energy levels are determined using Sommerfeld's formula [7]

$$E_n = \sqrt{m^2 - \lambda_n^2},\ \ \lambda_n = \frac{1}{aN_n},\ a = \frac{1}{Z\alpha m},\ n_{\mathrm{r}} = n - \kappa_n, \tag{2.40}$$

$$N_n = \sqrt{n^2 - 2n_{\mathrm{r}}\left(|\kappa_n| - \gamma_n\right)}, \;\; \gamma_n = \sqrt{\kappa_n^2 - (Z\alpha)^2}. \tag{2.41}$$

The quantum number κ_n is defined by

$$\kappa_n = \begin{cases} l, & j = l - \frac{1}{2} \\ -\left(l+1\right), & j = l + \frac{1}{2} \end{cases}. \tag{2.42}$$

and the spherical spinors

$$\begin{aligned} \chi_{jlM}(\hat{\mathbf{r}}) &= \sum_{m\mu} C^{jM}_{lm\frac{1}{2}\mu} Y_l^m\left(\hat{\mathbf{r}}\right)\eta_\mu \\ &= (-1)^{-l+\frac{1}{2}-M}\sqrt{2j+1}\sum_{m\mu}\begin{pmatrix} l & \frac{1}{2} & j \\ m & \mu & -M \end{pmatrix} Y_l^m(\hat{\mathbf{r}})\eta_\mu. \end{aligned} \tag{2.43}$$

3 1S Wave Functions

In our work, we will confine ourselves only to the wave-function in $1S$ state. They can be obtained from Eq. (2.38) and Eq. (2.39) by setting the following values of the quantum numbers: $n = 1$, $l = 0$, $j = \frac{1}{2}$, and

$$k_n = \begin{cases} l, & if\, j = l - \frac{1}{2} \\ -\left(l+1\right), & if\, j = l + \frac{1}{2} \end{cases},$$

correspondingly

$$n_{\mathrm{r}} = 0, \; \gamma = \gamma_1 = \sqrt{1-(Z\alpha)^2}, N_1 = 1, E = m\gamma, \; a = \frac{1}{Z\alpha m}.$$

Thus we have

$$\begin{aligned} g_{1S_{\frac{1}{2}}}\left(\mathrm{r}\right) \equiv g\left(\mathrm{r}\right) &= \frac{\left(\frac{2}{a}\right)^{\frac{3}{2}+\gamma-1}}{\Gamma\left(2\gamma+1\right)}\left[\frac{\left(1+\gamma\right)\Gamma\left(2\gamma+1+0\right)}{4\left(1\right)2\left(0\right)!}\right]^{\frac{1}{2}}\left(\mathrm{r}\right)^{\gamma-1}e^{-\frac{\mathrm{r}}{a}}, \\ g_{1S_{\frac{1}{2}}}\left(\mathrm{r}\right) \equiv g\left(\mathrm{r}\right) &= \left(\frac{2}{a}\right)^{\gamma+\frac{1}{2}}\sqrt{\frac{1+\gamma}{2\Gamma\left(2\gamma+1\right)}}\exp\left(-\frac{\mathrm{r}}{a}\right)\mathrm{r}^{\gamma-1}. \end{aligned} \tag{3.1}$$

Similarly for

$$\begin{aligned} f_{1S_{\frac{1}{2}}}\left(\mathrm{r}\right) \equiv f\left(\mathrm{r}\right) &= -\left(\frac{2}{a}\right)^{\gamma+\frac{1}{2}}\sqrt{\frac{1-\gamma}{2\Gamma\left(2\gamma+1\right)}}exp\left(-\frac{\mathrm{r}}{a}\right)\left(\mathrm{r}\right)^{\gamma-1}, \\ f_{1S_{\frac{1}{2}}}\left(\mathrm{r}\right) \equiv f\left(\mathrm{r}\right) &= -\sqrt{\frac{1-\gamma}{1+\gamma}}g_{1S_{\frac{1}{2}}}\left(\mathrm{r}\right). \end{aligned} \tag{3.2}$$

The spherical spinors with the spin-up $\left(M=\frac{1}{2}\right)$ are

$$\chi_{\frac{1}{2}0\frac{1}{2}}\left(\hat{\mathbf{r}}\right)=(-1)^{\frac{1}{2}-\frac{1}{2}}\sqrt{2}\begin{pmatrix}0 & \frac{1}{2} & \frac{1}{2}\\ 0 & \frac{1}{2} & -\frac{1}{2}\end{pmatrix}Y_0^0(\hat{\mathbf{r}})\begin{pmatrix}1\\ 0\end{pmatrix}=\frac{1}{\sqrt{4\pi}}, \tag{3.3}$$

$$\begin{aligned}\chi_{\frac{1}{2}1\frac{1}{2}}\left(\hat{\mathbf{r}}\right)&=-\sum_{0,\frac{1}{2}}C^{\frac{1}{2}\frac{1}{2}}_{10\frac{1}{2}\frac{1}{2}}Y_1^0\left(\hat{\mathbf{r}}\right)\begin{pmatrix}1\\0\end{pmatrix}+\sum_{1,-\frac{1}{2}}C^{\frac{1}{2}\frac{1}{2}}_{11\frac{1}{2}-\frac{1}{2}}Y_1^1\left(\hat{\mathbf{r}}\right)\begin{pmatrix}0\\1\end{pmatrix},\\ &=\left(-\frac{1}{\sqrt{3}}\right)\left(\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta\right)+\left(\sqrt{\frac{2}{3}}\right)\left(-\frac{1}{2}e^{i\phi}\sqrt{\frac{3}{2\pi}}\sin\theta\right),\\ &=-\frac{1}{\sqrt{4\pi}}\begin{pmatrix}\cos\theta\\ \sin\theta e^{i\phi}\end{pmatrix}.\end{aligned} \tag{3.4}$$

Therefore, the ground state wave function for the bound state in position space can be expressed as

$$\begin{aligned}\Phi\left(\mathbf{r}\right)&=\begin{pmatrix}g_{1S_{\frac{1}{2}}}(r)\chi_{\frac{1}{2}0\frac{1}{2}}(\hat{\mathbf{r}})\\ if_{1S_{\frac{1}{2}}}(r)\chi_{\frac{1}{2}1\frac{1}{2}}(\hat{\mathbf{r}})\end{pmatrix}\\ &=\frac{(2mZ\alpha)^{\gamma+\frac{1}{2}}}{\sqrt{4\pi}}\sqrt{\frac{1+\gamma}{2\Gamma\left(1+2\gamma\right)}}r^{\gamma-1}\exp\left(-mZ\alpha r\right)\begin{pmatrix}1\\ 0\\ \frac{i(1-\gamma)}{Z\alpha}\cos\theta\\ \frac{i(1-\gamma)}{Z\alpha}\sin\theta e^{i\phi}\end{pmatrix}.\end{aligned} \tag{3.5}$$

Chapter III

Bound Muon Decay into Electron And Neutrinos

This chapter discuss the detailed study of bound muon decay into bound electron, neutrino and antineutrino in position space. We have used the standard Casimir's trick method [5] to find decay rate. Moreover, we have discussed two cases namely equal masses ($m_e \simeq m_\mu$) and general cases (considering actual masses). The decay rate for the bound muon is define as [4]

$$\Gamma = \int \frac{d^3q}{(2\pi)^3} |A|^2 . \tag{3.6}$$

In Eq. (3.6) $|A|^2$ is an invariant amplitude, which can be written as

$$|A|^2 = \frac{1}{2} \sum J^{\alpha\beta} N_{\alpha\beta}. \tag{3.7}$$

In above expression $N_{\alpha\beta}$ is neutrino current and $J^{\alpha\beta}$ is the particle current tensor, which can be express as

$$J^{\alpha\beta} = J^\alpha (j^\beta)^\dagger, \quad J^\alpha \equiv \int d^3r \bar{\Phi}_e(r) \Phi_\mu(r) \exp(-\iota q.r). \tag{3.8}$$

But if we take nearly equal masses of electron and muon, the neutrino momentum $q \approx 0$, giving

$$J^\alpha \equiv \int d^3r \bar{\Phi}_e(r) \Phi_\mu(r).$$

Now let's consider the ground state wave function for the bound state in the position space as

$$\phi(r) = \psi_{n=1,j=1/2,\uparrow}(r,\theta,\phi) = \frac{(2mZ\alpha)^{3/2}}{\sqrt{4\pi}} \sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}} (2mrZ\alpha)^{\gamma-1} \exp(-mrZ\alpha) \begin{pmatrix} 1 \\ 0 \\ \frac{\iota(1-\gamma)}{Z\alpha} \cos\theta \\ \frac{\iota(1-\gamma)}{Z\alpha} \sin\theta \exp(\iota\phi) \end{pmatrix}. \tag{3.9}$$

We can write Eq. (3.9) in a more convenient form

$$\phi(r) = \frac{(2mZ\alpha)^{3/2}}{\sqrt{4\pi}} \sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}} (2mrZ\alpha)^{\gamma-1} \exp(-mrZ\alpha) \begin{pmatrix} 1 \\ 0 \\ \frac{\iota(1-\gamma)}{Z\alpha} \cos\theta \\ \frac{\iota(1-\gamma)}{Z\alpha} \sin\theta \exp(\iota\phi) \end{pmatrix},$$

$$\phi(r) = f(r)u_+. \tag{3.10}$$

Here in Eq. (3.10).

$$u_+ = \begin{pmatrix} 1 \\ 0 \\ \frac{\iota(1-\gamma)}{Z\alpha} \cos\theta \\ \frac{\iota(1-\gamma)}{Z\alpha} \sin\theta \exp(\iota\phi) \end{pmatrix}, \tag{3.11}$$

and

$$f(r) = \frac{(2mZ\alpha)^{3/2}}{\sqrt{4\pi}} \sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}} (2mrZ\alpha)^{\gamma-1} \exp(-mrZ\alpha).$$

Using spherical coordinates

$$x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta,$$

We can write a position unit vector as

$$\hat{r} = \frac{\vec{r}}{r} = \sin\theta\cos\varphi\vec{i} + \sin\theta\sin\varphi\vec{j} + \cos\theta\vec{k}. \tag{3.12}$$

Let's introduce Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.13}$$

and inner product of Eq. (3.12) and Eq. (3.13) gives.

$$\vec{\sigma}.\vec{r} = \sigma_x \sin\theta\cos\varphi + \sigma_y \sin\theta\sin\varphi + \sigma_z\cos\theta,$$
$$= \begin{pmatrix} \cos\theta & \sin\theta\exp(-\iota\phi) \\ \sin\theta\exp(\iota\phi) & -\cos\theta \end{pmatrix}. \tag{3.14}$$

Let's define

$$\phi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \phi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.15}$$

Using Eq. (3.14) and Eq. (3.15) we can write Eq. (3.11) as

$$u_+ = \begin{pmatrix} \phi_+ \\ \frac{\iota(1-\gamma)\vec{\sigma}.\vec{r}\phi_+}{Z\alpha} \end{pmatrix}. \tag{3.16}$$

To check our result, let us calculate

$$\vec{\sigma}.\vec{r}\phi_+ = \begin{pmatrix} \cos\theta & \sin\theta\exp(-\iota\phi) \\ \sin\theta\exp(\iota\phi) & -\cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} \sin\theta \\ \sin\theta\exp\iota\phi \end{pmatrix}. \tag{3.17}$$

Thus, we can see that Using Eq. (3.16) and Eq. (3.17) the expression of u_+ is retrieved. In Dirac representation Gamma matrices are express as

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \gamma_{0=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma_{i=}\gamma_0\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix},$$

Therefore, we can write

$$u_+ = \begin{pmatrix} \phi_+ \\ 0 \end{pmatrix} - \iota\frac{(1-\gamma)}{Z\alpha}\hat{r}. \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} \phi_+ \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \phi_+ \\ 0 \end{pmatrix} - \iota\frac{(1-\gamma)}{Z\alpha} \begin{pmatrix} 0 & \vec{r}.\sigma_i \\ -\vec{r}.\sigma_i & 0 \end{pmatrix} \begin{pmatrix} \phi_+ \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \phi_+ \\ 0 \end{pmatrix} - \iota\frac{(1-\gamma)}{Z\alpha}\vec{r}.\vec{\gamma} \begin{pmatrix} \phi_+ \\ 0 \end{pmatrix}$$
$$= \rho^\mu\gamma_\mu \begin{pmatrix} \phi_+ \\ 0 \end{pmatrix} = \not{\rho} \begin{pmatrix} \phi_+ \\ 0 \end{pmatrix} \tag{3.18}$$

In the above expression $\rho^{\mu} = (\rho^0, \vec{\rho})= (1,\, \iota\frac{(1-\gamma)}{Z\alpha}\hat{r})$, and

$$\begin{aligned}
u_+^{\dagger} &= \left[\rho^{\mu}\gamma_{\mu}\begin{pmatrix}\phi_+\\ 0\end{pmatrix}\right]^{\dagger}\\
u_+^{\dagger} &= \begin{pmatrix}\phi_+ & 0\end{pmatrix}\not{\rho}^{\dagger}\\
u_+^{\dagger}\gamma_0 &= \begin{pmatrix}\phi_+ & 0\end{pmatrix}\rho^{\dagger}\gamma_0\\
\overline{u}_+ &= \begin{pmatrix}\phi_+ & 0\end{pmatrix}\gamma_0\gamma_0\rho^{\dagger}\gamma_0\\
\overline{u}_+ &= \begin{pmatrix}\phi_+ & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\rho^{'}\\
\overline{u}_+ &= \begin{pmatrix}\phi_+ & 0\end{pmatrix}\not{\rho}'. \qquad (3.19)
\end{aligned}$$

Remember, we need to take the conjugate of ρ' also and in above equation it is

$$\rho'^{\mu} = \left(1, -\iota\frac{(1-\gamma)}{Z\alpha}\hat{r'}\right)$$

where, we have already taken the complex conjugate of the elements. Hence we can write

$$u_+\bar{u}_+ = \not{\rho}\begin{pmatrix}\phi_+\\ 0\end{pmatrix}\begin{pmatrix}\phi_+ & 0\end{pmatrix}\not{\rho}' = \not{\rho}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\not{\rho}' = \not{\rho}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\not{\rho}'. \qquad (3.20)$$

Now for ground state wave function for spin-down is

$$\begin{aligned}
\phi(r) &= \psi_{n=1,j=1/2,\downarrow}(r,\theta,\phi)\\
&= \frac{(2mZ\alpha)^{3/2}}{\sqrt{4\pi}}\sqrt{\frac{1+\gamma}{2\Gamma(1+2\gamma)}}(2mrZ\alpha)^{\gamma-1}\exp(-mrZ\alpha)\begin{pmatrix}0\\ 1\\ \frac{\iota(1-\gamma)}{Z\alpha}\sin\theta\exp\left(-\iota\phi\right)\\ \frac{\iota(1-\gamma)}{Z\alpha}\cos\theta\end{pmatrix}
\end{aligned}$$

Following the same line of action

$$\psi_{n=1,j=1/2,\downarrow}(r,\theta,\phi) = f(r)u_-$$

where $f(r)$ is define above, and

$$u_- = \begin{pmatrix} 0 \\ 1 \\ \frac{\iota(1-\gamma)}{Z\alpha} \sin\theta \exp -\iota\phi \\ \frac{\iota(1-\gamma)}{Z\alpha} \cos\theta \end{pmatrix} \cong \begin{pmatrix} \phi_- \\ \frac{\iota(1-\gamma)\vec{\sigma}.\vec{r}}{Z\alpha}\phi_- \end{pmatrix},$$

with $\phi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the last expression. Thus

$$\begin{aligned} \vec{\sigma}.\vec{r}\phi_- &= \begin{pmatrix} \cos\theta & \sin\theta \exp(-\iota\phi) \\ \sin\theta \exp(\iota\phi) & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \sin\theta \exp -\iota\phi \\ -\cos\theta \end{pmatrix}. \end{aligned}$$

We can find

$$\begin{aligned} u_- &= \begin{pmatrix} \phi_- \\ 0 \end{pmatrix} - \frac{\iota(1-\gamma)}{Z\alpha}\hat{r}. \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \phi_- \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \phi_- \\ 0 \end{pmatrix} - \frac{\iota(1-\gamma)}{Z\alpha} \begin{pmatrix} 0 & \hat{r}.\vec{\sigma} \\ -\hat{r}.\vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \phi_- \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \phi_- \\ 0 \end{pmatrix} - \frac{\iota(1-\gamma)}{Z\alpha}\hat{r}.\vec{\gamma} \begin{pmatrix} \phi_- \\ 0 \end{pmatrix} \\ &= \rho^\mu \gamma_\mu \begin{pmatrix} \phi_- \\ 0 \end{pmatrix} = \not{\rho} \begin{pmatrix} \phi_- \\ 0 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} u_-^\dagger &= \begin{pmatrix} \phi_-^\dagger & 0 \end{pmatrix} \not{\rho} \\ u_-^\dagger \gamma_0 &= \begin{pmatrix} \phi_- & 0 \end{pmatrix} \rho^\dagger \gamma_0 \\ \bar{u}_- &= \begin{pmatrix} \phi_- & 0 \end{pmatrix} \bar{\not{\rho}} \end{aligned}$$

Thus

$$u_-\bar{u}_- = \not{p}\begin{pmatrix}\phi_-\\0\end{pmatrix}\begin{pmatrix}\phi_- & 0\end{pmatrix}\not{p} = \not{p}\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}\not{p}$$
$$= \not{p}\begin{pmatrix}0 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}\not{p}. \tag{3.21}$$

Hence, making sum over the spins, we have

$$\sum_{spin} u\bar{u} = u_+\bar{u_+} + u_-\bar{u}_- = \not{p}\begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}\not{p}.$$

which in a convenient 2×2 notation can be written as

$$\frac{1}{2}(1+\gamma_0) = 1/2\left[\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right]$$
$$= \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}.$$

Now, let us evaluate the particle current tensor $J^{\alpha\beta}$, we have already defined Eq. (3.8). In the case of equal masses of electron and muon i.e., $\vec{q} = 0$,

$$J_\alpha \equiv \int d^3r\bar{\Phi}_e(r)\gamma_\alpha L\Phi_\mu(r), \tag{3.22}$$

where $\phi(r)$ is defined above and $\bar{\phi}(r)$ is its complex conjugate. Upon substituting the expression of $\bar{\Phi}_e(r)$ and $\Phi_\mu(r)$ into Eq. (3.22) we will get

$$J_\alpha \equiv \int d^3r f_\mu(r) f_e(r)\bar{u}_e\gamma_\alpha L u_\mu,$$

where $L = \frac{1-\gamma_5}{2}$. To calculate tensor $J^{\alpha\beta}$ we have to multiply J^α with $(J^\beta)^\dagger$ and separate radial part

$$\begin{aligned} J_{\alpha\beta} &\equiv 1/2\sum_{spin}\bar{u}_e\gamma_\alpha L u_\mu(\bar{u}_e\gamma_\beta L u_\mu)^\dagger \\ &\equiv 1/2\sum_{spin}\bar{u}_e\gamma_\alpha L u_\mu\bar{u}_\mu\gamma_\beta L u_e \\ &= 1/2\mathrm{Tr}\left[u_e\bar{u}_e\gamma_\alpha L u_\mu\bar{u}_\mu\gamma_\beta L\right] \\ &= 1/2\mathrm{Tr}\left[\not{\rho}\left(\frac{1+\gamma_0}{2}\right)\not{\rho}'\gamma_\alpha L\not{\rho}_1'\left(\frac{1+\gamma_0}{2}\right)\not{\rho}_1\gamma_\beta L\right], \end{aligned} \tag{3.23}$$

where

$$\rho_1^{'} = (1, \frac{\iota(1-\gamma)}{Z\alpha}\hat{r}') = (1, -\vec{\rho'}) \text{ and } \rho 1 = (1, -\frac{\iota(1-\gamma)}{Z\alpha}\hat{r}) = (1, -\vec{\rho})$$

We can solve trace of the above equation using Mathematica.Remember the three components of ρ and ρ_1 are same except the sign and the situation is same for ρ' and ρ_1'. However, the zeroth component of all of them is same.

4 Equal mass case

In this case we have considered mass of electron is equal to mass of muon, therefore neutrino momentum q is nearly equal to zero. We collect all non zero terms arises corresponding to q_0^2 and $\vec{q}^2$and the terms linear in vectors vanishes.

TERMS CORRESPONDING TO q_0^2

In the results below we will use $\vec{\rho} = a\hat{r}, \vec{\rho}' = a\hat{r}'$, where $a = \iota\frac{(1-\gamma)}{Z\alpha}$

$$-8\vec{\rho}\vec{\rho}^{2'} + 32(\vec{\rho}.\vec{\rho}^{'})(\vec{\rho}.\vec{\rho}^{'}) + 8\vec{\rho^2} + 8\vec{\rho'^2} + 24$$

$$\begin{aligned} &= -8a^4 + \frac{32}{3}a^4 + 8a^2 + 8a^2 + 24 \\ &= \frac{8}{3}a^4 + 16a^2 + 24 \\ &= 8\left[\frac{1}{3}a^4 + 2a^2 + 3\right] \end{aligned}$$

In the above equation we have used $\hat{r}_i\hat{r}_j = \hat{r}_i'\hat{r}_j' = \frac{\delta_{ij}}{3}$.

CORRESPONDING TO $\vec{q}^2$

$$\vec{q}^2\left[24\vec{\rho}\vec{\rho}^{2'} - 32(\vec{\rho}.\vec{\rho}^{'})(\vec{\rho}.\vec{\rho}^{'}) - 8\vec{\rho^2} - 8\vec{\rho'^2} - 8\right],$$

$$-32\vec{\rho}^2(\vec{q}.\vec{\rho})(\vec{q}.\vec{\rho}) + 32(\vec{q}.\vec{\rho})(\vec{q}.\vec{\rho}')(\vec{\rho}.\vec{\rho}\prime) - 32(\vec{q}.\vec{\rho}')(\vec{q}.\vec{\rho}'),$$

$$= \vec{q}^2\left[24a^4 - \frac{32}{3}a^4 - \frac{32}{3}a^4 + \frac{32}{9}a^4 - 16a^2 + \frac{32}{3}a^2 - 8\right],$$
$$= \vec{q}^2\left[\frac{56}{9}a^4 - \frac{80}{3}a^2 - 8\right]$$
$$= \vec{q}^2\left[\frac{7}{9}a^4 - \frac{10}{3}a^2 - 1\right].$$

Contracting neutrino tensor [4]

$$N^{\alpha\beta} = \frac{2G_F^2}{3\pi}(q^\alpha q^\beta - q^2\eta^{\alpha\beta}) \tag{4.1}$$

with the angular part of $J_{\alpha\beta}$ gives invariant amplitude

$$|A|^2 = \frac{1}{2}\sum J_{\alpha\beta}N^{\alpha\beta}$$
$$= \frac{G_F^2}{6\pi}\left[q_0^2(\frac{1}{3}a^4 + 2a^2 + 3) + |\vec{q}|^2(\frac{7}{9}a^4 - \frac{10}{3}a^2 - 1)\right]|\langle f_e(r)f_\mu(r)\rangle|^2.$$

Now we can calculate decay rate, using

$$\Gamma = \int \frac{d^3q}{(2\pi)^3}|A|^2$$

$$\Gamma = \int \frac{d^3q}{(2\pi)^3}\left[\frac{G_F^2}{6\pi}\left\{q_0 2(\frac{1}{3}a^4 + 2a^2 + 3) + |\vec{q}|^2(\frac{7}{9}a^4 - \frac{10}{3}a^2 - 1)\right\}\right]|\langle f_e(r)f_\mu(r)\rangle|^2$$

Integrating over the q leads to

$$\Gamma = \int \frac{d|\vec{q}|(4\pi)|\vec{q}|^2}{(2\pi)^3}\left[\frac{G_F^2}{6\pi}\left\{q_0 2(\frac{1}{3}a^4 + 2a^2 + 3) + \vec{q}^2(\frac{7}{9}a^4 - \frac{10}{3}a^2 - 1)\right\}\right]|\langle f_e(r)f_\mu(r)\rangle|^2$$
$$= \frac{G_F^2}{12\pi^3}\left[\frac{|\vec{q}|^3}{3}q_0 2(\frac{1}{3}a^4 + 2a^2 + 3) + \frac{|\vec{q}|^5}{5}(\frac{7}{9}a^4 - \frac{10}{3}a^2 - 1)\right]|\langle f_e(r)f_\mu(r)\rangle|^2$$
$$= \frac{G_F^2}{12\pi^3}\left[\frac{|\vec{q}|^3}{3}q_0 2(\frac{1}{3}a^4 + 2a^2 + 3) + \frac{|\vec{q}|^5}{5}(\frac{7}{9}a^4 - \frac{10}{3}a^2 - 1)\right]|\langle f_e(r)f_\mu(r)\rangle|^2.$$

Here in the nearly equal mass limit $q_0 = |\vec{q}| = \epsilon\gamma m_\mu$ therefore, above equation becomes

$$
\begin{aligned}
\Gamma &= \frac{G_F^2}{12\pi^3}\left[\frac{\epsilon^5\gamma^5 m_\mu^5}{3}(\frac{1}{3}a^4 + 2a^2 + 3) + \frac{\epsilon^5\gamma^5 m_{mu}^5}{5}(\frac{7}{9}a^4 - \frac{10}{3}a^2 - 1)\right] |\langle f_e(r) f_\mu(r)\rangle|^2 \\
&= \frac{G_F^2}{12\pi^3}\epsilon^5\gamma^5 m_\mu^5\left[(\frac{1}{9} + \frac{7}{45})a^4 + (\frac{2}{3} - \frac{10}{15})a^2 + (1 - \frac{1}{5})\right] |\langle f_e(r) f_\mu(r)\rangle|^2 \\
&= \frac{G_F^2}{12\pi^3}\epsilon^5\gamma^5 m_\mu^5\left[\frac{12}{45}a^4 + \frac{4}{5}\right] |\langle f_e(r) f_\mu(r)\rangle|^2
\end{aligned}
$$

and $a = \iota\sqrt{\frac{(1-\gamma)}{(1+\gamma)}}$, giving

$$
\begin{aligned}
\Gamma &= \frac{G_F^2}{15\pi^3}\epsilon^5\gamma^5 m_\mu^5\left[\frac{1}{3}\frac{(1-\gamma)^2}{(1+\gamma)^2} + 1\right] |\langle f_e(r) f_\mu(r)\rangle| \\
&= \frac{G_F^2}{15\pi^3}\epsilon^5\gamma^5 m_\mu^5\left[\frac{(1-\gamma)^2}{3} + (1+\gamma)^2\right] \frac{1}{(1+\gamma)^2} |\langle f_e(r) f_\mu(r)\rangle| \\
&= \frac{G_F^2}{15\pi^3}\epsilon^5\gamma^5 m_\mu^5\left[\frac{(1+\gamma+\gamma^2)}{3}\right] \frac{4}{(1+\gamma)^2} |\langle f_e(r) f_\mu(r)\rangle| \\
&= \frac{G_F^2}{15\pi^3}\epsilon^5\gamma^5 m_\mu^5\left[\frac{1+\gamma+\gamma^2}{3}\right]
\end{aligned} \tag{4.2}
$$

The decay rate of free muon decay is [4]

$$
\Gamma_0 = \frac{G_F^2}{192\pi^3} m_\mu^5. \tag{4.3}
$$

In order to find decay ratio, divide Eq. (4.2)and Eq. (4.3), which gives

$$
\frac{\Gamma}{\Gamma_0} = \frac{64}{5}\epsilon^5\gamma^5\left[\frac{1+\gamma+\gamma^2}{3}\right]. \tag{4.4}
$$

We can evaluate branching ratio for different values of Z using Eq. (4.4). For nearly equal masses case, the numerical value calculated by this equation is same as found by Alchemy's formalism [9] where the rate is calculated in the momentum space instead of the position space.

	Alchemy's formalism	Eq. (4.4)
$Z = 10$	1.25×10^{-9}	1.26×10^{-9}
$Z = 80$	3.85×10^{-10}	3.80×10^{-10}

5 General Case

In this case we have considered the actual mass of muon and electron, therefore the exponential factor $\exp(-\iota q.r)$ takes part in muon to electron current, i.e.,

$$J_\alpha = \int d^3r \bar{\Phi}_e(r)\gamma_\alpha L \Phi_\mu(r) \exp(-\iota \vec{q}.\vec{r}). \tag{5.1}$$

Just following the method devised in the last section, we can write $J_{\alpha\beta}$ an calculate the trace using Mathematics. Just like the previous case, we will solve different terms one by one.

Corresponding to q_0^2

First we will gather the terms that we have collected for equal muon and electron masses. Then we will add the term which are not zero in the general case. In the results below we will use $\vec{\rho} = a\hat{r}, \vec{\rho}' = a\hat{r}'$,where $a = \iota\frac{(1-\gamma)}{Z\alpha}$ as we did above. Taking

$$-8\vec{\rho}\vec{\rho}^{2'} + 32(\vec{\rho}.\vec{\rho}')(\vec{\rho}.\vec{\rho}') + 8\vec{\rho^2} + 8\vec{\rho'^2} + 24. \tag{5.2}$$

Let us solve it term by term. In the exponential factors $\exp(-\iota\vec{q}.\vec{r})$ and $\exp(-\iota\vec{q}.\vec{r}')$, $\vec{q}$ is the neutrinos momentum and we can take it along the $z-$axis. Hence, the angular integration gives

$$\begin{aligned}\vec{\rho}^2\vec{\rho}^{2'} &= a^4 \int d\Omega' \exp(\iota qr\cos\theta')\hat{r}^2 \int d\Omega \exp(\iota qr\cos\theta)\hat{r}'^2 \\ &= (4\pi)^2 a^2 j_0(qr) j_0(qr)\end{aligned} \tag{5.3}$$

In the above expression $j_0(qr)$ is Spherical Bessel function of order zero. Upon substituting $\hat{r}^2 = \hat{r}'^2 = 1$ and after angular integration we have taken $\hat{r} = \hat{r}'$. Considering the term

$$\begin{aligned}(\vec{\rho}.\vec{\rho}')(\vec{\rho}.\vec{\rho}') &= a^4 \int d\Omega' d\Omega \exp(-\iota qr\cos\theta')\exp(\iota qr\cos\theta)\hat{r}_i\hat{r}_j\hat{r}_i'\hat{r}_j' \\ &= a^4 \int d\Omega' \exp(-\iota qr\cos\theta')\hat{r}_i'\hat{r}_j' \int d\Omega \exp(\iota qr\cos\theta)\hat{r}_i\hat{r}_j.\end{aligned} \tag{5.4}$$

Here we can see that there is a summation on i and j indices, and for each value of i there are three value of j, and vice a verse. But only same values of i and j term survive due to ϕ integration. Let us consider $i = 1$ and j run from 1,..., 3,

$$\int d\Omega' \exp(-\iota qr\cos\theta')\hat{r}_i'\hat{r}_j' \int d\Omega \exp(\iota qr\cos\theta)\hat{r}_i\hat{r}_j$$

$$= \int d\Omega' \exp(\iota qr\cos\theta') \int d\Omega \exp(\iota qr\cos\theta)\left(\hat{r}_1\hat{r}_1\hat{r}_1'\hat{r}_1' + \hat{r}_1\hat{r}_2\hat{r}_2'\hat{r}_2' + \hat{r}_1\hat{r}_3\hat{r}_1'\hat{r}_3'\right).$$

Remember

$$\int d\Omega \exp(\iota qr\cos\theta)\hat{r}_1\hat{r}_2 = \int d\Omega \exp(\iota qr\cos\theta)\hat{r}_1\hat{r}_3 = 0,$$

and

$$\begin{aligned}
\int d\Omega \exp(\iota qr\cos\theta)\hat{r}_1\hat{r}_1 &= \int d\Omega^{'} \exp(-\iota qr\cos\theta^{,})\hat{r}_1{}'\hat{r}_1' \\
&= \int d\Omega \exp(\iota qr\cos\theta) Sin^2\theta Cos^2\phi \\
&= \int_0^{\pi} d\theta \exp(\iota qr\cos\theta) Sin^3\theta \int_0^{2\pi} d\phi \cos^2\phi \\
&= 4\pi\frac{j_1(qr)}{qr} = \frac{4\pi}{3}\left[j_2(qr) + j_0(qr)\right].
\end{aligned}$$

Similarly

$$\begin{aligned}
\int d\Omega \exp(\iota qr\cos\theta)\hat{r}_2\hat{r}_2 &= \int d\Omega^{'} \exp(-\iota qr\cos\theta')\hat{r}_2{}'\hat{r}_2' \\
&= 4\pi\frac{j_1(qr)}{qr} = \frac{4\pi}{3}\left[j_2(qr) + j_0(qr)\right],
\end{aligned}$$

and

$$\begin{aligned}
\int d\Omega \exp(\iota qr\cos\theta)\hat{r}_3\hat{r}_3 &= \int d\Omega^{'} \exp(\iota qr\cos\theta')\hat{r}_3{}'\hat{r}_3' \\
&= -8\pi\frac{j_1(qr)}{qr} + 4\pi j_0(qr).
\end{aligned}$$

Thus, we can write

$$\int d\Omega^{'} \exp(-\iota qr\cos\theta^{,})\hat{r}_i{}'\hat{r}_j' \int d\Omega \exp(\iota qr\cos\theta)\hat{r}_i\hat{r}_j$$

$$= (4\pi)^2 \left[\frac{j_1(qr)}{qr}\frac{j_1(qr')}{qr'} + \frac{j_1(qr)}{qr}\frac{j_1(qr')}{qr'} + (j_0(qr) - 2\frac{j_1(qr)}{qr})(j_0(qr') - 2\frac{j_1(qr')}{qr'}) \right]$$
$$= (4\pi)^2 \left[6\frac{j_1(qr)}{qr}\frac{j_1(qr')}{qr'} + j_0(qr)j_0(qr') - 2j_0(qr)\frac{j_1(qr')}{qr'} - 2j_0(qr')\frac{j_1(qr)}{qr}) \right]$$
$$= (4\pi)^2 \left[\frac{6}{9}\left(j_2(qr) + j_0(qr)\right) 2j_0(qr)j_0(qr) - \frac{4}{3}j_0(qr)\left(j_1(qr) + j_0(qr)\right) \right]$$
$$= (4\pi)^2 \left[\frac{6}{9}(j_2(qr)j_2(qr) + j_0(qr)j_0(qr) + 2j_0(qr)j_2(qr) + j_0(qr)j_0(qr) - \frac{4}{3}j_0(qr)(j_2(qr) - \frac{4}{3}j_0(qr)j_0(qr)) \right]$$
$$= (4\pi)^2 \left[\frac{2}{3}j_2(qr)j_2(qr) + \left(\frac{2}{3} + 1 - \frac{4}{3}\right) j_0(qr)j_0(qr) + \left(\frac{4}{3} - \frac{4}{3}\right) j_0(qr)j_2(qr) \right]$$
$$= (4\pi)^2 \left[\frac{2}{3}j_2(qr)j_2(qr) + \frac{1}{3}j_0(qr)j_0(qr) \right]$$

Therefore, Eq. (5.4) becomes

$$(\vec{\rho}.\vec{\rho}')(\vec{\rho}.\vec{\rho}') = (4\pi)^2 a^4 \left[\frac{2}{3}j_2(qr)j_2(qr) + \frac{1}{3}j_0(qr)j_0(qr) \right]. \tag{5.5}$$

Finally

$$\vec{\rho}'^2 = \vec{\rho}^2 = a^2 \int d\Omega \exp(\iota qr\cos\theta) \int d\Omega \exp(\iota qr\cos\theta)$$
$$= (4\pi)^2 a^2 j_0(qr)j_0(qr) \tag{5.6}$$

Substituting Eq. (5.3), Eq. (5.5) and Eq. (5.6) in Eq. (5.2), we get

$$-8\vec{\rho}\vec{\rho}^{2'} + 32(\vec{\rho}.\vec{\rho}')(\vec{\rho}.\vec{\rho}') + 8\vec{\rho^2} + 8\vec{\rho'^2} + 24$$

$$= (4\pi)^2 \left[-8a^4 j_0(qr)j_0(qr) + \frac{32}{3}a^4(2j_2(qr)j_2(qr) + j_0(qr)j_0(qr) + 16a^2 j_0(qr)j_0(qr) + 24j_0(qr)j_0(qr) \right]$$
$$= (4\pi)^2 a^4 \left[8(\frac{1}{3}a^4 + 2a^2 + 3)j_0(qr)j_0(qr) + \frac{64}{3}a^4(2j_2(qr)j_2(qr) \right]. \tag{5.7}$$

In the above expression $j_i(qr)$ is Spherical Bessel function of order i. Here we would like to point out that we have started with the expression of the amplitude, prior to angular integration that are derived for the equal mass. In those expressions we put the odd terms in $\hat{r}, \hat{r}'$ to be zero. However, it is not the case when we consider actual masses. of electron and muon. Therefore, the remaining term we are going to calculate below.

Corresponding to q_0^2, the term, which otherwise was zero for equal mass is $-64q_0^2\vec{\rho}.\vec{\rho}$. In this case, it gives

$$-64q_0^2\int d\Omega'\exp(-\iota qr\cos\theta')\int d\Omega\exp(\iota qr\cos\theta) = -\,64a^2q_0^2(4\pi)^2\left(\iota j_1(qr)-\iota j_1(qr)\right)$$
$$= -\,64a^2q_0^2(4\pi)^2 j_1(qr)j_1(qr)$$

Hence altogether, the term corresponding to q_0^2 is

$$(4\pi)^2\left\{8(\frac{1}{3}a^4+2a^2+3)j_0(qr)j_0(qr)+\frac{64}{3}a^4(2j_2(qr)j_2(qr)-64a^2q_0^2(4\pi)^2j_1(qr)j_1(qr)\right\}. \tag{5.8}$$

Corresponding to $\vec{q}^2$

Again we have collected the terms that we have calculated for equal muon and electron masses. Then we will add the terms which were zero in that case but not in general case. The non zero terms are

$$\vec{q}^2[24\vec{\rho}\vec{\rho}^{2'}-32(\vec{\rho}.\vec{\rho}')(\vec{\rho}.\vec{\rho}')-8\vec{\rho^2}-8\vec{\rho'^2}-8]$$
$$-32\vec{\rho}^2(\vec{q}.\vec{\rho})(\vec{q}.\vec{\rho})+32(\vec{q}.\vec{\rho})(\vec{q}.\vec{\rho'})(\vec{\rho}.\vec{\rho}')-32(\vec{q}.\vec{\rho}')(\vec{q}.\vec{\rho}') \tag{5.9}$$

The first line terms gives

$$\vec{q}^2[24\vec{\rho}\vec{\rho}^{2'}-32(\vec{\rho}.\vec{\rho}')(\vec{\rho}.\vec{\rho}')-8\vec{\rho^2}-8\vec{\rho'^2}-8]$$

$$=\vec{q}^2(4\pi)^2[a^4(24-\frac{32}{3})j_0(qr)j_0(qr)-\frac{64}{3}a^4j_2(qr)j_2(qr)-16a^2j_0(qr)j_0(qr)-8j_0(qr)j_0(qr)]$$
$$=\vec{q}^2(4\pi)^2[a^4\frac{40}{3}j(qr)j_0(qr)-\frac{64}{3}a^4j_2(qr)j_2(qr)-16a^2j_0(qr)j_0(qr)-8j_0(qr)j_0(qr)] \tag{5.10}$$

Now consider the term

$$\vec{\rho}^2(\vec{q}.\vec{\rho})(\vec{q}.\vec{\rho}) = a^4\vec{q}^2\hat{r}_3\hat{r}_3$$

and its angular integration over $d\Omega$ and $d\Omega'$gives

$$\int d\Omega'\exp(-\iota qr\cos\theta')\int d\Omega\exp(\iota qr\cos\theta)\vec{\rho}^2(\vec{q}.\vec{\rho})(\vec{q}.\vec{\rho}) = \vec{q}^2(4\pi)^2a^4j_0(qr)\left\{-2\frac{j_1(qr)}{qr}+j_0(qr)\right\}$$

$$= \vec{q}^2(4\pi)^2 a^4 j_0(qr)\left[-\frac{2}{3}(j_2(qr)+j_0(qr))+j_0(qr)\right]$$
$$= \vec{q}^2(4\pi)^2 a^4 j_0(qr)\left[-\frac{2}{3}j_2(qr)+\frac{1}{3}j_0(qr)\right]. \qquad (5.11)$$

Similarly,

$$\int (\vec{q}.\vec{\rho'})(\vec{q}.\vec{\rho'})d\Omega' \exp(-\iota qr\cos\theta') \int d\Omega \exp(\iota qr\cos\theta)$$

$$= \vec{q}^2(4\pi)^2 a^2 j_0(qr)\left\{-2(j_2(qr)+j_0(qr)\right\}$$
$$= \vec{q}^2(4\pi)^2 a^2 j_0(qr)\left\{-\frac{2}{3}j_2(qr)+\frac{1}{3}j_0(qr)\right\}. \qquad (5.12)$$

Let's look at the term

$$\int (\vec{q}.\vec{\rho})(\vec{q}.\vec{\rho'})(\vec{\rho}.\vec{\rho'})d\Omega' \exp(-\iota qr\cos\theta') \int d\Omega \exp(\iota qr\cos\theta)$$

$$=\vec{q}^2 a^2 \int \hat{r}_3\hat{r'_3}(\vec{\rho}.\vec{\rho})d\Omega' \exp(-\iota qr\cos\theta') \int d\Omega \exp(\iota qr\cos\theta)$$
$$=\vec{q}^2 a^4 \int cos\theta cos\theta' d\Omega' \exp(-\iota qr\cos\theta') \int d\Omega \exp(\iota qr\cos\theta)$$
$$=\vec{q}^2 a^4 \int d\Omega' \exp(-\iota qr\cos\theta')\cos^2\theta' \int \cos^2\theta d\Omega \exp(\iota qr\cos\theta)$$
$$=\vec{q}^2 a^4(4\pi)^2\left(-2\frac{j_1(qr)}{qr}+j_0(qr)\right)\left(-2\frac{j_1(qr)}{qr}+j_0(qr)\right)$$
$$=\vec{q}^2 a^4(4\pi)^2\left(-\frac{2}{3}j_2(qr)+\frac{1}{3}j_0(qr)\right)\left(-\frac{2}{3}j_2(qr)+\frac{1}{3}j_0(qr)\right)$$
$$=\vec{q}^2 a^4(4\pi)^2\frac{1}{9}\left(-2j_2(qr)+j_0(qr)\right)\left(-2j_2(qr)+j_0(qr)\right)$$
$$=\vec{q}^2 a^4(4\pi)^2\frac{1}{9}\left(4j_2(qr)j_2(qr)-4j_0(qr)j_2(qr)+j_0(qr)j_0(qr)\right). \qquad (5.13)$$

Thus we have

$$-32\vec{\rho}^2(\vec{q}.\vec{\rho})(\vec{q}.\vec{\rho})+32(\vec{q}.\vec{\rho})(\vec{q}.\vec{\rho'})(\vec{\rho}.\vec{\rho\prime})-32(\vec{q}.\vec{\rho'})(\vec{q}.\vec{\rho'})$$

$$
\begin{aligned}
&= \vec{q}^2 a^4 (4\pi)^2 \left(\frac{64}{3} j_2(qr) j_0(qr) - \frac{32}{3} j_0(qr) j_0(qr) + \frac{128}{9} j_2(qr) j_2(qr) - \frac{128}{9} j_0(qr) j_2(qr) + \frac{32}{3} j_0(qr) j_0(qr) \right) \\
&+ \vec{q}^2 a^2 (4\pi)^2 \left(\frac{64}{3} j_2(qr) j_0(qr) - \frac{32}{3} j_0(qr) j_0(qr) \right) \\
&= \vec{q}^2 a^4 (4\pi)^2 \left(\frac{64}{9} j_2(qr) j_0(qr) - \frac{64}{9} j_0(qr) j_0(qr) + \frac{128}{9} j_2(qr) j_2(qr) \right) \\
&+ \vec{q}^2 a^2 (4\pi)^2 \left(\frac{64}{3} j_2(qr) j_0(qr) - \frac{32}{3} j_0(qr) j_0(qr) \right) \qquad (5.14)
\end{aligned}
$$

Combing Eqs. ((5.9)- (5.14)), we have

$$
\vec{q}^2 \left[24\vec{\rho}\vec{\rho}^{2'} - 32(\vec{\rho}.\vec{\rho}^{'})(\vec{\rho}.\vec{\rho}^{'}) - 8\vec{\rho^2} - 8\vec{\rho'^2} - 8 \right] - 32\vec{\rho}^2 (\vec{q}.\vec{\rho})(\vec{q}.\vec{\rho}) + 32(\vec{q}.\vec{\rho})(\vec{q}.\vec{\rho'})(\vec{\rho}.\vec{\rho\prime}) - 32(\vec{q}.\vec{\rho}')(\vec{q}.\vec{\rho}')
$$

$$
\begin{aligned}
&= \vec{q}^2 (4\pi)^2 [a^4 \frac{40}{3} j_0(qr) j_0(qr) - \frac{64}{3} a^4 j_2(qr) j_2(qr) - 16a^2 j_0(qr) j_0(qr) - 8 j_0(qr) j_0(qr) \\
&+ \frac{64}{9} a^4 j_2(qr) j_0(qr) - \frac{64}{9} a^4 j_0(qr) j_0(qr) + \frac{128}{9} a^4 j_2(qr) j_2(qr) + \frac{64}{3} a^2 j_2(qr) j_0(qr) - \frac{32}{3} a^2 j_0(qr) j_0(qr)] \\
&= \vec{q}^2 (4\pi)^2 [\frac{56}{9} a^4 j_0(qr) j_0(qr) - \frac{64}{9} a^4 j_2(qr) j_2(qr) + \frac{64}{9} a^4 j_0(qr) j_2(qr) \\
&- \frac{80}{3} a^2 j_0(qr) j_0(qr) + \frac{64}{3} a^2 j_2(qr) j_0(qr) - 8 j_0(qr) j_0(qr)] \\
&= \vec{q}^2 (4\pi)^2 \left\{ 8 \left(\frac{7}{9} a^4 - \frac{10}{3} - 1 \right) j_0(qr) j_0(qr) - \frac{64}{9} a^4 j_2(qr) j_2(qr) + \frac{64}{9} \left(a^4 + 3a^2 \right) j_0(qr) j_2(qr) \right\}.
\end{aligned}
$$

Now the term which was zero in equal masses case corresponding to $\vec{q}^2$ is $32\vec{q}^2 \vec{\rho}.\vec{\rho} + 32(\vec{q}.\vec{\rho})(\vec{q}.\vec{\rho}')$. Solving it

$$
32 \int d\Omega^{'} \exp(-\iota qr \cos\theta') \int d\Omega \exp(\iota qr \cos\theta) \left(\vec{q}^2 \vec{\rho}.\vec{\rho} + (\vec{q}.\vec{\rho})(\vec{q}.\vec{\rho}') \right)
$$

$$
\begin{aligned}
&= 64\vec{q}^2 a^2 \int d\Omega^{'} \exp(-\iota qr \cos\theta') \int d\Omega \exp(\iota qr \cos\theta) \\
&= 64 a^2 \vec{q}^2 (4\pi)^2 j_1(qr) j_1(qr).
\end{aligned}
$$

Hence altogether, the terms corresponding to $\vec{q}^2$ are

$$(4\pi)^2\Bigg[8\left(\frac{7}{9}a^4-\frac{10}{3}a^2-1\right)j_0(qr)j_0(qr)-\frac{64}{9}a^4j_2(qr)j_2(qr)+\frac{64}{9}\left(a^4+3a^2\right)j_0(qr)j_2(qr)$$
$$+64a^2(4\pi)^2j_1(qr)j_1(qr)\Bigg]$$

Therefore the corresponding invariant amplitude becomes (factor of $(4\pi)^2$ cancel out with the spin part of $1S$ wave function of electron and muon)

$$\begin{aligned}|A|^2=&\frac{G_F^2}{6\pi}q_0^2\left((\frac{1}{3}a^4+2a^2+3)\left\langle j_0(qr)f_ef_\mu\right\rangle^2+\frac{8}{3}a^4\left\langle 2j_2(qr)f_ef_\mu\right\rangle^2-8a^2\left\langle 2j_1(qr)f_ef_\mu\right\rangle^2\right)\\&+\frac{G_F^2}{6\pi}\vec{q}^2\Bigg[\left(\frac{7}{9}a^4-\frac{10}{3}a^2-1\right)\left\langle j_0(qr)f_ef_\mu\right\rangle^2-\frac{8}{9}a^4\left\langle j_2(qr)f_ef_\mu\right\rangle^2\\&+\frac{8}{9}\left(a^4+3a^2\right)\left\langle j_0(qr)f_ef_\mu\right\rangle\left\langle j_2(qr)f_ef_\mu\right\rangle+8a\left\langle j_1(qr)f_ef_\mu\right\rangle^2\Bigg].\end{aligned}\tag{5.15}$$

Substitute Eq. (5.15) in Eq. (3.6) the decay rate becomes

$$\begin{aligned}\Gamma=&\frac{1}{2\pi^2}\int d\left|\vec{q}\right|\vec{q}^2\left|A\right|^2\\=&\frac{G_F^2}{12\pi^3}\int d\left|\vec{q}\right|\vec{q}^2\Bigg[q_0^2\left((\frac{1}{3}a^4+2a^2+3)\left\langle j_0(qr)f_ef_\mu\right\rangle^2+\frac{8}{3}a^4\left\langle 2j_2(qr)f_ef_\mu\right\rangle^2-8a^2\left\langle 2j_1(qr)f_ef_\mu\right\rangle^2\right)\\&+\vec{q}^2\Bigg((\frac{7}{9}a^4-\frac{10}{3}a^2-1)\left\langle j_0(qr)f_ef_\mu\right\rangle^2-\frac{8}{9}a^4\left\langle j_2(qr)f_ef_\mu\right\rangle^2\\&+\frac{8}{9}\left(a^4+3a^2\right)\left\langle j_0(qr)f_ef_\mu\right\rangle\left\langle j_2(qr)f_ef_\mu\right\rangle+8a\left\langle j_1(qr)f_ef_\mu\right\rangle^2\Bigg)\Bigg]\end{aligned}$$

The ratio of bound to free muon decay is

$$
\begin{aligned}
\frac{\Gamma}{\Gamma_0} = \frac{16}{m_\mu} \int d\,|\vec{q}|\,\vec{q}^{\,2} \Bigg[q_0^2 \left((\frac{1}{3}a^4 + 2a^2 + 3) \langle j_0(qr) f_e f_\mu \rangle^2 + \frac{8}{3} a^4 \langle 2 j_2(qr) f_e f_\mu \rangle^2 - 8a^2 \langle 2 j_1(qr) f_e f_\mu \rangle^2 \right) \\
+ \vec{q}^{\,2} \Bigg((\frac{7}{9}a^4 - \frac{10}{3}a^2 - 1) \langle j_0(qr) f_e f_\mu \rangle^2 - \frac{8}{9} a^4 \langle j_2(qr) f_e f_\mu \rangle^2 \\
+ \frac{8}{9}\left(a^4 + 3a^2\right) \langle j_0(qr) f_e f_\mu \rangle \langle j_2(qr) f_e f_\mu \rangle + 8a \langle j_1(qr) f_e f_\mu \rangle^2 \Bigg) \Bigg]. \qquad (5.16)
\end{aligned}
$$

In the last expression $j_0(qr)$, $j_1(qr)$ and $j_2(qr)$ are the Spherical Bessel function of zero, first and second kind respectively. After some simplification, we can find the decay ratio for different values of Z. In Eq. (5.16), we can see that it is quite complicated and to be sure if it is correct, it will be worthy to check if in the equal mass limit, we can retrieve the result derived in Eq. (4.4). To see this, let us write $a = \iota\sqrt{\frac{1-\gamma}{1+\gamma}}$

$$
\begin{aligned}
\frac{\Gamma}{\Gamma_0} &= \frac{16}{m_\mu} \int d\,|\vec{q}|\,\vec{q}^{\,2} \left\{ q_0^2 (\frac{1}{3}a^4 + 2a^2 + 3) + \vec{q}^{\,2} \left(\frac{7}{9}a^4 - \frac{10}{3}a^2 - 1 \right) \right\} \langle f_e f_\mu \rangle^2 \\
&= \frac{16}{m_\mu} \left\{ \frac{\vec{q}^{\,3}}{3} q_0^2 \left(\frac{1}{3}a^4 + 2a^2 + 3 \right) + \frac{\vec{q}^{\,5}}{5} \left(\frac{7}{9}a^4 - \frac{10a^2}{3} - 1 \right) \langle f_e f_\mu \rangle^2 \right\},
\end{aligned}
$$

where, in equal masses limit $q_0 = |\vec{q}| = \epsilon\gamma m_\mu$, giving

$$
\begin{aligned}
\frac{\Gamma}{\Gamma_0} &= \frac{16}{m_\mu} \left\{ \frac{\epsilon^5\gamma^5 m_\mu^5}{3} \left(\frac{1}{3}\frac{(1-\gamma)^2}{(1+\gamma)^2} - 2\frac{(1-\gamma)}{(1+\gamma)} + 3 \right) + \frac{\epsilon^5\gamma^5 m_\mu^5}{5} \left(\frac{7}{9}\frac{(1-\gamma)^2}{(1+\gamma)^2} + \frac{10}{3}\frac{(1-\gamma)}{(1+\gamma)} - 1 \right) \right\} \langle f_e f_\mu \rangle^2 \\
&= \frac{16}{m_\mu}\epsilon^5\gamma^5 m_\mu^5 \left\{ \left(\frac{1}{9} + \frac{7}{45} \right) \frac{(1-\gamma)^2}{(1+\gamma)^2} + \left(\cancel{\frac{2}{3}} + \cancel{\frac{10}{15}} \right) \frac{(1-\gamma)}{(1+\gamma)} + \left(1 - \frac{1}{5} \right) \right\} \langle f_e f_\mu \rangle^2 \qquad (5.17) \\
&= \frac{16}{m_\mu}\epsilon^5\gamma^5 m_\mu^5 \left\{ \frac{4}{15}\frac{(1-\gamma)^2}{(1+\gamma)^2} + \frac{4}{5} \right\} \langle f_e f_\mu \rangle^2 \qquad (5.18) \\
&= \frac{16}{5}\epsilon^5\gamma^5 4 \left(\frac{(1-\gamma)^2 + 3(1+\gamma)^2}{3(1+\gamma)^2} \right) \langle f_e f_\mu \rangle^2 \\
&= \frac{16}{5}\epsilon^5\gamma^5 \left(\frac{1}{3}(1-\gamma)^2 + (1+\gamma)^2 \right) \cancel{\frac{4}{(1+\gamma)^2}\frac{(1+\gamma)^2}{4}} \\
&= \frac{16}{5}\epsilon^5\gamma^5 \left(\frac{1}{3}(1-\gamma)^2 + (1+\gamma)^2 \right) \\
&= \frac{64}{5}\epsilon^5\gamma^5 \frac{1+\gamma+\gamma^2}{3}. \qquad (5.19)
\end{aligned}
$$

We can see that the result matches with Eq. (4.4).

Chapter IV

Bound Muon Decay To Bound Electron And Scalar Majoron

In the previous chapter, we have discussed the method to solve bound muon decay in position space. We have considered the situations in which bound muon decay into a bound electron, muon neutrino and electron antineutrino. In this section, we consider a scalar massless Majoron (J) instead of neutrinos, and the rest of the method is the same as we used previously.

The interaction Lagrangian $\mathcal{L}$ for Majoron emission in decay $\mu \to eJ$ is [10]

$$\mathcal{L} = \bar{\mu} g1 P_R e J + \bar{\mu} g2 P_L e J. \tag{5.20}$$

Here $P_R = \frac{1+\gamma^5}{2}$, $P_L = \frac{1-\gamma^5}{2}$ and g_1 and g_2 are dimensionless coupling constants. The particles current becomes

$$J \equiv \int d^3 r f_\mu(r) f_e(r) \bar{u}_e (g1 P_R + g2 P_L) u_\mu. \tag{5.21}$$

To calculate tensor ($JJ^\dagger$), we have to multiple Eq. (5.21) with $(J)^\dagger$ and neglecting radial part for brevity. We get in the last expression

$$\begin{aligned}
(J)(J)^\dagger = \bar{J} = 1\frac{1}{2}\sum_{spin} &\left[\bar{u}_e (g1 P_R + g2 P_L) u_\mu \left\{\bar{u}_e (g1 P_R + g2 P_L) u_\mu\right\}^\dagger\right] \\
= \frac{1}{2}\sum_{spin} &\left[\bar{u}_e \left\{g1(\frac{1+\gamma^5}{2}) + g2(\frac{1-\gamma^5}{2})\right\} u_\mu \left(\bar{u}_e \left\{(g1(\frac{1+\gamma^5)}{2} + g2(\frac{1-\gamma^5}{2})\right\} u_\mu\right)^\dagger\right] \\
= \frac{1}{8}\sum_{spin} &\left[\bar{u}_e \left\{g1(1+\gamma^5) + g2(1-\gamma^5)\right\} u_\mu \left(\bar{u}_\mu \left\{(g1(1+\gamma^5) + g2(1-\gamma)\right\} u_e\right)\right] \\
= \frac{1}{8}\sum_{spin} &\left[\bar{u}_e \left\{(g_1+g_2) + (g_1-g_2)\gamma^5\right\} u_\mu \bar{u}_\mu \left\{(g_1+g_2) + (g_1-g_2)\gamma^5\right\} u_e\right] \\
= \frac{1}{8}\text{Tr} &\left[u_e \bar{u}_e \left\{(g_1+g_2) + (g_1-g_2)\gamma^5\right\} u_\mu \bar{u}_\mu \left\{(g_1+g_2) + (g_1-g_2)\gamma^5\right\}\right] \\
= \frac{1}{8}\text{Tr} &\left[\not{p}\left(\frac{1+\gamma_0}{2}\right) \not{p}' \left((g_1+g_2) + (g_1-g_2)\gamma^5\right) \not{p}_1' \left(\frac{1+\gamma_0}{2}\right) \not{p}_1 \left((g_1+g_2) + (g_1-g_2)\gamma^5\right)\right]
\end{aligned} \tag{5.22}$$

$$\rho_1^{'} = (1, \frac{\iota(1-\gamma)}{Z\alpha}\hat{r}') = (1, -\vec{\rho'}), and, \rho 1 = (1, -\frac{\iota(1-\gamma)}{Z\alpha}\hat{r}) = (1, -\vec{\rho}).$$

Again we will consider the two cases.

6 Equal mass case

The trace of Eq. (5.22) is evaluated with the Mathematica. We collected all the non-zero terms and dropping all the terms corresponding to Levi Civita. Remember, we have considered the masses of muon and electron to be nearly equal so the momentum of Majoron must approach zero.

In the result below $\vec{\rho} = a\hat{r}, \vec{\rho'} = a\hat{r}'$m where $a = \sqrt{\frac{(1-\gamma)}{(1+\gamma)}}$. This gives

$$\begin{aligned}
\frac{1}{2}(g_1+g_2)^2\left[1+\vec{\rho}^2\vec{\rho}^{2'}+\vec{\rho^2}+\vec{\rho^{2'}}\right]+2(g_1-g_2)\vec{\rho}.\vec{\rho}' \\
= \frac{1}{2}(g_1^2+g_2^2)\left[1+a^4+a^2+a^2\right]+2g_1g_2\left[0\right] \\
= \frac{1}{2}(g_1+g_2)^2\left[1+2a^2+a^4\right] \\
= \frac{1}{2}(g1+g2)^2\frac{4\gamma^2}{(1+\gamma)^2}. \\
= 2(g1+g2)^2\frac{\gamma^2}{(1+\gamma)^2}
\end{aligned}$$

Here, we have used $\hat{r_i}\hat{r}j = \hat{r}i'\hat{r}_j' = \frac{\delta_{ij}}{3}$. Finally the expression of invariant amplitude becomes

$$|A|^2 = 2(g1+g2)^2\frac{\gamma^2}{(1+\gamma)^2}\left|f_e(r)f_\mu(r)\right|.^2 \tag{6.1}$$

The corresponding decay rate is define as

$$\frac{\Gamma}{\Gamma_0} = \frac{4}{m_\mu(g_1^2+g_2^2)}(E_\mu - E_e)\left|A\right|^2. \tag{6.2}$$

To evaluate decay ratio, let us substitute the expression of amplitude in Eq. (6.2). This gives

$$\begin{aligned}
\frac{\Gamma}{\Gamma_0} &= \frac{4}{m_\mu(g_1^2+g_2^2)}(E_\mu - E_e)\left|A\right|^2 \\
&= \frac{4}{m_\mu(g_1^2+g_2^2)}(E_\mu - E_e)[(g1+g2)^2\frac{2\gamma^2}{(1+\gamma)^2}\left|f_e(r)f_\mu(r)\right|^2,
\end{aligned}$$

where, $(E_\mu - E_e) = \gamma(1-b)m_\mu$ and $b = 0.99$, gives

$$\begin{aligned}\frac{\Gamma}{\Gamma_0} &= 2\frac{(g1+g2)^2}{m_\mu(g_1^2+g_2^2)}\gamma^3(1-b)m_\mu \cancel{\frac{4}{(1+\gamma)^2}|f_e(r)f_\mu(r)|^2} \\ &= 2\frac{(g1+g2)^2}{(g_1^2+g_2^2)}\gamma^3(1-b) \qquad (6.3)\end{aligned}$$

Finally, considering case when $g_1 = g_2 = 1$ Eq. (6.3), calculated decay ratio for $Z = 10$ and $Z = 80$ is

$$\frac{\Gamma}{\Gamma_0} = 3.96809 \times 10^{-2}, Z = 10.$$

$$\frac{\Gamma}{\Gamma_0} = 2.14081 \times 10^{-2}, Z = 80.$$

7 General case

In the general case, we considered the actual masses of muon and electron, so exponential factor take part in angular integration over $d\Omega$ and $d\Omega'$. Let us evaluated it step by step as follows

$$\bar{J} = \frac{1}{8}\mathrm{Tr}\left[\not{\rho}\left(\frac{1+\gamma_0}{2}\right)\not{\rho}'\left((g_1+g_2)+(g_1-g_2)\gamma^5\right)\not{\rho}_1'\left(\frac{1+\gamma_0}{2}\right)\not{\rho}_1\left((g_1+g_2)+(g_1-g_2)\gamma^5\right)\right].$$

The trace of above expression is already obtained.

Terms Corresponding to $(g_1+g_2)^2$

$$\left[1+\vec{\rho}^2\vec{\rho}^{2'}+\vec{\rho^2}+\vec{\rho^{2'}}\right]. \qquad (7.1)$$

We have

$$\begin{aligned}\vec{\rho^{2'}} = \vec{\rho}^2 &= a^2\int d\Omega'\exp(\iota qr\cos\theta')\int d\Omega\exp(\iota qr\cos\theta) \\ &= (4\pi)^2a^2j_0(qr)j_0(qr), \qquad (7.2)\end{aligned}$$

$$\begin{aligned}\vec{\rho}^2\vec{\rho}^{2'} &= a^4\int d\Omega'\exp(\iota qr\cos\theta')\int d\Omega\exp(\iota qr\cos\theta) \\ &= (4\pi)^2a^4j_0(qr)j_0(qr), \qquad (7.3)\end{aligned}$$

and

$$
\begin{aligned}
(\vec{\rho}.\vec{\rho}^{'}) &= \int d\Omega^{'} \exp(\iota qr\cos\theta') \int d\Omega \exp(\iota qr\cos\theta) \\
&= (4\pi)^2 a^2 (\iota j_1(qr))(-\iota j_1(qr)) \\
&= (4\pi)^2 a^2 j_1(qr) j(qr). \qquad (7.4)
\end{aligned}
$$

Using Eq. (7.2) and (7.3) in Eq. (7.1), give

$$
\begin{aligned}
\left[1 + \vec{\rho}^2 \vec{\rho}^{2'} + \rho\vec{^2} + \rho\vec{^{2'}}\right] &= \Big[(4\pi)^2 j_0(qr) j_0(qr) + (4\pi)^2 a^4 j_0(qr) j_0(qr) + (4\pi)^2 a^2 j_0(qr) j_0(qr) \\
&+ (4\pi)^2 a^2 j_0(qr) j_0(qr)\Big] \\
&= (4\pi)^2 \Big[[j_0(qr) j_0(qr) + a^4 j_0(qr) j_0(qr) + 2a^2 j_0(qr) j_0(qr)\Big]. \\
&= (4\pi)^2 \{1 + 2a^2 + a^4\} j_0(qr) j_0(qr) \qquad (7.5)
\end{aligned}
$$

Corresponding to $(g_1 - g_2)2$

$$
(\vec{\rho}.\vec{\rho})^{'} = (4\pi)^2 a^2 j_1(qr) j_1(qr). \qquad (7.6)
$$

Combining Eq. (7.5) and Eq. (7.6) leads to

$$
\frac{1}{2}\left(g_1 + g_2\right)^2 (4\pi)^2 \{1 + 2a^2 + a^4\} j_0(qr) j_0(qr) - 2\left(g_1 - g_2\right)^2 (4\pi)^2 a^2 j_1(qr) j_1(qr).
$$

Remember, $j_0(qr)$ and $j_1(qr)$ are Spherical Bessel function of 0^{th} and 1^{st} kind, respectively. The factor of $(4\pi)^2$ cancels out with the spin part in the $1S$ wave function of electron and muon. Collecting all the terms gives the invariant amplitude to be

$$
|A|^2 = \frac{(g_1 + g_2)^2}{2}\left(1 + 2a^2 + a^4\right)\langle j_0(qr) f_e(r) f_\mu(r)\rangle^2 - 2(g_1 - g_2)^2 a^2 \langle j_1(qr) f_e(r) f_\mu(r)\rangle^2. \qquad (7.7)
$$

The corresponding decay ratio is

$$
\frac{\Gamma}{\Gamma_0} = \frac{4}{m_\mu (g_1^2 + g_2^2)} (E_\mu - E_e)\, |A|^2. \qquad (7.8)
$$

In order to evaluate decay ratio put the Eq. (7.7) into Eq. (7.8), giving

$$\frac{\Gamma}{\Gamma_0} = \frac{4}{m_\mu(g_1^2+g_2^2)}(E_\mu - E_e)\frac{(g1+g2)^2}{2}\left[\{1+2a^2+a^4\}\langle j_0(qr)f_e(r)f_\mu(r)\rangle^2 - 2(g1-g2)^2a^2\langle j_1(qr)f_e(r)f_\mu(r)\rangle^2\right].$$

After putting $a = \iota\sqrt{\frac{(1-\gamma)}{(1+\gamma)}}$ and $(E_\mu - E_e) = \gamma(m_\mu - m_e)$, in the last expression, we will get

$$\begin{aligned}\frac{\Gamma}{\Gamma_0} &= 4\frac{\gamma(m_\mu - m_e)}{m_\mu(g_1^2+g_2^2)}\left[4\frac{(g_1+g_2)^2}{2}\frac{\gamma^2}{(1+\gamma)^2}\langle j_0(qr)f_e(r)f_\mu(r)\rangle^2 + 2(g_1-g_2)^2\frac{(1-\gamma)^2}{(1+\gamma)^2}\langle j_1(qr)f_e(r)f_\mu(r)\rangle^2\right] \\ &= 4\frac{(m_\mu - m_e)}{m_\mu(g_1^2+g_2^2)}\left[2\frac{(g_1+g_2)^2\gamma^3}{(1+\gamma)^2}\langle j_0(qr)f_e(r)f_\mu(r)\rangle^2 + 2(g_1-g_2)^2\frac{\gamma(1-\gamma)^2}{(1+\gamma)^2}\langle j_1(qr)f_e(r)f_\mu(r)\rangle^2\right]\end{aligned} \tag{7.9}$$

Using the numerical values of the electron and muon masses, and by considering $Z = 10$ and $Z = 80$, the numerical values of the branching ratio are

$$\frac{\Gamma}{\Gamma_0} = 4.73457\times 10^{-7}, Z = 10$$

$$\frac{\Gamma}{\Gamma_0} = 2.35375\times 10^{-6}, Z = 80.$$

If we would like to reproduce the result of equal mass, we need to put $j_0(qr) \to 1$,$j_1(qr) \to 0$ and $(E_\mu - E_e) = \gamma(1-b)m_\mu$ in Eq. (7.9), which gives

$$\begin{aligned}\frac{\Gamma}{\Gamma_0} &= \frac{4\gamma(1-b)\cancel{m_\mu}}{\cancel{m_\mu}(g_1^2+g_2^2)}\left[2(g_1+g_2)^2\frac{\gamma^2}{(1+\gamma)^2}\langle f(r)_e f_\mu(r)\rangle^2\right] \\ &= \frac{2\gamma^3(1-b)}{(g_1^2+g_2^2)}\left[(g_1+g_2)^2\cancel{\frac{4}{(1+\gamma)^2}\langle f(r)ef_\mu(r)\rangle^2}\right] \\ &= 2\frac{(g_1+g_2)^2}{(g_1^2+g_2^2)}(1-b)\gamma^3.\end{aligned}$$

This result is the same as given in Eq. (6.3).

Chapter V

Conclusion

In this work, we have calculated decay rates of two types of decay mode of $\mu \to e$

conversion using Casimir's trick in position space. As a first step we have calculated the decay rate of bound muon decay into bound electron and neutrinos $(Z\mu) \rightarrow (Ze)\nu_\mu\bar{\nu}_e$. In order to find it, we have have evaluated Dirac wave-functions for Coulomb potential and using them we have calculated invariant amplitude. Using it we have computed the value of the decay rate for different values of Z and in equal mass limit we find that the value of the bound to free muon decay rate is equal to the one calculated in Alchemy's formalism [9].

In the second part, we have studied a neutrino-less conversion of a bound muon into bound electron. Here, we have calculated decay ratio of $\mu \rightarrow eJ$, where J is a Majoron by using the same method as we did for $(Z\mu) \rightarrow (Ze)\nu_\mu\bar{\nu}_e$ decay, where instead of neutrinos we have consider mass-less scalar Majoron (J). We find that the value of decay branching ratio in this case is of the order of $10^{-7}$and 10^{-6} for Z =10 and Z = 80, respectively. We hope that the present study is useful for the future CLFV experiments such as Mu2e and COMET at JPARC.

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Chapter VI

Appendix

8 A:Integration over position space

Let's evaluate integration of position part of Dirac wave function.

$$
\begin{aligned}
\int f(r)_e f_\mu(r) &= \frac{(2Z\alpha)^{2\gamma+1}(m_e m_\mu)^{\gamma-1}}{4\pi}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\int dr^3.r^{2\gamma-2}\exp[-(m_e+m_\mu)rZ\alpha] \\
&= \frac{(2Z\alpha)^{2\gamma+1}(m_e m_\mu)^{\gamma-1}}{4\pi}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\int r^2\sin\theta d\theta d\phi dr.r^{2\gamma-2}\exp[-(m_e+m_\mu)rZ\alpha] \\
&= \frac{(2Z\alpha)^{2\gamma+1}(m_e m_\mu)^{\gamma-1}}{4\pi}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\int dr.r^{2\gamma}\exp[-(m_e+m_\mu)rZ\alpha]\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\phi \\
&= \frac{(2Z\alpha)^{2\gamma+1}(m_e m_\mu)^{\gamma-1}}{\cancel{4\pi}}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\int dr.r^{2\gamma}\exp[-(m_e+m_\mu)rZ\alpha]\cancel{4\pi} \\
&= (2Z\alpha)^{2\gamma+1}(m_e m_\mu)^{\gamma-1}\frac{1+\gamma}{2\Gamma(1+2\gamma)}\frac{\Gamma(1+2\gamma)}{[(m_e+m_\mu)Z\alpha]^{1+2\gamma}} \\
&= \left(\frac{2}{m_e+m_\mu}\right)^{2\gamma+1}(m_e m_\mu)^{\gamma-1/2}\left(\frac{1+\gamma}{2}\right) \qquad (8.1)
\end{aligned}
$$

Upon substituting $m_e = (1-\epsilon)m_\mu$in Eq (8.1) we will get

$$
\begin{aligned}
\int f(r)_e f_\mu(r) &= \left(\frac{2}{(1-\epsilon)m_\mu+m_\mu}\right)^{2\gamma+1}((1-\epsilon)m_\mu^2)^{\gamma-1/2}\left(\frac{1+\gamma}{2}\right) \\
&= \left(\frac{1+\gamma}{2}\right)\left\{\frac{(1-\epsilon)^{\gamma+1/2}}{\left(1-\frac{\epsilon}{2}\right)^{2\gamma+1}}\right\} \\
&\approx \left(\frac{1+\gamma}{2}\right)
\end{aligned}
$$

9 B:Evaluation Of angular integration

Let's evaluate one by one the angular integration over $d\Omega$ and $d\Omega'$ we used above.

$$
\vec{\rho^{2'}} = \vec{\rho}^2 = a^2\int d\Omega'\exp(\iota qr\cos\theta')\int d\Omega\exp(\iota qr\cos\theta)
$$

Thus.

$$\begin{aligned}
\int d\Omega \exp(\iota qr\cos\theta) &= \int \sin\theta d\theta d\phi \exp(\iota qr\cos\theta) \\
&= \int\limits_0^{\pi} \sin\theta \exp(\iota qr\cos\theta) d\theta \int\limits_0^{2\pi} d\phi \\
&= -\left[\frac{\exp(\iota qr\cos\theta)}{\iota qr}\right]_0^{\pi} .2\pi \\
&= -2\pi\left[\frac{\exp(-\iota qr)}{\iota qr} - \frac{\exp(\iota qr)}{\iota qr}\right] \\
&= \frac{2\pi}{\iota qr}\left[\exp(\iota qr) - \exp(-\iota qr)\right] \\
&= \frac{4\pi}{qr}\left[\frac{\exp(\iota qr) - \exp(-\iota qr)}{2\iota}\right] \\
&= \frac{4\pi}{qr} Sin(qr) = 4\pi J_0(qr)x
\end{aligned}$$

and

$$
\begin{aligned}
\int d\Omega \exp(\iota qr\cos\theta)\hat{r}_1\hat{r}_1 &= \int d\Omega \exp(\iota qr\cos\theta)\sin^2\theta\cos^2\phi \\
&= \int \sin\theta d\theta d\phi \exp(\iota qr\cos\theta)\sin^2\theta\cos^2\phi \\
&= \int \sin^3\theta \exp(\iota qr\cos\theta)d\theta \int \cos^2\phi d\phi \\
&= \int \sin^2\theta . \sin\theta \exp(\iota qr\cos\theta)d\theta . \pi \\
&= \pi\left[\sin^2\theta \int \sin\theta\exp(\iota qr\cos\theta)d\theta - \int\int \sin\theta\exp(\iota qr\cos\theta)d\theta . \frac{d\sin^2\theta}{d\theta}d\theta\right] \\
&= \pi\left[\sin^2\theta\left\{-\frac{\exp(\iota qr\cos\theta)}{\iota qr}\right\} - \int\left\{-\frac{\exp(\iota qr\cos\theta)}{\iota qr}\right\}2\sin\theta\cos\theta d\theta\right] \\
&= \pi\left[-\sin^2\theta\frac{\exp(\iota qr\cos\theta)}{\iota qr} + \frac{2}{\iota qr}\int \exp(\iota qr\cos\theta)\sin\theta . \cos\theta d\theta\right] \\
&= \pi\left[-\sin^2\theta\frac{\exp(\iota qr\cos\theta)}{\iota qr} + \frac{2}{\iota qr}\left\{\cos\theta\int \exp(\iota qr\cos\theta)\sin\theta d\theta\right.\right. \\
&\left.\left. - \int\int \exp(\iota qr\cos\theta)\sin\theta d\theta . \frac{d\cos\theta}{d\theta}d\theta\right\}\right] \\
&= \pi\left[-\sin^2\theta\frac{\exp(\iota qr\cos\theta)}{\iota qr} + \frac{2}{\iota qr}\left\{-\cos\theta\frac{\exp(\iota qr\cos\theta)}{\iota qr}\right.\right. \\
&\left.\left. - \int -\frac{\exp(\iota qr\cos\theta)}{\iota qr} - \sin\theta d\theta\right\}\right] \\
&= \pi\left[-\sin^2\theta\frac{\exp(\iota qr\cos\theta)}{\iota qr} + \frac{2}{\iota qr}\left\{-\cos\theta\frac{\exp(\iota qr\cos\theta)}{\iota qr} - \frac{\exp(\iota qr\cos\theta)}{q^2r^2}\right\}\right] \\
&= \pi\left[-\sin^2\theta\frac{\exp(\iota qr\cos\theta)}{\iota qr} - 2\cos\theta\frac{\exp(\iota qr\cos\theta)}{q^2r^2} - \frac{2\exp(\iota qr\cos\theta)}{\iota q^3r^3}\right]_0^\pi \\
&= \left[\pi - \pi\sin^2\theta\frac{\exp(\iota qr Cos\theta)}{\iota qr} - 2\left\{-\frac{\exp(-\iota qr)}{q^2r^2} - \frac{\exp(\iota qr)}{q^2r^2}\right\}\right. \\
&\left. - 2\left\{\frac{\exp(-\iota qr)}{\iota q^3r^3} - \frac{\exp(\iota qr)}{\iota q^3r^3}\right\}\right] \\
&= \pi\left[-\frac{4Cos(qr)}{q^2r^2} + \frac{4Sin(qr)}{q^3r^3}\right] \\
&= 4\pi\left[-\frac{Cos(qr)}{q^2r^2} + \frac{Sin(qr)}{q^3r^3}.\right] = 4\pi\frac{J_1(qr)}{qr}
\end{aligned}
$$

Similarly for $\hat{r}_2\hat{r}_2$ and $\hat{r}_3\hat{r}_3$.

$$
\begin{aligned}
\int d\Omega \exp(\iota qrCos\theta)\hat{r}_3\hat{r}_3 &= \int d\Omega \exp(\iota qrCos\theta).Cos^2\theta \\
&= \int Cos^2\theta Sin\theta \exp(\iota qrCos\theta)d\theta \int d\phi \\
&= \int Cos^2\theta.Sin\theta \exp(\iota qrCos\theta)d\theta.2\pi \\
&= 2\pi\left[Cos^2\theta.\int Sin\theta \exp(\iota qrCos\theta)d\theta - \int\int Sin\theta \exp(\iota qrCos\theta)d\theta.\frac{dCos^2\theta}{d\theta}d\theta\right] \\
&= 2\pi\left[Cos^2\theta\left\{-\frac{\exp(\iota qrCos\theta)}{\iota qr}\right\} - \int\left\{-\frac{\exp(\iota qrCos\theta)}{\iota qr}\right\} - 2Cos\theta Sin\theta d\theta\right] \\
&= \pi\left[Cos^2\theta\left\{-\frac{\exp(\iota qrCos\theta)}{\iota qr}\right\} - \int\left\{\frac{\exp(\iota qrCos\theta)}{\iota qr}\right\}2Sin\theta Cos\theta d\theta\right] \\
&= \pi\left[-Cos^2\theta\frac{\exp(\iota qrCos\theta)}{\iota qr} - \frac{2}{\iota qr}\int \exp(\iota qrCos\theta)Sin\theta.Cos\theta d\theta\right] \\
&= \pi\Bigg[-Cos^2\theta\frac{\exp(\iota qrCos\theta)}{\iota qr} - \frac{2}{\iota qr}\Bigg\{Cos\theta\int \exp(\iota qrCos\theta)Sin\theta d\theta \\
&- \int\int \exp(\iota qrCos\theta)Sin\theta d\theta.\frac{dCos\theta}{d\theta}d\theta\Bigg\}\Bigg] \\
&= 2\pi\left[-Cos^2\theta\frac{\exp(\iota qrCos\theta)}{\iota qr} - \frac{2}{\iota qr}\left\{-Cos\theta\frac{\exp(\iota qrCos\theta)}{\iota qr} - \frac{\exp(\iota qrCos\theta)}{q^2r^2}\right\}\right] \\
&= 2\pi\left[-Cos^2\theta\frac{\exp(\iota qrCos\theta)}{\iota qr} - 2Cos\theta\frac{\exp(\iota qrCos\theta)}{q^2r^2} - \frac{2\exp(\iota qrCos\theta)}{\iota q^3r^3}\right]_0^\pi \\
&= \Bigg[2\pi - \left\{\frac{\exp(-\iota qr)}{\iota qr} - \frac{\exp(-\iota qr)}{\iota qr}\right\} - 2\left\{-\frac{\exp(-\iota qr)}{q^2r^2} - \frac{\exp(\iota qr)}{q^2r^2}\right\} \\
&- 2\left\{\frac{\exp(-\iota qr)}{\iota q^3r^3} - \frac{\exp(\iota qr)}{\iota q^3r^3}\right\}\Bigg] \\
&= 2\pi\left[2\frac{Sin(qr)}{qr} + 4\frac{Cos(qr)}{q^2r^2} - 4\frac{Sin(qr)}{q^3r^3}\right] \\
&= 4\pi\frac{Sin(qr)}{qr} + 8\frac{Cos(qr)}{q^2r^2} - 8\frac{Sin(qr)}{q^3r^3} \\
&= 4\pi j_0(qr) - 8\pi\frac{j_1(qr)}{qr}
\end{aligned}
$$