A New Approach to Γ-semihypergroups through Applications of Intuitionistic Fuzzy Sets

Ph.D Thesis

By

Saleem Abdullah

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
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A THESIS SUBMITTED IN THE PARTIAL FULFILMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
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Supervised By

Dr. Muhammad Aslam

Department of Mathematics
Quaid-i-Azam University
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CERTIFICATE

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF
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We accept this dissertation as conforming to the required standard

1. ________________________
Dr. Muhammad Aslam
(Supervisor)

2. ________________________
Prof. Dr. Tasawar Hayat
(Chairman)

3. ________________________
Prof. Dr Akbar Azam
(External Examiner)

4. ________________________
Dr. Muhammad Irfan Ali
(External Examiner)

Department of Mathematics
Quaid-i-Azam University
Islamabad, Pakistan
2015

Dedicated

to

My Parents

Who are the most precious gems of my life.
Who've always given me perpetual love, care, and cheers.
Whose prayers have always been a source of great inspiration for me and whose sustained hope in me led me to where I stand today.

Foreign Reviewer List

1) Professor Dr. John N. Mordeson

Department of Mathematics, Creighton University

Omaha, nebraska 68178, USA.

2) Professor Dr. Young B. jun

Department of Mathematics Education

Gyeongsang national University

Jinju 600-701, Korea.

3) Professor Dr. Florentin Smarandache

Department of Mathematics & Science,

705 gurley Ave., Gallup NM 87301, USA.

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Saleem Abdullah

0.2 Research profile

The following research papers are the out come of my Ph.D. Thesis.

1. S. Abdullah, M. Aslam and T. Anwar, A note on M-hypersystems and N-hypersystems in Γ-semihypergroups, Quasigroups and Related Systems **19** (2011), 169-172.

2. S. Abdullah, K. Hila and M.Aslam, On bi-Γ-hyperideals of Γ-semihypergroups, UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, 74 (2012), 4, 79-90.

3. M. Aslam, S. Abdullah, B. Davvaz and N. Yaqoob, Rough M-hypersystems and fuzzy M-hypersystems in Γ-semihypergroups, Neural Comput. Applic., **21** (2012), 281-287.

4. K. Hila, S. Abdullah and J. Dine, On intuitionistic fuzzy hyperideals in Γ-semihypergroups through left operator semihypergroup, Accepted in Utilitas Mathematica, 2013.K.

5. Hila and S. Abdullah, A study on intuitionistic fuzzy sets in Γ-semihypergroups, J. Intell. Fuzzy Systems, 26 (2014) 1695–1710.

6. S. Abdullah and M. Aslam, A Note on left regular and intra-regular Γ-semihypergroups, submitted.

7. S. Abdullah, M. Aslam and B. Davvaz, Bi-Γ-hyperideals of Γ-semihypergroups based on intuitionistic fuzzy points, submitted.

8. S. Abdullah and M. Aslam, Interior Γ-hyperideals of Γ-semihypergroups based on intuitionistic fuzzy points, submitted.

9. S. Abdullah, Characterization of Γ-semihypergroups by intuitionistic fuzzy Γ-hyperideals, submitted.

10. K. Hila and S. Abdullah, Study on Quasi-Γ-hyperideals in Γ-semihypergroups.

I have published 50 research papers during my PhD study in different peer review journals (ISI, Scopus, etc).

0.3 Introduction

Hyperstructure is a natural generalization of the classical algebraic structure. Marty has defined a new novel concept so called hyperstructure in 1934, when he introduced the notion of a hypergroup based on a hyperoperation [37]. In the last few decades and now a days the scientists introduced so many different types of algebraic hyperstructures. They studied these hyper structures from the theoretical point of view, and also studied their applications in many subjects of pure and applied mathematics. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Different authors have been written many books on hyperstructures [23, 19, 20, 49]. Applications of hyperstructures have been studied in lattices, rough set theory, probability, coding theory, binary relations, graphs, hypergraphs automata and geometry [20]. A detail study of the theory of semihypergroups can be found in [24, 17].

Recently, Anvariyeh, et. al. [48], defined the notion of a Γ-semihypergroup and the notion of a Γ-hyperideal, a bi-Γ-hyperideal and a quasi-Γ-hyperideal of a Γ-semihypergroup. A Γ-semihypergroup is a generalization of the notions of a semigroup, a semihypergroup and a Γ-semigroup. In [29] Heidari et. al. further extended the theory of a Γ-semihypergroup. They introduced the notions of a prime Γ-hyperideal, extension of a Γ-hyperideal in Γ-semihypergroups. They proved some results and present many examples of a Γ-semihypergroup. Also, they studied the notion of a quotient Γ-semihypergroup by using a congruence relation, and gave the concept of a right Noetherian Γ-semihypergroup [29]. In [30], Heidari and Davvaz studied further the notion of semiprime hyperideals in a Γ-semihypergroup and also, they defined the concept of a Γ-hypergroup and a closed Γ-subhypergroup. Finally, they studied the concept of a Γ-semihypergroup associated to binary relations. They gave necessary and sufficient conditions on a set of binary relations Γ on a non-empty set H such that H becomes a Γ-semihypergroup or a Γ-hypergroup. In 2011 [2], Abdullah et. al. introduced the concept of M-hypersystems and N-hypersystems of a Γ-semihypergroup and they studied different relations of M-hypersystems and N-hypersystems with quasi-prime hyperideals of a Γ-semihypergroup. Mirvakili et. al [39] provided more canonic properties and confronted various examples of a Γ-semihypergroup. Also, they prevailed some results of regular and strongly regular relations on a Γ-semihypergroup. They fabricated a Γ-semigroup from a Γ-semihypergroup by using the concept of fundamental relation. Hila et. al., presented many interesting examples and obtained several characterizations of a Γ-semihypergroup [35].

After the introduction of the concept of a fuzzy set by Zadeh, several researchers conducted research on the generalizations of the notions of a fuzzy set with huge applications in computer, logic and many branches of pure and applied mathematics. In 1971, Rosenfeld [41] defined the concept of a fuzzy group. Since then many papers have been published in the field of a fuzzy algebra. Recently fuzzy set theory has been well developed in the context of hyperalgebraic structure theory. A recent book [20] contains a wealth of applications. In [26], Davvaz introduced the concept of a fuzzy hyperideal in a semihypergroup. Several papers have been written on a fuzzy set in several algebraic hyperstructures. Corsini et. al studied semihypergroups in term of fuzzy prime hyperideals. They characterized semihypergroups by the properties of fuzzy prime hyperideals [21]. Recently Aslam et. al studied the fuzzy Γ-hyperideals in Γ-semihypergroups and also defined fuzzy M-hypersystems and N-hypersystems [11]. Recently in [33], Hila and Galani have studied the structure of a Γ-semihypergroup through a fuzzy set and they studied Γ-semihypergroups through left operater semihypergroups. In [22], Davvaz and Leoreanu-Fotea, studied some structure properties of a fuzzy Γ-hyperideal in a Γ-semihypergroup. In 1984, Atanassov introduced an important generalization of the notion of a fuzzy set, so called an intuitionistic fuzzy set [14]. The concept of an intuitionistic fuzzy set on a non-empty set X gives both a membership degree and a non-membership degree. The relations between intuitionistic fuzzy sets and algebraic structures have been already considered by many mathematicians. In [27], Davvaz established the intuitionistic fuzzification of the concept of a hyperideal in a semihypergroup and investigated some of their properties. Recently, in [28], Ersoy and Davvaz initiated a study on intuitionistic fuzzy sets in Γ-semihypergroups and they gave some properties of intuitionistic fuzzy Γ-hyperideals of Γ-semihypergroups.

0.4 Chapterwise Study

This thesis consists of nine chapters. Throughout this thesis, H will denote a Γ-semihypergroup, unless otherwise stated.

In chapter one, we provide the basic concept and definitions of hyperoperations, semihypergroups and gives examples of these notions. we give the concepts of γ-hyperoperations, Γ-semihypergroups. We also provide some definitions of theory of Γ-hyperideal of a Γ-semihypergroup. We write some definitions of fuzzy sets and intuitionistic fuzzy sets. This chapter is introductory nature, provides some basic notion which are needed for subsequent chapters.

In chapter two, we introduce the concept of the strongly prime, prime, semiprime, strongly irreducible and irreducible bi-Γ-hyperideals of Γ-semihypergroups. The space of strongly prime bi-Γ-hyperideals is topologized. Also, we characterize those Γ-semihypergroups for which each bi-Γ-hyperideal is strongly prime. We study some different relations among these concepts and characterized Γ-semihypergroups by the properties of prime bi-Γ-hyperideals.

In chapter three, we introduce the notion of a quasi-Γ-hyperideal in Γ-semihypergroups, and moreover, we introduce the notion of an (m, n)-quasi-Γ-hyperideal, n-right Γ-hyperideal and m-left Γ-hyperideal in Γ-semihypergroups and relations between them are studied. We obtain different characterizations concerning different properties of (m, n)-quasi-Γ-hyperideals, minimal (m, n)-quasi-Γ-hyperideals, minimal m-left Γ-hyperideals, minimal n-right Γ-hyperideals and relations between them are investigated. Also, some intersection properties and characterizations of (m, n)-quasi-Γ-hyperideals of Γ-semihypergroups and regular Γ-semihypergroups have been studied. In sequel, m-left simple, n-right simple and (m, n)-quasi-simple Γ-semihypergroups are defined, and some properties of them are investigated.

In chapter four, we use the notion of an intuitionistic fuzzy set to theory of Γ-hyperideal of Γ-semihypergroups. We use Atanassov idea, we continue the study on intuitionistic fuzzy sets in Γ-semihypergroups initiated recently by Ersoy and Davvaz.

We give some further properties of intuitionistic fuzzy Γ-hyperideals and intuitionistic fuzzy bi-Γ-hyperideals in a Γ-semihypergroup. We use the intuitionistic fuzzy left, right, two-sided and bi-Γ-hyperideals to characterize some classes of Γ-semihypergroups. We introduce and study (λ, μ)-intuitionistic fuzzy Γ-hyperideals. We define intuitionistic fuzzy prime(semiprime) Γ-hyperideals, intuitionistic fuzzy M-hypersystem and N-hypersystem of a Γ-semihypergroup and intuitionistic fuzzy semisimple Γ-semihypergroups and some properties of them are investigated. We characterize Artinian and Noetherian Γ-semihypergroups in term of intuitionistic fuzzy Γ-hyperideals

In chapter five, we characterize Γ-semihypergroups by the properties of intuitionistic fuzzy Γ-hyperideals. We characterize regular Γ-semihypergroup, intra-regular Γ-semihypergroup by intuitionistic fuzzy Γ-hyperideal, intuitionistic fuzzy bi-Γ-hyperideal, intuitionistic fuzzy generalized bi-Γ-hyperideal, intuitionistic fuzzy interior Γ-hyperideal and intuitionistic fuzzy quasi-Γ-hyperideal. We also define intuitionistic fuzzy quasi-Γ-hyperideal of a Γ-semihypergroup. We give some relations among these notions.

In chapter six, we introduce the concept of an (α, β)-intuitionistic fuzzy bi-Γ-hyperideal, (α, β)-intuitionistic fuzzy $(1, 2)$-Γ-hyperideal of a Γ-semihypergroup by using the notion of intuitionistic fuzzy point to intuitionistic fuzzy set. (α, β)-intuitionistic fuzz bi-Γ-hyperideal, (α, β)-intuitionistic fuzzy $(1, 2)$-Γ-hyperideal of a Γ-semihypergroup is a generalization of ordinary intuitionistic fuzzy bi-hyperideals. We can construct twelve different types of intuitionistic fuzzy bi-Γ-hyperideal and intuitionistic fuzzy $(1, 2)$-Γ-hyperideal of a Γ-semihypergroup. We also define $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal, $(\in, \in \vee q)$-intuitionistic fuzzy $(1, 2)$-Γ-hyperideal of a Γ-semihypergroup. We characterize $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal, $(\in, \in \vee q)$-intuitionistic fuzzy $(1, 2)$-Γ-hyperideal by the properties of $\in$-level set, q-level set and $(\in, \in \vee q)$-level set.

In chapter seven, we introduce the concept of an (α, β)-intuitionistic fuzzy interior Γ-hyperideal of a Γ-semihypergroup. Also we study some properties of (α, β)-intuitionistic fuzzy interior Γ-hyperideal. The concept of (α, β)-intuitionistic fuzzy interior Γ-hyperideal is a generalization of intuitionistic fuzzy interior Γ-ideals of Γ-semihypergroups.

We characterize $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of a Γ-semihypergroup by the properties of level set.

In chapter eight, we obtain a series of lemmas and theorems which are mainly on various relationships between a Γ-semihypergroup and its operator semihypergroups in term of intuitionistic fuzzy subsets showing so the effectiveness of using operator semihypergroups in extending several results of semigroups to Γ-semihypergroups as well as to Γ-semigroups. Among other results, we obtain an inclusion preserving bijection between the set of all intuitionistic fuzzy hyperideal of a Γ-semihypergroup H and that of its left operator semihypergroup S. Throughout in this chapter, unless otherwise mentioned, H will denote a Γ-semihypergroup and S will denote the left operator semihypergroup of H.

In chapter nine, we apply the concept of an interval valued intuitionistic fuzzy set to theory of Γ-hyperideals, interval valued intuitionistic fuzzy (1,2) Γ-hyperideal of Γ-semihypergroup and obtain some basic results. We give some further properties of interval valued intuitionistic fuzzy Γ-hyperideals and interval valued intuitionistic fuzzy bi-Γ-hyperideals in a Γ-semihypergroup. We define an interval valued intuitionistic fuzzy prime(semiprime) Γ-hyperideals, intuitionistic fuzzy M-hypersystem and N-hypersystem of a Γ-semihypergroup and intuitionistic fuzzy semisimple Γ-semihypergroups and some properties of them are investigated

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Contents

Chapter 1

Introduction

1.1 Hyperstructures and Semihypergroups

In this section we give the brief discussion of hyperstructures. We denote by H a non-empty set and $\wp^*(H)$ is the set of all non-empty subsets of H, otherwise we shall mention.

A hyperoperation "$\circ$" on H is a map $\circ : H \times H \longrightarrow \wp^*(H)$ [37]. This means that a hyperoperation is different from a binary operation. In hyper algebraic structures the product of two elements is a set and in algebraic structure the product of two elements is an element. A non-empty set H with a hyperoperation is called a *hyperstructure* and is denoted by $(H, \circ)$, also $(H, \circ)$ is called a *hypergroupoid*. Let P and Q be non-empty subsets of a hypergroupoid H. Then, a hyperproduct of P and Q is denoted by $P \circ Q$ and defined as [37]:

$$P \circ Q = \bigcup_{p\in P, q\in Q} p \circ q,\ \ a \circ P = \{a\} \circ P \text{ and } P \circ a = P \circ \{a\}\,.$$

A hyperstructure $(H, \circ)$ is called a *semihypergroup* if assocaitive property holds [17] i.e,

$$(x \circ y) \circ z = x \circ (y \circ z) \text{ for all } x, y, z \in H.$$

Example 1 *Let $H = \{1, 2, 3, 4, 5\}$ be a semihypergroup defined by the following Cayley* *table;*

$\circ$	1	2	3	4	5
1	$\{1\}$	$\{1,2,4\}$	$\{1\}$	$\{1,2,4\}$	$\{1,2,4\}$
2	$\{1\}$	$\{2\}$	$\{1\}$	$\{1,2,4\}$	$\{1,2,4\}$
3	$\{1\}$	$\{1,2,4\}$	$\{1,3\}$	$\{1,2,4\}$	$\{1,2,3,4,5\}$
4	$\{1\}$	$\{1,2,4\}$	$\{1\}$	$\{1,2,4\}$	$\{1,2,4\}$
5	$\{1\}$	$\{1,2,4\}$	$\{1,3\}$	$\{1,2,4\}$	$\{1,2,3,4,5\}$

An element e in a semihypergroup H is called a left(right) hyperidentity if for all $x \in H$, $x \in e \circ x$ ($x \in x \circ e$). An element e in a semihypergroup is called an identity if e is a left hyperidentity and a right hyperidentity. An element e of a semihypergroup is called a scalar left (right) identity if $\{x\} = e \circ x$ ($\{x\} = x \circ e$) for all $x \in H$. A semihypergroup with a scalar identity e is called a *hypermonoid*. If a semihypergroup holds reproduction axiom, $x \circ H = H \circ x = H$ for all $x \in H$ is said to be a *hypergroup*. A *sub-semihypergroup* A is a non-empty subset of a semihypergroup such that $x \circ y \subseteq A$ for all $x, y \in A$ [24, 17, 25].

A non-empty subset I of a semihypergroup H is called a right(left) hyperideal of H if $x \in I \Rightarrow x \circ y \subseteq I$ ($x \in I \Rightarrow y \circ x \subseteq I$) for all $y \in H$. A hyperideal I is a non-empty subset of a semihypergroup H such that $x \in I \Rightarrow x \circ y \subseteq I$ and $x \in I \Rightarrow y \circ x \subseteq I$ for all $y \in H$.

1.2 Γ-semihypergroups

This section deals with the definitions of γ-hyperoperations, Γ-semihypergroups and basic properties of Γ-semihypergroups. We discuss the theory of Γ-hyperideals.

A γ-hyperoperation on H is a mapping from $H \times \Gamma \times H$ to $\wp^*(H)$ i.e for every $\gamma \in \Gamma$ and $x, y \in H$, $x\gamma y \subseteq H$.

Let H and Γ be two non-empty sets. We denote by the English alphabets, the elements of H and by the letters of the Greek alphabets, the elements of Γ. Then H is called a

Γ-*semihypergroup* if [45]

1. $x\gamma y \subseteq H$, for all $x, y \in H$ and $\gamma \in \Gamma$.

2. $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in H$ and for all $\alpha, \beta \in \Gamma$.

3. If $h_1, h_2, h_3, h_4 \in H, \gamma_1, \gamma_2 \in \Gamma$ such that $h_1 = h_3, \gamma_1 = \gamma_2$ and $h_2 = h_4$, then $h_1\gamma_1 h_2 = h_3\gamma_2 h_4$.

H is called a Γ-hypergroupoid if only the assertions (1) and (3) are satisfied in the above definition. An element e in a Γ-semihypergroup H is called a *left(right) hyperidentity* if for all $x \in H$ and $\gamma \in \Gamma$, $x \in e\gamma x$ $(x \in x\gamma e)$. An element e in a Γ-semihypergroup is called an *hyperidentity* if e is a left *hyperidentity* and a right *hyperidentity*. An element e of a Γ-semihypergroup is called a *scalar left (right) identity* if $\{x\} = e\gamma x$ $(\{x\} = x\gamma e)$ for all $x \in H$ and $\gamma \in \Gamma$. A Γ-semihypergroup with scalar identity e is called a Γ-*hypermonoid*. If a Γ-semihypergroup holds reproduction axiom, $x\gamma H = H\gamma x$ for all $x \in H$ and $\gamma \in \Gamma$ is said to be a Γ-*hypergroup. Also,* H is called a Γ-*hypergroup* if for each $\gamma \in \Gamma$, (H, γ) is a hypergroup. *A* Γ-semihypergroup is called *commutative* if $x\gamma y = y\gamma x$ for all $x, y \in H$ and $\gamma \in \Gamma$ [45].

Example 2 *Consider a semihypergroup $(H, \circ)$ and Γ a non-empty set. We define a γ-hyperoperation on H as: $x\gamma y = x \circ y$ for all $x, y \in H$ and $\gamma \in \Gamma$. Then, clearly H is a Γ-semihypergroup. Also, if we define a γ-hyperoperation on H as: $x\gamma y = x \circ \gamma \circ y$ if $\Gamma \subseteq H$, then H is a Γ-semihypergroup.*

From above example we can say that Γ-semihypergroup is a suitable extension and a generalization of a semihypergroup.

Example 3 *Let $H = [-1, 0]$ and $\Gamma = \{-1, -2, -3, -4, -5, ... - n\}$, where $n < \infty$, be two non-empty sets. For every $a, b \in [-1, 0]$ and $\gamma \in \Gamma$, we define $\gamma : H \times H \longrightarrow \wp^*(H)$ by $a\gamma b = \left[\frac{ab}{\gamma}, 0\right]$. Then, every $\gamma \in \Gamma$ is a γ-hyperoperation. For every $a, b, c \in H$ and $\beta, \gamma \in \Gamma$ we have*

$$(a\beta b)\,\gamma c = \left[\frac{abc}{\beta\gamma}, 0\right] = a\beta\,(b\gamma c)\,.$$

Thus, H is a Γ-semihypergroup.

Also, H is not a Γ-hypergroup because if $-0.1 \in [-1, 0]$, then $-0.1\gamma H \neq H$.

If P and Q are subsets of Γ-hypergroupoid H and γ any element of Γ, then we can define

$$P\gamma Q = \bigcup_{p \in P, q \in Q} p\gamma q,\ a\gamma P = \{a\}\gamma P,\ P\gamma a = P\gamma\{a\} \text{ and } P\Gamma Q = \bigcup_{\gamma \in \Gamma} P\gamma Q.$$

Let K be a non-empty subset of a Γ-semihypergroup H. Then, K is called a sub-Γ-semihypergroup of H if $a\gamma b \subseteq K$ for all $a, b \in K$ and $\gamma \in \Gamma$.

Let (H_1, Γ_1) be a Γ_1-semihypergroup and (H_2, Γ_2) a Γ_2-semihypergroup. A function $\Psi : H_1 \longrightarrow H_2$ is said to be a *homomorphism,* if we have a bijective function $g : \Gamma_1 \longrightarrow \Gamma_2$ such that for all $a, b \in H_1$ and $\gamma \in \Gamma_1$, $\Psi(a\gamma b) \subseteq \Psi(a)\, g(\gamma)\, \Psi(b)$.

A non-empty subset A of a Γ-semihypergroup H is called a right(left) Γ-hyperideal of H if $x \in I \Rightarrow x\gamma y \subseteq I$ ($x \in I \Rightarrow y\gamma x \subseteq I$) for all $y \in H$ and $\gamma \in \Gamma$. A Γ-hyperideal I is a non-empty subset of a Γ-semihypergroup H such that $x \in I \Rightarrow x\gamma y \subseteq I$ and $x \in I \Rightarrow y\gamma x \subseteq I$ for all $y \in H$ and $\gamma \in \Gamma$ [45].

Example 4 *Let $H = \{a, b, c, d\}$ and $\Gamma = \{\gamma, \delta\}$ be two non-empty sets and γ-hyperoperations defined by the following Cayley tables;*

γ	a	b	c	d
a	$\{a, c\}$	$\{b, d\}$	$\{a, c\}$	$\{d\}$
b	$\{b, d\}$	$\{b\}$	$\{b, d\}$	$\{d\}$
c	$\{a, c\}$	$\{b, d\}$	$\{a, c\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

δ	a	b	c	d
a	$\{a\}$	$\{b, d\}$	$\{c\}$	$\{d\}$
b	$\{b, d\}$	$\{b\}$	$\{b, d\}$	$\{d\}$
c	$\{c\}$	$\{b, d\}$	$\{a\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

Then, clearly, (H, Γ) is a Γ-semihypergroup. The subsets $I = \{d\}$ and $\{b, d\}$ of H are only proper Γ-hyperideals of H.

A non-empty subset B of a Γ-semihypergroup H is called a bi-Γ-hyperideal of H if

$B\Gamma B \subseteq B$ and $B\Gamma H\Gamma B \subseteq B$. A non-empty subset Q of a Γ-semihypergroup H is called a quasi-Γ-hyperideal if $Q\Gamma H \cap H\Gamma Q \subseteq Q$. Every Γ-hyperideal of H is a quasi-Γ-hyperideal of H. It means that a quasi-Γ-hyperideal of a Γ-semihypergroup H is a generalization of a Γ-hyperideal. Intersection of quasi-Γ-hyperideals of a Γ-semihypergroup H is a quasi-Γ-hyperideal [45].

A non-empty subset M of a Γ-semihypergroup H is called an M-hypersystem if for all $a, b \in M$, there exist $x \in H$ and $\alpha, \beta \in \Gamma$ such that $a\alpha x\beta b \subseteq M$ [2].

A non-empty subset N of Γ-semihypergroup H is called an N-hypersystem if for all $a \in N$, there exist $x \in H$ and $\alpha, \beta \in \Gamma$ such that $a\alpha x\beta a \subseteq N$ [2].

We define a relation ρ on $H \times \Gamma$ as fallows, where H is a Γ-semihypergroup.

$$(x, \alpha)\rho(y, \beta) \Leftrightarrow x\alpha s = y\beta s, \forall s \in H \text{ and } \alpha, \beta \in \Gamma.$$

Then clearly, ρ is an equivalence relation. The equivalence class containing (x, α) is denoted by $[x, \alpha]$. Let $S = \{[x, \alpha] : x \in H, \alpha \in \Gamma\}$ be the collection of all the equivalence classes. A hyperoperation "$\circ$" on S is defined as follows: $[x, \alpha] \circ [y, \beta] = \{[z, \beta] : z \in x\alpha y\}$, for all $x, y, z \in H$ and $\alpha, \beta \in \Gamma$. Since H is a Γ-semihypergroup, so

$$[x, \alpha] \circ ([y, \beta] \circ [z, \gamma]) = ([x, \alpha] \circ [y, \beta]) \circ [z, \gamma].$$

Thus, hyperoperation "$\circ$" is associative, so $(S, \circ)$ is a semihypergroup. This semihypergroup S is called a *left operator semihypergroup* of H [29].

For $A \subseteq S$ we define $A^+ = \{x \in H : [x, \alpha] \in S, \forall \alpha \in \Gamma\}$. Similarly, for $I \subseteq H$ we define $I^{+'} = \{[x, \alpha] \in S : x\alpha s \subseteq I, \forall s \in H\}$.

Theorem 5 *[29] Let S be a left operator semihypergroup of a Γ-semihypergroup H. Then, the following properties hold:*

1. *If A is a right hyperideal of S, then A^+ is a right Γ-hyperideal of H.*

2. *If I is a right Γ-hyperideal of H, then $I^{+'}$ is a right hyperideal of S.*

Let H be a Γ-semihypergroup and S the left operator semihypergroup of H. Let I be a Γ-hyperideal of H and A a hyperideal of S. Then, it is easy to see that $I \subseteq (I^{+})^{+'}$ and $A \subseteq (A^{+'})^{+}$ [29].

Theorem 6 *[29] Let H be a Γ-semihypergroup and S be the left operator semihypergroup of H. Let P be a prime right Γ-hyperideal of H. Then $P = (P^{+'})^{+}$.*

Let H be a Γ-semihypergroup. If there exist an element $e \in H$ and $\delta \in \Gamma$ such that $e\delta x = x$ for every $x \in H$, then H is said to have a left unity. If H has a left unity, then $[e, \delta]$ is a left unity of the left operator semihypergroup S.

Theorem 7 *[29] Let H be a Γ-semihypergroup and S its left operator semihypergroup. If I is a right Γ-hyperideal of H, then $I = (I^{+'})^{+}$..*

Lemma 8 *[29]Let H be a Γ-semihypergroup and S its left operator semihypergroup. If P is a prime hyperideal of S, then P^{+} is a prime Γ-hyperideal of H.*

Lemma 9 *[29]Let H be a Γ-semihypergroup. If Q is a prime Γ-hyperideal of H, then $Q^{+'}$ is a prime hyperideal of S.*

Theorem 10 *[29]Let H be a Γ-semihypergroup and S its left operator semihypergroup. Then there exists an inclusion preserving bijection $Q \to Q^{+'}$ between the set of all prime Γ-hyperideals of H and the set of all prime hyperideals of S.*

1.3 Fuzzy sets and Intuitionistic fuzzy sets

Zadeh introduced the notion of a fuzzy set [49] in 1965 and also a fuzzy set is called a fuzzy logic. A fuzzy logic is a generalization of mathematical logic. In 1986, Atanassov introduced the concept of an intuitionistic fuzzy set [14, 15].

A mapping from a non-empty set X to unit closed interval of 0 and 1 i.e, $\mu : X \longrightarrow [0, 1]$, is called a fuzzy set. The set of all fuzzy sets is denoted by $FP(X)$ i.e, $FP(X) =$

$\{\mu \mid \mu : X \longrightarrow [0,1]\}$. The complement of a fuzzy set μ is denoted by μ^c and defined by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$. Let μ and λ be two fuzzy sets in a non-empty set X. Then, $\mu \leq \lambda$ if and only if $\mu(x) \leq \lambda(x)$ for all $x \in X$. The fuzzy subsets $\mu \wedge \nu$ and $\mu \vee \nu$ of X are defined as follows:

$$\begin{aligned} (\mu \wedge \nu)(a) &= \mu(x) \wedge \nu(a) \\ (\mu \vee \nu)(a) &= \mu(x) \vee \nu(a) \end{aligned}$$

for all $a \in X$. The set $\mu_t = \{x \in X \mid \mu(x) \geq \alpha\}$, where $\alpha \in (0,1]$ is called an α-cut or the level set. The product of two fuzzy subsets μ and ν of a groupoid H is denoted by $\mu \circ \nu$ and defined as:

$$(\mu \circ \nu)(x) = \begin{cases} \bigvee\limits_{x=yz} \{\mu(y) \wedge \nu(z)\}, & \\ \quad \text{if there exist } y, z \in H, \text{ such that } x = yz. & \\ 0 & \text{otherwise.} \end{cases}$$

Let X be a nonempty fixed set. An *intuitionistic fuzzy set* (briefly, IFS) A is an object having the form

$$A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$$

where the functions $\mu_A : X \longrightarrow [0,1]$ and $\lambda_A : X \longrightarrow [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\lambda_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$ For the sake of simplicity, we use the symbol $A = \langle \mu_A, \lambda_A \rangle$ for an intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \lambda_A(x) \rangle \mid x \in X\}$ [14].

Let A and B be two intuitionistic fuzzy sets on X. The following expressions are defined in [14, 15].

1. $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in X$.

2. $A = B$ if and only if if $A \subseteq B$ and $B \subseteq A$.

3. $A^c = \{\langle x, \lambda_A(x), \mu_A(x)\rangle \, | x \in X\}$.

4. $A \cap B = \{\langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\}\rangle \, | x \in X\}$.

5. $A \cup B = \{\langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\}\rangle \, | x \in X\}$.

6. $\Box A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \, | x \in X\}$.

7. $\Diamond A = \{\langle x, 1 - \lambda_A(x), \lambda_A(x)\rangle \, | x \in X\}$.

Let A, B be two intuitionistic fuzzy sets in a Γ-hypergroupoid H. Then,
$A * B = \{\langle x, \mu_{A*B}(x), \lambda_{A*B}\rangle \, | x \in H\}$,
where

$$\mu_{A*B}(x) = \begin{cases} \sup\limits_{x \in y\gamma z} \{\min\{\mu_A(y), \mu_B(z)\}\} & \text{if } x \in y\gamma z, \text{ for some } \gamma \in \Gamma \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_{A*B}(x) = \begin{cases} \inf\limits_{x \in y\gamma z} \{\max\{\lambda_A(y), \lambda_B(z)\}\} & \text{if } x \in y\gamma z, \text{ for some } \gamma \in \Gamma \\ 1 & \text{otherwise} \end{cases}$$

Let c be a point in a non-empty set X. If $t \in (0, 1]$ and $s \in [0, 1)$ are two real numbers such that $0 \leq t + s \leq 1$, then the IFS

$$c(t, s) = \langle x, c_t, 1 - c_{1-s}\rangle$$

is called an *intuitionistic fuzzy point* (IFP for short) in X, where t (resp., s) is the degree of membership (resp., non-membership) of $c(t, s)$ and $c \in X$ is the support of $c(t, s)$. Let $c(t, s)$ be an IFP in X and let $A = \langle x, \mu_A, \lambda_A\rangle$ be an IFS in X. Then, $c(t, s)$ is said to *belong* to A, written $c(t, s) \in A$, if $\mu_A(c) \geq t$ and $\lambda_A(c) \leq s$ [18].

By an *interval number* D we mean an interval $[\alpha^-, \alpha^+]$, where $0 \leq \alpha^- \leq \alpha^+ \leq 1$. For *interval numbers* $D_1 = [\alpha^-, \alpha^+]$, $D_2 = \left[\beta^-, \beta^+\right] \in D\,[0, 1]$, where $D\,[0, 1]$ denotes the set of all closed subintervals of the interval $[0,1]$, we define

$$\begin{aligned}
D_1 \cap D_2 &= \min(D_1,\ D_2) = \min\left(\left[\alpha^-, \alpha^+\right],\ \left[\beta^-, \beta^+\right]\right) \\
&= \left[\min\left\{\alpha^-, \beta^-\right\},\ \min\left\{\alpha^+, \beta^+\right\}\right], \\
D_1 \cup D_2 &= \max(D_1,\ D_2) = \max\left(\left[\alpha^-, \alpha^+\right],\ \left[\beta^-, \beta^+\right]\right) \\
&= \left[\max\left\{\alpha^-, \beta^-\right\},\ \max\left\{\alpha^+, \beta^+\right\}\right], \\
D_1 &\le D_2 \Leftrightarrow \alpha^- \le \beta^- \text{ and } \alpha^+ \le \beta^+, \\
D_1 &< D_2 \Leftrightarrow \alpha^- < \beta^- \text{ and } \alpha^+ < \beta^+, \\
D_1 &= D_2 \Leftrightarrow \alpha^- = \beta^- \text{ and } \alpha^+ = \beta^+, \\
mD &= m\left[\alpha^-, \alpha^+\right] = \left[m\alpha^-, m\alpha^+\right], \text{ where } 0 \le m \le 1, \\
D_1 + D_2 &= \left[\alpha^- + \beta^- - \alpha^-.\beta^-,\ \alpha^+ + \beta^+ - \alpha^+.\beta^+\right].
\end{aligned}$$

Let X be a given non-empty set. An interval valued fuzzy set $\widetilde{\lambda}$ on X is defined by

$$B = \left\{\left(x,\ \left[\lambda_B^-(x), \lambda_B^+(x)\right]\right) : x \in X\right\},$$

where $\lambda_B^-(x)$ and $\lambda_B^+(x)$ are fuzzy sets of X such that $\lambda_B^-(x) \le \lambda_B^+(x)$ for all $x \in X$. Let $\widetilde{\lambda}_B(x) = \left[\lambda_B^-(x), \lambda_B^+(x)\right]$. Then,

$$B = \left\{\left(x,\ \widetilde{\lambda}_B(x)\right) : x \in X\right\},$$

where $\widetilde{\lambda}_B : X \longrightarrow D[0,1]$.

A mapping $A = \langle \widetilde{\mu}_A,\ \widetilde{\lambda}_A \rangle : X \longrightarrow D[0,1] \times D[0,1]$ is called an interval valued intuitionistic fuzzy set (IIF set, in short) in X if $0 \le \mu_A^+(x) + \lambda_A^+(x) \le 1$ and $0 \le \mu_A^-(x) + \lambda_A^-(x) \le 1$ for all $x \in X$, where the mapping $\widetilde{\mu}_A = \left[\mu_A^-(x), \mu_A^+(x)\right] : X \longrightarrow D[0,1]$ denotes the *degree of membership* (namely $\widetilde{\mu}_A(x)$) and $\widetilde{\lambda}_A = \left[\lambda_A^-(x), \lambda_A^+(x)\right] : X \longrightarrow D[0,1]$ denotes the *degree of non-membership* (namely $\widetilde{\lambda}_A(x)$) for each element $x \in X$ to A, respectively.

Throughout in this thesis, the following notions will be used:

(1) We use $\widetilde{0}$ to denote the interval valued fuzzy empty set and $\widetilde{1}$ to denote the interval valued fuzzy whole set in a set X, and define $\widetilde{0}(x) = [0, 0]$ and $\widetilde{1}(x) = [1, 1]$, for all $x \in X$.

(2) We write $\widetilde{t} = [t_1, t_2]$ and $\widetilde{s} = [s_1, s_2] \in D[0, 1]$.

(3) We write $m\{t_1, t_2\} = \min\{t_1, t_2\}$ and $M\{s_1, s_2\} = \max\{s_1, s_2\}$.

1.4 Intuitionistic Fuzzy Γ-hyperideals in Γ-semihypergroups

In [28], the authors defined the image and inverse image of an intuitionistic fuzzy subset in a Γ-semihypergroup and property on the image and inverse image of an intuitionistic fuzzy Γ-hyperideal was obtained.

Let Φ be a mapping from a set X to a set Y. Let $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy subset of X and $B = \langle \mu_B, \lambda_B \rangle$ be an intuitionistic fuzzy subset of Y. Then the inverse image $\Phi^{-1}(B)$ of B, is the intuitionistic fuzzy set of X defined by $\Phi^{-1}(B) = (\Phi^{-1}(\mu_B), \Phi^{-1}(\lambda_B))$ where $\Phi^{-1}(\mu_B)(x) = \mu_B(\Phi(x))$ and $\Phi^{-1}(\lambda_B) = \lambda_B(\Phi(x))$ [28].

The image $\Phi(B)$ of the intuitionistic fuzzy subset $B = \langle \mu_B, \lambda_B \rangle$ is the intuitionistic fuzzy set in Y defined by $\Phi(B) = (\Phi(\mu_B), \Phi(\lambda_B))$, where for every $y \in Y$

$$\Phi(\mu_B)(y) = \begin{cases} \sup\{\mu_B(t) | t \in \Phi^{-1}(y)\} & \text{if } \Phi^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$\Phi^*(\lambda_B)(y) = \begin{cases} \inf\{\lambda_B(t) | t \in \Phi^{-1}(y)\} & \text{if } \Phi^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Chapter 2

Prime Bi-Γ-hyperideals

2.1 Introduction

In this chapter, we study some properties of prime bi-Γ-hyperideals in a Γ-semihypergroup. We introduce the concept of the strongly prime, prime, semiprime, strongly irreducible and irreducible bi-Γ-hyperideals of Γ-semihypergroups. The space of strongly prime bi-Γ-hyperideals is topologized. Also, we characterize those Γ-semihypergroups for which each bi-Γ-hyperideal is strongly prime. We study some different relations among these concepts and characterize Γ-semihypergroups by the properties of prime bi-Γ-hyperideals.

2.2 Prime Bi-Γ-hyperideals

A bi-Γ-hyperideal B of a Γ-semihypergroup H is called a *prime (strongly prime) bi-Γ-hyperideal* if $B_1\Gamma B_2 \subseteq B(B_1\Gamma B_2 \cap B_2\Gamma B1 \subseteq B)$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-Γ-hyperideals B_1 and B_2 of H. A bi-Γ-hyperideal B of a Γ-semihypergroup H is called a *semiprime bi-Γ-hyperideal* if $B_1\Gamma B_1 \subseteq B$ implies $B_1 \subseteq B$ for any bi-Γ- hyperideal B_1 of H.

Every strongly prime bi-Γ-hyperideal of a Γ-semihypergroup H is a prime bi-Γ-hyperideal and every prime bi-Γ-hyperideal of a Γ-semihypergroup H is a semiprime bi-Γ-hyperideal. A prime bi-Γ-hyperideal is not necessarily strongly prime and a semi-prime bi-Γ-hyperideal is not necessarily prime.

A bi-Γ-hyperideal B of a Γ-semihypergroup H is called an *irreducible (strongly irreducible) bi-Γ-hyperideal* if $B_1 \cap B_2 = B(B_1 \cap B_2 \subseteq B)$ implies $B_1 = B$ or $B_2 = B$ $(B1 \subseteq B$ or $B_2 \subseteq B)$.

Every strongly irreducible bi-Γ-hyperideal of a Γ- semihypergroup is an irreducible bi-Γ-hyperideal but the converse is not true in general.

Example 11 *Let $H = \{1, 2, 3, 4\}$ and $\Gamma = \{\gamma, \delta\}$be a Γ-semihypergroup with the following Cayley's tables.*

γ	1	2	3	4
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1\}$	$\{1,4\}$	$\{1\}$
3	$\{1\}$	$\{1\}$	$\{3\}$	$\{1\}$
4	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

δ	1	2	3	4
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1\}$	$\{2,4\}$	$\{1\}$
3	$\{1\}$	$\{1\}$	$\{3\}$	$\{1\}$
4	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

Bi-Γ-hyperideals of H are $\{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,2,4\}$,$\{1,3,4\}$ and H. Bi-Γ-hyperideals $\{1,2,4\}$ and H of H are prime bi-Γ-hyperideals of H. The bi-Γ-hyperideal $\{1,2,4\}$ and H are also strongly prime bi-Γ-hyperideals of H. Bi-Γ-hyperideals $\{1,2\}, \{1,3\}, \{1,2,4\}$, $\{1,3,4\}$ and H are irreducible and $\{1,2,4\}$, $\{1,3,4\}$ and H are strongly irreducible:

Example 12 *Let $H = \{1, 2, 3, 4, 5\}$ and $\Gamma = \{\gamma, \delta\}$ be two non-empty sets. Then, H is a Γ-semihypergroup, where Γ-hyperoperations are defined by the following Cayley tables.*

γ	1	2	3	4	5
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
3	$\{1\}$	$\{1\}$	$\{3\}$	$\{3\}$	$\{3\}$
4	$\{1\}$	$\{1\}$	$\{3,4\}$	$\{3,4\}$	$\{5\}$
5	$\{1\}$	$\{1\}$	$\{3,4\}$	$\{3,4\}$	$\{5\}$

δ	1	2	3	4	5
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
3	$\{1\}$	$\{1\}$	$\{3\}$	$\{3\}$	$\{3\}$
4	$\{1\}$	$\{1\}$	$\{3\}$	$\{3,4\}$	$\{5\}$
5	$\{1\}$	$\{1\}$	$\{3\}$	$\{3,4\}$	$\{5\}$

Example 13 *Let $H = \{1, 2, 3\}$ and $\Gamma = \{\gamma, \delta\}$ be two non-empty sets. Then, clearly H is a Γ-semihypergroup by the following Cayley's tables of Γ-hyperoperations:*

γ	1	2	3
1	$\{1\}$	$\{1,2\}$	$\{1,3\}$
2	$\{1,2\}$	$\{2\}$	$\{2,3\}$
3	$\{1,3\}$	$\{2,3\}$	$\{2,3\}$

δ	1	2	3
1	$\{1\}$	$\{1,2\}$	$\{1,3\}$
2	$\{1,2\}$	$\{2\}$	$\{2,3\}$
3	$\{1,3\}$	$\{2,3\}$	$\{3\}$

The prime-bi-Γ-hyperideal of H is $\{1\}$.

Lemma 14 *The intersection of any family of prime bi-Γ-hyperideals of a Γ-semihypergroup is a semiprime bi-Γ-hyperideal.*

Proof. Straightforward. ■

Theorem 15 *Every strongly irreducible, semiprime bi-Γ-hyperideal of a Γ-semihypergroup H is a strongly prime bi-Γ-hyperideal.*

Proof. Let B be a strongly irreducible semiprime bi-Γ-hyperideal of H. Let B_1, B_2 be any bi-Γ-hyperideals of H such that $B_1\Gamma B_2 \cap B_2\Gamma B1 \subseteq B$. Since $(B_1 \cap B_2)^2 \subseteq B_1\Gamma B_2$ and $(B_1 \cap B_2)^2 \subseteq B_2\Gamma B_1$, $(B_1 \cap B_2)^2 \subseteq B_1\Gamma B_2 \cap B_2\Gamma B1 \subseteq B$. Since B is a semiprime bi-Γ-hyperideal, $B_1 \cap B_2 \subseteq B$. Because B is a strongly irreducible bi-Γ-hyperideal of H, so either $B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is a strongly prime bi-Γ-hyperideal of H. ■

Theorem 16 *Let B be a bi-Γ-hyperideal of a Γ-semihypergroup H and $a \in H$ be such that $a \notin B$. Then there exists an irreducible bi-Γ-hyperideal I of H such that $B \subseteq I$ and $a \notin I$.*

Proof. Let $\mathcal{A}$ be the collection of all bi-Γ-hyperideals of H which contain B and do not contain a. Then, it is non-empty, because $B \in \mathcal{A}$. The collection $\mathcal{A}$ is a partially ordered set under inclusion. If $\mathcal{C}$ is any totally ordered subset of A, then $\cup\mathcal{C}$ is a bi-Γ-hyperideal of H containing B. Hence by Zorn's Lemma, there exists a maximal element

I in $\mathcal{A}$. We show that I is an irreducible bi-Γ-hyperideal. Let C and D be two bi-Γ-hyperideals of H such that $I = C \cap D$. If both C and D properly contain I, then $a \in C$ and $a \in D$. Hence $a \in C \cap D = I$. This contradicts the fact that $a \notin I$. Thus $I = C$ or $I = D$. ■

2.3 Characterizations of Γ-semihypergroups by Prime Bi-Γ-hyperideals

All Γ-semihypergroups considered in this section are Γ-semihypergroups with zero.

An element 0 of a Γ-semihypergroup H, with at least two elements, is called a zero element of H if

$$0\gamma x = x\gamma 0 = \{0\} \text{ for all } x \in H \text{ and } \gamma \in \Gamma.$$

A Γ-semihypergroup is called a Γ-semihypergroup with zero if it contains a zero element.

An element a of a Γ-semihypergroup H is called *regular* if there exists $x \in H$ and $\alpha, \beta \in \Gamma$ such that $a \in a\alpha x\beta a$. If every element of a Γ-semihypergroup H is *regular*, then H is called a *regular* Γ*-semihypergroup*. The following properties are equivalent

(1) For every $A \subseteq H$, $A \subseteq A\Gamma H\Gamma A$.

(2) For every element $a \in H$, $a \in a\Gamma H\Gamma a$

A Γ-semihypergroup H is called *intra-regular* if for every $a \in H$ there exist $x, y \in H$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a \in x\alpha a\gamma a\beta y$. For a Γ-semihypergroup H, we denote by $R(a)$ (resp. $L(a)$, $Q(a)$, $B(a)$) the right (resp. left, quasi-,bi-) Γ-hyperideal of H generated by $a(a \in H)$ and define as following;

$$\begin{aligned} R(a) &= \{a\} \cup a\Gamma H,\ L(a) = \{a\} \cup H\Gamma a, \\ Q(a) &= \{a\} \cup ((a\Gamma H) \cap (H\Gamma a)),\ B(a) = \{a\} \cup a\Gamma H\Gamma a). \end{aligned}$$

The Γ-hyperideal generated by a is the set $\{a\} \cup H\Gamma a \cup a\Gamma H \cup H\Gamma a\Gamma H)$.

A Γ-semihypergroup H is called a *left(resp., right)strongly regular* if for every $a \in H$

and for all $\gamma \in \Gamma$ there exists $x \in H$ such that

$$a \in x\gamma a\gamma a(resp,\ a \in a\gamma a\gamma x).$$

We define a relation "$\mathcal{L}$" on H as follows:

$$a\mathcal{L}b \iff L(a) = L(b).$$

Lemma 17 *Let H be a Γ-semihypergroup. Then $\mathcal{L}$ is an equivalence relation.*

Proof. The proof is straightforward. ■

Theorem 18 *Let H be a Γ-semihypergroup. Then the following are equivalent:*

1) *H is left regular.*

2) *$a\mathcal{L}(a\gamma a)$ for all $a \in H$* and *for all $\gamma \in \Gamma$.*

Proof. 1) $\implies$ 2): Let H be left strongly regular and $a \in H$. If $t \in L(a)$, then

$$t = a \text{ or } t \in x\gamma a \text{ for some } x \in H \text{ and } \gamma \in \Gamma.$$

Since H is left strongly regular, $a \in y\gamma a\gamma a$ for some $y \in H$ and for all $\gamma \in \Gamma$. If $t = a$, then $t = a \in y\gamma a\gamma a$ for some $y \in H$ and for all $\gamma \in \Gamma$. If $t \in x\gamma a$, then $t \in x\gamma a \subseteq x\gamma\,(y\gamma a\gamma a) = (x\gamma y)\,\gamma\,(a\gamma a)$. This implies that

$$t \in (x\gamma y)\,\gamma\,(a\gamma a) \subseteq H\Gamma(a\gamma a) \subseteq ((a\gamma a) \cup H\Gamma(a\gamma a) = L(a\gamma a)$$

Hence, $t \in L(a\gamma a)$ for all $\gamma \in \Gamma$. Hence $L(a) \subseteq L(a\gamma a)$.

Let $a \in H$ and $t \in L(a\gamma a)$. Then $t \in a\gamma a$ or $t \in x\gamma(a\gamma a)$ for some $x \in H$. In both cases, $t \in H\Gamma a \subseteq \{a\} \cup H\Gamma a = L(a)$. Thus $L(a\gamma a) \subseteq L(a)$, so $L(a) = L(a\gamma a)$. Therefore, $a\mathcal{L}(a\gamma a)$ for all $a \in H$ and $\gamma \in \Gamma$.

2) $\implies$ 1): Assume that $a\mathcal{L}(a\gamma a)$ for all $a \in H$ and for all $\gamma \in \Gamma$. Then $a \in L(a) = L(a\gamma a) = \{a\gamma a\} \cup H\Gamma(a\gamma a)$. Thus,

$$a \in a\gamma a \text{ or } a \in x\Gamma(a\gamma a) \text{ for some } x \in H.$$

From both cases, we have $a \in x\gamma(a\gamma a)$ for some $x \in H$ and for all $\gamma \in \Gamma$. Thus, a is a left strongly regular, and so H is a left strongly regular. ∎

Example 19 *Let $H = \{a, b, c, d\}$ and $\Gamma = \{\gamma, \delta\}$ be two non-empty sets. Then H is a Γ-semihypergroup, where and hyperoperations are defined by the following Cayley tables;*

γ	a	b	c	d
a	$\{a,c\}$	$\{b,d\}$	$\{a,c\}$	$\{d\}$
b	$\{b,d\}$	$\{b\}$	$\{b,d\}$	$\{d\}$
c	$\{a,c\}$	$\{b,d\}$	$\{a,c\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

δ	a	b	c	d
a	$\{a\}$	$\{b,d\}$	$\{c\}$	$\{d\}$
b	$\{b,d\}$	$\{b\}$	$\{b,d\}$	$\{d\}$
c	$\{c\}$	$\{b,d\}$	$\{a\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

Thus, H is a left strongly regular, i.e,

$$\begin{aligned} a &\in c\gamma a\gamma a, a \in a\delta a\delta a, a \in a\gamma a\delta a, a \in a\delta a\gamma a \\ b &\in c\gamma b\gamma b, b \in b\delta b\delta b, b \in b\gamma b\delta b, b \in b\delta b\gamma b \\ c &\in a\gamma c\gamma c, c \in a\delta c\delta c, c \in c\gamma c\delta c, c \in a\delta c\gamma c \\ d &\in b\gamma d\gamma d, d \in b\delta d\delta d, d \in d\gamma d\delta d, d \in d\delta d\gamma d \end{aligned}$$

Now

$$\begin{aligned} L(a) &= H = L(a\gamma a) = L(a\delta a),\ L(b) = \{b,d\} = L(b\gamma b) = L(b\delta b), \\ L(c) &= H = L(c\gamma c) = L(c\delta c),\ , L(d) = \{d\} = L(d\gamma d) = L(d\delta d). \end{aligned}$$

$a\mathcal{L}(a\gamma a)$ for all $a \in H$ and for all $\gamma \in \Gamma$.

Theorem 20 *Let H be a Γ-semihypergroup. Then H is left strongly regular if and only if $L(a\gamma a)$ is a semiprime.*

Proof. Let H be a left strongly regular Γ-semihypergroup. Then by Theorem 18, $L(a) = L(a\gamma a)$ for all $a \in H$ and $\gamma \in \Gamma$. Let for all $a \in H$ and $\gamma \in \Gamma$ such tha $a\gamma a \subseteq L(a\gamma a)$ and also $a \in L(a)$ this implies $a \in L(a\gamma a)$.

Conversely, suppose that $L(a\gamma a)$ is semiprime. Let $a \in H$. Since $L(a\gamma a)$ is a semiprime. Then $a\gamma a \subseteq L(a\gamma a)$ for all $\gamma \in \Gamma$ implies $a \in L(a\gamma a)$, where $a \in L(a)$. Thus, $L(a) \subseteq L(a\gamma a)$. Now let $t \in L(a\gamma a)$. Then, $t \in a\gamma a$ or $t \in x\alpha a\gamma a$ for some $x \in H$. Thus, $t \in H\Gamma a$ implies $t \in \{a\} \cup H\Gamma a$. Hence $L(a\gamma a) \subseteq L(a)$ this implies $L(a) = L(a\gamma a)$. By Theorem 18. H is a left strongly regular. ∎

Theorem 21 *Let H be a Γ-semihypergroup. $\{H_\varepsilon : \varepsilon \in \Omega\}$ be a family of left simple sub-Γ-semihypergroups of H such that $H = \cup\{H_\varepsilon : \varepsilon \in \Omega\}$. Then, H is left strongly regular.*

Proof. Suppose every H_ε is a left simple sub-Γ-semihypergroup of H and let $H = \cup\{H_\varepsilon : \varepsilon \in \Omega\}$. If L is a left Γ-hyperideal of H and $a\Gamma a \subseteq L$ for $a \in H$, then $a \in H_\varepsilon$ for some $\varepsilon \in \Omega$. Consider a subset $L \cap H_\varepsilon$ of H. Since H_ε is a sub-Γ-semihypergroup of H, so $a\Gamma a \subseteq H_\varepsilon$. Thus $L \cap H_\varepsilon \neq \emptyset$. Furthermore,

$$H_\varepsilon\Gamma(L \cap H_\varepsilon) \subseteq H_\varepsilon\Gamma L \cap H_\varepsilon\Gamma H_\varepsilon \subseteq H\Gamma L \cap H_\varepsilon \subseteq L \cap H_\varepsilon.$$

Hence, $L \cap H_\varepsilon$ is a left Γ-hyperideal of H_ε. Since H_ε is left simple, we have $L \cap H_\varepsilon = H_\varepsilon$. Thus $a \in L$. Hence, L is semiprime. By Theorem 18, H is a left strongly regular. ∎

Lemma 22 *A Γ-semihypergroup H is completely regular if and only if $A \subseteq (A\Gamma A)\,\Gamma H\Gamma\,(A\Gamma A)$ for every $A \subseteq H$. Equivalently, a Γ-semihypergroup H is completely regular if and only if $a \in a\Gamma a\Gamma H\Gamma a\Gamma a$ for all $a \in H$.*

Theorem 23 *A Γ-semihypergroup H is a completely regular if and only if every bi-Γ-hyperideal of H is semiprime.*

Proof. Suppose H is completely regular Γ-semihypergroup. Let B be a bi-Γ-hyperideal, $a \in H$ and $a\Gamma a \subseteq B$. Then for some $x, y, z \in H$ and $\alpha, \beta, \gamma, \rho, \tau, \eta, \mu \in \Gamma$,

$$\begin{aligned} a &\in a\alpha x\beta a = (a\gamma a\rho y)\alpha x\beta(z\tau a\eta a) = a\gamma(a\rho y\alpha x\beta z\tau a)\eta a \subseteq B\Gamma H\Gamma B \subseteq B \\ &\implies a \in B. \end{aligned}$$

Thus, B is semiprime.

Conversely, let $a \in H$. Then $a\Gamma a\Gamma H\Gamma a\Gamma a$ is a non-empty subset of H. Let $x, y \in (a\Gamma a)\,\Gamma H\Gamma\,(a\Gamma a)$ and $z \in H$. Then, for some $s, t \in H$ and $\alpha, \beta, \gamma, \rho, \tau, \eta, \mu \in \Gamma$,

$$\begin{aligned} x\alpha z\beta y &\subseteq (a\gamma a\rho u\tau a\gamma a)\alpha z\beta(a\gamma a\eta v\mu a\gamma a) \\ &= a\gamma a\rho(u\tau a\gamma a\alpha z\beta a\gamma a\eta v)\mu a\gamma a \subseteq (a\Gamma a)\,\Gamma H\Gamma\,(a\Gamma a) \end{aligned}$$

Thus, $x\alpha z\beta y \subseteq a\Gamma a\Gamma H\Gamma a\Gamma a$. Then,

$$\begin{aligned} ((a\Gamma a)\,\Gamma H\Gamma\,(a\Gamma a))\,\Gamma\,((a\Gamma a)\,\Gamma H\Gamma\,(a\Gamma a)) &\subseteq ((a\Gamma a)\,\Gamma H\Gamma\,(a\Gamma a)) \\ \text{and } ((a\Gamma a)\,\Gamma H\Gamma\,(a\Gamma a))\Gamma H\Gamma((a\Gamma a)\,\Gamma H\Gamma\,(a\Gamma a)) &\subseteq (a\Gamma a)\,\Gamma H\Gamma\,(a\Gamma a) \end{aligned}$$

Hence, $(a\Gamma a)\,\Gamma H\Gamma\,(a\Gamma a)$ is a bi-Γ-hyperideal of H for all $a \in H$. Since

$$a\Gamma a\Gamma a\Gamma a\Gamma a\Gamma a\Gamma a\Gamma a = (a\Gamma a)\Gamma\,(a\Gamma a\Gamma a\Gamma a)\,\Gamma\,(a\Gamma a) \subseteq ((a\Gamma a)\,\Gamma H\Gamma\,(a\Gamma a))$$

and $(a\Gamma a)\,\Gamma H\Gamma\,(a\Gamma a)$ is semiprime, we get $a\Gamma a\Gamma a\Gamma a$, $a\Gamma a \subseteq a\Gamma a\Gamma H\Gamma a\Gamma a$ and so $a \in (a\Gamma a)\,\Gamma H\Gamma\,(a\Gamma a)$. Hence, by Lemma 22, H is a completely regular Γ-semihypergroup. ■

Theorem 24 *For a Γ-semihypergroup H, the following assertions are equivalent:*

(i) H is both regular and intra-regular.

(ii) $B\Gamma B = B$ for every bi-Γ-hyperideal B of H.

(iii) $B_1 \cap B_2 = B_1\Gamma B_2 \cap B_2\Gamma B_1$ *for all bi-*Γ*-hyperideals* B_1 *and* B_2 *of* H.

(iv) Each bi-Γ*-hyperideal of* H *is semiprime.*

(v) Each proper bi-Γ*-hyperideal of* H *is the intersection of irreducible semiprime bi-*Γ*-hyperideals of* H *which contain it.*

Proof. $(i) \iff (ii)$

The proof is easy, we omit.

$(ii) \Longrightarrow (iii)$

Let B_1 and B_2 be any two bi-Γ-hyperideals of H. Then, by our hypothesis,

$$\begin{aligned} B_1 \cap B_2 &= (B_1 \cap B_2)\,\Gamma\,(B_1 \cap B_2) \\ B_1 \cap B_2 &\subseteq B_1\Gamma B_2 \\ \text{Similarly,} \qquad B_1 \cap B_2 &\subseteq B_2\Gamma B_1 \end{aligned}$$

Thus,

$$B_1 \cap B_2 \subseteq B_1\Gamma B_2 \cap B_2\Gamma B_1 \tag{2.1}$$

Now, $B_1\Gamma B_2$ and $B_2\Gamma B_1$ are bi-Γ-hyperideals being the products of bi-Γ-hyperideals. Also $B_1\Gamma B_2 \cap B_2\Gamma B_1$ is a bi-Γ-hyperideal. Then,

$$\begin{aligned} B_1\Gamma B_2 \cap B_2\Gamma B_1 &= (B_1\Gamma B_2 \cap B_2\Gamma B_1)\,\Gamma\,(B_1\Gamma B_2 \cap B_2\Gamma B_1) \\ &\subseteq (B_1\Gamma B_2)\,\Gamma\,(B_2\Gamma B_1) \\ &\subseteq B_1\Gamma S\Gamma B_1 \subseteq B_1 \\ \text{Similarly,} \qquad B_1\Gamma B_2 \cap B_2\Gamma B_1 &\subseteq B_2 \end{aligned}$$

Thus,

$$B_1\Gamma B_2 \cap B_2\Gamma B_1 \subseteq B_1 \cap B_2 \tag{2.2}$$

Hence from (2.1) and (2.2)

$$B_1 \cap B_2 = B_1\Gamma B_2 \cap B_2\Gamma B_1$$

$(iii) \Longrightarrow (iv)$

Let B_1 and B be bi-Γ-hyperideals of H such that $B_1\Gamma B_1 \subseteq B$. By hypothesis,

$$B_1 = B_1 \cap B_1 = B_1\Gamma B_1 \cap B_1\Gamma B_1 = B_1\Gamma B_1$$

Thus,

$$B_1 \subseteq B$$

Hence, every bi-Γ-hyperideal of H is semiprime.

$(iv) \Longrightarrow (v)$

Let B be a proper bi-Γ-hyperideal of H. Then, B is contained in the intersection of all irreducible bi-Γ- hyperideals of H which contain B. Theorem 16 guarantees the existence of such irreducible bi-Γ-hyperideals. If $a \notin B$, then there exists an irreducible bi-Γ-hyperideal of H which contains B but does not contain a. Hence B is the intersection of all bi-Γ-hyperideals of H which contain it. By our hypothesis, every bi-Γ-hyperideal is semiprime, and so each bi-Γ-hyperideal is the intersection of irreducible semiprime bi-Γ-hyperideals of H containing it.

$(v) \Longrightarrow (ii)$

Let B be a bi-Γ-hyperideal of H. If $B\Gamma B = H$, then clearly B is idempotent, that is, $B\Gamma B = B$. If $B\Gamma B \neq H$, then $B\Gamma B$ is a proper bi-Γ-hyperideal of H containing $B\Gamma B$ and so by our hypothesis

$$B\Gamma B = \bigcap_{\alpha} \{B_\alpha : B_\alpha \text{ is irreducible semiprime bi-}\Gamma\text{-hyperideal of } H\}$$

Since each B_α is a semiprime bi-Γ-hyperideal, $B \subseteq B_\alpha$ for all α and so $B \subseteq \bigcap_{\alpha} B_\alpha = B\Gamma B$. Hence each bi-$\Gamma$-hyperideal in H is idempotent. ■

Theorem 25 *Let H be a regular and an intra-regular Γ-semihypergroup. Then, the following assertions, for a bi-Γ-hyperideal B of H, are equivalent:*

(i) *B is strongly irreducible.*

(ii) *B is strongly prime.*

Proof. Straightforward. ■

Next, we characterize those Γ-semihypergroups in which each bi-Γ-hyperideal is a strongly prime and also those Γ-semihypergroups in which each bi-Γ-hyperideal is a strongly irreducible.

Theorem 26 *Each bi-Γ-hyperideal of a Γ-semihypergroups H is a strongly prime if and only if H is a regular, intra-regular and the set of bi-Γ-hyperideals of H is totally ordered by inclusion.*

Proof. Suppose that each bi-Γ-hyperideal of H is strongly prime. Then each bi-Γ-hyperideal of H is semiprime. Thus, by Theorem 22, H is both regular and intra-regular. We show that the set of bi-Γ-hyperideals of H is totally ordered. Let B_1 and B_2 be any two bi-Γ-hyperideals of H. Then, by Theorem 22, $B_1 \cap B_2 = B_1 \Gamma B_2 \cap B_2 \Gamma B_1$. As each bi-Γ-hyperideal is strongly prime, $B_1 \cap B_2$ is strongly prime. Hence, either $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$. If $B_1 \subseteq B_1 \cap B_2$, then $B_1 \subseteq B_2$. If $B_2 \subseteq B_1 \cap B_2$, then $B_2 \subseteq B_1$.

Conversely, assume that H is regular, intra-regular and since the set of bi-Γ-hyperideals of H is totally ordered under inclusion. Then, we want to show that each bi-Γ-hyperideal of H is strongly prime. Let B be an arbitrary bi-Γ-hyperideal of H and B_1, B_2 be bi-Γ-hyperideals of H such that

$$B_1 \Gamma B_2 \cap B_2 \Gamma B_1 \subseteq B$$

Since H is both regular and intra-regular, by Theorem 22,

$$B_1 \cap B_2 = B_1 \Gamma B_2 \cap B_2 \Gamma B_1$$

Also $B_1 \Gamma B_2 \cap B_2 \Gamma B_1 \subseteq B$ implies $B_1 \cap B_2 \subseteq B$. Since the set of all bi-Γ-hyperideals of H is totally ordered under inclusion, so either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$, that is, either $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Thus either $B_1 \subseteq B$ or $B_2 \subseteq B$. ■

Theorem 27 *If the set of all bi-Γ-hyperideals of a Γ-semihypergroup H is totally ordered, then H is both regular and intra-regular if and only if each bi-Γ-hyperideal of H is prime.*

Proof. Suppose that H is both regular and intra-regular. Let B be any bi-Γ-hyperideal of H and B_1, B_2 be bi-Γ-hyperideals of H such that $B_1 \Gamma B_2 \subseteq B$. Since the set of all bi-Γ-hyperideals of H is totally ordered, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Suppose $B_1 \subseteq B_2$. Then $B_1 \Gamma B_1 \subseteq B_1 \Gamma B_2 \subseteq B$. By Theorem 22, B is semiprime so $B_1 \subseteq B$. Hence, B is a semiprime bi-Γ- hyperideal of H.

Conversely, assume that every bi-Γ-hyperideal of H is prime. Since the set of all bi-Γ-hyperideals of H is totally ordered so the concepts of prime and strongly prime coincide. Now, by Theorem 26, we see that H is both regular and intra-regular. ■

Theorem 28 *For a Γ-semihypergroup H the following assertions are equivalent:*

(i) *The set of all bi-Γ-hyperideals of H is totally ordered under inclusion.*

(ii) *Each bi-Γ-hyperideal of H is strongly irreducible.*

(iii) *Each bi-Γ-hyperideal of H is irreducible.*

Proof. $(i)) \Longrightarrow (ii)$ Let B be an arbitrary bi-Γ-hyperideal of H and B_1, B_2 be two bi-Γ-hyperideals of H such that $B_1 \cap B_2 \subseteq B$. Since the set of all bi-Γ-hyperideals of H is totally ordered, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Thus either $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Hence $B_1 \cap B_2 \subseteq B$ implies either $B_1 \subseteq B$ or $B_2 \subseteq B$. This shows that B is strongly irreducible bi-Γ- hyperideal.

$(ii) \Longrightarrow (iii)$ Let B be an arbitrary bi-Γ-hyperideal of H and B_1, B_2 two bi-Γ-hyperideals of H such that $B_1 \cap B_2 = B$. Then $B \subseteq B_1$ and $B \subseteq B_2$. By hypothesis, either $B_1 \subseteq B$ or $B_2 \subseteq B$. Hence either $B_1 = B$ or $B_2 = B$. That is, B is irreducible bi-Γ-hyperideal.

$(iii) \Longrightarrow (i)$ Let B_1 and B_2 be any two bi-Γ- hyperideals of H. Then, $B_1 \cap B_2$ is a bi-Γ-hyperideal of H. Also, $B_1 \cap B_2 = B_1 \cap B_2$. So by hypothesis, either $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$, that is, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. ■

Let H be a Γ-semihypergroup, $\mathcal{B}$ be the set of all bi-Γ-hyperideals of H and $\mathcal{P}$ be the set of all strongly prime proper bi-Γ-hyperideals of H: Define for each $B \in \mathcal{B}$

$$\begin{aligned}
\Theta_B &= \{J \in \mathcal{P} : B \nsubseteq J\} \\
\Im(\mathcal{P}) &= \{\Theta_B : B \text{ is a bi-}\Gamma\text{-hyperideal of } H\}
\end{aligned}$$

Theorem 29 *If H is a regular and an intra-regular Γ-semihypergroup. Then $\Im(\mathcal{P})$ forms a topology on the set $\mathcal{P}$.*

Proof. Since $\{0\}$ is a bi-Γ-hyperideal of H,

$$\begin{aligned}
\Theta_0 &= \{J \in \mathcal{P} : \{0\} \nsubseteq J\} \\
&= \emptyset \text{ because 0 belongs to every bi-}\Gamma\text{-hyperideal of } H
\end{aligned}$$

Also, since H is a bi-Γ-hyperideal of H, $\Theta_H = \{J \in \mathcal{P} : H \nsubseteq J\} = \mathcal{P}$ because $\mathcal{P}$ is the collection of all strongly prime proper bi-Γ-hyperideal of H. Let $\{\Theta_{B_i} : i \in I\} \subseteq \Im(\mathcal{P})$. Then,

$$\begin{aligned}
\bigcup_{i \in I} \Theta_{B_i} &= \{J \in \mathcal{P} : B_i \nsubseteq J \text{ for some } i \in I\} \\
&= \{J \in \mathcal{P} : \overset{\Lambda}{\bigcup_{i \in I}} B_i \nsubseteq J\} = \Theta_{\overset{\Lambda}{\bigcup}_{i \in I} B_i},
\end{aligned}$$

where $\overset{\Lambda}{\bigcup_{i \in I}} B_i$ is the bi-Γ-hyperideal of H generated by $\bigcup_{i \in I} B_i$.

Now let Θ_{B_1} and $\Theta_{B_2} \in \Im(\mathcal{P})$. If $J \in \Theta_{B_1} \cap \Theta_{B_2}$, then $J \in \mathcal{P}$ and $B_1 \nsubseteq J$,

$B_2 \nsubseteq J$. Suppose $B_1 \cap B_2 \subseteq J$. Since H is both regular and intra-regular, $B_1 \cap B_2 = B_1 \Gamma B_2 \cap B_2 \Gamma B_1$. Hence, $B_1 \Gamma B_2 \cap B_2 \Gamma B_1 \subseteq J$. This implies either $B_1 \subseteq J$ or $B_2 \subseteq J$, a contradiction. Consequently, $B_1 \cap B_2 \nsubseteq J$, which implies that $J \in \Theta_{B_1 \cap B_2}$. Thus $\Theta_{B_1} \cap \Theta_{B_2} \subseteq \Theta_{B_1 \cap B_2}$. If $J \in \Theta_{B_1 \cap B_2}$, then we have $J \in \mathcal{P}$ and $B_1 \cap B_2 \nsubseteq J$. This implies that $B_1 \nsubseteq J, B_2 \nsubseteq J$. Thus $J \in \Theta_{B_1}$ and $J \in \Theta_{B_2}$, and therefore $J \in \Theta_{B_1} \cap \Theta_{B_2}$. Hence $\Theta_{B_1 \cap B_2} \subseteq \Theta_{B_1} \cap \Theta_{B_2}$. Consequently, $\Theta_{B_1 \cap B_2} = \Theta_{B_1} \cap \Theta_{B_2}$. This shows that $\Im(\mathcal{P})$ is a topology on $\mathcal{P}$. ■

Theorem 30 *Let H be a Γ-semihypergroup. Then, the following are true.*

(1) *H is intra-regular if and only if for a bi-Γ-hyperideal B and a quasi-Γ-hyperideal Q of H, we have $B \cap Q \subseteq (H\Gamma B\Gamma Q\Gamma H)$.*

(2) *H is intra-regular if and only if for a bi-Γ-hyperideal B and a quasi-Γ-hyperideal Q of H, we have $B \cap Q \subseteq (H\Gamma Q\Gamma B\Gamma H)$.*

Proof. Let $a \in B \cap Q$. Since H is intra-regular, so there exist x, $y \in H$ and α, $\beta, \gamma \in \Gamma$ such that

$$\begin{aligned} a &\in x\alpha a\gamma a\beta y \subseteq x\alpha a\alpha(x\alpha a\gamma a\beta y)\beta y = x\alpha(a\alpha x\alpha a)\gamma a\beta y\beta y \\ &= x\alpha(a\alpha x\alpha a)\gamma a\beta y\beta y \subseteq H\Gamma(B\Gamma H\Gamma B)\Gamma Q\Gamma H \subseteq (H\Gamma B\Gamma Q\Gamma H) \end{aligned}$$

Therefore, $B \cap Q \subseteq (H\Gamma B\Gamma Q\Gamma H)$.

Conversely, let $a \in H$ and let $B(a)$ be a bi-Γ-hyperideal of H generated by a and

$Q(a)$ be a quasi-Γ-hyperideal of H generated by a. Then, by hypothesis,

$$\begin{aligned} a &\in B(a) \cap Q(a) \subseteq H\Gamma B(a)\Gamma Q(a)\Gamma H \\ &= H\Gamma(a \cup a\Gamma H\Gamma a)\Gamma(a \cup a\Gamma H \cap H\Gamma a)\Gamma H \\ &\subseteq (H\Gamma a \cup H\Gamma a\Gamma a \cup H\Gamma a\Gamma H\Gamma a)\,\Gamma(a \cup H\Gamma a)\Gamma H \\ &\subseteq H\Gamma a\Gamma(a \cup H\Gamma a)\Gamma H \subseteq H\Gamma a\Gamma\,(a\Gamma H \cup H\Gamma a\Gamma H) \\ &\subseteq H\Gamma a\Gamma a\Gamma H \cup H\Gamma a^2\Gamma H = H\Gamma a^2\Gamma H \\ &\Rightarrow a \in H\Gamma a\Gamma a\Gamma H. \end{aligned}$$

Therefore, H is intra-regular.

(2) Let $a \in B\cap Q$. Since H is an intra-regular, so there exist $x, y \in H$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a \in x\alpha a\gamma a\beta y$. Thus,

$$\begin{aligned} a &\in x\alpha a\gamma a\beta y \subseteq x\alpha(x\alpha a\gamma a\beta y)\alpha a\beta y = x\alpha x\alpha a\alpha(a\beta y\alpha a)\beta y \\ &\subseteq H\Gamma Q\Gamma(B\Gamma H\Gamma B)\Gamma H \subseteq H\Gamma Q\Gamma B\Gamma H. \end{aligned}$$

Therefore, $B \cap Q \subseteq (H\Gamma Q\Gamma B\Gamma H)$.

Conversely, let $a \in H$ and let $B(a)$ be a bi-Γ-hyperideal of H generated by a and $Q(a)$ be a quasi-Γ-hyperideal of H generated by a. Then, by hypothesis,

$$\begin{aligned} a &\in B(a) \cap Q(a) \subseteq H\Gamma Q(a)\Gamma B(a)\Gamma H \\ &= H\Gamma(a \cup (a\Gamma H) \cap (H\Gamma a))\Gamma\,(a \cup a\Gamma H\Gamma a)\,\Gamma H \\ &\subseteq (H\Gamma(a \cup H\Gamma a)\Gamma(a\Gamma H \cup a\Gamma a\Gamma H \cup a\Gamma H\Gamma a\Gamma H) \\ &\subseteq (H\Gamma a \cup H\Gamma a)\,(a\Gamma H) \subseteq (H\Gamma a)\,\Gamma(a\Gamma H) \\ &\subseteq H\Gamma a\Gamma a\Gamma H \Rightarrow a \in H\Gamma a\Gamma a\Gamma H. \end{aligned}$$

Therefore, H is an intra-regular. ■

Theorem 31 *Let H be a Γ-semihypergroup. Then the following are true.*

(1) *H is intra-regular if and only if for a left* Γ*-hyperideal L and a bi-*Γ-hyperideal B *of H, we have $L \cap B \subseteq (L\Gamma B\Gamma H)$.*

(2) *H is intra-regular if and only if for a right* Γ-hyperideal R *and a bi-*Γ-hyperideal B *of H, we have $B \cap R \subseteq (H\Gamma B\Gamma R)$.*

Proof. (1) Let $a \in L \cap B$. Since H is intra-regular, so there exist x, $y \in H$ and α, $\beta, \gamma \in \Gamma$ such that $a \in x\alpha a\gamma a\beta y$. Then,

$$\begin{aligned} a &\in x\alpha a\gamma a\beta y \subseteq x\alpha(x\alpha a\gamma a\beta y)\alpha a\beta y = x\alpha x\alpha a\alpha(a\beta y\alpha a)\beta y \\ &\subseteq H\Gamma L\Gamma(B\Gamma H\Gamma B)\Gamma H \subseteq L\Gamma B\Gamma H. \end{aligned}$$

Thus, $L \cap B \subseteq (L\Gamma B\Gamma H)$.

Conversely, let $a \in H$ and let $B(a)$ be a bi-Γ-hyperideal of H generated by a and $L(a)$ be a left Γ-hyperideal of H generated by a of H. Then, by hypothesis,

$$\begin{aligned} a &\in L(a) \cap B(a) \subseteq (L(a)\Gamma B(a)\Gamma H) \\ &= (a \cup H\Gamma a)\,\Gamma\,(a \cup a\Gamma H\Gamma a)\,\Gamma H \\ &\subseteq (a \cup H\Gamma a)\,\Gamma\,(a\Gamma H \cup a\Gamma a\Gamma H \cup a\Gamma H\Gamma a\Gamma H) \\ &\subseteq (a \cup H\Gamma a)\,\Gamma\,(a\Gamma H) = a\Gamma a\Gamma H \cup H\Gamma a\Gamma a\Gamma H \\ &\Rightarrow a \in a\Gamma a\Gamma H \cup H\Gamma a\Gamma a\Gamma H \Rightarrow a \in a\Gamma a\Gamma H \text{ or } a \in H\Gamma a\Gamma a\Gamma H. \end{aligned}$$

If $a \in a\Gamma a\Gamma H$, then $a \in a\gamma a\alpha x$ for some $x \in H$ and $\alpha, \gamma \in \Gamma$.

$$\begin{aligned} a &\in a\gamma a\alpha x \subseteq a\gamma(a\gamma a\alpha x)\alpha x = a\alpha a\gamma a\alpha x\alpha x \subseteq H\Gamma a\Gamma a\Gamma H \\ a &\in H\Gamma a\Gamma a\Gamma H. \end{aligned}$$

Hence, H is intra-regular.

(2) Let $a \in B \cap R$. Since H is intra-regular, so there exist $x, y \in H$ and α, $\beta, \gamma \in \Gamma$

such that

$$\begin{aligned} a &\in x\alpha a\gamma a\beta y \subseteq x\alpha a\gamma(x\alpha a\gamma a\beta y)\beta y = x\alpha(a\gamma x\beta a)\alpha a\beta y\beta y \\ &\subseteq H\Gamma(B\Gamma H\Gamma B)\Gamma R\Gamma H \subseteq H\Gamma B\Gamma R. \end{aligned}$$

Thus, $B \cap R \subseteq (H\Gamma B\Gamma R)$.

Conversely, let $a \in H$ and let $B(a)$ be a bi-Γ-hyperideal of H generated by a and $R(a)$ be a right Γ-hyperideal of H generated by a. Then, by hypothesis,

$$\begin{aligned} a &\in B(a) \cap R(a) \subseteq H\Gamma B(a)\Gamma R(a) \\ &= H\Gamma\,(a \cup a\Gamma H\Gamma a)\,\Gamma\,(a \cup a\Gamma H) \\ &\subseteq (H\Gamma a \cup H\Gamma a\Gamma a \cup H\Gamma a\Gamma H\Gamma a)\,\Gamma\,(a \cup a\Gamma H) \\ &\subseteq (H\Gamma a)\,\Gamma\,(a \cup a\Gamma H) = H\Gamma a\Gamma a\Gamma H \cup a\Gamma a\Gamma H \\ &\Rightarrow a \in H\Gamma a\Gamma a\Gamma H \cup a\Gamma a\Gamma H \Rightarrow a \in H\Gamma a\Gamma a\Gamma H \text{ or} \Rightarrow a \in a\Gamma a\Gamma H. \end{aligned}$$

If $a \in a\Gamma a\Gamma H$, then $a \in a\gamma a\alpha x$ for some $x \in H$ and $\alpha, \gamma \in \Gamma$.

$$\begin{aligned} a &\in a\gamma a\alpha x \subseteq a\gamma(a\gamma a\alpha x)\alpha x = a\gamma a\gamma a\alpha x\alpha x \subseteq H\Gamma a\Gamma a\Gamma H \\ a &\in H\Gamma a\Gamma a\Gamma H. \end{aligned}$$

Hence, H is intra-regular. ■

2.4 Left(Right) Filters of Γ-semihypergroups

Let H be a Γ-semihypergroup. A non-empty sub-Γ-semihypergroup F of H is called a left(resp. right) filter of H if

1) $a\alpha b \subseteq F$ for $a, b \in H$ and $\alpha \in \Gamma \implies a \in F$ (resp. $b \in F$).

A sub-Γ-semihypergroup F of H is called a filter of H if F is a left and a right filter.

In this section, we give the characterizations of a left(right) filter of H in terms of the right(left) prime Γ-hyperideals.

Theorem 32 *Let H be a Γ-semihypergroup and F a non-empty subset of H. Then the following are equivalent:*

1) *F is a left filter of H.*

2) *$H \backslash F = \emptyset$ or $H \backslash F$ is a prime right Γ-hyperideal.*

Proof. 1) $\Longrightarrow$ 2): Assume that $H \backslash F \neq \varphi$. Let $x \in H \backslash F$ and $y \in H$. Then $x\alpha y \subseteq H \backslash F$. Indeed, if $x\alpha y \nsubseteq H \backslash F$, then $x\alpha y \subseteq F$. Since F is a left filter, so $x \in F$. It is impossible. Thus, $x\alpha y \subseteq H \backslash F$, so $(H \backslash F)\Gamma H \subseteq F$.

Next, we shall prove that $H \backslash F$ is prime. Let $x\alpha y \subseteq H \backslash F$ for $x, y \in H$ and $\alpha \in \Gamma$. Suppose that $x \notin H \backslash F$ and $y \notin H \backslash F$. Then $x \in F$ and $y \in F$. Since F is a sub-Γ-semihypergroup of H, $x\alpha y \subseteq F$. It is impossible. Thus, $x \in H \backslash F$ or $y \in H \backslash F$. Hence, $H \backslash F$ is prime, and so $H \backslash F$ is a prime right Γ-hyperideal.

2) $\Longrightarrow$ 1). If $H \backslash F = \emptyset$, then $H = F$. Thus, F is a left filter of H.

Next, assume that $H \backslash F$ is a prime right Γ-hyperideal of H. Then, F is a sub-Γ-semihypergroup of H. Indeed, suppose that $x\alpha y \nsubseteq F$ for $x, y \in F$ and $\alpha \in \Gamma$. Then, $x\alpha y \subseteq H \backslash F$ for $x, y \in F$ and $\alpha \in \Gamma$. Since $H \backslash F$ is prime, so x or $y \in H \backslash F$. It is impossible. Thus, $x\alpha y \subseteq F$, so F is a sub-Γ-semihypergroup of H.

Let $x\alpha y \subseteq F$ for $x, y \in H$. Then $x \in F$. Indeed, if $x \notin F$, then $x \in H \backslash F$. Since $H \backslash F$ is a right Γ-hyperideal of H. Then, $x\alpha y \subseteq (H \backslash F)\Gamma H \subseteq (H \backslash F)$. It is impossible. Thus, $x \in F$. Therefore, F is a left filter of H. ■

Theorem 33 *Let H be a Γ-semihypergroup and F a non-empty subset of H. The following are equivalent:*

1) *F is a right filter of H.*

2) *$H \backslash F = \emptyset$ or $H \backslash F$ is a prime left Γ-hyperideal.*

From above two Theorems, we have the following:

Corollary 34 *Let H be a Γ-semihypergroup and F a non-empty subset of H. The following are equivalent:*

1) F *is a filter of* H.

2) $H \setminus F = \emptyset$ *or* $H \setminus F$ *is a prime* Γ-*hyperideal.*

Chapter 3

Quasi-Γ-hyperideals of Γ-semihypergroups

3.1 Introduction

Recently Davvaz et al. introduced a *quasi-*Γ*-hyperideal of a* Γ-semihypergroup and studied some basic properties of it. In this chapter we introduce the concept of a generalization of a quasi-Γ-hyperideal in a Γ-semihypergroup which is called (m, n)-quasi-Γ-hyperideal. We also introduce the concept of a m-left Γ-hyperideal and an n-right Γ-hyperideal which is a generalization of the concept of a left Γ-hyperideal and a right Γ-hyperideal in a Γ-semihypergroup. We introduce the concept of minimal (m, n)-quasi-Γ-hyperideals, minimal m-left Γ-hyperideals and minimal n-right Γ-hyperideals in Γ-semihypergroups. We give some different characterizations concerning different properties of (m, n)-quasi-Γ-hyperideals, minimal (m, n)-quasi-Γ-hyperideals, minimal m-left Γ-hyperideals and minimal n-right Γ-hyperideals and relations among them are investigated. Also, some intersection properties and characterizations of (m, n)-quasi-Γ-hyperideals of Γ-semihypergroups and regular Γ-semihypergroups have been studied. In sequel, m-left simple, n-right simple and (m, n)-quasi-simple Γ-semihypergroups are defined, and some properties of them are investigated.

3.2 (m, n)-quasi-Γ-hyperideals

In this section, we introduce the notion of an (m, n)-quasi-Γ-hyperideal, n-right Γ-hyperideal and m-left Γ-hyperideal in Γ-semihypergroups and relations between them are studied.

Let H be a Γ-semihypergroup and Q a non-empty subset of H. Then Q is called a *quasi-Γ-hyperideal* of H if $Q\Gamma H \cap H\Gamma Q \subseteq Q$. Let H be a Γ-semihypergroup and Q be a non-empty subset of H. Then Q is called an *(m, n)-quasi-Γ-hyperideal* of H if $H^m\Gamma Q \cap Q\Gamma H^n \subseteq Q$, where m and n are positive integers.

Example 35 *Let $H = [0, 1]$ and $\Gamma = N$. Define a Γ-hyperoperation on H as: $x\gamma y = [0, \frac{xy}{\gamma}]$ for all $x, y \in H$ and $\gamma \in \Gamma$. Then, clearly H is a Γ-semihypergroup. Let $Q_1 = [0, 0.9]$, $Q_2 = [0, 0.8]$, $Q_3 = [0, 0.7]$, $Q_4 = [0, 0.6]$, $Q_5 = [0, 0.5]$. Then, clearly Q_i is a (m, n)-quasi-Γ-hyperideal for $i = 1, 2, 3, 4, 5$.*

Example 36 *Let $H =$ [-1, 0] and $\Gamma = \{-1, -2, -3\}$. Define a Γ-hyperoperation on H as: $x\gamma y = [\frac{xy}{\gamma}, 0]$ for all $x, y \in H$ and $\gamma \in \Gamma$. Then, clearly H is a Γ-semihypergroup. Let $Q_1 = [-0.9, 0]$, $Q_2 = [-0.8, 0]$, $Q_3 = [-0.7, 0]$, $Q_4 = [-0.6, 0]$, $Q_5 = [-0.5, 0]$. Then, clearly Q_i is a (m, n)-quasi-Γ-hyperideal for $i = 1, 2, 3, 4, 5$.*

Example 37 *Let $H = N$ and $\Gamma = \{\gamma_i | i \in N\}$. Define a Γ-hyperoperation on H as: $x\gamma_i y = x + iN + y$ for all $x, y \in H$ and $\gamma_i \in \Gamma$. Then, clearly H is a Γ-semihypergroup. Let $Q_i = \{i, i + 1, i + 2, ...\}$, $i \in N$. Then, clearly Q_i is a (m, n)-quasi-Γ-hyperideal for $i \in N$.*

Example 38 *Let $H = [-1, 1]$ and $\Gamma = [0, 1]$. Let $x, y \in H$ and $\gamma \in \Gamma$. Then the Γ-hyperoperation on H is defined as follows:*

$$x\gamma y = \begin{cases} [x\gamma y, 0] & \text{if } x < 0, y > 0 \text{ or } x > 0, y < 0 \\ [0, x\gamma y] & \text{if } x > 0, y > 0 \text{ or } x < 0, y < 0 \\ 0 & \text{if } x = 0 \text{ or } y = 0. \end{cases}$$

Then, clearly H is a Γ-semihypergroup. Let $Q_1 = [-0.9, 0.9]$, $Q_2 = [-0.8, 0.8]$, $Q_3 = [-0.7, 0.7]$, $Q_4 = [-0.6, 0.6]$, $Q_5 = [-0.5, 0.5]$, $Q_6 = [-0.4, 0.4]$, $Q_7 = [-0.3, 0.3]$. Then, clearly Q_i is a (m, n)-quasi-Γ-hyperideal of H for $i = 1, 2, 3, 4, 5, 6, 7$.

Example 39 *Let $H = \{x, y, z\}$ and $\Gamma = \{\alpha, \beta\}$ be two non-empty sets. Then, it can be easily verified that H is a Γ-semihypergroup, where the Γ-hyperoperations defined by*

α	x	y	z
x	$\{x\}$	$\{x, y\}$	$\{z\}$
y	$\{x, y\}$	$\{x, y\}$	$\{z\}$
z	$\{z\}$	$\{z\}$	$\{z\}$

β	x	y	z
x	$\{x, y\}$	$\{x, y\}$	$\{z\}$
y	$\{x, y\}$	$\{y\}$	$\{z\}$
z	$\{z\}$	$\{z\}$	$\{z\}$

It can be easily shown that $Q_1 = \{x, y, z\}$ and $Q_2 = \{z\}$ are (m, n)-quasi-Γ-hyperideals of H.

It is clear that a quasi-Γ-hyperideal Q of a Γ-semihypergroup H is a $(1, 1)$-quasi-Γ-hyperideal of H. Moreover, an (m, n)-quasi-Γ-hyperideal of H is a (k, l)-quasi-Γ-hyperideal of H for all $k \geq m$ and $l \geq n$. We know that an (m, n)-quasi-Γ-hyperideal of a Γ-semihypergroup H need not to be a quasi-Γ-hyperideal of H. The following example shows the same.

Example 40 *Let $H = N$ and $\Gamma = \{\gamma_i | i \in N\}$. Define a Γ-hyperoperation on H as: $x\gamma_i y = x + iN + y$ for all $x, y \in H$ and $\gamma_i \in \Gamma$. Then, clearly H is a Γ-semihypergroups. Let $Q = \{2\} \cup \{k \in N | k > 3 + s\}$. Then, clearly for $m, n > s$, Q is a (m, n)-quasi-Γ-hyperideal but it is not a quasi-Γ-hyperideal of H.*

If we consider the class of quasi-Γ-hyperideals in a Γ-semihypergroup, we observe that it is the generalization of the class of one-sided Γ-hyperideal in a Γ-semihypergoup. It is clear that every one sided Γ-hyperideal of a Γ-semihypergroup is a quasi-Γ-hyperideal of H.

Lemma 41 *Let H be a Γ-semihypergroup and A_i be a sub-Γ-semihypergroup of H for all $i \in I$. If $\bigcap_{i\in I} A_i \neq \emptyset$, then $\bigcap_{i\in I} A_i$ is a sub-Γ-semihypergroup of H.*

Proof. Assume that $\bigcap_{i\in I} A_i \neq \emptyset$. Let $a, b \in \bigcap_{i\in I} A_i$. Then, $a, b \in A_i$ for all $i \in I$. Since A_i is a sub-Γ-semihypergroup of H for all $i \in I$, so $a\gamma b \subseteq A_i$ for all $i \in I$ and $\gamma \in \Gamma$. Hence, $a\gamma b \subseteq \bigcap_{i\in I} A_i$ for all $\gamma \in \Gamma$. Thus, $\bigcap_{i\in I} A_i$ is a sub-Γ-semihypergroup of H. ■

Proposition 42 *Let H be a Γ-semihypergroup. Let Q_i and A_i be an (m, n)-quasi-Γ-hyperideal and sub-Γ-semihypergroup of H, respectively, for all $i \in I$. Then, $A_i \cap Q_i$ is either empty or an (m, n)-quasi-Γ-hyperideal of A_i.*

Proof. If $A_i \cap Q_i$ is not empty, then $A_i \cap Q_i$ is a subset of A_i, such that $((A_i^m \cap Q_i)\Gamma A_i^m) \cap (A_i^n\Gamma(A_i^n \cap Q_i)) \subseteq A_i\Gamma A_i \subseteq A_i$ and $((A_i^m \cap Q_i)\Gamma A_i^m) \cap (A_i^n\Gamma(A_i^n \cap Q_i)) \subseteq Q_i\Gamma H^m \cap H^n\Gamma Q_i \subseteq Q_i$. This shows that $A_i \cap Q_i$ is an (m, n)-quasi-Γ-hyperideal of A_i for all $i \in I$. ■

Proposition 43 *Let H be a Γ-semihypergroup and $\{Q_i, i \in I\}$ be a set of (m, n)-quasi-Γ-hyperideals of H. If $\bigcap_{i\in I} Q_i \neq \emptyset$, then $\bigcap_{i\in I} Q_i$ is an (m, n)-quasi-Γ-hyperideal of H.*

Proof. Let Q_i be an (m, n)-quasi-Γ-hyperideal of H for $i \in I$. Assume that $\bigcap_{i\in I} Q_i \neq \emptyset$. Then, for every Q_j, $j \in I$, we have $(H^m\Gamma(\bigcap_{i\in I} Q_i)) \cap ((\bigcap_{i\in I} Q_i)\Gamma H^m) \subseteq H^m\Gamma Q_j \cap Q_j\Gamma H^n \subseteq Q_j$. This shows that $\bigcap_{i\in I} Q_i$ is an (m, n)-quasi-Γ-hyperideal of H. ■

Theorem 44 *Let H be a Γ-semihypergroup and Q_i be an (m, n)-quasi-Γ-hyperideal for all $i \in I$. If $\bigcap_{i\in I} Q_i \neq \emptyset$, then $\bigcap_{i\in I} Q_i$ is an (m, n)-quasi-Γ-hyperideal of H.*

Proof. Assume $\bigcap_{i\in I} Q_i \neq \emptyset$. By Lemma 41 and Proposition 43, we have that $\bigcap_{i\in I} Q_i$ is a sub-Γ-semihypergroup of H. Let $a \in H^m\Gamma(\bigcap_{i\in I} Q_i) \cap (\bigcap_{i\in I} Q_i)\Gamma H^n$. Then, we have $a \in x\gamma q = p\gamma y$ for some $x \in H^m, y \in H^n$, for all $\gamma \in \Gamma$ and $p, q \in \bigcap_{i\in I} Q_i$. But since $p, q \in \bigcap_{i\in I} Q_i$, we have $p, q \in Q_i$ for all $i \in I$. Thus, $a \in H^m\Gamma Q_i \cap Q_i\Gamma H^n$ for all $i \in I$. Since we know that Q_i is an (m, n)-quasi-Γ-hyperideal of H for all $i \in I$, so we have $a \in \bigcap_{i\in I} Q_i$. Hence $\bigcap_{i\in I} Q_i$ is an (m, n)-quasi-Γ-hyperideal of H. ■

Let H be a Γ-semihypergroup and A be a non-empty subset of H. We denote $\mathcal{F} = \{Q | Q$ is an $(m,n) -$ quasi-Γ-hyperideal of H containing $A\}$. It is clear that $\mathcal{F}$ is not empty since $H \in \mathcal{F}$. Let $(A)_{q(m,n)} = \bigcap\limits_{Q \in \mathcal{F}} Q$. It is clear that $(A)_{q(m,n)}$ is non-empty since $A \subseteq (A)_{q(m,n)}$. By Theorem 44, $(A)_{q(m,n)}$ is an (m,n)-quasi-Γ-hyperideal of H and moreover, it is the smallest (m,n)-quasi-Γ-hyperideal of H containing A. The (m,n)-quasi-Γ-hyperideal $(A)_{q(m,n)}$ is called the *(m,n)-quasi-Γ-hyperideal of H generated by A.*

Theorem 45 *Let H be a Γ-semihypergroup and $\emptyset \neq A \subseteq H$. Then*
$(A)_{q(m,n)} = \left(\bigcup\limits_{i=1}^{\max\{m,n\}} A^i \right) \cup (H^m \Gamma A \cap A \Gamma H^n)$.

Proof. Let $k = \max\{m,n\}$ and $Q = \left(\bigcup\limits_{i=1}^{k} A^i \right) \cup (H^m \Gamma A \cap A \Gamma H^n)$. It is clear that $A \subseteq Q$. Let $x, y \in Q$ and $\gamma \in \Gamma$. Then, we have the following cases:

Case 1.Let $x, y \in \bigcup\limits_{i=1}^{k} A^i$. Then, $x\gamma y \subseteq A^t$ for some positive integer t. If $t \leq k$, then $x\gamma y \subseteq \bigcup\limits_{i=1}^{k} A^i$. If $t \geq k$, then $x\gamma y \subseteq H^m \Gamma A \cap A \Gamma H^n$.

Case 2. Let $x \in H^m \Gamma A \cap A \Gamma H^n$ or $y \in H^m \Gamma A \cap A \Gamma H^n$. It can be easily seen that $x\gamma y \subseteq H^m \Gamma A \cap A \Gamma H^n$. Therefore, $x\gamma y \subseteq Q$. Then, Q is a sub-Γ-semihypergroup of H containing A. We have

$$\begin{aligned} H^m \Gamma Q \cap Q \Gamma H^n &= H^m \Gamma ((\bigcup_{i=1}^{k} A^i) \cup (H^m \Gamma A \cap A \Gamma H^n)) \cap ((\bigcup_{i=1}^{k} A^i) \cup \\ &\quad \cup (H^m \Gamma A \cap A \Gamma H^n)) \Gamma H^n \\ &\subseteq H^m \Gamma ((\bigcup_{i=1}^{k} A^i) \cup H^m \Gamma A) \cap ((\bigcup_{i=1}^{k} A^i) \cup A \Gamma H^n) \Gamma H^n \\ &\subseteq H^m \Gamma A \cap A \Gamma H^n \subseteq Q. \end{aligned}$$

Thus we have that Q is an (m,n)-quasi-Γ-hyperideal of H containing A.

Now, we show that Q is the smallest. Let Q' be any (m,n)-quasi-Γ-hyperideal of H containing A. Then, $A^i \subseteq Q'$ for all positive integers i and $H^m \Gamma A \cap A \Gamma H^n \subseteq H^m \Gamma Q' \cap Q'^n \subseteq Q'$. Therefore, $Q = (\bigcup\limits_{i=1}^{k} A^i) \cup (H^m \Gamma A \cap A \Gamma H^n) \subseteq Q'$. Hence, Q is the smallest

(m,n)-quasi-Γ-hyperideal of H containing A. Therefore, we obtain the requested result. ■

3.3 m-left Γ-hyperideals and n-right Γ-hyperideals

Let H be a Γ-semihypergroup and L be a sub-Γ-semihypergroup of H. Then, L is called an *m-left Γ-hyperideal* of H if $H^m\Gamma L \subseteq L$ where m is any positive integer. Dually, if $R\Gamma H^n \subseteq R$, then R is called an *n-right Γ-hyperideal* of H, where n is any positive integer. In the following theorems we prove some results concerning an m-left and an n-right Γ-hyperideal of a Γ-semihypergroup H.

Theorem 46 *Let H be a Γ-semihypergroup. Then, the following statements hold:*

1. *Let L_i be an m-left Γ-hyperideal of H for all $i \in I$. If $\bigcap\limits_{i\in I} L_i \neq \emptyset$, then $\bigcap\limits_{i\in I} L_i$ is an m-left Γ-hyperideal of H.*

2. *Let R_i be an n-right Γ-hyperideal of H for all $i \in I$. If $\bigcap\limits_{i\in I} R_i \neq \emptyset$, then $\bigcap\limits_{i\in I} R_i$ is an n-right Γ-hyperideal of H.*

Proof. (1). Assume that $\bigcap\limits_{i\in I} L_i \neq \emptyset$. Let $a \in H^m\Gamma(\bigcap\limits_{i\in I} L_i)$. It follows that $a \in x\Gamma l$ for some $x \in H^m$ and $l \in \bigcap\limits_{i\in I} L_i$. Then $l \in L_i$ for all $i \in I$. This implies that $a \in H^m\Gamma L_i$ for all $i \in I$. Since L_i is an m-left Γ-hyperideal of H for all $i \in I$ and $a \in L_i$ for all $i \in I$, so, $a \in \bigcap\limits_{i\in I} L_i$. Therefore, $\bigcap\limits_{i\in I} L_i$ is an m-left Γ-hyperideal of H.

(2). It can be proved in the similar way with (1). ■

Let H be a Γ-semihypergroup and A a non-empty subset of H. We denote $\mathcal{F} = \{L | L \text{ is an } m-\text{left } \Gamma\text{-hyperideal of H containing } A\}$. It is clear that $\mathcal{F}$ is non-empty since $H \in \mathcal{F}$. Let $(A)_{l(m)} = \bigcap\limits_{L\in\mathcal{F}} L$. It is clear that $(A)_{l(m)}$ is non-empty since $A \subseteq (A)_{l(m)}$. By Theorem 46(1), $(A)_{l(m)}$ is an m-left Γ-hyperideal of H and moreover, it is the smallest m-left Γ-hyperideal of H containing A. The m-left Γ-hyperideal $(A)_{l(m)}$ is called the *m-left Γ-hyperideal of H generated by A*. The *n-right Γ-hyperideal $(A)_{r(n)}$ of H generated by A* is defined analogously.

Theorem 47 *Let H be a Γ-semihypergroup and $\emptyset \neq A \subseteq H$. Then, the following statements hold:*

1. $(A)_{l(m)} = (\bigcup_{i=1}^{m} A^i) \cup H^m \Gamma A.$

2. $(A)_{r(n)} = (\bigcup_{i=1}^{n} A^i) \cup A \Gamma H^n.$

Proof. It is similar to the proof of Theorem 45. ■

Theorem 48 *Let H be a Γ-semihypergroup and L, R be an m-left Γ-hyperideal, and n-right Γ-hyperideal of H, respectively. Then, $L \cap R$ is an (m, n)-quasi-Γ-hyperideal of H.*

Proof. By the properties of L and R, we have $R^m \Gamma L^n \subseteq H^m \Gamma L \cap R \Gamma H^n \subseteq L \cap R$. Then $L \cap R$ is non-empty. By Lemma 41 it follows that $L \cap R$ is a sub-Γ-semihypergroup of H. Now we have

$$(H^m \Gamma (L \cap R)) \cap ((L \cap R) \Gamma H^n) \subseteq H^m \Gamma L \cap R \Gamma H^n) \subseteq L \cap R.$$

Which proves that $L \cap R$ is an (m, n)-quasi-Γ-hyperideal of H. ■

We say that an (m, n)-quasi-Γ-hyperideal Q has the (m, n) intersection property if Q is the intersection of an m-left Γ-hyperideal and an n-right Γ-hyperideal of a Γ-semihypergroup H. In this case every m-left Γ-hyperideal and every n-right Γ-hyperideal have the (m, n) intersection property. If arbitrary family of an (m, n)-quasi-Γ-hyperideal of H has the (m, n) intersection property, then H is said to have intersection property of an (m, n)-quasi-Γ-hyperideal. The following theorem characterizes (m, n)-quasi-Γ-hyperideals having the (m, n) intersection property.

Theorem 49 *Let H be a Γ-semihypergroup and Q be an (m, n)-quasi-Γ-hyperideal of H. Then, the following statements are equivalent:*

1. *Q has the (m, n) intersection property.*

2. $(Q \cup H^m\Gamma Q) \cap (Q \cup Q\Gamma H^n) = Q$.

3. $H^m\Gamma Q \cap (Q \cup Q\Gamma H^n) = Q$.

4. $Q\Gamma H^n \cap (Q \cup Q\Gamma H^n) \subseteq Q$.

Proof. (1) $\Rightarrow$ (2). Suppose that Q has the (m, n) intersection property. It is obvious that $Q \subseteq (Q \cup H^m\Gamma Q) \cap (Q \cup Q\Gamma H^n)$ (*). Since Q has the (m, n) intersection property, so there exist an m-left Γ-hyperideal L and an n-right Γ-hyperideal R of H in such a way that $Q = L \cap R$. Thus, $Q \subseteq L$ and $Q \subseteq R$. Also, we have that $H^m\Gamma Q \subseteq H^m\Gamma L \subseteq L$ and in the similar way $Q\Gamma H^n \subseteq R\Gamma H^n \subseteq R$ which implies that $Q \cup H^m\Gamma Q \subseteq L$ and $Q \cup Q\Gamma H^n \subseteq R$. Hence we have that $(Q \cup H^m\Gamma Q) \cap (Q \cup Q\Gamma H^n) \subseteq L \cap R \subseteq Q$ (**). Thus, by (*) and (**), we have $(Q \cup H^m\Gamma Q) \cap (Q \cup Q\Gamma H^n) = Q$.

(2) $\Rightarrow$ (1). Let $(Q \cup H^m\Gamma Q) \cap (Q \cup Q\Gamma H^n) = Q$. Then, we have to show that $Q \cup H^m\Gamma Q$ is an m-left Γ-hyperideal of H and $Q \cup Q\Gamma H^n$ is an n-right Γ-hyperideal of H. Let $L = Q \cup H^m\Gamma Q$ and $R = Q \cup Q\Gamma H^n$. We show first that L is a sub-Γ-semihypergroup of H. Let $a, b \in L$. Then, we have the following cases:

Case 1. If $a, b \in Q$, then since Q is a sub-Γ-semihypergroup of H, $a\gamma b \subseteq Q \subseteq L$ for all $\gamma \in \Gamma$.

Case 2. If $a \in Q$ and $b \in H^m\Gamma Q$, then $a\gamma b \subseteq Q\Gamma H^m\Gamma Q \subseteq H^m\Gamma Q \subseteq L$ for all $\gamma \in \Gamma$.

Case 3. If $a \in H^m\Gamma Q$ and $b \in Q$, then $a\gamma b \subseteq H^m\Gamma Q\Gamma Q \subseteq H^m\Gamma Q \subseteq L$ for all $\gamma \in \Gamma$.

Case 4. If $a \in H^m\Gamma Q$ and $b \in H^m\Gamma Q$, then $a\gamma b \subseteq H^m\Gamma Q\Gamma H^m \subseteq H^m\Gamma Q \subseteq L$ for all $\gamma \in \Gamma$. Therefore, L is a sub-Γ-semihypergroup of H. We have

$$H^m\Gamma L = H^m\Gamma(Q \cup H^m\Gamma Q) = H^m\Gamma Q \cup H^{2m}\Gamma Q \subseteq H^m\Gamma Q \subseteq L.$$

Hence, L is an m-left Γ-hyperideal of H. In the similar way, R is an n-right Γ-hyperideal of H. Since, $H^m\Gamma Q \cap Q\Gamma H^m \subseteq Q$, so

$$L \cap R = (Q \cup H^m\Gamma Q) \cap (Q \cup Q\Gamma H^n) = Q \cup (H^m\Gamma Q \cap Q\Gamma H^n) = Q.$$

Therefore, $Q = L \cap R$.

(2) $\Rightarrow$ (3). Let $(Q \cup H^m\Gamma Q) \cap (Q \cup Q\Gamma H^n) = Q$. Then, since $H^m\Gamma Q \subseteq Q \cup H^n\Gamma Q$, so we have

$$H^m\Gamma Q \cap (Q \cup Q\Gamma H^n) \subseteq (Q \cup H^m\Gamma Q) \cap (Q \cup Q\Gamma H^n) = Q.$$

(3) $\Rightarrow$ (2). Let $H^m\Gamma Q \cap (Q \cup Q\Gamma H^n) = Q$. Then, $Q \subseteq (Q \cup H^m\Gamma Q) \cap (Q \cup Q\Gamma H^n)$ (*). Let $x \in (Q \cup H^m\Gamma Q) \cap (Q \cup Q\Gamma H^n)$. Then, since $H^m\Gamma Q \cap (Q \cup Q\Gamma H^n) \subseteq Q$, so we have $x \in Q$, so $(Q \cup H^m\Gamma Q) \cap (Q \cup Q\Gamma H^n) \subseteq Q$ (**). By (*) and (**) we have the requested result.

The proofs for (2) $\Rightarrow$ (4) and (4) $\Rightarrow$ (2) are similar to the proofs of (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2), respectively. ■

Proposition 50 *Let H be a Γ-semihypergroup and Q be an (m, n)-quasi-Γ-hyperideal of H. If $H^m\Gamma Q \subseteq Q\Gamma H^n$ or $Q\Gamma H^m \subseteq H^n\Gamma Q$, then Q has the (m, n) intersection property.*

Proof. Let $H^m\Gamma Q \subseteq Q\Gamma H^n$. Then, $H^m\Gamma Q = H^m\Gamma Q \cap Q\Gamma H^n \subseteq Q$ which shows that Q is an m-left Γ-hyperideal of H. Thus, Q has the (m, n) intersection property. In the similar way if we assume $Q\Gamma H^m \subseteq H^n\Gamma Q$, then Q has an n-right Γ-hyperideal of H. In this case also Q has the (m, n) intersection property. ■

The following theorem is dealing with the intersection property of (m, n)-quasi-Γ-hyperideal of a regular Γ-semihypergroup.

Theorem 51 *Every regular Γ-semihypergroup H has the intersection property of (m, n)-quasi-Γ-hyperideals for any $m, n \in N$.*

Proof. Let Q be an (m, n)-quasi-Γ-hyperideal of a regular Γ-semihypergroup H. Then, it can be easily shown that $Q \subseteq Q\Gamma H^n$. Thus $Q \cup Q\Gamma H^n = Q\Gamma H^n$. Therefore, $H^m\Gamma Q \cap (Q \cup Q\Gamma H^n) = H^m\Gamma Q \cap Q\Gamma H^n \subseteq Q$. By the Theorem 48 it follows that Q has the intersection property. ■

Let H be a Γ-semihypergroup and A be a non-empty subset of H. Then A is called an (m,n)-Γ-*hyperideal* if $A^m\Gamma H\Gamma A^n \subseteq A$. A $(m,0)$-Γ-hyperideal of H is $A^m\Gamma H \subseteq A$ and $(0,n)$-Γ-hyperideal is $H\Gamma A^n \subseteq A$.

Proposition 52 *Let H be a regular Γ-semihypergroup. Then, a non-empty subset Q of H is an (m,n)-quasi-Γ-hyperideal of H if and only if it is the intersection of a $(m,0)$-right Γ-hyperideal and a $(0,n)$-left Γ-hyperideal of H.*

Proof. Let $\emptyset \neq Q \subseteq H$ be an (m,n)-quasi-Γ-hyperideal of H, i.e. $Q^m\Gamma H \cap H\Gamma Q^n \subseteq Q$, which is possible only when Q is the intersection of $(m,0)$-Γ-hyperideal and $(0,n)$-Γ-hyperideal of H which is obvious as $Q^m\Gamma H \cap H\Gamma Q^n \subseteq Q$.

Conversely, suppose that $Q^m\Gamma H \cap H\Gamma Q^n \subseteq Q$. Since $Q^m\Gamma H \subseteq Q$ and $H\Gamma Q^n \subseteq Q$, it follows that $Q^m\Gamma H\Gamma H\Gamma Q^n = Q^m\Gamma H\Gamma Q^n \subseteq Q$. Hence Q is an (m,n)-Γ-hyperideal of H. ■

Theorem 53 *The collection L of all (m,n)-quasi-Γ-hyperideals with non-empty intersection of a Γ-semihypergroup H is a complete hyperlattice.*

Proof. It is clear that L is partially ordered by inclusion. By the Theorem 44, the infimum of any collection of (m,n)-quasi-Γ-hyperideals $Q_i (i \in I)$ is obvious the $\bigcap\limits_{i\in I} Q_i$.

Similarly, we set

$$\bigvee_{i\in I} Q_i = \left\langle \bigcup_{i\in I} Q_i \right\rangle_R \cap \left\langle \bigcup_{i\in I} Q_i \right\rangle_L,$$

where $\left\langle \bigcup\limits_{i\in I} Q_i \right\rangle_R = \bigcup\limits_{i\in I} Q_i \cup H^m\Gamma(\bigcup\limits_{i\in I} Q_i)$, $\left\langle \bigcup\limits_{i\in I} Q_i \right\rangle_L = \bigcup\limits_{i\in I} Q_i \cup (\bigcup\limits_{i\in I} Q_i)\Gamma H^n$. By Theorem 48 and Theorem 49, this is obviously a (m,n)-quasi-Γ-hyperideal which bounds from above all the (m,n)-quasi-Γ-hyperideals $Q_i (i \in I)$. It is the supremum of L. Indeed, for any (m,n)-quasi-Γ-hyperideal Q containing all the $Q_i(i \in I)$, we have

$$\begin{aligned}
\bigvee_{i\in I} Q_i &= \left[\bigcup_{i\in I} Q_i \cup H^m\Gamma(\bigcup_{i\in I} Q_i)\right] \cap \left[\bigcup_{i\in I} Q_i \cup (\bigcup_{i\in I} Q_i)\Gamma H^n\right] = \\
&= \bigcup_{i\in I} Q_i \cup \left[H^m\Gamma(\bigcup_{i\in I} Q_i) \cap (\bigcup_{i\in I} Q_i)\Gamma H^n\right] \\
&\subseteq H^m\Gamma Q \cap Q\Gamma H^n \\
&\subseteq Q.
\end{aligned}$$

Hence, proved. ■

3.4 Minimal (m,n)-quasi-Γ-hyperideals

In this section, we obtain different characterizations concerning different properties of (m,n)-quasi-Γ-hyperideals, minimal (m,n)-quasi-Γ-hyperideals, minimal m-left Γ-hyperideals, minimal n-right Γ-hyperideals and relations of these concept are investigated. Also, some intersection properties and characterizations of (m,n)-quasi-Γ-hyperideals of Γ-semihypergroups and regular Γ-semihypergroups have been studied. In sequel, m-left simple, n-right simple and (m,n)-quasi-simple Γ-semihypergroups are defined, and some properties of them are investigated.

Let H be a Γ-semihypergroup and L be an m-left Γ-hyperideal of H. Then L is called a *minimal m-left Γ-hyperideal* of H if L does not properly contain any m-left Γ-hyperideal of H. Let H be a Γ-semihypergroup and R be an n-right Γ-hyperideal of H. Then R is called a *minimal n-right Γ-hyperideal* of H if R does not properly contain any n-right Γ-hyperideal of H.

Example 54 *Let $H = \{a,b,c\}$ and $\Gamma = \{\alpha,\gamma\}$. Then, Γ-hyperoperations are defined by* *the following Cayley's tables*

α	a	b	c
a	$\{b\}$	$\{b,c\}$	$\{b,c\}$
b	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$
c	$\{b,c\}$	$\{b,c\}$	$\{c\}$

γ	a	b	c
a	$\{c\}$	$\{b,c\}$	$\{b,c\}$
b	$\{b,c\}$	$\{b\}$	$\{b,c\}$
c	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$

Then, H is a Γ-semihypergroup. Let $I = \{b,c\}$ be a m-left Γ-hyperideal of H and also I is a minimal n-right Γ-hyperideal of H.

Let H be a Γ-semihypergroup and Q be an (m,n)-quasi-Γ-hyperideal of H. Then Q is called a *minimal (m,n)-quasi-Γ-hyperideal* of H if Q does not properly contain any (m,n)-quasi-Γ-hyperideal of H.

Example 55 *Let H be a Γ-semihypergroup in Example 39. Then, $Q = \{z\}$ is a minimal (m,n)-quasi-Γ-hyperideal*

Lemma 56 *Let H be a Γ-semihypergroup and $a \in H$. Then, the following statements hold true:*

1. *$H^m\Gamma a$ is an m-left Γ-hyperideal of H.*
2. *$a\Gamma H^n$ is an n-right Γ-hyperideal of H.*
3. *$H^m\Gamma a \cap a\Gamma H^n$ is an (m,n)-quasi-Γ-hyperideal of H.*

Proof. (1). We have that $(H^m\Gamma a)\Gamma(H^m\Gamma a) \subseteq H^m\Gamma a$ and $H^m\Gamma(H^m\Gamma a) \subseteq H^m\Gamma a$. Hence, (1) holds true.

(2). It is similar to (1).

(3). It follows by (1), (2) and Theorem 48 ■

Theorem 57 *Let H be a Γ-semihypergroup and Q be an (m,n)-quasi-Γ-hyperideal of H. Then, Q is minimal if and only if Q is the intersection of some minimal m-left Γ-hyperideal L and some minimal n-right Γ-hyperideal R of H.*

Proof. Assume that Q is a minimal (m,n)-quasi-Γ-hyperideal of H. Let $a \in Q$. Then, by Lemma 56, it follows that $H^m\Gamma a, a\Gamma H^n$ and $H^m\Gamma a \cap a\Gamma H^n$ are an m-left Γ-hyperideal, an n-right Γ-hyperideal and an (m,n)-quasi-Γ-hyperideal of H, respectively. By the minimality of Q, since $H^m\Gamma a \cap a\Gamma H^n \subseteq H^m\Gamma Q \cap Q\Gamma H^n \subseteq Q$, we have $H^m\Gamma a \cap a\Gamma H^n = Q$.

We have to show the minimality of the m-left Γ-hyperideal $H^m\Gamma a$ and the minimality of the n-right Γ-hyperideal $a\Gamma H^n$. Let L be an m-left Γ-hyperideal of H contained in $H^m\Gamma a$. Then, we have $L \cap a\Gamma H^n \subseteq H^m\Gamma a \cap a\Gamma H^n = Q$. By the minimality of Q, since $L \cap a\Gamma H^n$ is an (m,n)-quasi-Γ-hyperideal of H, it follows that $L \cap a\Gamma H^n = Q$. Then, $Q \subseteq L$. Therefore, $H^m\Gamma a \subseteq H^m\Gamma Q \subseteq H^m\Gamma L \subseteq L$. Therefore, $H^m\Gamma a \subseteq H^m\Gamma Q \subseteq H^m\Gamma L \subseteq L$. This implies $L = H^m\Gamma a$. Thus, the m-left Γ-hyperideal $H^m\Gamma a$ is minimal. In similar way, dually it can be proved the minimality of the n-right Γ-hyperideal $a\Gamma H^n$.

Conversely, assume that $Q = L \cap R$ for some minimal m-left Γ-hyperideal L and some minimal n-right Γ-hyperideal R of H. Let Q' be an (m,n)-quasi-Γ-hyperideal of H contained in Q. Then we have $H^m\Gamma Q'^m\Gamma Q \subseteq H^m\Gamma L \subseteq L$ and $Q'^n \subseteq Q\Gamma H^n \subseteq R\Gamma H^n \subseteq R$.

It can be easily proved that $H^m\Gamma Q'$ and Q'^n is an m-left Γ-hyperideal and an n-right Γ-hyperideal of H, respectively. The minimality of L and R implies $H^m\Gamma Q' = L$ and $Q'^n = R$. Hence, $Q = L \cap R = H^m\Gamma Q' \cap Q'^n \subseteq Q'$. Then, $Q = Q'$. Therefore, Q is a minimal (m,n)-quasi-Γ-hyperideal of H. ∎

The following propositions give necessary and sufficient conditions for the existence of a minimal (m,n)-quasi-Γ-hyperideal of a Γ-semihypergroup.

An immediate corollary of the Theorem 57 is the following.

Corollary 58 *Let H be a Γ-semihypergroup. Then H has at least one minimal (m,n)-quasi-Γ-hyperideal if and only if H has at least one minimal m-left Γ-hyperideal and at least one minimal n-right Γ-hyperideal.*

Theorem 59 *Let H be a Γ-semihypergroup. The following statements hold:*

1. *An m-left Γ-hyperideal L is minimal if and only if $H^m\Gamma a = L$ for all $a \in L$.*

2. *An n-right Γ-hyperideal R is minimal if and only if $a\Gamma H^n = R$ for all $a \in R$.*

3. *An (m, n)-quasi-Γ-hyperideal Q is minimal if and only if $H^m\Gamma a \cap a\Gamma H^n = Q$ for all $a \in Q$.*

Proof. (1). Assume that L is minimal. Let $a \in L$. Then $H^m\Gamma a \subseteq H^m\Gamma L \subseteq L$. By Lemma 56(1) it follows that $H^m\Gamma a$ is an m-left Γ-hyperideal of H. Since L is a minimal m-left Γ-hyperideal of H, we have $H^m\Gamma a = L$.

Conversely, assume that $H^m\Gamma a = L$ for all $a \in L$. Let L' be an m-left Γ-hyperideal of H contained in L. Let $x \in L' \subseteq L$. Then $H^m\Gamma x = L$. We have:

$L = H^m\Gamma x \subseteq H^m\Gamma L' \subseteq L'$.

This implies that $L = L'$. Therefore L is minimal.

(2) and (3) can be proved similar as (1). ■

Let H be a Γ-semihypergroup. Then, H is called *m-left simple Γ-semihypergroup* if H is a unique m-left Γ-hyperideal of H. Let H be a Γ-semihypergroup. Then, H is called *n-right simple Γ-semihypergroup* if H is a unique n-right Γ-hyperideal of H. Let H be a Γ-semihypergroup. H is called *(m, n)-quasi-simple Γ-semihypergroup* if H is a unique (m, n)-quasi-Γ-hyperideal of H.

Theorem 60 *Let H be a Γ-semihypergroup. The following statements hold true:*

1. *H is an m-left simple Γ-semihypergroup if and only if $H^m\Gamma a = H$ for all $a \in H$.*

2. *H is an n-right simple Γ-semihypergroup if and only if $a\Gamma H^n = H$ for all $a \in H$.*

3. *H is an (m, n)-quasi-simple Γ-semihypergroup if and only if $H^m\Gamma a \cap a\Gamma H^n = H$ for all $a \in H$.*

Proof. (1). Since H is an m-left simple Γ-semihypergroup, we have that H is a minimal m-left Γ-hyperideal of H. By the Theorem 59(1), $H^m\Gamma a = H$ for all $a \in H$.

Conversely, assume that $H^m\Gamma a = H$ for all $a \in H$. By the Theorem 59(1), H is a minimal m-left Γ-hyperideal of H and therefore, H is an m-left simple Γ-semihypergroup.

(2) and (3) can be proved similarly to (1). ■

Theorem 61 *Let H be a Γ-semihypergroup. The following statements hold:*

1. *If an m-left Γ-hyperideal L of H is an m-left simple Γ-semihypergroup, then L is a minimal m-left Γ-hyperideal of H.*

2. *If an n-right Γ-hyperideal R of H is an n-right simple Γ-semihypergroup, then R is a minimal n-right Γ-hyperideal of H.*

3. *If an (m,n)-quasi-Γ-hyperideal Q of H is an (m,n)-quasi-simple Γ-semihypergroup, then Q is a minimal (m,n)-quasi-Γ-hyperideal of H.*

Proof. (1). Let L be an m-left simple Γ-semihypergroup. Then, by the Theorem 60(1), we have $L^m\Gamma a = L$ for all $a \in L$. For every $a \in L$, we have $L = L^m\Gamma a \subseteq H^m\Gamma a \subseteq H^m\Gamma L \subseteq L$. Then $H^m\Gamma a = L$ for all $a \in L$. By the Theorem 59(1), we have L is minimal.

(2) and (3) can be proved similarly to (1). ■

Chapter 4

Intuitionistic Fuzzy Sets in Γ-semihypergroups

4.1 Introduction

In this chapter, by using Atanassov idea, we continue the study of intuitionistic fuzzy sets in Γ-semihypergroups which was initiated recently by Ersoy and Davvaz [28]. We define intuitionistic fuzzy bi-Γ-hyperideals, intuitionistic fuzzy interior Γ-hyperideals and intuitionistic fuzzy prime(semiprime) Γ-hyperideals in a Γ-semihypergroup. We give some further properties of intuitionistic fuzzy Γ-hyperideals and intuitionistic fuzzy bi-Γ-hyperideals in a Γ-semihypergroup. We use the intuitionistic fuzzy left, right, two-sided and bi-Γ-hyperideals to characterize some classes of Γ-semihypergroups. We introduce and study (λ, μ)-intuitionistic fuzzy Γ-hyperideals. We also introduce the notion of an intuitionistic fuzzy M(resp. N)-hypersystem of a Γ-semihypergroup and some properties of them are investigated.

4.2 Intuitionistic Fuzzy Γ-hyperideals of Γ-semihypergroup

Definition 62 *[28] Let H be a Γ-semihypergroup. An intuitionistic fuzzy set $A = \langle \mu_A, \lambda_A \rangle$ in H is called an intuitionistic fuzzy sub-Γ-semihypergroup in H if for all $x, y \in H, \gamma \in \Gamma$,*

$$\inf_{z \in x\gamma y}\{\mu_A(z)\} \geq \min\{\mu_A(x), \mu_A(y)\} \text{ and } \sup_{z \in x\gamma y}\{\lambda_A(z)\} \leq \max\{\lambda_A(x), \lambda_A(y)\}.$$

Definition 63 *[28]Let H be a Γ-semihypergroup. An intuitionistic fuzzy set $A = \langle \mu_A, \lambda_A \rangle$ in H is called an intuitionistic fuzzy left (resp. right) Γ-hyperideal of H if for all $x, y \in H, \gamma \in \Gamma$,*

1. $\mu_A(y) \leq \inf_{z \in x\gamma y}\{\mu_A(z)\}$ *(resp.* $\mu_A(x) \leq \inf_{z \in x\gamma y}\{\mu_A(z)\}$*).*
2. $\sup_{z \in x\gamma y}\{\lambda_A(y)\} \leq \lambda_A(y)$ *(resp.* $\sup_{z \in x\gamma y}\{\lambda_A(z)\} \leq \lambda_A(x)$*).*

An intuitionistic fuzzy set A in H is called an intuitionistic fuzzy two-sided Γ-hyperideal of H if it is both an intuitionistic fuzzy left and an intuitionistic right Γ-hyperideal of H. Similarly, we can define an intuitionistic fuzzy left (resp. right) Γ-hyperideal of H as follows:

$A(z) \supseteq A(y)$ *(resp.* $A(z) \supseteq A(x)$*) for each* $z \in x\gamma y$.

Definition 64 *Let H be a Γ-semihypergroup. An intuitionistic fuzzy sub-Γ-semihypergroup $A = \langle \mu_A, \lambda_A \rangle$ in H is called an intuitionistic fuzzy bi-Γ-hyperideal of H if for all $x, y, z \in H, \alpha, \beta \in \Gamma$,*

$$\inf_{t \in x\alpha y\beta z}\{\mu_A(t)\} \geq \min\{\mu_A(x), \mu_A(z)\} \text{ and } \sup_{t \in x\alpha y\beta z}\{\lambda_A(t)\} \leq \max\{\lambda_A(x), \lambda_A(z)\}.$$

Similarly, we can define an intuitionistic fuzzy bi-Γ-hyperideal of H as follows:

$A(z) \supseteq A(x) \cap A(y)$ *for each* $z \in x\gamma y$.

$A(t) \supseteq A(x) \cap A(z)$ *for each* $t \in x\gamma y\beta z$.

Definition 65 *Let H be a Γ-semihypergroup. An intuitionistic fuzzy sub-Γ-semihypergroup $A = \langle \mu_A, \lambda_A \rangle$ of H is called an intuitionistic fuzzy $(1, 2)$-Γ-hyperideal of H if*

1. $\inf_{a \in x\alpha w\beta(y\gamma z)} \{\mu_A(a)\} \geq \min\{\mu_A(x), \mu_A(y), \mu_A(z)\}$,

2. $\sup_{a \in x\alpha w\beta(y\gamma z)} \{\lambda_A(a)\} \leq \max\{\lambda_A(x), \lambda_A(y), \lambda_A(z)\}$,

for all $w, x, y, z \in H$ and $\alpha, \beta, \gamma \in \Gamma$.

Example 66 *Let $H = \{1, 2, 3, 4, 5\}$ and $\Gamma = \{\gamma, \delta\}$ be two non-empty sets. Then, H is a Γ-semihypergroup and is defined by the following Cayley tables.*

γ	1	2	3	4	5
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
3	$\{1\}$	$\{1\}$	$\{3\}$	$\{3\}$	$\{3\}$
4	$\{1\}$	$\{1\}$	$\{3,4\}$	$\{3,4\}$	$\{5\}$
5	$\{1\}$	$\{1\}$	$\{3,4\}$	$\{3,4\}$	$\{5\}$

δ	1	2	3	4	5
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
3	$\{1\}$	$\{1\}$	$\{3\}$	$\{3\}$	$\{3\}$
4	$\{1\}$	$\{1\}$	$\{3\}$	$\{3,4\}$	$\{5\}$
5	$\{1\}$	$\{1\}$	$\{3\}$	$\{3,4\}$	$\{5\}$

1) Let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS in a Γ-semihypergroup H and is defined by $\mu_A(1) = 0.7$, $\mu_A(2) = 0.5$, $\mu_A(3) = \mu_A(4) = \mu_A(5) = 0.3$, and $\lambda_A(1) = 0.2$, $\lambda_A(2) = 0.4$, $\lambda_A(3) = \lambda_A(4) = \lambda_A(5) = 0.6$. Then, $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy Γ-hyperideal of H.

2) Let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS in a Γ-semihypergroup H and is defined by $\mu_A(1) = 0.85$, $\mu_A(2) = 0.7$, $\mu_A(3) = 0.5$, $\mu_A(4) = \mu_A(5) = 0.3$, and $\lambda_A(1) = 0.1$, $\lambda_A(2) = 0.25$, $\lambda_A(3) = 0.45$, $\lambda_A(4) = \lambda_A(5) = 0.7$. Then, $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy bi-Γ-hyperideal of H.

3) Let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS in a Γ-semihypergroup H and is defined by $\mu_A(1) = 0.9$, $\mu_A(2) = 0.4$, $\mu_A(3) = 0.7$, $\mu_A(4) = 0.6$, $\mu_A(5) = 0.3$, and $\lambda_A(1) = 0.1$, $\lambda_A(2) = 0.5$, $\lambda_A(3) = 0.2$, $\lambda_A(4) = 0.3$ $\lambda_A(5) = 0.6$. Then, $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy (1,2)-Γ-hyperideal of H.

For any $t \in [0, 1]$ and an intuitionistic fuzzy set $A = \langle \mu_A, \lambda_A \rangle$ of H, the sets [14]

$$U(\mu_A; t) = \{x \in H | \mu_A(x) \geq t\} \text{ and } L(\lambda_A; t) = \{x \in H | \lambda_A(x) \leq t\}.$$

are called respectively an upper and lower t-level cut of $A = \langle \mu_A, \lambda_A \rangle$.

Some interesting results concerned with intuitionistic fuzzy Γ-hyperideals of Γ-semihypergroups are obtained in [28]. We continue this study with further results.

Proposition 67 *Let H be a Γ-semihypergroup. Then a non-empty subset B of H is a bi-Γ-hyperideal of H if and only if the intuitionistic fuzzy characteristic function $\widetilde{B} = \langle \chi_B, \chi_B^c \rangle$ is an intuitionistic fuzzy bi-Γ-hyperideal of H.*

Proposition 68 *Let $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy bi-Γ-hyperideal and $B = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy sub-Γ-semihypergroup of a Γ-semihypergroup H. Then, $A \cap B$ is an intuitionistic fuzzy bi-Γ -hyperideal of H.*

Proof. The proof is straightforward and we omit it. ■

Proposition 69 *Let $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy right Γ-hyperideal of H and $B = \langle \mu_B, \lambda_B \rangle$ be an intuitionistic fuzzy left Γ-hyperideal of H. Then, $A * B \subseteq A \cap B$.*

Proof. Let $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy right Γ-hyperideal of H and $B = \langle \mu_B, \lambda_B \rangle$ be an intuitionistic fuzzy left Γ-hyperideal of H, let $x \in H$. Let us suppose that there exist $u, v \in H$ and $\gamma \in \Gamma$ such that $x \in u\gamma v$. Then,

$$\begin{aligned}
(\mu_A * \mu_B)(x) &= \sup_{x \in u\gamma v} \min\{\mu_A(u), \mu_B(v)\} \\
&\leq \sup_{x \in u\gamma v} \min\{\inf_{x \in u\gamma v} \mu_A(x), \inf_{x \in u\gamma v} \mu_B(x)\} \\
&= \min\{\mu_A(x), \mu_B(x)\} = (\mu_A \cap \mu_B)(x)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_A * \lambda_B)(x) &= \inf_{x \in u\gamma v} \max\{\lambda_A(u), \lambda_B(v)\} \\
&\geq \inf_{x \in u\gamma v} \max\{\sup_{x \in u\gamma v} \lambda_A(x), \sup_{x \in u\gamma v} \lambda_B(x)\} \\
&= \max\{\lambda_A(x), \lambda_B(x)\} = (\lambda_A \cup \lambda_B)(x).
\end{aligned}$$

Let us suppose there do not exist $u, v \in H$ such that $x \in u\gamma v$. Then $(\mu_A * \mu_B)(x) = 0 \leq (\mu_A \cap \mu_B)(x)$ and $(\lambda_A * \lambda_B)(x) = 1 \geq (\lambda_A \cup \lambda_B)(x)$. Hence, the proof is completed. ■

By the above proposition and the definition of $A \cap B$ the following proposition follows.

Proposition 70 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle, B = \langle \mu_B, \lambda_B \rangle$ be intuitionistic fuzzy Γ-hyperideals of H. Then, $A * B \subseteq A \cap B \subseteq A, B$.*

Proposition 71 *Let H be a regular Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ and $B = \langle \mu_B, \lambda_B \rangle$ be two intuitionistic fuzzy sets in H. Then, $A * B \supseteq A \cap B$.*

Proof. Let $c \in H$. Since H is regular, so there exists an element $x \in H$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $c \in c\gamma_1 x \gamma_2 c \subseteq c\gamma c$ for some $\gamma \in \Gamma$. Then

$$\begin{aligned}(\mu_A * \mu_B)(c) &= \sup_{c \in c\gamma c}\{\min\{\mu_A(c), \mu_B(c)\}\} \\ &\geq \min\{\mu_A(c), \mu_B(c)\} = (\mu_A \cap \mu_B)(c)\end{aligned}$$

and

$$\begin{aligned}(\lambda_A * \lambda_B)(c) &= \inf_{c \in c\gamma c}\{\max\{\lambda_A(c), \lambda_B(c)\}\} \\ &\leq \max\{\lambda_A(c), \lambda_B(c)\} = (\lambda_A \cup \lambda_B)(c).\end{aligned}$$

Hence, $A * B \supseteq A \cap B$. ■

Theorem 72 *Let H be a Γ-semihypergroup. Then, the following statements are equivalent:*

1. *H is regular.*

2. *$A * B = A \cap B$, where $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy right Γ-hyperideal of H and $B = \langle \mu_B, \lambda_B \rangle$ is an intuitionistic fuzzy left Γ-hyperideal of H.*

Proof. Let H be a regular Γ-semihypergroup. Then, by [28, Theorem 3.13], $A * B \supseteq A \cap B$ and by [28, Theorem 3.12], $A * B \subseteq A \cap B$. Hence, $A * B = A \cap B$.

Conversely, let H be a Γ-semihypergroup and $A * B = A \cap B$, where $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy right Γ-hyperideal of H and $B = \langle \mu_B, \lambda_B \rangle$ be an intuitionistic fuzzy left Γ-hyperideal of H. Let L and R be a left Γ-hyperideal and a right Γ-hyperideal of H, respectively. Let $x \in R \cap L$. Then, $x \in R$ and $x \in L$. Hence, $(\chi_L(x), \chi_L^c(x)) = (\chi_R(x), \chi_R^c(x)) = (1, 0)$ (where $\chi_L(x)$ and $\chi_R(x)$ are respectively the characteristic functions of L and R). Thus,

$(\chi_R \cap \chi_L)(x) = \min\{\chi_R(x), \chi_L(x)\} = 1$ and $(\chi_R^c \cup \chi_L^c)(x) = \max\{\chi_R^c(x), \chi_L^c(x)\} = 0$.

By [28, Theorem 3.11], (χ_L, χ_L^c) and (χ_R, χ_R^c) are respectively an intuitionistic fuzzy left Γ-hyperideal and an intuitionistic fuzzy right Γ-hyperideal of H. Hence, by hypothesis, $\chi_R * \chi_L = \chi_R \cap \chi_L$ and $\chi_R^c * \chi_L^c = \chi_R^c \cup \chi_L^c$. Hence,

$(\chi_R * \chi_L)(x) = 1$

since $\sup\limits_{x \in y\gamma z} [\min\{\chi_R(y), \chi_L(z)\} : y, z \in H, \gamma \in \Gamma] = 1$, and

$(\chi_R^c * \chi_L^c)(x) = 0$

since $\inf\limits_{x \in y\gamma z} [\max\{\chi_R^c(y), \chi_L^c(z)\} : y, z \in H, \gamma \in \Gamma] = 0$.

This implies that there exist some $r, s \in H$ and $\gamma_1 \in \Gamma$ such that $x \in r\gamma_1 s$ and $(\chi_R(r), \chi_R^c(r)) = (1, 0) = (\chi_L(s), \chi_L^c(s))$. Hence, $r \in R$ and $s \in L$. Hence, $x \in R\Gamma L$. Thus, $R \cap L \subseteq R\Gamma L$. Also, $R\Gamma L \subseteq R \cap L$. Hence, $R\Gamma L = R \cap L$. Consequently, the Γ-semihypergroup H is regular. ■

Theorem 73 *Let H be a Γ-semihypergroup. Then, every intuitionistic fuzzy bi-Γ-hyperideal of H is an intuitionistic fuzzy $(1, 2)$-Γ-hyperideal of H.*

Proof. Let $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy bi-Γ-hyperideal of H and let $w, x, y, z \in H$ and $\alpha, \beta, \gamma \in \Gamma$. Then, for all $a \in x\alpha w\beta(y\gamma z)$, we have

$$\begin{aligned}
\inf_{a \in x\alpha w\beta(y\gamma z)} \{\mu_A(a)\} &= \inf_{a \in (x\alpha w\beta y)\gamma z} \{\mu_A(a)\} \\
&= \inf_{a \in c\gamma z} \{\mu_A(a)\} \text{ for every } c \in x\alpha w\beta y \\
&\geq \min\{\mu_A(c), \mu_A(z)\} \\
&\geq \min\{\min\{\mu_A(x), \mu_A(y)\}, \mu_A(z)\} \\
&\quad (\text{ because } \inf_{c \in x\alpha w\beta y} \{\mu_A(c)\} \geq \min\{\mu_A(x), \mu_A(y)\}) \\
&= \min\{\mu_A(x), \mu_A(y), \mu_A(z)\}
\end{aligned}$$

and

$$\begin{aligned}
\sup_{a \in x\alpha w\beta(y\gamma z)} \{\lambda_A(a)\} &= \sup_{a \in (x\alpha w\beta y)\gamma z} \{\lambda_A(a)\} \\
&= \sup_{a \in c\gamma z} \{\mu_A(a)\} \text{ for every } c \in x\alpha w\beta y \\
&\leq \max\{\lambda_A(c), \lambda_A(z)\} \\
&\leq \max\{\max\{\lambda_A(x), \lambda_A(y)\}, \lambda_A(z)\} \\
&\quad (\text{ because } \sup_{c \in x\alpha w\beta y} \{\lambda_A(c)\} \leq \max\{\lambda_A(x), \lambda_A(y)\}) \\
&= \max\{\lambda_A(x), \lambda_A(y), \lambda_A(z)\}.
\end{aligned}$$

Hence, $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy $(1, 2)$-Γ-hyperideal of H. ■

Theorem 74 *Let H be a regular Γ-semihypergroup. Then, every intuitionistic fuzzy $(1, 2)$-Γ-hyperideal of H is an intuitionistic fuzzy bi-Γ-hyperideal of H.*

Proof. Let us assume that a Γ-semihypergroup H is regular and let $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy $(1, 2)$-Γ-hyperideal of H. Let $w, x, y \in H$ and $\alpha, \beta, \gamma \in \Gamma$. Then, since H is regular, so we have for every $w \in x\gamma y \subseteq (x\alpha a\beta x)\gamma y = x\alpha(a\beta x)\gamma y$ for some

$a \in H$. Thus, for every $c \in a\beta x, w \in x\alpha c\gamma y$, we have

$$\begin{aligned}\inf_{w\in x\alpha c\gamma y\subseteq x\alpha(a\beta x)\gamma y}\{\mu_A(w)\} &\geq \min\{\mu_A(x),\mu_A(x),\mu_A(y)\}\\ &= \min\{\mu_A(x),\mu_A(y)\}\end{aligned}$$

and

$$\begin{aligned}\sup_{w\in x\alpha c\gamma y\subseteq x\alpha(a\beta x)\gamma y}\{\lambda_A(w)\} &\leq \max\{\lambda_A(x),\lambda_A(x),\lambda_A(y)\}\\ &= \max\{\lambda_A(x),\lambda_A(y)\}.\end{aligned}$$

Hence, $A = \langle\mu_A, \lambda_A\rangle$ is an intuitionistic fuzzy bi-Γ-hyperideal of H. ■

Theorem 75 *Let H be a regular Γ-semihypergroup. Then, the following statements hold:*

1. *$A \cap B \subseteq A * B$ for every intuitionistic fuzzy bi-Γ-hyperideal A and intuitionistic fuzzy left Γ-hyperideal B of H.*
2. *$C \cap A \cap B \subseteq C * A * B$, for every intuitionistic fuzzy bi-Γ-hyperideal A , intuitionistic fuzzy left Γ-hyperideal B and intuitionistic fuzzy right Γ-hyperideal C of H, respectively.*

Proof. (1). Let us suppose that H is regular and $a \in H$. Then, there exists $x \in H$, and $\alpha, \beta \in \Gamma$ such that $a \in a\alpha x\beta a$. Then, $(A * B)(a) \neq (0, 1)$. Thus,

$$\begin{aligned}\mu_{A*B}(a) &= \sup_{a\in(a\alpha x\beta a)}\min\{\mu_A(a), \inf_{t\in x\beta a}\mu_B(t)\}\\ &\geq \min\{\mu_A(a), \inf_{t\in x\beta a}\mu_B(t)\}\\ &\geq \min\{\mu_A(a),\mu_B(a)\}\\ &= \mu_{A\cap B}(a)\end{aligned}$$

and

$$\begin{aligned}
\lambda_{A*B}(a) &= \inf_{a\in(a\alpha x\beta a)} \max\{\lambda_A(a), \sup_{t\in x\beta a} \lambda_B(t)\} \\
&\leq \max\{\lambda_A(a), \sup_{t\in x\beta a} \lambda_B(t)\} \\
&\leq \max\{\lambda_A(a), \lambda_B(a)\} \\
&= \lambda_{A\cap B}(a).
\end{aligned}$$

Hence, $A \cap B \subseteq A * B$.

(2). Let us suppose that H is regular and $a \in H$. Then there exists $x \in H$, and $\alpha, \beta \in \Gamma$ such that $a \in a\alpha x\beta a$. Then

$$\begin{aligned}
\mu_{C*A*B}(a) &= \sup_{a\in(a\alpha x\beta a)} \min\{\inf_{t\in a\alpha x} \mu_C(t), \mu_{A*B}(a)\} \\
&\geq \min\{\inf_{t\in a\alpha x} \mu_C(t), \mu_{A*B}(a)\} \\
&\geq \min\{\mu_C(a), \sup_{a\in a\alpha x\beta a} \min\{\mu_A(a), \inf_{h\in x\beta a} \mu_B(h)\}\} \\
&\geq \min\{\mu_C(a), \mu_A(a), \inf_{h\in x\beta a} \mu_B(h)\} \\
&\geq \mu_{C\cap A\cap B}(a)
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{C*A*B}(a) &= \inf_{a\in(a\alpha x\beta a)} \max\{\sup_{t\in a\alpha x} \lambda_C(t), \lambda_{A*B}(a)\} \\
&\leq \max\{\sup_{t\in a\alpha x} \lambda_C(t), \lambda_{A*B}(a)\} \\
&\leq \max\{\lambda_C(a), \inf_{a\in a\alpha x\beta a} \max\{\lambda_A(a), \sup_{h\in x\beta a} \lambda_B(h)\}\} \\
&\leq \max\{\lambda_C(a), \lambda_A(a), \sup_{h\in x\beta a} \lambda_B(h)\} \\
&\leq \lambda_{C\cap A\cap B}(a)
\end{aligned}$$

Hence, $C \cap A \cap B \subseteq C * A * B$. ■

Theorem 76 *Let H be a completely regular Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an* *intuitionistic fuzzy bi-*Γ*-hyperideal of* H*. Then, for every* $r \in a\gamma a$*, we have* $A(a) = A(r)$ *for all* $a \in H, \gamma \in \Gamma$*.*

Proof. Let $a \in H, \gamma \in \Gamma$. Then, there exist $x \in H, \alpha, \beta, \rho, \delta \in \Gamma$ such that $a \in a\alpha a\beta x\rho a\delta a$. Hence, for every $r \in a\gamma a, a \in r\beta x\rho r$. Hence,

$$\begin{aligned}
\inf_{a \in r\beta x\rho r}\{\mu_A(a)\} &\geq \min\{\mu_A(r), \mu_A(r)\} \\
&= \mu_A(r) \geq \min\{\mu_A(a), \mu_A(a)\} \text{ because } r \in a\gamma a, \forall \gamma \in \Gamma \\
&= \mu_A(a)
\end{aligned}$$

and

$$\begin{aligned}
\sup_{a \in r\beta x\rho r}\{\lambda_A(a)\} &\leq \max\{\lambda_A(r), \lambda_A(r)\} \\
&= \lambda_A(r) \leq \max\{\lambda_A(a), \lambda_A(a)\} \text{ because } r \in a\gamma a, \forall \gamma \in \Gamma \\
&= \lambda_A(a).
\end{aligned}$$

It follows that $\mu_A(a) = \mu_A(r)$ and $\lambda_A(a) = \lambda_A(r)$, so that $A(a) = A(r)$ for all $a \in H$. ■

Theorem 77 *Let* H *be an intra-regular* Γ*-semihypergroup and* $A = \langle \mu_A, \lambda_A \rangle$ *be an intuitionistic fuzzy* Γ*-hyperideal of* H*. Then, for every* $r \in a\gamma a$*, we have* $A(a) = A(r)$ *for all* $a \in H, \gamma \in \Gamma$*.*

Proof. Let $a \in H, \gamma \in \Gamma$. Then, since H is intra-regular, so there exist $x, y \in H, \alpha, \beta, \delta \in \Gamma$ such that $a \in x\alpha a\beta a\delta y$. Then, for every $r \in a\gamma a$ and $s \in x\alpha r$ we have $a \in s\delta y$. Hence, since $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy Γ-hyperideal of H, so

$\inf\limits_{a \in s\delta y}\{\mu_A(a)\} \geq \mu_A(s)$

and

$\inf\limits_{s \in x\alpha r}\{\mu_A(s)\} \geq \mu_A(r)$

and

$\inf\limits_{r \in a\gamma a}\{\mu_A(r)\} \geq \mu_A(a)$.

Hence $\mu_A(a) = \mu_A(r)$.

Also

$\sup\limits_{a \in s\delta y} \{\lambda_A(a)\} \leq \lambda_A(s)$

and

$\sup\limits_{s \in x\alpha r} \{\lambda_A(s)\} \leq \lambda_A(r)$

and

$\sup\limits_{r \in a\gamma a} \{\lambda_A(r)\} \leq \lambda_A(a)$.

Hence, $\lambda_A(a) = \lambda_A(r)$. Thus, $A(a) = A(r)$ for all $a \in H$. ■

Theorem 78 *Let H be a Γ-semihypergroup. Then, the following statements are equivalent:*

1. *H is left quasi-regular*
2. *$A * A = A$, where $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy left Γ-hyperideal of H.*

Proof. Let H be a left quasi-regular Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy left Γ-hyperideal of H. Then, it is clear that $A * A \subseteq A$. Let $a \in H$. Then, by definition of left quasi-regular, there exist $x, y \in H, \alpha, \beta, \gamma \in \Gamma$ such that $a \in x\alpha a\beta y\gamma a$. We have

$$\begin{aligned} \mu_{A*A}(a) &= \sup_{a \in (x\alpha a\beta y\gamma a)} \min\{\inf_{t \in x\alpha a} \mu_A(t), \inf_{s \in y\gamma a} \mu_A(s)\} \\ &\geq \min\{\inf_{t \in x\alpha a} \mu_A(t), \inf_{s \in y\gamma a} \mu_A(s)\} \\ &\geq \min\{\mu_A(a), \mu_A(a)\} \\ &= \mu_A(a) \end{aligned}$$

and

$$\begin{aligned}
\lambda_{A*A}(a) &= \inf_{a\in(x\alpha a\beta y\gamma a)} \max\{\sup_{t\in x\alpha a}\lambda_A(t), \sup_{s\in y\gamma a}\lambda_A(s)\} \\
&\leq \max\{\sup_{t\in x\alpha a}\lambda_A(t), \sup_{s\in y\gamma a}\lambda_A(s)\} \\
&\leq \max\{\lambda_A(a), \lambda_A(a)\} \\
&= \lambda_A(a)
\end{aligned}$$

This implies that $A \subseteq A * A$. Therefore, we have $A * A = A$.

Conversely, let us suppose that the condition (2) holds and let T be a left Γ-hyperideal of H. Then (χ_T, χ_T^c) is an intuitionistic fuzzy left Γ-hyperideal of H. Indeed, (χ_T, χ_T^c) is an intuitionistic fuzzy left Γ-hyperideal of H if and only if for any $x, y \in H, \alpha \in \Gamma$, $\inf\limits_{t\in x\alpha y}\chi_T(t) \geq \chi_T(y)$ and $\sup\limits_{s\in x\alpha y}\chi_T^c(s) \leq \chi_T^c(y)$ if and only if for every $x \in H, y \in T, \alpha \in \Gamma$, $x\alpha y \subseteq T$ if and only if L is a left Γ-hyperideal of H. Let $a \in T$. Then, by (2), we have $(\chi_T, \chi_T^c) * (\chi_T, \chi_T^c) = (\chi_T, \chi_T^c)$. Thus,

$\mu_{[(\chi_T,\chi_T^c)*(\chi_T,\chi_T^c)]}(a) = \chi_T(a) = 1$

and

$\lambda_{[(\chi_T,\chi_T^c)*(\chi_T,\chi_T^c)]}(a) = \chi_T(a) = 0.$

So, $[(\chi_T, \chi_T^c) * (\chi_T, \chi_T^c)](a) \neq (0, 1)$. This implies that $\sup\limits_{a\in x\alpha y}\min\{\chi_T(x), \chi_T(y)\} = 1$ and $\inf\limits_{a\in x\alpha y}\min\{\chi_T^c(x), \chi_T^c(y)\} = 0$. Thus, there exist $b, d \in H, \alpha \in \Gamma$ with $a \in b\alpha d$ such that $\chi_T(b) = 1, \chi_T^c(b) = 0$ and $\chi_T(d) = 1, \chi_T^c(d) = 0$. So we have $b \in T$, that is $a \in b\alpha d \subseteq T\Gamma T$. Then, $T \subseteq T\Gamma T$ and since clearly $T\Gamma T \subseteq T$, we have $T = T\Gamma T$. Now, we prove that this fact implies H is left quasi-regular. We have

$$\begin{aligned}
T &\subseteq L(T) = L(T)\Gamma L(T) = (T \cup H\Gamma T)\Gamma(T \cup H\Gamma T) \\
&= T\Gamma T \cup H\Gamma T\Gamma T \cup T\Gamma H\Gamma T \cup H\Gamma T\Gamma H\Gamma T \\
&\subseteq (T\Gamma H\Gamma T \cup H\Gamma T\Gamma H\Gamma T) \cup (H\Gamma T\Gamma H\Gamma T) \cup (T\Gamma H\Gamma T \cup H\Gamma T\Gamma H\Gamma T) \\
&\subseteq T\Gamma H\Gamma T \cup H\Gamma T\Gamma H\Gamma T \\
&\subseteq H\Gamma T\Gamma H\Gamma T,
\end{aligned}$$

where $L(T)$ is the left Γ-hyperideal generated by T. This shows that H is left quasi-regular. ■

The following theorem characterizes the Γ-semihypergroups that are intra-regular and left quasi-regular through intuitionistic fuzzy sets.

Theorem 79 *Let H be a Γ-semihypergroup. If H is both intra-regular and left quasi-regular, then $B \cap C \cap A \subseteq B * C * A$, for every A intuitionistic fuzzy bi-Γ-hyperideal, B intuitionistic fuzzy left Γ-hyperideal and C intuitionistic fuzzy right Γ-hyperideal of H, respectively.*

Proof. Let us suppose that H is intra-regular and left quasi-regular. Let A be an intuitionistic fuzzy bi-Γ-hyperideal, B an intuitionistic fuzzy left Γ-hyperideal and C be an intuitionistic fuzzy right Γ-hyperideal of H, respectively. Let $a \in H$. Then, since H is left quasi-regular, so there exist $u, v \in H, \alpha, \beta, \gamma \in \Gamma$ such that $a \in u\alpha a\beta v\gamma a$. Then, for some $x, y \in H, \delta, \rho, \omega \in \Gamma$ we have

$a \in u\alpha a\beta v\gamma a \subseteq u\alpha(x\delta a\rho a\omega y)\beta v\gamma a \subseteq ((u\alpha x)\delta a)\rho((a\omega(y\beta v)\gamma a)$.

Thus, we have

$$\begin{aligned}
\mu_{B*C*A}(a) &= \sup_{a\in(p\psi q)} \min\{\mu_B(p), \mu_{C*A}(q)\} \\
&\geq \min\{\inf_{t\in((u\alpha x)\delta a)} \mu_B(t), \inf_{s\in((a\omega(y\beta v)\gamma a)} \mu_{C*A}(s)\} \\
&\geq \min\{\mu_B(a), \sup_{((a\omega(y\beta v)\gamma a)\subseteq p\psi q} [\mu_C(p), \mu_A(q)]\} \\
&\geq \min\{\mu_B(a), \inf_{h\in(a\omega(y\beta v))} \mu_C(h), \mu_A(a)\} \\
&\geq \min\{\mu_B(a), \mu_C(a), \mu_A(a)\} \\
&= \mu_{B\cap C\cap A}(a)
\end{aligned}$$

and

$$
\begin{aligned}
\lambda_{B*C*A}(a) &= \inf_{a\in(p\psi q)} \max\{\lambda_B(p), \lambda_{C*A}(q)\} \\
&\leq \max\{\sup_{t\in((u\alpha x)\delta a)} \mu_B(t), \sup_{s\in((a\omega(y\beta v)\gamma a)} \mu_{C*A}(s)\} \\
&\leq \max\{\lambda_B(a), \inf_{((a\omega(y\beta v)\gamma a)\subseteq p\psi q} [\lambda_C(p), \lambda_A(q)]\} \\
&\leq \max\{\lambda_B(a), \sup_{h\in(a\omega(y\beta v))} \lambda_C(h), \lambda_A(a)\} \\
&\leq \max\{\lambda_B(a), \lambda_C(a), \lambda_A(a)\} \\
&= \lambda_{B\cap C\cap A}(a).
\end{aligned}
$$

Hence, $C \cap B \cap A \subseteq C * B * A$. ■

Theorem 80 *Let H_1 be a Γ_1-semihypergroup and H_2 be a Γ_2-semihypergroup. Let (Φ, f) be a homomorphism from H_1 to H_2. If $B = \langle \mu_B, \lambda_B \rangle$ is an intuitionistic fuzzy bi-Γ-hyperideal of H_2, then $\Phi^{-1}(B)$ is an intuitionistic fuzzy bi-Γ-hyperideal of H_1.*

Proof. Let us suppose that $B = \langle \mu_B, \lambda_B \rangle$ is an intuitionistic fuzzy bi-Γ-hyperideal of H_2 and let $x, y \in H_1, \alpha \in \Gamma_1$. Then, we have

$$
\begin{aligned}
\inf_{z\in x\alpha y}\{\Phi^{-1}(\mu_B)(z)\} &= \inf_{z\in x\alpha y}\{\mu_B(\Phi(z)\} \\
&\geq \inf_{\Phi(z)\in\Phi(x\alpha y)}\{\mu_B(\Phi(z))\} \\
&\geq \inf_{\Phi(z)\in\Phi(x)f(\alpha)\Phi(y)}\{\mu_B(\Phi(z))\} \\
&\geq \min\{\mu_B(\Phi(x)), \mu_B(\Phi(y))\} \\
&= \min\{\Phi^{-1}(\mu_B(x)), \Phi^{-1}(\mu_B(y)\},
\end{aligned}
$$

and

$$\begin{aligned}
\sup_{z\in x\alpha y}\{\Phi^{-1}(\lambda_B)(z)\} &= \sup_{z\in x\alpha y}\{\lambda_B(\Phi(z)\} \\
&\leq \sup_{\Phi(z)\in\Phi(x\alpha y)}\{\lambda_B(\Phi(z))\} \\
&\leq \sup_{\Phi(z)\in\Phi(x)f(\alpha)\Phi(y)}\{\lambda_B(\Phi(z))\} \\
&\leq \max\{\lambda_B(\Phi(x)),\lambda_B(\Phi(y))\} \\
&= \max\{\Phi^{-1}(\lambda_B(x)),\Phi^{-1}(\lambda_B(y)\}.
\end{aligned}$$

Therefore, $\Phi^{-1}(B)$ is an intuitionistic fuzzy sub-Γ-semihypergroup of H. For any $a, x, y \in H, \alpha, \beta \in \Gamma$, we have

$$\begin{aligned}
\inf_{z\in x\alpha a\beta y}\{\Phi^{-1}(\mu_B)(z)\} &= \inf_{z\in x\alpha a\beta y}\{\mu_B(\Phi(z)\} \\
&\geq \inf_{\Phi(z)\in\Phi(x\alpha a\beta y)}\{\mu_B(\Phi(z))\} \\
&\geq \inf_{\Phi(z)\in\Phi(x)f(\alpha)\Phi(a)f(\beta)\Phi(y)}\{\mu_B(\Phi(z))\} \\
&\geq \min\{\mu_B(\Phi(x)),\mu_B(\Phi(y))\} \\
&= \min\{\Phi^{-1}(\mu_B(x)),\Phi^{-1}(\mu_B(y)\},
\end{aligned}$$

and

$$\begin{aligned}
\sup_{z\in x\alpha a\beta y}\{\Phi^{-1}(\lambda_B)(z)\} &= \sup_{z\in x\alpha a\beta y}\{\lambda_B(\Phi(z)\} \\
&\leq \sup_{\Phi(z)\in\Phi(x\alpha a\beta y)}\{\lambda_B(\Phi(z))\} \\
&\leq \sup_{\Phi(z)\in\Phi(x)f(\alpha)\Phi(a)f(\beta)\Phi(y)}\{\lambda_B(\Phi(z))\} \\
&\leq \max\{\lambda_B(\Phi(x)),\lambda_B(\Phi(y))\} \\
&= \max\{\Phi^{-1}(\lambda_B(x)),\Phi^{-1}(\lambda_B(y)\},
\end{aligned}$$

Therefore, $\Phi^{-1}(B)$ is an intuitionistic fuzzy bi-Γ-hyperideal of H_1. ■

4.3 Γ-semihypergroups Characterized by their Intuitionistic Fuzzy Prime (Semiprime) Γ-hyperideals

In this section, we define intuitionistic fuzzy prime(semiprime) Γ-hyperideals, intuitionistic fuzzy M-hypersystem and N-hypersystem of a Γ-semihypergroup and intuitionistic fuzzy semisimple Γ-semihypergroups and some properties of them are investigated.

Let $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy Γ-hyperideal in H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is called an *intuitionistic fuzzy prime* Γ*-hyperideal* of H if for all intuitionistic fuzzy Γ-hyperideals $B = \langle \mu_B, \lambda_B \rangle$ and $C = (\mu_C, \lambda_C)$ of H, such that $B * C \subseteq A$ this implies $B \subseteq A$ or $C \subseteq A$. Let $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy Γ-hyperideal in H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is called an *intuitionistic fuzzy semiprime* Γ*-hyperideal* of H if for all intuitionistic fuzzy Γ-hyperideals $B = \langle \mu_B, \lambda_B \rangle$ of H, such that $B^2 \subseteq A$ this implies $B \subseteq A$. An intuitionistic fuzzy Γ-hyperideal $A = \langle \mu_A, \lambda_A \rangle$ of H is called idempotent if $A^2 = A$ and a Γ-semihypergroup is called *intuitionistic fuzzy semisimple* if all of its intuitionistic fuzzy Γ-hyperideals are idempotent.

Proposition 81 *If $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy left Γ-hyperideal and $B = \langle \mu_B, \lambda_B \rangle$ is an intuitionistic fuzzy right Γ-hyperideal of a Γ-semihypergroup H, then $A*B$ is an intuitionistic fuzzy Γ-hyperideal of H.*

Proof. Let x, y and $z \in H$. Then,

$$\begin{aligned}
(\mu_A * \mu_B)(x) &= \bigvee_{x \in y\beta z} \{\mu_A(y) \wedge \mu_B(z)\} \\
&\leq \bigvee_{x \in y\beta z} \{\mu_A(\alpha) \wedge \mu_B(z)\}, \text{ for each } \alpha \in h\gamma y, \inf_{\alpha \in h\gamma y} \lambda(\alpha) \geq \lambda(\alpha) \\
&\leq \bigvee_{y' \in h\delta x \subseteq (h\gamma y)\delta z} \{\mu_A(y') \wedge \mu_B(z')\}, \text{ where } \alpha \in h\gamma y,\ y' \in h\delta x \text{ and } z = z' \\
&\leq \inf_{y' \in h\delta x} (\mu_A * \mu_B)(y').
\end{aligned}$$

and

$$
\begin{aligned}
(\lambda_A * \lambda_B)(x) &= \bigwedge_{x \in y\beta z} \{\lambda_A(y) \vee \lambda_B(z)\} \\
&\geq \bigwedge_{x \in y\beta z} \{\lambda_A(\alpha) \vee \lambda_B(z)\}, \text{ for each } \alpha \in h\gamma y, \inf_{\alpha \in h\gamma y} \lambda(\alpha) \leq \lambda(\alpha) \\
&\geq \bigwedge_{y' \in h\delta x \subseteq (h\gamma y)\delta z} \{\lambda_A(y') \vee \lambda_B(z')\}, \text{ where } \alpha \in h\gamma y, \ y' \in h\delta x \text{ and } z = z' \\
&\geq \sup_{y' \in h\delta x} (\lambda_A * \lambda_B)(y').
\end{aligned}
$$

Similarly,

$$
(\lambda * \mu)(x) \leq \inf_{y' \in x\delta h} (\lambda * \mu)(y') \text{ and } (\lambda_A * \lambda_B)(x) \leq \sup_{y' \in x\delta h} (\lambda_A * \lambda_B)(y').
$$

Hence, $A * B$ is an intuitionistic fuzzy Γ-hyperideal of H. ■

Proposition 82 *Let $c(t,s)$ be an intuitionistic fuzzy point of a Γ-semihypergroup H. Then, the intuitionistic fuzzy left (right) Γ-hyperideal of H generated by $c(t,s)$ is denoted by $l_{c(t,s)}(\xi_{c(t,s)})$ and defined as:*

$$
l_{c(t,s)}(x) = \begin{cases} (t,s) & \text{if } x \in H\Gamma a \\ (0,1) & \text{if } x \notin H\Gamma a \end{cases}
$$

and

$$
\xi_{c(t,s)}(x) = \begin{cases} (t,s) & \text{if } x \in a\Gamma H \\ (0,1) & \text{if } x \notin a\Gamma H. \end{cases}
$$

Proof. For $x, y \in H$ if $x\gamma y \subseteq H\Gamma a$ for each $\gamma \in \Gamma$, then for each $a \in x\gamma y$, $l_{c(t,s)}(a) = (t,s) \supseteq l_{c(t,s)}(y)$. If $x\gamma y \nsubseteq H\Gamma a$, then $y \notin H$ so for all $a \in x\gamma y$, $l_{c(t,s)}(a) = (0,1) = l_{c(t,s)}(y)$. Hence, in any case, $l_{c(t,s)}(a) \supseteq l_{c(t,s)}(y)$ for each $a \in x\gamma y$. So, $l_{c(t,s)}$ is an intuitionistic fuzzy left Γ-hyperideal of H. By definition of $l_{c(t,s)}$, we find that $c(t,s) \subseteq l_{c(t,s)}$. If $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy left Γ-hyperideal of H containing $c(t,s)$ and if $x \in H\Gamma a$, then as $(t,s) = c(t,s)(a) \subseteq A(a)$ implies that $(t,s) \subseteq A(a) \subseteq A(b)$, for $b \in h\gamma a$ and $h \in H$ which implies that $A(x) \supseteq (t,s) = l_{c(t,s)}(x)$. If $x \notin H\Gamma a$, then $l_{c(t,s)}(x) = (0,1) \subseteq A(x)$. So, $l_{c(t,s)} \subseteq A$. Thus, $l_{c(t,s)}$ is an intuitionistic fuzzy left Γ-hyperideal of H generated by $c(t,s)$. Similarly, we can prove $\xi_{c(t,s)}(x)$ is an intuitionistic fuzzy right Γ-hyperideal of H. ■

Proposition 83 *$l_{c(t,s)} * H$ and $H * \xi_{c(t,s)}$ are intuitionistic fuzzy Γ-hyperideals of H generated by $c(t,s)$, where $H = (\chi_H, \chi_H^c)$ and χ_H is the characteristic function of H.*

Proof. Straighforward. ■

Theorem 84 *An intuitionistic fuzzy sets $A = \langle \mu_A, \lambda_A \rangle$ of a Γ-semihypergroup H is an intuitionistic fuzzy left(resp. right) Γ-hyperideal of H if and only if it satisfies for all $x, y \in H$ and $t \in (0,1]$ and $s \in [0,1)$, $y(t,s) \in A \Longrightarrow (z)(t,s) \in A$ (resp. $(z)(t,s) \in A$ for each $z \in x\gamma y$.*

Proof. Assume that $x \in H$ and $y(t,s) \in A$, where $t \in (0,1]$ and $s \in [0,1)$. Then, $\mu_A(y) \geq t$ and $\lambda_A(y) \leq s$. It follows from definition 63 that

$$\begin{aligned} \inf_{z \in x\gamma y} \mu_A(z) &\geq \mu_A(y) \geq t \Rightarrow \inf_{z \in x\gamma y} \mu_A(z) \geq t \\ \sup_{z \in x\gamma y} \lambda_A(xy) &\leq \lambda_A(y) \leq s \Rightarrow \sup_{z \in x\gamma y} \lambda_A(xy) \leq s \end{aligned}$$

Thus, $(z)(t,s) \in A$ for each $z \in x\gamma y$.

$(b) \Longrightarrow (a)$ Suppose that (b) is valid. Note that $x \in H$ and $y(\mu_A(y), \lambda_A(y)) \in A$, then by (b) we have $(z)(\mu_A(x), \lambda_A(y)) \in A$ for each $z \in x\gamma y$ implies that $\mu_A(z) \geq \mu_A(y)$ and $\lambda_A(z) \leq \lambda_A(y)$ for each $z \in x\gamma y$. This completes the proof. ■

Proposition 85 *If $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy left(right) Γ-hyperideal of a Γ-semihypergroup H and $c(t_1, s_1), d(t_2, s_2)$ are intuitionistic fuzzy points of H such that $c(t_1, s_1) * H * d(t_2, s_2) \subseteq A$ then $l_{c(t_1,s_1)} *\, l_{d(t_2,s_2)} \subseteq A$ $\left(\xi_{c(t_1,s_1)} * \xi_{d(t_2,s_2)} \subseteq A\right)$.*

Proof. Straightforward. ■

A Γ-semihypergroup H is called *fully intuitionistic fuzzy prime (semiprime)* if each of its intuitionistic fuzzy Γ-hyperideal is prime (semiprime).

Proposition 86 *A* Γ*-semihypergroup* H *is fully intuitionistic fuzzy semiprime if and only if* H *is an intuitionistic fuzzy semisimple.*

Proof. Let H be an intuitionistic fuzzy semisimple Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy Γ-hyperideal of H. If for an intuitionistic fuzzy Γ-hyperideal $B = (\mu_B, \lambda_B)$ of H such thst $B * B \subseteq A$ then $B \subseteq A$ (since H is intuitionistic fuzzy semisimple). Hence, $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy semiprime Γ-hyperideal of H. Thus, H is a fully intuitionistic fuzzy semiprime.

Conversely, let H be a fully intuitionistic fuzzy semiprime Γ-semihypergroup. Let $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy Γ-hyperideal of H. Then, A^2 is also an intuitionistic fuzzy Γ-hyperideal of H. As $A^2 \subseteq A^2$ implies $A \subseteq A^2$ (because A is an intuitionistic fuzzy semiprime Γ-hyperideal of H). But $A^2 \subseteq A$ always. Hence, $A^2 = A$. Thus, each intuitionistic fuzzy Γ-hyperideal of H is idempotent. So, H is semisimple. ■

Proposition 87 *A* Γ*-semihypergroup* H *is fully intuitionistic fuzzy prime if and only if* H *is an intuitionistic fuzzy semisimple and the set of all intuitionistic fuzzy* Γ*-hyperideals of* H *is totally ordered under inclusion.*

Proof. Suppose H is an intuitionistic fully fuzzy prime. Let $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy Γ-hyperideal of H. Then, A^2 is also an intuitionistic fuzzy Γ-hyperideal of H. Since $A^2 \subseteq A^2$ implies $A \subseteq A^2$. But $A^2 \subseteq A$ always. Hence, $A^2 = A$. Thus, each intuitionistic fuzzy Γ-hyperideal of H is idempotent. So, H is semisimple. Now, suppose that $A = \langle \mu_A, \lambda_A \rangle$ and $B = (\mu_B, \lambda_B)$ are intuitionistic fuzzy Γ-hyperideals of H. Since, $A * B \subseteq A \cap B$ and $A \cap B$ is an intuitionistic fuzzy Γ-hyperideal of H, so $A \cap B$ is an intuitionistic fuzzy prime Γ-hyperideal. Thus, either $A \subseteq A \cap B$ or $B \subseteq A \cap B$ implies either $A \subseteq B$ or $B \subseteq B$.

Conversely, let H be an intuitionistic fuzzy semisimple Γ-semihypergroup and the set of all intuitionistic fuzzy Γ-hyperideals of H is totally ordered under inclusion. Let $A = \langle \mu_A, \lambda_A \rangle, B = (\mu_B, \lambda_B)$ and $C = (\mu_C, \lambda_C)$ be intuitionistic fuzzy Γ-hyperideals of H such that $A * B \subseteq C$. Since the set of all intuitionistic fuzzy Γ-hyperideals of H is totally ordered under inclusion, so either $A \subseteq B$ or $B \subseteq A$. If $A \subseteq B$. Now, $A = A^2 \subseteq A * B \subseteq C$. Hence, $A \subseteq C$ so A is an intuitionistic fuzzy prime Γ-hyperideal of H. Similarly, if $B \subseteq A$. ■

Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy subset of H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is called an *intuitionistic fuzzy M-hypersystem* of H if for all $x, y, z \in H$ and $\alpha, \beta \in \Gamma$ we have

$$\begin{aligned} \min \{\mu_A(x), \mu_A(z)\} &\leq \inf_{a \in x\alpha y\beta z} \mu_A(a) \\ \max \{\lambda_A(x), \lambda_A(z)\} &\geq \sup_{a \in x\alpha y\beta z} \lambda_A(a) \end{aligned}$$

Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy subset of H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is called an *intuitionistic fuzzy N-hypersystem* of H if for all $x, y \in H$ and $\alpha, \beta \in \Gamma$, we have

$$\begin{aligned} \mu(x) &\leq \inf_{a \in x\alpha y\beta x} \mu(a) \\ \lambda(x) &\geq \sup_{a \in x\alpha y\beta x} \lambda(a) \end{aligned}$$

Remark 88 *Every intuitionistic fuzzy M-hypersystem of a Γ-semihypergroup H is an intuitionistic fuzzy N-hypersystem.*

Theorem 89 *If $\{A_i\}_{i \in \Lambda}$ is a family of intuitionistic fuzzy M (resp. N)-hypersystems of Γ-semihypergroup H, then $\bigcap_{i \in \Lambda} A_i$ is also an intuitionistic fuzzy M(resp. N)-hypersystem of H.*

Proof. Let $\mathcal{A} = \bigcap_{i \in \Lambda} A_i = (\bigwedge_{i \in \Lambda} \mu_{A_i}, \bigvee_{i \in \Lambda} \lambda_{A_i}) = \langle \mu_A, \lambda_A \rangle$ and $x, y, z \in H$ and $\alpha, \beta \in \Gamma$.

Then, we have

$$\begin{aligned}
\inf_{a\in x\alpha y\beta z}\mu_{\mathcal{A}}(a) &= \inf_{a\in x\alpha y\beta z}\left\{\bigwedge_{i\in\Lambda}\{\mu_{A_i}(a)\}\right\} \\
&\geq \left\{\bigwedge_{i\in\Lambda}\left(\min\{\mu_{A_i}(x),\mu_{A_i}(z)\}\right)\right\} \\
&= \min\left\{\bigwedge_{i\in\Lambda}\mu_{A_i}(x),\bigwedge_{i\in\Lambda}\mu_{A_i}(z)\right\} \\
&= \min\{\mu_{\mathcal{A}}(x),\mu_{\mathcal{A}}(z)\} \\
\inf_{a\in x\alpha y\beta z}\mu_{\mathcal{A}}(a) &\geq \min\{\mu_{\mathcal{A}}(x),\mu_{\mathcal{A}}(z)\}
\end{aligned}$$

and

$$\begin{aligned}
\sup_{a\in x\alpha y\beta z}\lambda_{\mathcal{A}}(a) &= \sup_{a\in x\alpha y\beta z}\left\{\bigvee_{i\in\Lambda}\{\lambda_{\mathcal{A}}(a)\}\right\} \\
&\leq \left\{\bigvee_{i\in\Lambda}\left(\min\{\lambda_{A_i}(x),\lambda_{A_i}(z)\}\right)\right\} \\
&= \max\left\{\bigwedge_{i\in\Lambda}\lambda_{A_i}(x),\bigwedge_{i\in\Lambda}\lambda_{A_i}(z)\right\} \\
&= \max\{\lambda_{\mathcal{A}}(x),\lambda_{\mathcal{A}}(z)\} \\
\sup_{a\in x\alpha y\beta z}\lambda_{\mathcal{A}}(a) &\leq \max\{\lambda_{\mathcal{A}}(x),\lambda_{\mathcal{A}}(z)\}
\end{aligned}$$

Hence $\bigcap_{i\in\Lambda} A_i$ is an intuitionistic fuzzy M-hypersystem of a Γ-semihypergroup H. This completes the proof. ■

Proposition 90 *Every intuitionistic fuzzy two sided Γ-hyperideal of a Γ-semihypergroup H is an intuitionistic fuzzy M-hypersystem of H.*

Proof. Let $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy two sided Γ-hyperideal of H. Let

$x, y, z \in H$ and $\alpha, \beta \in \Gamma$. Then,

$$\begin{aligned}
\inf_{a \in x\alpha y\beta z} \mu_A(a) &\geq \inf_{a \in x\alpha y} \mu_A(a) \geq \mu_A(x) \text{ and} \\
\inf_{a \in x\alpha y\beta z} \mu_A(a) &\geq \inf_{a \in y\beta z} \mu_A(a) \geq \mu_A(z) \\
\inf_{a \in x\alpha y\beta z} \mu_A(a) &\geq \min\{\mu_A(x), \mu_A(z)\}
\end{aligned}$$

and

$$\begin{aligned}
\sup_{a \in x\alpha y\beta z} \lambda_A(a) &\leq \sup_{a \in x\alpha y} \lambda_A(a) \leq \lambda_A(x) \text{ and} \\
\sup_{a \in x\alpha y\beta z} \lambda_A(a) &\leq \sup_{a \in x\alpha z} \lambda_A(a) \leq \lambda_A(z) \\
\sup_{a \in x\alpha y\beta z} \lambda_A(a) &\leq \max\{\lambda_A(x), \lambda_A(z)\}
\end{aligned}$$

This completes the proof. ■

Corollary 91 *Every intuitionistic fuzzy one sided Γ-hyperideal of a Γ-semihypergroup H is an intuitionistic fuzzy N-hypersystem of H.*

4.4 (λ, μ)-Intuitionistic Fuzzy Γ-hyperideals in Γ-semihypergro

In this section we define and study the concept of (λ, μ)-intuitionistic fuzzy Γ-hyperideals and intuitionistic fuzzy interior Γ-hyperideal in Γ-semihypergroups.

Definition 92 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy set in H. If for all $x, y \in H, \gamma \in \Gamma$, the following conditions hold*

(P1) $\max\{\inf_{z \in x\gamma y} \mu_A(z), \lambda\} \geq \min\{\mu_A(x), \mu_A(y), \mu\}$.

(P2) $\min\{\sup_{z \in x\gamma y} \lambda_A(z), \mu\} \leq \max\{\lambda_A(x), \lambda_A(y), \lambda\}$.

then $A = \langle \mu_A, \lambda_A \rangle$ is called a (λ, μ)-intuitionistic fuzzy sub-Γ-semihypergroup of H.

It is clear that for $\lambda = 0$ and $\mu = 1$ we obtain the intuitionistic fuzzy sub-Γ-semihypergroups.

Definition 93 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy set in H. If for all $x, y \in H, \gamma \in \Gamma$, the following conditions hold*

$$\begin{array}{lrcl} L1 & \max\{\inf\limits_{z \in x\gamma y} \mu_A(z), \lambda\} & \geq & \min\{\mu_A(y), \mu\} \\ L2 & \min\{\sup\limits_{z \in x\gamma y} \lambda_A(z), \mu\} & \leq & \max\{\lambda_A(y), \lambda\}. \end{array}$$

Then $A = \langle \mu_A, \lambda_A \rangle$ is called a first (resp. second) (λ, μ)-intuitionistic fuzzy left Γ-hyperideal of H if it satisfies (L1) (resp. L2). $A = \langle \mu_A, \lambda_A \rangle$ is called an (λ, μ)-intuitionistic fuzzy left Γ-hyperideal of H if it is both a first and a second (λ, μ)-intuitionistic fuzzy left Γ-hyperideal of H.

Definition 94 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy set in H. If for all $x, y \in H, \gamma \in \Gamma$, the following conditions hold*

$$\begin{array}{lrcl} R1 & \max\{\inf\limits_{z \in x\gamma y} \mu_A(z), \lambda\} & \geq & \min\{\mu_A(x), \mu\} \\ R2 & \min\{\sup\limits_{z \in x\gamma y} \lambda_A(z), \mu\} & \leq & \max\{\lambda_A(x), \lambda\}. \end{array}$$

then $A = \langle \mu_A, \lambda_A \rangle$ is called a first (resp. second) (λ, μ)-intuitionistic fuzzy right Γ-hyperideal of H if it satisfies (R1) (resp. R2). $A = \langle \mu_A, \lambda_A \rangle$ is called a (λ, μ)-intuitionistic fuzzy right Γ-hyperideal of H if it is both a first and a second (λ, μ)-intuitionistic fuzzy right Γ-hyperideal of H.

Theorem 95 *Let H be a Γ-semihypergroup and $\emptyset \neq S \subseteq H$. If S is a sub-Γ-semihypergroup of H, then $\tilde{S} = (\chi_S, \chi_S^c)$ is a (λ, μ)-intuitionistic fuzzy sub-Γ-semihypergroup of H.*

Proof. Let $x, y \in H$ and $\gamma \in \Gamma$. We have the two following cases:

Case 1. $x, y \in S$. Then $x\gamma y \subseteq S$ since S is sub-Γ-semihypergroup of H. Thus we have

$\max\{\inf_{z\in x\gamma y}\chi_S(z),\lambda\} = \max\{1,\lambda\} = 1 \geq \min\{\chi_S(x),\chi_S(y),\mu\}$

and

$\min\{\sup_{z\in x\gamma y}\chi_S^c(z),\mu\} = \min\{1-\sup_{z\in x\gamma y}\chi_S(z),\mu\} = \min\{1-1,\mu\} = 0 \leq \max\{\chi_S^c(x),\chi_S^c(y),\lambda\}$.

Case 2. $x \notin S$ or $y \notin S$. Then $\chi_S(x) = 0$ or $\chi_S(y) = 0$. Thus we have

$\max\{\inf_{z\in x\gamma y}\chi_S(z),\lambda\} \geq 0 = \min\{\chi_S(x),\chi_S(y),\mu\}$

and

$\min\{\sup_{z\in x\gamma y}\chi_S^c(z),\mu\} \leq 1 = \max\{\chi_S^c(x),\chi_S^c(y),\lambda\}$

This completes the proof. ■

Theorem 96 *Let H be a Γ-semihypergroup and S be a non-empty subset of H. If $\tilde{S} = (\chi_S,\chi_S^c)$ is a (λ,μ)-intuitionistic fuzzy sub-Γ-semihypergroup of H then S is a sub-Γ-semihypergroup of H.*

Proof. 1. Let us suppose first that $\tilde{S} = (\chi_S,\chi_S^c)$ is a first (λ,μ)-intuitionistic fuzzy sub-Γ-semihypergroup of H. For any $x, y \in S, \gamma \in \Gamma$, by (P1), we have

$\max\{\inf_{z\in x\gamma y}\chi_S(z),\lambda\} \geq \min\{\chi_S(x),\chi_S(y),\mu\} = \min\{1,1,\mu\} = \mu$.

We have $\inf_{z\in x\gamma y}\chi_S(z) \geq \mu > 0$ since $\lambda < \mu$. Therefore we have $\inf_{z\in x\gamma y}\chi_S(z) = 1$. This implies $x\gamma y \subseteq S$ and thus S is a sub-Γ-semihypergroup of H.

2. Let us assume that $\tilde{S} = (\chi_S,\chi_S^c)$ is a second (λ,μ)-intuitionistic fuzzy sub-Γ-semihypergroup of H. For any $x, y \in S, \gamma \in \Gamma$, by (P2), we have

$\min\{\sup_{z\in x\gamma y}\chi_S^c(z),\mu\} \leq \max\{\chi_S^c(x),\chi_S^c(y),\lambda\} = \max\{0,0,\lambda\} = \lambda$.

We have $\sup_{z\in x\gamma y}\chi_S^c(z) \leq \lambda$ since $\lambda < \mu$. Therefore we have $\sup_{z\in x\gamma y}\chi_S^c(z) = 0$, that is $\sup_{z\in x\gamma y}\chi_S(z) = 1$. This implies $x\gamma y \subseteq S$ and thus S is a sub-Γ-semihypergroup of H. ■

Proposition 97 *Let H be a Γ-semihypergroup and L be a left Γ-hyperideal of H. Then, $\tilde{L} = (\chi_L,\chi_L^c)$ is an (λ,μ)-intuitionistic fuzzy left Γ-hyperideal of H.*

Proof. Let $x, y \in H, \gamma \in \Gamma$ and L be a Γ-hyperideal of H. Then, we consider the following two cases.

Case 1. $y \in L$. Then $x\gamma y \subseteq L$. Thus, we have

$\max\{\inf_{z\in x\gamma y} \chi_L(z), \lambda\} = \max\{1, \lambda\} = 1 \geq \min\{\chi_L(y), \mu\}.$

and

$\min\{\sup_{z\in x\gamma y} \chi_L^c(z), \mu\} = \min\{0, \mu\} = 0 \leq \max\{\chi_L^c(y), \lambda\}.$

Case 2. $y \notin L$. Then, $\chi_L(y) = 0$. Thus, we have

$\max\{\inf_{z\in x\gamma y} \chi_L(z), \lambda\} \geq 0 = \min\{\chi_L(y), \mu\}.$

and

$\min\{\sup_{z\in x\gamma y} \chi_L^c(z), \mu\} \leq 1 = \max\{\chi_L^c(y), \lambda\}.$

Therefore, $\tilde{L} = (\chi_L, \chi_L^c)$ is an (λ, μ)-intuitionistic fuzzy left Γ-hyperideal of H. ■

Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ an intuitionistic fuzzy sub-Γ-semihypergroup of H. If for all $x, s, y \in H, \beta, \gamma \in \Gamma$, the following conditions hold

$$\begin{array}{lrcl} \text{I1} & \max\{\inf_{z\in x\beta s\gamma y} \mu_A(z), \lambda\} & \geq & \min\{\mu_A(s), \mu\} \\ \text{I2} & \min\{\sup_{z\in x\beta s\gamma y} \lambda_A(z), \mu\} & \leq & \max\{\lambda_A(s), \lambda\}. \end{array}$$

Then, $A = \langle \mu_A, \lambda_A \rangle$ is called a *first* (resp. *second*) (λ, μ)-*intuitionistic fuzzy interior* Γ-*hyperideal* of H if it satisfies (I1) (resp. I2). $A = \langle \mu_A, \lambda_A \rangle$ is called an (λ, μ)-*intuitionistic fuzzy interior* Γ-*hyperideal* of H if it is both a first and a second (λ, μ)-intuitionistic fuzzy interior Γ-hyperideal of H.

Theorem 98 *Let H be a Γ-semihypergroup. Then, every (λ, μ)-intuitionistic fuzzy Γ-hyperideal is a (λ, μ)-intuitionistic fuzzy interior Γ-hyperideal.*

Proof. Let $A = \langle \mu_A, \lambda_A \rangle$ be a (λ, μ)-intuitionistic fuzzy Γ-hyperideal of H and let $x, s, y \in H, \beta, \gamma \in \Gamma$, we have

$$\begin{aligned} \max\{\inf_{z\in x\beta s\gamma y} \mu_A(z), \lambda\} &= \max\{\inf_{z\in x\beta s\gamma y} \mu_A(z), \lambda, \lambda\} \geq \max\{\min\{\inf_{t\in x\beta s} \mu_A(t), \mu\}, \lambda\} \\ &= \min\{\max\{\inf_{t\in x\beta s} \mu_A(t), \lambda\}, \max\{\mu, \lambda\}\} \geq \min\{\mu_A(s), \mu\}. \end{aligned}$$

In similar way, we show $\min\{\sup_{z\in x\beta s\gamma y}\lambda_A(z),\mu\}\le\max\{\lambda_A(s),\lambda\}$. So, $A=\langle\mu_A,\lambda_A\rangle$ is a (λ,μ)-intuitionistic fuzzy interior Γ-hyperideal of H. ■

Theorem 99 *Let H be a regular Γ-semihypergroup. Then every (λ,μ)-intuitionistic fuzzy interior Γ-hyperideal is a (λ,μ)-intuitionistic fuzzy Γ-hyperideal.*

Proof. Let $A=\langle\mu_A,\lambda_A\rangle$ be a (λ,μ)-intuitionistic fuzzy interior Γ-hyperideal of H and $x,y\in H$. Then, there exist $s,s'\in H,\beta,\beta'\gamma,\gamma'\in\Gamma$, such that $x\in x\beta s\gamma x$ and $y\in y\beta's'\gamma'y$. Thus, for all $\alpha\in\Gamma$, we have

$$\max\{\inf_{t\in x\alpha y}\mu_A(t),\lambda\}\ge\max\{\inf_{t\in x\alpha(y\beta's'\gamma'y)}\mu_A(t),\lambda\}\ge\max\{\inf_{t\in x\alpha y\beta'(s'\gamma'y)}\mu_A(t),\lambda\}\ge\min\{\mu_A(y),\mu\}$$

and

$$\min\{\sup_{t\in x\alpha y}\lambda_A(t),\mu\}\le\max\{\inf_{t\in x\alpha(y\beta's'\gamma'y)}\lambda_A(t),\mu\}\le\min\{\inf_{t\in x\alpha y\beta'(s'\gamma'y)}\lambda_A(t),\mu\}\le\max\{\lambda_A(y),\lambda\}.$$

This shows that $A=\langle\mu_A,\lambda_A\rangle$ is a (λ,μ)-intuitionistic fuzzy left Γ-hyperideal of H. In similar way, we show that $A=\langle\mu_A,\lambda_A\rangle$ is a (λ,μ)-intuitionistic fuzzy right Γ-hyperideal of H and so $A=\langle\mu_A,\lambda_A\rangle$ is a (λ,μ)-intuitionistic fuzzy Γ-hyperideal of H. ■

Theorem 100 *Let H be a Γ-semihypergroup and I an interior Γ-hyperideal of H. Then $\tilde{I}=(\chi_I,\chi_I^c)$ is a (λ,μ)-intuitionistic fuzzy interior Γ-hyperideal of H.*

Proof. Let $x,y,s\in H,\beta,\gamma\in\Gamma$. We consider the two following cases.

Case 1. $s\in I$. Then, $x\beta s\gamma y\subseteq I$. Thus, we have

$$\max\{\inf_{z\in x\beta s\gamma y}\chi_I(z),\lambda\}=\max\{1,\lambda\}=1\ge\min\{\chi_I(s),\mu\}.$$

and

$$\min\{\sup_{z\in x\beta s\gamma y}\chi_I^c(z),\mu\}=\min\{0,\mu\}=0\le\max\{\chi_I^c(s),\lambda\}.$$

Case 2. $s\notin I$. Then, $\chi_I(s)=0$. Thus, we have

$$\max\{\inf_{z\in x\beta s\gamma y}\chi_I(z),\lambda\}\ge 0=\min\{\chi_I(s),\mu\}.$$

and

$$\min\{\sup_{z\in x\beta s\gamma y}\chi_I^c(s),\mu\}\le 1=\max\{\chi_I^c(s),\lambda\}.$$

Therefore, $\tilde{I}=(\chi_I,\chi_I^c)$ is a (λ,μ)-intuitionistic fuzzy interior Γ-hyperideal of H. ■

Theorem 101 *Let H be a Γ-semihypergroup and S be a non-empty subset of H. If $\tilde{S} = (\chi_S, \chi_S^c)$ is a first (λ, μ)-intuitionistic fuzzy interior Γ-hyperideal of H or a second (λ, μ)-intuitionistic fuzzy interior Γ-hyperideal of H, then S is an interior Γ-hyperideal of H.*

Proof. By Theorem 96 it follows that S is a sub-Γ-semihypergroup of H. We consider the two following cases.

Case 1. Assume that $\tilde{S} = (\chi_S, \chi_S^c)$ is a first (λ, μ)-intuitionistic fuzzy interior Γ-hyperideal of H and let $x \in H\Gamma S\Gamma H$. Thus $x \in u\beta s\gamma t$ for some $u, t \in H, s \in S, \beta, \gamma \in \Gamma$. By the condition (I1) follows

$\max\{\chi_S(x), \lambda\} = \max\{\inf\limits_{z \in u\beta s\gamma t} \chi_S(z), \lambda\} \geq \min\{\chi_S(s), \mu\} = \min\{1, \mu\} = \mu.$

Since $\lambda < \mu$, we have $\chi_S(x) \geq \mu$, i.e. $\chi_S(x) = 1$. So, $x \in S$ and thus S is an interior Γ-hyperideal of H.

Case 2. Assume that $\tilde{S} = (\chi_S, \chi_S^c)$ is a second (λ, μ)-intuitionistic fuzzy interior Γ-hyperideal of H and let $x \in H\Gamma S\Gamma H$. Thus, $x \in u\beta s\gamma t$ for some $u, t \in H, s \in S, \beta, \gamma \in \Gamma$. By the condition (I2) follows

$\min\{\chi_S^c(x), \mu\} = \min\{\sup\limits_{z \in u\beta s\gamma t} \chi_S^c(z), \mu\} \leq \min\{\chi_S^c(s), \lambda\} = \max\{0, \lambda\} = \lambda.$

Since $\lambda < \mu$, we have $\chi_S^c(x) \leq \lambda$, i.e. $\chi_S^c(x) = 0$. So, $x \in S$ and thus S is an interior Γ-hyperideal of H. ■

Recall that a Γ-semihypergroup is called left simple if H has no left Γ-hyperideals other then itself. A Γ-semihypergroup is left simple if and only if $H = H\Gamma a$ for all $a \in H$.

A Γ-semihypergroup H is called a *first* (resp. *second*) (λ, μ)-*intuitionistic fuzzy left simple* Γ-*semihypergroup* if for any first (resp. second) (λ, μ)-intuitionistic fuzzy left Γ-hyperideal $A = \langle \mu_A, \lambda_A \rangle$ of H, for all $a, b \in H$, we have $\max\{\mu_A(a), \lambda\} \geq \min\{\mu_A(b), \mu\}$ (resp. $\min\{\lambda_A(a), \mu\} \leq \max\{\lambda_A(b), \lambda\}$. H is called a (λ, μ)-*intuitionistic fuzzy left simple* Γ-*semihypergroup* if it is both first and second (λ, μ)-intuitionistic fuzzy left simple Γ-semihypergroup.

Theorem 102 *If a Γ-semihypergroup H is left simple, then H is (λ, μ)-intuitionistic fuzzy left simple Γ-semihypergroup.*

Proof. Assume that $A = \langle \mu_A, \lambda_A \rangle$ is (λ, μ)-intuitionistic fuzzy left Γ-hyperideal of H and $x, x' \in H$. Since H is left simple, so there exist $s, s' \in H, \gamma, \gamma' \in \Gamma$ such that $x \in s\gamma x'$ and $x' \in s'\gamma' x$. Thus, by hypothesis for A, we have

$$\begin{aligned}\max\{\mu_A(x), \lambda\} &= \max\{\inf_{z \in s\gamma x'} \mu_A(z), \lambda\} \geq \min\{\mu_A(x'), \mu\},\\ \max\{\mu_A(x'), \lambda\} &= \max\{\inf_{z \in s'\gamma' x} \mu_A(z), \lambda\} \geq \min\{\mu_A(x), \mu\}\end{aligned}$$

and

$$\begin{aligned}\min\{\lambda_A(x), \mu\} &= \min\{\sup_{z \in s\gamma x'} \lambda_A(z), \mu\} \leq \max\{\lambda_A(x'), \lambda\},\\ \min\{\lambda_A(x'), \mu\} &= \min\{\sup_{z \in s'\gamma' x} \lambda_A(z), \mu\} \leq \max\{\lambda_A(x), \lambda\}.\end{aligned}$$

Therefore, H is a (λ, μ)-intuitionistic fuzzy left simple Γ-semihypergroup. ■

Theorem 103 *Let H be a Γ-semihypergroup. If H is first or second (λ, μ)-intuitionistic fuzzy left simple Γ-semihypergroup, then H is left simple Γ-semihypergroup.*

Proof. Let us suppose that H is first (or second) (λ, μ)-intuitionistic fuzzy left simple Γ-semihypergroup and let L be a left Γ-hyperideal of H. By Proposition 5.6, we have that $\tilde{L} = (\chi_L, \chi_L^c)$ is (λ, μ)-intuitionistic fuzzy left Γ-hyperideal of H. We have for all $a, b \in H$, $\max\{\chi_L(a), \lambda\} \geq \min\{\chi_L(b), \mu\}$. Let $a \in H$ and $b \in L$. Then, we have $\max\{\chi_L(a), \lambda\} \geq \min\{\chi_L(b), \mu\} = \min\{1, \mu\} = \mu$. So, $\max\{\chi_L(a), \lambda\} \geq \mu$. But $\lambda < \mu$, so $\chi_L(a) \geq \mu < 0$. This implies $\chi_L(a) = 1$, that is, $a \in L$. Therefore, $H = L$ and so H is left simple Γ-semihypergroup. ■

4.5 Intuitionistic Fuzzy Γ-hyperideals in Artinian and Noetherian Γ-semihypergroups

In this section we characterize Artinian and Noetherian Γ-semihypergroups in terms of intuitionistic fuzzy Γ-hyperideals.

Definition 104 *[29] Let H be a Γ-semihypergroup. Then, H is called Noetherian (Artinian resp.) if H satisfies the asceding (descending) chain condition on Γ-hyperideals. That is, for any sequence of Γ-hyperideals $\{I_i\}_1^\infty$ of H such that $I_1 \subseteq I_2 \subseteq I_3 \subseteq ...I_i \subseteq ...(I_1 \supseteq I_2 \supseteq I_3 \supseteq ...I_i...$ resp.) there exists $n \in \mathbb{N}$ such that $I_m = I_n$ for each $m \in \mathbb{N}; m \geq n$.*

Theorem 105 *[29] Let H be a Γ-semihypergroup. Then, the following are equivalent:*

1. *H is Noetherian;*
2. *H satisfies the maximum condition for Γ-hyperideals;*
3. *Every Γ-hyperideal of H is finitely generated.*

Theorem 106 *Let H be a Γ-semihypergroup. If every intuitionistic fuzzy Γ-hyperideal of a Γ-semihypergroup has finite number of values, then H is Artinian.*

Proof. Let assume that every intuitionistic fuzzy Γ-hyperideal of a Γ-semihypergroup H has finite number of values and H is not Artinian. Then, there exists a strictly descending chain

$H = I_0 \supset I_1 \supset I_2 \supset ...$

of Γ-hyperideals of H. Now, we define the intuitionistic fuzzy set A by

$$\mu_A(x) = \begin{cases} \frac{n}{n+1} & \text{if } x \in I_n \backslash I_{n+1}, n = 0, 1, 2, ... \\ 1 & \text{if } x \in \bigcap_{n=0}^{\infty} I_n \end{cases}$$

$$\lambda_A(x) = \begin{cases} \frac{1}{n+1} & \text{if } x \in I_n \backslash I_{n+1}, n = 0, 1, 2, ... \\ 0 & \text{if } x \in \bigcap\limits_{n=0}^{\infty} I_n \end{cases}$$

It can be easily seen that A is an intuitionistic fuzzy Γ-hyperideal of H which has infinite number of different values because of the infinitely descending chain of Γ-hyperideals $I_0, I_1, I_2, ...$ of H. It is impossible. So, H is Artinian. ■

Theorem 107 *Let H be a Γ-semihypergroup. Then, the following statements are equivalent:*

1. *H is Noetherian;*
2. *The set of values of any intuitionistic fuzzy Γ-hyperideal of H is a well-ordered subset of $[0; 1]$.*

Proof. $(1) \Rightarrow (2)$. Let A be an intuitionistic fuzzy Γ-hyperideal of a Γ-semihypergroup H. Let us suppose that the set of values of A is not a well-ordered subset of $[0, 1]$. Then, there exists a strictly infinite decreasing sequence $\{t_n\}$ such that $\mu_A(x_n) = t_n$ and $\lambda_A(x_n) = 1 - t_n$. Let $I_n = \{x \in H | \mu_A(x) \geq t_n\}$ and $J_n = \{x \in H | \lambda_A(x) \leq 1 - t_n\}$. Then, $I_1 \subset I_2 \subset ...$ and $J_1 \subset J_2 \subset ...$ are strictly infinite ascending chains of Γ-hyperideals of H. It is impossible.

$(2) \Rightarrow (1)$. Let us suppose that there exists a strictly infinite ascending chain $I_1 \subset I_2 \subset ...$ (*) of Γ-hyperideals of H. Let $I = \bigcup\limits_{n \in \mathbb{N}} I_n$. It can be easily seen that I is a Γ-hyperideal of H. We define the intuitionistic fuzzy set $A = \langle \mu_A, \lambda_A \rangle$ by

$$\mu_A(x) = \begin{cases} 0 & \text{if } x \notin I \\ \frac{1}{k} & \text{where } k = \min\{n \in \mathbb{N} | x \in I_n\} \end{cases}$$

$$\lambda_A(x) = \begin{cases} 1 & \text{if } x \notin I \\ \frac{k-1}{k+1} & \text{where } k = \max\{n \in \mathbb{N} | x \in I_n\} \end{cases}$$

It can be easily seen that A is an intuitionistic fuzzy Γ-hyperideal of H. Since the chain (*) is not finite, A has strictly infinite ascending sequence of values and so the value set of the intuitionistic fuzzy Γ-hyperideal A of H is not well-ordered. It is impossible. ■

Theorem 108 *Let H be a Γ-semihypergroup. If H is both Artinian and Noetherian, then every intuitionistic fuzzy Γ-hyperideal of H is finite valued.*

Proof. Let us suppose that A is an intuitionistic fuzzy Γ-hyperideal of H and $Im(\mu_A), Im(\lambda_A)$ are not finite. By the previous theorem, we consider the two following cases:

Case 1. Assume that $t_1 < t_2 < t_3 < ...$ is strictly increasing sequence in $Im(\mu_A)$ and $s_1 > s_2 > s_3 > ...$ is strictly decreasing sequence in $Im(\lambda_A)$. Then we have

$U(\mu_A; t1) \supset U(\mu_A; t_2) \supset U(\mu_A; t_3) \supset ...,$

$L(\lambda_A; s_1) \supset L(\lambda_A; s_2) \supset L(\lambda_A; s_3) \supset ...$

are strictly descending and ascending chains of Γ-hyperideals of H, respectively. Since H is both Artinian and Noetherian there exists $i \in \mathbb{N}$, such that $U(\mu_A; t_i) = U(\mu_A; t_{i+n})$ and $L(\lambda_A; s_i) = L(\lambda_A; s_{i+n})$ for $n \geq 1$. This implies that $t_i = t_{i+n}$ and $s_i = s_{i+n}$. It is impossible.

Case 2. Assume that $t_1 > t_2 > t_3 > ...$ is strictly decreasing sequence in $Im(\mu_A)$ and $s_1 < s_2 < s_3 < ...$ is strictly increasing sequence in $Im(\lambda_A)$. Then we have

$U(\mu_A; t1) \subset U(\mu_A; t_2) \subset U(\mu_A; t_3) \subset ...,$

$L(\lambda_A; s_1) \subset L(\lambda_A; s_2) \subset L(\lambda_A; s_3) \subset ...$

are strictly ascending and descending chains of Γ-hyperideals of H, respectively. Since H is both Artinian and Noetherian there exists $j \in \mathbb{N}$, such that $U(\mu_A; t_j) = U(\mu_A; t_{j+n})$ and $L(\lambda_A; s_j) = L(\lambda_A; s_{j+n})$ for $n \geq 1$. This implies that $t_j = t_{j+n}$ and $s_j = s_{j+n}$. It is impossible. ■

Chapter 5

Characterizations of Γ-semihypergroups by Intuitionistic Fuzzy Γ-hyperideals

5.1 Introduction

In this chapter, we characterize Γ-semihypergroups by the properties of intuitionistic fuzzy Γ-hyperideals. We characterize regular Γ-semihypergroup by intuitionistic fuzzy Γ-hyperideal, intuitionistic fuzzy bi-Γ-hyperideal, intuitionistic fuzzy generalized bi-Γ-hyperideal, intuitionistic fuzzy interior Γ-hyperideal and intuitionistic fuzzy quasi-Γ-hyperideal. We also define an intuitionistic fuzzy quasi-Γ-hyperideal of a Γ-semihypergroup.

5.2 Intuitionistic Fuzzy Quasi-Γ-hyperideals

In this section we define an intuitionistic fuzzy quasi-Γ-hyperideal in a Γ-semihypergroup. We discuss different properties of intuitionistic fuzzy quasi-Γ-hyperideal.

Definition 109 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy set in H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is called an intuitionistic fuzzy quasi-Γ-hyperideal of*

H, if the following conditions hold.

$$\begin{aligned}\mu_A(x) &\geq (\mu_A * 1)(x) \wedge (1 * \mu_A)(x) \\ \lambda_A(x) &\leq (\lambda_A * 0)(x) \vee (0 * \lambda_A)(x),\end{aligned}$$

for all $x \in H$. Also, we can define an intuitionistic fuzzy quasi-Γ-hyperideal of H as: $A * \mathcal{S} \cap \mathcal{S} * A \subseteq A$ for an intuitionistic fuzzy set $A = \langle \mu_A, \lambda_A \rangle$ and $\mathcal{S} = \langle 1, 0 \rangle$.

Example 110 *Let $H = \{1, 2, 3\}$ and $\Gamma = \{\gamma, \delta\}$ be two non-empty sets. Then, hyperoperations are defined by the following Cayley tables.*

γ	1	2	3
1	$\{1\}$	$\{1,2\}$	$\{3\}$
2	$\{1,2\}$	$\{1,2\}$	$\{3\}$
3	$\{3\}$	$\{3\}$	$\{3\}$

δ	1	2	3
1	$\{1,2\}$	$\{1,2\}$	$\{3\}$
2	$\{1,2\}$	$\{2\}$	$\{3\}$
3	$\{3\}$	$\{3\}$	$\{3\}$

Clearly, H is a Γ-semihypergroup. Now, we define intuitionistic fuzzy set $A = \langle \mu_A, \lambda_A \rangle$ in H as:

$$\mu_A(1) = 0.4 = \mu_A(2),\ \mu_A(3) = 0.7 \text{ and } \lambda_A(1) = 0.5 = \lambda_A(2),\ \lambda_A(3) = 0.2.$$

Then, $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy quasi-Γ-hyperideal of H.

Example 111 *Let $H = \{a, b, c, d\}$, $\Gamma = \{\gamma, \delta\}$ be two non-empty sets and γ-hyperoperations defined by the following Cayley tables;*

γ	a	b	c	d
a	$\{a,c\}$	$\{b,d\}$	$\{a,c\}$	$\{d\}$
b	$\{b,d\}$	$\{b\}$	$\{b,d\}$	$\{d\}$
c	$\{a,c\}$	$\{b,d\}$	$\{a,c\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

δ	a	b	c	d
a	$\{a\}$	$\{b,d\}$	$\{c\}$	$\{d\}$
b	$\{b,d\}$	$\{b\}$	$\{b,d\}$	$\{d\}$
c	$\{c\}$	$\{b,d\}$	$\{a\}$	$\{d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

Then, clearly, (H,Γ) *is a* Γ*-semihypergroup.* $I=\{d\}$ *and* $\{b,d\}$ *are only proper* Γ*-hyperideals of* H*.*

Proposition 112 *Let* H *be a* Γ*-semihypergroup and* Q *be a non-empty subset of* H*. Then,* Q *is a quasi-*Γ*-hyperideal of* H *if and only if the intuitionistic characteristic function* $\widetilde{Q}=\langle\chi_Q,\chi_Q^c\rangle$ *is an intuitionistic fuzzy quasi-*Γ*-hyperideal of* H*.*

Proof. straightforward. ■

Proposition 113 *Let* $A=\langle\mu_A,\lambda_A\rangle$ *and* $B=\langle\mu_B,\lambda_B\rangle$ *be an intuitionistic fuzzy right* Γ*-hyperideal and an intuitionistic fuzzy left* Γ*-hyperideal of* Γ*-semihypergroup* H*, respectively. Then,* $A\cap B$ *is an intuitionistic fuzzy quasi-*Γ*-hyperideal of* H*.*

Proof. Let $x\in H$. Then, we have

$$\begin{aligned}((\mu_A\wedge\mu_B)*1)(x)\wedge(1*(\mu_A\wedge\mu_B))(x) &\le (\mu_A*1)(x)\wedge(1*\mu_B)(x)\\ &\le (\mu_A\wedge\mu_B)(x)\end{aligned}$$

and

$$\begin{aligned}((\lambda_A\vee\lambda_B)*1)(x)\vee(1*(\lambda_A\vee\lambda_B))(x) &\ge (\lambda_A*1)(x)\vee(1*\lambda_B)(x)\\ &\ge (\lambda_A\vee\lambda_B)(x).\end{aligned}$$

This implies that $A\cap B$ is an intuitionistic fuzzy quasi-Γ-hyperideal of H. ■

Proposition 114 *Every intuitionistic fuzzy quasi-*Γ*-hyperideal of a* Γ*-semihypergroup* H *is an intuitionistic fuzzy bi-*Γ*-hyperideal of* H*.*

Proof. Let $A=\langle\mu_A,\lambda_A\rangle$ be an intuitionistic fuzzy quasi-Γ-hyperideal of H. Since, $\mu_A(x)\le 1$ and $\lambda_A\ge 0$ i.e, $A\subseteq\mathcal{S}$. This implies that $A*A\subseteq\mathcal{S}*A$ and $A*A\subseteq A*\mathcal{S}$. Hence, $A*A\subseteq A*\mathcal{S}\cap\mathcal{S}*A\subseteq A$ this implies that $A*A\subseteq A$. Thus, $A=\langle\mu_A,\lambda_A\rangle$ is an intuitionistic fuzzy sub-Γ-semihypergroup of H. Since $\mathcal{S}*A\subseteq\mathcal{S}$, so we have

$A * \mathcal{S} * A \subseteq A * \mathcal{S}$. Also, since $A * \mathcal{S} \subseteq \mathcal{S}$ this implies $A * \mathcal{S} * A \subseteq \mathcal{S} * A$. Thus, we have $A * \mathcal{S} * A \subseteq A * \mathcal{S} \cap \mathcal{S} * A \subseteq A$ this implise that $A * \mathcal{S} * A \subseteq A$. Therefore, $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy bi-Γ-hyperideal of H. ■

Proposition 115 *Every one sided intuitionistic fuzzy Γ-hyperideal of a Γ-semihypergroup H is an intuitionistic fuzzy quasi-Γ-hyperideal of H.*

Proof. Let $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy left Γ-hyperideal of H. Then, $\mathcal{S} * A \subseteq A$ and also $A * \mathcal{S} \subseteq \mathcal{S}$. Thus, this implies that $A * \mathcal{S} \cap \mathcal{S} * A \subseteq \mathcal{S} * A \subseteq A$. ■

Proposition 116 *Let $\{A_i\}_{i \in \Lambda}$ be a family of intuitionisitic fuzzy quasi-Γ-hyperideals of a Γ-semihypergroup H. Then $\bigcap_{i \in \Lambda} A_i$ is an intuitionistic fuzzy quasi-Γ-hyperideal of H.*

Proof. Strightforward. ■

Proposition 117 *The product of two intuitionistic fuzzy quasi-Γ-hyperideals of a Γ-semihypergroup H is an intuitionistic fuzzy bi-Γ-hyperideal of H.*

Proof. Strightforward. ■

Theorem 118 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy set in H. Then,*

1. *$A = \langle \mu_A, \lambda_A \rangle$ is an intuionistic fuzzy sub-Γ-semihypergroup of H if and only if $A * A \subseteq A$.*

2. *$A = \langle \mu_A, \lambda_A \rangle$ is an intuionistic fuzzy left (right) Γ-hyperideal of H if and only if $\mathcal{S} * A \subseteq A$ $(A * \mathcal{S} \subseteq A)$.*

Proof. 1) Let $A = \langle \mu_A, \lambda_A \rangle$ be an intuionistic fuzzy sub-Γ-semihypergroup of H. Then, $A * A \subseteq A$. Indeed, let $x \in H$. Then,

$$\begin{array}{rcl}
(\mu_A * \mu_A)(x) & = & \bigvee\limits_{x \in y\gamma z} \{\mu_A(y) \wedge \mu_A(z)\} \\
& \leq & \bigvee\limits_{x \in y\gamma z} \{\mu_A(y\gamma z)\} \\
& = & \inf\limits_{x \in y\gamma z} \mu_A(x) \\
(\mu_A * \mu_A)(x) & \leq & \inf\limits_{x \in y\gamma z} \mu_A(x)
\end{array}$$

and

$$\begin{array}{rcl}
(\lambda_A * \lambda_A)(x) & = & \bigwedge\limits_{x \in y\gamma z} \{\lambda_A(y) \vee \lambda_A(z)\} \\
& \geq & \bigwedge\limits_{x \in y\gamma z} \{\lambda_A(y\gamma z)\} \\
& = & \sup\limits_{x \in y\gamma z} \lambda_A(x) \\
(\lambda_A * \lambda_A)(x) & \geq & \sup\limits_{x \in y\gamma z} \lambda_A(x).
\end{array}$$

This implies that $A * A \subseteq A$.

Conversely, suppose that $A * A \subseteq A$ holds. Let $x, y \in H$ and $\gamma \in \Gamma$. Then,

$$\begin{array}{rcl}
\inf\limits_{x \in y\gamma z} \mu_A(x) & \geq & (\mu_A * \mu_A)(x) = \bigvee\limits_{x \in y\gamma z} \{\mu_A(y) \wedge \mu_A(z)\} \\
& \geq & \mu_A(y) \wedge \mu_A(z)
\end{array}$$

and

$$\begin{array}{rcl}
\sup\limits_{x \in y\gamma z} \lambda_A(x) & \leq & (\lambda_A * \lambda_A)(x) = \bigwedge\limits_{x \in y\gamma z} \{\lambda_A(y) \vee \lambda_A(z)\} \\
& \leq & \lambda_A(y) \vee \lambda_A(z)
\end{array}$$

Thus, $A = \langle \mu_A, \lambda_A \rangle$ is an intuionistic fuzzy sub-Γ-semihypergroup of H.

2) Suppose that $A = \langle \mu_A, \lambda_A \rangle$ is an intuionistic fuzzy left Γ-hyperideal of H. Let $x \in H$. Then, we have

$$\begin{aligned}
(1 * \mu_A)(x) &= \bigvee_{x \in y\gamma z} \{1(y) \wedge \mu_A(z)\} \\
&\leq \bigvee_{x \in y\gamma z} \left\{1(y) \wedge \inf_{x \in y\gamma z} \mu_A(z)\right\} \\
&= \bigvee_{x \in y\gamma z} \left\{1 \wedge \inf_{x \in y\gamma z} \mu_A(x)\right\} = \inf_{x \in y\gamma z} \mu_A(x) \\
(1 * \mu_A)(x) &\leq \inf_{x \in y\gamma z} \mu_A(x) \\
(1 * \mu_A)(x) &\leq \mu_A(x) \text{ for each } x \in H
\end{aligned}$$

and

$$\begin{aligned}
(0 * \lambda_A)(x) &= \bigvee_{x \in y\gamma z} \{0(y) \vee \lambda_A(z)\} \\
&\geq \bigvee_{x \in y\gamma z} \left\{0(y) \vee \sup_{x \in y\gamma z} \lambda_A(z)\right\} \\
&= \bigvee_{x \in y\gamma z} \left\{0 \vee \sup_{x \in y\gamma z} \lambda_A(x)\right\} = \sup_{x \in y\gamma z} \lambda_A(x) \\
(1 * \lambda_A)(x) &\geq \sup_{x \in y\gamma z} \lambda_A(x) \\
(1 * \lambda_A)(x) &\geq \lambda_A(x) \text{ for each } x \in H.
\end{aligned}$$

Hence, $\mathcal{S} * A \subseteq A$. For an intuionistic fuzzy left Γ-hyperideal of H can be proved in a similar way. ■

Theorem 119 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy sub-Γ-semihypergroup of H. Then,*

1. *$A = \langle \mu_A, \lambda_A \rangle$ is an intuionistic fuzzy bi-Γ-hyperideal of H if and only if $A * \mathcal{S} * A \subseteq A$.*

2. *$A = \langle \mu_A, \lambda_A \rangle$ is an intuionistic fuzzy (1,2)-Γ-hyperideal of H if and only if $A * \mathcal{S} * A * A \subseteq A$.*

Proof. 1) Assume that $A = \langle \mu_A, \lambda_A \rangle$ is an intuionistic fuzzy bi-Γ-hyperideal of H. Then, $A * \mathcal{S} * A \subseteq A$. Indeed, let $x \in H$. Then, in the case $(\mu_A * 1 * \mu_A)(x) = 0$, and $(\lambda_A * 0 * \lambda_A)(x) = 1$ it is clear $(\mu_A * 1 * \mu_A)(x) \subseteq \mu_A(x)$ and $(\lambda_A * 0 * \lambda_A)(x) \supseteq \lambda_A(x)$. If $(\mu_A * 1 * \mu_A)(x) \neq 0$, then there exist a, b, p, q and $\beta, \gamma \in \Gamma$ such that $x \in a\beta b$ and $a \in p\gamma q$. Since $A = \langle \mu_A, \lambda_A \rangle$ is an intuionistic fuzzy bi-Γ-hyperideal of H, so we have

$$\begin{aligned}
(\mu_A * 1 * \mu_A)(x) &= \bigvee_{x \in a\beta b} \{(\mu_A * 1)(a) \wedge \mu_A(b)\} \\
&= \bigvee_{x \in a\beta b} \left\{ \bigvee_{a \in p\gamma q} \{\mu_A(p) \wedge 1(q)\} \wedge \mu_A(b) \right\} \\
&= \bigvee_{x \in a\beta b} \left\{ \bigvee_{a \in p\gamma q} \{\mu_A(p) \wedge 1\} \wedge \mu_A(b) \right\} \\
&= \bigvee_{x \in p\gamma q\beta b} \{\mu_A(p) \wedge \mu_A(b)\} \\
&\leq \bigvee_{x \in p\gamma q\beta b} \{\mu_A(p\gamma q\beta b)\} = \inf_{x \in p\gamma q\beta b} \{\mu_A(x)\} \\
(\mu_A * 1 * \mu_A)(x) &\leq \inf_{x \in p\gamma q\beta b} \{\mu_A(x)\} \leq \mu_A(x) \\
(\mu_A * 1 * \mu_A)(x) &\leq \mu_A(x)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_A * 0 * \lambda_A)(x) &= \bigwedge_{x \in a\beta b} \{(\lambda_A * 0)(a) \vee \lambda_A(b)\} \\
&= \bigwedge_{x \in a\beta b} \left\{ \bigwedge_{a \in p\gamma q} \{\lambda_A(p) \vee 0(q)\} \vee \lambda_A(b) \right\} \\
&= \bigwedge_{x \in a\beta b} \left\{ \bigwedge_{a \in p\gamma q} \{\lambda_A(p) \vee 0\} \vee \lambda_A(b) \right\} \\
&= \bigwedge_{x \in p\gamma q\beta b} \{\lambda_A(p) \vee \lambda_A(b)\} \\
&\geq \bigwedge_{x \in p\gamma q\beta b} \{\lambda_A(p\gamma q\beta b)\} = \sup_{x \in p\gamma q\beta b} \{\lambda_A(x)\} \\
(\lambda_A * 1 * \lambda_A)(x) &\geq \sup_{x \in p\gamma q\beta b} \{\lambda_A(x)\} \geq \lambda_A(x) \\
(\lambda_A * 1 * \lambda_A)(x) &\geq \lambda_A(x).
\end{aligned}$$

Thus, this implies that $A * \mathcal{S} * A \subseteq A$.

Conversely, suppose that $A * \mathcal{S} * A \subseteq A$ holds. Let $x, y, z \in H$ and $\beta, \gamma \in \Gamma$ such that $a \in x\beta y\gamma z$. Then,

$$\begin{aligned}
\mu_A(a) &\geq (\mu_A * 1 * \mu_A)(a) \text{ for each } a \in x\beta y\gamma z \\
&= \bigvee_{a \in p\beta q} \{(\mu_A * 1)(p) \wedge \mu_A(a)\} \\
&\geq \{(\mu_A * 1)(p) \wedge \mu_A(z)\} \quad \text{for each } p \in x\beta y \\
&= \bigvee_{p \in x\beta y = m\gamma n} \{\mu_A(m) \wedge 1(n) \wedge \mu_A(z)\} \\
&\geq \mu_A(x) \wedge 1(y) \wedge \mu_A(z) = \mu_A(x) \wedge 1 \wedge \mu_A(z) \\
\mu_A(a) &\geq \mu_A(x) \wedge \mu_A(z) \text{ for each } a \in x\beta y\gamma z
\end{aligned}$$

and

$$\begin{aligned}
\lambda_A(a) &\leq (\lambda_A * 0 * \lambda_A)(a) \text{ for each } a \in x\beta y\gamma z \\
&= \bigwedge_{a \in p\beta q} \{(\lambda_A * 0)(p) \vee \lambda_A(a)\} \\
&\leq \{(\lambda_A * 0)(p) \vee \lambda_A(z)\} \quad \text{for each } p \in x\beta y \\
&= \bigvee_{p \in x\beta y = m\gamma n} \{\lambda_A(m) \vee 0(n) \wedge \lambda_A(z)\} \\
&\leq \lambda_A(x) \vee 0(y) \vee \lambda_A(z) = \lambda_A(x) \vee 0 \vee \lambda_A(z) \\
\lambda_A(a) &\leq \lambda_A(x) \vee \lambda_A(z) \text{ for each } a \in x\beta y\gamma z
\end{aligned}$$

Thus,

$$\begin{aligned}
\inf_{a \in x\beta y\gamma z} \mu_A(a) &\geq \min\{\mu_A(x), \mu_A(z)\} \\
\sup_{a \in x\beta y\gamma z} \lambda_A(a) &\leq \max\{\lambda_A(x), \lambda_A(z)\}.
\end{aligned}$$

Therefore, $A = \langle \mu_A, \lambda_A \rangle$ is an intuionistic fuzzy bi-Γ-hyperideal of H.

2) This part follows from part (1). ■

5.3 Regular Γ-semihypergroups

In this section, we characterize regular Γ-semihypergroups in terms of intuitionistic fuzzy Γ-hyperideals.

Theorem 120 *Let H be a Γ-semihypergroup. Then, the following are equivalent:*

1. *H is regular.*

2. *$A = A * \mathcal{S} * A$ for every intuitionistic fuzzy quasi-Γ-hyperideal $A = \langle \mu_A, \lambda_A \rangle$ of H.*

3. *$A = A * \mathcal{S} * A$ for every intuitionistic fuzzy generalized bi-Γ-hyperideal $A = \langle \mu_A, \lambda_A \rangle$ of H.*

4. *$A = A * \mathcal{S} * A$ for every intuitionistic fuzzy generalized bi-Γ-hyperideal $A = \langle \mu_A, \lambda_A \rangle$ of H.*

Proof. 1) ⇒ 4). Suppose that (1) holds. Let $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy generalized bi-Γ-hyperideal of H amd let $x \in H$. Then, since H is a regular Γ-semihypergroup, so there exist $y \in H$ and $\alpha, \beta \in \Gamma$ such that $x \in x\alpha y\beta x$. Now, we have

$$
\begin{aligned}
(\mu_A * 1 * \mu_A)(x) &= \bigvee_{x \in a\alpha b} \{(\mu_A * 1)(a) \wedge \mu_A(b)\} \\
&\geq \{(\mu_A * 1)(z) \wedge \mu_A(x)\}, \text{ for all } z \in x\alpha y \\
&= \bigvee_{z \in s\alpha t} \{\mu_A(s) \wedge 1(t)\} \wedge \mu_A(x) \text{ for all } z \in x\alpha y \\
&\geq \mu_A(x) \wedge 1(y) \wedge \mu_A(x) \\
&= \mu_A(x) \\
(\mu_A * 1 * \mu_A)(x) &\geq \mu_A(x)
\end{aligned}
$$

and

$$\begin{aligned}
(\lambda_A * 1 * \lambda_A)(x) &= \bigwedge_{x\in a\alpha b}\{(\lambda_A * 0)(a) \vee \lambda_A(b)\} \\
&\leq \{(\lambda_A * 0)(z) \vee \lambda_A(x)\}, \text{ for all } z \in x\alpha y \\
&= \bigwedge_{z\in s\alpha t}\{\lambda_A(s) \vee 0(t)\} \vee \lambda_A(x) \text{ for all } z \in x\alpha y \\
&\leq \lambda_A(x) \vee 0(y) \vee \lambda_A(x) \\
&= \lambda_A(x) \\
(\lambda_A * 1 * \lambda_A)(x) &\leq \lambda_A(x).
\end{aligned}$$

Thus, $(\mu_A * 1 * \mu_A) \geq \mu_A$ and $(\lambda_A * 1 * \lambda_A) \leq \lambda_A$ this implies that $A \subseteq A * \mathcal{S} * A$. Since $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy generalized bi-Γ-hyperideal of H, so $A \supseteq A * \mathcal{S} * A$. Hence, $A = A * \mathcal{S} * A$ and so 1) $\Rightarrow$ 4). This is very easy to prove that 4) $\Rightarrow$ 3) $\Rightarrow$ 2).

2) $\Rightarrow$ 1): Suppose that (2) holds. Let Q be a quasi-Γ-hyperideal of H. Then, we have

$$\begin{aligned}
Q\Gamma H\Gamma Q &\subseteq Q\Gamma H\Gamma H) \cap (H\Gamma H)\Gamma Q \subseteq Q\Gamma H \cap H\Gamma Q \subseteq Q. \\
Q\Gamma H\Gamma Q &\subseteq Q.
\end{aligned}$$

Let $x \in Q$. Since $\widetilde{Q} = (\chi_Q, \chi_Q^c)$ is an intuitionistic fuzzy quasi-Γ-hyperideal of H, so by Proposition 112. We have

$$\begin{aligned}
\bigvee_{x\in a\alpha b}\left\{\left(\chi_Q * 1\right)(a) \wedge \chi_Q(b)\right\} &= \left(\left(\chi_Q * 1\right) * \chi_Q\right)(x) \\
&= \chi_Q(x) = 1
\end{aligned}$$

This implies that there exist $y, z \in H$ such that $\left(\chi_Q * 1\right)(y) = 1$ and $\chi_Q(b) = 1$ with $x \in y\alpha z$. Thus, we have

$$\bigvee_{y\in s\gamma t}\left\{\left(\chi_Q\right)(s) \wedge 1(t)\right\} = \left(\chi_Q * 1\right)(y) = 1$$

This implies that there exist $m, n \in H$ such that $\left(\chi_Q\right)(m) = 1$ and $1(n) = 1$ with $y \in m\beta n$. Hence, $m, z \in Q$ and $n \in H$. This implies that $x \in y\alpha z \subseteq m\beta n\alpha z \subseteq Q\Gamma H\Gamma Q$, this implies that $Q \subseteq Q\Gamma H\Gamma Q$. Hence, $Q = Q\Gamma H\Gamma Q$. ■

Theorem 121 *Let H be a regular Γ-semihypergroup. Then, the following statements are equivalent.*

1. *Every bi-Γ-hyperideal of H is a left(right, two sided) Γ-hyperideal of H.*

2. *Every intuitionistic fuzzy bi-Γ-hyperideal of H is an intuitionistic fuzzy left(right, two sided) Γ-hyperideal of H.*

Proof. 1) $\Rightarrow$ 2): Suppose that (1) holds. Let $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy bi-Γ-hyperideal of H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy left Γ-hyperideal of H. Indeed, let $x, y \in H$ and $\alpha \in \Gamma$. Since the set $y\Gamma H\Gamma y$ is a bi-Γ-hyperideal of H, so by (1) $y\Gamma H\Gamma y$ is left Γ-hyperideal of H. Since H is regular, so $x\alpha y \subseteq (y\Gamma H\Gamma y)\,\Gamma H \subseteq y\Gamma H\Gamma y$. This implies that there exist $a \in H$ and $\beta,\gamma \in \Gamma$ such that $x\alpha y \subseteq y\beta a\gamma y$. Since $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy bi-Γ-hyperideal of H, so

$$\begin{aligned} \inf_{z e x\alpha y} \mu_A(z) &= \inf_{z e y\beta a\gamma y} \mu_A(z) \geq \mu_A(y) \wedge \mu_A(y) = \mu_A(y) \\ \inf_{z e x\alpha y} \mu_A(z) &\geq \mu_A(y) \end{aligned}$$

and

$$\begin{aligned} \sup_{z e x\alpha y} \lambda_A(z) &= \sup_{z e y\beta a\gamma y} \lambda_A(z) \geq \lambda_A(y) \wedge \lambda_A(y) = \lambda_A(y) \\ \sup_{z e x\alpha y} \lambda_A(z) &\geq \lambda_A(y) \end{aligned}$$

Thus, $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy left Γ-hyperideal of H.

Conversely, 2) $\Rightarrow$ 1): Let I be a bi-Γ-hyperideal of H. Then, I is a left Γ-hyperideal of H. Indeed, By 67 $\widetilde{I} = (\chi_I, \chi_I^c)$ is an intuitionistic fuzzy bi-Γ-hyperideal of H. Hence, by given assumption $\widetilde{I} = (\chi_I, \chi_I^c)$ is an intuitionitic fuzzy left Γ-hyperideal of H. Thus, by [28, Theorem 3.11] I is left Γ-hyperideal of H. ■

Theorem 122 *Let H be a Γ-semihypergroup. Then the following statements are equivalent:*

1. *H is regular.*
2. *$A \cap B = A * B * A$ for evey intuitionistic fuzzy quasi-Γ-hyperideal of H and for every intuitionistic fuzzy Γ-hyperideal of H.*
3. *$A \cap B = A * B * A$ for evey intuitionistic fuzzy quasi-Γ-hyperideal of H and for every intuitionistic fuzzy interior Γ-hyperideal of H.*
4. *$A \cap B = A * B * A$ for evey intuitionistic fuzzy bi-Γ-hyperideal of H and for every intuitionistic fuzzy Γ-hyperideal of H.*
5. *$A \cap B = A * B * A$ for evey intuitionistic fuzzy bi-Γ-hyperideal of H and for every intuitionistic fuzzy interior Γ-hyperideal of H.*
6. *$A \cap B = A * B * A$ for evey intuitionistic fuzzy generalized bi-Γ-hyperideal of H and for every intuitionistic fuzzy Γ-hyperideal of H.*
7. *$A \cap B = A * B * A$ for evey intuitionistic fuzzy generalized bi-Γ-hyperideal of H and for every intuitionistic fuzzy interior Γ-hyperideal of H.*

Proof. 1) $\Rightarrow$ 2) Assume that (1) holds. Let A and B be an intuitionistic fuzzy generalized bi-Γ-hyperideal and an intuitionistic fuzzy interior Γ-hyperideal of H, respectively. Then,

$$A * B * A \subseteq A * \mathcal{S} * A \subseteq A$$

and

$$A * B * A \subseteq \mathcal{S} * B * \mathcal{S} \subseteq B.$$

This implies that

$$A * B * A \subseteq A \cap B.$$

Let $x \in H$. Then, since H is regular, so there exist $y \in H$ and $\alpha,\beta \in H$ such that $x \in x\alpha y\beta x \subseteq x\alpha y\gamma x\delta y\beta x$. Since B is an intuitionistic fuzzy interior Γ-hyperideal of H, so we have

$$\begin{aligned}
(\mu_A * \mu_B * \mu_A)(x) &= \bigvee_{x \in a\alpha b} \{\mu_A(a) \wedge (\mu_B * \mu_A)(b)\} \\
&\geq \mu_A(x) \wedge (\mu_B * \mu_A)(z) \text{ for each } z \in y\gamma x\delta y\beta x \\
&= \mu_A(x) \wedge \left(\bigvee_{z \in p\beta q} \{\mu_B(p) \wedge \mu_A(q)\} \right) \\
&\geq \mu_A(x) \wedge \mu_B(u) \wedge \mu_A(x) \text{ for each } u \in y\gamma x\delta y \\
&\geq \mu_A(x) \wedge \mu_B(x) \wedge \mu_A(x) \\
&\geq \mu_A(x) \wedge \mu_B(x) \\
(\mu_A * \mu_B * \mu_A)(x) &\geq (\mu_A \wedge \mu_B)(x)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_A * \lambda_B * \lambda_A)(x) &= \bigwedge_{x \in a\alpha b} \{\lambda_A(a) \vee (\lambda_B * \lambda_A)(b)\} \\
&\leq \lambda_A(x) \vee (\lambda_B * \lambda_A)(z) \text{ for each } z \in y\gamma x\delta y\beta x \\
&= \lambda_A(x) \vee \left(\bigvee_{z \in p\beta q} \{\lambda_B(p) \vee \lambda_A(q)\} \right) \\
&\leq \lambda_A(x) \vee \lambda_B(u) \vee \lambda_A(x) \text{ for each } u \in y\gamma x\delta y \\
&\leq \lambda_A(x) \vee \lambda_B(x) \vee \lambda_A(x) \\
&\leq \lambda_A(x) \vee \lambda_B(x) \\
(\lambda_A * \lambda_B * \lambda_A)(x) &\leq (\lambda_A \vee \lambda_B)(x).
\end{aligned}$$

This implies that $A \cap B \subseteq A * B * A$. Hence, $A \cap B = A * B * A$ and so (1) imlies (7).

This is easy to verify (7) $\Rightarrow$ (5) $\Rightarrow$ (3) $\Rightarrow$ (2) and (7) $\Rightarrow$ (6) $\Rightarrow$ (4) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). Assume that (2) holds. Let A be an intuitionistic fuzzy quasi-Γ-hyperideal of H and $\mathcal{S}$ be itself an intuitionistic fuzzy Γ-hyperideal of H. Then, we have

$$A = A \cap \mathcal{S} = A * \mathcal{S} * A.$$

Thus, by Theorem 120, H is regular. Hence, (2) $\Rightarrow$ (1). ■

Theorem 123 *Let H be a Γ-semihypergroup. Then, the following statements are equivalent:*

1. *H is regular*
2. *$A \cap B = A * B$ for evey intuitionistic fuzzy quasi-Γ-hyperideal of H and for every intuitionistic fuzzy left Γ-hyperideal of H.*
3. *$A \cap B = A * B$ for evey intuitionistic fuzzy bi-Γ-hyperideal of H and for every intuitionistic fuzzy left Γ-hyperideal of H.*
4. *$A \cap B = A * B$ for evey intuitionistic fuzzy generalized bi-Γ-hyperideal of H and for every intuitionistic fuzzy left Γ-hyperideal of H.*

Proof. 1) $\Rightarrow$ 2) Suppose that (1) holds. Let A and B be an intuitionistic fuzzy generalized bi-Γ-hyperideal of H and an intuitionistic fuzzy left Γ-hyperideal of H, respectively. Let $x \in H$. Since H is regular, so there exist $y \in H$ and $\alpha,\beta \in \Gamma$ such that $x \in x\alpha y\beta x$. Thus, we have

$$\begin{aligned}
(\mu_A * \mu_B)(x) &= \bigvee_{x \in a\alpha b} \{\mu_A(a) \wedge \mu_B(b)\} \\
&\geq \mu_A(x) \wedge \mu_B(z) \text{ for all } z \in y\beta x \\
&\geq \mu_A(x) \wedge \mu_B(x) \\
(\mu_A * \mu_B)(x) &\geq (\mu_A \wedge \mu_B)(x)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_A * \lambda_B)(x) &= \bigwedge_{x \in a\alpha b} \{\lambda_A(a) \vee \lambda_B(b)\} \\
&\leq \lambda_A(x) \vee \lambda_B(z) \text{ for all } z \in y\beta x \\
&\leq \lambda_A(x) \vee \lambda_B(x) \\
(\lambda_A * \lambda_B)(x) &\leq (\lambda_A \vee \lambda_B)(x).
\end{aligned}$$

This implies that $A \cap B \subseteq A * B$. Thus, (1) $\Rightarrow$ (4). It is easy to verify (4) $\Rightarrow$ (3) $\Rightarrow$ (2).

2) $\Rightarrow$ 1) Assume that (2) holds. Since every intuitionistic fuzzy right Γ-hyperideal of H, $A \cap B \subseteq A * B$ for every intuitionistic fuzzy right Γ-hyperideal of H and for every intuitionistic fuzzy left Γ-hyperideal of H. Since $A \cap B \supseteq A * B$ always holds. Thus, $A \cap B = A * B$ for every intuitionistic fuzzy right Γ-hyperideal of H and for every intuitionistic fuzzy left Γ-hyperideal of H. Thus, by Theorem 72, H is regular. ■

The following Theorem is a left-right dual of Theorem 123.

Theorem 124 *Let H be a Γ-semihypergroup. Then, the following statements are equivalent:*

1. *H is regular*

2. *$A \cap B = A * B$ for evey intuitionistic fuzzy quasi-Γ-hyperideal of H and for every intuitionistic fuzzy left Γ-hyperideal of H.*

3. *$A \cap B = A * B$ for evey intuitionistic fuzzy bi-Γ-hyperideal of H and for every intuitionistic fuzzy left Γ-hyperideal of H.*

4. *$A \cap B = A * B$ for evey intuitionistic fuzzy generalized bi-Γ-hyperideal of H and for every intuitionistic fuzzy left Γ-hyperideal of H.*

Proof. The proof follows from Theorem 123. ■

Theorem 125 *Let H be a Γ-semihypergroup. Then, the following statements are equivalent:*

1. *H is regular.*

2. *$A \cap B \cap C \subseteq A * B * C$ for evey intuitionistic fuzzy right Γ-hyperideal A of H, for every intuitionistic fuzzy quasi-Γ-hyperideal B of H and for evey intuitionistic fuzzy left Γ-hyperideal C of H.*

3. *$A \cap B \cap C \subseteq A * B * C$ for evey intuitionistic fuzzy right Γ-hyperideal A of H, for every intuitionistic fuzzy bi-Γ-hyperideal B of H and for evey intuitionistic fuzzy left Γ-hyperideal C of H.*

4. *$A \cap B \cap C \subseteq A * B * C$ for evey intuitionistic fuzzy right Γ-hyperideal A of H, for every intuitionistic fuzzy generalized bi-Γ-hyperideal B of H and for evey intuitionistic fuzzy left Γ-hyperideal C of H.*

Proof. (1) $\Rightarrow$ (4) : Suppose that (1) holds. Let $A = \langle \mu_A, \lambda_A \rangle$, $B = \langle \mu_B, \lambda_B \rangle$ and $C = \langle \mu_B, \lambda_B \rangle$ be an intuitionistic fuzzy right Γ-hyperideal, an intuitionistic fuzzy quasi-Γ-hyperideal and an intuitionistic fuzzy left Γ-hyperideal of H, respectively. Let $x \in H$. Then, $x \in x\gamma y\delta x$ for some $y \in H$ and γ, $\delta \in \Gamma$, since H is regular. We have

$$\begin{aligned}
(\mu_A * \mu_B * \mu_C)(x) &= \bigvee_{x \in a\alpha b} \{\mu_A(a) \wedge (\mu_B * \mu_C)(b)\} \\
&\geq \{\mu_A(z) \wedge (\mu_B * \mu_C)(x)\} \text{ for all } z \in x\gamma y \\
&\geq \mu_A(x) \wedge \bigvee_{x \in s\alpha t} \{\mu_B(s) \wedge \mu_C(t)\} \\
&\geq \mu_A(x) \wedge \mu_B(x) \wedge \mu_C(p) \text{ for all } p \in y\delta x \\
&\geq \mu_A(x) \wedge \mu_B(x) \wedge \mu_C(x) \\
(\mu_A * \mu_B * \mu_C)(x) &\geq (\mu_A \wedge \mu_B \wedge \mu_C)(x)
\end{aligned}$$

and

$$
\begin{aligned}
(\lambda_A * \lambda_B * \lambda_C)(x) &= \bigwedge_{x \in a\alpha b} \{\lambda_A(a) \vee (\lambda_B * \lambda_C)(b)\} \\
&\leq \{\lambda_A(z) \vee (\lambda_B * \lambda_C)(x)\} \text{ for all } z \in x\gamma y \\
&\leq \mu_A(x) \vee \bigwedge_{x \in s\alpha t} \{\lambda_B(s) \vee \lambda_C(t)\} \\
&\leq \lambda_A(x) \vee \lambda_B(x) \vee \lambda_C(p) \text{ for all } p \in y\delta x \\
&\leq \lambda_A(x) \vee \lambda_B(x) \vee \lambda_C(x) \\
(\lambda_A * \lambda_B * \lambda_C)(x) &\leq (\lambda_A \vee \lambda_B \vee \lambda_C)(x).
\end{aligned}
$$

This implies that $A \cap B \cap C \subseteq A * B * C$. Thus, (1) implies (4). It is easy to verify (4) implies (3) implies (2).

(2) $\Rightarrow$ (1) : Suppose that (2) is true. Let $A = \langle \mu_A, \lambda_A \rangle$ and $C = \langle \mu_B, \lambda_B \rangle$ be an intuitionistic fuzzy right Γ-hyperideal and an intuitionistic fuzzy left Γ-hyperideal of H, respectively. Since $\mathcal{S} = \langle 1, 0 \rangle$ itself is an intuitionistic fuzzy quasi Γ-hyperideal of H. Using the given assumption, we have

$$
\begin{aligned}
A \cap C &= A \cap \mathcal{S} \cap C \subseteq A * \mathcal{S} * C \subseteq A * C \\
A \cap C &\subseteq A * C \text{ and } A \cap C \supseteq A * C \text{ always true.}
\end{aligned}
$$

Hence, this implies that $A \cap C = A * C$ for every intuitionistic fuzzy right Γ-hyperideal A and an intuitionistic fuzzy left Γ-hyperideal C of H. Hence, by using Theorem 72, H is regular. ■

Chapter 6

Bi-Γ-hyperideals of Γ-semihypergroups based on Intuitionistic Fuzzy Points

6.1 Introduction

In this chapter, we introduce the concept of an (α, β)-intuitionistic fuzzy left (right) Γ-hyperideal, an (α, β)-intuitionistic fuzzy bi-Γ-hyperideal, an (α, β)-intuitionistic fuzzy $(1, 2)$-Γ-hyperideal of a Γ-semihypergroup by using the notion of intuitionsitic fuzzy point to intuitionistic fuzzy set. The concepts of an (α, β)-intuitionistic fuzzy left (right) Γ-hyperideal, an (α, β)-intuitionistic fuzz bi-Γ-hyperideal, an (α, β)-intuitionistic fuzzy $(1, 2)$-Γ-hyperideal of a Γ-semihypergroup is a generalization of ordinary intuitionitic fuzzy left(right, bi, (1,2))-Γ-hyperideals. We can construct twelve different types of an intuitionistic fuzzy bi-Γ-hyperideal and an intuitionistic fuzzy $(1, 2)$-Γ-hyperideal of a Γ-semihypergroup. We also define an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal, an $(\in, \in \vee q)$-intuitionistic fuzzy $(1, 2)$-Γ-hyperideal of a Γ-semihypergroup. We charcterize an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal, an $(\in, \in \vee q)$-intuitionistic fuzzy $(1, 2)$-Γ-hyperideal by the properties of an $\in$-level set, a q-level set and an $(\in, \in \vee q)$-level set.

6.2 (α, β)-Intuitionistic Fuzzy Left(right, (1,2), Bi) Γ-hyperideals

Recently, in [28, 31] the authors introduced the notions of an intuitionistic fuzzy Γ-hyperideal, intuitionistic fuzzy bi-Γ-hyperideal and intuitionistic fuzzy interior Γ-hyperideal in a Γ-semihypergroup and studied some fundamental properties. In this section we introduce the concept of an (α, β)-intuitionistic fuzzy bi-Γ-hyperideal, an (α, β)-intuitionistic fuzzy $(1, 2)$-Γ-hyperideal of a Γ-semihypergroup by using the notion of an intuitionsitic fuzzy point to an intuitionistic fuzzy set. The concept of an (α, β)-intuitionistic fuzzy bi-Γ-hyperideal, an (α, β)-intuitionistic fuzzy $(1, 2)$-Γ-hyperideal of a Γ-semihypergroup is a generalization of ordinary intuitionitic fuzzy bi-Γ-hyperideals.

Let H be a Γ-semihypergroup. An intuitionistic fuzzy set $A = \langle \mu_A, \lambda_A \rangle$ in H is called an (α, β)-intuitionistic fuzzy left (resp. right) Γ-hyperideal of H, where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, if for all $x, y, z \in H$, $\gamma \in \Gamma$, $t \in (0, 0.5]$ and $s \in [0.5, 1)$ or $t \in (0.5, 1]$ and $s \in [0, 0.5)$, the following condition holds:

(IFI) $y(t, s)\alpha A \Longrightarrow (z)(t, s)\beta A$ $(x(t, s)\alpha A \Longrightarrow (z)(t, s)\beta A)$, for each $z \in x\gamma y$.

An intuitionistic fuzzy set A in H is called an (α, β)-intuitionistic fuzzy two-sided Γ-hyperideal of H if it is both an (α, β)-intuitionistic fuzzy left and an (α, β)-intuitionistic right Γ-hyperideal of H.

An IFS $A = \langle \mu_A, \lambda_A \rangle$ in a Γ-semihypergroup H is said to be an (α, β)-*intuitionistic fuzzy bi-Γ-hyperideal* of a Γ-semihypergroup H, where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, if for all $x, y, z \in H$, $\gamma, \delta \in \Gamma$, $t_1, t_2 \in (0, 0.5]$ and $s_1, s_2 \in [0.5, 1)$ or $t_1, t_2 \in (0.5, 1]$ and $s_1, s_2 \in [0, 0.5)$, the following conditions hold:

$(IFB1)$ $x(t_1, s_1)\alpha A$ and $y(t_2, s_2)\alpha A \Longrightarrow (z)(m\{t_1, t_2\}, M\{s_1, s_2\})\beta A$, for each $z \in x\gamma y$,

$(IFB2)$ $x(t_1, s_1)\alpha A$ and $z(t_2, s_2)\alpha A \Longrightarrow (z_1)(m\{t_1, t_2\}, M\{s_1, s_2\})\beta A$, for each $z \in x\gamma y\delta z$.

An IFS $A = \langle \mu_A, \lambda_A \rangle$ in a Γ-semihypergroup H is said to be an (α, β)-*intuitionistic* *fuzzy* (1, 2) Γ-*hyperideal* of a Γ-semihypergroup H, if for all $x, y, z, a \in H$, $\gamma, \delta, \tau \in \Gamma$, $t_1, t_2 \in (0, 0.5]$ and $s_1, s_2 \in [0.5, 1)$ or $t_1, t_2 \in (0.5, 1]$ and $s_1, s_2 \in [0, 0.5)$, the following conditions hold.

$(IFS2)$ $x(t_1, s_1)\alpha A$ and $y(t_2, s_2)\alpha A \Rightarrow (z)(m\{t_1, t_2\}, M\{s_1, s_2\})\beta A$, for each $z \in x\gamma y$,

$(IFS2)$ $x(t_1, s_1)\alpha A$ and $z(t_2, s_2)\alpha A \Rightarrow (z_1)(m\{t_1, t_2\}, M\{s_1, s_2\})\beta A$, for each $z \in x\gamma a\tau(y\delta z)$.

Theorem 126 *Let $A = \langle \mu_A, \lambda_A \rangle$ be a non-zero (α, β)-intuitionistic fuzzy sub-Γ-semihypergroup of H. Then, the set $A_{(0,1)} = \{x \in H : \mu_A(x) > 0$ and $\lambda_A(x) < 1\}$ is a sub-Γ-semihypergroup of H.*

Proof. Let $x, y \in A_{(0,1)}$. Then, $\mu_A(x) > 0$ and $\lambda_A(x) < 1$, and $\mu_A(y) > 0$ and $\lambda_A(y) < 1$. Let us suppose that $\mu_A(z) = 0$ and $\lambda_A(z) = 1$ for each $z \in x\gamma y$. If $\alpha \in \{\in, \in \vee q\}$, then

$$x(\mu_A(x), \lambda_A(x))\alpha A \text{ and } y(\mu_A(y), \lambda_A(y))\alpha A \text{ but}$$
$$\mu_A(z) = 0 < m\{\mu_A(x), \mu_A(y)\} \text{ and } \lambda_A(z) = 1 > M\{\lambda_A(x), \lambda_A(y)\}$$
$$\text{for each } z \in x\gamma y.$$

So, $(z)(m\{\mu_A(x), \mu_A(y)\}, M\{\lambda_A(x), \lambda_A(y)\})\overline{\beta}A$ for $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contradiction. Now, let $x(1,0)qA$ and $y(1,0)qA$ but $(z)(1,0)\overline{\beta}A$, for each $z \in x\gamma y$, for $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, which is a contaradiction. Hence $\mu_A(z) > 0$ and $\lambda_A(z) < 1$, for each $z \in x\gamma y$, that is $z \in A_{(0,1)}$ for each $z \in x\gamma y$. Thus, $A_{(0,1)}$ is a sub-Γ-semihypergroup of H. ■

Theorem 127 *Let $A = \langle \mu_A, \lambda_A \rangle$ be a non-zero (α, β)-intuitionistic fuzzy bi-Γ-hyperideal of H. Then, the set $A_{(0,1)} = \{x \in H : \mu_A(x) > 0$ and $\lambda_A(x) < 1\}$ is a bi-Γ-hyperideal of H.*

Proof. Let $A = \langle \mu_A, \lambda_A \rangle$ be a non-zero (α, β)-intuitionistic fuzzy bi-Γ-hyperideal of H. Then, by Theorem 126, $A_{(0,1)}$ is a sub-Γ-semihypergroup of H. Now, let $x, z \in A_{(0,1)}$,

$y \in H$ and $\gamma, \delta \in \Gamma$. Then, $\mu_A(x) > 0$ and $\lambda_A(x) < 1$, and $\mu_A(z) > 0$ and $\lambda_A(z) < 1$. Suppose that $\mu_A(u) = 0$ and $\lambda_A(u) = 1$ for each $u \in x\gamma y\delta z$. If $\alpha \in \{\in, \in \vee q\}$, then

$$x(\mu_A(x), \lambda_A(x))\, \alpha A \text{ and } z(\mu_A(z), \lambda_A(z))\, \alpha A \text{ but}$$
$$\mu_A(u) = 0 < m\{\mu_A(x), \mu_A(z)\} \text{ and } \lambda_A(xyz) = 1 > M\{\lambda_A(x), \lambda_A(z)\},$$
$$\text{for each } u \in x\gamma y\delta z.$$

which implies that, for each $u \in x\gamma y\delta z$, $(u)(m\{\mu_A(x), \mu_A(z)\}, M\{\lambda_A(x), \lambda_A(z)\})\,\overline{\beta}A$ for $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, this is a contradiction. Now, let $x(1,0)\,qA$ and $z(1,0)\,qA$ but for each $u \in x\gamma y\delta z$, $(u)(1,0)\,\overline{\beta}A$ for $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$, this is again contaradiction. Hence, $\mu_A(u) > 0$ and $\lambda_A(u) < 1$ for each $u \in x\gamma y\delta z$, that is, $x\gamma y\delta z \subseteq A_{(0,1)}$. Thus, $A_{(0,1)}$ is a bi-Γ-hyperideal of H. ■

Theorem 128 *Let $A = \langle \mu_A, \lambda_A \rangle$ be a non-zero (α, β)-intuitionistic fuzzy* (1, 2) Γ-*hyperideal of H. Then, the set $A_{(0,1)} = \{x \in H : \mu_A(x) > 0$ and $\lambda_A(x) < 1\}$ is a* (1, 2) Γ-*hyperideal of H.*

Proof. Straightforward. ■

Theorem 129 *Let L be a left (resp. right) Γ-hyperideal of H and let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS such that*

(a) $(\forall x \in H \setminus L)$ $(\mu_A(x) = 0$ and $\lambda_A(x) = 1)$,

(b) $(\forall x \in L)$ $(\mu_A(x) \geq 0.5$ and $\lambda_A(x) \leq 0.5)$.

Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\alpha, \in \vee q)$-intuitionistic fuzzy left (resp. right) Γ-hyperideal of H.

Proof. (i) (For $\alpha = q$), let $x, y \in H$, $\gamma \in \Gamma$, $t \in (0, 0.5]$ and $s \in [0.5, 1)$ or $t \in (0.5, 1]$ and $s \in [0, 0.5)$ be such that $y(t,s)\,qA$. Then, $\mu_A(y) + t > 1$ and $\lambda_A(y) + s < 1$. So, $y \in L$. Therefore, $x\gamma y \subseteq L$. Thus, if $t \leq 0.5$ and $s \geq 0.5$, then $\inf_{u \in x\gamma y} \mu_A(u) \geq 0.5 \geq t$ and $\sup_{u \in x\gamma y} \lambda_A(u) \leq 0.5 \leq s$ and so $(u)(t,s) \in A$, for each $u \in x\gamma y$. If $t > 0.5$ and $s < 0.5$, then $\inf_{u \in x\gamma y} \mu_A(u) + t > 0.5 + 0.5 = 1$ and $\sup_{u \in x\gamma y} \lambda_A(y) + s < 0.5 + 0.5 = 1$ this implies that $(u)(t,s)\,qA$, for each $u \in x\gamma y$. Since $0 \leq t+s \leq 1$. Therefore, $u(t,s) \in \vee qA$, for each $u \in x\gamma y$. Hence, $A = \langle \mu_A, \lambda_A \rangle$ is an $(q, \in \vee q)$-intuitionistic fuzzy left Γ-hyperideal of H.

(ii) (For $\alpha = \in$), let $x, y \in H$, $\gamma \in \Gamma$, $t \in (0, 0.5]$ and $s \in [0.5, 1)$ or $t \in (0.5, 1]$ and $s \in [0, 0.5)$ be such that $y(t,s) \in A$. Then, $\mu_A(y) \geq t$ and $\lambda_A(y) \leq s$. So, $y \in L$. Therefore, $x\gamma y \subseteq L$. Thus, if $t \leq 0.5$ and $s \geq 0.5$, then $\inf_{u \in x\gamma y} \mu_A(u) \geq 0.5 \geq t$ and $\sup_{u \in x\gamma y} \lambda_A(u) \leq 0.5 \leq s$ this implies that $(u)(t,s) \in A$ for each $u \in x\gamma y$. If $t > 0.5$ and $s < 0.5$, then $\inf_{u \in x\gamma y} \mu_A(u) + t > 0.5 + 0.5 = 1$ and $\sup_{u \in x\gamma y} \lambda_A(u) + s < 0.5 + 0.5 = 1$ this implies that $(u)(t,s)\,qA$ for each $u \in x\gamma y$. Since $0 \leq t+s \leq 1$. Therefore, $(u)(t,s) \in \vee qA$ for each $u \in x\gamma y$. Hence, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy left Γ-hyperideal of H.

(iii) (For $\alpha = \in \vee q$), follows from (i) and (ii). ■

Theorem 130 *Let B be a sub-Γ-semihypergroup of H and let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS such that*

(a) $(\forall x \in H \setminus B)$ $(\mu_A(x) = 0$ and $\lambda_A(x) = 1)$,

(b) $(\forall x \in B)$ $(\mu_A(x) \geq 0.5$ and $\lambda_A(x) \leq 0.5)$.

Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\alpha, \in \vee q)$-intuitionistic fuzzy sub-Γ-semihypergroup of H.

Proof. Straightforward. ■

Theorem 131 *Let B be a bi-Γ-hyperideal of a Γ-semihypergroup H and let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS of H such that*

(a) $(\forall x \in H \setminus B)$ $(\mu_A(x) = 0$ and $\lambda_A(x) = 1)$,

(b) $(\forall x \in B)$ $(\mu_A(x) \geq 0.5$ and $\lambda_A(x) \leq 0.5)$.

Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\alpha, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H.

Proof. (i) (For $\alpha = q$), let $x, y \in H$, $\gamma \in \Gamma$, $t_1, t_2 \in (0, 0.5]$ and $s_1, s_2 \in [0.5, 1)$ or $t_1, t_2 \in (0.5, 1]$ and $s_1, s_2 \in [0, 0.5)$ be such that $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A$. Then, $x, y \in B$. Since B is sub-Γ-semihypergroup, so, $x\gamma y \subseteq B$. If $m(t_1, t_2) > 0.5$ and

$M(s_1, s_2) < 0.5$, then $\inf_{u \in x\gamma y} \mu_A(u) + m(t_1, t_2) > 1$ and $\sup_{u \in x\gamma y} \lambda_A(u) + M(s_1, s_2) < 1$. Thus, $(u)(m(t_1, t_2), M(s_1, s_2))qA$ for each $u \in x\gamma y$. If $m(t_1, t_2) \leq 0.5$ and $M(s_1, s_2) \geq 0.5$, then $(u)(m(t_1, t_2), M(s_1, s_2)) \in A$ for each $u \in x\gamma y$. Since $0 \leq t_1 + s_1 \leq 1$ and $0 \leq t_2 + s_2 \leq 1$. Hence, $A = \langle \mu_A, \lambda_A \rangle$ is a $(q, \in \vee q)$-intuitionistic fuzzy sub-Γ-semihypergroup of H. Let $x, y, z \in H$ $\gamma, \delta \in \Gamma$ $t_1, t_2 \in (0, 0.5]$ and $s_1, s_2 \in [0.5, 1)$ or $t_1, t_2 \in (0.5, 1]$ and $s_1, s_2 \in [0, 0.5)$ be such that $x(t_1, s_1) \in A$ and $z(t_2, s_2) \in A$. Then, $x, z \in B$. Since B is a bi-Γ-ideal, so $x\gamma y\delta z \subseteq B$. If $m(t_1, t_2) > 0.5$ and $M(s_1, s_2) < 0.5$, then $\mu_A(u) + m(t_1, t_2) > 1$ and $\lambda_A(u) + M(s_1, s_2) < 1$, for each $u \in x\gamma y\delta z$. So, $(u)((m(t_1, t_2), M(s_1, s_2))qA$. If $m(t_1, t_2) \leq 0.5$ and $M(s_1, s_2) \geq 0.5$, then $(u)(m(t_1, t_2), M(s_1, s_2)) \in A$ for each $u \in x\gamma y\delta z$. Therefore, $(xyz)(m(t_1, t_2), M(s_1, s_2)) \in \vee qA$. Hence, $A = \langle \mu_A, \lambda_A \rangle$ is a $(q, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H.

(ii) (For $\alpha = \in$ and $\in \vee q$), straightforward. ■

Theorem 132 *Let B be a $(1, 2)$ Γ-hyperideal of a Γ-semihypergroup H and let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS of H such that*

(a) $(\forall x \in H \setminus B)$ $(\mu_A(x) = 0$ and $\lambda_A(x) = 1)$,

(b) $(\forall x \in B)$ $(\mu_A(x) \geq 0.5$ and $\lambda_A(x) \leq 0.5)$.

Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\alpha, \in \vee q)$-intuitionistic fuzzy $(1,\ 2)$ Γ-hyperideal of H.

Proof. Proof follows from Theorem 131. ■

6.3 Intutionistic Fuzzy Left(right, (1,2), Bi) Γ-hyperideal of type $(\in, \in \vee q)$

The concepts of $(\in, \in \vee q)$-intutionistic fuzzy left (right) Γ-hyperideals, $(\in, \in \vee q)$-intutionistic fuzzy bi-Γ-hyperideals, $(\in, \in \vee q)$-intutionistic fuzzy (1,2)-Γ-hyperideals in a Γ-semihypergroup plays a vital role in the theory of (α, β)-intutionistic fuzzy left (right) Γ-hyperideals, (α, β)-intutionistic fuzzy bi-Γ-hyperideals, (α, β)-intutionistic fuzzy (1,2)-Γ-hyperideals.

We give some different characterizations of $(\in, \in \vee q)$-intutionistic fuzzy left (right) Γ-hyperideals, $(\in, \in \vee q)$-intutionistic fuzzy bi-Γ-hyperideals, $(\in, \in \vee q)$-intutionistic fuzzy (1,2)-Γ-hyperideals in a Γ-semihypergroup.

Definition 133 *Let H be a Γ-semihypergroup. An intuitionistic fuzzy set $A = \langle \mu_A, \lambda_A \rangle$ in H is called an $(\in, \in \vee q)$-intuitionistic fuzzy left (resp. right) Γ-hyperideal of H, if for all $x, y, z \in H$, $\gamma \in \Gamma$,$t \in (0, 0.5]$ and $s \in [0.5, 1)$ or $t \in (0.5, 1]$ and $s \in [0, 0.5)$, the following condition holds:*

(IFI) $y(t, s)\alpha A \Longrightarrow (z)(t, s)\beta A$ $(x(t, s)\alpha A \Longrightarrow (z)(t, s)\beta A)$, for each $z \in x\gamma y$.

An intuitionistic fuzzy set A in H is called an $(\in, \in \vee q)$-intuitionistic fuzzy two-sided Γ-hyperideal of H if it is both an $(\in, \in \vee q)$-intuitionistic fuzzy left and an $(\in, \in \vee q)$-intuitionistic right Γ-hyperideal of H.

Definition 134 *An IFS $A = \langle \mu_A, \lambda_A \rangle$ in a Γ-semihypergroup H is said to be an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of a Γ-semihypergroup H if $\forall x, y, a \in H$, $\gamma \in \Gamma$, $t_1, t_2 \in (0, 0.5]$ and $s_1, s_2 \in [0.5, 1)$ or $t_1, t_2 \in (0.5, 1]$ and $s_1, s_2 \in [0, 0.5)$, the following conditions hold.*

$(IFB3)$ $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A \Longrightarrow (z)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee qA$ for each $z \in x\gamma y$.

$(IFB4)$.$x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A \Longrightarrow (z)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee qA$ for each $z \in x\gamma y$.

Definition 135 *An IFS $A = \langle \mu_A, \lambda_A \rangle$ in a Γ-semihypergroup S is said to be an $(\in, \in \vee q)$-intuitionistic fuzzy $(1, 2)$ Γ-hyperideal of a Γ-semihypergroup S if for all $x, y, z, w, a \in H$, $\gamma, \delta, \rho \in \Gamma$, $t_1, t_2 \in (0, 0.5]$ and $s_1, s_2 \in [0.5, 1)$ or $t_1, t_2 \in (0.5, 1]$ and $s_1, s_2 \in [0, 0.5)$, the following conditions hold:*

$(IFB3)$ $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A \Rightarrow (w)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee qA$ for each $w \in x\gamma y$.

$(IFB4)$.$x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A \Rightarrow (w)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee qA$ for each $w \in x\gamma a\delta(y\rho z)$.

Example 136 *Let $H = \{1, 2, 3, 4, 5\}$ and $\Gamma = \{\gamma, \delta\}$ be two non-empty sets. Then, H is a Γ-semihypergroup defined by the following Cayley tables.*

γ	1	2	3	4	5
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
3	$\{1\}$	$\{1\}$	$\{3\}$	$\{3\}$	$\{3\}$
4	$\{1\}$	$\{1\}$	$\{3, 4\}$	$\{3, 4\}$	$\{5\}$
5	$\{1\}$	$\{1\}$	$\{3, 4\}$	$\{3, 4\}$	$\{5\}$

δ	1	2	3	4	5
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
3	$\{1\}$	$\{1\}$	$\{3\}$	$\{3\}$	$\{3\}$
4	$\{1\}$	$\{1\}$	$\{3\}$	$\{3, 4\}$	$\{5\}$
5	$\{1\}$	$\{1\}$	$\{3\}$	$\{3, 4\}$	$\{5\}$

1) Let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS in a Γ-semihypergroup H defined as:

$$\begin{aligned} \mu_A(1) &= 0.8, \mu_A(2) = 0.75, \mu_A(3) = 0.7, \mu_A(4) = \mu_A(5) = 0.65, \\ \lambda_A(1) &= 0.1, \lambda_A(2) = 0.2, \lambda_A(3) = 0.25, \lambda_A(4) = \lambda_A(5) = 0.4 \end{aligned}$$

Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy right Γ-hyperideal of H.

2) Let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS in a Γ-semihypergroup H defined as:

$$\begin{aligned} \mu_A(1) &= 0.9, \mu_A(2) = 0.8, \mu_A(3) = \mu_A(4) = 0.75, \mu_A(5) = 0.7, \\ \lambda_A(1) &= 0.1, \lambda_A(2) = 0.2, \lambda_A(3) = \lambda_A(4) = 0.25\lambda_A(5) = 0.3 \end{aligned}$$

Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy left Γ-hyperideal of H.

3) Let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS in a Γ-semihypergroup H defined as:

$$\begin{aligned} \mu_A(1) &= 0.9, \mu_A(2) = \mu_A(4) = 0.8, \mu_A(3) = 0.7, \mu_A(5) = 0.6, \\ \lambda_A(1) &= 0.1, \lambda_A(2) = \lambda_A(4) = 0.15, \lambda_A(3) = 0.2, \lambda_A(5) = 0.3 \end{aligned}$$

Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H.

4) Let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS in a Γ-semihypergroup H defined as:

$$\begin{aligned} \mu_A(1) &= \mu_A(2) = 0.9, \mu_A(3) = \mu_A(4) = 0.7, \mu_A(5) = 0.6, \\ \lambda_A(1) &= \lambda_A(2) = 0.1, \lambda_A(3) =, \lambda_A(4) = 0.2, \lambda_A(5) = 0.3 \end{aligned}$$

Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy (1,2)-Γ-hyperideal of H

Proposition 137 *An IFS $A = \langle \mu_A, \lambda_A \rangle$ of a Γ-semihypergroup H is an intuitionistic fuzzy sub-Γ-semihypergroup if and only if it satisfy for all $x, y \in H$, $\gamma \in \Gamma$, $t_1, t_2 \in (0, 1]$ and $s_1, s_2 \in [0, 1)$, $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A \Longrightarrow (z)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in A$ for each $z \in x\gamma y$.*

Proof. Let us suppose that $A = \langle \mu_A, \lambda_A \rangle$ is an intuitionistic fuzzy sub-Γ-semihypergroup of H. Let $x, y \in H$, $t_1, t_2 \in (0, 1]$ and $s_1, s_2 \in [0, 1)$ and let $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A$. Then, $\mu_A(x) \geq t_1$ and $\lambda_A(x) \leq s_1$, and $\mu_A(y) \geq t_2$ and $\lambda_A(y) \leq s_2$. Since by given condition

$$\begin{aligned} \inf_{z \in x\gamma y} \mu_A(z) &\geq \min\{\mu_A(x), \mu_A(y)\} \text{ and } \sup_{z \in x\gamma y} \lambda_A(z) \leq \max\{\lambda_A(x), \lambda_A(y)\} \\ \inf_{z \in x\gamma y} \mu_A(z) &\geq m\{t_1, t_2\} \text{ and } \sup_{z \in x\gamma y} \lambda_A(z) \leq \{s_1, s_2\}. \end{aligned}$$

So, $(z)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in A$ for all $z \in x\gamma y$. Thus, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in)$-intuitionistic fuzzy sub-Γ-semihypergroup of H.

Conversely, suppose that $A = \langle \mu_A, \lambda_A \rangle$ satisfies the given condition. We show that $\inf_{z \in x\gamma y} \mu_A(z) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\sup_{z \in x\gamma y} \lambda_A(z) \leq \max\{\lambda_A(x), \lambda_A(y)\}$. On contrary assume that there exist $x, y \in H$ and $\gamma \in \Gamma$ such that $\inf_{z \in x\gamma y} \mu_A(z) < \min\{\mu_A(x), \mu_A(y)\}$ and $\sup_{z \in x\gamma y} \lambda_A(z) > \max\{\lambda_A(x), \lambda_A(y)\}$. Let $t \in (0, 1]$ and $s \in [0, 1)$ be such that $\inf_{z \in x\gamma y} \mu_A(z) < t < \min\{\mu_A(x), \mu_A(y)\}$ and $\sup_{z \in x\gamma y} \lambda_A(z) > s > \max\{\lambda_A(x), \lambda_A(y)\}$. Then, $x(t, s) \in A$ and $y(t, s) \in A$ but $(z)(t, s)\overline{\in} A$ for some $z \in x\gamma y$, which is a contradiction to the hypothesis. Hence, $\inf_{z \in x\gamma y} \mu_A(z) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\sup_{z \in x\gamma y} \lambda_A(z) \leq \max\{\lambda_A(x), \lambda_A(y)\}$. ■

Proposition 138 *An IFS $A = \langle \mu_A, \lambda_A \rangle$ of a Γ-semihypergroup H is an intuitionistic fuzzy bi-Γ-hyperideal of H if and only if it satisfies for all $x, y, z \in H$, $\gamma, \delta \in \Gamma$ and $t_1, t_2 \in (0, 1]$ and $s_1, s_2 \in [0, 1)$,*

(a) $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A \Longrightarrow (z)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in A$, for each $z \in x\gamma y$

(b) $x(t_1, s_1) \in A$ and $z(t_2, s_2) \in A \Longrightarrow (u)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in A$ for each $z \in x\gamma y\delta z$.

Proof. Proof follows from Proposition 137. ■

Theorem 139 *Let $A = \langle \mu_A, \lambda_A \rangle$ be IFS in Γ-semihypergroup H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of a Γ-semihypergroup H if and only if the following conditions hold for all $x, y, z, u \in H$ and $\gamma, \delta \in \Gamma$;*

(a) $\inf_{z \in x\gamma y} \mu_A(z) \geq \min\{\mu_A(x), \mu_A(y), 0.5\}$ *and* $\sup_{z \in x\gamma y} \lambda_A(u) \leq \max\{\lambda_A(x), \lambda_A(y), 0.5\}$.

(b) $\inf_{u \in x\gamma y\delta z} \mu_A(u) \geq \min\{\mu_A(x), \mu_A(y), 0.5\}$ *and* $\sup_{u \in x\gamma y\delta z} \lambda_A(u) \leq \max\{\lambda_A(x), \lambda_A(y), 0.5\}$.

Proof. Suppose that $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of a Γ-semihypergroup H.

(a) Let $x, y \in H$. We consider the following cases:

(1) $\min\{\mu_A(x), \mu_A(y)\} < 0.5$ and $\max\{\lambda_A(x), \lambda_A(y)\} > 0.5$

(2) $\min\{\mu_A(x), \mu_A(y)\} \geq 0.5$ and $\max\{\lambda_A(x), \lambda_A(y)\} \leq 0.5$

(3) $\min\{\mu_A(x), \mu_A(y)\} < 0.5$ and $\max\{\lambda_A(x), \lambda_A(y)\} < 0.5$

Case (1) Assume that

$$\inf_{z \in x\gamma y} \mu_A(z) < \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \sup_{z \in x\gamma y} \lambda_A(z) > \max\{\lambda_A(x), \lambda_A(y), 0.5\}.$$

Then, $\inf_{z \in x\gamma y} \mu_A(z) < \min\{\mu_A(x), \mu_A(y)\}$ and $\sup_{z \in x\gamma y} \lambda_A(z) > \max\{\lambda_A(x), \lambda_A(y)\}$

Choose $t \in (0, 1]$ and $s \in [0, 1)$ such that

$$\inf_{z \in x\gamma y} \mu_A(z) < t < \min\{\mu_A(x), \mu_A(y)\} \text{ and } \sup_{z \in x\gamma y} \lambda_A(z) > s > \max\{\lambda_A(x), \lambda_A(y)\}$$

Case 1: If $\min\{\mu_A(x), \mu_A(y)\} < 0.5$ and $\max\{\lambda_A(x), \lambda_A(y)\} > 0.5$, then $x(t,s) \in A$ and $y(t,s) \in A$, but $(z)(t,s)\overline{\in \vee q}A$, a contradictsfor all $z \in x\gamma y$.

Case (2) If $\min\{\mu_A(x), \mu_A(y)\} \geq 0.5$ and $\max\{\lambda_A(x), \lambda_A(y)\} \leq 0.5$, then $\inf_{z\in x\gamma y} \mu_A(z) < 0.5$ and $\sup_{z\in x\gamma y} \lambda_A(z) > 0.5$. Thus, $x(0.5, 0.5) \in A$ and $y(0.5, 0.5) \in A$, but $(z)(0.5, 0.5)\overline{\in \vee q}A$ for all $z \in x\gamma y$, a contradicts. Case (3) If $\min\{\mu_A(x), \mu_A(y)\} < 0.5$ and $\max\{\lambda_A(x), \lambda_A(y)\} < 0.5$, then $\mu_A(z) < \min\{\mu_A(x), \mu_A(y)\}$ and $\lambda_A(xy) > 0.5$. Thus, $x(t,s) \in A$ and $y(t,s) \in A$, but $(z)(t,s)\overline{\in \vee q}A$ for all $z \in x\gamma y$, a contradicts. Therefore, $\inf_{z\in x\gamma y} \mu_A(z) \geq \min\{\mu_A(x), \mu_A(y), 0.5\}$ and $\sup_{z\in x\gamma y} \lambda_A(z) \leq \max\{\lambda_A(x), \lambda_A(y), 0.5\}$.

(b) Now, let $x, y, z \in H$. We consider the following cases

(1) $\min\{\mu_A(x), \mu_A(z)\} < 0.5$ and $\max\{\lambda_A(x), \lambda_A(z)\} > 0.5$

(2) $\min\{\mu_A(x), \mu_A(z)\} \geq 0.5$ and $\max\{\lambda_A(x), \lambda_A(z)\} \leq 0.5$

(3) $\min\{\mu_A(x), \mu_A(z)\} < 0.5$ and $\max\{\lambda_A(x), \lambda_A(z)\} < 0.5$

(1) Assume that for some

$$\inf_{u\in x\gamma y\delta z} \mu_A(z) < \min\{\mu_A(x), \mu_A(z), 0.5\} \text{ and } \sup_{u\in x\gamma y\delta z} \lambda_A(z) > \max\{\lambda_A(x), \lambda_A(z), 0.5\}$$
$$\inf_{u\in x\gamma y\delta z} \mu_A(z) < \min\{\mu_A(x), \mu_A(z)\} \text{ and } \sup_{u\in x\gamma y\delta z} \lambda_A(z) > \max\{\lambda_A(x), \lambda_A(z)\}$$

Choose $t \in (0,1]$ and $s \in [0,1)$ such that

$$\inf_{u\in x\gamma y\delta z} \mu_A(z) < t < \min\{\mu_A(x), \mu_A(z)\} \text{ and } \sup_{u\in x\gamma y\delta z} \lambda_A(z) > s > \max\{\lambda_A(x), \lambda_A(z)\}$$

Then $x(t,s) \in A$ and $z(t,s) \in A$, but $(u)(t,s)\overline{\in \vee q}A$ for all $u \in x\gamma y\delta z$, which is a contradiction.

Case (2) If $\min\{\mu_A(x), \mu_A(z)\} \geq 0.5$ and $\max\{\lambda_A(x), \lambda_A(z)\} \leq 0.5$, then $\mu_A(u) < 0.5$ and $\lambda_A(u) > 0.5$ for some $u \in x\gamma y\delta z$. Since, $x(0.5, 0.5) \in A$ and $z(0.5, 0.5) \in A$, but $(u)(0.5, 0.5)\overline{\in \vee q}A$ for all $u \in x\gamma y\delta z$, which is a contradiction. Case (3) If $\min\{\mu_A(x), \mu_A(z)\} < 0.5$ and $\max\{\lambda_A(x), \lambda_A(z)\} < 0.5$, then $\mu_A(z) < \min\{\mu_A(x), \mu_A(z)\}$ and $\lambda_A(z) > 0.5$. Thus, $x(t,s) \in A$ and $z(t,s) \in A$, but $(u)(t,s)\overline{\in \vee q}A$ for all $u \in x\gamma y\delta z$, which is again a contradiction. Therefore,

$$\inf_{u\in x\gamma y\delta z}\mu_A(u)\geq \min\{\mu_A(x),\mu_A(z),0.5\} \text{ and } \sup_{u\in x\gamma y\delta z}\lambda_A(u)\leq \max\{\lambda_A(x),\lambda_A(z),0.5\}.$$

Conversely, assume that $A=\langle\mu_A,\lambda_A\rangle$ satisfy (a) and (b). Let for any $x,y\in S$ and $t_1,t_2\in(0,1]$ and $s_1,s_2\in[0,1)$, such that $x(t_1,s_1)\in A$ and $y(t_2,s_2)\in A$. Then, $\mu_A(x)\geq t_1$ and $\lambda_A(x)\leq s_1$, and $\mu_A(y)\geq t_2$ and $\lambda_A(y)\leq s_2$. Now we have for each $z\in x\gamma y$

$$\begin{aligned}\inf_{u\in x\gamma y}\mu_A(z) &\geq \min\{\mu_A(x),\mu_A(y),0.5\} \text{ and } \sup_{u\in x\gamma y}\lambda_A(u)\leq \max\{\lambda_A(x),\lambda_A(y),0.5\}\\ \inf_{u\in x\gamma y}\mu_A(z) &\geq \min\{t_1,t_2,0.5\} \text{ and } \sup_{u\in x\gamma y\delta z}\lambda_A(z)\leq \max\{s_1,s_2,0.5\}.\end{aligned}$$

Then, we have the following cases

(1) $\min\{t_1,t_2\}\leq 0.5$ and $\max\{s_1,s_2\}\geq 0.5$

(2) $\min\{t_1,t_2\}>0.5$ and $\max\{s_1,s_2\}<0.5$

Case (1) If $\min\{t_1,t_2\}\leq 0.5$ and $\max\{s_1,s_2\}\geq 0.5$. Then, $\mu_A(u)\geq\min\{t_1,t_2\}$ and $\lambda_A(u)\leq\max\{s_1,s_2\}$, which implies that $(u)(m\{t_1,t_2\},M\{s_1,s_2\})\in A$

Case(2) If $\min\{t_1,t_2\}>0.5$ and $\max\{s_1,s_2\}<0.5$, then $\mu_A(u)\geq 0.5$ and $\lambda_A(u)\leq 0.5$ for each $u\in x\gamma y$, which implies that $\mu_A(u)+\min\{t_1,t_2\}>0.5+0.5=1$ and $\lambda_A(u)+\max\{s_1,s_2\}<0.5+0.5=1$ for each $u\in x\gamma y$. Therefore, $(u)(m\{t_1,t_2\},M\{s_1,s_2\})qA$. Hence, $(u)(m\{t_1,t_2\},M\{s_1,s_2\})\in\vee qA$, for each $u\in x\gamma y$.

Let $x,y,z\in H$, $\gamma,\delta\in\Gamma$, $t_1,t_2\in(0,1]$ and $s_1,s_2\in[0,1)$ such that $x(t_1,s_1)\in A$ and $y(t_2,s_2)\in A$. Then, $\mu_A(x)\geq t_1$ and $\lambda_A(x)\leq s_1$, and $\mu_A(y)\geq t_2$ and $\lambda_A(y)\leq s_2$. Now we have

$$\begin{aligned}\inf_{u\in x\gamma y\delta z}\mu_A(u) &\geq \min\{\mu_A(x),\mu_A(z),0.5\} \text{ and } \sup_{u\in x\gamma y\delta z}\lambda_A(u)\leq \max\{\lambda_A(x),\lambda_A(z),0.5\}\\ \inf_{u\in x\gamma y\delta z}\mu_A(u) &\geq \min\{t_1,t_2,0.5\} \text{ and } \sup_{u\in x\gamma y\delta z}\lambda_A(u)\leq \max\{s_1,s_2,0.5\} \text{ for each}\end{aligned}$$

Then, we have the following cases

(3) $\min\{t_1,t_2\}\leq 0.5$ and $\max\{s_1,s_2\}\geq 0.5$

(4) $\min\{t_1, t_2\} > 0.5$ and $\max\{s_1, s_2\} < 0.5$

Case (3) If $\min\{t_1, t_2\} \leq 0.5$ and $\max\{s_1, s_2\} \geq 0.5$, then $\mu_A(u) \geq \min\{t_1, t_2\}$ and $\lambda_A(u) \leq \max\{s_1, s_2\}$ for each $u \in x\gamma y\delta z$, which implies that $(z)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in A$ for each $u \in x\gamma y\delta z$.

Case(4) If $\min\{t_1, t_2\} > 0.5$ and $\max\{s_1, s_2\} < 0.5$, then $\mu_A(u) \geq 0.5$ and $\lambda_A(u) \leq 0.5$ for each $u \in x\gamma y\delta z$, which implies that $\mu_A(z) + \min\{t_1, t_2\} > 0.5 + 0.5 = 1$ and $\lambda_A(z) + \max\{s_1, s_2\} < 0.5 + 0.5 = 1$. Therefore, $(u)(m\{t_1, t_2\}, M\{s_1, s_2\})qA$ for each $u \in x\gamma y\delta z$. Hence, $(u)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee qA$. This completes the proof. ■

Remark 140 *Every intuitionistic fuzzy bi-*Γ*-hyperideal of a* Γ*-semihypergroup* H *is an* $(\in, \in \vee q)$*-intuitionistic fuzzy bi-*Γ*-hyperideal of* H*. But the converse is not true in general.*

Example 141 *Let* $H = \{1, 2, 3, 4, 5\}$ *and* $\Gamma = \{\gamma, \delta\}$ *be two non-empty sets. Then,* H *is a* Γ*-semihypergroup defined by the following Cayley tables.*

γ	1	2	3	4	5
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
3	$\{1\}$	$\{1\}$	$\{3\}$	$\{3\}$	$\{3\}$
4	$\{1\}$	$\{1\}$	$\{3,4\}$	$\{3,4\}$	$\{5\}$
5	$\{1\}$	$\{1\}$	$\{3,4\}$	$\{3,4\}$	$\{5\}$

δ	1	2	3	4	5
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
3	$\{1\}$	$\{1\}$	$\{3\}$	$\{3\}$	$\{3\}$
4	$\{1\}$	$\{1\}$	$\{3\}$	$\{3,4\}$	$\{5\}$
5	$\{1\}$	$\{1\}$	$\{3\}$	$\{3,4\}$	$\{5\}$

Let $A = \langle \mu_A, \lambda_A \rangle$ *be an IFS in a* Γ*-semihypergroup* H *defined by* $\mu_A(1) = 0.9$, $\mu_A(2) = \mu_A(4) = 0.8$, $\mu_A(3) = 0.7$, $\mu_A(5) = 0.6$, *and* $\lambda_A(1) = 0.1$, $\lambda_A(2) = \lambda_A(4) = 0.15$, $\lambda_A(3) = 0.2$, $\lambda_A(5) = 0.3$*. Then,* $A = \langle \mu_A, \lambda_A \rangle$ *is an* $(\in, \in \vee q)$*-intuitionistic fuzzy bi-*Γ*-hyperideal of* S *but* $A = \langle \mu_A, \lambda_A \rangle$ *is not an intuitionistic fuzzy bi-*Γ*-hyperideal.*

Remark 142 *From above Remark and Example, we can say that an* $(\in, \in \vee q)$*-intuitionistic fuzzy bi-*Γ*-hyperideal of* H *is a generalization of an intuitionistic fuzzy bi-*Γ*-hyperideal of* H*.*

Theorem 143 *Let $A = \langle \mu_A, \lambda_A \rangle$ be IFS in a Γ-semihypergroup H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy $(1,\ 2)$ Γ-hyperideal of a Γ-semihypergroup H if and only if the following conditions hold;*

(a) $\inf_{w \in x\gamma y} \mu_A(u) \geq \min\{\mu_A(x), \mu_A(y), 0.5\}$ *and* $\lambda_A(u) \leq \max\{\lambda_A(x), \lambda_A(y), 0.5\}$,

(b) $\inf_{w \in x\gamma a\delta(y\rho z)} \mu_A(w) \geq \min\left\{\begin{array}{c} \mu_A(x), \mu_A(y), \\ \mu_A(z), 0.5 \end{array}\right\}$ *and* $\lambda_A(w) \leq \max\left\{\begin{array}{c} \lambda_A(x), \lambda_A(y), \\ \lambda_A(z), 0.5 \end{array}\right\}$.

Proof. Straightforward. ■

Remark 144 *If $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H, then $A = \langle \mu_A, \lambda_A \rangle$ need not to be an $(\in, \in \vee q)$-intuitionistic fuzzy left (right) Γ-hyperideal of H.*

Example 145 *Let $H = \{1, 2, 3, 4\}$ and $\Gamma = \{\gamma, \delta\}$be a Γ-semihypergroup with the following Cayley table.*

γ	1	2	3	4	δ	1	2	3	4
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1\}$	$\{1,4\}$	$\{1\}$	2	$\{1\}$	$\{1\}$	$\{2,4\}$	$\{1\}$
3	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	3	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
4	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	4	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

(1) *Let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS defined as; $\mu_A(1) = \mu_A(2) = 0.3$, $\mu_A(3) = \mu_A(4) = 0.1$ and $\lambda_A(1) = \lambda_A(2) = 0.5$, $\lambda_A(3) = \lambda_A(4) = 0.8$. Then, clearly $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H but does not $(\in, \in \vee q)$-intuitionistic fuzzy right Γ-hyperideal of S because*

$$\inf \mu_A(2\gamma 3) \ngeq \min\{\mu_A(2), 0.5\}$$
$$\sup \lambda_A(2\gamma 3) \nleq \max\{\mu_A(2), 0.5\}$$

(1) *Let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS defined as; $\mu_A(1) = \mu_A(3) = 0.4$, $\mu_A(2) = \mu_A(4) = 0.2$ and $\lambda_A(1) = \lambda_A(3) = 0.5$, $\mu_A(2) = \mu_A(4) = 0.7$. Then, clearly $A = \langle \mu_A, \lambda_A \rangle$ is an*

$(\in, \in \vee q)$-*intuitionistic fuzzy bi-*Γ*-hyperideal of* H *but not* $(\in, \in \vee q)$-*intuitionistic fuzzy left* Γ*-hyperideal of* H *because*

$$\begin{aligned} \inf \mu_A (2\gamma 3) &\ngeq \min \{\mu_A (3), 0.5\} \\ \sup \lambda_A (2\gamma 3) &\nleq \max \{\mu_A (3), 0.5\} \end{aligned}$$

Proposition 146 (1)*Every* $(\in \vee q, \in \vee q)$-*intuitionistic fuzzy bi-*Γ*-hyperideal of* H *is* $(\in, \in \vee q)$-*intuitionistic fuzzy bi-*Γ*-hyperideal.*

(2) *Every* $(\in, \in)$-*intuitionistic fuzzy bi-*Γ*-hyperideal of* H *is* $(\in, \in \vee q)$-*intuitionistic fuzzy bi-*Γ*-hyperideal.*

Proof. Straightforward. ■

Example 141 shows that the converse of Proposition 146, is not true in general.

Theorem 147 *If* $\{A\}_{i\in\Lambda}$ *is a family of* $(\in, \in \vee q)$-*intuitionistic fuzzy bi-*Γ*-hyperideals of* H*, then* $\bigcap_{i\in\Lambda} A_i$ *is an* $(\in, \in \vee q)$-*intuitionistic fuzzy bi-*Γ*-hyperideal of* H*, where* $\bigcap_{i\in\Lambda} A_i = \langle \bigwedge_{i\in\Lambda} \mu_{A_i}, \bigvee_{i\in\Lambda} \lambda_{A_i} \rangle$.

Proof. Let $x, y \in H$, $\gamma \in \Gamma$and for each $z \in x\gamma y$. Then, we have

$$\begin{aligned} \inf_{z\in x\gamma y} \left(\bigwedge_{i\in\Lambda} \mu_{A_i} \right)(z) &= \bigwedge_{i\in\Lambda} \left(\inf_{z\in x\gamma y} \mu_{A_i}(z) \right) \geq \bigwedge_{i\in\Lambda} \left(\min \{\mu_{A_i}(x), \mu_{A_i}(y), 0.5\} \right) \\ &= \min \left\{ \bigwedge_{i\in\Lambda} \mu_{A_i}(x), \bigwedge_{i\in\Lambda} \mu_{A_i}(y), 0.5 \right\} \\ \inf_{z\in x\gamma y} \left(\bigwedge_{i\in\Lambda} \mu_{A_i} \right)(z) &\geq \min \left\{ \left(\bigwedge_{i\in\Lambda} \mu_{A_i} \right)(x), \left(\bigwedge_{i\in\Lambda} \mu_{A_i} \right)(y), 0.5 \right\} \text{ and} \\ \sup_{z\in x\gamma y} \left(\bigvee_{i\in\Lambda} \lambda_{A_i} \right)(z) &= \bigvee_{i\in\Lambda} \left(\sup_{z\in x\gamma y} \lambda_{A_i}(z) \right) \leq \bigvee_{i\in\Lambda} \left(\max \{\lambda_{A_i}(x), \lambda_{A_i}(y), 0.5\} \right) \\ &= \max \left\{ \bigvee_{i\in\Lambda} \lambda_{A_i}(x), \bigvee_{i\in\Lambda} \lambda_{A_i}(y), 0.5 \right\} \\ \sup_{z\in x\gamma y} \left(\bigvee_{i\in\Lambda} \lambda_{A_i} \right)(z) &\leq \max \left\{ \left(\bigvee_{i\in\Lambda} \lambda_{A_i} \right)(x), \left(\bigvee_{i\in\Lambda} \lambda_{A_i} \right)(y), 0.5 \right\} \end{aligned}$$

Now, let $x, y, z \in H$ and $u \in x\gamma y\delta z$. Then, we have

$$\begin{aligned}
\inf_{u\in x\gamma y\delta z}\left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(u) &= \bigwedge_{i\in\Lambda}\left(\inf_{u\in x\gamma y\delta z}\mu_{A_i}(z)\right) \geq \bigwedge_{i\in\Lambda}\left(\min\left\{\mu_{A_i}(x), \mu_{A_i}(z), 0.5\right\}\right) \\
&= \min\left\{\bigwedge_{i\in\Lambda}\mu_{A_i}(x), \bigwedge_{i\in\Lambda}\mu_{A_i}(z), 0.5\right\} \\
\inf_{u\in x\gamma y\delta z}\left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(u) &\geq \min\left\{\left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(x), \left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(z), 0.5\right\} \text{ and} \\
\sup_{u\in x\gamma y\delta z}\left(\bigvee_{i\in\Lambda}\lambda_{A_i}\right)(u) &= \bigvee_{i\in\Lambda}\left(\sup_{u\in x\gamma y\delta z}\lambda_{A_i}(u)\right) \leq \bigvee_{i\in\Lambda}\left(\max\left\{\lambda_{A_i}(x), \lambda_{A_i}(z), 0.5\right\}\right) \\
&= \max\left\{\bigvee_{i\in\Lambda}\lambda_{A_i}(x), \bigvee_{i\in\Lambda}\lambda_{A_i}(z), 0.5\right\} \\
\sup_{u\in x\gamma y\delta z}\left(\bigvee_{i\in\Lambda}\lambda_{A_i}\right)(u) &\leq \max\left\{\left(\bigvee_{i\in\Lambda}\lambda_{A_i}\right)(x), \left(\bigvee_{i\in\Lambda}\lambda_{A_i}\right)(z), 0.5\right\}
\end{aligned}$$

Hence $\bigcap_{i\in\Lambda} A_i = \langle\bigwedge_{i\in\Lambda}\mu_{A_i}, \bigvee_{i\in\Lambda}\lambda_{A_i}\rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H. ■

Remark 148 *If $\{A\}_{i\in\Lambda}$ is a family of $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideals of a Γ-semihypergroup H, then $\bigcup_{i\in\Lambda} A_i$ is not an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H, where $\bigcup_{i\in\Lambda} A_i = \langle\bigvee_{i\in\Lambda}\mu_{A_i}, \bigwedge_{i\in\Lambda}\lambda_{A_i}\rangle$. This shows in the following example..*

Example 149 *Let $H = \{1, 2, 3, 4\}$ and $\Gamma = \{\gamma, \delta\}$be a Γ-semihypergroup as defined in Example 145.*

Let $A = \langle\mu_A, \lambda_A\rangle$ and $B = \langle\mu_B, \lambda_B\rangle$ be IFS's of a Γ-semihypergroup H defined by $\mu_A(1) = \mu_A(2) = 0.4$, $\mu_A(3) = \mu_A(4) = 0$ and $\lambda_A(1) = \lambda_A(2) = 0.5$, $\lambda_A(3) = \lambda_A(4) = 0.8$ and $\mu_B(1) = \mu_B(3) = 0.4$, $\mu_B(2) = \mu_B(4) = 0$ and $\lambda_B(1) = \lambda_B(3) = 0.5$, $\mu_B(2) = \mu_B(4) = 0.7$. Then, both $A = \langle\mu_A, \lambda_A\rangle$ and $B = \langle\mu_B, \lambda_B\rangle$ are $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-ideals of H. But $A \cup B$ is not an $(\in, \in \vee q)$-intuitionistic fuzzy *bi-*Γ*-hyperideal of* H*. i.e.*

$$\begin{aligned}(\mu_A \vee \mu_B)(2\gamma 3) &\ngeq \min\{(\mu_A \vee \mu_B)(2), (\mu_A \vee \mu_B)(3), 0.5\} \\ (\lambda_A \vee \lambda_B)(2\gamma 3) &\nleq \min\{(\lambda_A \vee \lambda_B)(2), (\lambda_A \vee \lambda_B)(3), 0.5\}.\end{aligned}$$

Theorem 150 *If* $\{A_i\}_{i\in\Lambda}$ *is a family of* $(\in, \in \vee q)$*-intuitionistic fuzzy bi-*Γ*-hyperideals of* H *such that* $A_i \subseteq A_j$ *or* $A_j \subseteq A_i$ *for all* i*,* $j \in I$*, then* $\bigcup_{i\in\Lambda} A_i = \langle \bigvee_{i\in\Lambda} \mu_{A_i}, \bigwedge_{i\in\Lambda} \lambda_{A_i} \rangle$ *is an* $(\in, \in \vee q)$*-intuitionistic fuzzy bi-*Γ*-hyperideal of* H*.*

Proof. For all $x, y \in H$ and for each $z \in x\gamma y$ we have

$$\begin{aligned}\inf_{z\in x\gamma y}\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(z) &= \bigvee_{i\in\Lambda}\left(\inf_{z\in x\gamma y} \mu_{A_i}(z)\right) \geq \bigvee_{i\in\Lambda}[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5\}] \\ &= \left[\bigvee_{i\in\Lambda} \mu_{A_i}(x) \wedge \bigvee_{i\in\Lambda} \mu_{A_i}(y) \wedge 0.5\right] \\ &= \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right] \\ \inf_{z\in x\gamma y}\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(z) &\geq \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right].\end{aligned}$$

It is clear that

$$\bigvee_{i\in\Lambda}[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5] \leq \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right].$$

Assume that

$$\bigvee_{i\in\Lambda}[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5] \neq \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right].$$

Then, there exists t such that

$$\bigvee_{i\in\Lambda}[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5] < t < \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right]$$

Since $\mu_{A_i} \subseteq \mu_{A_j}$ or $\mu_{A_j} \subseteq \mu_{A_i}$ for all i, $j \in I$, so there exists $k \in I$ such that $t < \mu_{A_k}(x) \wedge \mu_{A_k}(y) \wedge 0.5$. On other hand $\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5 < t$ for all $i \in I$, a contradiction. Hence,

$$\bigvee_{i\in\Lambda} \left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5\right] = \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right]$$

and

$$\begin{aligned}
\sup_{z\in x\gamma y}\left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(z) &= \bigwedge_{i\in\Lambda}\left(\sup_{z\in x\gamma y} \lambda_{A_i}(z)\right) \leq \bigwedge_{i\in\Lambda} \left[\lambda_{A_i}(x) \vee \lambda_{A_i}(y) \vee 0.5\}\right] \\
&= \left[\bigwedge_{i\in\Lambda} \lambda_{A_i}(x) \vee \bigwedge_{i\in\Lambda} \lambda_{A_i}(y) \vee 0.5\right] \\
&= \left[\left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(y) \vee 0.5\right] \\
\sup_{z\in x\gamma y}\left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(z) &\leq \left[\left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(y) \vee 0.5\right].
\end{aligned}$$

It is clear that

$$\bigwedge_{i\in\Lambda} \left[\lambda_{A_i}(x) \vee \lambda_{A_i}(y) \vee 0.5\right] \geq \left[\left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda} \mu_{A_i}\right)(y) \vee 0.5\right].$$

Assume that

$$\bigwedge_{i\in\Lambda} \left[\lambda_{A_i}(x) \vee \lambda_{A_i}(y) \vee 0.5\right] \neq \left[\left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(y) \vee 0.5\right].$$

Then there exists t such that

$$\bigwedge_{i\in\Lambda} \left[\lambda_{A_i}(x) \vee \lambda_{A_i}(y) \vee 0.5\right] > t > \left[\left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(y) \vee 0.5\right]$$

Since $\lambda_{A_i} \subseteq \lambda_{A_j}$ or $\lambda_{A_j} \subseteq \lambda_{A_i}$ for all $i, j \in I$, so there exists $k \in I$ such that $k > \mu_{A_k}(x) \wedge \mu_{A_k}(y) \wedge 0.5$. On other hand $\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5 > t$ for all $i \in I$, a contradiction. Hence,

$$\bigwedge_{i\in\Lambda} [\lambda_{A_i}(x) \wedge \lambda_{A_i}(y) \wedge 0.5] = \left[\left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(x) \wedge \left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(y) \wedge 0.5\right]$$

For all $x, y, z \in H$ $\gamma, \delta \in \Gamma$ and $u \in x\gamma y\delta z$, we obtain

$$\begin{aligned}
\inf_{u\in x\gamma y\delta z} \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(u) &= \bigvee_{i\in\Lambda} (\mu_{A_i}(z)) \geq \bigvee_{i\in\Lambda} [\mu_{A_i}(x) \wedge \mu_{A_i}(z) \wedge 0.5\}] \\
&= \left[\bigvee_{i\in\Lambda} \mu_{A_i}(x) \wedge \bigvee_{i\in\Lambda} \mu_{A_i}(z) \wedge 0.5\right] \\
&= \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(z) \wedge 0.5\right] \\
\inf_{u\in x\gamma y\delta z} \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(z) &\geq \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(z) \wedge 0.5\right]
\end{aligned}$$

and

$$\begin{aligned}
\sup_{u\in x\gamma y\delta z} \left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(z) &= \bigwedge_{i\in\Lambda} \left(\sup_{u\in x\gamma y\delta z} \lambda_{A_i}(u)\right) \leq \bigwedge_{i\in\Lambda} [\lambda_{A_i}(x) \vee \lambda_{A_i}(z) \vee 0.5\}] \\
&= \left[\bigwedge_{i\in\Lambda} \lambda_{A_i}(x) \vee \bigwedge_{i\in\Lambda} \lambda_{A_i}(z) \vee 0.5\right] \\
&= \left[\left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(z) \vee 0.5\right] \\
\left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(z) &\leq \left[\left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(x) \vee \left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(z) \vee 0.5\right]
\end{aligned}$$

Hence, $\bigcap_{i\in\Lambda} A_i = \langle \bigvee_{i\in\Lambda} \mu_{A_i}, \bigwedge_{i\in\Lambda} \lambda_{A_i} \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H. ■

Definition 151 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ and $B = \langle \mu_B, \lambda_B \rangle$ be*

IFSs of H. Then, the 0.5-product of A and B is defined by:

$$\begin{aligned} A \circ_{0.5} B &= \langle \mu_A \circ_{0.5} \mu_B, \lambda_A \circ_{0.5} \lambda_B \rangle \\ (\mu_A \circ_{0.5} \mu_B)(x) &= \left\{ \begin{array}{ll} \bigvee_{x \in y\gamma z} \{\mu_A(y) \wedge \mu_B(z) \wedge 0.5\} & \text{if } x \in y\gamma z \\ 0 & \text{if } x \notin y\gamma z \end{array} \right\} \\ (\lambda_A \circ_{0.5} \lambda_B)(x) &= \left\{ \begin{array}{ll} \bigwedge_{x \in y\gamma z} \{\lambda_A(y) \vee \lambda_B(z) \vee 0.5\} & \text{if } x \in y\gamma z \\ 1 & \text{if } x \notin y\gamma z \end{array} \right\}. \end{aligned}$$

Let $A = \langle \mu_A, \lambda_A \rangle$ and $B = \langle \mu_B, \lambda_B \rangle$ be IFSs in H. Then,

$$\begin{aligned} A \cap_{0.5} B &= \langle \mu_A \wedge_{0.5} \mu_B, \lambda_A \vee_{0.5} \lambda_B \\ (\mu_A \wedge_{0.5} \mu_B)(x) &= \mu_A(x) \wedge \mu_B(x) \wedge 0.5 \text{ and} \\ (\lambda_A \vee_{0.5} \lambda_B)(x) &= \lambda_A(x) \vee \lambda_B(x) \vee 0.5. \end{aligned}$$

Remark 152 *If H is a Γ-semihypergroup and A, B, C, D are IFSs of H such that $A \subseteq B$ and $C \subseteq D$, then $A \circ_{0.5} B \subseteq C \circ_{0.5} D$.*

Proposition 153 *Let H be a Γ-semihypergroup, $A = \langle \mu_A, \lambda_A \rangle$ and $B = \langle \mu_B, \lambda_B \rangle$ be $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-ideals of H. Then, $A \cap_{0.5} B$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H.*

Proof. Straightforward. ■

Definition 154 *An $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H is called 0.5-idemoptent if $A \circ_{0.5} A = A$.*

Proposition 155 *Let H be a Γ-semihypergroup and A be an $(\in, \in \vee q)$-intuitionistic fuzzy sub-Γ-semihypergroup of H. Then, $A \circ_{0.5} A \subseteq A$.*

Proof. Let $A = \langle \mu_A, \lambda_A \rangle$ be an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H. Then, for each $x \in H$, we have two cases: (1) If $x \notin y\gamma z$. (2) If $x \in y\gamma z$.

Case 1 : If $x \notin y\gamma z$, then clearly

$$\begin{aligned} (\mu_A \circ_{0.5} \mu_A)(x) &= 0 \leq \mu_A(x) \text{ and } (\lambda_A \circ_{0.5} \lambda_A)(x) = 1 \geq \lambda_A \\ \text{Thus, } A \circ_{0.5} A &\subseteq A. \end{aligned}$$

Case 2 : If $x \in y\gamma z$, then

$$\begin{aligned} (\mu_A \circ_{0.5} \mu_A)(x) &= \bigvee_{x \in y\gamma z} \{\min\{\mu_A(y), \mu_A(z), 0.5\}\} \\ &\leq \bigvee_{x \in y\gamma z} \left\{\inf_{x \in y\gamma z} \mu_A(x)\right\} = \mu_A(x) \text{ for each } x \in y\gamma z \\ (\mu_A \circ_{0.5} \mu_A)(x) &\leq \mu_A(x) \end{aligned}$$

and

$$\begin{aligned} (\lambda_A \circ_{0.5} \lambda_A)(x) &= \bigwedge_{x = y\gamma z} \{\max\{\lambda_A(y), \lambda_A(z), 0.5\}\} \\ &\geq \bigwedge_{x \in y\gamma z} \left\{\sup_{x \in y\gamma z} \lambda_A(x)\right\} = \lambda_A(x) \text{ for each } x \in y\gamma z \\ (\lambda_A \circ_{0.5} \lambda_A)(x) &\geq \lambda_A(x) \end{aligned}$$

Thus, $A \circ_{0.5} A \subseteq A$. ■

Lemma 156 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ and $B = \langle \mu_B, \lambda_B \rangle$ be $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-ideals of H. Then, $A \circ_{0.5} B \subseteq \mathcal{S} \circ_{0.5} B$ (resp. $A \circ_{0.5} B \subseteq A \circ_{0.5} \mathcal{S}$).*

Theorem 157 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H. Then, $A \circ_{0.5} \mathcal{S} \circ_{0.5} A \subseteq A$, where $\mathcal{S} = \langle 1, 0 \rangle$, and always $1(x) = 1$ and $0(x) = 0$ for all $x \in H$.*

Proof. Let $x \in H$. Then, we have two cases. (1) If $x \notin y\gamma z$ for every $y, z \in H$ and $\gamma \in \Gamma$. (2) If $x \in y\gamma z$ for some $y, z \in H$ and $\gamma \in \Gamma$.

Case 1 : If $x \notin y\gamma z$, then clearly

$$\begin{aligned}
(\mu_A \circ_{0.5} 1 \circ_{0.5} \mu_A)(x) &= 0 \leq \mu_A(x) \text{ and } (\lambda_A \circ_{0.5} 0 \circ_{0.5} \lambda_A) = 1 \geq \lambda_A \\
\text{Thus, } A \circ_{0.5} \mathcal{S} \circ_{0.5} A &\subseteq A.
\end{aligned}$$

Case 2 :If $x \in y\gamma z$ for some $y, z \in H$, then we have

$$\begin{aligned}
(\mu_A \circ_{0.5} 1 \circ_{0.5} \mu_A)(x) &= \bigvee_{x \in y\gamma z} \{\min\{\mu_A(y), (1 \circ_{0.5} \mu_A)(z), 0.5\}\} \\
&= \bigvee_{x \in y\gamma z} \left\{\min\left\{\mu_A(y), \bigvee_{z \in t\gamma r} \{\min\{1(t), \mu_A(r), 0.5\}\}, 0.5\right\}\right\} \\
&= \bigvee_{x \in y\gamma z} \bigvee_{z \in t\gamma r} \{\min\{\mu_A(y), 1, \mu_A(r), 0.5\}\} \\
&= \bigvee_{x \in y\alpha t\gamma r} \{\min\{\mu_A(y), \mu_A(r), 0.5\}\}
\end{aligned}$$

Since $x \in yz \subseteq y\alpha(t\gamma r) = y\alpha t\gamma r$ and $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H, so we have $\mu_A(ytr) \geq \min\{\mu_A(y), \mu_A(r), 0.5\}$. Thus,

$$\begin{aligned}
\bigvee_{x \in y\alpha t\gamma r} \{\min\{\mu_A(y), \mu_A(r), 0.5\}\} &\leq \bigvee_{x \in y\alpha t\gamma r} \left\{\inf_{x \in y\alpha t\gamma r} \mu_A(x)\right\} = \mu_A(x) \text{ for all } x \in y\alpha t\gamma r \\
(\mu_A \circ_{0.5} 1 \circ_{0.5} \mu_A)(x) &\leq \mu_A(x)
\end{aligned}$$

and

$$\begin{aligned}
(\lambda_A \circ_{0.5} 0 \circ_{0.5} \lambda_A)(x) &= \bigwedge_{x \in y\gamma z} \{\max\{\lambda_A(y), (0 \circ_{0.5} \lambda_A)(z), 0.5\}\} \\
&= \bigwedge_{x \in y\gamma z} \left\{\max\left\{\mu_A(y), \bigwedge_{z \in t\gamma r} \{\max\{0(t), \lambda_A(r), 0.5\}\}, 0.5\right\}\right\} \\
&= \bigwedge_{x \in y\gamma z} \bigwedge_{z \in t\gamma r} \{\max\{\lambda_A(y), 0, \lambda_A(r), 0.5\}\} \\
&= \bigwedge_{x \in y\gamma t\gamma r} \{\max\{\lambda_A(y), \lambda_A(r), 0.5\}\}.
\end{aligned}$$

$A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H, so we have $\sup_{x \in y\gamma t\gamma r} \lambda_A(x) \leq \max\{\lambda_A(y), \lambda_A(r), 0.5\}$. Thus,

$$\begin{aligned} \bigwedge_{x \in y\gamma t\gamma r} \{\min\{\lambda_A(y), \lambda_A(r), 0.5\}\} &\geq \bigwedge_{x \in y\gamma t\gamma r} \{\lambda_A(x)\} = \lambda_A(x) \\ (\lambda_A \circ_{0.5} 0 \circ_{0.5} \lambda_A)(x) &\leq \lambda_A(x) \text{ for all } x \in y\alpha t\gamma r. \end{aligned}$$

Hence, $A \circ_{0.5} \mathcal{S} \circ_{0.5} A \subseteq A$. ■

Theorem 158 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an IFS of H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy sub-Γ-semihypergroup of H if and only if $A \circ_{0.5} A \subseteq A$.*

Theorem 159 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an IFS of H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H if and only if the following hold:*

(1) $A \circ_{0.5} A \subseteq A$,

(2) $A \circ_{0.5} \mathcal{S} \circ_{0.5} A \subseteq A$.

Proof. Let $A = \langle \mu_A, \lambda_A \rangle$ be an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H. Then, by Proposition 155 and Theorem 157, we have $A \circ_{0.5} A \subseteq A$ and $A \circ_{0.5} \mathcal{S} \circ_{0.5} A \subseteq A$.

Conversely, suppose that the given conditions hold. Now, let $x, y \in H$ such that $a \in x\gamma y$. Then, we have

$$\begin{aligned} \inf_{a \in x\gamma y} \mu_A(a) &\geq (\mu_A \circ_{0.5} \mu_A)(a) = \bigvee_{a \in s\gamma t} \{\min\{\mu_A(s), \mu_A(t), 0.5\}\} \\ &\geq \min\{\mu_A(x), \mu_A(y), 0.5\} \\ \inf_{a \in x\gamma y} \mu_A(a) &\geq \min\{\mu_A(x), \mu_A(y), 0.5\} \end{aligned}$$

and

$$
\begin{aligned}
\sup_{a\in x\gamma y}\lambda_A(a) &\le (\lambda_A\circ_{0.5}\lambda_A)(a)=\bigwedge_{a\in s\gamma t}\{\max\{\lambda_A(s),\lambda_A(t),0.5\}\}\\
&\le \max\{\lambda_A(x),\lambda_A(y),0.5\}\\
\sup_{a\in x\gamma y}\lambda_A(a) &\le \min\{\lambda_A(x),\lambda_A(y),0.5\}.
\end{aligned}
$$

Now, let $x,y,z\in H$ such that $a\in x\gamma y\gamma z$. Then, we have

$$
\begin{aligned}
\inf_{a\in x\gamma y\gamma z}\mu_A(a) &\ge (\mu_A\circ_{0.5}1\circ_{0.5}\mu_A)(a)=\bigvee_{a\in s\gamma t}\{\min\{\mu_A(s),(1\circ_{0.5}\mu_A)(t),0.5\}\}\\
&= \bigvee_{a\in s\gamma t}\left\{\min\left\{\mu_A(s),\bigvee_{t\in p\gamma q}\{\min\{1(p),\mu_A(q),0.5\}\},0.5\right\}\right\}\\
&= \bigvee_{a\in s\gamma t}\left\{\min\left\{\mu_A(s),\bigvee_{t\in p\gamma q}\{\min\{1,\mu_A(q),0.5\}\},0.5\right\}\right\}\\
&\ge \bigvee_{a\in s\gamma t}\bigvee_{t\in p\gamma q}\{\min\{\mu_A(s),\mu_A(q),0.5\}\}\ge\bigvee_{a\in s\alpha p\gamma q}\{\min\{\mu_A(s),\mu_A(q),0.5\}\}\\
&\ge \min\{\mu_A(x),\mu_A(z),0.5\}\\
\inf_{a\in x\gamma y\gamma z}\mu_A(a) &\ge \min\{\mu_A(x),\mu_A(z),0.5\}.
\end{aligned}
$$

and

$$
\begin{aligned}
\sup_{a\in x\gamma y\gamma z}\lambda_A(a) &\ge (\lambda_A\circ_{0.5}0\circ_{0.5}\lambda_A)(a)=\bigwedge_{a\in s\alpha t}\{\max\{\lambda_A(s),(0\circ_{0.5}\lambda_A)(t),0.5\}\}\\
&= \bigwedge_{a\in s\alpha t}\left\{\max\left\{\lambda_A(s),\bigwedge_{t\in p\gamma q}\{\max\{0(p),\lambda_A(q),0.5\}\},0.5\right\}\right\}\\
&= \bigwedge_{a\in s\alpha t}\left\{\max\left\{\lambda_A(s),\bigwedge_{t\in p\gamma q}\{\max\{0,\lambda_A(q),0.5\}\},0.5\right\}\right\}\\
&\le \bigwedge_{a\in s\alpha t}\bigwedge_{t\in p\gamma q}\{\max\{\lambda_A(s),\lambda_A(q),0.5\}\}\ge\bigwedge_{a=spq}\{\max\{\lambda_A(s),\lambda_A(q),0.5\}\}\\
&\le \max\{\lambda_A(x),\lambda_A(z),0.5\}\\
\sup_{a\in x\gamma y\gamma z}\lambda_A(a) &\le \max\{\lambda_A(x),\lambda_A(z),0.5\}.
\end{aligned}
$$

Hence, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H. ■

Theorem 160 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an IFS of H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy left (resp. right, two sided) Γ-hyperideal of H if and only if the following hold: $\mathcal{S} \circ_{0.5} A \subseteq A$ (resp. $A \circ_{0.5} \mathcal{S} \subseteq A$, $A \circ_{0.5} \mathcal{S} \subseteq A$ and $\mathcal{S} \circ_{0.5} A \subseteq A$).*

Proof. Let $A = \langle \mu_A, \lambda_A \rangle$ be an $(\in, \in \vee q)$-intuitionistic fuzzy left Γ-hyperideal of H and let $x \in H$. Then, we have two cases (1) If $x \notin y\gamma z$ for every $y, z \in H$ and (2) If $x \in y\gamma z$ for some $y, z \in H$.

Case (1) If $x \notin y\gamma z$ for every $y, z \in H$, then clearly $(1 \circ_{0.5} \mu_A)(x) = 0 \leq \mu_A(x)$ and $(0 \circ_{0.5} \lambda_A)(x) = 1 \leq \mu_A(x)$.

Case (2) If $x \in y\gamma z$ for some $y, z \in H$, then we have

$$\begin{aligned} (1 \circ_{0.5} \mu_A)(x) &= \bigvee_{x \in y\gamma z} \{\min\{1(y), \mu_A(z), 0.5\}\} \\ &= \bigvee_{x \in y\gamma z} \{\min\{1, \mu_A(z), 0.5\}\} \\ &= \bigvee_{x \in y\gamma z} \{\min\{\mu_A(z), 0.5\}\} \end{aligned}$$

Since $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy left Γ-hyperideal of H, so

$$\inf_{x \in y\gamma z} \mu_A(x) \geq \min\{\mu_A(z), 0.5\}.$$

Thus,

$$\begin{aligned} \bigvee_{x \in y\gamma z} \{\min\{\mu_A(z), 0.5\}\} &\leq \bigvee_{x \in y\gamma z} \left(\inf_{x \in y\gamma z} \mu_A(x)\right) = \inf_{x \in y\gamma z} \mu_A(x) \\ (1 \circ_{0.5} \mu_A)(x) &\leq \inf_{x \in y\gamma z} \mu_A(x) \leq \mu_A(x) \end{aligned}$$

and

$$\begin{aligned}(0 \circ_{0.5} \lambda_A)(x) &= \bigwedge_{x \in y\gamma z} \{\max\{0(y), \lambda_A(z), 0.5\}\} \\ &= \bigwedge_{x \in y\gamma z} \{\max\{0, \lambda_A(z), 0.5\}\} \\ &= \bigwedge_{x \in y\gamma z} \{\max\{\lambda_A(z), 0.5\}\}\end{aligned}$$

Since $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy left Γ-hyperideal of H, so

$$\lambda_A(x) \leq \max\{\lambda_A(z), 0.5\}.$$

Thus,

$$\begin{aligned}\bigwedge_{x \in y\gamma z} \{\max\{\lambda_A(z), 0.5\}\} &\geq \bigwedge_{x \in y\gamma z} \left(\sup_{x \in y\gamma z} \lambda_A(x)\right) = \lambda_A(x) \\ (0 \circ_{0.5} \lambda_A)(x) &\geq \sup_{x \in y\gamma z} \lambda_A(x) \geq \lambda_A(x)\end{aligned}$$

Hence, $\mathcal{S} \circ_{0.5} A \subseteq A$.

Conversely, suppose that the given condition holds and let $x, y \in H$ such that $a \in x\gamma y$. Then,

$$\begin{aligned}\inf_{a \in x\gamma y} \mu_A(a) &\geq (1 \circ_{0.5} \mu_A)(a) = \bigvee_{x \in p\alpha q} \{\min\{1(p), \mu_A(q), 0.5\}\} \\ &\geq \bigvee_{x \in p\alpha q} \{\min\{1, \mu_A(q), 0.5\}\} = \bigvee_{x \in p\alpha q} \{\min\{\mu_A(q), 0.5\}\} \\ &\geq \min\{\mu_A(y), 0.5\}, \text{ because } a \in x\gamma y.\end{aligned}$$

and

$$\begin{aligned}
\sup_{a\in x\gamma y}\lambda_A(a) &\geq (0\circ_{0.5}\lambda_A)(a)=\bigwedge_{x\in p\alpha q}\{\max\{0(p),\lambda_A(q),0.5\}\} \\
&\leq \bigwedge_{x\in p\alpha q}\{\max\{0,\lambda_A(q),0.5\}\}=\bigwedge_{x\in p\alpha q}\{\max\{\lambda_A(q),0.5\}\} \\
&\leq \max\{\mu_A(y),0.5\}, \text{ because } a\in x\gamma y.
\end{aligned}$$

Hence, $A=\langle\mu_A,\lambda_A\rangle$ is an $(\in,\in\vee q)$-intuitionistic fuzzy left Γ-hyperideal of H. This completes the proof. ■

Theorem 161 *Let $A=\langle\mu_A,\lambda_A\rangle$ and $B=\langle\mu_B,\lambda_B\rangle$ be $(\in,\in\vee q)$-intuitionistic fuzzy bi-Γ-hyperideals of H. Then, $A\circ_{0.5}B$ is an $(\in,\in\vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H.*

Proof. Let $A=\langle\mu_A,\lambda_A\rangle$ and $B=\langle\mu_B,\lambda_B\rangle$ be $(\in,\in\vee q)$-intuitionistic fuzzy bi-Γ-ideals of H and let $x\in H$. Then, we have two cases (1) If $x\notin y\gamma z$ for any $y,z\in H$ and $\gamma\in\Gamma$. (2) If $x\in y\gamma z$ for some $y,z\in H$ and $\gamma\in\Gamma$.

Case 1 : If $x\notin y\gamma z$ for any $y,z\in H$ and $\gamma\in\Gamma$, then

$$((\mu_A\circ_{0.5}\mu_B)\circ_{0.5}(\mu_A\circ_{0.5}\mu_B))(x)=0\leq(\mu_A\circ_{0.5}\mu_B)(x)$$

and

$$((\lambda_A\circ_{0.5}\lambda_B)\circ_{0.5}(\lambda_A\circ_{0.5}\lambda_B))(x)=1\geq(\lambda_A\circ_{0.5}\lambda_B)(x)$$

Thus, $A\circ_{0.5}A\subseteq A$ in this case.

Case 2 : If $x \in y\gamma z$ for some $y, z \in H$ and $\gamma \in \Gamma$, then

$$
\begin{aligned}
\left(\left(\mu_A \circ_{0.5} \mu_B\right) \circ_{0.5} \left(\mu_A \circ_{0.5} \mu_B\right)\right)(a) &= \bigvee_{x \in y\gamma z} \left\{ \begin{array}{c} \left(\mu_A \circ_{0.5} \mu_B\right)(y) \wedge \left(\mu_A \circ_{0.5} \mu_B\right)(z) \\ \wedge 0.5 \end{array} \right\} \\
&= \bigvee_{x \in y\gamma z} \left\{ \begin{array}{c} \bigvee_{y \in a\gamma b} \left\{\mu_A(a) \wedge \mu_B(b) \wedge 0.5\right\} \\ \wedge \bigvee_{z \in p\beta q} \left\{\mu_A(p) \wedge \mu_B(q) \wedge 0.5\right\} \end{array} \right\} \\
&= \bigvee_{x \in y\gamma z} \bigvee_{y \in a\gamma b} \bigvee_{z \in p\beta q} \left\{ \begin{array}{c} \left\{\mu_A(a) \wedge \mu_B(b) \wedge 0.5\right\} \\ \wedge \left\{\mu_A(p) \wedge \mu_B(q) \wedge 0.5\right\} \end{array} \right\} \\
&= \bigvee_{x \in y\gamma z} \bigvee_{y \in a\gamma b} \bigvee_{z \in p\beta q} \left\{ \begin{array}{c} \left\{\mu_A(a) \wedge \mu_A(p) \right. \\ \left. \wedge \mu_B(b) \wedge \mu_B(q) \wedge 0.5\right\} \end{array} \right\} \\
&\leq \bigvee_{x \in y\gamma z} \bigvee_{y \in a\gamma b} \bigvee_{z \in p\beta q} \left\{\mu_A(a) \wedge \mu_A(p) \wedge 0.5 \wedge \mu_B(q)\right\}
\end{aligned}
$$

Since $x \in y\gamma z$, $y \in a\gamma b$ and $z \in p\beta q$, so, $x \in (a\gamma b)\,\alpha\,(p\beta q) = (a\gamma b\alpha p)\,\beta q$ and we have

$$
\begin{aligned}
&\bigvee_{x \in y\gamma z} \bigvee_{y \in a\gamma b} \bigvee_{z \in p\beta q} \left\{\mu_A(a) \wedge \mu_A(p) \wedge 0.5 \wedge \mu_B(q)\right\} \\
\leq &\bigvee_{x \in (a\gamma b\alpha p)\beta q} \left\{\mu_A(a) \wedge \mu_A(p) \wedge 0.5 \wedge \mu_B(q) \wedge 0.5\right\}
\end{aligned}
$$

Since $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H, so we have

$$\inf_{m \in a\gamma b\alpha p} \mu_A(x) \geq \mu_A(a) \wedge \mu_A(p) \wedge 0.5.$$

So,

$$\begin{aligned}
&\bigvee_{x\in m\beta q}\{\mu_A(a)\wedge\mu_A(p)\wedge 0.5\wedge\mu_B(q)\wedge 0.5\}\\
\leq &\bigvee_{x\in m\beta q}\left\{\inf_{m\in a\gamma b\alpha p}\mu_A(m)\wedge\mu_B(q)\wedge 0.5\right\}\\
\leq &\bigvee_{x\in m\delta dq}\{\mu_A(m)\wedge\mu_B(q)\wedge 0.5\}=(\mu_A\circ_{0.5}\mu_B)(x)
\end{aligned}$$

Therefore, $((\mu_A\circ_{0.5}\mu_B)\circ_{0.5}(\mu_A\circ_{0.5}\mu_B))(x)\leq(\mu_A\circ_{0.5}\mu_B)(x)$. Now,

$$\begin{aligned}
((\lambda_A\circ_{0.5}\lambda_B)\circ_{0.5}(\lambda_A\circ_{0.5}\lambda_B))(a) &= \bigwedge_{x\in y\gamma z}\left\{\begin{array}{c}(\lambda_A\circ_{0.5}\lambda_B)(y)\vee(\lambda_A\circ_{0.5}\lambda_B)(z)\\ \vee 0.5\end{array}\right\}\\
&= \bigwedge_{x\in y\gamma z}\left\{\begin{array}{c}\bigwedge_{y\in a\gamma b}\{\lambda_A(a)\vee\lambda_B(b)\vee 0.5\}\\ \vee\bigwedge_{z\in p\beta q}\{\lambda_A(p)\vee\lambda_B(q)\vee 0.5\}\end{array}\right\}\\
&= \bigwedge_{x\in y\gamma z}\bigwedge_{y\in a\gamma b}\bigwedge_{z\in p\beta q}\left\{\begin{array}{c}\{\lambda_A(a)\vee\lambda_B(b)\vee 0.5\}\\ \vee\{\lambda_A(p)\vee\lambda_B(q)\vee 0.5\}\end{array}\right\}\\
&= \bigwedge_{x\in y\gamma z}\bigwedge_{y\in a\gamma b}\bigwedge_{z\in p\beta q}\left\{\begin{array}{c}\lambda_A(a)\vee\lambda_A(p)\\ \vee\lambda_B(b)\vee\lambda_B(q)\vee 0.5\end{array}\right\}\\
&\geq \bigwedge_{x\in y\gamma z}\bigwedge_{y\in a\gamma b}\bigwedge_{z\in p\beta q}\{\lambda_A(a)\vee\lambda_A(p)\vee 0.5\vee\lambda_B(q)\}
\end{aligned}$$

Since $x\in y\gamma z$, $y\in a\gamma b$ and $z\in p\beta q$, so $x\in(a\gamma b)\alpha(p\beta q)=(a\gamma b\alpha p)\beta q$ and we have

$$\begin{aligned}
&\bigwedge_{x\in y\gamma z}\bigwedge_{y\in a\gamma b}\bigwedge_{z\in p\beta q}\{\lambda_A(a)\vee\lambda_A(p)\vee 0.5\vee\lambda_B(q)\}\\
\geq &\bigwedge_{x=(a\gamma b\alpha p)\beta q}\{\lambda_A(a)\vee\lambda_A(p)\vee 0.5\vee\lambda_B(q)\}
\end{aligned}$$

Since $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H, so we have

$$\sup_{w \in a\gamma b\alpha p} \lambda_A(w) \leq \lambda_A(a) \vee \lambda_A(p) \vee 0.5.$$

Thus,

$$\begin{aligned}
& \bigwedge_{x \in (w)\beta q} \{\lambda_A(a) \vee \lambda_A(p) \vee 0.5 \vee \lambda_B(q)\} \\
\geq & \bigwedge_{x \in (w)\beta q} \left\{\sup_{w \in a\gamma b\alpha p} \lambda_A(w) \vee \lambda_B(q) \vee 0.5\right\} \\
\geq & \bigwedge_{x \in w\beta q} \{\lambda_A(w) \vee \lambda_B(q) \vee 0.5\} = (\lambda_A \circ_{0.5} \lambda_B)(x).
\end{aligned}$$

Therefore, $((\lambda_A \circ_{0.5} \lambda_B) \circ_{0.5} (\lambda_A \circ_{0.5} \lambda_B))(a) \geq (\lambda_A \circ_{0.5} \lambda_B)(x)$ and so $A \circ_{0.5} A \subseteq A$. Thus, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy sub-Γ-semihypergroup of H.

Now, let $x, y, z \in H$. Then,

$$\begin{aligned}
(\mu_A \circ_{0.5} \mu_B)(x) \wedge (\mu_A \circ_{0.5} \mu_B)(z) \wedge 0.5 &= \begin{array}{c} \left[\bigvee_{z \in a\alpha b} \{\mu_A(a) \wedge \mu_B(b) \wedge 0.5\}\right] \wedge \\ \left[\bigvee_{z \in p\beta q} \{\mu_A(p) \wedge \mu_B(q) \wedge 0.5\}\right] \wedge 0.5 \end{array} \\
&= \bigvee_{z \in a\alpha b} \bigvee_{z \in p\beta q} \left[\begin{array}{c} \{\mu_A(a) \wedge \mu_B(b) \wedge 0.5\} \\ \wedge \{\mu_A(p) \wedge \mu_B(q) \wedge 0.5\} \\ \wedge 0.5 \end{array}\right] \\
&\leq \bigvee_{z \in a\alpha b} \bigvee_{z \in p\beta q} \left[\begin{array}{c} \mu_A(a) \wedge \mu_A(p) \wedge \mu_B(b) \\ \wedge \mu_B(q) \wedge 0.5 \end{array}\right] \\
&\leq \bigvee_{z \in a\alpha b} \bigvee_{z \in p\beta q} [\mu_A(a) \wedge \mu_A(p) \wedge \mu_B(q) \wedge 0.5]
\end{aligned}$$

Since $x \in y\gamma z$, $y \in a\gamma b$ and $z \in p\beta q$, so $x \in (a\gamma b)\,\gamma\,(p\beta q) = (a\gamma b\alpha p)\,\beta q$ and we have

$$\begin{aligned} & \bigvee_{y \in a\gamma b} \bigvee_{z \in p\beta q} \left[\mu_A(a) \wedge \mu_A(p) \wedge \mu_B(q) \wedge 0.5\right] \\ \leq & \bigvee_{x\alpha y\gamma z = (a\gamma(b\alpha p)\beta q)} \left[\{\mu_A(a) \wedge \mu_A(p) \wedge 0.5\} \wedge \mu_B(q)\right] \end{aligned}$$

Since $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H, so we have

$$\mu_A\left((a\gamma\,(b\alpha p)\,\beta q)\right) \geq \mu_A(a) \wedge \mu_A(p) \wedge 0.5.$$

Thus,

$$\begin{aligned} & \bigvee_{x\alpha y\gamma z = (a\gamma(b\alpha p)\beta q)} \left[\{\mu_A(a) \wedge \mu_A(p) \wedge 0.5\} \wedge \mu_B(q)\right] \\ \leq & \bigvee_{x\alpha y\gamma z = (a\gamma(b\alpha p)\beta q)} \left[\mu_A(a\,(by)\,p) \wedge \mu_B(q)\right] = (\mu_A \circ_{0.5} \mu_B)(xyz)\,. \end{aligned}$$

Thus,

$$\inf_{u \in x\alpha y\gamma z} (\mu_A \circ_{0.5} \mu_B)(u) \geq (\mu_A \circ_{0.5} \mu_B)(x) \wedge (\mu_A \circ_{0.5} \mu_B)(z) \wedge 0.5$$

and

$$
\begin{aligned}
(\lambda_A \circ_{0.5} \lambda_B)(x) \vee (\lambda_A \circ_{0.5} \lambda_B)(z) \vee 0.5 &= \begin{array}{c} \left[\bigwedge_{y\in a\gamma b} \{\lambda_A(a) \vee \lambda_B(b) \vee 0.5\}\right] \vee \\ \left[\bigwedge_{z\in p\beta q} \{\lambda_A(p) \vee \lambda_B(q) \vee 0.5\}\right] \vee 0.5 \end{array} \\
&= \bigwedge_{y\in a\gamma b} \bigwedge_{z\in p\beta q} \left[\begin{array}{c} \{\lambda_A(a) \vee \lambda_B(b) \vee 0.5\} \\ \vee \{\lambda_A(p) \vee \lambda_B(q) \vee 0.5\} \\ \vee 0.5 \end{array}\right] \\
&\geq \bigwedge_{y\in a\gamma b} \bigwedge_{z\in p\beta q} \left[\begin{array}{c} \lambda_A(a) \vee \lambda_A(p) \vee \lambda_B(b) \\ \vee \lambda_B(q) \vee 0.5 \end{array}\right] \\
&\geq \bigwedge_{y\in a\gamma b} \bigwedge_{z\in p\beta q} [\lambda_A(a) \vee \lambda_A(p) \vee \lambda_B(q) \vee 0.5]
\end{aligned}
$$

Since $x \in y\gamma z$, $y \in a\gamma b$ and $z \in p\beta q$, so $x \in (a\gamma b)\gamma(p\beta q) = (a\gamma b\alpha p)\beta q$ and we have

$$
\begin{aligned}
&\bigwedge_{y\in a\gamma b} \bigwedge_{z\in p\beta q} [\lambda_A(a) \vee \lambda_A(p) \vee \lambda_B(q) \vee 0.5] \\
\geq &\bigwedge_{x\alpha y\gamma z=(a\gamma b\alpha p)\beta q} [\{\lambda_A(a) \vee \lambda_A(q) \vee 0.5\} \vee \lambda_B(q)]
\end{aligned}
$$

Since $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H, so we have

$$
\inf_{x\in a\gamma(b\alpha p)\beta q} \lambda_A(x) \geq \lambda_A(a) \vee \lambda_A(q) \vee 0.5.
$$

So,

$$
\begin{aligned}
&\bigwedge_{x\alpha y\gamma z=(a\gamma b\alpha p)\beta q} [\{\lambda_A(a) \vee \lambda_A(q) \vee 0.5\} \vee \lambda_B(q)] \\
\geq &\bigwedge_{x\alpha y\gamma z=(a\gamma b\alpha p)\beta q} \left[\inf_{x\in a\gamma(b\alpha p)\beta q} \lambda_A(x) \vee \lambda_B(q)\right] = (\lambda_A \circ_{0.5} \lambda_B)(x\alpha y\gamma z) \\
= &\sup_{u\in x\alpha y\gamma z} (\lambda_A \circ_{0.5} \lambda_B)(u).
\end{aligned}
$$

Thus,

$$\sup_{u \in x\alpha y\gamma z} (\lambda_A \circ_{0.5} \lambda_B)(u) \leq (\lambda_A \circ_{0.5} \lambda_B)(x) \vee (\lambda_A \circ_{0.5} \lambda_B)(z) \vee 0.5.$$

Hence, $A \circ_{0.5} B$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H. ■

For any intuitionistic fuzzy set $A = \langle \mu_A, \lambda_A \rangle$ in H and $t \in (0,1]$, $s \in [0,1)$, we denote $A_{(t,s)} = \{x \in H : x(t,s)\, qA\}$ and $[A]_{(t,s)} = \{x \in H : x(t,s) \in \vee qA\}$.

Obviously, $[A]_{(t,s)} = A_{(t,s)} \cup U_{(t,s)}$, where $U_{(t,s)}$, $A_{(t,s)}$ and $[A]_{(t,s)}$ are called $\in$-level set, q-level set and $\in \vee q$-level set of $A = \langle \mu_A, \lambda_A \rangle$, respectively [3].

Theorem 162 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an IFS of H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy left (resp. right) Γ-hyperideal of H if and only if for all $t \in (0, 0.5]$ and $s \in [0.5, 1)$, the set $U_{(t,s)} \neq \emptyset$ is a left (resp. right) Γ-hyperideal of H.*

Proof. Let $A = \langle \mu_A, \lambda_A \rangle$ be an $(\in, \in \vee q)$-intuitionistic fuzzy left Γ-hyperideal of H and $U_{(t,s)} \neq \emptyset$ for any $t \in (0, 0.5]$ and $s \in [0.5, 1)$. Let $y \in U_{(t,s)} \neq \emptyset$ and $x \in H$. Then, $\mu_A(y) \geq t$ and $\lambda_A(y) \leq s$. Since

$$\begin{aligned}
\inf_{z \in x\gamma y} \mu_A(z) &\geq \mu_A(y) \wedge 0.5 \geq t \wedge 0.5 \geq t \text{ and} \\
\sup_{z \in x\gamma y} \lambda_A(z) &\leq \lambda_A(y) \vee 0.5 \leq t \vee 0.5 \leq s,
\end{aligned}$$

so, $z \in U_{(t,s)}$ for all $z \in x\gamma y$. Hence, $x\gamma y \subseteq U_{(t,s)}$, is a left Γ-hyperideal of H.

Conversely, let us suppose that $A = \langle \mu_A, \lambda_A \rangle$ is an IFS of H such that $U_{(t,s)} \neq \emptyset$ is a left Γ-hyperideal of H. Suppose on contrary there exist $x, y \in H$ such that

$$\begin{aligned}
\inf_{z \in x\gamma y} \mu_A(z) &< \mu_A(y) \wedge 0.5 \\
\text{and } \sup_{z \in x\gamma y} \lambda_A(z) &> \lambda_A(y) \wedge 0.5.
\end{aligned}$$

Let us choose $t \in (0, 0.5]$ and $s \in [0.5, 1)$. Then,

$$\begin{aligned} \inf_{z \in x\gamma y} \mu_A(z) &< t < \mu_A(y) \wedge 0.5 \\ \text{and } \sup_{z \in x\gamma y} \lambda_A(z) &> s > \lambda_A(y) \wedge 0.5. \end{aligned}$$

Thus, $y \in U_{(t,s)}$ but $z \notin U_{(t,s)}$ for all $z \in x\gamma y$, which is a contradiction. Hence,

$$\begin{aligned} \inf_{z \in x\gamma y} \mu_A(z) &\geq \mu_A(y) \wedge 0.5 \\ \text{and } \sup_{z \in x\gamma y} \lambda_A(z) &\leq \lambda_A(y) \wedge 0.5. \end{aligned}$$

This completes the proof. ■

Theorem 163 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an IFS of H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H if and only if for all $t \in (0, 0.5]$ and $s \in [0.5, 1)$, the set $U_{(t,s)} \neq \emptyset$ is a bi-Γ-hyperideal of H.*

Proof. The proof follows from Theorem 162. ■

Theorem 164 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an IFS of H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy $(1,2)$-Γ-hyperideal of H if and only if for all $t \in (0, 0.5]$ and $s \in [0.5, 1)$, the set $U_{(t,s)} \neq \emptyset$ is a $(1,2)$-Γ-hyperideal of H.*

Proof. The proof follows from Theorem 162. ■

Theorem 165 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an IFS of H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy sub-Γ-semihypergroup of H if and only if for all $t \in (0, 1]$ and $s \in [0, 1)$, the set $[A]_{(t,s)} \neq \emptyset$ is a sub-Γ-semihypergroup of H.*

Proof. Let $x, y \in [A]_{(t,s)}$. Then, $\mu_A(x) \geq t$ and $\lambda_A(x) \geq s$ or $\mu_A(x) + t > 1$ and $\lambda_A(x) + s < 1$, and $\mu_A(y) \geq t$ and $\lambda_A(y) \geq s$ or $\mu_A(y) + t > 1$ and $\lambda_A(y) + s < 1$. We can consider four cases:

(*i*) $\mu_A(x) \geq t$ and $\lambda_A(x) \leq s$, and $\mu_A(y) \geq t$ and $\lambda_A(y) \leq s$,

(ii) $\mu_A(x) \geq t$ and $\lambda_A(x) \leq s$, and $\mu_A(y) + t > 1$ and $\lambda_A(y) + s < 1$,

(iii) $\mu_A(x) + t > 1$ and $\lambda_A(x) + s < 1$, and $\mu_A(y) \geq t$ and $\lambda_A(y) \leq s$,

(iv) $\mu_A(x) + t > 1$ and $\lambda_A(x) + s < 1$, and $\mu_A(y) + t > 1$ and $\lambda_A(y) + s < 1$.

For the first case, by Theorem 139 (a), implies that

$$\inf_{z \in x\gamma y} \mu_A(z) \geq \min\{\mu_A(x), \mu_A(y), 0.5\} = \min\{t, 0.5\} = \begin{cases} 0.5 & \text{if } t > 0.5 \\ t & \text{if } t \leq 0.5 \end{cases}$$

and

$$\sup_{z \in x\gamma y} \lambda_A(z) \leq \max\{\lambda_A(x), \lambda_A(y), 0.5\} = \max\{s, 0.5\} = \begin{cases} 0.5 & \text{if } s < 0.5 \\ s & \text{if } s \geq 0.5 \end{cases}$$

and so $\inf_{z \in x\gamma y} \mu_A(z) + t > 0.5 + 0.5 = 1$ and $\sup_{z \in x\gamma y} \lambda_A(z) + s < 0.5 + 0.5 = 1$, *i.e.*, $(z)(s,t)\, qA$, or $z \in A_{(t,s)}$ for all $z \in x\gamma y$. Therefore, $x\gamma y \subseteq U_{(t,s)} \cup A_{(t,s)} = [A]_{(t,s)}$. For the case (ii), assume that $t > 0.5$ and $s < 0.5$. Then, $1 - t < 0.5$ and $1 - s > 0.5$. If $\min\{\mu_A(y), 0.5\} \leq \mu_A(x)$ and $\max\{\lambda_A(y), 0.5\} \geq \lambda_A(x)$, then

$$\begin{aligned} \inf_{z \in x\gamma y} \mu_A(z) &\geq \min\{\mu_A(y), 0.5\} > 1 - t \text{ and} \\ \sup_{z \in x\gamma y} \lambda_A(z) &\leq \max\{\lambda_A(y), 0.5\} < 1 - s \end{aligned}$$

and if $\min\{\mu_A(y), 0.5\} > \mu_A(x)$ and $\max\{\lambda_A(y), 0.5\} < \lambda_A(x)$, then $\inf_{z \in x\gamma y} \mu_A(z) \geq \mu_A(x) \geq t$ and $\sup_{z \in x\gamma y} \lambda_A(z) \leq \lambda_A(x) \leq s$. Hence, $z \in U_{(t,s)} \cup A_{(t,s)} = [A]_{(t,s)}$for all $z \in x\gamma y$. Therefore, $x\gamma y \subseteq U_{(t,s)} \cup A_{(t,s)} = [A]_{(t,s)}$ for $t > 0.5$ and $s < 0.5$. Suppose that $t \leq 0.5$ and $s \geq 0.5$. Then, $1 - t \geq 0.5$ and $1 - s \leq 0.5$. If $\min\{\mu_A(x), 0.5\} \leq \mu_A(y)$ and $\max\{\lambda_A(x), 0.5\} \geq \lambda_A(y)$, then

$$\begin{aligned} \inf_{z \in x\gamma y} \mu_A(z) &\geq \min\{\mu_A(x), 0.5\} \geq t \text{ and} \\ \sup_{z \in x\gamma y} \lambda_A(z) &\leq \max\{\lambda_A(x), 0.5\} \leq s \end{aligned}$$

and if $\min\{\mu_A(x), 0.5\} > \mu_A(y)$ and $\max\{\lambda_A(x), 0.5\} < \lambda_A(y)$, then $\inf_{z\in x\gamma y} \mu_A(z) \geq \mu_A(y) > 1 - t$ and $\sup_{z\in x\gamma y} \lambda_A(z) \leq \lambda_A(y) < 1 - s$. Thus, $z \in U_{(t,s)} \cup A_{(t,s)} = [A]_{(t,s)}$ for all $z \in x\gamma y$. Therefore, $x\gamma y \subseteq U_{(t,s)} \cup A_{(t,s)} = [A]_{(t,s)}$ for $t \leq 0.5$ and $s \geq 0.5$. We have similar result for the case (iii). For final case, if $t > 0.5$ and $s < 0.5$, then $1-t < 0.5$ and $1 - s > 0.5$. Hence,

$$\begin{aligned}\inf_{z\in x\gamma y} \mu_A(z) &\geq \min\{\mu_A(x), \mu_A(y), 0.5\} \\ &= \begin{cases} 0.5 > 1 - t & \text{if } \min\{\mu_A(x), \mu_A(y)\} \geq 0.5, \\ \min\{\mu_A(x), \mu_A(y)\} > 1 - t & \text{if } \min\{\mu_A(x), \mu_A(y)\} < 0.5, \end{cases}\end{aligned}$$

and

$$\begin{aligned}\sup_{z\in x\gamma y} \lambda_A(z) &\leq \max\{\lambda_A(x), \lambda_A(y), 0.5\} \\ &= \begin{cases} 0.5 < 1 - s & \text{if } \max\{\lambda_A(x), \lambda_A(y)\} \leq 0.5, \\ \max\{\lambda_A(x), \lambda_A(y)\} < 1 - s & \text{if } \max\{\lambda_A(x), \lambda_A(y)\} > 0.5, \end{cases}\end{aligned}$$

and so $x\gamma y \subseteq A_{(t,s)} \subseteq [A]_{(t,s)}$. If $t \leq 0.5$ and $s \geq 0.5$, then $1 - t \geq 0.5$ and $1 - s \leq 0.5$. Thus,

$$\begin{aligned}\inf_{z\in x\gamma y} \mu_A(z) &\geq \min\{\mu_A(x), \mu_A(y), 0.5\} \\ &= \begin{cases} 0.5 \geq t & \text{if } \min\{\mu_A(x), \mu_A(y)\} \geq 0.5, \\ \min\{\mu_A(x), \mu_A(y)\} > 1 - t & \text{if } \min\{\mu_A(x), \mu_A(y)\} < 0.5, \end{cases}\end{aligned}$$

and

$$\begin{aligned}\sup_{z\in x\gamma y} \lambda_A(z) &\leq \max\{\lambda_A(x), \lambda_A(y), 0.5\} \\ &= \begin{cases} 0.5 \leq s & \text{if } \max\{\lambda_A(x), \lambda_A(y)\} \leq 0.5, \\ \max\{\lambda_A(x), \lambda_A(y)\} < 1 - s & \text{if } \max\{\lambda_A(x), \lambda_A(y)\} > 0.5, \end{cases}\end{aligned}$$

which implies that $x\gamma y \subseteq U_{(t,s)} \cup A_{(t,s)} = [A]_{(t,s)}$.

Conversely, suppose that $A = \langle \mu_A, \lambda_A \rangle$ is an IFS in H such that $[A]_{(t,s)}$ is a sub-Γ-semihypergroup of H. Suppose that $A = \langle \mu_A, \lambda_A \rangle$ is not an $(\in, \in \vee q)$-intuitionistic fuzzy sub-Γ-semihypergroup of H. Then, there exist $x, y \in H$ such that

$$\inf_{z \in x\gamma y} \mu_A(z) < \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \sup_{z \in x\gamma y} \lambda_A(z) > \max\{\lambda_A(x), \lambda_A(y), 0.5\}.$$

Let

$$\begin{aligned} t &= \frac{1}{2}\left[\inf_{z \in x\gamma y} \mu_A(z) + \min\{\mu_A(x), \mu_A(y), 0.5\}\right] \text{ and} \\ s &= \frac{1}{2}\left[\sup_{z \in x\gamma y} \lambda_A(z) + \max\{\lambda_A(x), \lambda_A(y), 0.5\}\right]. \end{aligned}$$

Then,

$$\begin{aligned} \inf_{z \in x\gamma y} \mu_A(z) &< t < \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and} \\ \sup_{z \in x\gamma y} \lambda_A(z) &> s > \max\{\lambda_A(x), \lambda_A(y), 0.5\}. \end{aligned}$$

this imply that $x, y \in [A]_{(t,s)}$ and $(x\gamma y) \subseteq [A]_{(t,s)}$. Hence, $\inf_{z \in x\gamma y} \mu_A(z) \geq t$ and $\sup_{z \in x\gamma y} \lambda_A(z) \leq s$ or $\inf_{z \in x\gamma y} \mu_A(z) + t > 1$ and $\sup_{z \in x\gamma y} \lambda_A(z) + s < 1$, which is a contradiction. Therefore, we have

$$\inf_{z \in x\gamma y} \mu_A(z) \geq \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \sup_{z \in x\gamma y} \lambda_A(z) \leq \max\{\lambda_A(x), \lambda_A(y), 0.5\}.$$

Thus, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy sub-Γ-semihypergroup of H. ■

Theorem 166 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ an IFS of H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy left (resp. right) Γ-hyperideal of H if and only if for all $t \in (0, 1]$ and $s \in [0, 1)$, the set $[A]_{(t,s)} \neq \emptyset$ is a left (resp. right) Γ-hyperideal of H.*

Proof. Assume that $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy left Γ-hyperideal of H and let $t \in (0,1]$ and $s \in [0,1)$ be such that $[A]_{(t,s)} \neq \emptyset$. Let $y \in [A]_{(t,s)}$ and $x \in H$. Then, $\mu_A(y) \geq t$ and $\lambda_A(y) \leq s$ or $\mu_A(y) + t > 1$ and $\lambda_A(y) + s < 1$. Assume that $\mu_A(y) \geq t$ and $\lambda_A(y) \leq s$ by Theorem **??** (a), implies that

$$\inf_{z \in x\gamma y} \mu_A(z) \geq \min\{\mu_A(y), 0.5\} \geq \min\{t, 0.5\} = \begin{cases} t & \text{if } t \leq 0.5, \\ 0.5 > 1 - t & \text{if } t > 0.5, \end{cases}$$

and

$$\sup_{z \in x\gamma y} \lambda_A(z) \leq \max\{\lambda_A(y), 0.5\} \geq \max\{s, 0.5\} = \begin{cases} s & \text{if } s \geq 0.5, \\ 0.5 < 1 - s & \text{if } s < 0.5, \end{cases}$$

so that $x\gamma y \subseteq U_{(t,s)} \cup A_{(t,s)} = [A]_{(t,s)}$. Suppose that $\mu_A(y) + t > 1$ and $\lambda_A(y) + s < 1$. If $t > 0.5$ and $s < 0.5$, then

$$\inf_{z \in x\gamma y} \mu_A(z) \geq \min\{\mu_A(y), 0.5\} = \begin{cases} 0.5 > 1 - t & \text{if } t \leq 0.5, \\ \mu_A(y) > 1 - t & \text{if } t > 0.5, \end{cases}$$

and

$$\sup_{z \in x\gamma y} \lambda_A(z) \geq \max\{\lambda_A(y), 0.5\} = \begin{cases} 0.5 < 1 - s & \text{if } s \geq 0.5, \\ \lambda_A(y) < 1 - s & \text{if } s < 0.5, \end{cases}$$

and thus $xy \in A_{(t,s)} \subseteq [A]_{(t,s)}$. Consequently, $[A]_{(t,s)}$ is a left Γ-hyperideal of H.

Conversely, suppose that $A = \langle \mu_A, \lambda_A \rangle$ is an IFS in H such that $[A]_{(t,s)}$ is a left Γ-hyperideal of H. Suppose that $A = \langle \mu_A, \lambda_A \rangle$ is not an $(\in, \in \vee q)$-intuitionistic fuzzy Γ-hyperideal of H. Then, there exist $x, y \in H$ such that

$$\inf_{z \in x\gamma y} \mu_A(z) < \min\{\mu_A(y), 0.5\} \text{ and } \lambda_A(xy) > \max\{\lambda_A(y), 0.5\}.$$

Let

$$\begin{aligned} t &= \frac{1}{2}\left[\inf_{z\in x\gamma y}\mu_A(z)+\min\{\mu_A(y),0.5\}\right] \text{ and} \\ s &= \frac{1}{2}\left[\sup_{z\in x\gamma y}\lambda_A(z)+\max\{\lambda_A(y),0.5\}\right]. \end{aligned}$$

Then,

$$\begin{aligned} \inf_{z\in x\gamma y}\mu_A(z) &< t<\min\{\mu_A(y),0.5\} \text{ and} \\ \sup_{z\in x\gamma y}\lambda_A(z) &> s>\max\{\lambda_A(y),0.5\}. \end{aligned}$$

this imply that $x, y \in U_{(t,s)} \subseteq [A]_{(t,s)}$ so that $z \in [A]_{(t,s)}$ for all $z \in x\gamma y$. Thus, $\inf_{z\in x\gamma y}\mu_A(z) \geq t$ and $\sup_{z\in x\gamma y}\lambda_A(z) \leq s$ or $\inf_{z\in x\gamma y}\mu_A(z)+t > 1$ and $\sup_{z\in x\gamma y}\lambda_A(z)+s < 1$, which is a contradiction. Therefore, we have

$$\inf_{z\in x\gamma y}\mu_A(z) \geq \min\{\mu_A(y),0.5\} \text{ and } \sup_{z\in x\gamma y}\lambda_A(z) \leq \max\{\lambda_A(y),0.5\},$$

for all $x, y \in H$. Hence, $A = \langle\mu_A, \lambda_A\rangle$ is an $(\in, \in\vee q)$-intuitionistic fuzzy left Γ-hyperideal of H. Similarly, the right case also follows. This completes the proof. ■

Theorem 167 *Let H be a Γ-semihypergroup and $A = \langle\mu_A, \lambda_A\rangle$ an IFS of H. Then, $A = \langle\mu_A, \lambda_A\rangle$ is an $(\in, \in\vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of H if and only if for all $t \in (0,1]$ and $s \in [0,1)$, the set $[A]_{(t,s)} \neq \emptyset$ is a bi-Γ-hyperideal of H.*

Theorem 168 *Let H be a Γ-semihypergroup and $A = \langle\mu_A, \lambda_A\rangle$ an IFS of H. Then, $A = \langle\mu_A, \lambda_A\rangle$ is an $(\in, \in\vee q)$-intuitionistic fuzzy $(1,2)$-Γ-hyperideal of H if and only if for all $t \in (0,1]$ and $s \in [0,1)$, the set $[A]_{(t,s)} \neq \emptyset$ is a $(1,2)$-Γ-hyperideal of H.*

Theorem 169 *Every $(\in, \in\vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of a Γ-semihypergroup H is an $(\in, \in\vee q)$-intuitionistic fuzzy $(1,\ 2)$ Γ-hyperideal of a Γ-semihypergroup H.*

Proof. Let $A = \langle \mu_A, \lambda_A \rangle$ be an $(\in, \in \vee q)$-intuitionistic fuzzy bi-Γ-hyperideal of a Γ-semihypergroup H. Then

$$\inf_{z \in x\gamma y} \mu_A(z) \geq \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \sup_{z \in x\gamma y} \lambda_A(z) \leq \max\{\lambda_A(x), \lambda_A(y), 0.5\}$$

Now let for any $x, y, z, a \in H$. Then

$$\begin{aligned}
\inf_{z \in x\gamma a\delta(y\beta z)} \mu_A(z) &= \inf_{z \in u\beta z} \mu_A(z) \geq \min\{\mu_A(u), \mu_A(z), 0.5\} \quad : u \in x\gamma a\delta y \\
\inf_{z \in (x\gamma a\delta y)\beta z} \mu_A(z) &\geq \min\left\{\inf_{u \in x\gamma a\delta y} \mu_A(u), \mu_A(z), 0.5\right\} \\
\inf_{z \in x\gamma a\delta(y\beta z)} \mu_A(z) &\geq \min\{\mu_A(x), \mu_A(y), \mu_A(z), 0.5\}. \\
\sup_{z \in x\gamma a\delta(y\beta z)} \lambda_A(xa(yz)) &= \sup_{z \in u\beta z} \lambda_A(z) \leq \max\{\lambda_A(u), \lambda_A(z), 0.5\} \quad : u \in x\gamma a\delta y \\
\sup_{z \in x\gamma a\delta(y\beta z)} \lambda_A(xa(yz)) &\leq \max\left\{\sup_{u \in x\gamma a\delta y} \lambda_A(u), \lambda_A(z), 0.5\right\} \\
\sup_{z \in x\gamma a\delta(y\beta z)} \lambda_A(z) &\leq \max\{\lambda_A(x), \lambda_A(y), \lambda_A(z), 0.5\}
\end{aligned}$$

Therefor $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy (1, 2) Γ-hyperideal of a Γ-semihypergroup H. ■

Chapter 7

Interior Γ-hyperideals of Γ-semihypergroups based on Intuitionistic Fuzzy Points

7.1 Introduction

In this chapter, we introduce the concept of an (α, β)-intuitionistic fuzzy interior Γ-hyperideal of a Γ-semihypergroup by using the notion of an intuitionsitic fuzzy point to an intuitionistic fuzzy set. An (α, β)-intuitionistic fuzz interior Γ-hyperideal of a Γ-semihypergroup is a generalization of ordinary intuitionitic fuzzy interior Γ-hyperideals. We also define an $(\in, \in \vee q)$-intuitionistic fuzzyinterior Γ-hyperideal a Γ-semihypergroup. We charcterize an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal by the properties of an $\in$-level set, a q-level set and an $(\in, \in \vee q)$-level set.

7.2 (α, β)-Intuitionistic Fuzzy Interior Γ-hyperideals

In this section, we introduce the concept of an (α, β)-intuitionistic fuzzy interior Γ-hyperideal of a Γ-semihypergroup. We also study some properties of an (α, β)-intuitionistic fuzzy interior Γ-hyperideal. The give concept in this section is a generalization of intuitionsitic fuzzy interior Γ-hyperideals of Γ-semihypergroups.

Definition 170 *An IFS $A = \langle \mu_A, \lambda_A \rangle$ in Γ-semihypergroup H is said to be an (α, β)-intuitionistic fuzzy interior Γ-hyperideal of H, where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, if the following are hold.*

$(IFI1)$ ($\forall x, y \in H$ and $\gamma \in \Gamma$, ($t_1, t_2 \in (0, 0.5]$ and $s_1, s_2 \in [0.5, 1)$) or ($t_1, t_2 \in (0.5, 1]$ and $s_1, s_2 \in [0, 0.5)$)) If $x(t_1, s_1)\alpha A$ and $y(t_2, s_2)\alpha A \Longrightarrow (z_1)(m\{t_1, t_2\}, M\{s_1, s_2\})\beta A$ for all $z_1 \in x\gamma y$.

$(IFI2)$ ($\forall x, a, y \in H$ and $\gamma, \delta \in \Gamma$, ($t \in (0, 0.5]$ and $s \in [0.5, 1)$) or ($t \in (0.5, 1]$ and $s \in [0, 0.5)$)) If $a(t, s)\alpha A \Longrightarrow (z_1)(t, s)\beta A$ for all $z_1 \in x\gamma a\delta y$.

Theorem 171 *Let $A = \langle \mu_A, \lambda_A \rangle$ be a non-zero (α, β)-intuitionistic fuzzy interior Γ-hyperideal of a Γ-semihypergroup H. Then , the set $I = \{x \in H : \mu_A(x) > 0$ and $\lambda_A(x) < 1\}$ is an interior Γ-hyperideal of H.*

Proof. Let $x, y \in I$ and $\gamma \in H$. Then, $\mu_A(x) > 0$ and $\lambda_A(x) < 1$ and $\mu_A(y) > 0$ and $\lambda_A(y) < 1$. Assume that $\mu_A(z) = 0$ and $\lambda_A(z) = 1$ for all $z \in x\gamma y$. If $\alpha \in \{\in, \in \vee q\}$, then $x(\mu_A(x), \lambda_A(x))\alpha A$ and $y(\mu_A(y), \lambda_A(y))\alpha A$ but for each $z \in x\gamma y$, $(z)(m\{\mu_A(x), \mu_A(y)\}, M\{\lambda_A(x), \lambda_A(y)\})\overline{\beta}A$ for every $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$, which is a contradiction. Since $x(1, 0)qA$ and $y(1, 0)qA$ but for each $z \in x\gamma y$, $(z_1)(1, 0)\overline{\beta}A$ for every $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$, which is a contradiction. Hence, for each $z \in x\gamma y$, $\mu_A(z) > 0$ and $\lambda_A(z) < 1$. This implies that $z \in I$. Thus, $x\gamma y \subseteq I$. Now, let $x, y \in S$, $a \in I$ and $\gamma, \delta \in H$. Then, assume that , $\mu_A(z) = 0$ and $\lambda_A(z_2) = 1$ for each $z \in x\gamma a\delta y$. If $\alpha \in \{\in, \in \vee q\}$, then $a(\mu_A(a), \lambda_A(a))\alpha A$ but for all $z \in x\gamma a\delta y$, $(z)(\mu_A(a), \lambda_A(a))\overline{\beta}A$ for every $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$, which is a contradiction. Since $a(1, 0)qA$ but for all $z \in x\gamma a\delta y$, $(z)(1, 0)\overline{\beta}A$ for every $\beta \in \{\in, q, \in \wedge q, \in \vee q\}$, which is a contradiction. Hence, for each $z \in x\gamma a\delta y$, $\mu_A(z) > 0$ and $\lambda_A(z) < 1$. This implies that $z \in I$ for each $z \in x\gamma a\delta y$. Thus, $x\gamma a\delta y \subseteq I$ and $I = \{x \in S : \mu_A(x) > 0$ and $\lambda_A(x) < 1\}$ is an interior Γ-hyperidealof H. ■

Corollary 172 *Let $A = \langle\mu_A, \lambda_A\rangle$ be a non-zero (α, β)-intuitionistic fuzzy interior Γ-hyperidealof a Γ-semihypergroup H. Then, the sets $I_1 = \{x \in H : \mu_A(x) > 0\}$ and $I_2 = \{x \in H : \lambda_A(x) < 1\}$ are interior Γ-hyperideals of H.*

Theorem 173 *Let H be a strong right(resp, left) zero Γ-semihypergroup and $A = \langle\mu_A, \lambda_A\rangle$ be a non-zero (q, q)-intuitionistic fuzzy interior Γ-hyperideal of H. Then, $A = \langle\mu_A, \lambda_A\rangle$ is constant on I.*

Proof. Let w be an element of H such that $\mu_A(w) = \sup_{x\in H}\{\mu_A(x)\}$ and $\lambda_A(w) = \inf_{x\in H}\{\lambda_A(x)\}$. Then, $w \in I$. Suppose that there exist x,$w \in I$ such that $t_x = \mu_A(x) \neq \mu_A(w) = t_w$ and $s_x = \lambda_A(x) \neq \lambda_A(w) = s_w$. Then $t_x < t_w$ and $s_x > s_w$. Choose $t_1, t_2 \in (0, 1]$ and $s_1, s_2 \in [0, 1)$ such that $1 - t_w < t_1 < 1 - t_x < t_2$ and $1 - s_w > s_1 > 1 - s_x > s_2$. Then $w(t_1, s_1)\, qA$ and $x(t_2, s_2)\, qA$ but $(w\gamma x)(m\{t_1, t_2\}, M\{s_1, s_2\}) =$

$(x)(t_1, s_1)\, \overline{q}A$ (resp, $(x\gamma w)(m\{t_1, t_2\}, M\{s_1, s_2\}) = (x)(t_1, s_1)\, \overline{q}A$) becouse H is a strong right (resp, left) zero, which is a contradiction. Hence, $\mu_A(x) = \mu_A(e)$ and $\lambda_A(x) = \lambda_A(e)$. Therefore $A(x) = A(e)$ for all $x \in I$. ■

7.3 Intuitionistic fuzzy interior Γ-hyperideals of type $(\in, \in \vee q)$

The concept of $(\in, \in \vee q)$-intutionistic fuzzy interior Γ-hyperideals in a Γ-semihypergroup plays a vital rule in the theory of (α, β)-intutionistic fuzzy interior Γ-hyperideals. We give some different characterization of $(\in, \in \vee q)$-intutionistic fuzzy interior Γ-hyperideals in a Γ-semihypergroup.

Definition 174 *An IFS $A = \langle\mu_A, \lambda_A\rangle$ in a Γ-semihypergroup is said to be an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of H if the following conditions hold.*

$(IFI3)$ ($\forall x, y \in H$, ($t_1, t_2 \in (0, 0.5]$ and $s_1, s_2 \in [0.5, 1)$) or ($t_1, t_2 \in (0.5, 1]$ and $s_1, s_2 \in [0, 0.5)$)) If $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A \Longrightarrow (z_1)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee qA$ for each $z_1 \in x\gamma y$.

$(IFI4)$ ($\forall x, a, y \in H$, ($t \in (0.5, 1]$ and $s \in [0.5, 1)$) or ($t \in (0.5, 1]$ and $s \in [0, 0.5)$)) If $a(t, s) \in A \Longrightarrow (z_2)(t, s) \in \vee q A$ for each $z_2 \in x\gamma y\delta z$.

Theorem 175 *Let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS in a Γ-semihypergroup H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of H if and only if the following conditions hold;*

$$\begin{aligned} \inf_{z_1 \in x\gamma y} \mu_A(z_1) &\geq \min\{\mu_A(x), \mu_A(y), 0.5\}, \sup_{z_1 \in x\gamma y} \lambda_A(z_1) \leq \max\{\lambda_A(x), \lambda_A(y), 0.5\} \\ \inf_{z_1 \in x\gamma a\delta y} \mu_A(z_2) &\geq \min\{\mu_A(a), 0.5\}, \sup_{z_1 \in x\gamma a\delta y} \lambda_A(z_2) \leq \max\{\lambda_A(a), 0.5\}, \end{aligned}$$

for all $x, a, y \in H$ and $\gamma, \delta \in \Gamma$ ($t_1, t_2 \in (0, 0.5]$ and $s_1, s_2 \in [0.5, 1)$) or ($t_1, t_2 \in (0.5, 1]$ and $s_1, s_2 \in [0, 0.5)$).

Proof. Since given that $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of a Γ-semihypergroup H. Suppose that

$$\inf_{z_1 \in x\gamma y} \mu_A(z_1) < \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \sup_{z_1 \in x\gamma y} \lambda_A(z_1) > \max\{\lambda_A(x), \lambda_A(y), 0.5\}.$$

Choose $t \in (0, 1]$ and $s \in [0, 1)$ such that

$$\inf_{z_1 \in x\gamma y} \mu_A(z_1) < t < \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \sup_{z_1 \in x\gamma y} \lambda_A(z_1) > s > \max\{\lambda_A(x), \lambda_A(y), 0.5\}.$$

If $\min\{\mu_A(x), \mu_A(y)\} < 0.5$ and $\max\{\lambda_A(x), \lambda_A(y)\} > 0.5$, then

$$\inf_{z_1 \in x\gamma y} \mu_A(z_1) < t < \min\{\mu_A(x), \mu_A(y)\} \text{ and } \sup_{z_1 \in x\gamma y} \lambda_A(z_1) > s > \max\{\lambda_A(x), \lambda_A(y)\}.$$

Then $x(t, s) \in A$ and $y(t, s) \in A$ but for each $z_1 \in x\gamma y$, $(z_1)(t, s)\overline{\in \vee q}A$, which is a contradiction. If $\min\{\mu_A(x), \mu_A(y)\} \geq 0.5$ and $\max\{\lambda_A(x), \lambda_A(y)\} \leq 0.5$, then

$$\inf_{z_1 \in x\gamma y} \mu_A(z_1) < t < 0.5 \text{ and } \sup_{z_1 \in x\gamma y} \lambda_A(z_1) > s > 0.5.$$

Since $x(0.5, 0.5) \in A$ and $y(0.5, 0.5) \in A$, but for each $z_1 \in x\gamma y$, $(z_1)(0.5, 0.5)\overline{\in \vee q}A$, which is a contradiction. If $\min\{\mu_A(x), \mu_A(y)\} < 0.5$ and $\max\{\lambda_A(x), \lambda_A(y)\} < 0.5$, then

$$\inf_{z_1 \in x\gamma y} \mu_A(z_1) < t < \min\{\mu_A(x), \mu_A(y)\} \text{ and } \sup_{z_1 \in x\gamma y} \lambda_A(z_1) > s > 0.5.$$

Thus, $x(t, s) \in A$ and $y(t, s) \in A$ but for each $z_1 \in x\gamma y$, $(z_1)(t, s)\overline{\in}A$. Also, $\inf_{z_1 \in x\gamma y} \mu_A(z_1) + t < 0.5 + 0.5 = 1$ and $\sup_{z_1 \in x\gamma y} \lambda_A(z_1) + s > 0.5 + 0.5 = 1$. This implies for each $z_1 \in x\gamma y$, $(z_1)(t, s)\overline{q}A$. Hence, for each $z_1 \in x\gamma y$, $(z_1)(t, s)\overline{\in \vee q}A$, which is contradiction. Hence,

$$\inf_{z_1 \in x\gamma y} \mu_A(z_1) \geq \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \sup_{z_1 \in x\gamma y} \lambda_A(z_1) \leq \max\{\lambda_A(x), \lambda_A(y), 0.5\}.$$

(b) Suppose that

$$\inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) < \min\{\mu_A(a), 0.5\} \text{ and } \sup_{z_1 \in x\gamma a\delta y} \lambda_A(z_1) > \max\{\lambda_A(a), 0.5\}.$$

Choose $t \in (0, 1]$ and $s \in [0, 1)$ such that

$$\inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) < t < \min\{\mu_A(a), 0.5\} \text{ and } \sup_{z_1 \in x\gamma y} \lambda_A(z_1) > s > \max\{\lambda_A(a), 0.5\}.$$

If $\mu_A(a) < 0.5$ and $\lambda_A(a) > 0.5$, then

$$\inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) < t < \mu_A(a) \text{ and } \sup_{z_1 \in x\gamma y} \lambda_A(z_1) > s > \lambda_A(a).$$

Then, $a(t, s) \in A$ but for each $z_1 \in x\gamma a\delta y$, $(z_1)(t, s)\overline{\in \vee q}A$, which is a contradiction. If $\mu_A(a) \geq 0.5$ and $\lambda_A(a) \leq 0.5$, then

$$\inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) < t < 0.5 \text{ and } \sup_{z_1 \in x\gamma a\delta y} \lambda_A(z_1) > s > 0.5.$$

Since $a(0.5,0.5) \in A$ but for each $z_1 \in x\gamma a\delta y$, $(z_1)(0.5,0.5)\overline{\in \vee q}A$, which is a contradiction. If $\mu_A(a) \leq 0.5$ and $\lambda_A(a) \leq 0.5$, then

$$\inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) < t < \mu_A(a) \text{ and } \sup_{z_1 \in x\gamma a\delta y} \lambda_A(z_1) > s > 0.5.$$

Thus, $a(t,s) \in A$ but for each $z_1 \in x\gamma a\delta y$, $(z_1)(t,s)\overline{\in}A$. Also, $\inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) + t < 0.5 + 0.5 = 1$ and $\sup_{z_1 \in x\gamma a\delta y} \lambda_A(z_1) + s > 0.5 + 0.5 = 1$. This imply for each $z_1 \in x\gamma a\delta y$, $(z_1)(t,s)\overline{q}A$. Thus, for each $z_1 \in x\gamma a\delta y$, $(z_1)(t,s)\overline{\in \vee q}A$, which is a contradiction. Hence,

$$\inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) \geq \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \sup_{z_1 \in x\gamma a\delta y} \lambda_A(z_1) \leq \max\{\lambda_A(x), \lambda_A(y), 0.5\}.$$

Conversely, assume that $A = \langle \mu_A, \lambda_A \rangle$ satisfies (a) and (b). Let $x, y \in H$ and $(t_1, t_2 \in (0, 0.5]$ and $s_1, s_2 \in [0.5, 1))$ or $(t_1, t_2 \in (0.5, 1]$ and $s_1, s_2 \in [0, 0.5))$ such that $x(t_1, s_1) \in A$ and $y(t_2, s_2) \in A \Rightarrow \mu_A(x) \geq t_1$ and $\lambda_A(x) \leq s_1$, $\mu_A(y) \geq t_2$ and $\lambda_A(y) \leq s_2$. Then,

$$\begin{aligned} \inf_{z_1 \in x\gamma y} \mu_A(z_1) &\geq \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \sup_{z_1 \in x\gamma y} \lambda_A(z_1) \leq \max\{\lambda_A(x), \lambda_A(y), 0.5\} \\ \inf_{z_1 \in x\gamma y} \mu_A(z_1) &\geq \min\{t_1, t_2, 0.5\} \text{ and } \sup_{z_1 \in x\gamma y} \lambda_A(z_1) \leq \max\{s_1, s_2, 0.5\} \end{aligned}$$

Then we have the following case's

(1) $\min\{t_1, t_2\} > 0.5$ and $\max\{s_1, s_2\} < 0.5$

(2) $\min\{t_1, t_2\} \leq 0.5$ and $\max\{s_1, s_2\} \geq 0.5$

Case (1) If $\min\{t_1, t_2\} > 0.5$ and $\max\{s_1, s_2\} < 0.5$, then $\inf_{z_1 \in x\gamma y} \mu_A(z_1) \geq 0.5$ and $\sup_{z_1 \in x\gamma y} \lambda_A(z_1) \leq 0.5$. This implies that $\inf_{z_1 \in x\gamma y} \mu_A(z_1) + \min\{t_1, t_2\} > 1$ and $\lambda_A(xy) + \max\{s_1, s_2\} < 1$. So, for each $z_1 \in x\gamma y$, $(z_1)(m\{t_1, t_2\}, M\{s_1, s_2\})qA$.

Case(2) if $\min\{t_1, t_2\} \leq 0.5$ and $\max\{s_1, s_2\} \geq 0.5$, then $\inf_{z_1 \in x\gamma y} \mu_A(z_1) \geq \min\{t_1, t_2\}$ and $\sup_{z_1 \in x\gamma y} \lambda_A(z_1) \leq \max\{s_1, s_2\}$. This implies that for each $z_1 \in x\gamma y$ $(z_1)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in A$. Therefore,

$$(z_1)(m\{t_1, t_2\}, M\{s_1, s_2\}) \in \vee qA$$

Let $x, y, a \in H$ and ($t \in (0.5, 1]$ and $s \in [0.5, 1)$) or ($t \in (0.5, 1]$ and $s \in [0, 0.5)$) such that $a(t, s) \in A \Rightarrow \mu_A(x) \geq t$ and $\lambda_A(x) \leq s$. Then,

$$\begin{aligned}
\inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) &\geq \min\{\mu_A(a), 0.5\} \text{ and } \sup_{z_1 \in x\gamma a\delta y} \lambda_A(z_1) \leq \max\{\lambda_A(a), 0.5\} \\
\inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) &\geq \min\{t, 0.5\} \text{ and } \sup_{z_1 \in x\gamma a\delta y} \lambda_A(z_1) \leq \max\{s, 0.5\}
\end{aligned}$$

Then we have the following case's

(3) $t > 0.5$ and $s < 0.5$

(4) $t \leq 0.5$ and $s \geq 0.5$

Case (3) If $t > 0.5$ and $s < 0.5$, then $\inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) \geq 0.5$ and $\sup_{z_1 \in x\gamma a\delta y} \lambda_A(z_1) \leq 0.5$. This implies that $\inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) + t > 1$ and $\sup_{z_1 \in x\gamma a\delta y} \lambda_A(z_1) + s < 1$. Then, for each $(z_1)(t, s)qA$.

Case(4) If $t \leq 0.5$ and $s \geq 0.5$, then $\inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) \geq t$ and $\sup_{z_1 \in x\gamma a\delta y} \lambda_A(z_1) \leq s$. This implies that $(z_1)(t, s) \in A$. Therefore, for each $z_1 \in x\gamma a\delta y$, $(z_1)(t, s) \in \vee qA$. ■

Remark 176 *Every intuitionistic fuzzy interior Γ-hyperideal of a Γ-semihypergroup H is an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of H. But the converse is not true.*

Example 177 *Let $H = \{1, 2, 3, 4, 5\}$ be a Γ-semihypergroup with the following Cayley table.*

γ	1	2	3	4	5	δ	1	2	3	4	5
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
3	$\{1\}$	$\{1\}$	$\{3\}$	$\{3\}$	$\{3\}$	3	$\{1\}$	$\{1\}$	$\{3,4\}$	$\{3,4\}$	$\{3,4\}$
4	$\{1\}$	$\{1\}$	$\{3\}$	$\{4,5\}$	$\{4,5\}$	4	$\{1\}$	$\{1\}$	$\{3,4\}$	$\{3,4\}$	$\{5\}$
5	$\{1\}$	$\{1\}$	$\{3,5\}$	$\{3\}$	$\{5\}$	5	$\{1\}$	$\{1\}$	$\{3,5\}$	$\{3,5\}$	$\{3,5\}$

Let $A = \langle \mu_A, \lambda_A \rangle$ be IFS in Γ-semihypergroup H define by $\mu_A(1) = \mu_A(2) = \mu_A(4) = 0.8$, $\mu_A(3) = 0.7$, $\mu_A(5) = 0.6$, *and* $\lambda_A(1) = \lambda_A(2) = \lambda_A(4) = 0.1$, $\lambda_A(3) = 0.2$, $\lambda_A(5) = 0.3$.

Then $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of S but not intuitionistic fuzzy interior Γ-hyperideal.

Proposition 178 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an $(\in, \in \vee q)$-intuitionistic fuzzy Γ-hyperideal of H. Then, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal.*

Proof. Let $x, y \in H$. Then,

$$\begin{aligned} \inf_{z_1 \in x\gamma y} \mu_A(z_1) &\geq \min\{\mu_A(y), 0.5\} \geq \min\{\{\mu_A(x), \mu_A(y), 0.5\} \\ \text{and} \sup_{z_1 \in x\gamma y} \lambda_A(z_1) &\leq \max\{\lambda_A(y), 0.5\} \leq \max\{\{\lambda_A(x), \lambda_A(y), 0.5\}. \end{aligned}$$

Now, let $x, a, y \in H$. Then

$$\begin{aligned} \inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) &\geq \min\{\inf_{t \in x\gamma a} \mu_A(t), 0.5\} \geq \min\{\mu_A(a), 0.5, 0.5\} = \min\{\mu_A(a), 0.5\} \\ \inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) &\geq \min\{\mu_A(a), 0.5\} \text{ and} \\ \sup_{z_1 \in x\gamma a\delta y} \lambda_A(z_1) &\leq \max\{\sup_{t \in x\gamma a} \lambda_A(t), 0.5\} \leq \max\{\lambda_A(a), 0.5, 0.5\} = \max\{\lambda_A(a), 0.5\} \\ \sup_{z_1 \in x\gamma a\delta y} \lambda_A(z_1) &\leq \max\{\lambda_A(a), 0.5\}. \end{aligned}$$

By Theorem 175, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of H. ■

Theorem 179 *If $\{A\}_{i \in \Lambda}$ is family of $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideals of H. Then, $\bigcap_{i \in \Lambda} A_i$ is an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of H, where $\bigcap_{i \in \Lambda} A_i = \langle \bigwedge_{i \in \Lambda} \mu_{A_i}, \bigvee_{i \in \Lambda} \lambda_{A_i} \rangle$.*

Proof. Let $x, y \in H$. Then we have

$$\begin{aligned}
\left(\bigwedge_{i\in\Lambda}\inf_{z_1\in x\gamma y}\mu_{A_i}\right)(z_1) &= \bigwedge_{i\in\Lambda}\left(\inf_{z_1\in x\gamma y}\mu_{A_i}(z_1)\right) \geq \bigwedge_{i\in\Lambda}\left(\min\left\{\mu_{A_i}(x), \mu_{A_i}(y), 0.5\right\}\right) \\
&= \min\left\{\bigwedge_{i\in\Lambda}\mu_{A_i}(x), \bigwedge_{i\in\Lambda}\mu_{A_i}(y), 0.5\right\} \\
\left(\bigwedge_{i\in\Lambda}\inf_{z_1\in x\gamma y}\mu_{A_i}\right)(z_1) &\geq \min\left\{\left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(x), \left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(y), 0.5\right\} \\
\text{and } \left(\sup_{z_1\in x\gamma y}\mu_{A_i}\right)(z_1) &= \bigvee_{i\in\Lambda}\left(\sup_{z_1\in x\gamma y}\lambda_{A_i}(z_1)\right) \leq \bigvee_{i\in\Lambda}\left(\max\left\{\lambda_{A_i}(x), \lambda_{A_i}(y), 0.5\right\}\right) \\
&= \max\left\{\bigvee_{i\in\Lambda}\mu_{A_i}(x), \bigvee_{i\in\Lambda}\mu_{A_i}(y), 0.5\right\} \\
\left(\bigvee_{i\in\Lambda}\sup_{z_1\in x\gamma y}\mu_{A_i}\right)(z_1) &\leq \max\left\{\left(\bigvee_{i\in\Lambda}\mu_{A_i}\right)(x), \left(\bigvee_{i\in\Lambda}\mu_{A_i}\right)(y), 0.5\right\}
\end{aligned}$$

Now, let for any $x, y, a \in H$. The we have

$$\begin{aligned}
\left(\bigwedge_{i\in\Lambda}\inf_{z_1\in x\gamma a\delta y}\mu_{A_i}\right)(z_1) &= \bigwedge_{i\in\Lambda}\left(\inf_{z_1\in x\gamma a\delta y}\mu_{A_i}(z_1)\right) \geq \bigwedge_{i\in\Lambda}\left(\min\left\{\mu_{A_i}(a), 0.5\right\}\right) \\
&= \min\left\{\bigwedge_{i\in\Lambda}\mu_{A_i}(a), 0.5\right\} \\
\left(\bigwedge_{i\in\Lambda}\inf_{z_1\in x\gamma a\delta y}\mu_{A_i}\right)(z_1) &\geq \min\left\{\left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(a), 0.5\right\} \\
\left(\bigvee_{i\in\Lambda}\sup_{z_1\in x\gamma a\delta y}\mu_{A_i}\right)(z_1) &= \bigvee_{i\in\Lambda}\left(\sup_{z_1\in x\gamma a\delta y}\lambda_{A_i}(z_1)\right) \leq \bigvee_{i\in\Lambda}\left(\max\left\{\lambda_{A_i}(a), 0.5\right\}\right) \\
&= \max\left\{\bigvee_{i\in\Lambda}\mu_{A_i}(a), 0.5\right\} \\
\left(\bigvee_{i\in\Lambda}\sup_{z_1\in x\gamma a\delta y}\mu_{A_i}\right)(z_1) &\leq \max\left\{\left(\bigvee_{i\in\Lambda}\mu_{A_i}\right)(a), 0.5\right\}
\end{aligned}$$

Hence, $\bigcap_{i\in\Lambda} A_i = \langle \bigwedge_{i\in\Lambda}\mu_{A_i}, \bigvee_{i\in\Lambda}\lambda_{A_i}\rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of H. ■

Remark 180 *The union of two $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideals of H is not necessary to an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of H.*

Example 181 *Let $H = \{a, b, c, d\}$ and $\Gamma = \{\gamma, \delta\}$ be two non-empty sets. Then, $(H,\ \Gamma)$*

is a Γ*-semihypergroup with the following multiplication tables:*

γ	1	2	3	4	δ	1	2	3	4
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1\}$	$\{1,4\}$	$\{1\}$	2	$\{1\}$	$\{1\}$	$\{2,4\}$	$\{1\}$
3	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	3	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
4	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	4	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

Let $A = \langle \mu_A, \lambda_A \rangle$ *and* $B = \langle \mu_B, \lambda_B \rangle$ *be two IFSs of* H *such that*

$$\begin{aligned} \mu_A(1) &= \mu_A(2) = 0.4,\ \mu_A(3) = \mu_A(4) = 0, \\ \lambda_A(1) &= \lambda_A(2) = 0.6,\ \lambda_A(3) = \lambda_A(4) = 0.8 \\ \text{and } \mu_B(1) &= 0.4,\ \mu_B(2) = 0,\ \mu_B(3) = 0.4,\ \mu_B(4) = 0, \\ \lambda_B(1) &= 0.6,\ \lambda_B(2) = 0.8,\ \lambda_B(3) = 0.6,\ \lambda_B(4) = 0.8. \end{aligned}$$

Then both $A = \langle \mu_A, \lambda_A \rangle$ *and* $B = \langle \mu_B, \lambda_B \rangle$ *are* $(\in, \in \vee q)$*-intuitionistic fuzzy interior* Γ*-hyperideals of* H*, but* $A \cup B = \langle \mu_A \vee \mu_B, \lambda_A \wedge \lambda_B \rangle$ *is not an* $(\in, \in \vee q)$*-intuitionistic fuzzy interior* Γ*-hyperideal of* H*.*

The following theorem can be obtained if we present a sufficient condition.

Theorem 182 *If* $\{A_i\}_{i\in\Lambda}$ *is a family of* $(\in, \in \vee q)$*-intuitionistic fuzzy interior* Γ*-hyperideals of* H *such that* $A_i \subseteq A_j$ *or* $A_j \subseteq A_i$ *for all* $i,\ j \in I$*, then* $\bigcup_{i\in\Lambda} A_i = \langle \bigvee_{i\in\Lambda} \mu_{A_i}, \bigwedge_{i\in\Lambda} \lambda_{A_i} \rangle$ *is an* $(\in, \in \vee q)$*-intuitionistic fuzzy interior* Γ*-hyperideal of* H*.*

Proof. For all $x, y, a \in H$ and $\gamma, \delta \in \Gamma$, we have

$$\begin{aligned}
\left(\bigvee_{i\in\Lambda} \inf_{z_1 \in x\gamma y} \mu_{A_i}\right)(z_1) &= \bigvee_{i\in\Lambda}\left(\inf_{z_1\in x\gamma a\delta y} \mu_{A_i}(z_1)\right) \geq \bigvee_{i\in\Lambda}\left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5\}\right] \\
&= \left[\bigvee_{i\in\Lambda} \mu_{A_i}(x) \wedge \bigvee_{i\in\Lambda} \mu_{A_i}(y) \wedge 0.5\right] \\
&= \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right] \\
\left(\bigvee_{i\in\Lambda} \inf_{z_1 \in x\gamma y} \mu_{A_i}\right)(z_1) &\geq \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right]
\end{aligned}$$

It is clear that

$$\bigvee_{i\in\Lambda}\left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5\right] \leq \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right].$$

Assume that

$$\bigvee_{i\in\Lambda}\left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5\right] \neq \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right]$$

Then there exists t such that

$$\bigvee_{i\in\Lambda}\left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5\right] < t < \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right]$$

Since $\mu_{A_i} \subseteq \mu_{A_j}$ or $\mu_{A_j} \subseteq \mu_{A_i}$ for all $i,\ j \in I$, so exists $k \in I$ such that $t < \mu_{A_k}(x) \wedge \mu_{A_k}(y) \wedge 0.5$. On other hand $\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5 < t$ for all $i \in I$, a contradiction. Hence

$$\bigvee_{i\in\Lambda}\left[\mu_{A_i}(x) \wedge \mu_{A_i}(y) \wedge 0.5\right] = \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(x) \wedge \left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(y) \wedge 0.5\right]$$

Now,

$$\begin{aligned}
\left(\bigwedge_{i\in\Lambda}\sup_{z_1\in x\gamma y}\lambda_{A_i}\right)(z_1) &= \bigwedge_{i\in\Lambda}\left(\sup_{z_1\in x\gamma y}\lambda_{A_i}(z_1)\right)\leq\bigwedge_{i\in\Lambda}[\lambda_{A_i}(x)\vee\lambda_{A_i}(y)\vee 0.5\}] \\
&= \left[\bigwedge_{i\in\Lambda}(\lambda_{A_i}(x))\vee\bigwedge_{i\in\Lambda}(\lambda_{A_i}(y))\vee 0.5\right] \\
&= \left[\left(\bigwedge_{i\in\Lambda}\lambda_{A_i}\right)(x)\vee\left(\bigwedge_{i\in\Lambda}\lambda_{A_i}\right)(y)\vee 0.5\right] \\
\left(\bigwedge_{i\in\Lambda}\sup_{z_1\in x\gamma y}\lambda_{A_i}\right)(z_1) &\leq \left[\left(\bigwedge_{i\in\Lambda}\lambda_{A_i}\right)(x)\vee\left(\bigwedge_{i\in\Lambda}\lambda_{A_i}\right)(y)\vee 0.5\right]
\end{aligned}$$

It is clear that

$$\bigwedge_{i\in\Lambda}[\lambda_{A_i}(x)\vee\lambda_{A_i}(y)\vee 0.5]\geq\left[\left(\bigwedge_{i\in\Lambda}\lambda_{A_i}\right)(x)\vee\left(\bigwedge_{i\in\Lambda}\mu_{A_i}\right)(y)\vee 0.5\right].$$

Assume that

$$\bigwedge_{i\in\Lambda}[\lambda_{A_i}(x)\vee\lambda_{A_i}(y)\vee 0.5]\neq\left[\left(\bigwedge_{i\in\Lambda}\lambda_{A_i}\right)(x)\vee\left(\bigwedge_{i\in\Lambda}\lambda_{A_i}\right)(y)\vee 0.5\right]$$

Then there exist s such that

$$\bigwedge_{i\in\Lambda}[\lambda_{A_i}(x)\vee\lambda_{A_i}(y)\vee 0.5]>s>\left[\left(\bigwedge_{i\in\Lambda}\lambda_{A_i}\right)(x)\vee\left(\bigwedge_{i\in\Lambda}\lambda_{A_i}\right)(y)\vee 0.5\right]$$

Since $\lambda_{A_i}\subseteq\lambda_{A_j}$ or $\lambda_{A_j}\subseteq\lambda_{A_i}$ for all i, $j\in I$, there exists $k\in I$ such that $s>\mu_{A_k}(x)\wedge\mu_{A_k}(y)\wedge 0.5$. On other hand $\mu_{A_i}(x)\wedge\mu_{A_i}(y)\wedge 0.5>s$ for all $i\in I$, a contradiction. Hence

$$\bigwedge_{i\in\Lambda}[\lambda_{A_i}(x)\wedge\lambda_{A_i}(y)\wedge 0.5]=\left[\left(\bigwedge_{i\in\Lambda}\lambda_{A_i}\right)(x)\wedge\left(\bigwedge_{i\in\Lambda}\lambda_{A_i}\right)(y)\wedge 0.5\right]$$

Let $x, a, y \in H$ and $\gamma, \delta \in \Gamma$. Then

$$\begin{aligned}
\left(\bigvee_{i\in\Lambda} \inf_{z_1 \in x\gamma a\delta y} \mu_{A_i}\right)(z_1) &= \bigvee_{i\in\Lambda}\left[\inf_{z_1\in x\gamma a\delta y} \mu_{A_i}(z_1)\right] \geq \bigvee_{i\in\Lambda}\left[\mu_{A_i}(a) \wedge \wedge 0.5\}\right] \\
&= \left[\bigvee_{i\in\Lambda} \mu_{A_i}(a) \wedge 0.5\right] \\
&= \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(a) \wedge \wedge 0.5\right] \\
\left(\bigvee_{i\in\Lambda} \inf_{z_1 \in x\gamma a\delta y} \mu_{A_i}\right)(z_1) &\geq \left[\left(\bigvee_{i\in\Lambda} \mu_{A_i}\right)(a) \wedge \wedge 0.5\right]
\end{aligned}$$

and

$$\begin{aligned}
\left(\bigwedge_{i\in\Lambda} \sup_{z_1 \in x\gamma a\delta y} \lambda_{A_i}\right)(z_1) &= \bigwedge_{i\in\Lambda}\left(\sup_{z_1\in x\gamma a\delta y} \lambda_{A_i}(xay)\right) \leq \bigwedge_{i\in\Lambda}\left[\lambda_{A_i}(a) \vee 0.5\}\right] \\
&= \left[\bigwedge_{i\in\Lambda} (\lambda_{A_i}(a)) \vee 0.5\right] \\
&= \left[\left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(a) \vee 0.5\right] \\
\left(\bigwedge_{i\in\Lambda} \sup_{z_1 \in x\gamma a\delta y} \lambda_{A_i}\right)(z_1) &\leq \left[\left(\bigwedge_{i\in\Lambda} \lambda_{A_i}\right)(a) \vee 0.5\right]
\end{aligned}$$

Hence $\bigcup_{i\in\Lambda} A_i = \langle \bigvee_{i\in\Lambda} \mu_{A_i}, \bigwedge_{i\in\Lambda} \lambda_{A_i}\rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of H. ■

Theorem 183 *An IFS $A = \langle \mu_A, \lambda_A\rangle$ in a Γ-semihypergroup H is an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of H if and only if the non-empty sets $U(\mu_A,\ t)$ and $L(\lambda_A,\ s)$ are interior Γ-hyperideals of H for all $t \in (0, 0.5]$ and $s \in [0.5, 1)$, where $U(\mu_A,\ t) = \{x \in H : \mu_A(x) \geq t\}$ and $L(\lambda_A,\ t) = \{x \in H : \lambda_A(x) \leq s\}$.*

Proof. Let $A = \langle \mu_A, \lambda_A\rangle$ be an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of H and the sets $U(\mu_A,\ t)$ and $L(\lambda_A,\ s)$ be non-empty for any $t \in (0, 0.5]$ and $s \in [0.5, 1)$.

Let $x, y \in U(\mu_A,\ t)$. Then, $\mu_A(x) \geq t$ and $\mu_A(y) \geq t$. Since

$$\begin{aligned}\inf_{z_1 \in x\gamma y} \mu_A(z_1) &\geq \mu_A(x) \wedge \mu_A(y) \wedge 0.5 \\ &\geq t \wedge t \wedge 0.5 = t\end{aligned}$$

This implies that $z_1 \in U(\mu_A,\ t)$ for each $z_1 \in x\gamma y$. Thus, $x\gamma y \subseteq U(\mu_A,\ t)$. Now let $a \in U(\mu_A,\ t)$, $x, y \in H$ and $\gamma, \delta \in \Gamma$. Then $\mu_A(a) \geq t$. Since

$$\begin{aligned}\inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) &\geq \mu_A(a) \wedge 0.5 \\ &\geq t \wedge 0.5 = t\end{aligned}$$

This implies that $z_1 \in U(\mu_A,\ t)$ for each $z_1 \in x\gamma a\delta y$. Thus, $x\gamma a\delta y \subseteq U(\mu_A,\ t)$. Therefore, $U(\mu_A,\ t)$ is an interior Γ-hyperideal of H. Similarly, we can prove $L(\lambda_A,\ s)$ is an interior Γ-hyperideal of H.

Conversely, let $A = \langle \mu_A, \lambda_A \rangle$ be an IFS in H such that $U(\mu_A,\ t)$ and $L(\lambda_A,\ s)$ be interior Γ-hyperideals of H. If there exist $x, y \in H$ and $\gamma \in \Gamma$ such that $\inf_{z_1 \in x\gamma y} \mu_A(z_1) < \mu_A(x) \wedge \mu_A(y) \wedge 0.5$ and $\sup_{z_1 \in x\gamma y} \lambda_A(z_1) > \lambda_A(x) \vee \lambda_A(y) \vee 0.5$, then there exist $t \in (0, 1]$ and $s \in [0, 1)$ such that $\inf_{z_1 \in x\gamma y} \mu_A(z_1) < t < \mu_A(x) \wedge \mu_A(y) \wedge 0.5$ and $\sup_{z_1 \in x\gamma y} \lambda_A(z_1) > s > \lambda_A(x) \vee \lambda_A(y) \vee 0.5$. This inmplies that $x, y \in U(\mu_A,\ t)$ and $x, y \in L(\lambda_A,\ s)$ but $x\gamma y \nsubseteq U(\mu_A,\ t_\circ)$ and $x\gamma y \nsubseteq L(\lambda_A,\ s)$, which is a contradiction. Hence

$$\begin{aligned}\inf_{z_1 \in x\gamma y} \mu_A(z_1) &\geq \mu_A(x) \wedge \mu_A(y) \wedge 0.5 \\ \text{and} \sup_{z_1 \in x\gamma y} \lambda_A(z_1) &\leq \lambda_A(x) \vee \lambda_A(y) \vee 0.5\end{aligned}$$

Also, if there exist $x, y, a \in H$ and $\gamma, \delta \in \Gamma$such that $\inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) < \mu_A(a) \wedge 0.5$ and $\sup_{z_1 \in x\gamma a\delta y} \lambda_A(z_1) > \lambda_A(a) \vee 0.5$, then choose $t_1 \in (0, 1]$ and $s_1 \in [0, 1)$ such that $\inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) < t_1 < \mu_A(a) \wedge 0.5$ and $\sup_{z_1 \in x\gamma a\delta y} \lambda_A(z_1) > s_1 > \lambda_A(a) \vee 0.5$. This implies that $a \in U(\mu_A,\ t_1)$ and $a \in L(\lambda_A,\ s_1)$ but $x\gamma a\delta y \nsubseteq U(\mu_A,\ t_1)$ and $x\gamma a\delta y \nsubseteq$

$L(\lambda_A,\ s_1)$, which is a contradiction. Hence,

$$\begin{aligned} \inf_{z_1 \in x\gamma a\delta y} \mu_A(z_1) &\leq \lambda_A(a) \wedge 0.5 \\ \text{and} \sup_{z_1 \in x\gamma a\delta y} \lambda_A(z_1) &\leq \lambda_A(a) \vee 0.5 \end{aligned}$$

Therefore, $A = \langle \mu_A, \lambda_A \rangle$ is an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of H. ■

Theorem 184 *Every $(\in \vee q, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of a Γ-semihypergroup H is an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of H.*

Every $(\in, \in,)$-intuitionistic fuzzy interior Γ-hyperideal of Γ-semihypergroup H is an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of H.

Now we give the condition for $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal to be an $(\in, \in,)$-intuitionistic fuzzy interior Γ-hyperideal of H.

Theorem 185 *Let $A = \langle \mu_A, \lambda_A \rangle$ be $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of H such that $\mu_A(x) < 0.5$ and $\lambda_A(x) > 0.5$. Then $A = \langle \mu_A, \lambda_A \rangle$ be $(\in, \in)$-intuitionistic fuzzy interior Γ-hyperideal of H.*

Theorem 186 *An IFS $A = \langle \mu_A, \lambda_A \rangle$ of H is an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of H if and only if $U_{(t,s)} = \{x \in H : x(t,s) \in A\}$ for all $t \in (0, 0.5]$ and $s \in [0.5, 1)$ is an interior Γ-hyperideal of H.*

Proof. Its follows from Theorem 183. ■

Theorem 187 *An IFS $A = \langle \mu_A, \lambda_A \rangle$ in a Γ-semihypergroup H is an $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of H if and only if $[A]_{(t,s)}$ is an interior Γ-hyperideal of H for all $t \in (0, 1]$, $s \in [0.1)$.*

Proof. Let $A = \langle \mu_A, \lambda_A \rangle$ be $(\in, \in \vee q)$-intuitionistic fuzzy interior Γ-hyperideal of H and let $x, y \in [A]_{(t,s)}$ for all $t \in (0, 0.5]$, $s \in [0.5.1)$. Then $x(t,s)$, $y(t,s) \in \vee qA$, that is

$$\begin{aligned} \mu_A(x) &\geq t \text{ and } \lambda_A(x) \leq s,\ \mu_A(y) \geq t \text{ and } \lambda_A(y) \leq s \text{ and} \\ \mu_A(x) + t &> 1 \text{ and } \lambda_A(x) + s < 1,\ \mu_A(y) + t > 1 \text{ and } \lambda_A(y) + s < 1. \end{aligned}$$

Since, by Theorem 175, we have

$$\begin{aligned} \inf_{z \in x\gamma y} \mu_A(z) &\geq \min\{\mu_A(x), \mu_A(y), 0.5\} \text{ and } \sup_{z \in x\gamma y} \lambda_A(z) \leq \max\{\lambda_A(x), \lambda_A(y), 0.5\}. \\ \inf_{z \in x\gamma y} \mu_A(z) &\geq \min\{t, 0.5\} \text{ and } \sup_{z \in x\gamma y} \lambda_A(z) \leq \max\{s, 0.5\}. \end{aligned}$$

For, if not, then $x(t,s)\overline{\in \vee q}A$ or $y(t,s)\overline{\in \vee q}A$, which is a contradiction. If $t \leq 0.5$ and $s \geq 0.5$, then $\inf_{z \in x\gamma y} \mu_A(z) \geq \min\{t, 0.5\} = t$ and $\sup_{z \in x\gamma y} \lambda_A(z) \leq \max\{s, 0.5\} = s$. Thus $(z)(t,s) \in A$ for each $z \in x\gamma y$. If $t > 0.5$ and $s < 0.5$, then $\inf_{z \in x\gamma y} \mu_A(z) \geq 0.5$ and $\sup_{z \in x\gamma y} \lambda_A(z) \leq 0.5$ and thus $\inf_{z \in x\gamma y} \mu_A(z) + t > 1$ and $\sup_{z \in x\gamma y} \lambda_A(z) + s < 1$. Thus $(z)(t,s)\, qA$ for each $z \in x\gamma y$. Hence $(z)(t,s) \in \vee qA$ for each $z \in x\gamma y$. Therefore $x\gamma y \subseteq [A]_{(t,s)}$. Thus, $[A]_{(t,s)}$ is sub-Γ-semihypergroup of H. Now, let $x, y \in H$ and $a \in [A]_{(t,s)}$. Then $a(t,s) \in \vee qA$, that is $\mu_A(a) \geq t$ and $\lambda_A(a) \leq s$ or $\mu_A(a) + t > 1$ and $\lambda_A(a) + s < 1$. Since from Theorem 175, we have

$$\begin{aligned} \inf_{z \in x\gamma a\delta y} \mu_A(z) &\geq \min\{\mu_A(a), 0.5\} \text{ and } \sup_{z \in x\gamma a\delta y} \lambda_A(z) \leq \max\{\lambda_A(a), 0.5\}. \\ \inf_{z \in x\gamma a\delta y} \mu_A(z) &\geq \min\{t, 0.5\} \text{ and } \sup_{z \in x\gamma a\delta y} \lambda_A(z) \leq \max\{s, 0.5\}. \end{aligned}$$

Otherwise, we get

$$\inf_{z \in x\gamma a\delta y} \mu_A(z) < \min\{t, 0.5\} \text{ and } \sup_{z \in x\gamma a\delta y} \lambda_A(z) > \max\{s, 0.5\}.$$

Which is a contradiction. If $t \leq 0.5$ and $s \geq 0.5$, then $\inf_{z \in x\gamma a\delta y} \mu_A(z) \geq \min\{t, 0.5\} = t$ and $\sup_{z \in x\gamma a\delta y} \lambda_A(z) \leq \max\{s, 0.5\} = s$ and thus $(z)(t,s) \in A$ for each $z \in x\gamma a\delta y$. If

$t > 0.5$ and $s < 0.5$, then $\inf_{z\in x\gamma a\delta y}\mu_A(z) \geq 0.5$ and $\sup_{z\in x\gamma a\delta y}\lambda_A(z) \leq 0.5$ and so, $\inf_{z\in x\gamma a\delta y}\mu_A(z) + t > 0.5 + 0.5 = 1$ and $\sup_{z\in x\gamma a\delta y}\lambda_A(z) + s < 0.5 + 0.5 = 1$. Thus $(z)(t,s)\,qA$ for each $z \in x\gamma a\delta y$. Therefore $(z)(t,s) \in \vee qA$ for each $z \in x\gamma a\delta y$ and so $x\gamma a\delta y \subseteq [A]_{(t,s)}$. Hence $[A]_{(t,s)}$ is an interior Γ-hyperideal of Γ-semihypergroup H. This is complete proof. ■

Chapter 8

Intuitionistic Fuzzy hyperideals in Γ-semihypergroups Through Left Operator Semihypergroup

8.1 Introduction

In this chapter, we obtain a series of lemmas and theorems which are mainly on various relationships between a Γ-semihypergroup and its operator semihypergroups in terms of intuitionistic fuzzy subsets showing so the effectiveness of using operator semihypergroups in extending several results of semigroups to Γ-semihypergroups as well as to Γ-semigroups. Among other results, we obtain an inclusion preserving bijection between the set of all intuitionistic fuzzy hyperideal of a Γ-semihypergroup H and that of its left operator semihypergroup S.

Throughout this section, unless otherwise mentioned, H will denote a Γ-semihypergroup and S will denote the left operator semihypergroup of H. For the sake of simplicity we shall use the follwoing abbreviated notations: $IFS(H)$, $IFH(H)$, $IFLH(H)$, $IFPH(H)$, $IFSPH(H)$, $RH(H)$, $H(H)$ respectively denote intuitionistic fuzzy subset(s) of H, intuitionistic fuzzy hyperideal(s) of H, intuitionistic fuzzy left hyperideal(s) of H, intuitionistic fuzzy prime hyperideal(s) of H, intuitionistic fuzzy semiprime hyperideal(s) of H, right hyperideal(s) of H, hyperideal(s) of H.

8.2 Intuitionistic Fuzzy Hyperideals in Γ-semihypergroups

We obtain a series of lemmas and theorems which are mainly on various relationships between a Γ-semihypergroup and its operator semihypergroups in terms of intuitionistic fuzzy subsets showing so the effectiveness of using operator semihypergroups in extending several results of semigroups to Γ-semihypergroups as well as to Γ-semigroups. Among other results, we obtain an inclusion preserving bijection between the set of all intuitionistic fuzzy hyperideal of a Γ-semihypergroup H and that of its left operator semihypergroup S

Let H be a Γ-semihypergroup and S a left operator semihypergroup of H. For an $IFS(S)$, $C = (\mu_C, \lambda_C)$ we define an $IFS(H)$, $C^+ = (\mu_C, \lambda_C)^+ = (\mu_C^+, \lambda_C^+)$ by $\mu_C^+(a) = \inf_{\gamma\in\Gamma} \mu_C([a,\gamma])$ and $\lambda_C^+(a) = \sup_{\gamma\in\Gamma} \lambda_C([a,\gamma])$, where $a \in H$. For an $IFS(H)$, $D = (\mu_D, \lambda_D)$ we define an $IFS(S)$, $D^{+'} = (\mu_D, \lambda_D)^{+'} = (\mu_D^{+'}, \lambda_D^{+'})$ by $\mu_D^{+'}([a,\gamma]) = \inf_{m\in H, t\in a\gamma m} \mu_D(t)$ and $\lambda_D^{+'}([a,\gamma]) = \sup_{m\in H, t\in a\gamma m} \lambda_D(t)$, where $[a,\gamma] \in S$.

The following examples illustrate the above notion.

Example 188 *Let $H = [0,1]$ be the real unit close interval, and $\Gamma = N$. Then, H together with the hyperoperation $x\gamma y = [0, \frac{xy}{\gamma}]$ is a Γ-semihypergroup. If we define the following relation ρ in $H \times \Gamma$ as follows:*

$(x,\alpha)\rho(y,\beta) \Leftrightarrow \frac{x}{\alpha} = \frac{y}{\beta}$

it is clear that ρ is an equivalence relation. Then $S = \{[x,\alpha] | x \in H, \alpha \in \Gamma\}$ with the hyperoperation

$[x,\alpha] \circ [y,\beta] = \{[z,\beta] : z \in x\alpha y\}$

is the left operator semihypergroup of H. We can define the following fuzzy subset of

S:

$$\mu_C([x,\alpha]) = \begin{cases} 0.7 & if\ 0 \leq x < \frac{1}{\alpha} \\ 0.4 & if\ \frac{1}{\alpha} \leq x \leq 1 \end{cases} \quad and\ \lambda_C([x,\alpha]) = \begin{cases} 0.3 & if\ 0 \leq x < \frac{1}{\alpha} \\ 0.5 & if\ \frac{1}{\alpha} \leq x \leq 1 \end{cases}$$

so

$$\mu_C^+(x) = \inf_{\alpha\in\Gamma} \mu_C([x,\alpha]) \ and\ \lambda_C^+(x) = \sup_{\gamma\in\Gamma} \lambda_C([a,\gamma])$$

So, for example, for $x = 0.5$ we have

$$\mu_C^+(0.5) = \inf_{\alpha\in\Gamma} \mu_C([0.5,\alpha]) = 0.4 \ and\ \lambda_C^+(0.5) = \sup_{\gamma\in\Gamma} \lambda_C([a,\gamma]) = 0.5$$

.

Example 189 *Let $H = \{1,2\}$ and $\Gamma = H$. Then, H together with hyperoperation $x\alpha y = \{x,\alpha,y\}$ is a Γ-semihypergroup. If we define the following relation ρ in $H \times \Gamma$ as follows: $x\alpha y = x\beta y$, then ρ is an equivalence relation. We have*

$$\begin{aligned} [1,1] &= [2,1] = \{(1,1),(2,1)\} \\ [2,2] &= [1,2] = \{(1,2),(2,2)\}. \end{aligned}$$

So, we have $S = \{[1,1],[1,2]\}$ is a semihypergroup with the following hyperoperation define

$\circ$	$[1,1]$	$[1,2]$
$[1,1]$	$[1,1]$	$[1,2]$
$[1,2]$	S	S

Now, we define IFS $A = (\mu_A, \gamma_A)$ in S as follows:

$$\begin{aligned} \mu_A([1,1]) &= 0.6,\ \mu_A([1,2]) = 0.4 \\ \gamma_A([1,1]) &= 0.4,\ \gamma_A([1,2]) = 0.5. \end{aligned}$$

We have $A^{+}=\left(\mu_A^{+},\gamma_A^{+}\right)$ *defined as* $\mu_A^{+}(x)=\inf\limits_{\alpha\in\Gamma}\mu_A([x,\alpha])$ *and* $\gamma_A^{+}(x)=\sup\limits_{\alpha\in\Gamma}\mu_C([x,\alpha])$

$$\begin{aligned}\mu_A^{+}(1) &= \mu_A^{+}(2)=0.4\\ \gamma_A^{+}(1) &= \gamma_A^{+}(2)=0.5.\end{aligned}$$

Proposition 190 *Let* H *be a* Γ*-semihypergroup and* S *be its left operator semihypergroup. Let* $A=\langle\mu_A,\lambda_A\rangle$ *be an* $IFS(S)$*, then* $[U(\mu_A;t)]^{+}=U((\mu_A)^{+};t)$ *and* $[L(\lambda_A;t)]^{+}=L((\lambda_A)^{+};t)$ *for all* $t\in[0,1]$*, provided the sets are non-empty.*

Proof. Let $m\in H$. Then,

$$\begin{aligned} m &\in [U(\mu_A;t)]^{+}\Leftrightarrow[m,\gamma]\subseteq U(\mu_A;t),\forall\gamma\in\Gamma\\ &\Leftrightarrow \mu_A([m,\gamma])\geq t,\forall\gamma\in\Gamma\\ &\Leftrightarrow \inf_{\gamma\in\Gamma}\mu_A([m,\gamma])\geq t\\ &\Leftrightarrow (\mu_A^{+}(m)\geq t\Leftrightarrow m\in U((\mu_A)^{+};t).\end{aligned}$$

Again, let $n\in H$. Then,

$$\begin{aligned} n &\in [L(\lambda_A;t)]^{+}\Leftrightarrow[n,\gamma]\subseteq L(\lambda_A;t),\forall\gamma\in\Gamma\\ &\Leftrightarrow \lambda_A([n,\gamma])\leq t,\forall\gamma\in\Gamma\\ &\Leftrightarrow \sup_{\gamma\in\Gamma}\lambda_A([n,\gamma])\leq t\\ &\Leftrightarrow (\lambda_A^{+}(n)\leq t\Leftrightarrow n\in L((\lambda_A)^{+};t).\end{aligned}$$

Hence, $[U(\mu_A;t)]^{+}=U((\mu_A)^{+};t)$ and $[L(\lambda_A;t)]^{+}=L((\lambda_A)^{+};t)$. ■

Proposition 191 *Let* H *be a* Γ*-semihypergroup and* S *be its left operator semihypergroup. Let* $B=\langle\mu_B,\lambda_B\rangle$ *be an* $IFS(H)$*. Then,* $[U(\mu_B;t)]^{+'}=U((\mu_B)^{+'};t)$ *and* $[L(\lambda_B;t)]^{+'}=L((\lambda_B)^{+'};t)$ *for all* $t\in[0,1]$*, provided the sets under consideration are non-empty.*

Proof. Let $[x,\alpha] \in S$ and t be as mentioned in the statement. Then,

$$\begin{aligned} [x,\alpha] &\in [U(\mu_B;t)]^{+'} \Leftrightarrow x\alpha m \subseteq U(\mu_B;t), \forall H \in H \\ &\Leftrightarrow \mu_B(x\alpha m) \geq t, \forall m \in H \Leftrightarrow \inf_{m\in H, t\in x\alpha H} \mu_B(t) \geq t \\ &\Leftrightarrow (\mu_B)^{+'}([x,\alpha]) \geq t \Leftrightarrow [x,\alpha] \in U((\mu_B^{+'};t). \end{aligned}$$

Again let $[y,\beta] \in S$ and t be as mentioned in the statement. Then,

$$\begin{aligned} [y,\beta] &\in [L(\lambda_B;t)]^{+'} \Leftrightarrow y\beta n \subseteq L(\lambda_B;t), \forall n \in H \\ &\Leftrightarrow \lambda_B(y\beta n) \leq t, \forall n \in H \Leftrightarrow \sup_{n\in H, t\in y\beta n} \lambda_B(t) \leq t \\ &\Leftrightarrow (\lambda_B)^{+'}([y,\beta]) \leq t \Leftrightarrow [y,\beta] \in U((\lambda_B)^{+'};t). \end{aligned}$$

Hence, $[U(\lambda_B;t)]^{+'} = U((\lambda_B)^{+'};t)$ and $[L(\lambda_B;t)]^{+'} = L((\lambda_B)^{+'};t)$. ■

In the following, H denotes a Γ-semihypergroup with unities.

Proposition 192 *Let H be a Γ-semihypergroup with unities and S be its left operator semihypergroup. If $A = \langle \mu_A, \lambda_A \rangle \in IFS(S)$, then $A^+ = \langle \mu_A, \lambda_A \rangle^+ = (\mu_A^+, \lambda_A^+) \in IFS(H)$.*

Proof. Let us suppose $A = \langle \mu_A, \lambda_A \rangle \in IFS(S)$. Then, $U(\mu_A;t)$ and $L(\lambda_A;t)$ are $I(S), \forall t \in [0,1]$. Hence, $[U(\mu_A;t)]^+$ and $[L(\lambda_A;t)]^+$ are $I(H), \forall t \in [0,1]$. Now, since $A = \langle \mu_A, \lambda_A \rangle$ is an $IFS(S)$, $A = \langle \mu_A, \lambda_A \rangle$ is a non-empty $IFS(S)$. Hence, for some $[m,\alpha] \in S, 0 < \mu_A([m,\alpha]) + \lambda_A([m,\alpha]) \leq 1$. Then, $U(\mu_A;t) \neq \emptyset$ and $L(\lambda_A;t) \neq \emptyset$, where $t = \mu_A([m,\alpha]) = \lambda([m,\alpha])$. So by the same argument applied above $[U(\mu_A;t)]^+ \neq \emptyset$ and $[L(\lambda_A;t)]^+ \neq \emptyset$. Let $u \in [U(\mu_A;t)]^+$. Then, $[u,\beta] \subseteq U(\mu_A;t)$ for all $\beta \in \Gamma$. Hence, $\mu_A([u,\beta]) \geq t$. This implies that $\inf_{\beta\in\Gamma} \mu_A([u,\beta]) \geq t$, i.e. $(\mu_A)^+(u) \geq t$. Hence, $u \in U((\mu_A)^+;t)$. Hence, $U((\mu_A)^+;t) \neq \emptyset$. By similar argument we can show that $[L(\lambda_A^+;t)] \neq \emptyset$. Consequently, $[U(\mu_A;t)]^+ = U((\mu_A)^+;t)$ and $[L(\lambda_A;t)]^+ = L((\lambda_A)^+;t)$. It follows that $U((\mu_A)^+;t)$ and $L((\lambda_A)^+;t)$ are $H(H)$ for all $t \in [0,1]$. Hence, $A^+ = \langle \mu_A, \lambda_A \rangle^+ = (\mu_A^+, \lambda_A^+)$ is an $IFH(H)$. ■

By using the property (ii) in section 2, the Proposition 192 and [31], we deduce the following proposition.

Proposition 193 *Let H be a Γ-semihypergroup with unities and S be its left operator semihypergroup. If $B = \langle \mu_B, \lambda_B \rangle \in IFH(H)(IFLH(H))$, then $B^{+'} = \langle \mu_B, \lambda_B \rangle^{+'} = (\mu_B^{+'}, \lambda_B^{+'}) \in IFH(S)$ (resp. $IFLH(S)$).*

Now we obtain the inclusion preserving bijection of mapping $()^{+'}$ for the intuitionistic fuzzy hyperideals and intuitionistic fuzzy right hyperideals of a Γ-semihypergroup H and its left operator semihypergroup S which is the analogue with the case of fuzzy sets [33]. The theorem also shows that the mappings $()^{+'}$ and $()^{+}$ are inverse to each other.

Theorem 194 *Let H be a Γ-semihypergroup with unities and S be its left operator semihypergroup. Then there exists an inclusion preserving bijection $A \to A^{+'}$ between the set of all $IFH(H)(IFRH(H))$ and the set of all $IFH(S)$ (resp. $IFRH(S)$), where $A = \langle \mu_A, \lambda_A \rangle$ is an $IFH(H)$ (resp. $IFRH(H)$).*

Proof. Let $A = \langle \mu_A, \lambda_A \rangle \in IFH(H)(IFRH(H))$ and $x \in H$. Then, we have

$(\mu_A^{+'})^{+}(x) = \inf\limits_{\gamma\in\Gamma} \mu_A^{+'}([x,\gamma]) = \inf\limits_{\gamma\in\Gamma}[\inf\limits_{m\in H, t\in x\gamma m} \mu_A(t)] \geq \mu_A(x)$.

Again

$(\lambda_A^{+'})^{+}(x) = \sup\limits_{\gamma\in\Gamma} \lambda_A^{+'}([x,\gamma]) = \sup\limits_{\gamma\in\Gamma}[\sup\limits_{m\in H, t\in x\gamma m} \lambda_A(t)] \geq \lambda_A(x)$.

Hence, $A \subseteq (A^{+'})^{+}$. Let $[\gamma, f]$ be the right unity of H. Then $x\gamma f = x, \forall x \in H$. This implies that

$\mu_A(x) = \mu_A(x\gamma f) \geq \inf\limits_{\alpha\in\Gamma}[\inf\limits_{m\in H, t\in x\alpha m} \mu_A(t)] = \inf\limits_{\alpha\in\Gamma} \mu_A^{+'}([x,\alpha]) = (\mu_A^{+'})^{+}(x)$.

Again

$\lambda_A(x) = \lambda_A(x\gamma f) \leq \sup\limits_{\alpha\in\Gamma}[\sup\limits_{m\in H, t\in x\alpha m} \lambda_A(t)] = \sup\limits_{\alpha\in\Gamma} \lambda_A^{+'}([x,\alpha]) = (\lambda_A^{+'})^{+}(x)$.

So $A \supseteq (A^{+'})^{+}$. Hence, we obtain $(A^{+'})^{+} = A$. Now, let $B = \langle \mu_B, \lambda_B \rangle \in IFH(S)(IFRH(H))$. Then,

$$\begin{aligned}(\mu_B^{+})^{+'}([x,\alpha]) &= \inf_{m\in H, h\in x\alpha m} \mu_B^{+}(h) = \inf_{m\in H}[\inf_{\gamma\in\Gamma} \mu_B([x\alpha m,\gamma])] \\ &= \inf_{m\in H}[\inf_{\gamma\in\Gamma, z\in x\alpha m} \mu_B([z,\gamma])] \geq \mu_B([x,\alpha]).\end{aligned}$$

Again

$$\begin{aligned}(\lambda_B^+)^{+'}([x,\alpha]) &= \sup_{m\in H, h\in x\alpha m} \lambda_B^+(h) = \sup_{m\in H}[\sup_{\gamma\in\Gamma} \lambda_B([x\alpha H,\gamma])] \\ &= \sup_{m\in H}[\sup_{\gamma\in\Gamma, z\in x\alpha m} \lambda_B([z,\gamma])] \geq \lambda_B([x,\alpha]).\end{aligned}$$

Thus, we obtain $B \subseteq (B^+)^{+'}$. Let $[e,\delta]$ be the left unity of S. Then,

$$\begin{aligned}\mu_B([x,\alpha]) &= \mu_B([x,\alpha]\circ[e,\delta]) \geq \inf_{m\in H}[\inf_{\gamma\in\Gamma} \mu_B([x,\alpha]\circ[m,\gamma])] = \\ &= (\mu_B^+)^{+'}([x,\alpha]).\end{aligned}$$

Again

$$\begin{aligned}\lambda_B([x,\alpha]) &= \lambda_B([x,\alpha]\circ[e,\delta]) \leq \sup_{m\in H}[\sup_{\gamma\in\Gamma} \lambda_B([x,\alpha]\circ[m,\gamma])] = \\ &= (\lambda_B^+)^{+'}([x,\alpha]).\end{aligned}$$

Thus, we have $B \supseteq (B^+)^{+'}$. Hence, we obtain $B = (B^+)^{+'}$. Thus, we see that the correspodence $A \longmapsto A^{+'}$ is a bijection. Now, let $C = (\mu_C,\lambda_C), D = (\mu_D,\lambda_D) \in IFH(H)(IFRH(H))$ be such that $C \subseteq D$, i.e., $\mu_C \subseteq \mu_D$ and $\lambda_C \supseteq \lambda_D$. Then, for all $[x,\alpha] \in S$,

$\mu_C^{+'}([x,\alpha]) = \inf_{m\in H, t\in x\alpha m} \mu_C(t) \leq \inf_{m\in H, t\in x\alpha m} \mu_D(t) = \mu_D^{+'}([x,\alpha])$,

and

$\lambda_C^{+'}([x,\alpha]) = \sup_{m\in H, t\in x\alpha m} \lambda_C(t) \leq \sup_{m\in H, t\in x\alpha m} \lambda_D(t) = \lambda_D^{+'}([x,\alpha])$.

Consequently, $\mu_C^{+'} \subseteq \mu_D^{+'}$ and $\lambda_C^{+'} \supseteq \lambda_D^{+'}$. Thus, we have $C^{+'} \subseteq D^{+'}$. Hence, we conclude that $A \longmapsto A^{+'}$ is an inclusion preserving bijection. ■

Lemma 195 *Let $I \in RH(S)(H(S))$ of a Γ-semihypergroup H and $P = (\chi_I, \chi_I^c)$ where χ_I is the characteristic function of I. Then $P^+ = (\chi_I,\chi_I^c)^+ = ((\chi_I)^+, (\chi_I^c)^+) = (\chi_{I^+}, \chi_{I^+}^c)$.*

Proof. Let us suppose that $H \in I^+$. Then, $[m,\beta] \in I, \forall\beta \in \Gamma$. This means $\inf_{\beta\in\Gamma}(\chi_I([m,\beta])) = 1$ and $\sup_{\beta\in\Gamma}(\chi_I^c([m,\beta])) = 0$. Also, $\chi_{I^+}(m) = 1$ and $\chi_{I^+}^c(m) = 0$.

Now, suppose that $m \notin I^+$. Then, there exists $\delta \in \Gamma$ such that $[m, \delta] \notin I$. Hence, $\chi_I([m, \delta]) = 0, \chi_I^c([m, \delta]) = 1$ and so $\inf_{\beta \in \Gamma}(\chi_I([m, \beta])) = 0, \sup_{\beta \in \Gamma}(\chi_I^c([m, \beta])) = 1$. Hence, $(\chi_I)^+(m) = 0$ and $(\chi_I^c)^+(m) = 1$. Again $(\chi_{I^+})(m) = 0$ and $(\chi_{I^+}^c)(m) = 1$. Thus, $P^+ = (\chi_I, \chi_I^c)^+ = ((\chi_I)^+, (\chi_I^c)^+) = (\chi_{I^+}, \chi_{I^+}^c)$. ■

In a similar way we obtain the following lemma.

Lemma 196 *Let $I \in LH(H)(H(H))$, $P = (\chi_I, \chi_I^c)$ and S be the left operator semihypergroup of a Γ-semihypergroup H. Then $P^{+'} = (\chi_I, \chi_I^c)^{+'} = ((\chi_I)^{+'}, (\chi_I^c)^{+'}) = (\chi_{I^{+'}}, \chi_{I^{+'}}^c)$, where χ_I is the characteristic function of I.*

Using the Lemma 195, Lemma 196, Theorem 194 and Proposition 193 we have the following result on Γ-semihypergroup [33].

Theorem 197 *Let H be a Γ-semihypergroup with unities. Then, there exists an inclusion preserving bijection between the set of all hyperideals (right hyperideals) of H and that of its left operator semihypergroup S via the mapping $I \to I^{+'}$.*

Proof. Let us denote the mapping $I \to I^{+'}$ by ϕ (cf. Proposition 193). Now let $\phi(I_1) = \phi(I_2)$. Then, $I_1^{+'} = I_2^{+'}$. This implies that $(\chi_{I_1^{+'}}, \chi_{I_1^{+'}}^c) = (\chi_{I_2^{+'}}, \chi_{I_2^{+'}}^c)$ (where χ_I is the characteristic function of I). Hence by Lemma 196, $(\chi_{I_1}, \chi_{I_1}^c)^{+'} = (\chi_{I_2}, \chi_{I_2}^c)^{+'}$. From this, by using Theorem 194, we obtain $(\chi_{I_1}, \chi_{I_1}^c) = (\chi_{I_2}, \chi_{I_2}^c)$, whence $I_1 = I_2$. Consequently, ϕ is one-one. Let I be a hyperideal (right hyperideal) of S. Then, $(\chi_{I_1}, \chi_{I_1}^c)$ is an $IFH(S)(IFRH(S))$. Hence by Theorem 194, $((\chi_I, \chi_I^c)^+)^{+'} = (\chi_I, \chi_I^c)$. Hence by noting that I^+ and $(I^+)^{+'}$ are hyperideals (right hyperideals) of H and S, respectively, (cf. (i) and (ii) of §2) we use Lemma 195 and Lemma 196 and deduce that $(\chi_{(I^+)^{+'}}, \chi_{(I^+)^{+'}}^c) = (\chi_I, \chi_I^c)$. This implies that $(I^+)^{+'}) = I$ whence $\phi(I^+) = I$. Hence, in view of (i) of §2 (cf. Proposition 193), ϕ is onto. Let I_1, I_2 be two hyperideals (right hyperideals) of H with $I_1 \subseteq I_2$. Then, $\chi_{I_1} \subseteq \chi_{I_2}$ and $\chi_{I_1}^c \supseteq \chi_{I_2}^c$. Hence, by using Theorem 194, we obtain $(\chi_{I_1})^{+'} \subseteq (\chi_{I_2})^{+'}$ and $(\chi_{I_1})^{+'} \supseteq (\chi_{I_2})^{+'}$, i.e. $\chi_{I_1^{+'}} \subseteq \chi_{I_2^{+'}}$ and $\chi_{I_1^{+'}}^c \supseteq \chi_{I_2^{+'}}^c$. This implies that, by Lemma 196, this implies that $\chi_{I_1^{+'}} \subseteq \chi_{I_2^{+'}}$ and $\chi_{I_1^{+'}}^c \supseteq \chi_{I_2^{+'}}^c$ and so we obtain $I_1^{+'} \subseteq I_2^{+'}$. Hence, we conclude the theorem. ■

Proposition 198 *Let H be a Γ-semihypergroup not necessary with unity and S be its left operator semihypergroup.*

If $A = \langle \mu_A, \lambda_A \rangle \in IFPH(S)(IFSPH(S))$, then $A^+ = (\mu_A^+, \lambda_A^+) \in IFPH(H)$(resp. $IFSPH(H)$).

Proof. Let $A = \langle \mu_A, \lambda_A \rangle \in IFPH(S)$. Then, $A = \langle \mu_A, \lambda_A \rangle \in IFH(S)$. By Proposition 192, $A^+ = (\mu_A^+, \lambda_A^+) \in IFPH(H)$. Since $A = \langle \mu_A, \lambda_A \rangle \in IFPH(S)$, so we have $U(\mu_A; t)$ and $L(\mu_A; t)$ are $PH(S)$. By Remark 2.12, for all $t \in [0, 1], [U(\mu_A; t)]^+$ and $[L(\lambda_A; t)]^+$ are $PH(H)$. By Remark 2.12, $[U(\mu_A; t)]^+ = U((\mu_A)^+; t)$ and $[L(\lambda_A; t)]^+ = L((\lambda_A)^+; t)$. So $U((\mu_A)^+; t)$ and $L((\lambda_A)^+; t)$ are $PH(H)$. Hence $A^+ = (\mu_A^+, \lambda_A^+) \in IFPH(H)$. Similarly we can prove the other case also. ■

In the similar way it can be shown the following proposition.

Proposition 199 *Let H be a Γ-semihypergroup not necessary with unities and S be its left operator semihypergroup.*

If $A = \langle \mu_A, \lambda_A \rangle \in IFPH(H)(IFSPH(H))$, then $A^{+'} = (\mu_A^{+'}, \lambda_A^{+'}) \in IFPH(S)$(resp. $IFSPH(S)$).

The following theorem is on the inclusion preserving bijection between the sets of intuitionistic fuzzy prime hyperideals of H and its left operator semihypergroup S.

Theorem 200 *Let H be a Γ-semihypergroup (not necessary with unity) and S be its left operator semihypergroup. Then, there exists an inclusion preserving bijection $A = \langle \mu_A, \lambda_A \rangle \to A^{+'} = (\mu_A^{+'}, \lambda_A^{+'})$ between the set of all intuitionistic fuzzy prime (resp. semiprime) hyperideals of H and set of all intuitionistic fuzzy prime (resp. semiprime) hyperideals of S.*

Proof. Let $x \in H$ and $A = \langle \mu_A, \lambda_A \rangle \in IFPH(S)$. Then, $(\mu_A^{+'})^+(x) = \inf\limits_{\alpha \in \Gamma} \mu_A^{+'}([x, \alpha]) = \inf\limits_{\alpha \in \Gamma} \inf\limits_{m \in H, h \in x\alpha m} \mu_A(h)$. Hence, as $\inf\limits_{\alpha \in \Gamma, m \in H, h \in x\alpha m} \mu_A(h) \geq \mu_A(x), \forall H \in H, \alpha \in \Gamma$ (A being a intuitionistic fuzzy hyperideal), we deduce that $\inf\limits_{\alpha \in \Gamma} \inf\limits_{m \in H, h \in x\alpha m} \mu_A(h) \geq \mu_A(x)$ and hence $(\mu_A^{+'})^+(x) \geq \mu_A(x)$. Consequently, $\mu_A \subseteq \mu_A^{+'}$. In similar way, we obtain $\lambda_A \subseteq \lambda_A^{+'}$.

Again for $x \in H, (\mu^{+'})^{+}(x) = \inf_{m\in H} \inf_{\alpha\in\Gamma, h\in x\alpha m} \mu_A(h) = \inf_{m\in H}(\max\{\mu_A(m), \mu_A(x)\})$ (since A is an intuitionistic fuzzy prime hyperideal)$\leq \max\{\mu_A(x), \mu_A(x)\} = \mu_A(x)$. Hence, $\mu_A \supseteq \mu_A^{+'}$. Thus, we obtain $\mu_A = \mu_A^{+'}$. In similar way, we obtain $\lambda_A \supseteq \lambda_A^{+'}$. Consequently, $(A^{+'})^{+} = A$. Hence, the mapping is one-one. Now, if A is an intuitionistic fuzzy prime hyperideal of S, then for $[x,\alpha] \in S, (\mu_A^{+})^{+'}([x,\alpha]) = \inf_{m\in H, h\in x\alpha m} \mu_A^{+}(h) = \inf_{m\in H} \inf_{\beta\in\Gamma} \mu_A([x\alpha H, \beta]) = \inf_{m\in H, z\in x\alpha m} \inf_{\beta\in\Gamma} \mu_A([z,\beta]) \geq \mu_A([x,\alpha])$ (A being an intuitionistic fuzzy hyperideal). This implies that $\mu_A \subseteq (\mu_A^{+})^{+'}$. Let $[x,\alpha] \in S$. Then, since μ_A is an intuitionistic fuzzy prime hyperideal, so $\inf_{z\in x\alpha m} \mu_A([z,\beta]) = \max\{\mu_A([x,\alpha]), \mu_A([m,\beta])\}, \forall [m,\beta] \in S$. Hence for $H = x$ and $\beta = \alpha$, $\inf_{z\in x\alpha m} \mu_A([z,\beta]) = \mu_A([x,\alpha])$. Hence $\inf_{m\in H, z\in x\alpha m} \inf_{\beta\in\Gamma} \mu_A([z,\beta]) \leq \mu_A([x,\alpha])$. Again
$(\mu_A^{+})^{+'}([x,\alpha]) = \inf_{m\in H, z\in x\alpha m} \inf_{\beta\in\Gamma} \mu_A([z,\beta])$. Hence $(\mu_A^{+})^{+'}([x,\alpha]) \leq$
$\mu_A([x,\alpha]), \forall [x,\alpha] \in S$. This implies that $\mu_A \supseteq (\mu_A^{+})^{+'}$. Consequently, we obtain $\mu_A = (\mu_A^{+})^{+'}$. In similar way, we obtain $\lambda_A = (\lambda_A^{+})^{+'}$. Consequently, $(A^{+})^{+'} = A$. Thus we deduce that the mapping is onto. Inclusion preserving property is similar as in Theorem 194. Hence, $A \longmapsto A^{+'}$ is an inclusion preserving bijection.

Similar results hold for $IFSPH(H)$. ■

As another illustration of the effectiveness of relationship between a Γ-semihypergroup and its left operator semihypergroup in terms of intuitionistic fuzzy subsets we obtain the following theorem on prime hyperideals [33, 35]. The proof of this theorem is omitted because it follows by using Theorem 200 and similar argument as applied in Theorem 197.

Theorem 201 *Let H be a Γ-semihypergroup and S be its left operator semihypergroup. Then, there exists an inclusion preserving bijection between the set of all prime hyperideals of H and set of all prime hyperideals of S via the mapping $I \rightarrow I^{+'}$.*

Definition 202 *Let H be a Γ-semihypergroup and $A = \langle \mu_A, \lambda_A \rangle$ be an intuitionistic fuzzy subset of H and $x \in H$. Then, the intuitionisic fuzzy subset $\langle x, A \rangle = \{(y, \langle x, \mu_A \rangle (y), \langle x, \lambda_A \rangle (y)) : x \in H\}$, where the functions $\langle x, \mu_A \rangle : H \rightarrow [0,1]$ and $\langle x, \lambda_A \rangle : H \rightarrow [0,1]$ defined by*

$\langle x, \mu_A \rangle (y) = \inf\limits_{\gamma \in \Gamma, h \in x\gamma y} \mu_A(h), \forall y \in H$ *and* $\langle x, \lambda_A \rangle (y) = \sup\limits_{\gamma \in \Gamma, h \in x\gamma y} \lambda_A(h), \forall y \in H$, *is called the intuitionistic fuzzy extension of* A *by* x.

By a routine verification we obtain the following propositions.

Proposition 203 *Let* H *be a commutative* Γ*-semihypergroup and* S *be its left semihypergroup. If* $A = \langle \mu_A, \lambda_A \rangle$ *is an* $IFLH(H)(IFH(H))$, *then* $\left\langle x, A^{+'} \right\rangle$ *is an* $IFLH(S)(IFH(S))$ *for all* $x \in S$.

Proposition 204 *Let* H *be a commutative* Γ*-semihypergroup and* S *be its left semihypergroup. If* $A = \langle \mu_A, \lambda_A \rangle$ *is an* $IFLH(S)(IFH(S))$, *then* $\langle x, A^{+} \rangle$ *is an* $IFLH(H)(IFH(H))$ *for all* $x \in H$.

The mappings $()^{+}, ()^{+'}$, do not commute with the operation of taking intuitionistic fuzzy extension. But, how they actually behave in this respect, are shown in the following lemmas which characterize also the relationship between a Γ-semihypergroup and its left operator semihypergroup in terms of intuitionistic fuzzy extension.

Lemma 205 *Let* $A = \langle \mu_A, \lambda_A \rangle$ *be an intuitionistic fuzzy subset of a commutative* Γ*-semihypergroup* H. *Then for all* $x \in H$

1. $\langle x, A \rangle^{+'} \subseteq \left\langle [x, \alpha], A^{+'} \right\rangle, \forall \alpha \in \Gamma$
2. $\langle x, A \rangle^{+'} = (\langle x, \mu_A \rangle^{+'}, \langle x, \lambda_A \rangle^{+'}) = (\inf\limits_{\alpha \in \Gamma} \left\langle [x, \alpha], \mu_A^{+'} \right\rangle, \sup\limits_{\alpha \in \Gamma} \left\langle [x, \alpha], \lambda_A^{+'} \right\rangle)$.

Proof. (1). Let $[y, \beta] \in S$. Then $\langle x, \mu_A \rangle^{+'} ([y, \beta]) = \inf\limits_{m \in H, h \in y\beta m} \langle x, \mu_A \rangle (h) = \inf\limits_{m \in H} \inf\limits_{\gamma \in \Gamma, h \in y\beta m\gamma x} \mu(h) = \inf\limits_{\gamma \in \Gamma} \inf\limits_{m \in H, h \in y\beta m\gamma x} \mu(h)$. Again $\left\langle [x, \alpha], \mu_A^{+'} \right\rangle ([y, \beta]) = \inf\limits_{z \in x\alpha y} \mu_A^{+'}([z, \beta]) = \mu_A^{+'}([x\alpha y, \beta]) = \inf\limits_{h \in x\alpha y\beta m, m \in H} \mu_A(h) = \inf\limits_{m \in H, h \in x\alpha m\beta y} \mu_A(h)$. Since, $\inf\limits_{\gamma \in \Gamma} \inf\limits_{m \in H, h \in x\gamma m\beta y} \mu_A(h) \leq \inf\limits_{m \in H, h \in x\alpha m\beta y} \mu_A(h)$, we deduce that
$\langle x, \mu_A \rangle^{+'} ([y, \beta]) \leq \left\langle [x, \alpha], \mu_A^{+'} \right\rangle ([y, \beta])$. Consequently, $\langle x, \mu_A \rangle^{+'} \subseteq \left\langle [x, \alpha], \mu_A^{+'} \right\rangle$, $\forall \alpha \in \Gamma$. In similar way we obtain $\langle x, \lambda_A \rangle^{+'} ([y, \beta]) \geq \left\langle [x, \alpha], \lambda_A^{+'} \right\rangle ([y, \beta])$. Consequently, $\langle x, A \rangle^{+'} \subseteq \left\langle [x, \alpha], A^{+'} \right\rangle, \forall \alpha \in \Gamma$.

(2). Let $[y,\beta]\in S$. Then

$$\begin{aligned}
\inf_{\alpha\in\Gamma}\left\langle [x,\alpha],\mu_A^{+'}\right\rangle([y,\beta]) &= \inf_{\alpha\in\Gamma, z\in x\alpha y}\mu_A^{+'}([z,\beta] \\
&= \inf_{\alpha\in\Gamma}\mu_A^{+'}([x\alpha y,\beta]) = \inf_{\alpha\in\Gamma}\inf_{m\in H, h\in x\alpha y\beta m}\mu_A(h) \\
&= \inf_{m\in H}\inf_{\alpha\in\Gamma, h\in x\alpha m\beta y}\mu_A(h) = \inf_{m\in H, h\in y\beta m}\langle x,\mu_A\rangle(h) = \langle x,\mu_A\rangle^{+'}([y,\beta]).
\end{aligned}$$

In similar way we obtain
$\sup_{\alpha\in\Gamma}\left\langle [x,\alpha],\lambda_A^{+'}\right\rangle([y,\beta]) = \langle x,\lambda_A\rangle^{+'}([y,\beta])$. Consequently,
$\langle x,A\rangle^{+'} = (\langle x,\mu_A\rangle^{+'},\langle x,\lambda_A\rangle^{+'}) = \left(\inf_{\alpha\in\Gamma}\left\langle [x,\alpha],\mu_A^{+'}\right\rangle, \sup_{\alpha\in\Gamma}\left\langle [x,\alpha],\lambda_A^{+'}\right\rangle\right)$. ■

Lemma 206 *Let $A=\langle\mu_A,\lambda_A\rangle$ be an intuitionistic fuzzy subset of the left operator semihypergroup S of a Γ-semihypergroup H. Then for all $x\in H$, $\langle [x,\beta],A\rangle^{+}\supseteq\langle x,A^{+}\rangle, \forall\beta\in\Gamma$.*

Proof. Let $p\in H$. Then $\langle [x,\beta],\mu_A\rangle^{+}(p) = \inf_{\gamma\in\Gamma}\langle [x,\beta],\mu_A\rangle([p,\gamma])=$
$\inf_{\gamma\in\Gamma, z\in p\gamma x}\mu_A[z,\beta]) = \inf_{\gamma\in\Gamma}\mu_A([p\gamma x,\beta])$. Again $\left\langle x,\mu_A^{+}\right\rangle(p) = \inf_{\gamma\in\Gamma}\mu_A^{+}(p\gamma x) = \inf_{\gamma\in\Gamma}\inf_{\alpha\in\Gamma}\mu_A([p\gamma x,\alpha]) =$
$\inf_{\alpha\in\Gamma}\inf_{\gamma\in\Gamma}\mu_A([p\gamma x,\alpha])$. Now since $\inf_{\gamma\in\Gamma}\mu_A([p\gamma x,\beta]) \geq \inf_{\alpha\in\Gamma}\inf_{\gamma\in\Gamma}\mu_A([p\gamma x,\alpha])$, we deduce that $\left\langle [x,\beta],\mu_A^{+}\right\rangle(p) \geq \left\langle x,\mu_A^{+}\right\rangle(p)$. In similar way we obtain $\left\langle [x,\beta],\lambda_A^{+}\right\rangle(p) \leq \left\langle x,\lambda_A^{+}\right\rangle(p)$. Consequently, $\langle [x,\beta],A\rangle^{+}\supseteq\langle x,A^{+}\rangle, \forall\beta\in\Gamma$. ■

Lemma 207 *Let I be a hyperideal of a Γ-semihypergroup H. Then $((\mu_I)^{+'},(\lambda_I)^{+'}) = (\mu_{I^{+'}},\lambda_{I^{+'}})$.*

Proof. Let $[y,\beta]\in S$. Then $(\mu_I)^{+'}([y,\beta]) = \inf_{m\in H, h\in y\beta m}\mu_I(h)$ and
$(\lambda_I)^{+'}([y,\beta]) = \sup_{m\in m, h\in y\beta m}\lambda_I(h)$. Now let us suppose $[y,\beta]\in I^{+'}$. Then $y\beta H\subseteq I, \forall m\in H$. Hence $\mu_I(h)=1, \forall m\in H, h\in y\beta H$ and $\lambda_I(h)=0, \forall m\in H, h\in y\beta m$ whence $\inf_{m\in H, h\in y\beta m}\mu_I(h)=1$ and $\sup_{m\in H, h\in y\beta m}\lambda_I(h)=1$. Thus we deduce that $(\mu_I)^{+'}([y,\beta])=1$ and $(\lambda_I)^{+'}([y,\beta])=0$. Also, $\mu_{I^{+'}}([y,\beta])=1$ and $\lambda_{I^{+'}}([y,\beta])=0$. On the other hand, if $[y,\beta]\notin I^{+'}$, then for some $t\in H, y\beta t\nsubseteq I$. Hence $\mu_I(h)=0, h\in y\beta t$ and $\lambda_I(h)=1, h\in y\beta t$, whence $\inf_{m\in H, h\in y\beta t}\mu_I(h)=0$ and $\sup_{m\in H, h\in y\beta t}\lambda_I(h)=1$. Thus

we see that $(\mu_I)^{+'}([y,\beta]) = 0$ and $(\lambda_I)^{+'}([y,\beta]) = 1$. Again $\mu_{I^{+'}}([y,\beta]) = 0$ and $\lambda_{I^{+'}}([y,\beta]) = 1$. Thus we deduce that $(\mu_I)^{+'}([y,\beta]) = \mu_{I^{+'}}([y,\beta]), \forall [y,\beta] \in S$ and $(\lambda_I)^{+'}([y,\beta]) = \lambda_{I^{+'}}([y,\beta]), \forall [y,\beta] \in S$. Hence we conclude that $(\mu_I)^{+'} = \mu_{I^{+'}}$ and $(\lambda_I)^{+'} = \lambda_{I^{+'}}$. Consequently,
$((\mu_I)^{+'}, (\lambda_I)^{+'}) = (\mu_{I^{+'}}, \lambda_{I^{+'}})$. ■

The following lemmas are on the commutativity of the mapping $()^{+'}$ with intersection and infimum.

Lemma 208 *[33] Let $\{A_i\}_{i\in I}$ be a family of hyperideals of a Γ-semihypergroup H. Then* $\left(\bigcap\limits_{i\in I} A_i\right)^{+'} = \bigcap\limits_{i\in I} A_i^{+'}$.

Lemma 209 *Let H be a Γ-semihypergroup, S its left operator semihypergroup and $\{A_i\}_{i\in I} = (\mu_{A_i}, \lambda_{A_i})_{i\in I}$ be a family of intuitionistic fuzzy subsets of H such that $A = \langle \mu_A, \lambda_A\rangle = \inf\limits_{i\in I} A_i = (\inf\limits_{i\in I} \mu_{A_i}, \sup\limits_{i\in I} \lambda_{A_i})$. Then $A^{+'} = ((\mu_A)^{+'}, (\lambda_A)^{+'}) = \inf\limits_{i\in I}(A_i)^{+'} = \left(\inf\limits_{i\in I}(\mu_{A_i})^{+'}, \sup\limits_{i\in I}(\lambda_{A_i})^{+'}\right)$.*

Proof. Let $[x,\alpha] \in S$. Then $(\mu_A)^{+'}([x,\alpha]) = (\inf\{\mu_{A_i})^{+'}([x,\alpha])=$
$\inf\limits_{i\in I, m\in H, h\in x\alpha H} \mu_{A_i}(h) = \inf\limits_{i\in I, m\in H, h\in x\alpha H} \mu_{A_i}(h) = \inf\limits_{i\in I}(\mu_{A_i})^{+'}([x,\alpha])$. In similar way we obtain $(\lambda_A)^{+'}([x,\alpha]) = \sup\limits_{i\in I}(\lambda_{A_i})^{+'}([x,\alpha])$. Hence we conclude that $A^{+'} = \inf\limits_{i\in I}(A_i)^{+'}$. ■

The following results are on the intuitionistic fuzzy subset extension which are obtained using mainly the operator semihypergroups technique.

Proposition 210 *Let H be a commutative Γ-semihypergroup with unities and $A = \langle \mu_A, \lambda_A\rangle$ be an intuitionistic fuzzy hyperideal of H. Then $\langle x, A\rangle = (\langle x, \mu_A\rangle, \langle x, \lambda_A\rangle)$ is an intuitionistic fuzzy hyperideal of H, for all $x \in H$.*

Proof. Let S be the left semihypergroup of H. Since H is commutative, S is commutative. Now by Proposition 193, $A^{+'} = (\mu_A^{+'}, \lambda_A^{+'})$ is an intuitionistic fuzzy hyperideal of S. Let $x \in H$. Then for any $\alpha \in \Gamma, \langle [x,\alpha], A^{+'}\rangle = \left(\left\langle [x,\alpha], \mu_A^{+'}\right\rangle, \left\langle [x,\alpha], \lambda_A^{+'}\right\rangle\right)$ is an intuitionistic fuzzy hyperideal of S and hence $\left(\inf\limits_{\alpha\in\Gamma}\left\langle [x,\alpha], \mu_A^{+'}\right\rangle, \sup\limits_{\alpha\in\Gamma}\left\langle [x,\alpha], \lambda_A^{+'}\right\rangle\right)$ is an intuitionistic fuzzy hyperdieal of S. Hence, by using Lemma 205(2), we see that

$\left(\langle x,\mu_A\rangle^{+'},\langle x,\lambda_A\rangle^{+'}\right)=\langle x,A\rangle^{+'}$ is an intuitionistic fuzzy hyperideal of S. Consequently, by Proposition 190, $\left(\langle x,A\rangle^{+'}\right)^{+}$ is an intuitionistic fuzzy hyperideal of H. Hence $\langle x,A\rangle$ is an intuitionistic fuzzy hyperideal of H (cf. Theorem 194). ■

Applying Proposition 199 and Theorem 200 we obtain the following proposition.

Proposition 211 *Let H be a commutative Γ-semihypergroup and $A=\langle\mu_A,\lambda_A\rangle$ be an intuitionistic fuzzy semiprime hyperideal of H. Then $\langle x,A\rangle=(\langle x,\mu_A\rangle,\langle x,\lambda_A\rangle)$ is an intuitionistic fuzzy semiprime hyperideal of H for all $x\in H$.*

Proposition 212 *Let H be a commutative Γ-semihypergroup, $\{A_i\}_{i\in I}=(\mu_{A_i},\lambda_{A_i})_{i\in I}$ be a non-empty family of intuitionistic fuzzy semiprime hyperideals of H and let $A=\langle\mu_A,\lambda_A\rangle=\inf\limits_{i\in I}A_i=(\inf\limits_{i\in I}\mu_{A_i},\sup\limits_{i\in I}\lambda_{A_i})$. Then for any $x\in H,\langle x,A\rangle$ is an intuitionistic fuzzy semiprime hyperideal of H.*

Proof. Let S be the left operator semihypergroup of H. Since H is commutative, S is commutative. Now, in view of Proposition 199, we have $\{A_i^{+'}\}_{i\in I}=\left(\mu_{A_i}^{+'},\lambda_{A_i}^{+'}\right)_{i\in I}$ is a non-empty family of intuitionistic fuzzy semiprime hyperideals of S. Hence $\inf\limits_{i\in I}\{A_i^{+'}\}$ is an intuitionistic fuzzy semiprime hyperideal of S. Again, by Lemma 209, $A^{+'}$ is an intuitionistif fuzzy semiprime hyperideal of S. This implies that for any $[x,\alpha]\in S,\left\langle[x,\alpha],A^{+'}\right\rangle=\left(\left\langle[x,\alpha],\mu_A^{+'}\right\rangle,\left\langle[x,\alpha],\lambda_A^{+'}\right\rangle\right)$ is an intuitionistic fuzzy semiprime hyperideal of S and hence $\left(\inf\limits_{\alpha\in\Gamma}\left\langle[x,\alpha],\mu_A^{+'}\right\rangle,\sup\limits_{\alpha\in\Gamma}\left\langle[x,\alpha],\lambda_A^{+'}\right\rangle\right)$ is an intuitionistic fuzzy semiprime hyperideal of S. Hence, by using Lemma 205(2), we immediately deduce that $\left(\langle x,\mu_A\rangle^{+'},\langle x,\lambda_A\rangle^{+'}\right)=\langle x,A\rangle^{+'}$ is an intuitionistic fuzzy semiprime hyperideal of S. Consequently, in view of Proposition 198, $(\langle x,A\rangle^{+'})^{+}$ is an intuitionistic fuzzy semiprime hyperideal of H. Hence, in view of Proposition 199 and Theorem 200, we conclude that $\langle x,A\rangle$ is an intuitionistic fuzzy semiprime hyperideal of H. ■

Theorem 213 *Let H be a commutative Γ-semihypergroup, $\{H_i\}_{i\in I}$ is a non-empty family of semiprime hyperideals of H, $A=\bigcap\limits_{i\in I}H_i\neq\emptyset$ and $B=(\mu_A,\mu_A^c)$. Then $\langle x,B\rangle$ is an intuitionistic fuzzy semiprime hyperideal of H for all $x\in H$.*

Proof. Since, $\forall i \in I$, H_i is a semiprime hyperideal of H, $H_i^{+'}$ is a semiprime hyperideal of the left operator semihypergroup S. Now, since $A = \bigcap\limits_{i\in I} H_i$, $A^{+'} = \left(\bigcap\limits_{i\in I} H_i\right)^{+'} = \bigcap\limits_{i\in I} H_i^{+'} \neq \emptyset$ (see Lemma 209). So by a suitable adoption of [47, Corollary 3.11], $\langle [x,\alpha], B\rangle = (\langle [x,\alpha], \mu_A\rangle, \langle [x,\alpha], \mu_A^c\rangle)$ is an intuitionistic fuzzy semiprime hyperideal of S, $\forall \alpha \in \Gamma$. Hence
$\left(\inf\limits_{\alpha\in\Gamma} \langle [x,\alpha], \mu_A\rangle, \sup\limits_{\alpha\in\Gamma} \langle [x,\alpha], \mu_A^c\rangle\right)$ is an intuitionistic fuzzy semiprime hyperideal of S and so $\left(\langle [x,\alpha]\mu_A\rangle^{+'}, \langle [x,\alpha]\mu_A^c\rangle^{+'}\right) = \langle x, B\rangle$ is an intuitionistic fuzzy semiprime hyperideal of S. Hence, $(\langle x, B\rangle^{+'})^{+}$ is an intuitionistic fuzzy semiprime hyperideal of H. Consequently, by Theorem 200, $\langle x, B\rangle$ is an intuitionistic fuzzy semiprime hyperideal of H. ■

Finally, we obtain, by using the concept of intuitionistic fuzzy extension and relationship between a Γ-semihypergroup and its operator semihypergroups in terms of intuitionistic fuzzy subsets, the following characterization of a prime hyperideal of a Γ-semihypergroup.

Theorem 214 *Let H be a Γ-semihypergroup, I be a hyperideal of H and $B = (\chi_I, \chi_I^c)$ where χ_I is the characteristic function of I. Then I is prime hyperideal of H if and only if for $x \in H$, with $x \notin I$, $\langle x, B\rangle = B$.*

Proof. Let I be a prime hyperideal of H and $x \notin I$. Then by Lemma 9, we see that $I^{+'}$ is a prime hyperideal of the left operator semihypergroup S. Also, as $x \notin I, [x,\alpha] \notin I^{+'}$ for some $\alpha \in \Gamma$. Hence $\left\langle [x,\alpha], B^{+'}\right\rangle = B^{+'}$, i.e., $\left(\left\langle [x,\alpha], \chi_I^{+'}\right\rangle, \left\langle [x,\alpha], (\chi_I^c)^{+'}\right\rangle\right) = ((\chi_I)^{+'}, (\chi_I^c)^{+'}) = B^{+'}$. Now by Lemma 195, we note that $\left(\left\langle [x,\alpha], \chi_I^{+'}\right\rangle, \left\langle [x,\alpha], (\chi_I^c)^{+'}\right\rangle\right) = (\chi_I, \chi_I^c)^{+'}$. Hence it immediately follows that
$\left(\left\langle [x,\alpha], B^{+'}\right\rangle\right)^{+} = \left(\left\langle [x,\alpha], \chi_I^{+'}\right\rangle, \langle [x,\alpha], (\chi_I^c)+'\rangle\right)^{+} = ((\chi_I, \chi_I^c)^{+'})^{+} = (\chi_I, \chi_I^c) = B$. By Lemma 205(1), $\langle x, B\rangle^{+'} \subseteq \left\langle [x,\alpha], B^{+'}\right\rangle, \forall \alpha \in \Gamma$. So $\left(\langle x, B\rangle^{+'}\right)^{+} \subseteq B$. Consequently by Theorem 200, $\langle x, B\rangle \subseteq B$. By a suitable adoption of [34, Proposition 2.5], we get $B \subseteq \langle x, B\rangle$. Hence $\langle x, B\rangle = B$.

Conversely, let us suppose $\langle z, B\rangle = B$ for all $z \in H$ with $z \notin I$. Let $x\Gamma y \subseteq I$ with $x, y \in H$. Then $\inf\limits_{h\in x\gamma y} \chi_I(h) = 1$ and $\inf\limits_{h\in x\gamma y} \chi_I^c(h) = 0$, $\forall \gamma \in \Gamma$. Let $x \notin I$. Then by hypothesis $\langle x, B\rangle = B$. This implies that $\langle x, \chi_I\rangle(y) = \chi_i(y)$ and $\langle x, \chi_I^c\rangle(y) = \chi_i^c(y)$. Then $\inf\limits_{\gamma\in\Gamma, h\in x\gamma y} \chi_I(h) = \chi_I(y)$ and $\sup\limits_{\gamma\in\Gamma, h\in x\gamma y} \chi_I^c(y)$ which implies that $\chi_I(y) = 1$ and $\chi_I^c(y) = 0$ whence $y \in I$. Consequently, I is a prime hyperideal of H. ■

Chapter 9

Interval Valued Intuitionistic Fuzzy Sets in Γ-semihypergroups

9.1 Introduction

In this chapter, we apply the concept of an interval valued intuionsitic fuzzy set to theory of Γ-hyperideals, interval valued intuitionistic fuzzy (1,2) Γ-hyperideal of Γ-semihypergroup and obtain some basic results. We give some further properties of interval valued intuitionistic fuzzy Γ-hyperideals and interval valued intuitionistic fuzzy bi-Γ-hyperideals in a Γ-semihypergroup. We define an interval valued intuitionistic fuzzy prime(semiprime) Γ-hyperideals, intuitionistic fuzzy M-hypersystem and N-hypersystem of a Γ-semihypergroup and intuitionistic fuzzy semisimple Γ-semihypergroups and some properties of them are investigated

9.2 Interval Valued Intuitionistic Fuzzy Γ-hyperideals

In this section we define an interval valued intuitionistic fuzzy left (right, two sided, bi) Γ-hyperideal and interval valued intuitionistic fuzzy (1,2) Γ-hyperideal of Γ-semihypergroup and obtained some basic results.

Definition 215 *Let H be a Γ-semihypergroup. An interval valued intuitionistic fuzzy set $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ in H is called an interval valued intuitionistic fuzzy sub-Γ-semihypergroup in H if for all $x, y \in H, \gamma \in \Gamma$,*

$$\inf_{z \in x\gamma y} \{\widetilde{\mu}_A(z)\} \geq \min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\} \text{ and } \sup_{z \in x\gamma y} \{\widetilde{\gamma}_A(z)\} \leq \max\{\widetilde{\gamma}_A(x), \widetilde{\gamma}_A(y)\}.$$

Definition 216 *Let H be a Γ-semihypergroup. An interval valued intuitionistic fuzzy set $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ in H is called an interval valued intuitionistic fuzzy left (resp. right) Γ-hyperideal of H if for all $x, y \in H, \gamma \in \Gamma$,*

1. $\widetilde{\mu}_A(y) \leq \inf_{z \in x\gamma y} \{\widetilde{\mu}_A(z)\}$ *(resp.* $\widetilde{\mu}_A(x) \leq \inf_{z \in x\gamma y} \{\widetilde{\mu}_A(z)\}$*).*
2. $\sup_{z \in x\gamma y} \{\widetilde{\gamma}_A(y)\} \leq \widetilde{\gamma}_A(y)$ *(resp.* $\sup_{z \in x\gamma y} \{\widetilde{\gamma}_A(z)\} \leq \widetilde{\gamma}_A(x)$*).*

An interval valued intuitionistic fuzzy set $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ in H is called an interval valued intuitionistic fuzzy Γ-hyperideal of H if it is both an interval valued intuitionistic fuzzy left and an interval valued intuitionistic right Γ-hyperideal of H.

Definition 217 *Let H be a Γ-semihypergroup. An interval valued intuitionistic fuzzy sub-Γ-semihypergroup $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ in H is called an interval valued intuitionistic fuzzy bi-Γ-hyperideal of H if for all $x, y, z \in H, \alpha, \beta \in \Gamma$,*

$$\inf_{t \in x\alpha y\beta z} \{\widetilde{\mu}_A(t)\} \geq \min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(z)\} \text{ and } \sup_{t \in x\alpha y\beta z} \{\widetilde{\gamma}_A(t)\} \leq \max\{\widetilde{\gamma}_A(x), \widetilde{\gamma}_A(z)\}.$$

Example 218 *Let $H = (0, 1)$, $\Gamma = \{\gamma_n \,|n \in \mathbb{N}\}$ and for every $n \in \mathbb{N}$ we define hyperoperation γ_n on S as follows*

$$x\gamma_n y = \left\{ \frac{xy}{2^k} \,|0 \leq k \leq n \right\}, \forall x, y \in H.$$

Then, $x\gamma_n y \subset H$ and for every $m, n \in \mathbb{N}$ and $x, y, z \in H$

$$(x\gamma_n y)\gamma_m z = \left\{ \frac{xyz}{2^k} \,|0 \leq k \leq n + m \right\} = x\gamma_n \left(y\gamma_m z\right).$$

So, H is a Γ-semihypergroup. Now we defined an interval valued IFS $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ on H as:

$$\widetilde{\mu}_A(x) = \begin{cases} [0.6, 0.6] & \text{if } 0 < x < \frac{1}{2^k} \\ [0.4, 0.5] & \text{if } \frac{1}{2^k} \le x < 1 \end{cases} \quad \text{where } k \in N$$

$$\widetilde{\gamma}_A(x) = \begin{cases} [0.2, 0.3] & \text{if } 0 < x < \frac{1}{2^k} \\ [0.4, 0.5] & \text{if } \frac{1}{2^k} \le x < 1 \end{cases} \quad \text{where } k \in N$$

Then, by routine calculation, $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ *is an interval valued intuitionistic fuzzy* Γ*-hyperideal of* H*.*

Example 219 *Let* H *be the* Γ*-semihypergroup in Example 218. We defined an interval valued IFS* $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ *by*

$$\widetilde{\mu}_A(x) = \begin{cases} [0.7, 0.8] & \text{if } 0 < x < \frac{1}{4^{10}} \\ [0.5, 0.7] & \text{if } \frac{1}{4^{10}} \le x < \frac{1}{4^5} \\ [0.2, 0.4] & \text{if } \frac{1}{4^5} \le x < 1 \end{cases} \quad \text{and } \widetilde{\gamma}_A(x) = \begin{cases} [0.1, 0.2] & \text{if } 0 < x < \frac{1}{4^{10}} \\ [0.3, 0.3] & \text{if } \frac{1}{4^{10}} \le x < \frac{1}{4^5} \\ [0.6, 0.5] & \text{if } \frac{1}{4^5} \le x < 1 \end{cases}$$

Then, clearly $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ *is an interval valued intuitionistic fuzzy bi-*Γ*-hyperideal of* H*.*

Example 220 *Let* $H = \{1, 2, 3, 4, 5\}$ *and* $\Gamma = \{\gamma, \delta\}$ *be two non-empty sets. Then,* H *is a* Γ*-semihypergroup defined by the following Cayley tables.*

γ	1	2	3	4	5
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
3	$\{1\}$	$\{1\}$	$\{3\}$	$\{3\}$	$\{3\}$
4	$\{1\}$	$\{1\}$	$\{3,4\}$	$\{3,4\}$	$\{5\}$
5	$\{1\}$	$\{1\}$	$\{3,4\}$	$\{3,4\}$	$\{5\}$

δ	1	2	3	4	5
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
3	$\{1\}$	$\{1\}$	$\{3\}$	$\{3\}$	$\{3\}$
4	$\{1\}$	$\{1\}$	$\{3\}$	$\{3,4\}$	$\{5\}$
5	$\{1\}$	$\{1\}$	$\{3\}$	$\{3,4\}$	$\{5\}$

1) Let $A = \langle \mu_A, \lambda_A \rangle$ *be an IFS in a* Γ*-semihypergroup* H *defined by* $\mu_A(1) = [0.7, 0.75]$, $\mu_A(2) = [0.5, 0.6], \mu_A(3) = \mu_A(4) = \mu_A(5) = [0,3, 0.4]$, *and* $\lambda_A(1) = [0.1, 0.15]$, $\lambda_A(2) = [0.3, 0.4], \lambda_A(3) = \lambda_A(4) = \lambda_A(5) = [0.4, 0.5]$. *Then,* $A = \langle \mu_A, \lambda_A \rangle$ *is an intuitionistic fuzzy* Γ*-hyperideal of* H*.*

2) Let $A = \langle \mu_A, \lambda_A \rangle$ *be an IFS in a* Γ*-semihypergroup* H *defined by* $\mu_A(1) = [0.8, 0.85]$, $\mu_A(2) = [0.6, 0.7]$, $\mu_A(3) = [0.45, 0.55]$, $\mu_A(4) = \mu_A(5) = [0.2, 0.3]$, *and* $\lambda_A(1) =$

$[0.05, 0.1]$, $\lambda_A(2) = [0.15, 0.2]$, $\lambda_A(3) = [0.4, 0.45]$, $\lambda_A(4) = \lambda_A(5) = [0.5, 0.6]$. *Then,* $A = \langle \mu_A, \lambda_A \rangle$ *is an intuitionistic fuzzy* Γ*-hyperideal of* H.

Proposition 221 *If* $\{A_i\}_{i \in I}$ *is a family of interval valued intuitionistic fuzzy left (resp. right)* Γ*-hyperideals of a* Γ*-semihypergroup* H*, then* $\bigcap_{i \in I} A_i$ *is an interval valued intuitionistic fuzzy left (resp. right)* Γ*-hyperideal of* H.

Proof. Let us suppose that $B = \bigcap_{i \in I} A_i$ and $x, y \in H$. Then for all $\gamma \in \Gamma$, we have:

$$\begin{aligned}\widetilde{\mu}_B(y) &= \inf_{i\in I}\{\widetilde{\mu}_{A_i}(y)\} \leq \inf_{i\in I}\{\inf_{z\in x\gamma y}\{\widetilde{\mu}_{A_i}(z)\}\} \\ &= \inf_{z\in x\gamma y}\{\inf_{i\in I}\{\widetilde{\mu}_{A_i}(z)\}\} = \inf_{z\in x\gamma y}\{\widetilde{\mu}_B(z)\}.\end{aligned}$$

$$\begin{aligned}\sup_{z\in x\gamma y}\{\widetilde{\gamma}_B(z)\} &= \sup_{z\in x\gamma y}\{\sup_{i\in I}\{\widetilde{\gamma}_{A_i}(z)\}\} = \sup_{i\in I}\{\sup_{z\in x\gamma y}\{\widetilde{\gamma}_{A_i}(z)\}\} \\ &\leq \sup_{i\in I}\{\widetilde{\gamma}_{A_i}(y)\} = \widetilde{\gamma}_B(y).\end{aligned}$$

Hence, $\bigcap_{i \in I} A_i$ is an interval valued intuitionistic fuzzy left (resp. right) Γ-hyperideal of H. ■

Proposition 222 *Let* $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ *be an interval valued intuitionistic fuzzy bi-*Γ*-hyperideal and* $B = \langle \widetilde{\mu}_B, \widetilde{\lambda}_B \rangle$ *be an interval valued intuitionistic fuzzy sub-*Γ*-semihypergroup of a* Γ*- semihypergroup* H*. Then,* $A \cap B$ *is an interval valued intuitionistic fuzzy bi-*Γ *-hyperideal of* H.

Proof. The proof is straightforward and we omit it. ■

Proposition 223 *Let* H *be a* Γ*-semihypergroup. An interval valued intuitionistic fuzzy set* $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ *is an interval valued intuitionistic fuzzy left (resp. right)* Γ*-hyperideal of* H *if and only if the interval valued fuzzy sets* μ_A *and* λ_A^c *are interval valued fuzzy left (resp. right)* Γ*-hyperideal.*

Proof. Let us assume that $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is an interval valued intuitionistic fuzzy left Γ-hyperideal of H. Clearly, $\widetilde{\mu}_A$ is an interval valued fuzzy left Γ-hyperideal of H. For $x, y \in H, \gamma \in \Gamma$, we have

$$\begin{aligned} \inf_{z \in x\gamma y} \{\widetilde{\gamma}_A^c(z)\} &= \inf_{z \in x\gamma y} \{\widetilde{1} - \widetilde{\gamma}_A(z)\} = \widetilde{1} - \sup_{z \in x\gamma y} \{\widetilde{\gamma}_A(z)\} \\ &\geq \widetilde{1} - \widetilde{\gamma}_A(y) = \widetilde{\gamma}_A^c(y). \end{aligned}$$

Hence $\widetilde{\gamma}_A^c$ is an interval valued fuzzy left Γ-hyperideal of H.

Conversely, let us suppose $\widetilde{\mu}_A$ and $\widetilde{\gamma}_A^c$ are interval valued fuzzy left Γ-hyperideals of H. For every $x, y \in H$, we have $\widetilde{\mu}_A(y) \leq \inf\limits_{z \in x\gamma y} \{\widetilde{\mu}_A(z)\}$ and we get

$$\begin{aligned} \sup_{z \in x\gamma y} \{\widetilde{\gamma}_A(z)\} &= \sup_{z \in x\gamma y} \{\widetilde{1} - \widetilde{\gamma}_A^c(z)\} = \widetilde{1} - \inf_{z \in x\gamma y} \{\widetilde{\gamma}_A^c(z)\} \\ &\leq \widetilde{1} - \widetilde{\gamma}_A^c(y) = \widetilde{\gamma}_A(y). \end{aligned}$$

Hence, $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is an interval valued intuitionistic fuzzy left Γ-hyperideal of H. ■

Corollary 224 *Let H be a Γ-semihypergroup and $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ be an interval valued intuitionistic fuzzy set in H. Then, $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is an interval valued intuitionistic fuzzy left Γ-hyperideal of H if and only if $\Box A$ and $\Diamond A$ are interval valued intuitionistic fuzzy left Γ-hyperideals of H. For interval valued intuitionistic right Γ-hyperideal similar result holds as well.*

For any $\widetilde{t} \in D[0,1]$ and an interval valued intuitionistic fuzzy set $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ of H, the sets

$$U(\widetilde{\mu}_A; \widetilde{t}) = \{x \in H | \widetilde{\mu}_A(x) \geq \widetilde{t}\} \text{ and } L(\widetilde{\gamma}_A; \widetilde{s}) = \{x \in H | \widetilde{\gamma}_A(x) \leq \widetilde{s}\}.$$

are called respectively, an upper and lower level cut of $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$.

Theorem 225 *Let H be a Γ-semihypergroup and $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ an interval valued intuitionistic fuzzy set in H. Then, $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is an interval valued intuitionistic fuzzy* *left (resp. right)* Γ*-hyperideal of* H *if and only if for all* $\widetilde{s},\widetilde{t} \in D[0,1]$*, the sets* $U(\widetilde{\mu}_A;\widetilde{t})$ *and* $L(\widetilde{\gamma}_A;\widetilde{s})$ *are either empty or left (resp. right)* Γ*-hyperideals of* H*.*

Proof. Let us assume that all non-empty level sets $U(\widetilde{\mu}_A;\widetilde{t})$ and $L(\widetilde{\gamma}_A;\widetilde{s})$ are left (resp. right) Γ-hyperideals of H. Let $x, y \in H, \gamma \in \Gamma$. If $\widetilde{t_1} = \widetilde{\mu}_A(y)$ and $\widetilde{s}_1 = \widetilde{\gamma}_A(y)$, then $y \in U(\widetilde{\mu}_A;\widetilde{t_1})$ and $y \in L(\widetilde{\gamma}_A;\widetilde{s}_1)$. So $x\gamma y \subseteq U(\widetilde{\mu}_A;\widetilde{t_1})$ and $x\gamma y \subseteq L(\widetilde{\gamma}_A;\widetilde{s}_1)$. Therefore, for all $z \in x\gamma y$, we have $\widetilde{\mu}_A(z) \geq t_1$ and $\widetilde{\gamma}_A(z) \leq s_1$, and so

$\inf\limits_{z\in x\gamma y}\{\widetilde{\mu}_A(z)\} \geq \widetilde{\mu}_A(y)$ and $\sup\limits_{z\in x\gamma y}\{\widetilde{\gamma}_A(z)\} \leq \widetilde{\gamma}_A(y)$.

Hence, $A = \langle\widetilde{\mu}_A, \widetilde{\lambda}_A\rangle$ is an interval valued intuitionistic fuzzy left Γ-hyperideal of H.

Conversely, let $A = \langle\widetilde{\mu}_A, \widetilde{\lambda}_A\rangle$ be an interval valued intuitionistic fuzzy left Γ-hyperideal of H. Let $x \in H, \gamma \in \Gamma$ and $y \in U(\widetilde{\mu}_A;\widetilde{t})$. We have $\inf\limits_{z\in x\gamma y}\{\widetilde{\mu}_A(z)\} \geq \widetilde{\mu}_A(y) \geq \widetilde{t}$. Therefore, for all $z \in x\gamma y$, we have $z \in U(\widetilde{\mu}_A;\widetilde{t})$, and so $x\gamma y \subseteq U(\widetilde{\mu}_A;\widetilde{t})$.

Now, let $y \in L(\widetilde{\gamma}_A;\widetilde{s})$. We have $\sup\limits_{z\in x\gamma y}\{\widetilde{\gamma}_A(z)\} \leq \widetilde{\gamma}_A(y) \leq \widetilde{s}$. Therefore, for all $z \in x\gamma y$, we have $z \in L(\widetilde{\gamma}_A;\widetilde{s})$, and so $x\gamma y \subseteq L(\widetilde{\gamma}_A;\widetilde{s})$. This completes the proof. ■

Corollary 226 *Let* H *be a* Γ*-semihypergroup and* I *a left (resp. right)* Γ*-hyperideal of* H*. If interval valued fuzzy sets* $\widetilde{\mu}_A$ *and* $\widetilde{\lambda}_A$ *are defined on* H *by*

$$\widetilde{\mu}_A(x) = \begin{cases} \widetilde{a}_0 \text{ if } x \in I \\ \widetilde{a_1} \text{ if } x \in S\backslash I \end{cases} \quad \text{and } \widetilde{\gamma}_A(x) = \begin{cases} b_0 \text{ if } x \in I \\ b_1 \text{ if } x \in S\backslash I \end{cases}$$

where $\widetilde{0} \leq \widetilde{a}_1 < \widetilde{a}_0, \widetilde{0} \leq \widetilde{b}_0 < \widetilde{b}_1$ *and* $\widetilde{a}_i + \widetilde{b}_i \leq \widetilde{1}$ *for* $i = 0, 1$*. Then* $A = \langle\widetilde{\mu}_A, \widetilde{\lambda}_A\rangle$ *is an interval valued intuitionistic fuzzy left(resp. right)* Γ*-hyperideal of* H *and* $U(\widetilde{\mu}_A, \widetilde{a}_0) = I = L(\widetilde{\gamma}_A, \widetilde{b}_0)$*.*

Corollary 227 *Let* H *be a* Γ*-semihypergroup and* χ_I *be the characteristic function of a left (resp. right, two-sided)* Γ*-hyperideal* I *of* H*. Then,* $I = (\widetilde{\chi}_I, \widetilde{\chi}_I^c)$ *is an interval valued intuitionistic fuzzy left (resp. right, two-sided)* Γ*-hyperideal of* H*.*

Theorem 228 *Let* H *be a* Γ*-semihypergroup and* $A = \langle\widetilde{\mu}_A, \widetilde{\lambda}_A\rangle$ *is an interval valued intuitionistic fuzzy left (resp. right)* Γ*-hyperideal of* H*. Then for all* $x \in H$*, we have*

$\widetilde{\mu}_A(x) = \sup\{\widetilde{t} \in D[0,1] | x \in U(\widetilde{\mu}_A; \widetilde{t})\}$

and

$\widetilde{\gamma}_A(x) = \inf\{\widetilde{s} \in D[0,1] | x \in L(\widetilde{\gamma}_A; \widetilde{s}))\}$.

Theorem 229 *Let H be a Γ-semihypergroup. The following statements are equivalent:*

1. *$A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is an interval valued intuitionistic fuzzy left (resp. right) Γ-hyperideal of H.*

2. *$\mathcal{S} * A \subseteq A$ (resp. $A * \mathcal{S} \subseteq A$), where $\mathcal{S} = \langle \widetilde{1}, \widetilde{0} \rangle$ and $\widetilde{1}(x) = \widetilde{1}$ and $\widetilde{0}(x) = \widetilde{0}$ for all $x \in H$.*

Proof. Let $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ be an interval valued intuitionistic fuzzy left Γ-hyperideal of H. Let $a \in H$. Let us suppose that there exist $x, y \in H$ and $\gamma \in \Gamma$ such that $a \in x\gamma y$. Then, since $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is an interval valued intuitionistic fuzzy left Γ-hyperideal of H, we have

$$\begin{aligned}(\widetilde{1} * \widetilde{\mu}_A)(a) &= \sup_{a \in x\gamma y} [\min\{\widetilde{1}(x), \widetilde{\mu}_A(y)\}] \\ &= \sup_{a \in x\gamma y} [\min\{\widetilde{1}, \widetilde{\mu}_A(y)\}] = \sup_{a \in x\gamma y} \widetilde{\mu}_A(y)\end{aligned}$$

and

$$\begin{aligned}(\widetilde{0} * \widetilde{\gamma}_A)(a) &= \inf_{a \in x\gamma y} [\max\{\widetilde{0}(x), \widetilde{\gamma}_A(y)\}] \\ &= \inf_{a \in x\gamma y} [\max\{\widetilde{0}, \widetilde{\gamma}_A(y)\}] = \inf_{a \in x\gamma y} \widetilde{\gamma}_A(y).\end{aligned}$$

in case of $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is an interval valued intuitionistic fuzzy left Γ-hyperideal of H, $\inf_{z \in x\gamma y} \widetilde{\mu}_A(z) \geq \widetilde{\mu}_A(y)$ and $\sup_{z \in x\gamma y} \widetilde{\gamma}_A(z) \leq \widetilde{\gamma}_A(y)$. So, in particular, $\widetilde{\mu}_A(y) \leq \widetilde{\mu}_A(a)$ and $\widetilde{\gamma}_A(y) \geq \widetilde{\gamma}_A(a)$ for all $a \in x\gamma y$. Hence $\sup_{a \in x\gamma y} \widetilde{\mu}_A(y) \leq \widetilde{\mu}_A(a)$ and $\inf_{a \in x\gamma y} \widetilde{\gamma}_A(y) \geq \widetilde{\gamma}_A(a)$. Thus, $\widetilde{\mu}_A(a) \geq (\widetilde{1} * \widetilde{\mu}_A)(a)$ and $\widetilde{\gamma}_A(a) \leq (\widetilde{0} * \widetilde{\gamma}_A)(a)$. If there do not exist $x, y \in H, \gamma \in \Gamma$ such that $a \in x\gamma y$, then $(\widetilde{1} * \widetilde{\mu}_A)(a) = \widetilde{0} \leq \widetilde{\mu}_A(a)$ and $(\widetilde{0} * \widetilde{\gamma}_A)(a) = \widetilde{1} \geq \widetilde{\gamma}_A(a)$. In similar way we can prove the other case also.

Conversely, let $x, y \in H, \gamma \in \Gamma$ and $a \in x\gamma y$. Then, $\inf\limits_{a\in x\gamma y} \widetilde{\mu}_A(a) \geq (\widetilde{1} * \widetilde{\mu}_A)(a)$ and $\sup\limits_{a\in x\gamma y} \widetilde{\gamma}_A(a) \leq (\widetilde{0} * \widetilde{\gamma}_A)(a)$. We have for all $\gamma \in \Gamma$,

$$\begin{aligned}(\widetilde{1} * \widetilde{\mu}_A)(a) &= \sup_{a\in x\gamma y} [\min\{\widetilde{1}(x), \widetilde{\mu}_A(y)\}] \geq \min\{\widetilde{1}(x), \widetilde{\mu}_A(y)\} \\ &= \min\{\widetilde{1}, \widetilde{\mu}_A(y)\} = \widetilde{\mu}_A(y)\end{aligned}$$

and

$$\begin{aligned}(\widetilde{0} * \widetilde{\gamma}_A)(a) &= \inf_{a\in x\gamma y} [\max\{\widetilde{0}(x), \widetilde{\gamma}_A(y)\}] \leq \max\{\widetilde{0}(x), \widetilde{\gamma}_A(y)\} \\ &= \max\{\widetilde{0}, \widetilde{\gamma}_A(y)\} = \widetilde{\gamma}_A(y).\end{aligned}$$

Consequently, $\inf\limits_{a\in x\gamma y} \widetilde{\mu}_A(a) \geq \widetilde{\mu}_A(y)$ and $\sup\limits_{a\in x\gamma y} \widetilde{\gamma}_A(a) \leq \widetilde{\gamma}_A(y)$. Hence, $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is an interval valued intuitionistic fuzzy left Γ-hyperideal of H. In similar way it can be proved in case of intuitionistic fuzzy right Γ-hyperideal of H. ■

From the above theorem it follows the following theorem.

Theorem 230 *Let H be a Γ-semihypergroup. The following statements are equivalent:*

1. *$A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is an interval valued intuitionistic fuzzy Γ-hyperideal of S.*

2. *$\mathcal{S} * A \subseteq A$ and $A * \mathcal{S} \subseteq A$, $\mathcal{S} = \langle \widetilde{1}, \widetilde{0} \rangle$ and $\widetilde{1}(x) = \widetilde{1}$ and $\widetilde{0}(x) = \widetilde{0}$ for all $x \in S$.*

Theorem 231 *Let H be a Γ-semihypergroup. The following statements are equivalent:*

1. *$A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is an interval valued intuitionistic fuzzy bi-Γ-hyperideal of H.*

2. *$A * A \subseteq A$ and $A * \mathcal{S} * A \subseteq A$, where $\mathcal{S} = \langle \widetilde{1}, \widetilde{0} \rangle$ and $\widetilde{1}(x) = \widetilde{1}$, and $\widetilde{0}(x) = \widetilde{0}$ for all $x \in H$.*

Proof. Same as Theorem 229 ■

Theorem 232 *If $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is an interval valued intuitionistic fuzzy left Γ-hyperideal and $B = \langle \widetilde{\mu}_B, \widetilde{\lambda}_B \rangle$ is an interval valued intuitionistic fuzzy right Γ-hyperideal of a Γ-semihypergroup H, then $A * B$ is an interval valued intuitionistic fuzzy Γ-hyperideal of H.*

Proof. Consider

$$\begin{aligned} \mathcal{S} * (A * B) &= (\mathcal{S} * A) * B \\ &\subseteq A * B \quad (\text{ by Theorem 229} \\ \mathcal{S} * (A * B) &\subseteq A * B. \end{aligned}$$

Hence, $A * B$ is an interval valued intuitionistic fuzzy left Γ-hyperideal of H.

Now,

$$\begin{aligned} (A * B) * \mathcal{S} &= A * (B * \mathcal{S}) \\ &\subseteq A * B \quad (\text{ by Theorem 229} \\ (A * B) * \mathcal{S} &\subseteq A * B. \end{aligned}$$

Hence, $A * B$ is an interval valued intuitionistic fuzzy right Γ-hyperideal of H. Thus, $A * B$ is an interval valued intuitionistic fuzzy Γ-hyperideal of H. ■

Proposition 233 *Let $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ be an interval valued intuitionistic fuzzy right Γ-hyperideal of H and $B = \langle \widetilde{\mu}_B, \widetilde{\lambda}_B \rangle$ be an interval valued intuitionistic fuzzy left Γ-hyperideal of H. Then $A * B \subseteq A \cap B$.*

Proof. Let $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ be an interval valued intuitionistic fuzzy right Γ-hyperideal of H and $B = \langle \widetilde{\mu}_B, \widetilde{\lambda}_B \rangle$ be an interval valued intuitionistic fuzzy left Γ-hyperideal of H.

Let $x \in H$ and suppose that there exist $u, v \in H$ and $\gamma \in \Gamma$ such that $x \in u\gamma v$. Then

$$\begin{aligned}
(\widetilde{\mu}_A * \widetilde{\mu}_B)(x) &= \sup_{x \in u\gamma v} \{\min\{\widetilde{\mu}_A(u), \widetilde{\mu}_B(v)\}\} \\
&\leq \sup_{x \in u\gamma v} \left\{ \min\{ \inf_{x \in u\gamma v} \{\widetilde{\mu}_A(x)\}, \inf_{x \in u\gamma v} \{\widetilde{\mu}_B(x)\}\} \right\} \\
&= \min\{\widetilde{\mu}_A(x), \widetilde{\mu}_B(x)\} = (\widetilde{\mu}_A \wedge \widetilde{\mu}_B)(x)
\end{aligned}$$

and

$$\begin{aligned}
(\widetilde{\gamma}_A * \widetilde{\gamma}_B)(x) &= \inf_{x \in u\gamma v} \{\max\{\widetilde{\gamma}_A(u), \widetilde{\gamma}_B(v)\}\} \\
&\geq \inf_{x \in u\gamma v} \left\{ \max\{ \sup_{x \in u\gamma v} \{\widetilde{\gamma}_A(x)\}, \sup_{x \in u\gamma v} \{\widetilde{\gamma}_B(x)\}\} \right\} \\
&= \max\{\widetilde{\gamma}_A(x), \widetilde{\gamma}_B(x)\} = (\widetilde{\gamma}_A \vee \widetilde{\gamma}_B)(x).
\end{aligned}$$

Let us suppose there do not exist $u, v \in H$ such that $x \in u\gamma v$. Then, $(\widetilde{\mu}_A * \widetilde{\mu}_B)(x) = 0 \leq (\widetilde{\mu}_A \wedge \widetilde{\mu}_B)(x)$ and $(\widetilde{\gamma}_A * \widetilde{\gamma}_B)(x) = 1 \geq (\widetilde{\gamma}_A \vee \widetilde{\gamma}_B)(x)$. Hence the proof is completed. ■

Proposition 234 *Let H be a Γ-semihypergroup and $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle, B = \langle \widetilde{\mu}_B, \widetilde{\lambda}_B \rangle$ be interval valued intuitionistic fuzzy Γ-hyperideals of S. Then, $A * B \subseteq A \cap B \subseteq A, B$.*

Proposition 235 *Let H be a regular Γ-semihypergroup and $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ and $B = \langle \widetilde{\mu}_B, \widetilde{\lambda}_B \rangle$ two interval valued intuitionistic fuzzy sets in H. Then $A * B \supseteq A \cap B$.*

Proof. Let $c \in H$. Since, H is regular, then there exists an element $x \in S$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $c \in c\gamma_1 x \gamma_2 c \subseteq c\gamma c$ for some $\gamma \in \Gamma$. Then,

$$\begin{aligned}
(\widetilde{\mu}_A * \widetilde{\mu}_B)(c) &= \sup_{c \in c\gamma c} \{\min\{\widetilde{\mu}_A(c), \widetilde{\mu}_B(c)\}\} \\
&\geq \min\{\widetilde{\mu}_A(c), \widetilde{\mu}_B(c)\} = (\widetilde{\mu}_A \wedge \widetilde{\mu}_B)(c)
\end{aligned}$$

and

$$
\begin{aligned}
(\widetilde{\gamma}_A * \widetilde{\gamma}_B)(c) &= \inf_{c \in c\gamma c} \{\max\{\widetilde{\gamma}_A(c), \widetilde{\gamma}_B(c)\}\} \\
&\leq \max\{\widetilde{\gamma}_A(c), \widetilde{\gamma}_B(c)\} = (\widetilde{\gamma}_A \vee \widetilde{\gamma}_B)(c).
\end{aligned}
$$

Hence, $A * B \supseteq A \cap B$. ■

Theorem 236 *Let H be a Γ-semihypergroup. The following statements are equivalent:*

1. *H is regular*
2. *$A * B = A \cap B$, where $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is an interval valued intuitionistic fuzzy right Γ-hyperideal of H and $B = \langle \widetilde{\mu}_B, \widetilde{\lambda}_B \rangle$ is an interval valued intuitionistic fuzzy left Γ-hyperideal of H.*

Proof. Let H be a regular Γ-semihypergroup. Then, by Proposition 235, $A * B \supseteq A \cap B$. By Proposition 233, $A * B \subseteq A \cap B$. Hence, $A * B = A \cap B$.

Conversely, let H be a Γ-semihypergroup and $A * B = A \cap B$ where $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is an interval valued intuitionistic fuzzy right Γ-hyperideal of H and $B = \langle \mu_B, \lambda_B \rangle$ is an interval valued intuitionistic fuzzy left Γ-hyperideal of H. Let L and R be respectively a left Γ-hyperideal and a right Γ-hyperideal of H and $x \in R \cap L$. Then $x \in R$ and $x \in L$. Hence $(\widetilde{\chi}_L(x), \widetilde{\chi}_L^c(x)) = (\widetilde{\chi}_R(x), \widetilde{\chi}_R^c(x)) = \langle \widetilde{1}, \widetilde{0} \rangle$ (where $\widetilde{\chi}_L(x)$ and $\widetilde{\chi}_R(x)$ are respectively the interval valued characteristic functions of L and R). Thus

$(\widetilde{\chi}_R \cap \widetilde{\chi}_L)(x) = \min\{\widetilde{\chi}_R(x), \widetilde{\chi}_L(x)\} = 1$ and $(\widetilde{\chi}_R^c \cup \widetilde{\chi}_L^c)(x) = \max\{\widetilde{\chi}_R^c(x), \widetilde{\chi}_L^c(x)\} = 0$.

By Corollary 227, $(\widetilde{\chi}_L, \widetilde{\chi}_L^c)$ and $(\widetilde{\chi}_R, \widetilde{\chi}_R^c)$ are respectively, an interval valued intuitionistic fuzzy left Γ-hyperideal and an interval valued intuitionistic fuzzy right Γ-hyperideal of H. Hence, by hypothesis, $\widetilde{\chi}_R * \widetilde{\chi}_L = \widetilde{\chi}_R \cap \widetilde{\chi}_L$ and $\widetilde{\chi}_R^c * \widetilde{\chi}_L^c = \widetilde{\chi}_R^c \cup \widetilde{\chi}_L^c$. Hence

$(\widetilde{\chi}_R * \widetilde{\chi}_L)(x) = 1$

since $\sup\limits_{x \in y\gamma z} [\min\{\widetilde{\chi}_R(y), \widetilde{\chi}_L(z)\} : y, z \in H, \gamma \in \Gamma] = \widetilde{1}$, and

$(\widetilde{\chi}_R^c * \widetilde{\chi}_L^c)(x) = \widetilde{0}$

since $\inf_{x \in y\gamma z}[\max\{\widetilde{\chi}_R^c(y), \widetilde{\chi}_L^c(z)\} : y, z \in H, \gamma \in \Gamma] = \widetilde{0}$.

This implies that there exist some $r, s \in H$ and $\gamma_1 \in \Gamma$ such that $x \in r\gamma_1 s$ and $(\widetilde{\chi}_R(r), \widetilde{\chi}_R^c(r)) = \langle\widetilde{1}, \widetilde{0}\rangle = (\widetilde{\chi}_L(s), \widetilde{\chi}_L^c(s))$. Thus, $r \in R$ and $s \in L$. Hence, $x \in R\Gamma L$. Thus $R \cap L \subseteq R\Gamma L$. Also $R\Gamma L \subseteq R \cap L$. Hence $R\Gamma L = R \cap L$. Consequently, the Γ-semihypergroup H is regular. ■

Lemma 237 *Let H be a Γ-semihypergroup. Then the following conditions are equivalent:*

(1) H is semisimple;

(2) $I_1 \cap I_2 = I_1\Gamma I_2$, for every Γ-hyperideals I_1 and I_2 of H,

(3) $I_1 = I_1 \cap I_1$, for all Γ-hyperideal I_1 of S.

Proof. (1)⇒(2). Let $a \in I_1 \cap I_2$. Then, $a \in I_1$ and $a \in I_2$. Since, H is semisimple so for each $a \in H$ there exist $x, y, z \in H$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that

$$a \in x\alpha a\beta y\gamma a\delta z = (x\alpha a\beta y)\gamma(a\delta z) \subseteq I_1\Gamma I_2.$$

Thus, $I_1 \cap I_2 \subseteq I_1\Gamma I_2$.

Conversely, let $a \in I_1\Gamma I_2 \subseteq A$ (because I_1 is a Γ-hyperideal of H) and $a \in I_1\Gamma I_2 \subseteq I_1$ (because I_2 is a Γ-hyperideal of H). So, we have $a \in I_1 \cap I_2$. Hence, $I_1\Gamma I_2 \subseteq I_1 \cap I_2$. Thus, $I_1 \cap I_2 = I_1\Gamma I_2$.

(2)⇒(3). Let $I_1 = I_2 = I_1\Gamma I_2$. Then, we get $I_1 \cap I_1 = I_1\Gamma I_1$ implies that $I_1 = I_1\Gamma I_1$.

(3)⇒(1). Let I_1 be a Γ-hyperideal of H such that $I_1 = I_1\Gamma I_1$. Let $a \in I_1 = I_1\Gamma I_1$ implies $a \in I_1\Gamma I_1$. Then, there exist $x, y, z \in H$, and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a \in x\alpha a\beta y \subseteq I_1$ and $a \in a\delta z \subseteq I_1$, because I_1 is a Γ-hyperideal of S.

Now, $a \in a\gamma a \subseteq (x\alpha a\beta y)\gamma(a\delta z)$ which means $a \in x\alpha a\beta y\gamma a\delta z$. Hence, H is semisimple. ■

Proposition 238 *Let H be a Γ-semihypergroup. Then, H is semisimple if and only if for each interval valued intuitionistic fuzzy Γ-hyperideal $A = \langle\widetilde{\mu}_A, \widetilde{\lambda}_A\rangle$ and $B = \langle\widetilde{\mu}_B, \widetilde{\lambda}_B\rangle$ of H, we have $A * B = A \cap B$.*

Proof. Let $A = \langle \widetilde{\mu}_A,\ \widetilde{\lambda}_A \rangle$ and $B = \langle \widetilde{\mu}_B,\ \widetilde{\lambda}_B \rangle$ be interval valued intuitionistic fuzzy Γ-hyperideals of a Γ-semihypergroup H. Suppose H is semisimple. Then there exist $x, y, z \in H$, and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a \in x\alpha a\beta y\gamma a\delta z = (x\alpha a\beta y)\gamma(a\delta z)$. So for each $s \in x\alpha a\beta y$ and $t \in a\delta z, (s,t) \in X_a$ i.e. $a \in s\gamma t \subseteq (x\alpha a\beta y)\gamma(a\delta z)$ and thus $X_a \neq \emptyset$. Hence

$$\begin{aligned}(\widetilde{\mu}_A * \widetilde{\mu}_B)(a) &= \bigvee_{(s,t)\in X_a} \min\{\widetilde{\mu}_A(s), \widetilde{\mu}_B(t)\} \\ &\geq \min\{\widetilde{\mu}_A(s), \widetilde{\mu}_B(t)\}.\end{aligned}$$

Since, $A = \langle \widetilde{\mu}_A,\ \widetilde{\lambda}_A \rangle$ and $B = \langle \widetilde{\mu}_B,\ \widetilde{\lambda}_B \rangle$ are interval valued intuitionistic fuzzy Γ-hyperideals, we have, $\widetilde{\mu}_A(s) \geq \widetilde{\mu}_A(s_1) \geq \widetilde{\mu}_A(a)$, for each $s \in s_1\beta y$ and $s_1 \in x\alpha a$ and $\widetilde{\mu}_B(t) \geq \widetilde{\mu}_B(a)$, for every $t \in a\delta z$.

Hence,

$$\min\{\widetilde{\mu}_A(s), \widetilde{\mu}_B(t)\} \geq \min\{\widetilde{\mu}_A(a), \widetilde{\mu}_B(a)\}.$$

Thus,

$$(\widetilde{\mu}_A * \widetilde{\mu}_B)(a) \geq (\widetilde{\mu}_A \wedge \widetilde{\mu}_B)(a).$$

Now, we have

$$\begin{aligned}(\widetilde{\gamma}_A * \widetilde{\gamma}_B)(a) &= \bigwedge_{(s,t)\in X_a} \max\{\widetilde{\gamma}_A(s), \widetilde{\gamma}_B(t)\} \\ &\leq \max\{\widetilde{\gamma}_A(s), \widetilde{\gamma}_B(t)\}.\end{aligned}$$

Since, $A = \langle \widetilde{\mu}_A,\ \widetilde{\lambda}_A \rangle$ and $B = \langle \widetilde{\mu}_B,\ \widetilde{\lambda}_B \rangle$ are interval valued intuitionistic fuzzy Γ-hyperideals, we have, $\widetilde{\gamma}_A(s) \leq \widetilde{\gamma}_A(s_1) \leq \widetilde{\gamma}_A(a)$, for each $s \in s_1\beta y$ and $s_1 \in x\alpha a$ and $\widetilde{\gamma}_B(t) \geq \widetilde{\gamma}_B(a)$, for every $t \in a\delta z$. Hence,

$$\max\{\widetilde{\gamma}_A(s), \widetilde{\gamma}_B(t)\} \geq \max\{\widetilde{\gamma}_A(a), \widetilde{\gamma}_B(a)\}.$$

Thus,

$$(\widetilde{\gamma}_A * \widetilde{\gamma}_B)(a) \leq (\widetilde{\gamma}_A \vee \widetilde{\gamma}_B)(a).$$

Hence, $A * B \supseteq A \cap B$.

On the other hand, by Proposition 233, we have $A * B \subseteq A \cap B$.

Thus, $A * B = A \cap B$.

Conversely, assume $A * B = A \cap B$. Let for any Γ-hyperideals I_1, I_2 of H, $x \in I_1 \cap I_2$. Then $x \in I_1$ and $x \in I_2$. Hence $(\widetilde{\chi}_{I_1}(x), \widetilde{\chi}^c_{I_1}(x)) = (\widetilde{\chi}_{I_2}(x), \widetilde{\chi}^c_{I_2}(x)) = \langle \widetilde{1}, \widetilde{0} \rangle$ (where $\widetilde{\chi}_{I_1}(x)$ and $\widetilde{\chi}_{I_2}(x)$ are respectively the interval valued characteristic functions of I_1 and I_2). Thus,

$(\widetilde{\chi}_{I_1} \cap \widetilde{\chi}_{I_2})(x) = \min\{\widetilde{\chi}_{I_1}(x), \widetilde{\chi}_{I_2}(x)\} = 1$ and $(\widetilde{\chi}^c_{I_1} \cup \widetilde{\chi}^c_{I_2})(x) = \max\{\widetilde{\chi}^c_{I_1}(x), \widetilde{\chi}^c_{I_2}(x)\} = 0$.

By Corollary 227, $(\widetilde{\chi}_{I_1}, \widetilde{\chi}^c_{I_1})$ and $(\widetilde{\chi}_{I_2}, \widetilde{\chi}^c_{I_2})$ are interval valued intuitionistic fuzzy Γ-hyperideals of H. Hence, by hypothesis, $\widetilde{\chi}_{I_1} * \widetilde{\chi}_{I_2} = \widetilde{\chi}_{I_1} \cap \widetilde{\chi}_{I_2}$ and $\widetilde{\chi}^c_{I_1} * \widetilde{\chi}^c_{I_2} = \widetilde{\chi}^c_{I_1} \cup \widetilde{\chi}^c_{I_2}$. Hence,

$(\widetilde{\chi}_{I_1} * \widetilde{\chi}_{I_2})(x) = \widetilde{1}$

since $\sup\limits_{x \in y\gamma z} [\min\{\widetilde{\chi}_{I_1}(y), \widetilde{\chi}_{I_2}(z)\} : y, z \in H, \gamma \in \Gamma] = \widetilde{1}$, and

$(\widetilde{\chi}^c_{I_1} * \widetilde{\chi}^c_{I_2})(x) = \widetilde{0}$

since $\inf\limits_{x \in y\gamma z} [\max\{\widetilde{\chi}^c_{I_1}(y), \widetilde{\chi}^c_{I_2}(z)\} : y, z \in H, \gamma \in \Gamma] = \widetilde{0}$.

This implies that there exist some $r, s \in H$ and $\gamma_1 \in \Gamma$ such that $x \in r\gamma_1 s$ and $(\widetilde{\chi}_{I_1}(r), \widetilde{\chi}^c_{I_2}(r)) = \langle \widetilde{1}, \widetilde{0} \rangle = (\widetilde{\chi}_{I_1}(s), \widetilde{\chi}^c_{I_2}(s))$. Thus, $r \in I_1$ and $s \in I_2$. Hence, $x \in I_1 \Gamma I_2$. Thus $I_1 \cap I_2 \subseteq I_1 \Gamma I_2$. Also $I_1 \Gamma I_2 \subseteq I_1 \cap I_2$. Hence, $I_1 \cap I_2 = I_1 \Gamma I_2$. Thus, by Lemma 237 , H is semisimple. ■

Definition 239 *Let H be a Γ-semihypergroup. An interval valued intuitionistic fuzzy sub-Γ-semihypergroup $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ of H is called an interval valued intuitionistic fuzzy $(1, 2)$ Γ-hyperideal of H if*

1. $\inf\limits_{a \in x\alpha w\beta(y\gamma z)} \{\widetilde{\mu}_A(a)\} \geq \min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y), \widetilde{\mu}_A(z)\}$,
2. $\sup\limits_{a \in x\alpha w\beta(y\gamma z)} \{\widetilde{\gamma}_A(a)\} \leq \max\{\widetilde{\gamma}_A(x), \widetilde{\gamma}_A(y), \widetilde{\gamma}_A(z)\}$,

for all $w, x, y, z \in H$ *and* $\alpha, \beta, \gamma \in \Gamma$.

Theorem 240 *Let* H *be a* Γ*-semihypergroup. Then, every interval valued intuitionistic fuzzy bi-*Γ*-hyperideal of* H *is an interval valued intuitionistic fuzzy* $(1,2)$*-*Γ*-hyperideal of* H*.*

Proof. Let $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ be an interval valued intuitionistic fuzzy bi-Γ-hyperideal of H and let $w, x, y, z \in H$ and $\alpha, \beta, \gamma \in \Gamma$. Then, for all $a \in x\alpha w\beta(y\gamma z)$, we have

$$\begin{aligned}
\inf_{a \in x\alpha w\beta(y\gamma z)} \{\widetilde{\mu}_A(a)\} &= \inf_{a \in (x\alpha w\beta y)\gamma z} \{\widetilde{\mu}_A(a)\} \\
&\geq \inf_{a \in c\gamma z} \{\widetilde{\mu}_A(a)\} \text{ for every } c \in x\alpha w\beta y \\
&\geq \min\{\widetilde{\mu}_A(c), \widetilde{\mu}_A(z)\} \\
&\geq \min\{\min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}, \widetilde{\mu}_A(z)\} \\
&\quad (\text{ because } \inf_{c \in x\alpha w\beta y} \{\widetilde{\mu}_A(c)\} \geq \min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}) \\
&= \min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y), \widetilde{\mu}_A(z)\}
\end{aligned}$$

and

$$\begin{aligned}
\sup_{a \in x\alpha w\beta(y\gamma z)} \{\widetilde{\gamma}_A(a)\} &= \sup_{a \in (x\alpha w\beta y)\gamma z} \{\widetilde{\gamma}_A(a)\} \\
&\leq \sup_{a \in c\gamma z} \{\widetilde{\gamma}_A(a)\} \text{ for every } c \in x\alpha w\beta y \\
&\leq \max\{\widetilde{\gamma}_A(c), \mu_A(z)\} \\
&\leq \max\{\max\{\widetilde{\gamma}_A(x), \mu_A(y)\}, \widetilde{\gamma}_A(z)\} \\
&\quad (\text{ because } \sup_{c \in x\alpha w\beta y} \{\widetilde{\gamma}_A(c)\} \leq \max\{\widetilde{\gamma}_A(x), \widetilde{\gamma}_A(y)\}) \\
&= \max\{\widetilde{\gamma}_A(x), \widetilde{\gamma}_A(y), \widetilde{\gamma}_A(z)\}
\end{aligned}$$

Hence $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is an interval valued intuitionistic fuzzy $(1,2)$-Γ-hyperideal of S. ■

Theorem 241 *Let* H *be a regular* Γ*-semihypergroup. Then, every interval valued intuitionistic fuzzy* $(1,2)$*-*Γ*-hyperideal of* H *is an interval valued intuitionistic fuzzy bi-*Γ*-hyperideal of* H*.*

Proof. Let us assume that a Γ-semihypergroup H is regular and let $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ be an interval valued intuitionistic fuzzy $(1,2)$-Γ-hyperideal of H. Let $w, x, y \in H, \gamma \in \Gamma$. Since, H is regular, we have for every $w \in x\gamma y \subseteq (x\alpha a\beta x)\gamma y = x\alpha a\beta\,(x\gamma y)$ for some $a \in H, \alpha, \beta \in \Gamma$. Thus, for every $c \in x\gamma y, w \in x\alpha a\beta c$, we have

$$\begin{aligned}\inf_{w \in x\alpha a\beta c \subseteq x\alpha a\beta(x\gamma y)} \{\widetilde{\mu}_A(w)\} &\geq \min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\} \\ &= \min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}\end{aligned}$$

and

$$\begin{aligned}\sup_{w \in x\alpha a\beta c \subseteq x\alpha a\beta(x\gamma y)} \{\widetilde{\gamma}_A(w)\} &\leq \max\{\widetilde{\gamma}_A(x), \widetilde{\gamma}_A(x), \widetilde{\gamma}_A(y)\} \\ &= \max\{\widetilde{\gamma}_A(x), \widetilde{\gamma}_A(y)\}.\end{aligned}$$

Hence, $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is an interval valued intuitionistic fuzzy bi-Γ-hyperideal of H. ■

Theorem 242 *Let H be a completely regular Γ-semihypergroup and $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ be an interval valued intuitionistic fuzzy bi-Γ-hyperideal of H. Then, for every $r \in a\gamma a$, we have $A(a) = A(r)$ for all $a \in H, \gamma \in \Gamma$.*

Proof. Let $a \in H, \gamma \in \Gamma$. Then, there exist $x \in H$ and $\alpha, \beta, \rho, \delta \in \Gamma$ such that $a \in a\alpha a\beta x\rho a\delta a$. Hence, for every $r \in a\gamma a, a \in r\beta x\rho r$. Hence,

$$\begin{aligned}\inf_{a \in r\beta x\rho r} \{\widetilde{\mu}_A(a)\} &\geq \min\{\widetilde{\mu}_A(r), \widetilde{\mu}_A(r)\} \\ &= \widetilde{\mu}_A(r) \geq \min\{\widetilde{\mu}_A(a), \widetilde{\mu}_A(a)\} \text{ because } r \in a\gamma a, \forall \gamma \in \Gamma \\ &= \widetilde{\mu}_A(a)\end{aligned}$$

and

$$\begin{aligned}\sup_{a \in r\beta x\rho r}\{\widetilde{\gamma}_A(a)\} &\leq \max\{\widetilde{\gamma}_A(r), \widetilde{\gamma}_A(r)\} \\ &= \widetilde{\gamma}_A(r) \leq \max\{\widetilde{\gamma}_A(a), \widetilde{\gamma}_A(a)\} \text{ because } r \in a\gamma a, \forall \gamma \in \Gamma \\ &= \widetilde{\gamma}_A(a).\end{aligned}$$

It follows that $\widetilde{\mu}_A(a) = \widetilde{\mu}_A(r)$ and $\widetilde{\gamma}_A(a) = \widetilde{\gamma}_A(r)$, so that $A(a) = A(r)$ for all $a \in H$. ■

Theorem 243 *Let H be an intra-regular Γ-semihypergroup and $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ be an interval valued intuitionistic fuzzy Γ-hyperideal of H. Then, for every $r \in a\gamma a$, we have $A(a) = A(r)$ for all $a \in H, \gamma \in \Gamma$.*

Proof. Let $a \in H, \gamma \in \Gamma$. Then, since H is intra-regular. Then there exist $x, y \in H, \alpha, \beta, \delta \in \Gamma$ such that $a \in x\alpha a\beta a\delta y$. Then, for every $r \in a\gamma a$ and $s \in x\alpha r$ we have $a \in s\delta y$. Hence, since $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is an interval valued intuitionistic fuzzy Γ-hyperideal of H, so

$\inf_{a \in s\delta y}\{\widetilde{\mu}_A(a)\} \geq \widetilde{\mu}_A(s)$
and

$\inf_{s \in x\alpha r}\{\widetilde{\mu}_A(s)\} \geq \widetilde{\mu}_A(r)$
and

$\inf_{r \in a\gamma a}\{\widetilde{\mu}_A(r)\} \geq \widetilde{\mu}_A(a)$.
Hence, $\widetilde{\mu}_A(a) = \widetilde{\mu}_A(r)$.

Also,

$\sup_{a \in s\delta y}\{\widetilde{\gamma}_A(a)\} \leq \widetilde{\gamma}_A(s)$
and

$\sup_{s \in x\alpha r}\{\widetilde{\gamma}_A(s)\} \leq \widetilde{\gamma}_A(r)$
and

$\sup_{r \in a\gamma a}\{\widetilde{\gamma}_A(r)\} \leq \widetilde{\gamma}_A(a)$.
Hence, $\widetilde{\gamma}_A(a) = \widetilde{\gamma}_A(r)$. Thus, $A(a) = A(r)$ for all $a \in H$. ■

9.3 Interval Valued Intuitionistic fuzzy M-hypersystems and N-hypersystems in Γ-semihypergroup

In this section we will define interval valued intuitionistic fuzzy M-hypersystems and N-hypersystems in Γ-semihypergroup and some basic properties are obtained.

Definition 244 *Let H be a Γ-semihypergroup and $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ be an interval valued intuitionistic fuzzy subset of H. Then, $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is called an interval valued intuitionistic fuzzy M-hypersystem of H if for all $x, y, z \in H$ and $\alpha, \beta \in \Gamma$ we have*

$$\begin{aligned} \min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(z)\} &\leq \inf_{a \in x\alpha y\beta z} \widetilde{\mu}_A(a) \\ \max\{\widetilde{\gamma}_A(x), \widetilde{\gamma}_A(z)\} &\geq \sup_{a \in x\alpha y\beta z} \widetilde{\gamma}_A(a) \end{aligned}$$

Definition 245 *Let H be a Γ-semihypergroup and $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ be an interval valued intuitionistic fuzzy subset of S. Then, $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ is called an interval valued intuitionistic fuzzy N-hypersystem of H if for all $x, y \in H$ and $\alpha, \beta \in \Gamma$, we have*

$$\begin{aligned} \widetilde{\mu}_A(x) &\leq \inf_{a \in x\alpha y\beta x} \widetilde{\mu}_A(a) \\ \widetilde{\gamma}_A(x) &\geq \sup_{a \in x\alpha y\beta x} \widetilde{\gamma}_A(a) \end{aligned}$$

Remark 246 *Every interval valued intuitionistic fuzzy M-hypersystem of a Γ-semihypergroup H is an interval valued intuitionistic fuzzy N-hypersystem.*

Theorem 247 *If $\{A_i\}_{i \in \Lambda}$ is a family of interval valued intuitionistic fuzzy M (resp. N)-hypersystems of a Γ-semihypergroup H, then $\bigcap_{i \in \Lambda} A_i$ is also an interval valued intuitionistic fuzzy M(resp. N)-hypersystem of H.*

Proof. Let $\mathcal{A} = \bigcap\limits_{i\in\Lambda} A_i = (\bigwedge\limits_{i\in\Lambda} \widetilde{\mu}_{A_i}, \bigvee\limits_{i\in\Lambda} \widetilde{\gamma}_A) = (\widetilde{\mu}_{\mathcal{A}}, \widetilde{\lambda}_{\mathcal{A}})$ and $x, y, z \in H$ and $\alpha, \beta \in \Gamma$. Then, we have

$$\begin{aligned}
\inf_{a\in x\alpha y\beta z} \widetilde{\mu}_{\mathcal{A}}(a) &= \inf_{a\in x\alpha y\beta z} \left\{ \bigwedge_{i\in\Lambda} \{\widetilde{\mu}_{A_i}(a)\} \right\} \\
&\geq \left\{ \bigwedge_{i\in\Lambda} \left(\min\{\widetilde{\mu}_{A_i}(x), \widetilde{\mu}_{A_i}(z)\}\right) \right\} \\
&= \min\left\{ \bigwedge_{i\in\Lambda} \widetilde{\mu}_{A_i}(x), \bigwedge_{i\in\Lambda} \widetilde{\mu}_{A_i}(z) \right\} \\
&= \min\{\widetilde{\mu}_{\mathcal{A}}(x), \widetilde{\mu}_{\mathcal{A}}(z)\} \\
\inf_{a\in x\alpha y\beta z} \widetilde{\mu}_{\mathcal{A}}(a) &\geq \min\{\widetilde{\mu}_{\mathcal{A}}(x), \widetilde{\mu}_{\mathcal{A}}(z)\}
\end{aligned}$$

and

$$\begin{aligned}
\sup_{a\in x\alpha y\beta z} \widetilde{\gamma}_{\mathcal{A}}(a) &= \sup_{a\in x\alpha y\beta z} \left\{ \bigvee_{i\in\Lambda} \{\widetilde{\gamma}_{A_i}(a)\} \right\} \\
&\leq \left\{ \bigvee_{i\in\Lambda} \left(\min\{\widetilde{\gamma}_{A_i}(x), \widetilde{\gamma}_{A_i}(z)\}\right) \right\} \\
&= \max\left\{ \bigwedge_{i\in\Lambda} \widetilde{\gamma}_{A_i}(x), \bigwedge_{i\in\Lambda} \widetilde{\gamma}_{A_i}(z) \right\} \\
&= \max\{\widetilde{\gamma}_{\mathcal{A}}(x), \widetilde{\gamma}_{\mathcal{A}}(z)\} \\
\sup_{a\in x\alpha y\beta z} \lambda_{\mathcal{A}}(a) &\leq \max\{\widetilde{\gamma}_{\mathcal{A}}(x), \widetilde{\gamma}_{\mathcal{A}}(z)\}.
\end{aligned}$$

Hence, $\bigcap\limits_{i\in\Lambda} A_i$ is an interval valued intuitionistic fuzzy M-hypersystem of Γ-semihypergroup H. This completes the proof. ■

Proposition 248 *Every interval valued intuitionistic fuzzy two sided Γ-hyperideal of Γ-semihypergroup H is an interval valued intuitionistic fuzzy M-hypersystem of H.*

Proof. Let $A = \langle \widetilde{\mu}_A, \widetilde{\lambda}_A \rangle$ be an interval valued intuitionistic fuzzy two sided Γ-hyperideal of H. Let $x, y, z \in H$ and $\alpha, \beta \in \Gamma$. Then,

$$\begin{aligned}
\inf_{a \in x\alpha y\beta z} \widetilde{\mu}_A(a) &\geq \inf_{a \in x\alpha y} \widetilde{\mu}_A(a) \geq \widetilde{\mu}_A(x) \text{ and} \\
\inf_{a \in x\alpha y\beta z} \widetilde{\mu}_A(a) &\geq \inf_{a \in y\beta z} \widetilde{\mu}_A(a) \geq \widetilde{\mu}_A(z) \\
\inf_{a \in x\alpha y\beta z} \widetilde{\mu}_A(a) &\geq \min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(z)\}
\end{aligned}$$

and

$$\begin{aligned}
\sup_{a \in x\alpha y\beta z} \widetilde{\gamma}_A(a) &\leq \sup_{a \in x\alpha y} \widetilde{\gamma}_A(a) \leq \widetilde{\gamma}_A(x) \text{ and} \\
\sup_{a \in x\alpha y\beta z} \widetilde{\gamma}_A(a) &\leq \sup_{a \in x\alpha z} \widetilde{\gamma}_A(a) \leq \widetilde{\gamma}_A(z) \\
\sup_{a \in x\alpha y\beta z} \widetilde{\gamma}_A(a) &\leq \max\{\widetilde{\gamma}_A(x), \widetilde{\gamma}_A(z)\}
\end{aligned}$$

This completes the proof. ■

Corollary 249 *Every interval valued intuitionistic fuzzy one sided Γ-hyperideal of a Γ-semihypergroup H is an interval valued intuitionistic fuzzy N-hypersystem of H.*

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Contents

 Chapter 1 Introduction 1.1 Hyperstructures and Semihypergroups In this section we give the brief discussion of hyperstructures. We denote by H a non- empty

1**set and } (H)** is **the** set **of all** non-empty **subsets of H,**

otherwise we shall mention. A hyperoperation "" on H is a map : H H ! } (H) [37]: This means that a hyperoperation is di¤erent from a binary operation. In hyper algebraic structures the product of two elements is a set and in algebraic structure the product of two elements is an element. A non-empty set H with a hyperoperation is called a hyperstructure and is denoted by (H;), also (H;) is called a hypergroupoid. Let P and Q be non-empty subsets of a hypergroupoid H. Then, a hyperproduct of

4**P and Q is denoted by P Q**

and de…ned as [37]:

11**P Q = p q;** a **P** = fag **P** and **P** a = **P**

fag : p2P;q2Q S A hyperstructure (H;) is called a semihypergroup if assocaitive property holds [17] i.e,

1**(x y) z = x (y z) for all x; y; z** 2 **H:** Example 1 Let **H**

= f1; 2; 3; 4; 5g be a semihypergroup de…ned by the following Cayley table; 1 2 3 4 5 1 f1g f1; 2; 4g f1g f1; 2; 4g f1; 2; 4g 2 f1g f2g f1g f1; 2; 4g f1; 2; 4g 3 f1g f1; 2; 4g f1; 3g f1; 2; 4g f1; 2; 3; 4; 5g 4 f1g f1; 2; 4g f1g f1; 2; 4g f1; 2; 4g 5 f1g f1; 2; 4g f1; 3g f1; 2; 4g f1; 2; 3; 4; 5g An element e in a semihypergroup H

6**is called a left(** right) hyperidentity **if for** all **x**

2 H, x 2 e x (x 2 x e). An element e in a semihypergroup is called an identity if e is a left hyperidentity and a right hyperidentity. An element e of a semihypergroup is called a scalar left (right) identity if fxg = e x (fxg = x e) for all x 2 H: A semihypergroup with a scalar identity e is called a hypermonoid. If a semihypergroup holds reproduction axiom, x H = H x = H for all x 2 H is said to be a hypergroup. A sub-semihypergroup A is a non-empty subset of a semihypergroup such that

3**x y A** for all **x; y** 2 **A**

[24, 17, 25]. A

39**non-empty subset I of a** semihypergroup H **is called a** right(left) hyperideal **of** H **if x** 2 I) x **y I**

63**(x 2 I) y x I)**

for all y 2 H. A hyperideal I is a non-empty subset of a semihypergroup H such that

33**x 2 I) x** y **I** and **x 2 I**) y **x I**

for all y 2 H: 1.2 -semihypergroups This section deals with the de…nitions of -hyperoperations, - semihypergroups and basic properties of -semihypergroups. We discuss the theory of -hyperideals. A - hyperoperation on H is a mapping from H H to } (H) i.e for every 2 and x; y 2 H , x y H . Let H and be two non-empty sets. We denote by the English alphabets, the elements of H and by the letters of the Greek alphabets, the elements of . Then H is called a -semihypergroup if [45]

1**1. x** y **H** , for **all x;** y **2 H**

and 2 . 2. (x

1**y) z = x (y z) for all x; y; z** 2 **H**

and for all ; 2 . 3. If h1; h2; h3; h4 2 H; 1; 2 2 such that h1 = h3; 1 = 2 and h2 = h4, then h1 1h2 = h3 2h4. H is called a -hypergroupoid if only the assertions (1) and (3) are satis…ed in the above de…nition. An element e in a -semihypergroup H

6**is called a left(** right) hyperiden- tity **if for** all **x**

2 H and 2 , x 2 e x (x 2 x e). An element e in a -semihypergroup is called an hyperidentity if e is a left hyperidentity and a right hyperidentity. An element e of a -semihypergroup is called a scalar left (right) identity if fxg = e x (fxg = x e) for all x 2 H and 2 : A -semihypergroup with scalar identity e is called a - hypermonoid. If a -semihypergroup holds reproduction axiom, x H = H x for all x 2 H and 2 is said to be a - hypergroup. Also, H is called a -hypergroup if for each 2 , (H;) is a hypergroup. A -semihypergroup is called commutative

1**if x y** = y **x for all x; y** 2 **H**

and 2 [45]. Example 2 Consider a semihypergroup (H;) and a non-empty set. We de…ne a - hyperoperation on H as:

1x y = x y for all x; y 2 H

and 2 . Then, clearly H is a -semihypergroup. Also, if we de…ne a -hyperoperation on H as: x y = x y if H, then H is a -semihypergroup. From above example we can say that -semihypergroup is a suitable extension and a generalization of a semihypergroup. Example 3 Let H = [1;0] and = f 1; 2; 3; 4; 5;::: ng, where n < 1, be two non-empty sets. For every a;b 2 [1;0] and 2 , we de…ne : H H ! } (H) by a b = ab ; 0 . Then, every 2 is a -hyperoperation. For every a; b; c 2 H and ; 2 whe havie (a b) c = abc ; 0 = a (b c) : Thus, H is a -semihypergroup. Also, H is not a -hypergroup because if 0:1 2 [1; 0], then 0:1 H6 = H . If P and Q are subsets of -hypergroupoid H and any element of , then we can de…ne

23P Q = p q; a P = fag P, P a = P fag and P Q = P Q. p2P;q2Q 2

S S Let K

2be a non-empty subset of a -semihypergroup H. Then, K is called a sub- -semihypergroup of H if

a b K for all a; b 2 K and 2 . Let (H1; 1) be a 1-semihypergroup and (H2; 2) a 2-semihypergroup. A function : H1 ! H2 is said to be a homomorphism, if we have a bijective function g : 1 ! 2 such that for all a; b 2 H1 and 2 1; (a b) (a) g ( )

2(b). A non-empty subset A of a -semihypergroup H is called a right(left) -hyperideal of H if

33x 2 I) x y I (x 2 I) y x I)

for all y 2 H and 2 . A -hyperideal I is a non-empty subset of a -semihypergroup H such that

33x 2 I) x y I and x 2 I

) y x I for all y 2 H and 2 [45]: Example 4 Let H = fa; b; c; dg and = f ; g be two non-empty sets and -hyperoperations de…ned by the following Cayley tables; a b c d a fa; cg fb; dg fa; cg fdg b fb; dg fbg fb; dg fdg c fa; cg fb; dg fa; cg fdg d fdg fdg fdg fdg a b c d a fag fb; dg fcg fdg b fb; dg fbg fb; dg fdg c fcg fb; dg fag fdg d fdg fdg fdg fdg Then, clearly, (H;) is a -semihypergroup. The subsets I = fdg and fb; dg of H are only proper -hyperideals of H. A non-empty

1subset B of a -semihypergroup H is called a bi- -hyperideal of H if

B

1**B B** and **B H B B. A** non-empty **subset Q of a** -semihypergroup **H is called a quasi- -hyperideal** if Q **H** \ H **Q Q.** Every -hyperideal of **H** is **a** quasi- -hyperideal **of H**

. It means that a

1**quasi- -hyperideal of** a -semihypergroup **H is a** generalization of a **-hyperideal.** Intersection **of**

quasi- -hyperideals

1**of a** -semihypergroup **H is a quasi- -hyperideal**

[45]. A non-empty subset M of a -semihypergroup H is called an M -hypersystem if for all a; b 2 M , there exist x 2 H and ; 2 such that a x b M [2]. A

3**non-empty subset** N **of** -semihypergroup H **is called an** N -hypersystem **if for**

all a 2 N , there exist x 2 H and ; 2 such that a x a N [2]. We de…ne a relation on H as fallows, where H is a -semihypergroup. (x;) (y;) , x s = y s; 8s 2 H and ; 2 . Then clearly, is an equivalence relation. The equivalence class containing (x;) is denoted by [x;]. Let S = f[x;] : x 2 H; 2 g be the collection of all the equivalence classes. A hyperoperation "" on S is de…ned as follows: [x;] [y;] = f[z;] : z 2 x yg;

1**for all x; y; z** 2 **H** and ; 2 . Since **H is**

a -semihypergroup, so [x;] ([y;] [z;]) = ([x;] [y;]) [z;]. Thus, hyperoperation "" is associative, so (S;) is a semihypergroup. This semihy- pergroup S is called a left operator semihypergroup of H [29]. For A S we de…ne A+ = fx 2 H : [x;] 2 S; 8 2 g. Similarly, for I H we de…ne I +0 = f[x;] 2 S : x s I ; 8s 2 H g. Theorem 5 [29] Let S be a left operator semihypergroup of a -semihypergroup H. Then, the following properties

1**hold: 1.** If **A is a right hyperideal of**

S, then A+

1**is a** right **-hyperideal of H.** 2. If I **is a right -hyperideal of H,**

then I+0 is a right hyperideal of S. Let H be a -semihypergroup and S the left operator semihypergroup of

H. Let I be a

1**-hyperideal of H and A** a **hyperideal of**

S. Then, it is easy to see that I (I+)+0 and A (A+0)+ [29]. Theorem 6 [29] Let H be a -semihypergroup and S be the left operator semihypergroup of H. Let P be a prime right -hyperideal of H. Then P = (P+0)+. Let H be a -semihypergroup. If there exist an element e 2 H and 2 such that e x = x for every x 2 H, then H is said to have a left unity. If H has a left unity, then [e;] is a left unity of the left operator semihypergroup S. Theorem 7 [29] Let H be a -semihypergroup and S its left operator semihypergroup. If I is a right -hyperideal of H, then I = (I+0)+.. Lemma 8 [29]Let H be a -semihypergroup and S its left operator semihypergroup. If P is a prime hyperideal of S, then P+ is a prime -hyperideal of H. Lemma 9 [29]Let H be a -semihypergroup. If Q

1**is a** prime **-hyperideal of H,** then Q+0 **is a** prime **hyperideal of**

S. Theorem 10 [29]Let H be a -semihypergroup and S its left operator semihypergroup. Then there exists an inclusion preserving bijection Q ! Q+0 between the set of all prime -hyperideals of H and the set of all prime hyperideals of S. 1.3 Fuzzy sets and Intuitionistic fuzzy sets Zadeh introduced

2**the notion of** a **fuzzy set** [49] **in**

1965 and also a fuzzy set

2**is called a fuzzy** logic. **A fuzzy**

logic is a generalization of mathematical logic. In 1986, Atanassov

2**introduced the concept of an intuitionistic fuzzy set**

[14, 15]. A mapping

2**from a non-empty set** X **to unit** closed **interval of**

0 and 1 i.e, : X ! [0;1], is called a fuzzy set.

2**The set of all fuzzy sets**

is denoted by FP(X) i.e, FP(X) = f j : X ! [0; 1]g : The complement of a fuzzy set is denoted by c and de… ned by c

5(x) = 1 (x) for all x 2

X: Let and be two fuzzy sets in a non-empty set X: Then,

2if and only if (x) (x) for all x 2 X

. The fuzzy subsets ^ and _ of X are de…ned as follows: (^) (a) =

5(x) ^ (a) (_) (a) = (x) _ (a) for all a 2 X:

The set t = fx 2 X j (x) g; where 2 (0; 1] is called an -cut or the level set. The product of

1two fuzzy subsets and of a groupoid H is denoted by

and de…ned as: f (y) ^ (z)g; ( ) (x) = 8 x=yz >< Wif there exist y; z 2 H; such that x = yz: 0 otherwise. >:

62Let X be a nonempty…xed set.

3An intuitionistic fuzzy set (brie‡y, IFS) A is an

object

2having the form A = f (x; A(x); A(x)) : x 2 Xg where the functions A : X ! [0; 1] and A : X ! [0; 1] denote the degree of membership (namely A(x)) and the degree of nonmembership (namely A(x)) of each element x 2 X to the set A, respectively, and 0 A(x) + A(x) 1 for all x

2 X For the sake of simplicity, we use the symbol A = h A; Ai for

3an intuitionistic fuzzy set A = fhx; A(x); A(

x)i jx 2 Xg [14]. Let A and B be two intuitionistic fuzzy sets on X. The following expressions are de…ned in [14, 15].

21. A B if and only if A(x) B(x) and A(x) B(x) for all x 2 X. 2. A

= B if and only if if A B and B A. 3. Ac=fhx; A(x); A(x)ijx2Xg. 4. A\B=fhx;minf A

44**(x); B(x)** g;maxf **A(x); B(x)** gijx2Xg. 5. A **[B=** fhx;maxf **A(**

44**x); B(x)** g;minf **A(x); B(x)**

gijx2Xg. 6. A=fhx; A(x);1 A(x)ijx2Xg. 7. A=fhx;1 A(x); A(x)ijx2Xg. Let A;B be

2**two intuitionistic fuzzy sets** in **a** -hypergroupoid H. Then, **A**

B = fhx; A B(x); A Bijx 2 Hg, where sup fminf A(y); B(z)gg if x 2 y z; for some A B (x) = >< 0 8 x2y z otherwise >: A B(x) = 8 x2y z inf fmaxf A(y); B(z)gg if x 2 y z; for some < 1 otherwise 2 2 Let c be a point i:na non-empty set X. If t 2 (0; 1] and s 2 [0; 1) are two real numbers such that 0 t + s 1, then the IFS c(t; s) = hx; ct; 1 c1 si is called an intuitionistic fuzzy point (IFP for short) in X, where t (resp., s) is the degree of membership (resp., non-membership) of c(t; s) and c 2 X is the support of c(t; s). Let c(t; s) be an IFP in X and let A = hx; A; Ai be an IFS in X. Then, c(t; s) is said to belong to A, written

24**c(t; s)** 2 **A,** if A **(c) t** and **A(c)**

s [18]. By an interval number D we mean an interval [; +], where 0 + 1. For interval numbers D1 = [; +], D2 = ; + 2 D [0; 1], where D [0; 1] denotes the set of all closed subintervals of the interval [0; 1] ; we de…ne D1 \ D2 = min (D1; D2) = min ; + ; ; + = min ; ; min +; + ; D1 [D2 = max (D1; D2) = max ; + ; ; + = max D1 D2 , ; ; max and + +; + ; +; D1 < D2 , < and + < +; D1 = D2 , = and + = +; mD = m ; + = m ; m + , where 0 m 1; Let X be a given non-empty set. An interval valued fuzzy set on X is de…ned by

34**B = x;** B **(x) ; +B (x) : x** 2 **X** ; e where **B (x)** and **+B (x)**

are fuzzy sets of X such that

34**B (x) +B (x)** for all **x** 2 **X** . Let **B (x) = B (x) ; +B (x)** . Then, e **B = x;** B **(x) : x** 2 **X** ; n where B : **X**

! D [0; 1] : e o A meapping A = h A ; Ai : X ! D [0; 1] D [0; 1] is called an interval valued intuitionistic fuzzy set (IIF set, in short) in X if 0 A (x) + A (x) 1 for all x 2 X; where the mapping

9**A = A (x)** ; +A **(x) : X e e +A** (x) + **+A**

(x) 1 and 0 ! D

2**[0; 1]** denotes **the degree of membership (namely A (x)) and**

A = A (x) ; +A (x) : X ! D [0; 1] denotes

2**the degree of non-membership (namely** eA **(x))** for **each element** e **x** 2 X **to A; respectively.**

e e Throughout in this thesis, the following notions will be used: (1) We use 0 to denote the interval valued fuzzy empty set and 1 to denote the interval valued

2**fuzzy whole set in a set X,** and de…ne **0 (x) = [0;** 0] **and 1 (x) = [1;** 1], **for** all **x** 2 **X:** e e (2) **We write**

t = [t1; t2] and s = [s1; s2] 2 D [0; 1]. e e (3) We write m ft1; t2g = min ft1; t2g and M fs1; s2g = max fs1; s2g. e e 1.4 Intuitionistic Fuzzy -hyperideals in -semihypergroups In [28], the authors de…ned the image and inverse image of an intuitionistic fuzzy subset in a -semihypergroup and property on the image and inverse image of an intuitionistic fuzzy -hyperideal was obtained. Let be a mapping from a set X to a set Y . Let A = h A; Ai be an intuitionistic fuzzy subset of X and B = h B; Bi be an intuitionistic fuzzy subset of Y . Then the inverse image 1(B) of B, is the intuitionistic fuzzy set of X de…ned by

54**1(B)** = (1(**B); 1(B))** where 1(**B)(** x) = **B(** (x)) and **1(B) = B(**

(x)) [28]. The image (B) of the intuitionistic fuzzy subset B = h B; Bi is the intuitionistic fuzzy set in Y de…ned by (B) = ((B); (B)), where for every y 2 Y (supf B (t)jt 2 B)(y) = 8 < 0 : (B)(y) = 8 inff B(t)jt 2 < 0 : 1(y)g if 1(y)6 = ; otherwise 1(y)g if 1(y)6 = ; otherwise. Chapter 2 Prime Bi- -hyperideals 2.1 Introduction In this chapter, we study some properties of prime bi- -hyperideals in a -semihypergroup.

2**We introduce the concept of** the strongly **prime, prime, semiprime,**

strongly irreducible and irreducible bi- -hyperideals of -semihypergroups. The space of strongly prime bi- -hyperideals is topologized. Also,

2**we characterize those** -semihypergroups **for which each bi-** -hyperideal **is strongly prime.**

We study some di¤erent relations among these concepts and characterize -semihypergroups by the properties of prime bi- -hyperideals. 2.2 Prime Bi- -hyperideals A bi- -hyperideal

1**B of a** -semihypergroup **H is called a** prime (strongly prime) **bi- -hyperideal if** B1 B2 **B(**

B1 B2 \ B2 B1 B) implies B1 B or B2 B for any bi- -hyperideals B1 and B2 of H. A bi- -hyperideal

1**B of a** -semihypergroup **H is called a** semiprime **bi- -hyperideal if** B1 B1 **B** implies B1 **B**

for any bi- - hyperideal B1 of H. Every strongly prime bi- -hyperideal

1**of a** -semihypergroup **H is a** prime **bi- - hyperideal**

and every prime bi- -hyperideal of a -semihypergroup H is a semiprime bi- -hyperideal. A prime bi- - hyperideal is not necessarily strongly prime and a semi- prime bi- -hyperideal is not necessarily prime. A bi- -hyperideal B of a -semihypergroup H is called an irreducible (strongly ir- reducible) bi- -hyperideal if B1 \ B2 = B(B1 \ B2 B) implies B1 = B or B2 = B (B1 B or B2 B). Every strongly irreducible bi- -hyperideal of a - semihypergroup is an irreducible bi- -hyperideal but the converse is not true in general. Example 11 Let H = f1; 2; 3; 4g and = f ; gbe a -semihypergroup with the follow- ing Cayley's tables. 1 2 3 4 1 2 3 4 1 f1g f1g f1g f1g 1 f1g f1g f1g f1g 2 f1g f1g f1; 4g f1g 2 f1g f1g f2; 4g f1g 3 f1g f1g f3g f1g 3 f1g f1g f3g f1g 4 f1g f1g f1g f1g 4 f1g f1g f1g f1g Bi- -hyperideals of H are f1g ; f1; 2g ; f1; 3g ; f1; 4g ; f1; 2; 4g,f1; 3; 4g and H . Bi- -hyperideals f1; 2; 4g and H of H are prime bi- -hyperideals of H . The bi- -hyperideal f1; 2; 4g and H are also strongly prime bi- -hyperideals of H . Bi- -hyperideals f1; 2g ; f1; 3g ; f1; 2; 4g, f1; 3; 4g and H are irreducible and f1; 2; 4g, f1; 3; 4g and H are strongly irreducible: Example 12 Let H = f1; 2; 3; 4; 5g and = f ; g be two non-empty sets. Then, H is a -semihypergroup, where -hyperoperations are de…ned by the following Cayley tables. 1 1 f1g 2 f1g 3 f1g 4 f1g 5 f1g 2 3 f1g f1g f1g f1g f1g f3g f1g f3; 4g f1g f3; 4g 4 2 3 4 5 5 1 f1g f1g 1 f1g f1g f1g f1g f1g f1g f1g 2 f1g f1g f1g f1g f1g f3g f3g 3 f1g f1g f3g f3g f3g f3; 4g f5g 4 f1g f1g f3g f3; 4g f5g f3; 4g f5g 5 f1g f1g f3g f3; 4g f5g Example 13 Let H = f1; 2; 3g and = f ; g be two non-empty sets. Then, clearly H is a -semihypergroup by the following Cayley's tables of - hyperoperations: 1 2 3 1 2 3 1 f1g f1; 2g f1; 3g 1 f1g f1; 2g f1; 3g 2 f1; 2g f2g f2; 3g 2 f1; 2g f2g f2; 3g 3 f1; 3g f2; 3g f2; 3g 3 f1; 3g f2; 3g f3g The prime-bi- -hyperideal of H is f1g. Lemma 14 The intersection of any family of prime bi- -hyperideals of a -semihypergroup is a semiprime bi- -hyperideal. Proof. Straightforward. Theorem 15 Every strongly irreducible, semiprime bi- -hyperideal of a -semihypergroup H is a strongly prime bi- -hyperideal. Proof. Let B be a strongly irreducible semiprime bi- -hyperideal of H. Let B1; B2 be any bi- -hyperideals of H such that

15**B1 B2 \B2 B1** B. Since **(B1 \ B2)** 2 **B1 B2** and **(B1 \ B2)** 2 **B2 B1, (B1 \ B2)** 2 **B1 B2 \ B2 B1**

B. Since B is a semiprime bi- -hyperideal, B1 \ B2 B. Because B is a strongly irreducible bi- -hyperideal of H, so either B1 B or B2 B. Thus B is a strongly prime bi- -hyperideal of H. Theorem 16

1**Let B be** a **bi- -hyperideal of** a -semihypergroup **H and**

a 2 H be such that a 2= B. Then there exists an irreducible bi- -hyperideal I of H such that B I and a 2= I. Proof. Let A be the collection of all bi- -hyperideals of H which contain B and do not contain a. Then, it is non-empty, because B 2 A. The collection A is a partially ordered set under inclusion. If C is any totally ordered subset of A, then [C is a bi- - hyperideal of H containing B. Hence by Zorn's Lemma, there exists a maximal element I in A. We show that I is an irreducible bi- -hyperideal. Let C and D be two bi- - hyperideals of H such that I = C \ D. If both C and D properly contain I, then a 2 C and a 2 D. Hence a 2 C \ D = I. This contradicts the fact that a 2= I. Thus I = C or I = D. 2.3 Characterizations of -semihypergroups by Prime Bi- -hyperideals All -semihypergroups considered in this section are -semihypergroups with zero.

2**An element 0 of a** -semihypergroup H, **with at least two elements, is called a zero element of** H **if** 0 x = x **0** = f0g **for all x**

2 H and 2 . A -semihypergroup is called a -semihypergroup with zero if it contains a zero element.

2**An element a of a** -semihypergroup H **is called regular if there exists x**

2 H and ; 2 such that a 2 a x a. If every element of a -semihypergroup H is regular, then H is called a regular -semihypergroup. The following properties are equivalent (1) For every A H, A A H A. (2) For every element a 2 H, a 2 a H a A -semihypergroup H is called intra-regular if for every a 2 H there exist x, y 2 H and ; ; 2 such that a 2 x a a y. For a -semihypergroup H, we denote by R(a) (resp. L(a), Q(a), B(a)) the right (resp. left, quasi-,bi-) -hyperideal of H generated by a(a 2 H) and de…ne as following; R(a) = fag [a H, L(a) = fag [H a, Q(a) = fag [((a H) \ (H a)), B(a) = fag [a H a). The -hyperideal generated by a is the set fag [

27**H a** [a **H** [H **a H). A** -semihypergroup **H** is called **a**

left(resp., right)strongly regular if for every a 2 H and for all 2 there exists x 2 H such that a 2 x a a(resp; a 2 a a x): We de…ne a relation "L" on H as follows: aLb () L(a) = L(b). Lemma 17 Let H be a - semihypergroup. Then L is an equivalence relation. Proof. The proof is straightforward. Theorem 18

1**Let H be a** -semihypergroup. **Then the following are equivalent: 1) H is** left regular. **2)** aL **(a a)**

for all a 2 H and for all 2 . Proof. 1) =) 2): Let H be left strongly regular and a 2 H. If t 2 L(a), then t = a or t 2 x a for some x 2 H and 2 . Since H is left strongly regular, a 2 y a a for some y 2 H and for all 2 . If

3**t = a,** then **t = a** 2 **y a a**

for some y 2 H and for all 2 . If t 2 x a, then t 2 x a

3**x (y a a) = (x** y) **(a a).** This **implies** that t 2 **(x y)**

(a a) H (a a) ((a a) [H (a a) = L(a a) Hence, t 2 L(a a) for all 2 . Hence L(a) L(a a). Let a 2 H and t 2 L(a a). Then t 2 a a or t 2 x (a a) for some x 2 H. In both cases, t 2 H a fag[H a =

16**L(a).** Thus **L(a** a) **L(a),** so **L(a) = L(a** a). Therefore, aL **(a** a) for all **a**

2 H and 2 . 2) =) 1): Assume that aL(a a) for all a 2 H and for all 2 . Then a 2 L(a) = L(a a) = fa ag [H (a a). Thus, a 2 a a or a 2 x (a a) for some x 2 H . From both cases, we have a 2 x (a a) for some x 2 H and for all 2 . Thus, a is a left strongly regular, and so H is a left strongly regular. Example 19 Let H = fa; b; c; dg and = f ; g be two non-empty sets. Then H is a -semihypergroup, where and hyperoperations are de… ned by the following Cayley tables; a b c d a fa; cg fb; dg fa; cg fdg b fb; dg fbg fb; dg fdg c fa; cg fb; dg fa; cg fdg d fdg fdg fdg fdg a b c d a fag fb; dg fcg fdg b fb; dg fbg fb; dg fdg c fcg fb; dg fag fdg d fdg fdg fdg fdg Thus, H is a left strongly regular, i.e, a 2 c

16**a a; a** 2 **a a a; a** 2 **a a a; a** 2 **a a** a **b** 2 c **b b; b** 2 **b b b; b** 2 **b b b; b** 2 **b b b**

41**c 2** a **c c; c 2** a **c c; c** 2 **c c c; c** 2 a **c c d** 2 b **d** d; d **2** b **d** d; d **2**

38**d d d; d** 2 **d d d** Now **L** (a) = **H = L** (a a) = **L** (a a) ; **L** (b) = fb; dg = **L** (b b) = **L**

48**(b b)** ; L **(c)** = H = L **(c c)** = L **(c c)** , ; L **(d)** = fdg = L **(d d)** = L **(d d)**

: aL (a a) for all a 2 H and for all 2 . Theorem 20 Let H be a -semihypergroup. Then H is left strongly regular if and only if L(a a) is a semiprime. Proof. Let H be a left strongly regular -semihypergroup. Then by Theorem 18, L (a) = L(a a) for all a 2 H and 2 . Let for all a 2 H and 2 such tha a a L(a a) and also a 2 L(a) this implies a 2 L(a a). Conversely, suppose that L(a a) is semiprime. Let a 2 H. Since L(a a) is a semi- prime. Then a a L(a a) for all 2 implies a 2

16**L(a a),** where **a** 2 **L(a).** Thus, **L(a) L(a a).**

Now let t 2 L(a a). Then, t 2 a a or t 2 x a a for some x 2 H. Thus, t 2 H a implies t 2 fag [H a. Hence

16**L(a** a) **L(a)** this implies **L(a) = L(a a).**

By Theorem 18. H is a left strongly regular. Theorem 21 Let H be a -semihypergroup. fH" : " 2 g be a family of left simple sub- -semihypergroups of H such that H = [fH" : " 2 g. Then, H is left strongly regular. Proof. Suppose every H" is a left simple sub- -semihypergroup of H and let H = [fH" : " 2 g. If L

1**is a left -hyperideal of H and**

a a L for a 2 H, then a 2 H" for some " 2 . Consider a subset L \ H" of H. Since H" is a sub- -semihypergroup of H, so a a H". Thus L \ H"6 = ;. Furthermore, H"

49**(L \ H") H" L \ H" H" H L \ H"** L \ **H".** Hence, **L \ H"**

1**is a left -hyperideal of H".**

Since H" is left simple, we have L \ H" = H". Thus a 2 L. Hence, L is semiprime. By Theorem 18, H is a left strongly regular. Lemma 22 A -semihypergroup H is completely regular if and only if A (A

57**A) H (A A)** for every **A H.** Equivalently, **a** -semihypergroup **H**

is completely regular if and only if a 2 a a H a a for all a 2 H. Theorem 23 A -semihypergroup H is a completely regular if and only if every bi- - hyperideal of H is semiprime. Proof. Suppose H is completely regular -semihypergroup.

1**Let B be** a **bi- - hyperideal,** a 2 **H and**

a a B. Then for some

5**x; y; z 2** H **and** ; ; ; ; ; ; 2 , **a**

2 a

8**x a = (a a** y) x **(z a a) = a (a** y x **z a) a** B **H**

B B =) a 2 B. Thus, B is semiprime. Conversely, let a 2 H . Then a a H a a is a non-empty subset of H . Let x; y 2 (a a) H (a a) and z 2 H . Then, for some s; t 2 H and ; ; ; ; ; ; 2 , x z y

7**(a a** u **a a)** z **(a a** v **a a) = a a** (u **a a** z **a a** v) **a a (a a)** H **(a a)** Thus, x **z** y **a a** H **a a.** Then, **((a a)** H **(a a)) ((a a)** H **(a a)) ((a a)** H **(a a))** and **((a a) H (a a))** H **((a a) H (a a))** (a a) H (a a) Hence, (a a) H (a a) is a bi- -hyperideal of H for all a 2 H. Since a a a a a a a a = (a a) (a a a a a) (a a) ((a a) H (a a)) and (a a) H (a a)

is semiprime, we get a a a a a, a a a a H a a and so a 2 (a a) H (a a). Hence, by Lemma 22, H is a completely regular -semihypergroup. Theorem 24 For a -semihypergroup H,

1the following assertions are equivalent: (i) H is

both regular and intra-regular. (ii) B B = B for

1every bi- -hyperideal B of H

. (iii) B1 \ B2 = B1 B2 \ B2 B1 for all bi- -hyperideals B1and B2 of H. (iv) Each bi- -hyperideal of H is semiprime. (v) Each proper bi- -hyperideal of H is the intersection of irreducible semiprime bi- -hyperideals of H which contain it. Proof. (i) () (ii) The proof is easy, we omit. (ii) =) (iii) Let B1 and B2 be any two bi- -hyperideals of H. Then, by our hypothesis,

15B1 \ B2 = (B1 \ B2) (B1 \ B2) B1 \ B2 B1 B2 Similarly, B1 \ B2 B2 B1 Thus, B1 \ B2 B1 B2 \ B2 B1 (2.1) Now, B1 B2 and B2 B1

are bi- -hyperideals being the products of bi- -hyperideals. Also B1 B2 \ B2 B1 is a bi- -hyperideal. Then, B1

15B2 \ B2 B1 = (B1 B2 \ B2 B1) (B1 B2 \ B2 B1) (B1 B2) (B2 B1) B1 S B1 B1 Similarly, B1 B2 \ B2 B1 B2 Thus, B1 B2 \ B2 B1 B1 \ B2

(2.2) Hence from (2.1) and (2.2) B1 \ B2 = B1 B2 \ B2 B1 (iii) =) (iv) Let B1 and B be bi- -hyperideals of H such that B1 B1 B. By hypothesis,

42B1 = B1 \ B1 = B1 B1 \ B1 B1 = B1 B1 Thus, B1

B Hence, every bi- -hyperideal of H is semiprime. (iv) =) (v) Let B be a proper bi- -hyperideal of H. Then, B is contained in the

2intersection of all irreducible bi- - hyperideals of H which contain

B. Theorem 16 guarantees the existence of such irreducible bi- -hyperideals. If a 2= B; then there exists an irreducible bi- -hyperideal of H which contains B but does not contain a. Hence B is the

2**intersection of all bi-** -hyperideals **of** H **which contain**

it. By our hypothesis, every bi- -hyperideal is semiprime, and so each bi- -hyperideal is the intersection of irreducible semiprime bi- -hyperideals of H containing it. (v) =) (ii)

1**Let B be** a **bi- -hyperideal of H.**

If B B = H, then clearly B is idempotent, that is, B B = B. If B B6 = H,

1**then B B is a** proper **bi- -hyperideal of H** containing B B **and**

so by our hypothesis B B = fB : B is irreducible semiprime bi- -hyperideal of Hg \ Since each B is a semiprime bi- -hyperideal, B B for all and so B B = B B. Hence each bi- -hyperideal in H is idempotent. T Theorem 25 Let H be a regular and an intra-regular -semihypergroup. Then, the following assertions, for a bi- -hyperideal B of H, are equivalent: (i) B is strongly irreducible. (ii) B is strongly prime. Proof. Straightforward. Next, we characterize those strongly prime and also those strongly irreducible. -semihypergroups in which each bi- -hyperideal is a -semihypergroups in which each bi- -hyperideal is a Theorem 26 Each bi- -hyperideal of a -semihypergroups H is a strongly prime if and only if H is a regular, intra-regular and the set of bi- -hyperideals of H is totally ordered by inclusion. Proof. Suppose that each bi- -hyperideal of H is strongly prime. Then each bi- - hyperideal of H is semiprime. Thus, by Theorem 22, H is both

2**regular and intra-regular.** We show that **the** set **of**

bi- -hyperideals of H is totally ordered. Let B1 and B2 be any two bi- -hyperideals of H. Then, by Theorem 22, B1 \ B2 = B1 B2 \ B2 B1. As each bi- -hyperideal is strongly prime, B1 \ B2 is strongly prime. Hence, either

26**B1 B1 \ B2** or **B2 B1 \ B2.** If **B1 B1 \ B2,** then **B1 B2.** If **B2** B1 \ **B2,** then **B2 B1.**

Conversely, assume that H is regular, intra-regular and since the set of bi- -hyperideals of H is totally ordered under inclusion. Then, we want to show that each bi- -hyperideal of H is strongly prime.

1**Let B be** an arbitrary **bi- -hyperideal of H and**

B1, B2 be bi- -hyperideals of H such that B1 B2 \ B2 B1 B Since H is both regular and intra-regular, by Theorem 22,

15**B1 \ B2 = B1 B2 \ B2 B1** Also **B1 B2 \ B2 B1** B implies **B1 \ B2**

B. Since the set of all bi- -hyperideals of H is totally ordered under inclusion, so either B1 B2 or B2 B1, that is, either

42**B1 \ B2 = B1** or **B1 \ B2 = B2.** Thus either **B1** B or **B2**

B. Theorem 27 If the set of all bi- -hyperideals of a -semihypergroup H is totally ordered, then H is both regular and intra-regular if and only if each bi- -hyperideal of H is prime. Proof. Suppose that H is both regular and intra-regular.

1**Let B be any bi- - hyperideal of H and**

B1, B2 be bi- -hyperideals of H such that B1 B2 B. Since the set of all bi- -hyperideals of H is totally ordered, either

53**B1 B2** or **B2 B1.** Sup- pose **B1 B2.** Then **B1 B1 B1 B2**

B. By Theorem 22, B is semiprime so B1 B. Hence, B is a semiprime bi- - hyperideal of H. Conversely, assume that every bi- -hyperideal of H is prime. Since the set of all bi- - hyperideals of H is totally ordered so the concepts of prime and strongly prime coincide. Now, by Theorem 26, we see that H is both regular and intra-regular. Theorem 28 For a -semihypergroup H the following assertions are equivalent: (i) The set of all bi- -hyperideals of H is totally ordered under inclusion. (ii) Each bi- -hyperideal of H is strongly irreducible. (iii) Each bi- -hyperideal of H is irreducible. Proof. (i)) =) (ii)

1**Let B be** an arbitrary **bi- -hyperideal of H and**

B1, B2 be two bi- -hyperideals of H such that B1 \ B2 B. Since the set of all bi- -hyperideals of H is totally ordered, either

26**B1 B2** or **B2 B1.** Thus either **B1 \ B2 = B1** or **B1 \ B2 = B2.** Hence **B1 \ B2** B implies either **B1** B or **B2**

B. This shows that B is strongly irreducible bi- - hyperideal. (ii) =) (iii)

1**Let B be** an arbitrary **bi- -hyperideal of H and**

B1, B2 two bi- - hyperideals of H such that B1 \ B2 = B. Then B B1 and B B2. By hypothesis, either B1 B or B2 B. Hence either B1 = B or B2 = B. That is, B is irreducible bi- -hyperideal. (iii) =) (i) Let B1 and B2 be any two bi- - hyperideals of H. Then, B1 \ B2 is a bi- -hyperideal of H. Also, B1 \ B2 = B1 \ B2. So by hypothesis, either

26**B1 = B1 \ B2** or **B2 = B1 \ B2,** that is, either **B1 B2** or **B2 B1.**

Let H be a -semihypergroup, B be the set of all bi- -hyperideals of H and P be the set of all strongly prime proper bi- -hyperideals of H: De…ne for each B 2 B B = fJ 2 P : B * Jg =(P) = f B : B is a bi- -hyperideal of Hg Theorem 29 If H is a regular and an intra-regular -semihypergroup. Then =(P) forms a topology on the set P . Proof. Since f0g

1**is a bi- -hyperideal of H,**

0 = fJ 2 P : f0g * J g = ; because 0 belongs to every

1**bi- -hyperideal of H** Also, since H **is a** bi- **-hyperideal of H,**

H = fJ 2 P : H * Jg = P because P is the collection of all strongly prime proper bi- -hyperideal of H. Let f Bi : i 2 Ig =(P). Then, Bi = fJ 2 P : Bi * J for some i 2 I g [i2I = fJ 2 P : Bi * J g = [i2I Bi ; iS2I where Bi is the bi- -hyperideal of H generated by Bi. i2I i2I NowS let B1 and B2 2 =(P). If J 2 B1 \ SB2 , then J 2 P and B1 * J , B2 * J . Suppose B1 \ B2 J . Since H is both regular and intra-regular,

52**B1 \ B2 = B1 B2 \ B2 B1.** Hence, **B1 B2 \ B2 B1**

J . This implies either B1 J or B2 J , a contradiction. Consequently, B1 \ B2 * J , which implies that J 2

26**B1\B2** . Thus **B1 \ B2 B1\B2** . If J 2 **B1\B2**

, then we have J 2 P and B1 \ B2 * J: This implies that B1 * J; B2 * J . Thus J 2 B1 and J 2 B2 , and therefore J 2

15**B1 \ B2** . Hence **B1\B2 B1 \ B2.** Consequently, **B1\B2 = B1 \ B2.**

This shows that =(P) is a topology on P. Theorem

1**30 Let H be a** -semihypergroup. **Then, the following are** true. **(1) H is**

intra-regular if and only if for a

1**bi- -hyperideal B and** a quasi- **-hyperideal** Q **of H**

,

1we have B \ Q (H B Q H

). (2) H is intra-regular if and only if for a

1bi- -hyperideal B and a quasi- -hyperideal Q of H

, we have B \ Q (H Q B H). Proof. Let a 2 B \ Q. Since H is

2intra-regular, so there exist x, y

2 H and , ; 2 such

4that a 2 x a a y x a (x a a y) y = x (a x a) a y y = x (a x a)

a y y H (B H

11B) Q H (H B Q H) Therefore, B \ Q (H B Q H).

Conversely, let a 2 H and

1let B(a) be a bi- -hyperideal of H generated by

a and Q(a) be a quasi- -hyperideal of H generated by a. Then, by hypothesis, a 2 B(a) \ Q

31(a) H B (a) Q (a) H = H (a [a H a) (a [a H \ H a) H (H a [H a a [H a H a) (a [H a) H H a (a [H a) H H a (a H [H a H) H a a H [H a2 H = H a2 H) a 2 H a a H: Therefore, H

is intra-regular. (2) Let a 2 B\Q. Since H is an intra-regular, so there exist x, y 2 H and , ; 2 such that a 2 x a a y. Thus, a 2 xaay x(xaay)ay=xxa(ay a)y H Q(BH B)H H QBH: Therefore, B \ Q (H Q B H). Conversely, let a 2 H and

1let B(a) be a bi- -hyperideal of H generated by

a and Q(a) be a quasi- -hyperideal of H generated by a. Then, by hypothesis, a 2

19B(a) \ Q (a) H Q (a) B (a) H = H (a [(a H) \ (H a)) (a [a H a) H (H (a [H a) (a H [a a H [a H a H) (H a [H a) (a H) (H a) (a H) H a a H) a 2 H a a H: Therefore, H

is an intra-regular.

1Theorem 31 Let H be a -semihypergroup. Then the following are true. (1) H is

intra-regular if and only if for a left -hyperideal L and a bi- -hyperideal B of H, we have L \ B (L B H). (2) H is intra-regular if and only if for a

1right -hyperideal R and a bi- -hyperideal B of H,

we have B \ R (H B R). Proof. (1) Let a 2 L \ B. Since H is intra-regular, so there exist x, y 2 H and , ; 2 such that a 2 x a a y. Then, a 2 xaay x(xaay)ay=xxa(ay a)y H L(BH B)H LBH: Thus, L \ B (L B H). Conversely, let a 2 H and

1let B(a) be a bi- -hyperideal of H generated by

a and L(a) be a left -hyperideal of H generated by a of H. Then, by hypothesis, a 2 L(a) \ B(a) (L(a)

19B(a) H) = (a [H a) (a [a H a) H (a [H a) (a H [a a H [a H a H) (a [H a) (a H) = a a H [H a a H) a 2 a a H [H a a H) a 2 a a H or a 2 H a a H: If a 2 a a H, then a 2 a

a x for some x 2 H and ; 2 . a 2

7a a x a (a a x) x = a a a x x H a a H a 2 H a a

H: Hence, H is intra-regular. (2) Let a 2 B \ R. Since H is

2intra-regular, so there exist x; y

2 H and , ; 2 such

4that a 2 x a a y x a (x a a y) y = x (a x a) a

y y H (B H

11B) R H H B R: Thus, B \ R (H B R).

Conversely, let a 2 H and

1let B(a) be a bi- -hyperideal of H generated by

a and R(a) be a right -hyperideal of H generated by a. Then, by hypothesis, a 2 B(a) \ R

7(a) H B (a) R (a) = H (a [a H a) (a [a H) (H a [H a a [H a H a) (a [a H) (H a) (a [a H) = H a a H [a a H) a 2 H a a H [a a H) a 2 H a a H or) a 2 a a H: If a 2 a a H, then a 2 a a

x for some x 2 H and ; 2 . a 2

8a a x a (a a x) x = a a a x x H a a H a 2 H a a

H: Hence, H is intra-regular. 2.4 Left(Right) Filters of -semihypergroups Let H be a -semihypergroup. A non-empty sub- -semihypergroup F of H

1is called a left(resp. right)…lter of H if

1) a b F for a; b 2 H and 2 =) a 2 F (resp. b 2 F). A sub- -semihypergroup F of H is called a …lter of H if F is a left and a right …lter. In this section, we give the characterizations of a left(right) …lter of H in terms of the right(left) prime -hyperideals. Theorem 32 Let H be a -semihypergroup and F a non-empty subset of H. Then the following are equivalent: 1) F is a left …lter of H. 2) H nF = ; or H nF is a prime right -hyperideal. Proof. 1) =) 2): Assume that H nF6 = '. Let x 2 H nF and y 2 H. Then x y H nF . Indeed, if x y * H nF , then x y F. Since F is a left …lter, so x 2 F. It is impossible. Thus, x y H nF , so (H nF) H F. Next, we shall prove that H nF is prime. Let x y H nF for x, y 2 H and 2 . Suppose that x 2= H nF and y 2= H nF .

5Then x 2 F and y 2 F. Since F

is a sub- - semihypergroup of H, x y F. It is impossible. Thus, x 2 H nF or y 2 H nF . Hence, H nF is prime, and so H nF is a prime right -hyperideal. 2) =) 1). If H nF = ;, then H = F. Thus, F is a left …lter of H. Next, assume that H nF is a prime

1right -hyperideal of H. Then, F is a sub- - semihypergroup of H.

Indeed, suppose that x y * F

5**for x, y 2** F and 2 . **Then, x y**

H nF for x, y 2 F and 2 . Since H nF is prime, so x or y 2 H nF . It is impossible. Thus, x y F, so F is a sub- -semihypergroup of H. Let x y F

5**for x, y 2** H. **Then x**

2 F. Indeed, if x 2= F, then x 2 H nF . Since H nF is a

1**right -hyperideal of H. Then, x y**

(H nF) H (H nF). It is impossible. Thus, x 2 F. Therefore, F is a left …lter of H. Theorem 33 Let H be a -semihypergroup and F a non-empty subset of H. The fol- lowing are equivalent: 1) F is a right …lter of H. 2) H nF = ; or H nF is a prime left -hyperideal. From above two Theorems, we have the following: Corollary 34 Let H be a -semihypergroup and F a non-empty subset of H. The fol- lowing are equivalent: 1) F is a …lter of H . 2) H nF = ; or H nF is a prime -hyperideal. Chapter 3 Quasi- -hyperideals of -semihypergroups 3.1 Introduction Recently Davvaz et al. introduced a quasi- -hyperideal of a -semihypergroup and stud- ied some basic properties of it. In this chapter we introduce the concept of a generalization of a quasi- -hyperideal in a -semihypergroup which is called (m; n)-quasi- -hyperideal. We also introduce the concept of a m-left -hyperideal and an n-right -hyperideal which is a generalization of the concept of a left -hyperideal and a right -hyperideal in a -semihypergroup. We introduce the concept of minimal (m; n)-quasi- -hyperideals, minimal m-left -hyperideals and minimal n-right -hyperideals in -semihypergroups. We give some di¤erent characterizations concerning di¤erent properties of (m; n)-quasi- - hyperideals, minimal (m; n)-quasi- -hyperideals, minimal m-left -hyperideals and mini- mal n-right -hyperideals and relations among them are investigated. Also, some intersec- tion properties and characterizations of (m; n)-quasi- -hyperideals of -semihypergroups and regular -semihypergroups have been studied. In sequel, m-left simple, n-right sim- ple and (m; n)-quasi-simple -semihypergroups are de…ned, and some properties of them are investigated. 3.2 (m; n)-quasi- -hyperideals In this section, we introduce the notion of an (m; n)-quasi- -hyperideal, n-right - hyperideal and m-left -hyperideal in -semihypergroups and relations between them are studied. Let H be a -semihypergroup and Q a non-empty subset of H . Then Q

1**is called a quasi- -hyperideal of H if** Q H \ H **Q Q.** Let **H** be **a**

-semihypergroup and Q

2**be a non-empty subset of** H . **Then** Q **is**

called an (m; n

1)-quasi- -hyperideal of H if H m Q \ Q H n Q,

where m and n are positive integers. Example 35 Let H = [0; 1] and = N . De…ne a -hyperoperation on H as: x y = [0;

5xy] for all x; y 2 H and

2 . Then, clearly H is a -semihypergroup. Let

51Q1 = [0; 0: 9], Q2 = [0; 0: 8], Q3 = [0; 0: 7], Q4 = [0; 0:

6], Q5 = [0; 0:5]. Then, clearly Qi is a (m; n)-quasi- -hyperideal for i = 1; 2; 3; 4; 5. Example 36 Let H = [-1; 0] and = f 1; 2; 3g. De…ne a -hyperoperation on H as: x y = [xy;0] for all x;y 2 H and 2 . Then, clearly H is a -semihypergroup. Let Q1 = [0:9; 0], Q2 = [0:8; 0], Q3 = [0:7; 0], Q4 = [0:6; 0], Q5 = [0:5; 0]. Then, clearly Qi is a (m; n)-quasi- -hyperideal for i = 1; 2; 3; 4; 5. Example 37 Let H = N and = f iji 2 N g. De…ne a -hyperoperation on H as: x iy = x + iN + y for all x; y 2 H and i 2 . Then, clearly H is a -semihypergroup. Let Qi = fi; i + 1; i + 2; :::g, i 2 N . Then, clearly Qi is a (m; n)-quasi- -hyperideal for i 2 N . Example 38 Let H = [1; 1] and = [0; 1]: Let x; y 2 H and 2 . Then the -hyperoperation on H is de…ned as follows: [x y; 0] if

20x < 0; y >0 or x > 0; y <0 x y = [0; x y] if x > 0; y >0 or x < 0; y <0

8 >< 0 if x = 0 or y = 0: : > Then, clearly H is a -semihypergroup. Let Q1 = [0:9; 0:9], Q2 = [0:8; 0:8], Q3 = [0:7; 0:7], Q4 = [0:6; 0:6], Q5 = [0:5; 0:5]; Q6 = [0:4; 0:4], Q7 = [0:3; 0:3]. Then, clearly Qi is a (m; n)-quasi- -hyperideal of H for i = 1; 2; 3; 4; 5; 6; 7. Example 39 Let H = fx; y; zg and = f ; g be two non-empty sets. Then, it can be easily veri…ed that H is a -semihypergroup, where the -hyperoperations de…ned by x y z x y z x fxg fx; yg fzg x fx; yg fx; yg fzg y fx; yg fx; yg fzg y fx; yg fyg fzg z fzg fzg fzg z fzg fzg fzg It can be easily shown that Q1 = fx;y;zg and Q2 = fzg are (m;n)-quasi- -hyperideals of H. It is clear that a quasi- -hyperideal Q

1of a -semihypergroup H is a (1;1)-quasi- -hyperideal of H.

Moreover, an (m;n

1)-quasi- -hyperideal of H is a (k;l)-quasi- - hyperideal of H

for all k m and l n. We know that an (m;n)-quasi- -hyperideal of a -semihypergroup H need not to be a quasi- -hyperideal of H. The following example shows the same. Example 40 Let H = N and = f iji 2 N g. De…ne a -hyperoperation on H as: x iy = x + iN + y for all x; y 2 H and i 2 . Then, clearly H is a -semihypergroups. Let Q = f2g [fk 2 N jk > 3 + sg. Then, clearly for m; n > s, Q is a (m; n)-quasi- -hyperideal but

1it is not a quasi- -hyperideal of H. If we consider the class of

quasi- -hyperideals in a -semihypergroup, we observe that it is the generalization of the class of one-sided -hyperideal in a -semihypergoup. It is clear that every one sided -hyperideal of a -semihypergroup is a

1quasi- -hyperideal of H. Lemma 41 Let H be a

-semihypergroup and Ai be a sub- -semihypergroup of H for all i 2 I. If Ai6 = ;, then Ai is a sub- -semihypergroup of H. i2I i2I Proof. Assume that Ai6 = ;. Let a; b 2 Ai. Then, a; b 2 Ai for all i 2 I. Since T T i2I i2I Ai is a sub- -semihypergTroup of H for all i 2TI, so a b Ai for all i 2 I and 2 . Hence, a b Ai for all 2 . Thus, Ai is a sub- -semihypergroup of H. i2I i2I Proposition 42 Let H be a -semihypergroup. Let Qi and Ai be an (m;n)-quasi- - T T hyperideal and sub- -semihypergroup of H, respectively, for all i 2 I. Then, Ai \ Qi is either empty or an (m; n)-quasi- -hyperideal of Ai. Proof. If Ai \ Qi is not empty, then Ai \ Qi is a subset of Ai, such that ((Ami \ Qi) Ami) \ (Ani (Ani \ Qi)) Ai Ai Ai and ((Ami \ Qi) Ami) \ (Ani (Ani \ Qi)) Qi Hm \ Hn Qi Qi. This shows that Ai \ Qi is an (m; n)-quasi- -hyperideal of Ai for all i 2 I . Proposition 43 Let H be a -semihypergroup and fQi;i 2 Ig be a set of (m;n)-quasi- -hyperideals of H . If Qi6 = ;, then Qi is an (m; n)-quasi- -hyperideal of H . i2I i2I Proof. Let Qi be an (m; n)-quasi- -hyperideal of H for i 2 I . Assume that Qi6 = ;. T T Then, for every Qj , j 2 I , we have (H m Qi)) \ ((Qi) H m) H m Qj \ TQj H n i2I (i2I i2I Qj. This shows that Qi is an (m;n)-quTasi- -hyperTideal of H. i2I Theorem 44 Let H be a -semihypergroup and Qi be an (m; n)-quasi- -hyperideal for T all i 2 I. If Qi6 = ;, then Qi is an (m;n)-quasi- -hyperideal of H. i2I i2I Proof. Assume Qi6 = ;. By Lemma 41 and Proposition 43, we have that Qi T T i2I i2I is a sub- -semihyperTgroup of H. Let a 2 Hm (Qi) \ (Qi) Hn. Then, weThave a 2 x q = p y for some x 2 Hm;y 2 Hn, for alTI i2I i2I 2 aTnd p;q 2 Qi. But since p;q 2 Qi, we have p;q 2 Qi for all i 2 I. Thus, a 2 Hm Qi \ Qi THn for all i 2 I. i2I i2I Since wTe know that Qi is an (m;n)-quasi- -hyperideal of H for all i 2 I, so we have a 2 Qi. Hence Qi is an (m;n)-quasi- -hyperideal of H. i2I i2I T T Let H be a -semihypergroup and A be a non-empty subset of H. We denote F = fQjQ is an (m; n) quasi- -hyperideal of H containing Ag. It is clear that F is not empty since H 2 F . Let (A)q(m;n) = Q. It is clear that (A)q(m;n) is non-empty Q2F since A (A)q(m;n). By Theorem 44, (A)Tq(m;n) is an (m; n)-quasi- -hyperideal of H and moreover, it is the smallest (m; n)-quasi- -hyperideal of H containing A. The (m; n)- quasi- -hyperideal (A)q(m;n) is called the (m; n)-quasi- -hyperideal of H generated by A. Theorem 45 Let H be a -semihypergroup and ;6 = A H . Then maxfm;ng (A)q(m;n) = Ai [(Hm A \ A Hn). i=1 S ! k Proof. Let k = maxfm;ng and Q = Ai [(Hm A\A Hn). It is clear that i=1 A Q. Let x; y 2 Q and 2 . Then, we hSave the following cases: Case 1.Let x; y 2 Ai. Then, x y At for some positive integer t. If t k, then k

59i=1 k x y Ai. If t k, then x y S H

m A \ A Hn. CaseS2. Let x 2 Hm A \ A Hn or y 2 Hm A \ A Hn. It can be easily seen that i=1 x y Hm A \ A Hn. Therefore, x y Q. Then, Q is a sub- -semihypergroup of H containing A. We have k k Hm Q \ Q Hn = Hm ((Ai) [(Hm A \ A Hn)) \ ((Ai) [[(Hm i[=A1\A Hn)) Hn [i=1 k k Hm ((Ai) [Hm A) \ ((Ai) [A Hn) Hn H m Ai[\=1A Hn Q: i[=1 Thus we have that Q is an (m; n)-quasi- -hyperideal of H containing A. Now, we show that Q is the smallest. Let Q0 be any (m; n)-quasi- -hyperideal of H containing A. Then, Ai Q0 for all positive integers i and Hm A \ A Hn Hm Q0 \ Q0n Q0. Therefore, Q = (Ai) [(Hm A \ A Hn) Q0. Hence, Q is the smallest k i=1 S (m; n)-quasi- -hyperideal of H containing A. Therefore, we obtain the requested result. 3.3 m-left -hyperideals and n-right -hyperideals Let H be a -semihypergroup and L be a sub- -semihypergroup of H. Then, L is called an m-left

1**-hyperideal of H if** Hm **L L**

where m is any positive integer. Dually, if R Hn R, then R is called an n-right -hyperideal of H, where n is any positive integer. In the following theorems we prove some results concerning an m-left and an n-right -hyperideal of a -semihypergroup H. Theorem 46 Let H be a -semihypergroup. Then, the following statements hold: 1. Let Li be an m-left -hyperideal of H for all i 2 I. If Li6 = ;, then Li is an i2I i2I m-left -hyperideal of H . T T 2. Let Ri be an n-right -hyperideal of H for all i 2 I. If Ri6 = ;, then Ri is an n-right -hyperideal of H . i2I i2I T T Proof. (1). Assume that Li6 = ;. Let a 2 Hm (Li). It follows that a 2 x l for i2I i2I some x 2 Hm and l 2 Li. TThen l 2 Li for all i 2 I. TThis implies that a 2 Hm Li for i2I all i 2 I. Since Li is anTm-left -hyperideal of H for all i 2 I and a 2 Li for all i 2 I, so, a 2 Li. Therefore, Li is an m-left -hyperideal of H. i2I i2I (T2). It can be provTed in the similar way with (1). Let H be a -semihypergroup and A a non-empty subset of H. We denote F = fLjL is an m left -hyperideal of H containing Ag. It is clear that F is non-empty since H 2 F. Let (A)l(m) = L.

22**It is clear that** (A)l **(m) is non-**

empty since A (A)l(m). By L2F Theorem 46(1), (A)l(m)Tis an m-left

1**-hyperideal of H and** moreover, **it is** the smallest m **-left -hyperideal of H** containing A. **The**

m-left -hyperideal (A)l(m) is called the m- left -hyperideal of H generated by A. The n-right -hyperideal (A)r(n) of H generated by A is de…ned analogously. Theorem 47 Let H be a -semihypergroup and ;6 = A H. Then, the following statements hold: 1. (A)l(m)=(Ai)[Hm A. m i=1 S 2. (A)r(n) = (Ai) [A Hn. n i=1 Proof. It is similar to the proof of Theorem 45. S Theorem 48 Let H be a -semihypergroup and L; R be an m-left -hyperideal, and n- right -hyperideal of H , respectively. Then, L \ R is an (m; n

1**)-quasi- -hyperideal of H** . Proof. **By the** properties of L **and**

R, we have Rm Ln H m L \ R H n L \ R. Then L \ R is non-empty. By Lemma 41 it follows that L \ R is a sub- -semihypergroup of H . Now we have (Hm

18**(L \ R)) \ ((L \ R)** Hn) Hm **L \ R** Hn) **L \ R:** Which proves that **L \ R**

is an (m; n)-quasi- -hyperideal of H . We say that an (m; n)-quasi- -hyperideal Q has the (m; n) intersection property if Q is the intersection of an m-left -hyperideal and an n-right -hyperideal of a -semihypergroup H . In this case every m-left -hyperideal and every n-right -hyperideal have the (m; n) intersection property. If arbitrary family of an (m; n)-quasi- -hyperideal of H has the (m; n) intersection property, then H is said to have intersection property of an (m; n)-quasi- -hyperideal. The following theorem characterizes (m; n)-quasi- - hyperideals having the (m; n) intersection property. Theorem 49 Let H be a -semihypergroup and Q be an (m;n)-quasi- -hyperideal of H.

46**Then, the following statements are equivalent: 1.** Q has **the**

(m; n) intersection property.

23**2. (Q[** Hm **Q)\(Q[Q** Hn **)=Q.** 3. Hm **Q\(Q[Q** Hn **)=Q.** 4. **Q** Hn **\(Q[Q** Hn) **Q.**

Proof. (1)) (2). Suppose that Q has the (m; n) intersection property. It is obvious that Q (Q [Hm Q) \ (Q [Q Hn) (*). Since Q has the (m; n) intersection property, so there exist an m-left -hyperideal L and an n-right -hyperideal R of H in such a way that Q = L \ R. Thus, Q L and Q R. Also, we have that Hm Q Hm L L and in the similar way Q Hn R Hn R which implies that Q [Hm Q L and Q [Q Hn R. Hence

1**we have** that **(Q** [Hm **Q) \ (Q [Q**

Hn) L \ R Q (**). Thus, by (*) and (**), we have (Q [Hm Q) \

23**(Q [Q** Hn) = **Q. (2)**) (1). Let **(Q** [Hm **Q) \ (Q [Q** Hn) = **Q.**

Then, we have to show that Q [Hm Q is an m-left -hyperideal of H and Q [Q Hn is an n-right -hyperideal of H. Let L = Q [Hm Q and R = Q [Q Hn. We show …rst that L is a sub- - semihypergroup of H. Let a; b 2 L. Then, we have the following cases: Case 1. If a;b 2 Q, then since Q is a sub- -semihypergroup of H, a b Q L for all 2 . 2 . Therefore, L is a sub- -semihypergroup of H. We have Case 4. If a 2 Hm Q and b 2 Hm Q, then a b Hm Q Hm Hm Case 2. If a 2 Q and b 2 Hm Q, then a b Q Hm Q Hm Q L for all Case 3. If a 2 Hm Q and b 2 Q, then a b Hm Q Q Hm Q L for all Q L for all 2 . 2 .

27**H m** L = **H m** (Q [**H m** Q) = **H m** Q [**H** 2m Q **H m**

Q L: Hence, L is an m-left -hyperideal of H. In the similar way, R is an n-right

1**-hyperideal of H.** Since, Hm **Q \ Q** Hm **Q;**

so L\R=(Q[Hm

11**Q)\(Q[Q** Hn **)=Q[(** Hm **Q\Q** Hn **)=Q:** Therefore, **Q** = L \ **R.** (2)) (3). Let **(Q** [Hm **Q)** **\ (Q [Q** Hn) = **Q.**

Then, since Hm Q Q [Hn Q, so we have Hm

23Q\(Q[Q Hn) (Q[Hm Q)\(Q[Q Hn)=Q: (3)) (2). Let Hm Q \ (Q [Q Hn) = Q. Then, Q (Q [Hm Q)

\ (Q [Q Hn) (*). Let x 2 (Q [Hm Q) \ (Q [Q Hn). Then, since Hm Q \ (Q [Q Hn) Q, so we have x 2 Q, so (Q [Hm Q) \ (Q [Q Hn) Q (**). By (*) and (**) we have the requested result. The proofs for (2)) (4) and (4)) (2) are similar to the proofs of (2)) (3) and (3)) (2), respectively. Proposition 50 Let H be a - semihypergroup and Q be an (m; n

1)-quasi- -hyperideal of H . If H m Q Q H n or Q H m H n Q, then Q

has the (m; n) intersection property. Proof. Let Hm Q Q Hn. Then, Hm Q = Hm Q \ Q Hn Q which shows that Q is an m-left -hyperideal of H. Thus, Q has the (m;n) intersection property. In the similar way if we assume Q Hm Hn Q, then Q has an n-right -hyperideal of H. In this case also Q has the (m;n) intersection property. The following theorem is dealing with the intersection property of (m;n)-quasi- - hyperideal of a regular -semihypergroup. Theorem 51 Every regular -semihypergroup H has the intersection property of (m; n)- quasi- -hyperideals for any m; n 2 N . Proof. Let Q be an (m; n)-quasi- -hyperideal of a regular - semihypergroup H. Then, it can be easily shown that Q Q Hn. Thus Q [Q Hn = Q Hn. Therefore, Hm Q \ (Q [Q Hn) = Hm Q \ Q Hn Q. By the Theorem 48 it follows that Q has the intersection property. Let H be a -semihypergroup and A be a non-empty subset of H. Then A is called an (m; n)- -hyperideal if Am H An A. A (m; 0)- -hyperideal of H is Am H A and (0; n)- -hyperideal is H An A. Proposition 52 Let H be a regular - semihypergroup. Then, a non-empty subset Q of H is an (m; n

1)-quasi- -hyperideal of H if and only if it is the intersection of a

(m; 0)-right -hyperideal and a (0; n

1)-left -hyperideal of H . Proof. Let ;6 = Q H be

an (m; n)-quasi- -hyperideal of H, i.e. Qm H \ H Qn Q, which is possible only when Q is the intersection of (m; 0)- -hyperideal and (0; n)- - hyperideal of H which is obvious as Qm H \ H Qn Q. Conversely, suppose that Qm H \ H Qn Q. Since Qm H Q and H Qn Q, it follows that Qm H H Qn = Qm H Qn Q. Hence Q is an (m;n)- -hyperideal of H. Theorem 53 The collection L of all (m; n)-quasi- -hyperideals with non-empty inter- section of a -semihypergroup H is a complete hyperlattice. Proof. It is clear that L is partially ordered by inclusion. By the Theorem 44, the in…mum of any collection of (m; n)-quasi- -hyperideals Qi(i 2 I) is obvious the Qi. i2I Similarly, we set T Qi = Qi \ Qi , i2I i2I i2I where W S R Qi = Qi [H m S L (Qi), Qi = Qi [(Qi) H n. By Theo- i2I R i2I i2I i2I L i2I i2I rem 48 and TSheorem 49,Sthis is obviouSsly a (m;Sn)-quasi- - ShyperideSal which bounds from above all the (m;n)-quasi- -hyperideals Qi(i 2 I). It is the supremum of L. Indeed, for any (m;n)-quasi- -hyperideal Q containing all the Qi(i 2 I), we have Qi = _i2I = "i2I Qi [Hm (Qi) # \ "i2I Qi [(Qi) Hn # = [i[2I [[i2I Qi [" Hm (Qi) \ (Qi) Hn [Hi2Im Q \ Q H ni[2I i[2I # Q: Hence, proved. 3.4

Minimal (m; n)-quasi- -hyperideals In this section, we obtain di¤erent characterizations concerning di ¤erent properties of (m; n)-quasi- -hyperideals, minimal (m; n)-quasi- -hyperideals, minimal m-left -hyperideals, minimal n-right -hyperideals and relations of these concept are investigated. Also, some intersection properties and characterizations of (m; n)-quasi- -hyperideals of - semihypergroups and regular -semihypergroups have been studied. In sequel, m-left simple, n-right simple and (m; n)-quasi-simple -semihypergroups are de…ned, and some properties of them are investigated. Let H be a -semihypergroup and L be an m-left -hyperideal of H . Then L is called a minimal m-left -hyperideal of H if L does not properly contain any m

1**-left -hyperideal of H . Let** H **be**

a -semihypergroup and R be an n-right -hyperideal of H . Then R

1**is called a** minimal n **-right -hyperideal of H if**

R does not properly contain any n-right -hyperideal of H . Example 54 Let H = fa; b; cg and = f ; g. Then, -hyperoperations are de…ned by the following Cayley's tables

1**a b c a b c a**

fbg fb; cg fb; cg a fcg fb; cg fb; cg b fb; cg fb; cg fb; cg b fb; cg fbg fb; cg c fb; cg fb; cg fcg c fb; cg fb; cg fb; cg Then, H is a -semihypergroup. Let I = fb; cg be a m-left

1**-hyperideal of H and** also I **is a** minimal n **-right -hyperideal of H:**

Let H be a -semihypergroup and Q be an (m; n

1**)-quasi- -hyperideal of H.** Then Q **is** called **a**

minimal (m; n

1**)-quasi- -hyperideal of H if Q**

does not properly contain any (m; n

1**)-quasi- -hyperideal of H.** Example 55 **Let H be a**

-semihypergroup in Example 39. Then, Q = fzg is a minimal (m; n)-quasi- -hyperideal Lemma 56 Let H be a -semihypergroup and a 2 H. Then, the following statements hold true: 1. H m a is an m-left -hyperideal of H . 2. a H n is an n

1-right -hyperideal of H . 3. Hm a \ a

Hn is an (m; n

1)-quasi- -hyperideal of H. Proof. (1). We have that

(Hm a) (Hm a) Hm a and Hm (Hm a) Hm a. Hence, (1) holds true. (2). It is similar to (1). (3). It follows by (1), (2) and Theorem 48 Theorem 57 Let H be a -semihypergroup and Q be an (m; n)-quasi- -hyperideal of H. Then, Q is minimal if and only if Q is the intersection of some minimal m-left -hyperideal L and some minimal n-right -hyperideal R of H. Proof. Assume that Q is a minimal (m;n

1)-quasi- -hyperideal of H. Let a

2 Q. Then, by Lemma 56, it follows that H m a; a H n and H m a \ a H n are an m-left - hyperideal, an n-right -hyperideal and an (m; n

1)-quasi- -hyperideal of H, respectively. By the

minimality of Q, since Hm a \ a Hn Hm Q \ Q Hn Q, we have Hm a \ a Hn = Q. We have to show the minimality of the m-left -hyperideal Hm a and the minimality of the n-right -hyperideal a Hn. Let L be an m-left -hyperideal of H contained in Hm a. Then, we have L \ a Hn Hm a \ a Hn = Q. By the minimality of Q, since L \ a Hn is an (m; n

1)-quasi- -hyperideal of H, it follows that L \ a

Hn = Q. Then, Q L. Therefore, Hm a Hm Q Hm L L. Therefore, Hm a Hm Q Hm L L. This implies L = Hm a. Thus, the m-left -hyperideal Hm a is minimal. In similar way, dually it can be proved the minimality of the n-right -hyperideal a Hn. Conversely, assume that Q = L \ R for some minimal m-left -hyperideal L and some minimal n-right -hyperideal R of H. Let Q0 be an (m;n)-quasi- -hyperideal of H contained in Q. Then we have H m Q0m Q H m L L and Q0n Q H n R H n R. It can be easily proved that Hm Q0 and Q0n is an m-left -hyperideal and an n

1-right -hyperideal of H, respectively. The minimality of

L and R implies Hm Q0 = L and Q0n = R. Hence, Q = L \ R = Hm Q0 \ Q0n Q0. Then, Q = Q0. Therefore, Q is a minimal (m; n)-quasi- -hyperideal of H. The following propositions give necessary and su¢ cient conditions for the existence of a minimal (m; n)-quasi- -hyperideal of a -semihypergroup. An immediate corollary of the Theorem 57 is the following. Corollary 58 Let H be a -semihypergroup. Then H has at least one minimal (m; n)- quasi- -hyperideal if and only if H has at least one minimal m-left -hyperideal and at least one minimal n-right -hyperideal. Theorem 59 Let H be a -semihypergroup. The following statements hold: 1. An m-left -hyperideal L is minimal if and only if Hm a = L for all a 2 L. 2. An n-right -hyperideal R is minimal if and only if a Hn = R for all a 2 R. 3. An (m; n)-quasi- -hyperideal Q is minimal if and only if Hm a \ a Hn = Q for all a 2 Q. Proof. (1). Assume that L is minimal. Let a 2 L. Then Hm a Hm L L. By Lemma 56(1) it follows that Hm a is an m-left -hyperideal of H. Since L is a minimal m-left -hyperideal of H, we have Hm a = L. Conversely, assume that Hm a = L for all a 2 L. Let L0 be an m

1**-left -hyperideal of H** contained in L. **Let x**

2 L0 L. Then Hm x = L. We have: L = Hm x Hm L0 L0. This implies that L = L0. Therefore L is minimal. (2) and (3) can be proved similar as (1). Let H be a -semihypergroup. Then, H is called m-left simple -semihypergroup if H is a unique m

1**-left -hyperideal of H. Let** H **be**

a -semihypergroup. Then, H is called n-right simple -semihypergroup if H is a unique n-right -hyperideal of H. Let H be a -semihypergroup. H is called (m; n)-quasi-simple -semihypergroup if H is a unique (m; n

1**)-quasi- -hyperideal of H.** Theorem 60 **Let H be a** -semihypergroup. **The following** statements hold true: **1. H is**

an m-left simple -semihypergroup if and only if Hm a = H for all a 2 H. 2. H is an n-right simple -semihypergroup if and only if a Hn = H for all a 2 H. 3. H is an (m; n)-quasi-simple -semihypergroup if and only if Hm a \ a Hn = H for all a 2 H. Proof. (1). Since H is an m-left simple -semihypergroup, we have that H is a minimal m-left -hyperideal of H. By the Theorem 59(1), Hm a = H for all a 2 H. Conversely, assume that Hm a = H for all a 2 H. By the Theorem 59(1), H is a minimal m-left

1**-hyperideal of H and** therefore, H **is** an m **-left**

simple -semihypergroup. (2) and (3) can be proved similarly to (1). Theorem 61 Let H be a -semihypergroup. The following statements hold: 1. If an m-left -hyperideal L of H is an m-left simple -semihypergroup, then L is a minimal m-left -hyperideal of H. 2. If an n-right -hyperideal R of H is an n-right simple -semihypergroup, then R is a minimal n-right -hyperideal of H. 3. If an (m;n

1**)-quasi- -hyperideal** Q **of H is an**

(m;n)-quasi-simple -semihypergroup, then Q is a minimal (m;n

1**)-quasi- -hyperideal of H. Proof.** (1). **Let** L **be** an m-left simple -semihypergroup. Then, by the Theorem 60(1), we have Lm a = L for all a 2 L. For every a 2 L, we have L = Lm a Hm a H m L L. Then H m a = L for all a 2 L. By the Theorem 59(1), we have L is minimal. (2) and (3) can be proved similarly to (1). Chapter 4 Intuitionistic Fuzzy Sets in -semihypergroups 4.1 Introduction In this chapter, by using Atanassov idea, we continue the study of intuitionistic fuzzy sets in -semihypergroups which was initiated recently by Ersoy and Davvaz [28]. We de…ne intuitionistic fuzzy bi- -hyperideals, intuitionistic fuzzy interior -hyperideals and intuitionistic fuzzy prime(semiprime) -hyperideals in a -semihypergroup. We give some further properties of intuitionistic fuzzy -hyperideals and intuitionistic fuzzy bi- - hyperideals in a -semihypergroup. We use the intuitionistic fuzzy left, right, two-sided and bi- -hyperideals to characterize some classes of -semihypergroups. We introduce and study (;)-intuitionistic fuzzy -hyperideals. We also

2**introduce the** notion **of** an **intuitionistic fuzzy** M **(resp.**

N)-hypersystem of a -semihypergroup and some properties of them are investigated. 4.2 Intuitionistic Fuzzy -hyperideals of -semihypergroup De…nition 62 [28] Let H be a -semihypergroup. An intuitionistic fuzzy set A = h A; Ai in H is called an intuitionistic fuzzy sub- -semihypergroup in H if for all x; y 2 H; 2 , inf f A(z)g minf A(x); A(y)g and sup f A(z)g maxf A(x); A(y)g. z2x y z2x y De…nition 63 [28]Let H be a -semihypergroup. An intuitionistic fuzzy set A = h A; Ai in H is called an intuitionistic

1**fuzzy left (resp. right) -hyperideal** of H **if for** all **x; y** 2 **H;**

2 , 1. A(y) z2inxfyf A(z)g (resp. A(x) z2inxfyf A(z)g): 2. sup f A(y)g A(y) (resp. sup f A(z)g A(x)). z2x y z2x y

3**An intuitionistic fuzzy set A** in H **is** called **an intuitionistic fuzzy**

2**two-sided** -hyperideal **of** H **if it is both** an intuitionistic fuzzy **left and** an intuitionistic **right** -hyperideal **of**

H. Similarly, we can de…ne an intuitionistic

1**fuzzy left (resp. right) -hyperideal of H** as follows: **A** (z) A **(y)**

(resp. A (z) A(x)) for each z 2 x y. De…nition 64 Let H be a -semihypergroup. An intuitionistic fuzzy sub- -semihypergroup A = h A; Ai in H

2**is called an intuitionistic fuzzy** bi- -hyperideal **of** H **if**

1**for all x; y; z** 2 **H;**

; 2 , t2x y z inf f A(t)g minf A(x); A(z)g and sup f A(t)g maxf A(x); A(z)g. t2x y z Similarly, we can de…ne an intuitionistic fuzzy bi- -hyperideal of H as follows:

20**A (z) A(x)** \ A **(y)** for each **z** 2 **x y.** A (t) A **(x)** \ A **(z)**

for each t 2 x y z. De…nition 65 Let H be a -semihypergroup. An intuitionistic fuzzy sub- -semihypergroup A = h A; Ai of H is called an intuitionistic fuzzy (1; 2)- -hyperideal of H if 1. a2x w (y z) inf f A(a)g minf A(x); A(y); A(z)g, 2. sup f A(a)g maxf A(x); A(y); A(z)g, a2x w

1**(y z) for all** w; **x; y; z** 2 **H**

and ; ; 2 . Example 66 Let H = f1; 2; 3; 4; 5g and = f ; g be two non-empty sets. Then, H is a -semihypergroup and is de…ned by the following Cayley tables. 1 2 3 4 5 1 2 3 4 5 1 f1g f1g f1g f1g f1g 1 f1g f1g f1g f1g f1g 2 f1g f1g f1g f1g f1g 2 f1g f1g f1g f1g f1g 3 f1g f1g f3g f3g f3g 3 f1g f1g f3g f3g f3g 4 f1g f1g f3; 4g f3; 4g f5g 4 f1g f1g f3g f3; 4g f5g 5 f1g f1g f3; 4g f3; 4g f5g 5 f1g f1g f3g f3; 4g f5g 1) Let A = h A; Ai be an IFS in a -semihypergroup H and is de…ned by A (1) = 0:7, A (2) = 0:5; A (3) = A (4) = A (5) = 0:3; and A (1) = 0:2, A (2) = 0:4; A (3) = A (4) = A (5) = 0:6. Then, A = h A; Ai is an intuitionistic fuzzy -hyperideal of H: 2) Let A = h A; Ai be an IFS in a -semihypergroup H and is de…ned by A (1) = 0:85, A (2) = 0:7; A (3) = 0:5; A (4) = A (5) = 0:3; and A (1) = 0:1, A (2) = 0:25; A (3) = 0:45; A (4) = A (5) = 0:7. Then, A = h A; Ai is an intuitionistic

1**fuzzy bi- -hyperideal of H: 3)** Let **A** = h **A;**

Ai be an IFS in a -semihypergroup H and is de…ned by A (1) = 0:9, A (2) = 0:4; A (3) = 0:7; A (4) = 0:6; A (5) = 0:3; and A (1) = 0:1, A (2) = 0:5; A (3) = 0:2; A (4) = 0:3 A (5) = 0:6. Then, A = h A; Ai is an intuitionistic fuzzy (1,2)- -hyperideal of H: For any t 2 [0; 1] and an intuitionistic fuzzy set A = h A; Ai of H, the sets [14] U(A;t) = fx 2 Hj A(x) tg and L(A;t) = fx 2 Hj A(x) tg.

3**are called** respectively **an upper and lower** t **-level cut of**

A = h A; Ai. Some interesting results concerned with intuitionistic fuzzy -hyperideals of -semihypergroups are obtained in [28]. We continue this study with further results. Proposition 67 Let H be a -semihypergroup. Then a non-empty subset B of H is a bi- -hyperideal of H if and only if the intuitionistic fuzzy characteristic function B = h B ; cB

2**i is an intuitionistic fuzzy** bi- -hyperideal **of**

H . e Proposition 68 Let A = h A; Ai be an intuitionistic fuzzy bi- -hyperideal and B = h A; Ai be an intuitionistic fuzzy sub- -semihypergroup of a -semihypergroup H. Then, A \

1**B is an** intuitionistic fuzzy **bi- -hyperideal of H.**

Proof. The proof is straightforward and we omit it. Proposition 69 Let A = h A; Ai be an intuitionistic fuzzy right -hyperideal of H and B = h B; Bi be an intuitionistic

1**fuzzy left -hyperideal of H. Then,**

A B A \ B. Proof. Let A = h A; Ai be an intuitionistic fuzzy right -hyperideal of H and B = h B; Bi be an intuitionistic fuzzy

1**left -hyperideal of H, let x**

2 H. Let us suppose that there exist u; v 2 H and 2 such that x 2 u v. Then, (A B)(x) = x2u v sup minf A(u); B(v)g x2u v sup minf inf A(x); inf B(x)gg x2u v x2u v = minf A(x);

5**B(x)** g = (**A \ B)(** x) **and (A B)(x)** = inf maxf **A(** u); **B(**

v)g x2u v inf maxf sup A(x); sup B(x)gg x2u v x2u v x2u v = maxf A(x); B(x)g = (A [B)(x): Let us suppose there do not exist u; v 2 H such that x 2 u v. Then (A

5**B)(x)** = 0 (**A \ B)(** x) **and (A B)(x)** = 1 (**A [B)(** x). Hence, **the**

proof is completed. By the above proposition and the de…nition of A\B the following proposition follows. Proposition 70 Let H be a -semihypergroup and A = h A; Ai; B = h B; Bi be intu- itionistic fuzzy -hyperideals of H.

4**Then, A B A \ B** A; **B.** Proposition 71 **Let**

H be a regular -semihypergroup and A = h A; Ai and B = h B; Bi be two intuitionistic fuzzy sets in H. Then, A B A \ B. Proof. Let c 2 H. Since H is regular, so there exists an element x 2 H and 1; 2 2 such that c 2 c 1x 2c c c for some 2 . Then (A B)(c) = sup fminf A

4**(c); B(c)** gg c2c **c** minf **A(** c); **B(c)** g = (**A \ B)(c)** and **(A B)(c)** = c2c **c** inf fmaxf **A(c); B(c)** gg maxf **A(** c); **B(c)** g = (**A** [B **)(c):** Hence, A **B A \ B.**

Theorem 72

1**Let H be a** -semihypergroup. **Then, the following** statements **are** equiva- lent: **1. H is** regular. **2.** A **B = A \ B,** where **A**

= h A; Ai is an intuitionistic fuzzy right -hyperideal of H and B = h B; Bi is an intuitionistic fuzzy

1**left -hyperideal of H. Proof. Let H be a**

regular -semihypergroup. Then, by [28, Theorem 3.13], A B A \ B and by [28, Theorem 3.12],

9**A B A \ B.** Hence, **A B = A \ B.**

Conversely, let H be a -semihypergroup and A B = A \ B, where A = h A; Ai be an intuitionistic fuzzy right - hyperideal of H and B = h B; Bi be an intuition- istic

1**fuzzy left -hyperideal of H.** Let **L** and **R** be a left **-hyperideal and**

a right

1**-hyperideal of H; respectively.** Let x 2 **R \ L.**

Then, x 2 R and x 2 L. Hence, (L(x); cL(x)) = (R(x); cR(x)) = (1; 0) (where L(x) and R(x) are respectively the characteristic functions of L and R). Thus, (R \ L)(x) = minf R(x); L(x)g = 1 and (cR [cL)(x) = maxf cR(x); cL(x)g = 0. By [28, Theorem 3.11], (L; cL) and (R; cR) are respectively an intuitionistic fuzzy left - hyperideal and an intuitionistic fuzzy right -hyperideal of H. Hence, by hypoth- esis, R L = R \ L and cR cL = cR [cL. Hence, (R L)(x) = 1 since sup [minf R(y); L(z)g : y; z 2 H; 2] = 1, and x2y z (Rc cL)(x) = 0 since xi2nyf z[maxf cR(y); cL(z)g : y; z 2 H; 2] = 0. This implies that there exist some r; s 2 H and 1 2 such that x 2 r 1s and (R(r); cR(r)) = (1; 0) = (L(s); cL(s)). Hence, r 2 R and s 2 L. Hence, x 2

18**R L.** Thus, **R \ L R L.** Also, **R L R \ L.** Hence, **R L = R \ L.** Consequently, the - semihypergroup **H**

is regular. Theorem 73 Let H be a -semihypergroup. Then, every intuitionistic fuzzy

1**bi- -hyperideal of H is an**

intuitionistic fuzzy (1; 2)- -hyperideal

1**of H. Proof. Let** A = **h A;**

Ai be an intuitionistic

1**fuzzy bi- -hyperideal of H and** let w; **x;**

y; z 2 H and ; ; 2 . Then, for all a 2 x w (y z), we have a2x w (y z) inf f A(a)g = = a2(x w y) z inf f A(a)g ai2ncfzf A(a)g for every c 2 x w y minf A(c); A(z)g minfminf A(x); A(y)g; A(z)g (because inf f A(c)g minf A(x); A(y)g) c2x w y = minf A(x); A(y); A(z)g and sup f A(a)g = a2x w (y z) = sup f A(a)g a2(x w y) z a2c z sup f A(a)g for every c 2 x w y maxf A(c); A(z)g maxfmaxf A(x); A(y)g; A(z)g (because sup f A(c)g maxf A(x); A(y)g) c2x w y = maxf A(x); A(y); A(z)g: Hence, A = h A; Ai is an intuitionistic fuzzy (1;2)- -hyperideal of H. Theorem 74 Let H be a regular -semihypergroup. Then, every intuitionistic fuzzy (1;2)-

1**-hyperideal of H is** an intuitionistic fuzzy **bi- -hyperideal of H.**

Proof. Let us assume that a -semihypergroup H is regular and let A = h A; Ai be an intuitionistic fuzzy (1; 2)- -hyperideal of H. Let w; x; y 2 H and ; ; 2 . Then, since H is regular, so we have for every w 2

55**x y (x** a **x) y = x** (a **x)** y **for** some **a**

2 H. Thus, for every c 2 a x;w 2 x c y, we have w2x c y x (a x) y inf f A(w)g minf A

5**(x); A(x); A(y)** g = minf **A(**

x); A(y)g and sup f A(w)g maxf A(x); A(x); A(y)g w2x c y

25**x (a** x) **y** = maxf **A(x); A(y)g:** Hence, A = **h A;**

Ai is an intuitionistic fuzzy bi- -hyperideal of H. Theorem 75 Let H be a regular -semihypergroup. Then, the following statements hold: 1. A \ B A B for every intuitionistic fuzzy bi- -hyperideal A and intuitionistic fuzzy left -hyperideal B of H. 2. C \ A \ B C A B, for every intuitionistic fuzzy bi- -hyperideal A , intuition- istic fuzzy left -hyperideal B and intuitionistic fuzzy right -hyperideal C of H; respectively. Proof. (1). Let us suppose that H is regular and a 2 H. Then, there exists x 2 H; and ; 2 such that a 2 a x a. Then, (A B)(a)6 = (0; 1). Thus, A B(a) = a2(saupx a)minf A(a);t2inxfa B(t)g = minf A(a); inf B(t)g t2x a minf A(a);

9**B(a)** g **A\B(a)** and **A B(a)** = inf maxf **A(a);** sup **B(**

t)g a2(a x a) t2x a maxf A(a); sup B(t)g t2x a maxf A(a);

9**B(a)** g = **A\B (a):** Hence, **A \ B A B.**

(2). Let us suppose that H is regular and a 2 H. Then there exists x 2 H; and ; 2 such that a 2 a x a. Then

4**C A B (a)** = a2(saupx **a)** minft2inafx **C(** t); **A B(a)** g minf inf **C(** t); **A**

B(a)g t2a x minf C(a); sup minf A(a); inf a2a x a h2x a B (h)gg minf C(a); A(a); inf B(h)g h2x a

21**C\A\B(a)** and **C A B (a)** = a2(a **x a)** inf maxf sup **C** (t); **A** B **(a)**

g t2a x maxf sup C (t); A B (a)g t2a x maxf C(a); inf a2a x a A(a); sup B (h)gg h2x a maxf C(a); A(a); sup B(h)g h2x a

2**C\A\B(a)** Hence, **C \ A \ B C A B.**

Theorem 76 Let H be a completely regular -semihypergroup and A = h A; Ai be an intuitionistic fuzzy bi- -hyperideal of H. Then, for every r 2 a a, we have A(a) = A(r) for all a 2 H; 2 . Proof. Let a 2 H; 2 . Then, there exist x 2 H; ; ; ; 2 such that a 2 a a x a a. Hence, for every r 2 a a; a 2 r x r. Hence, inf f A(a)g a2r x r = = and minf

35**A(r); A(r)** g **A(** r) minf **A(a); A(a)** g because **r**

2 a a; 8 2 A(a) sup f A(a)g maxf A(r); A(r)g a2r x

35**r = A(r)** maxf **A(a); A(a)** g because **r**

2 a a; 8 2 = A(a): It follows that A(a) = A(r) and A(a) = A(r), so that A(a) = A(r) for all a 2 H. Theorem 77 Let H be an intra-regular -semihypergroup and A = h A; Ai be an intu- itionistic fuzzy -hyperideal of H. Then, for every r 2 a a, we have A(a) = A(r) for all a 2 H; 2 . Proof. Let a 2 H; 2 . Then, since H is

2**intra-regular,** so **there exist x; y**

2 H; ; ; 2 such that a 2 x a a y. Then, for every r 2 a a and s 2 x r we have a 2 s y. Hence, since A = h A; Ai is an intuitionistic fuzzy -hyperideal of H, so inf f A(a)g A(s) a2s y and inf f A(s)g s2x r and A(r) inf f A(r)g r2a a A (a). Hence A(a) = A(r). Also sup f A(a)g A(s) a2s y and sup f A(s)g s2x r and A(r) sup f A(r)g A(a). r2a a Hence, A(a) = A(r). Thus, A(a) = A(r) for all a 2 H. Theorem 78

1**Let H be a** -semihypergroup. **Then, the following** statements **are** equiva- lent: **1. H**

is left quasi-regular **2. A A**

= A, where A = h A; Ai is an intuitionistic fuzzy

1**left -hyperideal of H. Proof. Let H be a**

left quasi-regular -semihypergroup and A = h A; Ai be an intuitionistic

1**fuzzy left -hyperideal of H. Then,**

it is clear that A A A. Let a 2 H. Then, by de…nition of left quasi-regular,

6**there exist x; y** 2 H; ; ; 2 **such that** a 2 **x** a **y**

a. We have A A(a) = = sup a2(x a y a) minft2inxfa A(t); s2inyfa A(s)g minf inf A(t); inf A(s)g t2x a s2y a minf A(a); A(a)g A(a) A A(a) = = a2(x a y a) inf maxf sup A(t); sup A(s)g t2x a s2y a maxf sup A(t); sup A(s)g t2x a s2y a maxf A(a); A(a)g A (a) This implies that A A A. Therefore, we have A A = A. Conversely,letussupposethatthecondition(2)holdsandletT bealeft -hyperideal of H. Then (T; cT) is an intuitionistic fuzzy left -hyperideal of H. Indeed, (T; cT) is an intuitionistic fuzzy left -hyperideal of H

1**if and only if for any x; y** 2 **H;**

2 , inf T (t) T (y) and sup t2x y s2x y cT(s) cT

39**(y) if and only if** for every **x** 2 H **;y** 2 T; **2**

, x y T if and only if L is a left -hyperideal of H. Let a 2 T. Then, by (2), we have (T;

64**cT) (T; cT) = (T; cT).** Thus, [(**T;**

cT) (T; cT)](a) = T(a) = 1 and [(T ; cT) (T ; cT)](a) = T (a) = 0. So, [(T ; cT) (T ; cT)](a)6 = (0; 1). This implies that sup minf T (x); T (y)g = 1 a2x y and a2inxf y minf cT (x); cT (y)g = 0. Thus, there exist b; d 2 H; 2 with a 2 b d such that T (b) = 1; cT (b) = 0 and T (d) = 1; cT (d) = 0. So we have b 2 T , that is a 2 b

11**d T T** . Then, **T T T** and since clearly **T T T** , we have **T = T T**

. Now, we prove that this fact implies H is left quasi-regular. We have T L(T) = L(T) L(T) =

18(T [H T) (T [H T) = T T [H T T [T H T [H T H T (T H T [H T H T) [(H T H T) [(T H T [H T H T) T H T [H T H T H T H T ; where L(T) is

the left -hyperideal generated by T . This shows that H is left quasi- regular. The following theorem characterizes the -semihypergroups that are intra-regular and left quasi-regular through intuitionistic fuzzy sets. Theorem 79 Let H be a -semihypergroup. If H is both intra-regular and left quasi- regular, then

2B \ C \ A B C A, for every A

intuitionistic fuzzy bi- -hyperideal, B intuitionistic fuzzy left -hyperideal and C intuitionistic fuzzy right -hyperideal of H , respectively. Proof. Let us suppose that H is intra-regular and left quasi-regular. Let A be an intuitionistic fuzzy bi- -hyperideal, B an intuitionistic fuzzy left -hyperideal and C be an intuitionistic fuzzy right -hyperideal of H; respectively. Let a 2 H. Then, since H is left quasi-regular, so there exist u; v 2 H; ; ; 2 such that a 2 u a v a. Then, for some x; y 2 H; ; ; ! 2 we have a 2 u a

3v a u (x a a !y) v a ((u x) a) ((a !(y

v) a). Thus, we have B C A(a) = = sup minf B(p); C A(q)g a2(p q) minf inf t2((u x) a) B(t); s2((a!(y v) a) inf C A(s)g minf B(a); sup [C(p); A(q)]g ((a!(y v) a) p q minf B(a); inf C(h); A(a)g h2(a!(y v)) minf

28B(a); C(a); A(a) g B \C\A(a) B C A(a)

= = a2(p q) inf maxf B(p); C A(q)g maxf sup B(t); sup C A(s)g t2((u x) a) s2((a!(y v) a) maxf

16B(a); ((a!(y v) a) p q inf [C (p); A(

q)]g maxf B(a); sup C(h); A(a)g h2(a!(y v)) maxf B(a); C(a); A(a)g

28B\C\A(a): Hence, C \ B \ A C B A.

Theorem 80 Let H1 be a 1-semihypergroup and H2 be a 2-semihypergroup. Let (;f) be a homomorphism from H1 to H2. If B = h B; Bi is an intuitionistic fuzzy bi- - hyperideal of H2, then 1(B) is an intuitionistic fuzzy bi- -hyperideal of H1. Proof. Let us suppose that B = h B; Bi is an intuitionistic fuzzy bi- -hyperideal of H2 and let x;y 2 H1; 2 1. Then, we have inf f 1(B)(z)g = inf f B ((z)g z2x y z2x y (z)2 (x y) inf f B(

32(z))g (z)2 (x) f( ) (y) inf f B((z))

g minf B ((x)); B ((y))g = minf 1(B (x)); 1(B (y)g; sup f 1(B)(z)g = sup f B((z)g z2x y z2x y sup f B((z))g (z)2 (x y) sup f B (

32(z))g (z)2 (x) f( ) (y) maxf B((x)); B((y))

g = maxf 1(B (x)); 1(B (y)g: Therefore, 1(B) is an intuitionistic fuzzy sub- -semihypergroup of H. For any a; x; y 2 H; ; 2 ; we have z2x a y inf f 1(B)(z)g = inf f B((z)g z2x a y (z)2 (x a y) inf f B((z))g (z)2 (x)f( ) (a)f( ) (y) inf f B((z))g minf B ((x)); B ((y))g = minf 1(B (x)); 1(B (y)g; and sup f 1(B)(z)g = sup f B ((z)g z2x a y z2x a y sup f B((z))g

29(z)2 (x a y) sup f B ((z))

g

29(z)2 (x) f ( ) (a) f ( ) (y) maxf B

((x)); B ((y))g = maxf 1(B (x)); 1(B (y)g; Therefore, 1(B) is

2an intuitionistic fuzzy bi- -hyperideal of H1. 4 .3

-semihypergroups Characterized by their Intu- itionistic Fuzzy Prime (Semiprime) -hyperideals In this section, we de…ne intuitionistic fuzzy prime(semiprime) -hyperideals, intuition- istic fuzzy M-hypersystem and N-hypersystem of a -semihypergroup and intuitionistic fuzzy semisimple -semihypergroups and some properties of them are investigated. Let A = h A; Ai be an intuitionistic fuzzy -hyperideal in H. Then, A = h A; Ai

2is called an intuitionistic fuzzy prime -hyperideal of H if

for all intuitionistic fuzzy - hyperideals B = h B; Bi and C = (C; C) of H, such that

2B C A this implies B A or C A: Let A = h A;

Ai be an intuitionistic fuzzy -hyperideal in H: Then, A = h A; Ai

2is called an intuitionistic fuzzy semiprime -hyperideal of H if

for all intuitionistic fuzzy -hyperideals B = h B; Bi of H, such that B2 A this implies B A. An intuitionistic fuzzy -hyperideal A = h A; Ai of H is called idempotent if A2 = A and a -semihypergroup is called intuitionistic fuzzy semisimple if all of its intuitionistic fuzzy -hyperideals are idempotent. Proposition 81 If A = h A; Ai is an intuitionistic fuzzy left -hyperideal and B = h B; Bi is an intuitionistic fuzzy

1right -hyperideal of a -semihypergroup H, then A

1B is an intuitionistic fuzzy -hyperideal of H.

3Proof. Let x; y and z 2 H: Then, (A B) (x)

= f A (y) ^ B (z)g x2_yz x2_y z f A ( ) ^ B (z)g , for each 2 h y; i2nhf y ( ) ( ) f A (y0) ^ B (z0)g ; where 2 h y; y0 2 h x and z = z0 y02h x_(h y) z y02h x inf (

29A B) (y0) : (A B) (x) = f A (y) _ B (z)

g x2^y z f A ( ) _ B (z)g , for each 2 h y; inf ( ) () x2^y z 2h y f A (y0) _ B (z0)g ; where 2 h y; y0 2 h x and z = z0 y02h x^(h y) z sup (A B) (y0) : y02h x Similarly, ( ) (x) inf (y02x h) (y0) and (A B) (x) sup (A B) (y0) : y02x h Hence, A B is an intuitionistic fuzzy -hyperideal of H. Proposition 82 Let c(t; s) be an intuitionistic fuzzy point of a -semihypergroup H: Then, the intuitionistic fuzzy left (right) -hyperideal of H generated by c(t; s) is denoted by Ic(t;s)(c(t;s)) and de…ned as: Ic(t;s) (x) = 8 (t; s) if x 2 H a < (0; 1) if x 2= H a and : (t; s) if x 2 a H c(t;s) (x) = 8< (0; 1) if x 2= a H: Proof. For x; y 2 H if x y H a:for each 2 , then for each a 2 x y; Ic

30(t;s) (a) = (t; s) Ic (t;s)

(y). If x y * H a, then y 2= H so for all a 2 x y, Ic(t;s) (a) = (0; 1) = Ic(t;s) (y). Hence, in any case, Ic(t;s) (a) Ic(t;s) (y) for each a 2 x y: So, Ic(t;s) is an intuitionistic fuzzy left -hyperideal of H: By de…nition of Ic(t;s); we …nd that c(t; s) Ic(t;s). If A = h A; Ai is an intuitionistic fuzzy left -hyperideal of H containing c(t; s) and if x 2 H a, then as (t; s) = c(t; s) (a) A (a) implies that (t; s) A (a) A (b) ; for b 2 h a and h 2 H

3which implies that A (x) (t; s) = Ic (t;s)

(x). If x 2= H a; then Ic(t;s) (x) = (0; 1) A (x). So, Ic(t;s) A. Thus, Ic(t;s) is an intuitionistic fuzzy left -hyperideal of H generated by c(t; s): Similarly, we can prove c(t;s) (x) is an intuitionistic fuzzy right -hyperideal of H . Proposition 83 Ic(t;s) H and H c(t;s) are intuitionistic fuzzy -hyperideals of H gen- erated by c(t;s), where H = (H; cH) and H is the characteristic function of H. Proof. Straighforward. Theorem 84 An

2intuitionistic fuzzy sets A = h A; Ai of a

-semihypergroup H is an intuitionistic

1**fuzzy left(resp. right) -hyperideal** of H **if and only if** it satis…es **for** all **x; y** 2 **H**

and

12**t 2 (0; 1]** and s **2 [0; 1),** y **(t; s)** 2 A =) (z **)(t; s)** 2 **A** (resp. (z **)(t;** s) **2**

A for each z 2 x y. Proof. Assume that x 2 H and y(t; s) 2 A, where

13**t 2 (0; 1] and s 2 [0; 1).** Then, A(y) **t**

and A(y) s. It follows from de…nition 63 that inf A(z) A(y) t) inf A(z) z2x y z2x y t sup A(xy) A(y) s) sup A(xy) z2x y z2x y s Thus, (z)(t; s) 2 A for each z 2 x y. (b) =) (a) Suppose that (b) is valid. Note that x 2 H

2**and y(A(y); A(y))** 2 **A,**

then by (b) we have (z)(

5**A(x); A(y))** 2 **A for** each z 2 **x y**

implies that A(z) A(y) and A(z) A(y) for each z 2 x y. This completes the proof. Proposition 85 If A = h A; Ai is an

2**intuitionistic fuzzy left(right)** -hyperideal **of a**

-semihypergroup H and c(t1; s1); d(t2; s2) are intuitionistic fuzzy points of H such that c(t1; s1) H d(t2; s2) A then Ic(t1;s1) Id(t2;s2) A c(t1;s1) d(t2;s2) A : Proof. Straightforward. A -semihypergroup H is called fully intuitionistic fuzzy prime (semiprime) if each of its intuitionistic fuzzy -hyperideal is prime (semiprime). Proposition 86 A -semihypergroup H is fully intuitionistic fuzzy semiprime if and only if H is an intuitionistic fuzzy semisimple. Proof. Let H be an intuitionistic fuzzy semisimple -semihypergroup and A = h A; Ai be an intuitionistic fuzzy -hyperideal of H. If for an intuitionistic fuzzy - hyperideal B = (B; B) of H such thst B B A then B A (since H is intuitionistic fuzzy semisimple): Hence, A = h A; Ai is an intuitionistic fuzzy semiprime -hyperideal of H. Thus, H is a fully intuitionistic fuzzy semiprime. Conversely, let H be a fully intuitionistic fuzzy semiprime -semihypergroup. Let A = h A; Ai

2**be an intuitionistic fuzzy** -hyperideal **of** H. **Then,** A2 **is** also **an intuitionistic fuzzy** -hyperideal **of**

H: As A2 A2 implies A A2 (because A is an intuitionistic fuzzy semiprime -hyperideal of H): But A2 A always. Hence, A2 = A: Thus, each intuitionistic fuzzy -hyperideal of H is idempotent. So, H is semisimple.

Proposition 87 A -semihypergroup H is fully intuitionistic fuzzy prime if and only if H is an intuitionistic fuzzy semisimple and

2**the set of all intuitionistic fuzzy** -hyperideals **of**

H is totally ordered under inclusion. Proof. Suppose H is an intuitionistic fully fuzzy prime. Let A = h A; Ai

2**be an intuitionistic fuzzy** -hyperideal **of** H. **Then,** A2 **is** also **an intuitionistic fuzzy** - hyperideal **of**

H: Since A2 A2 implies A A2. But A2 A always. Hence, A2 = A: Thus, each intuitionistic fuzzy -hyperideal of H is idempotent. So, H is semisimple. Now, suppose that A = h A; Ai and B = (B; B) are intuitionistic fuzzy -hyperideals of H: Since,

2**A B A \ B** and **A \ B**

is

2**an intuitionistic fuzzy** -hyperideal **of** H; so **A** B **is an intuitionistic fuzzy**

prime -hyperideal. Thus, either A

11**A\B** or **B A\B** implies either **A B** or **B B:**

Conversely, let H be an intuitionistic fuzzy semisimple -semihypergroup and

2**the set of all intuitionistic fuzzy** -hyperideals **of**

H is totally ordered under inclusion. Let A = h A; Ai;B = (B; B) and C = (C; C) be intuitionistic fuzzy -hyperideals of H such that A B C: Since

2**the set of all intuitionistic fuzzy** -hyperideals **of**

H is totally ordered under inclusion, so either A B or

28**B A:** If **A B:** Now, **A** = A2 **A** B **C:** Hence, **A C** so **A** is an intuitionistic fuzzy prime -hyperideal of H: Similarly, if B A: Let H be a -semihypergroup and A = h A; Ai

2**be an intuitionistic fuzzy** subset **of** H. **Then, A** = h A; Ai **is called an intuitionistic fuzzy** M-hypersystem **of** H **if**

for all x; y; z 2 H and ; 2 we have min f A (x) ; A (z)g max f

8**A (x) ; A (z)** g a2x y **z** inf **A (a)** sup **A (a)**

a2x y z Let H be a -semihypergroup and A = h A; Ai

2**be an intuitionistic fuzzy** subset **of** H. **Then, A** = h A; Ai **is called an intuitionistic fuzzy** N -hypersystem **of** H **if**

for all x; y 2 H and ; 2 ,

3**we have (x)** a2x **y** x inf **(a) (x)** sup **(a)** a2x **y**

x Remark 88 Every intuitionistic fuzzy M -hypersystem of a -semihypergroup H is an intuitionistic fuzzy N -hypersystem. Theorem 89 If fAigi2 is a family of intuitionistic fuzzy M (resp. N)-hypersystems of -semihypergroup H, then Ai is also an intuitionistic fuzzy M(resp. N)-hypersystem i2 of H . T Proof. Let A = Ai = (Ai; Ai) = h A; Ai and x;y;z 2 H and ; 2 . i2 i2 i2 T V W Then, we have a2x y z inf a2x y z inf and A (a) = inf a2x y z (i2 Ai (a)) ^ min Ai (x) ; Ai (z) (i2) ^ = min Ai (x) ; Ai (z) = min (fi^2A(x) ; A (zi^2)g) A (a) min f A (x) ; A (z)g sup a2x y z A (a) = sup f A (a)g a2x y z (i2) _ (min f Ai (x) ; Ai (z)g) (i2) _ = max Ai (x) ; Ai (z) = max (fi^2A(x) ; A (zi^2)g) sup a2x y z A (a) max f A (x) ; A (z)g Hence Ai is an intuitionistic fuzzy M-hypersystem of a -semihypergroup H. This i2 completes Tthe proof. Proposition 90 Every intuitionistic fuzzy two sided -hyperideal of a -semihypergroup H is an intuitionistic fuzzy M-hypersystem of H. Proof. Let A = h A; Ai be an intuitionistic fuzzy two sided -hyperideal of H. Let x; y; z 2 H and ; 2 . Then, inf a2x y z a2x y z inf a2x y z inf and

7**A (a) A (a) A (a)** a2inxf y **A (a) A (x)** and a2y **z** inf **A (a)** A **(z)** min f **A** (x) ; **A (z)**

g sup a2x y z sup a2x y z sup a2x y z This completes the proof. A (a) A (a) A (a) sup A (a) A (x) and a2x y sup

8**A (a) A (z)** a2x **z** max f **A** (x) ; **A (z)**

g Corollary 91 Every intuitionistic fuzzy one sided -hyperideal of a -semihypergroup H is an intuitionistic fuzzy N -hypersystem of H . 4.4 (;)-Intuitionistic Fuzzy -hyperideals in -semihypergro In this section we de…ne and study the concept of (;)-intuitionistic fuzzy -hyperideals and intuitionistic fuzzy interior -hyperideal in -semihypergroups. De…nition 92 Let H be a -semihypergroup and A = h A; Ai be an intuitionistic fuzzy set in H. If for all x; y 2 H; 2 , the following conditions hold (P1) maxf inf A(z); g minf A(x); A(y); g. z2x y (P2) minf sup A(z); g maxf A(x); A(y); g. z2x y then A = h A; Ai is called a (;)-intuitionistic fuzzy sub- -semihypergroup of H. It is clear that for = 0 and = 1 we obtain the intuitionistic fuzzy sub- -semihypergroups. De…nition 93 Let H be a -semihypergroup and A = h A; Ai be an intuitionistic fuzzy set in H. If for all x; y 2 H; 2 , the following conditions hold L1 maxf inf z2x y L2 minf sup z2x y A(z);

27**g A(** z); **g** minf **A(** y); **g** maxf **A(** y); **g:** Then **A** = h **A;**

Ai is called a …rst (resp. second) (;)-intuitionistic fuzzy left - hyperideal of H if it satis…es (L1) (resp. L2). A = h A; Ai is called an (;)- intuitionistic fuzzy left -hyperideal of H if it is both a …rst and a second (;)-intuitionistic fuzzy

1**left -hyperideal of H.** De…nition 94 **Let** H **be**

a -semihypergroup and A = h A; Ai be an intuitionistic fuzzy set in H. If

1**for all x; y** 2 **H;** 2 , **the**

following conditions hold R1 maxf inf z2x y A(z); g R2 minf sup z2x y A(z); g minf A(x); g maxf A(x); g: then A = h A; Ai is called a …rst (resp. second) (;)-intuitionistic fuzzy

1**right - hyperideal of H if it**

satis…es (R1) (resp. R2). A = h A; Ai

1**is called a** (;)-intuitionistic fuzzy **right -hyperideal of H if it**

is both a …rst and a second (;)-intuitionistic fuzzy right -hyperideal of H. Theorem 95 Let H be a -semihypergroup and ;6 = S H . If S is a sub- -semihypergroup of H , then S~ = (S ; cS)

3**is a (;)-intuitionistic fuzzy** sub- -semihypergroup **of** H . Proof. **Let x;y** 2 H **and**

2 . We have the two following cases: Case 1. x;y 2 S. Then x y S since S is sub- -semihypergroup of H. Thus we have maxf inf S(z); g = maxf1; g = 1 minf S(x); S(y); g z2x y and minf sup cS(z); g = minf1 sup S(z); g = minf1 1; g = 0 maxf cS(x); cS(y); g. z2x y z2x y Case 2. x 2= S or y 2= S. Then S(x) = 0 or S(y) = 0. Thus we have maxf inf S(z); g 0 = minf S(x); S(y); g z2x y and minf sup cS(z); g 1 = maxf cS(x); cS(y); g z2x y This completes the proof. Theorem 96 Let H be a -semihypergroup and S be a non-empty subset of

H . If S~ = (S ; cS) is a (;)-intuitionistic fuzzy sub- -semihypergroup of H then S is a sub- -semihypergroup of H . Proof. 1. Let us suppose …rst that S~ = (S; cS) is a …rst (;)-intuitionistic fuzzy sub- -semihypergroup of H. For any x;y 2 S; 2 , by (P1), we have maxf inf S(z); g minf S(x); S(y); g = minf1;1; g = . z2x y We have inf S(z) z2x y > 0 since < . Therefore we have z2inxfy S(z) = 1. This implies x y S and thus S is a sub- -semihypergroup of H. 2. Let us assume that S~ = (S; cS) is a second (;)-intuitionistic fuzzy sub- - semihypergroup of H. For any x;y 2 S; 2 , by (P2), we have minf sup cS(z); g maxf cS(x); cS(y); g = maxf0; 0; g = . z2x y We have sup cS(z) since < . Therefore we have sup cS(z) = 0, that is z2x y z2x y sup S(z) = 1. This implies x y S and thus S is a sub- -semihypergroup of H. z2x y Proposition 97 Let H be a -semihypergroup and L be a left -hyperideal of H. Then, L~ = (L; cL) is an (;)-intuitionistic fuzzy

1**left -hyperideal of H. Proof. Let**

x; y 2 H; 2 and L be a -hyperideal of H. Then, we

46**consider the following two cases. Case 1. y**

2 L. Then x y L. Thus, we have maxf inf L(z); g = maxf1; g = 1 minf L(y); g. z2x y and minf sup cL(z); g = minf0; g = 0 maxf cL(y); g. z2x y Case 2. y 2= L. Then, L(y) = 0. Thus, we have maxf inf L(z); g 0 = minf L(y); g. z2x y and minf sup cL(z); g 1 = maxf cL(y); g. z2x y Therefore, L~ = (L; cL) is an (;)-intuitionistic fuzzy

1**left -hyperideal of H. Let** H **be**

a -semihypergroup and A = h A; Ai an intuitionistic fuzzy sub- -semihypergroup of H. If

1**for all x;** s; **y** 2 **H;** ; 2 , **the**

following conditions hold I1 maxf inf A(z); g z2x s y I2 minf sup A(z); g z2x s y minf A(s); g maxf A(s); g: Then, A = h A; Ai is called a …rst (resp. second) (;)-intuitionistic fuzzy interior - hyperideal of H if it satis…es (I1) (resp. I2). A = h A; Ai is called an (;)-intuitionistic fuzzy interior -hyperideal of H if it is both a …rst and a second (;)-intuitionistic fuzzy interior -hyperideal of H. Theorem 98 Let H be a -semihypergroup. Then, every (;)-intuitionistic fuzzy - hyperideal is a (;)-intuitionistic fuzzy interior -hyperideal. Proof. Let A = h A; Ai be a (;)-intuitionistic fuzzy -hyperideal of H and let x; s; y 2 H; ; 2 , we have maxf inf A(z); g = maxf inf A(z); ; g maxfminf inf A(t); g; g z2x s y z2x s y t2x s = minfmaxf inf A(t); g; maxf ; gg minf A(s); g: t2x s In similar way, we show minf sup A(z); g maxf A(s); g. So, A = h A; Ai is a z2x s y (;)-intuitionistic fuzzy interior -hyperideal of H. Theorem 99 Let H be a regular -semihypergroup. Then every (;)-intuitionistic fuzzy interior -hyperideal is a (;)-intuitionistic fuzzy -hyperideal. Proof. Let A = h A; Ai be a (;)-intuitionistic fuzzy interior -hyperideal of H and x; y 2 H. Then, there exist s; s0 2 H; ; 0 ; 0 2 , such that x 2 x s x and y 2 y 0s0 0y. Thus, for all 2 , we have maxf inf t2x y A(t); g maxf inf t2x (y 0s0 0y) A(t); g maxf inf t2x y 0(s0 0y) A(t); g minf A(y); g and minf sup t2x y A(t); g maxf inf t2x (y 0s0 0y) A(t); g minf inf t2x y 0(s0 0y) A(t); g maxf A(y); g. This shows that A = h A; Ai is a (;)-intuitionistic fuzzy left -hyperideal of H. In similar way, we show that A = h A; Ai is a (;)-intuitionistic fuzzy

1**right -hyperideal of H** and so **A** = h **A;**

Ai is a (;)-intuitionistic fuzzy -hyperideal of H. Theorem 100 Let H be a -semihypergroup and I an interior -hyperideal of H . Then I~ = (I ; cI)

3**is a (;)-intuitionistic fuzzy** interior -hyperideal **of** H . Proof. **Let x; y;**

s 2 H; ; 2 . We consider the two following cases. Case 1. s 2 I . Then, x s y I . Thus, we have maxf inf I(z); g = maxf1; g = 1 minf I(s); g. z2x s y and minf sup cI(z); g = minf0; g = 0 maxf cI(s); g. z2x s y Case 2. s 2= I. Then, I(s) = 0. Thus, we have maxf inf I(z); g 0 = minf I(s); g. z2x s y and minf sup cI(s); g 1 = maxf cI(s); g. z2x s y Therefore, I~= (I; cI) is a (;)-intuitionistic fuzzy interior -hyperideal of H. Theorem 101 Let H be a -semihypergroup and S be a non-empty subset of H. If S~ = (S; cS) is a …rst (;)-intuitionistic fuzzy interior -hyperideal of H or a second (;)-intuitionistic fuzzy interior -hyperideal of H, then S is an interior -hyperideal of H. Proof. By Theorem 96 it follows that S is a sub- -semihypergroup of H. We consider the two following cases. Case 1. Assume that S~ = (S ; cS) is a …rst (;)-intuitionistic fuzzy interior -hyperideal of H and let x 2 H S H. Thus x 2 u s t for some u; t 2 H; s 2 S; ; 2 . By the condition (I1) follows maxf S(x); g = maxf inf S(z); g minf S(s); g = minf1; g = . z2u s t Since < , we have S(x) , i.e. S(x) = 1. So, x 2 S and thus S is an interior -hyperideal of H. Case 2. Assume that S~ = (S ; cS) is a second (;)-intuitionistic fuzzy interior - hyperideal of H and let x 2 H S H. Thus, x 2 u s t for some u;t 2 H;s 2 S; ; 2 . By the condition (I2) follows minf cS(x); g = minf sup cS(z); g minf cS(s); g = maxf0; g = . z2u s t Since < , we have cS(x) , i.e. cS(x) = 0. So, x 2 S and thus S is an interior -hyperideal of H. Recall that a -semihypergroup is called left simple if H has no left -hyperideals other then itself. A -semihypergroup is left simple if and only if H = H a for all a 2 H. A -semihypergroup H is called a …rst (resp. second) (;)-intuitionistic fuzzy left simple -semihypergroup if for any …rst (resp. second) (;)-intuitionistic fuzzy left -hyperideal A = h A; Ai of H, for all a; b 2 H, we have maxf A(a); g minf A(b); g (resp. minf A(a); g maxf A(b); g. H is called a (;)-intuitionistic fuzzy left simple -semihypergroup if it is both …rst and second (;)-intuitionistic fuzzy left simple - semihypergroup. Theorem 102 If a -semihypergroup H is left simple, then H is (;)-intuitionistic fuzzy left simple -semihypergroup. Proof. Assume that A = h A; Ai is (;)-intuitionistic fuzzy left -hyperideal of H and x; x0 2 H. Since H is left simple, so there exist s; s0 2 H; ; 0 2 such that x 2 s x0 and x0 2 s0 0x. Thus, by hypothesis for A, we have and maxf A(x); g = maxf inf A(z); g z2s x0 maxf A(x0); g = maxfz2isn0f0x A(z); g minf A(x0); g; minf A(x); g minf A(x); g = minf sup A(z); g maxf A(x0); g; z2s x0 minf A(x0); g = minf sup A(z); g maxf A(x); g: z2s0 0x Therefore, H is a (;)-intuitionistic fuzzy left simple -semihypergroup. Theorem 103 Let H be a -semihypergroup. If H is …rst or second (;)-intuitionistic fuzzy left simple -semihypergroup, then H is left simple -semihypergroup. Proof. Let us suppose that H is …rst (or second) (;)-intuitionistic fuzzy left simple -semihypergroup and let L be a left -hyperideal of H. By Proposition 5.6, we have that L~ = (L; cL) is (;)-intuitionistic

1**fuzzy left -hyperideal of H.** We have **for**

all a; b 2 H, maxf L(a); g minf L(b); g. Let a 2 H and b 2 L. Then, we have maxf L(a); g minf L(b); g = minf1; g = . So, maxf L(a); g . But < , so L(a) < 0. This implies

22L(a) = 1, that is, a 2 L. Therefore, H = L and

so H is left simple -semihypergroup. 4.5 Intuitionistic Fuzzy -hyperideals in Artinian and Noetherian -semihypergroups In this section we characterize Artinian and Noetherian -semihypergroups in terms of intuitionistic fuzzy -hyperideals. De…nition 104 [29] Let H be a -semihypergroup. Then, H is called Noetherian (Ar- tinian resp.) if H satis…es the asceding (descending) chain condition on -hyperideals. That is, for any sequence of -hyperideals flig11 of H such that I1 I2 I3 :::Ii :::(I1 I2 I3 :::Ii::: resp.) there exists n 2 N such that Im = In for each m 2 N; m n. Theorem 105 [29]

1Let H be a -semihypergroup. Then, the following are equivalent: 1. H is Noetherian; 2. H

satis…es the maximum condition for -hyperideals; 3. Every -hyperideal of H is …nitely generated. Theorem 106 Let H be a -semihypergroup. If every intuitionistic fuzzy -hyperideal of a -semihypergroup has …nite number of values, then H is Artinian. Proof. Let assume that every intuitionistic fuzzy -hyperideal of a -semihypergroup H has …nite number of values and H is not Artinian. Then, there exists a strictly descending chain H = I0 I1 I2 ::: of -hyperideals of H. Now, we de…ne the intuitionistic fuzzy set A by n+1 if x 2 InIn+1; n = 0; 1; 2; ::: n A(x) = 8 1 >< 1 if x 2 I n n=0 : > T n+1 if x 2 InIn+1; n = 0; 1; 2; ::: 1 A(x) = 8 1 >< 0 if x 2 I n n=0 It can be easily seen that A is >:an intuitionisticTfuzzy -hyperideal of H which has in…nite number of di¤erent values because of the in…nitely descending chain of -hyperideals I0; I1; I2; ::: of H. It is impossible. So, H is Artinian. Theorem 107

1Let H be a -semihypergroup. Then, the following statements are equiv- alent: 1. H is Noetherian; 2.

The set of values of any intuitionistic fuzzy -hyperideal of H is a well-ordered subset of [0; 1]. Proof. (1)) (2). Let A be an intuitionistic fuzzy -hyperideal of a -semihypergroup H. Let us suppose that the set of values of A is not a well-ordered subset of [0; 1]. Then, there exists a strictly in…nite decreasing sequence ftng such that A(xn) = tn and A(xn) = 1 tn. Let In = fx 2 Hj A(x) tng and Jn = fx 2 Hj A(x) 1 tng. Then, I1 I2 ::: and J1 J2 ::: are strictly in…nite ascending chains of -hyperideals of H. It is impossible. (2)) (1). Let us suppose that there exists a strictly in…nite ascending chain I1 I2 ::: (*) of -hyperideals of H. Let I = In. It can be easily seen that I is a -hyperideal of H. We de…ne the intuitionistic fSuzzy set A = h A; Ai by n2N 0 ifx2=I A(x) = 8 1 < k where k = minfn 2 Njx 2 Ing : A(x) = 8 k 1 1 < k+1 : if x 2= I where k = maxfn 2 Njx 2 Ing It can be easily seen that A is an intuitionistic fuzzy -hyperideal of H. Since the chain (*) is not …nite, A has strictly in…nite ascending sequence of values and so the value set of the intuitionistic fuzzy -hyperideal A of H is not well-ordered. It is impossible. Theorem 108 Let H be a -semihypergroup. If H is both Artinian and Noetherian, then every intuitionistic fuzzy -hyperideal of H is …nite valued. Proof. Let us

3suppose that A is an intuitionistic fuzzy -hyperideal of

H and Im(A); Im(A) are not …nite. By the previous theorem, we consider the two follow- ing cases: Case 1. Assume that t1 < t2 < t3 < ::: is strictly increasing sequence in Im(A) and s1 > s2 > s3 > ::: is strictly decreasing sequence in Im(A). Then we have U (A; t1) U (A; t2) U (A; t3) :::, L(A; s1) L(A; s2) L(A; s3) ::: are strictly descending and ascending chains of -hyperideals of H, respectively. Since H is both Artinian and Noetherian there exists i 2 N, such that U (A; ti) = U (A; ti+n) and L(A; si) = L(A; si+n) for n 1. This implies that ti = ti+n and si = si+n. It is impossible. Case 2. Assume that t1 > t2 > t3 > ::: is strictly decreasing sequence in Im(A) and s1 < s2 < s3 < ::: is strictly increasing sequence in Im(A). Then we have U (A; t1) U (A; t2) U (A; t3) :::, L(A; s1) L(A; s2) L(A; s3) ::: are strictly ascending and descending chains of -hyperideals of H, respectively. Since H is both Artinian and Noetherian there exists j 2 N, such that U (A; tj) = U (A; tj+n) and L(A; sj) = L(A; sj+n) for n 1. This implies that tj = tj+n and sj = sj+n. It is impossible. Chapter 5 Characterizations of -semihypergroups by Intuitionistic Fuzzy -hyperideals 5.1 Introduction In this chapter, we characterize -semihypergroups by the properties of intuitionistic fuzzy -hyperideals. We characterize regular -semihypergroup by intuitionistic fuzzy -hyperideal, intuitionistic fuzzy bi- -hyperideal, intuitionistic fuzzy generalized bi- - hyperideal, intuitionistic fuzzy interior -hyperideal and intuitionistic fuzzy quasi- - hyperideal. We also de…ne an intuitionistic fuzzy quasi- -hyperideal of a -semihypergroup. 5.2 Intuitionistic Fuzzy Quasi- -hyperideals In this section we de…ne an intuitionistic fuzzy quasi- -hyperideal in a -semihypergroup. We discuss di¤erent properties of intuitionistic fuzzy quasi- -hyperideal. De…nition 109 Let H be a -semihypergroup and A = h A; Ai

3**be an intuitionistic fuzzy set in** H. **Then, A** = h A; Ai **is** called **an intuitionistic fuzzy**

quasi- -hyperideal of H, if the following conditions hold.

9**A (x) (A** 1) **(x)** ^ (1 **A)** (x) **A (x) (A** 0) **(x)** _ (0 **A) (x)**

; for all x 2 H. Also, we can de…ne an intuitionistic fuzzy quasi- -hyperideal of H as: A S \ S A A for an intuitionistic fuzzy set A = h A; Ai and S = h1; 0i. Example 110 Let H = f1; 2; 3g and = f ; g be two non-empty sets. Then, hyperop- erations are de…ned by the following Cayley tables. 1 2 3 1 2 3 1 f1g f1; 2g f3g 1 f1; 2g f1; 2g f3g 2 f1; 2g f1; 2g f3g 2 f1; 2g f2g f3g 3 f3g f3g f3g 3 f3g f3g f3g Clearly, H is a -semihypergroup. Now, we de…ne intuitionistic fuzzy set A = h A; Ai in H as: A (1) = 0:4 = A (2) , A (3) = 0:7 and A (1) = 0:5 = A (2) , A (3) = 0:2: Then, A = h A; Ai is an intuitionistic fuzzy quasi- -hyperideal of H. Example 111 Let H = fa; b; c; dg, = f ; g be two non-empty sets and -hyperoperations de…ned by the following Cayley tables; a

9**b c d a b c d a**

fa; cg fb; dg fa; cg fdg a fag fb; dg fcg fdg b fb; dg fbg fb; dg fdg b fb; dg fbg fb; dg fdg c fa; cg fb; dg fa; cg fdg c fcg fb; dg fag fdg d fdg fdg fdg fdg d fdg fdg fdg fdg Then, clearly, (H;) is a -semihypergroup. I = fdg and fb; dg are only proper - hyperideals of H. Proposition 112 Let H be a -semihypergroup and Q

2**be a non-empty subset of** H. **Then,** Q **is a** quasi- -hyperideal **of** H **if and only if**

the intuitionistic characteristic func- tion Q = h Q; cQi is an intuitionistic fuzzy quasi- -hyperideal of H:

Preoof. straightforward. Proposition 113 Let A = h A; Ai and B = h B; Bi

6**be an** intuitionistic **fuzzy right** -hyperideal **and an** intuitionistic **fuzzy left** -hyperideal of -semihypergroup **H,**

respec- tively. Then, A \ B is an intuitionistic

1**fuzzy quasi- -hyperideal of H.** Proof. **Let**

x 2 H. Then, we have ((A ^ B) 1) (x) ^

5**(1 (A** ^ B)) **(x) (A 1) (x) ^ (1** B) **(x) (A** ^ B) **(x)** and ((**A** _ B) 1) **(x)** _ (1 (**A** _ B)) (x) (**A**

1) (x) _

60**(1 B) (x)** (A _ **B)** (x) : **This** implies **that A** \ B **is**

an intuitionistic

1**fuzzy quasi- -hyperideal of H.** Proposition 114 **Every**

intuitionistic

1**fuzzy quasi- -hyperideal of** a -semihypergroup **H is an**

intuitionistic

1**fuzzy bi- -hyperideal of H:** Proof. Let **A** = h **A;**

Ai be an intuitionistic

1**fuzzy quasi- -hyperideal of H.** Since, **A(**

x) 1 and A 0 i.e, A S. This implies that A A S A and A A A S. Hence, A A A S \ S A A this implies that A A A. Thus, A = h A; Ai is an intuitionistic fuzzy sub- -semihypergroup of H. Since S A S, so we have A S A A S. Also, since A S S this implies A S A S A. Thus, we have A S A A S \ S A A this implise that A S A A.

Therefore, A = h A; Ai is an intuitionistic fuzzy bi- -hyperideal of H. Proposition 115 Every one sided intuitionistic fuzzy -hyperideal of a -semihypergroup H is an intuitionistic

1**fuzzy quasi- -hyperideal of H:** Proof. **Let A** = h A; Ai **be**

an intuitionistic

1**fuzzy left -hyperideal of H. Then,**

S A A and also A S S . Thus, this implies that A S \ S A S A A. Proposition 116 Let fAigi2 be a family of intuitionisitic fuzzy quasi- -hyperideals of a -semihypergroup H . Then Ai is an intuitionistic fuzzy quasi- -hyperideal of H . i2 T Proof. Strightforward. Proposition 117 The product of two intuitionistic fuzzy quasi- -hyperideals of a - semihypergroup H is an intuitionistic fuzzy bi- -hyperideal of H. Proof. Strightforward. Theorem 118 Let H be a -semihypergroup and A = h A; Ai

3**be an intuitionistic fuzzy set in** H. **Then,** 1. **A** = h A; Ai **is an** intuionistic **fuzzy**

sub- -semihypergroup of H if and only if A A A. 2. A = h A; Ai is an intuionistic fuzzy left

1**(right) -hyperideal of H if and only if** S **A**

A (A S A): Proof. 1) Let A = h A; Ai be an intuionistic fuzzy sub- -semihypergroup of H: Then, A A A. Indeed, let x 2 H. Then, (A A) (x) = x2y z f A (y) ^ A (z)g W x2y z f A (y z)g = x2y z iWnf A (x) (A A) (x) x2y z inf

8**A (x)** and (**A A)** (x) = f **A** (y) _ **A (z)** g x2y **z**

V f A (y z)g x2y z = sVup x2y z A (x) (A A) (x) sup A (x) : x2y z This implies that A A A. Conversely, suppose that A A A holds. Let x; y 2 H and 2 . Then, x2y z inf A(x) (A

8**A) (x)** = f **A** (y) ^ **A (z)** g x2y **z A** (y) ^ **A (z)**

W and sup x2y

8**z A(** x) (**A** A) **(x)** = f **A** (y) _ **A (z)** g x2y **z A** (y) _ **A (z)** V Thus, **A = h A;**

Ai is an intuionistic fuzzy sub- -semihypergroup of H. 2) Suppose that A = h A; Ai is an intuionistic fuzzy

1**left -hyperideal of H. Let x**

2 H. Then, we have (1

5**(1 (1** and **A) (x) = = A) (x)** A) **(x)**

x2y z W x2y z W x2y z iWnf x2y z f1 (y) ^ A (z)g 1 (y) ^ inf A (z) x2y z 1 ^ inf A (x) = inf A (x) x2y z x2y z A (x) A (x) for each x 2 H Hence, S A similar way. (0 A) (x) = f0 (y) _ A (z)g x2y z W 0 (y) _ sup A (z) x2y z x2y z = W 0 _ sup A (x) = sup A (x) x2y z x2y z x2y

5**z (1 A) (x)** sWup **A (x)**

x2y

5**z (1 A) (x) A (x)**

for each x 2 H. A. For an intuionistic fuzzy left -hyperideal of H can be proved in a Theorem 119 Let H be a -semihypergroup and A = h A; Ai be an intuitionistic fuzzy sub- -semihypergroup of H. Then, 1. A = h A; Ai is an intuionistic

1**fuzzy bi- -hyperideal** of H **if and only if**

A S A A. 2. A = h A; Ai is an intuionistic fuzzy (1,2)- -hyperideal of H if and only if A S A A A. Proof. 1) Assume that A = h A; Ai is an intuionistic fuzzy bi- -hyperideal of H. Then, A (A 0 S A A. Indeed, let x 2 H: Then, in the case (A A) (x) = 1 it is clear (A 1

9**A) (x) A (x)** and (**A** 1 0 **A) (x)** = 0, and **A) (x) A (x).**

If (A 1 A) (x)6 = 0, then there exist a; b; p; q and ; 2 such that x 2 a b and a 2 p q: Since A = h A; Ai is an intuionistic fuzzy bi- -hyperideal of H, so we have (A (A (A and

9**1 1** 1 **A) (x) = A) (x) A)** (x) **f(A** 1) **(a)**

^ A (b)g x2a b W = x2a b f A (p) ^ 1 (q)g ^ A (b) a2p q W W = x2a b f A (p) ^ 1g ^ A (b) a2p q = W f WA (p) ^ A (b)g x2p q b W f A (p q b)g = inf f A (x)g x2p q b x2p q b x2p q b iWnf f

9**A (x)** g **A (x) A (x) (A** 0 **A) (x)** = f(**A** 0) **(a)**

_ A (b)g x2a b V = f A(p)_0(q)g_ A(b) x2a b a2p q V V = f A(p)_0g_ A(b) x2a b a2p q = V fVA(p)_ A(b)g x2p q b V f A (p q b)g = sup f A (x)g x2p q b x2p q b (A 1 A) (x) sVup

5f A (x) g A (x) x2p q b (A 1 A) (x)

A (x) : Thus, this implies that A S A A. Conversely, suppose that A S A A holds. Let x; y; z 2 H and ; 2 such that a 2 x y z. Then,

8A (a) A (a) and = = (A 1 A) (a) for each a 2 x y z f(A 1) (p) ^ A(a)

g a2p q f(WA 1) (p) ^ A(z)g for each p 2 x y f A (m) ^ 1 (n) ^ A(z)g p2x y=m n A (Wx) ^ 1 (y) ^

8A(z) = A (x) ^ 1 ^ A(z) A (x) ^ A(z) for each a 2 x y z A (a) = = A (a) (A 0 A) (a) for each a 2 x y z f(A 0) (p) _ A(a)

g a2p q f(VA 0) (p) _ A(z)g for each p 2 x y f A (m) _ 0 (n) ^ A(z)g p2x y=m n A (Wx) _ 0 (y) _

8A(z) = A (x) _ 0 _ A(z) A (x) _ A(z) for each a

2 x y z Thus, a2x y z inf A (a) sup A (a) a2x y z min f

8A (x) ; A(z) g max f A (x) ; A(z) g : Therefore, A = h A;

Ai is an intuionistic fuzzy bi- -hyperideal of H: 2) This part follows from part (1). 5.3 Regular -semihypergroups In this section, we characterize regular -semihypergroups in terms of intuitionistic fuzzy -hyperideals. Theorem 120

1Let H be a -semihypergroup. Then, the following are equivalent: 1. H is regular. 2. A = A S A for every intuitionistic fuzzy quasi- -hyperideal

A = h A; Ai of H. 3. A = A S A for every intuitionistic fuzzy generalized bi- -hyperideal A = h A; Ai of H. 4. A = A S A for every intuitionistic fuzzy generalized bi- -hyperideal A = h A; Ai of H. Proof. 1)) 4). Suppose that (1) holds. Let A = h A; Ai be an intuitionistic fuzzy generalized bi- -hyperideal of H amd let x 2 H. Then, since H is a regular - semihypergroup, so there exist y 2 H and have ; 2 such that x 2 x y x. Now, we (A 1 (A

301 A) (x) = = = A) (x) f(A 1) (a) ^ A (b) g x2a b

f(WA 1) (z) ^ A (x)g ; for all z 2 x y f A (s) ^ 1 (t)g ^ A (x) for all z 2 x y z2s t WA(x) ^ 1 (y) ^ A (x) A

5**(x) A (x)** and **(A 1 (A 1 A) (x) = = = A) (x)** f(**A** 0) **(a) _ A**

(b)g x2a b f(VA 0) (z) _ A (x)g ; for all z 2

3**x y** f **A (s)** _ 0 **(t)** g _ **A**

(x) for all z 2 x y z2s t VA(x) _ 0 (y) _

3**A (x)** A **(x) A (x)** : Thus, (**A** 1 **A) A** and (**A** 1 **A) A** this implies that **A A** S **A.** Since **A** = h **A;**

Ai is an intuitionistic fuzzy generalized bi- -hyperideal of H; so A A S A. Hence, A = A S A and so 1)) 4). This is very easy to prove that 4)) 3)) 2). 2)) 1): Suppose that (2) holds. Let Q be a

1**quasi- -hyperideal of H.** Then, we have **Q** H **Q** Q **H H**) \ (H H) **Q Q H** \ H **Q Q:** Q **H**

Q Q: Let x 2 Q. Since Q = (Q; cQ) is an intuitionistic fuzzy quasi- -hyperideal of H; so by Proposition 112. We have e x2a b Q 1 (a) ^ Q (b) = Q 1 Q (x) W = Q (x) = 1 This implies that there exist y; z 2 H such that Q 1 (y) = 1 and Q (b) = 1 with x 2 y z. Thus, we have 1 (y) = 1 y2s t Q (s) ^ 1 (t) = Q W This implies that there exist m; n 2 H such that Q (m) = 1 and 1(n) = 1 with y 2 m n. Hence, m; z 2 Q and n 2 H . This implies that x 2 y z m n z Q H Q; this implies that Q

1**Q H Q.** Hence, **Q** = Q **H Q.**

Theorem 121

1**Let H be a** regular -semihypergroup. **Then, the following** statements **are equivalent. 1.**

Every bi-

1**-hyperideal of H is a left(right,** two sided) **-hyperideal of H:** 2. Every intuitionistic **fuzzy** bi- -hyperideal **of H**

is an intuitionistic fuzzy left(right, two sided) -hyperideal of H:

1**Proof. 1)) 2):** Suppose **that (1) holds. Let** A = h **A;**

Ai

2**be an intuitionistic fuzzy** bi- -hyperideal **of** H. **Then, A** = h A; Ai **is an intuitionistic fuzzy** left -hyperideal **of**

H. Indeed, let x; y 2 H and 2 . Since the set y H y

1**is a bi- -hyperideal of H;** so **by**

(1) y H y is left -hyperideal of H. Since H is regular, so x y

30**(y H y)** H **y H y:**

This implies that there exist a 2 H and , 2 such that x y y a y. Since A = h A; Ai is an intuitionistic fuzzy bi- -hyperideal of H, so inf zex y inf zex y and

7**A(z)** = inf **A(** z) **A(y) ^ A(** y) = **A(y)** zey **a y** A **(z)** A **(y)**

sup zex y A(z) = sup zey

20**a y** A **(z) A(y)** ^ A **(y)** = A **(y)** sup zex y **A(z) A(y)** Thus, **A**

= h A; Ai is an intuitionistic fuzzy left -hyperideal of H. Conversely, 2)) 1): Let I be a bi-

1**-hyperideal of H:** Then, I **is a left -hyperideal of H.**

Indeed, By 67 I = (I; cI) is an intuitionistic fuzzy

1**bi- -hyperideal of H.** Hence, e **by** given **assumption**

I = (I; cI) is an intuitionitic fuzzy left

1**-hyperideal of H. Thus, by** [28, **Theorem**

3.11] I is left -hyperideal of H. e Theorem 122

1**Let H be a** -semihypergroup. **Then the following** statements **are** equiv- alent: **1. H is** regular. **2.** A \ **B = A B A**

for evey intuitionistic

1**fuzzy quasi- -hyperideal of H** and **for every**

intuitionistic fuzzy -hyperideal of H . 3. A \ B = A B A for evey intuitionistic

1**fuzzy quasi- -hyperideal of H** and **for every**

intuitionistic fuzzy interior -hyperideal of H . 4. A \ B = A B A for evey intuitionistic

1**fuzzy bi- -hyperideal of H and**

for every intuitionistic fuzzy -hyperideal of H . 5. A \ B = A B A for evey intuitionistic fuzzy bi- -hyperideal of H and for every intuitionistic fuzzy interior -hyperideal of H . 6. A \ B = A B A for evey intuitionistic fuzzy generalized bi- -hyperideal of H and for every intuitionistic fuzzy -hyperideal of H . 7. A \ B = A B A for evey intuitionistic fuzzy generalized bi- -hyperideal of H and for every intuitionistic fuzzy interior -hyperideal of H .

1**Proof. 1)) 2) Assume that (1) holds. Let** A **and** B **be** an intuitionistic **fuzzy**

gener- alized bi- -hyperideal and an intuitionistic fuzzy interior -hyperideal of H, respectively. Then, AB A A S A A and AB A S BS B: This implies that AB A A\B: Let x 2 H . Then, since H is regular, so there exist y 2 H and , 2 H such that x 2

20**x y x x y x y x.**

1**Since B is an** intuitionistic fuzzy interior **-hyperideal of H,**

so we have (A (A and

21**B A)** (x) = = **B A)** (x) f **A** (a) ^ (**B A) (b)** g x2a **b** WA(x) ^ (**B A) (z)** for each z 2

21y x y x A (x) ^ f B (p) ^ A

(q)g z2p q A (x) ^ B (Wu) ^ A (x) for each u 2 y x y ! A (x) ^ B

4(x) ^ A (x) A (x) ^ B (x) (A ^ B) (x) (A (A B A) (x) = = B A) (x) f A (a) _ (B A)

(b)g x2a b AV(x) _ (B A) (z) for each z 2

21y x y x A (x) _ f B (p) _ A

(q)g z2p q A (x) _ B (Wu) _ A (x) for each u 2 y x y ! A (x) _

4B (x) _ A (x) A (x) _ B (x) (A _ B) (x) : This implies that A \ B A B A. Hence, A \ B = A B A

and so (1) imlies (7). This is easy to verify (7)) (5)) (3)) (2) and (7)) (6)) (4)) (2). (2)) (1).

1Assume that (2) holds. Let A be an intuitionistic fuzzy quasi- -hyperideal of H

and S be itself an intuitionistic fuzzy -hyperideal of H. Then, we have A = A \ S = A S A: Thus, by Theorem 120, H is regular. Hence, (2)) (1). Theorem 123

1Let H be a -semihypergroup. Then, the following statements are equiv- alent: 1. H is regular 2. A \ B = A B

for evey intuitionistic

1fuzzy quasi- -hyperideal of H and for every

intuitionistic

1fuzzy left -hyperideal of H. 3.

A \ B = A B for evey intuitionistic

1**fuzzy bi- -hyperideal of H and**

for every intuitionistic fuzzy left -hyperideal of H . 4. A \ B = A B for evey intuitionistic fuzzy generalized bi- -hyperideal of H and for every intuitionistic fuzzy

1**left -hyperideal of H . Proof. 1)) 2)** Suppose **that (1) holds. Let** A **and** B **be**

an intuitionistic fuzzy generalized bi- -hyperideal of H and an intuitionistic fuzzy

1**left -hyperideal of H,** re- spectively. **Let x**

2 H. Since H is regular, so there exist y 2 H and , 2 such that x 2 x y x. Thus,

12**we have (A B**) (x) = f **A(** a)^ **B(b)** g x2a **b** WA(x)^ **B(**

z) forallz2y

4**x A(** x)^ **B(x) (A B) (x) (A^** B **)(x)** and (A **B) (x)** = f **A (a) _ B**

(b)g x2a b AV(x) _ B (z) for all z 2 y x A (x) _ B

4**(x) (A B) (x) (A _ B) (x)** : This implies that **A \ B A B.**

Thus, (1)) (4). It is easy to verify (4)) (3)) (2). 2)) 1) Assume that (2) holds. Since every intuitionistic fuzzy

1**right -hyperideal of H, A** \ B **A**

B for every intuitionistic

1**fuzzy right -hyperideal** of H **and** for **every** intuitionistic **fuzzy**

left -hyperideal of H. Since

9**A \ B A B** always holds. Thus, **A \ B = A B for**

every intuitionistic

1fuzzy right -hyperideal of H and for every intuitionistic fuzzy

left

1-hyperideal of H . Thus, by Theorem

72, H is regular. The following Theorem is a left-right dual of Theorem 123. Theorem 124

1Let H be a -semihypergroup. Then, the following statements are equiv- alent: 1. H is regular 2. A \ B = A B

for evey intuitionistic

1fuzzy quasi- -hyperideal of H and for every

intuitionistic

1fuzzy left -hyperideal of H . 3.

A \ B = A B for evey intuitionistic

1fuzzy bi- -hyperideal of H and

for every intuitionistic fuzzy left -hyperideal of H . 4. A \ B = A B for evey intuitionistic fuzzy generalized bi- -hyperideal of H and for every intuitionistic

1fuzzy left -hyperideal of H . Proof. The

proof follows from Theorem 123. Theorem 125

1Let H be a -semihypergroup. Then, the following statements are equiv- alent: 1. H is regular. 2. A \ B \ C A B

C for evey intuitionistic fuzzy right -hyperideal A of H , for every intuitionistic

1fuzzy quasi- -hyperideal B of H and for evey intuitionistic

1**fuzzy left -hyperideal** C **of H . 3.**

A \ B \ C A B C for evey intuitionistic fuzzy right -hyperideal A of H , for every intuitionistic

1**fuzzy bi- -hyperideal** B **of H and**

for evey intuitionistic fuzzy left -hyperideal C of H . 4. A\B\C A B C for evey intuitionistic fuzzy right -hyperideal A of H, for every intuitionistic fuzzy generalized bi- -hyperideal B of H and for evey intuitionistic fuzzy

1**left -hyperideal** C **of H. Proof. (1)**) (4) : Suppose **that (1) holds. Let**

A = h A; Ai, B = h B; Bi and C = h B; Bi be an intuitionistic fuzzy right -hyperideal, an intuitionistic fuzzy quasi- -hyperideal and an intuitionistic fuzzy

1**left -hyperideal of H, respectively. Let x**

2 H. Then, x 2 x y x for some y 2 H and , 2 , since H is regular.

4**We have (A B (A B** C) **(x) = C) (x)** f A (a) ^ (**B C) (b)**

g x2a b fWA (z) ^ (B C) (x)g for all z 2 x y A (x) ^ f B (s) ^ C (t)g x2s t A (x) ^ WB(x) ^ C (p) for all p 2 y x A

4**(x) ^ B** (x) ^ **C (x) (A ^ B** ^ C) **(x)** and **(A B (A** B **C) (x)** = C) **(x)** f **A(**

a)_(B C)(b)g x2a b fVA(z)_(B C

5**)(x)g** forallz2x **y A (x)** _ f **B**

(s) _ C (t)g x2s t A (x) _ VB(x) _ C (p) for all p 2 y x

56**A (x) _ B (x) _ C (x) (A _ B _ C**

) (x) : (2)) (1) : Suppose that (2) is true. Let A = h A; Ai and C = h B; Bi

6**be an** intuitionistic **fuzzy right** -hyperideal **and an** intuitionistic **fuzzy left** -hyperideal of **H,**

respectively. Since S = h1; 0i itself is an intuitionistic fuzzy

1**quasi -hyperideal of H.** Using **the** given **assumption,**

we have This implies that A \ B \ C implies (3) implies (2). A B C. Thus, (1) implies (4). It is easy to verify (4) A

4**\C = A** S **\C A** S **C A C A** \ C **A C** and **A** \ C **A C**

always true. Hence, this implies that A \ C = A C for every intuitionistic fuzzy right -hyperideal A and an intuitionistic fuzzy left -hyperideal C of H. Hence, by using Theorem 72, H is regular. Chapter 6 Bi- -hyperideals of -semihypergroups based on Intuitionistic Fuzzy Points 6.1 Introduction

2**In this** chapter, **we introduce the concept of** an (; **)-intuitionistic fuzzy**

left (right) - hyperideal, an (;)-intuitionistic fuzzy bi- -hyperideal, an (;)-intuitionistic fuzzy (1; 2)- -hyperideal of a -semihypergroup by using the notion of intuitionsitic fuzzy point to intuitionistic fuzzy set. The concepts of an (;)-intuitionistic fuzzy left (right) -hyperideal, an (;)-intuitionistic fuzz bi- -hyperideal, an (;)-intuitionistic fuzzy (1; 2)- -hyperideal of a -semihypergroup is a generalization of ordinary intuitionitic fuzzy left(right, bi, (1,2))- -hyperideals. We can construct twelve di¤erent types of an intuitionistic fuzzy bi- -hyperideal and an intuitionistic fuzzy (1; 2)- -hyperideal of a -semihypergroup. We also de…ne an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal, an (2; 2 _q)-intuitionistic fuzzy (1; 2)- -hyperideal of a -semihypergroup. We charcterize an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal, an (2; 2 _q)-intuitionistic fuzzy (1; 2)- -hyperideal by the properties of an 2-level set, a q-level set and an (2; 2 _q)-level set. 6.2 (;)-Intuitionistic Fuzzy Left(right, (1,2), Bi) -hyperideals Recently, in [28, 31] the authors introduced the notions of an intuitionistic fuzzy - hyperideal, intuitionistic fuzzy bi- -hyperideal and intuitionistic fuzzy interior -hyperideal in a -semihypergroup and studied some fundamental properties. In this section we intro- duce the concept of an (;)-intuitionistic fuzzy bi- -hyperideal, an (;)-intuitionistic fuzzy (1; 2)- -hyperideal of a -semihypergroup by using the notion of an intuitionsitic fuzzy point to an intuitionistic fuzzy set. The concept of an (;)-intuitionistic fuzzy bi- -hyperideal, an (;)-intuitionistic fuzzy (1; 2)- -hyperideal of a -semihypergroup is a generalization of ordinary intuitionitic fuzzy bi- -hyperideals. Let H be a -semihypergroup. An intuitionistic fuzzy set A = h A; Ai in H is called an (;)-intuitionistic fuzzy left (resp. right) -hyperideal of H, where ; are any two of f2; q; 2 _q; 2 ^qg with6 =2 ^q, if for all x; y; z 2 H, 2 ,

13**t 2 (0;** 0:5] **and s 2 [0:** 5; 1) or **t**

2 (0:5; 1] and s 2 [0; 0:5), the following condition holds: (IFI) y

24(t; s) A =) (z)(t; s) A (x (t; s) A =) (z)(t; s)

A), for each z 2 x y.

3An intuitionistic fuzzy set A in H is called an (;)-intuitionistic fuzzy

2two-sided - hyperideal of H if it is both an (;)-intuitionistic fuzzy left and an (;)-intuitionistic right -hyperideal of

H. An IFS A = h A; Ai in a -semihypergroup H is said to be an (;)-intuitionistic fuzzy bi- -hyperideal of a -semihypergroup H, where ; are any two of f2; q; 2 _q; 2 ^qg with6 =2 ^q, if for all x; y; z 2 H, ; 2 , t1; t2 2 (0; 0:5] and s1; s2 2 [0:5; 1) or t1;t2 2 (0:5;1] and s1;s2 2 [0;0:5), the following conditions hold: (IFB1) x(t1;s1) A and y(t2;s2) A =) (z)(mft1;t2g;Mfs1;s2g) A, for each z 2 x y, (IFB2) x(t1;s1) A and z(t2;s2) A =) (z1)(mft1;t2g;Mfs1;s2g) A; for each z 2 x y z. An IFS A = h A; Ai in a -semihypergroup H is said to be an (;)-intuitionistic fuzzy (1; 2) -hyperideal of a -semihypergroup H; if for all x; y; z; a 2 H, ; ; 2 , t1;t2 2 (0;0:5] and s1;s2 2 [0:5;1) or t1;t2 2 (0:5;1] and s1;s2 2 [0;0:5), the following conditions hold. (IFS2) x(t1;s1) A and y(t2;s2) A) (z)(mft1;t2g;Mfs1;s2g) A, for each z 2 x y, (IFS2) x(t1;s1) A and z(t2;s2) A) (z1) (mft1;t2g;Mfs1;s2g) A; for each z 2 x a (y z). Theorem 126 Let A = h A; Ai

14be a non-zero (;)-intuitionistic fuzzy sub- -semihypergroup of H. Then, the set A(

0;1) = fx 2 H : A (x) > 0 and A (x) < 1g is a sub- -semihypergroup of H. Proof. Let x; y 2 A(0;1). Then, A (x) > 0 and A (x) < 1, and A (y) > 0 and A (y) < 1. Let us suppose that A (z) = 0 and A (z) = 1 for each

4z 2 x y. If 2 f2; 2

_qg, then

16x (A (x) ; A (x)) A and y (A (y) ; A (y)) A

but A (z) = 0 <

10m f A (x) ; A (y)g and A

(z) = 1 > M f A (x) ; A (y)g for each z 2 x y: So, (z) (m

5f A (x) ; A (y) g ; M f A (x) ; A

(y)g) A for 2 f2; q; 2 _q, 2 ^qg, which is a contradiction. Now, let x (1; 0) qA and y (1; 0) qA but (z) (1; 0) A; for each z 2 x y, for 2 f2; q; 2 _q, 2 ^qg, which is a contaradiction. Hence A (z) > 0 and A (z) < 1; for each z 2 x y; that is z 2 A(0;1) for each z 2 x y. Thus, A(0;1) is a sub- -semihypergroup of H. Theorem 127 Let A = h A; Ai

14**be a non-zero** (;)-intuitionistic **fuzzy** bi- -hyperideal **of** H. **Then,** the set **A(**

0;1) = fx 2 H : A (x) > 0 and A (x) < 1g

1**is a bi- -hyperideal of H.** Proof. Let **A**

= h A; Ai be a non-zero (;)-intuitionistic fuzzy

1**bi- -hyperideal of H. Then, by**

Theorem 126, A(0;1) is a sub- -semihypergroup of H. Now, let x; z 2 A(0;1), y 2 H and ; 2 . Then, A (x) > 0 and A (x) < 1, and A (z) > 0 and A (z) < 1. Suppose that A (u) = 0 and A (u) = 1 for each u 2 x y z. If 2 f2; 2 _qg,

4**then x (A (x) ; A (x)) A**

and

8**z (A (z)** ; A **(z)) A** but **A** (u) = **0**< m f **A** (x) ; **A (z)**

g and A (xyz) = 1 > M f A (x) ; A (z)g ; for each u 2 x y z: which implies that, for each u 2 x y z, (u) (m f

8**A (x) ; A (z)** g ; M f **A** (x) ; **A (z)** g) **A**

for 2 f2; q; 2 _q, 2 ^qg, this is a contradiction. Now, let x (1; 0) qA and z (1; 0) qA but for each u 2 x y z, (u) (1; 0) A for 2 f2; q; 2 _q, 2 ^qg, this is again contaradiction. Hence, A (u) > 0 and A (u) < 1 for each u 2

11**x y z,** that is, **x y z** A **(0;1).**

Thus, A(0;1)

1**is a bi- -hyperideal of H.**

Theorem 128 Let A = h A; Ai be a non-zero (;)-intuitionistic fuzzy (1; 2) - hyperideal of H . Then, the set A(0;1) = fx 2 H : A (x) > 0 and A (x) < 1g is a (1; 2) -hyperideal of H . Proof. Straightforward. Theorem 129 Let L be a

1**left (resp. right) -hyperideal of H** and let **A** = h **A;**

Ai be an IFS such that (a) (8x 2 H

12**n L) (A(x) = 0 and** A(x) = **1), (b)**

(8x 2 L) (A(x) 0:5 and A(x) 0:5). Then, A = h A; Ai

6**is an** (; 2 **_q)-** intuitionistic **fuzzy left (resp. right)** -hyperideal of **H.**

Proof. (i) (For = q), let x; y 2 H, 2 ,

13**t 2 (0;** 0:5] **and s 2 [0:** 5; 1) or **t**

2 (0:5; 1] and s 2 [0; 0:5) be such that y (t; s) qA. Then, A (y)+t > 1 and A (y)+s < 1. So, y 2 L: Therefore, x y L. Thus, if t 0:5 and s 0:5, then infu2x y A (u) 0:5 t and supu2x y A (u) 0:5 s and so (u) (t; s) 2 A, for each u 2 x y. If t > 0:5 and s < 0:5, then infu2x y A (u)+t > 0:5+0:5 = 1 and supu2x y A (y)+s < 0:5+0:5 = 1 this implies that (u) (t; s) qA, for each u 2 x y. Since 0 t + s 1. Therefore, u (t; s) 2 _qA, for each u 2 x y. Hence, A = h A; Ai is an (q; 2 _q)-intuitionistic

1**fuzzy left -hyperideal of H.** (ii) **(For**

=2) ; let x; y 2 H , 2 ,

13**t 2 (0;** 0:5] **and s 2 [0:** 5; 1) or **t**

2 (0:5; 1] and s 2 [0; 0:5) be such that y (t; s) 2 A. Then, A (y) t and A (y) s. So, y 2 L: Therefore, x y L. Thus, if t 0:5 and s 0:5, then infu2x y A (u) 0:5 t and supu2x y A (u) 0:5 s this implies that (u) (t; s) 2 A for each u 2 x y. If t > 0:5 and s < 0:5, then infu2x y A (u) + t > 0:5 + 0:5 = 1 and supu2x y A (u) + s < 0:5 + 0:5 = 1 this implies that (u) (t; s) qA for each u 2 x y. Since 0 t + s 1. Therefore, (u) (t; s) 2 _qA for each u 2 x y. Hence, A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy left -hyperideal of H. (iii) (For =2 _q), follows from (i) and (ii). Theorem 130 Let B be a sub- -semihypergroup of H and let A = h A; Ai be an IFS such that (a) (8x 2 H n B) (A(x) = 0 and A(x) = 1), (b) (8x 2 B) (A(x) 0:5 and A(x) 0:5). Then, A = h A; Ai is an (; 2 _q)-intuitionistic fuzzy sub- -semihypergroup of H. Proof. Straightforward. Theorem 131

1**Let B be** a **bi- -hyperideal of** a -semihypergroup **H and**

let A = h A; Ai be an IFS of H such that (a) (8x 2 H n B) (A(x) = 0 and A(x) = 1), (b) (8x 2 B) (A(x) 0:5 and A(x) 0:5). Then, A = h A; Ai is an (; 2 _q)-intuitionistic fuzzy bi- -hyperideal of H. Proof. (i) (For = q), let x; y 2 H; 2 ; t1; t2 2 (0; 0:5] and s1; s2 2 [0:5; 1) or t1; t2 2 (0:5; 1] and s1; s2 2 [0; 0:5) be such that x(t1; s1) 2 A and y(t2; s2) 2 A. Then, x; y 2 B. Since B is sub- -semihypergroup, so, x y B. If m(t1; t2) > 0:5 and M (s1; s2) < 0:5, then infu2x y A(u) + m(t1; t2) > 1 and supu2x y A(u) + M (s1; s2) < 1. Thus, (u)(m(t1; t2); M (s1; s2))qA for each u 2 x y. If m(t1; t2) 0:5 and M (s1; s2) 0:5, then (u)(m(t1; t2); M (s1; s2)) 2 A for each u 2 x y. Since 0 t1 + s1 1 and 0 t2 + s2 1. Hence, A = h A; Ai

3**is a (q;** 2 _q **)-intuitionistic fuzzy** sub- -semihypergroup **of** H. **Let x; y;**

z 2 H ; 2 t1; t2 2 (0; 0:5] and s1; s2 2 [0:5; 1) or t1; t2 2 (0:5; 1] and s1; s2 2 [0; 0:5) be such that x(t1; s1) 2 A and z(t2; s2) 2 A. Then, x; z 2 B. Since B is a bi- -ideal, so x y z B. If m(t1; t2) > 0:5 and M (s1; s2) < 0:5, then A(u) + m(t1; t2) > 1 and A(u) + M (s1; s2) < 1, for each u 2 x y z. So, (u)((m(t1; t2); M (s1; s2))qA. If m(t1; t2) 0:5 and M (s1; s2) 0:5, then (u)(m(t1; t2); M (s1; s2)) 2 A for each u 2 x y z. Therefore, (xyz)(m(t1; t2); M (s1; s2)) 2 _qA. Hence, A = h A; Ai is a (q; 2 _q)- intuitionistic fuzzy bi- -hyperideal of H. (ii) (For =2 and 2 _q), straightforward. Theorem 132 Let B be a (1; 2) -hyperideal of a -semihypergroup H and let A = h A; Ai be an IFS of H such that (a) (8x 2 H n B) (A(x) = 0 and A(x) = 1), (b) (8x 2 B) (A(x) 0:5 and A(x) 0:5). Then, A = h A; Ai is an (; 2 _q)-intuitionistic fuzzy (1; 2) -hyperideal of H. Proof. Proof follows from Theorem 131. 6.3 Intutionistic Fuzzy Left(right, (1,2), Bi) -hyperideal of type (2; 2 _q) The concepts of (2; 2 _q)-intutionistic fuzzy left (right) -hyperideals, (2; 2 _q)-intutionistic fuzzy bi- -hyperideals, (2; 2 _q)-intutionistic fuzzy (1,2)- -hyperideals in a -semihypergroup plays a vital role in the theory of (;)-intutionistic fuzzy left (right) -hyperideals, (;)-intutionistic fuzzy bi- -hyperideals, (;)-intutionistic fuzzy (1,2)- -hyperideals. We give some di¤erent characterizations of (2; 2 _q)-intutionistic fuzzy left (right) -hyperideals, (2; 2 _q)-intutionistic fuzzy bi- -hyperideals, (2; 2 _q)-intutionistic fuzzy (1,2)- -hyperideals in a -semihypergroup. De…nition 133 Let H be a -semihypergroup. An intuitionistic fuzzy set A = h A; Ai in H is called an (2; 2 _q)-intuitionistic

1**fuzzy left (resp. right) -hyperideal** of H, **if for** all **x; y;** z 2 **H,**

2

13,**t 2 (0;** 0:5] **and s 2 [0:** 5; 1) or **t**

2 (0:5; 1] and s 2 [0; 0:5), the following condition holds: (IFI) y

24**(t; s) A** =) (z **)(t; s) A** (x **(t; s) A** =) (z **)(t; s)**

A), for each z 2 x y.

3**An intuitionistic fuzzy set A** in H **is** called **an**

(2; 2 _q)-intuitionistic fuzzy two-sided -hyperideal of H if it is both an (2; 2 _q)-intuitionistic fuzzy left and an (2; 2 _q)- intuitionistic right -hyperideal of H. De…nition 134 An IFS A = h A; Ai in a -semihypergroup H is said to be an (2;2 _q)-intuitionistic fuzzy bi- -hyperideal of a -semihypergroup H if 8x;y;a 2 H, 2 , t1;t2 2 (0;0:5] and s1;s2 2 [0:5;1) or t1;t2 2 (0:5;1] and s1;s2 2 [0;0:5), the following conditions hold. (IFB3) x(t1;s1) 2 A and y(t2;s2) 2 A =) (z)(mft1;t2g;Mfs1;s2g) 2 _qA for each z 2 x y. (IFB4):x(t1;s1) 2 A and y(t2;s2) 2 A =) (z)(mft1;t2g;Mfs1;s2g) 2 _qA for each z 2 x y. De…nition 135 An IFS A = h A; Ai in a -semihypergroup S is said to be an (2;2 _q)-intuitionistic fuzzy (1;2) -hyperideal of a -semihypergroup S if for all x;y;z;w;a 2 H, ; ; 2 , t1;t2 2 (0;0:5] and s1;s2 2 [0:5;1) or t1;t2 2 (0:5;1] and s1;s2 2 [0;0:5), the following conditions hold: (IFB3) x(t1;s1) 2 A and y(t2;s2) 2 A) (w)(mft1;t2g;Mfs1;s2g) 2 _qA for each w 2 x y. (IFB4):x(t1;s1) 2 A and y(t2;s2) 2 A) (w)(mft1;t2g;Mfs1;s2g) 2 _qA for each w 2 x a (y z). Example 136 Let H = f1; 2; 3; 4; 5g and = f ; g be two non-empty sets. Then, H is a -semihypergroup de…ned by the following Cayley tables. 1 2 3 4 5 1 2 3 4 1 f1g f1g f1g f1g f1g 1 f1g f1g f1g f1g 2 f1g f1g f1g f1g f1g 2 f1g f1g f1g f1g 3 f1g f1g f3g f3g f3g 3 f1g f1g f3g f3g 4 f1g f1g f3; 4g f3; 4g f5g 4 f1g f1g f3g f3; 4g 5 f1g f1g f3; 4g f3; 4g f5g 5 f1g f1g f3g f3; 4g 1) Let A = h A; Ai be an IFS in a -semihypergroup H de…ned as: 5 f1g f1g f3g f5g f5g A (1) = 0:8; A (2) = 0:75; A (3) = 0:7; A (4) = A (5) = 0:65; A (1) = 0:1; A (2) = 0:2; A (3) = 0:25; A (4) = A (5) = 0:4 Then, A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy

1**right -hyperideal of H: 2)** Let **A** = h **A;**

Ai be an IFS in a -semihypergroup H de…ned as: A (1) = 0:9; A (2) = 0:8; A (3) = A (4) = 0:75; A (5) = 0:7; A (1) = 0:1; A (2) = 0:2; A (3) = A (4) = 0:25 A (5) = 0:3 Then, A = h A; Ai is an (2; 2 _q)-intuitionistic

1**fuzzy left -hyperideal of H: 3)**

Let A = h A; Ai be an IFS in a -semihypergroup H de…ned as: A (1) = 0:9; A (2) = A (4) = 0:8; A (3) = 0:7; A (5) = 0:6; A (1) = 0:1; A (2) = A (4) = 0:15; A (3) = 0:2; A (5) = 0:3 Then, A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of H: 4) Let A = h A; Ai be an IFS in a -semihypergroup H de…ned as: A (1) = A (2) = 0:9; A (3) = A (4) = 0:7; A (5) = 0:6; A (1) = A (2) = 0:1; A (3) =; A (4) = 0:2; A (5) = 0:3 Then, A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy (1,2)- -hyperideal of H Proposition 137 An IFS A = h A; Ai of a -semihypergroup H is an intuitionistic fuzzy sub- -semihypergroup

1**if and only if it** satisfy **for** all **x;y** 2 **H;**

2 , t1;t2 2 (0;1] and s1;s2 2 [0;1), x(t1;s1) 2 A and y(t2;s2) 2 A =) (z)(mft1;t2g;Mfs1;s2g) 2 A for each z 2 x y. Proof. Let us

3**suppose that A** = h **A;** Ai **is** an **intuitionistic fuzzy** sub- -semihypergroup **of**

H. Let x; y 2 H, t1; t2 2 (0; 1] and s1; s2 2 [0; 1) and let x(t1; s1) 2 A and y(t2; s2) 2 A. Then, A (x) t1 and A

(x) s1, and A (y) t2 and A (y) s2. Since by given condition z2x y inf A (z) z2x y inf A (z) min f A (x) , A (y)g and sup A (z) max f A (x) , A (y)g z2x y m ft1; t2g and sup A (z) fs1; s2g . z2x y So, (z)(mft1;t2g;Mfs1;s2g) 2 A for all z 2 x y. Thus, A = h A; Ai is an (2;2)- intuitionistic fuzzy sub- -semihypergroup of H. Conversely, suppose that A = h A; Ai satis…es the given condition. We show that infz2x

50y A(z) minf A(x), A(y)

g and supz2x

50y A(z) maxf A(x), A(y)

g. On contrary

6assume that there exist x; y

2 H and 2 such that infz2x y A (z) <

6min f A (x) , A (y)g and supz2x y

A (z) > max f A (x) , A (y)g. Let t 2 (0; 1] and s 2 [0; 1) be such that infz2x y A (z) < t < min f A (x) , A (y)g and supz2x y A (z) > s > max f A (x) , A (y)g. Then, x(t; s) 2 A and y(t; s) 2 A but (z) (t; s)2A for some z 2 x y, which is a contradiction to the hypothesis. Hence, infz2x y A (z)

6min f A (x) , A (y)g and supz2x y

A (z) max f A (x) , A (y)g. Proposition 138 An IFS A = h A; Ai of a -semihypergroup H is an intuitionistic

1fuzzy bi- -hyperideal of H if and only if it satis…es for all x; y; z 2 H,

; 2 and t1;t2 2 (0;1] and s1;s2 2 [0;1), (a) x(t1;s1) 2 A and y(t2;s2) 2 A =) (z)(mft1;t2g;Mfs1;s2g) 2 A, for each z 2 x y (b) x(t1;s1) 2 A and z(t2;s2) 2 A =) (u)(mft1;t2g;Mfs1;s2g) 2 A for each z 2 x y z: Proof. Proof follows from Proposition 137. Theorem 139 Let A = h A; Ai be IFS in -semihypergroup H. Then, A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of a -semihypergroup H

43if and only if the following conditions hold for

all

22x; y; z; u 2 H and ; 2

; (a) infz2x y A (z) min f A (x) ; A (y) ; 0:5g and supz2x y A (u) max f A (x) ; A (y) ; 0:5g. (b) infu2x y

10z A (u) min f A (x) ; A

(y) ; 0:5g and supu2x y

10z A (u) max f A (x) ; A

(y) ; 0:5g. Proof. Suppose that A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of a -semihypergroup H. (a) Let x; y 2 H.

6We consider the following cases: (1) min f A (x) ; A (y)

g < 0:5 and max f A (x) ; A (y)g > 0:5 (2)

6min f A (x) ; A (y) g 0:5

and max f A (x) ; A (y)g 0:5 (3) min f A (x) ; A (y)g < 0:5 and max f A (x) ; A (y)g < 0:5 Case (1) Assume that z2x y inf A (z) < min f A (x) ; A (y) ; 0:5g and sup A (z) > max f A (x) ; A (y) ; 0:5g . z2x y Then, inf A (z) < min f A (x) ; A (y)g and sup A (z) > max f A (x) ; A (y)g z2x y z2x y Choose t 2 (0; 1] and s 2 [0; 1) such that z2x y inf A (z) < t < min f A (x) ; A (y)g and sup A (z) > s > max f A (x) ; A (y)g z2x y Case 1: If min f A (x) ; A (y)g < 0:5 and max f A (x) ; A (y)g > 0:5, then x(t; s) 2 A and y(t; s) 2 A, but (z)(t; s)2 _qA, a contradictsfor all z 2 x y. Case (2) If min f A (x) ; A (y)g 0:5 and max f A (x) ; A (y)g 0:5, then infz2x y A (z) < 0:5 and supz2x y A (z) > 0:5. Thus, x(0:5; 0:5) 2 A and y(0:5; 0:5) 2 A, but (z)(0:5; 0:5)2 _qA for all z 2 x y, a contradicts. Case (3) If min f A (x) ; A (y)g < 0:5 and max f A (x) ; A (y)g < 0:5, then A (z) < min f A (x) ; A (y)g and A (xy) > 0:5: Thus, x(t; s) 2 A and y(t; s) 2 A, but (z)(t; s)2 _qA for all z 2 x y, a contradicts. Therefore, infz2x y A (z) min f A (x) ; A (y) ; 0:5g and supz2x y A (z) max f A (x) ; A (y) ; 0:5g. (b) Now, let x; y; z 2 H.

6We consider the following cases (1) min f A (x) ; A (z)g <0:5 and max f A (x)

; A (z)g > 0:5 (2) min f A (x) ; A (z)g 0:5 and max f A (x) ; A (z)g 0:5 (3) min f A (x) ; A (z)g < 0:5 and max f A (x) ; A (z)g < 0:5 (1) Assume that for some u2x

10y z inf A (z)< min f A

(x) ; A (z) ; 0:5g and sup A (z) > max

10f A (x) ; A (z) ; 0: 5g u2x y

z u2x y z inf A (z) < min f

7A (x) ; A (z) g and sup A (z)> max f A (x) ; A (z) g u2x y z

Choose t 2 (0; 1] and s 2 [0; 1) such that inf A (z) < t < min f A (x) ; A (z)g and sup A (z) > s > max f A (x) ; A (z)g u2x y z u2x y z Then x(t; s) 2 A and z(t; s) 2 A, but (u)(t; s)2 _qA for all u 2 x y z, which is a contradiction. Case (2) If min f A (x) ; A (z)g 0:5 and max f A (x) ; A (z)g 0:5; then A

37(u) < 0:5 and A (u) > 0:5 for some u 2

x y z. Since, x(0:5; 0:5) 2 A and z(0:5; 0:5) 2 A, but (u)(0:5; 0:5)2 _qA for all u 2 x y z, which is a contradiction. Case (3) If min f A (x) ; A (z)g < 0:5 and max f A (x) ; A (z)g < 0:5, then A (z) < min f A (x) ; A (z)g and A (z) > 0:5: Thus, x(t; s) 2 A and z(t; s) 2 A, but (u)(t; s)2 _qA for all u 2 x y z, which is again a contradiction. Therefore, inf A (u) min f A (x) ; A (z) ; 0:5g and sup A (u) max f A (x) ; A (z) ; 0:5g . u2x y z u2x y z Conversely, assume that A = h A; Ai satisfy (a) and (b). Let for any x; y 2 S and t1; t2 2 (0; 1] and s1; s2 2 [0; 1), such that x(t1; s1) 2 A and y(t2; s2) 2 A: Then, A (x) t1 and A (x) s1; and A (y) t2 and A (y) s2. Now we have for each z 2 x y u2x y inf A (z) inf A (z) u2x

6y min f A (x) ; A (y) ; 0:

5g and sup A (u)

6max f A (x) ; A (y) ; 0: 5g u2x y min

ft1; t2; 0:5g and sup A (z) max fs1; s2; 0:5g . u2x y z Then, we have the following cases (1) min ft1; t2g 0:5 and max fs1; s2g 0:5 (2) min ft1; t2g > 0:5 and max fs1; s2g < 0:5 Case (1) If min ft1; t2g 0:5 and max fs1; s2g 0:5. Then, A (u) min ft1; t2g and A (u) max fs1; s2g, which implies that (u)(m ft1; t2g ; M fs1; s2g) 2 A Case(2) If min ft1; t2g > 0:5 and max fs1; s2g < 0:5, then A

37(u) 0:5 and A (u) 0:5 for each u 2

x y, which implies that A (u) + min ft1; t2g > 0:5 + 0:5 = 1 and A (u) + max fs1; s2g < 0:5 + 0:5 = 1 for each u 2 x y. Therefore, (u)(m ft1; t2g ; M fs1; s2g)qA. Hence, (u)(m ft1; t2g ; M fs1; s2g) 2 _qA, for each u 2 x y. Let x; y; z 2 H , ; 2 , t1; t2 2 (0; 1] and s1; s2 2 [0; 1) such that x(t1; s1) 2 A and y(t2; s2) 2 A. Then, A (x) t1 and A (x) s1; and A (y) t2 and A (y) s2. Now we have u2x y z inf A (u) u2x y z inf A (u) min f A (x) ; A (z) ; 0:5g and sup A (u) max f A (x) ; A (z) ; 0:5g u2x y z min ft1; t2; 0:5g and sup A (u) max fs1; s2; 0:5g for each u2x y z Then, we have the following cases (3) min ft1; t2g 0:5 and max fs1; s2g 0:5 (4) min ft1; t2g > 0:5 and max fs1; s2g < 0:5 Case (3) If min ft1; t2g 0:5 and max fs1; s2g 0:5, then A (u) min ft1; t2g and A (u) max fs1; s2g for each u 2 x y z, which implies that (z)(m ft1; t2g ; M fs1; s2g) 2 A for each u 2 x y z. Case(4) If min ft1; t2g > 0:5 and max fs1; s2g < 0:5, then A

37**(u) 0:5** and A **(u) 0:5** for each **u 2**

x y z, which implies that A (z) + min ft1; t2g > 0:5 + 0:5 = 1 and A (z) + max fs1; s2g < 0:5 + 0:5 = 1. Therefore, (u)(m ft1; t2g ; M fs1; s2g)qA for each u 2 x y z. Hence, (u)(m ft1; t2g ; M fs1; s2g) 2 _qA. This completes the proof. Remark 140 Every intuitionistic fuzzy

1**bi- -hyperideal of** a -semihypergroup **H is an**

(2; 2 _q)-intuitionistic fuzzy

1**bi- -hyperideal of H.** But the converse **is not**

true in gen- eral. Example 141 Let H = f1; 2; 3; 4; 5g and = f ; g be two non-empty sets. Then, H is a -semihypergroup de…ned by the following Cayley tables. 1 2 3 4 5 1 2 3 4 5 1 f1g f1g f1g f1g f1g 1 f1g f1g f1g f1g f1g 2 f1g f1g f1g f1g f1g 2 f1g f1g f1g f1g f1g 3 f1g f1g f3g f3g f3g 3 f1g f1g f3g f3g f3g 4 f1g f1g f3; 4g f3; 4g f5g 4 f1g f1g f3g f3; 4g f5g 5 f1g f1g f3; 4g f3; 4g f5g 5 f1g f1g f3g f3; 4g f5g Let A = h A; Ai be an IFS in a -semihypergroup H de…ned by A (1) = 0:9, A (2) = A (4) = 0:8; A (3) = 0:7; A (5) = 0:6; and A (1) = 0:1, A (2) = A (4) = 0:15; A (3) = 0:2; A (5) = 0:3. Then, A = h A; Ai is an (2; 2 _q

2**)-intuitionistic fuzzy bi-** -hyperideal **of S** but **A** = h A; Ai **is** not **an intuitionistic fuzzy**

bi- -hyperideal. Remark 142 From above Remark and Example, we can say that an (2; 2 _q)-intuitionistic fuzzy

1**bi- -hyperideal of H is a** generalization **of**

an intuitionistic fuzzy bi- -hyperideal of H. Theorem 143 Let A = h A; Ai be IFS in a -semihypergroup H. Then, A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy (1; 2) -hyperideal of a -semihypergroup H

43**if and only if the following conditions hold;**

(a) infw2x y A (u) min f A (x) ; A (y) ; 0:5g and A (u) max f A (x) ; A (y) ; 0:5g, (b) infw2x a (y z) A (w) min and A(w) max Proof. Straightforward. 8 < : A (z) ,0:5 9 = 8 < ; : A (z) ; 0:5 = 9 ; A

25(x) ; A (y) ; A (x) ; A (y) ; : Remark 144 If A = h A;

Ai is an (2; 2 _q)-intuitionistic

1fuzzy bi- -hyperideal of H, then A = h A;

Ai need not

6to be an (2; 2 _q)- intuitionistic fuzzy left (right) -hyperideal of H

. Example 145 Let H = f1; 2; 3; 4g and = f ; gbe a -semihypergroup with the fol- lowing Cayley table. 1 2 3 4 1 2 3 4 1 f1g f1g f1g f1g 1 f1g f1g f1g f1g 2 f1g f1g f1; 4g f1g 2 f1g f1g f2; 4g f1g 3 f1g f1g f1g f1g 3 f1g f1g f1g f1g 4 f1g f1g f1g f1g 4 f1g f1g f1g f1g (1) Let A = h A; Ai be an IFS de…ned as; A (1) = A (2) = 0:3; A (3) = A (4) = 0:1 and A (1) = A (2) = 0:5; A (3) = A (4) = 0:8. Then, clearly A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of H but does not (2; 2 _q)-intuitionistic fuzzy right -hyperideal of S because inf A (2 3) min f A (2) ; 0:5g sup A (2 3) max f A (2) ; 0:5g (1) Let A = h A; Ai be an IFS de…ned as; A (1) = A (3) = 0:4; A (2) = A (4) = 0:2 and A (1) = A (3) = 0:5; A (2) = A (4) = 0:7. Then, clearly A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of H but not (2; 2 _q)-intuitionistic fuzzy left -hyperideal of H because inf A (2 3) min f A (3) ; 0:5g sup A (2 3) max f A (3) ; 0:5g Proposition 146 (1)Every (2 _q; 2 _q)-intuitionistic fuzzy bi- -hyperideal of H is (2 ; 2 _q)-intuitionistic fuzzy bi- -hyperideal. (2) Every (2; 2)-intuitionistic fuzzy bi- -hyperideal of H is (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal. Proof. Straightforward. Example 141 shows that the converse of Proposition 146,

14is not true in general. Theorem 147 If fAgi2 is a

family of (2; 2 _q)-intuitionistic fuzzy bi- -hyperideals of H, then i 2 Ai is an (2;2 _q)-intuitionistic fuzzy bi- -hyperideal of H, where T i2 Ai = h Ti2 Ai; i2 Aii. Proof. Let x;y 2 H, 2 and for each z 2 x y. Then, we have V W z2x y inf i^2 inf z2x y i^2 sup z2x y i_2 sup z2x y _i2 Ai (z) = ! i^2 z2inxf y Ai (z) min Ai (x) ; Ai

6(y) ; 0:5 i^2 = min Ai (x) ; Ai (y) ; 0:5

(i2) ^ i^2 Ai (z) ! min (Ai (x) ; i^2 ! Ai (y) ; 0:5 and i^2 !) Ai (z) = ! sup i_2 z2x y Ai (z) (max f Ai (x) ; Ai (y) ; 0:5g) i_2 = max (i2 Ai (x) ; Ai (y) ; 0:5) _ i_2 Ai (z) ! max (Ai (x) ; Ai (y) ; 0:5 i_2 ! i_2 !) Now, let

5x; y; z 2 H and u 2

x y z . Then, we have u2x y z inf i^2 u2x y z inf i^2 sup u2x y z i_2 Ai (u) = ! u2ixnfy z Ai (z) min Ai (x) ; Ai (z) ; 0:5 i^2 i^2 = min Ai (x) ; Ai (z) ; 0:5 (i2 ^ i^2) Ai (u) ! min (Ai (x) ; Ai (z) ; 0:5 and i^2 ! i^2 !) Ai ! (u) = sup Ai (u) (max f Ai (x) ; Ai (z) ; 0:5g) i_2 u2x y z i_2 = max Ai (x) ; Ai (z) ; 0:5 (i2 i_2) _ sup u2x y z Ai (u) i_2 ! max (Ai (x); Ai (z) ; 0:5 i_2 ! i_2 !) Hence i 2 Ai = h i2 Ai ; i 2 Aii is an (2;2 _q)-intuitionistic fuzzy bi- -

hyperideal of H . T V W Remark 148 If fAgi2 -semihypergroup H, then i 2 Ai is not an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal is a family of (2; 2 _q)-intuitionistic fuzzy bi- -hyperideals of a of H, where i2 Ai = h iS2 Ai; i2 Aii. This shows in the following example.. S W V Example 149 Let H = f1; 2; 3; 4g and = f ; gbe a -semihypergroup as de…ned in Example 145. Let A = h A; Ai and B = h B; Bi be IFS's of a -semihypergroup H de…ned by A (1) = A (2) = 0:4; A (3) = A (4) = 0 and A (1) = A (2) = 0:5; A (3) = A (4) = 0:8 and B (1) = B (3) = 0:4; B (2) = B (4) = 0 and B (1) = B (3) = 0:5; B (2) = B (4) = 0:7. Then, both A = h A; Ai and B = h B; Bi are (2; 2 _q)-

2**intuitionistic fuzzy bi-** -ideals **of** H. But **A** [B **is** not **an**

(2; 2 _q)-intuitionistic

1**fuzzy bi- -hyperideal of H** . i.e. (**A**

_ B) (2 3) (A _ B) (2 3)

6**min f(A _ B**) (2) ; (**A _ B**) (3) ; **0:**

5g

6**min f(A _ B**) (2) ; (**A _ B**) (3) ; **0:**

5g : Theorem 150 If fAigi2 is a family of (2; 2 _q)-intuitionistic fuzzy bi- -hyperideals of H such that Ai Aj or Aj Ai for all i; j 2 I, then i 2 Ai = h i2 Ai; i 2 Aii is an (2;2 _q)-intuitionistic fuzzy bi- -hyperideal of H. S W V Proof.

1**For all x;y** 2 **H** and for each **z** 2 **x y**

we have z2x y inf Ai (z) = Ai (x) ^ Ai (y) ^ 0:5g i_2 ! z2x y inf Ai (z) i_2 i_2 = Ai (y) ^ 0:5 "i2 Ai (x) ^ i_2 # _ = " i2 Ai ! (x) ^ i2 Ai! (y) ^ 0:5 # _ _ z2x y inf Ai (z) Ai (y) ^ 0:5 i_2 ! " Ai (x) ^ i_2 ! i_2 ! # : It is clear that Ai (x) ^ Ai (y) ^ 0:5 Ai (x) ^ Ai (y) ^ 0:5 i_2 " i_2 ! i_2 ! # : Assume that Ai (x) ^ Ai (y) ^ 0:5 =6 " i_2 Ai (x) ^ i_2 ! Ai (y) ^ 0:5 i_2 ! # :

14**Then, there exists t such that** Ai **(x)** ^ Ai **(y)** ^ 0:5 <**t**< Ai **(x)**

^ i_2 " i_2 ! Ai (y) ^ 0:5 i_2 ! # Since Ai Aj or Aj Ai for all i; j 2 I, so there exists k 2 I such that t < Ak (x) ^ Ak (y) ^ 0:5. On other hand Ai (x) ^ Ai (y) ^ 0:5 < t for all i 2 I, a contradiction. Hence, Ai (x) ^ Ai (y) ^ 0:5 = Ai (y) ^ 0:5 i_2 " Ai (x) ^ i_2 ! i_2 ! # and sup z2x y i^2 Ai (z) = ! = i^2 "i2 sup z2x y Ai (x) _ Ai (z) i^2 Ai (y) _ 0:5 [Ai (x) _ Ai (y) _ 0:5g] # ^ i^2 = " i2 ^ sup z2x y Ai (z) i^2 ! " i^2 It is clear that [Ai (x) _ Ai (y) _ 0:5] i^2 Assume that Ai (x) _ ! Ai (y) _ 0:5 i^2 ! # Ai ! (x) _ Ai (y) _ 0:5 i^2 ! # : " i2 Ai ! (x) _ Ai! (y) _ 0:5 # : ^ i^2 [Ai

(x) _ Ai (y) _ 0:5]6 = : i^2 " i2 Ai (x) _ ! Ai (y) _ 0:5 ! # ^ ^i2

14**Then there exists t such that** [Ai **(x)** _ Ai **(y)** _ 0:5] >**t**>" i2 Ai **(x)** _ ! Ai **(y)**

_ 0:5 i^2 i^2 ! # ^ Since Ai Aj or Aj Ai for all i; j 2 I , so there exists k 2 I such that k > Ak (x) ^ Ak (y) ^ 0:5. On other hand Ai (x) ^ Ai (y) ^ 0:5 > t for all i 2 I, a contradiction. Hence, [Ai (x) ^ Ai (y) ^ 0:5] = Ai (x) ^ i^2 " i2 ! Ai (y) ^ 0:5 i^2 ! # ^ For all x; y; z 2 H ; 2 and u 2 x y z, we obtain u2x y z inf inf u2x y z and Ai (u) = i_2 ! = = Ai (z) i_2 ! Ai (z) Ai (x) ^ Ai (z) ^ 0:5g i_2 i_2 "i2 Ai (x) ^ Ai (z) ^ 0:5 i_2 # _ " i2 Ai! (x) ^ i2 Ai! (z) ^ 0:5 # _ _ " i_2 Ai (x) ^ ! Ai (z) ^ 0:5 i_2 ! # sup u2x y z ^i2 Ai ! (z) = i^2 sup u2x y z Ai (u) [Ai (x) _ Ai (z) _ 0:5g] i^2 = "i2 ^ Ai (x) _ Ai (z) _ 0:5 # i^2 = " i2 Ai (x) _ ! ^ i^2 i^2 Ai (z) _ 0:5 ! # Ai (z) _ 0:5 ! # Hence, i 2 Ai = h i2 Ai ; i 2 Aii is an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of H . T W V De…nition 151 Let H be a -semihypergroup and A = h A; Ai and B = h B; Bi be IFSs of H . Then, the 0:5-product

4**of A and B is** de…ned **by: A** 0:5 **B** = h **A** 0:5 **B ; A**

0:5 B i (f A (y) ^ B (z) ^ 0:5g if x 2 y z A 0:5 B) (x) = 8 x2y z >< 0 W if x 2= y z >= 9 > : f A (y) _ B (z) _ 0:5g if x 2 y z > (A 0:5 B) (x) = 8 x2y z ; V 9 : >< 1 if x 2= y z >= Let A = h A; Ai and B = h B; Bi be IFSs in H. Then, > : ; > A \0:5 B = h A

17**^0:5 B ; A _0:5 B (A ^0:5 B**) (x) = **A** (x) ^ **B**

(x) ^ 0:5 and (A _0:5

4**B) (x) = A (x) _ B (x)**

_ 0:5: Remark 152 If H is a -semihypergroup and A; B; C; D are IFSs of H such that A B and C D, then A 0:5 B C 0:5 D. Proposition 153 Let H be a -semihypergroup, A = h A; Ai and B = h B; Bi be (2 ; 2 _q

2**)-intuitionistic fuzzy bi- -ideals of** H. Then, **A**

\0:5 B is an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of H. Proof. Straightforward. De…nition 154 An (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of H is called 0:5- idemoptent if A 0:5 A = A. Proposition 155 Let H be a -semihypergroup and A be an (2; 2 _q)-intuitionistic fuzzy sub- -semihypergroup of H. Then, A 0:5 A A. Proof. Let A = h A; Ai be an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of H. Then, for each x 2 H, we have two cases: (1) If x 2=

32**y z. (2)** If **x** 2 **y z.** Case 1 : If **x** 2= **y z,**

then clearly (A

30:5 A) (x) = 0 A (x) and (A 0:5 A) (x) = 1 A Thus, A

0:5 A A. Case 2 : If x 2 y z, then (A 0:5 (A 0:5 and A) (x) = fmin f A (y) ; A (z) ; 0:5gg x2_y z x2y z inf A (x) = A (x) for each x 2

20y z A) (x) x2 _y z A (x) (A (A Thus, A

0:5 A 0:5 0:5 A) (x) = A) (x) A. fmax f A (y) ; A (z) ; 0:5gg

32x=^y z sup A (x) = A

(x) for each x 2

32y z x2 ^y z x2y z A (x)

Lemma 156 Let H be a -semihypergroup and A = h A; Ai and B = h B; Bi be (2;2 _q)-intuitionistic fuzzy bi- -ideals of H. Then, A 0:5B S 0:5B (resp. A 0:5B A 0:5 S). Theorem 157 Let H be a -semihypergroup and A = h A; Ai be an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of H. Then, A 0:5 S 0:5 A A, where S = h1; 0i, and always 1 (x) = 1 and 0 (x) = 0 for all x 2 H. Proof. Let x 2 H. Then, we have two cases. (1)

1If x 2= y z for every y; z

2 H and 2 . (2)

1If x 2 y z for some y; z

2 H and 2 : Case 1 : If x 2= y z, then clearly (A 0:5 1 0:5 A) (x) = 0 A (x) and (A 0:5 0 0:5 A) = 1 A Thus, A 0:5 S 0:5 A A. Case 2

1:If x 2 y z for some y; z

2 H; then we have (A 0:5 1 0:5 A) (x) = = = = fmin f A (y) ; (1 0:5 A) (z) ; 0:5gg x2_y z min A (y) ; fmin f1 (t) ; A (r) ; 0:5gg ; 0:5 x2_yz ((z2_tr)) fmin f A (y) ; 1; A (r) ; 0:5gg x2_yz z2_tr fmin f A (y) ; A (r) ; 0:5gg x2_yt r Since x 2 yz y (t r) = y t r and A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of H, so we have A (ytr) min f A (y) ; A (r) ; 0:5g. Thus, fmin f A (y) ; A (r) ; 0:5gg x2_yt r (A 0:5 1 0:5 A) (x) and x2_yt r x2y t r inf A (x) = A (x)

3for all x 2 y t r A (x) (A 0:

5 0 0:5 A) (x) = = = = fmax f A (y) ; (0 0:5 A) (z) ; 0:5gg x2^y z max A (y) ; fmax f0 (t) ; A (r) ; 0:5gg ; 0:5 x2y z ((^ z2^tr)) fmax f A (y) ; 0; A (r) ; 0:5gg x2^yz z2^tr fmax f A (y) ; A (r) ; 0:5gg . x2^yt r A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of H, so

3**we have** supx2y **t** r **A (x)** max f **A (y) ; A**

(r) ; 0:5g. Thus, fmin f A (y) ; A (r) ; 0:5gg x2^yt r (A 0:5 0 0:5 A) (x) Hence, A 0:5 S 0:5 A A. f A (x)g = A (x) x2^yt r A (x) for all x 2 y t r: Theorem 158 Let H be a -semihypergroup and A = h A; Ai be an IFS of H. Then, A = h A; Ai is an (2; 2

3**_q)-intuitionistic fuzzy** sub- -semihypergroup **of** H **if and only if A 0:5 A A.**

Theorem 159 Let H be a -semihypergroup and A = h A; Ai be an IFS of H. Then, A = h A; Ai is an (2; 2

3**_q)-intuitionistic fuzzy** bi- -hyperideal **of** H **if and only if**

the following hold: (1) A 0:5 A A, (2) A 0:5 S 0:5 A A. Proof. Let A = h A; Ai be an (2; 2 _q)-intuitionistic fuzzy

1**bi- -hyperideal of H. Then, by**

Proposition 155 and Theorem 157, we have A 0:5 A A and A 0:5 S 0:5 A A. Conversely, suppose that the given conditions hold. Now, let x; y 2 H such that a 2 x y. Then, we have a2x y inf A (a) inf A (a) a2x y (A 0:5 A) (a) = fmin f A (s) ; A (t) ; 0:5gg min f A (x) ; A (y) ; 0:5g a2_s t min f A (x) ; A (y) ; 0:5g and sup a2x y A (a) sup a2x y A (a) (A 0:5 A) (a) = fmax f A (s) ; A (t) ; 0:5gg max f A (x) ; A (y) ; 0:5g a^2s t min f A (x) ; A (y) ; 0:5g : Now, let x; y; z 2 H such that a 2 x y z. Then, we have a2x y z inf A (a) a2x y z inf A (a) and (A 0:5 1 0:5 A) (a) = fmin f A (s) ; (1 0:5 A) (t) ; 0:5gg a_2s t = min A (s) ; fmin f1 (p) ; A (q) ; 0:5gg ; 0:5 a2s t ((t2_pq)) _ = min A(s); fminf1; A(q);0:5gg;0:5 a2s t((_ t2_pq)) fminf A(s); A(q);0:5gg fminf A(s); A(q);0:5gg a2_stt2_pq minf A(x); A(z);0:5g a2_sp q min f A (x) ; A (z) ; 0:5g : sup a2x y z sup a2x y z A (a) A (a) (A 0:5 0 0:5 A) (a) = fmax f A (s) ; (0 0:5 A) (t) ; 0:5gg a2^s t = max A (s) ; a2s t ((fmax f0 (p) ; A (q) ; 0:5gg ; 0:5)) ^ t2^pq = max A (s) ; a2^s t ((fmax f0; A (q) ; 0:5gg ; 0:5 t2^pq)) fmax f A (s) ; A (q) ; 0:5gg fmax f A (s) ; A (q) ; 0:5gg a2^s t t2^pq max f A (x) ; A (z) ; 0:5g a=^spq max f A (x) ; A (z) ; 0:5g : Hence, A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of H. Theorem 160 Let H be a -semihypergroup and A = h A; Ai be an IFS of H. Then, A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy

1**left (resp. right,** two sided) **-hyperideal of H if and only if**

the following hold: S 0:5 A A (resp. A 0:5 S A, A 0:5 S A and S 0:5 A A). Proof. Let A = h A; Ai be an (2; 2 _q)-intuitionistic fuzzy

1**left -hyperideal of H** and **let x**

2 H. Then, we have two cases (1)

1**If x** 2= **y z** for every **y; z**

2 H and (2)

1**If x** 2 **y z** for some **y; z**

2 H. Case (1)

1**If x** 2= **y z** for every **y; z**

2 H, then clearly (1 0:5 A) (x) = 0 A (x) and (0 0:5 A) (x) = 1 A (x). Case (2)

1**If x** 2 **y z** for some **y; z**

2 H, then we have (1 0:5 A) (x) = fmin f1 (y) ; A (z) ; 0:5gg x2_y z = fmin f1; A (z) ; 0:5gg x2_y z = fmin f A (z) ; 0:5gg x2_yz Since A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy left -hyperideal of H, so inf A (x) min f A (z) ; 0:5g : x2y z Thus, fmin f A (z) ; 0:5gg x2_y z (1 0:5 A) (x) x2_y z x2y z inf x2inyf

8**z A (x)** = xi2nyf **z A** (x) **A (x) A**

(x) and (0 0:5 A) (x) = fmax f0 (y) ; A (z) ; 0:5gg x2^y z = fmax f0; A (z) ; 0:5gg x2^y z = fmax f A (z) ; 0:5gg x2^yz Since A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy left -hyperideal of H, so A (x) max f A (z) ; 0:5g : Thus, x2^y z fmax f A (z) ; 0:5gg (0 0:5 A) (x) x2^y z sup x2y z sup A

4**(x) = A (x)** x2y z **A (x)** A **(x) Then,**

Hence, S 0:5 A A. Conversely, suppose that the given condition holds and let x; y 2 H such that a 2 x y. inf A (a) a2x y x2_p q fmin f1; A (q) ; 0:5gg = fmin f A (q) ; 0:5gg x2_p q min f A (y) ; 0:5g , because a 2 x y. x2_p q (1 0:5 A) (a) = fmin f1 (p) ; A (q) ; 0:5gg and sup A (a) a2x y (0 0:5 A) (a) = fmax f0 (p) ; A (q) ; 0:5gg x2^p q fmax f0; A (q) ; 0:5gg = fmax f A (q) ; 0:5gg x2^p q max f A (y) ; 0:5g , because a 2 x y. x2^p q Hence, A = h A; Ai is an (2; 2 _q)-intuitionistic

1**fuzzy left -hyperideal of H.** This completes **the** proof. Theorem 161 Let A = h A; Ai and B = h B; Bi be (2; 2 _q)-intuitionistic fuzzy bi- -hyperideals of H. Then, A 0:5 B is an (2; 2 _q)-intuitionistic

1**fuzzy bi- -hyperideal of H.** Proof. Let **A** = h **A;**

Ai and B = h B; Bi be (2; 2 _q)-intuitionistic

1**fuzzy bi-** - ideals **of H and** let **x**

2 H. Then, we have two cases (1)

1**If x** 2= **y z** for any **y; z**

2 H and 2 : (2)

1**If x** 2 **y z** for some **y; z**

2 H and 2 . Case 1 :

1**If x** 2= **y z** for any **y; z**

2 H and 2 ; then ((A 0

17**:5 B) 0:5 (A 0:5 B**)) (x) = **0** (A **0:5** B) (x) and ((**A 0:5 B) 0:5 (A 0:5 B**

)) (x) = 1 (A 0:5 B) (x) Thus, A 0:5 A A in this case. Case 2 :

1**If x** 2 **y z** for some **y; z**

2 H and 2 , then ((A 0:5 B) 0:5 (0:5 B)) (a) = A = = = (A 0:5 B) (y) ^ (A 0:5 B) (z) x2_y z < 8 ^0:5 = 9 : f A (a) ^ B (b) ^ 0:5g ; 8 y2a b x2_yz <> ^ z2p q A (p) ^ B (q) ^ 0:5g >= W 9 f > W : f A(a)^ B(b)^0>:;5g x2_yzy2_abz2_pq< ^f A(p)^ B(q)^0:5g 8 9 = : f A(a)^ A(p) ; x2_yzy2_abz2_pq< B(b)^ B(q)^0:5g 8 ^ 9 = :f A (a) ^ A (p) ^ 0:5 ^ B ;(q)g x2_y z y2_a b z2_p q Since x 2 y z, y 2 a b and z 2 p q, so, x 2 (a b) (p q) = (a b p) q and we have f A(a)^ A(p)^0:5^ B(q)g x2_yzy2_abz2_pq f A(a)^ A(p)^0:5^ B(q)^0:5g x2(a_bp) q Since A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of H; so we have inf A (x) A (a) ^ A (p) ^ 0:5: m2a b p So, f A (a) ^ A (p) ^ 0:5 ^ B (q) ^ 0:5g x2_m q inf A (m) ^ B (q) ^ 0:5 x2_m q m2a b p f A(m)^ B(q)^0:5g=(A 0:5 B)(x) x2_mdq Therefore,((

17A 0:5 B) 0:5(A 0:5 B))(x) (A 0:5 B)(

x). Now, ((A

170:5 B) 0:5 (A 0:5 B)) (a) = = = = (A 0:5 B) (y) _ (A 0:5 B)

(z) x2^y z < 8 _0:5 9 = : f A (a) _ B (b) _ 0:5g ; 8 y2a b 9 x2^yz <> _ z2p q A (p) _ B (q) _ 0:5g >= V f > : V f A(a)_ B(b)_0>:;5g x2^yzy2^abz2^pq< _f A(p)_ B(q)_0:5g 8 9 = : A(a)_ A(p) ; x2^yzy2^abz2^pq< _ B(b)_ B(q)_0:5 8 9 = :f A (a) _ A (p) _ 0:5 _ B;(q)g x2^y z y2^a b z2^p q Since x 2 y z, y 2 a b and z 2 p q, so x 2 (a b) (p q) = (a b p) q and we have f A(a)_ A(p)_0:5_ B(q)g x^2yzy2^abz2^pq f A(a)_ A(p)_0:5_ B(q)g x= (a^bp) q Since A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of H; so we have sup A (w) A (a) _ A (p) _ 0:5: w2a b p Thus, f A (a) _ A (p) _ 0:5 _ B (q)g x2^(w) q sup A (w) _ B (q) _ 0:5 x2^(w) q w2a b p f A (w) _ B (q) _ 0:5g = (A 0:5 B) (x) : x2^w q Therefore, ((A 0

17:5 B) 0:5 (A 0:5 B))(a) (A 0:5 B)(

x) and so A 0:5 A A. Thus, A = h A; Ai is an (2;2 _q)-intuitionistic fuzzy sub- -semihypergroup of H. Now, let x;y;z 2 H. Then, f A (a) ^ B (b) ^ 0:5g ^ (A 0:5 B) (x) ^ (A 0:5 B) (z) ^ 0:5 = z2a b W "z2p q f A (p) ^ B (q) ^ 0:5g W # ^ 0:5 f A (a) ^ B (b) ^ 0:5g = 2 ^ f A (p) ^ B (q) ^ 0:5g 3 z2_a b z2_p q 6 ^0:5 4 7 A (a) ^ A (p) ^ B (b) 5 z2_a b z2_p q 2 ^ B (q) ^ 0:5 3 [4 A (a) ^ A (p) ^ B (q) ^ 05:5] z2_a b z2_p q Since x 2 y z, y 2 a b and z 2 p q, so

12x 2 (a b) (p q) = (a b

p) q and we have [A(a)^ A(p)^ B(q)^0:5] y2_abz2_pq [f A(a)^ A(p)^0:5g^ B(q)] x y z=(_a(b p) q) SinceA=h A; Aiisan(2;2_q)-intuitionisticfuzzybi- -hyperidealofH;sowehave A ((a (b p) q)) A (a) ^ A (p) ^ 0:5: Thus, [f A (a) ^ A (p) ^ 0:5g ^ B (q)] x y z=(_a(b p) q) [A (a (by) p) ^ B (q)] = (A 0:5 B) (xyz) : x y z=(_a(b p) q) Thus, u2x y z inf (A 0:5 B) (u) (A 0:5 B) (x) ^ (A 0:5 B) (z) ^ 0:5 and f A (a) _ B (b) _ 0:5g (A 0:5 B) (x) _ (A 0:5 B) (z) _ 0:5 = "y2a b V # _ "z2p q f A (p) _ B (q) _ 0:5g _ 0:5 V # f A (a) _ B (b) _ 0:5g = 2 _ f A (p) _ B (q) _ 0:5g 3 y2^a b z2^p q 6 _0:5 4 7 A (a) _ A (p) _ B (b) 5 y2^a b z2^p q 2 _ B (q) _ 0:5 3 [4 A (a) _ A (p) _ B (q) _ 05:5] y2^abz2^pq Since x 2 y z, y 2 a b and z 2 p q, so x 2 (a b) (p q) = (a b p) q and we have [A(a)_ A(p)_ B(q)_0:5] y2^abz2^pq [f A(a)_ A(q)_0:5g_ B(q)] x y z=^(ab p) q SinceA=h A; Aiisan(2;2_q)-intuitionisticfuzzybi- -hyperidealofH,sowehave x2a (b p) q inf A (x) A (a) _ A (q) _ 0:5: So, [f A (a) _ A (q) _ 0:5g _ B (q)] x y z=^(a b p) q x2a (b p) q inf A (x) _ B (q) = (A 0:5 B)

21(x y z) = x y z=^(a b

p) q sup (A 0:5 B) (u) : u2x y z Thus, sup (A 0:5 B) (u) (A 0:5 B) (x) _ (A 0:5 B) (z) _ 0:5: u2x y z Hence, A 0:5 B is an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of H. For any intuitionistic fuzzy set A = h A; Ai in H and

12**t 2 (0; 1], s 2 [0; 1),** we denote A **(t;s)**

= fx 2 H : x (t; s) qAg and [A](t;s) = fx 2 H : x (t; s) 2 _qAg. Obviously, [A

47**](t;s)** = A **(t;s)** [U **(t;s),** where U **(t;s),** A **(t;s)** and [A **](t;s)** are called **2-**

level set, q-level set and 2 _q-level set of A = h A; Ai, respectively [3]: Theorem 162 Let H be a -semihypergroup and A = h A; Ai be an IFS of H. Then, A = h A; Ai is an (2; 2 _q)-intuitionistic

1**fuzzy left (resp. right) -hyperideal** of H **if and only if for**

all t 2 (0; 0:5] and s 2 [0:5; 1), the set U(t;s)6 = ;

1**is a left (resp. right) -hyperideal of H.**

Proof. Let A = h A; Ai be an (2; 2 _q)-intuitionistic fuzzy left -hyperideal of H and U

12**(t;s)** 6 = ; **for** any **t 2 (0;** 0:5] **and s 2 [0:**

5; 1). Let y 2 U(t;s)6 = ; and x 2 H. Then, A (y) t and A (y) s. Since inf z2x y A (z) sup z2x y A (z) A (y) ^ 0:5

6**t ^ 0:5 t and** A **(y)** _ 0:5 **t**

_ 0:5 s, so, z 2 U(t;s) for all z 2 x y. Hence, x y U(t;s); is a left -hyperideal of H. Conversely, let us suppose that A = h A; Ai is an IFS of H

6**such that U(t;** s)6 = ; **is a left** -hyperideal of **H.**

Suppose on contrary

6**there exist x; y** 2 H **such that** z2x **y**

inf and sup z2x y A (z) < A (y) ^ 0:5 A (z) > A (y) ^ 0:5. Let us choose t 2 (0; 0:5] and s 2 [0:5; 1). Then, z2x y inf A (z) < t < A (y) ^ 0:5 and sup A (z) > s > A (y) ^ 0:5. z2x y Thus, y 2 U(t;s) but z 2= U(t;s) for all z 2 x y, which is a contradiction. Hence, inf z2x y and sup z2x y This completes the proof. A (z) A (z) A (y) ^ 0:5 A (y) ^ 0:5. Theorem 163 Let H be a -semihypergroup and A = h A; Ai be an IFS of H. Then, A = h A; Ai is an (2; 2

3_q)-intuitionistic fuzzy bi- -hyperideal of H if and only if for all t 2 (0;

0:5] and s 2 [0:5; 1), the set U(t;s)6 = ; is a bi- -hyperideal of H. Proof. The proof follows from Theorem 162. Theorem 164 Let H be a -semihypergroup and A = h A; Ai be an IFS of H. Then, A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy (1; 2)- -hyperideal of H if and only if for all t 2 (0; 0:5] and s 2 [0:5; 1), the set U(t;s)6 = ; is a (1; 2)- -hyperideal of H. Proof. The proof follows from Theorem 162. Theorem 165 Let H be a -semihypergroup and A = h A; Ai be an IFS of H. Then, A = h A; Ai is an (2; 2

3_q)-intuitionistic fuzzy sub- -semihypergroup of H if and only if for

all

13t 2 (0; 1] and s 2 [0; 1), the set [A](t;s)

=6 ; is a sub- -semihypergroup of H.

3Proof. Let x; y 2 [A](t;s). Then, A (x) t and A

(x) s or A (x) + t > 1 and A (x) + s <

221, and A (y) t and A (y) s or A (y) + t

> 1 and A (y) + s < 1. We can consider four cases: (i) A (x) t and A (x) s; and A (y) t and A (y) s; (ii) A (x) t and A (x) s, and A (y) + t > 1 and A (y) + s < 1; (iii) A (x) + t > 1 and A (x) + s <

221, and A (y) t and A (y) s; (iv) A (x) + t> 1 and A

(x) + s <

221, and A (y) + t> 1 and A (y) + s

< 1. For the …rst case, by Theorem 139 (a), implies that inf A (z) minf A (x) ; A (y) ; 0:5g = minft; 0:5g = 0:5 if t > 0:5 z2x y < 8 t if t 0:5 and : sup A (z) maxf A (x) ; A (y) ; 0:5g = maxfs; 0:5g = 0:5 if s < 0:5 z2x y < 8 s if s 0:5 and so infz2x y A (z) + t > 0:5 + 0:5 = 1 and supz2x y A (z) + s < 0:5 + 0:5 = 1; i:e:; : (z) (s; t) qA, or z 2 A(t;s) for all

30z 2 x y. Therefore, x y

12**U(t;s) [A(t;** s) = [A **](t;s). For**

the case (ii), assume that t > 0:5 and s < 0:5. Then, 1 t < 0:5 and 1 s > 0:5. If minf A (y) ; 0:5g A (x) and maxf A (y) ; 0:5g A (x), then inf z2x y A (z) sup z2x y A (z) minf A (y) ; 0:5g > 1 t and maxf A (y) ; 0:5g < 1 s and if minf A (y) ; 0:5g > A (x) and maxf A (y) ; 0:5g < A (x) ; then infz2x y A (z) A (x) t and supz2x y A (z) A (x) s. Hence, z 2

12**U(t;s) [A(t;** s) = [A **](t;s)for** all z **2**

x y. Therefore, x y

12**U(t;s) [A(t;** s) = [A **](t;s) for t**

> 0:5 and s < 0:5. Suppose that t 0:5 and s 0:5. Then, 1 t 0:5 and 1 s 0:5. If minf A (x) ; 0:5g A (y) and maxf A (x) ; 0:5g A (y), then z2x y inf A (z) sup A (z) z2x y minf A (x) ; 0:5g t and maxf A (x) ; 0:5g s and if minf A (x) ; 0:5g > A (y) and maxf A (x) ; 0:5g < A (y) ; then infz2x y A (z) A (y) > 1 t and supz2x y A (z) A (y) < 1 s. Thus, z 2

12**U(t;s) [A(t;** s) = [A **](t;s)for** all z **2**

x y. Therefore, x y

12**U(t;s) [A(t;** s) = [A **](t;s) for t**

0:5 and s 0:5. We have similar result for the case (iii): For …nal case, if t > 0:5 and s < 0:5; then 1 t < 0:5 and 1 s > 0:5. Hence, inf z2x y A (z) = and minf A (x) ; A (y) ; 0:5g

6**0:5**> 1 **t**< 8 **min f** A **(x)** ; A **(y)**

g > 1 : if

6**min f** A **(x)** ; A **(y)** g **0:5; t** if min **f** A **(x)** ; A **(y)**

g < 0:5; sup z2x y A (z) maxf A (x) ; A (y) ; 0:5g = 0:5 < 1 s if max f A (x) ; A (y)g 0:5; < 8 max f A (x) ; A (y)g < 1 s if max f A (x) ; A (y)g > 0:5; and so x y A(t;s) [A](t;s). If t 0:5 and s : 0:5; then 1 t 0:5 and 1 s 0:5. Thus, inf z2x y A (z) = and minf A (x) ; A (y) ; 0:5g

6**0:5 t**< 8 **min f** A **(x)** ; A **(y)** g > 1 : if

6**min f** A **(x)** ; A **(y)** g **0:5; t** if min **f** A **(x)** ; A **(y)**

g < 0:5; sup z2x y A (z) maxf A (x) ; A (y) ; 0:5g = 8 0:5 max f A (x) ; A (y)g < 1 s if max f A (x) ; A (y)g 0:5; s if max f A (x) ; A (y)g > 0:5; < : which implies that x y

12**U(t;s)** [A **(t;s)** = [A **](t;s).** Conversely, suppose that **A** = h **A;**

Ai is an IFS in H such that [A](t;s) is a sub- - semihypergroup of H. Suppose that A = h A; Ai is not an (2; 2 _q)-intuitionistic fuzzy sub- -semihypergroup of H. Then, there exist x; y 2 H such that inf A (z) < minf A (x) ; A (y) ; 0:5g and sup A (z) > maxf A (x) ; A (y) ; 0:5g. z2x y z2x y Let t= s= Then, 1 2 z2x y inf 1 sup 2 z2x y A (z) + minf A (x) ; A (y) ; 0:5g and A (z) + maxf A (x) ; A (y) ; 0:5g . inf z2x y A (z) < t < minf A (x) ; A (y) ; 0:5g and sup z2x y A (z) > s > maxf A (x) ; A (y) ; 0:5g. this imply that x; y 2 [A](t;s) and (x y) [A](t;s). Hence, infz2x y A (z) t and supz2x y A (z) s or infz2x y A (z) + t > 1 and supz2x y A (z) + s < 1, which is a contradiction. Therefore, we have z2x y inf A (z) minf A (x) ; A (y) ; 0:5g and sup A (z) maxf A (x) ; A (y) ; 0:5g. z2x y Thus, A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy sub- -semihypergroup of H: Theorem 166 Let H be a -semihypergroup and A = h A; Ai an IFS of H. Then, A = h A; Ai is an (2; 2 _q)-intuitionistic

1**fuzzy left (resp. right) -hyperideal** of H **if and only if for**

all

13**t 2 (0; 1] and s 2 [0; 1),** the set [A **](t;s)**

=6 ;

1**is a left (resp. right) -hyperideal of H.**

Proof. Assume that A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy left -hyperideal of H and let t 2 (0; 1] and s 2 [0; 1) be such that [A](t;s) =6 ;. Let y 2 [A](t;s) and x 2 H. Then, A (y) t and A (y) s or A (y) + t > 1 and A (y) + s < 1. Assume that A (y) t and A (y) s by Theorem ?? (a), implies that inf A (z) minf A (y) ; 0:5g minft; 0:5g = t if t 0:5; z2x y < 8 0:5 > 1 t if t > 0:5; and : sup A (z) maxf A (y) ; 0:5g maxfs; 0:5g = s if s 0:5; z2x y < 8 0:5 < 1 s if s < 0:5; so that x y U

24**(t;s) [A(t;s) = [A](t;s).** Suppose that **A** (y) + **t**

> 1 and A (y) + s < 1. If : t > 0:5 and s < 0:5, then z2x y inf A (z) minf A (y) ; 0:5g = < 8 and : 0:5 > 1 t if t

0:5; A (y) > 1 t if t > 0:5; sup A (z) maxf A (y) ; 0:5g = 0:5 < 1 s if s 0:5; z2x y < 8 A (y) < 1 s if s < 0:5; and thus xy 2 A

24**(t;s) [A](t;s).** Consequently, **[:A](t;s)** is **a**

left -hyperideal of H: Conversely, suppose that A = h A; Ai is an IFS in H such that [A](t;s)

1**is a left -hyperideal of H.**

Suppose that A = h A; Ai is not an (2; 2 _q)-intuitionistic fuzzy -hyperideal of H. Then, there exist x; y 2 H such that inf A (z) < minf A (y) ; 0:5g and A (xy) > maxf A (y) ; 0:5g. z2x y Let t= s= Then, 1 2 z2x y inf 1 sup 2 z2x y A (z) + minf A (y) ; 0:5g and A (z) + maxf A (y) ; 0:5g . inf z2x y A (z) < t < minf A (y) ; 0:5g and sup z2x y A (z) > s > maxf A (y) ; 0:5g. this imply that x;y 2 U

30**(t;s) [A](t;s)** so that z **2 [A](t;s)**

for all z 2 x y. Thus, infz2x

7**y A(z)** tandsupz2x **y A(z)** sorinfz2x **y A(z)+** t>1andsupz2x **y A(z)+**

s<1,whichisacontradiction. Therefore,wehave z2x y inf A (z) minf A (y) ; 0:5g and sup A (z) maxf A (y) ; 0:5g, z2x y

1**for all x; y** 2 **H.** Hence, **A = h** A; Ai **is**

an (2; 2 _q)-intuitionistic

1**fuzzy left -hyperideal of H.** Similarly, **the** right **case**

also follows. This completes the proof. Theorem 167 Let H be a -semihypergroup and A = h A; Ai an IFS of H. Then, A = h A; Ai is an (2; 2

3**_q)-intuitionistic fuzzy** bi- -hyperideal **of** H **if and only if for**

all

13**t 2 (0; 1] and s 2 [0; 1),** the set [A **](t;s)**

=6 ;

1is a bi- -hyperideal of H.

Theorem 168 Let H be a -semihypergroup and A = h A; Ai an IFS of H. Then, A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy (1; 2)- -hyperideal of H if and only if for all

13t 2 (0; 1] and s 2 [0; 1), the set [A](t;s) =6 ; is a (1;

2)- -hyperideal of H. Theorem 169 Every (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of a -semihypergroup H is an (2; 2 _q)-intuitionistic fuzzy (1; 2) -hyperideal of a -semihypergroup H. Proof. Let A = h A; Ai be an (2; 2 _q)-intuitionistic fuzzy bi- -hyperideal of a -semihypergroup H. Then z2x y inf A (z) min f A (x) ; A (y) ; 0:5g and sup A (z) max f A (x) ; A (y) ; 0:5g z2x y Now let for any x; y; z; a 2 H. Then inf z2x a (y

8z) A(z) = z2 (x a y) z inf A(z) inf z2x a (y z) A(z) sup A (xa(yz)) = z2x a (y z) sup A

(xa(yz)) z2x

20a (y z) sup A(z) z2x a (y z) z2u z inf min A(z)

min f A (u) ; A (z) ; 0:5g : u 2 x a y inf A (u) ; A (z) ; 0:5 u2x a y min f A (x) ; A (y) ; A (z) ; 0:5g . sup A(z) max f A (u) ; A (z) ; 0:5g : u 2 x a y z2u z max sup A (u) ; A (z) ; 0:5 u2x a y max f A (x) ; A (y) ; A (z) ; 0:5g Therefor A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy (1; 2) -hyperideal of a - semihypergroup H. Chapter 7 Interior -hyperideals of -semihypergroups based on Intuitionistic Fuzzy Points 7.1 Introduction

2In this chapter, we introduce the concept of an (;)-intuitionistic fuzzy

interior - hyperideal of a -semihypergroup by using the notion of an intuitionsitic fuzzy point to an intuitionistic fuzzy set. An (;)-intuitionistic fuzz interior -hyperideal of a - semihypergroup is a generalization of ordinary intuitionitic fuzzy interior -hyperideals. We also de…ne an (2; 2 _q)-intuitionistic fuzzyinterior -hyperideal a -semihypergroup. We charcterize an (2; 2 _q)-intuitionistic fuzzy interior - hyperideal by the properties of an 2-level set, a q-level set and an (2; 2 _q)-level set. 7.2 (;)-Intuitionistic Fuzzy Interior -hyperideals

2In this section, we introduce the concept of an (;)-intuitionistic fuzzy

interior - hyperideal of a -semihypergroup. We also study some properties of an (;)-intuitionistic fuzzy interior -hyperideal. The give concept in this section is a generalization of intu- itionsitic fuzzy interior - hyperideals of -semihypergroups. De…nition 170 An IFS A = h A; Ai in -semihypergroup H is said to be an (;)- intuitionistic fuzzy interior -hyperideal of H, where ; are any two of f2; q; 2 _q; 2 ^qg with6 =2 ^q, if the following are hold. (IFI1) (8x;y 2 H and 2 , (t1;t2 2 (0;0:5] and s1;s2 2 [0:5;1)) or (t1;t2 2 (0:5; 1] and s1; s2 2 [0; 0:5))) If x(t1; s1) A and y(t2; s2) A =) (z1)(mft1; t2g; Mfs1; s2g) A for all z1 2 x y. (IF I2) (8x; a; y 2 H and ; 2 ,

13(t 2 (0; 0:5] and s 2 [0: 5; 1)) or (t

2 (0:5; 1] and s 2 [0; 0:5))) If a(t; s) A =) (z1)(t; s) A for all z1 2 x a y. Theorem 171 Let A = h A; Ai

3be a non-zero (;)-intuitionistic fuzzy interior - hyperideal of a -semihypergroup H . Then , the

set I = fx 2 H : A (x) > 0 and A (x) < 1g is an interior -hyperideal of H . Proof. Let x;y 2 I and 2 H. Then, A(x) > 0 and A(x) < 1 and A(y) > 0 and A (y) < 1. Assume that A (z) = 0 and A (z) = 1 for all

4z 2 x y. If 2 f2; 2

_qg, then

25x (A (x) ; A (x)) A and y (A (y) ; A (y)) A but for each z

2 x y, (z) (m

5f A (x) ; A (y) g ; M f A (x) ; A

(y)g) A for every 2 f2; q; 2 ^q; 2 _qg, which is a contradiction. Since x (1; 0) qA and y (1; 0) qA but for each z 2 x y, (z1) (1; 0) A for every 2 f2; q; 2 ^q; 2 _qg, which is a contradiction. Hence, for each z 2 x y, A (z) > 0 and A (z) < 1. This implies that z 2 I. Thus, x y I. Now, let x; y 2 S, a 2 I and ; 2 H . Then, assume that , A (z) = 0 and A (z2) = 1 for each z 2 x a y. If 2 f2; 2 _qg, then

7a (A (a) ; A (a)) A but for all z 2 x a y, (z) (A (a) ; A (a)) A

for every 2 f2; q; 2 ^q; 2 _qg, which is a contradiction. Since a (1; 0) qA but for all z 2 x a y, (z) (1; 0) A for every 2 f2; q; 2 ^q; 2 _qg, which is a contradiction. Hence, for each

29z 2 x a y, A (z)> 0 and A

(z) < 1. This implies that z 2 I for each

29z 2 x a y. Thus, x a

y I and I = fx 2 S : A (x) > 0 and A (x) < 1g is an interior -hyperidealof H. Corollary 172 Let A = h A; Ai be a non-zero (;)-intuitionistic fuzzy interior - hyperidealof a -semihypergroup H. Then, the sets I1 = fx 2 H : A (x) > 0g and I2 = fx 2 H : A (x) < 1g are interior -hyperideals of H. Theorem 173 Let H be a strong right(resp, left) zero -semihypergroup and A = h A; Ai

14be a non-zero (q; q)- intuitionistic fuzzy interior -hyperideal of H. Then,

A = h A; Ai is constant on I. Proof. Let w be an element of H such that A (w) = supx2H f A (x)g and A (w) = infx2H f A (x)g. Then, w 2 I. Suppose that there exist x,w 2 I such that tx = A (x)6 = A (w) = tw and sx = A (x)6 = A (w) = sw. Then tx < tw and sx > sw. Choose t1; t2 2 (0; 1] and s1; s2 2 [0; 1) such that 1 tw < t1 < 1 tx < t2 and 1 sw > s1 > 1 sx > s2. Then w (t1; s1) qA and x (t2; s2) qA but (w x) (m ft1; t2g ; M fs1; s2g) = (x) (t1; s1) qA (resp, (x w) (m ft1; t2g ; M fs1; s2g) = (x) (t1; s1) qA) becouse H is a strong right (resp, left) zero, which is a contradiction. Hence,

9A (x) = A (e) and A (x) = A (e). Therefore A (x) = A (e) for all x

2 I. 7.3 Intuitionistic fuzzy interior -hyperideals of type (2; 2 _q) The concept of (2; 2 _q)-intutionistic fuzzy interior -hyperideals in a -semihypergroup plays a vital rule in the theory of (;)-intutionistic fuzzy interior -hyperideals. We give some di¤erent characterization of (2; 2 _q)-intutionistic fuzzy interior -hyperideals in a -semihypergroup. De…nition 174 An IFS A = h A; Ai in a -semihypergroup

6is said to be an (2; 2 _q)- intuitionistic fuzzy interior -hyperideal of H

if the following conditions hold. (IF I3) (8x; y 2 H, (t1; t2 2 (0; 0:5] and s1; s2 2 [0:5; 1)) or (t1; t2 2 (0:5; 1] and s1; s2 2 [0; 0:5))) If x(t1; s1) 2 A and y(t2; s2) 2 A =) (z1)(mft1; t2g; Mfs1; s2g) 2 _qA for each z1 2 x y: (IFI4) (8x;a;y 2 H,

13(t 2 (0: 5;1] and s 2 [0: 5;1)) or (t

2 (0:5;1] and s 2 [0;0:5))) If a(t;s) 2 A =) (z2)(t;s) 2 _qA for each z2 2 x y z. Theorem 175 Let A = h A; Ai be an IFS in a -semihypergroup H. Then, A = h A; Ai is an (2; 2

3_q)-intuitionistic fuzzy interior -hyperideal of H if and only if

the following conditions hold; z12x y inf A (z1) min f A (x) ; A (y) ; 0:5g , sup A (z1) maxf A (x) ; A (y) ; 0:5g z12x y z12x a y inf A (z2) min f A (a) ; 0:5g , sup A (z2) max f A (a) ; 0:5g , z12x a y for all x; a; y 2 H and ; 2 (t1; t2 2 (0; 0:5] and s1; s2 2 [0:5; 1)) or (t1; t2 2 (0:5; 1] and s1; s2 2 [0; 0:5)). Proof. Since given that A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy interior -hyperideal of a -semihypergroup H. Suppose that z12x y inf A (z1) < min f A (x) ; A (y) ; 0:5g and sup A (z1) > max f A (x) ; A (y) ; 0:5g . z12x y Choose t 2 (0; 1] and s 2 [0; 1) such that z12x y inf A (z1) < t < min f A (x) ; A (y) ; 0:5g and sup A (z1) > s >

6**max f** A **(x)** ; A **(y) ; 0:** 5g . z12x **y** If **min f** A **(x)** ; A **(y)**

g < 0:5 and max f A (x) ; A (y)g > 0:5, then z12x y inf A (z1) < t < min f A (x) ; A (y)g and sup A (z1) > s > max f A (x) ; A (y)g . z12x y Then x(t; s) 2 A and y(t; s) 2 A but for each z1 2 x y, (z1)(t; s)2 _qA,

14**which is a contradiction. If min** f A **(x)** ; A **(y)**

g 0:5 and max f A (x) ; A (y)g 0:5, then z12x y inf A (z1) < t < 0:5 and sup A (z1) > s > 0:5. z12x y Since x(0:5; 0:5) 2 A and y(0:5; 0:5) 2 A, but for each z1 2 x y, (z1)(0:5; 0:5)2 _qA,

14**which is a contradiction. If min** f A **(x)** ; A **(y)**

g < 0:5 and max f A (x) ; A (y)g < 0:5, then z12x y inf A (z1) < t < min f A (x) ; A (y)g and sup A (z1) > s > 0:5. z12x y Thus, x(t; s) 2 A and y(t; s) 2 A but for each z1 2 x y, (z1)(t; s)2A. Also, infz12x y A (z1)+ t < 0:5+0:5 = 1 and supz12x y A (z1)+s > 0:5+0:5 = 1. This implies for each z1 2 x y, (z1)(t; s)qA. Hence, for each z1 2 x y, (z1)(t; s)2 _qA, which is contradiction. Hence, z12x y inf A (z1) min f A (x) ; A (y) ; 0:5g and sup A (z1) max f A (x) ; A (y) ; 0:5g . z12x y (b) Suppose that z12x a y inf A (z1) < min f A (a) ; 0:5g and sup A (z1) > max f A (a) ; 0:5g . z12x a y Choose t 2 (0; 1] and s 2 [0; 1) such that z12x a y inf A (z1) < t < min f A (a) ; 0:5g and sup z12x y If A (a) < 0:5 and A (a) > 0:5, then z12x a y inf A (z1) < t < A (a) and sup z12x y A (z1) > s > max f A (a) ; 0:5g . A (z1) > s > A (a) . Then, a(t; s) 2 A but for each z1 2 x a y, (z1)(t; s)2 _qA, which is a contradiction. If A (a) 0:5 and A (a) 0:5, then z12x a y inf A (z1) < t < 0:5 and sup A (z1) > s > 0:5. z12x a y Since a(0:5; 0:5) 2 A but for each z1 2 x a y, (z1)(0:5; 0:5)2 _qA, which is a contradic- tion. If A (a) 0:5 and A (a) 0:5, then z12x a y inf A (z1) < t < A (a) and sup A (z1) > s > 0:5. z12x a y Thus, a(t; s) 2 A but for each z1 2 x a y, (z1)(t; s)2A. Also, infz12x a y A (z1) + t < 0:5 + 0:5 = 1 and supz12x a y A (z1) + s > 0:5 + 0:5 = 1. This imply for each z1 2 x a y, (z1)(t; s)qA. Thus, for each z1 2 x a y, (z1)(t; s)2 _qA, which is a contradiction. Hence, z12x a y inf A (z1) min f A (x) ; A (y) ; 0:5g and sup A (z1) max f A (x) ; A (y) ; 0:5g . z12x a y Conversely, assume that A = h A; Ai satis…es (a) and (b). Let x; y 2 H and (t1; t2 2 (0; 0:5] and s1; s2 2 [0:5; 1)) or (t1; t2 2 (0:5; 1] and s1; s2 2 [0; 0:5)) such that x(t1; s1) 2 A and y(t2; s2) 2 A) A (x) t1 and A (x) s1, A (y) t2 and A (y) s2. Then, z12x y inf A (z1) z12x y inf A (z1) min f A (x) ; A (y) ; 0:5g and sup A (z1) max f A (x) ; A (y) ; 0:5g z12x y min ft1; t2; 0:5g and sup A (z1) max fs1; s2; 0:5g z12x y Then we have the following case's (1) min ft1; t2g > 0:5 and max fs1; s2g < 0:5 (2) min ft1; t2g 0:5 and max fs1; s2g 0:5 Case (1) If min ft1; t2g > 0:5 and max fs1; s2g < 0:5, then infz12x y A (z1) 0:5 and supz12x y A (z1) 0:5. This implies that infz12x y A (z1) + min ft1; t2g > 1 and A (xy) + max fs1; s2g < 1. So, for each z1 2 x y, (z1)(mft1; t2g; M fs1; s2g)qA. Case(2) if min ft1; t2g 0:5 and max fs1; s2g 0:5, then infz12x y A (z1) min ft1; t2g and supz12x y A (z1) max fs1; s2g. This implies that for each z1 2 x y (z1)(mft1; t2g; M fs1; s2g) 2 A. Therefore, (z1)(mft1; t2g; M fs1; s2g) 2 _qA Let x; y; a 2 H and

13**(t 2 (0:** 5; 1] **and s 2 [0:** 5; 1)) or **(t**

2 (0:5; 1] and s 2 [0; 0:5)) such that a(t; s) 2 A) A (x) t and A (x) s. Then, z12x a y inf A (z1) z12x a y inf A (z1) min f A (a) ; 0:5g and sup A (z1) max f A (a) ; 0:5g z12x a y min ft; 0:5g and sup A (z1) max fs; 0:5g z12x a y Then we have the following case's (3) t > 0:5 and s < 0:5 (4) t 0:5 and s 0:5 Case (3) If t > 0:5 and s < 0:5, then infz12x a y A (z1) 0:5 and supz12x a y A (z1) 0:5. This implies that infz12x a y A (z1) + t > 1 and supz12x a y A (z1) + s < 1. Then, for each (z1)(t; s)qA. Case(4) If t 0:5 and s 0:5, then infz12x a y A (z1) t and supz12x a y A (z1) s. This implies that (z1)(t; s) 2 A. Therefore, for each z1 2 x a y; (z1)(t; s) 2 _qA. Remark 176 Every intuitionistic fuzzy interior -hyperideal of a -semihypergroup H is an (2;2 _q)-intuitionistic fuzzy interior true. -hyperideal of H. But the converse is not Example 177 Let H = f1; 2; 3; 4; 5g be a table. -semihypergroup with the following Cayley 1 2 3 4 5 1 2 3 4 5 1 f1g f1g f1g f1g f1g 1 f1g f1g f1g f1g f1g 2 f1g f1g f1g f1g f1g 2 f1g f1g f1g f1g f1g 3 f1g f1g f3g f3g f3g 3 f1g f1g f3; 4g f3; 4g f3; 4g 4 f1g f1g f3g f4; 5g f4; 5g 4 f1g f1g f3; 4g f3; 4g f5g 5 f1g f1g f3; 5g f3g f5g 5 f1g f1g f3; 5g f3; 5g f3; 5g Let A = h A; Ai be IFS in -semihypergroup H de…ne by A (1) = A (2) = A (4) = 0:8; A (3) = 0:7; A (5) = 0:6; and A (1) = A (2) = A (4) = 0:1; A (3) = 0:2; A (5) = 0:3. Then A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy interior -hyperideal of S but not intuitionistic fuzzy interior -hyperideal. Proposition 178 Let H be a -semihypergroup and A = h A; Ai be an (2; 2 _q)- intuitionistic fuzzy -hyperideal of H. Then, A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy interior -hyperideal: Proof. Letx;y2H. Then, z12x y inf A (z1) and sup A (z1) z12x y Now, let x; a; y 2 H . Then minf A (y) ; 0:5g minff A (x) ; A (y) ; 0:5g maxf A (y) ; 0:5g maxff A (x) ; A (y) ; 0:5g. inf z12x a y A (z1) z12x a y inf A (z1) sup A (z1) z12x a y sup z12x a y A (z1) minf inf t2x a A (t) ; 0:5g minf A (a) ; 0:5g and maxf sup A (t) ; 0:5g t2x a maxf A (a) ; 0:5g. minf A (a) ; 0:5; 0:5g = minf A (a) ; 0:5g maxf A (a) ; 0:5; 0:5g = maxf A (a) ; 0:5g By Theorem 175, A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy interior -hyperideal of H. Theorem 179 If fAgi2 is family of (2;2 _q)-intuitionistic fuzzy interior -hyperideals of H. Then, i2 Ai is an (2;2 _q)-intuitionistic fuzzy interior -hyperideal of H, where i2 Ai = h Ti2 Ai; i2 Aii. T V W Proof. Letx;y2H. Thenwehave inf i2 z12x y Ai (z1) = i2 V = min V inf min i2 z12x y Ai (z1) and sup V z12x y Ai (z1) = i2 z1i2nxf y Ai (z1) i2 min Ai (x) ; Ai (y) ; 0:5 i2 Ai (x) ; i2 Ai (y) ; 0:5 V V Ai (x) ; V i2 i2 Ai (y) ; 0:5 sup Ai (z1) V V z12x y (max f Ai (x) ; Ai (y) ; 0:5g) i2 Now, let for any x;y;a 2 H. The we have sup i2 z12x y W Ai (z1) = max W max i2 W i2 W Ai (x) ; i2 W Ai (x) ; Ai (y) ; 0:5 W i2 Ai (y) ; 0:5 W inf i2 z12x a y V Ai (z1) = i2 = min V z12inxfa y Ai (z1) i2 Ai (a) ; 0:5 V min Ai (a) ; 0:5 i2 inf V i2 z12x a y Ai (z1) min Ai (a) ; 0:5 i2 V sup i2 z12x a y Ai (z1) = i2 z12x a y sup Ai (z1) V (max f Ai (a) ; 0:5g) i2 W = max W W i2 Ai (a) ; 0:5 sup (z1) max W i2 z12x a y Ai i2 Ai (a);0:5 W W Hence, i 2 Ai = h i2 Ai; i 2 Aii is an (2;2 _q)-intuitionistic fuzzy interior - hyperideTal of H. V W Remark 180 The union of two (2; 2 _q)-intuitionistic fuzzy interior -hyperideals of H is not necessary to an (2;2 _q)-intuitionistic fuzzy interior -hyperideal of H. Example 181 Let H = fa; b; c; dg and = f ; g be two non-empty sets. Then, (H;) is a -semihypergroup with the following multiplication tables: 1 2 3 4 1 2 3 4 1 f1g f1g f1g f1g 1 f1g f1g f1g f1g 2 f1g f1g f1; 4g f1g 2 f1g f1g f2; 4g f1g 3 f1g f1g f1g f1g 3 f1g f1g f1g f1g 4 f1g f1g f1g f1g 4 f1g f1g f1g f1g Let A = h A; Ai and B = h B; Bi be two IFSs of H such that A (1) = A (2) = 0:4; A (3) = A (4) = 0; A (1) = A (2) = 0:6; A (3) = A (4) = 0:8 and B (1) = 0:4; B (2) = 0; B (3) = 0:4; B (4) = 0; B (1) = 0:6, B (2) = 0:8; B (3) = 0:6, B (4) = 0:8: Then both A = h A; Ai and B = h B; Bi are (2;2 _q)-intuitionistic fuzzy interior -hyperideals of H, but A [B = h A _ B; A ^ Bi is not an (2;2 _q)-intuitionistic fuzzy interior -hyperideal of H. The following theorem can be obtained if we present a su¢ cient condition. Theorem 182 If fAigi2 is a family of (2; 2 _q)-intuitionistic fuzzy interior -hyperideals of H such that Ai Aj or Aj Ai for all i; j 2 I, then i 2 Ai = h i2 Ai; i 2 Aii is an (2;2 _q)-intuitionistic fuzzy interior -hyperideal oSfH. W V Proof. For all x; y; a 2 H and ; 2 , we have inf i2 z12x y W inf i2 z12x y W It is clear that Ai (z1) = = = Ai (z1) i2 z12x a y inf Ai (z1) i2 Ai (x) ^ Ai (y) ^ 0:5g W Ai (x) ^ i2 Ai (y) ^ 0:5 W i2 W (x) ^ W (y) ^ 0:5 i2 Ai i2 Ai W (x) ^ W i2 Ai i2 Ai (y) ^ 0:5 W W Ai (x) ^ Ai (y) ^ 0:5 Ai (x) ^ i2 i2 i2 Ai (y) ^ 0:5 : W W W Assume that i2 Ai (x) ^ Ai (y) ^ 0:5 =6 i2 Ai (x) ^ i2 Ai (y) ^ 0:5 W W W

14**Then there exists t such that** i2 Ai **(x)** ^ Ai **(y)** ^ 0:5 <**t**< i2 Ai **(x)**

^ i2 Ai (y) ^ 0:5 W W W Since Ai Aj or Aj Ai for all i; j 2 I, so exists k 2 I such that t < Ak (x) ^ Ak (y) ^ 0:5. On other hand Ai (x) ^ Ai (y) ^ 0:5 < t for all i 2 I, a contradiction. Hence i2 Ai (x) ^ Ai (y) ^ 0:5 = i2 Ai (x) ^ i2 Ai (y) ^ 0:5 W W W Now, sup i2 z12x y V sup i2 z12x y V It is clear that Ai (z1) = = = Ai (z1) sup Ai (z1) [Ai (x) _ Ai (y) _ 0:5g] i2 z12x y i2 V V i2 (Ai (x)) _ (Ai (y)) _ 0:5 i2 V Ai (x) _ V Ai (y) _ 0:5 i2 i2 V V Ai (x) _ Ai (y) _ 0:5 i2 i2 V V [Ai (x) _ Ai (y) _ 0:5] Ai (x) _ i2 i2 i2 Ai (y) _ 0:5 : V V V Assume that [Ai (x) _ Ai (y) _ 0:5]6 = Ai (x) _ Ai (y) _ 0:5 i2 i2 i2 V V V Then there exist s such that i2 [Ai (x) _ Ai (y) _ 0:5] > s > Ai (x) _ Ai (y) _ 0:5 i2 i2 V V V Since Ai Aj or Aj Ai for all i; j 2 I, there exists k 2 I such that s > Ak (x) ^ Ak (y) ^ 0:5. On other hand Ai (x) ^ Ai (y) ^ 0:5 > s for all i 2 I, a contradiction. Hence i2 [Ai (x) ^ Ai (y) ^ 0:5] = i2 Ai (x) ^ i2 Ai

3**(y) ^ 0:5 V** V **V** Let **x; a; y**

2 H and ; 2 . Then i2 z12x a y inf i2 z12x a y W inf Ai Ai (z1) = (z1) = = i2 z12ixnfa y Ai (z1) W i2 Ai (a) ^ 0:5 W (a) ^ ^0:5 i2 Ai W (a) ^ ^0:5 i2 Ai i2 W Ai (a) ^ ^0:5g W W and sup i2 z12x a y V sup i2 z12x a y V Ai Ai Hence i 2 Ai = h i2 hyperidSeal of H. W (z1) = sup i2 z12x a y Ai (xay) V i2 V [Ai (a) _ 0:5g] = i2 (Ai (a)) _ 0:5 (z1) = V i2 V i2 V Ai (a) _ 0:5 Ai (a) _ 0:5 Ai; i 2 Aii is an (2; 2 _q)-intuitionistic fuzzy interior - V Theorem 183 An IFS A = h A; Ai in a -semihypergroup H is an (2; 2 _q)-intuitionistic fuzzy interior - hyperideal of H

6**if and only if** the **non-empty** sets **U** (A; **t)** and L (**A;**

s) are interior -hyperideals of H for all t 2 (0; 0:5] and s 2 [0:5; 1), where U (A; t) = fx 2 H : A (x) tg and L (A; t) = fx 2 H : A (x) sg. Proof. Let A = h A; Ai be an (2; 2 _q)-intuitionistic fuzzy interior -hyperideal of H and the sets U (A; t) and L (A; s) be non-empty for any t 2 (0; 0:5] and s 2 [0:5; 1).

6**Let x; y** 2 **U** (A; **t). Then,** A **(x) t and** A **(y) t.**

Since z12x y inf A(z1) A (x) ^ A (y) ^

6**0:5 t ^ t ^ 0:5 = t**

This implies that z1 2 U (A; t) for each z1 2 x y. Thus, x y U (A; t). Now let a 2 U (A; t), x; y 2 H and ; 2 . Then A (a) t. Since z12x a y inf A(z1) A (a) ^ 0:5 t ^ 0:5 = t This implies that z1 2 U (A; t) for each z1 2 x a y. Thus, x a y U (A; t). There- fore, U (A; t) is an interior -hyperideal of H. Similarly, we can prove L (A; s)

1**is an** interior **-hyperideal of H.** Conversely, let A = **h**

A; Ai be an IFS in H such that U (A; t) and L (A; s) be interior -hyperideals of H. If there exist x; y 2 H and 2 such that infz12x

3y A(z1) <A (x)^ A (y)^

0:5 and supz12x

3y A(z1) >A (x)_ A (y)_

0:5, then there exist t 2 (0; 1] and s 2 [0; 1) such that infz12x y A(z1) < t < A (x)^ A (y)^0:5 and supz12x y A(z1) > s > A (x) _ A (y) _ 0:5. This inmplies that x; y 2 U (A; t) and x; y 2 L (

3A; s) but x y * U (A; t) and x y * L (A;

s), which is a contradiction. Hence z12x y inf A(z1) and sup A(z1) z12x

25y A (x) ^ A (y) ^ 0:5 A (x) _ A (y)

_ 0:5 Also, if there exist x; y; a 2 H and ; 2 such that infz12x a y A(z1) < A (a) ^ 0:5 and supz12x a y A(z1) > A (a) _ 0:5, then choose t1 2 (0; 1] and s1 2 [0; 1) such that infz12x a y A(z1) < t1 < A (a) ^ 0:5 and supz12x a y A(z1) > s1 > A (a) _ 0:5. This implies that a 2 U (A; t1) and a 2 L (A; s1) but

25x a y * U (A; t1) and x a y * L (A;

s1), which is a contradiction. Hence, z12x a y inf A(z1) and sup A(z1) z12x a y A (a) ^ 0:5 A (a) _ 0:5 Therefore, A = h A; Ai is an (2; 2 _q)-intuitionistic fuzzy interior -hyperideal of H. Theorem 184 Every (2 _q; 2 _q)-intuitionistic fuzzy interior -hyperideal of a - semihypergroup H is an (2;2 _q)-intuitionistic fuzzy interior -hyperideal of H. Every (2;2;)-intuitionistic fuzzy interior -hyperideal of -semihypergroup H is an (2;2 _q)-intuitionistic fuzzy interior -hyperideal of H. Now we give the condition for (2;2 _q)-intuitionistic fuzzy interior -hyperideal to be an (2;2;)-intuitionistic fuzzy interior -hyperideal of H. Theorem 185 Let A = h A; Ai be (2; 2 _q)-intuitionistic fuzzy interior -hyperideal of H such that A (x) < 0:5 and A (x) > 0:5. Then A = h A; Ai be (2; 2)-intuitionistic fuzzy interior -hyperideal of H. Theorem 186 An IFS A = h A; Ai of H is an (2; 2

3_q)-intuitionistic fuzzy interior -hyperideal of H if and only if

U(t;s) = fx 2 H : x(t; s) 2 Ag for all t 2 (0; 0:5] and s 2 [0:5; 1) is an interior -hyperideal of H. Proof. Its follows from Theorem 183. Theorem 187 An IFS A = h A; Ai in a -semihypergroup H is an (2; 2

3_q)-intuitionistic fuzzy interior -hyperideal of H if and only if [A](t;s) is an interior -hyperideal of

H for all t 2 (0; 1], s 2 [0:1). Proof. Let A = h A; Ai be (2; 2 _q)-intuitionistic fuzzy interior -hyperideal of H and let x; y 2 [A

13](t;s) for all t 2 (0; 0:5], s 2 [0: 5:1). Then

x (t; s), y (t; s) 2 _qA, that is A (x) t and A

3(x) s, A (y) t and A

(y) s and A (x) + t > 1 and A

3(x) + s< 1, A (y) + t> 1 and A

(y) + s < 1. Since, by Theorem 175, we have inf A (z) z2x y z2x y inf A (z) min f A (x) ; A (y) ; 0:5g and sup A (z) max f A (x) ; A (y) ; 0:5g : z2x y min ft; 0:5g and sup A (z) max fs; 0:5g . z2x y For, if not, then x (t; s) 2 _qA or y (t; s) 2 _qA, which is a contradiction. If t 0:5 and s 0:5, then infz2x y A (z) minft; 0:5g = t and supz2x y A (z) maxfs; 0:5g = s. Thus (z) (t; s) 2 A for each z 2 x y. If t > 0:5 and s < 0:5, then infz2x y A (z) 0:5 and supz2x y A (z) 0:5 and thus infz2x y A (z) + t > 1 and supz2x y A (z) + s < 1. Thus (z) (t; s) qA for each

4z 2 x y. Hence (z) (t; s) 2

_qA for each

4z 2 x y. Therefore x y

[A](t;s). Thus, [A](t;s) is sub- -semihypergroup of H. Now, let x; y 2 H and a 2 [A](t;s). Then a (t; s) 2 _qA, that is A (a) t and A (a) s or A (a) + t > 1 and A (a) + s < 1. Since from Theorem 175, we have z2x a y inf A (z) z2x a y inf A (z) Otherwise, we get min f A (a) ; 0:5g and sup A (z) max f A (a) ; 0:5g : z2x a y min ft; 0:5g and sup A (z) max fs; 0:5g . z2x a y inf A (z) < min ft; 0:5g and sup A (z) > max fs; 0:5g . z2x a y z2x a y Which is a contradiction. If t 0:5 and s 0:5, then infz2x a y A (z) min ft; 0:5g = t and supz2x a y A (z) max fs; 0:5g = s and thus (z) (t; s) 2 A for each z 2 x a y. If t > 0:5 and s < 0:5, then infz2x a y A (z) 0:5 and supz2x a y A (z) 0:5 and so, infz2x a y A (z) + t > 0:5 + 0:5 = 1 and supz2x a y A (z) + s < 0:5 + 0:5 = 1. Thus (z) (t; s) qA for each

29z 2 x a y. Therefore (z)

(t; s) 2 _qA for each z 2

5x a y and so x a y [A](

t;s). Hence [A](t;s) is an interior -hyperideal of -semihypergroup H. This is complete proof. Chapter 8 Intuitionistic Fuzzy hyperideals in -semihypergroups Through Left Operator Semihypergroup 8.1 Introduction In this chapter, we obtain a series of lemmas and theorems which are mainly on various relationships between a -semihypergroup and its operator semihypergroups in terms of intuitionistic fuzzy subsets showing so the e¤ectiveness of using operator semihyper- groups in extending several results of semigroups to -semihypergroups as well as to -semigroups. Among other results, we obtain an inclusion preserving bijection between

2the set of all intuitionistic fuzzy hyperideal of a

-semihypergroup H and that of its left operator semihypergroup S . Throughout this section, unless otherwise mentioned, H will denote a -semihypergroup and S will denote the left operator semihypergroup of H . For the sake of simplicity we shall use the follwoing abbreviated notations: I F S (H), I F H (H), I F LH

11(H), I F P H (H), I F S P H

(H), RH (H), H (H) respectively denote intuitionistic fuzzy subset(s) of H , intu- itionistic fuzzy hyperideal(s) of H , intuitionistic fuzzy left hyperideal(s) of H , intuition- istic fuzzy prime hyperideal(s) of H, intuitionistic fuzzy semiprime hyperideal(s) of H, right hyperideal(s) of H, hyperideal(s) of H. 8.2 Intuitionistic Fuzzy Hyperideals in -semihypergroups We obtain a series of lemmas and theorems which are mainly on various relationships between a -semihypergroup and its operator semihypergroups in terms of intuitionistic fuzzy subsets showing so the e¤ectiveness of using operator semihypergroups in extending several results of semigroups to -semihypergroups as well as to -semigroups. Among other results, we obtain an inclusion preserving bijection between the set of all intuitionis- tic fuzzy hyperideal of a -semihypergroup H and that of its left operator semihypergroup S Let H be a -semihypergroup and S a left operator semihypergroup of H. For an IFS(S), C = (C; C) we de…ne an IFS(H),

28C+ = (C; C)+ = (+C; +C) by +C(a) = inf C ([a;]) and +C(a) = sup C([a;]), where a

2 H. For an IFS(H), D = (D; D) 2 2 we de…ne an IFS(S), D+0 = (D; D)+0 = (+D0; +D0) by +D0([a;]) = inf D(t) and m2H;t2a m +D0([a;]) = sup D(t), where [a;] 2 S. m2H;t2a m The following examples illustrate the above notion. Example 188 Let H = [0; 1] be the real unit close interval, and = N . Then, H together with the hyperoperation x y = [0; xy] is a -semihypergroup. If we de…ne the following relation in H as follows: (x;) (y;) , x = y it is clear that is an equivalence relation. Then S = f[x;]jx 2 H; 2 g with the hyperoperation [x;] [y;] = f[z;] : z 2 x yg is the left operator semihypergroup of H. We can de…ne the following fuzzy subset of S: 0:7 if 0 x < 1 0:3 if 0 x < 1 C ([x;]) = 8 and C ([x;]) = < 0:4 if 1 x 1 < 8 0:5 if 1 x 1 so : : +C(x) = inf C([x;]) and +C(x) = sup C([a;]) 2 2 So, for example, for x = 0:5 we have +C(0:5) = inf C([0:5;]) = 0:4 and +C(0:5) = sup C([a;]) = 0:5 2 2 : Example 189 Let H = f1; 2g and = H . Then, H together with

hyperoperation x y = fx; ; yg is a -semihypergroup. If we de…ne the following relation in H as follows: x y = x y, then is an equivalence relation. We have [1; 1] = [2; 1] = f(1; 1) ; (2; 1)g [2; 2] = [1; 2] = f(1; 2) ; (2; 2)g : So, we have S = f[1; 1] ; [1; 2]g is a semihypergroup with the following hyperoperation de…ne [1; 1] [1; 2] [1; 1] [1; 1] [1; 2] [1; 2] S S Now, we de…ne IFS A = (A; A) in S as follows: A ([1; 1]) = 0:6; A ([1; 2]) = 0:4 A ([1; 1]) = 0:4; A ([1; 2]) = 0:5: We have A+ = +A; +A de…ned as +A(x) = inf A([x;]) and +A(x) = sup C([x;]) 2 2 +A (1) = +A (2) = 0:4 +A (1) = +A (2) = 0:5: Proposition 190 Let H be a -semihypergroup and S be its left operator semihyper- group. Let A = h A; Ai be an IFS(S), then [U(A

38**;t)]+ = U((** A **)+;t)** and **[L(** A **;t)]+ = L((** A **)+;t)** for all **t**

2 [0;1], provided the sets are non-empty. Proof. Let m 2 H. Then, m 2 [U(A;t)]+ , [m;] U(A;t);8 2 , A([m;]) t; 8 2 , inf A([m;]) 2 t , (+A(m) t , m 2 U((A)+;t): Again, let n 2 H. Then, n 2 [L(A; t)]+ , [n;] L(A; t); 8 2 , A([n;]) t; 8 2 , sup A([n;]) t 2 , (+A(n) t , n 2 L((A)+; t): Hence, [U (A; t)]+ = U ((A)+; t) and [L(A; t)]+ = L((A)+; t). Proposition 191 Let H be a -semihypergroup and S be its left operator semihyper- group. Let B = h B ; B i be an I F S (H). Then, [U (B ;

45**t)]+0 = U** ((B **)+0** ; t) and **[L(** B ; **t)]+0** = L((B **)+0**

; t) for all t 2 [0; 1], provided the sets under consideration are non-empty. Proof. Let [x;] 2 S and t be as mentioned in the statement. Then, [x;] 2 [U (B ; t)]+0 , x m U (B ; t); 8H 2 H , B (x m) t; 8m 2 H , inf B (t) t m2H;t2x H , (B)+0([x;]) t , [x;] 2 U((+B0;t): Again let [y;] 2 S and t be as mentioned in the statement. Then, [y;] 2 [L(B;t)]+0 , y n L(B;t);8n 2 H , B(y n) t;8n 2 H , sup B(t) t n2H;t2y n , (B)+0([y;]) t , [y;] 2 U((B)+0;t): Hence, [U(B

45**;t)]+0 = U((** B **)+0;** t) and **[L(** B **;t)]+0** = L((B **)+0;**

t). In the following, H denotes a -semihypergroup with unities. Proposition 192 Let H be a -semihypergroup with unities and S be its left opera- tor semihypergroup. If A = h A; Ai 2 IFS(S), then A+ = h A; Ai+ = (+A; +A) 2 IFS(H). Proof. Let us suppose A = h A; Ai 2 IF S(S). Then, U(A; t) and L(A; t) are I(S); 8t 2 [0; 1]. Hence, [U(A; t)]+ and [L(A; t)]+ are I(H); 8t 2 [0; 1]. Now, since A = h A; Ai is an IF S(S), A = h A; Ai is a non-empty IF S(S). Hence, for some [m;] 2 S; 0 < A([m;]) + A([m;]) 1. Then, U(A; t)6 = ; and L(A; t)6 = ;, where t = A([m;]) = ([m;]). So by the same argument applied above [U(A; t)]+6 = ; and [L(A;t)]+6 = ;. Let u 2 [U(A;t)]+. Then, [u;] U(A;t) for all 2 . Hence, A([u;]) t. This implies that inf A([u;]) t,

10**i.e. (A)+(u) t.** Hence, 2 **u** 2 **U((**

A)+;t). Hence, U((A)+;t)6 = ;. By similar argument we can show that [L(+A;t)]6 = ;. Consequently, [U(A;t)]+ = U((A)+;t) and [L(A;t)]+ = L((A)+;t). It follows that U((A)+;t) and L((A)+;t) are H(H) for all t 2 [0;1]. Hence, A+ = h A; Ai+ = (+A; +A) is an IFH(H). By using the property (ii) in section 2, the Proposition 192 and [31], we deduce the following proposition. Proposition 193 Let H be a -semihypergroup with unities and S be its left operator semihypergroup. If B = h B; Bi 2 IFH(H)(IFLH(H)), then B+0 = h B; Bi+0 = (+B0; +B0) 2 IFH(S) (resp. IFLH(S)). Now we obtain the inclusion preserving bijection of mapping ()+0 for the intuitionistic fuzzy hyperideals and intuitionistic fuzzy right hyperideals of a -semihypergroup H and its left operator semihypergroup S which is the analogue with the case of fuzzy sets [33]. The theorem also shows that the mappings ()+0 and ()+ are inverse to each other. Theorem 194 Let H be a -semihypergroup with unities and S be its left operator semi- hypergroup. Then there exists an inclusion preserving bijection A ! A+0 between the set of all IFH(H)(IFRH(H)) and the set of all IFH(S) (resp. IFRH(S)), where A = h A; Ai is an IFH(H) (resp. IFRH(H)). Proof. Let A = h A; Ai 2 IFH(H)(IFRH(H)) and x 2 H. Then, we have (A+0)+(x) = inf +A0([x;]) = inf[m2Hi;nt2fx m 2 2 A(t)] A(x). Again (+A0)+(x) = sup +A0([x;]) = sup[sup A(t)] A(x). 2 2 m2H;t2x m Hence, A (A+0)+. Let [;f] be the right unity of H. Then x f = x;8x 2 H. This implies that A(x) = A(x f) inf[2 m2Hi;nt2fx m A(t)] = inf +A0([x;]) = (+A0)+(x). 2 Again A(x) = A(x f) sup[sup A(t)] = sup +A0([x;]) = (A+0)+(x). 2 m2H;t2x m 2 So A (A+0)+. Hence, we obtain (A+0)+ = A. Now, let B = h B ; B i 2 I F H (S)(I F RH (H)). Then, (+B)+0([x;]) = m2Hi;nh2fx m +B(h) = min2Hf[inf B([x m;])] 2 = inf [inf B ([z;])] B ([x;]): m2H 2 ;z2x m Again (+B)+0([x;]) = sup m2H;h2x m +B(h) = sup[sup B([x H;])] m2H 2 = sup[sup B([z;])] B([x;]): m2H 2 ;z2x m Thus, we obtain B (B+)+0. Let [e;] be the left unity of S. Then, B ([x;]) = B ([x;] [e;]) = (+B)+0 ([x;]): Again inf [inf B([x;] [m;])] = m2H 2 B ([x;]) = B ([x;] [e;]) sup [sup B ([x;] [m;])] = m2H 2 = (+B)+0 ([x;]): Thus, we have B (B+)+0. Hence, we obtain B = (B+)+0. Thus, we see that the correspodence A7 ! A+0 is a bijection. Now, let C = (C; C); D = (D; D) 2 IFH(H)(IFRH(H)) be such that C D, i.e., C D and C D. Then, for all [x;] 2 S, +C0([x;]) = inf C(t) m2H;t2x m and +C0 ([x;]) = sup C (t) m2H;t2x m inf m2H;t2x m sup m2H;t2x m D(t) = +D0([x;]), D(t) = +D0([x;]). Consequently, +C0 +D0 and +C0 +D0. Thus, we have C+0 D+0. Hence, we conclude that A7 ! A+0 is an inclusion preserving bijection. Lemma 195 Let I 2 RH(S)(H(S)) of a -semihypergroup H and P = (I; cI) where I is the characteristic function of I. Then P+ = (I; cI)+ = ((I)+;(cI)+) = (I+; cI+). Proof. Let us suppose that H 2 I+. Then, [m;] 2 I;8 2 . This means inf (I ([m;])) = 1 and sup(cI ([m;])) = 0. Also, I+ (m) = 1 and cI+ (m) = 0. 2 2 Now, suppose that m 2= I+. Then, there exists 2 such that [m;] 2= I. Hence, I([m;]) = 0; Ic([m;]) = 1 and so inf(I([m;])) = 0;sup(Ic([m;])) = 1. Hence, 2 2 (I)+(m) = 0 and (cI)+(m) = 1. Again (I+)(m) = 0 and (cI+)(m) = 1. Thus, P+=(I; cI)+=((I)+;(cI)+)=(I+; cI+). Inasimilarwayweobtainthefollowinglemma. Lemma 196 Let I 2 LH(H)(H(H)), P = (I; cI) and S be the left operator semihyper- group of a -semihypergroup H. Then P+0 = (I; cI)+0 = ((I)+0;(cI)+0) = (I+0; cI+0), where I is the characteristic function of I. Using the Lemma 195, Lemma 196, Theorem 194 and Proposition 193 we have the following result on -semihypergroup [33]. Theorem 197 Let H be a -semihypergroup with unities. Then, there exists an inclu- sion preserving bijection between the set of all hyperideals (right hyperideals) of H and that of its left operator semihypergroup S via the mapping I ! I+0. Proof. Let us denote the mapping I ! I +0 by (cf. Proposition 193). Now let (I1) = (I2). Then, I1+0 = I2+0 . This implies that (I1+0 ; cI1+0) = (I2+0 ; cI2+0) (where I is the characteristic function of I). Hence by Lemma 196, (I1 ; cI1)+0 = (I2 ; Ic2)+0 . From this, by using Theorem 194, we obtain (I1 ; cI1) = (I2 ; cI2), whence I1 = I2. Consequently, is one-one. Let I be a hyperideal (right hyperideal) of S . Then, (I1 ; cI1) is an I F H (S)(I F RH (S)). Hence by Theorem 194, ((I ; cI)+)+0 = (I ; cI). Hence by noting that I + and (I +)+0 are hyperideals (right hyperideals) of H and S; respectively, (cf. (i) and (ii) of §2) we use Lemma 195 and Lemma 196 and deduce that ((I +)+0 ; c(I +)+0) = (I ; cI). This implies that (I +)+0) = I whence (I +) = I . Hence, in view of (i) of §2 (cf. Proposition 193), is onto. Let I1; I2 be two hyperideals (right hyperideals) of H with I1 I2. Then, I 1 I 2 and cI1 cI2 . Hence, by using Theorem 194, we obtain (I1)+0 (I2)+0 and (I1)+0 (I2)+0 , i.e. I1+0 +0 and I2 cI1+0 cI2+0 . This implies that, by Lemma 196, this implies that +0 +0 and and so we obtain I1 I2 I1+0 I2+0 . Hence, we conclude the theorem. cI1+0 cI2+0 Proposition 198 Let H be a -semihypergroup not necessary with unity and S be its left operator semihypergroup. If A = h A; Ai 2 IFPH(S)(IFSPH(S)), then A+ = (+A; +A) 2 IFPH(H)(resp. IFSPH(H)). Proof. Let A = h A; Ai 2 IFPH(S). Then, A = h A; Ai 2 IFH(S). By Propo- sition 192, A+ = (+A; +A) 2 IFPH(H). Since A = h A; Ai 2 IFPH(S), so we have U(A;t) and L(A;t) are PH(S). By Remark 2.12, 12**for all t 2 [0;1];[** U(A **;t)]+** and [L(A **;t)]+**

are PH(H). By Remark 2.12, [U(A;t)]+ = U((A)+;t) and [L(A

16**;t)]+ = L((A)+;t).** So U((**A)+;t)** and L((**A)+;t)**

are PH(H). Hence A+ = (+A; +A) 2 IFPH(H). Similarly we can prove the other case also. In the similar way it can be shown the following proposition. Proposition 199 Let H be a -semihypergroup not necessary with unities and S be its left operator semihypergroup. If A = h A; Ai 2 IFPH(H)(IFSPH(H)), then A+0 = (+A0; +A0) 2 IFPH(S)(resp. IFSPH(S)). The following theorem is on the inclusion preserving bijection between the sets of intuitionistic fuzzy prime hyperideals of H and its left operator semihypergroup S. Theorem 200 Let H be a -semihypergroup (not necessary with unity) and S be its left operator semihypergroup. Then, there exists an inclusion preserving bijection A = h A; Ai ! A+0 = (+A0; +A0) between the set of all intuitionistic fuzzy prime (resp. semi- prime) hyperideals of H and set of all intuitionistic fuzzy prime (resp. semiprime) hy- perideals of S. Proof. Let x 2 H and A = h A; Ai 2 IFPH(S). Then, (+A0)+(x) = inf +A0([x;]) = 2 inf inf 2 m2H;h2x m A(h). Hence, as 2 ;m2H;h2x m inf A(h) A(x);8H 2 H; 2 (A being a intuitionistic fuzzy hyperideal), we deduce that inf inf A(h) A(x) and hence (+A0)+(x) 2 m2H;h2x m A(x). Consequently, A +A0. In similar way, we obtain A +A0. Again for x 2 H; (+0)+(x) = min2fH 2 i;hn2fx m A(h) = min2fH(maxf A(m); A(x)g) (since A is an intuitionistic fuzzy prime hyperideal) maxf A(x); A(x)g = A(x). Hence, A +A0 . Thus, we obtain A = +A0 . In similar way, we obtain A +A0 . Conse- quently, (A+0)+ = A. Hence, the mapping is one-one.

3**Now, if A is** an **intuitionistic fuzzy** prime hyperideal **of** S, **then** for **[x;**

] 2 S; (+A)+0([x;]) = m2Hi;nh2fx m +A(h) = inf inf A([x H;]) = inf inf A([z;]) A([x;]) (A being an intuitionis- m2H 2 m2H;z2x m 2 tic fuzzy hyperideal). This implies that A (+A)+0. Let [x;] 2 S. Then, since A is an intuitionistic fuzzy prime hyperideal, so z2ixnfm A([z;]) = maxf A([x;]), A([m;])g;8[m;] 2 S. Hence for H = x and = , inf A([z;]) = A([x;]). z2x m Hence inf inf A([z;]) A([x;]). Again m2H;z2x m 2 (+A)+0([x;]) = m2Hi;nz2fx m in2f A([z;]). Hence (+A)+0([x;]) A([x;]); 8[x;] 2 S. This implies that A (+A)+0. Consequently, we obtain A = (+A)+0. In similar way, we obtain A = (+A)+0. Consequently, (A+)+0 = A. Thus we deduce that the mapping is onto. Inclusion preserving property is similar as in Theorem 194. Hence, A7 ! A+0 is an inclusion preserving bijection. Similar results hold for IFSPH(H). As another illustration of the e¤ectiveness of relationship between a -semihypergroup and its left operator semihypergroup in terms of intuitionistic fuzzy subsets we obtain the following theorem on prime hyperideals [33, 35]. The proof of this theorem is omitted because it follows by using Theorem 200 and similar argument as applied in Theorem 197. Theorem 201 Let H be a -semihypergroup and S be its left operator semihypergroup. Then, there exists an inclusion preserving bijection between the set of all prime hyperideals of H and set of all prime hyperideals of S via the mapping I ! I+0. De…nition 202 Let H be a -semihypergroup and A = h A; Ai be an intuitionistic fuzzy subset of H and x 2 H. Then, the intuitionisic fuzzy subset hx; Ai = f(y; hx; Ai (y); hx; Ai (y)) : x 2 Hg, where the functions hx; Ai : H ! [0; 1] and hx; Ai : H ! [0; 1] de…ned by hx; Ai (y) = 2 i;nh2fx y A(h); 8y 2 H and hx; Ai (y) = 2 s;uh2px y A(h); 8y 2 H, is called the intuitionistic fuzzy extension of A by x. By a routine veri…cation we obtain the following propositions. Proposition 203 Let H be a commutative -semihypergroup and S be its left semihyper- group. If A = h A ; A i is an I F LH (H)(I F H (H)), then x; A+0 is an I F LH (S)(I F H (S)) for al l x 2 S . Proposition 204 Let H be a commutative -semihypergroup and S be its left semihyper- group. If A = h A; Ai is an I F LH (S)(I F H (S)), then hx; A+i is an I F LH (H)(I F H (H)) for all x 2 H . The mappings ()+; ()+0, do not commute with the operation of taking intuitionistic fuzzy extension. But, how they actually behave in this respect, are shown in the following lemmas which characterize also the relationship between a -semihypergroup and its left operator semihypergroup in terms of intuitionistic fuzzy extension. Lemma 205 Let A = h A; Ai be an intuitionistic fuzzy subset of a commutative - semihypergroup H. Then for all x 2 H 1. hx; Ai+0 [x;]; A+0 ; 8 2 2. hx; Ai+0 = (hx; Ai+0 ; hx; Ai+0) = (inf [x;]; +A0 ; sup [x;]; +A0): 2 D E 2 D E Proof. (1). Let [y;] 2 S. Then hx; Ai+0 ([y;]) = inf hx; Ai (h) = m2H;h2y m m2H 2 ;h2y m x inf inf (h) = inf inf 2 m2H;h2y m x (h). Again [x;]; +A0 ([y;]) = z2inxfy +A0([z;]) = +A0([x y;]) = h2x yinmf;m2H A(h) = m2H;hin2xf m y AD(h). SinceE, in2f m2H;hin2xf m y A(h) m2H;h2x m y inf A(h), we deduce that hx; Ai+0 ([y;]) [x;]; +A0 ([y;]). Consequently, hx; Ai+0 [x;]; +A0 , 8 2 . In similar way Dwe obtainEhx; Ai+0 ([y;]) [x;]; A+0 ([y; D]). ConseEquently, hx; Ai+0 [x;]; A+0 ; 8 2 . D E (2). Let [y;] 2 S. Then inf [x;]; +A0 ([y;]) = 2 inf +A0 ([z;] D E 2 ;z2x y = in2f +A0([x y;]) = in2f m2H;hin2fx y m A (h) = inf inf m2H 2 ;h2x m y A(h) = m2Hi;nhf2y m hx; Ai (h) = hx; Ai+0 ([y;]): In similar way we obtain sup [x;]; +A0 ([y;]) = hx; Ai+0 ([y;]). Consequently, hx; ADi+0 = (hx; Ai+0 ; hx; Ai+0) = inf [x;]; +A0 ; sup [x;]; +A0 2 E 2 . D E 2 D E Lemma 206 Let A = h A; Ai be an intuitionistic fuzzy subset of the left operator semi- hypergroup S of a -semihypergroup H. Then for all x 2 H; h[x;]; Ai+ hx; A+i ; 8 2 . Proof. Let p 2 H. Then h[x;]; Ai+ (p) = inf h[x;]; Ai ([p;])= 2 inf 2 ;z2p x A[z;]) = in2f A

4**([p x;**]). Again **x; +A (p)** = in2f +A(p **x)**

= in2f in2f A([p x;]) = inf inf A([p x;]). Now since inf A([p x;]) inf inf A([p x;]), we deduce that 2 2 2 2 2 2 [x;]; +A (p) x; +A (p). In similar way we obtain [x;]; +A (p) x; +A (p). Consequently, h[x;]; Ai+ hx; A+i ; 8 2 . Lemma 207 Let I be a hyperideal of a -semihypergroup H. Then ((I)+0;(I)+0) = (I+0; I+0). Proof. Let [y;] 2 S. Then (I)+0([y;]) = inf m2H;h2y m I(h) and (I)+0([y;]) = sup I(h). Now let us suppose [y;] 2 I+0. Then y H I;8m 2 m2m;h2y m H. Hence I

27**(h) = 1;** 8m **2 H;h 2** y **H** and I **(h)**

= 0;8m 2 H;h 2 y m whence m2H;h2y m inf I(h) = 1 and sup I(h) = 1. Thus we deduce that (I)+0([y;]) = 1 m2H;h2y m and (I)+0([y;]) = 0. Also, I+0([y;]) = 1 and I+0([y;]) = 0. On the other hand, if [y;] 2= I+0, then for some t 2 H; y t * I. Hence I(h) = 0; h 2 y t and I (h) = 1; h 2 y t, whence inf I (h) = 0 and sup I (h) = 1. Thus m2H;h2y t m2H;h2y t we see that (I)+0([y;]) = 0 and (I)+0([y;]) = 1. Again I+0([y;]) = 0 and I+0([y;]) = 1. Thus we deduce that (I)+0([y;]) = I+0([y;]);8[y;] 2 S and (I)+0([y;]) = I+0([y;]); 8[y;] 2 S. Hence we conclude that (

11**I)+0 = I+0** and (**I)+0 = I+** 0. Consequently, ((**I)+** 0;(**I)+** 0) = (**I+** 0; **I+**

0). The following lemmas are on the commutativity of the mapping ()+0 with intersection and in…mum. Lemma 208 [33] Let fAigi2I be a family of hyperideals of a -semihypergroup H. Then +0 Ai = i2I A+i0 . i2I Lemma 209 Let H be a -semihypergroup, S its left operator semihypergroup and fAigi2I = T T (Ai; Ai)i2I

2**be a family of intuitionistic fuzzy subsets of** H such that **A**

= h A; Ai = iin2If Ai = (iin2If Ai; siu2pI Ai). Then A+0 = ((A)+0; (A)+0) = iin2If(Ai)+0 = ii2nIf(Ai)+0; siu2pI(Ai)+0 . Proof. Let [x;] 2 S. Then (A)+0([x;]) = (inff Ai)+0([x;])= inf i2I;m2H;h2x H Ai(h) = i2I;m2iHn;fh2x H Ai(h) = ii2nIf(Ai)+0([x;]). In similar way we obtain (A)+0([x;]) = sup(Ai)+0([x;]). Hence we conclude that A+0 = inf(Ai)+0. i2I i2I The following results are on the intuitionistic fuzzy subset extension which are obtained using mainly the operator semihypergroups technique. Proposition 210 Let H be a commutative -semihypergroup with unities and A = h A; Ai be an intuitionistic fuzzy hyperideal of H. Then hx; Ai = (hx; Ai ; hx; Ai) is an intuitionistic fuzzy hyperideal of H, for all x 2 H. Proof. Let S be the left semihypergroup of H. Since H is commutative, S is com- mutative. Now by Proposition 193, A+0 = (+A0; +A0)

2**is an intuitionistic fuzzy** hyperideal **of S.**

Let x 2 H. Then for any 2 ; [x;];A+0 = [x;]; +A0 ; [x;]; +A0

2**is an intuitionistic fuzzy** hyperideal **of S**

and hence in2f [x;]; +A0 E;suDp [x;]; EA+0 D

2**is an intuitionistic fuzzy** hyperdieal **of S.**

Hence, by usiDng LemmaE205(2)D, we see thEat 2 hx; Ai+0 ; hx; Ai+0 = hx; Ai+0 is an intuitionistic fuzzy hyperideal of S. Consequently, by Proposition 190, hx;Ai+0 is an intuitionistic fuzzy hyperideal of H. Hence hx;Ai + is an intuitionistic fuzzy hyperideal of H (cf. Theorem 194). Applying Proposition 199 and Theorem 200 we

58**obtain the following proposition. Proposition** 211 **Let** H **be a**

commutative -semihypergroup and A = h A; Ai be an intuitionistic fuzzy semiprime hyperideal of H. Then hx; Ai = (hx; Ai ; hx; Ai) is an intuitionistic fuzzy semiprime hyperideal of H for all x 2 H. Proposition 212 Let H be a commutative -semihypergroup, fAigi2I = (Ai; Ai)i2I be a non-empty family of intuitionistic fuzzy semiprime hyperideals of H and let A = h A; Ai = iin2IfAi = (iin2If Ai;siu2pI Ai). Then for any x 2 H;hx;Ai is an intuitionistic fuzzy semiprime hyperideal of H. Proof. Let S be the left operator semihypergroup of H. Since H is commutative, S is commutative. Now, in view of Proposition 199, we have fA+i0gi2I = +A0i; +A0i is i2I a non-empty

2**family of intuitionistic fuzzy** semiprime hyperideals **of S.**

Hence inffA+i0g i2I is

2**an intuitionistic fuzzy** semiprime hyperideal **of S.** Again, by **Lemma**

209, A+0 is an intuitionistif fuzzy semiprime hyperideal of S. This implies that for any [x;] 2 S; [x;]; A+0 = [x;]; +A0 ; [x;]; +A0 is an intuitionistic fuzzy semiprime hy- perideal of S and hence inf E [xD;]; +A0 ; sup [x;]; +A0 is an intuitionistic fuzzy D E semiprime hyperideal of S. HenDce, by usinEg LemDma 205(2),Ewe immediately deduce that 2 2 hx; Ai+0 ; hx; Ai+0 = hx; Ai+0 is an intuitionistic fuzzy semiprime hyperideal of S. Consequently, in view of Proposition 198, (hx; Ai+0)+ is an intuitionistic fuzzy semiprime hyperideal of H. Hence, in view of Proposition 199 and Theorem 200, we conclude that hx;Ai is an intuitionistic fuzzy semiprime hyperideal of H. Theorem 213 Let H be a commutative -semihypergroup, fHigi2I is a non-empty fam- ily of semiprime hyperideals of H, A = Hi6 = ; and B = (A; cA). Then hx;Bi is an i2I intuitionistic fuzzy semiprime hyperidealTof H for all x 2 H. Proof. Since, 8i 2 I , Hi

1**is a** semiprime **hyperideal of H** , Hi+0 **is a** semiprime **hyperideal of the**

left operator semihypergroup S. Now, since A = Hi, A+0 = +0 i2I Hi = Hi6+0 = ; (see Lemma 209). So by a suitable adoption of [47, Corollary T i2I i2I 3.1T1], h[x;]; BTi = (h[x;]; Ai ; h[x;]; cAi) is

2**an intuitionistic fuzzy** semiprime hyper- **ideal of S,**

8 2 . Hence inf h[x;]; Ai ; sup h[x;]; cAi is

2**an intuitionistic fuzzy** semiprime hyperideal **of S and**

2 2 so h[x;] Ai+0 ; h[x;] cAi+0 = hx; Bi is an intuitionistic fuzzy semiprime hyperideal of S. Hence, (hx; Bi+0)+ is an intuitionistic fuzzy semiprime hyperideal of H. Consequently, by Theorem 200, hx; Bi is an intuitionistic fuzzy semiprime hyperideal of H . Finally, we obtain, by using the concept of intuitionistic fuzzy extension and re- lationship between a -semihypergroup and its operator semihypergroups in terms of intuitionistic fuzzy subsets, the following characterization of a prime hyperideal of a -semihypergroup. Theorem 214 Let H be a -semihypergroup, I be a hyperideal of H and B = (I ; cI) where I is the characteristic function of I . Then I is prime hyperideal of H if and only if for x 2 H , with x 2= I , hx; Bi = B. Proof. Let I be a prime hyperideal of H and x 2= I . Then by Lemma 9, we see that I +0 is a prime hyperideal of the left operator semihypergroup S . Also, as

33**x 2= I ; [x;] 2= I** +0 for some **2** . Hence **[x;**

]; B+0 = B+0 , i.e., [x;]; +I0 ; [x;]; (cI)+0 = ((I)+0 ; (Ic)+0) = B+0 . Now by Lemma 195, we note thatD [x;]; EI+0 ; [x;]; (Ic)+0 = (I ; cI)+0 . Hence it immediately follows that D E [x;]; B+0 + = [x;]; +I0 ; h[x;]; (cI)+0i = ((I ; cI)+0)+ = (I ; cI) = B. By + Lemma 205(1), hx; BDi+0 [x;E]; B+0 ; 8 2 . So hx; Bi+0 + B. Consequently by Theorem 200, hx; Bi B. By a suitable adoption of [34, Proposition 2.5], we get B hx; Bi. Hence hx; Bi = B. Conversely, let us suppose hz;Bi = B for all z 2 H with

61**z 2= I.** Let **x** y **I** with x **;y**

2 H. Then inf I(h) = 1 and inf cl(h) = 0, 8 2 . Let x 2= I. Then by hy- h2x y h2x y pothesis hx; Bi = B. This implies that hx; Ii (y) = i(y) and hx; cIi (y) = ci(y). Then 2 ;h2x y inf I(h) = I(y) and sup cI(y) which implies that I(y) = 1 and cI

22(y) = 0 2 ;h2x y whence y 2 I. Consequently, I is a

prime hyperideal of H. Chapter 9 Interval Valued Intuitionistic Fuzzy Sets in -semihypergroups 9.1 Introduction In this chapter, we apply the concept of an interval valued intuionsitic fuzzy set to theory of -hyperideals, interval valued intuitionistic fuzzy (1,2) -hyperideal of - semihypergroup and obtain some basic results. We give some further properties of in- terval valued intuitionistic fuzzy -hyperideals and interval valued intuitionistic fuzzy bi- -hyperideals in a -semihypergroup. We de…ne an interval valued intuitionistic fuzzy prime(semiprime) -hyperideals, intuitionistic fuzzy M -hypersystem and N -hypersystem of a -semihypergroup and intuitionistic fuzzy semisimple -semihypergroups and some properties of them are investigated 9.2 Interval Valued Intuitionistic Fuzzy -hyperideals In this section we de…ne an interval valued intuitionistic fuzzy left (right, two sided, bi) - hyperideal and interval valued intuitionistic fuzzy (1,2) -hyperideal of -semihypergroup and obtained some basic results. De…nition 215 Let H be a -semihypergroup. An interval valued intuitionistic fuzzy set A = h A; Ai in H is called an interval valued intuitionistic fuzzy sub- -semihypergroup in H if fore all x; y 2 H; 2 , inf f A(z)g minf A(x); A(y)g and sup f A(z)g maxf A(x); A(y)g. e z2x y z2x y De…nition 216 Let H be a -semihypergroup. An interval valued intuitionistic fuzzy set A = h A; Ai in H is called an interval valued intuitionistic

1fuzzy left (resp. right) -hyperideal of H if for all x; y 2 H;

2 , e e 1. A(y) z2inxfyf A(z)g (resp. A(x) z2inxfyf A(z)g). 2. sup f A(y)g A(y) (resp. sup f A(z)g A(x)). z2x y z2x y An interval valued intuitionistic fuzzy set A = h A; Ai in H is called an interval valued intuitionistic fuzzy -hyperideal of H if it is both an interval valued intuitionistic e e fuzzy left and an interval valued intuitionistic right -hyperideal of H . De…nition 217 Let H be a -semihypergroup. An interval valued intuitionistic fuzzy sub- -semihypergroup A = h A; Ai in H is called an interval valued intuitionistic fuzzy bi- -hyperideal of H if for all x;ey; z 2 H; ; 2 , inf f A(t)g minf A(x); A(z)g and sup f A(t)g maxf A(x); A(z)g. e t2x y z t2x y z Example 218 Let H = (0; 1), = f n jn 2 N g and for every n 2 N we de…ne hyperop- eration n on S as follows xy x ny = 2k j0 k n ; 8x; y 2 H: n o Then, x ny H and for every

21m; n 2 N and x; y; z 2

H xyz (x ny) mz = 2k j0 k n + m = x n (y mz) . n o So, H is a -semihypergroup. Now we de…ned an interval valued IFS A = h A; Ai on H as: e e A (x) = [0:6; 0:6] if 0 < x < 21k 8 < [0:4; 0:5] if 2k x < 1 1 where k 2 N [0:2; 0:3] A (x) = : if 0 < x < 21k 8 < [0:4; 0:5] if 2k x < 1 1 where k 2 N Then, by routine calculation, A = h : A ; Ai is an interval valued intuitionistic fuzzy -hyperideal of H: e e Example 219 Let H be the -semihypergroup in Example 218. We de…ned an interval valued IFS A = h A; Ai by [0:7; 0:8] if 0 < x < 4110 [0:1; 0:2] if 0 < x < 4110 A (x) = [0:5; 0:7] if 4110 x < 415 and A (x) = [0:3; 0:3] if 4110 x < 415 8 e e 8 >< [0:2; 0:4] if 415 x < 1 >< [0:6; 0:5] if 415 x < 1 Then, cle>:arly A = h A; Ai is an interval valued intuit>:ionistic fuzzy bi- -hyperideal of H . e e Example 220 Let H = f1; 2; 3; 4; 5g and = f ; g be two non- empty sets. Then, H is a -semihypergroup de…ned by the following Cayley tables. 1 2 3 4 5 1 2 3 4 5 1 f1g f1g f1g f1g f1g 1 f1g f1g f1g f1g f1g 2 f1g f1g f1g f1g f1g 2 f1g f1g f1g f1g f1g 3 f1g f1g f3g f3g f3g 3 f1g f1g f3g f3g f3g 4 f1g f1g f3; 4g f3; 4g f5g 4 f1g f1g f3g f3; 4g f5g 5 f1g f1g f3; 4g f3; 4g f5g 5 f1g f1g f3g f3; 4g f5g 1) Let A = h A; Ai be an IFS in a -semihypergroup H de…ned by A (1) = [0:7; 0:75], A (2) = [0:5; 0:6] ; A (3) = A (4) = A (5) = [0; 3; 0:4] ; and A (1) = [0:1; 0:15], A (2) = [0:3; 0:4] ; A (3) = A (4) = A (5) = [0:4; 0:5]. Then, A = h A; Ai is an intuitionistic fuzzy -hyperideal of H: 2) Let A = h A; Ai be an IFS in a - semihypergroup H de…ned by A (1) = [0:8; 0:85], A (2) = [0:6; 0:7] ; A (3) = [0:45; 0:55] ; A (4) = A (5) = [0:2; 0:3] ; and A (1) = [0:05; 0:1], A (2) = [0:15; 0:2] ; A (3) = [0:4; 0:45] ; A (4) = A (5) = [0:5; 0:6]. Then, A = h A; Ai is an intuitionistic fuzzy -hyperideal of H: Proposition 221 If fAigi2I is a family of interval valued intuitionistic

1**fuzzy left (resp. right)** -hyperideals **of** a -semihypergroup **H , then**

Ai is an interval valued intuition- istic

1**fuzzy left (resp. right) -hyperideal of H**

. i2I T Proof. Let us suppose that B = Ai and x;y 2 H. Then for all 2 , we have: i2I T B(y) = iin2Iff Ai(y)g ii2nIffz2inxfyf Ai(z)gg = z2inxfyfiin2Iff Ai(z)gg = zi2nxfyf B(z)g: sup f B(z)g = z2x y z2x y i2I sup fsupf Ai(z)gg = supf sup f Ai(z)gg i2I z2x y supf Ai(y)g = B(y): i2I Hence, Ai is an interval valued intuitionistic fuzzy left (resp. right) -hyperideal of H . i2I T Proposition 222 Let A = h A; Ai be an interval valued intuitionistic fuzzy bi- - hyperideal and B = h B ; B i be an ineterval valued intuitionistic fuzzy sub- - semihypergroup of a - semihypergroup H . Then, A \ B is an interval valued intuitionistic fuzzy bi- e - hyperideal of H . Proof. The proof is straightforward and we omit it. Proposition 223 Let H be a - semihypergroup. An interval valued intuitionistic fuzzy set A = h A; Ai is an interval valued intuitionistic fuzzy

1**left (resp. right) -hyperideal of H if and only if**

the interval valued fuzzy sets and cA are interval valued

1**fuzzy left (resp. right) -hyperideal.** e e **A**

Proof. Let us assume that A = h A; Ai is an interval valued intuitionistic fuzzy left -hyperideal of H. Clearly, A is an inteerval valued

1**fuzzy left -hyperideal of H. For**

x;y 2 H; 2 , we have e e z2inxf yf cA(z)g = inf f1 A(z)g = 1 sup f A(z)g z2x y z2x y e 1 Ae(y)e= cA(y): e e Hence cA is an interval valued fuzzy lefet -hypeerideal of H. e Conversely, let us suppose A and cA are interval valued fuzzy left -hyperideals of H. For every x; y 2 H, we have A(y) inf f A(z)g and we get e e e z2x y sup f A(z)g = e sup f1 z2x y z2x y cAe(z)g = 1 z2inxfyf cA(z)g e 1 cAe(y

10)e= A(y): e e Hence, A = h A;

Ai is an interval valued intuitionistic fuzzy left -hyperideal of H. e e e Corollary 22e4 Leet H be a -semihypergroup and A = h A; Ai be an interval valued intuitionistic fuzzy set in H. Then, A = h A; Ai is an inteerval valued intuitionistic fuzzy

1left -hyperideal of H if and only if A

and A are interval valued intuitionistic e e e fuzzy left -hyperideals of H. For interval valued intuitionistic right -hyperideal similar result holds as well. For any t 2 D[0; 1] and an interval valued intuitionistic fuzzy set A = h A; Ai of H, the sets e e e U(A; t) = fx 2 Hj A(x) tg and L(A; s) = fx 2 Hj A(x) sg. are called erespeectively, an uepper ande lower leveel ceut of A = h eA; Ai. e Theorem 225 Let H be a -semihypergroup and A = h A ; Ai ean ienterval valued intu- itionistic fuzzy set in H. Then, A = h A ; Ai is an intervael valued intuitionistic fuzzy e e 17e3

1left (resp. right) -hyperideal of H

14if and only if for all s ;t 2 D [0;1], the sets U(A ;t) and L(A;

s) are either empty or left (resp. right) -hypereideeals of H. e e Proeof.eLet us assume that all non-empty level sets U(A;t) and L(A;s) are left (resp. right) -hyperideals of H. Let x;y 2 H; 2 . If t1 = A(ye) and s1 e= eA(y), then e y 2 U(A;t1) and y 2 L(A;s1). So x y U(A;t1) and x y L(e for all z 2 xe y, we have eA(z)e t1 and A(z) s1,eand so e Ae;s1). Therefore, e e

10e e e z2inxfyf A(z)g A(y) and zs2uxpyf A(

z)g A(y). e e Hencee,A = h Ae; Ai is an interveal valued ientuitionistic fuzzy left -hyperideal of H. Conversely,letA=eh A ; Aibeanintervalvaluedintuitionisticfuzzyleft -hyperideal ofH. Letx2H; e 2 andye2U(A;t). Wehavez2inxfyf A(z)g A(y) t. Therefore, forallz2x y,wehavez2U(A;t),aendsox y U(A;t). e Now,lety2L(A;s). Wehaveezs2uxpyf A(z)g A(y) s. Therefore,forallz2x y,

10e e e e wehavez2L(A; s),andsox y L(A;

s). Thiscompletestheproof.

11e e e e e e e e Corollary 226e Leet H

be a -semihypeergreoup and I a left (resp. right) -hyperideal of H . If interval valued fuzzy sets A and A are de…ned on H by a0 if xe2 I e b0 if x 2 I A(x) = e < ea1 if x 2 SnI 8 and A(x) = 8 < b1 if x 2 SnI e where 0 a1 < a0;0 b0e< b1 and ai + bi 1 for i = 0;1. Then A = h A; Ai is an : : interval valued inetuietionistic

1**fuzzy left(resp. right) -hyperideal of H**

and U(eA;a0) =

7**e e e e e e e e** I = L(**A;** b0). **e e**

Corollearye227 Let H be a -semihypergroup and be the characteristic function of a I

1**left (resp. right,** two-sided) **-hyperideal** I **of H. Then,**

I = (I; cI) is an interval valued intuitionistic

1**fuzzy left (resp. right,** two-sided) **-hyperideal of H.**

e e Theorem 228 Let H be a -semihypergroup and A = h A; Ai is an interval valued intuitionistic

1**fuzzy left (resp. right) -hyperideal of H. Then** foer all **x**

2 H, we have e A(x) = supft 2 D[0;1]jx 2 U(A;t)g and e e A(x) = inffs 2 D[0; 1]jx 2 L(A; s))g. e e Theeorem 229 Leet H be a -semiehypeergroup.

1**The following** statements **are equivalent: 1.** A = **h** A; Ai **is**

an interval valued intuitionistic fuzzy

1**left (resp. right) -hyperideal** e e **2.** S **A A**

(resp. A S A), where S = h1; 0i and 1(x) = 1 and 0(x) = 0 for all x 2

11**H. e e e e e e** of **H.** Proof. Let A = **h**

A; Ai be an interval valued intuitionistic fuzzy left -hyperideal of H. Let a 2 H. Let us seuppose that there exist x;y 2 H and e 2 such that a 2 x y. Then, since A = h A; Ai is an interval valued intuitionistic fuzzy left -hyperideal of H , we have e e (1 A)(a) = sup [minf1(x); A(y)g] a2x y e e = sup [minf1e; A(ey)g] = sup A(y) a2x y a2x y and ee e (0 A)(a) = ee = a2inxf y[maxf0(x); A(y)g] a2inxf y[maxfe0; A(ey)g] = ai2nxf y A(y): in case of A = h A; Ai is an interval valued intuitionistic fuzzy left -hyperideal of

10**e e e** H, inf **A(** z) **A(z) A(**

y). So, in particular, A(y) A(a) and z2x

2**y A(y) A(** a) **for all** a 2 **x y.**

Hence sup A(y) A(a) and inf A(y) A(a). e A(ye) and zs2uxpy e e e a2ex y a2ex y e Thus, A(ea) (1 A)(a) and A(a) (0 Ae)(a). If there do not exist x; y 2 H; e e e e 2 such that a 2 x y, then (1 A)(a) =

10**0 e e e e e** Ae **(a)** and (0 **A)(a)**

= 1 A(a). In similar way we can prove the other case also.

9**e e e e e e e e** Conversely, let **x;**

y 2 H; 2 and a 2 x y. Then, inf A(a) (1 A)(a) and a2x y a2x y sup A(a) (0 A)(a). We have for all 2 ,

10**e e e e** (1 **A)(a)**

= sup [minf1(x); A(y)g] e e a2x y minf1(x); A(y

10**)g e e** = minf1; **A(** ey)g **=e A(y) e**

e and ee e (0 A)(a) = inf [maxf0

5**(x); A(y)** g] maxf0 **(x); A(y)**

g a2x y e e = maxf0; A(ey)g

10**=e A(** y): **e e** Consequently, inf **A(a)** a2x y **A(**

y) and sup A(a)

10**e e e A(** y). Hence, A = **h A;**

Ai is a2x y an interval valued intuitionistic fuzzy left -hyperideal of H. In similar way it can be proved in case of intuitionistic fuzzy right -hyperideal of H.

5e e e e e e

From the above theorem it follows the following theorem. Theorem 230

1Let H be a -semihypergroup. The following statements are equivalent: 1. A = h A; Ai is

an interval valued intuitionistic fuzzy -hyperideal of S. 2. S Ae Aeand A S A, S = h1; 0i and 1(x) = 1 and 0(x) = 0 for all x 2 S. Theorem 231 Let H be a -semihypeergreoup. Tehe folloewing steatemenets are equivalent: 1. A = h A; Ai is an interval valued intuitionistic fuzzy bi- -hyperideal of H. 2. A Ae Aeand A S A A, where S = h1; 0i and 1(x) = 1, and 0(x) = 0 for all x 2 H. e e e e e e Proof. Same as Theorem 229 Theorem 232 If A = h A; Ai is an interval valued intuitionistic fuzzy left -hyperideal and B = h B; Bi is an ineterval valued intuitionistic fuzzy right - semihypergroup H; then A B is an interval valued intuitionistic fuzzy -hyperideal of e -hyperideal of a e H: e Proof. Consider S (A B) = (S A)

4B A B (by Theorem 229 S (A B) A B. Hence, A B

is an interval valued intuitionistic

1fuzzy left -hyperideal of H. Now,

9(A B) S = A (B S) A B (by Theorem 229 (A B) S A B. Hence, A B

is an interval valued intuitionistic fuzzy

1right -hyperideal of H. Thus, A B is

36an interval valued intuitionistic fuzzy -hyperideal of

H. Proposition 233 Let A = h A; Ai be

36an interval valued intuitionistic fuzzy right - hyperideal of H and

B = h B; eBi be an interval valued intuitionistic

1**fuzzy left hyperideal of H. Then**

A e B e A \ B. e - Proof. Let A = h A; Ai be

36**an interval valued intuitionistic fuzzy** right -hyperideal **of** H **and**

B = h B ; Bi bee

36**an interval valued intuitionistic fuzzy** left -hyperideal **of**

H. e e e Let x 2 H and suppose that there exist u;v 2 H and 2 such that x 2 u v. Then (A B)(x) = sup fminf A(u); B(v)gg x2u v e e sup x2u v minfe inf fe A (x)g ; inf f B (x)gg x2u v x2u v = minf A(x); B(x)g = (

5**e A** ^ B **)(x) e** and **e e e e (A** B **)(x)**

= x2inufvfmaxf A(u); B(v)gg e e x2u v inf maxfesup feA(x)g; sup f B(x)gg x2u v x2u v = maxf A(x); B(x)g = (e A _ B)(x): e Let us suppose there do not exist u;v 2 H such that x 2 u v. Then, (A B)(x) = 0 e e e e (A ^

5**B)(x)** and (**A B)(x)** = 1 (**A _ B)(x).**

Hence the proof is completed. e e Peropoesition 234 eLet He be a -semiheypergeroup and A = h A; Ai;B = h B; Bi be interval valued

2**intuitionistic fuzzy** -hyperideals **of S. Then, A B A\B A;B.**

e e e e Proposition 235 Let H be a regular -semihypergroup and A = h A; Ai and B = h B; Bi two interval valued intuitionistic fuzzy sets in H. Then A B Ae\ B. e e e Proof. Let c 2 H. Since, H is regular, then there exists an element x 2 S and 1; 2 2 such that c 2 c 1x 2c c c for some 2 .

4**Then, (A B)(c)** = sup fminf **A(** c); **B(c)** gg c2c **c**

e e minf A

10**(c);e** B **(c)** eg = (**A** ^ B **)(c) e** e e e and (A B)(c) = c2incf cfmaxf

27**A (c);** B **(c)** gg **e e** maxf **A(c)e;**

B(c)g = (

4**e A _ B)(c):** Hence, **A B A \ B.**

e e e e Theorem 236

1**Let H be a** -semihypergroup. **The following** statements **are equivalent: 1. H is** regular **2.** A **B = A \ B,** where **A**

= h A; Ai is an interval valued intuitionistic fuzzy right -hyperideal of H and B = h B;e Bi is an interval valued intuitionistic fuzzy left -hyperideal of H. e e e Proof. Let H be a regular -semihypergroup. Then, by Proposition 235,

17**A B A \ B.** By Proposition 233, **A B A \ B.** Hence, **A B**

= A \ B. Conversely, let H be a -semihypergroup and A B = A \ B where A = h A; Ai is an interval valued intuitionistic fuzzy right -hyperideal of H and B = h B; Bi isean interval valued intuitionistic

1**fuzzy left -hyperideal of H.** Let **L** and **R**

be respectively a e left -hyperideal and a right -hyperideal of H and x 2 R \ L. Then x 2 R and x 2 L. Hence (L(x); cL(x)) = (R(x); cR(x)) = h1; 0i (where L(x) and R(x) are respectively the interval valued characteristic functions of L and R). Thus

9**e e e e e e** (R \ L **)(x)** = minf R **(x);** L **(x)**

g = 1 and (cR [cL)(x) = maxf cR(x); cL(x)g = 0. e e By Corollary 227, (L; cL) and (R; cR) are respectively, an interval valued intuition- istic fuzzy left -hyperideal and an interval valued intuitionistic fuzzy right -hyperideal

11**e e e e e e e e** of **H.** Hence, by hypothesis, **e e e e** R **L** = R \ **L**

and cR cL = cR [cL. Hence (R L)(x) = 1 since sup [minf R(y); L(z)g : y; z 2 H; 2] = 1, and

8e e e e e e e e e xe2y z

(cR cL)(x) = 0

5e e e e e e

since xi2nyf z[maxf cR(y); cL(z)g : y; z 2 H; 2] = 0. This implies that there exist some r; s 2 H and 1 2 such that x 2 r 1s and (R(r); cR(r)) = h1; 0i = (L(s); cL(s)). Thus, r 2 R and s 2 L. Hence, x 2 R L. e e e Thus

18**R \ L R L.** Also **R L R \ L.** Hence **R L = R \ L.**

Consequently, the e -semihypergroup H is regular. e e e e e Lemma 237

1**Let H be a** -semihypergroup. **Then the following conditions are equivalent: (1) H is** semisimple; **(2)**

I1 \ I2 = I1 I2; for every -hyperideals I1 and I2 of H; (3) I1 = I1 \ I1; for all -hyperideal I1 of S: Proof. (1))(2). Let a 2 I1 \ I2. Then, a 2 I1 and a 2 I2: Since, H is semisimple so for each a 2 H there exist

5**x; y; z 2** H **and** ; ; ; 2 such that **a**

2

5**x a y a** z = **(x a y) (a**

z) I1 I2: Thus, I1 \ I2 I1 I2: Conversely, let a 2 I1 I2 A (because I1 is a -hyperideal of H) and a 2 I1 I2 I1 (because I2 is a -hyperideal of H). So, we have a 2

40**I1 \ I2:** Hence, **I1 I2 I1 \ I2:**

Thus, I1 \ I2 = I1 I2: (2))(3). Let I1 = I2 = I1 I2. Then, we get I1 \ I1 = I1 I1 implies that I1 = I1 I1: (3))(1). Let I1 be a -hyperideal of H such that I1 = I1 I1: Let a 2 I1 = I1 I1 implies a 2 I1 I1. Then, there exist x; y; z 2 H, and ; ; ; 2 such that a 2 x a y I1 and a 2 a z I1; because I1 is a -hyperideal of S: Now, a 2

8**a a (x a** y) **(a z)** which means **a** 2 **x a** y **a z:** Hence, **H**

is semisim- ple. Proposition 238 Let H be a -semihypergroup. Then, H is semisimple if and only if for each interval valued intuitionistic fuzzy -hyperideal A = h A; Ai and B = h B; Bi of H; we have A B = A \ B: e e e e Proof. Let A = h A; Ai and B = h B; Bi be interval valued intuitionistic fuzzy -hyperideals of a -semihypergroup H: Suppose H is semisimple. Then there exist e

5**x; y; z 2** H, **and** ; ; ; e such that **a**

2

5**x a y** a **z = (x a y)** (a **z):** So **for** each e e **2**

s 2 x a y and t 2 a z; (s; t) 2 Xa i.e. a 2 s t (x a y) (a z) and thus Xa6 = ;: Hence (A B)(a) = minf A(s); B (t)g e e minf A(s); eB(t)g:e (s;t_)2Xa Since, A = h A; Ai and B = h B; Bi are interval valued intuitionistic fuzzy e e - hyperideals; we have, A(s) e e A(s1) e e A(a); for each s 2 s1 y and s1 2 x a and B(t) B(a); for every t 2 a z: Hence,

10**e e e e e** minf **A(**

s); B(t)g minf A(a); B(a)g: Thus,

10**e e e e (A** B **)(a) (A** ^ B **)(a):**

Now, we have

10**e e e e (A** B)(a) = **e** e Since, **A** = h **A;**

Ai and B = h B; maxf A(s); B (t)g maxf A(s); eB(t)g:e (s;^t)2Xa B i are interval valued intuitionistic fuzzy e e - hyperideals; we have, A(s) A(s1) e B(t) B(a); for every t 2 a z: Hence, e e e A(a); for each s 2 s1 y and s1 2 x a and

10**e e e e e** maxf **A(**

s); B(t)g maxf A(a); B(a)g: e e e e Thus, (

9**A B)(a) (A _ B)(a):** Hence, **A B A \ B:**

e e e e On the other hand, by Proposition 233,

4**we have A B A \ B: Thus, A B = A \ B:** Conversely,assumeA B **=A\B:**

Letforany -hyperidealsI1;I2ofH;x2I1\I2. Then x 2 I1 and x 2 I2. Hence (I1(x); cI1(x)) = (I2(x); cI2(x)) = h1; 0i (where I1(x) and I2(x) are respectively the interval valued characteristic functioenseof I1 and I2). Thus,

9**e e e e e** (I1\ I2 **)(x)** = minf I1 **(x);** I2 **(x)**

g = 1 and (cI1[cI2)(x) = maxf cI1(x); cI2(x)g = 0. e By Corollary 227, (eI1; cI1) and (I2; cI2) are interval valued intuitionistic fuzzy e e e e = I1 \ I2 and cI1 cI2 = cI1 [cI2. e e e - hyperideals of H. Hence, by hypothesis, eI1 e e e I 2 Hence, (I1 I2)(x) =

111 **e e e e e e e e**

since xse2uypz[minfeI1(y); I2(z)g : y;z 2 H; e 2] = 1, and (cI1 cI2)(x) = 0 e e e since xei2nyfz[maxfecI1(y); cI2(z)g : y;z 2 H; e 2] = 0. This implies that there exist some r;s 2 H and e1 2 such that x 2 r 1s and (I1(r); cI2(r)) = h1;0i = (I1(s); cI2(s)). Thus, r 2 I1 and s 2 I2. Hence, x 2 I1 I2. e e Thus I1 \ I2 I1 Ie2.eAlso

40**I1 I2 I1 \ I2.** Hence, **I1 \ I2**

= I1 I2. Thus, by Lemma 237 , H is semisimple. e e e e De…nition 239 Let H be a -semihypergroup. An interval valued intuitionistic fuzzy sub- -semihypergroup A = h A; Ai of H is called an interval valued intuitionistic fuzzy (1; 2) -hyperideal of H if e e 1. a2x w (y z) inf f A(a)g minf A(x); A(y); A(z)g, 2. sup feA(a)g maxfeA(x);eA(y);eA(z)g, a2x w (y z) e e e e

1**for all** w; **x; y; z** 2 **H**

and ; ; 2 . Theorem 240 Let H be a -semihypergroup. Then, every interval valued intuitionistic fuzzy

1**bi- -hyperideal of H is an**

interval valued intuitionistic fuzzy (1; 2)- -hyperideal

1**of H. Proof. Let** A = **h A;**

Ai be an interval valued intuitionistic

1**fuzzy bi- -hyperideal of H and** let w; **x;**

y; z 2 H and ; ; 2 . Then, for all a 2 x w (y z), we have e e a2x w (y z) inf f A(a)g = a2(x w y) z inf f A(a)g e ai2ncfzf A(a)ge for every c 2 x w y minf eA

10(c); A(z)g minfminf Ae(x); A(y)

g; A(z)g e (because ec2xinwf yef A(c)ge minf A(x); A(y)g) = minf A(x); A(y); eA(z

40)g e e and e e e sup f A(a)g = sup f A(a)g

a2x w (y z) a2(x w y) z e as2ucpzf A(a)ge for every c 2 x w y maxf eA(c); A(z)g maxfmeaxf A(x); A(y)g; A(z)g (because sup f A(c)g maxf A(x); A(y)g) ce2x w y e = maxf A(x); A(y);eA(z)g e e Hence A = h A; Ai is an interval valued intuitionistic fuzzy (1; 2)- -hyperideal of S. e e e Theorem 24e1 Leet H be a regular -semihypergroup. Then, every interval valued in- tuitionistic fuzzy (1; 2)- -hyperideal of H is an interval valued intuitionistic fuzzy bi- - hyperideal of H . Proof. Let us assume that a -semihypergroup H is regular and let A = h A; Ai be an interval valued intuitionistic fuzzy (1; 2)- -hyperideal of H. Let w; x; y 2 H; e 2 . Since, H is regular, we have for every w 2 x y

5(x a x) y = x a (x y) for some e a 2

H; ; 2 . Thus, for every c 2 x y; w 2 x a c, we have w2x a c x a (x y) inf f A(w)g minf A(x); A(x); A(y)g e = minfeA(x); eA(y)ge and e e sup f A(w)g maxf A

4(x); A(x); A(y)g w2x a c x a

(x y) e = maxfeA(x); eA(y)g:e Hence, A = h A; Ai is an interval valued intuitionistic fuzzy bi- -hyperideal of H. e e Theorem 24e2 Leet H be a completely regular -semihypergroup and A = h A; Ai be an interval valued intuitionistic fuzzy bi- -hyperideal of H. Then, for every r 2 aea, we have A(a) = A(r) for all a 2 H; 2 . e Proof. Let a 2 H; 2 . Then, there exist x 2 H and ; ; ; 2 such that a 2 a a x a a. Hence, for every r 2 a a; a 2 r x r. Hence, inf f A(a)g minf A(r); A(r)g a2r x r e = = A(r) minef A(a); A(a)g because r 2 a a; 8 2 e eA(a) e e e and a2r x r sup f A(a)g e = = maxf A

35(r); A(r) g A(r) maxef A(a); A(a) g because r

2 a a; 8 2 e eA(a): e e It follows that A(a) = A(r) and A(a) = A(r), so that A(a) = A(r) for all a 2 H. e Theorem 243eLet H bee an intrae-regular e -semihypergroup and A = h A; Ai be an interval valued intuitionistic fuzzy -hyperideal of H. Then, for every r 2 a a, we have e e A(a) = A(r) for all a 2 H; 2 . Proof. Let a 2 H; 2 . Then, since H is

2intra-regular. Then there exist x; y

2 H; ; ; 2 such that a 2 x a a y. Then, for every r 2 a a and s 2 x r we have a 2 s y. Hence, since A = h A; Ai is an interval valued intuitionistic fuzzy -hyperideal of H, so e ai2nsfyf A(a)g A(s) e and inf f A(s)g A(r) e e

s2x r and r2inafaf A(r)g A(a). e e Hence, A(a) = A(r). Also,

10**e e e e** a2s **y** sup **f A(**

a)g A(s) and s2x r sup f A(s)g A(r) e e and sup

10**f A(r)** g **A(a). e**

e r2a a Hence, A(a) = A(r). Thus, A(a) = A(r) for all a 2 H. e e e e 9.3 Interval Valued Intuitionistic fuzzy M -hypersystems and N -hypersystems in -semihypergroup In this section we will de…ne interval valued intuitionistic fuzzy M -hypersystems and N -hypersystems in -semihypergroup and some basic properties are obtained. De…nition 244 Let H be a -semihypergroup and A = h A; Ai be an interval val- ued intuitionistic fuzzy subset of H. Then, A = h A ; Ai is caelled an interval valued intuitionistic fuzzy M-hypersystem of H if for all x;y;ze2 H and e e ; 2 we have min f A (x) ; A (z)g max feA (x) ; eA (z)g inf A (a) a2x y z a2sxupy z eA (a) De…nition 245 Let H be a -semihypergroup and A = h A; Ai be an interval val- e e e ued

2**intuitionistic fuzzy** subset **of S. Then, A** = h A ; Ai **is** caelled **an**

interval valued intuitionistic fuzzy N-hypersystem of H if for all x;y 2eH and e e ; 2 , we have A (x) inf A (a) a2x y x eA (x) a2sxupy x eA (a) Remark 246 Every interval valued intuitionistic fuzzy M -hypersystem of a -semihypergroup e e H is an interval valued intuitionistic fuzzy N -hypersystem. Theorem 247 If fAigi2 is a family of interval valued intuitionistic fuzzy M (resp. N)- hypersystems of a -semihypergroup H, then Ai is also an interval valued intuitionistic fuzzy M (resp. N)-hypersystem of H . i2 T Proof. Let A = Ai = (Ai; A) = (A; A) and x; y; z 2 H and ; 2 . i2 i2 i2 Then, we have T V e W e e e a2x y z inf A (a) = inf a2x y z (i2 Ai (a)) e ^ min Ai (x) ; Ai (z) e (i2) ^ = min Ai (x) ; Ai (z) e e = min (fi^2A(ex) ; A (zi^2)g e) a2x y z inf A (a) min feA (x) ; eA (z)g and e e e a2x y z sup A (a) = sup a2x y z (i_2 Ai (a)) e min Ai (x) ; Ai (z) e (i2) _ = max Ai (x) ; Ai (z) e e = max f A (ex) ; A (z)g e (i^2 i^2) sup A (a) a2x y z max feA (x) ; eA (z)g : Hence, Ai is an interval valued intuitionistic fuzzy M -hypersystem of -semihypergroup e e i2 H. ThisTcomplets the proof. Proposition 248 Every interval valued intuitionistic fuzzy two sided -hyperideal of - semihypergroup H is an interval valued intuitionistic fuzzy M -hypersystem of H . Proof. Let A = h A; Ai be an interval valued intuitionistic fuzzy two sided - hyperideal of H. Let x;y;z 2 H and ; 2 . Then, e e inf A (a) a2x y z a2xinfy z eA (a) a2xinfy z eA (a) and e a2inxf y A (a) A (x) and ai2nyf z eA (a) eA (z) min fe A (x) ; Ae (z)g e e a2x y z sup A (a) a2sxupy z eA (a) a2sxupy z eA (a) This completes the proof. e a2x y sup A (a) A (x) and as2uxpz eA (a) eA (z) max fe A (x) ; eA (z)g e e Corollary 249 Every interval valued intuitionistic fuzzy one sided -hyperideal of a - semihypergroup H is an interval valued intuitionistic fuzzy N -hypersystem of H . Bibliography [1] S. Abdullah, K. Hila and M.Aslam, On bi- -hyperideals of -semihypergroups, UPB Scienti…c Bulletin, Series A: Applied Mathematics and Physics, 74 (2012), 4, 79-90. [2] S. Abdullah, M. Aslam and T. Anwar, A note on M -hypersystems and N - hypersystems in -semihypergroups, Quasigroups and Related Systems 19 (2011), 169-172. [3] S. Abdullah, B. Davvaz and M. Aslam, (,)-intuitionistic fuzzy ideals in hemirings, Computer and Mathematics with Applications, 62 (2011), 3077-3090. [4] S. Abdullah, M. Aslam and H. 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