Rough Approximations in Γ-Semihypergroups

Ph.D Thesis

By

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بسم الله الرحمن الرحيم

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A THESIS SUBMITTED IN THE PARTIAL FULFILMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
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Supervised By

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CERTIFICATE

A DISSERTATION SUBMITTED IN THE PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF THE
DOCTOR OF PHILOSOPHY

We accept this dissertation as conforming to the required standard

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2016

Dedicated

to

My Father and Mother

Who are the most precious gems of my life.
Who've always given me perpetual love, care, and cheers.
Whose prayers have always been a source of great inspiration for me and whose sustained hope in me led me to where I stand today.

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0.2 Research Profile

1. On rough (m,n) bi-Γ-hyperideals in Γ-semihypergroups, UPB Scientific Bulletin, Series A: Applied Mathematics and Physics, 75(1) (2013) 119-128.

2. Prime (m,n) bi-Γ-hyperideals in Γ-semihypergroups, Applied Mathematics and Information Sciences, 8(5) (2014) 2243-2249.

3. Generalized rough approximations in Γ-semihypergroups, Journal of Intelligent and Fuzzy Systems, 27(5) (2014) 2445-2452.

4. Generalized rough Γ-hyperideals in Γ-semihypergroups, Journal of Applied Mathematics, Article ID 658252 (2014) 6 pages.

5. Structures of bipolar fuzzy Γ-hyperideals in Γ-semihypergroups, Journal of Intelligent and Fuzzy Systems, 27(6) (2014) 3015-3032.

6. On rough Quasi-Γ-hyperideals in Γ-semihypergroups, Afrika Matematika, 26(3-4) (2015) 303-315.

7. New types of bipolar fuzzy sets in Γ-semihypergroups, Songklanakarin Journal of Science and Technology, in press.

8. Rough prime bi-Γ-hyperideals and fuzzy prime bi-Γ-hyperideals of Γ- semihypergroups, submitted.

9. A study on rough bipolar fuzzy sets in Γ-semihypergroups, submitted.

0.3 Introduction

The hyperstructure theory (also known as multialgebras) was introduced in 1934 by Marty [56] at the 8th congress of Scandinavian Mathematicians, where he presented some notes on hypergroups, as a generalization of groups, and gave their applications to non-commutative groups, algebraic functions and rational functions. The hyperstructure theory represents an extension of a classical algebraic structure. In algebraic hyperstructure, the composition of two elements is a set, while in a classical algebraic structure, the composition of two elements is an element. Algebraic hyperstructure theory has many applications in other disciplines. Because of new viewpoints and importance of this theory, many authors applied this theory in nuclear physics and chemistry. Moosavi et al. [58] applied this theory to particle physics and nuclear physics and presented the new concept of the algebraic hyperstructures. Davvaz and Nezad [19] provided examples of a hypergroup associated with chemistry, also see [20]. Around 1940's, several mathematicians explored the theory of hypergroups, especially in United States, France, Russia, Italy, Japan and Iran. Several papers and books have been written on hypergroups. One of these is "Prolegomena of Hypergroup Theory" written by Corsini [16]. Another book "Hyperstructures and Their Representations" written by Vougiouklis [69], was published in 1994. Anvariyeh et al. [9] introduced and studied Γ-semihypergroup which is a generalization of semihypergroup. Hila et al. [34] studied some structural properties of Γ-semihypergroups.

Fuzzy set theory introduced by Zadeh [89] has been well developed in many disciplines. There are several authors who studied the theory of fuzzy sets in hyperstructures, for instance, Corsini et al. [17] introduced fuzzy hyperideals in semihypergroups. Davvaz [22] defined intuitionistic hyperideals of semihypergroups. Ersoy and Davvaz [30] and Hila et al. [35, 36] studied some results on intuitionistic fuzzy Γ-hyperideals of Γ-semihypergroups. He et al. [32] studied fuzzy hyperlattices and fuzzy preordered lattices. Ma et al. [55] proved (fuzzy) isomorphism theorems of gama-hyperrings. Shabir and Mahmood [67] characterized semihypergroups by the properties of $(\in, \in \vee q_k)$-fuzzy hyperideals. Zhan et al. applied the fuzzy set theory to Γ-hyperrings [90], algebraic hypersystems [91] and hyperquasigroups [92]. Also see [93, 94, 95, 96]. Bipolar fuzzy set [97] is an extension of a fuzzy set, which attracted several mathematicians. Researchers like, Akram et al. [4, 5], Jun et al. [40, 41], Lee [49], Lee and Jun [50] and Shabir and Iqbal [65] have studied bipolar fuzzy sets in different algebraic structures.

Rough set theory, introduced in 1982 by Pawlak [59], is a mathematical approach to imperfect knowledge. The methodology of rough set is concerned with the classification and analysis of imprecise, uncertain or incomplete information and knowledge. The subset generated by lower approximations is characterized by objects that will definitely form part of an interest subset, whereas the upper approximation is characterized by objects that will possibly form part of an interest subset. Every subset defined through upper and lower approximation is known as Rough set. After Pawlak's work, Yao [77, 78] and Zhu [101, 102] provided some new views on rough set theory. Ali et al. [7] studied some properties of generalized rough sets. The applications of rough set theory used today is much wider than in the past, principally in the areas of cognitive sciences, medicine, knowledge acquisition, analysis of database attributes, automata theory, machine learning, pattern recognition and process control. There are several authors who considered the theory of rough sets in algebraic structures, for instance, Biswas and Nanda [14] developed some results on rough subgroups. Kuroki [46] introduced the notion of rough ideal in a semigroup. Kuroki and Wang [44] gave some properties of the lower and upper approximations with respect to the normal subgroups. Kuroki and Mordeson [47] studied the structure of rough sets and rough groups. Jun [38] applied the rough set theory to BCK-algebras. Shabir and Irshad [68] defined roughness in ordered semigroups. Davvaz applied rough set theory to rings [23], hyperrings [24], Hv-groups [25], and Hv-modules [26]. Xiao et al. [70] presented some results on rough prime ideals and rough fuzzy prime ideals in semigroups. Chinram [15] and Jun [39] defined roughness in gamma-semigroups.

Davvaz [27] introduced a new view of generalized rough sets or T-rough sets in which he used set-valued homomorphisms instead of congruence relations. Ali et al. [6] studied hemirings in terms of generalized rough sets. Xiao [71] defined T-roughness in semigroups. Yamak et al. [72] applied generalized rough sets to the theory of rings.

0.4 Chapter-wise Study

This thesis contains six chapters. Throughout this thesis, H will denote a Γ- semihypergroup, unless otherwise mentioned.

In Chapter 1, some of the basic definitions and examples on the theory of Γ- semihypergroups, fuzzy sets, bipolar fuzzy sets and rough sets are given. These definitions will be needed for the subsequent chapters.

In Chapter 2, some results on prime bi-Γ-hyperideals, (m, n) bi-Γ-hyperideals and prime (m, n) bi-Γ-hyperideals have been provided which are the extended notions of bi-Γ-hyperideals of a Γ-semihypergroup.

In Chapter 3, the notion of bipolar fuzzy Γ-hyperideals is introduced. Here study some properties of bipolar fuzzy Γ-hyperideals (bi-Γ-hyperideals and $(1, 2)$ Γ-hyperideals) and prove some results in regular Γ-semihypergroups. Further, the notion of bipolar fuzzy points is introduced. We consider the Γ-semihypergroup $\underline{\mathcal{H}}$ of the bipolar fuzzy points of a Γ-semihypergroup H to discuss the relation between the bipolar fuzzy sub Γ-semihypergroup (left, right, bi-, interior, $(1, 2)$-) Γ-hyperideal) and the subsets of $\underline{\mathcal{H}}$ in a (regular) Γ-semihypergroup. Homomorphic images and preimages are also discussed. In the end of this chapter, we establish some results on bipolar (λ, δ)-fuzzy Γ-hyperideals (bi-Γ-hyperideals and $(1, 2)$ Γ-hyperideals) and provided some characterizations of different classes of Γ-semihypergroups by the properties of their bipolar (λ, δ)-fuzzy Γ-hyperideals (bi-Γ-hyperideals and $(1, 2)$ Γ-hyperideals).

In Chapter 4, we study the rough set theory to the ideal theory of Γ-semihypergroup. Some properties of rough sets in Γ-semihypergroups are introduced. We prove that the lower and upper approximation of a bi-Γ-hyperideal (resp., (m, n) bi-Γ-hyperideal, prime (m, n) bi-Γ-hyperideal and (m, n) quasi Γ-hyperideal) is a bi-Γ-hyperideal (resp., (m, n) bi-Γ-hyperideal, prime (m, n) bi-Γ-hyperideal and (m, n) quasi Γ-hyperideal). At the end, some results on rough bi-Γ-hyperideal ((m, n) bi-Γ-hyperideal, prime (m, n) bi-Γ-hyperideal and (m, n) quasi Γ-hyperideal) in the quotient Γ-semihypergroups are given.

In Chapter 5, the notion of generalized rough Γ-hyperideals is introduced and some related properties are provided. Some properties of rough sets in Γ-semihypergroups are offered and then we establish some new results on rough Γ-hyperideals in terms of set-valued homomorphisms, which is, in fact, the extension of rough Γ-hyperideals introduced in [10]. We introduce the concept of generalized rough M-hypersystems and generalized rough N-hypersystems in Γ-semihypergroups.

In Chapter 6, we study some properties of rough bipolar fuzzy sets in Γ-semihypergroups and then study the properties of rough bipolar fuzzy (bi-)Γ-hyperideals.

Chapter 1

Preliminaries

In this introductory chapter we present a brief summary of basic definitions and preliminary results which will be of great help in our further pursuits.

1.1 Semihypergroups

The hyperstructure is an algebraic structure equipped with at least one multi-valued operation, called hyperoperation. A map $\circ : S \times S \to P^*(S)$ is called hyperoperation or join operation on the set S, where S is a non-empty set and $P^*(S) = P(S)\backslash\{\emptyset\}$ denotes the set of all non-empty subsets of S. A hyperstructure is a pair $(S, \circ)$, where $\circ$ is a hyperoperation on the set S. A hypergroupoid is a set S together with a (binary) hyperoperation. If U and V are two non-empty subsets of S, then we define

$$U \circ V = \bigcup_{u \in U, v \in V} u \circ v, \qquad u \circ V = \{u\} \circ V \ \text{ and } U \circ v = U \circ \{v\}.$$

A hypergroupoid $(S, \circ)$ is called a semihypergroup if for all $x, y, z \in S$,

$$(x \circ y) \circ z = x \circ (y \circ z).$$

Example 1 *Let $S = \{a, b, c, d, e\}$ with the binary hyperoperation defined below:*

$\circ$	a	b	c	d	e
a	$\{a,b\}$	$\{b,e\}$	c	$\{c,d\}$	e
b	$\{b,e\}$	e	c	$\{c,d\}$	e
c	c	c	c	c	c
d	$\{c,d\}$	$\{c,d\}$	c	d	$\{c,d\}$
e	e	e	c	$\{c,d\}$	e

Here $(S, \circ)$ is a semihypergroup because elements of S satisfy the associative law.

A non-empty subset A of a semihypergroup S is called a sub semihypergroup of S if $A \circ A \subseteq A$. A sub semihypergroup A of S is called a bi-hyperideal of S if $A \circ S \circ A \subseteq A$. Let $(S, \circ)$ and $(S', \circ')$ be semihypergroups. A function $f : S \longrightarrow S'$ is called a homomorphism if it satisfies $f(x \circ y) = f(x) \circ' f(y)$ for all $x, y \in S$.

1.2 Γ-Semihypergroups: Definitions and Examples

The theory of Γ-semihypergroup, which is a generalization of semigroup, Γ-semigroup and semihypergroup, was first introduced in the paper [9]. They introduced the notion of Γ-hyperideal, bi-Γ-hyperideal and quasi Γ-hyperideal of a Γ-semihypergroup. The Γ-hyperideal (resp., bi-Γ-hyperideal and quasi Γ-hyperideal) is a generalization of an ideal (resp., bi-ideal and quasi ideal) of a semigroup and a generalization of a hyperideal (resp., bi-hyperideal and quasi hyperideal) of a semihypergroup. Also they defined simple quasi Γ-hyperideals and minimal Γ-hyperideals in a Γ-semihypergroup.

Here we present some definitions and examples related to the basics of the theory of Γ-semihypergroups.

Definition 2 *[9] Let H and Γ be two non-empty sets. Then H is called a Γ- semihypergroup if every $\gamma \in \Gamma$ is a hyperoperation on H, i.e., $x\gamma y \subseteq H$ for every $x, y \in H$, and for every $\alpha, \beta \in \Gamma$ and $x, y, z \in H$ we have $x\alpha(y\beta z) = (x\alpha y)\beta z$.*

Let A and B be two non-empty subsets of H. Then we define

$$A\Gamma B = \bigcup_{\gamma \in \Gamma} A\gamma B = \bigcup \{a\gamma b \mid a \in A,\ b \in B \text{ and } \gamma \in \Gamma\}.$$

Example 3 *[9] Let* $(H, \circ)$ *be a semihypergroup and* Γ *be a non-empty subset of* H*. We define* $x\gamma y = x \circ y$ *for every* $x, y \in H$ *and* $\gamma \in \Gamma$*. Then,* H *is a* Γ*-semihypergroup.*

Example 4 *[9] Let* $H = [0, 1]$ *and* $\Gamma = \mathbb{N}$*. For every* $x, y \in H$ *and* $\gamma \in \Gamma$*, we define* $\gamma : H \times H \longrightarrow \wp^*(H)$ *by* $x\gamma y = \left[0, \frac{xy}{\gamma}\right]$*. Then* γ *is a hyperoperation. For every* $x, y, z \in H$ *and* $\alpha, \beta \in \Gamma$*, we have* $(x\alpha y)\beta z = \left[0, \frac{xyz}{\alpha\beta}\right] = x\alpha(y\beta z)$*. This means that* H *is a* Γ*-semihypergroup.*

Let $(H, \circ)$ be a semihypergroup and let $\Gamma = \{\circ\}$. Then H is Γ-semihypergroup. So every semihypergroup is Γ-semihypergroup.

Example 5 *[33] Let* H *be a non-empty set and let* Γ *be a non-empty subset of* H*. If we define* $x\gamma y = \{x, \gamma, y\}$ *for every* $x, y \in H$ *and* $\gamma \in \Gamma$*, then* H *is a* Γ*-semihypergroup.*

Example 6 *[33] Let* H *be a semigroup and* $N_1, N_2, ..., N_k$ *be non-empty subsets of* H*. Let* $\Gamma = \{\alpha_1, \alpha_2, ..., \alpha_k\}$*. We define*

$$x\alpha_i y = xN_i y,$$

for every $x, y \in H$ *and* $\alpha_i \in \Gamma$*,* $1 \leq i \leq k$*. Then* H *is a* Γ*-semihypergroup.*

Example 7 *[33] Let* $(H, \leq)$ *be a totally ordered set and* Γ *be a non-empty subset of* H*. We define*

$$x\gamma y = \{z \in H | z \geq \max\{x, \gamma, y\}\},$$

for every $x, y \in H$ *and* $\gamma \in \Gamma$*. Then* H *is a* Γ*-semihypergroup.*

Definition 8 *[9] Let* H *be a* Γ*-semihypergroup and* $\gamma \in \Gamma$*. A non-empty subset* A *of* H *is called a sub* Γ*-semihypergroup of* H *if* $x\gamma y \subseteq A$ *for every* $x, y \in A$*.*

Definition 9 *[9] A* Γ*-semihypergroup* H *is called commutative if for all* $x, y \in H$ *and* $\gamma \in \Gamma$*, we have* $x\gamma y = y\gamma x$*.*

Definition 10 *[3] A* Γ*-semihypergroup* H *is called regular* Γ*-semihypergroup if for every* $x \in H$*, there exist* $\alpha, \beta \in \Gamma$*, such that* $x \in x\alpha y\beta x$*, for some* $y \in H$*.*

Definition 11 *[3] A* Γ*-semihypergroup* H *is called an intra-regular* Γ*-semihypergroup if for every* $x \in H$*, there exist* $\alpha, \beta, \gamma \in \Gamma$*, such that* $x \in y\alpha x\beta x\gamma z$*, for some* $y, z \in H$*.*

Definition 12 *[3] A Γ-semihypergroup H is called a completely regular Γ- semihypergroup if for every $a \in H$, there exist $\alpha, \beta, \gamma, \delta \in \Gamma$, such that $a \in a\alpha a\beta x\gamma a\delta a$, for some $x \in H$.*

If H is a Γ-semihypergroup and $\rho \subseteq H \times H$ is an equivalence relation on H, then for all pairs (A, B) of non-empty subsets of H, we set $A\bar{\rho}B$ if and only if for all $a \in A$ there exist $b \in B$ such that $a\rho b$ and for $b \in B$ there exist $a \in A$ such that $a\rho b$.

Definition 13 *[10] Let H be a Γ-semihypergroup. An equivalence relation ρ on H is called regular on H if*

$$(a, b) \in \rho \ \ \textit{implies} \ \ (a\gamma x, b\gamma x) \in \rho \ \ \textit{and} \ \ (x\gamma a, x\gamma b) \in \rho,$$

for all $x \in H$ and $\gamma \in \Gamma$.

If ρ is a regular relation on H, then, for every $x \in H$, $[x]_\rho$ stands for the class of x with respect to ρ.

Lemma 14 *[10] Let H be a Γ-semihypergroup and ρ be a regular relation on H. If $a, b \in H$, then $[a]_\rho\gamma[b]_\rho \subseteq [a\gamma b]_\rho$ for every $\gamma \in \Gamma$.*

A regular relation ρ on H is called complete if $[a]_\rho\gamma[b]_\rho = [a\gamma b]_\rho$ for all $a, b \in H$ and $\gamma \in \Gamma$. In addition, ρ on H is called congruence if, for every $(a, b) \in H$ and $\gamma \in \Gamma$, we have $c \in [a]_\rho\gamma[b]_\rho \Longrightarrow [c]_\rho \subseteq [a]_\rho\gamma[b]_\rho$.

Definition 15 *[57] Let H and $\dot{H}$ be Γ and $\dot{\Gamma}$-semihypergroups, respectively and $f : \Gamma \longrightarrow \dot{\Gamma}$ be a bijection. The map $\Phi : H \longrightarrow \dot{H}$ is called a homomorphism, if for every $x, y \in H$ and $\gamma \in \Gamma$, we have*

$$\Phi(x\gamma y) = \Phi(x) f(\gamma) \Phi(y).$$

If we set $f(\gamma) = \dot{\gamma}$, then $\Phi(x\gamma y) = \Phi(x)\dot{\gamma}\Phi(y)$.

Lemma 16 *[57] Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup, and $\Phi : H \longrightarrow \dot{H}$ be a homomorphism. Then*

1. *If A is a sub Γ-semihypergroup of H, then $\Phi(A)$ is a sub $\dot{\Gamma}$-semihypergroup of $\dot{H}$.*

2. *If $\dot{A}$ is a sub $\dot{\Gamma}$-semihypergroup of $\dot{H}$ and $\Phi^{-1}(A) \neq \emptyset$, then $\Phi^{-1}(A)$ is a sub Γ-semihypergroup of H.*

Let $(\Phi, f) : (H, \Gamma) \longrightarrow (\dot{H}, \dot{\Gamma})$ be a homomorphism from a Γ-semihypergroup H to a $\dot{\Gamma}$-semihypergroup $\dot{H}$. Then for $x, y \in H$, we define the relation $\Phi^{-1} * \Phi$ as

$$\begin{aligned} \Phi^{-1} * \Phi &= \{(x,y) \in H \times H : (\text{there exist } z \in H)\ (x,z) \in \Phi,\ (y,z) \in \Phi\} \\ &= \{(x,y) \in H \times H : \Phi(x) = \Phi(y)\}. \end{aligned}$$

The relation $\Phi^{-1} * \Phi$ is called the kernel of f, and we write $\Phi^{-1} * \Phi = \ker \Phi$.

Proposition 17 *[10]* $\ker \Phi$ *is a regular relation on H.*

Theorem 18 *[9] Let Φ be a homomorphism from a Γ-semihypergroup H into a $\dot{\Gamma}$-semihypergroup $\dot{H}$ and $f : \Gamma \longrightarrow \dot{\Gamma}$ be a bijection. Let $\rho = \Phi^{-1} * \Phi$ i.e., $\Phi(x) = \Phi(y)$ if and only if $x\rho y$. Then, ρ is a congruence on H and there exists a homomorphism $\phi : H/\rho \longrightarrow \dot{H}$ such that $\phi T = \Phi$, where T is the canonical homomorphism from H into H/ρ.*

1.3 Γ-Hyperideals in Γ-Semihypergroups

Here we will provide some definitions and results about the ideal theory of Γ- semihypergroups.

Definition 19 *[9] A non-empty subset A of a Γ-semihypergroup H is a right (left) Γ-hyperideal of H if $A\Gamma H \subseteq A$ ($H\Gamma A \subseteq A$), and is a Γ-hyperideal of H if it is both a right and a left Γ-hyperideal.*

Definition 20 *[11, 33] A Γ-hyperideal A of a Γ-semihypergroup H is a prime if $x\gamma y \subseteq A$ for some $x, y \in H$ and $\gamma \in \Gamma$, implies $x \in A$ or $y \in A$.*

Definition 21 *[11, 33] A Γ-hyperideal A of a Γ-semihypergroup H is a semiprime if $x\gamma x \subseteq A$ for some $x \in H$ and $\gamma \in \Gamma$, implies $x \in A$.*

Example 22 *Let $H = \{a, b, c, d\}$ and $\Gamma = \{\beta, \gamma\}$ be the sets of binary hyperoperations defined below:*

β	a	b	c	d
a	a	b	$\{a, c\}$	d
b	b	b	b	d
c	a	b	$\{a, c\}$	d
d	d	d	d	d

γ	a	b	c	d
a	a	b	$\{a, c\}$	d
b	b	b	b	d
c	$\{a, c\}$	b	c	d
d	d	d	d	d

Here H is a Γ-semihypergroup. Here $\{d\}$ and $\{b, d\}$ are prime Γ-hyperideals of H.

Theorem 23 *[33] Let H be a Γ-semihypergroup and P be a left Γ-hyperideal of H. Then P is prime if and only if for all $x, y \in H$,*

$$x\Gamma H\Gamma y \subseteq P \quad \text{implies } x \in P \quad \text{or} \quad y \in P.$$

Definition 24 *[2] Let P be a left Γ-hyperideal of Γ-semihypergroup H. For any left Γ-hyperideals A and B of H, P is called quasi-prime if $A\Gamma B \subseteq P \Rightarrow A \subseteq P$ or $B \subseteq P$.*

Definition 25 *[2] Let P be a left Γ-hyperideal of Γ-semihypergroup H. For any left Γ-hyperideal A of H, P is called quasi-semiprime if $A\Gamma A \subseteq P \Rightarrow A \subseteq P$.*

Definition 26 *[9] A sub Γ-semihypergroup B of a Γ-semihypergroup H is called a bi-Γ-hyperideal of H if $B\Gamma H\Gamma B \subseteq B$.*

Example 27 *[9] Let $H = [0, 1]$ and $\Gamma = \mathbb{N}$. Then with hyperoperation $x\gamma y = \left[0, \frac{xy}{\gamma}\right]$ is Γ-semihypergroup. Let $t \in [0, 1]$ and set $T = [0, t]$. Then T is left (right, bi)-Γ-hyperideal of H.*

Proposition 28 *[9] Let H be a Γ-semihypergroup and A_i be a bi-Γ-hyperideal of H, for all $i \in I$. If $\bigcap_{i \in I} A_i \neq \emptyset$, then $\bigcap_{i \in I} A_i$ is a bi-Γ-hyperideal of H.*

Let A be a non-empty subset of a Γ-semihypergroup H. Let

$$\Im = \{B | B \text{ is a bi-}\Gamma\text{-hyperideal of } H \text{ containing } A\}.$$

Then $\Im \neq \emptyset$, because $H \in \Im$. Let $(A)_b = \bigcap_{B \in \Im} B$. It is clearly seen that $A \subseteq (A)_b$. Where $(A)_b$ is the smallest bi-Γ-hyperideal of H containing A ([9]).

Theorem 29 *[9] Let A be a non-empty subset of a Γ-semihypergroup of H. Then*

$$(A)_b = A \cup A\Gamma A \cup A\Gamma H\Gamma A.$$

Theorem 30 *[9] Let H be Γ-semihypergroup, B be a bi-Γ-hyperideal of H and A be a non-empty subset of H. Then*

1. *$B\Gamma A$ is a bi-Γ-hyperideal of H.*
2. *$A\Gamma B$ is a bi-Γ-hyperideal of H.*

Corollary 31 *[9] Let H be a Γ-semihypergroup and for a positive integer n, $B_1, B_2, ..., B_n$ be bi-Γ-hyperideals of H. Then $B_1\Gamma B_2\Gamma ...\Gamma B_n$ is a bi-Γ-hyperideal of H.*

Definition 32 *[9] A non-empty subset I of a Γ-semihypergroup H is called an interior Γ-hyperideal of H if $H\Gamma I\Gamma H \subseteq I$.*

Definition 33 *[9] A sub Γ-semihypergroup B of a Γ-semihypergroup H is called a $(1,2)$-Γ-hyperideal of H if $B\Gamma H\Gamma B^2 \subseteq B$.*

Definition 34 *[9] A sub Γ-semihypergroup Q of H is called a quasi Γ-hyperideal of H if $Q\Gamma H \cap H\Gamma Q \subseteq Q$.*

A quasi Γ-hyperideal is a generalization of right and left Γ-hyperideals.

Proposition 35 *[9] Let H be a Γ-semihypergroup and Q_i be a quasi Γ-hyperideal of H, for all $i \in I$. If $\bigcap_{i\in I} Q_i \neq \emptyset$, then $\bigcap_{i\in I} Q_i$ is a quasi Γ-hyperideal of H.*

Theorem 36 *[9] Let H be a Γ-semihypergroup and R, L be a right and a left Γ-hyperideal of H, respectively. Then $R \cap L$ is a quasi Γ-hyperideal of H.*

Theorem 37 *[9] Every quasi Γ-hyperideal of H is the intersection of a right Γ-hyperideal and a left Γ-hyperideal of H.*

Definition 38 *[37] Let H be a Γ-semihypergroup and L a sub Γ-semihypergroup of H. Then L is called an m-left Γ-hyperideal of H if $H^m\Gamma L \subseteq L$, where m is any positive integer.*

Definition 39 *[37] Let H be a Γ-semihypergroup and R a sub Γ-semihypergroup of H. Then R is called an n-right Γ-hyperideal of H if $R\Gamma H^n \subseteq R$, where n is any positive integer.*

Theorem 40 *[37] Let H be a Γ-semihypergroup. The following statements hold true:*

1. *Let L_i be an m-left Γ-hyperideal of H for all $i \in I$. If $\bigcap_{i\in I} L_i \neq \emptyset$, then $\bigcap_{i\in I} L_i$ is an m-left Γ-hyperideal of H.*
2. *Let R_i be an n-right Γ-hyperideal of H for all $i \in I$. If $\bigcap_{i\in I} R_i \neq \emptyset$, then $\bigcap_{i\in I} R_i$ is an n-right Γ-hyperideal of H.*

Definition 41 *[37] Let H be a Γ-semihypergroup and Q be a non-empty subset of H. Then Q is called an (m,n) quasi Γ-hyperideal of H if $H^m\Gamma Q \cap Q\Gamma H^n \subseteq Q$, where m and n are positive integers.*

Proposition 42 *[37] Let H be a Γ-semihypergroup. Let Q_i and A_i be an (m,n) quasi Γ-hyperideal and sub Γ-semihypergroup of H, respectively, for all $i \in I$. Then $A_i \cap Q_i$ is either empty or an (m,n) quasi Γ-hyperideal of A_i.*

Theorem 43 *[37] Let H be a Γ-semihypergroup and Q_i be an (m,n) quasi Γ-hyperideal for all $i \in I$. If $\bigcap_{i\in I} Q_i \neq \emptyset$, then $\bigcap_{i\in I} Q_i$ is an (m,n) quasi Γ-hyperideal of H.*

Theorem 44 *[37] Let H be a Γ-semihypergroup and L, R be m-left Γ-hyperideal and n-right Γ-hyperideal of H, respectively. Then $L \cap R$ is an (m,n) quasi Γ-hyperideal of H.*

Definition 45 *[2] A subset M of Γ-semihypergroup H is called M-hypersystem if for all $a, b \in M$, there exist $x \in H$ and $\alpha, \beta \in \Gamma$ such that $a\alpha x\beta b \subseteq M$.*

Definition 46 *[2] A subset N of Γ-semihypergroup H is called N-hypersystem if for all $a \in N$, there exist $x \in H$ and $\alpha, \beta \in \Gamma$ such that $a\alpha x\beta a \subseteq N$.*

Example 47 *[2] Let $H = (0,1)$, $\Gamma = \{\gamma_n \,|n \in \mathbb{N}\}$ and for every $n \in \mathbb{N}$ we define hyperoperation γ_n on H as follows*

$$x\gamma_n y = \left\{\frac{xy}{2^k} \,|0 \leq k \leq n\right\}, \forall x, y \in H.$$

Let $H_i = \left(0, \frac{1}{2^i}\right)$ be subset of H, where $i \in \mathbb{N}$. It is easy to see that H_i is an M-hypersystem of H. Let $T_i = \left(0, \frac{1}{4^i}\right)$ be subset of H, where $i \in \mathbb{N}$. It is easy to see that T_i is an N-hypersystem of H.

Example 48 *[2] Let $H = [0, 1]$ and $\Gamma = N$. Then H with hyperoperation $x\gamma y = \left[0, \frac{xy}{\gamma}\right]$ is a Γ-semihypergroup. Let $t \in [0, 1]$ and set $T = [0, t]$. Then clearly T is an M-hypersystem and N-hypersystem of H.*

Proposition 49 *[2] A left Γ-hyperideal P of Γ-semihypergroup H is quasi-prime if and only if$H \setminus P$ is an M-hypersystem.*

Proposition 50 *[2] A left Γ-hyperideal P of Γ-semihypergroup H is quasi-semiprime if and only if$H \setminus P$ is an N-hypersystem.*

Theorem 51 *[2] Let H be Γ-semihypergroup and P a proper left Γ-hyperideal of H. Then the following are equivalent:*

1. *P is quasi-prime.*
2. *For every $a, b \in H$ such that $a\Gamma H\Gamma b \subseteq P$, then $a \in P$ or $b \in P$.*
3. *$H \setminus P$ is an M-hypersystem.*

Theorem 52 *[2] Let H be Γ-semihypergroup and P a proper left Γ-hyperideal of H. The following are equivalent:*

1. *P is quasi-semiprime.*
2. *For every $a, b \in H$ such that $a\Gamma H\Gamma b \subseteq P$, then $a \in P$ or $b \in P$.*
3. *$H \setminus P$ is an N-hypersystem.*

1.4 Fuzzy Sets and Fuzzy Γ-Hyperideals

In 1965, Zadeh [89] introduced the concept of fuzzy sets which attracted several mathematicians. The elements of these sets have degree of membership. Fuzzy sets generalize classical sets, since the characteristic functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1.

A fuzzy subset A of X is characterized through a membership function $\mu_A : X \longrightarrow [0,1]$ which associates with every point $x \in X$ its grade or degree of membership $\mu_A(x) \in [0,1]$. If A and B are two fuzzy subsets of X, then

(1) $A = B$ if and only if $\mu_A(x) = \mu_B(x)$, for all $x \in X$,

(2) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$, for all $x \in X$,

(3) $C = A \cup B$ if and only if $\mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}$, for all $x \in X$,

(4) $D = A \cap B$ if and only if $\mu_D(x) = \min\{\mu_A(x), \mu_B(x)\}$, for all $x \in X$.

The complement of A, denoted by A^c, is defined by $\mu_{A^c}(x) = 1 - \mu_A(x)$, for all $x \in X$.

Definition 53 *[21] Let X be a non-empty set. For any $A \subseteq X$ and $r \in (0,1]$, the fuzzy subset r_A of X is defined by*

$$r_A(x) = \begin{cases} r & \text{if } x \in A \\ 0 & \text{if } x \notin A, \end{cases}$$

for all $x \in X$. In particular, when $r = 1$, r_A is said to be the characteristic function of A, denoted by χ_A; when $A = \{x\}$, r_A is said to be a fuzzy point with support x and value r and is denoted by x_r.

Definition 54 *[21] Let μ be a fuzzy subset of a Γ-semihypergroup H. Then μ is called*

1. *a fuzzy left Γ-hyperideal of H if*

$$\mu(y) \leq \inf_{z \in x\gamma y}\{\mu(z)\}, \text{ for all } x, y \in H, \ \gamma \in \Gamma;$$

2. *a fuzzy right Γ-hyperideal of H if*

$$\mu(x) \leq \inf_{z \in x\gamma y}\{\mu(z)\}, \text{ for all } x, y \in H, \ \gamma \in \Gamma;$$

3. *a fuzzy Γ-hyperideal or fuzzy two-sided Γ-hyperideal if it is both a fuzzy left Γ-hyperideal and fuzzy right Γ-hyperideal.*

Example 55 *[21] Let $H = \mathbb{N}$ with natural order and Γ be a non-empty subset of H. We define*

$$x\gamma y = \{z \in H | z \geq \max\{x, \gamma, y\}\}$$

for all $x, y \in H$ *and* $\gamma \in \Gamma$. *Then* H *is a* Γ*-semihypergroup. Now, we define the fuzzy subset* μ_n *of* H *as follows:*

$$\mu_n(x) = \begin{cases} 0 & \text{if } x < n \\ 1 - \frac{1}{n+i} & \text{if } x = n + i, \quad i = 1, 2, ... \end{cases}$$

Then for every n, μ_n *is a fuzzy* Γ*-hyperideal of* H.

Example 56 *[2, 11] Let* $H = (0, 1)$, $\Gamma = \{\gamma_n \, |n \in \mathbb{N}\}$ *and for every* $n \in \mathbb{N}$ *we define hyperoperation* γ_n *on* H *as follows*

$$x\gamma_n y = \left\{ \frac{xy}{2^k} \, |0 \leq k \leq n \right\}, \forall x, y \in H.$$

Then, $x\gamma_n y \subset H$ *and for every* $m, n \in \mathbb{N}$ *and* $x, y, z \in H$

$$(x\gamma_n y)\gamma_m z = \left\{ \frac{xyz}{2^k} \, |0 \leq k \leq n + m \right\} = x\gamma_n \left(y\gamma_m z \right).$$

So H *is a* Γ*-semihypergroup. Now we define a fuzzy subset on* H *as;*

$$\mu\left(x\right) = \begin{cases} t_1 & \text{if } 0 < x < \frac{1}{2^k} \\ t_2 & \text{if } \frac{1}{2^k} \leq x < 1 \end{cases} \quad \text{where } k \in N \text{ and } t_1 > t_2$$

Then by routine calculation μ *is fuzzy* Γ*-hyperideal of* H.

Lemma 57 *[21] If* $\{\mu_i\}_{i \in \lambda}$ *is the collection of fuzzy* Γ*-hyperideals of* H, *then* $\bigcap_{i \in \lambda} \mu_i$ *and* $\bigcup_{i \in \lambda} \mu_i$ *are fuzzy* Γ*-hyperideal of* H, *too.*

Proposition 58 *[11] A fuzzy subset* μ *of a* Γ*-semihypergroup* H *is a fuzzy left (resp., right)* Γ*-hyperideal of* H *if and only if* $\mathcal{H}\Gamma\mu \subseteq \mu$ *(resp.,* $\mu\Gamma\mathcal{H} \subseteq \mu$*). Where* H *is a fuzzy subset of* H, *which mapped on* 1.

Theorem 59 *[11] Let* A *be a left (right)* Γ*-hyperideal of a* Γ*-semihypergroup* H. *Then for any* $t \in (0, 1]$, *there exists a fuzzy left (resp., right)* Γ*-hyperideal of* H *such that* $\mu_t = A$.

Theorem 60 *[11] Let* m *be a fixed element in* H. *If* μ *is a fuzzy left (resp., right)* Γ*-hyperideal of* H, *then*

$$\mu^m = \{x \in H : \mu\left(x\right) \geq \mu\left(m\right)\} \neq \emptyset$$

is a left (resp., right) Γ*-hyperideal of* H.

Proposition 61 *[11] If μ is a fuzzy left (resp., right) Γ-hyperideal of a Γ- semihypergroup H, then μ^c is anti fuzzy Γ-hyperideal of H.*

For any $t \in [0,1]$ and fuzzy set μ of H, the sets

$$U(\mu, t) = \{x \in H : \mu(x) \geq t\} \quad \text{and } U^s(\mu, t) = \{x \in H : \mu(x) > t\}$$

are called an upper t-level cut and upper strong t-level cut of μ.

Theorem 62 *[11] A fuzzy subset μ of a Γ-semihypergroup H is a fuzzy left (resp., right) Γ-hyperideal of H if and only if $U(\mu, t)$ is a left (resp., right) Γ-hyperideal of H.*

Proposition 63 *[11] Let I be a subset of a Γ-semihypergroup H. Then I is a left (resp., right) Γ-hyperideal of H if and only if the characteristic function $\mathcal{X}_I$ is a fuzzy left (resp., right) Γ-hyperideal of H.*

Definition 64 *[11] Let H be a Γ-semihypergroup and μ be a fuzzy left Γ-hyperideal of H. Then μ is called a fuzzy quasi prime if for all fuzzy left Γ-hyperideals λ, δ of H*

$$\lambda\Gamma\delta \subseteq \mu \Rightarrow \text{either } \lambda \subseteq \mu \text{ or } \delta \subseteq \mu$$

Definition 65 *[11] Let H be a Γ-semihypergroup and μ be a fuzzy left Γ-hyperideal of H. Then μ is called a fuzzy quasi semi-prime if for all fuzzy left Γ-hyperideals λ of H*

$$\lambda\Gamma\lambda \subseteq \mu \Rightarrow \lambda \subseteq \mu$$

Proposition 66 *[11] A fuzzy subset μ of Γ-semihypergroup H is a fuzzy quasi prime (semi-prime) Γ-hyperideal of H if and only if $U(\mu, t)$ is quasi prime (semi-prime) Γ-hyperideal of H.*

1.5 Bipolar Fuzzy Sets

Bipolar fuzzy set [97] is an extension of fuzzy set whose membership degree range is enlarged from the interval $[0,1]$ to $[-1,1]$. Bipolar fuzzy sets have membership degrees that represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter-property. In a bipolar fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $(0,1]$ indicate that elements somewhat satisfy the property, and the membership degrees on $[-1,0)$ indicate that elements somewhat satisfy the implicit counter property.

Definition 67 *[48] A bipolar fuzzy subset $\mathcal{B}$ of a non-empty set X is an object having the form*

$$\mathcal{B} = \left\{(x, \mu^{+}_{\mathcal{B}}(x), \mu^{-}_{\mathcal{B}}(x))|x \in X\right\},$$

where $\mu^{+}_{\mathcal{B}} : X \longrightarrow [0,1]$ and $\mu^{-}_{\mathcal{B}} : X \longrightarrow [-1,0]$.

The positive membership degree $\mu^{+}_{\mathcal{B}}$ denote the satisfaction degree of an element x to the property corresponding to a bipolar fuzzy subset $\mathcal{B}$, and the negative membership degree $\mu^{-}_{\mathcal{B}}$ denotes the satisfaction degree of x to some implicit counter property of bipolar fuzzy subset $\mathcal{B}$. Bipolar fuzzy sets and intuitionistic fuzzy sets look similar to each other. However, they are different from each other [48]. For the sake of simplicity, we shall use the symbol $\mathcal{B} = \left\langle \mu^{+}_{\mathcal{B}}, \mu^{-}_{\mathcal{B}} \right\rangle$ for a bipolar fuzzy subset $\mathcal{B} = \left\{(x, \mu^{+}_{\mathcal{B}}(x), \mu^{-}_{\mathcal{B}}(x))|x \in X\right\}$.

Definition 68 *[48] Let $\mathcal{A}$ and $\mathcal{B}$ be any two bipolar fuzzy subsets of a non-empty set X, then $\mathcal{A} \subseteq \mathcal{B}$ means that*

$$\mu^{+}_{\mathcal{A}}(x) \leq \mu^{+}_{\mathcal{B}}(x) \quad \text{and} \quad \mu^{-}_{\mathcal{A}}(x) \geq \mu^{-}_{\mathcal{B}}(x)$$

for all x in X.

Definition 69 *[48] Let $\mathcal{A} = \left\langle \mu^{+}_{\mathcal{A}}, \mu^{-}_{\mathcal{A}} \right\rangle$ and $\mathcal{B} = \left\langle \mu^{+}_{\mathcal{B}}, \mu^{-}_{\mathcal{B}} \right\rangle$ be two bipolar fuzzy subsets of a non-empty set X. Then their intersection $\mathcal{A} \cap \mathcal{B}$ is defined by*

$$\mathcal{A} \cap \mathcal{B} = \left\{\left\langle x, \mu^{+}_{\mathcal{A}\cap\mathcal{B}}(x), \mu^{-}_{\mathcal{A}\cap\mathcal{B}}(x)\right\rangle : x \in H\right\},$$

where

$$\begin{aligned} \mu^{+}_{\mathcal{A}\cap\mathcal{B}}(x) &= \left(\mu^{+}_{\mathcal{A}} \wedge \mu^{+}_{\mathcal{B}}\right)(x) = \mu^{+}_{\mathcal{A}}(x) \wedge \mu^{+}_{\mathcal{B}}(x) \\ \text{and} \quad \mu^{-}_{\mathcal{A}\cap\mathcal{B}}(x) &= \left(\mu^{-}_{\mathcal{A}} \vee \mu^{-}_{\mathcal{B}}\right)(x) = \mu^{-}_{\mathcal{A}}(x) \vee \mu^{-}_{\mathcal{B}}(x). \end{aligned}$$

Their union $\mathcal{A} \cup \mathcal{B}$ is defined by

$$\mathcal{A} \cup \mathcal{B} = \left\{\left\langle x, \mu^{+}_{\mathcal{A}\cup\mathcal{B}}(x), \mu^{-}_{\mathcal{A}\cup\mathcal{B}}(x)\right\rangle : x \in H\right\},$$

where

$$\begin{aligned} \mu^+_{\mathcal{A}\cup\mathcal{B}}(x) &= \left(\mu^+_{\mathcal{A}} \vee \mu^+_{\mathcal{B}}\right)(x) = \mu^+_{\mathcal{A}}(x) \vee \mu^+_{\mathcal{B}}(x) \\ \text{and} \quad \mu^-_{\mathcal{A}\cup\mathcal{B}}(x) &= \left(\mu^-_{\mathcal{A}} \wedge \mu^-_{\mathcal{B}}\right)(x) = \mu^-_{\mathcal{A}}(x) \wedge \mu^-_{\mathcal{B}}(x). \end{aligned}$$

for all $x \in X$.

Definition 70 *[48] Let* X *be a non-empty set and let* $\emptyset \neq W \subseteq X$, *then the bipolar characteristic function* $\Omega_W = (\mu^+_{\Omega_w}, \mu^-_{\Omega_w})$ *of* W *is defined as*

$$\mu^+_{\Omega_w}(x) = \begin{cases} 1, & \text{if } x \in W \\ 0, & \text{if } x \notin W \end{cases} \quad \text{and} \quad \mu^-_{\Omega_w}(x) = \begin{cases} -1, & \text{if } x \in W \\ 0, & \text{if } x \notin W. \end{cases}$$

Definition 71 *[48] Let* $\mathcal{B} = \left\langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \right\rangle$ *be a bipolar fuzzy set and* $(s,\, t) \in [-1,\, 0] \times [0,\, 1]$. *Define:*

1. *the sets* $\mathcal{B}^+_t = \{x \in X \mid \mu^+_{\mathcal{B}}(x) \geq t\}$ *and* $\mathcal{B}^-_s = \{x \in X \mid \nu^-_{\mathcal{B}}(x) \leq s\}$, *which are called positive* t*-cut of* $\mathcal{B} = \left\langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \right\rangle$ *and the negative* s*-cut of* $\mathcal{B} = \left\langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \right\rangle$, *respectively,*

2. *the sets* ${}^{>}\mathcal{B}^+_t = \{x \in X \mid \mu^+_{\mathcal{B}}(x) > t\}$ *and* ${}^{<}\mathcal{B}^-_s = \{x \in X \mid \mu^-_{\mathcal{B}}(x) < s\}$, *which are called strong positive* t*-cut of* $\mathcal{B} = \left\langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \right\rangle$ *and the strong negative* s*-cut of* $\mathcal{B} = \left\langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \right\rangle$, *respectively,*

3. *the set* $X^{(t,s)}_{\mathcal{B}} = \{x \in X \mid \mu^+_{\mathcal{B}}(x) \geq t, \quad \mu^-_{\mathcal{B}}(x) \leq s\}$ *is called an* (t, s)*-level subset of* $\mathcal{B}$,

4. *the set* ${}^{S}X^{(t,s)}_{\mathcal{B}} = \{x \in X \mid \mu^+_{\mathcal{B}}(x) > t, \quad \mu^-_{\mathcal{B}}(x) < s\}$ *is called a strong* (t, s)*-level subset of* $\mathcal{B}$,

5. *the set of all* $(s, t) \in Im(\mu^+_{\mathcal{B}}) \times Im(\mu^-_{\mathcal{B}})$ *is called the image of* $\mathcal{B} = (\mu^+_{\mathcal{B}}, \lambda^-_{\mathcal{B}})$.

1.6 Rough Sets: Definitions and Examples

In this section, we will provide some concepts related to rough set theory offered by Zdzisław Pawlak [59, 60].

Definition 72 *Suppose U be the set of objects, called the universe and an indiscernibility relation $R \subseteq U \times U$, for the sake of simplicity we assume that R such that xRy if and only if (x, y) is in R. R is an equivalence relation if it satisfies the following properties:*

1. *Reflexive Property: (x, x) is in R for all x in U.*
2. *Symmetric Property: if (x, y) is in R, then (y, x) is in R.*
3. *Transitive Property: if (x, y) and (y, z) are in R, then (x, z) is in R.*

The equivalence class of $x \in U$, such that $[x]_R = \{y \in U : xRy\}$, consists of objects indiscernible from x. The indiscernibility relation $IND(P)$, is defined as follows

$$IND(P) = \{(x, y) \in U \times U : \text{ for all } a \in P,\ a(x) = a(y)\},$$

here P is partition P of U.

Definition 73 *[59] The lower approximation of a set $X \subseteq U$ with respect to R is the set of all objects, which can be for certain classified as X with respect to R (are certainly X with respect to R).*

From the different representations of an equivalence relation, we obtain three constructive definitions of lower approximation

1. $R_-(X) = \{x \in U : [x]_R \subseteq X\}$, (element based definition)
2. $R_-(X) = \bigcup_{[x]_R \subseteq X} [x]_R$, (granule based definition)
3. $R_-(X) = \bigcup \{A \in U/R : A \subseteq X\}$, (subsystem based definition)

where $[x]_R = \{y : xRy\}$.

Definition 74 *[59] The upper approximation of a set X with respect to R is the set of all objects which can be possibly classified as X with respect to R (are possibly X in view of R).*

From the different representations of an equivalence relation, we obtain three constructive definitions of upper approximation

1. $R^{-}(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}$, (element based definition)
2. $R^{-}(X) = \bigcup\limits_{[x]_R \cap X \neq \emptyset} [x]_R$, (granule based definition)
3. $R^{-}(X) = \bigcap\{A \in U/R : A \cap X \neq \emptyset\}$, (subsystem based definition)

where $[x]_R = \{y : xRy\}$

The lower and upper approximations, $R_{-}, R^{-} : 2^U \longrightarrow 2^U$, can be interpreted as a pair of unary set-theoretic operators. They are dual operators in the sense that $R_{-}(X) = (R^{-}(X^c))^c$ and $R^{-}(X) = (R_{-}(X^c))^c$, where X^c is set complement of X. The pair (U, R) is called approximation space.

Definition 75 *[76] The boundary region is the collection of elementary sets defined by*

$$BND(X) = R^{-}(X) - R_{-}(X).$$

These sets are included in R-upper but not in R-lower approximations.

Based on the lower and upper approximations of a set $X \subseteq U$, the universe U can be divided into three disjoint regions:

1. the positive region $\text{POS}(X) = R_{-}(X)$;
2. the negative region $\text{NEG}(X) = U - R^{-}(X) = (R^{-}(X))^c$;
3. the boundary region $\text{BND}(X) = R^{-}(X) - R_{-}(X)$.

As we can see from the granule based definition, approximations are expressed in terms of granules of knowledge. The lower approximation of a set is union of all granules which are entirely included in the set, the upper approximation is union of all granules which have non-empty intersection with the set, the boundary region of set is the difference between the upper and the lower approximation. This definition is clearly depicted in Figure 1

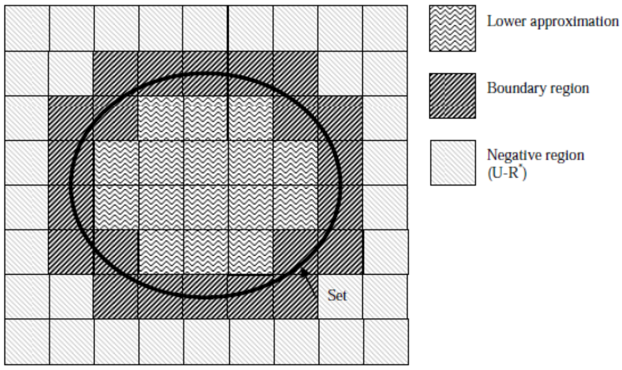


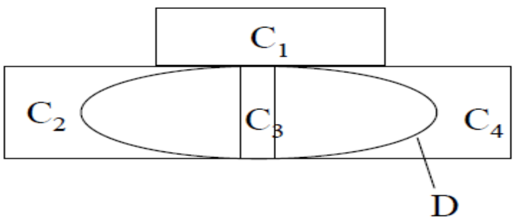
Fig. 1: *Illustration of the boundary region of rough set*

Figure 1 illustrates the approximation of a set X, and the positive, negative and boundary regions. Each small square represents an equivalence class. The upper approximation of a set X is the union of the positive and boundary regions, namely, $R^{-}(X) = \text{POS}(X) \cup \text{BND}(X)$.

Example 76 *Consider a set* $U = \{1, 2, 3, 4, 5, 6\}$ *as a universal set. Define* R *to be an equivalence relation such that, for an equivalence relation* R *on* U :

$$1R1,\quad 2R2,\quad 2R3,\quad 3R2,\quad 3R3,\quad 4R4,\quad 5R6,\quad 6R5,\quad 5R5,\quad 6R6$$

The equivalence relation induces four equivalence classes, which are the subsets $C_1 = \{1\}$, $C_2 = \{2, 3\}$, $C_3 = \{4\}$, $C_4 = \{5, 6\}$, *here we want to characterize the set* $D = \{3, 4, 5\}$ *with respect to* R. *For this we have*



Approximation of D *by* $\{C_1, C_2, C_3, C_4\}$

C_1 *is definitely outside*

$R_{-}(D) = \{4\} = C_3$, C_3 *definitely inside.*

$R^{-}(D) = \{2, 3, 4, 5, 6\} = C_2 \cup C_3 \cup C_4$

$C_2 \cup C_4 = \{2, 3, 5, 6\}$ *is the boundary region of* D.

Approximations have the following properties: [59]

1. $R_{-}(X) \subseteq X \subseteq R^{-}(X)$
2. $R_{-}(\emptyset) = \emptyset = R^{-}(\emptyset)$; $R_{-}(U) = U = R^{-}(U)$
3. $R^{-}(X \cup Y) = R^{-}(X) \cup R^{-}(Y)$
4. $R_{-}(X \cup Y) \supseteq R_{-}(X) \cup R_{-}(Y)$
5. $R^{-}(X \cap Y) \subseteq R^{-}(X) \cap R^{-}(Y)$
6. $R_{-}(X \cap Y) = R_{-}(X) \cap R_{-}(Y)$
7. $X \subseteq Y$ implies $R_{-}(X) \subseteq R_{-}(Y)$, $R^{-}(X) \subseteq R^{-}(Y)$
8. $R_{-}(-X) = -R^{-}(X)$
9. $R^{-}(-X) = -R_{-}(X)$
10. $R_{-}R_{-}(X) = R^{-}R_{-}(X) = R_{-}(X)$
11. $R^{-}R^{-}(X) = R_{-}R^{-}(X) = R^{-}(X)$

It is easily seen that approximations are in fact interior and closure operations in a topology generated by data.

Definition 77 *A subset X of U is called Crisp when its boundary region is empty i.e. $R_{-}(X) = R^{-}(X)$.*

Definition 78 *[59] Let U be a universal set and let R be an equivalence relation on U, then the set $X \subseteq U$ is called a rough with respect to R if $R_{-}(X) \neq R^{-}(X)$.*

Another definition is

Definition 79 *[76] A subset defined through its lower and upper approximations is called a Rough set. That is, when the boundary region is a non-empty set ($R_{-}(X) \neq R^{-}(X)$).*

Example 80 *Let* (U, R) *is an approximation space, where* $U = \{x_1, x_2, x_3, ..., x_8\}$ *and an equivalence relation* R *with the following equivalence classes:*

$E_1 = \{x_1, x_4, x_8\}$

$E_2 = \{x_2, x_5, x_7\}$

$E_3 = \{x_3\}$

$E_4 = \{x_6\}$

Let $X = \{x_3, x_5\}$ *and* $Y = \{x_3, x_6\}$

$R_-(X) = \{x_3\}$ *and* $R^-(X) = \{x_2, x_3, x_5, x_7\}$

$R_-(Y) = \{x_3, x_6\}$ *and* $R^-(Y) = \{x_3, x_6\}$

So $R(X) = (\{x_3\}, \{x_2, x_3, x_5, x_7\})$ *is a rough set and* $R(Y)$ *is a crisp set.*

Example 81 *[76] This example illustrates the main ideas developed so far, consider a universe consisting of three elements* $U = \{1, 2, 3\}$ *and an equivalence relation* R *on* U*:*

$$1R1, \quad 2R2, \quad 1R3, \quad 3R1, \quad 3R3$$

The equivalence relation induces two equivalence classes $[1]_R = [3]_R = \{1, 3\}$, $[2]_R = \{2\}$, *now*

$$P(U) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, U\}$$

is the set of all subsets of U*. The following table summarizes the lower and upper approximations, the positive, negative and boundary regions for all subsets of* U*.*

X	$R_-(X)$	$R^-(X)$	$POS(X)$	$NEG(X)$	$BND(X)$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	U	$\emptyset$
$\{1\}$	$\emptyset$	$\{1,3\}$	$\emptyset$	$\{2\}$	$\{1,3\}$
$\{2\}$	$\{2\}$	$\{2\}$	$\{2\}$	$\{1,3\}$	$\emptyset$
$\{3\}$	$\emptyset$	$\{1,3\}$	$\emptyset$	$\{2\}$	$\{1,3\}$
$\{1,2\}$	$\{2\}$	U	$\{2\}$	$\emptyset$	$\{1,3\}$
$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{1,3\}$	$\{2\}$	$\emptyset$
$\{2,3\}$	$\{2\}$	U	$\{2\}$	$\emptyset$	$\{1,3\}$
U	U	U	U	$\emptyset$	$\emptyset$

Now it is clear from the table that

$$\begin{aligned} R_{-}(\{1\}) &\neq R^{-}(\{1\}) \\ R_{-}(\{3\}) &\neq R^{-}(\{3\}) \\ R_{-}(\{1,2\}) &\neq R^{-}(\{1,2\}) \\ R_{-}(\{2,3\}) &\neq R^{-}(\{2,3\}) \end{aligned}$$

So $\{1\}$, $\{3\}$, $\{1,2\}$, $\{2,3\}$ *are rough sets with respect to* R, *and* $\{2\}$, $\{1,3\}$ *are crisp sets with respect to* R.

Two subsets X and Y of the universe U will be equal if $R_{-}(X) = R_{-}(Y)$ and $R^{-}(X) = R^{-}(Y)$.

1.7 Lower and Upper Approximations in Γ-Semihypergroups

Let A be a non-empty subset of a Γ-semihypergroup H and ρ be a regular relation on H. Then, the sets

$$\underline{Apr}_{\rho}(A) = \left\{x \in H : [x]_{\rho} \subseteq A\right\} \quad \text{and} \quad \overline{Apr}_{\rho}(A) = \left\{x \in H : [x]_{\rho} \cap A \neq \emptyset\right\}$$

are called ρ-lower and ρ-upper approximations of A, respectively. For a non-empty subset A of H, $Apr_{\rho}(A) = (\underline{Apr}_{\rho}(A), \overline{Apr}_{\rho}(A))$ is called a rough set with respect to ρ if $\underline{Apr}_{\rho}(A) \neq \overline{Apr}_{\rho}(A)$.

Theorem 82 *[10] Let* ρ *be a regular relation on a* Γ*-semihypergroup* H *and let* A *and* B *be non-empty subsets of* H. *Then,*

1. $\overline{Apr}_{\rho}(A)\,\Gamma\overline{Apr}_{\rho}(B) \subseteq \overline{Apr}_{\rho}(A\Gamma B)$;
2. *if* ρ *is complete, then* $\underline{Apr}_{\rho}(A)\,\Gamma\underline{Apr}_{\rho}(B) \subseteq \underline{Apr}_{\rho}(A\Gamma B)$.

Definition 83 *[10] Let* ρ *be a regular relation on* H. *Then a non-empty subset* A *of* H *is called a* ρ*-upper rough sub* Γ*-semihypergroup (resp., right* Γ*-hyperideal, left* Γ*-hyperideal) of* H *if* $\overline{Apr}_{\rho}(A)$ *is a sub* Γ*-semihypergroup (resp., right* Γ*-hyperideal, left* Γ*-hyperideal) of* H.

Theorem 84 *[10] Let* ρ *be a regular relation on a* Γ*-semihypergroup* H. *Then,*

1. *Every sub Γ-semihypergroup of H is a ρ-upper rough sub Γ-semihypergroup of H.*

2. *Every right (left) Γ-hyperideal of H is a ρ-upper rough right (left) Γ-hyperideal of H.*

Definition 85 *[10] Let ρ be a regular relation on H. Then a non-empty subset A of H is called a ρ-lower rough sub Γ-semihypergroup (resp., right Γ-hyperideal, left Γ-hyperideal) of H if $\underline{Apr}_{\rho}(A)$ is a sub Γ-semihypergroup (resp., right Γ-hyperideal, left Γ-hyperideal) of H.*

Theorem 86 *[10] Let $\emptyset \neq A \subseteq H$ and let ρ be a complete regular relation on H such that the ρ-lower approximation of A is non-empty. Then,*

1. *If A is a sub Γ-semihypergroup of H, then A is a ρ-lower rough sub Γ-semihypergroup of H.*

2. *If A is a right (left) Γ-hyperideal of H, then A is a ρ-lower rough right (left) Γ-hyperideal of H.*

Definition 87 *[10] A subset A of a Γ-semihypergroup H is called a ρ-upper (ρ-lower) rough bi-Γ-hyperideal of H if $\overline{Apr}_{\rho}(A)$ ($\underline{Apr}_{\rho}(A)$) is a bi-$\Gamma$-hyperideal of H.*

Theorem 88 *[10] Let ρ be a regular relation on H and A be a bi-Γ-hyperideal of H. Then,*

1. *A is a ρ-upper rough bi-Γ-hyperideal of H.*

2. *If ρ is complete such that the ρ-lower approximation of A is non-empty, then A is a ρ-lower rough bi-Γ-hyperideal of H.*

Theorem 89 *[10] Let ρ be a regular relation on H. If A and B are a right Γ-hyperideal and a left Γ-hyperideal of H, respectively, then*

$$\overline{Apr}_{\rho}(A\Gamma B) \subseteq \overline{Apr}_{\rho}(A) \cap \overline{Apr}_{\rho}(B) \quad \text{and} \quad \underline{Apr}_{\rho}(A\Gamma B) \subseteq \underline{Apr}_{\rho}(A) \cap \underline{Apr}_{\rho}(B).$$

Definition 90 *[11] Let ρ be a regular relation on H. A subset M of a Γ-semihypergroup H is called a ρ-upper (ρ-lower) rough M-hypersystem in H if $\overline{Apr}_{\rho}(M)$ ($\underline{Apr}_{\rho}(M)$) is an M-hypersystem in H.*

Theorem 91 *[11] Let ρ be a complete regular relation on a Γ-semihypergroup H. If M is an M-hypersystem in H, then M is a ρ-upper rough M-hypersystem in H.*

Definition 92 *[11] Let ρ be a regular relation on H. A subset N of a Γ-semihypergroup H is called a ρ-upper (ρ-lower) rough N-hypersystem in H if $\overline{Apr}_\rho(N)$ ($\underline{Apr}_\rho(N)$) is an N-hypersystem in H.*

Theorem 93 *[11] Let ρ be a complete regular relation on a Γ-semihypergroup H. If N is an N-hypersystem in H, then N is a ρ-upper rough N-hypersystem in H.*

Theorem 94 *[11] Let ρ be a regular relation on a Γ-semihypergroup H. If A is a sub Γ-semihypergroup of H, then*

1. *$\overline{Apr}_\rho(A)$ is an M-hypersystem in H.*
2. *If ρ is complete then, $\underline{Apr}_\rho(A)$ is an M-hypersystem in H.*

Theorem 95 *[11] Let ρ be a regular relation on a Γ-semihypergroup H. If A is a sub Γ-semihypergroup of H, then*

1. *$\overline{Apr}_\rho(A)$ is an N-hypersystem in H.*
2. *If ρ is complete then, $\underline{Apr}_\rho(A)$ is an N-hypersystem in H.*

Definition 96 *[11] Let ρ be a regular relation on a Γ-semihypergroup H. Then a subset A of H is called a ρ-upper (ρ-lower) rough prime Γ-hyperideal of H, if $\overline{Apr}_\rho(A)$ ($\underline{Apr}_\rho(A)$) is a prime Γ-hyperideal of H.*

Theorem 97 *[11] Let ρ be a complete regular relation on a Γ-semihypergroup H and P be a prime Γ-hyperideal of H. Then $\overline{Apr}_\rho(A)$ is a prime Γ-hyperideal of H.*

Theorem 98 *[11] Let ρ be a complete regular relation on a Γ-semihypergroup H and P be a prime Γ-hyperideal of H. Then $\underline{Apr}_\rho(P)$ is, if it is non-empty, a prime Γ-hyperideal of H.*

Let ρ be a regular relation on a Γ-semihypergroup H. We put $\widehat{\Gamma} = \{\widehat{\gamma} \ : \ \gamma \in \Gamma\}$. For every $[a]_\rho, [b]_\rho \in H/\rho$, we define $[a]_\rho \widehat{\gamma} [b]_\rho = \{[z]_\rho : z \in a\gamma b\}$. A pair $(H/\rho, \widehat{\Gamma})$ is called a Γ-quotient hyperalgebra, where $\widehat{\Gamma} = \{\widehat{\gamma} \ |\gamma \in \Gamma\}$.

Theorem 99 *[10, Theorem 4.1] If H is a Γ-semihypergroup, then H/ρ is a $\widehat{\Gamma}$- semihypergroup.*

Definition 100 *Let ρ be a regular relation on a Γ-semihypergroup H. The ρ-lower approximation and ρ-upper approximation of a non-empty subset A of H can be presented in an equivalent form as shown below:*

$$\underline{\underline{Apr}}_{\rho}(A) = \left\{[x]_{\rho} \in H/\rho : [x]_{\rho} \subseteq A\right\} \quad \text{and} \quad \overline{\overline{Apr}}_{\rho}(A) = \left\{[x]_{\rho} \in H/\rho : [x]_{\rho} \cap A \neq \emptyset\right\},$$

respectively.

Theorem 101 *[10, Theorems 4.3, 4.4] Let ρ be a regular relation on a Γ-semihypergroup H. If A is a sub Γ-semihypergroup of H, then,*

1. *$\overline{\overline{Apr}}_{\rho}(A)$ is a sub $\widehat{\Gamma}$-semihypergroup of H/ρ.*
2. *$\underline{\underline{Apr}}_{\rho}(A)$ is, if it is non-empty, a sub $\widehat{\Gamma}$-semihypergroup of H/ρ.*

Theorem 102 *[11] Let ρ be a regular relation a Γ-semihypergroup H. If A is a left (right) Γ-hyperideal of H, then $\overline{\overline{Apr}}_{\rho}(A)$ is left (right) $\widehat{\Gamma}$-hyperideal of H/ρ.*

Theorem 103 *[11] Let ρ be a regular relation a Γ-semihypergroup H. If A is a left (right) Γ-hyperideal of H, then $\underline{\underline{Apr}}_{\rho}(A)$ is, if it is non-empty, a left (right) $\widehat{\Gamma}$-hyperideal of H/ρ.*

Theorem 104 *[11] Let ρ be a complete regular relation on a Γ-semihypergroup H. Then*

1. *$\overline{\overline{Apr}}_{\rho}(M)$ is an M-hypersystem of H/ρ, if M is a ρ-upper rough M-hypersystem of H.*
2. *$\underline{\underline{Apr}}_{\rho}(M)$ is an M-hypersystem of H/ρ, if M is a ρ-lower rough M-hypersystem of H.*

Theorem 105 *[11] Let ρ be a complete regular relation on a Γ-semihypergroup H. Then*

1. *$\overline{\overline{Apr}}_{\rho}(N)$ is an N-hypersystem in H/ρ, if N is a ρ-upper rough N-hypersystem in H.*

2. *$\underline{\underline{Apr}}_{\rho}(N)$ is an N-hypersystem in H/ρ, if N is a ρ-lower rough N-hypersystem in H.*

Chapter 2

Generalized Γ-Hyperideals in Γ-Semihypergroups

In this chapter, we define prime bi-Γ-hyperideals, (m,n) bi-Γ-hyperideals and prime (m,n) bi-Γ-hyperideals in a Γ-semihypergroup.

2.1 Prime Bi-Γ-Hyperideals

The results presented in this section are a part of our submitted paper [79].

Definition 106 *A bi-Γ-hyperideal B of a Γ-semihypergroup H is called a prime bi-Γ-hyperideal of H if for $x, y \in H$, $x\Gamma H\Gamma y \subseteq B$ (or $x\beta a\gamma y \subseteq B$, for all $a \in H$ and $\beta, \gamma \in \Gamma$) implies $x \in B$ or $y \in B$. A bi-Γ-hyperideal B of a Γ-semihypergroup H is called semiprime if for $x \in H$, $x\Gamma H\Gamma x \subseteq B$ (or $x\beta a\gamma x \subseteq B$, for all $a \in H$ and $\beta, \gamma \in \Gamma$) implies $x \in B$.*

Here we will denote $(x)_l = \{x\} \cup H\Gamma x$ (resp., $(x)_r = \{x\} \cup x\Gamma H$) is the principal left (resp., right) Γ-hyperideal of H generated by x.

Theorem 107 *A bi-Γ-hyperideal B of a Γ-semihypergroup H is prime if and only if for a right Γ-hyperideal R and a left Γ-hyperideal L of H, $R\Gamma L \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$.*

Proof. Suppose that $R\Gamma L \subseteq B$ for a right Γ-hyperideal R and a left Γ-hyperideal L of H and $R \not\subseteq B$. Then there exists $x \in R \backslash B$. Let $y \in L$. Then $x\Gamma H\Gamma y \subseteq R\Gamma H\Gamma L \subseteq R\Gamma L \subseteq B$. Since B is a prime bi-Γ-hyperideal and $x \notin B$, we have $y \in B$. Thus $L \subseteq B$.

Conversely, suppose that for a right Γ-hyperideal R and a left Γ-hyperideal L of H, $R\Gamma L \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$. If $x\Gamma H\Gamma y \subseteq B$ for $x, y \in H$, then $(x\Gamma H)\Gamma(H\Gamma y) \subseteq x\Gamma H\Gamma y \subseteq B$. Since $x\Gamma H$ is a right Γ-hyperideal and $H\Gamma y$ is a left Γ-hyperideal of H, by hypothesis, $x\Gamma H \subseteq B$ or $H\Gamma y \subseteq B$.

If $x\Gamma H \subseteq B$, $x^2 \subseteq x\Gamma H \subseteq B$. Thus

$$\begin{aligned}(x)_r\Gamma(x)_l &= (\{x\} \cup x\Gamma H)\Gamma(\{x\} \cup H\Gamma x) \\ &= x^2 \cup x\Gamma H\Gamma x \cup x\Gamma H^2\Gamma x \subseteq x^2 \cup x\Gamma H \subseteq B.\end{aligned}$$

Since $(x)_r$ is a right Γ-hyperideal and $(x)_l$ is a left Γ-hyperideal of H containing x, $(x)_r \subseteq B$ or $(x)_l \subseteq B$ by hypothesis. Hence $x \in B$. If $H\Gamma y \subseteq B$, $y \in B$ by the similar method. Therefore B is prime. ■

Proposition 108 *If a bi-Γ-hyperideal B of a Γ-semihypergroup H is prime, then B is a left or a right Γ-hyperideal of H.*

Proof. Since $B\Gamma H$ is a right Γ-hyperideal of H and $H\Gamma B$ a left Γ-hyperideal of H such that $(B\Gamma H)\Gamma(H\Gamma B) \subseteq B\Gamma H\Gamma B \subseteq B$, we get $B\Gamma H \subseteq B$ or $H\Gamma B \subseteq B$ by Theorem 107. Hence B is a left Γ-hyperideal or a right Γ-hyperideal of H. ■

Now let L_B, R_B, I_L and I_R for a bi-Γ-hyperideal B of a Γ-semihypergroup H as follows:

$$L_B = \{x \in B : H\Gamma x \subseteq B\} \text{ and } R_B = \{x \in B : x\Gamma H \subseteq B\}$$

$$I_L = \{y \in L_B : y\Gamma H \subseteq L_B\} \text{ and } I_R = \{x \in R_B : H\Gamma y \subseteq R_B\}.$$

Then we have the following results.

Proposition 109 *Let B be a prime bi-Γ-hyperideal of a Γ-semihypergroup H. Then L_B (resp., R_B) is a left (resp., right) Γ-hyperideal of H contained in B if L_B (resp., R_B) is non-empty.*

Proof. Let $x \in L_B \neq \emptyset$ and $z \in H$. Then for $\gamma \in \Gamma$, $z\gamma x \subseteq H\Gamma x \subseteq B$. Since $H\Gamma z\Gamma x \subseteq H^2\Gamma x \subseteq H\Gamma x \subseteq B$, we have $z\gamma x \subseteq L_B$. Thus $H\Gamma L_B \subseteq L_B$, so L_B is a left Γ-hyperideal of H.

Also, by the similar method we can prove that R_B is a right Γ-hyperideal of H contained in B. ■

Theorem 110 *Let B be a bi-Γ-hyperideal of a Γ-semihypergroup H. Then I_L (resp., I_R) is the largest Γ-hyperideal of H contained in B if I_L (resp., I_R) is non-empty. Furthermore I_L coincides with I_R.*

Proof. Let $x \in I_L$. Then $x\Gamma H \subseteq L_B$. Thus for any $z \in H$ and $\gamma \in \Gamma$, $x\gamma z \subseteq x\Gamma H \subseteq L_B$. Hence $x\Gamma z\Gamma H \subseteq x\Gamma H^2 \subseteq x\Gamma H \subseteq L_B$. Therefore $x\gamma z \subseteq I_L$, and so $I_L\Gamma G \subseteq I_L$. So I_L is a right Γ-hyperideal of H. Since $I_L \subseteq L_B \subseteq B$, we have $x \in L_B$ and $x \in B$. Thus $H\Gamma x \subseteq B$ and $H\Gamma z\Gamma x \subseteq H^2\Gamma x \subseteq H\Gamma x \subseteq B$ for any $z \in G$. Hence $z\gamma x \subseteq L_B$. Since L_B is a left Γ-hyperideal of H by Proposition 109, and $x\Gamma G \subseteq L_B$, we get $z\Gamma x\Gamma H \subseteq H\Gamma L_B \subseteq L_B$. Therefore $z\gamma x \subseteq I_L$, and so $H\Gamma I_L \subseteq I_L$. So I_L is a left Γ-hyperideal of H. It follows that I_L is a Γ-hyperideal of H contained in B.

Let A be any Γ-hyperideal of H contained in B. Then $H\Gamma A \subseteq A \subseteq B$, and so $A \subseteq L_B$. Since $A\Gamma H \subseteq A \subseteq L_B$, we get $A \subseteq I_L$. Therefore I_L is the largest Γ-hyperideal of H contained in B.

Also we can prove that I_R is the largest Γ-hyperideal of H contained in B by the similar method. Furthermore, since I_L and I_R are the largest Γ-hyperideals of H contained in B, I_L coincides with I_R. ■

Remark 111 *By Theorem 110 we denote I_B as $I_B = I_L = I_R$.*

Proposition 112 *Let H be a Γ-semihypergroup. If B is a prime bi-Γ-hyperideal of H, then I_B is a prime Γ-hyperideal of H contained in B.*

Proof. Let B be a bi-Γ-hyperideal of H. Then by Theorem 110, I_B is a Γ-hyperideal of H. Let us suppose that $X\Gamma Y \subseteq I_B$ for any Γ-hyperideals X, Y of H. Since X is a left Γ-hyperideal, Y a right Γ-hyperideal of H and $I_B \subseteq L_B \subseteq B$, by Theorem 107, we get $X \subseteq B$ or $Y \subseteq B$. Since I_B is the largest Γ-hyperideal contained in B, we get $X \subseteq I_B$ or $Y \subseteq I_B$. It follows that I_B is a prime Γ-hyperideal of H. ■

Corollary 113 *Let H be a Γ-semihypergroup. If B is a semi-prime bi-Γ-hyperideal of H, then I_B is a semi-prime Γ-hyperideal of H if I_B is non-empty.*

Proposition 114 *If a bi-*Γ*-hyperideal* B *of a* Γ*-semihypergroup* H *is semiprime, then for any left (resp., right)* Γ*-hyperideal* L *(resp.,* R*) of* H*,* $L^2 \subseteq B$ *implies* $L \subseteq B$ *(resp.,* $R^2 \subseteq B$ *implies* $R \subseteq B$*).*

Proof. Suppose that $L^2 \subseteq B$ for a left Γ-hyperideal L of H. If $L \nsubseteq B$, then there exists $x \in L \backslash B$. Since $x\Gamma H\Gamma x \subseteq L\Gamma H\Gamma L \subseteq L^2 \subseteq B$ and B is semiprime, we get $x \in B$. It is impossible. Hence $L \subseteq B$. We can prove that if $R^2 \subseteq B$, then $R \subseteq B$ by the similar method. ■

Proposition 115 *If a bi-*Γ*-hyperideal* B *of a* Γ*-semihypergroup* H *is semiprime, then* B *is a quasi* Γ*-hyperideal of* H*.*

Proof. Let $y \in B\Gamma H \cap H\Gamma B$. Then $y\Gamma H\Gamma y \subseteq (B\Gamma H)\Gamma H\Gamma(H\Gamma B) \subseteq B\Gamma H\Gamma B \subseteq B$. Since B is semiprime, we have $y \in B$. Thus $B\Gamma H \cap H\Gamma B \subseteq B$. Therefore B is a quasi Γ-hyperideal. ■

Proposition 116 *Let* H *be a* Γ*-semihypergroup. A prime bi-*Γ*-hyperideal* B *of* H *is a prime one-sided* Γ*-hyperideal.*

Proof. It will be sufficient to show that B is a one-sided Γ-hyperideal. So, let us suppose that B is not a one-sided Γ-hyperideal of H. Then $B\Gamma H \nsubseteq B$ and $H\Gamma B \nsubseteq H$. Since B is prime, $(B\Gamma H)\Gamma H\Gamma(H\Gamma B) \nsubseteq H$, and since B is a bi-Γ-hyperideal of H, $B\Gamma H\Gamma B \subseteq B$, then $(B\Gamma H)\Gamma H\Gamma(H\Gamma B) \subseteq B$, a contradiction. Hence, $B\Gamma H \subseteq B$ or $H\Gamma B \subseteq B$, that is, B is a one-sided Γ-hyperideal of H. ■

For a bi-Γ-hyperideal B, let us put

$$J(B) = \{x \in H | H\Gamma x\Gamma H \subseteq B\}.$$

It is not difficult to show that $J(B)$ is the (unique) largest two-sided Γ-hyperideal contained in B.

Proposition 117 *Let* H *be a* Γ*-semihypergroup. If the bi-*Γ*-hyperideal* B *is prime, then* $J(B)$ *is a prime* Γ*-hyperideal.*

Proof. Let B be prime and suppose that $I_1\Gamma I_2 \subseteq J(B)$ for (two-sided) Γ-hyperideals I_1 and I_2. Then, since $I_1\Gamma I_2 \subseteq B$, by Theorem 107, $I_1 \subseteq B$ or $I_2 \subseteq B$. Now $J(B)$ is the largest Γ-hyperideal in B, so this implies $I_1 \subseteq J(B)$ or $I_2 \subseteq J(B)$. Therefore $J(B)$ is prime. ■

2.2 (m, n) Bi-Γ-Hyperideals

The results presented in this section are a part of our published paper [80].

Lemma 118 *In a* Γ*-semihypergroup* H*,* $(A\Gamma B)^m = A^m\Gamma B^m$ *holds if* $A\Gamma B = B\Gamma A$ *for all* $A, B \in H$ *and* m *is a positive integer.*

Proof. We prove the result $(A\Gamma B)^m = A^m\Gamma B^m$ by induction on m. For $m = 1$, $A\Gamma B = A\Gamma B$, which is true. For $m = 2$, $(A\Gamma B)^2 = (A\Gamma B)\Gamma(A\Gamma B) = A\Gamma(B\Gamma A)\Gamma B = A^2\Gamma B^2$. Suppose that the result is true for $m = k$. That is, $(A\Gamma B)^k = A^k\Gamma B^k$. Now for $m = k + 1$, we have

$$\begin{aligned}(A\Gamma B)^{k+1} &= (A\Gamma B)^k\Gamma(A\Gamma B) = (A^k\Gamma B^k)\Gamma(A\Gamma B) = A^k\Gamma(B^k\Gamma A)\Gamma B \\ &= (A^k\Gamma A)\Gamma(B^k\Gamma B) = A^{k+1}\Gamma B^{k+1}.\end{aligned}$$

Thus, the result is true for $m = k + 1$. By induction hypothesis the result $(A\Gamma B)^m = A^m\Gamma B^m$ is true for all positive integers m. ■

A subset A of a Γ-semihypergroup H is called an $(m, 0)$ Γ-hyperideal ($(0, n)$ Γ-hyperideal) if $A^m\Gamma H \subseteq A$ ($H\Gamma A^n \subseteq A$). A sub Γ-semihypergroup A of a Γ-semihypergroup H is called (m, n) bi-Γ-hyperideal of H, if A satisfies the condition

$$A^m\Gamma H\Gamma A^n \subseteq A,$$

where m, n are non-negative integers (A^m is suppressed if $m = 0$). Here if $m = n = 1$ then A is called bi-Γ-hyperideal of H. By a proper (m, n) bi-Γ-hyperideal we mean an (m, n) bi-Γ-hyperideal, which is a proper subset of H.

Example 119 *Let* $(H, \circ)$ *be a semihypergroup and* Γ *be a non-empty subset of* H*. Define a mapping* $H \times \Gamma \times H \to \mathcal{P}^*(H)$ *by* $x\gamma y = x \circ y$ *for every* $x, y \in H$ *and* $\gamma \in \Gamma$*. By Example 3, we know that* H *is a* Γ*-semihypergroup. Let* B *be an* (m, n) *bi-hyperideal of the semihypergroup* H*. Then,* $B^m \circ H \circ B^n \subseteq B$*. So,* $B^m\Gamma H\Gamma B^n = B^m \circ H \circ B^n \subseteq B$*. Hence,* B *is an* (m, n) *bi-*Γ*-hyperideal of* H*.*

Example 120 *Let* $H = [0, 1]$ *and* $\Gamma = \mathbb{N}$*. Then,* H *together with the hyperoperation* $x\gamma y = \left[0, \frac{xy}{\gamma}\right]$ *is a* Γ*-semihypergroup. Let* $t \in [0, 1]$ *and set* $T = [0, t]$*. Then, clearly it can be seen that* T *is a sub* Γ*-semihypergroup of* H*. Since* $T^m\Gamma H = [0, t^m] \subseteq [0, t] = T$ ($H\Gamma T^n = [0, t^n] \subseteq [0, t] = T$)*, so* T *is an* $(m, 0)$ Γ*-hyperideal* ($(0, n)$ Γ*-hyperideal*) *of* H*. Since* $T^m\Gamma H\Gamma T^n = [0, t^{m+n}] \subseteq [0, t] = T$*, then* T *is an* (m, n) *bi-*Γ*-hyperideal of* Γ*-semihypergroup* H*.*

Proposition 121 *Let H be a Γ-semihypergroup, B be a sub Γ-semihypergroup of H and let A be an (m,n) bi-Γ-hyperideal of H. Then, the intersection $A\cap B$ is an (m,n) bi-Γ-hyperideal of Γ-semihypergroup B.*

Proof. The intersection $A\cap B$ evidently is a sub Γ-semihypergroup of H. We show that $A\cap B$ is an (m,n) bi-Γ-hyperideal of B, for this

$$(A\cap B)^m\Gamma B\Gamma(A\cap B)^n \subseteq A^m\Gamma H\Gamma A^n \subseteq A, \tag{1}$$

because of A is an (m,n) bi-Γ-hyperideal of H. Secondly

$$(A\cap B)^m\Gamma B\Gamma(A\cap B)^n \subseteq B^m\Gamma B\Gamma B^n \subseteq B. \tag{2}$$

Therefore, (1) and (2) imply that $(A\cap B)^m\Gamma B\Gamma(A\cap B)^n \subseteq A\cap B$, that is, the intersection $A\cap B$ is an (m,n) bi-Γ-hyperideal of B. ■

Theorem 122 *Suppose that $\{A_i : i\in I\}$ be a family of (m,n) bi-Γ-hyperideals of a Γ-semihypergroup H. Then, the intersection $\bigcap\limits_{i\in I} A_i \neq \emptyset$ is an (m,n) bi-Γ-hyperideal of H.*

Proof. Let $\{A_i : i\in I\}$ be a family of (m,n) bi-Γ-hyperideals in a Γ-semihypergroup H. We know that the intersection of sub Γ-semihypergroups is a sub Γ-semihypergroup. Let $B=\bigcap\limits_{i\in I} A_i$. Now we have to show that $B=\bigcap\limits_{i\in I} A_i$ is an (m,n) bi-Γ-hyperideal of H. Here we need only to show that $B^m\Gamma H\Gamma B^n \subseteq B$. Let $x\in B^m\Gamma H\Gamma B^n$. Then, $x=a_1^m\alpha s\beta a_2^n$ for some $a_1^m, a_2^n \subseteq B$, $s\in H$ and $\alpha,\beta\in\Gamma$. Thus, for any arbitrary $i\in I$ as $a_1^m, a_2^n\subseteq B_i$. So, $x\in B_i^m\Gamma H\Gamma B_i^n$. Since B_i is an (m,n) bi-Γ-hyperideal so $B_i^m\Gamma H\Gamma B_i^n\subseteq B_i$ and therefore $x\in B_i$. Since i was chosen arbitrarily so $x\in B_i$ for all $i\in I$ and hence $x\in B$. So, $B^m\Gamma H\Gamma B^n\subseteq B$ and hence $B=\bigcap\limits_{i\in I} A_i$ is an (m,n) bi-Γ-hyperideal of H. ■

It is obvious that the intersection of two or more $(m,0)$ Γ-hyperideals ($(0,n)$ Γ-hyperideals) is an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal). Similarly, the union of two or more $(m,0)$ Γ-hyperideals ($(0,n)$ Γ-hyperideals) is an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal).

Theorem 123 *Let H be a Γ-semihypergroup. If A is an $(m,0)$ Γ-hyperideal and also $(0,n)$ Γ-hyperideal of H, then A is an (m,n) bi-Γ-hyperideal of H.*

Proof. Suppose that A is an $(m, 0)$ Γ-hyperideal and also $(0, n)$ Γ-hyperideal of H. Then,

$$A^m \Gamma H \Gamma A^n \subseteq A \Gamma A^n \subseteq H \Gamma A^n \subseteq A,$$

which implies that A is an (m, n) bi-Γ-hyperideal of H. ■

Theorem 124 *Let m, n be arbitrary positive integers. Let H be a Γ-semihypergroup, B be an (m, n) bi-Γ-hyperideal of H and A be a sub Γ-semihypergroup of H. Suppose that $A\Gamma B = B\Gamma A$. Then,*

1. *$B\Gamma A$ is an (m, n) bi-Γ-hyperideal of H.*
2. *$A\Gamma B$ is an (m, n) bi-Γ-hyperideal of H.*

Proof. (1) The suppositions of the theorem imply that

$$(B\Gamma A)\Gamma(B\Gamma A) = (B\Gamma A\Gamma B)\Gamma A = B\Gamma A.$$

This shows that $B\Gamma A$ is a sub Γ-semihypergroup of H. On the other hand, as B is an (m, n) bi-Γ-hyperideal of H, so

$$(B\Gamma A)^m \Gamma H \Gamma (B\Gamma A)^n = (B^m \Gamma A^m \Gamma H \Gamma B^n)\Gamma A^n \subseteq B\Gamma A^n \subseteq B\Gamma A.$$

Hence, the product $B\Gamma A$ is an (m, n) bi-Γ-hyperideal of H.

(2) The proof is similar to (1). ■

Theorem 125 *Let H be a Γ-semihypergroup and for a positive integer n, $B_1, B_2, \cdots, B_n$ be (m, n) bi-Γ-hyperideals of H. Then, $B_1\Gamma B_2\Gamma \cdots \Gamma B_n$ is an (m, n) bi-Γ-hyperideal of H.*

Proof. We prove the theorem by induction. By Theorem 124, $B_1\Gamma B_2$ is an (m, n) bi-Γ-hyperideal of H. Next, for $k \leq n$, suppose that $B_1\Gamma B_2\Gamma ... \Gamma B_k$ is an (m, n) bi-Γ-hyperideal of H. Then, $B_1\Gamma B_2\Gamma ... \Gamma B_k \Gamma B_{k+1} = (B_1\Gamma B_2\Gamma ... \Gamma B_k)\Gamma B_{k+1}$ is an (m, n) bi-Γ-hyperideal of H by Theorem 124. ■

Theorem 126 *Let H be a Γ-semihypergroup, A be an (m, n) bi-Γ-hyperideal of H, and B be an (m, n) bi-Γ-hyperideal of the Γ-semihypergroup A such that $B^2 = B\Gamma B = B$. Then, B is an (m, n) bi-Γ-hyperideal of H.*

Proof. It is trivial that B is a sub Γ-semihypergroup of H. Secondly, since $A^m\Gamma H\Gamma A^n \subseteq A$ and $B^m\Gamma A\Gamma B^n \subseteq B$, we have

$$B^m\Gamma H\Gamma B^n = B^m\Gamma(B^m\Gamma H\Gamma B^n)\Gamma B^n \subseteq B^m\Gamma(A^m\Gamma H\Gamma A^n)\Gamma B^n \subseteq B^m\Gamma A\Gamma B^n \subseteq B.$$

Therefore, B is an (m,n) bi-Γ-hyperideal of H. ■

2.3 Prime (m,n) Bi-Γ-Hyperideals

The results offered in this section are a part of our published paper [81].

Definition 127 *An (m,n) bi-Γ-hyperideal B of a Γ-semihypergroup H is called prime if for $x, y \in H$, $x^m\alpha H\beta y^n \subseteq B$ (or $x^m\alpha z\beta y^n \subseteq B$, for all $z \in H$) implies $x \in B$ or $y \in B$, for all $\alpha, \beta \in \Gamma$.*

Definition 128 *An (m,n) bi-Γ-hyperideal B of a Γ-semihypergroup H is called semi-prime if for $x \in H$, $x^m\alpha H\beta x^n \subseteq B$ (or $x^m\alpha z\beta x^n \subseteq B$, for all $z \in H$) implies $x \in B$, for all $\alpha, \beta \in \Gamma$.*

Example 129 *Let $H = M_2(\mathbb{Z})$ be the set of all 2×2 matrices, then H is a semigroup under usual multiplication. Let*

$$T_1 = \left\{ \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z} \right\}, \quad T_2 = \left\{ \begin{pmatrix} a & b \\ -c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z} \right\},$$
$$T_3 = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a,b,d \in \mathbb{Z} \right\}, \quad T_4 = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : a,c,d \in \mathbb{Z} \right\},$$

be non-empty subsets of H. Let $\Gamma = \{\beta_1, \beta_2, \beta_3, \beta_4\}$. We define $A_1\beta_iA_2 = A_1T_iA_2$ for every $A_1, A_2 \in H$, and $\beta_i \in \Gamma$, $1 \le i \le 4$. Then H is a Γ-semihypergroup. Let

$$B = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in 2\mathbb{Z} \right\}.$$

Since $B^m\Gamma H\Gamma B^n \subseteq B$. Then B is a prime (m,n) bi-Γ-hyperideal of H.

Example 130 *Let $H = \{a,b,c,d,e,f\}$ and $\Gamma = \{\gamma, \beta\}$ be the sets of binary hyperop-*

erations defined below:

γ	a	b	c	d	e	f
a	a	b	a	a	a	a
b	b	b	b	b	b	b
c	a	b	$\{a,c\}$	a	a	$\{a,f\}$
d	a	b	$\{a,e\}$	a	a	$\{a,d\}$
e	a	b	$\{a,e\}$	a	a	$\{a,d\}$
f	a	b	$\{a,c\}$	a	a	$\{a,f\}$

β	a	b	c	d	e	f
a	a	b	a	a	a	a
b	b	b	b	b	b	b
c	a	b	a	a	a	a
d	a	b	a	$\{a,d\}$	$\{a,e\}$	a
e	a	b	a	a	a	a
f	a	b	a	$\{a,f\}$	$\{a,c\}$	a

Then H is a Γ-semihypergroup. The $(m,0)$ Γ-hyperideals are $\{a,b\}$, $\{b\}$, $\{a,b,c,f\}$, $\{a,b,d,e\}$ and H. The $(0,n)$ Γ-hyperideals are $\{a,b\}$, $\{b\}$, $\{a,b,c,e\}$, $\{a,b,d,f\}$ and H. The (m,n) bi-Γ-hyperideals are $\{a,b\}$, $\{b\}$, $\{a,b,c\}$, $\{a,b,f\}$, $\{a,b,d\}$, $\{a,b,c,e\}$, $\{a,b,d,f\}$, $\{a,b,c,f\}$, $\{a,b,d,e\}$ and H.

The only prime (m,n) bi-Γ-hyperideals of H are $\{b\}$ and H, and hence these are semiprime.

Furthermore $\{a,b\}$, $\{a,b,c\}$, $\{a,b,f\}$, $\{a,b,d\}$, $\{a,b,c,e\}$, $\{a,b,d,f\}$, $\{a,b,c,f\}$, $\{a,b,d,e\}$ are not prime (m,n) bi-Γ-hyperideals. Indeed

$e^m\Gamma H\Gamma f^n \subseteq \{a,b\}$, *but* $e,f \notin \{a,b\}$,

$e^m\Gamma H\Gamma f^n \subseteq \{a,b,c\}$, *but* $e,f \notin \{a,b,c\}$,

$c^m\Gamma H\Gamma d^n \subseteq \{a,b,f\}$, *but* $c,d \notin \{a,b,f\}$,

$e^m\Gamma H\Gamma f^n \subseteq \{a,b,d\}$, *but* $e,f \notin \{a,b,d\}$,

$d^m\Gamma H\Gamma f^n \subseteq \{a,b,c,e\}$, *but* $d,f \notin \{a,b,c,e\}$,

$c^m\Gamma H\Gamma e^n \subseteq \{a,b,d,f\}$, *but* $c,e \notin \{a,b,d,f\}$,

$d^m\Gamma H\Gamma e^n \subseteq \{a,b,c,f\}$, *but* $d,e \notin \{a,b,c,f\}$,

$c^m\Gamma H\Gamma f^n \subseteq \{a,b,d,e\}$, *but* $c,f \notin \{a,b,d,e\}$.

Theorem 131 *If an (m,n) bi-Γ-hyperideal B of a Γ-semihypergroup H is prime, then for an $(m,0)$ Γ-hyperideal R and a $(0,n)$ Γ-hyperideal L of H, $R\Gamma L \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$.*

Proof. Suppose that $R\Gamma L \subseteq B$ for an $(m,0)$ Γ-hyperideal R and a $(0,n)$ Γ-hyperideal L of H and $R \nsubseteq B$. Then there exists $x \in R\backslash B$. Let $y \in L$. Then

$$x^m \Gamma H \Gamma y^n \subseteq R\Gamma H\Gamma L \subseteq R\Gamma L \subseteq B.$$

Since B is a prime (m,n) bi-Γ-hyperideal and $x \notin B$, we have $y \in B$. Thus $L \subseteq B$. ■

Proposition 132 *If an (m,n) bi-Γ-hyperideal B of a Γ-semihypergroup H is prime, then B is a $(0,n)$ Γ-hyperideal or an $(m,0)$ Γ-hyperideal of H.*

Proof. Since $B^m\Gamma H$ is an $(m,0)$ Γ-hyperideal of H and $H\Gamma B^n$ a $(0,n)$ Γ-hyperideal of H such that

$$(B^m\Gamma H)\Gamma(H\Gamma B^n) \subseteq B^m\Gamma H\Gamma B^n \subseteq B,$$

we get $B^m\Gamma H \subseteq B$ or $H\Gamma B^n \subseteq B$ by Theorem 131. Hence B is a $(0,n)$ Γ-hyperideal or an $(m,0)$ Γ-hyperideal of H. ■

Definition 133 *An (m,n) bi-Γ-hyperideal B of a Γ-semihypergroup H is called a strongly prime (m,n) bi-Γ-hyperideal if $B_1\Gamma B_2 \cap B_2\Gamma B_1 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any (m,n) bi-Γ-hyperideals B_1 and B_2 of H.*

Every strongly prime (m,n) bi-Γ-hyperideal of a Γ-semihypergroup H is a prime (m,n) bi-Γ-hyperideal and every prime (m,n) bi-Γ-hyperideal is a semiprime (m,n) bi-Γ-hyperideal. A prime (m,n) bi-Γ-hyperideal is not necessarily strongly prime.

Example 134 *Let $H = \{e,a,b,c,d\}$ and $\Gamma = \{\gamma,\beta\}$ be the sets of binary hyperoperations defined below:*

γ	e	a	b	c	d
e	e	e	e	e	e
a	e	$\{a,b\}$	b	b	b
b	e	b	b	b	b
c	e	c	c	c	c
d	e	d	d	d	d

β	e	a	b	c	d
e	e	e	e	e	e
a	e	a	a	a	a
b	e	a	$\{a,b\}$	a	a
c	e	c	c	c	c
d	e	d	d	d	d

Then H is a Γ-semihypergroup. The (m,n) bi-Γ-hyperideals of H are $\{e\}$, $\{e,c\}$, $\{e,d\}$, $\{e,a,b\}$, $\{e,c,d\}$ and H. Here all (m,n) bi-Γ-hyperideals of H are prime and *hence semiprime. However, the prime (m,n) bi-Γ-hyperideal $\{e\}$ is not strongly prime (m,n) bi-Γ-hyperideal of H because*

$$\{e,c\}\Gamma\{e,d\}\cap\{e,d\}\Gamma\{e,c\}=\{e\}\subseteq\{e\},$$

but neither $\{e,c\}$ nor $\{e,d\}$ is contained in $\{e\}$.

Definition 135 *An (m,n) bi-Γ-hyperideal B of a Γ-semihypergroup H is called an irreducible (resp., strongly irreducible) (m,n) bi-Γ-hyperideal if $B_1\cap B_2=B$ (resp., $B_1\cap B_2\subseteq B$) implies $B_1=B$ or $B_2=B$ (resp., $B_1\subseteq B$ or $B_2\subseteq B$).*

In Example 134, the irreducible (m,n) bi-Γ-hyperideals of H are $\{e,c\}$, $\{e,d\}$, $\{e,a,b\}$, $\{e,c,d\}$ and H. But the (m,n) bi-Γ-hyperideal $\{e\}$ is not irreducible, because $\{e,c\}\cap\{e,d\}=\{e\}$ but neither $\{e,c\}=\{e\}$ nor $\{e,d\}=\{e\}$.

Lemma 136 *The intersection of any family of prime (m,n) bi-Γ-hyperideals of a Γ-semihypergroup is a semiprime (m,n) bi-Γ-hyperideal.*

Proof. The proof is straightforward. ■

Theorem 137 *Every strongly irreducible, semiprime (m,n) bi-Γ-hyperideal of a Γ-semihypergroup H is a strongly prime (m,n) bi-Γ-hyperideal.*

Proof. Let B be a strongly irreducible semiprime (m,n) bi-Γ-hyperideal of H. Let B_1,B_2 be any (m,n) bi-Γ-hyperideal of H such that $B_1\Gamma B_2\cap B_2\Gamma B_1\subseteq B$. Since

$$(B_1\cap B_2)^2\subseteq B_1\Gamma B_2 \text{ and } (B_1\cap B_2)^2\subseteq B_2\Gamma B_1,$$

$$(B_1\cap B_2)^2\subseteq B_1\Gamma B_2\cap B_2\Gamma B_1\subseteq B.$$

Since B is a semiprime (m,n) bi-Γ-hyperideal, $B_1\cap B_2\subseteq B$. Because B is a strongly irreducible (m,n) bi-Γ-hyperideal of H, so either $B_1\subseteq B$ or $B_2\subseteq B$. Thus B is a strongly prime (m,n) bi-Γ-hyperideal of H. ■

Theorem 138 *Let B be an (m,n) bi-Γ-hyperideal of a Γ-semihypergroup H and $a\in H$ such that $a\notin B$ for a positive integer m. Then there exists an irreducible (m,n) bi-Γ-hyperideal I of H such that $B\subseteq I$ and $a\notin I$.*

Proof. Let $\mathcal{A}$ be the collection of all (m,n) bi-Γ-hyperideals of H which contain B and do not contain a. Then $\mathcal{A}$ is non-empty, because $B \in \mathcal{A}$. The collection $\mathcal{A}$ is a partially ordered set under inclusion. If $\mathcal{C}$ is any totally ordered subset of $\mathcal{A}$ then $\cup\mathcal{C}$ is an (m,n) bi-Γ-hyperideal of H containing B. Hence by Zorn's Lemma, there exists a maximal element I in $\mathcal{A}$. We show that I is an irreducible (m,n) bi-Γ-hyperideal. Let C and D be two (m,n) bi-Γ-hyperideals of H such that $I = C \cap D$. If both C and D properly contain I then $a \in C$ and $a \in D$. Hence $a \in C \cap D = I$. This contradicts the fact that $a \notin I$. Thus $I = C$ or $I = D$. ■

Definition 139 *An element $x \in H$ is called regular if there exist a in H and $\alpha, \beta \in \Gamma$ such that $x \in x\alpha a\beta x$. If every element of a Γ-semihypergroup H is regular then H called regular Γ-semihypergroup.*

Definition 140 *An element a of a Γ-semihypergroup H is called an intra-regular if there exist $x, y \in H$ such that $a \in x\alpha a^2\beta y$, for all $\alpha, \beta \in \Gamma$ and H is called intra-regular, if every element of H is intra-regular.*

Theorem 141 *Let H be a regular and intra-regular Γ-semihypergroup. Then the following assertions, for an (m,n) bi-Γ-hyperideal B of H, are equivalent:*

1. *B is strongly irreducible.*
2. *B is strongly prime.*

Proof. The proof is straightforward. ■

Theorem 142 *For a Γ-semihypergroup H the following assertions are equivalent:*

1. *The set of (m,n) bi-Γ-hyperideals of H is totally ordered under inclusion,*
2. *Each (m,n) bi-Γ-hyperideal of H is strongly irreducible.*
3. *Each (m,n) bi-Γ-hyperideal of H is irreducible.*

Proof. (1) $\Longrightarrow$ (2) Let B be an arbitrary (m,n) bi-Γ-hyperideal of H and B_1, B_2 be two (m,n) bi-Γ-hyperideals of H such that $B_1 \cap B_2 \subseteq B$. Since the set of (m,n) bi-Γ-hyperideals is totally ordered, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Thus either $B_1 \cap B_2 =$

B_1 or $B_1 \cap B_2 = B_2$. Hence $B_1 \cap B_2 \subseteq B$ implies either $B_1 \subseteq B$ or $B_2 \subseteq B$. This shows that B is a strongly irreducible (m, n) bi-Γ-hyperideal.

(2) $\Longrightarrow$ (3) Let B be an arbitrary (m, n) bi-Γ-hyperideal of H and B_1, B_2 be two (m, n) bi-Γ-hyperideals of H such that $B_1 \cap B_2 = B$. Then $B \subseteq B_1$ and $B \subseteq B_2$. By hypothesis, either $B_1 \subseteq B$ or $B_2 \subseteq B$. Hence either $B_1 = B$ or $B_2 = B$. That is, B is an irreducible (m, n) bi-Γ-hyperideal.

(3) $\Longrightarrow$ (1) Let B_1 and B_2 be any two (m, n) bi-Γ-hyperideals of H. Then $B_1 \cap B_2$ is an (m, n) bi-Γ-hyperideal of H. So by hypothesis, either $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$, that is, either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. ■

Chapter 3

Bipolar Fuzzy Sets in Γ-Semihypergroups

Zhang [97] introduced the theory of bipolar fuzzy sets which has been applied to many branches of mathematics. After the introduction of this theory, several authors applied this theory to different algebraic structures, for instance, Akram et al. [4, 5], Jun et al. [40, 41], Lee [49], Lee and Jun [50] and Shabir and Iqbal [65]. In this section, we initiate a study on bipolar fuzzy sets in Γ-semihypergroups. We define bipolar fuzzy left (right, bi-, interior, $(1, 2)$-) Γ-hyperideals and explore some related properties. We use these bipolar fuzzy Γ-hyperideals to characterize some classes of Γ-semihypergroups.

3.1 Bipolar Fuzzy Γ-Hyperideals

The results presented in this section are a part of our published paper [82].

Definition 143 *Let $\mathcal{A} = \langle \mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-} \rangle$ and $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ be two bipolar fuzzy subsets of a Γ-semihypergroup H. Then their product $\mathcal{A} \circ_{\Gamma} \mathcal{B}$ is defined by*

$$\mathcal{A} \circ_{\Gamma} \mathcal{B} = \left\{ \left\langle x, \mu_{\mathcal{A}\circ_{\Gamma}\mathcal{B}}^{+}(x), \mu_{\mathcal{A}\circ_{\Gamma}\mathcal{B}}^{-}(x) \right\rangle : x \in H \right\},$$

where

$$\mu_{\mathcal{A}\circ_{\Gamma}\mathcal{B}}^{+} : H \longrightarrow [0,1] | x \longmapsto \mu_{\mathcal{A}\circ_{\Gamma}\mathcal{B}}^{+}(x) := \begin{cases} \sup\limits_{x \in y\gamma z} \{\mu_{\mathcal{A}}^{+}(y) \wedge \mu_{\mathcal{B}}^{+}(z)\} & \text{if } x \in y\gamma z,\ \forall \gamma \in \Gamma \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mu^-_{\mathcal{A}\circ_\Gamma\mathcal{B}} : H \longrightarrow [-1,0] | x \longmapsto \mu^-_{\mathcal{A}\circ_\Gamma\mathcal{B}}(x) := \begin{cases} \inf\limits_{x\in y\gamma z}\{\mu^-_{\mathcal{A}}(y) \vee \mu^-_{\mathcal{B}}(z)\} & \text{if } x \in y\gamma z,\ \forall\gamma\in\Gamma \\ 0 & \text{otherwise,} \end{cases}$$

for some $x, y, z \in H$.

Definition 144 *Let* H *be a* Γ*-semihypergroup. A bipolar fuzzy subset* $\mathcal{B} = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \rangle$ *of* H *is called*

1. *a bipolar fuzzy sub* Γ*-semihypergroup of* H *if*

$$\inf_{x\in y\gamma z} \mu^+_{\mathcal{B}}(x) \geq \min\{\mu^+_{\mathcal{B}}(y), \mu^+_{\mathcal{B}}(z)\} \quad \text{and} \quad \sup_{x\in y\gamma z} \mu^-_{\mathcal{B}}(x) \leq \max\{\mu^-_{\mathcal{B}}(y), \mu^-_{\mathcal{B}}(z)\},$$

for all $x, y, z \in H$ *and* $\gamma \in \Gamma$.

2. *a bipolar fuzzy left* Γ*-hyperideal of* H *if*

$$\inf_{x\in y\gamma z} \mu^+_{\mathcal{B}}(x) \geq \mu^+_{\mathcal{B}}(z) \quad \text{and} \quad \sup_{x\in y\gamma z} \mu^-_{\mathcal{B}}(x) \leq \mu^-_{\mathcal{B}}(z),$$

for all $x, y, z \in H$ *and* $\gamma \in \Gamma$.

3. *a bipolar fuzzy right* Γ*-hyperideal of* H *if*

$$\inf_{x\in y\gamma z} \mu^+_{\mathcal{B}}(x) \geq \mu^+_{\mathcal{B}}(y) \quad \text{and} \quad \sup_{x\in y\gamma z} \mu^-_{\mathcal{B}}(x) \leq \mu^-_{\mathcal{B}}(y),$$

for all $x, y, z \in H$ *and* $\gamma \in \Gamma$.

4. *a bipolar fuzzy* Γ*-hyperideal of* H *if*

$$\inf_{x\in y\gamma z} \mu^+_{\mathcal{B}}(x) \geq \max\{\mu^+_{\mathcal{B}}(y), \mu^+_{\mathcal{B}}(z)\} \quad \text{and} \quad \sup_{x\in y\gamma z} \mu^-_{\mathcal{B}}(x) \leq \min\{\mu^-_{\mathcal{B}}(y), \mu^-_{\mathcal{B}}(z)\},$$

for all $x, y, z \in H$ *and* $\gamma \in \Gamma$.

Example 145 *Let* $H = \{1, -1, i, -i\}$ *be the group with respect to multiplication and* $\Gamma = \{\gamma_1, \gamma_2\}$. *Let* $A_1 = \{1, -1\}$ *and* $A_2 = \{1\}$ *be non-empty subsets of* H. *We define* $x\gamma_k y = xA_k y$, *for every* $\gamma_k \in \Gamma$ *and* $x, y \in H$. *Then* H *is a* Γ*-semihypergroup. Now, we define a bipolar fuzzy set* $\mathcal{B} = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \rangle$ *on* H *as follows:*

$$\mu^+_{\mathcal{B}}(x) = \begin{cases} 0.7 & \text{if } x = 1 \\ 0.5 & \text{if } x = -1 \\ 0.2 & \text{if } x = i, -i \end{cases} \quad \text{and} \quad \mu^-_{\mathcal{B}}(x) = \begin{cases} -0.9 & \text{if } x = 1 \\ -0.6 & \text{if } x = -1 \\ -0.3 & \text{if } x = i, -i \end{cases}$$

It is easy to see that $\mathcal{B}$ *is a bipolar fuzzy sub* Γ*-semihypergroup of* H.

Example 146 *Let $H = (0,1)$, $\Gamma = \{\gamma_n \,|n \in \mathbb{N}\}$ and for every $n \in \mathbb{N}$ we define hyperoperation γ_n on H as follows*

$$x\gamma_n y = \left\{\frac{xy}{2^k} \,|0 \leq k \leq n\right\}, \forall x, y \in H.$$

Then, $x\gamma_n y \subset H$ and for every $m, n \in \mathbb{N}$ and $x, y, z \in H$

$$(x\gamma_n y)\gamma_m z = \left\{\frac{xyz}{2^k} \,|0 \leq k \leq n+m\right\} = x\gamma_n \left(y\gamma_m z\right).$$

So H is a Γ-semihypergroup. Now we define a bipolar fuzzy set $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \right\rangle$ on H as:

$$\mu_{\mathcal{B}}^{+}(x) = \begin{cases} 0.8 & \textit{if } 0 < x < \frac{1}{2^k} \\ 0.5 & \textit{if } \frac{1}{2^k} \leq x < 1 \end{cases} \quad \textit{and} \quad \mu_{\mathcal{B}}^{-}(x) = \begin{cases} -0.7 & \textit{if } 0 < x < \frac{1}{2^k} \\ -0.2 & \textit{if } \frac{1}{2^k} \leq x < 1 \end{cases}$$

where $k \in \mathbb{N}$. Then by routine calculations, $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \right\rangle$ is a bipolar fuzzy Γ-hyperideal of H.

Definition 147 *Let H be a Γ-semihypergroup. A bipolar fuzzy sub Γ-semihypergroup $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \right\rangle$ of H is called a bipolar fuzzy bi-Γ-hyperideal of H if*

$$\inf_{a \in x\alpha y\beta z} \mu_{\mathcal{B}}^{+}(a) \geq \min\{\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(z)\} \quad \textit{and} \quad \sup_{a \in x\alpha y\beta z} \mu_{\mathcal{B}}^{-}(a) \leq \max\{\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(z)\},$$

for all $x, y, z \in H$ and $\alpha, \beta \in \Gamma$.

Example 148 *Let $H = \{a, b, c, d, e\}$ and $\Gamma = \{\gamma, \beta\}$ be the sets of binary hyperoperations defined below:*

γ	a	b	c	d	e
a	$\{a,b\}$	$\{b,e\}$	c	$\{c,d\}$	e
b	$\{b,e\}$	e	c	$\{c,d\}$	e
c	c	c	c	c	c
d	$\{c,d\}$	$\{c,d\}$	c	d	$\{c,d\}$
e	e	e	c	$\{c,d\}$	e

β	a	b	c	d	e
a	$\{b,e\}$	e	c	$\{c,d\}$	e
b	e	e	c	$\{c,d\}$	e
c	c	c	c	c	c
d	$\{c,d\}$	$\{c,d\}$	c	d	$\{c,d\}$
e	e	e	c	$\{c,d\}$	e

Here H is a Γ-semihypergroup. Now we define a bipolar fuzzy set $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \right\rangle$ on H as:

$$\mu_{\mathcal{B}}^{+}(x) = \begin{cases} 0.2 & \textit{if } x \in \{a,b\} \\ 0.9 & \textit{if } x = c \\ 0.7 & \textit{if } x = d \\ 0.3 & \textit{if } x = e \end{cases} \quad \textit{and} \quad \mu_{\mathcal{B}}^{-}(x) = \begin{cases} -0.5 & \textit{if } x \in \{a,b,e\} \\ -0.8 & \textit{if } x = c \\ -0.7 & \textit{if } x = d \end{cases}$$

Then by routine calculations, $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \right\rangle$ is a bipolar fuzzy bi-Γ-hyperideal of H.

Definition 149 *Let H be a Γ-semihypergroup. A bipolar fuzzy subset $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \right\rangle$ of H is called a bipolar fuzzy interior Γ-hyperideal of H if*

$$\inf_{a \in x\beta y\gamma z} \mu_{\mathcal{B}}^{+}(a) \geq \mu_{\mathcal{B}}^{+}(y) \text{ and } \sup_{a \in x\beta y\gamma z} \mu_{\mathcal{B}}^{-}(a) \leq \mu_{\mathcal{B}}^{-}(y),$$

for all $x, y, z \in H$ and $\beta, \gamma \in \Gamma$.

Example 150 *Let $H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}^{+} \right\}$, $A_i = \left\{ \begin{pmatrix} ie & 0 \\ 0 & if \end{pmatrix} : e, f \in 2\mathbb{Z}^{+} \right\}$, for all $i \in \mathbb{N}$ and $\Gamma = \{\gamma_i : i \in \mathbb{N}\}$. Then H is a Γ-semihypergroup under the hyperoperation $x\gamma_i y = xA_i y$, for every $\gamma_i \in \Gamma$ and $x, y \in H$. If we define:*

$$\mu_{\mathcal{B}}^{+}(x) = \begin{cases} 0.8 & \text{if } a, b, c, d \in 2\mathbb{Z}^{+} \\ 0.3 & \text{otherwise} \end{cases} \quad \text{and} \quad \mu_{\mathcal{B}}^{-}(x) = \begin{cases} -0.9 & \text{if } a, b, c, d \in 2\mathbb{Z}^{+} \\ -0.5 & \text{otherwise} \end{cases}$$

It is easy to see that $\mathcal{B}$ is a bipolar fuzzy interior Γ-hyperideal of H.

Definition 151 *Let H be a Γ-semihypergroup. A bipolar fuzzy sub Γ-semihypergroup $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \right\rangle$ of H is called a bipolar fuzzy $(1,2)$-Γ-hyperideal of H if*

$$\begin{aligned} \inf_{a \in x\alpha w\beta(y\gamma z)} \mu_{\mathcal{B}}^{+}(a) &\geq \min\{\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(y), \mu_{\mathcal{B}}^{+}(z)\} \\ \text{and} \quad \sup_{a \in x\alpha w\beta(y\gamma z)} \mu_{\mathcal{B}}^{-}(a) &\leq \max\{\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(y), \mu_{\mathcal{B}}^{-}(z)\}, \end{aligned}$$

for all $w, x, y, z \in H$ and $\alpha, \beta, \gamma \in \Gamma$.

Example 152 *Let $H = \{a, b, c, d, e\}$ and $\Gamma = \{\gamma, \beta\}$ be the sets of binary hyperoperations defined in Example 148. If we define a bipolar fuzzy set $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \right\rangle$ on H as:*

$$\mu_{\mathcal{B}}^{+}(x) = \begin{cases} 0.2 & \text{if } x = a, b, e \\ 0.7 & \text{if } x = c, d \end{cases} \quad \text{and} \quad \mu_{\mathcal{B}}^{-}(x) = \begin{cases} -0.5 & \text{if } x = a, b, e \\ -0.7 & \text{if } x = c, d \end{cases}$$

Then by routine calculations, $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \right\rangle$ is a bipolar fuzzy $(1,2)$-Γ-hyperideal of H.

Theorem 153 *Let $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \right\rangle$ be a bipolar fuzzy subset of H such that the least upper bound t_0 of $Im(\mu_{\mathcal{B}}^{+})$ and the greatest lower bound s_0 of $Im(\mu_{\mathcal{B}}^{-})$ exist. Then the following conditions are equivalent:*

1. *$\mathcal{B}$ is a bipolar fuzzy sub Γ-semihypergroup of H,*

2. *For all* $(s,t) \in Im(\mu_{\mathcal{B}}^{-}) \times Im(\mu_{\mathcal{B}}^{+})$, *the non-empty level subset* $H_{\mathcal{B}}^{(t,s)}$ *of* $\mathcal{B}$ *is a sub* Γ*-semihypergroup of* H.

3. *For all* $(s,t) \in Im(\mu_{\mathcal{B}}^{-}) \times Im(\mu_{\mathcal{B}}^{+}) \setminus (s_0, t_0)$, *the non-empty strong level subset* ${}^{S}H_{\mathcal{B}}^{(t,s)}$ *of* $\mathcal{B}$ *is a sub* Γ*-semihypergroup of* H.

4. *For all* $(s,t) \in [-1,0] \times [0,1]$, *the non-empty strong level subset* ${}^{S}H_{\mathcal{B}}^{(t,s)}$ *of* $\mathcal{B}$ *is a sub* Γ*-semihypergroup of* H.

5. *For all* $(s,t) \in [-1,0] \times [0,1]$, *the non-empty level subset* $H_{\mathcal{B}}^{(t,s)}$ *of* $\mathcal{B}$ *is a sub* Γ*-semihypergroup of* H.

Proof. $(1 \to 4)$ Let $\mathcal{B}$ be a bipolar fuzzy sub Γ-semihypergroup of H, $(s,t) \in [-1,0] \times [0,1]$ and $x, y \in {}^{S}H_{\mathcal{B}}^{(t,s)}$. Then we have $\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(y) > t$ and $\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(y) < s$. Thus,

$$\min\{\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(y)\} > t \quad \text{and} \quad \max\left\{\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(y)\right\} < s.$$

Since $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ is a sub Γ-semihypergroup of H, so for $\gamma \in \Gamma$, $\inf\limits_{z \in x\gamma y} \mu_{\mathcal{B}}^{+}(z) > t$ and $\sup\limits_{z \in x\gamma y} \mu_{\mathcal{B}}^{-}(z) < s$. Thus $x\gamma y \subseteq {}^{S}H_{\mathcal{B}}^{(t,s)}$. Hence ${}^{S}H_{\mathcal{B}}^{(t,s)}$ is a sub Γ-semihypergroup of H.

$(4 \to 3)$ It is clear.

$(3 \to 2)$ Let $(s,t) \in Im(\mu_{\mathcal{B}}^{-}) \times Im(\mu_{\mathcal{B}}^{+})$. Then $H_{\mathcal{B}}^{(t,s)}$ is non-empty. Since $H_{\mathcal{B}}^{(t,s)} = \bigcap\limits_{t>\beta, s<\alpha}^{H} H_{\mathcal{B}}^{(\beta,\alpha)}$, where $\beta \in Im(\mu_{\mathcal{B}}^{+}) \setminus s_0$ and $\alpha \in Im(\mu_{\mathcal{B}}^{-}) \setminus t_0$. Then by (3) we get that $H_{\mathcal{B}}^{(t,s)}$ is a sub Γ-semihypergroup of H.

$(2 \to 5)$ Let $(s,t) \in [-1,0] \times [0,1]$ and $H_{\mathcal{B}}^{(t,s)}$ be non-empty. Suppose that $x, y \in H_{\mathcal{B}}^{(t,s)}$. Then we have $\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(y) \geq t$ and $\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(y) \leq s$. Let $\alpha = \min\{\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(y)\}$ and $\beta = \max\{\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(y)\}$. It is clear that $\alpha \geq t$ and $\beta \leq s$. Thus $x, y \in H_{\mathcal{B}}^{(t,s)}$ and $\alpha \in Im(\mu_{\mathcal{B}}^{+})$ and $\beta \in Im(\mu_{\mathcal{B}}^{-})$, by (2) $H_{\mathcal{B}}^{(\alpha,\beta)}$ is a sub Γ-semihypergroup of H, hence for all $\gamma \in \Gamma$, $x\gamma y \subseteq H_{\mathcal{B}}^{(\alpha,\beta)}$. Then we have

$$\inf_{z \in x\gamma y} \mu_{\mathcal{B}}^{+}(z) \geq \alpha \geq s \quad \text{and} \quad \sup_{z \in x\gamma y} \mu_{\mathcal{B}}^{-}(z) \leq \beta \leq t.$$

Therefore $x\gamma y \subseteq H_{\mathcal{B}}^{(t,s)}$. Then $H_{\mathcal{B}}^{(t,s)}$ is a sub Γ-semihypergroup of H.

$(5 \to 1)$ Assume that the non-empty set $H_{\mathcal{B}}^{(t,s)}$ is a sub Γ-semihypergroup of H, for any $(s,t) \in [-1,0] \times [0,1]$. Let $x, y \in H$. Let us take $t = \min\{\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(y)\}$ and $s = \max\{\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(y)\}$. Then $\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(y) \geq t$ and $\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(y) \leq s$. Thus

$x, y \in H_{\mathcal{B}}^{(t,s)}$. Since $H_{\mathcal{B}}^{(t,s)}$ is a sub Γ-semihypergroup of H, so for $\gamma \in \Gamma$, $x\gamma y \subseteq H_{\mathcal{B}}^{(t,s)}$. Thus,

$$\inf_{z \in x\gamma y} \mu_{\mathcal{B}}^{+}(z) \geq t = \min\{\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(y)\} \text{ and } \sup_{z \in x\gamma y} \mu_{\mathcal{B}}^{-}(z) \leq s = \max\{\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(y)\}.$$

This shows that $\mathcal{B}$ is a bipolar fuzzy sub Γ-semihypergroup of H. This completes the proof. ■

Theorem 154 *Let $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ be a bipolar fuzzy subset of H such that the least upper bound t_0 of $Im(\mu_{\mathcal{B}}^{+})$ and the greatest lower bound s_0 of $Im(\mu_{\mathcal{B}}^{-})$ exist. Then the following conditions are equivalent:*

1. *$\mathcal{B}$ is a bipolar fuzzy Γ-hyperideal (resp., left Γ-hyperideal, right Γ-hyperideal, bi-Γ-hyperideal, interior Γ-hyperideal) of H,*

2. *For all $(s, t) \in Im(\mu_{\mathcal{B}}^{-}) \times Im(\mu_{\mathcal{B}}^{+})$, the non-empty level subset $H_{\mathcal{B}}^{(t,s)}$ of $\mathcal{B}$ is a Γ-hyperideal (resp., left Γ-hyperideal, right Γ-hyperideal, bi-Γ-hyperideal, interior Γ-hyperideal) of H.*

3. *For all $(s, t) \in Im(\mu_{\mathcal{B}}^{-}) \times Im(\mu_{\mathcal{B}}^{+}) \setminus (s_0, t_0)$, the non-empty strong level subset ${}^{S}H_{\mathcal{B}}^{(t,s)}$ of $\mathcal{B}$ is a Γ-hyperideal (resp., left Γ-hyperideal, right Γ-hyperideal, bi-Γ-hyperideal, interior Γ-hyperideal) of H.*

4. *For all $(s, t) \in [-1, 0] \times [0, 1]$, the non-empty strong level subset ${}^{S}H_{\mathcal{B}}^{(t,s)}$ of $\mathcal{B}$ is a Γ-hyperideal (resp., left Γ-hyperideal, right Γ-hyperideal, bi-Γ-hyperideal, interior Γ-hyperideal) of H.*

5. *For all $(s, t) \in [-1, 0] \times [0, 1]$, the non-empty level subset $H_{\mathcal{B}}^{(t,s)}$ of $\mathcal{B}$ is a Γ-hyperideal (resp., left Γ-hyperideal, right Γ-hyperideal, bi-Γ-hyperideal, interior Γ-hyperideal) of H.*

Proof. The proof is similar to the proof of Theorem 153. ■

Definition 155 *Let X be a non-empty set. For any $A \subseteq X$ and $(q^{-}, r^{+}) \in [-1, 0) \times (0, 1]$, the bipolar fuzzy subset $A^{(r^{+}, q^{-})} = \langle r_A^{+}, q_A^{-} \rangle$ of X is defined by*

$$r_A^{+} : H \longrightarrow (0, 1] | A \longmapsto r_A^{+}(x) := \begin{cases} r^{+} & \text{if } x \in A \\ 0 & \text{if } x \notin A, \end{cases}$$

and

$$q_A^- : H \longrightarrow [-1,0)|A \longmapsto q_A^-(x) := \begin{cases} q^- & \text{if } x \in A \\ 0 & \text{if } x \notin A, \end{cases}$$

for all $x \in X$. *In particular, when* $r^+ = 1$ *and* $q^- = -1$, $A^{(r^+,q^-)}$ *is said to be the characteristic function of* A, *denoted by* $\chi_A = \left\langle \mu^+_{\chi_A}, \mu^-_{\chi_A} \right\rangle$. *When* $A = \{x\}$, $A^{(r^+,q^-)}$ *is said to be a bipolar fuzzy point with support* x *and values* r^+ *and* q^- *and is denoted by* $x_{(r^+,q^-)}$.

Theorem 156 *Let* A *be a non-empty subset of a* Γ*-semihypergroup* H. *Then* A *is a sub* Γ*-semihypergroup (resp.,* Γ*-hyperideal, left* Γ*-hyperideal, right* Γ*-hyperideal, bi-*Γ*-hyperideal, interior* Γ*-hyperideal and* $(1,2)$*-*Γ*-hyperideal) of* H *if and only if* $A^{(r^+,q^-)} = \left\langle r_A^+, q_A^- \right\rangle$ *is a bipolar fuzzy sub* Γ*-semihypergroup (resp.,* Γ*-hyperideal, left* Γ*-hyperideal, right* Γ*-hyperideal, bi-*Γ*-hyperideal, interior* Γ*-hyperideal and* $(1,2)$*-*Γ*-hyperideal) of* H.

Proof. Let A be a sub Γ-semihypergroup of H. For any $x, y \in H$, we have the following cases:

Case (1) : If $x, y \in A$. Since A is a sub Γ-semihypergroup of H, for all $\gamma \in \Gamma$, we have $x\gamma y \subseteq A$. Then $r_A^+(x) = r^+$ and $r_A^+(y) = r^+$. Therefore $\inf\limits_{z \in x\gamma z} r_A^+(z) = r^+ = \min\{r_A^+(x), r_A^+(y)\}$. And $q_A^-(x) = q^-$ and $q_A^-(y) = q^-$. Therefore $\sup\limits_{z \in x\gamma z} q_A^-(z) = q^- = \max\{q_A^-(x), q_A^-(y)\}$.

Case (2) : If $x, y \notin A$. Then $r_A^+(x) = 0$ and $r_A^+(y) = 0$. Therefore $\inf\limits_{z \in x\gamma z} r_A^+(z) \geq 0 = \min\{r_A^+(x), r_A^+(y)\}$. And $q_A^-(x) = 0$ and $q_A^-(y) = 0$. Therefore $\sup\limits_{z \in x\gamma z} q_A^-(z) \leq 0 = \max\{q_A^-(x), q_A^-(y)\}$.

Case (3) : If $x \in A$ or $y \in A$. Then $\inf\limits_{z \in x\gamma z} r_A^+(z) \geq 0 = \min\{r_A^+(x), r_A^+(y)\}$. And $\sup\limits_{z \in x\gamma z} q_A^-(z) \leq 0 = \max\{q_A^-(x), q_A^-(y)\}$.

Hence $A^{(r^+,q^-)} = \left\langle r_A^+, q_A^- \right\rangle$ is a bipolar fuzzy sub Γ-semihypergroup of H.

Conversely, suppose that $A^{(r^+,q^-)} = \left\langle r_A^+, q_A^- \right\rangle$ is a bipolar fuzzy sub Γ-semihypergroup of H. Let $x, y \in A$ and $\gamma \in \Gamma$. Then we have

$$\begin{aligned} \inf_{z \in x\gamma y} r_A^+(z) &\geq \min\{r_A^+(x), r_A^+(y)\} = r^+ \wedge r^+ = r^+ \\ \inf_{z \in x\gamma y} r_A^+(z) &\geq r^+, \text{ but } \inf_{z \in x\gamma y} r_A^+(z) \leq r^+ \\ \inf_{z \in x\gamma y} r_A^+(z) &= r^+, \end{aligned}$$

and

$$\begin{aligned}\sup_{z\in x\gamma y} q_A^-(z) &\leq \max\{q_A^-(x), q_A^-(y)\} = q^- \vee q^- = q^- \\ \sup_{z\in x\gamma y} q_A^-(z) &\leq q^-, \text{ but } \sup_{z\in x\gamma y} q_A^-(z) \geq q^- \\ \sup_{z\in x\gamma y} q_A^-(z) &= q^-.\end{aligned}$$

Hence $x\gamma y \subseteq A$. Therefore A is a sub Γ-semihypergroup of H. The other cases can be seen in a similar way. ■

Corollary 157 *Let A be a non-empty subset of a Γ-semihypergroup H. Then A is a sub Γ-semihypergroup (resp., Γ-hyperideal, left Γ-hyperideal, right Γ-hyperideal, bi-Γ-hyperideal, interior Γ-hyperideal and $(1,2)$-Γ-hyperideal) of H if and only if $\chi_A = \left\langle \mu^+_{\chi_A}, \mu^-_{\chi_A} \right\rangle$ is a bipolar fuzzy sub Γ-semihypergroup (resp., Γ-hyperideal, left Γ-hyperideal, right Γ-hyperideal, bi-Γ-hyperideal, interior Γ-hyperideal and $(1,2)$-Γ-hyperideal) of H.*

Proof. The proof is straightforward. ■

Theorem 158 *A bipolar fuzzy subset $\mathcal{B} = \left\langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \right\rangle$ of a Γ-semihypergroups H is a bipolar fuzzy*

1. *sub Γ-semihypergroup of H if and only if $\mathcal{B} \circ_\Gamma \mathcal{B} \subseteq \mathcal{B}$,*
2. *left Γ-hyperideal of H if and only if $\mathcal{H} \circ_\Gamma \mathcal{B} \subseteq \mathcal{B}$,*
3. *right Γ-hyperideal of H if and only if $\mathcal{B} \circ_\Gamma \mathcal{H} \subseteq \mathcal{B}$,*
4. *Γ-hyperideal of H if and only if $\mathcal{H} \circ_\Gamma \mathcal{B} \subseteq \mathcal{B}$ and $\mathcal{B} \circ_\Gamma \mathcal{H} \subseteq \mathcal{B}$,*
5. *bi-Γ-hyperideal of H if and only if $\mathcal{B} \circ_\Gamma \mathcal{B} \subseteq \mathcal{B}$ and $\mathcal{B} \circ_\Gamma \mathcal{H} \circ_\Gamma \mathcal{B} \subseteq \mathcal{B}$,*
6. *interior Γ-hyperideal of H if and only if $\mathcal{H} \circ_\Gamma \mathcal{B} \circ_\Gamma \mathcal{H} \subseteq \mathcal{B}$,*
7. *$(1,2)$-Γ-hyperideal of H if and only if $\mathcal{B} \circ_\Gamma \mathcal{B} \subseteq \mathcal{B}$ and $\mathcal{B} \circ_\Gamma \mathcal{H} \circ_\Gamma \mathcal{B} \circ_\Gamma \mathcal{B} \subseteq \mathcal{B}$,*

where $\mathcal{H} = \left\langle \mu^+_{\mathcal{H}}, \mu^-_{\mathcal{H}} \right\rangle$, such that $\mu^+_{\mathcal{H}}(x) = 1$ and $\mu^-_{\mathcal{H}}(x) = -1$ for all $x \in H$.

Proof. (1) Let $\mathcal{B} = \left\langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \right\rangle$ be a bipolar fuzzy sub Γ-semihypergroup of H and $z \in H$. Let us suppose that $z \in x\gamma y$ for $x, y \in H$ and $\gamma \in \Gamma$. Then

$$\begin{aligned}\mu^+_{\mathcal{B}\circ_\Gamma\mathcal{B}}(z) &= \bigvee_{z\in x\gamma y} \left\{\mu^+_{\mathcal{B}}(x) \wedge \mu^+_{\mathcal{B}}(y)\right\} \leq \bigvee_{z\in x\gamma y} \left\{\inf_{z\in x\gamma y} \mu^+_{\mathcal{B}}(z)\right\} \\ &\leq \bigvee_{z\in x\gamma y} \left\{\mu^+_{\mathcal{B}}(x\gamma y)\right\} = \mu^+_{\mathcal{B}}(z),\end{aligned}$$

and

$$\begin{aligned}\mu^{-}_{\mathcal{B}\circ_\Gamma\mathcal{B}}(z) &= \bigwedge_{z\in x\gamma y}\left\{\mu^{-}_{\mathcal{B}}(x)\vee\mu^{-}_{\mathcal{B}}(y)\right\}\geq\bigwedge_{z\in x\gamma y}\left\{\sup_{z\in x\gamma y}\mu^{-}_{\mathcal{B}}(z)\right\}\\ &\geq \bigwedge_{z\in x\gamma y}\left\{\mu^{-}_{\mathcal{B}}(x\gamma y)\right\}=\mu^{-}_{\mathcal{B}}(z).\end{aligned}$$

Therefore $\mu^{+}_{\mathcal{B}\circ_\Gamma\mathcal{B}}\subseteq\mu^{+}_{\mathcal{B}}$ and $\mu^{-}_{\mathcal{B}\circ_\Gamma\mathcal{B}}\supseteq\mu^{-}_{\mathcal{B}}$. If there do not exist any $x,y\in H$ and $\gamma\in\Gamma$ such that $z\in x\gamma y$, then

$$\mu^{+}_{\mathcal{B}\circ_\Gamma\mathcal{B}}(z)=0\leq\mu^{+}_{\mathcal{B}}(z)\ \text{ and }\ \mu^{-}_{\mathcal{B}\circ_\Gamma\mathcal{B}}(z)=0\geq\mu^{-}_{\mathcal{B}}(z).$$

Hence for all cases $\mathcal{B}\circ_\Gamma\mathcal{B}\subseteq\mathcal{B}$.

Conversely, let us assume that $\mathcal{B}\circ_\Gamma\mathcal{B}\subseteq\mathcal{B}$ holds for all bipolar fuzzy subsets of H. Let $x,y\in H$ and $\gamma\in\Gamma$. Then, we have

$$\mu^{+}_{\mathcal{B}}(x\gamma y)\geq\mu^{+}_{\mathcal{B}\circ_\Gamma\mathcal{B}}(x\gamma y)\ \text{ and }\ \mu^{-}_{\mathcal{B}}(x\gamma y)\leq\mu^{-}_{\mathcal{B}\circ_\Gamma\mathcal{B}}(x\gamma y).$$

If there exist $p,q\in H$ and $\beta\in\Gamma$ such that $x\gamma y\subseteq p\beta q$, then

$$\begin{aligned}\mu^{+}_{\mathcal{B}}(x\gamma y) &\geq \mu^{+}_{\mathcal{B}\circ_\Gamma\mathcal{B}}(x\gamma y)=\bigvee_{x\gamma y\subseteq p\beta q}\left\{\mu^{+}_{\mathcal{B}}(p)\wedge\mu^{+}_{\mathcal{B}}(q)\right\}\\ &\geq \mu^{+}_{\mathcal{B}}(x)\wedge\mu^{+}_{\mathcal{B}}(y).\\ \inf_{z\in x\gamma y}\mu^{+}_{\mathcal{B}}(z) &\geq \min\{\mu^{+}_{\mathcal{B}}(x),\mu^{+}_{\mathcal{B}}(y)\},\end{aligned}$$

and

$$\begin{aligned}\mu^{-}_{\mathcal{B}}(x\gamma y) &\leq \mu^{-}_{\mathcal{B}\circ_\Gamma\mathcal{B}}(x\gamma y)=\bigwedge_{x\gamma y\subseteq p\beta q}\left\{\mu^{-}_{\mathcal{B}}(p)\vee\mu^{-}_{\mathcal{B}}(q)\right\}\\ &\leq \mu^{-}_{\mathcal{B}}(x)\vee\mu^{-}_{\mathcal{B}}(y).\\ \sup_{z\in x\gamma y}\mu^{-}_{\mathcal{B}}(z) &\leq \max\{\mu^{-}_{\mathcal{B}}(x),\mu^{-}_{\mathcal{B}}(y)\}.\end{aligned}$$

This means that $\mathcal{B}=\left\langle\mu^{+}_{\mathcal{B}},\mu^{-}_{\mathcal{B}}\right\rangle$ is a bipolar fuzzy sub Γ-semihypergroup of H. The other cases can be seen in a similar way. ■

Theorem 159 *If $\{\mathcal{B}_i\}_{i\in I}$ is a family of bipolar fuzzy sub Γ-semihypergroups (resp., left Γ-hyperideals, right Γ-hyperideals, Γ-hyperideals, bi-Γ-hyperideals, interior Γ-hyperideals and (1,2)-Γ-hyperideals) of H. Then $\bigcap_{i\in I}\mathcal{B}_i$ is a bipolar fuzzy sub Γ-semihypergroup (resp., left Γ-hyperideal, right Γ-hyperideal, Γ-hyperideal, bi-Γ-hyperideal, interior Γ-hyperideal and (1,2)-Γ-hyperideal) of H, where $\bigcap_{i\in I}\mathcal{B}_i=(\bigwedge_{i\in I}\mu^{+}_{\mathcal{B}_i},\bigvee_{i\in I}\mu^{-}_{\mathcal{B}_i})$ and*

$$\begin{aligned}\bigwedge_{i\in I}\mu^{+}_{\mathcal{B}_i} &: H\longrightarrow[0,1]|A\longmapsto\bigwedge_{i\in I}\mu^{+}_{\mathcal{B}_i}(x):=\inf_{i\in I}\left\{\mu^{+}_{\mathcal{B}_i}(x):\ x\in H\right\},\\ \bigvee_{i\in I}\mu^{-}_{\mathcal{B}_i} &: H\longrightarrow[-1,0]|A\longmapsto\bigvee_{i\in I}\mu^{-}_{\mathcal{B}_i}(x):=\sup_{i\in I}\left\{\mu^{-}_{\mathcal{B}_i}(x):\ x\in H\right\}.\end{aligned}$$

Proof. Consider $\{\mathcal{B}_i\}_{i\in I}$ is a family of bipolar fuzzy sub Γ-semihypergroups of H. Let $x, y, z \in H$. Then for every $\gamma \in \Gamma$ and $z \in x\gamma y$, we have

$$\begin{aligned}\inf_{z\in x\gamma y}\left\{\bigwedge_{i\in I}\mu^{+}_{\mathcal{B}_i}(z)\right\} &= \bigwedge_{i\in I}\left\{\inf_{z\in x\gamma y}\left\{\mu^{+}_{\mathcal{B}_i}(z)\right\}\right\} \geq \bigwedge_{i\in I}\left\{\min\left\{\mu^{+}_{\mathcal{B}_i}(x), \mu^{+}_{\mathcal{B}_i}(y)\right\}\right\} \\ &= \min\left\{\bigwedge_{i\in I}\mu^{+}_{\mathcal{B}_i}(x), \bigwedge_{i\in I}\mu^{+}_{\mathcal{B}_i}(y)\right\},\end{aligned}$$

and

$$\begin{aligned}\sup_{z\in x\gamma y}\left\{\bigvee_{i\in I}\mu^{-}_{\mathcal{B}_i}(z)\right\} &= \bigvee_{i\in I}\left\{\sup_{z\in x\gamma y}\left\{\mu^{-}_{\mathcal{B}_i}(z)\right\}\right\} \leq \bigvee_{i\in I}\left\{\max\left\{\mu^{-}_{\mathcal{B}_i}(x), \mu^{-}_{\mathcal{B}_i}(y)\right\}\right\} \\ &= \max\left\{\bigvee_{i\in I}\mu^{-}_{\mathcal{B}_i}(x), \bigvee_{i\in I}\mu^{-}_{\mathcal{B}_i}(y)\right\}.\end{aligned}$$

Hence this shows that $\bigcap_{i\in I}\mathcal{B}_i$ is a bipolar fuzzy sub Γ-semihypergroup of H. The other cases can be seen in a similar way. ■

Proposition 160 *Let $\mathcal{B} = \langle\mu^{+}_{\mathcal{B}}, \mu^{-}_{\mathcal{B}}\rangle$ be a bipolar fuzzy bi-Γ-hyperideal and $\mathcal{A} = \langle\mu^{+}_{\mathcal{A}}, \mu^{-}_{\mathcal{A}}\rangle$ be a bipolar fuzzy sub Γ-semihypergroup of H. Then, $\mathcal{A} \cap \mathcal{B}$ is a bipolar fuzzy bi-Γ-hyperideal of H.*

Proof. The proof is straightforward. ■

Proposition 161 *Let $\mathcal{A} = \langle\mu^{+}_{\mathcal{A}}, \mu^{-}_{\mathcal{A}}\rangle$ be a bipolar fuzzy right Γ-hyperideal of H and $\mathcal{B} = \langle\mu^{+}_{\mathcal{B}}, \mu^{-}_{\mathcal{B}}\rangle$ be a bipolar fuzzy left Γ-hyperideal of H. Then $\mathcal{A} \circ_{\Gamma} \mathcal{B} \subseteq \mathcal{A} \cap \mathcal{B}$.*

Proof. The proof is straightforward. ■

Theorem 162 *Let $\mathcal{A} = \langle\mu^{+}_{\mathcal{A}}, \mu^{-}_{\mathcal{A}}\rangle$ be a bipolar fuzzy right Γ-hyperideal and $\mathcal{B} = \langle\mu^{+}_{\mathcal{B}}, \mu^{-}_{\mathcal{B}}\rangle$ be a bipolar fuzzy left Γ-hyperideal of a Γ-semihypergroup H. Then $\mathcal{A} \circ_{\Gamma} \mathcal{H} \circ_{\Gamma} \mathcal{B} \subseteq \mathcal{A} \cap \mathcal{B}$.*

Proof. The proof is straightforward. ■

Proposition 163 *Let H be a regular Γ-semihypergroup and $\mathcal{A} = \langle\mu^{+}_{\mathcal{A}}, \mu^{-}_{\mathcal{A}}\rangle$, $\mathcal{B} = \langle\mu^{+}_{\mathcal{B}}, \mu^{-}_{\mathcal{B}}\rangle$ be two bipolar fuzzy sets in H. Then $\mathcal{A} \circ_{\Gamma} \mathcal{B} \supseteq \mathcal{A} \cap \mathcal{B}$.*

Proof. Let $a \in H$. Since H is regular, then there exist an element $x \in H$ and $\alpha, \beta \in \Gamma$ such that $a \in a\alpha x\beta a \subseteq a\gamma a$ for some $\gamma \in \Gamma$. Then,

$$\mu^+_{\mathcal{A}\circ_\Gamma\mathcal{B}}(a) = \sup_{a\in a\gamma a}\{\mu^+_{\mathcal{A}}(a) \wedge \mu^+_{\mathcal{B}}(a)\} \geq \mu^+_{\mathcal{A}}(a) \wedge \mu^+_{\mathcal{B}}(a) = \mu^+_{\mathcal{A}\cap\mathcal{B}}(a),$$

and

$$\mu^-_{\mathcal{A}\circ_\Gamma\mathcal{B}}(a) = \inf_{a\in a\gamma a}\{\mu^-_{\mathcal{A}}(a) \vee \mu^-_{\mathcal{B}}(a)\} \leq \mu^-_{\mathcal{A}}(a) \vee \mu^-_{\mathcal{B}}(a) = \mu^-_{\mathcal{A}\cap\mathcal{B}}(a).$$

Hence $\mathcal{A} \circ_\Gamma \mathcal{B} \supseteq \mathcal{A} \cap \mathcal{B}$. ■

Theorem 164 *Let H be a Γ-semihypergroup. The following statements are equivalent:*

1. *H is regular*
2. *$\mathcal{A} \circ_\Gamma \mathcal{B} = \mathcal{A} \cap \mathcal{B}$, where $\mathcal{A} = \langle \mu^+_{\mathcal{A}}, \mu^-_{\mathcal{A}} \rangle$ is a bipolar fuzzy right Γ-hyperideal of H and $\mathcal{B} = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \rangle$ is a bipolar fuzzy left Γ-hyperideal of H.*

Proof. The proof is straightforward. ■

Theorem 165 *Let H be a regular Γ-semihypergroup. Then the following statements hold:*

1. *$\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \circ_\Gamma \mathcal{B}$, for every bipolar fuzzy bi-Γ-hyperideal $\mathcal{B}$ and bipolar fuzzy right Γ-hyperideal $\mathcal{A}$ of H,*
2. *$\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \circ_\Gamma \mathcal{B}$, for every bipolar fuzzy bi-Γ-hyperideal $\mathcal{A}$ and bipolar fuzzy left Γ-hyperideal $\mathcal{B}$ of H.*
3. *$\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} \subseteq \mathcal{A} \circ_\Gamma \mathcal{B} \circ_\Gamma \mathcal{C}$, for every bipolar fuzzy right Γ-hyperideal $\mathcal{A}$, bipolar fuzzy bi-Γ-hyperideal $\mathcal{B}$ and bipolar fuzzy left Γ-hyperideal $\mathcal{C}$ of H respectively.*

Proof. (1) Let H be regular and $a \in H$. Then there exist $x \in H$ and $\alpha, \beta \in \Gamma$ such that $a \in a\alpha x\beta a$. Then

$$\begin{aligned}\mu^+_{\mathcal{A}\circ_\Gamma\mathcal{B}}(a) &= \sup_{a\in(a\alpha x)\beta a} \min\left\{\inf_{t\in a\alpha x} \mu^+_{\mathcal{A}}(t), \mu^+_{\mathcal{B}}(a)\right\} \geq \min\left\{\inf_{t\in a\alpha x} \mu^+_{\mathcal{A}}(t), \mu^+_{\mathcal{B}}(a)\right\} \\ &\geq \min\{\mu^+_{\mathcal{A}}(a), \mu^+_{\mathcal{B}}(a)\} = (\mu^+_{\mathcal{A}} \wedge \mu^+_{\mathcal{B}})(a) = \mu^+_{\mathcal{A}\cap\mathcal{B}}(a),\end{aligned}$$

and hence $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \circ_\Gamma \mathcal{B}$.

(2) The proof is similar to (1).

(3) Let us suppose that H is regular and $a \in H$. Then there exist $x \in H$ and $\alpha, \beta \in \Gamma$ such that $a \in a\alpha x\beta a$. Then

$$\begin{aligned}
\mu^{+}_{\mathcal{A}\circ_\Gamma\mathcal{B}\circ_\Gamma\mathcal{C}}(a) &= \sup_{a\in a\alpha x\beta a} \min\left\{\inf_{t\in a\alpha x} \mu^{+}_{\mathcal{A}}(t), (\mu^{+}_{\mathcal{B}} \circ_\Gamma \mu^{+}_{\mathcal{C}})(a)\right\} \\
&\geq \min\left\{\inf_{t\in a\alpha x} \mu^{+}_{\mathcal{A}}(t), (\mu^{+}_{\mathcal{B}} \circ_\Gamma \mu^{+}_{\mathcal{C}})(a)\right\} \\
&\geq \min\left\{\mu^{+}_{\mathcal{A}}(a), \sup_{a\in a\alpha x\beta a} \min\left\{\mu^{+}_{\mathcal{B}}(a), \inf_{h\in x\beta a} \mu^{+}_{\mathcal{C}}(h)\right\}\right\} \\
&\geq \min\left\{\mu^{+}_{\mathcal{A}}(a), \mu^{+}_{\mathcal{B}}(a), \inf_{h\in x\beta a} \mu^{+}_{\mathcal{C}}(h)\right\} \geq \min\left\{\mu^{+}_{\mathcal{A}}(a), \mu^{+}_{\mathcal{B}}(a), \mu^{+}_{\mathcal{C}}(a)\right\} \\
&= (\mu^{+}_{\mathcal{A}} \wedge \mu^{+}_{\mathcal{B}} \wedge \mu^{+}_{\mathcal{C}})(a) = \mu^{+}_{\mathcal{A}\cap\mathcal{B}\cap\mathcal{C}}(a),
\end{aligned}$$

and

$$\begin{aligned}
\mu^{-}_{\mathcal{A}\circ_\Gamma\mathcal{B}\circ_\Gamma\mathcal{C}}(a) &= \inf_{a\in a\alpha x\beta a} \max\left\{\sup_{t\in a\alpha x} \mu^{-}_{\mathcal{A}}(t), (\mu^{-}_{\mathcal{B}} \circ_\Gamma \mu^{-}_{\mathcal{C}})(a)\right\} \\
&\leq \max\left\{\sup_{t\in a\alpha x} \mu^{-}_{\mathcal{A}}(t), (\mu^{-}_{\mathcal{B}} \circ_\Gamma \mu^{-}_{\mathcal{C}})(a)\right\} \\
&\leq \max\left\{\mu^{-}_{\mathcal{A}}(a), \inf_{a\in a\alpha x\beta a} \max\left\{\mu^{-}_{\mathcal{B}}(a), \sup_{h\in x\beta a} \mu^{-}_{\mathcal{C}}(h)\right\}\right\} \\
&\leq \max\left\{\mu^{-}_{\mathcal{A}}(a), \mu^{-}_{\mathcal{B}}(a), \sup_{h\in x\beta a} \mu^{-}_{\mathcal{C}}(h)\right\} \leq \max\left\{\mu^{-}_{\mathcal{A}}(a), \mu^{-}_{\mathcal{B}}(a), \mu^{-}_{\mathcal{C}}(a)\right\} \\
&= (\mu^{-}_{\mathcal{A}} \vee \mu^{-}_{\mathcal{B}} \vee \mu^{-}_{\mathcal{C}})(a) = \mu^{-}_{\mathcal{A}\cap\mathcal{B}\cap\mathcal{C}}(a).
\end{aligned}$$

Hence $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} \subseteq \mathcal{A} \circ_\Gamma \mathcal{B} \circ_\Gamma \mathcal{C}$. ■

Theorem 166 *Let H be a Γ-semihypergroup. Then every bipolar fuzzy bi-Γ-hyperideal of H is a bipolar fuzzy $(1,2)$-Γ-hyperideal of H.*

Proof. Let $\mathcal{B} = \langle \mu^{+}_{\mathcal{B}}, \mu^{-}_{\mathcal{B}} \rangle$ be a bipolar fuzzy bi-Γ-hyperideal of H and let $w, x, y, z \in H$. Then for all $a \in x\alpha w\beta(y\gamma z)$ and $\alpha, \beta, \gamma \in \Gamma$, we have

$$\begin{aligned}
\inf_{a\in x\alpha w\beta(y\gamma z)} \mu^{+}_{\mathcal{B}}(a) &= \inf_{a\in(x\alpha w\beta y)\gamma z} \mu^{+}_{\mathcal{B}}(a) = \inf_{c\in x\alpha w\beta y} \inf_{a\in c\gamma z} \mu^{+}_{\mathcal{B}}(a), \text{ for every } c \in x\alpha w\beta y \\
&\geq \inf_{c\in x\alpha w\beta y} \{\min\{\mu^{+}_{\mathcal{B}}(c), \mu^{+}_{\mathcal{B}}(z)\}\} = \min\left\{\inf_{c\in x\alpha w\beta y} \mu^{+}_{\mathcal{B}}(c), \mu^{+}_{\mathcal{B}}(z)\right\} \\
&\geq \min\{\min\{\mu^{+}_{\mathcal{B}}(x), \mu^{+}_{\mathcal{B}}(y)\}, \mu^{+}_{\mathcal{B}}(z)\} = \min\{\mu^{+}_{\mathcal{B}}(x), \mu^{+}_{\mathcal{B}}(y), \mu^{+}_{\mathcal{B}}(z)\},
\end{aligned}$$

and

$$\begin{aligned}\sup_{a\in x\alpha w\beta(y\gamma z)}\mu_{\mathcal{B}}^{-}(a) &= \sup_{a\in(x\alpha w\beta y)\gamma z}\mu_{\mathcal{B}}^{-}(a)=\sup_{c\in x\alpha w\beta y}\sup_{a\in c\gamma z}\mu_{\mathcal{B}}^{-}(a),\ \text{for every } c\in x\alpha w\beta y\\ &\leq \sup_{c\in x\alpha w\beta y}\{\max\{\mu_{\mathcal{B}}^{-}(c),\mu_{\mathcal{B}}^{-}(z)\}\}=\max\left\{\sup_{c\in x\alpha w\beta y}\mu_{\mathcal{B}}^{-}(c),\mu_{\mathcal{B}}^{-}(z)\right\}\\ &\leq \max\{\max\{\mu_{\mathcal{B}}^{-}(x),\mu_{\mathcal{B}}^{-}(y)\},\mu_{\mathcal{B}}^{-}(z)\}=\max\{\mu_{\mathcal{B}}^{-}(x),\mu_{\mathcal{B}}^{-}(y),\mu_{\mathcal{B}}^{-}(z)\}.\end{aligned}$$

Hence $\mathcal{B}=\left\langle\mu_{\mathcal{B}}^{+},\mu_{\mathcal{B}}^{-}\right\rangle$ is a bipolar fuzzy $(1,2)$-Γ-hyperideal of H. ■

Theorem 167 *Let H be a regular Γ-semihypergroup. Then every bipolar fuzzy $(1,2)$-Γ-hyperideal of H is a bipolar fuzzy bi-Γ-hyperideal of H.*

Proof. Let us assume that a Γ-semihypergroup H is regular and let $\mathcal{B}=\left\langle\mu_{\mathcal{B}}^{+},\mu_{\mathcal{B}}^{-}\right\rangle$ be a bipolar fuzzy $(1,2)$-Γ-hyperideal of H. Let $w,x,y\in H$ and $\gamma\in\Gamma$. Since H is regular, we have for every $w\in x\gamma y\subseteq(x\alpha a\beta x)\gamma y=x\alpha(a\beta x)\gamma y$ for some $a\in H$ and $\alpha,\beta\in\Gamma$. Thus for every $c\in a\beta x, w\in x\alpha c\gamma y$, we have

$$\inf_{w\in x\alpha c\gamma y\subseteq x\alpha(a\beta x)\gamma y}\mu_{\mathcal{B}}^{+}(w)\geq\min\{\mu_{\mathcal{B}}^{+}(x),\mu_{\mathcal{B}}^{+}(x),\mu_{\mathcal{B}}^{+}(y)\}=\min\{\mu_{\mathcal{B}}^{+}(x),\mu_{\mathcal{B}}^{+}(y)\},$$

and

$$\sup_{w\in x\alpha c\gamma y\subseteq x\alpha(a\beta x)\gamma y}\mu_{\mathcal{B}}^{-}(w)\leq\max\{\mu_{\mathcal{B}}^{-}(x),\mu_{\mathcal{B}}^{-}(x),\mu_{\mathcal{B}}^{-}(y)\}=\max\{\mu_{\mathcal{B}}^{-}(x),\mu_{\mathcal{B}}^{-}(y)\}.$$

Hence $\mathcal{B}=\left\langle\mu_{\mathcal{B}}^{+},\mu_{\mathcal{B}}^{-}\right\rangle$ is a bipolar fuzzy bi-Γ-hyperideal of H. ■

Theorem 168 *Let H be a completely regular Γ-semihypergroup and $\mathcal{B}=\left\langle\mu_{\mathcal{B}}^{+},\mu_{\mathcal{B}}^{-}\right\rangle$ be a bipolar fuzzy bi-Γ-hyperideal of H. Then for every $r\in a\gamma a$, we have $\mathcal{B}(a)=\mathcal{B}(r)$ for all $a\in H$ and $\gamma\in\Gamma$.*

Proof. The proof is straightforward. ■

Theorem 169 *Let H be an intra-regular Γ-semihypergroup and $\mathcal{B}=\left\langle\mu_{\mathcal{B}}^{+},\mu_{\mathcal{B}}^{-}\right\rangle$ be a bipolar fuzzy Γ-hyperideal of H. Then for every $r\in a\gamma a$, we have $\mathcal{B}(a)=\mathcal{B}(r)$ for all $a\in H$ and $\gamma\in\Gamma$.*

Proof. The proof is straightforward. ■

3.2 Γ-Semihypergroups in Terms of Bipolar Fuzzy Points

The results presented here are a part of our published paper [82].

In [61], Pu and Liu introduced the notion of fuzzy points. Kim [43] characterized fuzzy ideals as fuzzy points of semigroups. Sardar et al. [64] defined some relations between the intuitionistic fuzzy ideals of a semigroup S and the set of all intuitionistic fuzzy points of S. In this section, we introduce the concept of bipolar fuzzy points in Γ-semihypergroup and obtained some related results. Here, we consider the Γ-semihypergroup $\underline{\mathcal{H}}$ of the bipolar fuzzy points of a Γ-semihypergroup H to discuss the relation between the bipolar fuzzy sub Γ-semihypergroup (left, right, bi-, interior, $(1,2)$-) Γ-hyperideal and the subsets of $\underline{\mathcal{H}}$ in a (regular) Γ-semihypergroup.

Let $\underline{\mathcal{H}}$ be the set of all bipolar fuzzy points of a Γ-semihypergroup H. Then for all $x_{(a^+,b^-)}, y_{(c^+,d^-)}, z_{(e^+,f^-)} \in \underline{\mathcal{H}}$ and $\gamma, \beta \in \Gamma$, we have

$$x_{(a^+,b^-)} \circ_\Gamma y_{(c^+,d^-)} = \{l_{(a^+\wedge c^+, b^-\vee d^-)} : \text{ for every } l \in x\gamma y\} \subseteq \underline{\mathcal{H}}.$$

Also it satisfies that

$$\begin{aligned}(x_{(a^+,b^-)} \circ_\Gamma y_{(c^+,d^-)}) \circ_\Gamma z_{(e^+,f^-)} &= ((x\gamma y)\beta z)_{(a^+\wedge c^+\wedge e^+, b^-\vee d^-\vee f^-)} \\ &= (x\gamma(y\beta z))_{(a^+\wedge c^+\wedge e^+, b^-\vee d^-\vee f^-)} \\ &= x_{(a^+,b^-)} \circ_\Gamma (y_{(c^+,d^-)} \circ_\Gamma z_{(e^+,f^-)}).\end{aligned}$$

Hence $\underline{\mathcal{H}}$ is a Γ-semihypergroup. For any bipolar fuzzy subset $\mathcal{B} = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \rangle$ of H, $\underline{\mathcal{B}}$ denotes the set of all bipolar fuzzy points contained in $\mathcal{B}$ that is

$$\underline{\mathcal{B}} = \{x_{(a^+,b^-)} \in \underline{\mathcal{H}} : \mu^+_{\mathcal{B}}(x) \geq a^+ \text{ and } \mu^-_{\mathcal{B}}(x) \leq b^-\}.$$

If $x_{(a^+,b^-)} \in \underline{\mathcal{H}}$, then $a^+ > 0$ and $b^- < 0$.

Proposition 170 *Let $\mathcal{A} = \langle \mu^+_{\mathcal{A}}, \mu^-_{\mathcal{A}} \rangle$ and $\mathcal{B} = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \rangle$ be two bipolar fuzzy subsets of a Γ-semihypergroup H. Then*

1. $\underline{\mathcal{A} \cup \mathcal{B}} = \underline{\mathcal{A}} \cup \underline{\mathcal{B}}$,

2. $\underline{\mathcal{A} \cap \mathcal{B}} = \underline{\mathcal{A}} \cap \underline{\mathcal{B}}$,

3. $\underline{\mathcal{A} \circ_\Gamma \mathcal{B}} \supseteq \underline{\mathcal{A}} \circ_\Gamma \underline{\mathcal{B}}$.

Proof. (1) Let $x_{(a^+,b^-)} \in \underline{\mathcal{A} \cup \mathcal{B}} \Longleftrightarrow \left\{x_{(a^+,b^-)} \in \underline{\mathcal{H}} : \mu^+_{\mathcal{A}\cup\mathcal{B}}(x) \geq a^+ \text{ and } \mu^-_{\mathcal{A}\cap\mathcal{B}}(x) \leq b^-\right\}$

$$\begin{aligned}
&\Longleftrightarrow \left\{x_{(a^+,b^-)} \in \underline{\mathcal{H}} : \max\{\mu^+_{\mathcal{A}}(x), \mu^+_{\mathcal{B}}(x)\} \geq a^+ \text{ and } \min\{\mu^-_{\mathcal{A}}(x), \mu^-_{\mathcal{B}}(x)\} \leq b^-\right\} \\
&\Longleftrightarrow \left\{x_{(a^+,b^-)} \in \underline{\mathcal{H}} : \{\mu^+_{\mathcal{A}}(x) \geq a^+ \text{ or } \mu^+_{\mathcal{B}}(x) \geq a^+\} \text{ and } \{\mu^-_{\mathcal{A}}(x) \leq b^- \text{ or } \mu^-_{\mathcal{B}}(x) \leq b^-\}\right\} \\
&\Longleftrightarrow \left\{ \begin{array}{c} \{x_{(a^+,b^-)} \in \underline{\mathcal{H}} : \mu^+_{\mathcal{A}}(x) \geq a^+ \text{ and } \mu^-_{\mathcal{A}}(x) \leq b^-\} \text{ or} \\ \{x_{(a^+,b^-)} \in \underline{\mathcal{H}} : \ \mu^+_{\mathcal{B}}(x) \geq a^+ \text{ and } \mu^-_{\mathcal{B}}(x) \leq b^-\} \end{array} \right\} \\
&\Longleftrightarrow x_{(a^+,b^-)} \in \underline{\mathcal{A}} \text{ or } x_{(a^+,b^-)} \in \underline{\mathcal{B}} \\
&\Longleftrightarrow x_{(a^+,b^-)} \in \underline{\mathcal{A}} \cup \underline{\mathcal{B}}.
\end{aligned}$$

Hence $\underline{\mathcal{A} \cup \mathcal{B}} = \underline{\mathcal{A}} \cup \underline{\mathcal{B}}$.

(2) The proof is similar to (1).

(3) For any $\gamma \in \Gamma$, $\underline{\mathcal{A}} \circ_\Gamma \underline{\mathcal{B}} = \left\{x_{(a^+,b^-)} \circ_\Gamma y_{(c^+,d^-)} : x_{(a^+,b^-)} \in \underline{\mathcal{A}} \text{ and } y_{(c^+,d^-)} \in \underline{\mathcal{B}}\right\}$

$$\begin{aligned}
&= \left\{(x\gamma y)_{(a^+\wedge c^+,b^-\vee d^-)} : \mu^+_{\mathcal{A}}(x) \geq a^+,\ \mu^-_{\mathcal{A}}(x) \leq b^- \text{ and } \mu^+_{\mathcal{B}}(y) \geq c^+,\ \mu^-_{\mathcal{B}}(y) \leq d^-\right\} \\
&= \left\{(x\gamma y)_{(a^+\wedge c^+,b^-\vee d^-)} : (\mu^+_{\mathcal{A}}(x) \geq a^+ \text{ and } \mu^+_{\mathcal{B}}(y) \geq c^+),\ (\mu^-_{\mathcal{A}}(x) \leq b^- \text{ and } \mu^-_{\mathcal{B}}(y) \leq d^-)\right\} \\
&= \left\{ \begin{array}{c} (x\gamma y)_{(a^+\wedge c^+,b^-\vee d^-)} : \min\{\mu^+_{\mathcal{A}}(x), \mu^+_{\mathcal{B}}(y)\} \geq a^+ \wedge c^+, \\ \max\{\mu^-_{\mathcal{A}}(x), \mu^-_{\mathcal{B}}(y)\} \leq b^- \vee d^- \end{array} \right\} \\
&\leq \left\{ \begin{array}{c} (x\gamma y)_{(a^+\wedge c^+,b^-\vee d^-)} : \sup\limits_{u\in x\gamma y} \min\{\mu^+_{\mathcal{A}}(x), \mu^+_{\mathcal{B}}(y)\} \geq a^+ \wedge c^+, \\ \inf\limits_{u\in x\gamma y} \max\{\mu^-_{\mathcal{A}}(x), \mu^-_{\mathcal{B}}(y)\} \leq b^- \vee d^- \end{array} \right\} \\
&= \left\{u_{(a^+\wedge c^+,b^-\vee d^-)} : \mu^+_{\mathcal{A}\circ_\Gamma\mathcal{B}}(u) \geq a^+ \wedge c^+,\ \mu^-_{\mathcal{A}\circ_\Gamma\mathcal{B}}(u) \leq b^- \vee d^-, \text{ for every } u \in x\gamma y \subseteq H\right\} \\
&= \underline{\mathcal{A} \circ_\Gamma \mathcal{B}}.
\end{aligned}$$

Hence $\underline{\mathcal{A} \circ_\Gamma \mathcal{B}} \supseteq \underline{\mathcal{A}} \circ_\Gamma \underline{\mathcal{B}}$. ■

Lemma 171 *Let $\mathcal{B} = \left\langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \right\rangle$ be a bipolar fuzzy subset of a Γ-semihypergroup H. Then $\mathcal{B}$ is a bipolar fuzzy*

1. *sub Γ-semihypergroup of H if and only if $\underline{\mathcal{B}}$ is a sub Γ-semihypergroup of $\underline{\mathcal{H}}$,*
2. *left Γ-hyperideal of H if and only if $\underline{\mathcal{B}}$ is a left Γ-hyperideal of $\underline{\mathcal{H}}$,*
3. *right Γ-hyperideal of H if and only if $\underline{\mathcal{B}}$ is a right Γ-hyperideal of $\underline{\mathcal{H}}$,*
4. *Γ-hyperideal of H if and only if $\underline{\mathcal{B}}$ is a Γ-hyperideal of $\underline{\mathcal{H}}$,*
5. *bi-Γ-hyperideal of H if and only if $\underline{\mathcal{B}}$ is a bi-Γ-hyperideal of $\underline{\mathcal{H}}$,*

6. *interior* Γ*-hyperideal of* H *if and only if* $\underline{\mathcal{B}}$ *is an interior* Γ*-hyperideal of* $\underline{\mathcal{H}}$,

7. $(1,2)$-Γ*-hyperideal of* H *if and only if* $\underline{\mathcal{B}}$ *is a* $(1,2)$-Γ*-hyperideal of* $\underline{\mathcal{H}}$.

Proof. (1) Let $\mathcal{B} = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \rangle$ be a bipolar fuzzy sub Γ-semihypergroup of H and $x_{(a^+,b^-)}$, $y_{(c^+,d^-)} \in \underline{\mathcal{B}}$. Then $\mu^+_{\mathcal{B}}(x) \geq a^+ > 0$, $\mu^+_{\mathcal{B}}(y) \geq c^+ > 0$ and $\mu^-_{\mathcal{B}}(x) \leq b^- < 0$, $\mu^-_{\mathcal{B}}(y) \leq d^- < 0$. Since

$$\begin{aligned} \inf_{u \in x\gamma y} \mu^+_{\mathcal{B}}(u) &\geq \min\{\mu^+_{\mathcal{B}}(x), \mu^+_{\mathcal{B}}(y)\} \geq a^+ \wedge c^+ \\ \text{and } \sup_{u \in x\gamma y} \mu^-_{\mathcal{B}}(u) &\leq \max\{\mu^-_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(y)\} \leq b^- \vee d^-. \end{aligned}$$

Thus, for all $\gamma \in \Gamma$ and $u \in x\gamma y$, $x_{(a^+,b^-)} \circ_\Gamma y_{(c^+,d^-)} = u_{(a^+ \wedge c^+, b^- \vee d^-)} \in \underline{\mathcal{B}}$, which implies that, $\underline{\mathcal{B}} \circ_\Gamma \underline{\mathcal{B}} \subseteq \underline{\mathcal{B}}$. Hence $\underline{\mathcal{B}}$ is a sub Γ-semihypergroup of $\underline{\mathcal{H}}$.

Conversely, we assume that $\underline{\mathcal{B}}$ is a sub Γ-semihypergroup of $\underline{\mathcal{H}}$. Let $x, y \in H$. If, $\mu^+_{\mathcal{B}}(x) = \mu^+_{\mathcal{B}}(y) = 0$ and $\mu^-_{\mathcal{B}}(x) = \mu^-_{\mathcal{B}}(y) = 0$, then

$$\begin{aligned} \min\{\mu^+_{\mathcal{B}}(x), \mu^+_{\mathcal{B}}(y)\} &= 0 \leq \inf_{u \in x\gamma y} \mu^+_{\mathcal{B}}(u) \\ \text{and } \max\{\mu^-_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(y)\} &= 0 \geq \sup_{u \in x\gamma y} \mu^-_{\mathcal{B}}(u). \end{aligned}$$

If $\mu^+_{\mathcal{B}}(x), \mu^+_{\mathcal{B}}(y) \neq 0$ and $\mu^-_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(y) \neq 0$, then $x_{(\mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(x))}, y_{(\mu^+_{\mathcal{B}}(y), \mu^-_{\mathcal{B}}(y))} \in \underline{\mathcal{B}}$. Since $\underline{\mathcal{B}}$ is a sub Γ-semihypergroup of $\underline{\mathcal{H}}$, so that for all $\gamma \in \Gamma$ and $u \in x\gamma y$,

$$\begin{aligned} u_{(\mu^+_{\mathcal{B}}(x) \wedge \mu^+_{\mathcal{B}}(y), \mu^-_{\mathcal{B}}(x) \vee \mu^-_{\mathcal{B}}(y))} &\in (x\gamma y)_{(\mu^+_{\mathcal{B}}(x) \wedge \mu^+_{\mathcal{B}}(y), \mu^-_{\mathcal{B}}(x) \vee \mu^-_{\mathcal{B}}(y))} \\ &= x_{(\mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(x))} \circ_\Gamma y_{(\mu^+_{\mathcal{B}}(y), \mu^-_{\mathcal{B}}(y))} \subseteq \underline{\mathcal{B}}, \end{aligned}$$

which implies that

$$\begin{aligned} \inf_{u \in x\gamma y} \mu^+_{\mathcal{B}}(u) &\geq \mu^+_{\mathcal{B}}(x) \wedge \mu^+_{\mathcal{B}}(y) = \min\{\mu^+_{\mathcal{B}}(x), \mu^+_{\mathcal{B}}(y)\} \\ \text{and } \sup_{u \in x\gamma y} \mu^-_{\mathcal{B}}(u) &\leq \mu^-_{\mathcal{B}}(x) \vee \mu^-_{\mathcal{B}}(y) = \max\{\mu^-_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(y)\}. \end{aligned}$$

Hence $\mathcal{B}$ is a bipolar fuzzy sub Γ-semihypergroup of H.

(2), (3) and (4) are similar to (1).

(5) Let $\mathcal{B} = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \rangle$ be a bipolar fuzzy bi-Γ-hyperideal of H. Then by (1), $\underline{\mathcal{B}}$ is a sub Γ-semihypergroup of $\underline{\mathcal{H}}$. Let $x_{(a^+,b^-)}, z_{(e^+,f^-)} \in \underline{\mathcal{B}}$ and $y_{(c^+,d^-)} \in \underline{\mathcal{H}}$. As $\mathcal{B}$ is a bipolar fuzzy bi-Γ-hyperideal, so for $\gamma, \beta \in \Gamma$, we have

$$\inf_{u \in x\gamma y\beta z} \mu^+_{\mathcal{B}}(u) \geq \min\{\mu^+_{\mathcal{B}}(x), \mu^+_{\mathcal{B}}(z)\} \geq a^+ \wedge e^+ \geq a^+ \wedge c^+ \wedge e^+,$$

and

$$\sup_{u\in x\gamma y\beta z} \mu_{\mathcal{B}}^{-}(u) \leq \max\{\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(z)\} \leq b^{-} \vee f^{-} \leq b^{-} \vee d^{-} \vee f^{-}.$$

Thus, for all $\gamma, \beta \in \Gamma$ and $u \in x\gamma y\beta z$,

$$x_{(a^{+},b^{-})} \circ_{\Gamma} y_{(c^{+},d^{-})} \circ_{\Gamma} z_{(c^{+},d^{-})} = u_{(a^{+}\wedge c^{+}\wedge e^{+},b^{-}\vee d^{-}\vee f^{-})} \in \underline{\mathcal{B}},$$

which implies that, $\underline{\mathcal{B}} \circ_{\Gamma} \underline{\mathcal{H}} \circ_{\Gamma} \underline{\mathcal{B}} \subseteq \underline{\mathcal{B}}$. Hence $\underline{\mathcal{B}}$ is a bi-Γ-hyperideal of $\underline{\mathcal{H}}$.

Conversely, we suppose that $\underline{\mathcal{B}}$ is a bi-Γ-hyperideal of $\underline{\mathcal{H}}$. Then by (1), $\mathcal{B}$ is a bipolar fuzzy sub Γ-semihypergroup of H. Let $x, y, z \in H$. If $\mu_{\mathcal{B}}^{+}(x) = \mu_{\mathcal{B}}^{+}(z) = 0$ and $\mu_{\mathcal{B}}^{-}(x) = \mu_{\mathcal{B}}^{-}(z) = 0$, then

$$\begin{aligned} \min\{\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(z)\} &= 0 \leq \inf_{u\in x\gamma y\beta z} \mu_{\mathcal{B}}^{+}(u) \\ \text{and} \quad \max\{\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(z)\} &= 0 \geq \sup_{u\in x\gamma y\beta z} \mu_{\mathcal{B}}^{-}(u). \end{aligned}$$

If $\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(z) \neq 0$ and $\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(z) \neq 0$, then $x_{(\mu_{\mathcal{B}}^{+}(x),\mu_{\mathcal{B}}^{-}(x))}, z_{(\mu_{\mathcal{B}}^{+}(z),\mu_{\mathcal{B}}^{-}(z))} \in \underline{\mathcal{B}}$. As $\underline{\mathcal{B}}$ is a bi-Γ-hyperideal of $\underline{\mathcal{H}}$, so for all $\gamma, \beta \in \Gamma$ and $u \in x\gamma y\beta z$,

$$\begin{aligned} u_{(\mu_{\mathcal{B}}^{+}(x)\wedge\mu_{\mathcal{B}}^{+}(z),\mu_{\mathcal{B}}^{-}(x)\vee\mu_{\mathcal{B}}^{-}(z))} &\in (x\gamma y\beta z)_{(\mu_{\mathcal{B}}^{+}(x)\wedge\mu_{\mathcal{B}}^{+}(x)\wedge\mu_{\mathcal{B}}^{+}(z),\mu_{\mathcal{B}}^{-}(x)\vee\mu_{\mathcal{B}}^{-}(x)\vee\mu_{\mathcal{B}}^{-}(z))} \\ &= x_{(\mu_{\mathcal{B}}^{+}(x),\mu_{\mathcal{B}}^{-}(x))} \circ_{\Gamma} y_{(\mu_{\mathcal{B}}^{+}(x),\mu_{\mathcal{B}}^{-}(x))} \circ_{\Gamma} z_{(\mu_{\mathcal{B}}^{+}(z),\mu_{\mathcal{B}}^{-}(z))} \subseteq \underline{\mathcal{B}}, \end{aligned}$$

which implies that

$$\begin{aligned} \inf_{u\in x\gamma y\beta z} \mu_{\mathcal{B}}^{+}(u) &\geq \mu_{\mathcal{B}}^{+}(x) \wedge \mu_{\mathcal{B}}^{+}(z) = \min\{\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(z)\} \\ \text{and} \quad \sup_{u\in x\gamma y\beta z} \mu_{\mathcal{B}}^{-}(u) &\leq \mu_{\mathcal{B}}^{-}(x) \vee \mu_{\mathcal{B}}^{-}(z) = \max\{\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(z)\}. \end{aligned}$$

Hence $\mathcal{B}$ is a bipolar fuzzy bi-Γ-hyperideal of H.

(6) and (7) are similar to (5). ■

Remark 172 *Any Γ-hyperideal of a Γ-semihypergroup H is an interior Γ-hyperideal of H and any bipolar fuzzy Γ-hyperideal of H is a bipolar fuzzy interior Γ-hyperideal of H.*

Theorem 173 *Let $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ be a bipolar fuzzy subset of a regular Γ-semihypergroup H, then the following conditions are equivalent:*

1. *$\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ is a bipolar fuzzy right Γ-hyperideal and a bipolar fuzzy left Γ-hyperideal of H.*

2. $\underline{\mathcal{B}}$ *is an interior* Γ*-hyperideal of* $\underline{\mathcal{H}}$.

Proof. (1 → 2) Follows easily from Remark 172 and Lemma 171.

(2 → 1) We suppose that $\underline{\mathcal{B}}$ is an interior Γ-hyperideal of $\underline{\mathcal{H}}$. Let $x, y \in H$. Since H is regular then $x \in x\beta a\gamma x$ for some $a \in H$ and $\beta, \gamma \in \Gamma$. If $\mu^+_{\mathcal{B}}(x) = 0$ and $\mu^-_{\mathcal{B}}(x) = 0$, then $\inf\limits_{u \in x\gamma y} \mu^+_{\mathcal{B}}(u) \geq 0 = \mu^+_{\mathcal{B}}(x)$ and $\sup\limits_{u \in x\gamma y} \mu^-_{\mathcal{B}}(u) \leq 0 = \mu^-_{\mathcal{B}}(x)$. If $\mu^+_{\mathcal{B}}(x) \neq 0$ and $\mu^-_{\mathcal{B}}(x) \neq 0$, then $x_{(\mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(x))} \in \underline{\mathcal{B}}$. Since $\underline{\mathcal{B}}$ is an interior Γ-hyperideal of $\underline{\mathcal{H}}$, then for $\beta, \gamma, \delta \in \Gamma$, we have

$$\begin{aligned}
(x\delta y)_{(\mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(x))} &\subseteq (x\beta a\gamma x\delta y)_{(\mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(x))} \\
&= (x\beta a\gamma x\delta y)_{(\mu^+_{\mathcal{B}}(x) \wedge \mu^+_{\mathcal{B}}(x) \wedge \mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(x) \vee \mu^-_{\mathcal{B}}(x) \vee \mu^-_{\mathcal{B}}(x))} \\
&= (x\beta a)_{(\mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(x))} \circ_\Gamma x_{(\mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(x))} \circ_\Gamma y_{(\mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(x))} \\
&= \{t_{(\mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(x))} \circ_\Gamma x_{(\mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(x))} \circ_\Gamma y_{(\mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(x))} : t \in x\beta a\} \\
&\subseteq \underline{\mathcal{B}}, \qquad \text{(for all } t \in x\beta a)
\end{aligned}$$

which implies that $\inf\limits_{u \in x\delta y} \mu^+_{\mathcal{B}}(u) \geq \mu^+_{\mathcal{B}}(x)$ and $\sup\limits_{u \in x\delta y} \mu^-_{\mathcal{B}}(u) \leq \mu^-_{\mathcal{B}}(x)$. Hence $\mathcal{B}$ is a bipolar fuzzy right Γ-hyperideal of H. The case for left Γ-hyperideal can be seen in a similar way. ■

Lemma 174 *A* Γ*-semihypergroup* H *is regular (resp., intra-regular, completely regular) if and only if the* Γ*-semihypergroup* $\underline{\mathcal{H}}$ *is regular (resp., intra-regular, completely regular).*

Proof. Let H be a regular Γ-semihypergroup and $x_{(a^+, b^-)} \in \underline{\mathcal{H}}$ for $x \in H$. As H regular so there exist $z \in H$ and $\gamma, \beta \in \Gamma$ such that $x \in x\gamma z\beta x$. Now for $z_{(a^+, b^-)} \in \underline{\mathcal{H}}$,

$$\begin{aligned}
x_{(a^+, b^-)} &\in (x\gamma z\beta x)_{(a^+, b^-)} = (x\gamma z\beta x)_{(a^+ \wedge a^+ \wedge a^+, b^- \vee b^- \vee b^-)} \\
&= x_{(a^+, b^-)} \circ_\Gamma z_{(a^+, b^-)} \circ_\Gamma x_{(a^+, b^-)}.
\end{aligned}$$

Hence $\underline{\mathcal{H}}$ is a regular Γ-semihypergroup.

Conversely, we suppose that $\underline{\mathcal{H}}$ is regular and $x \in H$. Then for any $(b^-, a^+) \in [-1, 0) \times (0, 1]$, there exist an element $z_{(c^+, d^-)} \in \underline{\mathcal{H}}$ such that

$$\begin{aligned}
x_{(a^+, b^-)} &\in x_{(a^+, b^-)} \circ_\Gamma z_{(c^+, d^-)} \circ_\Gamma x_{(a^+, b^-)} = (x\gamma z\beta x)_{(a^+ \wedge c^+ \wedge a^+, b^- \vee d^- \vee b^-)} \\
&= (x\gamma z\beta x)_{(a^+ \wedge c^+, b^- \vee d^-)}.
\end{aligned}$$

This implies that $x \in x\gamma z\beta x$. Hence H is a regular Γ-semihypergroup. The other cases can be seen in a similar way. ■

Theorem 175 *Let H be a Γ-semihypergroup. The following conditions are equivalent:*

1. *H is regular,*
2. *$\underline{\mathcal{A}} \cap \underline{\mathcal{B}} = \underline{\mathcal{A}} \circ_\Gamma \underline{\mathcal{B}}$, for every bipolar fuzzy right Γ-hyperideal $\mathcal{A} = \langle \mu^+_{\mathcal{A}}, \mu^-_{\mathcal{A}} \rangle$ and bipolar fuzzy left Γ-hyperideal $\mathcal{B} = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \rangle$ of H.*

Proof. $(1 \to 2)$ Let H be a regular Γ-semihypergroup, $\mathcal{A}$ a bipolar fuzzy right Γ-hyperideal of H and $\mathcal{B}$ a bipolar fuzzy left Γ-hyperideal of H. Then by Lemma 174, $\underline{\mathcal{H}}$ is regular and by Lemma 171, $\underline{\mathcal{A}}$ is a bipolar fuzzy right Γ-hyperideal and $\underline{\mathcal{B}}$ is a bipolar fuzzy left Γ-hyperideal of $\underline{\mathcal{H}}$. Hence by Theorem 164, $\underline{\mathcal{A}} \cap \underline{\mathcal{B}} = \underline{\mathcal{A}} \circ_\Gamma \underline{\mathcal{B}}$.

$(2 \to 1)$ We suppose that $\underline{\mathcal{A}} \cap \underline{\mathcal{B}} = \underline{\mathcal{A}} \circ_\Gamma \underline{\mathcal{B}}$, for every bipolar fuzzy right Γ-hyperideal $\mathcal{A}$ and bipolar fuzzy left Γ-hyperideal $\mathcal{B}$ of H. Let $x \in H$. If $\mu^+_{\mathcal{A}}(x) = 0$ or $\mu^+_{\mathcal{B}}(x) = 0$ and $\mu^-_{\mathcal{A}}(x) = 0$ or $\mu^-_{\mathcal{B}}(x) = 0$, then

$$\begin{aligned} \min\{\mu^+_{\mathcal{A}}(x), \mu^+_{\mathcal{B}}(x)\} &= 0 \leq \mu^+_{\underline{\mathcal{A}} \circ_\Gamma \underline{\mathcal{B}}}(x) \Longrightarrow \mu^+_{\underline{\mathcal{A}} \cap \underline{\mathcal{B}}}(x) \leq \mu^+_{\underline{\mathcal{A}} \circ_\Gamma \underline{\mathcal{B}}}(x) \\ \text{and } \max\{\mu^-_{\mathcal{A}}(x), \mu^-_{\mathcal{B}}(x)\} &= 0 \geq \mu^-_{\underline{\mathcal{A}} \circ_\Gamma \underline{\mathcal{B}}}(x) \Longrightarrow \mu^-_{\underline{\mathcal{A}} \cup \underline{\mathcal{B}}}(x) \geq \mu^-_{\underline{\mathcal{A}} \circ_\Gamma \underline{\mathcal{B}}}(x). \end{aligned}$$

This implies that $\underline{\mathcal{A}} \cap \underline{\mathcal{B}} \subseteq \underline{\mathcal{A}} \circ_\Gamma \underline{\mathcal{B}}$. Now if $\mu^+_{\mathcal{A}}(x), \mu^+_{\mathcal{B}}(x) \neq 0$ and $\mu^-_{\mathcal{A}}(x), \mu^-_{\mathcal{B}}(x) \neq 0$, then

$$\begin{aligned} x_{(\mu^+_{\mathcal{A}}(x) \wedge \mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{A}}(x) \vee \mu^-_{\mathcal{B}}(x))} &\in \underline{\mathcal{A}} \text{ and } x_{(\mu^+_{\mathcal{A}}(x) \wedge \mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{A}}(x) \vee \mu^-_{\mathcal{B}}(x))} \in \underline{\mathcal{B}} \\ &\Longrightarrow x_{(\mu^+_{\mathcal{A}}(x) \wedge \mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{A}}(x) \vee \mu^-_{\mathcal{B}}(x))} \in \underline{\mathcal{A}} \cap \underline{\mathcal{B}} = \underline{\mathcal{A}} \circ_\Gamma \underline{\mathcal{B}}. \end{aligned}$$

By Proposition 170, $\underline{\mathcal{A}} \circ_\Gamma \underline{\mathcal{B}} \subseteq \underline{\mathcal{A} \circ_\Gamma \mathcal{B}}$, so we have

$$x_{(\mu^+_{\mathcal{A}}(x) \wedge \mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{A}}(x) \vee \mu^-_{\mathcal{B}}(x))} \in \underline{\mathcal{A} \circ_\Gamma \mathcal{B}}.$$

Then

$$\mu^+_{\mathcal{A} \circ_\Gamma \mathcal{B}}(x) \geq \mu^+_{\mathcal{A}}(x) \wedge \mu^+_{\mathcal{B}}(x) = \mu^+_{\underline{\mathcal{A}} \cap \underline{\mathcal{B}}}(x) \text{ and } \mu^-_{\mathcal{A} \circ_\Gamma \mathcal{B}}(x) \leq \mu^-_{\mathcal{A}}(x) \vee \mu^-_{\mathcal{B}}(x) = \mu^-_{\underline{\mathcal{A}} \cup \underline{\mathcal{B}}}(x).$$

Which implies that $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \circ_\Gamma \mathcal{B}$. Also obviously $\mathcal{A} \circ_\Gamma \mathcal{B} \subseteq \mathcal{A} \cap \mathcal{B}$. Hence $\mathcal{A} \circ_\Gamma \mathcal{B} = \mathcal{A} \cap \mathcal{B}$ and by Theorem 164, H is regular. ■

Definition 176 *A bipolar fuzzy* Γ*-hyperideal* $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ *of a* Γ*-semihypergroup* H *is weakly prime if*

$$\left\{ \inf_{x \in y\gamma z} \mu_{\mathcal{B}}^{+}(x) = \mu_{\mathcal{B}}^{+}(y) \text{ or } \inf_{x \in y\gamma z} \mu_{\mathcal{B}}^{+}(x) = \mu_{\mathcal{B}}^{+}(z) \right\}$$
$$\text{and } \left\{ \sup_{x \in y\gamma z} \mu_{\mathcal{B}}^{-}(x) = \mu_{\mathcal{B}}^{-}(y) \text{ or } \sup_{x \in y\gamma z} \mu_{\mathcal{B}}^{-}(x) = \mu_{\mathcal{B}}^{-}(z) \right\}.$$

Alternatively we can define

Definition 177 *A bipolar fuzzy* Γ*-hyperideal* $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ *of a* Γ*-semihypergroup* H *is weakly prime if*

$$\inf_{x \in y\gamma z} \mu_{\mathcal{B}}^{+}(x) = \max\{\mu_{\mathcal{B}}^{+}(y), \mu_{\mathcal{B}}^{+}(z)\} \text{ and } \sup_{x \in y\gamma z} \mu_{\mathcal{B}}^{-}(x) = \min\{\mu_{\mathcal{B}}^{-}(y), \mu_{\mathcal{B}}^{-}(z)\}.$$

Definition 178 *A bipolar fuzzy* Γ*-hyperideal* $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ *of a* Γ*-semihypergroup* H *is weakly semiprime if*

$$\inf_{x \in y\gamma y} \mu_{\mathcal{B}}^{+}(x) = \mu_{\mathcal{B}}^{+}(y) \text{ and } \sup_{x \in y\gamma y} \mu_{\mathcal{B}}^{-}(x) = \mu_{\mathcal{B}}^{-}(y).$$

Theorem 179 *Let* $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ *be a bipolar fuzzy subset of a* Γ*-semihypergroup* H*. Then* $\mathcal{B}$ *is a bipolar fuzzy weakly prime* Γ*-hyperideal of* H *if and only if* $\underline{\mathcal{B}}$ *is a weakly prime* Γ*-hyperideal of* $\underline{\mathcal{H}}$*.*

Proof. Let $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ be a bipolar fuzzy weakly prime Γ-hyperideal of H. Then

$$\inf_{x \in y\gamma z} \mu_{\mathcal{B}}^{+}(x) = \max\{\mu_{\mathcal{B}}^{+}(y), \mu_{\mathcal{B}}^{+}(z)\} \text{ and } \sup_{x \in y\gamma z} \mu_{\mathcal{B}}^{-}(x) = \min\{\mu_{\mathcal{B}}^{-}(y), \mu_{\mathcal{B}}^{-}(z)\},$$

for all $x, y, z \in H$. Let $y_{(a^+,b^-)} \circ_{\Gamma} z_{(a^+,b^-)} \subseteq \underline{\mathcal{B}}$, but

$$y_{(a^+,b^-)} \circ_{\Gamma} z_{(a^+,b^-)} = (y\gamma z)_{(a^+ \wedge a^+, b^- \vee b^-)} = (y\gamma z)_{(a^+,b^-)},$$

implies that $(y\gamma z)_{(a^+,b^-)} \subseteq \underline{\mathcal{B}}$, for all $\gamma \in \Gamma$. Then $\inf_{x \in y\gamma z} \mu_{\mathcal{B}}^{+}(x) \geq a^{+}$ and $\sup_{x \in y\gamma z} \mu_{\mathcal{B}}^{-}(x) \leq b^{-}$, which implies that

$$\max\{\mu_{\mathcal{B}}^{+}(y), \mu_{\mathcal{B}}^{+}(z)\} \geq a^{+} \text{ and } \min\{\mu_{\mathcal{B}}^{-}(y), \mu_{\mathcal{B}}^{-}(z)\} \leq b^{-}.$$

Then

$$\mu_{\mathcal{B}}^{+}(y) \geq a^{+} \text{ or } \mu_{\mathcal{B}}^{+}(z) \geq a^{+} \text{ and } \mu_{\mathcal{B}}^{-}(y) \leq b^{-} \text{ or } \mu_{\mathcal{B}}^{-}(z) \leq b^{-},$$

$$\Longrightarrow (\mu^{+}_{\mathcal{B}}(y) \geq a^{+} \text{ and } \mu^{-}_{\mathcal{B}}(y) \leq b^{-}) \text{ or } (\mu^{+}_{\mathcal{B}}(z) \geq a^{+} \text{ and } \mu^{-}_{\mathcal{B}}(z) \leq b^{-}).$$

$\Longrightarrow y_{(a^{+},b^{-})} \in \underline{\mathcal{B}}$ or $z_{(a^{+},b^{-})} \in \underline{\mathcal{B}}$. Hence $\underline{\mathcal{B}}$ is a weakly prime Γ-hyperideal of $\underline{\mathcal{H}}$.

Conversely, we suppose that $\underline{\mathcal{B}}$ is a weakly prime Γ-hyperideal of $\underline{\mathcal{H}}$. Let $\inf\limits_{x \in y\gamma z} \mu^{+}_{\mathcal{B}}(x) = a^{+}$ and $\sup\limits_{x \in y\gamma z} \mu^{-}_{\mathcal{B}}(x) = b^{-}$. Then for $\gamma \in \Gamma$,

$$y_{(a^{+},b^{-})} \circ_{\Gamma} z_{(a^{+},b^{-})} = (y\gamma z)_{(a^{+},b^{-})} \subseteq \underline{\mathcal{B}},$$

implies that $y_{(a^{+},b^{-})} \in \underline{\mathcal{B}}$ or $z_{(a^{+},b^{-})} \in \underline{\mathcal{B}}$, as $\underline{\mathcal{B}}$ is a weakly prime Γ-hyperideal of $\underline{\mathcal{H}}$. Which implies that

$$(\mu^{+}_{\mathcal{B}}(y) \geq a^{+} \text{ and } \mu^{-}_{\mathcal{B}}(y) \leq b^{-}) \text{ or } (\mu^{+}_{\mathcal{B}}(z) \geq a^{+} \text{ and } \mu^{-}_{\mathcal{B}}(z) \leq b^{-}),$$

$$\Longrightarrow (\mu^{+}_{\mathcal{B}}(y) \geq a^{+} \text{ or } \mu^{+}_{\mathcal{B}}(z) \geq a^{+}) \text{ and } (\mu^{-}_{\mathcal{B}}(y) \leq b^{-} \text{ or } \mu^{-}_{\mathcal{B}}(z) \leq b^{-}),$$

$$\Longrightarrow \max\{\mu^{+}_{\mathcal{B}}(y), \mu^{+}_{\mathcal{B}}(z)\} \geq a^{+} \quad \text{and} \quad \min\{\mu^{-}_{\mathcal{B}}(y), \mu^{-}_{\mathcal{B}}(z)\} \leq b^{-}.$$

This implies that

$$\max\{\mu^{+}_{\mathcal{B}}(y), \mu^{+}_{\mathcal{B}}(z)\} \geq a^{+} = \inf_{x \in y\gamma z} \mu^{+}_{\mathcal{B}}(x) \quad \text{and} \quad \min\{\mu^{-}_{\mathcal{B}}(y), \mu^{-}_{\mathcal{B}}(z)\} \leq b^{-} = \sup_{x \in y\gamma z} \mu^{-}_{\mathcal{B}}(x).$$

Since $\underline{\mathcal{B}}$ is a weakly prime Γ-hyperideal of $\underline{\mathcal{H}}$, so $\underline{\mathcal{B}}$ is a Γ-hyperideal of $\underline{\mathcal{H}}$. Thus by Lemma 171, $\mathcal{B} = \langle \mu^{+}_{\mathcal{B}}, \mu^{-}_{\mathcal{B}} \rangle$ is a bipolar fuzzy Γ-hyperideal of H. Then

$$\inf_{x \in y\gamma z} \mu^{+}_{\mathcal{B}}(x) \geq \max\{\mu^{+}_{\mathcal{B}}(y), \mu^{+}_{\mathcal{B}}(z)\} \quad \text{and} \quad \sup_{x \in y\gamma z} \mu^{-}_{\mathcal{B}}(x) \leq \min\{\mu^{-}_{\mathcal{B}}(y), \mu^{-}_{\mathcal{B}}(z)\},$$

for all $x, y, z \in H$. Hence we have

$$\inf_{x \in y\gamma z} \mu^{+}_{\mathcal{B}}(x) = \max\{\mu^{+}_{\mathcal{B}}(y), \mu^{+}_{\mathcal{B}}(z)\} \quad \text{and} \quad \sup_{x \in y\gamma z} \mu^{-}_{\mathcal{B}}(x) = \min\{\mu^{-}_{\mathcal{B}}(y), \mu^{-}_{\mathcal{B}}(z)\},$$

for all $x, y, z \in H$. Which implies that $\mathcal{B} = \langle \mu^{+}_{\mathcal{B}}, \mu^{-}_{\mathcal{B}} \rangle$ is a bipolar fuzzy weakly prime Γ-hyperideal of a Γ-semihypergroup H. ∎

Theorem 180 *Let $\mathcal{B} = \langle \mu^{+}_{\mathcal{B}}, \mu^{-}_{\mathcal{B}} \rangle$ be a bipolar fuzzy subset of a Γ-semihypergroup H. Then $\mathcal{B}$ is a bipolar fuzzy weakly semiprime Γ-hyperideal of H if and only if $\underline{\mathcal{B}}$ is a weakly semiprime Γ-hyperideal of $\underline{\mathcal{H}}$.*

Proof. The proof is straightforward. ∎

3.3 Homomorphic Images and Preimages

The results obtained in this section are a part of our published paper [82].

In [21], the authors defined the image and inverse image of a fuzzy subset in a Γ-semihypergroup. Here we obtain some properties concerning the image and inverse image of bipolar fuzzy Γ-hyperideals.

Definition 181 *[21] Let Φ be a mapping from a set X to Y and μ be a fuzzy subset in X. The image $\Phi(\mu)$ of μ is the fuzzy subset of Y defined by*

$$\Phi(\mu)(y) = \begin{cases} \sup\{\mu(t) | t \in \Phi^{-1}(y)\} & \text{if } \Phi^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Definition 182 *[21] Let Φ be a mapping from a set X to Y and μ be a fuzzy subset in Y, then the preimage of μ under Φ is denoted by $\Phi^{-1}(\mu)$ and is defined as $\Phi^{-1}(\mu(x)) = \mu\left(\Phi(x)\right)$, for all $x \in X$.*

Definition 183 *[21] Let H be a Γ-semihypergroup and $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup. If there exists a mapping $\Phi : H \to \dot{H}$ and a bijection $f : \Gamma \to \dot{\Gamma}$ such that $\Phi(x\gamma y) = \Phi(x)f(\gamma)\Phi(y)$, for all $x, y \in H$ and $\gamma \in \Gamma$, then we say (Φ, f) is a homomorphism from H to $\dot{H}$. Also, if Φ is a bijection, then (Φ, f) is called an isomorphism, and H and $\dot{H}$ are isomorphic.*

Definition 184 *Let Φ be a mapping from a Γ-semihypergroup H to a $\dot{\Gamma}$-semihypergroup $\dot{H}$ and $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ be a bipolar fuzzy subset in H. The image $\Phi(\mathcal{B}) = \left\langle \mu_{\Phi(\mathcal{B})}^{+}, \mu_{\Phi(\mathcal{B})}^{-} \right\rangle$ of $\mathcal{B}$ is the bipolar fuzzy subset of $\dot{H}$ which is defined by*

$$\mu_{\Phi(\mathcal{B})}^{+}(y) = \begin{cases} \sup\{\mu_{\mathcal{B}}^{+}(t) | t \in \Phi^{-1}(y)\} & \text{if } \Phi^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_{\Phi(\mathcal{B})}^{-}(y) = \begin{cases} \inf\{\mu_{\mathcal{B}}^{-}(t) | t \in \Phi^{-1}(y)\} & \text{if } \Phi^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Definition 185 *Let Φ be a mapping from a Γ-semihypergroup H to a $\dot{\Gamma}$-semihypergroup $\dot{H}$ and $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ be a bipolar fuzzy subset in $\dot{H}$. Then the inverse image of $\mathcal{B}$ under Φ is denoted by $\Phi^{-1}(\mathcal{B}) = \left\langle \mu_{\Phi^{-1}(\mathcal{B})}^{+}, \mu_{\Phi^{-1}(\mathcal{B})}^{-} \right\rangle$ and is defined as*

$$\mu_{\Phi^{-1}(\mathcal{B})}^{+}(x) = \mu_{\mathcal{B}}^{+}(\Phi(x)) \text{ and } \mu_{\Phi^{-1}(\mathcal{B})}^{-}(x) = \mu_{\mathcal{B}}^{-}(\Phi(x)).$$

Theorem 186 *Let* $(\Phi, f) : (H, \Gamma) \longrightarrow (\dot{H}, \dot{\Gamma})$ *be a homomorphism from a* Γ*- semihypergroup* H *to a* $\dot{\Gamma}$*-semihypergroup* $\dot{H}$*. If* $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ *is a bipolar fuzzy sub* $\dot{\Gamma}$*-semihypergroup (resp.,* $\dot{\Gamma}$*-hyperideal, left* $\dot{\Gamma}$*-hyperideal, right* $\dot{\Gamma}$*-hyperideal, bi-*$\dot{\Gamma}$*-hyperideal, interior* $\dot{\Gamma}$*-hyperideal,* $(1,2)$*-*$\dot{\Gamma}$*-hyperideal) of* $\dot{H}$*, then the preimage* $\Phi^{-1}(\mathcal{B})$ *of* $\mathcal{B}$ *under* Φ *is a bipolar fuzzy sub* Γ*-semihypergroup (resp.,* Γ*-hyperideal, left* Γ*-hyperideal, right* Γ*-hyperideal, bi-*Γ*-hyperideal, interior* Γ*-hyperideal,* $(1,2)$*-*Γ*-hyperideal) of* H*.*

Proof. Let $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ be a bipolar fuzzy bi-$\dot{\Gamma}$-hyperideal of $\dot{H}$ and let $x, y \in H$ and $\gamma \in \Gamma$. Then we have

$$\begin{aligned}
\inf_{z \in x\gamma y} \left\{ \mu^{+}_{\Phi^{-1}(\mathcal{B})}(z) \right\} &= \inf_{z \in x\gamma y} \{ \mu^{+}_{\mathcal{B}}(\Phi(z) \} \geq \inf_{\Phi(z) \in \Phi(x\gamma y)} \{ \mu^{+}_{\mathcal{B}}(\Phi(z)) \} \\
&\geq \inf_{\Phi(z) \in \Phi(x) f(\gamma) \Phi(y)} \{ \mu^{+}_{\mathcal{B}}(\Phi(z)) \} \\
&\geq \min\{ \mu^{+}_{\mathcal{B}}(\Phi(x)), \mu^{+}_{\mathcal{B}}(\Phi(y)) \} \\
&= \min\left\{ \mu^{+}_{\Phi^{-1}(\mathcal{B})}(x), \mu^{+}_{\Phi^{-1}(\mathcal{B})}(y) \right\},
\end{aligned}$$

and

$$\begin{aligned}
\sup_{z \in x\gamma y} \left\{ \mu^{-}_{\Phi^{-1}(\mathcal{B})}(z) \right\} &= \sup_{z \in x\gamma y} \{ \mu^{-}_{\mathcal{B}}(\Phi(z) \} \leq \sup_{\Phi(z) \in \Phi(x\gamma y)} \{ \mu^{-}_{\mathcal{B}}(\Phi(z)) \} \\
&\leq \sup_{\Phi(z) \in \Phi(x) f(\gamma) \Phi(y)} \{ \mu^{-}_{\mathcal{B}}(\Phi(z)) \} \\
&\leq \max\{ \mu^{-}_{\mathcal{B}}(\Phi(x)), \mu^{-}_{\mathcal{B}}(\Phi(y)) \} \\
&= \max\left\{ \mu^{-}_{\Phi^{-1}(\mathcal{B})}(x), \mu^{-}_{\Phi^{-1}(\mathcal{B})}(y) \right\}.
\end{aligned}$$

Therefore, $\Phi^{-1}(\mathcal{B})$ is a bipolar fuzzy sub Γ-semihypergroup of H. For any $a, x, y \in H$ and $\gamma, \beta \in \Gamma$, we have

$$\begin{aligned}
\inf_{z \in x\gamma a\beta y} \left\{ \mu^{+}_{\Phi^{-1}(\mathcal{B})}(z) \right\} &= \inf_{z \in x\gamma a\beta y} \{ \mu^{+}_{\mathcal{B}}(\Phi(z) \} \geq \inf_{\Phi(z) \in \Phi(x\gamma a\beta y)} \{ \mu^{+}_{\mathcal{B}}(\Phi(z)) \} \\
&\geq \inf_{\Phi(z) \in \Phi(x) f(\gamma) \Phi(a) f(\beta) \Phi(y)} \{ \mu^{+}_{\mathcal{B}}(\Phi(z)) \} \\
&\geq \min\{ \mu^{+}_{\mathcal{B}}(\Phi(x)), \mu^{+}_{\mathcal{B}}(\Phi(y)) \} \\
&= \min\left\{ \mu^{+}_{\Phi^{-1}(\mathcal{B})}(x), \mu^{+}_{\Phi^{-1}(\mathcal{B})}(y) \right\},
\end{aligned}$$

and

$$\begin{aligned}
\sup_{z \in x\gamma a\beta y} \left\{ \mu^{-}_{\Phi^{-1}(\mathcal{B})}(z) \right\} &= \sup_{z \in x\gamma a\beta y} \{ \mu^{-}_{\mathcal{B}}(\Phi(z) \} \leq \sup_{\Phi(z) \in \Phi(x\gamma a\beta y)} \{ \mu^{-}_{\mathcal{B}}(\Phi(z)) \} \\
&\leq \sup_{\Phi(z) \in \Phi(x) f(\gamma) \Phi(a) f(\beta) \Phi(y)} \{ \mu^{-}_{\mathcal{B}}(\Phi(z)) \} \\
&\leq \max\{ \mu^{-}_{\mathcal{B}}(\Phi(x)), \mu^{-}_{\mathcal{B}}(\Phi(y)) \} \\
&= \max\left\{ \mu^{-}_{\Phi^{-1}(\mathcal{B})}(x), \mu^{-}_{\Phi^{-1}(\mathcal{B})}(y) \right\}.
\end{aligned}$$

Therefore, $\Phi^{-1}(\mathcal{B})$ is a bipolar fuzzy bi-Γ-hyperideal of H. The other cases can be seen in a similar way. ■

Theorem 187 *Let* $(\Phi, f) : (H, \Gamma) \longrightarrow (\dot{H}, \dot{\Gamma})$ *be a homomorphism from a* Γ*- semihypergroup* H *to a* $\dot{\Gamma}$*-semihypergroup* $\dot{H}$*. If a bipolar fuzzy subset* $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ *of* $\dot{H}$ *is a*

1. *sub* $\dot{\Gamma}$*-semihypergroup of* $\dot{H}$*, then* $\Phi^{-1}(\mathcal{B}) \circ_{\Gamma} \Phi^{-1}(\mathcal{B}) \subseteq \Phi^{-1}(\mathcal{B})$,
2. *right* $\dot{\Gamma}$*-hyperideal of* $\dot{H}$*, then* $\Phi^{-1}(\mathcal{B}) \circ_{\Gamma} \Phi^{-1}(\mathcal{H}) \subseteq \Phi^{-1}(\mathcal{B})$,
3. *left* $\dot{\Gamma}$*-hyperideal of* $\dot{H}$*, then* $\Phi^{-1}(\mathcal{H}) \circ_{\Gamma} \Phi^{-1}(\mathcal{B}) \subseteq \Phi^{-1}(\mathcal{B})$,
4. $\dot{\Gamma}$*-hyperideal of* $\dot{H}$*, then* $\Phi^{-1}(\mathcal{H}) \circ_{\Gamma} \Phi^{-1}(\mathcal{B}) \subseteq \Phi^{-1}(\mathcal{B})$ *and* $\Phi^{-1}(\mathcal{B}) \circ_{\Gamma} \Phi^{-1}(\mathcal{H}) \subseteq \Phi^{-1}(\mathcal{B})$,
5. *bi-*$\dot{\Gamma}$*-hyperideal of* $\dot{H}$*, then* $\Phi^{-1}(\mathcal{B}) \circ_{\Gamma} \Phi^{-1}(\mathcal{B}) \subseteq \Phi^{-1}(\mathcal{B})$ *and* $\Phi^{-1}(\mathcal{B}) \circ_{\Gamma} \Phi^{-1}(\mathcal{H}) \circ_{\Gamma} \Phi^{-1}(\mathcal{B}) \subseteq \Phi^{-1}(\mathcal{B})$,
6. *interior* $\dot{\Gamma}$*-hyperideal of* $\dot{H}$*, then* $\Phi^{-1}(\mathcal{H}) \circ_{\Gamma} \Phi^{-1}(\mathcal{B}) \circ_{\Gamma} \Phi^{-1}(\mathcal{H}) \subseteq \Phi^{-1}(\mathcal{B})$,
7. $(1, 2)$*-*$\dot{\Gamma}$*-hyperideal of* $\dot{H}$*, then* $\Phi^{-1}(\mathcal{B}) \circ_{\Gamma} \Phi^{-1}(\mathcal{B}) \subseteq \Phi^{-1}(\mathcal{B})$ *and* $\Phi^{-1}(\mathcal{B}) \circ_{\Gamma} \Phi^{-1}(\mathcal{H}) \circ_{\Gamma} (\Phi^{-1}(\mathcal{B}) \circ_{\Gamma} \Phi^{-1}(\mathcal{B})) \subseteq \Phi^{-1}(\mathcal{B})$,

where $\mathcal{H} = \langle \mu_{\mathcal{H}}^{+}, \mu_{\mathcal{H}}^{-} \rangle$*, such that* $\mu_{\mathcal{H}}^{+}(x) = 1$ *and* $\mu_{\mathcal{H}}^{-}(x) = -1$ *for all* $x \in H$*.*

Proof. (1) Let $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ be a bipolar fuzzy sub $\dot{\Gamma}$-semihypergroup of $\dot{H}$ and $z \in H$. Let us suppose that $z \in x\gamma y$ for $x, y \in H$ and $\gamma \in \Gamma$. Then

$$\begin{aligned}
\mu^{+}_{\Phi^{-1}(\mathcal{B}) \circ_{\Gamma} \Phi^{-1}(\mathcal{B})}(z) &= \bigvee_{z \in x\gamma y} \left\{ \mu^{+}_{\Phi^{-1}(\mathcal{B})}(x) \wedge \mu^{+}_{\Phi^{-1}(\mathcal{B})}(y) \right\} \\
&= \bigvee_{z \in x\gamma y} \left\{ \mu^{+}_{\mathcal{B}}(\Phi(x)) \wedge \mu^{+}_{\mathcal{B}}(\Phi(y)) \right\} \\
&\leq \bigvee_{z \in x\gamma y} \left\{ \inf_{\Phi(z) \in \Phi(x) f(\gamma) \Phi(y)} \mu^{+}_{\mathcal{B}}(\Phi(z)) \right\} \\
&\leq \bigvee_{z \in x\gamma y} \left\{ \inf_{\Phi(z) \in \Phi(x\gamma y)} \mu^{+}_{\mathcal{B}}(\Phi(z)) \right\} \\
&\leq \bigvee_{z \in x\gamma y} \left\{ \inf_{z \in x\gamma y} \mu^{+}_{\mathcal{B}}(\Phi(z)) \right\} \leq \bigvee_{z \in x\gamma y} \mu^{+}_{\mathcal{B}}(\Phi(x\gamma y)) \\
&= \mu^{+}_{\mathcal{B}}(\Phi(z)) = \mu^{+}_{\Phi^{-1}(\mathcal{B})}(z),
\end{aligned}$$

and

$$\begin{aligned}
\mu^{-}_{\Phi^{-1}(\mathcal{B})\circ_{\Gamma}\Phi^{-1}(\mathcal{B})}(z) &= \bigwedge_{z\in x\gamma y}\left\{\mu^{-}_{\Phi^{-1}(\mathcal{B})}(x)\vee\mu^{-}_{\Phi^{-1}(\mathcal{B})}(y)\right\} \\
&= \bigwedge_{z\in x\gamma y}\left\{\mu^{-}_{\mathcal{B}}(\Phi(x))\vee\mu^{-}_{\mathcal{B}}(\Phi(y))\right\} \\
&\geq \bigwedge_{z\in x\gamma y}\left\{\sup_{\Phi(z)\in\Phi(x)f(\gamma)\Phi(y)}\mu^{-}_{\mathcal{B}}(\Phi(z))\right\} \\
&\geq \bigwedge_{z\in x\gamma y}\left\{\sup_{\Phi(z)\in\Phi(x\gamma y)}\mu^{-}_{\mathcal{B}}(\Phi(z))\right\} \\
&\geq \bigwedge_{z\in x\gamma y}\left\{\sup_{z\in x\gamma y}\mu^{-}_{\mathcal{B}}(\Phi(z))\right\}\geq\bigwedge_{z\in x\gamma y}\mu^{-}_{\mathcal{B}}(\Phi(x\gamma y)) \\
&= \mu^{-}_{\mathcal{B}}(\Phi(z))=\mu^{-}_{\Phi^{-1}(\mathcal{B})}(z).
\end{aligned}$$

Therefore $\mu^{+}_{\Phi^{-1}(\mathcal{B})\circ_{\Gamma}\Phi^{-1}(\mathcal{B})}\subseteq\mu^{+}_{\Phi^{-1}(\mathcal{B})}$ and $\mu^{-}_{\Phi^{-1}(\mathcal{B})\circ_{\Gamma}\Phi^{-1}(\mathcal{B})}\supseteq\mu^{-}_{\Phi^{-1}(\mathcal{B})}$. If there do not exist $x,y\in H$ and $\gamma\in\Gamma$ such that $z\in x\gamma y$, then

$$\mu^{+}_{\Phi^{-1}(\mathcal{B})\circ_{\Gamma}\Phi^{-1}(\mathcal{B})}(z)=0\leq\mu^{+}_{\Phi^{-1}(\mathcal{B})}(z)\ \text{ and }\ \mu^{-}_{\Phi^{-1}(\mathcal{B})\circ_{\Gamma}\Phi^{-1}(\mathcal{B})}(z)=0\geq\mu^{-}_{\Phi^{-1}(\mathcal{B})}(z).$$

Hence $\Phi^{-1}(\mathcal{B})\circ_{\Gamma}\Phi^{-1}(\mathcal{B})\subseteq\Phi^{-1}(\mathcal{B})$.

(2) Let $\mathcal{B}=\left\langle\mu^{+}_{\mathcal{B}},\mu^{-}_{\mathcal{B}}\right\rangle$ be a bipolar fuzzy right $\dot{\Gamma}$-hyperideal of $\dot{H}$ and $z\in H$. Let us suppose that $z\in x\gamma y$ for $x,y\in H$ and $\gamma\in\Gamma$. Then

$$\begin{aligned}
\mu^{+}_{\Phi^{-1}(\mathcal{B})\circ_{\Gamma}\Phi^{-1}(\mathcal{H})}(z) &= \bigvee_{z\in x\gamma y}\left\{\mu^{+}_{\Phi^{-1}(\mathcal{B})}(x)\wedge\mu^{+}_{\Phi^{-1}(\mathcal{H})}(y)\right\} \\
&= \bigvee_{z\in x\gamma y}\left\{\mu^{+}_{\mathcal{B}}(\Phi(x))\wedge\mu^{+}_{\mathcal{H}}(\Phi(y))\right\} \\
&= \bigvee_{z\in x\gamma y}\left\{\mu^{+}_{\mathcal{B}}(\Phi(x))\wedge 1\right\}=\bigvee_{z\in x\gamma y}\left\{\mu^{+}_{\mathcal{B}}(\Phi(x))\right\} \\
&\leq \bigvee_{z\in x\gamma y}\left\{\inf_{\Phi(z)\in\Phi(x)f(\gamma)\Phi(y)}\mu^{+}_{\mathcal{B}}(\Phi(z))\right\} \\
&\leq \bigvee_{z\in x\gamma y}\left\{\inf_{\Phi(z)\in\Phi(x\gamma y)}\mu^{+}_{\mathcal{B}}(\Phi(z))\right\} \\
&\leq \bigvee_{z\in x\gamma y}\left\{\inf_{z\in x\gamma y}\mu^{+}_{\mathcal{B}}(\Phi(z))\right\}\leq\bigvee_{z\in x\gamma y}\mu^{+}_{\mathcal{B}}(\Phi(x\gamma y)) \\
&= \mu^{+}_{\mathcal{B}}(\Phi(z))=\mu^{+}_{\Phi^{-1}(\mathcal{B})}(z),
\end{aligned}$$

and

$$\begin{aligned}
\mu^-_{\Phi^{-1}(\mathcal{B})\circ_\Gamma \Phi^{-1}(\mathcal{H})}(z) &= \bigwedge_{z\in x\gamma y}\left\{\mu^-_{\Phi^{-1}(\mathcal{B})}(x)\vee\mu^-_{\Phi^{-1}(\mathcal{H})}(y)\right\}\\
&= \bigwedge_{z\in x\gamma y}\left\{\mu^-_{\mathcal{B}}(\Phi(x))\vee\mu^-_{\mathcal{H}}(\Phi(y))\right\}\\
&= \bigwedge_{z\in x\gamma y}\left\{\mu^-_{\mathcal{B}}(\Phi(x))\vee -1\right\} = \bigwedge_{z\in x\gamma y}\left\{\mu^-_{\mathcal{B}}(\Phi(x))\right\}\\
&\geq \bigwedge_{z\in x\gamma y}\left\{\sup_{\Phi(z)\in\Phi(x)f(\gamma)\Phi(y)}\mu^-_{\mathcal{B}}(\Phi(z))\right\}\\
&\geq \bigwedge_{z\in x\gamma y}\left\{\sup_{\Phi(z)\in\Phi(x\gamma y)}\mu^-_{\mathcal{B}}(\Phi(z))\right\}\\
&\geq \bigwedge_{z\in x\gamma y}\left\{\sup_{z\in x\gamma y}\mu^-_{\mathcal{B}}(\Phi(z))\right\}\geq \bigwedge_{z\in x\gamma y}\mu^-_{\mathcal{B}}(\Phi(x\gamma y))\\
&= \mu^-_{\mathcal{B}}(\Phi(z)) = \mu^-_{\Phi^{-1}(\mathcal{B})}(z).
\end{aligned}$$

Therefore $\mu^+_{\Phi^{-1}(\mathcal{B})\circ_\Gamma \Phi^{-1}(\mathcal{H})}\subseteq\mu^+_{\Phi^{-1}(\mathcal{B})}$ and $\mu^-_{\Phi^{-1}(\mathcal{B})\circ_\Gamma \Phi^{-1}(\mathcal{H})}\supseteq\mu^-_{\Phi^{-1}(\mathcal{B})}$. If there do not exist $x, y\in H$ and $\gamma\in\Gamma$ such that $z\in x\gamma y$, then

$$\mu^+_{\Phi^{-1}(\mathcal{B})\circ_\Gamma \Phi^{-1}(\mathcal{H})}(z) = 0 \leq \mu^+_{\Phi^{-1}(\mathcal{B})}(z) \text{ and } \mu^-_{\Phi^{-1}(\mathcal{B})\circ_\Gamma \Phi^{-1}(\mathcal{H})}(z) = 0 \geq \mu^-_{\Phi^{-1}(\mathcal{B})}(z).$$

Hence $\Phi^{-1}(\mathcal{B})\circ_\Gamma\Phi^{-1}(\mathcal{H})\subseteq\Phi^{-1}(\mathcal{B})$.

The proofs of (3), (4), (5), (6) and (7) are similar to (1) and (2). ■

Theorem 188 *Let $(\Phi, f) : (H,\Gamma)\longrightarrow(\dot{H},\dot{\Gamma})$ be a homomorphism from a Γ- semihypergroup H to a $\dot{\Gamma}$-semihypergroup $\dot{H}$. Let Φ be a surjective homomorphism. If $\mathcal{B} = \langle\mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}}\rangle$ is a bipolar fuzzy sub Γ-semihypergroup (resp., Γ-hyperideal, left Γ-hyperideal, right Γ-hyperideal, bi-Γ-hyperideal, interior Γ-hyperideal, $(1,2)$-Γ-hyperideal) of H, then the image $\mathcal{B}$ under Φ is a bipolar fuzzy sub $\dot{\Gamma}$-semihypergroup (resp., $\dot{\Gamma}$-hyperideal, left $\dot{\Gamma}$-hyperideal, right $\dot{\Gamma}$-hyperideal, bi-$\dot{\Gamma}$-hyperideal, interior $\dot{\Gamma}$-hyperideal, $(1,2)$-$\dot{\Gamma}$-hyperideal) of $\dot{H}$.*

Proof. Let $\mathcal{B} = \langle\mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}}\rangle$ be a bipolar fuzzy bi-Γ-hyperideal of H and let $\dot{x}, \dot{y}, \dot{z}\in\dot{H}$ and $\dot{\gamma}\in\dot{\Gamma}$. Then there exist $x, y, z\in H$ and $\gamma\in\Gamma$ such that $\Phi(x) = \dot{x}$, $\Phi(y) = \dot{y}$,

$\Phi(z) = \dot{z}$ and $f(\gamma) = \dot{\gamma}$. Now, we have

$$\begin{aligned}
\inf_{\dot{z}\in\dot{x}\dot{\gamma}\dot{y}} \left\{\mu^{+}_{\Phi(\mathcal{B})}(\dot{z})\right\} &= \inf_{\dot{z}\in\dot{x}\dot{\gamma}\dot{y}} \left\{\sup_{t\in\Phi^{-1}(\dot{z})} \mu^{+}_{\mathcal{B}}(t)\right\} \geq \inf_{\dot{z}\in\dot{x}\dot{\gamma}\dot{y}} \mu^{+}_{\mathcal{B}}(z) = \inf_{\Phi(z)\in\Phi(x)f(\gamma)\Phi(y)} \mu^{+}_{\mathcal{B}}(z) \\
&= \inf_{\Phi(z)\in\Phi(x\gamma y)} \mu^{+}_{\mathcal{B}}(z) = \inf_{z\in x\gamma y} \mu^{+}_{\mathcal{B}}(z) \geq \mu^{+}_{\mathcal{B}}(x) \wedge \mu^{+}_{\mathcal{B}}(y) \\
&\geq \sup_{x\gamma y\subseteq\Phi^{-1}(\dot{x})f^{-1}(\dot{\gamma})\Phi^{-1}(\dot{y})} \left\{\mu^{+}_{\mathcal{B}}(x) \wedge \mu^{+}_{\mathcal{B}}(y)\right\} \\
&\geq \min\left\{\mu^{+}_{\Phi(\mathcal{B})}(\dot{x}), \mu^{+}_{\Phi(\mathcal{B})}(\dot{y})\right\},
\end{aligned}$$

and

$$\begin{aligned}
\sup_{\dot{z}\in\dot{x}\dot{\gamma}\dot{y}} \left\{\mu^{-}_{\Phi(\mathcal{B})}(\dot{z})\right\} &= \sup_{\dot{z}\in\dot{x}\dot{\gamma}\dot{y}} \left\{\inf_{t\in\Phi^{-1}(\dot{z})} \mu^{-}_{\mathcal{B}}(t)\right\} \leq \sup_{\dot{z}\in\dot{x}\dot{\gamma}\dot{y}} \mu^{-}_{\mathcal{B}}(z) = \sup_{\Phi(z)\in\Phi(x)f(\gamma)\Phi(y)} \mu^{-}_{\mathcal{B}}(z) \\
&= \sup_{\Phi(z)\in\Phi(x\gamma y)} \mu^{-}_{\mathcal{B}}(z) = \sup_{z\in x\gamma y} \mu^{-}_{\mathcal{B}}(z) \leq \mu^{-}_{\mathcal{B}}(x) \vee \mu^{-}_{\mathcal{B}}(y) \\
&\leq \inf_{x\gamma y\subseteq\Phi^{-1}(\dot{x})f^{-1}(\dot{\gamma})\Phi^{-1}(\dot{y})} \left\{\mu^{-}_{\mathcal{B}}(x) \vee \mu^{-}_{\mathcal{B}}(y)\right\} \\
&\leq \max\left\{\mu^{-}_{\Phi(\mathcal{B})}(\dot{x}), \mu^{-}_{\Phi(\mathcal{B})}(\dot{y})\right\}.
\end{aligned}$$

Therefore, $\Phi(\mathcal{B})$ is a bipolar fuzzy sub $\dot{\Gamma}$-semihypergroup of $\dot{H}$. For any $\dot{a}, \dot{x}, \dot{y}, \dot{z} \in \dot{H}$ and $\dot{\gamma}, \dot{\beta} \in \dot{\Gamma}$ we have

$$\begin{aligned}
\inf_{\dot{z}\in\dot{x}\dot{\gamma}\dot{a}\dot{\beta}\dot{y}} \left\{\mu^{+}_{\Phi(\mathcal{B})}(\dot{z})\right\} &= \inf_{\dot{z}\in\dot{x}\dot{\gamma}\dot{a}\dot{\beta}\dot{y}} \left\{\sup_{t\in\Phi^{-1}(\dot{z})} \mu^{+}_{\mathcal{B}}(t)\right\} \geq \inf_{\dot{z}\in\dot{x}\dot{\gamma}\dot{a}\dot{\beta}\dot{y}} \mu^{+}_{\mathcal{B}}(z) \\
&= \inf_{\Phi(z)\in\Phi(x)f(\gamma)\Phi(a)f(\beta)\Phi(y)} \mu^{+}_{\mathcal{B}}(z) = \inf_{\Phi(z)\in\Phi(x\gamma a\beta y)} \mu^{+}_{\mathcal{B}}(z) \\
&= \inf_{z\in x\gamma a\beta y} \mu^{+}_{\mathcal{B}}(z) \geq \mu^{+}_{\mathcal{B}}(x) \wedge \mu^{+}_{\mathcal{B}}(y) \\
&\geq \sup_{x\gamma y\subseteq\Phi^{-1}(\dot{x})f^{-1}(\dot{\gamma})\Phi^{-1}(\dot{y})} \left\{\mu^{+}_{\mathcal{B}}(x) \wedge \mu^{+}_{\mathcal{B}}(y)\right\} \\
&\geq \min\left\{\mu^{+}_{\Phi(\mathcal{B})}(\dot{x}), \mu^{+}_{\Phi(\mathcal{B})}(\dot{y})\right\},
\end{aligned}$$

and

$$\begin{aligned}
\sup_{\dot{z}\in\dot{x}\dot{\gamma}\dot{a}\dot{\beta}\dot{y}} \left\{\mu^{-}_{\Phi(\mathcal{B})}(\dot{z})\right\} &= \sup_{\dot{z}\in\dot{x}\dot{\gamma}\dot{a}\dot{\beta}\dot{y}} \left\{\inf_{t\in\Phi^{-1}(\dot{z})} \mu^{-}_{\mathcal{B}}(t)\right\} \leq \sup_{\dot{z}\in\dot{x}\dot{\gamma}\dot{a}\dot{\beta}\dot{y}} \mu^{-}_{\mathcal{B}}(z) \\
&= \sup_{\Phi(z)\in\Phi(x)f(\gamma)\Phi(a)f(\beta)\Phi(y)} \mu^{-}_{\mathcal{B}}(z) = \sup_{\Phi(z)\in\Phi(x\gamma a\beta y)} \mu^{-}_{\mathcal{B}}(z) \\
&= \sup_{z\in x\gamma a\beta y} \mu^{-}_{\mathcal{B}}(z) \leq \mu^{-}_{\mathcal{B}}(x) \vee \mu^{-}_{\mathcal{B}}(y) \\
&\leq \inf_{x\gamma y\subseteq\Phi^{-1}(\dot{x})f^{-1}(\dot{\gamma})\Phi^{-1}(\dot{y})} \left\{\mu^{-}_{\mathcal{B}}(x) \vee \mu^{-}_{\mathcal{B}}(y)\right\} \\
&\leq \max\left\{\mu^{-}_{\Phi(\mathcal{B})}(\dot{x}), \mu^{-}_{\Phi(\mathcal{B})}(\dot{y})\right\}.
\end{aligned}$$

Therefore, $\Phi(\mathcal{B})$ is a bipolar fuzzy bi-$\dot{\Gamma}$-hyperideal of $\dot{H}$. The other cases can be seen in a similar way. ■

3.4 Bipolar (λ, δ)-Fuzzy Γ-Hyperideals

The results presented in this section are a part of our paper [83].

Yao introduced a new type of fuzzy sets called (λ, θ)-fuzzy sets, and studied (λ, θ)-fuzzy normal subfields [75]. Several authors extended Yao's idea and continued their research works in applying (λ, θ)-fuzzy sets on different algebraic structures. Coumaressane [18] characterized near-rings by their (λ, θ)-fuzzy quasi-ideals. Shabir et al. [66] characterized semigroups by the properties of their fuzzy ideals with thresholds and Khan et al. [42] characterized ordered semigroups by their (λ, θ)-fuzzy bi-ideals. Li and Feng [53] extended the idea of (λ, θ)-fuzzy sets in intuitionistic fuzzy sets and studied intuitionistic fuzzy (λ, μ)-fuzzy sets in Γ-semigroups. Yaqoob and Ansari [88] also extended the idea of (λ, θ)-fuzzy sets in bipolar fuzzy sets and studied bipolar (λ, δ)-fuzzy ideals in ternary semigroups. Here, we introduce and study the notion bipolar (λ, δ)-fuzzy Γ-hyperideals (resp., interior Γ-hyperideals and bi-Γ-hyperideals) in Γ-semihypergroups and discuss some related properties.

In what follows, let $\lambda^+, \delta^+ \in [0,1]$ be such that $0 \leq \lambda^+ < \delta^+ \leq 1$ and $\lambda^-, \delta^- \in [-1,0]$ be such that $-1 \leq \delta^- < \lambda^- \leq 0$. Both $\lambda, \delta \in [0,1]$ are arbitrary but fixed.

Definition 189 *A bipolar fuzzy subset $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^+, \mu_{\mathcal{B}}^- \right\rangle$ of a Γ-semihypergroup H is called*

1. *a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup of H if*

$$\begin{aligned} \max\left\{\inf_{x\in y\gamma z} \mu_{\mathcal{B}}^+(x), \lambda^+\right\} &\geq \min\{\mu_{\mathcal{B}}^+(y), \mu_{\mathcal{B}}^+(z), \delta^+\} \\ \text{and} \quad \min\left\{\sup_{x\in y\gamma z} \mu_{\mathcal{B}}^-(x), \lambda^-\right\} &\leq \max\{\mu_{\mathcal{B}}^-(y), \mu_{\mathcal{B}}^-(z), \delta^-\} \end{aligned}$$

   for all $x, y, z \in H$ and $\gamma \in \Gamma$.

2. *a bipolar (λ, δ)-fuzzy left Γ-hyperideal of H if*

$$\begin{aligned} \max\left\{\inf_{x\in y\gamma z} \mu_{\mathcal{B}}^+(x), \lambda^+\right\} &\geq \min\{\mu_{\mathcal{B}}^+(z), \delta^+\} \\ \text{and} \quad \min\left\{\sup_{x\in y\gamma z} \mu_{\mathcal{B}}^-(x), \lambda^-\right\} &\leq \max\{\mu_{\mathcal{B}}^-(z), \delta^-\} \end{aligned}$$

   for all $x, y, z \in H$ and $\gamma \in \Gamma$.

3. a bipolar (λ, δ)-fuzzy right Γ-hyperideal of H if

$$\begin{aligned} \max\left\{\inf_{x\in y\gamma z}\mu_{\mathcal{B}}^{+}(x), \lambda^{+}\right\} &\geq \min\{\mu_{\mathcal{B}}^{+}(y), \delta^{+}\} \\ \text{and} \quad \min\left\{\sup_{x\in y\gamma z}\mu_{\mathcal{B}}^{-}(x), \lambda^{-}\right\} &\leq \max\{\mu_{\mathcal{B}}^{-}(y), \delta^{-}\} \end{aligned}$$

for all $x, y, z \in H$ and $\gamma \in \Gamma$.

Example 190 *Let $H = (0,1)$, $\Gamma = \{\gamma_n \,|n \in \mathbb{N}\}$ and for every $n \in \mathbb{N}$ we define hyperoperation γ_n on H as follows*

$$x\gamma_n y = \left\{\frac{xy}{2^k} \,|0 \leq k \leq n\right\}, \forall x, y \in H.$$

Then, $x\gamma_n y \subset H$ and for every $m, n \in \mathbb{N}$ and $x, y, z \in H$

$$(x\gamma_n y)\gamma_m z = \left\{\frac{xyz}{2^k} \,|0 \leq k \leq n+m\right\} = x\gamma_n\left(y\gamma_m z\right).$$

So H is a Γ-semihypergroup. Now we define a bipolar fuzzy set $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-}\right\rangle$ on H as:

$$\mu_{\mathcal{B}}^{+}(x) = \begin{cases} 0.7 & \text{if } 0 < x < \frac{1}{2^k} \\ 0.6 & \text{if } \frac{1}{2^k} \leq x < 1 \end{cases} \quad \text{where } k \in \mathbb{N}$$

$$\mu_{\mathcal{B}}^{-}(x) = \begin{cases} -0.9 & \text{if } 0 < x < \frac{1}{2^k} \\ -0.5 & \text{if } \frac{1}{2^k} \leq x < 1 \end{cases} \quad \text{where } k \in \mathbb{N}.$$

Then by routine calculations, $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-}\right\rangle$ is a bipolar $(0.2, 0.35)$-fuzzy Γ-hyperideal of H.

Definition 191 *A bipolar fuzzy subset $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-}\right\rangle$ of H is called a bipolar (λ, δ)-fuzzy interior Γ-hyperideal of H if*

$$\begin{aligned} \max\left\{\inf_{a\in x\beta y\gamma z}\mu_{\mathcal{B}}^{+}(a), \lambda^{+}\right\} &\geq \min\{\mu_{\mathcal{B}}^{+}(y), \delta^{+}\} \\ \text{and} \min\left\{\sup_{a\in x\beta y\gamma z}\mu_{\mathcal{B}}^{-}(a), \lambda^{-}\right\} &\leq \max\{\mu_{\mathcal{B}}^{-}(y), \delta^{-}\}, \end{aligned}$$

for all $x, y, z \in H$ and $\beta, \gamma \in \Gamma$.

Definition 192 *A bipolar (λ, δ)-fuzzy sub Γ-semihypergroup $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-}\right\rangle$ of H is called a bipolar (λ, δ)-fuzzy bi-Γ-hyperideal of H if*

$$\begin{aligned} \max\left\{\inf_{a\in x\beta y\gamma z}\mu_{\mathcal{B}}^{+}(a), \lambda^{+}\right\} &\geq \min\{\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(z), \delta^{+}\} \\ \text{and} \min\left\{\sup_{a\in x\beta y\gamma z}\mu_{\mathcal{B}}^{-}(a), \lambda^{+}\right\} &\leq \max\{\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(z), \delta^{-}\}, \end{aligned}$$

for all $x, y, z \in H$ and $\beta, \gamma \in \Gamma$.

Example 193 *Let $H = \{x, y, z, w\}$ and $\Gamma = \{\beta, \gamma\}$ be the sets of binary hyperoperations defined below:*

β	x	y	z	w
x	x	$\{x, y\}$	$\{z, w\}$	w
y	$\{x, y\}$	$\{x, y\}$	$\{z, w\}$	w
z	$\{z, w\}$	$\{z, w\}$	z	w
w	w	w	w	w

γ	x	y	z	w
x	$\{x, y\}$	$\{x, y\}$	$\{z, w\}$	w
y	$\{x, y\}$	y	$\{z, w\}$	w
z	$\{z, w\}$	$\{z, w\}$	z	w
w	w	w	w	w

Here H is a Γ-semihypergroup. Now we define a bipolar fuzzy set $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \right\rangle$ on H as:

$$\mu_{\mathcal{B}}^{+}(t) = \begin{cases} 0.5 & \text{if } t \in \{x, y\} \\ 0.6 & \text{if } t = z \\ 0.8 & \text{if } t = w \end{cases} \quad \text{and} \quad \mu_{\mathcal{B}}^{-}(t) = \begin{cases} -0.6 & \text{if } t \in \{x, y\} \\ -0.8 & \text{if } t = z \\ -0.9 & \text{if } t = w \end{cases}$$

Then by routine calculation, $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \right\rangle$ is a bipolar $(0.3, 0.4)$-fuzzy bi-Γ-hyperideal of H.

Definition 194 *A bipolar (λ, δ)-fuzzy sub Γ-semihypergroup $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \right\rangle$ of H is called a bipolar $(1, 2)$-fuzzy bi-Γ-hyperideal of H if*

$$\begin{aligned} \max\left\{ \inf_{a \in x\alpha w\beta(y\gamma z)} \mu_{\mathcal{B}}^{+}(a), \lambda^{+} \right\} &\geq \min\{\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(y), \mu_{\mathcal{B}}^{+}(z), \delta^{+}\} \\ \text{and } \min\left\{ \sup_{a \in x\alpha w\beta(y\gamma z)} \mu_{\mathcal{B}}^{-}(a), \lambda^{-} \right\} &\leq \max\{\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(y), \mu_{\mathcal{B}}^{-}(z), \delta^{-}\}, \end{aligned}$$

for all $x, y, z, w \in H$ and $\alpha, \beta, \gamma \in \Gamma$.

Theorem 195 *A bipolar fuzzy subset $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \right\rangle$ of a Γ-semihypergroup H is a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup (resp., left Γ-hyperideal, right Γ-hyperideal, interior Γ-hyperideal, bi-Γ-hyperideal, $(1, 2)$-Γ-hyperideal) of H if and only if $\emptyset \neq H_{\mathcal{B}}^{(t,s)}$ is a sub Γ-semihypergroup (resp., left Γ-hyperideal, right Γ-hyperideal, interior Γ-hyperideal, bi-Γ-hyperideal, $(1, 2)$-Γ-hyperideal) of H for all $(s, t) \in [\delta^{-}, \lambda^{-}) \times (\lambda^{+}, \delta^{+}]$.*

Proof. Let $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \right\rangle$ be a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup of H. Let $x, y \in H$, $(s, t) \in [\delta^{-}, \lambda^{-}) \times (\lambda^{+}, \delta^{+}]$ and $x, y \in H_{\mathcal{B}}^{(t,s)}$. Then $\mu_{\mathcal{B}}^{+}(x) \geq t$ and $\mu_{\mathcal{B}}^{+}(y) \geq t$, also $\mu_{\mathcal{B}}^{-}(x) \leq s$ and $\mu_{\mathcal{B}}^{-}(y) \leq s$. As $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ is a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup of H. Therefore,

$$\max\left\{\inf_{z\in x\gamma y} \mu_{\mathcal{B}}^{+}(z), \lambda^{+}\right\} \geq \min\{\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(y), \delta^{+}\} \geq \min\left\{t, t, \delta^{+}\right\} = t,$$

and

$$\min\left\{\sup_{z\in x\gamma y} \mu_{\mathcal{B}}^{-}(z), \lambda^{-}\right\} \leq \max\{\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(y), \delta^{-}\} \leq \max\left\{s, s, \delta^{-}\right\} = s.$$

This implies that $\inf_{z\in x\gamma y} \mu_{\mathcal{B}}^{+}(z) \geq t$ and $\sup_{z\in x\gamma y} \mu_{\mathcal{B}}^{-}(z) \leq s$. Thus $x\gamma y \subseteq H_{\mathcal{B}}^{(t,s)}$. Hence $H_{\mathcal{B}}^{(t,s)}$ is a sub Γ-semihypergroup of H.

Conversely, suppose that $H_{\mathcal{B}}^{(t,s)}$ is a sub Γ-semihypergroup of H. Let $x, y \in H$ such that

$$\max\left\{\inf_{z\in x\gamma y} \mu_{\mathcal{B}}^{+}(z), \lambda^{+}\right\} < \min\{\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(y), \delta^{+}\},$$

and

$$\min\left\{\sup_{z\in x\gamma y} \mu_{\mathcal{B}}^{-}(z), \lambda^{-}\right\} > \max\{\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(y), \delta^{-}\}.$$

Then there exist $(s, t) \in [\delta^{-}, \lambda^{-}) \times (\lambda^{+}, \delta^{+}]$ such that

$$\max\left\{\inf_{z\in x\gamma y} \mu_{\mathcal{B}}^{+}(z), \lambda^{+}\right\} < t \leq \min\{\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(y), \delta^{+}\},$$

and

$$\min\left\{\sup_{z\in x\gamma y} \mu_{\mathcal{B}}^{-}(z), \lambda^{-}\right\} > s \geq \max\{\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(y), \delta^{-}\}.$$

This shows that $\mu_{\mathcal{B}}^{+}(x) \geq t$, $\mu_{\mathcal{B}}^{+}(y) \geq t$ and $\inf_{z\in x\gamma y} \mu_{\mathcal{B}}^{+}(z) < t$, also $\mu_{\mathcal{B}}^{-}(x) \leq s$, $\mu_{\mathcal{B}}^{-}(y) \leq s$ and $\sup_{z\in x\gamma y} \mu_{\mathcal{B}}^{-}(z) > s$. Thus $x, y \in H_{\mathcal{B}}^{(t,s)}$, since $H_{\mathcal{B}}^{(t,s)}$ is a sub Γ-semihypergroup of H. Therefore $x\gamma z \subseteq H_{\mathcal{B}}^{(t,s)}$ for some $z \in x\gamma y$, but this is a contradiction to $\inf_{z\in x\gamma y} \mu_{\mathcal{B}}^{+}(z) < t$ and $\sup_{z\in x\gamma y} \mu_{\mathcal{B}}^{-}(z) > s$. Thus,

$$\max\left\{\inf_{z\in x\gamma y} \mu_{\mathcal{B}}^{+}(z), \lambda^{+}\right\} \geq \min\{\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(y), \delta^{+}\},$$

and

$$\min\left\{\sup_{z\in x\gamma y} \mu_{\mathcal{B}}^{-}(z), \lambda^{-}\right\} \leq \max\{\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(y), \delta^{-}\}.$$

Hence $\mathcal{B} = (\mu^{+}, \mu_{\mathcal{B}}^{-})$ is a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup of H. The other cases can be seen in a similar way. ■

Corollary 196 *Every bipolar fuzzy sub* Γ*-semihypergroup (resp., left* Γ*-hyperideal, right* Γ*-hyperideal, interior* Γ*-hyperideal, bi-*Γ*-hyperideal,* $(1,2)$-Γ*-hyperideal)* $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ *is a bipolar* (λ, δ)*-fuzzy sub* Γ*-semihypergroup (resp., left* Γ*-hyperideal, right* Γ*-hyperideal, interior* Γ*-hyperideal, bi-*Γ*-hyperideal,* $(1,2)$-Γ*-hyperideal) of* H *with* $\lambda^{+} = 0$, $\delta^{+} = 1$ *and* $\lambda^{-} = 0$, $\delta^{-} = -1$.

Theorem 197 *If a bipolar fuzzy subset* $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ *is a bipolar* (λ, δ)*-fuzzy sub* Γ*-semihypergroup (resp., left* Γ*-hyperideal, right* Γ*-hyperideal, interior* Γ*-hyperideal, bi-*Γ*-hyperideal,* $(1,2)$-Γ*-hyperideal) of* H*. Then the set* $\mathcal{B}_{\lambda} = ({}^{>}\mathcal{B}_{\lambda^{+}}^{+}, {}^{<}\mathcal{B}_{\lambda^{-}}^{-})$ *is a sub* Γ*-semihypergroup (resp., left* Γ*-hyperideal, right* Γ*-hyperideal, interior* Γ*-hyperideal, bi-*Γ*-hyperideal,* $(1,2)$-Γ*-hyperideal) of* H*, where* ${}^{>}\mathcal{B}_{\lambda^{+}}^{+} = \{x \in H \mid \mu_{\mathcal{B}}^{+}(x) > \lambda^{+}\}$ *and* ${}^{<}\mathcal{B}_{\lambda^{-}}^{-} = \{x \in H \mid \mu_{\mathcal{B}}^{-}(x) < \lambda^{-}\}$.

Proof. Suppose that $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ is a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup of H. Let $x, y \in H$ such that $x, y \in \mathcal{B}_{\lambda}$. Then $\mu_{\mathcal{B}}^{+}(x) > \lambda^{+}$, $\mu_{\mathcal{B}}^{+}(y) > \lambda^{+}$ and $\mu_{\mathcal{B}}^{-}(x) < \lambda^{-}$, $\mu_{\mathcal{B}}^{-}(y) < \lambda^{-}$. Since $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ is a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup therefore,

$$\max\left\{\inf_{z \in x\gamma y} \mu_{\mathcal{B}}^{+}(z), \lambda^{+}\right\} \geq \min\{\mu_{\mathcal{B}}^{+}(x), \mu_{\mathcal{B}}^{+}(y), \delta^{+}\} > \min\left\{\lambda^{+}, \lambda^{+}, \delta^{+}\right\} = \lambda^{+},$$

and

$$\min\left\{\sup_{z \in x\gamma y} \mu_{\mathcal{B}}^{-}(z), \lambda^{-}\right\} \leq \max\{\mu_{\mathcal{B}}^{-}(x), \mu_{\mathcal{B}}^{-}(y), \delta^{-}\} < \max\left\{\lambda^{-}, \lambda^{-}, \delta^{-}\right\} = \lambda^{-}.$$

Hence $\inf_{z \in x\gamma y} \mu_{\mathcal{B}}^{+}(z) > \lambda^{+}$ and $\sup_{z \in x\gamma y} \mu_{\mathcal{B}}^{-}(z) < \lambda^{-}$. This shows that $x\gamma y \subseteq \mathcal{B}_{\lambda}$ for $z \in x\gamma y$ and $\gamma \in \Gamma$. Hence $\mathcal{B}_{\lambda}$ is a sub Γ-semihypergroup of H. The other cases can be seen in a similar way. ■

Theorem 198 *A non-empty subset* A *of* H *is a sub* Γ*-semihypergroup (resp., left* Γ*-hyperideal, right* Γ*-hyperideal, interior* Γ*-hyperideal, bi-*Γ*-hyperideal,* $(1,2)$-Γ*-hyperideal) of* H *if and only if the bipolar fuzzy subset* $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ *of* H *defined as follows:*

$$\mu_{\mathcal{B}}^{+}(x) = \begin{cases} \geq \delta^{+} & \text{if } x \in A \\ \lambda^{+} & \text{if } x \notin A, \end{cases} \quad \text{and} \quad \mu_{\mathcal{B}}^{-}(x) = \begin{cases} \leq \delta^{-} & \text{if } x \in A \\ \lambda^{-} & \text{if } x \notin A, \end{cases}$$

is a bipolar (λ, δ)*-fuzzy sub* Γ*-semihypergroup (resp., left* Γ*-hyperideal, right* Γ*-hyperideal, interior* Γ*-hyperideal, bi-*Γ*-hyperideal,* $(1,2)$-Γ*-hyperideal) of* H.

Proof. Suppose that A is a sub Γ-semihypergroup of H. Let $x, y \in H$ be such that $x, y \in A$ then $x\gamma y \subseteq A$ for $\gamma \in \Gamma$. Hence $\inf\limits_{z \in x\gamma y} \mu^+_{\mathcal{B}}(z) \geq \delta^+$ and $\sup\limits_{z \in x\gamma y} \mu^-_{\mathcal{B}}(z) \leq \delta^-$. Therefore

$$\max\left\{\inf_{z \in x\gamma y} \mu^+_{\mathcal{B}}(z), \lambda^+\right\} \geq \delta^+ = \min\left\{\mu^+_{\mathcal{B}}(x), \mu^+_{\mathcal{B}}(y), \delta^+\right\},$$

and

$$\min\left\{\sup_{z \in x\gamma y} \mu^-_{\mathcal{B}}(z), \lambda^-\right\} \leq \delta^- = \max\left\{\mu^-_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(y), \delta^-\right\}.$$

If $x \notin A$ or $y \notin A$, then $\min\left\{\mu^+_{\mathcal{B}}(x), \mu^+_{\mathcal{B}}(y), \delta^+\right\} = \lambda^+$ and $\max\left\{\mu^-_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(y), \delta^-\right\} = \lambda^-$. Thus

$$\max\left\{\inf_{z \in x\gamma y} \mu^+_{\mathcal{B}}(z), \lambda^+\right\} \geq \lambda^+ = \min\left\{\mu^+_{\mathcal{B}}(x), \mu^+_{\mathcal{B}}(y), \delta^+\right\},$$

and

$$\min\left\{\sup_{z \in x\gamma y} \mu^-_{\mathcal{B}}(z), \lambda^-\right\} \leq \lambda^- = \max\left\{\mu^-_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(y), \delta^-\right\}.$$

Consequently $\mathcal{B} = \left\langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \right\rangle$ is a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup of H.

Conversely: Let $x, y \in A$. Then $\mu^+_{\mathcal{B}}(x) \geq \delta^+$, $\mu^+_{\mathcal{B}}(y) \geq \delta^+$ and $\mu^-_{\mathcal{B}}(x) \leq \delta^-$, $\mu^-_{\mathcal{B}}(y) \leq \delta^-$. As $\mathcal{B} = \left\langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \right\rangle$ is a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup of H, therefore

$$\max\left\{\inf_{z \in x\gamma y} \mu^+_{\mathcal{B}}(z), \lambda^+\right\} \geq \min\left\{\mu^+_{\mathcal{B}}(x), \mu^+_{\mathcal{B}}(y), \delta^+\right\} \geq \min\left\{\delta^+, \delta^+, \delta^+\right\} = \delta^+,$$

and

$$\min\left\{\sup_{z \in x\gamma y} \mu^-_{\mathcal{B}}(z), \lambda^-\right\} \leq \max\left\{\mu^-_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(y), \delta^-\right\} \leq \max\left\{\delta^-, \delta^-, \delta^-\right\} = \delta^-.$$

This implies that $x\gamma y \subseteq A$. Hence A is a sub Γ-semihypergroup of H. The other cases can be seen in a similar way. ■

Theorem 199 *A non-empty subset A of H is a sub Γ-semihypergroup (resp., left Γ-hyperideal, right Γ-hyperideal, interior Γ-hyperideal, bi-Γ-hyperideal, $(1,2)$-Γ-hyperideal) of H if and only if $\mathcal{B}_A = \left\langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \right\rangle$ is a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup (resp., left Γ-hyperideal, right Γ-hyperideal, interior Γ-hyperideal, bi-Γ-hyperideal, $(1,2)$-Γ-hyperideal) of H.*

Proof. Let A be a sub Γ-semihypergroup of H. Then $\mathcal{B}_A = \left\langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \right\rangle$ is a bipolar fuzzy sub Γ-semihypergroup of H and by Corollary 196, $\mathcal{B}_A$ is a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup of H.

Conversely, let $x, y \in H$ be such that $x, y \in A$. Then $\mu^+_{\mathcal{B}_A}(x) = \mu^+_{\mathcal{B}_A}(y) = 1$ and $\mu^-_{\mathcal{B}_A}(x) = \mu^-_{\mathcal{B}_A}(y) = -1$. Since $\mathcal{B}_A$ is a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup of H. Therefore

$$\begin{aligned} \max\left\{\inf_{z\in x\gamma y}\mu^+_{\mathcal{B}}(z), \lambda^+\right\} &\geq \min\left\{\mu^+_{\mathcal{B}_A}(x), \mu^+_{\mathcal{B}_A}(y), \delta^+\right\} \\ &= \min\left\{1, 1, 1, \delta^+\right\} = \delta^+, \end{aligned}$$

and

$$\begin{aligned} \min\left\{\sup_{z\in x\gamma y}\mu^-_{\mathcal{B}}(z), \lambda^-\right\} &\leq \max\left\{\mu^-_{\mathcal{B}_A}(x), \mu^-_{\mathcal{B}_A}(y), \delta^-\right\} \\ &= \max\left\{-1, -1, -1, \delta^-\right\} = \delta^-. \end{aligned}$$

It implies that $\inf_{z\in x\gamma y}\mu^+_{\mathcal{B}}(z) \geq \delta^+$ and $\sup_{z\in x\gamma y}\mu^-_{\mathcal{B}}(z) \leq \delta^-$. Thus $z \in x\gamma y \subseteq A$ for $z \in x\gamma y$ and $\gamma \in \Gamma$. Therefore $\mathcal{B}_A = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \rangle$ is a sub Γ-semihypergroup of H. The other cases can be seen in a similar way. ■

Theorem 200 *Let $\mathcal{A} = \langle \mu^+_{\mathcal{A}}, \mu^-_{\mathcal{A}} \rangle$ be a bipolar (λ, δ)-fuzzy left Γ-hyperideal and $\mathcal{B} = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \rangle$ be a bipolar (λ, δ)-fuzzy right Γ-hyperideal of a Γ-semihypergroup H, then $\mathcal{A} \circ_\Gamma \mathcal{B}$ is a bipolar (λ, δ)-fuzzy Γ-hyperideal of H.*

Proof. The proof is straightforward. ■

Lemma 201 *Intersection of any family of bipolar (λ, δ)-fuzzy sub Γ-semihypergroups (resp., left Γ-hyperideals, right Γ-hyperideals, interior Γ-hyperideals, bi-Γ-hyperideals, $(1,2)$-Γ-hyperideals) of a Γ-semihypergroup H is a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup (resp., left Γ-hyperideal, right Γ-hyperideal, interior Γ-hyperideal, bi-Γ-hyperideal, $(1,2)$-Γ-hyperideal) of H.*

Proof. The proof is straightforward. ■

Now we prove that if $\mathcal{B} = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \rangle$ is a bipolar (λ, δ)-fuzzy Γ-hyperideal of H, then $\mathcal{B}^\delta_\lambda = \left\langle \mu^+_{\mathcal{B}^\delta_\lambda}, \mu^-_{\mathcal{B}^\delta_\lambda} \right\rangle$ is also a bipolar (λ, δ)-fuzzy Γ-hyperideal of H.

Definition 202 *Let $\mathcal{B} = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \rangle$ be a bipolar fuzzy subset of a Γ-semihypergroup H, $\lambda^+, \delta^+ \in (0, 1]$ such that $\lambda^+ < \delta^+$. We define the bipolar fuzzy subset $\mathcal{B}^\delta_\lambda = \left\langle \mu^+_{\mathcal{B}^\delta_\lambda}, \mu^-_{\mathcal{B}^\delta_\lambda} \right\rangle$ of H as follows,*

$$\mu^+_{\mathcal{B}^\delta_\lambda}(x) = \left(\mu^+_{\mathcal{B}}(x) \wedge \delta^+\right) \vee \lambda^+ \quad \textit{and} \quad \mu^-_{\mathcal{B}^\delta_\lambda}(x) = \left(\mu^-_{\mathcal{B}}(x) \vee \delta^-\right) \wedge \lambda^-$$

for all $x \in H$.

Definition 203 *Let* $\mathcal{A} = \langle \mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-} \rangle$ *and* $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ *be bipolar fuzzy subsets of a* Γ*-semihypergroup* H*. Then we define,*

1. *the bipolar fuzzy subset* $\mathcal{A} \wedge_{\lambda}^{\delta} \mathcal{B} = \left\langle \mu_{\mathcal{A}\wedge_{\lambda}^{\delta}\mathcal{B}}^{+}, \mu_{\mathcal{A}\vee_{\lambda}^{\delta}\mathcal{B}}^{-} \right\rangle$ *as follows:*

   $$\mu_{\mathcal{A}\wedge_{\lambda}^{\delta}\mathcal{B}}^{+}(x) = (\mu_{\mathcal{A}\wedge\mathcal{B}}^{+}(x) \wedge \delta^{+}) \vee \lambda^{+} \quad and \quad \mu_{\mathcal{A}\vee_{\lambda}^{\delta}\mathcal{B}}^{-}(x) = (\mu_{\mathcal{A}\vee\mathcal{B}}^{-}(x) \vee \delta^{-}) \wedge \lambda^{-}$$

2. *the bipolar fuzzy subset* $\mathcal{A} \vee_{\lambda}^{\delta} \mathcal{B} = \left\langle \mu_{\mathcal{A}\vee_{\lambda}^{\delta}\mathcal{B}}^{+}, \mu_{\mathcal{A}\wedge_{\lambda}^{\delta}\mathcal{B}}^{-} \right\rangle$ *as follows:*

   $$\mu_{\mathcal{A}\vee_{\lambda}^{\delta}\mathcal{B}}^{+}(x) = (\mu_{\mathcal{A}\vee\mathcal{B}}^{+}(x) \wedge \delta^{+}) \vee \lambda^{+} \quad and \quad \mu_{\mathcal{A}\wedge_{\lambda}^{\delta}\mathcal{B}}^{-}(x) = (\mu_{\mathcal{A}\wedge\mathcal{B}}^{-}(x) \vee \delta^{-}) \wedge \lambda^{-}$$

3. *the bipolar fuzzy subset* $\mathcal{A} \circ_{\lambda}^{\delta} \mathcal{B} = \left\langle \mu_{\mathcal{A}\circ_{\lambda}^{\delta}\mathcal{B}}^{+}, \mu_{\mathcal{A}\circ_{\lambda}^{\delta}\mathcal{B}}^{-} \right\rangle$ *as follows:*

   $$\mu_{\mathcal{A}\circ_{\lambda}^{\delta}\mathcal{B}}^{+}(x) = (\mu_{\mathcal{A}\circ_{\Gamma}\mathcal{B}}^{+}(x) \wedge \delta^{+}) \vee \lambda^{+} \quad and \quad \mu_{\mathcal{A}\circ_{\lambda}^{\delta}\mathcal{B}}^{-}(x) = (\mu_{\mathcal{A}\circ_{\Gamma}\mathcal{B}}^{-}(x) \vee \delta^{-}) \wedge \lambda^{-}$$

   for all $x \in H$*.*

Lemma 204 *Let* $\mathcal{A} = \langle \mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-} \rangle$ *and* $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ *be bipolar fuzzy subsets of a* Γ*-semihypergroup* H*. Then the following holds:*

1. $(\mathcal{A} \wedge_{\lambda}^{\delta} \mathcal{B}) = (\mathcal{A}_{\lambda}^{\delta} \wedge \mathcal{B}_{\lambda}^{\delta})$
2. $(\mathcal{A} \vee_{\lambda}^{\delta} \mathcal{B}) = (\mathcal{A}_{\lambda}^{\delta} \vee \mathcal{B}_{\lambda}^{\delta})$
3. $(\mathcal{A} \circ_{\lambda}^{\delta} \mathcal{B}) \geq (\mathcal{A}_{\lambda}^{\delta} \circ_{\Gamma} \mathcal{B}_{\lambda}^{\delta})$.

Proof. The proof is straightforward. ■

Theorem 205 *A bipolar fuzzy subset* $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ *of a* Γ*-semihypergroup* H *is a bipolar* (λ, δ)*-fuzzy,*

1. *sub* Γ*-semihypergroup of* H *if and only if* $\mathcal{B} \circ_{\lambda}^{\delta} \mathcal{B} \subseteq \mathcal{B}_{\lambda}^{\delta}$,
2. *left* Γ*-hyperideal of* H *if and only if* $\mathcal{H} \circ_{\lambda}^{\delta} \mathcal{B} \subseteq \mathcal{B}_{\lambda}^{\delta}$,
3. *right* Γ*-hyperideal of* H *if and only if* $\mathcal{B} \circ_{\lambda}^{\delta} H \subseteq \mathcal{B}_{\lambda}^{\delta}$,
4. *bi-*Γ*-hyperideal of* H *if and only if* $\mathcal{B} \circ_{\lambda}^{\delta} \mathcal{B} \subseteq \mathcal{B}_{\lambda}^{\delta}$ *and* $\mathcal{B} \circ_{\lambda}^{\delta} \mathcal{H} \circ_{\lambda}^{\delta} \mathcal{B} \subseteq \mathcal{B}_{\lambda}^{\delta}$,
5. *interior* Γ*-hyperideal of* H *if and only if* $\mathcal{H} \circ_{\lambda}^{\delta} \mathcal{B} \circ_{\lambda}^{\delta} \mathcal{H} \subseteq \mathcal{B}_{\lambda}^{\delta}$,

6. $(1,2)$-Γ-*hyperideal of* H *if and only if* $\mathcal{B} \circ^{\delta}_{\lambda} \mathcal{B} \subseteq \mathcal{B}^{\delta}_{\lambda}$ *and* $\mathcal{B} \circ^{\delta}_{\lambda} \mathcal{H} \circ^{\delta}_{\lambda} \mathcal{B} \circ^{\delta}_{\lambda} \mathcal{B} \subseteq \mathcal{B}^{\delta}_{\lambda}$.

Proof. (1) Let $\mathcal{B} = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \rangle$ be a bipolar (λ,δ)-fuzzy sub Γ-semihypergroup of H and $z \in H$. Let us suppose that $z \in x\gamma y$ for $x, y \in H$ and $\gamma \in \Gamma$. Then

$$\begin{aligned}
\mu^+_{\mathcal{B}\circ^{\delta}_{\lambda}\mathcal{B}}(z) &= (\mu^+_{\mathcal{B}\circ_{\Gamma}\mathcal{B}}(z) \wedge \delta^+) \vee \lambda^+ \\
&= \left(\left\{\bigvee_{z\in x\gamma y} \{\mu^+_{\mathcal{B}}(x) \wedge \mu^+_{\mathcal{B}}(y)\}\right\} \wedge \delta^+\right) \vee \lambda^+ \\
&= \left(\left\{\bigvee_{z\in x\gamma y} \{\mu^+_{\mathcal{B}}(x) \wedge \mu^+_{\mathcal{B}}(y) \wedge \delta^+\}\right\} \wedge \delta^+\right) \vee \lambda^+ \\
&\leq \left(\left\{\bigvee_{z\in x\gamma y} \left(\inf_{z\in x\gamma y} \mu^+_{\mathcal{B}}(z) \vee \lambda^+\right)\right\} \wedge \delta^+\right) \vee \lambda^+ \\
&= \left(\left\{\bigvee_{z\in x\gamma y} \left(\inf_{z\in x\gamma y} \mu^+_{\mathcal{B}}(z) \wedge \delta^+\right)\right\} \vee \lambda^+\right) \vee \lambda^+ \\
&\leq \left(\bigvee_{z\in x\gamma y} (\mu^+_{\mathcal{B}}(x\gamma y) \wedge \delta^+)\right) \vee \lambda^+ \vee \lambda^+ \\
&= (\mu^+_{\mathcal{B}}(z) \wedge \delta^+) \vee \lambda^+ = \mu^+_{\mathcal{B}^{\delta}_{\lambda}}(z).
\end{aligned}$$

and

$$\begin{aligned}
\mu^-_{\mathcal{B}\circ^{\delta}_{\lambda}\mathcal{B}}(z) &= (\mu^-_{\mathcal{B}\circ_{\Gamma}\mathcal{B}}(z) \vee \delta^-) \wedge \lambda^- \\
&= \left(\left\{\bigwedge_{z\in x\gamma y} \{\mu^-_{\mathcal{B}}(x) \vee \mu^-_{\mathcal{B}}(y)\}\right\} \vee \delta^-\right) \wedge \lambda^- \\
&= \left(\left\{\bigwedge_{z\in x\gamma y} \{\mu^-_{\mathcal{B}}(x) \vee \mu^-_{\mathcal{B}}(y) \vee \delta^-\}\right\} \vee \delta^-\right) \wedge \lambda^- \\
&\geq \left(\left\{\bigwedge_{z\in x\gamma y} \left(\sup_{z\in x\gamma y} \mu^-_{\mathcal{B}}(z) \wedge \lambda^-\right)\right\} \vee \delta^-\right) \wedge \lambda^- \\
&= \left(\left\{\bigwedge_{z\in x\gamma y} \left(\sup_{z\in x\gamma y} \mu^-_{\mathcal{B}}(z) \vee \delta^-\right)\right\} \wedge \lambda^-\right) \wedge \lambda^- \\
&\geq \left(\bigwedge_{z\in x\gamma y} (\mu^-_{\mathcal{B}}(x\gamma y) \vee \delta^-)\right) \wedge \lambda^- \wedge \lambda^- \\
&= (\mu^-_{\mathcal{B}}(z) \vee \delta^-) \wedge \lambda^- = \mu^-_{\mathcal{B}^{\delta}_{\lambda}}(z).
\end{aligned}$$

If there do not exist any $x, y \in H$ and $\gamma \in \Gamma$ such that $z \in x\gamma y$, then

$$\mu^+_{\mathcal{B}\circ^{\delta}_{\lambda}\mathcal{B}}(x) = \lambda^+ \leq \mu^+_{\mathcal{B}^{\delta}_{\lambda}}(x),$$

and

$$\mu^-_{\mathcal{B}\circ^\delta_\lambda\mathcal{B}}(x) = \lambda^- \geq \mu^-_{\mathcal{B}^\delta_\lambda}(x).$$

Hence $\mathcal{B}\circ^\delta_\lambda \mathcal{B} \subseteq \mathcal{B}^\delta_\lambda$.

Conversely, assume that $\mathcal{B}\circ^\delta_\lambda \mathcal{B} \subseteq \mathcal{B}^\delta_\lambda$. If there exist $m, n \in H$ and $\beta \in \Gamma$ such that $x\gamma y \subseteq m\beta n$, then

$$\begin{aligned}
\max\left\{\mu^+_{\mathcal{B}}(x\gamma y), \lambda^+\right\} &\geq (\mu^+_{\mathcal{B}}(x\gamma y) \wedge \delta^+) \vee \lambda^+ = \mu^+_{\mathcal{B}^\delta_\lambda}(x\gamma y) \geq \mu^+_{\mathcal{B}\circ^\delta_\lambda\mathcal{B}}(x\gamma y) \\
&= \left(\left\{\bigvee_{x\gamma y\subseteq m\beta n}\left\{\mu^+_{\mathcal{B}}(m) \wedge \mu^+_{\mathcal{B}}(n)\right\}\right\} \wedge \delta^+\right) \vee \lambda^+ \\
&\geq \left((\mu^+_{\mathcal{B}}(x) \wedge \mu^+_{\mathcal{B}}(y)) \wedge \delta^+\right) \vee \lambda^+ \\
&\geq \mu^+_{\mathcal{B}}(x) \wedge \mu^+_{\mathcal{B}}(y)) \wedge \delta^+ \\
\max\left\{\inf_{z\in x\gamma y}\mu^+_{\mathcal{B}}(z), \lambda^+\right\} &\geq \min\left\{\mu^+_{\mathcal{B}}(x), \mu^+_{\mathcal{B}}(y)), \delta^+\right\},
\end{aligned}$$

and

$$\begin{aligned}
\min\left\{\mu^-_{\mathcal{B}}(x\gamma y), \lambda^-\right\} &\leq (\mu^-_{\mathcal{B}}(x\gamma y) \vee \delta^-) \wedge \lambda^- = \mu^-_{\mathcal{B}^\delta_\lambda}(x\gamma y) \leq \mu^-_{\mathcal{B}\circ^\delta_\lambda\mathcal{B}}(x\gamma y) \\
&= \left(\left\{\bigwedge_{x\gamma y\subseteq m\beta n}\left\{\mu^-_{\mathcal{B}}(m) \vee \mu^-_{\mathcal{B}}(n)\right\}\right\} \vee \delta^-\right) \wedge \lambda^- \\
&\leq \left((\mu^-_{\mathcal{B}}(x) \vee \mu^-_{\mathcal{B}}(y)) \vee \delta^-\right) \wedge \lambda^- \\
&\leq \mu^-_{\mathcal{B}}(x) \vee \mu^-_{\mathcal{B}}(y) \vee \delta^- \\
\min\left\{\sup_{z\in x\gamma y}\mu^-_{\mathcal{B}}(z), \lambda^-\right\} &\leq \max\left\{\mu^-_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(y), \delta^-\right\}.
\end{aligned}$$

Hence $\mathcal{B} = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}}\rangle$ is a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup of H. The proofs of (2), (3), (4), (5) and (6) are similar to the proof of (1). ■

Proposition 206 *If $\mathcal{B} = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}}\rangle$ is a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup (resp., left Γ-hyperideal, right Γ-hyperideal, interior Γ-hyperideal, bi-Γ-hyperideal, $(1, 2)$-Γ-hyperideal) of a Γ-semihypergroup H, then $\mathcal{B}^\delta_\lambda = \left\langle \mu^+_{\mathcal{B}^\delta_\lambda}, \mu^-_{\mathcal{B}^\delta_\lambda}\right\rangle$ is also a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup (resp., left Γ-hyperideal, right Γ-hyperideal, interior Γ-hyperideal, bi-Γ-hyperideal, $(1, 2)$-Γ-hyperideal) of H.*

Proof. Suppose $\mathcal{B} = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}}\rangle$ is a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup of H.

Let $x, y \in H$ and $\gamma \in \Gamma$. Then

$$\begin{aligned}
\max\left\{\inf_{z\in x\gamma y}\mu^{+}_{\mathcal{B}^{\delta}_{\lambda}}(z), \lambda^{+}\right\} &= \left(\inf_{z\in x\gamma y}\left\{\left(\mu^{+}_{\mathcal{B}}(z)\wedge\delta^{+}\right)\vee\lambda^{+}\right\}\right)\vee\lambda^{+} \\
&= \left(\inf_{z\in x\gamma y}\mu^{+}_{\mathcal{B}}(z)\wedge\delta^{+}\right)\vee\lambda^{+} \\
&= \left(\inf_{z\in x\gamma y}\mu^{+}_{\mathcal{B}}(z)\vee\lambda^{+}\right)\wedge\left(\delta^{+}\vee\lambda^{+}\right) \\
&= \left(\inf_{z\in x\gamma y}\mu^{+}_{\mathcal{B}}(z)\vee\lambda^{+}\right)\wedge\delta^{+} \\
&= \left(\left(\inf_{z\in x\gamma y}\mu^{+}_{\mathcal{B}}(z)\vee\lambda^{+}\right)\vee\lambda^{+}\right)\wedge\delta^{+} \\
&\geq \left(\left(\mu^{+}_{\mathcal{B}}(x)\wedge\mu^{+}_{\mathcal{B}}(y)\wedge\delta^{+}\right)\vee\lambda^{+}\right)\wedge\delta^{+} \\
&= \left(\left(\mu^{+}_{\mathcal{B}}(x)\wedge\mu^{+}_{\mathcal{B}}(y)\wedge\delta^{+}\wedge\delta^{+}\right)\vee\lambda^{+}\vee\lambda^{+}\right)\wedge\delta^{+} \\
&= \left(\left(\mu^{+}_{\mathcal{B}}(x)\wedge\delta^{+}\right)\vee\lambda^{+}\right)\wedge\left(\left(\mu^{+}_{\mathcal{B}}(y)\wedge\delta^{+}\right)\vee\lambda^{+}\right)\wedge\delta^{+} \\
&= \mu^{+}_{\mathcal{B}^{\delta}_{\lambda}}(x)\wedge\mu^{+}_{\mathcal{B}^{\delta}_{\lambda}}(y)\wedge\delta^{+} \\
&= \min\left\{\mu^{+}_{\mathcal{B}^{\delta}_{\lambda}}(x), \mu^{+}_{\mathcal{B}^{\delta}_{\lambda}}(y), \delta^{+}\right\}
\end{aligned}$$

and

$$\begin{aligned}
\min\left\{\sup_{z\in x\gamma y}\mu^{-}_{\mathcal{B}^{\delta}_{\lambda}}(z), \lambda^{-}\right\} &= \left(\sup_{z\in x\gamma y}\left\{\left(\mu^{-}_{\mathcal{B}}(z)\vee\delta^{-}\right)\wedge\lambda^{-}\right\}\right)\wedge\lambda^{-} \\
&= \left(\sup_{z\in x\gamma y}\mu^{-}_{\mathcal{B}}(z)\vee\delta^{-}\right)\wedge\lambda^{-} \\
&= \left(\sup_{z\in x\gamma y}\mu^{-}_{\mathcal{B}}(z)\wedge\lambda^{-}\right)\vee\left(\delta^{-}\wedge\lambda^{+}\right) \\
&= \left(\sup_{z\in x\gamma y}\mu^{-}_{\mathcal{B}}(z)\wedge\lambda^{-}\right)\vee\delta^{-} \\
&= \left(\left(\sup_{z\in x\gamma y}\mu^{-}_{\mathcal{B}}(z)\wedge\lambda^{-}\right)\wedge\lambda^{-}\right)\vee\delta^{-} \\
&\geq \left(\left(\mu^{-}_{\mathcal{B}}(x)\vee\mu^{-}_{\mathcal{B}}(y)\vee\delta^{-}\right)\wedge\lambda^{-}\right)\vee\delta^{-} \\
&= \left(\left(\mu^{-}_{\mathcal{B}}(x)\vee\mu^{-}_{\mathcal{B}}(y)\vee\delta^{-}\vee\delta^{-}\right)\wedge\lambda^{-}\wedge\lambda^{-}\right)\vee\delta^{-} \\
&= \left(\left(\mu^{-}_{\mathcal{B}}(x)\vee\delta^{-}\right)\wedge\lambda^{-}\right)\vee\left(\left(\mu^{-}_{\mathcal{B}}(y)\vee\delta^{-}\right)\wedge\lambda^{-}\right)\vee\delta^{-} \\
&= \mu^{-}_{\mathcal{B}^{\delta}_{\lambda}}(x)\vee\mu^{-}_{\mathcal{B}^{\delta}_{\lambda}}(y)\vee\delta^{-} \\
&= \max\left\{\mu^{-}_{\mathcal{B}^{\delta}_{\lambda}}(x), \mu^{-}_{\mathcal{B}^{\delta}_{\lambda}}(y), \delta^{-}\right\}
\end{aligned}$$

Hence $\mathcal{B}^{\delta}_{\lambda} = \left\langle \mu^{+}_{\mathcal{B}^{\delta}_{\lambda}}, \mu^{-}_{\mathcal{B}^{\delta}_{\lambda}} \right\rangle$ is a bipolar (λ, δ)-fuzzy sub Γ-semihypergroup of H. The other cases can be seen in a similar way. ■

Theorem 207 *Let* $\mathcal{A} = \langle \mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-} \rangle$ *be a bipolar* (λ, δ)*-fuzzy right* Γ*-hyperideal and* $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ *be a bipolar* (λ, δ)*-fuzzy left* Γ*-hyperideal of* H*. Then* $\mathcal{A} \circ_{\lambda}^{\delta} \mathcal{B} \subseteq \mathcal{A} \wedge_{\lambda}^{\delta} \mathcal{B}$.

Proof. Let $\mathcal{A} = \langle \mu_{\mathcal{A}}^{+}, \mu_{\mathcal{A}}^{-} \rangle$ be a bipolar (λ, δ)-fuzzy right Γ-hyperideal and $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ be a bipolar (λ, δ)-fuzzy left Γ-hyperideal of H. Then for all $z \in H$, we have

$$\begin{aligned}
\mu^{+}_{\mathcal{A}\circ^{\delta}_{\lambda}\mathcal{B}}(z) &= \left(\mu^{+}_{\mathcal{A}\circ_{\Gamma}\mathcal{B}}(z) \wedge \delta^{+}\right) \vee \lambda^{+} \\
&= \left(\left\{\bigvee_{z\in x\gamma y} \{\mu^{+}_{\mathcal{A}}(x) \wedge \mu^{+}_{\mathcal{B}}(y)\}\right\} \wedge \delta^{+}\right) \vee \lambda^{+} \\
&= \left(\bigvee_{z\in x\gamma y} \{\mu^{+}_{\mathcal{A}}(x) \wedge \mu^{+}_{\mathcal{B}}(y) \wedge \delta^{+}\}\right) \vee \lambda^{+} \\
&= \left(\bigvee_{z\in x\gamma y} \{(\mu^{+}_{\mathcal{A}}(x) \wedge \delta^{+}) \wedge (\mu^{+}_{\mathcal{B}}(y) \wedge \delta^{+}) \wedge \delta^{+}\}\right) \vee \lambda^{+} \\
&\leq \left(\bigvee_{z\in x\gamma y} \left\{\left(\inf_{z\in x\gamma y}\mu^{+}_{\mathcal{A}}(z) \vee \lambda^{+}\right) \wedge \left(\inf_{z\in x\gamma y}\mu^{+}_{\mathcal{B}}(z) \vee \lambda^{+}\right) \wedge \delta^{+}\right\}\right) \vee \lambda^{+} \\
&= \left((\mu^{+}_{\mathcal{A}}(z) \vee \lambda^{+}) \wedge (\mu^{+}_{\mathcal{B}}(z) \vee \lambda^{+}) \wedge \delta^{+}\right) \vee \lambda^{+} \\
&= \left\{((\mu^{+}_{\mathcal{A}}(z) \wedge \mu^{+}_{\mathcal{B}}(z)) \vee \lambda^{+}) \wedge \delta^{+}\right\} \vee \lambda^{+} \\
&= \left\{(\mu^{+}_{\mathcal{A}} \wedge \mu^{+}_{\mathcal{B}})(z) \wedge \delta^{+}\right\} \vee \lambda^{+} \\
&= \mu^{+}_{\mathcal{A}\wedge^{\delta}_{\lambda}\mathcal{B}}(z),
\end{aligned}$$

and

$$\begin{aligned}
\mu^{-}_{\mathcal{A}\circ^{\delta}_{\lambda}\mathcal{B}}(z) &= \left(\mu^{-}_{\mathcal{A}\circ_{\Gamma}\mathcal{B}}(z) \vee \delta^{-}\right) \wedge \lambda^{-} \\
&= \left(\left\{\bigwedge_{z\in x\gamma y} \{\mu^{-}_{\mathcal{A}}(x) \vee \mu^{-}_{\mathcal{B}}(y)\}\right\} \vee \delta^{-}\right) \wedge \lambda^{-} \\
&= \left(\bigwedge_{z\in x\gamma y} \{\mu^{-}_{\mathcal{A}}(x) \vee \mu^{-}_{\mathcal{B}}(y) \vee \delta^{-}\}\right) \wedge \lambda^{-} \\
&= \left(\bigwedge_{z\in x\gamma y} \{(\mu^{-}_{\mathcal{A}}(x) \vee \delta^{-}) \vee (\mu^{-}_{\mathcal{B}}(y) \vee \delta^{-}) \vee \delta^{-}\}\right) \wedge \lambda^{-} \\
&\geq \left(\bigwedge_{z\in x\gamma y} \left\{\left(\sup_{z\in x\gamma y}\mu^{-}_{\mathcal{A}}(z) \wedge \lambda^{-}\right) \vee \left(\sup_{z\in x\gamma y}\mu^{-}_{\mathcal{B}}(z) \wedge \lambda^{-}\right) \vee \delta^{-}\right\}\right) \wedge \lambda^{-} \\
&= \left((\mu^{-}_{\mathcal{A}}(z) \wedge \lambda^{-}) \vee (\mu^{-}_{\mathcal{B}}(z) \wedge \lambda^{-}) \vee \delta^{-}\right) \wedge \lambda^{-} \\
&= \left\{((\mu^{-}_{\mathcal{A}}(z) \vee \mu^{-}_{\mathcal{B}}(z)) \wedge \lambda^{-}) \vee \delta^{-}\right\} \wedge \lambda^{-} \\
&= \left\{(\mu^{-}_{\mathcal{A}} \vee \mu^{-}_{\mathcal{B}})(z) \vee \delta^{-}\right\} \wedge \lambda^{-} \\
&= \mu^{-}_{\mathcal{A}\wedge^{\delta}_{\lambda}\mathcal{B}}(z).
\end{aligned}$$

If there do not exist any $x, y \in H$ and $\gamma \in \Gamma$ such that $z \in x\gamma y$, then

$$\begin{aligned} \mu^{+}_{\mathcal{A}\circ^{\delta}_{\lambda}\mathcal{B}}(x) &= \left(\mu^{+}_{\mathcal{A}\circ_{\Gamma}\mathcal{B}}(x) \wedge \delta^{+}\right) \vee \lambda^{+} = 0 \vee \lambda^{+} = \lambda^{+} \\ &\leq \left(\mu^{+}_{\mathcal{A}\wedge\mathcal{B}}(x) \wedge \delta^{+}\right) \vee \lambda^{+} = \mu^{+}_{\mathcal{A}\wedge^{\delta}_{\lambda}\mathcal{B}}(x), \end{aligned}$$

and

$$\begin{aligned} \mu^{-}_{\mathcal{A}\circ^{\delta}_{\lambda}\mathcal{B}}(x) &= \left(\mu^{-}_{\mathcal{A}\circ_{\Gamma}\mathcal{B}}(x) \vee \delta^{-}\right) \wedge \lambda^{-} = 0 \wedge \lambda^{-} = \lambda^{-} \\ &\geq \left(\mu^{-}_{\mathcal{A}\vee\mathcal{B}}(x) \vee \delta^{-}\right) \wedge \lambda^{-} = \mu^{-}_{\mathcal{A}\vee^{\delta}_{\lambda}\mathcal{B}}(x). \end{aligned}$$

Hence we get $\mathcal{A} \circ^{\delta}_{\lambda} \mathcal{B} \subseteq \mathcal{A} \wedge^{\delta}_{\lambda} \mathcal{B}$. ■

Chapter 4

Rough Approximations in Γ-Semihypergroups

The results offered in this chapter are a part of our papers [79, 80, 81, 84].

The rough set theory introduced by Pawlak [59] has been applied to many branches, especially in cellular automata and neural networks. Several authors applied the theory of rough sets in different algebraic hyperstructures. Ameri et al. [8] developed some results on approximations of bi-hyperideals of semihypergroups. Dehkordi and Davvaz [28] applied the rough set theory to the ideal theory of Γ-semihyperrings. He et al. [31] studied the ideal theory of hyperlattices in terms of rough sets. Leoreanu-Fotea [51] introduced the notion of lower and upper approximations in a hypergroup. Leoreanu-Fotea and Davvaz [52] applied roughness to n-ary hypergroups.

4.1 Rough Subsets in Γ-Semihypergroups

In this section we will present some results on rough subsets in a Γ-semihypergroup.

Proposition 208 *Let ρ be a regular relation on a Γ-semihypergroup H. If A and B are non-empty subsets of H, then the following hold:*

1. $\underline{Apr}_\rho(A) \subseteq A \subseteq \overline{Apr}_\rho(A)$;
2. $\overline{Apr}_\rho(A \cup B) = \overline{Apr}_\rho(A) \cup \overline{Apr}_\rho(B)$;

3. $\underline{Apr}_{\rho}(A \cap B) = \underline{Apr}_{\rho}(A) \cap \underline{Apr}_{\rho}(B)$;

4. $A \subseteq B$ *implies* $\underline{Apr}_{\rho}(A) \subseteq \underline{Apr}_{\rho}(B)$;

5. $A \subseteq B$ *implies* $\overline{Apr}_{\rho}(A) \subseteq \overline{Apr}_{\rho}(B)$;

6. $\underline{Apr}_{\rho}(A \cup B) \supseteq \underline{Apr}_{\rho}(A) \cup \underline{Apr}_{\rho}(B)$;

7. $\overline{Apr}_{\rho}(A \cap B) \subseteq \overline{Apr}_{\rho}(A) \cap \overline{Apr}_{\rho}(B)$.

Proof. (1) If $a \in \underline{Apr}_{\rho}(A)$, then $a \in [a]_{\rho} \subseteq A$. Hence $\underline{Apr}_{\rho}(A) \subseteq A$. Next, if $a \in A$, then, since $a \in [a]_{\rho}$, we have $[a]_{\rho} \cap A \neq \emptyset$, and so $a \in \overline{Apr}_{\rho}(A)$. Thus $A \subseteq \overline{Apr}_{\rho}(A)$.

(2) Note that

$$\begin{aligned} a \in \overline{Apr}_{\rho}(A \cup B) &\iff [a]_{\rho} \cap (A \cup B) \neq \emptyset \\ &\iff ([a]_{\rho} \cap A) \cup ([a]_{\rho} \cap B) \neq \emptyset \\ &\iff [a]_{\rho} \cap A \neq \emptyset \quad \text{or} \quad [a]_{\rho} \cap B \neq \emptyset \\ &\iff a \in \overline{Apr}_{\rho}(A) \quad \text{or} \quad a \in \overline{Apr}_{\rho}(B) \\ &\iff a \in \overline{Apr}_{\rho}(A) \cup \overline{Apr}_{\rho}(B). \end{aligned}$$

Thus

$$\overline{Apr}_{\rho}(A \cup B) = \overline{Apr}_{\rho}(A) \cup \overline{Apr}_{\rho}(B).$$

(3) Note that

$$\begin{aligned} a \in \underline{Apr}_{\rho}(A \cap B) &\iff [a]_{\rho} \subseteq A \cap B \\ &\iff [a]_{\rho} \subseteq A \quad \text{and} \quad [a]_{\rho} \subseteq B \\ &\iff a \in \underline{Apr}_{\rho}(A) \quad \text{and} \quad a \in \underline{Apr}_{\rho}(B) \\ &\iff a \in \underline{Apr}_{\rho}(A) \cap \underline{Apr}_{\rho}(B). \end{aligned}$$

Thus

$$\underline{Apr}_{\rho}(A \cap B) = \underline{Apr}_{\rho}(A) \cap \underline{Apr}_{\rho}(B).$$

(4) Since $A \subseteq B$ if and only if $A \cap B = A$, by (3) we have

$$\underline{Apr}_{\rho}(A) = \underline{Apr}_{\rho}(A \cap B) = \underline{Apr}_{\rho}(A) \cap \underline{Apr}_{\rho}(B).$$

This implies that

$$\underline{Apr}_{\rho}(A) \subseteq \underline{Apr}_{\rho}(B).$$

(5) Since $A \subseteq B$ if and only if $A \cup B = B$, by (2) we have

$$\overline{Apr}_{\rho}(B) = \overline{Apr}_{\rho}(A \cup B) = \overline{Apr}_{\rho}(A) \cup \overline{Apr}_{\rho}(B).$$

This implies that

$$\overline{Apr}_{\rho}(A) \subseteq \overline{Apr}_{\rho}(B).$$

(6) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by (4) we have

$$\underline{Apr}_{\rho}(A) \subseteq \underline{Apr}_{\rho}(A \cup B) \quad \text{and} \quad \underline{Apr}_{\rho}(B) \subseteq \underline{Apr}_{\rho}(A \cup B).$$

Which yields

$$\underline{Apr}_{\rho}(A) \cup \underline{Apr}_{\rho}(B) \subseteq \underline{Apr}_{\rho}(A \cup B).$$

(7) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by (5) we have

$$\overline{Apr}_{\rho}(A \cap B) \subseteq \overline{Apr}_{\rho}(A) \quad \text{and} \quad \overline{Apr}_{\rho}(A \cap B) \subseteq \overline{Apr}_{\rho}(B).$$

Which yields

$$\overline{Apr}_{\rho}(A \cap B) \subseteq \overline{Apr}_{\rho}(A) \cap \overline{Apr}_{\rho}(B).$$

This completes the proof. ■

Lemma 209 *Let ρ be a regular relation on a Γ-semihypergroup H. Then, for a non-empty subset A of H*

1. $\left(\overline{Apr}_{\rho}(A)\right)^n \subseteq \overline{Apr}_{\rho}(A^n)$ *for all* $n \in \mathbb{N}$.
2. *If ρ is complete, then* $\left(\underline{Apr}_{\rho}(A)\right)^n \subseteq \underline{Apr}_{\rho}(A^n)$ *for all* $n \in \mathbb{N}$.

Proof. (1) Let A be a non-empty subset of H, then for $n = 2$, and by Theorem 82(1), we get

$$\left(\overline{Apr}_{\rho}(A)\right)^2 = \overline{Apr}_{\rho}(A)\Gamma\overline{Apr}_{\rho}(A) \subseteq \overline{Apr}_{\rho}(A\Gamma A) = \overline{Apr}_{\rho}(A^2).$$

Now for $n = 3$, we get

$$\begin{aligned} \left(\overline{Apr}_{\rho}(A)\right)^3 &= \overline{Apr}_{\rho}(A)\Gamma\left(\overline{Apr}_{\rho}(A)\right)^2 \subseteq \overline{Apr}_{\rho}(A)\Gamma\overline{Apr}_{\rho}(A^2) \\ &\subseteq \overline{Apr}_{\rho}(A\Gamma A^2) = \overline{Apr}_{\rho}(A^3). \end{aligned}$$

Suppose that the result is true for $n = k-1$, such that $(\overline{Apr}_{\rho}(A))^{k-1} \subseteq \overline{Apr}_{\rho}(A^{k-1})$, then for $n = k$, we get

$$\begin{aligned} \left(\overline{Apr}_{\rho}(A)\right)^k &= \overline{Apr}_{\rho}(A)\Gamma\left(\overline{Apr}_{\rho}(A)\right)^{k-1} \subseteq \overline{Apr}_{\rho}(A)\Gamma\overline{Apr}_{\rho}(A^{k-1}) \\ &\subseteq \overline{Apr}_{\rho}(A\Gamma A^{k-1}) = \overline{Apr}_{\rho}(A^k). \end{aligned}$$

Hence, this shows that $\left(\overline{Apr}_{\rho}(A)\right)^{k} \subseteq \overline{Apr}_{\rho}(A^{k})$. This implies that $\left(\overline{Apr}_{\rho}(A)\right)^{n} \subseteq \overline{Apr}_{\rho}(A^{n})$ is true for all $n \in \mathbb{N}$. By using Theorem 82(2), the proof of (2) can be seen in a similar way. This completes the proof. ■

4.2 Rough Prime Bi-Γ-Hyperideals

Let ρ be a regular relation on a Γ-semihypergroup H. Then a subset A of H is called a ρ-lower rough prime bi-Γ-hyperideal of H if $\underline{Apr}_{\rho}(A)$ is a prime bi-Γ-hyperideal of H. A ρ-upper rough prime bi-Γ-hyperideal of H is defined analogously. A is called a rough prime bi-Γ-hyperideal of H if A is a ρ-lower and a ρ-upper rough prime bi-Γ-hyperideal of H.

Theorem 210 *Let ρ be a complete regular relation on a Γ-semihypergroup H and A be a prime bi-Γ-hyperideal of H. Then A is a ρ-upper rough prime bi-Γ-hyperideal of H.*

Proof. Since A is a bi-Γ-hyperideal of H, then by Theorem 88(1), $\overline{Apr}_{\rho}(A)$ is bi-Γ-hyperideal of H. Let a be any arbitrary element of H, then for $\beta, \gamma \in \Gamma$, we have

$$x\beta a\gamma y \subseteq \overline{Apr}_{\rho}(A), \text{ for some } x, y \in H.$$

Then

$$[x]_{\rho}\beta[a]_{\rho}\gamma[y]_{\rho} \cap A = [x\beta a\gamma y]_{\rho} \cap A \neq \emptyset.$$

Thus there exist $x' \in [x]_{\rho}$, $a' \in [a]_{\rho}$ and $y' \in [y]_{\rho}$ such that $x'\beta a'\gamma y' \subseteq A$. Since A is a prime bi-Γ-hyperideal, we have $x' \in A$ or $y' \in A$. Thus

$$x' \in [x]_{\rho} \cap A \quad \text{or} \quad y' \in [y]_{\rho} \cap A.$$

Thus

$$[x]_{\rho} \cap A \neq \emptyset \quad \text{or} \quad [y]_{\rho} \cap A \neq \emptyset,$$

and so $x \in \overline{Apr}_{\rho}(A)$ or $y \in \overline{Apr}_{\rho}(A)$. Therefore $\overline{Apr}_{\rho}(A)$ is a prime bi-Γ-hyperideal of H. Here a was any arbitrary element of H, so this theorem holds for all $a \in H$. ■

Theorem 211 *Let ρ be a complete regular relation on a Γ-semihypergroup H and A be a prime bi-Γ-hyperideal of H. Then $\underline{Apr}_{\rho}(A)$ is, if it is non-empty, a prime bi-Γ-hyperideal of H.*

Proof. Since A is a bi-Γ-hyperideal of H, by Theorem 88(2), we know that $\underline{Apr}_\rho(A)$ is a bi-Γ-hyperideal of H. Let a be any arbitrary element of H, then for $\beta, \gamma \in \Gamma$, we have

$$x\beta a\gamma y \subseteq \underline{Apr}_\rho(A), \text{ for some } x, y \in H.$$

Then

$$[x]_\rho\beta[a]_\rho\gamma[y]_\rho = [x\beta a\gamma y]_\rho \subseteq A.$$

We suppose that $\underline{Apr}_\rho(A)$ is not a prime bi-Γ-hyperideal, then there exist $x, y \in H$ and an arbitrary element $a \in H$, such that $x\beta a\gamma y \in \underline{Apr}_\rho(A)$ but $x \notin \underline{Apr}_\rho(A)$ and $y \notin \underline{Apr}_\rho(A)$ for $\beta, \gamma \in \Gamma$. Thus

$$[x]_\rho \nsubseteq A \text{ and } [y]_\rho \nsubseteq A.$$

Then there exist

$$x' \in [x]_\rho,\ x' \notin A \text{ and } y' \in [y]_\rho,\ y' \notin A.$$

Thus for $\beta, \gamma \in \Gamma$,

$$x'\beta a\gamma y' \subseteq [x]_\rho\beta[a]_\rho\gamma[y]_\rho \subseteq A.$$

Since A is a prime bi-Γ-hyperideal, we have $x' \in A$ or $y' \in A$. It contradicts the supposition. This means that $\underline{Apr}_\rho(A)$ is, if it is non-empty, a prime bi-Γ-hyperideal of H. Here a was any arbitrary element of H, so this theorem holds for all $a \in H$. ■

The following example shows that the converse of Theorem 210 and Theorem 211 does not hold.

Example 212 *Let $H = \{1, 2, 3, 4\}$ and $\Gamma = \{\beta, \gamma\}$ be the sets of binary hyperoperations defined below:*

β	1	2	3	4
1	1	2	$\{1,3\}$	4
2	2	2	2	4
3	1	2	$\{1,3\}$	4
4	4	4	4	4

γ	1	2	3	4
1	1	2	$\{1,3\}$	4
2	2	2	2	4
3	$\{1,3\}$	2	3	4
4	4	4	4	4

Here H is a Γ-semihypergroup. Let ρ be a complete regular relation on H such that the ρ-regular classes are the subsets $\{1, 2, 3\}$, $\{4\}$. Now for $A = \{2, 3, 4\} \subseteq H$, $\overline{Apr}_\rho(A) = \{1, 2, 3, 4\}$ and $\underline{Apr}_\rho(A) = \{4\}$. It is clear that $\overline{Apr}_\rho(A)$ and $\underline{Apr}_\rho(A)$ are prime bi-Γ-hyperideals of H, but A is not a prime bi-Γ-hyperideal of H.

4.3 Rough (m,n) Bi-Γ-Hyperideals

Let ρ be a regular relation on a Γ-semihypergroup H. A subset A of H is called a ρ-upper rough $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of H if $\overline{Apr}_\rho(A)$ is an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of H. Similarly, a subset A of a Γ-semihypergroup H is called a ρ-lower rough $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of H if $\underline{Apr}_\rho(A)$ is an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of H.

Theorem 213 *Let ρ be a regular relation on a Γ-semihypergroup H and A be an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of H. Then,*

1. *$\overline{Apr}_\rho(A)$ is an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of H.*
2. *If ρ is complete, then $\underline{Apr}_\rho(A)$ is, if it is non-empty, an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of H.*

Proof. (1) Let A be an $(m,0)$ Γ-hyperideal of H, that is, $A^m\Gamma H \subseteq A$. Note that $\overline{Apr}_\rho(H) = H$. Then, by Theorem 82(1) and Lemma 209(1), we have

$$\begin{aligned}\left(\overline{Apr}_\rho(A)\right)^m \Gamma H &= \left(\overline{Apr}_\rho(A)\right)^m \Gamma\overline{Apr}_\rho(H) \subseteq \overline{Apr}_\rho(A^m)\Gamma\overline{Apr}_\rho(H)\\ &\subseteq \overline{Apr}_\rho(A^m\Gamma H) \subseteq \overline{Apr}_\rho(A).\end{aligned}$$

This shows that $\overline{Apr}_\rho(A)$ is an $(m,0)$ Γ-hyperideal of H, that is, A is a ρ-upper rough $(m,0)$ Γ-hyperideal of H. Similarly, we can show that the ρ-upper approximation of a $(0,n)$ Γ-hyperideal is a $(0,n)$ Γ-hyperideal of H.

(2) Let A be an $(m,0)$ Γ-hyperideal of H, that is, $A^m\Gamma H \subseteq A$. Note that $\underline{Apr}_\rho(H) = H$. Then, by Theorem 82(2) and Lemma 209(2), we have

$$\begin{aligned}\left(\underline{Apr}_\rho(A)\right)^m \Gamma H &= \left(\underline{Apr}_\rho(A)\right)^m \Gamma\underline{Apr}_\rho(H) \subseteq \underline{Apr}_\rho(A^m)\Gamma\underline{Apr}_\rho(H)\\ &\subseteq \underline{Apr}_\rho(A^m\Gamma H) \subseteq \underline{Apr}_\rho(A).\end{aligned}$$

This shows that $\underline{Apr}_\rho(A)$ is an $(m,0)$ Γ-hyperideal of H, that is, A is a ρ-lower rough $(m,0)$ Γ-hyperideal of H. Similarly, we can show that the ρ-lower approximation of a $(0,n)$ Γ-hyperideal is a $(0,n)$ Γ-hyperideal of H. This completes the proof. ■

A subset A of a Γ-semihypergroup H is called a ρ-upper [ρ-lower] rough (m,n) bi-Γ-hyperideal of H if $\overline{Apr}_\rho(A)$ [$\underline{Apr}_\rho(A)$] is an (m,n) bi-Γ-hyperideal of H.

Theorem 214 *Let ρ be a regular relation on a Γ-semihypergroup H. If A is an (m,n) bi-Γ-hyperideal of H, then it is a ρ-upper rough (m,n) bi-Γ-hyperideal of H.*

Proof. Let A be an (m,n) bi-Γ-hyperideal of H. Then, by Theorem 82(1) and Lemma 209(1), we have

$$\begin{aligned}(\overline{Apr}_\rho(A))^m\Gamma H\Gamma(\overline{Apr}_\rho(A))^n &= (\overline{Apr}_\rho(A))^m\Gamma\overline{Apr}_\rho(H)\Gamma(\overline{Apr}_\rho(A))^n\\ &\subseteq \overline{Apr}_\rho(A^m)\Gamma\overline{Apr}_\rho(H)\Gamma\overline{Apr}_\rho(A^n)\\ &\subseteq \overline{Apr}_\rho(A^m\Gamma H)\Gamma\overline{Apr}_\rho(A^n)\\ &\subseteq \overline{Apr}_\rho(A^m\Gamma H\Gamma A^n)\subseteq\overline{Apr}_\rho(A).\end{aligned}$$

From this and Theorem 84(1), we obtain that $\overline{Apr}_\rho(A)$ is an (m,n) bi-Γ-hyperideal of H, that is, A is a ρ-upper rough (m,n) bi-Γ-hyperideal of H. This completes the proof. ■

Theorem 215 *Let ρ be a complete regular relation on a Γ-semihypergroup H. If A is an (m,n) bi-Γ-hyperideal of H, then $\underline{Apr}_\rho(A)$ is, if it is non-empty, an (m,n) bi-Γ-hyperideal of H.*

Proof. Let A be an (m,n) bi-Γ-hyperideal of H. Then, by Theorem 82(2) and Lemma 209(2), we have

$$\begin{aligned}(\underline{Apr}_\rho(A))^m\Gamma H\Gamma(\underline{Apr}_\rho(A))^n &= (\underline{Apr}_\rho(A))^m\Gamma\underline{Apr}_\rho(H)\Gamma(\underline{Apr}_\rho(A))^n\\ &\subseteq \underline{Apr}_\rho(A^m)\Gamma\underline{Apr}_\rho(H)\Gamma\underline{Apr}_\rho(A^n)\\ &\subseteq \underline{Apr}_\rho(A^m\Gamma H)\Gamma\underline{Apr}_\rho(A^n)\\ &\subseteq \underline{Apr}_\rho(A^m\Gamma H\Gamma A^n)\subseteq\underline{Apr}_\rho(A).\end{aligned}$$

From this and Theorem 86(1), we obtain that $\underline{Apr}_\rho(A)$ is, if it is non-empty, an (m,n) bi-Γ-hyperideal of H. This completes the proof. ■

The following example shows that the converse of Theorem 214 and Theorem 215 does not hold.

Example 216 *Let $H=\{x,y,z\}$ and $\Gamma=\{\beta,\gamma\}$ be the sets of binary hyperoperations defined below:*

β	x	y	z
x	x	$\{x,y\}$	z
y	$\{x,y\}$	$\{x,y\}$	z
z	z	z	z

γ	x	y	z
x	$\{x,y\}$	$\{x,y\}$	z
y	$\{x,y\}$	y	z
z	z	z	z

Here H is a Γ-semihypergroup. Let ρ be a complete regular relation on H such that the ρ-regular classes are the subsets $\{x, y\}$, $\{z\}$. Now for $A = \{x, z\} \subseteq H$, $\overline{Apr}_\rho(A) = \{x, y, z\}$ and $\underline{Apr}_\rho(A) = \{z\}$. It is clear that $\overline{Apr}_\rho(A)$ and $\underline{Apr}_\rho(A)$ are (m, n) bi-Γ-hyperideals of H, but A is not an (m, n) bi-Γ-hyperideal of H. Because $A^m\Gamma H\Gamma A^n = H \nsubseteq A$.

4.4 Rough Prime (m, n) Bi-Γ-Hyperideals

Let ρ be a regular relation on a Γ-semihypergroup H. Then a subset A of H is called a ρ-lower rough prime (m, n) bi-Γ-hyperideal of H if $\underline{Apr}_\rho(A)$ is a prime (m, n) bi-Γ-hyperideal of H. A ρ-upper rough prime (m, n) bi-Γ-hyperideal of H is defined analogously. A is called a rough prime (m, n) bi-Γ-hyperideal of H if A is a ρ-lower and a ρ-upper rough prime (m, n) bi-Γ-hyperideal of H.

Theorem 217 *Let ρ be a complete regular relation on a Γ-semihypergroup H. If A is a prime (m, n) bi-Γ-hyperideal of H, then A is a ρ-upper rough prime (m, n) bi-Γ-hyperideal of H.*

Proof. Since A is an (m, n) bi-Γ-hyperideal of H, then by Theorem 214, $\overline{Apr}_\rho(A)$ is an (m, n) bi-Γ-hyperideal of H. Let w be any element of H. Let $x, y \in H$ and $\beta, \gamma \in \Gamma$ such that $x^m\beta w\gamma y^n \subseteq \overline{Apr}_\rho(A)$. Thus

$$[x^m\beta w\gamma y^n]_\rho \cap A = [x^m]_\rho\beta[w]_\rho\gamma[y^n]_\rho \cap A \neq \emptyset.$$

Thus there exist $a^m \subseteq [x^m]_\rho = [x]_\rho^m$, $w' \in [w]_\rho$ and $b^n \subseteq [y^n]_\rho = [y]_\rho^n$ such that $a^m\beta w'\gamma b^n \subseteq A$. Since A is a prime (m, n) bi-Γ-hyperideal, we have $a \in A$ or $b \in A$. Now

$$a^m \subseteq [x]_\rho^m \Longrightarrow a \in [x]_\rho \text{ also } b^n \subseteq [y]_\rho^n \Longrightarrow b \in [y]_\rho.$$

Thus $a \in [x]_\rho \cap A$ or $b \in [y]_\rho \cap A$. So $[x]_\rho \cap A \neq \emptyset$ or $[y]_\rho \cap A \neq \emptyset$, and so $x \in \overline{Apr}_\rho(A)$ or $y \in \overline{Apr}_\rho(A)$. Therefore $\overline{Apr}_\rho(A)$ is a prime (m, n) bi-Γ-hyperideal of H. ■

Theorem 218 *Let ρ be a complete regular relation on a Γ-semihypergroup H and A is a prime (m, n) bi-Γ-hyperideal of H. Then $\underline{Apr}_\rho(A)$ is, if it is non-empty, a prime (m, n) bi-Γ-hyperideal of H.*

Proof. Since A is an (m,n) bi-Γ-hyperideal of H, by Theorem 215, we know that $\underline{Apr}_\rho(A)$ is an (m,n) bi-Γ-hyperideal of H. We suppose that $\underline{Apr}_\rho(A)$ is not a prime (m,n) bi-Γ-hyperideal, then for $\beta,\gamma\in\Gamma$ there exists $x,y\in H$ and any element $w\in H$, such that $x^m\beta w\gamma y^n\subseteq\underline{Apr}_\rho(A)$, but $x\notin\underline{Apr}_\rho(A)$ and $y\notin\underline{Apr}_\rho(A)$. Thus $[x]_\rho\nsubseteq A$ and $[y]_\rho\nsubseteq A$. Then there exist

$$a\in[x]_\rho \text{ but } a\notin A \quad\text{and}\quad b\in[y]_\rho \text{ but } b\notin A.$$

We have for all $w\in H$ and $\beta,\gamma\in\Gamma$,

$$\begin{aligned}a^m\beta w\gamma b^n &\subseteq [x]_\rho^m\beta[w]_\rho\gamma[y]_\rho^n=[x^m]_\rho\beta[w]_\rho\gamma[y^n]_\rho\\ &= [x^m\beta w\gamma y^n]_\rho\subseteq A.\end{aligned}$$

This implies that $a^m\beta w\gamma b^n\subseteq A$. Since A is a prime (m,n) bi-Γ-hyperideal, we have $a\in A$ or $b\in A$. It contradicts the supposition. This means that $\underline{Apr}_\rho(A)$ is, if it is non-empty, a prime (m,n) bi-Γ-hyperideal of H. ■

The following example shows that the converse of Theorem 217 and Theorem 218 does not hold.

Example 219 *Let $H=\{a,b,c,d,e\}$ and $\Gamma=\{\gamma,\beta\}$ be the sets of binary hyperoperations defined below:*

γ	a	b	c	d	e
a	$\{a,b\}$	$\{b,c\}$	c	$\{d,e\}$	e
b	$\{b,c\}$	c	c	$\{d,e\}$	e
c	c	c	c	$\{d,e\}$	e
d	$\{d,e\}$	$\{d,e\}$	$\{d,e\}$	d	e
e	e	e	e	e	e

β	a	b	c	d	e
a	$\{b,c\}$	c	c	$\{d,e\}$	e
b	c	c	c	$\{d,e\}$	e
c	c	c	c	$\{d,e\}$	e
d	$\{d,e\}$	$\{d,e\}$	$\{d,e\}$	d	e
e	e	e	e	e	e

Then H is a Γ-semihypergroup. Let ρ be a complete regular relation on H such that ρ-regular classes are the subsets $\{a,b,c\}$, $\{d,e\}$. Then for $A=\{c,d,e\}\subseteq H$, $\overline{Apr}_\rho(A)=\{a,b,c,d,e\}$, and $\underline{Apr}_\rho(A)=\{d,e\}$. It is clear that $\overline{Apr}_\rho(A)$, $\underline{Apr}_\rho(A)$ are prime (m,n) bi-Γ-hyperideals of H. The (m,n) bi-Γ-hyperideal A is not a prime (m,n) bi-Γ-hyperideal for $b^m\Gamma c\Gamma a^n=c\in A$ but $b\notin A$ and $a\notin A$.

4.5 Rough Quasi Γ-Hyperideals

In this section we prove some results on rough quasi Γ-hyperideal in Γ-semihypergroup.

Definition 220 *Let ρ be a regular relation on a Γ-semihypergroup H. A subset Q of H is called a ρ-upper rough quasi Γ-hyperideal of H if $\overline{Apr}_{\rho}(Q)$ is a quasi Γ-hyperideal of H. Similarly, a subset Q of a Γ-semihypergroup H is called a ρ-lower rough quasi Γ-hyperideal of H if $\underline{Apr}_{\rho}(Q)$ is a quasi Γ-hyperideal of H.*

Theorem 221 *Let ρ be a complete regular relation on a Γ-semihypergroup H and Q be a quasi Γ-hyperideal of H. Then $\underline{Apr}_{\rho}(Q)$ is, if it is non-empty, a quasi Γ-hyperideal of H.*

Proof. Let Q be a quasi Γ-hyperideal of H, that is, $H\Gamma Q \cap Q\Gamma H \subseteq Q$. Note that $\underline{Apr}_{\rho}(H) = H$. Then by Proposition 208(3) and Theorem 82(2), we have

$$\begin{aligned} H\Gamma\underline{Apr}_{\rho}(Q) \cap \underline{Apr}_{\rho}(Q)\Gamma H &= \underline{Apr}_{\rho}(H)\Gamma\underline{Apr}_{\rho}(Q) \cap \underline{Apr}_{\rho}(Q)\Gamma\underline{Apr}_{\rho}(H) \\ &\subseteq \underline{Apr}_{\rho}(H\Gamma Q) \cap \underline{Apr}_{\rho}(Q\Gamma H) \\ &= \underline{Apr}_{\rho}(H\Gamma Q \cap Q\Gamma H) \\ &\subseteq \underline{Apr}_{\rho}(Q). \end{aligned}$$

This shows that $\underline{Apr}_{\rho}(Q)$ is a quasi Γ-hyperideal of H, that is, Q is a ρ-lower rough a quasi Γ-hyperideal of H. ■

The following example shows that the converse of Theorem 221 does not hold.

Example 222 *Let $H = \{0, 1, 2, 3\}$ and $\Gamma = \{\beta, \gamma\}$ be the sets of binary hyperoperations defined below:*

β	0	1	2	3
0	$\{0,2\}$	$\{1,3\}$	$\{0,2\}$	3
1	$\{1,3\}$	1	$\{1,3\}$	3
2	$\{0,2\}$	$\{1,3\}$	$\{0,2\}$	3
3	3	3	3	3

γ	0	1	2	3
0	0	$\{1,3\}$	2	3
1	$\{1,3\}$	1	$\{1,3\}$	3
2	2	$\{1,3\}$	0	3
3	3	3	3	3

Here H is a Γ-semihypergroup. Let ρ be a complete regular relation on H such that the ρ-regular classes are the subsets $\{0, 2\}$, $\{1, 3\}$. Now for $A = \{1, 2, 3\} \subseteq H$, $\underline{Apr}_{\rho}(A) = \{1, 3\}$. It is clear that $\underline{Apr}_{\rho}(A)$ is a quasi Γ-hyperideal of H, but A is not a quasi Γ-hyperideal of H. Because $H\Gamma A \cap A\Gamma H = \{0, 1, 2, 3\} \nsubseteq A$.

Theorem 223 *Let ρ be a complete regular relation on a Γ-semihypergroup H. Let L and R be a ρ-lower rough left Γ-hyperideal and a ρ-lower rough right Γ-hyperideal of H, respectively. Then $L \cap R$ is a ρ-lower rough quasi Γ-hyperideal of H.*

Proof. The proof is straightforward. ■

Theorem 224 *Let ρ be a complete regular relation on a Γ-semihypergroup H and let Q be a quasi Γ-hyperideal of H. Then there exist a ρ-lower rough left Γ-hyperideal L and a ρ-lower rough right Γ-hyperideal R of H such that $Q = L \cap R$.*

Proof. The proof is straightforward. ■

Theorem 225 *Let ρ be a complete regular relation on a Γ-semihypergroup H. Then the following statements are true.*

1. *Every ρ-lower rough quasi Γ-hyperideal of H is a ρ-lower rough bi-Γ-hyperideal of H.*
2. *Every ρ-upper rough quasi Γ-hyperideal of H is a ρ-upper rough bi-Γ-hyperideal of H.*

Proof. (1) Let Q be a ρ-lower rough quasi Γ-hyperideal of H, that is, $H\Gamma\underline{Apr}_\rho(Q) \cap \underline{Apr}_\rho(Q)\Gamma H \subseteq \underline{Apr}_\rho(Q)$. Then $\underline{Apr}_\rho(Q)$ is a sub Γ-semihypergroup of H. Now we have

$$\underline{Apr}_\rho(Q)\Gamma H\Gamma\underline{Apr}_\rho(Q) \subseteq H\Gamma\underline{Apr}_\rho(Q) \cap \underline{Apr}_\rho(Q)\Gamma H \subseteq \underline{Apr}_\rho(Q).$$

Hence this shows that $\underline{Apr}_\rho(Q)$ is a bi-Γ-hyperideal of H.

(2) is similar to (1). ■

Theorem 226 *Let ρ be a complete regular relation on a Γ-semihypergroup H. If Q is a quasi Γ-hyperideal of H, then Q is a ρ-upper rough bi-Γ-hyperideal of H.*

Proof. Let Q be a quasi Γ-hyperideal of H, that is, $H\Gamma Q \cap Q\Gamma H \subseteq Q$. By Proposition 208(1), $Q \subseteq \overline{Apr}_\rho(Q)$, $\overline{Apr}_\rho(Q) \neq \emptyset$. We have

$$\overline{Apr}_\rho(Q)\Gamma\overline{Apr}_\rho(Q) \subseteq \overline{Apr}_\rho(Q\Gamma Q) \subseteq \overline{Apr}_\rho(Q).$$

This shows that $\overline{Apr}_\rho(Q)$ is a sub Γ-semihypergroup of H. Now we have

$$\begin{aligned} \overline{Apr}_\rho(Q)\Gamma H\Gamma\overline{Apr}_\rho(Q) &= \overline{Apr}_\rho(Q)\Gamma\overline{Apr}_\rho(H)\Gamma\overline{Apr}_\rho(Q) \\ &\subseteq \overline{Apr}_\rho(Q\Gamma H\Gamma Q) \\ &\subseteq \overline{Apr}_\rho(H\Gamma Q \cap Q\Gamma H) \\ &\subseteq \overline{Apr}_\rho(Q). \end{aligned}$$

Therefore $\overline{Apr}_\rho(Q)$ is a bi-Γ-hyperideal of H. ■

4.6 Rough (m,n) Quasi Γ-Hyperideals

In this section we prove that the ρ-upper and ρ-lower approximation of an m-left (resp., n-right) Γ-hyperideal is an m-left (resp., n-right) Γ-hyperideal. Also we will prove that the ρ-lower approximation of an (m,n) quasi Γ-hyperideal is an (m,n) quasi Γ-hyperideal.

Definition 227 *Let ρ be a regular relation on a Γ-semihypergroup H. A subset L of H is called a ρ-upper (resp., ρ-lower) rough m-left Γ-hyperideal of H if $\overline{Apr}_\rho(L)$ (resp., $\underline{Apr}_\rho(L)$) is an m-left Γ-hyperideal of H. Similarly, a subset R of H is called a ρ-upper (resp., ρ-lower) rough n-right Γ-hyperideal of H if $\overline{Apr}_\rho(R)$ (resp., $\underline{Apr}_\rho(R)$) is an n-right Γ-hyperideal of H.*

Theorem 228 *Let ρ be a regular relation on a Γ-semihypergroup H and A be an m-left (resp., n-right) Γ-hyperideal of H. Then,*

1. *$\overline{Apr}_\rho(A)$ is an m-left (resp., n-right) Γ-hyperideal of H.*
2. *If ρ is complete, then $\underline{Apr}_\rho(A)$ is, if it is non-empty, an m-left (resp., n-right) Γ-hyperideal of H.*

Proof. (1) Let A be an m-left Γ-hyperideal of H, that is, $H^m\Gamma A \subseteq A$. Note that $\overline{Apr}_\rho(H) = H$. Then, by Theorem 82(1) and Lemma 209(1), we have

$$\begin{aligned} H^m\Gamma\overline{Apr}_\rho(A) &= \left(\overline{Apr}_\rho(H)\right)^m \Gamma\overline{Apr}_\rho(A) \subseteq \overline{Apr}_\rho(H^m)\Gamma\overline{Apr}_\rho(A) \\ &\subseteq \overline{Apr}_\rho(H^m\Gamma A) \subseteq \overline{Apr}_\rho(A). \end{aligned}$$

This shows that $\overline{Apr}_\rho(A)$ is an m-left Γ-hyperideal of H, that is, A is a ρ-upper rough m-left Γ-hyperideal of H. Similarly, we can show that the ρ-upper approximation of an n-right Γ-hyperideal is an n-right Γ-hyperideal of H.

(2) Let A be an m-left Γ-hyperideal of H, that is, $H^m\Gamma A \subseteq A$. Note that $\underline{Apr}_\rho(H) = H$. Then, by Theorem 82(2) and Lemma 209(2), we have

$$\begin{aligned} H^m\Gamma\underline{Apr}_\rho(A) &= \left(\underline{Apr}_\rho(H)\right)^m \Gamma\underline{Apr}_\rho(A) \subseteq \underline{Apr}_\rho(H^m)\Gamma\underline{Apr}_\rho(A) \\ &\subseteq \underline{Apr}_\rho(H^m\Gamma A) \subseteq \underline{Apr}_\rho(A). \end{aligned}$$

This shows that $\underline{Apr}_{\rho}(A)$ is an m-left Γ-hyperideal of H, that is, A is a ρ-lower rough m-left Γ-hyperideal of H. Similarly, we can show that the ρ-lower approximation of an n-right Γ-hyperideal is an n-right Γ-hyperideal of H. ■

The following example shows that the converse of Theorem 228 does not hold.

Example 229 *Let $H = \{0, 1, 2\}$ and $\Gamma = \{\beta, \gamma\}$ be the sets of binary hyperoperations defined below:*

β	0	1	2
0	0	$\{0,1\}$	2
1	$\{0,1\}$	$\{0,1\}$	2
2	2	2	2

γ	0	1	2
0	$\{0,1\}$	$\{0,1\}$	2
1	$\{0,1\}$	1	2
2	2	2	2

Here H is a Γ-semihypergroup. Let ρ be a complete regular relation on H such that the ρ-regular classes are the subsets $\{0, 1\}$, $\{2\}$. Now for $A = \{0, 2\} \subseteq H$, $\overline{Apr}_{\rho}(A) = \{0, 1, 2\}$ and $\underline{Apr}_{\rho}(A) = \{2\}$. It is clear that $\overline{Apr}_{\rho}(A)$ and $\underline{Apr}_{\rho}(A)$ are m-left (resp., n-right) Γ-hyperideals of H, but A is not an m-left (resp., n-right) Γ-hyperideal of H.

Definition 230 *Let ρ be a regular relation on a Γ-semihypergroup H. A subset Q of H is called a ρ-upper rough (m, n) quasi Γ-hyperideal of H if $\overline{Apr}_{\rho}(Q)$ is an (m, n) quasi Γ-hyperideal of H. Similarly, a subset Q of a Γ-semihypergroup H is called a ρ-lower rough (m, n) quasi Γ-hyperideal of H if $\underline{Apr}_{\rho}(Q)$ is an (m, n) quasi Γ-hyperideal of H.*

Theorem 231 *Let ρ be a complete regular relation on a Γ-semihypergroup H and Q be an (m, n) quasi Γ-hyperideal of H. Then $\underline{Apr}_{\rho}(Q)$ is, if it is non-empty, an (m, n) quasi Γ-hyperideal of H.*

Proof. Let Q be an (m, n) quasi Γ-hyperideal of H, that is, $H^m\Gamma Q \cap Q\Gamma H^n \subseteq Q$. Note that $\underline{Apr}_{\rho}(H) = H$. Then by Proposition 208(3), Theorem 82(2) and Lemma 209(2), we have

$$\begin{aligned}
H^m\Gamma\underline{Apr}_{\rho}(Q) \cap \underline{Apr}_{\rho}(Q)\Gamma H^n &= \left(\underline{Apr}_{\rho}(H)\right)^m \Gamma\underline{Apr}_{\rho}(Q) \cap \underline{Apr}_{\rho}(Q)\Gamma\left(\underline{Apr}_{\rho}(H)\right)^n \\
&\subseteq \underline{Apr}_{\rho}(H^m)\Gamma\underline{Apr}_{\rho}(Q) \cap \underline{Apr}_{\rho}(Q)\Gamma\underline{Apr}_{\rho}(H^n) \\
&\subseteq \underline{Apr}_{\rho}(H^m\Gamma Q) \cap \underline{Apr}_{\rho}(Q\Gamma H^n) \\
&= \underline{Apr}_{\rho}(H^m\Gamma Q \cap Q\Gamma H^n) \\
&\subseteq \underline{Apr}_{\rho}(Q).
\end{aligned}$$

This shows that $\underline{Apr}_{\rho}(Q)$ is an (m,n) quasi Γ-hyperideal of H, that is, Q is a ρ-lower rough (m,n) quasi Γ-hyperideal of H. ■

The next theorem shows that the intersection of a ρ-lower rough m-left ideal and a ρ-lower rough n-right ideal of a Γ-semihypergroup H is a ρ-lower rough (m,n) quasi Γ-hyperideal of H.

Theorem 232 *Let ρ be a complete regular relation on a Γ-semihypergroup H. Let L and R be a ρ-lower rough m-left Γ-hyperideal and a ρ-lower rough n-right Γ-hyperideal of H, respectively. Then $\underline{Apr}_{\rho}(L \cap R)$ is, if it is non-empty, an (m,n) quasi Γ-hyperideal of H.*

Proof. Let L and R be a ρ-lower rough m-left Γ-hyperideal and a ρ-lower rough n-right Γ-hyperideal of H, respectively. Then

$$H^m\Gamma\underline{Apr}_{\rho}(L) \subseteq \underline{Apr}_{\rho}(L) \text{ and } \underline{Apr}_{\rho}(R)\Gamma H^n \subseteq \underline{Apr}_{\rho}(R).$$

Now we have

$$\begin{aligned} H^m\Gamma\underline{Apr}_{\rho}(L\cap R) \cap \underline{Apr}_{\rho}(L\cap R)\Gamma H^n &\subseteq H^m\Gamma\underline{Apr}_{\rho}(L) \cap \underline{Apr}_{\rho}(R)\Gamma H^n \\ &\subseteq \underline{Apr}_{\rho}(L) \cap \underline{Apr}_{\rho}(R) \\ &= \underline{Apr}_{\rho}(L\cap R). \end{aligned}$$

Hence this shows that $\underline{Apr}_{\rho}(L\cap R)$ a ρ-lower rough (m,n) quasi Γ-hyperideal of H. ■

Theorem 233 *Let ρ be a complete regular relation on a Γ-semihypergroup H and let Q be an (m,n) quasi Γ-hyperideal of H. Then there exist a ρ-lower rough m-left Γ-hyperideal L and a ρ-lower rough n-right Γ-hyperideal R of H such that $Q = L \cap R$.*

Proof. Let Q be an (m,n) quasi Γ-hyperideal of H, that is, $H^m\Gamma Q \cap Q\Gamma H^n \subseteq Q$. Let

$$L = Q \cup \underline{Apr}_{\rho}(H^m\Gamma Q) \;\text{ and } R = Q \cup \underline{Apr}_{\rho}(Q\Gamma H^n).$$

It is easy to see that $Q \subseteq L \cap R$. We have

$$\begin{aligned} L\cap R &= \Big(Q \cup \underline{Apr}_{\rho}(H^m\Gamma Q)\Big) \cap \Big(Q \cup \underline{Apr}_{\rho}(Q\Gamma H^n)\Big) \\ &= Q \cup \Big(\underline{Apr}_{\rho}(H^m\Gamma Q) \cap \underline{Apr}_{\rho}(Q\Gamma H^n)\Big) \\ &\subseteq Q \cup \underline{Apr}_{\rho}(H^m\Gamma Q \cap Q\Gamma H^n) \\ &\subseteq Q \cup \underline{Apr}_{\rho}(Q) \\ &= Q. \end{aligned}$$

This shows that $Q = L \cap R$. Now to show that L is a ρ-lower rough m-left Γ-hyperideal of H. We have

$$\begin{aligned}
H^m\Gamma\underline{Apr}_\rho(L) &= H^m\Gamma\underline{Apr}_\rho(Q \cup \underline{Apr}_\rho(H^m\Gamma Q)) \\
&= \left(\underline{Apr}_\rho(H)\right)^m \Gamma\underline{Apr}_\rho(Q \cup \underline{Apr}_\rho(H^m\Gamma Q)) \\
&\subseteq \underline{Apr}_\rho(H^m)\Gamma\underline{Apr}_\rho(Q \cup \underline{Apr}_\rho(H^m\Gamma Q)) \\
&\subseteq \underline{Apr}_\rho(H^m\Gamma(Q \cup \underline{Apr}_\rho(H^m\Gamma Q)) \\
&= \underline{Apr}_\rho(H^m\Gamma Q \cup H^m\Gamma\underline{Apr}_\rho(H^m\Gamma Q) \\
&\subseteq \underline{Apr}_\rho(H^m\Gamma Q \cup \underline{Apr}_\rho(H^m\Gamma H^m\Gamma Q) \\
&\subseteq \underline{Apr}_\rho(H^m\Gamma Q \cup H^m\Gamma H^m\Gamma Q) \\
&\subseteq \underline{Apr}_\rho(H^m\Gamma Q \cup H^m\Gamma Q) = \underline{Apr}_\rho(H^m\Gamma Q) \\
&= \underline{Apr}_\rho\left(\underline{Apr}_\rho(H^m\Gamma Q)\right) \subseteq \underline{Apr}_\rho(L).
\end{aligned}$$

Thus L is a ρ-lower rough m-left Γ-hyperideal of H. Similarly, R is a ρ-lower rough n-right Γ-hyperideal of H. This completes the proof. ■

4.7 Rough Γ-Hyperideals in the Quotient Γ-Semihypergroup

Here we presented some results on rough Γ-hyperideals in the quotient Γ-semihypergroups and prove that the lower and upper approximations of Γ-hyperideals (resp., bi-Γ-hyperideals, (m, n) bi-Γ-hyperideals, quasi Γ-hyperideals and (m, n) quasi Γ-hyperideals) in the quotient Γ-semihypergroup are Γ-hyperideals (resp., bi-Γ-hyperideals, (m, n) bi-Γ-hyperideals, quasi Γ-hyperideals and (m, n) quasi Γ-hyperideals).

Theorem 234 *Let ρ be a regular relation on a Γ-semihypergroup H and A a prime Γ-hyperideal of H. Then $\overline{\overline{Apr}}_\rho(A)$ is a prime $\widehat{\Gamma}$-hyperideal of H/ρ.*

Proof. Since A is a Γ-hyperideal of H, by Theorem 102, we know that $\overline{\overline{Apr}}_\rho(A)$ is a $\widehat{\Gamma}$-hyperideal of H/ρ. Suppose

$$[x]_\rho\widehat{\gamma}[y]_\rho \subseteq \overline{\overline{Apr}}_\rho(A), \text{ for some } [x]_\rho, [y]_\rho \in H/\rho \text{ and } \widehat{\gamma} \in \widehat{\Gamma}.$$

Now, there exist z, such that $z \in x\gamma y \subseteq \overline{Apr}_\rho(A)$. We obtain $[x\gamma y]_\rho \cap A \neq \emptyset$. Since A is a prime Γ-hyperideal of H. Then by Theorem 97, $\overline{Apr}_\rho(A)$ is a prime Γ-hyperideal of H. So

$$x \in \overline{Apr}_\rho(A) \text{ or } y \in \overline{Apr}_\rho(A).$$

Now, as $z \in x\gamma y$, we obtain

$$[z]_\rho \in [x]_\rho \widehat{\gamma} [y]_\rho.$$

On the other hand, since $z \in \overline{Apr}_\rho(A)$, we have $[z]_\rho \cap A \neq \emptyset$. Thus,

$$[x]_\rho \in \overline{\overline{Apr}}_\rho(A) \ \text{ or } \ [y]_\rho \in \overline{\overline{Apr}}_\rho(A).$$

Therefore $\overline{\overline{Apr}}_\rho(A)$ is a prime $\widehat{\Gamma}$-hyperideal of H/ρ. ■

Theorem 235 *Let ρ be a regular relation on a Γ-semihypergroup H and A be a prime Γ-hyperideal of H. Then $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, a prime $\widehat{\Gamma}$-hyperideal of H/ρ.*

Proof. Since A is a Γ-hyperideal of H, by Theorem 103, we know that $\underline{\underline{Apr}}_\rho(A)$ is a $\widehat{\Gamma}$-hyperideal of H/ρ. Suppose

$$[x]_\rho \widehat{\gamma} [y]_\rho \subseteq \underline{\underline{Apr}}_\rho(A), \text{ for some } \ [x]_\rho, [y]_\rho \in H/\rho \text{ and } \widehat{\gamma} \in \widehat{\Gamma}.$$

Now, there exist z, such that $z \in x\gamma y \subseteq \underline{Apr}_\rho(A)$. We obtain $[x\gamma y]_\rho \subseteq A$. Since A is a prime Γ-hyperideal of H. Then by Theorem 98, $\underline{Apr}_\rho(A)$ is prime Γ-hyperideal of H. So

$$x \in \underline{Apr}_\rho(A) \ \text{ or } \ y \in \underline{Apr}_\rho(A).$$

Now, as $z \in x\gamma y$, we obtain

$$[z]_\rho \in [x]_\rho \widehat{\gamma} [y]_\rho.$$

On the other hand, since $z \in \underline{Apr}_\rho(A)$, we have $[z]_\rho \subseteq A$. Thus,

$$[x]_\rho \in \underline{\underline{Apr}}_\rho(A) \ \text{ or } \ [y]_\rho \in \underline{\underline{Apr}}_\rho(A).$$

Therefore $\underline{\underline{Apr}}_\rho(A)$ is a prime $\widehat{\Gamma}$-hyperideal of H/ρ. ■

Theorem 236 *Let ρ be a regular relation on a Γ-semihypergroup H and A be a bi-Γ-hyperideal of H. Then $\overline{\overline{Apr}}_\rho(A)$ is a bi-$\widehat{\Gamma}$-hyperideal of H/ρ.*

Proof. Let $[x]_\rho$ and $[y]_\rho$ be any elements of $\overline{\overline{Apr}}_\rho(A)$ and $[s]_\rho$ be any element of H/ρ. Then,

$$[x]_\rho \cap A \neq \emptyset \quad \text{and} \quad [y]_\rho \cap A \neq \emptyset.$$

Hence, $x \in \overline{Apr}_\rho(A)$ and $y \in \overline{Apr}_\rho(A)$. By Theorem 88(1), $\overline{Apr}_\rho(A)$ is a bi-Γ-hyperideal of H. So, for every $\alpha, \beta \in \Gamma$, we have

$$x\alpha s\beta y \subseteq \overline{Apr}_\rho(A).$$

Now, for every $t \in x\alpha s\beta y$, we obtain

$$[t]_\rho \in [x]_\rho \widehat{\alpha} s \widehat{\beta} [y]_\rho.$$

On the other hand, since $t \in \overline{Apr}_\rho(A)$, we have $[t]_\rho \cap A \neq \emptyset$. Thus,

$$[x]_\rho \widehat{\alpha} s \widehat{\beta} [y]_\rho \subseteq \overline{\overline{Apr}}_\rho(A).$$

Therefore, $\overline{\overline{Apr}}_\rho(A)$ is a bi-$\widehat{\Gamma}$-hyperideal of H/ρ. ■

Theorem 237 *Let ρ be a regular relation on a Γ-semihypergroup H and A be a bi-Γ-hyperideal of H. Then $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, a bi-$\widehat{\Gamma}$-hyperideal of H/ρ.*

Proof. Let $[x]_\rho$ and $[y]_\rho$ be any elements of $\underline{\underline{Apr}}_\rho(A)$ and $[s]_\rho$ be any element of H/ρ. Then,

$$[x]_\rho \subseteq A \text{ and } [y]_\rho \subseteq A.$$

Hence, $x \in \underline{Apr}_\rho(A)$ and $y \in \underline{Apr}_\rho(A)$. By Theorem 88(2), $\underline{Apr}_\rho(A)$ is a bi-Γ-hyperideal of H. So, for every $\alpha, \beta \in \Gamma$, we have

$$x\alpha s\beta y \subseteq \underline{Apr}_\rho(A).$$

Then, for every $t \in x\alpha s\beta y$, we obtain

$$[t]_\rho \in [x]_\rho \widehat{\alpha} a \widehat{\beta} [y]_\rho.$$

On the other hand, since $t \in \underline{Apr}_\rho(A)$, we have $[t]_\rho \subseteq A$. So,

$$[x]_\rho^m \widehat{\alpha} a \widehat{\beta} [y]_\rho^n \subseteq \underline{\underline{Apr}}_\rho(A).$$

Therefore, $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, a bi-$\widehat{\Gamma}$-hyperideal of H/ρ. ■

Theorem 238 *Let ρ be a regular relation on a Γ-semihypergroup H and A be a prime bi-Γ-hyperideal of H. Then $\overline{\overline{Apr}}_\rho(A)$ is a prime bi-$\widehat{\Gamma}$-hyperideal of H/ρ.*

Proof. Since A is a bi-Γ-hyperideal of H, by Theorem 236, we know that $\overline{\overline{Apr}}_\rho(A)$ is a bi-$\widehat{\Gamma}$-hyperideal of H/ρ. For all $s \in H$, suppose

$$[x]_\rho \widehat{\beta} s \widehat{\gamma} [y]_\rho \subseteq \overline{\overline{Apr}}_\rho(A), \text{ for some } [x]_\rho, [s]_\rho, [y]_\rho \in H/\rho \text{ and } \widehat{\beta}, \widehat{\gamma} \in \widehat{\Gamma}.$$

Now, there exist z, such that $z \in x\beta s\gamma y \subseteq \overline{Apr}_\rho(A)$. We obtain $[x\beta s\gamma y]_\rho \cap A \neq \emptyset$. Since A is a prime bi-Γ-hyperideal of H. Then by Theorem 210, $\overline{Apr}_\rho(A)$ is a prime bi-Γ-hyperideal of H. So

$$x \in \overline{Apr}_\rho(A) \text{ or } y \in \overline{Apr}_\rho(A).$$

Now, as $z \in x\beta s\gamma y$, we obtain

$$[z]_\rho \in [x]_\rho \widehat{\beta} s \widehat{\gamma} [y]_\rho.$$

On the other hand, since $z \in \overline{Apr}_\rho(A)$, we have $[z]_\rho \cap A \neq \emptyset$. Thus,

$$[x]_\rho \in \overline{\overline{Apr}}_\rho(A) \text{ or } [y]_\rho \in \overline{\overline{Apr}}_\rho(A).$$

Therefore $\overline{\overline{Apr}}_\rho(A)$ is a prime bi-$\widehat{\Gamma}$-hyperideal of H/ρ. ■

Theorem 239 *Let ρ be a regular relation on a Γ-semihypergroup H and A be a prime bi-Γ-hyperideal of H. Then $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, a prime bi-$\widehat{\Gamma}$-hyperideal of H/ρ.*

Proof. Since A is a bi-Γ-hyperideal of H, by Theorem 237, we know that $\underline{\underline{Apr}}_\rho(A)$ is a bi-$\widehat{\Gamma}$-hyperideal of H/ρ. For all $s \in H$, suppose

$$[x]_\rho \widehat{\beta} s \widehat{\gamma} [y]_\rho \subseteq \underline{\underline{Apr}}_\rho(A), \text{ for some } [x]_\rho, [s]_\rho, [y]_\rho \in H/\rho \text{ and } \widehat{\beta}, \widehat{\gamma} \in \widehat{\Gamma}.$$

Now, there exist z, such that $z \in x\beta s\gamma y \subseteq \underline{Apr}_\rho(A)$. We obtain $[x\beta s\gamma y]_\rho \subseteq A$. Since A is prime bi-Γ-hyperideal of H. Then by Theorem 211, $\underline{Apr}_\rho(A)$ is prime bi-Γ-hyperideal of H. So

$$x \in \underline{Apr}_\rho(A) \text{ or } y \in \underline{Apr}_\rho(A).$$

Now, as $z \in x\beta s\gamma y$, we obtain

$$[z]_\rho \in [x]_\rho \widehat{\beta} s \widehat{\gamma} [y]_\rho.$$

On the other hand, since $z \in \underline{Apr}_\rho(A)$, we have $[z]_\rho \subseteq A$. Thus,

$$[x]_\rho \in \underline{\underline{Apr}}_\rho(A) \text{ or } [y]_\rho \in \underline{\underline{Apr}}_\rho(A).$$

Therefore $\underline{\underline{Apr}}_\rho(A)$ is a prime bi-$\widehat{\Gamma}$-hyperideal of H/ρ. ■

Theorem 240 *Let ρ be a complete regular relation on a Γ-semihypergroup H. The following statements are true.*

1. *If Q is a ρ-upper rough quasi Γ-hyperideal of H, then $\overline{\overline{Apr}}_\rho(Q)$ is a quasi $\widehat{\Gamma}$-hyperideal of H/ρ.*

2. *If Q is a ρ-lower rough quasi Γ-hyperideal of H, then $\underline{\underline{Apr}}_\rho(Q)$ is, if it is non-empty, a quasi $\widehat{\Gamma}$-hyperideal of H/ρ.*

Proof. Let $[a]_\rho \in (H/\rho)\,\Gamma\overline{\overline{Apr}}_\rho(Q)\cap\,\overline{\overline{Apr}}_\rho(Q)\Gamma\,(H/\rho)$. Then there exist $[x_1]_\rho, [x_2]_\rho \in \overline{\overline{Apr}}_\rho(Q)$, $\beta, \gamma \in \Gamma$ and $[y_1]_\rho, [y_2]_\rho \in H/\rho$ such that

$$[a]_\rho = [y_1]_\rho\widehat{\beta}[x_1]_\rho = [x_2]_\rho\widehat{\gamma}[y_2]_\rho.$$

Then $[x_1]_\rho \cap Q \neq \emptyset$ and $[x_2]_\rho \cap Q \neq \emptyset$. This implies that $x \in \overline{Apr}_\rho(Q)$ for all $x \in [x_1]_\rho \cup [x_2]_\rho$. We have

$$[a]_\rho \subseteq H\Gamma\overline{Apr}_\rho(Q) \cap \overline{Apr}_\rho(Q)\Gamma H \subseteq \overline{Apr}_\rho(Q).$$

Then $a \in \overline{Apr}_\rho(Q)$. Thus $[a]_\rho \cap Q \neq \emptyset$. This implies that $[a]_\rho \in \overline{\overline{Apr}}_\rho(Q)$. Therefore $\overline{\overline{Apr}}_\rho(Q)$ is a quasi $\widehat{\Gamma}$-hyperideal of H/ρ.

(2) Let $[a]_\rho \in (H/\rho)\,\Gamma\underline{\underline{Apr}}_\rho(Q)\cap\,\underline{\underline{Apr}}_\rho(Q)\Gamma\,(H/\rho)$. Then there exist $[x_1]_\rho, [x_2]_\rho \in \underline{\underline{Apr}}_\rho(Q)$, $\beta, \gamma \in \Gamma$ and $[y_1]_\rho, [y_2]_\rho \in H/\rho$ such that

$$[a]_\rho = [y_1]_\rho\widehat{\beta}[x_1]_\rho = [x_2]_\rho\widehat{\gamma}[y_2]_\rho.$$

Then $[x_1]_\rho \subseteq Q$ and $[x_2]_\rho \subseteq Q$. This implies that $x \in \underline{Apr}_\rho(Q)$ for all $x \in [x_1]_\rho \cup [x_2]_\rho$. Thus $[x_1]_\rho \subseteq \underline{Apr}_\rho(Q)$ and $[x_2]_\rho \subseteq \underline{Apr}_\rho(Q)$. Since $\underline{Apr}_\rho(Q)$ is a quasi Γ-hyperideal of H,

$$[a]_\rho \subseteq H\Gamma\underline{Apr}_\rho(Q) \cap \underline{Apr}_\rho(Q)\Gamma H \subseteq \underline{Apr}_\rho(Q).$$

So $[a]_\rho \in \underline{\underline{Apr}}_\rho(Q)$. Therefore $\underline{\underline{Apr}}_\rho(Q)$ is a quasi $\widehat{\Gamma}$-hyperideal of H/ρ. □ ■

Theorem 241 *Let ρ be a regular relation on a Γ-semihypergroup H. If A is an $(m,0)$ Γ-hyperideal ($(0,n)$ Γ-hyperideal) of H. Then,*

1. *$\overline{\overline{Apr}}_\rho(A)$ is an $(m,0)$ $\widehat{\Gamma}$-hyperideal ($(0,n)$ $\widehat{\Gamma}$-hyperideal) of H/ρ.*

2. *$\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, an $(m,0)$ $\widehat{\Gamma}$-hyperideal ($(0,n)$ $\widehat{\Gamma}$-hyperideal) of H/ρ.*

Proof. (1) Assume that A is a $(0,n)$ Γ-hyperideal of H. Let $[x]_\rho$ and $[s]_\rho$ be any elements of $\overline{\overline{Apr}}_\rho(A)$ and H/ρ, respectively. Then, $[x]_\rho \cap A \neq \emptyset$. Hence, $x \in \overline{Apr}_\rho(A)$. Since A is a $(0,n)$ Γ-hyperideal of H, by Theorem 213, $\overline{Apr}_\rho(A)$ is a $(0,n)$ Γ-hyperideal of H. So, for $\gamma \in \Gamma$, we have $s\gamma x^n \subseteq \overline{Apr}_\rho(A)$. Now, for every $t \in s\gamma x^n$, we have $[t]_\rho \cap A \neq \emptyset$. On the other hand, from $t \in s\gamma x^n$, we obtain $[t]_\rho \in [s]_\rho \widehat{\gamma}[x]_\rho^n$. Therefore, $[s]_\rho \widehat{\gamma}[x]_\rho^n \subseteq \overline{\overline{Apr}}_\rho(A)$. This means that $\overline{\overline{Apr}}_\rho(A)$ is a $(0,n)$ $\widehat{\Gamma}$-hyperideal of H/ρ.

(2) Let A be a $(0,n)$ Γ-hyperideal of H. Let $[x]_\rho$ and $[s]_\rho$ be any elements of $\underline{\underline{Apr}}_\rho(A)$ and H/ρ, respectively. Then, $[x]_\rho \subseteq A$, which implies $x \in \underline{Apr}_\rho(A)$. Since A is a $(0,n)$ Γ-hyperideal of H, by Theorem 213, $\underline{Apr}_\rho(A)$ is a $(0,n)$ Γ-hyperideal of H. Thus, for every $\gamma \in \Gamma$, we have $s\gamma x^n \subseteq \underline{Apr}_\rho(A)$. Now, for every $t \in s\gamma x^n$, we have $t \in \underline{Apr}_\rho(A)$, which implies that $[t]_\rho \subseteq A$. Hence, $[t]_\rho \in \underline{\underline{Apr}}_\rho(A)$. On the other hand, from $t \in s\gamma x^n$, we have $[t]_\rho \in [s]_\rho \widehat{\gamma}[x]_\rho^n$. Therefore, $[s]_\rho \widehat{\gamma}[x]_\rho^n \subseteq \underline{\underline{Apr}}_\rho(A)$. This means that $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, a $(0,n)$ $\widehat{\Gamma}$-hyperideal of H/ρ. The other cases can be seen in a similar way. This completes the proof. ■

Theorem 242 *Let ρ be a regular relation on a Γ-semihypergroup H. If A is an m-left Γ-hyperideal (resp., n-right Γ-hyperideal) of H. Then,*

1. *$\overline{\overline{Apr}}_\rho(A)$ is an m-left $\widehat{\Gamma}$-hyperideal (resp., n-right $\widehat{\Gamma}$-hyperideal) of H/ρ.*
2. *$\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, an m-left $\widehat{\Gamma}$-hyperideal (resp., n-right $\widehat{\Gamma}$-hyperideal) of H/ρ.*

Proof. (1) Assume that A is an m-left Γ-hyperideal of H. Let $[x]_\rho$ and $[s]_\rho$ be any elements of $\overline{\overline{Apr}}_\rho(A)$ and H/ρ, respectively. Then, $[x]_\rho \cap A \neq \emptyset$. Hence, $x \in \overline{Apr}_\rho(A)$. Since A is an m-left Γ-hyperideal of H, by Theorem 228(1), $\overline{Apr}_\rho(A)$ is an m-left Γ-hyperideal of H. So, for $\gamma \in \Gamma$, we have

$$s^m \gamma x \subseteq \overline{Apr}_\rho(A).$$

Now, for every $t \in s^m\gamma x$, we have $[t]_\rho \cap A \neq \emptyset$. On the other hand, from $t \in s^m\gamma x$, we obtain $[t]_\rho \in [s]_\rho^m \widehat{\gamma}[x]_\rho$. Therefore,

$$[s]_\rho^m \widehat{\gamma}[x]_\rho \subseteq \overline{\overline{Apr}}_\rho(A).$$

This means that $\overline{\overline{Apr}}_\rho(A)$ is an m-left $\widehat{\Gamma}$-hyperideal of H/ρ.

(2) Let A be an m-left Γ-hyperideal of H. Let $[x]_\rho$ and $[s]_\rho$ be any elements of $\underline{\underline{Apr}}_\rho(A)$ and H/ρ, respectively. Then, $[x]_\rho \subseteq A$, which implies $x \in \underline{Apr}_\rho(A)$. Since A is an m-left Γ-hyperideal of H, by Theorem 228(2), $\underline{Apr}_\rho(A)$ is an m-left Γ-hyperideal of H. Thus, for every $\gamma \in \Gamma$, we have

$$s^m \gamma x \subseteq \underline{Apr}_\rho(A).$$

Now, for every $t \in s^m\gamma x$, we have $t \in \underline{Apr}_\rho(A)$, which implies that $[t]_\rho \subseteq A$. Hence, $[t]_\rho \in \underline{\underline{Apr}}_\rho(A)$. On the other hand, from $t \in s^m\gamma x$, we have $[t]_\rho \in [s]_\rho^m \widehat{\gamma}[x]_\rho$. Therefore,

$$[s]_\rho^m \widehat{\gamma}[x]_\rho \subseteq \underline{\underline{Apr}}_\rho(A).$$

This means that $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, an m-left $\widehat{\Gamma}$-hyperideal of H/ρ. The other cases can be seen in a similar way. This completes the proof. ■

Theorem 243 *Let ρ be a regular relation on a Γ-semihypergroup H. If A is an (m, n) bi-Γ-hyperideal of H. Then,*

1. *$\overline{\overline{Apr}}_\rho(A)$ is an (m, n) bi-$\widehat{\Gamma}$-hyperideal of H/ρ.*
2. *$\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, an (m, n) bi-$\widehat{\Gamma}$-hyperideal of H/ρ.*

Proof. (1) Let $[x]_\rho$ and $[y]_\rho$ be any elements of $\overline{\overline{Apr}}_\rho(A)$ and $[s]_\rho$ be any element of H/ρ. Then,

$$[x]_\rho \cap A \neq \emptyset \quad \text{and} \quad [y]_\rho \cap A \neq \emptyset.$$

Hence, $x \in \overline{Apr}_\rho(A)$ and $y \in \overline{Apr}_\rho(A)$. By Theorem 214, $\overline{Apr}_\rho(A)$ is an (m, n) bi-Γ-hyperideal of H. So, for every $\alpha, \beta \in \Gamma$, we have $x^m\alpha s\beta y^n \subseteq \overline{Apr}_\rho(A)$. Now, for every $t \in x^m\alpha s\beta y^n$, we obtain $[t]_\rho \in [x]_\rho^m \widehat{\alpha} s \widehat{\beta}[y]_\rho^n$. On the other hand, since $t \in \overline{Apr}_\rho(A)$, we have $[t]_\rho \cap A \neq \emptyset$. Thus,

$$[x]_\rho^m \widehat{\alpha} s \widehat{\beta}[y]_\rho^n \subseteq \overline{\overline{Apr}}_\rho(A).$$

Therefore, $\overline{\overline{Apr}}_\rho(A)$ is an (m, n) bi-$\widehat{\Gamma}$-hyperideal of H/ρ.

(2) Let $[x]_\rho$ and $[y]_\rho$ be any elements of $\underline{\underline{Apr}}_\rho(A)$ and $[s]_\rho$ be any element of H/ρ. Then,

$$[x]_\rho \subseteq A \quad \text{and} \quad [y]_\rho \subseteq A.$$

Hence, $x \in \underline{Apr}_\rho(A)$ and $y \in \underline{Apr}_\rho(A)$. By Theorem 215, $\underline{Apr}_\rho(A)$ is an (m, n) bi-Γ-hyperideal of H. So, for every $\alpha, \beta \in \Gamma$, we have $x^m\alpha s\beta y^n \subseteq \underline{Apr}_\rho(A)$. Then, for every

$t \in x^m \alpha s \beta y^n$, we obtain $[t]_\rho \in [x]_\rho^m \widehat{\alpha} a \widehat{\beta} [y]_\rho^n$. On the other hand, since $t \in \underline{Apr}_\rho(A)$, we have $[t]_\rho \subseteq A$. So,

$$[x]_\rho^m \widehat{\alpha} a \widehat{\beta} [y]_\rho^n \subseteq \underline{\underline{Apr}}_\rho(A).$$

Therefore, $\underline{\underline{Apr}}_\rho(A)$ is, if it is non-empty, an (m, n) bi-$\widehat{\Gamma}$-hyperideal of H/ρ. This completes the proof. ■

Theorem 244 *Let ρ be a complete regular relation on a Γ-semihypergroup H. If A is a ρ-upper rough prime (m, n) bi-Γ-hyperideal of H, then $\overline{\overline{Apr}}_\rho(A)$ is a prime (m, n) bi-$\widehat{\Gamma}$-hyperideal of H/ρ.*

Proof. Let A be a ρ-upper rough prime (m, n) bi-Γ-hyperideal of H, by Theorem 243(1), we know that $\overline{\overline{Apr}}_\rho(A)$ is an (m, n) bi-Γ-hyperideal of H/ρ. Suppose for any $w \in H$, $\beta, \gamma \in \Gamma$ and $[x]_\rho, [y]_\rho \in H/\rho$, such that

$$\begin{aligned} [x]_\rho^m \widehat{\beta} w \widehat{\gamma} [y]_\rho^n &= [x^m]_\rho \widehat{\beta} w \widehat{\gamma} [y^n]_\rho \\ &= [x^m \beta w \gamma y^n]_\rho \subseteq \overline{\overline{Apr}}_\rho(A), \end{aligned}$$

for $\widehat{\beta}, \widehat{\gamma} \in \widehat{\Gamma}$. Thus $[x^m \beta w \gamma y^n]_\rho \cap A \neq \emptyset$. Now there exist t, such that $t \in x^m \beta w \gamma y^n \subseteq \overline{Apr}_\rho(A)$. Since A is a ρ-upper rough prime (m, n) bi-Γ-hyperideal of H, that is $\overline{Apr}_\rho(A)$ is a prime (m, n) bi-Γ-hyperideal, we have $x \in \overline{Apr}_\rho(A)$ or $y \in \overline{Apr}_\rho(A)$. Now as $t \in x^m \beta w \gamma y^n$, we obtain $[t]_\rho \in [x]_\rho^m \widehat{\beta} w \widehat{\gamma} [y]_\rho^n$. On the other hand, since $t \in \overline{Apr}_\rho(A)$, we have $[t]_\rho \cap A \neq \emptyset$. So $[x]_\rho \cap A \neq \emptyset$ or $[y]_\rho \cap A \neq \emptyset$. Hence $[x]_\rho \in \overline{\overline{Apr}}_\rho(A)$ or $[y]_\rho \in \overline{\overline{Apr}}_\rho(A)$. Therefore $\overline{\overline{Apr}}_\rho(A)$ is a prime (m, n) bi-$\widehat{\Gamma}$-hyperideal of H/ρ. ■

Theorem 245 *Let ρ be a complete regular relation on a Γ-semihypergroup H. If A is a ρ-lower rough prime (m, n) bi-Γ-hyperideal of H, then $\underline{\underline{Apr}}_\rho(A)$ is a prime (m, n) bi-$\widehat{\Gamma}$-hyperideal of H/ρ.*

Proof. Let A be a ρ-lower rough prime (m, n) bi-Γ-hyperideal of H, by Theorem 243(2), we know that $\underline{\underline{Apr}}_\rho(A)$ is an (m, n) bi-$\widehat{\Gamma}$-hyperideal of H/ρ. Suppose for any $w \in H$, $\beta, \gamma \in \Gamma$ and $[x]_\rho, [y]_\rho \in H/\rho$, such that

$$\begin{aligned} [x]_\rho^m \widehat{\beta} w \widehat{\gamma} [y]_\rho^n &= [x^m]_\rho \widehat{\beta} w \widehat{\gamma} [y^n]_\rho \\ &= [x^m \beta w \gamma y^n]_\rho \subseteq \underline{\underline{Apr}}_\rho(A). \end{aligned}$$

for $\widehat{\beta}, \widehat{\gamma} \in \widehat{\Gamma}$. Thus $[x^m \beta w \gamma y^n]_\rho \subseteq A$. Now there exist t, such that $t \in x^m \beta w \gamma y^n \subseteq \underline{Apr}_\rho(A)$. Since A is a ρ-lower rough prime (m, n) bi-Γ-hyperideal of H, that is

$\underline{Apr}_{\rho}(A)$ is a prime (m,n) bi-Γ-hyperideal, we have $x \in \underline{Apr}_{\rho}(A)$ or $y \in \underline{Apr}_{\rho}(A)$. Now as $t \in x^m \beta w \gamma y^n$, we obtain $[t]_\rho \in [x]_\rho^m \widehat{\beta} w \widehat{\gamma} [y]_\rho^n$. On the other hand, since $t \in \underline{Apr}_{\rho}(A)$, we have $[t]_\rho \subseteq A$. So $[x]_\rho \subseteq A$ or $[y]_\rho \subseteq A$. Hence $[x]_\rho \in \underline{\underline{Apr}}_{\rho}(A)$ or $[y]_\rho \in \underline{\underline{Apr}}_{\rho}(A)$. Therefore $\underline{\underline{Apr}}_{\rho}(A)$ is a prime (m,n) bi-$\widehat{\Gamma}$-hyperideal of H/ρ. ∎

Theorem 246 *Let ρ be a complete regular relation on a Γ-semihypergroup H. The following statements are true.*

1. *If Q is a ρ-upper rough (m,n) quasi Γ-hyperideal of H, then $\overline{\overline{Apr}}_{\rho}(Q)$ is an (m,n) quasi $\widehat{\Gamma}$-hyperideal of H/ρ.*

2. *If Q is a ρ-lower rough (m,n) quasi Γ-hyperideal of H, then $\underline{\underline{Apr}}_{\rho}(Q)$ is, if it is non-empty, an (m,n) quasi $\widehat{\Gamma}$-hyperideal of H/ρ.*

Proof. (1) Let $[a]_\rho \in (H/\rho)^m\, \Gamma \overline{\overline{Apr}}_{\rho}(Q) \cap \overline{\overline{Apr}}_{\rho}(Q) \Gamma\, (H/\rho)^n$. Then there exist $[x_1]_\rho, [x_2]_\rho \in \overline{\overline{Apr}}_{\rho}(Q)$, $\beta, \gamma \in \Gamma$, $[y_1]_\rho \in (H/\rho)^m$ and $[y_2]_\rho \in (H/\rho)^n$ such that $[a]_\rho = [y_1]_\rho \widehat{\beta} [x_1]_\rho = [x_2]_\rho \widehat{\gamma} [y_2]_\rho$. Then $[x_1]_\rho \cap Q \neq \emptyset$ and $[x_2]_\rho \cap Q \neq \emptyset$. This implies that $x \in \overline{Apr}_{\rho}(Q)$ for all $x \in [x_1]_\rho \cup [x_2]_\rho$. We have

$$[a]_\rho \subseteq H^m \Gamma \overline{Apr}_{\rho}(Q) \cap \overline{Apr}_{\rho}(Q) \Gamma H^n \subseteq \overline{Apr}_{\rho}(Q).$$

Then $a \in \overline{Apr}_{\rho}(Q)$. Thus $[a]_\rho \cap Q \neq \emptyset$. This implies that $[a]_\rho \in \overline{\overline{Apr}}_{\rho}(Q)$. Therefore $\overline{\overline{Apr}}_{\rho}(Q)$ is an (m,n) quasi $\widehat{\Gamma}$-hyperideal of H/ρ.

(2) Let $[a]_\rho \in (H/\rho)^m\, \Gamma \underline{\underline{Apr}}_{\rho}(Q) \cap \underline{\underline{Apr}}_{\rho}(Q) \Gamma\, (H/\rho)^n$. Then there exist $[x_1]_\rho, [x_2]_\rho \in \underline{\underline{Apr}}_{\rho}(Q)$, $\beta, \gamma \in \Gamma$, $[y_1]_\rho \in (H/\rho)^m$ and $[y_2]_\rho \in (H/\rho)^n$ such that $[a]_\rho = [y_1]_\rho \widehat{\beta} [x_1]_\rho = [x_2]_\rho \widehat{\gamma} [y_2]_\rho$. Then $[x_1]_\rho \subseteq Q$ and $[x_2]_\rho \subseteq Q$. This implies that $x \in \underline{Apr}_{\rho}(Q)$ for all $x \in [x_1]_\rho \cup [x_2]_\rho$. Thus $[x_1]_\rho \subseteq \underline{Apr}_{\rho}(Q)$ and $[x_2]_\rho \subseteq \underline{Apr}_{\rho}(Q)$. Since $\underline{Apr}_{\rho}(Q)$ is an (m,n) quasi Γ-hyperideal of H,

$$[a]_\rho \subseteq H^m \Gamma \underline{Apr}_{\rho}(Q) \cap \underline{Apr}_{\rho}(Q) \Gamma H^n \subseteq \underline{Apr}_{\rho}(Q).$$

So $[a]_\rho \in \underline{\underline{Apr}}_{\rho}(Q)$. Therefore $\underline{\underline{Apr}}_{\rho}(Q)$ is an (m,n) quasi $\widehat{\Gamma}$-hyperideal of H/ρ. ∎

Chapter 5

Generalized Rough Sets in Γ-Semihypergroups

Here, we study the roughness of sub Γ-semihypergroups, Γ-hyperideals and bi-Γ-hyperideals in terms of set valued homomorphisms. We study generalized lower and upper approximation operators to Γ-semihypergroups. We prove that the generalized lower and upper approximations of a Γ-hyperideal, by mean of a set valued mapping, is a Γ-hyperideal which is an extended notion of rough Γ-hyperideals introduced in [10]. Also we introduce the notion of generalized rough M-hypersystems and generalized rough N-hypersystems in Γ-semihypergroups and prove that the generalized upper rough approximation of an M-hypersystem (resp., N-hypersystem) is an M-hypersystem (resp., N-hypersystem). The results presented in this chapter are a part of our published papers [85, 86].

5.1 Some Notions in Generalized Rough Sets.

Here we will present some basics of the theory of generalized rough sets.

Definition 247 *[71] Let X and Y be two non-empty universes. Let T be a set-valued mapping given by $T : X \to P(Y)$. Then the triple (X, Y, T) is referred to as a generalized approximation space or generalized rough set. Any set-valued function from X to $P(Y)$ defines a binary relation from X to Y by setting $R_T = \{(x, y)|y \in T(x)\}$. Obviously, if R is an arbitrary relation from X to Y, then it can be defined as a set-valued* *mapping* $T_R : X \to P(Y)$ *by* $T_R(x) = \{y \in Y | (x, y) \in R\}$ *where* $x \in X$. *For any set* $A \subseteq Y$, *the lower and upper approximations* $\underline{T}(A)$ *and* $\overline{T}(A)$, *are defined by*

$$\begin{aligned} \underline{T}(A) &= \{x \in X | T(x) \subseteq A\} \\ \overline{T}(A) &= \{x \in X | T(x) \cap A \neq \emptyset\}. \end{aligned}$$

The pair $(\underline{T}(A), \overline{T}(A))$ *is referred to as a generalized rough set, and* $\underline{T}$ *and* $\overline{T}$ *are referred to as lower and upper generalized approximation operators, respectively.*

If a subset $A \subseteq Y$ satisfies that $\underline{T}(A) = \overline{T}(A)$, then A is called a definable set of (X, Y, T). We denote all the definable sets of (X, Y, T) by Def(T).

Theorem 248 *[71] Let* (X, Y, T) *be a generalized approximation space, its lower and upper approximation operators satisfy the following properties: For all* $A, B \in P(Y)$,

$(L1)$ $\underline{T}(A) = (\overline{T}(A^c))^c$,

$(L2)$ $\underline{T}(Y) = X$,

$(L3)$ $\underline{T}(A \cap B) = \underline{T}(A) \cap \underline{T}(B)$,

$(L4)$ $A \subseteq B \Rightarrow \underline{T}(A) \subseteq \underline{T}(B)$,

$(L5)$ $\underline{T}(A \cup B) \supseteq \underline{T}(A) \cup \underline{T}(B)$,

$(U1)$ $\overline{T}(A) = (\underline{T}(A^c))^c$,

$(U2)$ $\overline{T}(\emptyset) = \emptyset$,

$(U3)$ $\overline{T}(A \cup B) = \overline{T}(A) \cup \overline{T}(B)$

$(U4)$ $A \subseteq B \Rightarrow \overline{T}(A) \subseteq \overline{T}(B)$,

$(U5)$ $\overline{T}(A \cap B) \subseteq \overline{T}(A) \cap \overline{T}(B)$,

where A^c *is the complement of the set* A.

Theorem 249 *[71] Let* (X, X, T) *be a generalized approximation space, its lower and upper generalized approximation operators satisfy the following properties: For all* $A \in$

$P(X)$,

$$\begin{array}{lll}
(1)\ R_T \textit{ is serial} & \Leftrightarrow (L0)\underline{T}(\emptyset) = \emptyset \\
& \Leftrightarrow (U0)\overline{T}(X) = X \\
& \Leftrightarrow (LU0)\underline{T}(A) \subseteq \overline{T}(A) \\
(2)\ R_T \textit{ is reflexive} & \Leftrightarrow (L6)\underline{T}(A) \subseteq A \\
& \Leftrightarrow (U6)A \subseteq \overline{T}(A) \\
(3)\ R_T \textit{ is symmetric} & \Leftrightarrow (L7)\overline{T}(\underline{T}(A)) \subseteq A \\
& \Leftrightarrow (U7)A \subseteq \underline{T}(\overline{T}(A)) \\
(4)\ R_T \textit{ is transitive} & \Leftrightarrow (L8)\underline{T}(A) \subseteq \underline{T}(\underline{T}(A)) \\
& \Leftrightarrow (U8)\overline{T}(\overline{T}(A)) \subseteq \overline{T}(A).
\end{array}$$

If R is an equivalence relation on X, then the pair (X, R) is the Pawlak approximation space. Therefore, a generalized rough set is an extended notion of Pawlak's rough sets.

5.2 Generalized Lower and Upper Approximations

In this section, we will discuss some results on generalized lower and upper approximations in Γ-semihypergroups.

Definition 250 *A set-valued homomorphism T from a Γ-semihypergroup H to a $\dot{\Gamma}$-semihypergroup $\dot{H}$ is a mapping from H to $\wp^*(\dot{H})$ that preserves the operation, that is, $T(a)\dot{\beta}T(b) \subseteq T(a\beta b)$ for all $a, b \in H$, $\beta \in \Gamma$ and $\dot{\beta} \in \dot{\Gamma}$. T is called a strong set-valued homomorphism, if $T(a)\dot{\beta}T(b) = T(a\beta b)$ for all $a, b \in H$, $\beta \in \Gamma$ and $\dot{\beta} \in \dot{\Gamma}$.*

Example 251 *Let $H = \{x, y, z\}$ and $\Gamma = \{\beta, \gamma\}$ be the sets of binary hyperoperations defined below:*

β	x	y	z
x	$\{x, y\}$	y	$\{y, z\}$
y	y	y	y
z	z	z	z

γ	x	y	z
x	$\{x, z\}$	$\{y, z\}$	z
y	y	y	y
z	z	z	z

Here H is a Γ-semihypergroup. Let $\dot{H} = \{a, b, c, d, e\}$ and $\dot{\Gamma} = \{\dot{\beta}, \dot{\gamma}\}$ be the sets of *binary hyperoperations defined below:*

$\dot{\beta}$	a	b	c	d	e
a	a	a	a	a	a
b	a	$\{b,d\}$	a	$\{a,d\}$	$\{a,b\}$
c	a	a	a	a	a
d	a	$\{a,d\}$	a	a	$\{a,b\}$
e	a	a	$\{a,c\}$	a	$\{a,e\}$

$\dot{\gamma}$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	$\{c,e\}$	$\{a,c\}$	$\{a,e\}$
d	a	$\{a,b\}$	a	$\{a,d\}$	a
e	a	a	$\{a,e\}$	$\{a,c\}$	a

Here $\dot{H}$ is a $\dot{\Gamma}$-semihypergroup. Assume $T(x) = \{a,b,d\}$, $T(y) = \{a\}$ and $T(z) = \{a,b,c,d,e\}$. Here T is a set-valued homomorphism from H to $\dot{H}$. But T is not a strong set-valued homomorphism from H to $\dot{H}$ because for all $\beta \in \Gamma$ and $\dot{\beta} \in \dot{\Gamma}$, we have

$$\{a\} = T(z)\dot{\beta}T(y) \neq T(z\beta y) = \{a,b,c,d,e\}.$$

Theorem 252 *Let θ be a (complete) regular relation on a Γ-semihypergroup H. Define $T_\theta : H \to P(H)$ by $T_\theta(x) = [x]_\theta$. Then T_θ is a (strong) set-valued homomorphism.*

Theorem 253 *Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup and $T : H \to \wp^*(\dot{H})$ be a set-valued homomorphism. If A, B are two non-empty subsets of $\dot{H}$. Then*

1. $\overline{T}(A)\Gamma\overline{T}(B) \subseteq \overline{T}(A\dot{\Gamma}B)$.
2. *If T is strong then,* $\underline{T}(A)\Gamma\underline{T}(B) \subseteq \underline{T}(A\dot{\Gamma}B)$.

Proof. (1) Assume that w be any element of $\overline{T}(A)\Gamma\overline{T}(B)$. Then $w \in x\beta y$ with $x \in \overline{T}(A)$, $y \in \overline{T}(B)$ and $\beta \in \Gamma$. Therefore $T(x) \cap A \neq \emptyset$ and $T(y) \cap B \neq \emptyset$. Thus there exist elements a, b such that $a \in T(x) \cap A$ and $b \in T(y) \cap B$. Since T is a set-valued homomorphism, it follows that

$$a\dot{\beta}b \subseteq T(x)\dot{\beta}T(y) \subseteq T(x\beta y), \text{ for all } \dot{\beta} \in \dot{\Gamma}.$$

Also, since $a\dot{\beta}b \subseteq A\dot{\Gamma}B$, we have $a\dot{\beta}b \subseteq T(x\beta y) \cap A\dot{\Gamma}B$, so $w \in x\beta y \subseteq \overline{T}(A\dot{\Gamma}B)$. This shows that $\overline{T}(A)\Gamma\overline{T}(B) \subseteq \overline{T}(A\dot{\Gamma}B)$.

(2) Assume that w be any element of $\underline{T}(A)\Gamma\underline{T}(B)$. Then $w \in x\beta y$ with $x \in \underline{T}(A)$, $y \in \underline{T}(B)$ and $\beta \in \Gamma$. It follows that $T(x) \subseteq A$ and $T(y) \subseteq B$. Since T is a strong set-valued homomorphism, we have

$$T(x\beta y) = T(x)\dot{\beta}T(y) \subseteq A\dot{\Gamma}B, \text{ for all } \dot{\beta} \in \dot{\Gamma}.$$

We have $w \in x\beta y \subseteq \underline{T}(A\dot{\Gamma}B)$. This shows that $\underline{T}(A)\Gamma\underline{T}(B) \subseteq \underline{T}(A\dot{\Gamma}B)$. ■

Let T_1 and T_2 be two set-valued mappings from H to $\wp^*(\dot{H})$, we define

$$(T_1 \cap T_2)(x) = T_1(x) \cap T_2(x).$$

Theorem 254 *Let T_1 and T_2 be two set-valued mappings from H to $\wp^*(\dot{H})$. If A is a non-empty subset of $\dot{H}$. Then*

1. $\overline{T_1 \cap T_2}(A) \subseteq \overline{T_1}(A) \cap \overline{T_2}(A)$;
2. $\underline{T_1 \cap T_2}(A) \supseteq \underline{T_1}(A) \cap \underline{T_2}(A)$.

Proof. (1) Assume that $x \in \overline{T_1 \cap T_2}(A)$, thus $(T_1 \cap T_2)(x) \cap A \neq \emptyset$. There exists $y \in (T_1 \cap T_2)(x) \cap A$. So,

$$y \in T_1(x) \cap A \text{ and } y \in T_2(x) \cap A,$$

which imply that $x \in \overline{T_1}(A)$ and $x \in \overline{T_2}(A)$. Thus $x \in \overline{T_1}(A) \cap \overline{T_2}(A)$.

(2) Assume that $x \in \underline{T_1}(A) \cap \underline{T_2}(A)$, thus $x \in \underline{T_1}(A)$ and $x \in \underline{T_2}(A)$. We have $T_1(x) \subseteq A$ and $T_2(x) \subseteq A$. So $T_1(x) \cap T_2(x) \subseteq A$. Thus $(T_1 \cap T_2)(x) \subseteq A$, which implies that $x \in \underline{T_1 \cap T_2}(A)$. ■

Theorem 255 *Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup and $T : H \to \wp^*(\dot{H})$ be a set-valued homomorphism. If A and B are a right $\dot{\Gamma}$-hyperideal and a left $\dot{\Gamma}$-hyperideal of $\dot{H}$, respectively. Then*

$$\overline{T}(A\dot{\Gamma}B) \subseteq \overline{T}(A) \cap \overline{T}(B) \quad \textit{and} \quad \underline{T}(A\dot{\Gamma}B) \subseteq \underline{T}(A) \cap \underline{T}(B).$$

Proof. Since A is a right $\dot{\Gamma}$-hyperideal of $\dot{H}$, so $A\dot{\Gamma}B \subseteq A\dot{\Gamma}\dot{H} \subseteq A$. Since B is a left $\dot{\Gamma}$-hyperideal of $\dot{H}$, so $A\dot{\Gamma}B \subseteq \dot{H}\dot{\Gamma}B \subseteq B$. Thus $A\dot{\Gamma}B \subseteq A \cap B$. Then by Theorem 248 (U4 and U5), we have

$$\overline{T}(A\dot{\Gamma}B) \subseteq \overline{T}(A \cap B) \subseteq \overline{T}(A) \cap \overline{T}(B).$$

Also by Theorem 248 (L4 and L3), we have

$$\underline{T}(A\dot{\Gamma}B) \subseteq \underline{T}(A \cap B) = \underline{T}(A) \cap \underline{T}(B).$$

This completes the proof. ■

Lemma 256 *Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup and $T : H \to \wp^*(\dot{H})$ be a set-valued homomorphism. Then, for a non-empty subset A of $\dot{H}$*

1. $\left(\overline{T}(A)\right)^n \subseteq \overline{T}(A^n)$ *for all* $n \in \mathbb{N}$.
2. *If T is strong, then* $(\underline{T}(A))^n \subseteq \underline{T}(A^n)$ *for all* $n \in \mathbb{N}$.

Proof. The proof is straightforward. ■

5.3 Generalized Rough Γ-Hyperideals

In this section we present some results of generalized rough Γ-hyperideals in Γ- semihypergroups.

Theorem 257 *Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup and $T : H \to \wp^*(\dot{H})$ be a set-valued homomorphism.*

1. *If A is a sub $\dot{\Gamma}$-semihypergroup of $\dot{H}$, then $\overline{T}(A)$ is, if it is non-empty, a sub Γ-semihypergroup of H.*
2. *If A is a left (resp., right) $\dot{\Gamma}$-hyperideal of $\dot{H}$ and $T(x) \neq \emptyset$ for all $x \in H$, then $\overline{T}(A)$ is, if it is non-empty, a left (resp., right) Γ-hyperideal of H.*

Proof. (1) Since A is a sub $\dot{\Gamma}$-semihypergroup of $\dot{H}$, we have $A\dot{\Gamma}A \subseteq A$. By Theorem 248(U4) and Theorem 253(1), we have

$$\overline{T}(A)\Gamma\overline{T}(A) \subseteq \overline{T}(A\dot{\Gamma}A) \subseteq \overline{T}(A).$$

Thus $\overline{T}(A)$ is a sub Γ-semihypergroup of H.

(2) Let A be a left $\dot{\Gamma}$-hyperideal of $\dot{H}$, we have $\dot{H}\dot{\Gamma}A \subseteq A$. By Theorem 248(U4), $\overline{T}(\dot{H}\dot{\Gamma}A) \subseteq \overline{T}(A)$. Since $\overline{T}(\dot{H}) = H$. So by Theorem 253(1), we have

$$H\Gamma\overline{T}(A) = \overline{T}(\dot{H})\Gamma\overline{T}(A) \subseteq \overline{T}(\dot{H}\dot{\Gamma}A) \subseteq \overline{T}(A).$$

Thus $\overline{T}(A)$ is, if it is non-empty, a left Γ-hyperideal of H. The other case can be seen in a similar way. ■

Theorem 258 *Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup and $T : H \to \wp^*(\dot{H})$ be a strong set-valued homomorphism.*

1. *If A is a sub $\dot{\Gamma}$-semihypergroup of $\dot{H}$, then $\underline{T}(A)$ is, if it is non-empty, a sub Γ-semihypergroup of H.*
2. *If A is a left (resp., right) $\dot{\Gamma}$-hyperideal of $\dot{H}$, then $\underline{T}(A)$ is, if it is non-empty, a left (resp., right) Γ-hyperideal of H.*

Proof. (1) Since A is a sub $\dot{\Gamma}$-semihypergroup of $\dot{H}$, we have $A\dot{\Gamma}A \subseteq A$. By Theorem 248(L4) and Theorem 253(2), we have

$$\underline{T}(A)\Gamma\underline{T}(A) \subseteq \underline{T}(A\dot{\Gamma}A) \subseteq \underline{T}(A).$$

Thus $\underline{T}(A)$ is a sub Γ-semihypergroup of H.

(2) Let A be a left $\dot{\Gamma}$-hyperideal of $\dot{H}$, we have $\dot{H}\dot{\Gamma}A \subseteq A$. By Theorem 248(L4), $\underline{T}(\dot{H}\dot{\Gamma}A) \subseteq \underline{T}(A)$. Since $\underline{T}(\dot{H}) = H$. So by Theorem 253(2), we have

$$H\Gamma\underline{T}(A) = \underline{T}(\dot{H})\Gamma\underline{T}(A) \subseteq \underline{T}(\dot{H}\dot{\Gamma}A) \subseteq \underline{T}(A).$$

Thus $\underline{T}(A)$ is, if it is non-empty, a Γ-hyperideal of H. The other case can be seen in a similar way. ■

Theorem 259 *Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup and $T : H \to \wp^*(\dot{H})$ be a strong set-valued homomorphism. If A is a prime $\dot{\Gamma}$-hyperideal of $\dot{H}$. Then*

1. *$\overline{T}(A)$ is, if it is non-empty, a prime Γ-hyperideal of H,*
2. *$\underline{T}(A)$ is, if it is non-empty, a prime Γ-hyperideal of H.*

Proof. By Theorem 257(2) and Theorem 258(2), we know $\overline{T}(A)$ and $\underline{T}(A)$ are Γ-hyperideals of H.

(1) Let $x\Gamma y \subseteq \overline{T}(A)$, for some $x, y \in H$. Then $T(x\Gamma y) \cap A \neq \emptyset$. Since T is a strong set-valued homomorphism, we have $T(x\Gamma y) = T(x)\dot{\Gamma}T(y)$. So there exist $u \in T(x)$, $v \in T(y)$ such that $u\dot{\Gamma}v \subseteq A$. Since A is a prime $\dot{\Gamma}$-hyperideal of $\dot{H}$, we have $u \in A$ or $v \in A$. Thus

$$T(x) \cap A \neq \emptyset \text{ or } T(y) \cap A \neq \emptyset.$$

So $x \in \overline{T}(A)$ or $y \in \overline{T}(A)$. Therefore, $\overline{T}(A)$ is, if it is non-empty, a prime Γ-hyperideal of H.

(2) Let $x\Gamma y \subseteq \underline{T}(A)$, for some $x, y \in H$. We suppose that $\underline{T}(A)$ is not a prime Γ-hyperideal of H, then $x\Gamma y \subseteq \underline{T}(A)$ but $x \notin \underline{T}(A)$ and $y \notin \underline{T}(A)$. Thus $T(x) \nsubseteq A$ and $T(y) \nsubseteq A$. So there exist $a \in T(x)$ and $b \in T(y)$, but $a, b \notin A$. Thus

$$a\dot{\Gamma}b \subseteq T(x)\dot{\Gamma}T(y) = T(x\Gamma y) \subseteq A.$$

Since A is a prime $\dot{\Gamma}$-hyperideal of $\dot{H}$, we have $a \in A$ or $b \in A$, which is a contradiction. Hence $\underline{T}(A)$ is, if it is non-empty, a prime Γ-hyperideal of H. ■

The following example shows that the converse of Theorem 259, does not hold.

Example 260 *Let $H = \{x, y, z\}$ and $\Gamma = \{\beta, \gamma\}$ be the sets of binary hyperoperations defined below:*

β	x	y	z
x	x	y	$\{x, z\}$
y	y	y	y
z	$\{x, z\}$	y	$\{x, z\}$

γ	x	y	z
x	x	y	$\{x, z\}$
y	y	y	y
z	$\{x, z\}$	y	z

Here H is a Γ-semihypergroup. Let $\dot{H} = \{a, b, c, d\}$ and $\dot{\Gamma} = \{\dot{\beta}, \dot{\gamma}\}$ be the sets of binary hyperoperations defined below:

$\dot{\beta}$	a	b	c	d
a	a	$\{a, b\}$	$\{c, d\}$	d
b	$\{a, b\}$	$\{a, b\}$	$\{c, d\}$	d
c	$\{c, d\}$	$\{c, d\}$	c	d
d	d	d	d	d

$\dot{\gamma}$	a	b	c	d
a	$\{a, b\}$	$\{a, b\}$	$\{c, d\}$	d
b	$\{a, b\}$	b	$\{c, d\}$	d
c	$\{c, d\}$	$\{c, d\}$	c	d
d	d	d	d	d

Here $\dot{H}$ is a $\dot{\Gamma}$-semihypergroup. Assume $T(x) = \{a, b\}$, $T(y) = \{d\}$ and $T(z) = \{a, b, c, d\}$. Here T is a strong set-valued homomorphism from H to $\dot{H}$. Now for $A = \{a, d\} \subseteq \dot{H}$, $\overline{T}(A) = \{x, y, z\}$ and $\underline{T}(A) = \{y\}$. It is clear that $\overline{T}(A)$ and $\underline{T}(A)$ are prime Γ-hyperideals of H. But A is not a $\dot{\Gamma}$-hyperideal of $\dot{H}$, hence A is not a prime $\dot{\Gamma}$-hyperideal of $\dot{H}$.

Theorem 261 *Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup and $T : H \to \wp^*(\dot{H})$ be a set-valued homomorphism. If A is a bi-$\dot{\Gamma}$-hyperideal of $\dot{H}$, then*

1. *$\overline{T}(A)$ is, if it is non-empty, a bi-Γ-hyperideal of H.*

2. *if T is strong, then $\underline{T}(A)$ is, if it is non-empty, a bi-Γ-hyperideal of H.*

Proof. (1) Since A is a bi-$\dot{\Gamma}$-hyperideal of $\dot{H}$, we have $A\dot{\Gamma}\dot{H}\dot{\Gamma}A \subseteq A$. We know that $\overline{T}(\dot{H}) = H$. By Theorem 253(1), we have

$$\begin{aligned}\overline{T}(A)\Gamma H\Gamma\overline{T}(A) &= \overline{T}(A)\Gamma\overline{T}(\dot{H})\Gamma\overline{T}(A)\\ &\subseteq \overline{T}(A\dot{\Gamma}\dot{H}\dot{\Gamma}A) \subseteq \overline{T}(A).\end{aligned}$$

By this and Theorem 257(1), $\overline{T}(A)$ is, if it is non-empty, a bi-Γ-hyperideal of H.

(2) Since A is a bi-$\dot{\Gamma}$-hyperideal of $\dot{H}$, we have $A\dot{\Gamma}\dot{H}\dot{\Gamma}A \subseteq A$. We know that $\underline{T}(\dot{H}) = H$. By Theorem 253(2), we have

$$\begin{aligned}\underline{T}(A)\Gamma H\Gamma\underline{T}(A) &= \underline{T}(A)\Gamma\underline{T}(\dot{H})\Gamma\underline{T}(A)\\ &\subseteq \underline{T}(A\dot{\Gamma}\dot{H}\dot{\Gamma}A) \subseteq \underline{T}(A).\end{aligned}$$

By this and Theorem 258(1), $\underline{T}(A)$ is, if it is non-empty, a bi-Γ-hyperideal of H. ■

Theorem 262 *Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup and $T : H \to \wp^*(\dot{H})$ be a strong set-valued homomorphism. If A is a prime bi-$\dot{\Gamma}$-hyperideal of $\dot{H}$, then*

1. *$\overline{T}(A)$ is, if it is non-empty, a prime bi-Γ-hyperideal of H.*

2. *$\underline{T}(A)$ is, if it is non-empty, a prime bi-Γ-hyperideal of H.*

Proof. The proof is similar to the proof of the Theorem 259. ■

5.4 Generalized Rough (m, n) (Bi-)Quasi Γ-Hyperideals

We will study here some properties of generalized lower and upper approximations of (m, n) bi-Γ-hyperideals in Γ-semihypergroups.

A subset A of a $\dot{\Gamma}$-semihypergroup $\dot{H}$ is called a generalized upper (resp., generalized lower) rough (m, n) bi-Γ-hyperideal of H if $\overline{T}(A)$ (resp., $\underline{T}(A)$) is an (m, n) bi-Γ-hyperideal of H.

Theorem 263 *Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup and $T : H \to \wp^*(\dot{H})$ be a set-valued homomorphism. If A is an (m,n) bi-$\dot{\Gamma}$-hyperideal of $\dot{H}$, then*

1. *$\overline{T}(A)$ is, if it is non-empty, an (m,n) bi-Γ-hyperideal of H.*

2. *if T is strong, then $\underline{T}(A)$ is, if it is non-empty, an (m,n) bi-Γ-hyperideal of H.*

Proof. (1) Let A be an (m,n) bi-$\dot{\Gamma}$-hyperideal of $\dot{H}$. Then, by Theorem 253(1) and Lemma 256(1), we have

$$\begin{aligned}(\overline{T}(A))^m\Gamma H\Gamma(\overline{T}(A))^n &= (\overline{T}(A))^m\Gamma\overline{T}(\dot{H})\Gamma(\overline{T}(A))^n \\ &\subseteq \overline{T}(A^m)\Gamma\overline{T}(\dot{H})\Gamma\overline{T}(A^n) \\ &\subseteq \overline{T}(A^m\dot{\Gamma}\dot{H})\Gamma\overline{T}(A^n) \\ &\subseteq \overline{T}(A^m\dot{\Gamma}\dot{H}\dot{\Gamma}A^n) \subseteq \overline{T}(A).\end{aligned}$$

From this and Theorem 257(1), we obtain that $\overline{T}(A)$ is an (m,n) bi-Γ-hyperideal of H.

(2) Let A be an (m,n) bi-$\dot{\Gamma}$-hyperideal of $\dot{H}$. Then, by Theorem 253(2) and Lemma 256(2), we have

$$\begin{aligned}(\underline{T}(A))^m\Gamma H\Gamma(\underline{T}(A))^n &= (\underline{T}(A))^m\Gamma\underline{T}(\dot{H})\Gamma(\underline{T}(A))^n \\ &\subseteq \underline{T}(A^m)\Gamma\underline{T}(\dot{H})\Gamma\underline{T}(A^n) \\ &\subseteq \underline{T}(A^m\dot{\Gamma}\dot{H})\Gamma\underline{T}(A^n) \\ &\subseteq \underline{T}(A^m\dot{\Gamma}\dot{H}\dot{\Gamma}A^n) \subseteq \underline{T}(A).\end{aligned}$$

From this and Theorem 258(1), we obtain that $\underline{T}(A)$ is, if it is non-empty, an (m,n) bi-Γ-hyperideal of H. This completes the proof. ■

Corollary 264 *Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup and $T : H \to \wp^*(\dot{H})$ be a set-valued homomorphism. If A is an $(m,0)$ $\dot{\Gamma}$-hyperideal (resp., $(0,n)$ $\dot{\Gamma}$-hyperideal, m-left $\dot{\Gamma}$-hyperideal, n-right $\dot{\Gamma}$-hyperideal) of $\dot{H}$, then*

1. *$\overline{T}(A)$ is, if it is non-empty, an $(m,0)$ Γ-hyperideal (resp., $(0,n)$ Γ-hyperideal, m-left Γ-hyperideal, n-right Γ-hyperideal) of H.*

2. *if T is strong, then $\underline{T}(A)$ is, if it is non-empty, an $(m,0)$ Γ-hyperideal (resp., $(0,n)$ Γ-hyperideal, m-left Γ-hyperideal, n-right Γ-hyperideal) of H.*

Proof. The proof is straightforward. ■

A subset A of a $\dot{\Gamma}$-semihypergroup $\dot{H}$ is called a generalized upper (resp., generalized lower) rough prime (m,n) bi-Γ-hyperideal of H if $\overline{T}(A)$ (resp., $\underline{T}(A)$) is a prime (m,n) bi-Γ-hyperideal of H.

Theorem 265 *Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup and $T : H \rightarrow \wp^*(\dot{H})$ be a strong set-valued homomorphism. If A is a prime (m,n) bi-$\dot{\Gamma}$-hyperideal of $\dot{H}$, then*

1. *$\overline{T}(A)$ is, if it is non-empty, a prime (m,n) bi-Γ-hyperideal of H,*
2. *$\underline{T}(A)$ is, if it is non-empty, a prime (m,n) bi-Γ-hyperideal of H.*

Proof. Since A is an (m,n) bi-$\dot{\Gamma}$-hyperideal of $\dot{H}$. By Theorem 263, we know that $\overline{T}(A)$ and $\underline{T}(A)$ are (m,n) bi-Γ-hyperideals of H.

(1) Let w be any element of H. Let $x,y \in H$ and $\beta,\gamma \in \Gamma$ such that $x^m\beta w\gamma y^n \subseteq \overline{T}(A)$. Thus

$$T(x^m\beta w\gamma y^n) \cap A = (T(x^m)\dot{\beta}T(w)\dot{\gamma}T(y^n)) \cap A \neq \emptyset,$$

where $\dot{\beta},\dot{\gamma} \in \dot{\Gamma}$. Thus there exist $a^m \subseteq T(x^m) = T(x)^m$, $w' \in T(w)$ and $b^n \subseteq T(y^n) = T(y)^n$ such that $a^m\dot{\beta}w'\dot{\gamma}b^n \subseteq A$. Since A is a prime (m,n) bi-$\dot{\Gamma}$-hyperideal, we have $a \in A$ or $b \in A$. Now

$$a^m \subseteq T(x)^m \Longrightarrow a \in T(x) \text{ also } b^n \subseteq T(y)^n \Longrightarrow b \in T(y).$$

Thus $a \in T(x) \cap A$ or $b \in T(y) \cap A$. So $x \in \overline{T}(A)$ or $y \in \overline{T}(A)$. Therefore $\overline{T}(A)$ is a prime (m,n) bi-Γ-hyperideal of H.

(2) We suppose that $\underline{T}(A)$ is not a prime (m,n) bi-Γ-hyperideal, then for $\beta,\gamma \in \Gamma$ there exists $x,y \in H$ and any element $w \in H$, such that $x^m\beta w\gamma y^n \subseteq \underline{T}(A)$, but $x \notin \underline{T}(A)$ and $y \notin \underline{T}(A)$. Thus $T(x) \nsubseteq A$ and $T(y) \nsubseteq A$. Then there exist

$$a \in T(x) \text{ but } a \notin A \quad \text{and} \quad b \in T(y) \text{ but } b \notin A.$$

Now for $w' \in H$, $\beta,\gamma \in \Gamma$ and $\dot{\beta},\dot{\gamma} \in \dot{\Gamma}$, we have

$$a^m\dot{\beta}w'\dot{\gamma}b^n \subseteq T(x)^m\dot{\beta}T(w)\dot{\gamma}T(y)^n = T(x^m)\dot{\beta}T(w)\dot{\gamma}T(y^n) = T(x^m\beta w\gamma y^n) \subseteq A.$$

This implies that $a^m \dot{\beta} w' \dot{\gamma} b^n \subseteq A$. Since A is a prime (m,n) bi-$\dot{\Gamma}$-hyperideal of $\dot{H}$, we have $a \in A$ or $b \in A$. It contradicts the supposition. This means that $\underline{T}(A)$ is, if it is non-empty, a prime (m,n) bi-Γ-hyperideal of H. ■

The following example shows that the converse of Theorem 265 does not hold.

Example 266 *Let $H = \{x, y, z\}$ and $\Gamma = \{\beta, \gamma\}$ be the sets of binary hyperoperations defined below:*

β	x	y	z
x	x	x	x
y	x	y	$\{y,z\}$
z	x	$\{y,z\}$	$\{y,z\}$

γ	x	y	z
x	x	x	x
y	x	y	$\{y,z\}$
z	x	$\{y,z\}$	z

Here H is a Γ-semihypergroup. Let $\dot{H} = \{a, b, c, d, e\}$ and $\dot{\Gamma} = \{\dot{\beta}, \dot{\gamma}\}$ be the sets of binary hyperoperations defined below:

$\dot{\beta}$	a	b	c	d	e
a	$\{a,b\}$	$\{b,c\}$	c	$\{d,e\}$	e
b	$\{b,c\}$	c	c	$\{d,e\}$	e
c	c	c	c	$\{d,e\}$	e
d	$\{d,e\}$	$\{d,e\}$	$\{d,e\}$	d	e
e	e	e	e	e	e

$\dot{\gamma}$	a	b	c	d	e
a	$\{b,c\}$	c	c	$\{d,e\}$	e
b	c	c	c	$\{d,e\}$	e
c	c	c	c	$\{d,e\}$	e
d	$\{d,e\}$	$\{d,e\}$	$\{d,e\}$	d	e
e	e	e	e	e	e

Here $\dot{H}$ is a $\dot{\Gamma}$-semihypergroup. Assume $T(x) = \{d, e\}$, $T(y) = \{a, b, c\}$ and $T(z) = \{a, b, c, d, e\}$. Here T is a strong set-valued homomorphism from H to $\dot{H}$. Now for $A = \{b, d, e\} \subseteq \dot{H}$, $\overline{T}(A) = \{x, y, z\}$ and $\underline{T}(A) = \{x\}$. It is clear that $\overline{T}(A)$ and $\underline{T}(A)$ are prime (m,n) bi-Γ-hyperideals of H. But A is not a sub $\dot{\Gamma}$-semihypergroup of $\dot{H}$, hence A is not a prime (m,n) bi-$\dot{\Gamma}$-hyperideal of $\dot{H}$.

A subset Q of a $\dot{\Gamma}$-semihypergroup $\dot{H}$ is called a generalized upper (resp., generalized lower) rough (m,n) quasi Γ-hyperideal of H if $\overline{T}(A)$ (resp., $\underline{T}(A)$) is an (m,n) quasi Γ-hyperideal of H.

Theorem 267 *Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup and $T : H \to \wp^*(\dot{H})$ be a strong set-valued homomorphism. If Q is an (m,n) quasi $\dot{\Gamma}$-hyperideal of $\dot{H}$, then $\underline{T}(A)$ is, if it is non-empty, an (m,n) quasi Γ-hyperideal of H.*

Proof. Let Q be an (m,n) quasi $\dot{\Gamma}$-hyperideal of $\dot{H}$, that is, $\dot{H}^m\dot{\Gamma}Q \cap Q\dot{\Gamma}\dot{H}^n \subseteq Q$. Note that $\underline{T}(\dot{H}) = H$. Then by Theorem 248(L3), Theorem 253(2) and Lemma 256(2), we have

$$\begin{aligned} H^m\Gamma\underline{T}(Q) \cap \underline{T}(Q)\Gamma H^n &= \left(\underline{T}(\dot{H})\right)^m \Gamma\underline{T}(Q) \cap \underline{T}(Q)\Gamma\left(\underline{T}(\dot{H})\right)^n \\ &\subseteq \underline{T}(\dot{H}^m)\Gamma\underline{T}(Q) \cap \underline{T}(Q)\Gamma\underline{T}(\dot{H}^n) \\ &\subseteq \underline{T}(\dot{H}^m\dot{\Gamma}Q) \cap \underline{T}(Q\dot{\Gamma}\dot{H}^n) \\ &= \underline{T}(\dot{H}^m\dot{\Gamma}Q \cap Q\dot{\Gamma}\dot{H}^n) \\ &\subseteq \underline{T}(Q). \end{aligned}$$

This shows that $\underline{T}(Q)$ is an (m,n) quasi Γ-hyperideal of H. ■

The next theorem shows that the intersection of a generalized lower rough m-left Γ-hyperideal and a generalized lower rough n-right Γ-hyperideal of a Γ-semihypergroup H is a generalized lower rough (m,n) quasi Γ-hyperideal of H.

Theorem 268 *Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup and $T : H \to \wp^*(\dot{H})$ be a strong set-valued homomorphism. Let L and R be a generalized lower rough m-left Γ-hyperideal and a generalized lower rough n-right Γ-hyperideal of H, respectively. Then $\underline{T}(L \cap R)$ is, if it is non-empty, an (m,n) quasi Γ-hyperideal of H.*

Proof. Let L and R be a generalized lower rough m-left Γ-hyperideal and a generalized lower rough n-right Γ-hyperideal of H, respectively. Then

$$H^m\Gamma\underline{T}(L) \subseteq \underline{T}(L) \text{ and } \underline{T}(R)\Gamma H^n \subseteq \underline{T}(R).$$

Now we have

$$\begin{aligned} H^m\Gamma\underline{T}(L\cap R) \cap \underline{T}(L\cap R)\Gamma H^n &\subseteq H^m\Gamma\underline{T}(L) \cap \underline{T}(R)\Gamma H^n \\ &\subseteq \underline{T}(L) \cap \underline{T}(R) \\ &= \underline{T}(L\cap R). \end{aligned}$$

Hence this shows that $\underline{T}(L\cap R)$ is a generalized lower rough (m,n) quasi Γ-hyperideal of H. ■

5.5 Generalized Rough M-Hypersystems

In this section we presented some results on generalized rough M-hypersystems and generalized rough N-hypersystems in Γ-semihypergroups.

Theorem 269 *Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup and $T : H \to \wp^*(\dot{H})$ be a set-valued homomorphism. If M is an M-hypersystem in $\dot{H}$, then $\overline{T}(M)$ is an M-hypersystem in H.*

Proof. Let M be an M-hypersystem in $\dot{H}$. Let $x, y \in \overline{T}(M)$, then $T(x) \cap M \neq \emptyset$ and $T(y) \cap M \neq \emptyset$. Let $a \in T(x) \cap M$ and $b \in T(y) \cap M$, then $a \in T(x)$, $a \in M$, $b \in T(y)$ and $b \in M$. Since M is an M-hypersystem in $\dot{H}$, so there exist $w \in \dot{H}$ and $\dot{\gamma}, \dot{\beta} \in \dot{\Gamma}$ such that $a\dot{\gamma}w\dot{\beta}b \subseteq M$. And also there exist some $u \in H$ such that $w \in T(u)$. Now we have

$$a\dot{\gamma}w\dot{\beta}b \subseteq T(x)\dot{\gamma}T(u)\dot{\beta}T(y) = T(x\gamma u\beta y).$$

Hence $a\dot{\gamma}w\dot{\beta}b \subseteq T(x\gamma u\beta y) \cap M$. This implies that $T(x\gamma u\beta y) \cap M \neq \emptyset$. Thus $x\gamma u\beta y \subseteq \overline{T}(M)$. Hence $\overline{T}(M)$ is an M-hypersystem in H. ■

Theorem 270 *Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup and $T : H \to \wp^*(\dot{H})$ be a set-valued homomorphism. If N is an N-hypersystem in $\dot{H}$, then $\overline{T}(N)$ is an N-hypersystem in H.*

Proof. The proof is similar to the proof of the Theorem 269. ■

Theorem 271 *Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup and $T : H \to \wp^*(\dot{H})$ be a set-valued homomorphism. If A is a sub $\dot{\Gamma}$-semihypergroup of $\dot{H}$, then*

1. *$\overline{T}(A)$ is an M-hypersystem in H.*
2. *if T is strong, then $\underline{T}(A)$ is an M-hypersystem in H.*

Proof. The proof is straightforward. ■

Theorem 272 *Let H be a Γ-semihypergroup, $\dot{H}$ be a $\dot{\Gamma}$-semihypergroup and $T : H \to \wp^*(\dot{H})$ be a set-valued homomorphism. If A is a sub $\dot{\Gamma}$-semihypergroup of H, then*

1. *$\overline{T}(A)$ is an N-hypersystem in H.*
2. *If T is strong, then $\underline{T}(A)$ is an N-hypersystem in H.*

Proof. The proof is straightforward. ■

Chapter 6

Rough Approximations of Bipolar Fuzzy Γ-Hyperideals

Combination of rough sets with fuzzy sets [89] and intuitionistic fuzzy sets [12] have been considered by many mathematicians. Dubois and Prade [29] developed some results on rough fuzzy sets and fuzzy rough sets. Li and Zhang [54] introduced the idea of rough fuzzy approximations on two universes of discourse. The rough fuzzy sets have been applied to information systems by Beaubouef and Petry [13]. Rizvi et al. [62] introduced the notion of rough intuitionistic fuzzy sets. Later, Zhou and Wu [98, 99, 100] further explored rough intuitionistic fuzzy sets and added many useful results to the theory of rough intuitionistic fuzzy sets. Samanta and Mondal [63] added some new results on intuitionistic fuzzy rough sets and rough intuitionistic fuzzy sets. Abd-Allah et al. [1] introduced the concept of rough intuitionistic fuzzy subgroup which is a generalization of the notion of rough subgroups and fuzzy subgroups. Yang et al. [73] introduced the notions of the union, the intersection, and the inverse of bipolar fuzzy approximation spaces. Yang et al. [74] discussed some properties of the bipolar fuzzy rough set model.

The results presented in this section are a part of our submitted paper [87]. Here in this chapter we combine rough set theory with bipolar fuzzy set theory and introduce the notion of rough bipolar fuzzy sets. We apply the notion of rough bipolar fuzzy sets to the theory of Γ-semihypergroups and obtain some results on rough bipolar fuzzy Γ-hyperideals and rough bipolar fuzzy bi-Γ-hyperideals.

6.1 Rough Bipolar Fuzzy Sets in Γ-Semihypergroups

We denote the set of all bipolar fuzzy subsets of H by $BF(H)$. For a regular relation ρ on a Γ-semihypergroup H and for any $\mathcal{B} \in BF(H)$, the upper and lower approximations of $\mathcal{B}$ are denoted by $\overline{Apr}_\rho(\mathcal{B})$ and $\underline{Apr}_\rho(\mathcal{B})$, are two bipolar fuzzy sets and are, respectively, defined as follows:

$$\begin{aligned}\overline{Apr}_\rho(\mathcal{B}) &= \left\{\left\langle x, \mu^+_{\overline{Apr}_\rho(\mathcal{B})}(x), \mu^-_{\overline{Apr}_\rho(\mathcal{B})}(x)\right\rangle : x \in H\right\},\\ \underline{Apr}_\rho(\mathcal{B}) &= \left\{\left\langle x, \mu^+_{\underline{Apr}_\rho(\mathcal{B})}(x), \mu^-_{\underline{Apr}_\rho(\mathcal{B})}(x)\right\rangle : x \in H\right\},\end{aligned}$$

where

$$\begin{aligned}\mu^+_{\overline{Apr}_\rho(\mathcal{B})}(x) &= \sup_{a\in[x]_\rho} \mu^+_{\mathcal{B}}(a), \quad \mu^-_{\overline{Apr}_\rho(\mathcal{B})}(x) = \inf_{a\in[x]_\rho} \mu^-_{\mathcal{B}}(a),\\ \mu^+_{\underline{Apr}_\rho(\mathcal{B})}(x) &= \inf_{a\in[x]_\rho} \mu^+_{\mathcal{B}}(a), \quad \mu^-_{\underline{Apr}_\rho(\mathcal{B})}(x) = \sup_{a\in[x]_\rho} \mu^-_{\mathcal{B}}(a).\end{aligned}$$

$\overline{Apr}_\rho(\mathcal{B})$ and $\underline{Apr}_\rho(\mathcal{B})$ are, respectively, called the upper and lower approximations of $\mathcal{B} \in BF(H)$ with respect to ρ. The pair $(\overline{Apr}_\rho(\mathcal{B}), \underline{Apr}_\rho(\mathcal{B}))$ is called rough bipolar fuzzy set of $\mathcal{B}$, and $\overline{Apr}_\rho(\mathcal{B})$, $\underline{Apr}_\rho(\mathcal{B}) : BF(H) \longrightarrow BF(H)$ are referred to as upper and lower rough bipolar fuzzy approximation operators, respectively.

Proposition 273 *Let ρ be a regular relation on a Γ-semihypergroup H. If $\mathcal{A}, \mathcal{B} \in BF(H)$, then the following holds:*

1. $\underline{Apr}_\rho(\mathcal{A}) \subseteq \mathcal{A} \subseteq \overline{Apr}_\rho(\mathcal{A})$,
2. $\overline{Apr}_\rho(A \cup \mathcal{B}) = \overline{Apr}_\rho(\mathcal{A}) \cup \overline{Apr}_\rho(\mathcal{B})$,
3. $\overline{Apr}_\rho(\mathcal{A} \cap \mathcal{B}) \subseteq \overline{Apr}_\rho(\mathcal{A}) \cap \overline{Apr}_\rho(\mathcal{B})$,
4. $\underline{Apr}_\rho(\mathcal{A} \cup \mathcal{B}) \supseteq \underline{Apr}_\rho(\mathcal{A}) \cup \underline{Apr}_\rho(\mathcal{B})$,
5. $\underline{Apr}_\rho(\mathcal{A} \cap \mathcal{B}) = \underline{Apr}_\rho(\mathcal{A}) \cap \underline{Apr}_\rho(\mathcal{B})$,
6. $\mathcal{A} \subseteq \mathcal{B}$ *implies* $\overline{Apr}_\rho(\mathcal{A}) \subseteq \overline{Apr}_\rho(\mathcal{B})$,
7. $\mathcal{A} \subseteq \mathcal{B}$ *implies* $\underline{Apr}_\rho(\mathcal{A}) \subseteq \underline{Apr}_\rho(\mathcal{B})$.

Proof. (1) For $x \in H$ and $\mathcal{A} \in BF(H)$, we have

$$\mu^{+}_{\underline{Apr}_{\rho}(\mathcal{A})}(x) = \inf_{a\in[x]_{\rho}} \mu^{+}_{\mathcal{A}}(a) \leq \mu^{+}_{\mathcal{A}}(x) \leq \sup_{a\in[x]_{\rho}} \mu^{+}_{\mathcal{A}}(a) = \mu^{+}_{\overline{Apr}_{\rho}(\mathcal{A})}(x),$$

and

$$\mu^{-}_{\underline{Apr}_{\rho}(\mathcal{A})}(x) = \sup_{a\in[x]_{\rho}} \mu^{-}_{\mathcal{A}}(a) \geq \mu^{-}_{\mathcal{A}}(x) \geq \inf_{a\in[x]_{\rho}} \mu^{-}_{\mathcal{A}}(a) = \mu^{-}_{\overline{Apr}_{\rho}(\mathcal{A})}(x).$$

This implies that $\underline{Apr}_{\rho}(\mathcal{A}) \subseteq \mathcal{A} \subseteq \overline{Apr}_{\rho}(\mathcal{A})$.

(2) For $x \in H$ and $\mathcal{A}, \mathcal{B} \in BF(H)$, we have

$$\begin{aligned}
\mu^{+}_{\overline{Apr}_{\rho}(\mathcal{A}\cup\mathcal{B})}(x) &= \sup_{a\in[x]_{\rho}} \mu^{+}_{\mathcal{A}\cup\mathcal{B}}(a) = \sup_{a\in[x]_{\rho}} \left(\mu^{+}_{\mathcal{A}}(a) \vee \mu^{+}_{\mathcal{B}}(a)\right) \\
&= \left(\sup_{a\in[x]_{\rho}} \mu^{+}_{\mathcal{A}}(a)\right) \vee \left(\sup_{a\in[x]_{\rho}} \mu^{+}_{\mathcal{B}}(a)\right) \\
&= \mu^{+}_{\overline{Apr}_{\rho}(\mathcal{A})}(x) \vee \mu^{+}_{\overline{Apr}_{\rho}(\mathcal{B})}(x) = \mu^{+}_{\overline{Apr}_{\rho}(\mathcal{A})\cup\overline{Apr}_{\rho}(\mathcal{B})}(x),
\end{aligned}$$

and

$$\begin{aligned}
\mu^{-}_{\overline{Apr}_{\rho}(\mathcal{A}\cup\mathcal{B})}(x) &= \inf_{a\in[x]_{\rho}} \mu^{-}_{\mathcal{A}\cup\mathcal{B}}(a) = \inf_{a\in[x]_{\rho}} \left(\mu^{-}_{\mathcal{A}}(a) \wedge \mu^{-}_{\mathcal{B}}(a)\right) \\
&= \left(\inf_{a\in[x]_{\rho}} \mu^{-}_{\mathcal{A}}(a)\right) \wedge \left(\inf_{a\in[x]_{\rho}} \mu^{-}_{\mathcal{B}}(a)\right) \\
&= \mu^{-}_{\overline{Apr}_{\rho}(\mathcal{A})}(x) \wedge \mu^{-}_{\overline{Apr}_{\rho}(\mathcal{B})}(x) = \mu^{-}_{\overline{Apr}_{\rho}(\mathcal{A})\cup\overline{Apr}_{\rho}(\mathcal{B})}(x).
\end{aligned}$$

Thus $\mu^{+}_{\overline{Apr}_{\rho}(\mathcal{A}\cup\mathcal{B})} = \mu^{+}_{\overline{Apr}_{\rho}(\mathcal{A})\cup\overline{Apr}_{\rho}(\mathcal{B})}$ and $\mu^{-}_{\overline{Apr}_{\rho}(\mathcal{A}\cup\mathcal{B})} = \mu^{-}_{\overline{Apr}_{\rho}(\mathcal{A})\cup\overline{Apr}_{\rho}(\mathcal{B})}$. This implies that $\overline{Apr}_{\rho}(\mathcal{A} \cup \mathcal{B}) = \overline{Apr}_{\rho}(\mathcal{A}) \cup \overline{Apr}_{\rho}(\mathcal{B})$.

(3) For $x \in H$, we have

$$\begin{aligned}
\mu^{+}_{\overline{Apr}_{\rho}(\mathcal{A}\cap\mathcal{B})}(x) &= \sup_{a\in[x]_{\rho}} \mu^{+}_{\mathcal{A}\cap\mathcal{B}}(a) = \sup_{a\in[x]_{\rho}} \left(\mu^{+}_{\mathcal{A}}(a) \wedge \mu^{+}_{\mathcal{B}}(a)\right) \\
&\leq \left(\sup_{a\in[x]_{\rho}} \mu^{+}_{\mathcal{A}}(a)\right) \wedge \left(\sup_{a\in[x]_{\rho}} \mu^{+}_{\mathcal{B}}(a)\right) \\
&= \mu^{+}_{\overline{Apr}_{\rho}(\mathcal{A})}(x) \wedge \mu^{+}_{\overline{Apr}_{\rho}(\mathcal{B})}(x) = \mu^{+}_{\overline{Apr}_{\rho}(\mathcal{A})\cap\overline{Apr}_{\rho}(\mathcal{B})}(x),
\end{aligned}$$

and

$$\begin{aligned}
\mu^{-}_{\overline{Apr}_{\rho}(\mathcal{A}\cap\mathcal{B})}(x) &= \inf_{a\in[x]_{\rho}} \mu^{-}_{\mathcal{A}\cap\mathcal{B}}(a) = \inf_{a\in[x]_{\rho}} \left(\mu^{-}_{\mathcal{A}}(a) \vee \mu^{-}_{\mathcal{B}}(a)\right) \\
&\geq \left(\inf_{a\in[x]_{\rho}} \mu^{-}_{\mathcal{A}}(a)\right) \vee \left(\inf_{a\in[x]_{\rho}} \mu^{-}_{\mathcal{B}}(a)\right) \\
&= \mu^{-}_{\overline{Apr}_{\rho}(\mathcal{A})}(x) \vee \mu^{-}_{\overline{Apr}_{\rho}(\mathcal{B})}(x) = \mu^{-}_{\overline{Apr}_{\rho}(\mathcal{A})\cap\overline{Apr}_{\rho}(\mathcal{B})}(x).
\end{aligned}$$

Thus $\mu^{+}_{\overline{Apr}_\rho(\mathcal{A}\cap\mathcal{B})} \subseteq \mu^{+}_{\overline{Apr}_\rho(\mathcal{A})\cap\overline{Apr}_\rho(\mathcal{B})}$ and $\mu^{-}_{\overline{Apr}_\rho(\mathcal{A}\cap\mathcal{B})} \supseteq \mu^{-}_{\overline{Apr}_\rho(\mathcal{A})\cap\overline{Apr}_\rho(\mathcal{B})}$. This implies that $\overline{Apr}_\rho(\mathcal{A}\cap\mathcal{B}) \subseteq \overline{Apr}_\rho(\mathcal{A}) \cap \overline{Apr}_\rho(\mathcal{B})$.

(4) The proof is similar to (3).

(5) The proof is similar to (2).

(6) Since $\mathcal{A} \subseteq \mathcal{B}$ if and only if $\mathcal{A} \cup \mathcal{B} = \mathcal{B}$, by (2) we have

$$\overline{Apr}_\rho(\mathcal{B}) = \overline{Apr}_\rho(\mathcal{A}\cup\mathcal{B}) = \overline{Apr}_\rho(\mathcal{A}) \cup \overline{Apr}_\rho(\mathcal{B}).$$

This implies that $\overline{Apr}_\rho(\mathcal{B}) = \overline{Apr}_\rho(\mathcal{A}) \cup \overline{Apr}_\rho(\mathcal{B})$. Hence $\overline{Apr}_\rho(\mathcal{A}) \subseteq \overline{Apr}_\rho(\mathcal{B})$.

(7) Since $\mathcal{A} \subseteq \mathcal{B}$ if and only if $\mathcal{A} \cap \mathcal{B} = \mathcal{A}$, by (4) we have

$$\underline{Apr}_\rho(\mathcal{A}) = \underline{Apr}_\rho(\mathcal{A}\cap\mathcal{B}) = \underline{Apr}_\rho(\mathcal{A}) \cap \underline{Apr}_\rho(\mathcal{B}).$$

This implies that $\underline{Apr}_\rho(\mathcal{A}) = \underline{Apr}_\rho(\mathcal{A}) \cap \underline{Apr}_\rho(\mathcal{B})$. Hence $\underline{Apr}_\rho(\mathcal{A}) \subseteq \underline{Apr}_\rho(\mathcal{B})$. ■

Let $\overline{RBF}(H)$ and $\underline{RBF}(H)$ denote the collection of all upper rough bipolar fuzzy subsets and lower rough bipolar fuzzy subsets of H, respectively.

Proposition 274 *Let H be a Γ-semihypergroup, then*

1. $\left(\overline{RBF}(H), \circ_\Gamma\right)$ *is a* Γ*-semihypergroup.*
2. $\left(\underline{RBF}(H), \circ_\Gamma\right)$ *is a* Γ*-semihypergroup.*

Proof. The proof is straightforward. ■

Lemma 275 *Let ρ be a regular relation on a Γ-semihypergroup H. Let $\mathcal{B} = \langle \mu^{+}_{\mathcal{B}}, \mu^{-}_{\mathcal{B}} \rangle$ be a bipolar fuzzy subset of H and $(s,\, t) \in [-1,\, 0] \times [0,\, 1]$. Then*

1. $\left(\underline{Apr}_\rho(H_{\mathcal{B}})\right)^{(t,s)} = \underline{Apr}_\rho\left(H_{\mathcal{B}}^{(t,s)}\right).$
2. ${}^{S}\left(\overline{Apr}_\rho(H_{\mathcal{B}})\right)^{(t,s)} = \overline{Apr}_\rho\left({}^{S}H_{\mathcal{B}}^{(t,s)}\right).$

Proof. (1). Let $x \in \left(\underline{Apr}_\rho(H_{\mathcal{B}})\right)^{(t,s)}$. Then $\mu^{+}_{\underline{Apr}_\rho(\mathcal{B})}(x) \geq t$ and $\mu^{-}_{\underline{Apr}_\rho(\mathcal{B})}(x) \leq s$. So $\inf\limits_{a\in[x]_\rho} \mu^{+}_{\mathcal{B}}(a) \geq t$ and $\sup\limits_{a\in[x]_\rho} \mu^{-}_{\mathcal{B}}(a) \leq s$. Therefore $\mu^{+}_{\mathcal{B}}(a) \geq t$ and $\mu^{-}_{\mathcal{B}}(a) \leq s$, for all $a \in [x]_\rho$. This implies $[x]_\rho \subseteq H_{\mathcal{B}}^{(t,s)}$. Therefore $x \in \underline{Apr}_\rho\left(H_{\mathcal{B}}^{(t,s)}\right)$.

Conversely, let us assume that $x \in \underline{Apr}_\rho\left(H_{\mathcal{B}}^{(t,s)}\right)$. Thus $[x]_\rho \subseteq H_{\mathcal{B}}^{(t,s)}$. Then $\mu_{\mathcal{B}}^{+}(a) \geq t$ and $\mu_{\mathcal{B}}^{-}(a) \leq s$, for all $a \in [x]_\rho$. This implies $\inf_{a\in[x]_\rho} \mu_{\mathcal{B}}^{+}(a) \geq t$ and $\sup_{a\in[x]_\rho} \mu_{\mathcal{B}}^{-}(a) \leq s$. Thus $\mu^{+}_{\underline{Apr}_\rho(\mathcal{B})}(x) \geq t$ and $\mu^{-}_{\underline{Apr}_\rho(\mathcal{B})}(x) \leq s$. Hence $x \in \left(\underline{Apr}_\rho\left(H_{\mathcal{B}}\right)\right)^{(t,s)}$.

(2). Let $x \in {}^{S}\left(\overline{Apr}_\rho\left(H_{\mathcal{B}}\right)\right)^{(t,s)}$. Then $\mu^{+}_{\underline{Apr}_\rho(\mathcal{B})}(x) > t$ and $\mu^{-}_{\underline{Apr}_\rho(\mathcal{B})}(x) < s$. So $\inf_{a\in[x]_\rho} \mu_{\mathcal{B}}^{+}(a) > t$ and $\sup_{a\in[x]_\rho} \mu_{\mathcal{B}}^{-}(a) < s$. Therefore $\mu_{\mathcal{B}}^{+}(a) > t$ and $\mu_{\mathcal{B}}^{-}(a) < s$, for some $a \in [x]_\rho$. This implies $[x]_\rho \cap {}^{S}H_{\mathcal{B}}^{(t,s)} \neq \emptyset$. Therefore $x \in \overline{Apr}_\rho\left({}^{S}H_{\mathcal{B}}^{(t,s)}\right)$.

Conversely, let us assume $x \in \overline{Apr}_\rho\left({}^{S}H_{\mathcal{B}}^{(t,s)}\right)$. Thus $[x]_\rho \cap {}^{S}H_{\mathcal{B}}^{(t,s)} \neq \emptyset$. Then $\mu_{\mathcal{B}}^{+}(a) > t$ and $\mu_{\mathcal{B}}^{-}(a) < s$, for some $a \in [x]_\rho$. This implies $\inf_{a\in[x]_\rho} \mu_{\mathcal{B}}^{+}(a) > t$ and $\sup_{a\in[x]_\rho} \mu_{\mathcal{B}}^{-}(a) < s$. Thus $\mu^{+}_{\underline{Apr}_\rho(\mathcal{B})}(x) > t$ and $\mu^{-}_{\underline{Apr}_\rho(\mathcal{B})}(x) < s$. Hence $x \in {}^{S}\left(\overline{Apr}_\rho\left(H_{\mathcal{B}}\right)\right)^{(t,s)}$. ■

Lemma 276 *Let ρ be a regular relation on a Γ-semihypergroup H. Let $\mathcal{B} = \left\langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-}\right\rangle$ be a bipolar fuzzy subset of H and $(s, t) \in [-1, 0] \times [0, 1]$. Then*

1. $\left(\underline{Apr}_\rho\left(\mathcal{B}^{+}\right)\right)_t = \underline{Apr}_\rho\left(\mathcal{B}_t^{+}\right)$ *and* $\left(\underline{Apr}_\rho\left(\mathcal{B}^{-}\right)\right)_s = \underline{Apr}_\rho\left(\mathcal{B}_s^{-}\right)$.
2. ${}^{>}\left(\overline{Apr}_\rho\left(\mathcal{B}^{+}\right)\right)_t = \overline{Apr}_\rho\left({}^{>}\mathcal{B}_t^{+}\right)$ *and* ${}^{<}\left(\overline{Apr}_\rho\left(\mathcal{B}^{-}\right)\right)_s = \overline{Apr}_\rho\left({}^{<}\mathcal{B}_s^{-}\right)$.

Proof. The proof is similar to the proof of Lemma 275. ■

6.2 Rough Bipolar Fuzzy Γ-Hyperideals

A bipolar fuzzy subset $\mathcal{B}$ of a Γ-semihypergroup H is called a ρ-upper (resp., ρ-lower) rough bipolar fuzzy sub Γ-semihypergroup of H if $\overline{Apr}_\rho(\mathcal{B})$ (resp., $\underline{Apr}_\rho(\mathcal{B})$) is a bipolar fuzzy sub Γ-semihypergroup of H.

Theorem 277 *Let ρ be a regular relation on a Γ-semihypergroup H and $\mathcal{B}$ a bipolar fuzzy sub Γ-semihypergroup of H. Then*

1. $\overline{Apr}_\rho(\mathcal{B})$ *is a bipolar fuzzy sub* Γ*-semihypergroup of* H,
2. $\underline{Apr}_\rho(\mathcal{B})$ *is a bipolar fuzzy sub* Γ*-semihypergroup of* H.

Proof. Let $\mathcal{B} = \langle \mu_{\mathcal{B}}^{+}, \mu_{\mathcal{B}}^{-} \rangle$ be a bipolar fuzzy sub Γ-semihypergroup of H.

(1) Let $x, y, z \in H$ and $\gamma \in \Gamma$. Then we have

$$\begin{aligned}
\inf_{z \in x\gamma y} \mu^{+}_{\overline{Apr}_{\rho}(\mathcal{B})}(z) &= \sup_{m \in [x\gamma y]_{\rho}} \mu^{+}_{\mathcal{B}}(m) \geq \sup_{m \in [x]_{\rho}\gamma[y]_{\rho}} \mu^{+}_{\mathcal{B}}(m) \\
&\geq \inf_{m \in p\gamma q} \sup_{p\gamma q \subseteq [x]_{\rho}\gamma[y]_{\rho}} \mu^{+}_{\mathcal{B}}(m) = \sup_{p \in [x]_{\rho}, q \in [y]_{\rho}} \inf_{m \in p\gamma q} \mu^{+}_{\mathcal{B}}(m) \\
&\geq \left(\sup_{p \in [x]_{\rho}} \mu^{+}_{\mathcal{B}}(p) \right) \wedge \left(\sup_{q \in [y]_{\rho}} \mu^{+}_{\mathcal{B}}(q) \right) \\
&= \min \left\{ \mu^{+}_{\overline{Apr}_{\rho}(\mathcal{B})}(x), \mu^{+}_{\overline{Apr}_{\rho}(\mathcal{B})}(y) \right\},
\end{aligned}$$

and

$$\begin{aligned}
\sup_{z \in x\gamma y} \mu^{-}_{\overline{Apr}_{\rho}(\mathcal{B})}(z) &= \inf_{m \in [x\gamma y]_{\rho}} \mu^{-}_{\mathcal{B}}(m) \leq \inf_{m \in [x]_{\rho}\gamma[y]_{\rho}} \mu^{-}_{\mathcal{B}}(m) \\
&\leq \sup_{m \in p\gamma q} \inf_{p\gamma q \subseteq [x]_{\rho}\gamma[y]_{\rho}} \mu^{-}_{\mathcal{B}}(m) = \inf_{p \in [x]_{\rho}, q \in [y]_{\rho}} \sup_{m \in p\gamma q} \mu^{-}_{\mathcal{B}}(m) \\
&\leq \left(\inf_{p \in [x]_{\rho}} \mu^{-}_{\mathcal{B}}(p) \right) \vee \left(\inf_{q \in [y]_{\rho}} \mu^{-}_{\mathcal{B}}(q) \right) \\
&= \max \left\{ \mu^{-}_{\overline{Apr}_{\rho}(\mathcal{B})}(x), \mu^{-}_{\overline{Apr}_{\rho}(\mathcal{B})}(y) \right\}.
\end{aligned}$$

Thus

$$\inf_{z \in x\gamma y} \mu^{+}_{\overline{Apr}_{\rho}(\mathcal{B})}(z) \geq \min \left\{ \mu^{+}_{\overline{Apr}_{\rho}(\mathcal{B})}(x), \mu^{+}_{\overline{Apr}_{\rho}(\mathcal{B})}(y) \right\}$$

and

$$\sup_{z \in x\gamma y} \mu^{-}_{\overline{Apr}_{\rho}(\mathcal{B})}(z) \leq \max \left\{ \mu^{-}_{\overline{Apr}_{\rho}(\mathcal{B})}(x), \mu^{-}_{\overline{Apr}_{\rho}(\mathcal{B})}(y) \right\}.$$

Therefore $\overline{Apr}_{\rho}(\mathcal{B})$ is a bipolar fuzzy sub Γ-semihypergroup of H.

(2) Let $x, y, z \in H$ and $\gamma \in \Gamma$. Then we have

$$\begin{aligned}
\inf_{z \in x\gamma y} \mu^{+}_{\underline{Apr}_{\rho}(\mathcal{B})}(z) &= \inf_{m \in [x\gamma y]_{\rho}} \mu^{+}_{\mathcal{B}}(m) \geq \inf_{m \in [x]_{\rho}\gamma[y]_{\rho}} \mu^{+}_{\mathcal{B}}(m) \\
&= \inf_{m \in p\gamma q} \inf_{p\gamma q \subseteq [x]_{\rho}\gamma[y]_{\rho}} \mu^{+}_{\mathcal{B}}(m) = \inf_{p \in [x]_{\rho}, q \in [y]_{\rho}} \inf_{m \in p\gamma q} \mu^{+}_{\mathcal{B}}(m) \\
&\geq \left(\inf_{p \in [x]_{\rho}} \mu^{+}_{\mathcal{B}}(p) \right) \wedge \left(\inf_{q \in [y]_{\rho}} \mu^{+}_{\mathcal{B}}(q) \right) \\
&= \min \left\{ \mu^{+}_{\underline{Apr}_{\rho}(\mathcal{B})}(x), \mu^{+}_{\underline{Apr}_{\rho}(\mathcal{B})}(y) \right\},
\end{aligned}$$

and

$$\begin{aligned}
\sup_{z\in x\gamma y}\mu^-_{\underline{Apr}_\rho(\mathcal{B})}(z) &= \sup_{m\in[x\gamma y]_\rho}\mu^-_{\mathcal{B}}(m) \le \sup_{m\in[x]_\rho\gamma[y]_\rho}\mu^-_{\mathcal{B}}(m)\\
&= \sup_{m\in p\gamma q}\ \sup_{p\gamma q\subseteq[x]_\rho\gamma[y]_\rho}\mu^-_{\mathcal{B}}(m) = \sup_{p\in[x]_\rho,q\in[y]_\rho}\ \sup_{m\in p\gamma q}\mu^-_{\mathcal{B}}(m)\\
&\le \left(\sup_{p\in[x]_\rho}\mu^-_{\mathcal{B}}(p)\right)\vee\left(\sup_{q\in[y]_\rho}\mu^-_{\mathcal{B}}(q)\right)\\
&= \max\left\{\mu^-_{\underline{Apr}_\rho(\mathcal{B})}(x),\mu^-_{\underline{Apr}_\rho(\mathcal{B})}(y)\right\}.
\end{aligned}$$

Thus

$$\inf_{z\in x\gamma y}\mu^+_{\underline{Apr}_\rho(\mathcal{B})}(z)\ge\min\left\{\mu^+_{\underline{Apr}_\rho(\mathcal{B})}(x),\mu^+_{\underline{Apr}_\rho(\mathcal{B})}(y)\right\},$$

and

$$\sup_{z\in x\gamma y}\mu^-_{\underline{Apr}_\rho(\mathcal{B})}(z)\le\max\left\{\mu^-_{\underline{Apr}_\rho(\mathcal{B})}(x),\mu^-_{\underline{Apr}_\rho(\mathcal{B})}(y)\right\}.$$

Therefore $\underline{Apr}_\rho(\mathcal{B})$ is a bipolar fuzzy sub Γ-semihypergroup of H. ■

The following example shows that the converse of Theorem 277 does not hold.

Example 278 *Let $H=\{1,-1,i,-i\}$ be the group with respect to multiplication and $\Gamma=\{\gamma_1,\gamma_2\}$. Let $A_1=\{1,-1\}$ and $A_2=\{1\}$ be the non-empty subsets of H. We define $x\gamma_k y=xA_k y$, for every $\gamma_k\in\Gamma$ and $x,y\in H$. Then H is a Γ-semihypergroup. Let ρ be a regular relation on H such that the ρ regular classes are the subsets $\{1,-1\}$ and $\{i,-i\}$. Now we define a bipolar fuzzy subset $\mathcal{B}=\langle\mu^+_{\mathcal{B}},\mu^-_{\mathcal{B}}\rangle$ of H as:*

$$\begin{aligned}
\mathcal{B} &= \{\langle x,\mu^+_{\mathcal{B}}(x),\mu^-_{\mathcal{B}}(x)\rangle : x\in H\}\\
&= \{\langle 1,0.9,-0.5\rangle,\ \langle -1,0.5,-0.7\rangle,\ \langle i,0.3,-0.4\rangle,\ \langle -i,0.1,-0.2\rangle\}.
\end{aligned}$$

Now

$$\overline{Apr}_\rho(\mathcal{B})=\{\langle 1,0.9,-0.7\rangle,\ \langle -1,0.9,-0.7\rangle,\ \langle i,0.3,-0.4\rangle,\ \langle -i,0.3,-0.4\rangle\},$$

also

$$\underline{Apr}_\rho(\mathcal{B})=\{\langle 1,0.5,-0.5\rangle,\ \langle -1,0.5,-0.5\rangle,\ \langle i,0.1,-0.2\rangle,\ \langle -i,0.1,-0.2\rangle\}.$$

By routine calculations we can easily check that $\overline{Apr}_\rho(\mathcal{B})$ and $\underline{Apr}_\rho(\mathcal{B})$ are bipolar fuzzy sub Γ-semihypergroups of H. But $\mathcal{B}$ is not a bipolar fuzzy sub Γ-semihypergroup of H, because for all $\gamma_k\in\Gamma$

$$\begin{aligned}
\inf_{t\in i\gamma_k-1}\mu^+_{\mathcal{B}}(t) &= 0.1<0.3=\min\{\mu^+_{\mathcal{B}}(i),\mu^+_{\mathcal{B}}(-1)\}\\
\text{and}\quad \sup_{t\in i\gamma_k-1}\mu^-_{\mathcal{B}}(t) &= -0.2>-0.4=\max\{\mu^-_{\mathcal{B}}(i),\mu^-_{\mathcal{B}}(-1)\}.
\end{aligned}$$

A bipolar fuzzy subset $\mathcal{B}$ of a Γ-semihypergroup H is called a ρ-upper (resp., ρ-lower) rough bipolar fuzzy Γ-hyperideal of H if $\overline{Apr}_\rho(\mathcal{B})$ (resp., $\underline{Apr}_\rho(\mathcal{B})$) is a bipolar fuzzy Γ-hyperideal of H.

Theorem 279 *Let ρ be a regular relation on a Γ-semihypergroup H. Let $\mathcal{B}$ be a bipolar fuzzy left (resp., right) Γ-hyperideal of H. Then*

1. *$\overline{Apr}_\rho(\mathcal{B})$ is a bipolar fuzzy left (resp., right) Γ-hyperideal of H,*
2. *$\underline{Apr}_\rho(\mathcal{B})$ is a bipolar fuzzy left (resp., right) Γ-hyperideal of H.*

Proof. Let $\mathcal{B} = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}} \rangle$ be a bipolar fuzzy left Γ-hyperideal of H.

(1) Let $x, y, z \in H$ and $\gamma \in \Gamma$. Then we have

$$\begin{aligned}
\inf_{z\in x\gamma y} \mu^+_{\overline{Apr}_\rho(\mathcal{B})}(z) &= \sup_{m\in[x\gamma y]_\rho} \mu^+_{\mathcal{B}}(m) \geq \sup_{m\in[x]_\rho\gamma[y]_\rho} \mu^+_{\mathcal{B}}(m) \\
&\geq \inf_{m\in p\gamma q} \sup_{p\gamma q\subseteq[x]_\rho\gamma[y]_\rho} \mu^+_{\mathcal{B}}(m) = \sup_{p\in[x]_\rho, q\in[y]_\rho} \inf_{m\in p\gamma q} \mu^+_{\mathcal{B}}(m) \\
&\geq \left(\sup_{q\in[y]_\rho} \mu^+_{\mathcal{B}}(q) \right) = \mu^+_{\overline{Apr}_\rho(\mathcal{B})}(y),
\end{aligned}$$

and

$$\begin{aligned}
\sup_{z\in x\gamma y} \mu^-_{\overline{Apr}_\rho(\mathcal{B})}(z) &= \inf_{m\in[x\gamma y]_\rho} \mu^-_{\mathcal{B}}(m) \leq \inf_{m\in[x]_\rho\gamma[y]_\rho} \mu^-_{\mathcal{B}}(m) \\
&\leq \sup_{m\in p\gamma q} \inf_{p\gamma q\subseteq[x]_\rho\gamma[y]_\rho} \mu^-_{\mathcal{B}}(m) = \inf_{p\in[x]_\rho, q\in[y]_\rho} \sup_{m\in p\gamma q} \mu^-_{\mathcal{B}}(m) \\
&\leq \left(\inf_{q\in[y]_\rho} \mu^-_{\mathcal{B}}(q) \right) = \mu^-_{\overline{Apr}_\rho(\mathcal{B})}(y).
\end{aligned}$$

Thus

$$\inf_{z\in x\gamma y} \mu^+_{\overline{Apr}_\rho(\mathcal{B})}(z) \geq \mu^+_{\overline{Apr}_\rho(\mathcal{B})}(y) \text{ and } \sup_{z\in x\gamma y} \mu^-_{\overline{Apr}_\rho(\mathcal{B})}(z) \leq \mu^-_{\overline{Apr}_\rho(\mathcal{B})}(y).$$

Therefore $\overline{Apr}_\rho(\mathcal{B})$ is a bipolar fuzzy left Γ-hyperideal of H.

(2) Let $x, y, z \in H$ and $\gamma \in \Gamma$. Then we have

$$\begin{aligned}
\inf_{z\in x\gamma y} \mu^+_{\underline{Apr}_\rho(\mathcal{B})}(z) &= \inf_{m\in[x\gamma y]_\rho} \mu^+_{\mathcal{B}}(m) \geq \inf_{m\in[x]_\rho\gamma[y]_\rho} \mu^+_{\mathcal{B}}(m) \\
&= \inf_{m\in p\gamma q} \inf_{p\gamma q\subseteq[x]_\rho\gamma[y]_\rho} \mu^+_{\mathcal{B}}(m) = \inf_{p\in[x]_\rho, q\in[y]_\rho} \inf_{m\in p\gamma q} \mu^+_{\mathcal{B}}(m) \\
&\geq \left(\inf_{q\in[y]_\rho} \mu^+_{\mathcal{B}}(q) \right) = \mu^+_{\underline{Apr}_\rho(\mathcal{B})}(y),
\end{aligned}$$

and

$$\begin{aligned}
\sup_{z\in x\gamma y}\mu^-_{\underline{Apr}_\rho(\mathcal{B})}(z) &= \sup_{m\in[x\gamma y]_\rho}\mu^-_{\mathcal{B}}(m) \le \sup_{m\in[x]_\rho\gamma[y]_\rho}\mu^-_{\mathcal{B}}(m) \\
&= \sup_{m\in p\gamma q}\ \sup_{p\gamma q\subseteq[x]_\rho\gamma[y]_\rho}\mu^-_{\mathcal{B}}(m) = \sup_{p\in[x]_\rho, q\in[y]_\rho}\ \sup_{m\in p\gamma q}\mu^-_{\mathcal{B}}(m) \\
&\le \left(\sup_{q\in[y]_\rho}\mu^-_{\mathcal{B}}(q)\right) = \mu^-_{\underline{Apr}_\rho(\mathcal{B})}(y).
\end{aligned}$$

Thus

$$\inf_{z\in x\gamma y}\mu^+_{\underline{Apr}_\rho(\mathcal{B})}(z) \ge \mu^+_{\underline{Apr}_\rho(\mathcal{B})}(y) \text{ and } \sup_{z\in x\gamma y}\mu^-_{\underline{Apr}_\rho(\mathcal{B})}(z) \le \mu^-_{\underline{Apr}_\rho(\mathcal{B})}(y).$$

Therefore $\underline{Apr}_\rho(\mathcal{B})$ is a bipolar fuzzy sub Γ-semihypergroup of H. The case for right Γ-hyperideal can be seen in a similar way. ■

The following example shows that the converse of Theorem 279 does not hold.

Example 280 *Let $H = \{a, b, c, d, e\}$ and $\Gamma = \{\gamma, \beta\}$ be the sets of binary hyperoperations defined below:*

γ	a	b	c	d	e
a	$\{a,b\}$	$\{b,e\}$	c	$\{c,d\}$	e
b	$\{b,e\}$	e	c	$\{c,d\}$	e
c	c	c	c	c	c
d	$\{c,d\}$	$\{c,d\}$	c	d	$\{c,d\}$
e	e	e	c	$\{c,d\}$	e

β	a	b	c	d	e
a	$\{b,e\}$	e	c	$\{c,d\}$	e
b	e	e	c	$\{c,d\}$	e
c	c	c	c	c	c
d	$\{c,d\}$	$\{c,d\}$	c	d	$\{c,d\}$
e	e	e	c	$\{c,d\}$	e

Here H is a Γ-semihypergroup. Let ρ be a regular relation on H such that the ρ regular classes are the subsets $\{a, b, e\}$ and $\{c, d\}$. Now we define a bipolar fuzzy set $\mathcal{B} = \langle \mu^+_{\mathcal{B}}, \mu^-_{\mathcal{B}}\rangle$ on H as:

$$\begin{aligned}
\mathcal{B} &= \{\langle x, \mu^+_{\mathcal{B}}(x), \mu^-_{\mathcal{B}}(x)\rangle : x \in H\} \\
&= \{\langle a, 0.3, -0.3\rangle,\ \langle b, 0.4, -0.3\rangle,\ \langle c, 0.7, -0.6\rangle,\ \langle d, 0.9, -0.8\rangle,\ \langle e, 0.1, -0.2\rangle\}.
\end{aligned}$$

Now

$$\overline{Apr}_\rho(\mathcal{B}) = \{\langle a, 0.4, -0.3\rangle,\ \langle b, 0.4, -0.3\rangle,\ \langle c, 0.9, -0.8\rangle,\ \langle d, 0.9, -0.8\rangle,\ \langle e, 0.4, -0.3\rangle\},$$

also

$$\underline{Apr}_\rho(\mathcal{B}) = \{\langle a, 0.1, -0.2\rangle,\ \langle b, 0.1, -0.2\rangle,\ \langle c, 0.7, -0.6\rangle,\ \langle d, 0.7, -0.6\rangle,\ \langle e, 0.1, -0.2\rangle\}.$$

By routine calculations we can easily check that $\overline{Apr}_\rho(\mathcal{B})$ *and* $\underline{Apr}_\rho(\mathcal{B})$ *are bipolar fuzzy left* Γ*-hyperideals of* H. *But* $\mathcal{B}$ *is not a bipolar fuzzy left* Γ*-hyperideal of* H, *because for all* $\gamma \in \Gamma$

$$\begin{aligned} \inf_{t\in b\gamma a} \mu_{\mathcal{B}}^{+}(t) &= 0.1 < 0.3 = \mu_{\mathcal{B}}^{+}(a) \\ \text{and} \quad \sup_{t\in b\gamma a} \mu_{\mathcal{B}}^{+}(t) &= -0.2 > -0.3 = \mu_{\mathcal{B}}^{+}(a). \end{aligned}$$

Also it is very easy to check that $\mathcal{B}$ *is not a bipolar fuzzy right* Γ*-hyperideal of* H.

Theorem 281 *Let* ρ *be a regular relation on a* Γ*-semihypergroup* H. *Let* $\mathcal{B}$ *be a bipolar fuzzy interior* Γ*-hyperideal (resp., bi-*Γ*-hyperideal,* $(1,2)$*-*Γ*-hyperideal) of* H. *Then*

1. $\overline{Apr}_\rho(\mathcal{B})$ *is a bipolar fuzzy interior* Γ*-hyperideal (resp., bi-*Γ*-hyperideal,* $(1,2)$*-*Γ*-hyperideal) of* H,
2. $\underline{Apr}_\rho(\mathcal{B})$ *is a bipolar fuzzy interior* Γ*-hyperideal (resp., bi-*Γ*-hyperideal,* $(1,2)$*-*Γ*-hyperideal) of* H.

Proof. The proof is similar to the proof of the Theorems 277 and 279. ■

Proposition 282 *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two bipolar fuzzy sub* Γ*-semihypergroups (resp., left* Γ*-hyperideals, right* Γ*-hyperideals, interior* Γ*-hyperideals, bi-*Γ*-hyperideals,* $(1,2)$*-*Γ*-hyperideals) of a* Γ*-semihypergroup* H. *Then*

1. $\overline{Apr}_\rho(\mathcal{A}) \cap \overline{Apr}_\rho(\mathcal{B})$ *is also a bipolar fuzzy sub* Γ*-semihypergroup (resp., left* Γ*-hyperideal, right* Γ*-hyperideal, interior* Γ*-hyperideal, bi-*Γ*-hyperideal,* $(1,2)$*-*Γ*-hyperideal) of* H.
2. $\underline{Apr}_\rho(\mathcal{A}) \cap \underline{Apr}_\rho(\mathcal{B})$ *is also a bipolar fuzzy sub* Γ*-semihypergroup (resp., left* Γ*-hyperideal, right* Γ*-hyperideal, interior* Γ*-hyperideal, bi-*Γ*-hyperideal,* $(1,2)$*-*Γ*-hyperideal) of* H.

Proof. The proof is straightforward. ■

Definition 283 *Let* H *be a* Γ*-semihypergroup. For any* $A \subseteq H$ *and* $(q^-, r^+) \in [-1,0) \times (0,1]$, *the bipolar fuzzy subset* $A^{(r^+,q^-)} = \langle r_A^+, q_A^- \rangle$ *of* H *is defined by*

$$r_A^+ : H \longrightarrow (0,1] | A \longmapsto r_A^+(x) := \begin{cases} r^+ & \text{if } x \in A \\ 0 & \text{if } x \notin A, \end{cases}$$

and

$$q_A^- : H \longrightarrow [-1,0)|A \longmapsto q_A^-(x) := \begin{cases} q^- & \text{if } x \in A \\ 0 & \text{if } x \notin A, \end{cases}$$

for all $x \in H$. In particular, when $r^+ = 1$ and $q^- = -1$, $A^{(r^+,q^-)}$ is said to be the characteristic function of A, denoted by $\chi_A = \left\langle \mu^+_{\chi_A}, \mu^-_{\chi_A} \right\rangle$. When $A = \{x\}$, $A^{(r^+,q^-)}$ is said to be a bipolar fuzzy point with support x and values r^+ and q^- and is denoted by $x_{(r^+,q^-)}$.

Theorem 284 *Let ρ be a regular relation on a Γ-semihypergroup H and A a non-empty subset of H. The following statements are true:*

1. *A is a ρ-upper rough sub Γ-semihypergroup of H if and only if $\overline{Apr}_\rho(A^{(r^+,q^-)}) = \left\langle r^+_{\overline{Apr}_\rho(A)}, q^-_{\overline{Apr}_\rho(A)} \right\rangle$ is a bipolar fuzzy sub Γ-semihypergroup of H.*
2. *A is a ρ-lower rough sub Γ-semihypergroup of H if and only if $\underline{Apr}_\rho(A^{(r^+,q^-)}) = \left\langle r^+_{\underline{Apr}_\rho(A)}, q^-_{\underline{Apr}_\rho(A)} \right\rangle$ is a bipolar fuzzy sub Γ-semihypergroup of H.*

Proof. (1) Let A be a ρ-upper rough sub Γ-semihypergroup of H. So $\overline{Apr}_\rho(A)$ is a sub Γ-semihypergroup of H. Let $x, y \in H$ and $\gamma \in \Gamma$. We claim that

$$\begin{aligned} \inf_{z \in x\gamma y} r^+_{\overline{Apr}_\rho(A)}(z) &\geq \sup_{a \in [x]_\rho, b \in [y]_\rho} \inf_{m \in a\gamma b} r^+_A(m) \\ &\geq \left(\sup_{a \in [x]_\rho} r^+_A(a) \right) \wedge \left(\sup_{b \in [y]_\rho} r^+_A(b) \right), \end{aligned}$$

and

$$\begin{aligned} \sup_{z \in x\gamma y} q^-_{\overline{Apr}_\rho(A)}(z) &\leq \inf_{a \in [x]_\rho, b \in [y]_\rho} \sup_{m \in a\gamma b} q^-_A(m) \\ &\leq \left(\inf_{a \in [x]_\rho} q^-_A(a) \right) \vee \left(\inf_{b \in [y]_\rho} q^-_A(b) \right). \end{aligned}$$

Case 1 : If $[x]_\rho \cap A \neq \emptyset$ and $[y]_\rho \cap A \neq \emptyset$. Then, $x \in \overline{Apr}_\rho(A)$ and $y \in \overline{Apr}_\rho(A)$. Since $\overline{Apr}_\rho(A)$ is a sub Γ-semihypergroup of H, so $x\gamma y \subseteq \overline{Apr}_\rho(A)$. Then, $\inf_{z \in x\gamma y} r^+_{\overline{Apr}_\rho(A)}(z) = r^+$ and $\sup_{z \in x\gamma y} q^-_{\overline{Apr}_\rho(A)}(z) = q^-$. This implies

$$\inf_{z \in x\gamma y} r^+_{\overline{Apr}_\rho(A)}(z) \geq \left(\sup_{a \in [x]_\rho} r^+_A(a) \right) \wedge \left(\sup_{b \in [y]_\rho} r^+_A(b) \right),$$

and

$$\sup_{z\in x\gamma y} q^-_{\overline{Apr}_\rho(A)}(z) \leq \left(\inf_{a\in[x]_\rho} q^-_A(a)\right) \vee \left(\inf_{b\in[y]_\rho} q^-_A(b)\right).$$

Case 2 : If $[x]_\rho \cap A = \emptyset$ or $[y]_\rho \cap A = \emptyset$. Then, there exists $a \in [x]_\rho$ but $a \notin A$ or $b \in [y]_\rho$ but $b \notin A$. So $\sup_{a\in[x]_\rho} r^+_A(a) = 0$ or $\sup_{b\in[y]_\rho} r^+_A(b) = 0$. This implies $\left(\sup_{a\in[x]_\rho} r^+_A(a)\right) \wedge \left(\sup_{b\in[y]_\rho} r^+_A(b)\right) = 0$, therefore

$$\inf_{z\in x\gamma y} r^+_{\overline{Apr}_\rho(A)}(z) \geq \left(\sup_{a\in[x]_\rho} r^+_A(a)\right) \wedge \left(\sup_{b\in[y]_\rho} r^+_A(b)\right),$$

and also $\inf_{a\in[x]_\rho} q^-_A(a) = 0$ or $\inf_{b\in[y]_\rho} q^-_A(b) = 0$. This implies $\left(\inf_{a\in[x]_\rho} q^-_A(a)\right) \vee \left(\inf_{b\in[y]_\rho} q^-_A(b)\right) = 0$, therefore

$$\sup_{z\in x\gamma y} q^-_{\overline{Apr}_\rho(A)}(z) \leq \left(\inf_{a\in[x]_\rho} q^-_A(a)\right) \vee \left(\inf_{b\in[y]_\rho} q^-_A(b)\right),$$

Thus $\overline{Apr}_\rho(A^{(r^+,q^-)})$ is a bipolar fuzzy sub Γ-semihypergroup of H.

Conversely, assume that $\overline{Apr}_\rho(A^{(r^+,q^-)})$ is a bipolar fuzzy sub Γ-semihypergroup of H. Then, there exists $a \in H$ such that $[a]_\rho \cap A \neq \emptyset$. This implies $\overline{Apr}_\rho(A) \neq \emptyset$. Let $x, y \in \overline{Apr}_\rho(A)$. We have $r^+_{\overline{Apr}_\rho(A)}(x) = r^+$, $r^+_{\overline{Apr}_\rho(A)}(y) = r^+$ and $q^-_{\overline{Apr}_\rho(A)}(x) = q^-$, $q^-_{\overline{Apr}_\rho(A)}(y) = q^-$. This implies that $\inf_{z\in x\gamma y} r^+_{\overline{Apr}_\rho(A)}(z) = r^+$ and $\sup_{z\in x\gamma y} q^-_{\overline{Apr}_\rho(A)}(z) = q^-$. So $x\gamma y \subseteq \overline{Apr}_\rho(A)$. Thus $\overline{Apr}_\rho(A)$ is a sub Γ-semihypergroup of H.

(2) Let A be a ρ-lower rough sub Γ-semihypergroup of H. So $\underline{Apr}_\rho(A)$ is a sub Γ-semihypergroup of H. Let $x, y \in H$ and $\gamma \in \Gamma$. We claim that

$$\begin{aligned}\inf_{z\in x\gamma y} r^+_{\underline{Apr}_\rho(A)}(z) &= \inf_{a\in[x]_\rho, b\in[y]_\rho} \inf_{m\in a\gamma b} r^+_A(m) \\ &\geq \left(\inf_{a\in[x]_\rho} r^+_A(a)\right) \wedge \left(\inf_{b\in[y]_\rho} r^+_A(b)\right),\end{aligned}$$

and

$$\begin{aligned}\sup_{z\in x\gamma y} q^-_{\underline{Apr}_\rho(A)}(z) &= \sup_{a\in[x]_\rho, b\in[y]_\rho} \sup_{m\in a\gamma b} q^-_A(m) \\ &\leq \left(\sup_{a\in[x]_\rho} q^-_A(a)\right) \vee \left(\sup_{b\in[y]_\rho} q^-_A(b)\right).\end{aligned}$$

Case 1 : If $[x]_\rho \subseteq A$ and $[y]_\rho \subseteq A$. Then, $x \in \underline{Apr}_\rho(A)$ and $y \in \underline{Apr}_\rho(A)$. Since $\underline{Apr}_\rho(A)$ is a sub Γ-semihypergroup of H, so $x\gamma y \subseteq \underline{Apr}_\rho(A)$. Then, $\inf_{z\in x\gamma y} r^+_{\underline{Apr}_\rho(A)}(z) = r^+$ and

$\sup\limits_{z\in x\gamma y} q^{-}_{\underline{Apr}_\rho(A)}(z) = q^{-}$. This implies

$$\inf_{z\in x\gamma y} r^{+}_{\underline{Apr}_\rho(A)}(z) \geq \left(\inf_{a\in[x]_\rho} r^{+}_{A}(a)\right) \wedge \left(\inf_{b\in[y]_\rho} r^{+}_{A}(b)\right),$$

and

$$\sup_{z\in x\gamma y} q^{-}_{\underline{Apr}_\rho(A)}(z) \leq \left(\sup_{a\in[x]_\rho} q^{-}_{A}(a)\right) \vee \left(\sup_{b\in[y]_\rho} q^{-}_{A}(b)\right).$$

Case 2 : If $[x]_\rho \nsubseteq A$ or $[y]_\rho \nsubseteq A$. Then, there exists $a \in [x]_\rho$ but $a \notin A$ or $b \in [y]_\rho$ but $b \notin A$. So $\inf\limits_{a\in[x]_\rho} r^{+}_{A}(a) = 0$ or $\inf\limits_{b\in[y]_\rho} r^{+}_{A}(b) = 0$. This implies $\left(\inf\limits_{a\in[x]_\rho} r^{+}_{A}(a)\right) \wedge \left(\inf\limits_{b\in[y]_\rho} r^{+}_{A}(b)\right) = 0$, therefore

$$\inf_{z\in x\gamma y} r^{+}_{\underline{Apr}_\rho(A)}(z) \geq \left(\inf_{a\in[x]_\rho} r^{+}_{A}(a)\right) \wedge \left(\inf_{b\in[y]_\rho} r^{+}_{A}(b)\right),$$

and also $\sup\limits_{a\in[x]_\rho} q^{-}_{A}(a) = 0$ or $\sup\limits_{b\in[y]_\rho} q^{-}_{A}(b) = 0$. This implies $\left(\sup\limits_{a\in[x]_\rho} q^{-}_{A}(a)\right) \vee \left(\sup\limits_{b\in[y]_\rho} q^{-}_{A}(b)\right) = 0$, therefore

$$\sup_{z\in x\gamma y} q^{-}_{\underline{Apr}_\rho(A)}(z) \leq \left(\sup_{a\in[x]_\rho} q^{-}_{A}(a)\right) \vee \left(\sup_{b\in[y]_\rho} q^{-}_{A}(b)\right),$$

Thus $\underline{Apr}_\rho(A^{(r^+,q^-)})$ is a bipolar fuzzy sub Γ-semihypergroup of H.

Conversely, assume that $\underline{Apr}_\rho(A^{(r^+,q^-)})$ is a bipolar fuzzy sub Γ-semihypergroup of H. Then, there exists $a \in H$ such that $[a]_\rho \subseteq A$. This implies $\underline{Apr}_\rho(A) \neq \emptyset$. Let $x, y \in \underline{Apr}_\rho(A)$. We have $r^{+}_{\underline{Apr}_\rho(A)}(x) = r^{+}$, $r^{+}_{\underline{Apr}_\rho(A)}(y) = r^{+}$ and $q^{-}_{\underline{Apr}_\rho(A)}(x) = q^{-}$, $q^{-}_{\underline{Apr}_\rho(A)}(y) = q^{-}$. This implies that $\inf\limits_{z\in x\gamma y} r^{+}_{\underline{Apr}_\rho(A)}(z) = r^{+}$ and $\sup\limits_{z\in x\gamma y} q^{-}_{\underline{Apr}_\rho(A)}(z) = q^{-}$. This implies that $x\gamma y \subseteq \underline{Apr}_\rho(A)$. Thus $\underline{Apr}_\rho(A)$ is a sub Γ-semihypergroup of H. ■

Theorem 285 *Let ρ be a regular relation on a Γ-semihypergroup H and A a nonempty subset of H. The following statements are true:*

1. *A is a ρ-upper rough left Γ-hyperideal (resp., right Γ-hyperideal, interior Γ-hyperideal, bi-Γ-hyperideal, $(1,2)$-Γ-hyperideal) of H if and only if $\overline{Apr}_\rho(A^{(r^+,q^-)})$ is a bipolar fuzzy left Γ-hyperideal (resp., right Γ-hyperideal, interior Γ-hyperideal, bi-Γ-hyperideal, $(1,2)$-Γ-hyperideal) of H.*

2. *A is a ρ-lower rough left Γ-hyperideal (resp., right Γ-hyperideal, interior Γ-hyperideal, bi-Γ-hyperideal, $(1,2)$-Γ-hyperideal) of H if and only if $\underline{Apr}_\rho(A^{(r^+,q^-)})$*

is a bipolar fuzzy left Γ*-hyperideal (resp., right* Γ*-hyperideal, interior* Γ*-hyperideal, bi-*Γ*-hyperideal,* $(1,2)$*-*Γ*-hyperideal) of* H*.*

Proof. The proof is similar to the proof of the Theorem 284. ■

Theorem 286 *Let* $\overline{RBF}(H)$ *denote the set of all* ρ*-upper rough bipolar fuzzy left (right)* Γ*-hyperideals of a* Γ*-semihypergroup* H*. Then* $\left(\overline{RBF}(H), \subseteq, \cup, \cap\right)$ *is a lattice.*

Proof. For all $\overline{Apr}_\rho(\mathcal{A})$, $\overline{Apr}_\rho(\mathcal{B})$, $\overline{Apr}_\rho(\mathcal{C}) \in \overline{RBF}(H)$, we have

(1) Reflexive: Since for $x \in H$

$$\mu^+_{\overline{Apr}_\rho(\mathcal{A})}(x) \geq \mu^+_{\overline{Apr}_\rho(\mathcal{A})}(x) \quad \text{and} \quad \mu^-_{\overline{Apr}_\rho(\mathcal{A})}(x) \leq \mu^-_{\overline{Apr}_\rho(\mathcal{A})}(x),$$

this gives $\overline{Apr}_\rho(\mathcal{A}) \subseteq \overline{Apr}_\rho(\mathcal{A})$.

(2) Antisymmetric: For all $\overline{Apr}_\rho(\mathcal{A})$, $\overline{Apr}_\rho(\mathcal{B}) \in \overline{RBF}(H)$, we have $\overline{Apr}_\rho(\mathcal{A}) \subseteq \overline{Apr}_\rho(\mathcal{B})$ and $\overline{Apr}_\rho(\mathcal{B}) \subseteq \overline{Apr}_\rho(\mathcal{A})$ then

$$\mu^+_{\overline{Apr}_\rho(\mathcal{A})}(x) \leq \mu^+_{\overline{Apr}_\rho(\mathcal{B})}(x) \quad \text{and} \quad \mu^-_{\overline{Apr}_\rho(\mathcal{A})}(x) \geq \mu^-_{\overline{Apr}_\rho(\mathcal{B})}(x),$$

and

$$\mu^+_{\overline{Apr}_\rho(\mathcal{B})}(x) \leq \mu^+_{\overline{Apr}_\rho(\mathcal{A})}(x) \quad \text{and} \quad \mu^-_{\overline{Apr}_\rho(\mathcal{B})}(x) \geq \mu^-_{\overline{Apr}_\rho(\mathcal{A})}(x).$$

Thus $\overline{Apr}_\rho(\mathcal{A}) = \overline{Apr}_\rho(\mathcal{B})$.

(3) Transitive: For all $\overline{Apr}_\rho(\mathcal{A})$, $\overline{Apr}_\rho(\mathcal{B})$, $\overline{Apr}_\rho(\mathcal{C}) \in \overline{RBF}(H)$, we have $\overline{Apr}_\rho(\mathcal{A}) \subseteq \overline{Apr}_\rho(\mathcal{B})$ and $\overline{Apr}_\rho(\mathcal{B}) \subseteq \overline{Apr}_\rho(\mathcal{C})$ then

$$\mu^+_{\overline{Apr}_\rho(\mathcal{A})}(x) \leq \mu^+_{\overline{Apr}_\rho(\mathcal{B})}(x) \quad \text{and} \quad \mu^-_{\overline{Apr}_\rho(\mathcal{A})}(x) \geq \mu^-_{\overline{Apr}_\rho(\mathcal{B})}(x),$$

and

$$\mu^+_{\overline{Apr}_\rho(\mathcal{B})}(x) \leq \mu^+_{\overline{Apr}_\rho(\mathcal{C})}(x) \quad \text{and} \quad \mu^-_{\overline{Apr}_\rho(\mathcal{B})}(x) \geq \mu^-_{\overline{Apr}_\rho(\mathcal{C})}(x),$$

it gives us

$$\mu^+_{\overline{Apr}_\rho(\mathcal{A})}(x) \leq \mu^+_{\overline{Apr}_\rho(\mathcal{C})}(x) \quad \text{and} \quad \mu^-_{\overline{Apr}_\rho(\mathcal{A})}(x) \geq \mu^-_{\overline{Apr}_\rho(\mathcal{C})}(x).$$

Thus $\overline{Apr}_\rho(\mathcal{A}) \subseteq \overline{Apr}_\rho(\mathcal{C})$. Hence $\left(\overline{RBF}(H), \subseteq\right)$ is a poset. Now for lattice we have to show that the sup and inf of any two rough bipolar fuzzy sets $\overline{Apr}_\rho(\mathcal{A})$, $\overline{Apr}_\rho(\mathcal{B})$ belongs to $\overline{RBF}(H)$.

Inf: For any $\overline{Apr}_\rho(\mathcal{A})$, $\overline{Apr}_\rho(\mathcal{B}) \in \overline{RBF}(H)$, we have

$$\overline{Apr}_\rho(\mathcal{A}) \cap \overline{Apr}_\rho(\mathcal{B}) = \left\{\mu^+_{\overline{Apr}_\rho(\mathcal{A})} \wedge \mu^+_{\overline{Apr}_\rho(\mathcal{B})},\ \mu^-_{\overline{Apr}_\rho(\mathcal{A})} \vee \mu^-_{\overline{Apr}_\rho(\mathcal{B})}\right\}.$$

Now we show that $\overline{Apr}_\rho(\mathcal{A}) \cap \overline{Apr}_\rho(\mathcal{B})$ is an bipolar fuzzy right Γ-hyperideal of Γ-semihypergroup H. For any $x, y \in H$ and $\gamma \in \Gamma$,

$$\begin{aligned}
\inf_{z\in x\gamma y} \mu^+_{\overline{Apr}_\rho(\mathcal{A})\cap\overline{Apr}_\rho(\mathcal{B})}(z) &= \inf_{z\in x\gamma y}\left\{\mu^+_{\overline{Apr}_\rho(\mathcal{A})}(z) \wedge \mu^+_{\overline{Apr}_\rho(\mathcal{B})}(z)\right\} \\
&= \inf_{z\in x\gamma y} \mu^+_{\overline{Apr}_\rho(\mathcal{A})}(z) \wedge \inf_{z\in x\gamma y} \mu^+_{\overline{Apr}_\rho(\mathcal{B})}(z) \\
&\geq \mu^+_{\overline{Apr}_\rho(\mathcal{A})}(x) \wedge \mu^+_{\overline{Apr}_\rho(\mathcal{B})}(x) = \mu^+_{\overline{Apr}_\rho(\mathcal{A})\cap\overline{Apr}_\rho(\mathcal{B})}(x),
\end{aligned}$$

and

$$\begin{aligned}
\sup_{z\in x\gamma y} \mu^-_{\overline{Apr}_\rho(\mathcal{A})\cap\overline{Apr}_\rho(\mathcal{B})}(z) &= \sup_{z\in x\gamma y}\left\{\mu^-_{\overline{Apr}_\rho(\mathcal{A})}(z) \vee \mu^-_{\overline{Apr}_\rho(\mathcal{B})}(z)\right\} \\
&= \sup_{z\in x\gamma y} \mu^-_{\overline{Apr}_\rho(\mathcal{A})}(z) \vee \sup_{z\in x\gamma y} \mu^-_{\overline{Apr}_\rho(\mathcal{B})}(z) \\
&\leq \mu^-_{\overline{Apr}_\rho(\mathcal{A})}(x) \vee \mu^-_{\overline{Apr}_\rho(\mathcal{B})}(x) = \mu^-_{\overline{Apr}_\rho(\mathcal{A})\cap\overline{Apr}_\rho(\mathcal{B})}(x).
\end{aligned}$$

Thus $\overline{Apr}_\rho(\mathcal{A}) \cap \overline{Apr}_\rho(\mathcal{B})$ is an bipolar fuzzy right Γ-hyperideal H. Thus $\overline{Apr}_\rho(\mathcal{A}) \cap \overline{Apr}_\rho(\mathcal{B}) \in \overline{RBF}(H)$, $\inf\{\overline{Apr}_\rho(\mathcal{A}), \overline{Apr}_\rho(\mathcal{B})\}$ exist in $\overline{RBF}(H)$.

Sup: For any $\overline{Apr}_\rho(\mathcal{A})$, $\overline{Apr}_\rho(\mathcal{B}) \in \overline{RBF}(H)$, we have

$$\overline{Apr}_\rho(\mathcal{A}) \cup \overline{Apr}_\rho(\mathcal{B}) = \{\mu^+_{\overline{Apr}_\rho(\mathcal{A})} \vee \mu^+_{\overline{Apr}_\rho(\mathcal{B})},\ \mu^-_{\overline{Apr}_\rho(\mathcal{A})} \wedge \mu^-_{\overline{Apr}_\rho(\mathcal{B})}\}.$$

Now we show that $\overline{Apr}_\rho(\mathcal{A}) \cup \overline{Apr}_\rho(\mathcal{B})$ is an bipolar fuzzy right Γ-hyperideal of Γ-semihypergroup H. For any $x, y \in H$

$$\begin{aligned}
\inf_{z\in x\gamma y} \mu^+_{\overline{Apr}_\rho(\mathcal{A})\cup\overline{Apr}_\rho(\mathcal{B})}(z) &= \inf_{z\in x\gamma y}\left\{\mu^+_{\overline{Apr}_\rho(\mathcal{A})}(z) \vee \mu^+_{\overline{Apr}_\rho(\mathcal{B})}(z)\right\} \\
&\geq \inf_{z\in x\gamma y} \mu^+_{\overline{Apr}_\rho(\mathcal{A})}(z) \vee \inf_{z\in x\gamma y} \mu^+_{\overline{Apr}_\rho(\mathcal{B})}(z) \\
&\geq \mu^+_{\overline{Apr}_\rho(\mathcal{A})}(x) \vee \mu^+_{\overline{Apr}_\rho(\mathcal{B})}(x) = \mu^+_{\overline{Apr}_\rho(\mathcal{A})\cup\overline{Apr}_\rho(\mathcal{B})}(x),
\end{aligned}$$

and

$$\begin{aligned}
\sup_{z\in x\gamma y} \mu^-_{\overline{Apr}_\rho(\mathcal{A})\cup\overline{Apr}_\rho(\mathcal{B})}(z) &= \sup_{z\in x\gamma y}\left\{\mu^-_{\overline{Apr}_\rho(\mathcal{A})}(z) \wedge \mu^-_{\overline{Apr}_\rho(\mathcal{B})}(z)\right\} \\
&\leq \sup_{z\in x\gamma y} \mu^-_{\overline{Apr}_\rho(\mathcal{A})}(z) \wedge \sup_{z\in x\gamma y} \mu^-_{\overline{Apr}_\rho(\mathcal{B})}(z) \\
&\leq \mu^-_{\overline{Apr}_\rho(\mathcal{A})}(x) \wedge \mu^-_{\overline{Apr}_\rho(\mathcal{B})}(x) = \mu^-_{\overline{Apr}_\rho(\mathcal{A})\cup\overline{Apr}_\rho(\mathcal{B})}(x).
\end{aligned}$$

Thus $\overline{Apr}_\rho(\mathcal{A}) \cup \overline{Apr}_\rho(\mathcal{B})$ is an bipolar fuzzy right Γ-hyperideal H. Thus $\overline{Apr}_\rho(\mathcal{A}) \cup \overline{Apr}_\rho(\mathcal{B}) \in \overline{RBF}(H)$, $\sup\{\overline{Apr}_\rho(\mathcal{A}), \overline{Apr}_\rho(\mathcal{B})\}$ exist in $\overline{RBF}(H)$. Hence $\left(\overline{RBF}(H), \subseteq, \cup, \cap\right)$ is a lattice. ■

Theorem 287 *Let* $\underline{RBF}(H)$ *denote the set of all* ρ*-lower rough bipolar fuzzy left (right)* Γ*-hyperideals of* Γ*-semihypergroup* H*. Then* $(\underline{RBF}(H), \subseteq, \cup, \cap)$ *is a lattice.*

Proof. The proof is similar to the proof of Theorem 286. ∎

Conclusion

In this thesis we contributed to the development of the theoretical background of Γ-hyperideals of Γ-semihypergroups. We provided the concept of generalized bi-Γ-hyperideals and introduced the notion of (m, n) bi-Γ-hyperideals and prime (m, n) bi-Γ-hyperideals in Γ-semihypergroups. Then we investigated some of the properties of these (m, n) bi-Γ-hyperideals and prime (m, n) bi-Γ-hyperideals.

As a generalization of fuzzy sets, the concept of bipolar fuzzy sets was introduced by Lee [48], and applications of bipolar fuzzy sets have been done by many researchers in algebra, graph theory, finite state machines, temporal databases, cognitive modeling, multiagent decision analysis and mathematical morphology. The combination of bipolar fuzzy set theory and algebraic systems have resulted in many interesting research topics. In chapter 3, we applied the theory of bipolar fuzzy sets to the ideal theory of Γ-semihypergroups. Regular Γ-semihypergroups are characterized by the properties of bipolar fuzzy Γ-hyperideals. Some results on homomorphic images and preimages are also discussed. After that we extended the idea of (λ, θ)-fuzzy sets and studied bipolar (λ, δ)-fuzzy Γ-hyperideals (resp., interior Γ-hyperideals and bi-Γ-hyperideals) in Γ-semihypergroups and discussed some related properties. This study is useful for the theory of Γ-semihypergroups and it can be extended in the context of several classes of Γ-semihypergroups.

The theory of rough sets is regarded as a generalization of the classical sets theory. In chapter 4, we substituted a universe set by a Γ-semihypergroup, and introduced the notions of rough (prime bi-Γ-hyperideals, (m, n) bi-Γ-hyperideals, prime (m, n) bi-Γ-hyperideals and (m, n) quasi Γ-hyperideals). We demonstrated that the notions of rough (prime bi-Γ-hyperideals, (m, n) bi-Γ-hyperideals, prime (m, n) bi-Γ-hyperideals and (m, n) quasi Γ-hyperideals) in a Γ-semihypergroup are extensions of the notions of prime bi-Γ-hyperideals, (m, n) bi-Γ-hyperideals, prime (m, n) bi-Γ-hyperideals and (m, n) quasi Γ-hyperideals.

In next chapter we studied the roughness of sub Γ-semihypergroups, Γ-hyperideals and bi-Γ-hyperideals in terms of set valued homomorphisms. We applied generalized lower and upper approximation operators to Γ-semihypergroups. We proved that the generalized lower (upper) approximation of a Γ-hyperideal, by mean of a set valued mapping, is a Γ-hyperideal which is an extended notion of rough Γ-hyperideal introduced in [10]. Also we introduced the notion of generalized rough M-hypersystems and generalized rough N-hypersystems in Γ-semihypergroups and proved that the generalized upper rough approximation of an M-hypersystem (resp., N-hypersystem) is an M-hypersystem (resp., N-hypersystem).

Next chapter was about the study of the notion of rough bipolar fuzzy sets in Γ-semihypergroups. Here we combined rough sets with bipolar fuzzy sets and then we discussed some basic properties related to unions, intersections and containments of rough bipolar fuzzy set in Γ-semihypergroups. We showed that the set of all upper (lower) approximations of a bipolar fuzzy Γ-semihypergroup is a bipolar fuzzy Γ-semihypergroup. Then we studied the ideal theory of Γ-semihypergroups in terms of rough bipolar fuzzy sets.

Note also that our results generalize the theorems of fuzzy Γ-hyperideals and rough Γ-hyperideals of Γ-semihypergroups. To extend this work, one might consider:

1. The bipolar fuzzification of regular and intra-regular classes of Γ-semihypergroups.
2. The roughness of bipolar (λ, δ)-fuzzy Γ-hyperideals of Γ-semihypergroups.
3. The bipolar fuzzification of Γ-hyperideals of Γ-semihypergroups through left operator semihypergroups.
4. The characterizations of Γ-semihypergroups in terms of bipolar $(\in, \in \vee q)$-fuzzy Γ-hyperideals.

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