Lovelock Black Holes and Nonlinear Electrodynamics

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2022

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2022

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A Thesis Submitted to the Department of Mathematics,
Quaid-i-Azam University, Islamabad, in partial fulfillment of
the requirement for the degree of

Doctorate of Philosophy

in

Mathematics

By

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2022

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Lovelock Black Holes and Nonlinear Electrodynamics

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

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To my Grandfather Ajum Khan and my Supervisor

Acknowledgements

Alhamdolillah! It is indeed the blessing of ALLAH (SWT) who gave me enough strength and consistency to accomplish this thesis. I want to take this opportunity to acknowldege the role of various people around me, without whom I would not have been able to acheive this goal. First of all this research work is the result of constant prayers and support of my family - my father, mother, sisters and brothers - to whome I am grateful and indebted. I am extremely thankful to them for tolerating my being non-social for this entire period and letting me carry on my work without any family constraints.

I would like to express my sincere gratitutde to my mentor, my supervisor Prof. Khalid Saifullah, who guided me in each and every moment of my PhD. I am very thankful to you Sir, for considering me first as your MPhil, and then as a PhD student, training me as a research student to conduct good research work and to study all the important books of general relativity, quantum field theory, black hole physics and string theory. I am thankful for your discussions, your concern in each and every matter and most importantly your financial support given to me throughout my PhD and MPhil. I remain truly indebted, Sir.

I also want to acknowledge the support of my PhD research colleauges in Quaid-i-Azam University and colleauges in FAST NUCES, Peshawar campus for their cooperation, their encouragement in tough times and also for their companionship in good times. A research grant from the Higher Education Commission of Pakistan under its Project No. 6151 is also gratefully acknowledged. Lastly, I am thankful to all my well wishers who helped me directly or indirectly.

List of Publications

As publication is one of the requirements of the Higher Education Commission of Pakistan, we give here a list of publications from the thesis:

- A. Ali and K. Saifullah, Asymptotic magnetically charged non-singular black hole and its thermodynamics, Phys. Lett. B **792** (2019) 276.
- A. Ali and K. Saifullah, Magnetized topological black holes of dimensionally continued gravity, Phys. Rev. D **99** (2019) 124052.
- A. Ali and K. Saifullah, Magnetized black holes surrounded by dark fluid in Lovelock-power-Yang-Mills gravity, J. Cosmol. Astropart. Phys. **10** (2021) 058.
- A. Ali and K. Saifullah, $(2+1)$-dimensional black holes of Einstein's theory with Born-Infeld type electrodynamic sources, Eur. Phys. J. C **82** (2022) 131.
- A. Ali and K. Saifullah, Charged black holes in 4D Einstein-Gauss-Bonnet gravity coupled to nonlinear electrodynamics with maximum allowable symmetries, Ann. Phys. **437** (2022) 168726.
- A. Ali and K. Saifullah, Power-Yang-Mills black holes and black branes in quartic quasi-topological gravity, (submitted).
- A. Ali and K. Saifullah, Effects of quintessence and configuration of strings on the black holes of Lovelock-scalar gravity, Eur. Phys. J. C **82** (2022) 408.
- A. Ali and K. Saifullah, Rotating black branes in Lovelock gravity with double-logarithmic electrodynamics, Ann. Phys. **446** (2022) 169094.

Abstract

This work is based on the investigation of nonlinearly charged and power-Yang-Mills black holes. In this setup, first we use the limiting curvature conjecture and derive spherically symmetric nonsingular black hole solutions in the framework of nonlinear electrodynamics. The metric describing asymptotic Hayward or sandwich black hole with magnetic charge is worked out in terms of the parameter of nonlinear electrodynamics. When this parameter vanishes our solution reduces to the Hayward metric. After this we study circularly symmetric $(2+1)$-dimensional black holes of Einstein's theory coupled with Born-Infeld type electrodynamic theories. In the first case we use exponential electrodynamics and find magnetically charged $(2+1)$-dimensional black hole solution in terms of magnetic charge q and nonlinearity parameter β. In the second case, we use arcsin electrodynamics and derive the associated $(2+1)$-dimensional black hole solution in terms of electric charge Q and the parameter β. After discussing the black holes of Einstein's theory, we study the power-Yang-Mills black holes surrounded by Chaplygin-like dark fluid in Lovelock gravity. In this context, we derive the polynomial equation generating Lovelock black hole solutions sourced by Chaplygin-like dark fluid and power-Yang-Mills field. In particular, we work out the metric functions in both d-dimensional Einstein and Gauss-Bonnet gravities as well. In addition to this, we also discuss the magnetized black holes of dimensionally continued gravity in the framework of exponential electrodynamics and power-Yang-Mills theory. These dimensionally continued black hole solutions are then further extended to the case of Lovelock-scalar gravity within the framework of these two matter sources. Recently the quartic quasi-topological black holes have also attracted much attention. Therefore, we also discuss the power-Yang-Mills black holes in the framework of this higher curvature gravity. Furthermore, the black holes of pure quasi-topological gravity with power-Yang-Mills source and rotating black branes are also studied. Recently a new model of nonlinear Maxwell electrodynamics known as the ModMAx model has been formulated. This formulation preserves both conformal and SO(2) duality-rotational invariance in four dimensions. We consider this model as a matter source and investigate black holes of the novel four-dimensional Einstein-Gauss-Bonnet gravity.

The study of thermodynamic properties is also one of the important issues in the area of black hole physics. Like Maxwellian charged black holes, thermodynamics of black holes has gained much interest in the context of nonlinear matter sources as well. Hence, we also discuss thermodynamic properties associated with the solutions obtained in this work under the effects of different matter sources, for example, nonlinear electromagnetic and power-Yang-Mills fields.

Preface

Einstein's theory can be considered as the most successful model in four-dimensional spacetimes. The recent detection of gravitational waves is actually the experimental verification of this theory. However, one cannot take it as a complete model if the effects of higher energies comparable to the Planck scale are taken into account. Therefore, it is important to consider the modifications of Einstein's theory. Recently, modified theories containing higher curvature terms, particularly Lovelock and quasi-topological gravities, have received much attention. The main reason behind this is the AdS/CFT correspondence where the higher curvature terms are needed on the gravity side for the investigation of conformal field theories (CFTs) associated with different central charges. Lovelock gravity forms a natural generalization of Einstein's theory where higher curvature terms retain the second order differential equations of motion. However, the presence of arbitrary coupling parameters in the action of Lovelock gravity produce difficulties for the construction of physically meaningful solutions. Recently, an interesting proposal has been made which reduces these parameters to two, i.e. the cosmological and Newtonian constants. On the basis of this specific choice, the Lovelock action generates the new theory called dimensionally continued gravity. Note that the dimensionally continued gravity reduces to the Born-Infeld theory in even dimensions, and to Euler-Chern-Simons theory in odd dimensions. Hence, one can say that this particular type of Lovelock gravity can be further classified into two other branches according to odd and even critical dimensions.

Quasi-topological gravity is also an interesting modified theory which contains higher curvature terms in its associated action. In the context of AdS/CFT correspondence, this higher curvature gravity generates a one-to-one duality between the central charges of CFT and the parameters describing the strength of gravitational field. The terms associated with quasi-topological gravity in the action are not truly topological, thus they can contribute to the gravitational dynamics in lesser dimensions. This is the advantage of this modified gravity on the other theories, for instance Lovelock. The linearized equations of quasi-topological theory assure that it is a physical and ghost-free theory. Thus it provides the simplest toy model for the holographic analysis of four- and higher- dimensional CFTs.

Recently, the novel four-dimensional Gauss-Bonnet gravity has been developed on the basis of dimensional regularization of the Gauss-Bonnet equations. This new formulation of modified gravity avoids the Lovelock theorem and Ostrogradsky instability. The field equations are still second order and the action function is also the diffeomorphism invariant. In this setup, the d-dimensional action contains the usual Einstein-Hilbert term and cosmological constant, however, the Gauss-Bonnet term appears with a scalar multiple $1/(d-4)$. The model has been developed in $d > 4$ and field equations are derived. Finally, the modified gravity in four dimensions is demonstrated as the four-dimensional limit of the associated higher dimensional model after the suitable rescaling of Gauss-Bonnet coupling parameter.

Nonlinear electrodynamics is highly inspirational theory as it introduces new ways for the solution of some physical problems. First, it has been put forth for doing the task to alleviate the divergences of electric field and potential at the central location. Second, many physical phenomena in nature can be modeled through nonlinear field equations, for example, the systems related to gravitational fields are highly nonlinear. Third, in the context of AdS/CFT correspondence, the effects of nonlinear electrodynamics when coupled with strongly dual theory are unavoidable. Another development in this direction came when the Einsteinian black holes were studied in the presence of nonlinear electrodynamic sources. Like nonlinear electrodynamics, Yang-Mills theory which belongs to the class of non-abelian gauge theories comes from the low energy limit of string theory. One expects that non-abelian Yang-Mills fields when coupled to gravity may be helpful in the investigation of the consequences produced by super-string models. Hence, these facts motivates us to investigate physical properties of the black holes with nonlinear electrodynamic and power-Yang-Mills sources.

The outline of this thesis is as follows. In Chapter 1, we review the historical background of Einstein's theory of gravity. Then we briefly describe the higher curvature gravities such as Lovelock, quartic quasi-topological and four-dimensional Einstein-Gauss-Bonnet theories. In addition to this, we also introduce the background of nonlinear electrodynamic and Yang-Mills theories and their importance in the study of black holes. Finally, we briefly study the aspects of non-singular black holes and Markov's limiting curvature principle.

In Chapter 2, we start with a derivation of new magnetically charged nonsingular black hole solutions of some modified theory of gravity in the framework of exponential electrodynamics. We also examine the thermodynamic properties and quantum radiation of this gravitating object. In Chapter 3, we investigate a new family of $(2+1)$-dimensional black holes of Einstein's theory in the framework of Born-Infeld type theories. In this chapter, we first used the exponential electromagnetic field as a matter source and derived a magnetically charged $(2+1)$-dimensional black hole solution. In the second case we used the model of arcsin electrodynamics and worked out the associated $(2 + 1)$-dimensional black hole solution dependent on the parameters of electric charge and arcsin electrodynamics.

Chapter 4 is devoted to the study of Lovelock black holes sourced by dark fluid and power-Yang-Mills field. In this chapter, we construct a new family of magnetized black hole solutions of the associated field equations. In particular, we work out the Lovelock polynomial and derive metric functions in both $d-$dimensional Einstein and Gauss-Bonnet gravities as well. Furthermore, thermodynamic stability of these black holes is also discussed. In Chapter 5, we couple exponential electrodynamics and power-Yang-Mills field with Lovelock gravity and derive new solutions describing the dimensionally continued black holes. Further, black hole solutions of dimensionally continued gravity with these two matter sources and a conformal scalar field are also derived.

Chapter 6 deals with the higher dimensional power-Yang-Mills black holes in the framework of quartic quasi-topological gravity. We derive new solutions representing black holes in both quartic and pure quasi-topological theories. The thermodynamic quantities for these black holes are computed and local stability in both canonical and grand canonical ensembles is checked. Additionally, we also discuss the thermodynamics of horizon flat power-Yang-Mills rotating black branes and analyze their thermodynamic and conserved quantities by using the counter-term method.

In Chapter 7, we consider the ModMax model of nonlinear electrodynamics containing SO(2) duality-rotational and conformal symmetries as a source of gravitational field and study black holes of novel four-dimensional Einstein-Gauss-Bonnet gravity. In this setup we work out the metric function that describes a black hole which generalizes the Maxwellian charged black hole of four-dimensional Einstein-Gauss-Bonnet gravity. Thermodynamics associated with the resulting black hole solution is also discussed.

Finally, Chapter 8 is devoted to the concluding remarks and a summary of this thesis.

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Chapter 1

Introduction

1.1 Historical background of Einstein's theory of gravity

Gravity is one of the fundamental interactions of nature, others being the electromagnetic, strong and weak nuclear forces. Each of these interactions possess its own peculiar nature. On the basis of their comparison of strength, it is shown that gravity is the weakest among them (nearly 10^{29} times weaker than the weak force, 10^{36} times weaker than the electromagnetic force and 10^{38} times weaker than the strong force). Despite the weakness of gravity, it has certain properties that make it different from the other ones. It is a long range interaction and has attractive nature. In contrast to strong and weak fields, there does not exist a length scale which can fix its range. The absence of negative gravitational charges makes this interaction to be unscreened. Furthermore, the effects related to this fundamental interaction can also be felt in everyday life. Gravity has gone through experimental observations also. At the end of the sixteenth century, it was Galileo Galilei who first developed pendulums and inclined planes for the investigation of gravity. It was widely believed that the experiments of Galileo greatly affected the modern thinking of physics. Nonetheless, the concept of gravitation was very complicated until 1665, when Issac Newton gave the celebrated "inverse square law of gravitation". Newton's theory depends on two concepts. First, space is absolute and should be considered as an unaffected structure with physical phenomena occurring in a rigid arena. Second, the gravitational and inertial masses are the same things. At first, Newton's theory was considered as a very helpful theory for the description of gravity. However, soon it was realized that it could be applicable only for the description of limited portions of the physical world.

In the 19^{th} century, there were many observational results which could not support Newton's theory of gravity. They include the Verrier's observation in 1855 of a 35 arc-second per century excess precession of Mercury's orbit and later on, Newcomb's more precise result of 43 arc-seconds. Also, the principle stated by Ernst Mach in 1893 was taken as the first constructive attack on Newton's theory. In the form of Mach's principle, the formulations were blurred but then Einstein made the things clearer in a well-designed way. According to him, "inertia appears due to the interaction between two bodies". This picture completely opposes Newton's view who thought that inertia was a property of the fixed space. In addition to this, Dicke said that the gravitational constant is dependent on the mass distribution while according to Newton, it should be taken as a universal constant. Because of these contradictory facts and attacks on Newton's theory, physicists started to study the phenomena of gravity in a more standard way.

In order to explain the non-gravitational phenomena in a better way, Einstein in 1905 postulated his special theory of relativity. It was then realized that this new way of thinking about relative motion totally contradicted the concepts of Galilean and Newtonian physics. After this, it was observed that this special theory can be generalized if one takes the non-inertial frames of reference in the background. In 1907, the gravitational redshift was predicted by Einstein from his equivalence principle of gravity and inertia. Then in 1915, he formulated his general relativity which gives narrative of gravitation in a modern way. The experimental observations, i.e., the results about the precision of Mercury's orbit and deflection of light from the sun, were also checked in the context of this theory. In each case, the observations were in excellent agreement with Einstein's predictions. Although the formulation of Einstein defines a well-established and successful theory for gravity, however, one cannot fully discard Newton's theory as well. This is because Newtonian physics is still valid in restricted cases and could be implemented in many practical problems. Moreover, it should be noted that it can be recovered from Einstein's theory if one imposes certain limits on field strengths and velocities. In other words, one can say that Einstein reconsidered Newton's ideas in a more appropriate way, for instance, his equivalence principle.

Nowadays, Einstein's theory is also facing some objections. Among these objections, one is the issue of being ineffective at giving explanation to specific observations. Also, the issues regarding its compatibility with launched theories and lack of uniqueness have been reported.

1.2 Current challenges of general relativity

The groundwork of modern Physics is based on the two famous discoveries of the 20^{th} century: Einstein's theory of gravity and quantum theory. Both of these theories were formulated to cover the major drawbacks of Newtonian physics. In fact, due to the discovery of these celebrated theories, our perception of observing physical phenomena is now radically changed. General relativity is completely dependent on classical viewpoint which provides a narrative for gravitational fields and non-inertial frames. This theory is very helpful in many areas related to astrophysics and cosmology. On the other side, quantum theory determines the nature of mysterious high energies on the atomic and subatomic scales where the classical viewpoint fails. Quantum field theory provides the unification of quantum mechanics and special relativity and is therefore considered as a major intellectual achievement of the 20^{th} century. In quantum field theory, the spacetime is assumed as flat whereas one can also make the generalization to a curved spacetime, however, quantum fields play in a rigid arena. On the other side, general relativity does not deal with the quantum nature of matter. In other words, the behaviour of gravitational phenomena at the small scales itself is unclear. Although these two revolutionary theories are almost correct as both of them are consistent with the experimental results. However, an interesting question still remains, i.e., how can general relativity be compatible with the laws of quantum theory and how can one develop the complete and self-consistent theory of quantum gravity.

The main difficulty is that no one has any idea about the nature of gravitational field at atomic and subatomic scales. This is because gravity is the weakest among all the forces and the characteristic scale on which one can encounter its significant non-classical effects is very small. This scale is known as the Planck scale and is of the order of 10^{-33} cm. There exists several reasons why the compatibility of general relativity with quantum theory becomes so important [1]. Curiosity is the main motivation for cutting edge scientific triumphs. In the present scenario, curiosity is the basis of modifications of general relativity which may be helpful in the approaches towards a theory of quantum gravity.

Some of the gravity theories which generalize general relativity are discussed in this thesis. In particular, those theories which are developed on the basis of adding higher curvature terms to the Einstein-Hilbert action are probed.

1.3 Higher curvature gravities

The well-known Einstein-Hilbert action specifies the dynamics of gravitational field in Einstein's theory of gravity. It is a diffeomorphism invariant action which contains all the important required properties. For instance, the equations of motion associated to it are of order two. This shows that the governing theory is ghost-free and hence physical.

Although general relativity becomes a widely accepted gravity theory due to its experimental verifications so far [2–4], however, it has still some shortcomings. For example, it is a non-renormalizable gravity theory. To avoid this inadequacy, loop quantum corrections were proposed [5]. This can be done by considering counter-terms in the action function. Note that, these counter terms are of higher order in curvature invariants. Moreover, the Einstein-Hilbert action can be reproduced from the α' expansion in string theory, where $\sqrt{\alpha'}$ stands for the string length. By taking higher derivative terms in this expansion, higher curvature corrections are produced. String theory also gives other strong predictions which support the assumption of higher curvature corrections in the action function of Einstein's gravity [6].

The existence of super-symmetry and higher dimensions predicted by string theory have also gained a lot of attention. For the existence of super-symmetry, one requirement is that the spacetime dimensions should be 10. With these motivations one considers modifications of Einstein's theory in higher dimensional spacetimes and tackle the problems of quantum gravity.

Einstein's theory of gravity works appropriately only in four spacetime dimensions. If the dimensions are greater than four, the Einstein field equations are not well suited to satisfy Einstein's assumptions. The idea of higher dimensions as suggested by string theory too plays a role in the AdS/CFT correspondence which explores a connection of the $(n-1)$-dimensional conformal field theories with the n-dimensional AdS black holes [7]. Hence, it is quite interesting to extend the study towards higher curvature gravities and investigate black hole solutions in these theories. A part of this thesis is devoted to the black holes of modified gravities in which the higher curvature corrections have been made. These additional terms also need to be physical and invariant under diffeomorphism so that the theory remains free of ghosts at least at a linear level.

1.4 $f(R)$ gravities

The simplest possible way to modify Einstein's gravity is to add the square of the Ricci scalar with the Einstein-Hilbert term in the action, i.e.,

$$\int (R + \tilde{\alpha} R^2). \tag{1.1}$$

This is called the Starobinsky mode [8] and is also invariant under diffeomorphism. It is a specific case of a more general family of Lagrangians, which are simply the polynomials of the Ricci scalar. The theories whose Lagrangians belong to this general family are known as $f(R)$ gravity theories. The laws of black hole thermodynamics can be successfully tested within the framework of these gravity theories. These $f(R)$ gravities when minimally coupled with scalar fields become equivalent to Einstein's theory. Because of this property, these theories are called the scalar-tensor theories. The natural constraint on a scalar field is that it would be proportional to the first derivative of the function. This guarantees the ghost-free nature of these theories. Note that the equations of motion in general $f(R)$ gravities are of order four and the ghosts will be propagating. However, if we run into scalar-tensor theories then the ghosts will be absent.

1.5 Lovelock theory of gravity

It is acknowledged that string theory can be declared as a theory of everything that works only in higher dimensions. One of the main issues, besides unification of all forces, is that it works as theory of quantum gravity. Thus, the exploration of gravitational fields in higher dimensions is quite interesting and important. The most natural modification of Einstein's theory in diverse dimensions is the Lovelock theory of gravity [9,10]. In this theory of gravity, the differential equations of motion in arbitrary spacetime dimensions are still second order. It is worth noting that it recovers Einstein's theory in four dimensions whereas in dimensions greater than four it behaves differently. Thus, Einstein's theory could be considered as a special case of Lovelock gravity. This is because the Einstein-Hilbert term is among the different terms that comprise the action function for Lovelock gravity. There also exist several other theories which are specific cases and can be recovered from Lovelock gravity, for instance, the Chern-Simons gravity. These theories are also free from ghosts [6,11]. The action function describing these theories is topologically invariant in even dimensions [12]. In general, the Lagrangian describing the Lovelock theory can be expressed as

$$\mathfrak{L} = \sqrt{-g}\sum_{p=0}^{N}\alpha_p\mathfrak{L}_p, \tag{1.2}$$

where N is the maximum integer defined as $N = [(d-1)/2]$ in which the bracket represents the integer part and g denotes the determinant of the metric tensor $g_{\mu\nu}$. On the other hand α_p's are arbitrary parameters which describe the couplings and $\mathfrak{L}_p$ is specified as

$$\mathfrak{L}_p = \frac{1}{2^p}\delta^{\mu_1\mu_2...\mu_{2p}}_{\nu_1\nu_2...\nu_{2p}}R^{\nu_1\nu_2}_{\mu_1\mu_2}...R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}, \tag{1.3}$$

in which $R^{\alpha\beta}_{\mu\nu}$ labels the Riemann tensor and $\delta^{\mu_1\mu_2...\mu_{2p}}_{\nu_1\nu_2...\nu_{2p}}$ denotes the generalized Kronecker delta with order $2p$. The first term, i.e., the one corresponding to $N = 1$ is the well-known Einstein-Hilbert term. Similarly, the second order term corresponding to $N = 2$

is called the Gauss-Bonnet term whose Lagrangian can be defined as

$$\mathfrak{L}_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{1.4}$$

The above Gauss-Bonnet Lagrangian is topological in four dimensional spacetime so it is non-dynamical. In other words, it will not effect the equations of motion in four dimensions. Hence, it behaves like a total derivative in four dimensions. In case of dimensions lower than or equal to four, one can show that the Gauss-Bonnet term vanishes. While in spacetime dimensions higher than four, it has contributions in the equations of motion which makes it dynamical. Hence, the equations of motion corresponding to this theory will be identically zero in dimensions lower than four. This is because the Bach-Lanczos identity [13]

$$C_{\mu\sigma\rho\kappa}C_{\nu}^{\sigma\rho\kappa} - \frac{1}{4}g_{\mu\nu}C_{\alpha\beta\sigma\rho}C^{\alpha\beta\sigma\rho} = 0, \tag{1.5}$$

associated with the Gauss-Bonnet Lagrangian holds in these dimensions. Note that, $C_{\mu\sigma\rho\kappa}$ refers to the Weyl tensor. It is important to note that among others, the Gauss-Bonnet term could also arise in the low energy limit of heterotic string theory [14–16]. This is perhaps the indication to us that Lovelock gravity may arise as the high energy limit of the classical theory of gravity. The inclusion of high energy effects in the framework of classical gravity needs higher order Riemannian curvature terms and if one demands to retain second order equations of motion then the consideration of Lovelock gravity and higher dimensions is more helpful [17]. Thus, it may work as an in-between state between classical and quantum frameworks.

Although the Lovelock gravity possesses similarity in content and structure with Einstein's gravity but still it has many redundant features. First, when $d \geq 5$, the time evolution of the fields comes out to be non-unique: For a given initial time instant $t = t_0$, the fields at a later time t cannot be explicitly determined from the field equations. It is due to the participation of nonlinear velocity terms in the action [18, 19]. Second, along with cosmological and Newtonian constants, the Lovelock action have $N = [(d+1)/2]$ additional coupling parameters as well. The arbitrariness of these coupling coefficients causes difficulties in the investigation regarding the behaviour and characteristics of solutions, for example, properties of black holes. Solutions of Lovelock gravity under the assumption of spherical symmetry were derived in literature [20–23]. In these references, the arbitrariness in the coefficients causes troubles in obtaining physical information from the metric function. The reason behind this difficulty is the roots of $(N-1)$-th degree polynomial equation which cannot yield the explicit solution. Apart from this issue, some physical consequences due to the generic choice for the coupling coefficients also arise: (i) There should be at most $[(d-1)/2]$ various solutions, (ii) these may possess horizons for both signs of energy, and (iii) the entropy does not increases and hence the second law of thermodynamics does not satisfy. These remarks suggest that the special choice for the coupling parameters should be made. Therefore, the authors of Ref. [24] introduced a suitable choice for these coefficients which gives a unique solution with a dressed singularity only for positive masses. Moreover, this choice makes the entropy monotonically increasing function as well. Based upon this specific choice, a new gravity theory known as dimensionally continued gravity (DCG) has been introduced. It is also important to note that for even and odd critical dimensions DCG reduces to the Born-Infeld theory and Euler-Chern-Simons theory, respectively [24, 25]. We will use this concept in our thesis for the possibility of new black hole solutions.

Although significant attention has been devoted to Lovelock and $f(R)$ theories for the extension of Einstein's theory, however, there also exists a family of theories, known as $f(Lovelock)$ gravities, which contains both of them as particular cases. The general form of the action necessary for the description of these theories can be written as

$$\mathcal{I}_{f(Lovelock)} = \frac{1}{16\pi G}\int d^dx\sqrt{|g|}f(\mathfrak{L}_0, \mathfrak{L}_1, \mathfrak{L}_2, ..., \mathfrak{L}_{d/2}), \tag{1.6}$$

where f is a function that depends upon dimensionally continued Euler densities defined in Eq. (1.2) and is differentiable. It should be noted that $\mathfrak{L}_1$ denotes the Einstein-Hilbert term while $\mathfrak{L}_2$ refers to the Gauss-Bonnet term. We will choose the form of $\mathfrak{L}_3$ and $\mathfrak{L}_4$ in later sections. It is also worth noting that $\mathfrak{L}_p$ should be equal to zero when p exceeds the value $d/2$. The usual Lovelock and $f(R)$ gravities can also be recovered from the general $f(Lovelock)$ theory if one considers f as a linear combination of Euler densities and arbitrary functions of Ricci scalar, respectively. As we mentioned earlier that Lovelock gravity forms the most natural generalization of Einstein's theory where the differential equations of motion are of order two, however, the $f(Lovelock)$ theory yields fourth order equations. If one confines the spacetime to four dimensions, then only the Ricci scalar and Gauss-Bonnet term would only contribute in the action and hence the new theories known as $f(R, G)$ gravities arise.

1.6 Quartic quasi-topological gravity

Quasi-topological gravity theory also belongs to the class of extended theories which can be used for the description of gravitational phenomena in higher dimensions. This theory is very significant in spherically symmetric conditions like Lovelock gravity. Under these conditions, the associated differential equations describing dynamical contribution are of order two and one can compute the exact solutions for the field equations. However, difficulties appear when one removes the condition of spherical symmetry because the order of equations of motion rises to four. Although Lovelock and quasi-topological gravities are equivalent in the background of spherical symmetry but quasi-topological gravity appears to be more significant as it encounters the gravitational effects in lesser dimensions than Lovelock theory. For instance, if one assumes third order curvature terms, Lovelock gravity generates gravitational effects in spacetime dimensions seven and higher while quasi-topological gravity produces these effects in dimensions higher than four. In more mathematical language, we can say that Lovelock gravity with curvature terms of order p possesses a constraint on spacetime dimensions, i.e., higher curvature terms contribute only when $p \leq [d/2]$ while in quasi-topological gravity no such restriction is applied [26]. In addition to this, the linearized quasi-topological equations are almost the same as that of Einstein's gravity up to an overall factor for the case of maximally symmetric spacetimes. This correspondence of linearized equations assures that the gravity theory is physical, ghost-free and contains no extra degrees of freedom. This linearity has a great importance, i.e, one can find a stable vacua in the ghost-free theory with no violation of unitarity principle [27]. Furthermore, the matching of linearized equations with Einstein's theory implies that the quasi-topological gravity provides a simplest toy model for the holographic analysis of four and higher dimensional conformal field theories [28, 29].

The quasi-topological gravities of various orders have been studied so far. The second order quasi-topological gravity coincides with the Gauss-Bonnet gravity and has been investigated by several authors [30–32]. Cubic quasi-topological gravity with third order curvature terms is investigated in Refs. [33–35]. Similarly, investigations related to quartic quasi-topological gravity have been done in Refs. [26, 36–41]. In this thesis, we have also focused on the study of quartic quasi-topological gravity in higher spacetime dimensions. The d-dimensional gravitational action which includes higher curvature terms of fourth order can be expressed as

$$\mathcal{I}_G = \frac{1}{16\pi} \int d^dx\sqrt{-g}\Big[R - 2\Lambda + \tilde{\mu}_2\mathfrak{L}_2 + \tilde{\mu}_3\mathfrak{L}_3 + \tilde{\mu}_4\mathfrak{L}_4\Big], \tag{1.7}$$

where Λ denotes the cosmological constant, R refers to the Ricci scalar and $\tilde{\mu}_2$, $\tilde{\mu}_3$ and $\tilde{\mu}_4$ are the coupling parameters of this gravity theory. Furthermore, $\mathfrak{L}_2$, $\mathfrak{L}_3$ and $\mathfrak{L}_4$ denote the Lagrangians for Gauss-Bonnet, the cubic and quartic quasi-topological theories, respectively and are expressed as [42]

$$\mathfrak{L}_2 = R_{\mu\nu\gamma\rho}R^{\mu\nu\gamma\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \tag{1.8}$$

$$\begin{aligned}\mathfrak{L}_3 = {} & R_{\mu\nu}^{\rho\sigma}R_{\rho\sigma}^{\alpha\beta}R_{\alpha\beta}^{\mu\nu} + \frac{1}{8(2d-3)(d-4)}\big(b_1R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}R + b_2R_{\mu\nu\rho\sigma}R_{\alpha}^{\mu\nu\rho}R^{\sigma\alpha} \\ & + b_3R_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} + b_4R_{\mu}^{\nu}R_{\nu}^{\rho}R_{\rho}^{\mu} + b_5R_{\mu}^{\nu}R_{\nu}^{\mu}R + b_6R^3\big),\end{aligned} \tag{1.9}$$

$$
\begin{aligned}
\mathfrak{L}_4 = {} & c_1 R_{\mu\nu\rho\sigma} R^{\rho\sigma\alpha\beta} R^{\kappa\gamma}_{\alpha\beta} R^{\mu\nu}_{\kappa\gamma} + c_2 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} R^{\alpha\beta}_{\alpha\beta} + c_3 R R_{\mu\nu} R^{\mu\rho} R^{\nu}_{\rho} \\
& + c_4 \big(R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\big)^2 + c_5 R_{\mu\nu} R^{\mu\rho} R_{\rho\sigma} R^{\sigma\nu} + c_6 R R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} \\
& + c_7 R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\alpha} R^{\sigma}_{\alpha} + c_8 R_{\mu\nu\rho\sigma} R^{\mu\rho\alpha\beta} R^{\nu}_{\alpha} R^{\sigma}_{\beta} + c_9 R_{\mu\nu\rho\sigma} R^{\mu\rho} R_{\alpha\beta} R^{\nu\alpha\sigma\beta} \\
& + c_{10} R^4 + c_{11} R^2 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + c_{12} R^2 R_{\alpha\beta} R^{\alpha\beta} + c_{13} R_{\mu\nu\rho\sigma} R^{\mu\nu\alpha\beta} R^{\nu}_{\alpha\beta\gamma} R^{\sigma\gamma} \\
& + c_{14} R_{\mu\nu\rho\sigma} R^{\mu\alpha\rho\beta} R_{\kappa\alpha\gamma\beta} R^{\kappa\nu\gamma\sigma},
\end{aligned} \tag{1.10}
$$

where the coefficients b_i's and c_i's have been determined as

$$
\begin{aligned}
b_1 &= 9d_1 - 15, \\
b_2 &= -24d_2, \\
b_3 &= 24d, \\
b_4 &= 48d_2, \\
b_5 &= -12(3d_1 - 1), \\
b_6 &= 3d,
\end{aligned} \tag{1.11}
$$

$$
\begin{aligned}
c_1 &= -d_2(d_1^7 - 3d_1^6 - 29d_1^5 + 170d_1^4 - 349d_1^3 + 348d_1^2 - 180d_1 + 36), \\
c_2 &= -4d_4(2d_1^6 - 20d_1^5 + 65d_1^4 - 81d_1^3 + 13d_1^2 + 45d_1 - 18), \\
c_3 &= -64d_2(3d_1^2 - 8d_1 + 3)(d_1^2 - 3d_1 + 3), \\
c_4 &= -(d_1^8 - 6d_1^7 + 12d_1^6 - 22d_1^5 + 114d_1^4 - 345d_1^3 + 468d_1^2 - 270d_1 + 54), \\
c_5 &= 16d_2(10d_1^4 - 51d_1^3 + 93d_1^2 - 72d_1 + 18), \\
c_6 &= -32d_2^2 d_4^2(3d_1^2 - 8n + 3), \\
c_7 &= 64d_3 d_2^2(4d_1^3 - 18d_1^2 + 27d_1 - 9), \\
c_8 &= -96d_2 d_3(2d_1^4 - 7d_1^3 + 4d_1^2 + 6d_1 - 3), \\
c_9 &= 16d_2^3(2d_1^4 - 26d_1^3 + 93d_1^2 - 117d_1 + 36),
\end{aligned} \tag{1.12}
$$

$$
\begin{aligned}
c_{10} &= d_1^5 - 31d_1^4 + 168d_1^3 - 360d_1^2 + 330d_1 - 90, \\
c_{11} &= 12d_1^6 - 134d_1^5 + 622d_1^4 - 1484d_1^3 + 1872d_1^2 - 1152d_1 + 252, \\
c_{12} &= 8(7d_1^5 - 47d_1^4 + 121d_1^3 - 141d_1^2 + 63d_1 - 9), \\
c_{13} &= 16d_1 d_2 d_3 d_4(3d_1^2 - 8d_1 + 3), \\
c_{14} &= 8d_1(d_1^7 - 4d_1^6 - 15d_1^5 + 122d_1^4 - 287d_1^3 + 297d_1^2 - 126d_1 + 18).
\end{aligned} \tag{1.13}
$$

Note that, for adopting the simplicity we have choose $d_i = d - i$ in above definitions.

1.7 Four-dimensional Einstein-Gauss-Bonnet gravity

The well-known Lovelock theorem [9, 10] says that Einstein's gravity along with cosmological constant forms a unique theory when one considers: (i) the spacetime dimensions are equal to four, (ii) the gravitational action is diffeomorphism invariant, (iii) metricity and (iv) differential equations of motion are of order two. Recently, a new way has been introduced which bypasses the Lovelock theorem and provides a model satisfying all the axioms (i)-(iv), but nonetheless displaying modified dynamics [43].

It is illustrious that the most general model in four-dimensional curved spacetimes contains the Einstein-Hilbert action with a cosmological constant Λ, i.e.,

$$\mathcal{I}_{EH}[g_{\alpha\beta}] = \int d^4x\sqrt{-g}\left[\frac{M_P^2 R}{2} - \Lambda\right], \tag{1.14}$$

where M_P represents the reduced Planck mass describing the strength of gravity coupling while Λ gives the description of vacuum energy. Moreover, for the gravitational phenomena in higher dimensions, the higher order curvature terms, for instance, Lovelock invariants satisfying the assumptions (ii)-(iv) should be taken into account. The first such term involved in five spacetime dimensions is given by

$$\mathcal{I}_{GB}[g_{\alpha\beta}] = \int d^dx\sqrt{-g}\alpha_2\left[R^{\alpha\beta}_{\sigma\kappa}R^{\sigma\kappa}_{\alpha\beta} - 4R^{\alpha}_{\beta}R^{\beta}_{\alpha} + R^2\right], \tag{1.15}$$

where α denotes the dimensionless coupling parameter. In the spacetime dimensions equal to four, the Gauss-Bonnet invariant becomes the total derivative and hence is not dynamical. This can be revealed by its contribution to the field equation

$$\begin{aligned}\frac{g_{\beta\rho}}{\sqrt{-g}}\frac{\delta\mathcal{I}_{GB}}{\delta g_{\alpha\rho}} &= -2R^{\alpha\sigma}_{\rho\kappa}R^{\rho\kappa}_{\beta\sigma} + 4R^{\alpha\sigma}_{\beta\rho}R^{\rho}_{\sigma} + 4R^{\alpha}_{\sigma}R^{\sigma}_{\beta} - 2RR^{\alpha}_{\beta} \\ &+ \frac{1}{2}\delta^{\alpha}_{\beta}\left(R^{\mu\nu}_{\sigma\kappa}R^{\sigma\kappa}_{\mu\nu} - 4R^{\mu}_{\nu}R^{\nu}_{\mu} + R^2\right).\end{aligned} \tag{1.16}$$

The above equation vanishes in four spacetime dimensions, however, it is nonzero for $d \geq 5$. This can be demonstrated from the trace of Eq. (1.16),

$$\frac{g_{\rho\sigma}}{\sqrt{-g}}\frac{\delta\mathcal{I}_{GB}}{\delta g_{\rho\sigma}} = (d-4)\times\frac{\alpha}{2}\left[R^{\mu\nu}_{\alpha\kappa}R^{\alpha\kappa}_{\mu\nu} - 4R^{\mu}_{\nu}R^{\nu}_{\mu} + R^2\right], \tag{1.17}$$

which is directly related to a vanishing factor $(d-4)$ in four dimensions. The question arises that whether this property is particularly related to the trace equation or it is generally present in the field equations. This question was discussed previously in Refs. [44, 45]. It has been observed that the Gauss-Bonnet invariant contributes to all the components of field equations with a proportionality factor $(d-4)$, in spite of the spacetime symmetries. The proportionality of this factor has also been examined in the ADM decomposition analysis for graviton [45]. In order to obtain the nonzero contributions of Gauss-Bonnet term for $d = 4$, the idea of rescaling the coupling parameter as $\alpha_2 \to \alpha_2/(d-4)$ has been recently introduced [43]. This is related to the technique in which finite terms are produced from dimensional regularization, when the counter-terms absorb all the divergences. It is also equivalent of the method where conformal anomaly arises due to quantum effects in curved spacetimes [46]. Nevertheless, in contrast to the dimensional regularization, in this case there do not exist any divergences that want to be removed, but instead, the singular coefficient should be taken into account which can extract finite contribution from Gauss-Bonnet invariant. Therefore, this treatment has been used for undertaking dynamical contributions of Gauss-Bonnet term in four spacetime dimensions [43].

Additionally, the interesting property which makes this rescaling procedure differ from conformal anomaly is that the number of degrees of freedom does not change in any dimensions as α_2 tends to zero. Due to this, it can be smoothly connected to Einstein's theory and hence it is free from the Ostrogradsky instability [47]. This fact is not valid for the conformal anomaly where the presence of higher order derivative terms increases the degrees of freedom.

To analyze the consequences of rescaling procedure in four dimensions, let us suppose the action function $\mathcal{I} = \mathcal{I}_{EH} + \mathcal{I}_{GB}$, which is a combination of Einstein-Hilbert and Gauss-Bonnet terms, describing a pure gravity theory in the form

$$\mathcal{I}[g_{\mu\nu}] = \int d^d x \sqrt{-g}\left[\frac{M_P^2}{2}R - \Lambda + \frac{\alpha}{d-4}\left(R^{\mu\nu}_{\alpha\kappa}R^{\alpha\kappa}_{\mu\nu} - 4R^{\mu}_{\nu}R^{\nu}_{\mu} + R^2\right)\right]. \tag{1.18}$$

In the case of maximally symmetric spacetime, the Riemann curvature tensor can be given as $M_P^2 R^{\alpha\gamma}_{\rho\sigma} = (\delta^{\alpha}_{\rho}\delta^{\gamma}_{\sigma} - \delta^{\alpha}_{\sigma}\delta^{\gamma}_{\rho})\Lambda_{eff}/(d-1)$, in which Λ_{eff} refers to the effective cosmological constant. One can easily compute the contribution of Gauss-Bonnet invariant as

$$\frac{g_{\gamma\rho}}{\sqrt{-g}}\frac{\delta\mathcal{I}_{GB}}{\delta g_{\alpha\rho}} = \frac{\alpha}{d-4} \times \frac{(d-2)(d-3)(d-4)\Lambda_{eff}^2\delta^{\alpha}_{\gamma}}{2(d-1)M_P^2}. \tag{1.19}$$

It can be clearly seen that the divergent factor $1/(d-4)$, as a result of the rescaling of coupling parameter α_2, cancels out the vanishing factor $(d-4)$ from the variation of S_{GB}. This property can also be revealed from all the equations of motion. Hence, using the limit $d \to 4$ in the above expression one can obtain the non-vanishing Gauss-Bonnet contribution as $\alpha\Lambda_{eff}^2\delta^{\alpha}_{\gamma}/3M_P^4$.

The spherically symmetric vacuum black hole solutions of this theory obtained in Ref. [43] possess many interesting properties, for example, the gravitational force has a repulsive nature at short distances and therefore an incoming particle will never reach the origin. This means that this theory does not contain the problem of central singularity. This contrasts with Einstein's theory of gravity, where an incoming particle will ultimately hit the singularity at which the effective theory fails.

Although the four-dimensional Einstein-Gauss-Bonnet gravity opens new doors for studying physical and thermodynamic properties of black holes in four-dimensional spacetimes, however, some objections on this theory have also been reported [48–54]. For instance, it has been proven [51, 52] that the solutions in this gravity theory are different from Einsteinian black hole solutions when they were coupled to strong scalar field. It is worth mentioning that the authors of Refs. [55–57] determine spherically symmetric solutions in the consistent theory of four-dimensional Einstein-Gauss-Bonnet gravity (EGB gravity). These solutions of the consistent theory could also be found in the framework of rescaling procedure of Refs. [43, 58].

1.8 Black hole solutions

The most interesting predictions of Einstein's theory of gravity are black holes. We can define a black hole as a place in spacetime where gravity pulls so much that no matter or radiation can escape from it. Here the gravitational field is very strong because matter has been squeezed into a tiny space. After the formulations of Einstein's theory, research for the physical properties of black holes attained a lot of attention. Significant development has been accomplished in the study of these gravitating objects, e.g., their astrophysical aspects and details of the different physical phenomena. The physical properties and the causal structure of black holes exhibit a profound connection with other theories such as thermodynamics, quantum theory and information theory.

Higher curvature theories exhibit new families of black hole solutions which generalize the fundamental solutions of Einstein's theory. Maximally symmetric vacuum solutions of higher curvature theory contain, for example, anti-de Sitter (AdS) vacua and also flat space [59]. Among these solutions, some are not physically meaningful as they are singular with respect to the coefficients of higher order interactions. To handle these difficulties, systematic techniques for the determination of analytic solutions have been proposed [60]. Since the domain for the validity of equations of motion associated with the effective field theory is likely to be restricted, therefore, it is useful to use the perturbative approach for the consideration of higher curvature terms. This approach has been implemented to carry out the modifications of the Schwarzschild solution in the background of renormalized Einstein's theory [61]. It is believed that the inclusion of second order curvature terms could not generate any modifications to Einstein's equations. Therefore, the effects as a result of third order curvature terms have been examined. The results obtained from this process [61] describe that the mass and associated thermodynamic quantities of the black hole should be modified. For instance, the entropy will be no more proportional to the surface area of the event horizon which contradicts the basic definition of entropy in Einstein's theory.

Furthermore, perturbative approaches were also used for studying black holes in string theory [62–64]. The authors of these references have studied spherically symmetric black holes under the effects of second order curvature terms in arbitrary spacetime dimensions. In addition to this, axially symmetric and charged black holes in four spacetime dimensions have also been investigated through this approach [64]. Apart from the modifications in thermodynamic quantities, these curvature terms also produce different scalar fields "hair" on the black holes. However, these scalar hairs are not primary because they can be entirely determined, e.g., mass and charge. Hence, the famous "no hair theorems" are satisfied for these solutions. Indeed, new hairs come up due to the scalar field being non-minimally coupled to higher curvature terms.

It is well-known that string theory and brane cosmology strongly support the importance of higher dimensions. Thus, it is highly motivating to focus on the study of higher dimensional black holes of Lovelock gravity. In this context, the metric describing spherically symmetric black holes were derived in second order Lovelock gravity [59, 65]. The black holes of Lovelock gravity with arbitrary order were studied in Refs. [24,65,66]. The explicit formulas for mass and free energy of these black holes were found in Ref. [67, 68]. Similarly, the charged Gauss-Bonnet black holes with Maxwell source were studied in Ref. [69]. It should be noted that these black holes reveal some strange properties such as multiple horizons and unusual thermodynamical aspects. For instance, the Hawking temperature vanishes for some of these black holes [70]. The effects of higher curvature terms on the objects like black strings and branes were also analyzed in Ref. [71]. Recently, solutions representing dimensionally continued black holes have also been worked out [72–75]. The thermodynamic and physical properties associated with these objects were studied in Refs. [72–79]. In addition to the black holes of Lovelock gravities, the quasi-topological black holes have also been discussed [26, 33, 34, 80, 81]. Two families of solutions for the neutral and Maxwellian charged quasi-topological black holes were derived in Refs. [33,36]. The solutions describing Lifshitz quartic quasi-topological black holes were also derived in Ref. [38].

Recently, black hole solutions in four-dimensional Einstein-Gauss-Bonnet gravity attracted much attention [43,55–57]. These black hole solutions in the presence of Maxwell source were found in Refs. [82,83]. A study related to a cloud of strings has been done in this theory as well [84]. Black hole solutions generated from Vaidya metric [85] were also derived in Ref. [86]. Regular black holes [87, 88] and black holes with angular momentum by employing Newman-Janis algorithm [89, 90] were also analyzed in this context. Similarly, a rotating black hole solution that worked as a particle accelerator [91] was also constructed. Furthermore, thermodynamic quantities of AdS black holes and their stability [92], phase transitions in de Sitter black holes [93] and quasi-normal modes (QNMs), thermodynamic stability and black hole shadows [94] have also been discussed. Gravitational lensing [95] and strong gravitational lensing in homogeneous plasma [96] were investigated. QNMs, strong cosmic censorship [97], stability of Einstein's static universe [98] have also been studied in Einstein-Gauss-Bonnet theory in four dimensions. In addition to this, wormholes in this new theory [99], thin shell wormholes [100] and relativistic stars [101] were also taken into account. The black hole as a heat engine [102] and the innermost stable circular orbits and shadow were also discussed [103]. Greybody factor [104] and power spectra associated to thermal radiation for de-Sitter black holes were also considered [106]. Super-radiance, thermodynamic stability and weak cosmic censorship conjecture for the charged black holes [107, 108] and modified thermodynamics and micro-structures in AdS space [109] were studied as well. Spinning test particle [110], perturbative and non-perturbative QNMs [111] were also investigated. Thermodynamic and physical properties of black holes of novel four-dimensional Einstein-Gauss-Bonnet theory in the scalar-tensor formulations have also been worked out [112]. Studies related to regularized Lovelock gravity [113], thin accretion disks [114] and other interesting aspects have also been undertaken recently [115–119].

1.9 Thermodynamics of black holes

Black hole thermodynamics deals with the study of reassembling the usual laws of thermodynamics with the existence of black hole horizons. The properties of black holes are almost independent of the features of collapsing matter, so this independence is eventually related to the fact that these gravitating objects could provide the thermodynamic limit of underlying quantum gravitational degrees of freedom. Hence, it can be imagined that the classical and semi-classical aspects of black holes could provide some insights into the development of quantum gravity. The momentous barrier for the construction of quantum gravity is the absence of its observational or experimental verifications. The only "test" that one can expect is the approach of theoretical and mathematical consistency. For determining consistent formulations, the consideration of the fundamental laws of black hole thermodynamics could be an important constraint on quantum gravity. Originally, the subject of black hole thermodynamics was build up in the background of Einstein's theory. Subsequently, it was straightforward to check this aspect in modified gravities. It was the work of Stephan Hawking who first introduced that, black holes could emit thermal radiation with a characteristic temperature called Hawking temperature which is related with surface gravity κ through [120]

$$k_B T_H = \frac{\hbar\kappa}{2\pi c}, \tag{1.20}$$

where k_B, $\hbar$ and c stand for the Boltzmann constant, reduced Planck's constant and speed of light, respectively. It is obvious from the presence of Planck's constant that the understanding of this result requires the formulation of quantum gravity. Infact, one can claim it as the leading order result of quantum gravity. From the Euclidean path integral approach [121], one can define the first law of black hole thermodynamics as

$$\frac{\kappa}{2\pi c}\delta\mathcal{S} = c^2\delta M - \Omega\delta J, \tag{1.21}$$

in which M is the mass of the gravitating object, J denotes the angular momentum and Ω refers to the angular velocity evaluated at the horizon. It should be noted that the first law can be generalized in different ways. For example, in the presence of matter such as electromagnetic field, the additional work terms should also be taken into account. However, a very interesting generalization would be made for the higher curvature gravities. Once the Hawking temperature and surface gravity corresponding to the black hole are obtained, the expression for entropy can be computed as well. In the background of Einstein's theory, the celebrated Bekenstein-Hawking entropy is defined through the relation [122, 123]

$$\mathcal{S}_{BH} = \frac{\kappa^3 c A_H}{4\hbar G}, \tag{1.22}$$

where A_H denotes the horizon area. In other words, this relation shows that the horizon's area can be interpreted as the thermodynamic entropy. Although this relation is valid in Einstein's theory but in higher curvature gravities it fails generally [124–128] where the entropy is proportional to a different local geometric quantity on the event horizon. In reality, the entropy in any diffeomorphism invariant gravity theory comes out to be the Noether charge associated with the Killing horizon [127, 128].

1.9.1 Wald entropy

The direct relation between the Hawking temperature and surface gravity not only holds in Einstein's gravity but in higher curvature theories as well. However, the entropy and horizon area are no more directly proportional to one another. It was Wald [127, 128] who found the generalized formula of entropy which is valid in the context of higher curvature gravities. To visualize this definition of entropy, consider the Lagrangian in the form $\mathcal{L}(g_{ab}, R_{abcd})$. Then the Wald entropy can be defined as

$$\mathcal{S} = -2\pi \int \frac{\partial \mathfrak{L}}{\partial R_{abcd}} \epsilon_{ab}\epsilon_{cd}, \tag{1.23}$$

in which $\epsilon_{cd} = k_c l_d - l_c k_d$ denotes the binormal to bifurcate the Killing horizon. In particular, for the case of Einstein's theory one can get

$$\frac{\partial \mathfrak{L}}{\partial R_{abcd}} = \frac{1}{32\pi G}(g^{ac}g^{bd} - g^{ad}g^{bc}). \tag{1.24}$$

Using this formula and the fact $\epsilon_{ab}\epsilon^{ab} = -2$, one can easily recover the Bekenstein-Hawking entropy $\mathcal{S}_{BH} = A_H/4$ (in gravitational units).

1.10 Nonlinear electrodynamic and Yang-Mills theories

In order to use the electromagnetic field as a source of gravity for the construction of charged solutions in Einstein's theory, one usually considers the standard Maxwell's theory of $U(1)$ gauge field. However, the linear Maxwell's theory becomes problematic and remains unsuitable when the field is strong. Thus, the nonlinearities of electromagnetic field should be taken into account to construct the nonlinear generalization of Maxwell's theory. The most general Lorentz-invariant action for the description of nonlinear electrodynamics (NED) in four dimensions is given by

$$\mathcal{I}_{NED} = \int d^4x \mathfrak{L}_{NED}(\digamma, P). \tag{1.25}$$

Here, $\digamma = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $P = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}$ are the two independent Lorentz-invariants. It should be noted that both of these invariants are quadratic in the Maxwell tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, A_ν being the electromagnetic potential. Typically, one should choose the Lagrangian density $\mathcal{L}_{NED}(\digamma, P)$ regular at $\digamma = P = 0$ and in weak field limit one should recover the classical Maxwell's theory from it. However, recently several interesting models, for instance the ModMax model, have also been developed which violate these assumptions. Other constraints on the generic model of NED were imposed by the satisfaction of causality and unitarity principles [129–131].

There are several NED models in the literature. However, the first relativistic and gauge invariant model was the Born-Infeld NED. It was formulated in 1934 by Max Born and Leopold Infeld [132–134] and was constructed as a modified form of Maxwell electrodynamics such that the electromagnetic self energy of a point charge remains finite at the central position of the charged particle. The vacuum polarization phenomenon in quantum electrodynamics has been observed experimentally since 1940s. This observation shows that vacuum polarization of the virtual electron-positron pairs gives an indication of the nonlinear interactions of electromagnetic field, for example, in the case of photon-photon scattering. This scattering process could be examined carefully with the use of Heisenburg-Euler electrodynamics [135]. Recently, many other NED models have also been proposed, for example, the logarithmic NED [136], the power-Maxwell theory [137–140], the rational [141, 142], exponential [143–145], arcsin [146] and double-logarithmic NED [147] models. Other formulations, that were recently introduced from a particular form of Born-Infeld model, are the Dirac-Born-Infeld inflation theory and Eddington-inspired Born-Infeld theory [148, 149]. These theories are useful in the study of dark energy, holographic entanglement entropy and holographic superconductors [150–152]. It was Hoffmann who found the first solution of Einstein's theory in the frame work of Born-Infeld electrodynamics [153]. Subsequently, different solutions asymptotic to the Reissner-Nordström metric were derived with the help of other NED models [154, 155]. For example, exponential electrodynamics [145] has been used to derive the asymptotic Reissner-Nordström black hole solution with a magnetic charge. Furthermore, black holes of higher curvature gravities have also been investigated, for example, the solution of Lovelock gravity with Born-Infeld source is derived in Ref. [74]. Similarly, nonlinearly charged black holes of quartic quasi-topological gravity were probed in Refs. [157, 158]. Recently, black holes of four-dimensional Gauss-Bonnet gravity were also studied within the framework of various NED models [159, 160]. It is worth-noting that classical electrodynamics has been recovered from all these NED models except power-Maxwell theory in the weak field limit. However, it is also worthwhile to mention that there exist NED models from which Maxwell electrodynamics cannot be recovered in this limit. Such type of models can be considered as matter sources for the derivation of a regular electrically charged black holes [161, 162].

It is well-known that a generic NED model may not contain the familiar symmetries of Maxwell electrodynamics, for instance, symmetries of conformal and $SO(2)$ duality-rotational invariance. Recently, a model of NED known as ModMax model is proposed which satisfies these symmetries [163]. In this very interesting model, a positive parameter γ describing the strength of nonlinear electromagnetic field is taken into account [163, 164]. In order to satisfy causality and unitarity principles, an estimate for the upper bound of this parameter has been found and a black hole solution in Einstein's theory has been derived [165]. Recently, a new class of chiral 2-form electrodynamics in six spacetime dimensions have also been developed [166]. This model gives a nonlinear generalization of electromagnetic fields which possess both the symmetries of duality rotation and conformal transformations. It is also shown that in the weak field limit, it can be related to the ModMax model by dimensional reduction. The charged black holes of Einstein's theory and exact gravitational waves in the framework of ModMax model have also been studied [167]. Furthermore, the novel charged Taub-NUT solutions in the presence of ModMax source were determined in Ref. [168]. The generalized ModMax model having four parameters has also been formulated [169]. This generalized model of NED contain both Born-Infeld type and ModMax formulations as its specific cases.

Historically, starting with the familiar Reissner-Nordström solution, its higher dimensional version known as the Einstein-Maxwell black hole solution is well-known by now. Similarly, the higher dimensional Einstein-Yang-Mills black holes have also acquired a lot of attention in recent works [170–172]. From physical standpoint, the effects of electromagnetic field can be observed outside the nuclei of natural matter. However, Yang-Mills field produces effects inside the nuclei, but since our universe contains the exotic and highly dense matter so the Yang-Mills field needs to be explored in a broader sense. The d-dimensional action describing the Yang-Mills field can be written as

$$\mathcal{I}_{YM} = -\frac{1}{2}\int_{\mathcal{M}} d^d x\sqrt{-g}\,\mathbf{Tr}(F^{(a)}_{\alpha\beta}F^{(a)\alpha\beta}), \tag{1.26}$$

where $\mathbf{Tr}(.) = \sum_{a=1}^{(d-1)(d-2)/2}(.)$ and $F^{(a)}_{\alpha\beta} = \partial_\alpha A^{(a)}_\beta - \partial_\beta A^{(a)}_\alpha + \frac{1}{2\sigma}C^{(a)}_{(b)(c)}A^{(b)}_\alpha A^{(c)}_\beta$ defines the Yang-Mills field. Here, $C^{(a)}_{(b)(c)}$ refer to the structure constants of $(d-1)(d-2)/2$-parameter Lie group, σ is the coupling parameter and $A^{(a)}_\beta$ denotes the Yang-Mills potential. It is worth noting that the indices {a,b,c,...} produce no effect whether they are placed at contravariant or covariant position. Through variation of the action with respect to $A^{(a)}_\beta$, the equations of motion corresponding to the Yang-Mills field can be obtained as

$$\nabla_\alpha F^{(a)\beta} + \frac{1}{\sigma}C^{(a)}_{(b)(c)}A^{(b)}_\alpha F^{(c)\alpha\beta} = 0. \tag{1.27}$$

Similarly, the stress-energy tensor corresponding to the above action can be found as

$$\mathcal{T}_{\alpha\beta} = \mathbf{Tr}\left[2F^{(a)\rho}_\alpha F^{(a)}_{\beta\rho} - \frac{1}{2}F^{(a)}_{\rho\lambda}F^{(a)\rho\lambda}g_{\alpha\beta}\right]. \tag{1.28}$$

The relaxed condition for the validity of no-hair theorem in asymptotic AdS spacetimes also encourages us to take Yang-Mills field as a matter source and check the existence of new black hole solutions. However, the equations of motion corresponding to Yang-Mills source are so ambiguous that earlier efforts for the derivation of new black hole solutions were only presented numerically. The first analytic solution describing Yang-Mills black hole was derived with the help of Wu-Yang ansatz [173]. The existence of higher dimensional Yang-Mills black holes in the framework of modified gravities has also been checked in Refs. [174,175]. Recently, regular non-minimal Yang-Mills solutions have also been determined [176]. Similarly, black holes of Einstein's gravity minimally coupled with both Yang-Mills and Born-Infeld theories have also been studied [177].

Analogous to the black holes with nonlinear electromagnetic sources, it is also inspiring to study black holes within the framework of nonlinear Yang-Mills theory. In this context, the existence of black holes with a power of Yang-Mills source is investigated in Ref. [178]. In other words, the matter source has been chosen as $\left(F^{(a)}_{\rho\lambda}F^{(a)\rho\lambda}\right)^q$ in which q is a real number such that the value $q = 1$ corresponds to the case of Yang-Mills theory. This assumption on the Yang-Mills invariant makes the metric function and the associated thermodynamic quantities dependent on the parameter q, which results in extra r-dependence. In this thesis, we study new spherically symmetric black holes of higher curvature gravities coupled to power-Yang-Mills theory. It will be shown that the power-Yang-Mills field affect the thermodynamic stability of these black holes. It is also worth noting that like the nonlinear electrodynamics, the fulfillment of conformal invariance needs further constraints on the parameter q and the spacetime dimensions come out to be a multiple of four.

1.11 Non-singular black holes and the limiting curvature principle

It is well-known that, in both classical and quantum realms, Einstein's theory of gravity is ultraviolet-incomplete (UV-incomplete). The existence of singularities is the main problem in this theory, e.g. solutions of Einstein's equations such as Schwarzschild, Reissner-Nordström and Kerr metric, have curvature singularities at the origin. So, in general, one expects that the modification of this theory is possible in those regions where the curvature is very high. Many proposals have been put forward to achieve such modifications. For example, it was also proposed that if higher order terms are included in curvature and those terms which contain higher order derivatives, then the theory of gravity can be made UV-complete [179]. However these theories contain non-physical degrees of freedom, the so-called ghosts. Recently a new UV-complete modification of the theory of gravity has been put forward in which this problem does not occur. This theory is called a ghost-free gravity theory [180–184]. This ghost-free theory is also applicable in the problem of singularities in black holes and cosmology [185–189]. If the unfailing fundamental theory is not known then the more naive, phenomenological approach for the description of physics in the regime of high curvature can also be useful. In this approach one can consider that the gravity is still described by a classical metric in this regime where the curvature is high. In view of describing gravity there exists a parameter μ for fundamental energy scale which is related to the fundamental scale length by $l = \mu^{-1}$. The classical Einstein's equations will be modified if the curvature is comparable with l^{-2}. Instead of correcting the field equations, we would impose a number of restrictions on the line element. More precisely we consider: (i) the field equations in the modified theory are approximately similar to Einstein's equations in the domain where the curvature is small i.e. $R \ll l^{-2}$; (ii) the metric functions are regular; (iii) the curvature invariants satisfy the limiting curvature condition, which means that their value is uniformly constrained by some fundamental value, $|R| \leq kl^{-2}$. Here R denotes any type of invariants which can be constructed from curvature tensor and its covariant derivatives and k is the dimensionless constant. This is called the limiting curvature principle which was first given by Markov [190, 191]. Those black hole metrics which satisfy the above conditions are called non-singular black holes. A large number of non-singular black hole models were proposed in Ref. [192]. For spherically symmetric objects, this principle says that the apparent horizon can never cross the origin $r = 0$. In simple words, other then the outer part of the inner horizon there also exists an inner part, separated from $r = 0$. Hence, after the complete evaporation of black hole the event horizon does not exist and the apparent one is closed. Such a model was formulated in Ref. [193], and later was thoroughly studied in Refs. [194–196]. In this thesis, we study physical and thermodynamic properties of these gravitating objects in the presence of nonlinear electrodynamics.

Chapter 2

Non-singular black holes and exponential electrodynamics

Some very interesting solutions of the field equations of Einstein's general theory of relativity have been constructed in the framework of nonlinear electrodynamics. In particular, magnetically charged black hole solutions in the framework of exponential nonlinear electrodynamics have been obtained in general relativity in Kruglov (2017) [145]. Using this approach a magnetically charged non-singular black hole spacetime in the framework of exponential electrodynamics in some modified theory of gravity has been constructed in this chapter. The metric describing asymptotic non-singular magnetized black hole is worked out in terms of the parameter of our model. When this parameter vanishes our solution reduces to the above mentioned black hole solution. Thermodynamics of the resulting solution is also discussed by calculating the Hawking temperature and heat capacity when the magnetic charge is constant. We also find out the point where the first order phase transition induced by temperature changes takes place. The quantum radiations from this black hole are also discussed and the mathematical expression for the rate of energy flux of these radiations has been obtained.

2.1 Asymptotic magnetically charged non-singular black hole solution

The solutions describing black holes in general relativity within the framework of nonlinear electrodynamics have been studied [153–155]. These solutions describe electrically charged black holes which asymptotically approach Reissner-Nordström solution at radial infinity. In order to describe magnetically charged black hole solutions we consider exponential nonlinear electrodynamics [145] for which the Lagrangian density is given by

$$\mathfrak{L}_{exp} = -\mathcal{F}\exp(-\beta\mathcal{F}), \tag{2.1}$$

where $\mathcal{F} = (1/4)\,F_{\mu\nu}F^{\mu\nu} = \left(\mathbf{B}^2 - \mathbf{E}^2\right)/2,\ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Here $F^{\mu\nu}$ is the electromagnetic field tensor, A^μ is the four-potential, $\mathbf{B}$ is the magnetic field, $\mathbf{E}$ is the electric field, and β is the parameter which has dimensions of $(\text{length})^4$ and its upper bound $\left(\beta \leq 1\times 10^{-23}\mathrm{T}^{-2}\right)$ which was found from PVLAS experiment. Now, the Euler-Lagrange equations are

$$\partial_\mu\left(\frac{\partial\mathfrak{L}_{exp}}{\partial\left(\partial_\mu A_v\right)}\right) - \frac{\partial\mathfrak{L}_{exp}}{\partial A_v} = 0, \quad \text{where } \mu, v = 0,1,2,3.$$

Thus the field equations become

$$\partial_\mu\left[\sqrt{-g}\,(\beta\mathcal{F}-1)\exp\left(-\beta\mathcal{F}\right)F^{\mu\nu}\right] = 0. \tag{2.2}$$

The stress-energy tensor is given by

$$\mathcal{T}^{\mu v} = H^{\mu\lambda}F^{v}_{\lambda} - g^{\mu v}\mathfrak{L}_{exp}, \tag{2.3}$$

where $g^{\mu v}$ is the reciprocal metric tensor and the quantity $H^{\mu\lambda}$ is given by

$$H^{\mu\lambda} = \frac{\partial\mathfrak{L}_{exp}}{\partial F_{\mu\lambda}} = -\left(1-\beta\mathcal{F}\right)\exp\left(-\beta\mathcal{F}\right)F^{\mu\lambda}. \tag{2.4}$$

Thus we can find the stress-energy tensor from the Lagrangian density (2.1) as

$$\mathcal{T}^{\mu v} = \exp\left(-\beta\mathcal{F}\right)\left[\left(\beta\mathcal{F}-1\right)F^{\mu\lambda}F^{v}_{\lambda} + g^{\mu v}\mathcal{F}\right], \tag{2.5}$$

from which its trace can be calculated as

$$\mathcal{T} = 4\beta\mathcal{F}^2\exp\left(-\beta\mathcal{F}\right). \tag{2.6}$$

For weak fields, or when $\beta\to 0$, we get the results of classical electrodynamics, i.e., $\mathfrak{L}_{exp}\to -\mathcal{F}$ and the trace of the stress-energy tensor becomes zero. In general, $\beta\neq 0$, and the non-zero trace of the stress-energy tensor means that the scale invariance is violated in the theory. So, any variants of nonlinear electrodynamics with the dimensional parameter give the breaking of scale invariance and so the divergence of dilation current does not vanish, i.e., $\partial_v D^v = \mathcal{T}$ where $D^v = x^\mu\mathcal{T}^v_\mu$.

If the general principles of causality and unitarity hold here then the theory is workable. According to this principle the group velocity of excitations over the background does not exceed the speed of light. This gives the requirement [145, 154, 155] that $\partial\mathfrak{L}_{exp}/\partial\mathcal{F}\leq 0$.

So, from Eq. (2.1) we get to the point that when $\beta\mathcal{F} \leq 1$ the causality principle holds. For the case of pure magnetic field we have the condition $B \leq \sqrt{\frac{2}{\beta}}$. The unitarity principle holds when $\partial\mathfrak{L}_{exp}/\partial\mathcal{F} + 2\mathcal{F}\partial^2\mathfrak{L}_{exp}/\partial\mathcal{F}^2 \leq 0$ and $\partial^2\mathfrak{L}_{exp}/\partial\mathcal{F}^2 \geq 0$, and thus with the help of Eq. (2.1) we get $\beta\mathcal{F} \leq 0.219$ which is the restriction for unitarity principle [145]. So, causality and unitarity both take place when $\beta\mathcal{F} \leq 0.219$ and for purely magnetic field this yields the requirement

$$\mathbf{B} \leq \sqrt{\frac{5-\sqrt{17}}{2\beta}} \simeq \frac{0.66}{\sqrt{\beta}}.$$

Now, we derive the metric which represents the static magnetic non-singular black hole. The invariant $\mathcal{F}$ for pure magnetic field in spherical spacetime is given by

$$\mathcal{F} = \frac{q^2}{2r^4}. \tag{2.7}$$

The most general spherically symmetric spacetime is defined by

$$dS^2 = \sigma^2 ds^2,$$

where

$$ds^2 = -\alpha^2 f dv^2 + 2\alpha dv dr + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{2.8}$$

Here σ is the conformal factor and α and f are functions of radial coordinate r. If we put

$$\alpha = 1, \quad f = 1 - \frac{2Mr^2}{r^3 + 2Ml^2}, \tag{2.9}$$

then Eq. (2.8) represents the Hayward metric [197] which describes uncharged static non-singular black hole and is the solution of some modified theory of gravity. Here, l is an extra parameter which describes the scale where modification of the solution of the Einstein's theory becomes important. The non-singular black hole defined by this line element is also called a sandwich black hole. In the assumption that the mass M of this black hole varies with r, we can write

$$M(r) = \int_0^r \rho(r)r^2 dr = m - \int_r^\infty \rho(r)r^2 dr. \tag{2.10}$$

In the above equation $m = \int_0^\infty \rho(r)r^2 dr$ represents the black hole's magnetic mass. The energy density, in the case of zero electric field, can be written from Eq. (5.61)

$$\rho = \frac{q^2}{2r^4}\exp\left(\frac{-\beta q^2}{2r^4}\right). \tag{2.11}$$

Thus the mass function becomes

$$M(r) = \frac{q^2}{2}\int_0^r \exp\left(\frac{-\beta q^2}{2r^4}\right)\frac{dr}{r^2}. \tag{2.12}$$

Or, using the incomplete gamma function

$$\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}dt, \tag{2.13}$$

this takes the form

$$M(r) = \frac{q^{\frac{3}{2}}\Gamma\left(\frac{1}{4},\frac{\beta q^2}{2r^4}\right)}{2^{\frac{11}{4}}\beta^{\frac{1}{4}}}. \tag{2.14a}$$

The magnetic mass of the black hole is then given by

$$m = M(\infty) = \frac{q^{\frac{3}{2}}\Gamma\left(\frac{1}{4}\right)}{2^{\frac{11}{4}}\beta^{\frac{1}{4}}} \simeq \frac{0.54q^{\frac{3}{2}}}{\beta^{\frac{1}{4}}}. \tag{2.15}$$

Thus the metric function becomes

$$f(r) = 1 - \frac{r^2 q^{\frac{3}{2}}\Gamma\left(\frac{1}{4},\frac{\beta q^2}{2r^4}\right)2^{\frac{-7}{4}}\beta^{\frac{-1}{4}}}{r^3 + l^2 q^{\frac{3}{2}}2^{\frac{-7}{4}}\beta^{\frac{-1}{4}}\Gamma\left(\frac{1}{4},\frac{\beta q^2}{2r^4}\right)}. \tag{2.16}$$

If we put $l = 0$, we obtain the metric function for Einstein's theory [145]. Using the above results we can write the asymptotic value of the metric function in the neighbourhood of radial infinity. For this we use the series expansion

$$\Gamma(s,z) = \Gamma(s) - z^s\left[\frac{1}{s} - \frac{z}{s+1} + \frac{z^2}{2(s+2)} + O\left(z^3\right)\right], \quad z \to 0. \tag{2.17a}$$

Thus the metric function $f(r)$ at $r \to \infty$ takes the following form

$$f(r) = 1 - \frac{r^2\left[2m - \frac{q^2}{r} + \frac{\beta q^4}{20r^5} + O(r^{-9})\right]}{r^3 + l^2\left[2m - \frac{q^2}{r} + \frac{\beta q^4}{20r^5} + O(r^{-9})\right]}. \tag{2.18}$$

In the numerator and denominator if we choose $\beta = 0$ i.e. by neglecting the nonlinear effects of magnetic field we get

$$f(r) = 1 - \frac{\left(2mr - q^2\right)r^2}{r^4 + l^2\left(2mr - q^2\right)}, \tag{2.19}$$

which corresponds to the metric of non-singular charged black hole [197]. In the limit $l \to 0$, a metric similar to the Reissner-Nordström solution is obtained. Further, from

(2.18) we see that

$$f(r) \approx 1 - \frac{2m}{r} + \frac{q^2}{r^2} + \frac{4m^2l^2}{r^4} - \frac{\left(40q^2l^2 + \beta q^4\right)}{r^5}. \tag{2.20}$$

By making corrections to the fourth order terms, a solution similar to the Reissner-Nordström metric is obtained. The limit $r \to \infty$, gives Minkowski spacetime.

We can also write the asymptotic values of the metric function at $r \to 0$. For doing this we will use the series expansion

$$\Gamma(s, z) = \exp(-z) z^s \left[\frac{1}{z} + \frac{s-1}{z^2} + \frac{s^2 - 3s + 2}{z^3} + O(z^{-4})\right], \quad z \to \infty, \tag{2.21}$$

and obtain the expression

$$f(r) \approx 1 + \exp\left(\frac{-\beta q^2}{2r^4}\right)\left[\frac{-l^2}{2\beta} - \frac{r^2}{2\beta} + \frac{3l^2r^4}{4\beta^2q^2} + \frac{3r^6}{4\beta^2q^2}\right]. \tag{2.22}$$

The above result shows that the metric function is finite at the origin. Let us define here a new variable x as a function of the radial coordinate r, by

$$x = \left(\frac{2}{\beta q^2}\right)^{\frac{1}{4}} r, \tag{2.23}$$

so that (2.19) becomes

$$f(x) = 1 - \frac{\Gamma(\frac{1}{4}, \frac{1}{x^4}) q x^2 \beta^{\frac{1}{2}}}{2^{\frac{3}{2}} x^3 \beta + \sqrt{2} l^2 \Gamma(\frac{1}{4}, \frac{1}{x^4})}. \tag{2.24}$$

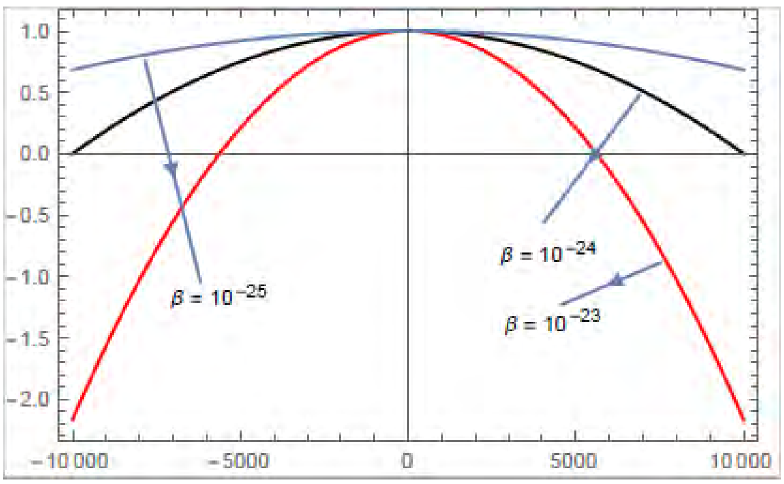

FIGURE 2.1: Three plots of function $f(x)$ for fixed values of $Q = 1$ and $l = 0.01$, with different values of β for each curve.

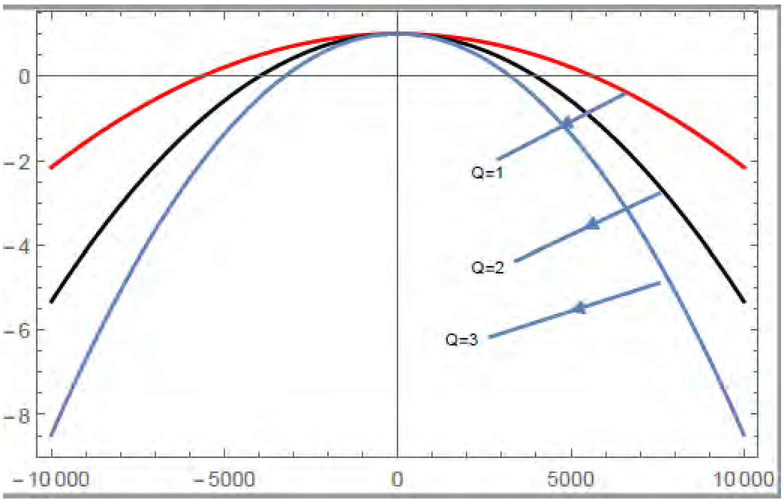

FIGURE 2.2: Three plots of function $f(x)$ for fixed values of $\beta = 1 \times 10^{-23}$ and $l = 0.01$, with different values of Q for each curve.

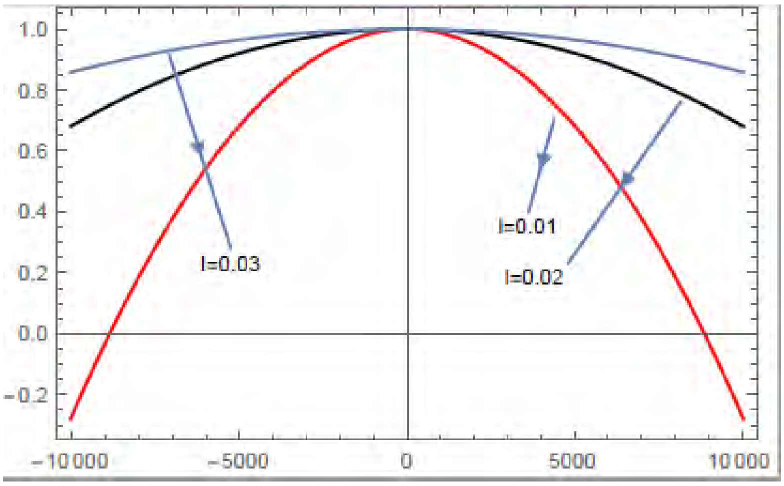

FIGURE 2.3: Three plots of function $f(x)$ for fixed values of $\beta = 1 \times 10^{-25}$ and $Q = 0.01$, with different values of l for each curve.

Clearly it can be seen that apparent horizons can be found by solving equation $f(x) = 0$. The function given by Eq. (2.24) is plotted in Figs. 2.1-2.3 for different values of β, l and $Q = q/\sqrt{2}$. The points where these curves intersect horizontal axes indicate the position of the apparent horizon.

Now, we want to confirm that our metric is asymptotically flat and regular at $r = 0$, i.e., we have indeed a non-singular object under discussion. For this purpose the Ricci scalar is given by

$$R = \frac{d^2 f}{dr^2} + \frac{4}{r}\frac{df}{dr} - 2\frac{f-1}{r^2}. \tag{2.25}$$

From the quadratic invariant $C = C_{\mu\upsilon\alpha\beta}C^{\mu\upsilon\alpha\beta}$, where $C_{\mu\upsilon\alpha\beta}$ is the Weyl tensor, we obtain

$$C = \frac{1}{\sqrt{3}}\left[\frac{d^2 f}{dr^2} - \frac{2}{r}\frac{df}{dr} + 2\frac{f-1}{r^2}\right]. \tag{2.26}$$

Differentiating (2.24) we obtain

$$\frac{df}{dx}=\frac{-q\left[8x^3\beta^{\frac{3}{2}}\exp\left(-\frac{1}{x^4}\right)+2x\beta^{\frac{1}{2}}l^2\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)^2-2x^4\beta^{\frac{3}{2}}\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)\right]}{2^{\frac{5}{2}}x^6\beta^2+\sqrt{2}l^4\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)^2+2^{\frac{5}{2}}l^2\beta x^3\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)}. \tag{2.27}$$

Differentiating again gives

$$\frac{d^2f}{dx^2}=-q\sqrt{\frac{\beta}{2}}\frac{H\left(x\right)+L\left(x\right)+N\left(x\right)}{S\left(x\right)+W\left(x\right)}, \tag{2.28}$$

where $H,\ L,\ N,\ S$ and W are given by

$$H\left(x\right)=16\beta^3x^9\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)-128\beta^3x^8\exp\left(-\frac{1}{x^4}\right)-48l^2\beta^2x^6\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)^2, \tag{2.29}$$

$$L\left(x\right)=2l^6\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)^4-64l^4\beta x\exp\left(-\frac{2}{x^4}\right)\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)+64l^4\beta x^2\exp\left(-\frac{1}{x^4}\right)\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)^2, \tag{2.30}$$

$$N\left(x\right)=64l^2\beta^2x^5\exp\left(-\frac{1}{x^4}\right)\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)-128l^2\beta^2x^4\exp\left(-\frac{2}{x^4}\right)-24l^4\beta x^3\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)^3, \tag{2.31}$$

$$S\left(x\right)=16\beta^4x^{12}+l^8\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)^4+24l^4\beta^2x^6\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)^2, \tag{2.32}$$

$$W\left(x\right)=32l^2\beta^3x^9\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)+8l^6\beta x^3\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)^3. \tag{2.33}$$

Using the expansion

$$\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)=\Gamma\left(\frac{1}{4}\right)-\frac{1}{x}\left[4-\frac{4}{5x^5}+O\left(x^{-8}\right)\right],\quad x\to\infty, \tag{2.34}$$

in the expression of Ricci scalar we note that

$$\lim_{r\to\infty}R(r)=0. \tag{2.35}$$

Similarly

$$\lim_{r\to\infty}C(r)=0. \tag{2.36}$$

By using the series expansion

$$\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)=\exp\left(-\frac{1}{x^4}\right)\left[x^3-\frac{3}{4}x^7+O\left(x^{11}\right)\right],\quad x\to 0, \tag{2.37}$$

we conclude that

$$\lim_{r\to 0}R(r)=0, \tag{2.38}$$

and

$$\lim_{r\to 0} C(r) = 0. \tag{2.39}$$

This clearly shows that the scalar curvature has no singularities. The spacetime becomes Minkowski at $r \to \infty$, while at the origin $r = 0$, finite curvature suggests that the black hole under consideration is regular. The curvature invariants are regular at $r = 0$, which indicates that our solution of some modified theory of gravity describes a non-singular black hole with exponential pure magnetic source. This is in contrast to Einstein's theory in the framework of exponential electrodynamics [145] where the Kretschmann scalar is singular at $r = 0$. It is worth mentioning that as the charged generalization of the Hayward metric in Maxwell's electrodynamics is non-singular [197], our solution in nonlinear electrodynamics also describes a non-singular black hole, where the curvature of the spacetime is finite everywhere [198].

2.2 Thermodynamics of magnetically charged non-singular black hole

Here we will investigate the thermal stability of magnetized non-singular black hole by working out the Hawking temperature and its heat capacity. The black hole is unstable where the temperature becomes negative. The Hawking temperature is described by the relation [199–201]

$$T_H = \frac{\kappa}{2\pi}, \tag{2.40}$$

where κ defines the surface gravity which is given by

$$\kappa = \frac{1}{2}\frac{df}{dr}\bigg|_H. \tag{2.41}$$

Thus if r_1 and r_2 are the inner and outer apparent horizons, respectively, then for the inner horizon the surface gravity is

$$\kappa_1 = \frac{1}{2}\frac{df}{dr}\bigg|_{r_1} = \frac{1}{2}\;\frac{df}{dx}\frac{dx}{dr}\bigg|_{x_1}. \tag{2.42}$$

By using the relation (2.27) in the above we get

$$\kappa_1 = \frac{\sqrt{q}\beta^{\frac{1}{4}}}{2^{\frac{5}{4}}}\frac{\left[2x_1^4\beta\Gamma\left(\frac{1}{4},\frac{1}{x_1^4}\right) - 8x_1^3\beta\exp\left(\frac{-1}{x_1^4}\right) - 2x_1 l^2\Gamma\left(\frac{1}{4},\frac{1}{x_1^4}\right)^2\right]}{4x_1^6\beta^2 + l^4\Gamma\left(\frac{1}{4},\frac{1}{x_1^4}\right)^2 + 4l^2\beta x_1^3\Gamma\left(\frac{1}{4},\frac{1}{x_1^4}\right)}. \tag{2.43}$$

Similarly, for the outer horizon the surface gravity is given by

$$\kappa_2 = \frac{\sqrt{q}\beta^{\frac{1}{4}}}{2^{\frac{5}{4}}} \frac{\left[2x_2^4\beta\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right) - 8x_2^3\beta\exp\left(\frac{-1}{x_2^4}\right) - 2x_2l^2\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)^2\right]}{4x_2^6\beta^2 + l^4\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)^2 + 4l^2\beta x_2^3\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)}, \tag{2.44}$$

so that the expression of Hawking temperature yields the result

$$T_H = \frac{\sqrt{q}\beta^{\frac{1}{4}}}{2^{\frac{5}{4}}\pi} \frac{\left[x_2^4\beta\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right) - 4x_2^3\beta\exp\left(\frac{-1}{x_2^4}\right) - x_2l^2\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)^2\right]}{4x_2^6\beta^2 + l^4\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)^2 + 4l^2\beta x_2^3\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)}. \tag{2.45}$$

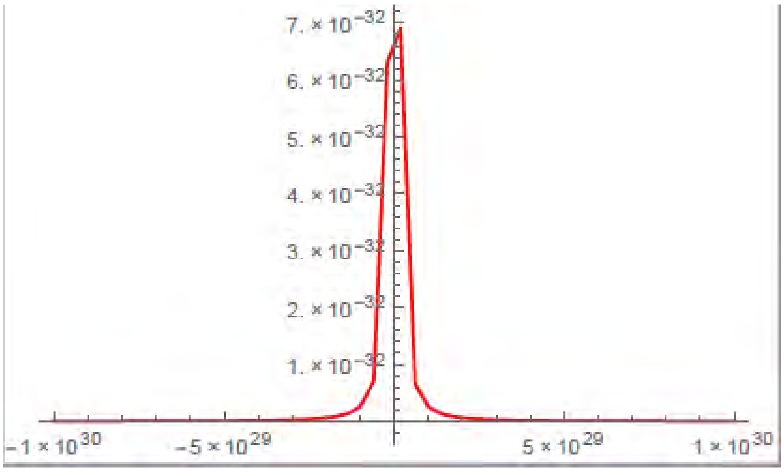

FIGURE 2.4: The curve of Hawking temperature for fixed $\beta = 1 \times 10^{-26}$ and $l = 0.02$.

This reduces to the result for the black hole solution of Einstein's theory with exponential magnetic source [145], if we put $l = 0$. From Fig. 2.4 it is clear that the first order phase transition of black hole occurs at $\pm 5 \times 10^{29}$ because the Hawking temperature is zero there. Hawking temperature gives the maximum value as $x_2 \to 0$. The entropy of the black hole is described by Hawking area law i.e. $S = A_h/4 = \pi r_2^2$. Then heat capacity is defined for constant charge as

$$C_H = T_H \frac{\partial S}{\partial T_H}\bigg|_q = T_H \frac{\partial S/\partial r_2}{\partial T_H/\partial r_2} = \frac{2\pi r_2 T_H}{\partial T_H/\partial r_2}. \tag{2.46}$$

Thus with the help of Eq. (2.45) we get the formula for heat capacity in the form

$$C_H = \frac{8\pi x_2 q\sqrt{\beta}\left[A(x_2) + B(x_2) + \beta x_2^3\exp\left(\frac{-1}{x_2^4}\right)C(x_2)\right]}{2^{\frac{3}{2}}D(x_2)\left[E(x_2) + 64\beta x_2\exp\left(\frac{-1}{x_2^4}\right)F(x_2)\right]}, \tag{2.47}$$

where the functions A, B, C, D, E and F are given by

$$\begin{aligned} A\left(x_2\right) &= 16l^2\beta^4x_2^{13}\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)\left[\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)-2\right]+8l^6\beta^2x_2^7\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)^3\left[3\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)-1\right] \\ &+l^8\beta x_2^4\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)^4\left[8\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)-1\right], \end{aligned} \tag{2.48}$$

$$B\left(x_2\right)=l^{10}x_2\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)^6-16\beta^5x_2^{16}+32l^4\beta^3x_2^{10}\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)^2\left[\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)-1\right], \tag{2.49}$$

$$\begin{aligned} C\left(x_2\right) &= 64\beta^4x_2^{12}+4l^8\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)^4+96l^4\beta^2x_2^6\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)^2+128l^2\beta^3x_2^9\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right) \\ &+32\beta l^6x_2^3\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)^3, \end{aligned} \tag{2.50}$$

$$D\left(x_2\right)=4x_2^6\beta^2+l^4\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)^2+4l^2\beta x_2^3\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right), \tag{2.51}$$

$$E\left(x_2\right)=2l^6\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)^4+16\beta^3x_2^9\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)-48l^2\beta^2x_2^6-24\beta l^4x_2^3\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)^3, \tag{2.52}$$

$$\begin{aligned} F\left(x_2\right) &= -2\beta^2x_2^7+l^2\beta x_2^4\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)-2l^2\beta x_2^3\exp\left(\frac{-1}{x_2^4}\right)-l^4x_2\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right)^2 \\ &-l^4\exp\left(\frac{-1}{x_2^4}\right)\Gamma\left(\frac{1}{4},\frac{1}{x_2^4}\right). \end{aligned} \tag{2.53}$$

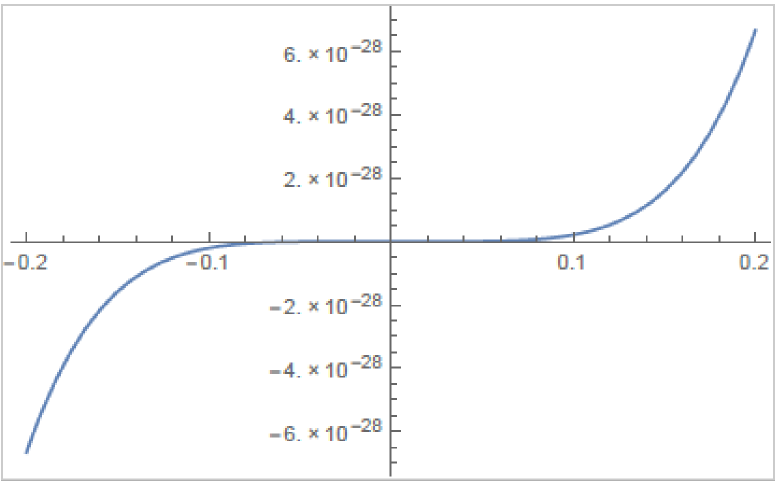

FIGURE 2.5: The graph representing heat capacity for fixed values of β and l.

For fixed values of $\beta = 1\times10^{-26}$ and $l = 0.02$, the graph in Fig. 2.5 of $2^{\frac{3}{2}}C_H/8\pi q\sqrt{\beta}$ vs x_2 shows that heat capacity is singular in the interval $(0, 0.05969)$, which says that second order phase transition occurs in this interval. Those values of x_2 at which the temperature and heat capacity are negative indicate the black hole is unstable. Fig. 5 shows that heat capacity diverges as T_H increases.

2.3 Quantum radiations from magnetically charged non-singular black hole

When a black hole is formed, it emits quantum radiation. An observer outside the black hole can register outgoing Hawking radiation from the black hole, due to which its mass decreases and it shrinks in size. One possibility is that the black hole disappears completely as a result of evaporation. For a spherically symmetric spacetime of non-singular black hole, this implies that the apparent horizon is closed, and there is no event horizon. Thus, according to the usual definition this object, in fact, is not a black hole, but its long-time analogue. However, we are using the same term black hole for these objects too. The most important property of such type of non-singular black holes is that not only Hawking radiation is emitted but in addition to that quantum radiation also comes from the interior spacetime [197]. We should expect that after the complete evaporation of this object, the total energy loss by it will be equal to its initial mass. Since, in this letter we coupled the non-singular black hole solution with the model of exponential nonlinear electrodynamics, so we consider the model

$$ds^2 = -fdv^2 + 2dvdr + r^2 d\omega^2, \tag{2.54}$$

where the function $f(r)$ is given by

$$f(r) = 1 - \frac{r^2 q^{\frac{3}{2}} \Gamma\left(\frac{1}{4}, \frac{\beta q^2}{2r^4}\right)}{2^{\frac{7}{4}} \beta^{\frac{1}{4}} r^3 + l^2 q^{\frac{3}{2}} \Gamma\left(\frac{1}{4}, \frac{\beta q^2}{2r^4}\right)}. \tag{2.55}$$

This function depends on r for some real interval $0 < v < p$, while the function is equal to unity outside this interval i.e. the spacetime becomes flat. Here also our assumption is that this black hole is formed as a result of collapse of the spherical null shell which has mass M [197]. This black hole exists for some time ΔV, and after that it completely disappears due to the collapse of some other shell which has mass $-M$. It is possible to find the gain function for such type of black hole whose interior spacetime is static. Let us assume that an incoming radial photon whose initial energy is E_1, reaches the first shell at distance r_-. It moves through the interior spacetime between the shells and after crossing the second shell at distance r_+, leaves the black hole with energy E_2. We call such a photon of radial type I, and r_-, r_+ are the points of entrance and exit of photon. Then the gain function is given by

$$\chi = \frac{f_-}{f_+} = \frac{dr_-}{dr_+}, \tag{2.56}$$

where $f_\pm$ are the quantity f evaluated at $r_\pm$. For the photons which propagate along the horizon $r_+ = r_-$, one gets $f_- = f_+ = 0$. Therefore

$$\chi_H = \exp\left(-\kappa_H p\right). \tag{2.57}$$

Now, consider type II, a beam of incoming radial photons having energy E_1. The radial type II photons cross the first shell in the interval $\left(r_-, r_- + \Delta r_-\right)$, and then the second shell between the interval $\left(r_+, r_+ + \Delta r_+\right)$ having energy E_2. Then

$$E_- \Delta r_- = E_- \frac{dr_-}{dr_+} \Delta r_+ = E_- \frac{f_-}{f_+} \Delta r_+ = E_+ \Delta r_+. \tag{2.58}$$

The rate of energy flux is then given by

$$\xi_\pm = \frac{E_\pm}{\Delta r_\pm}, \tag{2.59}$$

so that we get

$$\xi_+ = \chi^2 \xi_-. \tag{2.60}$$

Since the radial type II photon starts motion in the interval $(0, p)$, the gain function can be obtained as

$$\varkappa = \exp\left[\frac{pq^{\frac{1}{2}}}{2^{\frac{3}{4}}\beta^{\frac{1}{4}}}\left(\frac{8x^3\beta^{\frac{3}{2}}\exp\left(-\frac{1}{x^4}\right) + 2x\beta^{\frac{1}{2}}l^2\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)^2 - 2x^4\beta^{\frac{3}{2}}\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)}{2^{\frac{5}{2}}x^6\beta^2 + \sqrt{2}l^4\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)^2 + 2^{\frac{5}{2}}l^2\beta x^3\Gamma\left(\frac{1}{4},\frac{1}{x^4}\right)}\right)\right], \tag{2.61}$$

where we use (2.23). For the double shell model, having a constant metric in the interior, the gain function is given by

$$\chi = \frac{1}{f_+}, \tag{2.62}$$

and $f\ (r)$ is given by Eq. (2.55). For type III null rays, that is, those rays which are outside the interval $(0, p)$ the gain function is equal to 1.

The quantum radiation from magnetized non-singular black hole can be estimated with the help of a result from Ref. [202], where conformal anomaly is used to work out two-dimensional quantum average of the stress-energy tensor. Now, massless particles are created from the initial vacuum state. So for type I rays, the rate of energy flux of these massless particles is given by the following expression

$$\xi = \frac{1}{192\pi}\left[-2\frac{d^2F}{dr_+^2} + \left(\frac{dF}{dr_+}\right)^2\right], \tag{2.63}$$

where $F = \ln|\chi|$. Using (2.56) this takes the form

$$\xi = \frac{1}{192\pi} \frac{G(r_-) - G(r_+)}{f^2(r_+)}, \quad r_- = r_-(r_+), \tag{2.64}$$

where the function G is introduced as

$$G(r) = -2f(r)\frac{d^2 f}{dr^2} + \left(\frac{df}{dr}\right)^2. \tag{2.65}$$

Again using Eq. (2.23) we get the result

$$\xi = \left(\frac{2}{\beta q^2}\right)^{\frac{1}{2}} \frac{1}{192\pi f^2(x_+)} \left[-2f(x_-)\frac{d^2 f}{dx_-^2} + \left(\frac{df}{dx_-}\right)^2 + 2f(x_+)\frac{d^2 f}{dx_+^2} - \left(\frac{df}{dx_+}\right)^2\right]. \tag{2.66}$$

Here $f(x)$ and its first and second order derivatives are given by Eqs. (2.24), (2.27) and (2.28), respectively. The values x_- and x_+ are related to r_- and r_+ where the photon crosses the first and second shells.

For type II rays, i.e., when $0 < v < p$, the corresponding outgoing null ray intersects the second shell only, with negative mass, and one has $f(r_-) = 1$, so that the expression for the energy flux becomes

$$\xi = \frac{1}{192\pi}\left(\frac{2}{\beta q^2}\right)^{\frac{1}{2}} \frac{2f(x_+)\frac{d^2 f}{dx_+^2} - \left(\frac{df}{dx_+}\right)^2}{f^2(x_+)}. \tag{2.67}$$

For type III rays we get $\xi = 0$, because in that region there are no quantum radiations.

Chapter 3

$(2+1)$-dimensional black holes of Einstein's theory with Born-Infeld type electrodynamic sources

A new family of $(2+1)$-dimensional black holes are investigated in the background of Born-Infeld type theories coupled to Riemannian curved spacetime. We know that both the scale and dual invariances are violated for these nonlinear electromagnetic theories. In this chapter, first we consider a pure magnetic source in a model of exponential electrodynamics and find a magnetically charged $(2+1)$-dimensional black hole solution in terms of magnetic charge q and nonlinearity parameter β. In the second case we consider a pure electric source of gravity in the framework of arcsin electrodynamics and derive the associated $(2+1)$-dimensional black hole solution in terms of electric charge Q and the parameter β. The asymptotic behaviour of the solutions at infinity as well as at $r \to 0$ in both the frameworks is discussed. The asymptotic expressions of curvature invariants in the case of exponential electrodynamics shows that there exists a finite value of curvature at the origin, while in arcsin electrodynamics, the corresponding asymptotic behaviour shows that there is a true curvature singularity at the centre of the charged object. Furthermore, thermodynamics of the resulting charged black holes within the context of both the models is studied. It is shown that the thermodynamic quantities corresponding to these objects satisfy the first law of black hole thermodynamics.

3.1 $(2+1)$-dimensional black holes in exponential electrodynamics

The action function describing $(2+1)$-dimensional Einstein's gravity in the presence of NED is given by [203]

$$\mathcal{I} = \int d^3x\sqrt{-g}\Bigg[\frac{1}{2\pi}\left(R - \frac{2}{3}\Lambda\right) + \mathfrak{L}_{NED}\Bigg]. \tag{3.1}$$

where the Lagrangian density of NED is taken as Eq. (2.1). Variation of action (3.1) with respect to the four potential A_μ gives the equations of motion corresponding to NED as

$$\partial_\mu\Bigg[\sqrt{-g}F^{\mu\nu}\big(1-\beta\mathcal{F}\big)\exp\left(-\beta\mathcal{F}\right)\Bigg] = 0. \tag{3.2}$$

Similarly, by varying (3.1) with respect to the metric tensor $g_{\mu\nu}$, we can obtain gravitational field equations as

$$R^\nu_\mu - \frac{1}{2}\delta^\nu_\mu R + \frac{1}{3}\Lambda\delta^\nu_\mu = \pi\mathcal{T}^\nu_\mu, \tag{3.3}$$

where R^ν_μ is the Ricci tensor, R is the Ricci scalar, Λ is the cosmological constant and $\mathcal{T}^{\nu\mu}$ represents the matter tensor of NED given by

$$\mathcal{T}^{\mu\nu} = \exp\left(-\beta\mathcal{F}\right)\Bigg[(\beta\mathcal{F}-1)F^{\mu\lambda}F^\nu_\lambda + \mathcal{F}g^{\mu\nu}\Bigg]. \tag{3.4}$$

The trace of the above matter tensor is

$$\mathcal{T} = -4\beta\mathcal{F}^2\exp\left(-\beta\mathcal{F}\right), \tag{3.5}$$

which clearly shows the breaking of conformal invariance in this theory. However, by applying the limit $\beta \to 0$, this trace vanishes which implies that Maxwell's theory can be recovered in this limit.

In order to derive the circularly symmetric magnetically charged $(2+1)$-black hole solution, first we choose the pure magnetic field such that $\mathbf{E} = 0$. This implies that Maxwell's invariant would be equal to $\mathcal{F} = \mathbf{B}^2/2 = q^2/2r^4$ [145], where q represents the magnetic charge. Now, the line element ansatz in $(2+1)$-dimensional spacetime can be taken as

$$ds^2 = -\psi(r)dt^2 + \frac{dr^2}{\psi(r)} + r^2d\theta^2. \tag{3.6}$$

Using the Lagrangian density of exponential NED, the components of matter tensor (3.4) can be calculated as follows:

$$\mathcal{T}_0^0 = \frac{q^2}{2r^4} \exp\left(- \frac{\beta q^2}{2r^4}\right), \tag{3.7}$$

$$\mathcal{T}_1^1 = \mathcal{T}_2^2 = \exp\left(- \frac{\beta q^2}{2r^4}\right) \left[\frac{q^2}{r^2 \psi(r)} \left(\frac{q^2 \beta}{2r^4} - 1\right) + \frac{q^2}{2r^4}\right]. \tag{3.8}$$

From the line element (3.6), the 00-component of Einstein's equations (3.3) gives

$$\frac{1}{2r}\frac{d\psi}{dr} + \frac{\Lambda}{3} = \frac{\pi q^2}{2r^4} \exp\left(- \frac{\beta q^2}{2r^4}\right). \tag{3.9}$$

Solving the above differential equation yields

$$\psi(r) = D + \frac{r^2}{3l^2} - \frac{q\pi^{\frac{3}{2}}}{\sqrt{\beta} 2^{\frac{3}{2}}} E\left(\frac{q\sqrt{\beta}}{\sqrt{2} r^2}\right), \tag{3.10}$$

where we choose $\Lambda = -1/l^2$, l being a real parameter and $E(x)$ is the error function. The constant D can be related to mass of the gravitating object at infinity by using the Brown-York formalism [145, 204, 205]. With the use of quasilocal mass formulation, a static circularly symmetric $(2+1)$-dimensional line element takes the form [205]

$$ds^2 = -f(r)^2 dt^2 + \frac{dr^2}{g(r)^2} + r^2 d\theta^2. \tag{3.11}$$

This yields the quasilocal mass M_{QL} in the form

$$M_{QL} = \lim_{r_b \to \infty} 2f(r_b)[g_r(r_b) - g(r_b)]. \tag{3.12}$$

In the above expression, $g_r(r_b)$ stands for a non-negative arbitrary reference function which allows the zeros of the energy in the spacetime. The value r_b denotes the radius of some spacelike hypersurface. From the form of our line element (3.6), it can be clearly seen that

$$f(r) = g(r) = \sqrt{D + \frac{r^2}{3l^2} - \frac{q\pi^{\frac{3}{2}}}{\sqrt{\beta} 2^{\frac{3}{2}}} E\left(\frac{q\sqrt{\beta}}{\sqrt{2} r^2}\right)}, \tag{3.13}$$

and

$$g_r(r) = \sqrt{\frac{r^2}{3l^2} - \frac{q\pi^{\frac{3}{2}}}{\sqrt{\beta} 2^{\frac{3}{2}}} E\left(\frac{q\sqrt{\beta}}{\sqrt{2} r^2}\right)}. \tag{3.14}$$

Thus, by substituting the above Eqs. (3.13)-(3.14) in Eq. (3.12) the metric function (3.10) for choosing $M > 0$ becomes [206]

$$\psi(r) = -M + \frac{r^2}{3l^2} - \frac{q\pi^{\frac{3}{2}}}{\sqrt{\beta}2^{\frac{3}{2}}} E\bigg(\frac{q\sqrt{\beta}}{\sqrt{2}r^2}\bigg). \tag{3.15}$$

The asymptotic expression of the above metric function in the vicinity of infinity can be obtained as

$$\psi(r) = -M + \frac{r^2}{3l^2} - \frac{\pi q^2}{2r^2} + \frac{\pi\beta q^4}{12r^6} + O(r^{-10}). \tag{3.16}$$

This shows that at infinity the spacetime is not Minkowskian and the solution (3.15) reduces to that of Einstein-Maxwell theory when $\beta \to 0$. Similarly the asymptotic value of the metric function when $r \to 0$ is calculated as

$$\psi(r) = -M + \frac{r^2}{3l^2} - \frac{q\pi^{\frac{3}{2}}}{\sqrt{\beta}2^{\frac{3}{2}}} + \exp\bigg(-\frac{\beta q^2}{2r^4}\bigg)\bigg[\frac{\pi r^2}{2\beta} - \frac{\pi r^6}{2\beta^2 q^2} + O(r^{10})\bigg]. \tag{3.17}$$

Note that in calculating the above expansion we used the relation

$$E(x) = 1 - \frac{1}{\sqrt{\pi}}\Gamma(1/2, x^2), \tag{3.18}$$

where $\Gamma(s,x)$ refers to the incomplete Gamma function. The asymptotic expansion (3.17) shows that the metric function is finite at the centre $r = 0$. This behaviour of metric functions is a consequence of the nonlinear electromagnetic nature of Lagrangian density (2.1). It is worth-noting that the limit $\beta \to 0$ cannot be taken for (3.17) and hence the apparent singularity in this case is fake.

The weak energy condition (WEC) would be satisfied if and only if the energy density $\rho = \mathcal{T}_0^0$ and principle pressures $p_m = -\mathcal{T}_m^m$ (there is no summation in the index m) satisfy

$$\rho \geq 0, \;\; \rho + p_m \geq 0, \;\; m = 1, 2. \tag{3.19}$$

The satisfaction of WEC ensures that any local observer will measure the non-negative energy density. It is also worthwhile to note that the above WEC is equivalent to the $\mathcal{T}_{\mu\nu}\zeta^\mu\zeta^\nu$ for any timelike vector ζ^μ [145, 155, 207]. Thus, from Eqs.(3.8) and (3.19), one can easily verify that WEC holds for any value of magnetic field $\mathbf{B}$ such that $\beta\mathbf{B}^2 \leq 2$. Validity of the dominant energy condition (DEC) [145, 155, 207] is governed by the conditions

$$\rho \geq 0, \;\; \rho + p_m \geq 0, \;\; \rho - p_m \geq 0, \;\; m = 1, 2. \tag{3.20}$$

Again from Eq. (3.8) one can prove that DEC also holds if $\beta\mathcal{F} \leq 1$ for the Lagrangian density (2.1). This shows that speed of sound would be always less than the speed of light. Similarly, validity of the strong energy condition (SEC) [145, 155, 207] is given by

$$\rho + p_1 + p_2 \geq 0. \tag{3.21}$$

Using the stress-energy tensor components one can conclude that SEC holds for this model of electrodynamics if the exponential magnetic field satisfies the inequality

$$\mathbf{B} \leq \sqrt{\frac{2}{\beta} - \frac{\psi(r)}{2r^2\beta}}. \tag{3.22}$$

The satisfaction of SEC implies that there is no acceleration of the universe in this model of nonlinear electromagnetic field coupled to the gravitational field [145, 155].

Now, we discuss the nature of singularity of our resulting solution described by Eqs. (3.6) and (3.15), for which we will calculate the curvature invariants. It is possible to find out the Ricci scalar from Einstein's equations describing gravitational field i.e. Eq. (3.3) so that

$$R = -\frac{2}{l^2} + \frac{2\pi q^4\beta}{r^4}\exp\left(-\frac{\beta q^2}{2r^4}\right). \tag{3.23}$$

This expression of the Ricci scalar implies that it is defined and finite at $r = 0$. The finiteness of Ricci scalar shows that at any point say r_0 for which $R(r_0) = 0$, one can find the change in curvature around such a point. So one may encounter a transition from negative curvature to positive. One can also find the Kretschmann scalar $\mathcal{K}$ for the metric (3.6) in the form

$$\mathcal{K} = \frac{2}{r^2}\left(\frac{d\psi}{dr}\right)^2 + \left(\frac{d^2\psi}{dr^2}\right)^2. \tag{3.24}$$

Now from Eq. (3.16) we have

$$\frac{d\psi}{dr} = \frac{2r}{3l^2} + \frac{q^2\pi}{r^3}\exp\left(-\frac{\beta q^2}{2r^4}\right), \tag{3.25}$$

and

$$\frac{d^2\psi}{dr^2} = \frac{2}{3l^2} - \frac{3q^2\pi}{r^4}\exp\left(-\frac{\beta q^2}{2r^4}\right) - \frac{2q^4\beta\pi}{r^8}\exp\left(-\frac{\beta q^2}{2r^4}\right). \tag{3.26}$$

Hence, the asymptotic expansion of Kretschmann scalar at $r \to \infty$ is given by

$$\mathcal{K} = \frac{4}{3l^4} - \frac{4\pi q^2}{3l^2r^4} + \frac{(11\pi^2q^4l^2 - 2\pi q^4\beta)}{l^2r^8} + O(r^{-11}). \tag{3.27}$$

The expression of the Ricci scalar (3.23) and the above expansion of Kretschmann scalar show that the spacetime is not asymptotically flat since $\lim_{r\to\infty} K(r) = 4/3l^4$. The asymptotic expression of Kretschmann scalar in the vicinity of $r = 0$ is given by

$$\begin{aligned}\mathcal{K} &= \frac{4}{3l^4} - \exp\left(\frac{-q^2\beta}{2r^4}\right)\left(\frac{4\pi q^2}{3l^2r^4} + \frac{8\pi\beta q^4}{3l^2r^8} + O(r^{11})\right) \\ &+ \exp\left(\frac{-q^2\beta}{r^4}\right)\left(\frac{4\pi^2q^8\beta^2}{r^{16}} + \frac{12\pi^2q^6\beta}{r^{12}} + \frac{11\pi^2q^4}{r^8} + O(r^3)\right),\end{aligned} \tag{3.28}$$

which is regular at the origin. Thus our resulting (2+1)-dimensional black hole solutions within exponential electrodynamics, i.e., Eq. (3.15) describe a family of regular black holes.

The event horizons can be obtained from the condition $\psi(r) = 0$, which implies that

$$M = \frac{r_h^2}{3l^2} - \frac{q\pi^{\frac{3}{2}}}{2\sqrt{2\beta}} E\left(\frac{q\sqrt{\beta}}{\sqrt{2}r_h^2}\right). \tag{3.29}$$

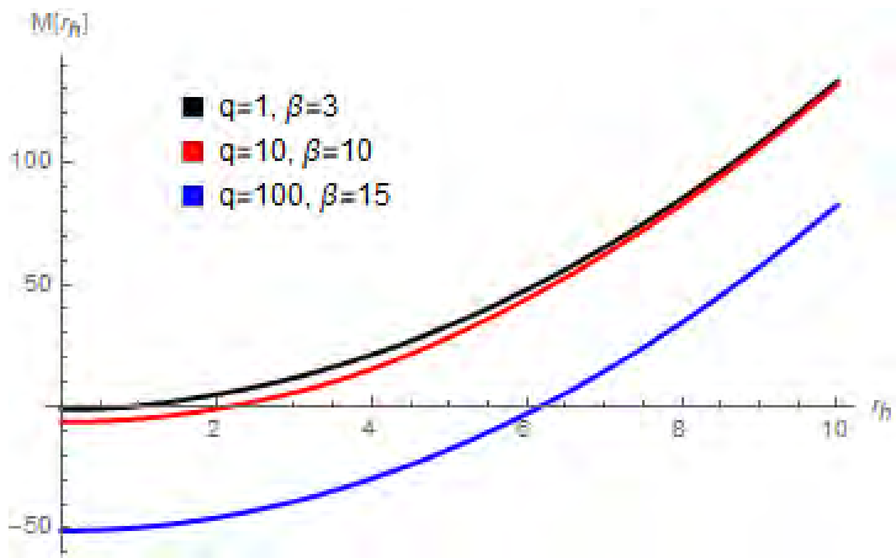

FIGURE 3.1: Plot of function M from Eq. (3.29) for fixed value of $l = 0.5$ and different values of q and β.

Fig. 3.1 shows clearly that there exists a critical value r_c below which no event horizon corresponding to the value of M can exist as mass becomes negative. However, for all values of r_h greater than the critical value, there exist event horizons associated with the positive value of mass M. Fig. 3.2 shows the plot of metric function (3.15) for different values of mass, magnetic charge and nonlinearity parameter. It can be clearly seen that the magnetic charge and parameter β affect the horizon structure of (2+1)-dimensional black hole.

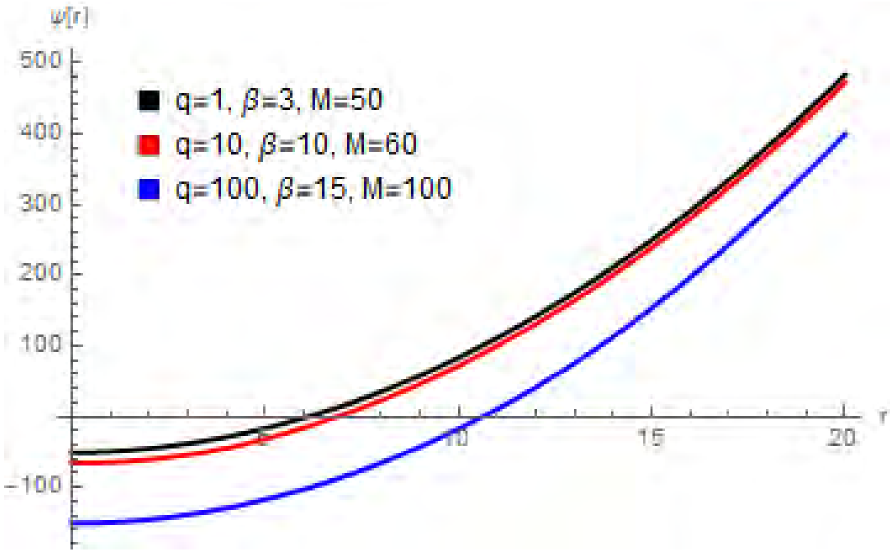

FIGURE 3.2: Plot of function $\psi(r)$ from Eq. (3.15) for fixed values of $l = 0.5$ and three different values of M, q and β.

3.2 Thermodynamics of magnetically charged black holes

Here we study thermodynamics of the black hole solutions described by Eq. (3.15). For doing this we use an alternative method to determine the local Hawking temperature with the help of the Unruh effect in curved spacetime [199, 208, 209]. A similar analysis was also introduced in Ref. [210] for d-dimensional black holes in the framework of Einstein-power-Maxwell theory. In the Unruh effect the observer in the spacetime exterior to the black hole observes a thermal state with local temperature given by

$$T_H(r) = \frac{2}{\pi\sqrt{-X^a X_a}}\frac{d\psi(r_h)}{dr_h}, \tag{3.30}$$

which on using Eq. (3.15) becomes

$$T_H(r) = \frac{2}{\pi\sqrt{\psi(r)}}\left[\frac{2r_h}{3l^2} + \frac{q^2\pi}{r_h^3}\exp\left(-\frac{\beta q^2}{2r_h^4}\right)\right]. \tag{3.31}$$

Here X^a represents a Killing vector field that generates the outer horizon r_h. This expression of local Hawking temperature implies that at $r \to r_h$, $T_H(r)$ is undefined and at $r \to \infty$, it is zero. This behaviour of temperature is expected since our resulting black hole solution (3.15) is not asymptotically flat, thus the local Hawking temperature vanishes at infinity. By using the method described in Refs. [203, 209], the re-energized temperature can be obtained as

$$T_\infty = \frac{2}{\pi}\left[\frac{2r_h}{3l^2} + \frac{q^2\pi}{r_h^3}\exp\left(-\frac{\beta q^2}{2r_h^4}\right)\right]. \tag{3.32}$$

By using the Brown-York quasilocal energy formalism [204, 205], we can express the internal energy of this system on a constant t hypersurface as

$$\Xi = -2\left(\sqrt{\psi(r_b)} - \frac{r_b}{\sqrt{3}l}\right), \tag{3.33}$$

where $r = r_b$ is a finite boundary of the black hole spacetime. By varying the internal energy $\Xi(r_b)$ with respect to r_b and q, one can easily arrive at the first law

$$d\Xi(r_b) = T_H(r_b)dS + \Psi(r_b)dq, \tag{3.34}$$

where

$$T_H(r_b) = \frac{-1}{\sqrt{\psi(r_b)}}\left[\frac{r_h}{6\pi l^2} + \frac{q^2}{4r_h^3}\exp\left(-\frac{\beta q^2}{2r_h^4}\right)\right], \tag{3.35}$$

is the Hawking temperature evaluated at r_b, and $S = 4\pi r_h$ in Eq. (3.34) is the entropy of the $(2+1)$-dimensional black hole. The function

$$\Psi(r_b) = \frac{1}{\sqrt{\psi(r_b)}}\left[\frac{\pi q}{2r_b^2}\exp\left(-\frac{\beta q^2}{2r_b^4}\right) + \frac{\pi\sqrt{\pi}}{2\sqrt{2\beta}}E\left(\frac{q\sqrt{\beta}}{\sqrt{2}r_b^2}\right)\right], \tag{3.36}$$

is the magnetic potential difference between the boundary of the black hole and at infinity. Similarly, the difference between the potential at the event horizon and boundary r_b can also be calculated as

$$\begin{aligned}\Phi(r_b) = \frac{\pi}{2\sqrt{\psi(r_b)}}\Bigg[&\frac{q}{r_b^2}\exp\left(-\frac{\beta q^2}{2r_b^4}\right) + \frac{\sqrt{\pi}}{\sqrt{2\beta}}E\left(\frac{q\sqrt{\beta}}{\sqrt{2}r_b^2}\right)\\ &- \frac{q}{r_h^2}\exp\left(-\frac{\beta q^2}{2r_h^4}\right) - \frac{\sqrt{\pi}}{\sqrt{2\beta}}E\left(\frac{q\sqrt{\beta}}{\sqrt{2}r_h^2}\right)\Bigg].\end{aligned} \tag{3.37}$$

Now, we compute the expression of heat capacity at a constant magnetic charge q of our magnetically charged black holes which are considered inside a box and bounded by $r = r_b$. The heat capacity is defined by

$$C_H = T_H\frac{\partial S}{\partial T_H}|_q. \tag{3.38}$$

Therefore, by using Eq. (3.35) in this we get

$$C_H = \frac{\pi r_h^5\left(8r_h^4 + 12l^2q^2\pi\exp\left(-\frac{\beta q^2}{2r_h^4}\right)\right)}{2r_h^8 + 6\pi q^4\beta l^2\exp\left(-\frac{\beta q^2}{2r_h^4}\right) - 9\pi q^2l^2r_h^4\exp\left(-\frac{\beta q^2}{2r_h^4}\right)}. \tag{3.39}$$

This represents the general expression for black hole's heat capacity for any value of nonlinear electrodynamics parameter β. Fig. 3.3 shows the plot of heat capacity for

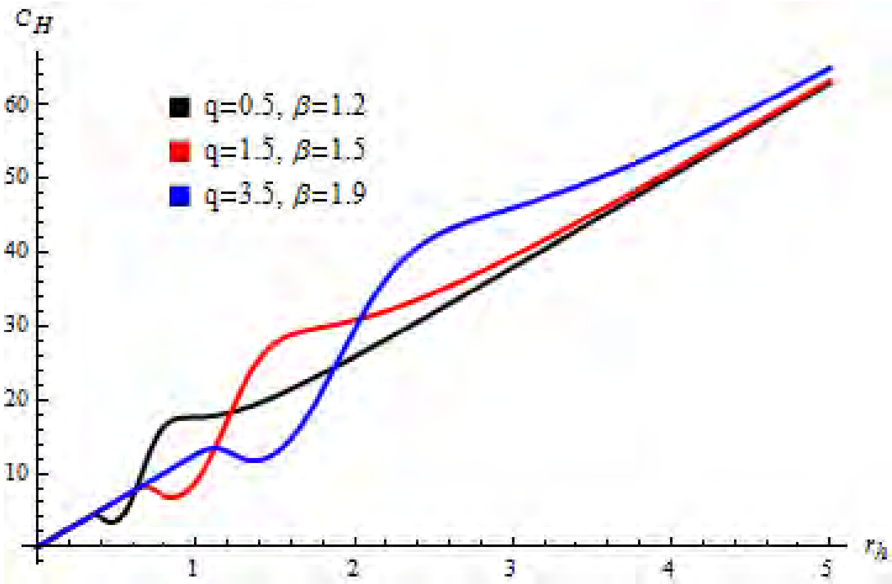

FIGURE 3.3: Plot of function C_H from Eq. (3.39) for fixed value of $l = 0.5$ and different values of q and β.

different values of NED parameter β and magnetic charge q. One can see that both the parameter β and magnetic charge q have effects on the local thermal stability of black holes. The value of r_h at which the heat capacity is positive implies the black hole of that horizon radius is stable and physical. Thus, we conclude that the mentioned (2+1)-dimensional black holes are enjoying thermal stability.

3.3 $(2+1)$-dimensional black holes in arcsin electrodynamics

The action function for the model of nonlinear electrodynamics coupled with general relativity in three dimensions is written in the form (3.1). Here we will take the Lagrangian density $\mathfrak{L}_{NED}(\mathcal{F})$ of the form

$$\mathfrak{L}_{NED}(\mathcal{F}) = -\frac{1}{\beta}\arcsin(\beta\mathcal{F}). \tag{3.40}$$

The nonlinear electromagnetic field equations can be obtained from variation of Eq. (3.1) with respect to A_μ as

$$\partial_\mu\left(\frac{\sqrt{-g}F^{\mu\nu}}{\sqrt{1-(\beta\mathcal{F})^2}}\right) = 0. \tag{3.41}$$

The gravitational field equations will have the same form as (3.3), however, the stress-energy tensor in this case will be written as

$$\mathcal{T}_{\mu}^{\nu} = -\frac{F^{\nu\lambda}F_{\mu\lambda}}{\sqrt{1-(\beta\mathcal{F})^2}} - \delta_{\mu}^{\nu}\mathfrak{L}_{NED}. \tag{3.42}$$

The trace of the above stress-energy tensor can be worked out as

$$\mathcal{T} = -\frac{4\mathcal{F}}{\sqrt{1-(\beta\mathcal{F})^2}} + \frac{4}{\beta}\arcsin(\beta\mathcal{F}). \tag{3.43}$$

Hence, like the exponential electrodynamics, the scale invariance is violated in arcsin electrodynamics too due to the above non-zero trace of the matter tensor. It should be noted that the trace (3.43) also vanishes in the limit $\beta \to 0$ which implies that the arcsin electrodynamics reduces to the Maxwell theory in the weak field limit. As we want to study electrically charged black hole solution, therefore, we should assume magnetic field $\mathbf{B} = 0$ which makes Maxwell's invariant equal to $\mathcal{F} = -(E(r))^2/2$. Thus from Eqs. (3.41) on using the $(2+1)$-dimensional line element (3.6), it is easy to obtain the value of electric field as

$$E(r) = \frac{\sqrt{2}}{Q\beta}\sqrt{\sqrt{r^4+\beta^2Q^4}-r^2}, \tag{3.44}$$

where the constant of integration Q represents the electric charge. The asymptotic

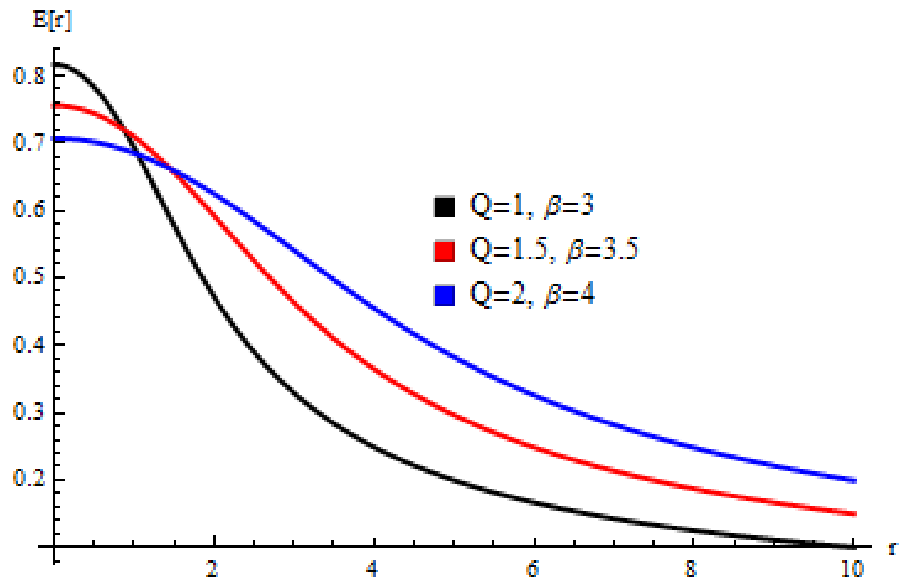

FIGURE 3.4: Plot of $E(r)$ from Eq. (3.44) for fixed values of Q and β.

expansion of electric field at $r \to \infty$ is given by

$$E(r) = \frac{Q}{r} - \frac{Q^5\beta^2}{8r^5} + O(r^{-7}). \tag{3.45}$$

Similarly, the asymptotic expansion of electric field at $r = 0$ becomes

$$E(r) = \frac{\sqrt{2}}{\sqrt{\beta}} - \frac{r^2}{\sqrt{2}Q^2\beta^{\frac{3}{2}}} + \frac{r^4}{4\sqrt{2}Q^4\beta^{\frac{5}{2}}} + O(r^5). \tag{3.46}$$

The electric potential $A(r)$ can be easily obtained through integration of Eq. (3.44) as

$$\begin{aligned} A(r) = \frac{1}{2\sqrt{2}}\Big[r\sqrt{2}\sqrt{\sqrt{r^4+\beta^2Q^4}-r^2} - Q^2\beta\log\big(\sqrt{r^4+\beta^2Q^4}-r^2\big)\Big] \\ + Q^2\beta\log\big[Q^2\beta(Q^2\beta+\sqrt{2}r\sqrt{\sqrt{r^4+\beta^2Q^4}-r^2})\big]. \end{aligned} \tag{3.47}$$

One can also find the asymptotic value of electric potential at $r \to \infty$ as

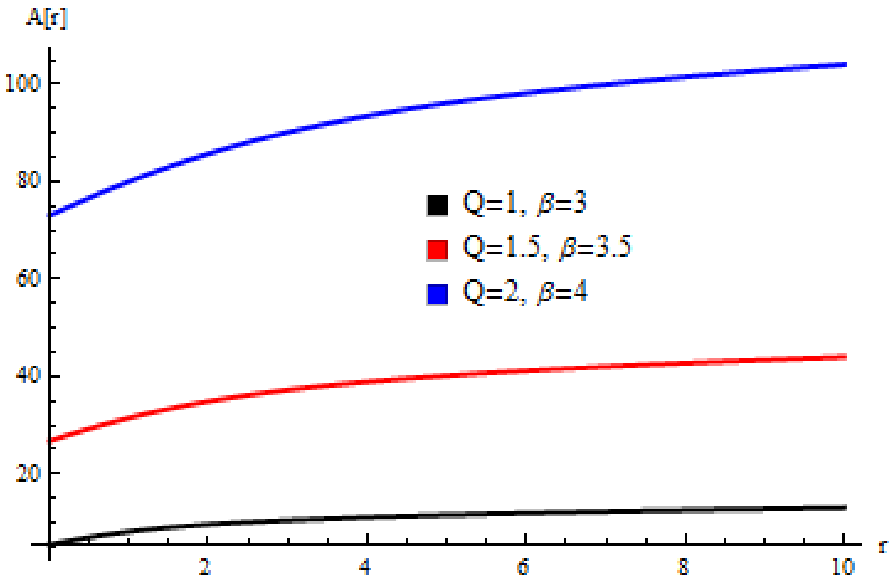

FIGURE 3.5: Plot of function $A(r)$ from Eq. (3.47) for fixed values of β and Q.

$$A(r) = Q\log(r) + \frac{Q^5\beta^2}{32r^4} + O(r^{-6}). \tag{3.48}$$

Similarly, the asymptotic value at $r \to 0$ becomes

$$A(r) = \frac{Q}{2}\log(\beta Q^2) + \frac{\sqrt{2}r}{\sqrt{\beta}} - \frac{r^3}{3\sqrt{2}Q^2\beta^{\frac{3}{2}}} + O(r^5). \tag{3.49}$$

Eqs. (3.46) and (3.49) show that the potential and electric field are both finite at the origin $r = 0$ unlike in Maxwell's theory where both of these quantities diverge at the origin. This non-Maxwellian behaviour of electric potential and electric field is due to the nonlinear electromagnetic nature of Lagrangian density (3.40). Using this Lagrangian density, the components of matter tensor (3.42) can be calculated as

$$\mathcal{T}_0^0 = \mathcal{T}_1^1 = \frac{2E^2}{\sqrt{4-\beta^2 E^4}} - \frac{\arcsin\left(\beta E^2/2\right)}{\beta},$$
$$\mathcal{T}_2^2 = -\frac{1}{\beta}\arcsin\left(\beta E^2/2\right). \tag{3.50}$$

Hence, by choosing the value of electric field as Eq. (3.44) in the above components we can show that WEC, SEC and DEC are valid for this model in $(2+1)$-dimensional geometry. Substitution of the metric ansatz (3.6) and the matter tensor components (3.50) into field equations (3.3) yields

$$\frac{1}{2r}\frac{d\psi}{dr} + \frac{\Lambda}{3} = \pi\left[\frac{2E^2}{\sqrt{4-\beta^2E^4}} - \frac{\arcsin\left(\beta E^2/2\right)}{\beta}\right]. \tag{3.51}$$

Upon solving the above differential equation, we can get the metric function as

$$\psi(r) = D + \frac{r^2}{3l^2} + \frac{\pi r(Q^2+2)}{\beta Q^2}\sqrt{\sqrt{r^4+\beta^2Q^4}-r^2} + \frac{\pi r^2}{\beta}\arcsin\left(\frac{r^2-\sqrt{r^4+\beta^2Q^4}}{\beta Q^2}\right)$$
$$+\pi\tanh^{-1}\left(\frac{r\sqrt{2}\sqrt{\sqrt{r^4+\beta^2Q^4}-r^2}}{Q^2\beta}\right) + \frac{Q^2\pi}{2}\log\left(\frac{\sqrt{Q^4\beta^2+2r^2(r^2-\sqrt{r^4+\beta^2Q^4})}}{Q^2\beta+r\sqrt{2}\sqrt{\sqrt{r^4+\beta^2Q^4}-r^2}}\right), \tag{3.52}$$

where D is the integration constant and $\Lambda = -1/l^2$. Now again with the help of quasilocal mass formalism, a static circularly symmetric three-dimensional line element yields a quasilocal mass M_{QL} in the form (3.12). And so, from line element (3.6) we obtain

$$f(r)^2 = g(r)^2 = D + \frac{r^2}{3l^2} + \frac{\pi r(Q^2+2)}{\beta Q^2}\sqrt{\sqrt{r^4+\beta^2Q^4}-r^2} + \frac{\pi r^2}{\beta}\arcsin\left(\frac{r^2-\sqrt{r^4+\beta^2Q^4}}{\beta Q^2}\right)$$
$$+\pi\tanh^{-1}\left(\frac{r\sqrt{2}\sqrt{\sqrt{r^4+\beta^2Q^4}-r^2}}{Q^2\beta}\right) + \frac{Q^2\pi}{2}\log\left(\frac{\sqrt{Q^4\beta^2+2r^2(r^2-\sqrt{r^4+\beta^2Q^4})}}{Q^2\beta+r\sqrt{2}\sqrt{\sqrt{r^4+\beta^2Q^4}-r^2}}\right), \tag{3.53}$$

and

$$g_r(r)^2 = \frac{r^2}{3l^2} + \frac{\pi r(Q^2+2)}{\beta Q^2}\sqrt{\sqrt{r^4+\beta^2Q^4}-r^2} + \frac{\pi r^2}{\beta}\arcsin\left(\frac{r^2-\sqrt{r^4+\beta^2Q^4}}{\beta Q^2}\right)$$
$$+\pi\tanh^{-1}\left(\frac{r\sqrt{2}\sqrt{\sqrt{r^4+\beta^2Q^4}-r^2}}{Q^2\beta}\right) + \frac{Q^2\pi}{2}\log\left(\frac{\sqrt{Q^4\beta^2+2r^2(r^2-\sqrt{r^4+\beta^2Q^4})}}{Q^2\beta+r\sqrt{2}\sqrt{\sqrt{r^4+\beta^2Q^4}-r^2}}\right). \tag{3.54}$$

Thus, by using the above Eqs. (3.54) and (3.55) in Eq. (3.12), the metric function (3.52) becomes [206]

$$\psi(r) = -M + \frac{r^2}{3l^2} + \frac{\pi r(Q^2+2)}{\beta Q^2}\sqrt{\sqrt{r^4+\beta^2Q^4}-r^2} + \frac{\pi r^2}{\beta}\arcsin\left(\frac{r^2-\sqrt{r^4+\beta^2Q^4}}{\beta Q^2}\right)$$
$$+\pi\tanh^{-1}\left(\frac{r\sqrt{2}\sqrt{\sqrt{r^4+\beta^2Q^4}-r^2}}{Q^2\beta}\right) + \frac{Q^2\pi}{2}\log\left(\frac{\sqrt{Q^4\beta^2+2r^2(r^2-\sqrt{r^4+\beta^2Q^4})}}{Q^2\beta+r\sqrt{2}\sqrt{\sqrt{r^4+\beta^2Q^4}-r^2}}\right). \tag{3.55}$$

The asymptotic expansion for the above metric function when $r\to\infty$ can be computed as

$$\begin{aligned}\psi(r) = {} & -M + \frac{r^2}{3l^2} + \sqrt{2}\pi\left(1+\frac{Q^2}{2}+\frac{Q^2}{2\sqrt{2}}\log\left(\frac{Q^2}{4}\right)\right) - \pi Q^2\log(r) \\ & + \frac{\beta^2Q^4\pi}{16r^4}\left(\frac{Q^2}{6}+3-\sqrt{2}(Q^2+2)\right) + O(r^{-5}).\end{aligned} \tag{3.56}$$

The above expression shows that the black hole is not asymptotically flat at infinity. Similarly, the asymptotic behaviour in the vicinity of $r=0$ is given by the series expansion

$$\psi(r) = -M + \frac{\pi}{\sqrt{\beta}}\left(\frac{\sqrt{2}}{Q}-\frac{Q}{\sqrt{2}}+\frac{Q^2+2}{Q}\right)r + \left(\frac{1}{3l^2}-\frac{\pi^2}{2\beta}\right)r^2 + O(r^3). \tag{3.57}$$

This shows that the metric function remains finite and regular at the origin $r=0$. It is possible to find out the Ricci scalar from Eq. (3.3) as

$$R = -\pi\mathcal{T} - \frac{4}{3l^2} = \frac{4\pi}{\beta}\arcsin\left(\frac{\sqrt{r^4+Q^4\beta^2}-r^2}{Q^2\beta}\right) - \frac{4}{3l^2} - \frac{4\pi\left(\sqrt{r^4+Q^4\beta^2}-r^2\right)}{\beta\sqrt{Q^4\beta^2-\left(\sqrt{r^4+Q^4\beta^2}-r^2\right)^2}}. \tag{3.58}$$

The asymptotic value of Ricci scalar at radial infinity is given by

$$R = -\frac{4}{3l^2} + O(r^{-5}). \tag{3.59}$$

This clearly indicates that at $r\to\infty$ the Ricci scalar is non-zero which means that the spacetime is not asymptotically flat. Similarly, the asymptotic value of Ricci scalar at $r\to 0$ is calculated as

$$R = -\frac{2\sqrt{2}Q\pi}{\sqrt{\beta}r} - \frac{4}{3l^2} + \frac{2\pi^2}{\beta} - \frac{3\sqrt{2}\pi r}{Q\beta^{\frac{3}{2}}} + \frac{5\pi r^3}{6\sqrt{2}Q^3\beta^{\frac{5}{2}}} + O(r^5). \tag{3.60}$$

This expansion shows that the Ricci scalar possesses singularity at $r = 0$. Thus, our resulting (2+1)-dimensional black hole solution given by Eq. (3.55) has a true curvature singularity at $r = 0$.

3.4 Thermodynamics of electrically charged black holes

In order to study thermodynamics corresponding to (2 + 1)-dimensional electrically charged black hole (3.55), we will calculate important thermodynamic quantities. Following the same technique as before we can obtain the local Hawking temperature in this case as

$$\begin{aligned} T_H(r) = \frac{2}{\pi\sqrt{\psi(r)}}\Bigg[\frac{2r_h}{3l^2} + 2\pi\Bigg(&\frac{2r_h(\sqrt{r_h^4+Q^4\beta^2}-r_h^2)}{\sqrt{\beta}Q^2\sqrt{Q^4\beta^3-(\sqrt{r_h^4+Q^4\beta^2}-r_h^2)^2}} \\ &-\frac{r_h}{\beta}\arcsin\left(\frac{\sqrt{r_h^4+Q^4\beta^2}-r_h^2}{Q^2\beta}\right)\Bigg)\Bigg]. \end{aligned} \tag{3.61}$$

Here also X^a is a Killing vector field which generates the event horizon r_h. The event horizon equation $\psi(r) = 0$ implies that

$$\begin{aligned} M = \frac{r_h^2}{3l^2} + \frac{\pi r_h(Q^2+2)}{\beta Q^2}\sqrt{\sqrt{r_h^4+\beta^2Q^4}-r_h^2} + \frac{\pi r_h^2}{\beta}\arcsin\left(\frac{r_h^2-\sqrt{r_h^4+\beta^2Q^4}}{\beta Q^2}\right) \\ +\pi\tanh^{-1}\left(\frac{r_h\sqrt{2}\sqrt{\sqrt{r_h^4+\beta^2Q^4}-r_h^2}}{Q^2\beta}\right) + \frac{Q^2\pi}{2}\log\left(\frac{\sqrt{Q^4\beta^2+2r_h^2(r_h^2-\sqrt{r_h^4+\beta^2Q^4})}}{Q^2\beta+r_h\sqrt{2}\sqrt{\sqrt{r_h^4+\beta^2Q^4}-r_h^2}}\right). \end{aligned} \tag{3.62}$$

Fig. 3.6 shows the behaviour of M in terms of horizon radius. Those values of r_h which corresponds to the positive values of M describes the horizons of black hole. Similarly, by choosing different values of mass M, electric charge Q and nonlinearity parameter β, we have plotted the metric function (3.55) in Fig. 3.7. Those values of r for which the curve intersects the r-axis indicate the location of horizons. The expression of local Hawking temperature (3.61) indicates that at $r \to r_h$, it is undefined and at $r \to \infty$ it is zero. This behaviour is expected because the black hole solution defined by (3.55) is not asymptotically flat, hence the temperature vanishes at infinity. By using the same

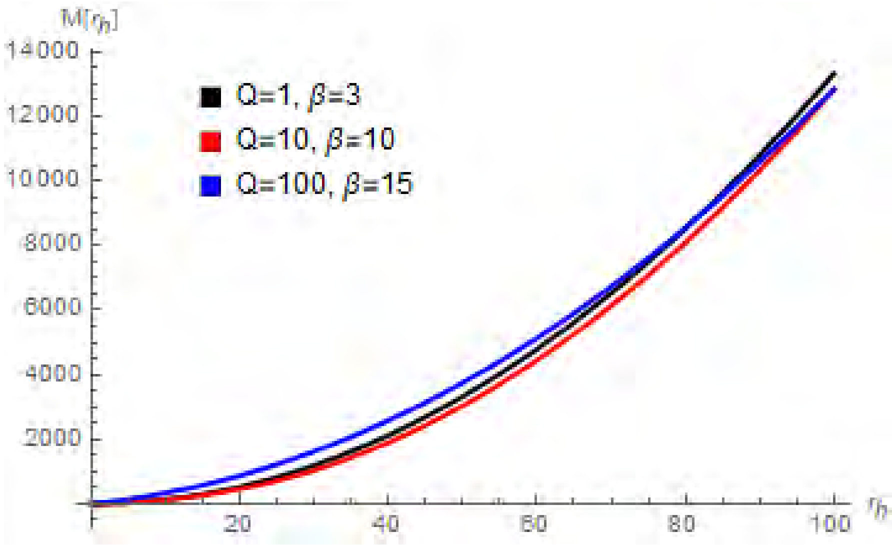

FIGURE 3.6: Plot of mass M from Eq. (3.62) for specific value of $l = 0.5$ and different values of Q and β.

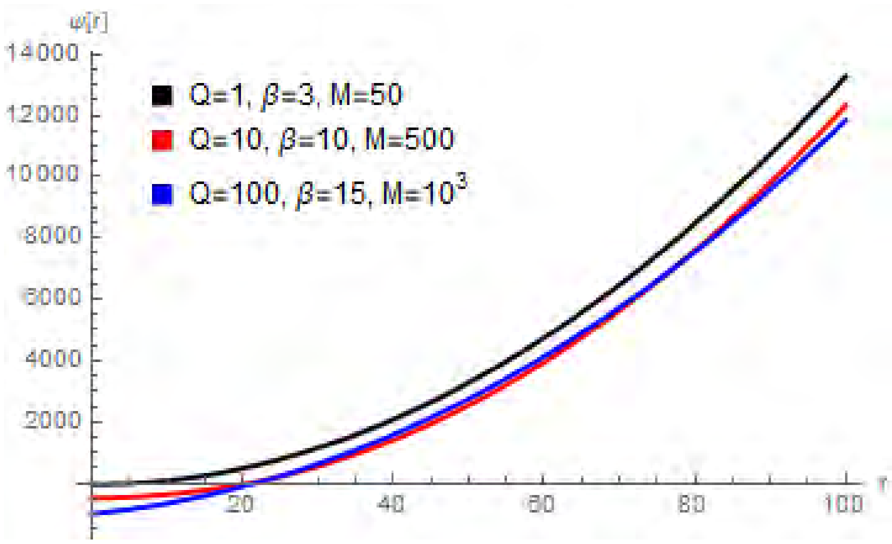

FIGURE 3.7: Plot of metric function $\psi(r)$ from Eq. (3.55) for specific value of $l = 1$ and different values of M, Q and β.

method as described in Refs. [203, 209], the re-energized temperature is defined by

$$T_\infty = \frac{2}{\pi}\left[\frac{2r_h}{3l^2} + 2\pi\left(\frac{2r_h(\sqrt{r_h^4 + Q^4\beta^2} - r_h^2)}{\sqrt{\beta}Q^2\sqrt{Q^4\beta^3 - (\sqrt{r_h^4 + Q^4\beta^2} - r_h^2)^2}} - \frac{r_h}{\beta}\arcsin\left(\frac{\sqrt{r_h^4 + Q^4\beta^2} - r_h^2}{Q^2\beta}\right)\right)\right]. \tag{3.63}$$

Now, from the expression of internal energy (3.33) on a constant t hypersurface, it is easy to obtain the first law of thermodynamics by varying the internal energy with respect to r_h and Q, so that

$$d\Xi(r_b) = T_H(r_b)dS + \Psi(r_b)dQ, \tag{3.64}$$

where

$$\begin{aligned} T_H(r_b) = \frac{1}{\sqrt{\psi(r_b)}}\Bigg[\frac{r_h}{2\beta}\arcsin\left(\frac{\sqrt{r_h^4+\beta^2Q^4}-r_h^2}{\beta Q^2}\right) - \frac{r_h}{6\pi l^2} - \\ \frac{r_h(\sqrt{r_h^4+\beta^2Q^4}-r_h^2)}{\sqrt{\beta}Q^2\sqrt{Q^4\beta^2-(\sqrt{r_h^4+\beta^2Q^4}-r_h^2)^2}}\Bigg] + \frac{1}{2\pi\sqrt{3}l}, \end{aligned} \tag{3.65}$$

denotes the Hawking temperature evaluated at the boundary $r = r_b$ and $S = 4\pi r_h$ in Eq. (3.64) is the entropy. Furthermore, the function

$$\Psi(r_b) = -\frac{\pi r_b H_1(Q, r_b)}{2\sqrt{\psi(r_b)}H_2(Q, r_b)}, \tag{3.66}$$

where

$$\begin{aligned} H_1(Q, r_b) = (2+\sqrt{2})Q^6(Q^2-2)\beta^3 + 4(1-Q^2)r_b^3(\sqrt{Q^4\beta^2+r_b^4}-r_b^2)^{\frac{3}{2}} + \\ 8\beta Q^2 r_b^2(\sqrt{Q^4\beta^2+r_b^4}-r_b^2) - 4(1+\sqrt{2})\beta^2Q^4 r_b\sqrt{\sqrt{Q^4\beta^2+r_b^4}-r_b^2} - \\ 2\beta^2Q^6 r_b\sqrt{\sqrt{Q^4\beta^2+r_b^4}-r_b^2}\left(\frac{\sqrt{2}r_b}{Q^2\beta}\sqrt{\sqrt{Q^4\beta^2+r_b^4}-r_b^2} - 1 - \sqrt{2}\right), \end{aligned} \tag{3.67}$$

and

$$\begin{aligned} H_2(Q, r_b) = \beta Q^2\sqrt{\beta^2Q^4+r_b^4}\sqrt{\sqrt{Q^4\beta^2+r_b^4}-r_b^2}\left(Q^2\beta+\sqrt{2}r_b\sqrt{\sqrt{Q^4\beta^2+r_b^4}-r_b^2}\right) \\ + 2Q\log\left(\frac{\sqrt{Q^4\beta^2+2r_b^4-2r_b^2\sqrt{\beta^2Q^4+r_b^4}}}{\beta Q^2+\sqrt{2}r_b\sqrt{\sqrt{Q^4\beta^2+r_b^4}-r_b^2}}\right), \end{aligned} \tag{3.68}$$

is the electric potential difference between the boundary r_b and infinity. Similarly, the electric potential difference between horizon r_h and boundary r_b can be expressed as

$$\begin{aligned} \Phi(r_b) = -\frac{\pi r_b H_1(Q, r_b)}{2\sqrt{\psi(r_b)}H_2(Q, r_b)} + \frac{1}{2\sqrt{2}}\Bigg[\sqrt{2}r_h\sqrt{\sqrt{Q^4\beta^2+r_h^4}-r_h^2} - Q^2\beta \\ \log\left(\sqrt{Q^4\beta^2+r_h^4}-r_h^2\right) + Q^2\beta\log\left[Q^2\beta\left(Q^2\beta+\sqrt{2}r_h\sqrt{\sqrt{Q^4\beta^2+r_h^4}-r_h^2}\right)\right]\Bigg]. \end{aligned} \tag{3.69}$$

Finally, the heat capacity in this case takes the form

$$C_H = \frac{4\pi \left[r_h T_0 - \frac{r_h}{6\pi l^2} \right]}{\left[T_0 + T_1 + 2r_h^2 T_2 - \frac{1}{6\pi l^2} \right]}, \tag{3.70}$$

where

$$T_0(r_h) = \frac{1}{2\beta} \arcsin\left(\frac{\sqrt{Q^4\beta^2 + r_h^4} - r_h^2}{Q^2\beta} \right) - \frac{\sqrt{Q^4\beta^2 + r_h^4} - r_h^2}{Q^2\sqrt{\beta}\sqrt{Q^4\beta^2 - \left(\sqrt{Q^4\beta^2 + r_h^4} - r_h^2 \right)^2}}, \tag{3.71}$$

$$T_1(r_h) = \frac{r_h^2 \left(\sqrt{Q^4\beta^2 + r_h^4} - r_h^2 \right)}{\sqrt{Q^4\beta^2 + r_h^4}\sqrt{Q^4\beta^2 - \left(\sqrt{Q^4\beta^2 + r_h^4} - r_h^2 \right)^2}} \left(\frac{2}{\sqrt{\beta}Q^2} - \frac{1}{\beta} \right), \tag{3.72}$$

and

$$T_2(r_h) = \frac{\left(\sqrt{Q^4\beta^2 + r_h^4} - r_h^2 \right)^3}{\sqrt{Q^4\beta^2 + r_h^4}\sqrt{\beta}Q^2 \left(Q^4\beta^2 - \left(\sqrt{Q^4\beta^2 + r_h^4} - r_h^2 \right)^2 \right)^{\frac{3}{2}}}. \tag{3.73}$$

Eq. (3.70) is the general expression for black hole's heat capacity for any value of nonlinear electrodynamics parameter β. The behaviour of heat capacity for different

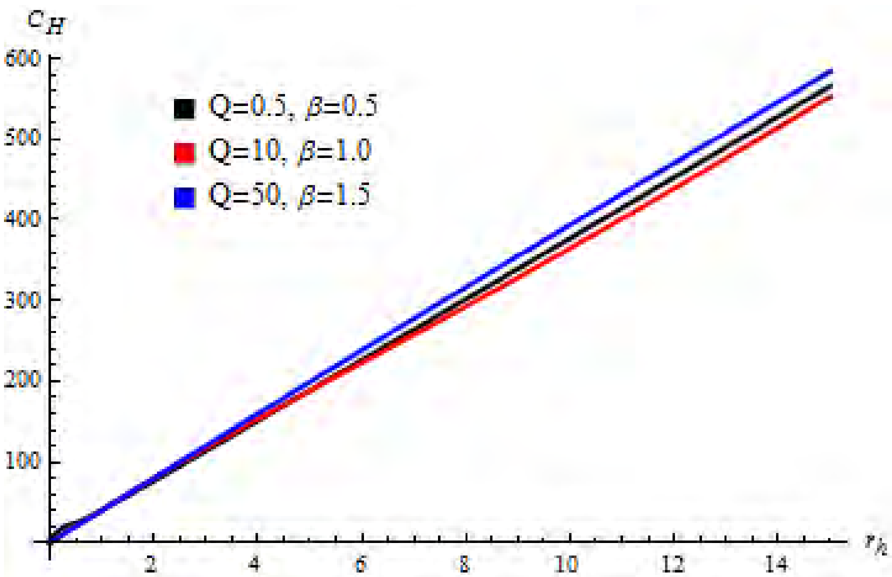

FIGURE 3.8: Plot of function C_H from Eq. (3.70) for fixed value of $l = 0.5$ and different values of Q and β.

values of charge Q and NED parameter β is shown in Fig. 3.8. The region where this quantity is positive guarantees local thermal stability. It can also be seen that both the charge Q and parameter β can affect the local thermodynamic stability and are giving us stable solutions.

Chapter 4

Magnetized black holes surrounded by dark fluid in Lovelock-power-Yang-Mills gravity

We consider a model where static spherically symmetric black hole, in the presence of power-Yang-Mills source, is surrounded by a dark fluid with a non-linear equation of state. In this set up we construct a new class of magnetized Lovelock black hole solutions of the gravitational field equations. In particular, we work out the metric functions in both d-dimensional Einstein and Gauss-Bonnet gravities. We study thermodynamics of these black holes also and show that the mass and associated thermodynamic quantities like Hawking temperature and heat capacity depend on parameters of the dark fluid and the power-Yang-Mills magnetic source. We note that the entropy does not satisfy the area law in Gauss-Bonnet and higher order Lovelock black holes. Furthermore, phase transitions of black holes in each case are also discussed.

4.1 Lovelock black holes, dark fluid and power-Yang-Mills theory

The action function for Lovelock gravity with matter sources in diverse dimensions can be written in the form

$$\mathcal{I} = \mathcal{I}_L + \mathcal{I}_M, \tag{4.1}$$

where $\mathcal{I}_M$ corresponds to the action function of matter contents in the spacetime i.e. the power-Yang-Mills field and dark fluid. In this chapter we want to obtain the Lovelock black hole solution of massive power-Yang-Mills gravity surrounded by dark fluid. The Lovelock action $\mathcal{I}_L$ is given by

$$\mathcal{I}_L = \frac{1}{2}\int d^dx\sqrt{-g}\left[\sum_{p=0}^{s}\frac{\alpha_p}{2^p}\delta^{\mu_1...\mu_{2p}}_{\nu_1...\nu_{2p}}R^{\nu_1\nu_2}_{\mu_1\mu_2}...R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right], \tag{4.2}$$

where the number s can be defined as $s = (d-1)/2$ in odd dimensions, and $s = (d/2)-1$ in even dimensions. The coefficients α_p in (4.1) are arbitrary constants in which $\alpha_0 = -2\Lambda$ means that it is related to the cosmological constant.

If we take the variation of action (4.1) relative to the metric tensor, $g_{\mu\nu}$, we obtain the equations of the gravitational field as

$$\sum_{p=0}^{s}\frac{\alpha_p}{2^{p+1}}\delta^{\nu\lambda_1...\lambda_{2p}}_{\mu\rho_1...\rho_{2p}}R^{\rho_1\rho_2}_{\lambda_1\lambda_2}...R^{\rho_{2p-1}\rho_{2p}}_{\lambda_{2p-1}\lambda_{2p}} = \mathcal{T}^{\nu}_{\mu}, \tag{4.3}$$

where $\mathcal{T}^{\nu}_{\mu}$ represents the stress-energy tensor given by

$$\mathcal{T}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{I}_M}{\delta g^{\mu\nu}}. \tag{4.4}$$

Now, the general spherically symmetric static line element in d-dimensions can be expressed as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(h_{ij}dx^i dx^j), \tag{4.5}$$

where $h_{ij}dx^i dx^j$ is the metric of $d-2$ dimensional hypersurface having constant scalar curvature k such that the values $k = 1, 0, -1$, represent spherical, flat and hyperbolic spaces, respectively.

Now the Lagrangian density corresponding to power-Yang-Mills field is given by

$$\mathfrak{L}_{PYM} = -(\Upsilon)^q, \tag{4.6}$$

where Υ denotes the Yang-Mills invariant given by

$$\Upsilon = \sum_{a=1}^{(d-2)(d-1)/2}\left(F^a_{\lambda\sigma}F^{(a)\lambda\sigma}\right), \tag{4.7}$$

q is a positive real parameter and the Yang-Mills field is defined by

$$F^{(a)} = dA^{(a)} + \frac{1}{2\eta}C^{(a)}_{(b)(c)}A^{(b)}\wedge A^{(c)}, \tag{4.8}$$

where, $C^{(a)}_{(b)(c)}$ are the structure constants of $(d-2)(d-1)/2$ -parameter Lie group G, η represents the coupling constant and $A^{(a)}$ are Yang-Mills potentials of the $SO(d-1)$ gauge group. These structure constants have been calculated in Ref. [211]. It should be noted that it does not matter whether we write the internal indices $[a, b, ...]$ in covariant or contravariant form. Thus, using the Lagrangian density (4.6), the stress-energy tensor of power-Yang-Mills field is given by

$$\mathcal{T}^{(a)\nu}_{\mu} = -\frac{1}{2}\left[\delta^{\nu}_{\mu}\Upsilon^{q} - 4q\sum_{a=1}^{(d-2)(d-1)/2}\left(F^{(a)}_{\mu\lambda}F^{(a)\nu\lambda}\right)\Upsilon^{q-1}\right]. \tag{4.9}$$

We obtain the equations of Yang-Mills field by varying the action (4.1) with respect to the gauge potentials $A^{(a)}$

$$d({}^{\star}F^{(a)}\Upsilon^{q-1}) + \frac{1}{\eta}C^{(a)}_{(b)(c)}\Upsilon^{q-1}A^{(b)} \wedge^{\star} F^{(c)} = 0, \tag{4.10}$$

where $\star$ is used to denote the duality of a field. If we take the static metric given in (4.5) and using the Wu-Yang ansatz in Einstein-Yang-Mills theory of gravity [173], then the power-Yang-Mills equations [177,178,211] are satisfied if the one-forms of the Yang-Mills magnetic gauge potential in this ansatz are described by

$$A^{(a)} = \frac{Q}{r^2}C^{(a)}_{(l)(j)}x^{l}dx^{j}, r^2 = \sum_{l=1}^{d-1}x_l^2, \tag{4.11}$$

where the constant Q is associated with the Yang-Mills magnetic charge and $2 \le j+1 \le l \le d-1$. Therefore, the symmetric stress-energy tensor (4.9), with

$$\Upsilon = \frac{n(n-1)Q^2}{r^4} = \frac{(d-2)(d-3)Q^2}{r^4}, \tag{4.12}$$

becomes

$$\mathcal{T}^{(a)\nu}_{\mu} = -\frac{1}{2}\Upsilon^{q}diag[1, 1, \zeta, \zeta, ..., \zeta], \zeta = \left(1 - \frac{4q}{(d-2)}\right). \tag{4.13}$$

Now we consider the dark fluid with non-linear equation of state $p_{df} = -B/\rho_{df}$, where B is a constant. For spherically symmetric d-dimensional spacetime geometry, the stress-energy tensor corresponding to the dark fluid can be defined as

$$\mathcal{T}^{(df)t}_{t} = \Pi(r), \mathcal{T}^{(df)i}_{t} = 0, \mathcal{T}^{(df)j}_{i} = \Delta(r)\frac{r_i r^j}{r_n r^n} + \Theta(r)\delta^{j}_{i}. \tag{4.14}$$

If we consider isotropic average over the angles

$$[r_i r^j]_A = \frac{r_n r^n \delta^j_i}{d-1}, \tag{4.15}$$

we can get

$$[\mathcal{T}_i^{(df)j}]_A = \left(\frac{\Delta(r)}{d-1} + \Theta(r)\right)\delta_i^j = p_{df}(r)\delta_i^j = -\frac{B}{\rho_{df}(r)}. \tag{4.16}$$

Using the condition of staticity and spherical symmetry, $\mathcal{T}_t^{(df)t} = \mathcal{T}_r^{(df)r} = -\rho_{df}(r)$, in Eq. (4.16) we have the following expressions

$$\begin{aligned}\Pi(r) &= -\rho_{df}(r),\\ \Delta(r) &= -\frac{(d-1)\rho_{df}}{(d-2)} + \frac{(d-1)B}{(d-2)\rho_{df}},\\ \Theta(r) &= \frac{\rho_{df}}{(d-2)} - \frac{(d-1)B}{(d-2)\rho_{df}}.\end{aligned} \tag{4.17}$$

So, the other angular components of the stress-energy tensor are calculated as

$$\mathcal{T}_{\theta_1}^{(df)\theta_1} = \mathcal{T}_{\theta_2}^{(df)\theta_2} = \cdots = \mathcal{T}_{\theta_{d-2}}^{(df)\theta_{d-2}} = \frac{\rho_{df}(r)}{d-2} - \frac{(d-1)B}{(d-2)\rho_{df}(r)}. \tag{4.18}$$

Let us introduce the metric function $f(r)$ in line element (4.5) as

$$f(r) = k - r^2 V(r), \tag{4.19}$$

where the function $V(r)$ is related to the polynomial

$$P[V(r)] = \sum_{p=0}^{s} \overline{\alpha}_p V^p(r), \tag{4.20}$$

whose coefficients $\overline{\alpha}_p$ are defined as

$$\begin{aligned}\overline{\alpha}_0 &= \frac{\alpha_0}{(d-1)(d-2)}, \overline{\alpha}_1 = 1,\\ \overline{\alpha}_p &= \prod_{i=3}^{2p}(d-i)\alpha_p.\end{aligned} \tag{4.21}$$

The general expression for $\overline{\alpha}_p$ in (4.21) holds only for $p > 1$. By using the above components of the stress-energy tensor, $\mathcal{T}_{\mu\nu} = \mathcal{T}_{\mu\nu}^{(a)} + \mathcal{T}_{\mu\nu}^{(df)}$, in the Lovelock field equations (4.3) we obtain the differential equation for energy density of the dark fluid in the form

$$\frac{d\rho_{df}}{dr} + \frac{(d-1)}{r}\rho_{df} = \frac{B(d-1)}{r\rho_{df}}, \tag{4.22}$$

which can be solved to give

$$\rho_{df}(r) = \sqrt{B + \frac{S^2}{r^{2d-2}}}, \tag{4.23}$$

where, $S > 0$ is the constant of integration. It is clearly seen that this is valid in any particular Lovelock gravity theory and the energy density of the dark fluid will still satisfy the same differential equation (4.22). Furthermore, we can see from Eq. (4.23) that in the limit $r \to \infty$, B approaches the value $\sqrt{B}$, which implies that at large distances from the gravitating object the dark fluid behaves like a cosmological constant. Similarly, when we use the above information of matter sources and the line element (4.5) in gravitational field equations (4.3) we can obtain the polynomial equation for the magnetized black hole as [212]

$$\begin{aligned} P[V(r)] = &\frac{2M}{(d-2)\Sigma_{d-2} r^{d-1}} + \frac{(d-2)^{q-1}(d-3)^q Q^{2q}}{(d-4q-1)r^{4q}} \\ &+ \frac{2r^{1-d}}{(d-2)(d-1)}\left[r^{d-1}\sqrt{B+\frac{S^2}{r^{2d-2}}} - S \arcsin\left(\frac{S}{\sqrt{B}r^{d-1}}\right)\right], \end{aligned} \tag{4.24}$$

which is satisfied by function $V(r)$ and hence by the metric function $f(r)$ as well. Note that M is an integration constant associated with the black hole mass and

$$\Sigma_{d-2} = \frac{2\pi^{\frac{(d-1)}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}, \tag{4.25}$$

is the volume of $d-2$-dimensional hypersurface. Next we will discuss thermodynamics of Lovelock black holes generated by polynomial equation (4.24). For this we will calculate different thermodynamic quantities as a function of the event horizon r_+ which satisfies the equation $f(r_+) = 0$. So, using Eq. (4.19) we can write

$$r_+^2 = \frac{k}{V(r_+)}. \tag{4.26}$$

Thus, the total mass in terms of horizon's radius r_+ is given by

$$\begin{aligned} M = &\frac{\Sigma_{d-2}}{2}\left[\sum_{p=0}^{s} \frac{\overline{\alpha}_p k^p (d-2)}{r_+^{-(d-2p-1)}} - \frac{(d-2)^q (d-3)^q Q^{2q}}{(d-4q-1)r_+^{4q-d+1}} - \frac{2}{d-1}\right. \\ &\left.\times\left(r_+^{d-1}\sqrt{B+\frac{S^2}{r_+^{2d-2}}} - S\arcsin\left(\frac{S}{\sqrt{B}r_+^{d-1}}\right)\right)\right]. \end{aligned} \tag{4.27}$$

The Hawking temperature of Lovelock black holes governed by the polynomial equation (4.24) in terms of surface gravity κ is given by $T_H = \kappa/2\pi$. Thus, we use the surface gravity to obtain the Hawking temperature

$$\begin{aligned} T_H = &\frac{1}{4\pi W(r_+)}\left[\sum_{p=0}^{s} \frac{\overline{\alpha}_p k^p (d-2p-1)}{r_+^{2p+1}} - \frac{(d-2)^{q-1}(d-3)^q Q^{2q}}{r_+^{4q+1}}\right. \\ &\left.- \frac{2}{(d-2)r_+}\sqrt{B+\frac{S^2}{r_+^{2d-2}}}\right], \end{aligned} \tag{4.28}$$

where $W(r_+)$ is defined by

$$W(r_+) = \sum_{p=0}^{s} \frac{p\overline{\alpha}_p k^{p-1}}{r_+^{2p}}. \tag{4.29}$$

The entropy of the black hole is given by

$$\mathcal{S} = \int \frac{dM}{T_H} = \int T_H^{-1} \frac{dM}{dr_+} dr_+. \tag{4.30}$$

Now, differentiating the mass (4.27) with respect to r_+ we obtain

$$\begin{aligned} \frac{dM}{dr_+} = \frac{\Sigma_{d-2}}{2} \Bigg[\sum_{p=0}^{s} \frac{\overline{\alpha}_p k^p (d-2p-1)(d-2)}{r_+^{-(d-2p-2)}} - \frac{(d-2)^q (d-3)^q Q^{2q}}{r_+^{4q-d+2}} \\ - 2r_+^{d-2} \sqrt{B + \frac{S^2}{r_+^{2d-2}}} \Bigg]. \end{aligned} \tag{4.31}$$

So, using the above value of Hawking temperature (4.28) and Eq. (4.31) we can obtain the entropy in the form as follows

$$\mathcal{S} = 2\pi(d-2)\Sigma_{d-2} \sum_{p=1}^{s} \frac{\overline{\alpha}_p k^{p-1} p}{(d-2p) r_+^{-(d-2p)}}. \tag{4.32}$$

This form of entropy shows that in Lovelock gravity, magnetically charged black hole in the presence of dark fluid does not obey the Hawking area theorem. This behaviour is very similar to the electrically charged black holes obtained recently in Lovelock gravity [213].

The heat capacity of the black hole can defined by the relation

$$C_H = \frac{dM}{dT_H} = \frac{dM}{dr_+} \left(\frac{dT_H}{dr_+} \right)^{-1}. \tag{4.33}$$

So, by using Eq. (4.31) and Hawking temperature we get the heat capacity in the form

$$C_H = \frac{2\pi \Sigma_{d-2} r_+^{d-1} \psi_1(r_+)}{W(r_+) \left(\frac{(4q+1)(d-2)^{q-1}(d-3)^q Q^{2q}}{r_+^{4q+2}} + \psi_3(r_+) \right) + 2\psi_2(r_+)}, \tag{4.34}$$

where

$$\begin{aligned} \psi_1(r_+) = \sum_{p=0}^{s} \frac{\overline{\alpha}_p k^p (d-2p-1)(d-2)}{r_+^{2p+1}} - \frac{(d-2)^q (d-3)^q Q^{2q}}{r_+^{4q+1}} \\ - \frac{2}{r_+} \sqrt{B + \frac{S^2}{r_+^{2d-2}}}, \end{aligned} \tag{4.35}$$

$$\psi_2(r_+) = \sum_{p=0}^{s} \frac{\overline{\alpha}_p p^2 k^{p-1}}{r_+^{2p+1}} \Bigg[- \frac{(d-2)^{q-1}(d-3)^q Q^{2q}}{r_+^{4q+1}} - \frac{2}{(d-2)r_+}\sqrt{B + \frac{S^2}{r_+^{2d-2}}} + \sum_{p=0}^{s} \frac{\overline{\alpha}_p k^p (d-2p-1)}{r_+^{2p+1}} \Bigg], \tag{4.36}$$

$$\psi_3(r_+) = \frac{2}{(d-2)r_+^2}\sqrt{B + \frac{S^2}{r_+^{2d-2}}} + \frac{2(d-1)S^2}{(d-2)r_+^{2d}} \left(\sqrt{B + \frac{S^2}{r_+^{2d-2}}}\right)^{-1} - \sum_{p=0}^{s} \frac{\overline{\alpha}_p k^p (d-2p-1)(2p+1)}{r_+^{2p+2}}. \tag{4.37}$$

Note that the heat capacity is significant in the sense that thermal stability of black holes is dependent on the sign of heat capacity. The region where this quantity is positive indicates that the black hole is stable whereas negativity of heat capacity corresponds to the unstable thermodynamic system. The point at which Hawking temperature and heat capacity change signs indicates the first order phase transition while the point at which heat capacity is not convergent corresponds to the second order phase transition of black hole.

4.2 Magnetized Einsteinian black holes surrounded by dark fluid

Here we determine the metric function corresponding to magnetized Einsteinian black holes in diverse dimensions where the higher curvature terms are neglected by considering the coefficients $\alpha_p = 0$ for $p \geq 2$ in the action function (4.1). Note that, these black holes are also surrounded by dark fluid and the source of gravity is again the power-Yang-Mills field. Therefore, using this assumption we obtain the metric function for $k = 1$ from the polynomial equation (4.24) as

$$f(r) = 1 - \frac{16\pi\overline{M}}{(d-2)\Sigma_{d-2} r^{d-3}} - \frac{(d-2)^{q-1}(d-3)^q Q^{2q}}{(d-4q-1)r^{4q-2}} + \frac{2r^{3-d}}{(d-2)(d-1)} \times \left(S \arcsin\left(\frac{S}{\sqrt{B} r^{d-1}}\right) - r^{d-1}\sqrt{B + \frac{S^2}{r^{2d-2}}} \right) + \frac{\alpha_0 r^2}{(d-1)(d-2)}, \tag{4.38}$$

where we have used the value $M = 8\pi\overline{M}$. Choosing $d = 4$ and $q = 1$, the above metric function becomes

$$f(r) = 1 - \frac{2\overline{M}}{r} - \frac{Q^2}{r^2} + \frac{\alpha_0 r^2}{6} + \frac{1}{3r}\left[S \arcsin\left(\frac{S}{\sqrt{B} r^3}\right) - r^3 \sqrt{B + \frac{S^2}{r^6}} \right], \tag{4.39}$$

which is now the metric function for an anti-de Sitter black hole of Einstein-Yang-Mills gravity surrounded by dark fluid which satisfies a non-linear equation of state. If we choose $B = S = \alpha_0 = 0$, then the metric function obtained corresponds to the familiar Reissner-Nordström like black hole with Yang-Mills magnetic charge. Eq. (4.38) shows that the resulting solution has the behaviour of a non-asymptotically flat spacetime, however, by choosing $\alpha_0 = 2\sqrt{B}$, the solution represents an asymptotically flat spacetime.

In general, the Ricci scalar and Kretschmann scalar for the line element (4.5) are given by

$$R(r) = \left[(d-2)(d-3)\left(\frac{k-f(r)}{r^2}\right) - \frac{d^2 f}{dr^2} - \frac{2(d-2)}{r}\frac{df}{dr}\right], \tag{4.40}$$

and

$$\mathcal{K}(r) = \left[2(d-2)(d-3)\left(\frac{k-f(r)}{r^2}\right)^2 - \left(\frac{d^2 f}{dr^2}\right)^2 + \frac{2(d-2)}{r^2}\left(\frac{df}{dr}\right)^2\right]. \tag{4.41}$$

So, by using the metric function (4.38), one can clearly see that both Ricci and Kretschmann scalars diverge at $r = 0$, which confirms that the resulting gravitating object is basically a black hole. In d dimensions the mass of the black hole as a function of the event horizon r_+ is given by

$$\begin{aligned} M = \frac{\Sigma_{d-2}}{2}\Bigg[&\frac{\alpha_0 r_+^{d-1}}{(d-1)} + (d-2)r_+^{d-3} - \frac{(d-2)^q(d-3)^q Q^{2q}}{(d-4q-1)r_+^{4q-d+1}} \\ &- \frac{2}{d-1}\left(r_+^{d-1}\sqrt{B + \frac{S^2}{r_+^{2d-2}}} - S\arcsin\left(\frac{S}{\sqrt{B}r_+^{d-1}}\right)\right)\Bigg]. \end{aligned} \tag{4.42}$$

The plots of mass as a function of horizon radius r_+ in different dimensions are shown in Fig. 4.1 which shows that for large values of r_+ it is increasing and for very small values it becomes undefined. Fig. 4.2 shows the plot of the metric function for different values of d. The point where the curve touches the horizontal axis indicates position of the black hole's horizon. Using Eq. (4.39), we obtain the Hawking temperature $T_H = \kappa/2\pi$ in this case also in the following form

$$T_H = \frac{r_+}{4\pi}\left[\frac{\alpha_0}{(d-2)} + \frac{(d-3)}{r_+^2} - \frac{(d-2)^{q-1}(d-3)^q Q^{2q}}{r_+^{4q-2}} - \frac{2}{(d-2)}\sqrt{B + \frac{S^2}{r_+^{2d-2}}}\right]. \tag{4.43}$$

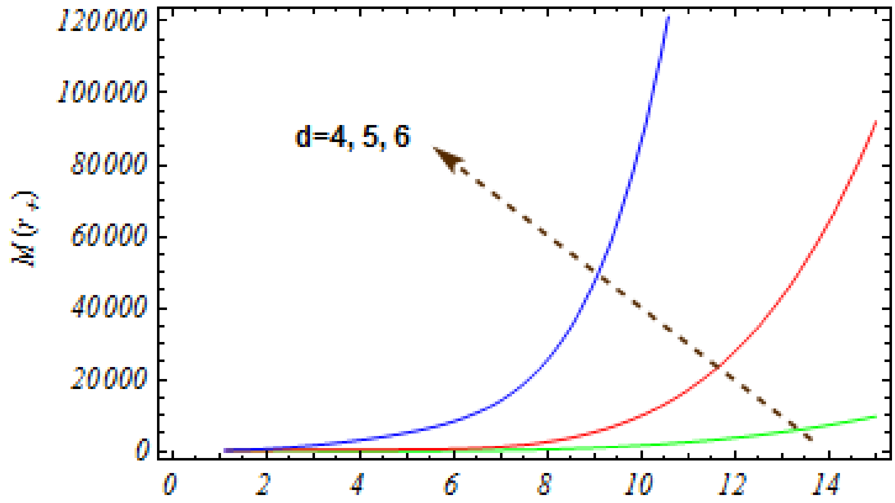

FIGURE 4.1: Plot of function M (Eq. (4.42)) vs r_+ for fixed values of $Q = 100$, $B = 0.1$, $\Sigma_{d-2} = 50$, $S = 0.5$, $\alpha_0 = 1$ and $q = 0.2$.

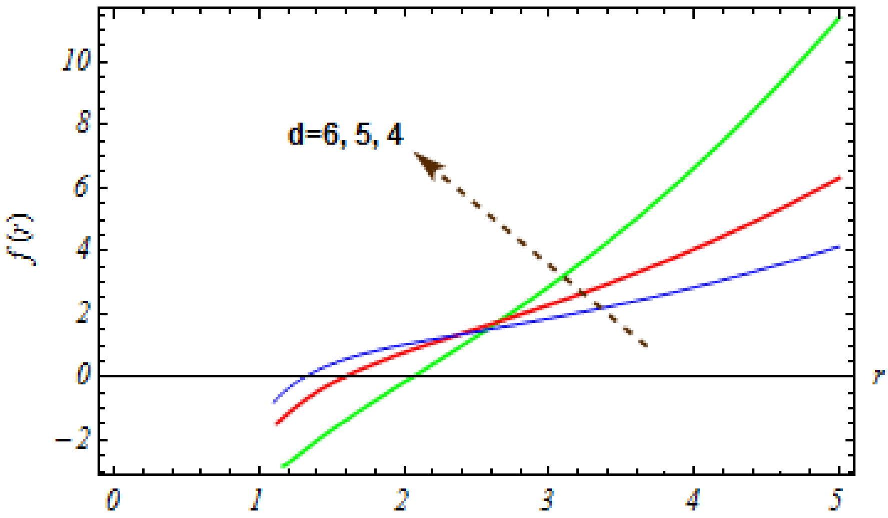

FIGURE 4.2: Plot of function $f(r)$ (Eq. (4.38)) for fixed values of $\overline{M} = 10$, $\Sigma_{d-2} = 50$, $\alpha_0 = 1$, $Q = 100$, $B = 0.1$, $S = 0.5$ and $q = 0.1$.

Fig. 4.3 shows the behaviour of Hawking temperature of Einsteinian black holes for different values of d in the indicated domain of r_+. The point at which the curve intersects r_+-axis indicates the first order phase transition of black hole and the region where temperature is negative shows that the black hole is unstable. Now, by differentiating Eq. (4.42) with respect to r_+ we have

$$\frac{dM}{dr_+} = \frac{\Sigma_{d-2}}{2}\Bigg[\alpha_0 r_+^{d-2} - 2r_+^{d-2}\sqrt{B + \frac{S^2}{r_+^{2d-2}}} + (d-2)(d-3)r_+^{d-4} - \frac{(d-2)^q(d-3)^q Q^{2q}}{r_+^{4q-d+2}}\Bigg]. \tag{4.44}$$

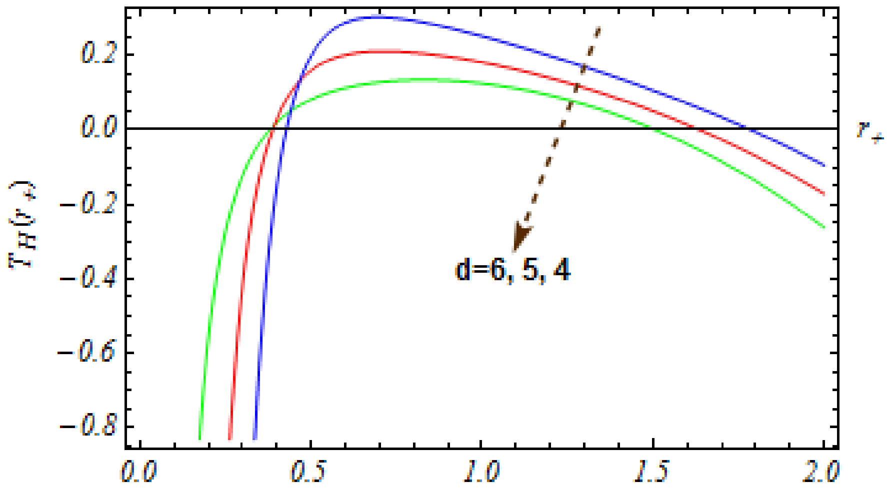

FIGURE 4.3: Plot of temperature T_H (Eq. (4.43)) vs r_+ in small domain of r_+ for fixed values of $\alpha_0 = 1$, $Q = 100$, $B = 0.1$, $S = 0.5$, $\Sigma_{d-2} = 50$ and $q = 0.1$.

So, by using Eq. (4.44) and the expression of Hawking temperature (4.43) we get the entropy in the form

$$\mathcal{S} = 2\pi\Sigma_{d-2}r_+^{d-2}, \tag{4.45}$$

which clearly indicates that unlike the case of Lovelock black holes, here entropy satisfies the area law. Differentiation of Hawking temperature yields

$$\begin{aligned}\frac{dT_H}{dr_+} = \frac{1}{4\pi}\Bigg[&\frac{\alpha_0}{(d-2)} - \frac{2}{d-2}\sqrt{B + \frac{S^2}{r_+^{2d-2}}} + \frac{2S^2(d-1)}{(d-2)r_+^{2d-2}\sqrt{B+\frac{S^2}{r_+^{2d-2}}}}\\ &+ \frac{3-d}{r_+^2} + \frac{(4q-1)(d-2)^{q-1}(d-3)^q Q^{2q}}{r_+^{4q}}\Bigg].\end{aligned} \tag{4.46}$$

The heat capacity can be obtained from the general formula (4.33), by using Eqs. (4.43)-(4.46), in the following form

$$\begin{aligned}C_H = \frac{2\pi(d-2)r_+^2}{\Psi(r_+)}\Bigg[&\alpha_0 r_+^{d-2} - 2r_+^{d-2}\sqrt{B+\frac{S^2}{r_+^{2d-2}}} + (d-3)(d-2)r_+\\ &- \frac{(d-2)^q(d-3)^q Q^{2q}}{r_+^{4q-D+2}}\Bigg],\end{aligned} \tag{4.47}$$

where

$$\begin{aligned}\Psi(r_+) = {}&\alpha_0(d-2)(3-d)r_+^2 - 2r_+^2\sqrt{B+\frac{S^2}{r_+^{2d-2}}} + \frac{2S^2(d-1)r_+^{4-2d}}{\sqrt{B+\frac{S^2}{r_+^{2d-2}}}}\\ &+ \frac{(4q-1)(d-2)^q(d-3)^q Q^{2q}}{r_+^{4q-2}}.\end{aligned} \tag{4.48}$$

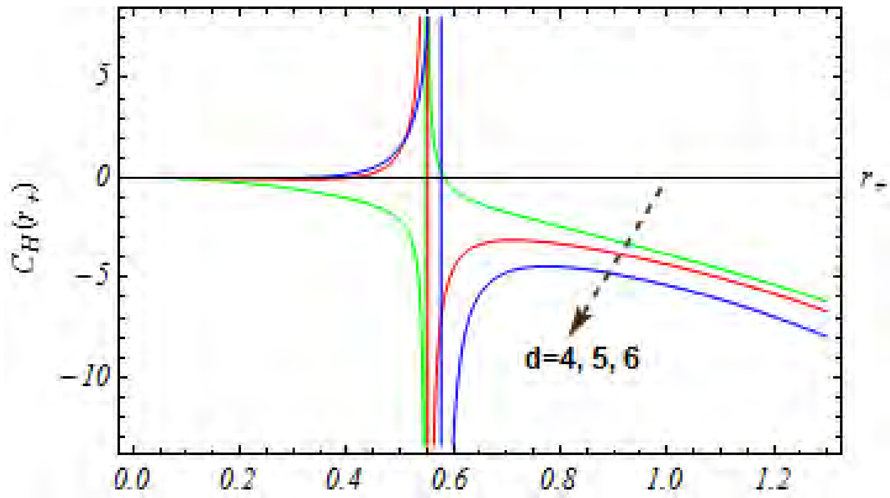

FIGURE 4.4: Plot of function C_H (Eq. (4.47)) vs r_+ for fixed values of $\alpha_0 = 1$, $Q = 100$, $B = 0.1$, $S = 0.5$ and $q = 0.1$.

The graph of heat capacity is plotted in Fig. 4.4. Note that the black hole is stable in the region where both Hawking temperature and heat capacity are positive. The points at which heat capacity gives indefinite values and diverges implies the second order phase transition of black holes.

4.3 Magnetized Gauss-Bonnet black holes surrounded by dark fluid

In this section we will work out magnetized black hole solutions of Gauss-Bonnet massive gravity. For this we consider α_2 to be non-zero and $\alpha_p = 0$ for $p \geq 3$ in the action function (4.1) and choose $k = 1$ in the polynomial equation (4.24) to obtain the expression for the metric function in two branches as

$$f_{\pm}(r) = 1 + \frac{r^2}{2\overline{\alpha}_2^2}\left(1 \pm \sqrt{H(r)}\right), \tag{4.49}$$

where $H(r)$ is given by

$$\begin{aligned} H(r) = 1 - \frac{4\alpha_0\overline{\alpha}_2}{(d-1)(d-2)} + \frac{8\overline{\alpha}_2 M}{\Sigma_{d-2}(d-2)r^{d-1}} + \frac{4\overline{\alpha}_2 Q^{2q}(d-2)^{q-1}(d-3)^q}{(d-4q-1)r^{4q}} \\ + \frac{8\overline{\alpha}_2}{(d-2)(d-1)r^{d-1}}\left(r^{d-1}\sqrt{B + \frac{S^2}{r^{2d-2}}} - S\arcsin\left(\frac{S}{\sqrt{B}r^{d-1}}\right)\right). \end{aligned} \tag{4.50}$$

The asymptotic value of $f_{\pm}(r)$ in the limit $r \to \infty$ is given by

$$f_{\pm}(r) = 1 + \frac{r^2}{2\overline{\alpha}_2}\left(1 \pm \sqrt{1 - \frac{4\overline{\alpha}_2(\alpha_0 - 2\sqrt{B})}{(d-1)(d-2)}}\right). \tag{4.51}$$

This expression shows that for any positive value of $\overline{\alpha}_2$ the function $f_+(r)$ represents anti-de Sitter spacetime for $\alpha_0 > 2\sqrt{B}$ and de Sitter for $\alpha_0 < 2\sqrt{B}$. For negative values of the coefficient $\overline{\alpha}_2$, this positive branch represents de Sitter spacetime for $\alpha_0 < 2\sqrt{B}$, and for the range $\alpha_0 > 2\sqrt{B}$ it yields anti-de Sitter spacetime. Similarly for positive values of $\overline{\alpha}_2$, the negative branch $f_-(r)$represents de Sitter for $\alpha_0 > 2\sqrt{B}$ and we get anti-de Sitter for $\alpha_0 < 2\sqrt{B}$. However, for negative values of $\overline{\alpha}_2$, the negative branch yields anti-de Sitter for $\alpha_0 > 2\sqrt{B}$ and de Sitter for $\alpha_0 < 2\sqrt{B}$. We choose the physically meaningful metric which approaches to the well known Schwarzschild metric of Einstein's theory when the effects of the cosmological constant and other gravitational sources are absent and there is only a central mass, then the positive branch $f_+(r)$ is not workable. Because the metric solution needs to be real-valued, we must take those values of r for which the function $H(r)$ in Eq. (51) is non-negative. Now it is possible to find the root r_b of equation $H(r) = 0$ for given values of M, Q and B. This root $r = r_b$ is called a branch singularity [214, 215] which makes the metric function $f_-(r)$ well defined in the interval $[r_+, \infty)$ because $H(r_\pm) = (1 + 2\overline{\alpha}_2/r_\pm^2)^2$. This leads us to the observation that there exists branch singularity r_b for the metric function $f_-(r)$ having inner horizon r_- and outer horizon r_+, for which the inequality $r_b < r_- \leq r_+$ is satisfied.

Like the previous case of Einsteinian black holes, here also one can easily verify that both the curvature invariants have singularity at $r = 0$. This can be done by using the metric function (4.49) in Eqs. (4.40)-(4.41). Thus, we conclude that our resulting solution (4.49) of Gauss-Bonnet massive gravity also describes black hole. The mass of this type of black hole in terms of horizon radius is given by

$$
\begin{aligned}
M = \frac{\Sigma_{d-2}}{2}\Bigg[&\frac{\overline{\alpha}_2(d-2)}{r_+^{5-d}} + \frac{\alpha_0 r_+^{d-1}}{(d-1)} + (d-2)r_+^{d-3} - \frac{(d-2)^q(d-3)^q Q^{2q}}{(d-4q-1)r_+^{4q-d+1}} \\
&- \frac{2}{d-1}\left(r_+^{d-1}\sqrt{B + \frac{S^2}{r_+^{2d-2}}} - S\arcsin\left(\frac{S}{\sqrt{B}r_+^{d-1}}\right)\right)\Bigg].
\end{aligned} \tag{4.52}
$$

Fig. 4.5 shows the graphs of the mass function as a function of the horizon radius in different dimensions. Each of the plots indicates that there exist possible negative values in the range of M associated with different values of r_+ which do not corresponds to the horizons of the black hole. However, there also exists an upper bound on mass below which the associated value of r_+ represents the black hole event horizon. The metric function $f(r)$ is plotted in Fig. 4.6 for fixed values of parameters in different dimensions; the point at which the indicated curve intersects the horizontal axis gives location of the event horizon dependent on the given fixed values of parameters in the metric function. Similarly, the Hawking temperature of the obtained Gauss-Bonnet black hole is given by

$$T_H = \frac{r_+^2}{4\pi\left(1+\frac{2\overline{\alpha}_2}{r_+^2}\right)}\left[\frac{\overline{\alpha}_2(d-1)}{r_+^5}+\frac{(d-1)}{r_+^2}+\frac{\alpha_0}{(d-2)r_+}-\frac{2}{(d-2)r_+}\right.$$
$$\left.\times\sqrt{B+\frac{S^2}{r_+^{2d-2}}}-\frac{(d-2)^{q-1}(d-3)^q Q^{2q}}{r_+^{1+4q}}\right]-\frac{1}{2\pi r_+}. \tag{4.53}$$

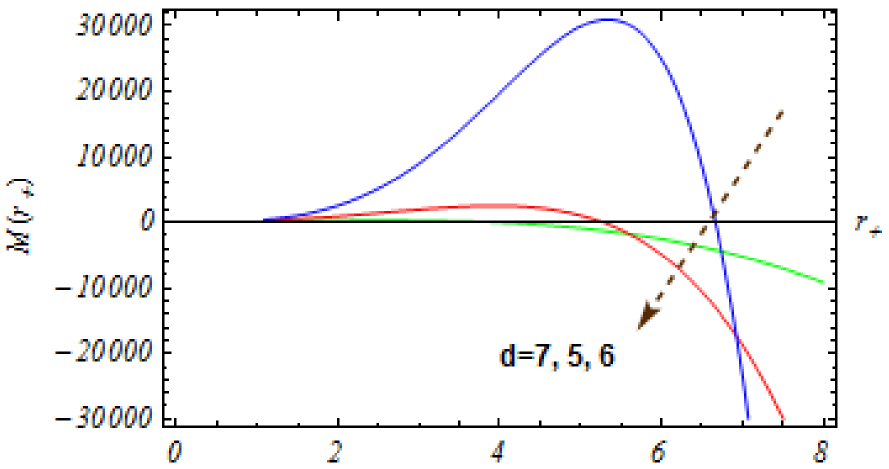

FIGURE 4.5: Plot of function $M(r_+)$ (Eq. (4.52)) for fixed values of $Q = 10$, $B = 0.1$, $\Sigma_{d-2} = 50$, $S = 0.5$, $\alpha_0 = 1$, $\overline{\alpha}_2 = 2$ and $q = 0.1$.

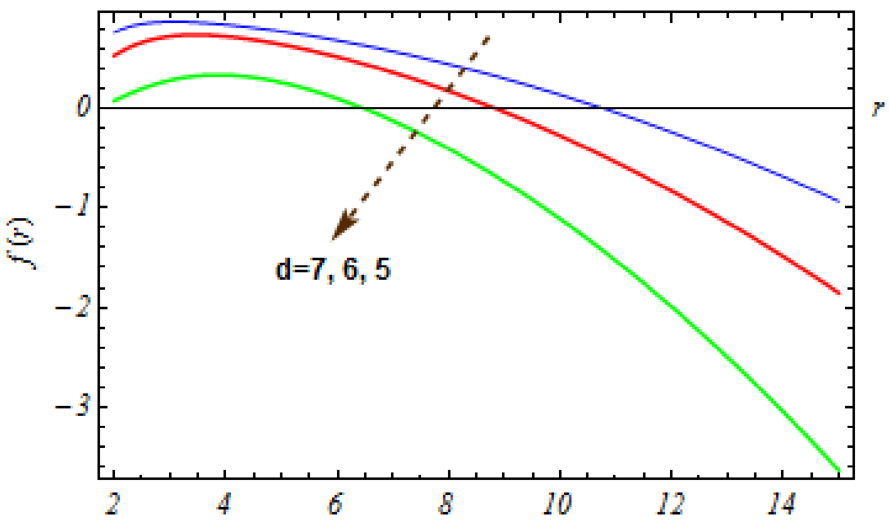

FIGURE 4.6: Plot of function $f(r)$ (Eq. (4.49)) for fixed values of $M = 100$, $Q = 0.10$, $B = 0.1$, $\Sigma_{d-2} = 50$, $S = 0.5$, $\alpha_0 = 0.1$, $\overline{\alpha}_2 = 2$ and $q = 5$.

The behaviour of Hawking temperature of Gauss-Bonnet black holes for different values of d is shown in Fig. 4.7. This graph shows that for the chosen values of parameters it is possible to obtain these types of black holes that are thermodynamically stable. Note that, for any fixed values of the parameters of matter sources in their respective ranges, the temperature as a function of r_+ can be plotted. The region where the temperature

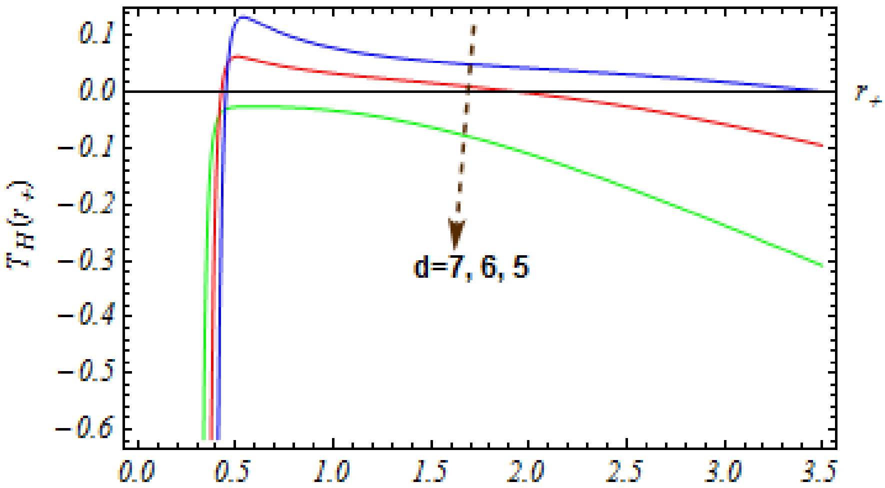

FIGURE 4.7: Plot of function $T_H(r_+)$ (Eq. (4.53)) for fixed values of $Q = 0.10$, $B = 0.1$, $\Sigma_{d-2} = 50$, $S = 0.5$, $\alpha_0 = -5$, $\overline{\alpha}_2 = 2$ (Solid line), $\overline{\alpha}_2 = 6$ (Dotted line), $\overline{\alpha}_2 = 12$ (Dashed line) and $q = 5$.

is positive for some values of these parameters the black hole is stable otherwise it is unstable. Further, it is clearly seen that stability increases with the dimensionality of spacetimes.

Now, differentiating Eq. (4.52) with respect to r_+ we have

$$\frac{dM}{dr_+} = \frac{\Sigma_{d-2}}{2}\Bigg[\overline{\alpha}_2(d-2)(d-5)r_+^{d-6} + (d-2)(d-3)r_+^{d-4} - \frac{2}{r_+^{2-d}}\sqrt{B + \frac{S^2}{r_+^{2d-2}}} - \frac{Q^{2q}(d-2)^q(d-3)^q}{r_+^{4q+2-d}}\Bigg], \tag{4.54}$$

So, by using Eqs. (4.53) and (4.54) in Eq. (4.30) we get entropy for the Gauss-Bonnet black hole as

$$S = 2(d-2)\pi\Sigma_{d-2}r_+^{d-4}\left(\frac{r_+^2}{d-2} + \frac{2\overline{\alpha}_2}{d-4}\right), \tag{4.55}$$

which clearly shows that in this case also the area law is not satisfied (as is the case in any other Lovelock black holes). Differentiation of Eq. (3.54) gives

$$\frac{dT_H}{dr_+} = \frac{r_+^2}{4\pi\left(1 + \frac{2\overline{\alpha}_2}{r_+^2}\right)}\frac{d\chi_1}{dr_+} + \frac{r_+\chi_1(r_+)}{2\pi\left(1 + \frac{2\overline{\alpha}_2}{r_+^2}\right)} + \frac{1}{4\pi r_+^2} + \frac{\overline{\alpha}_2\chi_1(r_+)}{\pi r_+\left(1 + \frac{2\overline{\alpha}_2}{r_+^2}\right)^2}, \tag{4.56}$$

where

$$\chi_1(r_+) = \frac{\overline{\alpha}_2(d-1)}{r_+^5} + \frac{d-1}{r_+^2} + \frac{\alpha_0}{(d-2)r_+} - \frac{2}{(d-1)r_+}\sqrt{B + \frac{S^2}{r_+^{2d-2}}} - \frac{(d-2)^{q-1}(d-3)^qQ^{2q}}{r_+^{4q+1}}, \tag{4.57}$$

$$\frac{d\chi_1}{dr_+} = \frac{5\overline{\alpha}_2(1-d)}{r_+^6} + \frac{2(1-d)}{r_+^3} - \frac{\alpha_0}{r_+^2(d-2)} + \frac{Q^{2q}(4q+1)(d-2)^{q-1}(d-3)^q}{r_+^{4q}}$$
$$+ \frac{2}{(d-1)r_+^2}\sqrt{B + \frac{S^2}{r_+^{2d-2}}} + \frac{2S^2}{r_+^{2d}\sqrt{B + \frac{S^2}{r_+^{2d-2}}}}. \tag{4.58}$$

Finally, using the quantities calculated above, the heat capacity of our Gauss-Bonnet black hole can be expressed in the following form

$$C_H = \frac{2\pi\Sigma_{d-2} r_+^2 \left(1 + \frac{2\overline{\alpha}_2}{r_+^2}\right)^2 \left[\chi_2(r_+) + (d-2)(d-3)r_+^{d-4} + \overline{\alpha}_2(d-2)(d-5)r_+^{d-6}\right]}{\left[r_+^4\left(1 + \frac{2\overline{\alpha}_2}{r_+^2}\right)\frac{d\chi_1}{dr_+} + 2r_+^3\chi_1(r_+)\left(1 + \frac{2\overline{\alpha}_2}{r_+^2}\right) + \left(1 + \frac{2\overline{\alpha}_2}{r_+^2}\right)^2 + 4\overline{\alpha}_2 r_+ \chi_1(r_+)\right]}, \tag{4.59}$$

where

$$\chi_2(r_+) = -\frac{2}{r_+^{2-d}}\sqrt{B + \frac{S^2}{r_+^{2d-2}}} - \frac{(d-2)^q(d-3)^q Q^{2q}}{r_+^{4q+2-d}}. \tag{4.60}$$

The graph of heat capacity is given in Fig. 4.8. The phase transition points are clearly

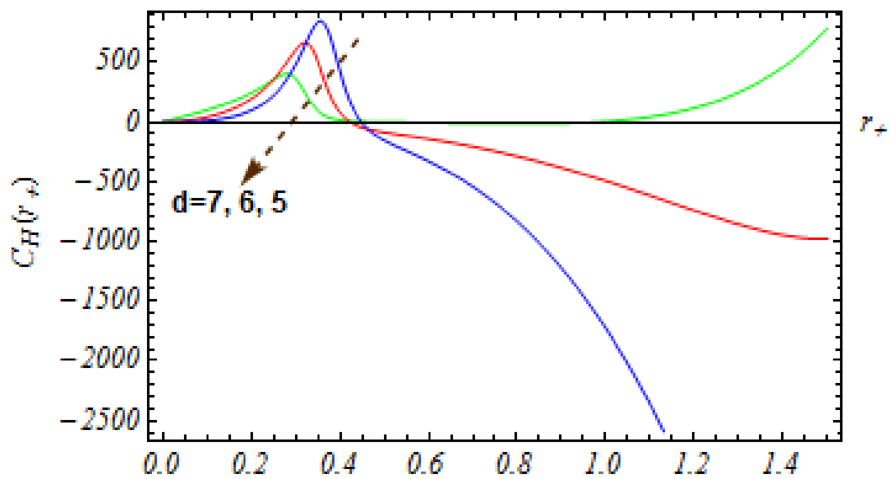

FIGURE 4.8: Plot of function C_H (Eq. (4.59)) vs r_+ for fixed values of $Q = 0.10$, $B = 0.1$, $\Sigma_{d-2} = 50$, $S = 0.5$, $\alpha_0 = -5$, $\overline{\alpha}_2 = 2$ (Solid line), $\overline{\alpha}_2 = 6$ (Dotted line), $\overline{\alpha}_2 = 12$ (Dashed line) and $q = 5$.

visible in each case, i.e., the points at which heat capacity changes sign represent the first order phase transition of the black hole and the point at which it diverges represents the second order phase transition.

Chapter 5

Magnetized topological black holes of dimensionally continued gravity

In this chapter, a large family of topological black hole solutions of dimensionally continued gravity are derived. The action of Lovelock gravity is coupled to the exponential electrodynamics and the equations of motion are solved in the presence of a pure magnetic source. We work out the metric functions in terms of the parameter β of exponential electrodynamics, and magnetic charge. Further, we couple Lovelock gravity to power-Yang-Mills theory and construct black holes, in diverse dimensions, having Yang-Mills magnetic charge. We also discuss the asymptotic bahaviour of metric functions and curvature invariants at the origin for both the models. The thermodynamics of resulting magnetized black hole solutions in the framework of two different models is also studied. The thermodynamic quantities like Hawking temperature, entropy and specific heat capacity at constant charge are found and we show that the resulting quantities satisfy the first law of black hole thermodynamics. We also study the magnetized hairy black holes of dimensionally continued gravity.

5.1 Topological black holes of DCG coupled to exponential electrodynamics

5.1.1 Magnetized black hole solution

The action function for Lovelock gravity with exponential electromagnetic source [145, 198, 216] in diverse dimensions is written in the form

$$\mathcal{I}=\frac{1}{16\pi G}\int d^dx\sqrt{-g}\left[\sum_{p=0}^{n-1}\frac{\alpha_p}{2^p}\delta^{\mu_1...\mu_{2p}}_{\nu_1...\nu_{2p}}R^{\nu_1\nu_2}_{\mu_1\mu_2}...R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}+\mathfrak{L}_{exp}(\mathcal{F})\right], \tag{5.1}$$

where $\mathfrak{L}_{exp}(\mathcal{F})$ is given by Eq. (2.1), G is the Newtonian constant while $d=n+2$ such that n is a natural number. The coefficients α_p in (5.1) represent arbitrary constants, however, in the particular case of DCG, they are chosen in the form

$$\alpha_p=C^{n-1}_p\frac{(d-2p-1)!}{(d-2)!l^{2(n-p-1)}}, \tag{5.2}$$

where l is related to the cosmological constant. It is important to note that DCG becomes Born-Infeld theory in even dimensions and Chern-Simons theory in odd dimensions [24, 25]. Varying (5.1) with respect to the electromagnetic potential A_μ yields the equations of motion of nonlinear electromagnetic field

$$\partial_\mu\left[\sqrt{-g}F^{\mu\nu}\Big(1-\beta\mathcal{F}\Big)\exp\left(-\beta\mathcal{F}\right)\right]=0. \tag{5.3}$$

After taking the variation of the action (5.1) with respect to the metric tensor, $g_{\mu\nu}$, one can arrive at the equations of gravitational field

$$\sum_{p=0}^{n-1}\frac{\alpha_p}{2^{p+1}}\delta^{\nu\lambda_1...\lambda_{2p}}_{\mu\rho_1...\rho_{2p}}R^{\rho_1\rho_2}_{\lambda_1\lambda_2}...R^{\rho_{2p-1}\rho_{2p}}_{\lambda_{2p-1}\lambda_{2p}}=\mathcal{T}^{\nu}_{\mu}, \tag{5.4}$$

where $\mathcal{T}^{\nu}_{\mu}$ takes the form similar as Eq. (2.3).

According to the causality principle [145, 198, 216] the group velocity of excitations in the background should not exceed the speed of light or we can say that tachyons will not appear. This principle will hold if $\partial L/\partial\mathcal{F}\leq 0$. Now from (2.1) we have

$$\frac{\partial\mathfrak{L}_{exp}}{\partial\mathcal{F}}=(\beta\mathcal{F}-1)\exp\left(-\beta\mathcal{F}\right), \tag{5.5}$$

which shows that causality is satisfied for our model if $(\beta\mathcal{F}-1)\leq 0$. This gives us the requirement for the case of pure magnetic field in d-dimensional spacetime

$$r\leq\left(\frac{2}{\beta Q^2}\right)^{\frac{1}{2d-4}}. \tag{5.6}$$

According to the unitarity principle the norm of every elementary vacuum excitation will be positive definite, which means $(\partial\mathfrak{L}_{exp}/\partial\mathcal{F}+2\mathcal{F}\partial^2\mathfrak{L}_{exp}/\partial\mathcal{F}^2)\leq 0$ and $\partial^2 L/\partial\mathcal{F}^2\geq 0$ [145]. Thus, this principle holds if and only if

$$r\leq\left(\frac{0.44}{\beta Q^2}\right)^{\frac{1}{2d-4}}. \tag{5.7}$$

The WEC is fulfilled if and only if the relations $\rho\geq 0$ and $(\rho+P_m)\geq 0$, where $m=1,2,3...,d-1$ are satisfied. Note that, ρ represents the energy density and P_m are the principle pressures for each m. Thus, using stress-energy tensor (2.3) it can easily be verified that WEC is satisfied if and only if (5.6) holds. From Eq. (2.3) it is also easy to verify that the DEC and SEC are also satisfied for our model if and only if

$$r^{2d-4}\leq\mid\frac{2}{\beta Q^2}\mid. \tag{5.8}$$

Note that, DEC guarantees that the speed of sound does not exceed the speed of light, while SEC says that there cannot be acceleration of the universe in the framework of exponential electrodynamics coupled with DCG.

As the action of Lovelock gravity (4.2) is the sum of the dimensionally extended Euler densities, it is shown that the equations of motion do not contain more than second order derivatives with respect to the metric tensor. Moreover, the Lovelock gravity has been shown to be a ghost-free theory when expanding on a flat space, avoiding any problems with unitarity [217]. Here, note that although the Lagrangian density corresponding to Lovelock gravity contains some curvature terms with higher order derivatives, in essence it is not a higher derivative gravity theory because its equations of motion do not involve terms higher than second derivatives of the metric. From this it becomes clear that the Lovelock gravity is a ghost-free theory [218]. The action function (4.2) looks very complicated due to the presence of so many terms. However, spherically symmetric static black hole solutions can be found [219] with the help of a real root of the corresponding polynomial equation. As the gravity (5.1) contains many arbitrary coefficients α_p it is not trivial to obtain physical information from the solution. Some authors [72–74] choose a special set of coefficients (5.2) which make the metric function simpler. These solutions could be described as spherically symmetric solutions. Further, the black hole solutions having nontrivial horizon topology in this theory with the special choice of coefficients have been studied [75, 220].

Here we determine the magnetically charged static black hole solution of DCG. For this we assume a pure magnetic field such that $\mathbf{E} = 0$, which yields from Maxwell's invariant, the form $\mathcal{F} = (B(r))^2/2 = Q^2/2r^{2d-4}$, where Q represents the magnetic charge. Now, the general static and spherically symmetric line element [221] can be considered as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(h_{ij}dx^i dx^j), \tag{5.9}$$

where $h_{ij}dx^i dx^j$ is the metric of n-dimensional hypersurface having constant curvature. Now using the above line element and substituting (2.3) in Eq. (5.4), the equation of motion becomes [219, 222]

$$\frac{d}{dr}\left[\sum_{p=0}^{n-1}\frac{\alpha_p(d-2)!}{2(d-2p-1)!}r^{d-1}\left(\frac{k-f(r)}{r^2}\right)^p\right] = 32\pi G\exp\left(\frac{-\beta Q^2}{2r^{2d-4}}\right)\left(\frac{\beta Q^4}{2r^{4d-8}} - \frac{Q^2}{2r^{2d-4}}\right), \tag{5.10}$$

where $k = 0, 1, -1$ associated to the codimensions-2 hypersurface with planar, spherical and hyperbolic topology respectively. If we chose α_p from Eq. (5.2), that is, for the case of DCG, the above equation becomes [219, 222]

$$\frac{d}{dr}\left[r^{d-1}\left(\frac{1}{l^2} + \frac{k-f(r)}{r^2}\right)^{n-1}\right] = 64\pi G\exp\left(\frac{-\beta Q^2}{2r^{2d-4}}\right)\left(\frac{\beta Q^4}{2r^{4d-8}} - \frac{Q^2}{2r^{2d-4}}\right). \tag{5.11}$$

Integration of the above equation with respect to r yields

$$f(r) = \frac{r^2}{l^2} + k - r^2\left[\frac{16\pi Gm}{r^{d-1}\Sigma_{d-2}} + \frac{\delta_d}{r^{d-1}} + \frac{\pi G Q^{\frac{1}{d-2}}\beta^{\frac{5-2d}{2d-4}}}{(d-2)2^{\frac{21-10d}{2d-4}}r^{d-1}}\left(\Gamma\left(\frac{2d-5}{2d-4}, \frac{\beta Q^2}{2r^{2d-4}}\right)\right.\right.$$
$$\left.\left. - 2\Gamma\left(\frac{4d-9}{2d-4}, \frac{\beta Q^2}{2r^{2d-4}}\right)\right)\right]^{\frac{1}{d-3}}, \tag{5.12}$$

where m is a constant of integration which is associated to the Arnowitt Deser Misner (ADM) mass of black hole, Σ_{d-2} represents the volume of n-dimensional hypersurface. The reason for the appearance of additional constant δ_d in Eq. (5.12) is that one can expect the horizon of the black hole to shrink to a single point when $m \to 0$ and the function $\Gamma(s, x)$ is the incomplete gamma function.

It is worth mentioning here that black strings and black branes can be thought of generalizations of black holes and they play a significant role in the AdS/CFT correspondence. It has been rightly pointed out [77] that while it is easy to construct black brane and black string solutions in the vacuum Einstein gravity by adding Ricci flat directions to Schwarzschild and Kerr solutions, it is nontrivial to obtain such solutions in Lovelock gravity, and numerical and other techniques have been used to construct such solutions.

Now we discuss the asymptotic behaviour of the metric function at $r = 0$. We take $k = 1$ here (the cases $k = 0, -1$ can be studied in a similar manner). Thus for even-dimensional spacetime, the asymptotic value of the metric function becomes

$$f(r) = 1 + \frac{r^2}{l^2} - m^{\frac{1}{2s-3}}\Bigg[r^{2k-5} + \frac{\delta_{d,2s}}{m(2s-3)}r^{2s-5} + \frac{\pi G}{m(2s-3)(2s-2)}\exp\left(\frac{-\beta Q^2}{2r^{4(s-1)}}\right) \times\left(\frac{32r^{4s-4}(3-2s)}{\beta(2s-2)} - \frac{Q^{\frac{1}{2s-2}}2^{\frac{20s-21}{4s-4}}\beta^{\frac{5-4s}{4s-4}}r^{6s-8}}{4s-4} - \frac{Q^2 2^{\frac{14s-15}{2s-2}}}{r^2 s} + O(r^{10s-12})\right)\Bigg], \quad d = 2s, s = 1, 2, 3.... \tag{5.13}$$

Similarly, for odd-dimensional spacetime we have

$$f(r) = 1 + \frac{r^2}{l^2} - m^{\frac{1}{2s-2}}\Bigg[r^{2s-4} + \frac{\delta_{d,2s+1}}{2m(s-1)}r^{2s-4} + \frac{\pi G}{2m(s-1)(2s-1)}\exp\left(\frac{-\beta Q^2}{2r^{4s-2}}\right) \times\left(\frac{64r^{2s-3}(1-s)}{\beta(2s-1)} - \frac{Q^{\frac{1}{2s-3}}2^{\frac{20s-11}{4s-2}}\beta^{\frac{3-4s}{4s-2}}r^{6s-5}}{4s-2} - \frac{Q^2 2^{\frac{14s-8}{2s-1}}}{r^{2s+1}} + O(r^{10s-7})\right)\Bigg], \quad d = 2s+1, s = 1, 2, 3.... \tag{5.14}$$

From the above asymptotic expressions, it is clearly seen that for the case of even-dimensions, metric function $f(r)$ is finite as $r \to 0$ for all values of $s \geq 3$. For the case of odd-dimensions, $f(r)$ yields finite value for all $s \geq 2$ at the origin, while at $s = 1$ the function is infinite. The event horizons can be found by solving the equation $f(r_+) = 0$. From Eq. (5.12), we can write the ADM mass of the black hole in the form as

$$m = \frac{r_+^{d-1}\Sigma_{d-2}}{16\pi G}\left(\frac{1}{l^2} + \frac{k}{r_+^2}\right)^{d-3} - \frac{\delta_d}{16\pi G} - \frac{Q^{\frac{1}{d-2}}\Sigma_{d-2}\beta^{\frac{5-2d}{2d-4}}}{(d-2)2^{\frac{5-2d}{2d-4}}}\left[\Gamma\left(\frac{2d-5}{2d-4}, \frac{\beta Q^2}{2r_+^{2d-4}}\right) - 2\Gamma\left(\frac{4d-9}{2d-4}, \frac{\beta Q^2}{2r_+^{2d-4}}\right)\right]. \tag{5.15}$$

From Fig. (5.1) it is clear that there exists a minimum value m_{ext} of m which corresponds to an extremal black hole. Extremal black holes are the black holes whose horizons coincide, which can be obtained by solving $f(r) = 0$ and $df/dr = 0$ simultaneously. For $r_+ = 0$, m takes a finite value m_0 for $d \geq 3$. If $m_{ext} < m < m_0$, there are two horizons, if $m \geq m_0$, there is a single horizon and if $m < m_{ext}$, there is no event horizon. In Fig. (5.2), the point where curve touches the horizontal axis indicates the position of the event horizon.

In general, the Ricci and Kretschmann scalars for the line element (5.9) are given by Eqs. (4.40)-(4.41), respectively. For arbitrary value of d the metric function (5.12) gives very lengthy and complicated expression for the Ricci scalar which we cannot write here. Avoiding these lengthy calculations we discuss the asymptotic behaviour of curvature

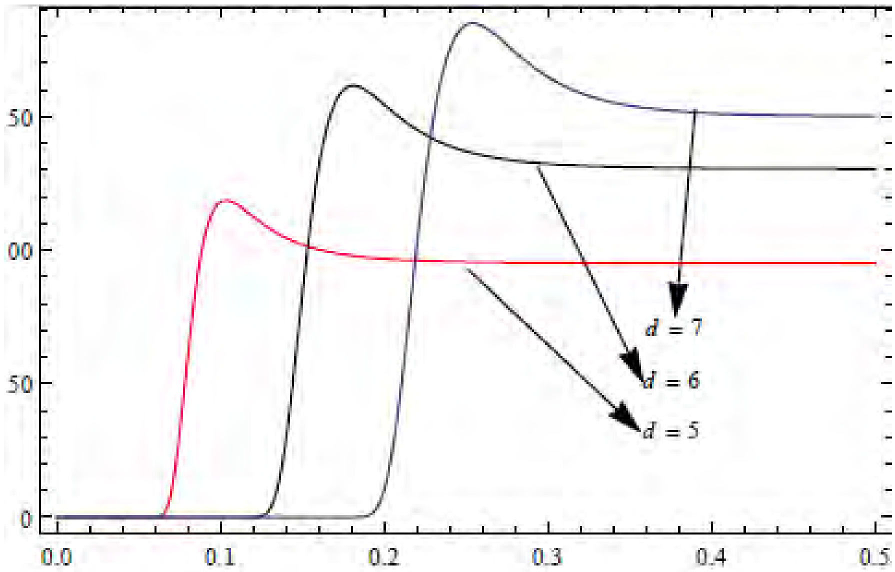

FIGURE 5.1: Plot of function m [Eq. (5.15)] vs r_+ for fixed values of $Q = 0.01$, $\beta = 0.01$ and $l = 1$.

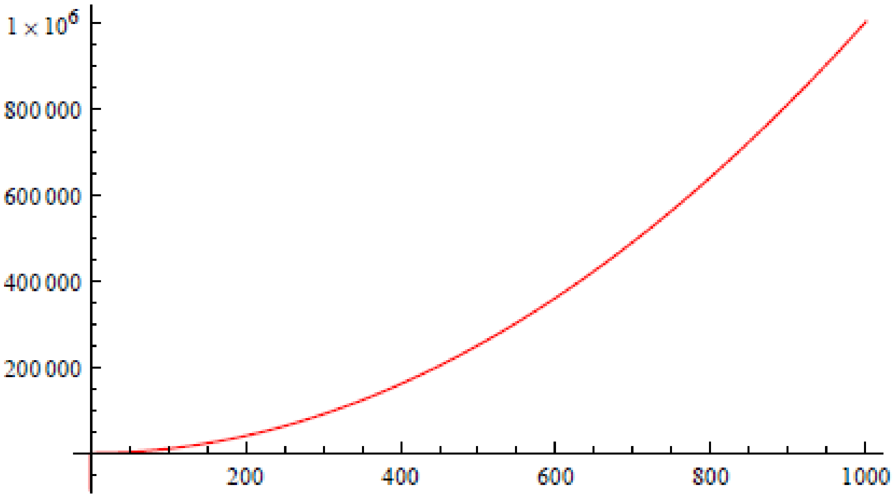

FIGURE 5.2: Plot of function $f(r)$ [Eq. (5.12)] vs r for fixed values of $m = 10$, $Q = 0.01$, $\beta = 0.01$ and $l = 1$.

invariants for $d = 5$. Differentiating (5.12) for the 5-dimensional case we can write

$$\begin{aligned}\frac{df}{dr} &= \frac{2r}{l^2} - \frac{2\beta^{\frac{1}{3}}}{r^{\frac{1}{3}}}\left[\delta_{d,5} + 16m\pi\left(\Gamma\left(\frac{5}{6},\frac{\beta Q^2}{2r^6}\right) - 2\Gamma\left(\frac{5}{6},\frac{\beta Q^2}{2r^6}\right)\right)\right]^{\frac{1}{3}} \\ &- \frac{4}{3^{\frac{2}{3}}r^{\frac{16}{3}}}\left[24\pi Q^2\beta^{\frac{5}{6}}\exp\left(\frac{-\beta Q^2}{2r^6}\right)\left(\Gamma\left(\frac{5}{6}\right)\right)^{-1} - \frac{Q^2\beta}{r^6}\left(\Gamma\left(\frac{11}{6}\right)\right)^{-1}\right) \\ &- (3\delta_{d,5} + 48m\pi)\beta^{\frac{5}{6}}r^5 - 2^{\frac{29}{6}}\pi Q^{\frac{1}{3}}r^5\left(\Gamma\left(\frac{5}{6},\frac{\beta Q^2}{2r^6}\right) - 2\Gamma\left(\frac{5}{6},\frac{\beta Q^2}{2r^6}\right)\right)\Bigg] \\ &\times\left[(3\delta_{d,5} + 48m\pi)\beta^{\frac{5}{6}} + 2^{\frac{29}{6}}\pi Q^{\frac{1}{3}}\left(\Gamma\left(\frac{5}{6},\frac{\beta Q^2}{2r^6}\right) - 2\Gamma\left(\frac{5}{6},\frac{\beta Q^2}{2r^6}\right)\right)\right]^{-1}. \qquad (5.16)\end{aligned}$$

Similarly, the second derivative of the metric function is

$$\frac{d^2 f}{dr^2} = \frac{2}{l^2} - \frac{2}{3^{\frac{1}{3}} r^{\frac{4}{3}}} \Pi_1^{\frac{1}{3}} + \frac{32\pi Q^2 \beta^{\frac{2}{9}} \Pi_2}{r^{\frac{55}{3}} \Pi_3} + \frac{32\Pi_4^2}{3^{\frac{7}{3}} r^{\frac{4}{3}} \Pi_5^{\frac{5}{3}}} + \frac{3^{\frac{-4}{3}} r^{\frac{-4}{3}} \Pi_6}{\Pi_5^{\frac{2}{3}}}, \tag{5.17}$$

where

$$\Pi_1 = 3\delta_{d,5} + 48m\pi + 2^{\frac{29}{6}} Q^{\frac{1}{3}} \beta^{\frac{-5}{6}} \left(\Gamma\left(\frac{5}{6}, \frac{\beta Q^2}{2r^6}\right) - 2\Gamma\left(\frac{5}{6}, \frac{\beta Q^2}{2r^6}\right) \right), \tag{5.18}$$

$$\Pi_2 = \exp\left(\frac{-\beta Q^2}{2r^6}\right) \left[Q^4 \beta^2 \Gamma\left(\frac{5}{6}\right) + 3r^{12} \Gamma\left(\frac{11}{6}\right) - Q^2 \beta r^6 \left(5\Gamma\left(\frac{5}{6}\right) + \Gamma\left(\frac{11}{6}\right) \right) \right], \tag{5.19}$$

$$\Pi_3 = \Gamma\left(\frac{5}{6}\right) \Gamma\left(\frac{11}{6}\right) \left[(3\delta_{d,5} + 48m\pi)\beta^{\frac{5}{6}} + 2^{\frac{29}{6}} \pi Q^{\frac{1}{3}} \left(\Gamma\left(\frac{5}{6}, \frac{\beta Q^2}{2r^6}\right) - 2\Gamma\left(\frac{5}{6}, \frac{\beta Q^2}{2r^6}\right) \right) \right], \tag{5.20}$$

$$\begin{aligned} \Pi_4 = {} & 3\delta_{d,5} + 48m\pi + \frac{84mQ^2\pi}{r} \left(\Gamma\left(\frac{11}{6}\right) \right)^{-1} \left(\Gamma\left(\frac{5}{6}\right) \right)^{-1} \exp\left(\frac{-\beta Q^2}{2r^6}\right) \\ & \times \left[Q^2 \beta \Gamma\left(\frac{5}{6}\right) - r^6 \Gamma\left(\frac{11}{6}\right) \right] + 2^{\frac{29}{6}} \pi Q^{\frac{1}{3}} \beta^{\frac{-5}{6}} \left[\Gamma\left(\frac{5}{6}, \frac{\beta Q^2}{2r^6}\right) - 2\Gamma\left(\frac{5}{6}, \frac{\beta Q^2}{2r^6}\right) \right], \end{aligned} \tag{5.21}$$

$$\Pi_5 = 3\delta_{d,5} + 48m\pi + 2^{\frac{29}{6}} \pi Q^{\frac{1}{3}} \beta^{\frac{-5}{6}} \left[\Gamma\left(\frac{5}{6}, \frac{\beta Q^2}{2r^6}\right) - 2\Gamma\left(\frac{5}{6}, \frac{\beta Q^2}{2r^6}\right) \right], \tag{5.22}$$

$$\begin{aligned} \Pi_6 = {} & 96\pi Q^2 \exp\left(\frac{-\beta Q^2}{2r^6}\right) \left[\frac{1}{r^5 \Gamma\left(\frac{5}{6}\right)} - \frac{Q^2 \beta}{r^{11} \Gamma\left(\frac{5}{6}\right)} \right] - 12\delta_{d,5} \\ & - 192m\pi - 2^{\frac{29}{6}} \pi Q^{\frac{1}{3}} \beta^{\frac{-5}{6}} \left(\Gamma\left(\frac{5}{6}, \frac{\beta Q^2}{2r^6}\right) - 2\Gamma\left(\frac{5}{6}, \frac{\beta Q^2}{2r^6}\right) \right). \end{aligned} \tag{5.23}$$

Using the above values in Eq. (4.40), we conclude that $R(r) \to \infty$ as r tends to zero. This shows that our resulting solution has a true curvature singularity at the origin $r = 0$, which indicates that our solution represents a black hole. Furthermore, the metric function $f(r)$ given by (5.12) becomes infinite as $r \to \infty$, which means that it represents non-asymptotically flat spacetime [216].

5.1.2 Thermodynamics of black holes of DCG with exponential magnetic source

The Hawking temperature of the black hole is given by

$$T_H(r_+) = \frac{r_+}{2\pi l^2} - \frac{r_+}{2\pi} [G(r_+)]^{\frac{1}{d-3}} - \frac{r_+^2}{4\pi} [G(r_+)]^{\frac{2-d}{d-3}} W(r_+), \tag{5.24}$$

where

$$G(r_+) = \frac{16\pi Gm}{\Sigma_{d-2} r_+^{d-1}} + \frac{\delta_d}{r_+^{d-1}} + \frac{\pi G Q^{\frac{1}{d-2}} \beta^{\frac{5-2d}{2d-4}}}{(d-2) 2^{\frac{21-10d}{2d-4}} r_+^{d-1}} H(r_+), \tag{5.25}$$

$$\begin{aligned} W(r_+) &= \frac{16\pi Gm(1-d)}{r_+^d \Sigma_{d-2}} + \frac{(1-d)\delta_d}{r_+^d} + \frac{\pi G Q^{\frac{1}{d-2}} (1-d) \beta^{\frac{5-2d}{2d-4}} H(r_+)}{(d-2) 2^{\frac{21-10d}{2d-4}} r_+^d} \\ &+ \frac{16\pi G Q^2}{(d-2) r_+^{4d-8}} \exp\left(-\frac{\beta Q^2}{2 r_+^{2d-4}}\right) \left((2d-4) r_+^{2d-4} + (4-2d) \beta^{\frac{3d-7}{2-d}} 2^{\frac{9-4d}{2-d}}\right) \end{aligned} \tag{5.26}$$

and

$$H(r_+) = \Gamma\left(\frac{2d-5}{2d-4}, \frac{\beta Q^2}{2 r_+^{2d-4}}\right) - 2\Gamma\left(\frac{4d-9}{2d-4}, \frac{\beta Q^2}{2 r_+^{2d-4}}\right). \tag{5.27}$$

To obtain the expression for heat capacity, we first find the black hole's entropy. A consistent approach which is very fruitful in getting conserved charges of a black hole was put forward by Wald [127, 128]. Black hole thermodynamics can be developed by using the derived conserved quantities. This Wald's formalism is applicable in general diffeomorphism-invariant theories and even in those types of theories where higher order derivatives are found. Wald's formalism has been helpful in studying thermodynamics of black holes in different theories, for example, Einstein-scalar theory [223, 224], Einstein-Proca theory [225], Einstein-Yang-Mills theory [226], gravity with quadratic curvature invariants [227], Lifschitz gravity [228] etc. In our work too i.e. the case of DCG we use the general formalism of Wald for obtaining the black hole's entropy. Hence the Wald entropy for our solution (5.12) is defined by

$$\begin{aligned} \mathcal{S} &= -2\pi \oint d^n x \sqrt{h} \frac{\partial \mathfrak{L}}{\partial R_{\mu\nu\rho\lambda}} \epsilon_{\mu\nu} \epsilon_{\rho\lambda} \\ &= \frac{(d-3)\Sigma_{d-2} r_+^d}{4kG(6-d)} \left(\frac{k}{r_+^2} + \frac{1}{l^2}\right)^{d-3} F_1\left[1, \frac{d}{2}, \frac{8-d}{2}, \frac{-r_+^2}{kl^2}\right], \end{aligned} \tag{5.28}$$

where $\epsilon_{\mu\nu}$ is the normal bivector of the $t = const$ and $r = r_+$ hypersurface such that $\epsilon_{\rho\lambda}\epsilon^{\rho\lambda} = -2$, F_1 is the Gaussian hypergeometric function. The magnetic potential conjugate to the magnetic charge Q is given by

$$\Phi_m = \frac{\partial m}{\partial Q} = \frac{\Sigma_{d-2}}{(d-2)} \exp\left(-\frac{\beta Q^2}{2 r_+^{2d-4}}\right) \left[\frac{2Q}{r_+^{2d-5}} - \frac{2\beta Q^3}{r_+^{4d-9}}\right] - \frac{\Sigma_{d-2} Q^{\frac{3-d}{d-2}} \beta^{\frac{5-2d}{2d-4}} H(r_+)}{(d-2)^2 2^{\frac{5-2d}{2d-4}}}. \tag{5.29}$$

Using the thermodynamic quantities obtained above, we can now easily confirm that the first law of thermodynamics

$$dm = T_H dS + \Phi_m dQ, \tag{5.30}$$

is satisfied. Now the specific heat capacity at a constant charge Q is given by

$$C_H = T_H \frac{\partial \mathcal{S}}{\partial T_H}|_Q. \tag{5.31}$$

Differentiating (5.24) we get

$$\begin{aligned}
\frac{\partial T_H}{\partial r_+} &= \frac{1}{2\pi l^2} - \frac{G(r_+)^{\frac{1}{d-3}}}{2\pi} - \frac{r_+}{2\pi(d-3)G(r_+)^{\frac{d-2}{d-3}}}\frac{dG(r_+)}{dr_+} \\
&- \left(\frac{r_+}{2\pi}G(r_+)^{\frac{d-2}{d-3}} + \frac{(d-2)r_+^2}{4\pi(3-d)G(r_+)^{\frac{2d-5}{d-3}}}\frac{dG(r_+)}{dr_+}\right)W(r_+) \\
&- \frac{r_+^2 G(r_+)^{\frac{d-2}{d-3}}}{4\pi}\frac{dW(r_+)}{dr_+},
\end{aligned} \tag{5.32}$$

where

$$\begin{aligned}
\frac{dW(r_+)}{dr_+} &= \frac{16\pi Gmd(d-1)}{\Sigma_{d-2}r_+^{d+1}} + \frac{d(d-1)\delta_d}{r_+^{d+1}} + \frac{\pi Gd(d-1)Q^{\frac{1}{d-2}}H(r_+)}{(d-2)2^{\frac{21-10d}{2d-4}}\beta^{\frac{2d-5}{2d-4}}r_+^{d+1}} \\
&+ \frac{\pi G(1-d)Q^{\frac{1}{d-2}}}{(d-2)2^{\frac{21-10d}{2d-4}}r_+^d\beta^{\frac{2d-5}{2d-4}}}\frac{dH(r_+)}{dr_+} + \frac{16\pi GQ^2(2d-4)^2}{(d-2)r_+^{2d-3}}\exp\left(\frac{-\beta Q^2}{2r_+^{2d-4}}\right) \\
&+ \left[\frac{8\pi GQ^4\beta(2d-4)^2}{r_+^{6d-11}(d-2)} - \frac{32\pi GQ^2(2d-4)^2}{(d-2)r_+^{4d-7}}\right] \\
&\times \exp\left(\frac{-\beta Q^2}{2r_+^{2d-4}}\right)\left(r_+^{2d-4} - \beta^{\frac{3d-7}{2-d}}Q^{\frac{6d-14}{2-d}}2^{\frac{9-4d}{2-d}}\right),
\end{aligned} \tag{5.33}$$

$$\frac{dG(r_+)}{dr_+} = \frac{16\pi Gm(1-d)}{\Sigma_{d-2}r_+^d} + \frac{(1-d)\delta_d}{r_+^d} + \frac{\pi GQ^{\frac{1}{d-2}}\beta^{\frac{5-2d}{2d-4}}}{(d-2)r_+^d 2^{\frac{21-10d}{2d-4}}}\left((1-d)H(r_+) + r_+\frac{dH(r_+)}{dr_+}\right), \tag{5.34}$$

and

$$\frac{dH(r_+)}{dr_+} = \frac{(4-2d)\beta^{\frac{4d-9}{2d-4}}Q^{\frac{4d-9}{d-2}}}{2^{\frac{2d-5}{2d-4}}r_+^{4d-8}}\exp\left(\frac{-\beta Q^2}{2r_+^{2d-4}}\right)\left[1 - \frac{\beta^{\frac{7-3d}{d-2}}r_+^{2d-4}}{Q^2}\right]. \tag{5.35}$$

Putting Eqs. (5.24), (5.28) and (5.32) in the general expression of heat capacity (5.31) we obtain

$$C_H = \left(\frac{r_+}{2\pi l^2} - \frac{r_+}{2\pi}[G(r_+)]^{\frac{1}{d-3}} - \frac{r_+^2}{4\pi}[G(r_+)]^{\frac{2-d}{d-3}}W(r_+)\right)\frac{(d-3)\Sigma_{d-2}r_+^d Z(r_+)}{4G(6-d)X(r_+)}\left(\frac{1}{r_+^2}+\frac{1}{l^2}\right)^{d-3}, \tag{5.36}$$

where

$$Z(r_+) = F_1\left[1, \frac{d}{2}, \frac{8-d}{2}, \frac{-r_+^2}{l^2}\right]\left(\frac{d}{r_+} + \frac{(d-3)r_+^2 l^2}{r_+^2+l^2}\right) - \frac{2r_+ d}{l^2(8-d)}F_1\left[2, \frac{d+2}{2}, \frac{10-d}{2}, \frac{-r_+^2}{l^2}\right], \tag{5.37}$$

and

$$\begin{aligned} X(r_+) &= \frac{1}{2\pi l^2} - \frac{G(r_+)^{\frac{1}{d-3}}}{2\pi} - \frac{r_+}{2\pi(d-3)G(r_+)^{\frac{d-2}{d-3}}}\frac{dG(r_+)}{dr_+} - \left(\frac{r_+}{2\pi}G(r_+)^{\frac{d-2}{d-3}}\right. \\ &+ \left.\frac{(d-2)r_+^2}{4\pi(3-d)G(r_+)^{\frac{2d-5}{d-3}}}\frac{dG(r_+)}{dr_+}\right)W(r_+) - \frac{r_+^2 G(r_+)^{\frac{d-2}{d-3}}}{4\pi}\frac{dW(r_+)}{dr_+}. \end{aligned} \tag{5.38}$$

The above Eq. (5.36) represents the general expression for black hole's heat capacity for any value of nonlinear electrodynamical parameter β. The black hole is stable if the Hawking temperature and heat capacity are positive. The black hole becomes unstable in the region where Hawking temperature or heat capacity become negative. The point at which the sign of Hawking temperature changes corresponds to the first order phase transition of black hole. The maximum of Hawking temperature identifies the second order phase transition since the heat capacity at that point is singular.

5.1.3 Hairy black holes of DCG with exponential magnetic source

The action describing Lovelock-scalar gravity with nonlinear exponential electrodynamics source [216] is defined by

$$\begin{aligned} \mathcal{I} &= \frac{1}{16\pi G}\int d^d x\sqrt{-g}\left[\sum_{p=0}^{n-1}\frac{\alpha_p}{2^p}\delta^{\mu_1\dots\mu_{2p}}_{\nu_1\dots\nu_{2p}}\left(\alpha_p R^{\nu_1\nu_2}_{\mu_1\mu_2}\dots R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\right.\right. \\ &+ \left.\left. 16\pi G b_p \phi^{d-4p} S^{\nu_1\nu_2}_{\mu_1\mu_2}\dots S^{\nu_{2p-1}\nu_{2p}}\right) + 4\pi G\mathfrak{L}_{exp}(\mathcal{F})\right], \end{aligned} \tag{5.39}$$

where $\mathfrak{L}_{exp}(\mathcal{F})$ corresponds to the Lagrangian density describing the exponential electromagnetic field [145, 198]. The second term in the integrand which contains ϕ^{d-4p}

denotes the Lagrangian density of the scalar field, where [229]

$$S_{\mu\nu}^{\rho\sigma} = \phi^2 R_{\mu\nu}^{\rho\sigma} - 2\delta_{[\mu}^{[\rho}\delta_{\nu]}^{\sigma]}\partial_\xi\phi\partial^\xi\phi - 4\phi\delta_{[\mu}^{[\rho}\partial_{\nu]}\partial^{\sigma]}\phi + 48\delta_{\mu}^{[\rho}\partial_\nu\phi\partial^{\sigma]}\phi. \tag{5.40}$$

Note that, by putting the scalar function ϕ equal to zero we get the action function defined in (5.1). The matter tensor corresponding to the scalar field is given by

$$[\mathcal{T}_\mu^\nu]^{(s)} = \sum_{p=0}^{d-3} \frac{b_p}{2^{p+1}} \phi^{d-4p} \delta_{\mu\rho_1\rho_2...\rho_{2p}}^{\nu\lambda_1\lambda_2...\lambda_{2p}} S_{\lambda_1\lambda_2}^{\rho_1\rho_2} ... S_{\lambda_{2p-1}\lambda_{2p}}^{\rho_{2p-1}\rho_{2p}}. \tag{5.41}$$

Thus the equation of motion for the scalar field becomes

$$\sum_{p=0}^{d-3} \frac{b_p(d-2p)}{2^p} \phi^{d-4p-1} \delta_{\rho_1\rho_2...\rho_{2p}}^{\lambda_1\lambda_2...\lambda_{2p}} S_{\lambda_1\lambda_2}^{\rho_1\rho_2} ... S_{\lambda_{2p-1}\lambda_{2p}}^{\rho_{2p-1}\rho_{2p}} = 0. \tag{5.42}$$

Taking variation of (5.39) with respect to the metric tensor, the field equations are obtained in the form [219, 222]

$$\sum_{p=0}^{n-1} \frac{\alpha_p}{2^{p+1}} \delta_{\mu\rho_1...\rho_{2p}}^{\nu\lambda_1...\lambda_{2p}} R_{\lambda_1\lambda_2}^{\rho_1\rho_2} ... R^{\rho_{2p-1}\rho_{2p}} = 16\pi G[\mathcal{T}_\mu^\nu]^{(M)} + 16\pi G[\mathcal{T}_\mu^\nu]^{(s)}, \tag{5.43}$$

where $[\mathcal{T}_\mu^\nu]^{(M)}$ corresponds to the stress-energy tensor of exponential electromagnetic field and α_p corresponds to the value defined as in Eq. (5.2) so that the Lovelock gravity becomes DCG. If we take the scalar configuration as [230]

$$\phi = \frac{N}{r}, \tag{5.44}$$

the equation describing the scalar field becomes

$$\sum_{p=0}^{d-3} \frac{b_p(d-1)!(d^2-d+4p^2)}{(d-2p-1)!} N^{-2p} = 0. \tag{5.45}$$

Using the assumption of pure magnetic field and substituting Eqs. (5.41) and (5.42) in Eq. (5.43) we get the metric function in the form

$$\begin{aligned} f(r) &= k + \frac{r^2}{l^2} - r^2 \Bigg[\frac{16\pi G m}{\Sigma_{d-2} r^{d-1}} + \frac{32\pi Y}{r^d} + \frac{\delta_d}{r^{d-1}} \\ &+ \frac{\pi G Q^{\frac{1}{d-2}} \beta^{\frac{5-2d}{2d-4}}}{(d-2) 2^{\frac{21-10d}{2d-4}}} \left(\Gamma\left(\frac{2d-5}{2d-4}, \frac{\beta Q^2}{2r^{2d-4}} \right) - 2\Gamma\left(\frac{4d-9}{2d-4}, \frac{\beta Q^2}{2r^{2d-4}} \right) \right) \Bigg]^{\frac{1}{d-3}}, \end{aligned} \tag{5.46}$$

where we have

$$Y = \sum_{p=0}^{d-3} b_p \frac{(d-2)!}{(d-2p-2)!} N^{d-2p}. \tag{5.47}$$

Now we investigate the asymptotic behaviour of metric function (5.46). In what follows, we take $k = 1$ and the cases $k = 0, -1$ can be studied in similar manners. Thus, for the odd-dimensional spacetimes we obtain

$$f(r) = 1 + \frac{r^2}{l^2} - m^{\frac{1}{2s-2}} \left[1 + \frac{32\pi Y r^{\frac{1}{1-s}}}{2m(s-1)} + \frac{\delta_{d,2s+1} r^{\frac{s-2}{s-1}}}{2m(s-1)} + \frac{\pi G Q^{\frac{1}{2s-1}} \beta^{\frac{3-4s}{4s-2}} r^{\frac{2s^2-s-2}{s-1}}}{(4s-2)(s-1)m2^{\frac{21-20s-10}{4s-2}}} \right.$$
$$\left. \times \exp\left(\frac{-\beta Q^2}{2r^{4s-2}}\right) \left(\frac{2^{\frac{1}{4s-2}}}{\beta^{\frac{1}{4s-2}} Q^{\frac{1}{2s-1}}} \left(2^{4s} - \frac{4s-3}{4s-2}\right) r - \frac{2^{\frac{1}{4s-2}} r^{3-4s}}{\beta^{\frac{3-4s}{4s-2}} Q^{\frac{3-4s}{2s-1}}} \right) \right],$$
$$d = 2s+1, s = 1, 2, 3.... \tag{5.48}$$

Similarly, for even-dimensional spacetime we have

$$f(r) = 1 + \frac{r^2}{l^2} - m^{\frac{1}{2s-3}} \left[1 + \frac{32\pi Y r^{\frac{2}{3-2s}}}{2s-3} + \frac{\delta_{d,2s} r^{\frac{2s-5}{2s-3}}}{m(2s-3)} + \frac{\pi G Q^{\frac{1}{2s-3}} \beta^{\frac{5-4s}{4(s-1)}} r^{\frac{4s^2-6s-2}{2s-3}}}{2(s-1)(2s-3)m2^{\frac{21-10s}{2(s-1)}}} \right.$$
$$\left. \times \exp\left(\frac{-\beta Q^2}{2r^{4(s-1)}}\right) \left(\frac{2^{\frac{1}{4(s-1)}}}{\beta^{\frac{1}{4(s-1)}} Q^{\frac{1}{2(s-1)}}} \left(2^{4s-2} - \frac{4s-5}{4s-4}\right) r - \frac{2^{\frac{1}{4(s-1)}} r^{5-4s}}{\beta^{\frac{5-4s}{4s-4}} Q^{\frac{5-4s}{2s-2}}} \right) \right],$$
$$d = 2s, s = 1, 2, 3.... \tag{5.49}$$

The above asymptotic expressions of metric functions show that in the case of both odd and even dimensions, metric functions are regular and finite at $r = 0$ for all values of $s > 1$. This finiteness of metric functions is due to the non-Maxwell behaviour of Lagrangian density (2.1) corresponding to exponential electrodynamics. Furthermore, at $r \to \infty$ the metric functions become infinite and hence this shows that the spacetime is nonasymptotically flat.

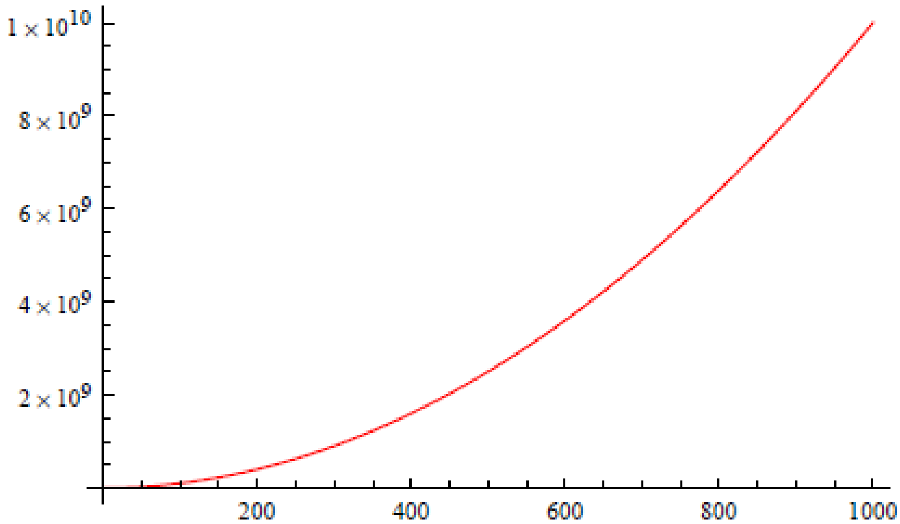

FIGURE 5.3: Plot of function $f(r)$ [Eq. (5.46)] vs r for fixed values of $m = 10$, $d = 6$, $Q = 10.0$, $\beta = 10.0$ and $l = 0.001$.

Fig. (5.3) shows the graph of function $f(r)$ vs r, for fixed values of β, m, Q and d. Note that the values of all other parameters are taken as unity. The point at which the curve touches the horizontal axis indicates the position of event horizon.

Furthermore, by using (5.46) with Eqs. (4.40) and (4.41), one can easily check that both the Ricci scalar and Kretschmann scalar give infinite values at $r = 0$ for any value of d. This shows that the metric function possesses true curvature singularity at the origin. Hence, the line element (5.9) with metric function given by (5.46) describes a hairy black hole solution of DCG sourced by exponential electrodynamics. For the four-dimensional case, it is seen from Eq. (5.45) that all the coefficients b_p vanish, thus in the case of $d = 4$, hairy black holes do not exist. It is easy to see that, by taking $Y = 0$, this black hole solution reduces to the black hole solution with no scalar hair, i.e. Eq. 5.12.

5.2 Topological black holes of DCG coupled to power-Yang-Mills theory

5.2.1 Black hole solution with power-Yang-Mills source

The action function describing the Lovelock-power-Yang-Mills theory is given by

$$\mathcal{I} = \frac{1}{16\pi G}\int d^d x\sqrt{-g}\Bigg[\sum_{p=0}^{n-1}\frac{\alpha_p}{2^p}\delta^{\mu_1\ldots\mu_{2p}}_{\nu_1\ldots\nu_{2p}}R^{\nu_1\nu_2}_{\mu_1\mu_2}\ldots R^{\nu_{2p-1}\nu_{2p}} + (\Upsilon)^q\Bigg], \tag{5.50}$$

where Υ is the Yang-Mills invariant defined by Eq. (4.7). Taking variation of the action defined in (5.50) with respect to the metric tensor gives the Lovelock field equations (5.4) with stress-energy tensor of power-Yang-Mills field given by (4.9). Similarly, the equations of Yang-Mills field (4.10) can be obtained if we vary the action with respect to gauge potentials $A^{(a)}$.

By taking the static metric in the form (5.9), it can easily be checked that power-Yang-Mills equations are satisfied for the choice of Yang-Mills gauge potential one-forms (4.11). Thus the symmetric stress-energy tensor (4.9) in d-dimensional spacetime can be computed in the form (4.13).

The causality principle is satisfied for power-Yang-Mills model if $(\partial\mathfrak{L}_{PYM}/\partial\Upsilon) \leq 0$ and unitarity principle is satisfied when $(\partial\mathfrak{L}_{PYM}/\partial\Upsilon + 2\Upsilon\partial^2\mathfrak{L}_{PYM}/\partial\Upsilon^2) \leq 0$ and $\partial^2\mathfrak{L}_{PYM}/\partial\Upsilon^2 \geq 0$. One can show that the causality does not hold in this model for all values of positive integer $d > 3$. However, for $d \leq 3$ this principle holds. By using the Lagrangian density of power-Yang-Mills theory it is shown that the unitarity principle holds for all values of $d \geq 3$ and $q \leq (1/2)$. When $d \leq 3$, this principle is satisfied for all values of $q \geq 3$ for power-Yang-Mills magnetic field.

The SEC in DCG coupled to power-Yang-Mills field holds for all values of $q \geq (d-1)/4$. This energy condition tells us that the universe did not experience any acceleration in this model. However WEC and DEC do not hold for all values of $d > 3$, in contrast to the exponential model coupled to DCG. Therefore, WEC and DEC hold only for (2+1)-dimensional black holes of DCG with power-Yang-Mills source. Now by using Eqs. (5.2), (5.9) and (4.13) in the field equations (5.4), we get

$$\frac{d}{dr}\left[r^{d-1}\left(\frac{1}{l^2}+\frac{k-f(r)}{r^2}\right)^{d-3}\right] = \frac{-32\pi G Q^{2q}[(d-2)(d-3)]^q}{r^{4q}}. \tag{5.51}$$

Direct integration with respect to r yields the metric function

$$f(r) = k + \frac{r^2}{l^2} - r^2\left[\frac{32\pi G Q^{2q}[(d-2)(d-3)]^q}{(4q-1)r^{4q+d-2}} + \frac{16\pi G m}{\Sigma_{d-2} r^{d-1}} + \frac{\delta}{r^{d-1}}\right]^{\frac{1}{d-3}}. \tag{5.52}$$

Now we discuss the asymptotic behaviour of our solution at $r = 0$. For this we will work out for $k = 1$ as follows:

$$\begin{aligned} f(r) &= 1 + \frac{r^2}{l^2} - M r^{\frac{2s-5}{2s-3}} - \frac{32\pi G Q^{2q}(2s-2)^q(2s-3)^{q-1}}{(4q-1)} r^{\frac{4s-8-8qs+12q}{2s-3}} \\ &+ \frac{512\pi^2 G^2 Q^{4q}(2s-2)^{2q}(2s-3)^{2q-2}}{M(4q-1)^2} r^{\frac{6s-11-16qs+24q}{2s-3}} + O(r^{\frac{8s-15-24qs+36q}{2s-3}}), \\ & d = 2s,\ s = 1, 2, 3..., \end{aligned} \tag{5.53}$$

$$\begin{aligned} f(r) &= 1 + \frac{r^2}{l^2} - M r^{\frac{s-2}{s-1}} - \frac{32\pi G Q^{2q}(2s-1)^q(2s-2)^{q-1}}{(4q-1)} r^{\frac{4s-6-8qs+8q}{2s-2}} \\ &+ \frac{512\pi^2 G^2 Q^{4q}(2s-1)^{2q}(2s-2)^{2q-2}}{M(4q-1)^2} r^{\frac{6s-8-16qs+16q}{2s-2}} + O(r^{\frac{8s-15-24qs+24q}{2s-2}}), \\ & d = 2s+1,\ s = 1, 2, 3.... \end{aligned} \tag{5.54}$$

These two expressions indicate that both for even and odd-dimensional spacetimes the metric function (5.52) is regular and finite for all values of $s > 1$. Note that in the above we assumed $(m + \delta_{d,2s,2s+1})^{1/d-3} = M$. Furthermore, at $r \to \infty$ the metric goes to infinity which shows that the spacetime is nonasymptotically flat.

Fig. (5.4) shows the graph of $f(r)$ vs r for fixed values of the parameters involved in metric function (5.52). The curve touching the horizontal axis indicates the position of the event horizon. We get extremal black holes when horizons coincide, and they are obtained by solving $f(r) = 0$ and $df/dr = 0$ simultaneously. This set of simultaneous

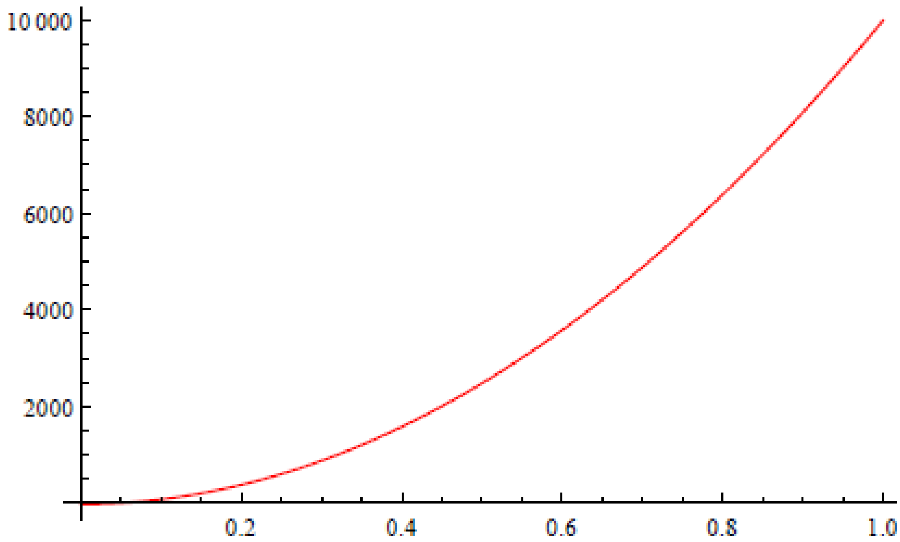

FIGURE 5.4: Plot of function $f(r)$ [Eq. (5.52)] vs r for fixed values of $m = 10$, $d = 5$, $Q = 0.01$, $q = 0.001$ and $l = 0.01$.

equations for metric function (5.52), without cosmological constant i.e. for $l \to \infty$, yields

$$M_e = \frac{k^{\frac{1}{d-3}}}{r_e^{4d-d^2-1}} - \frac{32\pi G(d-2)^q(d-3)^q Q^{2q} r_e^{1-4q}}{4q-1}, \tag{5.55}$$

$$\begin{aligned} Q_e = & \frac{(d-3)^{1-q}(1-4q)k^{\frac{d-4}{d-3}} r_e^{d^3-9d^2+23d+4q-10}}{16\pi G(d-2)^q(4q+1)} \\ & + \frac{r_e^{d+4q-2}(1-4q)}{32\pi G(d-2)^q(d-3)^q(4q+1)}\left(\frac{(1-d)k^{\frac{1}{d-3}}}{r_e^{5d-d^2-2}}\right), \end{aligned} \tag{5.56}$$

where $M_e = 16\pi G m_e/\Sigma_{d-2} + \delta_d$, related to the extremal mass m_e, r_e is the extremal radius and Q_e is the extremal charge.

Now, we confirm that our resulting solution possesses a central essential singularity. For this, we discuss the asymptotic behaviour of curvature scalars at $r = 0$ as follows. Differentiating (5.52) we obtain

$$\begin{aligned} \frac{df}{dr} = & \frac{2r}{l^2} - 2r\left[\frac{16\pi Gm}{\Sigma_{d-2}r^{d-1}} + \frac{\delta_d}{r^{d-1}} + \frac{32\pi G Q^{2q}(d-2)^q(d-3)^q}{(4q-1)r^{4q+d-2}}\right]^{\frac{1}{d-3}} \\ & - \frac{r^2}{d-3}\left[\frac{16\pi Gm}{\Sigma_{d-2}r^{d-1}} + \frac{\delta_d}{r^{d-1}} + \frac{32\pi G Q^{2q}(d-2)^q(d-3)^q}{(4q-1)r^{4q+d-2}}\right]^{\frac{4-d}{d-3}} \\ & \times\left(\frac{16\pi Gm(1-d)}{\Sigma_{d-2}r^d} + \frac{(1-d)\delta_d}{r^d} + \frac{32\pi G Q^{2q}(d-2)^q(d-3)^q(2-d-4q)}{(4q-1)r^{4q+d-1}}\right). \end{aligned} \tag{5.57}$$

Differentiating again we get

$$\frac{d^2 f}{dr^2} = \frac{2}{l^2} - 2P^{\frac{1}{d-3}} - \frac{4rP^{\frac{4-d}{d-3}}}{d-3}\frac{dP}{dr} - \frac{r^2 P^{\frac{4-d}{d-3}}}{(d-3)}\frac{d^2P}{dr^2} - \frac{r^2(4-d)P^{\frac{7-2d}{d-3}}}{(d-3)^2}\left(\frac{dP}{dr}\right)^2, \tag{5.58}$$

where

$$P(r)=\frac{16\pi Gm}{\Sigma_{d-2}r^{d-1}}+\frac{\delta_d}{r^{d-1}}+\frac{32\pi GQ^{2q}(d-2)^q(d-3)^q}{(4q-1)r^{4q+d-2}}, \tag{5.59}$$

$$\frac{dP}{dr}=\frac{16\pi Gm(1-d)}{\Sigma_{d-2}r^{d}}+\frac{(1-d)\delta_d}{r^{d}}+\frac{32\pi GQ^{2q}(d-2)^q(d-3)^q(2-d-4q)}{(4q-1)r^{4q+d-1}}, \tag{5.60}$$

and

$$\frac{d^2P}{dr^2}=\frac{16\pi Gmd(d-1)}{\Sigma_{d-2}r^{d+1}}+\frac{d(d-1)\delta_d}{r^{d+1}}+\frac{32\pi GQ^{2q}(d-2)^q(d-3)^q(2-d-4q)(1-d-4q)}{(4q-1)r^{4q+d}}. \tag{5.61}$$

We can work out the Ricci scalar for any value of d and, therefore, with the help of Eq. (4.40) and the above equations of metric function and its derivatives, we find that $R\to\infty$ as $r\to 0$. Similarly, Kretschmann scalar (4.41) also possess singularity at $r=0$. Thus the line element (5.9) with the metric function (5.52) represents higher dimensional magnetized black hole solution of DCG for any value of the parameter q. However, it indicates that $q=1$ will give the solution of DCG coupled to the standard Yang-Mills theory and for $q=1/4$ the solution does not exist. By substituting the Yang-Mills magnetic charge $Q=0$, we get the neutral solution in DCG.

5.2.2 Thermodynamics of black holes of DCG with power-Yang-Mills magnetic source

It is observed from the metric function (5.52) that the physical properties of such type of black holes depend on the parameter q. The black hole horizons are given by $f(r_+)=0$, where r_+ represents the location of horizon. Thus

$$m=\frac{\Sigma_{d-2}r_+^{d-1}\left(\frac{1}{l^2}+\frac{k}{r_+^2}\right)^{d-3}}{16\pi G}-\frac{\Sigma_{d-2}\delta_d}{16\pi G}-\frac{2Q^{2q}\Sigma_{d-2}[(d-2)(d-3)]^q}{(4q-1)r_+^{4q-1}}. \tag{5.62}$$

Equation (5.62) gives the ADM mass of the black hole in terms of the horizon radius and Yang-Mills magnetic charge Q. The black hole's Hawking temperature can be computed as

$$
\begin{aligned}
T_H &= \frac{1}{4\pi}\frac{df}{dr}|_{r=r_+} \\
&= \frac{r_+}{2\pi l^2} - \frac{r_+}{2\pi}\left[\frac{32\pi G Q^{2q}(d-2)^q(d-3)^q}{(4q-1)r_+^{4q+d-2}} + \frac{16\pi G m}{\Sigma_{d-2}r_+^{d-1}} + \frac{\delta_d}{r_+^{d-1}}\right]^{\frac{1}{d-3}} \\
&- \frac{r_+^2}{4\pi(d-3)}\left[\frac{32\pi G Q^{2q}(d-2)^q(d-3)^q}{(4q-1)r_+^{4q+d-2}} + \frac{16\pi G m}{\Sigma_{d-2}r_+^{d-1}} + \frac{\delta_d}{r_+^{d-1}}\right]^{\frac{4-d}{d-3}} \\
&\times \left[\frac{32\pi G(2-d-4q)Q^{2q}(d-2)^q(d-3)^q}{(4q-1)r_+^{4q+d-1}} + \frac{(1-d)16\pi G m}{\Sigma_{d-2}r_+^{d}} + \frac{(1-d)\delta_d}{r_+^{d}}\right] \quad (5.63)
\end{aligned}
$$

The entropy of the black hole in this case using Wald's method [127, 128] becomes

$$
\mathcal{S} = \frac{(d-3)\Sigma_{d-2}r_+^d}{4kG(6-d}\left(\frac{k}{r_+^2} + \frac{1}{l^2}\right)^{d-3} F_1\left[1, \frac{d}{2}, \frac{8-d}{2}, \frac{-r_+^2}{kl^2}\right]. \quad (5.64)
$$

The Yang-Mills magnetic potential conjugate to Yang-Mills magnetic charge Q is given by

$$
\Phi_m = \frac{-64\pi G q(d-2)^q(d-3)^q Q^{2q-1}}{(4q-1)r_+^{4q-1}}. \quad (5.65)
$$

Thus, using the above thermodynamic quantities, it can be seen that the first law of black hole thermodynamics given by (5.30) is satisfied. The free energy density of the resulting black hole can also be calculated as

$$
\Xi = m - T_H S, \quad (5.66)
$$

where m, T_H and S are the ADM mass, Hawking temperature and entropy of the black hole given by Eqs. (5.62), (5.63) and (5.64), respectively. Similarly, the heat capacity at constant charge is written as

$$
C_H = \frac{\left[\frac{r_+^{d+1}}{2\pi l^2} - \frac{r_+^{d+1}H_1^{\frac{1}{d-3}}}{2\pi} - \frac{r_+^{d+2}}{4\pi(d-3)H_1^{\frac{d-4}{d-3}}}\frac{dH_1}{dr_+}\right](d-3)\Sigma_{d-2}H_2(r_+)\left(\frac{k}{r_+^2} + \frac{1}{l^2}\right)^{d-3}}{4kG\left[\frac{1}{2\pi l^2} - \frac{H_1^{\frac{1}{d-3}}}{2\pi} - \frac{r_+H_1^{\frac{4-d}{d-3}}}{\pi(d-3)}\frac{dH_1}{dr_+} - \frac{r_+^2H_1^{\frac{4-d}{d-3}}}{4\pi(d-3)}\frac{d^2H_1}{dr_+^2} + \frac{r_h^2(d-4)H_1^{\frac{7-2d}{d-3}}}{4\pi(d-3)^2}\left(\frac{dH_1}{dr_+}\right)^2\right]}, \quad (5.67)
$$

where

$$
H_1(r_+) = \frac{32\pi G Q^{2q}[(d-2)(d-3)]^q}{(4q-1)r_+^{4q+d-2}} + \frac{16\pi G m}{\Sigma_{d-2}r_+^{d-1}} + \frac{\delta_d}{r_+^{d-1}}, \quad (5.68)
$$

$$\frac{dH_1(r_+)}{dr_+} = \frac{32\pi G Q^{2q}(2-4q-d)[(d-2)(d-3)]^q}{(4q-1)r_+^{4q+d-1}} + \frac{16\pi G(1-d)m}{\Sigma_{d-2}r_+^d} + \frac{(1-d)\delta_d}{r_+^d}, \tag{5.69}$$

$$\begin{aligned}\frac{d^2H_1(r_+)}{dr_+^2} &= \frac{32\pi G Q^{2q}(2-4q-d)(1-4q-d)[(d-2)(d-3)]^q}{(4q-1)r_+^{4q+d}} \\ &+ \frac{16\pi G d(d-1)m}{\Sigma_{d-2}r_+^{d+1}} + \frac{d(d-1)\delta_d}{r_+^{d+1}},\end{aligned} \tag{5.70}$$

and

$$H_2(r_+) = \left[\frac{d}{r_+} + \frac{r_+^2 l^2(d-3)}{kl^2+r_+^2}\right] F_1\left(1, \frac{d}{2}, \frac{8-d}{2}, \frac{-r_+^2}{kl^2}\right) + \frac{2r_+ d}{kl^2(d-2)} F_1\left(2, \frac{d+2}{2}, \frac{10-d}{2}, \frac{-r_+^2}{kl^2}\right). \tag{5.71}$$

5.2.3 Hairy black holes of DCG with power-Yang-Mills magnetic source

The action function for Lovelock-scalar gravity with power-Yang-Mills source [178, 216] is defined by

$$\begin{aligned}\mathcal{I} = \frac{1}{16\pi G}\int d^dx\sqrt{-g}\Bigg[\sum_{p=0}^{n-1}\frac{\alpha_p}{2^p}\delta^{\mu_1...\mu_{2p}}_{\nu_1...\nu_{2p}}\Big(a_p R^{\nu_1\nu_2}_{\mu_1\mu_2}...R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}} \\ + 16\pi G b_p \phi^{d-4p} S^{\nu_1\nu_2}_{\mu_1\mu_2}...S^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}\Big) + 4\pi G(\Upsilon)^q\Bigg],\end{aligned} \tag{5.72}$$

where Υ corresponds to the power-Yang-Mills invariant defined by (4.7) and (4.8) and we have used the Lagrangian density of the scalar field given by

$$\mathfrak{L}_s = \sum_{p=0}^{n-1}\frac{b_p}{2^p}\phi^{d-4p}\delta^{\mu_1...\mu_{2p}}_{\nu_1...\nu_{2p}} S^{\nu_1\nu_2}_{\mu_1\mu_2}...S^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}}, \tag{5.73}$$

where $S^{\rho\sigma}_{\mu\nu}$ is given by Eq. (5.40). Note that, by putting scalar function ϕ equal to zero, the above action function reduces to the case of DCG coupled to power-Yang-Mills field, i.e., (5.50).

Taking variation of (5.72) with respect to the metric tensor, we obtain the field equations as

$$\sum_{p=0}^{n-1}\frac{\alpha_p}{2^{p+1}}\delta^{\nu\lambda_1...\lambda_{2p}}_{\mu\rho_1...\rho_{2p}} R^{\rho_1\rho_2}_{\lambda_1\lambda_2}...R^{\rho_{2p-1}\rho_{2p}}_{\mu_{2p-1}\mu_{2p}} = 16\pi G[\mathcal{T}^\nu_\mu]^{(M)} + 16\pi G[\mathcal{T}^\nu_\mu]^{(s)}, \tag{5.74}$$

where $[\mathcal{T}^\nu_\mu]^{(M)}$ corresponds to the matter tensor of power-Yang-Mills field. while α_p corresponds to the value defined as in Eq. (5.2) so that the Lovelock gravity becomes DCG. The stress-energy tensor corresponding to the scalar field is given by (5.41) and the equation of motion of scalar field is given by (5.42). Choosing the scalar configuration (5.44) and using Eqs. (5.41) and (5.42) in Eq. (5.74), we get the metric function in the form

$$f(r)=k+\frac{r^2}{l^2}-r^2\left[\frac{16\pi Gm}{\Sigma_{d-2}r^{d-1}}+\frac{32\pi Y}{r^d}+\frac{\delta_d}{r^{d-1}}+\frac{32\pi GQ^{2q}(d-2)^q(d-3)^q}{(4q-1)r^{4q+d-2}}\right]^{\frac{1}{d-3}}, \tag{5.75}$$

where Y is given by (5.47).

Now, we discuss the asymptotic behaviour of metric function (5.75) at $r=0$. So, assuming $M=16\pi Gm/\Sigma_{d-2}$ and $\mu=\delta_d+M$ we can write the asymptotic expressions as follows:

$$\begin{aligned}f(r)=1+\frac{r^2}{l^2}-(32\pi Y)^{\frac{1}{2s-3}}\Bigg[&r^{\frac{2s-6}{2s-3}}+\frac{\mu r^{\frac{4s-9}{2s-3}}}{32\pi Y(2s-3)}\\&+\frac{GQ^{2q}(2s-2)^q(2s-3)^{q-1}}{Y(4q-1)}r^{\frac{6s-12-8qs+12q}{2s-3}}+O(r^{\frac{6s-12}{2s-3}})\Bigg], d=2s, s=1,2,3...,\end{aligned} \tag{5.76}$$

$$\begin{aligned}f(r)=1+\frac{r^2}{l^2}-(32\pi Y)^{\frac{1}{2s-2}}\Bigg[&r^{\frac{2s-5}{2s-2}}+\frac{\mu r^{\frac{4s-7}{2s-3}}}{32\pi Y(2s-2)}\\&+\frac{GQ^{2q}(2s-1)^q(2s-2)^{q-1}}{Y(4q-1)}r^{\frac{6s-9-8qs+8q}{2s-2}}+O(r^{\frac{6s-9}{2s-2}})\Bigg], d=2s+1, s=1,2,3....\end{aligned} \tag{5.77}$$

The above asymptotic expansions show that for the even-dimensional spacetime, the metric function is finite and regular for all values of s at the origin. However, for the case of odd dimensions, the metric function is regular and finite at the origin when $s>1$. Note that this finiteness is due to the nonlinear behaviour of Lagrangian density corresponding to power-Yang-Mills theory. Furthermore, the solution given by (5.75) is nonasymptotically flat as it grows infinitely for large values of r.

Fig. 5.5 shows the graph of $f(r)$ vs r, for fixed values of parameters involved in metric function (5.75). The event horizon is the value of r at which the curve intersects the horizontal axis. Extremal quantities corresponding to the extremal black hole solution can be obtained by solving $f(r)=0$ and $df/dr=0$ simultaneously. This set of simultaneous equations for our metric function (5.75), without cosmological constant i.e. for

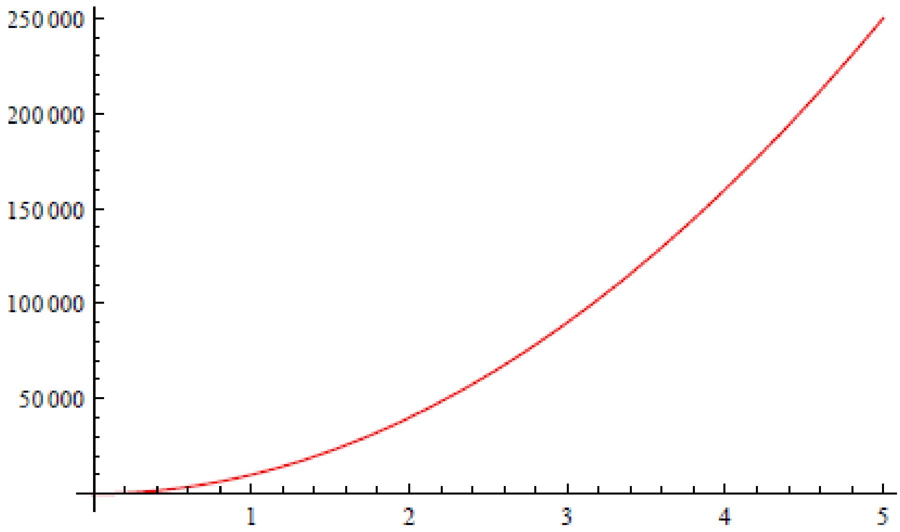

FIGURE 5.5: Plot of function $f(r)$ [Eq. (5.75)] vs r for fixed values of $m = 100$, $d = 5$, $Q = 0.10$, $q = 0.001$ and $l = 0.01$.

$l \to \infty$, is given by

$$M_e = \frac{k^{\frac{1}{d-3}}}{r_e^{4d-d^2-1}} - \frac{32\pi Y}{r_e} - \frac{32\pi G(d-2)^q(d-3)^q Q^{2q} r_e^{1-4q}}{4q-1}, \tag{5.78}$$

$$Q_e = \frac{(d-3)^{1-q}(1-4q)k^{\frac{d-4}{d-3}} r_e^{d^3-9d^2+23d+4q-10}}{16\pi G(d-2)^q(4q+1)} + \frac{r_e^{d+4q-2}(1-4q)}{32\pi G(d-2)^q(d-3)^q(4q+1)}\left(\frac{(1-d)k^{\frac{1}{d-3}}}{r_e^{5d-d^2-2}} - \frac{32\pi Y}{r_e^d}\right), \tag{5.79}$$

where $M_e = 16\pi G m_e/\Sigma_{d-2} + \delta_d$, related to the extremal mass m_e, r_e is the extremal radius and Q_e is the extremal charge.

Now we confirm that our resulting solution represents a black hole. This can be done by studying the asymptotic behaviour of curvature invariants at the origin. For this we differentiate Eq. (5.75) and obtain

$$\frac{df}{dr} = \frac{2r}{l^2} - 2r\left[\frac{16\pi Gm}{\Sigma_{d-2}r^{d-1}} + \frac{32\pi Y}{r^d} + \frac{\delta_d}{r^{d-1}} + \frac{32\pi G Q^{2q}(d-2)^q(d-3)^q}{(4q-1)r^{4q+d-2}}\right]^{\frac{1}{d-3}} - \frac{r^2}{d-3}\left[\frac{16\pi Gm}{\Sigma_{d-2}r^{d-1}} + \frac{32\pi Y}{r^d} + \frac{\delta_d}{r^{d-1}} + \frac{32\pi G Q^{2q}(d-2)^q(d-3)^q}{(4q-1)r^{4q+d-2}}\right]^{\frac{4-d}{d-3}} \times \left(\frac{16\pi Gm(1-d)}{\Sigma_{d-2}r^d} - \frac{32\pi dY}{r^{d+1}} + \frac{(1-d)\delta_d}{r^d} + \frac{32\pi G Q^{2q}(d-2)^q(d-3)^q(2-d-4q)}{(4q-1)r^{4q+d-1}}\right). \tag{5.80}$$

Similarly, by differentiating again we get

$$\frac{d^2f}{dr^2} = \frac{2}{l^2} - 2P^{\frac{1}{d-3}} - \frac{4rP^{\frac{4-d}{d-3}}}{d-3}\frac{dP}{dr} - \frac{r^2P^{\frac{4-d}{d-3}}}{(d-3)}\frac{d^2P}{dr^2} - \frac{r^2(4-d)P^{\frac{7-2d}{d-3}}}{(d-3)^2}\left(\frac{dP}{dr}\right)^2, \tag{5.81}$$

where

$$P(r) = \frac{16\pi Gm}{\Sigma_{d-2}r^{d-1}} + \frac{32\pi Y}{r^d} + \frac{\delta_d}{r^{d-1}} + \frac{32\pi GQ^{2q}(d-2)^q(d-3)^q}{(4q-1)r^{4q+d-2}}, \tag{5.82}$$

$$\frac{dP}{dr} = \frac{16\pi Gm(1-d)}{\Sigma_{d-2}r^d} - \frac{32\pi dY}{r^{d+1}} + \frac{(1-d)\delta_d}{r^d} + \frac{32\pi GQ^{2q}(d-2)^q(d-3)^q(2-d-4q)}{(4q-1)r^{4q+d-1}}, \tag{5.83}$$

and

$$\frac{d^2P}{dr^2} = \frac{16\pi Gmd(d-1)}{\Sigma_{d-2}r^{d+1}} + \frac{32\pi d(d+1)Y}{r^{d+2}} + \frac{d(d-1)\delta_d}{r^{d+1}} + \frac{32\pi GQ^{2q}(d-2)^q(d-3)^q(2-d-4q)(1-d-4q)}{(4q-1)r^{4q+d}}. \tag{5.84}$$

Thus using Eqs. (5.75), (5.80) and (5.81) and the general expression of Ricci scalar (4.40), one can easily confirm that in the limit $r \to 0$ Ricci scalar becomes infinite. Thus, we conclude that metric function (5.75) represents an object which has true curvature singularity at $r = 0$. Therefore, line element (5.9) with metric function given by Eq. (5.75) represents a hairy black hole solution of DCG in the background of power-Yang-Mills field. Hence, we derive a large family of black hole solutions for any value of real parameter q except for the case $q = 1/4$. It is easy to see that, by taking $Y = 0$, this black hole solution reduces to the black hole with no scalar hair, i.e. to Eq. (5.52).

5.3 Black holes of DCG and Yang-Mills hierarchies

Here we study the possible black holes of DCG whose gravitational field is sourced by the superposition of different power-Yang-Mills field. The Yang-Mills hierarchies in diverse dimensions have been discussed in the literature [231]. Here we begin with an action defined as

$$\mathcal{I} = \frac{1}{16\pi G}\int d^dx\sqrt{-g}\left[\sum_{p=0}^{n-1}\frac{\alpha_p}{2^p}\delta^{\mu_1...\mu_{2p}}_{\nu_1...\nu_{2p}}R^{\nu_1\nu_2}_{\mu_1\mu_2}...R^{\nu_{2p-1}\nu_{2p}}_{\mu_{2p-1}\mu_{2p}} + \sum_{j=0}^{q}c_j(\Upsilon)^j\right], \tag{5.85}$$

where Υ represents the Yang-Mills invariant defined in (4.7), $c_j, j \geq 1$ is a coupling constant. The variation of the above action with respect to the metric tensor gives (4.4) with the stress-energy tensor given by

$$\mathcal{T}_{\mu}^{\nu} = -\frac{1}{2}\sum_{j=0}^{q} c_j \left(\delta_{\mu}^{\nu}\Upsilon^{j} - 4jTr(F_{\mu\sigma}^{(a)\nu\sigma})\right). \tag{5.86}$$

The Yang-Mills equations are determined by varying (5.85) with respect to the gauge potential $A^{(a)}$

$$\sum_{j=0}^{q} c_j \left[d(^{\star}F^{(a)}\Upsilon^{j-1}) + \frac{1}{\eta}C_{(b)(c)}^{(a)}\Upsilon^{j-1}A^{(b)} \wedge^{\star} F^{(c)}\right] = 0. \tag{5.87}$$

Our $d = n + 2$-dimensional line element ansatz is given by (5.9) and the power-Yang-Mills field ansatz would be chosen as before such that the stess-energy tensor takes the form

$$\mathcal{T}_{\mu}^{\nu} = -\frac{1}{2}\sum_{j=0}^{q} c_j\Upsilon^{j} diag[1, 1, \xi, \xi, ..., \xi], \tag{5.88}$$

where $\xi = 1 - \frac{4j}{(d-2)}$. Thus using Eqs. (5.2), (5.9) and (5.88) in Eq. (5.4) we get the metric function

$$f(r) = k + \frac{r^2}{l^2} - r^2\left[32\pi G\sum_{j=0}^{q}\frac{(d-2)^j(d-3)^jQ^{2j}}{(4j-1)r^{4j+d-2}} + \frac{16\pi mG}{\Sigma_{d-2}r^{d-1}} + \frac{\delta_d}{r^{d-1}}\right]^{\frac{1}{d-3}}. \tag{5.89}$$

The above equation indicates that for $j = 0$ the neutral black hole solution is obtained, for $j = 4$ the solution is undefined and for taking a unique value of $j = q$ we get the solution given by Eq. (5.52). The finite mass in terms of the event horizon radius r_+ is

$$m = \frac{\Sigma_{d-2}r_+^{d-1}\left(\frac{1}{l^2} + \frac{k}{r_+^2}\right)^{d-3}}{16\pi G} - \frac{\Sigma_{d-2}\delta_d}{16\pi G} - \sum_{j=0}^{q}\frac{2Q^{2j}\Sigma_{d-2}[(d-2)(d-3)]^j}{(4j-1)r_+^{4j-1}}. \tag{5.90}$$

When superposition of different power-Yang-Mills sources are taken into account, the metric function corresponding to the hairy black hole solution of DCG takes the form

$$f(r) = k + \frac{r^2}{l^2} - r^2\left[32\pi G\sum_{j=0}^{q}\frac{(d-2)^j(d-3)^jQ^{2j}}{(4j-1)r^{4j+d-2}} + \frac{16\pi mG}{\Sigma_{d-2}r^{d-1}} + \frac{\delta_d}{r^{d-1}} + \frac{32\pi Y}{r^d}\right]^{\frac{1}{d-3}}, \tag{5.91}$$

where Y is given by (5.47). The asymptotic behaviour of these solutions is similar to those obtained earlier in the case of DCG coupled to power-Yang-Mills source. Also, one can easily check that for metric functions (5.89) and (5.91), the curvature invariants possess an essential singularity at the origin $r = 0$ which is the mathematical aspect of the black hole.

Chapter 6

Power-Yang-Mills black holes and black branes in quartic quasi-topological gravity

We study higher dimensional quartic quasi-topological black holes in the framework of non-abelian power-Yang-Mills theory. It is shown that real solutions of the gravitational field equations exist only for positive values of quartic quasi-topological coefficient. Depending on the values of the mass parameter and Yang-Mills charge, they can be interpreted as black holes with one horizon, two horizons and naked singularity. It is also shown that the solution associated with these black holes has an essential curvature singularity at the centre $r = 0$. Thermodynamic and conserved quantities for these black holes are computed and we show that the first law has been verified. We also check thermodynamic stability in both canonical and grand canonical ensembles. In addition to this, we also formulate new power-Yang-Mills black hole solutions in pure quasi-topological gravity. The physical and thermodynamic properties of these black holes are discussed as well. It is concluded that unlike Yang-Mills black holes there exist stability regions for smaller power-Yang-Mills black holes in grand canonical ensemble. Finally, we discuss the thermodynamics of horizon flat power-Yang-Mills rotating black branes and analyze their thermodynamic and conserved quantities by using the counter-term method inspired by AdS/CFT correspondence.

6.1 Quartic quasi-topological black holes with power-Yang-Mills source

The action describing the quartic quasi-topological gravity coupled to Yang-Mills theory is given in Ref. [42]. Here, we use the power-Yang-Mills field as a source and work for new quartic quasi-topological solutions. In this setup, we consider the N-parameters gauge group $\mathfrak{G}$ whose structure constants $C^{(k)}_{(i)(j)}$ are defined as

$$\gamma_{ij} = -\frac{\Gamma_{(i)(j)}}{|\Gamma|^{1/N}}, \tag{6.1}$$

where, $i.j, k$ run from 1 to N, $\Gamma_{(i)(j)} = C^{(k)}_{(i)(l)} C^{(l)}_{(j)(k)}$ and Γ is its determinant. Thus, the action for the quartic quasi-topological gravity coupled with the power-Yang-Mills theory in higher spacetime dimensions is given by

$$\mathcal{I}_{bulk} = \frac{1}{16\pi} \int d^d x \sqrt{-g} \Big[R - 2\Lambda + \tilde{\mu}_2 \mathfrak{L}_2 + \tilde{\mu}_3 \mathfrak{L}_3 + \tilde{\mu}_4 \mathfrak{L}_4 - \Upsilon^q \Big], \tag{6.2}$$

where Υ is the Yang-Mills invariant (4.7) such that

$$\Upsilon = \gamma_{ab} F^{(a)}_{\alpha\beta} F^{(b)\alpha\beta}. \tag{6.3}$$

Also, $\tilde{\mu}_2$, $\tilde{\mu}_3$ and $\tilde{\mu}_4$ are the coefficients of quasi-topological gravity. Furthermore, the Lagrangian densities $\mathfrak{L}_2$, $\mathfrak{L}_3$ and $\mathfrak{L}_4$ are defined in Eqs. (1.8)-(1.10). Similarly, the coefficients b_i's and c_i's are given in Eqs. (1.11)-(1.12). We take the metric ansatz in d-dimensional spacetime as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_k^2, \tag{6.4}$$

where

$$d\Omega_k^2 = \begin{cases} d\theta_1^2 + \sum_{j=2}^{d-2} \prod_{l=1}^{j-1} \sin^2\theta_l d\theta_j^2, & k=1, \\ d\theta_1^2 + \sinh^2\theta_1 d\theta_2^2 + \sinh^2\theta_1 \sum_{j=3}^{d-2} \prod_{l=2}^{j-1} \sin^2\theta_l d\theta_j^2, & k=-1, \\ \sum_{j=1}^{d-2} d\phi_j^2, & k=0, \end{cases} \tag{6.5}$$

stands for the metric of a $(d-2)$-dimensional hyper-surface of constant curvature $(d-2)(d-3)k$ and volume $\mathcal{V}_{d-2}$. Now, using the line element (6.4) and the Lagrangian density of power-Yang-Mills model i.e. $\mathfrak{L}_{PYM} = -\Upsilon^q$, it is possible to write the stress-energy tensor associated with power-Yang-Mills field in the form as Eq. (4.9). The variation of action (6.2) with respect to the gauge potentials $A^{(a)}$ gives the equations of motion (4.10). Now, using the line element (6.4) and the Wu-Yang ansatz introduced in Refs. [173, 178, 212, 216], these equations will be satisfied provided the gauge potential one-forms satisfy Eq. (4.11). For simplicity it is convenient to use $d_i = d-i$ and redefine the quasi-topological coefficients as

$$\begin{aligned}
\mu_2 &= d_3 d_4 \tilde{\mu}_2, \\
\mu_3 &= \frac{d_3 d_6 (3d_1^2 - 9d_1 + 4)}{8(2d-3)} \tilde{\mu}_3, \\
\mu_4 &= d_1 d_2 d_4 d_8 d_3^2 (d_1^4 - 15d_1^4 + 17d_1^3 - 156d_1^2 + 150d_1 - 42) \tilde{\mu}_4.
\end{aligned} \tag{6.6}$$

The equations of motion describing gravitational field can be obtained if we vary the action (6.2) with respect to metric tensor $g_{\mu\nu}$. Thus, using Eqs. (4.10), (4.11) and (6.6) in Eq. (6.2) one can get the following fourth-order equation

$$\mu_4 \Phi^4 + \mu_3 \Phi^3 + \mu_2 \Phi^2 + \Phi + \mathfrak{X} = 0, \tag{6.7}$$

where $\Phi = (k - f(r))/r^2$ and

$$\mathfrak{X} = \begin{cases} -\frac{2\Lambda}{d_1 d_2} - \frac{m}{r^{d_1}} - \frac{d_3^q d_2^{q-1} Q^{2q}}{(d_1 - 4q) r^{4q}}, & q \neq \frac{d_1}{4} \\ -\frac{2\Lambda}{d_1 d_2} - \frac{m}{r^{d_1}} - \frac{Q^{d_1/2} d_3}{r^{d_1}} \ln r, & q = \frac{d_1}{4}. \end{cases} \tag{6.8}$$

The constant of integration m in the above equation refers to the mass of gravitating object with power-Yang-Mills magnetic charge. The power-Yang-Mills quasi-topological solution can be explicitly expressed from the polynomial equation (6.7) as

$$f(r) = k - r^2 \times \begin{cases} -\frac{\mu_3}{4\mu_4} - \frac{1}{2}\big(H_1 - \sqrt{\frac{2A_2}{H_1} - 2\overline{y} - 3A_1}\big), & \mu_4 > 0 \\ -\frac{\mu_3}{4\mu_4} + \frac{1}{2}\big(H_1 - \sqrt{-\frac{2A_2}{H_1} - 2\overline{y} - 3A_1}\big), & \mu_4 < 0, \end{cases} \tag{6.9}$$

where

$$\begin{aligned}
H_1 &= \big(A_1 + 2\overline{y}\big)^{\frac{1}{2}}, \\
A_1 &= \frac{\mu_2}{4} - \frac{3\mu_3^2}{8\mu_4^2}, \\
H_2 &= -\frac{A_1^3}{108} + \frac{A_1 \mathcal{Z}}{3} - \frac{A_2^2}{8}, \\
A_2 &= \frac{\mu_3^3}{8\mu_4^3} - \frac{\mu_3 \mu_2}{2\mu_4^2} + \frac{1}{\mu_4}, \\
\mathcal{Z} &= -\frac{3\mu_3^4}{256\mu_4^4} + \frac{\mu_2 \mu_3^2}{16\mu_4^3} - \frac{\mu_3}{4\mu_4^2} + \frac{\mathfrak{X}}{\mu_4}, \\
\mathcal{W} &= \left(-\frac{H_2}{2} \pm \sqrt{\frac{H_2^2}{4} + \frac{\overline{P}^3}{27}}\right)^{\frac{1}{3}}, \\
\overline{P} &= -\frac{A_1^2}{12} - \mathcal{Z},
\end{aligned} \tag{6.10}$$

and

$$\overline{y} = \begin{cases} -\frac{5}{6}A_1 + \mathcal{W} - \frac{\overline{P}}{3\mathcal{W}}, & W \neq 0 \\ -\frac{5}{6}A_1 + \mathcal{W} - H_2^{\frac{1}{3}}, & \mathcal{W} = 0. \end{cases} \tag{6.11}$$

It can be easily understood from Eq. (6.9) that the metric function describes solutions of the gravitational field equations of two types for $\mu_4 > 0$ and $\mu_4 < 0$. Similar to the case of Yang-Mills black holes [42] in this theory, in this situation too, the parameter $\mathfrak{X}$ obtained in Eq. (6.8) becomes highly negative for small values of the coordinate r. This makes the fourth term in parameter $\mathcal{Z}$ as well as the parameter $\overline{P}$ of Eq. (6.10), very large. In case $\mu_4 < 0$, this gives negatively large value for $\overline{P}$ which yields an imaginary solution for $\mathcal{W}$ for small values of r. Hence, we would not consider $\mu_4 < 0$. In order to get the metric function (6.9) in simpler form, we assume the special case $\overline{\mu}_2 = \overline{\mu}_3 = 0$. Thus, the power-Yang-Mills quasi-topological solution for $\mu_4 \neq 0$ becomes

$$f(r) = k - \frac{r^2}{2}\left[\mp\sqrt{2\Delta^{\frac{1}{3}} + \frac{2\mathfrak{X}}{3\mu_4\Delta^{\frac{1}{3}}}} \pm \sqrt{-2\Delta^{\frac{1}{3}} \pm \frac{2}{\mu_4}\left(2\Delta^{\frac{1}{3}} + \frac{2\mathfrak{X}}{3\mu_4\Delta^{\frac{1}{3}}}\right)^{-\frac{1}{2}} - \frac{2\mathfrak{X}}{3\mu_4\Delta^{\frac{1}{3}}}}\right], \tag{6.12}$$

where

$$\Delta = \frac{1}{16\mu_4^2} + \sqrt{\frac{1}{256\mu_4^4} - \frac{\mathfrak{X}^3}{27\mu_4^3}}. \tag{6.13}$$

It should be noted that the upper sign in Eq. (6.12) corresponds to the case $\mu_4 > 0$ while the lower sign is for $\mu_4 < 0$. In the limit $\mu_4 \to 0$, we can write the series expansion of metric function (6.12) as

$$f(r) = k + \mathfrak{X}r^2 + \mu_4\mathfrak{X}^4r^2 + 4\mu_4^2\mathfrak{X}^7r^2 + O((\mu_4)^{8/3}), \tag{6.14}$$

where $\mathfrak{X}$ is given in Eq. (6.8). The above expansion implies the Einstein-power-Yang-Mills solutions with some corrections in μ_4. The Ricci and Kretschmann invariants for the metric ansatz (6.4) respectively take the forms as Eqs. (4.40) and (4.41). So, by using the metric function obtained for the case $\mu_4 > 0$, it can easily be shown that both the scalars diverge at the center $r = 0$. Hence, there is a true curvature singularity at $r = 0$ for our power-Yang-Mills solutions. The horizons of the black hole can be described from the condition $f(r_+) = 0$. Fig. 6.1 shows the plot of the metric function for different values of the mass parameter m. The values of r for which the curve touches the horizontal axis correspond to the horizon's location. It may be noted that the metric function (6.9) can be interpreted as power-Yang-Mills quasi-topological black hole with two horizons when $m > m_{ext}$; extremal black hole when $m = m_{ext}$ and naked singularities otherwise. Fig. 6.2 describes the behaviour of solution (6.9) for various values of μ_4. It can be observed that for the fixed values of parameters d, m, Λ, k,

Q, and q, the values of horizons are affected by the parameter μ_4. However, at infinity the behaviour of the metric function does not depend on parameter μ_4. Furthermore, the dependence of the solution on the Yang-Mills charge Q is presented in Fig. 6.3. It is easily seen that for the chosen fixed values of other parameters, the outer horizon is independent of the charge parameter Q whereas the inner one increases with the increases of Q. Similarly, the behaviour of the resulting metric function associated with dS black holes for different values of Yang-Mills charge Q in spacetime dimensions such that $q = d_1/4$ is shown in Fig. 6.4. Also, the behaviour of the metric function corresponding to asymptotically flat black holes i.e. with $\Lambda = 0$ for different values of Q is shown in Fig. 6.5. It is also worthwhile to note that, for $q = 1$, the solution (6.9) reduces to the metric function of Yang-Mills quasi-topological black hole [42].

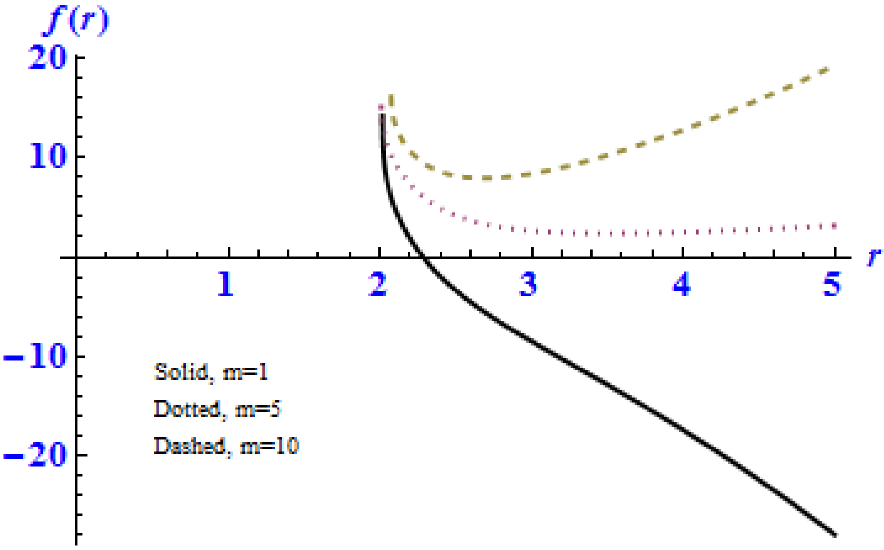

FIGURE 6.1: Dependence of function $f(r)$ (Eq. (6.9)) on the mass for fixed values of $d = 7$, $Q = 2$, $q = 2$, $k = 1$, $\mu_2 = -0.09$, $\mu_3 = -0.006$, $\mu_4 = 0.0004$ and $\Lambda = -1$.

6.2 Thermodynamics of quartic quasi-topological power-Yang-Mills black holes

Now, we study thermodynamic properties of the black holes described by equations (6.7)-(6.11). We can compute the ADM mass density with the help of subtraction method [232] as follows

$$M = \frac{d_2}{16\pi} m, \tag{6.15}$$

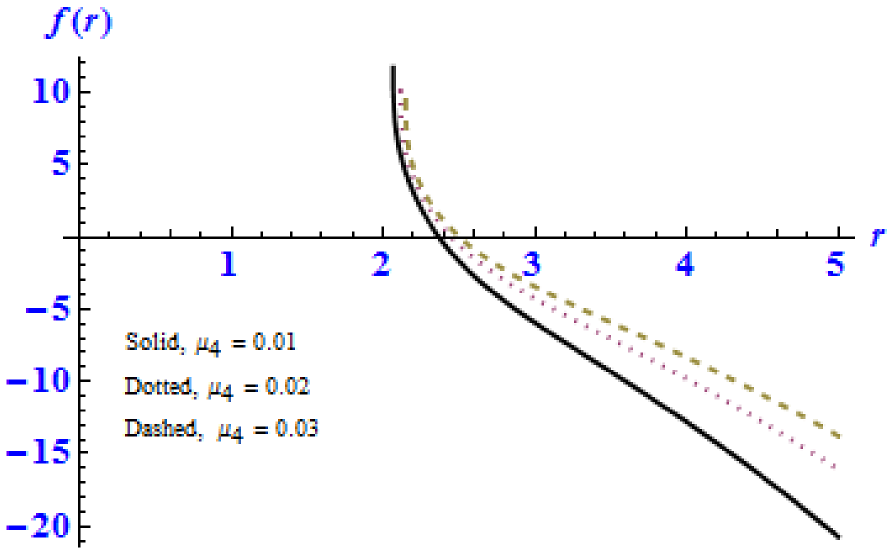

FIGURE 6.2: Dependence of function $f(r)$ (Eq. (6.9)) on the parameter μ_4 for fixed values $d = 7$, $m = 1$, $Q = 2$, $q = 2$, $k = 1$, $\mu_2 = -0.09$, $\mu_3 = -0.006$ and $\Lambda = -1$.

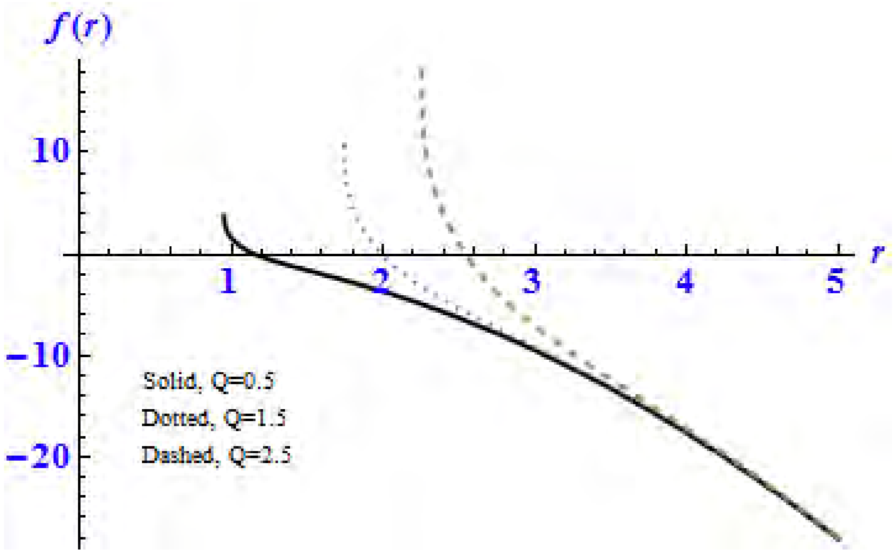

FIGURE 6.3: Dependence of function $f(r)$ (Eq. (6.9)) on the Yang-Mills charge Q for fixed values of $d = 7$, $m = 1$ $q = 2$, $k = 1$, $\mu_2 = -0.09$, $\mu_3 = -0.006$, $\mu_4 = 0.0004$ and $\Lambda = -1$.

where the parameter m is given in Eq. (6.8). Hence, we can write the value of M in terms of the outer horizon radius as

$$M = \begin{cases} \frac{d_2}{16\pi}\left(\mu_4 k^4 r_+^{d_9} + \mu_3 k^3 r_+^{d_7} + \mu_2 k^2 r_+^{d_5} + k r_+^{d_3} - \frac{2\Lambda r_+^{d_1}}{d_1 d_2} - \frac{d_2^{q-1} d_3^q Q^{2q} r_+^{d_1-4q}}{(d_1-4q)}\right), & q \neq \frac{d_1}{4}, \\ \frac{d_2}{16\pi}\left(\mu_4 k^4 r_+^{d_9} + \mu_3 k^3 r_+^{d_7} + \mu_2 k^2 r_+^{d_5} + k r_+^{d_3} - \frac{2\Lambda r_+^{d_1}}{d_1 d_2} - Q^{d_1/2} d_3 \ln r_+\right), & q = \frac{d_1}{4}. \end{cases} \tag{6.16}$$

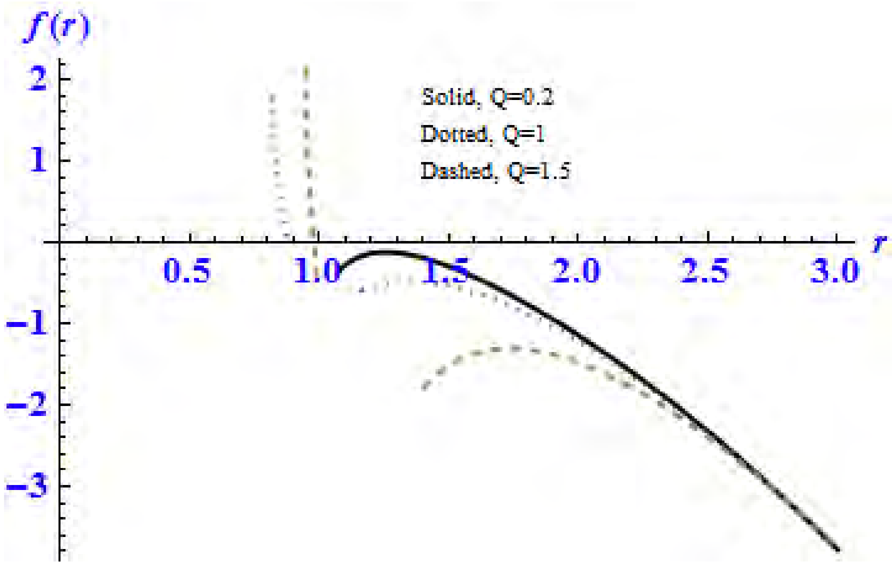

FIGURE 6.4: Dependence of function $f(r)$ (Eq. (6.9)) on the Yang-Mills charge Q for fixed values of $q = d_1/4$, $m = 1$, $q = 2$, $k = 1$, $\mu_2 = -0.06$, $\mu_3 = -0.1$, $\mu_4 = 0.03$ and $\Lambda = 1$.

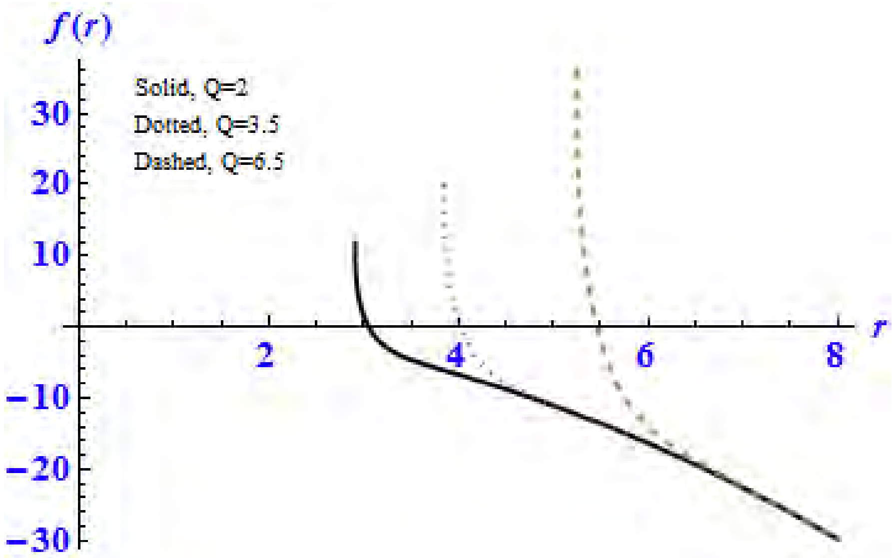

FIGURE 6.5: Dependence of function $f(r)$ (Eq. (6.9)) on the Yang-Mills charge Q for fixed values of $d = 9$, $q = 5$, $m = 10$, $k = 1$, $\mu_2 = -0.06$, $\mu_3 = -0.1$, $\mu_4 = 0.03$ and $\Lambda = 0$.

The Yang-Mills charge corresponding to this black hole can be computed through the Gauss law as

$$\tilde{Q} = \frac{1}{4\pi\sqrt{d_2 d_3}} \int d^{d-2}r \sqrt{\sum_{a=1}^{d_2 d_1/2} \left(F_{\mu\lambda}^{(a)} F^{(a)\nu\lambda} \right)} = \frac{Q}{4\pi}. \tag{6.17}$$

Using the condition $f(r_+) = 0$ and differentiating the polynomial equation (6.7) yields the Hawking temperature as

$$T_H(r_+) = \frac{f'(r_+)}{4\pi} = \begin{cases} \frac{r_+^8}{\mathcal{W}_2}\left(\frac{d_1\mathcal{W}_1}{r_+^9} - \frac{2\Lambda}{d_2 r_+} - \frac{d_2^{q-1}d_3^q Q^{2q}}{r_+^{4q+1}}\right) - \frac{2k}{r_+}, & q \neq \frac{d_1}{4}, \\ \frac{r_+^8}{\mathcal{W}_2}\left(\frac{d_1\mathcal{W}_1}{r_+^9} - \frac{2\Lambda}{d_2 r_+} - \frac{d_3 Q^{d_1/2}}{r_+^d}\right) - \frac{2k}{r_+}, & q = \frac{d_1}{4}, \end{cases} \tag{6.18}$$

where

$$\begin{aligned} \mathcal{W}_1 &= \mu_4 k^4 + \mu_3 k^3 r_+^2 + \mu_2 k^2 r_+^4 + k r_+^6, \\ \mathcal{W}_2 &= 4\mu_4 k^3 + 3\mu_3 k^2 r_+^2 + 2k\mu_2 r_+^4 + r_+^6. \end{aligned} \tag{6.19}$$

Fig. 6.6 describes the behaviour of temperature for several values of charge Q when

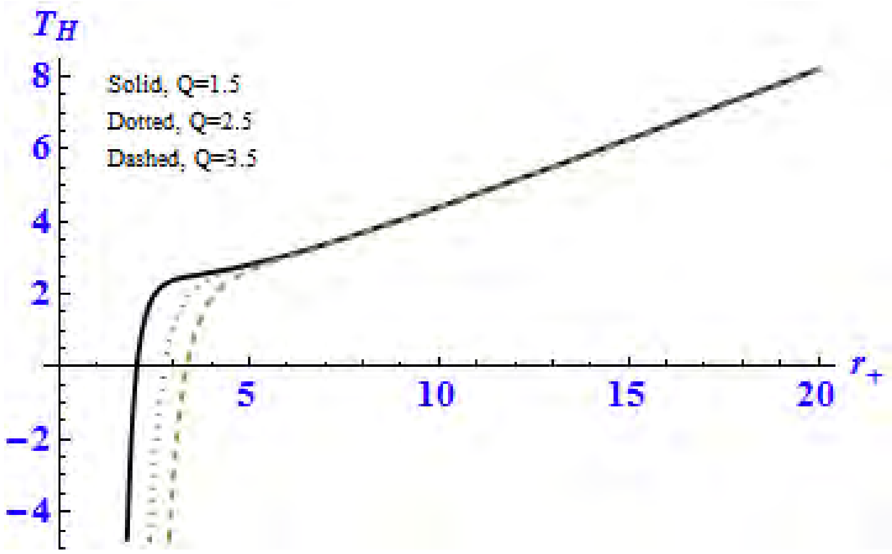

FIGURE 6.6: Dependence of temperature $T_H(r)$ (Eq. (6.18)) on the charge parameter Q for fixed values of $d = 7$, $q = 2$, $k = 1$, $\mu_2 = -0.06$, $\mu_3 = -0.1$, $\mu_4 = 0.03$ and $\Lambda = -1$.

$\Lambda = -1$. The positivity of temperature indicates that the black hole solution is physical. Similarly, the behaviour of Hawking temperature for the case $q = d_1/4$ and $\Lambda = 1$ can be observed from Fig. 6.7. It can be easily seen from these graphs that the solutions for which Λ is negative may have a larger range of parameters with positive temperature as compared to the ones with positive Λ. Following Ref. [127, 128], we compute the entropy density as

$$\mathcal{S} = \frac{r_+^{d_2}}{4} + \frac{d_2 k\mu_2 r_+^{d_4}}{2d_4} + \frac{3d_2 k^2\mu_3 r_+^{d_6}}{4d_6} + \frac{d_2 k^3\mu_4 r_+^{d_8}}{4d_8}. \tag{6.20}$$

Consideration of mass M as a function of entropy $\mathcal{S}$ and charge $\tilde{Q}$ enables us to construct the first law as

$$dM = T_H d\mathcal{S} + \mathcal{U} d\tilde{Q}, \tag{6.21}$$

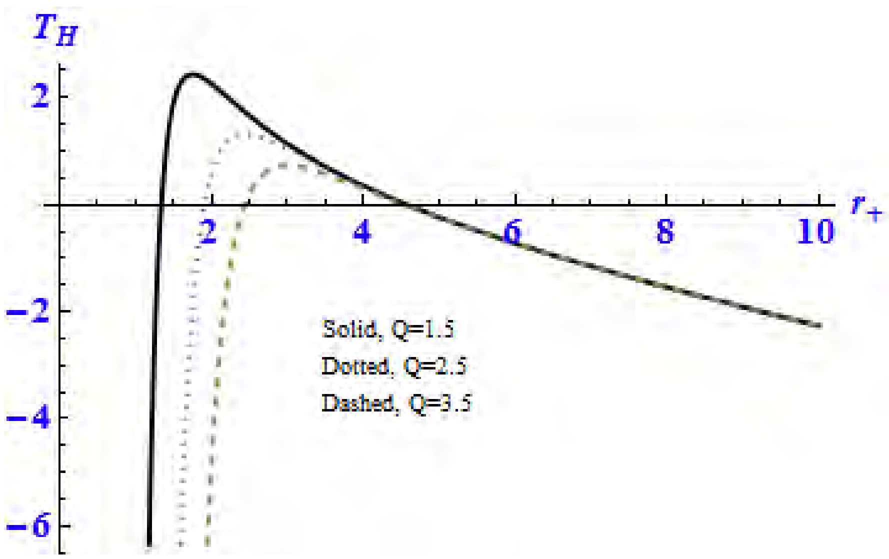

FIGURE 6.7: Dependence of temperature $T_H(r)$ (Eq. (6.18)) on the charge parameter Q for fixed values of $d = 9$, $q = \frac{d_1}{4} = 2$, $k = 1$, $\mu_2 = -0.06$, $\mu_3 = -0.1$, $\mu_4 = 0.03$ and $\Lambda = 1$.

where $T_H = \left(\frac{\partial M}{\partial \mathcal{S}}\right)_{\tilde{Q}}$ and $\mathcal{U} = \left(\frac{\partial M}{\partial \tilde{Q}}\right)_{\mathcal{S}}$. The calculations show that the same form of temperature i.e. Eq. (6.18) can be obtained from this relation. Moreover, the power-Yang-Mills potential can be obtained as follows:

$$\mathcal{U} = \left(\frac{\partial M}{\partial \tilde{Q}}\right)_{\mathcal{S}} = \begin{cases} -\frac{q d_2^q d_3^q (4\pi\tilde{Q})^{2q-1}}{8\pi(d_1-4q)} r_+^{d_1-4q}, & q \neq \frac{d_1}{4}, \\ -\frac{d_1 d_2 d_3 (4\pi\tilde{Q})^{d_3/2}}{32\pi} \ln r_+, & q = \frac{d_1}{4}, \end{cases} \tag{6.22}$$

The charge $\tilde{Q}$ should be fixed in the canonical ensemble and the thermodynamic stability can be examined by assuming small variations of the entropy. Hence, thermodynamic stability would be guaranteed if the specific heat is positive. The specific heat capacity at a constant Yang-Mills charge $\tilde{Q}$ can be computed from $C_H = T_H(\frac{\partial \mathcal{S}}{\partial T_H})_{\tilde{Q}}$. So, from Eqs. (6.18)-(6.20), we can work out the heat capacity in the form

$$C_H = \begin{cases} \frac{\mathcal{W}_2\mathcal{W}_3\left(d_1 d_2 \mathcal{W}_1 r_+^{4q} - 2\Lambda r_+^{4q+8} - (4\pi\tilde{Q})^{2q} d_2^q d_3^q r_+^8 - 2k d_2 \mathcal{W}_2 r_+^{4q}\right)}{4 d_2 r_+^7 \left(r_+^{4q-6}\mathcal{W}_2^2 G(r_+) + ((4q-7)\mathcal{W}_2 + r\mathcal{W}_2'(r_+))(4\pi\tilde{Q})^{2q} d_2^{q-1} d_3^q\right)}, & q \neq \frac{d_1}{4}, \\ \frac{\mathcal{W}_2\mathcal{W}_3\left(d_1 d_2 \mathcal{W}_1 r_+^{d} - 2\Lambda r_+^{d+8} - d_2 d_3 (4\pi\tilde{Q})^{d_1/2} r_+^9 - 2k\mathcal{W}_2 d_2 r_+^{d}\right)}{4 d_2 r_+^8 \left(r_+^{d7}\mathcal{W}_2^2 G(r_+) + (r_+\mathcal{W}_2' + d_8\mathcal{W}_2) d_3 (4\pi\tilde{Q})^{d_1/2}\right)}, & q = \frac{d_1}{4}, \end{cases} \tag{6.23}$$

where

$$\begin{aligned} \mathcal{W}_1'(r_+) &= 2\mu_3 k^3 r_+ + 4\mu_2 k^2 r_+^3 + 6k r_+^5, \\ \mathcal{W}_2'(r_+) &= 6\mu_3 k^2 r_+ + 8k\mu_2 r_+^3 + 6r_+^5, \\ \mathcal{W}_3(r_+) &= d_2 r_+^{d_3} + 2d_2 k\mu_2 r_+^{d_5} + 3d_2 k^2 \mu_3 r_+^{d_7} + d_2 k^3 \mu_4 r_+^{d_9} \end{aligned} \tag{6.24}$$

and

$$G(r_+) = \frac{1}{\mathcal{W}_2}\left(\frac{d_1\mathcal{W}_1'}{r_+} - \frac{d_1\mathcal{W}_1}{r_+^2} - \frac{14\Lambda r_+^6}{d_2}\right) + \frac{2k}{r_+^2} - \frac{\mathcal{W}_2'}{\mathcal{W}_2^2}\left(\frac{d_1\mathcal{W}_1}{r_+} - \frac{2\Lambda r_+^7}{d_2}\right). \tag{6.25}$$

The behaviour of heat capacity depending on outer horizon r_+ for various values of

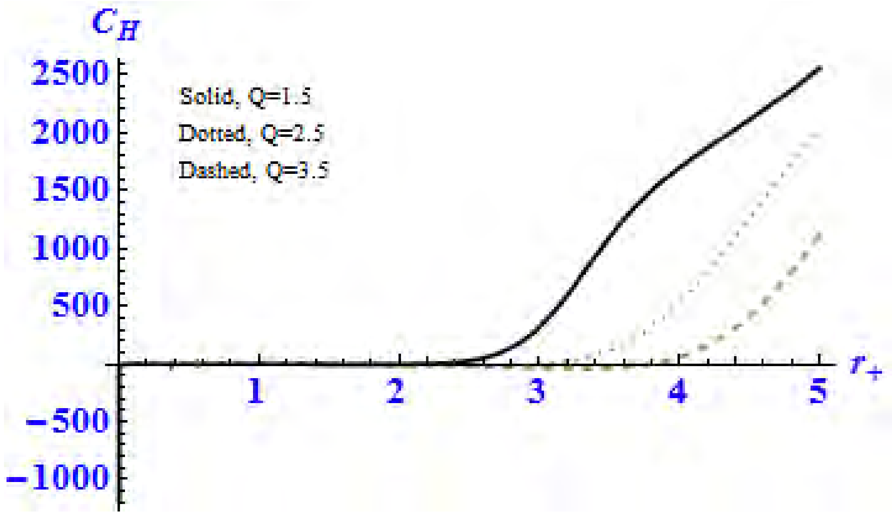

FIGURE 6.8: Dependence of heat capacity C_H (Eq. (6.23)) on the Yang-Mills charge $\tilde{Q} = Q/4\pi$ for fixed values of $d = 7$, $q = 2$, $k = 1$, $\mu_2 = -0.06$, $\mu_3 = -0.1$, $\mu_4 = 0.03$ and $\Lambda = -1$.

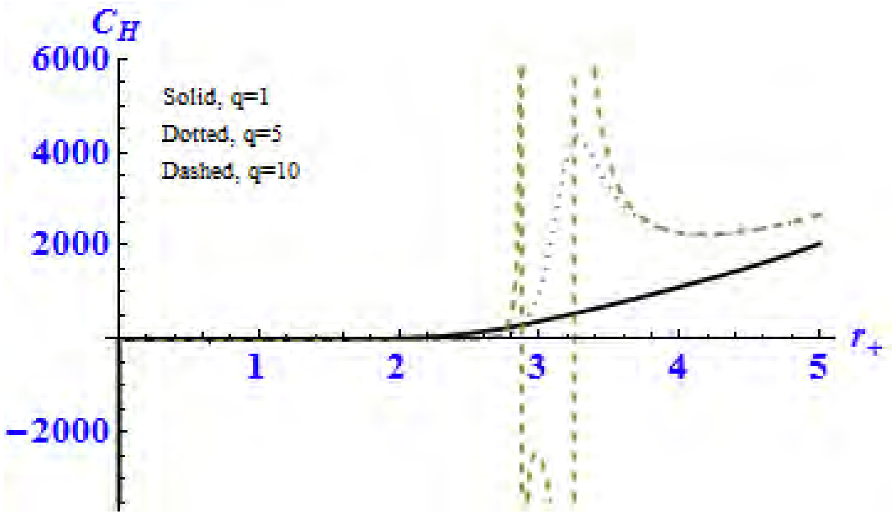

FIGURE 6.9: Dependence of heat capacity C_H (Eq. (6.23)) on the parameter q for fixed values of $d = 7$, $Q = 1.5$, $k = 1$, $\mu_2 = -0.06$, $\mu_3 = -0.1$, $\mu_4 = 0.03$ and $\Lambda = -1$.

charge $\tilde{Q} = Q/4\pi$ is given in Fig. 6.8. The region where this quantity is positive implies black hole stability in this ensemble. It can also be observed that as charge Q increases, the outer horizon for stable black hole increases. Similarly, Fig. 6.9 shows the corresponding plot for different values of parameter q. The case $q = 1$ corresponds to the heat capacity for Yang-Mills black hole in this gravity theory. It can be observed that this

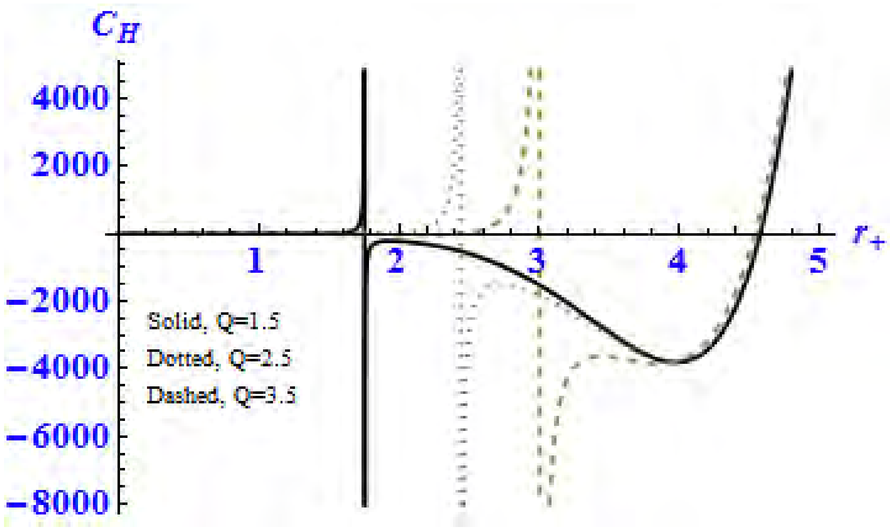

FIGURE 6.10: Dependence of heat capacity C_H (Eq. (6.23)) on the Yang-Mills charge $\tilde{Q} = Q/4\pi$ for fixed values of $d = 9$, $q = d_1/4 = 2$, $k = 1$, $\mu_2 = -0.06$, $\mu_3 = -0.1$, $\mu_4 = 0.03$ and $\Lambda = 1$.

parameter of nonlinear Yang-Mills field affects the thermodynamic local stability of black holes. The plot of heat capacity for different values of charge Q with positive cosmological constant and spacetime dimensions satisfying $q = d_1/4$ is shown in Fig. 6.10. The region of local stability and instability can easily be seen from it. It is also worthwhile to note that the points at which this quantity vanishes indicate the possibility of first order phase transitions. However, those values for which it is infinite correspond to the possibility of second order phase transitions. When it comes to the grand canonical ensemble, both the charge Q and entropy $\mathcal{S}$ should be treated as variables. In addition to specific heat and Hawking temperature, the local thermodynamic stability can be guaranteed from the positivity of $\partial^2 M/\partial\tilde{Q}^2$ and the determinant of the Hessian matrix [233, 234]. The determinant of the Hessian matrix is given by

$$det\mathbf{H} = \left(\frac{\partial^2 M}{\partial \mathcal{S}^2}\right)\left(\frac{\partial^2 M}{\partial \tilde{Q}^2}\right) - \left(\frac{\partial^2 M}{\partial \mathcal{S}\partial \tilde{Q}}\right)^2. \tag{6.26}$$

If we compute $\left(\frac{\partial^2 M}{\partial \tilde{Q}^2}\right)$ for our resulting solution, we have

$$\left(\frac{\partial^2 M}{\partial \tilde{Q}^2}\right) = \begin{cases} -\frac{q(2q-1)d_2^q d_3^q (4\pi)^{2q}(\tilde{Q})^{2q-2}}{8\pi(d_1-4q)} r_+^{d_1-4q}, & q \neq \frac{d_1}{4}, \\ -\frac{d_1 d_2 d_3^2 (4\pi)^{d_3/2}(\tilde{Q})^{d_5/2}}{64\pi} \ln r_+, & q = \frac{d_1}{4}. \end{cases} \tag{6.27}$$

This equation shows that the parameter $\left(\frac{\partial^2 M}{\partial \tilde{Q}^2}\right)$ is negative for $d_1 > 4q$ when $2q \geq 1$ and so the black hole would be unstable in this ensemble. However, for the spacetime dimensions satisfying $d_1 < 4q$ it is positive and so thermal stability will be determined from the behaviour of the Hessian matrix. It should be noted that, when the spacetime dimensions satisfy $q = d_1/4$, then the black hole is unstable in grand canonical ensemble. Furthermore, the case $q = 1$ in Eq. (6.27) corresponds to the instability of Yang-Mills quasi-topological black hole [42]. One can compute the Hessian matrix determinant as

$$det\mathbf{H} = \begin{cases} \frac{(4\pi)^{2q}(2q)(2q-1)d_2^q d_3^q \tilde{Q}^{2q-2} r_+^{d_1-4q} A_q(r_+)}{\left(d_2 r_+^{d_3}+2d_2 k\mu_2 r_+^{d_5}+3d_2\mu_3 k^2 r_+^{d_7}+d_2 k^3\mu_4 r_+^{d_9}\right)} - \frac{4q^2(4\pi)^{2q} d_2^{2q-2} d_3^{2q}\tilde{Q}^{4q-2}}{\mathcal{W}_2^2 r_+^{8q-14}}, & q > \frac{d_1}{4}, \\ \frac{d_1 d_2 d_3 (4\pi)^{d_1/2}\tilde{Q}^{d_3/2}\ln r_+ A_q(r_+)}{32\pi\left(d_2 r_+^{d_3}+2d_2 k\mu_2 r_+^{d_5}+3d_2\mu_3 k^2 r_+^{d_7}+d_2 k^3\mu_4 r_+^{d_9}\right)} - \frac{d_1^2 d_3^2 (4\pi)^{d_1}\tilde{Q}^{d_3}}{4 r_+^{2d_8}\mathcal{W}_2^2}, & q = \frac{d_1}{4}, \end{cases} \tag{6.28}$$

where

$$A_q(r_+) = \begin{cases} \frac{\mathcal{W}_2^2 G(r_+) r_+^{4q-6} + (4\pi)^{2q} d_2^{q-1} d_3^q \tilde{Q}^{2q}\left((4q-7)\mathcal{W}_2(r_+) + r_+\mathcal{W}_2'(r_+)\right)}{4\pi\mathcal{W}_2^2(4q-d_1) r_+^{4q-6}}, & q > \frac{d_1}{4}, \\ -\frac{\mathcal{W}_2^2 G(r_+) r_+^{d_7} + (4\pi)^{d_1/2} d_3\tilde{Q}^{d_1/2}\left(r_+\mathcal{W}_2'(r_+)+) + d_8\mathcal{W}_2(r_+)\right)}{8\pi r_+^{d_7}\mathcal{W}_2^2(r_+)}, & q = \frac{d_1}{4}. \end{cases} \tag{6.29}$$

Figs. 6.11-6.12 describe the plots of $det\mathbf{H}$ in terms of the outer horizon when $q > d_1/4$.

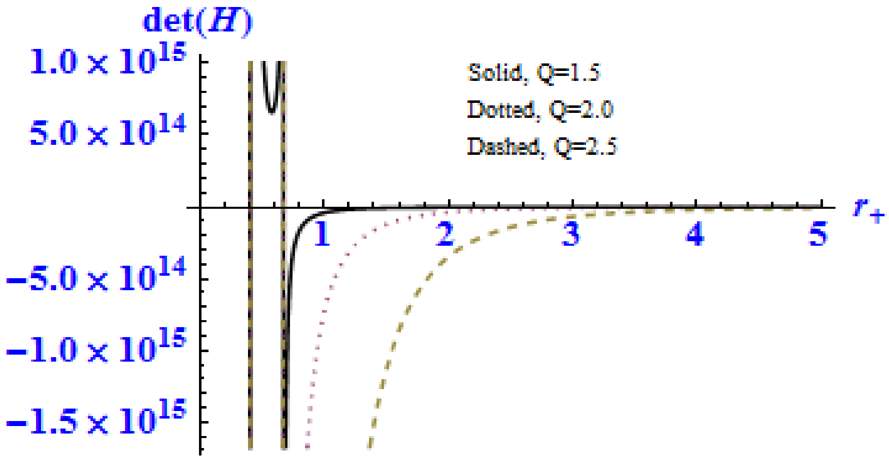

FIGURE 6.11: Plot of $det\mathbf{H}$ (Eq. (6.28)) for different values of Yang-Mills charge $\tilde{Q} = Q/4\pi$ for fixed values of $d = 11$, $q = 3$, $k = 1$, $\mu_2 = -0.06$, $\mu_3 = -0.1$, $\mu_4 = 0.03$ and $\Lambda = -1$.

These plots indicate that the power-Yang-Mills black holes of smaller outer horizons can be thermodynamically stable in the grand canonical ensemble because the associated determinant of the Hessian matrix could be positive when $q > d_1/4$. However, as r_+ increases this quantity is negative and so we have instability of black holes. Fig. 6.13 shows the plot of this quantity as a function of the nonlinearity parameter q. It may be noted that the parameter q has a great influence on the stability of smaller black holes. One can also confirm from this plot that the Yang-Mills black holes [42] (i.e. the case corresponding to $q = 1$ in power-Yang-Mills black hole solution (6.9)) are thermally unstable in this ensemble.

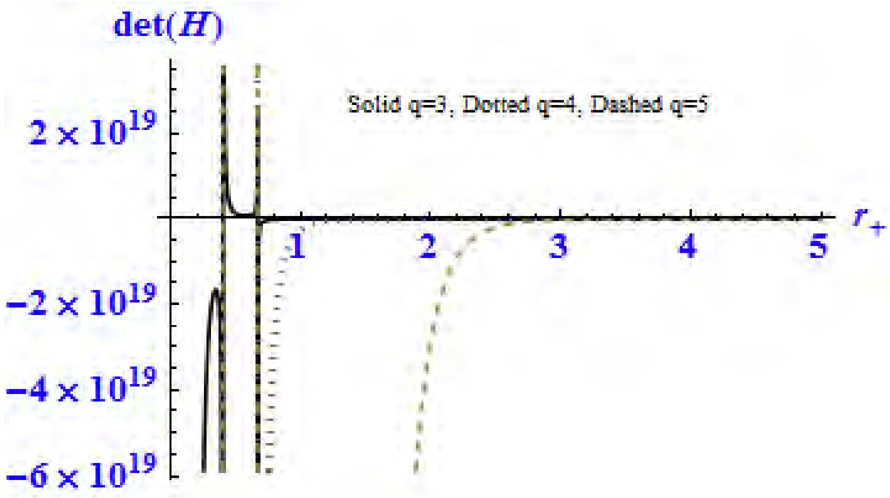

FIGURE 6.12: Dependence of $det\mathbf{H}$ (Eq. (6.28)) on the parameter q for fixed values of $d = 11$, $Q = 3$, $k = 1$, $\mu_2 = -0.06$, $\mu_3 = -0.1$, $\mu_4 = 0.03$ and $\Lambda = -1$.

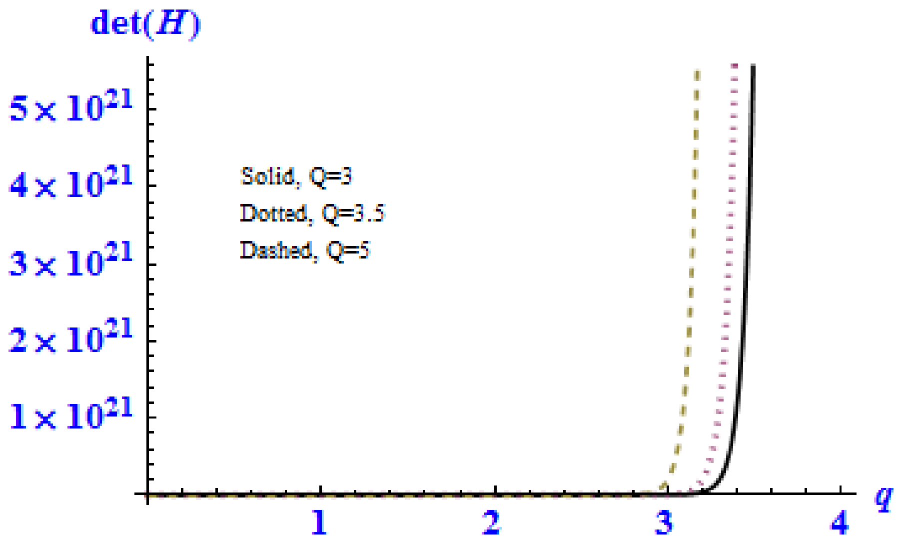

FIGURE 6.13: Dependence of $det\mathbf{H}$ (Eq. (6.28)) on the Yang-Mills charge $\tilde{Q}$ for fixed values of $d = 11$, $r_+ = 0.5$, $k = 1$, $\mu_2 = -0.06$, $\mu_3 = -0.1$, $\mu_4 = 0.03$ and $\Lambda = -1$.

6.3 Pure quasi-topological black holes with power-Yang-Mills source

Now, we want to determine a new class of power-Yang-Mills black hole solutions in pure quasi-topological gravity. In order to do this, we set $R = \mathfrak{L}_2 = \mathfrak{L}_3 = 0$, so that the action becomes

$$\mathcal{I}_{bulk} = \frac{1}{16\pi} \int d^d x \sqrt{-g} \Big[-2\Lambda + \tilde{\mu}_4 \mathfrak{L}_4 - \Upsilon^q \Big], \tag{6.30}$$

while, the field equation (6.7) reduces to

$$\mu_4 \Phi^4 + \mathfrak{X} = 0, \tag{6.31}$$

where $\Phi = (k - f(r))/r^2$ and $\mathfrak{X}$ was defined in (6.8). Hence, the solution in this case can be obtained as

$$f_p(r) = \begin{cases} k \mp \frac{r^{\frac{3}{2}}}{\mu_4}\big[\mu_4^3\big(\frac{2\Lambda r^2}{d_1 d_2} + \frac{m}{r^{d_3}} + \frac{d_3^q d_2^{q-1} Q^{2q}}{(d_1-4q)r^{4q-2}}\big)\big]^{\frac{1}{4}}, & q \neq \frac{d_1}{4}, \\ k \mp \frac{r^{\frac{3}{2}}}{\mu_4}\big[\mu_4^3\big(\frac{2\Lambda r^2}{d_1 d_2} + \frac{m}{r^{d_3}} + \frac{Q^{d_1/2} d_3 \ln r}{r^{d-3}}\big)\big]^{\frac{1}{4}}, & q = \frac{d_1}{4}. \end{cases} \tag{6.32}$$

For obtaining real solutions, we take $\Lambda > 0$ and $\mu_4 > 0$. Since the spacetime dimension $d = 9$ produces negative value for μ_4 while other choices of d lead to positive values, so it is convenient to ignore the case of $d = 9$. Our numerical calculations show that for the determination of black hole solution, one needs to take $d > 9$. In the limit $r \to \infty$, the metric function corresponding to pure quasi-topological-power-Yang-Mills solution tends to

$$f_p(r) = k \mp \left(\frac{2\Lambda}{\mu_4 d_1 d_2}\right)^{\frac{1}{4}} r^2. \tag{6.33}$$

Note that, the minus and plus signs are defined respectively for $k = 1$ and $k = -1$, whereas the other cases lead to naked singularity. Thus, the choice $\Lambda > 0$ in this metric function may describe asymptotically AdS and dS pure quasi-topological black holes with $k = -1$ and $k = 1$, respectively. These power-Yang-Mills black holes possess horizons, if the metric function satisfies the condition $f_p(r_+) = 0$. Hence, on the basis of appropriate choices for the parameters d, m, Q, Λ and μ_4, the pure quasi-topological solution can describe a black hole having horizons. In this regard, we plot metric function (6.32) as a function of r with $\Lambda = 1$ in Figs. 6.14-6.17. Those points for which the curve intersects the horizontal axis correspond to the location of horizons. It may be noted that for $k = -1$ and $d > 9$, the solution (6.32) describes AdS black hole having two horizons, an extreme dS black hole and a naked singularity. It is also shown that the Yang-Mills charge Q and nonlinearity parameter q affect the horizon structure of the black hole. The case $q = 1$ in Fig. 6.16 corresponds to the behaviour of the black hole solution of quasi-topological gravity with Yang-Mills source. Fig. 6.17 shows the behaviour of metric function (6.32) in those spacetime dimensions which satisfy the condition $q = d_1/4$. It can also be verified that for a positive value of quasi-topological parameter μ_4, the Kretschmann scalar (4.41) associated with pure quasi-topological solution in the vicinity of $r = 0$ takes the form as

$$K \propto \big(\frac{m}{\mu_4}\big)^{1/2} r^{-(d_1)/2}, \tag{6.34}$$

which diverges at $r = 0$. Therefore, the pure quasi-topological power-Yang-Mills black hole has an essential central singularity.

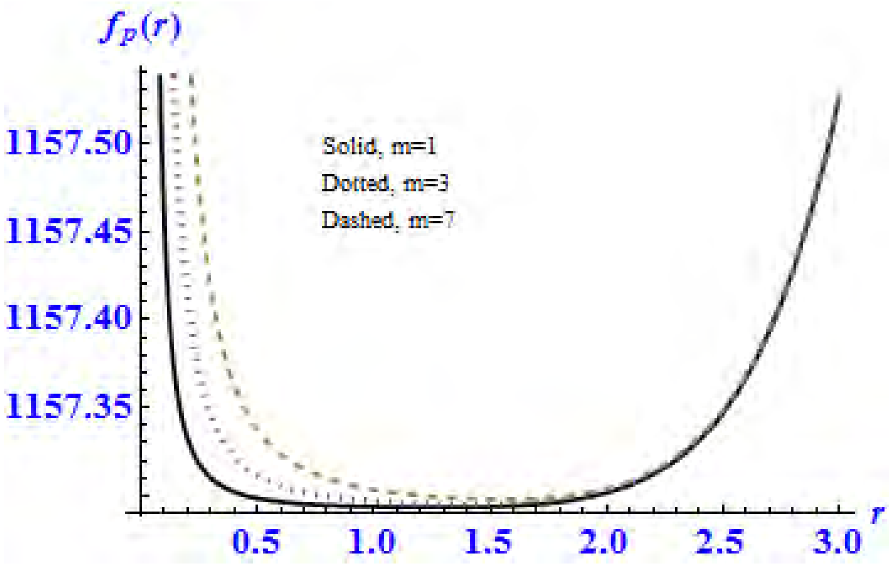

FIGURE 6.14: Dependence of f_p (Eq. (6.32)) on the mass when $d = 11$, $q = 2$, $k = 1$, $Q = 5$, $\mu_4 = 10^{-7}$ and $\Lambda = 1$.

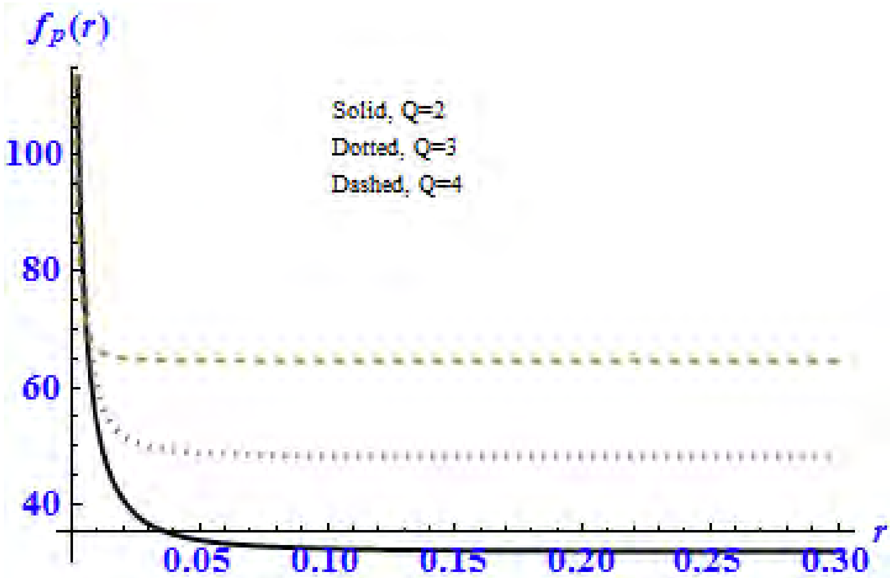

FIGURE 6.15: Dependence of f_p (Eq. (6.32)) on the Yang-Mills charge parameter Q when $d = 11$, $q = 2$, $k = -1$, $m = 3$, $\mu_4 = 0.004$ and $\Lambda = 1$.

6.4 Thermodynamics of pure quasi-topological power-Yang-Mills black holes

In order to study thermodynamic properties of pure quasi-topological black holes described by metric function (6.32) we will work out various thermodynamic quantities. In this case too, the mass density follows from Eq. (6.15) and, as a function of outer

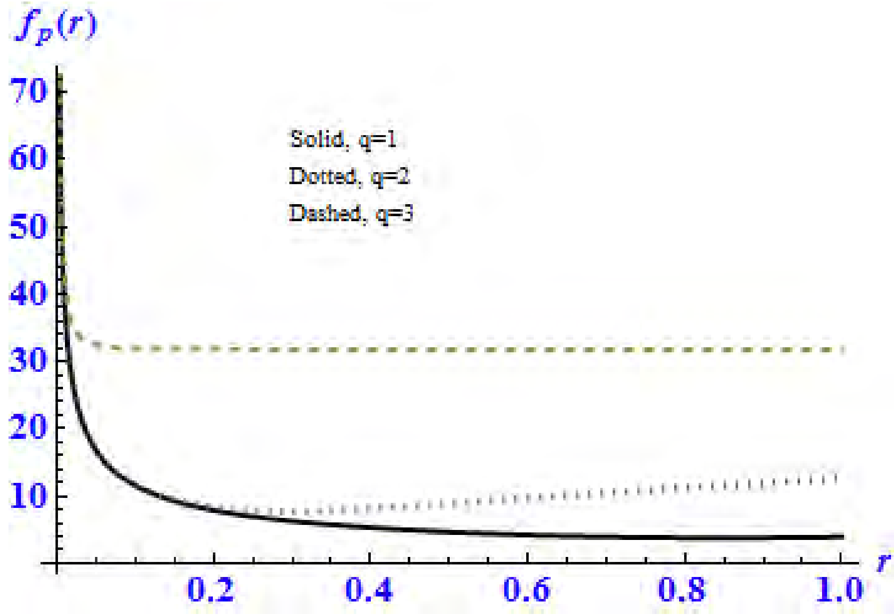

FIGURE 6.16: Dependence of f_p (Eq. (6.32)) on the parameter q when $d = 11$, $Q = 2$, $k = -1$, $m = 1$, $\mu_4 = 0.004$ and $\Lambda = 1$.

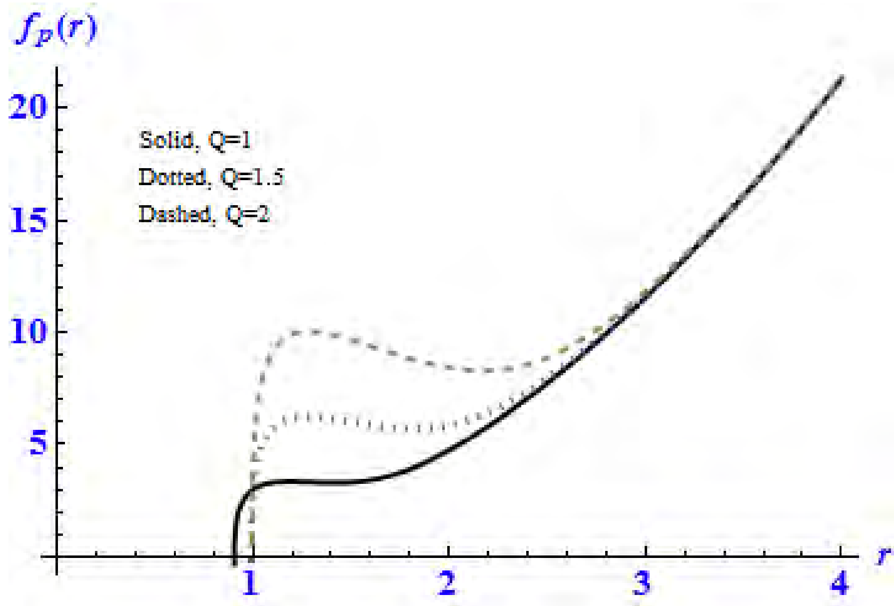

FIGURE 6.17: Dependence of f_p (Eq. (6.32)) on the Yang-Mills charge parameter Q when $d = 13$, $q = d_1/4 = 3$, $k = -1$, $m = 1$, $\mu_4 = 0.004$ and $\Lambda = 1$.

horizon, it can be obtained as

$$M = \begin{cases} \frac{d_2}{16\pi}\left(\mu_4 k^4 r_+^{d_9} - \frac{2\Lambda r_+^{d_1}}{d_1 d_2} - \frac{d_3^q d_2^{q-1} Q^{2q} r_+^{d_1-4q}}{d_1-4q}\right), & q \neq \frac{d_1}{4}, \\ \frac{d_2}{16\pi}\left(\mu_4 k^4 r_+^{d_9} - \frac{2\Lambda r_+^{d_1}}{d_1 d_2} - d_3 Q^{d_1/2} \ln r_+\right), & q = \frac{d_1}{4}. \end{cases} \quad (6.35)$$

Using the condition $f_p(r_+) = 0$ and the polynomial equation (6.31), it is straightforward to obtain Hawking temperature as

$$T_H = \begin{cases} \frac{1}{4\pi}\left[\frac{r_+^8}{4\mu_4 k^3}\left(\frac{\mu_4 k^4 d_1}{r_+^9} - \frac{2\Lambda}{d_2 r_+} - \frac{d_3^q d_2^{q-1} Q^{2q}}{r_+^{4q+1}}\right) - \frac{2k}{r_+}\right] & q \neq \frac{d_1}{4}, \\ \frac{1}{4\pi}\left[\frac{r_+^8}{4\mu_4 k^3}\left(\frac{\mu_4 k^4 d_1}{r_+^9} - \frac{2\Lambda}{d_2 r_+} - \frac{d_3 Q^{d_1/2}}{r_+^d}\right) - \frac{2k}{r_+}\right], & q = \frac{d_1}{4}. \end{cases} \tag{6.36}$$

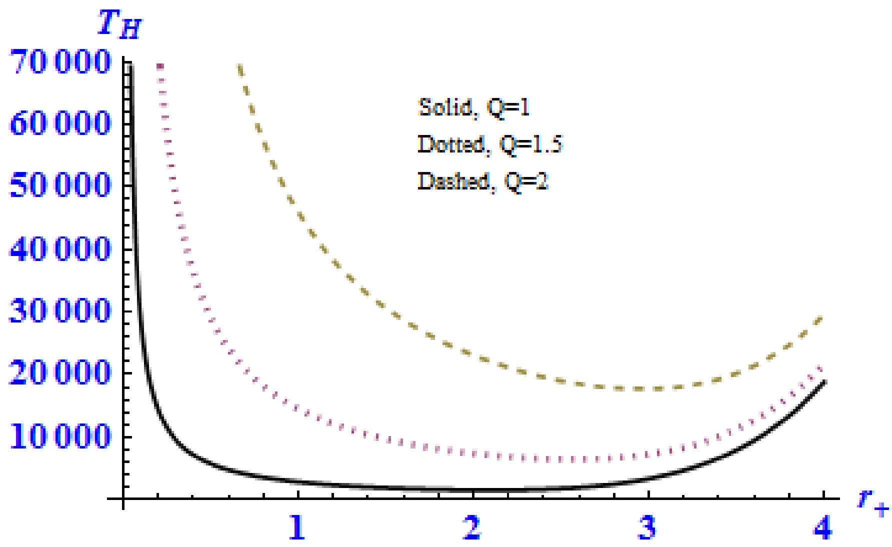

FIGURE 6.18: Plot of temperature T_H (Eq. (6.36)) for different values of the Yang-Mills charge Q and fixed values of $d = 11$, $q = 2$, $k = -1$, $\mu_4 = 0.004$ and $\Lambda = 1$.

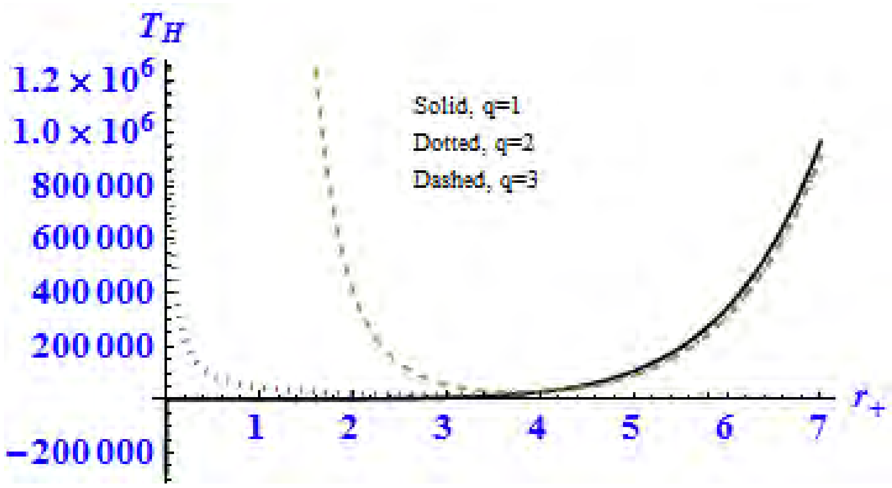

FIGURE 6.19: Dependence of temperature T_H (Eq. (6.36)) on the parameter q for fixed values of $d = 11$, $Q = 2$, $k = -1$, $\mu_4 = 0.004$ and $\Lambda = 1$.

Fig. 6.18 describes the behaviour of Hawking temperature as a function of the outer horizon for some values of Q. It can be clearly seen that for smaller black holes when Yang-Mills charge increases, the temperature also increases while for the larger ones the

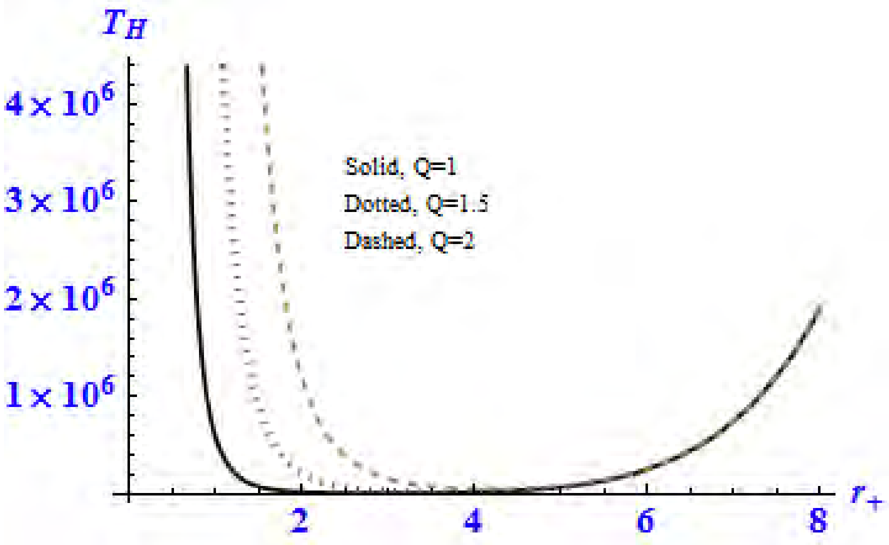

FIGURE 6.20: Plot of temperature T_H (Eq. (6.36)) for different values of the Yang-Mills charge Q and fixed values of $d = 13$, $q = d_1/4 = 3$, $k = -1$, $\mu_4 = 0.004$ and $\Lambda = 1$.

Hawking temperature does not change with this increase in Q. The choice $q = 1$ in Fig. 6.19 shows the temperature of Yang-Mills quasi-topological black holes. It can also be observed that for smaller black holes the nonlinearity parameter q possesses greater influence on the temperature. However, for larger black holes, the effect of parameter q is negligible. Similarly, Fig. 6.20 shows the plot of Hawking temperature when the spacetime dimensions satisfy the case $q = d_1/4$. For this choice too, the Yang-Mills charge affects the temperature of quasi-topological black holes for small values of r_+ while for large horizon radii, the temperature although increases but the effect of Q is negligible.

Again, using the same technique as in Ref. [127, 128], the entropy density of pure quasi-topological black hole can be calculated as

$$\mathcal{S} = \frac{d_2 k^3 \mu_4}{d_8} r_+^8. \tag{6.37}$$

Our calculations show that the temperature (6.36) is equal to $\left(\frac{\partial M}{\partial \mathcal{S}}\right)$. Thus, the power-Yang-Mills black holes in pure quasi-topological gravity also satisfy the first law of thermodynamics (6.21) if the power-Yang-Mills potential associated with solution (6.32) is given by

$$\mathcal{U} = \frac{\partial M}{\partial \mathcal{S}} = \begin{cases} -\frac{q d_3^q d_2^q Q^{2q-1} r_+^{d_1 - 4q}}{8\pi (d_1 - 4q)}, & q \neq \frac{d_1}{4}, \\ -\frac{d_1 d_2 d_3}{32\pi} Q^{d_3/2} \ln r_+, & q = \frac{d_1}{4}. \end{cases} \tag{6.38}$$

The heat capacity can be obtained as

$$C_H = T_H\left(\frac{\partial \mathcal{S}}{\partial T_H}\right)_{\tilde{Q}} = \begin{cases} \frac{d_2k^3\mu_4 r_+^{d_8}\left(d_2k^4\mu_4 d_9 r_+^{4q-8} - 2\Lambda r_+^{4q} - d_3^q d_2^q Q^{2q}\right)}{\left((4q-7)d_2^q d_3^q Q^{2q} - 14\Lambda r_+^{4q} - \mu_4 k^4 d_2 d_9 r_+^{4q-8}\right)}, & q \neq \frac{d_1}{4}, \\ \frac{d_2k^3\mu_4 r_+^{d_8}\left(\mu_4 k^4 d_2 d_9 r_+^{d_9} - 2\Lambda r_+^{d_1} - d_2 d_3 Q^{d_1/2}\right)}{\left(Q^{d_1/2} d_2 d_3 d_8 - 14\Lambda r_+^{d_1} - k^4\mu_4 d_2 d_9 r_+^{d_9}\right)}, & q = \frac{d_1}{4}. \end{cases} \tag{6.39}$$

Note that, the parameter Q is related to Yang-Mills charge $\tilde{Q}$ through Eq. (6.17).

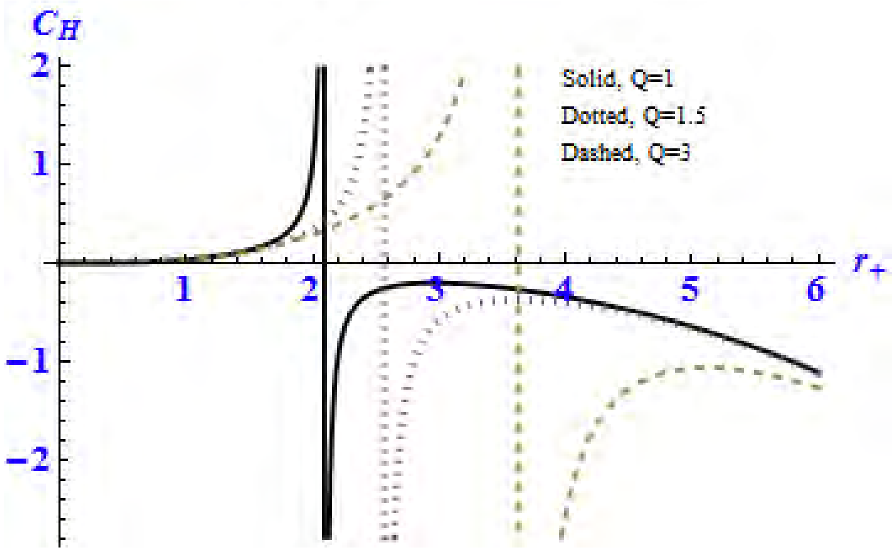

FIGURE 6.21: Dependence of heat capacity C_H (Eq. (6.39)) on the parameter $\tilde{Q} = Q/4\pi$ for fixed values of $d = 11$, $q = 2$, $k = -1$, $\mu_4 = 0.004$ and $\Lambda = 1$
.

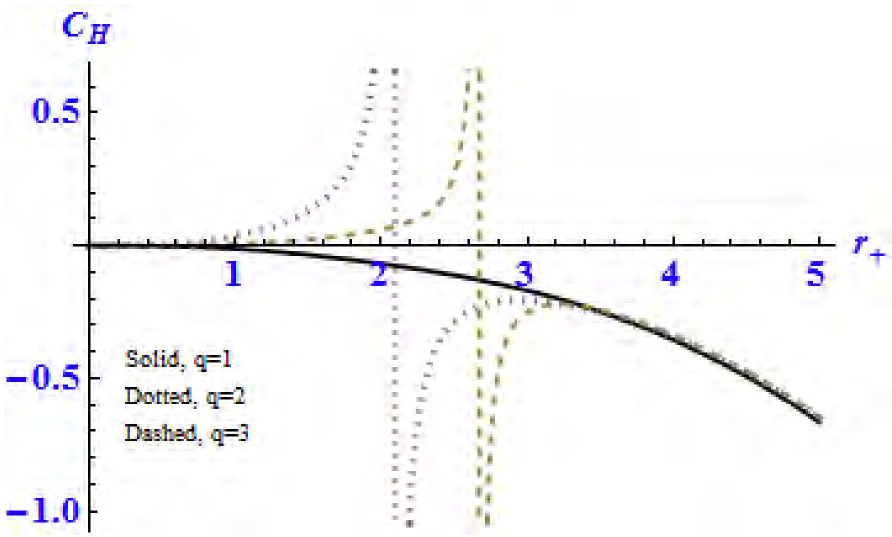

FIGURE 6.22: Dependence of heat capacity C_H (Eq. (6.39)) on the nonlinearity parameter q for fixed values of $d = 11$, $Q = 2$, $k = -1$, $\mu_4 = 0.004$ and $\Lambda = 1$
.

The above expression reduces to the heat capacity of Yang-Mills black holes in pure quasi-topological gravity [42] when $q = 1$. One can also observe that for positive values

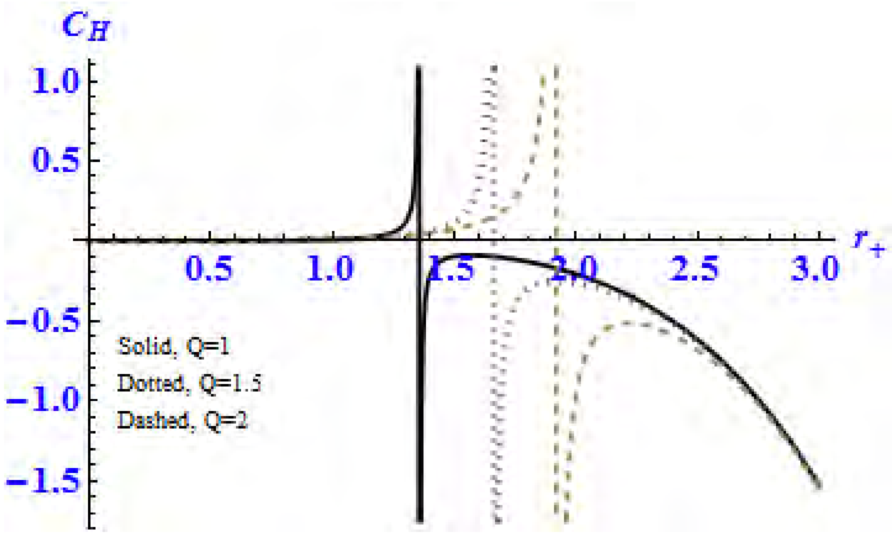

FIGURE 6.23: Dependence of heat capacity C_H (Eq. (6.39)) on the parameter $\tilde{Q} = Q/4\pi$ for fixed values of $d = 13$, $q = d_1/4 = 3$, $k = -1$, $\mu_4 = 0.004$ and $\Lambda = 1$
.

of μ_4 and Λ, the above heat capacity is negative when $q = 1$. Hence, it follows that the Yang-Mills black holes are thermodynamically unstable in the canonical ensemble. However, the local stability of power-Yang-Mills black holes (i.e. when $q \neq 1$) can be described from the plot of heat capacity as a function of r_+. Fig. 6.21 shows the plot of heat capacity for different values of Q and fixed values of other parameters involved in (6.39). The region in which this thermodynamic quantity is positive corresponds to local stability while its negativity implies local instability of pure quasi-topological black holes. Fig. 6.22 depicts the behaviour of this quantity for different values of the parameter q. One can clearly see that the Yang-Mills black hole (i.e. when $q = 1$) is unstable in the canonical ensemble. However, for $q \neq 1$, there exist regions of local stability in this ensemble. Hence, the nonlinearity of Yang-Mills field induces certain affects on the local thermodynamic stability of black holes. Finally, the local stability of pure quasi-topological black holes in spacetime dimensions satisfying $q = d_1/4$ can be examined from Fig. 6.23. One can also analyze the possibility of first and second order transitions from these plots of heat capacity. The points at which heat capacity vanishes correspond to first order transitions of black hole. The points which make this quantity divergent imply the possibility of second order phase transitions.

In the grand canonical ensemble, the entropy $\mathcal{S}$ and charge $\tilde{Q}$ should be considered as variables. In this ensemble, the local thermodynamic stability can be determined from the positivity of both $\left(\partial^2 M/\partial \tilde{Q}^2\right)$ and the determinant of the Hessian matrix. Using the above value of mass in Eq. (6.35) we can compute

$$\left(\frac{\partial^2 M}{\partial Q^2}\right) = \begin{cases} -\frac{q(2q-1)d_3^q d_2^q (4\pi)^{2q}\tilde{Q}^{2q-2}}{8\pi(d_1-4q)} r_+^{d_1-4q}, & q \neq \frac{d_1}{4}, \\ -\frac{d_1 d_2 d_3^2 (4\pi)^{2q}\tilde{Q}^{d_5/2}}{64} \ln r_+, & q = \frac{d_1}{4}. \end{cases} \tag{6.40}$$

The above quantity is negative when $q = 1$ and $q \leq d_1/4$. Thus, for these two choices the black hole is unstable in this ensemble. However, for $q > d_1/4$, the above quantity is positive and so we need to check the behaviour of the Hessian matrix determinant for the investigation of thermal stability. The determinant of Hessian matrix in terms of the outer horizon can be computed as

$$det\mathbf{H} = \begin{cases} \frac{q(2q-1)d_3^q d_2^q (4\pi)^{2q}\tilde{Q}^{2q-2} r_+^{d_1-4q}}{32\pi^2 d_2 \mu_4 k^3 (d_1-4q)} A_p(r_+) - \frac{4q^2 d_3^{2q} d_2^{2q-2}(4\pi)^{2q}\tilde{Q}^{4q-2}}{256\pi^2 \mu_4^2 k^6 r_+^{8q-14}}, & q > \frac{d_1}{4}, \\ \frac{d_1 d_2 d_3^2 (4\pi)^{d_1/2}\tilde{Q}^{d_1/2}\ln r_+}{256\pi^2 d_2 \mu_4 k^3} A_p(r_+) - \frac{d_1^2 (4\pi)^{d_1/2}\tilde{Q}^{d_3} d_3^2}{1624\pi^2 \mu_4^2 k^6 r_+^{2d_8}}, & q = \frac{d_1}{4}, \end{cases} \tag{6.41}$$

where

$$A_p(r_+) = \begin{cases} \frac{kd_9}{4r_+^{d_7}} + \frac{7\Lambda}{2\mu_4 d_2 k^3 r_+^{d_{15}}} + \frac{(7q-4)d_3^q d_2^{q-1}(4\pi)^{2q}\tilde{Q}^{2q}}{4\mu_4 k^3 r_+^{d_{15}+4q}}, & q > \frac{d_1}{4}, \\ \frac{kd_9}{r_+^{d_7}} + \frac{7\Lambda}{2\mu_4 d_2 k^3 r_+^{d_{15}}} - \frac{d_3 d_8 (4\pi)^{d_1/2}\tilde{Q}^{d_1/2}}{4\mu_4 k^3 r_+^{2d_8}}, & q = \frac{d_1}{4}. \end{cases} \tag{6.42}$$

The plot of $det\mathbf{H}$ for various values of charge $\tilde{Q} = Q/4\pi$ is shown in Fig. 6.24. One can

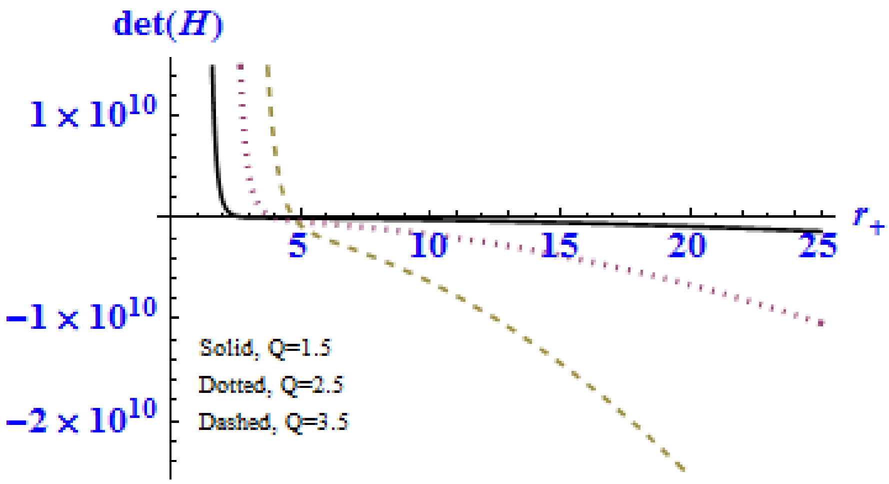

FIGURE 6.24: Dependence of the determinant $det\mathbf{H}$ (Eq. (6.41)) on the parameter Q for fixed values of $d = 11$, $q = 3$, $k = -1$, $\mu_2 = -0.06$, $\mu_3 = -0.1$, $\mu_4 = 0.03$ and $\Lambda = 1$.

see that there exists a value r_0 such that $det\mathbf{H}$ is positive when $r_+ < r_0$. This indicates the region of black hole's stability. However, as r_+ increases its value from r_0, this determinant becomes negative and we have thermodynamic instability. Similarly, the

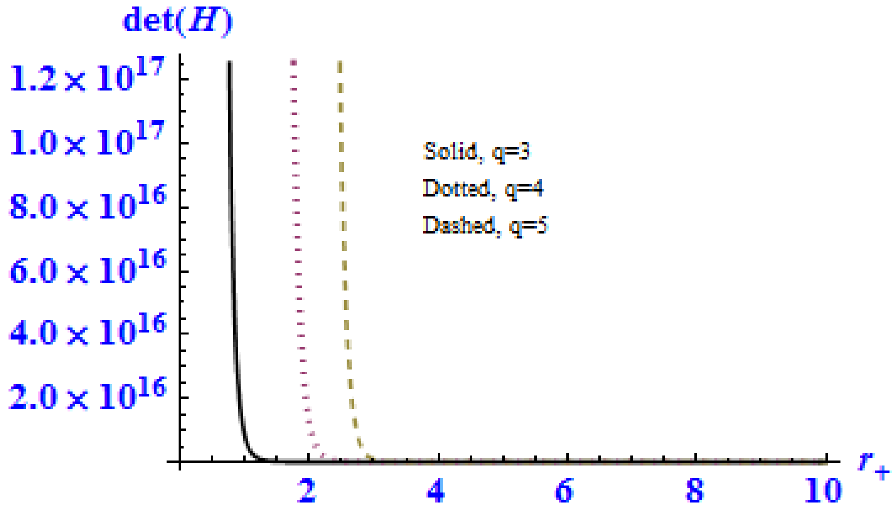

FIGURE 6.25: Plot of the determinant $det\mathbf{H}$ (Eq. (6.41)) for different values of the parameter q and fixed values of $d = 11$, $Q = 3$, $k = -1$, $\mu_2 = -0.06$, $\mu_3 = -0.1$, $\mu_4 = 0.03$ and $\Lambda = 1$.

behaviour of this determinant for different values of nonlinearity parameter q is given in Fig. 6.25. One can observe that as the parameter q increases, the horizon radius of stable black hole also increases. Hence, we conclude that unlike Yang-Mills black holes [42] of pure quasi-topological gravity, the power-Yang-Mills black holes in this theory could be thermodynamically stable in the grand canonical ensemble.

6.5 Thermodynamics of quartic quasi-topological rotating black branes with power-Yang-Mills source

In this section we will implement solution (6.9) for $k = 0$ with a global rotation. This can be done, if we use the transformation describing the rotation boost in the $t - \phi_i$ planes, i.e.,

$$\begin{aligned} t &\longmapsto \Xi t - \sum_{i=1}^{p} a_i \phi_i, \\ \phi_i &\longmapsto \Xi \phi_i - \frac{a_i}{l^2} t. \end{aligned} \tag{6.43}$$

The $SO(d_1)$ rotation group in d-dimensions contains the maximum number of rotational parameters. Hence, the independent parameters of rotation are of number $[d_1/2]$, where [...] stands for the integer part. Thus, the line element for the rotating spacetime with flat horizon and $p \leq [d_1/2]$ rotation parameters can be given as

$$\begin{aligned} ds^2 =& -f(r)\big(\Xi dt - \sum_{i=1}^{p} a_i d\phi_i\big)^2 + \frac{dr^2}{f(r)} + \frac{r^2}{l^4}\sum_{i=1}^{p}\big(a_i dt - \Xi l^2 d\phi_i\big)^2 \\ & - \frac{r^2}{l^2}\sum_{i<j}^{p}\big(a_i d\phi_j - a_j d\phi_i\big)^2 + r^2 \sum_{i=1}^{d_2-p} dx_i^2, \end{aligned} \tag{6.44}$$

where $\Xi = \sqrt{1+\sum_{i=1}^{p} a_i^2/l^2}$, l is a scale factor related to cosmological constant and a_i's are the p parameters of rotation. It should be noted that static line element (6.4) and rotating metric (6.44) can be mapped locally onto each other, not globally. In order to investigate the physical properties of the solutions obtained for $k=0$ in quartic quasi-topological gravity coupled to power-Yang-Mills theory, we plot the metric function $f(r)$ for suitable values of parameters involved in it. In Figs. 6.26-6.29, it is clear that the metric function possesses divergences at the central position $r=0$. One can also verify that the Kretschmann scalar diverges at this point. However, as r becomes larger and larger, the behaviour of $f(r)$ is dependent on the value of the cosmological constant Λ. Hence, we observe from these graphs that the metric function approaches towards $+\infty$ when $\Lambda < 0$ while it tends to $-\infty$ when $\Lambda > 0$. Moreover, Fig. 6.26 shows the behaviour of the metric function for different values of parameter Q in AdS spacetime. It may be noted from here, that for fixed values of other parameters there exists a value Q_{ext} which gives us an extreme black brane, while for the case $Q < Q_{ext}$, there may be black brane having two horizons and for $Q > Q_{ext}$ we will arrive at a naked singularity. The dependence of nonlinearity parameter q on the metric function with negative cosmological constant can be visualized from Fig. 6.27. One can see that this parameter is also affecting the values of the horizons. The solution in dS spacetimes has also been plotted in Fig. 6.28 for different values of Yang-Mils charge Q. Similarly, the behaviour of $f(r)$ for all the three cases of the cosmological constant has been compared in Fig. 6.29, when the spacetime dimensions satisfy $q = d_1/4$. Now, the Killing vector associated with the rotating black brane metric can be defined as

$$\mathcal{X} = \partial_t + \sum_{j=1}^{p} \Omega_j \partial_{\phi_j}, \tag{6.45}$$

where Ω_j refers to the angular velocity and is given by

$$\Omega_j = -\left(\frac{g_{t\phi_j}}{g_{\phi_j\phi_j}}\right) = \frac{a_j}{\Xi l^2}. \tag{6.46}$$

The expression for Hawking temperature associated with this rotating black brane takes the form

$$T_H(r_+) = \frac{f'(r_+)}{4\pi\Xi} = \frac{r_+^2}{4\pi\Xi}\mathfrak{X}'(r_+). \tag{6.47}$$

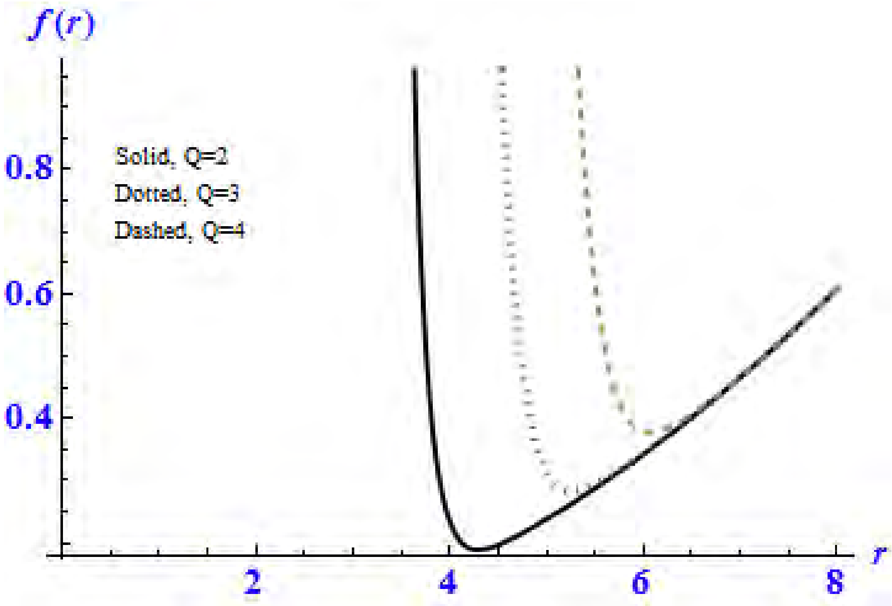

FIGURE 6.26: Dependence of the solution $f(r)$ (Eq. (6.9)) on the parameter Q for fixed values of $d = 11$, $m = 1$, $q = 7$, $k = 0$, $\mu_2 = -0.06$, $\mu_3 = -0.1$, $\mu_4 = 2^{10}$ and $\Lambda = -1$.

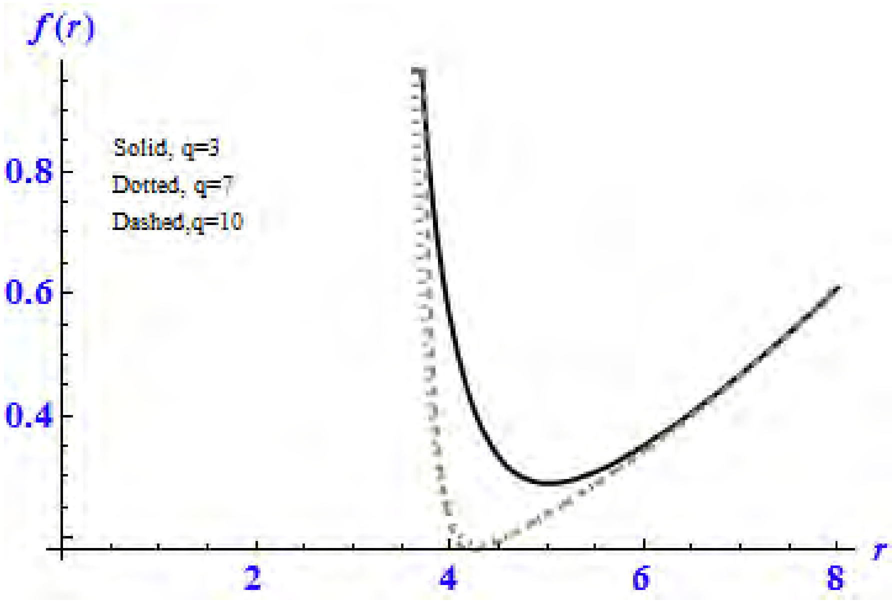

FIGURE 6.27: Plot of the solution $f(r)$ (Eq. (6.9)) for different values of the parameter q and fixed values of $d = 11$, $m = 1$, $Q = 3$, $k = 0$, $\mu_2 = -0.06$, $\mu_3 = -0.1$, $\mu_4 = 2^{10}$ and $\Lambda = -1$.

During the computation of thermodynamic quantities from the variation of action (6.2) with respect to metric tensor, one gets a total derivative surface term containing the derivatives of $\delta g_{\mu\nu}$ normal to the boundary. Since these derivative terms do not cancel with each other, so, the variation of the action is not well-defined. To handle this issue, it is convenient to add the Gibbons-Hawking surface term $\mathcal{I}_b$ with the bulk action (6.2). Thus, the variational principle would be well-defined if this boundary term can

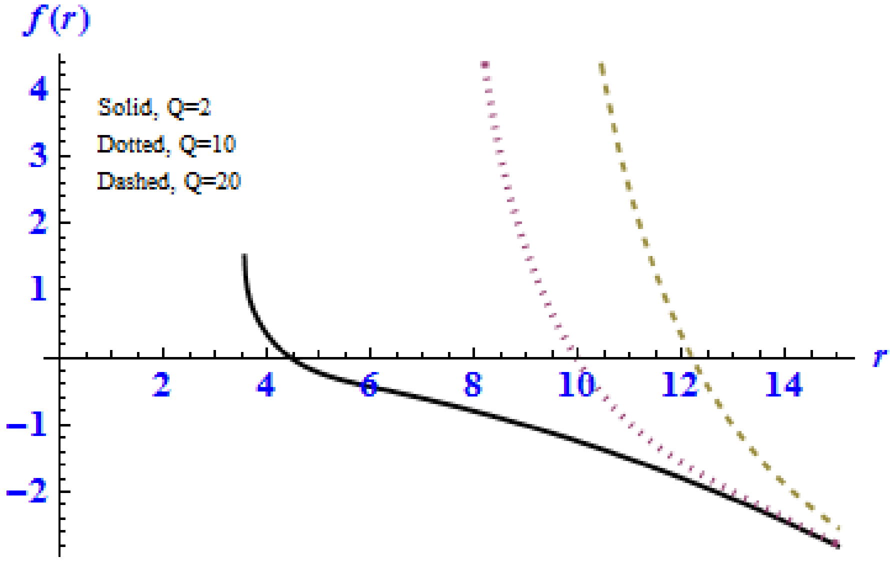

FIGURE 6.28: Dependence of the solution $f(r)$ (Eq. (6.9)) on the parameter Q for fixed values of $d = 11$, $m = 1$, $q = 7$, $k = 0$, $\mu_2 = -0.06$, $\mu_3 = -0.1$, $\mu_4 = 2^{10}$ and $\Lambda = 1$.

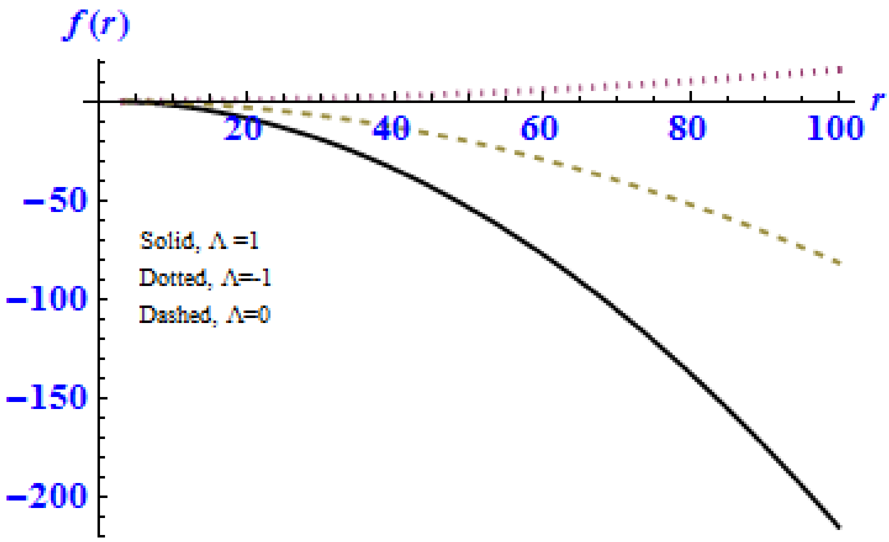

FIGURE 6.29: Plot of the solution $f(r)$ (Eq. (6.9)) for different values of the cosmological constant and fixed values of $d = 13$, $q = d_1/4 = 3$, $Q = 4$, $k = 0$, $\mu_2 = -0.06$, $\mu_3 = -0.1$ and $\mu_4 = 2^{10}$

.

be included in the following form

$$\mathcal{I}_b = \mathcal{I}_b^{(I)} + \mathcal{I}^{(II)} + \mathcal{I}^{(III)} + \mathcal{I}_b^{(IV)}, \tag{6.48}$$

where $\mathcal{I}_b^{(I)}$, $\mathcal{I}_b^{(II)}$, $\mathcal{I}_b^{(III)}$ and $\mathcal{I}_b^{(IV)}$ respectively, stand for the surface terms corresponding to Einstein [121], second order Lovelock (Gauss-Bonnet) [235], cubic quasi-topological

[81] and quartic quasi-topological [236] gravities. These terms are obtained as

$$\mathcal{I}_b^{(I)} = \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^{d_1}x\sqrt{-\gamma}\mathcal{K}, \tag{6.49}$$

$$\mathcal{I}_b^{(II)} = \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^{d_1}x\sqrt{-\gamma}\frac{2\tilde{\mu}_2 l^2}{3d_3d_4}\big(3\mathcal{K}\mathcal{K}_{ad}\mathcal{K}^{ad} - 2\mathcal{K}_{ac}\mathcal{K}^{cd}\mathcal{K}_d^a - \mathcal{K}^3\big), \tag{6.50}$$

$$\begin{aligned}\mathcal{I}_b^{(III)} =& \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^{d_1}x\sqrt{-\gamma}\Bigg[\frac{3\tilde{\mu}_3 l^4}{5d_1d_2^2d_3d_6}\Big(d_1\mathcal{K}^5 - 2\mathcal{K}^3\mathcal{K}_{ad}\mathcal{K}^{ad} + 4d_2\mathcal{K}_{ab}\mathcal{K}^{ab}\mathcal{K}_{cd}\mathcal{K}_e^d\mathcal{K}^{ec} \\ &- (5d_1-6)\mathcal{K}\mathcal{K}_{ab}\big(d_1\mathcal{K}^{ab}\mathcal{K}^{cd}\mathcal{K}_{cd} - d_2\mathcal{K}^{ac}\mathcal{K}^{bd}\mathcal{K}_{cd}\big)\Big)\Bigg],\end{aligned} \tag{6.51}$$

and

$$\begin{aligned}\mathcal{I}_b^{(IV)} =& \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^{d_1}x\sqrt{-\gamma}\Bigg[\frac{2\tilde{\mu}_4 l^6}{7d_1d_2d_3d_8(d_1^2-3d_1+3)}\Big(\alpha_1\mathcal{K}^3\mathcal{K}^{ab}\mathcal{K}_{ac}\mathcal{K}_{bd}\mathcal{K}^{cd} \\ &+ \alpha_2\mathcal{K}^2\mathcal{K}^{ab}\mathcal{K}_{ab}\mathcal{K}^{cd}\mathcal{K}_c^e\mathcal{K}_{de} + \alpha_3\mathcal{K}^2\mathcal{K}^{ab}\mathcal{K}_{ac}\mathcal{K}_{bd}\mathcal{K}^{ce}\mathcal{K}_e^d \\ &+ \alpha_4\mathcal{K}\mathcal{K}^{ab}\mathcal{K}_{ab}\mathcal{K}^{cd}\mathcal{K}_c^e\mathcal{K}_d^f\mathcal{K}_{ef} + \alpha_5\mathcal{K}\mathcal{K}^{ab}\mathcal{K}_a^c\mathcal{K}_{bc}\mathcal{K}^{de}\mathcal{K}_d^f\mathcal{K}_{ef} \\ &+ \alpha_6\mathcal{K}\mathcal{K}^{ab}\mathcal{K}_{ac}\mathcal{K}_{bd}\mathcal{K}^{ce}\mathcal{K}^{df}\mathcal{K}_{ef} + \alpha_7\mathcal{K}^{ab}\mathcal{K}_a^c\mathcal{K}_{bc}\mathcal{K}^{de}\mathcal{K}_{df}\mathcal{K}_{eg}\mathcal{K}^{fg}\Big)\Bigg],\end{aligned} \tag{6.52}$$

where γ_{ab} refers to the induced metric tensor on the boundary $\partial\mathcal{M}$ while $\mathcal{K}$ stands for the trace of the extrinsic curvature $\mathcal{K}^{ab}$ of this boundary. It is worthwhile to note that the value of the total action $\mathcal{I}_{bulk} + \mathcal{I}_b$ is infinite on the solutions. However, this divergence can be removed with the help of the counter-term method [237–241]. The counter-term action needed for the removal of this divergence can be written as

$$\mathcal{I}_{count} = -\frac{1}{8\pi}\int_{\partial\mathcal{M}} d^{d_1}x\sqrt{-\gamma}\left(\frac{d_2}{L_{eff}}\right), \tag{6.53}$$

where L_{eff} describes the effective scale length factor related to l and parameters $\tilde{\mu}_2$, $\tilde{\mu}_3$ and $\tilde{\mu}_4$. Note that, L_{eff} reduces to l when the coupling constants i.e. $\tilde{\mu}_2$, $\tilde{\mu}_3$ and $\tilde{\mu}_4$ approach zero. Using the counter-term method, the overall action $\mathcal{I}_{bulk} + \mathcal{I}_b + \mathcal{I}_{count}$ becomes finite and can be used to compute the conserved and thermodynamic quantities.

The conserved quantities related to the timelike ∂_t and rotational ∂_{ϕ_j} Killing vector fields can be computed as

$$M = \frac{(d_1\Xi^2 - 1)m}{16\pi d_2 l^{d_2}}, \tag{6.54}$$

$$J_i = \frac{d_1\Xi a_i m}{16\pi d_2 l^{d_2}}. \tag{6.55}$$

One can clearly see that by choosing the rotation parameter a_i equal to zero or $\Xi = 1$, the angular momentum $\mathbf{J}$ vanishes and (6.54) then describes the mass of the static black hole. The Yang-Mills charge per unit volume $\mathcal{V}_{d_2}$ in this case can be obtained as

$$\tilde{Q} = \frac{\Xi Q}{4\pi l^{d_4}}. \tag{6.56}$$

It is well-known that entropy is the quarter of horizon area [127, 128]. Using this, the entropy density for the power-Yang-Mills black brane can be obtained as

$$\mathcal{S} = \frac{\Xi r_+^{d_2}}{4l^{d_4}}. \tag{6.57}$$

In order to check the validity of the first law, it is more convenient to calculate the mass in terms of extensive variables $\mathcal{S}$, $\tilde{Q}$ and $\mathbf{J}$. Therefore, by taking $\mathcal{Z} = \Xi^2$ and using Eqs. (6.54)-(6.55), we construct the Smarr-type formula as

$$M(\mathcal{S}, \tilde{Q}, J) = \frac{(d_1\mathcal{Z} - 1)J}{d_1 l\sqrt{\mathcal{Z}(\mathcal{Z} - 1)}}. \tag{6.58}$$

It should be noted that the parameter $\mathcal{Z}$ should be dependent on the extensive parameters. Using Eqs. (6.56)-(6.57) and the condition for event horizon i.e. $f(r_+) = 0$, it is possible to obtain an equation $\mathcal{E}(\mathcal{S}, \tilde{Q}, J) = 0$ [67], whose positive real root is $\mathcal{Z} = \Xi^2$ and

$$\mathcal{E}(\mathcal{S}, \tilde{Q}, J) = \begin{cases} \frac{16\pi l^{d_3} d_2 J \mathcal{Z}^{d_1/2d_2}}{d_1\sqrt{\mathcal{Z}(\mathcal{Z}-1)}} + \frac{2\Lambda l^{d_1 d_4/d_2}(4)^{d_1/d_2}}{d_1 d_2 \mathcal{S}^{-d_1/d_2}} + \frac{(\pi\tilde{Q})^{2q} d_2^{q-1} d_3^q \left(4l^{d_4}\right)^{(d_1+2qd_4)/d_2}}{(d_1-4q)\mathcal{Z}^{d_4 q/d_2}\mathcal{S}^{(4q-d_1)/d_2}}, & q \neq \frac{d_1}{4}, \\ \frac{16\pi l^{d_3} d_2 J \mathcal{Z}^{d_1/2d_2}}{d_1\sqrt{\mathcal{Z}(\mathcal{Z}-1)}} + \frac{2\Lambda l^{d_1 d_4/d_2}(4)^{d_1/d_2}}{d_1 d_2 \mathcal{S}^{-d_1/d_2}} + \frac{\left(4l^{d_4}\pi\tilde{Q}\right)^{d_1/d_2} d_3}{d_2\mathcal{Z}^{d_1/4}} \ln\left(\frac{4l^{d_4} S}{\mathcal{Z}^{1/2}}\right), & q = \frac{d_1}{4}. \end{cases} \tag{6.59}$$

Now, it is straightforward to write the mass $M(\mathcal{S}, \tilde{Q}, J)$ in terms of the extensive parameters and compute the intensive parameters conjugate to them as follows

$$\begin{aligned} T_H &= \left(\frac{\partial M}{\partial \mathcal{S}}\right)_{J,\tilde{Q}}, \\ \Omega_k &= \left(\frac{\partial M}{\partial J_k}\right)_{\mathcal{S},\tilde{Q}}, \end{aligned} \tag{6.60}$$

while, the power-Yang-Mills potential is given by

$$\mathcal{U} = \left(\frac{\partial M}{\partial \tilde{Q}}\right)_{\mathcal{S},J} = \begin{cases} \frac{2qJ(d_3\mathcal{Z}+1)d_2^{q-1}d_3^q(\pi\tilde{Q})^{2q}\left(4l^{d_4}\right)^{(2qd_4+1)/d_2}\mathcal{S}^{(d_1-4q)/d_2}}{2d_1 l(d_1-4q)\mathcal{Z}^{qd_4/d_2}(\mathcal{Z}(\mathcal{Z}-1))^{3/2}\mathfrak{Y}(\mathcal{S},\tilde{Q},J)}, & q \neq \frac{d_1}{4}, \\ \frac{d_3(d_3\mathcal{Z}+1)(4\pi)^{d_1/d_2} l^{d_1 d_4/d_2} J\tilde{Q}^{1/d_2}}{2ld_2^2\mathcal{Z}^{d_1/4}(\mathcal{Z}(\mathcal{Z}-1))^{3/2}\mathfrak{Y}(\mathcal{S},\tilde{Q},J)} \ln\left(\frac{4\mathcal{S}l^{d_4}}{\sqrt{\mathcal{Z}}}\right), & q = \frac{d_1}{4}, \end{cases} \tag{6.61}$$

in which

$$\mathfrak{Y}(\mathcal{S},\tilde{Q},J)=\begin{cases}\frac{q(\pi\tilde{Q})^{2q}d_2^q d_3^q d_4\left(4l^{d_4}\right)^{(d_1+2qd_4)/d_2}\mathcal{S}^{(d_1-4q)/d_2}}{(d_1-4q)\mathcal{Z}^{1+d_4q/d_2}}+\frac{8\pi l^{d_3}J(d_3\mathcal{Z}+1)\mathcal{Z}^{d_1/2d_2}}{d_1(\mathcal{Z}(\mathcal{Z}-1))^{3/2}}, & q\neq\frac{d_1}{4},\\ \frac{8\pi l^{d_3}J(d_3\mathcal{Z}+1)\mathcal{Z}^{d_1/2d_2}}{d_1(\mathcal{Z}(\mathcal{Z}-1))^{3/2}}+\frac{d_3(4\pi\tilde{Q})^{d_1/d_2}l^{d_1d_4/d_2}}{2d_2\mathcal{Z}^{1+(d_1/4)}}\left(1+\frac{d_1}{2}\ln\left(\frac{4\mathcal{S}l^{d_4}}{\sqrt{\mathcal{Z}}}\right)\right), & q=\frac{d_1}{4}.\end{cases} \tag{6.62}$$

Our calculations showed that the angular velocity and Hawking temperature in (6.60) are same as (6.46) and (6.47), respectively. Thus, our power-Yang-Mills rotating black brane satisfy the first law as

$$dM=T_H dS+\sum_{i=1}^{p}\Omega_i dJ_i+\mathcal{U}d\tilde{Q}. \tag{6.63}$$

Chapter 7

Charged black holes in 4D Einstein-Gauss-Bonnet gravity coupled to nonlinear electrodynamics with maximum allowable symmetries

Recently a new model of nonlinear Maxwell electrodynamics which preserves both conformal and SO(2) duality-rotational invariance has been proposed. In this chapter, we consider this model as a source of gravitational field and study black holes of novel four-dimensional Einstein-Gauss-Bonnet gravity. In this framework we solve the gravitational field equations and work out the metric function that describe a black hole which generalizes the Maxwellian charged black hole of four-dimensional Einstein-Gauss-Bonnet gravity. Some basic thermodynamic quantities corresponding to the new black hole solution are computed and thermodynamic stability as well as thermal phase transitions are investigated.

7.1 Black holes of novel four-dimensional EGB gravity

The action function corresponding to a d-dimensional EGB gravity minimally coupled with NED is given by

$$\mathcal{I}=\frac{1}{16\pi}\int d^dx\sqrt{-g}\Big[R-2\Lambda+\frac{\alpha}{d-4}\Big(R^2-4R_{\mu\nu}R^{\mu\nu}+R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\Big)+\mathfrak{L}_{NED}(\mathcal{F},P)\Big], \quad (7.1)$$

where $\mathfrak{L}_{NED}(\mathcal{F}, P)$ is the nonlinear Maxwell's Lagrangian proposed as [163]

$$\mathfrak{L}_{NED}(\mathcal{F}, P) = -\frac{1}{2}\bigg(\mathcal{F}\cosh\gamma - \sqrt{\mathcal{F}^2 + P^2}\sinh\gamma\bigg). \tag{7.2}$$

Note that we are using the units in which Newton's constant G is equal to unity. In the above action, Λ is the cosmological constant given by $\Lambda = -(d-1)(d-2)/2l^2$, $\mathcal{F} = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$ and $P = \frac{1}{2}F_{\mu\nu}\tilde{F}^{\mu\nu}$ and $F_{\mu\nu}$ is the Maxwell tensor while $\tilde{F}^{\mu\nu}$ is its dual. It is also to be noted that parameter γ is positive and possesses an upper bound from the causality and unitarity conditions satisfied by NED model [163–165].

The equations of motion corresponding to the gravitational field in the presence of NED source can be obtained by variation of the action with respect to the metric field $g_{\mu\nu}$ as

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} + \frac{\alpha}{d-4}\bigg[2RR_{\mu\nu} + 2R_{\mu}^{\rho\sigma\lambda}R_{\nu\rho\sigma\lambda} - 4R_{\mu\lambda}R_{\nu}^{\lambda} \\ - 4R^{\rho\sigma}R_{\mu\rho\nu\sigma} - \frac{1}{2}g_{\mu\nu}\bigg(R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma}\bigg)\bigg] = \mathcal{T}_{\mu\nu}, \end{aligned} \tag{7.3}$$

where $\mathcal{T}_{\mu\nu}$ is the stress-energy tensor corresponding to the nonlinear electromagnetic field defined as

$$\mathcal{T}_{\mu\nu} = \frac{1}{2}\bigg[\bigg(\mathfrak{L}_{NED} - P\frac{\partial\mathfrak{L}_{NED}}{\partial P}\bigg)g_{\mu\nu} - 2F_{\mu\lambda}F_{\nu}^{\lambda}\frac{\partial\mathfrak{L}_{NED}}{\partial\mathcal{F}}\bigg]. \tag{7.4}$$

Similarly, by varying the action with respect to the electromagnetic four-potential A_μ we get the nonlinear-Maxwell equations as

$$d\bigg(\frac{\partial\mathfrak{L}_{NED}}{\partial\mathcal{F}}\tilde{\mathbf{F}} + \frac{\partial\mathfrak{L}_{NED}}{\partial P}\mathbf{F}\bigg) = 0, \tag{7.5}$$

where the electromagnetic 2-form is given by

$$\mathbf{F} = Edt\wedge dr + Br^{d-2}\sum_{i=2}^{d-2}\prod_{j=1}^{i-1} sin\theta_j d\theta_j \wedge d\theta_i, \tag{7.6}$$

in which E and B are the radial components of static electric and magnetic fields respectively. Similarly we can get the dual form $\tilde{\mathbf{F}}$ as

$$\tilde{\mathbf{F}} = -Bdt\wedge dr + Er^{d-2}\sum_{i=2}^{d-2}\prod_{j=1}^{i-1} sin\theta_j d\theta_j \wedge d\theta_i. \tag{7.7}$$

Note that, the d-dimensional static line element ansatz is chosen for the above electromagnetic 2-form in the following form

$$ds^2 = -a(r)\exp\left[-2b[r]\right]dt^2 + \frac{dr^2}{a(r)} + r^2 d\Omega_{d-2}^2, \tag{7.8}$$

where $d\Omega_{d-2}^2$ denotes as usual the line element corresponding to $(d-2)$-dimensional sphere. Having the above expressions of $\mathbf{F}$ and $\tilde{\mathbf{F}}$, it can easily be shown that $\digamma = B^2 - E^2$ and $P = 2EB$. These values of $\digamma$ and P simplify Eq. (7.5) into

$$d\left[\left(-B\frac{\partial L}{\partial \digamma} + E\frac{\partial L}{\partial \Upsilon}\right)dt \wedge dr\right] = 0, \tag{7.9}$$

and

$$d\left[\left(E\frac{\partial L}{\partial \digamma} + B\frac{\partial L}{\partial \Upsilon}\right)r^{d-2}\sum_{i=2}^{d-2}\prod_{j=1}^{i-1} sin\theta_j d\theta_j \wedge d\theta_i\right] = 0. \tag{7.10}$$

Now, the Bianchi identity $d\mathbf{F} = 0$ implies that

$$d\left[\left(Edt \wedge dr + Br^{d-2}\sum_{i=2}^{d-2}\prod_{j=1}^{i-1} sin\theta_j d\theta_j \wedge d\theta_i\right] = 0, \tag{7.11}$$

from which one can find that the radial electric and magnetic fields as well as the invariants $\digamma$ and P depend only on coordinate r, and thus Eq. (7.9) is trivially satisfied and (7.10) yields

$$\left(E\frac{\partial \mathfrak{L}_{NED}}{\partial \digamma} + B\frac{\partial \mathfrak{L}_{NED}}{\partial P}\right)r^{d-2} = C, \tag{7.12}$$

where C is the integration constant. We also have the Bianchi identity which implies that

$$B(r) = \frac{q_m}{r^{d-2}}, \tag{7.13}$$

where q_m is the magnetic charge. By redefining $\mathfrak{L}_{NED}(\digamma, P) = -\digamma y(z)$ such that $z = P/\digamma$ and

$$y(z) = \frac{1}{2}\left(\cosh\gamma - \sqrt{1+z^2}\sinh\gamma\right), \tag{7.14}$$

we can get from Eqs. (7.13) and (7.12)

$$E(r) = \frac{q_e}{r^{d-2}}, \tag{7.15}$$

where q_e is the electric charge satisfying

$$\left(q_m - \frac{2q_e^2 q_m}{q_m^2 - q_e^2}\right)\frac{q_m q_e \sinh\gamma}{(q_m^2 + q_e^2)} - \frac{q_e}{2}\left(\cosh\gamma - \sinh\gamma\frac{(q_m^2 + q_e^2)}{(q_m^2 - q_e^2)}\right) = C. \tag{7.16}$$

Similarly, $z = P/\mathcal{F} = 2q_e q_m/q_m^2 - q_e^2$ while the invariants $\mathcal{F}$ and P in terms of q_m and q_e take the forms $\mathcal{F} = q_m^2 - q_e^2/r^{2d-4}$ and $P = 2q_m q_e/r^{2d-4}$, respectively. Using the above values of electric and magnetic fields, one can find the stress-energy tensor associated to NED as

$$T_\mu^\nu = \frac{w^2(q_m^2 + q_e^2)}{2r^{2d-4}} diag(-1, -1, 1, ..., 1), \tag{7.17}$$

where

$$w^2 = \frac{\cosh\gamma}{2} - \frac{(q_m^2 - q_e^2)}{2(q_m^2 + q_e^2)}\sinh\gamma. \tag{7.18}$$

One can easily verify that the energy conditions are satisfied here and w^2 is also positive definite because of causality and unitarity conditions which are satisfied when $q_e < q_m$. The trace of the stress-energy tensor is given by

$$\mathcal{T} = \frac{w^2(q_m^2 + q_e^2)(d-4)}{2r^{2d-4}}, \tag{7.19}$$

which implies that in higher $d-$dimensional spacetimes, conformal invariance has been broken down. However, for $d = 4$ conformal symmetry is preserved because the trace is zero in four dimensions. Instead of solving the gravitational field equations with NED source given by Lagrangian density (7.2) directly [83, 242, 243], one can also consider the following reduced form of action [82, 244] for the solution

$$\begin{aligned} I = \frac{\pi^{\frac{d-1}{2}}}{8\pi\Gamma[\frac{d-1}{2}]}\int dtdr(d-2)\exp(-b)\Bigg[\frac{d}{dr}\left(r^{d-1}\psi(1+\alpha(d-3)\psi) + \frac{r^{d-1}}{l^2}\right) \\ - \frac{r^{d-2}}{2(d-2)}\left(\frac{(q_m^2 - q_e^2)\cosh\gamma}{r^{2d-4}} - \frac{(q_m^2 + q_e^2)\sinh\gamma}{r^{2d-4}}\right)\Bigg], \end{aligned} \tag{7.20}$$

where for simplicity we have chosen $\psi(r) = (1 - a(r))/r^2$. By taking the variation of the reduced action with respect to $a(r)$ and $b(r)$ one can easily obtain the solution as

$$b(r) = 0, \tag{7.21}$$

and

$$a(r) = 1 + \frac{r^2}{2\alpha(d-3)}\left[1 \pm \sqrt{1 + 4\alpha(d-3)\left(\frac{2M}{r^{d-1}} - \frac{1}{l^2} - \frac{w^2(q_m^2 - q_e^2)}{(2d-4)(d-3)r^{2d-4}}\right)}\right], \tag{7.22}$$

where M is the constant of integration. Since we are interested in the four-dimensional solution of EGB gravity, thus for the case $d = 4$ we have the solution [245]

$$b(r) = 0, E(r) = \frac{q_e}{r^2}, B(r) = \frac{q_m}{r^2}, \tag{7.23}$$

and

$$a(r) = 1 + \frac{r^2}{2\alpha}\left[1 \pm \sqrt{1 + 4\alpha\left(\frac{2M}{r^3} - \frac{1}{l^2} - \frac{w^2(q_m^2 - q_e^2)}{4r^4}\right)}\right]. \tag{7.24}$$

It should be noted that the constant of integration M in Eq. (7.24) corresponds to

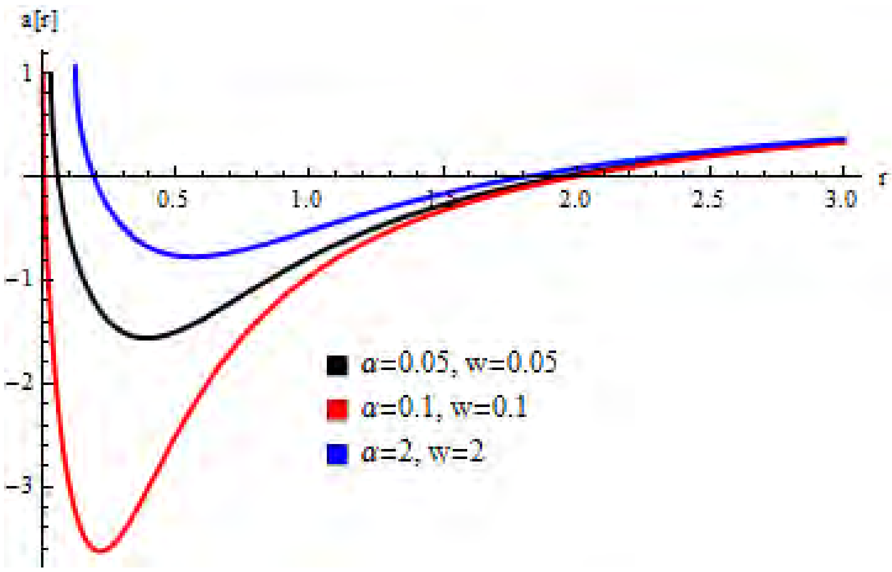

FIGURE 7.1: Plots of metric function $a(r)$ (Eq. (7.24)) vs r for fixed values of $l = 0.1$, $M = 10$, $q_e = 1$, $q_m = 10$.

the mass of black hole and is chosen such that the proper asymptotic limit can be recovered from the solution. One can checked that the Weyl tensor for the spatial part of the general spherically symmetric metric (7.8) vanishes. Hence, our obtained metric function (7.24) determined in the framework of [43] is also the new solution for consistent theory [55–57]. Fig. 7.1 shows the plot of metric function (7.24), and the values of coordinate r for which the corresponding curves touch the horizontal axis indicate the location of horizons. It should be emphasized that the resulting metric function gives Maxwellian charged AdS black hole solution for $w = 1$ and $\gamma = 0$ in the novel four-dimensional EGB [82]. Furthermore, the Schwarzschild-AdS like black hole solutions of novel four-dimensional EGB gravity [43] can also be recovered from the resulting metric function by closing the charges contribution in solution (7.24). By considering large distances from the black hole, the obtained metric function takes the following asymptotic form

$$a(r) = 1 + \frac{r^2}{2\alpha}\left(1 \pm \sqrt{1 - \frac{4\alpha}{l^2}}\right) \pm \frac{2Mr - (q_m^2 - q_e^2)}{r^2\sqrt{1 - \frac{4\alpha}{l^2}}} + O(r^{-4}). \tag{7.25}$$

This behaviour is in complete aggrement with the four dimensional EGB solution obtained in Ref. [244]. Moreover, for the determination of real valued metric function at large distance limit, it is obvious that one needs to consider $0 < \alpha \leq l^2/4$ or $\alpha < 0$. Note that, for small values of α, the solution associated with the plus sign is asymptotic to Reissner-Nordström- AdS like solution with negative gravitational mass, an imaginary magnetic charge and real electric charge. However, the solution corresponding to the minus sign is asymptotic to proper Reissner-Nordström solution with negative cosmological constant

$$a(r) = 1 - \frac{2M}{r} - \frac{q_m^2}{r^2} + \frac{q_e^2}{r^2} + \frac{r^2}{l^2} + O(\alpha), \tag{7.26}$$

which for $q_m = 0$ and $\Lambda = 0$, gives the familiar Reissner-Nordström solution of Einstein gravity. In general, the Ricci and Kretschmann scalars for the ansatz (7.8) can be determined for $d = 4$ and $b(r) = 0$ as

$$R(r) = \left[2\left(\frac{1 - a(r)}{r^2}\right) - \frac{d^2a}{dr^2} - \frac{4}{r}\frac{da}{dr}\right], \tag{7.27}$$

and

$$K(r) = \left[4\left(\frac{1 - a(r)}{r^2}\right)^2 - \left(\frac{d^2a}{dr^2}\right)^2 + \frac{4}{r^2}\left(\frac{da}{dr}\right)^2\right]. \tag{7.28}$$

So, by using the metric function (7.24), one can easily verify that both the Ricci and Kretschmann scalars have a singularity at $r = 0$, which confirms that the resulting gravitating object is a black hole.

7.2 Thermodynamics of novel four-dimensional EGB black holes

In this section we explore some basic thermodynamics corresponding to the nonlinearly charged black hole solution obtained in the last section. In order to do this, we compute basic thermodynamic quantities in terms of the horizon radius r_+. Thus, by using the condition $a(r_+) = 0$, the mass of the charged black hole can be calculated as

$$M = \frac{r_+}{2}\left(\frac{\alpha}{r_+^2} + 1 + \frac{r_+^2}{l^2} + \frac{w^2(q_m^2 - q_e^2)}{4r_+^2}\right). \tag{7.29}$$

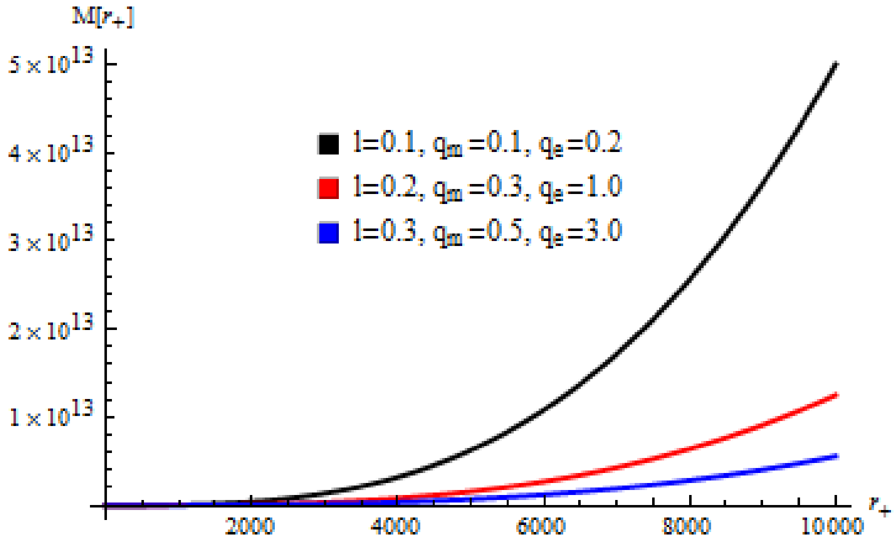

FIGURE 7.2: Plots of mass M (Eq. (7.29)) vs r_+ for fixed values of $\alpha = 10$, and $w = 0.05$.

The behaviour of mass in terms of horizon radius is plotted in Fig. 7.2. It is shown that for the types of black hole solutions given by (7.24), there exist one or more horizons. Those values of r_+ for which this mass gives positive values in its range correspond to horizon of the black hole. But the values of r_+ which assign negative values of M do not correspond to the horizon radius. It is seen that the number of horizons depend on the parameter l, electric charge q_e and magnetic charge q_m. Similarly, one can also find the critical horizon radius for the extremal black hole from the condition $a(r_{ext}) = 0$ and $\frac{da}{dr}|_{r=r_{ext}} = 0$. Hence, the extremal mass can be obtained as

$$M_{ext} = \frac{r_{ext}}{3}\left(1 + \frac{2\alpha}{r_{ext}^2}\right) + \frac{w^2(q_m^2 - q_e^2)}{6r_{ext}}, \tag{7.30}$$

where the extremal horizon is given by the equation

$$2r_{ext}^3 - 2\alpha r_{ext} + \frac{6r_{ext}^5}{l^2} = w^2(q_m^2 - q_e^2). \tag{7.31}$$

The Hawking temperature [120, 121] is defined in terms of surface gravity κ_s as $T_H = \kappa_s/2\pi$, where the surface gravity is given by $\kappa_s = -\frac{1}{2}\frac{\partial g_{tt}}{\partial r}|_{r=r_+}$. Thus, for the solution obtained in this chapter, we can write Hawking temperature in the form

$$T_H = \frac{1}{4\pi}\left[-\frac{2}{r_+} + \frac{r_+^2}{\left(1 + \frac{2\alpha}{r_+^2}\right)}\left(\frac{3}{r_+ l^2} + \frac{3\alpha}{r_+^5} + \frac{3}{r_+^3} - \frac{w^2(q_m^2 - q_e^2)}{4r_+^5}\right)\right]. \tag{7.32}$$

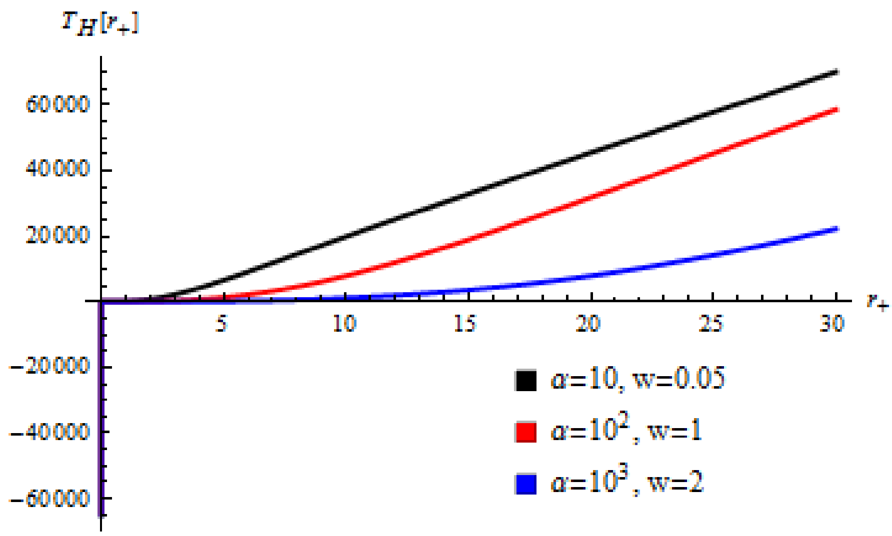

FIGURE 7.3: Plots of temperature T_H (Eq. (7.32)) vs r_+ for fixed values of $l = 0.01$, $q_m = 1$ and $q_e = 20$.

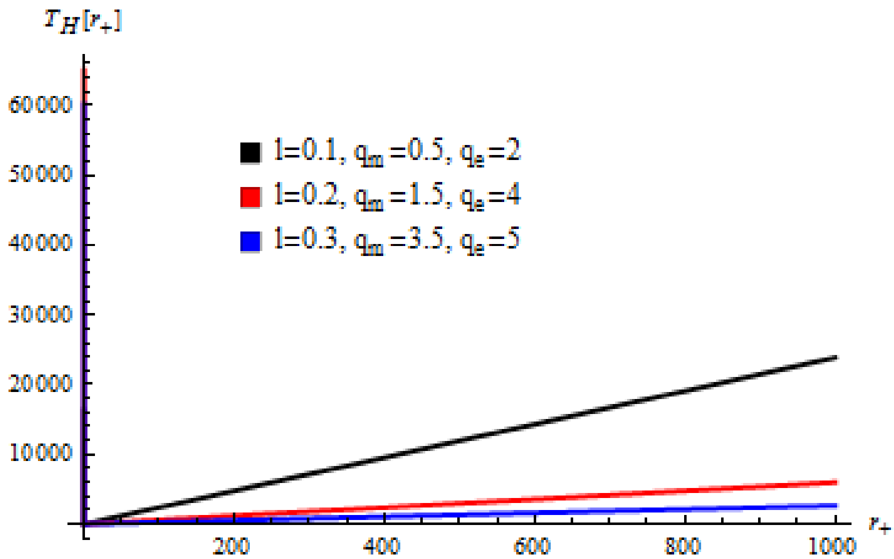

FIGURE 7.4: Plots of temperature T_H (Eq. (7.32)) vs r_+ for fixed values of $\alpha = 10$ and $w = 10$.

The plots of Hawking temperature for different values of parameters l, q_m, q_e, w and α are shown in Figs. 7.3 and 7.4. The value of r_+ for which Hawking temperature touches the horizontal axis implies the first order phase transition of the black hole. One can also easily check that for the extremal black hole, the expression of Hawking temperature (7.32) vanishes. The entropy corresponding to the black hole solution defined by Eqs. (7.23) and (7.24) can be computed with the help of the method similar to the one discussed in Ref. [75]. According to this approach, we have

$$S = \pi r_+^2 + 2\pi\alpha \ln(r_+^2) + S_0 = \frac{A_H}{4} + 2\pi\alpha \ln\left(\frac{A_H}{A_0}\right), \tag{7.33}$$

in which A_H is the black hole's horizon area and A_0 is the constant having dimension of area, which cannot be computed from the first principle [246]. However, by assuming that the entropy of black hole, in general, is independent of the charges and cosmological constant involved in the associated metric function, one can simply fix [247] the undetermined constant as $A_0 = 4\pi|\alpha|$. It is to be noted that the associated logarithmic term with Gauss-Bonnet coupling parameter α appears as a subleading correction to the Bekenstein-Hawking area formula. This subleading correction is universal in some quantum gravity theories [248]. Although, novel EGB gravity theory is considered as classical [43] but still the logarithmic correction arises at classical level in this theory.

The heat capacity at constant electric and magnetic charges is defined by the relation

$$C_H = \left(\frac{\partial M}{\partial T_H}\right)_{q_e, q_m}. \tag{7.34}$$

Differentiation of Hawking temperature with respect to r_+ implies

$$\begin{aligned}\frac{\partial T_H}{\partial r_+} = \frac{1}{4\pi}\Bigg[\frac{2}{r_+^2} + \frac{4\alpha r_+^3}{(r_+^2+2\alpha)^2}\left(\frac{3}{r_+ l^2} + \frac{3\alpha}{r_+^5} + \frac{3}{r_+^3} - \frac{w^2(q_m^2-q_e^2)}{4r_+^5}\right) \\ + \frac{r_+^2}{r_+^2+2\alpha}\left(\frac{3}{l^2} - \frac{9\alpha}{r_+^4} - \frac{3}{r_+^2} + \frac{3w^2(q_m^2-q_e^2)}{4r_+^4}\right)\Bigg].\end{aligned} \tag{7.35}$$

Similarly, differentiation of Eq. (7.29) gives

$$\frac{\partial M}{\partial r_+} = \frac{1}{2}\left(1 - \frac{\alpha}{r_+^2} + \frac{3r_+^2}{l^2} - \frac{w^2(q_m^2-q_e^2)}{4r_+^2}\right). \tag{7.36}$$

Hence, by using Eqs. (7.35) and (7.36) in the general expression of heat capacity (7.34), we will obtain

$$C_H = \frac{\pi(1+2\alpha/r_+^2)^2\left(4r_+^2 - 4\alpha + 12r_+^4 l^{-2} - w^2(q_m^2-q_e^2)\right)}{8(1+2\alpha/r_+^2)^2 + 16\alpha r_+\Delta_1(r_+) + 4r_+^2(1+2\alpha/r_+^2)\Delta_2(r_+)}, \tag{7.37}$$

where

$$\Delta_1(r_+) = \frac{3}{r_+ l^2} + \frac{3\alpha}{r_+^5} + \frac{3}{r_+^3} - \frac{w^2(q_m^2-q_e^2)}{4r_+^5}. \tag{7.38}$$

and

$$\Delta_2(r_+) = \frac{3}{l^2} - \frac{9\alpha}{r_+^4} - \frac{3}{r_+^2} + \frac{3w^2(q_m^2-q_e^2)}{4r_+^4}. \tag{7.39}$$

Heat capacity is very significant because it plays a crucial role in local thermodynamic stability or instability of black holes. Fig. 7.5 shows the associated plot of heat capacity

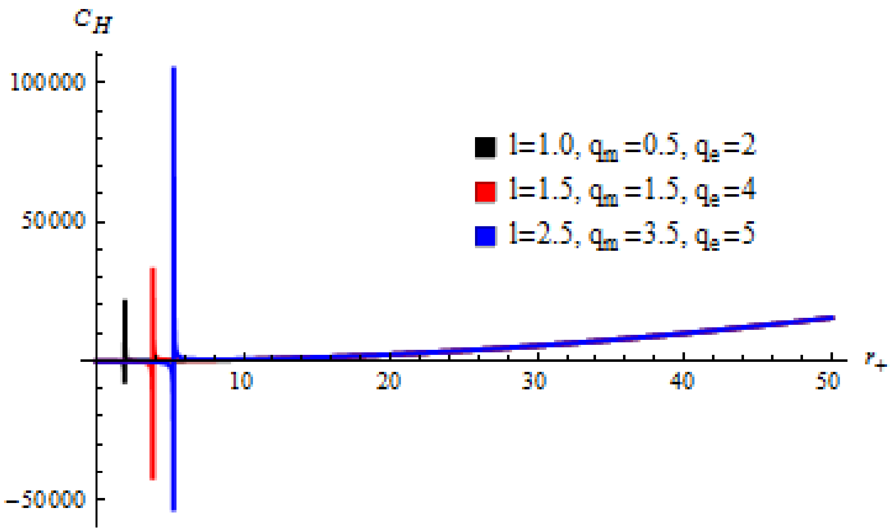

FIGURE 7.5: Plots of temperature C_H (Eq. (7.37)) vs r_+ for fixed values of $\alpha = 10$ and $w = 10$.

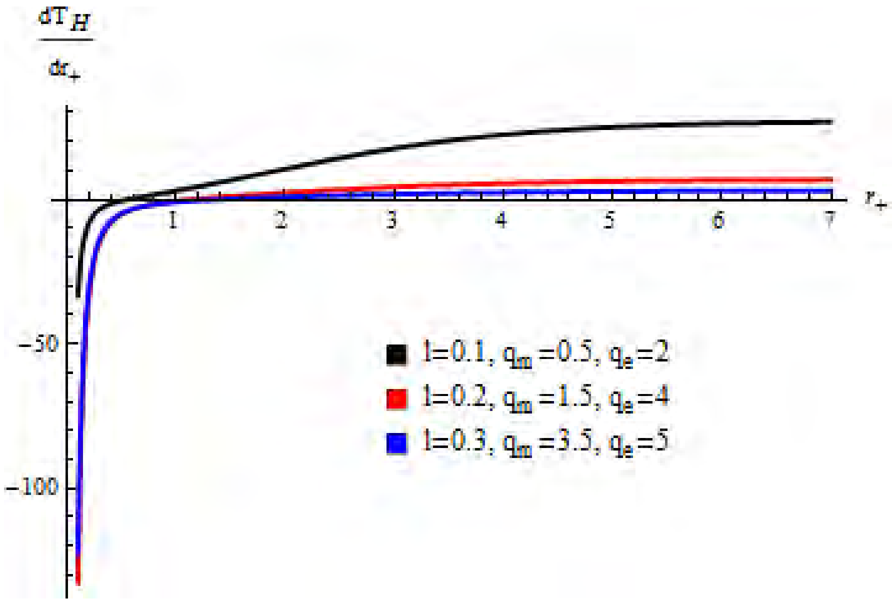

FIGURE 7.6: Plots of dT_H/dr_+ (Eq. (7.35)) vs r_+ for fixed values of $\alpha = 10$ and $w = 10$.

C_H in terms of r_+, the event horizon. The region in which heat capacity is positive indicates that the black hole is locally stable and the region where it is negative implies instability. The first order phase transitions are determined from those values of r_+ at which heat capacity C_H changes its sign, however, those values for which it is divergent correspond to the second order phase transitions of the gravitating object. It is worthwhile to say that the second order phase transitions of black holes can be described from Fig. 7.6 as well, since the roots of $dT_H/dr_+ = 0$, give the singularities of heat capacity. So, those values of r_+ at which the curve of dT_H/dr_+ touches the horizontal axis correspond to the second order thermal phase transition points.

In order to explore global stability, we calculate Gibb's free energy for the black hole solution which can be defined in a canonical ensemble as

$$F = M - T_H S. \tag{7.40}$$

So, by using the expressions corresponding to mass, Hawking temperature and entropy we get

$$\begin{aligned} F = {} & \frac{r_+}{2}\left(\frac{\alpha}{r_+^2} + 1 + \frac{r_+^2}{l^2} + \frac{w^2(q_m^2 - q_e^2)}{4r_+^2}\right) - \left(r_+^2 + 2\alpha \ln \frac{r_+^2}{|\alpha|}\right) \\ & \times \left[-\frac{1}{2r_+} + \frac{r_+^3}{4(r_+^2 + 2\alpha)}\left(\frac{3}{l^2} + \frac{3\alpha}{r_+^4} + \frac{3}{r_+^2} - \frac{w^2(q_m^2 - q_e^2)}{4r_+^4}\right)\right]. \end{aligned} \tag{7.41}$$

The behaviour of free energy is illustrated in Fig. 7.7 in terms of the event horizon and for several choices of parameters involved in its expression. Those values of horizon r_+ for which Gibb's energy is negative implies that the black holes having such horizons are globally stable while black holes having positive Gibb's energy are globally unstable.

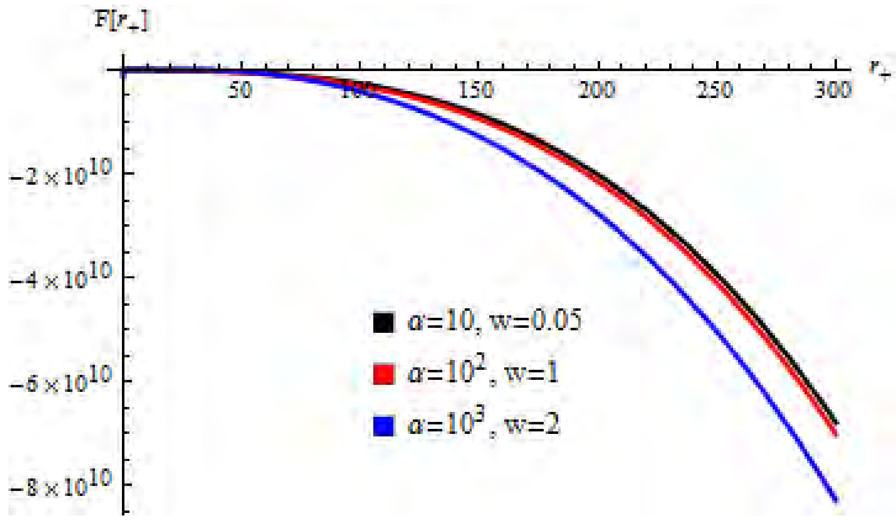

FIGURE 7.7: Plots of F (Eq. (7.41)) vs r_+ for fixed values of $l = 0.01$, $q_m = 1$ and $q_e = 20$.

Chapter 8

Summary and Conclusion

For almost 90 years the phenomenon of NED has been widely studied. It is believed that this theory can provide new ways for tackling different problems appearing in cosmology, for example, the big bang singularity and inflation can be properly investigated through it. NED models are illustrated by classical theory and therefore may be coupled with Einstein as well as extended gravity theories. All (or most of) the formulations of NED are viewed as effective field theories. However, associated with a few models, for example, the Born-Infeld and Euler-Heisenberg formulations, the effective theories such as the string theory and quantum electrodynamics are known. Many NED models can be recovered in the low energy limit of string theory through the Kaluza-Klein dimensional reduction. Born-Infeld NED possesses several desirable characteristics, for instance, the electric field and potential yield finite values at the centre $r = 0$. On the other hand the values of electric field and potential in Maxwell's theory are singular at this position. For finiteness of electric field Born and Infeld chose a particular nonlinear action having maximum field strength; the field equations thus obtained were coupled with the gravitational field equations and then the equations were solved to determine the solution corresponding to a point charge which was static and spherically symmetric. Thus we have used different models of NED in the framework of gravity theories and explored new nonlinearly charged black hole solutions.

After introducing the main ideas of the thesis in the first chapter, in Chapter 2, we used a model of exponential electrodynamics which is reducible to the Maxwell electrodynamics in the weak field limit. In this model birefriengence phenomenon holds and also weak energy condition is satisfied, and similarly, unitarity and causality principles are also satisfied. A regular black hole solution has been obtained in exponential nonlinear electrodynamics in the framework of Einstein's general relativity [145]. Using this method we coupled an exponential electrodynamics with a UV regulating modified theory of gravity to obtain an asymptotic magnetically charged non-singular black hole metric. The resulting metric is regular at the origin as the curvature invariants are finite there. Moreover, the asymptotic values of the metric functions at $r \to 0$ and $r \to \infty$ have been worked out. This metric is similar to the electrically charged non-singular metric [192]. Further, we calculated the exact expressions of its temperature and specific heat. We note that the first order phase transition occurs at the outer horizon $r_2 = \pm 2^{-1/4}\beta^{1/4}\sqrt{q}(5 \times 10^{29})$, as Hawking temperature is zero at these values, for particular values of β, q and l. Here $r_2 < -0.050194\sqrt{q}\beta^{1/4}$, the heat capacity comes out to be negative so the black hole is unstable. The heat capacity diverges as the Hawking temperature increases, so the second order phase transition occurs in the interval $x_2 \in (-0.05969, 0.05969)$. Note that x_2 is associated with the horizon radius r_2 through Eq. (2.23). Furthermore, quantum radiations from this non-singular black hole have also been discussed. Here we found the expression for the quantum energy flux of the massless particles emitted from the interior of such a non-singular black hole.

In the limit $\beta \to 0$, our results correspond to the case of magnetically charged non-singular black holes where Maxwell's electrodynamics is coupled with the modified gravity [197]. For $l = 0$, these results coincide with the case of black hole solutions of Einstein's gravity in the presence of exponential electromagnetic field [145]. When both the parameters β and l vanish, our results reduce to those for the Reissner-Nordström black hole of Einstein's theory. Similarly, neutral black hole solutions are obtained if we put $q = 0$ and $l = 0$ in the formulae derived in this chapter.

In Chapter 3, we have investigated $(2+1)$-dimensional black holes in the framework of two different NED models. In the background of these NED models, the scale invariance has been violated because the associated stress-energy tensor possesses non-zero trace. First, we derived magnetically charged $(2+1)$-dimensional black hole solution in Einstein's theory coupled to exponential electrodynamics. It is shown that the consideration of exponential electromagnetic field as a source of gravity makes the metric function (3.15) regular and finite at the origin $r = 0$. It is also shown that in the limit $r \to \infty$ or $\beta \to 0$, the obtained metric function (3.15) describes $(2+1)$-dimensional black hole with a Maxwellian magnetic charge. The convergence of curvature invariants associated to (3.15) shows that the black hole is regular and there is no true curvature singularity at $r = 0$. In the second case, we derived electrically charged $(2+1)$-dimensional black hole solution in Einstein's theory coupled to arcsin electrodynamics. In this case too, the equations of motion are solved and the metric function (3.55) is derived. The asymptotic behaviour of the resulting metric function at both radial infinity and at $r = 0$ are also discussed. Like the case of exponential electrodynamics, the arcsin electromagnetic source on the right hand side of gravitational field equations makes the metric function finite at the origin, which can be seen from the asymptotic value (3.57). Moreover, the asymptotic expression of the metric function at infinity shows that for large values of r, the contributions of NED could be negligible and the resulting solution reduces to Maxwellian electrically charged $(2+1)$-dimensional black hole. The curvature invariant, i.e., the Ricci scalar was also calculated for the electrically charged black hole. The series expansions of these curvature invariants about $r \to \infty$ shows that the object is not asymptotically flat while the corresponding series expansion about $r \to 0$ indicates that there exists an essential curvature singularity at the central position.

The thermodynamic properties associated with the black hole solutions derived within the backgrounds of exponential and arcsin models of NED are also analyzed. For doing this, we have computed entropy, temperature and heat capacity for both the black holes. It is shown that these quantities satisfy the first law.

Higher dimensional gravities sometimes give rise to more interesting possibilities than the four-dimensional theories. This major leap can help solve the problem of "hierarchy of scales". Many higher dimensional models have been formulated in recent years. The well-known generalization of Einstein's gravity in higher dimensions is the Lovelock gravity. The equations of motion obtained in this theory are still second order. Besides, it is ghost-free, which implies that the principle of unitarity is not violated. In Chapter 4 of this thesis, we studied spherically symmetric Lovelock black holes in the presence of power-Yang-Mills field and Chaplygin-like dark fluid. We obtained a polynomial equation (4.24) which generates the metric function for the Lovelock-power-Yang-Mills black holes surrounded by dark fluid. We considered only the solutions with Yang-Mills magnetic charges because the problem with the electric solutions is that Υ^q may not be real for any q, which is not the case with pure magnetic type Yang-Mills field. We note that this is true for the power-Maxwell case as well. We then studied thermodynamics of these objects and worked out different thermodynamic quantities associated with the polynomial equation in terms of the event horizon. After this we discussed two special cases of Eq. (4.24) and derived metric functions in terms of parameters q, B and S for magnetized Einsteinian and Gauss-Bonnet black holes. It is shown that the resulting black holes are nonasymptotically flat and the associated spacetime geometries have true curvature singularities at $r = 0$. In both the cases, thermodynamic properties are studied and quantities such as mass, Hawking temperature and heat capacity are worked out and plotted for fixed values of different parameters arising in the corresponding metric functions. We discussed the behaviour of these quantities and pointed out regions of thermodynamic stability. Furthermore, it is also shown that phase transitions are also possible for the black hole solutions that we have obtained. The existence of the first order phase transition is associated with the value of r_+ at which either the temperature or heat capacity or both change signs, while the second order phase transitions of black holes are related to the points at which the heat capacity diverges.

It should be noted that when one puts $q = 1$ in Eq. (4.24), the corresponding metric functions for black holes surrounded by dark fluid with the standard Yang-Mills source can be obtained, but for $d = 4q + 1$ the solutions do not exist. For $B = S = 0$ we get the solutions corresponding to Lovelock, Einsteinian and Gauss-Bonnet black holes, where only the power-Yang-Mills field is the gravitational source. By choosing $Q = 0$ the corresponding neutral black holes in such gravities are obtained. It should also be noted that the Einsteinian black hole solutions obtained are valid even for $q = d/4$ unlike the black holes of Einstein gravity in the presence of only Yang-Mills or power-Yang-Mills sources. This behaviour is due to the presence of dark fluid as a source of gravity which does not make the scalar curvature zero even when the trace of the stress-energy tensor is zero at $q = d/4$. It is also important to note that our Gauss-Bonnet and Lovelock black hole solutions are valid only for $d \geq 5$ provided that $d \neq 4q + 1$. However, the Einsteinian black hole solution is valid in any dimensions such that $d \neq 4q + 1$.

Although the natural generalization of Einstein's theory is the Lovelock gravity where the problems related to appearance of ghosts and unitarity are not present. However, beyond the perturbation theory, the presence of higher curvature terms make it totally different from Einstein's theory. If the Lovelock coupling parameters are selected arbitrarily, the Lovelock theory has some limitations. For example, the time evolution of the fields comes out to be non-unique and the Hamiltonian is not well-defined [249]. Additionally, the negative energy solutions with event horizons or positive energy solutions with naked singularities appeared [20]. To overcome these deficiencies, an appropriate choice has been made for the coupling parameters and the dimensionally continued gravity has been formulated. Thus, in Chapter 5 we investigate magnetized black holes of DCG in the framework of both exponential electrodynamics and power-Yang-Mills theory. First, we derived both topological black hole solutions and hairy black hole solutions of DCG with pure exponential magnetic source. These solutions depend on the parameter β of exponential electrodynamics. For the model of exponential electrodynamics, there is no need to impose the condition on the stress-energy tensor for making it traceless, so we conclude that the scale and dual invariances are completely violated here. Further, the components of the stress-energy tensor obtained from the Lagrangian density of this model satisfy all the energy conditions along with causality and unitarity principles in some specified region of the radial coordinate. For any value of parameter β, it is possible to obtain a solution which is regular at the origin. Moreover, these solutions are nonasymptotically flat for nonzero values of constant l while in the limit l approaching to infinity one can get asymptotically flat solutions. It is shown that the metric functions are finite at the origin; this finiteness property of metric functions is due to the nonlinear behaviour of electromagnetic field characterized by the Lagrangian density of exponential NED. Second, we use a model of power-Yang-Mills theory and derive a large family of topological black hole solutions in DCG. These solutions depend on the parameter q and are also nonasymptotically flat for any nonzero values of l. The case $q = 1$ gives the solutions of black holes with the standard Yang-Mills source and for $q = 1/4$ the solution does not exist. The hairy black hole solutions are also derived in the framework of power-Yang-Mills theory which are reducible to black holes with no scalar hair for $Y = 0$ and to neutral black holes for $Q = 0$.

The thermodynamics of topological black holes is also studied within both the exponential electrodynamics and power-Yang-Mills theory. Thermodynamic quantities such as entropy, Hawking temperature and specific heat capacity at constant charge of the resulting magnetized higher dimensional black holes of DCG have been worked out. The first law of black hole thermodynamics has also been shown to hold for these black hole solutions. Recently, we have also followed this method and studied the hairy black holes surrounded by cloud of strings and quintessence in Lovelock-scalar gravity [250].

Quasi-topological gravity also belongs to the class of extended gravity theories which may be fruitful for the investigation of gravitational fields in higher dimensional spherically symmetric geometries. Although Lovelock and quasi-topological gravities are equivalent under the assumption of spherical symmetry, however, quasi-topological gravity is way more significant because it encounters the gravitational effects in lesser dimensions than the Lovelock theory. For example, if the third order curvature terms are included in the action, Lovelock gravity generates gravitational effects in spacetime dimensions $d \geq 7$ while quasi-topological gravity produces these effects in dimensions higher than four. Therefore, it is highly motivating to study physical properties of black holes in the framework of quasi-topological gravity. Hence, in Chapter 6 of this thesis, we mainly focused on the physical and thermodynamic properties of quartic quasi-topological black holes with power-Yang-Mills source. First, we considered the fourth order quasi-topological gravity and coupled it with the power-Yang-Mills theory. From the Wu-Yang ansatz, the gauge potentials are defined and the gravitational field equations are solved. In this context, two analytic power-Yang-Mills black hole solutions are derived for $\mu_4 > 0$ and $\mu_4 < 0$ in this theory. It is shown that real solutions exist only when $\mu_4 > 0$. We have also written the two expressions separately for the metric function valid in spacetime dimensions when $q \neq d_1/4$ and $q = d_1/4$. We also studied physical properties of these black holes and plotted the associated solution $f(r)$ given in Eq. (6.9) for various values of the parameters m, q, Q, Λ and μ_i's. Depending on the suitable choices for these parameters, either the solution describes a black hole which can possesses one or more horizons or it can describe a naked singularity. It is shown that variations in quasi-topological parameter μ_4 affect the position of the horizon. Similarly, it can also be concluded that the value of the outer horizon is not affected by the charge parameter Q, however, the inner horizon increases with the increase of Yang-Mills magnetic charge. It should be noted that for $q = 1$, the solution (6.9) yields the metric function of Yang-Mills black hole [42] in this theory. In addition to this, we have also studied thermodynamics of these power-Yang-Mills black holes. In this study, we worked out different thermodynamic quantities and plotted them as well. The region where positive temperature arises implies that the black hole is physical. It is also shown that the AdS power-Yang-Mills black holes may have larger range of parameters with positive temperature than the dS black holes. In the canonical ensemble, the positivity of specific heat capacity implies local thermodynamic stability. Hence, from the plots of heat capacity we have concluded that the outer horizon of stable power-Yang-Mills black holes increases with the increase in charge Q. The effects of nonlinearity parameter q on the stability of black holes in this ensemble have also been shown. One can also observe that the stable power-Yang-Mills black holes have larger outer horizons than the Yang-Mills black holes. Thermodynamic stability in grand canonical ensemble have also been investigated. It is shown that the black holes with $q \leq d_1/4$ are unstable in this ensemble. However, for $q > d_1/4$ there exist stability regions for black holes. From the plots of the Hessian matrix, we have concluded that the smaller black holes could be stable in this ensemble. However, when the outer horizon r_+ increases then $det\mathbf{H}$ is negative and so we have thermodynamic instability of the black holes. Thus, it can be concluded that the power-Yang-Mills field produces the possibility for local stability in the grand canonical ensemble. This behaviour is in contrast to the Yang-Mills theory [42], where quasi-topological black holes are locally unstable in this ensemble.

The physics of black holes in pure Lovelock gravity has attracted much attention from theoretical physicists. Recently, black holes of this theory with different matter sources have been studied in the literature [42, 77, 251]. In Chapter 6, we also derived a new family of black hole solutions in pure quasi-topological theory within the framework of power-Yang-Mills sources. The associated plots of pure quasi-topological black hole solution (6.32) for suitable values of parameters show that for $k = -1$ and $d > 9$, this solution describes AdS black hole with two horizons, an extreme dS black hole and a naked singularity. It is shown that by choosing $\Lambda > 0$, the asymptotic expression of metric function (6.33) describes the asymptotically AdS and dS power-Yang-Mills black holes for $k = -1$ and $k = 1$, respectively. The effects of Yang-Mills charge and parameter q on the horizon structure of black holes can also be observed from these plots. Note that, the case $q = 1$ in solution (6.32) gives the black hole solution of pure quasi-topological gravity with Yang-Mills source. The local thermodynamic stability in both canonical and grand canonical ensembles have also been probed. Our results show that there exist regions of local stability for the power-Yang-Mills black holes in the canonical ensemble. This can be seen from the corresponding plots of heat capacity and Hawking temperature. Moreover, from the plots of the determinant of Hessian matrix, one can identify the regions of local stability for these black holes in the grand canonical ensemble. It should be noted that it is the nonlinearity of Yang-Mills field that plays the main role in the stability of black holes. For $q = 1$, our results correspond to those of the Yang-Mills black holes in pure quasi-topological gravity which are unstable in both canonical and grand canonical ensembles.

In the last part of Chapter 6, we have assumed a general rotating line element with $p \leq [(d-1)/2]$ rotational parameters and studied the rotating black branes of quartic quasi-topological gravity coupled to power-Yang-Mills theory. The plots of metric function (6.9) with $k = 0$ show that for fixed values of parameters d, q, m, μ_2, μ_3 and μ_4, it can describe the power-Yang-Mills black brane with inner and outer horizons for $Q < Q_{ext}$, extremal black brane for $Q = Q_{ext}$ and naked singularity for $Q > Q_{ext}$. These plots of the metric function show the effects of Yang-Mills charge Q and nonlinearity parameter q in all the three cases for cosmological constant Λ. Here, we included the generalized Gibbons-Hawking surface terms for the quasi-topological gravity which made the action well-defined. We calculated the Hawking temperature and angular velocities by taking analytic continuation of the metric. In order to derive the finite action and conserved quantities, we have used the counter-term method. It is shown that the conserved quantities of these power-Yang-Mills black branes are independent of the coupling coefficients i.e. μ_2, μ_3 and μ_4 for fixed values of mass, Yang-Mills charge and rotation parameters a_i. However, one can notice that the thermodynamic quantities depend indirectly on these coupling coefficients through the value of the outer horizon r_+. Furthermore, we have also derived the Smarr-type formula which describes the mass density in terms of entropy, Yang-Mills magnetic charge and angular momenta. We showed that the first law is satisfied for the power-Yang-Mills black branes obtained in this chapter. It is worthwhile to note that for $q = 1$, these results correspond to the Yang-Mills black branes of quartic quasi-topological gravity.

We know that Lovelock introduced [9] a natural modification of Einstein's gravity in higher dimensions where higher curvature terms are added to the usual Einstein-Hilbert Lagrangian. Besides the Einstein-Hilbert and cosmological constant terms, there also exists a Gauss-Bonnet term involved in Lovelock's action. Note that, Gauss-Bonnet term is quadratic in curvature and it contributes to ultraviolet corrections of Einstein's theory. On the other hand we know that since it is a total derivative in four dimensions, it has no contribution for gravitational dynamics. In order to encounter the dynamical contribution in four dimensions, a new theory known as four-dimensional novel Einstein-Gauss-Bonnet gravity has been introduced [43]. This theory has very interesting properties, for example, it avoids Ostrogradsky instability and completely dodged the Lovelock theorem. Chapter 7 of this thesis contain the investigation of charged black holes in the framework of this theory coupled to the ModMax model of NED. It should be noted that this model of NED satisfies the two fundamental duality rotational and conformal symmetries. In this setup, the metric function given by Eq. (7.24) describing nonlinearly charged static and spherically symmetric black hole is worked out. It is shown that the solution obtained in the novel four-dimensional gravity reduces to the generalized AdS Schwarzschild solution of EGB if we put electromagnetic charges equal to zero, while for small values of GB parameter α, the AdS Reissner-Nordström solution can be obtained. It is known from the literature that the charged solution in the presence of a Maxwell source is real only when $\alpha < 0$, however, in our case the solution can be real for both positive and negative values. This behaviour of our solution is similar to the one obtained in Ref. [165]. It is also shown that the black hole obtained may have zero, one or more horizons depending on the parameters involved in the metric function. In addition to this, thermodynamic properties associated with the resulting black hole solution have been explored and thermodynamic quantities such as Hawking temperature, heat capacity and Gibb's free energy have been calculated. Thermodynamic stability and thermal phase transitions have been studied and the results are illustrated in the corresponding graphs of these thermodynamic quantities.

It would be very interesting to study Hawking radiation, thermal fluctuations and greybody factors for the black holes we have obtained. Further, it is expected that investigation of the causal structure and causality conditions will give useful insights into the black hole solutions obtained in this thesis. Similarly one can also investigate the properties of nonlinearly charged Lovelock and quasi-topological black holes in the presence of quintessence and cloud of strings. In addition to this, the study of higher dimensional solutions in the context of Lovelock gravities sourced by differential three-form fields could also be interesting.

Bibliography

[1] D. R. Brill and R. H. Gowdy, Rep. Prog. Phys. **33** (1973) 413.

[2] C. M. Will, Living Rev. Rel. **17** (2014) 4.

[3] LIGO Scientific, Virgo collaboration, B. P. Abbott et al., Phys. Rev. Lett. **116** (2016) 061102.

[4] P. Dayal and A. Ferrara, Phys. Rep. **780** (2018) 1.

[5] C. Rovelli, Living Rev. Rel. **11** (2008) 5.

[6] B. Zwiebach, Phys. Lett. B **156** (1985) 315.

[7] O. Aharony, S. S. Gubser, J. M. Maldecena, H. Ooguri and Y. Oz, Phys. Rep. **323** (2000) 183.

[8] A. A. Starobinsky, Phys. Lett. B **91** (1980) 99.

[9] D. Lovelock, J. Math. Phys. **12** (1971) 498.

[10] D. Lovelock, J. Math. Phys. **13** (1972) 874.

[11] E. Dyer and K. Hinterbichler, Phys. Rev. D **79** (2009) 024028.

[12] J. Madore, Class. Quantum Grav. **3** (1986) 361.

[13] C. Lanczos, Ann. Math. (1938) 842.

[14] C. G. Callan, I. R. Klebanov and M. Perry, Nucl. Phys. B **278** (1986) 78.

[15] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B **258** (1985) 46.

[16] D. J. Gross, J. A. Harvey, E. Martinec and R. Rohlm, Nucl. Phys. B **256** (1985) 253.

[17] N. Dadhich, Int. J. Mod. Phys. D **20** (2011) 1739.

[18] C. Teitelboim and J. Zanelli, Class. Quantum Grav. **4** (1987) L125.

[19] M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. A **36** (1987) 4417.

[20] J. T. Wheelar, Nucl. Phys. B **268** (1986) 737.

[21] B. Whitt, Phys. Rev. D **38** (1988) 3001.

[22] R. C. Myers and J. Simon, Phys. Rev. D **38** (1988) 2434.

[23] R. C. Myers and J. Simon, Phys. Rev. D **38** (1988) 2445.

[24] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. D **49** (1994) 975.

[25] J. Crisostomo, R. Troncoso and J. Zanelli, Phys. Rev. D **62** (2000) 084013.

[26] M. H. Dehghani, A. Bazrafshan, R. B. Mann, M. R. Mehdizadeh, M. Ghanaatian and M. H. Vahidinia, Phys. Rev. D **85** (2012) 104009.

[27] T. C. Sisman, I. Gullu and B. Tekin, Class. Quantum Grav. **28** (2011) 195004.

[28] R. C. Myers, M. F. Paulos and A. Sinha, J. High Energy Phys. **08** (2010) 035.

[29] R. C. Myers and A. Sinha, J. High Energy Phys. **01** (2011) 125.

[30] M. H. Dehghani, Phys. Rev. D **70** (2004) 064019.

[31] M. H. Dehghani, Phys. Rev. D **67** (2003) 064017.

[32] M. H. Dehghani and S. H. Hendi, Int. J. Mod. Phys. D **16** (2007) 1829.

[33] R. C. Myers and B. Robinson, J. High Energy Phys. **08** (2010) 067.

[34] W. G. Brenna and R. B. Mann, Phys. Rev. D **86** (2012) 064035.

[35] M. Ghanaatian, F. Naeimipour, A. Bazrafshan, M. Eftekharian and A. Ahmadi, Phys. Rev. D **99** (2019) 024006.

[36] M. Ghanaatian, F. Naeimipour, A. Bazrafshan and M. Akbar, Phys. Rev. D **97** (2018) 104054.

[37] A. Bazrafshan, F. Naeimipour, M. Ghanaatian, G. Forozani and A. Alizadeh, Phys. Rev. D **99** (2019) 124009.

[38] M. Ghanaatian, A. Bazrafshan and W. G. Brenna, Phys. Rev. D **89** (2014) 124012.

[39] A. Cristerna, L. Guajardo, M. Hassaine and J. Oliva, J. High Energy Phys. **02** (2017) 010.

[40] J. Ahmed, R. A. Hennigar, R. B. Mann and M. Mir, J. High Energy Phys. **2017** (2017) 134.

[41] M. Mir, R. A. Hennigar, J. Ahmed and R. B. Mann, J. High Energy Phys. **2019** (2019) 68.

[42] F. Naeimipour, B. Mirza and F. M. Jahromi, Eur. Phys. J. C **81** (2021) 455.

[43] D. Glavan and C. Lin, Phys. Rev. Lett. **124** (2020) 081301.

[44] A. Mardones and J. Zanelli, Class. Quantum Grav. **8** (1991) 1545.

[45] T. Torri and H. A. Shinkai, Phys. Rev. D **78** (2008) 084037.

[46] L. S. Brown, Phys. Rev. D **15** (1977) 1469.

[47] R. P. Woodard, Ostrogradsky's theorem on Hamiltonian instabilities, Scholarpedia **10** (2015) 32243.

[48] M. Gurses, T. C. Sisman and B. Tekin, Phys. Rev. Lett. **125** (2020) 149001.

[49] M. Gurses, T. C. Sisman and B. Tekin, Eur. Phys. J. C **80** (2020) 647.

[50] S. Mahapatra, Eur. Phys. J. C **80** (2020) 992.

[51] T. Kobayashi, J. Cosm. Astropart. Phys. **7** (2020) 013.

[52] J. Bonifacio, K. Hinterbichler and L. A. Johnson, Phys. Rev. D **102** (2020) 024029.

[53] J. Arrechea, A. Delhom and A. Jiménez-Cano, Chin. Phys. C **45** (2021) 013107.

[54] M. Hohmann, C. Pfeifer and N. Voicu, Eur. Phys. J. Plus 136 (2021) 180.

[55] K. Aoki, M. A. Gorji and S. Mukohyama, Phys. Lett. B **810** (2020) 135843.

[56] K. Aoki, M. A. Gorji and S. Mukohyama, J. Cosmol. Astropart. Phys. **09** (2020) 014.

[57] K. Aoki, M. A. Gorji, S. Mizuno and S. Mukohyama, J. Cosmol. Astropart. Phys. **01** (2021) 054.

[58] K. E. Jafarzade, M. K. Zangeneh and F. S. N. Lobo, J. Cosmol. Astropart. Phys. **04** (2021) 008.

[59] D. G.Boulware, S. Deser and K. S. Stelle, Phys. Lett. B **168** (1986) 336.

[60] J. Z. Simon, Phys. Rev. D **41** (1990) 3720.

[61] M. Lu and M. B. Wise, Phys. Rev. D **47** (1993) R3095.

[62] R. C. Myers, Nucl. Phys. B **289** (1987) 701.

[63] C. Callan, R. C. Myers and M. J. Perry, Nucl. Phys. B **311** (1989) 673.

[64] S. Mignemi and N. Stewart, Phys. Rev. D **47** (1993) 5259.

[65] J. T. Wheeler, Nucl. Phys. B **268** (1986) 737.

[66] R. C. Myers and J. Z. Zimon, Phys. Rev. D **38** (1988) 2434.

[67] D. Kastor, S. Ray and J. Traschen, Class. Quantum Grav. **27** (2010) 235014.

[68] D. Kastor, S. Ray and J. Traschen, Class. Quantum Grav. **28** (2011) 195022.

[69] D. Wiltshire, Phys. Lett. B **169** (1986) 36.

[70] R. C. Myers and J. Z. Simon, Gen. Rel. Grav. **21** (1989) 761.

[71] D. Kastor and R. Mann, J. High Energy Phys. **04** (2006) 048.

[72] M. Ba$\tilde{n}$ados, C. Teitelboim and J. Zanelli, Phys. Rev. D **49** (1994) 975.

[73] J. Crisostomo, R. Troncoso and J. Zanelli, Phys. Rev. D **62** (2000) 084013.

[74] K. Meng and D. B. Yang, Phys. Lett. B **780** (2018) 363.

[75] R. G. Cai, K. S. Soh, Phys. Rev. D **59** (1999) 044013.

[76] X. M. Kuang, O. Miskovic, Phys. Rev. D **95** (2017) 046009.

[77] R. G. Cai, N. Ohta, Phys. Rev. D **74** (2006) 064001.

[78] M. Aiello, R. Ferraro and G. Giribet, Phys. Rev. D **70** (2004) 104014.

[79] O. Miskovic, R. Olea, Phys. Rev. D **77** (2008) 124048.

[80] W. G. Brenna, M. H. Dehghani and R. B. Mann, Phys. Rev. D **84** (2011) 024012.

[81] M. H. Dehghani and M. H. Vahidinia, Phys. Rev. D **84** (2011) 084044.

[82] P. G. S. Fernandes, Phys. Lett. B **805** (2020) 135468.

[83] R. A. Konoplya and A. Zhidenko, Phys. Rev. D **101** (2020) 084038.

[84] D. V. Singh, S. G. Ghosh and S. D. Maharaj, “Cloud of strings in 4D Einstein-Gauss-Bonnet black holes”, arXiv:2003.14136.

[85] S. G. Ghosh and S. D. Maharaj, Phys. Dark Univ. **30** (2020) 100687.

[86] S. G. Ghosh and R. Kumar, “Generating black holes in 4D Einstein-Gauss-Bonnet gravity”, arXiv:2003.12291.

[87] A. Kumar and S. G. Ghosh, “Hayward black holes in the novel 4D Einstein-Gauss-Bonnet gravity”, arXiv:2004.01131.

[88] A. Kumar and R. Kumar, “Bardeen black holes in the novel 4D Einstein-Gauss-Bonnet gravity” arXiv:2003.13104.

[89] S. W. Wei and Y. X. Liu, “Testing the nature of Gauss-Bonnet gravity by four-dimensional rotating black hole shadow”, arXiv:2003.07769.

[90] R. Kumar and S. G. Ghosh, J. Cosmol. Astropart. Phys. **2020** (2020) 053.

[91] A. N. Kumara, C. L. A. Rizwan, K. Hegde, M. S. Ali and K. M. Ajith, Anns. Phys. **434** (2021) 168599.

[92] S. A. H. Mansoori, Phys. Dark Univ. **31** (2021) 100776.

[93] G. A. Marks, F. Simovic, R. B. Mann “Phase Transitions in 4D Gauss-Bonnet-de Sitter Black Holes”, arXiv:2107.11352.

[94] R. A. Konoplya and A. F. Zinhailo, Eur. Phys. J. C **80** (2020) 1049.

[95] S. U. Islam, R. Kumar and S. G. Ghosh, J. Cosmol. Astropart. Phys. **09** (2020) 030.

[96] X. h. Jin, Y. x. Gao and D. j. Liu, “Strong gravitational lensing of a 4D Einstein-Gauss-Bonnet black hole in homogeneous plasma”, arXiv:2004.02261.

[97] A. K. Mishra, Gen. Relativ. Gravit. **52** (2020) 106.

[98] S. L. Li, P. Wu and H. Yu, “Stability of the Einstein static universe in 4D Gauss-Bonnet gravity” arXiv:2004.02080 [gr-qc].

[99] K. Jusufi, A. Banerjee and S. G. Ghosh, Eur. Phys. J. C **80** (2020) 698.

[100] P. Liu, C. Niu, X. Wang and C-Y Zhang, “Traversable thin-shell wormhole in the novel 4D Einstein-Gauss-Bonnet theory”, arXiv:2004.14267.

[101] D. D. Doneva and S. S. Yazadjiev, “Relativistic stars in 4D Einstein-Gauss-Bonnet gravity” arXiv:2003.10284.

[102] B. E. Panah and K. Jafarzade, Nucl. Phys. B **961** (2020) 115269.

[103] M. Guo and P. Li, Eur. Phys. J. C **80** (2020) 588.

[104] J. Ahmed and K. Saifullah, Eur. Phys. J. C **77** (2017) 885.

[105] J. Ahmed and K. Saifullah, Eur. Phys. J. C **78** (2018) 316.

[106] C. Zhang, P. Li and M. Guo, Eur. Phys. J. C **80** (2020) 874.

[107] C. Zhang, S. Zhang, P. Li and M. Guo, "Superradiance and stability of the novel 4D charged Einstein-Gauss-Bonnet black hole", arXiv:2004.03141.

[108] S. Yang, J. Wan, J. Chen, J. Yang and Y. Wang, Eur. Phys. J. C **80** (2020) 937.

[109] S. Wei and Y. Liu, Phys. Rev. D **101** (2020) 104018.

[110] Y. P. Zhang, S. Wei and Y. Liu, "Spinning test particle in four-dimensional Einstein-Gauss-Bonnet black hole", arXiv: 2003.10960.

[111] A. Aragon, R. Becar, P. Gonzalez and Y. Vasquez, "Perturbative and non-perturbative quasinormal modes of 4D Einstein-Gauss-Bonnet black holes", arXiv:2004.05632.

[112] P. G. S. Fernandes, P. Carrilho, T. Clifton and D. J. Mulryne, Phys. Rev. D **104** (2021) 044029.

[113] A. Casalino, A. Colleaux, M. Rinaldi and S. Vicentini, Phys. Dark Univ. **31** (2020) 100770.

[114] C. Liu, T. Zhu and Q. Wu, Chin. Phys. C **45** (2021) 015105.

[115] S. Odintsov and V. Oikonomou, Phys. Lett. B **805** (2020) 135437.

[116] J. Bonifacio, K. Hinterbichler and L. A. Johnson, Phys. Rev. D **102** (2020) 024029.

[117] F. W. Shu, Phys. Lett. B **811** (2020) 135907.

[118] M. H. Fard and H. R. Sepangi, Eur. Phys. Lett. **133** (2021) 50006.

[119] A. Casalino and L. Sebastiani, Phys. Dark Univ. **31** (2021) 100771.

[120] S. W. Hawking, Math. Phys. **43** (1975) 199.

[121] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15** (1977) 2752.

[122] J. D. Bekenstein, Lett. Nuovo Cim. **4** (1972) 737.

[123] J. D. Bekenstein, Phys. Rev. D **7** (1973) 2333.

[124] M. Visser, Phys. Rev. D **48** (1993) 583.

[125] T. Jacobson and R. C. Myers, Phys. Rev. Lett. **70** (1993) 3684.

[126] M. Visser, Phys. Rev. D **48** (1993) 5697.

[127] R. M. Wald, Phys. Rev. D **48** (1993) R3427.

[128] M. Visser and R. M. Wald, Phys. Rev. D **50** (1994) 846.

[129] G. W. Gibbons and C. A. R. Herdeiro, Phys. Rev. D **63** (2001) 064006.

[130] A. E. Shabad and V. V. Usov, Phys. Rev. D **83** (2011) 105006.

[131] V. I. Denisov, E. E. Dolgaya, V. A. Sokolov and I. P. Denisova, Phys. Rev. D **96** (2017) 036008.

[132] M. Born and L. Infeld, Proc. Roy. Soc. Lond. A **144** (1934) 425.

[133] M. Born and L. Infeld, Proc. Roy. Soc. Lond. A **143** (1934) 410.

[134] M. Born and L. Infeld, Proc. Roy. Soc. Lond. A **147** (1934) 522.

[135] W. Heisenburg and H. Euler, Z. Phys. **98** (1936) 714.

[136] H. H. Soleng, Phys. Rev. D **52** (1995) 6178.

[137] M. Hassaine and C. Martinez, Phys. Rev. D **75** (2007) 027502.

[138] M. Hassaine and C. Martinez, Class. Quantum Grav. **25** (2008) 195023.

[139] H. A. Gonzalez, M. Hassaine and C. Martinez, Phys. Rev. D **80** (2009) 104008.

[140] H. Maeda, M. Hassaine and C. Martinez, Phys. Rev. D **79** (2009) 044012.

[141] S. I. Kruglov, Phys. Lett A **379** (2015) 623.

[142] S. I. Kruglov, Ann. Phys. (Belin) **529** (2017) 1700073.

[143] S. H. Hendi, J. High Energy Phys. **03** (2012) 065.

[144] S. H. Hendi and A. Sheykhi, Phys. Rev. D **88** (2013) 044044.

[145] S. I. Kruglov, Ann. Phys. **378** (2017) 59.

[146] S. I. Kruglov, Ann. Phys. **527** (2015) 397.

[147] I. Gullu and S. H. Mazharimousavi, "Double-Logarithmic electrodynamics", arXiv:2009.08665.

[148] M. Banados and P. G. Ferreira, Phys. Rev. Lett. **105** (2010) 011101.

[149] M. Alishahiha, E. Silverstein and D. Tong, Phys. Rev. D **70** (2004) 123505.

[150] E. Elizald, J. E. Lidsey, S. Nojiri and S. D. Odintsov, Phys. Lett. B **574** (2003) 1.

[151] C. Lai, Q. Pan, J. Jing and Y. Wang, Phys. Lett. B **49** (2015) 437.

[152] W. Yao and J. Jing, J. High Energy Phys. **1405** (2014) 058.

[153] B. Hoffmann, Phys. Rev. **47** (1935) 877.

[154] S. I. Kruglov, Phys. Rev. D **92** (2015) 123523.

[155] S. I. Kruglov, Phys. Rev. D **94** (2016) 044026.

[156] H. Maeda, M. Hassaine and C. Martinez, Phys. Rev. D **79** (2009) 044012.

[157] A. Bazrafshan, M. Ghanaatian, F. Naeimipour and A. Khajeh, Phys. Rev. D **100** (2019) 064062.

[158] M. Ghanaatian, Gen. Relativ. Gravit. **47** (2015) 105.

[159] S. I. Kruglov, Ann. Phys. **428** (2021) 168449.

[160] S. I. Kruglov, Eur. Phys. Lett. **133** (2021) 69001.

[161] E. Ayon-Beato and A. Garcia, Phys. Rev. Lett. **80** (1998) 5056.

[162] E. Ayon-Beato and A. Garcia, Phys. Lett. B **493** (2000) 149.

[163] I. Bandos, K. Lechner, D. Sorokin and P. Townsend, "A non-linear duality-invariant conformal extension of Maxwell's equations" arXiv:2007.09092.

[164] B. P. Kosyakov, Phys. Lett. B **810** (2020) 135840.

[165] Z. Amirabi and S. H. Mazharimousavi, Eur. Phys. J. C **81** (2021) 207.

[166] I. Bandos, K. Lechner, D. Sorokin and P. K. Townsend, J. High Energy Phys. **03** (2021) 022.

[167] D. Flores-Alfonso, B. A. González-Morales, R. Linares and M. Maceda, Phys. Lett. B **812** (2021) 136011.

[168] A. B. Bordo, D. Kubizňák and T. R. Perche, Phys. Lett. B **817** (2021) 136312.

[169] S. I. Kruglov, Phys. Lett. B **822** (2021) 136633.

[170] N. Okuyama and K. I. Maeda, Phys. Rev. D **67** (2003) 104012.

[171] G. W. Gibbons and P. K. Townsend, Class. Quantum Grav. **23** (2006) 4873.

[172] S. H. Mazharimousavi and M. Halilsoy, Phys. Lett. B **659** (2008) 471.

[173] P. B. Yasskin, Phys. Rev. D **12** (1975) 2212.

[174] S. H. Mazharimousavi and M. Halilsoy, Phys. Rev. D **76** (2007) 087501.

[175] S. H. Mazharimousavi and M. Halilsoy, J. Cosmol. Astropart. Phys. **2008** (2008) 12.

[176] A. B. Balakin, J. P. Lemos and A. E. Zayats, Phys. Rev. D **93** (2016) 024008.

[177] K. Meng, D. Yang and Z. Hu, Advances in High Energy Phys. **2017** (2017) 2038202.

[178] S. H. Mazharimousavi and M. Halilsoy, Phys. Lett. B **681** (2009) 190.

[179] K. S. Stelle, Phys. Rev. D **16** (1977) 953.

[180] T. Biswas, E. Gerwick, T. Koivisto and A. Mazumdar, Phys. Rev. Lett. **108** (2012) 031101.

[181] L. Modesto and L. Rachwal, Nucl. Phys. D **889** (2014) 228.

[182] S. Talaganis, T. Biswas and A. Mazumdar, Class. Quant. Grav. **32** (2015) 215017.

[183] E. T. Tomboulis, "Nonlocal and quasi-local field theories," (2015).

[184] E. T. Tomboulis, Mod. Phys. Lett. A **30** (2015) 1540005.

[185] E. Spallucci, A. Smailagic and P. Nicolini, Phys. Rev. D **73** (2006) 084004.

[186] T. Biswas, T. Koivisto and A. Mazumdar, JCAP **1011** (2010) 008.

[187] L. Modesto, W. M. John and P. Nicolini, Phys. Lett. B **695** (2011) 397.

[188] S. Hossenfelder, L. Modesto and I. P. Schwarz, Phys. Rev. D **81** (2010) 044036.

[189] A. Conroy, A. Mazumdar and A. Teimouri, Phys. Rev. Lett. **114** (2015) 201101.

[190] M. Markov, JETP Letters **36** (1982) 265.

[191] M. Markov, Ann. Phys. **155** (1984) 333.

[192] V. P. Frolov, Phys. Rev. D **94** (2016) 104056.

[193] V. P. Frolov and G. Vilkovisky, Phys. Lett. B **106** (1981) 307.

[194] A. Borde, Phys. Rev. D **55** (1997) 7615.

[195] S. A. Hayward, Phys. Rev. Lett. **96** (2006) 031103.

[196] V. P. Frolov, J. High Energy Phys. **05** (2014) 049.

[197] V. P. Frolov and A. Zelnikov, Phys. Rev. D **95** (2017) 044042.

[198] A. Ali and K. Saifullah, Phys. Lett. B **792** (2019) 276.

[199] S. W. Hawking, Commun. Math. Phys. **43** (1975) 199.

[200] U. A. Gillani and K. Saifullah, Phys. Lett. B **699** (2011) 15.

[201] M. Rehman and K. Saifullah, J. Cosmol. Astropart. Phys. **03** (2011) 001.

[202] S. M. Christensen and S. A. Fulling, Phys. Rev. D **15** (1977) 2088.

[203] O. Gurtug, S. H. Mazharimousavi and M. Halilsoy, Phys. Rev. D **85** (2012) 104004.

[204] M. Cataldo, N. Cruz, S. D. Campo and A. Garcia, Phys. Lett. B **484** (2000) 154.

[205] J. D. Brown, J. Creighton and R. B. Mann, Phys. Rev. D **50** (1994) 6394.

[206] A. Ali and K. Saifullah, Eur. Phys. J. C **82** (2022) 131.

[207] S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Spacetime, Cambridge University Press, Cambridge, (1973).

[208] W. G. Unruh, Phys. Rev. D **14** (1976) 3251.

[209] H.A. Gonzalez, M. Hassaine and C. Martinez, Phys. Rev. D **80** (2009) 104008.

[210] K. C. K. Chan and R. B. Mann, Phys. Rev. D **50** (1994) 6385.

[211] S. H. Mazharimousavi, M. Halilsoy and Z. Amirabi, Gen. Relativ. Gravit. **42** (2010) 261.

[212] A. Ali and K. Saifullah, J. Cosmol. Astropart. Phys. **10** (2021) 058.

[213] X. Qian Li, B. Chen and L. Xing, Eur. Phys. J. Plus **135** (2020) 175.

[214] T. Torii and H. Maeda, Phys. Rev. D **71** (2005) 124002.

[215] T. Torii and H. Maeda, Phys. Rev. D **72** (2005) 064007.

[216] A. Ali and K. Saifullah, Phys. Rev. D **99** (2019) 124052.

[217] D. G. Boulware and S. Deser, Phys. Rev. Lett **55** (1985) 2656.

[218] B. Zwiebach, Phys. Lett. B **156** (1985) 315.

[219] J. T. Wheeler, Nucl. Phys. B **268** (1986) 737; Nucl. Phys. B **273** (1986) 732.

[220] R. Aros, R. Troncoso and J. Zanelli, Phys. Rev. D **63** (2001) 084015.

[221] R. C. Myers and J. Z. Simon, Phys. Rev. D **38** (1988) 2434.

[222] R. G. Cai, Phys. Lett. B **582** (2004) 237.

[223] H. S. Liu and H. Lu, Phys. Lett. B **730** (2014) 267.

[224] H. Lu, C. N. Pope and Q. Wen, J. High Energy Phys. **1503** (2015) 165.

[225] H. S. Liu, H. Lu and C. N. Pope, J. High Energy Phys. **1406** (2014) 109.

[226] Z. Y. Fan and H. Lu, J. High Energy Phys. **1502** (2015) 013.

[227] Z. Y. Fan and H. Lu, Phys. Rev. D **91** (2015) 064009.

[228] H. S. Liu and H. Lu, J. High Energy Phys. **1412** (2014) 071.

[229] J. Oliva and S. Ray, Class. Quantum Grav. **29** (2012) 205008.

[230] K. Meng, Phys. Lett. B **784** (2018) 56.

[231] E. Radu, C. Stelea and D. H. Tchrakian, Phys. Rev. D **73** (2006) 084015.

[232] J. Brown and J. York, Phys. Rev. D **47** (1993) 1407.

[233] M. Cvetic and S. S. Gubser, J. High Energy Phys. **04** (1999) 024.

[234] M. M. Caldarelli, G. Cognola and D. Klemm, Class. Quantum Grav. **17** (2000) 399.

[235] S. C. Davis, Phys. Rev. D **67** (2003) 024030.

[236] A. Bazrafshan, M. H. Dehghani and M. Ghanaatian, Phys. Rev. D **86** (2012) 104043.

[237] J. D. Brown, J. Creighton and R. B. Mann, Phys. Rev. D **50** (1994) 6394.

[238] S. Nojiri and S. D. Odintsov, Phys. Lett. B **444** (1998) 92.

[239] M. Henningson and K. Skenderis, J. High Energy Phys. **07** (1998) 023.

[240] V. Balasubramanian and P. Kraus, Commun. Math. Phys. **208** (1999) 413.

[241] A. Ali and K. Saifullah, Ann. Phys. **446** (2022) 169094.

[242] S. I. Kruglov, Symmetry **13** (2021) 204.

[243] S. I. Kruglov, Symmetry **13** (2021) 944.

[244] K. Yang, B. M. Gu, S. W. Wei and Y. X. Liu, Eur. Phys. J. C **80** (2020) 662.

[245] A. Ali and K. Saifullah, Ann. Phys. **437** (2022) 168726.

[246] H. Lu and Y. Pang, "Horndeski gravity as D limit of Gauss-Bonnet", arXiv:2003.11552.

[247] S. Wei and Y. Liu, Phys. Rev. D **101** (2020) 104018.

[248] R. G. Cai, L. M. Cao and N. Ohta, J. High Energy Phys. **04** (2010) 082.

[249] C. Teitelboim and J. Zanelli, Class. Quantum Grav. **4** (1987) L125.

[250] A. Ali and K. Saifullah, Eur. Phys. J. C **82** (2022) 408.

[251] J. M. Toledo and V. B. Bezerra, Eur. Phys. J. C **79** (2019) 117.

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